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LA THÈSE A ÉTÉ
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NUMERICAL ANALYSIS OF IN-VIVO DIELECTRIC SENSORS

by

Gregory B. Gajda

A thesis submitted to the School of Graduate Studies and Research
of the University of Ottawa
as partial fulfillment of the
requirements for the M.A.Sc. degree
in Electrical Engineering

Ottawa, Ontario, 1982

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ABSTRACT

Numerical methods are applied in the analysis of coaxial structures used as sensors for in-vivo permittivity studies of biological substances. The methods used for the solution of the resulting static conductor-dielectric problems are the Finite Element Method and the Method of Moments applied to a pair of coupled integral equations. A linear model which relates the sample permittivity to the fringe-field capacitance of the sensor is discussed and values of the model parameters are calculated for different types of commonly used sensors. In addition, a method of increasing the sensor capacitance for use at lower frequencies, is investigated.

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Chapter I

INTRODUCTION

The need for accurate determination of the dielectric properties of biological materials is evidenced by the increasing use of microwave energy in the fields of medicine, biology and the agricultural and food industries. The dielectric properties determine the absorption and propagation of electromagnetic energy in a material and are in general, functions of frequency and temperature.

Medical applications of microwave energy such as hyperthermia treatment of cancer [1] requires a knowledge of the distribution of permittivity of healthy and diseased tissue. This knowledge is used to predict heating patterns in the tissue due to specific applicator configurations. Often, preferential heating of tumors is achieved due to differences in the dielectric properties, particularly the losses, between healthy and diseased tissue [2].

Dielectric measurements may have the potential to be used as investigative or diagnostic tools in the study of physiological processes such as organ activity or inactivity and disease. A correlation between the physiological state and the dielectric properties would provide a method of monitoring changes in the state of an organ [11].

Accurate dielectric information is also required for the design of sensors for the imaging of soft tissue and impedance monitoring applications using microwaves [4].

In the field of radiofrequency and microwave dosimetry, knowledge of the spatial dependence of the permittivity is necessary for the calculation of the local and average specific absorption rate (SAR) in the tissues [5]. This information is helpful in setting safe radiofrequency and microwave exposure standards.

Dielectric information also finds use in studies of water and molecular structure in biological materials [6]. Since the complex permittivity of water is relatively large at microwave frequencies, the dielectric properties of tissue are greatly influenced by the water content. Also, dielectric relaxation mechanisms which predominate at various frequencies provide information on molecular size and structure [6].

Permittivity measurements are used to monitor local or spatial distribution of moisture content for industrial applications. Also, knowledge of the dielectric properties of food is necessary for the estimation of energy deposition from microwave thawing and cooking in the food processing industry [7].

For these applications, a dielectric measurement method is required which is capable of: 1) performing in-vivo measurements on small sample volumes; 2) performing measure-

ments non-destructively to allow the normal functioning of a given organ; and 3) allowing temperature control of the sample.

A method which seems to satisfy these requirements is the one which utilizes an open-ended coaxial line as a sensor [8,9] (Fig. 1).

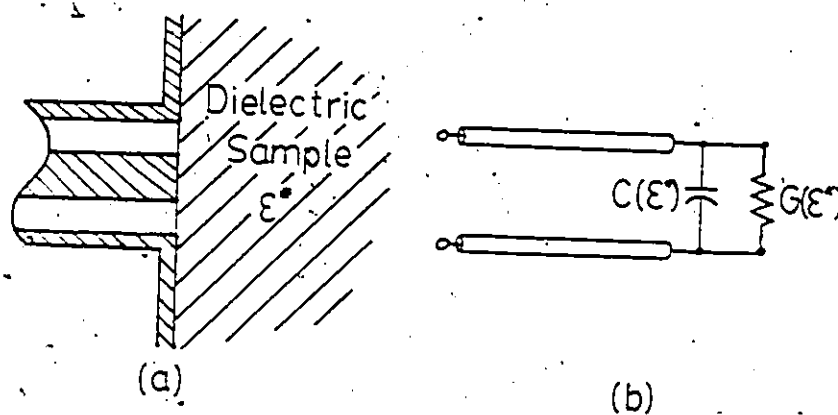


Figure 1: (a) Open-ended coaxial line, (b) generalized equivalent circuit.

In this method, the sample permittivity ϵ^* may be determined from measurement of the input reflection coefficient of the sensor. Some advantages of using an open-ended coaxial structure as a sensor over other waveguiding structures are: a wide frequency range (typically from 100 MHz to several GHz); the ability to use time domain, frequency domain or resonant measurement techniques; and ease of fabrication.

Other types of open-ended coaxial structures have also been used for dielectric measurements on biological substances including short monopoles [10,11] and coaxial lines open-ended into cut-off circular waveguide [12].

1.1 EQUIVALENT CIRCUIT OF AN OPEN-ENDED COAXIAL SENSOR

In general, the equivalent circuit of a one-port coaxial structure may be represented by a shunt lumped admittance terminating a transmission line with a characteristic admittance Y_0 . It is assumed that the coaxial line operates in the TEM mode and does not support propagation of higher-order modes.

For the configuration consisting of an open-ended coaxial line in contact with a dielectric sample, the equivalent circuit consists of the parallel combination of a capacitance $C(\epsilon^*)$ and a conductance $G(\epsilon^*)$ as shown in Fig. 1. Both the capacitance and the conductance are functions of the sample permittivity and frequency in general. The former represents the storage of energy in the fringing fields at the end of the line and the latter represents losses due to radiation. The conductance due to losses in the dielectric is included in $C(\epsilon^*)$.

For the purposes of this work a linear relationship between the terminal capacitance $C(\epsilon^*)$ and the sample permittivity ϵ^* is assumed and the conductance is neglected. The input admittance of the sensor may be written as [8,10]:

$$\begin{aligned} Y &= j\omega C(\epsilon^*) \\ &= j\omega(C_f + \epsilon^*C_0) \end{aligned} \quad (1.1)$$

where ω represents the angular frequency, $j = \sqrt{-1}$ and $\epsilon^* = \epsilon' - j\epsilon''$ is the complex relative permittivity of the test dielectric.

The capacitance C_0 may be interpreted as the external capacitance whose value, when multiplied by ϵ^* , represents the storage of energy in the dielectric sample. The internal capacitance C_f represents the storage of energy in fringing fields not affected by the dielectric. If, for the general case, $C(\epsilon^*)$ does not vary linearly with ϵ^* , then C_f and C_0 will both be functions of the sample permittivity. However, if a range of values of ϵ^* exists where C_f and C_0 are relatively constant, the linear model will be a good approximation.

When no dielectric sample is present the measured capacitance of the sensor, defined as the total capacitance C_T , will equal the sum of C_f and C_0 (i.e. $C(\epsilon_0) = C_T = C_f + C_0$, where ϵ_0 is the permittivity of free space).

The scattering parameter S_{11} (or the input reflection coefficient) of the sensor is given in terms of the input admittance by:

$$S_{11} \exp(j\theta_{11}) = \frac{Y_0 - Y}{Y_0 + Y} \quad (1.2)$$

where S_{11} represents the magnitude of the reflection coefficient and θ_{11} represents the phase of the reflection coefficient referenced to the plane of the open circuit of the sensor. Upon substitution of (1.1) into (1.2) and after rearranging, the real and imaginary parts of the permittivity, denoted by the relative dielectric constant ϵ' and the loss factor ϵ'' respectively, can be written as [10]:

$$\epsilon' = \frac{2 S_{11} \sin(-\theta_{11})}{\omega C_0 Z_0 [1 + 2 S_{11} \cos(-\theta_{11}) + S_{11}^2]} - \frac{C_f}{C_0} \quad (1.3)$$

$$\epsilon'' = \frac{1 - S_{11}^2}{\omega C_0 Z_0 [1 + 2 S_{11} \cos(-\theta_{11}) + S_{11}^2]} \quad (1.4)$$

where $Z_0 = 1/Y_0$ is the characteristic impedance of the transmission line.

The use of a linear model for the capacitance results in simple closed-form expressions for the real and imaginary parts of the permittivity in terms of the reflection coefficient of the sensor. They may be easily incorporated into an automatic network analyser routine to provide a fast and accurate dielectric measurement system. Notice from (1.3) and (1.4) that the external capacitance C_0 has the greatest influence in the measured values of ϵ' and ϵ'' while the va-

7
lue of C_f has a much lesser influence especially if
 $C_f/C_0 \ll \epsilon'$.

1.2 DESCRIPTION OF THE PROBLEM AND PURPOSE OF THE INVESTIGATION

In order to use (1.3) and (1.4) to calculate the relative dielectric constant and loss factor from the measured reflection coefficient, it is necessary to know the values of the two capacitances C_0 and C_f for a given sensor. An analytical expression for the total fringing capacitance C_T , defined as the sum of C_f and C_0 , has been derived [13,15] for the case of an open-ended coaxial line with an infinite groundplane (Fig. 2(a)). However, this expression does not indicate the relative values of C_f and C_0 , nor does it include the effects of a finite or no groundplane at all.

The purpose of this investigation is to use numerical methods which allow these two parameters, C_f and C_0 , to be calculated for the non-idealized cases of open-ended coaxial structures shown in Figs. 2(b) to (e).

The two numerical methods which were chosen to carry out this investigation are the Finite Element Method (FEM) and a technique based upon the Method of Moments (MOM). Since interest is mainly in the low-frequency or static capacitance

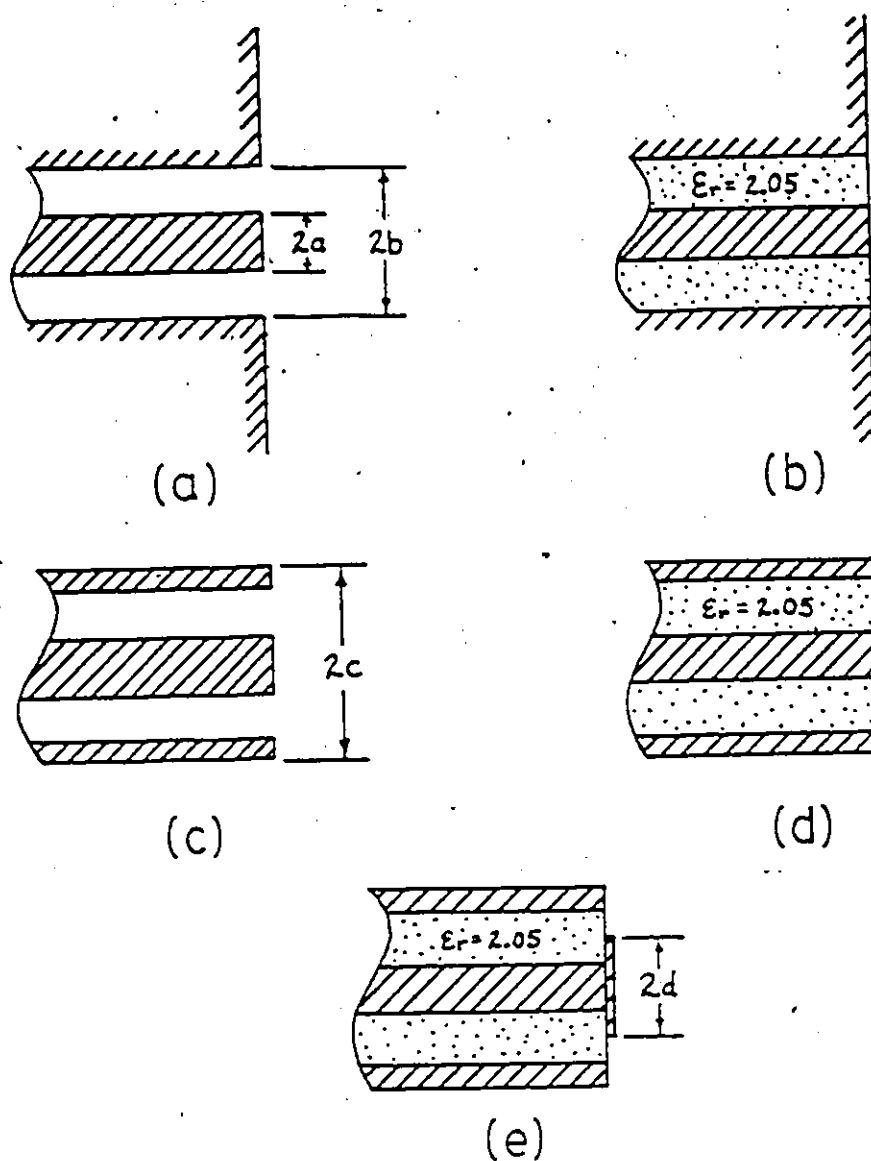


Figure 2: Five configurations of open-ended coaxial sensors.

of the sensors, it is required that these methods solve Laplace's equation in cylindrical coordinates with rotational symmetry, subject to the boundary conditions imposed by the conductors and dielectrics.

In addition to computing values for the two capacitances C_o and C_f , these methods should provide information on other quantities such as the depth of penetration of the fields away from the aperture. This information is important from the standpoint of determining how localized the measurement of permittivity is using these sensors. It must be stressed that since these methods provide a static analysis, only lossless dielectrics can be modelled.

The configurations of Figs. 2(a), (b) and (d) have been used in actual dielectric measurements on biological substances [9,10,11]. The configurations of Figs. 2(a) and (c) require a mica or mylar window placed over the line opening to prevent samples from penetrating into the line. This poses a problem when measuring biological substances possessing appreciable d.c. conductivities since the window insulates the conductors from the test substance. This problem is alleviated by use of the configurations of Figs. 2(b) and (d) where penetration into the line of the test substance is prevented by the presence of the teflon dielectric of the line. The use of teflon dielectric-filled lines also enables very small diameter sensors to be fabricated using commercially available semi-rigid coaxial line (potentially as small as 0.5-mm in diameter).

Sensors with larger capacitances are required for work at lower frequencies in order that the input admittance is com-

parable in magnitude to the characteristic admittance of the line. It has been shown by Stuchly et. al. [14] that this condition results in the smallest relative errors in δ' and δ'' when using (1.3) and (1.4). The configuration of Fig. 2(e) represents an attempt to increase the fringing capacitance of the sensor while maintaining a small diameter of the line. It is similar to the configuration of Fig. 2(d) except that a concentric conducting disk is placed over the centre conductor, partially covering the aperture.

1.3 STATE OF PRESENT KNOWLEDGE

An equation for the total capacitance C_T of the idealized configuration of open-ended coaxial line with an infinite groundplane and homogeneous air dielectric (Fig. 2(a)) is given by [13, 15]:

$$C_T = \frac{2 \epsilon_0}{k [\ln(b/a)]^2} \int_0^\pi [2 \text{Si}(k\sqrt{a^2 + b^2 - 2ab \cos \phi}) - \text{Si}(2ka \sin(\phi/2)) - \text{Si}(2kb \sin(\phi/2))] d\phi \quad (1.5)$$

where $\text{Si}(x)$ is the sine integral function, $k = 2\pi/\lambda$, λ is the wavelength and ϵ_0 is the permittivity of free space. Dimensions b and a represent the outer-conductor inner-radius and the inner-conductor outer-radius respectively.

The radiation conductance, G in the equivalent circuit of Fig. 1 may be calculated from [13,15]:

$$G = \frac{2\pi}{\eta_p [\ln(b/a)]^2} \int_0^{\pi/2} [J_0(ka \sin\theta) - J_0(kb \sin\theta)]^2 \frac{d\theta}{\sin\theta} \quad (1.6)$$

where $J_0(x)$ is the Bessel function of the first kind of order zero and η_p is the intrinsic impedance of free space.

Equations (1.5) and (1.6) were obtained from a variational expression for the input admittance of a coaxial waveguide opening into an infinite groundplane and half-space. The variational expression was written in terms of an integral equation for the unknown electric field distribution in the aperture and was solved using the principal (TEM) mode of the coaxial guide to represent the aperture field. Thus the solution given by (1.5) and (1.6) does not take into account the presence of higher-order modes in the aperture.

Computations of G versus frequency using (1.6) have shown that the conductance in free space is negligible over a wide range of frequencies [8]. The effect of the dielectric sample on the radiation conductance is treated by Stuchly et. al. [16] who shows that the radiation conductance may be assumed equal to zero, except at higher frequencies and for high permittivities. Even for cases where the assumption of zero radiation is no longer valid, ϵ^* may be computed from (1.3) and (1.4) and later, correction factors to take into account the non-zero conductance, may be applied.

Calculations of (1.5) have been performed by Galejs [15] and Gajda [8] over a wide range of frequencies. Figure 3 shows two plots of the normalized capacitance versus normalized frequency, one for 50- Ω air lines ($b/a=2.303$) and the other for 50- Ω teflon-dielectric lines ($b/a=3.268$) opening into a teflon-dielectric half-space. These values were computed from (1.5) and apply to lines of any size. The actual value of C_T for a particular line is obtained by multiplying the normalized capacitance $C_T/\epsilon_0(b-a)$ by the aperture dimension $(b-a)$ and the free space permittivity ϵ_0 (8.842×10^{-12} farads/m).

From Fig. 3 it is apparent that the capacitance is constant at low frequencies and increases slowly with frequency beyond a certain point. The point at which the capacitance begins to increase with frequency (say 1 %) is approximately 1 GHz for 14-mm, 50- Ω air line and higher for smaller diameter lines.

An analytical expression relating the low-frequency or static capacitance to the dimensions of the line is given by [13]:

$$\lim_{\lambda \rightarrow \infty} C_T = \frac{8\epsilon_0(b+a)}{[\ln(b/a)]^2} \left[E\left(\frac{2\sqrt{ab}}{b+a}\right) - 1 \right] \quad (1.7)$$

where $E(x)$ is the complete elliptic integral of the second kind.

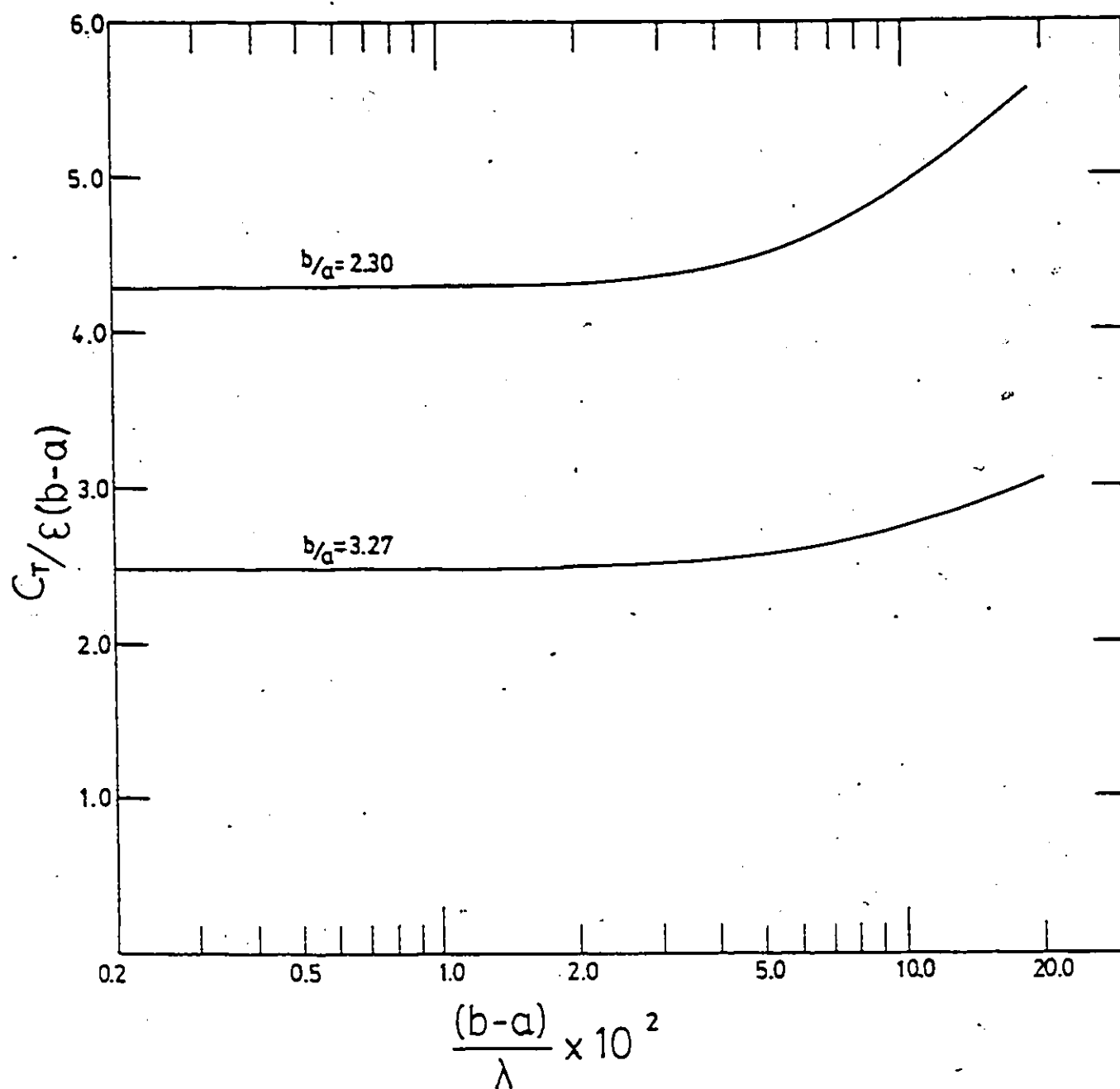


Figure 3: Normalized capacitance of open-ended coaxial line versus normalized frequency computed from (1.5) .

Equation (1.7) is approximately valid for $(b-a)/\lambda < 2 \times 10^{-2}$, based on a 1 % increase in capacitance with frequency and may be obtained from (1.5) by noting that for $x \ll 1$, the sine integral function may be approximated by its truncated power series $Si(x) \approx x$. A plot of the normalized capacitance $C_T/\epsilon_0(b-a)$ versus the ratio of outer to inner-conductor radii b/a is shown in Fig. 4.

Limitations of (1.7) for the computation of the capacitance of practical coaxial sensors are: 1) the presence of an infinite groundplane in the plane of the aperture is assumed; 2) its derivation does not take into account the effect of dielectric discontinuities in the aperture plane; and 3) it gives no information on how the total capacitance is divided between C_0 and C_f .

A similar formulation of the problem from which (1.5) was derived, is given by Mosig et. al. [17], this time taking into account the discontinuity of the dielectric in the aperture plane and the presence of higher-order modes. An integral equation for the unknown aperture field was derived by matching at the aperture, the transverse magnetic fields in the half-space (written in terms of the aperture field) to the fields in the coaxial guide. The fields in the aperture and in the line were expanded in a series consisting of an incident and reflected TEM mode plus higher-order TM modes, each weighted by an unknown reflection coefficient.

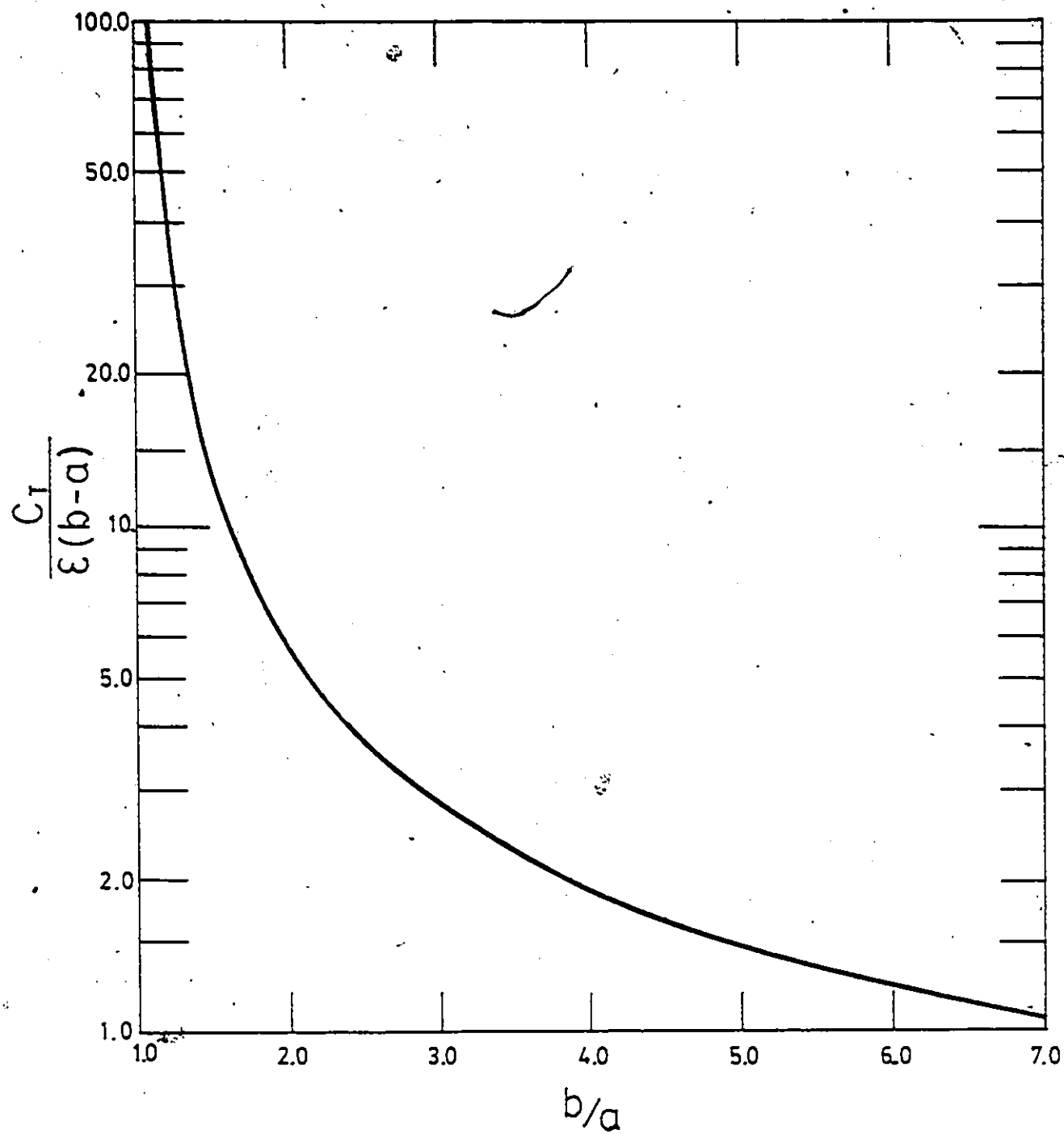


Figure 4: Static or low-frequency capacitance versus outer to inner-conductor radii b/a computed from (1.7).

The resulting expression was enforced at a discrete number of points in the aperture, corresponding to the number of modes used in the series, and the resulting linear system of equations was solved for the principal mode (TEM) reflection coefficient.

Published results from this analysis, for the idealized sensor configuration with a homogeneous dielectric, differ from values of reflection coefficient computed using (1.5) and (1.6) at high frequencies due to the inclusion of higher-order modes in the analysis. However, for low frequencies, the results from both methods are in agreement [17].

As a dielectric measurement technique, the above approach is limited since extensive nomograms of relative dielectric constant and loss factor versus reflection coefficient are required at each measurement frequency. Also an idealized sensor configuration is assumed in order to provide well defined boundaries in the analysis.

A measurement of the relative values of C_0 and C_f for a 14-mm, 50 Ω air line with a groundplane was attempted by Gajda [8]. Measurements of the admittance of the sensor with methanol and water as test dielectrics were compared with measurements of the sensor in air at frequencies ranging from 200 MHz to 500 MHz. Since the relative permittivities of the test dielectrics were known at each frequency, the ratio of C_f to C_0 could be computed using (1.1) and the measured admittances. It was found that for both the water and methanol, the ratio C_f/C_0 was approximately 0.20.

1.4 CONCLUDING REMARKS

In conclusion, the use of open-ended coaxial sensors for dielectric measurement provides a fast and efficient means of making non-destructive or in-vivo measurements. The sensors can be fabricated easily from commercially available coaxial lines and can be used over a wide range of frequencies.

In order to take advantage of the simple relationships between the permittivity and the measured reflection coefficient (1.3) and (1.4), it is necessary to accurately know the parameters of the sensor, either through measurement or calculation. In this work, the static or low-frequency parameters of the different sensor configurations depicted in Fig. 2 are calculated using two different numerical methods. The static parameters give rise to the linear circuit model (1.1), from which the effect of the dielectric sample on the input admittance (reflection coefficient) can be easily determined.

Chapter II

METHODS OF SOLUTION

This chapter gives a brief overview of the some of the methods used for the solution of linear operator equations such as those encountered in electrostatic field problems. The principles behind variational methods and their application to differential and integral operators is reviewed and a short discussion of moment methods and their relation to variational methods is given.

Two well known methods for the solution of conductor-dielectric problems are also described. They are the Finite Element variational method for differential operators and a version of the Method of Moments applied to an integral equation formulation of the problem.

2.1 DEFINITIONS

Given a deterministic field problem of the form:

$$Lu = f \quad (2.1)$$

where L is a linear operator (L can be any one of integral, differential or difference operators) and f is a known forc-

ing function, it is desired to determine the unknown function u . (Both f and u are functions of position.)

Accompanying the operator L is a set of boundary conditions which the unknown functions must satisfy. The set of all possible functions satisfying these boundary conditions is called the domain of L and the functions f resulting from the operation are called the range of L [18].

An inner product is usually defined for the problem (2.1), where the process of taking the inner product of two functions results in a scalar. The inner product is defined here as the integral of the product of two functions over the region of solution and is analogous to the Euclidean vector "dot" product. In the mathematical notation of function spaces the inner product of u and f is expressed as:

$$\langle u, f \rangle = \alpha = \int_{\Omega} u * f \, d\Omega \quad (2.2)$$

where the pair of pointed brackets denotes the inner product operation resulting in the scalar α . Note that throughout this discussion only real scalar functions are considered.

In electrostatic field problems the linear operator L may represent the Laplacian operator $-\nabla \cdot (\epsilon \nabla)$ while u and f represent the potential and volume charge distribution respectively. The inner product of u and f would be the integral of the potential times the charge distribution over the region in which the field exists, bounded by conductors and/or electric field lines.

Associated with each operator L is an adjoint operator L^a defined by:

$$\langle Lu, f \rangle = \langle u, L^a f \rangle \quad (2.3)$$

An operator is said to be self-adjoint if $\langle Lu, f \rangle = \langle u, Lf \rangle$ or $L^a = L$. For differential operators, self-adjointness is largely determined by the associated homogeneous boundary conditions [19] while for integral operators, self-adjointness is assured if the kernel of the integral equation is symmetric [20].

An operator is said to be positive-definite if: $\langle Lu, u \rangle > 0$ whenever the function u is not identically zero and $\langle Lu, u \rangle = 0$ if and only if u is identically zero. Positive-definiteness of the linear operator L insures that the solution of (2.1) is unique [19].

2.1.1 Boundary Conditions

Three types of boundary conditions generally occur. The first, called Dirichlet boundary conditions, exist wherever the value of the unknown function is specified, while the second type, called Neumann boundary conditions exist wherever the normal gradient of the unknown function is specified. The third type are called "mixed" boundary conditions and consist of a linear combination of Dirichlet and Neumann conditions.

In electrostatics, Dirichlet boundaries correspond to equipotential surfaces or conductors held at specified po-

tentials. Homogeneous Neumann boundaries correspond to lines or surfaces of the electric field since the normal derivative of the potential across an electric field line is equal to zero.

2.2 VARIATIONAL METHODS

Variational methods for finding approximate solutions of (2.1) make use of a functional (a mapping from a function space to the real numbers) which possesses certain properties at the solution point given by (2.1). These properties usually include stationarity and a minimum value of the functional [19]. Since the functional is normally given in the form of inner products, variational methods essentially consist of finding a function which minimizes a certain integral. In many cases the functional or integral to be minimized represents a fundamental physical quantity such as energy [19].

The quadratic functional pertaining to the deterministic problem (2.1) is given by:

$$F = \langle Lu, u \rangle - 2\langle u, f \rangle \quad (2.4)$$

If the operator is self-adjoint and positive-definite, the function u which minimizes (2.4) is also a solution of (2.1) or conversely a solution of (2.1) will cause the func-

tional to be a minimum. In mathematical terms, the problem (2.1) is said to be the "Euler equation" of the functional (2.4). Of course, trial functions must be in the domain of definition of L , i.e. they must satisfy the associated boundary conditions, otherwise smaller values of the functional may be found for functions which are not the solution of (2.1) [19].

If L is non-positive-definite, the functional will only be stationary about the solution point u and not necessarily minimal. For a positive-definite operator, an approximating procedure such as the Rayleigh-Ritz method, which uses a set of known basis functions along with a set of unknown variational parameters, will converge to the true solution as the number of variational parameters increase. Convergence of this procedure is not guaranteed if the operator is non-positive-definite [20].

If the operator is non-self-adjoint, stationarity of (2.4) cannot be proven, however stationarity of the dual functional given by:

$$F = \langle Lu, u^a \rangle - \langle f, u^a \rangle - \langle u, g \rangle \quad (2.4a)$$

can be proven about a solution point given by (2.1) and the adjoint problem:

$$L^a u^a = g \quad (2.1a)$$

where g is the adjoint inhomogeneity and u^* is the adjoint function which lies in the domain of the adjoint operator L^* [22,23].

2.2.1 Rayleigh-Ritz Procedure

In the Rayleigh-Ritz approximating procedure for locating the stationary point of a functional, the trial function u is expanded in a linear combination of basis functions:

$$u = \sum_{i=1}^n a_i u_i, \quad \underline{a}^T \underline{u} = \underline{u}^T \underline{a} \quad (2.5)$$

where each basis function u_i satisfies the boundary conditions and the amplitudes or variational parameters a_i are yet to be determined. The underscore notation in (2.5) is used to denote a column vector while the superscript T notation is used to denote the matrix transpose operation.

The trial function is inserted into the functional which now becomes an ordinary function of the n variational parameters:

$$F = F(a_1, a_2, \dots, a_n) = \underline{a}^T \langle L\underline{u}, \underline{u}^T \rangle \underline{a} - 2\underline{a}^T \langle \underline{u}, \underline{f} \rangle \quad (2.6)$$

At the stationary point, the partial derivative of the functional with respect to each of the variational parameters vanishes, i.e. $\partial F / \partial a_i = 0$, yielding the system of equations:

$$\langle L\underline{u}, \underline{u}^T \rangle \underline{a} = \langle \underline{u}, \underline{f} \rangle \quad (2.7)$$

This procedure yields an $n \times n$ set of simultaneous equations where the element in row "i" and column "j" on the left hand side is given by $\langle Lu_i, u_j \rangle$ and on the right hand side row by $\langle u_i, f \rangle$.

If the basis functions form an orthonormal set, all but the diagonal elements disappear on the left hand side and the Rayleigh-Fitz procedure reduces to the Fourier series method of solution [19].

2.2.2. Variational Methods for Differential Operators

The distribution of potential in a region is governed by the Poisson equation:

$$-\nabla \cdot (\epsilon \nabla \phi) = \rho \quad (2.8)$$

where ϕ represents the potential, ϵ the permittivity of the medium and ρ the volume charge density. All three are in general functions of position. The ∇ symbol denotes the gradient operation and the $\cdot$ denotes the vector dot product. In a source-free region bounded by conductors at specified potentials, $\rho = 0$ and Laplace's equation holds:

$$-\nabla \cdot (\epsilon \nabla \phi) = 0 \quad (2.9)$$

In either case the operator is the Laplacian operator and the functional may be written as [19,25]:

$$F = \int_{\Omega} \epsilon (\nabla \phi)^2 d\Omega - 2 \int_{\Omega} \phi \rho d\Omega \quad (2.10)$$

or for the case where $\rho = 0$ as:

$$F = \int_{\Omega} \epsilon (\nabla \phi)^2 d\Omega \quad (2.11)$$

where the integrations are taken over the entire region of solution.

All trial functions used in (2.10) or (2.11) must satisfy the Dirichlet conditions on the corresponding part of the boundary. While homogeneous Neumann conditions, i.e. $\partial \phi / \partial n = 0$, need not be imposed due to the nature of the Laplacian operator [19]. Homogeneous Neumann conditions are called "natural" boundary conditions for this operator and are satisfied automatically by minimizing the functionals (2.10) and (2.11).

At the solution point given by (2.8) or (2.9), the magnitude of the functional (2.10) or (2.11) is equal to twice the amount of stored electric energy in the system. In other words, the potential assumes a distribution in such a way that the stored electric energy is minimized.

2.2.3 Variational Methods for Integral Operators

For integral operators the unknown function is contained within the integrand of an integral equation. The Poisson or Laplace equation can be converted to an integral equation by the use of an appropriate Green's function and Green's identity [21].

The potential at an observation point $\bar{x}$ in $\bar{S}$ space due to a known charge distribution $\sigma(\bar{\xi})$ is given by:

$$\phi(\bar{x}) = \int_R \sigma(\bar{\xi}) G(\bar{x}|\bar{\xi}) d\bar{\xi} \quad (2.12)$$

where $\bar{\xi}$ represents a source point, $G(\bar{x}|\bar{\xi})$ is a Green's function and the integration is carried out over the entire charge distribution. The scalar potential Green's function represents the potential at $\bar{x}$ due to a unit point-charge located at $\bar{\xi}$.

If charges are considered to exist on surfaces having specified potentials, an integral equation for the unknown surface charge density $\sigma(s)$ may be written as:

$$\int_S \sigma(s') G(s|s') ds' = \phi(s) \quad (2.13)$$

where $\phi(s)$ is the known potential on the surface S .

Equation (2.13) is known as a Fredholm integral equation of the first kind and may be expressed in operator notation as:

$$L \sigma = \phi \quad (2.14)$$

Because the above integral operator is self-adjoint (due to the symmetry of the Green's function kernel) and positive-definite, the functional (2.4) may be used to find approximate solutions for $\sigma(s)$ with the assurance that they will converge as the degree of approximation increases and that the functional will be a minimum [20]. The functional in integral form is:

$$F = \iint_S \sigma(s) \sigma(s') G(s|s') ds' ds - 2 \int_S \sigma(s) \phi(s) ds \quad (2.15)$$

where the inner product is a surface integral over all charge.

As with the case of differential operators, the minimum value of the functional (2.15) equals twice the amount of stored electric energy in the system. This fact seems intuitively obvious since the same functional is used as for differential operators and the integral equation may be obtained from the partial differential equations (2.8) or (2.9) through the use of Green's identity and a Green's function.

2.2.3.1 Interface Condition

In the partial differential equation formulation, the spatial dependence of permittivity is incorporated directly into the differential equation and its associated functional. The integral equation formulation however, requires an extra equation to account for the distribution of polarization charge on the interface between different dielectric media.

On the interface between two differing media, the normal derivative of the potential on either side is given by [20,21]:

$$\frac{\partial \phi}{\partial n_1}(s) = \frac{\sigma(s)}{2} + \int_S \sigma(s') \frac{\partial G}{\partial n}(s|s') ds' \quad (2.16)$$

$$\frac{\partial \phi}{\partial n_2}(s) = -\frac{\sigma(s)}{2} + \int_S \sigma(s') \frac{\partial G}{\partial n}(s|s') ds' \quad (2.17)$$

where $\sigma(s)$ is the polarization charge density and $\partial G(s|s')/\partial n$ is the normal derivative of the Green's function across the interface. The normals n_1 and n_2 point towards and away from the interface respectively, as shown in Figure 5.

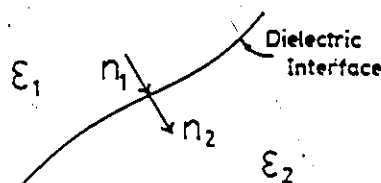


Figure 5: Orientation of normal vectors on the interface between different dielectric media.

Subtracting (2.16) from (2.17) yields the polarization charge $\sigma(s)$:

$$\frac{\partial \phi}{\partial n_1}(s) - \frac{\partial \phi}{\partial n_2}(s) = \sigma(s) \quad (2.18)$$

which is equal to the difference between the normal components of electric polarization $\vec{P}$ across the interface i.e. $\sigma(s) = (\vec{P}_1(s) - \vec{P}_2(s)) \cdot \vec{n}$. Since the polarization equals the ef-

fective dipole moment per unit volume in a dielectric, the polarization charge density may be interpreted as the difference in the dipole moment per unit volume between the two media at the interface.

The integral equation governing the distribution of charge at the interface may be obtained by substituting (2.16) and (2.17) into the continuity condition for the electric flux density, $\epsilon_1 \partial \phi / \partial n_1 = \epsilon_2 \partial \phi / \partial n_2$. It is given here as:

$$\frac{\epsilon_1 + \epsilon_2}{2} \sigma(s) + (\epsilon_1 - \epsilon_2) \int_S \sigma(s') \frac{\partial G}{\partial n}(s|s') ds' = 0 \quad (2.19)$$

where ϵ_1 and ϵ_2 are the permittivities on either side of the interface and the normal vector points from region 1 to region 2 as in Fig. 5.

Thus for the general problem of conductor surfaces at specified potentials, denoted by S_C , and interfaces between media of different permittivities, denoted by S_I , the pair of coupled integral equations to be solved are:

$$\int_{S_C + S_I} \sigma(s') G(s|s') ds' = \phi(s), \quad s \text{ on } S_C \quad (2.20)$$

$$\frac{\epsilon_1 + \epsilon_2}{2} \sigma(s) + (\epsilon_1 - \epsilon_2) \int_{S_C + S_I} \sigma(s') \frac{\partial G}{\partial n}(s|s') ds' = 0, \quad s \text{ on } S_I \quad (2.21)$$

where the integrations in (2.20) and (2.21) are over all free and polarization charge [20,21].

Straightforward variational formulation of the interface problem (2.19) or (2.21) is hampered by the fact that the kernel of the integral operator is non-symmetric. In general $\partial G(s|s')/\partial n \neq \partial G(s'|s)/\partial n'$ for arbitrary curves, the equality depending on the angle made by the normal of the line joining s to s' to the curve at s and s' . Self-adjointness of the interface operator can only be proven for some specific surface configurations.

McDonald et. al. [20] proposes a modified interface operator denoted by L' which is formed by taking the inner product of the interface problem (2.21) with the Green's function:

$$L' \sigma = \langle L\sigma, G \rangle = 0 \quad (2.22)$$

where $L\sigma = 0$ is the operator notation of (2.19) or (2.21).

They show that it is self-adjoint and positive-definite, thus making it a viable candidate for use in the functional (2.4). However, Jeng et. al. [22] argue that a possible solution σ_0 which causes $L'\sigma_0 = 0$ does not necessarily mean a solution of $L\sigma_0 = 0$ since for certain choices of trial functions, $L\sigma_0$ and G may be orthogonal. Instead, Jeng et. al. propose a generalized matrix operator based on the operator and its adjoint and proceed to derive a functional from (2.4) which is identical to the dual functional (2.4a).

2.2.4 Finite Element Method for Differential Operators

In a problem where it is difficult to find sets of functions satisfying all the boundary conditions or where boundary shapes are too complicated for the purposes of integration, it may be convenient to divide the total region into smaller subregions and solve for the field in each subregion. This is the rationale behind Finite Element variational methods.

The functional pertaining to the entire region is subdivided into a number of "sub-functionals" where the inner product corresponding to each sub-functional is defined as the integration over that particular subregion. The Rayleigh-Ritz procedure is used to find an approximate solution in each subregion, taking care to ensure that the solution is continuous from subregion to subregion. The resulting solution is the union of the solutions for each subregion.

The region is usually divided into triangles for two-dimensional problems and tetrahedrons for three-dimensional problems. If Laplace's equation is to be solved, subject to certain Dirichlet and homogeneous Neumann conditions, the

functional (2.11) is written as the summation over all of the sub-functionals for each triangle or subregion:

$$F = \sum_t F_t = \sum_t \int_{\Omega_t} \epsilon_t (\nabla \phi)^2 d\Omega \quad (2.23)$$

where the permittivity ϵ_t is assumed constant in each subregion but may vary from subregion to subregion.

For two-dimensional problems, the potential in each triangle is approximated by a polynomial in the two space coordinates u and v :

$$\phi(u, v) = \sum_k \sum_l c_{kl} u^k v^l \quad (2.24)$$

where the c_{kl} are unknown coefficients. For a linear approximation, the trial function becomes:

$$\phi(u, v) = c_{00} + c_{10} u + c_{01} v \quad (2.25)$$

where there are a total of three variational parameters per triangle. To ensure the continuity of the potential along the common sides of adjacent triangles, (2.25) is enforced at the three vertices of the triangle and the potentials at the vertices (or commonly called nodes) are used as the variational parameters [25].

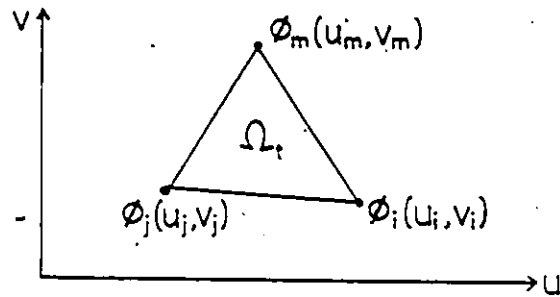


Figure 6: The general triangle for the two-dimensional Finite Element method.

For the general triangle shown in Fig. 6, having vertices located at (u_i, v_i) , (u_j, v_j) and (u_m, v_m) , the node potentials are given by:

$$\begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix} = \begin{bmatrix} 1 & u_i & v_i \\ 1 & u_j & v_j \\ 1 & u_m & v_m \end{bmatrix} \begin{bmatrix} C_{00} \\ C_{10} \\ C_{01} \end{bmatrix} \quad (2.26)$$

or in matrix notation:

$$\underline{\phi} = \underline{A} \underline{c} \quad (2.27)$$

The unknown coefficients are determined from $\underline{c} = \underline{A}^{-1} \underline{\phi}$ and the trial function may be expressed in terms of the new variational parameters ϕ_i , ϕ_j and ϕ_m as:

$$\begin{aligned} \phi(u, v) &= a_i(u, v) \phi_i + a_j(u, v) \phi_j + a_m(u, v) \phi_m \\ &= \underline{a}^T \underline{\phi} \end{aligned} \quad (2.28)$$

where $a_k(u,v)$, $k=i,j,m$ are linear interpolatory functions, sometimes referred to as shape functions because they depend on the shape of the triangle, and are determined from the solution of (2.27).

The shape functions, with reference to the general triangle shown in Fig. 6, have the following properties:

$$\left. \begin{aligned} a_i(u,v) &= 1 \\ a_j(u,v) &= 0 \\ a_m(u,v) &= 0 \end{aligned} \right\} \text{ at } (u,v) = (u_i, v_i) \quad (2.29)$$

and along the triangle side joining node "i" to node "j" :

$$a_m(u,v) = 0 \quad (2.30)$$

The second property ensures that the potential is continuous between adjacent triangles since the potential along the triangle side joining nodes "i" to "j" is given by:

$$\phi = a_i(u,v) \phi_i + a_j(u,v) \phi_j \quad (2.31)$$

Since the shape functions are linear in u and v , the potential in (2.31) is a linear interpolate of ϕ_i and ϕ_j alone. Also, since equation (2.31) is valid for either triangle possessing these two nodes, the potential is continuous along their common side.

The first and second properties allow the trial function to satisfy any Dirichlet conditions. This is apparent if

nodes "i" and "j" lie on a Dirichlet boundary since the linear interpolation (2.31) implies that the potential is constant along the triangle side joining these two nodes.

Having defined the trial function (2.28), the functional for each triangle is assembled and variations with respect to the variational parameters or node potentials are taken. The functional for the general triangle becomes:

$$\begin{aligned} F_t &= \varepsilon_t \int_{\Omega_t} \underline{\vartheta}^T \underline{\nabla} \underline{a} \cdot \underline{\nabla} \underline{a}^T \underline{\vartheta} \, d\Omega \\ &= \varepsilon_t \underline{\vartheta}^T \underline{\nabla} \underline{a} \cdot \underline{\nabla} \underline{a}^T \underline{\vartheta} \int_{\Omega_t} d\Omega \end{aligned} \quad (2.32)$$

where the gradient of each shape function is a constant and depends only on the shape of the triangle. The Rayleigh-Ritz minimizing condition gives:

$$\frac{\partial F_t}{\partial \vartheta_i} = 0, \quad \frac{\partial F_t}{\partial \vartheta_j} = 0, \quad \frac{\partial F_t}{\partial \vartheta_m} = 0 \quad (2.33)$$

or in vector form:

$$\frac{\partial F_t}{\partial \underline{\vartheta}} = 2\varepsilon_t \int_{\Omega_t} \underline{\nabla} \underline{a} \cdot \underline{\nabla} \underline{a}^T \underline{\vartheta} \, d\Omega = 0 \quad (2.34)$$

If there are a total of n unknown node potentials, (2.34) constitutes an $n \times n$ system of equations with 3×3 non-zero entries. Dirichlet nodes have fixed potentials and thus are not varied as parameters. Instead these nodes contribute terms to the right hand side of (2.34) and subsequently, to the final system of equations [25].

The final system of equations is obtained by summing the contribution of the Rayleigh-Ritz condition $\partial F_t / \partial \underline{\theta}$ due to each triangle:

$$\frac{\partial F}{\partial \underline{\theta}} = \sum_t \frac{\partial F_t}{\partial \underline{\theta}} = 0 \quad (2.35)$$

which yields the system:

$$S \underline{\theta} = \underline{b} \quad (2.36)$$

The system matrix S is symmetric [25] while the right hand column vector $\underline{b}$ in (2.36) contains information about the Dirichlet nodes. If there are a total of n nodes in the system, of which m are Dirichlet nodes, $n-m$ linear algebraic equations result which can be solved by standard techniques such as Gauss elimination.

The interface condition, which states that the normal component of electric flux density is continuous between adjacent triangles, is satisfied as a natural consequence of minimizing the functional (2.11) [26].

2.3 MOMENT METHODS

The general Method of Moments is a direct procedure for transforming a linear operator equation, which in many cases is difficult to solve, into a system of linear algebraic equations, which can be solved by standard methods. Under certain conditions, Moment Methods resemble variational

methods, or, are equivalent to them in the sense that the resulting system of equations is similar or identical. Under these conditions the same conclusions about convergence, uniqueness and the minimization of energy may be extended to Moment Methods [18,23].

In the Method of Moments, the unknown function u in the operator equation (2.1) is expanded in a linear combination of basis or expansion functions u_j :

$$u = \sum_{j=1}^n a_j u_j = \underline{a}^T \underline{u} \quad (2.37)$$

The basis functions are selected such that they lie in the domain of the operator (i.e. satisfy the boundary conditions) while the coefficients a_j are unknown constants.

The expansion for the unknown function is substituted into the operator equation resulting in:

$$\sum_{j=1}^n a_j Lu_j = \underline{a}^T L \underline{u} = f \quad (2.38)$$

where the property of linearity of the operator is used.

A set of weighting or testing functions w_i are defined and the inner product of (2.38) with each weighting function is taken, producing the system of equations:

$$\sum_{j=1}^n a_j \langle Lu_j, w_i \rangle = \langle f, w_i \rangle \quad (2.39)$$

$$i = 1, \dots, n$$

The system is solved for the unknown coefficients a_j and the solution is given by (2.37).

If the same functions are used for expansion and weighting, the method is known as Galerkin's method, and the resulting system of equations are identical to those obtained by the Rayleigh-Ritz variational method (if applied to integral operators) [18].

2.3.1 Moment Method for Integral Operators - Method of Subsections

For the Moment Method solution of integral equations, where the unknown charge is distributed over complicated conductor and dielectric surfaces, it is more convenient to define the basis functions over a subsection of the surface rather than to define them over the entire domain of the problem.

With reference to the coupled integral equations (2.20) and (2.21), the conductor surfaces S_C and dielectric interface S_I are divided into subsections:

$$S_C = \Delta s_1 \cup \Delta s_2 \cup \dots \cup \Delta s_n \quad (2.40)$$

$$S_I = \Delta s_{n+1} \cup \Delta s_{n+2} \cup \dots \cup \Delta s_{n+m}$$

where S_C is the union of the first n non-overlapping subsections, located on conductor surfaces, and S_I is the union of the remaining $n+1$ to $n+m$ subsections, located on the dielectric interface. The general subsection is denoted by Δs_j .

The unknown charge distribution is expanded in a series of subsectional basis functions:

$$\sigma(s) = \sum_{j=1}^{n+m} \sigma_j f_j(s - \Delta s_j) \quad (2.41)$$

where the function $f_j(s - \Delta s_j)$ is defined only over Δs_j and σ_j is the unknown amplitude. The basis functions should be chosen such that, in combination, they are able to represent the charge distribution satisfactorily.

Substitution of (2.41) into the coupled integral equations (2.20) and (2.21) gives:

$$\sum_{j=1}^{n+m} \sigma_j \int_{\Delta s_j} f_j(s' - \Delta s_j) G(s|s') ds' = \vartheta(s) \quad (2.42)$$

$$\frac{(\epsilon_1 + \epsilon_2)}{2} \sigma(s) + (\epsilon_1 - \epsilon_2) \sum_{j=1}^{n+m} \sigma_j \int_{\Delta s_j} f_j(s' - \Delta s_j) \frac{\partial G}{\partial n}(s|s') ds' = 0 \quad (2.43)$$

Weighting functions, each defined over only one subsection, are denoted by $w_i(s - \Delta s_i)$, $i=1, \dots, n+m$. The inner product of (2.42) is taken with the first n weighting func-

tions since $\vartheta(s)$ is given over S_C while the inner product of (2.43) is taken with the remaining $n+1$ to $n+m$ weighting functions since (2.43) is satisfied on S_I . This procedure produces a system of $n+m$ linear equations:

$$\sum_{j=1}^{n+m} \sigma_j \int_{\Delta S_i} \int_{\Delta S_j} w_i(s - \Delta s_i) f_j(s' - \Delta s_j) G(s|s') ds' ds = \int_{\Delta S_i} w_i(s - \Delta s_i) \vartheta(s) ds \quad (2.44)$$

$$i = 1, \dots, n$$

$$\begin{aligned} & \frac{(\varepsilon_1 + \varepsilon_2)}{2} \sigma_i \int_{\Delta S_i} w_i(s - \Delta s_i) f_i(s - \Delta s_i) ds \\ & + (\varepsilon_1 - \varepsilon_2) \sum_{j=1}^{n+m} \sigma_j \int_{\Delta S_i} \int_{\Delta S_j} w_i(s - \Delta s_i) f_j(s' - \Delta s_j) \frac{\partial G}{\partial n}(s|s') ds' ds = 0 \end{aligned} \quad (2.45)$$

$$i = n+1, \dots, n+m$$

in the $n+m$ unknowns σ_j .

In general, the basis and weighting functions need not be all alike. For instance, where a charge singularity is expected to occur such as on the edge of a conductor, a basis function possessing a singularity may be used for that particular subsection. Also, weighting and basis functions may be selected to reduce the amount of numerical computation required. For example, if pulse functions defined as:

$$P(s - \Delta s_j) = \begin{cases} 1 & , s \text{ on } \Delta s_j \\ 0 & , \text{ elsewhere} \end{cases} \quad (2.46)$$

are used as basis functions and Dirac or impulse functions, $\delta(s-s_i)$, are used for weighting, (2.44) and (2.45) reduce to the particularly simple form:

$$\sum_{j=1}^{n+m} \sigma_j \int_{\Delta s_j} G(s_i | s') ds' = \theta(s_i) \quad (2.47)$$

$$i = 1, \dots, n$$

$$\frac{(\varepsilon_1 + \varepsilon_2)}{2} \sigma_i + (\varepsilon_1 - \varepsilon_2) \sum_{j=1}^{n+m} \sigma_j \int_{\Delta s_j} \frac{\partial G}{\partial n}(s_i | s') ds' = 0 \quad (2.48)$$

$$i = n+1, \dots, n+m$$

where s_i denotes the mid-point of the "i" th subsection.

The use of Dirac or impulse functions as weighting functions is equivalent to enforcing or collocating (2.42) and (2.43) at the mid-points of the first n and $n+1$ to $n+m$ subsections respectively and is known in the literature as the method of collocation or point-matching [18].

If pulses are used as basis and weighting functions (Galerkin's method), (2.44) and (2.45) become:

$$\sum_{j=1}^{n+m} \sigma_j \int_{\Delta s_i} \int_{\Delta s_j} G(s | s') ds' ds = \int_{\Delta s_i} \theta(s) ds \quad (2.49)$$

$$i = 1, \dots, n$$

$$\frac{(\varepsilon_1 + \varepsilon_2)}{2} \sigma_i \Delta s_i + (\varepsilon_1 - \varepsilon_2) \sum_{j=1}^{n+m} \sigma_j \int_{\Delta s_i} \int_{\Delta s_j} \frac{\partial G}{\partial n}(s | s') ds' ds = 0 \quad (2.50)$$

$$i = n+1, \dots, n+m$$

To summarize, the method of subsections using point-matching or Galerkin's method, reduces the pair of integral equations (2.20) and (2.21) to a system of $n+m$ linear algebraic equations. If pulse expansion functions are used, the solution is a step-wise approximation to the true solution. In order that the pulse approximation should closely follow the exact charge distribution, pulses should be made narrow at corners and abrupt discontinuities while far away from such discontinuities, they may be made broader.

2.4 CONCLUDING REMARKS

Electrostatic problems may be classed into two broad categories based on whether the problem is bounded or unbounded. A bounded problem is one in which the region of solution is bounded completely by either Dirichlet or Neumann boundaries while an unbounded problem may be defined as one in which part of the boundary is at infinity, which in most cases is taken to be a homogeneous Dirichlet boundary.

Generally, differential equation formulations are used to solve bounded problems while integral equation formulations are used for unbounded problems. There are exceptions to this rule however. Finite Element methods for differential operators have been adapted to unbounded problems with a corresponding increase in complexity [27].

Certain generalizations may be made about the two methods of solution outlined here (FEM for differential operators and MOM for integral operators). First, the integral equation formulation possesses singular kernels which make integration difficult. Also, modelling of the spatial distribution of permittivity requires an additional integral equation for each dielectric interface. The variational differential equation formulation on the other hand, is computationally simpler and requires no extra equations for the interface problem. However, it is usually unable to accurately model unbounded problems.

The problems under investigation in this work are of the unbounded type. It was decided, nevertheless to use the Finite Element Method for differential operators (FEM) due to the ease in handling the interface problem. A second method where the coupled integral equations (2.20) and (2.21) are solved using the Method of Moments and a subsectional basis (MOM) is presented as well.

Chapter III

NUMERICAL PROCEDURE

3.1 IMPLEMENTATION OF FEM

In this section, the implementation of the Finite Element Method (FEM) for analysing open-ended coaxial sensors is described. The application of the method to cylindrical coordinate systems with rotational symmetry is given and details of applying the method to the unbounded sensor problem is explained. Also the method by which the external and internal capacitances, C_0 and C_i respectively, were determined is elaborated.

3.1.1 FEM for Rotationally Symmetric Systems

In Chapter II the basic Finite Element Method was presented for a general two-dimensional coordinate system. It is applicable to rotationally-symmetric systems since the fields have no variation in the angular coordinate when a cylindrical coordinate system is used. The independent space coordinates u and v used in the trial function (2.25) become the axial coordinate z and the radial coordinate r respectively.

The elemental area for rotationally-symmetric systems is $d\Omega = r dr dz$ and the integral over the domain of the element in (2.34) is given by:

$$\int_{\Omega_t} d\Omega = \int \int_{\text{triangle "t"}} r dr dz \quad (3.1)$$

For the general triangle depicted in Fig. 6, (3.1) can be evaluated as:

$$\begin{aligned} \int \int_{\text{triangle "t"}} r dr dz &= \frac{(r_i + r_j + r_m)}{6} [z_i (r_m - r_j) + z_j (r_i - r_m) + z_m (r_j - r_i)] \\ &= \frac{(r_i + r_j + r_m)}{3} \times \text{Area of triangle "t"} \end{aligned} \quad (3.2)$$

where the u and v coordinates have been replaced by z and r respectively.

Equation (3.1) is actually the first moment of the area of the general triangle about the z -axis while the term in brackets in (3.2) may be interpreted as the mean radial distance of the general triangle from the z -axis.

3.1.2 Application to Coaxial Sensor Problem

The configuration of an air line with a groundplane (Fig. 2(a)) is treated as an example of the application of FEM to the coaxial sensor problem. The region of solution is depicted two-dimensionally in Fig. 7, while the actual

physical region is the volume of revolution about the z -axis.

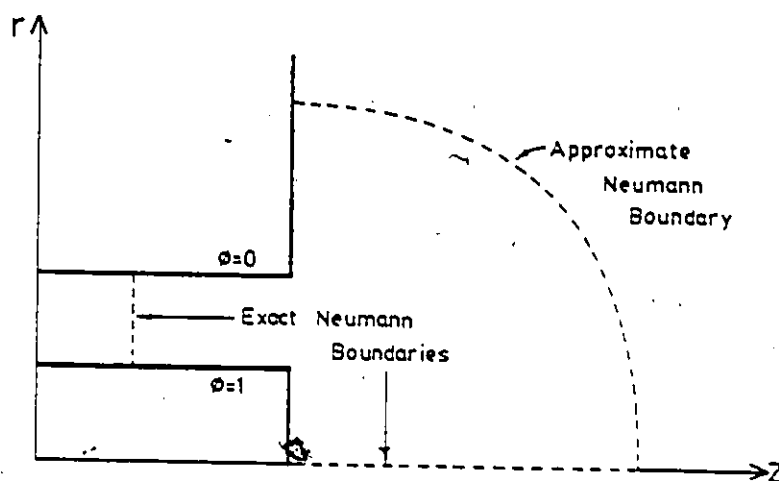


Figure 7: Region of solution for coaxial sensor with groundplane.

Dirichlet boundaries are provided by the inner conductor, held at a potential of 1 volt, and the outer conductor and groundplane held at 0 volts. Since the line is assumed to be infinite in the negative z -direction, a vertical Neumann boundary is placed far to the left of the aperture in Fig. 7. This boundary represents an electric field line resulting from the transverse electric and magnetic (TEM) fields of an infinitely long coaxial line. The z -axis represents another homogeneous Neumann boundary (or electric field line) due to the axial symmetry of the problem. A fi-

nal boundary is required to close the region completely before dividing it into triangles.

The problem is unbounded (theoretically, the final boundary is a homogeneous Dirichlet one at infinity), however an approximate Neumann boundary enclosing a sufficient amount of the region may be used. This approximate boundary is found by making an initial guess at the location and orientation of an electric field line originating on the z-axis and terminating at right angles on a conductor, in this case the groundplane. An example of an approximate Neumann boundary is shown in Fig. 7.

To ensure that a sufficient area around the aperture is enclosed so that almost all of the stored energy is included, several separate runs are made, each with a progressively larger region of solution. When the difference between the stored energy from consecutive runs becomes less than a specified amount, say 1 %, the required area is determined. Also, once the orientation of the electric field has been calculated, a further refinement of the approximate boundary may be made.

To model the effect of different dielectrics inside and outside the line, a dielectric interface is placed in the aperture plane (see Fig. 7) separating the two dielectric regions. Each dielectric region is divided into triangular elements separately making sure that adjacent triangle sides match on the interface.

Where the electric field is expected to be stronger, the triangularization should be denser and the triangle size should increase approximately logarithmically with radius to ensure accurate solutions [25].

3.1.3 Calculation of Capacitance

The functional (2.23), written as the sum over all the elements, is proportional to the stored electric energy at the solution point. In a rotationally-symmetric system, the stored electric energy W_E is given by:

$$W_E = \frac{1}{2} \int_{\Omega} \epsilon (V\phi)^2 2\pi r dr dz \quad (3.3)$$

Comparing (3.3) with the functional (2.23) and noting that the capacitance is given in terms of the stored energy by: $C = 2W_E/V^2$, the capacitance may be calculated from the functional by:

$$C = \frac{2\pi F}{V^2} \quad (3.4)$$

where F corresponds to the value of the functional at the solution point and V is the potential difference between the two conductors. Computation of the functional is straightforward once the unknown potentials have been solved.

A portion of the capacitance calculated by FEM in this problem results from the TEM fields inside the line. The

net fringe-field capacitance is determined by subtracting the contribution due to the TEM fields which is given by:

$$C_{\text{TEM}} = \frac{2\pi\epsilon_L \ell}{\ln(b/a)} \quad (3.5)$$

where ℓ is the length of coaxial line in the FEM analysis, ϵ_L is the permittivity inside the line and a and b are the inner and outer-conductor radii respectively.

C_0 and C_f :

Determination of the values of the external and internal capacitances C_0 and C_f , defined in Chapter I, follows from the linear circuit model (1.1) given again by:

$$C(\epsilon) = C_f + \epsilon C_0 \quad (3.6)$$

where $C(\epsilon)$ is the fringing capacitance that is calculated when a dielectric sample of relative permittivity ϵ is present.

With no sample present the calculated fringing capacitance corresponds to $C_T = C_f + C_0$ and, from these two expressions, the external capacitance may be calculated by:

$$C_0 = \frac{C(\epsilon) - C_T}{\epsilon - 1} \quad (3.7)$$

Once the external capacitance has been found for a given value of ϵ , the internal capacitance is calculated from:

$$C_f = C_T - C_o \quad (3.8)$$

The values of C_o and C_f determined in the above manner will in general, depend on ϵ . However, for a range of values of ϵ where they are constant, the linear model implied by (1.1) or (3.6) will be approximately valid. In this case C_o may be interpreted as the constant slope of a plot of $C(\epsilon)$ versus ϵ .

3.2 IMPLEMENTATION OF MOM

This section deals with the application of the Method of Moments (MOM) to solve the coupled integral equations (2.20) and (2.21). First a brief derivation of the Green's function and its derivatives is given for rotationally-symmetric systems. The procedure for calculating the capacitance is presented, and finally, the application of the method to the coaxial sensor problem is discussed.

3.2.1 Green's Function for Rotationally Symmetric Systems

For rotationally-symmetric conductors and dielectrics, the charge is distributed uniformly about the circumference for a given r and z . To obtain the two-dimensional Green's function, valid for rotationally-symmetric problems, consider the infinitesimally thin circular ring with axis coincident with the z -axis and charged uniformly with a density of 1 coulomb/m, as shown in Fig. 8.

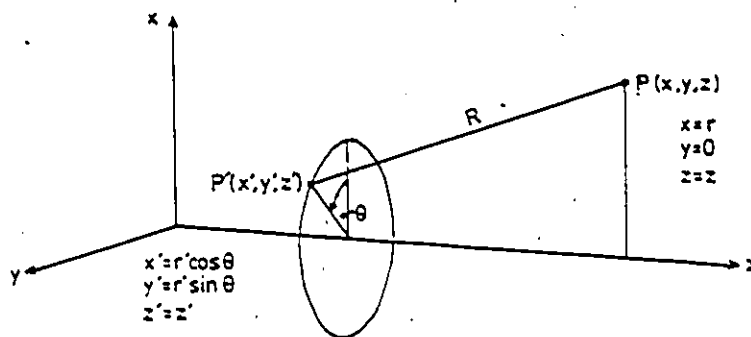


Figure 8: Infinitesimal ring used in the derivation of the Green's function.

The three-dimensional form of the Green's function is:

$$G(\mathbf{x}|\mathbf{x}') = \frac{1}{4\pi R} = \frac{1}{4\pi |\bar{\mathbf{x}} - \bar{\mathbf{x}}'|} \quad (3.9)$$

where the primed coordinate $\bar{\mathbf{x}}' = (x', y', z')$ denotes a source point and the unprimed coordinate $\bar{\mathbf{x}} = (x, y, z)$ denotes an observation point. Because of rotational symmetry, the observation point may be placed in the x - z plane without loss of generality. The distance between the observation point P and the source point P' is given by:

$$\begin{aligned} R &= \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \\ &= \sqrt{(r+r')^2 + (z-z')^2 - 4rr' \cos^2 \theta/2} \end{aligned} \quad (3.10)$$

while the increment of potential at P due to the increment of charge at P' is:

$$d\phi(P) = \frac{d\theta}{4\pi\sqrt{(r+r')^2 + (z-z')^2 - 4rr'\cos^2\theta/2}} \quad (3.11)$$

The potential at P due to the entire ring of charge is the integral of (3.11) over the interval 0 to 2π , thus the rotationally-symmetric Green's function is given by [21]:

$$G(r,z|r',z') = \int_0^{2\pi} \frac{d\theta}{4\pi\sqrt{(r+r')^2 + (z-z')^2 - 4rr'\cos^2\theta/2}} \quad (3.12)$$

Equation (3.12) can be recast into the form [21]:

$$G(r,z|r',z') = \frac{F(k)}{\sqrt{(r+r')^2 + (z-z')^2}}, \quad k^2 = \frac{4rr'}{[(r+r')^2 + (z-z')^2]} \quad (3.13)$$

where $F(k)$ is the complete elliptic integral of the first kind and is defined by [29,30]:

$$F(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad (3.14)$$

The derivatives of (3.13) are required in order to calculate $\partial G/\partial n$, which occurs in the integral equation governing the polarization charge. They may be obtained by differentiating (3.13) with respect to the unprimed coordinates r and z and are given by:

$$\frac{\partial G}{\partial z}(r, z | r', z') = \frac{-(z-z')}{\pi [(r+r')^2 + (z-z')^2]^{3/2}} [F(k) + kF'(k)] \quad (3.15)$$

$$\begin{aligned} \frac{\partial G}{\partial r}(r, z | r', z') &= \frac{1}{2\pi r} \frac{kF'(k)}{\sqrt{(r+r')^2 + (z-z')^2}} \\ &\quad - \frac{(r+r')}{\pi [(r+r')^2 + (z-z')^2]^{3/2}} [F(k) + kF'(k)] \end{aligned} \quad (3.16)$$

where $F'(k)$ is the derivative of (3.14) with respect to k .

Both G and $\partial G/\partial r$ possess logarithmic singularities at $r=r'$ and $z=z'$ since $F(k)$ is singular for $k=1$ [30], however it can be shown¹ that $\partial G/\partial z$ vanishes at $z=z'$ for all r and r' including $r=r'$.

Evaluation of the normal derivative of the Green's function to an arbitrary surface proceeds as follows. The normal derivative of a function may be expressed as the vector dot product of the function gradient with the unit normal vector to the surface:

$$\frac{\partial G}{\partial n} = \nabla G \cdot \bar{n} \quad (3.17)$$

where $\bar{n}$ denotes the unit normal vector.

If the surface, given by $f(r, z) = \text{constant}$, can be expressed two dimensionally as $r=h(z)$, the unit normal may be expressed as:

¹ See Appendix A.

$$\vec{n} = \frac{+ (\vec{a}_r - h'(z) \vec{a}_z)}{\sqrt{1 + h'(z)^2}} \quad (3.18)$$

where $\vec{a}_r$ and $\vec{a}_z$ are unit component vectors in the r and z -direction and the sign in (3.18) depends on the orientation of $\vec{n}$.

After substituting (3.18) into (3.17) and performing the indicated gradient operation, the general form of the normal derivative of G may be written as:

$$\frac{\partial G}{\partial n} = \pm \frac{1}{\sqrt{1 + h'(z)^2}} \left(\frac{\partial G}{\partial r} - h'(z) \frac{\partial G}{\partial z} \right) \quad (3.19)$$

Equation (3.19) takes on a particularly simple form if the surface of rotation is approximated by frustrums of a right circular cone or, in two dimensions, as a series of straight line segments as shown in Fig. 9.

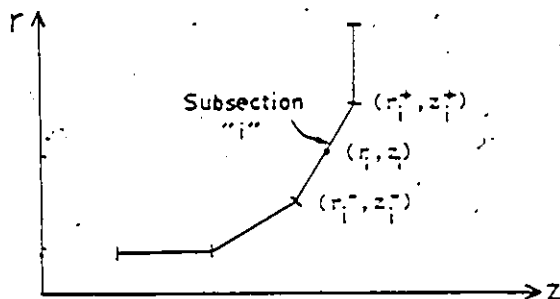


Figure 9: Surface represented two dimensionally by straight line segments showing the "i"th segment.

If the endpoints of the i th segment are denoted as (r_i^+, z_i^+) and (r_i^-, z_i^-) , the slope of the segment is $h^i(z) = (r_i^+ - r_i^-) / (z_i^+ - z_i^-)$ and the normal derivative of the Green's function becomes:

$$\frac{\partial G}{\partial n} = \frac{+ [(z_i^+ - z_i^-) (\partial G / \partial r) - (r_i^+ - r_i^-) (\partial G / \partial z)]}{\sqrt{(z_i^+ - z_i^-)^2 + (r_i^+ - r_i^-)^2}} \quad (3.20)$$

If the segment is strictly annular ($z_i^+ = z_i^-$) or strictly cylindrical ($r_i^+ = r_i^-$), the normal derivative of G reduces to $\partial G / \partial z$ or $\partial G / \partial r$ respectively.

3.2.2 Calculation of Capacitance

The capacitance is evaluated by integrating the charge over one of the conductor surfaces and dividing by the potential difference V [20,26]:

$$C = \frac{1}{V} \int_{S_C} \epsilon(s) \frac{\partial \phi}{\partial n}(s) ds \quad (3.21)$$

where $\partial \phi / \partial n$ is the potential gradient terminating on the conductor which corresponds to the charge density $\sigma(s)$ in (2.13) and $\epsilon(s)$ is the permittivity of the dielectric contacting the surface. The integration in (3.21) is performed over either the positively or negatively-charged conductor.

For the pulse function expansion of the charge, given by:

$$\sigma(s) = \frac{\partial \phi}{\partial n}(s) = \sum_{i=1}^{n+m} \sigma_i P(s - \Delta s_i) \quad (3.22)$$

where $P(s-\Delta s_i)$ is defined by (2.46) and σ_i is the pulse amplitude in units of coulombs/m², (3.21) becomes:

$$C = \frac{1}{V} \sum_{i=1}^k \sigma_i \varepsilon_i \int_{\Delta s_i} ds \quad (3.23)$$

where the summation is over all positively or negatively-charged subsections lying on conductors.

If the surface is represented by frustrums of a right circular cone, the integral term in (3.23) is the area of the "i"th frustrum. In general, the elemental surface for rotationally symmetric systems is given by:

$$ds = 2\pi r \, dl \quad (3.24)$$

where $dl = \sqrt{dr^2 + dz^2}$.

Referring to Fig. 9, the inverse slope of the frustrum is given by $dz/dr = (z_i^+ - z_i^-)/(r_i^+ - r_i^-)$ therefore the frustrum surface area is given by:

$$\begin{aligned} \int_{\Delta s_i} ds &= \int_{r_i^-}^{r_i^+} \sqrt{1 + (dz/dr)^2} \, 2\pi r \, dr \\ &= 2\pi \frac{(r_i^+ + r_i^-)}{2} \sqrt{(r_i^+ - r_i^-)^2 + (z_i^+ - z_i^-)^2} \end{aligned} \quad (3.25)$$

Equation (3.23) and (3.25) may be used to calculate the capacitance once the unknown charge pulse amplitudes have been solved. The same procedures for determining the net fringing capacitance and C_0 and C_f , given for the PEM analysis (section 3.1.3), are used in the MOM calculations.

3.2.3 Application to the Coaxial Sensor Problem

The sensor configuration of Fig. 2(a) will again be treated as an example to illustrate the application of the method of subsections to the coaxial sensor problem. The outline of the conductors and dielectric interface for which the charge distribution is to be solved is shown in Fig. 7.

The outer conductor and groundplane are held at 0 volts while the inner conductor is held at a potential of 1 volt. The dielectric interface is placed in the plane of the aperture while the homogeneous Neumann boundary, which arises due to the assumption that the line is infinite in the negative z -direction, is located at $z=0$.

The assumption of an infinite line implies that the charge distribution on both the inner and outer conductors is uniform far from the aperture. If the integral equations are solved for the line truncated at $z=0$, as in Fig. 7, a charge singularity will occur where the conductors terminate and the result will not be the desired solution. In order to simulate a homogeneous Neumann boundary at $z=0$, the method of images [20,28] was used.

In the method of images, the problem that is solved consists of the actual problem, shown in Fig. 7, and its "image" problem formed by the mirror image of the conductors and the interface about the plane $z=0$. The resulting configuration is shown in Fig. 10.

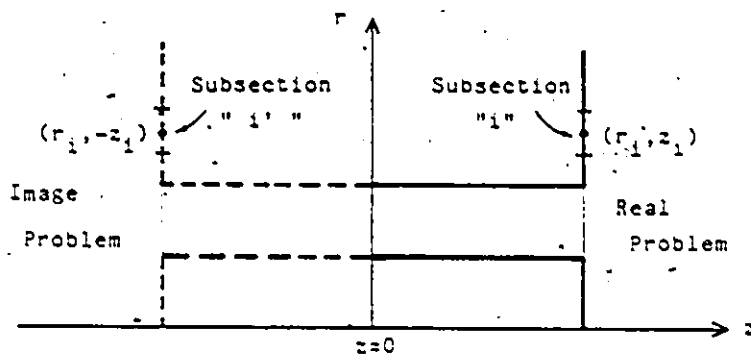


Figure 10: Configuration of real and image problems for MOM solution.

Given a sufficient length of coaxial line, this technique will insure that the charge distribution on both inner and outer conductors will be approximately uniform in the vicinity of $z=0$ (Note that this may be taken as a criterion for the required length of line). Because of the symmetrical distribution of charge however, no additional unknowns are introduced. This is apparent since the charge densities at $(r, -z)$ and (r, z) have the same values thus requiring only one of them to be solved for.

To implement a subsectional solution, the real conductors and interface in Fig. 10 are divided into straight line segments (subsections) which are numbered from $j=1$ to $j=n$ for conductors and $j=n+1$ to $j=n+m$ for the interface. The image conductors and interface are divided and numbered identically, in the mirror image of the real ones. Image subsections are differentiated from real ones by writing their indexes with a prime.

For expansion by pulse functions and weighting by Dirac functions (point matching), the system governing the distribution of free charge (2.45) becomes:

$$\sum_{j=1}^{n+m} \sigma_j K_{i,j} + \sum_{j'=1}^{n+m} \sigma_{j'} K_{i,j'} = \phi_i \quad (3.26)$$

$$i = 1, \dots, n$$

where the potential ϕ_i on Δs_i is due to contributions from both real and image charges and $\sigma_{j'}$ denotes the charge pulse amplitude of the "j"th image subsection.

The element $K_{i,j}$ represents the potential at the mid-point of Δs_i due to a uniform charge distribution on Δs_j and from (2.43) is given by:

$$K_{i,j} = \int_{\Delta s_j} G(s_i | s') ds' \quad (3.27)$$

Similarly, the system governing the distribution of polarization charge (2.44) may be written as:

$$\sum_{j=1}^{n+m} \sigma_j L_{i,j} + \sum_{j'=1}^{n+m} \sigma_{j'} L_{i,j'} = - \frac{(\epsilon_1 + \epsilon_2)}{2} \sigma_i \quad (3.28)$$

$$i = n+1, \dots, n+m$$

where the general element $L_{i,j}$ is given from (2.44) by:

$$L_{i,j} = (\epsilon_1 - \epsilon_2) \int_{\Delta s_j} \frac{\partial G}{\partial n} (s_i | s') ds' \quad (3.29)$$

Recall that the normal n in (3.29) is directed from the region with permittivity ϵ_1 to the region with permittivity ϵ_2 .

Because of symmetry, the real and image charge distributions are identical, i.e. $\sigma_j' = \sigma_j$, thus (3.26) and (3.28) may be written as:

$$\sum_{j=1}^{n+m} \sigma_j (K_{i,j} + K_{i,j}') = q_i \quad (3.30)$$

$$i = 1, \dots, n$$

$$\sum_{j=1}^{n+m} \sigma_j (L_{i,j} + L_{i,j}' + \frac{\epsilon_1 + \epsilon_2}{2} \delta_{i,j}) = 0 \quad (3.31)$$

$$i = n+1, \dots, n+m$$

where the Kronecker delta $\delta_{i,j}$ is defined as:

$$\delta_{i,j} = \begin{cases} 1 & , i=j \\ 0 & , i \neq j \end{cases} \quad (3.32)$$

The image elements $K_{i,j}$ and $L_{i,j}'$ in (3.30) and (3.31) are computed by integrating the Green's function or it's normal derivative over the interval $(r_j^+, -z_j^+)$ to $(r_j^-, -z_j^-)$, since corresponding points on a real subsection and it's image are given by (r, z) and $(r, -z)$ respectively. Thus, there is no need to actually subsection and number the image problem.

The dimension of the resulting system of equations $S\bar{q} = \bar{b}$ is $n+m$ and the general element is given by:

$$\begin{aligned} S_{i,j} &= K_{i,j} + K_{i,j}', \quad i=1, \dots, n \\ &= L_{i,j} + L_{i,j}' + \frac{\varepsilon_1 + \varepsilon_2}{2} \delta_{i,j}, \quad i=n+1, \dots, n+m \end{aligned} \quad (3.33)$$

The elements of the right hand side are given by:

$$\begin{aligned} b_i &= \phi_i, \quad i=1, \dots, n \\ &= 0, \quad i=n+1, \dots, n+m \end{aligned} \quad (3.34)$$

If Galerkin's method is implemented (pulses for basis and weighting functions), the same notation for the system matrix elements (3.33) and (3.34) may be used. The components making up the left hand side are double surface integrals defined by (2.49) and (2.50), and, using the notation of (3.33), they are given by:

$$K_{i,j} = \int_{\Delta S_i} \int_{\Delta S_j} G(s|s') ds' ds \quad (3.35)$$

$$L_{i,j} = (\varepsilon_1 - \varepsilon_2) \int_{\Delta S_i} \int_{\Delta S_j} \frac{\partial G}{\partial n} (s|s') ds' ds \quad (3.36)$$

For the same notation as in (3.33), the Kronecker delta must be redefined as:

$$\delta_{i,j} = \begin{cases} \Delta S_i & , i=j \\ 0 & , i \neq j \end{cases} \quad (3.37)$$

The right hand elements for Galerkin's method become:

$$\begin{aligned} b_i &= \phi_i \Delta S_i, & i=1, \dots, n \\ &= 0 & i=n+1, \dots, n+m \end{aligned} \quad (3.38)$$

Both the single and double integrations may be performed numerically using Gaussian quadrature techniques [31] or simple integration schemes such as trapezoidal or Simpson's rule.² However, great care must be taken when computing the diagonal components $K_{i,i}$ and $L_{i,i}$ since the Green's function and it's derivative with respect to r are singular over the interval of integration. These elements should be calculated analytically [21] or numerically, with a large number of integration points to ensure accuracy. Fortunately for the application at hand, the normal derivative of the Green's function is always $\partial G / \partial z$, which is not singular³ and therefore computation of $L_{i,i}$ requires no special consideration.

² Appendix B gives a general procedure for evaluating single or double surface integrals which is compatible with Gaussian quadrature techniques.

³ see Appendix A

Chapter IV

NUMERICAL RESULTS

This chapter presents the numerical results of the Finite Element and Moment Method analyses. Also, brief descriptions of the two FORTRAN programs used in the calculations are given. The sensor configurations of Figs. 2(a), (d) and (e) were analysed using both FEM and MOM while only the Moment Method was used in the calculations for the configurations of Figs. 2(b) and (c).

4.1 FINITE ELEMENT METHOD

4.1.1 Program Description

A two-dimensional cartesian Finite Element program was modified for use in rotationally-symmetric problems using the ideas presented in Section 3.1.1. The FORTRAN program consists of three major parts, the first part constructs the system of equations, the second part solves the system and finally the third part calculates the stored energy and capacitance. The system is solved using Gauss elimination [31]. All operations were performed in double precision (8-word) arithmetic on an Amdahl 470/V7A computer. The main program which is made up of the first and third parts is 5,000 words in length while the Gauss elimination routine is

approximately 1300 words long. Both routines are reproduced in Appendix C.

4.1.2 Results

Air Lines:

For the analysis of the sensor configuration of Fig. 2(a), the region of solution (shown in Fig. 7) was divided into 125 triangular elements. The total number of nodes was 79 which included 16 Dirichlet nodes. Referring to Fig. 7, the necessary area of the half-space required to contain most of the stored energy was found to have a radius of approximately 2.5 times the outer-conductor radius. The area was determined by progressively increasing the radius of the quadrant until the change in the stored energy between consecutive runs became less than 1%.

Using the relative permittivity of air both inside and outside the line, the total capacitance (normalized to the free-space permittivity and aperture dimension $b-a$) was calculated to be $C_T/\epsilon_0(b-a) = 4.3108$. Recall that this is a dimensionless quantity which is valid for any size of line provided the ratio of outer to inner-conductor radii is $b/a = 2.303$, as required for all 50- Ω air lines.

The internal and external fringing capacitances, C_f and C_o respectively, were determined using the procedure outlined in Section 3.1.3. Calculations of the fringing capacitance were performed for different values of the half-

space permittivity, ϵ ranging from 2.0 to 60.0. From these results and the computed value of C_T given earlier, C_0 and C_f were determined for each value of ϵ . In all calculations, the permittivity inside the line was maintained at 1.0. Table 1 gives computed values of the ratio of C_f/C_0 versus ϵ for the 50 Ω air line with a groundplane depicted in Fig. 2(a).

TABLE 1

C_f/C_0 vs. ϵ for 50- Ω air lines with groundplanes (FEM).

ϵ	C_f/C_0
2.0	0.0360
5.0	0.0474
10.0	0.0531
20.0	0.0565
40.0	0.0584
60.0	0.0590

Teflon-Dielectric Lines:

FEM calculations were performed for the configurations of Figs. 2(d) and (e). They consist of open-ended teflon-dielectric lines without groundplanes with the second one possessing a conducting disk which partially covers the aperture. Figure 11 illustrates the general configuration of these sensors showing all dimensions as they are defined. The coaxial line dimensions were taken from standard manufacturers data and are given by $c/a=3.93$ and

$b/a=3.27$. The relative permittivity of the dielectric filling the line (teflon) was taken to be $\epsilon_{\text{Line}}=2.05$. Where the disk radius d equals the inner-conductor radius, the general configuration reduces to that of Fig. 2(d), an open-ended teflon-dielectric line without a groundplane.

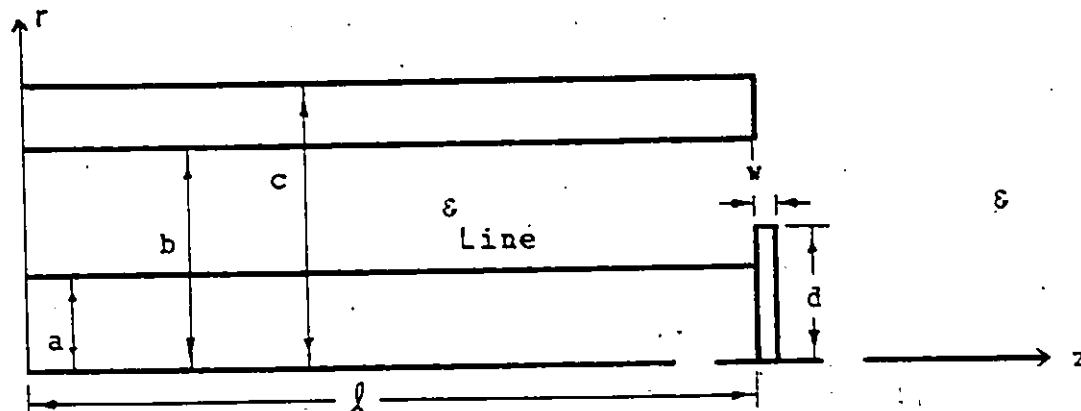


Figure 11: General configuration of an open-ended teflon-dielectric line with a capacitive disk.

For the FEM analysis, the total region was divided into 156 triangles with 96 nodes. A length of line equal to twice the outer-conductor radius was used while the half-space region was approximately contained in a half-circle with a radius 3 times the outer-conductor outer-radius.

For the case where $d/a=1.0$, i.e. the case of an open-ended teflon-dielectric line without a groundplane, the total capacitance (normalized) in air was calculated to be

$C_T/\epsilon_0(b-a)=2.4808$. Values of the ratio C_f/C_0 for this configuration are given in Table 2 for different values of the half-space permittivity ϵ .

TABLE 2

C_f/C_0 vs. ϵ for open-ended teflon-dielectric lines without groundplanes (FEM).

ϵ	C_f/C_0
2.0	0.0577
5.0	0.0804
10.0	0.0948
20.0	0.1047
40.0	0.1106
60.0	0.1128

For values of disk radius given by $1.0 < d/a < 3.27$, the general configuration is that of an open-ended teflon-dielectric line with a capacitive disk (Fig. 2(e)). For simplicity, the disk was modelled as being infinitesimally thin ($w=0.0$).

Table 3 gives computed values of the normalized capacitance in air and the ratio C_f/C_0 corresponding to a half-space permittivity of $\epsilon=60.0$ for different values of the disk radius.

The data for the total capacitance (normalized) is also plotted in Fig. 12 versus d/a where the values of $C_f/\epsilon_0(b-a)$ for $d/a > 1.0$ have been normalized to the value corresponding to $d/a=1/0$ (i.e. case of no disk). Figure 13 shows plots of C_f/C_0 versus the half-space permittivity ϵ for the different values of d/a .

TABLE 3

$C_T/\epsilon_0(b-a)$ and C_f/C_0 vs. d/a for teflon-dielectric lines
with a capacitive disk (FEM).

d/a	$C_T/\epsilon_0(b-a)$	$C_f/C_0 (\epsilon=60.0)$
1.0	2.4808	0.1128
1.2	3.1431	0.1262
1.6	5.5605	0.4265
2.0	9.1806	0.7938
2.5	17.460	1.3340

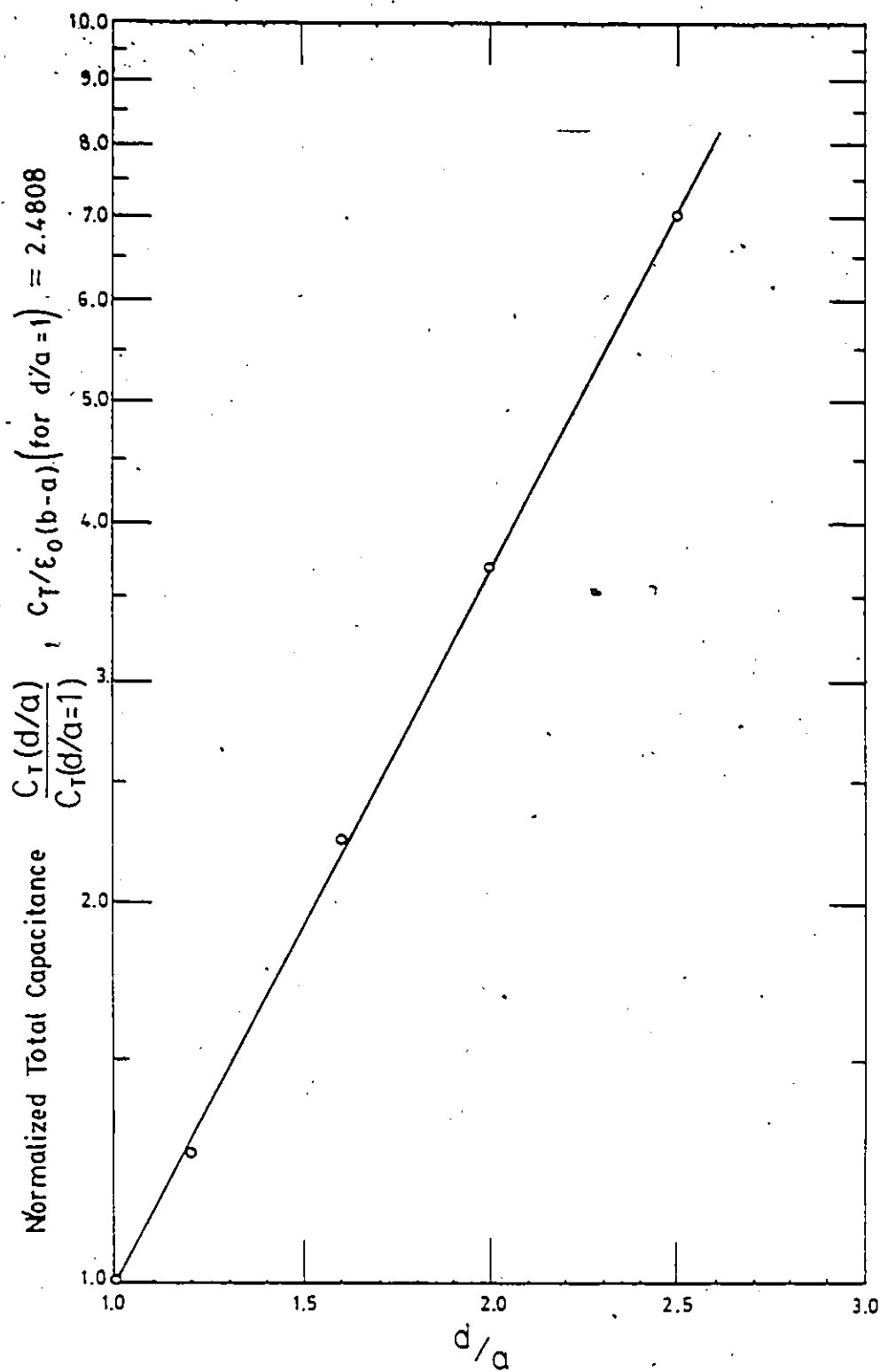


Figure 12: Total capacitance $C_T/C_T(d/a=1)$ versus d/a for teftlon-dielectric lines with capacitive disk (FEM).

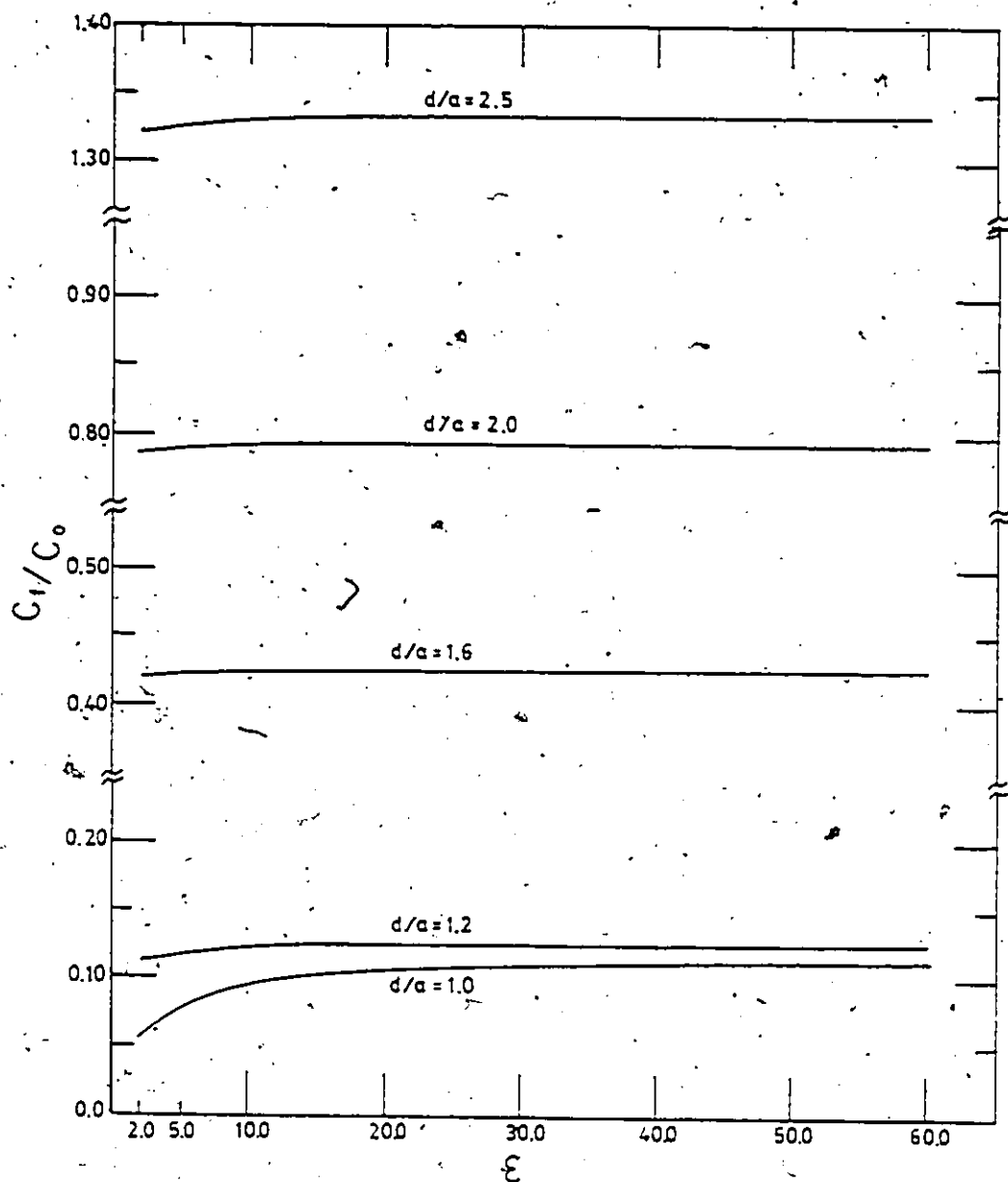


Figure 13: C_1/C_0 versus ϵ for tefflon-dielectric lines with capacitive disk (FEM).

4.2 METHOD OF MOMENTS

4.2.1 Program Description

FORTRAN programs which solve the coupled integral equations (2.20) and (2.21) using a subsectional basis and point-matching or Galerkin's method were written for arbitrary rotationally-symmetric conductors and dielectrics. All surfaces are represented by straight line segments which are defined by the coordinates of their end-points.

The system of equations is assembled by performing the integrations outlined in Appendix B for each of the subsections. Off-diagonal elements were computed using Legendre-Guass quadrature [31] for both single and double integrals. The number of abscissae in the quadrature subroutine, called GAUSS ranges from 6 to 16 for a single integration and may be selected by the user. For diagonal elements where the kernel is singular, various methods for performing the integration were attempted. They consisted of: 1) using a large number of abscissae, say 1000, with a low-order rule such as the trapezoidal rule; 2) using a procedure (outlined in Appendix A) for extracting the singularity and integrating analytically; and 3) using a combination of analytical and numerical integration. In the latter method, the Green's function, written as in (3.12), is first integrated analytically over the subsection and finally integrated numerically over the angular coordinate θ using a low-order rule and a large number of abscissae.

Numerical computations using the third approach gave the best results when applied to test problems. This may be due to the fact that the truncation error inherent in the numerical integration is independent of the size of the subsection.

The Green's function was computed with the aid of an IMSL subroutine package [32] which was used to calculate values of the complete elliptic integral. Derivatives of the elliptic integral were computed using backward difference approximations. The required step length for the backward difference was determined by using successively smaller step lengths on consecutive runs of a test problem. When no apparent difference (up to 7 decimal places) in the total stored energy between consecutive runs was observed, the necessary step length was found (approximately $10^{-6} \approx 10^{-7}$).

The FORTRAN program consists of several parts. The main program constructs the system of equations and computes the stored energy once all charge densities are known. Separate subroutines are provided for: solution of the system using Gauss elimination; single or double surface integration; evaluation of the Green's function and its normal derivatives; and for special handling of the singularities. All programs use double precision (8-word) arithmetic and are reproduced in Appendix C.

4.2.2 Results

Air Lines:

Solutions of the fringing capacitance of open-ended 50Ω air lines with groundplanes (Fig.3(a)) were carried out for a range of relative permittivity in the half-space between 1.0 and 60.0. A length of line approximately equal to twice the outer-conductor radius was used and the groundplane radius was taken to be three times the outer-conductor radius. The conductors were divided into 65 Dirichlet pulses and the interface was divided into 11 Neumann pulses, resulting in a total of 76 unknowns. Calculations of C_f and C_0 were performed using the same procedure as in the FEM analysis (Section 3.1.3). For all solutions, the relative dielectric constant inside the line was maintained constant at a value of 1.0.

For a point-matching solution, the total capacitance in air was computed to be $C_T/\epsilon_0(b-a)=4.0759$. Table 4 gives values of the ratio of C_f/C_0 for different values of the half-space permittivity obtained from the point-matching solution.

For comparison, a solution of the same problem was performed using Galerkin's method. Only minor modifications were required to adapt the main program for constructing the system matrix and the right hand side elements (3.38). The double surface integration subroutine performed the integration using Legendre-Gauss quadrature with 9×10 abscissae.

TABLE 4

C_f/C_0 vs. ϵ for 50- Ω air lines with groundplanes (MOM).

ϵ	C_f/C_0
2.0	0.0739
5.0	0.0962
10.0	0.1079
20.0	0.1150
40.0	0.1190
60.0	0.1201

Diagonal elements were computed using the trapezoidal rule with 20 x 20 abscissae.

The value of the total capacitance found using Galerkin's method was $C_T/\epsilon_0(b-a) = 4.0660$. Values of C_f/C_0 obtained with Galerkin's method were very close to the values obtained with the point-matching solution (within 1%).

Solutions for the capacitance of the configuration of Fig. 2(c), a 50- Ω air line without a groundplane, were performed for different values of outer-conductor outer-radius. This was done to observe the effects of outer-conductor thickness on the capacitance. Table 5 gives values of the total capacitance and values of C_f/C_0 versus ϵ for three different values of the ratio of outer-conductor outer-radius to inner-conductor radius c/a . Note that in all cases, the ratio of b/a was fixed at 2.303 as required for 50- Ω air lines.

Teflon-Dielectric Lines:

TABLE 5

C_f/C_0 vs. ϵ and $C_T/\epsilon_0(b-a)$ for 50- Ω air lines without groundplanes (MOM).

	$c/a=3.5$	$c/a=3.0$	$c/a=2.5$
$C_T/\epsilon_0(b-a)$	4.0697	4.0538	4.0112
ϵ	C_f/C_0	C_f/C_0	C_f/C_0
2.0	0.0677	0.0675	0.0666
5.0	0.0882	0.0878	0.0866
10.0	0.0989	0.0985	0.0970
20.0	0.1055	0.1050	0.1034
40.0	0.1092	0.1087	0.1069
60.0	0.1105	0.1091	0.1082

Solutions were obtained for the configuration of Fig. 2(b), a 50- Ω teflon-dielectric line with a groundplane. The radius of the groundplane was taken to be approximately 3 times the radius of the outer-conductor. The coaxial line dimensions were the same as those given in the FEM analysis of open-ended teflon-dielectric lines and the line dielectric constant was maintained at 2.05.

The computed total capacitance was found to be $C_T/\epsilon_0(b-a)=2.3906$. Table 6 gives values of C_f/C_0 versus ϵ for this configuration.

For teflon lines without groundplanes (Figs. 2(e) and (f)), the line dimensions were taken to be the same as in the FEM analysis. They are defined in Fig. 11 where the case $d/a=1.0$ corresponds to an open-ended teflon line without a groundplane or a capacitive disk. The total capacitance calculated for this case was $C_T/\epsilon_0(b-a)=2.3830$. Table 7 gives values of C_f/C_0 versus ϵ for this configuration.

TABLE 6

C_f/C_o vs. ϵ for 50-n teflon-dielectric lines with ground-planes (MOM).

ϵ	C_f/C_o
2.0	0.0844
5.0	0.1284
10.0	0.1589
20.0	0.1815
40.0	0.1957
60.0	0.2010

TABLE 7

C_f/C_o vs. ϵ for teflon-dielectric lines without ground-planes (MOM).

ϵ	C_f/C_o
2.0	0.0971
5.0	0.1441
10.0	0.1769
20.0	0.2011
40.0	0.2150
60.0	0.2206

Table 8 gives values of the total capacitance $C_T/\epsilon_o(b-a)$ and the ratio C_f/C_o (for a half-space permittivity $\epsilon=60.0$) versus the ratio of disk radius to inner-conductor radius d/a .

Also, values of the total capacitance (normalized to the value of capacitance corresponding to $d/a=1.0$) are plotted logarithmically against values of d/a in Fig. 14 while the ratio C_f/C_o is plotted versus the half space permittivity ϵ for different values of d/a in Fig. 15

TABLE 8

$C_T/\epsilon_0(b-a)$ and C_f/C_0 (for $\epsilon=60.0$) versus b/a for teflon-dielectric lines with a capacitive disk (MOM).

d/a	$C_T/\epsilon_0(b-a)$	C_f/C_0 ($\epsilon=60.0$)
1.0	2.3830	0.2206
1.2	2.9686	0.3458
1.4	3.8899	0.4907
1.6	5.1214	0.6575
1.8	6.6755	0.8292
2.0	8.5917	0.9976
2.5	15.602	1.380
3.0	30.320	1.696

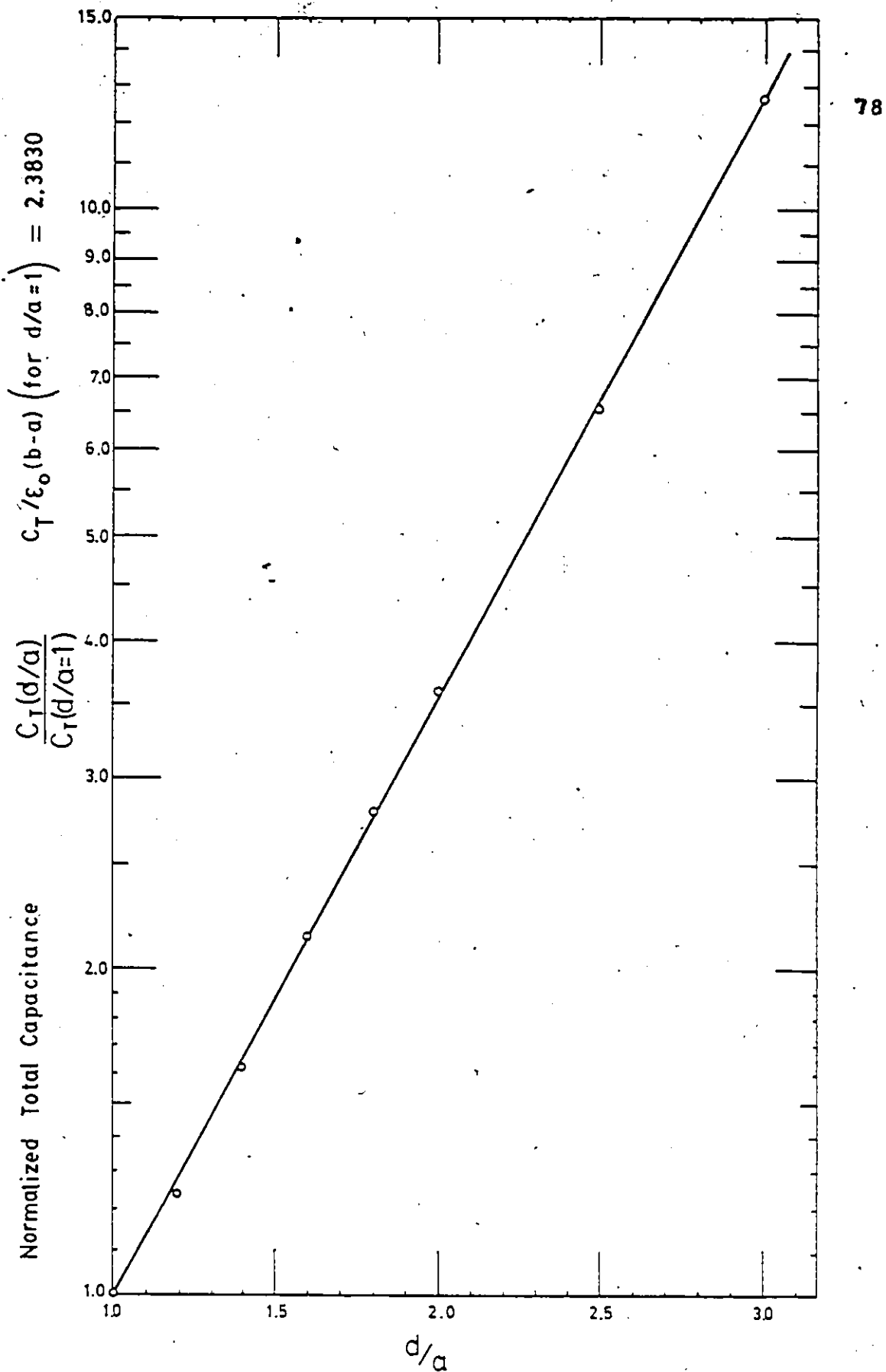


Figure 14: Total capacitance $C_T/C_T(d/a=1)$ versus d/a for teflon-dielectric lines with capacitive disk (MOM).

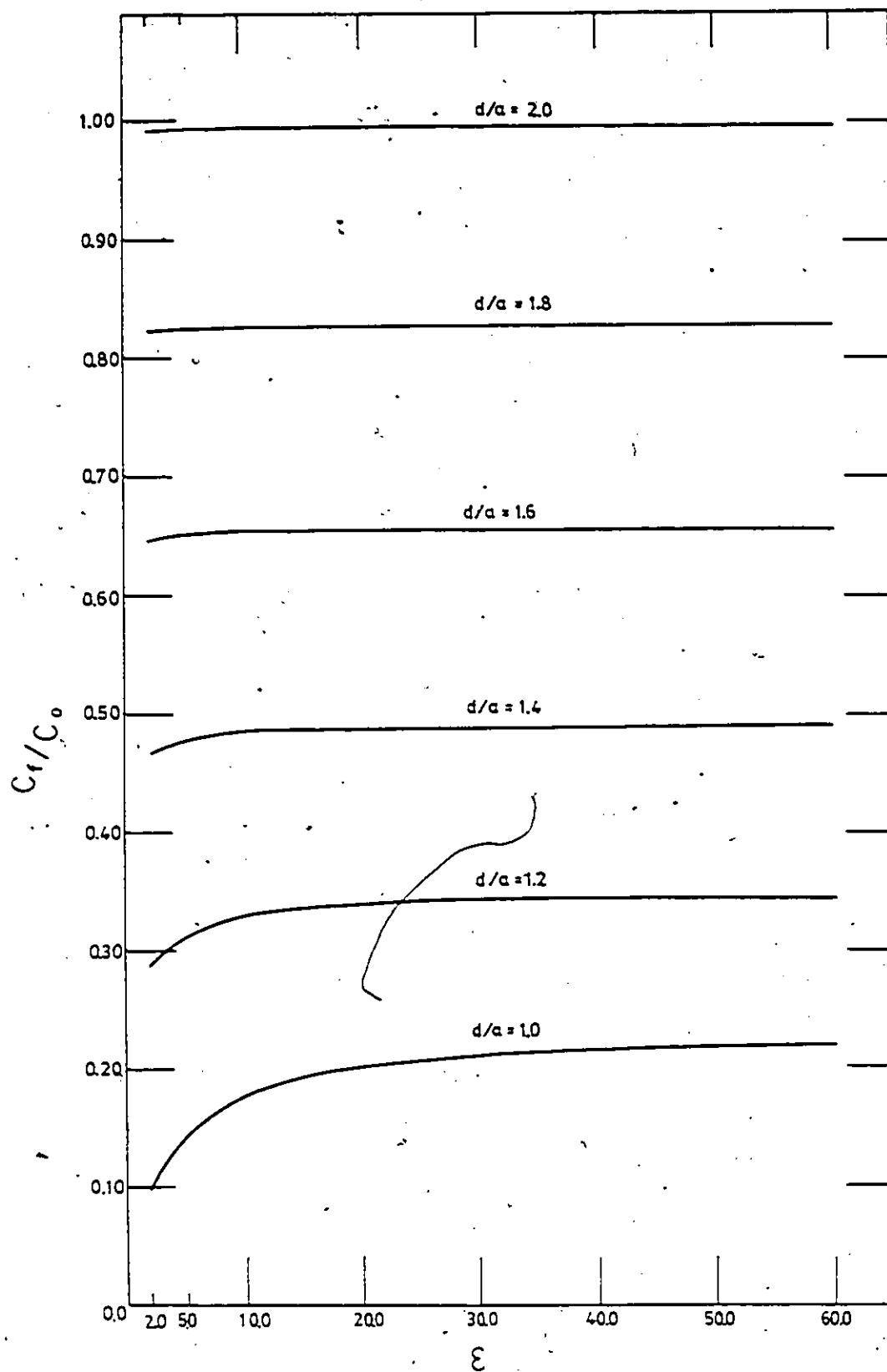


Figure 15: C_f/C_0 versus ϵ for tefflon-dielectric lines with capacitive disk (MOM).

Chapter V

DISCUSSION AND CONCLUSION

In this thesis five configurations of open-ended coaxial structures have been analysed in terms of their static or low frequency equivalent circuits. These structures are intended for use as sensors for the measurement of in-vivo biological permittivities. Two general numerical methods were used in the analysis, the Finite Element Method (FEM) and the Method of Moments.

Chapter I gave a description of the equivalent "lumped" circuit of the sensor. A linear model relating the sample permittivity to the fringing capacitance of the sensor yields closed-form expressions for the relative dielectric constant and loss factor as a function of the input reflection coefficient and frequency. This model is valid at low frequencies where the capacitances are frequency invariant and for a range of sample permittivities where the fringing capacitance varies linearly with ϵ .

Analytical results corresponding to an idealized sensor configuration are also examined in Chapter I. These results, although useful for indicating frequency limitations do not give any information about the effect of the sample permittivity on the equivalent circuit and hence the param-

ters of the linear model. This was the justification for proceeding to an analysis using numerical methods which are capable of solving conductor-dielectric problems.

Chapter II gave a brief overview of the general numerical methods used to solve linear operator equations of which the static conductor-dielectric problem is a member. The two approaches selected for the analysis were the Finite Element Method (FEM) and the Method of Moments (MOM). The former solves Laplace's equation in a region divided into triangular elements subject to the boundary conditions surrounding the region. A set of linear algebraic equations is constructed by employing the Rayleigh-Ritz procedure for each element, making sure that solutions match on the boundaries between elements.

The latter method solves the set of coupled integral equations which describe the conductor-dielectric problem. The unknown quantity is the distribution of the free and polarization charge residing on conductors held at fixed potentials and dielectric interfaces respectively. The solution proceeds by dividing all charged surfaces into subsections and using simple functions to approximate the charge distribution on each. A system of equations is obtained by utilizing the Method of Moments over the assumed charge distribution.

In essence, both methods transform linear operator equations into a system of algebraic equations which can be

solved by standard matrix methods on a digital computer. Both are characterized by their ability to handle complicated conductor and dielectric shapes since trial functions are defined over only a small portion of the problem.

Chapter III gave details of the application of FEM and MOM to problems having two-dimensional rotational symmetry. The coaxial sensor problem, which is bounded on one side and unbounded on the other, posed special problems to the application of both methods. The modifications required in order to use the Finite Element Method, which is best suited for bounded problems, and the integral equation method (MOM), which is suited for unbounded problems, are presented in this chapter. Also the methods for calculating capacitances and in particular for determining the internal and external fringing capacitances C_i and C_o respectively, were discussed.

Chapter IV presented the numerical results as well as giving brief descriptions of the two FORTRAN programs.

5.1 DISCUSSION

Since the primary purpose of this work was to perform the analyses and not to investigate the numerical methods in general, no attempt was made at estimating the accuracy of either method. The accuracy of a particular solution depends on a number of factors such as the size of elements or subsections, the degree of the approximating functions and computational errors such as round-off. Due to the irregu-

lar shape of the boundaries and the non-uniform size of elements or subsections, extrapolation methods similar to those used in finite difference solutions would be difficult to implement. Also, the gross errors incurred in the Finite Element Method by arbitrarily bounding the problem and in the Moment Method by the use of images contribute to the difficulty in estimating accuracies.

In the Finite Element Method this difficulty is compounded by a cancellation of errors which may occur due to the artificial imposition of a Neumann boundary. Ordinarily a variational method such as FEM gives a value of capacitance greater than the true value. However, since the region of solution is truncated, the amount of stored energy calculated will be less, thus yielding a smaller capacitance. Therefore the FEM results cannot be considered as an upper bound on the true value of capacitance.

Throughout this work certain observations were made on the convergence and accuracy of solutions. It was found that solutions converged rather slowly for FEM as the number of triangles increased. For example, in the configuration of a 50- Ω air line with a groundplane, a doubling of the number of triangles from 62 to 125 decreased the total capacitance by only 20%. In the MCM solutions, it was observed that small changes in the size and number of subsections produced negligible change in the capacitance. As an example, a change in the capacitance of 0.05% was observed when

the number subsections was increased from 74 to 78 for the configuration of a 50- Ω air line with a groundplane. Also, for the same solution, the calculated charge densities at a location far from the aperture (a region of almost uniform charge density) differed from the theoretical charge density of an infinite line by 0.15% for the inner conductor and 0.07% for the outer conductor. On the other hand for FEM solutions, slight variations of the size and distribution of triangles gave more noticeable changes in the capacitance.

In terms of programming, the Finite Element Method produces a simpler and more efficient program than the Moment Method. Running times for a typical problem with approximately the same number of unknowns were 5 seconds for FEM compared to 90 seconds for MOM. Since both programs used the same subroutine for solution of the system of equations, the difference in running times is mostly due the time spent constructing the system.

When using a large number of elements or subsections, it was found that coding and inputting data for FEM was tedious and time consuming since for each triangle, the location and identifying number of all three vertices must be coded. On the other hand, coding and inputting data to MOM was relatively easy even for large numbers of subsections. This is due to the fact that only the end-points of each subsection are inputted to the program. This permits easy modification of the problem configuration and checking of the input data.

A comparison of numerical results for the 50- Ω air line with a groundplane between FEM, MOM and the analytical expression (1.5) is given in Table 9.

TABLE 9

Comparison of total capacitance obtained from FEM, MOM and (1.5).

	Equation (1.5)	FEM	MOM Collocation	Galerkin
$C_T/\epsilon_0(b-a)$ Fig. 2(a)	4.295	4.311	4.076	4.066

Keeping in mind that (1.5) was derived in an approximate manner, the FEM value is very close to the analytical value while the MOM (collocation) result is approximately 5.0% below it. The closeness of the FEM result may be misleading due to the possible cancellation of errors mentioned previously.

A comparison of the values obtained from the collocation (or point-matching) and Galerkin's method indicate that they are within 0.3% of each other. Also it was mentioned in Section 4.2 that the ratio of C_T/C_0 calculated from Galerkin's method did not vary appreciably from the point-matching results. On the basis of these results it was subsequently decided to concentrate on the point-matching scheme throughout the rest of the work since the algorithm is computationally more efficient than Galerkin's scheme.

The relative values of C_f and C_o also show a discrepancy between MOM and FEM solutions. A comparison of Tables 1 and 4 indicates that the ratio of C_f/C_o (for air lines with groundplanes) determined from MOM is approximately twice as large as for FEM for all values of ϵ . This is also true for teflon lines without groundplanes.

If the absolute values of C_f and C_o (both normalized to ϵ_o and $(b-a)$) computed from FEM and MOM are compared as in Tables 10 and 11 it may be seen that FEM gives smaller values of C_f and larger values of C_o than MOM.

TABLE 10

C_f and C_o (normalized to ϵ_o and $(b-a)$) for 50- Ω air lines with groundplanes. (FEM and MOM)

ϵ	$C_f/\epsilon_o (b-a)$		$C_o/\epsilon_o (b-a)$		ϵ
	FEM	MOM	FEM	MOM	
2.0	0.1497	0.2806	4.1611	3.7953	
5.0	0.1949	0.3579	4.1159	3.7180	
10.0	0.2172	0.3969	4.0936	3.6790	
20.0	0.2305	0.4205	4.0803	3.6554	
40.0	0.2378	0.4335	4.0730	3.6424	
60.0	0.2403	0.4370	4.0705	3.6389	

The larger values of C_o obtained from FEM (if the MOM values are taken to be more accurate) may be due to the large error incurred by arbitrarily bounding the problem. On the other hand, the larger values of C_f obtained from MOM may be due to the artificial unbounding of the problem i.e. the use of images, in order to apply an integral equation type solu-

TABLE 11

C_f and C_o (normalized to ϵ_o and $(b-a)$) for 50 Ω teflon-dielectric lines without groundplanes (FEM and MOM)

ϵ	$C_f/\epsilon_o (b-a)$		$C_o/\epsilon_o (b-a)$	
	FEM	MOM	FEM	MOM
2.0	0.1354	0.2091	2.3455	2.1731
5.0	0.1846	0.2982	2.2962	2.0840
10.0	0.2147	0.3560	2.2661	2.0261
20.0	0.2350	0.3967	2.2458	1.9855
40.0	0.2471	0.4215	2.2338	1.9607
60.0	0.2514	0.4306	2.2294	1.9516

tion. However, several observations lead one to suspect that is not the case. The fact that in the MOM solutions, the calculated charge densities far from the aperture were close to the theoretical values for an infinite line and that little change in the value of C_f/C_o was observed as the length of coaxial line was extended leads one to suspect that overall, the MOM values are more accurate than the FEM results. Also, values of C_f/C_o obtained by MOM are more in agreement with measured values reported in [8].

In the solutions for teflon lines with a capacitive disk, it is believed that the FEM results are less accurate than the MOM results although they do show the same trends in the capacitance as the disk radius increases. The erratic results of C_f/C_o obtained from FEM (shown in Fig. 13) are probably due to the fact that not enough Dirichlet nodes were used to define the disk. It appears that FEM is poorly suited for this problem because of the inordinate number of

triangles required to model the large variations of the fields in the vicinity of the disk and the fields in the half-space.

In terms of the circuit model of the sensor and the two permittivity expressions for ϵ' and ϵ'' ((1.3) and (1.4)) the value of C_0 has the greatest effect on the measured values of ϵ' and ϵ'' . This is especially true if C_f/C_0 is much less than unity or the value of ϵ' being measured. An ideal situation occurs if C_0 is independent of the permittivity being measured. From Table 11 it may be seen that C_0 , for air lines with groundplanes, changes by less than 2.2% when going from a half-space permittivity of 2.0 to 60.0 (for MOM solutions). The change is under 1.2% in the range $10.0 < \epsilon < 60.0$ which corresponds to biological materials. Thus the linear circuit model given by (1.1) is a good approximation for this configuration. The variation of C_f with ϵ (approximately 40%) contributes little error in the calculation of ϵ' from (1.3) since C_f/C_0 is very small compared to most values of ϵ' encountered in biological materials.

Likewise, the variation in C_0 for teflon lines without groundplanes is 11.3% for $2.0 < \epsilon < 60.0$ and 3.7% for $10.0 < \epsilon < 60.0$ (for MOM solutions).

For teflon lines with a capacitive disk, the variation in C_f/C_0 with ϵ decreases as the disk radius increases as shown by Fig. 16. This implies that the value of C_0 becomes more independent of ϵ as the disk radius increases. For example,

the variation in C_0 for $d/a=1.2$ is 4.5% for $2.0 < \epsilon < 60.0$ and 1.4% for $10.0 < \epsilon < 60.0$. For the case where $d/a=2.0$, the total variation is under 0.3% for the entire range of ϵ . Table 12 gives values of C_f and C_0 (normalized to ϵ_0 and $(b-a)$) for different disk radii. Since these quantities change little over the range of ϵ , especially for larger values of d/a , only values for $\epsilon=60.0$ are given.

TABLE 12

C_f and C_0 (normalized) versus d/a for teflon lines with a capacitive disk for $\epsilon=60.0$ (FEM and MOM).

d/a	$C_f/\epsilon_0(b-a)$		$C_0/\epsilon_0(b-a)$	
	FEM	MOM	FEM	MOM
1.0	0.2514	0.4306	2.2294	1.9516
1.2	0.3522	0.7627	2.7909	2.2058
1.4	----	1.2805	----	2.6095
1.6	1.6625	2.0315	3.8980	3.0898
1.8	----	3.0260	----	3.6495
2.0	4.0626	4.2906	5.1180	4.3011
2.5	9.9779	9.0466	7.4818	6.5554
3.0	----	19.0700	----	11.2495

The MOM data in Table 12 is also plotted in Fig. 16. The values of C_f and C_0 are plotted versus d/a and are normalized to the values of C_f and C_0 corresponding to $d/a=1.0$ (i.e. the case of no disk).

The effect of the groundplane on the total capacitance may be observed from the MOM solutions for air lines without groundplanes (Table 5). The results show that the total capacitance $C_T/\epsilon_0(b-a)$ decreases as the ratio c/a decreases

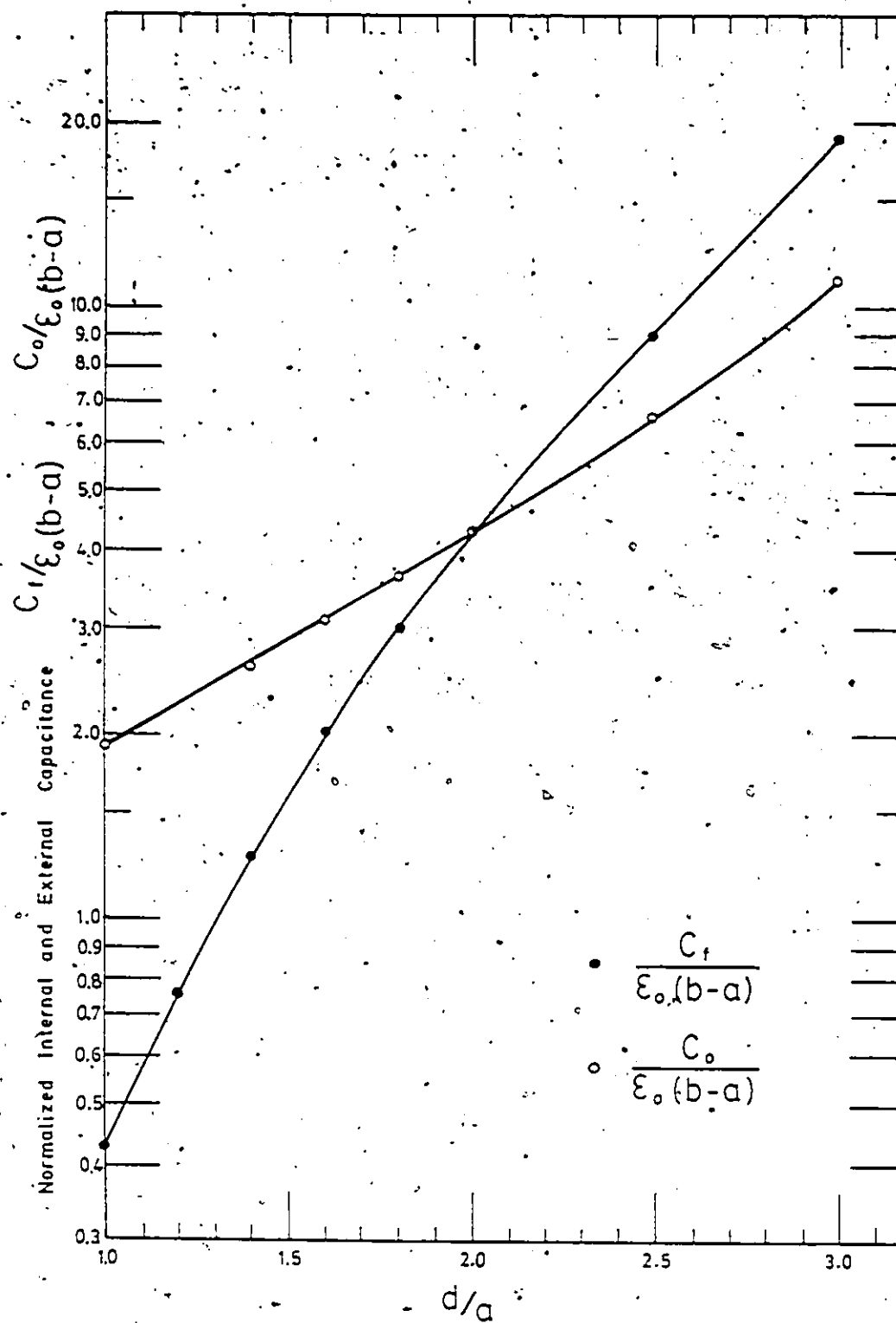


Figure 16: C_t and C_o (normalized) versus d/a for tefflon lines with capacitive disk (corresponding to $\epsilon=60.0$) (MOM).

although the total change is very small. In fact the normalized capacitance changes by only 1.6% when going from a full groundplane ($c/a = 7$) to the case where $c/a=2.5$ for 50- Ω air lines having $b/a=2.3$. A similar result occurs with teflon lines where the change is less than 1% for a similar reduction in the groundplane size. Although these results are approximate, they indicate that the groundplane has very little effect on the fringing fields and subsequently on the capacitance. From Tables 4 and 5 it is also apparent that the groundplane size has very little effect on the ratio of C_f/C_o .

5.2 CONCLUSION

It is hoped that these results will provide useful additional data for the selection and design of sensors for dielectric measurement. It must be kept in mind however that they are approximate and were obtained from static solutions.

The sensor configuration with a capacitive disk may find use in measurements at lower frequencies where larger capacitances are required. The addition of the disk increases the capacitance while maintaining the dimensions of the coaxial line constant. Another desirable consequence of using a disk to increase the capacitance appears to be that the values of C_f and C_o become more insensitive to the sample permittivity. Qualitatively, this may be due to the re-

duced area of the dielectric interface that results from the presence of the disk. Also, because of the reduced aperture area of these sensors it is expected that radiation losses will be less than for an equivalent-sized line without a disk.

From a design standpoint, increases in C_0 of up to 100% may be achieved for values of d/a up to 2.0 (recall that for teflon lines $b/a=3.27$). At this value of d/a , the two capacitances C_f and C_0 become approximately equal.

Information about the depth of penetration of the fields away from the aperture may be obtained from consideration of the FEM calculations. For air lines with groundplanes it was found that most of the stored energy was contained in an approximate hemisphere with a radius equal to 2.5 times the outer-conductor radius (for $\epsilon=1.0$). This may be taken as a general guideline for the minimum size of the sample or where spatial variations of the permittivity are being measured, the degree of resolution. With lossy dielectrics it is expected that the fields will decay more rapidly hence the minimum sample size would be smaller.

Numerical results from both the Finite Element Method and the Method of Moments show the linear model, which relates the fringing capacitance of the sensor to the sample permittivity, to be a good approximation especially over the range of permittivities encountered in biological materials.

An interesting extension of this work would be to perform a dynamic analysis of these sensors using a general numerical method. The variations of the external and internal capacitances with frequency for different sensor configurations could then be examined. Also, since the programs used in the present static analysis are able to solve problems with arbitrary conductors and dielectrics, other configurations of sensors could be investigated.

Appendix A

GREEN'S FUNCTION SINGULARITIES

The Green's function for rotationally symmetric systems, given by (3.12) or (3.13), has a singularity which behaves logarithmically as the field point approaches the source point. This may be seen by the fact that k in (3.13), approaches unity as (r, z) approaches (r', z') and it may be shown [30, p.591] that:

$$\lim_{k \rightarrow 1} F(k) = \frac{1}{2} \ln \left| \frac{16}{1 - k^2} \right| \quad (A.1)$$

The derivatives of the Green's function, given by (3.15) and (3.16) contain terms corresponding to the derivative of the elliptic integral $F'(k)$. The behaviour of $F'(k)$ as k approaches unity may be determined by differentiating the right hand side of (A.1) with respect to k :

$$\lim_{k \rightarrow 1} F'(k) = \frac{k}{1 - k^2} \quad (A.2)$$

The evaluation of the diagonal elements $K_{i,i}$, for the pulse expansion and point matching solution (given by (3.37)) requires the integration of the Green's function

over an interval in which it is singular. That is, when the variable of integration s' equals the mid-point of the subsection Δs_j . Three methods of handling the integration of the singular Green's function were attempted.

The first method involved integrating the Green's function numerically using the trapezoidal rule [31] with a large number of abscissae or integration points. An even number of abscissae must be used so that none

lie on the singular point. With this method, the accuracy of the evaluation of $K_{i,i}$, for a given number of abscissae, depends on the length of the subsection.

The second method uses the well known technique of extracting the singularity from the integrand and integrating it analytically [20]. The procedure for accomplishing this consists of adding and subtracting from the integrand, the function which represents the behaviour of the integrand near the singularity. Denoting the singular part of the Green's function as $G_S(s|s')$, the diagonal element $K_{i,i}$ may be written as:

$$K_{i,i} = \int_{\Delta s_j} [G(s_i|s') - G_S(s_i|s')] ds' + \int_{\Delta s_j} G_S(s_i|s') ds' \quad (A.3)$$

The second integral in (A.3) may be evaluated analytically while the first integral, which is no longer singular, may be evaluated numerically.

Using (A.1) and (3.13), the singular part of the Green's function may be formulated as:

$$G_S(r_i, z_i | r', z') = - \frac{1}{2\pi r_i} \ln \left| \frac{\sqrt{(r_i - r')^2 + (z_i - z')^2}}{8r_i} \right| \quad (A.4)$$

where the primed coordinates have been allowed to approach (r_i, z_i) except in the argument of the natural logarithm.

The integration of $G_S(s_i | s')$ may be performed by using the surface integration procedure outlined in Appendix B. Using this procedure, the singular integral may be expressed as:

$$\begin{aligned} \int_{\Delta s_i} G_S(s_i | s') ds' &= - \frac{\Delta l}{4\pi r_i} \int_{-1}^1 \ln \left| \frac{u \Delta l}{16r_i} \right| \left(r_i + u \frac{\Delta r_i}{2} \right) du \\ &= \frac{\Delta l}{2\pi} \left[1 - \ln \left| \frac{\Delta l}{16r_i} \right| \right] \end{aligned} \quad (A.5)$$

where $\Delta l = \sqrt{(r_i^+ - r_i^-)^2 + (z_i^+ - z_i^-)^2}$ represents the length of the "ith subsection in two dimensions.

The non-singular integral in (A.3) was evaluated using Legendre-Gauss quadrature with 16 abscissae. The accuracy using this technique is still somewhat sensitive to the length of the subsection for a given number of abscissae.

If double surface integrals are required, the singular part of the Green's function may be first integrated analytically over one variable and integrated numerically over the second variable. The evaluation of the non-singular part is

performed numerically in the same manner as the single surface integration.

The third method which was attempted consists of writing the Green's function as in (3.12) and directly applying the procedure for surface integration given in Appendix B. The resulting expression for $K_{i,i}$ is:

$$K_{i,i} = \frac{\Delta l}{4\pi} \int_{-1}^1 \int_0^\pi \frac{(r_i + u \Delta r_i / 2) d\theta du}{\sqrt{4r_i (r_i + u \Delta r_i / 2) \sin^2 \theta + u^2 \Delta l^2 / 4}} \quad (A.6)$$

Integrating analytically with respect to u gives:

$$K_{i,i} = \frac{1}{\pi} \int_0^{\pi/2} \frac{\Delta r_i}{\Delta l} [a(r_i^+, \theta) - a(r_i^-, \theta)] d\theta \\ + \frac{1}{\pi} \int_0^{\pi/2} r_i \left(1 - 2 \frac{\Delta r_i^2}{\Delta l^2} \sin^2 \theta\right) \ln \left| \frac{a(r_i^+, \theta) + \frac{\Delta l}{2} + 2r_i \frac{\Delta r_i}{\Delta l} \sin^2 \theta}{a(r_i^-, \theta) - \frac{\Delta l}{2} + 2r_i \frac{\Delta r_i}{\Delta l} \sin^2 \theta} \right| d\theta \quad (A.7)$$

where:

$$a(x, \theta) = \sqrt{\Delta l^2 / 4 + 4r_i x \sin^2 \theta} \quad (A.8)$$

and

$$\Delta l = \sqrt{\Delta z_i^2 + \Delta r_i^2}$$

$$\Delta z_i = z_i^+ - z_i^-$$

$$\Delta r_i = r_i^+ - r_i^-$$

The expression (A.7) may be integrated with respect to θ numerically using the rectangle rule to give a value for

$K_{i,j}$. This approach does not eliminate the singularity problem since the integrand of the second integral is singular at $\theta=0$. However, using a large number of integration points, the element $K_{i,j}$ may be calculated with an accuracy which is independent of the length of the subsection. This property is useful when very long subsections are used in regions where the charge density is almost uniform. Equation (A.7) appears at first hand, to be very complicated however it reduces to simpler expressions when only annular or cylindrical subsections are considered.

The above three techniques for handling the singularity problem were implemented in a test problem using a pulse expansion and point matching solution. The problem consisted of a length of 50Ω coaxial air line opening into free space. In all three cases, off-diagonal elements were computed using 16-point Legendre-Gauss quadrature. Images were used to simulate a Neumann boundary at one end of the line, effectively doubling its length.

The computed charge density in the middle of the line (actually the line and its image) was used as a criterion for accuracy since in this region, the charge density is uniform and is easily calculated analytically.

The first method (trapezoidal rule) and the third method (direct integration over the subsection) gave similar results for the same number of integration points (≈ 800). The differences between the computed inner conductor charge

densities and the theoretical values were less than 0.4% for both methods. The second method, which involved extraction of the singularity, gave results which fluctuated between adjacent subsections, about the value of charge density calculated by the other two methods. The magnitude of this ripple in the charge density was approximately 2% of the theoretical value.

Evaluation of the diagonal elements $L_{i,i}$ given by (3.39) involves integration of the normal derivative of the Green's function which in general, possesses a singularity in the mid-point of the interval. For the problems encountered in this work however, the normal derivative equals the derivative with respect to z since all interface subsections are annular (on planes of constant z).

For an annular interface subsection, where $z=z'$ in (3.15), the diagonal element $L_{i,i}$ is given by:

$$L_{i,i} = (\epsilon_1 - \epsilon_2) \int_{r_i^-}^{r_i^+} \frac{\partial G}{\partial z}(r_i, z_i | r', z_i) r' dr' \quad (A.9)$$

From (3.15) it may be seen that the integrand vanishes at all points in the integration interval except perhaps in the neighbourhood of $r'=r$ where $F(k)$ and $F'(k)$ possess singularities of the form (A.1) and (A.2). On the other hand, by writing $\partial G/\partial z$ in terms of finite differences, it may be seen that the element $L_{i,i}$ vanishes.

Using finite differences, $L_{i,i}$ may be written as:

$$L_{i,j} = (\epsilon_1 - \epsilon_2) \lim_{\delta \rightarrow 0} \int_{r_i^-}^{r_i^+} \frac{G(r_i, z_i + \delta/2 | r', z_i) - G(r_i, z_i - \delta/2 | r', z_i)}{\delta} r' dr' \quad (A.10)$$

Since z and z' appear in the Green's function as the factor $(z - z')$, the following holds at each point in the interval of integration:

$$G(r_i, z_i + \delta/2 | r', z_i) = G(r_i, z_i - \delta/2 | r', z_i) \quad (A.11)$$

and the diagonal element vanishes.

From finite difference considerations, it may be seen that diagonal elements $L_{i,i}$ for strictly cylindrical subsections do not vanish since $G(r_i + \delta/2, z_i | r_i, z_i) \neq G(r_i - \delta/2, z_i | r_i, z_i)$. This is due to the fact r and r' appear in the Green's function in a more complicated manner than z and z' .

Appendix B

SURFACE INTEGRATION PROCEDURE

All of the different applications of the method of subsections require the computation of single or double surface integrals. Since surfaces are represented two dimensionally by plane curves in rotationally symmetric problems, the surface integral reduces to a line integral. The elementary surface may be written as:

$$ds = r \, dl \quad (B.1)$$

where the factor 2π does not appear since it has been accounted for in the derivation of the Green's function (3.12).

Consider the general surface integral over the "i"th subsection:

$$I = \int_{\Delta S_i} K(r, z) \, r \, dl \quad (B.2)$$

where the integrand, usually the Green's function or its normal derivative, is a function of the surface coordinates r and z . The end points of ΔS_i are located at (r_i^+, z_i^+) and (r_i^-, z_i^-) .

The term r appearing alone in (B.2) may be lumped with the integrand to produce a new integrand $L(r, z) = rK(r, z)$. The surface (line) integral may now be written:

$$I = \int_{\Delta S_i} L(r, z) \, dl \quad (B.3)$$

The line integral (B.3) may be evaluated by replacing the r coordinate by its z dependence $r=h(z)$, provided r can be expressed explicitly by a function of z on the surface. The elementary line length becomes:

$$dl = \sqrt{1 + h'(z)^2} \, dz \quad (B.4)$$

and upon substitution of (B.4) into (B.3), the surface integral over the interval (r_i^+, z_i^+) to (r_i^-, z_i^-) may be written as:

$$I = \int_{z_i^-}^{z_i^+} L(h(z), z) \sqrt{1 + h'(z)^2} \, dz \quad (B.5)$$

If the surface is represented by frustrums of a right circular cone, $h(z)$ may be written as (for the i th subsection shown in Fig. 9):

$$h(z) = \left(\frac{r_i^+ - r_i^-}{z_i^+ - z_i^-} \right) (z - z_i^-) + r_i^- \quad (B.6)$$

The line integral (B.3) becomes an integral over z given by:

$$I = \int_{z_i^-}^{z_i^+} \sqrt{1 + \left(\frac{r_i^+ - r_i^-}{z_i^+ - z_i^-} \right)^2} F(h(z), z) \, dz \quad (B.7)$$

Equation (B.7) may be put into a convenient form for use with high order numerical quadrature formulas if the change of variables from z to u is made:

$$z = \left(\frac{z_i^+ + z_i^-}{2} \right) + u \left(\frac{z_i^+ - z_i^-}{2} \right), \quad dz = \left(\frac{z_i^+ - z_i^-}{2} \right) du \quad (B.8)$$

so that the limits of integration are now $u = -1$ to $u = 1$ for the general subsection. Equation (B.6) is now a function of u given by:

$$\begin{aligned} h(z) = h(u) &= \left(\frac{r_i^+ - r_i^-}{z_i^+ - z_i^-} \right) \left(\frac{z_i^+ + z_i^-}{2} + u \frac{z_i^+ - z_i^-}{2} - z_i^- \right) + r_i^- \\ &= \left(\frac{r_i^+ + r_i^-}{2} \right) + u \left(\frac{r_i^+ - r_i^-}{2} \right) \end{aligned} \quad (B.9)$$

Upon substitution of (B.8) and (B.9) into (B.7), the surface integral may be written as:

$$I = \frac{\sqrt{(r_i^+ - r_i^-)^2 + (z_i^+ - z_i^-)^2}}{2} \int_{-1}^1 L \left(\frac{r_i^+ + r_i^-}{2} + u \frac{r_i^+ - r_i^-}{2}, \frac{z_i^+ + z_i^-}{2} + u \frac{z_i^+ - z_i^-}{2} \right) du \quad (B.10)$$

Equation (B.10) is in a convenient form for use with Gaussian quadrature formulas such as Legendre-Guass quadrature, where the weights and abscissas are normally given for an interval of integration between -1 and 1 [21]. The sur-

face integral (B.10) is valid for the limiting cases of annular or cylindrical subsections. In the case of annular subsections, where $z_i^+ = z_i^-$, (B.10) reduces to an integration over r . For cylindrical subsections, where $r_i^+ = r_i^-$, (B.10) reduces to an integration over z .

To evaluate double surface integrals such as those encountered in Galerkin's method, the above procedure is applied twice, once for each set of coordinates.

Appendix C

COMPUTER PROGRAM LISTINGS

This appendix contains listings of the two computer programs used in this work. The first is a listing of the Finite Element Method program and the second is a listing of the Method of Moments program. Since both programs utilize the same subroutine for the solution of a system of equations (called SIST), it is reproduced here only in the FEM program.

C.1 FINITE ELEMENT METHOD

```

C *****
C * Program name: FEM
C *
C * Purpose: Solve rotationally symmetric conductor-
C *           dielectric problems using the Finite
C *           Element Method.
C *
C *           Gregory Gajda
C *           Dept. Electrical Engineering
C *           University of Ottawa
C *****
C   IMPLICIT REAL*8 (A-H,O-Z, )
C   REAL*8 F(3,3),S(70,70),SI(3),B(70),X(72),Y(72),
C   SPOT(72),IV1(113),IV2(113),IV3(113),IV(113,3)
C   REAL*8 EPS(113)
C
C   Coordinates (X(i),Y(i)) represent the z and r
C   coordinates of the "i"th triangle vertice.
C
C   DATA X/.75,.9,4*1.0,.75,3*1.0,.6,0.0,.75,.9/
C   DATA Y/3*.456,.25,.125,0.0,2*1.49,1.65,3*1.79/
C
C   POT(I) - potential of node (vertice) "i".
C   Set Dirichlet nodes to their assigned potentials
C   and Neumann and unknown nodes to zero potential.
C   DATA POT/6*1.0D0,66*0.0D0/
C
C   Each node (vertice) is assigned a number beginning
C   with the Dirichlet nodes.
C   IV1(I), IV2(I) and IV3(I) represent the three vertice
C   code numbers of the three vertices of triangle "i".
C
C   DATA IV1/7,7,8,13,14,14,16,16,17,18,19,19,21,21,22/
C   DATA IV2/13,8,14,14,16,15,18,17,2*19,21,20,23,22,24/
C   DATA IV3/2*14,15,16,2*17,2*19,20,21,2*22,2*24,25,3/
C
C   Functions Q,A1,A2,VOL used in computation
C   Q(X1,Y1,X2,Y2,X3,Y3)=X1*Y2-X2*Y1+X2*Y3-X3*Y2+X3*Y1-X1*Y3
161  FORMAT(5E17.7)
1   FORMAT(10I4//)
      DO 1000 I=1,113
        IV(I,1)=IV1(I)
        IV(I,2)=IV2(I)
        IV(I,3)=IV3(I)
1000  CONTINUE
C
C   NOPTL - total number of nodes
C   NOTR - total number of triangles
C   NDIR - total number of nodes with Dirichlet values
C   NIN - total number of triangles inside line
C   NIN=19
C   NOPTL=72

```

```

NOTR=113
NDIR=12

C
C   Enter values of the dielectric constant:
C   EPS3 - dielectric constant outside the line
C   EPS2 - dielectric constant inside the line
C   EPS3=1.0D0
C   EPS2=2.05D0
C   DO 71 I=1,NOTR
C       IF(I.LE.NIN) EPS(I)=EPS2
C       IF(I.GT.NIN) EPS(I)=EPS3
71  CONTINUE
C   Scale the coordinates:
C   SCALE=10.0D0
C   DO 72 I=1,NOPTL
C       X(I)=X(I)*SCALE
C       Y(I)=Y(I)*SCALE
72  CONTINUE
C   PRINT 1,NOPTL,NOTR,NDIR
C   NN=NOPTL-NDIR
C   Initialize computation matrices S and B
C   DO 6 I=1,63
C       B(I)=0.
C   DO 6 J=1,63
C       S(I,J)=0.
6   DO 16 ITR=1,NOTR
C       I1=IV(ITR,1)
C       I2=IV(ITR,2)
C       I3=IV(ITR,3)
C       PRINT 1,I1,I2,I3
C       DET=Q(X(I1),X(I1),X(I2),Y(I2),X(I3),Y(I3))
C       IF(DET.EQ.0.) GO TO 100
C       F(1,1)=(X(I2)*Y(I3)-X(I3)*Y(I2))/DET
C       F(1,2)=(X(I3)*Y(I1)-X(I1)*Y(I3))/DET
C       F(1,3)=(X(I1)*Y(I2)-X(I2)*Y(I1))/DET
C       F(2,1)=(Y(I2)-Y(I3))/DET
C       F(2,2)=(Y(I3)-Y(I1))/DET
C       F(2,3)=(Y(I1)-Y(I2))/DET
C       F(3,1)=(X(I3)-X(I2))/DET
C       F(3,2)=(X(I1)-X(I3))/DET
C       F(3,3)=(X(I2)-X(I1))/DET
C       ARR=(Y(I1)+Y(I2)+Y(I3))*(X(I1)*(Y(I3)-Y(I2))+X(I2)*
C       8(Y(I1)-Y(I3))+X(I3)*(Y(I2)-Y(I1)))/6.0D0
C       AR=DABS(ARR)
C       DO 23 I=1,3
C           L=IV(ITR,I)-NDIR
C           IF(IV(ITR,I).LE.NDIR) GO TO 23
C           DO 24 K=1,3
C               SI(K)=AR*(F(2,K)*F(2,I)+F(3,K)*F(3,I))*EPS(ITR)
24  CONTINUE
C       DO 14 K=1,3
C           IF(IV(ITR,K).LE.NDIR) GO TO 15
C           LC=IV(ITR,K)-NDIR
C           S(L,LC)=S(L,LC)+SI(K)

```

```

15  B(L)=B(L)-POT (IV (ITR,K) ) *SI (K)
14  CONTINUE
23  CONTINUE
16  CONTINUE
C
C    Solve the system of equations:
    CALL SIST(S,B,NN)
C
C-   Printout the node potentials:
    PRINT 161,(B(I),I=1,NN)
    DO 20 K=1,NN
      I=NDIR+K
20   POT(I)=B(K)
      PRINT 162,(I,POT(I),I=1,NOPTL)
162  FORMAT(4(' NODE',I4,D17.7//))
C
C    Calculation of stored energy:
C    ENER1 - total energy stored inside line
C    ENER2 - total energy stored outside line
    ENER1=0.0D0
    ENER2=0.0D0
    DO 51 ITR=1,NOTR
      I1=IV (ITR,1)
      I2=IV (ITR,2)
      I3=IV (ITR,3)
      DET=Q (X (I1) ,Y (I1) ,X (I2) ,Y (I2) ,X (I3) ,Y (I3) )
      AXT=(Y (I1) +Y (I2) +Y (I3) ) *(X (I1) *(Y (I3) -Y (I2) ) +X (I2) *
& (Y (I1) -Y (I3) ) +X (I3) *(Y (I2) -Y (I1) ) ) /6.0D0
      AR=DABS (AXT)
C    DELI - z-component of electric field
C    DELJ - r-component of electric field
      DELI=((Y (I2) -Y (I3) ) *POT (I1) + (Y (I3) -Y (I1) ) *POT (I2)
& + (Y (I1) -Y (I2) ) *POT (I3) ) /DET
      DELJ=((X (I3) -X (I2) ) *POT (I1) + (X (I1) -X (I3) ) *POT (I2)
& + (X (I2) -X (I1) ) *POT (I3) ) /DET
      RMODSQ=DELI**2+DELJ**2
C    RMOD - magnitude of electric field
      RMOD=DSQRT (RMODSQ)
      ENER=RMODSQ*AF*3.1415926535898D0*EPS (ITR)
      IF (ITR.LE.NIN) ENER1=ENER1+ENER
      IF (ITR.GT.NIN) ENER2=ENER2+ENER
51  CONTINUE
      ENTOT=ENER1+ENER2
      61  FORMAT (1X,'ENERGY IN LINE = ',D17.7)
262  FORMAT (1X,'ENERGY OUTSIDE LINE = ',D17.7)
263  FORMAT (1X,'TOTAL ENERGY = ',D17.7)
      PRINT 61,ENER1
      PRINT 262,ENER2
      PRINT 263,ENTOT
C
C    GO TO 99
91   FORMAT (20X,'SINGULAR MATRIX',I4)
100  PRINT 91,ITR
99   CONTINUE

```

```

      STOP
      END

C
C      SUBROUTINE SECTION
C      -----
C      SUBROUTINE SIST(A,B,N)
C
C      *****
C      Solution of a system by triangularization
C      and back substitution.
C      A-matrix of the system
C      B-right hand side ,contains solution
C      N-dimension of the system
C
C      *****
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 A(63,63),B(63)
      NM1=N-1
      D=1.
      DO 7 K=1,NM1
        BA=A(K,K)
        J=K
        KP1=K+1
C      Find max term in column
        DO 1 I=KP1,N
          IF(DABS(BA).GE.DABS(A(I,K))) GO TO 1
          BA=A(I,K)
          J=I
1        CONTINUE
        D=D*BA
        IF(J.LE.K) GO TO 3
        IF(DABS(D).EQ.0.) GO TO 9
C      Interchange rows
        DO 2 I=K,N
          H=A(K,I)
          A(K,I)=A(J,I)/BA
2        A(J,I)=H
          H=B(K)
          B(K)=B(J)/BA
          B(J)=H
        GO TO 5
C      Divide row by max coefficient
3      DO 4 I=KP1,N
4        A(K,I)=A(K,I)/BA
          B(K)=B(K)/BA
5      DO 7 I=KP1,N
        DO 6 J=KP1,N
6      A(I,J)=A(I,J)-A(I,K)*A(K,J)
7      B(I)=B(I)-A(I,K)*B(K)
        D=D*A(N,N)
        IF(DABS(D).EQ.0.0D0) GO TO 9
        B(N)=B(N)/A(N,N)
C      Back substitution
C

```



```
DO 8 I=1,NM1
K=N-I
KP1=K+1
DO 8 J=KP1,N
8  B(K)=B(K)-A(K,J)*B(J)
RETURN
9  PRINT 10
10 FORMAT(' SINGULAR SOLUTION')
RETURN
END
```

C.2 METHOD OF MOMENTS

```

C *****
C * Program name: PHOM *
C * Purpose: To solve conductor-dielectric problems *
C * using the Method of Moments (point matching) *
C * *
C * Greg Gajda *
C * Dept. of Electrical Engineering *
C * University of Ottawa *
C *****
C
C IMPLICIT REAL*8 (A-H,O-Z, )
C REAL*8 K(82,82),R(82),Z(82),RI1(82),RI2(82),
C &ZI1(82),ZI2(82),POT(82)
C INTEGER OUT PAR
C
C Coordinates ( RI1(I),ZI1(I) ) and ( RI2(I),ZI2(I) )
C represent the end points of the "i" th subsection.
C Number the subsections beginning with the those
C lying on conductor surfaces.
C DATA RI1/19*1.0,1.0,.95,.9,.85,.8,.7,.6,.5,.4,.3,/
C DATA RI2/19*1.0,.95,.9,.85,.8,.7,.6,.5,.4,.3,.15/
C DATA ZI1/0.0,.5,1.0,1.5,1.9,2.3,2.6,2.8,3.0,3.1/
C DATA ZI2/.5,1.0,1.5,1.9,2.3,2.6,2.8,3.0,3.1,3.2/
C
C POT(I) - value of potential on subsection "i":
C DATA POT/30*1.0D0,52*0.0D0/
C
C NPULT - total no. of subsections including interface.
C NDPUL - no. of subsections lying on conductors.
C NUNI - no. of subsections being held at 1 volt.
C NPULT=82
C NDPUL=70
C NUNI=30
C
C EPS3 - dielectric constant outside the line
C EPS2 - dielectric constant inside the line.
C EPS3=1.0D0
C EPS2=1.0D0
C
C If OUT PAR equals 1 do not print charge densities:
C OUT PAR=0
C
C Scaling of the dimensions:
C SCALE=10.0D0
C DO 87 I=1,NPULT
C RI1(I)=RI1(I)*SCALE
C RI2(I)=RI2(I)*SCALE
C ZI1(I)=ZI1(I)*SCALE
C ZI2(I)=ZI2(I)*SCALE
C 87 CONTINUE
C
C Calculation of coordinates of pulse centres:

```

```

DO 1 I=1, NPULT
  R(I) = (RI1(I) + RI2(I)) / 2.0D0
  Z(I) = (ZI1(I) + ZI2(I)) / 2.0D0
1 CONTINUE

C
C   Construction of upper system by numerical integration:
C   Set KCODE=1 for regular Green's function kernel:
  KCODE=1
  DO 2 I=1, NDPUL
    DO 3 J=1, NPULT
      IF((I.EQ.J).AND.(RI1(J).EQ.RI2(J))) GOTO 50
      CALL
    GAUSS(R(I), Z(I), RI1(J), RI2(J), ZI1(J), ZI2(J), SUM, KCODE)
    GOTO 51
  50 CALL DIRZ(R(I), Z(I), R(J), ZI1(J), ZI2(J), SUM)
C   Calculation of the "image" element:
  51 CALL
  GAUSS(R(I), Z(I), RI1(J), RI2(J), -ZI1(J), -ZI2(J), SMP, KCODE)
  K(I, J) = SUM + SMP
  3 CONTINUE
  2 CONTINUE

C
C   Calculation of the lower system:
  NNN = NDPUL + 1
  DO 32 I = NNN, NPULT
C   Set KCODE depending on orientation of "i"th element:
  KCODE=2
  IF(RI1(I).EQ.RI2(I)) KCODE=3
  DO 33 J=1, NPULT
    CALL
  GAUSS(R(I), Z(I), RI1(J), RI2(J), ZI1(J), ZI2(J), SUM, KCODE)
C   Calculation of the "image" element:
  CALL
  GAUSS(R(I), Z(I), RI1(J), RI2(J), -ZI1(J), -ZI2(J), SMP, KCODE)
  K(I, J) = (SUM + SMP) * (EPS3 - EPS2)
  IF(I.EQ.J) K(I, J) = K(I, J) - (EPS3 + EPS2) / 2.0D0
  33 CONTINUE
  32 CONTINUE

C
C   Solve the system for the charge pulse heights:
  CALL SIST(K, POT, NPULT)

C
C   Output Section:
  WRITE(6, 100) NPULT
100 FORMAT(1X, 'TOTAL NUMBER OF PULSES: NPULT=', I5)
  WRITE(6, 107) NDPUL
107 FORMAT(1X, 'NO. OF DIRICHLET PULSES: NDPUL =', I5)
  IF(OUT PAR.EQ.1) GOTO 305
  WRITE(6, 101)
101 FORMAT(1X, 'PULSE NO.  R-COORD.  Z-COORD.  CHARGE
DENSITY')
  WRITE(6, 102) (I, R(I), Z(I), POT(I), I=1, NDPUL)
102 FORMAT(1X, I4, 2X, F8.4, 2X, F8.4, 2X, D17.7)
  WRITE(6, 122)

```

```

NDIEL=NPULT-NDPUL
WRITE(6,108) NDIEL
108 FORMAT(1X,'NO. OF INTERFACE PULSES: NDIEL =',I5)
WRITE(6,102) (I,R(I),Z(I),POT(I),I=NNN,NPULT)
305 WRITE(6,122)
WRITE(6,351) EPS2,EPS3
351 FORMAT(1X,'EPS IN LINE = ',F8.4,' EPS OUTSIDE =
',F8.4)
C
C Calculation of the total charge and capacitance:
CAPOUT=0.0D0
CAPIN=0.0D0
DO 5 I=1,NUNI
CHAR=POT(I)*R(I)*DSQRT((RI2(I)-RI1(I))**2+(ZI2(I)-ZI1(I))**2)
IF(I.GT.NIN) CAPOUT=CAPOUT+CHAR*EPS3
IF(I.LE.NIN) CAPIN=CAPIN+CHAR*EPS2
5 CONTINUE
CAPOUT=CAPOUT*2.0D0*3.14159265359D0
CAPIN=CAPIN*2.0D0*3.14159265359D0
CAP=CAPIN+CAPOUT
WRITE(6,122)
WRITE(6,123)
123 FORMAT(1X,'CAPACITANCES NORMLZD TO FREE SPACE PERM :')
WRITE(6,124)
124 FORMAT(8X,'INSIDE',12X,'OUTSIDE',12X,'TOTAL')
WRITE(6,125) CAPIN,CAPOUT,CAP
125 FORMAT(1X,3D18.7)
122 FORMAT(1X,' ')
STOP
END

```

Subroutine Section:

```

C
C
C
C SUBROUTINE GAUSS(RI,ZI,RJM,RJP,ZJM,ZJP,S,KCODE)
C *****
C Single integration subroutine: utilizes Legendre-
C Gauss quadrature applied to a single integral.
C *****
C IMPLICIT REAL*8 (A-H,O-Z)
C REAL*8 F(16),U4(4),H4(4),U6(6),H6(6),U8(8),H8(8),
C U10(10),H10(10),U16(16),H16(16)
C COMMON U4,H4,U6,H6,U8,H8,U10,H10,U16,H16
C NIP - no. of integration points:
C NIP=16
C Calculate integrand values at each abscissa:
C DO 1 I=1,NIP
C RJ=((RJP+RJM)+U16(I)*(RJP-RJM))/2.0D0
C ZJ=((ZJP+ZJM)+U16(I)*(ZJP-ZJM))/2.0D0
C CALL KERNEL(RI,ZI,RJ,ZJ,FF,KCODE)
C F(I)=FF*RJ
C 1 CONTINUE
C Sum the integrand values multiplied by the weights:
C S=0.0D0

```

```

      DO 2 I=1,NIP
      S=S+H16(I)*F(I)
2 CONTINUE
      S=S*DSQRT((RJP-RJM)**2+(ZJP-ZJM)**2)/2.0D0
      RETURN
      END

```

```

C
      SUBROUTINE
DGAUSS(RIM,RIP,ZIM,ZIP,RJM,RJP,ZJM,ZJP,S,KCODE)
C
*****
C      Double integration subroutine: utilizes Legendre-
C      Gauss quadrature applied to a double integral.
C
*****
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 F(9,10),U10(10),H10(10),V9(9),M9(9)
      COMMON U10,H10,V9,M9
C      Calculate integrand values at each abscissa:
      DO 1 I=1,9
      RI=((RIP+RIM)+V9(I)*(RIP-RIM))/2.0D0
      ZI=((ZIP+ZIM)+V9(I)*(ZIP-ZIM))/2.0D0
      DO 2 J=1,10
      RJ=((RJP+RJM)+U10(J)*(RJP-RJM))/2.0D0
      ZJ=((ZJP+ZJM)+U10(J)*(ZJP-ZJM))/2.0D0
      CALL KERNEL(RI,ZI,RJ,ZJ,FF,KCODE)
      F(I,J)=FF*RI*RJ
2 CONTINUE
1 CONTINUE
C      Sum the integrand values multiplied by the weights:
      S=0.0D0
      DO 3 I=1,9
      DO 4 J=1,10
      S=S+M9(I)*H10(J)*F(I,J)
4 CONTINUE
3 CONTINUE
      S=S*DSQRT((RIP-RIM)**2+(ZIP-ZIM)**2)*DSQRT((RJP-RJM)**2
&(ZJP-ZJM)**2)/4.0D0
      RETURN
      END
C
      SUBROUTINE DIRZ(RI,ZI,RJ,Z1,Z2,SUM)
C
*****
C      Direct integration of the regular Green's function.
C
*****
      IMPLICIT REAL*8 (A-H,O-Z)
      F(X)=DLOG(DABS(((ZI-Z2)+DSQRT((RI+RJ)**2-4.D0*RI*RJ*X*X
&+(ZI-Z2)**2))/((ZI-Z1)+DSQRT((RI+RJ)**2-4.D0*RI*RJ*X*X
&+(ZI-Z1)**2))))
C      N - number of integration steps
      N=1000
      YN=N
      SUM=0.0D0
      DO 1 I=1,N

```

```

      YI=I
      X=DCOS((2.0D0*YI-1.0D0)*3.1415926535898D0/(4.0D0*YN))
      SUM=SUM+DABS(F(X))
1  CONTINUE
      SUM=SUM*RJ/(2.0D0*YN)
      RETURN
      END

C
      SUBROUTINE KERNEL(R1,Z1,R2,Z2,G,KCODE)
C *****
C This routine evaluates the three different types
C of kernel according to the value of KCODE.
C KCODE=1: regular Green's function kernel
C KCODE=2: z-derivative of Green's function,
C KCODE=3: r-derivative of Green's function.
C *****
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MMDELK,ARG,ARG2,G
      IOPT=1
C DEL - backward difference step size.
      DEL=1.0D-7
C ARG - argument of the elliptic integral of 1st kind
      ARG=4.0D0*R1*F2/((R1+R2)**2+(Z1-Z2)**2)
C MMDELK - IBM-IMSL elliptic int. subroutine.
      H=MMDELK(IOPT,ARG,IER)
      G=H/DSQRT((R1+R2)**2+(Z1-Z2)**2)
      IF(KCODE.EQ.1) GOTO 1
      A=DSQRT(4.0D0*R1*R2)
      ARG2=(DSQRT(ARG)-DEL)**2
      F=MMDELK(IOPT,ARG2,IER)
      F=(H-F)*ARG/(A*DEL)
      E=(F+G)*ARG/(A*A)
      IF(KCODE.EQ.2) GOTO 2
      G=0.5D0*F/R1-(R1+F2)*E
C G is now r-derivative of Green's function
      GOTO 1
2  G=(Z2-Z1)*E
C G is now z-derivative of Green's function
1  G=G/3.14159265358979D0
      RETURN
      END

C
      BLOCK DATA
C *****
C Data block containing Gaussian quadrature weights
C and abscissae.
C U,V - abscissae
C H,M - weights
C *****
C
      REAL*8 U6(6),H6(6),U8(8),H8(8),U10(10),H10(10),
      EU16(16),H16(16),V9(9),M9(9)
      COMMON U6,H6,U8,H8,U10,H10,U16,H16,V9,M9
C

```

DATA U6/- .9324695142D0,
 &- .6612093865D0, -.2386191861D0,
 &.2386191861D0, .6612093865D0, .9324695142D0/

C

DATA H6/.1713244924D0,
 &.3607615730D0, .4679139346D0,
 &.4679139346D0, .3607615730D0, .1713244924D0/

C

DATA U8/- .1834346424957D0, -.5255324099163D0,
 &- .7966664774136D0, -.9602898564975D0, .9602898564975D0,
 &.7966664774136D0, .5255324099163D0, .1834346424957D0/

C

DATA H8/.3626837833784D0, .3137066458779D0,
 &.2223810344534D0, .1012285362904D0, .1012285362904D0,
 &.2223810344534D0, .3137066458779D0, .3626837833784D0/

C

DATA U10/- .1488743389816D0,
 &- .43339539413D0, -.67940956830D0,
 &- .865063366689D0, -.9739065285172D0, .9739065285172D0,
 &.865063366689D0, .679409568299D0, .4333953941292D0,
 &.1488743389816D0/

C

DATA H10/.29552422471475D0,
 &.269266719310D0, .219086362516D0,
 &.1494513491506D0, .0666713443087D0, .0666713443087D0,
 &.1494513491506D0, .219086362516D0, .269266719310D0,
 &.29552422471475D0/

C

DATA U16/- .9894009349916D0, -.9445750230732D0,
 &- .8656312023878D0, -.755404408355D0, -.6178762444026D0,
 &- .4580167776572D0, -.2816035507793D0, -.0950125098376D0,
 &.0950125098376D0, .2816035507793D0, .4580167776572D0,
 &.6178762444026D0, .755404408355D0, .8656312023878D0,
 &.9445750230732D0, .9894009349916D0/

C

DATA H16/.0271524594118D0, .0622535239386D0,
 &.0951585116825D0, .1246289712555D0, .1495959888166D0,
 &.169156519395D0, .1826034150449D0, .1894506104551D0,
 &.1894506104551D0, .1826034150449D0, .169156519395D0,
 &.1495959888166D0, .1246289712555D0, .0951585116825D0,
 &.0622535239386D0, .0271524594118D0/

C

DATA V9/- .9681602395076D0, -.8360311073266D0,
 &- .6133714327006D0, -.3242534234038D0, 0.0D0,
 &.3242534234038D0, .6133714327006, .8360311073266,
 &.9681602395076D0/

C

DATA M9/.0812743883616D0, .1806481606949D0,
 &.2606106964029D0, .31234707704D0, .3302393550013D0,
 &.31234707704D0, .2606106964029D0, .1806481606949D0,
 &.0812743883616D0/

END

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