CONTRIBUTIONS TO RELIABILITY PREDICTION THEORY

by

John Bruce Arthur de Mercado

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Department of Electrical Engineering
Faculty of Pure and Applied Science
The University of Ottawa
Ottawa 2, Canada

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"Science is only useful if it tells you about some experiment that has not been done; it is no good if it only tells you what just went on".

Richard P. Feynman
The Messenger Lectures
Cornell University
November 1964.

ABSTRACT

In this dissertation we present methods for the reliability modelling of systems with many failed states, as well as methods for the reliability modelling of systems that exhibit degradation failure. These methods are presented in two chapters.

In Chapter I elements of the theory of discrete Markov processes are used to develop methods for predicting the reliability and moments of the first time to failure of systems, which operate in a repair environment with known constant failure and repair rates.

Specifically, we consider systems composed of any finite number of subsystems. The system has r ($r \geq 1$) acceptable states and m ($m \geq 1$) failed states. The methods that are presented for obtaining a probabilistic description of such systems rest on the assumption, that the state transition behaviour of the system can be characterized by a stationary Markov process (also called Markov chain) with finite-dimensional state space and a discrete time set.

It is shown that once the matrix of the constant failure and repair rates is known, and the state assignment is made, then it is a straight forward matter to obtain the complete probabilistic description of the system.

In Chapter II elements of the theory of continuous Markov processes are utilized to develop methods for predicting the reliability and moments of the first time to failure for systems that exhibit degradation failure.

Specifically, we consider systems comprised of k degradation-prone parameters, each of which is characterizable by a Feller process. It is shown that once the particular Feller processes have been chosen to characterize the degradation-prone parameters, then it is a straight forward matter to obtain the probabalistic description of the k parameter system.

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INTRODUCTION

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INTRODUCTION

0.1. Discussion of Reliability Prediction Theory and Its History.

Reliability Prediction Theory, is the science which forecasts the occurrence of failures in systems. Since any prediction about the future operation of a given system involves uncertainty it is not surprising that the theory of stochastic processes furnishes the mathematical models for the theory of reliability prediction. The real challenge arises in developing general and systematic methods for applying a given class of stochastic processes to the reliability modeling of a class of systems.

This approach for developing the theory of reliability was not followed until recently (the mid 1960's), and workers in this field relied on particular results from probability theory for the treatment of a restricted class of reliability problems[11, 12]. The idea of using the theory of stochastic processes to develop a general theory of reliability prediction did not appear (as far as this author is aware) in the literature up to this time.

However the intense technical progress over the last decade has given rise to many extremely costly, complex and one-of-a-type systems, and with this has been created an urgent need for bonafide mathematical methods for the reliability modelling of such systems.

In his Ph.D. dissertation [1], Ronald Howard presented an application oriented and readable version of the classical theory of

stationary Markov chains with discrete and continuous time parameters and finite dimensional state spaces. Although Howard's dissertation was not devoted specifically to any problems of reliability prediction theory, Dolazza[2] following Howard's thinking recognized that the state transition behaviour of a system, having many subsystems, each with known constant failure and repair rates, could be characterized by the methods given by Howard. Dolazza developed methods for the reliability modelling for this class of systems. Briefly, his results may be summarized as follows: Let the system have k acceptable states A_i, i = 1,..., k which forms the set A. Let all failed states be lumped into a single failed state F. Dolazza presented a method for computing the time dependent reliability function R(t), defined as the probability that the system is in some acceptable state in A at time t. He also gave a method for computing the moments of the first time to failure, that is, the moments of the time it takes the system to pass from any acceptable state in A to the failed state F. His method suffers from a number of obvious limitations. First of all the lumping of failed states into a single failed state, conceals the relative importance of the different types of failure modes that are present in any system. Secondly, no methods are provided for computing the important moments of the first time the system passes from specified acceptable states to specified failed states. Dolazza's paper, although written in 1964 was not published until the end of 1966, in the mean time Thin Htun [5,6] gave exactly the same results as Dolozza.

Somewhat earlier (1963), Udagawa and Inagaki[2,3], had recognized that the expression for the state transition equation for a

sequential machine *) was identical with that of a stationary Markov process (chain) with finite dimensional state space. They gave a method for the reliability modelling of sequential machines with imperfect "gates". Glinski [8] presented a systematic survey of the results in probabilistic logic circuits, in a 1967 course. de Mercado's thesis [9] gave a survey of Markov techniques used in reliability prediction theory up until that time (1967).

0.2 Contributions of This Research To Reliability Prediction Theory.

When this research was begun (1967), the theory of reliability prediction still lacked the general mathematical techniques for treating systems with many failed states, as well as systems with a continuum of failed states, (such as systems that exhibit degradation failure).

The reliability prediction techniques for systems with many failed states as given in this dissertation, appear (as far as the author is aware) for the first time. Likewise the use of the theory of continuous Markov processes to develop a general theory of reliability prediction for degradation-prone systems appears for the first time, and is a major contribution of this dissertation **).

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*) The reliability modelling of finite probabilistic automata dates back to as far as the early 1950's. References to earlier works can be found in a 1962 survey paper by Glinski [7] Reference (20) in this paper is the first serious mathematical work in this field.

**) The author wishes to emphasize that this dissertation attempts no, and makes no contribution to the mathematical theory of Markov processes. In fact all the mathematical techniques used are known, and finer points of the mathematical theory have either been assumed, stated or neglected.

From a mathematical point of view the unifying feature between these two seemingly disjoint classes of problems is that the first, namely the reliability modelling of systems with many failed states, is accomplished by using methods of the theory of Markov processes having discrete state spaces. On the other hand, the reliability modelling of the second, namely those systems that exhibit degradation failure, is accomplished by using methods of the theory of Markov processes having continuous state spaces. This natural mathematical extension from discrete to continuous Markov processes is the basis for unifying and treating what might appear from an engineering point of view to be two distinct classes of problems, and is the explanation of the division of this dissertation into two separate chapters.

In chapter I, elements of the theory of Markov processes with discrete state spaces as presented by Girault,[10] are utilized to develop methods for predicting the reliability and moments of the first time to failure of systems which operate in a repair environment and whose subsystems have known constant failure and repair rates. Specifically, the system can have r, $(r \geq 1)$ acceptable states, and m, $(m \geq 1)$ failed states. The methods of reliability prediction that are given for such systems rest on the assumption that the state transition behavior of the system is characterizable by a stationary Markov process (Markov Chain) with finite dimensional state space and discrete time set. In this chapter, it is shown that once the matrix of the constant failure and repair rates is known, and the state assignment is made, then it is a straightforward and purely computational matter to obtain the probabilistic description of the system.

In chapter II, elements of the theory of continuous Markov processes (specifically Diffusion processes) as given, for example, in Dynkin[13,14] or Itô[3] and Mackean[15] are utilized to develop methods for predicting the reliability and moments of the first time of failure of degradation-prone systems. Specifically, the system can have k, ($k \geq 1$) degradation-prone parameters each of which is characterizable by a Feller process. It is shown that once the particular Feller process has been chosen to characterize each degradation-prone parameter, then it is again a straightforward matter to obtain the probabilistic description of the system.

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CHAPTER I

RELIABILITY PREDICTION TECHNIQUES FOR SYSTEMS WITH MANY FAILED STATES

INTRODUCTION

It is well known [1], [2], that the reliability modelling of systems that operate in a repair environment, and whose subsystems have known constant failure and repair rates, can be affected via a linear matrix calculus using elements of the theory of stationary Markov processes

In what follows, this approach is utilized to develop for such systems methods for obtaining a time dependent reliability function. Methods are also presented for obtaining the moments of first time to (a particular) failed state, as well as the moments of the first time to failure (any state).

The construction of a model for predicting the behavior of such a system poses three distinct problems. The first two are in effect specification problems. The first of these is the "state assignment" problem, that is the enumeration of the states that suffice to characterize the various modes of the system. The method for making such a state assignment will depend on the specification of the operation of a given system. The second problem involves the determination of meaningful numerical estimates of the often unknown failure and repair rates (ie, of the one step state transition probabilities*). This is the, so called "general inference" problem [3, pp 69-70]** for Markov processes. The third problem which is the one we are considering, involves the application of techniques from the theory of stationary Markov process to develop methods for obtaining apriori state probability functions, reliability functions, and estimates of the moments of the first time that it takes the system to go from one given state to another. We have assumed that the solution to the "state assignment" problem as well as the "general inference" problem is known. That is there exists a state characterization of the system together with a matrix [M] of the failure and repair rates (that is, the matrix of one step state transition probabilities).

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*) The one step state transition probabilities are dimensionless and are obtained by multiplying the failure or repair rates, (whichever are appropriate) by the "unit of time" (for example, 1 hour, 1 day, etc).

**) Numbers in square brackets refer to the references.

In section 1.2, it is shown that in any Markov model for characterizing the system, the acceptable states form a transient class. We also admit the possibility that the system has m($m \geq 1$) failed states. This is an extension of the current practice of lumping [2] all the failed states into a single failed state. This lumping of failed states introduces unnecessary restrictions as it conceals the relative importance of the various failure modes that are present in any system. It is also shown that matrix [M] of one step state transition probabilities can be partitioned into four matrices [A], [B], [ϕ], [$\bar{\phi}$]. In later sections it is shown that such partitioning is sufficient to use all the methods presented herein.

In section 1.3, it is shown that the state probability functions*) $\bar{\pi}(n)$ are obtained simply by taking the n^{th} power of the matrix [M]. Then for the set A of acceptable states, the time dependent reliability function R(n) is shown to be the sum of the state probability functions over the set A. The reliability function R(n) is the probability that at time n the system is operating acceptably; (that is, is in some acceptable state in A).

In section 1.4, the steady state transition failure probabilities p_{ij} are derived. p_{ij} is the probability that the system, will eventually go from an acceptable state A_i to a failed state**) F_j. A theorem is presented which shows that the r x m matrix [P], of these steady state failure probabilities is obtainable directly in terms of the matrices [B], [$\bar{\phi}$] and [A], which are the partitions of [M]. The concept of an evolution diagram, as introduced by Girault [6], is utilized to prove this theorem. These evolution diagrams provide a useful conceptual aid for proving many

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*) Capital letters in square brackets denote matrices; the bar on top of a letter denotes a vector. An explanation of the notation is given after the references.

**) The set of m, ($m \geq 1$) failed states are said to form the failure class F.

interesting results in the theory of stationary Markov processes. The theorem and evolution diagram in the section appears (as far as we are aware) for the first time in the context of reliability theory.

In section 1.5, the steady state transition probability failure functions $p_{ij}(n)$ are derived. $p_{ij}(n)$ is the probability that the system will go after n units of time from acceptable state A_i to failed state F_j. A theorem is presented which shows that the r x m matrix $[P(n)]$ of these transition probability failure functions is expressible directly in terms of the matrices $[A]$, and $[B]$. This theorem and its proof, which again involves the use of an evolution diagram, appears (as far as we are aware) for the first time in the reliability context.

In section 1.6, it is shown that the concepts of defective probability distributions [7, pg 112] and pseudo random variables arise naturally in the reliability modelling of systems with more than one failed state.

In section 1.7, a method is presented for computing the pseudo generating functions $g_{ij}(z)$ that give the time moments $\tau_{ij}(k)$ of the random variables τ_{ij}, "first time to failure state". It is shown that these moments are obtained by differentiating the pseudo generating functions. A theorem is presented, which shows that the r x m matrix of these generating functions is directly expressible as a simple linear function of the matrices $[A]$, $[B]$ and $[\tilde{\varphi}]$. The theorem and its proof, appears (as far as we are aware) for the first time in the reliability context.

In section 1.8 we prove the theorem that all probability distribution functions arising in connection with the reliability modelling of such systems, are of the simple exponential type [6 page 79].

In section 1.9, the exit probability functions $w_i(n)$ are derived. $w_i(n)$ is the probability that the system will go from the successful state A_i, into any failed state in F in n units of time. A theorem is presented which shows that the 1 x r column vector $\bar{\Omega}(n)$ of the exit

probability functions is directly expressible as a simple function of the matrices [A] and [B] . This theorem and its proof which involves the use of an evolution diagram, appears (as far as we are aware) for the first time in the reliability context.

In section 1.10, a method is presented for computing the generating functions $\psi_i(z)$ that give the moments $\tau_i(k)$ of the random variables τ_i - "first exit time from acceptable state A_i into the class F ". A theorem is presented which relates the generating function $g_{ij}(z)$ of section (1.7) to the generating function $\psi_i(z)$. Another theorem is presented which shows that the 1 x r vector $\overline{\psi}(z)$ of generating functions is expressible as a simple linear function of the matrices [A], [B] and [ϕ]. This theorem, appears (as far as we are aware) for the first time in the reliability context.

SECTION 1.1

MATHEMATICAL PRELIMINARIES

In developing methods for characterizing the time behavior of any given system, we shall utilize a stationary Markov process $S(\cdot)$. The process *) $S(\cdot)$, has a finite dimensional state space Λ, and discrete time set T.

The object will be to derive for each state such systems a state probability function $\pi_v(n)$. The function $\pi_v(n)$ is defined as

$$\pi_v(n) \equiv \text{Prob}\{S(n) = a_v \in \Lambda\}, \; n \in T, \tag{1}$$

To completely characterize the process $S(\cdot)$, it is sufficient to know the matrix [M] of the one step transition probabilities between the states of Λ. That is for every pair of states a_v, $a_y \in \Lambda$, there corresponds an entry m_{vy}, which is defined as

$$m_{vy} \equiv \text{Prob}\{S(t+1) = a_y \mid S(t) = a_v\} \tag{2}$$

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*) Instead of $S(\cdot)$, we really should write $S(\cdot, \xi)$, where ξ is a "sample" path in some abstract space. However, since this is no need to emphasize ξ in what follows, we shall suppress it.

for the stationary process m_{vy} as given in (2), is independent of the choice of $t \in T$. That is the entries of $[M]$ are constants.

It can be shown [8, page 405], that the matrix $[M]$ corresponding to a Markov process with (absorbing) failed state can always be partitioned as

$$[M] = \begin{bmatrix} [A], & [B] \\ [\phi], & [\Phi] \end{bmatrix} \tag{3}$$

Matrix $[A] \equiv [a_{ik}]$, is the r x r matrix of the one step transition probabilities between the r acceptable states of transient class $A \subset \Lambda$. Let I' be the index set for states in $A \subset \Lambda$. Then, for every a_i, $a_k \in \Lambda$, we have a_i, $a_k \in A$ whenever i, k $\in$ I' and

$$a_{ik} \equiv \text{Prob}\ \{ S(t+1) = a_k \mid S(t) = a_i \}$$

Matrix $[\Phi]$ is an m x m matrix of the one step transition probabilities between the m fail states of failure class $F \subset \Lambda$. Let I" be the index set for states in $F \subset \Lambda$. Matrix $[\Phi]$ is always the unit (identity) matrix and this follows from the following formula, which holds for every a_u, a_j, whenever j, u $\in$ I".

Namely

$$\text{Prob}\ \{ S(t+1) = a_u \mid S(t) = a_j \} = \delta_{ju} \quad \begin{cases} =1, & u = j \\ =0, & u \neq j \end{cases} \tag{5}$$

This equation states that evolution between states of F is not permitted. That is, once the process reaches a failed state, it stays there with probability one.

Matrix $[B] \equiv [b_{ij}]$, $\forall i \in I'$ and $\forall j \in I''$, is the r x m matrix of the one step transition probabilities, from states of A to states of F. That is for every a_i, a_j, we have

$$b_{ij} \equiv \text{Prob}\ \{ S(t+1) = a_j \mid S(t) = a_i \} \qquad \forall i \in I',\ \forall j \in I''$$

Matrix $[\phi]$, is the m x r null matrix, that is the matrix with all entries zero. This follows, because transitions from failed (absorbing) states to acceptable states, cannot occur.

SECTION 1.2

THE RELIABILITY MODEL - PRELIMINARY DEFINITIONS

In order to develop a model for predicting the reliability and moments of the time to first failure for systems with constant failure and repair rates, we introduce the following definitions.

Definition 1.2.1 "Acceptable state". a state $a_i \in \Lambda$, $i \in I'$ is called an acceptable state, if it characterizes some acceptable working mode of the system. Acceptable states are re-labled A_i, $i \in I'$.

In developing the model, we shall assume that the system has r ($r \geq 1$) acceptable states, which are collected into a single transient class $A \subset \Lambda$.

Definition 1.2.2 "Consequent states". Two states a_i, $a_k \in \Lambda$ are said to be consequent to each other, if it is possible to go from a_i to a_k and a_k to a_i. Obviously all acceptable states in A are consequent, and failed states of F are consequent only to themselves.

Letting

$$m_{iv}(n) \equiv \text{Prob}\ \{ S(t+n) = a_v \mid S(t) = a_i \} \tag{7}$$

we have the following definition

Definition 1.2.3 "Transient Class A". The set of acceptable states $a_i \in \Lambda$, $i \in I'$ are consequent to each other, and are said to form the transient class $A \subset \Lambda$. That is

$$A \equiv \{ a_i \in \Lambda \mid m_{ik}(n_1) > 0, m_{ki}(n_2) > 0 \text{ for some } n_1\ n_2, \forall ik \in I' \}$$

$$\equiv \{ A_i \},\ i \in I'$$

and the one step transition probabilities $[a_{ik}]$ between the r states of A, form the square matrix $[A]$.

Definition 1.2.4 "Failed State". A state a_j, $j \in I''$, $a_j \in F \subset \Lambda$, that is consequent to no other state except itself, and which describes one or more unacceptable modes of the system, is called a failed state. Then the set

F of failed states can be defined as

$$F \equiv \{ a_j \in \Lambda \mid \forall n,\ m_{ji}(n) = 0,\ m_{ju}(n) = \delta_{ju} : \text{for some } n_1, m_{ij}(n_1) > 0;\ i \in I', j, u \in I'' \}$$

$$\equiv \{ F_j \},\ j \in I''$$

At this point, we summarize the above definitions by using the [M] matrix as given by (3). We can write [M] as

$$[M] \equiv \begin{array}{c|c|c|c|} & & \text{time } t+1 & & \\ & & A_1 \ \ A_i \ \ A_r & F_1 \ \ F_j \ \ F_m & \\ \hline \text{time } t & A_1 & a_{11} \ldots a_{1i} \ldots a_{1r} & b_{11} \ldots b_{1j} \ldots b_{1m} & \\ A \to A & A_i & a_{i1} \ldots a_{ii} \ldots a_{ir} & b_{i1} \ldots b_{ij} \ldots b_{im} & A \to F \\ & A_r & a_{r1} \ldots a_{ri} \ldots a_{rr} & b_{r1} \ldots b_{rj} \ldots b_{rm} & \text{-----(3)} \\ \hline & F_1 & 0 \ldots 0 \ldots 0 & 1 \ldots 0 \ldots 0 & \\ F \to A & F_j & 0 \ldots 0 \ldots 0 & 0 \ldots 1 \ldots 0 & F \to F \\ & F_m & 0 \ldots 0 \ldots 0 & 0 \ldots 0 \ldots 1 & \\ \hline \end{array}$$

SECTION 1.3

THE RELIABILITY MODEL - STATE PROBABILITY FUNCTIONS

In this section, a method is presented for computing the state probability functions $\pi_v(n)$. Specifically, let $\overline{\pi}(n) \equiv [\pi_1(n), \ldots, \pi_r(n), \ldots \pi_{r+m}(n)]^T$ be the 1 x(r+m) vector of state probability functions. Then the following theorem shows that $\overline{\pi}(n)$ is a simple function of the matrix [M], which is the matrix of failure and repair rates between the states of the system, (that is the one step state transition probabilities).

Theorem 1.3.1 The state transition probability vector $\overline{\pi}(n)$ satisfies

$$\overline{\pi}(n) = [M]^n \overline{\pi}(0) \tag{9}$$

where $\overline{\pi}(0)$ is the initial state probability *) vector .

Proof (This can also be found in any elementary text on Markov processes [6, page 56]). It is reproduced below.

We have from the definitions of $\overline{\pi}(0)$ and m_{uv}, that **)

$$\pi_v(1) = \sum_{u \in I' \cup I''} m_{uv} \pi_u(0), \quad \forall u \in I' \cup I''$$

Then from $\pi_v(1)$, we can compute $\pi_v(2)$ for the stationary case

$$\pi_v(2) = \sum_{u \in I' \cup I''} m_{uv} \pi_u(1)$$

substituting for $\pi_v(1)$, in matrix form we have

$$\overline{\pi}(1) = [M]\,\overline{\pi}(0)$$

$$\overline{\pi}(2) = [M]^2 \overline{\pi}(0) = [M]\overline{\pi}(1)$$

carrying on by induction completes the proof. QED.

Definition 1.3.1

RELIABILITY FUNCTION

The reliability function R(n) is defined [1,2], as the probability that at time $n \in T$, the system is in a acceptable state. That is

$$R(n) = \text{Prob}\{S(n) \in A\}$$

alternatively

$$R(n) = \sum_{i \in I'} \pi_i(n)$$

or using (9), R(n) may be written as

$$R(n) = \sum_{i \in I'} \sum_{k \in I' \cup I''} m_{ik}^{(n)} \pi_k(0) \tag{10}$$

*) $\overline{\pi}(0)$ is the probability of being in any of the v + m states, initially, that is at time n = 0.

**) Capital U is used to denote set union.

where $m_{ij}^{(n)}$ denotes the ij^{th} entry of $[M]^n$. Thus in order to obtain R(n), it is necessary only to raise the matrix [M] to the n^{th} power, multiply it by the initial state vector $\overline{\pi}(0)$ and then sum the elements of the set. $\{ \pi_i(n) ; i \in I' \}$. In the appendix, we have included a Fortran program, which computes the powers $[M]^k$ for k = 1, 2,..., n where n is a variable.

SECTION 1.4

THE STEADY STATE TRANSITION FAILURE PROBABILITIES

In this section, a method is presented for computing the steady state transition probability p_{ij}, $i \in I'$, $i \in I''$, that a system starting in acceptable state A_i will eventually end up in a failed state F_j. In what follows, it will be shown that once the partition of [M] has been completed as shown in (3), then it is a simple matter to obtain these probabilities.

Formally, p_{ij} is defined as

$$p_{ij} \equiv \text{Prob} \{ S(\infty) = F_j \mid S(t) = A_i \} \tag{11}$$

Then, denoting the r x m matrix $[p_{ij}]$, $i \in I'$, $j \in I''$ by [P], we have the following theorem.

Theorem 1.4.1. Consider a system with r acceptable states and m failed states operating in a repair environment. Furthermore, assume the system has known constant failure and repair rates, that is, has a known matrix [M], which can be partitioned as in (3). Then the steady state r x m matrix [P] of transition failure probabilities defined by (11), satisfies*

$$[P] = [[\phi] - [A]]^{-1} [B] \tag{12}$$

Proof. The proof of this and other theorems presented herein, is facilitated by formerly introducing evolution diagrams [6 pp 74-78]. Consider therefore evolution diagram I, which shows the possible evolutions from state $A_i \in A$ to any state $F_j \in F$.

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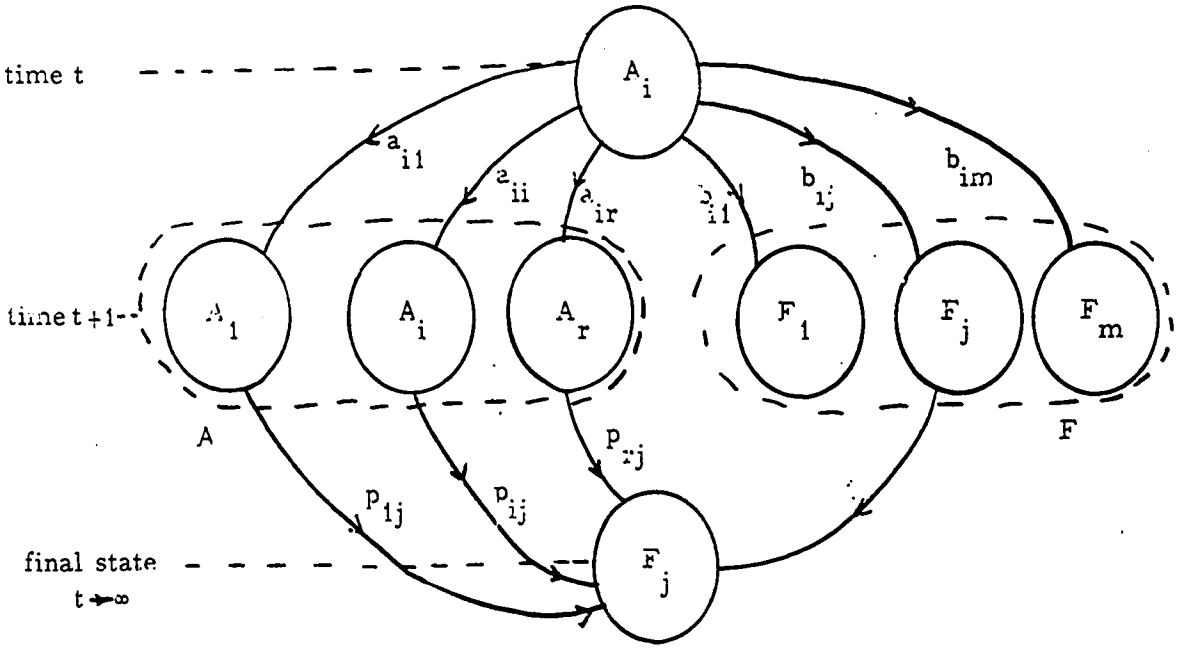
* In equation (12) $[\phi]$ is a m x m unit matrix.



Evolution diagram I.

From the above diagram, summing up the transmittances of the paths incident on node F_j from A_i, we have

$$p_{ij} = \sum_{k \in I'} a_{ik} p_{kj} + b_{ij} , \qquad j \in I''$$

we can obviously construct such a diagram for every $A_i \in A$ and every $F_j \in F$ and we therefore have in matrix form

$$[P] = [A][P] + [B]$$

or

$$[[\hat{e}] - [A]][P] = [B] \tag{13}$$

which completes the proof. QED.

SECTION 1.5

THE TRANSITION PROBABILITY FAILURE FUNCTIONS

In the section, a method is presented for computing the transition probability failure functions $p_{ij}(n)$, $i \in I'$, $j \in I''$. Specifically $p_{ij}(n)$ is the probability that at time $n \in T$, the system is in failed state $F_j \in F$. Formally

$$p_{ij}(n) = \text{Prob}\{ S(t+n) = F_j \mid S(t) = A_i \} \tag{14}$$

(obviously $\lim_{n=\infty} p_{ij}(n) \equiv p_{ij}$). Denoting the r x m matrix $[p_{ij}(n)]$, $i \in I'$, $j \in I''$ of transition probability failure functions by $[P(n)]$, we have the following theorem

Theorem 1.5.1 Consider a system with r acceptable states and m failed states. The system has known constant failure and repair rates, that is, has a known matrix $[M]$ which is partitioned as in (3). Then the matrix $[P(n)]$ of transition probability failure functions defined by (14), satisfies

$$[P(n)] = [A]^{n-1}[B] \tag{15}$$

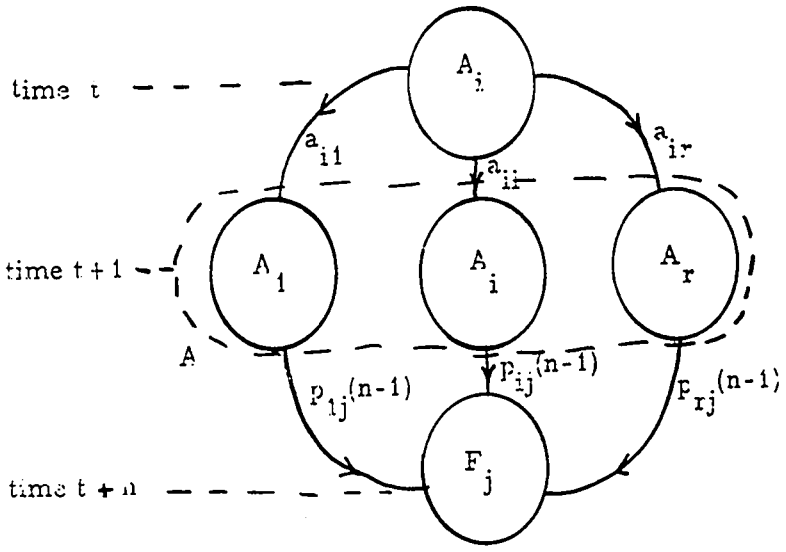
where

$$[P(1)] = [B] \tag{16}$$

and

$$[A]^{0} \equiv [\Phi] \tag{17}$$

Proof Equation (16) follows directly from (6). To prove (15) consider evolution diagram II. This diagram enumerates the possible evolutions in n steps from any state $A_i \in A$ to any state $F_j \in F$.



Evolution diagram II

In the above diagram, summing up the transmittances of the paths incident on node F_j from A_i, we have

$$p_{ij}(n) = \sum_{k \in I'} a_{ik} p_{kj}(n-1) \tag{18}$$

We can construct such an evolution diagram for every $A_i \in A$ and every $F_j \in F$. Therefore equation (18) can be written in matrix form as

$$[P(n)] = [A]^{n-1} [P(1)]$$

using (16) completes the proof. QED.

Comments

a) Equation (15) can be computed using the program given in the appendix.

b) $p_{ij}(n)$ is the probability that the system will go from acceptable state A_i to failed state F_j in n units of time. Thus letting τ_{ij} be the pseudo random variable "time taken to go from state A_i to state F_j". Then, formally

$$p_{ij}(n) = \text{Prob}\{\tau_{ij} = n\} \tag{19}$$

SECTION 1.6

DEFECTIVE PROBABILITY DISTRIBUTIONS AND PSEUDO RANDOM VARIABLES

In this section we show that the concepts of defective [7, page 112] probability distribution functions and pseudo [6 page 79] random variables arise naturally in the reliability modelling of systems with more than one failed state.

Definition 1.6.1. Defective distribution - The probability distribution function $p_{ij}(n)$ is called defective if

$$\sum_{n=1}^{\infty} p_{ij}(n) < 1 \tag{20}$$

Definition 1.6.2. Pseudo random variable - The random variable τ_{ij}, "first time from A_i to F_j", is called pseudo random if its probability distribution function $p_{ij}(n)$ defined as in (19), is defective.

Theorem 1.6.1 The probability distribution functions $p_{ij}(n)$, for systems with more than one failed state, are defective. That is,

$$\sum_{n=1}^{\infty} p_{ij}(n) < 1 , \quad \forall i \in I', \ \forall j \in I''$$

Proof

Comparing (11) with (14) we have that, p_{ij}, the probability of eventually going from acceptable state A_i to failed state F_j, is related to $p_{ij}(n)$ by

$$p_{ij} = \sum_{n=1}^{\infty} p_{ij}(n) \tag{21}$$

All physical systems eventually fail. Mathematically, this fact may be written as

$$\text{Prob}\{S(\infty) \in F\} = 1$$

alternately, this can be written as

$$\sum_{j \in I''} p_{ij} = 1 \tag{22}$$

and this holds every $i \in I'$. Comparing (21) and (22) we have

$$\sum_{n=1}^{\infty} p_{ij}(n) < 1$$

and the distribution function $p_{ij}(n)$ are defective in the sense of definition 1.6.1

Corollary. The random variable τ_{ij} is pseudo random.

Comment. In the familiar case treated by Thin Htun[2], in which there is a single failed state, say F_j, then the summation in (22) contains only a single term, that is

$$p_{ij} = 1$$

and from (21) for the single failed state case we have

$$\sum_{n=1}^{\infty} p_{ij}(n) = 1 .$$

which is well known property of the ordinary probability distribution function.

SECTION 1.7

MOMENTS OF THE FIRST TIME TO FAILED STATE

In this section we derive expressions for the pseudo generating functions for computing the moments of the pseudo random variables τ_{ij}, $\forall i \in I'$, $\forall j \in I''$.

These moments are the moments of the first time it takes the system to pass from the state $A_i \in A$ to failed state $F_j \in F$. In the reliability context, these moments of τ_{ij}, will be called moments of first time to failed state F_j.

Since we are using the discrete time approach, it is standard practice to define the generating function in terms of its one sided z-transform. That is, the generating function $g_{ij}(z)$ for these moments is defined as

$$g_{ij}(z) = \sum_{n=1}^{\infty} z^n p_{ij}(n) \tag{23}$$

Then letting $\tau_{ij}(k)$ be the k^{th} moment of τ_{ij}, we have

Definition. Moments of First Time to Failed State. The moments of the first time to failed state, are defined as the moments of the first time the system passes from acceptable state $A_i \in A$ to failed state $F_j \in F$. These moments are obtained from the generating function (23) as

$$\tau_{ij}(k) = \frac{d^k}{dz^k} (g_{ij}(z)) \Big|_{z=1}, \quad k = 1, 2, \ldots$$

We now prove the following important theorem which is given for the first time in the reliability context. The theorem shows that it is possible to obtain these pseudo generating functions in terms of matrices

[A] and [B], without the need for evaluating infinite series of the form (23).

Let [G(z)] be the r x m matrix of the generating functions $g_{ij}(z)$. Then we have

Theorem 1.7.1 Let $[\tau(k)]$ be the r x m matrix of the k^{th} moments $\tau_{ij}(k)$. For a system with r acceptable and m failed states operating in a repair environment, with known matrix [M], these moments are given by

$$[\tau(k)] = \frac{d^k}{dz^k} \left([G(z)] \right) \Big|_{z=1} , \qquad k = 1, 2, \ldots\ldots$$

and

$$[G(z)] = z \left[[\bar{\varrho}] - z [A] \right]^{-1} [B] \tag{24}$$

where [A] and [B] are the partitions of the [M] matrix, as given in (3), and $[\bar{\varrho}]$ is the identity matrix.

Proof

Equation (23) can be written in matrix form as

$$[G(z)] = \sum_{n=1}^{\infty} z^n [P(n)] \tag{25}$$

expanding (25) and using (16) we have

$$[G(z)] = [B] z + z \left[[P(2)] z + P(3) z^2 + \ldots\ldots \right] \tag{26}$$

Combining (15) and (16) we have

$$[P(n)] = [A]^{n-1} [P(1)]$$

inserting the above in (26) gives

$$[G(z)] = [B] z + z [A] [[P(1)] z + [P(2)] z^2 + \ldots] \tag{27}$$

from comparison of (27) and (25):

$$[G(z)] = [B] z + z [A] [G(z)]$$

rearranging, gives

$$\left[[\bar{\varrho}] - z [A] \right] [G(z)] = z [B]$$

which is (24)

QED.

Comment

Combining (11) and (14), we can relate the steady state transitional failure probabilities and the transition probability functions as follows:

$$p_{ij} = \sum_{n=1}^{\infty} p_{ij}(n)$$

or, using (23),

$$p_{ij} = \lim_{z=1} g_{ij}(z)$$

or, in matrix form

$$[P] = \text{limit}_{z=1} [G(z)]$$

Taking limit for $z = 1$ in equation (24), we find

$$[P] = [[\mathfrak{s}] - [A]]^{-1} [B]$$

which is equation (12) as it should be.

SECTION 1.8

THE EXPONENTIAL PROPERTY OF THE PROBABILITY DISTRIBUTION FUNCTIONS

In this section, we prove that the probability distribution functions $p_{ij}(n)$ are of the simple exponential type. This result follows directly from the semi-group property given by the Chapman-Kolmogorov equation, or alternatively from the simple proof presented below, which is taken from Girault[6, page 79].

Definition 1.8.1 The probability distribution function $p_{ij}(n)$ is said to be of the simple exponential type, if there can be found from linear operations on the entries of $[A]$ and $[B]$, two constants α and β such that for n variable

$$p_{ij}(n+1) < \beta (1 - \alpha)^n, \quad \beta > 0, \quad 0 \leq \alpha < 1 \qquad (28)$$

We now prove Girault's result as theorem 1.8.1.

Theorem 1.8.1. The probability distribution function $p_{ij}(n)$ are of the simple exponential type. That is, they satisfy (28).

Proof

From (15) for n = 2, we have

$$p_{ij}(2) = \sum_{k \in I'} a_{ik} b_{kj} \tag{29}$$

now in (29) we can obviously choose a β such that $\beta > b_{kj}$ for every $k \in I'$. Then (29) can be written as

$$p_{ij}(2) < \beta \sum_{k \in I'} a_{ik} \tag{30}$$

similarly in (30), we can obviously choose a α, such that $1 - \sum_{k \in I'} a_{ik} > \alpha$. Therefore (30) can be made to satisfy the inequality

$$p_{ij}(2) < \beta(1-\alpha) \tag{31}$$

Again, using (15), we can write

$$p_{ij}(3) = \sum_{k \in I'} a_{ik} p_{kj}(2) \tag{32}$$

substituting (31) into (32) yields the inequality

$$p_{ij}(3) < \beta(1-\alpha) \sum_{k \in I'} a_{ik}$$

Choosing again α as indicated above, we have

$$p_{ij}(3) < \beta(1-\alpha)^2$$

Continuing the process by induction leads to

$$p_{ij}(n+1) < \beta(1-\alpha)^n$$

which completes the proof QED.

SECTION 1.9

THE EXIT PROBABILITY FUNCTIONS

In this section we derive expressions for the exit probability functions $w_i(n)$, defined as

$$w_i(n) = \text{Prob}\{ S(t+n) \in F \mid S(t) = A_i \} \tag{33}$$

$$= \sum_{j \in I''} \text{Prob}\{S(t+n) = F_j \mid S(t) = A_i\} \tag{34}$$

$w_i(n)$, is the probability that the system will go from acceptable state A_i, into a failed state in n units of time, that is in the interval $(t, t+n)$.

Using (14), we can write (34) as

$$w_i(n) = \sum_{j \in I''} p_{ij}(n) \tag{35}$$

that is, the probability of going from an acceptable state A_i, into any failed state, in n units of time, is the sum of the probabilities $p_{ij}(n)$ of going from A_i to every state of F in the time internal $(t, t+n)$.

Letting $\overline{\omega}(n) = [w_1(n), \ldots w_r(n)]^T$ be the $r \times 1$ column vector*) of the exit probability functions from the r acceptable states of A. Then we have the following theorem.

Theorem 1.9.1. The exit probability functions defined by (33) satisfy

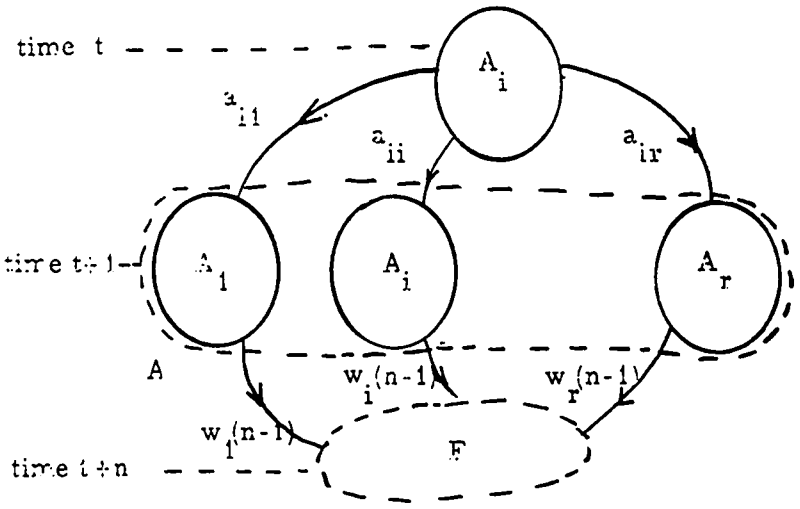
$$w_i(1) = \sum_{j \in I''} b_{ij}, \qquad \forall\, i \in I' \tag{36}$$

$$\overline{\omega}(n) = [A]\,\overline{\omega}(n-1) \tag{37}$$

*) The T in $[\ldots\ldots]^T$, means the transpose of a row vector, that is, a column vector.

Proof

Result (36) follows from equating (6) and (34). To prove (37), consider evolution diagram III



Evolution diagram III

The above diagram enumerates the possible evolutions in n steps from any state $A_i \in A$ into the set F of failed states. Then, summing up the transmittances of the paths incident on F, we have for each of the r states $A_i \in A$, an expression of the form

$$w_i(n) = \sum_{k \in I'} a_{ik} w_k(n-1), \quad \forall i \in I' \text{ which is (37).} \quad \text{QED.}$$

SECTION 1.10

MOMENTS OF THE FIRST EXIT TIMES FROM A

In this section we derive expressions for the generating functions for computing the moments of the random variables τ_i, "first exit time from acceptable state A_i into failed class F". Then, formally

$$w_i(n) = \text{Prob}\{\tau_i = n\}$$

These moments, are the moments of the first time it takes the system to pass from any state $A_i \in A$ into the set of failed states F. In the reliability literature, the moments of τ_i, which we denote by $\tau_i(k)$, $k = 1, 2, \ldots$ are the moments of the first time to failure.

Since we are using the discrete time approach, we define again the generating function in terms of its one sided z transform. The generating function $\psi_i(z)$ for the moments of τ_i, is defined as

$$\psi_i(z) = \sum_{n=1}^{\infty} z^n w_i(n) \tag{38}$$

Definition Moments of the First Time to Failure*). The moments of the first time to failure, are defined as the moments of the first time the system passes from acceptable state A_i to any failed state in F. These moments are obtained from the generating function (38) as

$$\tau_i(k) = \frac{d^k}{dz^k} (\psi_i(z)) \Big|_{z=1}, \quad k = 1, 2, \ldots$$

The following important theorem establishes the relationship between the generating functions $g_{ij}(z)$ and $\psi_i(z)$ or, equivalently, the relationship between $\tau_{ij}(k)$ and $\tau_i(k)$.

Theorem 1.10.1 The generating functions $g_{ij}(z)$ of section (1.7) and $\psi_i(z)$ of section (1.10) are related by

$$\psi_i(z) = \sum_{j \in I''} g_{ij}(z) \tag{39}$$

or equivalently

$$\tau_i(k) = \sum_{j \in I''} \tau_{ij}(k) \tag{40}$$

Proof

Substituting (35) in (38) gives

$$\psi_i(z) = \sum_{n=1}^{\infty} \sum_{j \in I''} z^n p_{ij}(n) \tag{41}$$

interchanging the order of summation in (41), we have

$$\psi_i(z) = \sum_{j \in I''} \sum_{n=1}^{\infty} z^n p_{ij}(n) \tag{42}$$

*) In particular, the first moment, is the mean time to first failure.

substituting (23) into (42) we obtain (39). Equation (40) follows, by definition, from (39). QED.

Comment

Equation (39), can be computed directly, once (24) has been evaluated, or directly, in terms of the matrices [A] and [B] , as given in theorem 1.10.2 below.

Let $\overline{\psi}(z) = [\psi_1(z), \ldots, \psi_r(z)]^T$ be the 1 x r column vector of the generating functions of the first time to failure. Then we have the following theorem

Theorem 1.10.2. Let $\overline{\tau}(k)$ be the 1 x r vector of the moments $\tau_i(k)$. For a system operating in a repair environment and having r acceptable states and known matrix [M] , these moments are given by

$$\overline{\tau}(k) = \frac{d^k}{dz^k}(\overline{\psi}(z))\Big|_{z=1} \tag{43}$$

and

$$\overline{\psi}(z) = z\,[\,[\hat{0}] - z\,[A]\,]^{-1}\,\overline{B}' \tag{43}$$

where

$$\overline{B}' = [\,b'_1, \ldots, b'_r\,]^T$$

and

$$b'_i = \sum_{j \in I''} b_{ij}, \qquad \forall\, i \in I' \tag{44}$$

Proof

Equation (38) can be written in vector form as

$$\overline{\psi}(z) = \sum_{n=1}^{\infty} z^n\,\overline{\Omega}(n) \tag{45}$$

expanding (45) and combining (36) and (44); we have

$$\overline{\psi}(z) = z\,\overline{B}' + z\,[\,\overline{\Omega}(2)z + \overline{\Omega}(3)\,z^2 + \ldots.] \tag{46}$$

now from (37), we obtain

$$\overline{\Omega}(n) = [A]^{n-1}\,\overline{\Omega}(1)$$

substituting this in (46), we have

$$\overline{\psi}(z) = z\,\overline{B}' + z\,[A]\,[\overline{\Omega}(1)\,z + \overline{\Omega}(2)\,z^2 + \ldots\,] \tag{47}$$

comparing (47) and (45), we have

$$\bar{\Psi}(z) = z\,\bar{B}' + z[A]\,\bar{\Psi}(z) \qquad (48)$$

rearranging (48), gives

$$\Gamma\,[\hat{5}] - z[A]]\,\bar{\Psi}(z) = z\,\bar{B}'$$

which is (43) QED.

Example

The following example illustrates how the theory presented can be used to obtain a complete probabilistic description of the system shown in figure (1)

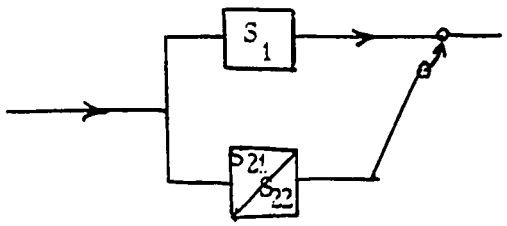


Figure 1

The system consists of an on line system S_1, and a standby system S_2. System S_2 acts as a perfect spare, that is, S_2 cannot fail while it is on standby. Although system S_2 performs the same functions as system S_1, it fails if its subsystems S_{21} or S_{22} fail. The operation of the system requires that on failure of S_1, the system S_2 immediately goes on line. Repairs crews may not be immediately available to start repairing S_1 and there is usually a delay before work starts. The system is considered to be in a failed state when either S_{21} or S_{22} fails before work has been started on S_1 or before work has been completed on S_1.

The two systems S_1 and S_2 are not identical and therefore have different failure and repair rates.

From the above verbal description, it is possible to make the state assignment as shown in figure (2). In keeping with the notation used above, the acceptable, or working states are labelled A_i, $i = 1, 2, 3$ and the failed states are labelled F_j, $j = 1, 2, 3, 4$.

State Assignment

State	Word Description
A_1	Both systems S_1 and S_2 are operating
A_2	S_1 fails and S_2 immediately goes on line
A_3	Service begins on S_1
F_1	S_{21} fails before service to S_1 begins
F_2	S_{22} fails before service to S_1 begins
F_3	S_{21} fails before service to S_1 is complete
F_4	S_{22} fails before service to S_1 is complete.

Figure 2

The transition graph [1] corresponding to this state assignment is shown in figure (3).

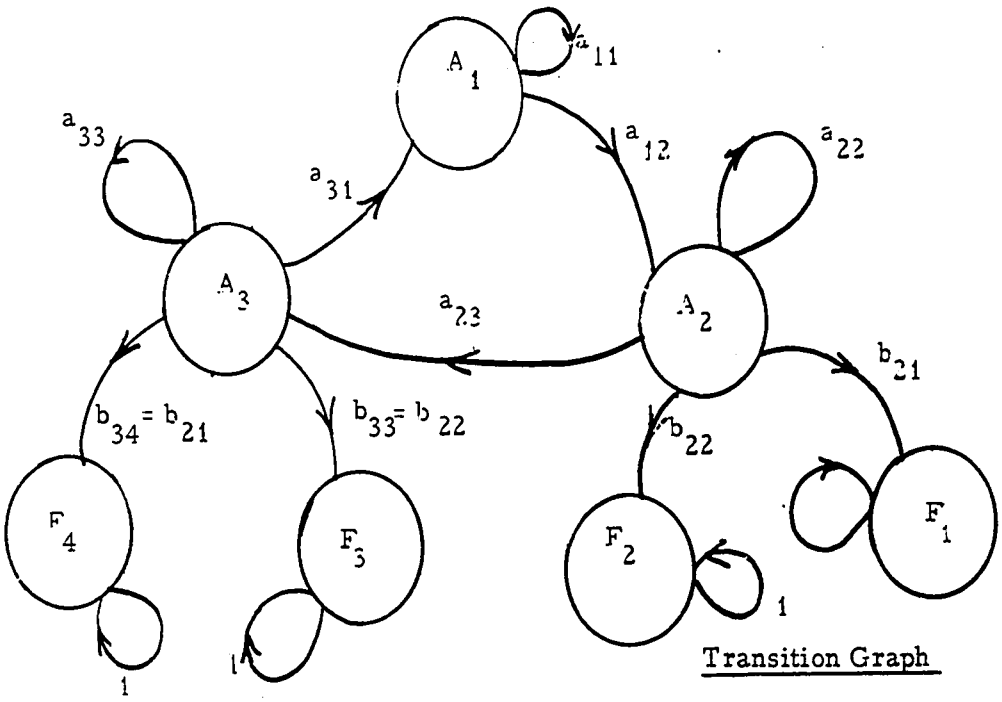


Figure 3.

Inspection of figure (3), yields the following [M] matrix (see also, eqn. (3)).

[M] =

	A_1	A_2	A_3	F_1	F_2	F_3	F_4
A_1	a_{11}	a_{12}	0	0	0	0	0
A_2	0	a_{22}	a_{23}	b_{21}	b_{22}	0	0
A_3	a_{31}	0	a_{33}	0	0	b_{33}	b_{34}
F_1	0	0	0	1	0	0	0
F_2	0	0	0	0	1	0	0
F_3	0	0	0	0	0	1	0
F_4	0	0	0	0	0	0	1

The methods given in sections (1.2)→(1.10) can now be utilized to obtain a complete probabilistic description of the system of figure (1).

SUMMARY

It has been shown, that the reliability modelling of systems with many failed states presents no particular problem, once the matrix [M] of failure and repair rates is known. The methods presented are ideally suited for digital computer computation. These methods are believed to be novel in the context of reliability theory.

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Explanations of Notation

$S(\cdot)$	Stationary Markov process characterizing the state of the system.
Λ	State space of $S(\cdot)$. The state space Λ contains r+m states, where r are acceptable states and m are failed states, $\Lambda = AUF$.
a_v	The state $a_v \in \Lambda$, $v = 1, \ldots, r+m$.
m_{uv}	The one step transition probability between states a_u and a_v of Λ.
$[M] \equiv [m_{uv}]$	The r + m square matrix of the one step transition probabilities.
$\in$	"is an element of"
$\subset$	Set inclusion
$\forall$	Quantifier meaning "for all".
$\cup$	Set union.
A	The class containing the r acceptable states.
I'	The index set for the r acceptable states.
A_i	The acceptable state $A_i \in A$, $i = 1, \ldots, r$. $i \in I'$.
a_{ik}	The one step transition probability between acceptable states A_i and A_k; $i, k \in I'$
$[A] \equiv [a_{ik}]$	The r square matrix of the one step transition probabilities between the r acceptable states.
F	The class containing the m failed states.
I"	The index set for the m failed states
F_j	The failed state $F_j \in F$, $j = 1, \ldots, m$. $j \in I"$
δ_{ju}	The one step transition probability between failed states F_j and F_u. $\delta_{ju} = \begin{cases} 1 & u = j \\ 0 & u \neq j \end{cases}$; $j, u \in I"$.

$[\delta] \equiv [\delta_{ju}]$ The m square matrix of the one step transition proabilities between the m failed states. This is by definition the unit m x m matrix.

b_{ij} The one step transition probability between acceptable state A_i and failed state F_j; $i \in I'$, $j \in I''$.

$[B] \equiv [b_{ij}]$ The r x m matrix of the one step transition probabilities from the class A of acceptable states to the class F of failed states.

$[\phi]$ The m x r null matrix.

$\pi_v(n)$ The probability that at time t = n, the system is in state $a_v \in \Lambda$. This is called the state probability function.

$\overline{\pi}(n)$ The r + m vector of the state probability functions.

$R(n)$ The reliability function. R(n) is the probability that at time t = n, the system is operating acceptable, that is, in some acceptable state in A.

p_{ij} The steady state transition failure probability. This is the probability of going eventually from acceptable state A_i to failed state F_j.

$[P] \equiv [p_{ij}]$ The r x m matrix of steady state transition failure probabilities.

$p_{ij}(n)$ The transition probability failure function. This is the probability of going from acceptable state A_i to failed state F_j in n units of time.

$[P(n)] \equiv [p_{ij}(n)]$ The r x m matrix of the transition probability failure functions.

τ_{ij} The random variable "first time the state of the system is failed state F_j given that the system initially started in acceptable state A_i.

$\tau_{ij}(k)$ The k^{th} moment of τ_{ij}; $k = 1, 2, \ldots$

$[\tau(k)] \equiv [\tau_{ij}(k)]$ The r x m matrix of the k^{th} moments of τ_{ij}.

$g_{ij}(z)$ The generating function for the moments $\tau_{ij}(k)$.

$[G(z)] \equiv [g_{ij}(z)]$ The r x m matrix of the above generating functions.

$w_i(n)$ The probability that after n units of time the system will be in some failed state in F, given that the system initially started in acceptable state A_i. This is called the exit probability function.

$\overline{\Omega}(n) = [w_1(n), .. w_r(n)]^T$ The 1 x r column vector of the exit probability functions.

τ_i The random variable "first exit time from acceptable state A_i into the failed class F".

$\tau_i(k)$ The k^{th} moment of τ_i; $k = 1, 2, \ldots$

$\overline{\tau}(k) = [\tau_1(k), .. \tau_r(k)]^T$ The 1 x r column vector of the k^{th} moments of τ_i.

$\psi_i(z)$ The generating function for the moments $\tau_i(k)$.

$\overline{\psi}(z) = [\psi_1(z), .. \psi_r(z)]^T$ The 1 x r vector of the above generating functions.

Appendix

This program has been successfully run on an IBM 360 Model 65 computer. It computes the powers $[A]^k$, k = 1,..., L of the m x m matrix [A] . As written, the value of m, must be less than or equal to 30, and the value of L, must be less than or equal to 60. The last data card should contain the number 61 in columns 2 and 3. The dimension statement can of course be modified to accommodate larger matrices, that is larger values of m, as well as larger values of L. For example, suppose [A] was a 40 x 40 matrix and it was desired to compute up to and including the 100^{th} power, than the statement DIMENSION (60,30,30) should be changed to DIMENSION (100,40,40).

```
FORTRAN IV G LEVEL 1,
C THIS I
C IT COMPUTES THE K POWERS, K=1,...,L
C THE VALUE OF L,MUST BE LESS THAN OR EQUAL T
C THE VALUE OF M,MUST BE LESS THAN OR EQUAL TO
      DIMENSION A(50,30,30)
      IOUT=3
      IN=1
  100 READ(IN,1) M,L
    1 FORMAT (2I2)
      IF(L-6)2,11,11
    2 READ(IN,3) ((A(1,I,J),J=1,M),I=1,M)
    3 FORMAT (20F4.4)
      GO TO 4
    4 N=2
   15 I=1
      J=1
    5 A(N,I,J)=0
      DO 6 K=1,M
    6 A(N,I,J)=A(N,I,J)+(A(N-1,I,K)*A(1,K,J))
      I=I+1
      IF(I-M)5,5,7
    7 WRITE (IOUT,8) (A(N,I,J),I=1,M)
    8 FORMAT (///5X,25F5.4)
      GO TO 9
    9 J=J+1
      I=1
      IF(J-M)5,5,10
   10 WRITE (IOUT,13) N
   13 FORMAT (5X,///5X,'ABOVE MATRIX IS THE N POWER OF ORIGNAL MATRIX',/
     1 10X,'N=',I2)
      GO TO 12
   12 N=N+1
      IF(N-L)15,15,100
   11 CALL EXIT
      END
```

CHAPTER II

RELIABILITY PREDICTION TECHNIQUES FOR DEGRADATION-PRONE SYSTEMS

INTRODUCTION

In what follows, elements of the theory of continuous Markov processes as developed by Feller, Itô, Doob and Dynkin, are utilized to develop a general reliability model for systems that exhibit degradation failure.

A system is said to exhibit degradation failure when some subset of its parameters change and assume values outside of their regions of acceptance[7]. In practice, the time behaviour of each of these degradation-prone parameters will be a sample function of some stochastic process.

Specifically, we consider degradation-prone parameters whose time behaviour can be characterized by Feller processes. A system is considered operable provided its parameters maintain their values in certain sets, called regions of acceptance.

Methods are presented for obtaining for each degradation-prone parameter a time dependent reliability function R(t). This function gives at a specified time t, the probability that the given system parameter has a value in its region of acceptance.

In section (2.2), it is shown, that this reliability function R(t) is expressible directly in terms of a semigroup $\{ \mathbb{P}_t , t \geq 0 \}$ of Feller transition operators.

In section (2.3), the concept of the infinitesimal generator Ψ of this semigroup is introduced. It is shown that $\mathbb{P}_t = e^{\Psi t}$, and that the infinitesimal generator is completely specified once the particular Feller process for characterizing the degradation-prone parameter is choosen. Thus the problem of obtaining the reliability

function R(t), is reduced to selecting a particular Feller process as a model for the degradation. This approach to the reliability modelling of systems exhibiting degradation failure, is given here for the first time.

In section (2.4), the generalized Laplacian $\mathbb{D}_m \mathbb{D}_s$ of Feller is introduced as an alternative representation of the infinitesimal generator Ψ. A simple method is given for determining whether there is a nonzero probability that any specified parameter will assume a value outside of its region of acceptance in a finite time.

In section (2.5), general theorems of Dynkin are utilized to develop the generating function for computing the moments of the first time, (called first time to failure) that a parameter has unacceptable values. It is also shown that this generating function is completely specified once a particular infinitesimal generator Ψ has been choosen. Thus the obtaining of the moments of the first time of failure, reduces to choosing a particular Feller process as a model for the degradation. These results appear for the first time in the reliability context.

In section (2.6), the concept of the Kac potential is introduced to develop a method for computing the mean time that a parameter has values that are in some specified set. It is shown that the method also depends on the selection of a particular Feller process, which in this case is the Brownian process. That is, the method depends only on the choice of a particular infinitesimal generator Ψ

In section (2.7), for the purposes of suggesting further research, a relation is indicated between the terminology of the general theory of Linear Diffusion, and the theory of reliability modelling of systems exhibiting degradation failure.

Since most of the material presented herein, appears for the first time in the context of reliability theory, there was no way of comparing it with existing work except possibly with the paper[7] which resulted from an earlier phase of this work. Several examples, (mostly hypothetical) are given to illustrate some of the subtleties involved in applying the theory.

It should be emphasized that the object of this presentation is to introduce with a minimum of mathematical sophistication, the methods of continuous Markov processes into reliability prediction theory. For this reason there has been no attempt to achieve mathematical rigour, but rather to present, in a heuristic style, those elements of the theory that can be used to model degradation-prone systems.

SECTION 2.1

MATHEMATICAL PRELIMINARIES

Borel Algebra

Let Ω be a set, and let $\mathbb{B}(\Omega)$ be the class of sets containing all subsets of Ω. If this class is closed under countable unions and complementations, then by slight abuse of notation, $\mathbb{B}(\Omega)$ is also called the Borel Algebra.

Measurable Space.

A pair $(\Omega, \mathbb{B}(\Omega))$ is called a measurable space and, in particular a set A is called measurable if $A \in \mathbb{B}(\Omega)$.

Measurable Functions.

Let $(\Omega_1, \mathbb{B}(\Omega_1))$ and $(\Omega_2, \mathbb{B}(\Omega_2))$ be measurable spaces: a mapping $X: \Omega_1 \to \Omega_2$ is called a measurable function on $\mathbb{B}(\Omega_1)$, or simply $\mathbb{B}$ - measurable if every $A \in \mathbb{B}(\Omega_2)$ has an inverse image $X^{-1}(A) \in \mathbb{B}(\Omega_1)$.

Let Σ denote the collection of sets $X^{-1}(A) \in \mathbb{B}(\Omega_1)$, for every $A \in \mathbb{B}(\Omega_2)$. Then Σ is the smallest Borel algebra on which the mapping X is measurable. We also denote by $\bar{\mathbb{B}}(\Omega)$, the set of all bounded $\mathbb{B}$ - measurable functions on Ω.

Probability Space.

The triple $(\Omega, \mathbb{B}(\Omega), P)$, where P is a probability measure on $\mathbb{B}(\Omega)$, is called a probability space.

Random Variable.

A mapping $X: (\Omega, \mathbb{B}(\Omega), P) \to (\Lambda, \mathbb{B}(\Lambda), P_X)$ is called a random variable*). That is, a mapping of the abstract space Ω to the reals Λ, such that $X^{-1}(A) \in \mathbb{B}(\Omega)$ for every $A \in \mathbb{B}(\Lambda)$ is called a random variable. Thus a random variable is simply a real valued measurable function.

For example, let $(\Omega, \mathbb{B}(\Omega), P)$ be a probability space, and

---

*) This is an unfortunate historical mistake, because X is neither random, nor a variable. It is a well defined function.

let $(\Lambda, \mathcal{B}(\Lambda))$ be the ℓ-dimensional real measurable space. The mapping $X: \Omega \to \Lambda$ is then a ℓ-dimensional vector random variable.

Probability Distribution.

The measure $P_X(A) \equiv P(X^{-1}(A))$, $A \in \Lambda$, is the measure induced on Λ by the random variable $X: \Omega \to \Lambda$. $P_X(A)$ is called the probability distribution of the random variable X on Λ.

For obvious practical reasons, we will be concerned here only with probability distributions on the reals, that is on Λ.

Stochastic Process.

A stochastic process, is a finite or infinite (countable or noncountable) collection of random variables, which we shall index by t, and denote by $X(t)$, with $t \in T$. Thus a stochastic process is a mapping *) $X: \Omega \otimes T \to \Lambda$.

Sample Path.

A mapping $\omega : T \to \Lambda$ is called a sample path if $\omega(\infty) = \infty$, and there exists for each path ω, a number $\tau(\omega) \in T$, depending on the particular ω, such that $\omega(t) = \infty$, for $t > \tau(\omega)$. $\tau(\omega)$ is called the killing time of the path ω.

The abstract space Ω of all sample paths ω, is best thought of as "nature's space", for reasons that will be explained below.

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*) The notation $\otimes$ will be used to denote the cartesian product. The time set T is the set of the reals $[o, \infty)$, and instead of writing $X(\cdot,\cdot)$ for the process, we adopt the convention that $X(t)$ is the process when t is a variable, and $X(t)$ is the random variable when t is specified.

For every path (mapping $\omega \in \Omega$), we denote the value of ω at fixed time t, by the mapping (random variable) $X(t, \omega) \equiv \omega(t)$. It is customary to suppress ω and write $X(t)$ for $X(t,\omega)$, but we will be obliged to use the latter notation in proving Dynkin's theorem.

The reason for calling Ω, "nature's space", is that in repeated runs of a black box type experiment, for every run, nature usually picks a different path ω. Hence we observe at each run, a time function called a sample function, which usually will be different for different runs; that is for each different ω path. The uncertainty arises from our ignorance of nature's choice of ω.

The mapping $X(t,\cdot)$: $\Omega \to \Lambda$, for each fixed $t \in T$, is called a random variable. For t variable $X(\cdot,\cdot)$ is called a stochastic process. We shall use the widely accepted convention that $X(t)$ is a random variable when t is fixed, and a stochastic process when t is variable. In both cases the path ω is not fixed, and the domain of the random variable and, therefore also of the stochastic process is the Ω sample space. Finally a sample function is what is observed on a single run of an experiment, that is, for a single fixed sample path ω. Therefore a sample function should not be confused with a stochastic process, a sample function is simply one of an infinite number of possible realizations of a stochastic process.

The Sample Space Ω.

As has been pointed out above, in the context of this presentation the map $X: \Omega \otimes T \to \Lambda$ is the stochastic process. In order for the space Ω of paths to be a sample space there must correspond to each path $\omega \in \Omega$, two paths ω_s^- and ω_s^+ called respectively stopped and shifted paths. These paths exibit the following properties.

Stopped path

The effect of operating on a stopped path ω_s^- with the mapping $X(\cdot, \cdot)$, is given by the following relationship which holds true for all $s, t \in T$, namely*)

$$X(t, \omega_s^-) = X(t \wedge s, \omega) \quad , \quad t, s < \infty$$

Shifted Path

The effect of operating on a shifted path ω_s^+ with the mapping $X(\cdot, \cdot)$, is given by the following relationship which holds true for all $s, t \in T$, namely

$$X(t, \omega_s^+) = X(t + s, \omega) \quad , \quad t, s < \infty$$

That is the effect of changing from a path ω to its stopped or shifted paths, produces translations backward or forward in time. This seemingly simple concept has far reaching applications in deducing many non trivial results (for example Dynkin's Formula).

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*) The notation $\wedge$ means "smaller of".

Markov Process

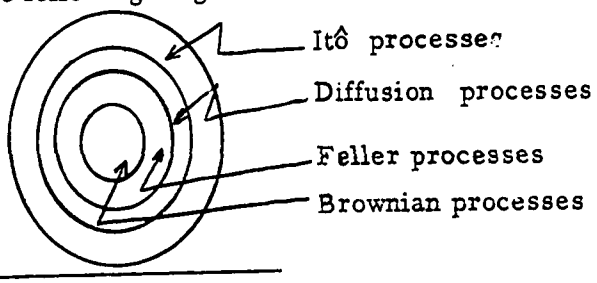
The stochastic process $X: \Omega \otimes T \rightarrow \Lambda$ is called a Markov process, if the probability distribution P_{x_o} of the random variable $X(\cdot, t)$ on Λ which is defined as

$$P_{x_o}\{X(t) \in A\} \equiv P(A,t / x_o,t_o), \quad A \subset \Lambda$$

has the Markov property, that is for every $\ldots t_k < \ldots < t_i < \ldots t_o < t$,

$$P(A, t/ x_o,t_o,\ldots x_i,t_i,\ldots x_k,t_k \ldots) \equiv P(A,t /x_o,t_o)$$

$P(A,t/x_o,t_o)$ is called the transition probability distribution function of the Markov process. In the context of reliability theory, a degradation-prone parameter X would be characterized by the process X(t).*) Then $P(A,t/x_o,t_o)$ is the probability that at time t, the parameter X, has a value in some set A, given that at time $t = t_o$ it had a value x_o. In what follows we shall utilize a special class of Markov processes called Feller processes. The relationship between the various types of Markov processes is illustrated by the following diagram.



*) We follow for the most part the usual convention of suppressing the Ω space unless it requires special emphasis, and understand that X(t) is the Markov process when t is not specified and the random variable when t is specified.

Markov Times

The concept of a Markov time, is perhaps the most important one in the theory of Markov process. Simply speaking, the mapping (random variable) $\tau : \Omega \rightarrow T$ is called a Markov time if it has a number of properties which are listed, for example, by Itô.[3, pages 56-58] As far as we are concerned, the Markov times that are of interest to us are the following random variables

$$\tau_A = \inf \{ t : X(t) \in A \}$$

That is, in the context of reliability prediction theory τ_A is the random variable "first time the parameter has a value in the set A". In the sections that follow we shall be mainly concerned with developing methods for obtaining the probability distributions and moments of these random variables (Markov times).

Strong Markov property*).

A Markov process $X(\cdot,\cdot)$ is said to have the strong Markov property with respect to the Markov time τ, if the following relationship (which we shall use later) is true [3, page 59] for all $f, g \in B(\Lambda)$

$$\mathbb{E}_{x_o}(f(X(\omega,t))\, g(X(\omega_\tau^+,t))) = \mathbb{E}_{x_o}(f(X(t,\omega))\, \mathbb{E}_{X(\tau)}(g(X(\omega,t))) \quad \ldots (1)$$

---

*) The importance of the strong Markov property cannot really be appreciated from the few lines given below, and the interested reader should consult Itô [3] on Dynkin [2, 23] for further explanations.

Diffusion process

A Markov process $X(\cdot,\cdot)$ is called a Diffusion process if it has the strong Markov property and if its underlying sample space Ω, contains only those sample paths ω that are continuous before the killing time.

Comment

As far as we are aware the most general type of Markov processes that have been used in any meaningful engineering applications have been Diffusion processes. This has also been pointed out by Kushner [6, page 28].

Conservative property

The Markov process $X(\cdot,\cdot)$ is called conservative if the killing time $\tau(\omega)$ of each path ω occurs only at time infinity with probability one. That is $X(\cdot,\cdot)$ is conservative if

$$\text{Prob}\{\tau(\omega) = \infty\} = 1.$$

Feller Processes

Time homogeneous Diffusion processes are called Feller processes. The Feller property of a Markov processes is reflected in the following property of its transition probability function which holds whenever $X(t)$ is a Feller process, namely

$$P(A,t/x_o,t_o) = P(A,t - t_o/x_o) \equiv P(A, t/x_o)$$

The transition probability function $P(A,t/x_o)$ is called a Feller transition function, [2, page 52] and has a number of important properties which are listed below:

Properties of $P(A,t/x_o)$

(i) $P(A,t/x_o) \leq 1 : \quad A \subset \Lambda$

(ii) $P(A,t/x_o) \in B(\Lambda)$; for fixed t, A

(iii) $\lim_{t \to o} P(x, t/x_o) = 1 \quad \{ x = x_o$
$= 0 \quad \{ x \neq x_o$

(iv) $P(A, t+s/x_o) = \int_\Lambda P(A,t/x,) P(dx, s/x_o)$

(v) $P(\Lambda, t/x_o) = 1$.

property (iv) is the Chapman-Kolmogorov equation [8, page 14]. The following theorem summarizes the above results.

Theorem 2.1.1 Every function $P(A,t/x_o)$ which satisfies properties (i) to (iv) is the transition probability function of a Feller process. If, in addition, property (v) also holds, then the process X(t) is conservative. For proof see Mandl. [12, page 11]

Instead of working with the transition function $P(A,t/x_o)$, the current practice [8, page 22] is to consider $P(A,t/x_o)$ as the kernel of an operator $\mathbb{P}_t$. This operator $\mathbb{P}_t$ is called a Feller transition operator, and is defined as

$$\mathbb{P}_t \cdot = \int_\Lambda P(dx, t/x_o) \cdot$$

and $\mathbb{P}_t$ maps $B(\Lambda)$ into itself. That is, for each $f \in B(\Lambda)$, $\mathbb{P}_t f \in B(\Lambda)$. It can be shown [8 page 40] that the set of operators $\{ \mathbb{P}_t, t \geq o \}$ have the semigroup property, and therefore satisfy the following conditions

(i) $\mathbb{P}_t \mathbb{P}_s = \mathbb{P}_{t+s} \quad (t, s \geq o)$

(ii) $\mathbb{P}_o = I$ (I is the identity operator)

(iii) $\lim_{t \to t_o} \left\| \mathbb{P}_t f - \mathbb{P}_{t_o} f \right\| = o$, for every $f \in B(\Lambda)$ and each $t_o > o$.

(iv) $\left\| \mathbb{P}_t \right\| \leq 1$ for all $(o \leq t < \infty)$ *)

Property (iv) above is called the contraction property, and in mathematical literature the set of operators, $\{ \mathbb{P}_t ; t \geq o \}$ is called a contraction semigroup of class C_o [9]. Feller [1] was the first to utilize the semigroup approach for the study of continuous Markov processes. His paper [1] appeared as recently as 1952 and the history of the semigroup approach dates only from then. In what follows this approach will be used to develop the reliability model, and although no attempt will be made to be mathematically rigorous, and concepts such as continuity and convergence will be either assumed or eliminated all together the results presented will be correct and generally applicable to reliability prediction problems involving degradation-prone systems.

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*) $\| \ \|$ means some suitable norm, usually the supremum.

SECTION 2.2

TRANSITION AND POTENTIAL OPERATORS

In this section we develop further the properties of the transition operator $\mathbb{P}_t$ and introduce its formal Laplace transform $\mathbb{U}_\alpha$ called the potential operator.[8, page 41] Some simple but useful properties of these operators are also given. The section concludes with the formulation of a general definition of reliability.

Using the conditional expectation operator $\mathbf{E}_{x_o}$, defined as

$$\mathbf{E}_{x_o} = \int_\Lambda \mathbf{P}(dx, t/x_o) .$$

we have for any $f \in B(\Lambda)$

$$\mathbf{E}_{x_o}[f(X(t))] = \int_\Lambda P(dx, t/x_o) f(x)$$

$$= \mathbb{P}_t f(x_o) \qquad (2)$$

as is given in Bluementhal[8, page 40]. The representation for $\mathbb{P}_t$ given by equation (2) is completely general and leads to many unifying and simplified concepts. A number of useful results is presented below.

Result 1 Let f_A be the indicator (characteristic) function for the set A, that is

$$f_A(X(t)) = 1 \quad \text{whenever } X(t) \in A$$

$$= 0 \quad \text{whenever } X(t) \notin A$$

Then $f_A \in B(\Lambda)$ and applying equation (2) we find

$$\mathbb{P}_t f_A(x_o) = \int_\Lambda P(dx, t/x_o) f_A(x)$$

$$= P(A, t/x_o)$$

$$= P_{x_o}\{X(t) \in A\} .$$

Thus the application of the transition operator $\mathbb{P}_t$ to the indicator function f_A gives back the transition function $P(A,t/x_o)$.

Result 2 The moments of the random variable $X(t)$ at specified time t, are obtained by letting $f_n(x) = x^n$ then

$$\mathbb{P}_t f_n(x_o) = \int_\Lambda P(dx, t/x_o)\, x^n \tag{3}$$

however, it should be emphasized that $f_n(x) = x^n \notin B(\Lambda)$, and the moments given by equation (3) may not always exist. The following theorem and its proof establish the semigroup property of the transition operators $\{\mathbb{P}_t;\ t \geq o\}$.

Theorem 2.1.1 The set of transition operators that map $B(\Lambda)$ to $B(\Lambda)$ has the semigroup property. In particular

$$\mathbb{P}_{t+s} = \mathbb{P}_t \mathbb{P}_s \quad ; \quad t, s \geq o \tag{4}$$

Proof[3, page 29] Using equation (2) we can write

$$\mathbb{P}_{t+s} f(x_o) = \mathbb{E}_{x_o}[\, f(X(t+s))\,]$$

$$= \int_\Lambda P(dx, t+s / x_o)\, f(x) \tag{5}$$

using the Chapman - Kolmogorov equation, equation (5) can be written as

$$\mathbb{P}_{t+s} f(x_o) = \int_\Lambda f(x) \int_\Lambda P(dy, t/x_o)\, P(dx, s/y)$$

$$= \int_\Lambda P(dy, t/x_o) \int_\Lambda P(dx, s/y)\, f(x)$$

$$= \int_\Lambda P(dy, t/x_o)\, \mathbb{P}_s f(y)$$

$$= \mathbb{P}_t \mathbb{P}_s f(x_o)$$

QED.

Comment

Equation (4) is nothing more than the general operator form of the Chapman-Kolmogorov equation. This fact can readily be established using Result 1. That is, using the indicator function f_A we find

$$\mathbb{P}_{t+s} f_A(x_o) = P(A, t+s / x_o)$$

$$= \int_\Lambda P(dy, t/ x_o) \mathbb{P}_s f_A(y)$$

$$= \int_\Lambda P(dy, t/ x_o) P(A, s/y)$$

which is, as was expected, the Chapman-Kolmogorov equation .

Potential Operators and Laplace Transform Theory

$\mathbb{P}_t$ is a linear operator, and it is possible to simplify the solution of many problems involving this operator, by taking its formal Laplace transform $\mathbb{U}_a$ defined as

$$\mathbb{U}_a \cdot = \int_o^\infty e^{-at} \mathbb{P}_t \cdot dt \tag{6}$$

The operator $\mathbb{U}_a$ also has as its domain the functions $f \in B(\Lambda)$. When $\mathbb{U}_a$ acts on a function $f \in B(\Lambda)$, it produces what is called, [8, page 41] the a-potential*) of f . This is written as

$$\mathbb{U}_a f(x_o) = \int_o^\infty e^{-at} \mathbb{P}_t f(x_o) dt \tag{7}$$

---

*) $\mathbb{U}_a$ is also called [3, page 30] the Green operator on $B(\Lambda)$

It is possible to obtain a convenient representation for $\mathbb{U}_a$ by substituting (2) in (7). Performing this substitution gives

$$\mathbb{U}_a f(x_o) = \int_o^\infty \int_\Lambda e^{-at} P(dx, t/x_o) f(x) dt$$

interchanging the order of integration in the above equation, we obtain the desired representation [8, page 41]

$$\mathbb{U}_a f(x_o) = \mathbb{E}_{x_o} [\int_o^\infty e^{-at} f(X(t)) dt] \tag{8}$$

A number of results analogous to those obtained as Result 1 and Result 2 for $\mathbb{P}_t$, hold for $\mathbb{U}_a$. We have, for example:

Result 3 Let f_A be the indicator function, then

$$\mathbb{U}_a f_A(x_o) = \int_o^\infty e^{-at} \mathbb{P}_t f_A(x_o) dt \tag{9}$$

but using (3) we can write (9) as

$$\mathbb{U}_a f_A(x_o) = \int_o^\infty e^{-at} \mathbb{P}(A,t / x_o) dt \equiv U(A, a / x_o) \tag{10}$$

where, as shown above, $U(A, a / x_o)$ is simply the ordinary Laplace transform of $P(A,t / x_o)$. $U(A, a /x_o)$ is also known as the a-potential measure on $B(\Lambda)$, or the Green's measure on $B(\Lambda)$.

It is not surprising to expect that just as $P(dx,t /x_o)$ was the kernel of $\mathbb{P}_t$ that $U(dx,a /x_o)$ is the kernel of $\mathbb{U}_a$. We show below that this is indeed the case. By definition

$$\mathbb{U}_a f(x_o) = \int_o^\infty e^{-at} \mathbb{P}_t f(x_o) dt \tag{7}$$

interchanging the order of integration in (7) gives

$$\mathbb{U}_a f(x_o) = \int_\Lambda f(x) \int_o^\infty e^{-at} P(dx, t /x_o) dt$$

but using (10) the above equation can also be written as

$$\mathbb{U}_a f(x_o) = \int_\Lambda U(dx, a/x_o) f(x) \tag{11}$$

which is the desired result, namely

$$\mathbb{U}_a \cdot = \int_{\Lambda} U(dx, a/x_o) .$$

The operator $\mathbb{U}_a$ has a number of properties which are analogous to those of $\mathbb{P}_t$ given earlier. These are listed below:

Properties of $\mathbb{U}_a$

(i) $\mathbb{U}_a$ is a linear operator

(ii) If f_n converges to f, where f_n and $f \in B(\Lambda)$; Then $\mathbb{U}_a f_n(x_o)$ converges to $\mathbb{U}_a f(x_o)$

(iii) If $f \geq o$, then $\mathbb{U}_a f \geq o$

(iv) $\|\mathbb{U}_a\| \leq \frac{1}{a}$ whenever $\mathbb{P}_t I \leq 1$

(v) $\mathbb{U}_a$ exists for complex a satisfying real $a > o$, and is analytic in a for every $f \in B(\Lambda)$ and $x_o \in \Lambda$

(vi) $\lim_{a \to \infty} a \mathbb{U}_a f(x_o) \to f(x_o)$

(vii) Let $\mathbb{U}_a f(x,t) \equiv F(a)$
Then $\mathbb{U}_a f(x, a + \lambda t) = \frac{1}{\lambda} e^{a a/\lambda} F(a/\lambda)$

Comments

(a) Property (vi) is really a generalized Tauberian theorem applied to Laplace transform theory. Its proof is presented below:

Proof of (vi) By equation (6), we can write

$$a \mathbb{U}_a f(x_o) = a \int_o^{\infty} e^{-at} \mathbb{P}_t f(x_o) \, dt$$

using the substitution $\tau = \alpha t$, the above equation can be written as

$$\alpha \mathbb{U}_\alpha f(x_o) = \int_o^\infty e^{-\tau} \mathbb{P}_{\tau/\alpha} f(x_o)\, d\tau$$

taking the limit for $\alpha \to \infty$, and using the property $\lim_{\alpha \to \infty} \mathbb{P}_{\tau/\alpha} = \mathbb{P}_o \equiv I$,

we obtain (vi) QED.

(b) Property (vii) is analogous to the shift and scale theorems of ordinary Laplace transform theory.

General Definition of Reliability

We are now in a position to use the results presented above to formulate a general definition of reliability. The previous paper [7] introduced a reliability function R(t) for systems with degradation-prone parameters. This function R(t) was defined for each degradation-prone parameter as

$$R(t) = \text{Prob}\{x_1 \leq X(t) \leq x_2\}$$

That is, R(t) is the probability that at time t, the degradation prone parameter X has a value in its "region of acceptance" $[x_1, x_2]$. In the paper cited above, only Brownian processes were utilized to characterize degradation-prone parameters. Brownian processes are the simplest type of Feller processes and therefore the theory presented now includes that presented in the already cited paper, [7] as a special case. The following definitions of reliability are the most general possible for systems with degradation-prone parameters.

Let a system have k degradation-prone parameters $(X^1, \ldots, X^k) \equiv X$; Let $X(t) = (X^1(t), \ldots, X^k(t))$ be the k-order vector Feller process

characterizing the time behaviour of X. The state space of X(t) is Λ^k, and the system is considered operable provided X(t) has values in some set of acceptable values $A^k \subset \Lambda^k$, that is, provided the parameters have values in their regions of acceptance. Then we can define

Definition Conditional Reliability Function. The conditional reliability function $R_{x_o}(t)$ is defined as

$$R_{x_o}(t) = P_{x_o}\{ X(t) \in A^k \} \tag{12}$$

$R_{x_o}(t)$ is the probability that at time t, the system parameters are in their regions of acceptance, conditioned on the fact that at time t = o, the initial parameter vector was x_o. That is

$$R_{x_o}(t) = \text{Prob}\{(X^1(t),\ldots\ldots,X^k(t)) \in A^k \Big/ X^1(o)=x_o^1,\ldots, X^k(o)=x_o^k\}$$

Alternatively, using Result 1 of this section, we have that equation (12) can be written as*)

$$R_{x_o}(t) = \mathbb{P}_t f_{A^k}(x_o) \tag{12}$$

Definition Unconditional Reliability Function. The unconditional reliability function R(t) is defined as

$$R(t) = \text{Prob}\{ X(t) \in A^k \}$$

R(t) is the probability that at time t, the systems parameters are in their regions of acceptance. That is

$$R(t) = \text{Prob}\{(X^1(t),\ldots\ldots, X^k(t)) \in A^k \}$$

---

*) The conditional reliability function for a single parameter is the same as (12) but with A^k replaced by A.

Letting $p(x_o)$ be the distribution of the values of the parameters at time t = o, that is the distribution of the random variable X(o), then from (12) we have

$$R(t) = \int_{\Lambda^k} R_{x_o}(t)\ p(x_o)\, dx_o$$

alternatively

$$R(t) = \int_{\Lambda^k} \mathbb{P}_t f_{A^k}(x_o)\ p(x_o) dx_o$$

$$= \int_{\Lambda^k} P(A^k, t/x_o)\ p(x_o)\, dx_o \qquad (13)$$

SECTION 2.3

PARTIAL DIFFERENTIAL EQUATIONS FOR DETERMINING THE TRANSITION OPERATOR

In section 2.2, it was shown that the problem of obtaining the reliability function R(t) for any degradation-prone system really consists in determining a transition operator $\mathbb{P}_t$. In this section, we present a method for determining $\mathbb{P}_t$. It is shown that $\mathbb{P}_t$ is obtained as the solution of a partial differential equation.

The following theorems are somewhat simplified versions of those originally given in Nelson [11], Doob [20], Itô [3], and are included as preparatory material for the presentation of the main results of this section.

Theorem 2.3.1

Let $X(t)$ (in this case, scalar) be a Feller process having state space Λ, and let $O(x_o) \subset \Lambda$ be a set whose elements x satisfy $|x-x_o| < \delta$ (arbitrary). Let the kernel of $\mathbb{P}_t$, namely the transition probability function $P(A, t/x_o)$ have the following properties

$$\text{(i)} \quad \lim_{t \to o} \frac{1}{t} \mathbb{P}_t f_{O(x_o)}(x_o) = P(O(x_o), t/x_o)$$

$$\text{(ii)} \quad \lim_{t \to o} \frac{1}{t} [\mathbb{E}_{x_o}(X(t+\Delta t) - X(t))] = \lim_{t \to o} \frac{1}{t} \int_{O(x_o)} (x-x_o) P(dx, t/x_o) = a(x_o)$$

$$\text{(iii)} \quad \lim_{t \to o} \frac{1}{t} [\mathbb{E}_{x_o}((X(t+\Delta t) - X(t))^2)] = \lim_{t \to o} \frac{1}{t} \int_{O(x_o)} (x-x_o)^2 P(dx, t/x_o) = b^2(x_o)$$

where $a(x_o)$ and $b^2(x_o)$ are finite $\forall\, x_o \in \Lambda$. Then the theorem states that the Feller process $X(t)$ can be constructed from the Wiener process $W(t)$ (the simplest type of Brownian process), as the solution of the stochastic integral equation*)

$$X(t) = X(o) + \int_o^t a(X(\xi))\, d\xi + \int_o^t b(X(\xi))\, dW(\xi)$$

Proof A proof of this can be found in Nelson [5, Chapter 11]

Comment. The conditions (i), (ii) and (iii) in the above theorem can be interpreted as stipulating that $X(t)$ has the following properties.

---

*) The integral $\int_o^t \ldots$, is called an Itô stochastic integral [11]. This is not a conventional integral, because the sample paths of the Wiener process are not of bounded variation. The approach of describing Feller stochastic processes as solutions to stochastic integral equations is an alternative approach to the one taken herein. However, the stochastic equation approach has many advantages, especially for multidimensional processes, and will prove valuable in many future investigations in reliability prediction theory.

(i) $\lim_{t \to o} P_{x_o} \{ / dx(t) / > o \} \lesssim o(dt)$

(ii) $\mathbb{E}_{x_o} (d\, x(t)) \approx a(x_o)dt,$ (called infinitesimal mean)

(iii) $\mathbb{E}_{x_o} (dx(t))^2) \approx b^2(x_o)dt,$ (called infinitesimal variance)

In preparation for what follows, we now prove the following Lemma [3, page 209].

Lemma 2.3.1 Let $f \in B(\Lambda)$ have continuous derivatives up to at least order two on some interval $[x_1, x_2]$. Let $\mathbb{P}_t$ be the transition operator of a Feller process and have as kernel $P(dx, t/x_o)$. Then the lemma states that

$$\lim_{t \to o} \frac{1}{t} (\mathbb{P}_t f(x_o) - f(x_o)) = \frac{b^2(x_o)}{2} f''(x_o) + a(x_o) f'(x_o) \qquad (14)$$

(where ' denotes the operator $\frac{d}{dx_o}$ etc.)

Proof By property (i) of theorem 2.3.1, we can write equation (14) as

$$\lim_{t \to o} \frac{1}{t} (\mathbb{P}_t f(x_o) - f(x_o)) = \lim_{t \to o} \frac{1}{t} \int_{O(x_o)} (f(x) - f(x_o)) P(dx, t/x_o) \qquad (15)$$

making a Taylor expansion of f about the point x_o, and noting that the integration in (15) is on the set $O(x_o)$ we neglect values of x that do not satisfy $/ x - x_o / < \delta$. Also since δ can be made arbitrarily small, we can neglect the terms of order $(x - x_o)^3$ and higher in the Taylor expansion. We can therefore write (15) as

$$\lim_{t \to o} \left[f'(x_o) \frac{1}{t} \int_{O(x_o)} (x - x_o) P(dx, t/x_o) + f''(x_o) \frac{1}{t} \int_{O(x_o)} (x - x_o)^2 P(dx, t/x_o \right]$$

On substituting (ii) and (iii) from Theorem 2.3.1 in the above we obtain (14) QED.

Theorem 2.3.2

Let u, u' and $u'' \in B(\Lambda)$, and let $a(x_o)$ and $b(x_o)$ be bounded $\forall x_o \in [x_1, x_2]$. Then u is contained in the domain of an operator Ψ, called the infinitesimal generator of the Feller process. This operator Ψ is given by

$$\Psi u = \frac{1}{2} b^2(x_o)\, u'' + a(x_o)\, u' \tag{16}$$

where

$$\lim_{h \to o} \left(\frac{\mathbb{P}_h - I}{h} \right) u = \Psi u \tag{17}$$

Proof

Follows directly from lemma 2.3.1.

The equation (17), establishes the 1: 1 correspondence between the transition operator $\mathbb{P}_t$ and its infinitesimal generator Ψ. We establish now thepartial differential equation for obtaining $\mathbb{P}_t$.

The Kolmogorov Backward Equation For $\mathbb{P}_t$

Let us select $f \in B(\Lambda)$ and $u = \mathbb{P}_t f \in B(\Lambda)$ where both u and f have derivatives up to order two. Using the semigroup property (4), equation (17) can be written as

$$\lim_{h \to o} \left(\frac{\mathbb{P}_{t+h} - \mathbb{P}_t}{h} \right) f = \Psi\, \mathbb{P}_t f \tag{17}$$

passing to the limit for $h \to o$ in equation (17) gives,

$$\frac{d\, \mathbb{P}_t f}{dt} = \Psi\, \mathbb{P}_t f \tag{18}$$

Equation (18) is the most general form of the Kolmogorov backward partial differential equation satisfied by the transition operator $\mathbb{P}_t$ of a Feller process. The usual familiar form in which (18) is often written, is obtained by taking $f = f_{[dx]}$ and performing the operation

$$\mathbb{P}_t f_{[dx]}(x_o) = P(dx, t/x_o) \qquad (19)$$

then, on substituting (16) and (19) in (18), we find the familiar equation*)

$$\frac{\partial P(dx,t/x_o)}{\partial t} = \frac{1}{2} b^2(x_o) \frac{\partial^2 P(dx,t/x_o)}{\partial x_o^2} + a(x_o) \frac{\partial P(dx,t/x_o)}{\partial x_o} \qquad \ldots\ldots (20)$$

The following example is intended to illustrate some of the methods presented above.

Example 1 (Reliability application).

In the paper [7], we considered the case of the Brownian process as characterizing the time behaviour of a degradation-prone parameter. For this case $a(x_o) \equiv C$ and $b^2(x_o) \equiv D$, were constants. The transition probability density function $P(dx,t/x_o)$ for this case satisfied

$$\frac{\partial P(dx,t/x_o)}{\partial t} = \frac{1}{2} D \frac{\partial^2 P}{\partial x_o^2}(dx,t/x_o) + C \frac{\partial P(dx,t/x_o)}{\partial x_o} \qquad (21)$$

---

*) Equation (20) is also called the Fokker-Planck backward equation.

It was shown there that the solution of (21) is

$$P(dx,t/x_o) = \frac{1}{\sqrt{4\pi\, Dt}} \exp - \left(\frac{(x-x_o-2\,Ct)^2}{4Dt} \right) \tag{22}$$

and using a region of acceptance $A \equiv [x_1,x_2]$ and an initial distribution $p(x_o)$ given by

$$p(x_o) = \frac{1}{\sqrt{2\pi\,\sigma(o)}} \exp - \left(\frac{(x_o-m(o))^2}{2\,\sigma(o)} \right) \tag{23}$$

We have substituting (22) and (23) in (13), that the reliability function

$$R(t) = \int_{x_1}^{x_2} \int_{-\infty}^{\infty} P(dx,t/x_o)\, p(x_o)\, dx\, dx_o$$

is *)

$$R(t) = G\left(\frac{(x_2-x_o-2Ct)}{\sqrt{\sigma(o)+2\,Dt}} \right) - G\left(\frac{(x_1-x_o-2\,Ct)}{\sqrt{\sigma(o)+2Dt}} \right) \tag{24}$$

Relationship Between $\mathbb{U}_a$ and Ψ

We now show that $\mathbb{U}_a$ and Ψ are directly related to each other. In purely operator form (18) can be written as

$$\frac{d\mathbb{P}_t}{dt} = \Psi\, \mathbb{P}_t \tag{25}$$

using the fact that $\mathbb{P}_o = I$, we find for the solution of (25)

$$\mathbb{P}_t = e^{\Psi t} \tag{26}$$

The a potential operator $\mathbb{U}_a$ is, by definition

$$\mathbb{U}_a \cdot = \int_o^{\infty} e^{-at}\, \mathbb{P}_t \cdot\, dt, \tag{27}$$

*) $G(\cdot)$ is the error function.

then on substituting (26) in (27) we find

$$\mathbb{U}_a = \frac{1}{a - \mathbb{\Psi}} \tag{28}$$

$\{\mathbb{U}_a;\ a \geq o\}$ is called [3,4,8,9] the resolvent of the semi-group $\{\mathbb{P}_t,\ t \geq o\}$. It is exactly analogous to the ordinary Laplace transform of the simple exponential function.

From (28), we can deduce that $u = \mathbb{U}_a f$, if and only if $f = \mathbb{U}_a^{-1} u = (a - \mathbb{\Psi})\, u$. Furthermore this solution is unique. It can also be shown [12, page 2], that the domain of the operator $\mathbb{\Psi}$ is dense [13] in $B(\Lambda)$, and this fact in turn makes the solution of the backward equation (18), well defined.

The following theorem summarizes the main results presented in this section.

Theorem 2.3.3. Let the coefficient functions of $\mathbb{\Psi}$, namely $a(x_o)$ and $b(x_o)$ satisfy Holder and Lipschitz conditions. Let $W(t)$ be Wiener process *), then the solution of the Itô stochastic integral equation

$$X(t) = X(o) + \int_o^t a(X(\xi))\, d\xi + \int_o^t b(X(\xi))\, dW(\xi)$$

is a Feller process with infinitesimal generator $\mathbb{\Psi}$ given by

$$\mathbb{\Psi}\cdot = a(x_o) \frac{d\cdot}{dx_o} + b^2 \frac{(x_o)}{2} \frac{d^2\cdot}{dx_o^2}$$

---

*) See section 2.6 of this presentation.

and with transition operator $\mathbb{P}_t = e^{\Psi t}$, which satisfies for each $f \in B(\Lambda)$, the Kolmogorov backward partial differential equation

$$\frac{d\mathbb{P}_t f}{dt} = \Psi \mathbb{P}_t f \tag{18}$$

and the kernel of $\mathbb{P}_t$, namely $P(dx,t/x_o)$, has the properties (i), (ii) and (iii) of theorem 2.3.1.

Comments

(a) This theorem summarizes (in a loose sense), a number of theorems which appear in Itô [3].

(b) In the language of Control systems, this theorem can be interpreted to mean that the output X(t) of a stochastic system whose parameters can be described by $a(x_o)$ and $b(x_o)$, and whose input is white noise,*) has a statistical description in terms of the semigroup $\{\mathbb{P}_t, t \geq o\}$. In control system language R(t) is the probability that the output of the system is in a certain set at time t. There is therefore a 1:1 correspondence between the theory of stability of stochastic control systems, and the reliability theory of systems exhibiting degradation failure. A purely analytical theory of stochastic stability can be derived using the methods presented herein. This stability theory will not depend on a stochastic lyapunov function approach as introduced by Kushner[6]. As control systems are not being treated here, this line of thought is not pursued any further.

---

*) In the engineering literature "white noise" is the name given to the derivative of the Wiener-Brownian motion process.

SECTION 2.4

GENERALIZED LAPLACIAN OF FELLER

[1][13 pages 404-]

In this section we derive an alternative representation of the infinitesimal generator Ψ. This representation permits us to derive a easily applied method of determining whether or not there is a non zero probability that a degradation-prone parameter will assume a value outside of its region of acceptance, $[x_1, x_2]$ in a finite time.

This method which was originally given in context of probability theory by Feller[1], will now be presented. Feller[1], considered the points (x_1, x_2) as lateral or boundary points of the infinitesimal generator. He classified them as regular, exit, entrance or natural, and gave them an interesting probabilistic interpretation which we shall make use of.

We now derive an expression for Feller's generalized Laplacian, denoted by $\mathbb{D}_m \mathbb{D}_s$. This operator is an alternative representation for the infinitesimal generator Ψ. The derivation is given as the following lemma.

Lemma 2.4.1

The operator Ψ has the alternative representation, called Feller's generalized Laplacian, defined by*)

$$\mathbb{D}_m \mathbb{D}_s \equiv \frac{1}{m'(x_o)} \frac{d}{dx_o} \left(\frac{1}{s'(x_o)} \frac{d}{dx_o} \right) \tag{29}$$

$s'(x_o)$ has been called by Feller, the canonical scale[13,14] of the process $X(t)$. The functions $s'(x_o)$ and $m'(x_o)$ satisfy the following equations.

$$s'(x_o) = \exp\left(-2 \int \frac{a(x_o)\, dx_o}{b^2(x_o)}\right) \tag{30}$$

$$m'(x_o) = \frac{2s'(x_o)}{b^2(x_o)} \tag{31}$$

---

*) The ' denotes differentiation once with respect to x_o. "denotes differentiation twice, etc. .

where $a(x_o)$ and $b(x_o)$ are the infinitesimal mean and variance functions respectively.

Proof

Equation (29) may be verified by substituting (30) and (31) in (29). Carrying out the differentiation of the term in the brackets in (29), gives

$$\mathbb{D}_m\mathbb{D}_s = \left[\frac{1}{m'(x_o)s'(x_o)}\frac{d^2}{dx_o^2}\right] - \left[\frac{s''(x_o)}{m'(x_o)(s'(x_o))^2}\frac{d}{dx_o}\right] \qquad (32)$$

but from (30)

$$s''(x_o) = \frac{d^2}{dx_o} s'(x_o)$$

$$= -\frac{2\,a(x_o)\,s'(x_o)}{b^2(x_o)} \qquad (33)$$

substituting (31) and (33) into (32) gives the result*)

$$\Psi \iff \mathbb{D}_m\mathbb{D}_s$$

QED.

Following Yosida[13, page 405], we introduce the four quantities ϕ_1, ϕ_2, θ_1 and θ_2. These are defined below, with choice of x_1^* and x_2^* arbitrary, as long as they are real numbers.

$$\phi_1 = \int\int_{x_1<y<x<x_1^*} dm(x)\,ds(y), \quad \theta_1 = \int\int_{x_1<y<x<x_1^*} ds(x)\,dm(y)$$

$$\phi_2 = \int\int_{x_2>y>x>x_2^*} dm(x)\,ds(y), \quad \theta_2 = \int\int_{x_2>y>x>x_2^*} ds(x)\,dm(y)$$

---

*) The notation $\iff$ means "equivalence".

The following "theorem" summarizes the contribution of this section. It appears for the first time in the reliability context. It summarizes a number of theorems in Yosida [13, pages 404].

"Theorem' 2.4.1

Let the degradation-prone parameter X be characterized by a Feller process X(t) with infinitesimal generator $\Psi \Leftrightarrow \mathbb{D}_m \mathbb{D}_s$. Then the problem of determining whether degradation-prone parameter X that initially has a value $x_o \in [x_1, x_2]$ can with a non zero probability leave its region of acceptance $[x_1, x_2]$ in a finite time, can be simply solved by computing from $\mathbb{D}_m \mathbb{D}_s$, the quantities θ_1, θ_2, ϕ_1, ϕ_2, and consulting the table 1 below. This table provides a classification and probabilistic interpretation of the lateral boundaries for the reliability problem.

Boundary $x_i, i=1,2$ is called	Coefficient θ_i $i=1,2$	Coefficient ϕ_i $i=1,2$	Probability that X can assume a value outside $[x_1, x_2]$ in a finite time.
regular	$< \infty$	$< \infty$	positive
entrance	∞	$< \infty$	zero
exit	$< \infty$	∞	positive
natural	∞	∞	zero

Table 1

Comment.

These results were first obtained by Feller[1]. Mandl's[12] monograph is an elegant and lucid analytical treatment of Feller processes using semi-groups. Mandl also gives these results.

Example 2 (hypothetical)

Let degradation-prone parameter X be characterized by the Wiener process. The Wiener process has infinitesimal mean and variance parameter $a(x_o) = 0$ and $\frac{b^2(x_o)}{2} = D$. To illustrate the use of our theorem 2.4.1, we take $[x_1, x_2] \equiv [-\infty, \infty]$. From (16), we find that the infinitesimal generator for the Wiener process is

$$\Psi = D \frac{d^2}{dx_o^2}$$

now, by using (30) and (31), this can also be written as $\mathbb{D}_m \mathbb{D}_s$, where

$$m(x_o) = D^{-1} x_o \quad , \quad s(x_o) = x_o$$

evaluating ϕ_1, ϕ_2, θ_1, θ_2 and applying the theorem 2.4.1, we obtain table II below

Boundary	Coefficients		Probability of escape from $[-\infty, \infty]$ in finite time
x_1 is natural	$\phi_1 = \infty$	$\theta_1 = \infty$	zero
x_2 is natural	$\phi_2 = \infty$	$\theta_2 = \infty$	zero

Table II

Example 3

Consider a Feller process X(t) with binomial variance $\frac{b^2(x_o)}{2} = \frac{x_o(1-x_o)}{4N}$, characterizing the variation in time of N identical degradation-prone parameters, with common region of acceptance $[x_1, x_2] = [0, 1]$. To illustrate the difficulty usually involved in obtaining a closed form expression for the reliability function (13),

we compute $P(dx,t/x_o)$. From (20), we have that this function satisfies

$$\frac{\partial P(dx,t/x_o)}{\partial t} = \frac{1}{4N} x_o(1-x_o) \frac{\partial^2 P(dx,t/x_o)}{\partial x_o^2} \qquad (34)$$

This seemingly simple equation has solution [15]:

$$P(dx,t/x_o) = \sum_{i=1}^{\infty} \frac{(2i+1)(4x-4x_o)^2}{i(i+1)} G_{i-1}(1-2x_o)G_{i-1}(1-2x)\exp(-\frac{i(i+1)t}{4N})$$

$$\simeq 6x_o(1-x_o)\exp(-t/2N) + 30x_o(1-x_o)(1-2x_o)(1-2x)\exp(\frac{-3t}{2N})$$

where $G_{i-1}(1-2x)$ is the Gegenbauer polynomial [16]. This polynomial can also be written in terms of the confluent hypergeometric function F as

$$G_{i-1}(1-2x) = \frac{i(i+1)}{2} F(i+2,\ 1-i;\ 2,\ x).$$

The object of this example, was to illustrate that even relatively simple models of degradation phenomena can lead to difficult computational problems. It is in such situations that theorem 2.4.1 provides an easy, qualitative answer to the important question of ultimate behaviour. For this example, it is a relatively simple matter to evaluate the coefficients ϕ_1, ϕ_2, θ_1 and θ_2, and thus obtain table III.

Boundary	Coefficients		Probability of escape from $[0,1]$ in finite time
x_1 is natural	$\phi_1 < \infty$	$\theta_1 = \infty$	positive
x_2 is natural	$\phi_2 < \infty$	$\theta_2 = \infty$	positive

Table III

The following table IV, lists the infinitesimal generators Ψ, corresponding to a number of well known processes. In all cases, Ψ has the form given in (16) with the specific function $a(x_o)$ and $b^2(x_o)$ as listed in the table below.

$b^2(x_o)$	$a(x_o)$	Name of Process
D	O	Wiener process
D	C	Brownian process. Linear Diffusion with drift.
D	$\frac{D(N-1)}{x_o}$	Rayleigh process. (Distance of N dimensional Brownian process from the origin)
D	$k x_o$	Orstein-Uhlenbeck process.
αx_o	βx_o	Branching process
$\frac{1}{4N} x_o(1-x_o)$	O	Genetic process [15]
$\alpha x_o^2(1-x_o^2)$	O	Genetic process [15]

Table IV

SECTION 2.5

FIRST PASSAGE TIME THEORY

In this section, we present a method for computing the moments of the first time that the Feller process X(t) has values inside some set A^c. In the context of reliability, the set A^c is a set of unacceptable

values, and the moments of first time of entry into A^c, would be the moments of the first time to failure of the degradation-prone system.

In preparation for what follows, we introduce the following theorem, due originally to Dynkin[2]. This theorem is one of the most important ones in the theory of continuous Markov processes.

Theorem 2.5.1

Let $X(t)$ be a Feller process with transition operator $\mathbf{P}_t$ and α-potential operator $\mathbf{U}_\alpha$. Let $u(x_o) = \mathbf{U}_\alpha f(x_o)$. Then the theorem states that

$$u(x_o) = \mathbb{E}_{x_o}\left(\int_0^\infty e^{-\alpha t} f(X(t))dt\right) \tag{8}$$

can also be written as

$$u(x_o) = \mathbb{E}_{x_o}\left(\int_0^\tau e^{-\alpha t} f(X(t))dt\right) + \mathbb{E}_{x_o}\left(e^{-\alpha\tau} u(X(\tau))\right) \tag{35}$$

where τ, is a Markov time.

Proof

We can expand (8) as

$$u(x_o) = \mathbb{E}_{x_o}\left(\int_0^\tau e^{-\alpha t} f(X(t))\, dt\right) + \mathbb{E}_{x_o}\left(\int_\tau^\infty e^{-\alpha t} f(X(t))\, dt\right) \tag{36}$$

to complete the proof of this theorem, we have to emphasize a sample path ω. Letting $t = t_1 + \tau$, we can write the integral on the right hand side of (36) as

$$\int_\tau^\infty e^{-\alpha t} f(X(t,\omega))\, dt = \int_0^\infty e^{-\alpha t_1} e^{-\alpha\tau} f(X(t_1+\tau,\omega))\, dt_1 \tag{37}$$

Since the dummy variable t_1 in (37) is unimportant, it can be rewritten as (where t_1 is replaced by t)

$$\int_\tau^\infty e^{-\alpha t} f(X(t,\omega))\, dt = \int_0^\infty e^{-\alpha t} e^{-\alpha\tau} f(X(t+\tau,\omega))\, dt \tag{37}$$

Now from section 2.1, we have that $X(t+\tau,\omega)$ can be written in terms of the shifted path ω_τ^+ as

$$X(t+\tau,\omega) = X(t,\omega_\tau^+)$$

and therefore (37) can be re-written as

$$\int_\tau^\infty e^{-at} f(X(t,\omega))dt = e^{-a\tau}\int_0^\infty e^{-at} f(X(t,\omega_\tau^+))\,dt \qquad (38)$$

then substituting (38) into the right hand side of (36), we find

$$\mathbb{E}_{x_0}\left(\int_\tau^\infty e^{-at} f(X(t))\,dt\right) = \int_0^\infty e^{-at}\,\mathbb{E}_{x_0}\left(e^{-a\tau} f(X(t,\omega_\tau^+))\right) dt \qquad (39)$$

substituting equation (1) of section (2.1) in the right hand side of (39), gives

$$\mathbb{E}_{x_0}\left(\int_\tau^\infty e^{-at} f(X(t))dt\right) = \int_0^\infty e^{-at}\,\mathbb{E}_{x_0}\left(e^{-a\tau}\mathbb{E}_{X(\tau)}(f(X(t)))\right) dt$$

rearranging, the above equation may be written as

$$\mathbb{E}_{x_0}\left(\int_0^\infty e^{-a\tau}\,\mathbb{E}_{X(\tau)}\left(e^{-at} f(X(t))\right) dt\right) \qquad (40)$$

but by (8) we can write

$$u(X(\tau)) = \mathbb{E}_{X(\tau)}\left(\int_0^\infty e^{-at} f(X(t))\,dt\right) \qquad (41)$$

substituting (41) into (40) we have the desired result, namely that

$$\mathbb{E}_{x_0}\left(\int_0^\infty e^{-at} f(X(t))\,dt\right) = \mathbb{E}_{x_0}\left(e^{-a\tau} u(X(\tau))\right)$$

which concludes the proof of the theorem.

QED.

The following theorem, which could be stated as a corallory of the theorem just proved, yields what is called Dynkin's formula.

This formula is the stochastic analog [6, chapter 1] of the basic property of the ordinaryRiemann integral. This theorem is needed in the derivation of the first passage time relationship that will be presented shortly

Theorem 2.5.2 Let τ be a Markov time satisfying $\mathbb{E}_{x_o}(\tau) < \infty$ *), and let u be contained in the domain of Ψ, then the theorem states

$$\mathbb{E}_{x_o}\left(\int_o^\tau \Psi u(X(t))\,dt\right) = \mathbb{E}_{x_o}(u(X(\tau))) - u(x_o) \tag{42}$$

Proof

$u(X(t)) = \mathbb{U}_\alpha f(X(t))$ implies, using (28), that $u(X(t))$ is the solution of $(\alpha - \Psi)\,u(X(t)) = f(X(t))$, and according to theorem 2.5.1, this solution is

$$u(x_o) = \mathbb{E}_{x_o}\left(\int_o^\tau e^{-\alpha t} f(X(t))dt\right) + \mathbb{E}_{x_o}(e^{-\alpha\tau}\,u(X(\tau))) \tag{43}$$

substituting $(\alpha - \Psi)\,u(X(t))$ for $f(X(t))$ in (43), gives

$$u(x_o) = \mathbb{E}_{x_o}\left(\int_o^\tau e^{-\alpha t}(\alpha - \Psi)\,u(X(t))dt\right) + \mathbb{E}_{x_o}(e^{-\alpha\tau}u(X(\tau))) \tag{44}$$

taking the limit for $\alpha \to o$ in (44), and rearranging we obtain (42).

QED.

The following theorem [17] has important applications in reliability theory and appears for the first time in this context.

Theorem 2.5.3 (Kac). Let $X(t)$ be a Feller process with infinitesimal generator Ψ and α-potential operator $\mathbb{U}_\alpha$ and let $k, f \in B(\Lambda)$. Then the theorem states that if

$$u = \mathbb{U}_\alpha f \tag{45}$$

satisfies

$$(\alpha - \Psi)\,u = f \tag{46}$$

---

*) That is, τ has finite mean value.

then

$$v = \mathbf{U}_a f e^{-\int k dt} \tag{47}$$

satisfies

$$(a + k - \Psi) v = f \tag{48}$$

Proof It is not difficult to show that

$$\begin{aligned} v &= \mathbf{U}_a f - \mathbf{U}_a kv \\ &= u - \mathbf{U}_a kv \end{aligned} \tag{49}$$

substituting (49) in (46) and remembering (see (20) that

$\Psi = a - \mathbf{U}_a^{-1}$, we obtain (48) QED.

Corallory

$$v \equiv v(a,x_o) = \mathbb{E}_{x_o} \left(\int_o^\infty e^{-at} f(X(t)) e^{-\int_o^t k(X(s))ds} dt \right) \tag{50}$$

is the solution of (48).

Proof

Expanding (47) (now $v \equiv v(a,x_o)$) yields

$$\begin{aligned} v(a,x_o) &= \mathbf{U}_a f(x_o) e^{-\int_o^t k(x)dt} \\ &= \int_o^\infty e^{-at} \int_\Lambda P(dx,t/x_o) f(x) e^{-\int_o^t k(x)dt} dt \end{aligned}$$

interchanging the order of integration gives (50) QED.

Example 4. [3, page 52] As an application of theorem 2.5.3, consider the case of degradation-prone parameter X, with initial value x_o, whose fluctuations are being characterized by the Feller process X(t) with infinitesimal mean and variance functions $a(x_o) = 0$ and $b^2(x_o)=1$. That is, by the unit variance Wiener process. Let U(t) be the portion of time out of a total time t, that parameter X has values greater than zero. We will now use theorem 2.5.3 to show that

$$P_{x_o}\{U(t) \le \tau\} = \int_o^{\tau} P_{x_o}\{U(t) \in d\tau\}$$

$$= {}^{2}/_{\pi} \text{ arc sin } \sqrt{{}^{\tau}/_{t}} \qquad , \ 0 \le \tau < t$$

To apply the theorem, we first define[18] a $k \in B(\Lambda)$ as

$$\gamma U(t) = \int_o^t k(X(t))\, dt$$

where

$$k(X(t)) = \gamma \qquad \text{if } X(t) > o$$

$$= 0 \qquad \text{if } X(t) \le o$$

then $\frac{\gamma U(t)}{t}$ is the portion of time out of the time t, that parameter X has values greater than zero. We next define

$$\Psi(\gamma, t, x_o) = \mathbb{E}_{x_o}(e^{-\gamma U(t)})$$

$$= \int_{-\infty}^{\infty} e^{-\gamma \xi} P_{x_o}\{U(t) \in d\xi\} \tag{51}$$

and take the Laplace transform (a-potential function) of $\Psi(\gamma,t,x_o)$, we find

$$v(a,x_o) = \mathbb{U}_a \Psi(\gamma,t,x_o) \tag{52}$$

$$= \mathbb{E}_{x_o}\left(\int_o^{\infty} e^{-at} e^{\gamma U(t)}\, dt\right)$$

$$= \mathbb{E}_{x_o}\left(\int_o^{\infty} e^{-at} e^{-\int_o^t k(X(\xi))\, d\xi}\, dt\right) \tag{53}$$

but equation (53) is nothing more than equation (50) with $f = 1$.

Now since $b^2(x_o) = 1$, and $a(x_o) = o$, the infinitesimal generator is $\mathbb{\Psi} = \frac{1}{2}\frac{d^2}{dx_o^2}$, and therefore $v(a, x_o)$ satisfies the ordinary differential equations

$$(a + \gamma - \frac{1}{2}\frac{d^2}{dx_o^2})\, v = 1 \quad , \quad \text{when parameter X has values } > o$$

$$(a - \frac{1}{2}\frac{d^2}{dx_o^2})\, v = 1 \quad , \quad \text{when parameter X has values } \le o$$

since parameter X, initially starts with a value $x_o = o$, The solution of the above equations is [3]

$$v(a,o) = \frac{1}{\sqrt{a\,(a+\gamma)}}$$

but

$$\int_o^\infty \frac{e^{-(a+\gamma)\xi}}{\sqrt{\pi\,\xi}}\, d\xi \int_o^\infty \frac{1}{\sqrt{\pi\,t}}\, e^{-at}\, dt = \frac{1}{\sqrt{a(a+\gamma)}}$$

and we have from (52)

$$\int_o^\infty e^{-at}\, \Psi(\gamma,t,o)\, dt = v(a,o) = \frac{1}{\sqrt{a(a+\gamma)}}$$

and therefore

$$\Psi(\gamma,t,o) = \int_o^t \frac{1}{\sqrt{\pi(t-\xi)}} \frac{1}{\sqrt{\pi\xi}}\, e^{-\gamma\xi}\, d\xi \qquad (55)$$

but comparing (51) with (55) we conclude

$$\int_o^\infty e^{-\gamma\,\xi}\, P_{x_o}\{U(t) \in d\,\xi\} = \int_o^t e^{-\gamma\xi}\, \frac{1}{\pi\sqrt{\xi(t-\xi)}}\, d\xi$$

therefore

$$P_{x_o}\{U(t) \in d\,\xi\} = \frac{1}{\pi\sqrt{\xi\,(t-\xi)}}$$

and

$$P_{x_o}\{U(t) \leq \tau\} = \int_o^\tau \frac{1}{\pi\sqrt{\xi(t-\xi)}}\, d\xi = \frac{2}{\pi}\, \arcsin\sqrt{\tau/t}\,.$$

Comment. Suppose the degradation-prone parameter of example (4), was described[7] by the process X(t) with infinitesimal mean and variance functions $b^2(x_o) = D$ and $a(x_o) = C$. Then the infinitesimal generator is

$$\Psi = \frac{D}{2} \frac{d^2}{dx_o^2} + C \frac{d}{dx_o}, \text{ and to compute } P_{x_o}\{U(t) \leq \tau\}$$

for this case, instead of (54), we would have to solve the following ordinary differential equations

$$(\alpha + \gamma - \frac{D}{2} \frac{d^2}{dx_o^2} + C\frac{d}{dx_o}) v = 1 \qquad \text{for } x_o > o$$

$$(\alpha - \frac{D}{2} \frac{d^2}{dx_o^2} + C\frac{d}{dx_o}) v = 1 \qquad \text{for } x_o \leq o$$

Thus although the procedure would be as indicated above, the analytical expressions would become decidedly unwieldy as more general $b(x_o)$ and $a(x_o)$ were used.

We now present a method for computing the moments of the first time that Feller process X(t) has values in some set $A^c \subset R$. In the reliability context, A is the set of acceptable values, and A^c, the set of unacceptable values. These moments, will be called the moments of the first time to failure of degradation-prone parameter X.

First Passage Time Relationships

Let us define a conditional joint probability distribution $q(x_o, dx, dt)$ induced on $\Lambda \otimes T$, by the mapping of $\omega \rightarrow (X(\tau), \tau)$, as

$$q(x_o, dx, dt) = P_{x_o}\{x \leq X(\tau) < x + dx \cap t \leq \tau < t+dt\} \qquad (56)$$

here τ is a Markov time. In the reliability context $q(x_o, dx, dt)$ is the conditional joint probability that degradation-prone parameter X, which intially has a value $x_o \in [x_1, x_2]$, will have a value $(x \le X(\tau) < x+dx) \subset A^c$ at time τ ("time of first entry into A^c")and that $(t \le \tau < t+dt)$.

The following lemma will play an important role later.

Lemma 2.5.1 Let $q(x_o, dx, dt)$ be defined as in (56). Then

$$\int_{[o,\infty)} \int_{\otimes \Lambda} e^{-\alpha t} q(x_o, dx, dt) \int_o^\infty e^{-\alpha s} \mathbb{P}_s f(x_o)\, ds$$

$$= \int_o^\infty e^{-\alpha t}\, dt \int_{[o,t)} \int_{\otimes \Lambda} \mathbb{P}_{t-s} f(x_o)\, q(x_o, dx, ds) \qquad (57)$$

Proof putting $\xi = t + s$, the left hand side of (57) may be written

$$\int_{[o,\infty)} \int_{\otimes \Lambda} e^{-\alpha t} q(x_o, dx, dt) \int_o^\infty e^{-\alpha s} \mathbb{P}_s f(x_o)\, ds = \int\int_{[o,\infty)\otimes\Lambda} q(x_o, dx, dt)$$

$$\int_t^\infty e^{-\alpha \xi} \mathbb{P}_{\xi - t} f(x_o) d\xi \qquad (58)$$

interchanging the order of integration in the right hand side of (58) and recalling that $\mathbb{P}_{\xi-t} f(x_o) = o$, for $t > \xi$, we can integrate for t, only in the interval (o, ξ), and get for the right hand side of (58) the expression

$$\int_o^\infty e^{-\alpha \xi}\, d\xi \int_{[o,\xi)\otimes\Lambda} \int \mathbb{P}_{\xi-t} f(x_o)\, q(x_o, dx, dt), \quad f \in B(\Lambda) \qquad (59)$$

Of course ξ and t are simply dummy variables of integration, and with no loss of generality, we can put $\xi - t$, and $t = s$ in (59) to get (57), which completes the proof of lemma 2.5.1.

QED.

In preparation for the proof of Theorem 2.5.4 we define the following functions

$$Q_2(A^c, t/x_o) = P_{x_o}\{ X(t) \in A^c \cap t \geq \tau \} \tag{60}$$

$$= P(A^c, t/x_o) - Q_1(A^c, t/x_o)$$

where

$$Q_1(A^c, t/x_o) = P_{x_o}\{X(t) \in A^c \cap \tau > t\}$$

$$= \mathbb{E}_{x_o}[f_{A^c}(X(t)) \cdot f_{[o,\tau]}(t)] \tag{61}$$

$$P(A^c, t/x_o) = P_{x_o}\{X(t) \in A^c\} \tag{62}$$

Theorem 2.5.4 (First Passage Theorem). Let X(t) be a Feller process with infinitesimal generator Ψ and transition operator P_t. Let X(t) characterize the time behaviour of degradation-prone parameter X. Let $Q_2(A^c, t/x_o)$ as defined in (60) be the probability distribution function of the Markov time (random variable) τ, ("first time parameter X has values in A^c") and of the random variable $X(\tau)$ in A^c. Then the theorem states that

$$Q_2(A^c, t/x_o) = \int_{[o,\infty)\otimes\Lambda} P(A^c, t-s/x)\, q(x_o, ds, dx) \tag{63}$$

Proof. We approach the proof of this theorem indirectly, using Theorem 2.5.1. We have from that theorem:

$$u(x_o) = \int_o^\infty e^{-\alpha t} P_t f(x_o)\, dt$$

$$= \mathbb{E}_{x_o}\left(\int_o^\tau e^{-\alpha t} f(X(t))dt\right) + \mathbb{E}_{x_o}(e^{-\alpha\tau} u(X(\tau))) \tag{35}$$

defining $f_{[o,\tau]}(t) = 1$, for $t \in [o,\tau]$, and zero for $t \notin [o,\tau]$.

(35) can be written as

$$\int_o^\infty e^{-at} \mathbb{P}_t f(x_o)\, dt = \int_o^\infty e^{-at} \mathbb{E}_{x_o} (f(X(t)): f(t))_{[o,\tau]}\, dt$$

$$+ \mathbb{E}_{x_o} (e^{-a\tau} u(X(\tau))) \qquad (64)$$

but the second expression on the right hand side of (64) is the expectation if the product of the functions of the random variable $X(\tau)$ and τ, whose density distribution function is $q(x_o, dx, dt)$. Therefore we can write

$$\mathbb{E}_{x_o} (e^{-a\tau} u X(\tau)) = \int_{[o,\infty)\otimes\Lambda} \int e^{-at} u(x) q(x_o, dx, dt) \qquad (65)$$

but by definition

$$u(x_o) = \int_o^\infty e^{-at} \mathbb{P}_t f(x_o)\, dt$$

Therefore, substituting this expression in (65) yields

$$\mathbb{E}_{x_o} (e^{-a\tau} u(X(\tau))) = \int_{[o,\infty)\otimes\Lambda} \int e^{-at} q(x_o, dx, dt) \int_o^\infty e^{-at} \mathbb{P}_t f(x_o)\, dt$$

but this is nothing more than equation (57) of lemma 2.5.1, therefore

$$\mathbb{E}_{x_o} (e^{-a\tau} u(X(\tau))) = \int_o^\infty e^{-at} dt \int_{[o,t)\otimes\Lambda} \int \mathbb{P}_{t-s} f(x_o) q(x_o, dx, ds) \qquad (66)$$

and substituting (66) in (64) we obtain

$$\int_o^\infty e^{-at} \mathbb{P}_t f(x_o) dt = \int_o^\infty e^{-at} \mathbb{E}_{x_o} (f(X(t)): f(t))_{[o,\tau]}\, dt$$

$$+ \int_o^\infty e^{-at} dt \int_{[o,t)\otimes\Lambda} \int \mathbb{P}_{t-s} f(x_o) q(x_o, dx, ds) \qquad (67)$$

pulling $f = f_{A^c}$, we have from (67)

$$\mathbb{P}_t f_{A^c}(x_o) = \mathbb{E}_{x_o}[f_{A^c}(X(t)) f_{[o,\tau]}(t)] + \int_{[o,t)} \int_{\Lambda} \mathbb{P}_{t-s} f_{A^c}(x) q(x_o, dx, ds) \quad (68)$$

but $\mathbb{P}_t f_{A^c}(x_o) = P(A^c, t/x_o)$, and substituting (60) and (61) into (68) yields (63). QED.

Corollary The following result, which is stated without proof in [7,18 page 66], follows from Theorem 2.5.4 and is called a renewal equation

$$\int P(x, t-s/a)\, q(x_o, ds, a) = P(x, t/x_o) \quad (69)$$

where $q(x_o, ds, a) \equiv P_{x_o}\{X(\tau) > a \cap t \leq \tau < t+dt\}$ is the probability density function of the "first time" parameter X has a value greater than a, given that it initially had a value x_o.

Proof In (63) put $x = a$ (fixed), for the state space Λ, take the interval $[o,a]$, and we obtain (69) QED.

Equation (69) is the key to the problem of computing the moments of the first time that the parameter X, has a value greater (less) that a, given that if initially had a value x_o less (greater) than a. It is immediately apparent that (69) is a convolution and we can obtain its solution by using the α-potential operator, that is, by using Laplace transform techniques. Accordingly, we can write

$$P(x, t/x_o) = \mathbb{P}_t f_{[o,x]}(x_o) = \int_o^{\infty} P(dx, t/x_o) f_{[o,x]}(x)$$

and, therefore, as shown previously

$$\mathbb{U}_\alpha f_{[o,x]}(x_o) = \int_o^{\infty} e^{-\alpha t} \mathbb{P}_t f_{[o,x]}(x_o)$$

$$= \int_o^{\infty} e^{-\alpha t} P(x, t/x_o)\, dt$$

is nothing more than the formal Laplace transform of $P(x,t/x_o)$. Letting the Laplace transform with respect to t, of $q(x_o,t,a)$ be $Z(x_o, a, a)$ we have immediately, that the Laplace transform of (69) is

$$\mathbf{U}_a \underset{[o,x]}{f(x_o)} = \mathbf{U}_a \underset{[o,x]}{f(a)}\ Z(x_o,a,a) \tag{70}$$

rearranging (70) we obtain the generating function for the moments of the first passage time (Markov time) τ. Namely

$$Z(x_o,a,a) = \frac{\mathbf{U}_a \underset{[o,x]}{f(x_o)}}{\mathbf{U}_a \underset{[o,x]}{f(a)}} \tag{71}$$

This is an interesting result because $\mathbf{U}_a$ is uniquely determined from the infinitesimal generator $\mathbf{\Psi}$, and $\mathbf{\Psi}$ in turn is uniquely determined once the particular Feller process for characterizing the degradation-prone parameter has been choosen. The following two cases are possible for a degradation-prone parameter X .

Case (1). Parameter X starts with value $x_o \leq a$

Case (2). Parameter X starts with value $x_o > a$

In order to obtain the specific form of the generating function (71) for these two cases we shall have to make use of the following theorem. Its proof can be found in a slightly less general form in Feller[1,Theorem 12].

Theorem 2.5.5 (Factorization Theorem). Let X(t) be a Feller process with transition probability function $\mathbf{P}_t \underset{[o,x]}{f(x_o)}$ that satisfies equation (18), namely

$$\frac{\partial}{\partial t} \mathbf{P}_t \underset{[o,x]}{f(x_o)} = \mathbf{\Psi}\, \mathbf{P}_t \underset{[0,x]}{f(x_o)} \tag{72}$$

Then, the Laplace transform $\mathbf{U}_a f(x_o)_{[o,x]}$ of $\mathbf{P}_t f(x_o)_{[o,x]}$ satisfies

$$(a - \Psi)\, \mathbf{U}_a f(x_o)_{[o,x]} = 0 \tag{73}$$

and this theorem states that the solution of (73) can always be factored as

$$\mathbf{U}_a f(x_o)_{[o,x]} = M^+(x,a)\, N^-(x_o,a), \qquad x_o \le x, \tag{74}$$

$$= M^-(x,a)\, N^+(x_o,a), \qquad x_o > x, \tag{75}$$

Proof Equation (73) is simply the Laplace transform with respect to t of (72). Equations (74) and (75) are given in Feller.[1, theorem 12]

For case (1), that is when parameter X initially has a value $x_o \le a$, on substituting (74) into (70), we obtain the generating function $Z_1(x_o, a, a)$ for computing the moments of the random variable τ_1 ("first time parameter X has a value greater than a, given that parameter X initially had the value $x_o \le a$"). Performing this substitution, we find that $Z_1(x_o, a, a)$ is

$$Z_1(x_o,a,a) = \frac{M^+(x,a)\, N^-(x_o,a)}{M^+(x,a) N^-(a,a)}$$

$$= \frac{N^-(x_o,a)}{N^-(a,a)} \qquad , \ldots x_o \le a, \ldots \tag{76}$$

Similarly for case (2), that is when parameter X initially has a value $x_o > a$, on substituting (75) into (70), we obtain the generating function $Z_2(x_o, a, a)$ for computing the moments of the random variable τ_2 -("first time that parameter X has a value less than a, given that

parameter X initially had the value $x_o > a$"). Performing the substituting we find that $Z_2(x_o, \alpha, a)$ is

$$Z_2(x_o, \alpha, a) = \frac{N^+(x_o, \alpha)}{N^+(a, \alpha)}, \quad x_o > a, \tag{77}$$

The following theorem summarizes the results of this section. This Theorem is given for the first time in the reliability context .

Theorem 2.5.6 Let X(t) be the Feller process with infinitesimal generator Ψ, and transition probability function $P_t f(x_o)$. Let this Feller process X(t) characterize the time behaviour of a degradation-prone parameter. Then the generating functions $Z_1(x_o, \alpha, a)$ and $Z_2(x_o, \alpha, a)$ for the moments of τ_1 and τ_2 corresponding to Case (1) and Case (2) are given by equations (76) and (77). These equations are completely determined once the solution of $(\alpha - \Psi) U_\alpha f(x_o) = 0$ is obtained and has been factored as shown in equations (74) and (75). In particular, for case (1), the n^{th} moment $\hat{\tau}_1(n, x_o, a)$ of the random variable τ_1, is

$$\hat{\tau}_1(n, x_o, a) = (-1)^n \frac{d^n}{d\alpha^n} [Z_1(x_o, \alpha, a)] \Big|_{\alpha=0}, \quad n = 1, \ldots \tag{78}$$

and, for case (2), the n^{th} moment $\hat{\tau}_2(n, x_o, a)$ of the random variable τ_2, is

$$\hat{\tau}_2(n, x_o, a) = (-1)^n \frac{d^n}{d\alpha^n} [Z_2(x_o, \alpha, a)] \Big|_{\alpha=0}, \quad n=1, \ldots \tag{79}$$

We are now in a position to define the mean time to first failure of degradation-prone parameter X.

Definition. Mean Time to First Failure $\hat{\tau}(1)$

The mean time $\hat{\tau}(1)$ to first failure of degradation-prone parameter X, is the mean of the first time that parameter X has a value outside of its region of acceptance $[x_1, x_2]$, given that parameter X initially had a value $x_o \in [x_1, x_2]$. Using (78) and (79) we have that the mean time is given by the following expression [7]

$$\hat{\tau}(1) = \hat{\tau}_1(1, x_o, x_2) \wedge \hat{\tau}_2(1, x_o, x_1) \tag{80}$$

The following example illustrates the procedure for applying the above theorem. The actual computational procedure is illustrated in [22].

Example 5.[7] Let us assume that a degradation-prone parameter X is characterized by the Brownian process X(t) with drift coefficient $a(x_o) = C$ and variance coefficient $b^2(x_o) = D$. We wish to obtain explicit expressions for the generating functions $Z_1(x_o, a, x_2)$ and $Z_2(x_o, a, x_1)$. Now, for this case, we have from (16) that the infinitesimal generator is

$$\Psi = C \frac{d}{dx_o} + \frac{D}{2} \frac{d^2}{dx_o^2}$$

and therefore from (18), $\mathbb{P}_t f_{[o,x]}(x_o) = P(x, t/x_o)$ satisfies

$$\frac{\partial P(x,t/x_o)}{\partial t} = C \frac{d}{\partial x_o} P(x,t/x_o) + \frac{D}{2} \frac{\partial^2 P(x, t/x_o)}{\partial x_o^2}$$

then as given in the Theorem 2.5.5, $\mathbb{U}_a f_{[o,x]}(t) \equiv p(x, a/x_o)$ satisfies the Laplace transform(with respect to t) of equation (20).

$$a\, p(x,a \mid x_o) = C \frac{d}{dx_o} p(x,a \mid x_o) + \frac{D}{2} \frac{d^2}{dx_o^2} p(x,a/x_o) \qquad (81)$$

solving (81) we obtain[7]

$$p(x, a/ x_o) = k \exp(-\lambda_1(a)(x-x_o)), \qquad x \geq x_o \qquad (82)$$

$$= k \exp(-\lambda_2(a)(x-x_o)), \qquad x < x_o \qquad (83)$$

where

$$\lambda_1(a) = \frac{\sqrt{C^2 + 2aD} - C}{D} \qquad (84)$$

$$\lambda_2(a) = \frac{-\sqrt{C^2 + 2aD} + C}{D} \qquad (85)$$

and k is a constant of integration, which need not be evaluated as it cancels out. Next using (74) and (75), we find

$$N^-(x_o,a) = k \exp(\lambda_1(a) x_o)$$
$$N^-(x_2,a) = k \exp(\lambda_2(a) x_2) \qquad (86)$$

and

$$N^+(x_o,a) = k \exp(\lambda_2(a) x_o)$$
$$N^+(x_1,a) = k \exp(\lambda_2(a) x_1) \qquad (87)$$

substituting (86) and (87) into (76) and (77) we obtain

$$Z_1(x_o, x_2, a) = \exp(\lambda_1(a)(x_o - x_2)) \qquad (88)$$

and

$$Z_2(x_o, x_1, a) = \exp(\lambda_2(a)(x_o - x_1)) \qquad (89)$$

It is now a straightforward matter to differentiate (88) and (89) and obtain the moments. In particular, the first moments are

for positive C

$$\hat{\tau}_1(1, x_o, x_2) = \frac{x_2 - x_o}{C}$$

$$\hat{\tau}_2(1, x_o, x_1) = \left(\frac{x_o - x_1}{C}\right) \exp\left(\frac{2C}{D}(x_o - x_1)\right)$$

Comment

There are many subtle points that have to be understood in attempting to apply the theory of Markov processes to problems of reliability prediction theory. The following example[19] illustrates one such case.

Example 6. Let degradation-prone parameter X be characterized by the Brownian motion process X(t) with drift coefficient C = o and diffusion coefficient D. We wish to compute the moments of the random variable τ_δ defined as

$$\tau_\delta = \inf \{t: X(t) - X(o) \geq x - x_o\}$$

that is, τ_δ is the random variable - "first time that the change in the value of parameter X is greater than $(x-x_o)$". It can be shown[19] that the probability density function of τ_δ is given by the following expression

$$\frac{x - x_o}{\sqrt{4\pi D t^3}} \exp\left(\frac{-(x-x_o)^2}{4Dt}\right)$$

and that the generating function for the moments of τ_δ is.

$$\mathbb{E}_{x_o}(e^{-a\tau\delta}) = \exp\left(\frac{-|x-x_o|\sqrt{2a}}{D}\right)$$

From the above expression we find that all the moments of τ_δ except the zero order one, are infinite.

SECTION 2.6

OCCUPATION TIMES - THE KAC POTENTIAL

In this section, a method is presented for computing themean time that the k dimensional Wiener - Brownian motion process $X(t) \equiv [X^1(t), \ldots . X^k(t)]$ has a value in some specified set $A^k \subset \Lambda^k$. It will be shown, that this mean time, is expressible as a potential function. This fact was first pointed out by Kac[17], and hence the potential is called the Kac potential.

In the reliability context, the set A^k would be the set of unacceptable values for the k parameter degradation-prone system. The mean time that the system spends in A^k, can then be interpreted as the mean time taken for the system to recover from its failure mode (many other interpretations are, of course, possible).

We begin by giving a brief review of the k dimensional Wiener Brownian motion process.

Definition 2.6.1 - k dimensional Wiener-Brownian motion process. The Feller process X(t) is called a k dimensional Wiener-Brownian motion process iff, its components

(a) $X^i(t)$, $i = 1, \ldots, k$ $\forall$ fixed t, are independent, identically distributed random variables with normal distribution function.

(b) $\mathbb{E}_{x_o} ((X(s) - X(t_o))^2) = 2Dt$

where $t = / s - t_o /$, and D is a constant called the diffusion coefficient.

The system considered, has k degradation-prone parameters. For the purposes of analytical simplicity, we assume*) that this k parameter system is characterizable by the k dimensional Wiener-Brownian motion process X(t), described above.

Accordingly, defining for each parameter a transition probability function

$$P^i(dx^i, t/ x_o^i) \equiv \text{Prob}\ \{x^i(t) \in dx^i / X^i(o) = x_o^i\}, \quad i = 1, \ldots, k.$$

as a consequence of definition 2.6.1, we have for the Wiener-Brownian motion process

$$P^i(dx^i,\ t/\ x_o^i) = \frac{1}{\sqrt{4\pi\ Dt}} \exp\left(-\frac{(x^i - x_o^i)^2}{4\ Dt}\right) dx^i$$

by adopting the convention $x_o = (x_o^1, \ldots, x_o^k)$ and $x \equiv (x^1, \ldots, x^k)$ with $x_o,\ x \in \Lambda^k$, we can write the k dimensional transition function for X(t) as the product of k one dimensional transition functions

$$P(dx, t\ /x_o) = P^1(dx^1, t\ /x_o^1) \ldots\ldots P^k(dx^k, t\ /x_o^k)$$

$$= \frac{1}{(4\pi\ Dt)^{k/2}} \exp\left(-\frac{/x-x_o/^2}{4\ Dt}\right) dx \qquad (90)$$

where $/\ x-x_o/^2 = \sum_{i=1}^{k} (x^i - x_o^i)^2$.

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*) For more general Feller processes and their potentials The reader is referred to Itô[3] or Bluementhal [8].

Equation (90) is the transition probability density function for the k dimensional Wiener-Brownian motion process. This expression is a function of $/x-x_o/$ and t, and this process is therefore said to be spatially and temporally homogeneous . That is, it has no preferred location in the state space and does not depend on the time at which x_o was measured, but only on the difference in time between x and x_o.

In general, for any $f \in B(\Lambda^k)$, the transition operator $\mathbb{P}_t$ as defined in (2), is

$$\mathbb{P}_t f(x_o) = \int_{\Lambda^k} P(dx, t/x_o) f(x_o) \tag{91}$$

substituting (90) into (91), we have for the Wiener-Brownian motion process

$$\mathbb{P}_t f(x_o) = \frac{1}{(4\pi Dt)^{k/2}} \int_{\Lambda^k} \exp\left(-\frac{/x/^2}{4Dt}\right) f(x+x_o)\, dx \tag{92}$$

To determine the generator Ψ for the Wiener Brownian motion process, we simply differentiate (92) and obtain

$$\frac{\partial \mathbb{P}_t f}{\partial t} = \frac{D}{2} \Delta \mathbb{P}_t f \tag{93}$$

where Δ is the Laplacian operator. Comparing (93) with (18), we have the well known result.

$$\Psi = \frac{D}{2} \Delta$$

$$= \frac{D}{2} \left(\sum_{i=1}^{k} \frac{\partial^2}{(\partial x_o^i)^2} \right)$$

The following theorem is important in the reliability modelling of degradation-prone systems. It is well known in the modern mathematical theory of Markov processes. However, it is given here for the first time in the reliability context. The theorem is stated only for the k dimensional Wiener-Brownian motion process; generalizations to arbitrary Feller processes can be found in Itô.[3,4]

Theorem 2.6.1 Let $X(t)$ be the k-dimensional Wiener process characterizing a k-parameter degradation-prone system. Let $f \in B(\Lambda^k)$ be bounded with compact support on $A^k \subset \Lambda^k$. Let $X(o) = x_o \in A^k$. Then

$$\text{a) } \mathbf{E}_{x_o}\left(\int_o^\infty f(X(t))\,dt\right) = \frac{\Gamma(k/2-1)}{2D\,(\pi)^{k/2}} \int_{\Lambda^k} \frac{f(x)}{|x-x_o|^{k-2}}\,dx$$

$$\text{b) } \mathbf{E}_{x_o}\left(\int_o^\infty f(X(t))\,dt\right) < \infty \quad , \quad k \geq 3.$$

Proof of a).

$$\mathbf{E}_{x_o}\left(\int_o^\infty f(X(t))\,dt\right) = \int_o^\infty \mathbf{E}_{x_o}\left(f(X(t))\,dt\right) \tag{95}$$

and using (90), we can write (95) as

$$= \int_o^\infty dt \int_{\Lambda^k} \frac{1}{(4\pi Dt)^{k/2}} \exp\left(-\frac{|x-x_o|^2}{4Dt}\right) f(x)\,dx$$

interchanging the order of integration this becomes

$$\int_{\Lambda^k} f(x)\,dx \int_o^\infty \frac{1}{(4\pi Dt)^{k/2}} \exp\left(-\frac{|x-x_o|^2}{4Dt}\right) dt \tag{96}$$

but the integral with respect to t in (96) is simply the α potential, $\alpha = 0$. Performing the integration in (96) and putting $B = \frac{(\pi)^{k/2} 2^k D}{\Gamma(k/2 - 1)}$, where Γ is the gamma function we obtain

$$E_{x_o} \left(\int_0^\infty f(X(t))dt \right) = \frac{1}{B} \int_{A^k} \frac{f(x)\, dx}{/x-x_o/^{k-2}} \qquad \text{QED.}$$

Proof of (b) Since the function f has compact support on A^k, (that is f vanishes outside of A^k.), we have

$$\left| \int_{A^k} \frac{f(x)\, dx}{/x-x_o/^{k-2}} \right| \leq \|f\| \int_{A^k} \frac{dx}{/x-x_o/^{k-2}}$$

but f is also bounded and $\int_{A^k} \frac{dx}{/x-x_o/^{k-2}} < \infty$, $k \geq 3$.

QED.

Application to Reliability

As stated in the beginning of this section, A^k could be interpreted as the unacceptable*) set of values for the degradation-prone k parameter system. Let the system be characterized by the k dimensional Wiener process X(t). We shall now compute T_{x_o}, which is defined as the mean time for which X(t) has values in A^k, given that it initially had a value $x_o \in A^k$. Accordingly define the single random variable

$$f_{A^k}[X(t)] \begin{cases} = 1 & X(t) \in A^k \\ = 0 & X(t) \notin A^k \end{cases}$$

---

*) A^k could also be interpreted as the set of acceptable values.

then

$$T_{x_o} \equiv \mathbb{E}_{x_o} \left(\int_o^\infty f_{A^k}(X(t))\, dt \right)$$

is obviously the mean time that $X(t)$ has values in A^k. Since f_{A^k} has compact support, is bounded, and belongs to $B(\Lambda^k)$, we can immediately apply theorem 2.6.1. part (a) and find

$$T_{x_o} = \frac{\Gamma(k/2-1)}{(\pi)^{k/2}\ 2D} \int_{\Lambda^k} \frac{f_{A^k}(x)\ dx}{/x-x_o/^{k-2}}$$

$$= \frac{\Gamma(k/2-1)}{(\pi)^{k/2}\ 2D} \int_{A^k} \frac{dx}{/x-x_o/^{k-2}} \qquad (97)$$

This expression is the Kac Potential which was mentioned before. Kac was the first to interpret it as the mean time the k dimension Brownian processes spends in a set $A^k \subset \Lambda^k$. From Part (b) of theorem 2.6.1 we know that this time is finite for $k \geq 3$.

The reason why the expression $\int_{A^k} \frac{dx}{/x-x_o/^{k-2}}$ is called a potential should also be obvious, since it is the solution of the Laplace equation from classical potential theory. Bearing in mind therefore the above probabilistic interpretation of the classical potential it will be possible in the future to make significant contributions to reliability theory by simply interpreting the results of classical potential theory in modern probabilistic terms.[24] Itô [3, page 111-] has given very general conditions under which this can be done.

Comment.

1) For $k = 1, 2$, we find $T_{x_o} = \infty$ which corresponds to recurrence[19]. For these cases we have to use logarithmic potential theory[3,4,24].

2) Putting k = 3 and using the fact that $\Gamma(1/2) = \pi$, we find the reason why Kac called the expression (97) a potential

$$T_{x_o} = \int_A \frac{c\ dx}{/x-x_o/}$$

where c is a constant. This is the classifical expression for the work done in moving a point charge a distance $/x-x_o/$ in an electric field.

SECTION 2.7

GENERALIZED LINEAR DIFFUSIONS

In this section, for the purposes of suggesting further research, we indicate a relation between the terminology of the general theory of Linear Diffusion, and the theory of reliability modelling of systems exhibiting degradation failure.

Definition 2.7.1 Linear Diffusion.

A one dimensional Diffusion process whose state space Λ is a linear connected set, is called a Linear Diffusion. Λ is therefore one of the following sets up to an isomorphism; $[0,1]$, $[0,1)$, $(0,1]$, $(0,1)$. Thus Λ is also admissable as the state space of a degradation-prone parameter.

Accordingly, let X(t) be the Linear Diffusion that characterizes the time behaviour of degradation-prone parameter X. Let τ_x be the random variable ("first time that parameter X has a value equal to x given that it initially had a value x_o"). That is τ_x is defined as

$$\tau_x = \inf \{t: X(t) = x\}$$

Let $P_{x_o}(\tau_x < \infty)$ be the probability that parameter X will assume a value equal to x in a finite time. Thus, following Itô [3 page 212-], the point x_o can be classified as belonging to one of the four sets. C_+, C_-, K_+, or K_-.

(i) $x_o \in C_+$ if $x > x_o$ and $P_{x_o}(\tau_x < \infty) > 0$

(ii) $x_o \in C_-$ if $x < x_o$ and $P_{x_o}(\tau_x < \infty) > 0$

(iii) $x_o \in K_+$ if $x > x_o$ and $P_{x_o}(\tau_x = \infty) = 1$

(iv) $x_o \in K_-$ if $x < x_o$ and $P_{x_o}(\tau_x = \infty) = 1$

The sets of points[3,4].

(a) $\{x_o\} \subset C_+ \cap C_-$
$\{x_o\} \subset K_+^c \cap K_-^c$
} are called regular points

(b) $\{x_o\} \subset C_+ - C_-$
$\{x_o\} \subset K_+ - K_-$
} are called pure right shunts

(c) $\{x_o\} \subset C_- - C_+$
$\{x_o\} \subset K_- - K_+$
} are called pure left shunts

(d) $\{x_o\} \subset C_-^c \cap C_+^c$
$\{x_o\} \subset K_- \cap K_+$
} are called traps.

Relation To Reliability Theory

The interpretation of sets (a), (b), (c) and (d) above in terms of reliability theory, is given below:

(a) If parameter X initially has a value x_o which is a regular point, then parameter X can assume positive or negative values with an equal probability

(b) If parameter X initially has a value x_o which is a pure right shunt, then parameter X can only assumes values greater than x_o with probability one.

(c) If parameter X initially has a value x_o which is a pure left shunt, then parameter X can only assume values less than x_o with probability one.

(d) If parameter X initially has a value x_o which is a trap, then parameter X maintains the value x_o with probability one. (This is obviously the deterministic case, and parameter X is not degradation-prone).

It should be apparent from the above discussion, that the concepts of traps and shunts are intimately related to the concepts of lateral conditions as introduced in section 2.4.

The concepts of shunts, trap points, etc. play an important role in the modern theory of Markov processes[2,3,4]. The object of this section was to indicate the role of these mathematical abstractions in the reliability modelling of degradation-prone systems.

The following theorem establishes the generator Ψ for Linear Diffusions and could be used as the starting point for a general study in reliability prediction theory.

Theorem 2.7.1 If point x_o is not a trap, then $E_{x_o}(\tau_x) < \infty$ for sufficiently small x in the neighborhood of x_o and the generator Ψ satisfies

$$\Psi u(x_o) = \lim_{x \to x_o} E_{x_o} \frac{(u(X(\tau_x))) - u(x_o)}{E_{x_o}(\tau_x)}$$

Proof

See Itô[3] page 216.

Summary and Suggestions For Further Research.

Several methods for the reliability modelling of degradation-prone systems characterizable by Feller processes have been presented. The methods which appear for the first time in the reliability context, rest on a firm mathematical basis, namely on the work of Itô, Feller, Doob, Dynkin, etc. . The contribution of this work, has been to show that the reliability modelling of degradation-prone systems, has a natural setting in the modern theory of Diffusion processes.

Experimental work must follow our mathematical modelling to verify the suitability of characterizing the degradation failures by Feller processes. More practical applications of the methods of sections 3.5 and 3.6, should prove extremely worthwhile.

Although Nelson[5], presents some results on the relative error that occurs when Wiener processes are used instead of Orstein - Uhlenbeck processes, the question of which generator Ψ to choose for a particular degradation mechanism remains an open problem. Positive results here would constitute the major contribution to further work in this area.

Finally, the use of the stochastic integral[10] approach to construct reliability models for degradation-prone systems has not been attempted. This approach appears to be very promising and should be initiated as soon as possible.

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EXPLANATION OF NOTATION

$\cap$	Set intersection, logical "and"
$\subset$	Set inclusision
$\in$	" is an element of"
$\equiv$	"defined as"
$\otimes$	Cartesian product
$\wedge$	"smaller of"
$\rightarrow$	"to"
$\Longleftrightarrow$	Equivalence
$\forall$	Quantifier meaning "for all"
Γ	The gamma function
Λ	State space (one dimensional) of process, (set of reals);
Λ^k	The k dimensional state space
ω	Sample path, (mapping of $T \rightarrow \Lambda$)
Ω	Sample space, (set of sample paths)
T	Time space, (set of reals $[o, \infty)$) .
$\mathbb{B}(\Omega)$	Borel algebra of sets of Ω.
$B(\Lambda)$	Set of bounded measurable functions defined on Λ
$X:$	Mapping
$X^{-1}:$	Inverse mapping

EXPLANATION OF NOTATION Cont'd....

$X(\cdot,\cdot)$	Stochastic process, (mapping of $X: \Omega \otimes T \rightarrow \Lambda$)
$X(\omega,\cdot)$	Sample function, (mapping $X : T \rightarrow \Lambda$)
$X(\cdot, t)$	Random variable, (mapping $X: \Omega \rightarrow \Lambda$)
X	Degradation-prone system parameter which is characterized by the stochastic process $X(t)$ *)
A	Set of acceptable values of degradation-prone parameter X . $A \subset \Lambda$
A^c	Set of unacceptable values of degradation-prone parameter X
f	A function $f \in B(\Lambda)$
f_A	Indicator function of the set A.
$f_{[dx]}$	Indicator function for the increment dx
$f_{[o,x]}$	Indicator function for the interval $[o,x]$.
τ	Markov time, (random variable; mapping $\tau : \Omega \rightarrow T$)
τ_A	Random variable - "first time parameter X has a value in set A".
T_{x_o}	Mean time for which X(t) has values in A^k given that it initially had a value $x_o \in A^k$
τ_1	Random variable, - "first time parameter X has a value greater than a, given that parameter X initially had the value $x_o \leq a$"

*) We also use the convention of suppressing the Ω space if there is no need to emphasize it, and write X(t) for the stochastic process when t is variable, and X(t) for the random variable when t is fixed.

EXPLANATION OF NOTATION Cont'd....

τ_2 — Random variable - "first time parameter X has a value less than a, given that parameter X initially had the value $x_o > a$"

τ_x — Random variable - "first time that parameter X has a value equal to x given that it initially had a value x_o.

P_{x_o} — The probability law for the process X(t)

$P(A,t/x_o)$ — Transition probability function. This is the probability that parameter X has a value in set A at time t, given that it initially had a value x_o. That is $P(A,t/x_o) = P_{x_o}\{X(t) \in A\}$

$\mathbb{P}_t$ — The Feller transition operator $\mathbb{P}_t = \int_\Lambda P(dx,t/x_o)\cdot$, with domain $B(\Lambda)$ and range $B(\Lambda)$.

$\mathbb{E}_{x_o}$ — The expectation operator.

$U(A,\alpha/x_o)$ — The Laplace transform with respect to t, of $P(A,t/x_o)$. That is $U(A,\alpha/x_o) = \int_o^\infty e^{-\alpha t} P(A,t/x_o)dt$

$\mathbb{U}_\alpha$ — The potential operator: $\mathbb{U}\cdot_\alpha = \int_o^\infty e^{-\alpha t}\,\mathbb{P}_t\cdot\,dt$. That is $\mathbb{U}_\alpha\cdot = \int_\Lambda U(dx, \alpha/x_o)\cdot$.

Ψ — The infinitesimal generator of the process X(t),

$$\Psi = \frac{b^2(x_o)}{2}\frac{d^2}{dx_o^2} + a(x_o)\frac{d}{dx_o} : (\mathbb{P}_t = e^{\Psi t})$$

$a(x_o)$ — The infinitesimal mean function; $a(x_o)\,dt = \mathbb{E}_{x_o}(dX(t))$

EXPLANATION OF NOTATION Cont'd....

$b^2(x_o)$	The infinitesimal variance function: $b^2(x_o)dt=E_{x_o}((dX(t))^2)$
$\mathbb{D}_m \mathbb{D}_s$	Feller's generalized Laplacian. $\mathbb{D}_m \mathbb{D}_s \Longleftrightarrow \Psi$
$R_{x_o}(t)$	Conditional reliability function. $R_{x_o}(t) = \mathbb{P}_t f_A(x_o)$
$R(t)$	Unconditional reliability function
$q(x_o, dx, dt)$	Conditional joint probability distribution function of the random variables $X(\tau)$ and τ, where τ is a Markov time.
$Z_1(x_o, a, a)$	Generating function for obtaining the moments of the random variable of τ_1
$Z_2(x_o, a, a)$	Generating function for obtaining the moments of the random variable of τ_2
$\hat{\tau}_1(n, x_o, a)$	The n^{th} moment, $n = 1,2,\ldots$, of the random variable τ_1
$\hat{\tau}_2(n, x_o, a)$	The n^{th} moment, $n = 1,2,\ldots$, of the random variable τ_2
$\hat{\tau}(1)$	Mean time to first failure of degradation-parameter X.

Curriculum Vitae

Name: John Bruce Arthur de Mercado

Place of Birth: Kingston, Jamaica, British West Indies

Date of Birth: August 30, 1941

Education

Primary: Campion College - Jamaica

Secondary: St. Georges College - Jamaica

University: University of Ottawa

B.A.Sc. (Electrical Engineering)

M.Sc. (First Class Honours - Electrical Engineering)