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Wittgenstein's Critique of the Logical Definition of Number in the *Tractatus Logico-Philosophicus*

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Wittgenstein's Critique of the Logicist Definition of Number in the *Tractatus Logico–Philosophicus*

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Abstract

The goal of this thesis is to present a coherent and exhaustive picture of Wittgenstein's critique of the logicist definition of number as presented by his philosophy of mathematics in the *Tractatus Logico-Philosophicus*. It does so by considering several critiques that Wittgenstein makes of the logicist's definition of number. Furthermore, this thesis compares Wittgenstein's operational definition of number—the central feature of Wittgenstein's theory—to Church's lambda calculus to demonstrate how Wittgenstein has a similar understanding of arithmetic.

1 Introduction

The goal of this thesis is to present a coherent and exhaustive picture of Wittgenstein's critique of the logicist definition of number as presented by his philosophy of mathematics in the *Tractatus Logico-Philosophicus*. Though there is a very good exegesis of the mathematical parts of the *Tractatus* (FRASCOLLA 1994) there is as yet no exhaustive study of Wittgenstein's philosophy of mathematics in the *Tractatus*. Thus, I shall attempt to provide not only an accurate analysis and commentary, but in doing so place Wittgenstein's philosophy of mathematics within the context of the *Tractatus* and point to both his position and his motivation. Though discussing how Wittgenstein's philosophy of mathematics shaped the *Tractatus* is beyond the scope of this thesis, I hope that my analysis will make evident that the importance of these passages has been often underestimated.

There are six major criticisms that Wittgenstein makes about the logicist programme. I will discuss these criticisms to explicate both Wittgenstein's objections to the Frege–Russell definition of mathematics and to describe Wittgenstein's more positive programme of defining natural numbers and arithmetic through what can be called his operational theory. In the second chapter I focus on the criticisms that undermine the logicists' foundationalism. In the last chapter I discuss the development of Wittgenstein's operational system and the criticisms which this brings to the fore.

The method then will be to analyze the *Tractatus* in order to highlight these two crucial aspects of this work. Thus the structure of the thesis will be such that it reflects the nature of both the motivation of Wittgenstein's critique and the details of his own system. Though the method will primarily be exegetical, I shall of course refer to the secondary literature on the topic, and discuss the various interpretations that are prevalent in the literature. This will highlight the gaps in the literature that my account of Wittgenstein's project identifies. Furthermore this thesis will compare Wittgenstein's operational definition of number to Church's lambda calculus to demonstrate how Wittgenstein has a similar understanding of arithmetic.

As I have stated above, what Wittgenstein has to say about mathematics in the *Tractatus* divides into two parts: his rejection of the logicist's programme of Russell and Frege, which I will call his 'negative' programme; and his description of the natural numbers and arithmetic using the theory of operations, which I will call his 'positive' programme.

The major criticisms that Wittgenstein makes about the logicist's programme can be divided into two groups: those that criticise what is wrong with a logicist definition; and those that are brought forth in the sketching out of a new interpretation. The first three are Wittgenstein's critique of type theory as transgressing the saying–showing distinction (3.331–3.333; 5.252; 6.123 [2]), Wittgenstein's rejection of the axioms of infinity (5.535 [2,3]) and reducibility (6.1232; 6.1233), and his critique of the logicist definition of a formal series for containing a vicious circle (4.1273).

While the first three are central to his rejection of logicism, the second three are revealed by his solution. As such the second three critiques are less the motivation for his rejection of logicism than its result. They are his critique of classes as being unnecessary for the foundation of mathematics (6.031), and his critique of the view that either mathematical propositions (6.2) or logical connectives (5.4–5.44) refer to specific entities. In his positive programme Wittgenstein demonstrates how number, and indeed arithmetic, need not be seen as referring to abstract objects, but should be defined in terms of the application of an operation to a base (6.002–6.02, 6.234–6.241). Wittgenstein understands mathematics to be not a description of independent mathematical objects, but rather a method of calculation (6.2331, 6.2341).

To help situate Wittgenstein's rejection of logicism, a brief summary of the various points which he criticised will preface my discussion of the *Tractatus*. In section 2.1 I will describe the development of the logicist's programme. The logicist's programme defined number in terms of a one-to-one correspondence of classes. Cardinal numbers were defined in terms of an equivalence relation between classes, and natural numbers were defined in terms of them. But Russell's paradox undermined Frege's attempt at such a system in the *Grundgesetze der Arithmetik* (FREGE 1893 [1964], pp.127-43). The paradox can be stated simply: Consider the class of all classes that are not members of themselves. If this class is a member of itself, then by definition it must not be a member of itself. If it is not a member of itself then by definition it is a member of itself. I will describe how the paradox applied to Frege's system in section 2.1.

In order to deal with this paradox, Russell introduced the theory of types (RUSSELL 1903, pp. 523-8; RUSSELL 1908; and WHITEHEAD and RUSSELL 1927 vol.1, pp. 37-66), which ultimately forced Russell to develop the axioms of reducibility and infinity. However Russell's use of type theory and the axioms of reducibility and infinity, to patch the holes in the logicist system, presented to Wittgenstein prime examples of why the logicist foundations of mathematics were problematic. Though he was not the only one to criticise Russell's type theory,[1] his criticism differed in that he was not seeking a refinement, but rather an entirely different approach to understanding mathematics. In sections 2.2 and 2.3 I will discuss

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[1] Ramsey, Hilbert and Ackermann removed orders from Russell's ramified type theory creating the simple theory of types, and thus they were refining rather than rejecting Russell's theory. A thorough account of this may be found in KAMAREDDINE et al. (2002).

Wittgenstein's critiques of type theory and these axioms, respectively. Wittgenstein's critique is tied to his terminology of the saying showing distinction, and I shall attempt to show how this distinction can be seen as making a division between the terms of intensional and extensional conceptions of logic.

In section 2.1 I shall discuss Russell's development of the theory of types. Type theory arranges all sentences into a hierarchy: the first level consisting of sentences about individuals, the next about such sentences, *et cetera.* It is thus possible to refer to all objects for a given predicate only if they are of the same type. In section 2.1.1 I will discuss Russell's early formulation of type theory; his 'substitutional' and 'zig-zag' theories; and finally his ramified type theory. The development of these various theories show that Russell's final position in the *Principia Mathematica* was a stop gap of a problem, not a certainty, a fact Wittgenstein would have, as his student, known. The tenuousness of the theory of types perhaps drew his criticism and led him to reject Russell's position as being based upon so shaky a ground.

Wittgenstein rejected the theory of types as transgressing the saying-showing distinction. This distinction between what can be expressed in language and the features of language that can only be shown is primary to the *Tractatus*. In section 2.2 I shall discuss how Wittgenstein saw the theory of types as transgressing this distinction. To do so first I will first in 2.2.1 discuss how the saying-showing distinction is bound up with issue of intensional and extensional logical concepts, and then attempt to make clear how these are similar distinctions. In 2.2.2 I will describe Wittgenstein's argument on how the theory of types transgresses this distinction, and so speaks extensionally about intensional concepts.

In section 2.3 I will discuss how the Russell's modification of the logicist approach created a problem in the definition of number and how this led to the introduction of the axioms of reducibility and infinity. The problem in the definition of number was that no longer could one have classes without restriction, but rather classes of various types and levels. Thus no longer do we have only one type of number defined by class, but numbers at each level of class we define. To deal with this further complication Russell introduced what is called the axiom of reducibility, which roughly states that anything definable at a higher level has an equivalent at the most basic level. (*cf.* WHITEHEAD and RUSSELL 1927: Part 1 Chapter II) First I will describe the axiom of reducibility and then discuss Wittgenstein's critique of the axiom of reducibility, which, like his rejection of type theory, is predicated on his view that the axiom speaks in an extensional manner about intensional, or in his terms logical, properties.

In addition to the axiom of reducibility, Russell and Whitehead introduced the axiom of infinity, which stated that an infinite number of classes exist. In the logicist account a number '*n*' is defined in terms of the class of classes with *n* objects in them. There must be therefore an infinite number of objects to define the natural numbers in such a manner. For if the universe had a maximum of *n* objects then the class of all classes with *n*+1 objects would be the empty set. I will describe the axiom of infinity and discuss Wittgenstein's critique of the axiom of infinity. As with the axiom of reducibility, the axiom of infinity speaks about logic in a manner which Wittgenstein saw as a transgression of the showing saying distinction.

In section 2.4 of my thesis I will discuss the third criticism in what I have called the negative programme of the *Tractatus*, this is Wittgenstein's rejection of the Russell–Frege definition of a formal series is a vicious circle (4.1273). First I shall attempt to describe the logicist description of a formal series and then I shall present a short account of Wittgenstein's critique of the logicist definition.

Wittgenstein's critique of Frege's and Russell's definition of a formal series, like his other critiques in what I call the negative programme, is dependent on a certain view about the nature of logic and the nature of formal languages. Wittgenstein never doubts that what he calls proper relations exist, but he makes clear a distinction between existential and logical language, and so between formal and proper relations and properties. Objects can be said to exist, but logic is shown. In Wittgenstein's view intensional concepts are shown in language, in the construction of complex objects. A formal series is a construction and so is constructed through a process. In his critique of Frege and Russell's series the point Wittgenstein is trying to make is that the formal concept must be understood to be different than a proper concept. Not to do so will lead one into circularity. The account of successorship rejected by Wittgenstein depends upon defining properties in terms of themselves. This critique, like Wittgenstein's critique of type theory and the axioms spawned by type theory, attempts to describe a problem with extensional descriptions of logical concepts better described as processes, or acts of construction.

The 'general term of a formal series' is according to Wittgenstein a 'variable,' as the concept 'term of a formal series' is a '*formal concept*.' A formal concept is given with the concept itself, it is 'shown not said' (4.12721). To Frege and Russell number was, in Wittgenstein's terms, a proper concept. The concept number was applied by Frege and Russell to certain classes of classes of type "0" and "the successor of 0." "The successor of" relation was defined in terms of the relation "the immediate successor of." This account of successorship in terms of hereditary properties requires one to define the property of having a certain ancestral relationship as a hereditary relationship. This is what opened the Frege–Russell description of a formal series up to claims of circularity (ANSCOMBE 1971, pp.126-8).

The cornerstone of Wittgenstein's positive programme is his theory of operations. In part three of this thesis I shall focus on giving an account of the theory of operations, especially in how it pertains to Wittgenstein definition of number and his critiques of logicism which such a construction entails. In section 3.1 I will present Wittgenstein's theory of operations showing how the operation is derived from Wittgenstein's understanding of what he calls formal relations. Wittgenstein first presents the notion of an operation as that which generates the internal relation of a formal series (5.21–5.2523). Wittgenstein uses truth operations to define connectives (5.3, 5.501, 5.51, 6) which define first order logic.

In section 3.2 I will discuss Fogelin's critique and the debate of the so–called central error in the *Tractatus*. Fogelin argues that the *Tractatus* description of the construction of complex propositions cannot work. I will review the debate and present the argument that the *Tractatus* view of the construction of logical propositions works well under a finitist interpretation. This supports an even bolder claim as to the constructive nature of logic and mathematics in the *Tractatus*, whether or not this was the intent of the author.

He uses the notion of operation to define natural numbers (6.02) and '+' and '×' (6.241) which make up arithmetic. In Wittgenstein's system numbers are not objects but "exponents of an operation" (6.022); *i.e.* references to applications of an operation to a base. Thus his definition is very much the same as that of Church numerals (MARION 1998, p.11 and p.37 n18, but *cf.* also MARION 2001, HANCOCK and MARTIN–LÖF 1975, and FRASCOLLA 1994 and 1997). In section 3.3 I will present an account of Frascolla's interpretation of the theory of operations and describe his modifications to Wittgenstein's notation. In section 3.4 I will present an argument that this view is equivalent to Church Numerals, and that the operation serves the same role in Wittgenstein's theory as the lambda calculus does in Church's.

In sections 3.5, 3.6 and 3.7 I will present the critiques of logicism in what I call the positive programme. These are, first, the superfluity of the theory of classes to the foundation of mathematics, second, the view that mathematical propositions refer to specific entities and, thirdly, the view that connectives refer to specific entities. These three critiques are all derived out of what I call the positive programme of the *Tractatus*. That is the theory of operations and the view of logic which motivates it.

In section 3.5 I will present Wittgenstein's fourth critique of logicism, that his method of defining number as a construction using the theory of operations brings him to reject the use of the theory of classes in the foundations of mathematics. Wittgenstein rejects classes as 'superfluous' to the foundation of mathematics (6.031). He states that this is related to "the fact that the generality which we need in mathematics is not the *accidental* one." By this he means that it is logical and not empirical. That he provides an account of mathematics without using classes may be taken as evidence that this is true. This provides a further argument for Wittgenstein's distinction between intensional and extensional concepts and their place in logic.

For Wittgenstein mathematics is not a description of objects, but rather a way of describing "internal," or "formal," relations through equations. In section 3.6 I show how Wittgenstein rejects the view that mathematical propositions (6.2) refer to specific entities. In 6.2 Wittgenstein asserts that mathematics is equational, and that the propositions of mathematics are 'pseudo-propositions.' Thus they express nothing about any particular entities in the world; they show the 'logic of the world' (6.22; *cf.* FRASCOLLA 1997, pp.353-362).

In section 3.7 I present Wittgenstein's rejection of the view that logical connectives (5.4–5.44) refer to specific entities. Rather he shows that they can be described both by logical operations or truth functions. They are not "relations in the sense of right and left" Wittgenstein writes (5.42), but internal relations. They are logical and part of that which may only be shown in Wittgenstein's system.

Wittgenstein's rejection of the theory of types as part of logic, the axioms of reducibility and infinity, the logicist account of the formal series, the use of classes in the foundation of mathematics, and the views that mathematical propositions and logical connectives refer to specific entities are all related to his rejection of an extensionalist and realist view of logic and mathematics. The rejection of the theory of types as violating the saying–showing distinction is primarily an objection to treating the rules of logic as if they stated facts. The saying–showing distinction is primarily a distinction between the way language works, which is shown, and what it is expresses, which is said. There could be almost no better description of intensional and extensional aspects of logic. Similarly the axioms of reducibility and infinity are rejected as being empirical statements. On an intensional reading logic is something one does, not a collection of objects the features of which can be described. Infinity and reducibility are thus properties that apply to objects and not to formal characteristics of objects. They could apply to objects but not to the rules for the combination of the objects, and it is the confusion of the object with these formal characteristics that is the root of this critique. The logicist account of the formal series is critiqued as being circular; but if one looks at this critique carefully, the reasoning behind it is that a distinction between syntactical rules and their objects is not properly drawn and therefore there is a definition of a relation in terms of itself. The treating of a syntactical—formal in Wittgenstein's terms—relation as a proper relation leads one into such errors. Wittgenstein's assertion of the superfluity of the theory of classes to the foundation of mathematics derives from his intensional interpretation of mathematical logic. If mathematics is to be understood algorithmically then classes need not be used to describe its basic features. Furthermore the views that neither mathematical propositions nor connectives refer to specific entities is also connected with this intensional, algorithmic understanding of mathematics. To see equations as processes rather than to see them as combinations of objects is central to Wittgenstein's theory.

Thus all of Wittgenstein's critiques of the logicist's programme are in line with a attempt to describe logic and mathematics in intensional as opposed to extensional terminology. The underlying structural rules of logic are not to be treated as objects, according to Wittgenstein; objects are the subjects of propositions. This account is very much along the lines of accounts of constructive semantics, except that Wittgenstein is much stricter in his admonitions about what he calls the saying–showing distinction. It is this extremism which makes it so hard to see the accomplishments of his work. As Frascolla points out, we have to break Wittgenstein's own strict rules to explicate his operation language theory (*cf.* FRASCOLLA 1994 pp.8 *ff.*). What is amazing is the success that Wittgenstein actually had. Later accounts of constructive logics realised that a distinction between rules and objects had to be made. These constructive descriptions of logic treat all parts equally, by describing them in combinatory rules. They provide as the semantic value—

the meaning of a sentence—the proof of the sentence; that is the construction, by logical rules, of that sentence, from other sentences. Wittgenstein's operational theory language provides a remarkably similar constructive apparatus.

Wittgenstein's attempt at providing the template of the general form of the proposition and of number show that he was not being purposefully obtuse, but that he lacked an approach to notation that enabled him to explain syntax well enough for it to be easily seen what he was trying to demonstrate. Had he been a more patient man, his place in the history of logic may have been as important as his place in philosophy.

1.1 Intensionality and Extensionality

Von Wright begins his paper "Modal Logic in the *Tractatus*" with a discussion of the terms "intensional(ist)" and "extensional(ist)." He writes that the original formulation of the "thesis of extensionality" is that "in any proposition about a given concept, this concept may be taken extensionally, that is, represented by its extension (class or relation)." And though he admits that truth-functions seem a "special case" of this principle, he notes that it has often been extended to the thesis that, "every meaningful proposition (sentence) can be construed as a truth-function of some elementary propositions (sentences) which are truth-functions of themselves." But though he admits that Wittgenstein's position in the *Tractatus* is extensionalist in regard to what can be said, he states that with regard to what cannot be spoken of, but only shown, the *Tractatus* is intensional (VON WRIGHT 1988, pp.185-6).

The 'principle of extensionality' is said, by several commentators, to be found in the *Tractatus*: in thesis 5 and in variants at 4.4 and 5.54. The idea that Wittgenstein defended the extensionality thesis can be said to come originally from Carnap. In *The Logical Syntax of*

Language, (CARNAP 1937 [1967] §67) refers to thesis 4.4, 5 and 5.54 as examples of the thesis of extensionality. In 4.4 Wittgenstein writes:

> 4.4 A proposition is an expression of agreement and disagreement with truth-possibilities of elementary propositions.

Since a proposition is a construction from elementary propositions the above statement must be true, whether or not it is an expression of the principle of extensionality. It certainly does express the fact that Wittgenstein though that all propositions are either true or false or that their extensions were truth or falsehood. For these reasons thesis 5 seems a better candidate for the thesis of extensionality:

> 5 A proposition is a truth-function of elementary propositions. (An elementary proposition is a truth-function of itself.)

Carnap in *The Logical Syntax of Language*, (CARNAP 1937 [1967] §67) refers to this thesis. According to Carnap this means that every sentence is "extensional in relation to partial sentence" he also notes that Russell "adopted the same view." Russell and Whitehead writing of mathematics state that it is "essentially extensional rather than intensional." (WHITEHEAD and RUSSELL 1927 Appendix C p. 659) thus their view pertains only to mathematical statements. The view held by Russell and Whitehead will be taken up in the discussion of Ramsey's interpretation shortly.

Carnap wrote that, "Wittgenstein, Russell and myself," argue not only the "possibility of an extensional language," rather than its necessity. While it is true that Wittgenstein never disputed the possibility of an extensional description of mathematics, but found that it was superfluous. As to the theory of propositions, Wittgenstein does have a truth functional understanding, but it has been argued by several (cf. VON WRIGHT and MAURY) that Wittgenstein's principle of bivalence does not entail the law of the excluded middle. That is that every proposition is true or false, which is a prerequisite for a totally extensional view of propositions. Maury writes:

> It does not *follow* from the fact that the significant proposition is such that it can be true and that it can be false that it *is* either true or false. This follows from the fact that it is a *proposition*. (The existence of logical propositions shows that it does not follow from the fact that a proposition is such that it is either true or false that it can be true and it can be false) (MAURY 1977, p. 27).

This modal, and thus intensional, understanding of the proposition is present in the *Tractatus* and it is the distinction between, what von Wright calls, the principle of bivalence and the law of the excluded middle that leads credence to the argument that the *Tractatus* version of the proposition is not simply extensional.

This idea of Carnap's that Wittgenstein's view of the proposition was entirely and essentially extensional was relayed in *Tractatus* scholarship mainly by Favrholdt and Black's commentaries. Favrholdt writes:

> The fundamental thesis of the *Tractatus* is the thesis of extensionality, i.e. the thesis that all sentences are truth functions of elementary propositions and that these in turn are truth functions of themselves (FAVRHOLDT 1967 p. 11).

This view is echoed by Black:

> With ... the 'thesis of extensionality', that propositions enter into the make-up of complex propositions only as arguments to truth functions (also stated at 5, 5.4) we arrive at a point of decisive importance for Wittgenstein's philosophy of language. By comparing 4.2, 4.3 and 4.4, we see that Wittgenstein takes the sense of a proposition to be 'agreement and disagreement' with the atomic possibilities (atomic situations) expressed in elementary propositions. Thus, every proposition signifies *via* the elementary propositions that can be uncovered by analysis. A truth–table is, therefore, a perspicuous rendering of the sense of a complex proposition (*cf.* 4.4) (BLACK 1964 pp. 219).

What these views ignore is the importance of the construction of the propositions through the theory of operations. It is not that the propositions aren't truth functional, but it is the manner in which they are constructed which determines their sense. This is why in addition to the truth–table Wittgenstein gives us the operation and the general form of the proposition. Black alludes to this when he continues, “The great stumbling-block for this conception is the existence of general propositions” (BLACK 1964 pp. 219). He takes this up later where he writes that the, “relation of general propositions to elementary propositions is hard to grasp” (BLACK 1964 pp. 281). Wittgenstein construes universal and existential sentences as concatenations and disjunctions of elementary propositions. In this way Black writes that, “Wittgenstein links the essence of generality with the notion of a variable, that than with that of a quantifier” (BLACK 1964 pp. 282). For Wittgenstein general sentences have to be constructed by the means of the operation, and thus such logical notation only has meaning because a construction is provided, through the concatenation of operations. The issue of general propositions will be taken up in detail in section 3.2 of my thesis where I discuss Fogelin’s criticism of what he calls the central error in the *Tractatus.*

One could raise the objection, that what we are talking about, i.e. the intensionality of mathematics or logic, is not in the *Tractatus* or even in the mind of people at the time, but rather the application of anachronistic concepts. However there is much discussion of the extensionality thesis at the time. Look at thesis 5.542 where Wittgenstein writes:

> 5.542 It is clear, however, that 'A believes that p', 'A has the thought p', and 'A says p' are of the form '“p” says p': and this does not involve a correlation of a fact with an object, but rather the correlation of facts by means of the correlation of their objects.

This was taken by many at the time as an attempt, albeit an obscure one, to extend the extensionalist thesis. We have already mentioned Carnap who in sections §43 to §45 of the *Aufbau* (CARNAP 1928 [1966] §43-45), mentions section 5.542 and Appendix C of *Principia Mathematica* where Whitehead and Russell present a related attempt at extending the extensionality thesis. Furthermore Ramsey’s 1923 critical notice of the *Tractatus* in *Mind* (RAMSEY 1923 [in COPI 1973] p.13 and p.16) mentions this as such an attempt.

There is no need to go into the details of any of these, but the fact that Russell, Ramsey and Carnap at the time all discussed the issue immediately after the publication of the *Tractatus*, shows us that we are on track.

In his 1926 paper 'The Foundations of Mathematics', Ramsey tried to argue explicitly for the thesis that mathematics is extensional. He claims that mathematical theorems are tautologies but also by modifying the basic notion of 'predicative function.' Wittgenstein famously objected to this.

Ramsey had tried to define mathematics, without identity as it appeared in the *Principia Mathematica, i.e.* as an "essential constituent of logical notation" (5.533). But he found now that "we can no longer use $x=y$ as a propositional function in defining classes," without identity as a basic part of logical notation he had to define classes through predicative functions (RAMSEY 1926 [1978] p.200). Wittgenstein had no problem with doing without identity to define classes, because to him classes were superfluous to the foundations of mathematics (6.031). Ramsey needed the theory of classes because he agreed with Russell that mathematics was extensional (MARION 1998, pp. 60-61). Ramsey defined $x=y$ as the tautology:

$$(\forall\phi\ \phi(x) \vee \neg\ \phi(x)\ ) \wedge (\forall\phi\ \phi(y) \vee \neg\ \phi(y))$$

and $x \neq y$ as the contradiction:

$$(\exists\phi\ \phi(x) \wedge \neg\ \phi(x)\ ) \wedge (\exists\phi\ \phi(y) \wedge \neg\ \phi(y))$$

(RAMSEY 1926 [1978] pp. 202-203)

In this manner Ramsey defined mathematical equations as tautologies (*cf.* MARION 1998, pp. 64-6). This bothered Wittgenstein terribly. He asserted, "calculus cannot have anything to do with tautology" (WAISMANN 1979, p. 218). Wittgenstein's assertion that mathematics is a logical method (6.2) implied that they both used the operation, not that propositions and equations were the same thing in any manner. In propositional logic the result of the operations was a proposition; in mathematics it was a number; they were not the same thing (WAISMANN 1979 , p. 218, *cf.* also MARION 1998, p. 65).

Though most of the Vienna Circle accepted Ramsey's view that tautologies and equations were the same (*cf.* HAHN 1980, p.25 and 34 and MARION 1998, pp. 64-65), but later Carnap clearly states that he realised that Wittgenstein did not believe theorems of mathematics to be tautologies, hence that they are not extensional (SCHILPP 1963 *p.* 47).

1.1.1 Definitions of Intensionality and Extensionality

Languages, logics or theories are more or less extensional or intensional in relation to other representations. Broadly speaking, an intensional understanding of a language focuses on how a term, or other aspect of language, is used, and extensional to what a term, or other aspect of language, refers. The thesis of extensionality, that every 'concept may be represented by its extension,' where that extension is a class or relation, thus treats concepts as things. Thus an extensional semantics attempts to understand the meaning of a concept by understanding what sort of thing it is.

A thesis of intensionality would state something like a 'concept is defined by process that creates it,' or a 'the meaning of a concept is how it is used.' Thus it treats concepts as processes or actions.

Notice that when one attempts to define the distinction between intensional and extensional vagueness seems inevitable. This is because the description of the extension of a concept always includes some aspect of its use, and the description of the intension of a concept always includes reference to what objects are affected by its use; *e.g.* a formal description of a proposition from an intensional point of view will still include terms. These terms themselves, even when treated wholly syntactically, still refer to something; be it a simple object or phenomena.

Wittgenstein's 'senseless' sentences—*e.g.* tautologies, contradictions and equations—are a means of dealing with this problem. They express nothing more than the form of the sentence as they have no real extensions in the world. In this manner they are profoundly intensional in purpose. As von Wright noted with regard to concepts which cannot be spoken of, but only shown, the *Tractatus* is intensional. So in regard to these 'senseless' sentences and as to what 'meaning' they express, the form that they present is intensional. Shifting our attention from the senseless to significant propositions, or those propositions which can be true or false, we see, in Wittgenstein, a more extensional treatment. These sentences refer to states of affairs. However the aspects, which Wittgenstein bought to light by investigating the senseless propositions, still apply to those with sense. Propositions with sense are still constructed, and still have a form. The method of construction—the application of the operation to a base—is where the intensional aspects of Wittgenstein treatment of propositions can be most clearly seen.

Not only do propositions have both intensional and extensional aspects but other logical concepts can be understood by reference to this tension. Let us look at four examples of intensional versus extensional explanations of concepts: words (terms), propositions, modal features, and functions:

Concept	**Intensional**	**Extensional**
Words	Use of the word in circumstances determines its meaning	Word refers to object in the word
Propositions	A construction whose elements are combined in a manner which mirrors the manner in which the elements of a state of affairs are combined	Proposition's reference is its truth value or state of affairs

Modality	Possibility is logical possibility	'fx is possible' means $(\exists x).fx$
Operation or Functions	Construction a process	Function is the mapping of one set to another.
Number	Exponent of an operation	Class of classes

1.1.1.1 Words and Propositions

Words, specifically nouns, seem the most obviously extensional concepts. A word refers to the object in the world that it represents. But this is surely not the only way to conceive of words. When speaking of borderline objects (e.g. is some object a table or a desk) the use of the object comes into the meaning of the word, and thus the use of the word must be considered at least part of its meaning. That is the word 'table' refers to table–objects partially because it is used to describe table–objects.

Propositions are often said to refer to a truth value, but they may refer to something else. In Wittgenstein's *Tractatus* propositions are said to refer to states of affairs, but are also said to be constructions made up of elementary propositions. The combinations of these elementary propositions mirror the combination states of affairs or objects or phenomenon in the world that make up the states of affairs described by complex propositions. What these elementary propositions are is not answered by Wittgenstein. In fact he later told Norman Malcolm that, at the time of writing the *Tractatus*, he neither knew what elementary propositions were, nor did he think it necessary, as a logician, to give examples of them (MALCOLM 1958, p.86).

The following table gives several non-classical interpretations of the propositional meaning. Each of theses interpretations is less extensional than the classical understanding of the meaning of a sentence being its truth value. Rather the 'value of a proposition' is the proof of that proposition or in other terms the construction of that proposition. Such supposedly intensional interpretations can be viewed extensionally if the propositions are viewed as sets of the elements and the propositions are thus represented as sets of solutions to problems, fulfillments of expectations or simply proofs of propositions.

Interpretation	**a $\in$ A means:**	**A true**
Gentzen	a is a proof of proposition A	A is true
Heyting	a fulfils the expectation A	A is fulfilled
Kolmogorov	a is a solution to problem A	A has a solution
Martin-Löf	a is an element of A	A has an element

(table from, RANTA, 1994, p. 40)

Each of these understandings of propositions, all developed after the *Tractatus*, attempts to speak of the meaning of a proposition as being more than its truth value. Each attempts to, in its way, express the meaning or rather the value of a proposition as something to do with the construction of that proposition.

In each of these views the construction of the proposition is seen as providing some sort of justification of that proposition. Martin-Löf's interpretation of the proposition is a set–theoretical interpretation—*i.e.* to say that a proposition has meaning is to say that there is an element of the set that represents that proposition, where that element is a construction which results in the proposition. This may be seen as going against several of Wittgenstein's admonishments—against what might be said and what might be shown, and the lack of a need for set theory—but all it is expressing is the meta–linguistic statement that '*p* is a significant proposition.' That is '*p* is a significant proposition' means that there is a concatenation of logical operations and elementary propositions that produce the proposition '*p*'. The 'set' in question is not something outside of the formalism, and hence does not refer the world. The 'set' in this view is a construction that may be performed by a person. It refers to the formalism.

The above reminds us that the notions 'intensionality' and 'extensionality' are conceptual tools that are to be seen in contrast; not absolute categories that describe one or another understanding of logic. Both in the classical view, in the relation between propositions and truth, and in the constructive, in the relation between constructions and propositions, we can speak of a many-one relation. Seeing proposition's extension as referring to truth or falsehood, demands that one 'truth' and 'falsehood' are things that exist. Understanding the meaning of a proposition as the construction that generates it—*i.e.* as its proof—does not require one to postulate the existence of entities. It is only when we attempt to create a notation to define this understanding, when we move to a meta–linguistic level, does this approach appeal to the extensions. The extensions in this understanding, however, are the propositions. The extension of the construction is the proposition.

1.1.1.2 Modality

Modal statements are a problematic part of language to represent logically. Once again the notions of extensionality and intensionality can be used as a conceptual tool to investigate various views of necessity, possibility and impossibility.

Von Wright, writing of the intensional-extensional distinction in modal logic, starts at the point of Frege's two definitions of modality. Frege separates two forms of possibility: the first is that of statistical possibility or empirical modality; the second is that of epistemic modality, knowledge that something can happen or that there is no logical contradiction in a certain possibility. Von Wright argues the *Tractatus* conception of modality is of the latter form.

Maury discusses Frege's treatment of possibility (MAURY 1977, p. 83) begins with Frege's description of to views of modality:

> If I term a proposition necessary then I am giving a hint as to my grounds of judgement. But does not affect the conceptual content of the judgement, and therefore the apodeictic form of a judgement has not for our purposes any significance (FREGE 1970 [1879], p.4).

Here Frege, according to Maury, rejects modality, because necessity has nothing to do with the conceptual content of a judgement. Frege then continues, "If a proposition is presented as possible, then either the speaker is refraining from judgement, and indicating at the same time he is not acquainted with any laws from which the negation of the proposition would follow; or else he is saying that the negation of the proposition is in general false ..." (FREGE 1970 [1879], p.4). And so Frege gives us two examples the first being "'it is possible that the Earth will one day collide with another celestial body'" and the second "'a chill may result in death'" (FREGE 1970 [1879], p.5).

Thus the 'it is possible' and 'may' express the kinds of grounds that the speaker has for assessing the content. According to Maury, Wittgenstein would so far agree with Frege's description, but disagree that "the notion of possibility, in its turn, is needed for an account of the notion of 'content'." We need instead "the notion of possibility" for the "*analysis* of propositions" (MAURY 1977, p.83). Wittgenstein has instead a notion of logical possibility presented along with his critique of Russell's use of the term possible. In 5.525 Wittgenstein states first that '$(\exists x) . fx$' is the incorrect rendering of the statement "fx is possible." After this he presents the view of logical possibility:

> The certainty, possibility, or impossibility of a situation is not expressed by a proposition, but by a proposition's being a tautology, a proposition with sense, or a contradiction (5.525).

Wittgenstein is criticizing Russell's view that a proposition is "possible" means "that there is at least one value x for which it is true." Russell states this concisely in *The Philosophy of Logical Atomism* where he writes:

> When you take any propositional function an assert of it that it is possible, that is it is sometimes true, that gives you the fundamental meaning of 'existence'. You may express it by saying that there is at least one value of x for which that propositional function is true (RUSSELL 1918 [1985] p.98).

Russell sees the property of possibility as being of propositional functions rather than of propositions. Wittgenstein takes it as being of propositions. Maury notes that Wittgenstein's is actually the more traditional view (MAURY 1977, p. 83). However though Wittgenstein would agree that propositions are possible he would not take possibility to be a property of propositions, rather the significant propositions "expresses a possibility." In this manner "the notion of possibility is irreducible" (MAURY 1977, p.84). The expression of a possibility is thus an intentional conception of modality, in comparison with Russell's extensional conception.

1.1.1.3 Operations and Functions

The operation in the *Tractatus* takes the place of the function in most standard interpretations of both mathematics and logic. It is in the distinction between the function and the operation that Wittgenstein makes most clear his understanding of the intensional–extensional distinction. The set theoretic function is still accepted by Wittgenstein; instead of redefining the function intensionally Wittgenstein refers to the intensional version of the function as the operation. Each has its place in his system.

The intensional definition of function is that a function is a process or calculation which determines a number (or output) from a input. This definition is older than the extensional. (FLOYD 2002, *passim*). It is also Frege's definition of function in the *Begriffsschrift*, though his formalization of the concept turned out to be extensional (FREGE 1879 [VAN HEIJENOORT 1967] p.56, on this *cf.* also SUNDHOLM 1992, p.52). The distinction between the constructive and the set theoretic definitions is one of focus. A constructive definition is one that focuses on how the input of the function—its base—is converted, or modified, to produce the output. The extensional definition of function focuses on what sort of things the inputs and outputs of a function are. The range of each is then considered a set and the mapping between them is the reference of the function.

2 The 'Negative' Programme: Wittgenstein's Critique of Logicism

Russell's paradox undermined Frege's attempt, in the *Grundgesetze der Arithmetik*, to ground mathematics in logic (FREGE 1893 [1964], pp.127-43). In order to deal with this paradox, and so rescue the logicist programme, Russell introduced the theory of types (RUSSELL 1903, pp. 523-8; RUSSELL 1908; and WHITEHEAD and RUSSELL 1927 vol.1, pp. 37-66). Type theory arranges sentences into a hierarchy: the lowest level consisting of sentences about individuals, the next lowest level about sets of individuals, the next about sets of sets of individuals *et cetera.* This means however that one can refer to all objects for a given predicate only if they are of the same type. This solved the problem of the paradox, but several axioms had to be added for there to be a satisfactory account of number. Wittgenstein rejected these axioms along with the theory of types; and so one can say that he pointed out that the logicist definition of number did not fulfill its own task: that arithmetic be defined in terms of logic.

In this chapter I will describe the parts of the logicist project which Wittgenstein rejects, focusing on Russell's Paradox, type theory as a solution to that paradox and the axioms that Russell added to deal with the problems caused by the very addition of type theory into the logicist account of number. Broadly speaking, Wittgenstein's criticism is that to him the logicist's account of number treats logic as if it were part of the world to which it refers and not a language that refers to objects in a world. This is another way of stating the saying–showing distinction, *i.e.* that there must be a distinction between the way language works, and what it expresses. Logic belongs to the former category, objects in the world to the latter.

The saying–showing distinction informs Wittgenstein's critique of the logicist project in general, and type theory in particular. Wittgenstein rejects the idea that logic can be at the same time about itself and the objects of the world to which it refers. The axioms of infinity and reducibility are critiqued for related reasons. Wittgenstein rejects them for treating the logical language, which defines natural number in a logicist system, in an empirical manner. The infinity of objects is not a question about logic, as the title 'axiom' would suggest, but about the world. The understanding of both the class and the object as real objects lead to the problem of Russell's paradox, which in turn required type theory as a fix. Wittgenstein's rejection of logicism was a rejection of this sort of realist interpretation. The construction by Wittgenstein of natural numbers and the construction of all of sensible language through the 'operation' show how a clear distinction between the posited objects and the language which describes them can be made.

Wittgenstein's showing–saying distinction is very much a distinction between extensional and intensional concepts of logic. As von Wright puts it:

> the *Tractatus* is 'extensionalist' as far as that which … can be *said* is concerned. What is 'intensional' cannot be spoken of in significant propositions. It may nevertheless *show* itself in language (VON WRIGHT 1988, p.186).

The critiques which are based on Wittgenstein saying–showing distinction, his critiques of type theory and the axioms of infinity and reducibility, are thus also expressible in terms of an extensional–intensional distinction. It is the intensional nature of the operation that will be highlighted in the chapter three, where we will consider the positive programme of the *Tractatus*. Likewise it is the extensional nature of the logicist's project that Wittgenstein criticises when he invokes the showing–saying distinction.

2.1 The Logicist Programme of Frege and Russell

Frege and Russell were both working on a reduction of mathematics to logic. That is they both defined number as classes of classes, such that the number 1 would be identified with the class of all singletons, the number 2 with all duals, the number 3 with all triples, the number n with all n–tuples, and so on.

Frege published the first volume of the *Grundgesetze der Arithmetik* in 1893. On June 16 1902 Russell wrote to Frege describing a paradox that showed that the axioms that Frege used in the first volume of the *Grundgesetze der Arithmetik* were inconsistent. To deal with this paradox Russell develops the theory of types from 1903 in the *Principles of Mathematics* through to the publication of *Principia Mathematica* in 1910. Russell developed several solutions to the paradox, as well as to several other paradoxes that he saw as preventing a logicist definition of arithmetic. The logicist project was thus saved by the theory of types, but according to Wittgenstein also destroyed by it because it required the addition of non-logical axioms and meta-theoretical concepts to work.

Russell's statement of his paradox works as follows. Assume a function "to be a predicate that cannot be predicated of itself" that is $f(x)$ is defined as $\neg x(x)$. Assume $f(f)$. Then by definition of f we get $\neg f(f)$ which is a contradiction, and so $\neg f(f)$ holds. Conversely, if we assume $\neg f(f)$ then again by definition of f, $f(f)$ holds. Thus $\neg f(f)$ and $f(f)$ hold which is a contradiction (KAMAREDDINE 2002, p. 196; RUSSELL 1902 [FREGE 1980] pp.130-133). Frege's response was quick, $f(f)$ is not possible in the *Begriffsschrift* he wrote, but in the *Grundgesetze* did allow a similar expression of the paradox.

Frege's reply included a restatement of the paradox using the course-of-values of functions. Russell's "a predicate is predicated of itself" did not appeal to Frege as a description of the paradox. He writes in his reply to Russell:

> A predicate is as a rule a first-level function which requires an object as argument and which cannot therefore have itself as argument (subject). Therefore I would rather say: 'a concept is predicated of its own extension'. If the function $\Phi(\xi)$ is a concept, I designate its extension (or the pertinent class) by '$\grave{\varepsilon}\Phi(\varepsilon)$' (though I now have some doubts about the justification for this). '$\Phi(\grave{\varepsilon}\Phi(\varepsilon))$' or '$\grave{\varepsilon}\Phi(\varepsilon) \cap \varepsilon'\Phi(\varepsilon)$' is then the predication of the concept $\Phi(\xi)$ of its own extension (FREGE 1902 [1980] pp.132-133).

Frege wrote the class defined by $\Phi(x)$, or in his terms the course-of-values for $\Phi(x)$, as $\grave{\varepsilon}\Phi\ (\varepsilon)$. The definition of equal courses-of-values is thus expressible as:

(1) $\grave{\varepsilon} f(\varepsilon) = \grave{\varepsilon} g\ (\varepsilon) \longleftrightarrow \forall a[f(a) = g\ (a)]$

That is, if the values of two functions are the same the two functions are equal and vice-versa.[2] Define $f(x)$ as: $\neg\forall\phi\ [(\grave{\alpha}\ \phi\ (\alpha) = x) \longrightarrow \phi(x)]$, a function that says of x that it is not that case that for every ϕ if the course of values of the function ϕ equals x then there is some function $\phi(x)$. We can write $\mathrm{K} = \varepsilon' f(\varepsilon)$ that is K is the shorthand for the course-of-values of the above defined function. Now by the definition of the equivalence of functions (1) we have for any $g\ (x)$ that if the course-of-value of g is equal to that of f then the functions are equal and we let $x = K$ and we get: $\grave{\varepsilon}\, g\ (\varepsilon) = \grave{\varepsilon} f(\varepsilon) \longrightarrow g\ (K) = f(K)$. By (1) we can for any function $g(x)$ say:

$$\grave{\varepsilon} g\ (\varepsilon) = \grave{\varepsilon} f(\varepsilon) \longrightarrow g\ (K) = f(K)$$

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[2] This is the fifth axiom of Frege's *Grundgesetze*. It asserts that there is a course–of–values for every function, and so a an object for every concept, and that a function is definable by that course of values.

Simplifying we get:

$$\grave{\varepsilon} g(\varepsilon) = \grave{\varepsilon} f(\varepsilon) \longrightarrow g(K)$$

And since $\grave{\varepsilon} f(\varepsilon) = K$ we have:

$$f(K) \longrightarrow ((\grave{\varepsilon} g(\varepsilon) = K) \rightarrow g(K))$$

Since we can do this by (1) for any function $g(x)$, we can write:

$$(3) \qquad f(K) \longrightarrow \forall \phi ((\grave{\varepsilon} \phi(\varepsilon) = K) \rightarrow \phi(K))$$

By the definition of course-of-values for any function g we can write:

$$\forall \phi [(\grave{\varepsilon} \phi(\varepsilon) = K) \rightarrow \phi(K)] \longrightarrow ((\grave{\varepsilon} g(\varepsilon) = K) \rightarrow g(K))$$

When we substitute $f(x)$ for $g(x)$ once again by the definition of course-of-values we obtain:

$$\forall \phi [(\grave{\varepsilon} \phi(\varepsilon) = K) \rightarrow \phi(K)] \longrightarrow ((\grave{\varepsilon} f(\varepsilon) = K) \rightarrow f(K))$$

Now since $\grave{\varepsilon} f(\varepsilon) = \mathrm{K}$ by definition of K we can now write:

$$\forall \phi [(\grave{\varepsilon} \phi(\varepsilon) = K) \rightarrow \phi(K)] \longrightarrow ((K = K) \rightarrow f(K)),$$

Which simplifies to:

$$\forall \phi [(\grave{\varepsilon} \phi(\varepsilon) = K) \rightarrow \phi(K)] \longrightarrow f(K)$$

Using the definition of f, we obtain:

$$\forall \phi [(\grave{\varepsilon} \phi(\varepsilon) = K) \rightarrow \phi(K)] \longrightarrow \neg \forall \phi [(\grave{\varepsilon} \phi(\varepsilon) = K) \rightarrow \phi(K)]$$

And hence by a *reductio ad absurdum*, $\neg \forall \phi [(\grave{\alpha} \phi(\alpha) = K) \rightarrow \phi(K)]$ or by the previously defined shorthand:

$$(4) \qquad f(K)$$

When we apply (3) this gives us $\forall \phi [(\grave{\alpha} \phi(\alpha) = K) \rightarrow \phi(K)]$, which implies:

$$\neg\neg \forall \phi [(\grave{\alpha} \phi(\alpha) = K) \rightarrow \phi(K)]$$

or,

$$(5) \qquad \neg f(\mathrm{K})$$

propositions (4) and (5) are obviously contradictory (KAMAREDDINE 2002, p. 195-6; and FREGE 1902 [1980] pp.130-133).

Though it is often noted that "Russell paradox showed the inconsistency of Frege's system," it is clear that Frege's system of logic was not the only one that is affected by Russell's paradox; Kamareddine points out that the paradox can also be formed in Peano's system (KAMAREDDINE 2002, p 199). Any naïve system of classes or sets falls to the same paradox. Russell wrote in his letter to Frege, that he himself was "...on the point of finishing a book on the principles of mathematics" (RUSSELL 1902 [FREGE 1980] pp.130-131) and it was this project, the *Principle of Mathematics*, that was stopped by the discovery of his paradox.

Frege's system was different from Russell's; it draws the eye to a tension in logic that Russell's passes over. As Hylton points out:

> For Frege, the distinction between concept and object ensures that no analogue of Russell's paradox threatens his philosophical view, in so far as this is independent of mathematics. The threat of paradox arises, for his view, only when he adds the notorious Axiom V of *Grundgesetze*, which asserts the existence of an object (*Wertverlauf*) corresponding to every concept. For Russell, by contrast, the object–based metaphysics gives the effect of Frege's Axiom without the use of any special assumption: a propositional function, in *Principles*, is already in the relevant sense an object (HYLTON 1990, p. 286).

The fifth axiom thus allows, in effect, the definition, rather than the construction of functions; it allows the arbitrary definition of function by defining a course of values. Frege's distinction between concept and object is similar to Wittgenstein's distinction between states–of–affairs and the propositions which mirror them. In both cases there is a parallel between things existing in the world and concepts or propositions. For Wittgenstein there is no conversion of one to another. States–of–affairs exist in the world, propositions are formal, and exist in language; though they are represented by objects in the world—*i.e.* by signs. You could not move from concepts to objects in Frege's philosophical system, it is only in Frege's *Grundgesetze* that this conversion slips in and make his system vulnerable to Russell's paradox. There are two of types solution to this problem: one like Russell's, add something to the theory of classes to prevent paradoxes, or, one like Wittgenstein's, abandon the programme of trying to define number using classes of classes.

As we shall see Wittgenstein defines natural number by iteration; *i.e.* by construction, rather than definition. If a function is defined as a mapping from one set, class, or course–of–values to another, such a definition says nothing about what the function does. For Wittgenstein functions, defined in this manner, cease being part of logic, rather they become objects. Wittgenstein's notion of the operation, as will be shown in the second half of this thesis, focuses on the process: *i.e.* the movement from one number to another number. Frege rejects the psychological definition of "shifting our attention" from one object to another (FREGE 1884 [1953] §80, p.92[e]-94[e]). But Wittgenstein's operation shows how this "shifting" can be seen logically, or formally, rather than psychologically. Wittgenstein is thus not a step away from Frege's rejection of psychologism in logic but a fine tuning of it.

2.1.1 The Development of the Theory of Types

Russell solved the problem that his paradox posed to the logicist definition of number by developing the theory of types. Type theory arranges all classes (or propositional functions or sentences) into a hierarchy, the lowest level consisting of classes whose values are individuals, the next level consisting of classes of classes of individuals. The third lowest level then consists of classes of classes of classes individuals, and so on. It is thus possible to refer to all objects for a given predicate only if they are of the same type. This is the core of the theory of types. In the *Principia Mathematica* Russell refers to this as order. Propositions that refer to individuals are of the first order; those that refer to propositions of the first order are of the second order and so on. Type in the *Principia Mathematica* is a broader notion, which Russell uses when talking about order, propositions of mixed generalities and propositional functions with various numbers of arguments. The combination of order and type is generally referred as the ramified type theory.[3]

However, Russell did not move directly to the ramified type theory from his simple theory of types, but rather when through two other theories on the way: the substitution theory and the zig-zag theory. Both were 'no-class' theories; they did not posit classes as logical entities. They instead replace classes with propositional functions.

2.1.1.1 Russell's Early Attempts.

Appendix B of the *Principles of Mathematics* is entitled "The Doctrine of Types" and it is here that Russell first devises the theory that enabled the logicist definition of number to stand without paradox. He puts it forth "tentatively, as affording a possible solution to the contradiction;" and describes it thus:

> Every proposition $\phi(x)$—so it is contended—has, in addition to its range of truth, a range of significance, *i.e.* a range within which x must lie if $\phi(x)$ is to be a proposition

[3] Hylton objects to this term as "tendentious" arguing that the presentation of it as a theory which "imposes complications (or ramifications) upon a simpler underlying theory" is not reflective of Russell's actual presentation. Copi, whose view Hylton is critiquing, to be fair, does note that Russell's view is not clearly presented and that he is extrapolating based on Russell's comments in the introduction, rather than just the text of the PM. Hylton takes issue with this method calling the passages in question (P.M., pp. 48-9) heuristic and taking his description from page 50 of the Introduction and *12. Hylton states that Russell's view that all of distinctions present in the hierarchy of types "have the same basis. All arise from the ... idea of the ambiguity of a propositional function." Furthermore Hylton argues that Copi's interpretation of Russell's view will work, because the type and order distinctions as Copi presents them are not independent (*cf.* COPI 1971, p.86; HYLTON 1990, pp.310-312, especially n36 and n37).

> at all, whether true or false. This is the first point in the theory of types; the second point is that the ranges of significance form *types*, *i.e.* if x belongs to the range of significance of $\phi(x)$, then there must be a class of objects, the *type* of x, all of which must belong to the range of significance of $\phi(x)$, however ϕ may be varied; and the range of significance is always either a single type of a sum of several whole types (RUSSELL 1903, p. 523).

Russell then goes on to define individuals as objects of the lowest type and so lacking their own range of values. Russell's early version of the theory of types is similar to what later became know as the simple theory of types, and it is instructive to set out the formalization of this system to see what Russell meant in the appendix of the *Principles of Mathematics*.

2.1.1.2 Russell's No Class Solutions

Russell found the conventions of type theory problematic. They seemed to go contrary to the universal character of logic (*cf.* URQUHART 1997, pp.83-84 and VAN HEIJENOORT 1967b pp.324-330). Russell came up with two intermediate solutions to the paradoxes that he saw as preventing a logicist definition of number: the "Substitutional Theory" and the "Zigzag Theory." Both of these were no-class solutions; that is, they did away with classes as entities, which solved the problem of Russell's paradox. But they were developed to solve other paradoxes, which Russell saw as blocking his project. Russell writes:

> In the zigzag theory we start form the suggestion that propositional functions determine classes when they are fairly simple and only fail to do so when they are complicated and recondite (RUSSELL 1905, pp. 145-6).

Russell does not give a clear criterion for simplicity. The explanation of the name 'zig-zag' comes form the manner in which this theory handles propositional functions and their extensions. For any propositional function $\varphi!x$ which does not determine a class then the extension of $\varphi!x$ must differ from any given class u. Likewise if there is some x such that x is a member of any given class u then $\sim\varphi!x$ obtains. This principle can be expressed in the following manner:

$$(\exists y)(\varphi!y \,.\, y \notin u) \vee (\exists y)(y \in u \,.\, \sim \varphi!y)$$

(URQUHART 1988 , p.85)

The idea presented here is that "recondite" propositional functions, those which do not have a class as an extension, have the above "zigzag" structure. Urquhart traces the development of this theory from comments made in the *Principles of Mathematics* to a MSS given to Whitehead in preparation for their collaboration on the *Principia Mathematica.* In this MSS, "On Functions" the primitives are "complex" and "constituent," propositions are complexes made by combining constituents. However, a complex may or may not be a function of its constituents. Russell's argument is that one can apprehend only those complexes of finite complexity. That is a complex which, "has a finite number of constituents that can be described in a finite number of steps" (URQUHART 1988, p.86). Under this interpretation $(x)\varphi(x)$ is a complex, but its constituents are not all the values of x, rather it is a function of φ'$\hat{x}$, written " $\vdash$.$Fo_{\varphi}\{(x)\ \varphi\text{'}x\}$" After several more attempts and refinements Russell gave up on the zigzag theory.

Russell's next attempt, the substitutional theory, is developed along with Russell's famous theory of descriptions (URQUHART 1988, p. 87). In Russell's confident manner he writes in a footnote, "I now feel hardly any doubt that the no–classes theory affords the complete solution of all the difficulties stated in the first section of this paper" (RUSSELL 1905 *cf.* URQUHART 1988). Russell expands on this theory in a paper "On the Substitutional Theory of Classes and Relations" (RUSSELL 1906 [1973] pp. 165-189). Russell describes a class of entities over which he allows quantification. The primitive notions of the theory being $p/a\text{'}x!q$ by which Russell means, "q results from p by substituting x for a in all those places (if any) where *a* occurs in *p*" (URQUHART 1988, p. 87). Classes are then disposed of by the following definition:

$x \in \alpha \;.{=}.\; (\exists p,a)(\alpha = p/a \;.\; p/a\,\dot{}\,x)$ Def.

This gets rid us of Russell's paradox "as $x \in x$ cannot be eliminated contextually if x is an entity, because $x = p/a$ must be false for all entities p and a" (URQUHART 1988, p. 88). However type theory re-emerges in an indirect manner, "Classes of classes are described as matrices which arise by performing a double substitution... On eliminating classes contextually, we find that "$p/a = p/(b/c)$" leads to nonsense..." (URQUHART 1988, p. 88). Russell is again must introduce a hierarchy which resembles his original theory of types.

2.1.1.3 Russell's Ramified Theory

The final theory presented in *Principia Mathematica* was more complicated than simple type theory[4]. Russell abandons the idea that there are "no ultimate ontological distinctions among entities," and also eliminates classes, replacing them with propositional functions[5] The

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[4] For the most part I follow Hylton's account (*cf.* HYLTON 1990, pp. 285-327).

[5] Copi's description of the ramified theory of types is that it denotes each sentence as being of a specific order and type. Type is handled, as in the simple theory of types, as a function is of type one if it acts on an individual of type zero. A proposition referring to individuals is of the 0th order, to propositions of the 1st, to propositions about propositions of the 2nd. Copi's ramified type theory is depicted in the following table:

	Order 1	**Order 2**	**Order 3**	...
Type 3:	$^1F^3, {}^1G^3, {}^1H^3,\ldots;$	$^2F^3, {}^2G^3, {}^2H^3,\ldots;$	$^3F^3, {}^3G^3, {}^3H^3,\ldots;$	...
Type 2:	$^1F^2, {}^1G^2, {}^1H^2,\ldots;$	$^2F^2, {}^2G^2, {}^2H^2,\ldots;$	$^3F^2, {}^3G^2, {}^3H^2,\ldots;$	...
Type 1:	$^1F^1, {}^1G^1, {}^1H^1,\ldots;$	$^2F^1, {}^2G^1, {}^2H^1,\ldots;$	$^3F^1, {}^3G^1, {}^3H^1,\ldots;$	...

(COPI 1971, p.87)

definition of class symbols "enables the system to simulate the theory of classes" (HYLTON 1990, pp. 286-7). Hylton makes the case that there are three reasons for this. First, the distinctions between entities seemed more palatable to Russell within the context of a theory of propositional functions, where one speaks of propositional functions and individuals, than one of classes. Second, Russell could more easily justify the theory of propositional functions as being logic, and finally, this enabled Russell to avoid questions of the ontological nature of classes (HYLTON 1990, pp.288-9).

The propositional functions alone do not dispense with Russell's paradox, for it can be restated in this theory. Russell's paradox now reformulated, assume a propositional function that *holds* of an object if the value of that propositional function for that object is a true proposition. Then let us consider a propositional function which *holds* of propositional function if and only if it does not hold of itself. This proposition holds and does not hold of itself (HYLTON 1990, p.299). Another paradox Russell now considers is that of the Cretan or liars paradox. Both paradoxes he holds are derived from what he calls the vicious circle principle. In the introduction to *Principia Mathematica* Russell and Whitehead describe a vicious circle in the following manner: "a vicious circle … arise[s] from supposing that a

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Copi presents ${}^{1}F^{1}(x)$ and "$y[{}^{1}F^{1}(y) \rightarrow {}^{1}G^{1}(x)]$ as examples of first order functions of the argument "*x*" and of type one, when "*x*" is of type zero. Second order functions of type one are still functions of "*x*" but make existence statements about propositions of the first order:

$$(\text{“}{}^{1}F^{1})[{}^{1}G^{2}({}^{1}F^{1}) \rightarrow {}^{1}F^{1}(x)] \text{ and } (\exists {}^{1}F^{1})[{}^{1}F^{1}(y) \vee {}^{1}F^{1}(x)]$$

These can written ${}^{2}H_{1}^{1}$ and ${}^{2}H_{2}^{1}$; that is second order type one statements (COPI 1971, p. 84).

collection of objects may contain members which can only be defined in terms of the collection as a whole" (WHITEHEAD and RUSSELL 1927, p.37).[6]

To avoid such problems Russell introduces the idea of order. There are two basic features of order as it pertains to propositional functions. The first is that any propositional function is of higher order than any object it may take as an argument. The second is that, "a propositional function is of higher order than any object within the range of a quantifier contained within the propositional function" (HYLTON 1990, pp. 300–1). An example of this second is that a proposition that contains a generality, *e.g.* Russell's example $\varphi\hat{x}$ where φ is "has all the properties of a great general" contains the propositional function "φ is a property of a great general."

This dispenses with both Russell's and the Liar's paradoxes. Russell's paradox falls by the first principle stated above. A propositional function must presuppose its values, as it is always of a greater order, and so must be independent of them. Thus it cannot have a value which contains it. The Liar's paradox falls by the second. The Liar's paradox consists of a statement, "there is a proposition I am affirming, which is false." But the second principle above, we note, states that proposition is a higher order than "any over which its quantified variable ranges" (HYLTON 1990, p. 302).

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6 Russell earlier described the vicious–circle principle in his 1906 paper "On the Substitutional Theory of Classes and Relations" in the following manner:

> 'Whatever involves an apparent variable must not be among the possible values of that variable.' Let us call this the 'vicious–circle principle'. The important case of this principle may be less exactly stated as: 'Whatever involves *all* must not be one of all which it involves.' Thus a statement about *all* propositions must be meaningless, or a statement of something which is not a proposition in the sense concerned (RUSSELL 1906 [1973] p. 204).

Type in the *Principia Mathematica*, it has been noted by both Copi and Hylton, is often a very flexible concept including order. It also includes other categorisations (cf. COPI 1971, p.80 n1 and HYLTON, pp. 305 and 310-12). Russell refers to type as the distinction as to the number of quantifiers in a propositional function, even if the range of the quantifiers is of the same order. Russell also refers to type as the number of argument places that a propositional function has. Thus we can note that in *21 it does the same thing as in *20, but for two variable rather than singular variable functions (HYLTON 1990, p.305).

2.2 Type Theory's Transgression of the Showing–Saying Distinction

Wittgenstein's objection to type theory was that it transgressed the showing–saying distinction (3.331–3.333; 5.252; 6.123 [2]). In this section I shall present Wittgenstein's formulation of the showing–saying distinction, and explain how Wittgenstein used it to critique the theory of types. This distinction is one of the central themes of the *Tractatus*. It proposes that there is a basic deference between what can be said in language and the structure of language itself which can only be shown.

Let us consider Wittgenstein's views on tautologies and contradictions to gain some insight into this distinction:

> 4.461 Propositions show what they say; tautologies and contradictions show that they say nothing. A tautology has no truth-conditions, since it is unconditionally true: and a contradiction is true on no condition. Tautologies and contradictions lack sense. (Like a point from which two arrows go out in opposite directions to one another.) (For example, I know nothing about the weather when I know that it is either raining or not raining.)

"It is or it is not raining" tells me nothing about the weather but instead reveals the nature of the symbols which are represented (KENNY 1973, p.45). Something can be said, can be understood, by the hearer, without knowing the truth value, only if the listener can ask "is p true?" Something can be said only if it passes on new information (KENNY 1973, p. 46).

Tautologies and contradictions do not extend to a set of facts about the world. Tautologies and contradictions are much like proofs. That is they can reflect the form of a two equivalent statements or two contradictory statements. A proof can be written in a form of either a tautology or contradiction, the concatenation of all the lines of a proof will take one of these two forms. Tautologies and contradictions thus show the structure of logic. A constructive reading of semantics means that the value of a proposition is how it is constructed, how it is proved, while the extensional view treats the meaning of a sentence as its extension as it would judge the empirical value of a sentence. Extensional semantics treats the meaning of a sentence as its extension, to its truth value, much as it extends to the state of affairs. A tautology has no extension in the world; it is always true. Tautologies are purely intensional; they have no state–of–affairs which they assert are the case:

> 4.462 Tautologies and contradictions are not pictures of reality

Tautologies show logical structure. Take, as an example, the distinction between signs and symbols (*cf.* 3.26 3.32 *ff.*). A word spoken or written is, from the point of view of its perceptible qualities, *e.g.* its shape or sound, a sign (KENNY 1973, p. 46). A sign is no more than a token that stands for something else, some other object. A symbol is the same thing from the point of view of its meaningful use, a rule for expression. For example, "bear" is an English sign for a noun and verb, and hence is two different symbols. To specify a symbol you do not have to say anything about how the symbol works, as Wittgenstein writes:

> …this is what is wrong with the theory of types: Russell in drawing up his syntactic rules was forced to speak about the things the signs mean (3.331).

On this passage Kenny writes:

> But that something belongs to a certain type is something that cannot be said: it cannot be said that M is a thing (as opposed to a class). This is something which must be shown by the symbol 'M' – not of course by the *sign* 'M' but by the symbol, the sign with syntactical rules. We cannot speak of types, we can speak only of symbols; but what the theory of types tried to say, the symbols can show (TLP 3.332—3.333; NB 108)" (KENNY 1973, p.47).

Wittgenstein speaks clearly about this in the *Notebooks* (NB 108-109). Take for example what he calls the nonsense sentences, "Socrates is a different type of thing as mortality," This sentence does not mean the same thing as the sentence "Socrates is not mortal." In the sentence "Socrates is mortal," we note that "Socrates" is a proper name and "mortal" is not. In this situation "is mortal" stands to the right of a proper name; whereas in "John loves Mary" both items are proper names. What this shows is what we are trying to express whenever we say that there is a difference in type between objects, relations and properties (KENNY 1973, p.48).

This example leaves us pretty close to using the sort of semantic language Wittgenstein bans. The verb "symbolises," Kenny points out, the sort of thing that crosses the boundary between language and the world. Wittgenstein rather states that we should focus instead upon the "accidental and essential features of the symbol" and thus try to develop a distinction between essence and accident within the realm of syntax (NB 109; TLP 3.334-41). For example the accidental features of Latin versus English words, the inflection of the Latin versus the word order of the English. Since every language must contain some feature to distinguish between the subject and the object then the essence of the symbol will of course be "what all symbols which can fulfil the same purpose have in common" (3.341). Wittgenstein uses the distinction between saying and showing in several arguments in the *Tractatus*. Not only in his rejection of the need for a meta-language to resolve paradoxes

(3.331); but also as a part of the picture theory in the explanation of the sense of a proposition (3.23, 2.14); the understanding and justification of the difference between sense, no-sense and nonsense (4.61-4.611); the question of how language attaches to reality (4.021); the limits of solipsism; and what Wittgenstein sees as the true nature of philosophy and the boundary between that which can and that which cannot be said (HARWARD 1974, p.2).

Stenius identifies two different senses of showing in the *Tractatus*: when show means "depicts" (4.022) and when it means "exhibits" (*aufweisen*). The latter is the meaning in 4.121-2, "what a sentence *exhibits* but cannot say is the 'logical form of reality'" (STENIUS 1960, p.179). The need for these two definitions is an appearance of a contradiction in the *Tractatus*. In 4.1212 Wittgenstein states, "what can be shown, cannot be said" while in 4.022 a "sentence shows how it stands, if it is true, *and says* that they do stand" (STENIUS 1960, p.178-180). The assertion of 4.1212 also seems at odds with 4.61; which indicates that some propositions, contradictions and tautologies, "show that they say nothing." As well 4.1212 and 4.022, which state that "a proposition shows its sense," seem to be at odd with 4.064 where a proposition's sense is said to be "just what is affirmed," *i.e.* what is said (HARWARD 1974, p.6). The need for two notions of showing is also noted by Harward, but he distinguishes "depicting" and "exhibiting," as the distinction between the demonstrative and the reflexive nature of the verb. The demonstrative notion is that of "I show *x* to *y*," while the reflexive is that of "*x* shows property *y*" (HARWARD 1974, pp. 6-15). Both solutions answer the question of whether the "shown" in the *Tractatus* is contradictory. However discussion of the differences between Harward and Stenius is beyond the scope of this thesis. Since the main use of "show" is exhibitory, or reflexive, especially in the critique of type theory, it will be the version discussed subsequently in this thesis.

Proceeding along the lines suggested above, in 4.461 Wittgenstein writes that the tautologies and contradictions lack sense, but points out at 4.4611 that they are not nonsensical. As noted above the tautology 'it is raining or it is not' does not tell us anything about the weather. Tautologies and contradictions thus show, or exhibit, the logical form of language and the world. They express the possible, rather than making statements about the world. Syntactical matters such as how notation combines are shown by their use. Thus we are discussing how things work; not what they are. This is an intensional mode; not an extensional one. It is about expressing the structure of a language, and how it reflects the world, or the manner in which it reflects the world; not the extension of the language to the world. In a constructive logic the value of a sentence is the proof that constructs it. In a classical logic, by contrast, the value of a sentence is said to be its truth or falsity. Showing, for Wittgenstein, amounts to how a sentence is generated by a formal series. Thus the intension of the iteration of the operation that creates the formal, or internal, relations between the propositions that each make a stage in the construction, or proof, of the final complex proposition. In sum, much of the logic of the *Tractatus* is about construction, and showing is how we perceive the construction.

2.2.1 Intensionality and the Tractatus

Von Wright's article "Model logic in the *Tractatus*" argues that the extensionality thesis, *i.e.* the thesis that in any proposition about a concept the concept may be represented by its extension, is correct for the *Tractatus*, only in reference to what can be said, but intensional in terms of what can be shown. Von Wright notes that the logical positivists did not take seriously the showing-saying distinction and they believed that modal concepts are either "bogus or translatable into extensional concepts" a view which von Wright refers to as "reductivist." This view would be contrary to Wittgenstein's account in the *Tractatus* giving as an example 5.525 which states:

> The certainty, possibility, or impossibility of a situation is not expressed by a proposition, but by an expression's being a tautology, a proposition with a sense, or a contradiction. The precedent to which we are constantly inclined to appeal must reside in the symbol itself. (5.525 [2])

Von Wright argues that modality in the *Tractatus* is connected to the nature of logic. For Wittgenstein possibility is "logical possibility" rather than statistical probability. This distinction is connected by von Wright to the intensional–extensional distinction in logic and thence to Frege's two definitions of modality. Frege separates two forms of possibility: the first is that of statistical possibility or is empirical modality; the second is that of epistemic modality; *i.e.* knowledge that something can happen, or that there is no logical contradiction in a certain possibility. Von Wright argues the *Tractatus* conception of modality is of the latter form.

Von Wright notes that the early readers of the *Tractatus* did not take seriously this distinction and that this has had the effect of them neglecting the non-extensional aspects of the work. Anscombe and Stenius were "early pioneers" of the study of the *Tractatus* for its own sake, but the topics on which they focused were the picture theory and Wittgenstein's concept of objects. By the 1960s some attention started to be paid to intensional notions in the *Tractatus*; Wolfgang Stegmüller, for example, considered the *Tractatus* "'soaked' in intensional concepts" (STEGMÜLLER 1966, p.181, trans. VON WRIGHT 1988, pp.186).

Von Wrights argues modality is one of the areas where intensional notions come to the forefront. One who holds an extensional view of logic will be "suspicious" of modal logic. This is because such a view of modality implies that "possible" means that there is at least one 'possible' object, whereas an intensional view considers possibility as meaning constructability. An extensionalist logician, von Wright continues, will either, "think it [modal logic] a bogus subject" or translate modal expressions to non-modal sentences.

Maury notes that propositional significance is a modal notion and thus an attempt to define necessity, possibility and impossibility in truth-functional terms cannot but end in failure (MAURY 1977 pp.14-15 and pp.26-40). This view of a modal notion is one that cannot be reduced through analysis to truth and falsehood, but depends upon a notion of possibility that is in some way primitive. To support this view von Wright describes Frege's analysis of possibility into epistemic and existential quantitative views. The first is that of not knowing the contrary, and is an intensional notion; the second was the statistical view of modality, and is the extensional view. Russell held the extensionalist view that the "doctrine of modality only applies to propositional functions not to propositions." But Wittgenstein writes, "it is incorrect to render the proposition '$(\exists x).fx$' in the words, 'fx is possible' as Russell does."

Two chief principles upon which logical reasoning can be said to depends are, first, that of 'bivalence' which, when extended to all propositions knowable or not, is termed the law of the excluded middle (*principium tertii exclusi*). The second principle is 'knowability', i.e. that if something is true then it is possible to know that it is true. The tension between these two principles is responsible for the split between classical and intuitionistic logic and mathematics. As Sundholm summarizes:

> It is only in 1908 that the tension between this principle of Knowability and that of Bivalence is felt, simultaneously in two different places, namely Cambridge and Amsterdam. G.E. Moore is the first to contemplate unknowably true propositions in view of the fact that as a Realist, he could not allow for the abolition of the law of Bivalence. L.E.J. Brouwer, on the other hand, took the opposite way out in his reaction to this dilemmatic tension and chose to retain the Knowability of truth but had then to refrain from acknowledging the law of Bivalence, and the ensuing law of Excluded Middle in their full generality (SUNDHOLM 1994, p.140).

To accept the primacy of bivalence means to give credence to the idea that every proposition's extension is either truth or falsehood, whether or not it is knowable. To focus on the knowability of truth rather than on the law of bivalence means that one must focus not on the extension of a proposition, but rather on its intension, not on its reference but on its use. That is, how to use a word, or a proposition, is a question of knowledge. The principle of knowability can be extended into a principle of correspondence; that is, a proposition represents something in the world in virtue of its form being similar to the structure of that thing in the world (SUNDHOLM 1994, pp.140-1). The principle of correspondence is presented in Wittgenstein's analysis of truth in the *Tractatus*:

> 2.202 The picture represents a possible state of affairs in logical space.
> 2.21 The picture agrees with reality or not; it is right or wrong true or false.
> 4.021 The proposition is a picture of reality, for I know the state of affairs presented by it, if I understand the proposition. And I understand the proposition, without its sense having been explained to me.

The principle of correspondence can be more simply defined as the principle that "if a proposition is true it is true in virtue of something." Wittgenstein demanded that if a proposition is true then there must be some state of affairs that show that it is true, and under his interpretation, it is false otherwise. The point that propositions are about the world, and thus must be grounded in the states of affairs that they represent. Otherwise the arbitrary definition of functions, as in the case the conversion of courses–of–values to functions in Frege, leads one into trouble. The tension between knowability and bivalence is evident in the problem with Frege's system. Frege's interpretation of function is not reflected in his actual formalization. Arend Heyting's anti-realist interpretation of correspondence, that if something is true then there must be a proof of the truth of the proposition, is not dissimilar to Wittgenstein account of correspondence just discussed immediately above (SUNDHOLM 1994, p.141-2.). In fact the construction by means of the operation from elementary propositions to complex propositions works in the same way that a proof does. That both describe an algorithm will become obvious when we discuss the operational definition of mathematics below. So it will be this primary tension between the intensional and extensional in logic that will separate the two views of natural number. Russell's continued adherence to a logicist view by definition forced him to more and more realist statements about logic in the theory of types. Wittgenstein's focus on the other aspect of logic drove him first to reject the logicist programme and second to develop an algorithmic, or in Wittgenstein's terms, "operational," view of both propositions and mathematical equations.

Like von Wright's description of intensional modality as epistemic, the concept of the principle of knowability is the epistemic and intensional part of logic. The principle of bivalence, the other basic conception of logic according to Sundholm, is the extensional. The intensional principle focuses on construction, on showing how something can be made. The extensional principle, states that an entity or a truth value exists. Here we see another expression of Wittgenstein's saying–showing distinction, both for logic and modality.

2.2.2 Wittgenstein's Showing–Saying Distinction and Type Theory

There are three sections where the showing–saying distinction is used by Wittgenstein to criticise Russell's type theory. In 3.331 to 3.333 where he asserts that the need for a meta-language is unnecessary; in 5.25 to 5.252 where he makes the distinction between a function proper and what he calls an operation; and, finally, in 6.123 where he argues that the "laws of logic" are not themselves part of logic. As Wittgenstein notes:

> 3.331 … It can be seen that Russell must be wrong, because he had to mention the meaning of signs when establishing the rules for them.

> 3.332 No proposition can make a statement about itself, because a propositional sign cannot be contained in itself (that is the whole of the 'theory of types').

Is this an accurate account of the theory of types; *i.e.* that no propositional sign can contain itself? As Wittgenstein continues:

> 3.333 The reason why a function cannot be its own argument is that the sign for a function already contains the prototype of its argument, and it cannot contain itself. For let us suppose that the function F(fx) could be its own argument: in that case there would be a proposition 'F(F(fx))', in which the outer function F and the inner function F must have different meanings, since the inner one has the form φ(f(x)) and the outer one has the form ψ(φ(fx)). Only the letter 'F' is common to the two functions, but the letter by itself signifies nothing. This immediately becomes clear if instead of 'F(Fu)' we write '(∃φ) : F(φu) . ψu = Fu'. That disposes of Russell's paradox.

What Wittgenstein is saying here is that the rules of the use of the notation describe the language that it represents. Naïve set theory is vulnerable to paradox. But if the use of the notation is carefully circumscribed, the nature of what they represent becomes shown in their use. No more can be said about them than an example of how they work. Type theory then is an attempt, from Wittgenstein's point of view, to do just this, *i.e.* tell us how a notation works by setting up an unnecessary set of rules, about the supposed objects that are the extension of functions.

To make clear the difference between the extensional and intensional views of function, Wittgenstein introduces the operation to apply to the latter:

> 5.251 A function cannot be its own argument, whereas an operation can take one of its own results as its base.

His distinction at this point, is curious, for the classical view of the function will be rejected as 'superfluous' in mathematics (6.031). He states in that passage that this is related to "the fact that the generality which we need in mathematics is not the *accidental* one." Classes being groups of objects are "in the world," and thus generality about these is accidental since the facts of the world could easily be other than they are. The generality needed for mathematics is logical, without reference to the world, but to the generating process.

2.3 Axioms of Infinity and Reducibility

The need for the axioms of Reducibility and Infinity arose out of Russell's modification of the logicist approach. The theory of types created a problem in the definition of number in that classes were not simple but of various types or orders. And so if number is defined by class, there are different numbers at each level of class we define. To deal with this complication Russell introduced what is called the axiom of reducibility (WHITEHEAD and RUSSELL 1927: Part 1 Chapter II).

In addition Russell introduced the axiom of infinity, which states that an infinite number of objects exist. As noted in the introduction, Russell's axiom of infinity states that there are an infinite number of objects. This is required because if there were not an infinite number of objects and the definition of number is the equivalence class, then the number defined by the largest class plus one would be the empty set and thus equivalent to zero. But, as we just saw, here we are making a definite claim about classes and thus about the world, and not just logical form. This is the basis of Wittgenstein's critique to be discussed below.

The axiom of infinity was not a logical axiom. It was as Black calls it a "[proposition] about the number of objects in the universe." In 5.535–5.5352 Wittgenstein develops his objection to this axiom. The infinity of objects could not be expressed as a law, as it is about the world, and it may be the case that there are not an infinite number of objects. If it could be different, then it would be an accidental generality and not a logical one.

Wittgenstein in 5.53-5.534 rejects identity statements, as pseudo-propositions. Black points out that in 4.242 he rejected identities as "expedients." He rejects them out right by showing that there is an alternate notation that does just as well. Establishing that the sign for identity in 5.533 "not an essential constituent" of an "adequate logical notation" (BLACK 1964, p.290-291). This culminates in his rejection of a series of pseudo-propositions in 5.534:

> 5.534 And now we see that in a correct conceptual notation pseudo-propositions like '$a = a$', '$a = b$. $b = c$. $\supset a = c$', '(x) . $x = x$', '($\exists x$) . $x = a$', etc. cannot even be written down.

The elimination of identity is "a matter of simple transformations" (MARION 1998, p.51; *cf.* also HINTIKKA 1956, pp. 225-45). This is seen above in 5.432 where he renders $\exists x\ \exists x\ (F(x) \wedge F(y) \wedge (x{\neq}y))$ in 5.532 as $\exists\ x,y\ (F(x) \wedge F(y))$. The x and the y are not objects but "signs of the pseudo–concept object ... Wittgenstein replaced identity with a convention for the reading of the quantifiers" (MARION 1998, p.52).

Having rejected identity, in 5.535 he approaches the problem of the axiom of infinity. This axiom, he states, would be expressed in language "through the existence of infinitely many names with different meanings." It is once again a case of saying what can only be shown. Black notes correctly that this causes several problems, as general propositions seem to limit the number of objects to the finite. As Wittgenstein notes:

> 5.535 This also disposes of all the problems that were connected with such pseudo-propositions.
> All the problems that Russell's 'axiom of infinity' brings with it can be solved at this point.
> What the axiom of infinity is intended to say would express itself in language through the existence of infinitely many names with different meanings.

Wittgenstein also notes on a similar point, about the proposition: 'there are no things':

> 5.5352 In the same way people have wanted to express, 'There are no things ', by writing '~($\exists$x) . x = x'. But even if this were a proposition, would it not be equally true if in fact 'there were things' but they were not identical with themselves?

The existence of things then is to be demonstrated by the use of their names in propositions. For Wittgenstein there can be no statement of their existence that is a statement with sense.

The axiom of reducibility says: anything which is definable at a higher level is coextensive with something which is introduced in the most basic level. This can be expressed in the following manner:[7]

$$\forall X^{(i)} \exists Y^{(0)} \forall n [n \in X^{(i)} \Leftrightarrow n \in Y^{(0)}]$$

where $X(i)$ is a variable of type 1 and level i and $Y(0)$ is a variable of type 1 and level 0 and n is a variable of type 0. That is, for every class X of any type there is some class Y of type zero such that for every *n*, *n* is a member of X if and only if it is also a member of Y. That is, for every class of a level above 0 there is some class of type zero which contains the same members; or, to put another way, for each element of a higher type there is one of a lower type. As we will see Wittgenstein's rejection of this axiom and the axiom of infinity are closely tied with his views about logic.

As Wittgenstein notes, the axiom of reducibility was likely true, but not a logical axiom for it could be false. Though the axiom of reducibility seems a reasonable conjecture Wittgenstein writes that it is possible that the world may be made in such a manner that it is not the case. In 6.1233 he asserts:

> 6.1233 It is possible to imagine a world in which the axiom of reducibility is not valid. It is clear, however, that logic has nothing to do with the question whether our world really is like that or not.

Therefore we see again that it is Wittgenstein's critique that such an axiom is a statement about the world.

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[7] This formulation is based upon Copi (Copi 1971, pp. 80*ff.*)

2.4 Circularity of Logicist's Definition of Number

In 4.1273 Wittgenstein criticises the logicist definition of a formal series for containing a vicious circle. The 'general term of a formal series' is according to Wittgenstein a 'variable,' as the concept 'term of a formal series' is a 'formal concept'. A formal concept is (4.12721) exhibited by the concept itself, it is 'shown not said,' as we saw earlier. To Russell and Frege number was, in Wittgenstein's terms, a proper concept. What opened the Frege–Russell description of a formal series to claims of circularity was the definition of "the successor of" relation. As Anscombe writes:

> The relation *successor of* was in turn defined by means of the relation *immediate successor of*; plainly these two related in the same way as *ancestor (in the male line) of* and *father of*—the on relation is, as Russell says, the ancestral of the other in each case. This brings us to the Frege–Russell account (independently devised by each of them in essentially the same form) of what it is for one relation to be the ancestral of another. For simplicity's sake, I shall merely explain how *ancestor* would be defined in terms of *parent*; the generalization of this account can readily be supplied.
>
> We first define the notion of a *hereditary* property: viz. a property which, if it belongs to one of a man's parents, belongs also to him. We then define 'a is and Ancestor of b' to mean:
>
> 'a is a parent of some human being, say x, all of whose hereditary properties belong to b.'
>
> Let us for the moment treat this as an arbitrary verbal stipulation of what the defined term 'Ancestor' is to mean; we must now enquire whether it is true that, on this definition, a is an Ancestor of b if and only if a is in the ordinary sense an ancestor of b; if so we have an adequate of ordinary ancestorship which does not introduce the 'and so on' brought into our ordinary explanations of the term (cf. the last paragraph but two). And I is quite easy to show intuitively that this equivalence between 'Ancestor' and 'ancestor' does hold.
>
> A. Suppose a is an ancestor of b. Then either (1) a is a parent of b, or (2) a is a parent of an ancestor b.
>
> (1) If a is a parent of b, then b himself fulfils the conditions of being a human being, x, whose parent is a and whose hereditary properties all belong to b. So a is an Ancestor of b by our definition.
>
> (2) If a is a parent of some human being x, who is b's ancestor, then the hereditary properties of x will all descend, through a finite number of generations, to b; so once again a will be a parent of some human being x whose hereditary properties all belong to b—*i.e.* will be an Ancestor of b.

> Hence, if a is an ancestor of b, a is an Ancestor of b.
>
> B. Suppose a is an Ancestor of b. Then there is some human being, x, whose parent is a and whose hereditary properties all belong to b. But *the property of having* a *as an ancestor is itself a hereditary property*, since any human being, one of whose parents has a as an ancestor, himself has a as an ancestor; hence since this hereditary property belongs to x, and all hereditary properties of x belong to b, this property belongs to b—i.e. b has a as an ancestor.
>
> Hence, if a is an Ancestor of b, a is an ancestor of b.
>
> The italicized assertion in proof B may well make the reader suspect a vicious circle; if we are attempting a definition ancestorship, how can we without circularity, in a proof that the definition is adequate, bring in properties that are themselves defined in terms of ancestorship?
>
> (ANSCOMBE 1971, pp.126-8)

The concept number is thus applied to certain classes of classes of type "0" and "the successor of 0." "The successor of" relation was defined in terms of the relation "the immediate successor of." This account of successorship in terms of hereditary properties requires one to define the property of having a certain ancestral relationship as a hereditary relationship (ANSCOMBE 1971, pp.126-8).

According to the logicist definition of number, number is the equivalence class of all equinumerous classes. Under this definition the natural number one is the class of all classes of one object, that of two as the class of all duals, that of three of the class of all triples and that of some natural number *n* as the class of all *n*–tuples.

The description of number as equivalence class is incomplete. The ordering of numbers by means of a formal series is required to complete the theory. The formal series in the Frege-Russell definition is obtained by means of a successor relation. Though the numbers can be each defined by equivalence classes the relation between each of them is described by a successor relation in the Frege-Russell definition. Anscombe notes that Frege's *Grundlagen der Arithmetik* §79–80 contain his account of what it takes for one relation to be ancestral to another; Russell and Whitehead's account is located in *Principia Mathematica* vol. 1 Part II Section E (ANSCOMBE 1971, pp.126-8).

Frege describes the series in the following manner: commenting on the description of the axiom that "every Number except 0 follows in the series of natural numbers directly after a Number." Frege writes that in order to prove that a number follows after every number in a series that we must "produce a concept to which this number belongs." He describes this concept as "member of the series of natural number ending with *n*," which he requires to be defined. He defines this in the following terms, in his *Begriffsschrift*:

> The proposition,
>
> > 'if every object to which x stands in the relation to ϕ falls under the concept F, and if from the proposition that *d* falls under the concept F it follows universally, whatever *d* may be, that every object to which *d* stands in the relation ϕ falls under the concept F, then *y* falls under the same concept F, whatever F may be.'
>
> is to mean the same as
>
> > '*y* follows in the ϕ–series after *x*'
>
> and again the same as
>
> > '*x* comes in the ϕ–series before *y*' (FREGE 1884 [1953] §79, p.92[e]).

And in this manner Frege defines the number series. In §80 he comments on his definition. He notes that ϕ is not definite and is not specifically spatial or temporal in nature. He rejects the psychological definition of "shifting our attention" from one object to another. His definition, he states is, not of what is "subjectively possible but of what is objectively definite." It is a criterion for determining,

> in every case the question: Does it follow after? Wherever it can be put; and however much in particular cases we may be prevented by extraneous difficulties from actually reaching a decision, that is irrelevant to the fact itself (FREGE 1884 [1953] §80, p.92[e]).

In the next section I will discuss why Wittgenstein rejects this account of number. As we shall see it relates to his earlier rejections of the axioms of infinity and reducibility.

Wittgenstein rejects the logicist definition of number on the basis of his critique of the logicist definition of a formal series for containing a vicious circle (4.1273). How is this circular? According to Anscombe's description, the definition of successor is done in terms of the "immediate successor of." This account of successorship in terms of hereditary properties requires one to define the property of having a certain ancestral relationship as a hereditary relationship. How does one then describe a hereditary relationship without reference to a series? Wittgenstein rejects this method of defining a series. He constructs rather than defines a series. There must be a distinction between the formal (or internal), or the logical, and the external (or proper or accidental), or the empirical. The descriptions of the series provided by Frege and Russell do not make these distinctions and thus his criticism is that they are circular. This is related to his criticisms of type theory and the axioms of infinity and reducibility. Both involve the error of confusing the logical with the empirical. Wittgenstein's rejection of the Frege-Russell account of number is that it traps one in a vicious circle. But part of the reason for that is the manner of the definition is existential, *i.e.* it tries to describe the properties of number as if they are objects. While Wittgenstein's construction of a series shows how a series works, it displays its syntax.

The series described by Frege and Russell is a proper concept in the Wittgenstein's terms that is it is defined by a relation like any other proper concept. There is no clear distinction between the language describing the relation between the objects of the series and the series itself.[8] In Wittgenstein's system the relations between members of a series are

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[8] Although this is certainly not the case in Frege's description of the distinction between concept and object, it is when, as described in section 2.1 of this thesis, we apply the fifth axiom of Frege's system to his definition. The intervening stage of concept collapses into the object.

internal relations and thus cannot be represented in the object language but only shown. It is in this distinction that we can see the beginnings of Wittgenstein critique of the circularity. Once again it is a question of reference. If a hierarchical relation is used to define a hierarchical relation in the same language then the concept is circular. Unless there is a language distinction the description cannot work. Wittgenstein rejected the type distinction and the order distinction. As noted above it is in treating the description of the relation of successorship as an object that one runs into the circularity (ANSCOMBE 1971, pp.126-8). When what Wittgenstein calls formal relations are treated as proper relations, as if they were between two mundane objects, rather than between two logical constructs, we end up with trying to define a construction in terms of itself.

3 The 'Positive' Programme: The Operation and Natural Number

The 'Positive' programme of the *Tractatus* is centered on the theory of operations and the consequences of this view for mathematics and logic. Wittgenstein rejects the logicist definition of natural number and replaces it with an operational definition. The goal of the first half of this chapter is to show how and why he does this. The theory of types was unacceptable to Wittgenstein. The operational theory does not suffer from the problems Wittgenstein found with the Frege and Russell's attempts. There is only one "type" distinction in the operational theory, between operation and base. An operation cannot be the base of another operation according to Wittgenstein theory, rather two operations written next to one another simply mean that the result of one is the base of the next. By syntactically defining one part of the language as object and one part as process, Wittgenstein prevents the problems that occur when one considers functions in a set theoretic manner. This means that natural number must be conceived of in an entirely different manner. No longer can we talk of equivalence classes. Number must now be constructed out of the application of an operation to a base.

The distinction between these two ways of looking at natural number is tied up with the logic of the *Tractatus*. The first goal of this chapter is to sketch an understanding of the theory of operations. This will be accomplished by understanding the two applications of the theory in the *Tractatus* and show how they are related to each other and to the understanding of logic and natural number respectively.

The second goal of this chapter will be to attempt to show why Wittgenstein introduces the theory of operations; understanding not only how it works but why it does. Once this is presented three further criticisms of Russell that follow from Wittgenstein's theory will be discussed: concerning classes, mathematical propositions, and logical connectives. The first and most obvious thing that can be said of the theory of operations is that it is intensional, and in contrast, the set-theoretic understanding of natural number is clearly extensional. As we saw in part two, Wittgenstein's showing–saying distinction is about these two ways of seeing semantics. Von Wright's paper "Modal Logic in the *Tractatus*" describes intensional and extensional understandings of meaning and applies them to the showing saying distinction. Hintikka's efforts to understand the centrality of proposition 6 of the *Tractatus* expand on von Wright's work, and Sundholm's dissection of the place of the operation in the *Tractatus* will set the place for an analysis of how the operational theory works and thence how the operational definition of natural number is central to the logic of the *Tractatus*.

3.1 Wittgenstein's Theory of Operations

The theory of operations in the *Tractatus* is at the basis of both logic and arithmetic. Wittgenstein first presents the notion of an operation as that which generates the internal relation of a formal series (5.21–5.2523). Wittgenstein then uses the notion of truth operation to define the logical connectives (5.3, 5.501, 5.51, 6) which in turn define first order logic. He then uses the notion of operation to define natural numbers (6.02) and '+' and '×' (6.241) for arithmetic.

Wittgenstein's definition of natural number is conceptually similar to Church Numerals. The operation serves the same purpose as the lambda function. It is an equational definition of function without resort to a set–theoretic metalanguage. The operation, like the lambda function, defines a function—operation in Wittgenstein's terms—as an algorithm rather than as a relation between domain and co–domain and thus refers to not the result but to the workings of a function—or operation. This similarity between Wittgenstein's operation theory language definition of natural numbers and Church numerals will be explored in section 3.4 below.

This section will attempt to trace the development of the operation in the *Tractatus*. It will start with his conceptions of formal relation and formal properties, and his description of formal series. Then we shall see how the construction of that series differs from the Frege–Russell series, and finally how internal relations within such series and operations are connected. This will provide us with a good understanding of how operations and functions differ in the *Tractatus*' terminology. The distinction between will be defended as being of the same extensional/intensional distinction made above in the two views of modality. Furthermore, I shall show how Wittgenstein's definition of operation is similar to Frege's definition of function, and relate the necessity of a clear distinction between extensional and intensional conceptions of function and operation to the susceptibility of the former to Russell's paradox.

3.1.1 Formal Relations and the Operation

Wittgenstein's definition of natural number is most explicitly found in his discussion of the general form of the operation. To understand his definition one must not only see what he is doing in these passages, something that may commentators have failed to do, but also one must understand his motives for so doing. His motive for describing natural number in the manner that he does is intrinsically linked to both the goals of the *Tractatus* and his philosophy as a whole. To understand exactly what he is doing in these passages, which are comments on thesis 6, one should start by following the development of the *Tractatus* up to this point. I begin with Wittgenstein's description of formal relations and properties, then the formal series and his views on the nature of propositions. All of these are intrinsically linked with his definition of the operations, and thence the general form of the proposition.

Operations are used as a formal method for both the construction of propositions out of elementary propositions, and for the construction of natural numbers out of a base. The former creates propositions about the world; they are true or false depending on the truth–values of the elementary propositions. The construction of natural numbers and propositions are both done by the construction using operations of internally related series. Why this is done in this matter is explained through several key *Tractatus* doctrines.

The rejection of extensional semantics follows from his saying–showing distinction. Wittgenstein made a distinction between internal relations and external relations and it is here that we see what can be called a rejection of semantics. If there is a non–reflexive relation between objects, then there is the problem of which side of the relation one is on. In *Theory of Knowledge* Russell deals with this problem by developing the theory of logical forms in which one has direct acquaintance with logical forms through what Russell calls 'logical data' (*cf.* RUSSELL 1984, pp. 97-118). The problem with such a theory is that the 'logical data' is itself a relation and thus ends up in the uncomfortable situation of defining relations in terms of themselves—much as Wittgenstein claims Russell and Frege do in their definition of a series. The saying–showing distinction was Wittgenstein's method of stopping this sort of circularity. In Wittgenstein's system, one is acquainted with the semantics by apprehending the syntax; *i.e.* by understanding the use of the notation, through having it shown unto one. Wittgenstein's point is to make perspicacious that which is shown when one proposition or natural number is converted into another, specifically all that changes is the structure of the complex, not the basic elements of it. As Robert Tully puts it, the *Tractatus,* "conceived logical forms and relations entirely in terms of structure rather than as objects of acquaintance" (TULLY 1994).

It is in the comments on 4.12 that Wittgenstein develops the saying–showing distinction, which will define his use of the operation and the thus both his understanding of logical and mathematical construction. In 4.12 he remarks that:

> In order to be able to represent logical form, we should have to be able to station ourselves with propositions somewhere outside logic, that is to say outside the world (4.12).

It is from this observation that he rejects the view that a logical form can be analyzed beyond a certain point. More accurately, he believes that it can be shown, in notation, but no more. That is not to say it may not be better shown in one notation than another, but instead that replacing one for another produces solely a restatement of what has been said already.

In 4.121 he remarks, "propositions cannot represent logical form." This is a clear rejection of any possibility of an extensional semantics to describe the form of a logical language.[9] But first he must clarify what formal properties and relations are (4.122-4.123), and since he claims such things are shown he must elaborate on how they are expressed (4.124-4.126). Finally he must deal with these consequences and it shapes his view of what a variable is, and leads him again in 4.1273 to reject the Frege–Russell view of mathematics (BLACK 1964, p. 201).

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[9] This is not to say that there are no extensional semantics in the *Tractatus*. Quite the opposite; sentences that deal with objects in the world have extensions, and the existence or non-existence of states of affairs extend to truth and falsehood. Rather my point is that the semantics of the *Tractatus* are profoundly intensional. Those which explain logical form and mathematics—languages which do not directly extend to objects in the world—are entirely so.

3.1.2 Formal Properties and Relations (4.122-4.123)

In 4.122 and 4.123 Wittgenstein discusses formal properties and formal relations. Objects and states of affairs have formal properties; facts have structural properties, or internal properties. Black notes that "formal" is a synonym for "internal" (BLACK 1964, p. 195). The distinction to be made is between formal properties and formal relations and genuine or external relations. Black asserts that Wittgenstein would be ready to call them pseudo-relations noting his use of the term pseudo-concept later (BLACK 1964, p. 195). This agrees well with what Wittgenstein writes in 4.122:

> It is impossible, however, to assert by means of propositions that such internal properties and relations obtain: rather, this makes itself manifest in the propositions that represent the relevant states of affairs and are concerned with the relevant objects.

That is, formal properties, and internal relations, are what is shown and not said. They are "manifest." These are the features for which no foundation that is not circular can be made.

3.1.3 Formal Series (4.124-4.126)

In 4.124 Wittgenstein writes that an internal property is not "expressed by means of a proposition" but "expresses itself." Furthermore, he notes that asserting presence or absence of a formal property is nonsensical. It is clearly not the having of formal properties that is nonsensical, but the asserting of them. They are part of the structure of the language and using language to define them is circular. This is further developed in 4.1241 where he notes that distinguishing formal properties is not possible by saying, or ascribing properties to forms. This implies that we can only know which has what property by showing, *i.e. by demonstrating or* giving examples as in a proof.

In 4.125 Wittgenstein asserts that "the existence of an internal relation expresses itself in language by means of an internal relation between the propositions representing them." The internal relation's existence is apparent when certain propositions are considered. So, like internal properties, they express themselves and can not be asserted by propositions. Rather the show themselves.

The introduction of internal relations is a rejection of several of Russell's views about logical forms and the ancestor relation. The internal relation is the thing that relates propositions, and internal properties are what define the form of complexes. Suppose two objects, *e.g.* a name and an external relation, together comprise a complex. The form of this complex is given structure by what Russell called logical form. Objects include relations and properties for it is between these that there are the internal relations. Relations are, as stated in 2.03, "like links in a chain." This metaphor is expanded and explained in 2.031, 2.032 and 2.033, where he makes clear that such a chain indicated a "determinate relation" (2.031), and that the "determinate way" objects fit together is the structure (2.032) and that: "Form is the possibility of structure" (2.033). This account does not make sense without such internal properties. In 4.1252 the internal relation is developed further. It is the internal series that orders the "series of forms," the *Formenreihe*, or as Black points out simply "series" in 5.232 (BLACK 1964, p.198). And it is in 5.232 where we learn, as Black notes that "an internal relation uniquely determines the operation and *vice versa*"(BLACK 1964, p. 260). Internal relations thus have an equivalent operation which orders a series. This will be dealt in detail later. The second paragraph of 4.1252 states that:

> 4.1252 (2) The order of the natural number-series is not governed by an external relation but by an internal relation.

So we know that a natural number series, or series of forms, is ordered by an internal relation determined by an operation. Thus a natural number series is not determined by a function, which can define external relations, but by the operation (*cf.* 5.501 for the ways of describing propositions). Furthermore, Wittgenstein will show us how he uses the operation $\Omega\text{‘}x$ to generate the natural number series (6.02). A series of forms once again draws us back to the metaphor of "links in a chain." The forms are the possibility of structure, and structure is the relation joining the links in the chain. It is clear that that link is an internal relation as we discussed above. A series is a group of propositions linked into a longer proposition, which is expanded by an operation generating internal relations. In the third paragraph of 4.1252 we have a description of the successor relation:

> The same is true of the series of propositions "*aRb*",
> "$(\exists x) : aRx \,.\, xRb$",
> "$(\exists x,y) : aRx \,.\, xRy \,.\, yRb$", and so forth.
> (If *b* stands in one of these relations to *a*, I call *b* a successor of *a*.)

As we noted in part two that the formal series depicted here is, in Russell's terms, the ancestor relation. Sundholm argues that the Wittgenstein series is closer to the Fregean series than the Russellian series (*cf.* SUNDHOLM 1992, p.59). There is a subtle difference in the two. One is a more intensional and the other a more extensional understanding of the function that creates the formal series and the relation that separates the objects in the formal series. Frege, Sundholm notes, has in his account of the formal series a relation that creates the series, whereas in Russell there is no distinction between the relation and the series that it forms:

> Originally, in thesis 4.1252, a series of forms is explained as a series ordered by an *internal* relation. This explanation permits two readings: one Russellian and one Fregean. The former is, *prima facie*, the natural one: the internal relation itself is a strict ordering of the terms in the series, that is, a series also in the terminology of the *Principia Mathematica*. The Fregean option, on the other hand, renders 'is ordered by,' as *gets its order from*. Thus the sequence of terms itself is a PM–series, but the internal relation is not an ordering

> relation for the series; it only generates such an ordering (SUNDHOLM 1992, p.58).

Sundholm argues that this is exactly the characterization that we should have of Wittgenstein's series. The Fregean reading, Sundholm states, is that a "series of forms is determined by laying down its first term and a general operation producing the next term from its predecessor" (SUNDHOLM 1992, p.58). The evidence Sundholm gives is in two parts: first, the relation generated by the operation is a many–one relation and thus, unlike a PM–series, is not transitive (SUNDHOLM 1992, p.59); and second, from 5.232 where Wittgenstein writes:

> The internal relation by which a series is ordered is equivalent to the operation that produces one term from another.

Sundholm continues that though, "ancestral Ω^* of Ω is a PM–series with the series S as its field, however, and this can be taken as a precise expression of the sense in which the internal relation Ω orders the terms S," (SUNDHOLM 1992, p. 60) the preference for the Fregean reading is established in Frege's description of the ancestral in the *Bregriffschrift*. Sundholm points out that Frege's description of the two place propositional function in the *Begriffsschrift* is very much the same as Wittgenstein's description of the operation at 5.232 (SUNDHOLM 1992, p.52).

Broadly speaking, in a realist account of mathematics, such as Russell's, the formal description of an entity, rather than the construction, establishes its existence. That is, one can specify a function $f(x)$ such that $f(x) = 1$ if x is a prime number, and $f(x) = 0$ if x is a composite number. In such a case the function is well defined as long as the conception of a prime number and a composite number are well defined. The function is not a construction, but rather a description of two sets. Most constructivists would not call this a function. In a constructive description of mathematics one must define how the result of a function is constructed. In the above $f(x)$ one would not only have to define the function, but also show how one determines the primacy of a number; that is provide a procedure.[10]

In this classical realist conception of mathematics the function is the object, *i.e.* there is not a clear distinction between the two. The formal description rather than calculation establishes the series under such a view; *i.e.* our realist defines the relation that describes the series, rather than a describing how one may construct a series.

Now we can clearly see the difference in Wittgenstein's account from Russell's. Wittgenstein's view sees the series as a whole, and the relations between its parts internal. The creation of the series is performed by an operation; thus we might say that Frege's series is half way there. Frege's separation between concept and object can then be seen as an attempt to allow for both intensional and extensional viewpoints. The problem comes in the allowing of the conversion. The conception of a function as a procedure emphasises its intensional element; it focuses on what a function does, not on what its extension is. Frege calls the extension of a function its course of values, as we saw in part two. However, as we noted above in section 2.1, the fifth axiom of Frege's *Grundgesetze der Arithmetik* allows one to define a function for any course of values. Frege's system, while drawing the distinction between the intensional and extensional conceptions of function, equates them, and thus falls into the Russell's paradox. Wittgenstein's strict distinction between the operation and the function is a method of avoiding this problem. The operation is still defined as a procedure, but all its values must be constructed. This means, of course, that

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[10] In this sense the constructive view of mathematics is from the bottom up and the realist from the top down.

Wittgenstein's description of a series of forms is not open to the charge of circularity, as was the Frege-Russell series as we saw in section 2.4 of this thesis.

Returning to the third paragraph of 4.1252 brings up our first look at a series that Wittgenstein will again examine at 4.1273. But first Wittgenstein must explain formal concepts, which he does in 4.126, for these formal concepts will explain how a proposition can be expanded into a series through the use of variables.

Wittgenstein himself notes that the reason he brings in the idea of a formal concept[11] is to differentiate it from the idea of concepts proper—a confusion which he states, "pervades the whole of traditional logic." Black notes that by traditional logic Wittgenstein means to include Frege and Russell (BLACK 1964, p.199), and continues that Wittgenstein was likely drawing our attention to what Frege said about concepts; *i.e.* "A concept is a function whose value is always a truth-value"(FREGE 1891 [1970], p.30). Not so for formal concepts as, "formal concepts cannot, in fact, be represented by means of a function, as concepts proper can" (4.126 [4]). They are shown by the sign for the object, as Wittgenstein writes parenthetically, "a name shows that it signifies an object, a sign for a natural number that it signifies a natural number, *etc*" (4.126 [3]).

Anscombe notes a similarity of the view Wittgenstein has of formal concepts and Frege's view that a concept cannot itself express the fact that it is a concept since, as an expression, a concept can only occur in the predicate position (ANSCOMBE 1971, p.122). This Anscombe notes is the same as the doctrine of 4.126: "when something falls under a formal concept as one of its objects, this cannot be expressed by means of a proposition. Instead it is shown in the very sign for this object." This is related to Frege's idea of an unsaturated

[11] Black also indicates that they are called pseudo-concepts in 4.1272.

versus a saturated part of a proposition (FREGE 1891 [1970], p. 31). A property has a certain natural number of argument spaces which are filled by concepts. The formal concept, for Wittgenstein, cannot be expressed by a predicate. Wittgenstein, Anscombe writes, is much more inclusive when talking about formal concepts than for example Carnap. For Wittgenstein, 'name', 'predicate', 'propositional', and 'relational expression' are formal concepts, not just concepts such as 'concept', 'function', 'object', 'natural number', 'fact' and 'complex' as in the case of Carnap. Continuing with 4.126, as noted above, formal properties are defined by operations not by functions, and formal properties are shown by the appropriate operations as Wittgenstein writes in 4.126:

> 4.126 We can now talk about formal concepts, in the same sense that we speak of formal properties. (I introduce this expression in order to exhibit the source of the confusion between formal concepts and concepts proper, which pervades the whole of traditional logic.) When something falls under a formal concept as one of its objects, this cannot be expressed by means of a proposition. Instead it is shown in the very sign for this object. (A name shows that it signifies an object, a sign for a number that it signifies a number, etc.) Formal concepts cannot, in fact, be represented by means of a function, as concepts proper can. For their characteristics, formal properties, are not expressed by means of functions. The expression for a formal property is a feature of certain symbols. So the sign for the characteristics of a formal concept is a distinctive feature of all symbols whose meanings fall under the concept. So the expression for a formal concept is a propositional variable in which this distinctive feature alone is constant.

We discover what Wittgenstein means by a "propositional variable" when we look ahead to 6.022:

> 6.022 The concept of natural number is simply what is common to all natural numbers, the general form of a natural number. The concept of natural number is the variable natural number. And the concept of numerical equality is the general form of all particular cases of numerical equality.

We must understand that the idea of the variable is different than that which is common now, *e.g.* the x in the function $f(x)$. Let us look at what Wittgenstein has to say about the general form of a proposition.

> 4.5 … The general form of a proposition is: This is how things stand.
> 4.53 The general propositional form is a variable.
> 6 The general form of a truth-function is $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$. This is the general form of a proposition.

So a propositional variable is the general form of a truth function $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$:

> 6.01 Therefore the general form of an operation $\Omega'(\bar{\eta})$ is $[\bar{\xi}\, N(\bar{\xi})]\, '(\bar{\eta})$ ($= [\bar{\eta}, \bar{\xi}, N(\bar{\xi})]$). This is the most general form of transition from one proposition to another.

And so we see that an operation transforms one proposition or one set of propositions into another. Thus when Wittgenstein writes in 4.126 [8] "So the expression for a formal concept is a propositional variable in which this distinctive feature alone is constant," he is including the variable in the proposition. It is, by 4.53, the form of the proposition.

In fact in the *Tractatus*' doctrine of the difference between the 'formal,' 'pseudo' or 'internal;' and the 'external' or 'proper' property, concept or relation we see reflections of Frege distinction between concept and object. Wittgenstein's philosophy, the offspring of Frege and Russell's logicism, deals with the paradox discovered by Russell by declaring their project impossible. Wittgenstein identified the tension between the intensional and extensional views of logic, and expressed it as the distinction between the structure and objects of propositions. This is the problem described in his distinction between showing and saying, formal or internal, and proper or external. This view of Russell's paradox means that if you distinguish between extension, in Frege's terms "course of values" and function, you avoid the problem. As seen above the problem only comes in when you cross the line between the two ways of talking, that is, when one speaks extensionally about intensional concepts. As we saw, Frege lets one do in Axiom 5.

In Russell's Paradox, classes, and by extension classes of classes, are treated as objects. As such they are treated extensionally. From Wittgenstein's viewpoint, one cannot talk of classes of classes. To do so would be speaking about the formal aspects of classes. Classes themselves are groupings of objects and thus must be seen as some form of objects, in relation to each other. In Wittgenstein's logical atomism relations are atomic (*cf.* HINTIKKA 1986, pp. 30-33) so a class of objects is a complex. A class is then a complex created by relations and objects—in Wittgenstein's system likely created by some manner of operation—a sorting operation perhaps. This same relation would not hold between the class created by the operation without defining the operation in language.

3.1.4 The Construction of a Formal Series (4.1273)

In 4.1273 Wittgenstein discusses how the general term of a formal series is a formal concept and is not thus vulnerable to Frege's complaint about the standard construction of a series. Frege's problem with the standard construction of the general series—*i.e.* why Frege refuses to accept, "0+1+1+1 ... and so on"—is that he sees a psychologistic element in the "and so on" (FREGE 1884 [1953] § 80, p.92^{e}-94^{e}). Wittgenstein criticises this in 4.1273 where he says that the "and so on" is a formal process. The psychologistic aspect is gone for Wittgenstein. Such statements end up being meta-linguistic statements that describe how the mathematics works. They do not describe a state of mind, or an object in the world, but a formal process. Such expressions say nothing about objects in the world, they do not refer; rather they specify a set of rules. As Wittgenstein notes:

> 4.1273 If we want to express in conceptual notation the general proposition, 'b is a successor of a', then we require an expression for the general term of the series of forms: *aRb*, $(\exists x)$: *aRx* . *xRb*, $(\exists x,y)$: *aRx* . *xRy* . *yRb*,... In order to express the general term of a series of forms, we must use a variable, because the concept 'term of that series of forms' is a formal concept. (This is what Frege and Russell overlooked: consequently the way in which they want to express general propositions like the one above is incorrect; it contains a vicious circle.)

> We can determine the general term of a series of forms by giving its first term and the general form of the operation that produces the next term out of the proposition that precedes it.

Wittgenstein's operation thus formally presents the "and so on," and thus avoids Frege's difficulty. Wittgenstein avoids the psychologistic element by formalising the procedural view of natural number and calling it the operation.

3.1.5 The Essence of the Proposition (4.5–5.01)

For Wittgenstein, the general form of the proposition is the 'essence' of the proposition. The essence is that which is not accidental; *i.e.* the logical features of the proposition as opposed to the empirical features (*cf.* 6.3 on logic and accidental). The essence, as Black paraphrases is, "its capacity to say something about how matters are in reality" (p. 263). He answers that the most 'general form' is the character of being a truth-function of elementary propositions (*cf.* BLACK 1964, p. 236). Black also notes that Wittgenstein is not truly successful here and attempts again at 5.47-5.472 to express the general form of the proposition. Wittgenstein writes:

> 4.5 It now seems possible to give the most general propositional form: that is, to give a description of the propositions of any sign-language whatsoever in such a way that every possible sense can be expressed by a symbol satisfying the description, and every symbol satisfying the description can express a sense, provided that the meanings of the names are suitably chosen.
> It is clear that only what is essential to the most general propositional form may be included in its description–for otherwise it would not be the most general form.
> The existence of a general propositional form is proved by the fact that there cannot be a proposition whose form could not have been foreseen (i.e. constructed). The general form of a proposition is: This is how things stand.

The proposition's essence in "any sign-language whatsoever" is "this is how things stand." But Wittgenstein indicates that the nature of any proposition that we can conceive of is that it is "constructed." Propositions are thus constructed following a general rule. They do not exist as objects do, but as constructions. Wittgenstein then describes from what they are constructed:

> 4.51 Suppose that I am given all elementary propositions: then I can simply ask what propositions I can construct out of them. And there I have all propositions, and that fixes their limits.

Elementary propositions are that out of which complex propositions are constructed. They are the base upon which operations operate. Operations place elementary propositions in internal relation to one another. Thus having all the elementary propositions means that one has all possible propositions; they are just not placed into their positions. Wittgenstein starts 4.52:

> Propositions comprise all that follows from the totality of all elementary propositions (and, of course, from its being the totality of them all) …

Once the elementary propositions are placed into all the possible configurations, through the application of an operation, one has all that follows from the elementary propositions. Thus Wittgenstein continues:

> 4.52 (Thus, in a certain sense, it could be said that all propositions were generalizations of elementary propositions.)

Generalizations of elementary propositions consist solely of elementary propositions in combination with one another. This obviously does not mean simply universal or existential quantifiers applied to elementary propositions. Rather this brings us to the next point:

> 4.53 The general propositional form is a variable.

Black notes that it is, strictly speaking, the form that is presented by a variable (Black 1964, p. 237). For Wittgenstein "$[\bar{p}, \bar{\xi}, N(\bar{\xi})]$" is the variable that is the general form of the proposition. Black is correct that the variable presents the general form, because it is shown and not said. That is there is no content to the variable, just formal structure. It seems that

Wittgenstein is breaking his own prohibition here. Wittgenstein now describes how propositions relate to truth functions:

> 5 A proposition is a truth-function of elementary propositions.
> (An elementary proposition is a truth-function of itself.)

That a complex proposition is a truth function of an elementary proposition seems obvious, as they are constructed out of them. However, Wittgenstein continues that the elementary propositions are truth-functions of themselves. Commenting on this Wittgenstein writes:

> 5.01 Elementary propositions are the truth-arguments of propositions.

This is because whatever elementary propositions are, and Wittgenstein is not clear on this, they are atomic. Complex propositions are constructions made up of elementary propositions. In 5.47 once again Wittgenstein returns to the general form of the proposition:

> 5.47 It is clear that whatever we can say in advance about the form of all propositions, we must be able to say all at once. An elementary proposition really contains all logical operations in itself. For '*fa*' says the same thing as '$(\exists x) . fx . x = a$'
>
> Wherever there is compositeness, argument and function are present, and where these are present, we already have all the logical constants.
>
> One could say that the sole logical constant was what all propositions, by their very nature, had in common with one another.
>
> But that is the general propositional form.

The general form of the proposition is thus the one logical constant that is common to all propositions. This general form is expressed by means of an operation. Wittgenstein reduces what he says in 5.47 in 5.471 to:

> 5.471 The general propositional form is the essence of a proposition.

The operational nature of the proposition then, *i.e.* that variable that defines the manipulation of elementary propositions into complex structures, is the essence of what it is to be a proposition. In 5.472 Wittgenstein continues the development of his operational concept of the proposition, where he states that:

> 5.472 The description of the most general propositional form is the description of the one and only general primitive sign in logic.

Thus all of logic is to be developed from one primitive operational sign. The whole of logic is seen through the algorithmic nature of the operational definition of logic. The details of how this works, specifically how certain operations generate series of propositions, we will discuss below.

3.1.6 Operations and Internal Relations (5.2-5.254)

As noted above, Sundholm (SUNDHOLM 1992, pp.58-59) argues for a Fregean, rather that the Russellian, reading of the formal series. That is, the series "gets its order from" the operation. In the Russellian version of the series, the series is the relation, whereas in part III of the *Begriffsschrift* Frege writes:

> "Δ is the result of an application of the procedure (*Verfahrens*) f to Γ" by … "Δ bears the relation (*Beziehung*) f to Γ "… these expressions are to be taken as equivalent." (FREGE 1879 [in VAN HEIJENOORT (1967)] p.56)

Sundholm points out that this is clearly the source of 5.232, if we take *Verfahrens* and being operation and *Beziehung* as internal relation.

> 5.232 The internal relation by which a series is ordered is equivalent to the operation that produces one term from another.

The equivalence between the internal relation and the operation described in 5.232 comes from the fact that the operation generates the series. For this we can see 5.23, 5.251:

> 5.23 The operation is what has to be done to the one proposition in order to make the other out of it.

> 5.251 A function cannot be its own argument, whereas an operation can take one of its own results as its base.

In type theory there have to be functions of various levels, so that a function can have a higher level function applied to it. Wittgenstein's operation is best conceived of as an algorithm. Thus when one applies one operation to another operation what happens is the result of the first operation is considered the base of the second. An operation is thus an action, or group of actions, performed on a base the output of which is a result. It is from 5.21 to 5.234 that there is talk of the result of the operational deed:

> 5.21 In order to give prominence to these internal relations we can adopt the following mode of expression: we can represent a proposition as the result of an operation that produces it out of other propositions (which are the bases of the operation).
> 5.22 An operation is the expression of a relation between the structures of its result and of its bases.

Wittgenstein's operation, like Frege's definition of function, is intensional, since it defines an operation as a procedure rather than as an extension to a set or class. The extensional definition—that a function is a type of relation between a domain and co-domain—is the one Russell's amounts to and is now the standard definition.[12] It is also that not of Frege's philosophy, but of his logic in the *Grundgesetze*. The courses–of–values and the function are equated, and so the functions are no longer constructed but defined. The function becomes equated with the course–of–values rather than being a method for generating them, and thus we end up with an extensional definition of function.

Black notes the importance of the operation to Wittgenstein, but does not seem to understand the importance of it himself. He is confused about the differences between the operation and the function, and finds Wittgenstein's contrast "hard to accept" (BLACK 1964,

[12] One could also say that relations are defined by functions, it really only matters which one is seen as primary.

p. 258). He lists the distinctive features of operations as follows, and comments that none of them seem "decisive":

i. operations are shown by variables 5.24 [1]
ii. they indicate differences between forms not the forms themselves 5.241
iii. their occurrence is not part of a proposition's sense 5.25 [1]
iv. operations unlike functions are self applicable 5.242 [1]
v. operations may cancel one another 5.253

Black makes at least one error in this list. In (iv) he should have 2.51 as the source, and it is the result of an operation to which that operation may be applied not the operation itself. Black seems not to understand that what Wittgenstein called an operation is often called a function in mathematics, but that there are competing versions of the definition of function and Wittgenstein has chosen to give each definition a different name. The set or class–theoretic he calls function, and the algorithmic he calls operation.

Let us look at how Wittgenstein develops his views. Propositional signs possess a structure, which is internally related to other structures and thus other propositional signs. The operation is thus carried out on the propositional signs. The operations act on signs:

> The concept of the operation is quite generally that according to which signs can be constructed according to a rule (NB 22.11.16).

Note that the operation is not conceived of here as something that is equivalent with that which the signs represent. Rather it describes the process for the manipulation of signs. Wittgenstein writes that:

> 5.2 The structures of propositions stand in internal relations to one another.

What does "stand in internal relations to one another" mean? The propositions are generated, according to 5.21, by operations:

> 5.21 In order to give prominence to these internal relations we can adopt the following mode of expression: we can represent a proposition as the result of an operation that produces it out of other propositions (which are the bases of the operation)

Complex propositions, then, are the final expression of a series of operations applied to other propositions. The series contains the internal relation, generated by the operations:

> 5.22 An operation is the expression of a relation between the structures of its result and of its bases.

The series is a group of internally related propositions each, except the base, the result of an operation.

> 5.23 The operation is what has to be done to the one proposition in order to make the other out of it.

The operation is thus something done to a proposition. The result is another proposition related to the previous one by an internal relation. Let us look at these points made by Black, but let us take seriously the distinction Wittgenstein is making in his definition of the operation. The first point Black notes is "(i) operations are shown by variables 5.24 [1]." Wittgenstein writes:

> 5.24 An operation manifests itself in a variable; it shows how we can get from one form of proposition to another. It gives expression to the difference between the forms. (And what the bases of an operation and its result have in common is just the bases themselves.)

Wittgenstein writes that the variable shows how one "gets from one form of proposition to another." Unlike a function, an operation acts on propositions, and thus the propositions in no way are the constituents of the operation. A definition of a function as domain and co-domain implies that it consists of theses sets; an operation is shown by a variable is thus not a thing but an action.

> 5.241 An operation is not the mark of a form, but only of a difference between forms.

The second of Black's points about operations is: "(ii) they indicate differences between forms not the forms themselves 5.241." This reasserts the previous point. The third point explains that, "(iii) their occurrence is not part of a proposition's sense 5.25 [1]" Being part of the sense of a proposition means that an operation's action on a proposition is formal. That is the operation does not modify the reference of the proposition, it changes propositions, but only by restructuring them. Propositions can be joined by an operation, but the propositions still refer to the same states of affairs. The propositions themselves hold all sense:

> 5.25 The occurrence of an operation does not characterize the sense of a proposition. Indeed, no statement is made by an operation, but only by its result, and this depends on the bases of the operation. (Operations and functions must not be confused with each other.)

The above leads to the forth point, *i.e.* "(iv) operations unlike functions are self applicable 5.242 [1]," which is to say that since operations only modify their bases they can be concatenated in a manner in which a conception of function as extension cannot. As Wittgenstein writes:

> 5.251 A function cannot be its own argument, whereas an operation can take one of its own results as its base.

The function, defined as a relation between two sets, is not able to apply to itself. That is because of the problems inherent in type hierarchy, as we saw in section 2.1. The last of Black's notes on the operation is "(v) operations may cancel one another 5.253." Wittgenstein write in the source propositions that:

> 5.253 One operation can counteract the effect of another. Operations can cancel one another.

That is for each operation an inverse operation can be defined, and this inverse operation can applied to the result of the application of the first to a base, the result of which would be the base. This is not distinctive of operations, but it does indicate that operations are one–to–one, rather than one–to–many. If a operation has an inverse it must always produce the same result. A one–to–one operation which converts a base uniformly into another base has an inverse, which converts the result back to the base. A one–to–many operation does readily invert from results to base, for the inverse of a one–to–many operation still produces many results from one base.

3.1.7 The Operation and the Truth Function (6-6.01)

Having discussed the nature of the operation and internal relations, let us turn to the example Wittgenstein gives in his text. The general form of the truth function is thesis 6. The general form expresses the concept that all truth functions can be constructed using the operation given in 6.001. It has been contested by Fogelin that this is not so for propositions with multiple generalities. Geach and Soames respond, though both of their responses require a modification of the strict wording of the *Tractatus*. However, if the sets represented by $\bar{\xi}$ are finite Cheung has shown that the *Tractatus* system can construct such propositions. These arguments will be presented in detail in section 3.2. Before discussing the debate it will profit us by looking at what Wittgenstein writes:

> 6 The general form of a truth-function is $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$. This is the general form of a proposition.
>
> 6.001 What this says is just that every proposition is a result of successive applications to elementary propositions of the operation $N(\bar{\xi})$

The operation presented in 6.001 is the operation of joint negation and works as follows: the joint negation of a single statement provides the negation of that statement, whereas the joint negation of a set of statements provides the negation of each joined by a conjunction:

$N(\bar{\xi})$ where $\bar{\xi}$: $\{p\} = {\sim}p$; $N(\bar{\xi})$ where $\bar{\xi}$: $\{p, q, r\} = {\sim}p \wedge {\sim}q \wedge {\sim}r$.

According to Wittgenstein all propositions may be generated by the action of the operation $N'(\bar{\xi})$:

> 6.002 If we are given the general form according to which propositions are constructed, then with it we are also given the general form according to

> which one proposition can be generated out of another by means of an operation.

At 6.01 the following definition of the general form of the operation is given:

> 6.01 Therefore the general form of an operation $\Omega'(\bar{\xi})$ is $[\bar{\xi}, N(\bar{\xi})]$ ' $(\bar{\xi})$ (= $[\bar{\xi}, \bar{\xi}, N(\bar{\xi})]$). This is the most general form of transition from one proposition to another.

It should be noted that the '$\Omega'(\bar{\xi})$' here is a variable for operations in general. This becomes obvious when the operation '$[x, \xi, \Omega' \xi]$' is considered in 6.02. Frascolla rejects Blacks interpretation of 6.01 as being absurd. The meaning that Black ascribes to Ω is that $\Omega'x$ stands for the set of truth functions of H, where H is a non-contradictory non-tautological proposition that is the sole member of the set represented by x, which, in turn, is the set consisting of H and ~H. Applying the operation again does nothing and thus:

$$\Omega'x = \Omega'\Omega'x = \Omega'\Omega'\Omega'x = \Omega'\Omega'\Omega'\Omega'x$$

This is not a satisfactory result. And Frascolla is not surprised when Black writes, "Wittgenstein's idea is hard to grasp." Under such an interpretation it is no wonder Black is confused (FRASCOLLA 1994, p.3). Frascolla's critique of Black will be dealt with in section 3.3.

Two points can be emphasised: first that joint negation is an example of an operation, and that secondly it is not the only operation. Wittgenstein argues that joint negation is enough to generate all propositions from elementary ones. Natural number is the repeated application of any operations. These are two separate logical methods. Turning to section 3.2 we will discuss the Fogelin debate about the "central error in the *Tractatus*"—*i.e.* the problem of propositions of multiple generality—before we draw our attention to the use of the operation in Wittgenstein's definition of arithmetic.

3.2 Fogelin's Critique and the Debate

In thesis 6 Wittgenstein gives the procedure by which all propositions might be constructed from elementary propositions: $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$ where '$\bar{p}$' is all atomic propositions, '$\bar{\xi}$' a selection of propositions including possibly elementary and constructed propositions, and '$N(\bar{\xi})$' the operation of construction. The operation N can be defined inductively as follows:

$N(\bar{\xi})$ where $\bar{\xi}$: $\{p\} = \sim p$; $N(\bar{\xi})$ where $\bar{\xi}$: $\{p, q, r \ldots\} = \sim p \wedge \sim q \wedge \sim r \ldots$

In *Wittgenstein* (First Edition), in the chapter called "The Fundamental error in the Logic of the *Tractatus*," Fogelin writes that "It seems that every proposition that can be constructed using the recipe in proposition 6 will admit of a decision procedure." Furthermore Fogelin notes that Church shows us that such decision procedures cannot be found to encompass logics of multiply generality for first order predicate logic, that is for propositions that contain more than one quantifier.

So while:

- $(\forall x)(\forall y)f(x)(y)$;$(\exists x)(\exists y)f(x)(y)$; $(\forall x)(\forall y)\sim f(x)(y)$;$(\exists x)(\exists y)\sim f(x)(y)$;

can be generated:

- $(\exists x)(\forall y)f(x)(y)$; $(\forall x)(\exists y)f(x)(y)$; $(\exists x)(\forall y)\sim f(x)(y)$; $(\forall x)(\exists y)\sim f(x)(y)$

cannot be constructed.

After making this claim Fogelin introduces what he calls Stalnaker's method, as a possible solution. You start with '$f(a)(y)$, $f(b)(y)$, $f(c)(y)$, $f(d)(y)$...' and apply N to each one. Thus we get '$\sim(\exists y)f(a)(y)$, $\sim(\exists y)f(b)(y)$, $(\exists y)f(c)(y)$, $(\exists y)f(d)(y)$...' apply N to this set and it gives us $(\forall x)(\exists y)f(x)(y)$. There are two possibilities: (1) The world has finitely many objects, or (2) There is nothing wrong with completing a task of infinitely many discrete steps (FOGELIN 1976 pp.78-79).

Fogelin now makes his objection to these two options. In the first case, if there were finitely many objects in the world, enumeration would be impractical but not impossible, *i.e.* "logically difficult." But Fogelin points out that Wittgenstein never mentions this. In fact Fogelin notes, in 4.463, that Wittgenstein uses the phrase "the whole infinite logical space." Fogelin further points out that in a finite world a logical natural number would be a special natural number, and that all would be associated with a definite truth function. Thus, he dismisses the first option.

But Fogelin does not deal with the option that Cheung will later raise. That is that while logical space can be infinite, logical space consists not of states of affairs but rather of propositions. In the same way an infinite natural number of sentences can be created from a finite natural number of letters; an infinite natural number of propositions can be created from a finite set of elementary propositions

Fogelin's second objection is that an infinite natural number of steps is not a logical problem. This he refers to as a "bold position," made ever bolder by the fact that Wittgenstein himself writes in 5.32 that all truth functions are the result of "successive applications to elementary propositions of a finite natural number of truth operations," not different kinds of operation, *i.e.* logical product, but rather a specific one the N operator.

Thus mixed multiple quantifier quantification can only be done if there are only a finite natural number of objects and an infinite application of the N operation. Fogelin asks if this is a "hitch" or a "flaw." Strictly Fogelin notes that not even $(\forall x)f(x)$ can be finitely constructed, as $\sim f(x)$ becomes '$\sim f(a) \sim f(b) \sim f(c) \sim f(d) \sim f(e)$..'. and each of these has to be constructed unless denials of elementary propositions are elementary. But Fogelin notes negative elementary propositions are not a problem, though it seems to offend the letter of the *Tractatus*.[13]

The operation N however "lets us down," and the logical product is needed to construct multiple generalities. N was given a privileged position because it was believed that "all propositions are the result of truth operations on elementary propositions" (5.3), which, according to Fogelin, should be moved up over 6 in prominence. But the truth operation is a logical product and logical sum for: $(\forall x)f(x)$and $(\exists x)f(x)$ unless the Stalnater's method is used, which Fogelin has already rejected. Fogelin notes another option, that he doesn't think bares much investigation, which is the objection that:

$(\exists x)(\forall y)f(x)(y)$; $(\forall x)(\exists y)f(x)(y)$; $(\exists x)(\forall y){\sim}f(x)(y)$; and $(\forall x)(\exists y){\sim}f(x)(y)$

are all things that can be shown and not said.

The thing he misses is that they are all easily said but only if each is bound over a finite set. Infinite sets can only be formal objects, *e.g.* natural numbers, which is why you cannot make propositions about them. They are things that can only be shown. This is the reason Wittgenstein rejects so much of mathematics, as well as formal logic, and is the key point behind the *Tractatus*. The only infinite sets are those of mathematics, and these are not things about which on can make propositions.

Soames writes that his goal is to identify the "logical hole" in the *Tractatus* and "show how it can be filled without disturbing central Tractarian doctrines" (SOAMES 1983, p.580). Soames states the *Tractatus* is radically defective, but he believes it can be fixed. Fogelin points out that $\forall x \exists y Fxy$ may not be expressible in the *Tractatus*'s logic. For with

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[13] This is a position Fogelin has held before *cf. Philosophical Studies* 15 1974 189-97.

$\forall x \exists y Fxy$ and applying N we get N(Fxy) or $\sim\exists x \exists y Fxy$ and more applications of N will never get us to $\forall x \exists y Fxy$.

Soames expands upon the problem. Wittgenstein, he notes, was not attempting to create such a weak system. The language of the *Tractatus*, Soames calls L_t is defined as:

If $F_1 \ldots F_n$ are formulas then $\ulcorner N(F_1 \ldots F_n) \urcorner$ are formulas and nothing else is.

Thesis 5.501 tells us that there are three methods to define a proposition, and L_t incorporates the first two; *i.e.* direct enumeration and giving a function $f(x)$ whose values for all values of x are propositions. The third method is the giving of a formal law that governs the construction of propositions.

To interpret the *Tractatus* as L_t is to accuse Wittgenstein of having made an error, but Soames asks, "what are the alternatives?" Soames first lists the inadequate ones, for example enriching L_t with a class of meta-linguistic descriptions of possible truth arguments for N. This he rejects for the obvious reason that it invokes meta-linguistics, and Wittgenstein bars semantics of this sort.

The second attempt to fix the problem that Soames presents interprets the general form of the proposition in an inaccurate manner (SOAMES 1983, p. 580). This method does not focus on the fact that an operation is not a function. Rather, an operation is a symbol that attaches itself to a base expression and can be iterated. This reinforces the primacy of notation in Wittgenstein's system. Wittgenstein explains his notion of a general term in a formal series in 5.2521, 5.2522:

a, O'a, O'O'a…

Under his schema p then may be a base and ξ an arbitrary set of term, but $\bar{p}$ and $\bar{\xi}$, as they occur indicate all members of a stipulated set that is being considered. The problem is that it does not tell us what expressions are available for representing arbitrary sets of propositions in the series. Here we look back at 5.501 and the three methods. But if this is the case we end up with L_t.

Soames sees two possible strategies. The first is that the types of expression available are limited to those explicitly mentioned in the *Tractatus*, and, by this description, Soames sees the *Tractatus* as being "radically defective." The second strategy is that those explicated are just examples of expressions, which could perform such a function. That is, that Wittgenstein is not defining exactly what he means, but just giving and example operation—a fairly standard procedure for Wittgenstein. On this second view Wittgenstein gives only a partial specification for his logical symbolism which could be completed in a manner consistent with the *Tractatus.* The second view, presented in such a manner by Soames, is obviously his preference. He has three reasons for preferring it:

(i) 5.501 and 3.317, where he states that he does not care how truth arguments are specified as long as it is a purely formal law which violates none of his dictums.

(ii) in 4.0411 Wittgenstein rejects an account of generality very close to the L_t one where he writes that, "we should not know the scope of the generality sign" This makes 5.501 and 5.52 (using the propositional function Fx to give the truth arguments of N) somewhat careless illustrations of how generality can be expressed.

(iii) to limit the means of representing truth arguments of N to L_t is to ignore Wittgenstein's intentions at 6 to allow the joint denials of arbitrary sets of propositions.

Soames states that a method which is true to (iii) is Wittgenstein's intention and also does not offend against other 'Tractatrian' doctrines. It is therefore to be preferred. Thus he requires that an extended language, L_{TE}, which must first express scope distinctions found in predicate calculus, and secondly must be consistent with the *Tractatus* (SOAMES 1983, p. 584).

Soames defines L_{TE} as: If $F_1 \ldots F_n$ are formulae, and G is a formula in which ν occurs free, then $\ulcorner(F_1 \ldots F_n)\urcorner$ and $\ulcorner(\nu[G])\urcorner$ are set representations; the ν to the left of the G being called a generality indicator. If S is a set representation then $\ulcorner N(S)\urcorner$ is a formula. Other free and bound occurrences of variable are defined as normal with generality indicators being the binding elements. L_{TE} sentences are formulas with no free variables. Soames notes that all propositions of first order predicate calculus are expressible in his expanded language L_{TE} (SOAMES 1983, p. 585).

Fogelin and Soames disagree on this: Fogelin believes that the doctrine that every proposition is constructible from elementary propositions implies that there is a decision procedure connecting every proposition with those elementary propositions which are its truth grounds. For Soames constructability has nothing to do with decision procedures. By avoiding decision procedure he can write, in L_{TE}, $N(N(x[N(N(Px),N(Qx))]$; where the inner set descriptors '$N(N(Px)$' and '$N(Qx)$' do not need to be analysed before the outer operations are applied to the set bound by the 'x' variable (SOAMES 1983, p. 585). The objection that Fogelin has is the same as the one he will have to Geach's solution presented below.

In his article Geach (GEACH 1981, pp.168-171) remarks that Russell erroneously conflates the operator N with binary joint denial. Furthermore, he notes many fail to see classes as the bases of the operation. Geach introduces the notation $N(\ddot{x} : F\ (x))$ to mean the joint denial of a class, with a class here represented by the variable 'x' with a diæresis, by substituting the names '$a,b,c,\dots$" for the variable 'x' in $F(x)$. Under this notation $(\exists x)F(x)$ and $(\forall x)F(x)$ are presented as $N(N(\ddot{x}\colon F(x))$ and $N(\ddot{x}\colon NF(x))$ respectively (GEACH 1981, p.169).

According to Geach, Wittgenstein was exaggerating when he said that the theory of classes is altogether superfluous in mathematics. Geach proposes that a rudimentary set theory is required, which would apply only to objects of the same type, and not be used as the definition by extensions of functions. Thus there would be no chance of generating paradoxes of the Russellian sort.

According to Geach, Fogelin makes the mistake of requiring the completion of an infinite set of steps to describe any function. In fact, Geach writes that even $\forall xF(x)$ would generate the same problem if the set generalized over were infinite. Geach writes, "*Nobody* has to run through an infinite series of propositions, negating them one by one, and then perform a further task of joint denial"(GEACH 1981, p.170). He sees the process of the application of the operation to a formal series in the following manner:

$[a\mathrm{R}b,\ a\mathrm{S}b,\ a\mathrm{R}/\mathrm{S}b]$

which provides the series of propositions:

$a\mathrm{R}b,\ a\mathrm{R}/\mathrm{R}b,\ a\mathrm{R}/(\mathrm{R}/\mathrm{R})b \dots$

then applying N operation to this series:

$\mathrm{N}[a\mathrm{R}b,\ a\mathrm{S}b,\ a\mathrm{R}/\mathrm{S}b]$

and apply N to the negation of the series, and so say that "*a* does bear *some* power of relation R to *b*," we write:

$\mathrm{N}(\mathrm{N}[a\mathrm{R}b,\ a\mathrm{S}b,\ a\mathrm{R}/\mathrm{S}b])$

(GEACH 1981, p.170).

But Geach is wrong. He is making a type error, or in the terms of the *Tractatus*, he is speaking where he should be showing. A proposition might be described by a rule and still form a finite set of states of affairs. The point to remember is that the possible ways of defining propositions does not mean that non-propositions can't be similarly described. In addition, Fogelin, in his response to Geach's article, writes that Geach's new notation differs from Wittgenstein's (FOGELIN 1982, pp.124-127). Fogelin notes that Wittgenstein's notation for $(\exists x)f(x)$ would be $N(N(\overline{f(x)}))$; where '$\overline{f(x)}$' would represent the possibly infinite set of propositions one would get by substituting names for the variable 'x'. Fogelin argues that Geach's proposals violate "*finiteness* and *successiveness*" of application found in general propositions.

> Wittgenstein: $N(N(\overline{f(x)})) = N(N(fa, fb, fc, \ldots)) = (\exists x)(fx)$
> Geach: $N(N(\ddot{x}: F(x)) = N(N(fa), N(fb), N(fc), \ldots) = (x)(fx)$
>
> (FOGELIN 1982, p.126)

Fogelin's problem is Wittgenstein's requirement that all truth functions "are results of successive applications to elementary proposition of a finite number of truth operations" (5.32). He does not fault Geach's version, or the inclusion of infinite sets, except for this fact. The sets of names may be infinite but the applications or operations may not be, since Wittgenstein explicitly denies this in 5.32.

Cheung defends Geach's and Soames's position that the *Tractatus* is expressively complete without being contradictory. He argues that Geach's and Soames's methods work, but shows that these methods only work under finite domains of elemental propositions (CHEUNG 2000, pp. 247–261). If this is the case, then the addition of Geach's and Soames's notation is unnecessary, since the finitude of the domains would make constructions possible, if not practical, by the method first rejected by Fogelin. Under the Cheung definition of the limitations of the logic of the *Tractatus* there is no need of any additional notation.

Cheung argues that the logic of the *Tractatus* must restrict itself to finite domains of elementary propositions; *i.e.* it must not offend Church's restrictions. He concludes that Wittgenstein most likely did not realise this, but that he would have had to agree if he did. While this is no doubt true, it is also the case that the elementary propositions would have to be finite for another reason. This reason is suggested, but (prematurely) rejected by Fogelin, which is that statements generated by the method of constructing propositions, in order to be propositions and not pseudo-propositions, must be about finite things. That is, they cannot be about natural number or other formal constructions, for in such cases they would cross the boundary between that which can be said and that which can only be shown.

3.3 Frascolla's Interpretation and Modifications

Frascolla's 1994 book *Wittgenstein's Philosophy of Mathematics* addressed, carefully and systematically, Wittgenstein's operation theory and his construction of natural number. Frascolla's sets for himself the task in the first chapter to clear up the exegesis of Wittgenstein's comments on natural number and arithmetic in the *Tractatus*. He first delves into the standard commentaries on the *Tractatus* and shows how they obviously misinterpret what Wittgenstein is saying. These misrepresentations marginalise Wittgenstein comments on natural number and mathematics, making them seem no more than *obiter dicta* of the true content of the work. Frascolla then goes on to carefully read the sections of the *Tractatus* dealing with the operation and natural number and attempts to provide a clear interpretation of Wittgenstein's reasoning. In so doing Frascolla finds it necessary to modify the *Tractatus*' notation to make more perspicuous Wittgenstein's meaning.

I will consider first Frascolla's criticism of Black and Anscombe, and then his interpretation of Wittgenstein's operations theory, including his modification of Wittgenstein's notation and the reasoning behind such modifications. This will lead well into section 3.4 where I shall show how the operation theory language, as presented by Frascolla, is similar to Church's lambda calculus.

3.3.1 Frascolla's Critique of 6.02-6.031

Frascolla critiques Anscombe's explanation of 6.02-6.031: Anscombe wonders why Wittgenstein did not take the simpler route of explaining the meaning of 0 and of the symbol +1 and then introduce 6.03 (FRASCOLLA 1994, pp.4-5). But Frascolla notes that Wittgenstein does exactly what she suggests in 6.02. Why then did Wittgenstein include the Ω operation, and not just explain 0 and +1 and proceed? Frascolla attempts to answer this by means of a systematic exposition.

According to Wittgenstein the meaning of arithmetical terms for 0+1+1+1+1...+1 "can be derived from the corresponding expression in the general theory of logical operations '$\Omega^{0+1+1+1...+1}$'" (FRASCOLLA 1994, p.7). This is why 6.021's statement that a natural number is the exponent of an operations is important. In the operation theory language, as presented by Frascolla, the variable n, or in Wittgenstein's terminology ν, indicates 0+1+1+1...+1 with n occurrences of +1. With the explicit definition of n, and the inductive definition of 6.02, the symbolic content of "Ω^{n}'x" is translatable into an expression in which neither "n" nor "0+1+1+1...+1 (with n occurrences of +1)" appear. That is, into a string of Ω'Ω'...Ω'x with n occurrences of " Ω' ". Furthermore:

> the expression "Ω^{t}'x" from the operation theory language will correspond to the arithmetical term 't' and the relevant definitions will make it possible to

> translate the former into a specific, complex arrangement of "Ω'" (FRASCOLA 1994, p.7).

And so the equation Ω^{t}'x = Ω^{s}'x will correlate with "t=s". Numerical calculation is thus defined down to strings of "Ω'", with the concatenation of more "Ω'" as well as subdivision and grouping based "on the properties of the application of a logical operation." That is:

> The relations between the configurations thus obtained within the theory of operations will *show* the possibilities of symbolic transformation corresponding to the arithmetical identities usually classified, in non-*Tractarian* jargon, as "true arithmetical identies." (FRASCOLLA 1994, p.7)

Frascolla then goes on to provide, in extraordinary detail, the exposition of the theory. He begins with:

> $x = \Omega^{0}{}'x$ Def., and
> $\Omega'\Omega^{v}{}'x = \Omega^{v+1}{}'x$ Def.

And considers the symbols one by one: "x" is an object variable, in the "peculiar sense" of an object in the *Tractatus*. Being an object, Frascolla notes it "is a symbol that shows the form of a name." "Ω" is the symbol for an operation variable; the symbol of the formal concept of operation.

> (i) an operation is a uniform procedure, by the application of which certain expressions can be generated from given expressions whenever the base and the result of the operation are linked by a certain formal relation (FRASCOLLA 1994, p.8).

This he gets from 5.23, 5.231 and the *Notebooks* (NB 22/11/16). Secondly:

> (ii) there is no *object* that corresponds to an operation sign as its fixed and distinguishable semantical value (FRASCOLLA 1994, p.8)

This he derives from 5.25. And finally from 5.251 he derives that:

> (iii) an operation can always be applied to the result of its own application : the result of one of its applications is again a possible base for a new application of the operation (FRASCOLLA 1994, p.8)

Returning to thesis 6.02:

$x = \Omega^{0}{}'x$ Def., and
$\Omega'\Omega^{\nu}{}'x = \Omega^{\nu+1}{}'x$ Def.

We can see that the variable x is used here by Wittgenstein as an object variable, and Ω is used as an operation variable. Wittgenstein has told us three things about operation variables. First, that they are uniform (NB 22/11/16; 5.23 5.231); second that there is no object that corresponds to them as their fixed semantic value (5.25); and thirdly that an operation can be applied to the result of its application (5.251). The apostrophe (') indicates the result of an operation (FRASCOLLA 1994, pp.8-16; and *cf.* also SUNDHOLM 1992, FLOYD 2000 and 2002). The general contention is that the apostrophe is modified from Russell's notation in *Principia Mathematica,* for the object of a relation: where R'x is the only object y having the relation R to x. That is where Rxy obtains y = R'x (FLOYD 2002 pp. 308-309).

The superscripted notation 0 and v+1 are, for Wittgenstein, the usual arithmetic terms with the series 0+1+1+1+...+1 derived from $\Omega^{0+1+1+1+\ldots+1}$ where n, or in Wittgenstein's terms v, is the natural number of occurrences of +1. Frascolla notes that $\Omega^{t}{}'x$ similarly corresponds to the arthmentic term t, and so $\Omega^{t}{}'x = \Omega^{s}{}'x$ correlates to t = s. The relations between the configurations in the theory of operations show the possibilities of symbolic transformation corresponding to arithmetical identities normally called "the arithmetical identities" (FRASCOLLA 1994, p.7). Thus 0+1+1+1+...+1 in the language of the general theory of logical operations must be assumed available; where 0 means no occurrence of +1, and v is the schematic letter for any natural number n≥0 occurrences of +1.

Frascolla replaces '0+1+1+...+1' with 'SSS...0' for ease of reading; the use of + for addition will be done using the successor operation, so the superscripted '0+1+1+...+1' will not be mistaken for the + operation of normal arithmetic terms. The definition given is construed by induction of the natural number of occurrences of the sign S in the example $\Omega^{SSS...0}$. In comparison Wittgenstein writes:

> 6.02 And this is how we arrive at natural numbers. I give the following definitions
> $x = \Omega^{0}{}'x$ Def., and
> $\Omega'\Omega^{v}{}'x = \Omega^{v+1}{}'x$ Def.
> So, in accordance with these rules, which deal with signs, we write the series
> $x\ \Omega'x,\ \Omega'\Omega'x,\ \Omega'\Omega'\Omega'x, \dots,$
> in the following way $\Omega^{0}{}'x,\ \Omega^{0+1}{}'x,\ \Omega^{0+1+1}{}'x,\ \Omega^{0+1+1+1}{}'x, \dots.$
> Therefore, instead of '$[x, \xi, \Omega'\xi]$', I write '$[\Omega^{0}{}'x, \Omega^{v}{}'x, \Omega^{v+1}{}'x]$'.
> And I give the following definitions:
> $0 + 1 = 1$ Def.,
> $0 + 1 + 1 = 2$ Def.,
> $0 + 1 + 1 +1 = 3$ Def.,
> (and so on).

Frascolla describes how Wittgenstein has presented the meaning of each term of arithmetical language which can be grasped by means of the definition: each term is shown by the *definiens* of the corresponding expression of the operations theory language in which 'SSS...S0' occurs as an exponent of Ω. The inductive definition in 6.02 is conceived of as a reduction of the notions of zero and successor to the abstract notion of the application of a logical operation (FRASCOLLA 1994, pp. 8-9).

This is the replacement Wittgenstein proposes to the logicist explanation of the arithmetic primitives by the means of class. Wittgenstein thus provides an intensional explanation of what the logicist gave an extensional explanation. Number is created through a process; not named by a set. It is not that Wittgenstein has shown that there is an error in the logicist definition, but rather he has shown that there is another way of defining number. The nature of the operational description is that natural number should be seen as a construction. This, in its way, is an earlier understanding of mathematics, and thus seems to point to the development of the logicist project as being a "wrong turn" (FLOYD 2002, *passim*) or philosophical invasion into mathematics.

Being constructed of variables each member of x, Ω'x, $\Omega\Omega$'x, $\Omega\Omega\Omega$'x, exhibits "a form." Thus each 'SSS...S0,' belonging to the operational theory language, is an abbreviated notion for an expression that shows the formal structure common to the elements of a certain class of linguistic expression (FRASCOLLA 1994, pp. 9-11). We have to, according to Frascolla, violate the Wittgensteinian prohibition of speaking of what may only be shown to be clear about what is going on it these passages from what he has said so far. Frascolla makes the following observations about Wittgenstein's examples:

- Ω^{0}'x is identical with x
- x is the initial form of the series
- 0 then stands for "formal property constituted by natural number of times the operation is applied"
- Ω^{S0}'x identical with Ω'x
- Ω'x shows "at some utmost level of generality, [as x] the form of an expression which is the result of the successive application of an operation consisting of one application to an initial symbol."
- "S0" stands for the formal property constituted by the natural number of times an operation is applied to form such an expression
- SS0 thus for Ω'Ω'x and so on.

In 6.02 Wittgenstein effects a reduction of the meaning of infinite arithmetical terms "SSSS...S0" to the general notion of the application of a logical operation. In 6.021 6.022 and 6.03 he then solves the problem of the definition of the natural numbers. In 6.021 'natural number' is defined as the exponent of an operation. This means that any numeral "n" can be reduced to the meaning of "n" as an exponent of Ω. For natural natural numbers there is another step required. In 6.022 Wittgenstein states that the concept of natural number is simply that which is common to all natural numbers; *i.e.* the general form of natural number. The method is the same as that used in the case of members of a formal series; namely they are joined by an internal, or formal, relation. Any term's relation to its successor is a constant formal relation. A formal concept must be expressed by a variable for an arbitrary term of the series. In 6.03 [0, ξ, ξ+1] a more "perspicuous variable" appears the meaning of which is defined from 0 and +1 in the theory of logical operations in 6.02. And in this way the introduction of the variable for the formal concept of natural number is completed (FRASCOLLA 1994, p.12).

The true importance of Wittgenstein's conceptions, according to Frascolla, is two fold. First he shows how arithmetical functions of sum and product are reduced to the "general notion of the application of a logical operation." Second, he shows that the relation between arithmetic and the world is obtained by mapping true arithmetical entities onto theorems of the general theory of operations (FRASCOLLA 1994, pp.12-13).

There are two languages that we are talking about. The first is what Frascolla calls the operations theory language. The second is what he calls the language of arithmetic. By the former he means terms of the form Ω^{v}'x, and by the latter terms in the form variables representing numerals (1,2,3,4...); *e.g.* "x" or arithmetical equations "x+y" or "x×y".

3.3.2 Sum and Product

In 6.241 Wittgenstein provides the evidence for a more in depth reconstruction of his theory of operations description of natural number. In 6.02 he requires numerals (represented by Frascolla as the superscripted SSS...S0) in the primitive notation of the arithmetic language of the theory of operations. Every numeral "n" is introduced by a definition which "strictly parallels," and is "analogous" to, the definition of standard numerals. Arithmetical terms "t" are also available. In the operation theory language "n" is an arithmetical term, and if "r" and "s" are arithmetical terms then (r+s) and (r×s) are also. If we let "t" be an arithmetic term in the language of arithmetic say some numeral then we know the meaning of Ω^{t}'x, and if "t" an arithmetic term in the language of arithmetic in the form (r+s) or (r×s), then we now understand $\Omega^{(r+s)}$'x and $\Omega^{(r\times s)}$'x in the operation theory language. In 6.241 Wittgenstein provides us a guide of how to understand such operation theory language terms and convert them to arithmetic terms in the language of arithmetic.

> 6.241 Thus the proof of the proposition $2 \times 2 = 4$ runs as follows:
> $(\Omega^{v})^{\mu}{}'x = \Omega^{v \times \mu}{}'x$ Def.,
> $\Omega^{2\times 2}{}'x = (\Omega^{2})^{2}{}'x = (\Omega^{2})^{1+1}{}'x = \Omega^{2}{}'\Omega^{2}{}'x = \Omega^{1+1}{}'\Omega^{1+1}{}'x = (\Omega'\Omega)'(\Omega'\Omega)'x = \Omega'\Omega'\Omega'\Omega'x = \Omega^{1+1+1+1}{}'x = \Omega^{4}{}'x.$

Terms, such as $\Omega^{(r+s)}$'x and $\Omega^{(r\times s)}$'x, that represent the term "t" can be shown by groupings of Ω', which, through definitions, can be converted in to a form Ω^{t}'x (FRASCOLLA 1994, p. 13). What is important about this is that Frascolla emphasises that the grouping "show[s] the arithmetical structure of a certain complex operational model of sign construction." And so the form:

> $(\Omega'\Omega)'(\Omega'\Omega)'(\Omega'\Omega)'x$ will correspond to the arithmetical term 2×3

and

> $((\Omega'\Omega)'(\Omega'\Omega'\Omega))'x$ will correspond to the arithmetical term $2+3$

Frascolla shows how these conversions work, and in so doing will introduce several modifications into Wittgenstein notation for purposes of "disambiguation." First the example in 6.241, Frascolla expands on Wittgenstein's example, giving the definition for both addition and for multiplication. He also chooses to use Ω^{2+3}'x and $\Omega^{2\times 3}$'x instead of $\Omega^{2\times 2}$'x as Wittgenstein does, one can assume for the very good reason that they denote two different numerals. $\Omega^{2\times 2}$'x and Ω^{2+2}'x reduce to the same arithmetic term in the language of arithmetic, that is the numeral "4", which makes the distinction in the notations less obvious.

In Wittgenstein's reconstruction of arithmetic, in the operations theory language, numerical calculations consist of strings of Ω', which are governed by the properties of the notion of the application of an operation. So, in Frascolla's example, $\Omega^{(2\times 3)}$'$x = \Omega^{6}$'x means the transformation of $\Omega^{(2\times 3)}$'x, written $(\Omega'\Omega)'(\Omega'\Omega)'(\Omega'\Omega)'x$, into Ω^{6}'x which is written $\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'x$. In the same way, $\Omega^{(2\times 3)}{}'x = \Omega^{(3\times 2)}{}'x$ is the transformation of $(\Omega'\Omega)'(\Omega'\Omega)'(\Omega'\Omega)'x$ into $(\Omega'\Omega'\Omega)'(\Omega'\Omega'\Omega)'x$.

Frascolla criticises the use of the numerical variables 'μ' and 'ν' in the definition of product. Their use is misleading, he proposes if "Wittgenstein's aim is the assessment of meaning to every expression of the operation theory language $\Omega^{t}{}'x$ for every arithmetical term 't.'" Frascolla thus replaces $\Omega^{\mu\times\nu}{}'x = (\Omega^{\mu})^{\nu}{}'x$ with $\Omega^{(r\times s)}{}'\mathrm{x} = (\Omega^{r})^{s}{}'\mathrm{x}$. That is μ and ν, numerical variables of the operation theory language, are replace with r and s, variables for arithmetical terms of the operation theory language (FRASCOLLA 1994, p.14). Frascolla is saying the definition of product has to be broader than that which uses only numerals. It should apply for all terms, rather than just numeral. This, of course, requires not only the replacement of 'μ' and 'ν' with 'r' and 's', but also the addition of the brackets to the expression of product so $\Omega^{\mu\times\nu}{}'x$ becomes $\Omega^{(r\times s)}{}'x$ not $\Omega^{r\times s}{}'x$, for 'r' and 's' could already be complexes themselves.

This modification of Wittgenstein's notation can't be easily rejected; it serves not only the purpose of disambiguation but also allows the conversion of Wittgenstein's operation theory language into standard arithmetical definitions of sum and product.[14]

The next problem to arise is Wittgenstein's use of the apostrophe ('), which as we noted above is generally seen to indicate the result of an operation. With $(\Omega^{r})^{s}{}'x$ the "r" applies to Ω, and the "s" applies to the Ω^{r}. But, Frascolla notes, that it should not be taken for granted that Ω^{r} is lacking the apostrophe and the "x" it is not $\Omega^{r}{}'x$ but just Ω^{r}.

[14] Frascolla expands on this project in FRASCOLLA 1997, where he proves an interpretability theorem between Wittgenstein's operation theory and the equational fragment of Peano's Arithmetic.

3.3.2.1 The Proof of 6.241

The proof provided in 6.241 is taken as a guide. In it $\Omega^{2\times2}$'x is converted into Ω^{1+1}'Ω^{1+1}'x—in Frascolla's terminology Ω^{SS0}'Ω^{SS0}'x. Frascolla invites us to look, then, at the conversion line by line.

[1] $\Omega^{2\times2}$'$x = (\Omega^2)^2$'x by definition of product (×)
[2] $= (\Omega^2)^{SS0}$'x by the definition of the numeral "2"
[3] $= \Omega^2$'Ω^2'x inductive definition from 6.02 with Ω^2 in place of Ω
[4] $= \Omega^{SS0}$'Ω^{SS0}'x

Observe that in the transition from [2] to [3], Ω^2 is used in place of Ω. That is, ^{SS0} is applied to Ω^2 instead of to just an omega, but without the apostrophe or the base variable "x". Also at [2] the brackets around (Ω^2) disappear. Now the next line of the proof:

[5] Ω^{SS0}'Ω^{SS0}'x = (Ω'Ω)'(Ω'Ω)'x

The question to be asked is where does the first (Ω'Ω)' come from? Should this not be Ω^{SS0}'Ω'Ω'. The (Ω'Ω)' obviously expresses the Ω^2' from [2] and [3], and the two occurrences in [3] and Ω^{SS0}' in [4]. What does (Ω'Ω) mean? Considering Ω'x, we can say that the Ω is the operation and the apostrophe in Ω' is the applied operation. But what is the (Ω'Ω) in (Ω'Ω)' mean? Where do the brackets come in, and what does it mean to apply such an operation so compounded? The (Ω'Ω), Frascolla answers, is the "Form of the operation resulting from the composition of a given operation with itself namely 'the second iteration of the given operation'." Therefore 2×2 corresponds to (Ω'Ω)'(Ω'Ω)'x. While symbols Ω'Ω'x show the formal result of the application to the result of one application to an initial symbol, (Ω'Ω)'x shows the form of the result of the application of an operation resulting from the composition of an operation with itself to an initial symbol.

Frascolla's Rule [A]

Frascolla's addition of the rules [A] through [D] to understand Wittgenstein's system is different than other attempts by some to 'fix' Wittgenstein's theory of operations, as his additions are solely interpretive. Unlike the additions by Soames and Geach, noted above, Frascolla's additions are explanations of, or rather expansions on, what Wittgenstein implies in his proof of 6.241. That is, they are conversion rules and definitions that provide a template for the syntactic rules shown by Wittgenstein's demonstration. They do not extend the language of the theory of operations; rather they describe the notation in a more definite manner.

Frascolla notes that the following rule is implied in Wittgenstein's proof in 6.241:

[A] $(\Omega'\Omega)'\xi = \Omega'\Omega'\xi$

But that the slightly different forms, " $\Omega'\Omega'\xi$ " and " $(\Omega'\Omega)'\xi$ ", show slightly different senses. Continuing with the proof of 6.241 we come to the application of rule [A]:

[6] $(\Omega'\Omega)'(\Omega'\Omega)'x = (\Omega'\Omega)'\Omega'\Omega'x$ by the definition [A]
[7] $= \Omega'\Omega'\Omega'\Omega'x$ also by definition [A]
[8] $= \Omega^{SSSS0}{}'x$ by 6.02
[9] $= \Omega^{4}{}'x$

And so we have Wittgenstein's proof that 2×2=4.

Frascolla's Rule [B]

Returning now to the form $\Omega^{(r\times s)}{}'x$, Frascolla notes that there is the problem of the distinction between "Ω^{r}'" in the context "(Ω^{r})", *i.e.* the context where there is no base directly attached to the operation and " $\Omega^{r}{}'x$ ". In 6.02 Wittgenstein establishes the meaning of $\Omega^{n}{}'x$ for every *n*, but what is the meaning of (Ω^{n}) without the apostrophe or the base? Here Frascolla introduces another modification to the notation; again to make perspicuous in the notation what Wittgenstein is doing.

$$
\begin{array}{ll}
[B] & {}^{0}\Omega = \mathrm{I} \quad \textit{where}\ \mathrm{I}'\xi = \xi \\
& {}^{1}\Omega = \Omega, \\
& {}^{2}\Omega = (\Omega'\Omega), \\
& {}^{3}\Omega = (\Omega'(\Omega'\Omega)), \\
& \ldots
\end{array}
$$

The justification of the adoption of this rule is Wittgenstein's use of a similar method in 6.02 for the introduction the numerals. The placing of the value on the left hand side "${}^{n}\Omega$" replaces Wittgenstein's notation of the bare "(Ω^{n})" without the apostrophe or the base. Thus it is the "nth iteration of an operation" (FRASCOLLA 1994, p.17) but not the result of the nth iteration of the base. Frascolla himself notes in footnote 31 that [B] should be properly described by induction, but that he will not fully describe it in such a manner because his purpose is to provide an exegesis rather than formally justifying it (*cf.* FRASCOLLA 1997).

Again an objection to Frascolla's method that the introduction of "${}^{n}\Omega$" is an extension of the operation theory does not hold water. Wittgenstein's symbol for this is "(Ω^{n})" and the modification by Frascolla only makes clear the distinction between "(Ω^{n})" and "$\Omega^{n}{}'x$". With this in mind the introduction of the rule [B], as a description, is acceptable because Wittgenstein does convert such base operations in his proof in 6.241 in the transition from [3] to [4].

In [B] the concept of identity is defined operationally, as it is in Churches lambda calculus. This too is a place where one might object—remembering Wittgenstein's objections—that identity was not an operation (WAISMANN 1979, p.218). However, Wittgenstein objected stating that identity was just showing that two applications operations had the same results. So while identity was not a function or a relation it may be shown by them. In Wittgenstein's original definition $\Omega^{0}{}'x = x$ shows how it might be done operationally. Though Frascolla has gone beyond what Wittgenstein would have likely realised his notation implied, it could easily be argued that Wittgenstein, in rejecting identity as a relation proper, would accept that "Ω^{0}" could be represented as "I". What the "I'$\xi = \xi$" really means is "Ω^{0}'Ω'$\xi = \Omega$'ξ". That is, "Ω^{0}" is the application of the operation 0 times.

Frascolla has thus made clear the meaning of (Ω^{r}) when it appears without a base, or apostrophe: "the *n*th iteration of operation." Only when it has a base attached to it can it be converted into a numeral.

Frascolla notes that the notation of [B] is needed to avoid making ambiguous the contexts of Ωn and $\Omega^{SS\ldots S0}$ he writes:

> Following Wittgenstein faithfully, we would have to add the definition of an expression "Ω^{n}", for every n, so as to determine the meaning of the occurrence of "Ω^{n}" in the context "(Ω^{n})". But this course has the considerable inconvenience of rendering the contexts "Ω^{n}'x" and "$\Omega^{SS\ldots S0}$'x" ambiguous, as has been seen scrutinizing the lines [3] and [4] of Wittgenstein's proof of the equation "$\Omega^{(2\times 2)}$'$x = \Omega^{4}$'x" (FRASCOLLA 1994, p.17).

In his discussion of lines [3] and [4] of Wittgenstein's proof, where there is the move from Ω^{2}'Ω^{2}'x to Ω^{SS0}'Ω^{SS0}'x, the problem that Frascolla brings up is that, while Ω^{2}'x is converted to Ω^{SS0}'x by definition, there is no definition for the first " Ω^{2}' ". The need becomes clear when we look at [5] Ω^{SS0}'Ω^{SS0}'x = (Ω'Ω)'(Ω'Ω)'x. What is going on here? Does " Ω^{SS0}' " convert into " (Ω'Ω)' " as " Ω^{SS0}'x [coverts] to " Ω'Ω'x "? If that interpretation is the case there are problems for the definition of product. Frascolla states that the ambiguity is solved in the transition from [4] to [5] by Wittgenstein's interpretation of Ω^{SS0}'x. In [5] the expression is first converted to (Ω'Ω)'Ω'Ω'x. That is, the second set of brackets are removed and only then in step [6] is the whole expression converted to Ω'Ω'Ω'Ω'x this indicates that rule [A], that Frascolla introduces, is being applied. But this also indicates that " (Ω'Ω)' " follows different rules, has a different meaning and hence rule [B]. The "Ω^{2}" of the expression $(\Omega^2)^{2}{}'x$ of [2] can now be followed as a separate operation through the whole proof. Instead of "Ω^2" Frascolla will write "${}^2\Omega$" to indicate this different role. Using Frascolla's new rule we could change the proof, and in the transition from [2]' to [3]', we get the application of "^{SS0}" to "${}^2\Omega$"; in transition from [3]' to [4]' the "${}^2\Omega$" more clearly converts into "$(\Omega'\Omega)'(\Omega'\Omega)'x$".

[2]' $= ({}^2\Omega)^{SS0}{}'x$
[3]' $= ({}^2\Omega'{}^2\Omega)'x$
[4]' $= (\Omega'\Omega)'(\Omega'\Omega)'x$ now by [B]

But the addition of the rule [B] creates problems. Rule [A] allows for the removal of brackets for an operation with a base in "$(\Omega'\Omega)'\xi$". The "ξ" variable is clearly interpreted as any base of the form:

$\xi ::= x$ or $\Omega'x$ or $\Omega'\ldots\Omega'x$

But [A] no longer solves the problem created by [B]. How would one convert the form now derived from "$\Omega^{(3\times2)}{}'x$", *i.e.* "$(\Omega'(\Omega'\Omega))'(\Omega'(\Omega'\Omega))'x$", using rule [A]? The problem is, of course, the brackets, and so Frascolla adopts another rule.

Frascolla's Rule [C]

[C] $(\Omega'\Psi)'\xi = \Omega'\Psi'\xi$ where "Ψ" like "Ω" is an operation variable.
(FRASCOLLA 1994, p.17)

In rule [C] we see an extension of rule [A]. Here Frascolla admits to differing from Wittgenstein, he writes that he is "differing from the line which Wittgenstein would certainly have taken, I will not adopt the exclusive interpretation of operation variables." He could not have achieved the same end with $(\Omega'{}^n\Omega)'\xi = \Omega'{}^n\Omega'\xi$. The reason that he could not is that the ${}^n\Omega'$ allows only for the combinations made in [B], and not some of the other combinations of operations and brackets which will be seen in proofs. The removal of brackets is the problem, and rule [C] addresses this problem. The "Ψ" is a more general operational variable, which allows for more combinations of variables and brackets.

It seems, in this case, an interpretation is required. It does not say that there is a need for more operations than that of Ω succession; rather, it is a syntactic assumption. In standard syntactic notation the operational variable " Ψ' " may be written:

$\Psi ::= \Omega, (\Psi), (\Psi' \ldots \Psi)$

Thus, it is reducible to the Ω notation.

Again one may object to Frascolla adding the " Ψ' " operation as being an unwelcome extension of Wittgenstein's theory. In this case, again, we could note that while " $(\Omega' n\Omega)'\xi = \Omega' n\Omega'\xi$ " might not be suitable—because there are complex operations not represented by it—some other form of induction might again be used here to generate all syntactically acceptable forms represented in the above syntactic read–write rules. A group of definitions can also represent, *i.e.* show, the same thing as the " Ψ' " operation, *e.g.* :

[C]′ $({}^{n}\Omega') = {}^{n}\Omega'$

[C]″ $(\underbrace{\Omega' \ldots}_{m \text{ times}}\ {}^{n}\Omega)' = \Omega^{m\prime} \ldots {}^{n}\Omega'$

[C]‴ $(\underbrace{{}^{p}\Omega' \ldots}_{m \text{ times}}\ {}^{n}\Omega)' = {}^{p}\Omega^{m\prime} \ldots {}^{n}\Omega'$

[C]⁗ $(\ldots(\underbrace{{}^{n}\Omega' \ldots\ \Omega'}_{m \text{ times}})\ldots) = (\ldots {}^{n}\Omega^{m\prime} \ldots \Omega')\ldots$

and so forth…

The " Ψ' " operation variable seems a simpler explanation of what Wittgenstein is doing in his proof presented in 6.241 when he removes brackets. And since it is just a explanation, and not part of the operational language itself, it can be understood, like the rest of Frascolla's rules, to be just a interpretive tool rather than an extension of the language.

Frascolla's Rule [D]

The next rule Frascolla introduces is [D] which describes induction for " $^{n}\Omega\text{'}x$ ":

> [D] for every $n \geq 0$, $^{n}\Omega\text{'}x = \Omega\text{'} \ldots \text{'}\Omega\text{'}x$, where " Ω' " occurs n times
> (FRASCOLLA 1994, p.17)

This is the counterpart to Wittgenstein's definition in 6.02 for $\Omega^{\nu}\text{'}x$. Since [B] introduces an notation " $^{n}\Omega\text{'}x$ " there must then be a left hand description of induction on it as well. Accepting [B] requires us then to accept [D] as its counterpart, once again a rule whose necessity is shown by Wittgenstein proof, and that conversion of [2] to [3] and [3] to [4].

Frascolla's Definitions [I]-[IV]

Frascolla now introduces four definitions which are made necessary by the introduction of his rules [A]–[D]. The notation changes he makes mean that Wittgenstein's original definitions are not enough. Now the notation includes " $n\Omega$ " as well as " $\Omega n\text{'}x$ " so it must include " $^{(r^{+}s)}\Omega$ " as well as " $\Omega^{(r^{\times}s)}\text{'}x$ ". The additional definitions are as follows:

[I] $\Omega^{(r\times s)}\text{'}x = {}^{r}\Omega^{s}\text{'}x$

that is, " $\Omega^{(r\times s)}\text{'}x$ " means that there are r applications of " Ω^{s}' " to the base 'x';

[II] $^{(r\times s)}\Omega = {}^{r_s}\Omega$

that is, " $^{(r\times s)}\Omega$ " means that there are r applications of the operation " $^{s}\Omega$ ";

[III] $\Omega^{(r+s)}\text{'}x = ({}^{r}\Omega\text{'}{}^{s}\Omega)\text{'}x$

that is, " $\Omega^{(r+s)}\text{'}x$ " means the concatenation of the " $^{r}\Omega$' " and " $^{s}\Omega$ " operations are applied to the base "x"; and finally:

[IV] $^{(r+s)}\Omega = ({}^{r}\Omega\text{'}{}^{s}\Omega)$

that is, " $^{(r+s)}\Omega$' " means the concatenation of the " $^{r}\Omega$' " and " $^{s}\Omega$ " operations are applied to no base (FRASCOLLA 1994, pp.17-18).

3.3.3 Frascolla's Proof

After introducing these rules and definitions we are ready to consider in detail Frascolla's systemization of Wittgenstein's theory of operations as applied to arithmetic. Frascolla uses his new interpretation of Wittgenstein's operational language to prove $(2+3)\times(2+1) = 15$. This has the advantage of showing an operational definition of both product and sum.

$$[1]\ \Omega^{((2+3)\times(2+1))'}x$$
$$= {}^{(2+3)}\Omega^{(2+1)'}x$$

The first step of the proof is done by definition [I], which uses the distinction between baseless and operations with bases first developed in [B].

$$[2] = ({}^{2(2+3)}\Omega'{}^{1(2+1)}\Omega)'x$$
$$[3] = (({}^{(2+3)}\Omega'{}^{(2+3)}\Omega)'\,{}^{(2+3)}\Omega)'x$$
$$[4] = ((({}^{2}\Omega'{}^{3}\Omega)'({}^{2}\Omega'{}^{3}\Omega))'({}^{2}\Omega'{}^{3}\Omega))'x$$
$$[5] = ((((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'((\Omega'\Omega)'\ (\Omega'(\Omega'\Omega))))'x$$

Now there are three instances of $(((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'$ which it is obvious from observation reduces to $\Omega'\Omega'\Omega'\Omega'\Omega'$.

$$[6] = (((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'((\Omega'\Omega)'\ (\Omega'(\Omega'\Omega)))'x$$

Now begins the reduction of brackets through [C]; note this is different than in [8], [9],[12], [13], [15], and [16] where [A] must be used as the base 'x' is present.

$$[7] = (((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'(\Omega'\Omega)'(\Omega'(\Omega'\Omega))'x$$
$$[8] = (((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'(\Omega'\Omega)'\Omega'\Omega'\Omega'x$$

Rule [C] is first used here to produce:

$$(((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega\Omega)))'(\Omega'\Omega)'(\Omega'\Omega'\Omega)'x$$

then [A] is used to remove the brackets from the last three " Ω' ".

[9] = $(((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega))))'\Omega'\Omega'\Omega'\Omega'\Omega'x$ again by [C]
[10] = $((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'\Omega'\Omega'\Omega'\Omega'\Omega'x$ again by [C]
[11] = $((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'(\Omega'\Omega)'(\Omega'(\Omega'\Omega))'\Omega'\Omega'\Omega'\Omega'\Omega'x$ again by [C]
[12] = $((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'(\Omega'\Omega)'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'x$ [C] then [A],
[13] = $((\Omega'\Omega)'(\Omega'(\Omega'\Omega)))'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'x$ by [A]
[14] = $(\Omega'\Omega)'(\Omega'(\Omega'\Omega))'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'x$ by [C]
[15] = $(\Omega'\Omega)'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'x$ again by [A] and [C]
[16] = $\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'\Omega'x$ by [A]
[17] = $\Omega^{15}{}'x$ by definition of 6.02

(FRASCOLLA 1994, pp. 18-19).

What Frascolla has done is expand on the proof example Wittgenstein gave us, and, in turn, provided an example of how one would go about defining product in the same manner. Furthermore, he has presented a series of rules that fully describe the notation that Wittgenstein is using; including presenting solutions to particular ambiguities that Wittgenstein did not describe carefully. To be fair to Wittgenstein the notation, and the examples he presents, are done so to sketch an alternate method of describing number and arithmetic. He as usual leaves a lot f the work to the reader. Some might argue that Black's criticism of the proof at 6.241 for being eccentric (BLACK 1964, p.343) is vindicated by the lengths that Frascolla has to go to explicate Wittgenstein's operational theory of arithmetic, but Frascolla shows that there is nothing wrong with the Wittgenstein's proof. With more examples the underlying rules of Wittgenstein's operational language would have been more obvious, but the example given was enough to establish those underlying rules clearly.

3.4 Equivalence to Lambda Calculus

This section discusses the equivalence that persists between Wittgenstein's theory of the operation and lambda calculus. There are two parts to this equivalence: first, the similarity of the operational definition of natural number and Church numerals, and, second, the broader similarity of the lambda calculus and the operational theory language.[15] The lambda calculus notation focuses on the algorithmic nature of functions rather than on the extension, or the domain and co-domain. Similar to how Wittgenstein understands operation, a lambda function is applied to a base. This means that the notation allows a function to be empty; *i.e.* it describes what would be done to a base rather than describing some object or another. The operation, in Wittgenstein's terminology, rather than the classic set theoretic understanding of function, is a good description. This is obviously important when dealing with natural number, for the function that generates any natural number set will have an infinite set. This, of course, is vulnerable to Wittgenstein's critique of the axiom of infinity as previously discussed. But an algorithmic understanding of natural number is entirely constructive, because we understand natural number through the processes used to generate them (MARION 1998, pp.21-29).

3.4.1 Lambda Calculus Syntax

The syntax of λ calculus is defined as follows:

> The following are λ–terms:
>
> *Variables*: any variable (*e.g.* $x,y,z\ldots$) is also a λ–term.
> *Application of λ–term*: If M and N are λ–terms, then so is $(M)(N)$.
> *λ –Abstraction*: If M is a λ–term and v is a variable, then $\lambda v.(M)$ is a λ-term.
> And nothing else is a λ–term.

For example, x, $\lambda z.(z)$, and $\lambda x.((x)(y)))(z)$ are λ-terms (HINDLEY 1986, pp.3-5).

[15] For example, the equivalence of the proposition as a member of a formal series and the lambda calculus description of proofs.

Now we should define three more or less obvious aspects of λ–terms: *length*, *occurrence* and *scope*. By *length* of a λ term what is meant is the total natural number of *occurrence* of atoms, atoms being understood as primitive syntactic elements. That is, within a λ-term the *length* of $\lambda x.x$ would be two, while the *length* of $\lambda x.M$ would be one plus the *length* of M, and the *length* of $(M)(N)$ would be the *length* of M plus the *length* of N (HINDLEY 1986, p. 5).

By *occurrence* within a λ–term we mean that for any two λ–terms P and Q, "P occurs in Q", or "P is a sub–term of Q", is defined in term of induction on the latter. We can define the induction as follows:

(a) P occurs in P;
(b) if P occurs in M or N then P occurs in (MN); and
(c) if P occurs in M of $P \equiv x$ then, P occurs in $(\lambda x.M)$

(HINDLEY 1986, p.6).

Scope in a λ–term is defined as follows: for any *occurrence* of $\lambda x.M$ in a lambda term P the *occurrence* of the M is the *scope* of λ on the left. An example would be that for some sentence P:

$$P \equiv (\lambda y.yx(\lambda x.y(\lambda y.z)x))vw$$

The *scope* of the left–most λ is $yx(\lambda x.y(\lambda y.z)x)$, that of the next λ is $y(\lambda y.z)x$, and that of the third is z. This is similar to the *scope* of quantifiers in first order propositional calculus (HINDLEY 1986, p.6).

Adopting the convention that application—*i.e.* the replacement of a bound variable in a lambda abstraction with a base—has higher precedence than the λ-abstraction, and associates from left to right, this means that one can omit parentheses. Thus λ-term $(\lambda x.((x)(y)))(z)$ can be written as $(\lambda x.xy)z$, the expression $\lambda x.(\lambda z(z))$ as $\lambda x.\lambda z.z$, and $((x)(y))(z)$

can be written as xyz. Furthermore, a succession of repeated λ-abstractions can be shortened by using a single λ *e.g.* $\lambda x.\lambda y.\lambda z.x(yz)$ may be shortened to $\lambda xyz.x(yz)$ (HINDLEY 1986, p.3).

Variables in λ–terms are either bound or free. In fact, the same variable can occur both free and bound in a λ–term. Precedence is the determiner here. For example, in $(\lambda x.xx)x$ the last occurrence of the variable x is free, while the other two are λ–bound. A variable in a λ-term is always free unless it is part of the formal argument (*e.g.* 'x' in $\lambda x.M$), or is within the scope of a λ–abstraction, That is, it is within the M, and has the same value as the formal argument. What this means is that those variables which can be substituted with values by applying the λ function to a base are not free. Any two λ–terms which differ only by the consistent renaming of the bound variables are not to be considered different. An example of this is that $(\lambda x.xx)x$ expresses the same equation as $(\lambda y.yy)x$. Of course this means that $\lambda x.xy$ is different from $\lambda y.yy$, since in the former the y is a free variable and it is not λ–bound to the formal argument. A λ–term M is closed if there are no free variables in M. If x is a bound, and y, z, w are free variables in P, we can define: $\mathrm{Fv}(P) = \{y,z,w\}$ (HINDLEY 1986, pp.6–7).

For any M, N, x, the substitution is defined in the following notation: $[N/x]M$ is said to be the result of substituting N for every free occurrence of x in M, with the notation of bound variables changed to avoid clashes (HINDLEY 1986, p. 7). Hindley presents the precise definition after Curry and Feys (CURRY and FEYS 1958, p.94):

(a) $[N/x]x \equiv N$;
(b) $[N/x]a \equiv a$ for all atoms $a \neq x$;
(c) $[N/x](PQ) \equiv ([N/x](P)\ [N/x](Q))$;
(d) $[N/x](\lambda x.P) \equiv (\lambda x.P)$;
(e) $[N/x](\lambda y.P) \equiv \lambda y.[N/x]P$ if $y \neq x$, and $y \notin \mathrm{Fv}(N)$ or $x \notin \mathrm{Fv}(P)$
(f) $[N/x](\lambda y.P) \equiv \lambda z.[N/x][z/y]P$ if $y \neq x$ and $y \in \mathrm{Fv}(N)$ and $x \in \mathrm{Fv}(P)$
in (f), z is chosen to be the first variable $\notin \mathrm{Fv}(NP)$

Beta reduction is the process by which lambda expressions are simplified. A term to be reduced is called the β-redux and the resulting term is called the *contractum*. The form of the β-redux is: $(\lambda x.M)N$, where M or N are any lambda terms, and the form of the *contractum* of this expression is: $[N/x]M$ to be read, "replace x with N in M." An example of β-reduction would be:

$$\lambda x.xy(y) \succ_\beta \lambda x.yy$$

(HINDLEY 1986, pp.11-13).

3.4.2 Church Numerals

The lambda calculus represents the natural numbers by the use of Church Numerals.

$$\bar{0} =_{\text{def}} \lambda xy.y$$
$$\bar{1} =_{\text{def}} \lambda xy.xy$$
$$\bar{2} =_{\text{def}} \lambda xy.x(xy)$$
$$\bar{3} =_{\text{def}} \lambda xy.x(x(xy))$$

$$\vdots$$

$$\bar{n} =_{\text{def}} \lambda xy.\underbrace{x(x \ldots}_{n \text{ times}} (xy))$$

This describes behaviour for any $n \geq 0$ and λ terms M and N such that:

$$\bar{n}\,MN \succ_\beta \underbrace{M(M \ldots}_{n \text{ times}} (M\,N)\ldots)$$

The above is referred to as the n-fold composition of M applied to the argument N.

Just as in Wittgenstein's system, outlined above in section 3.3, where normal arithmetic operators are defined in terms of the theory of operations, here the normal arithmetic operators are here definable in λ terms. First we, like Wittgenstein, define the successor operator as the λ-term:

$S = \lambda n.\lambda xy.(nxy)$

Such that:

$S\bar{4} = \lambda n.\lambda xy.(nxy)\ \bar{4}$
$\succ_\beta\ \lambda xy.(\bar{4}xy)$
$=\ \lambda xy.x((\lambda xy.x(x(x(xy))))xy)$
$\succ_\beta\ \lambda xy.x((\lambda y.x(x(x(xy))))y)$
$\succ_\beta\ \lambda xy.x(x(x(x(xy))))$
$= \bar{5}$

and so we can define the arithmetic operator for addition as:

$A = \lambda mn.mSn$

So that we apply the successor function *m* times to *n*. And we can define product as:

$M = \lambda mn.\lambda xy.m(nx)y$

The successor function can be compared with the definitions given in 6.02 of the *Tractatus*:

$x = \Omega^{0}{}'x$ Def., and
$\Omega'\Omega^{\nu}{}'x = \Omega^{\nu+1}{}'x$ Def.

And Wittgenstein's definition of product in 6.241:

6.241 Thus the proof of the proposition $2 \times 2 = 4$ runs as follows:
$(\Omega^{\nu})^{\mu}{}'x = \Omega^{\nu\times\mu}{}'x$ Def.,
$\Omega^{2\times2}{}'x = (\Omega^{2})^{2}{}'x = (\Omega^{2})^{1+1}{}'x = \Omega^{2}{}'\Omega^{2}{}'x = \Omega^{1+1}{}'\Omega^{1+1}{}'x = (\Omega'\Omega)'(\Omega'\Omega)'x = \Omega'\Omega'\Omega'\Omega'x = \Omega^{1+1+1+1}{}'x = \Omega^{4}{}'x.$

The variable x is used here by Wittgenstein as an object variable, while Ω is used as an operation variable. Wittgenstein has told us three things about operation variables. First that they are uniform (NB 22/11/16; 5.23 5.231); second that there is no object that corresponds to them as its fixed semantic value (5.25); and third that an operation can be applied to the result of its application (5.251). For Frascolla the use of + for addition will be done using the successor operation, so the superscripted 0+1+1+...+1 not be mistaken for the + operation of normal arithmetic terms (FRASCOLLA 1994, p.9). The superscripted notation 0 and ν+1 are for Wittgenstein the usual arithmetic terms with the series 0+1+1+1+...+1 derived from

$\Omega 0+1+1+1+...+1$ where n, or in Wittgenstein's terms ν, is the natural number of occurrences of +1. Frascolla notes that $\Omega^{t}{}'x$ similarly corresponds to the arithmetic term t and so $\Omega^{t}{}'x = \Omega^{s}{}'x$ correlates to $t = s$. The definition given is construed by induction of the natural number of occurrences of the sign S in the example $\Omega^{SSS...0}$ in Frascolla's notation. Looking at 6.02 more completely we have:

> 6.02 And this is how we arrive at natural numbers. I give the following definitions
>
> $$x = \Omega^{0}{}'x \text{ Def.,}$$
> $$\Omega'\Omega^{\nu}{}'x = \Omega^{\nu+1}{}'x \text{ Def.}$$
>
> So, in accordance with these rules, which deal with signs, we write the series
>
> $$x,\ \Omega'x,\ \Omega'\Omega'x,\ \Omega'\Omega'\Omega'x,\ \dots,$$
>
> in the following way
>
> $$\Omega^{0}{}'x,\ \Omega^{0+1}{}'x,\ \Omega^{0+1+1}{}'x,\ \Omega^{0+1+1+1}{}'x,\ \dots .$$
>
> Therefore, instead of '$[x, \xi, \Omega'\xi]$',
> I write '$[\Omega^{0}{}'x, \Omega^{\nu}{}'x, \Omega^{\nu+1}{}'x]$'.
> And I give the following definitions:
> $0 + 1 = 1$ Def.,
> $0 + 1 + 1 = 2$ Def.,
> $0 + 1 + 1 +1 = 3$ Def.,
> (and so on).

As described above Frascolla's analysis makes several comments about the operation theory language. The method we shall use is to compare the features of the operation theory language identified by Frascolla to Church's lambda calculus definition of natural number. Before this let us look at what Wittgenstein has told us about operation variables. They are uniform (NB 22/11/16; 5.23 5.231); there is no object that corresponds to them as its fixed semantic value (5.25); and an operation can be applied to the result of its application (5.251). Let us compare the two notations by applying Wittgenstein and Frascolla's list of features of operations and of "what is going on" to the lambda calculus.

That operations are uniform is the first point that Frascolla makes concerning what Wittgenstein says about operations. Wittgenstein writes in 5.23–5.231

> 5.23 The operation is what has to be done to the one proposition in order to make the other out of it.
>
> 5.231 And that will, of course, depend on their formal properties, on the internal similarity of their forms.

What this means is that each operation acts in the same manner on propositions— the effect they have on the structure of propositions follows a rule—and the rule applies in the same manner each time. Thus as Wittgenstein later writes:

> 5.253 One operation can counteract the effect of another. Operations can cancel one another.

Each operation is uniform, and thus it is possible to construct the opposite operation. This is the case with any lambda function as well.

Secondly, let us consider the point Wittgenstein makes that, "no object attaches to… [operations] as a fixed semantic value" in 5.35:

> No statement is made by an operation, but only by its result, and this depends upon the bases of the operation. (Operations and functions must not be confused with one another)

Compare this to a lambda function:

$$\lambda x.\lambda y.(x+y)$$

Much like the operation there is no assertion made here. As with Wittgenstein's operations until a base is applied, there is no sentence.

$$\lambda x.\lambda y.(x+y)(3)(4) = 7$$

When a base, or two in this case, is applied there is a result. Frascolla then notes that "an operation can be applied to the result of its application." This too is obviously the case with lambda calculus; the symbol $\succ_\beta$ is used to apply one lambda term to another. Returning to Church numerals, as above, we define the successor operator as the λ-term:

$$S = \lambda n.\lambda xy.x(nxy)$$

Unlike functions which are defined in a set theoretic manner the lambda function contains only the algorithm and not the entities applied to it. Thus the result of one can be applied to another.

The more difficult comparison, Frascolla asks, is what is going on in 6.02. First he notes that "Ω^{0}'x is identical with x." The expression for the operation theory language of the natural number 0 applied to a base has the same value as the base. In lambda calculus we define the church numeral $\bar{0}$ in the following manner:

$$\bar{0} = \lambda xy.y$$

Applying the Church numeral $\bar{1}$,that is $\lambda xy.xy$, provides the following result.

$$(\lambda \mathrm{n}.\lambda xy.x(nxy))(\lambda xy.y)$$

Beta reduction replaces the variable bound to λ furthest to the left.

$$\succ_\beta \lambda xy.x(\lambda xy.y(\mathrm{x}y))$$
$$\succ_\beta \lambda \mathrm{y}.(\lambda xy.y(\mathrm{x}y))$$
$$\succ_\beta \lambda xy.xy$$
$$=_{\mathrm{def}} \bar{1}$$

The second point that Frascolla notes about the operational definition of natural number is that "x is the initial form of the series," that the beginning of the series defined is one before the application of the operation. If we look above to the successor operation the second half is the equivalence to the second member in the series, because that is what is applied i.e. one more is added. Thus:

$$S = \lambda n.\lambda \mathrm{xy}.(nxy)$$

The first member is $\bar{0} = \lambda xy.y$ that is expression that represents the base alone when the base is applied. The series x, Ω'*x*, $\Omega\Omega$'*x*, … expresses the base as *x* (in the Church numeral the x is not expressly stated.)

The third point Frascolla makes is that '0 then stands for 'formal property constituted by natural number of times the operation is applied'." So 0 stands for the formal property constituted by the application of the operation no times, 1 stands for the formal property constituted by the application of the operation one time, 2 *et cetera.* Church numerals are certainly generated by the application of the succession lambda function [S = $\lambda n.\lambda xy.x(n(x(y)))$], but what do they then stand for? Since they are lambda functions they require a base to refer to any object. Thus they refer only to the algorithm that created them. They are purely formal, in the sense of the word as used by Wittgenstein in the *Tractatus.* They represent only a process, and not objects.

Recall that Frascolla also notes that Wittgenstein's definition of natural numbers implies that "Ω^{S0}'*x* is identical with Ω'*x*." The application of the successor operation to the base 0 is equivalent to the sign for the natural number 1. In lambda calculus we would write this:

$$
\begin{aligned}
&S\bar{0} = \lambda n.\lambda xy.x(nxy)\bar{0} \\
&\succ\beta\ \lambda xy.x(\bar{0}xy) \\
&\succ\beta\ \lambda xy.(x(\lambda xy.y)xy) \\
&\succ\beta\ \lambda xy.(x(\lambda y.y)y) \\
&\succ\beta\ \lambda xy.xy \\
&= \bar{1}
\end{aligned}
$$

The application of the successor operation to the base numeral $\bar{0}$ does provide the form of the Church numeral $\bar{1}$.

Recall from above Frascolla's analysis of 6.02, which states that Ω'x shows:

> at some utmost level of generality, [as x] the form of an expression which is the result of the successive application of an operation consisting of one application to an initial symbol (FRASCOLLA 1994, p.11).

The expression Ω'x is the application of the initial symbol Ω to x. This is how $\lambda xy.xy$ can express this "utmost level of generality" of the form of one application to an initial symbol. The form $\bar{n} = \lambda xy.x(x \ldots (xy))$ with n of x in the bound expression describes Church numerals at a basic level. The $\lambda x.\lambda y.x$ in the expression taking the place of Ω', and the y the place of Wittgenstein's base x, to carry the conversion one further the Ω is the 'x' of the lambda expression and the apostrophe of several apostrophes is the binding '$\lambda x.\lambda y$' that indicates the application of x to y. By this scheme the description of the series Ω'Ω'Ω'Ω'… Ω'x , becomes understood as $\lambda x\lambda y.x(x \ldots (xy))$. The most general form of the single application is then $\lambda x\lambda y.(xy)$ or Ω'x.

Finally, as we have seen, Frascolla notes that "'S0' stands for the formal property constituted by the natural number of times an operation is applied to form such an expression." The 'S0' of operational theory language is the short hand that Wittgenstein presents in 6.02 "Ω^{0}'x, Ω^{0+1}'x, Ω^{0+1+1}'x, $\Omega^{0+1+1+1}$'x, ..." is written for the expanded form "x, Ω'x, Ω'Ω'x, Ω'Ω'Ω'x, ...". If there must be an equivalent of this in the lambda calculus form it is the n applications of the S expression to the base $\bar{0}$ that is $SSS...S\bar{0}$, where S is the lambda expression $\lambda n.\lambda xy.x(nxy)$. Expanded it is written:

$$(\ldots(((\lambda n.\lambda xy.x(nxy))\ \lambda n.\lambda xy.x(nxy))\ \lambda n.\lambda xy.x(nxy))\ldots\ \lambda n.\lambda xy.x(nxy))(\lambda xy.y)$$

Thus the lambda expressions of Church numerals, and the operation theory language of the *Tractatus*, express very similar if not equivalent understandings of natural number. That is, as series of operations, to use Wittgenstein's terms, they are not true sentences or functions, rather they only express a variable (or lambda term) that produces another form when another term of the same type is applied as a base. Additionally, although I cannot here describe this at length, just as in Wittgenstein's system where normal arithmetic operators are defined in terms of the theory of operations the normal arithmetic operators are also here definable in λ terms.

3.4.3 Series of Forms as Proofs

This section develops the similarity noted in the previous section by looking at the virtual reading of a formal series as proof. Though we cannot go into much detail, this section shall point out how the similarities of Wittgenstein's operations and the lambda calculus apply not only for natural number and arithmetic but also for logic. For both a formal series that defines a proposition can be seen as a construction of that proposition. And a construction of a proposition can be converted into a proof of a proposition.[16] It is by focusing on the intensional aspect, as we defined it earlier, of the proposition rather than its extensional referent. The proposition becomes more than a truth table. The propositions meaning can in this sense be seen as a function of its proof. Constructive logic, specifically constructive semantics can help us understand this process. The description of proof as algorithm is common place in logic. Each step of a proof is conceived of as the application of a rule that generates from other lines of a proof the next step. The lambda calculus is a common tool for describing algorithms, and has been applied also to the typing of proofs (*cf.* MARTIN-LÖF 1975 1985 1987 1988, NORDSTROM *et al.*1990). Thus proof can be described as a lambda function much the same way calculations, as shown above, can. Just as Wittgenstein applied the operation to the formal series of propositions, and all complex propositions are seen by Wittgenstein as the outcome of the concatenation of various operations in a formal series, so

[16] Using Gentzen's sequence calculus a proof can be converted into axiom statements and logical rules of construction.

the outcome of a proof, the complex proposition in the form of X, Y, Z, $\vdash$ W, can be written in a lambda function, or in a series of operations that start from the propositions X, Y and Z and produce the complex proposition W. Although this cannot be developed fully here, the point to be made is that both lambda calculus and Wittgenstein's operations provide a general method of constructing formal languages that focuses on the procedural or intensional aspects rather than defining the abstractions as extensions such as classes or sets.

This brings us to the final part of the thesis: the three critiques that follow from what we called the positive programme. The first of these is Wittgenstein's critique of the superfluity of classes. As noted above, the operation provides a method of constructing for both mathematics and logical systems that does not depend on the use of classes as extensions. That is, Wittgenstein's operational definition presents mathematics as a group of processes and not entities. The superfluity of classes follows from the above methodology.

3.5 The Superfluity of Classes

The next three sections consider briefly the criticisms Wittgenstein develops as a result of the views outlined above. The first criticism is Wittgenstein's rejection of classes (6.031). Wittgenstein criticises classes as being unnecessary for the foundation of mathematics:

> 6.031 The theory of classes is completely superfluous in mathematics. This is connected with the fact that the generality required in mathematics is not accidental generality.

The operational definition of natural number does away with any need to conceive of natural number defined in terms of classes. The description of arithmetic that he has given with the operational theory has not disproved Whitehead and Russell's analysis in the *Principia Mathematica*, rather it is an attempt to show that there is not one definition of natural number. It is an analysis of logic and mathematics that disagrees with the logicist position that mathematics is reducible to logic not by rejecting that there is a relation.

The statement "all men are mortals" (6.1232), Wittgenstein states, is not a truth of logic but "the result of a fortunate accident." The set theoretic definition of function is similarly "accidental." It defines a function not by the action that takes place, but by describing the way things are in the world. The operational definition of both logic and mathematics is something different. Rather than pointing to an extension, some group of "mortal men," it goes about breaking down the proposition to its elementary propositions, and displays that there is some process at work; some formula or algorithm that produces such complex propositions. In logic, with propositions, it is hard to establish this intensional understanding. The question that Wittgenstein does not answer remains: "What are these elementary propositions?" This is not answered because this question is not the point of the *Tractatus*. The intensional nature of the construction of the complex is not further elucidated by knowing what exactly the beginning point is. For Wittgenstein this was an empirical question and not a logical one. Thus it was not his topic. In 5.01 Wittgenstein notes three forms of description:

> We can distinguish three kinds of description:
>
> 1.Direct enumeration, in which case we can simply substitute for the variable the constants that are its values;
>
> 2. giving a function *fx* whose values for all values of *x* are the propositions to be described;
>
> 3. giving a formal law that governs the construction of the propositions, in which case the bracketed expression has as its members all the terms of a series of forms.

In mathematics, for Wittgenstein, only the third method applies. The first method applies for objects. The second is what Russell attempts to use in his description of mathematics. But, for Wittgenstein, the first two talk about the objects of the world, and the third provides for the intensional nature of logic.

In Wittgenstein's description of arithmetic there are no facts, and no elementary propositions. Rather the equations of mathematics, like logical tautologies, lack a distinct subject matter. They are shown by the operation in Wittgenstein's view of mathematics, and the above comparison with the lambda calculus demonstrates that this, constructivist, view of natural number is primary to an algorithmic understanding of mathematics.

The lambda calculus is used not only to define natural number, as in Church numerals, but also used to define other functions algorithmically. It is the formal language commonly used in computation for this reason. If something is to actually be computed, if a function or a program is to be described, it cannot be done so practically in a set theoretical model. The model that is use is one that focuses on the method for computation. This leads us to how Wittgenstein describes mathematics; he calls it a "logical method."

3.5.1 What does "Logical Method" Mean?

"Logical method" must be taken as a view in opposition to that of the existence of logical entities. The emphasis, of the model Wittgenstein uses, must be on method and not logical entities, Wittgenstein is not saying that mathematics is definable or the same as logic, but rather that it is a method, and furthermore that logic can too be described as a method. What, then, is a method? In the case of arithmetic it is the manner of construction. In logic it is the application of the logical operators which are, in turn, defined from simpler ones. And, as we have seen, this is exemplified by Wittgenstein's theory of operations and is implicit in the presentation of function in the lambda calculus. Thus we can see that the positive programme stands as an alternative view of logic and mathematics and not just as a negative critique of the logicist programme. This view, as presented above, takes mathematics to be a method—it provides an algorithmic description of natural number and arithmetic.

3.6 Mathematic Propositions

On the basis of the view of mathematics outlined above we are now in a position to fully appreciate Wittgenstein's critique of the logicist view of mathematical propositions. In thesis 6.2 Wittgenstein rejects the logicist view of mathematical propositions because for him mathematics is not a description of objects, but rather a way of describing "internal" or "formal" relations through equations.

> 6.2 Mathematics is a logical method. The propositions of mathematics are equations, and therefore pseudo-propositions.

Wittgenstein expands on his view in 6.211 where he rejects the idea that mathematical propositions are about the world. Rather, mathematical propositions are used as intermediaries. They are used to clarify inferences from "propositions that do not belong to mathematics" (6.211) to other non-mathematical propositions. As an aside, Wittgenstein also points out the 'use' of a proposition is a clear guide understanding meaning:

> In philosophy the question, 'What do we actually use this word or this proposition for?' repeatedly leads to valuable insights (6.211).

As with so much of Wittgenstein's philosophy 'use' is the key to understanding his view of natural number. The 'use' of natural number in what Wittgenstein calls "real life" is as an intermediary; it is a method of reasoning and not a thing in and of itself. That the logicists treat it as such is the root of their error.

Wittgenstein refers to mathematical propositions as pseudo–propositions because they have no reference to what he calls 'accidental truths;' i.e. those that describe the facts of the world. This is so exactly because he takes mathematics to be a method of applying operations to a base. Thus mathematics is defined as the act of calculation rather than being dependant on what he calls accidental truths, *i.e.* truths about the world. Mathematics is about the expression of a rule, and not an accounting of objects. Thus mathematics is not about the world, but is the method used to generate one proposition, or pseudo–proposition, from another. It is thus a method and not a thing.

3.7 Connectives

Wittgenstein's third criticism that follows from his positive programme is of the logicist view that logical connectives (4.0312, 5.4–5.44) refer to specific entities. Rather, he shows that they can be described both by logical operations or truth functions. Wittgenstein writes in 4.0312:

> 4.0312 The possibility of propositions is based on the principle that objects have signs as their representatives. My fundamental idea is that the 'logical constants' are not representatives; that there can be no representatives of the logic of facts.

The logical constants are not going to be realised in Wittgenstein's system as "logical data"(*cf.* RUSSELL 1984, pp. 97-104). The logical connectives are determined intensionally as the internal relations between objects in propositions. They are not "relations in the sense of right and left" Wittgenstein writes (5.42), but internal relations. They are logical and thus part of that which may only be shown in Wittgenstein's system. The logical connectives are defined by operations, which modify the structure of complex propositions. The distinction is the same as noted above, *i.e.* that of intensional and extensional descriptions of logic.

Once the difference between the extensional and the intensional definition of logic is grasped understanding the difference between logical or formal and accidental generalities can be easily understood. If the nature of x is not the reason that $\forall(x)f(x)$, then this generality is just an observation, rather than a necessity; it is no more than an accidental generality. If the nature of x is such that for every x then $f(x)$, then this is a logical or formal generality. As Wittgenstein writes:

> 6.3 The exploration of logic means the exploration of everything that is subject to law. And outside logic everything is accidental.

Thus if mathematics is formal, and logical, it does not contain accidental generalities, it does not say anything about the world. This is why Wittgenstein writes that the mathematical method is the method of substitution. This does not mean we cannot use mathematics to describe the world, but rather when propositions use mathematics they use it to expresses structure.

4 Conclusion

This thesis is divided broadly into two main parts, or two discussions, of what I call the negative and the positive programmes of the *Tractatus*. In the first half of the thesis I described how Wittgenstein critiques the logicist definition of number. In the second I show how the understanding of logic and mathematics Wittgenstein develops are connected with the overall project of the *Tractatus*; in fact how the distinctions that inform his understanding of number are central and develop clearly out of the whole of his project. Both discussions are informed by a consideration of the distinctions between intensional and extensional conceptions of logic and mathematics. The argument put forth is that the logic and mathematics, specifically the theory of arithmetic, that Wittgenstein presents has a profoundly intensional interpretation. At the root of this is the theory of operations presented in detail in chapter 3. It is not that Wittgenstein did not think that propositions were in some sense extensional; he did, but he asserted that they had intensional aspects as well. Since the operation formalised those intensional aspects of sentence, and the theory of operations is the method by which Wittgenstein describes the natural numbers, it is shown that the description of the numbers in the *Tractatus* is wholly intensional. Propositions refer to the world, specifically to states of affairs, but mathematics is entirely formal in the *Tractatus*. It therefore refers not to the objects of the world, but derives its meaning from the processes by which its statements are constructed.

4.1 The Negative Programme

Chapter 2 of the thesis begins with an account of parts of the logicist system. I briefly describe the logicist programme, and Russell and Whitehead's attempts to fix the problems of Russell's paradox by introducing the theory of types. I then describe Wittgenstein's criticisms of the programme.

In section 2.1 I began by discussing the beginnings of the logicist programme, in an attempt to show how Russell's paradox caused problems with the extensional definition of numbers. I described the logicist project under Frege and I showed how Russell's Paradox undermined his system. While Frege's definition of function in the *Begriffsschrift* sounds much like Wittgenstein's definition of operation, since it focuses on the procedural nature of the function, his system in the *Grundgesetze der Arithmetik* allowed for a arbitrary conversion of functions to classes and so was susceptible to Russell's paradox..

Type theory, Russell's solution to his paradox, did not spring into existence in the final form that it appears in the *Principia Mathematica.* Wittgenstein would have known that its development was tumultuous. The final form of the logicism in the *Principia Mathematica* was by no means the only method that Russell considered. His earlier attempts tried to reconcile several motivations. Although type theory today seems a basic, and important, part of mathematical logic it was not always the case. Wittgenstein's critique is not to be seen as a heresy against a generally acceptable form of mathematics, but rather as taking part in an ongoing philosophical discussion.

Wittgenstein's rejection of type theory on the grounds that it was a transgression of the showing–saying distinction. That is, type theory as it pertains to understanding the logic of sensible propositions, and their construction. Wittgenstein never rejects it as being part of an extensional language; rather he rejects that language as not being able to express the structural nature of logic.

This leads us to my discussion of how intensionality and the showing–saying distinction are intertwined. Von Wright makes the comment that the *Tractatus* is intensional as it pertains to formal, or logical, concepts, and extensional as it pertains to proper, or empirical, concepts. The former are those which Wittgenstein holds may only be shown, the latter are those which may be said. With the above in mind in I discussed the relation between Wittgenstein's showing–saying distinction and critique of type theory. The intensional nature of logic described by the showing–saying distinction does not permit any sort of meta-linguistic theory akin to Russell's theory of types to describe logic. Logic, and mathematics, all formal parts of language, are shown, demonstrated, proven, rather than delimitated by a theory.

Section 2.3 dealt with Wittgenstein's criticism of the axioms of infinity and reducibility. Both are rejected as not logical axioms. Wittgenstein's criticism that they are empirical laws if they are true, but not logical axioms, is rooted in his intensional conception of the workings of logic.

In section 2.4 I discussed Wittgenstein's argument for the circularity of the logicist's definition of number. The circularity is bound in the nature of treating functions as abstract properties; the ancestor relation thus must be defined in terms of ancestry, which Wittgenstein argues is circular.

Thus the 'negative' programme of the *Tractatus* entail the criticisms of the logicist programme that Wittgenstein based, for the most part, on his understanding of what the logicists were doing in their own terms. The problem is the underlying understanding of what is and what is not logical. For Wittgenstein logical axioms, *e.g.* the axiom of infinity, are about the number of items in the world. But, as Wittgenstein writes, that sort of generalization does not apply to formal languages (5.01).

After dealing with the criticisms that open Wittgenstein's rejection of logicism I move on to his 'positive' programme, where I describe Wittgenstein's theory of operations and how it pertains specifically to his view of natural numbers and arithmetic.

4.2 The Positive Programme

In chapter 3 of the thesis the 'positive' programme is presented; highlighting Wittgenstein's theory of operations and his understanding of natural numbers and arithmetic. In section 3.1 I presented an exegesis of Wittgenstein's theory of operations, and in section 3.2 I dealt with Fogelin's critique of what he calls the central error in the *Tractatus*. In 3.3 I described Frascolla's exegesis of number and arithmetic in the *Tractatus*, and in section 3.4 I showed how the operations language theory of Wittgenstein is equivalent to lambda calculus; both of which are used to describe intensional understandings of the natural numbers. After these sections I then end the chapter by describing, what I call, Wittgenstein's 'positive' criticisms of the superfluity of classes, the logicist view of mathematical propositions and the logicist view of connectives. These seem to develop out of the views that he has presented about logic and mathematics.

In section 3.1 I investigated how Wittgenstein's views about the nature of logic are presented in the *Tractatus* by working through his views about the theory of operations. Section 3.1.1 considers formal relations and the operation. In thesis 4.12 Wittgenstein writes that we cannot do more than show the logical form of a statement because to represent it, "we should have to be able to station ourselves with propositions somewhere outside logic, that is to say outside the world." This establishes the break, that Wittgenstein argues for, between that which can be said and that which can be shown.

In section 3.1.2 of this thesis I discussed sections 4.122-4.123 of the *Tractatus* where Wittgenstein describes formal properties and relations. In this section it was established that 'formal' and 'internal' are synonymous for Wittgenstein. Furthermore, we saw that for Wittgenstein formal properties and internal relations are what are shown and not said, and that an internal property is the most primary property of an object; the structure of the object rather than the content of the object.

In section 3.1.3 I considered how this view of the formal aspects of the proposition led to Wittgenstein's conception of the formal series. In sections 4.124-4.126 we learn that a formal series is ordered by an internal relation determined by an operation. In 4.1252 Wittgenstein describes the successor relation. In section 3.1.4 the construction of a formal series, which Wittgenstein presents in 4.1273, is examined. I began by explaining how Frege's position that the "and so on" of standard construction of the general series "0+1+1+1 ... and so on," is psychologistic is rejected by Wittgenstein in 4.1273. Wittgenstein position is that "and so on" is a formal process, an operation. Wittgenstein avoids the psychologistic element by formalising the algorithmic the understanding of the successor function, through the notion of the operation.

I then turned, in section 3.1.5, to sections 4.5 through 5.01 of the *Tractatus* where Wittgenstein discusses the essence of the proposition. The essence of the proposition is, as Wittgenstein states, "this is how things stand." This general form of the proposition, Wittgenstein writes in 4.5 "is proved by the fact that there cannot be a proposition whose form could not have been foreseen (i.e. constructed)." It is this construction that is central to

Wittgenstein views of logic and mathematics. All propositions, Wittgenstein writes, are "generalizations of elementary propositions"(4.52). That is, they are constructed out of elementary propositions. How this happened brings us to the next section of the thesis.

In section 3.1.6 I discussed Wittgenstein's understanding of this construction, the operation and how the operation defines internal relations. I then turn, in section 3.1.7, to the example Wittgenstein gives in his text. This brings us to the general form of the truth function which is the 6th thesis of the *Tractatus*. There is a significant debate over this part of the *Tractatus*. This debate centers over the issue of whether or not Wittgenstein's method of construction, really allows one to construct propositions of multiple generalities. As a preliminary to my presentation of this debate I attempt to present what Wittgenstein means in his notation in sections 6 through 6.01.

My presentation of Wittgenstein's notation in section 3.1.7 leads into section 3.2 where I discuss in more detail Fogelin's critique of what he calls "the central error in the Tractatus" and the debate which his comments engendered. This debate has been brought to an end by Cheung's work which shows that *Tractatus*' logic works but must restrict itself to finite domains of elementary propositions. That is, it must not offend Church's restrictions. After presenting this debate we are prepared to look at Wittgenstein's presentation of number.

Frascolla's interpretation of, and his modifications to, Wittgenstein's system of arithmetic is the topic of section 3.3 of this thesis. Frascolla addresses, carefully and systematically, Wittgenstein's operation theory and his construction of number. He begins by attempting to clear up the exegesis of Wittgenstein's comments on number and arithmetic in the *Tractatus*. I consider first Frascolla's criticism of Black and Anscombe, and then his interpretation of Wittgenstein's operations theory.

In section 3.3.1 of this thesis Frascolla's critique of the sections 6.02-6.031 of the *Tractatus* are presented. Frascolla first deals with the views of Black and Anscombe, both of whom present questions that show they have not understood Wittgenstein's view of operation. Frascolla provides a clear exegesis of thesis 6.02 through 6.031 and makes several modifications to Wittgenstein's notation to make clear his meaning. However the modifications do not extend the language of Wittgenstein's theory of operations but instead are extrapolated from it.

Frascolla's exegesis of Wittgenstein's description of the operations of sum and product were presented with commentary in section 3.3.2. I described how in 6.241 Wittgenstein provides us a guide to understand operation theory language terms. Frascolla's modification of Wittgenstein notation, it is argued, must be accepted, as it serves the purpose of disambiguation and allows the conversion of Wittgenstein's operation theory language into standard arithmetical definitions of sum and product.

The proof of 6.241 of the *Tractatus*, wherein in "$\Omega^{2\times 2}{}'x$" is converted into "$\Omega^{1+1}{}'\Omega^{1+1}{}'x$" was presented in section 3.3.2.1. The proof provided in 6.241 is taken as a guide by Frascolla in his proof. By looking at it we understand how Frascolla develops his several rules and definitions. Frascolla presents several syntactic rules, much akin to Wittgenstein's general form of the operation, which clarify what is going on in the proof in 6.241.

In sections 3.3.3 I then discussed Frascolla's proof. After introducing these rules and definitions Frascolla uses his interpretation of Wittgenstein's operational language to prove the simple arithmetical statement $(2+3)\times(2+1) = 15$. This proof provides both product and sum to calculate, and shows how Frascolla's new rules clearly describe the process of the proof in 6.241.

The equivalence of the operation language theory to lambda calculus were then considered in section 3.4. Both lambda calculus and the operation language theory notation focus on the algorithmic nature of functions, or an intensional rather than extensional understanding of function.

Lambda calculus syntax and then Church numerals were introduced in sections 3.4.1 and 3.4.2. Once the syntax of lambda calculus is presented I showed how lambda calculus represents the natural numbers by the use of Church Numerals. Just as in Wittgenstein's system, where the theory of operations defines the normal arithmetic operators, in Church's system λ terms define the normal arithmetic.

In my comparison between the operation theory language the Church's lambda calculus, I argued that the lambda calculus has similar features to the operation theory language. First they are uniform, second they construct the natural number series similarly, and third they define 0 similarly.

In section 3.4.3 I discussed how a formal series can be seen to be equivalent to a proof. Wittgenstein saw all complex propositions as the outcome of the concatenation of various operations in a formal series. In the same way, it is argued, that the out come of a proof is a complex proposition. In this matter both arithmetic calculation and the construction of elementary propositions can be seen as proofs.

In final part of the thesis, I elaborate on the three critiques that follow from what I call the 'positive' programme. In section 3.5 I discuss Wittgenstein's rejection of the necessity of the theory of classes in the foundation of mathematics (6.031). Wittgenstein's operational definition of natural number renders unnecessary the need to conceive of natural number defined in terms of classes. In this spirit Wittgenstein makes a distinction between logic and mathematics, and propositions about the world. This is reflected in the two forms of generality: the one required in mathematics and logic, and accidental generality. Logical generality is defined by a formal rule, and accidental generality describes objects in the world. Wittgenstein identifies the set theoretic definition of function as accidental. Wittgenstein presents in 5.01 three forms of description of functions or operations—direct enumeration, giving a function whose values are the propositions to be described, and the giving of a formal law that governs the construction of the propositions—and it is the third method that provides us with logical generality, and thus mathematics.

In section 3.6 I discussed Wittgenstein's critique of the logicist view of mathematical propositions. Equations are not propositions for Wittgenstein, and are not the same as tautologies. They are both pseudo-propositions, as they don't speak about the world. The operation in propositional logic results in a proposition and in mathematics it results in numbers. The tautology is analogous, but in mathematics the operation never results in a proposition; only pseudo-propositions. Wittgenstein made it clear that the truth functional operations of logic do not apply to mathematics and that equity was not an operation. The equal sign just shows that "different [mathematical] operations lead to the same result" (WAISMANN 1979, p. 218 *cf.* also MARION 1998, p.65).

In the final section of this thesis, 3.7, I discussed Wittgenstein's criticism of the logicist view of connectives. This is that logical connectives (4.0312, 5.4–5.44) refer to specific entities. For Wittgenstein the logical connectives are determined intensionally, as the internal relations between objects in propositions. In this way the logical connectives are defined by operations that modify the structure of complex propositions. This final criticism of logicism is, like the previous five, once again rooted in the intensional conception of the construction of propositions.

Bibliography

Primary Sources

BENACERRAF, Paul and PUTNAM, Hilary eds. (1964), *Philosophy of Mathematics: Selected Readings*, Englewood Cliffs, New Jersey: Prentice Hall.

MANCOSU, Paolo (1998), *From Brouwer to Hilbert: The debate on the Foundation of Mathematics in the 1920s*, Oxford: Oxford University Press.

FREGE, Gottlob (1884)*Die Grundlagen der Arithmetik, Eine logish-mathematische Untersuchung über den Begriff der Zahl*, Verlag von Wilhelm Koeber, Breslau. Translated as *The Foundations of Arithmetic*, (1953), J.L. Austin trans. Oxford: Basil Blackwell.

FREGE, Gottlob (1964), *The Basic Laws of Arithmetic: Exposition of the System*, Montgomery Furth trans., Berkeley: University of California Press.

FREGE, Gottlob (1879), *Begriffsschrift, Eine der arithmetischen nachgeblidete Formelsprache des reinen Denkens*, Verlag von L. Nerbert, Halle. Translated as "*Begriffsschrift*, a Formal Language, Modeled upon that of Arithmetic, for Pure Thought," in VAN HEIJENOORT (1967): 5-82.

FREGE, Gottlob (1980), *Philosophical and Mathematical Correspondence*, H. Kaal trans., Brian McGuinness ed., Chicago: University of Chicago Press.

FREGE, Gottlob (1984), *Collected Papers On Mathematics, Logic, And Philosophy*, Black, *et al.* trans., McGuinness ed., Oxford: Basil Blackwell.

GEACH, Peter and BLACK, Max eds. (1970), *Translations from the Philosophical Writings of Gottlob Frege*, Oxford: Basil Blackwell.

VAN HEIJENOORT, Jean ed. (1967a), *From Frege to Godel. A Sourcebook in Mathematical Logic, 1879–1931*, Cambridge Mass.: Harvard University Press.

HINDELY, J. Roger and SELDIN, Jonathan (1986), *Introduction to Combinators and λ-Calculus*, Cambridge: Cambridge University Press.

RAMSEY, Frank Plumpton (1950), *The Foundations of Mathematics and Other Logical Essays*, R.B Braithwaite ed., New York: The Humanities Press.

RUSSELL, Bertrand (1903), *The Principles of Mathematics*, Cambridge: Cambridge University Press.

RUSSELL, Bertrand (1905), "On Denoting" in RUSSELL 1973.

RUSSELL, Bertrand (1906), "On the Substitutional Theory of Classes and Relations" in RUSSELL 1973.

RUSSELL, Bertrand (1908), "Mathematical Logic as Based on the Theory of Types," *American Journal of Mathematics*, 30: 222-262. Reprinted in VAN HEIJENOORT 1967: 152-82.

RUSSELL, Bertrand (1918), *The Philosophy of Logical Atomism*, in RUSSELL (1985).

RUSSELL, Bertrand (1919), *An Introduction to Mathematical Philosophy*, London: George Allen and Unwin Ltd.

RUSSELL, Bertrand (1973), *Essays In Analysis*, Douglas Lackey ed., New York: George Braziller.

RUSSELL, Bertrand (1984), *Theory of Knowledge: The 1913 Manuscript*, Elizabeth Ramsden Eames and Kenneth Blackwell ed., London: Routledge.

RUSSELL, Bertrand (1985), *The Philosophy of Logical Atomism*, David Pears ed., London: Open Court Classics.

RUSSELL, Bertrand (1992), *Logical and Philosophical Papers 1909-13*, John G. Slater ed., London and New York: Routledge.

RUSSELL, Bertrand (1997), *Problems of Philosophy*, NewYork, Oxford: Oxford University Press.

SKOLEM , Thoralf (1967), "The Foundations of Elementary Arithmetic established by means of the Recursive Mode of Thought, without Use of Apparent Variables ranging over Infinite Domains" in VAN HEIJENOORT (1967).

WHITEHEAD, Alfred North and RUSSELL, Bertrand (1927), *Principia Mathematica*, vols. I, II, III Second Editon, Cambridge: Cambridge University Press.

WITTGENSTEIN, Ludwig (1961), *Notebooks 1914–1916*, G.H. von Wright and G.E.M. Anscombe eds., G.E.M. Anscombe trans., Oxford: Basil Blackwell.

WITTGENSTEIN, Ludwig (1974), *Philosophical Remarks*, Rush Rhees ed., Raymond Hargreaves and Roger White trans., Oxford: Basil Blackwell.

WITTGENSTEIN, Ludwig (1974), *Philosophical Grammar*, Rush Rhees ed., Anthony Kenny trans., Oxford: Basil Blackwell.

WITTGENSTEIN, Ludwig (1974), *Tractatus Logico-Philosophicus*, David Pears and Brian McGuinness trans., London: Routledge.

WITTGENSTEIN, Ludwig (1989), *Wittgenstein's Cambridge Lectures on the Foundations of Mathematics Cambridge 1939 from the notes of R.G. Bosanquet, Norman Malcom, Rush Rhees, and Yorick Smythies*, Cora Diamond ed., Chicago: University of Chicago Press.

WITTGENSTEIN, Ludwig (1992), *Tractatus Logico-Philosophicus*, C.K. Ogden trans., London: Routledge.

WITTGENSTEIN, Ludwig (1992), *Philosophical Investigations*, G.E.M. Anscombe trans., Oxford: Basil Blackwell.

WITTGENSTEIN, Ludwig (1996), *Remarks on the Foundations of Mathematics*, Revised Edition G.H. von Wright *et al.* ed., G.E.M. Anscombe trans., Cambridge Mass.: MIT Press.

WITTGENSTEIN, Ludwig *et al.* (1997), *Ludwig Wittgenstein: Cambridge Letters Correspondence with Russell Keynes, Moore, Ramsey and Sraffa*, Brian McGuinness and Georg Henrik von Wright eds., Oxford: Blackwell.

WITTGENSTEIN, Ludwig (2001), *Logish–philosophische Abhandlung Tractatus Logico–philosophicus*, Kritische Edition, Brian McGuinness and Joachim Schutle eds., Frankfurt: Suhrkamp Verlag.

Secondary Sources

ANSCOMBE, Gertrude E. M. (1971), *An Introduction to Wittgenstein's Tractatus*, 4th edn., London: Hutchinson.

AMBROSE Alice and LAZEROWITZ, M. eds. (1972), *Ludwig Wittgenstein: Philosophy and Language*, London: Allen, and Unwin.

BLACK, Max (1964), *A Companion to Wittgenstein's Tractatus*, Cornell, New York: Cornell University Press.

BLACK, Max (1933), *The Nature of Mathematics: A Critical Survey*, New York: Harcourt.

BENACERRAF, Paul and PUTNAM, Hilary eds. (1964), *Philosophy of Mathematics: Selected Readings*, Englewood Cliffs, New Jersey: Prentice Hall.

CARNAP, Rudolf (1967), *The Logical Syntax of Language*, London: Routledge and Kegan Paul Ltd.

CARNAP, Rudolf (1966), *Der logische Aufbau der Welt*, Hamburg: Felix Meiner Verlag.

CHEUNG , Leo K.C. (2000), "The Tractarian Operation *N* and Expressive Completeness" in *Synthese* 123: 247–261.

CHEUNG , Leo K.C. (1999), "The Proofs of the Grundgedanke in Wittgenstein's *Tractatus*" in *Synthese* 120: 395–410.

COPI, Irving M. (1971), *The Theory of Logical Types*, London: Routledge and K. Paul.

COPI, Irving M. and BEARD, Robert eds. (1973), *Essays on Wittgenstein's Tractatus*, NewYork: Hafner Press.

CRARY, Alice and READ, Rupert eds. (2000), *The New Wittgenstein*, London: Routledge.

DEMOPOULOS, William ed. (1995), *Frege's Philosophy of Mathematics*, Cambridge Mass.:Harvard University Press.

DUMMETT, Michael (1981), *Frege Philosophy of Language*, 2nd edn., Cambridge Mass.:Harvard University Press.

DUMMETT, Michael (1959), "Wittgenstien's Philosophy of Mathematics" in *Philosophical Review* 68: 324-348.

FAVRHOLDT (1967), *An Interpretation and Critique of Wittgenstein's Tractatus*, Copenhagen: Munksgaard.

FINCH, Henry Leroy (1971), *Wittgenstein The Early Philosophy: An Expostion of the Tractatus*, NewYork: Humanities Press.

FLOYD, Juliet (2002), "Inside the Funhouse: Number and Ascriptions of Number in the Tractatus" in RECK (2002).

FLOYD, Juliet (2000), "Wittgenstein, Mathematics and Philosophy" in CRARY (2000).

FOGELIN, Robert J. (1976), *Wittgenstein*, London: Routledge and Kegan Paul.

FOGELIN, Robert J. (1982), "Wittgenstein's Operator *N*" in *Analysis* 42 no.3: 124–127.

FOGELIN, Robert J. (1987), *Wittgenstein*, 2nd edn. London: Routledge and Kegan Paul.

FRASCOLLA, Pasquale (1994), *Wittgenstein's Philosophy of Mathematics*, London: Routledge.

FRASCOLLA, Pasquale (1997), "The *Tractatus* System of Arithmetic" in *Synthese* 112: 353–378.

GEACH, Peter (1981), "Wittgenstein's Operator *N*" in *Analysis* 41 no. 4: 168–171.

GERRARD, Steve (1991), "Wittgenstein's Philosophy of Mathematics" in *Synthese* 87: 125–142.

GOODSTEIN, Reuben Louis (1945), "Function Theory in an Axiom Free Equation Calculus" in *Proceedings of the London Mathematical Society*, ser.2, vol. 48: 401-34.

GOODSTEIN, Reuben Louis (1951), *Constructive Formalism: Essays on the Foundations of Mathematics*, University College Leicester.

GOODSTEIN, Reuben Louis (1961), *Recursive Analysis*, Amsterdam: North-Holland.

GOODSTEIN, Reuben Louis (1972), "Wittgenstein's Philosophy of Mathematics" in AMBROSE and LAZEROWITZ (1972): 271–86.

GRATTAN–GUINNESS, I. (2000), *The Search for Mathematical Roots 1870–1940: Logics Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel*, Princeton, New Jersey: Princeton University Press.

GRIFFIN, James (1969), *Wittgenstein's Logical Atomism*, Seattle: University of Washington Press.

HANCOCK, Peter and MARTIN–LÖF, Per (1975), "Syntax and Semantics of the Language Primitive Recursive Functions", University of Stockholm, 'Preprint #3'.

HARWARD, Donald (1976), *Wittgenstein's Saying and Showing Themes*, Bonn: Bouvier Verlag Herbert Grundmann.

VAN HEIJENNOORT, Jean (1967b), "Logic as Calculus and Logic as Language" in *Synthese*, vol. 17: 324–330.

HINTIKKA, Jaakko (1956), "Identity, Variables and Impredicative Definitions" in *Journal of Symbolic Logic* 21: 225-45.

HINTIKKA, Jaakko (1995a), "Standard vs. Nonstandard Distinction: A watershed in the Foundations of Mathematics" in HINTIKKA 1995b.

HINTIKKA, Jaakko (1995b), *From Dedekind to Gödel*, Dordrecht: Kluwer Academic Publishers.

HINTIKKA, Jaakko (1996), *Ludwig Wittgenstein Half-Truths and One-and-a-Half-Truths*, Dordrecht: Kluwer Academic Publishers.

HINTIKKA, Merrill B. and HINTIKKA, Jaakko (1986), *Investigating Wittgenstein*, Oxford: Basil Blackwell.

HYDER, David (2002), *The Mechanics of Meaning: Propositional Content and the Logical Space of Wittgenstein's* Tractatus, Berlin: Walter de Gruyter.

HYLTON, Peter (1990), *Russell, Idealism, and The Emergence of Analytic Philosophy*, Oxford: Clarendon Press.

JOSEPH, Marc A. (2001), "Review of *Wittgenstein Finitism and the Foundations of Mathematics*" in *Mind* vol. 110, Issue 438: 501-504.

JOSEPH, Marc A. (1998), "Mathematics, Mind, and Necessity in Wittgenstein's Later Philosophy" in *The Southern Journal of Philosophy*, vol. 36: 197–213.

JOSEPH, Marc A. (1998), "Wittgenstein's Philosophy of Arithmetic" in *Dialogue* vol. 37: 83-106.

HAHN, H. (1980), *Empiricism, Logic and Mathematics: Philosophical Papers*, Brian McGuinness ed., Dordrecht: Reidel.

KAMAREDDINE, Fairouz *et al.* (2002), "Types in Logic and Mathematics Before 1940" in *The Bulletin of Symbolic Logic*, vol. 8 no.2: 185–245.

KÖRNER, Stephen (1968), *The Philosophy of Mathematics*, London : Hutchinson.

MALCOLM, Norman (1958), *Ludwig Wittgenstein, A Memoir*; *with biographical sketch by von Wright*, Oxford University Press, London.

MARION, Mathieu (1995), "Wittgenstein and Finitism" in *Synthese* 105: 141–176.

MARION, Mathieu (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, Oxford: Clarendon Press.

MARION, Mathieu (2000), "Operations and Numbers in the *Tractatus*" in *Wittgenstein Studien*, vol. 2, 105-123.

MARION, Mathieu (2003), "Wittgenstein and Brouwer" in *Synthese*, vol. 137, 103-127.

MARTIN-LÖF, Per (1975), "An Intuitionistic Theory of Types: Predicative Part" in *Logic Colloquium '73*, H.E. Rose and J.C. Shepherdson eds., Amsterdam: North-Holland Publishing Company.

MARTIN-LÖF, Per (1985), "On the Meanings of the Logical Constants and the Justification of the Logical Laws" in *Atti degli incontri di logica mathematica*, vol. 2, pages 203-281, Università di Siena: Siena Scuola di Specializzazione in Logica Mathematica.

MARTIN-LÖF, Per (1987), "Truth of a Proposition, Evidence of a judgement, Validity of a Proof" in *Synthese*, vol. 73: 407-20.

MARTIN-LÖF, Per (1988), "An Intuitionistic Theory of Types" in *Twenty-Five Years of Constructive Type Theory*, Giovanni Sambin and Jan Smith eds., Oxford: Clarendon Press: 127-172.

MAURY, André (1977), *The Concepts of* Sinn *and* Gegenstand *in Wittgenstein's Tractatus*, Amsterdam: North-Holland.

McDONOUGH, Richard M (1986), *The Argument of the* Tractatus*: its Relevance to Contemporary Theories of Logic, Language, Mind, and Philosophical Truth*, Albany: SUNY Press.

McGUINNESS, Brian (1988), *Wittgenstein A Life: Young Ludwig (1889–1921)*, London: Duckworth.

MONK, Ray (1990), *Ludwig Wittgenstein: The Duty of Genius*, London: Jonathan Cape.

NORDSTROM, Bengt; PETERSON, Kent and SMITH, Jan (1990), *Programming in Martin-Löf's Type Theory*, Clarendon Press, Oxford.

PEARS, David (1987), *The False Prison: A Study of the Development of Wittgenstein's Philosophy,* vols. 1 & 2, Oxford: Clarendon Press.

RANTA, Aarne (1994), *Type-Theoretical Grammar Clarendon Press*, Oxford: Oxford University Press.

RAMSEY, Frank Plumpton (1923) "Critical Notice: *Tractatus Logico-Philosophicus* by Ludwig Wittgenstein" in *Mind*, vol. 32: 465-478. Reprinted in COPI 1973: 9-23.

RECK, Erich ed. (2002), *From Frege to Wittgenstein: Perspectives on Early Analytic Philosophy*, New York: Oxford University Press.

SCHILPP, P. A. ed. (1963), *The Philosophy of Rudolf Carnap*, La Salle Ill.: Open Court.

SCHMITZ, François (1988), *Wittgenstein: la philosophie et les mathématiques*, Paris: Presses Universitaires de France.

SHANKER, Stuart G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, Albany: SUNY Press.

SOAMES, Scott (1983), "Generality Truth and Expressive Capacity in the *Tractatus*" in *The Philosophical Review* 92 no. 4: 573-589.

STENIUS, Erik (1959), *Wittgenstein's* Tractatus*; A Critical Exposition of its Main Lines of Thought*, Oxford: Blackwell.

STERN, David (1991), "The Middle Wittgenstein From Logical Atomism to Practical Holism" in *Synthese* 87: 203-226.

STERN, David (1995), *Wittgenstein on Mind and Language*, New York : Oxford University Press.

SUNDHOLM, Göran (1992), "The General Form of the Operation in Wittgenstein's *Tractatus*" in *Grazer Philosophische Studien*, vol. 42 1992.

SUNDHOLM, Göran (1994), "Vestiges of Realism" in *The Philosophy of Michael Dummett*, Brian McGuinness and Gainluigi Oliveri eds., Kluwer Academic Publishers Dordrecht:137-166.

TULLY, R.E. (1994), "Sub-Tractatus" in *Wittgenstein Studien* , 1 (1).

URQUHART, Aladair (1988), "Russell's Zigzag Path to the Ramified Theory of Types" in *Russell*, 8: 82-91.

WAISMANN, F. (1979), *Ludwig Wittgenstein and the Vienna Circle*, Oxford: Blackwell.

VON WRIGHT, G. H. (1988a), "Modal Logic and the *Tractatus*" in VON WRIGHT (1988b).

VON WRIGHT, G. H. (1988b), *Wittgenstein*, Basil Blackwell, Oxford.

WRIGHT, Crispin (1980), *Wittgenstein on the Foundations of Mathematics*, London: Duckworth.