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Finite Element Formulations for Lateral Torsional Buckling of Shear Deformable Planar Frames

By

Liping Wu

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the Faculty of Graduate Studies and Research
in partial fulfillment of
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Abstract

Laterally unsupported steel frames commonly occur in industrial structures and pipe racks. In these applications, frames are typically unsupported in the lateral direction at the beam to column junctions, a key difference from typical details in commercial applications. Present design procedures in structural steel standards are not intended for the design of these types of structures. It is thus of interest to develop analysis and design tools for such structures that are reliable, systematic, and simple enough to be implemented in structural analysis software and used in design offices.

Towards this goal, this doctoral thesis contributes to two aspects of importance to the lateral torsional buckling analysis of laterally unsupported plane frames: (a) the incorporation of shear deformation effects in the buckling analysis of thin walled members, and (b) the quantification and modelling of partial warping restraints provided by moment connections in typical moment resisting plane frames to adjoining beams and columns.

The theory and finite element developed under item (a) deviate from conventional theories in three respects: 1) It is based on general non-orthogonal coordinates, which makes it particularly suited for the effective modeling of members with eccentric supports (i.e., supports that do not coincide with the location of the shear centre; 2) It captures shear deformation effects due to bending and warping; 3) It incorporates the effect of load position relative to the shear centre.

A new variational principle for the problem is developed from first principle. The variational principle is used to formulate the governing differential equations for thin-walled members based on the principle of stationary total potential energy. The closed form solutions of the resulting field equations and boundary conditions are developed for the cases of column flexural buckling, column torsional buckling, and lateral torsional buckling of beams under uniform moments, and comparisons to classical solutions were conducted.

A finite element is then developed for more general cases of loading schemes and boundary conditions and compared to predictions based on classical solutions and general finite element analysis results undertaken. In all cases examined, the present solution provides buckling resistance predictions lower than those based on the classical solution that do not include the

effect of shear deformation. Shear deformation effects are found negligible when predicting buckling resistances for columns of typical geometries, but it can be significant when predicting the buckling resistance of beams under high moment gradients.

Towards item (b), a formulation of the partial warping restraint provided by common moment connections to adjoining members is developed based on a shell formulation. The finite element developed is designed to interface with other collinear elements such as that developed under task (a), to accurately and efficiently predict the lateral buckling resistance of moment resisting plane frames. Comparisons between the predictions based on the present formulation and those based on established shell finite element analysis illustrate the ability of the model to efficiently and reliably predict the lateral buckling resistance of laterally unsupported frames.

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Chapter 1 Introduction

1.1 Background

Open thin-walled members are commonly used in steel structures. Examples are shown in Fig. 1.1. These frames are either planar, or three dimensional which can be analyzed as a series of plane frames. In order to determine the resistance of a plane frame, its out-of-plane buckling resistance needs to be determined. Yet, while the present design codes recognize this limit state for individual members (columns, beams and beam-columns) and provide suitable design rules against their out-plane buckling modes of failure, they do not provide guidance in designing plane frames for out-of-plane resistance. This is in part due to the complexities of developing relevant simplified design rules. Yet, with the advent of high strength steel production, steel members are expected to become more and more slender, will become more prone to buckling, which will necessitate the development of rigorous methods for analyzing plane frame buckling.

1.2 Motivation

With the increasing flexibility of modern steel structures, research activities tend to focus on developing better models capable of reliably predicting structural buckling behaviour. In turn, issues affecting out-of-plane stability for plane frames, such as, shear deformation effects on the lateral-torsional buckling of members, and the accurate characterization and modeling of warping continuity in common structural beam to column connections, which have generally been neglected in developing lateral buckling solutions for collinear members, become important when developing solutions for non-collinear frame structures. Within this context, this Ph.D study aims at investigating two aspects of lateral buckling of plane frames. These are:

(1) Developing a lateral-torsional buckling finite element formulation which incorporates the effect of shear deformation. A shell finite element buckling analysis by Dabbas (2002) has demonstrated that this effect gains significance for short members. Yet, the resistance of a short span single (collinear) member tends to be governed by its yield strength in general. An accurate determination of the lateral-torsional resistance for such a member is not needed from a practical

design viewpoint, and thus has received little or no attention by previous researchers. In a frame configuration such as a short cantilever supported by a long flexible column (Fig. 1.4 (b)), an accurate determination of the lateral buckling resistance for the short cantilever portion becomes important (Dabbas 2002, and Mohareb and Dabbas 2003). Adopting conventional lateral buckling solutions which neglect shear deformation effects were observed to grossly underestimate the lateral buckling of such a frame. Thus, the first aim of the present study is to develop a lateral buckling solution which incorporates the effect of shear deformation;

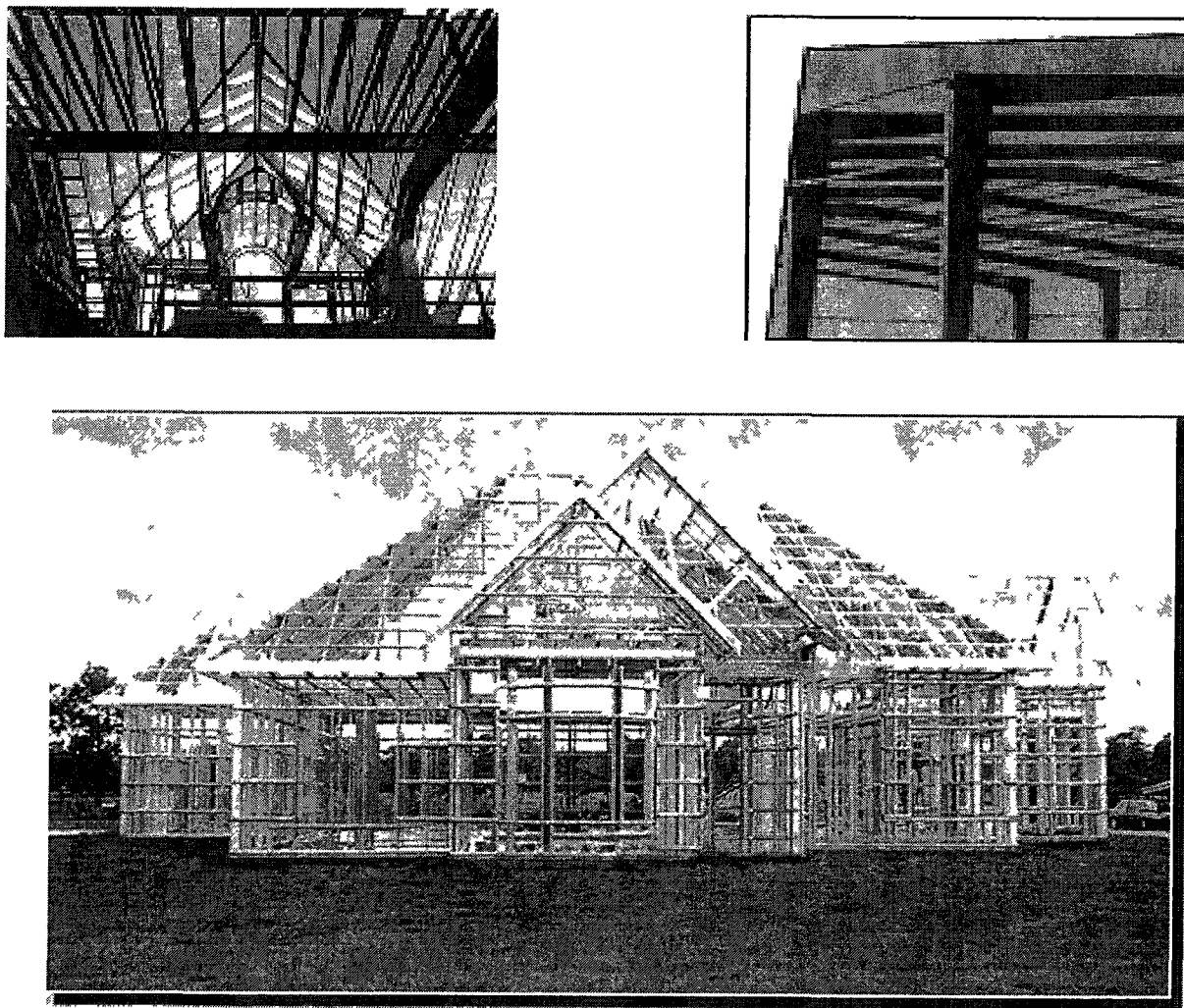


Figure 1.1 Thin-Walled Steel Structures

- (2) Common details for beam to column connections provide complex restraints to adjacent beams and columns (Fig. 1.2). They include coupled partial torsional, lateral, and

warping restraints. A few studies (Sections 2.5 and 2.6 in the literature review) have attempted to model the effect of such restraints. The problem remains largely unresolved. A major goal of the present study is to develop a mathematical description of connection stiffness in a finite element context. The developed finite element for the connection has to properly interface with established lateral-torsional buckling elements (e.g. Barsoum and Gallagher (1970), Trahair (1993)). Specifically, the finite element to be developed has to successfully interface with the degrees of freedom of connecting elements (i.e., lateral displacements u_{b1} and u_{b2} , lateral rotation θ_{yb1} and θ_{yb2} , angle of twist θ_{zb1} and θ_{zb2} , and warping deformation ψ_{b1} and ψ_{b2} , where subscript b denotes the buckling displacements, while subscripts 1 and 2 denote member 1 and 2, respectively).

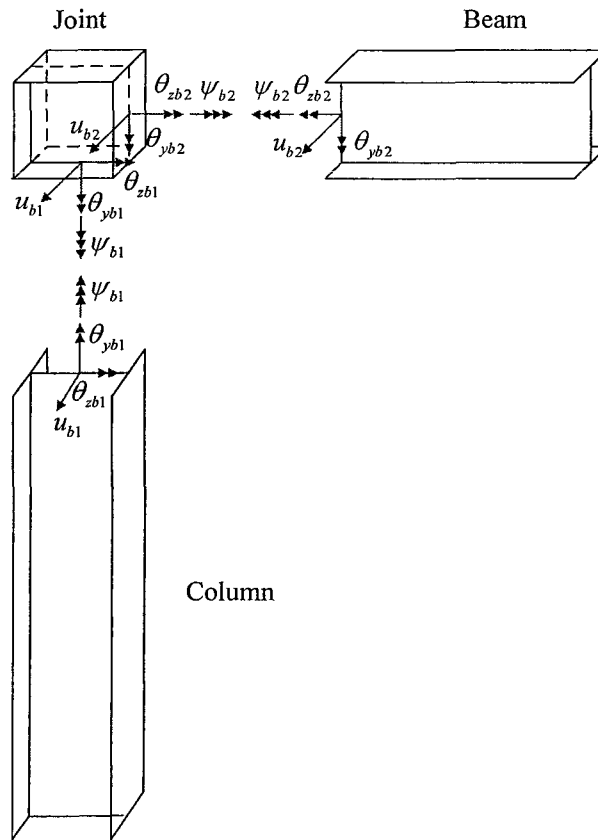


Figure 1.2 Degrees of freedom of a joint during lateral-torsional deformation

The well known Vlasov theory (1961) is the most extensively used in analyzing flexural-torsional problems of open thin-walled structures. It is based on two kinematic assumptions: 1) the in-plane configuration and geometrical dimensions of the cross-section remain unchanged during deformation; 2) the shear strain vanishes at the middle surface of the cross-section. Many researchers (e.g Barsoum and Ghallagher (1970), Yang and McGuire (1986), Trahair (1993)) applied the Vlasov theory for lateral-torsional buckling problems. Nevertheless, the second assumption of Vlasov theory, neglecting the shear strain in the middle surface, can over predict the buckling capacities when members are relatively short (Dabbas 2002, Mohareb and Dabbas 2003). The lateral-torsional buckling resistance is not critical for the design of a single short member since the resistance of such members is usually governed by the yield strength of the material. However, for an open thin-walled structure consisting of a long column supporting a short cantilever such as the one shown in Fig. 1.4 (b), neglecting shear effects may over predict the buckling capacity of the system leading to grossly un-conservative predictions of the buckling load (Dabbas, 2002). In other load deformation analyses, the Vlasov theory is reported to yield un-conservative errors in predicting the torsional response of open cross-sections when the member is subjected to a twisting moment, which induces large shear stresses in the cross-section (Pavazza 2005, Erkmén and Mohareb, 2006).

Secondly, in analyzing open thin-walled frames for out-of-plane stability, it is common to make one of two limiting assumptions: either 1) each member is assumed to be rigidly jointed at its end connections, so that warping is considered to be completely prevented (Hartmann and Munse 1966, Magued 1971, Nethercot 1974, Papangelis, et. al. 1998); or 2) zero warping restraint is assumed to be provided at the joint (Magued 1971). The first limiting assumption is valid for connections involving three pairs of stiffeners (Fig. 1.3 a), while the second assumption may be realistic in a simple connection (Fig. 1.3 b). Both assumptions satisfy the overall equilibrium of the connecting joint, but ignore the partial warping restraint provided by a joint. In reality, many joints provide partial restraints to warping displacements, for instance, the joints in Fig. 1.3 c, d, and e partially restrict warping of adjacent members. Some efforts were made to study diagonally stiffened joints (Fig. 1.3 d) by Tong et al. (2005) and Krenk and Damkilde (1991). A

few studies (Vacharajittiphan and Trahair 1974, Mohareb and Dabbas 2003, Zinoviev and Mohareb 2004) have attempted to quantify the warping restraint for the joint details shown in Fig. 1.3 (c). The present study will focus on developing a thorough description of the stiffness of this particular type of connection.

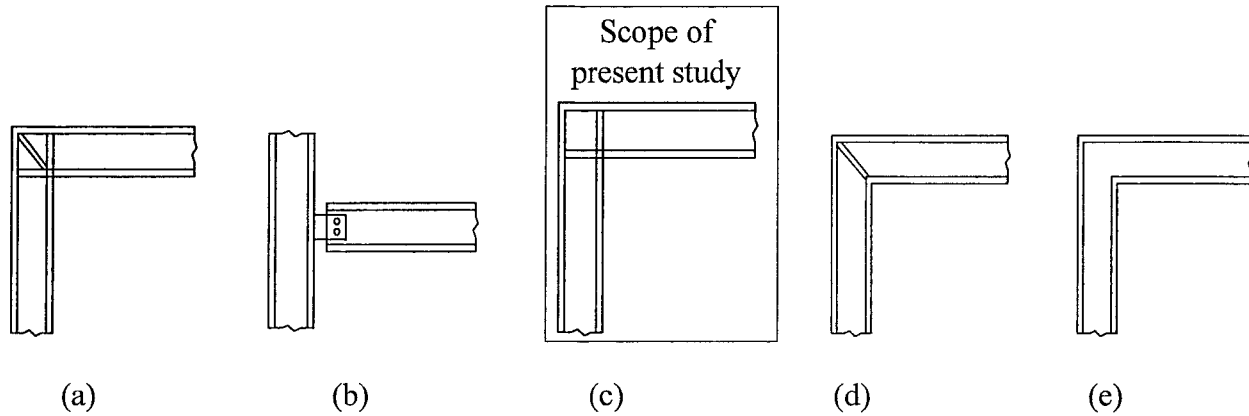


Figure 1.3 (a) Rigid Joint (no warping), (b) Simple Connection (free warping for horizontal members), (c) Box Joint, d) Diagonally Stiffened Joint, (e) Non-Stiffened Joint

1.3 Overview of the Thesis

The first objective (Task 1) is to develop a mathematical derivation and a finite element formulation for predicting the elastic flexural-torsional buckling resistance of open thin-walled members. The model sought will capture shear deformation effects induced by warping and bending, so as to precisely predict the buckling resistance in co-linear and non co-linear structural assemblies involving short cantilevers. In contrast to conventional solutions, the developments will be based on non-orthogonal coordinates in order to facilitate modelling of eccentric supports. Towards this goal the following work is performed.

- (a) A review of the relevant literature with emphasis on variational principles, and thin-walled beam kinematics is conducted (Section 2.1 through 2.4 of Chapter 2),
- (b) The kinematic relations for lateral-torsional buckling for both the pre-buckling and buckling stages are developed in Chapter 3,
- (c) The displacement fields in the pre-buckling and buckling stages along with the notations used throughout the thesis are identified in Chapter 4,

- (d) The differential equations and a finite element formulation consistent with the kinematic relations developed in subtask (b) for pre-buckling are developed and solved both in a closed form and finite element formulation context and are presented in Chapter 5. This work is aimed to provide the pre-buckling solutions including the internal forces within the structure based on the theory developed,
- (e) The variational principle for the condition of neutral stability for lateral-torsional buckling is developed in Chapter 6. The closed form solutions for simple structural problems and comparison of the results with other established results are provided as well,
- (f) The finite element formulation for lateral-torsional buckling based on the variational principle developed in subtask (e) is formulated. The comparison with the results based on ABAQUS shell FEA models and other established numeric solution is conducted in Chapter 7. The parametric study aimed at quantifying shear deformation effects, and the effect of eccentric supports are presented in Chapter 7 as well, and
- (g) An extension of the work to include the analysis of mono-symmetric cross-sections is presented in Appendix A.

It is noted that Chapters 5, 6, and 7 are written in a paper format.

The second objective of the present study (Task 2) is to develop a finite element model of a joint including two pairs of stiffeners (Fig. 1.3 c) by developing a stiffness matrix for the joint which relates the translational, rotational, and warping degrees of freedom of the two elements interfacing with the element (e.g., the column and the beam in Fig. 1.4) to the corresponding generalized forces, (i.e., shear forces, twisting moments, bending moments, and bimoments). The joint element proposed features pre-buckling and out-of-plane buckling analyses of frame structures, (e.g. portal frame in Fig. 1.4 a), while properly interfacing with the lateral-torsional beam-column elements developed in the literature. This includes the formulation of Barsoum and Gallagher (1970), Trahair (1993), McGuire et al (2000), and other more advanced elements which capture the shear deformation effects, such as the one proposed under Task 1. The following subtasks are conducted in order to reach the objective.

- (a) A literature review focusing on the representation and modelling of partial warping restraint of the beam-to-column connections in open thin-walled structures is presented in Sections 2.5 and 2.6.

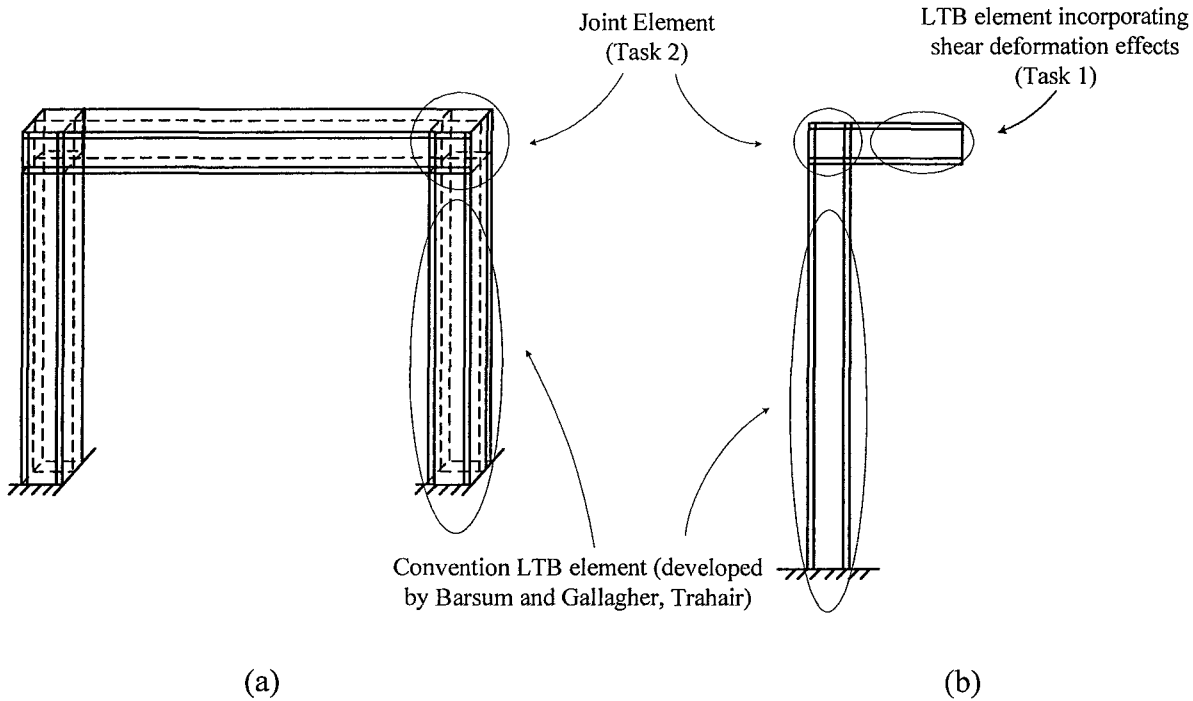


Figure 1.4 (a) Portal Frame, (b) L-shaped structure

- (b) The behaviour of a beam-to-column joint is assumed rigid in pre-buckling stage, which is consistent with conventional analysis techniques of plane frames. During buckling, the joint is assumed to undergo rigid body motion consisting of out-of-plane displacement u_b and two rotations θ_{yb} and θ_{zb} , which are related to the adjacent member buckling deformations. Warping deformation ψ_b of the joint depends on the rigidity of the plate members consisting of the joint to the corresponding deformations, such as plate bending, torsion and in-plane translation. The mathematical descriptions and finite element formulations of the joint behaviour during buckling are presented in Chapter 9.
- (c) The joint constraints to the out-of-plane deformation obtained in subtask (b) are enforced to a planer structure by using Lagrange Multipliers. In addition, the partial

warping restraint of the joint determined in subtask (b) is inserted in the structural buckling analysis. These mathematical and finite element techniques are described in Chapter 9 as well.

For all finite element developments, the implementations are coded in FORTRAN. MATLAB is used to perform the mathematical formulation. For the joint element deformed configuration, C++ and OPENGL are used. All the numerical results are compared with ABAQUS shell element analyses.

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List of Symbols

$\theta_{xb}, \theta_{yb}, \theta_{zb}$: Rotation angles about x, y, z axes, respectively, due to buckling deformation

$\psi_b(z)$: Rate of change of the longitudinal displacement due to warping ($1/Length$)

Chapter 2 Literature Review

This chapter reviews the theories and previous research work relevant to the current research. A brief review of common variational formulations is first presented in Section 2.1 since they form the basis of the finite element developments proposed. Since the problem under investigation is a buckling problem, the necessary and sufficient conditions of stability problems are presented in Section 2.2. Theories for the analysis of thin-walled members are presented in Section 2.3, in which emphasis is placed on kinematic assumptions made in various theories. Studies aimed at the stability of thin-walled members are reviewed in Section 2.4. A review of previous attempts to model and quantify warping restraints provided by frame joints to the adjacent members is then presented in Section 2.5. Investigations on the buckling resistance of open thin-walled frame structures are summarized in Section 2.6.

2.1 Variational Principle and Finite Element Method

Variational principles play an important role in applied mechanics as they provide powerful means of developing approximate solutions. The finite element method is based on variational principles, which enable the analyst to idealize the general differential equations of complex behaviour of a structure as a discrete system of algebraic equations.

2.1.1 Variational Principle

A variational principle extremizes (i.e., minimizes or maximizes) a functional with respect to a set of argument functions. According to Reddy (2002), Euler (1732) and Lagrange (1788) were among the first to formulate the basic equation of variational problems. A variational principle is described in the following (Qian 1980).

A functional π is defined as

$$\pi = \int_a^b F(x; y_1, y_2, \dots, y_i; y'_1, y'_2, \dots, y'_i; \dots; y_1^{(n)}, y_2^{(n)}, \dots, y_i^{(n)}) dx \quad (i = 1, 2, \dots, k) \quad (2.1)$$

in which x is an independent variable, $y_i = y_i(x)$ ($i = 1, 2, \dots, k$) are unknown functions (depending on variable of x), and $y'_i = dy_i/dx$, $y''_i = d^2 y_i/dx^2$, $\dots$, $y^{(n)}_i = d^n y_i/dx^n$ are the first, second and n^{th} derivatives of these unknown functions, respectively. Meanwhile, these functions meet the boundary conditions at ends ($x = a$, and $x = b$):

$$\left. \begin{array}{l} \text{At } x = a: \\ y_i(a) = y_{i1}, y'_i(a) = y'_{i1}, y''_i(a) = y''_{i1}, \dots, y^{(n)}_i(a) = y^{(n)}_{i1} \\ \text{At } x = b: \\ y_i(b) = y_{i2}, y'_i(b) = y'_{i2}, y''_i(b) = y''_{i2}, \dots, y^{(n)}_i(b) = y^{(n)}_{i2} \end{array} \right\} \quad (i = 1, 2, \dots, k) \quad (2.2)$$

If the functional $\pi(y_1, y_2, \dots, y_i)$ is to be stationary, the condition $\delta\pi = 0$ has to be satisfied:

$$\delta\pi = \int_a^b \left\{ \frac{\partial F}{\partial y_1} \delta y_1 + \frac{\partial F}{\partial y'_1} \delta y'_1 + \frac{\partial F}{\partial y''_1} \delta y''_1 + \dots + \frac{\partial F}{\partial y^{(n)}_1} \delta y^{(n)}_1 + \dots \right. \\ \left. + \frac{\partial F}{\partial y_i} \delta y_i + \frac{\partial F}{\partial y'_i} \delta y'_i + \frac{\partial F}{\partial y''_i} \delta y''_i + \dots + \frac{\partial F}{\partial y^{(n)}_i} \delta y^{(n)}_i \right\} dx = 0 \quad (i = 1, 2, \dots, k) \quad (2.3)$$

in which $\delta y_1, \delta y_2, \dots, \delta y_i$ are arbitrarily chosen functions such that they meet the boundary conditions. After applying integration by parts and combining terms with the boundary conditions (Eq. 2.2),

$$\delta y^{(j)}_i(a) = 0, \delta y^{(j)}_i(b) = 0 \quad (i = 1, 2, \dots, k), (j = 1, 2, \dots, n) \quad (2.4)$$

Equation 2.3 can be re-written as

$$\delta\pi = \int_a^b \left\{ \frac{\partial F}{\partial y_1} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'_1} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''_1} \right) - \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}_1} \right) \right\} \delta y_1 dx \\ + \int_a^b \left\{ \frac{\partial F}{\partial y_2} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'_2} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''_2} \right) - \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}_2} \right) \right\} \delta y_2 dx \\ + \dots \\ + \int_a^b \left\{ \frac{\partial F}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'_i} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''_i} \right) - \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}_i} \right) \right\} \delta y_i dx \\ + \dots \\ + \int_a^b \left\{ \frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'_k} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''_k} \right) - \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial F}{\partial y^{(n)}_k} \right) \right\} \delta y_k dx + B.C. = 0 \quad (2.5)$$

in which the last term (B.C.) denotes the boundary conditions terms obtained after applying integration by parts. The field terms in Eq. 2.5 lead to the governing equations. This leads to a system of partial differential equations, called Euler-Lagrange equations.

$$\left. \begin{aligned} \frac{\partial F}{\partial y_1} - \frac{d}{dx} \frac{\partial F}{\partial y_1'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y_1''} - \dots + (-1)^n \frac{d^n}{dx^n} \frac{\partial F}{\partial y_1^{(n)}} &= 0 \\ \dots\dots\dots \\ \frac{\partial F}{\partial y_i} - \frac{d}{dx} \frac{\partial F}{\partial y_i'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y_i''} - \dots + (-1)^n \frac{d^n}{dx^n} \frac{\partial F}{\partial y_i^{(n)}} &= 0 \end{aligned} \right\} \quad (i = 1, 2, \dots, k) \quad (2.6)$$

the unknown functions y_i can be solved by solving the set of differential equations.

In bucking applications, the second variation of the total potential has to be stationary for the equilibrium condition to be in a neutral state, $\delta\left(\frac{1}{2}\delta^2\pi\right)=0$. The stationary condition for the variation of the second variation of functional π is expressed as

$$\delta\left(\frac{1}{2}\delta^2\pi\right) = \delta \int_a^b \left\{ \begin{aligned} &\sum_{i=1}^k \sum_{j=1}^k \left(\frac{\partial^2 F}{\partial y_i \partial y_j} (\delta y_i \delta y_j) + \frac{\partial^2 F}{\partial y_i \partial y_j'} (\delta y_i \delta y_j') + \frac{\partial^2 F}{\partial y_i \partial y_j''} (\delta y_i \delta y_j'') + \dots + \frac{\partial^2 F}{\partial y_i \partial y_j^{(n)}} (\delta y_i \delta y_j^{(n)}) \right) \\ &+ \sum_{i=1}^k \sum_{j=1}^k \left(\frac{\partial^2 F}{\partial y_i' \partial y_j'} (\delta y_i' \delta y_j') + \frac{\partial^2 F}{\partial y_i' \partial y_j''} (\delta y_i' \delta y_j'') + \frac{\partial^2 F}{\partial y_i' \partial y_j'''} (\delta y_i' \delta y_j''') + \dots + \frac{\partial^2 F}{\partial y_i' \partial y_j^{(n)}} (\delta y_i' \delta y_j^{(n)}) \right) \\ &+ \sum_{i=1}^k \sum_{j=1}^k \left(\frac{\partial^2 F}{\partial y_i'' \partial y_j'} (\delta y_i'' \delta y_j') + \frac{\partial^2 F}{\partial y_i'' \partial y_j''} (\delta y_i'' \delta y_j'') + \frac{\partial^2 F}{\partial y_i'' \partial y_j'''} (\delta y_i'' \delta y_j''') + \dots + \frac{\partial^2 F}{\partial y_i'' \partial y_j^{(n)}} (\delta y_i'' \delta y_j^{(n)}) \right) \\ &\dots\dots\dots \\ &+ \sum_{i=1}^k \sum_{j=1}^k \left(\frac{\partial^2 F}{\partial y_i^{(n)} \partial y_j'} (\delta y_i^{(n)} \delta y_j') + \frac{\partial^2 F}{\partial y_i^{(n)} \partial y_j''} (\delta y_i^{(n)} \delta y_j'') + \frac{\partial^2 F}{\partial y_i^{(n)} \partial y_j'''} (\delta y_i^{(n)} \delta y_j''') + \dots + \frac{\partial^2 F}{\partial y_i^{(n)} \partial y_j^{(n)}} (\delta y_i^{(n)} \delta y_j^{(n)}) \right) \end{aligned} \right\} dx = 0 \quad (2.7)$$

2.1.2 Imposing Constraints

In a large class of problems, it is required to find a stationary value of a functional, subject to certain constraint conditions (Reddy 1984). In mechanics, these constraints can be kinematic relations that restrict displacements.

$$\left. \begin{aligned} G(y(x=c), y'(x=c)) &= 0 && \text{for a discrete system at a point } x=c \\ G(x, y, y') &= 0 && \text{for a continuous system} \end{aligned} \right\} \quad (2.8)$$

In the constrained minimization problems, the admissible functions should not only satisfy the specified boundary conditions and be sufficiently continuous, but also should satisfy the constraint conditions. Furthermore, the admissible variations should be such that the constraint

conditions are not violated. The Lagrange Multiplier Method is an approach to solve such problems (Hughes 1987, Park 2000). As an example, a functional problem as in Eq. 2.1 subjects to constraint conditions $G_j(x, y_i, y'_i) = 0$ ($j=1, \dots, l$). The given functional π is modified by adding the constraint conditions $G_j(x, y_i, y'_i) = 0$ ($j=1, \dots, l$), by multiplying each condition $G_j = 0$ with an unknown parameter μ_j , called a Lagrange Multiplier. The problem can then be treated as an extremization of a modified functional as

$$\pi^* = \int_a^b F^* dx = \int_a^b \left\{ F(x; y_1, y_2, \dots, y_i; y'_1, y'_2, \dots, y'_i; \dots; y_1^{(n)}, y_2^{(n)}, \dots, y_i^{(n)}) + \sum_{j=1}^l \mu_j(x) G_j(x, y_i, y'_i) \right\} dx \quad (2.9)$$

in which the $\mu_j(x)$ are additional unknown functions of x (Lanczos 1970).

The stationary condition $\delta\pi^* = 0$ has to be satisfied. Thus, the Euler-Lagrange equations are

$$\frac{\partial F^*}{\partial y_i} - \frac{d}{dx} \frac{\partial F^*}{\partial y'_i} + \frac{d^2}{dx^2} \frac{\partial F^*}{\partial y''_i} - \dots + (-1)^n \frac{d^n}{dx^n} \frac{\partial F^*}{\partial y^{(n)}_i} + \frac{\partial F^*}{\partial \mu_j} = 0$$

in which

$$F^* = F + \sum_{j=1}^l \mu_j G_j \quad (i=1, 2, \dots, k) \quad (2.10)$$

or

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \frac{\partial F}{\partial y'_i} + \dots + (-1)^n \frac{d^n}{dx^n} \frac{\partial F}{\partial y^{(n)}_i} + \sum_{j=1}^l \left[\mu_j \left(\frac{\partial G_j}{\partial y_i} - \frac{d}{dx} \frac{\partial G_j}{\partial y'_i} \right) + \frac{\partial (\mu_j G_j)}{\partial \mu_j} \right] = 0$$

It is noted that the additional stationary conditions $\partial F^* / \partial \mu_j = 0$ yield the constraint conditions $G_j(x, y_i, y'_i) = 0$.

2.1.3 Approximation Methods

In many complex structural problems, it is difficult to solve the Euler-Lagrange equation (or modified Euler-Lagrange equation). Rayleigh (1870) and Ritz (1909) proposed a powerful method of determining approximate solutions to the functional by directly using the variational statements (e.g. virtual work principles, or the principle of total potential energy). The method bypasses the derivation of the Euler-Lagrange equations and goes directly from a variational statement of the problem to the formulation of a discretized approximation. This is achieved by

assuming approximate equations for the unknown functions prior evoking the stationary conditions. Weighted residual (Galerkin 1915), least squares, and collocation methods are common types of approximate methods. These various methods differ in the selection of weighting function and approximation functions. However, since the approximate solutions are continuous functions over the domain and have to satisfy to the boundary conditions, it is sometimes difficult to select approximate functions continually over the whole domain of the continuum when it is geometrically complicated, and/or when the loads applied and boundary conditions are discontinuous. This problem is circumvented in the finite element method.

2.1.4 Finite Element Method

The finite element method is an established approximate solution method which approximately extremizes a functional. In contrast to other approximate methods of solution, the finite element method discretizes the domain as a collection of simple domains, called finite elements, so that it is possible to systematically approximate the unknown field variables independently within each element. Thus, geometric or material discontinuities can be naturally and effectively represented (Reddy 1993). The coefficient matrices of the assumed approximation functions (often polynomials) are the unknown to be determined. The accuracy of a finite element analysis much depends on the choice of the approximation function and size of elements.

2.2 Potential Energy and Buckling Equilibrium

The total potential π of a structure and its loads is defined as the sum of the internal strain energy U of the structure and the loss of potential energy of the loads ($-V$) (e.g., Trahair 1993), i.e.,

$$\pi = U - V \quad (2.11)$$

The equilibrium of a structure can be studied by examining its total potential energy π (Murray, 1984), Trahair, 1993). A deformed equilibrium position of a structure under loads may be stable, neutral, or unstable. If a structure is in an equilibrium state after undergoing a generic pre-buckling displacement u_p , its first variation of the total potential energy has to be stationary i.e., $\delta\pi = (\partial\pi/\partial u_p)\delta u_p = 0$ or $\partial\pi/\partial u_p = 0$, for all virtual displacements δu_p , where the subscript p denotes pre-buckling. In order to determine whether the equilibrium is in a stable, neutral, or

unstable state, the second variation of the total potential has to be examined. If the structure is in a neutral equilibrium state, the variation of the second variation of the total potential energy has to be stationary i.e., $\delta\left(\frac{1}{2}\delta^2\pi\right)=0$ for any set of buckling deformations δu_b , where the subscript b denotes buckling. Figure 2.1 provides the necessary and sufficient conditions of the three equilibrium types for a single degree of freedom system.

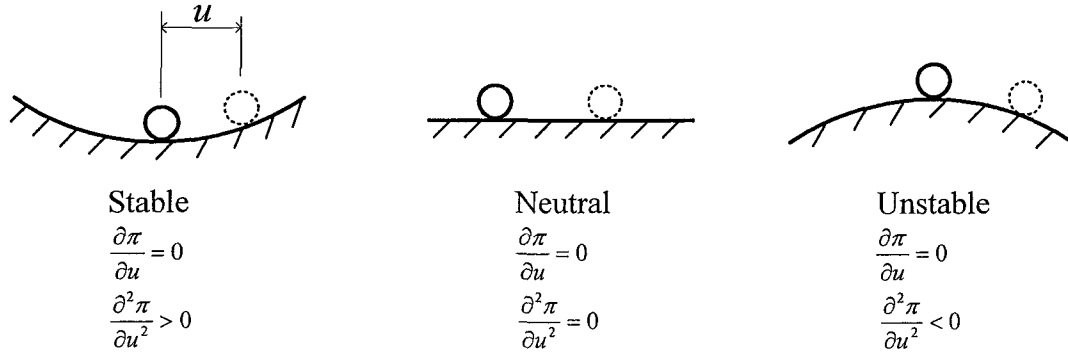


Figure 2.1 Types of Equilibrium

For a multiple degrees of freedom system, the Lagrange-Dirichlet theorem (Bazant & Cedolin, 1991), provides the stability conditions

$$\delta^2\pi = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 \pi}{\partial u_i \partial u_j} \delta u_i \delta u_j > 0 \quad (2.12)$$

for any δu_i . At the limit of stability, the second variation of the total potential energy ceases to be positive definite. This requires that the variation of $\delta^2\pi$ must turn to zero i.e., $\delta\left(\frac{1}{2}\delta^2\pi\right)=0$. This is the condition for neutral stability (also called the critical state).

2.3 Theory of Thin-Walled Members

The general theory of flexure and torsion of thin-walled elastic beams was developed relatively late to mechanics. Historically, a comprehensive theory of combined flexure and torsion of open thin-walled beams was developed by Vlasov in 1930, but his work became generally known to the western world upon the English translation of his work in 1961. Timoshenko (1945) also developed a general theory for flexure and torsion buckling of open thin-walled beams. In his study, flexure and torsion were considered separately. According to Vlasov's theory, the out-of-plane displacement of a cross-section is assumed to be proportional to the product of a warping

function and the gradient of the angle of the rotation. Benscoter (1954) presented a theory for sections consisting of closed cells. Instead of using the gradient of the angle of the rotation as a measure of warping, he defined a deformation function $\chi(x)$, which is distinct from rate of the angle of twist, and used it to describe the out-of-plane displacement, so that shear strain of the mid-surface was taken into consideration of the formulations. The flexural-torsional theories of thin-walled beams combined the bending theory of solid beams and torsional features of thin-walled beam. For example, the Vlasov theory combines of the concepts of Euler-Bernoulli beam theory and non-uniform torsion of open thin-walled beams.

2.3.1 Beam Kinematics

As mentioned, bending theories of solid beams were used to establish the flexural-torsional theories for open thin-walled beams. Thus, it is necessary to review some important bending theories for solid beams which are commonly extended for studying flexural-torsional behaviour of thin-walled beams. These are discussed in Section 2.3.1.1. The more general flexural-torsional theories are presented in Section 2.3.1.2.

2.3.1.1 Bending Theories of Beams

In order to describe various beam theories, a right-hand coordinate system is introduced. The z -axis is oriented along the longitudinal axis of the beam, the y -axis is along the height of the cross-section with the positive direction downward, and the x -axis is along the width of the beam, and the corresponding displacements along three axes are w , v , and u , respectively. There are a number of beam theories that are used to represent the kinematics of deformation. These were summarized by Saade (2005). In the following, these theories are reviewed and elaborated upon.

Euler-Bernoulli Beam Theory

In engineering analysis, the Euler-Bernoulli beam theory is extensively used. It assumes that a cross-section initially normal to the beam centerline remains planar and normal to the neutral axis after deformation. Therefore, the rotation angle of the cross-section is equal to the slope angle dv/dz , as illustrated in Fig. 2.2. The transverse shear strain effects are implicitly neglected in this assumption.

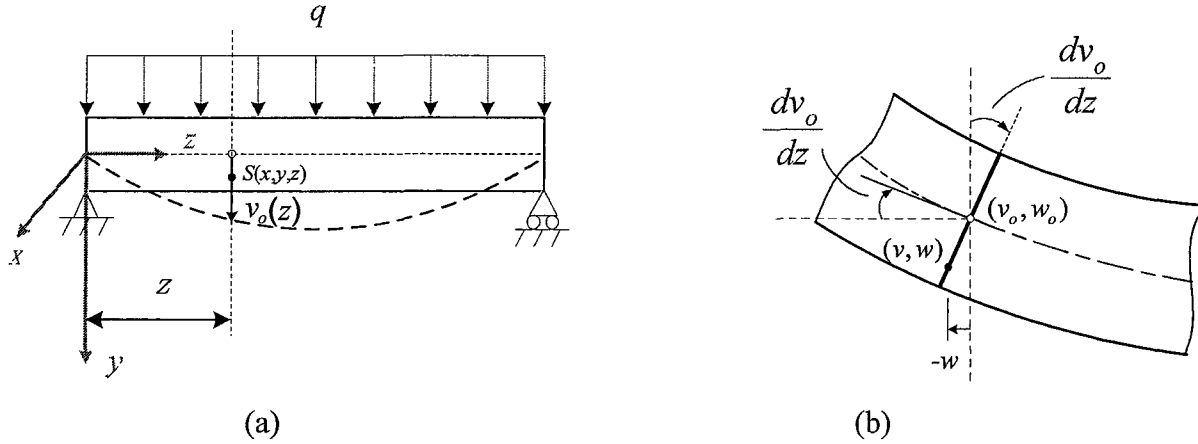


Figure 2.2 Bernoulli beam theory (a) Centreline deflection, (b) Kinematics

Referring to Fig. 2.2, the displacements u , v , and w at any point $S(x, y, z)$ on a cross-section are related to the centreline displacement $v_0(z)$ of the cross-section located at the point $(0, 0, z)$ on the centroidal axis of the beam. The strain displacement fields for the Euler-Bernoulli beam are

$$\left. \begin{aligned} u(x, y, z) &= 0 \\ v(x, y, z) &= v_0(z) \\ w(x, y, z) &= -yv'_0 \end{aligned} \right\} \quad (2.13)$$

The shear and normal strains associated with these displacements are

$$\left. \begin{aligned} \varepsilon_{xz} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0 \\ \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0 \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} = -y \frac{\partial^2 v}{\partial z^2} = -yv''_0 \end{aligned} \right\} \quad (2.14)$$

The bending moment, M_x , can be determined by integrating over the cross-section area A as the moments of the internal normal stresses $\sigma_{zz} = E\varepsilon_{zz}$, in which E is Young's modulus

$$M_x(z) = \int_A y \sigma_{zz} dA = E \int_A y \varepsilon_{zz} dA = -EI_x v''_0 \quad (2.15)$$

where A and I_x are the area and moment of inertia of the cross-section, respectively. An attempt to calculate the shearing force $Q_y = \int_A \sigma_{yz} dA = \int_A G(2\varepsilon_{yz}) dA$ will lead to $Q_y = 0$. Thus, in the

Euler- Bernoulli theory, the shear force must be found from equilibrium. This signifies that the shear strain fields postulated in Eq. 2.13 are inconsistent with equilibrium. Nevertheless, the theory gives good results in most practical situations.

Timoshenko Beam Theory

Timoshenko's beam theory is an improvement over the Euler-Bernoulli theory, in that it incorporates shear effects. It assumes that the cross-section remains planar, but the rotation angle of the cross-section, θ_x , is no longer equal to the slope dv_0/dz . In other words, the cross-section is no longer perpendicular to the neutral axis, as illustrated in Fig. 2.3.

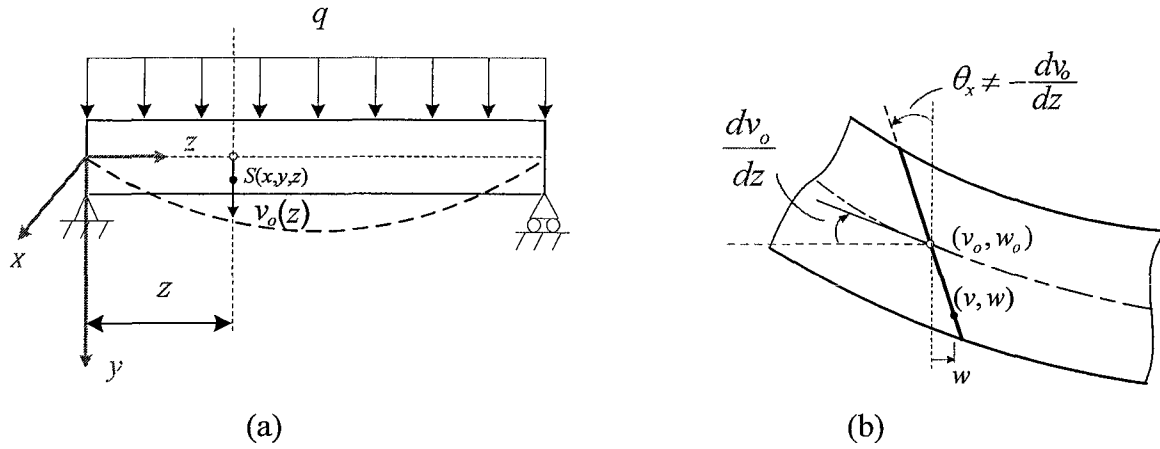


Figure 2.3 Timoshenko beam theory (a) Centreline deflection, (b) Kinematics

The displacement fields based on the Timoshenko beam assumptions are

$$\left. \begin{aligned} u(x, y, z) &= 0 \\ v(x, y, z) &= v_0(z) \\ w(x, y, z) &= y\theta_x(z) \end{aligned} \right\} \quad (2.16)$$

and the components of the strain tensor are

$$\left. \begin{aligned} \varepsilon_{xz} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0 \\ \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} (v'_0 + \theta'_x) \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} = y\theta'_x \end{aligned} \right\} \quad (2.17)$$

The bending moment M_x , can be determined directly from the stress and strain field.

$$M_x(z) = \int_A y \sigma_{zz} dA = E \int_A y \varepsilon_{zz} dA = EI_x \theta'_x \quad (2.18)$$

According to the second identity in Eq. 2.17, the shear strain and the correspondent shear stress are uniformly distributed over the cross-section. Yet, this postulate violates the loading boundary condition on the two extreme fibers of the top edge and the bottom edge of the beam, where the shear stresses are known to vanish. Thus, a more realistic kinematic assumption for the shear strains is to have a non-uniform distribution over the height of the cross-section and vanish at the free edges. In turn, the cross-section will no longer remain plane. One way of remedying this contradiction is the adoption of the shear correction factor k_s when calculating shear forces to compensate the incompatibility between the assumed kinematics and actual traction boundary conditions at the free surface. Another way to avoid this contradiction is to develop higher order beam theories. Using the first approach, the shear force over a cross-section is

$$Q_y(z) = k_s \int_A \tau_{yz} dA = k_s GA (\theta_x + v'_0) \quad (2.19)$$

where G is the shear modulus and A is the area of the cross-section. Several theories were developed to determine the reduced area factor k_s . According to Cowper (1966) and Pilkey (1994), the shear correction factor k_s can be evaluated by

$$k_s = \frac{A}{I_x} \int_A \left(\frac{\bar{S}_x}{t} \right)^2 dA \quad (2.20)$$

in which t is the thickness, $\bar{S}_x$ and I_x are static moment of inertia and the moment of inertia of the cross-section, respectively. Equation 2.19 is valid when a beam is loaded in the yz -plane.

Bickford-Reddy High-Order Beam Theory

The high-order beam theory (Reddy 1997) for rectangular sections attempts to resolve the contradictions outlined in the Euler Bernoulli and Timoshenko beams. The displacement fields based on this theory are illustrated in Fig. 2.4.

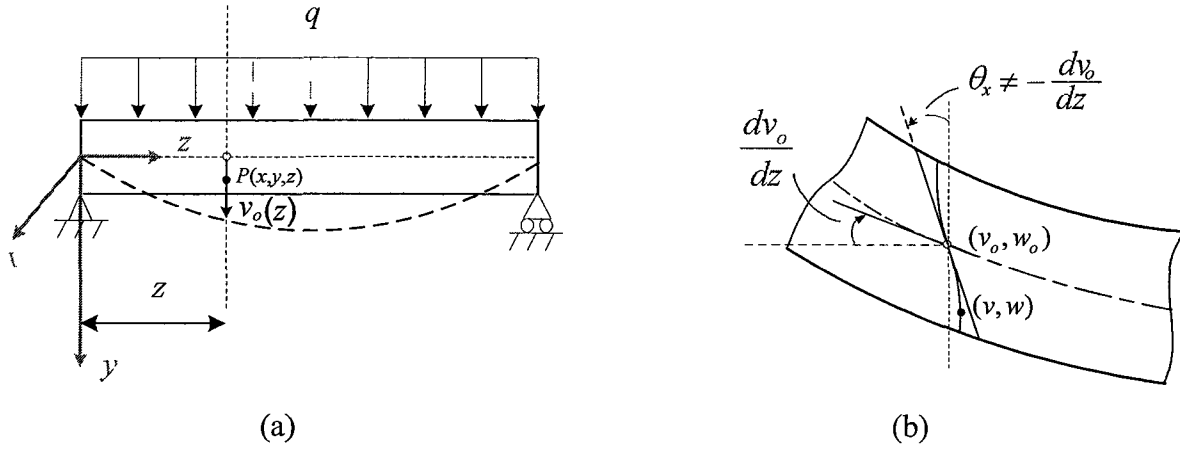


Figure 2.4 Reddy third-order beam theory (a) Centerline deflection, (b) Kinematics

The longitudinal displacement w is assumed to take a cubic distribution. The assumed displacement fields are:

$$\left. \begin{aligned} u(x, y, z) &= 0 \\ v(x, y, z) &= v_0(z) \\ w(x, y, z) &= y\theta_x(z) - \frac{4}{3h^2}y^3[\theta_x'(z) + v_0''(z)] \end{aligned} \right\} \quad (2.21)$$

where h is the height of the rectangular cross-section, θ_x is the rotation of line OP with respect to the vertical plane (Fig.2.4b). The corresponding strain expressions are

$$\left. \begin{aligned} \varepsilon_{xz} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0 \\ \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} \left(1 - \frac{4y^2}{h^2} \right) (v_0'(z) + \theta_x'(z)) \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} = y \left(1 - \frac{4y^2}{3h^2} \right) \theta_x''(z) - \frac{4y^3}{3h^2} v_0'''(z) \end{aligned} \right\} \quad (2.22)$$

Thus, the assumed displacement fields lead to a quadratic variation of the transverse shear strains, and hence the shear stresses are quadratic. Shear strains and stresses vanish at the top and bottom fibres of the beam ($y = \pm h/2$). Therefore, no correction factor is needed.

The governing equations of the third-order theory are $Q_y - (dM_x/dz) = 0$ and $(dQ_y/dz) = -q_y(z)$.

The bending moment is determined (Reddy 1997) as

$$M_x = \int_A \sigma_{zz} y dA = \int_A E \varepsilon_{zz} y dA = E \int_A \left[y^2 - \frac{4y^4}{3h^2} \right] \theta'_x - \frac{4y^4}{3h^2} v''_0 dA = \frac{4}{5} EI_x \theta'_x - \frac{1}{5} EI_x v''_0 \quad (2.23a)$$

The shear force does not require the shear correction factor because the transverse shear strain is represented as quadratic through the beam height. Under the plate theory, the effective shear force is expressed as (Reddy 1997),

$$\begin{aligned} Q_y &= \int_A \tau_{yz} dA - \frac{4}{h^2} \int_A \tau_{yz} y^2 dA + \frac{4}{3h^2} \int_A \frac{d\sigma_{zz}}{dz} y^3 dA \\ &= \int_A \tau_{yz} \left(1 - \frac{4}{h^2} y^2 \right) dA + \frac{4}{3h^2} \int_A y^3 \frac{d\sigma_{zz}}{dz} dA \\ &= G \left(A - \frac{28}{5h^2} I_x \right) (\theta'_x + v'_0) + \frac{EI_x}{21} (\theta''_x + v''_0) \end{aligned} \quad (2.23b)$$

2.3.1.2 Uniform Torsion for Beams

If a beam of uniform cross-section, which is free to warp out of its plane, is twisted by torques T at each end (Fig. 2.5), it is said to be in a state of uniform twist. Whenever a beam is twisted uniformly along its longitudinal axis, the fibres must undergo in-plane shear strains to accommodate the twist. Associated with these strains are shear stresses. This problem was first solved by Saint Venant and the stresses are denoted as St. Venant shear stresses τ_{sv} . Based on the uniform twist assumption, the displacement field of the beam can be expressed (Timoshenko, 1970) as

$$\left. \begin{aligned} u(x, y, z) &= -(y - y_s) \theta_z(z) \\ v(x, y, z) &= (x - x_s) \theta_z(z) \\ w(x, y, z) &= -\omega \theta'_z(z) \end{aligned} \right\} \quad (2.24)$$

where $\omega = \omega(x, y)$ is a warping function; θ_z is the twist angle (linear). In turn, $d\theta_z/dz$ is constant; (x_s, y_s) is the shear center of the cross-section. From Eq. 2.24, the out-of-plane displacement, w , due to warping is given by the product of warping function and the first derivative of the twisting angle. The corresponding strains are

$$\left. \begin{aligned} \varepsilon_{zx} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} \left(-y + y_s - \frac{\partial \omega}{\partial x} \right) \theta'_z(z) \\ \varepsilon_{zy} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} \left(x - x_s - \frac{\partial \omega}{\partial y} \right) \theta'_z(z) \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0 \end{aligned} \right\} \quad (2.25)$$

As the second derivative of the twisting angle is zero, the normal stress and strain of the cross-section vanish. By assuming elastic and isotropic material properties, the shear stresses are $\sigma_{zx} = 2G\varepsilon_{zx}$ and $\sigma_{zy} = 2G\varepsilon_{zy}$, where G is the elastic shear modulus, the torque T_{sv} can be calculated by the integration as

$$\begin{aligned} T_{sv} &= \int_A [\sigma_{yz}(x-x_s) - \sigma_{xz}(y-y_s)] dA = G \int_A [2\varepsilon_{zy}(x-x_s) - 2\varepsilon_{zx}(y-y_s)] dA \\ &= G\theta'_z(z) \int_A \left[\left(x - x_s - \frac{\partial \omega}{\partial y} \right) (x - x_s) - \left(-y + y_s - \frac{\partial \omega}{\partial x} \right) (y - y_s) \right] dA \\ &= G\theta'_z(z) \int_A \left[(x - x_s)^2 + (y - y_s)^2 - (x - x_s) \frac{\partial \omega}{\partial y} + (y - y_s) \frac{\partial \omega}{\partial x} \right] dA \end{aligned} \quad (2.26)$$

The area integral in Eq. 2.26 is that of the torsional constant J under the present theory (Saade, 2005),

$$J = \int_A \left[(x - x_s)^2 + (y - y_s)^2 - (x - x_s) \frac{\partial \omega}{\partial y} + (y - y_s) \frac{\partial \omega}{\partial x} \right] dA \quad (2.27)$$

Combining Eqs. 2.26 and 2.27, a relationship between the derivative of the twist angle and the external torque is established as:

$$T_{sv} = GJ\theta'_z(z) \quad (2.28)$$

The strain fields in Eq. 2.25 suggest that for an open cross-section, shear stresses due to uniform torsion occur linearly across the thickness of the wall with a zero value along the middle surface, and a maximum value at the outer and inner fibres (Fig. 2.5b).

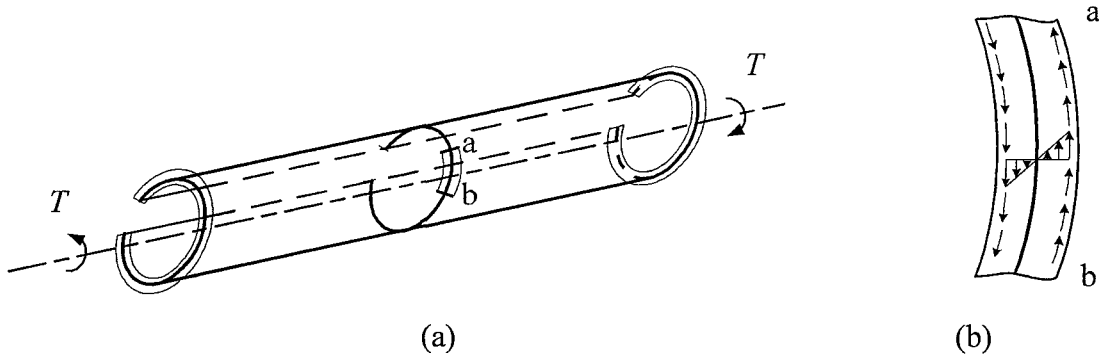


Figure 2.5 (a) A thin-walled beam of open cross-section under pure torsion,
(b) Shear stresses distribution along the wall thickness

2.3.1.3 Flexural-Torsional Theories of Thin-Walled Beams

When a thin-walled beam is twisted in a non-uniform manner, i.e. $\theta_z(z)$ is a nonlinear function, additional warping effects will take place. These will induce longitudinal warping stresses σ_ω and shear warping stresses τ_ω . The longitudinal warping stresses are very significant in thin-walled cross-sections. They should be superimposed to normal stresses arising from bending. The analysis of non uniform torsional behavior of open thin-walled sections is presented in various theories, such as those of Vlasov (1961), Gjelsvik (1981), and Prokic (1996).

Vlasov open thin-walled beam theory for non-uniform torsion

Under general loading and boundary conditions, beams undergo non-uniform torsion. In turn, non-uniform warping occurs and the first derivative of the twisting angle is not constant. Under such cases, the St. Venant torsion theory is no longer valid. Vlasov (1961) developed a theory to study open thin-walled beams under the combination of bending and non uniform torsion. This theory is based on two assumptions: 1) the cross-section of a thin-walled beam is assumed rigid (un-deformable) in its plane; 2) the shear strain of the middle surface (characterizing the change in the angle between the coordinate lines $z = \text{constant}$, $s = \text{constant}$) can be assumed to vanish, where s is a coordinate along the profile of the mid-surface. A mid-surface of an arbitrary open cross-section and its in-plane displacements are shown in Fig. 2.6.

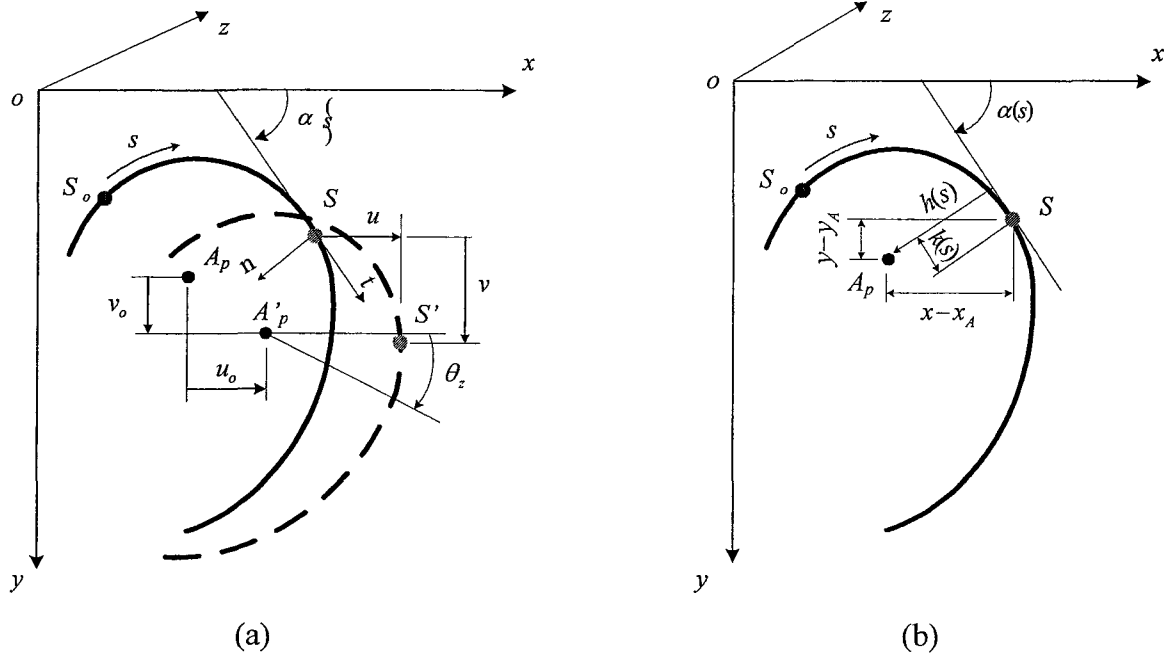


Figure 2.6 (a) Coordinates and displacements of a mid-surface,
(b) Distances between pole A_p and point S in tangential and normal directions

Here, an arbitrary point $A_p(x_A, y_A)$ is assumed to lie in the plane of the cross-section. Point A_p is called the pole. A line passing through the pole A parallel to the z axis is called pole axis. The displacements of point $S(x(s), y(s), z)$ lying on the middle surface of the cross-section has three components (Fig. 2.6a). These are:

1. The longitudinal displacement w which is positive when oriented along the positive direction of the axis z .
2. The transverse tangent displacement t which is directed along the tangent to the profile line and is positive when the displacement is in the direction of increasing s .
3. The transverse normal displacement n which is positive when acting along the direction determined by the condition that the displacements of w, t and n form a right-hand coordinate system.

When Pole $A_p(x_A, y_A)$ undergoes displacements u_o, v_o along axes of x and y , respectively, and the section undergoes an angle of rotation θ_z around z axis, Point $S(x(s), y(s))$ on the mid-surface moves to position S' through displacements u and v shown in Fig. 2.6a. The

displacements are expressed in terms of the pole displacements u_0 , v_0 and the rotation θ_z through

$$\begin{cases} u(z, s) = u_0(z) - [y(s) - y_A] \theta_z(z) \\ v(z, s) = v_0(z) + [x(s) - x_A] \theta_z(z) \end{cases} \quad (2.29)$$

The corresponding tangential and normal displacements, denoted by t and n respectively (Fig. 2.6a), are

$$\begin{cases} t(z, s) = u \cos \alpha(s) + v \sin \alpha(s) = u_0 \cos \alpha + v_0 \sin \alpha + [(x - x_A) \sin \alpha - (y - y_A) \cos \alpha] \theta_z \\ n(z, s) = v \cos \alpha(s) - u \sin \alpha(s) = v_0 \cos \alpha - u_0 \sin \alpha + [(x - x_A) \cos \alpha + (y - y_A) \sin \alpha] \theta_z \end{cases} \quad (2.30)$$

in which $\alpha(s)$ is the angle between the tangent to the profile at Point S and the x -axis. It is noted that $\alpha(s)$ is positive in the clockwise direction looking into the positive direction of z axis.

The terms in the two square brackets of Eq. 2.30 are denoted as $h(s)$ and $k(s)$, respectively,

$$\begin{cases} h(s) = (x - x_A) \sin \alpha - (y - y_A) \cos \alpha \\ k(s) = (x - x_A) \cos \alpha + (y - y_A) \sin \alpha \end{cases} \quad (2.31)$$

It is easily observed from Fig. 2.6b that $h(s)$ is the perpendicular distance from Pole A_p to the tangent at the point in the direction of increase s on the mid-surface and $k(s)$ is the perpendicular distance from Point S to $h(s)$. According to the second Vlasov assumption, the shear strains γ at the middle surface should vanish, i.e.,

$$\gamma = \frac{\partial w}{\partial s} + \frac{\partial t}{\partial z} = 0 \quad (2.32)$$

By integrating Eq. 2.32, the longitudinal displacement $w(z, s)$ can be determined

$$w = w_0(z) - \int_{s_0}^s \frac{\partial t}{\partial z} ds \quad (2.33)$$

Inserting Eq. 2.31 into Eq. 2.30 and differentiate with respect to z , we get,

$$\frac{\partial t}{\partial z} ds = u'_0(z) \cos \alpha(s) ds + v'_0(z) \sin \alpha(s) ds + \theta'_z(z) h(s) ds \quad (2.34)$$

From Fig. 2.7a, we have,

$$\begin{cases} dx = \cos \alpha ds \\ dy = \sin \alpha ds \\ d\omega = h(s) ds \end{cases} \quad (2.35)$$

where ds is an infinitesimal arc element along the mid-surface, and dx and dy are projections of ds along the Cartesian coordinates; $d\omega(s)$ is twice the area of the elementary triangle (sector) whose base is the arc element ds , and height $h(s)$ is the length of the perpendicular dropped from Pole A to the tangent to the mid-surface at the point of interest S (Fig. 2.7a).

Substituting Eqs. 2.34 and 2.35 into the right-hand side of Eq. 2.33, and integrating with respect to s , one obtains,

$$w = w_o(z) - x(s)u'_o(z) - y(s)v'_o(z) - \omega(s)\theta'_z(z) \quad (2.36)$$

where $x(s)$, $y(s)$ are the Cartesian coordinates of Point S , and $\omega(s)$ is twice the area of the sector enclosed between the arc S_oS of the mid-surface line and the two lines AS_o , AS , joining the ends of this segment with A (Fig. 2.7 b). The area $\omega(s)$ is called the sectorial area, point $A_p(x_A, y_A)$ is called the Pole of the sectorial area and Point S_o is called the sectorial origin. The sectorial area is positive when the mobile radius A_pS moves clockwise relative to the fixed radius A_pS_o as observed into the positive z -direction. Therefore, the determination of sectorial area of the profile as in Fig. 2.7 for the Pole A_p and starting from Point S_o can be generated by moving Point S along the profile and calculated by the definite integral

$$\omega_s = \int_{s_o}^s d\omega = 2 \int_{s_o}^s \frac{1}{2} h(s) ds = \int_{s_o}^s h(s) ds \quad (2.37)$$

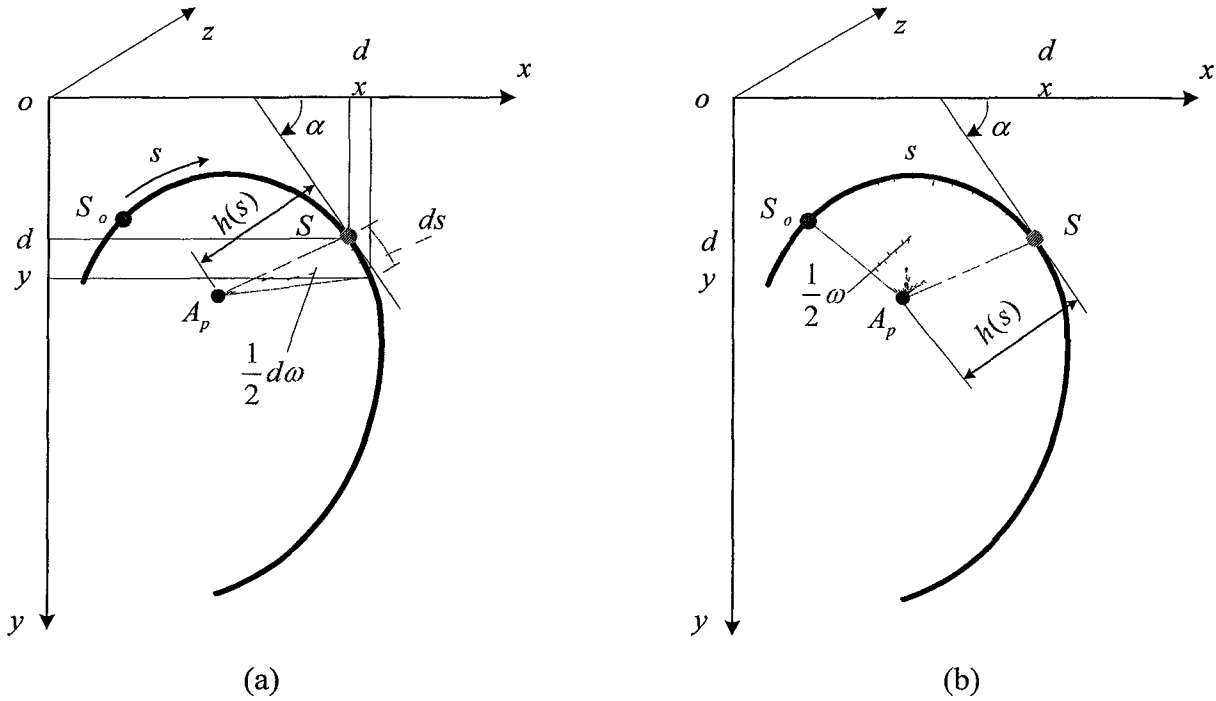


Figure 2.7 (a) Relations between Cartesian and sectorial coordinates, (b) Sectorial area

The sectorial area diagram is sketched in Fig. 2.8 based on an arbitrary Pole A_p and sectorial origin S_o . It is observed that the sectorial area diagrams rely on the selection of Pole A_p and Sectorial Origin S_o . In other words, different sectorial area diagrams for the same cross-section will be obtained for different selections of Pole A_p and Sectorial Origin S_o . Other authors such as Timoshenko (1945) referred to $\omega(s)$ as the warping function.

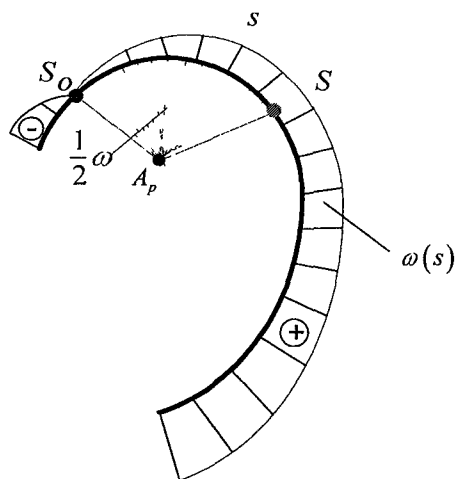


Figure 2.8 The sectorial area diagram

Based on Vlasov second assumption, the shear strains at the middle surface vanish. This is reflected in the first equation in Eq. 2.38. This assumption is not violated by the presence of the additional St.Venant torsion since the shear stresses and strains due to St. Venant torsion induce zero shear strain at the middle surface (Fig. 2.6). The strains associated with the displacement fields described in Eqs. 2.32 and 2.36 are expressed

$$\left. \begin{aligned} \gamma_{zs} &= \frac{\partial w}{\partial s} + \frac{\partial t}{\partial z} = 0 \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} = w'_0 - xu''_0 - yv''_0 - \omega\theta''_z \end{aligned} \right\} \quad (2.38)$$

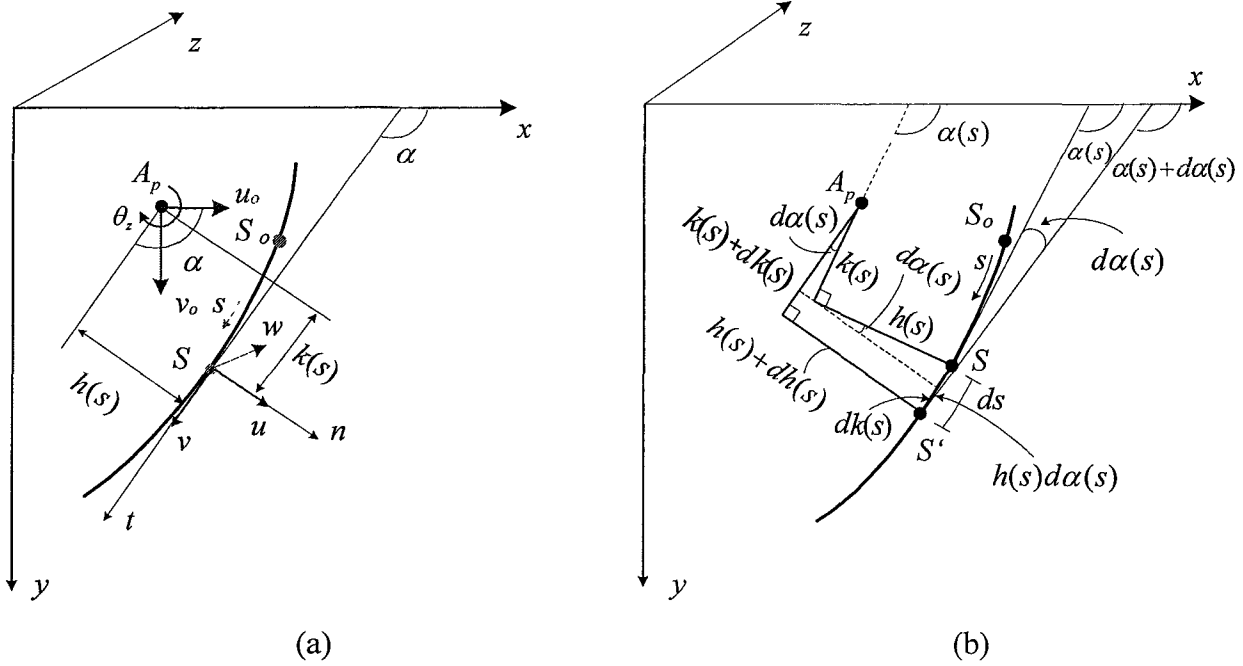
An arbitrary selection of the coordinate origin O and Pole A_p will result in a set of coupled differential equations between the angle of twist and the flexural deformations. In order to eliminate the coupling, origin O is commonly chosen to coincide with the section centroid, so that $\int_A x dA = 0$ and $\int_A y dA = 0$; axes x and y are oriented along the two principal directions, then $\int_A xy dA = 0$; Pole A_p is chosen coincidently with the shear center of the cross-section, so as to $\int_A \omega x dA = 0$ and $\int_A \omega y dA = 0$; and the principal sectorial origin is chosen as $\int_A \omega dA = 0$. These issues are clearly explained in Erkmen and Mohareb (2006 b).

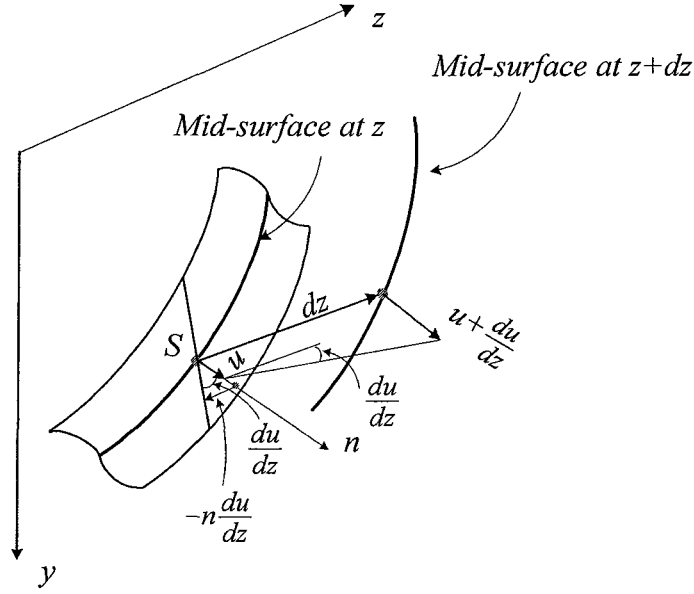
Gjelsvik Theory

Gjelsvik (1981) developed a theory in which the warping function consisted of a contour warping component and a thickness warping component. The Gjelsvik theory generalizes the Vlasov theory by extending it to relatively thick cross-sections and, unlike the Vlasov theory, results in non-zero warping functions for sections that are rectangular, T-shaped, etc. Three assumptions formed the basis of the method: (1) The section contour does not deform in its own plane; (2) The shear strain of the middle surface is negligible; (3) the Kirchhoff assumption (commonly used in plate bending theories) is adopted, which states that a straight line initially perpendicular to plate middle surface remains normal to the middle surface after deformation. Assumptions (1) and (2) are identical to those of the Vlasov theory.

In Fig. 2.9 a, two right-hand Cartesian coordinate systems (x, y, z) and (n, t, z) are depicted. The origin of the (n, t, z) coordinate system is placed at a generic point $S(x(s), y(s), z)$ located on the mid-surface, where s is the coordinate along the mid-surface line starting from an origin S_0 . Coordinate t is oriented along the tangent to the section contour at point S , while coordinate n is normal to coordinate t . Also, the figure shows the in-plane displacements u_o, v_o , of Pole $A_p(x_A, y_A, z)$ and the angle of twist of the cross-section θ_z . The displacements of Point S along the coordinates n, t, z are denoted as u, v and w , respectively. From the geometric relationships illustrated in Fig. 2.9 b, the following relations are obtained.

$$\left. \begin{aligned} dh(s) &= k(s) d\alpha(s) \\ ds &= h(s) d\alpha(s) + dk(s) \end{aligned} \right\} \quad (2.39)$$





(c)

Figure 2.9 (a) Coordinates and displacements, (b) Geometrical relationships
(c) Longitudinal displacement of the points offset from the mid-surface

Applying the first two assumptions with similar manipulations as those made in the Vlasov theory, lead to the displacements u , v , and $\bar{w}$ for a point on the middle surface

$$\left. \begin{aligned} u(z, s) &= u_o(z) \sin \alpha(s) - v_o(z) \cos \alpha(s) - k(s) \theta_z(z) \\ v(z, s) &= u_o(z) \cos \alpha(s) + v_o(z) \sin \alpha(s) + h(s) \theta_z(z) \\ \bar{w}(z, s) &= w_o(z) - x(s) u'_o(z) - y(s) v'_o(z) - \bar{w}(s) \theta'_z(z) \end{aligned} \right\} \quad (2.40)$$

in which $w_o(z)$ is an integration function that represents an average axial displacement of the cross-section and

$$\bar{w} = \int_{M_o}^M h(s) ds \quad (2.41)$$

The axial displacement w of a point within the cross-section is offset from the middle surface and follows from the third assumption (i.e. the Kirchhoff assumption),

$$w = \bar{w} - n \frac{\partial u}{\partial z} \quad (2.42)$$

Substituting Eq. 2.40 into Eq. 2.42, one obtains

$$w(z, n, s) = w_o(z) - [x(s) + n \sin \alpha(s)] u'_o(z) - [y(s) - n \cos \alpha(s)] v'_o(z) - [\bar{w}(s) - nk(s)] \theta'_z(z) \quad (2.43)$$

in which, the term $-nk(s)$ can be regarded as an additional thickness warping function

$$\bar{\bar{\omega}}(n, s) = -nk(s) \quad (2.44)$$

Therefore, the total warping function consists of two parts, a contour warping function (identical to that defined in the Vlasov theory) and a thickness warping function $\bar{\bar{\omega}}(n, s)$, or

$$\omega = \bar{\omega} + \bar{\bar{\omega}} \quad (2.45)$$

It is obvious that the terms $[x(s) + n \sin \alpha(s)]$ and $[y(s) - n \cos \alpha(s)]$ are X and Y coordinates of a point on the cross-section with curvilinear coordinates s and offset by a distance n from the mid-surface, where $-t(s)/2 \leq n \leq t(s)/2$, i.e.,

$$\begin{cases} X(n, s) = x(s) + n \sin \alpha(s) \\ Y(n, s) = y(s) - n \cos \alpha(s) \end{cases} \quad (2.46)$$

Substituting Eqs. 2.45 and 2.46 into Eq. 2.43, gives,

$$w(z, n, s) = w_o(z) - X(s, n) \frac{du_o(z)}{dz} - Y(s, n) \frac{dv_o(z)}{dz} - \omega(s, n) \frac{d\theta_z(z)}{dz} \quad (2.47)$$

For some sections, such as wide flange sections, the contour warping $\bar{\omega}$ is much larger than the local warping $\bar{\bar{\omega}}$, but occasionally $\bar{\bar{\omega}}$ is dominant. For example, a narrow rectangular cross-section has a zero contour warping function $\bar{\omega}$; however, warping deformations based on $\bar{\bar{\omega}}$ exist and is the dominant component. Lin and Hsiao (2003) presented an analytical formulation for the torsional warping function of a thin-walled section built on the combination of the Vlasov assumption and the Kirchhoff assumption of plate theory. The results were in agreement with Gjelsvik theory (1981).

Based on the above kinematics, Gjelsvik theory (1981) naturally formulates the expression of the St. Venant torsional constant as

$$J = \frac{1}{3} \int_I t^3(s) ds \quad (2.48)$$

Thus, unlike the Vlasov theory, Gjelsvik theory avoids the inclusion of the St. Venant torsion effect as an ad-hoc effect.

Prokic Theory

Prokic (1996) developed a thin-walled beam theory for the analysis of straight thin-walled beams with open or closed (multi-cellular) cross-sections. This theory uses both contour warping and thickness warping functions. The mid-surface of the cross-section is divided into a number of line segments depending on the desired accuracy. Points at which two or more polygonal sides jointed are called the nodal points of the cross-section, (e.g. there are 5 nodes in Fig. 2.10 a). A right-hand Cartesian coordinate system (x, y, z) is chosen at the centroid of the cross-section. The x -axis and y -axis are the two principal axes of the cross-section, and the z -axis is oriented along the longitudinal direction of the beam. The theory is based on the following assumptions.

1. The cross-section is perfectly rigid in its own plane.
2. The longitudinal displacements caused by warping vary linearly between any two adjacent nodal points on the mid-surface.
3. The thickness warping in relation to the mid-surface of a section follows Gjelsvik's theory (1981).

According to these assumptions, the displacement fields (u, v, w) of an arbitrary point (x, y) on the section contour are

$$\left. \begin{aligned} u &= u_o - y\theta_z \\ v &= v_o + x\theta_z \\ w &= w_o - y\theta_x - x\theta_y - \bar{\bar{\omega}}\theta'_z + \sum_i \Omega'_i w_i \end{aligned} \right\} \quad (2.49)$$

in which u_o , v_o , and w_o are the displacements of the centroid, θ_x and θ_y are the rotations of the cross-section around x and y axes (similar to the Timoshenko beam), respectively, and the angle of twist of the cross-section is θ_z . Unlike the Vlasov theory, u_o , v_o are the displacements of the centroid (as opposed to the shear center, as commonly used in most solutions). The axial displacement w consists of the translations due to flexure resulting from the Timoshenko beam kinematic assumption, the longitudinal warping as determined by the thickness warping function $\bar{\bar{\omega}} = k(s)n$ which linearly varies across the thickness and vanishes at the mid-surface (as defined

in Gjelsvik's theory (1981)), and the contour warping $\sum_{i=1}^n \Omega' w_i$, ($i=1, 2, 3, \dots, n$) which represents the warping along the mid-surface of the cross-section. Here, the unknown parameters w_i are chosen as the displacements of the nodal points. Functions Ω' are linear. They have a unit value at node i and vanish at adjacent nodes. In Fig. 2.10 a, we can see that the mid-surface of the cross-section has been divided into six segments jointed by five connecting nodes.

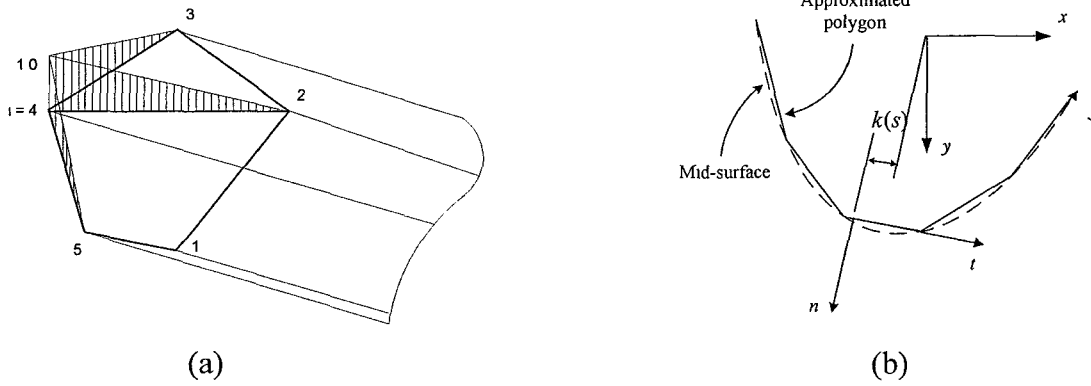


Figure 2.10 (a) Linear interpolation function Ω at the node $i = 4$, (b) Idealization of curved mid-surface of cross-section by approximated polygon

An advantage of this theory is the fact that it releases the assumption that shear strains vanish at the mid-surface, commonly adopted in other theories (Vlasov (1961), Gjelsvik (1981), et.). The strain fields based on the displacement fields postulated in Eq. 2.49 are

$$\left. \begin{aligned} \varepsilon_{xz} &= \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} \left[\frac{\partial u_o}{\partial z} - \left(y + \frac{\partial \omega}{\partial x} \right) \theta'_z - \theta_y + \sum_i \frac{\partial \Omega'_i}{\partial x} w_i \right] \\ \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} \left[\frac{\partial v_o}{\partial z} + \left(x - \frac{\partial \omega}{\partial y} \right) \theta'_z + \theta_x + \sum_i \frac{\partial \Omega'_i}{\partial y} w_i \right] \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} = \frac{dw_o}{dz} - xu''_o - yv''_o - \bar{\omega} \theta'_z + \sum_i \Omega'_i \frac{\partial w_i}{\partial z} \end{aligned} \right\} \quad (2.50)$$

In the first two equations in Eq. 2.50, the shear strains consist of contributions due to a) flexure, b) uniform torsion and c) warping shear. The shear strains do not vanish (in contrast with the Vlasov theory (in which $\gamma_{sz} = 0$)).

Jonsson Theory

Jonsson (1998, 1999) studied the warping displacement induced by distortion modes assumed in the plane of the cross-section. He determined warping displacements based on the longitudinal equilibrium condition of a small element taken from the member wall.

2.3.2 FEA Formulations for Flexural-Torsional Behavior of Open Thin-Walled Beams

Many finite element formulations were developed for the analysis of open thin-walled cross-sections. Tralli (1986) developed a hybrid flexural-torsional finite element model for thin-walled beams including shear deformation effects. The longitudinal displacement field postulated in their work is similar to Prokic's theory, i.e., a set of points on the contour of the cross-section were selected and the longitudinal displacement functions at these points were taken as the unknown displacements. The stiffness matrix was then derived on the basis of a hybrid finite element formulation starting with postulated stress distributions and kinematics, and then applying the principle of Hellinger-Reissner (Pian et al. 1982, 1983).

Gendy et al. (1992) developed a general finite element model for analyzing thin-walled beams under general loading and boundary conditions. They used a kinematic description which captures flexural, torsional and warping deformations and interpolated the displacement and strain fields separately (normal and shear strains, curvatures and warping functions) and applied a modified Hellinger-Reissner functional (Washizu, 1982). Thus, shear locking was circumvented in their C^0 element. This approach allowed using simple interpolation functions.

Shakourzadeh et al. (1995) formulated a finite element for the torsional problems of thin-walled beams. The element was based on Benscolter's theory (Benscolter 1954). They adopted a warping displacement function that was different from the derivative of angle twist. Thus, their formulation captured shear deformation effects due to warping.

Park et al. (1997) developed a general displacement field of thin-walled cross-section beams considering the uniform and nonuniform shear warping deformation. The longitudinal displacement included the components of shear deformation. The principle of virtual work was

used to develop a C^1 -continuous finite element for the general flexural-torsional analysis of beams with unsymmetrical cross-sections.

Back and Will (1998) derived a finite element for the analysis of thin-walled beams with arbitrary cross-sections by combining Timoshenko's beam theory and Vlasov's thin-walled beam theory $w(x, y, z) = w_c(z) - x(s)\theta_y - y(s)\theta_x - \omega(s)\psi$. By adopting an orthogonal Cartesian coordinate system (refer to the part of Vlasov's theory in the literature) and applying the principle of virtual work, a finite element formulation was developed.

Mohareb and Nowzartash (2003) developed an exact finite element combining both St. Venant torsion and warping torsion effects of open sections. The element had two end nodes, each having two degrees of freedom, featuring C^1 -continuity. The principle of virtual work was used to develop the differential equations of equilibrium. Hyperbolic functions were adopted as interpolation function for the angle of twist and developing a finite element.

Pavazza (2005) studied relatively short thin-walled beams with open cross-sections where the shear influence on torsion must be taken into account. He presented an approximate analytical approach to model torsion of open cross-section while accounting for warping shear effects. In the study, St. Venant torsion was considered negligible compared to warping torsion. The study analyzed the displacements along the contour direction for a beam undergoing bending and torsion. The governing partial differential equations of equilibrium were obtained based on direct equilibrium consideration of an infinitesimal element taken from the beam wall of equilibrium in an orthogonal coordinate system.

Saade (2005) modeled flexural and warping shear effects in thin-walled beams based on an improved version of Prokic's theory, in which she assumed the pole to coincide with the shear centre. The principle of virtual work was used to develop the finite element formulations.

Mohareb et al. (2006) analyzed the torsional behaviour of wide flange beams including shear deformation effects. In their formulations, they conceived the flanges as Timoshenko beams and

thus incorporated shear deformation effects. The formulation is based on the principle of minimum total potential energy. The field equations were derived and solved and the solution was adopted to develop exact shape functions in a finite element context.

Erkmen and Mohareb (2006) developed a variational formulation for the torsional analysis of prismatic thin-walled beams with open sections by using the principle of complementary energy, which was based on the assumed normal and shear stress fields. Based on the variational expression, general finite element formulations were formed and FEA solutions were shown to provide lower bound representations of the stiffness when compared to those based on conventional beam theories founded on postulated kinematic assumptions.

2.4 Lateral-Torsional Buckling of Thin-Walled Members

In the study of the lateral-torsional buckling resistance of beams, the governing differential equations can be formulated for members based on the direct equilibrium approach and subsequently solved. Closed-form solutions only exist for very simple structural problems. For most cases, energy based solutions are used to develop approximate solutions. Energy formulations are established by evoking the stationary conditions of the total potential energy throughout buckling i.e., setting the variation of the second variation to zero to yield the governing differential equations. Alternatively, displacement fields can be assumed prior to taking the variation of the second variation of the total potential energy in order to transfer the problem into a set of algebraic equations. The approach provides the basis of numerical solutions.

Timoshenko (1945) and Timoshenko and Gere (1963) derived the differential equations of torsional, flexural-torsional, and lateral buckling for open-thin-walled members based on the equilibrium of a finite segment cut from the wall under the deformed configurations. These equations were solved by applying the boundary conditions.

Krajcinovic (1969) used the principle of virtual displacement to develop the discretized buckling eigenvalue problems $\left([k_e] - \lambda[k_g]\right)\{u_i\} = \{0\}$. The developed finite element had two nodes with 4 degrees-of-freedom at each node. The elastic and geometric stiffness matrices ($[k_e]$ and $[k_g]$)

were formulated by postulating approximation functions of the displacements. In turn, the critical buckling load was obtained.

Barsoum and Gallagher (1970) developed a finite element model of thin-walled beams based on the Vlasov theory. The finite element has two end nodes with 7 degrees-of-freedom per node. The principle of stationary of virtual work was used to formulate a finite element for the stability problems of open thin-walled beams.

Wekezer (1985) studied thin-walled bars of variable, open cross-sections based on the theory of membrane shells and finite element method. Based on the assumption of large angle of twist but small strains and stresses, only the linear terms of the Green-Lagrange strain tensor were retained. Based on assumed strains, the elastic and geometric stiffness matrices were obtained from the principle of stationary total potential energy.

Saade (2004, 2005) also studied thin-walled beams of open or closed cross-section based on Prokic's theory. She developed a nonlinear stability formulation for thin-walled beams based on the first variation of the potential energy to formulate the critical loads.

Machado and Cortinez (2005a) numerically investigated the effects of pre-buckling displacements and shear deformation on the lateral-torsional buckling of bisymmetric thin-walled composite beams. The study was based on a geometrically nonlinear theory and assumed displacement fields which accounts for large displacements and rotations. The principle of virtual work was used in conjunction with the Ritz variational method to discretize the governing equations. The effects of the spans and the load position relative to the shear centre were investigated.

Machado and Cortinez (2005b) numerically studied the flexural-torsional, lateral-torsional and post buckling behaviour of simply supported beams. They pointed out the influence of shear deformation on different laminate stacking sequences in composite beams. Also, they investigated the effect of pre-buckling deformations on the predicted buckling loads. Their study was based on a geometrically nonlinear theory. The Ritz variational method was applied in

formulating the tangent stiffness matrix. The resulting discretized equilibrium equations were solved by means of the incremental Newton-Rapshon method.

Erkmen and Mohareb (2008 a, b) formulated a lateral-torsional stability solution for thin-walled beams of open cross-section based on the principle of stationary complementary energy. In the study, shear effects were included. The effects of the position of the transverse loads and different boundary conditions on the critical loads were studied.

Machado (2007) investigated the stability and free vibration of composite beams based on a geometrically nonlinear analysis accounting for large displacements and rotations. The variational principle of virtual work was used to develop the governing equations.

Displacement Fields Based on the Formulation of Pi et al. (1992)

Pi et al. (1992) presented the formulation of displacements u , v and w of any point on the section contour as a function of the displacements (u_o , v_o , w_o) at the section centroid and the angle of twist of the cross-section including second order effects. On this basis, they established the energy expression for lateral-torsional buckling of beams with doubly symmetric thin-walled cross-sections. The derivation of the displacement fields was based on the Finite Rotation equations developed by Euler (1775) and relied on the two assumptions of Vlasov's theory. The displacements of a Point $S(x, y, z)$ were related to the centroidal displacement u_o , v_o and w_o , and the rotations u' , v' and θ_z , i.e.,

$$\begin{Bmatrix} u(s,z) \\ v(s,z) \\ w(s,z) \end{Bmatrix} = \begin{Bmatrix} u_o \\ v_o \\ w_o \end{Bmatrix} + [R] \begin{Bmatrix} x \\ y \\ -\omega\theta_z \end{Bmatrix} - \begin{Bmatrix} x \\ y \\ 0 \end{Bmatrix} = \begin{Bmatrix} u_o - y\theta_z \\ v_o + x\theta_z \\ w_o - xu' - yv' - \omega\theta_z' \end{Bmatrix} + \begin{Bmatrix} -\frac{x}{2}[u'^2 + \theta_z'^2] - \frac{y}{2}(u'v') - \omega u'\theta_z' \\ -\frac{x}{2}u'v' - \frac{y}{2}[v'^2 + \theta_z'^2] - \omega v'\theta_z' \\ -xv'\theta_z' + yu'\theta_z' \end{Bmatrix} \quad (2.51)$$

in which

$$[R] \approx \begin{bmatrix} 1 - \frac{1}{2}(u'^2 + \theta_z'^2) & -\theta_z' - \frac{1}{2}u'v' & u' \\ \theta_z' - \frac{1}{2}u'v' & 1 - \frac{1}{2}(v'^2 + \theta_z'^2) & v' \\ -u' - v'\theta_z' & -v' + u'\theta_z' & 1 - \frac{1}{2}(u'^2 + v'^2) \end{bmatrix} \quad (2.52)$$

The rotation matrix $[R]$ is an approximation of the finite rotation matrix initially developed by Euler (1775).

If the rotations (u', v', θ_z) are considered small, the second order terms in Eq. 2.52 can possibly be neglected. The resulting displacement vector of Point S is thus approximated as

$$\begin{Bmatrix} u(s, z) \\ v(s, z) \\ w(s, z) \end{Bmatrix} \approx \begin{Bmatrix} u - y\theta_z \\ v + x\theta_z \\ w - xu' - yv' - \omega\theta'_z - xv'\theta_z + yu'\theta_z \end{Bmatrix} \quad (2.53)$$

The displacement fields of Eq. 2.53 are limited to doubly symmetric cross-sections, in which the location of the centroid and shear center coincide.

Displacement Fields based on the Formulation in Trahair (1993)

Trahair (1993) formulated the displacement fields of a point on a section profile as function of the displacements (u, v) of shear center SC , longitudinal displacement w at section centroid C and the angle of twist of the cross-section. In Fig. 2.12, the subscript “ I ” denotes the positions of the corresponding points after displacements). The formulations were based on the Vlasov theory and included second order effects due to twist, and the resulting displacements of a Point $S(x, y)$ in a contour of an arbitrary cross-section were expressed as Eq. 2.54.

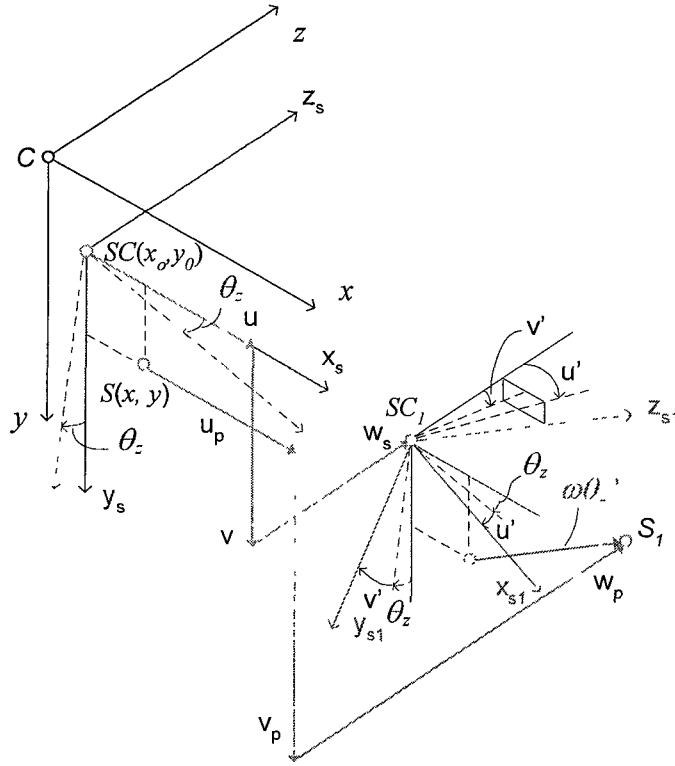


Figure 2.12 Displacements and rotations (adapted from Trahair 1993)

$$\left. \begin{aligned} u_s(s, z) &\approx u - (y - y_o)\theta_z \\ v_s(s, z) &\approx v + (x - x_o)\theta_z \\ w_s(s, z) &\approx w - xu' - yv' - \omega\theta'_z + (-xv'\theta_z + yu'\theta_z) \end{aligned} \right\} \quad (2.54a-c)$$

Equation 2.54 was formed under the assumption that the shear center undergoes displacements (u , v , w_s), and the longitudinal displacement w_s was expressed as

$$\begin{aligned} w_s &= w_{sc} - u'[(x - x_o) - (y - y_o)\theta_z] - v'[(y - y_o) + (x - x_o)\theta_z] + \omega\theta'_z \\ &= w - yv' - xu' - \omega\theta'_z + (-xv'\theta_z + yu'\theta_z) \end{aligned} \quad (2.55)$$

where $w = w_{sc} + x_o(u' + v'\theta_z) + y_o(v' - u'\theta_z)$.

The normal and shear strains associated with the displacement fields were expressed as

$$\left. \begin{aligned} \varepsilon_{zz_s} &\approx w'_p + \frac{u'^2_p + v'^2_p}{2} = \left[w' - xu'' - yv'' + \omega\theta''_z \right] + (x - x_o)(\theta'^2_z - v''\theta_z) + (y - y_o)(\theta'^2_z + u''\theta_z) \\ &\quad - x_o v'\theta'_z + y_o u'\theta'_z + \frac{1}{2} \left[u'^2 + v'^2 + (x_o^2 + y_o^2 + x^2 + y^2)\theta'^2_z \right] \\ \gamma_{zs_s} &= 2n_s\theta'_z \end{aligned} \right\} \quad (2.56)$$

where n_s is the perpendicular distance from the mid-thickness line to Point S within the cross-section. Shear strains due to bending and warping are neglected.

After establishing the expression for the displacement fields, the authors applied the principle of stationary total potential energy to determine the buckling load for the member. The total potential energy is

$$\pi = U - V = \frac{1}{2} \int_0^L \int_A (\varepsilon_{zz} \sigma_p + \gamma_p \tau_p) dA dz - \int_0^L (q_y v_q + q_z w_q) dz - \sum (Q_y v_Q + Q_z w_Q - M_x v'_M) \quad (2.57)$$

in which q_y and q_z denote the external distributed loads along y and z directions, while Q_y and Q_z are the concentrated loads along y and z axes, respectively; M_x is the external moment around x axis. In Eq. 2.57, v_q and w_q denote the displacements at lines of application of the loads q_y and q_z , respectively. Displacements v_Q and w_Q are those of the points of application of the loads Q_y and Q_z . The slope v'_M is that of the point of application of the bending moment M_x .

The subscripts q , Q and M indicate the corresponding displacement fields which are in line with the external loads (q_y , q_z , Q_y , Q_z , and M_x) at the points where the external loads are applied to form the terms of external potential energy.

As mentioned in Section 2.2, for equilibrium, the principle of virtual work requires $\delta\pi = 0$, which provides the pre-buckling configuration of the member. For neutral equilibrium (buckling position), the condition $\delta\left(\frac{1}{2}\delta^2\pi\right) = 0$ is enforced for any set of buckling deformations δu , δv , δw , $\delta\theta_z$, i.e.,

$$\begin{aligned} \delta\left(\frac{1}{2}\delta^2\pi\right) &= \delta\left(\frac{1}{2}\delta^2U - \frac{1}{2}\delta^2V\right) = 0 \\ &= \frac{1}{2} \int_0^L \int_A (\delta\varepsilon_{zz} \delta\sigma_{zz} + \delta\gamma_{zs} \delta\tau_{zs} + \sigma_{zz} \delta^2\varepsilon_{zz} + \tau_{zs} \delta^2\gamma_{zs}) dA dz \\ &\quad - \frac{1}{2} \int_0^L (q_y \delta^2 v_q + q_z \delta^2 w_q) dz - \sum (Q_y \delta^2 v_Q + Q_z \delta^2 w_Q - M_x \delta^2 v'_M) \end{aligned} \quad (2.58)$$

in which

$$\left. \begin{aligned} \delta \varepsilon_{zz} &= \delta w' - x \delta u'' - y \delta v'' + \omega \delta \theta_z'' \\ \delta \gamma_{zs} &= 2t \delta \theta_z' \\ \delta \sigma_{zz} &= E \delta \varepsilon_{zz} \\ \delta \tau_{zs} &= G \delta \gamma_{zs} \end{aligned} \right\} \quad (2.59)$$

and

$$\left. \begin{aligned} \delta^2 \varepsilon_{zz} &= \left\{ \delta u'^2 + \delta v'^2 + (x_o^2 + y_o^2) \delta \theta_z'^2 - 2x_o \delta v' \delta \theta_z' + 2y_o \delta u' \delta \theta_z' \right. \\ &\quad \left. - x \{ 2x_o \delta \theta_z'^2 + 2\delta \theta_z \delta v'' \} - y \{ 2y_o \delta \theta_z'^2 - 2\delta \theta_z \delta u'' \} + (x^2 + y^2) \delta \theta_z'^2 \right\} \\ \delta^2 \gamma_{zs} &= 0 \\ \sigma_{zz} &= \frac{N}{A} + \frac{M_x y}{I_{xx}} \end{aligned} \right\} \quad (2.60)$$

For an axially loaded column, $M_x = q_y = Q_y = 0$ and $\delta^2 w_q = \delta^2 w_Q = 0$. Under these conditions,

Eq. 2.58 becomes

$$\begin{aligned} & \frac{1}{2} \int_0^L E \left(I_{yy} \delta u''^2 + I_{xx} \delta v''^2 + I_{\omega\omega} \delta \theta_z''^2 + GJ \delta \theta_z'^2 \right) dz \\ & + \frac{1}{2} \int_0^L \bar{N} \left\{ \delta u'^2 + \delta v'^2 + (r_o^2 + x_A^2 + y_A^2) \delta \theta_z'^2 - 2x_A \delta v' \delta \theta_z' + 2y_A \delta u' \delta \theta_z' \right\} dz = 0 \end{aligned} \quad (2.61)$$

in which $r_o^2 = \int_A (x^2 + y^2) dA / A$

For a beam-column with a mono-symmetric cross-section, $x_A = 0$. Assuming that the member buckles in an inextensional buckling mode, we have $\delta^2 w_q = \delta^2 w_Q = 0$ similar to axially loaded columns, and $\delta^2 v'_M = 0$ which can be verified from taking the second variation of Eq. 2.54b (note that v'_M is the slope at the point of application of moment M_x).

$$\left. \begin{aligned} v_q &= -\frac{1}{2} (y_q - y_A) (\theta_{zb})^2 \\ v_Q &= -\frac{1}{2} (y_Q - y_A) (\theta_{zb})^2 \end{aligned} \right\} \quad (2.62)$$

The neutral equilibrium condition (Eq. 2.58) for the beam-column with mono-symmetric cross-section can be expressed as

$$\begin{aligned}
& \frac{1}{2} \int_0^L E \left(I_y \delta u''^2 + I_\omega \delta \theta_z'^2 + GJ \delta \theta_z'^2 \right) dz + \frac{1}{2} \int_0^L \bar{N} \left(\delta u'^2 + 2y_A \delta u' \delta \theta_z' + (y_A^2 + r_o^2) \delta \theta_z'^2 \right) dz \\
& + \frac{1}{2} \int_0^L \bar{M}_x \left(\beta_x \delta \theta_z'^2 + 2\delta u'' \delta \theta_z \right) dz + \frac{1}{2} \int_0^L q_y (y_q - y_A) \delta \theta_z'^2 dz \\
& + \frac{1}{2} \left[Q_y(0)(y_Q - y_A) \delta \theta_z'^2(0) + Q_y(L)(y_Q - y_A) \delta \theta_z'^2(L) \right] = 0
\end{aligned} \tag{2.63}$$

in which

$$r_o^2 = \int_A (x^2 + y^2) dA / A \text{ and } \beta_x = \int_A y(x^2 + y^2) dA / I_{xx} - 2y_A = I_{px} / I_{xx} - 2y_A$$

It is noted that the destabilizing term $\int_L \bar{M}_x \delta u'' \delta \theta_z dz$ in Eq. 2.63 has a critical importance for the lateral buckling analysis. Several researchers (Bleich 1952, Timoshenko and Gere 1961, Vlasov 1961, Rajasekaran 1977, Roberts 1983, Yang and McGuire 1986, Attard 1986, Kitipornchai and Chan 1987, Van Erp et al. 1988, Bild et al. 1988, and Chan 1990) have derived this destabilizing term either directly or indirectly. The use of the alternative expression $\int_L \bar{M}_x \delta u' \delta \theta_z' dz$ in the literature (Gellin and Lee 1988, Pandey and Sherbourne 1990, and De Jong 1990) resulting from neglecting the nonlinear terms in the displacement fields has led to incorrect buckling predictions (Pi et al. 1992, 2001). Another destabilizing expression based on considering shear straining due to bending, in which case additional terms must be included in the strain energy (Rajasekaran 1977, Yang and McGuire 1986, Kitipornchai and Chan 1987, Chan 1990) has been developed. These lead to an additional term $-\int_L \bar{M}_x' \delta u' \delta \theta_z dz$, which, when combined with $-\int_L \bar{M}_x \delta u' \delta \theta_z' dz$ leads to the correct expression $\int_L \bar{M}_x \delta u'' \delta \theta_z dz$ only when $\bar{M}_x, \delta u'$ or $\delta \theta_z$ vanishes at both ends of the beam. This is clarified using the following manipulations.

$$\begin{aligned}
& -\int_L (\bar{M}_x \delta u') \delta \theta_z' dz - \int_L \bar{M}_x' \delta u' \delta \theta_z dz \\
& = -\left[\bar{M}_x \delta u' \delta \theta_z \right]_0^L + \int_L (\bar{M}_x \delta u')' \delta \theta_z dz - \int_L \bar{M}_x' \delta u' \delta \theta_z dz \\
& = -\left[\bar{M}_x \delta u' \delta \theta_z \right]_0^L + \int_L \bar{M}_x \delta u'' \delta \theta_z dz + \int_L \bar{M}_x' \delta u' \delta \theta_z dz - \int_L \bar{M}_x' \delta u' \delta \theta_z dz \\
& = -\left[\bar{M}_x \delta u' \delta \theta_z \right]_0^L + \int_L \bar{M}_x \delta u'' \delta \theta_z dz
\end{aligned} \tag{2.64}$$

Thus, only when $\bar{M}_x, \delta u'$ or $\delta \theta_z$ vanish at both ends, does the destabilizing expression $\left(- \int_L \bar{M}_x \delta u' \delta \theta'_z dz - \int_L \bar{M}'_x \delta u' \delta \theta'_z dz \right)$ coincide with the expression $\int_L \bar{M}_x \delta u'' \delta \theta_z dz$.

2.5 Analysis of Warping Restraint of Joints to Thin-Walled Members

As discussed in Chapter 1, when analyzing non-collinear planar structures for lateral-torsional buckling, researchers normally adopt one of two extreme assumptions. At one end of the spectrum, the joint is assumed to provide zero warping restraints. This case is approached for the detail in Fig. 1.3b. At the other end of the spectrum, the joint is assumed to provide full warping restraints. This case is approached in the detail in Fig. 1.3 a. In reality, the connection details in Figs. 1.3 c-e would provide partial warping restraints to the connected members.

Vacharajittiphan and Trahair (1974) studied the warping and distortion of several stiffened and unstiffened I-section joints. The results of the four types of joints in Fig. 1.3 (a, c, d, e) were discussed. They found that the web distortion was effectively restrained by using the joint types a, c and d. The end warping was effectively restrained by joint type a. Furthermore, FEA investigations were conducted. The configuration of one of the joint modes is illustrated in Fig. 2.14 as an example. The lateral load P and boundary conditions were arranged as in the figure. The flanges at both ends were free to warp. Each model was composed of two identical I-section members, which were rigidly connected at the joint, and the angle α between the two jointed members was selected as 60° , 90° , 120° and 150° . Force P was applied at one end to induce a constant twisting moment in member AB, and a linear bending moment in member BC. The twist of member AB was restrained by the joint between members AB and BC.

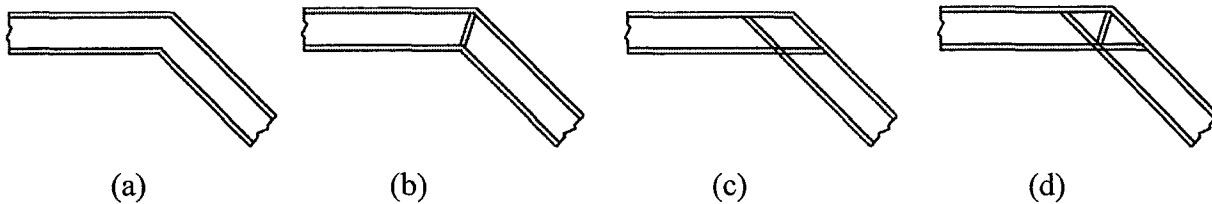


Figure 2.13 Joints stiffening arrangements, (a) Unstiffened, (b) A diagonal stiffener (c) Two pairs of stiffeners, (d) Three pairs of stiffeners.

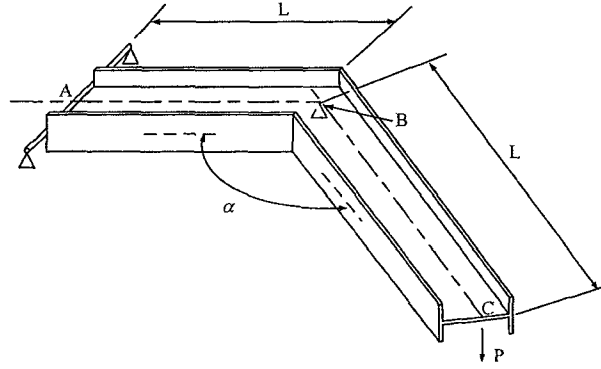


Figure 2.14 Example of the angle joint under load in the study of Vacharajittiphan and Trahair (1974)

The effect of warping restraint on twisting of member AB (Fig. 2.14) was conveniently expressed in terms of the ratio $\phi_B/\phi_{\max}$, where ϕ_B was the angle of twist of member AB at joint B also and $\phi_{\max} = (pL^2 \sin \alpha / GJ)$, which occurs with no warping restraint.

The effectiveness of the warping restraint K on the twisting member AB was judged by comparing the values of $\phi_B/\phi_{\max}$ obtained from the finite element analysis with the value (Kollbrunner and Basler 1969),

$$\frac{\phi_B}{\phi_{\max}} = 1 - \frac{K \frac{EI_{\omega}}{GJ}}{L \left(1 + K \sqrt{\frac{EI_{\omega}}{GJ}} \coth \sqrt{\frac{L^2 GJ}{EI_{\omega}}} \right)} \quad (2.65)$$

For stiffened right angle joints of type *c* in Fig. 2.13, warping is restrained by the torsional resistance of the I-section element within the joint. The warping restraint stiffness K at the joint for member AB was obtained as

$$K = \frac{\frac{2GJ}{EI_y}}{\sqrt{\frac{EI_{\omega}}{GJ}} \left(\sqrt{\frac{GJ}{EI_y}} - \frac{\tanh \sqrt{\frac{GJ}{EI_y}}}{1 + \frac{\sqrt{EI_y GJ}}{2GJ_f}} \tanh \sqrt{\frac{GJ}{EI_y}} \right)} \quad (2.66)$$

where J_f is the St. Venant torsion constant of the flanges.

Morrel (1979) investigated the influence of joint details on the torsional behaviour of thin-walled structures having an axial discontinuity. His study had a theoretical and an experimental component. The members connected consisted of two channel members having different lengths. The angle α between the two members were chosen as 135° and 90° (refer to Fig. 2.14). An external torque was applied at the middle-span of the longer member. The joints were configured in three types as in Figs. 2.13 a, b and c. The differential equations used for the analysis were the ones based on Vlasov theory. They were solved under the known boundary conditions. The boundary conditions and continuity at the joint for torsion and warping transmission or constraint were determined based on the bimoment continuity condition $B_l = B_r$ (the subscript l and r denote left-hand-side and right-hand-side of the joint) for any angle of the joints; the warping continuity $(\theta'_z)_l = (\theta'_z)_r$ in the cases where the angle of the connection was 90° . For the cases when the connection angle was 135° , the warping continuity condition was $[(\theta_z)_r - (\theta_z)_l] + (v'_r + v'_l) \tan(\phi/2) + \tan(\phi/2)[(\theta'_z)_r + (\theta'_z)_l] = 0$. The results of the experiments showed that the details of the joints significantly affected both the magnitude and sign of the twist within the unloaded member. However, the theoretical approach used was judged unlikely to develop an accurate prediction of the warping continuity for all aspects of joint configurations.

Ettouney and Kirby (1981) developed a new finite element formulation based on a two-node element which accounted for warping as an additional degree-of-freedom (total 7 degree-of-freedoms) at each node. Two warping conditions were introduced to one node, continuous warping or partial restrained warping. To specify the degree of partial warping restraint, they introduced a warping restraint factor defined as the ratio of the bimoment in the partially restrained case to the corresponding bimoment in the totally restrained case. Utilizing the warping restraint factors, the warping degree-of-freedoms were expressed in terms of other degree-of-freedoms and eliminated from the element matrix operations before the element stiffness matrix was assembled in the structure stiffness matrix. Unfortunately, the elimination process was not valid for the case where partial warping restraint was present at both ends.

Yang and McGuire (1984) modified Ettouney's work by deriving the warping stiffness matrix of an element with two nodes. Assuming that the warping boundary conditions could be specified on a member-by-member basis, an elastic warping support was introduced to represent conditions of partial warping restraint at the member ends. The partial warping restraint was expressed by elastic warping springs. In order to determine the warping spring constant, end A ($z=0$) of an element was prevented from twist, while end B ($z=l$) was free to twist, but the warping deformation at both ends were assumed to be partially restrained. Under this arrangement and based on the Vlasov differential equations, the bimoments at both ends were defined as $M_{\omega A} = -EI_{\omega} \left(d^2\theta_z/dz^2 \right)_{z=0} = -s_A \left(d\theta_z/dz \right)_{z=0}$, $M_{\omega B} = -EI_{\omega} \left(d^2\theta_z/dz^2 \right)_{z=l} = -s_B \left(d\theta_z/dz \right)_{z=l}$, where E was the modulus of elasticity, I_{ω} denotes the warping constant, s_A and s_B are the warping spring constants. Subsequently the warping degree-of freedom was eliminated from the stiffness matrix. The condensed stiffness matrices for the elements were then assembled in the structure stiffness matrix.

Sharman (1985) developed a finite element model for the thin-walled structures of general open cross-sections, in which the centroid and shear center do not necessarily coincide. In order to quantify the twist rate from one member to the other through a joint, three joint configurations (Fig. 2.13 a, b and c) were modeled by using an 8-node flat shell element in the PAFEC finite element program. The results showed that joint type "c" was able to transfer 92% of the twist from one member to the other compared to 83% for joint type "b", and 19% for joint "a".

Krenk and Damkilde (1991) studied the warping and distortion of joints in I-beam assemblages. They investigated four types of joint modes (in Fig. 2.13 a-d). In each case, the continuity conditions for the flanges were used to express the distortion in terms of the warping parameters of the two members at the joint. The stiffeners in joint types "c" and "d" were observed to enforce warping of equal magnitude in both members, while type "a" joints were observed to fully restrained warping. They also proposed expressions for warping restraint for all four types of joints.

Morrel et al. (1996) analyzed a frame consisting of two channel members connected at a right angle using ANSYS. The three types sketched in Fig. 2.13 (a, b and c) were investigated. A twisting moment was applied to one member and the warping and twisting response of the other member was measured. The results showed that in Joint type “c”, the warping deformation of the loaded member was the dominant factor in determining the magnitude and direction of the angle of twist in the unloaded member. For joint type “b”, twist in the unloaded member was dominant. For joint type “a”, the results showed that the warping and St. Venant torsion effects canceled each other in the unloaded member. The results confirmed the research done by Sharman (1985).

Pi and Trahair (2000) investigated the effects of end-support conditions on the lateral-distortional buckling of I-beams and warping restraints. They studied the effect of joint details on the elastic end-warping restraint for various common support conditions. The end-warping restraint α_ω is determined by $\alpha_\omega = (-M_\omega / \theta'_z)$, where the bimoment $M_\omega = EI_\omega \theta''_z$. In order to determine the bimoment M_ω for the various end-supports, the energy method was applied based on the postulated joint deformation models. They developed a methodology for quantifying the warping restraint for different end support conditions, for example, the end-warping restraint $\alpha_\omega = |M_\omega / \theta'_z| = GJh$, where J is the St. Venat torsion constant of the joint members and h is the height of the joint. The elastic-distortional buckling results for beams were compared to those based on finite-element programs, including ABAQUS as well as finite-strip programs. The comparison showed that the proposed equations provided good approximations to the warping restraint stiffness.

Masarira (2002) presented the results of investigations on the effects of joints with various stiffening arrangements on the lateral-torsional behaviour of steel frames. The study was based on the German standards (1992, 1952), which use two coefficients, warping restraint coefficient β_0 and bending restraint coefficient β_z to model the joint effects on lateral-torsional buckling of steel frames. The commercial program ANSYS was used for determining coefficients β_0 and β_z for the various type of joints. For joints with two pairs of stiffeners (Fig. 2.13 c) the degree of warping and bending restraints was found to be 80%.

Tong et al. (2005) studied the deformation of the joint with one diagonal stiffener (Fig.2.13 b). They presented a new model of warping and bimoment transmission for diagonally stiffened joints according to the geometrical relations between the members within the joint. In the new model bending and twisting deformations of the diagonal stiffener are allowed, which is different from the work done by Krenk (1991) who assumed the diagonal stiffener having infinite in plane stiffness. The new model has shown good agreement with the FEA results, and was better than Krenk and Damkilde's model.

2.6 Analysis of Out-of-Plane Behavior of Frames Consisting of Thin-Walled Members

While the previous section has focused on the force deformation modeling for joints in plane frames undergoing out-of-plane deformation, the present section focuses on the lateral buckling of such frames.

Hartmann and Munse (1966) analyzed lateral-torsional buckling of planar frames under planar loads by using the principle of stationary potential energy. Two parameters, the continuity of the members and lateral bracing stiffness, were concluded to have significant influence on the buckling load of the frame. The study assumed that the joints were infinitely rigid with respect to warping.

Magued (1971) investigated the elastic lateral-torsional buckling for portal frames by using the finite element method. The elastic and geometric stiffness matrices were derived by using the principle of virtual work. The frame element had two end nodes, each having seven degrees of freedom (three translations, three rotations and one warping). The warping degree of freedom was assumed either completely restrained or completely free.

Nethercot (1974) investigated lateral-torsional buckling of portal frames. The common line-element derived by Barsoum and Gallagher (1971) was adopted to model the structure under the assumption that warping was continuous at the junction of non-collinear members. Thus, for connected non-collinear members, the lateral and warping displacements were assumed identical while the rotation and twist were transformed in the conventional manner. The paper also

showed that for some configurations, lateral-torsional buckling predictions based on isolated members lead to acceptable estimates of frame lateral buckling loads.

Vacharajittiphan and Trahair (1973) proposed an analytical model for flexural-torsional buckling for rigidly jointed portal frames. The out-of-plane displacements at the beam to column joints were prevented. They proposed the following relations to quantify the partial warping restraint

$$\left. \begin{aligned} (\theta'_z)_b &= -L_b \frac{1-\beta_b}{4\beta_b} (\theta''_z)_b \\ (\theta'_z)_c &= L_c \frac{1-\beta_c}{2\beta_c} (\theta''_z)_c \end{aligned} \right\} \quad (2.67)$$

in which the subscript b and c indicated the beam and column; L denoted the length of the member the warping restraint parameter β varies from 0 (free warping) to 1 (fully restrained warping).

Vacharajittiphan and Trahair (1975) analyzed the lateral-torsional buckling of plane frames based on the results of two previous investigations (Vacharajittiphan and Trahair, 1973 and 1974). A general and systematic method of analyzing the elastic flexural-torsional buckling of rigid-jointed plane frames composed of I-section members and subjected to loads acting in the plane of the webs were presented. The differential equations for in-plane bending and flexural-torsional buckling were formulated. Buckling deformations were assumed infinitesimally small, so that the differential equations describing the in-plane pre-buckling equilibrium position of the frame could be separated from those describing lateral-torsional buckling, and thus were solved first. These equations were numerically integrated by the method of finite integrals, and the resulting sets of equations, together with the boundary, compatibility, warping and joint equilibrium equations were solved interactively for the critical load factor.

Krenk (1990) studied the lateral buckling of gable frames of I-shaped cross-section, which were laterally restrained at the top flange. Thus, the origin of coordinate was selected to coincide with the middle of the top-flange. The corresponding displacement fields were assumed as

$$\left. \begin{aligned} u(x, y, z) &= -y\theta_z \\ v(x, y, z) &= x\theta_z \\ w &= -xy\theta'_z \end{aligned} \right\} \quad (2.68)$$

The associated strain fields were expressed and substituted into the total potential energy functional to determine the lateral buckling load. Warping restraint was studied for the case of a joint with a diagonal stiffener (type B in Figure 2.13). Here the diagonal stiffener was assumed to be undeformable in its plane. Based on the geometric relations between the adjacent beam and column, the transformation of the displacements from the column to beam was obtained and substituted into the frame stiffness matrix.

Panangelis et al. (1998) developed the finite element software, PRFELB, to implement the analysis of elastic flexural-torsional buckling of thin-walled frames. The principle of stationary potential energy was used to formulate the stiffness and geometrical stiffness matrices for each frame member in its own local coordinate system. The two nodes line-element with four degrees of freedom (u, u', θ, θ') at each node was chosen to model the frame members. The end warping restraints can be modelled as free, fully restrained, or continuous.

Trahair and Chan (2003) studied in-plane strength and lateral buckling of steel structures based on plastic analyses. Their solution involves two steps, the first consisted of conducting an in-plane analysis using the plastic hinge method; and the second conducted an inelastic lateral buckling analysis which accounted for the effect of residual stresses, initial crookedness and twist. The warping restraint at the frame joints were taken from the results from the research done by Pi and Trahair (2000).

Zinoviev and Mohareb (2004) presented a methodology of the analysis and design of laterally unsupported portal frames for out-of-plane stability. The eigen-value problem of portal frames was solved using the principle of stationary potential energy. The problem was first analyzed by using finite element model in ABAQUS, so that the buckling load and associate buckling modes were generated. Regression analysis was then conducted to develop relatively simple expressions of the lateral displacement and angle of twist. These functions were then substituted into the total potential energy expression to approximately predict the lateral buckling loads of the frame.

MacPhedran and Grondin (2009) studied the lateral stability of plane frames considering member interaction and compatibility of warping deformations. They quantified the warping restraint capacity of the joints in frame structures by using the ABAQUS S4 shell element. A warping stiffness matrix was obtained and inserted into the planar frame structure model at the beam to column connections by using three spring elements oriented along horizontal, vertical and diagonal directions, while other structure members were modeled as beam elements. Their study concluded that neglecting warping displacement continuity appears to lead to conservative designs.

2.7 Closure

This chapter has presented a detailed reviewed the aspects necessary to develop solutions for buckling problem identified in Fig. 1a. Among the theories presented in Section 2.3, the Vlasov theory is the most extensively used in analyzing flexural, torsional, and lateral-torsional buckling of thin-walled members with open cross-sections. However, the second Vlasov assumption which neglects shear deformation within the middle surface can over-estimate the buckling resistance for members with a relatively short span. In this respect, the following developments contribute to the improvement of the Vlasov theory by providing a new solution which incorporates shear deformation. The variational principles for the conditions of equilibrium and neutral stability presented in Sections 2.1 and 2.2 will be used to formulate the governing differential equations and finite element formulations of the problem. The subtleties presented in Section 2.4 between various formulations related to lateral torsional buckling are the key to the correct implementation of buckling solutions presented in Chapters 6 and 7. Section 2.5 provided an overview of the various means developed of incorporating the warping restraints provided by joints to adjacent members. The joint element developed in Chapter 8 introduces a new technique for tackling the problem.

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List of Symbols

A :	area
A_p	a pole
C :	centroid of a cross-section
E :	Young's modulus
G :	shear modulus
$h(s)$:	perpendicular distance from a pole to the tangent of a point on the mid-surface of a cross-section
I_{xx}, I_{yy}, I_o :	moment of inertia of a cross-section about x -axis, y -axis and the origin, respectively
$I_{\omega\omega}$:	warping constant
J :	torsional constant
$k(s)$:	distance between pole A_p and an arbitrary point on the mid-surface along tangential direction
$[k]$:	elastic stiffness matrix
$[k_g]$:	geometrical stiffness matrix
k_s :	the shear correction factor used in Timoshenko beam theory
L :	length of a member
M_x, M_y, M_z :	bending moment about x, y, z axes
M_ω :	Bimoment
n :	an axis along the normal direction at a point on a contour profile of a cross-section
$\bar{N}(z)$:	axial resultant of normal stresses in pre-buckling range
O :	origin of Cartesian coordinates x, y, z
O' :	origin of Cartesian coordinates x', y', z'
q_y, q_z :	line loads applying on a member along y and z direction, respectively
Q_{yy}, Q_{zz} :	concentrated loads applying on a member along y and z direction, respectively
r_o	radius of gyration in orthogonal coordinates
$[R]$:	a rotation matrix
s :	curvilinear coordinate along mid-surface

S :	an arbitrary point
S_o :	sectorial origin
S_x, S_y, S_ω :	first moments of area of coordinates x, y , and ω
$S_{x\omega}, S_{y\omega}$:	section products of inertia
SC :	shear center of a cross-section
t :	an axis along the tangential direction at a point on the mid-surface
T_{sv} :	torque of Saint Venant torsion
u, v, w :	displacements along x, y, z axes, respectively
u', v', w' :	derivative of displacements u, v, w with respect to z , respectively
$\{u\}$:	general displacements
$\{u_N\}$:	nodal displacements
U :	strain energy
V :	potential energy
$\bar{V}_y$:	component (in y -direction) of shear resultant
x, y, z :	variables; or axes of Cartesian coordinates
π, π^* :	total potential, or a functional
$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$:	normal components of a strain tensor
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$:	shear components of a strain tensor
γ_{zs} :	shear strain along the mid-surface of a cross-section
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$:	normal components of a stress tensor
σ_ω :	normal stress of a cross-section due to warping
$\tau_{xy}, \tau_{xz}, \tau_{yz}$:	shear components of a stress tensor
τ_{zs} :	shear stress at the mid-surface of a cross-section
τ_{st} :	shear stress due to St. Venant torsion
τ_ω :	shear stress of a cross-section due to warping
α :	angle between the tangent to the contour and the x axis

ϕ :	an angle
κ :	curvature of a plate
λ :	eigenvalue
μ :	Lagrange multiplier
$\theta_x, \theta_y, \theta_z$:	rotation angles around x, y, z axes, respectively
θ_ω :	warping degree-of-freedom
ν :	Poisson's ratio
ω :	warping function or sectorial area of a cross-section
$\bar{\omega}$:	contour warping function
$\bar{\bar{\omega}}$:	thickness warping function
Ω :	contour warping function
$\psi(z)$:	a change rate of the longitudinal displacement due to warping along the axis z and has a unit of $1/Length$

Chapter 3 Development of Displacement Fields for Generalized Timoshenko Beam

As discussed in Chapter 2, Vlasov (1961) developed his theory for open thin-walled beams based on the combination of bending (based on the Euler-Bernoulli beam theory) and non uniform torsion. In a pre-buckling analysis, Vlasov's theory results into four coupled differential equations. Conventionally, adopting an orthogonal coordinate system can eliminate the coupling between the four differential equations (Murray 1984). However, this conventional approach limits the solution to simple structural problems. It leads to difficulties in modeling (1) structural members with eccentric boundary conditions (support at the top or bottom of the members); (2) members with cross-section discontinuities along the member (e.g., with holes, flange copes, discontinuous longitudinal stiffeners) (Ekmen and Mohareb 2006). Also, Vlasov theory omits shear deformation effects. This can over-estimate the buckling resistance of some structures, such as long column supporting a short cantilever subject to a vertical load at the cantilever tip (Dabbas 2002), and could lead to un-conservative designs.

This chapter develops the kinematic relations for a general open thin-walled beam including shear deformation effects. The assumptions made in the Timoshenko beam theory (1921) in bending are combined with the Vlasov assumptions which capture the non-uniform torsional warping. Also, a general non-orthogonal coordinate system is adopted in order to avoid the modeling limitations in conventional solutions based on orthogonal coordinate system. The kinematic relations developed in this chapter can be applied to pre-buckling and buckling analyses including lateral torsional buckling of open thin-walled members.

3.1 Hypotheses

The main assumptions in the formulation are:

1. The formulation is restricted to prismatic open thin-walled members.

2. The cross-section remains rigid in its own plane during deformation, in agreement with the first Vlasov assumption.
3. The generalized Timoshenko beam assumptions are adopted, i.e., a cross-section remains planar, but it does not remain perpendicular to the member axis after deformation. A similar kinematic assumption is extended for warping deformation.
4. Rotations are moderate.

3.2 Kinematics

An open thin-walled member is assumed to undergo displacements $(u, v, w, \theta_x, \theta_y, \theta_z, \psi)$, in which $u(z)$ and $v(z)$ are displacements of a general pole A_p along x and y axes; $w(z)$ is the longitudinal displacement of a general origin O' located on the z axis initially; $\theta_x(z)$, $\theta_y(z)$ and $\theta_z(z)$ are rotations of the cross-section around axes x , y and z , respectively and $\psi(z)$ is defined as the change rate of the longitudinal displacement due to warping and has a unit of $1/Length$. Two sets of right-hand Cartesian axes are used to describe the deformations (Fig. 3.1). The first set of axes x , y and z have an origin O in the plane of the cross-section, are fixed in space and associated with the undeformed beam configuration. The z axis is oriented along the axial direction; the x and y axes are along the major and minor principal directions. The second set of axes x' , y' and z' is originally parallel to the x , y and z axes, respectively. These axes follow the beam as it undergoes deformations. The z' axis is oriented along the direction of the normal to the deformed cross-section at point O' . It is required to determine the position of a general point on the mid-surface of the member as the member undergoes deformations $u, v, w, \theta_x, \theta_y, \theta_z$ and ψ .

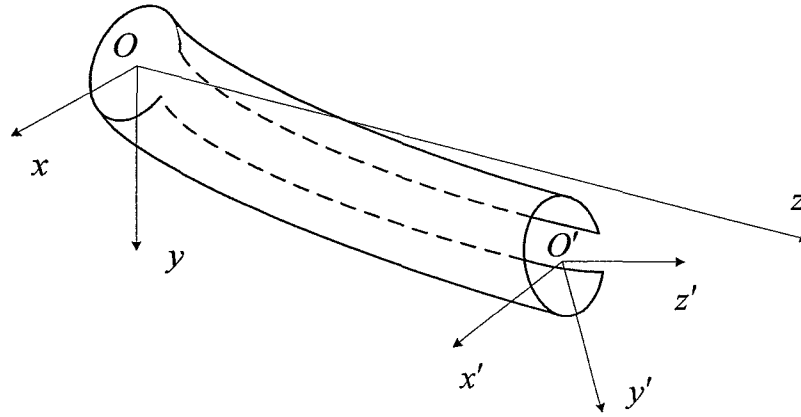


Figure 3.1 Deformations of a structural open thin-walled member

3.3 Finite Rotation Equation

The directions of axes x' , y' and z' relative to the fixed axes x , y and z (Fig. 3.2), can be determined by applying the finite rotation equation developed by Euler in 1775 (McGuire et al., 2000). The finite rotation equation provides a relationship between the locations of a vector before and after a rotation. The location of the vector in the deformed configuration is determined as the vector rotates through an angle $\{\phi\}$ about an arbitrary axis through the origin as illustrated in Fig. 3.2. The treatment provided in the following was presented by McGuire et al. (2000). Here, more detailed interpretations are provided.

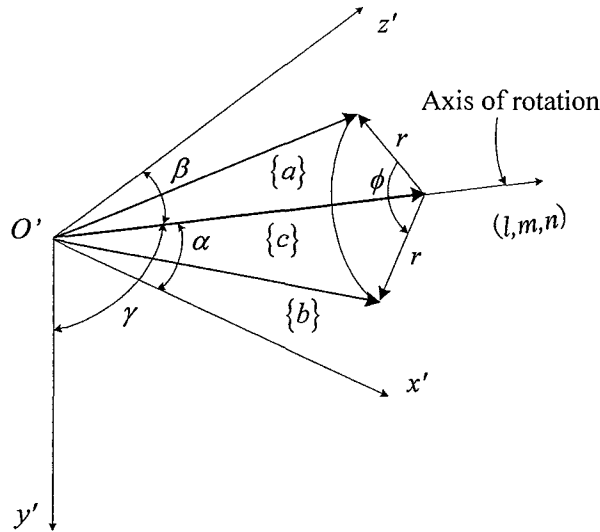


Figure 3.2 Finite rotation (adapted from McGuire et al., 2000)

In Fig. 3.2, vectors $\{a\}$ and $\{b\}$ denote the locations of the vector before and after rotation about a fixed vector $\{c\}$, respectively. Angles α , β and γ are the direction angles of the axis of rotation defined by vector $\{c\}$, and ϕ is the magnitude of angle of rotation. The finite rotation equation is

$$\begin{aligned}\frac{b_x}{|b|} &= \frac{1}{|a|} \left\{ a_x \left[l^2 + (1-l^2) \cos \phi \right] + a_y \left[lm(1-\cos \phi) - n \sin \phi \right] + a_z \left[ln(1-\cos \phi) + m \sin \phi \right] \right\} \\ \frac{b_y}{|b|} &= \frac{1}{|a|} \left\{ a_y \left[m^2 + (1-m^2) \cos \phi \right] + a_z \left[mn(1-\cos \phi) - l \sin \phi \right] + a_x \left[lm(1-\cos \phi) + n \sin \phi \right] \right\} \\ \frac{b_z}{|b|} &= \frac{1}{|a|} \left\{ a_z \left[n^2 + (1-n^2) \cos \phi \right] + a_x \left[ln(1-\cos \phi) - m \sin \phi \right] + a_y \left[mn(1-\cos \phi) + l \sin \phi \right] \right\}\end{aligned}\quad (3.1)$$

in which

$$\begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} \cos \alpha \\ \cos \beta \\ \cos \gamma \end{Bmatrix}, \quad \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = |a| \begin{Bmatrix} \cos \zeta \\ \cos \eta \\ \cos \vartheta \end{Bmatrix}, \quad \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = |b| \begin{Bmatrix} \cos \zeta' \\ \cos \eta' \\ \cos \vartheta' \end{Bmatrix}\quad (3.2)$$

in which (ζ, η, ϑ) and $(\zeta', \eta', \vartheta')$ are the direction angles of vector $\langle a \rangle^T = \langle a_x \ a_y \ a_z \rangle$ and $\langle b \rangle^T = \langle b_x \ b_y \ b_z \rangle$ before and after rotation.

Given that $|b| = |a|$, and using the identity $(1 - \cos \phi) = 2 \sin^2 (\phi/2)$, one obtains

$$\begin{aligned}b_x &= a_x \left[l^2 \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) + 1 - 2 \sin^2 \left(\frac{\phi}{2} \right) \right] + a_y \left[lm \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) - n \sin \phi \right] + a_z \left[ln \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) + m \sin \phi \right] \\ b_y &= a_y \left[m^2 \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) + 1 - 2 \sin^2 \left(\frac{\phi}{2} \right) \right] + a_z \left[mn \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) - l \sin \phi \right] + a_x \left[lm \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) + n \sin \phi \right] \\ b_z &= a_z \left[n^2 \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) + 1 - 2 \sin^2 \left(\frac{\phi}{2} \right) \right] + a_x \left[ln \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) - m \sin \phi \right] + a_y \left[mn \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) + l \sin \phi \right]\end{aligned}\quad (3.3)$$

Rearranging, Eq. (3.3) becomes

$$\begin{aligned}
b_x &= a_x \left[1 + (l^2 - 1) \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) \right] + a_y \left[-n \sin \phi + lm \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) \right] + a_z \left[m \sin \phi + ln \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) \right] \\
b_y &= a_x \left[n \sin \phi + lm \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) \right] + a_y \left[1 + (m^2 - 1) \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) \right] + a_z \left[-l \sin \phi + mn \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) \right] \\
b_z &= a_x \left[-m \sin \phi + ln \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) \right] + a_y \left[l \sin \phi + mn \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) \right] + a_z \left[1 + (n^2 - 1) \cdot 2 \sin^2 \left(\frac{\phi}{2} \right) \right]
\end{aligned} \tag{3.4}$$

In a matrix form, one has

$$\begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin \phi \begin{bmatrix} 0 & -n & m \\ n & 0 & -l \\ -m & l & 0 \end{bmatrix} + 2 \sin^2 \left(\frac{\phi}{2} \right) \begin{bmatrix} l^2 - 1 & lm & ln \\ lm & m^2 - 1 & mn \\ ln & mn & n^2 - 1 \end{bmatrix} \right) \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} \tag{3.5}$$

Given the identity $l^2 + m^2 + n^2 = 1$, Eq. 3.5 simplifies to

$$\{b\} = \left[[I] + \sin \phi [H] + 2 \sin^2 \left(\frac{\phi}{2} \right) [H]^2 \right] \{a\} \tag{3.6}$$

in which matrix $[H]$ is defined as

$$[H] = \begin{bmatrix} 0 & -n & m \\ n & 0 & -l \\ -m & l & 0 \end{bmatrix} \tag{3.7}$$

Under the moderate rotation assumption, one can scale the direction cosine vector $\langle l \ m \ n \rangle^T$ of the rotation axis by the magnitude of the rotation angle ϕ to recover the rotation vector $\langle \phi_x \ \phi_y \ \phi_z \rangle^T$, i.e.,

$$\{\phi\}^T \approx \phi \langle l \ m \ n \rangle^T = \langle \phi_x \ \phi_y \ \phi_z \rangle^T \tag{3.8}$$

Using a similar definition, one can define the rotation matrix $[R]$ as

$$[R] = \{\phi\}^T [H] = \begin{bmatrix} 0 & -\phi_z & \phi_y \\ \phi_z & 0 & -\phi_x \\ -\phi_y & \phi_x & 0 \end{bmatrix} \tag{3.9}$$

From Eq. 3.9 by substituting into Eq. 3.7, one obtains

$$\{b\} \approx \left[[I] + \frac{\sin \phi}{\phi} [R] + \frac{1}{2} \left(\frac{\sin(\phi/2)}{\phi/2} \right)^2 [R]^2 \right] \{a\} \tag{3.10}$$

Equation 3.10 is identical to Eq. (A.42) in McGuire et al. (2000). The approximation $\sin \phi \approx \phi$ and $\sin(\phi/2) \approx (\phi/2)$ is satisfactory when angle ϕ is small to moderate. Hence,

$$\{b\} = \left[[I] + [R] + \frac{1}{2}[R]^2 \right] \{a\} = [T] \{a\} \quad (3.11)$$

in which the rotation matrix $[T]$ is defined as,

$$\begin{aligned} [T] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\phi_z & \phi_y \\ \phi_z & 0 & -\phi_x \\ -\phi_y & \phi_x & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -(\phi_y^2 + \phi_z^2) & \phi_x \phi_y & \phi_x \phi_z \\ \phi_x \phi_y & -(\phi_x^2 + \phi_z^2) & \phi_y \phi_z \\ \phi_x \phi_z & \phi_y \phi_z & -(\phi_x^2 + \phi_y^2) \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{1}{2}(\phi_y^2 + \phi_z^2) & -\phi_z + \frac{1}{2}\phi_x \phi_y & \phi_y + \frac{1}{2}\phi_x \phi_z \\ \phi_z + \frac{1}{2}\phi_x \phi_y & 1 - \frac{1}{2}(\phi_x^2 + \phi_z^2) & -\phi_x + \frac{1}{2}\phi_y \phi_z \\ -\phi_y + \frac{1}{2}\phi_x \phi_z & \phi_x + \frac{1}{2}\phi_y \phi_z & 1 - \frac{1}{2}(\phi_x^2 + \phi_y^2) \end{bmatrix} \end{aligned} \quad (3.12)$$

Therefore, the displacement vector $\{u\}$ is calculated by $\{u\} = \{b\} - \{a\} = [R + (1/2)R^2] \{a\}$, or

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} -\frac{1}{2}(\phi_y^2 + \phi_z^2) & -\phi_z + \frac{1}{2}\phi_x \phi_y & \phi_y + \frac{1}{2}\phi_x \phi_z \\ \phi_z + \frac{1}{2}\phi_x \phi_y & -\frac{1}{2}(\phi_x^2 + \phi_z^2) & -\phi_x + \frac{1}{2}\phi_y \phi_z \\ -\phi_y + \frac{1}{2}\phi_x \phi_z & \phi_x + \frac{1}{2}\phi_y \phi_z & -\frac{1}{2}(\phi_x^2 + \phi_y^2) \end{bmatrix} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} \quad (3.13)$$

Although Eq. 3.13 was used by Pi et al. (1992), McGuire et al. (2000), and Erkmen et al (2008), the physical meaning of components ϕ_x , ϕ_y and ϕ_z of rotation vector $\{\phi\}$ was not clearly identified by them. In order to develop a clear relationship between the components ϕ_x , ϕ_y and ϕ_z of the angle of rotation $\{\phi\}$ about a vector $\langle l \ m \ n \rangle^T$ and the average angles of rotation $\theta_x, \theta_y, \theta_z$ of the plane of the cross-section about the x , y and z axes, Eq. 3.13 is used in the following to describe the displacement fields for a Timoshenko beam. The interpretation is generalized later on for a modified Vlasov theory which accounts for shear deformations.

3.4 Formulation of Rotation Matrix for a Timoshenko Beam

Consider two unit vectors (1,0,0) and (0,1,0) originally parallel to the x axis and y axis on a cross-section, respectively (Fig. 3.3, Pi et al. 1992). After undergoing a rotation $\{\phi\}$ about an arbitrary axis (l, m, n) , the displacements of the two vectors become

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} -\frac{1}{2}(\phi_y^2 + \phi_z^2) & -\phi_z + \frac{1}{2}\phi_x\phi_y & \phi_y + \frac{1}{2}\phi_x\phi_z \\ \phi_z + \frac{1}{2}\phi_x\phi_y & -\frac{1}{2}(\phi_x^2 + \phi_z^2) & -\phi_x + \frac{1}{2}\phi_y\phi_z \\ -\phi_y + \frac{1}{2}\phi_x\phi_z & \phi_x + \frac{1}{2}\phi_y\phi_z & -\frac{1}{2}(\phi_x^2 + \phi_y^2) \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{2}(\phi_y^2 + \phi_z^2) \\ \phi_z + \frac{1}{2}\phi_x\phi_y \\ -\phi_y + \frac{1}{2}\phi_x\phi_z \end{Bmatrix} \quad (3.14)$$

and

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} -\frac{1}{2}(\phi_y^2 + \phi_z^2) & -\phi_z + \frac{1}{2}\phi_x\phi_y & \phi_y + \frac{1}{2}\phi_x\phi_z \\ \phi_z + \frac{1}{2}\phi_x\phi_y & -\frac{1}{2}(\phi_x^2 + \phi_z^2) & -\phi_x + \frac{1}{2}\phi_y\phi_z \\ -\phi_y + \frac{1}{2}\phi_x\phi_z & \phi_x + \frac{1}{2}\phi_y\phi_z & -\frac{1}{2}(\phi_x^2 + \phi_y^2) \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\phi_z + \frac{1}{2}\phi_x\phi_y \\ -\frac{1}{2}(\phi_x^2 + \phi_z^2) \\ \phi_x + \frac{1}{2}\phi_y\phi_z \end{Bmatrix} \quad (3.15)$$

Thus, the projection of the unit vector (0,1,0) in the rotated configuration along x direction is $(-\phi_z + \phi_x\phi_y/2)$, which corresponds to an angle of twist $\left[\tan^{-1}(-\phi_z + \phi_x\phi_y/2) \approx (-\phi_z + \phi_x\phi_y/2) \right]$, and that of the unit vector (1,0,0) in the y direction is $(\phi_z + \phi_x\phi_y/2)$, which corresponds to an angle of twist is $\left[\tan^{-1}(\phi_z + \phi_x\phi_y/2) \approx (\phi_z + \phi_x\phi_y/2) \right]$ as depicted in Fig. 3.4. The angle between the original vector (1,0,0) and its position vector after rotation is $\left[\tan^{-1}(\phi_z + \phi_x\phi_y/2) \approx (\phi_z + \phi_x\phi_y/2) \right]$. In a similar manner, for vector (0,1,0), the angle of rotation is $\left[\tan^{-1}(-\phi_z + \phi_x\phi_y/2) \approx (-\phi_z + \phi_x\phi_y/2) \right]$. In turn, the mean angle of twist based on the two vectors (defined as the twisting angle θ_z of the cross-section around the z axis) is

$$\theta_z = \frac{1}{2} \left[\left(\phi_z + \frac{\phi_x\phi_y}{2} \right) - \left(-\phi_z + \frac{\phi_x\phi_y}{2} \right) \right] = \phi_z \quad (3.16)$$

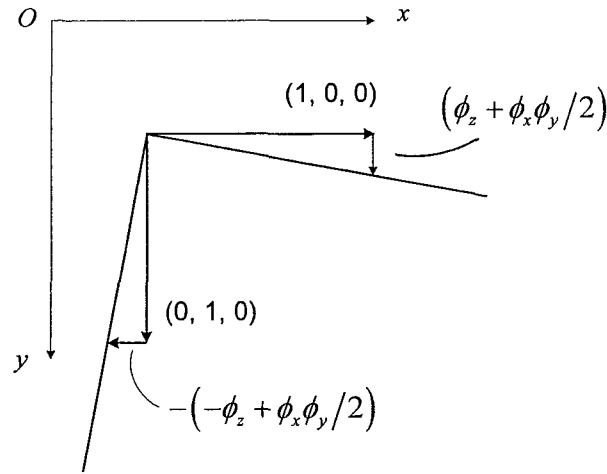
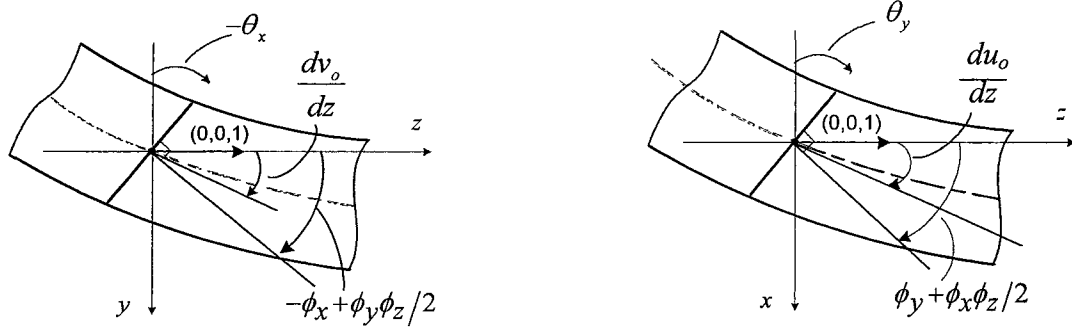


Figure 3.3 Angle of twist (Pi et. al., 1992)

Similar definitions for angles ϕ_x and ϕ_y can be obtained by considering a unit vector $(0,0,1)$ normal to the cross-section, i.e., originally parallel to the z axis undergoing angle $\{\phi\}$. The displacements of the unit vector $(0, 0, 1)$ are

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} -\frac{1}{2}(\phi_y^2 + \phi_z^2) & -\phi_z + \frac{1}{2}\phi_x\phi_y & \phi_y + \frac{1}{2}\phi_x\phi_z \\ \phi_z + \frac{1}{2}\phi_x\phi_y & -\frac{1}{2}(\phi_x^2 + \phi_z^2) & -\phi_x + \frac{1}{2}\phi_y\phi_z \\ -\phi_y + \frac{1}{2}\phi_x\phi_z & \phi_x + \frac{1}{2}\phi_y\phi_z & -\frac{1}{2}(\phi_x^2 + \phi_y^2) \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \phi_y + \frac{1}{2}\phi_x\phi_z \\ -\phi_x + \frac{1}{2}\phi_y\phi_z \\ -\frac{1}{2}(\phi_x^2 + \phi_y^2) \end{Bmatrix} \quad (3.17)$$

The direction cosines of the unit vector in the rotated configuration are $[\phi_y + \phi_x\phi_z/2, -\phi_x + \phi_y\phi_z/2, 1]$. The corresponding component of the rotation angle θ_x in the yz -plane is $\tan(-\phi_x + \phi_y\phi_z/2) \approx (-\phi_x + \phi_y\phi_z/2)$, and in xz -plane is $\tan(\phi_y + \phi_x\phi_z/2) \approx (\phi_y + \phi_x\phi_z/2)$. These rotations are illustrated as in Fig. 3.4.

Figure 3.4 Rotations θ_x and θ_y of a Timoshenko beam

Therefore, the rotation angles of the vector normal to plane of the cross-section may be expressed as

$$\theta_y = \phi_y + \frac{1}{2}\phi_x\phi_z = \phi_y + \frac{1}{2}\phi_x\theta_z \quad (3.18a)$$

$$\theta_x = \phi_x - \frac{1}{2}\phi_y\phi_z = \phi_x - \frac{1}{2}\phi_y\theta_z \quad (3.18b)$$

From Eq. 3.18, one can express ϕ_x and ϕ_y as functions of θ_x and θ_y yielding,

$$\phi_y = \theta_y - \frac{1}{2}\phi_x\theta_z \quad (3.19a)$$

$$\phi_x = \theta_x + \frac{1}{2}\phi_y\theta_z \quad (3.19b)$$

Substituting Eq. 3.19b into Eq. 3.19a for ϕ_x and substituting Eq. 3.19a into Eq. 3.19b transformation for ϕ_y , yields,

$$\phi_y = \theta_y - \theta_z(\theta_x + \phi_y\theta_z/2)/2 = \theta_y - \frac{1}{2}\theta_x\theta_z - \frac{1}{4}\phi_y\theta_z^2 \quad (3.20a)$$

$$\phi_x = \theta_x + \theta_z(\theta_y - \phi_x\theta_z/2)/2 = \theta_x + \frac{1}{2}\theta_y\theta_z - \frac{1}{4}\phi_x\theta_z^2 \quad (3.20b)$$

By neglecting third order terms in Eq. 3.20 and substituting it into Eq. 3.12, the rotation matrix $[T]$ is expressed in terms of the average angles of rotation $\theta_x, \theta_y, \theta_z$ about the axes x, y, z , respectively, as

$$[T] = \begin{bmatrix} \langle T_1 \rangle_{1 \times 3} \\ \langle T_2 \rangle_{1 \times 3} \\ \langle T_3 \rangle_{1 \times 3} \end{bmatrix} \approx \begin{bmatrix} 1 - \frac{1}{2}(\theta_y^2 + \theta_z^2) & -\theta_z + \frac{1}{2}\theta_x\theta_y & \theta_y \\ \theta_z + \frac{1}{2}\theta_x\theta_y & 1 - \frac{1}{2}(\theta_x^2 + \theta_z^2) & -\theta_x \\ -\theta_y + \theta_x\theta_z & \theta_x + \theta_y\theta_z & 1 - \frac{1}{2}(\theta_x^2 + \theta_y^2) \end{bmatrix}_{3 \times 3} \quad (3.21)$$

By setting $\theta_y = u'$ and $\theta_x = v'$, one obtains Eq. 10 in Pi et al. (1992), which is valid for the Euler-Bernoulli beam. In the present formulation, the more general conditions $\theta_y \neq u'$ and $\theta_x \neq v'$ are retained in order to capture shear deformation effects.

For a general point $S(x(s), y(s))$ on a mid-surface of a cross-section illustrated in Fig. 3.6, the displacements $u_s(s, z), v_s(s, z), w_s(s, z)$ of point S due to the rotations of the cross-section $\theta_x(z), \theta_y(z)$ and $\theta_z(z)$ are

$$\begin{Bmatrix} u_s(z, s) \\ v_s(z, s) \\ w_s(z, s) \end{Bmatrix} = [T] \begin{Bmatrix} x(s) \\ y(s) \\ 0 \end{Bmatrix} - \begin{Bmatrix} x(s) \\ y(s) \\ 0 \end{Bmatrix} \quad (3.22a-c)$$

$$= \begin{Bmatrix} -y(s)\theta_z(z) \\ x(s)\theta_z(z) \\ -x(s)\theta_y + y(s)\theta_x(z) \end{Bmatrix} + \begin{Bmatrix} -\frac{x(s)}{2}(\theta_y^2(z) + \theta_z^2(z)) + \frac{y(s)}{2}(\theta_x(z)\theta_y(z)) \\ \frac{x(s)}{2}(\theta_x(z)\theta_y(z)) - \frac{y(s)}{2}(\theta_x^2(z) + \theta_z^2(z)) \\ x(s)\theta_x(z)\theta_z(z) + y(s)\theta_y(z)\theta_z(z) \end{Bmatrix}$$

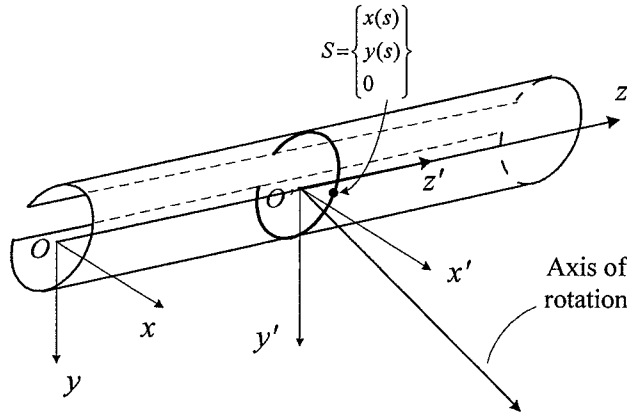


Figure 3.5 Finite rotation of a cross-section

3.5 Displacements of a General Point S on the Cross-section

In this section, our objective is to express the displacements of point S on the profile due to cross-section rotations $\theta_x, \theta_y, \theta_z$, displacements u, v of pole A_p , and longitudinal displacement w of origin O' . In Fig. 3.6, axes (x, y) are parallel to the directions of the major and minor principal axes. Their origin is point $O'(0, 0)$, which is neither necessarily coinciding with the coordinates of the shear centre $SC(x_{sc}, y_{sc})$, nor those of the centroid $C(\bar{x}, \bar{y})$. Since coordinates x, y are principal, one has $\int_A (y - \bar{y})(x - \bar{x}) dA = 0$. Pole A_p has coordinates (x_A, y_A) .

Thus, the relative distance between pole $A_p(x_A, y_A)$ point $S(x(s), y(s))$ in the plane of the cross-section is $(x(s) - x_A, y(s) - y_A)$.

When the cross-section exhibits rotations $\theta_x(z), \theta_y(z)$ and $\theta_z(z)$, displacements $u(z)$ and $v(z)$ at pole A_p , the displacements $u_s(s, z), v_s(s, z)$ of point S are given by

$$\begin{aligned} \begin{Bmatrix} u_s(z, s) \\ v_s(z, s) \end{Bmatrix} &= \begin{Bmatrix} u(z) \\ v(z) \end{Bmatrix} + \begin{Bmatrix} \langle T_1 \rangle_{\text{b3}} \\ \langle T_2 \rangle_{\text{b3}} \end{Bmatrix} \begin{Bmatrix} x(s) - x_A \\ y(s) - y_A \\ 0 \end{Bmatrix} - \begin{Bmatrix} x(s) - x_A \\ y(s) - y_A \end{Bmatrix} \\ &= \begin{Bmatrix} u(z) \\ v(z) \end{Bmatrix} + \begin{bmatrix} 1 - \frac{1}{2}(\theta_y(z)^2 + \theta_z(z)^2) & -\theta_z(z) + \frac{1}{2}\theta_x(z)\theta_y(z) & \theta_y(z) \\ \theta_z(z) + \frac{1}{2}\theta_y(z)\theta_x(z) & 1 - \frac{1}{2}(\theta_x(z)^2 + \theta_z(z)^2) & \theta_x(z) \end{bmatrix} \begin{Bmatrix} x(s) - x_A \\ y(s) - y_A \\ 0 \end{Bmatrix} - \begin{Bmatrix} x(s) - x_A \\ y(s) - y_A \end{Bmatrix} \quad (3.23) \\ &= \begin{Bmatrix} u - (y - y_A)\theta_z \\ v + (x - x_A)\theta_z \end{Bmatrix} + \begin{Bmatrix} -\frac{(x - x_A)}{2}(\theta_y^2 + \theta_z^2) + \frac{(y - y_A)}{2}(\theta_x\theta_y) \\ \frac{(x - x_A)}{2}(\theta_x\theta_y) - \frac{(y - y_A)}{2}(\theta_x^2 + \theta_z^2) \end{Bmatrix} \end{aligned}$$

We recall that row vectors $\langle T_1 \rangle$ and $\langle T_2 \rangle$ which appear in Eq. 3.23 are defined in Eq. 3.21.

Meanwhile, longitudinal displacement $w_s(s, z)$ due to displacements $w(z)$ of origin O' , and rotations $\theta_x(z), \theta_y(z)$ and $\theta_z(z)$ is

$$\begin{aligned}
w_s(z,s) &= w(z) + \langle T_3 \rangle_{3 \times 3} \begin{Bmatrix} x(s) \\ y(s) \\ 0 \end{Bmatrix} - 0 = w(z) + \left\langle \begin{matrix} -\theta_y + \theta_x \theta_z & \theta_x + \theta_y \theta_z & 1 - \frac{1}{2}(\theta_y^2 + \theta_x^2) \end{matrix} \right\rangle \begin{Bmatrix} x(s) \\ y(s) \\ 0 \end{Bmatrix} - 0 \\
&= w - x\theta_y + y\theta_x + x\theta_x\theta_z + y\theta_y\theta_z
\end{aligned} \tag{3.24}$$

in which sub-matrix $\langle T_3 \rangle$ is defined in Eq. 3.21. The complete displacement vector of point S due to generalized displacements u , v , w , θ_x , θ_y and θ_z can be expressed by consolidating Eqs. 3.23 and 3.24 in a single matrix equation.

$$\begin{Bmatrix} u_s(z,s) \\ v_s(z,s) \\ w_s(z,s) \end{Bmatrix} = \begin{Bmatrix} u - (y - y_A)\theta_z \\ v + (x - x_A)\theta_z \\ w - x\theta_y + y\theta_x \end{Bmatrix} + \begin{Bmatrix} -\frac{(x - x_A)}{2}(\theta_y^2 + \theta_z^2) + \frac{(y - y_A)}{2}(\theta_x\theta_y) \\ \frac{(x - x_A)}{2}(\theta_x\theta_y) - \frac{(y - y_A)}{2}(\theta_x^2 + \theta_z^2) \\ x\theta_x\theta_z + y\theta_y\theta_z \end{Bmatrix} \tag{3.25}$$

The displacements of point S along the tangential direction t and the normal direction n (Fig. 3.6) are obtained as

$$t_s(z,s) = u_s \cos \alpha + v_s \sin \alpha = u \cos \alpha + v \sin \alpha + [(x - x_A) \sin \alpha - (y - y_A) \cos \alpha] \theta_z \tag{3.26a}$$

$$n_s(z,s) = v_s \cos \alpha - u_s \sin \alpha = v \cos \alpha - u \sin \alpha + [(x - x_A) \cos \alpha + (y - y_A) \sin \alpha] \theta_z \tag{3.26b}$$

in which $\alpha(s)$ is the angle between the tangent line to the profile at point S and the x -axis (Fig. 3.6).

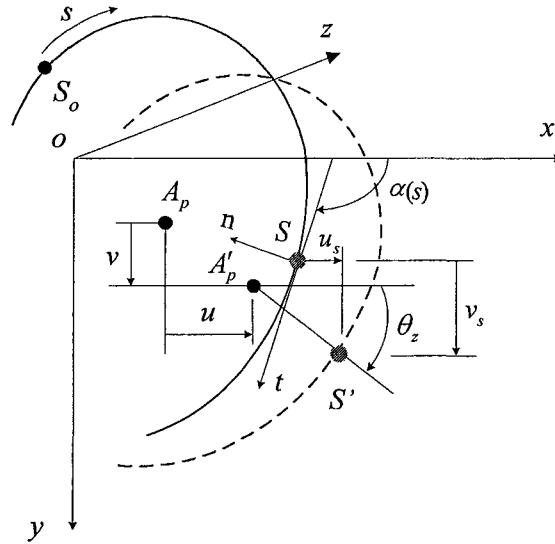


Figure 3.6 In-plane displacements of a cross-section

3.6 Displacements Due to Warping

Assume that the longitudinal displacement of the cross-section due to warping as $\omega(s)\psi(z)$, where function $\psi(z)$ is defined as the rate of change of the warping deformation along axis z of the beam and has a unit of $1/Length$ and the sectorial area $\omega(s)$ is defined (Vlasov 1961) as

$$\omega(s) = \int_{S_0}^s d\omega(s) = \int_{S_0}^s h(s) ds \quad (3.27)$$

In Eq. 3.27 point S_0 is the origin of coordinate s . At this point, S_0 is arbitrarily chosen on the mid-surface (Fig. 3.7) and $h(s)$ is the perpendicular distance from pole $A_p (x_A, y_A)$ to the tangent line to the contour at the point of interest $S(x(s), y(s))$. In Figure 3.7, $k(s)$ is defined as the distance between Pole $A_p (x_A, y_A)$ to point $S(x(s), y(s))$ along the direction of the tangent line to the contour at point S (Fig. 3.7). For Figure 3.6, one has

$$h(s) = [x(s) - x_A] \sin \alpha(s) - [y(s) - y_A(s)] \cos \alpha(s) \quad (3.28a)$$

$$k(s) = [x(s) - x_A] \cos \alpha(s) + [y(s) - y_A(s)] \sin \alpha(s) \quad (3.28b)$$

In Eq. 3.28, we recall that angle $\alpha(s)$ is measured from the positive x axis direction clockwise to the tangent line to the contour at point $S(x(s), y(s))$ looking from the negative to positive direction of the z axis. At point S on the middle surface, the rate of change of warping deformation is the local warping angle $\theta_\omega(s, z)$ i.e.,

$$\theta_\omega = \frac{d}{dz} [\omega(s)\psi(z)] = h(s)\psi(z) \quad (3.29)$$

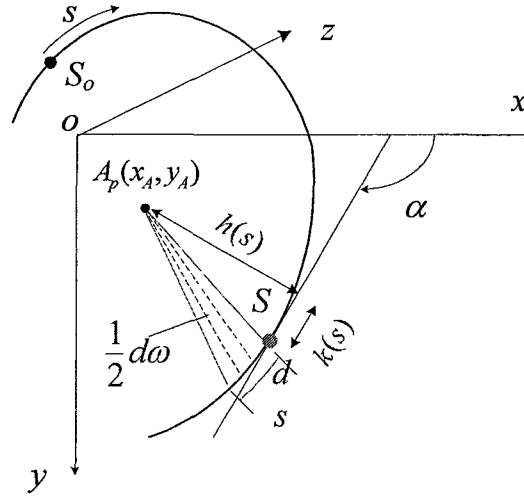


Figure 3.7 Illustration of the distances between Pole A_p and an arbitrary point S

When a point on the cross-section undergoes a longitudinal displacement $\omega(s)\psi(z)$ due to warping normal to the plane of the cross-section, it also induces the displacements in x and y directions. For an arbitrary point $S(x(s), y(s), 0)$ on a cross-section undergoing displacements u and v of pole A_p , rotations $(\theta_x(z), \theta_y(z), \theta_z(z))$ and longitudinal warping displacement $\omega_s(s)\psi(z)$, the displacements u_s and v_s of point S are

$$\begin{aligned} \begin{Bmatrix} u_s(s, z) \\ v_s(s, z) \end{Bmatrix} &= \begin{Bmatrix} u(z) \\ v(z) \end{Bmatrix} + \begin{bmatrix} \langle T_1 \rangle_{1 \times 3} \\ \langle T_2 \rangle_{1 \times 3} \end{bmatrix} \begin{Bmatrix} (x(s) - x_A) \\ (y(s) - y_A) \\ \omega_s(s)\psi(z) \end{Bmatrix} - \begin{Bmatrix} (x(s) - x_A) \\ (y(s) - y_A) \end{Bmatrix} \\ &= \begin{Bmatrix} u(z) - (y(s) - y_A)\theta_z(z) \\ v(z) + (x(s) - x_A)\theta_z(z) \end{Bmatrix} + \begin{Bmatrix} -\frac{(x(s) - x_A)}{2}(\theta_y(z)^2 + \theta_z(z)^2) + \frac{(y - y_A)}{2}(\theta_x(z)\theta_y(z)) + \omega_s(s)\psi(z)\theta_y(z) \\ \frac{(x(s) - x_A)}{2}(\theta_x(z)\theta_y(z)) - \frac{(y - y_A)}{2}(\theta_x(z)^2 + \theta_z(z)^2) - \omega_s(s)\psi(z)\theta_x(z) \end{Bmatrix} \end{aligned} \quad (3.30)$$

The longitudinal displacement $w_s(s, z)$ due to displacements $w(z)$ of origin O' , rotations $\theta_x(z), \theta_y(z), \theta_z(z)$ and the longitudinal warping displacement $\omega_s(s)\psi(z)$ is

$$\begin{aligned}
w_s(s, z) &= w(z) + \langle T_3 \rangle_{1 \times 3} \begin{Bmatrix} x(s) \\ y(s) \\ \omega_s(s)\psi(z) \end{Bmatrix} - 0 \\
&= w(z) + x(s)(-\theta_y(z) + \theta_x(z)\theta_z(z)) + y(s)(\theta_x(z) + \theta_y(z)\theta_z(z)) + \omega_s(s)\psi(z) \left[1 - \frac{1}{2}(\theta_y^2(z) + \theta_x^2(z)) \right]
\end{aligned} \tag{3.31}$$

in which sub-matrix $\langle T_3 \rangle$ is defined in Eq. 3.21. In developing Eq. 3.31, all the third order terms are ignored.

It is common to ignore the second order terms in Eq. 3.30 and third order terms in Eq. 3.31 in the analysis of thin-walled members (Pi et al. 1992, Trahair 1993 and Pi et al. 2001). Based on this simplification, the displacements of point $S [x(s), y(s), z]$ lying on the middle surface of the cross-section are determined by

$$u_s(z, s) \approx u(z) - (y(s) - y_A)\theta_z(z) \tag{3.32a}$$

$$v_s(z, s) \approx v(z) + (x(s) - x_A)\theta_z(z) \tag{3.32b}$$

$$w_s(z, s) = w(z) - x(s)\theta_y(z) + y(s)\theta_x(z) + \omega(s)\psi(z) + x(s)\theta_x(z)\theta_z(z) + y(s)\theta_y(z)\theta_z(z) \tag{3.32c}$$

In Eq. 3.32c, the second and third terms are the longitudinal displacements due to bending around the two reference axes; the fourth term results from the warping of the cross-section; the last two nonlinear terms resulted from the interactions of the rotations and are essential to consider in a buckling analysis. Exclusion of these two terms ignores the nonlinear interactions between the three rotations $(\theta_x, \theta_y, \theta_z)$ and results in an incorrect buckling formulation. This issue was recognized by Pi et al. (1992) (refer to Section 2.3.2) and investigated by Pi et al. (2001), who studied simply supported beams, cantilever beams with the tip loads, and continuous beams based on two different energy formulations (with and without the non-linear terms), and compared the predicted buckling loads with experimental results and ABAQUS shell element analyses. They found that the buckling loads predicted using the energy formulations including the non-linear terms are in good agreement with the results obtained from experimental studies and ABAQUS analyses. In contrast, the buckling loads predicted by the energy formulation excluding the non-linear terms is found significantly higher than the ones obtained from the experimental studies and ABAQUS analyses. Their investigation was

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List of Symbols

A_p :	a pole
C :	centroid of a cross-section
$h(s)$:	perpendicular distance from a pole to the tangent of a point on the mid-surface of a cross-section
$k(s)$:	distance between pole A_p and an arbitrary point on the mid-surface along tangential direction
L :	length of a member
n :	an axis along the normal direction at a point on a contour profile of a cross-section
O :	origin of Cartesian coordinates x, y, z
O' :	origin of Cartesian coordinates x', y', z'
s :	curvilinear coordinate along mid-surface
S :	an arbitrary point
S_o :	sectorial origin
SC :	shear center of a cross-section
t :	an axis along the tangential direction at a point on the mid-surface $[T]$: a rotation matrix
u, v, w :	displacements along x, y, z axes, respectively
x, y, z :	variables; or axes of Cartesian coordinates
α :	angle between the tangent to the contour and the x axis
ϕ :	an angle
$\theta_x, \theta_y, \theta_z$:	rotation angles around x, y, z axes, respectively
ω :	warping function or sectorial area of a cross-section
$\psi(z)$:	a change rate of the longitudinal displacement due to warping along the axis z and has a unit of $1/Length$

Chapter 4 Displacement Fields in Lateral-Torsional Buckling of Open Thin -Walled Members

This chapter clearly identifies the displacement fields arising in the pre-buckling stage, buckling stage, and the final state based on the kinematic developments in Chapter 3. This distinction is the key to the developments presented in Chapter 5 (Pre-buckling analysis) and Chapter 6 (Buckling analysis). Also, conventional notations are introduced in this chapter.

4.1 Statement of the Problem and Scope

A prismatic member is subjected to loads which induce combinations of axial forces, bending about the major principal axis (assumed to coincide with the x axis) of the section and the associated shear forces (Fig. 4.1). Under these loads, the member will undergo lateral-torsional buckling in which sudden twist and lateral displacement (i.e., out-of-plane displacement) take place.

The presence of additional lateral forces inducing bending about the minor axis and associated shear, torsion, or bimoments would transform the problem into a non-linear force-deformation analysis. In the buckling analysis sought in the current study, such effects do not appear and thus are excluded from the study. Figure 4.1 presents the coordinates and loads configuration for a prismatic member. The member is subjected to the reference loads consisting of

- (1) A longitudinal line load $q_z(z)$ lying within Plane 1 which is parallel to the yz -plane that includes the centroidal axis. The coordinates of the centroidal axis are $\bar{x}$ and $\bar{y}$ and are referred to an arbitrary origin O . The curve describing the line of application of line load $q_z(z)$ is described by $x = \bar{x}$ and $y_{qz}(z)$.
- (2) A transverse line load $q_y(z)$ acting in the direction of the minor principal axis of the section (along the direction of the y -axis) and lying within vertical Plane 2 which is parallel to the yz -plane that includes the shear centre axis. The coordinates of the shear center axis are x_{sc} and y_{sc} (in Fig. 4.1). The curve describing the line of

application of the line load $q_y(z)$ is described by $x = x_{sc}$ and $y = y_{qy}(z)$.

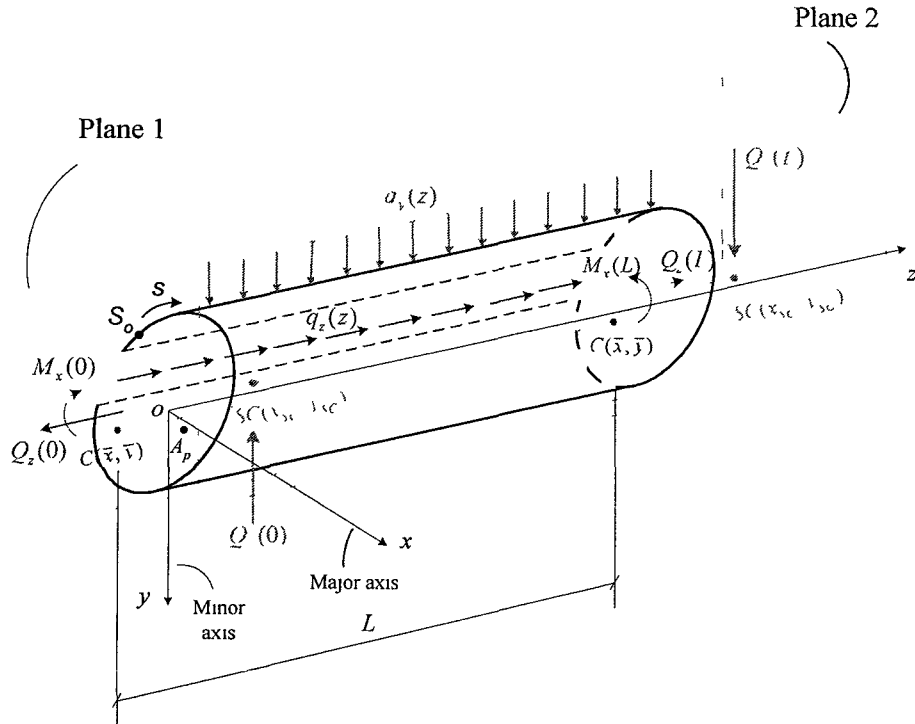


Figure 4.1 Loading Conditions in Buckling Analyses (Load $q_z(z)$ and Internal Forces $Q_z(0)$ and $Q_z(L)$, and Bending Moments $M_x(0)$ and $M_x(L)$ lie in Plane 1, while Transverse Loads $q_z(z)$, $Q_y(0)$ and $Q_y(L)$ lie in Plane 2

- (3) Two concentrated axial loads $Q_z(0)$, $Q_z(L)$ and two end moments $M_x(0)$, $M_x(L)$ are lying within Plane 1, which includes the centroidal axis, acting at the two ends ($z = 0$ and $z = L$) of the element, respectively.
- (4) Two concentrated transverse loads $Q_y(0)$ and $Q_y(L)$ are lying in Plane 2, which includes the shear center axis, and acting at the two ends ($z = 0$ and $z = L$) of the element, respectively.

The member is neither subjected to external twisting, nor bending about the minor axis. A coordinate s along the profile of the mid-surface, measured from an arbitrary sectorial origin S_o ,

identifies coordinates $x(s)$ and $y(s)$ of point S on the mid-surface of the cross section. The right hand coordinates x , y and z have an origin O . Pole A_p is a generator within the plane of the cross section. In general, origin O and pole A_p are outside of the two loading planes 1 and 2.

4.2 Hypotheses

The main assumptions made in the formulation are:

1. The cross section remains rigid in its own plane during deformation, which is in agreement with the first Vlasov assumption;
2. The generalized Timoshenko beam assumptions are adopted, i.e., the cross section remains planar, but the cross section does not remain perpendicular to the neutral axis after deformation. A similar kinematic assumption is extended for warping deformation;
3. The strains and rotations are small;
4. The material is linearly elastic and obeys Hooke's law;
5. Members are in inextensional buckling mode. This means that during buckling, the centroidal strain and the curvature in the principal yz -plane remain zero, so that the beam-column buckles under constant axial force and bending moment M_x (Trahair, 1993);
6. The formulations are restricted to prismatic open thin-walled members.

4.3 Displacement Fields and Notations

Under the reference loads described in Section 4.1, the reference axis (z axis) of the member will move from the initial configuration 1 to the equilibrium configuration 2 (Fig. 4.2). Its reference z axis undergoes displacements $v_p(z)$, $w_p(z)$ within the yz -plane. Also, a plane originally normal to the z axis will undergo a rotation $\theta_{xp}(z)$ about the x axis. As a notation convention, any variables (displacements, strains, stresses, or stress resultants) induced in the pre-buckling stage will have the subscript p .

The applied reference loads are assumed to increase. The associated pre-buckling displacements are assumed to linearly increase with the increase in the reference loads (i.e., second order effects in the pre-buckling analysis are assumed negligible). At the onset of buckling, the applied loads

are $\lambda Q_z(z), \lambda Q_y(z)$ and the corresponding displacements associated with equilibrium configuration 3 become $\lambda v_p(z), \lambda w_p(z), \lambda \theta_{x_p}(z)$ (Fig. 4.2).

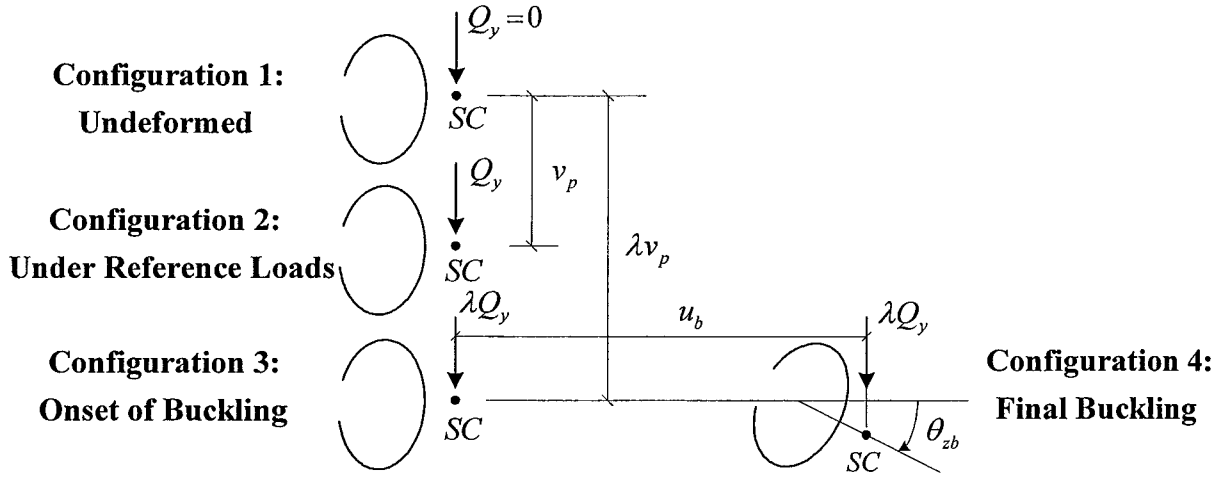


Figure 4.2 Four configurations from un-deformed member to buckling state

As the member buckles (i.e. goes from configuration 3 to configuration 4 in Fig. 4.2), the applied loads $\lambda Q_z(z), \lambda Q_y(z)$ remain unchanged, and the reference axis undergoes lateral displacement $u_b(z)$. Also, a plane originally normal to the y axis, undergoes rotations about the y and z axes by angles $\theta_{yb}(z), \theta_{zb}(z)$ and undergoes warping deformation characterized by function $\psi_b(z)$. As a convention, subscript b denotes all field variables (displacements, strains, stresses, stress resultants) during the buckling state, (i.e., in going from configuration 3 to configuration 4).

At configuration 4, the total displacement and rotation fields are obtained by summing the displacement fields at the onset of buckling to those during buckling. One obtains the following total displacements $u^*(z) = u_b(z)$, $v^*(z) = \lambda v_p(z) + v_b(z)$, $w^*(z) = \lambda w_p(z) + w_b(z)$, $\theta_x^*(z) = \lambda \theta_{x_p}(z)$, $\theta_y^*(z) = \theta_{yb}(z)$, $\theta_z^*(z) = \theta_{zb}(z)$ and $\psi^*(z) = \psi_b(z)$. As a convention, the superscript $*$ denotes all field variables (displacements, strains, stresses, stress resultants) in the final buckling state, (i.e., in going from configuration 1 to configuration 4).

Based on the kinematics developed in Chapter 3, the expressions of displacement fields in pre-buckling state and final buckling state can be determined.

4.3.1 In the Pre-Buckling State (Configuration 2)

1. Generalized displacement of cross sections $v_p(z)$, $w_p(z)$ and $\theta_{xp}(z)$.
2. Based on Eq. 3.32, the displacements of point S on the mid-surface are

$$u_{sp}(s, z) = 0 \quad (4.1a)$$

$$v_{sp}(s, z) = v_p(z) \quad (4.1b)$$

$$w_{sp}(s, z) = w_p(z) + y(s)\theta_{xp}(z) \quad (4.1c)$$

3. From Eq. 3.33a, the tangential displacement is

$$t_{sp}(z, s) = v_p(z) \frac{dy(s)}{ds} \quad (4.2)$$

4.3.2 At Onset of Buckling (Configuration 3)

At the onset of buckling, the applied loads and the displacements are at a critical state, but still in a pre-buckling equilibrium state. Thus, Eqs. 4.1 and 4.2 can be used to develop displacement expressions at the onset of buckling.

1. The displacements of the cross section where point S is located undergoes $\lambda v_p(z)$, $\lambda w_p(z)$ and $\lambda \theta_{xp}(z)$.
2. The displacements of point S on the mid-surface are

$$\lambda u_{sp}(s, z) = 0 \quad (4.3a)$$

$$\lambda v_{sp}(s, z) = \lambda v_p(z) \quad (4.3b)$$

$$\lambda w_{sp}(s, z) = \lambda w_p(z) + \lambda y(s)\theta_{xp}(z) \quad (4.3c)$$

3. The tangential displacement is

$$\lambda t_{sp}(z, s) = \lambda v_p(z) \frac{dy(s)}{ds} \quad (4.4)$$

4.3.3 In the Final Buckling Configuration (Configuration 4)

1. The displacements of the cross section due to the lateral-torsional buckling are: $u_b(z)$, $\theta_{yb}(z)$, $\theta_{zb}(z)$, and $\psi_b(z)$.
2. From Eq. 3.32, the total displacements of point S on the mid-surface are

$$u_s^*(s, z) = u_b(z) - (y(s) - y_A) \theta_{zb}(z) \quad (4.5a)$$

$$v_s^*(s, z) = \lambda v_p(z) + (x(s) - x_A) \theta_{zb}(z) \quad (4.5b)$$

$$w_s^*(s, z) = \lambda w_p(z) + y(s) \lambda \theta_{xp}(z) - x(s) \theta_{yb}(z) + \omega(s) \psi_b(z) + x(s) \lambda \theta_{xp}(z) \theta_{zb}(z) + y(s) \theta_{yb}(z) \theta_{zb}(z) \quad (4.5c)$$

3. The tangential and normal displacements along the directions of t and n (Fig. 3.6) are determined based on Eq. 3.33.

$$t_s^*(z, s) = t_{s_p}(z, s) + t_{s_b}(z, s) = \lambda v_p(z) \frac{dy(s)}{ds} + u_b(z) \frac{dx(s)}{ds} + h(s) \theta_{zb}(z) \quad (4.6a)$$

$$n_s^*(z, s) = n_{s_p}(z, s) + n_{s_b}(z, s) = \lambda v_p(z) \frac{dx(s)}{ds} - u_b(z) \frac{dy(s)}{ds} + k(s) \theta_{zb}(z) \quad (4.6b)$$

The displacement fields developed in the present chapter form an important basis for the developments presented in the following chapters.

List of Symbols

A_p :	a pole
$C(\bar{x}, \bar{y})$:	centroid of a cross-section
$h(s)$:	perpendicular distance from a pole to the tangent of a point on the mid-surface of a cross-section
$k(s)$:	distance between pole A_p and an arbitrary point on the mid-surface along tangential direction
L :	length of a member
n :	an axis along the normal direction at a point on a contour profile of a cross-section
O :	origin of Cartesian coordinates x, y, z
O' :	origin of Cartesian coordinates x', y', z'
q_y, q_z :	line loads applying on a member along y and z direction, respectively
Q_y, Q_z :	concentrated loads applying on a member along y and z direction, respectively
r_o :	radius of gyration in orthogonal coordinates
s :	curvilinear coordinate along mid-surface
S :	an arbitrary point
S_o :	sectorial origin
SC :	shear center of a cross-section
t :	an axis along the tangential direction at a point on the mid-surface
u, v, w :	displacements along x, y, z axes, respectively
x, y, z :	variables; or axes of Cartesian coordinates
α :	angle between the tangent to the contour and the x axis
ϕ :	an angle
λ :	eigenvalue
$\theta_x, \theta_y, \theta_z$:	rotation angles around x, y, z axes, respectively
ω :	warping function or sectorial area of a cross-section
$\psi(z)$:	a change rate of the longitudinal displacement due to warping along the axis z and has a unit of $1/Length$

Chapter 5 Finite Element Formulation for Shear Deformable Thin-Walled Beams

Abstract:

Starting with the principle of stationary potential energy, this paper develops the governing differential equations of equilibrium and boundary conditions for shear deformable thin-walled beams with open cross-sections. Unlike conventional solutions, the formulation is based on a non-orthogonal coordinate system, in which the selected origin is generally offset from the section centroid. The exact solution of the resulting coupled differential equations of equilibrium is developed and used to develop exact shape functions. A finite element based on the exact shape functions is then formulated. Through a series of examples, the adoption of non-orthogonal coordinates is shown to enable the seamless modelling of structural members with eccentric boundary conditions and/or stepwise cross-sectional variations.

Keywords:

Shear deformation, thin-walled beams, finite element, exact solution, coped beams, eccentric supports.

5.1 Introduction

In structural analysis, the well known Euler-Bernoulli and Vlasov beam theories (1961) are most extensively used in predicting displacements and stresses in members. These theories neglect the shear deformation effect and are suitable for members with spans that are significantly larger than those of the characteristic dimensions of the cross-sections. For short span members, shear deformation effects become significant and the adoption of shear deformable beam theories becomes imperative. Timoshenko's beam theory is an improvement over the Euler-Bernoulli theory, in that it incorporates shear deformation effects. It assumes that the cross-section remains planar during deformation, but rotates by angle different from that of the slope. Finite elements on shear deformable beams include the developments by Severn (1970), Davis et al (1972), Tessler and Dong (1981), Friedman and Kosmatka (1993), Vinayak et al (1996), Taylor et al (2003) and Kadoli et al (2008).

Shear deformable thin-walled beam finite element formulations involve the work of Tralli (1986) who developed a hybrid flexural-torsional finite element using the Hellinger-Reissner variational principle. Using a modified Hellinger-Reissner variational principle, Gendy et al. (1992) developed a finite element for thin-walled beams. Prokic (1996) developed a theory for the analysis of straight thin-walled beams with open and closed multi-cellular cross-sections. Back and Will (1998) formulated a finite element for the analysis of thin-walled beams with arbitrary cross-sections by combining Timoshenko's beam theory and Vlasov's thin-walled beam theory. Saade (2005) formulated a thin-walled beam solution based on an improved version of Prokic's theory (1996). Back and Will (2008) derived a shear-flexible thin-walled element for composite I-beams based on linear interpolation. Vo and Lee (2008) formulated the governing equations for flexural-torsional behavior of thin-walled composite box beams using a shear deformable theory. Kim and Shin (2009) formulated a solution for laminated composite thin-walled beams with mono-symmetric cross-sections. A feature common to the above formulations, is the adoption of orthogonal coordinate systems, in which the origin is selected to coincide with the section centroid. When this is the case, the orthogonality conditions $\int_A x dA = 0$, $\int_A y dA = 0$ are satisfied

(x, y being the coordinates of a general point on the cross sections and the integrals are performed over the member cross-sectional area A). While this arrangement simplifies the

resulting governing differential equations of equilibrium, it introduces difficulties in modelling members with a) eccentric boundary conditions (supports offset from the centroid), and b) members with stepwise cross-sectional variations along the span (e.g., the member with holes, flange copes, or discontinuous longitudinal stiffeners). Within this context, Erkmén and Mohareb (2006a) developed the solution for Vlasov thin-walled beam theory under non-orthogonal coordinate systems. The non-orthogonal solution developed was successfully applied to a number of practical load-deformation problems (Erkmén and Mohareb 2006b). The present development extends the developments to a shear deformable thin-walled beam theory in order to a) realize similar modelling advantages for short span beams by seamlessly modelling short beams with eccentric supports and flange copes, and b) to formulate the pre-buckling solution for such members, a necessary pre-requisite to develop a buckling solution under non-orthogonal coordinates. The chapter develops an efficient finite element based on the shape functions which exactly satisfy the homogeneous form of the governing differential equations of equilibrium.

5.2 Assumptions

The assumptions of the formulation are:

1. The generalized Timoshenko beam assumptions are adopted, i.e., the cross-section remains planar throughout deformations, but does not remain perpendicular to the member axis;
2. The strains are assumed small;
3. Material is isotropic linearly elastic and obeys Hooke's law;
4. The formulation is restricted to prismatic open thin-walled members;
5. The z axis is parallel to the longitudinal direction of the member. The y axis is oriented along one principal direction and passes through the section centroid while the x axis is taken parallel to the other principal direction and is assumed to be offset from section origin O , i.e., is non-centroidal (Fig. 5.1a). This deviation from conventional solutions will be shown to be advantageous when modelling a number of practical problems; and
6. The member is subjected to longitudinal distributed forces $q_z(z)$ acting at a height $y_{qz}(z)$ and transverse distributed forces $q_y(z)$ acting at a height $y_{qy}(z)$. These are assumed to induce only longitudinal displacements, bending about the x axis, and associated shear so that the member deforms only within the yz -plane (Fig. 5.1). For this to be realized, the resultant

longitudinal forces have to lie within the yz -plane containing the section centroidal axis. Also, the line of action of the resultant of transverse forces has to lie in the plane parallel to yz -plane containing the section shear centre axis.

5.3 Kinematics

A prismatic member of a thin-walled cross-section is illustrated in Fig. 5.1a. Only the middle surface of the thin-walled beam is depicted. Coordinate s along the profile of the mid-surface, is measured from an arbitrary origin S_o and identifies coordinates $x(s)$ and $y(s)$ of point S on the mid-surface of the cross-section. Under loading, the member is assumed to undergo displacements $v(z)$ along the y axis, $w(z)$ along the z axis and a rotation about x axis. The line tangent to the contour at point S is denoted as direction $t(s)$ and the angle between the tangent line and the x -axis is $\alpha(s)$ (Fig. 5.1b).

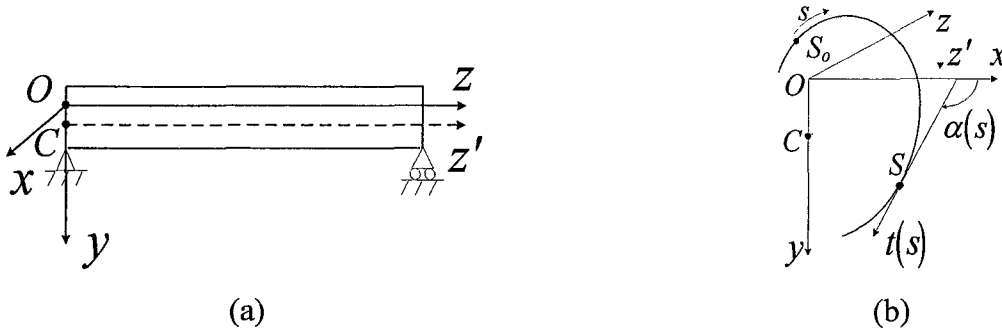


Figure 5.1 Non-orthogonal coordinates for a thin-walled member (a) Elevation, and (b) Cross-section

The displacements $v_s(s, z)$ and $w_s(s, z)$ of point S on the mid-surface are related to the displacements $v(z)$ and $w(z)$ at the origin through

$$\begin{aligned} v_s(s, z) &= v(z) \\ w_s(s, z) &= w(z) + y(s)\theta_x(z) \end{aligned} \quad (5.1)$$

The displacement of point S along the tangential direction is

$$t_s(s, z) = v_s(z) \frac{dy(s)}{ds} = v(z) \sin \alpha(s) \quad (5.2)$$

5.4 Formulation of Governing Equations

The linear components of the longitudinal and shear strains at point S on the mid-surface are

$$\begin{aligned}\varepsilon_{zz} &= \frac{\partial w_s}{\partial z} = w'(z) + y(s)\theta'_x(z) \\ \gamma_{zs} &= \frac{\partial t_s}{\partial z} + \frac{\partial w_s}{\partial s} = v'(z) + \theta_x(z)\end{aligned}\quad (5.3)$$

For an isotropic linearly elastic material, the normal and shear stresses can be related to the strains through $\sigma_{zz} = E\varepsilon_{zz}$ and $\tau_{zs} = G\gamma_{zs}$. The strain displacement relations in Eq. 5.3 at the stress strain relations are used to express the variation of the total potential energy of a prismatic thin-walled member is given by

$$\begin{aligned}\delta\pi &= 0 = \int_0^L \int_A (\sigma_{zz}\delta\varepsilon_{zz} + \tau_{zs}\delta\gamma_{zs}) dA dz - \int_0^L \left[q_y(z)\delta v_s(y_{qy}(z)) + q_z(z)\delta w_s(y_{qz}(z)) \right] dz \\ &= \int_0^L (EA w'\delta w' + EI_{xx}\theta'_x\delta\theta'_x + ES_x w'\delta\theta'_x + ES_x\theta'_x\delta w') dz \\ &\quad + \int_0^L (GD_{yy}v'\delta v' + GD_{yy}\theta_x\delta\theta_x + GD_{yy}v'\delta\theta_x + GD_{yy}\theta_x\delta v') dz \\ &\quad - \int_0^L [q_y(z)\delta v(z) + q_z(z)\delta w(z) + q_z(z)y_{qz}(z)\delta\theta_x(z)] dz\end{aligned}\quad (5.4)$$

in which L is the length of the member and the following section constants have been defined

$$A = \int_A dA, \quad I_{xx} = \int_A y^2 dA, \quad S_x = \int_A y dA, \quad D_{yy} = \int_A \left(\frac{dy}{ds} \right)^2 dA = \int_A \sin^2 \alpha dA \quad (5.5a-d)$$

After integration by parts of Eq. (5.4), one obtains the following equilibrium conditions

$$EA w'' + ES_x \theta''_x = -q_z(z) \quad (5.6a)$$

$$GD_{yy}(v'' + \theta'_x) = -q_y(z) \quad (5.6b)$$

$$EI_{xx}\theta''_x + ES_x w'' - GD_{yy}(v' + \theta_x) = -q_z(z)y_{qz}(s) \quad (5.6c)$$

and the possible boundary conditions $(EA w' + ES_x \theta'_x)\delta w = 0$, $(EI_{xx}\theta'_x + ES_x w')\delta\theta_x = 0$, and $(GD_{yy}v' + GD_{yy}\theta_x)\delta v = 0$ at $z = 0$ and $z = L$, in which either the bracketed terms provide the possible natural boundary condition, or the variation of the corresponding displacement provides the displacement boundary conditions. When the load terms are omitted from Eqs. (5.6a-c), the field equations take a form similar to that reported under Section 2.2 in Agarwal et al (2006)

after a) neglecting all time dependent terms, and b) considering a plain isotropic material in lieu of a composite orthotropic material. The difference between the two sets of field equations is while constant D_{yy} in the present formulation is given by Eq. 5.5d, that in Agarwal et al (2006) is that of the total cross-sectional area A instead. Additionally, the present treatment provides the possible boundary conditions of the problem.

5.5 Closed Form Solution

The closed form solution $\langle u(z) \rangle_{1 \times 3}^T = \langle w(z) \ v(z) \ \theta_x(z) \rangle_{1 \times 3}$ of the homogeneous part of Eqs. (5.6a-c) takes the form

$$\{u(z)\}_{3 \times 1} = [H(z)]_{3 \times 6} \{c\}_{6 \times 1}, \quad [H(z)]_{3 \times 6} = \begin{bmatrix} 3s_1 z^2 & 0 & 0 & 0 & z & 1 \\ z^3 & z^2 & z & 1 & 0 & 0 \\ -3z^2 + s_2 & -2z & -1 & 0 & 0 & 0 \end{bmatrix} \quad (5.7a-b)$$

in which $s_1 = S_x/A$, and $s_2 = 6E(S_x^2 - I_{xx}A)/GD_{yy}A$ and the vector of integration constants $\langle c \rangle^T = \langle c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6 \rangle$ is to be determined from the boundary conditions of the problem.

5.6 Finite Element Formulation

A two node finite element is developed with three degrees of freedom per node is to be developed. In order to relate the displacement fields to the nodal displacements $\langle u_n \rangle_{1 \times 6}^T =$

$\langle w_1, v_1, \theta_{x1}, w_2, v_2, \theta_{x2} \rangle^T$, shape functions which exactly satisfy the homogeneous form of the field equations are to be developed. This approach has two advantages: a) it eliminates discretization errors arising in other interpolation schemes. Thus, the resulting FEA convergences to the solution using a minimal number of elements, and b) it circumvents shear locking phenomena arising in other interpolation schemes. Towards this goal, the vector of the nodal displacements is related to the vector of integration constants $\langle c \rangle^T$ by enforcing the conditions $w(0) = w_1, v(0) = v_1, \theta_x(0) = \theta_{x1}, w(L) = w_2, v(L) = v_2, \theta_x(L) = \theta_{x2}$, and expressing the results in a matrix form, yielding

$$\{u_n\}_{6 \times 1} = [R]_{6 \times 6} \{c\}_{6 \times 1} \quad (5.8a)$$

in which

$$[R]_{6 \times 6} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ s_2 & 0 & -1 & 0 & 0 & 0 \\ 3s_1 L^2 & 0 & 0 & 0 & L & 1 \\ L^3 & L^2 & L & 1 & 0 & 0 \\ -3L^2 + s_2 & -2L & -1 & 0 & 0 & 0 \end{bmatrix} \quad (5.8b)$$

Equation (5.8a) is inverted and the resulting expression for vector $\{c\}$ is substituted into Eq. (5.7a), yielding

$$\{u(z)\}_{3 \times 1} = [H(z)]_{3 \times 6} [R]_{6 \times 6}^{-1} \{u_n\}_{6 \times 1} = [L(z)]_{3 \times 6} \{u_n\}_{6 \times 1} \quad (5.9)$$

in which matrix $[L(z)]_{3 \times 6} = [H(z)]_{3 \times 6} [R]_{6 \times 6}^{-1}$ contains the exact shape functions given by

$$[L(z)]_{3 \times 6} = \begin{bmatrix} 1 - \frac{z}{L} & \frac{6s_1(z^2 - Lz)}{L(L^2 - 2s_2)} & \frac{-3s_1 z^2 + 3s_1 Lz}{(L^2 - 2s_2)} & \frac{z}{L} & \frac{-6s_1 z^2 + 6s_1 Lz}{L(L^2 - 2s_2)} & \frac{-3s_1 z^2 + 3s_1 Lz}{(L^2 - 2s_2)} \\ 0 & \frac{6s_1(z^2 - Lz)}{L(L^2 - 2s_2)} & \frac{-Lz^3 + (2L^2 - s_2)z^2 - L(L^2 - s_2)z}{L(L^2 - 2s_2)} & 0 & \frac{-2z^3 + 3Lz^2 - 2s_2 z}{L(L^2 - 2s_2)} & \frac{-Lz^3 + (L^2 + s_2)z^2 - s_2 Lz}{L(L^2 - 2s_2)} \\ 0 & \frac{-6z^2 + 6Lz}{L(L^2 - 2s_2)} & \frac{L(3z^2 - 2s_2 + L^2) - 2(2L^2 - s_2)z}{L(L^2 - 2s_2)} & 0 & \frac{6z^2 - 6Lz}{L(L^2 - 2s_2)} & \frac{3Lz^2 - 2(L^2 + s_2)z}{L(L^2 - 2s_2)} \end{bmatrix} \quad (5.10)$$

From Eq. (5.10), by substituting into Eq. (5.4), and noting that the variation of the nodal displacements are arbitrary, one obtains $[K]_{6 \times 6} \{u_n\}_{6 \times 1} = \{F\}_{6 \times 1}$ in which the entries of the stiffness matrix $[K]_{6 \times 6}$ are given in Appendix 1, and the energy equivalent load vector $\{F\}_{6 \times 1}$ is given by

$$\{F\}_{6 \times 1} = \int_0^L [L]_{6 \times 3}^T \{Q(z)\}_{3 \times 1} dz \quad (5.11)$$

$$\text{and } \langle Q(z) \rangle_{1 \times 3}^T = \langle q_y(z) \quad q_z(z) \quad q_z(z) y_{qz}(z) \rangle.$$

5.7 Examples

In this section, the examples are investigated in order to illustrate the features of the present

formulation. In all examples, the material is assumed linear elastic with a modulus of elasticity $E=200,000 \text{ MPa}$ and shear modulus $G=77,000 \text{ MPa}$.

5.7.1 Example 1: Modelling using Non-Orthogonal Coordinates

A cantilever (Fig. 5.2a) has a span $L=4,000 \text{ mm}$ and a doubly symmetric wide-flange cross-section (Fig. 5.2b). The beam is subjected to a transverse load $Q_y=100 \text{ kN}$ acting at the cantilever tip. The centerline of the dimensions of the cross-section and the thicknesses of the web and flanges are given on Fig. 5.2b. In order to examine the validity of the formulation developed, the displacements of the cantilever due to the applied loads are analyzed under orthogonal and non-orthogonal coordinate systems. Under the orthogonal coordinates, origin O is located at the centroid of the cross-section (Fig. 5.2b), while under the non-orthogonal ones, origin O is selected to coincide with the midpoint of the top flange (Fig. 5.2c). The section properties under orthogonal coordinates, are $A = 7,389 \text{ mm}^2$, $I_{xx}=87.07 \times 10^6 \text{ mm}^4$, $S_x=0 \text{ mm}^3$ and $D_{yy}=1,908 \text{ mm}^2$, while under non-orthogonal coordinates, they are $A = 7,389 \text{ mm}^2$, $I_{xx}=192.14 \times 10^6 \text{ mm}^4$, $S_x=88.11 \times 10^4 \text{ mm}^3$, and $D_{yy}=1,908 \text{ mm}^2$.

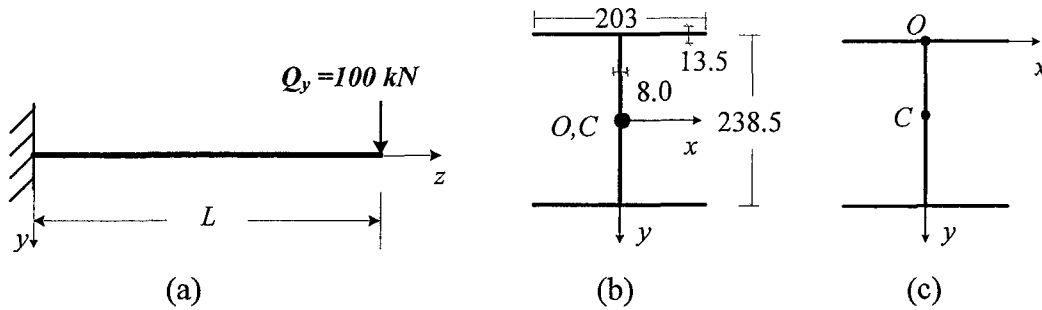


Figure 5.2 (a) Geometry and Loading, (b) Axes for orthogonal coordinates and dimensions
(c) Axes for non-orthogonal coordinates

Since the FEA formulation is based on exact shape function, the nodal degrees of freedom can be exactly predicted (within the limitations of the assumptions) using a single finite element. This has been verified by solving the problem using a single element and multiple elements and the tip deflection and rotation were found to exactly match (a similar observation was made for examples 2 and 3 below, while for example 4, three finite elements were needed to account for cross-sectional changes). The deflection, rotation angle and longitudinal

displacement curves based on the equations developed are plotted in Fig. 5.3 against coordinate z and compared with the ones based on the Euler-Bernoulli beam theory (E.B. Beam). The deflection and rotation angle curves obtained based on both solutions are found identical (Figs. 5.3a and 5.3b). The longitudinal displacement based on the orthogonal coordinates is found to vanish since the reference axis coincides with the neutral axis of the beam (Fig. 5.3c). This is schematically illustrated in Fig. 5.4a, in which point 1 in the undeformed configuration does not undergo longitudinal displacements under load Q_y . In contrast, the longitudinal displacement based on the non-orthogonal coordinates is non-zero. This is a direct result of the fact that the reference axis coincides with the top flange midpoint axis (as a result of selecting the origin to coincident with the top flange). Thus, under the chosen orthogonal coordinates, the longitudinal displacement w is that of the top flange. It can be verified that the longitudinal displacement w is the product of the angle of rotation θ_x at the cantilever tip by the web mid-height distance, i.e., $w = -\theta_x h/2$ as illustrated in Fig. 5.4b, in line with the kinematic assumption made. The agreement between the present formulation and that based on the conventional Euler-Bernoulli beam theory (E.B. Beam) on the vertical displacements is a result of the fact that the long span beam considered, shear deformation effects are negligible for the present problem.

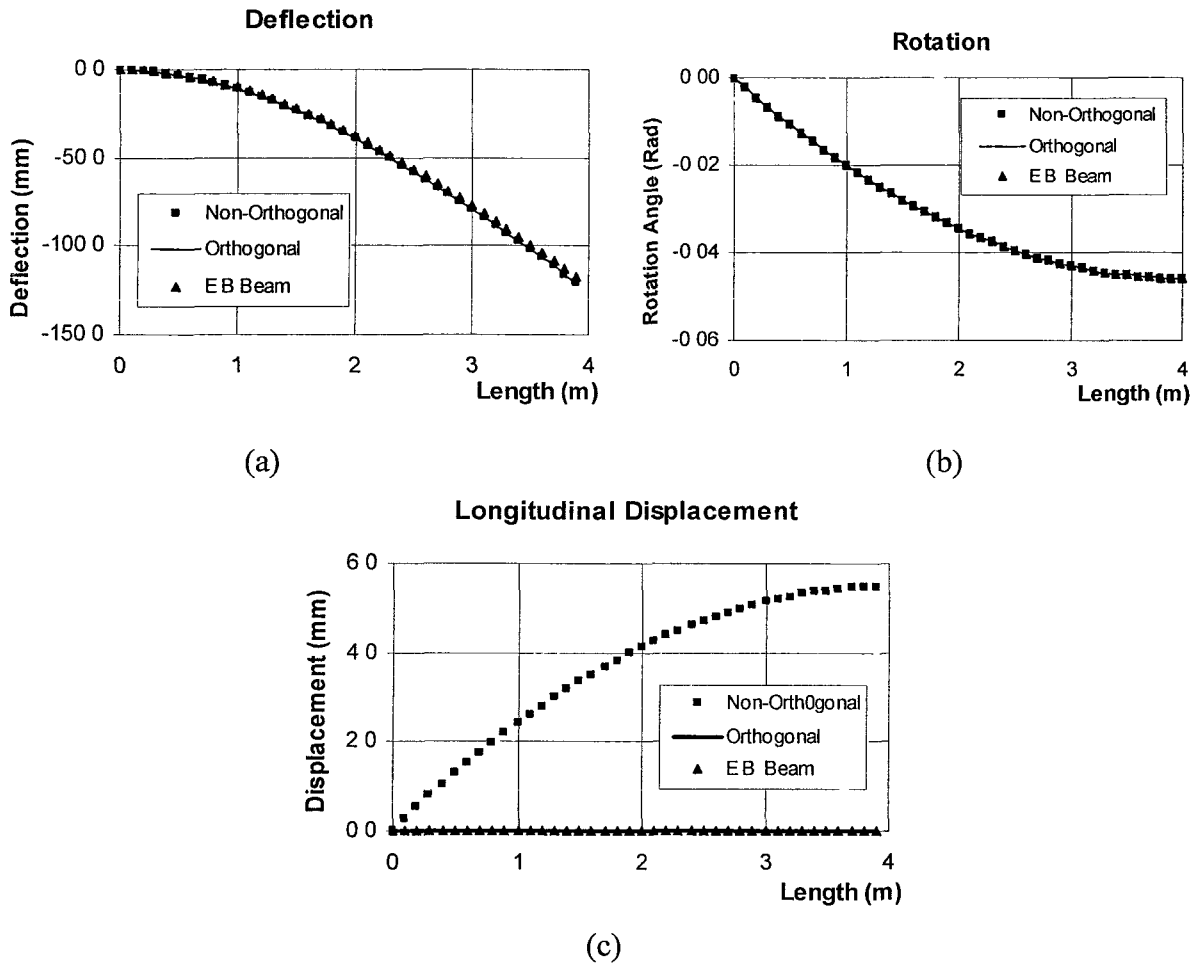


Figure 5.3 Deformation of 4,000 mm span cantilever (a) deflections, (b) rotations, and (c) longitudinal displacements

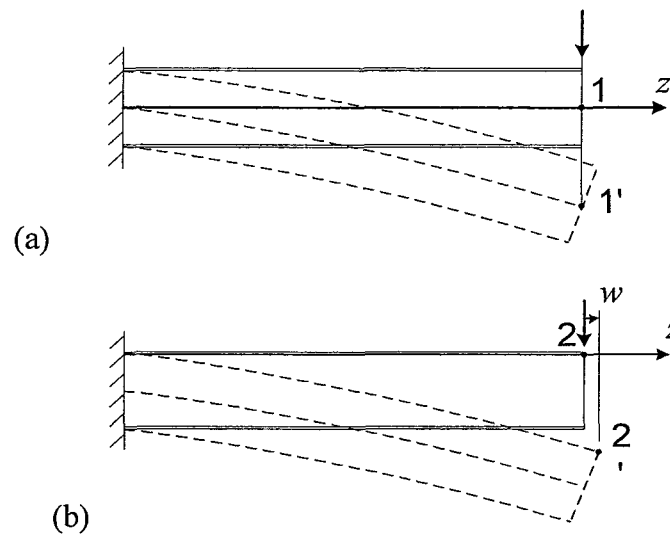


Figure 5.4 Analysis under (a) orthogonal coordinates, (b) non-orthogonal coordinates

5.7.2 Example 2: Effect of Shear Deformation

A cantilever with the same cross-section as given in Example 1 is re-analyzed for a shorter span ($L=1,000\text{mm}$). The deflection curve for the one meter cantilever is plotted in Fig. 5.5. In contrast to the long span cantilever in Example 1 (Fig. 5.3a), it is observed that the deflections obtained based on the current study are larger than those based on the Euler-Bernoulli beam theory (Fig. 5.5). This is because shear deformation plays an important role on the deflections of short members.

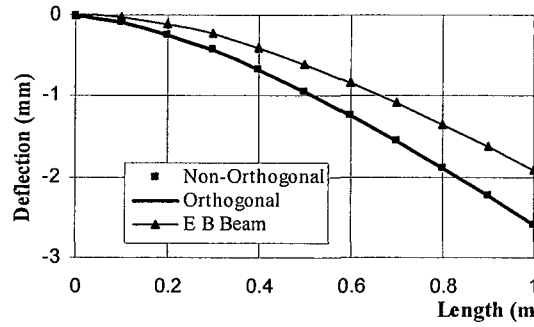


Figure 5.5 Cantilever Deflection for (a) one meter span cantilever

5.7.3 Example 3: Analysis of Beam with Eccentric Supports

A beam with a mono-symmetric cross-section (Fig. 5.6) is fixed at end ($z=0$) and pin jointed at the other end ($z=L$). Three locations for the pin support are considered: Case 1) at the centroid of the cross-section, Case 2) at the top flange midpoint, and Case 3) at the bottom flange midpoint (Fig. 5.6a). The beam cross-sectional mid-surface dimensions are given on Fig. 5.6b. The beam is subjected to a transverse load $Q_y=100\text{kN}$ acting at mid-span. Cases 2 and 3 can often arise in structural applications. However, they cannot be directly solved using conventional solutions based on orthogonal coordinates. In contrast, under the current development, they can be seamlessly solved by using the non-orthogonal coordinates illustrated in Figs. 5.6b through 5.6d for Cases 1 through 3, respectively. The sectional properties $A=8,000\text{ mm}^2$ and $D_{yy}=2,400\text{ mm}^2$, are common to all three coordinate systems. The moment inertia I_{xx} and the statical moment S_x differ between the three cases. In Case 1 $I_{xx}=57.0 \times 10^6\text{ mm}^4$, $S_x=0\text{ mm}^3$, in Case 2, $I_{xx}=96.2 \times 10^6\text{ mm}^4$, $S_x=5.6 \times 10^5\text{ mm}^3$, while in Case 3, $I_{xx}=192.2 \times 10^6\text{ mm}^4$, $S_x=10.4 \times 10^5\text{ mm}^3$.

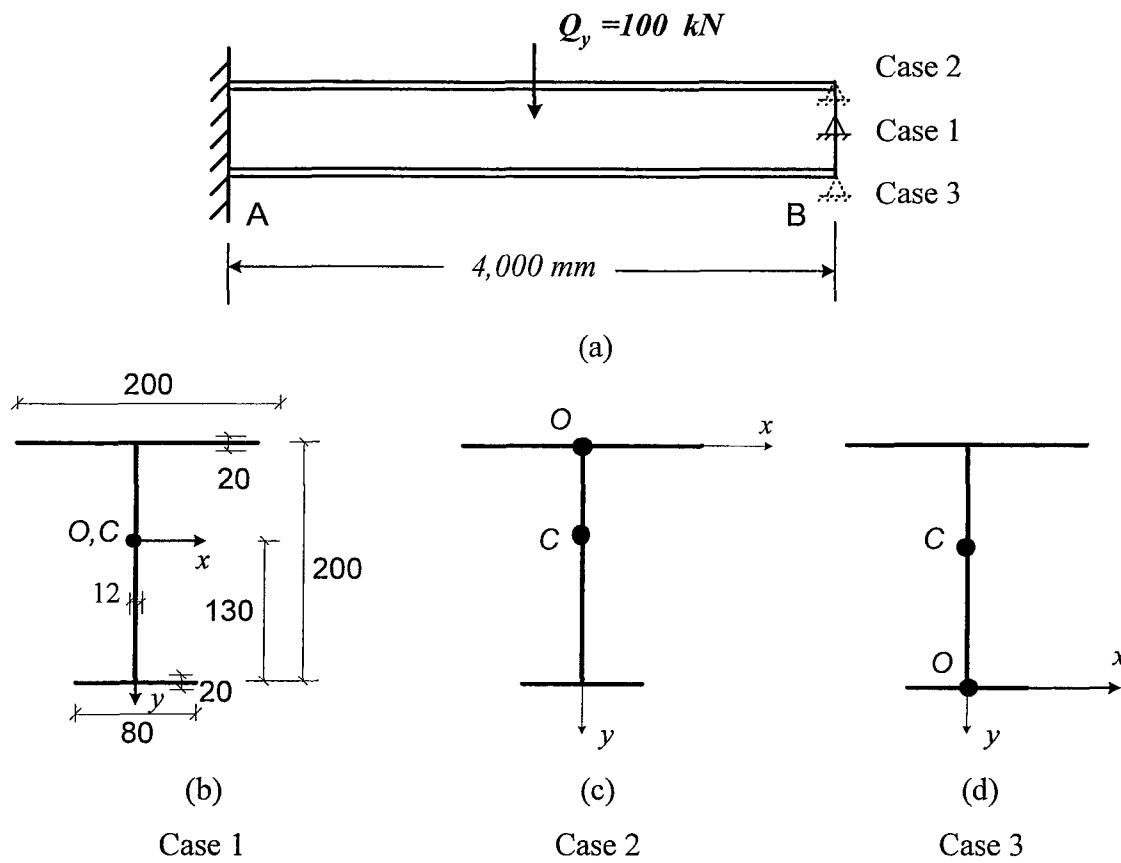


Figure 5.6 Mono-symmetric beam with support at different heights a) Elevation, b) Cross-sectional dimensions and location of axes for Case1 (Supported at centroid), c) Location of axes for Case 2 (Support at top flange), and d) Location of axes for Case 3 (Support at bottom flange)

The beam deflection curves for Cases 1 through 3 are plotted in Fig. 5.7 and compared with the results based on ABAQUS shell analysis. In the ABAQUS analysis, the member is subdivided into 200 S4R elements along the longitudinal direction, 10 elements and 4 elements along the width of top and bottom flanges, respectively, and 10 elements along the height of the web. Consequently, the model consists of 4,800 shell elements. The difference between the maximum displacement predicted by the current formulation and that based on ABAQUS are 1.5%, 2.3% and 3.7% for Case 1, 2 and 3, respectively. From Fig. 5.7, it is observed that the beam with the lower support (Case 3) has the smallest deflections compared to the other two cases. It is noticed that the deflections obtained from the present study are always smaller than the corresponding ones based on ABAQUS shell element analysis. This is because the current developed finite element model is based on the approximate Timoshenko assumption while that in ABAQUS

shell analysis such as constraint is not made. The larger number of nodes and degrees-of-freedom involved imply a more flexible representation of the problem.

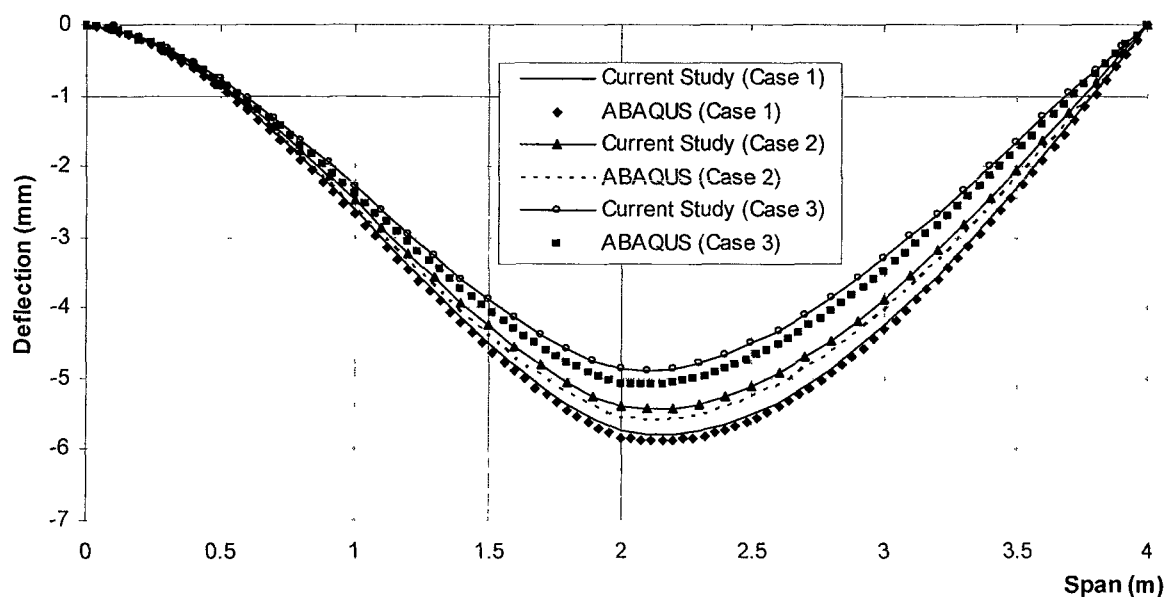


Figure 5.7 Deflection of beam with eccentric supports

The support reactions obtained from the current study are observed to be in close agreement with those based on ABAQUS analysis. A comparison of reactions for Case 1 based on the present analysis and based on the ABAQUS model is presented in Table 1. Both sets of results agree within a 0.5% difference.

Table 5.1 Comparison of reactions for Case 1

Reactions	ABAQUS	Current study	Difference
V_{yA} (kN)	68.45	68.54	0.13%
V_{yB} (kN)	31.55	31.46	0.28%
M_{xA} (kNm)	73.80	74.15	0.47%

The reactions at both supports based on the present analysis for all three cases are presented in Table 5.2. It is observed that the eccentric supports (Cases 2 and 3) induce an axial force in the member. This effect is easily captured using the model developed in the present study. Support eccentricity is thus an important factor in the analysis of structural members. Due to presence of the axial forces at the supports, the shear force and bending moment reactions depend on the

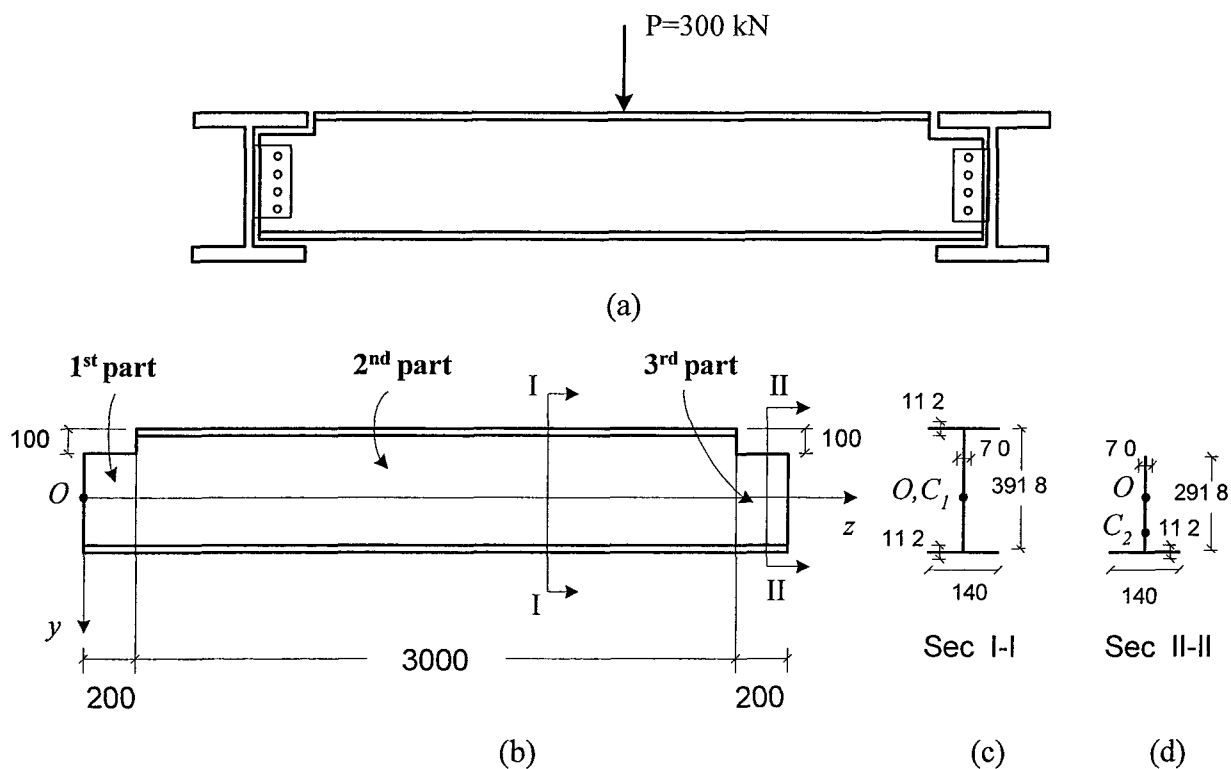
eccentricity of the support at end B.

Table 5.2 Reactions based on current study

Reactions	Case 1	Case 2	Case 3
V_{yA} (kN)	68.5	65.7	61.5
V_{yB} (kN)	31.5	34.3	38.5
M_{xA} (kNm)	74.2	62.0	46.0
N_z (kN)	0	107.9	146.2

5.7.4 Example 4: Analysis of Beam with Coped Ends

Beams with coped ends (Fig. 5.8a) are commonly used in steel structures. Conventional beam solutions based on orthogonal coordinates have difficulties in accurately modeling beams with end copes. Thus, such beams are commonly treated as prismatic un-coped beams. The present example is aimed at quantifying the effect of such a modelling approximation.



The beam is subdivided into three parts as shown in Fig. 5.8b. The middle part of the beam has cross-section type I (Fig. 5.8c), for which origin O of the coordinates coincides with the centroid C_1 of the cross-section, while the coped parts have cross-section type II (Fig. 5.8d), where the location of the origin is offset from the centroid C_2 . The present formulation allows the modelling of cross-section type I using orthogonal coordinates and cross-section Type II using non-orthogonal coordinates.

The deflections along the beam obtained from current study are plotted in Fig. 5.9 and compared against the results based on a) the Euler-Bernoulli theory and b) the ABAQUS shell element (S4R) analysis. In the ABAQUS modeling, the model is subdivided into 68 elements in the longitudinal direction, eight elements along the height of the web, and four elements along the width of each flange. The beam thus is modeled using 1,088 shell elements in total. The deflection curve based on the Euler-Bernoulli theory shows a significantly stiffer response than that obtained from ABAQUS analysis (the difference in maximum deflection values is 19.6%). In contrast, the deflection curve based on the current study is observed to be closer to the one from ABAQUS (the difference between the maximum deflections being only 4.3%).

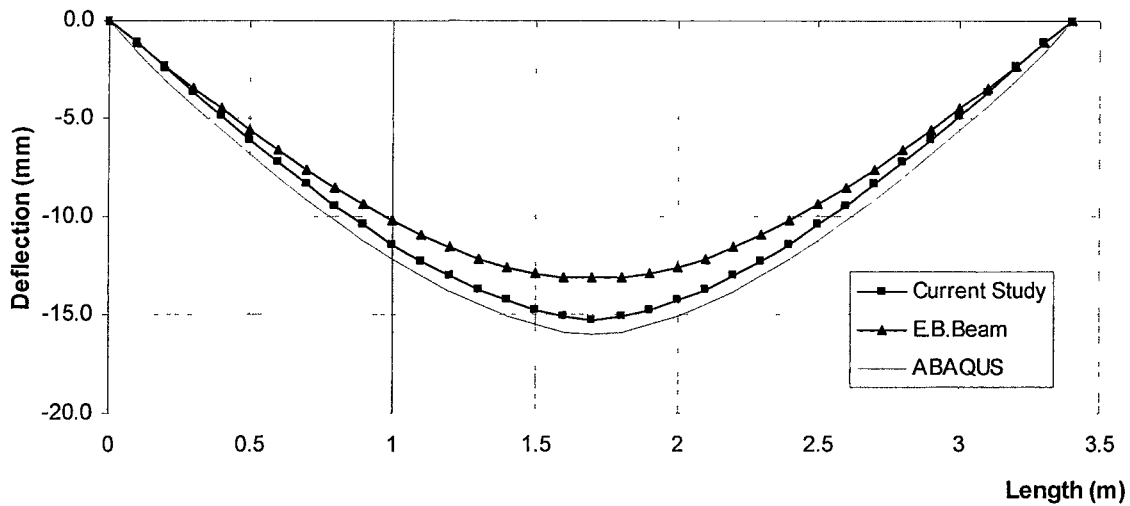


Figure 5.9 Deflection for coped beam

5.8 Summary and Conclusions

1. The governing differential equations and boundary conditions for thin-walled beams with open cross-sections including shear deformation effects are developed under non-orthogonal

coordinates.

2. The exact solution of the field equation was obtained, and used to develop exact shape functions. A finite element based on the exact shape function was formulated. The element is free from shear locking and converges to the solution with a minimal number of degrees of freedom
3. The numerical solutions of various practical examples illustrate that shear deformation effects on the deflections for relatively short beam segment are significant.
4. The developed theory is able to seamlessly model beams with eccentric boundary conditions and stepwise variations in cross-sections.
5. Results based on eccentric supports are shown to differ from those based on concentric supports. The formulation was successfully used to model coped beams. The response of a typical coped beam is shown to be significantly softer than a similar un-coped beam.
6. The present formulation provides the internal forces in a nonorthogonal coordinate system, which are pre-requisites prior conducting the buckling analysis presented in Chapters 7 and 8.

Appendix 5.1 Entries of Stiffness Matrix

The entries of the stiffness matrix are given by

$$[K]_{6 \times 6} = \begin{bmatrix} \frac{EA}{L} & 0 & \frac{ES_x}{L} & -\frac{EA}{L} & 0 & -\frac{ES_x}{L} \\ 0 & \frac{12E}{L^3(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & \frac{-6E}{L^2(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & 0 & \frac{-12E}{L^3(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & \frac{-6E}{L^2(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) \\ \frac{ES_x}{L} & \frac{-6E}{L^2(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & \frac{(4+\alpha)EI_{xx} - 3E\frac{S_x^2}{A}}{L(1+\alpha)} & -\frac{ES_x}{L} & \frac{6E}{L^2(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & \frac{(2-\alpha)EI_{xx} - 3E\frac{S_x^2}{A}}{L(1+\alpha)} \\ -\frac{EA}{L} & 0 & \frac{ES_x}{L} & \frac{EA}{L} & 0 & \frac{ES_x}{L} \\ 0 & \frac{-12E}{L^3(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & \frac{6E}{L^2(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & 0 & \frac{12E}{L^3(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & \frac{6E}{L^2(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) \\ -\frac{ES_x}{L} & \frac{-6E}{L^2(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & \frac{(2-\alpha)EI_{xx} - 3E\frac{S_x^2}{A}}{L(1+\alpha)} & \frac{ES_x}{L} & \frac{6E}{L^2(1+\alpha)} \left(I_{xx} - \frac{S_x^2}{A} \right) & \frac{(4+\alpha)EI_{xx} - 3E\frac{S_x^2}{A}}{L(1+\alpha)} \end{bmatrix}$$

A5.1-1

where constant $\alpha = 12E \left(I_{xx} - \frac{S_x^2}{A} \right) / GD_{yy} L^2$ has been defined.

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5-22

List of Symbols

A :	cross-sectional area,
C :	centroid of the cross-section,
D_{yy} :	cross-sectional property of a cross-section,
E :	Young's modulus,
$\{F\}$:	energy equivalent load vector,
G :	shear modulus,
I_{xx} :	moment of inertia of cross-section about the x -axis,
$[K]$:	stiffness matrix of an element,
L :	length of a member,
$[L(z)]$:	matrix of interpolation functions,
$M_x(z)$:	moment resultant of normal stresses,
$N(z)$:	axial force resultant of normal stresses,
O :	origin of Cartesian coordinates x, y, z ,
q_y, q_z :	line loads applied to a member along y and z direction, respectively,
s :	curvilinear coordinate along the cross-section mid-surface,
S :	an arbitrary point on the section mid-surface,
S_x :	first moments of area of y coordinate,
SC :	shear center of a cross-section,
t :	an axis along the tangential direction at a point located on the mid-surface,
$\{u\}$:	displacement vector,
$\{u_N\}$:	nodal displacement vector,
v, w :	displacements along y, z axes, respectively,
$V_y(z)$:	shear force resultant along the y -direction,
α :	angle between the tangent to the contour and the x axis,
$\delta\pi$:	variation of total potential energy,

$\varepsilon_{zz} :$	longitudinal strain,
$\gamma_{zs} :$	shear strain along the mid-surface of a cross-section,
$\sigma_{zz} :$	normal stress,
$\tau_{zs} :$	shear stress at the mid-surface of a cross-section, and
$\theta_x :$	rotation angle around x axis.

Chapter 6 Buckling of Shear Deformable Thin-Walled Members - I. Variational Principle and Analytical Solutions

Abstract:

A variational formulation for the buckling analysis of thin-walled members is developed based on the principle of stationary potential energy. The formulation is based on non-orthogonal coordinates and captures shear deformation effects due to bending and warping. It is applicable to members of doubly symmetric cross-sections subject to general axial and transverse forces and naturally incorporates the effect of load position relative to the shear centre. By applying the conditions of neutral stability to the variational expression, the governing differential equations of neutral stability and associated boundary conditions are formulated. The resulting field equations are exactly solved for benchmark cases involving column flexural buckling, column torsional buckling, lateral torsional buckling for beams, and the results are compared to closed form solutions based on classical and other modern theories.

Keywords:

Thin-walled members; Non-orthogonal coordinates; Shear deformation effect; Lateral-torsional buckling; Eccentric supports; Closed form solutions

6.1 Literature Review

The well known Vlasov theory [6-1] is extensively used in analyzing flexural, torsional, and lateral-torsional buckling of thin-walled members with open cross-sections. The theory is based on two kinematic assumptions: 1) the configuration of the cross-section in its own plane remains unchanged during deformation; 2) the shear strain vanishes at the middle surface of the cross-section. Many researchers (e.g., Barsoum and Ghallagher [6-2], Yang and McGuire [6-3], Trahair [6-4]) formulated buckling solutions based on the Vlasov kinematics. However, the second assumption which neglects the shear strain within the middle surface can over-estimate the buckling resistance for members with a relatively short span (Mohareb and Dabbas [6-5], Erkmén and Mohareb [6-6, 6-7]). Improved buckling theories for thin-walled members which account for shear deformation effects include the work of Reissner [6-8] who formulated the buckling solution for deep cantilever beams with no warping rigidity and Wekezer [6-9] who investigated thin-walled members of variable open cross-sections based on the theory of membrane shells and finite element method. Sherbourne and Kabir [6-10] investigated the shear deformation effects on the lateral stability of thin-walled fibrous composite beams and reported considerably lower buckling loads in short-span composite beams. Wang [6-11] developed a simple and approximate method to predict the effect of shear strains on the lateral buckling resistance of thin-walled members. Kollar [6-12] formulated a closed form solution based on the shear deformable theory for the buckling analysis of composite columns of open cross-sections. Machado and Cortinez [6-13] formulated a lateral buckling solution for members with doubly symmetric cross-sections. Their formulation included the effects of pre-buckling, shear deformation and load height relative to the shear centre. Machado extended his work to investigate free vibrations [6-14] and the post-buckling behaviour of composite beams [6-15] undergoing large displacements and rotations. Using infinite power series, Kim [6-16] developed a formulation for the lateral buckling of thin-walled composite beams. Sapountzakis and Dourakopoulos [6-17] derived a boundary element formulation for the flexural-torsional buckling of composite beams. Vo and Lee [6-18] developed a general analytical model based on shear-deformable beams to study the flexural-torsional buckling of thin-walled composite columns under axial load.

Apart from the work of Erkmén and Mohareb [6-6, 6-7], a common feature in the above formulations is the use of orthogonal coordinate systems, in which the section pole is chosen to coincide with the shear centre of the section, the origin is chosen to coincide with the section

centroid, and the sectorial origin is chosen to coincide with the principal sectorial origin. These selections simplify the resulting field equations, but lead to limitations in modelling members with eccentric boundary conditions and those with discontinuous cross-sections (e.g., members with holes, flange copes, or discontinuous longitudinal stiffeners). In order to preserve the versatility of the formulation, the present developments adopt a non-orthogonal coordinate system.

The treatment provided in the present paper differs from that in Erkmén and Mohareb [6-6, 6-7] in several respects: 1) The present formulation is based on the principle of stationary potential energy and adopts kinematically admissible strains as starting point in the formulation. This contrasts with that in Erkmén and Mohareb [6-6, 6-7], which is based on the principle of stationary complementary energy and starts with statistically admissible stress fields; 2) The resulting conditions of neutral stability and boundary conditions based on both theories are different, although it will be shown that both theories predict close buckling loads for benchmark problems; 3) A finite element formulation based on the present variational principle in the paper [6-19] is found to lead to a simpler and more systematic stiffness matrix assembly than that presented in [6-6, 6-7]; and 4) The present formulation provides more flexibility when modelling members with eccentric supports subject to transverse loads that are offset from the height of the supports (such as Example 6 in Chapter 7).

6.2 Description of the Problem

1. The formulation is restricted to prismatic thin-walled members with doubly symmetric cross-sections (Fig. 6.1). As a result, the shear centre SC coincides with the centroid C ,
2. A right-handed coordinate system x, y and z with origin O is chosen to lie on the axis of symmetry $x = 0$ of the cross section but does not necessarily coincide with the cross-section centroid $C(x_c = 0, y_c)$. This departure from conventional treatments will be shown in the paper [6-19] to facilitate the modelling of eccentric supports,
3. A sectorial origin S_o , identifies the origin of the circumferential coordinate s on the mid-surface of the cross section,
4. Pole A_p and sectorial origin S_o are assumed to coincide at point $A_p(x_A = 0, y_A)$, and

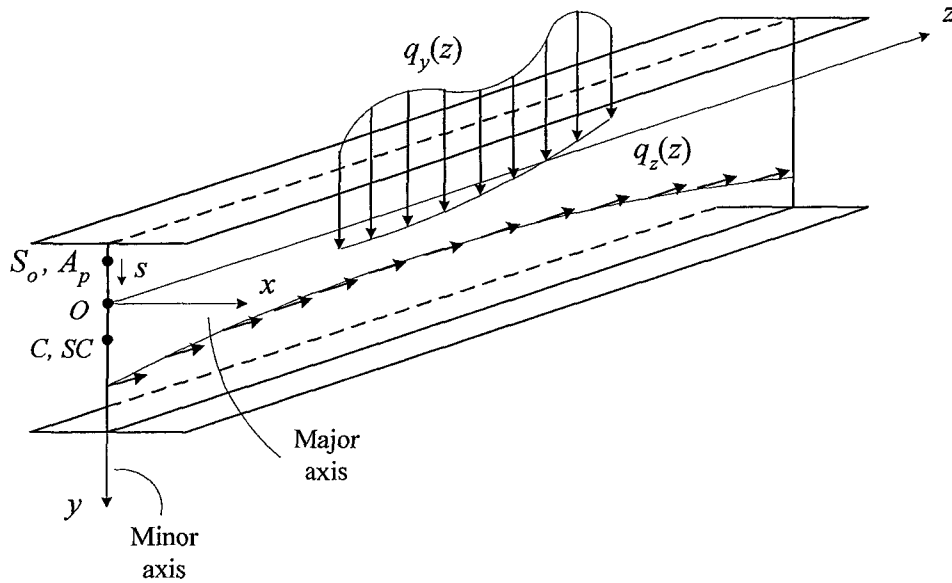


Figure 6.1 Coordinates and loading

5. The member is subjected to the following reference forces:
 - a. Distributed longitudinal force $q_z(z)$ acting along the z direction at distances $y_{qz}(z)$ from the origin, thus inducing axial force. When the line of action of the force does not coincide with the centroidal axis, $q_z(z)$ induces additional bending moments;
 - b. Distributed transverse force $q_y(z)$ located in the yz plane and passing through the shear centre at a distance $y_{qy}(z)$ from the origin, thus inducing strong axis bending about the x direction.

The above reference loads induce combinations of axial forces, bending about the major principal axis of the section, and associated shear forces. Under these loads, the member undergoes displacements within the yz plane in the pre-buckling state. The reference loads are assumed to gradually increase until the member attains a critical state, at which lateral-torsional buckling characterized by sudden twist and lateral displacements is assumed to occur.

Under the reference loads, the reference z axis of the member will move from initial configuration 1 to equilibrium configuration 2 (Fig. 6.2). Its reference z axis undergoes

displacements $v_p(z), w_p(z)$ along y and z directions, respectively. Also, a plane originally normal to the z axis undergoes a rotation $\theta_{xp}(z)$ about the x -axis. As a notation convention, all variables (displacements, strains, stresses, or stress resultants) induced in the pre-buckling stage will have the subscript p . The associated pre-buckling displacements are assumed to linearly increase with the increase in the reference loads (i.e., second order effects throughout pre-buckling deformation are assumed negligible). At the onset of buckling, the applied loads are $\lambda q_z(z), \lambda q_y(z)$ and the corresponding displacements associated with equilibrium configuration 3 become $\lambda v_p(z), \lambda w_p(z), \lambda \theta_{xp}(z)$ in which λ is the eigen value sought.

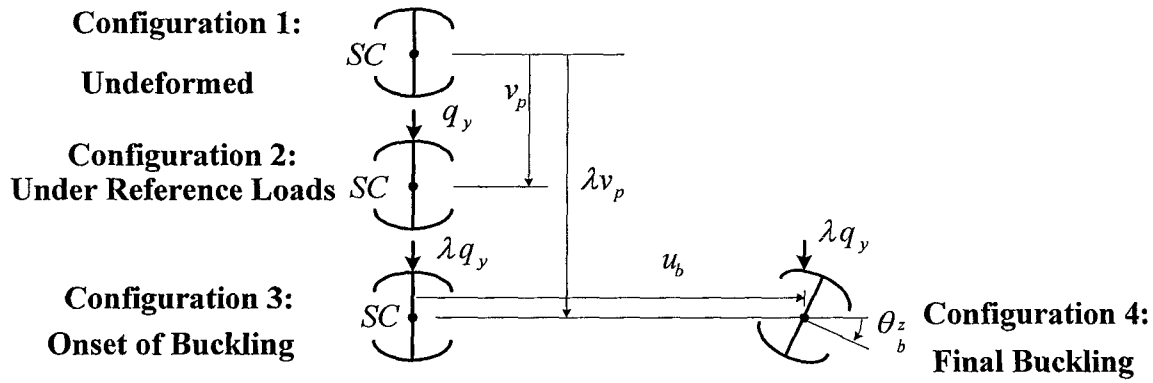


Figure 6.2 Stages of Deformation

As the member buckles (i.e. goes from configuration 3 to configuration 4 in Fig. 6.2), the applied loads $\lambda q_z(z)$ and $\lambda q_y(z)$ remain unchanged, and the reference axis undergoes lateral displacement $u_b(z)$ along the x direction. Also, a plane originally normal to the z axis, undergoes rotation $\theta_{yb}(z)$ about the y axis, rotation $\theta_{zb}(z)$ about the z axes, and warping deformation characterized by function $\psi_b(z)$. As a convention, subscript b denotes all field variables (displacements, strains, stresses, stress resultants) during the buckling state, (i.e., in going from Configuration 3 to Configuration 4).

At configuration 4, the total displacement and rotation fields are obtained by summing the displacement fields at the onset of buckling to those during buckling, yielding $u^*(z) = u_b(z)$, $v^*(z) = \lambda v_p(z) + v_b(z)$, $w^*(z) = \lambda w_p(z) + w_b(z)$, $\theta_x^*(z) = \lambda \theta_{xp}(z)$,

$\theta_y^*(z) = \theta_{yb}(z)$, $\theta_z^*(z) = \theta_{zb}(z)$, and $\psi^*(z) = \psi_b(z)$. As a convention, the superscript * denotes all field variables (displacements, strains, stresses, stress resultants) in the final buckling state (i.e., in going from configuration 1 to configuration 4).

The presence of other effects such as lateral forces inducing bending about the minor axis and associated shear, torsion, or bi-moments would transform the problem into a non-linear force-deformation analysis. Since the present formulation seeks a buckling analysis, such effects are discarded.

6.3 Assumptions

The main assumptions of the formulation are:

1. The cross-section remains rigid during deformation, in agreement with the first Vlasov assumption,
2. The generalized Timoshenko beam assumption is adopted, i.e., the cross section remains planar, but the cross section does not remain perpendicular to the neutral axis after deformation. A similar kinematic assumption is extended for warping deformation;
3. The strains are small and rotations are moderate,
4. The material is assumed linearly elastic and obeys Hooke's law,
5. The member buckles in an inextensional mode. This means that during buckling, the centroidal strain and the curvature in the principal yz -plane remain unchanged, so that the member buckles under constant axial force and bending moment. A similar assumption was adopted in Trahair [6-4],
6. Pre-buckling deformation effects are neglected [6-20], and
7. The cross-section consists of straight segments parallel to the x and y axes

6.4 Displacement Fields

Based on the above assumptions, and starting with the Euler-Rodriguez equations for moderate rotations, the total displacements (i.e., those associated with configuration 4), of an arbitrary point $S(x(s), y(s), z)$ on the mid-surface can be shown in Chapter 3 to take the form

$$u_s^*(s, z) = u_b(z) - (y(s) - y_A)\theta_{zb}(z) + \left[-\frac{1}{2}x(s)(\theta_{yb}^2(z) + \theta_{zb}^2(z)) + \frac{1}{2}(y(s) - y_A)\theta_{xp}(z)\theta_{yb}(z) + \omega(s)\psi_b(z)\theta_{yb}(z) \right] \quad (6.1)$$

$$v_s^*(s, z) = \lambda v_p(z) + x(s)\theta_{zb}(z) + \left[\frac{1}{2}x(s)(\theta_{xp}(z)\theta_{yb}(z)) - \frac{1}{2}(y(s) - y_A)(\theta_{xp}^2(z) + \theta_{zb}^2(z)) - \omega(s)\psi_b(z)\theta_{xp}(z) \right] \quad (6.2)$$

$$w_s^*(s, z) = \lambda w_p(z) + y(s)\lambda\theta_{xp}(z) - x(s)\theta_{yb}(z) + \omega(s)\psi_b(z) + x(s)\lambda\theta_{xp}(z)\theta_{zb}(z) + (y(s) - y_C)\theta_{yb}(z)\theta_{zb}(z) + \left[-\frac{1}{2}\omega(s)\psi_b(z)(\theta_{yb}^2(z) + \theta_{xp}^2(z)) \right] \quad (6.3)$$

in which coordinate s denotes a generic point S on the mid-surface (Fig. 6.3) and

$\omega(s) = \int_{s_0}^s h(s)ds$ is the sectorial coordinate as defined in Eq. 3.27. In the analysis of thin-walled

members (Trahair [6-4], Pi et al. [6-21], Pi et. al. [6-22], etc.), it is common to ignore the terms in the square brackets in Eqs. (6.1-6.3). In the present treatment, these terms are thus omitted when developing the internal strain energy terms. When expressing the load potential term, the inclusion of the square bracketed terms provides a natural way to incorporate the load position effect and thus these terms were retained in the formulation. The first four terms in Eq. (6.3) are identical to those used in the shear deformable finite element formulation for thin-walled beams in Back [6-23]. However, the present theory differs from Back's theory in that it incorporates the additional nonlinear terms $x(s)\lambda\theta_{xp}(z)\theta_{zb}(z) + (y(s) - y_C)\theta_{yb}(z)\theta_{zb}(z)$.

Tangential coordinate t is defined along the tangent to the section profile (Fig. 6.3) and is taken positive in the direction of increasing s and has a unit vector $(dx/ds, dy/ds)$. Normal coordinate n is taken positive when acting along the direction determined by the condition that the coordinates of z , t and n form a right-hand coordinate system. Its unit direction vector is $(-dy/ds, dx/ds)$. Displacements $u_s^*(s, z)$, $v_s^*(s, z)$ are resolved into tangential $t_s^*(z, s)$ and normal $n_s^*(z, s)$ displacement components given by

$$t_s^*(z, s) = u_b(z)\frac{dx(s)}{ds} + \lambda v_p(z)\frac{dy(s)}{ds} + h(s)\theta_{zb}(z) \quad (6.4)$$

$$n_s^*(z, s) = \lambda v_p(z)\frac{dx(s)}{ds} - u_b(z)\frac{dy(s)}{ds} + k(s)\theta_{zb}(z) \quad (6.5)$$

where

$$\begin{aligned} h(s) &= x(s) \sin \alpha(s) - [y(s) - y_A] \cos \alpha(s) \\ k(s) &= x(s) \cos \alpha(s) + [y(s) - y_A] \sin \alpha(s) \end{aligned} \quad (6.6)$$

and $\alpha(s)$ is the angle between the tangent to the profile at point S and the x axis (Fig 6.3).

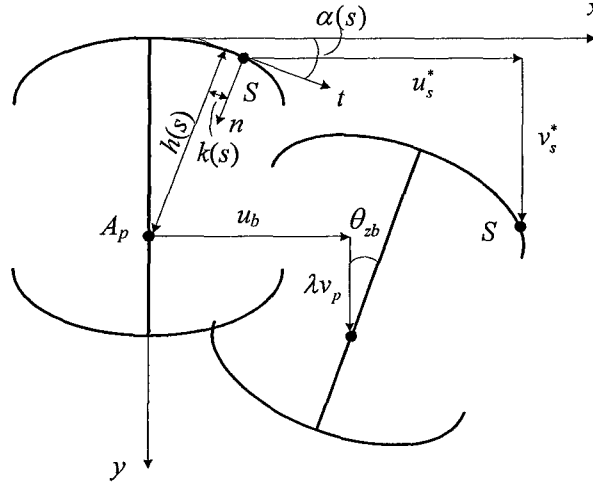


Figure 6.3 Displacements and tangential and normal directions at a generic point S

6.5 Pre-Buckling Analysis

Prior to conducting a buckling analysis, the configuration of the structure under the reference loads (Configuration 2 in Fig. 6.2) needs to be determined from equilibrium considerations (i.e., through a pre-buckling analysis). Under the conventional approach, orthogonal coordinates are adopted and the corresponding pre-buckling analysis can be conducted based on the well known Timoshenko beam formulation [6-24]. Under a non-orthogonal coordinate system, such a pre-buckling finite element formulation is not known and needs to be developed from first principles.

6.5.1 Equilibrium Conditions

The normal and shear stresses are related to the longitudinal and shear strains through the generalized Hooke's law and then the strains are expressed in terms of the displacement fields yielding

$$\sigma_{zxp} = E \varepsilon_{zxp} = E \begin{pmatrix} 1 & y \end{pmatrix} \begin{Bmatrix} w'_p \\ \theta'_{xp} \end{Bmatrix} \quad (6.7)$$

$$\tau_{zsp} = G\gamma_{zsp} = G\left[v'_p(z) + \theta_{xp}(z)\right]\frac{dy(s)}{ds} \quad (6.8)$$

in which E is Young's modulus and G is the shear modulus. The principle of virtual work $\delta\pi = 0$ is then invoked to formulate the equilibrium conditions (Chapter 5) yielding

$$\begin{aligned} \delta\pi = & \int_0^L \left(EA w'_p \delta w'_p + EI_{xx} \theta'_{xp} \delta \theta'_{xp} + ES_x w'_p \delta \theta'_{xp} + ES_x \theta'_{xp} \delta w'_p \right) dz \\ & + \int_0^L \left(GD_{yy} v'_p \delta v'_p + GD_{yy} \theta_{xp} \delta \theta_{xp} + GD_{yy} v'_p \delta \theta_{xp} + GD_{yy} \theta_{xp} \delta v'_p \right) dz \\ & - \int_0^L q_y y_{qz}(z) \delta v_{qp} + q_z y_{qz}(z) \delta w_{qp} dz = 0 \end{aligned} \quad (6.9)$$

in which the first integral arises from the normal stresses undergoing virtual normal strains and the second integral arises from the shear stresses undergoing virtual shear strains and the last integral arises from the external loads undergoing virtual displacements. The resulting field equations, boundary conditions, solution, formulation of shape functions, and finite element formulation are given in Chapter 5.

6.5.2 Relating Stress Resultants to Displacement Fields under Non-Orthogonal Coordinates

The condition of neutral stability (to be formulated under Section 6), will be shown to depend on the pre-buckling stresses. It is customary in lateral torsional buckling formulations to express the neutral stability condition in term of pre-buckling stress resultants. Therefore, the following subsection aims at relating the pre-buckling stresses to the pre-buckling stress resultants.

6.5.2.1 Stress Resultants Induced by Normal Stresses

The stress resultants induced by normal stresses at a given cross-section are obtained by integrating the stresses and their moments on the cross-section yielding

$$\begin{Bmatrix} N_p(z) \\ M_{xp}(z) \end{Bmatrix} = \int_A \begin{Bmatrix} 1 \\ y \end{Bmatrix} \sigma_{zsp} dA = E \int_A \begin{bmatrix} 1 & y \\ y & y^2 \end{bmatrix} \begin{Bmatrix} w'_p \\ \theta'_{xp} \end{Bmatrix} dA = E \begin{bmatrix} A & S_x \\ S_x & I_{xx} \end{bmatrix} \begin{Bmatrix} w'_p \\ \theta'_{xp} \end{Bmatrix} \quad (6.10)$$

in which the section properties $A = \int_A dA$, $I_{xx} = \int_A y^2 dA$, and $S_x = \int_A y dA$. The matrix relations in Eq. (6.10) is then inverted to relate the derivatives of the displacement fields w'_p, θ'_{xp} can be expressed in terms of stress resultants $N_p(z), M_{xp}(z)$ yielding

$$\begin{Bmatrix} w'_p \\ \theta'_{xp} \end{Bmatrix} = \frac{1}{E(AI_{xx} - S_x^2)} \begin{bmatrix} I_{xx} & -S_x \\ -S_x & A \end{bmatrix} \begin{Bmatrix} N_p(z) \\ M_{xp}(z) \end{Bmatrix} \quad (6.11)$$

From Eq. (6.11) by substituting into Eq. (6.7), one obtains

$$\sigma_{zsp} = \frac{1}{(AI_{xx} - S_x^2)} \begin{bmatrix} I_{xx} - yS_x & yA - S_x \end{bmatrix} \begin{Bmatrix} N_p(z) \\ M_{xp}(z) \end{Bmatrix} \quad (6.12)$$

It is observed that for an orthogonal system of coordinate, $S_x = 0$, and Eq. (6.12) reverts to $\sigma_{zsp} = N_p(z)/A + M_{xp}(z)y/I_{xx}$, which is the well known expression for normal stresses in beams under an orthogonal coordinate system.

6.5.2.2 Stress Resultant Induced by Shear Stresses

The shear stress resultant in y -direction is determined by

$$V_{yp} = \int_A \tau_{zsp} \sin \alpha dA = \int_A G \left[v'_p(z) + \theta_{xp}(z) \right] \left(\frac{dy(s)}{ds} \right)^2 dA = GD_{yy} \left[v'_p(z) + \theta_{xp}(z) \right] \quad (6.13)$$

in which the sectional property $D_{yy} = \int_A \left(\frac{dy}{ds} \right)^2 dA = \int_A \sin^2 \alpha(s) dA$ has been defined. From Eq. (6.13),

by expressing the shear strain $[v'_p(z) + \theta_{xp}(z)]$ in terms of the shear resultant $V_{yp}(z)$ and substituting into Eq. (6.8), the shear stress τ_{zsp} is related to the shear resultant $V_{yp}(z)$ as

$$\tau_{zsp} = \frac{V_y}{D_{yy}} \frac{dy}{ds} \quad (6.14)$$

6.6 Conditions of Neutral Stability

When the member is in a neutral stability state (Configuration 3 in Fig. 6.2), the variation of the second variation of the total potential energy has to vanish (Bazant [6-25]) for an arbitrary set of buckling deformations, i.e.,

$$\delta\left(\frac{1}{2}\delta^2\pi\right)=\delta\left[\frac{1}{2}(\delta^2U+\delta^2V)\right]=0 \quad (6.15)$$

in which, δ^2 denotes the second variation of the argument functional. The internal strain energy is expressed as

$$U=\int_0^L\int_A\left(\frac{1}{2}\sigma_{zz}^*\varepsilon_{zz}^*+\frac{1}{2}\tau_{zs}^*\gamma_{zs}^*\right)dAdz \quad (6.16)$$

At the buckled configuration (Configuration 4), Eq. (6.16) takes the form

$$U=\frac{1}{2}\int_0^L\int_A\left[(\sigma_{zzb}+\lambda\sigma_{zzp})(\varepsilon_{zzb}+\lambda\varepsilon_{zzp})+(\tau_{zsp}+\lambda\tau_{zsp})(\gamma_{zsb}+\lambda\gamma_{zsp})\right]dAdz+\frac{1}{2}\int_0^L GJ\theta_{zb}'^2 dz \quad (6.17)$$

where the last term represents the strain energy due to the St. Venant torsion, and J is the St. Venant torsional constant. By expanding the brackets in Eq. (6.17) and noting that $\sigma_{zsp}=E\varepsilon_{zsp}$,

$\sigma_{zzb}=E\varepsilon_{zzb}$, $\tau_{zsp}=G\gamma_{zsp}$ and $\tau_{zsb}=G\gamma_{zsb}$, one obtains

$$U=\frac{1}{2}\int_0^L\int_A\left\{E\left[(\varepsilon_{zzb})^2+2\varepsilon_{zzb}\lambda\varepsilon_{zsp}+(\lambda\varepsilon_{zsp})^2\right]+G\left[(\gamma_{zsb})^2+2\gamma_{zsb}\lambda\gamma_{zsp}+(\lambda\gamma_{zsp})^2\right]\right\}dAdz+\frac{1}{2}\int_0^L GJ(\delta\theta_{zb}')^2 dz \quad (6.18)$$

Under the inextensional buckling assumption, both the centroidal strain and the curvature in the principal yz plane remain unchanged throughout buckling. As a result, the member undergoes buckling under constant normal stress resultant $N_p(z)$ and moment resultant $M_{xp}(z)$, and associated shear stress resultant $V_{yp}(z)$ as determined from the pre-buckling analysis. Thus, the second variation of the internal strain energy can be expressed as

$$\frac{1}{2}\delta^2U=\frac{1}{2}\int_0^L\int_A\left\{E\left[(\delta\varepsilon_{zzb})^2+\lambda\varepsilon_{zsp}\delta^2\varepsilon_{zzb}\right]+G\left[(\delta\gamma_{zsb})^2+\lambda\gamma_{zsp}\delta^2\gamma_{zsb}\right]\right\}dAdz+\frac{1}{2}\int_0^L GJ(\delta\theta_{zb}')^2 dz \quad (6.19)$$

The second variation of the potential energy of the externally applied loads is

$$\frac{1}{2}\delta^2V=-\frac{1}{2}\int_0^L\left[q_y(z)\delta^2v_b(y_{qy}(z))+q_z(z)\delta^2w_b(y_{qz}(z))\right]dz \quad (6.20)$$

in which $v_b(y_{qy}(z))$ is the transverse displacement of the line of application of the transverse distributed axial load q_y and $w_b(y_{qz}(z))$ is the longitudinal displacement of the line of application of the distributed longitudinal load q_z , both displacements induced throughout buckling. Using Eqs. (6.2) and (6.3) while retaining the terms in square brackets, and recalling

that the second variation of pre-buckling deformations vanish, i.e., $\delta^2\theta_{xp} = 0$, the second variation of $v_b(y_{qv}, z)$ and $w_b(y_{qz}, z)$ can be expressed as

$$\delta^2 v_b(y_{qv}, z) = -(y_{qv} - y_A) [\delta\theta_{zb}(z)]^2 \quad (6.21)$$

$$\delta^2 w_b(y_{qz}, z) = 2(y_{qz} - y_C) \delta\theta_{yb}(z) \delta\theta_{zb}(z) \quad (6.22)$$

and the second variation of the potential energy (Eq. (6.20)) is rewritten as

$$\frac{1}{2} \delta^2 V = - \int_0^L -\frac{1}{2} q_y (y_{qv} - y_A) (\delta\theta_{zb})^2 + q_z [(y_{qz} - y_C) \delta\theta_{yb} \delta\theta_{zb}] dz \quad (6.23)$$

The condition of neutral stability is obtained by substituting Eqs. (6.19) and (6.23) into Eq. (6.15), yielding

$$\begin{aligned} \delta \left(\frac{1}{2} \delta^2 \pi \right) &= \delta \left[\frac{1}{2} (\delta^2 U + \delta^2 V) \right] = 0 \\ &= \delta \left\{ \frac{1}{2} \int_0^L \int_A \left[E (\delta\epsilon_{zzb})^2 + \lambda \sigma_{zzp} \delta^2 \epsilon_{zzb} \right] + \left[G (\delta\gamma_{zsb})^2 + \lambda \tau_{zsp} \delta^2 \gamma_{zsb} \right] dA dz + \frac{1}{2} \int_0^L GJ (\delta\theta'_{zb})^2 dz \right. \\ &\quad \left. - \int_0^L -\frac{1}{2} q_y (y_{qv} - y_A) (\delta\theta_{zb})^2 + q_z [(y_{qz} - y_C) \delta\theta_{yb} \delta\theta_{zb}] dz \right\} \end{aligned} \quad (6.24)$$

6.7 Expressions for Strains and their Variations

6.7.1 Longitudinal Strain

Total Longitudinal Strain

From Eqs. (6.1)-(6.3), the total longitudinal strain $\epsilon_{zz}^* \approx w'^* + (u'^{*2} + v'^{*2})/2$ can be expressed as

$$\epsilon_{zz}^* \approx \epsilon_{zz}^{*L} + \epsilon_{zz}^{*N} \quad (6.25)$$

in which ϵ_{zz}^{*L} and ϵ_{zz}^{*N} denote the linear and nonlinear components of the longitudinal strain, given by

$$\epsilon_{zz}^{*L} = \lambda (w'_p + y\theta'_{xp}) - x\theta'_{yb} + \omega\psi'_b + x(\lambda\theta_{xp}\theta'_{zb} + \lambda\theta'_{xp}\theta_{zb}) + \frac{1}{2} \left[(\lambda v'_p)^2 + 2x(\lambda v'_p)\theta'_{zb} \right] \quad (6.26)$$

$$\epsilon_{zz}^{*N} = (y - y_C)(\theta_{yb}\theta'_{zb} + \theta'_{yb}\theta_{zb}) + \frac{1}{2} (u_b'^2 - 2yu'_b\theta'_{zb} + y^2\theta_{zb}'^2) + \frac{1}{2} x^2\theta_{zb}'^2 \quad (6.27)$$

Since v'_p , w'_p , θ_{xp} and θ'_{xp} remain constant during buckling, the terms $x(\lambda\theta_{xp}\theta'_{zb} + \lambda\theta'_{xp}\theta_{zb}) + \frac{1}{2}\left[(\lambda v'_p)^2 + 2x(\lambda v'_p)\theta'_{zb}\right]$ are linear in the buckling displacements and were included in the linear component of the longitudinal strain (i.e., Eq. (6.26)) as opposed to the nonlinear strain (Eq. (6.27)).

First Variation of Total and Buckling Longitudinal Strains

Noting that the first variations of all pre-buckling displacements vanish (Assumption 5), the first variation of the total longitudinal strain is

$$\delta\epsilon_{zz}^* = \delta\epsilon_{zz}^{*L} + \delta\epsilon_{zz}^{*N} \quad (6.28)$$

in which $\delta\epsilon_{zz}^{*L}$ and $\delta\epsilon_{zz}^{*N}$ are given by

$$\delta\epsilon_{zz}^{*L} = -x\delta\theta'_{yb} + \omega\delta\psi'_b + x\lambda(\theta_{xp} + v'_p)\delta\theta'_{zb} + x\lambda\theta'_{xp}\delta\theta_{zb} \quad (6.29)$$

$$\begin{aligned} \delta\epsilon_{zz}^{*N} = & (y - y_C)(\delta\theta_{yb}\theta'_{zb} + \theta_{yb}\delta\theta'_{zb} + \delta\theta'_{yb}\theta_{zb} + \theta'_{yb}\delta\theta_{zb}) + u'_b\delta u'_b - y(\delta u'_b\theta'_{zb} + u'_b\delta\theta'_{zb}) \\ & + y^2\theta'_{zb}\delta\theta'_{zb} + \frac{1}{2}x^2\theta'_{zb}\delta\theta'_{zb} \end{aligned} \quad (6.30)$$

In a buckling formulation such as the one sought in the present study, the first variation of the total longitudinal strain $\delta\epsilon_{zz}^{*N}$ is negligible compared to that of the linear component $\delta\epsilon_{zz}^{*L}$ (Pi and Trahair [6-26], and Trahair [6-4]). Only in a post-buckling analysis does the term $\delta\epsilon_{zz}^{*N}$ need to be retained. This leads to the simplification $\delta\epsilon_{zz}^* = \delta\epsilon_{zz}^{*L} + \delta\epsilon_{zz}^{*N} \approx \delta\epsilon_{zz}^{*L}$.

The linear components of the variation of the strain in Eq. (6.29) include pre-buckling displacements $\lambda v'_p$, $\lambda\theta_{xp}$ and $\lambda\theta'_{xp}$. These would be retained if the effects of pre-buckling deformations were sought in the analysis, leading to a quadratic eigen-value problem upon substitution in the variational expression (Eq. (6.24)). In a classical buckling analysis, the pre-buckling stresses and strains are assumed to linearly vary with the applied load (i.e., the second order moments due to axial forces undergoing transverse displacement are neglected in the pre-buckling stage) and the terms $\lambda v'_p$, $\lambda\theta_{xp}$ and $\lambda\theta'_{xp}$ are conventionally neglected (Pi and Trahair [6-27]). Of the above two methods, the quadratic eigen value provides more accurate predictions while the classical analysis leads to a simpler and conservative buckling load predictions (Pi and Trahair [6-27]). The present study aims at investigating shear deformation effects on the lateral-

torsional buckling resistance of thin-walled members in a classical sense. Thus, pre-buckling deformations are neglected. Recalling that the first variation of the pre-buckling displacements vanishes, the variation of the buckling strain is equal to the variation of the total strain, (i.e., $\delta\epsilon_{zz_b} = \delta\epsilon_{zz}^*$), and

$$\delta\epsilon_{zz_b} = \delta\epsilon_{zz}^* \approx -x\delta\theta'_{y_b} + \omega\delta\psi'_b \quad (6.31)$$

Second Variation of Total and Buckling Longitudinal Strain

Since the first variation of the pre-buckling strain vanishes, the second variation of buckling strain is equal to the second variation of the total strain (i.e., $\delta^2\epsilon_{zz_b} = \delta^2\epsilon_{zz}^*$), yielding

$$\begin{aligned} \delta^2\epsilon_{zz_b} = \delta^2\epsilon_{zz}^* = & (\delta u'_b)^2 - 2(y - y_A)\delta u'_b\delta\theta'_{z_b} + \left[x^2 + (y - y_A)^2\right](\delta\theta'_{z_b})^2 \\ & + 2y(\delta\theta'_{y_b}\delta\theta_{z_b} + \delta\theta_{y_b}\delta\theta'_{z_b}) - 2y_C(\delta\theta_{y_b}\delta\theta'_{z_b} + \delta\theta'_{y_b}\delta\theta_{z_b}) \end{aligned} \quad (6.32)$$

6.7.2 Shear Strain

Total Shear Strain

The shear strain γ_{zs}^* at the mid-surface of a thin-walled member is given by

$$\gamma_{zs}^* = \frac{\partial t_s^*}{\partial z} + \frac{\partial w_s^*}{\partial s} + \frac{\partial t_s^*}{\partial z} \frac{\partial t_s^*}{\partial s} + \frac{\partial w_s^*}{\partial s} \frac{\partial w_s^*}{\partial z} + \frac{\partial n_s^*}{\partial z} \frac{\partial n_s^*}{\partial s} \quad (6.33)$$

From Eqs. (6.3)-(6.4), by substituting into Eq. (6.33), the shear strain γ_{zs}^* is expressed in terms of the displacements. The strain can be divided into two components

$$\gamma_{zs}^* = \gamma_{zs}^{*L} + \gamma_{zs}^{*N} \quad (6.34)$$

in which γ_{zs}^{*L} is the linear component and γ_{zs}^{*N} is the nonlinear component of the shear strain,

For the case of a section consisting of multiple straight segments (Appendix 6-1) these components are expressed as

$$\begin{aligned} \gamma_{zs}^{*L} = & \frac{dx}{ds}(u'_b - \theta_{y_b} + \lambda\theta_{xp}\theta_{z_b}) + \frac{dy}{ds}(\lambda v'_p + \lambda\theta_{xp}) + h(\theta'_{z_b} + \psi_b) + \frac{dy}{ds}(\lambda\theta_{xp})(\lambda w'_p) - \frac{dx}{ds}(\lambda w'_p)\theta_{y_b} \\ & + h(\lambda w'_p)\psi_b + \frac{dx}{ds}\lambda w'_p\lambda\theta_{xp}\theta_{z_b} + y\frac{dy}{ds}(\lambda\theta_{xp})(\lambda\theta'_{xp}) - y\frac{dx}{ds}(\lambda\theta'_{xp})\theta_{y_b} + yh(\lambda\theta'_{xp})\psi_b \\ & + y\frac{dx}{ds}\lambda\theta'_{xp}\lambda\theta_{xp}\theta_{z_b} - x\frac{dy}{ds}(\lambda\theta_{xp})\theta'_{y_b} + \omega\frac{dy}{ds}(\lambda\theta_{xp})\psi'_b + x\frac{dy}{ds}\lambda\theta_{xp}(\lambda\theta'_{xp}\theta_{z_b} + \lambda\theta_{xp}\theta'_{z_b}) \\ & + \left(\frac{dx}{ds}\right)\left(\frac{dk}{ds}\right)\lambda v'_p\theta_{z_b} \end{aligned} \quad (6.35)$$

and

$$\begin{aligned}
 \gamma_{zs}^{*N} = & \frac{dy}{ds} \theta_{yb} \theta_{zb} + \frac{dy}{ds} \lambda w'_p \theta_{yb} \theta_{zb} + y \frac{dy}{ds} \lambda \theta'_{xp} \theta_{yb} \theta_{zb} + x \frac{dx}{ds} \theta_{yb} \theta'_{yb} - x h \psi'_b \theta'_{yb} - x \frac{dx}{ds} \lambda \theta_{xp} \theta'_{yb} \theta_{zb} \\
 & - x \frac{dy}{ds} \theta_{yb} \theta'_{yb} \theta_{zb} - \omega \frac{dx}{ds} \theta_{yb} \psi'_b + \omega h \psi'_b \psi'_b + \omega \frac{dx}{ds} \lambda \theta_{xp} \theta_{zb} \psi'_b - x \frac{dx}{ds} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb} \\
 & + \omega \frac{dy}{ds} \theta_{yb} \theta_{zb} \psi'_b + x h (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \psi_b + x \frac{dx}{ds} \lambda \theta_{xp} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{zb} \\
 & + x \frac{dy}{ds} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb} \theta_{zb} + (y - y_C) \frac{dy}{ds} \lambda \theta_{xp} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \\
 & - (y - y_C) \frac{dx}{ds} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{yb} + (y - y_C) h (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \psi_b \\
 & + (y - y_C) \frac{dx}{ds} \lambda \theta_{xp} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{zb} + (y - y_C) \frac{dy}{ds} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{yb} \theta_{zb} \\
 & - \left(\frac{dy}{ds} \right) \left(\frac{dk}{ds} \right) u'_b \theta_{zb} + k \left(\frac{dk}{ds} \right) \theta'_{zb} \theta_{zb}
 \end{aligned} \tag{6.36}$$

First Variation of Total and Buckling Shear Strains

Given that the pre-buckling shear strains remain constant throughout buckling, the first variation of the total shear strain is equal to the first variation of the buckling shear strain, i.e., $\delta\gamma_{zsb} = \delta\gamma_{zs}^*$. By neglecting a) the non-linear components and b) pre-buckling deformations effects in line with the classical buckling analysis, the first variation of the shear strain is expressed as

$$\delta\gamma_{zsb} = \delta\gamma_{zs}^* \approx \frac{dx}{ds} (\delta u'_b - \delta\theta_{yb}) + h (\delta\theta'_{zb} + \delta\psi_b) \tag{6.37}$$

Second Variation of Total and Buckling Shear Strain

Again, because the pre-buckling shear strain is constant throughout buckling, the second variation of the total shear strain is equal to the second variation of the buckling shear strain $\delta^2\gamma_{zsb} = \delta^2\gamma_{zs}^*$. By neglecting a) all pre-buckling terms, and b) the variations of third and fourth order buckling terms, the second variation of the buckling shear strain simplifies to

$$\begin{aligned}
 \delta^2\gamma_{zsb} = \delta^2\gamma_{zs}^* \approx & 2 \left[\frac{dy}{ds} \delta\theta_{yb} \delta\theta_{zb} + x \frac{dx}{ds} (\delta\theta_{yb} \delta\theta'_{yb}) - x h (\delta\psi_b \delta\theta'_{yb}) - \omega \frac{dx}{ds} (\delta\theta_{yb} \delta\psi'_b) \right. \\
 & \left. + \omega h (\delta\psi_b \delta\psi'_b) - \left(\frac{dy}{ds} \right) \left(\frac{dk}{ds} \right) (\delta u'_b \delta\theta_{zb}) + k \left(\frac{dk}{ds} \right) (\delta\theta'_{zb} \delta\theta_{zb}) \right]
 \end{aligned} \tag{6.38}$$

6.8 Condition of Neutral Stability

The condition of neutral stability is obtained by setting the variation of the second variation to zero, i.e. $\delta[(1/2)\delta^2\pi]=0$. The second variation of the internal strain energy is obtained by substituting Eqs. (6.12), (6.14), (6.31), (6.32), (6.37) and (6.38) into Eq. (6.24). For a doubly symmetric section consisting of straight segments parallel to the x and y axes, one can show that

$$\begin{aligned} \int_A \left(x \frac{dx}{ds} \frac{dy}{ds} \right) dA &= \int_A \left(xh \frac{dy}{ds} \right) dA = \int_A \left(\omega \frac{dx}{ds} \frac{dy}{ds} \right) dA = \int_A \left(\omega h \frac{dy}{ds} \right) dA = 0 \text{ and Eq. (6.24) takes the form} \\ \delta \left(\frac{1}{2} \delta^2 \pi \right) &= 0 \\ &= \delta \left(\frac{1}{2} \int_0^L \left\{ EI_{yy} (\delta\theta'_{yb})^2 - 2ES_{x\omega} \delta\theta'_{yb} \delta\psi'_b + EI_{\omega\omega} (\delta\psi'_b)^2 + GJ \delta\theta'^2_{zb} \right\} dz \right. \\ &\quad + \frac{1}{2} G \int_0^L \left\{ D_{xx} (\delta u'_b)^2 - 2D_{xx} \delta u'_b \delta\theta_{yb} + 2D_{xh} \delta u'_b \delta\theta'_{zb} + 2D_{xh} \delta u'_b \delta\psi_b + D_{xx} (\delta\theta_{yb})^2 \right. \\ &\quad \left. - 2D_{xh} \delta\theta_{yb} \delta\theta'_{zb} - 2D_{xh} \delta\theta_{yb} \delta\psi_b + D_{hh} (\delta\theta'_{zb})^2 + 2D_{hh} \delta\theta'_{zb} \delta\psi_b + D_{hh} (\delta\psi_b)^2 \right\} dz \\ &\quad + \frac{1}{2} \lambda \int_0^L \left(\frac{N_p(z)I_{xx} - M_{xp}(z)S_x}{AI_{xx} - S_x^2} \right) \left\{ A\delta u_b'^2 + 2(y_A A - S_x) \delta u'_b \delta\theta'_{zb} + (I_{xx} + I_{yy} + y_A^2 A - 2y_A S_x) \delta\theta_{zb}'^2 \right. \\ &\quad \left. + 2(S_x - y_C A) (\delta\theta'_{yb} \delta\theta_{zb} + \delta\theta_{yb} \delta\theta'_{zb}) \right\} dz \\ &\quad - \frac{1}{2} \lambda \int_0^L \left(\frac{N_p(z)S_x - M_{xp}(z)A}{AI_{xx} - S_x^2} \right) \left\{ S_x \delta u_b'^2 + 2(y_A S_x - I_{xx}) \delta u'_b \delta\theta'_{zb} + (y_A^2 S_x + I_{px} - 2y_A I_{xx}) \delta\theta_{zb}'^2 \right. \\ &\quad \left. + 2(I_{xx} - y_C S_x) (\delta\theta'_{yb} \delta\theta_{zb} + \delta\theta_{yb} \delta\theta'_{zb}) \right\} dz \\ &\quad + \frac{\lambda}{2} \int_0^L \frac{V_{yp}(z)}{D_{yy}} \left\{ 2D_{yy} \delta\theta_{yb} \delta\theta_{zb} - 2D_{yyk} \delta u'_b \delta\theta_{zb} + 2D_{yk} \delta\theta'_{zb} \delta\theta_{zb} \right\} dz \\ &\quad \left. - \lambda \int_0^L \left(-\frac{1}{2} q_y (y_{qy} - y_A) (\delta\theta_{zb})^2 + q_z [(y_{qz} - y_C) \delta\theta_{yb} \delta\theta_{zb}] \right) dz \right) \end{aligned} \quad (6.39)$$

In Eq. (6.39), the following cross-sectional properties are defined

$$\begin{aligned}
 I_{yy} &= \int_A x^2 dA & I_{xy} &= \int_A xy dA & I_{\omega\omega} &= \int_A \omega^2 dA & I_{px} &= \int_A y(x^2 + y^2) dA \\
 S_x &= \int_A y dA & S_{x\omega} &= \int_A x\omega dA & J &= \sum_{i=1}^n \frac{l_i r_i^3}{3} & D_{hh} &= \int_A h^2 dA \\
 D_{xx} &= \int_A \left(\frac{dx}{ds} \right)^2 dA & D_{xh} &= \int_A h \left(\frac{dx}{ds} \right) dA & D_{yk} &= \int_A k \frac{dy}{ds} \frac{dk}{ds} dA & D_{yjk} &= \int_A \left(\frac{dy}{ds} \right)^2 \frac{dk}{ds} dA
 \end{aligned} \tag{6.40}$$

in which, l_i and r_i are the width and thickness for segments $i = 1..n$ of the cross-section. Once the pre-buckling internal forces in the member are determined through the equilibrium requirement in Eq. (6.9), the variational expression in Eq. (6.39) can be solved for the critical load factor λ , thus yielding the sought critical load combination. The variational principle developed in Eq. (6.39) has the following features: (a) it is based on a general non-orthogonal coordinate system, which facilitates the modelling of eccentric supports. The use of this feature will be illustrated in the Chapter 7); (b) it incorporates shear deformation effects due to bending and warping; and (c) it incorporates the load position effect and thus is capable of modelling transverse loads that are offset from shear center and longitudinal loads offset from the centroid.

Equation (6.39) can be used either to derive the field equations and boundary conditions, leading to closed form solutions. This approach is adopted in the present paper. Alternatively, it can be used to develop a finite element formulation applicable to general loading combinations and boundary conditions. This approach is developed in Chapter 7).

Integrating Eq. (6.39) by parts with respect to z yields the following governing differential equations

$$\begin{aligned}
 -GD_{xx}\delta u_b'' + GD_{xx}\delta \theta_{yb}' - GD_{xh}\delta \theta_{zb}'' - GD_{xh}\delta \psi_b' - \lambda \left\{ \frac{(N_p(z)I_{xx} - M_{xp}(z)S_x)}{AI_{xx} - S_x^2} [A\delta u_b' + (y_A A - S_x)\delta \theta_{zb}'] \right\}' \\
 + \lambda \left\{ \frac{(N_p(z)S_x - M_{xp}(z)A)}{AI_{xx} - S_x^2} [S_x\delta u_b' + (y_A S_x - I_{xx})\delta \theta_{zb}'] \right\}' + \lambda \frac{D_{yjk}}{D_{yy}} (V_{yp}(z)\delta \theta_{zb}') = 0
 \end{aligned} \tag{6.41}$$

$$\begin{aligned}
 & -EI_{yy}\delta\theta''_{yb} + ES_{x\omega}\delta\psi''_b - GD_{xx}\delta u'_b + GD_{xx}\delta\theta_{yb} - GD_{xh}\delta\theta'_{zb} - GD_{xh}\delta\psi_b \\
 & + \lambda \left[\frac{(N_p(z)I_{xx} - M_{xp}(z)S_x)}{AI_{xx} - S_x^2} (S_x - y_C A) \delta\theta'_{zb} - \frac{(N_p(z)S_x - M_{xp}(z)A)}{AI_{xx} - S_x^2} (I_{xx} - y_C S_x) \delta\theta'_{zb} + V_{yp}(z) \delta\theta_{zb} \right] \\
 & - \lambda \left[\frac{(N_p(z)I_{xx} - M_{xp}(z)S_x)}{AI_{xx} - S_x^2} (S_x - y_C A) \theta_{zb} + \frac{(N_p(z)S_x - M_{xp}(z)A)}{AI_{xx} - S_x^2} (I_{xx} - y_C S_x) \theta_{zb} + \frac{D_{yh}}{D_{yy}} V_{yp}(z) \psi_b \right] \\
 & - \lambda q_z(z) (y_{qz} - y_C) \delta\theta_{zb} = 0
 \end{aligned} \tag{6.42}$$

$$\begin{aligned}
 & -GD_{xh}\delta u''_b + GD_{xh}\delta\theta'_{yb} - GJ\delta\theta''_{zb} - GD_{hh}\delta\theta''_{zb} - GD_{hh}\delta\psi'_b \\
 & + \lambda \frac{1}{AI_{xx} - S_x^2} (N_p(z)I_{xx} - M_{xp}(z)S_x) (S_x - y_C A) \delta\theta'_{yb} \\
 & - \lambda \left\{ \frac{1}{AI_{xx} - S_x^2} (N_p(z)S_x - M_{xp}(z)A) (I_{xx} - y_C S_x) \delta\theta'_{yb} + \frac{V_{yp}(z)}{D_{yy}} (D_{yy}\delta\theta_{yb} - D_{yyk}\delta u'_b + D_{yk}\delta\theta'_{zb}) \right\} \\
 & - \lambda \left\{ \frac{(N_p(z)I_{xx} - M_{xp}(z)S_x)}{AI_{xx} - S_x^2} \left[(I_{xx} + I_{yy} + y_A^2 A - y_A S_x) \delta\theta'_{zb} + (S_x - y_C A) \delta\theta_{yb} + (y_A A - S_x) \delta u'_b \right] \right\} \\
 & + \lambda \left\{ \frac{(N_p(z)S_x - M_{xp}(z)A)}{AI_{xx} - S_x^2} \left[I_{px}\delta\theta'_{zb} + (I_{xx} - y_C S_x) \delta\theta_{yb} - I_{xx}\delta u'_b \right] - \frac{V_{yp}(z)}{D_{yy}} D_{yk}\delta\theta_{zb} \right\} \\
 & + \lambda [q_y(z) (y_{qy} - y_A) \delta\theta_{zb} - q_z(z) (y_{qz} - y_C) \delta\theta_{yb}] = 0
 \end{aligned} \tag{6.43}$$

$$ES_{x\omega}\delta\theta'_{yb} - EI_{\omega\omega}\delta\psi''_b + GD_{xh}\delta u'_b - GD_{xh}\delta\theta_{yb} + GD_{hh}\delta\theta'_{zb} + GD_{hh}\delta\psi_b = 0 \tag{6.44}$$

The boundary terms resulting from integration by parts lead to the following boundary conditions

At $z = 0$ and $z = L$.

Either δu_b is specified or $\tag{6.45a}$

Or

$$GD_{xx}(\delta u'_b - \delta \theta_{yb}) + \lambda \left[-\frac{D_{yyk}}{D_{yy}} V_{yp}(z) \delta \theta_{zb} + \left(\frac{N_p(z) I_{xx} - M_{xp} S_x}{AI_{xx} - S_x^2} \right) (A \gamma u'_b - S_x \delta \theta'_{zb}) - \left(\frac{N_p(z) S_x - M_{xp}(z) A}{AI_{xx} - S_x^2} \right) (S_x u'_b - I_{xx} \theta'_{zb}) \right] = 0 \quad (6.45b)$$

Also, either

$$\delta \theta_{yb} \text{ is specified, or} \quad (6.46a)$$

$$EI_{yy} \delta \theta'_{yb} - 2ES_{x\omega} \delta \psi_b + \lambda \left[\frac{D_{yh}}{D_{yy}} V_{yp} \delta \psi_b + (N_p I_{xx} - M_{xp} S_x) \frac{(S_x - y_C A)}{AI_{xx} - S_x^2} \delta \theta_{zb} - (N_p S_x - M_{xp} A) \frac{(I_{xx} - y_C S_x)}{AI_{xx} - S_x^2} \delta \theta_{zb} \right] = 0 \quad (6.46b)$$

$$\text{Either } \delta \theta_{zb} \text{ is specified, or} \quad (6.47a)$$

$$\left[G \left\{ J \delta \theta'_{zb} + D_{hh} \delta \theta'_{zb} + D_{hh} \delta \psi_b + D_{xh} \delta u'_b - D_{xh} \delta \theta_{yb} \right\} + \lambda \frac{V_{yp}(z)}{D_{yy}} (D_{yhh} + D_{yk}) \delta \theta_{zb} + \lambda \frac{(N_p(z) I_{xx} - M_{xp}(z) S_x)}{AI_{xx} - S_x^2} \left\{ (I_{xx} + I_{yy}) \delta \theta'_{zb} + (S_x - 2y_C A) \delta \theta_{yb} - S_x \delta u'_b \right\} - \lambda \frac{(N_p(z) S_x - M_{xp}(z) A)}{AI_{xx} - S_x^2} \left\{ I_{px} \theta'_{zb} + (I_{xx} - 2y_C S_x) \theta_{yb} - I_{xx} u'_b \right\} \right] = 0 \quad (6.47b)$$

$$\text{Either } \delta \psi_b \text{ is specified, or} \quad (6.48a)$$

$$[-2ES_{x\omega} \delta \theta'_{yb} + EI_{\omega\omega} \delta \psi'_b] = 0 \quad (6.48b)$$

6.9 Column Flexural Buckling

A prismatic column with a doubly symmetric cross-section has a span L loaded with an axial force Q_z ($M_{xp} = V_{yp} = 0$) applied at the centroid and is analyzed by using the general differential equations of equilibrium (Eqs. (6.41)-(6.44)) and boundary conditions (Eqs. (6.45)-(6.48)). The origin of the coordinates is arranged to coincide with the centroid, so that $I_{xy} = 0$, $S_x = 0$, $S_y = 0$, $S_{x\omega} = 0$, $D_{xh} = 0$. Under these conditions, the governing differential equations for the problem simplify to

$$(GD_{xx}\delta u_b'' + N\delta u_b'') - GD_{xx}\delta\theta_{yb}' = 0 \quad (6.49)$$

$$-GD_{xx}\delta u_b' + (-EI_{yy}\delta\theta_{yb}'' + GD_{xx}\delta\theta_{yb}') = 0 \quad (6.50)$$

The general solution of these coupled differential equations is

$$\delta u_b(z) = A_1 + A_2 z + A_3 \sin \alpha z + A_4 \cos \alpha z \quad (6.51)$$

$$\delta\theta_{yb}(z) = A_2 - \alpha \left(1 + \frac{N}{GD_{xx}}\right) A_4 \sin \alpha z + \alpha \left(1 + \frac{N}{GD_{xx}}\right) A_3 \cos \alpha z \quad (6.52)$$

6.9.1 Example 1-Simply Supported Column

The relevant boundary conditions are

$$\delta u_b(0) = \delta u_b(L) = 0, \delta\theta_{yb}'(0) = \delta\theta_{yb}'(L) = 0 \quad (6.53)$$

Starting with the displacement expressions in Eqs. (6.51-6.52), enforcing the boundary conditions in Eq. (6.53), and solving the resulting eigenvalue problem (the details of obtaining the critical loads are presented in Appendix 6-2), the buckling load is found to be

$$N_{cr} = \frac{-N_{cr}^o}{1 + kN_{cr}^o}; \quad N_{cr}^o = \frac{\pi^2 EI_{yy}}{L^2} \quad (6.54)$$

in which $k = 1/GD_{xx}$ in the present formulation. In other theories, constant k takes different values. For instance, the conventional Euler-Bernoulli beam, $k = 0$. Based on the shear deformation theory developed in Erkmén and Mohareb [6-6], it can be shown that $k = 6/5GD_{xx}$. In the Timoshenko column solution by Alfutov [6-28], $k = 1/GA$, while Wang et al. [6-29] propose $k = k_s/GA$ where k_s is section dependent shear correction factor.

6.9.2 Example 2-Fixed-Free Column

Based on Eqs. (6.45-6.48), the boundary conditions for the fixed-free problem are

$$\delta u_b(0) = 0, \quad \delta\theta_{yb}(0) = 0, \quad GD_{xx}(\delta u_b'(L) - \delta\theta_{yb}(L)) + N\delta u_b'(L) = 0, \quad \delta\theta_{yb}'(L) = 0 \quad (6.55)$$

Substituting the field solutions into the boundary equations, the corresponding critical load is determined as (the details of the procedure is presented in Appendix 6-2, A6-2.1.2 Example 2),

$$N_{cr} = \frac{-N_{cr}^o}{1 + kN_{cr}^o}; \quad N_{cr}^o = \frac{\pi^2 EI_{yy}}{4L^2} \quad (6.56)$$

Coefficient k has the same values as those given for the pin-pin column (Example 1).

6.9.3 Example 3-Column Torsional Buckling

A column of length L with I-shaped wide-flange cross-section is prevented from twisting, but free to warp and bend at the two ends, the field equations for the particular problem are obtained by applying the axial loads and the cross-sectional properties to the general differential equations of equilibrium (Eqs. (6.41-6.44)).

$$(GJ + GD_{hh} + Nr_o^2) \delta\theta_{zb}'' + GD_{hh} \delta\psi_b' = 0 \quad (6.57)$$

$$GD_{hh} \delta\theta_{zb}' + (-EI_{\omega\omega} \delta\psi_b'' + GD_{hh} \delta\psi_b) = 0 \quad (6.58)$$

Equations (6.57) and (6.58) have the solution

$$\delta\theta_{zb}(z) = C_1 + C_2 z + C_3 \sin \beta z + C_4 \cos \beta z \quad (6.59)$$

$$\delta\psi_b(z) = -C_2 + \frac{\beta(GJ + r_o^2 N + GD_{hh})}{GD_{hh}} C_4 \sin \beta z - \frac{\beta(GJ + r_o^2 N + GD_{hh})}{GD_{hh}} C_3 \cos \beta z \quad (6.60)$$

and the relevant boundary conditions are

$$\delta\theta_{zb}(0) = \delta\theta_{zb}(L) = 0, \quad \delta\psi_b'(0) = \delta\psi_b'(L) = 0 \quad (6.61)$$

Based on the displacement fields and boundary conditions, the critical load is found to take the form (the detailed solution is given in Appendix 6-2, A6-2.2.1 Example 3),

$$N_{cr_{tor}} = -\left[\frac{N_{cr_{tor}}^o}{1 + kN_{cr_{tor}}^o} + GJ \right] \frac{1}{r_o^2}; \quad N_{cr_{tor}}^o = \frac{\pi^2 EI_{\omega\omega}}{L^2} \quad (6.62)$$

in which $r_o^2 = (I_{xx} + I_{yy})/A$. Coefficient k in the present study takes the value $1/GD_{hh}$. This compares with $k = (6/5)GD_{hh}$ in the shear deformable theory in Erkmén and Mohareb [6-6], while $k = 0$ in classical solution which neglects shear deformation effects (e.g, Wang et al [6-29]). It is noted that as the span of the column L increases, the term $kN_{cr_{tor}}^o$ becomes negligible compared to unity and Eq. (6.62) approaches the classical torsional buckling solution $N_{cr_{tor}} = -(GJ + \pi^2 EI_{\omega\omega}/L^2)/r_o^2$.

6.9.4 Example 4 - Lateral Torsional Buckling of Beam under Uniform Bending

A simply supported wide-flange beam of length L is subject to end bending moments M_x about the x axis. Since the beam is subjected to a uniform bending moment M_x and the lateral-torsional buckling is assumed to take place in an inextensional mode, one has $N_z = V_y = q_y = q_z = 0$ and $\delta^2 w_q = \delta^2 w_Q = 0$.

When solving the problem, origin O of the coordinate systems is conveniently selected to coincide with the centroid, the shear center SC and the principal sectorial origin of the cross-section, under these selections, the coordinate system is orthogonal (i.e., $y_c = S_x = S_{x\omega} = 0$) and the resulting governing equilibrium equations are considerably simplified. The resulting differential equations as extracted from (Eqs. (6.41-6.44)) are

$$-GD_{xx}\delta u''_b + GD_{xx}\delta\theta'_{yb} + M_{xp}\delta\theta''_{zb} = 0 \quad (6.63)$$

$$-EI_{yy}\delta\theta''_{yb} - GD_{xx}\delta u'_b + GD_{xx}\delta\theta_{yb} = 0 \quad (6.64)$$

$$GJ\delta\theta''_{zb} + GD_{hh}\delta\theta''_{zb} + GD_{hh}\delta\psi'_b - M_{xp}\delta u''_b = 0 \quad (6.65)$$

$$EI_{\omega\omega}\delta\psi''_b - G(D_{hh}\delta\theta'_{zb} + D_{hh}\delta\psi_b) = 0 \quad (6.66)$$

and the relevant boundary conditions as extracted from Eqs. (6.45-6.48) are

$$\delta u_b(0) = 0, \delta\theta_{zb}(0) = 0, \delta\psi'_b(0) = 0, \text{ and } EI_{yy}\delta\theta'_{yb}(0) = 0$$

$$\delta u_b(L) = 0, \delta\theta_{zb}(L) = 0, \delta\psi'_b(L) = 0, \text{ and } EI_{yy}\delta\theta'_{yb}(L) = 0 \quad (6.67a-h)$$

By developing the closed form solution (Appendix 6-3) of the coupled system of differential equations (Eqs. 6.63-6.66), enforcing the eight boundary conditions in Eqs. (6.67a-h), and setting the determinant of the resulting system of algebraic equations to zero, one obtains the following expression for the critical moment

$$M_{cr} = \pm \frac{\pi}{L} \sqrt{\frac{EGI_{yy}J + \left(\frac{E\pi}{L}\right)^2 I_{yy}I_{\omega\omega} \left(1 + \frac{J}{D_{hh}}\right)}{I + \frac{E}{G} \left(\frac{I_{yy}}{D_{xx}} + \frac{I_{\omega\omega}}{D_{hh}}\right) \left(\frac{\pi}{L}\right)^2 + \left(\frac{E}{G}\right)^2 \left(\frac{I_{yy}I_{\omega\omega}}{D_{xx}D_{hh}}\right) \left(\frac{\pi}{L}\right)^4}} \quad (6.68)$$

It is of interest to note that, starting with the compatibility equations and boundary conditions developed in Erkmén and Mohareb [6.6], and after considerable manipulations, the critical moment M_{cr-EM} for the current problem is found to take the form

$$M_{cr-EM} = \pm \frac{\pi}{L} \sqrt{\frac{EGI_{yy}J + \left(\frac{E\pi}{L}\right)^2 I_{yy}I_{\omega\omega} \left(1 + \frac{J}{\chi_{22}}\right)}{1 + \frac{E}{G} \left(\frac{I_{yy}}{\chi_{11}} + \frac{I_{\omega\omega}}{\chi_{22}}\right) \left(\frac{\pi}{L}\right)^2 + \left(\frac{E}{G}\right)^2 \left(\frac{I_{yy}I_{\omega\omega}}{\chi_{11}\chi_{22}}\right) \left(\frac{\pi}{L}\right)^4}} \quad (6.69)$$

Equations (6.68) and (6.69) are similar but not identical. The difference lies in the definition of the section constants D_{xx} and D_{hh} in the present theory and the analogous section constants χ_{11} and χ_{22} defined in [6.6]. For a wide flange section with a flange width b , flange thickness r , web height h , and web thickness w , $D_{xx}=2br$, while the corresponding constant in [6-6] is $\chi_{11} = 5br/3$. Also, constant $D_{hh}=h^2br/2$ compares to $\chi_{22} = 5h^2br/12$ in [6-6].

Recalling that the classical critical lateral torsional buckling moment for the present problem (i.e., without considering shear deformation effects) is $M_{cl} = (\pi/L) \sqrt{EGI_{yy}J + (\pi E/L)^2 I_{yy}I_{\omega\omega}}$, one can express Eq. (6.68) in the normalized form

$$\frac{M_{cr}}{M_{cl}} = \sqrt{\frac{1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GD_{hh}L^2}\right)}{1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2}\right)}} \times \frac{1}{\left[1 + \frac{E}{G} \left(\frac{I_{yy}}{D_{xx}} + \frac{I_{\omega\omega}}{D_{hh}}\right) \left(\frac{\pi}{L}\right)^2 + \left(\frac{E}{G}\right)^2 \left(\frac{I_{yy}I_{\omega\omega}}{D_{xx}D_{hh}}\right) \left(\frac{\pi}{L}\right)^4\right]} \quad (6.70)$$

For a long span beam $1 + \left(\frac{\pi E}{L}\right)^2 \left(\frac{I_{yy}I_{\omega\omega}}{EGJ_{yy}}\right) \rightarrow 1$, $1 + \left(\frac{E\pi}{L}\right)^2 \left(\frac{I_{yy}I_{\omega\omega}}{EGJ_{yy}}\right) \left(\frac{J}{D_{hh}}\right) \rightarrow 1$, and

$1 + \frac{E}{G} \left(\frac{I_{yy}}{D_{xx}} + \frac{I_{\omega\omega}}{D_{hh}}\right) \left(\frac{\pi}{L}\right)^2 + \left(\frac{E}{G}\right)^2 \left(\frac{I_{yy}I_{\omega\omega}}{D_{xx}D_{hh}}\right) \left(\frac{\pi}{L}\right)^4 \rightarrow 1$, and the right hand side of Eq. (6.70)

approaches unity, i.e., the critical moment based on the present theory approaches that of the classical formulation for long spans. A numeric comparison of critical moments based on the present theory, that based on the shear deformable theory in [6.6] and that based on the classical solution M_{cl} is provided by considering a simply supported beam with a W250x58 wide-flange cross-section (cross-section dimensions are given in Fig. 6.5). The cross-sectional properties are $I_{yy} = 1.88 \times 10^7 \text{ mm}^2$, $I_{xx} = 8.71 \times 10^7 \text{ mm}^4$, $I_{\omega\omega} = 2.68 \times 10^{11} \text{ mm}^6$, $J = 3.74 \times 10^5 \text{ mm}^4$, $D_{xx} = 5,481 \text{ mm}^2$, $D_{hh} = 7.79 \times 10^7 \text{ mm}^4$, $\chi_{11} = 4,568 \text{ mm}^2$, and $\chi_{22} = 6.50 \times 10^7 \text{ mm}^4$.

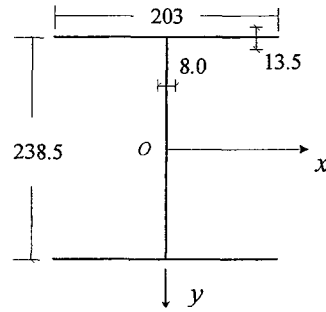


Figure 6.5 W250x58 cross-section

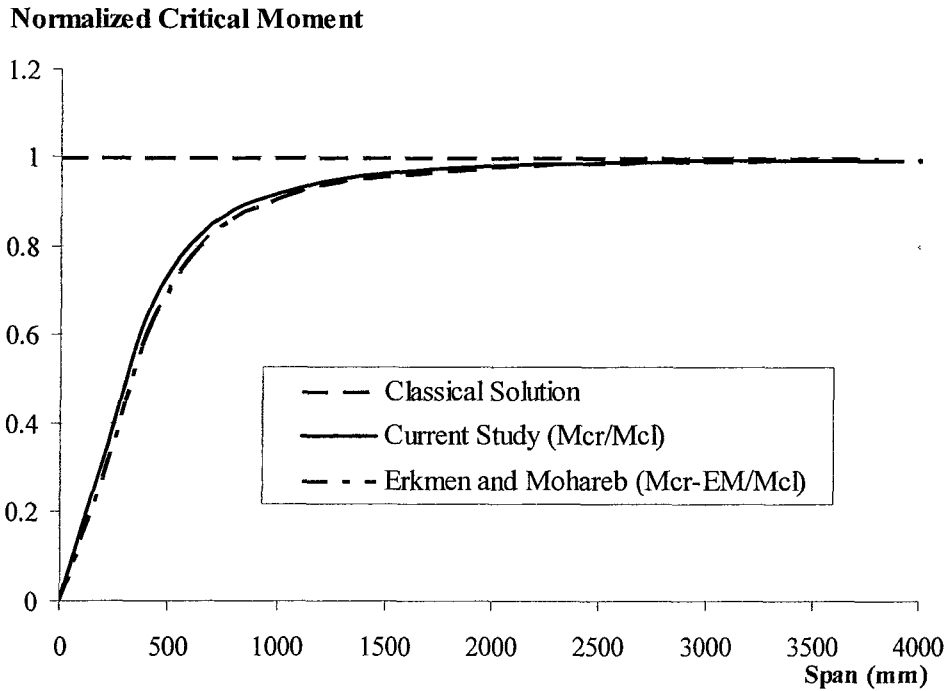


Figure 6.6 Normalized critical moment versus span (Normalization based on the classical critical moment expression)

It is observed that for beams with shorter span, (e.g., $L < 1000\text{mm}$ in the present example), the critical moment based on the current study is smaller than the one based on the classical solution, but agrees well with the shear deformable theory in [6-6]. The difference between the classical and present solution is due to the dominance of shear deformation effect in short beams. This effect is not captured in the classical solution. As the span increases, the critical moment approaches that based on the classical solution. At a 4,000 mm span, the critical moment based on the current study is 99.97% of that based on the classical solution. It should be noted that for

this section, the characteristic length is about 3,400 mm for the 300 MPa yield strength. Therefore, for a W250x58 cross-section of 300W material, the difference between the critical moment predicted by Eq. 6.68 and that based on the classical solution is negligible.

It is observed that for the case of uniform moment, the critical moment based on the present theory is consistently lower than that predicted by the classical theory. In a pre-buckling analysis for beams under uniform strong axis moment M (and thus zero shear force along the weak axis), whether one adopts the conventional Euler-Bernoulli beam theory or the Timoshenko shear deformable theory, one would obtain zero shear stresses under uniform moments because of the absence of shear force. Conversely, in a buckling analysis, the beam undergoes non-uniform twist $\theta_{zb}(z)$ throughout buckling and the corresponding weak axis moment in the deformed configuration is $M\theta_{zb}(z)$. This weak axis moment is clearly non-uniform. Thus, the associated shear force along the beam strong axis is non-zero and induces shear stresses in the cross-section. These shear stresses are not captured in the classical buckling formulation but captured in the present shear deformable formulation. Their inclusion in the present formulation leads to a more flexible representation of the stiffness of the beam and thus the buckling moments predicted by the present formulation should be lower than those based on the classical solution.

6.10 Summary and Conclusions

1. A variational principle based on the principle of stationary potential energy was derived for the buckling analysis of thin-walled members of doubly symmetric sections. In contrast to conventional solutions, the principle is applicable to non-orthogonal coordinate systems, and captures shear deformation and load position effects.
2. The variational principle was used to formulate the conditions of neutral stability and associated boundary conditions.
3. Closed form solutions of the resulting field equations and boundary conditions were developed for the cases of column flexural buckling, column torsional buckling, and laterally unsupported beams under uniform moments.
4. The buckling loads/moments based on the current theory are found to agree well with those based on recently developed buckling formulation which account for shear deformation.
5. Comparisons with classical solutions that do not include the effect of shear deformation illustrate the effect of shear deformation for members of short spans.

6. The variational formulation developed provides a foundation for developing finite element formulations for more general loading cases, boundary conditions, etc. An example of such a formulation is provided in Chapter 7.

Appendix 6-1 Transverse Shear Strain in Terms of Displacements

This appendix shows the steps of formulation of the shear strain – displacement relationship. The final expression is valid for sections consisting of multiple straight (as opposed to curved) segments. The starting point is Eq. (6.33):

$$\gamma_{zs}^* = \underbrace{\frac{\partial t_s^*}{\partial z} + \frac{\partial w_s^*}{\partial s}}_{\langle 1 \rangle} + \underbrace{\frac{\partial t_s^*}{\partial z} \frac{\partial t_s^*}{\partial s}}_{\langle 2 \rangle} + \underbrace{\frac{\partial w_s^*}{\partial s} \frac{\partial w_s^*}{\partial z}}_{\langle 3 \rangle} + \underbrace{\frac{\partial n_s^*}{\partial z} \frac{\partial n_s^*}{\partial s}}_{\langle 4 \rangle} \quad (\text{A6-1.1})$$

From Eqs. (6.3-6.5), by substituting into Eq. (A1.1), the four terms on the right hand side of Eq. (A1.1) are expressed as:

$$\begin{aligned} \langle 1 \rangle: \frac{\partial t_s^*}{\partial z} + \frac{\partial w_s^*}{\partial s} &= \frac{\partial}{\partial z} \left[u_b \frac{dx}{ds} + \lambda v_p \frac{dy}{ds} + h \theta_{zb} \right] + \frac{\partial}{\partial s} \left[\lambda w_p + y \lambda \theta_{xp} - x \theta_{yb} + \omega \psi_b + x \lambda \theta_{xp} \theta_{zb} + (y - y_c) \theta_{yb} \theta_{zb} \right] \\ &= \frac{dx}{ds} (u'_b - \theta_{yb} + \lambda \theta_{xp} \theta_{zb}) + \frac{dy}{ds} (\lambda v'_p + \lambda \theta_{xp} + \theta_{yb} \theta_{zb}) + h (\theta'_{zb} + \psi'_b) \end{aligned} \quad (\text{A6-1.2})$$

$$\langle 2 \rangle: \frac{\partial t_s^*}{\partial z} \frac{\partial t_s^*}{\partial s} = \left(u'_b \frac{dx}{ds} + \lambda v'_p \frac{dy}{ds} + h \theta'_{zb} \right) \left(u_b \frac{d^2 x}{ds^2} + \lambda v_p \frac{d^2 y}{ds^2} + \theta_{zb} \frac{dh}{ds} \right) \quad (\text{A6-1.3})$$

For a section consisting of linear segments, all the entries in the second bracket of Eq. (A1.3) vanish.

$$\begin{aligned}
 \langle 3 \rangle: \frac{\partial w_s^*}{\partial z} \frac{\partial w_s^*}{\partial s} = & \left[\lambda w_p' + y \lambda \theta_{xp}' - x \theta_{yb}' + \omega \psi_b' + x \left(\lambda \theta_{xp}' \theta_{zb} + \lambda \theta_{xp}' \theta_{zb}' \right) + (y - y_c) \left(\theta_{yb}' \theta_{zb} + \theta_{yb}' \theta_{zb}' \right) \right] \\
 & \times \left(\frac{dy}{ds} \lambda \theta_{xp} - \frac{dx}{ds} \theta_{yb} + h \psi_b + \frac{dx}{ds} \lambda \theta_{xp} \theta_{zb} + \frac{dy}{ds} \theta_{yb} \theta_{zb} \right) \\
 = & \frac{dy}{ds} (\lambda \theta_{xp}) (\lambda w_p') - \frac{dx}{ds} (\lambda w_p') \theta_{yb} + h (\lambda w_p') \psi_b + \frac{dx}{ds} (\lambda w_p') \lambda \theta_{xp} \theta_{zb} + \frac{dy}{ds} (\lambda w_p') \theta_{yb} \theta_{zb} \\
 & + y \frac{dy}{ds} (\lambda \theta_{xp}) (\lambda \theta_{xp}') - y \frac{dx}{ds} (\lambda \theta_{xp}') \theta_{yb} + y h (\lambda \theta_{xp}') \psi_b + y \frac{dx}{ds} (\lambda \theta_{xp}') \lambda \theta_{xp} \theta_{zb} + y \frac{dy}{ds} \lambda \theta_{xp}' \theta_{yb} \theta_{zb} \\
 & - x \frac{dy}{ds} (\lambda \theta_{xp}) \theta_{yb}' + x \frac{dx}{ds} \theta_{yb} \theta_{yb}' - x h \psi_b \theta_{yb}' - x \frac{dx}{ds} \lambda \theta_{xp} \theta_{yb}' \theta_{zb} - x \frac{dy}{ds} \theta_{yb} \theta_{yb}' \theta_{zb} + \omega \frac{dy}{ds} (\lambda \theta_{xp}) \psi_b' \\
 & - \omega \frac{dx}{ds} \theta_{yb} \psi_b' + \omega h \psi_b \psi_b' + \omega \frac{dx}{ds} \lambda \theta_{xp} \theta_{zb} \psi_b' + \omega \frac{dy}{ds} \theta_{yb} \theta_{zb} \psi_b' + x \frac{dy}{ds} \lambda \theta_{xp} (\lambda \theta_{xp}' \theta_{zb} + \lambda \theta_{xp}' \theta_{zb}') \\
 & - x \frac{dx}{ds} (\lambda \theta_{xp}' \theta_{zb} + \lambda \theta_{xp}' \theta_{zb}') \theta_{yb} + x h (\lambda \theta_{xp}' \theta_{zb} + \lambda \theta_{xp}' \theta_{zb}') \psi_b + x \frac{dx}{ds} \lambda \theta_{xp} (\lambda \theta_{xp}' \theta_{zb} + \lambda \theta_{xp}' \theta_{zb}') \theta_{zb} \\
 & + x \frac{dy}{ds} (\lambda \theta_{xp}' \theta_{zb} + \lambda \theta_{xp}' \theta_{zb}') \theta_{yb} \theta_{zb} + (y - y_c) \frac{dy}{ds} \lambda \theta_{xp} (\theta_{yb}' \theta_{zb} + \theta_{yb}' \theta_{zb}') - (y - y_c) \frac{dx}{ds} (\theta_{yb}' \theta_{zb} + \theta_{yb}' \theta_{zb}') \theta_{yb} \\
 & + (y - y_c) h (\theta_{yb}' \theta_{zb} + \theta_{yb}' \theta_{zb}') \psi_b + (y - y_c) \frac{dx}{ds} \lambda \theta_{xp} (\theta_{yb}' \theta_{zb} + \theta_{yb}' \theta_{zb}') \theta_{zb} \\
 & + (y - y_c) \frac{dy}{ds} (\theta_{yb}' \theta_{zb} + \theta_{yb}' \theta_{zb}') \theta_{yb} \theta_{zb}
 \end{aligned} \tag{A6-1.4}$$

$$\langle 4 \rangle: \frac{\partial n_s^*}{\partial z} \frac{\partial n_s^*}{\partial s} = \left(\lambda v_p' \frac{dx}{ds} - u_b' \frac{dy}{ds} + k \theta_{zb}' \right) \left(\lambda v_p \frac{d^2 x}{ds^2} - u_b \frac{d^2 y}{ds^2} + \theta_{zb} \frac{dk}{ds} \right) \tag{A6-1.5}$$

For a section consisting of linear segments, the expression $\frac{d^2 x}{ds^2}$ and $\frac{d^2 y}{ds^2}$ in the second bracket of

Eq. (A6-1.5) vanish.

$$\frac{\partial n_s^*}{\partial z} \frac{\partial n_s^*}{\partial s} = \left(\frac{dx}{ds} \right) \left(\frac{dk}{ds} \right) \lambda v_p' \theta_{zb} - \left(\frac{dy}{ds} \right) \left(\frac{dk}{ds} \right) u_b' \theta_{zb} + k \left(\frac{dk}{ds} \right) \theta_{zb}' \theta_{zb} \tag{A6-1.6}$$

In summary, the shear strain is expressed as

$$\begin{aligned}
 \gamma_{zs}^* = & \frac{dx}{ds} (u'_b - \theta_{yb} + \lambda \theta_{xp} \theta_{zb}) + \frac{dy}{ds} (\lambda v'_p + \lambda \theta_{xp} + \theta_{yb} \theta_{zb}) + h(\theta_{zb} + \psi_b) + \frac{dy}{ds} (\lambda \theta_{xp}) (\lambda w'_p) - \frac{dx}{ds} (\lambda w'_p) \theta_{yb} \\
 & + h(\lambda w'_p) \psi_b + \frac{dx}{ds} \lambda w'_p \lambda \theta_{xp} \theta_{zb} + \frac{dy}{ds} \lambda w'_p \theta_{yb} \theta_{zb} \\
 & + y \frac{dy}{ds} (\lambda \theta_{xp}) (\lambda \theta'_{xp}) - y \frac{dx}{ds} (\lambda \theta'_{xp}) \theta_{yb} + y h(\lambda \theta'_{xp}) \psi_b + y \frac{dx}{ds} \lambda \theta'_{xp} \lambda \theta_{xp} \theta_{zb} + y \frac{dy}{ds} \lambda \theta'_{xp} \theta_{yb} \theta_{zb} \\
 & - x \frac{dy}{ds} (\lambda \theta_{xp}) \theta'_{yb} + x \frac{dx}{ds} \theta_{yb} \theta'_{yb} - x h \psi_b \theta'_{yb} - x \frac{dx}{ds} \lambda \theta_{xp} \theta'_{yb} \theta_{zb} - x \frac{dy}{ds} \theta_{yb} \theta'_{yb} \theta_{zb} + \omega \frac{dy}{ds} (\lambda \theta_{xp}) \psi'_b \\
 & - \omega \frac{dx}{ds} \theta_{yb} \psi'_b + \omega h \psi_b \psi'_b + \omega \frac{dx}{ds} \lambda \theta_{xp} \theta_{zb} \psi'_b + \omega \frac{dy}{ds} \theta_{yb} \theta_{zb} \psi'_b + x \frac{dy}{ds} \lambda \theta_{xp} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \\
 & - x \frac{dx}{ds} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb} + x h (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \psi_b + x \frac{dx}{ds} \lambda \theta_{xp} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{zb} \\
 & + x \frac{dy}{ds} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb} \theta_{zb} + (y - y_c) \frac{dy}{ds} \lambda \theta_{xp} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) - (y - y_c) \frac{dx}{ds} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{yb} \\
 & + (y - y_c) h (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \psi_b + (y - y_c) \frac{dx}{ds} \lambda \theta_{xp} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{zb} \\
 & + (y - y_c) \frac{dy}{ds} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{yb} \theta_{zb} + \left(\frac{dx}{ds} \right) \left(\frac{dk}{ds} \right) \lambda v'_p \theta_{zb} - \left(\frac{dy}{ds} \right) \left(\frac{dk}{ds} \right) u'_b \theta_{zb} + k \left(\frac{dk}{ds} \right) \theta'_{zb} \theta_{zb}
 \end{aligned} \tag{A6-1.7}$$

The linear terms of the right hand side of Eq. A6-1.7 lead to Eq. (6. 35) while the nonlinear terms lead to Eq. (6.36).

Appendix 6-2 Critical Loads for Columns under Different Boundary Conditions

This appendix presents the detailed procedures of formulating the buckling solution for a column with a doubly symmetric cross-section under an orthogonal coordinate system. For a doubly symmetric section under pure compression ($M_{xp} = 0, V_{yp} = 0, N_p = N$), (no moments and no transverse shear) and under orthogonal coordinates $S_x = 0, S_{x\omega} = 0$. Starting with the variational formulation (Eq. (6.39)) and integrating it by parts, one has

$$\begin{aligned}
 \delta \left(\frac{1}{2} \delta^2 \pi \right) = & \frac{1}{2} \left[GD_{xx} (\delta u'_b - \delta \theta_{yb}) \delta u_b + \lambda N(z) \delta u'_b \right] \delta u_b \Big|_0^L + \frac{1}{2} (EI_{\omega\omega} \delta \psi'_b) \delta \psi_b \Big|_0^L \\
 & + \frac{1}{2} (EI_{yy} \delta \theta'_{yb}) \delta \theta_{yb} \Big|_0^L + \frac{1}{2} \left(GJ \delta \theta'_{zb} + GD_{hh} (\delta \theta'_{zb} + \delta \psi_b) + \lambda N(z) \frac{(I_{xx} + I_{yy})}{A} \delta \theta'_{zb} \right) \delta \theta_{zb} \Big|_0^L \\
 & - \frac{1}{2} \int_0^L [GD_{xx} (\delta u''_b - \delta \theta'_{yb}) + \lambda (N(z) \delta u''_b)] \delta u_b dz + \frac{1}{2} \int_0^L [-EI_{yy} \delta \theta''_{yb} + GD_{xx} (-\delta u'_b + \delta \theta_{yb})] \delta \theta_{yb} dz \\
 & - \frac{1}{2} \int_0^L \left[GJ \delta \theta''_{zb} + GD_{hh} (\delta \theta''_{zb} + \delta \psi'_b) + \lambda \left(N(z) \frac{(I_{xx} + I_{yy})}{A} \delta \theta'_{zb} \right)' \right] \delta \theta_{zb} dz \\
 & + \frac{1}{2} \int_0^L [-EI_{\omega\omega} \delta \psi''_b + GD_{hh} (\delta \theta'_{zb} + \delta \psi_b)] \delta \psi_b dz = 0
 \end{aligned} \tag{A6-2.1}$$

The axial force $N(z)$ is assumed constant, i.e. $N(z) = N$, for a doubly symmetric section and principal centroidal axes, $D_{xh} = \int_A h \left(\frac{dx}{ds} \right) dA = 0$. Under these conditions the variation of the total potential energy is:

$$\begin{aligned}
 \delta \left(\frac{1}{2} \delta^2 \pi \right) = & \frac{1}{2} \left[GD_{xx} (\delta u'_b - \delta \theta_{yb}) \delta u_b + \lambda N \delta u'_b \right] \delta u_b \Big|_0^L + \frac{1}{2} (EI_{\omega\omega} \delta \psi'_b) \delta \psi_b \Big|_0^L + \frac{1}{2} (EI_{yy} \delta \theta'_{yb}) \delta \theta_{yb} \Big|_0^L \\
 & + \frac{1}{2} \left(GJ \delta \theta'_{zb} + GD_{hh} (\delta \theta'_{zb} + \delta \psi_b) + \lambda N r_o^2 \delta \theta'_{zb} \right) \delta \theta_{zb} \Big|_0^L - \frac{1}{2} \int_0^L [GD_{xx} (\delta u''_b - \delta \theta'_{yb}) + \lambda (N \delta u''_b)] \delta u_b dz \\
 & + \frac{1}{2} \int_0^L [-EI_{yy} \delta \theta''_{yb} + GD_{xx} (-\delta u'_b + \delta \theta_{yb})] \delta \theta_{yb} dz - \frac{1}{2} \int_0^L [GJ \delta \theta''_{zb} + GD_{hh} (\delta \theta''_{zb} + \delta \psi'_b) + \lambda N r_o^2 \delta \theta''_{zb}] \delta \theta_{zb} dz \\
 & + \frac{1}{2} \int_0^L [-EI_{\omega\omega} \delta \psi''_b + GD_{hh} (\delta \theta'_{zb} + \delta \psi_b)] \delta \psi_b dz = 0
 \end{aligned} \tag{A6-2.2}$$

in which $r_0^2 = (I_{xx} + I_{yy})/A$

A6-2.1 Flexural Buckling of Columns

The field terms related to flexural buckling are

$$\begin{aligned} -(GD_{xx} + N)u'' + GD_{xx}\theta' &= 0 \\ -GD_{xx}u' + (GD_{xx}\theta_y - EI_{yy}\theta_y'') &= 0 \end{aligned} \quad (\text{A6-2.3})$$

assuming the solution

$$u = Ce^{mz}, \theta = De^{mz} \quad (\text{A6-2.4a,b})$$

and substituting Eqs. A6-2.4 into the field equations (Eq. A6-2.3), one obtains

$$\begin{aligned} -(GD_{xx} + N)m^2C + (GD_{xx}m)D &= 0 \\ -(GD_{xx}m)C + (GD_{xx} - EI_{yy}m^2)D &= 0 \end{aligned}$$

In a matrix form

$$\begin{bmatrix} -(GD_{xx} + N)m^2 & (GD_{xx}m) \\ -(GD_{xx}m) & (GD_{xx} - EI_{yy}m^2) \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{A6-2.5})$$

The determinant of Eq. A6-2.5 has to vanish for a non-zero solution. This gives

$$\begin{aligned} (GD_{xx} - EI_{yy}m^2)(GD_{xx} + N)m^2 - (GD_{xx}m)^2 &= 0 \\ m^2[GD_{xx}N - EI_{yy}(GD_{xx} + N)m^2] &= 0 \end{aligned}$$

The four roots for m are

$$m_1 = m_2 = 0, \quad \text{or} \quad (\text{A6-2.6a,b})$$

$$GD_{xx}N - EI_{yy}(GD_{xx} + N)m^2 = 0 \quad \Rightarrow$$

$$m_{3,4} = \pm i\alpha = \pm i \sqrt{\frac{-GD_{xx}N}{EI_{yy}(GD_{xx} + N)}} \quad (\text{A6-2.6c,d})$$

and the displacement fields take the following form:

$$u(z) = C_1 + C_2z + C_3 \sin \alpha z + C_4 \cos \alpha z \quad (\text{A6-2.7a})$$

$$\theta(z) = D_1 + D_2z + D_3 \sin \alpha z + D_4 \cos \alpha z \quad (\text{A6-2.7b})$$

in which constants C_i and D_i are related. In the following, the relationship between constants C_i and D_i will be established. Substituting the displacement field expressions (Eq. A6-2.7a,b) into Eq. A6-2.3, yields

$$GD_{xx}[D_2 + D_3\alpha \cos \alpha z - \alpha D_4 \sin \alpha z] + \alpha^2 (GD_{xx} + N)(C_3 \sin \alpha z + C_4 \cos \alpha z) = 0 \quad (A6-2.8)$$

Since the above relation is to be valid for any z , the following three conditions must be met.

$$\boxed{D_2 = 0} \quad (A6-2.9)$$

$$[GD_{xx}D_3\alpha + \alpha^2 (GD_{xx} + N)C_4] \cos \alpha z = 0 \quad \Rightarrow \quad D_3 = -\alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \quad (A6-2.10)$$

$$[-GD_{xx}D_4\alpha + \alpha^2 (GD_{xx} + N)C_3] \sin \alpha z = 0 \quad \Rightarrow \quad D_4 = \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_3 \quad (A6-2.11)$$

The displacement fields can now be re-written as:

$$u(z) = C_1 + C_2 z + C_3 \sin \alpha z + C_4 \cos \alpha z \quad (A6-2.12)$$

$$\theta_y(z) = D_1 - \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \sin \alpha z + \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_3 \cos \alpha z \quad (A6-2.13)$$

It is now required to relate constant D_1 to constants C_i . This is achieved by substituting Eq. A6-2.13 into the second differential equation, yielding

$$\begin{aligned} & -GD_{xx}(C_2 + C_3\alpha \cos \alpha z - C_4\alpha \sin \alpha z) \\ & + GD_{xx} \left(D_1 - \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \sin \alpha z + \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_3 \cos \alpha z \right) \\ & - EI_{yy} \left(\alpha^3 \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \sin \alpha z - \alpha^3 \frac{(GD_{xx} + N)}{GD_{xx}} C_3 \cos \alpha z \right) = 0 \end{aligned}$$

Noting that the above relation should be valid for any z , the following three conditions have to be met:

$$GD_{xx}D_1 - GD_{xx}C_2 = 0 \quad \Rightarrow \quad D_1 = C_2 \quad (A6-2.14)$$

which provides the relationship required, and

$$\left(-\alpha (GD_{xx} + N) - \alpha^3 \frac{EI_{yy} (GD_{xx} + N)}{GD_{xx}} + GD_{xx} \alpha \right) C_4 \sin \alpha z = 0$$

The bracketed term can be shown to vanish since

$$\left(-\alpha (GD_{xx} + N) - \alpha^3 \frac{EI_{yy} (GD_{xx} + N)}{GD_{xx}} + GD_{xx} \alpha \right) = 0$$

Also,

$$\alpha \left(-GD_{xx} + (GD_{xx} + N) - EI_{yy} \left(-\alpha^2 \frac{(GD_{xx} + N)}{GD_{xx}} \right) \right) C_3 \cos \alpha z = 0$$

Again, the bracketed term can be shown to vanish.

In summary,

$$u(z) = C_1 + C_2 z + C_3 \sin \alpha z + C_4 \cos \alpha z \quad (\text{A6-2.15a})$$

$$\theta_y(z) = C_2 - \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \sin \alpha z + \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_3 \cos \alpha z \quad (\text{A6-2.15b})$$

The four integration constants C_i are to be determined from the relevant boundary conditions appearing in expression of the variation of total potential energy. These are

$$\left\{ \left[GD_{xx} (u' - \theta_y) + Nu' \right] \delta u \right\} \Big|_0^L = 0 \quad (\text{A6-2.16a})$$

$$\left\{ \left[(EI_{yy} \theta_y') \delta \theta_y \right] \right\} \Big|_0^L = 0 \quad (\text{A6-2.16b})$$

A6-2.1.1 Example 1: Flexural Buckling of Columns Simply Supported at Both Ends

The relevant boundary conditions are:

$$u(0) = u(L) = 0, \theta'(0) = \theta'(L) = 0 \quad (\text{A6-2.17})$$

From the displacement field expressions Eqs. (A6-2.15a and b), by substituting into the boundary condition (A6-2.17) of the problem, one obtains

$$\theta_y'(z) = -\alpha^2 \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \cos \alpha z - \alpha^2 \frac{(GD_{xx} + N)}{GD_{xx}} C_3 \sin \alpha z$$

$$\theta_y'(0) = -\alpha^2 \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \Rightarrow \boxed{C_4 = 0} \quad (\text{A6-2.18a})$$

$$u(0) = C_1 + C_4 = 0 \Rightarrow \boxed{C_1 = 0} \quad (\text{A6-2.18b})$$

$$\theta_y'(L) = -\alpha^2 \frac{(GD_{xx} + N)}{GD_{xx}} C_3 \sin \alpha L = 0 \quad (\text{A6-2.18c})$$

$$u(L) = C_2 L + C_3 \sin \alpha L \quad (\text{A6-2.18d})$$

From Eq. (A6-2.18c), for a non-trivial solution, $C_3 \neq 0$ and $\sin \alpha L = 0 \Rightarrow \alpha L = \pi$, then

$$\begin{aligned}
 \frac{-GD_{xx}NL^2}{EI_{yy}(GD_{xx}+N)} &= \pi^2 \\
 -GD_{xx}NL^2 &= \pi^2 EI_{yy}GD_{xx} + \pi^2 EI_{yy}N \\
 -N(GD_{xx}L^2 + \pi^2 EI_{yy}) &= \pi^2 EI_{yy}GD_{xx} \\
 N = -\frac{\pi^2 EI_{yy}GD_{xx}}{(GD_{xx}L^2 + \pi^2 EI_{yy})}, &\quad \Rightarrow \quad N = -\frac{\pi^2 EI_{yy}}{L^2 \left(1 + \frac{\pi^2 EI_{yy}}{GD_{xx}L^2}\right)} \quad (A6-2.19)
 \end{aligned}$$

One recalls that since N is positive when tensile, the buckling compressive force is negative. Also, as the column span L increases, one recovers the classical solution

$$1 + \frac{\pi^2 EI_{yy}}{GD_{xx}L^2} \rightarrow 1, \quad \Rightarrow \quad N = -\frac{\pi^2 EI_{yy}}{L^2} \quad (A6-2.20)$$

A6-2.1.2 Example 2: Flexural Buckling of Column Fixed at $z = 0$, and free at $z = L$

The boundary conditions are

$$\begin{aligned}
 u(0) &= 0, \quad \theta_y(0) = 0, \\
 GD_{xx}(u'(L) - \theta_y(L)) + Nu'(L) &= 0, \quad \theta_y'(L) = 0 \quad (A6-2.21)
 \end{aligned}$$

From the displacement field equations Eqs. (A6-2.15a and b), substitute into the boundary conditions (Eq. (A6-2.21)) yields,

$$\begin{aligned}
 u'(z) &= C_2 + \alpha C_3 \cos \alpha z - \alpha C_4 \sin \alpha z \\
 \theta_y(z) &= C_2 - \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \sin \alpha z + \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_3 \cos \alpha z \quad (A6-2.22)
 \end{aligned}$$

$$\begin{aligned}
 GD_{xx}(u'(z) - \theta_y(z)) + Nu'(z) &= \\
 C_2 N + \alpha C_3 [GD_{xx} - (GD_{xx} + N) + N] \cos \alpha z + \alpha C_4 [(GD_{xx} + N) - GD_{xx} - N] \sin \alpha z &= C_2 N \\
 GD_{xx}(u'(L) - \theta_y(L)) + Nu'(L) = 0 &\Rightarrow \boxed{C_2 = 0}
 \end{aligned}$$

$$\theta_y(0) = C_2 + \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_3 = 0$$

$$C_2 + \alpha \frac{(GD_{xx} + N)}{GD_{xx}} C_3 = 0 \Rightarrow \boxed{C_3 = 0}$$

$$u(0) = C_1 + C_4 = 0 \Rightarrow \boxed{C_1 = -C_4}$$

$$\theta'_y(z) = -\alpha^2 \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \cos \alpha z - \alpha^2 \frac{(GD_{xx} + N)}{GD_{xx}} \cancel{C_3} \sin \alpha z$$

$$\theta'_y(L) = -\alpha^2 \frac{(GD_{xx} + N)}{GD_{xx}} C_4 \cos \alpha L = 0$$

For a non-trivial solution,

$$\cos \alpha L = 0 \Rightarrow \alpha L = \frac{\pi}{2}$$

$$\frac{-GD_{xx}NL^2}{EI_{yy}(GD_{xx} + N)} = \frac{\pi^2}{4}$$

$$-GD_{xx}NL^2 = \frac{\pi^2}{4} EI_{yy}GD_{xx} + \frac{\pi^2}{4} EI_{yy}N$$

$$-N \left(GD_{xx}L^2 + \frac{\pi^2}{4} EI_{yy} \right) = \frac{\pi^2}{4} EI_{yy}GD_{xx}$$

$$N = -\frac{\frac{\pi^2}{4} EI_{yy}GD_{xx}}{\left(GD_{xx}L^2 + \frac{\pi^2}{4} EI_{yy} \right)} \Rightarrow \boxed{N = -\frac{\pi^2 EI_{yy}}{4L^2 \left(1 + \frac{\pi^2 EI_{yy}}{4GD_{xx}L^2} \right)}} \quad (\text{A6-2.23})$$

The solution approaches the classical solution as L increases

$$1 + \frac{\pi^2 EI_{yy}}{4GD_{xx}L^2} \rightarrow 1 \Rightarrow N = -\frac{\pi^2 EI_{yy}}{4L^2} \quad (\text{A6-2.24})$$

A6-2.2 Torsional Buckling of Columns

The relevant field equations are

$$(GJ + r_0^2 N + GD_{hh})\theta'_z + GD_{hh}\psi' = 0 \quad (\text{A6-2.25a})$$

$$GD_{hh}\theta'_z + (GD_{hh}\psi - EI_{\omega\omega}\psi'') = 0 \quad (\text{A6-2.25b})$$

Assuming exponential field solutions

$$\theta_z = Ce^{mz}, \psi = De^{mz} \quad (\text{A6-2.26a,b})$$

and substituting the solution into the field equations, one obtains

$$C(GJ + r_0^2 N + GD_{hh})m^2 + D(GD_{hh})m = 0$$

$$C(GD_{hh})m + D(GD_{hh} - EI_{\omega\omega}m^2) = 0$$

In a matrix form, one has

$$\begin{bmatrix} (GJ + r_0^2 N + GD_{hh})m^2 & (GD_{hh}m) \\ (GD_{hh}m) & (GD_{hh} - EI_{\omega\omega}m^2) \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The determinant has to vanish for a non-zero solution. This gives

$$(GJ + r_0^2 N + GD_{hh})m^2 (GD_{hh} - EI_{\omega\omega}m^2) - (GD_{hh}m)^2 = 0$$

$$\left[(GJ + r_0^2 N)GD_{hh} \right] m^2 - (GJ + r_0^2 N + GD_{hh})EI_{\omega\omega}m^4 = 0$$

$$m^2 \left[(GJ + r_0^2 N)GD_{hh} - (GJ + r_0^2 N + GD_{hh})EI_{\omega\omega}m^2 \right] = 0$$

The four roots for m are

$$m_{1,2} = 0 \quad (\text{A6-2.27a})$$

$$m_{3,4} = \pm i\beta = \pm i \sqrt{\frac{-(GJ + r_0^2 N)GD_{hh}}{(GJ + r_0^2 N + GD_{hh})EI_{\omega\omega}}} \quad (\text{A6-2.27b})$$

and the displacement fields take the following form

$$\theta_z(z) = C_1 + C_2 z + C_3 \sin \beta z + C_4 \cos \beta z \quad (\text{A6-2.28a})$$

$$\psi(z) = D_1 + D_2 z + D_3 \sin \beta z + D_4 \cos \beta z \quad (\text{A6-2.28b})$$

in which constants C_i and D_i are related. In the following the relationship between constants C_i and D_i will be determined by substituting the displacement field expressions into the first differential equation (Eq. (A6-2.25a)) yielding

$$-\beta^2 (GJ + r_0^2 N + GD_{hh}) (C_3 \sin \beta z + C_4 \cos \beta z) + GD_{hh} (D_2 + D_3 \beta \cos \beta z - D_4 \beta \sin \beta z) = 0$$

The above relation should be valid for any z , the following three conditions have to be met.

$$\boxed{D_2 = 0} \quad (\text{A6-2.29a})$$

$$\left[-\beta^2 (GJ + r_0^2 N + GD_{hh}) C_4 + GD_{hh} \beta D_3 \right] \cos \beta z = 0 \Rightarrow \boxed{D_3 = \frac{\beta (GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_4} \quad (\text{A6-2.29b})$$

$$\left[-\beta^2 (GJ + r_0^2 N + GD_{hh}) C_3 - GD_{hh} D_4 \beta \right] \sin \beta z = 0 \Rightarrow \boxed{D_4 = -\frac{\beta (GJ + r_0^2 N + GD_{hh}) C_3}{GD_{hh}}} \quad (\text{A6-2.29c})$$

The displacement fields can now be re-written as:

$$\theta_z(z) = C_1 + C_2 z + C_3 \sin \beta z + C_4 \cos \beta z \quad (\text{A6-2.30a})$$

$$\psi(z) = D_1 + \frac{\beta(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_4 \sin \beta z - \frac{\beta(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_3 \cos \beta z \quad (\text{A6-2.30b})$$

It is now required to relate constant D_1 to constants C_i . This is achieved by substituting the above two expressions into the second differential equation, yielding

$$\begin{aligned} & GD_{hh} \theta'_z + (GD_{hh} \psi - EI_{\omega\omega} \psi'') = 0 \\ & GD_{hh} (C_2 + C_3 \beta \cos \beta z - C_4 \beta \sin \beta z) \\ & + GD_{hh} \left(D_1 + \frac{\beta(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_4 \sin \beta z - \frac{\beta(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_3 \cos \beta z \right) \\ & - EI_{\omega\omega} \left(-\frac{\beta^3(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_4 \sin \beta z + \frac{\beta^3(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_3 \cos \beta z \right) = 0 \end{aligned}$$

Rearranging, one obtains

$$\begin{aligned} & [GD_{hh} C_2 + GD_{hh} D_1] \\ & + \left[GD_{hh} - GD_{hh} \frac{(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} - EI_{\omega\omega} \frac{\beta^2(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} \right] C_3 \beta \cos \beta z \\ & + \left[-GD_{hh} + GD_{hh} \frac{(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} + EI_{\omega\omega} \frac{\beta^2(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} \right] C_4 \beta \sin \beta z = 0 \end{aligned}$$

Noting that the above relation should be valid for any z , the three bracketed terms have to vanish, i.e.,

$$[GD_{xx} D_1 + GD_{xx} C_2] = 0 \Rightarrow [D_1 = -C_2]$$

which provides the remaining relationship, and the second and third bracketed terms can be shown to vanish.

In summary,

$$\theta_z(z) = C_1 + C_2 z + C_3 \sin \beta z + C_4 \cos \beta z \quad (\text{A6-2.31a})$$

$$\psi(z) = -C_2 + \frac{\beta(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_4 \sin \beta z - \frac{\beta(GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_3 \cos \beta z \quad (\text{A6-2.31b})$$

The four integration constants C_i are to be determined from the relevant boundary conditions appearing in expression of the variation of total potential energy

$$\left[\left(G \left(J \theta'_z + D_{hh} (\theta'_z + \psi') \right) + r_0^2 N \theta'_z \right) \delta \theta_z \right]_0^L = 0 \quad (\text{A6-2.32a})$$

$$\left[(EI_{\omega\omega} \psi') \delta \psi \right]_0^L = 0 \quad (\text{A6-2.32b})$$

A6-2.2.1 Example 3: Torsional Buckling of a Column Simply Supported at Both Ends

The relevant boundary conditions are:

$$\theta_z(0) = \theta_z(L) = 0, \psi'(0) = \psi'(L) = 0 \quad (\text{A6-2.33})$$

From the displacement field expressions (Eqs. A6-2.31a and b), by substituting into the boundary condition of the problem, one obtains

$$\begin{aligned} \psi'(z) &= \frac{\beta^2 (GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_4 \cos \beta z + \frac{\beta^2 (GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_3 \sin \beta z \\ \psi'(0) &= \frac{\beta^2 (GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_4 \Rightarrow \boxed{C_4 = 0} \end{aligned} \quad (\text{A6-2.34a})$$

$$\begin{aligned} \theta_z(z) &= C_1 + C_2 z + C_3 \sin \beta z + C_4 \cos \beta z \\ \theta_z(0) &= C_1 + C_4 = 0 \Rightarrow \boxed{C_1 = 0} \end{aligned} \quad (\text{A6-2.34b})$$

$$\begin{aligned} \theta_z(z) &= C_2 z + C_3 \sin \beta z \\ \psi'(z) &= \frac{\beta^2 (GJ + r_0^2 N + GD_{hh})}{GD_{hh}} C_3 \sin \beta z \\ \theta_z(L) &= C_2 L + C_3 \sin \beta L = 0 \quad (\text{A6-2.34c}) \\ \psi'(L) &= C_3 \sin \beta L \quad (\text{A6-2.34d}) \end{aligned}$$

Subtracting the last two equations (Eqs. A6-2.34c and d) obtains $\boxed{C_2 = 0}$

The non-trivial solution is obtained from Eq. (A6-2.34d) as

$$\begin{aligned} \sin \beta L = 0 &\Rightarrow \beta L = \pi \\ \frac{-(GJ + r_0^2 N) GD_{hh} L^2}{(GJ + r_0^2 N + GD_{hh}) EI_{\omega\omega}} &= \pi^2 \\ -(GJ + r_0^2 N) GD_{hh} L^2 &= \pi^2 EI_{\omega\omega} (GJ + r_0^2 N + GD_{hh}) \\ -GJGD_{hh} L^2 - r_0^2 NGD_{hh} L^2 &= \pi^2 EI_{\omega\omega} G(J + D_{hh}) + \pi^2 EI_{\omega\omega} r_0^2 N \end{aligned}$$

$$\begin{aligned}
 -r_0^2 N (GD_{hh} L^2 + \pi^2 EI_{\omega\omega}) &= \pi^2 EI_{\omega\omega} G (J + D_{hh}) + GJGD_{hh} L^2 \\
 -r_0^2 N &= \frac{\pi^2 EI_{\omega\omega} G (J + D_{hh}) + GJGD_{hh} L^2}{GD_{hh} L^2 + \pi^2 EI_{\omega\omega}} = \frac{\frac{\pi^2 EI_{\omega\omega}}{L^2} \left(\frac{J}{D_{hh}} + 1 \right) + GJ}{1 + \frac{\pi^2 EI_{\omega\omega}}{GD_{hh} L^2}} \\
 \frac{-r_0^2 N}{GJ} &= \frac{\left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) \left(\frac{J}{D_{hh}} \right) + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) + 1}{1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) \left(\frac{J}{D_{hh}} \right)} \\
 &= \frac{1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) \left(\frac{J}{D_{hh}} \right)}{1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) \left(\frac{J}{D_{hh}} \right)} + \frac{\left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right)}{1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) \left(\frac{J}{D_{hh}} \right)} = 1 + \frac{\left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right)}{1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) \left(\frac{J}{D_{hh}} \right)} \\
 \frac{r_0^2 N}{GJ} &= - \left[1 + \frac{\left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right)}{1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) \left(\frac{J}{D_{hh}} \right)} \right] \tag{A6-2.35}
 \end{aligned}$$

This compares to the classical solution

$$N = - \left(\frac{1}{r_0^2} \right) \left[\frac{\pi^2 EI_{\omega\omega}}{L^2} + GJ \right] \Rightarrow Nr_0^2 = - \left[GJ + \frac{\pi^2 EI_{\omega\omega}}{L^2} \right]$$

Since $\left[1 + \frac{\left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right)}{1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) \left(\frac{J}{D_{hh}} \right)} \right] < 1 + \left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right)$, the buckling load based on the present

solution is always lower than the one based on the classical solution. Also, as L increases,

$$\left(\frac{\pi^2 EI_{\omega\omega}}{GJL^2} \right) \rightarrow 0, \text{ both solutions converge to the solution } \frac{r_0^2 N}{GJ} = -1 \Rightarrow N = - \frac{GJ}{r_0^2}.$$

Appendix 6-3 Critical Moment for Simply Supported Beam under Uniform Moment

This appendix presents the procedure of obtaining the general solutions of the governing equations (Eq. 6.41 to 6.44) for a simply supported beam under pure bending ($N_p = 0, V_{yp} = 0, M_{xp} = M_x$).

Equations 6.41 to 6.44 can be expressed in a matrix form:

$$\begin{bmatrix} -GD_{xx}D^2 & GD_{xx}D & M_xD^2 & 0 \\ -GD_{xx}D & GD_{xx}-EI_{yy}D^2 & 0 & 0 \\ -M_xD^2 & 0 & G(J+D_{hh})D^2 & GD_{hh}D \\ 0 & 0 & -GD_{hh}D & EI_{\omega\omega}D^2-GD_{hh} \end{bmatrix} \begin{Bmatrix} u \\ \theta_y \\ \theta_z \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (\text{A6-3.1})$$

in which $D = \partial/\partial z, D^2 = \partial^2/\partial^2 z$.

Assuming the displacement fields take the form

$$u = Ae^{mz}, \quad \theta_y = Be^{mz}, \quad \theta_z = Ce^{mz}, \quad \psi = De^{mz} \quad (\text{A6-3.2})$$

the differential equations (Eqs. 6.49 to 6.52) can be expressed as the functions of m and coefficients A, B, C and D as

$$\begin{bmatrix} -GD_{xx}m^2 & GD_{xx}m & M_xm^2 & 0 \\ -GD_{xx}m & GD_{xx}-EI_{yy}m^2 & 0 & 0 \\ -M_xm^2 & 0 & G(J+D_{hh})m^2 & GD_{hh}m \\ 0 & 0 & -GD_{hh}m & EI_{\omega\omega}m^2-GD_{hh} \end{bmatrix} \begin{Bmatrix} Ae^{mz} \\ Be^{mz} \\ Ce^{mz} \\ De^{mz} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (\text{A6-3.3})$$

For the left hand side of Eq. A6-3.3 to vanish, either all of the constants A, B, C and D should vanish or the determinant of the matrix must be zero. Setting $A = B = C = D = 0$ leads to a trivial solution which is of no interest. Therefore, the determinant of the matrix must vanish, i.e.,

$$\begin{vmatrix} -GD_{xx}m^2 & GD_{xx}m & M_xm^2 & 0 \\ -GD_{xx}m & GD_{xx}-EI_{yy}m^2 & 0 & 0 \\ -M_xm^2 & 0 & G(J+D_{hh})m^2 & GD_{hh}m \\ 0 & 0 & -GD_{hh}m & EI_{\omega\omega}m^2-GD_{hh} \end{vmatrix} = 0 \quad (\text{A6-3.4})$$

Expanding the determinant (Eq. A6-3.4) results in an 8th order polynomial in m ,

$$(G^2 D_{xx} J E^2 I_{yy} I_{\omega\omega} + G^2 D_{xx} D_{hh} E^2 I_{yy} I_{\omega\omega} - \bar{M}_x^2 E^2 I_{yy} I_{\omega\omega}) m^8 + (-G^3 D_{xx} D_{hh} J E I_y + \bar{M}_x^2 E I_{yy} G D_{hh} + \bar{M}_x^2 G D_{xx} E I_{\omega\omega}) m^6 - \bar{M}_x^2 G^2 D_{xx} D_{hh} m^4 = 0 \quad (\text{A6-3.5})$$

which has 4 zero roots. The remaining four roots are from the solutions of the 4th order polynomial.

$$[(G^2 D_{xx} J E^2 I_{yy} I_{\omega\omega} + G^2 D_{xx} D_{hh} E^2 I_{yy} I_{\omega\omega} - \bar{M}_x^2 E^2 I_{yy} I_{\omega\omega}) m^4 + (-G^3 D_{xx} D_{hh} J E I_y + \bar{M}_x^2 E I_{yy} G D_{hh} + \bar{M}_x^2 G D_{xx} E I_{\omega\omega}) m^2 - \bar{M}_x^2 G^2 D_{xx} D_{hh}] = 0 \quad (\text{A6-3.6})$$

Furthermore, the 4th order equation (Eq. A6-3.6) can be converted into a second order equation with the substitution $n = m^2$.

$$an^2 + bn + c = 0 \quad (\text{A6-3.7})$$

where

$$\begin{aligned} a &= E^2 I_{yy} I_{\omega\omega} (G^2 D_{xx} J + G^2 D_{xx} D_{hh} - \bar{M}_x^2) \\ b &= GE (-G^2 D_{xx} D_{hh} J I_y + \bar{M}_x^2 I_{yy} D_{hh} + \bar{M}_x^2 D_{xx} I_{\omega\omega}) \\ c &= -\bar{M}_x^2 G^2 D_{xx} D_{hh} \end{aligned} \quad (\text{A6-3.8})$$

Solving Eq. A6-3.7 yields 4 roots, and they are

$$m_{5,6,7,8} = \pm \sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \quad (\text{A6-3.9})$$

Four of the roots of Eq. A6-3.4 are $m_1 = m_2 = m_3 = m_4 = 0$. The solutions of the differential equations (Eqs. 6.49 to 6.52) become

$$u = A_1 + A_2 z + A_3 z^2 + A_4 z^3 + A_5 e^{m_5 z} + A_6 e^{m_6 z} + A_7 e^{m_7 z} + A_8 e^{m_8 z} \quad (\text{A6-3.10})$$

$$\theta_y = B_1 + B_2 z + B_3 z^2 + B_4 z^3 + B_5 e^{m_5 z} + B_6 e^{m_6 z} + B_7 e^{m_7 z} + B_8 e^{m_8 z} \quad (\text{A6-3.11})$$

$$\theta_z = C_1 + C_2 z + C_3 z^2 + C_4 z^3 + C_5 e^{m_5 z} + C_6 e^{m_6 z} + C_7 e^{m_7 z} + C_8 e^{m_8 z} \quad (\text{A6-3.12})$$

$$\psi = D_1 + D_2 z + D_3 z^2 + D_4 z^3 + D_5 e^{m_5 z} + D_6 e^{m_6 z} + D_7 e^{m_7 z} + D_8 e^{m_8 z} \quad (\text{A6-3.13})$$

Substituting the solutions (Eq. A6-3.10 to A6-3.13) and their derivatives into the differential equations (Eq. A6-3.1) some of the coefficients can be eliminated because the determinant is equal to zero, these differential equations are not independent.

$$\begin{aligned}
 & -GD_{xx} \left(2A_3 + 6A_4z + A_5m_5^2e^{m_5z} + A_6m_6^2e^{m_6z} + A_7m_7^2e^{m_7z} + A_8m_8^2e^{m_8z} \right) \\
 & +GD_{xx} \left(B_2 + 2B_3z + 3B_4z^2 + B_5m_5e^{m_5z} + B_6m_6e^{m_6z} + B_7m_7e^{m_7z} + B_8m_8e^{m_8z} \right) \\
 & +\bar{M}_x \left(2C_3 + 6C_4z + C_5m_5^2e^{m_5z} + C_6m_6^2e^{m_6z} + C_7m_7^2e^{m_7z} + C_8m_8^2e^{m_8z} \right) = 0
 \end{aligned} \tag{A6-3.14}$$

$$\begin{aligned}
 & -EI_{yy} \left(2B_3 + 6B_4z + B_5m_5^2e^{m_5z} + B_6m_6^2e^{m_6z} + B_7m_7^2e^{m_7z} + B_8m_8^2e^{m_8z} \right) \\
 & -GD_{xx} \left(A_2 + 2A_3z + 3A_4z^2 + A_5m_5e^{m_5z} + A_6m_6e^{m_6z} + A_7m_7e^{m_7z} + A_8m_8e^{m_8z} \right) \\
 & +GD_{xx} \left(B_1 + B_2z + B_3z^2 + B_4z^3 + B_5e^{m_5z} + B_6e^{m_6z} + B_7e^{m_7z} + B_8e^{m_8z} \right) = 0
 \end{aligned} \tag{A6-3.15}$$

$$\begin{aligned}
 & G(J + D_{hh}) \left(2C_3 + 6C_4z + C_5m_5^2e^{m_5z} + C_6m_6^2e^{m_6z} + C_7m_7^2e^{m_7z} + C_8m_8^2e^{m_8z} \right) \\
 & +GD_{hh} \left(D_2 + 2D_3z + 3D_4z^2 + D_5m_5e^{m_5z} + D_6m_6e^{m_6z} + D_7m_7e^{m_7z} + D_8m_8e^{m_8z} \right) \\
 & -\bar{M}_x(z) \left(2A_3 + 6A_4z + A_5m_5^2e^{m_5z} + A_6m_6^2e^{m_6z} + A_7m_7^2e^{m_7z} + A_8m_8^2e^{m_8z} \right) = 0
 \end{aligned} \tag{A6-3.16}$$

$$\begin{aligned}
 & EI_{\omega\omega} \left(2D_3 + 6D_4z + D_5m_5^2e^{m_5z} + D_6m_6^2e^{m_6z} + D_7m_7^2e^{m_7z} + D_8m_8^2e^{m_8z} \right) \\
 & -GD_{hh} \left(C_2 + 2C_3z + 3C_4z^2 + C_5m_5e^{m_5z} + C_6m_6e^{m_6z} + C_7m_7e^{m_7z} + C_8m_8e^{m_8z} \right) \\
 & +GD_{hh} \left(D_1 + D_2z + D_3z^2 + D_4z^3 + D_5e^{m_5z} + D_6e^{m_6z} + D_7e^{m_7z} + D_8e^{m_8z} \right) = 0
 \end{aligned} \tag{A6-3.17}$$

If Eqs. A6-3.14 to A6-3.17 must be valid for any value of z , the coefficients of z^0, z^1, z^2, z^3 and $e^{m_i z}$ have to vanish, i.e.,

from Eq. A6-3.14, one has

$$-2GD_{xx}A_3 + GD_{xx}B_2 + 2M_xC_3 = 0 \tag{A6-3.18a}$$

$$(-6GD_{xx}A_4 + 2GD_{xx}B_3 + 6M_xC_4)z = 0 \tag{A6-3.18b}$$

$$3GD_{xx}B_4z^2 = 0 \tag{A6-3.18c}$$

$$(-GD_{xx}m_i^2A_i + GD_{xx}m_iB_i + M_xm_i^2C_i)e^{m_iz} = 0, \quad i=5, 6, 7, 8 \tag{A6-3.18d}$$

also, from Eq. A6-3.15, one has

$$-2EI_{yy}B_3 - GD_{xx}A_2 + GD_{xx}B_1 = 0 \tag{A6-3.19a}$$

$$(-6EI_{yy}B_4 - 2GD_{xx}A_3 + GD_{xx}B_2)z = 0 \tag{A6-3.19b}$$

$$(-3GD_{xx}A_4 + GD_{xx}B_3)z^2 = 0 \quad (\text{A6-3.19c})$$

$$GD_{xx}B_4z^3 = 0 \quad (\text{A6-3.19d})$$

$$(-GD_{xx}m_iA_i - EI_{yy}m_i^2B_i + GD_{xx}B_i)e^{m_i z} = 0, \quad i=5, 6, 7, 8 \quad (\text{A6-3.19e})$$

Equation A6-3.16 gives,

$$-2G(J + D_{hh})C_3 - GD_{hh}D_2 + 2M_xA_3 = 0 \quad (\text{A6-3.20a})$$

$$(-6G(J + D_{hh})C_4 - 2GD_{hh}D_3 + 6M_xA_4)z = 0 \quad (\text{A6-3.20b})$$

$$-3GD_{hh}D_4z^2 = 0 \quad (\text{A6-3.20c})$$

$$(-G(J + D_{hh})m_i^2C_i - GD_{hh}m_iD_i + M_xm_i^2A_i)e^{m_i z} = 0, \quad i=5, 6, 7, 8 \quad (\text{A6-3.20d})$$

and Equation A6-3.17 yields,

$$-2EI_{\omega\omega}D_3 + GD_{hh}C_2 + GD_{hh}D_1 = 0 \quad (\text{A6-3.21a})$$

$$(-6EI_{\omega\omega}D_4 + 2GD_{hh}C_3 + GD_{hh}D_2)z = 0 \quad (\text{A6-3.21b})$$

$$(3GD_{hh}C_4 + GD_{hh}D_3)z^2 = 0 \quad (\text{A6-3.21c})$$

$$GD_{hh}D_4z^3 = 0 \quad (\text{A6-3.21d})$$

$$(-EI_{\omega\omega}m_i^2D_i + GD_{hh}m_iC_i + GD_{hh}D_i)e^{m_i z} = 0, \quad i=5, 6, 7, 8 \quad (\text{A6-3.21e})$$

Equations A6-3.18, A6-3.19, A6-3.20 and A6-3.21 lead to

$$\text{Eqs. A6-3.c, A6-3.19d} \Rightarrow B_4 = 0 \quad (\text{A6-3.22})$$

$$\text{Eqs. A6-3.20c and A6-3.21d} \Rightarrow D_4 = 0 \quad (\text{A6-3.23})$$

$$\text{Eq. A6-3.29c} \Rightarrow B_3 = 3A_4 \quad (\text{A6-3.24})$$

$$\text{Substituting Eq. A6-3.24 into Eq. A6-3.18b} \Rightarrow C_4 = 0 \quad (\text{A6-3.25})$$

$$\text{Eq. A6-3.19b} \Rightarrow B_2 = 2A_3 \quad (\text{A6-3.26})$$

$$\text{Inserting Eq. A6-3.26 into Eq. A6-3.18a} \Rightarrow C_3 = 0 \quad (\text{A6-3.27})$$

$$\text{Substituting Eq. A6-3.25 into Eq. A6-3.21c} \Rightarrow D_3 = 0 \quad (\text{A6-3.28})$$

$$\text{Substituting Eqs. A6-3.25 and A6-3.28 into Eq. A6-3.20b} \Rightarrow A_4 = 0 \quad (\text{A6-3.29})$$

$$\text{Substituting Eq. A6-3.29 into Eq. A6-3.24} \Rightarrow B_3 = 0 \quad (\text{A6-3.30})$$

$$\text{Substituting Eq. A6-3.30 into Eq. A6-3.19a} \Rightarrow B_1 = A_2 \quad (\text{A6-3.31})$$

$$\text{Substituting Eqs. A6-3.23 and A6-3.27 into Eq. A6-3.21b} \Rightarrow D_2 = 0 \quad (\text{A6-3.32})$$

$$\text{Substituting Eq. A6-3.28 into Eq. A6-3.21a} \Rightarrow D_1 = C_2 \quad (\text{A6-3.33})$$

$$\text{Substituting Eqs. A6-3.27 and A6-3.32 into Eq. A6-3.20a} \Rightarrow A_3 = 0 \quad (\text{A6-3.34})$$

$$\text{Substituting Eq. A6-3.34 into Eq. A6-3.26} \Rightarrow B_2 = 0 \quad (\text{A6-3.35})$$

In summary, the coefficients are expressed in terms of A_i and C_i as

$$A_1, \quad A_2, \quad A_3 = 0, \quad A_4 = 0$$

$$B_1 = A_2, \quad B_2 = 0, \quad B_3 = 0, \quad B_4 = 0$$

$$C_1, \quad C_2, \quad C_3 = 0, \quad C_4 = 0$$

$$D_1 = C_2, \quad D_2 = 0, \quad D_3 = 0, \quad D_4 = 0$$

From Eq. A6-3.19e:

$$B_i = \frac{GD_{xx}m_i}{(GD_{xx} - EI_{yy}m_i^2)} A_i \quad (i = 5, \dots, 8) \quad (\text{A6-3.36})$$

Substituting Eq. A6-3.36 into A6-3.d, one has

$$(-GD_{xx}m_i^2 A_i + GD_{xx}m_i B_i + \bar{M}_x m_i^2 C_i) = 0, \quad i = 5, 6, 7, 8$$

$$-GD_{xx}m_i^2 A_i + GD_{xx}m_i \frac{GD_{xx}m_i}{(GD_{xx} - EI_{yy}m_i^2)} A_i + \bar{M}_x m_i^2 C_i = 0$$

$$-GD_{xx}A_i + \frac{(GD_{xx})^2}{(GD_{xx} - EI_{yy}m_i^2)} A_i + \bar{M}_x C_i = 0$$

$$C_i = \frac{-EI_{yy}GD_{xx}m_i^2}{\bar{M}_x (GD_{xx} - EI_{yy}m_i^2)} A_i \quad (\text{A6-3.37})$$

From Eq. A6-3.21e:

$$D_i = \frac{-GD_{hh}m_i}{(GD_{hh} - EI_{\omega\omega}m_i^2)} C_i$$

$$D_i = \frac{EI_{yy}G^2D_{hh}D_{xx}m_i^3}{\bar{M}_x (GD_{xx} - EI_{yy}m_i^2)(GD_{hh} - EI_{\omega\omega}m_i^2)} A_i \quad (i = 5, \dots, 8) \quad (\text{A6-3.38})$$

The solution takes the form

$$u = A_1 + A_2 z + \sum_{i=5}^8 A_i e^{m_i z} \quad (\text{A6-3.39})$$

$$\theta_y = A_2 + \sum_{i=5}^8 \frac{GD_{xx} m_i}{(GD_{xx} - EI_{yy} m_i^2)} A_i e^{m_i z} \quad (\text{A6-3.40})$$

$$\theta_z = C_1 + C_2 z + \sum_{i=5}^8 \frac{-EI_{yy} GD_{xx} m_i^2}{\bar{M}_x (GD_{xx} - EI_{yy} m_i^2)} A_i e^{m_i z} \quad (\text{A6-3.41})$$

$$\psi = C_2 + \sum_{i=5}^8 \frac{EI_{yy} G^2 D_{hh} D_{xx} m_i^3}{\bar{M}_x (GD_{xx} - EI_{yy} m_i^2) (GD_{hh} - EI_{\omega\omega} m_i^2)} A_i e^{m_i z} \quad (\text{A6-3.42})$$

From Eq. A6-3.9, recall that

$m_5 = -m_6$ and $m_7 = -m_8$. Thus,

$$\begin{aligned} \sum_{i=5}^8 A_i e^{m_i z} &= (A_5 e^{m_5 z} + A_6 e^{m_6 z} + A_7 e^{m_7 z} + A_8 e^{m_8 z}) \\ &= A_5 e^{m_5 z} + A_6 e^{-m_5 z} + A_7 e^{m_7 z} + A_8 e^{-m_7 z} \\ &= (A_3^* \cosh m_I z + A_4^* \sinh m_I z) + (A_5^* \cosh m_{II} z + A_6^* \sinh m_{II} z) \end{aligned}$$

in which

$$m_I = m_5, \quad m_{II} = m_7$$

$$A_3^* = A_5 + A_6$$

$$A_4^* = A_5 - A_6$$

$$A_5^* = A_7 + A_8$$

$$A_6^* = A_7 - A_8$$

$$A_7^* = C_1$$

$$A_8^* = C_2$$

Equations A6-3.39 to A6-3.42 can be rewritten as

$$u = A_1 + A_2 z + A_3^* \cosh m_I z + A_4^* \sinh m_I z + A_5^* \cosh m_{II} z + A_6^* \sinh m_{II} z \quad (\text{A6-3.44})$$

$$\begin{aligned} \theta_y &= A_2 + \frac{GD_{xx} m_I}{(GD_{xx} - EI_{yy} m_I^2)} (A_3^* \sinh m_I z - A_4^* \cosh m_I z) \\ &\quad + \frac{GD_{xx} m_{II}}{(GD_{xx} - EI_{yy} m_{II}^2)} (A_5^* \sinh m_{II} z - A_6^* \cosh m_{II} z) \end{aligned} \quad (\text{A6-3.45})$$

$$\begin{aligned} \theta_z = & A_7^* + A_8^* z - \frac{EI_{yy} GD_{xx} m_I^2}{M_x (GD_{xx} - EI_{yy} m_I^2)} (A_3^* \cosh m_I z + A_4^* \sinh m_I z) \\ & - \frac{EI_{yy} GD_{xx} m_{II}^2}{M_x (GD_{xx} - EI_{yy} m_{II}^2)} (A_5^* \cosh m_{II} z + A_6^* \sinh m_{II} z) \end{aligned} \quad (A6-3.46)$$

$$\begin{aligned} \psi = & A_8^* + \frac{EI_{yy} G^2 D_{hh} D_{xx} m_I^3}{M_x (GD_{xx} - EI_{yy} m_I^2) (GD_{hh} - EI_{\omega\omega} m_I^2)} (A_3^* \sinh m_I z - A_4^* \cosh m_I z) \\ & + \frac{EI_{yy} G^2 D_{hh} D_{xx} m_{II}^3}{M_x (GD_{xx} - EI_{yy} m_{II}^2) (GD_{hh} - EI_{\omega\omega} m_{II}^2)} (A_5^* \sinh m_{II} z - A_6^* \cosh m_{II} z) \end{aligned} \quad (A6-3.47)$$

Up to now, eight coefficients remain in the solutions (Eqs. A6-3.44 to A6-3.47). These are determined from the boundary conditions of the problem. The boundary conditions for this problem are:

$$\left[(G \{D_{xx} u' - D_{xx} \theta_y\} - M_x \theta_z') \delta u \right]_0^L = 0 \quad (A6-3.48)$$

$$\left[(EI_{yy} \theta_y' + M_x \theta_z) \delta \theta_y \right]_0^L = 0 \quad (A6-3.49)$$

$$\left[(G \{J \theta_z' + D_{hh} \theta_z' + D_{hh} \psi\} + M_x \{\theta_y - u'\}) \delta \theta_z \right]_0^L = 0 \quad (A6-3.50)$$

$$\left[(EI_{\omega\omega} \psi') \delta \psi \right]_0^L = 0 \quad (A6-3.51)$$

Enforcing the boundary condition at $z = 0$:

$$u(0) = 0, \quad \Rightarrow \quad u = A_1 + A_2 z + A_3^* \cosh m_I z + A_4^* \sinh m_I z + A_5^* \cosh m_{II} z + A_6^* \sinh m_{II} z = 0$$

$$A_1 = - (A_3^* + A_5^*) \quad (A6-3.52)$$

$$EI_{yy} \theta_y'(0) + M_x \theta_z(0) = 0, \quad \Rightarrow$$

$$\begin{aligned} & \frac{EI_{yy} GD_{xx} m_I^2}{(GD_{xx} - EI_{yy} m_I^2)} A_3^* + \frac{EI_{yy} GD_{xx} m_{II}^2}{(GD_{xx} - EI_{yy} m_{II}^2)} A_5^* + M_x A_7^* - \frac{EI_{yy} GD_{xx} m_I^2}{(GD_{xx} - EI_{yy} m_I^2)} A_3^* - \frac{EI_{yy} GD_{xx} m_{II}^2}{(GD_{xx} - EI_{yy} m_{II}^2)} A_5^* = 0 \\ & A_7^* = 0 \end{aligned} \quad (A6-3.53)$$

$$\theta_z(0) = 0 \quad \Rightarrow$$

$$\begin{aligned}
 \theta_z = 0 + 0 - \frac{EI_{yy}GD_{xx}m_I^2}{M_x(GD_{xx}-EI_{yy}m_I^2)}(A_3^* \cosh m_I z + A_4^* \sinh m_I z) \\
 - \frac{EI_{yy}GD_{xx}m_{II}^2}{M_x(GD_{xx}-EI_{yy}m_{II}^2)}(A_5^* \cosh m_{II} z + A_6^* \sinh m_{II} z) = 0 \\
 A_3^* = -\frac{m_{II}^2(GD_{xx}-EI_{yy}m_I^2)}{m_I^2(GD_{xx}-EI_{yy}m_{II}^2)}A_5^*
 \end{aligned} \tag{A6-3.54}$$

$$\psi'(0) = 0 \Rightarrow$$

$$\begin{aligned}
 \psi' = \frac{EI_{yy}G^2D_{hh}D_{xx}m_I^4}{M_x(GD_{xx}-EI_{yy}m_I^2)(GD_{hh}-EI_{\omega\omega}m_I^2)}(A_3^* \cosh m_I z - A_4^* \sinh m_I z) \\
 + \frac{EI_{yy}G^2D_{hh}D_{xx}m_{II}^4}{M_x(GD_{xx}-EI_{yy}m_{II}^2)(GD_{hh}-EI_{\omega\omega}m_{II}^2)}(A_5^* \cosh m_{II} z - A_6^* \sinh m_{II} z) = 0 \\
 A_3^* = -\frac{m_3^4(GD_{xx}-EI_{yy}m_I^2)(GD_{hh}-EI_{\omega\omega}m_I^2)}{m_1^4(GD_{xx}-EI_{yy}m_{II}^2)(GD_{hh}-EI_{\omega\omega}m_{II}^2)}A_5^*
 \end{aligned} \tag{A6-3.55}$$

Examining Eqs. A6-3.54 and A6-3.55, a possible solution is

$$A_3^* = A_5^* = 0 \tag{A6-3.56}$$

Substituting Eq. A6-3.56 into Eq. A6-3.52, gives

$$A_I = 0 \tag{A6-3.57}$$

In summary, after enforcing the boundary conditions at $z = 0$, the field equations take the form

$$u(z) = A_2 z + A_4^* \sinh m_I z \tag{A6-3.58}$$

$$\theta_y(z) = A_2 + \frac{GD_{xx}m_I}{(GD_{xx}-EI_{yy}m_I^2)}(-A_4^* \cosh m_I z) + \frac{GD_{xx}m_{II}}{(GD_{xx}-EI_{yy}m_{II}^2)}(-A_6^* \cosh m_{II} z) \tag{A6-3.59}$$

$$\theta_z(z) = A_8^* z - A_4^* \frac{EI_{yy}GD_{xx}m_I^2}{M_x(GD_{xx}-EI_{yy}m_I^2)} \sinh m_I z - A_6^* \frac{EI_{yy}GD_{xx}m_{II}^2}{M_x(GD_{xx}-EI_{yy}m_{II}^2)} (\sinh m_{II} z) \tag{A6-3.60}$$

$$\begin{aligned} \psi(z) = & A_8^* - A_4^* \frac{EI_{yy} G^2 D_{hh} D_{xx} m_I^3}{M_x (GD_{xx} - EI_{yy} m_I^2) (GD_{hh} - EI_{\omega\omega} m_I^2)} (\cosh m_I z) \\ & - A_6^* \frac{EI_{yy} G^2 D_{hh} D_{xx} m_{II}^3}{M_x (GD_{xx} - EI_{yy} m_{II}^2) (GD_{hh} - EI_{\omega\omega} m_{II}^2)} (\cosh m_{II} z) \end{aligned} \quad (A6-3.61)$$

Enforcing the boundary conditions at $z = L$

$$u(L) = 0, \quad \Rightarrow \quad u = A_2 L + A_4^* \sinh m_I L = 0$$

$$A_2 = \frac{-A_4^* \sinh m_I L}{L} \quad (A6-3.62)$$

$$EI_{yy} \theta'_y(L) + M_x \theta_z(L) = 0, \quad \Rightarrow$$

$$M_x A_8^* L - 2A_4^* \frac{EI_{yy} G D_{xx} m_I^2}{(GD_{xx} - EI_{yy} m_I^2)} \sinh m_I L - 2A_6^* \frac{EI_{yy} G D_{xx} m_{II}^2}{(GD_{xx} - EI_{yy} m_{II}^2)} (\sinh m_{II} L) = 0 \quad (A6-3.63)$$

$$\theta_z(L) = 0 \quad \Rightarrow$$

$$A_8^* L - A_4^* \frac{EI_{yy} G D_{xx} m_I^2}{M_x (GD_{xx} - EI_{yy} m_I^2)} \sinh m_I L - A_6^* \frac{EI_{yy} G D_{xx} m_{II}^2}{M_x (GD_{xx} - EI_{yy} m_{II}^2)} (\sinh m_{II} L) = 0 \quad (A6-3.64)$$

$$\psi'(L) = 0 \quad \Rightarrow$$

$$\begin{aligned} & A_4^* \frac{EI_{yy} G^2 D_{hh} D_{xx} m_I^4}{M_x (GD_{xx} - EI_{yy} m_I^2) (GD_{hh} - EI_{\omega\omega} m_I^2)} (\sinh m_I L) \\ & - A_6^* \frac{EI_{yy} G^2 D_{hh} D_{xx} m_{II}^4}{M_x (GD_{xx} - EI_{yy} m_{II}^2) (GD_{hh} - EI_{\omega\omega} m_{II}^2)} (\sinh m_{II} L) = 0 \end{aligned} \quad (A6-3.65)$$

Finally, the field equation can be expressed as

$$u(z) = A_4^* \left(\sinh m_I z - \frac{\sinh m_I L}{L} z \right) \quad (A6-3.66)$$

$$\theta_y(z) = -A_4^* \left(\frac{GD_{xx} m_I}{(GD_{xx} - EI_{yy} m_I^2)} (\cosh m_I z) - \frac{\sinh m_I L}{L} \right) - A_6^* \frac{GD_{xx} m_{II}}{(GD_{xx} - EI_{yy} m_{II}^2)} (\cosh m_{II} z) \quad (A6-3.67)$$

$$\theta_z(z) = -A_4^* \frac{EI_{yy} G D_{xx} m_I^2}{M_x (GD_{xx} - EI_{yy} m_I^2)} \sinh m_I z - A_6^* \frac{EI_{yy} G D_{xx} m_{II}^2}{M_x (GD_{xx} - EI_{yy} m_{II}^2)} (\sinh m_{II} z) + A_8^* z \quad (A6-3.68)$$

$$\begin{aligned} \psi(z) = & A_8^* - A_4^* \frac{EI_{yy} G^2 D_{hh} D_{xx} m_I^3}{M_x (GD_{xx} - EI_{yy} m_I^2) (GD_{hh} - EI_{\omega\omega} m_I^2)} (\cosh m_I z) \\ & - A_6^* \frac{EI_{yy} G^2 D_{hh} D_{xx} m_{II}^3}{M_x (GD_{xx} - EI_{yy} m_{II}^2) (GD_{hh} - EI_{\omega\omega} m_{II}^2)} (\cosh m_{II} z) \end{aligned} \quad (\text{A6-3.69})$$

From Eqs. A6-3.63 to A6-3.65, the cross-sectional properties are not possible to be zero, only can the hyperbolic function be zero. Thus, the critical moment can be determined by solving the equations below

$$\text{Case 1: } (\sinh m_I L) = 0 \quad (\text{A6-3.70})$$

$$\text{Case 2: } (\sinh m_{II} L) = 0 \quad (\text{A6-3.71})$$

$$\text{Case 3: } \left[\frac{m_{II}^2}{M_x (GD_{hh} - EI_{\omega\omega} m_{II}^2)} + \frac{m_I^2}{M_x (GD_{hh} - EI_{\omega\omega} m_I^2)} \right] = 0 \quad (\text{A6-3.72})$$

$$\text{Case 1: } (\sinh m_I L) = 0$$

Based on the expressions of the roots (Eq. A6-3.43 and Eq. A6-3.9), take the root as

$$m_I = m_5 = \sqrt{\frac{-b + \sqrt{b^2 - 4ac}}{2a}} \quad (\text{A6-3.73})$$

The converting equation is

$$(\sinh m_I L) = -i \sin(im_I L) \quad (\text{A6-3.74})$$

$$(\sinh m_I L) = -i \sin \left(\sqrt{\frac{b - \sqrt{b^2 - 4ac}}{2a}} L \right) = 0 \quad (\text{A6-3.75})$$

$$\sqrt{\frac{b - \sqrt{b^2 - 4ac}}{2a}} L = n\pi \quad (n=0, 1, 2, \dots, n) \quad (\text{A6-3.76})$$

where the constants a , b and c are related to the properties of cross-section and given by Eq. A6-3.8 as,

$$\begin{aligned} a &= E^2 I_{yy} I_{\omega\omega} (G^2 D_{xx} J + G^2 D_{xx} D_{hh} - M_x^2) \\ b &= GE (-G^2 D_{xx} D_{hh} J I_y + M_x^2 I_{yy} D_{hh} + M_x^2 D_{xx} I_{\omega\omega}) \\ c &= -M_x^2 G^2 D_{xx} D_{hh} \end{aligned} \quad (\text{A6-3.77})$$

$n = 0$:

$$b - \sqrt{b^2 - 4ac} = 0 \quad (\text{A6-3.78})$$

Substituting Eq. A6-3.77 into Eq. A6-3.78 for the constants a , b and c , yields,

$$\left[E^2 I_{yy} I_{\omega\omega} (G^2 D_{xx} J + G^2 D_{xx} D_{hh} - M_x^2) \right] \left[-M_x^2 G^2 D_{xx} D_{hh} \right] = 0$$

$$\boxed{M_x = \pm \sqrt{G^2 D_{xx} J + G^2 D_{xx} D_{hh}} = \pm G \sqrt{D_{xx} (J + D_{hh})}} \quad (\text{A6-3.79})$$

$n = 1$:

$$\sqrt{\frac{b - \sqrt{b^2 - 4ac}}{2a}} = \frac{\pi}{L}$$

$$\frac{b - \sqrt{b^2 - 4ac}}{2a} = \left(\frac{\pi}{L} \right)^2$$

$$a \left(\frac{\pi}{L} \right)^4 - b \left(\frac{\pi}{L} \right)^2 + c = 0 \quad (\text{A6-3.80})$$

Substituting Eq. A6-3.77 into Eq. A6-3.80 for constants a , b and c , yields,

$$\bar{M}_x^2 G^2 D_{xx} D_{hh} + GE (-G^2 D_{xx} D_{hh} J I_y + \bar{M}_x^2 I_{yy} D_{hh} + \bar{M}_x^2 D_{xx} I_{\omega\omega}) \left(\frac{\pi}{L} \right)^2 - E^2 I_{yy} I_{\omega\omega} (G^2 D_{xx} J + G^2 D_{xx} D_{hh} - \bar{M}_x^2) \left(\frac{\pi}{L} \right)^4 = 0$$

$$\boxed{\bar{M}_{x_{5,6}} = \pm \frac{\pi}{L} \sqrt{\frac{EG^3 D_{xx} D_{hh} J I_y + E^2 I_{yy} I_{\omega\omega} (G^2 D_{xx} J + G^2 D_{xx} D_{hh}) \left(\frac{\pi}{L} \right)^2}{G^2 D_{xx} D_{hh} + GE (I_{yy} D_{hh} + D_{xx} I_{\omega\omega}) \left(\frac{\pi}{L} \right)^2 + E^2 I_{yy} I_{\omega\omega} \left(\frac{\pi}{L} \right)^4}}} \quad (\text{A6-3.81})$$

Case 2: $(\sinh m_{II} L) = 0$

Based on the expression of the roots (Eqs. A6-3.9 and A6-3.43), take the root as

$$m_{II} = m_7 = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}} \quad (\text{A6-3.82})$$

$$(\sinh m_{II} L) = -i \sin(im_{II} L) = -i \sin \left(\sqrt{\frac{b + \sqrt{b^2 - 4ac}}{2a}} L \right) = 0$$

$$\sqrt{\frac{b + \sqrt{b^2 - 4ac}}{2a}} L = n\pi \quad (n=0, 1, 2, \dots, n) \quad (\text{A6-3.83})$$

$n = 0$:

$$b + \sqrt{b^2 - 4ac} = 0$$

$$b = -\sqrt{b^2 - 4ac} \quad \text{no solution!}$$

$$b^2 = (b^2 - 4ac)i$$

$n = 1$:

$$\sqrt{\frac{b + \sqrt{b^2 - 4ac}}{2a}} L = \pi$$

$$b + \sqrt{b^2 - 4ac} = 2a \left(\frac{\pi}{L} \right)^2$$

$$b^2 - 4ac = 4a^2 \left(\frac{\pi}{L} \right)^4 - 4ab \left(\frac{\pi}{L} \right)^2 + b^2$$

$$a \left(\frac{\pi}{L} \right)^4 - b \left(\frac{\pi}{L} \right)^2 + c = 0 \quad (\text{A6-3.84})$$

Equation A6-3.84 has the same roots as when $(\sinh m_l L) = 0$.

$$\bar{M}_{x_6} = \pm \frac{\pi}{L} \sqrt{\frac{EG^3 D_{xx} D_{hh} J I_{yy} + E^2 I_{yy} I_{\omega\omega} (G^2 D_{xx} J + G^2 D_{xx} D_{hh}) \left(\frac{\pi}{L} \right)^2}{G^2 D_{xx} D_{hh} + GE (I_{yy} D_{hh} + D_{xx} I_{\omega\omega}) \left(\frac{\pi}{L} \right)^2 + E^2 I_{yy} I_{\omega\omega} \left(\frac{\pi}{L} \right)^4}} \quad (\text{A6-3.85})$$

Case 3:
$$\left[\frac{m_{II}^2}{\bar{M}_x (GD_{hh} - EI_{\omega\omega} m_{II}^2)} + \frac{m_I^2}{\bar{M}_x (GD_{hh} - EI_{\omega\omega} m_I^2)} \right] = 0$$

$$\frac{(m_{II}^2 + m_I^2)}{m_I^2 m_{II}^2} = 2 \frac{EI_{\omega\omega}}{GD_{hh}} \quad (\text{A6-3.86})$$

Based on the expressions of the roots (Eq. A6-3.43 and Eq. A6-3.9), we have

$$m_I = \sqrt{\frac{-b + \sqrt{b^2 - 4ac}}{2a}} \quad \text{and} \quad m_{II} = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

$$\begin{aligned}
 m_I^2 + m_{II}^2 &= -\frac{b}{a} \\
 m_I^2 m_{II}^2 &= \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a} \\
 \frac{m_1^2 + m_3^2}{m_1^2 m_3^2} &= -\frac{b}{c}
 \end{aligned} \tag{A6-3.87}$$

Substituting Eqs. A6-3.77 and A6-3.82 into Eq. A6-3.87 for constants a , b and c , yields,

$$\begin{aligned}
 a &= E^2 I_{yy} I_{\omega\omega} (G^2 D_{xx} J + G^2 D_{xx} D_{hh} - \bar{M}_x^2) \\
 b &= GE(-G^2 D_{xx} D_{hh} J I_y + \bar{M}_x^2 I_{yy} D_{hh} + \bar{M}_x^2 D_{xx} I_{\omega\omega}) \\
 c &= -\bar{M}_x^2 G^2 D_{xx} D_{hh} \\
 \frac{GE(-G^2 D_{xx} D_{hh} J I_y + \bar{M}_x^2 I_{yy} D_{hh} + \bar{M}_x^2 D_{xx} I_{\omega\omega})}{\bar{M}_x^2 G^2 D_{xx} D_{hh}} &= 2 \frac{EI_{\omega\omega}}{GD_{hh}} \\
 -G^2 D_{xx} D_{hh} J I_y + \bar{M}_x^2 I_{yy} D_{hh} + \bar{M}_x^2 D_{xx} I_{\omega\omega} &= 2 \bar{M}_x^2 D_{xx} I_{\omega\omega} \\
 \bar{M}_x^2 (I_{yy} D_{hh} - D_{xx} I_{\omega\omega}) &= G^2 D_{xx} D_{hh} J I_y \\
 \boxed{\bar{M}_{x1,2}} &= \pm \sqrt{\frac{G^2 D_{xx} D_{hh} J I_y}{(I_{yy} D_{hh} - I_{\omega\omega} D_{xx})}}
 \end{aligned} \tag{A6-3.88}$$

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List of Symbols

A :	Cross-sectional area
A_p :	Pole
C :	Cross-section centroid
$D_{xx}, D_{hh}, D_{xh}, D_{yy}, D_{yk}, D_{yyk}$:	Properties of cross-section
E :	Young's modulus
G :	Shear modulus
$h(s)$:	Distance perpendicular from the pole to the tangent of a point on the mid-surface denoted by coordinate s
I_{xx}, I_{yy} :	Moments of inertia of the cross-section about x axis, y axis, respectively
$I_{\omega\omega}$:	Second moment of area of sectorial coordinate $I_{\omega\omega} = \int_A \omega^2 dA$
J :	St. Venant torsional constant
$k(s)$:	Tangential projection of the distance between pole A_p and an arbitrary point on the mid-surface denoted by coordinate s
L :	Length of the member
$M_{xp}(z)$:	Resultant of the moments of the normal stresses obtained from pre-buckling analysis
$N_p(z)$:	Resultant of normal stresses obtained from pre-buckling analysis
O :	Origin of Cartesian coordinates x, y, z
r :	Thickness of the cross-section
q_y, q_z :	Line loads applied to a member acting along the y and z directions, respectively
s :	Curvilinear coordinate along mid-surface
S :	Arbitrary point on the mid-surface
S_o :	Sectorial origin
S_x, S_y, S_ω :	First moments of area for coordinates x, y and ω , respectively
$S_{x\omega}$:	Section product of inertia
SC :	Shear center of a cross-section

t :	Tangential coordinate to Point S on a section profile
u, v, w :	Displacements along x, y, z directions, respectively
U :	Internal strain energy
V :	Load potential energy
V_{yp} :	Shear force component (in y direction) obtained from pre-buckling analysis
x, y, z :	Cartesian coordinates
α :	Angle between the tangent to the contour and the x axis
ε_{zz} :	Longitudinal strain along the z direction
γ_{zs} :	Tangential shear strain along the mid-surface
λ :	Load factor (eigen-value)
π :	Total potential energy functional
$\theta_x, \theta_y, \theta_z$:	Rotation angles about x, y, z axes, respectively
σ_{zz} :	Longitudinal normal stress
τ_{zs} :	Tangential shear stress along the mid-surface
$\omega(s)$:	Warping function (sectorial area) of a point s
$\psi(z)$:	Rate of change of the longitudinal displacement due to warping ($1/Length$ dimension)

Chapter 7 Buckling Formulation for Shear Deformable Thin-Walled Members - II. Finite Element Formulation

Abstract:

In Chapter 6, a variational principle based on the principle of stationary potential energy was developed for the buckling analysis of thin-walled members with doubly symmetric cross-sections. In this chapter, the variational principle is adopted to formulate a buckling solution in a finite element context. The formulation has three features: 1) it is based on general non-orthogonal coordinates suitable for modeling members with supports that do not coincide with the shear centre; 2) it captures shear deformation effects due to bending and warping; and 3) it incorporates the effect of load position relative to the shear centre. A series of examples demonstrate the validity of the finite element solution and illustrates its applicability to a wide variety of buckling problems. Examples include column flexural buckling, lateral torsional buckling of beams subjected to moment gradients, buckling of members under combinations of axial force and bending moment, and eccentrically supported structural members. In all cases, the validity of the solutions is assessed through comparisons to well-established closed-form and/or other numerical solutions.

Keywords:

Thin-walled members; Non-orthogonal coordinates; Shear deformation effect; Lateral-torsional buckling; Finite element

7.1 Literature Review

The present review focuses on the buckling solution of thin-walled members of open cross-sections with emphasis on finite element formulations. Early finite element formulations for the buckling analysis of thin-walled members were developed by Krajcinovic [7-1] and Barsoum and Gallagher [7-2]. Bazant and Nimeri [7-3] developed a finite element formulation for the post-buckling solutions for thin-walled members. Johnson and Will [7-4] and Akay et al. [7-5] used shell elements to model cross-sectional distortion effects in the buckling analysis of thin-walled beams. Hancock et al. [7-6], Bradford and Trahair [7-7], and Bradford [7-8] formulated solutions of flexural-torsional buckling incorporating the effect of web distortion of doubly symmetric and mono-symmetric I-beams. Wekezer [7-9] developed a buckling finite element solution of thin-walled non-prismatic members based on a shell membrane theory. Attard [7-10] and Pi and Trahair [7-11, 7-12] formulated a finite element buckling solution which captures the pre-buckling deformation effects. Kitipornchai et al. [7-13] investigated the effects of moment gradient on lateral-torsional buckling for mono-symmetric members. Pi et al. [7-14] and Pi and Bradford [7-15] investigated the effect of nonlinear terms in the longitudinal displacements on the resulting energy formulation and resulting buckling predictions.

Sherbourne and Kabir [7-16] analyzed shear strain effects on the lateral stability of thin-walled fibrous composite beams and observed that shear effects caused a considerable drop in the buckling loads for short-span beams.

Machado and Cortinez [7-17] developed kinematic relations for the geometric nonlinear analysis of members of doubly symmetric cross-sections. In their study, the effects of pre-buckling deformation, shear deformation and load position effect were numerically investigated. Later on, based on their kinematic assumptions, Machado [7-18, 7-19] studied the buckling and post-buckling behaviour of composite beams. Kim [7-20] presented an exact stiffness matrix formulation for the lateral buckling of thin-walled composite beams under uniform moment based on the energy functional. Erkmén and Mohareb [7-21, 7-22] formulated a lateral-torsional stability solution for thin-walled beams based on the principle of stationary complementary energy which incorporated shear deformation and load position effects. Sapountzakis and Dourakopoulos [7-23] developed a boundary element solution for the flexural-torsional buckling of composite beams. Vo and Lee [7-24] also developed a finite element model based on shear deformable thin walled beams to investigate the flexural buckling of members under axial load.

7.2 Objective

This chapter aims at developing a buckling finite element formulation for doubly symmetric open cross-sections which captures the effects of shear deformation, load position, and eccentric boundary conditions. In these respects, the present formulation has the features in Erkmén and Mohareb [7-21, 7-22]. The present work however departs from that in [7-21, 7-22] in three respects; 1) the formulation is based on the stationary potential energy variational principle developed in Chapter 6 and thus contrasts with that in [7-21,7-22] which is based on a complementary energy principle; 2) The formulation lends itself to more flexibility when modelling combinations of eccentric supports and offset transverse loads (e.g., supports at the bottom flange and loads applied to the top flange); and 3) results in a more conventional and systematic stiffness matrix assembly than the one reported in [7-21, 7-22].

7.3 Conventions and Notation

A right hand coordinate system x, y and z is chosen such that the z axis is oriented along the axial direction, the x axis is parallel to the major principal axis of the section, and the y axis is parallel to the minor principal axis. Origin O lies on the axis of symmetry $x = 0$ of the cross-section but does not necessarily coincide with the cross-sectional centroid $C(x_c = 0, y_c)$. A sectorial origin S_o , identifies coordinates $x(s)$ and $y(s)$ of an arbitrary point S on the mid-surface of the cross section. The member is subjected to: a) Distributed axial force $q_z(z)$ acting longitudinally along the z direction at distance $y_{qz}(z)$ from the origin, and b) Distributed transverse loading $q_y(z)$ acting in the plane of the shear centre along the y direction (minor principal axis) at distance $y_{qy}(z)$ from the origin. Thus, reference loads q_y and q_z induce combinations of axial forces, bending about the major principal axis of the section and associated shear forces in the member. Under the applied reference loads, the member deforms from Configuration 1 to Configuration 2 in Fig. 7.1 by undergoing displacements $v_p(z), w_p(z)$, and rotation $\theta_{xp}(z)$ within the yz plane from the initial un-deformed state, where subscript p denotes the pre-buckling fields (displacements, strains, stresses, or stress resultants). As the loads increase to λq_y and λq_z , the member reaches the state of onset of buckling state (Configuration 3 in Fig. 7.1), and the associated pre-buckling

displacements are assumed to linearly increase to $\lambda v_p(z)$, $\lambda w_p(z)$ and $\lambda \theta_{xp}(z)$, where λ is a constant to be determined. When configuration 3 is reached, the member undergoes out-of-plane buckling displacements (i.e., lateral displacement $u_b(z)$, weak axis rotation $\theta_{yb}(z)$, angle of twist $\theta_{zb}(z)$ and warping deformation $\psi_b(z)$) to reach Configuration 4. Subscript b denotes all field variables (displacements, strains, stresses, stress resultants) taking place during the buckling state.

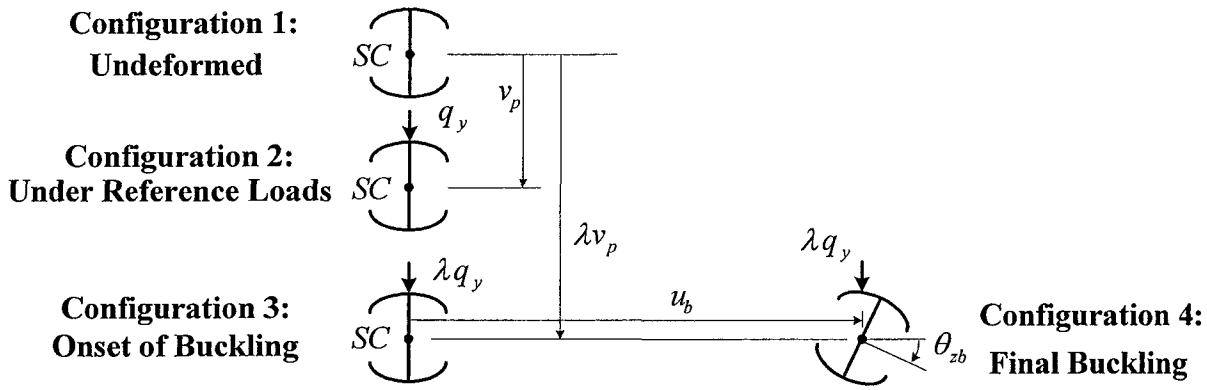


Figure 7.1 Stages of Deformation

7.4 Hypotheses

As stated in the Chapter 6, the formulation is restricted to prismatic thin-walled members with doubly symmetric cross-sections. The main assumptions of the formulation are:

1. The cross-section is assumed to remain rigid in its own plane throughout deformation, but does not remain perpendicular to the neutral axis after deformation. A similar kinematic assumption is extended for warping deformation;
2. Strains are assumed small and rotations moderate;
3. The material is assumed to be linearly elastic isotropic and obeys Hooke's law; and
4. The member is assumed to buckle in an inextensional mode [7-26]. This means that during buckling, the centroidal strain and curvature in the principal yz -plane remain zero, so that the member buckles under constant axial force and bending moment

7.5 Variational Principles for Neutral Equilibrium

In the critical equilibrium state, the first variation of the total potential $\delta\pi$ must vanish to satisfy

equilibrium and the variation of the second variation of the total potential energy $\delta^2\pi$ has vanish to satisfy the condition of neutral stability. The following two sections provide the governing variational principles for pre-buckling and buckling, while the implementation of both principles into a finite element context are presented under Section 6.

7.5.1 Equilibrium Condition

The equilibrium state (Configuration 2 in Fig. 7.1) is obtained by setting the first variation of the total potential energy to zero (Chapter 5), leading to the stationary condition

$$\begin{aligned}\delta\pi = & \int_0^l \left(EA w_p' \delta w_p' + EI_{xx} \theta_{xp}' \delta \theta_{xp}' + ES_x w_p' \delta \theta_{xp}' + ES_x \theta_{xp}' \delta w_p' \right) dz \\ & + \int_0^l \left(GD_{yy} v_p' \delta v_p' + GD_{yy} \theta_{xp} \delta \theta_{xp} + GD_{yy} v_p' \delta \theta_{xp} + GD_{yy} \theta_{xp} \delta v_p' \right) dz \\ & - \int_0^l \left[q_y \delta v_p + q_z \delta w_p + q_z y_{qz} \delta \theta_{xp} \right] dz = 0\end{aligned}\quad (7.1)$$

in which l is the length of the element, and the following sectional constants have been defined

$$A = \int_A dA, \quad I_{xx} = \int_A y^2 dA, \quad S_x = \int_A y dA, \quad D_{yy} = \int_A \sin^2 \alpha(s) dA \quad (7.2)$$

and $\alpha(s)$ is the angle between the tangent to the profile at point S and the x axis.

7.5.2 Condition of Neutral Stability

When the equilibrium state attained in Eq. (7.1) is in a neutral stability state (Configuration 3 in Fig. 7.1), the variation of the second variation of the total potential energy must also vanish [7-25], i.e.,

$$\delta \frac{1}{2} (\delta^2 \pi) = \delta \frac{1}{2} (\delta^2 U_{en} + \delta^2 U_{es} + \delta^2 V_{gN} + \delta^2 V_{gM} + \delta^2 V_{gV} + \delta^2 V_{gqy} + \delta^2 V_{gqz}) = 0 \quad (7.3)$$

in which $\delta^2 U_{en}$, the second variation of the internal strain energy due normal stress, is

$$\delta^2 U_{en} = E \int_0^l \left\{ I_{yy} (\delta \theta'_{yb})^2 - 2S_{xw} \delta \theta'_{yb} \delta \psi'_b + I_{ww} (\delta \psi'_b)^2 \right\} dz \quad (7.4a)$$

Also, $\delta^2 U_{es}$, the second variation of the internal strain energy due shear stress, is

$$\delta^2 U_{es} = G \int_0^l \left\{ J \delta \theta'_{zb}{}^2 + D_{xx} (\delta u'_b)^2 - 2D_{xx} \delta u'_b \delta \theta'_{yb} + 2D_{xh} \delta u'_b \delta \theta'_{zb} + 2D_{xh} \delta u'_b \delta \psi_b + D_{xx} (\delta \theta'_{yb})^2 \right. \\ \left. - 2D_{xh} \delta \theta'_{yb} \delta \theta'_{zb} - 2D_{xh} \delta \theta'_{yb} \delta \psi_b + D_{hh} (\delta \theta'_{zb})^2 + 2D_{hh} \delta \theta'_{zb} \delta \psi_b + D_{hh} (\delta \psi_b)^2 \right\} dz \quad (7.4b)$$

The terms $\delta^2 V_{gN}$, $\delta^2 V_{gM}$, $\delta^2 V_{gV}$, $\delta^2 V_{gqy}$, $\delta^2 V_{gqz}$, and $\delta^2 V_{gqz}$ are the second variations of the destabilizing terms of the total potential energy due to normal forces, bending moments, shearing forces, transverse load position effect, and longitudinal load position effect, respectively. They are given by

$$\delta^2 V_{gN} = -\lambda \int_0^l \left(\frac{N_p(z)}{AI_{xx} - S_x^2} \right) \left\{ (S_x^2 - AI_{xx}) \delta u'_b{}^2 + (y_A^2 S_x^2 + S_x I_{px} - I_{xx}^2 - I_{xx} I_{yy} - y_A^2 AI_{xx}) \delta \theta'_{zb}{}^2 \right. \\ \left. + 2y_A (S_x^2 - AI_{xx}) \delta u'_b \delta \theta'_{zb} + 2y_C (AI_{xx} - S_x^2) (\delta \theta'_{yb} \delta \theta'_{zb} + \delta \theta'_{yb} \delta \theta'_{zb}) \right\} dz \quad (7.4c)$$

$$\delta^2 V_{gM} = -\lambda \int_0^l \left(\frac{M_{xp}(z)}{AI_{xx} - S_x^2} \right) \left\{ 2(AI_{xx} - S_x^2) \delta u'_b \delta \theta'_{zb} + 2(S_x^2 - AI_{xx}) (\delta \theta'_{yb} \delta \theta'_{zb} + \delta \theta'_{yb} \delta \theta'_{zb}) \right. \\ \left. + (S_x I_{xx} + S_x I_{yy} - AI_{px} - 2y_A S_x^2 + 2y_A AI_{xx}) \delta \theta'_{zb}{}^2 \right\} dz \quad (7.4d)$$

$$\delta^2 V_{gV} = -\lambda \int_0^l \frac{V_{yp}(z)}{D_{yy}} \left\{ -2D_{yy} \delta \theta'_{yb} \delta \theta'_{zb} + 2D_{yyk} \delta u'_b \delta \theta'_{zb} - 2D_{yk} \delta \theta'_{zb} \delta \theta'_{zb} \right\} dz \quad (7.4e)$$

$$\delta^2 V_{gqy} = -\lambda \int_0^l \left\{ -q_y(z) (y_{qy} - y_A) (\delta \theta'_{zb})^2 \right\} dz \quad (7.4f)$$

$$\delta^2 V_{gqz} = -\lambda \int_0^l 2q_z(z) (y_{qz} - y_C) \delta \theta'_{yb} \delta \theta'_{zb} dz \quad (7.4g)$$

In Eq. (7.4a-g), the following cross-sectional properties are defined as

$$\begin{aligned} I_{yy} &= \int_A x^2 dA & I_{xy} &= \int_A xy dA & I_{\omega\omega} &= \int_A \omega^2(s) dA & I_{px} &= \int_A y(x^2 + y^2) dA \\ S_{x\omega} &= \int_A x\omega(s) dA & J &= \sum_{i=1}^n \frac{l_i t_i^3}{3} & D_{hh} &= \int_A h^2(s) dA & D_{xx} &= \int_A \left(\frac{dx}{ds} \right)^2 dA \\ D_{xh} &= \int_A h(s) \left(\frac{dx}{ds} \right) dA & D_{yk} &= \int_A k(s) \frac{dy}{ds} \frac{dk(s)}{ds} dA & D_{yyk} &= \int_A \left(\frac{dy}{ds} \right)^2 \frac{dk(s)}{ds} dA \end{aligned} \quad (7.5)$$

Also, in Eq. (7.4a-g), stress resultants $N_p(z)$, $M_{xp}(z)$ and $V_{yp}(z)$ are the pre-buckling axial forces, bending moment and transverse shear forces, respectively. Their nodal values are obtained from a pre-buckling analysis.

7.6 Finite Element Formulation

The finite element to be developed has two nodes. Each node $i=1,2$ has three pre-buckling degrees of freedom ($v_{ip}, w_{ip}, \theta_{xip}$) and four buckling degrees of freedom ($u_{ib}, \theta_{yib}, \theta_{zib}, \psi_{ib}$).

7.6.1 Pre-buckling Analysis

The pre-buckling solution sought differs from conventional treatments in two respects. It is to be based on a) non-orthogonal coordinate system and b) a shear deformable thin walled beam theory. In this section, only an overview of the pre-buckling formulation is presented and the reader is referred to Chapter 5 for the mathematical details.

In order to avoid shear locking stemming from conventional interpolation schemes, shape functions based on the exact solution of the equilibrium equations are formulated. Integration by parts is performed on the functional Eq. (7.1) to obtain the equilibrium equations. The exact solution for the resulting coupled system of differential equations is then solved for the displacement fields $\langle U_p(z) \rangle_{3 \times 1}^T = \langle w_p(z), v_p(z), \theta_{xp}(z) \rangle_{3 \times 1}^T$. The six integration constants arising in the solution are then related to the nodal displacements $\langle u_{np} \rangle^T = \langle w_{1p} \quad v_{1p} \quad \theta_{x1p} \quad w_{2p} \quad v_{2p} \quad \theta_{x2p} \rangle^T$ by enforcing the relations $\langle u_{np} \rangle^T = \langle w_p(0) \quad v_p(0) \quad \theta_{xp}(0) \quad w_p(L) \quad v_p(L) \quad \theta_{xp}(L) \rangle^T$. This allows the displacement fields $\{U_p(z)\}_{3 \times 1}$ to be expressed in terms of the nodal displacement $\{u_{np}\}_{6 \times 1}$ through $\{U_p(z)\}_{3 \times 1} = [H_p(z)]_{3 \times 6} \{u_{np}\}_{6 \times 1}$ in which $[H_p(z)]_{3 \times 6}$ is the matrix of exact shape functions for pre-buckling displacements. Next, the relation $\{U_p(z)\}_{3 \times 1} = [H_p(z)]_{3 \times 6} \{u_{np}\}_{6 \times 1}$ is substituted into Eq. (7.1), all integrations are performed, and the stationary conditions are evoked yielding $[K_p]_{6 \times 6} \{u_{NP}\}_{6 \times 1} = \{Q_{ml}\}_{6 \times 1}$ in which $[K_p]_{6 \times 6}$ is the exact stiffness matrix and $\{Q_{ml}\}_{6 \times 1}$ is the energy equivalent load vector. After the conventional assembly process and solution, the pre-buckling

nodal displacements are determined and used to determine the nodal forces N_{p1} , V_{yp1} , M_{xp1} , M_{xp2} , for each member, where N_{p1} , V_{yp1} , M_{xp1} , are the axial force, shear force and bending moments at node 1 and M_{xp2} is the bending moment at node 2. As will be discussed under Section 7.2, these nodal forces appear in the buckling formulation.

7.6.2 Buckling Analysis

7.6.2.1 Interpolation for Buckling Displacements

The vector of the nodal buckling displacements is $\langle \delta u_{nb} \rangle^T = \langle \delta u_{1b} \quad \delta \theta_{y1b} \quad \delta \theta_{z1b} \quad \delta \psi_{1b} \quad \delta u_{2b} \quad \delta \theta_{y2b} \quad \delta \theta_{z2b} \quad \delta \psi_{2b} \rangle^T$. A linear interpolation scheme is used to relate the displacement fields of the buckling nodal displacements, i.e.,

$$\langle \delta u_b(z), \delta \theta_{yb}(z), \delta \theta_{zb}(z), \delta \psi(z) \rangle^T = \langle H_b(z) \rangle_{1 \times 2}^T \left[\begin{Bmatrix} \delta u_1 \\ \delta u_2 \end{Bmatrix}_{2 \times 1}, \begin{Bmatrix} \delta \theta_{y1} \\ \delta \theta_{y2} \end{Bmatrix}_{2 \times 1}, \begin{Bmatrix} \delta \theta_{z1} \\ \delta \theta_{z2} \end{Bmatrix}_{2 \times 1}, \begin{Bmatrix} \delta \psi_1 \\ \delta \psi_2 \end{Bmatrix}_{2 \times 1} \right] \quad (7.6a-d)$$

where the interpolation functions for the buckling displacements are $\langle H_b(z) \rangle_{1 \times 2}^T = \langle (l-z)/l \quad (z/l) \rangle^T$.

7.6.2.2 Interpolation of Pre-Buckling Internal Forces

In the buckling analysis, the internal stress resultants N_p , V_{yp} , are assumed constant within an element, while the bending moment $M_{xp}(z)$ is assumed to have a linear distribution (Fig. 7.2). These conditions are approached when short span elements are adopted in the model. The stress resultants are interpolated through

$$N_p(z) = N_{p1}, \quad V_{yp}(z) = V_{yp1}, \quad M_{xp}(z) = \left(1 - \frac{z}{l}\right) M_{xp1} - \left(\frac{z}{l}\right) M_{xp2} \quad (7.7-7.9)$$

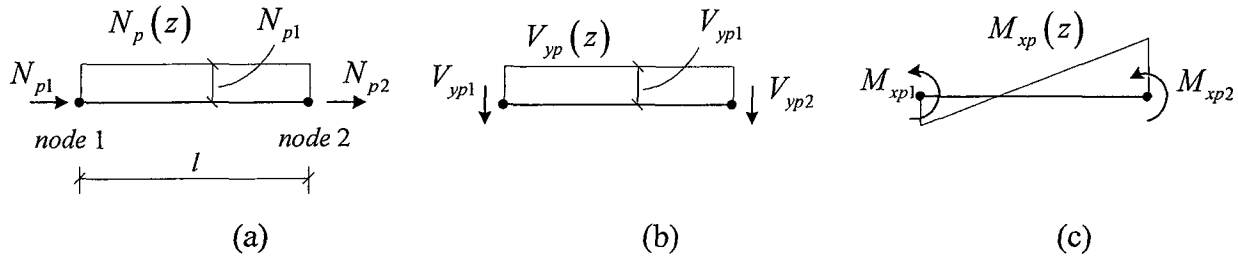


Figure 2 Pre-buckling internal stress resultants (a) Normal force,

(b) Shearing force, and (c) Bending moments

7.6.2.3 Formulation of the Elastic and Geometric Stiffness Matrices

Starting with the variational expression in Eqs. 4a-4g, and the interpolation schemes (Eqs. (7.6a-d) and Eqs. (7.7-7.9)), the following section formulates the matrix contributions appearing in the condition of neutral stability (Eq. (7.3)).

7.6.2.3.1 Elastic Stiffness Contribution due to Normal Stresses

The elastic stiffness matrix contribution due to normal stresses arises from the term $\delta^2 U_{en}$ in Eq. 7.3. Based on Eqs. (7.6a-d), the displacement derivatives are expressed in terms of the nodal displacements as

$$\begin{Bmatrix} \delta \theta'_{yb} \\ \delta \psi'_b \end{Bmatrix}_{2 \times 1} = [B_n]_{2 \times 8} \{ \delta u_{nb} \}_{8 \times 1}, \quad [B_n]_{2 \times 8} = \frac{1}{l} \begin{bmatrix} 0 & -1 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & | & 0 & 0 & 0 & -1 \end{bmatrix}_{2 \times 8} \quad (7.10, 7.11)$$

From Eqs. (7.10) and (7.11), by substituting into Eq. (7.4a), one obtains

$$\frac{1}{2} \delta^2 U_{en} = \frac{1}{2} \langle \delta u_{nb} \rangle_{1 \times 8}^T [K_{en}]_{8 \times 8} \{ \delta u_{nb} \}_{8 \times 1} \quad (7.12)$$

in which the elastic stiffness matrix contribution $[K_{en}]_{8 \times 8}$ arising from normal stresses is

$$[K_{en}]_{8 \times 8} = \frac{E}{l} \begin{bmatrix} 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ & I_{yy} & 0 & -S_{x\omega} & | & 0 & -I_{yy} & 0 & S_{x\omega} \\ & & 0 & 0 & | & 0 & 0 & 0 & 0 \\ & & & I_{\omega\omega} & | & 0 & S_{x\omega} & 0 & -I_{\omega\omega} \\ \hline & & & & | & 0 & 0 & 0 & 0 \\ & & & & | & & I_{yy} & 0 & -S_{x\omega} \\ & & & & | & & & 0 & 0 \\ & & & & | & & & & I_{\omega\omega} \end{bmatrix}_{8 \times 8} \quad (7.13)$$

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7.6.2.3.2 Elastic Stiffness Contribution due to Shear Stresses

The elastic stiffness matrix contribution arising from shear stresses results from the term $\delta^2 U_{es}$ in Eq. (7.3). The displacement fields and their derivatives can be expressed in terms of the nodal displacements from Eqs. (7.6a-d) as

$$\begin{Bmatrix} \delta u'_b \\ \delta \theta_{yb} \\ \delta \theta'_{zb} \\ \delta \psi_b \end{Bmatrix}_{4 \times 1} = [B_s]_{4 \times 8} \{\delta u_{nb}\}_{8 \times 1}, \quad [B_s]_{4 \times 8} = \frac{1}{l} \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & l-z & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & l-z & 0 & 0 & 0 & z \end{bmatrix}_{4 \times 8} \quad (7.14, 7.15)$$

From Eq. (7.14), by substituting into Eq. (7.4b), one obtains

$$\frac{1}{2} \delta^2 U_{es} = \frac{1}{2} \langle \delta u_{nb} \rangle_{1 \times 8}^T [K_{es}]_{8 \times 8} \{\delta u_{nb}\}_{8 \times 1} \quad (7.16)$$

in which the elastic stiffness matrix contribution $[K_{es}]_{8 \times 8}$ arising from shear stresses is

$$[K_{es}]_{8 \times 8} = \frac{G}{6l} \begin{bmatrix} 6D_{xx} & 3D_{xx} & 6D_{xh} & -3D_{xh}l & -6D_{xx} & 3D_{xx}l & -6D_{xh} & -3D_{xh}l \\ & 2D_{xx}l^2 & 3D_{xh}l & -2D_{xx}l^2 & -3D_{xx}l & D_{xx}l^2 & -3D_{xh}l & -D_{xx}l^2 \\ & & 6(D_{hh}+J) & -3D_{hh}l & -6D_{xh} & 3D_{xh}l & -6(D_{hh}+J) & -3D_{hh}l \\ & & & 2D_{hh}l^2 & 3D_{xh}l & -D_{hh}l^2 & 3D_{hh}l & D_{hh}l^2 \\ \hline & & & & 6D_{xx} & -3D_{xx}l & 6D_{xh} & 3D_{xh}l \\ & & & & & 2D_{xx}l^2 & -3D_{xh}l & -2D_{xx}l^2 \\ & & & & & & 6(D_{hh}+J) & 3D_{hh}l \\ & & & & & & & 2D_{hh}l^2 \end{bmatrix} \quad (7.17)$$

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7.6.2.3 Geometric Stiffness Contribution due to Axial Force

The geometric stiffness matrix contribution arising from axial force resultant results from the term $\delta^2 V_{gN}$ in Eq. (7.3). Expressing the displacement fields and their derivatives in terms of the nodal displacements based on Eqs. (7.6a-d) yields

$$\begin{Bmatrix} \delta u'_b \\ \delta \theta_{yb} \\ \delta \theta'_{yb} \\ \delta \theta_{zb} \\ \delta \theta'_{zb} \end{Bmatrix}_{5 \times 1} = [B_{gN}]_{5 \times 8} \{\delta u_{nb}\}_{8 \times 1}, \quad [B_{gN}]_{5 \times 8} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & l-z & 0 & 0 & 0 & z & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & l-z & 0 & 0 & 0 & z & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{5 \times 8} \quad (7.18, 7.19)$$

From the field interpolations (Eq. (7.19)) and axial stress resultant interpolation (Eq. (7.7)), by substituting into the energy term (Eq. (7.4c)) and integrating with respect to z , one obtains

$$\frac{1}{2} \delta^2 V_{gN} = -\frac{\lambda}{2} \langle \delta u_{nb} \rangle_{1 \times 8}^T [K_{gN}]_{8 \times 8} \{ \delta u_{nb} \}_{8 \times 1} \quad (7.20)$$

where the geometric stiffness matrix $[K_{gN}]$ contribution of the axial forces is expressed as

$$[K_{gN}]_{8 \times 8} = \frac{N_{p1}}{l} \begin{bmatrix} 1 & 0 & y_A & 0 & -1 & 0 & -y_A & 0 \\ & 0 & y_C l & 0 & 0 & 0 & 0 & 0 \\ & & -\beta & 0 & -y_A & 0 & \beta & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & 1 & 0 & y_A & 0 \\ & & & & & 0 & -y_C l & 0 \\ & & & & & & -\beta & 0 \\ & & & & & & & 0 \end{bmatrix}_{8 \times 8} \quad (7.21)$$

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and constant $\beta = (I_{xx}^2 + I_{xx} I_{yy} + y_A^2 A I_{xx} - y_A^2 S_x^2 - S_x I_{px}) / (A I_{xx} - S_x^2)$ has been defined.

7.6.2.3.4 Geometric Stiffness Contribution due to Bending Moments

The geometric stiffness contribution due to bending moments arises from the term $\delta^2 V_{gM}$ in Eq. (7.3). From the expressions of the nodal displacements (Eq. (7.18)) and the moment resultant interpolation (Eq. (8)), by substituting into Eq. 4d and integrating with respect to z , one obtains

$$\frac{1}{2} \delta^2 V_{gM} = -\frac{\lambda}{2} \langle \delta u_{nb} \rangle_{1 \times 8}^T [K_{gM}]_{8 \times 8} \{ \delta u_{nb} \}_{8 \times 1} \quad (7.22)$$

where the geometric stiffness matrix contribution due to bending moments $[K_{gM}]$ is

$$[K_{gM}]_{8 \times 8} = \frac{1}{6l} \begin{bmatrix} 0 & 0 & 3(-M_{xp1} + M_{xp2}) & 0 & 0 & 0 & 3(M_{xp1} - M_{xp2}) & 0 \\ & 0 & 2(-2M_{xp1} + M_{xp2})l & 0 & 0 & 0 & (M_{xp1} + M_{xp2})l & 0 \\ & & -3(M_{xp1} - M_{xp2})\chi & 0 & 3(M_{xp1} - M_{xp2}) & (M_{xp1} + M_{xp2})l & 3(M_{xp1} - M_{xp2})\chi & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & 0 & 0 & 3(-M_{xp1} + M_{xp2}) & 0 \\ & & & & & 0 & 2(M_{xp1} - 2M_{xp2})l & 0 \\ & & & & & & 3(-M_{xp1} + M_{xp2})\chi & 0 \\ & & & & & & & 0 \end{bmatrix} \quad (7.23)$$

sym

and constant $\chi = (S_x I_{xx} + S_x I_{yy} - A I_{px} - 2y_A S_x^2 + 2y_A A I_{xx}) / (A I_{xx} - S_x^2)$ has been defined.

7.6.2.3.5 Geometric Stiffness Contribution due to Shear Force

The geometric stiffness contribution arising due to shearing forces result from the term $\delta^2 V_{gV}$ in Eq. (7.3). Eqs. (7.6a-d) are used to express the displacement fields and their derivatives in terms of the nodal displacements yielding

$$\begin{Bmatrix} \delta u'_b \\ \delta \theta_{yb} \\ \delta \theta_{zb} \\ \delta \theta'_{zb} \end{Bmatrix}_{4 \times 1} = [B_{gV}]_{4 \times 8} \{\delta u_{nb}\}_{8 \times 1}, \quad [B_{gV}]_{4 \times 8} = \frac{1}{l} \begin{bmatrix} -1 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & l-z & 0 & 0 & | & 0 & z & 0 & 0 \\ 0 & 0 & l-z & 0 & | & 0 & 0 & z & 0 \\ 0 & 0 & 0 & -1 & | & 0 & 0 & 0 & 1 \end{bmatrix}_{4 \times 8} \quad (7.24, 7.25)$$

From Eqs. (7.9) and (7.24), by substituting into Eq. (7.4e) one obtains

$$\delta^2 V_{gV} = -\frac{\lambda}{2} \langle \delta u_{nb} \rangle_{1 \times 8}^T [K_{gV}]_{8 \times 8} \{\delta u_{nb}\}_{8 \times 1} \quad (7.26)$$

in which the geometric stiffness matrix $[K_{gV}]$ resulting from the shear resultant is expressed as

$$[K_{gV}]_{8 \times 8} = \frac{V_{yp1}}{6} \begin{bmatrix} 0 & 0 & 3 & 0 & | & 0 & 0 & 3 & 0 \\ & 0 & 2l & 0 & | & 0 & 0 & l & 0 \\ & & -6\xi & 0 & | & -3 & l & 0 & 0 \\ & & & 0 & | & 0 & 0 & 0 & 0 \\ \hline & & & & | & 0 & 0 & -3 & 0 \\ & & & & | & & 0 & 2l & 0 \\ & sym & & & | & & & 6\xi & 0 \\ & & & & | & & & & 0 \end{bmatrix}_{8 \times 8} \quad (7.27)$$

in which constant $\xi = D_{yk} / D_{yy}$ has been defined.

7.6.2.3.6 Geometric Stiffness Contribution due to Position of Distributed Transverse Load

The geometric stiffness contribution arising from the transverse load position results from the term $\delta^2 V_{gy}$ in Eq. (7.3). Eqs. (7.6a-d) is used to express the displacement field in terms of the nodal displacements, i.e.,

$$\delta \theta_{zb} = \langle B_{gy} \rangle_{1 \times 8} \{\delta u_{nb}\}_{8 \times 1}, \quad \langle B_{gy} \rangle_{1 \times 8}^T = \frac{1}{l} \langle 0 \quad 0 \quad (l-z) \quad 0 \quad | \quad 0 \quad 0 \quad z \quad 0 \rangle_{1 \times 8}^T \quad (7.28, 7.29)$$

From Eq. (7.28), by substituting into the energy term (Eq. (7.4f)), one obtains

$$\frac{1}{2} \delta^2 V_{gqy} = -\frac{\lambda}{2} \langle \delta u_{nb} \rangle_{1 \times 8}^T [K_{gqy}]_{8 \times 8} \{ \delta \theta_{zb} \}_{8 \times 1} \quad (7.30)$$

in which

$$[K_{gqy}]_{8 \times 8} = \int_0^l -q_y(z) \frac{(y_{qy}(z) - y_A)}{l^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & (l-z)^2 & 0 & 0 & 0 & z(l-z) & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & z^2 & 0 & 0 \\ & & & & & & 0 & 0 \end{bmatrix} dz \quad (7.31)$$

sym

For a given transverse load distribution $q_y(z)$ and load position function $y_{qy}(z)$, the transverse load position matrix $[K_{gqy}]$ can be determined.

7.6.2.3.7 Geometric Stiffness Contribution due to Position of Distributed Axial Force

The geometric stiffness contribution arising from the axial load position results from the term $\delta^2 V_{gqz}$. Expressing the displacement fields in terms of the nodal displacements based on Eqs. (7.6a-d) yields

$$\begin{Bmatrix} \delta \theta_{yb} \\ \delta \theta_{zb} \end{Bmatrix}_{2 \times 1} = [B_{qz}]_{2 \times 8} \{ \delta u_{nb} \}_{8 \times 1}, \quad [B_{qz}]_{2 \times 8} = \frac{1}{l} \begin{bmatrix} 0 & l-z & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & l-z & 0 & 0 & 0 & z & 0 \end{bmatrix}_{2 \times 8} \quad (7.32, 7.33)$$

From the expressions of the nodal displacements (Eq. (7.32)), by substituting into the energy term (Eq. (7.4g)), one obtains

$$\frac{1}{2} \delta^2 V_{gqz} = -\frac{\lambda}{2} \langle \delta u_{nb} \rangle_{1 \times 8}^T [K_{gqz}]_{8 \times 8} \{ \delta u_{nb} \}_{8 \times 1} \quad (7.34)$$

in which

$$[K_{gqz}]_{8 \times 8} = \int_0^l q_z(z) \frac{(y_{qz}(z) - y_c)}{l^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & (l-z)^2 & 0 & 0 & 0 & z(l-z) & 0 \\ & & 0 & 0 & 0 & z(l-z) & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & z^2 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix} dz \quad (7.35)$$

sym

For a given transverse load distribution $q_z(z)$ and a load position function $y_{qz}(z)$, the axial load position matrix $[K_{gqz}]$ can be determined.

7.6.3 Concentrated Load Effects

For a member subjected to a concentrated transverse load Q_y acting at $z = z_{Qy}$ and position y_{Qy} relative to the pole, the loading function can be expressed as $q_y(z) = Q_y \text{Dirac}(z - z_{Qy})$. By substituting into Eq. (7.4h), the geometric stiffness matrix due to the load position effect relative to the shear centre A takes the form

$$[K_{gy}]_{8 \times 8} = -Q_y \frac{(y_{Qy} - y_A)}{l^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & (l - z_{Qy})^2 & 0 & 0 & 0 & z_{Qy}(l - z_{Qy}) & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & z_{Qy}^2 & 0 \\ & & & & & & & 0 \end{bmatrix}_{8 \times 8} \quad (36)$$

sym

For the particular case where a concentrated load is applied at end $z_{Qy} = 0$, the only nonzero term of matrix $[K_{gy}]_{8 \times 8}$ in Eq. (7.46) is $K_{gy}(3,3) = -Q_y(y_{Qy} - y_A)$. Also, when a concentrated load is applied at end $z_{Qy} = l$, $K_{gy}(7,7) = -Q_y(y_{Qy} - y_A)$. Also, a concentrated axial load Q_z acting at $z = z_{Qz}$ and position y_{Qz} relative to the centroid C of the cross-section, the load can be expressed

as $q_z(z) = Q_z \text{Dirac}(z - z_{Qz})$. By substituting into Eq. (7.45), the geometric stiffness matrix due to the load position effect relative to the section centroid C is

$$[K_{gqz}]_{8 \times 8} = Q_z \frac{(y_{Qz} - y_c)}{l^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & (l - z_{Qz})^2 & 0 & 0 & 0 & z_{Qz}(l - z_{Qz}) & 0 \\ & & 0 & 0 & 0 & z_{Qz}(l - z_{Qz}) & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ \text{---} & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & & z_{Qz}^2 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix} dz \quad (7.37)$$

sym

For the case of a concentrated axial load is applied at end $z_{Qz} = 0$, the nonzero terms of matrix $[K_{gqz}]_{8 \times 8}$ in Eq. (7.37) are $K_{gqz}(2,3) = K_{gqz}(3,2) = Q_z(y_{Qz} - y_c)$. Also, when a concentrated axial load is applied at end $z_{Qz} = l$, $K_{gqz}(6,7) = K_{gqz}(7,6) = Q_z(y_{Qz} - y_c)$.

7.6.4 Summary

After determining the elastic stiffness contributions $[K_{en}]_{8 \times 8}$ and $[K_{es}]_{8 \times 8}$, and the geometric stiffness contributions $[K_{gN}]_{8 \times 8}$, $[K_{gM}]_{8 \times 8}$, $[K_{gV}]_{8 \times 8}$, $[K_{gqy}]_{8 \times 8}$ and $[K_{gqz}]_{8 \times 8}$, by substituting into the condition neutral equilibrium (Eq. 7.3), one recovers the discretized form of the neutral stability condition as

$$\delta \left[\frac{1}{2} \langle \delta u_{nb} \rangle_{1 \times 8}^T \left\{ [K_{en}]_{8 \times 8} + [K_{es}]_{8 \times 8} - \lambda \left([K_{gN}]_{8 \times 8} + [K_{gM}]_{8 \times 8} + [K_{gV}]_{8 \times 8} + [K_{gqy}]_{8 \times 8} + [K_{gqz}]_{8 \times 8} \right) \right\} \{ \delta u_{nb} \}_{8 \times 1} \right] = 0 \quad (7.38)$$

By setting the variation of the Eq. (7.38) to zero, the eigen value problem

$$\left([K_{en}]_{8 \times 8} + [K_{es}]_{8 \times 8} - \lambda \left([K_{gN}]_{8 \times 8} + [K_{gM}]_{8 \times 8} + [K_{gV}]_{8 \times 8} + [K_{gqy}]_{8 \times 8} + [K_{gqz}]_{8 \times 8} \right) \right) \{ \delta u_{nb} \}_{8 \times 1} = 0 \quad (7.39)$$

is recovered.

7.7 Examples

In this section, several examples are investigated in order to illustrate the features of the present formulation. In all examples, the material is assumed linear elastic with a modulus of elasticity $E=200,000 \text{ MPa}$ and shear modulus $G=77,000 \text{ MPa}$.

It is recalled that the present formulation is confined to doubly symmetric sections, i.e., the section centroid C coincides with the shear centre SC , and pole A_p was chosen to coincide with sectorial origin S_o , while the y coordinate of origin O was chosen to have an arbitrary value until this stage [7-28]. In all except Example 6, an orthogonal coordinate system is chosen. This is achieved by choosing origin O and pole A_p to coincide with centroid C . This selection is in line with classical solutions and leads to simplifications. In Example 6, we deviate from this convention by selecting non-orthogonal coordinates to illustrate the ability of the present formulation to model eccentric supports.

7.7.1 Example 1: Mesh Density Study

In order to determine the proper mesh density for the numerical studies, three cantilevers with spans L of 2m, 4m, and 6m are modeled using the present formulation. The cantilevers are subjected to a vertical load acting at the centroid of the free end (Fig. 7.3a). The section investigated is W250X58 designated in CISC Handbook of Steel Construction [7-27]. The section dimensions are provided in Fig. 3b.

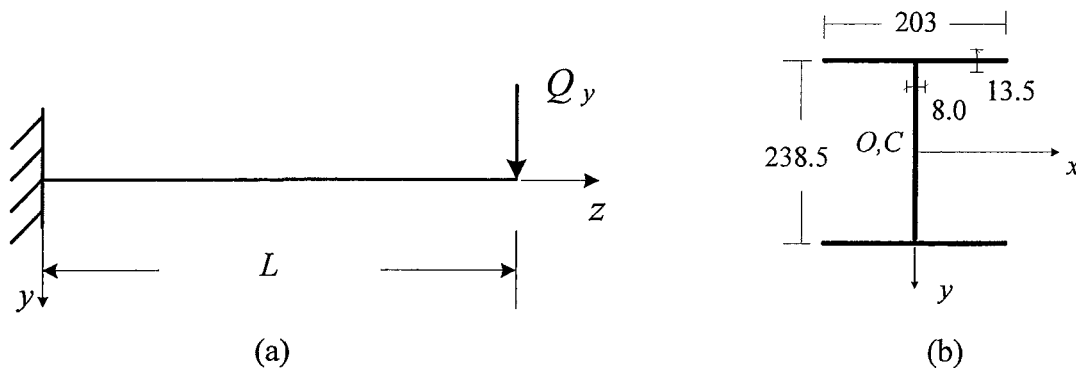


Figure 7.3 Cantilever with a wide-flange cross-section under an end load

Because the applied load is acting at the section shear centre, an orthogonal coordinate system is simplest to model the problem. Based on orthogonal coordinates, the sectional properties are I_{xx}

$$= 87.07 \times 10^6 \text{ mm}^4, I_{yy} = 18.83 \times 10^6 \text{ mm}^4, A = 7,389 \text{ mm}^2, I_{\omega\omega} = 2.677 \times 10^{11} \text{ mm}^6, J = 3.737 \times 10^5 \text{ mm}^4, D_{xx} = 5,481 \text{ mm}^2, D_{hh} = 77.60 \times 10^6 \text{ mm}^4, \text{ and } D_{yy} = 1,908 \text{ mm}^2.$$

Each cantilever is analyzed by using three meshes. The critical loads corresponding to each mesh are listed in Table 7.1. For the present cross-section, convergence is attained for the 64 element per meter mesh. Finer meshes were judged unnecessary while coarser meshes are observed to over-predict the critical load. The number of elements needed to achieve convergence is higher than that based on the classical solution by Barsoum and Gallagher (1970), in which 8 elements were found to be enough to achieve convergence, irrespective of the element span. The difference in convergence behavior is due to the fact that the shape functions in the present formulation are linear with C^0 continuity while the ones used in the classical solution were hermitian with $C1$ continuity. For longer spans, both solutions closely predict the buckling load while for the shorter 2m span, the buckling load predicted by the present study is found 9.5% less than that predicted by the classical solution.

This study was used as a guideline on the meshes to be adopted in the subsequent examples.

Table 7.1: Critical load Q_y (kN) predicted based on different meshes

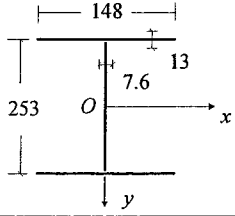
Span (m)	Present Study			Classical Solution	
	Number of Elements (per meter)			Number of Elements (total)	
	32	64	128	8	16
2	933.4	928.3	928.3	977.0	976.9
4	161.5	160.7	160.7	162.7	162.7
6	59.70	59.47	59.47	59.70	59.70

7.7.2 Example 2: Comparison to Other Solutions

A cantilever of a W200x45 cross-section is subject to a transverse load Q_y at the free end as shown in Fig. 7.3a. The load is applied at the shear centre of the cross-section. Two different spans L are

investigated: 1,000 mm and 5,000 mm. An orthogonal coordinate system is adopted and relevant sectional properties are given in Table 7.2.

Table 7.2 Sectional properties for Example 2

	$I_{xx} (mm^4)$	71.887×10^6	$J (mm^4)$	2.538×10^5
	$I_{yy} (mm^4)$	7.033×10^6	$D_{xx} (mm^2)$	3,848
	$A (mm^2)$	5,770.8	$D_{hh} (mm^4)$	61.577×10^6
	$I_{\omega\omega} (mm^6)$	1.011×10^{11}	$D_{yy} (mm^2)$	1,922.8

The buckling load is predicted based on three solutions: 1) The classical solution by Barsoum and Gallagher [7-2] which neglects shear deformation and distortional effects, 2) the present solution which accounts for shear deformation effects but neglects distortional effects, and 3) ABAQUS shell analysis which accounts for both effects. All three solutions neglect pre-buckling deformation effects.

Under the present solution, 64 elements were used per meter. In the abaqus model, six element per flange, and eight elements along the web height were used and 20 elements per meter in the longitudinal direction of the member (amounting to 400 elements per meter in total).

The results are summarized in Table 7.3. For the 5m span cantilever, the critical load based on the present study coincides with that based on the classical solution, but both solutions are 3.2% higher than that based on ABAQUS. The difference is likely due to distortional effects which are only captured in the ABAQUS solution. For the 1 m span cantilever, the critical load based on the present study is 4.7% higher than that based on ABAQUS, the difference being attributed to distortional effects, but 4.9% lower than that of the classical solution. The later difference is attributed to shear deformation effects which are expected to be more pronounced for cantilevers with shorter spans.

Table 7.3 Critical Load for cantilever under tip load

Span (mm)	1,000	5,000
Formulation		
ABAQUS (kN)	2,396	39.98
Present Study (kN)	2,515	41.29
Classical Solution (kN)	2,638	41.29

The buckling mode shapes for out-of-plane displacement u_b , y axis rotation θ_{yb} , z axis rotation θ_{zb} and warping deformation ψ_b obtained from the different solutions are plotted and compared in Fig. 7.4. It is noticed that for the one meter long cantilever, the buckling mode shapes obtained from ABAQUS solutions are in very good agreement with those based on the present study, but differ from the ones based on the classical solution. Conversely, for the five meter long cantilevers, the buckling mode shapes based on all three solutions agreed very well. The present theory more accurately predicts the buckling loads and modes than the classical solution for short span members but essentially provides results identical to those based on the classical for longer span members.

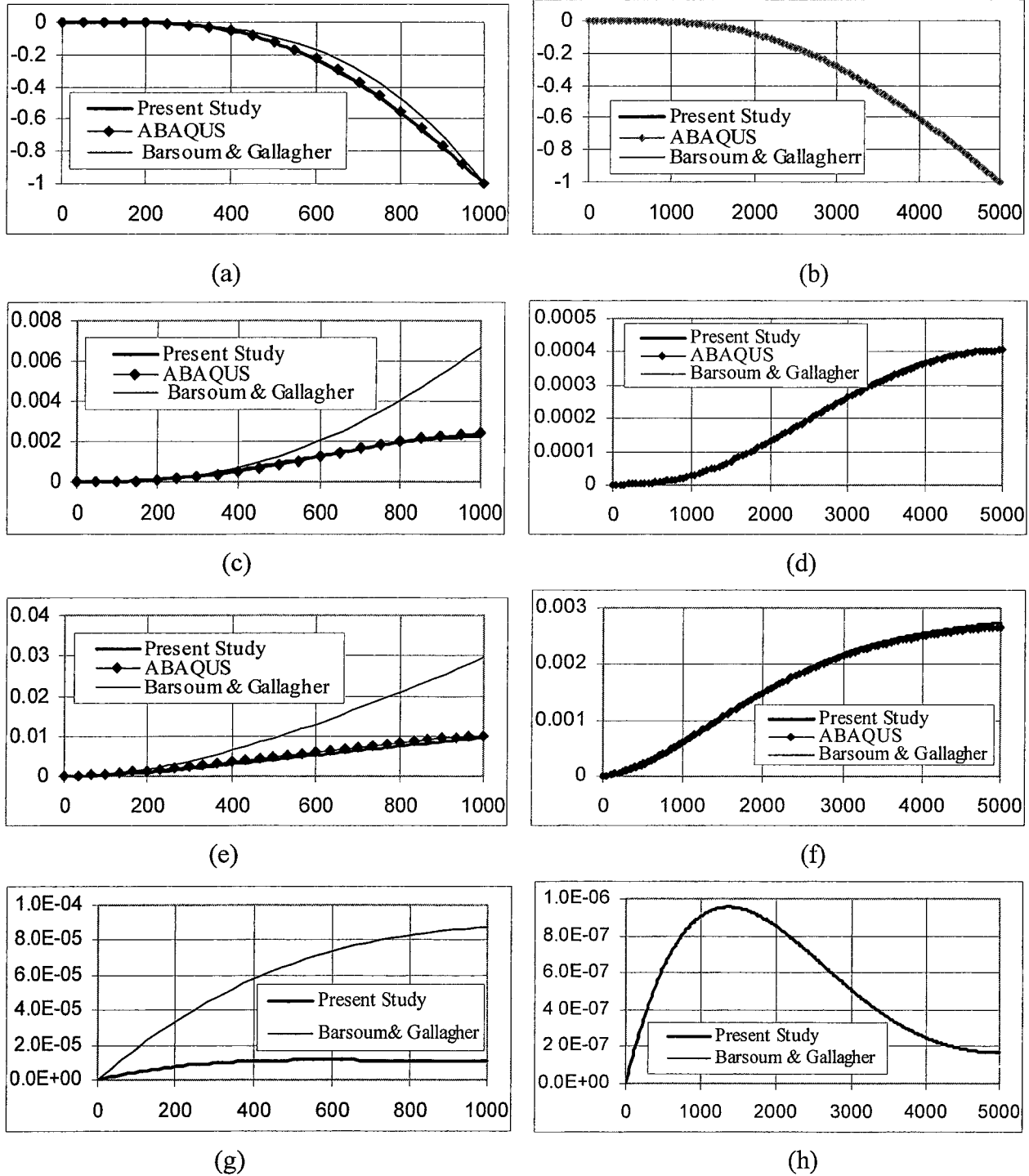


Figure 4 Buckling mode shape for Example 2, (a) Out-of-plane displacement (mm) for $L=1,000\text{mm}$, (b) Out-of-plane displacement (mm) for $L=5,000\text{mm}$, (c) Rotation (rad) about y-axis for $L=1,000\text{mm}$, (d) Rotation (rad) about y-axis for $L=5,000\text{mm}$, (e) Rotation about z-axis for $L=1,000\text{mm}$, (f) Rotation (rad) about z-axis for $L=5,000\text{mm}$, (g) Warping deformation (1/mm) for $L=1,000\text{mm}$, and (h) Warping deformation (1/mm) for $L=5,000\text{mm}$

7.7.3 Example 3: Members under Axial Force

A simply supported column is loaded by axial force Q_z (Fig. 7.5) acting at the centroid of the end cross-section. The dimensions of the cross-section are shown in Fig. 3b, and the cross-sectional properties are identical to those in Example 1. The span L of the column is varied from one to eight meters, and the critical load for each span is determined and presented in Table 7.5 based on four theories: the well known classical solution $Q_{zcr} = (\pi^2 EI_{xx} / L^2)$, and the closed form solutions based on the three shear deformable theories $Q_{zcr} = \left[\pi^2 EI_{yy} / L^2 \left(1 + k \pi^2 EI_{yy} / L^2 \right) \right]$, in which $k = 1/GD_{xx}$ from the derivation of the present study, $k = 6/5GD_{xx}$ from the Erkmén and Mohareb's theory [7-21], $k = 1/GA$ proposed by Alfutov [7-28] in the formulation of critical load of Timoshenko columns, $k = 0.253/GA$ determined by Wang et al [7-29].

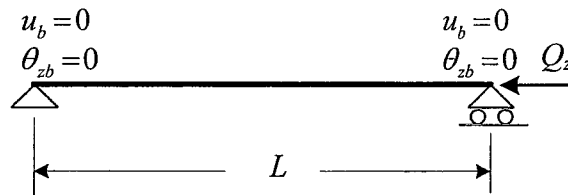


Figure 7.5 Simply supported column under axial force

Table 7.4 Critical buckling loads for columns

	Critical Load (kN)							
	1 m	2 m	3 m	4 m	5 m	6 m	7 m	8 m
Present Study	34,160	9,093	4,090	2,311	1,482	1,030	757.3	580.0
Erkmén and Mohareb [7-22]	33,620	9,054	4,082	2,308	1,481	1,030	757.0	580.0
Alfutov [7-28]	34,890	9,144	4,101	2,314	1,483	1,031	757.6	580.3
Wang et al [7-29]	29,540	8,730	4,015	2,286	1,472	1,025	754.7	578.5
Classical Solution	37,170	9,293	4,130	2,323	1,487	1,033	758.6	580.8
Present/Classical	91.9%	97.8%	99.0%	99.5%	99.7%	99.7%	99.8%	99.9%

The comparison provided at the bottom row of Table 7.4 indicates that, for the cross-section investigated, shear deformation effect on buckling loads is negligible for spans larger than 2m.

7.7.4 Example 4: Beam under Linear Moment Distribution

A simply supported beam with a W250x58 cross-section (cross-sectional properties based on an orthogonal coordinate system were given in Example 1) is subject to strong axis moments at both ends. The ratios of end moments are varied between 1 (case of uniform moment) and -1 (case where both moments are equal and opposite. Beam span is varied from one to eight meters.

The critical moments based on the present solution is presented in Table 7.5. They are compared to a) the ones obtained from the classical solution by Barsoum and Ghallagher which neglects shear

deformation effects, b) the AISC [7-30] solution $M_{xcrp} = C_b (\pi/L) \sqrt{EG I_{yy} J + (\pi E/L)^2 I_{yy} I_{\omega\omega}}$, in which C_b is the AISC [7-30] moment gradient factor,

$C_b = (12.5|M_{\max}|) / (2.5|M_{\max}| + 3|M_{L/4}| + 4|M_{L/2}| + 3|M_{3L/4}|)$, and $M_{\max}$ is the maximum bending moment within the member, and $M_{L/4}, M_{L/2}, M_{3L/4}$ are the bending moments at the quarter, midpoint and three quarter points of the spans, respectively, c) the solution based on the moment gradient factor proposed by Wong and Driver [7-31]

$C_b = (4M_{\max}) / \sqrt{((M_{\max})^2 + 4(M_{L/4})^2 + 7(M_{L/2})^2 + 4(M_{3L/4})^2)}$ and d) that based on the moment gradient factor proposed by Serna et al. (2006)

$$C_b = \sqrt{35(M_{\max})^2 / ((M_{\max})^2 + 9(M_{L/4})^2 + 16(M_{L/2})^2 + 9(M_{3L/4})^2)}.$$

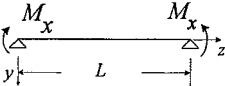
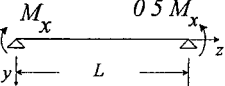
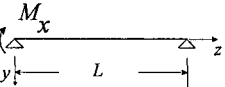
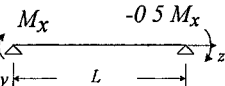
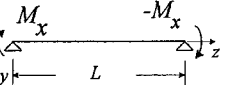
The results are presented in Table 7.5. As a general observation, the AISC moment gradient factor provides close approximations of the classical solution, consistently erring on the conservative side. The critical moment based on the present study is observed to always be lower than that based on the classical solution, a direct result of the fact that the present model allows for shear deformation and thus results in a more flexible model. The difference between the results based on the present formulation and those based on the classical and AISC is observed to be more pronounced for beams with a) shorter spans, and b) higher moment gradients. Both factors

increase the pre-buckling shear force V_{yp} . As a result, the shear destabilizing term reflected in $[K_{gV}]$ (present in the present formulation, and non-existent in the classical solution) plays a more predominant role for shorter beams with higher moment gradients.

It is observed that for the case of uniform moment, the critical moment based on the present theory is consistently lower than that predicted by the classical theory. This is due to the fact that beams are under uniform strong axis moment M (and thus zero shear force along the weak axis). In a pre-buckling analysis, whether one adopts the conventional Euler Bernoulli beam theory or the Timoshenko shear deformable theory, one would obtain zero shear stresses under uniform moments because of the absence of shear force.

Conversely, in a buckling analysis, the beam undergoes non-uniform twist $\theta_{zb}(z)$ throughout buckling and the corresponding weak axis moment in the deformed configuration is $M\theta_{zb}(z)$. This weak axis moment is clearly non-uniform. Thus, the associated shear force along the beam strong axis is non-zero and induces shear stresses in the cross-section. These shear stresses are not captured in the classical buckling formulation but captured in the present shear deformable formulation. Their inclusion in the present formulation leads to a more flexible representation of the stiffness of the beam and thus the buckling moments predicted by the present formulation should be lower than those based on the classical solution. A similar result was obtained in the formulation by Erkmén and Mohareb (2008a-b).

Table 7.5: Critical Moments for Beams under Moment Gradient

Critical Load (kNm)		1 m	2 m	3 m	4 m	5 m	6 m	7 m	8 m
	Present	4,234	1,209	597.1	377.9	272.1	211.7	173.3	146.5
	Classical[7-2]	4,554	1,223	601.3	379.0	272.5	211.9	173.3	146.7
	AISC [7-30]	4,551	1,223	601.1	378.9	272.4	211.8	173.3	146.7
	Wong [7-31]	4,551	1,223	601.1	378.9	272.4	211.8	173.3	146.7
	Serna [7-32]	4,551	1,223	601.1	378.9	272.4	211.8	173.3	146.7
	Present	5,533	1,588	788.6	499.5	357.8	278.4	227.7	192.8
	Classical[7-2]	6,016	1,616	794.0	500.3	359.5	279.5	228.3	193.2
	AISC [7-30]	5,916	1,590	781.4	492.6	354.1	275.4	225.2	190.7
	Wong [7-31]	5,887	1,582	777.5	490.1	352.4	274.0	224.1	189.7
	Serna [7-32]	5,960	1,601	787.2	496.2	358.8	277.4	226.9	192.1
	Present	7,251	2,135	1,068	676.2	486.9	378.6	308.9	262.1
	Classical[7-2]	8,447	2,266	1,111	698.2	500.3	387.8	316.1	266.8
	AISC [7-30]	7,964	2,140	1,052	663.1	476.7	370.7	303.2	256.4
	Wong [7-31]	7,945	2,135	1,049	661.5	475.6	369.8	302.4	256.0
	Serna [7-32]	8,260	2,219	1,091	687.7	494.4	384.4	311.4	266.2
	Present	8,259	2,584	1,310	833.6	601.5	468.4	383.2	324.3
	Classical[7-2]	11,922	3,190	1,559	975.4	695.6	536.5	435.4	365.9
	AISC [7-30]	10,694	2,873	1,413	890.4	640.2	497.8	407.1	344.6
	CSA[7-31]	10,402	2,795	1,374	866.1	622.7	484.1	396.0	335.2
	Serna [7-32]	11,320	3,041	1,495	942.5	677.6	526.9	430.9	364.8
	Present	7,133	2,304	1,178	752.4	544.1	424.2	347.5	294.4
	Classical[7-2]	12,490	3,350	1,644	1,033	740.8	574.1	467.9	394.7
	AISC [7-30]	11,380	3,057	1,503	947.3	681.0	529.6	433.1	366.6
	Wong [7-31]	10,510	2,824	1,388	875.0	629.1	489.2	400.1	338.7
	Serna [7-32]	11,480	3,084	1,516	955.8	687.2	534.3	437.0	370.0

7.7.5 Example 5: Axial Force-Bending Moment Interaction

A four meter span member of a W250X58 [7-27] cross-section is axially loaded by force Q_z at the right end and end moments M_x at both ends (Fig. 7.6). From Example 3, the buckling load Q_{z0} in the absence of bending moments is 2,311 kN, while (from Example 4) the buckling moment M_{x0} in the absence of axial force is 377.9 kNm. It is required to generate a Q_z - M_x buckling interaction diagram for the member. The diagram is generated by applying $i=1\dots n$ reference load combinations (Q_{zi}, M_{xi}) and obtaining the buckling eigen value for each reference case. The critical load combination $\lambda_i(Q_{zi}, M_{xi})$ provides a point on the interaction diagram. The process is repeated for n points to trace the interaction curve sought. The curve is normalized by dividing the buckling axial force $\lambda_i Q_{zi}$ by Q_{z0} and the associated buckling moment $\lambda_i M_{xi}$ by M_{x0} . The resulting interaction curve is plotted on Fig.7.7. The negative number of Critical Compressive Force Ratio ($\lambda_i Q_{zi} / Q_{z0}$) on Fig. 7.6 denotes a tensile force. As the compressive force ratio increases, the corresponding critical moment ratio decreases.

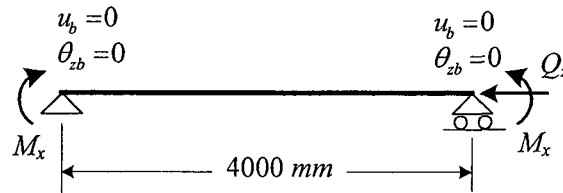


Figure 7.6 Simply supported beam under axial force and bending moment

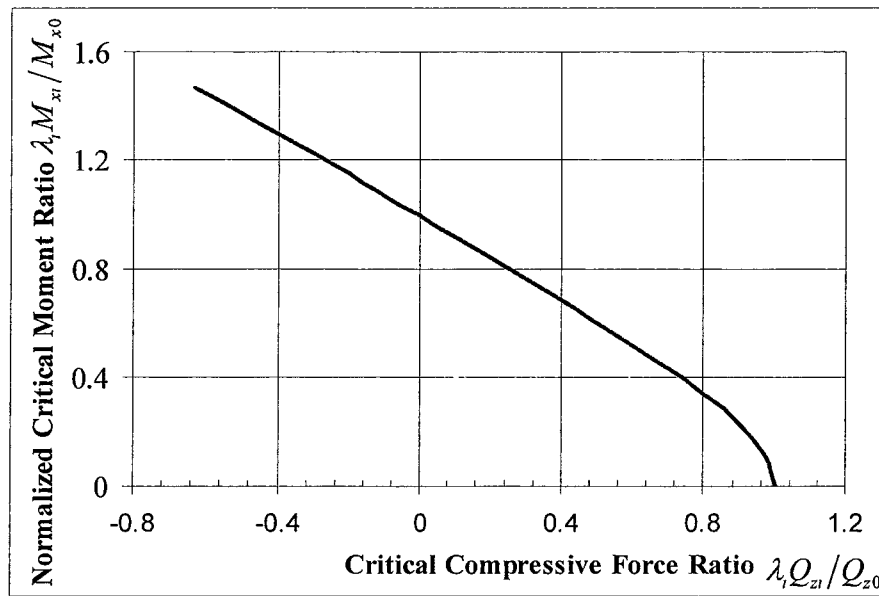


Figure 7.7 Normalized Interaction Diagram

7.7.6 Example 6: Lateral Buckling of Eccentrically Supported Beam

7.7.6.1 Motivation: In industrial structures, flexural steel members can sometimes be eccentrically supported (e.g., Fig. 7.8a), i.e., the support location is offset from the member centroid. In the absence of simplified solutions for such cases, the analyst may be forced to either neglect the effect of support eccentricity under the classical solution, or conduct a shell FEA analysis to capture this effect. The present formulation provides a simpler alternative solution which accounts for support eccentricity through the use of non-orthogonal coordinates illustrated in Fig. 7.8b.

7.7.6.2 Description: A five meter span beam is supported at both ends at the bottom flange. The beam cross-section is W250X58 [7-27] and the dimensions of the cross-section are given in Fig. 7.3b. A concentrated transverse load Q_y is applied at mid-span. Three cases are investigated for the load position: 1) top flange loading, 2) shear center loading, and 3) bottom flange loading. In order to prevent possible web distortion at the supports, transverse stiffeners are provided at both ends of the beam as shown. The stiffeners provide negligible warping resistance at the end of the member.

7.7.6.3 Solution: A seamless means of modelling the given eccentric support condition is to select origin O to coincide with the bottom flange to web junction. The sectional properties based on the non-orthogonal coordinates selected are provided in Table 7.6. The corresponding pre-buckling boundary conditions at both ends are $v_p = w_p = 0$, while the buckling boundary conditions are $u_b = \theta_{yb} = \theta_{zb} = 0$. It is noted that the formulation in [7-22] is able to handle only the case where the point of application of the load matches the support level (i.e., bottom flange loading in the present example). In this respect, the present formulation is superior in the sense that it is able to additionally model top flange and shear centre loading cases.

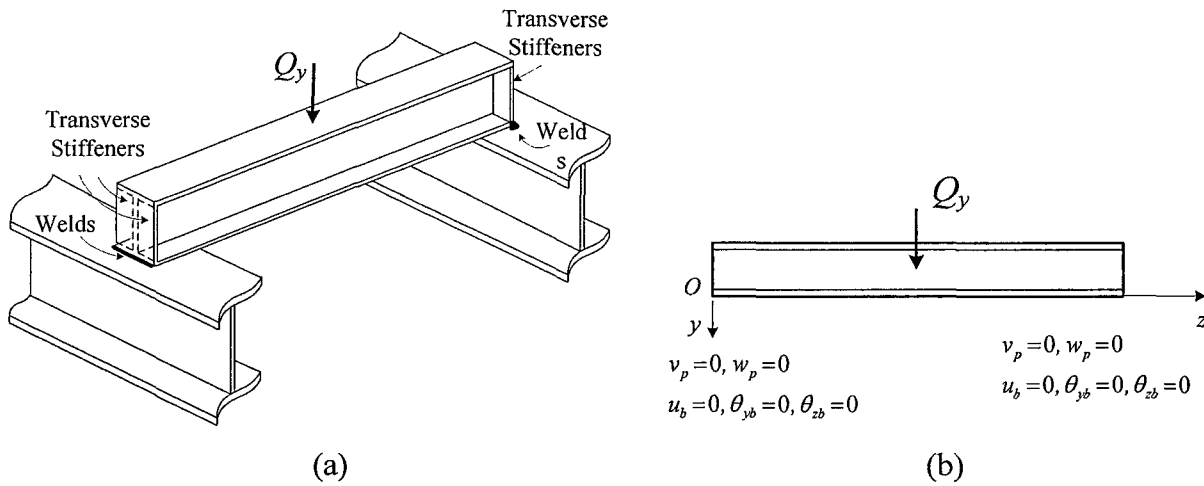


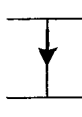
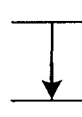
Figure 7.8 Beam with eccentric supports, (a) a thin-walled beam with transverse stiffeners at its two ends supported at the two end of the bottom flange, (b) Boundary conditions

Table 7.6 Sectional properties based on non-orthogonal coordinates

$I_{xx} (mm^4)$	1.921×10^8	$J (mm^4)$	3.737×10^5	$I_{px} (mm^5)$	-4.589×10^{10}
$I_{yy} (mm^4)$	18.83×10^6	$D_{xx} (mm^2)$	5,481	$S_{x\omega} (mm^5)$	2.245×10^9
$A (mm^2)$	7,389	$D_{hh} (mm^4)$	1.559×10^8	$S_x (mm^3)$	-8.811×10^5
$I_{\omega\omega} (mm^6)$	5.353×10^{11}	$D_{yy} (mm^2)$	1,908	$D_{xh} (mm^3)$	-6.536×10^5

The problem is also modeled based on the ABAQUS shell model (S4R element). As illustrated in Table 7.7, for all three cases excellent agreement is obtained between the ABAQUS shell analysis and present solution.

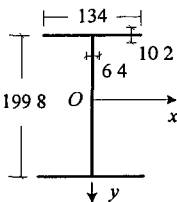
Table 7.7: Critical loads for eccentrically supported beam

			
ABAQUS	200.0 kN	290.0 kN	432.0 kN
Present Solution	206.0 kN	298.0 kN	432.0 kN

7.7.7 Example 7: Beam with Overhangs

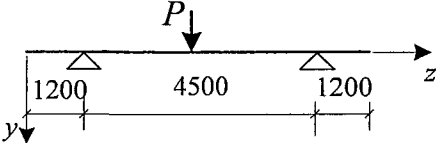
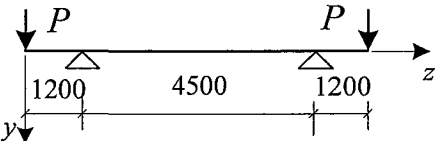
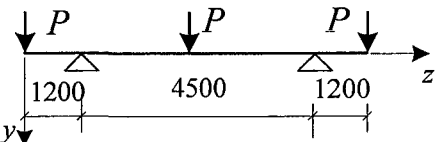
The buckling loads for 4,500 mm span beam with two 1,200 mm span overhangs are to be determined under three loading patterns: 1) mid-span load, 2) two tip loads of equal magnitude, and 3) a mid-span load and two tip loads of equal magnitude. The cross-sectional dimensions referred to W200X15 [7-27] and properties are present in Table 7.8.

Table 7.8 Sectional properties based on orthogonal coordinates

	$I_{xx} (mm^4)$	31.63×10^6	$J (mm^4)$	1.123×10^5
	$I_{yy} (mm^4)$	4.095×10^6	$D_{xx} (mm^2)$	2,734
	$A (mm^2)$	4,014	$D_{hh} (mm^4)$	27.34×10^6
	$I_{\omega\omega} (mm^6)$	4.09×10^{10}	$D_{yy} (mm^2)$	1,280

The critical loads obtained from the present study are compared with those based on ABAQUS shell solution (Table 7.9). It is noted that of all three solutions, Case 3 involving three loads is found less critical than that of Case 1 involving a single mid-span load and Case 2 involving two tip loads. In all three cases, the buckling load based on the present solution is found closer to the ABAQUS solution than that based on the classical solution. Also of interest is the fact that all three analyses indicate that the lowest buckling load is that based on Case 2 involving two tip loads. The present solution captures the load position effect. In case 2, the predicted buckling loads are less than 8% higher than those predicted by ABAQUS and quite close to those predicted by the classical finite element (Barsoum & Gallagher).

Table 7.9 Buckling Loads for beams with overhangs

	Loading Position	ABAQUS (kN)	Present Study (kN)	Classical (kN)
Case 1-Midspan load 		122.3	130.7	130.7
		91.0	99.2	99.2
		70.2	74.8	74.8
Case 2-Tip loads 		68.1	72.4	72.4
		64.0	69.4	69.5
		56.9	61.4	61.4
Case 3-Midspan and Tip loads 		252.7	271.7	273.6
		159.2	172.6	172.9
		72.0	79.8	79.9

7.8 Summary and Conclusions

1. A shear deformable buckling element was formulated based on non-orthogonal coordinates.
2. A mesh density study was conducted to establish the convergence characteristics of the solution. The study indicates that a mesh finer than that of the classical buckling element is needed to achieve convergence.
3. In all cases examined, the present solution provides buckling load predictions lower than those based on the classical solution, but higher than the shell FEA analysis.
4. The present model was found to predict buckling modes in excellent agreement with those based on ABAQUS shell analysis. For the short cantilever case, the mode shapes based on the present solution and ABAQUS were found to differ significantly from that based on the classical solution.

5. A comparison with the classical solution and shell FEA suggests that shear deformation effects are not significant in columns but important for beams with shorter spans and/or steep moment gradients.
6. The non-orthogonal coordinate feature of the solution was successful in predicting the buckling loads for eccentrically supported beams.

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List of Symbols

A :	Cross-sectional area
A_p	Pole
C :	Cross-section centroid
$D_{hh}, D_{xx}, D_{xh}, D_{yh}, D_{yy}, D_{yyk}$:	Cross-section properties accounting for shear deformation effects
E :	Young's modulus
G :	Shear modulus
$h(s)$:	Perpendicular distance from the pole to the tangent of a point on the profile
$\langle H_b(z) \rangle$:	Vector of shape functions used to interpolate buckling displacement fields
$[H_p(z)]$:	Matrix of shape functions used to interpolate pre-buckling displacement fields
I_{xx}, I_{yy} :	Moment of inertias about the x and y axes, respectively
$I_{\omega\omega}$:	Second moment of area of the sectorial coordinate
J :	St. Venant torsional constant
$k(s)$:	Distance between pole A_p and a point on the profile along tangential direction
$[K_{en}]$:	Elastic stiffness matrix arising from normal stresses
$[K_{es}]$:	Elastic stiffness matrix arising from shear stresses
$[K_{gN}]$:	Geometric stiffness matrix arising from axial resultant
$[K_{gM}]$:	Geometric stiffness matrix arising from moment resultant
$[K_{gV}]$:	Geometric stiffness matrix arising from shear resultant
$[K_{gqy}]$:	Geometric stiffness matrix arising from transverse force offset from pole
$[K_{gqz}]$:	Geometric stiffness matrix arising from longitudinal force offset from pole
l :	Length of finite element
L :	Member span
$M_{xp}(z)$:	Moment resultant of normal stresses obtained from pre-buckling analysis
$N_p(z)$:	Normal force resultant of normal stresses obtained from pre-buckling analysis
O :	Origin of Cartesian coordinates x , y , and z
q_y, q_z :	Line loads acting on a member along the y and z direction, respectively
s :	Curvilinear coordinate along cross-section mid-surface
S_o :	Sectorial origin

S_x :	First moments of area of coordinates x
$S_{x\omega}$:	Section products of inertia
SC :	Shear center of a cross-section
t :	Thickness of cross-section
$\{u_{nb}\}$:	Vector of buckling nodal displacements
$\{u_{np}\}$:	Vector of pre-buckling nodal displacements
u_b :	Lateral Buckling displacement
U :	Internal strain energy
v_p, w_p :	Pre-buckling displacements along the y and z directions
V :	Potential energy
$V_{yp}(z)$:	Shear force acting along the y direction as obtained from pre-buckling
x, y, z :	Cartesian coordinates
α :	Angle between the tangent to the contour at point S and x axis
ε_{zz} :	Longitudinal strain along z direction
γ_{zs} :	Shear strain along the profile of a cross-section
λ :	Load multiplier (Eigen-value)
π :	Total potential energy
θ_{xp} :	Pre-buckling rotation angle about the x axis
θ_{yb}, θ_{zb} :	Buckling rotation angles about the y, z axes, respectively
σ_{zz} :	normal stress along z direction
τ_{zs} :	shear stress on the cross-section midsurface
$\omega(s)$:	warping function or sectorial area of a cross-section
$\psi(z)$:	warping deformation ($1/Length$)

Chapter 8 Numerical Study of Buckling Behavior of Frame Joints

The buckling element developed in Chapter 7 was shown to successfully predict the buckling resistance of co-linear structures. The objective of the present chapter is to extend the formulation to apply for planar non co-linear structures such as portal frames and Γ -shaped frames by formulating a stiffness matrix which captures the partial warping restraint behaviour of joints. A mathematical description of the joint restraint to the displacements of the adjacent members will be formulated in a finite element context. The developed joint element is designed to properly interface with established lateral torsional buckling finite elements, such as the element developed in Chapter 7, and the conventional classical buckling element based on (e.g., Barsoum and Gallagher (1970), Trahair (1993)).

8.1 Assumptions and Limitations

The main assumptions and limitations of the formulation are:

1. The formulation is applicable to planar frames consisting of I shaped doubly symmetric cross-sections,
2. All members are connected at right angles through welded moment connections consisting of two pairs of horizontal and two pairs of vertical stiffeners,
3. The material is assumed linearly elastic isotropic and obeys Hooke's law,
4. Throughout buckling, the joint is assumed to undergo two actions: (a) rigid body displacements and rotations, (b) warping elastically along the sides of adjoining members such that it matches their warping deformations,
5. All rotations under item 4(a) are assumed small,
6. The warping response of the joint (item 4(b)) is characterized by the plate bending and plane stress resistances of its plates,
7. When characterizing the plate bending resistance of the joint (item 4b), second order effects due to large displacements are neglected, and

8. Second order effects in the pre-buckling state are neglected.

8.2 Coordinate Systems

A planar frame structure is considered. The frame is assumed to consist of n members, m joints, and o nodes. Each member element $i=1, \dots, n$ has its own local coordinate system (x_{ei}, y_{ei}, z_{ei}) in which the subscript e refers to an element. Also, each joint member $k=1, \dots, m$ has its own local coordinate system (x_{jk}, y_{jk}, z_{jk}) and subscript j refers to a joint. Coordinate systems (x_{ei}, y_{ei}, z_{ei}) define the positive directions of the nodal displacements within the member elements while coordinate systems (x_{jk}, y_{jk}, z_{jk}) define the positive directions of the nodal displacements within joint element k . In contrast, the nodal degrees of freedom at nodes $1, 2, \dots, o$ are oriented along the global coordinate system (x_g, y_g, z_g) .

8.3 Displacements of a Joint Element and Notations

When a laterally unsupported planar frame is subject to in-plane loading, joint k will undergo a movement from the initial state (Configuration 1 in Fig. 8.1) to an equilibrium state (Configuration 2). The pre-buckling movement is characterized by the generalized in-plane pre-buckling displacements ${}^j_k v_{pl}$, ${}^j_k w_{pl}$, and ${}^e_k \theta_{xpl}$ (Fig. 8.1). The generalized displacements are defined at a reference point (arbitrarily selected as node 2 here). Subscript p denotes all generalized pre-buckling displacements. As the loads are magnified by a factor λ , the structure is assumed to reach the state of onset of buckling (Configuration 3), and the displacements associated with this configuration become $\lambda({}^j_k v_{pl})$, $\lambda({}^j_k w_{pl})$, $\lambda({}^j_k \theta_{xpl})$ (Fig. 8.1). This assumption neglects second order effects throughout the pre-buckling state. In the pre-buckling stage, the joint is assumed to behave rigidly. When the structure undergoes lateral-torsional buckling, the joint is assumed to undergo an additional out-of-plane rigid body motion consisting of out-of-plane displacement ${}^j_k u_{bl}$ and two rotations ${}^j_k \theta_{yb}$ and ${}^j_k \theta_{zbl}$ (in Fig. 8.1). In addition, the joint undergoes elastic warping deformations ${}^j_k \psi_{bl}$ (Fig. 8.2) at Face 1 of the joint (connected to Element 1) and ${}^j_k \psi_{bl}$ at Face 2 (connected to Element 2) as illustrated in Fig. 8.2. The warping deformation ${}^j_k \psi_{bl}$ induces an angle of rotation ${}^j_k \theta_{\psi bl}$ at edge 3'-3'' and an equal but opposite

angle of rotation at edge 5'-5'', while the warping deformation ${}^k_2\psi_{bl}$ induces an angle of rotation ${}^k_2\theta_{\psi bl}$ at edge 3'-3'' and an equal but opposite angle of rotation at edge 4'-4''.

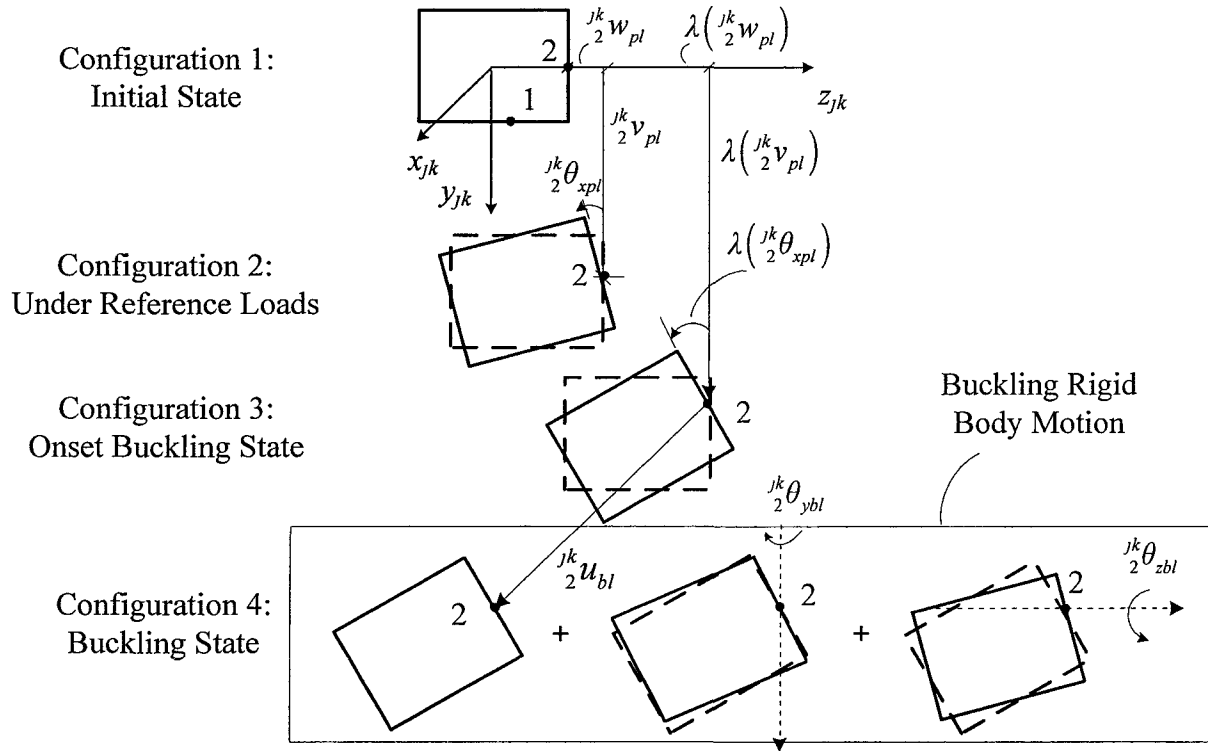


Figure 8.1 Rigid body motion of a joint from initial state to buckled configuration

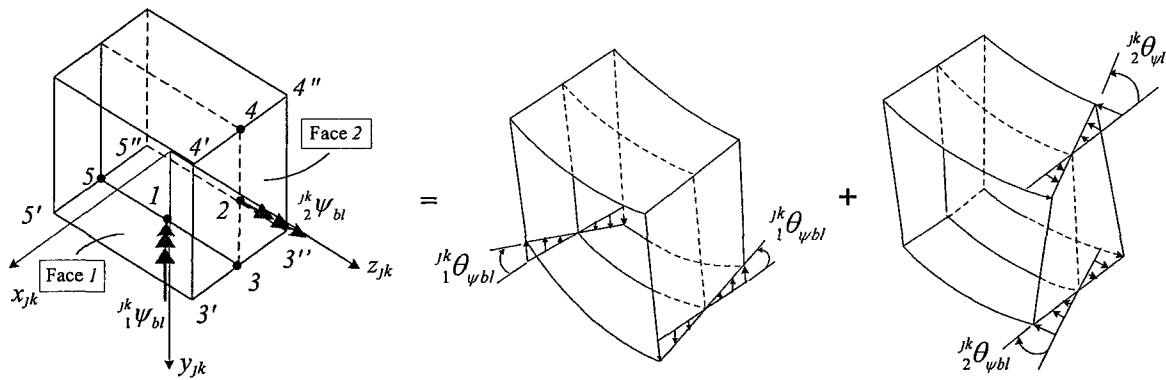


Figure 8.2 Warping deformations of joint k – Sign convention

8.4 Variational Formulation for Plane Frame

When the frame is in a neutral stability state, the variation of the second variation of the total potential energy π has to vanish for an arbitrary set of buckling deformations, i.e.,

$$\delta\left(\frac{1}{2}\delta^2\pi\right)=\delta\left[\frac{1}{2}\left(\delta^2\pi_m+\delta^2\pi_j\right)\right]=0 \quad (8.1)$$

in which π_m denotes the summation of the total potential energy of all the member elements, while π_j denotes the summation of the total potential energy of the joint elements (Fig. 8.3). The second variation of the total potential energy of the members takes the form

$$\begin{aligned} \frac{1}{2}\delta^2\pi_m &= \sum_{i=1}^n \frac{1}{2} \left\langle \delta\left({}^{ei}U_{nl}\right) \right\rangle^T \left[{}^{ei}K - \lambda\left({}^{ei}K_G\right) \right] \left\{ \delta\left({}^{ei}U_{nl}\right) \right\} \\ &= \sum_{i=1}^n \frac{1}{2} \left\langle \delta\left({}^{ei}U_{ng}\right) \right\rangle^T \left[{}^{ei}T \right]^T \left[{}^{ei}K - \lambda\left({}^{ei}K_G\right) \right] \left[{}^{ei}T \right] \left\{ \delta\left({}^{ei}U_{ng}\right) \right\} \end{aligned} \quad (8.2)$$

in which $\left\langle \delta\left({}^{ei}U_{nl}\right) \right\rangle^T = \left\langle {}^e_1u_{bl} \quad {}^e_1\theta_{ybl} \quad {}^e_1\theta_{zbl} \quad {}^e_1\psi_{bl} \mid {}^e_2u_{bl} \quad {}^e_2\theta_{ybl} \quad {}^e_2\theta_{zbl} \quad {}^e_2\psi_{bl} \right\rangle$ denotes the vector of buckling nodal displacements expressed in the local coordinates of the member. As a notation convention, the left superscript ei denotes member element i , and the left subscript takes the values 1 or 2 to denote the first or second nodes of the element, respectively. The identifier l in the right subscript indicates that nodal values are defined in local coordinates while the identifier g denotes they are defined in the global directions. In Eq. 8.2, ${}^{ei}K$ is the elastic stiffness matrix for element i and ${}^{ei}K_G$ is the geometric stiffness matrix. These were formulated in Chapter 7.

As a general convention, all subscripts b denote field variables throughout the buckling stage (i.e., in going from configuration 3 to 4). This is the case for instance for nodal displacement ${}^e_1u_{bl}$. Also, subscripts p denote field variables throughout the pre-buckling stage (i.e., in going from configuration 1 to 2). Total field variables (i.e., in going from configuration 1 to 4) will be denoted superscripts $*$, while variables without subscripts b , p , nor superscript $*$ are generic and are applicable to pre-buckling, buckling, or total field variables.

In the right hand side of Eq. 8.2, the variations of the local nodal displacements $\left\{ \delta\left({}^{ei}U_{nl}\right) \right\}$ are related to those of the global nodal displacements $\left\{ \delta\left({}^{ei}U_{ng}\right) \right\}$ through the transformation

$$\left\{ \delta \left({}^e U_{nl} \right) \right\}_{8 \times 1} = \left[{}^e T \right]_{8 \times 8} \left\{ \delta \left({}^e U_{ng} \right) \right\}_{8 \times 1} \quad (8.3)$$

in which

$$\left[{}^e T \right]_{8 \times 8} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \phi_{ei} & \sin \phi_{ei} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin \phi_{ei} & \cos \phi_{ei} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos \phi_{ei} & \sin \phi_{ei} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sin \phi_{ei} & \cos \phi_{ei} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{8 \times 8} \quad (8.4)$$

and angle ϕ_{ei} is the counterclockwise rotation angle from the global z_g direction to the local z_{ei} direction for member i (Fig. 8.3a). In the class of problem interested in the present study ϕ_{ei} takes one of the values $0, \pi/2, \pi, 3\pi/2$.

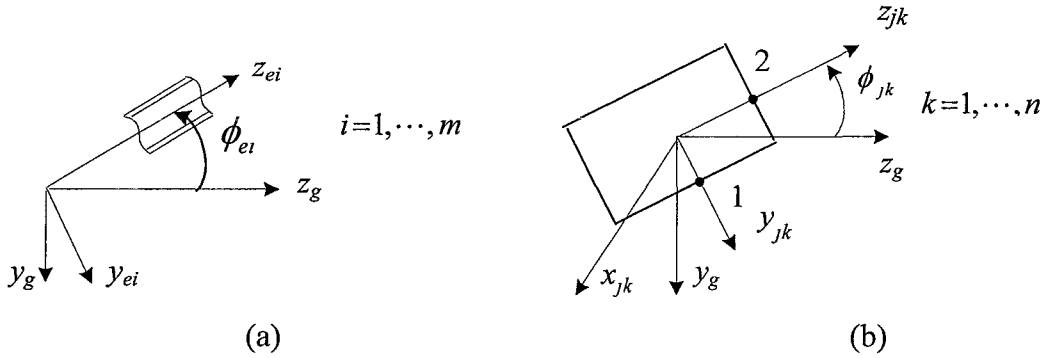


Figure 8.3 Definition of orientation angles ϕ (a) for the member element i ,
(b) for the joint element k

8.5 Finite Element Formulation

The joint element as described, has three pre-buckling and four buckling generalized displacements per node. The joint element has a total of eight buckling degrees of freedom $\langle {}^{jk}_1 u_{bl} \quad {}^{jk}_1 \theta_{ybl} \quad {}^{jk}_1 \theta_{zbl} \quad {}^{jk}_1 \psi_{bl} \mid {}^{jk}_2 u_{bl} \quad {}^{jk}_2 \theta_{ybl} \quad {}^{jk}_2 \theta_{zbl} \quad {}^{jk}_2 \psi_{bl} \rangle$. Each node is intended to interface with that of a buckling finite element (either the classical buckling element in Barsoum and Gallagher (1970), or the shear deformable element developed in Chapter 7 in the present study), i.e., the first node connects to Element 1 through the global Node 1 with generalized buckling nodal displacements ${}^{N1}u_{bg}$, ${}^{N1}\theta_{ybg}$, ${}^{N1}\theta_{zbg}$ and ${}^{N1}\psi_{bg}$ all defined in the global coordinates, while

the second node connects to the end of Element 2 through Node 2 with the generalized buckling displacements $N^2 u_{bg}$, $N^2 \theta_{ybg}$, $N^2 \theta_{zbg}$ and $N^2 \psi_{bg}$ (Fig. 8.4). All nodal displacements are defined along the global degrees of freedom.

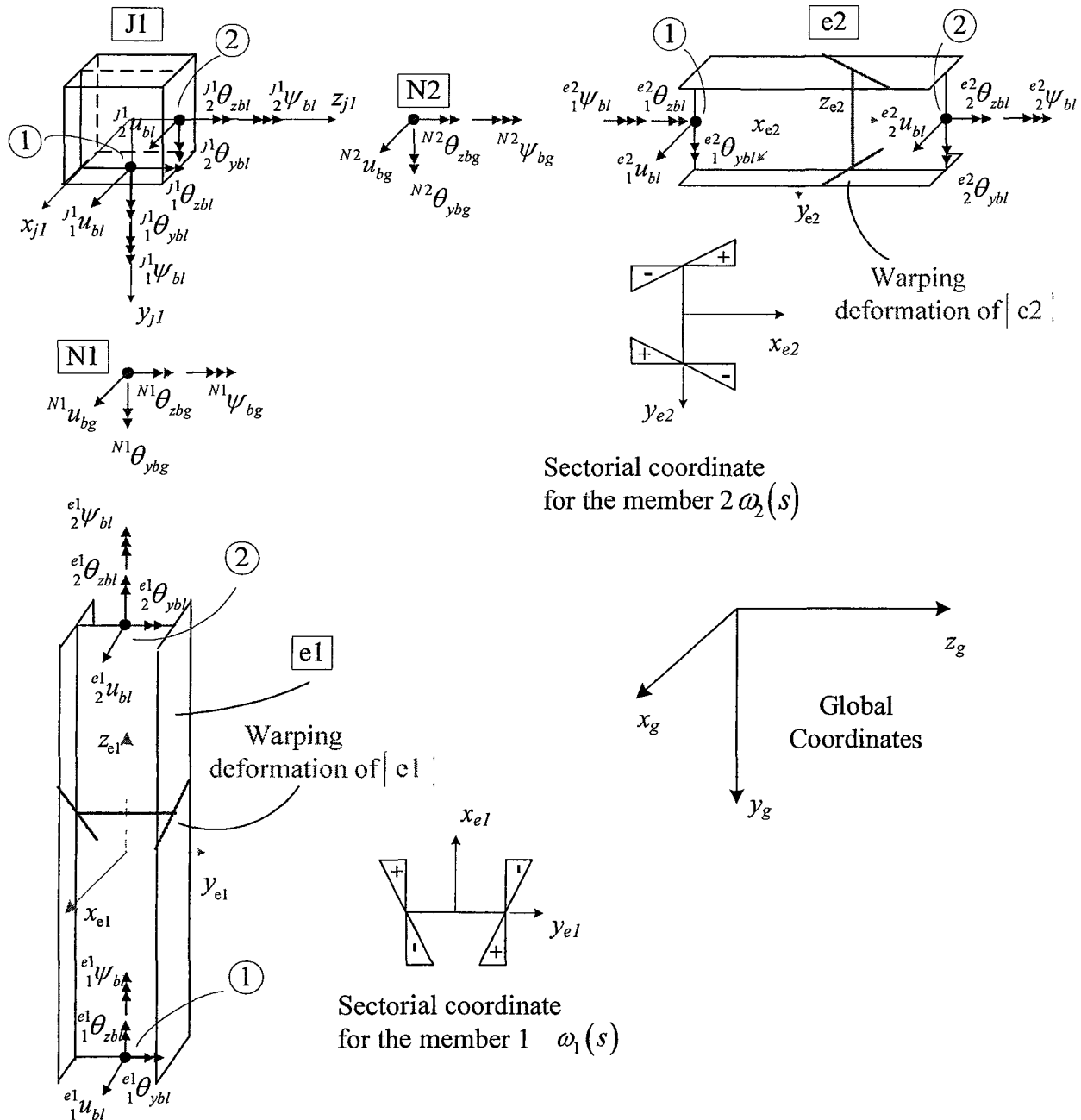


Figure 8.4 Example of a frame structure coordinates and notations

8.5.1 Constraints based on Buckling Rigid Body Motion

8.5.1.1 Rotations at Point 1

For a given joint k , based on the rigid body assumption (Assumption 4), the buckling rotations at node 1 and node 2 must match (Fig. 8.5), i.e.,

$$\left(-{}^j\theta_{1ybl} + {}^j\theta_{2ybl}\right) = 0, \quad \left(-{}^j\theta_{1zbl} + {}^j\theta_{2zbl}\right) = 0 \quad (8.5a,b)$$

8.5.1.2 Displacements at Point 1 due to Rigid Body Motion

In this section, the out-of-plane displacement ${}^j\mathbf{u}_{bl}$ at node 1 will be expressed in terms of nodal displacements $({}^j\mathbf{u}_{bl}, {}^j\theta_{ybl}, {}^j\theta_{zbl})$ at node 2 and the warping deformations ${}^j\psi_{bl}$ and ${}^j\psi_{bl}$. In the joint local coordinate system (x_{jk}, y_{jk}, z_{jk}) , position vectors $\{x_1\}$, $\{x_2\}$ and $\{x_3\}$ for nodes 1, 2 and point 3 (Fig. 8.5) in the undeformed configuration are

$$\{x_1\} = \begin{Bmatrix} 0 \\ h_2/2 \\ 0 \end{Bmatrix}, \quad \{x_2\} = \begin{Bmatrix} 0 \\ 0 \\ h_1/2 \end{Bmatrix}, \quad \{x_3\} = \begin{Bmatrix} 0 \\ h_2/2 \\ h_1/2 \end{Bmatrix} \quad (8.6a-c)$$

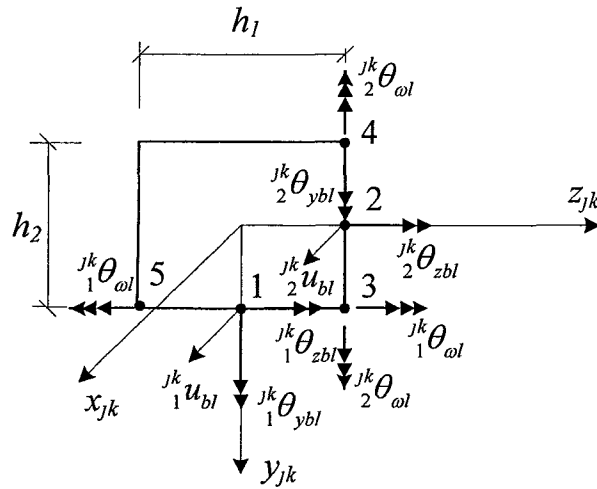


Figure 8.5 Buckling degrees of freedom for a joint element in local coordinates (displacements are shown as single headed arrows, rotations as double headed arrows, and rotations induced by warping a triple headed arrows)

The position vector $\langle x_{12} \rangle^T$ of Point 1 relative to Point 2 is

$$\langle x_{12} \rangle^T = \langle x_1 \rangle^T - \langle x_2 \rangle^T = \langle 0 \quad h_2/2 \quad -h_1/2 \rangle \quad (8.7a)$$

Also the position vectors $\langle x_{13} \rangle^T$ and $\langle x_{23} \rangle^T$ for of nodes 1 and 2 relative to point 3 are

$$\langle x_{13} \rangle^T = \langle x_1 \rangle^T - \langle x_3 \rangle^T = \langle 0 \quad 0 \quad -h_1/2 \rangle \quad (8.7b)$$

$$\langle x_{23} \rangle^T = \langle x_2 \rangle^T - \langle x_3 \rangle^T = \langle 0 \quad -h_2/2 \quad 0 \rangle \quad (8.7c)$$

The displacement vector $\langle u_{1l} \rangle^T = \langle {}^{jk}_1 u_l^* \quad {}^{jk}_1 v_l^* \quad {}^{jk}_1 w_l^* \rangle^T$ at node 1 due to the rigid body displacements $\langle u_{2l} \rangle^T = \langle {}^{jk}_2 u_l^* \quad {}^{jk}_2 v_l^* \quad {}^{jk}_2 w_l^* \rangle^T$ and rotations at node 2 are

$$\{u_{1l}\}_{3 \times 1} = \{u_{2l}\}_{3 \times 1} + [R]_{3 \times 3} \{x_{12}\}_{3 \times 1} \quad (8.8)$$

in which $[R]$ is the moderate rotation matrix already defined in Eq. 3.13.

$$[R]_{3 \times 3} = \begin{bmatrix} -\frac{1}{2} \left[\left({}^{jk}_2 \theta_{ybl} \right)^2 + \left({}^{jk}_2 \theta_{zbl} \right)^2 \right] & -{}^{jk}_2 \theta_{zbl} + \frac{1}{2} \lambda \left({}^{jk}_2 \theta_{xpl} \right) {}^{jk}_2 \theta_{ybl} & {}^{jk}_2 \theta_{ybl} + \frac{1}{2} \lambda \left({}^{jk}_2 \theta_{xpl} \right) {}^{jk}_2 \theta_{zbl} \\ {}^{jk}_2 \theta_{zbl} + \frac{1}{2} \lambda \left({}^{jk}_2 \theta_{xpl} \right) {}^{jk}_2 \theta_{ybl} & -\frac{1}{2} \left[\lambda^2 \left({}^{jk}_2 \theta_{xpl} \right)^2 + \left({}^{jk}_2 \theta_{zbl} \right)^2 \right] & -\lambda \left({}^{jk}_2 \theta_{xpl} \right) + \frac{1}{2} {}^{jk}_2 \theta_{ybl} {}^{jk}_2 \theta_{zbl} \\ -{}^{jk}_2 \theta_{ybl} + \frac{1}{2} \lambda \left({}^{jk}_2 \theta_{xpl} \right) {}^{jk}_2 \theta_{zbl} & \lambda \left({}^{jk}_2 \theta_{xpl} \right) + \frac{1}{2} \left({}^{jk}_2 \theta_{ybl} \right) {}^{jk}_2 \theta_{zbl} & -\frac{1}{2} \left[\lambda^2 \left({}^{jk}_2 \theta_{xpl} \right)^2 + \left({}^{jk}_2 \theta_{ybl} \right)^2 \right] \end{bmatrix} \quad (8.9)$$

From the Eqs. 8.7a and 8.9, by substituting into Eq. 8.8, the first row of the resulting identity gives

$${}^{jk}_1 u_l^* = {}^{jk}_2 u_l^* + \frac{h_2}{2} \left[-{}^{jk}_2 \theta_{zbl} + \frac{1}{2} \lambda \left({}^{jk}_2 \theta_{xpl} \right) {}^{jk}_2 \theta_{ybl} \right] - \frac{h_1}{2} \left[{}^{jk}_2 \theta_{ybl} + \frac{1}{2} \lambda \left({}^{jk}_2 \theta_{xpl} \right) {}^{jk}_2 \theta_{zbl} \right] \quad (8.10)$$

8.5.1.3 Additional displacement due to warping

According to the sign convention of the sectorial coordinate introduced in Eq. 2.37, Figs. 2.7 and

8.3, a positive warping deformation ${}^{jk}_2 \psi_{bl}$ at face 2 induces an angle of rotation

${}^{jk}_2 \theta_{\psi b2} = + (h_2/2) {}^{jk}_2 \psi_{bl}$ about local axis y_{jk} at point 3 (in addition to the weak axis rotation

${}^{jk}_2 \theta_{ybl}$ identified in the previous section) and an equal and opposite angle of rotation $-{}^{jk}_2 \theta_{\psi bl}$ at

point 4. The angle ${}^{jk}_2 \theta_{\psi b2}$ acting at point 3 induces additional displacements

$\langle {}^{kj}_1 U_{\psi 2}^* \rangle^T = \langle {}^{jk}_1 u_l^* \quad {}^{jk}_1 v_l^* \quad {}^{jk}_1 w_l^* \rangle^T$ at point 1 given by

$$\{ {}^{kj}_1 U_{\psi 2}^* \} = [R_{\psi 2}] \{ x_{13} \} \quad (8.11)$$

in which the rotation matrix induced by warping ${}^{jk}_2 \psi_{bl}$ is given by

$$[R_{\psi 2}] = \begin{bmatrix} -\frac{1}{2}\left(\frac{h_2}{2}\right)^2 \left({}^{jk}\psi_{bl}\right)^2 & \frac{h_2}{4}\lambda\left({}^{jk}\theta_{xpl}\right)\left({}^{jk}\psi_{bl}\right) & \left(\frac{h_2}{2}\right)\left({}^{jk}\psi_{bl}\right) \\ \frac{h_2}{4}\lambda\left({}^{jk}\theta_{xpl}\right)\left({}^{jk}\psi_{bl}\right) & -\frac{1}{2}\lambda^2\left({}^{jk}\theta_{xpl}\right)^2 & -\lambda\left({}^{jk}\theta_{xpl}\right) \\ -\left(\frac{h_2}{2}\right)\left({}^{jk}\psi_{bl}\right) & \lambda\left({}^{jk}\theta_{xpl}\right) & -\frac{1}{2}\left[\lambda^2\left({}^{jk}\theta_{xpl}\right)^2 + \left(\frac{h_2}{2}\right)^2\left({}^{jk}\psi_{bl}\right)^2\right] \end{bmatrix} \quad (8.12)$$

From Eqs. 8.7b and 8.12, by substituting into Eq. 8.11, the first row of the resulting displacements is,

$${}^{jk}_1 u_l^* = -(h_1 h_2 / 4) {}^{jk}_2 \psi_{bl} \quad (8.13)$$

In a similar way, the additional out-of-plane displacements at point 1 due to warping deformation

${}^{jk}_1 \psi_{bl}$ acting at face 1 induces angles ${}^{jk}_1 \theta_{\psi bl} = (h_1 / 2) {}^{jk}_1 \psi_{bl}$ at the point 3 and $-{}^{jk}_1 \theta_{\psi bl}$ at the point 5,

both rotating about x_{jk} axis. The rotation at Point 3 induces additional displacements

$$\left\langle {}^k_2 U_{\psi 2}^* \right\rangle^T = \left\langle {}^{jk}_2 u_l^* \quad {}^{jk}_2 v_l^* \quad {}^{jk}_2 w_l^* \right\rangle^T \text{ at Point 2 determined by}$$

$$\left\{ {}^k_2 U_{\psi 2}^* \right\} = [R_{\psi 1}] \{x_{23}\} \quad (8.14)$$

in which

$$[R_{\psi 1}]_{3 \times 3} = \begin{bmatrix} -\frac{1}{2}(h_1/2)^2 \left({}^{jk}_1 \psi_{bl}\right)^2 & -\frac{1}{2}h_1 \left({}^{jk}_1 \psi_{bl}\right) & \frac{h_1}{4}\lambda\left({}^{jk}_1 \theta_{xpl}\right)\left({}^{jk}_1 \psi_{bl}\right) \\ \frac{1}{2}h_1 \left({}^{jk}_1 \psi_{bl}\right) & -\frac{1}{2}(h_1/2)^2 \left({}^{jk}_1 \psi_{bl}\right)^2 & -\lambda\left({}^{jk}_1 \theta_{xpl}\right) \\ \frac{h_1}{4}\lambda\left({}^{jk}_1 \theta_{xpl}\right)\left({}^{jk}_1 \psi_{bl}\right) & \lambda\left({}^{jk}_1 \theta_{xpl}\right) & -\frac{1}{2}\lambda^2\left({}^{jk}_1 \theta_{xpl}\right)^2 \end{bmatrix}_{3 \times 3} \quad (8.15)$$

From Eqs. 8.7c and 8.15, by substituting into Eq. 8.14, the first row of the resulting displacement vector is

$${}^{jk}_2 u_l^* = (h_1 h_2 / 4) {}^{jk}_1 \psi_{bl} \quad (8.16)$$

Finally, the total out-of-plane displacement at point 1 ${}^{jk}_1 u_l^*$ is obtained by summation of Eqs. 8.10,

8.13 and 8.16 yielding

$$\begin{aligned} {}^{jk}_1 u_l^* = & \left({}^{jk}_2 u_{bl}\right) - \frac{h_1}{2} \left[\left({}^{jk}_2 \theta_{ybl}\right) + \frac{1}{2}\lambda\left({}^{jk}_2 \theta_{xpl}\right)\left({}^{jk}_2 \theta_{zbl}\right) \right] \\ & + \frac{h_2}{2} \left[-\left({}^{jk}_2 \theta_{zbl}\right) + \frac{1}{2}\lambda\left({}^{jk}_2 \theta_{xpl}\right)\left({}^{jk}_2 \theta_{ybl}\right) \right] + \frac{h_1 h_2}{4} \left({}^{jk}_1 \psi_{bl} - {}^{jk}_2 \psi_{bl}\right) \end{aligned} \quad (8.17)$$

It is recalled in Chapter 6 that pre-buckling rotations such $\lambda \left(\begin{smallmatrix} jk \\ 2 \end{smallmatrix} \theta_{xpl} \right)$ have been neglected in a linear buckling solution context. Thus, the buckling displacement ${}^{jk}u_{bl}$ at point 1 simplifies to

$$-{}^{jk}u_{bl} + \frac{h_1 h_2}{4} \left(\begin{smallmatrix} jk \\ 1 \end{smallmatrix} \psi_{bl} \right) + {}^{jk}u_{bl} - \frac{h_1}{2} \left(\begin{smallmatrix} jk \\ 2 \end{smallmatrix} \theta_{ybl} \right) - \frac{h_2}{2} \left(\begin{smallmatrix} jk \\ 2 \end{smallmatrix} \theta_{zbl} \right) - \frac{h_1 h_2}{4} \left(\begin{smallmatrix} jk \\ 2 \end{smallmatrix} \psi_{bl} \right) \approx 0 \quad (8.18)$$

The three constraint conditions in Eqs. 8.5a, 8.5b and 8.18 can be expressed in a matrix form as

$$\left[\begin{smallmatrix} jk \\ L \end{smallmatrix} \right]_{3 \times 8} \left\{ \begin{smallmatrix} jk \\ U_{nl} \end{smallmatrix} \right\}_{8 \times 1} = \{0\}_{8 \times 1} \quad (8.19)$$

in which

$$\left[\begin{smallmatrix} jk \\ L \end{smallmatrix} \right]_{3 \times 8} = \begin{bmatrix} -1 & 0 & 0 & (h_1 h_2 / 4) & 1 & -(h_1 / 2) & -(h_2 / 2) & -(h_1 h_2 / 4) \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & (h_2 / 2) \\ 0 & 0 & -1 & (h_1 / 2) & 0 & 0 & 1 & 0 \end{bmatrix} \quad (8.20)$$

and

$$\left\langle \begin{smallmatrix} jk \\ U_{nl} \end{smallmatrix} \right\rangle_{1 \times 8}^T = \left\langle \begin{smallmatrix} jk \\ 1 \end{smallmatrix} u_{bl} \quad \begin{smallmatrix} jk \\ 1 \end{smallmatrix} \theta_{ybl} \quad \begin{smallmatrix} jk \\ 1 \end{smallmatrix} \theta_{zbl} \quad \begin{smallmatrix} jk \\ 1 \end{smallmatrix} \psi_{bl} \quad \vdots \quad \begin{smallmatrix} jk \\ 2 \end{smallmatrix} u_{bl} \quad \begin{smallmatrix} jk \\ 2 \end{smallmatrix} \theta_{ybl} \quad \begin{smallmatrix} jk \\ 2 \end{smallmatrix} \theta_{zbl} \quad \begin{smallmatrix} jk \\ 2 \end{smallmatrix} \psi_{bl} \right\rangle_{1 \times 8}^T \quad (8.21)$$

The joint nodal displacements $\left\langle \begin{smallmatrix} jk \\ U_{nl} \end{smallmatrix} \right\rangle_{1 \times 8}^T$ in local coordinates are related to nodal displacements

$\left\langle \begin{smallmatrix} jk \\ U_{ng} \end{smallmatrix} \right\rangle_{1 \times 8}^T$ in the global coordinates through

$$\left\{ \begin{smallmatrix} jk \\ U_{nl} \end{smallmatrix} \right\}_{8 \times 1} = \left[\begin{smallmatrix} jk \\ T \end{smallmatrix} \right]_{8 \times 8} \left\{ \begin{smallmatrix} jk \\ U_{ng} \end{smallmatrix} \right\}_{8 \times 1} \quad (8.22)$$

in which the joint element transformation matrix $\left[\begin{smallmatrix} jk \\ T \end{smallmatrix} \right]_{8 \times 8}$ is given as

$$\left[\begin{smallmatrix} jk \\ T \end{smallmatrix} \right]_{8 \times 8} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \phi_{jk} & \sin \phi_{jk} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin \phi_{jk} & \cos \phi_{jk} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos \phi_{jk} & \sin \phi_{jk} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sin \phi_{jk} & \cos \phi_{jk} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{8 \times 8} \quad (8.23)$$

8.5.2 Warping Stiffness Matrix of Joint Element in Local Coordinates

8.5.2.1 Formulation

As a member in a frame undergoes buckling, the joints connected to it provide elastic partial warping restraint to the member. The restraint provided depends on the sectional and material

properties of the plates consisting of the joints. In this section, we aim at devising a simplified warping stiffness matrix for a joint element. Recalling that in addition to the rigid body motion characterized under section 8.5.2, the joint element has two warping deformations ${}^{jk}_1\psi_{bl}$, ${}^{jk}_2\psi_{bl}$ respectively acting on faces 1 and 2 (Fig. 8.2). They are related to the corresponding bimoments ${}^{jk}_1B_{bl}$, ${}^{jk}_2B_{bl}$ through a 2x2 warping restraint matrix which is determined by the joint element stiffness. A four nodal shell element, combining a plane stress element (Bathe, 1997) and the Hermite cubic element for thin-plate bending (Reddy, 1993), is developed and implemented in a FORTRAN routine to determine the partial warping restraint of the joint. The underlying shell formulation is presented in Appendix 8.1.

In order to obtain the 2x2 warping stiffness matrix of a joint ^{jk} (Fig. 8.6), the joint element is subjected to a unit warping deformation ${}^{jk}_1\psi_{bl}=1$ at face 1, i.e., applying the corresponding rotation angle ${}^{jk}_1\theta_{\psi bl} = (h_1/2) {}^{jk}_1\psi_{bl} = (h_1/2) \times 1$ about axis z_{jk} at the flange 3'-3'' and an equal but opposite angle $(-{}^{jk}_1\theta_{\psi bl} = -h_1/2)$ at edge 5'-5'', Fig. 8.6a) and zero warping deformation ${}^{jk}_2\psi_{bl}=0$ at face 2. The corresponding bimoments ${}^{jk}_{11}B_{bl}$ and ${}^{jk}_{21}B_{bl}$ at faces 1 and 2, respectively, are computed from the resulting reactions. The reactions ${}^{jk}_1R_{x1}$, ${}^{jk}_1R_{x2}$, ${}^{jk}_1R_{zr3'-3''}$, ${}^{jk}_1R_{zr4'-4''}$, ${}^{jk}_1R_{yr3'-3''}$, ${}^{jk}_1R_{yr5'-5''}$ ($r=1, \dots, n$) are extracted from the model (Fig. 8.6b), in which the left subscript 1 indicates the reactions at Face 1. These reactions are self balancing, i.e. $\sum ({}^{jk}_1R_{x1} + {}^{jk}_1R_{x2}) = 0$, $\sum_{r=1}^n ({}^{jk}_1R_{yr3'-3''} + {}^{jk}_1R_{yr5'-5''}) = 0$, $\sum_{r=1}^n ({}^{jk}_1R_{zr3'-3''} + {}^{jk}_1R_{zr4'-4''}) = 0$, $\sum_{r=1}^n (x_{r3'-3''} {}^{jk}_1R_{zr3'-3''} + x_{r4'-4''} {}^{jk}_1R_{zr4'-4''}) = 0$, $\sum_{r=1}^n (z_{r3'-3''} {}^{jk}_1R_{yr3'-3''} + z_{r5'-5''} {}^{jk}_1R_{yr5'-5''}) = 0$, in which x_r and z_r are indicated in Fig. 8.6b. Reactions ${}^{jk}_1R_{yr3'-3''}$ and ${}^{jk}_1R_{yr5'-5''}$ induce a bimoment given by ${}^{jk}_{11}B_{bl} = \sum_{r=1}^n [\omega_{13'-3''}(s) {}^{jk}_1R_{yr3'-3''} + \omega_{15'-5''}(s) {}^{jk}_1R_{yr5'-5''}]$ acting at Face 1, where the warping function $\omega_1(s)$ is illustrated in Fig. 8.4. For a point S along the flanges 3'-3'' and 5'-5'', the warping function is $\omega_1(s) = h_1 x_r(s)$.

$${}^{jk}_{11}B_{bl} = \int_{A_1} {}^{jk}\sigma_1 \omega_1(s) dA \approx \sum_{r=1}^{nr} ({}^{jk}_1R_{yr3'-3''}) \left(\frac{h_1}{2} x_{r3'-3''} \right) + \sum_{r=1}^{nr} ({}^{jk}_1R_{yr5'-5''}) \left(-\frac{h_1}{2} x_{r5'-5''} \right) \quad (8.24)$$

Reactions ${}^{jk}R_{zr3'-3''}$ and ${}^{jk}R_{zr4'-4''}$ acting along the edges $3'-3''$ and $4'-4''$ induce another non-zero bimoment ${}^{jk}B_{bl}$ given by ${}^{jk}B_{bl} = \sum_{r=1}^n \omega_2(s) R_{zr}$ at Face 2, where the warping function $\omega_2(s)$ (Fig. 8.4) of a point S along the flanges $3'-3''$ and $4'-4''$ is $\omega_2(s) = h_2 x_r(s)$. The reactive bimoment is determined by

$${}^{jk}B_{bl} = \int_{A_1} {}^{jk}\sigma_2 \omega_2(s) dA \approx \sum_{r=1}^{nr} ({}^{jk}R_{zr3'-3''}) \left(\frac{h_2}{2} x_{r3'-3''} \right) + \sum_{r=1}^{nr} ({}^{jk}R_{zr4'-4''}) \left(-\frac{h_2}{2} x_{r4'-4''} \right) \quad (8.25)$$

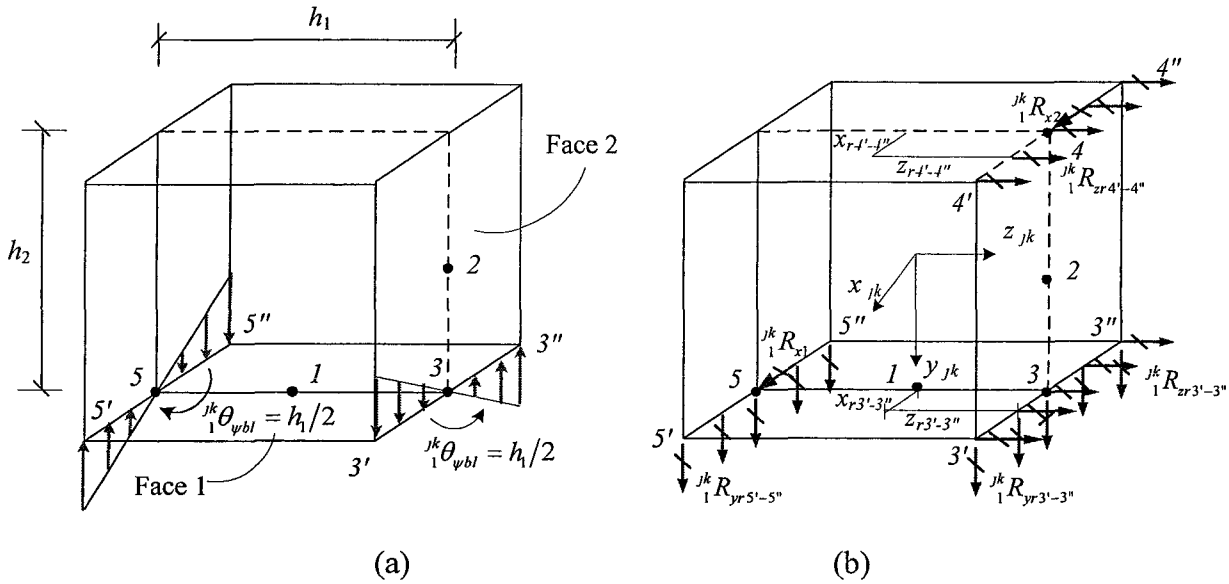


Figure 8.6 (a) Applied unit warping deformation at face 1, (b) Boundary conditions

The following step is to subject the joint to a unit warping deformation ${}^{jk}\psi_{bl} = 1$ and ${}^{jk}\psi_{bl} = 0$ (Fig. 8.7a), which induce a rotation about the z_{jk} axis ${}^{jk}\theta_{\psi bl} = (h_2/2) {}^{jk}\psi_{bl} = (h_2/2) \times 1$ at edge $3'-3''$ and an equal but opposite rotation ${}^{jk}\theta_{\psi bl} = -(h_2/2) {}^{jk}\psi_{bl} = -(h_2/2) \times 1$ at edge $4'-4''$. These rotations are imposed to the FEA model of the joint. The zero warping deformation at face 1 implies zero rotations about the y_{jk} axis at edges $3'-3''$ and $5'-5''$. Under the boundary conditions (Fig. 8.7b), the corresponding bimoments ${}^{jk}B_{bl}$ and ${}^{jk}B_{bl}$ at faces 2 and 1 respectively can be computed from the resulting reactions. Reactions ${}^{jk}R_{x1}$, ${}^{jk}R_{x2}$, ${}^{jk}R_{zr3'-3''}$, ${}^{jk}R_{zr4'-4''}$, ${}^{jk}R_{yr3'-3''}$, ${}^{jk}R_{yr5'-5''}$ ($r=1, \dots, n$) are extracted from the model (Fig. 8.7b), in which the left subscript 2 indicates the reactions are due to applying a unit warping deformation at face 2. These reactions

are self balancing, i.e. $\sum \left({}^j_k R_{x1} + {}^j_k R_{x2} \right) = 0$, $\sum_{r=1}^n \left({}^j_k R_{yr3'-3''} + {}^j_k R_{yr5'-5''} \right) = 0$, $\sum_{r=1}^n \left({}^j_k R_{zr3'-3''} + {}^j_k R_{zr4'-4''} \right) = 0$, $\sum_{r=1}^n \left(x_{r3'-3''} {}^j_k R_{zr3'-3''} + x_{r4'-4''} {}^j_k R_{zr4'-4''} \right) = 0$, $\sum_{r=1}^n \left(z_{r3'-3''} {}^j_k R_{yr3'-3''} + z_{r5'-5''} {}^j_k R_{yr5'-5''} \right) = 0$, in which x_r and z_r are indicated in Fig. 8.7b. But, the reaction ${}^j_k R_{yr3'-3''}$ and ${}^j_k R_{yr5'-5''}$ induce a reacted bimoment ${}^j_k B_{bl}$ defined by ${}^j_k B_{bl} = \sum_{r=1}^n \left[\omega_{13'-3''}(s) {}^j_k R_{yr3'-3''} + \omega_{15'-5''}(s) {}^j_k R_{yr5'-5''} \right]$ acting at the face 1, in which the warping function of a point S along the flanges 3'-3'' and 5'-5'' is $\omega_1(s) = h_1 x_r(s)$ shown in Fig. 8.3.

$${}^j_k B_{bl} = \int_{A_1} {}^j_k \sigma_1 \omega_1(s) dA \approx \sum_{r=1}^{nr} \left({}^j_k R_{yr3'-3''} \right) \left(\frac{h_1}{2} x_{r3'-3''} \right) + \sum_{r=1}^{nr} \left({}^j_k R_{yr5'-5''} \right) \left(-\frac{h_1}{2} x_{r5'-5''} \right) \quad (8.26)$$

The reactions ${}^j_k R_{zr3'-3''}$ and ${}^j_k R_{zr4'-4''}$ acting along edges 3'-3'' and 4'-4'' induce another non-zero

reactive bimoment ${}^j_k B_{bl}$ given by ${}^j_k B_{bl} = \sum_{r=1}^n \left[\omega_{23'-3''}(s) {}^j_k R_{zr3'-3''} + \omega_{24'-4''}(s) {}^j_k R_{zr4'-4''} \right]$ reacted at

the face 2, and is determined as

$${}^j_k B_{bl} = \int_{A_1} {}^j_k \sigma_2 \omega_2(s) dA \approx \sum_{r=1}^{nr} \left({}^j_k R_{zr3'-3''} \right) \left(\frac{h_2}{2} x_{r3'-3''} \right) + \sum_{r=1}^{nr} \left({}^j_k R_{zr4'-4''} \right) \left(-\frac{h_2}{2} x_{r4'-4''} \right) \quad (8.27)$$

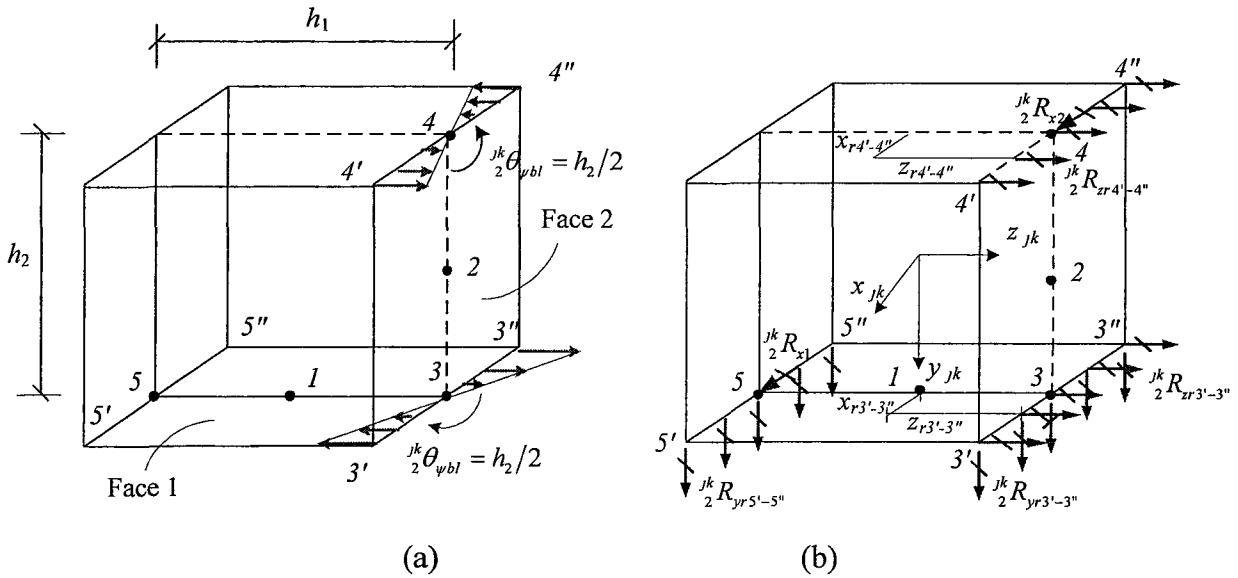


Figure 8.7 (a) Applied a unit warping deformation at face 2, (b) Reactions

Using the principle of superposition, the total bimoments ${}^{jk}_1 B_{bl}$ and ${}^{jk}_2 B_{bl}$ acting on faces 1 and 2 due to warping deformations ${}^{jk}_1 \psi_{bl}$ and ${}^{jk}_2 \psi_{bl}$ are obtained from Eqs. 8.24, 8.25, 8.26 and 8.27 as,

$$\begin{Bmatrix} {}^{jk}_1 B_{bl} \\ {}^{jk}_2 B_{bl} \end{Bmatrix} = [{}^{jk} K_{\psi bl}] \begin{Bmatrix} {}^{jk}_1 \psi_{bl} \\ {}^{jk}_2 \psi_{bl} \end{Bmatrix} = \begin{bmatrix} {}^{jk}_{11} B_{bl} & {}^{jk}_{21} B_{bl} \\ {}^{jk}_{12} B_{bl} & {}^{jk}_{22} B_{bl} \end{bmatrix} \begin{Bmatrix} {}^{jk}_1 \psi_{bl} \\ {}^{jk}_2 \psi_{bl} \end{Bmatrix} \quad (8.28)$$

Equation 8.28 defines the warping matrix $[{}^{jk} K_{\psi bl}]$ which relates the bimoments to warping deformations, both defined in local coordinates.

8.5.2.2 Verification

A wide-flange cross-section (W200x59) is presented in Fig. 8.8. The material and sectional properties of the cross-section are presented in Table 8.1.

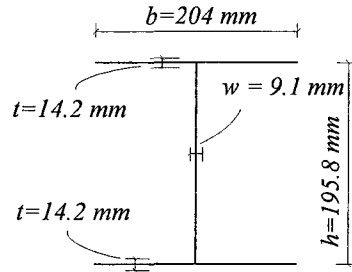


Figure 8.8 Cross-sectional dimension

Table 8.1 Sectional properties (W200x59) and material properties

Sectional Properties		Material Properties	
Principal moment inertia I_{xx} (mm ⁴)	61.32×10^6	Modulus of elasticity E (MPa)	2.0×10^5
Principal moment inertia I_{yy} (mm ⁴)	20.10×10^6	Shear Modulus G (MPa)	7.7×10^4
Torsional Constant J (mm ⁴)	4.39×10^5		
Warping Constant $I_{\omega\omega}$ (mm ⁶)	1.93×10^{11}		
Area A (mm ²)	7,575.4		

Three mesh densities, 80 elements, 320 elements and 1,280 elements, are investigated based on the shell element implemented in this study (Appendix 8.1). The results are sent to a post processor programmed in C++ and OpenGL to check the validity of the calculation. The deformed joints are depicted in Fig. 8.9 as examples obtained from the post processor.

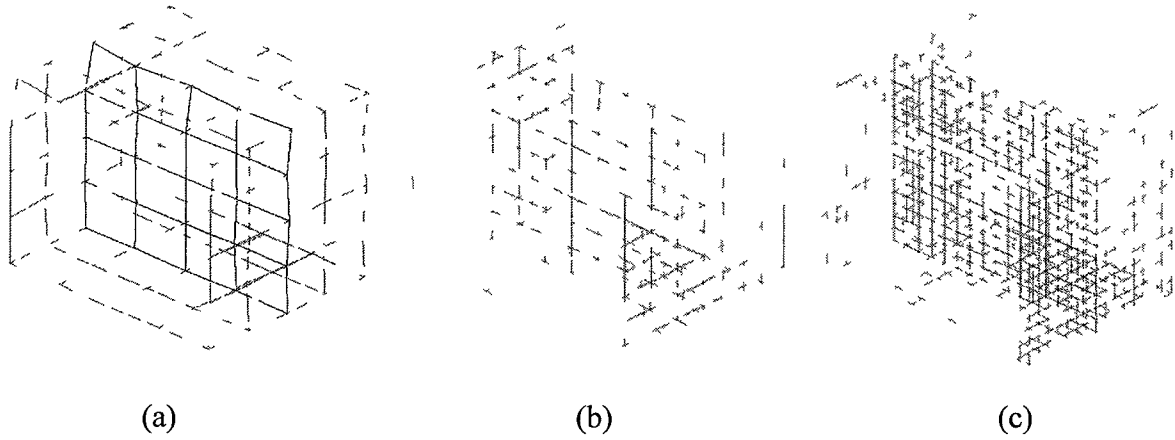


Figure 8.9 Different meshes of the deformed joint model (a) 80 elements, (b) 320 elements, and (c) 1280 elements

The results obtained from the shell element developed in the present study are compared to the ones obtained from ABAQUS shell element (S4R) analysis as shown in Figs. 8 10 and 8 11. The very good agreements are achieved in the results of bimoments and rotation angles.

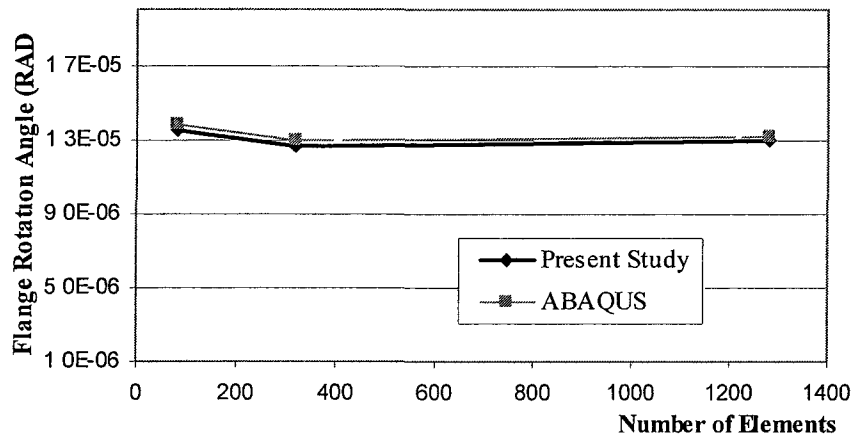


Figure 8.10 Reacting bimoment ${}^j_k B_{bl}$ with respect to the number of elements

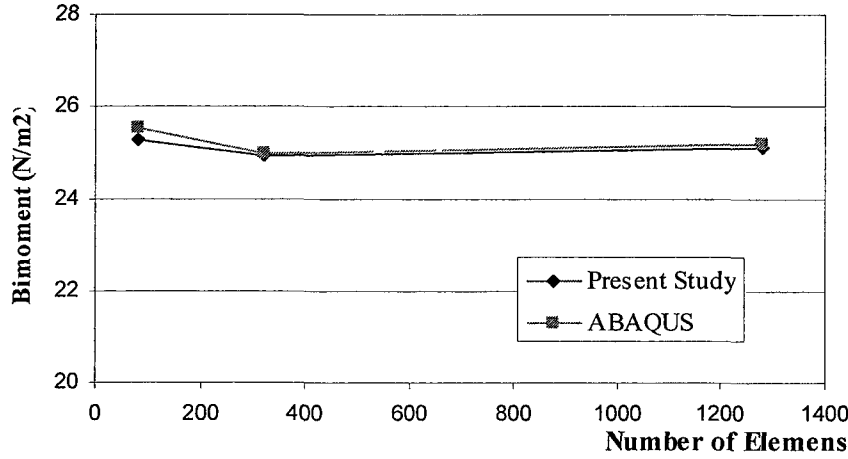


Figure 8.11 Rotation angle ${}^{jk}_1\theta_{\psi bl}$ with respect to the number of elements

Taking the results obtained from 1,280 elements mesh, the warping restraint matrix for the connected members is established as

$$[K_{\omega}] = \begin{bmatrix} \frac{B_{11}}{\theta_{\omega 1}} & -\frac{B_{12}}{\theta_{\omega 2}} \\ -\frac{B_{21}}{\theta_{\omega 1}} & \frac{B_{22}}{\theta_{\omega 2}} \end{bmatrix} = \begin{bmatrix} 2.111 \times 10^{12} & -1.932 \times 10^{12} \\ -1.932 \times 10^{12} & 2.111 \times 10^{12} \end{bmatrix} \text{ (Nmm}^2\text{/rad)}$$

8.5.2.3 Warping Stiffness Matrix in Global Coordinates

In Section 8.5.1, the warping stiffness matrix for a joint is formulated in the local coordinates. This section aims at developing a transformation scheme which generalizes the formulation for other cases in which the local coordinates do not coincide with the global coordinates (Fig. 8.12 b, c and d).

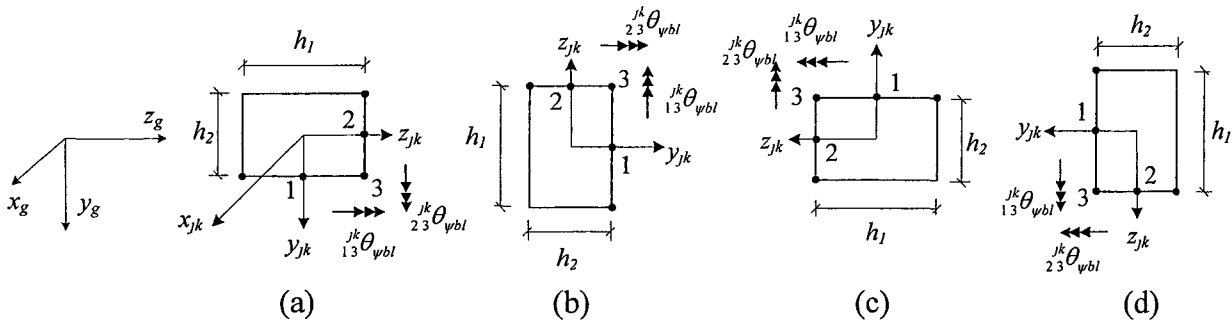


Figure 8.12 Joints with angles ϕ_{jk} to the global coordinates (a) $\phi_{jk} = 0$, (b) $\phi_{jk} = \pi/2$,

(c) $\phi_{jk} = \pi$, and (d) $\phi_{jk} = 3\pi/2$

Thus, we generically consider a joint inclined at an angle ϕ_{jk} relative to the global coordinates. Considering the angles of rotation at Point 3 due to warping deformation at the faces 1 and 2, both acting in the local directions of the joint, angle of rotation ${}^{jk}\theta_{\psi bl}$ is induced by warping deformation ${}^{jk}\psi_{bl}$ and rotates about local coordinate z_{jk} , while angle ${}^{jk}\theta_{\psi bl}$ is induced by warping deformation ${}^{jk}\psi_{bl}$ and rotates about local coordinate y_{jk} (Fig. 8.13). Both rotations ${}^{jk}\theta_{\psi zg}$, ${}^{jk}\theta_{\psi yg}$ can be resolved along the global directions z_g and y_g through

$$\begin{Bmatrix} {}^{jk}\theta_{\psi zg} \\ {}^{jk}\theta_{\psi yg} \end{Bmatrix} = \begin{bmatrix} \cos \phi_{jk} & \sin \phi_{jk} \\ -\sin \phi_{jk} & \cos \phi_{jk} \end{bmatrix} \begin{Bmatrix} {}^{jk}\theta_{\psi bl} \\ {}^{jk}\theta_{\psi bl} \end{Bmatrix} \quad (8.29)$$

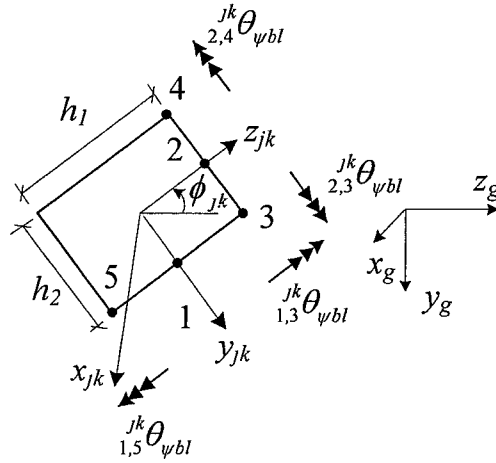


Figure 8.13 Warping transformations at a joint k

As discussed under section 8.5.2.3, angles of rotations ${}^{jk}\theta_{\psi bl}$ and ${}^{jk}\theta_{\psi bl}$ are related to the local warping deformation ${}^{jk}\psi_{bl}$ and ${}^{jk}\psi_{bl}$ through

$$\begin{Bmatrix} {}^{jk}\theta_{\psi bl} \\ {}^{jk}\theta_{\psi bl} \end{Bmatrix} = \begin{bmatrix} (2/h_1) & 0 \\ 0 & (2/h_2) \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{bl} \\ {}^{jk}\psi_{bl} \end{Bmatrix} \quad (8.30)$$

The warping deformation at Point 3 along the global z_g and y_g axes ${}^{jk}\psi_{zg}$ and ${}^{jk}\psi_{yg}$ can be related to the local warping deformation ${}^{jk}\psi_{bl}$ and ${}^{jk}\psi_{bl}$ as

$$\begin{Bmatrix} {}^{jk}\psi_{bl} \\ {}^{jk}\psi_{bl} \end{Bmatrix}_{2 \times 1} = [{}^{jk}T_{\psi}]_{2 \times 2} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} = \begin{bmatrix} \cos \phi_{jk} & -\sin \phi_{jk} \\ \sin \phi_{jk} & \cos \phi_{jk} \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix}_{2 \times 1} \quad (8.31)$$

in which $\begin{bmatrix} {}^{jk}T_{\psi} \end{bmatrix}_{2 \times 2}$ is the transformation matrix for joint warping restraint.

From Eqs. 8.30 and 8.31, by substituting into Eq. 8.29, one obtains the relations of angle of rotations ${}^{jk}\theta_{\psi zg}$ and ${}^{jk}\theta_{\psi yg}$ in the global coordinates and the global warping deformation ${}^{jk}\psi_{zbg}$ and ${}^{jk}\psi_{ybg}$ as

$$\begin{Bmatrix} {}^{jk}\theta_{\psi zg} \\ {}^{jk}\theta_{\psi yg} \end{Bmatrix} = \begin{bmatrix} \cos \phi_{jk} & \sin \phi_{jk} \\ -\sin \phi_{jk} & \cos \phi_{jk} \end{bmatrix} \begin{bmatrix} (2/h_1) & 0 \\ 0 & (2/h_2) \end{bmatrix} \begin{bmatrix} \cos \phi_{jk} & -\sin \phi_{jk} \\ \sin \phi_{jk} & \cos \phi_{jk} \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} \quad (8.32)$$

In the following, four examples of joints at different orientation angles (as in Fig. 8.12) show the validity of Eq. 8.32.

$$\begin{aligned} \text{(a)} \quad \phi_{jk} = 0^\circ, \quad \begin{Bmatrix} {}^{jk}\theta_{\psi zg} \\ {}^{jk}\theta_{\psi yg} \end{Bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (2/h_1) & 0 \\ 0 & (2/h_2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} = \begin{bmatrix} (2/h_1) & 0 \\ 0 & (2/h_2) \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} \\ \text{(b)} \quad \phi_{jk} = \frac{\pi}{2}, \quad \begin{Bmatrix} {}^{jk}\theta_{\psi zg} \\ {}^{jk}\theta_{\psi yg} \end{Bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} (2/h_1) & 0 \\ 0 & (2/h_2) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} = \begin{bmatrix} (2/h_2) & 0 \\ 0 & (2/h_1) \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} \\ \text{(c)} \quad \phi_{jk} = \pi, \quad \begin{Bmatrix} {}^{jk}\theta_{\psi zg} \\ {}^{jk}\theta_{\psi yg} \end{Bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} (2/h_1) & 0 \\ 0 & (2/h_2) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} = \begin{bmatrix} (2/h_1) & 0 \\ 0 & (2/h_2) \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} \\ \text{(d)} \quad \phi_{jk} = \frac{3\pi}{2}, \quad \begin{Bmatrix} {}^{jk}\theta_{\psi zg} \\ {}^{jk}\theta_{\psi yg} \end{Bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (2/h_1) & 0 \\ 0 & (2/h_2) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} = \begin{bmatrix} (2/h_2) & 0 \\ 0 & (2/h_1) \end{bmatrix} \begin{Bmatrix} {}^{jk}\psi_{zg} \\ {}^{jk}\psi_{yg} \end{Bmatrix} \end{aligned}$$

8.6 Finite Element Model of Partial Restraint Frame Structures

In the study of buckling resistance of portal frames (Fig. 8.14) under the classical element by Barsoum and Ghallagher (1970), researchers assumed that the beam is rigidly jointed to the column at the connections. In other words, the joints connecting beams and columns are idealized by a single node in the finite element model of such a structure, as illustrated in Fig. 8.14a. However, this assumption overestimates the buckling resistance of the frame. It is noticed in Sections 8.3 and 8.4 that there are relative displacements and warping transformations between the two connections of member 1 to the joint and member 2 to the joint. If the joints are assumed partially rigid jointed, the frame should be modeled as shown in Fig. 8.14 (b).

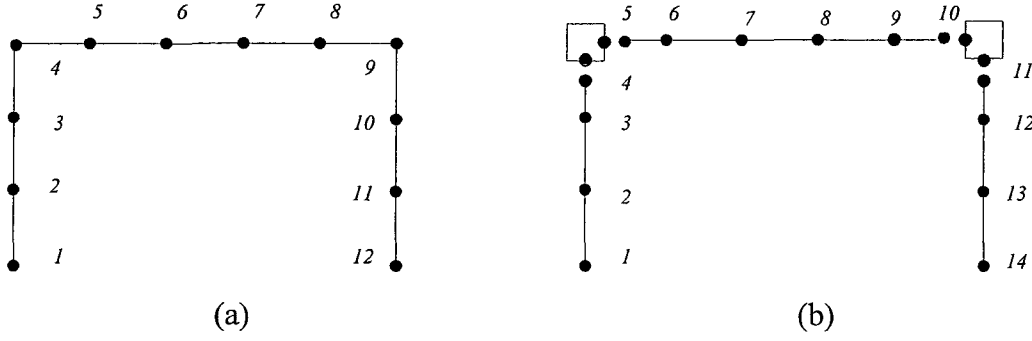


Figure 8.14 Finite element models of a portal frame (a) Rigid jointed model,
(b) Partial rigid jointed model

The neutral stability condition of such a frame structure states that the variation of the second variation of the total potential energy has to vanish for an arbitrary set of buckling deformations (Eq. 8.1). The total potential energy consists of two contributions, one is from the member elements, and the other is from the joint elements. As discussed under Sections 8.3 and 8.4, the total potential energy of the joints consists of a) buckling deformation due to rigid joint movement and b) the elastic warping stiffness. The rigid joint displacements can be treated as kinematic constraints to be enforced into the total potential energy functional using the Lagrange multiplier method, so that the solutions of the variational problem should not only satisfy the specified boundary conditions but also the constraint conditions. The neutral stability condition (Eq. 8.1) is

$$\begin{aligned}
 \delta \left(\frac{1}{2} \delta^2 \pi \right) &= \delta \left[\frac{1}{2} \left(\delta^2 \pi_m + \delta^2 \pi_j \right) \right] \\
 &= \sum_m \frac{1}{2} \left\langle \delta \left({}^e U_{ng} \right) \right\rangle^T \left[{}^e T \right]^T \left[{}^e K - \lambda \left({}^e K_G \right) \right] \left[{}^e T \right] \left\{ \delta \left({}^e U_{ng} \right) \right\} \\
 &\quad + \sum_j \frac{1}{2} \mu_j \left\langle \delta \left({}^j U_{ng} \right) \right\rangle^T \left[{}^j T \right]^T [L] \left[{}^j T \right] \left\{ \delta \left({}^j U_{ng} \right) \right\} \\
 &\quad + \sum_j \frac{1}{2} \left\langle \delta \left({}^j U_{\psi g} \right) \right\rangle^T \left[{}^j T_{\psi} \right]^T \left[{}^j K_{\psi bl} \right] \left[{}^j T_{\psi} \right] \left\{ \delta \left({}^j U_{\psi g} \right) \right\}
 \end{aligned} \tag{8.33}$$

in which λ denotes the eigenvalues, μ_j are Lagrange Multipliers, matrix $[L]$ is the joint kinematic constraint equations given in Eq. 8.16. The element transformation matrix $[{}^e T]$ for element i is given in Eq. 8.4 and the nodal displacement vector

$\langle {}^e U_{ng} \rangle_{1 \times 8}^T = \langle {}^e_1 u_{bg} \quad {}^e_1 \theta_{ybg} \quad {}^e_1 \theta_{zbg} \quad {}^e_1 \psi_{bg} \mid {}^e_2 u_{bg} \quad {}^e_2 \theta_{ybg} \quad {}^e_2 \theta_{zbg} \quad {}^e_2 \psi_{bg} \rangle^T$, while the joint transformation matrix $[{}^j T]$ is provided in Eq. 8.19 and the joint displacement vector $\langle {}^j U_{ng} \rangle^T = \langle {}^j_1 u_{bg} \quad {}^j_1 \theta_{ybg} \quad {}^j_1 \theta_{zbg} \mid {}^j_2 u_{bg} \quad {}^j_2 \theta_{ybg} \quad {}^j_2 \theta_{zbg} \rangle^T$. The warping restraint matrix $[{}^j K_{\psi bl}]$ is given in Eq. 8.28 for joint element k . The joint warping transformation matrix $[{}^j T_{\psi}]$ is provided by Eq. 8.36 and the joint warping displacement vector $\langle {}^j U_{\psi g} \rangle_{1 \times 2}^T = \langle {}^j_1 \psi_{bg} \mid {}^j_2 \psi_{bg} \rangle$.

The neutral stability condition of the frame structure (Eq. 8.33) can be expressed in a matrix form as

$$\left(\left[\begin{array}{c|c} K_{n \times n} & L_{n \times l}^T \\ \hline L_{l \times n} & 0_{l \times l} \end{array} \right] - \lambda \left[\begin{array}{c|c} K_{Gn \times n} & 0_{n \times l} \\ \hline 0_{l \times n} & 0_{l \times l} \end{array} \right] \right) \left\{ \begin{array}{c} U_{ng \ n \times 1} \\ \hline \mu_{l \times 1} \end{array} \right\} = \left\{ \begin{array}{c} 0_{n \times 1} \\ \hline 0_{l \times 1} \end{array} \right\} \quad (8.34)$$

in which subscript n is the size of the rows and columns of the elastic stiffness matrices $[K]$ and geometric stiffness matrix $[K_G]$. Subscript l is the number of the constraint conditions (Lagrange Multiplier). The vector of the total nodal displacements

$$\langle U_{ng} \rangle_{1 \times n}^T = \langle \langle {}^e_1 U_{ng} \rangle_{1 \times 8}^T \mid \langle {}^e_2 U_{ng} \rangle_{1 \times 8}^T \mid \dots \mid \langle {}^e_l U_{ng} \rangle_{1 \times 8}^T \dots \mid \langle {}^e_n U_{ng} \rangle_{1 \times 8}^T \rangle.$$

Since the rigid displacements of Node 1 $\langle {}^j_1 u_g \quad {}^j_1 \theta_{yg} \quad {}^j_1 \theta_{zg} \rangle^T$ at the joint k are expressed by the displacements at Node 2 $\langle {}^j_2 u_g \quad {}^j_2 \theta_{yg} \quad {}^j_2 \theta_{zg} \rangle^T$ and the warping deformation $\langle {}^j_1 \psi_g \quad {}^j_2 \psi_g \rangle^T$ (Eq.

8.19 and 8.20). For each joint element, the displacement vector can be expressed in two parts

$$\langle {}^j U_{ng} \rangle_{1 \times 8}^T = \langle {}^j_1 u_{bg} \quad {}^j_1 \theta_{ybg} \quad {}^j_1 \theta_{zbg} \mid {}^j_1 \psi_{bg} \quad {}^j_2 u_{bg} \quad {}^j_2 \theta_{ybg} \quad {}^j_2 \theta_{zbg} \quad {}^j_2 \psi_{bg} \rangle^T = \langle {}^j_1 U_{ng} \mid {}^j_2 U_{ng} \rangle^T.$$

The rigid nodal displacements at Node 1 at all the joint elements will be eliminated from the expression of neutral stability condition (Eq. 8.38). The total displacement vector can be

expressed by two sub-vector $\langle U_{ng} \rangle_{1 \times n}^T = \langle \langle U_{ng1} \rangle_{1 \times (n-l)}^T \mid \langle U_{ng2} \rangle_{1 \times l}^T \rangle$, where

$\langle U_{ng2} \rangle_{1 \times l}^T = \left\langle \begin{bmatrix} \langle U_{ng1}^1 \rangle_{1 \times 3} & \langle U_{ng1}^2 \rangle_{1 \times 3} & \dots & \langle U_{ng1}^k \rangle_{1 \times 3} & \dots & \langle U_{ng1}^m \rangle_{1 \times 3} \end{bmatrix}^T \right\rangle$ and the vector $\langle U_{ng1} \rangle_{1 \times (n-l)}$ is the remaining entries after eliminating the vector $\langle U_{ng2} \rangle_{1 \times l}^T$ from the total displacement vector $\langle U_{ng} \rangle_{1 \times n}^T$.

From Eq. 8.34, the elastic stiffness matrix $[K]$ and geometric stiffness matrix $[K_G]$ can be divided into sub-matrices corresponding to the two partitions of the structures displacement vectors $\left\langle \begin{bmatrix} \langle U_{ng1} \rangle_{1 \times (n-l)}^T & \langle U_{ng2} \rangle_{1 \times l}^T \end{bmatrix} \right\rangle$.

$$\left(\begin{bmatrix} K_{11(n-l) \times (n-l)} & K_{12(n-l) \times l} & L_1^T_{(n-l) \times l} \\ K_{21l \times (n-l)} & K_{22l \times l} & L_2^T_{l \times l} \\ \hline L_{1l \times (n-l)} & L_{2l \times l} & 0_{l \times l} \end{bmatrix} - \lambda \begin{bmatrix} K_{G11(n-l) \times (n-l)} & K_{G12(n-l) \times l} & 0_{(n-l) \times l} \\ K_{G21l \times (n-l)} & K_{G22l \times l} & 0_{l \times l} \\ \hline 0_{l \times (n-l)} & 0_{l \times l} & 0_{l \times l} \end{bmatrix} \right) \begin{Bmatrix} U_{ng1(n-l) \times 1} \\ U_{ng2l \times 1} \\ \mu_{l \times 1} \end{Bmatrix} = \begin{Bmatrix} 0_{(n-l) \times 1} \\ 0_{l \times 1} \\ 0_{l \times 1} \end{Bmatrix} \quad (8.35)$$

Applying matrix transformation technique, the displacement vector $\langle U_{ng2l} \rangle^T$ can be eliminated from Eq. (8.35). The detailed elimination procedure is presented in Appendix 8.2. After this operation, the condensed formulation of Eq. 8.35 is obtained as

$$\left([K^*]_{(n-l) \times (n-l)} - \lambda [K_G^*]_{(n-l) \times (n-l)} \right) \{U_{ng1}\}_{(n-l) \times 1} = \{0\}_{(n-l) \times 1} \quad (8.36)$$

where

$$[K^*]_{(n-l) \times (n-l)} = [K_{11}]_{(n-l) \times (n-l)} + [K_{12}]_{(n-l) \times l} [L_3]_{l \times (n-l)} + [L_3]^T_{(n-l) \times l} [K_{21}]_{l \times (n-l)} + [L_3]^T_{(n-l) \times l} [K_{22}]_{l \times l} [L_3]_{l \times (n-l)}, \text{ and}$$

$$[K_G^*]_{(n-l) \times (n-l)} = [K_{G11}]_{(n-l) \times (n-l)} + [K_{G12}]_{(n-l) \times l} [L_3]_{l \times (n-l)} + [L_3]^T_{(n-l) \times l} [K_{G21}]_{l \times (n-l)} + [L_3]^T_{(n-l) \times l} [K_{G22}]_{l \times l} [L_3]_{l \times (n-l)}$$

in which matrix $[L_3]_{l \times (n-l)} = [-L_2]_{l \times l}^{-1} [L_1]_{l \times (n-l)}$. Thus, the neutral stability condition for a partially rigid jointed frame is formulated.

8.7 Examples

8.7.1 Example 1 - Portal Frames

The portal frame structure studied is fixed at the two bases and free to translate out of the plane of the frame (Fig. 8.14). Two external loads P acting on the beam are applied at $L_b/3$ from the center of the joints. The material is assumed linear elastic steel ($E = 200,000 \text{ MPa}$ and Poisson's ratio $\nu = 0.3$). The beam and columns are assumed to have identical cross-sections. The beam to

column moment connection is assumed to consist of two pairs of perpendicular stiffeners (Fig. 1.3c). The cross-sectional properties investigated are W200x36, W200x27, and W200x59 and the column height L_c and beam span L_b are listed in Table 8.3. Cross-sectional properties are listed in Table 8.2 except those of the W200x59 which were listed in Table 8.1.

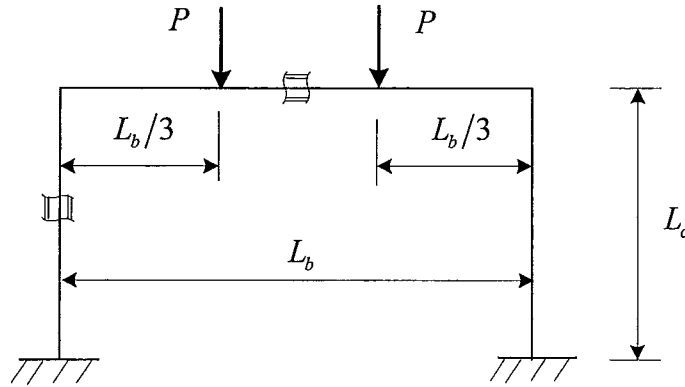


Figure 8.14 Portal frame under two concentrated loads

Table 8.2 Sectional properties

Sectional Properties	W200x36	W200x27
Width b (mm)	165	133
Height h (mm)	190.8	198.6
Flange thickness t (mm)	10.2	8.4
Web thickness w (mm)	6.2	5.8
Principle moment inertia I_{xx} (mm ⁴)	34.43×10^6	25.83×10^6
Principle moment inertia I_{yy} (mm ⁴)	7.64×10^6	3.30×10^6
Torsional Constant J (mm ⁴)	1.34×10^5	6.55×10^4
Warping Constant $I_{\omega\omega}$ (mm ⁶)	6.95×10^{10}	3.25×10^{10}
Area A (mm ²)	4606.20	3386.28

The buckling load is predicted by using three solutions: 1) ABAQUS shell element analysis. These values were provided based on the model in Mohareb and Zinoviev [8-5], 2) the present solution based on a partially restrained joint representation (based on the formulation developed in this chapter) and shear deformable elements (based on the formulation in Chapter 6 and 7), and

3) the traditional one based on a rigid representation of the joint and the classical finite element. A comparison of the results is obtained in Table 8.3. The average ratio between the buckling load as predicted by the present study (including the joint element) to that based on the shell FEA (Pr2/Pr1) is 1.09 with a standard deviation of 0.45. In contrast, the average ratio between the buckling load as predicted by assuming full warping restraint at the joint to that based on the shell FEA analysis is 1.23 with a standard deviation of 0.40. While the joint element and the rigid joint representation both overestimate the buckling capacity of the frame, the element based on the present study yields closer results to the shell FEA.

Table 8.3 Critical loads (kN) for portal frames

Cross-section	L _c (m)	L _b (m)	Pcr1 (ABAQUS)	Pcr2 (Present Study)	Pcr3 (Rigid joint)	Pcr2/Pcr1	Pcr3/Pcr1
W200x59	4.0	6.0	219.7	236.1	268.5	1.075	1.22
	4.0	5.0	314.0	324.4	386.1	1.033	1.23
	4.0	4.0	460.2	493.6	543.2	1.073	1.18
	5.0	5.0	275.4	299.8	327.3	1.089	1.19
	5.0	7.0	147.3	165.1	185.1	1.121	1.26
W200x36	4.0	6.0	77.2	85.0	95.5	1.101	1.24
	4.0	5.0	111.5	118.0	138.7	1.058	1.24
	4.0	4.0	174.5	183.6	199.7	1.052	1.14
	5.0	5.0	98.7	110.3	122.3	1.118	1.24
	5.0	7.0	50.2	58.9	65.4	1.173	1.30
W200x27	4.0	6.0	35.3	39.3	43.2	1.113	1.22
	4.0	5.0	50.5	51.5	62.5	1.020	1.24
	4.0	4.0	75.8	82.2	91.5	1.084	1.21
	5.0	5.0	43.6	49.8	55.4	1.142	1.27
	5.0	7.0	23.3	27.2	29.7	1.167	1.27
Average						1.09	1.23
Standard Deviation						0.45	0.40

8.7.2 Example 2 Γ -Shaped Frame

A Γ -shaped frame structure (Fig. 8.15) consists of a column of height $L_c = 4,000 \text{ mm}$ which is connected to a cantilever of span $L_b = 800 \text{ mm}$ through a moment connection consisting of two pairs of stiffeners (Fig. 1.3c) at the beam-to column-connection. A gravity load P acting at the tip of the cantilever is applied at the shear center. Material is steel ($E = 200,000 \text{ MPa}$ and a Poisson's ratio $\nu = 0.3$) and is assumed to deform within its elastic range. The cross-section for the column and the beam is W200x59 (properties listed in Table 8.1). It is required to determine the lateral buckling load of the frame based on the various solutions developed in this thesis.

Five solutions for the problem are presented in Table 8.4. Solution 1 is based on a partially restrained joint representation (based on the formulation developed in this chapter) and shear deformable elements (based on the formulation in Chapter 6 and 7). Solution 2 is based on a partially restrained joint representation and the classical element by Barsoum and Gallagher (1970). Solution 3 is based on a rigid representation of the joint (by assuming the warping deformation at the top of the column and that at the left end of the beam to vanish) and the shear deformable elements. Solution 4 is the traditional one based on a rigid representation of the joint and the classical finite element, while Solution 5 is based on a shell finite element representation. A comparison of the results is presented Table 8.5.

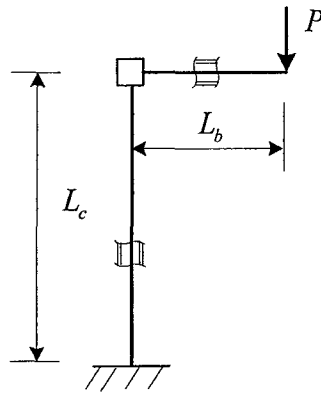


Figure 8.15 Γ -shaped frame

Given the large number of degrees of freedom in the shell solution, the buckling load of 170.8 kN should be the most accurate. Solution 1, based on the present study provides a very close prediction of the buckling load, only 2% higher than that based on the shell solution. A

comparison Solutions 1 and 2 reveals a minor influence for the shear deformation effect for the present problem. A similar conclusion is reached by comparing solutions 3 and 4 which are based on a rigid joint representation.

An accurate representation of the joint partial warping restraint is found to be important. This is evidenced by comparing solutions 1 and 3 (or solutions 2 and 4), in which a rigidly jointed representation is found to increase the predicted buckling load by about 25% in both cases.

Table 8.4 Critical load (kN) for Γ -shaped frame based on different solutions

Type of Solution	Beam Member Representation	Joint Representation	Column Member Representation	Critical Load	Critical load relative to Pcr1
Present Solution -Pcr1 Partially restrained joint and shear deformable elements	256 Shear deformable elements	1 Partially restrained	50 Shear deformable elements	174.4	
Pcr2 Partially restrained joint, shear deformation effects neglected	20 Classical elements	1 Partially restrained	4 Classical elements	176.9	Pcr1/Pcr2 = 0.985
Pcr3 Rigidly jointed with shear deformable elements	256 Shear deformable elements	Rigidly jointed	50 Shear deformable elements	224.7	Pcr1/Pcr3 = 0.776
Traditional Solution - Pcr4 Rigidly jointed, shear deformation effects neglected	20 Classical elements	Rigidly jointed	4 Classical elements	227.8	Pcr1/Pcr4 = 0.765
Shell Solution - Pcr5 (based on ABAQUS S4R analysis)	2400 Shell elements	320 Shell elements	480 Shell elements	170.8	Pcr1/Pcr5 = 1.02

8.8 Summary and Conclusion

A joint finite element was developed which accurately and systematically quantifies the partial warping restraint provided by common moment connections to adjoining members framing at 90 degrees. The finite element is based on an underlying shell formulation and is tailored to interface

with the classical 14 degree of freedom beam finite element by Barsoum and Ghallagher (1970) and that based on the shear deformable theory developed in Chapter 7. The element successfully extends the functionality of these elements to non-collinear structures and enables the accurate prediction of buckling resistance of portal frames and Γ shaped frames.

Appendix 8.1 Development of Shell Element for Determination of Warping Restraint Stiffness of Joints

A shell element with four nodes and five degrees-of-freedom ($u, v, w, \theta_x, \theta_y$) at each node (Fig. A8.1) is developed to study the behavior as the plate is subjected to bending, and in-plane loads. Here, the degree of in-plane rotation isn't included because the thickness of the plates is so small compared to the other two dimensions that the shear deformation along the thickness is negligible.

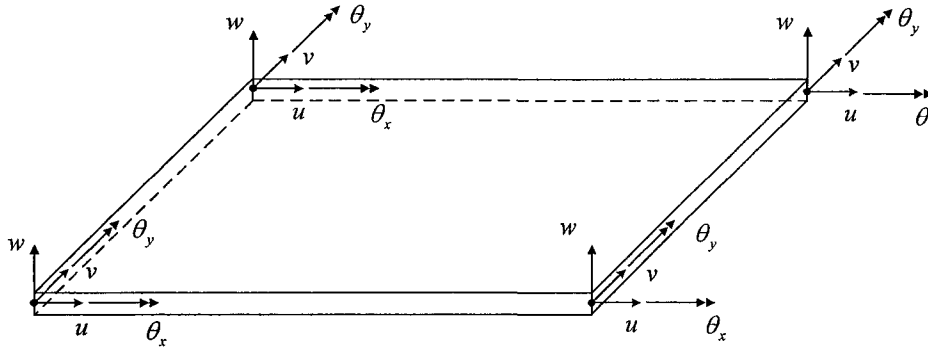


Figure A8.1-1 A plate element

The developed shell element can be decomposed into conventional plane stress element and plate bending element. In the development of the plate bending element, Kirchhoff plate theory is applied because the thickness of the plates is very small so that the in-plane rotation is negligible.

A8.1-1 Plane Stress Element

A four node rectangular element is sketched in Fig. A8.1-2. The element has a length $2a$, a width $2b$ and a thickness t . The displacements (u, v) at any point within the element along the x and y coordinates are

$$\begin{aligned} u &= N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 = N_i u_i \\ v &= N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4 = N_i v_i \end{aligned} \quad (i = 1, 2, \dots, 4) \quad (\text{A8.1-1})$$

in which N_i are shape functions and u_i and v_i are the nodal displacements along x and y directions at nodes ($i = 1, \dots, 4$), respectively. The interpolation functions in (Reddy, 1993) are adopted.

These are:

$$N_1 = \frac{1}{4} \left(\frac{x}{a} - 1 \right) \left(\frac{y}{b} - 1 \right) \quad (\text{A8.1-2a})$$

$$N_2 = -\frac{1}{4} \left(\frac{x}{a} + 1 \right) \left(\frac{y}{b} - 1 \right) \quad (\text{A8.1-2b})$$

$$N_3 = \frac{1}{4} \left(\frac{x}{a} + 1 \right) \left(\frac{y}{b} + 1 \right) \quad (\text{A8.1-2c})$$

$$N_4 = -\frac{1}{4} \left(\frac{x}{a} - 1 \right) \left(\frac{y}{b} + 1 \right) \quad (\text{A8.1-2c})$$

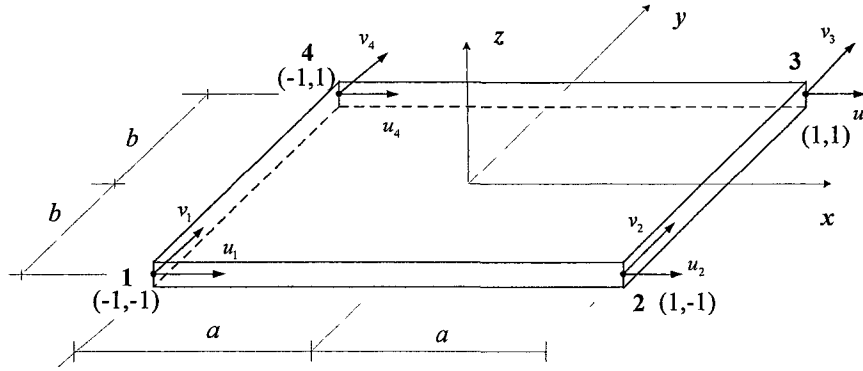


Figure A8.1-2 Plane Stress Four-Node Rectangular Element

The strain vector $\langle \varepsilon \rangle^T = \langle \varepsilon_{xx} \quad \varepsilon_{yy} \quad 2\varepsilon_{xy} \rangle^T$ is obtained by taking the partial derivative of the displacement with respect to the corresponding coordinates.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad (\text{A8.1-3})$$

The strain vector can be expressed in terms of the nodal displacement vector $\langle u_n \rangle^T = \langle u_1 \quad v_1 \mid u_2 \quad v_2 \mid u_3 \quad v_3 \mid u_4 \quad v_4 \rangle^T$ by substituting Eq. 8.27 into Eq. 8.29. It yields,

$$\{\varepsilon\}_{3 \times 1} = [B]_{3 \times 8} \{u_n\}_{8 \times 1} \quad (\text{A8.1-4})$$

in which

$$\begin{aligned}
[B] &= [B_1 \mid B_2 \mid B_3 \mid B_4] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \\
[B] &= \frac{1}{4ab} \begin{bmatrix} (y-b) & 0 & (y-b) & 0 & (y+b) & 0 & (y+b) & 0 \\ 0 & (x-a) & 0 & (x+a) & 0 & (x+a) & 0 & (x-a) \\ (x-a) & (y-b) & (x+a) & (y-b) & (x+a) & (y+b) & (x-a) & (y+b) \end{bmatrix}
\end{aligned} \tag{A8.1-5}$$

The stresses are expressed in terms of the nodal displacements $\{u_n\}$ as

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = [D]\{\varepsilon\} = [D][B]\{u_n\} \tag{A8.1-6}$$

For an elastic isotropic material in plane stress problem, the constitutive matrix $[D]$ is

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{A8.1-7}$$

in which, E is the Young's modulus, and ν is the Poisson's ratio. The internal strain energy is

$$\begin{aligned}
U &= \frac{1}{2} \int_V \langle \varepsilon \rangle^T \{\sigma\} dV = \frac{1}{2} \langle u_N \rangle^T \left[t \int_{-b-a}^b \int_{-a}^a [B]_{8 \times 3}^T [D]_{3 \times 3} [B]_{3 \times 8} dx dy \right] \{u_N\} \\
&= \frac{t}{2} \langle u_N \rangle^T [K^{PS}]_{8 \times 8} \{u_N\}
\end{aligned} \tag{A8.1-8}$$

$$\text{in which the stiffness matrix due to plane stress is } [K^{PS}]_{8 \times 8} = \left[t \int_{-b-a}^b \int_{-a}^a [B]_{8 \times 3}^T [D]_{3 \times 3} [B]_{3 \times 8} dx dy \right].$$

A8.1-2 Plate Bending Element

Kirchhoff plate bending theory is applicable to develop the plate bending element. The element has 4 nodes and each node has three degrees-of-freedom $\langle w \ \theta_x \ \theta_y \rangle^T$ as sketched in Fig. 8.11, where w is the displacement along the z -axis; θ_x and θ_y are the rotations around x and y axes, respectively. The rotations are determined as $\theta_x = \partial w / \partial y$ and $\theta_y = \partial w / \partial x$.

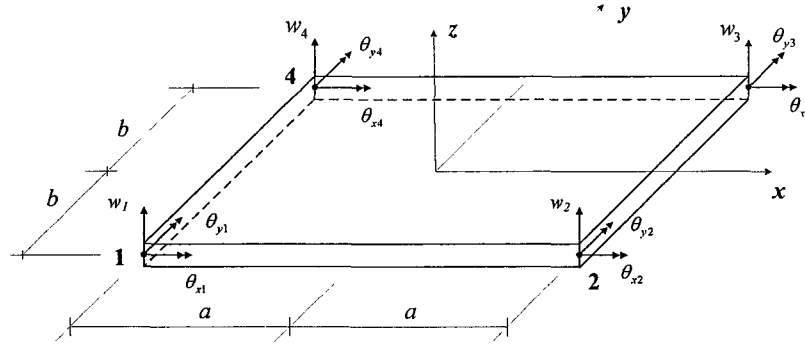


Figure A8.1-3 Plate bending element

Based on the Kirchhoff plate theory, the curvature $\{\kappa\}$ of the plate (Bathe, 1997) is

$$\{\kappa\}_{3 \times 1} = \begin{Bmatrix} \kappa_{xx} & \kappa_{yy} & \kappa_{xy} \end{Bmatrix}_{1 \times 3}^T = \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} & -\frac{\partial^2 w}{\partial y^2} & -2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}_{1 \times 3}^T \quad (\text{A8.1-9})$$

The strain tensor of the shell element is determined as

$$\{\varepsilon\}_{3 \times 1} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}_{3 \times 1} = z \{\kappa\}_{3 \times 1} = z \begin{Bmatrix} -\frac{\partial^2}{\partial x^2} \\ -\frac{\partial^2}{\partial y^2} \\ -2\frac{\partial^2}{\partial x \partial y} \end{Bmatrix}_{3 \times 1} \{w\}_{1 \times 1} \quad (\text{A8.1-10})$$

in which the variable z is the coordinate of a point along the thickness. The deflection $\{w\}$ of any point on the plate can be expressed in terms of the nodal displacement as:

$$\{w\}_{1 \times 1} = \langle N \rangle_{1 \times 12} \{w_n\}_{12 \times 1} = \langle N_{1 \times 3} \mid N_{2 \times 3} \mid N_{3 \times 3} \mid N_{4 \times 3} \rangle_{1 \times 12} \begin{Bmatrix} w_{n1 \times 3 \times 1} \\ w_{n2 \times 3 \times 1} \\ w_{n3 \times 3 \times 1} \\ w_{n4 \times 3 \times 1} \end{Bmatrix}_{12 \times 1} \quad (\text{A8.1-11})$$

in which the nodal displacement vector $\{w_{ni}\}$ is defined as:

$$\{w_{ni}\}_{3 \times 1} = \begin{Bmatrix} w_i & \theta_{xi} & \theta_{yi} \end{Bmatrix}_{1 \times 3}^T \quad (i = 1, \dots, 4) \quad (\text{A8.1-12})$$

and the interpolation matrix $[N]$ at each node it has a form:

$$[N_i]_{1 \times 3} = \begin{Bmatrix} N_i & N_{\theta_{xi}} & N_{\theta_{yi}} \end{Bmatrix}_{1 \times 3} \quad (i = 1, \dots, 4) \quad (\text{A8.1-13})$$

The Hermite cubic element (Reddy, 1993) is used and the interpolations $[N_i]$ are assumed as

$$N_1 = \frac{1}{8} \left(1 - \frac{x}{a}\right) \left(1 - \frac{y}{b}\right) \left[2 - \left(\frac{x}{a}\right) - \left(\frac{y}{b}\right) - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right] \quad (\text{A8.1-14a})$$

$$N_{\theta x1} = \frac{1}{8} b \left(1 - \frac{x}{a}\right) \left(1 - \frac{y}{b}\right) \left[1 - \left(\frac{y}{b}\right)^2\right] \quad (\text{A8.1-14b})$$

$$N_{\theta y1} = \frac{1}{8} a \left(1 - \frac{x}{a}\right) \left(1 - \frac{y}{b}\right) \left[1 - \left(\frac{x}{a}\right)^2\right] \quad (\text{A8.1-14c})$$

$$N_2 = \frac{1}{8} \left(1 + \frac{x}{a}\right) \left(1 - \frac{y}{b}\right) \left[2 + \left(\frac{x}{a}\right) - \left(\frac{y}{b}\right) - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right] \quad (\text{A8.1-14d})$$

$$N_{\theta x2} = \frac{1}{8} b \left(1 + \frac{x}{a}\right) \left(1 - \frac{y}{b}\right) \left[1 - \left(\frac{y}{b}\right)^2\right] \quad (\text{A8.1-14e})$$

$$N_{\theta y2} = -\frac{1}{8} a \left(1 + \frac{x}{a}\right) \left(1 - \frac{y}{b}\right) \left[1 - \left(\frac{x}{a}\right)^2\right] \quad (\text{A8.1-14f})$$

$$N_3 = \frac{1}{8} \left(1 + \frac{x}{a}\right) \left(1 + \frac{y}{b}\right) \left[2 + \left(\frac{x}{a}\right) + \left(\frac{y}{b}\right) - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right] \quad (\text{A8.1-14g})$$

$$N_{\theta x3} = -\frac{1}{8} b \left(1 + \frac{x}{a}\right) \left(1 + \frac{y}{b}\right) \left[1 - \left(\frac{y}{b}\right)^2\right] \quad (\text{A8.1-14h})$$

$$N_{\theta y3} = -\frac{1}{8} a \left(1 + \frac{x}{a}\right) \left(1 + \frac{y}{b}\right) \left[1 - \left(\frac{x}{a}\right)^2\right] \quad (\text{A8.1-14i})$$

$$N_4 = \frac{1}{8} \left(1 - \frac{x}{a}\right) \left(1 + \frac{y}{b}\right) \left[2 - \left(\frac{x}{a}\right) + \left(\frac{y}{b}\right) - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right] \quad (\text{A8.1-14j})$$

$$N_{\theta x4} = -\frac{1}{8} b \left(1 - \frac{x}{a}\right) \left(1 + \frac{y}{b}\right) \left[1 - \left(\frac{y}{b}\right)^2\right] \quad (\text{A8.1-14k})$$

$$N_{\theta y4} = \frac{1}{8} a \left(1 - \frac{x}{a}\right) \left(1 + \frac{y}{b}\right) \left[1 - \left(\frac{x}{a}\right)^2\right] \quad (\text{A8.1-14l})$$

The strain vector (Eq. A8.1-4) can be expressed in terms of the nodal displacement vector

$$\begin{aligned}
\{\varepsilon\}_{3 \times 1} &= z \begin{bmatrix} -\frac{\partial^2}{\partial x^2} \\ -\frac{\partial^2}{\partial y^2} \\ \frac{2\partial^2}{\partial x \partial y} \end{bmatrix}_{3 \times 1} \left\langle N_{1 \times 3} \mid N_{2 \times 3} \mid N_{3 \times 3} \mid N_{4 \times 3} \right\rangle_{1 \times 12} \begin{Bmatrix} w_{n1 \times 3 \times 1} \\ w_{n2 \times 3 \times 1} \\ w_{n3 \times 3 \times 1} \\ w_{n4 \times 3 \times 1} \end{Bmatrix}_{12 \times 1} \\
&= z \begin{bmatrix} B_{1 \times 3 \times 3} \mid B_{2 \times 3 \times 3} \mid B_{3 \times 3 \times 3} \mid B_{4 \times 3 \times 3} \end{bmatrix}_{3 \times 12} \begin{Bmatrix} w_{n1 \times 3 \times 1} \\ w_{n2 \times 3 \times 1} \\ w_{n3 \times 3 \times 1} \\ w_{n4 \times 3 \times 1} \end{Bmatrix}_{12 \times 1} = z [B]_{3 \times 12} \{w_n\}_{12 \times 1}
\end{aligned} \tag{A8.1-15}$$

in which the matrix $[B]_{3 \times 12}$ presents the relations between the strain and the nodal displacements, and it can be established by applying the displacement interpolations, for example, the sub-matrix $[B_i]_{3 \times 3}$ is determined by

$$[B_i]_{3 \times 3} = \begin{bmatrix} -\frac{\partial^2 N_i^T}{\partial x^2} & -\frac{\partial^2 N_{xi}^T}{\partial x^2} & -\frac{\partial^2 N_{yi}^T}{\partial x^2} \\ -\frac{1}{b^2} \frac{\partial^2 N_i^T}{\partial y^2} & -\frac{1}{b^2} \frac{\partial^2 N_{xi}^T}{\partial y^2} & -\frac{\partial^2 N_{yi}^T}{\partial y^2} \\ -2\frac{\partial^2 N_i^T}{\partial x \partial y} & -2\frac{\partial^2 N_{xi}^T}{\partial x \partial y} & -2\frac{\partial^2 N_{yi}^T}{\partial x \partial y} \end{bmatrix}_{3 \times 3} \quad (i=1, \dots, 4) \tag{A8.1-16}$$

The stresses

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = [D] \{\varepsilon\} = [D][B] \{w_n\} \tag{A8.1-17}$$

For an elastic isotropic material in plane stress problem, the constitutive matrix $[D]$ is given in Eq. 8.33. The internal strain energy is

$$\begin{aligned}
U &= \frac{1}{2} \int_V \langle \varepsilon \rangle^T \{\sigma\} dV = \frac{1}{2} \langle w_N \rangle^T \left[\int_{-t/2}^{t/2} \int_{-b}^b \int_{-a}^a z^2 [B]_{12 \times 3}^T [D]_{3 \times 3} [B]_{3 \times 12} dx dy dz \right] \{w_N\} \\
&= \frac{1}{2} \langle u_N \rangle^T [K^{PB}]_{12 \times 12} \{u_N\}
\end{aligned} \tag{A8.1-18}$$

From Eqs. A8.1-7 and A8.1-16, substituting into Eq. A8.1-18, the element stiffness matrix can be obtained.

A8.1-3 Transformation Matrix

Now consider a plate is inclined to xy -plane in the global coordinates, the shell element should have 6 degrees-of-freedom $\langle u \ v \ w \ \theta_x \ \theta_y \ \theta_z \rangle^T$ at each node as illustrated in Fig. A8.1-4.

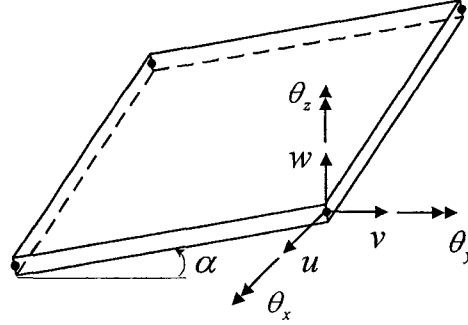


Figure A8.1-4 Degrees-of-freedom of the shell element in global coordinates

Therefore, a transformation matrix, transforming twenty degrees-of-freedom in local coordinates to the 24 degrees-of-freedom in global coordinates for the element is expressed as

$$\begin{Bmatrix} u_{ng1} \\ u_{ng2} \\ u_{ng3} \\ u_{ng4} \end{Bmatrix}_{1 \times 24} = \begin{bmatrix} T_{6 \times 6} & 0 & 0 & 0 \\ 0 & T_{6 \times 6} & 0 & 0 \\ 0 & 0 & T_{6 \times 6} & 0 \\ 0 & 0 & 0 & T_{6 \times 6} \end{bmatrix}_{24 \times 24} \begin{Bmatrix} u_{n1} \\ u_{n2} \\ u_{n3} \\ u_{n4} \end{Bmatrix}_{1 \times 24} \quad (\text{A8.1-19})$$

in which the global nodal displacement vector $\{u_{ngi}\} = \langle u_i \ v_i \ w_i \ \theta_{xi} \ \theta_{yi} \ \theta_{zi} \rangle^T$ where $(i=1, \dots, 4)$, the local nodal displacement vector $\{u_{ni}\} = \langle u_{ni} \ v_{ni} \ w_{ni} \ \theta_{xni} \ \theta_{yni} \ 0 \rangle^T$ where $(i=1, \dots, 4)$, and the transformation matrix $[T]_{6 \times 6}$ is determined as

$$[T]_{6 \times 6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 & 0 & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \alpha & -\sin \alpha \\ 0 & 0 & 0 & 0 & \sin \alpha & \cos \alpha \end{bmatrix}_{6 \times 6} \quad (\text{A8.1-20})$$

Since the transformation matrix $[T]$ is orthogonal, it has some important features, i.e.: $[T]^T = [T]^{-1}$ and $[T][T]^T = [T]^T[T] = [I]$.

The stiffness matrix of the shell element in the global coordinates is

$$[k^e]_{24 \times 24} = [T]_{24 \times 24}^T [k^e]_{24 \times 24}^l [T]_{24 \times 24} \quad (\text{A8.1-21})$$

and the force vector in the global coordinates can be calculated by

$$\{F\}_{24 \times 1} = [T]_{24 \times 24}^T [k^e]_{24 \times 24}^l [T]_{24 \times 24} \{u\}_{24 \times 1} \quad (\text{A8.1-22})$$

Appendix 8.2 Enforcing Kinematic Constraints

This appendix presents the procedures of eliminating the zero diagonal entries to obtain the condensed stiffness matrices, so that the Lagrange multiplier Method (the kinematic constraint conditions) can be enforced to the frame structures. Starting from Eq.8.34 and applying matrix transformation technique, the condensed matrix is formulated.

$$\left(\left[\begin{array}{c|c} K_{n \times n} & L^T_{n \times l} \\ \hline L_{l \times n} & 0_{l \times l} \end{array} \right] - \lambda \left[\begin{array}{c|c} K_{Gn \times n} & 0_{n \times l} \\ \hline 0_{l \times n} & 0_{l \times l} \end{array} \right] \right) \begin{Bmatrix} U_{ng \ n \times 1} \\ \mu_{l \times 1} \end{Bmatrix} = \begin{Bmatrix} 0_{n \times 1} \\ 0_{l \times 1} \end{Bmatrix} \quad (\text{A8.2-1})$$

where subscript l is the number of kinematic constraints to be imposed; n is the number of degrees of freedom before imposing the constraints. From Eq. A8.2-1, multiplying by $\langle U_{ng} \rangle_{l \times n}^T$ in left side, one has

$$\langle U_{ng} \rangle_{l \times n}^T [K_{n \times n} - \lambda K_{Gn \times n}] \{U_{ng}\}_{n \times 1} + \langle U_{ng} \rangle_{l \times n}^T [L]^T_{n \times l} \{\mu\}_{l \times 1} = \{0\}_{n \times 1} \quad (\text{A8.2-2})$$

$$\text{where: } \langle U_{ng} \rangle_{l \times n}^T [L]^T_{n \times l} = \langle 0 \rangle_{l \times n}^T$$

The joint displacement vector for a joint element k can be expressed as

$$\langle {}^{ik}U_{ng} \rangle_{1 \times 8}^T = \left\langle \begin{bmatrix} \langle {}^{jk}U_{ng} \rangle_1^T \\ \langle {}^{jk}U_{ng} \rangle_2^T \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} {}^{ik}u_{bg} & {}^{jk}\theta_{ybg} & {}^{jk}\theta_{zbg} & | & {}^{ik}\psi_{bg} & {}^{ik}u_{bg} & {}^{ik}\theta_{ybg} & {}^{ik}\theta_{zbg} & {}^{jk}\psi_{bg} \end{bmatrix} \right\rangle^T.$$

in which the nodal displacements at $\langle {}^{ik}U_{ng} \rangle_1^T$ are to be eliminated from the expression of neutral stability condition (Eq. A8.2-1).

The displacement vector $\langle U_{ng} \rangle_{1 \times n}^T$ for the structure can be partitioned in sub-vectors, i.e.,

$$\langle U_{ng} \rangle_{1 \times n}^T = \left\langle \begin{bmatrix} \langle U_{ng1} \rangle_{1 \times (n-l)}^T \\ \langle U_{ng2} \rangle_{1 \times l}^T \end{bmatrix} \right\rangle \text{ where}$$

$$\langle U_{ng2} \rangle_{1 \times l}^T = \left\langle \begin{bmatrix} \langle {}^{j1}U_{ng} \rangle_{1 \times 3}^T \\ \langle {}^{j2}U_{ng} \rangle_{1 \times 3}^T \\ \vdots \end{bmatrix} \right\rangle, \text{ and the vector } \langle U_{ng1} \rangle_{1 \times (n-l)}^T \text{ is}$$

the remaining entries after eliminating the vector $\langle U_{ng2} \rangle_{1 \times l}^T$ from the total displacement vector

$$\langle U_{ng} \rangle_{1 \times n}^T.$$

From Eq. A8.2-1, the elastic stiffness matrix $[K]$ and geometric stiffness matrix $[K_G]$ can be divided into sub-matrices corresponding to the two partitions of the structures displacement vectors $\left\langle \begin{matrix} \langle U_{ng1} \rangle_{1 \times (n-l)}^T \\ \langle U_{ng2} \rangle_{1 \times l}^T \end{matrix} \right\rangle$.

$$\left(\begin{bmatrix} [K_{11}]_{(n-l) \times (n-l)} & [K_{12}]_{(n-l) \times l} & [L_1^T]_{(n-l) \times l} \\ [K_{21}]_{l \times (n-l)} & [K_{22}]_{l \times l} & [L_2^T]_{l \times l} \\ \hline [L_1]_{l \times (n-l)} & [L_2]_{l \times l} & [0]_{l \times l} \end{bmatrix} - \lambda \begin{bmatrix} [K_{G11}]_{(n-l) \times (n-l)} & [K_{G12}]_{(n-l) \times l} & [0]_{(n-l) \times l} \\ [K_{G21}]_{l \times (n-l)} & [K_{G22}]_{l \times l} & [0]_{l \times l} \\ \hline [0]_{l \times (n-l)} & [0]_{l \times l} & [0]_{l \times l} \end{bmatrix} \right) \times \begin{Bmatrix} \{U_{ng1}\}_{(n-l) \times 1} \\ \{U_{ng2}\}_{l \times 1} \\ \hline \{\mu\}_{l \times 1} \end{Bmatrix} = \begin{Bmatrix} \{0\}_{(n-l) \times 1} \\ \{0\}_{l \times 1} \\ \hline \{0\}_{l \times 1} \end{Bmatrix} \quad (\text{A8.2-4})$$

From Eq. A8.2-4, one has,

$$\begin{bmatrix} [L_1]_{l \times (n-l)} & [L_2]_{l \times l} \end{bmatrix} \begin{Bmatrix} \{U_{ng1}\}_{(n-l) \times 1} \\ \{U_{ng2}\}_{l \times 1} \end{Bmatrix} = 0$$

$$\{U_{ng2}\}_{l \times 1} = [L_2]_{l \times (n-l)} \{U_{ng1}\}_{(n-l) \times 1} = [-L_2]_{l \times l}^{-1} [L_1]_{l \times (n-l)} \{U_{ng1}\}_{(n-l) \times 1} \quad (\text{A8.2-5})$$

Where matrix $[L_2]$ has to be square and non-singular.

By left multiplying Eq.A8.2-4 by $\left\langle \begin{matrix} U_{ng1} \\ U_{ng2} \end{matrix} \right\rangle^T$, one obtains

$$\left\langle \begin{matrix} U_{ng1} \\ U_{ng2} \end{matrix} \right\rangle^T \left\langle \begin{matrix} U_{ng1} \\ U_{ng2} \end{matrix} \right\rangle \left(\begin{bmatrix} [K_{11} - \lambda K_{G11}]_{(n-l) \times (n-l)} & [K_{12} - \lambda K_{G12}]_{(n-l) \times l} \\ [K_{21} - \lambda K_{G21}]_{l \times (n-l)} & [K_{22} - \lambda K_{G22}]_{l \times l} \\ \hline [L_1]_{l \times (n-l)} & [L_2]_{l \times l} & [0]_{l \times l} \end{bmatrix} \right) \begin{Bmatrix} \{U_{ng1}\}_{(n-l) \times 1} \\ \{U_{ng2}\}_{l \times 1} \\ \hline \{\mu\}_{l \times 1} \end{Bmatrix} = \begin{Bmatrix} \{0\}_{(n-l) \times 1} \\ \{0\}_{l \times 1} \\ \hline \{0\}_{l \times 1} \end{Bmatrix} \quad (\text{A8.2-6})$$

Replacing $\langle U_{ng2} \rangle_{l \times 1}^T$ in Eq.A8.2-6 by Eq.A8.2-5, one has

$$\begin{aligned}
& \left\langle \left\langle U_{ng1} \right\rangle_{b(n-l)}^T \mid \left\langle U_{ng1} \right\rangle_{b(n-l)}^T [L_3]_{(n-l) \times l}^T \right\rangle \left[\frac{[K_{11} - \lambda K_{G11}]_{(n-l) \times (n-l)}}{[K_{21} - \lambda K_{G21}]_{b(n-l)}} \mid \frac{[K_{12} - \lambda K_{G12}]_{(n-l) \times l}}{[K_{22} - \lambda K_{G22}]_{l \times l}} \right] \left\{ \frac{\{U_{ng1}\}_{(n-l) \times 1}}{[L_3]_{b(n-l)} \{U_{ng1}\}_{(n-l) \times 1}} \right\} \\
& = \left\{ \frac{\{0\}_{(n-l) \times 1}}{\{0\}_{l \times 1}} \right\}
\end{aligned} \tag{A8.2-7}$$

$$\begin{aligned}
& \left\langle \left\langle U_{ng1} \right\rangle_{b(n-l)}^T \mid \left\langle U_{ng1} \right\rangle_{b(n-l)}^T \right\rangle \left[\frac{[K_{11} - \lambda K_{G11}]_{(n-l) \times (n-l)}}{[L_3]_{(n-l) \times l}^T [K_{21} - \lambda K_{G21}]_{b(n-l)}} \mid \frac{[K_{12} - \lambda K_{G12}]_{(n-l) \times l} [L_3]_{b(n-l)}}{[L_3]_{(n-l) \times l}^T [K_{22} - \lambda K_{G22}]_{l \times l} [L_3]_{b(n-l)}} \right] \left\{ \frac{\{U_{ng1}\}_{(n-l) \times 1}}{\{U_{ng1}\}_{(n-l) \times 1}} \right\} \\
& = \left\{ \frac{\{0\}_{(n-l) \times 1}}{\{0\}_{l \times 1}} \right\}
\end{aligned} \tag{A8.2-8}$$

$$\begin{aligned}
& \left\langle \left\langle U_{ng1} \right\rangle_{b(n-l)}^T \mid \left\langle U_{ng1} \right\rangle_{b(n-l)}^T \right\rangle \left(\frac{[[K_{11} - \lambda K_{G11}] + [K_{12} - \lambda K_{G12}] [L_3]]_{(n-l) \times (n-l)} \{U_{ng1}\}_{(n-l) \times 1}}{[[L_3]^T [K_{21} - \lambda K_{G21}] + [L_3]^T [K_{22} - \lambda K_{G22}] \mathbf{I} [L_3]]_{(n-l) \times (n-l)} \{U_{ng1}\}_{(n-l) \times 1}} \right) \tag{A8.2-9} \\
& = \left\{ \frac{0_{(n-l) \times 1}}{0_{(n-l) \times 1}} \right\}
\end{aligned}$$

$$\begin{aligned}
& \left\langle U_{ng1} \right\rangle_{l \times (n-l)}^T \left([[L_3]^T [K_{21} - \lambda K_{G21}] + [L_3]^T [K_{22} - \lambda K_{G22}] \mathbf{I} [L_3]]_{(n-l) \times (n-l)} \{U_{ng1}\}_{(n-l) \times 1} \right)_{(n-l) \times (n-l)} \{U_{ng1}\}_{(n-l) \times 1} \tag{A8.2-10} \\
& = \{0\}_{l \times 1}
\end{aligned}$$

$$\begin{aligned}
& \left\langle U_{ng1} \right\rangle_{l \times (n-l)}^T \left[\begin{array}{c} [K_{11}] + [K_{12}] [L_3] + [L_3]^T [K_{21}] + [L_3]^T [K_{22}] [L_3] \\ -\lambda [[K_{G11}] + [K_{G12}] [L_3] + [L_3]^T [K_{G21}] + [L_3]^T [K_{G22}] [L_3]] \end{array} \right]_{(n-l) \times (n-l)} \{U_{ng1}\}_{(n-l) \times 1} = \{0\}_{l \times 1}
\end{aligned} \tag{A8.2-11}$$

$$\begin{aligned}
& \left(\underbrace{\left[\begin{aligned} & [K_{11}]_{(n-l) \times (n-l)} + [K_{12}]_{(n-l) \times l} [L_3]_{l \times (n-l)} + [L_3]^T_{(n-l) \times l} [K_{21}]_{l \times (n-l)} + [L_3]^T_{(n-l) \times l} [K_{22}]_{l \times l} [L_3]_{l \times (n-l)} \end{aligned} \right]}_{\text{symmetric}} \right)_{(n-l) \times (n-l)} \\
& - \lambda \left(\underbrace{\left[\begin{aligned} & [K_{G11}]_{(n-l) \times (n-l)} + [K_{G12}]_{(n-l) \times l} [L_3]_{l \times (n-l)} + [L_3]^T_{(n-l) \times l} [K_{G21}]_{l \times (n-l)} + [L_3]^T_{(n-l) \times l} [K_{G22}]_{l \times l} [L_3]_{l \times (n-l)} \end{aligned} \right]}_{\text{symmetric}} \right)_{(n-l) \times (n-l)} \left\{ U_{ngl} \right\}_{(n-l) \times 1} \\
& = \{0\}_{(n-l) \times 1}
\end{aligned} \tag{A8.2-12}$$

References

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List of Symbols

B :	Bimoment
$[K]$:	Elastic stiffness matrix of a member
$[K_G]$:	Geometric stiffness matrix of a member
$[K_\psi]$:	Warping restraint stiffness matrix defined by a joint
$[L]$:	Constraint conditions for enforcing Lagrange Multiplier to a frame Structure
S :	Arbitrary point on the mid-surface
$[T]$:	Transformation matrix
u_g, v_g, w_g :	Displacements along the global x_g, y_g, z_g directions, respectively
u_l, v_l, w_l :	Displacements along the local x, y, z axes, respectively
$\{U_n\}$:	Nodal displacement vector
$\{U^*\}$:	Total displacement vector
x, y, z :	Cartesian coordinates
ϕ :	Rotation angle
λ :	Load factor (eigen-value)
μ :	Lagrange Multiplier
π :	Total potential energy functional
$\theta_x, \theta_y, \theta_z$:	Rotation angles around x, y, z axes, respectively
θ_ψ :	Rotation of a flange due to warping deformation
$\omega(s)$:	Warping function (sectorial area) of a point S
$\psi(z)$:	Rate of change of the longitudinal displacement due to warping along the z axis (1/Length dimension)

Chapter 9 Summary, Conclusions and Recommendations

9.1 Summary

The study has focused on the following

1. A set of nonlinear kinematic relationships for general shear deformable open thin-walled members are developed based on the finite rotation equation (Chapter 3). The kinematic relations are applicable to pre-buckling and buckling analyses.
2. Variational principles for the pre-buckling and buckling analyses of thin-walled members with doubly symmetric and mono-symmetric sections were developed. The variational principles are applicable under non-orthogonal coordinate systems, and captures shear deformation effects and load position effects (Chapters 5, 6, and Appendix A).
3. The governing field equations and associated boundary conditions are formulated from the developed variational principles developed (Chapter 6) and closed form solutions for the cases of column flexural buckling, column torsional buckling, and laterally torsional buckling for beams under uniform moments are derived. The buckling loads/moments are found to agree well with those based on recently developed buckling formulation which account for shear deformation. However, when compared with classical solutions that do not include the effect of shear deformation the comparisons show that shear deformation effects tend to reduce the predicted buckling loads for short span members (Chapter 6).
4. A shear deformable beam element is formulated based on non-orthogonal coordinates (Chapter 7). The element developed is versatile and was successfully able to model columns, beams and beam-columns under a variety of loading and boundary conditions. The finite element is found to predict buckling modes in excellent agreement with those based on ABAQUS shell analysis. In all cases examined, the present solution provides buckling load predictions lower than those based on the classical solution, but higher than the shell FEA (Chapter 7).
5. The non-orthogonal coordinate feature of the solution is successful in predicting the buckling loads for eccentrically supported beams (Chapter 7).

6. A finite element formulation which captures the partial warping restraint of common moment connections in planar frame is formulated (Chapter 8). The formulation of the joint element can accurately predict the buckling resistance of portal frames and Γ shaped frames. It is limited to joints connecting two members at 90 degrees.
7. Four programs were coded in FORTRAN to implement the formulations developed in this thesis. These programs conduct the following analyses
 - (a) Pre-buckling analysis of shear deformable thin-walled members under non-orthogonal coordinates, based on the formulation developed in Chapter 5,
 - (b) Lateral-torsional buckling analysis of shear deformable thin walled members under non-orthogonal coordinates, based on the formulation developed in Chapter 7
 - (c) An extension of the solution was also implemented for mono-symmetric sections based on the formulation in Appendix A, and
 - (d) Lateral-torsional buckling analysis of planar frames involving the joint element which captures partial joint warping restraints, based on the formulation in Chapter 8, and Appendix 8.1

9.2 Conclusions

Among the conclusions outlined under each chapter, the most significant ones are:

1. The study in Chapter 5 indicates that the response of a typical coped beam is significantly softer than a similar un-coped beam. The formulation developed in Chapter 5 is successful in predicting the difference between these two cases.
2. The buckling solution developed in Chapter 7 indicates that the neglect of shear deformation effects in classical solutions (e.g. Barsoum & Gallagher, 1970) can over-predict the buckling resistance of beams with short spans and/or large moment gradients. Since code equations are based on the classical solution predictions, more attention to these problems should be given through additional finite element runs and full scale experiments.
3. The joint finite element developed in Chapter 8 in conjunction with the shear deformable element developed in Chapter 7 are able to reliably predict the lateral buckling resistance of laterally unsupported portal and Γ shaped frames when compared to shell FEA results.

9.3 Research Recommendations

1. The warping matrix developed in Chapter 8 can be used to conduct the elastic analyses of planar frames with moment connections under more general load conditions including weak axis bending, torsion and bimoments. The methodology introduced can be extended to joint connecting up to four members.
2. The FEA implementation developed can be extended to beams with other cross-sections (e.g., C-shaped cross-sections).
3. The strain expressions developed in Chapter 6 are quite general and could account for pre-buckling deformation effects in future developments.
4. The analysis can be extended to geometric non-linear analysis and dynamic analysis of thin-walled members.

Appendix A Lateral-Torsional Buckling Analysis of Open Thin-Walled Members with Mono-Symmetric Cross-Sections

This appendix presents the developed formulations for predicting the lateral-torsional buckling resistance of open thin-walled members with mono-symmetric cross-sections. The governing differential equations of equilibrium for the buckling deformations ($u_b, \theta_{yb}, \theta_{zb}, \psi_b$) and associated boundary conditions are formulated under a non-orthogonal coordinate system. The theory deviates from conventional theories in the following: 1) It is based on general non-orthogonal coordinates, a feature that is particularly suited for the effective modeling of eccentric supports (i.e., supports that do not coincide with the location of the shear centre; 2) It captures shear deformation effects due to bending and warping; 3) It naturally incorporates the effect of load position relative to the shear centre. The formulation is based on the principle of stationary potential energy.

A-1 Hypotheses

Based on the statement of the problem and hypotheses made in Chapter 4, also, we restrict our formulation to

- 1) Mono-symmetric I-shaped sections in which the centroid and shear centre lie on the axis of symmetry $x = 0$. (This makes the solution applicable for I-shaped sections with unequal flanges, crane beams consisting of I-shaped sections with a flange welded to a horizontal channel section, and doubly symmetric sections, and T section (when local warping effects are added).
- 2) The pole, origin, and sectorial origin are assumed to coincide at Point O . (This makes all degrees of freedom u, v, w defined at the same Point O , unlike orthogonal formulations where the longitudinal displacement is defined at the centroid while displacements in the plane of the cross-section are defined at the shear centre).
- 3) Point $O (0,0)$ is assumed to lie on the axis of symmetry $x = 0$ (but not necessarily on the shear centre, section centroid, nor the principal sectorial origin). This means that the coordinates of the centroid are $C(x=0, y_c)$. Also, $\int_A x dA = 0$, $\int_A xy dA = 0$, $\int_A \omega dA = 0$,

$$\int_A \omega y dA = 0, \text{ while the properties } \int_A y dA \neq 0, \int_A \omega x dA \neq 0.$$

4) The member is subject to

- a. Distributed axial force $q_z(z)$ and a set of concentrated axial forces $Q_{zi}(z)$ acting longitudinally along the z direction at respective distances $s_{qz}(z)$ and $s_{Qzi}(z)$ from the origin, thus inducing axial force and possibly bending moments when $s_q(z) \neq y_C$ or $s_{Q_i} \neq y_C$ (i.e., when the line of action of the force does not coincide with the centroidal axis).
- b. Distributed transverse loading $q_y(z)$ and a set of concentrated transverse forces $Q_{yi}(z)$ acting in the plane of the shear centre along the y direction (minor principal axis), thus inducing strong axis bending about the x direction. The transverse loads $q_y(z)$ and $Q_{yi}(z)$ are applied at distances $s_{qy}(z)$ and $s_{Qyi}(z)$ from the origin, respectively.

A-2 Displacement Fields

According to Eq. 4.5, the total displacements $u_s^*(s, z), v_s^*(s, z), w_s^*(s, z)$ of Point S at the mid-surface are

$$\begin{aligned} u_s^*(s, z) &= u_b(z) - y(s)\theta_{zb}(z) \\ v_s^*(s, z) &= \lambda v_p(z) + x(s)\theta_{zb}(z) \\ w_s^*(s, z) &= w_b(z) + \lambda w_p(z) + y(s)\lambda\theta_{xp}(z) - x(s)\theta_{yb}(z) + \omega(s)\psi_b(z) + x(s)\lambda\theta_{xp}(z)\theta_{zb}(z) + y(s)\theta_{yb}(z)\theta_{zb}(z) \end{aligned} \quad (\text{A-1a-c})$$

in which we have a longitudinal buckling displacement $w_b(z)$, which should appear when the coordinates are non-orthogonal. It is noted that in expressing Eq. A-1a and b the coordinates of the pole $A(x_A, y_A)$ in the present formulation are $A(0, 0)$ in line with assumptions made in the present formulation.

A-2.1 Enforcing the Inextensional Buckling Assumption

According to Trahair (1993), “for an axially loaded column, it is assumed that during buckling, the centroidal (longitudinal) strain remains zero, so that the column buckles under constant force

N (this is not the case for a latticed column for instance). For this inextensional buckling, δw_{sb} (the first variation of the longitudinal buckling displacement) is small compared to $\delta u_b, \delta v_b, \delta \phi_b$ and the term $\frac{1}{2} \int_L EA (\delta w'_b)^2 dz$ arising in the variation of the second variation of the total potential energy can be ignored”.

Also, according to Trahair (1993), “It is assumed that during buckling, the centroidal strain and the curvature in the principal yz plane remain zero, so that the beam-column buckles under constant axial force $\bar{N}(z)$ and (strong axis) moment $\bar{M}_x(z)$ (this is the beam-column equivalent of inextensional buckling). In this case, the buckling displacements $\delta w_b, \delta v_b$ are of a smaller order compared to the buckling deformations $\delta u_b, \delta \phi_b$ ”.

In an orthogonal coordinate system in which the section centroid $(0, y_c)$ coincides with the origin $(0, 0)$, the total longitudinal displacement at the centroid is $w_s^*(s_c, z) = \lambda w_p(z) + \omega(s_c) \psi_b(z)$. The longitudinal displacement of the centroid during buckling is $w_{sb}(s_c, z) = \omega(s_c) \psi_b(z)$ and the inextensional buckling assumption (i.e., no longitudinal displacement takes place at the centroid throughout buckling) $w_{sb}(s_c, z) = 0$ leads to the condition $\omega(s_c) \psi_b(z) = 0$, which is realized for a section that is mono-symmetric with respect to the y axis, since the condition $\omega(s_c) = 0$ is satisfied everywhere along y axis.

In a non-orthogonal coordinate system, the section centroid $(0, y_c(s_c))$ is offset from the origin $(0, 0)$. The total longitudinal displacement at the centroid is

$$w_s^*(s_c = 0, z) = w_b(z) + \lambda w_p(z) + y_c(s_c) \lambda \theta_{xp}(z) + y_c(s_c) \theta_{yb}(z) \theta_{zb}(z) \quad (\text{A-2})$$

Determination of $w_b(z)$

It will be assumed that the longitudinal displacement at the centroidal axis to vanish during buckling (in line with the inextensional buckling assumption). During buckling, the longitudinal

displacement at the centroid is $w_{sb}(s_c, z) = w_b(z) + y_c \theta_{yb}(z) \theta_{zb}(z)$. By enforcing the inextensional buckling assumption, $w_{sb}(s_c, z) = 0$, one obtains

$$w_b(z) = -y_c \theta_{yb}(z) \theta_{zb}(z) \quad (\text{A-3})$$

From Eq. A-3, substitute into Eq. A-1c to yield the longitudinal displacement of an arbitrary Point S at the mid-surface

$$\begin{aligned} w_s^*(s, z) = & \lambda w_p(z) - x(s) \theta_{yb}(z) + y(s) \lambda \theta_{xp}(z) + \omega(s) \psi_b(z) + x(s) \lambda \theta_{xp}(z) \theta_{zb}(z) \\ & + (y(s) - y_c) \theta_{yb}(z) \theta_{zb}(z) \end{aligned} \quad (\text{A-4})$$

Based on Eq. 4.6, the displacements along the tangential and normal directions of point S in the buckling stage are

$$\begin{aligned} t_s^*(z, s) = & u_b(z) \frac{dx(s)}{ds} + \lambda v_p(z) \frac{dy(s)}{ds} + h(s) \theta_{zb}(z) \\ n_s^*(z, s) = & \lambda v_p(z) \frac{dx(s)}{ds} - u_b(z) \frac{dy(s)}{ds} + k(s) \theta_{zb}(z) \end{aligned} \quad (\text{A-5})$$

where

$$\begin{aligned} h(s) = & x(s) \sin \alpha(s) - y(s) \cos \alpha(s) \\ k(s) = & x(s) \cos \alpha(s) + y(s) \sin \alpha(s) \end{aligned} \quad (\text{A-6})$$

A-3 Condition of Neutral Equilibrium

The total potential energy π of a member is the sum of the internal strain energy U (induced by normal stresses and shear stresses) and the gain of potential energy V due to externally applied loads and tractions undergoing displacements, i.e.,

$$\pi = U + V \quad (\text{A-7})$$

As discussed in Section 2.2, when the member is in a neutral equilibrium state, the variation of the second variation of the total potential energy has to be stationary (i.e., vanish for any set of buckling deformations $u_b(z)$, $\theta_{yb}(z)$, $\theta_{zb}(z)$, and $\psi_b(z)$).

$$\delta \left(\frac{1}{2} \delta^2 \pi \right) = \delta \left[\frac{1}{2} (\delta^2 U + \delta^2 V) \right] = 0 \quad (\text{A-8})$$

The internal strain energy of a prismatic open thin-walled member is given by

$$U = \int_0^L \int_A \left(\frac{1}{2} \sigma_{zz} \epsilon_{zz} + \frac{1}{2} \tau_{zs} \gamma_{zs} \right) dA dz \quad (\text{A-9})$$

At the buckling configuration (Configuration 4 in Fig. 4.2), the internal strain energy accumulates two contributions, one is due to pre-buckling deformations (and associated strains and stresses), and the other one arises in going from configuration 3 to configuration 4 (buckling deformation). Thus, Eq. A-9 can be expressed as

$$U = \frac{1}{2} \int_0^L \int_A \left[(\sigma_{zz_b} + \lambda \sigma_{zz_p}) (\varepsilon_{zz_b} + \lambda \varepsilon_{zz_p}) + (\tau_{zsb} + \lambda \tau_{zsp}) (\gamma_{zsb} + \lambda \gamma_{zsp}) \right] dA dz + \frac{1}{2} \int_0^L GJ \theta'_{zb}{}^2 dz \quad (A-10)$$

in which the last term represents the strain energy due to the St. Venant torsion .

By expanding the brackets in Eq. 6.10 and noting that $\sigma_{zz_p} = E \varepsilon_{zz_p}$, $\sigma_{zz_b} = E \varepsilon_{zz_b}$, $\tau_{zsp} = G \gamma_{zsp}$ and $\tau_{zsb} = G \gamma_{zsb}$, one obtains

$$U = \frac{1}{2} \int_0^L \int_A \left\{ E \left[(\varepsilon_{zz_b})^2 + 2 \varepsilon_{zz_b} \lambda \varepsilon_{zz_p} + (\lambda \varepsilon_{zz_p})^2 \right] + G \left[(\gamma_{zsb})^2 + 2 \gamma_{zsb} \lambda \gamma_{zsp} + (\lambda \gamma_{zsp})^2 \right] \right\} dA dz + \frac{1}{2} \int_0^L GJ \theta'_{zb}{}^2 dz \quad (A-11)$$

In inextensional buckling, the centroidal strain and the curvature in the principal yz -plane of the member remains unchanged. As a result, the member undergoes buckling under constant axial force $\bar{N}$ and bending moment $\bar{M}_x$ as obtained during pre-buckling. Therefore, the variations of the pre-buckling displacements $\lambda \delta v_p$, $\lambda \delta w_p$, $\lambda \delta \theta_x$ vanish and the second variation of the internal strain energy (Eq. A-11) is

$$\frac{1}{2} \delta^2 U = \frac{1}{2} \int_0^L \int_A \left\{ E \left[(\delta \varepsilon_{zz_b})^2 + \lambda \varepsilon_{zz_p} \delta^2 \varepsilon_{zz_b} \right] + G \left[(\delta \gamma_{zsb})^2 + \lambda \gamma_{zsp} \delta^2 \gamma_{zsb} \right] \right\} dA dz + \frac{1}{2} \int_0^L GJ (\delta \theta'_{zb})^2 dz \quad (A-12)$$

Noting that $\sigma_{zz_p} = E \varepsilon_{zz_p}$ and $\tau_{zsp} = G \gamma_{zsp}$, one obtains

$$\frac{1}{2} \delta^2 U = \frac{1}{2} \int_0^L \int_A \left\{ E \left[(\delta \varepsilon_{zz_b})^2 + \lambda \sigma_{zz_p} \delta^2 \varepsilon_{zz_b} \right] + \left[G (\delta \gamma_{zsb})^2 + \lambda \tau_{zsp} \delta^2 \gamma_{zsb} \right] \right\} dA dz + \frac{1}{2} \int_0^L GJ (\delta \theta'_{zb})^2 dz \quad (A-13)$$

The second variation of the potential energy of the externally applied loads is

$$\begin{aligned} \frac{1}{2} \delta^2 V = & - \left(\sum_i Q_i(z) \delta^2 w_{Q_i}(s_{Q_i}, z) + \sum_j Q_j(z) \delta^2 v_{Q_j}(s_{Q_j}, z) + \sum_k M_k(z) \delta^2 \theta_{xk}(s_{Q_k}, z) \right) \\ & - \int_0^L \left(q_y \delta^2 v_{q_y}(s_{q_y}, z) + q_z(z) \delta^2 w_{q_z}(s_{q_z}, z) \right) dz \end{aligned} \quad (A-14)$$

In Eq. A-14, w_{qzb} and w_{Qzbi} are the longitudinal displacements of the points where the axial loads q_z and Q_{zi} are applied. From Eq. A-4, the second variations of the displacements are obtained

$$\begin{aligned}\delta^2 w_{qzb}(s, z) &= 2(y_{qz}(s) - y_C) \delta\theta_{yb}(z) \delta\theta_{zb}(z) \\ \delta^2 w_{Qzbi}(s, z) &= 2(y_{Qzi}(s) - y_C) \delta\theta_{yb}(z) \delta\theta_{zb}(z)\end{aligned}\quad (A-15)$$

Recall Eq. 3.30, the total vertical displacement of any Point S at the mid-surface is nonlinearly expressed as

$$\begin{aligned}v_s^*(s, z) &= \lambda v_p(z) + x(s) \theta_{zb}(z) + \frac{x(s)}{2} (\lambda \theta_{xp}(z) \theta_{yb}(z)) \\ &\quad - \frac{y(s)}{2} (\lambda^2 \theta_{xp}(z)^2 + \theta_{zb}(z)^2) - \omega_s(s) \psi_b(z) \lambda \theta_{xp}(z)\end{aligned}\quad (A-16)$$

The second variation of the vertical displacements v_{Qybj} and v_{qyjb} of the points the transverse loads q_y and Q_{yj} applied are obtained

$$\begin{aligned}\delta^2 v_{qyjb}(s, z) &= -y_{qy}(s) (\delta\theta_{zb}(z))^2 \\ \delta^2 v_{Qybj}(s, z) &= -y_{Qy}(s) (\delta\theta_{zb}(z))^2\end{aligned}\quad (A-17)$$

The second variation of the potential energy (Eq. A-14) is rewritten as

$$\begin{aligned}\frac{1}{2} \delta^2 V &= - \int_0^L \left[-q_y y_{qy} (\delta\theta_{zb})^2 + q_z \left[2(y_{qz} - y_C) \delta\theta_{yb} \delta\theta_{zb} \right] \right] dz \\ &\quad - \left(\sum_i Q_{zi} \left[2(y_{Qzi} - y_C) \delta\theta_{yb} \delta\theta_{zb} \right] - \sum_j Q_{yj} y_{Qyj} (\delta\theta_{zb})^2 \right)\end{aligned}\quad (A-18)$$

Thus, the condition of neutral stability for the prismatic open thin-walled member under the reference external loads is obtained by substituting Eqs. A-13 and A-18 into Eq. A-8.

$$\begin{aligned}\delta \left(\frac{1}{2} \delta^2 \pi \right) &= \delta \left[\frac{1}{2} (\delta^2 U + \delta^2 V) \right] = 0 \\ &= \frac{1}{2} \int_0^L \int_A \left\{ \left[E (\delta\epsilon_{zsb})^2 + \lambda \sigma_{zsp} \delta^2 \epsilon_{zsb} \right] + \left[G (\delta\gamma_{zsb})^2 + \lambda \tau_{zsp} \delta^2 \gamma_{zsb} \right] \right\} dA dz + \frac{1}{2} \int_0^L G J (\delta\theta'_{zb})^2 dz \\ &\quad - \int_0^L \left[-q_y y_{qy} (\delta\theta_{zb})^2 + q_z \left[2(y_{qz} - y_C) \delta\theta_{yb} \delta\theta_{zb} \right] \right] dz \\ &\quad - \left(\sum_i Q_{zi} \left[2(y_{Qzi} - y_C) \delta\theta_{yb} \delta\theta_{zb} \right] - \sum_j Q_{yj} y_{Qyj} (\delta\theta_{zb})^2 \right)\end{aligned}\quad (A-19)$$

A-4 Strains and Their Variations**A-4.1 Longitudinal Strain and Variations****Total Longitudinal Strain**

The total longitudinal strain is

$$\begin{aligned}\varepsilon_{zz}^* &\approx w'^* + \frac{1}{2}(u'^{*2} + v'^{*2}) \\ &= \left[\lambda(w'_p + y\theta'_{xp}) - x\theta'_{yb} + \omega\psi'_b + x(\lambda\theta_{xp}\theta'_{zb} + \lambda\theta'_{xp}\theta_{zb}) + (y - y_c)(\theta_{yb}\theta'_{zb} + \theta'_{yb}\theta_{zb}) \right] \\ &\quad + \frac{1}{2}[u_b'^2 - 2yu'_b\theta'_{zb} + y^2\theta_{zb}'^2] + \frac{1}{2}[(\lambda v'_p)^2 + 2x(\lambda v'_p)\theta'_{zb} + x^2\theta_{zb}'^2]\end{aligned}\quad (A-20)$$

The longitudinal strain can be decomposed into pre-buckling strain ε_{zzp} and buckling strain ε_{zzb}

$$\varepsilon_{zzp} = \lambda(w'_p + y\theta'_{xp}) + \frac{1}{2}(\lambda v'_p)^2 \quad (A-21)$$

$$\begin{aligned}\varepsilon_{zzb} &\approx \left[-x\theta'_{yb} + \omega\psi'_b + x(\lambda\theta_{xp}\theta'_{zb} + \lambda\theta'_{xp}\theta_{zb}) + (y - y_c)(\theta_{yb}\theta'_{zb} + \theta'_{yb}\theta_{zb}) \right] \\ &\quad + \frac{1}{2}[u_b'^2 - 2yu'_b\theta'_{zb} + y^2\theta_{zb}'^2] + \frac{1}{2}[2x(\lambda v'_p)\theta'_{zb} + x^2\theta_{zb}'^2]\end{aligned}\quad (A-22)$$

First Variation of Total and Buckling Longitudinal Strain

Based on the expression of total longitudinal strain (Eq. A-20) and noting that the first variations of all pre-buckling displacements vanish, the first variation of the total longitudinal strain is

$$\begin{aligned}\delta\varepsilon_{zz}^* &= \left[-x\delta\theta'_{yb} + \omega\delta\psi'_b + x(\lambda\theta_{xp}\delta\theta'_{zb} + \lambda\theta'_{xp}\delta\theta_{zb}) + (y - y_c)(\delta\theta_{yb}\theta'_{zb} + \theta_{yb}\delta\theta'_{zb} + \delta\theta'_{yb}\theta_{zb} + \theta'_{yb}\delta\theta_{zb}) \right] \\ &\quad + \left[u'_b\delta u'_b - y(\delta u'_b\theta'_{zb} + u'_b\delta\theta'_{zb}) + y^2\theta'_{zb}\delta\theta'_{zb} \right] + \frac{1}{2}[2x(\lambda v'_p)\delta\theta'_{zb} + x^2\theta'_{zb}\delta\theta'_{zb}]\end{aligned}\quad (A-23)$$

The non-linear components of the first variation of the total longitudinal strain (Eq. A-23) are assumed negligible compared to the linear components. This leads to a buckling formulation that does capture post-buckling effects (Pi and Trahair 1992a, and Trahair 1993), which are outside of the scope of the present study. This assumption leads to the following simplification.

$$\delta\varepsilon_{zz}^* \approx \left[-x\delta\theta'_{yb} + \omega\delta\psi'_b + \lambda(\theta_{xp} + v'_p)x\delta\theta'_{zb} + \lambda x\theta'_{xp}\delta\theta_{zb} \right] \quad (A-24)$$

The first variation of the longitudinal strain in Eq. A-20 includes the pre-buckling displacements $\lambda v'_p$, $\lambda\theta_{xp}$ and $\lambda\theta'_{xp}$. These terms would be retained if the effects of pre-buckling deformations were to be included in the analysis leading to a quadratic eigen-value problem (a nonlinear analysis) upon substitution in the variational expression (Eq. A-19). In contrast, in a classical

buckling analysis, the stresses and strains are assumed to linearly vary with the applied load (i.e., the second order moments due to axial force undergoing transverse displacement are neglected in the pre-buckling stage) and the $\lambda v'_p$, $\lambda \theta_{xp}$ and $\lambda \theta'_{xp}$ terms are normally neglected. Another line of analysis is possible in which the pre-buckling displacements are substituted in variational expression, resulting in a quadratic eigen value analysis, but subsequently omitting the quadratic terms to linearize the system, leading to the so-called linearized formulation (Pi and Trahair 1992 b, c). Of all three methods, the nonlinear analysis provides the most accurate results. The classical analysis is the simplest and appears to provide the smallest buckling loads (Pi and Trahair 1992 c), while the linearized analysis tends to over-predict buckling loads.

The focus of the present study is to investigate shear deformation effects on lateral-torsional buckling resistance of thin-walled members in a classical sense. Thus, all pre-buckling deformations are neglected, leading to the simplification

$$\delta \varepsilon_{zz}^* \approx -x \delta \theta'_{yb} + \omega \delta \psi'_b \quad (\text{A-25})$$

Recalling that the first variation of the pre-buckling vanishes, one obtains

$$\delta \varepsilon_{zzb} \approx \delta \varepsilon_{zz}^* = -x \delta \theta'_{yb} + \omega \delta \psi'_b \quad (\text{A-26})$$

Second Variation of Total and Buckling Longitudinal Strain

Since the first variation of the pre-buckling strain vanish, the second variation of buckling strain is equal to the second variation of the total strain, $\delta^2 \varepsilon_{zzb} = \delta^2 \varepsilon_{zz}^*$

$$\begin{aligned} \delta^2 \varepsilon_{zzb} &= \delta^2 \varepsilon_{zz}^* \\ &= (\delta u'_b)^2 + 2y \left(\delta \theta'_{yb} \delta \theta'_{zb} + \delta \theta_{yb} \delta \theta'_{zb} - \delta u'_b \delta \theta'_{zb} \right) + (x^2 + y^2) (\delta \theta'_{zb})^2 - 2y_c \left(\delta \theta_{yb} \delta \theta'_{zb} + \delta \theta'_{yb} \delta \theta_{zb} \right) \end{aligned} \quad (\text{A-27})$$

A-4.2 Shear Strain

Total Shear Strain

The shear strain γ_{zs}^* at the mid-surface of a thin-walled member is

$$\gamma_{zs}^* = \frac{\partial t_s^*}{\partial z} + \frac{\partial w_s^*}{\partial s} + \frac{\partial t_s^*}{\partial z} \frac{\partial t_s^*}{\partial s} + \frac{\partial w_s^*}{\partial s} \frac{\partial w_s^*}{\partial z} + \frac{\partial n_s^*}{\partial z} \frac{\partial n_s^*}{\partial s} \quad (\text{A-28})$$

From Eqs. A-4 and A-5 by substituting into Eq. A-28, the shear strain γ_{zs}^* is expressed in terms of

the displacements (Appendix 6-1) yielding

$$\begin{aligned}
 \gamma_{zs}^* = & \frac{dx}{ds} (u'_b - \theta_{yb} + \lambda \theta_{xp} \theta_{zb}) + \frac{dy}{ds} (\lambda v'_p + \lambda \theta_{xp} + \theta_{yb} \theta_{zb}) + h (\theta'_{zb} + \psi'_b) \\
 & + \frac{dy}{ds} (\lambda \theta_{xp}) (\lambda w'_p) - \frac{dx}{ds} (\lambda w'_p) \theta_{yb} + h (\lambda w'_p) \psi'_b + \frac{dx}{ds} \lambda w'_p \lambda \theta_{xp} \theta_{zb} + \frac{dy}{ds} \lambda w'_p \theta_{yb} \theta_{zb} \\
 & + y \frac{dy}{ds} (\lambda \theta_{xp}) (\lambda \theta'_{xp}) - y \frac{dx}{ds} (\lambda \theta'_{xp}) \theta_{yb} + y h (\lambda \theta'_{xp}) \psi'_b + y \frac{dx}{ds} \lambda \theta'_{xp} \lambda \theta_{xp} \theta_{zb} + y \frac{dy}{ds} \lambda \theta'_{xp} \theta_{yb} \theta_{zb} \\
 & - x \frac{dy}{ds} (\lambda \theta_{xp}) \theta'_{yb} + x \frac{dx}{ds} \theta_{yb} \theta'_{yb} - x h \psi'_b \theta'_{yb} - x \frac{dx}{ds} \lambda \theta_{xp} \theta'_{yb} \theta_{zb} - x \frac{dy}{ds} \theta_{yb} \theta'_{yb} \theta_{zb} \\
 & + \omega \frac{dy}{ds} (\lambda \theta_{xp}) \psi'_b - \omega \frac{dx}{ds} \theta_{yb} \psi'_b + \omega h \psi'_b \psi'_b + \omega \frac{dx}{ds} \lambda \theta_{xp} \theta_{zb} \psi'_b + \omega \frac{dy}{ds} \theta_{yb} \theta_{zb} \psi'_b \\
 & + x \frac{dy}{ds} \lambda \theta_{xp} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) - x \frac{dx}{ds} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb} + x h (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \psi'_b \\
 & + x \frac{dx}{ds} \lambda \theta_{xp} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{zb} + x \frac{dy}{ds} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb} \theta_{zb} \\
 & + (y - y_C) \frac{dy}{ds} \lambda \theta_{xp} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) - (y - y_C) \frac{dx}{ds} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{yb} \\
 & + (y - y_C) h (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \psi'_b + (y - y_C) \frac{dx}{ds} \lambda \theta_{xp} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{zb} \\
 & + (y - y_C) \frac{dy}{ds} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{yb} \theta_{zb} + \left(\frac{dx}{ds} \right) \left(\frac{dk}{ds} \right) \lambda v'_p \theta_{zb} - \left(\frac{dy}{ds} \right) \left(\frac{dk}{ds} \right) u'_b \theta_{zb} + k \left(\frac{dk}{ds} \right) \theta'_{zb} \theta_{zb}
 \end{aligned} \tag{A-29}$$

The total shear strain in Eq. A-29 can be expressed as a pre-buckling and a buckling component as

$$\gamma_{zsp} = \frac{dy}{ds} (\lambda v'_p + \lambda \theta_{xp}) + \frac{dy}{ds} (\lambda \theta_{xp}) (\lambda w'_p) + y \frac{dy}{ds} (\lambda \theta_{xp}) (\lambda \theta'_{xp}) \tag{A-30}$$

$$\begin{aligned}
\gamma_{zsb} = & \frac{dx}{ds} (u'_b - \theta_{yb} + \lambda \theta_{xp} \theta_{zb}) + \frac{dy}{ds} (\theta_{yb} \theta_{zb}) + h (\theta'_{zb} + \psi_b) \\
& - \frac{dx}{ds} (\lambda w'_p) \theta_{yb} + h (\lambda w'_p) \psi_b + \frac{dx}{ds} \lambda w'_p \lambda \theta_{xp} \theta_{zb} + \frac{dy}{ds} \lambda w'_p \theta_{yb} \theta_{zb} \\
& - y \frac{dx}{ds} (\lambda \theta'_{xp}) \theta_{yb} + y h (\lambda \theta'_{xp}) \psi_b + y \frac{dx}{ds} \lambda \theta'_{xp} \lambda \theta_{xp} \theta_{zb} + y \frac{dy}{ds} \lambda \theta'_{xp} \theta_{yb} \theta_{zb} \\
& - x \frac{dy}{ds} (\lambda \theta_{xp}) \theta'_{yb} + x \frac{dx}{ds} \theta_{yb} \theta'_{yb} - x h \psi_b \theta'_{yb} - x \frac{dx}{ds} \lambda \theta_{xp} \theta'_{yb} \theta_{zb} - x \frac{dy}{ds} \theta_{yb} \theta'_{yb} \theta_{zb} \\
& + \omega \frac{dy}{ds} (\lambda \theta_{xp}) \psi'_b - \omega \frac{dx}{ds} \theta_{yb} \psi'_b + \omega h \psi_b \psi'_b + \omega \frac{dx}{ds} \lambda \theta_{xp} \theta_{zb} \psi'_b + \omega \frac{dy}{ds} \theta_{yb} \theta_{zb} \psi'_b \\
& + x \frac{dy}{ds} \lambda \theta_{xp} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) - x \frac{dx}{ds} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb} + x h (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \psi_b \\
& + x \frac{dx}{ds} \lambda \theta_{xp} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{zb} + x \frac{dy}{ds} (\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb} \theta_{zb} \\
& + (y - y_C) \frac{dy}{ds} \lambda \theta_{xp} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) - (y - y_C) \frac{dx}{ds} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{yb} \\
& + (y - y_C) h (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \psi_b + (y - y_C) \frac{dx}{ds} \lambda \theta_{xp} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{zb} \\
& + (y - y_C) \frac{dy}{ds} (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{yb} \theta_{zb} + \left(\frac{dx}{ds} \right) \left(\frac{dk}{ds} \right) \lambda v'_p \theta_{zb} - \left(\frac{dy}{ds} \right) \left(\frac{dk}{ds} \right) u'_b \theta_{zb} + k \left(\frac{dk}{ds} \right) \theta'_{zb} \theta_{zb}
\end{aligned} \tag{A-31}$$

First Variation of Total and Buckling Shear Strain

Given that the pre-buckling shear strains are constant throughout buckling, the first variation of the total shear strain is equal to the first variation of the buckling shear strain, $\delta\gamma_{zs}^* = \delta\gamma_{zsb}$.

Neglecting the second and higher order terms, the first variation of the shear strain is obtained

$$\begin{aligned}
\delta\gamma_{zsb} \approx & \frac{dx}{ds} (\delta u'_b - \delta \theta_{yb} + \lambda \theta_{xp} \delta \theta_{zb}) + h (\delta \theta'_{zb} + \delta \psi_b) - \frac{dx}{ds} (\lambda w'_p) \delta \theta_{yb} + h (\lambda w'_p) \delta \psi_b \\
& + \frac{dx}{ds} \lambda w'_p \lambda \theta_{xp} \delta \theta_{zb} - y \frac{dx}{ds} (\lambda \theta'_{xp}) \delta \theta_{yb} + y h (\lambda \theta'_{xp}) \delta \psi_b + y \frac{dx}{ds} \lambda \theta'_{xp} \lambda \theta_{xp} \delta \theta_{zb} \\
& - x \frac{dy}{ds} (\lambda \theta_{xp}) \delta \theta'_{yb} + \omega \frac{dy}{ds} (\lambda \theta_{xp}) \delta \psi'_b + x \frac{dy}{ds} \lambda \theta_{xp} (\lambda \theta'_{xp} \delta \theta_{zb} + \lambda \theta_{xp} \delta \theta'_{zb}) + \left(\frac{dx}{ds} \right) \left(\frac{dk}{ds} \right) \lambda v'_p \delta \theta_{zb}
\end{aligned} \tag{A-32}$$

Neglecting pre-buckling deformations effects in line of a classical buckling analysis in Eq. A-32, simplifies,

$$\delta\gamma_{zsb} \approx \frac{dx}{ds} (\delta u'_b - \delta \theta_{yb}) + h (\delta \theta'_{zb} + \delta \psi_b) \tag{A-33}$$

Second Variation of Total and Buckling Shear Strain

Since the pre-buckling shear strain is constant throughout buckling, the second variation of the total shear strain $\delta^2 \gamma_{zs}^*$ is equal to the second variation of the buckling shear strain $\delta^2 \gamma_{zsb}$.

$$\begin{aligned}
\delta^2 \gamma_{zsb} = & \frac{dy}{ds} \lambda w'_p \delta^2 (\theta_{yb} \theta_{zb}) + y \frac{dy}{ds} \lambda \theta'_{xp} \delta^2 (\theta_{yb} \theta_{zb}) + x \frac{dx}{ds} \delta^2 (\theta_{yb} \theta'_{yb}) - x h \delta^2 (\psi_b \theta'_{yb}) \\
& - x \frac{dx}{ds} \lambda \theta_{xp} \delta^2 (\theta'_{yb} \theta_{zb}) - x \frac{dy}{ds} \delta^2 (\theta_{yb} \theta'_{yb} \theta_{zb}) - \omega \frac{dx}{ds} \delta^2 (\theta_{yb} \psi'_b) + \omega h \delta^2 (\psi_b \psi'_b) \\
& + \omega \frac{dx}{ds} \lambda \theta_{xp} \delta^2 (\theta_{zb} \psi'_b) + \omega \frac{dy}{ds} \delta^2 (\theta_{yb} \theta_{zb} \psi'_b) - x \frac{dx}{ds} \delta^2 [(\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb}] \\
& + x h \delta^2 [(\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \psi_b] + x \frac{dx}{ds} (\lambda \theta_{xp}) (\lambda \theta_{xp}) \delta^2 (\theta'_{zb} \theta_{zb}) \\
& + x \frac{dy}{ds} \delta^2 (\lambda \theta'_{xp} \theta_{zb} \theta_{yb} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb} \theta_{yb} \theta_{zb}) + (y - y_C) \frac{dy}{ds} \lambda \theta_{xp} \delta^2 (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \\
& - (y - y_C) \frac{dx}{ds} \delta^2 [(\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \theta_{yb}] + (y - y_C) h \delta^2 (\theta'_{yb} \theta_{zb} \psi_b + \theta_{yb} \theta'_{zb} \psi_b) \\
& + (y - y_C) \frac{dx}{ds} \lambda \theta_{xp} \delta^2 (\theta'_{yb} \theta_{zb} \theta_{yb} + \theta_{yb} \theta'_{zb} \theta_{yb}) + (y - y_C) \frac{dy}{ds} \delta^2 (\theta'_{yb} \theta_{zb} \theta_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb} \theta_{yb} \theta_{zb}) \\
& - \left(\frac{dy}{ds} \right) \left(\frac{dk}{ds} \right) \delta^2 (u'_b \theta_{zb}) + k \left(\frac{dk}{ds} \right) \delta^2 (\theta'_{zb} \theta_{zb})
\end{aligned} \tag{A-34}$$

The variations of third and fourth order buckling terms are neglected from Eq. A-34, yielding

$$\begin{aligned}
\delta^2 \gamma_{zsb} = & \frac{dy}{ds} \lambda w'_p \delta^2 (\theta_{yb} \theta_{zb}) + y \frac{dy}{ds} \lambda \theta'_{xp} \delta^2 (\theta_{yb} \theta_{zb}) + x \frac{dx}{ds} \delta^2 (\theta_{yb} \theta'_{yb}) - x h \delta^2 (\psi_b \theta'_{yb}) \\
& - x \frac{dx}{ds} \lambda \theta_{xp} \delta^2 (\theta'_{yb} \theta_{zb}) - \omega \frac{dx}{ds} \delta^2 (\theta_{yb} \psi'_b) + \omega h \delta^2 (\psi_b \psi'_b) + \omega \frac{dx}{ds} \lambda \theta_{xp} \delta^2 (\theta_{zb} \psi'_b) \\
& - x \frac{dx}{ds} \delta^2 [(\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \theta_{yb}] + x h \delta^2 [(\lambda \theta'_{xp} \theta_{zb} + \lambda \theta_{xp} \theta'_{zb}) \psi_b] \\
& + x \frac{dx}{ds} (\lambda \theta_{xp}) (\lambda \theta_{xp}) \delta^2 (\theta'_{zb} \theta_{zb}) + (y - y_C) \frac{dy}{ds} \lambda \theta_{xp} \delta^2 (\theta'_{yb} \theta_{zb} + \theta_{yb} \theta'_{zb}) \\
& - \left(\frac{dy}{ds} \right) \left(\frac{dk}{ds} \right) \delta^2 (u'_b \theta_{zb}) + k \left(\frac{dk}{ds} \right) \delta^2 (\theta'_{zb} \theta_{zb})
\end{aligned} \tag{A-35}$$

By using classical analysis approach (neglecting all pre-buckling terms), the second variation of the buckling shear strain simplifies to

$$\begin{aligned} \delta^2 \gamma_{zb} \approx & 2 \left[\frac{dy}{ds} \delta \theta_{yb} \delta \theta_{zb} + x \frac{dx}{ds} (\delta \theta_{yb} \delta \theta'_{yb}) - x h (\delta \psi_b \delta \theta'_{yb}) - \omega \frac{dx}{ds} (\delta \theta_{yb} \delta \psi'_b) \right. \\ & \left. + \omega h (\delta \psi_b \delta \psi'_b) - \left(\frac{dy}{ds} \right) \left(\frac{dk}{ds} \right) (\delta u'_b \delta \theta_{zb}) + k \left(\frac{dk}{ds} \right) (\delta \theta'_{zb} \delta \theta_{zb}) \right] \end{aligned} \quad (\text{A-36})$$

A-5 Stress and Stress Resultants

The normal and shear pre-buckling stresses at a general Point S can be related to the pre-buckling strains. Neglecting the non-linear terms in Eqs. A-17 and A-26 leads to the following approximation

$$\varepsilon_{zsp} \approx \lambda (w'_p + y \theta'_{xp}) \quad (\text{A-37})$$

$$\gamma_{zsp} \approx \frac{dy}{ds} (\lambda v'_p + \lambda \theta_{xp}) \quad (\text{A-38})$$

From Eqs. 5.9 and 5.13, we recall that the normal stress σ_{zsp} and shear stress τ_{zsp} are related to the stress resultants through

$$\begin{aligned} \sigma_{zsp} &= \frac{1}{(AI_{xx} - S_x^2)} \left\langle \begin{pmatrix} I_{xx} - yS_x & yA - S_x \end{pmatrix} \right\rangle \begin{Bmatrix} \bar{N}_p(z) \\ \bar{M}_{xp}(z) \end{Bmatrix} \\ &= \frac{(I_{xx} - yS_x) \bar{N}_p(z) + (-S_x + yA) \bar{M}_{xp}(z)}{(AI_{xx} - S_x^2)} \end{aligned} \quad (\text{A-39})$$

where stress resultants $\bar{N}_p$ and $\bar{M}_{xp}$ are defined as

$$\begin{aligned} N_p &= \int_A \sigma_{zp} dA, \\ M_{xp} &= \int_A y \sigma_{zp} dA \end{aligned} \quad (\text{A-40})$$

Similarly, shear stress τ_{zsp} is obtained from Eq. 5.10 as

$$\tau_{zsp} = G \gamma_{zsp} = G \left[v'_p(z) + \theta_{xp}(z) \right] \frac{dy}{ds} \quad (\text{A-41})$$

The shear stress resultant is

$$V_{yp} = \int_A \tau_{zsp} \sin \alpha dA = \int_A G \left[v'_p(z) + \theta_{xp}(z) \right] \left(\frac{dy}{ds} \right)^2 dA = GD_y \left[v'_p(z) + \theta_{xp}(z) \right] \quad (\text{A-42})$$

A-6 Condition of Neutral Stability

The condition of neutral stability is determined by Eq. A-19. From Eqs. A-26, A-27, A-33, A-36, A-39 and A-41, by substituting into Eq. A-19, one obtains

$$\begin{aligned}
\delta \left(\frac{1}{2} \delta^2 \pi \right) &= 0 \\
&= \frac{1}{2} \int_0^L \left\{ EI_{yy} (\delta \theta'_{yb})^2 - 2ES_{x\omega} \delta \theta'_{yb} \delta \psi'_b + EI_{\omega\omega} (\delta \psi'_b)^2 + GJ \delta \theta'^2_{zb} \right\} dz \\
&\quad + \frac{1}{2} \int_0^L G \left\{ (D_{xx} \delta u'_b - D_{xx} \delta \theta_{yb} + D_{xh} \delta \theta'_{zb} + D_{xh} \delta \psi_b) \delta u'_b \right. \\
&\quad \quad + (-D_{xx} \delta u'_b + D_{xx} \delta \theta_{yb} - D_{xh} \delta \theta'_{zb} - D_{xh} \delta \psi_b) \delta \theta_{yb} \\
&\quad \quad + (D_{xh} \delta u'_b - D_{xh} \delta \theta_{yb} + D_{hh} \delta \theta'_{zb} + D_{hh} \delta \psi_b) \delta \theta'_{zb} \\
&\quad \quad \left. + (D_{xh} \delta u'_b - D_{xh} \delta \theta_{yb} + D_{hh} \delta \theta'_{zb} + D_{hh} \delta \psi_b) \delta \psi_b \right\} dz \\
&\quad + \frac{1}{2} \lambda \int_0^L \left(\frac{N_p(z) I_{xx} - M_{xp}(z) S_x}{AI_{xx} - S_x^2} \right) \left\{ A \delta u_b'^2 + 2S_x (\delta \theta'_{yb} \delta \theta_{zb} + \delta \theta_{yb} \delta \theta'_{zb}) - 2S_x \delta u'_b \delta \theta'_{zb} \right. \\
&\quad \quad \left. + (I_{xx} + I_{yy}) \delta \theta_{zb}'^2 - 2y_C A (\delta \theta_{yb} \delta \theta'_{zb} + \delta \theta'_{yb} \delta \theta_{zb}) \right\} dz \\
&\quad - \frac{1}{2} \lambda \int_0^L \left(\frac{N_p(z) S_x - M_{xp}(z) A}{AI_{xx} - S_x^2} \right) \left\{ S_x \delta u_b'^2 + 2I_{xx} (\delta \theta'_y \delta \theta_{zb} + \delta \theta_{yb} \delta \theta'_z) - 2I_{xx} \delta u'_b \delta \theta'_{zb} \right. \\
&\quad \quad \left. + I_{px} \delta \theta_{zb}'^2 - 2y_C S_x (\delta \theta_{yb} \delta \theta'_{zb} + \delta \theta'_{yb} \delta \theta_{zb}) \right\} dz \\
&\quad + \frac{\lambda}{2} \int_0^L \frac{V_{yp}}{D_{yy}} \left\{ 2D_{yy} \delta \theta_{yb} \delta \theta_{zb} - 2D_{yh} \delta \theta'_{yb} \delta \psi_b + 2D_{y\omega h} \delta \psi_b \delta \psi' \right. \\
&\quad \quad \left. - 2D_{yyk} \delta u'_b \delta \theta_{zb} + 2D_{yk} \delta \theta'_{zb} \delta \theta_{zb} \right\} dz \\
&\quad - \lambda \left(\sum_i Q_{zi} \left[2(y_{Q_{zi}} - y_C) \delta \theta_{yb} \delta \theta_{zb} \right] - \sum_j Q_{yj} y_{Q_{yj}} (\delta \theta_{zb})^2 \right) \\
&\quad - \lambda \int_0^L \left(-\frac{1}{2} q_y (y_{qy} - y_A) (\delta \theta_{zb})^2 + q_z \left[(y_{qz} - y_A) \delta \theta_{yb} \delta \theta_{zb} \right] \right) dz \tag{A-43}
\end{aligned}$$

in which some terms containing $\frac{dy}{ds} \frac{dx}{ds} = \sin \alpha \cdot \cos \alpha$ vanish for cross-sections entirely consisting of segments parallel to the x or y axes. In Eq. A-43, the following cross-sectional properties are defined

$$\begin{aligned}
D_{yk} &= \int_A k \frac{dy}{ds} \frac{dk}{ds} dA & D_{yyk} &= \int_A \left(\frac{dy}{ds} \right)^2 \frac{dk}{ds} dA & D_{yh} &= \int_A xh \frac{dy}{ds} dA \\
D_{y\omega h} &= \int_A h\omega \frac{dy}{ds} dA & D_{yhh} &= \int_A h \frac{dy}{ds} \frac{dh}{ds} dA
\end{aligned} \tag{A-44}$$

A-6.1 Differential Equations

Integrating Eq. A-43 by parts yields the differential equations of equilibrium and boundary conditions for the stability problem. In Eq. A-43, the integral term containing δu , $\delta\theta_y$, $\delta\theta_z$ and $\delta\psi$ should satisfy the condition of $\delta\left(\frac{1}{2}\delta^2\pi\right)=0$. Since δu , $\delta\theta_y$, $\delta\theta_z$ and $\delta\psi$ are arbitrarily chosen functions, their coefficients must vanish. This leads to the following differential equations are obtained

$$\begin{aligned}
& -GD_{xx}u_b'' + GD_{xx}\theta_y' - GD_{xh}\theta_z'' - GD_{xh}\psi_b' - \lambda \left(\frac{[N_p(z)I_{xx} - M_{xp}(z)S_x]}{AI_{xx} - S_x^2} [Au_b' - S_x\theta_z'] \right)' \\
& + \lambda \left(\frac{[N_p(z)S_x - M_{xp}(z)A]}{AI_{xx} - S_x^2} [S_xu_b' - I_{xx}\theta_z'] \right)' + \lambda \left(\frac{V_{yp}(z)}{D_{yy}} D_{yyk}\theta_z \right)' = 0
\end{aligned} \tag{A-45}$$

$$\begin{aligned}
& \left\{ -EI_{yy}\theta_y'' + ES_{x\omega}\psi_b'' - GD_{xx}u_b' + GD_{xx}\theta_y - GD_{xh}\theta_z' - GD_{xh}\psi_b + \lambda \frac{(N_p(z)I_{xx} - M_{xp}(z)S_x)}{AI_{xx} - S_x^2} (S_x - 2y_c A)\theta_z' \right. \\
& \quad \left. - \lambda \frac{(N_p(z)S_x - M_{xp}(z)A)}{AI_{xx} - S_x^2} (I_{xx} - 2y_c S_x)\theta_z' + \lambda \frac{V_{yp}(z)}{D_{yy}} D_{yy}\theta_z \right. \\
& \quad \left. - \lambda \left(\frac{[N_p(z)I_{xx} - M_{xp}(z)S_x]}{AI_{xx} - S_x^2} (S_x - 2y_c A)\theta_z \right)' + \lambda \left(\frac{[N_p(z)S_x - M_{xp}(z)A]}{AI_{xx} - S_x^2} (I_{xx} - 2y_c S_x)\theta_z \right)' \right. \\
& \quad \left. + \lambda \left(\frac{V_{yp}(z)}{D_{yy}} D_{yhh}\psi \right)' - q_z [2(y_{qz} - y_c)\theta_z] \right\} = 0
\end{aligned} \tag{A-46}$$

$$\begin{aligned}
& -GD_{xh}u_b'' - GJ\theta_{zb}'' + GD_{xh}\theta_{yb}' - GD_{hh}\theta_{zb}'' - GD_{hh}\psi_b' + \lambda \frac{(N_p(z)I_{xx} - M_{xp}(z)S_x)}{AI_{xx} - S_x^2} (S_x - 2y_c A) \theta_{yb}' \\
& - \lambda \frac{(N_p(z)S_x - M_{xp}(z)A)}{AI_{xx} - S_x^2} (I_{xx} - 2y_c S_x) \theta_{yb}' + \lambda \frac{V_{yp}(z)}{D_{yy}} [D_{yy}\theta_{yb} - D_{yyk}u_b' + (D_{yhh} + D_{yk})\theta_{zb}'] \\
& - \lambda \left(\frac{(N_p(z)I_{xx} - M_{xp}(z)S_x)}{AI_{xx} - S_x^2} [(I_{xx} + I_{yy})\theta_{zb}' + (S_x - 2y_c A)\theta_{yb} - S_x u_b'] \right)' \\
& + \lambda \left(\frac{[N_p(z)S_x - M_x(z)A]}{AI_{xx} - S_x^2} [I_{px}\theta_{zb}' + (I_{xx} - 2y_c S_x)\theta_{yb} - I_{xx}u_b'] \right)' \\
& - \lambda \left(\frac{V_{yp}(z)}{D_{yy}} (D_{yhh} + D_{yk})\theta_z \right)' + q_y y_{qy}\theta_{zb} - q_z (y_{qz} - y_c)\theta_{yb} = 0
\end{aligned} \tag{A-47}$$

$$\begin{aligned}
& ES_{x\omega}\theta_{yb}' - EI_{\omega\omega}\psi_b'' + GD_{xh}u_b' - GD_{xh}\theta_{yb} + GD_{hh}\theta_{zb}' + GD_{hh}\psi_b + \lambda \frac{V_{yp}(z)}{D_{yy}} [-D_{yh}\theta_{yb}' + D_{y\omega h}\psi_b'] \\
& - \lambda \left(\frac{V_{yp}(z)}{D_{yy}} D_{y\omega h}\psi_b \right)' = 0
\end{aligned} \tag{A-48}$$

A-6.2 Boundary Conditions

The remaining terms are grouped to recover the boundary conditions of the stability problem as

$$\begin{aligned}
& \left[G \{ D_{xx}u_b' - D_{xx}\theta_{yb} + D_{xh}\theta_{zb}' + D_{xh}\psi_b \} + \lambda \left\{ \frac{(\bar{N}(z)I_{xx} - \bar{M}_x(z)S_x)}{AI_{xx} - S_x^2} (Au_b' - S_x\theta_{zb}') \right. \right. \\
& \left. \left. - \frac{(\bar{N}(z)S_x - \bar{M}_x(z)A)}{AI_{xx} - S_x^2} (S_x u_b' - I_{xx}\theta_z') \delta u - \frac{\bar{V}_y}{D_{yy}} D_{yyk}\theta_z \right\} \delta u_b \right]_0^L = 0
\end{aligned} \tag{A-49}$$

$$\begin{aligned}
& \left[\left(EI_{yy}\theta_{yb}' - 2ES_{x\omega}\psi_b + \lambda \left\{ \frac{(\bar{N}(z)I_{xx} - \bar{M}_x(z)S_x)}{AI_{xx} - S_x^2} (S_x - 2y_c A)\theta_{zb} - \frac{(\bar{N}(z)S_x - \bar{M}_x(z)A)}{AI_{xx} - S_x^2} (I_{xx} - 2y_c S_x)\theta_{zb} \right. \right. \right. \\
& \left. \left. \left. + \frac{\bar{V}_y}{D_{yy}} D_{yh}\psi_b \right\} - Q_{zi}(y_{Qzi} - y_c)\theta_{zb} \right) \delta \theta_{yb} \right]_0^L = 0
\end{aligned} \tag{A-50}$$

$$\begin{aligned}
& \left[\left(G \{ J\theta'_{zb} + D_{xh}u' - D_{xh}\theta_{yb} + D_{hh}\theta'_{zb} + D_{hh}\psi_b \} - \{ Q_y(L)y_Q\theta_{zbQ}(L) + Q_y(0)y_Q\theta_{zbQ}(0) \} \right. \right. \\
& \quad + \lambda \frac{(\bar{N}_p(z)I_{xx} - \bar{M}_{xp}(z)S_x)}{AI_{xx} - S_x^2} \{ (I_{xx} + I_{yy})\theta'_{zb} + (S_x - 2y_C A)\theta_{yb} - S_x u'_b \} \\
& \quad - \lambda \frac{(\bar{N}_p(z)S_x - \bar{M}_{xp}(z)A)}{AI_{xx} - S_x^2} \{ I_{px}\theta'_{zb} + (I_{xx} - 2y_C S_x)\theta_{yb} - I_{xx}u'_b \} \\
& \quad \left. \left. + \lambda \frac{\bar{V}_{yp}}{D_{yy}} (D_{yhh} + D_{yk})\theta_{zb} - Q_{zi}(y_{Qzi} - y_C)\theta_{yb} + Q_{yj}y_{Qyj}\theta_z \right) \delta\theta_{zb} \right]_0^L = 0
\end{aligned} \tag{A-51}$$

$$\left[\left(-2ES_{x\omega}\theta'_{yb} + 2EI_{\omega\omega}\psi'_b + \lambda \frac{\bar{V}_y}{D_{yy}} D_{y\omega h}\psi_b \right) \delta\psi_b \right]_0^L = 0 \tag{A-52}$$

A-7 Finite Element Formulation

The finite element to be developed has two nodes and the methodology has been presented in Chapter 7. The condition of neutral stability (Eq. A-43) can be expressed as

$$\frac{1}{2} \delta^2 \pi = \frac{1}{2} \langle \delta(u_n) \rangle^T [K - \lambda(K_g)] \{ \delta(u_n) \} \tag{A-53}$$

in which the vector of buckling nodal displacements is denoted as

$$\langle \delta(u_n) \rangle^T = \langle u_{b1} \quad \theta_{yb1} \quad \theta_{zb1} \quad \psi_{b1} \mid u_{b2} \quad \theta_{yb2} \quad \theta_{zb2} \quad \psi_{b2} \rangle, \text{ and}$$

$$\begin{aligned}
[K] = & \int_0^L \left\{ EI_{yy} (\delta\theta'_{yb})^2 - 2ES_{x\omega} \delta\theta'_{yb} \delta\psi'_b + EI_{\omega\omega} (\delta\psi'_b)^2 + GJ \delta\theta'^2_{zb} \right\} dz \\
& + G \int_0^L \left\{ (D_{xx}\delta u'_b - D_{xx}\delta\theta_{yb} + D_{xh}\delta\theta'_{zb} + D_{xh}\delta\psi_b) \delta u'_b + (-D_{xx}\delta u'_b + D_{xx}\delta\theta_{yb} - D_{xh}\delta\theta'_{zb} - D_{xh}\delta\psi_b) \delta\theta_{yb} \right. \\
& \left. + (D_{xh}\delta u'_b - D_{xh}\delta\theta_{yb} + D_{hh}\delta\theta'_{zb} + D_{hh}\delta\psi_b) \delta\theta'_{zb} + (D_{xh}\delta u'_b - D_{xh}\delta\theta_{yb} + D_{hh}\delta\theta'_{zb} + D_{hh}\delta\psi_b) \delta\psi_b \right\} dz
\end{aligned} \tag{A-54}$$

$$\begin{aligned}
[K_g] = & \lambda \int_0^L \left(\frac{(N_p(z)I_{xx} - M_{xp}(z)S_x)}{AI_{xx} - S_x^2} \right) \left\{ A\delta u_b'^2 + 2S_x(\delta\theta'_{yb}\delta\theta_{zb} + \delta\theta_{yb}\delta\theta'_{zb}) - 2S_x\delta u_b'\delta\theta'_{zb} \right. \\
& \left. + (I_{xx} + I_{yy})\delta\theta_{zb}'^2 - 2y_C A(\delta\theta_{yb}\delta\theta'_{zb} + \delta\theta'_{yb}\delta\theta_{zb}) \right\} dz \\
& - \frac{1}{2} \lambda \int_0^L \left(\frac{(N_p(z)S_x - M_{xp}(z)A)}{AI_{xx} - S_x^2} \right) \left\{ S_x\delta u_b'^2 + 2I_{xx}(\delta\theta'_y\delta\theta_{zb} + \delta\theta_{yb}\delta\theta'_{zb}) - 2I_{xx}\delta u_b'\delta\theta'_{zb} \right. \\
& \left. + I_{px}\delta\theta_{zb}'^2 - 2y_C S_x(\delta\theta_{yb}\delta\theta'_{zb} + \delta\theta'_{yb}\delta\theta_{zb}) \right\} dz \\
& + \frac{\lambda}{2} \int_0^L \frac{V_{yp}}{D_{yy}} \left\{ 2D_{yy}\delta\theta_{yb}\delta\theta_{zb} - 2D_{yh}\delta\theta'_{yb}\delta\psi_b + 2D_{yoh}\delta\psi_b\delta\psi' - 2D_{yyk}\delta u_b'\delta\theta_{zb} + 2D_{yk}\delta\theta'_{zb}\delta\theta_{zb} \right\} dz \\
& - \lambda \left(\sum_i \mathcal{Q}_{zi} \left[2(y_{\mathcal{Q}zi} - y_C)\delta\theta_{yb}\delta\theta_{zb} \right] - \sum_j \mathcal{Q}_{yj} y_{\mathcal{Q}yj} (\delta\theta_{zb})^2 \right) \\
& - \lambda \int_0^L \left(-\frac{1}{2} q_y (y_{qy} - y_A) (\delta\theta_{zb})^2 + q_z \left[(y_{qz} - y_A) \delta\theta_{yb}\delta\theta_{zb} \right] \right) dz \tag{A-55}
\end{aligned}$$

Substituting the assumed interpolation of the pre-buckling resultants $N_p(z) = N_{p1}$, $V_{yp}(z) = V_{yp1}$

and $M_{xp}(z) = \left(1 - \frac{z}{l}\right) M_{xp1} - \left(\frac{z}{l}\right) M_{xp2}$ (Section 7.6.2.2) into Eqs. A-54 and A-55 and integrating

by parts, yields,

$$[K]_{8 \times 8} =$$

$$\frac{EG}{6I} \begin{bmatrix} \frac{6D_{xx}}{E} & \frac{3D_{xx}}{E} & \frac{6D_{xh}}{E} & \frac{-3D_{xh}l}{E} & \frac{-6D_{xx}}{E} & \frac{3D_{xx}l}{E} & \frac{-6D_{xh}}{E} & \frac{-3D_{xh}l}{E} \\ & \frac{2D_{xx}l^2}{E} + \frac{I_{yy}}{G} & \frac{3D_{xh}l}{E} & \frac{-2D_{xx}l^2}{E} + \frac{S_{x\omega}}{G} & \frac{-3D_{xx}l}{E} & \frac{D_{xx}l^2}{E} + \frac{I_{yy}}{G} & \frac{-3D_{xh}l}{E} & \frac{-D_{xx}l^2}{E} + \frac{S_{x\omega}}{G} \\ & & \frac{6(D_{hh}+J)}{E} & \frac{-3D_{hh}l}{E} & \frac{-6D_{xh}}{E} & \frac{3D_{xh}l}{E} & \frac{-6(D_{hh}+J)}{E} & \frac{-3D_{hh}l}{E} \\ & & & \frac{2D_{hh}l^2}{E} + \frac{I_{\omega\omega}}{G} & \frac{3D_{xh}l}{E} & \frac{-D_{hh}l^2}{E} + \frac{S_{x\omega}}{G} & \frac{3D_{hh}l}{E} & \frac{D_{hh}l^2}{E} + \frac{I_{\omega\omega}}{G} \\ \hline & & & & \frac{6D_{xx}}{E} & \frac{-3D_{xx}l}{E} & \frac{6D_{xh}}{E} & \frac{3D_{xh}l}{E} \\ & & & & & \frac{2D_{xx}l^2}{E} & \frac{-3D_{xh}l}{E} + \frac{I_{yy}}{G} & \frac{-2D_{xx}l^2}{E} + \frac{S_{x\omega}}{G} \\ & & & & & & \frac{6(D_{hh}+J)}{E} & \frac{3D_{hh}l}{E} \\ & & & & & & & \frac{2D_{hh}l^2}{E} + \frac{I_{\omega\omega}}{G} \end{bmatrix}$$

sym

(A-56)

$$[K_g]_{8 \times 8} = \frac{1}{6I} \times$$

$$\begin{bmatrix} 6N_{p1} & 0 & 6N_{p1}y_A - 3(M_{xp1} - M_{xp2}) + 3V_{yp1}l & 0 & -6N_{p1} & 0 & -6N_{p1}y_A + 3(M_{xp1} - M_{xp2}) + 3V_{yp1}l & 0 \\ & 0 & 6N_{p1}y_C - 2(M_{xp1} - M_{xp2})l + 2V_{yp1}l^2 & 0 & 0 & 0 & (M_{xp1} + M_{xp2})l + V_{yp1}l^2 & 0 \\ & & -6N_{p1}\beta - 3(M_{xp1} - M_{xp2})\chi - 6V_{yp1}\xi l & 0 & -6N_{p1}y_A + 3(M_{xp1} - M_{xp2}) - 3V_{yp1}l & (M_{xp1} + M_{xp2})l + V_{yp1}l^2 & 6N_{p1}\beta + 3(M_{xp1} - M_{xp2})\chi & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & 6N_{p1} & 0 & 6N_{p1}y_A + 3(-M_{xp1} + M_{xp2}) - 3V_{yp1}l & 0 \\ & & & & & 0 & -6N_{p1}y_C + 2(M_{xp1} - 2M_{xp2})l + 2V_{yp1}l^2 & 0 \\ & & & & & & -6N_{p1}\beta + 3(-M_{xp1} + M_{xp2})\chi + 6V_{yp1}\xi l & 0 \\ & & & & & & & 0 \end{bmatrix}$$

sym

(A-57)

A-8 Examples

A cantilever with a mono-symmetric cross-section (Fig. A-1) is fixed at the end ($z = 0$) and subjected to a transverse load $Q_y = 100 \text{ kN}$ acting at the other end ($z = 4000 \text{ mm}$). Three locations of the concentrated load are considered, mid-of-top flange, shear center and mid-of-bottom

flange. Two sets of coordinate systems are taken into studied: 1) the origin O of the coordinates is located at the centroid of the cross-section (Orthogonal coordinates), 2) at the midpoint of the top flange (Non-orthogonal coordinates). Meanwhile, in each coordinate system, Pole A and the sectorial origin S_o is assumed positioned at the mid-of -top flange, or the shear center of the cross-section, as illustrated in Table A-1 and Fig. A-2.

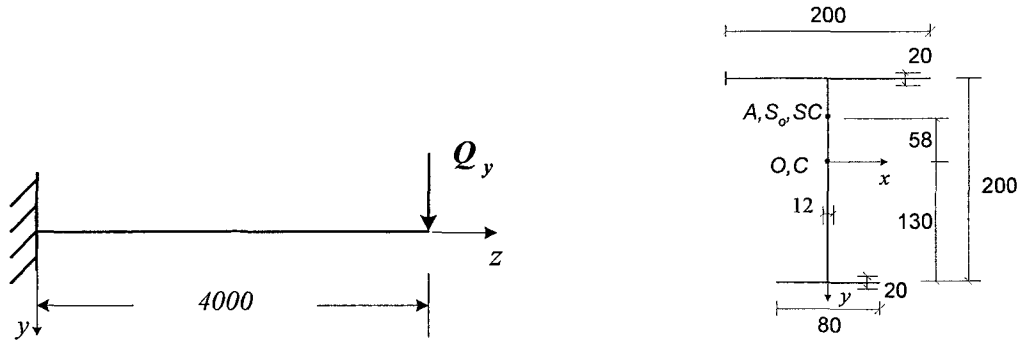
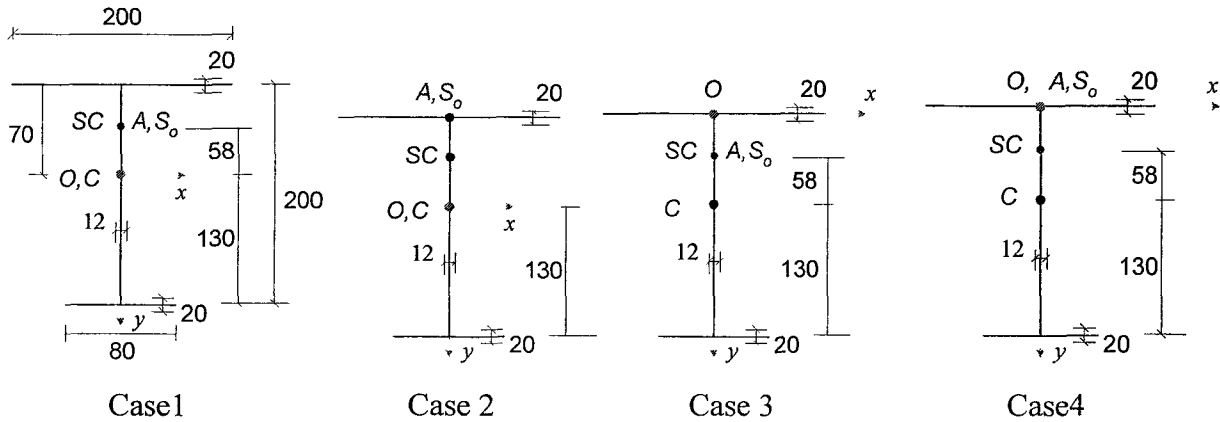


Figure A-1 A cantilever with mono-symmetric cross-section

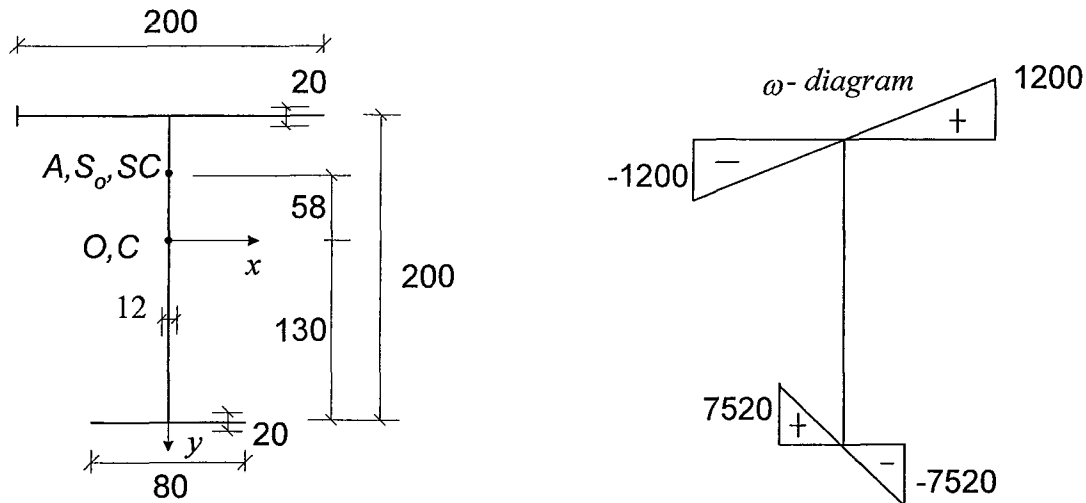
Table A-1 Description:

		Case 1	Case2	Case3	Case4
Location of origin O of axes from centroid		0.0	0.0	-70.0	-70.0
Location of Pole A and S_o from shear center SC		0.0	12.0	0.0	12.0
Location of loads from the shear center SC	Top	12.0	12.0	12.0	12.0
	SC	0.0	0.0	0.0	0.0
	Bottom	188.0	188.0	188.0	188.0



A-7.1 Cross-sectional properties of the mono-symmetric cross-section under orthogonal coordinates

If the origin O is placed at the centroid C , the pole A and the sectorial origin S_o are placed at the shear center SC of the mono-symmetric cross-section as shown in the following figure, the beam is arranged under the orthogonal coordinates.



The cross-section properties:

$$A = 20 \times 200 + 20 \times 80 + 12 \times 200 = 8000 \text{ mm}^2$$

$$I_{xx} = \frac{20^3(200+80)}{12} + \frac{12 \times 200^3}{12} + 200 \times 20 \times 70^2 + 80 \times 20 \times 130^2 + 12 \times 200 \times 30^2 = 56.987 \times 10^6 \text{ mm}^4$$

$$I_{yy} = \frac{20 \times (200^3 + 80^3)}{12} + \frac{12^3 \times 200}{12} = 14.2155 \times 10^6 \text{ mm}^4$$

$$J = \frac{20^3(200+80) + 12^3 \times 200}{3} = 86.1867 \times 10^4 \text{ mm}^4$$

$$I_{\omega\omega} = 2 \times 20 \times \frac{100}{6} \times (2 \times 0 + 2 \times 1200^2 + 0 + 0) + 2 \times 20 \times \frac{40}{6} \times (2 \times 0 + 2 \times 7520^2 + 0 + 0) = 3.192 \times 10^{10} \text{ mm}^6$$

$$D_{xx} = \int_A \cos^2 \alpha(s) dA = 20 \times (200 + 80) = 5600 \text{ mm}^2$$

$$D_{hh} = \int_A h^2(s) dA = 12^2 \times 200 \times 20 + 188^2 \times 80 \times 20 = 5.71264 \times 10^7 \text{ mm}^4$$

$$D_{xh} = \int_A h(s) \cos \alpha(s) dA = -(-70 + 58) \times 200 \times 20 - (130 + 58) \times 80 \times 20 = -252800 \text{ mm}^3$$

$$I_{px} = \int_A y(x^2 + y^2) dA = \int_A x^2 y dA + \int_A y^3 dA$$

$$\begin{aligned} \int_A x^2 y dA &= \int_{A-top} x^2 y dA + \int_{A-web} x^2 y dA + \int_{A-bot} x^2 y dA \\ &= \int_{-100}^{100} (-70x^2) t dx + 0 + \int_{-40}^{40} 130x^2 t dx = \left. \frac{-70 \times 20x^3}{3} \right|_{-100}^{100} + \left. \frac{130 \times 20x^3}{3} \right|_{-40}^{40} \\ &= -9.33 \times 10^8 + 1.1093 \times 10^8 = -8.2240 \times 10^8 \end{aligned}$$

$$\begin{aligned} \int_A y^3 dA &= \int_{A-top} y^3 dA + \int_{A-bot} y^3 dA + \int_{A-web} y^3 dA = 200 \times 20 \times (-70)^3 + 80 \times 20 \times 130^3 + \int_{-70}^{130} 12y^3 dy \\ &= -1.372 \times 10^9 + 3.5152 \times 10^9 + 7.848 \times 10^8 = 2.928 \times 10^9 \end{aligned}$$

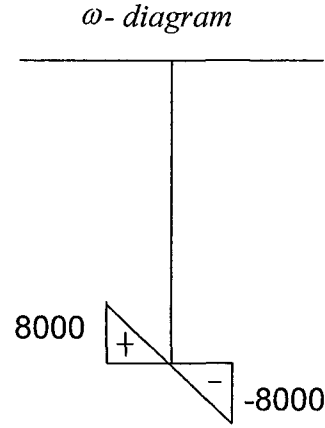
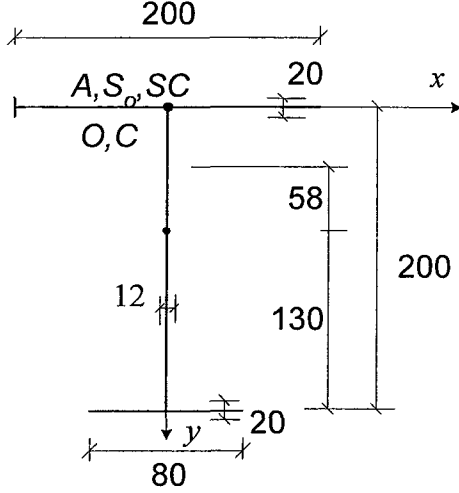
$$I_{px} = 2.928 \times 10^9 - 8.2240 \times 10^8 = 2.1056 \times 10^9 \text{ mm}^5$$

$$x_A = 0.0 \text{ mm}$$

$$y_A = -58.0 \text{ mm}$$

A-7.2 Cross-sectional properties of the mono-symmetric cross-section under orthogonal coordinates

If the origin O , the pole A and the sectorial origin S_o are placed at the joint of the top flange and the web of the mono-symmetric cross-section as shown in the following figure, the beam is arranged under the non-orthogonal coordinates.



The cross-section properties:

$$A = 20 \times 200 + 20 \times 80 + 12 \times 200 = 8000 \text{ mm}^2$$

$$I_{xx} = \frac{20^3(200+80)}{12} + \frac{12 \times 200^3}{12} + 80 \times 20 \times 200^2 + 12 \times 200 \times 100^2 = 96.1867 \times 10^6 \text{ mm}^4$$

$$I_{yy} = \frac{20 \times (200^3 + 80^3)}{12} + \frac{12^3 \times 200}{12} = 14.2155 \times 10^6 \text{ mm}^4$$

$$J = \frac{20^3(200+80) + 12^3 \times 200}{3} = 86.1867 \times 10^4 \text{ mm}^4$$

$$I_{\omega\omega} = 2 \times 20 \times \frac{40}{6} \times (2 \times 0 + 2 \times 8000^2 + 0 + 0) = 3.413333 \times 10^{10} \text{ mm}^6$$

$$D_{xx} = \int_A \cos^2 \alpha(s) dA = 20 \times (200 + 80) = 5600 \text{ mm}^2$$

$$D_{hh} = \int_A h^2(s) dA = 200^2 \times 20 \times 80 = 6.4 \times 10^7 \text{ mm}^4$$

$$D_{xh} = \int_A h(s) \cos \alpha(s) dA = -200 \times 80 \times 20 = -320000 \text{ mm}^3$$

$$I_{px} = \int_A y(x^2 + y^2) dA = \int_A x^2 y dA + \int_A y^3 dA$$

$$\int_A x^2 y dA = \int_{A-top} x^2 y dA + \int_{A-web} x^2 y dA + \int_{A-bot} x^2 y dA$$

$$= 0 + 0 + \int_{-40}^{40} 200x^2 t dx = \frac{200 \times 20x^3}{3} \Big|_{-40}^{40} = 1.70667 \times 10^8$$

$$\int_A y^3 dA = \int_{A-top} y^3 dA + \int_{A-bot} y^3 dA + \int_{A-web} y^3 dA = 0 + 80 \times 20 \times 200^3 + \int_0^{200} 12y^3 dy = 1.76 \times 10^{10}$$

$$I_{px} = 1.7773547 \times 10^{10} \text{ mm}^5$$

$$I_{xy} = \int_A xy dA = \int_{A-top} xy dA + \int_{A-bot} xy dA + \int_{A-web} xy dA = 0 + \int_{-40}^{40} 200 \times 20 x dx + \int_0^{200} \int_{-6}^6 xy dx dy = 0.0 \text{ mm}^4$$

$$S_x = \int_A y dA = \int_{A-top} y dA + \int_{A-bot} y dA + \int_{A-web} y dA = 0 + 200 \times 20 \times 80 + 12 \times \int_0^{200} y dy = 560000 \text{ mm}^3$$

$$S_y = \int_A x dA = \int_{A-top} x dA + \int_{A-bot} x dA + \int_{A-web} x dA = 0.0 \text{ mm}^3$$

$$S_{x\omega} = \int_A x \omega dA = 2 \times 20 \times \frac{40}{6} (2 \times (-40) \times 8000) = -1.70667 \times 10^8 \text{ mm}^4$$

$$x_A = 0.0 \text{ mm}$$

$$y_A = 0.0 \text{ mm}$$

A-7.3 Critical Loads

The critical load for each case is calculated by using the developed finite element program and compared with the results from ABAQUS. The comparisons are listed in Table A-2. From the comparison it is noticed that changing the position of Pole A and the sectorial origin S_o does not change the critical load.

Table A-2 Critical load Q_y (kN)

Load Position	ABAQUS	Case 1	Case 2	Case 3	Case 4
Top	81.47	88.6	88.6	88.76	88.76
Shear Center	83.2	90.6	90.6	90.7	90.7
Bottom	98.75	100.5	100.5	100.4	100.4

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