Geodesics in the Heisenberg group

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Objective:
The goal is to obtain the geodesic equations in the Heisenberg group. These will allow to map surfaces such that each point on the surface is the end of a geodesic of constant length.

Heisenberg group and its operation:

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$ which is the Heisenberg group

Now, we will define the following function ϕ as follows:

$$\phi : \mathbb{R}^3 \to H$$

$$(x_1, x_2, x_3) \mapsto \begin{pmatrix} 1 & x_1 & x_3 + \frac{1}{2}x_1x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

One can show that

$$\phi(x,y,z)\phi(a,b,c) = \phi(x+a, y+b, z+c+(xb-ya)/2)$$

We will multiply two vectors in $\mathbb{R}^3$ as follows:
$(x,y,z)\odot(a,b,c) = (x+a, y+b, z+c+(xb-ya)/2)$

The function ϕ was obtained by taking the exponential map of the matrix $\begin{bmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{bmatrix}$

In order to obtain geodesics, we need to define a metric on the Heisenberg group. We will first define two maps which will allow us to do so.

Let us define the following functions:

$$\mu_x : H \to H$$
$$y \mapsto x \odot y$$

Now, let γ be a curve in H with $\gamma(0)$ =y and $\dot{\gamma}(0) = v \in T_yH$. We define the function $(\mu_x)_*$ as follows:

$$(\mu_x)_* : T_yH \to T_{x\odot y}H$$
$$(\mu_x)_*(\gamma) = \frac{d}{dt}\big(x \odot \gamma(t)\big)$$

The metric defined in H will allow us to measure lengths. Using the previous maps we define our metric as follows:

Let $v, w \in T_yH$, then

$$g_y(v,w) = \left\langle \left(\mu_{y^{-1}}\right)_*(v) \middle| \left(\mu_{y^{-1}}\right)_*(w) \right\rangle$$

Recall that in $\mathbb{R}^n$ the length of a curve γ is given by $l(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt$. On an arbitrary manifold M with a metric :

$$l(\gamma) = \int_a^b g_{\gamma(t)}\big(\dot{\gamma}(t), \dot{\gamma}(t)\big)^{1/2} dt$$

Geodesics:

Geodesics are defined to be locally the shortest path between two points. Hence, in order to obtain geodesics in H, we must minimize the formula for the length of a curve. Using techniques of the calculation of variations, we obtain the geodesic equation given by:

$$\ddot{\gamma}_i(t) + \Gamma^i_{jk}\dot{y}^j(t)\dot{y}^k(t) = 0$$

Where $\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\partial_j g_{kl} - \partial_l g_{jk} + \partial_k g_{lj}\right)$

With $g_y(\partial_i, \partial_j) = g_{ij}(y)$

and$g^{ij}(y)g_{jk}(y) = \delta^i_k$

The geodesic equation can be solved in H using the metric that we defined previously. We must also construct left invariant vector fields as follows:

$$\begin{aligned} X_1 &= \tfrac{d}{dt}\big((x,y,z)\odot(t,0,0)\big)_{t=0} \\ &= \frac{d}{dt}\left(x+t, y, z-\frac{yt}{2}\right)_{t=0} \\ &= \left(1,0,-\frac{y}{2}\right) \\ &= \partial_1 - \frac{y}{2}\partial_3 \end{aligned}$$

$$\begin{aligned} X_2 &= \tfrac{d}{dt}\big((x,y,z)\odot(0,t,0)\big)_{t=0} \\ &= \frac{d}{dt}\left(x, y+t, z+\frac{xt}{2}\right)_{t=0} \\ &= \left(0,1,\frac{x}{2}\right) \\ &= \partial_2 + \frac{x}{2}\partial_3 \end{aligned}$$

$$\begin{aligned} X_3 &= \tfrac{d}{dt}\big((x,y,z)\odot(0,0,t)\big)_{t=0} \\ &= \frac{d}{dt}(x,y,z+t)_{t=0} \\ &= (0,0,1) \\ &= \partial_3 \end{aligned}$$

After differentiating γ with respect to t, we get:
$\gamma' = x'\partial_1 + y'\partial_2 + z'\partial_3$ so that we have:

$$\gamma' = x'X_1 + y'X_2 + \left(\frac{x'y}{2} - \frac{y'x}{2} + z'\right)X_3$$

With $\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^i}\right) := \Gamma^i_{jk}\frac{\partial}{\partial x^k}$ we have that $\nabla_{\gamma'}(\gamma') = 0$ which is equivalent to the geodesic equation.

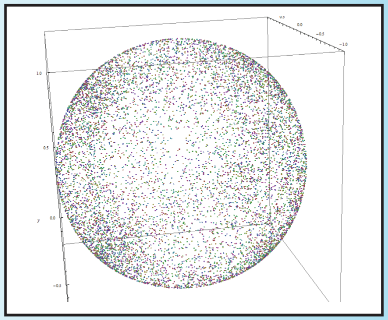
Geodesic surfaces:

Solving the geodesic equation, we obtain the following set of differential equations:

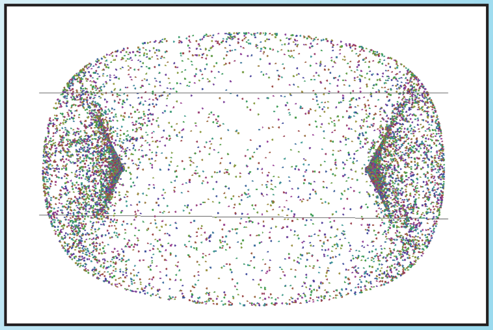
$$x'' + y'z_0' = 0$$
$$x'' + (y_0' + xz_0')z_0' = 0$$
$$x'' + y_0'z_0' + xz_0'^{\,2} = 0$$
$$x'' + xz_0'^{\,2} + y_0'z_0' = 0$$

The following illustrations are surfaces traced by the ends of geodesics of equal length. We let L denote the length of the geodesic curves.

L = 1



L = 7



L = 25