# A Künneth theorem for Lagrangian quantum homology

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# Abstract

In 1996, Yong-Geun Oh introduced the pearl complex as a means of computing the Lagrangian intersection Floer homology of a monotone Lagrangian with itself. This complex and its homology, the Lagrangian quantum homology, were later studied in detail by Paul Biran and Octav Cornea. We explain the construction of the pearl complex and, under a genericity assumption, prove a Künneth theorem for its homology. The proof consists of two parts: An algebraic part, which is a Künneth theorem for differential graded modules, and a geometric part, whose proof closely resembles the proof of the Künneth theorem for Morse homology. We present the algebraic part at the outset and the geometric part at the end after establishing the necessary prerequisites from local and global symplectic geometry.

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# Contents

# Introduction

In 1912, in the course of studying the restricted three-body problem, Poincaré was led to conjecture his "last geometric theorem"[Poi12]. This theorem states that a diffeomorphism of the annulus which rotates the two boundary components in opposite directions and preserves the standard area form must have at least two fixed points. Proved by Birkhoff in 1913 [Bir13], this theorem is widely considered to be the first theorem of global symplectic geometry [MS98].

In the 1960's, V.I. Arnold noticed that, by considering the torus $\mathbb{T}^2$ as two annuli glued along their boundaries, the Poincaré-Birkhoff theorem would follow from a fixed-point conjecture for diffeomorphisms of $\mathbb{T}^2$ preserving the area form [HZ11]. In dimension two, a symplectic form is simply a volume form, but Arnold proposed his fixed-point conjecture for general compact symplectic manifolds [Arn78, Arn65]. In its most well-known form, Arnold's conjecture can be stated as follows:

**The strong Arnold Conjecture** *Let $M$ be a compact symplectic manifold and let* $\mathrm{Crit}(M)$ *be the minimum number of critical points that a smooth function on $M$ must possess. Then the number of 'contractible' fixed points of a Hamiltonian diffeomorphism $\psi$ of $M$ is greater than or equal to* $\mathrm{Crit}(M)$.

We clarify below what is meant by a 'contractible' fixed point. This version of the Arnold conjecture also has a 'non-degenerate' form in which one requires all contractible fixed points of the Hamiltonian diffeomorphism to be non-degenerate, i.e. for any contractible fixed point $p$, 1 is not an eigenvalue of $(d\psi)_p$. In this case, the number of fixed points of $\psi$ is conjectured to be bounded below by the minimum number of critical points of a *Morse* function on $M$, $\mathrm{MCrit}(M)$.

The significance of the Arnold Conjecture to dynamics is clearer if it is reformulated using the definition of a Hamiltonian diffeomorphism. Hamiltonian diffeomorphisms are simply the time-one flows of time-dependent Hamiltonian functions. Moreover, every Hamiltonian diffeomorphism is the time-one map of a one-periodic Hamiltonian function, and the fixed points of a Hamiltonian diffeomorphism correspond to the one-periodic orbits of the Hamiltonian [MS98, p. 297]. A contractible fixed point of a Hamiltonian diffeomorphism is simply a fixed point whose corresponding periodic orbit is null-homotopic. In this language, the Arnold conjecture says that the number of contractible one-periodic orbits of a one-periodic Hamiltonian on a compact symplectic manifold $M$ is bounded below by $\mathrm{Crit}(M)$.

Using the Lusternik-Schnirelmann theorem, it is possible to weaken the version of the Arnold conjecture which allows for degenerate fixed points. By the Lusternik-Schnirelmann theorem, $\mathrm{Crit}(M)$ is greater than or equal to $\mathrm{cat}(M)+1$, where $\mathrm{cat}(M)$ is the Lusternik-Schnirelmann category of $M$ [LŠ47]. The weaker version of the degenerate Arnold conjecture states that the number of contractible periodic orbits is bounded below by $\mathrm{cat}(M) + 1$.

If the smooth function on $M$ is required to be Morse, a lower bound on its minimum number of critical points is given not by the Lusternik-Schnirelmann theorem, but by one of the Morse inequalites. Namely, $\mathrm{MCrit}(M)$ is bounded below by the sum of the Betti numbers of $M$ [BH04, p. 74]. This leads to the following weakened version of the non-degenerate Arnold conjecture:

**The weak non-degenerate Arnold Conjecture** *Let $M$ be a compact symplectic manifold and let $\psi$ be a Hamiltonian diffeomorphism whose contractible fixed points are all non-degenerate. Then the number of contractible fixed points of $\psi$ is bounded* *below by the sum*

$$\sum_{i=0}^{\dim(M)} \operatorname{rank}(H_i(M)). \tag{0.0.1}$$

This version of the Arnold Conjecture has now been proved, although the strong versions of the Arnold conjecture remain largely open [MS12, p. 297]. Most of the credit is due to Floer who solved the monotone case [Flo88, Flo89a, Flo89b].

In his approach, Floer actually used a reformulation of the Arnold conjecture in terms of the intersection of Lagrangian submanifolds. The graph of a symplectomorphism of a symplectic manifold $(M, \omega)$ is a Lagrangian submanifold of the product symplectic manifold $(M \times M, \omega \oplus (-\omega))$. Therefore the question about fixed points of Hamiltonian symplectomorphisms can be reformulated as a question about the intersection of the graph of $\psi$ with the diagonal in $M \times M$.

More generally, one can consider intersections of arbitrary Lagrangians in symplectic manifolds which are not necessarily products. The non-degeneracy assumption corresponds to the requirement that the Lagrangians intersect transversally. This line of thinking suggests the following generalization of the weak Arnold conjecture, which however is false in the absence of additional assumptions:

**The weak Arnold conjecture (Lagrangian intersection version)** *Let $L$ be a Lagrangian submanifold of a compact symplectic manifold $M$ and let $\psi$ be a Hamiltonian diffeomorphism of $M$. Under appropriate additional assumptions,*

$$\#(L \cap \psi(L)) \geq \operatorname{cat}(L) + 1 \text{ when } L \text{ and } \psi(L) \text{ intersect non-transversally,}$$

$$\#(L \cap \psi(L)) \geq \sum_{i=0}^{\dim(L)} \operatorname{rank}(H_i(L)) \text{ when } L \text{ and } \psi(L) \text{ intersect transversally.}$$

The transversal version of the above conjecture is implied by Floer's work under the assumption that the symplectic manifold is monotone [Flo89a], i.e. the first Chern class evaluated on spherical 2-cycles is related to the symplectic area of the cycles by a positive proportionality constant. Floer showed that, under this condition, one can define a complex generated by the intersection points of $L$ and $\psi(L)$. Moreover, the homology of this complex, the Lagrangian intersection Floer homology of $L$ with itself, is isomorphic to the singular homology of $L$.

Y.-G. Oh later generalized Floer's construction to intersections of pairs of Lagrangians which are not necessarily related by a Hamiltonian diffeomorphism [Oh95, Oh93a, Oh93b, Oh96]. He worked under the assumption that the Lagrangians are monotone with minimal Maslov number at least 2. In this more general picture, one can still use a Hamiltonian diffeomorphism to define the Lagrangian intersection Floer homology of a Lagrangian with itself, but if there are discs with boundary on $L$ that have non-trivial symplectic area, this homology will not necessarily be isomorphic to the singular homology of $L$.

Although the construction of the Floer homology of a monotone Lagrangian with itself relies on a Hamiltonian perturbation of the Lagrangian, the homology is independent of this choice. In [Oh96], Oh showed that this homology can also be defined intrinsically, i.e. independently of a choice of Hamiltonian perturbation, using what is now known as the pearl complex. The properties of the pearl complex and its homology, the Lagrangian quantum homology, were established rigourously by Biran and Cornea in [BC07]. Their later work [BC09a] provides a survey.

We give an account of the construction of the pearl complex and prove a Künneth theorem for the Lagrangian quantum homology of the product of two monotone Lagrangian submanifolds under some basic assumptions. Chapter 1 covers the non-symplectic foundations: locally trivial fibre bundles and vector bundles, differential graded modules, and Morse theory. The pearl complex is essentially an adaptation of the Morse complex. In the section on Morse theory, we describe the Morse complex and prove the Künneth theorem for Morse homology. The proof of the Künneth theorem for the homology of the pearl complex uses the same idea.

Chapters 2 and 3 contain the rather extensive background material in symplectic geometry needed to construct the pearl complex. Chapter 2 deals with symplectic vector spaces and Chapter 3 with symplectic manifolds. An important theme in modern symplectic topology, which is also relevant for the construction of the pearl complex, is the interplay of almost complex and symplectic structures. Accordingly we also provide an introduction to almost complex structures and $J$-holomorphic curves. The final chapter, Chapter 4, is devoted to the study of Lagrangian submanifolds, culminating in the proof of the Künneth theorem for the Lagrangian quantum homology.

# Chapter 1

# Preliminaries

In this chapter we provide a brief account of the non-symplectic background material we will need. We begin with a description of general locally trivial fibre bundles which we will use mainly in Chapter 2 when computing the fundamental group of the linear symplectic group and the Lagrangian Grassmannian. We then describe vector bundles and the vector bundle constructions we will use heavily throughout the text. Indeed we have chosen to approach the definition of a symplectic manifold in Chapter 3 via the more general concept of a symplectic vector bundle.

In Section 1.2, we establish the algebraic foundations used in defining the Lagrangian quantum homology. We also prove an algebraic Künneth theorem for differential graded modules which we will use later to prove the Künneth theorem for the pearl complex.

In the final section of this chapter, we introduce the concepts from Morse theory that are required to define the pearl complex. We also explain the construction of the Morse complex and prove a Künneth theorem for Morse homology. This proof will serve as a guide in proving the Künneth theorem for Lagrangian quantum homology.

## 1.1 Fibre bundles

### 1.1.1 Locally trivial fibre bundles

We begin with basic definitions and constructions and then cite the relevant homotopy theoretical results for locally trivial fibre bundles. Finally we mention two scenarios involving Lie groups which give rise to locally trivial fibre bundles. We will use these in the coming chapter.

**Definition 1.1.1.** *[MS98, p. 197] A **locally trivial fibre bundle**, also called a **locally trivial fibration**, consists of a triple of smooth manifolds $(F, E, B)$ called the **(model) fibre**, the **total space** and the **base space**, together with a smooth surjective map $\pi : E \to B$ that satisfies the following condition: There exists an open cover $\{U_\alpha\}_\alpha$ of $B$ and diffeomorphisms*

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

*such that the following diagram commutes:*

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\mathrm{pr}_\alpha"] \\ & U_\alpha & \end{tikzcd}$$

*Here $\mathrm{pr}_\alpha$ is simply the projection from $U_\alpha \times F$ onto $U_\alpha$. The open sets $U_\alpha$ are called **trivializing neighbourhoods**, the cover $\{U_\alpha\}_\alpha$ is called a **trivializing cover** of $B$, the pairs $(U_\alpha, \phi_\alpha)$ are called **local trivializations** of $E$ and the set $\{(U_\alpha, \phi_\alpha)\}_\alpha$ is called a **trivializing atlas** for $E$. For any point $b \in B$, the subset $\pi^{-1}(b)$ of $E$ is called **the fibre over** $b$ and is denoted by $E_b$. It is clear from the existence of local trivializations that $\pi$ is of constant rank. Therefore $E_b$ is a closed embedded submanifold of $E$. If $U_\alpha$ is a trivializing neighbourhood containing $b$, the map*

$$(\phi_\alpha)_b := \mathrm{pr}_F \circ \phi_\alpha|_{E_b}$$

*is a diffeomorphism from* $E_b$ *to* $F$*. Here* $\mathrm{pr}_F$ *is the projection from* $U_\alpha \times F$ *onto* $F$*. For* $U_\alpha$ *and* $U_\beta$ *trivializing neighbourhoods with non-empty intersection, we define maps*

$$\phi_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F), \quad \phi_{\beta\alpha}(b) := (\phi_\beta)_b \circ (\phi_\alpha)_b^{-1}$$

*called* ***transition functions****. The group* $\mathrm{Diff}(F)$ *is called the* ***structure group*** *of the locally trivial fibre bundle. A locally trivial fibre bundle is called* ***trivial*** *if* $\{B\}$ *is a trivializing cover.*

We will denote a locally trivial fibre bundle by $\pi : E \to B$, or simply $E \to B$ if it is not necessary to indicate the projection map. If we want to make the fibre explicit, we may write $F \to E \to B$. We may also refer to the total space $E$ as a locally trivial fibre bundle when the remaining data is clear from the context.

**Definition 1.1.2.** *A* ***smooth section*** *of a locally trivial fibre bundle* $\pi : E \to B$ *is a smooth function* $s : B \to E$ *satisfying* $\pi \circ s = \mathrm{Id}_B$*. The set of smooth sections of* $E$ *is denoted by* $\Gamma(E)$*.*

The next proposition establishes the concept of the pullback of a locally trivial fibre bundle.

**Proposition 1.1.3.** *[Hus94, p. 21] Let* $\pi : E \to B$ *be a locally trivial fibration with fibre* $F$*, let* $B'$ *be a smooth manifold and let* $f : B' \to B$ *be a smooth map. Define the set*

$$f^*E := \{(b, e) |\ b \in B', e \in E_{f(b)}\}$$

*and the projection map*

$$\pi_f : f^*E \to B', \quad (b, e) \mapsto b.$$

*With this projection map,* $f^*E$ *is a locally trivial fibre bundle over* $B'$ *with fibre* $F$*.*

**Definition 1.1.4.** *The locally trivial fibre bundle* $\pi_f : f^*E \to B'$ *is called the* ***induced bundle*** *or the* ***pullback bundle****. In the special case where* $B'$ *is a submanifold of* $B$

*and $f$ is the inclusion map, we call the induced bundle the* ***restriction*** *of the bundle to $B'$ and denote it by $E|_{B'} \to B'$.*

The following theorem establishes a relationship between the homotopy groups of the fibre, of the total space and of the base in a locally trivial fibration.

**Theorem 1.1.5.** *[Hat02, p. 344] If $\pi : E \to B$ is a locally trivial fibre bundle with fibre $F$ and path-connected base, there exists a long exact sequence of homotopy groups of the form*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots$$
$$\cdots \to \pi_0(E, x_0) \to 0$$

*where $b_0 \in B$ and $x_0 \in F_b$.*

Covering spaces of connected manifolds are simply locally trivial fibrations over the manifold where the fibre over each point is discrete. Much like covering spaces, locally trivial fibrations satisfy a homotopy lifting property, as we establish in the next proposition. However, when the fibre is not required to be discrete, we lose the uniqueness of lifts.

**Proposition 1.1.6.** *[Hat02, p. 379] Every locally trivial fibration $\pi : E \to B$ enjoys the following property: Given a topological space $X$ and a homotopy*

$$H : [0, 1] \times X \to B,$$

*if there is a lift $\widetilde{H}_0 : X \to E$ of $H_0$, then $\widetilde{H}_0$ can be extended to a lift $\widetilde{H}$ of $H$,*

$$\widetilde{H} : [0, 1] \times X \to E, \quad \pi \circ \widetilde{H} = H.$$

If $X$ is a single point, this property implies that a path $\gamma$ in $B$ can be lifted to a path in $E$ starting at any point in the fibre over $\gamma(0)$.

The following theorem and corollary indicate two situations where locally trivial fibrations arise from homomorphisms of Lie groups.

**Theorem 1.1.7.** *[Ste51, p. 33] The inclusion of a closed subgroup $H$ in a Lie group $G$ gives rise to a locally trivial fibration $G \to G/H$ with fibre $H$.*

**Corollary 1.1.8.** *A surjective homomorphism $f : G \to K$ of Lie groups yields a locally trivial fibration $G \to K$ with fibre* $\ker(f)$.

**Proof:** Since the map $f$ is continuous, $\ker(f)$ is a closed subgroup of $G$. Therefore we have a locally trivial fibration $G \to G/\ker(f) \cong K$ with fibre $\ker(f)$. ∎

### 1.1.2 Vector bundles

Locally trivial fibrations where the fibre is a real or complex vector space will be of particular interest, which leads to the following definition.

**Definition 1.1.9.** *[MS74, p. 13] A **real vector bundle** is a locally trivial fibre bundle $\pi : E \to B$ whose model fibre is $\mathbb{R}^n$ and which possesses a trivializing atlas $\{(U_\alpha, \phi_\alpha)\}_\alpha$ such that the images of the transition maps $\phi_{\beta\alpha}$ are subsets of* $\mathrm{GL}(n, \mathbb{R})$. *Here $n$ is a non-negative integer, which we call the **rank** of the vector bundle. The group* $\mathrm{GL}(n, \mathbb{R})$ *is called the **structure group** of the real vector bundle. Similarly, a **complex vector bundle** is a locally trivial fibre bundle whose fibre is $\mathbb{C}^n$ and which possesses a trivializing atlas where the images of the transition maps are subsets of the group* $\mathrm{GL}(n, \mathbb{C})$. *We say that the structure group of a complex vector bundle is* $\mathrm{GL}(n, \mathbb{C})$.

**Example 1.1.10.** The tangent bundle $TM$ of a smooth manifold $M$ is a real vector bundle of rank equal to the dimension of $M$. A smooth section of $TM$ is simply a vector field on $M$.

**Remark 1.1.11.** For each point $b \in B$ in the base space of a real (complex) vector bundle $E$ of rank $n$, we have a diffeomorphism $(\phi_\alpha)_b : E_b \to \mathbb{R}^n$, where $(U_\alpha, \phi_\alpha)$ is a local trivialization of $E$ with $b \in U_\alpha$. This diffeomorphism induces the structure of a real (complex) vector space on $E_b$. Because the transition maps are linear isomorphisms, this structure does not depend on the choice of trivialization. Therefore each fibre $E_b$ is a real (complex) vector space in a well-defined sense.

**Remark 1.1.12.** If $E \to B$ is a real (complex) vector bundle of rank $n$, $B'$ is a smooth manifold and $f : B' \to B$ is a smooth map, then the induced bundle $\pi_f : f^*E \to B'$ is a real (complex) vector bundle of rank $n$. In particular, if $B'$ is a submanifold of $B$, then the restriction $E|_{B'} \to B'$ is a real (complex) vector bundle of rank $n$.

**Definition 1.1.13.** *[MS74, p. 14] A* ***morphism*** *of real (complex) vector bundles $\pi : E \to B$ and $\pi' : E' \to B'$ is a smooth map $\eta : E \to E'$ that carries each fibre $E_b \subset E$ by a linear map onto a fibre $E'_{b'} \subset E'$. A morphism of vector bundles is called an* ***isomorphism*** *if it has an inverse which is also a morphism of vector bundles. Notice that a real (complex) vector bundle $\pi : E \to B$ of rank $n$ is* ***trivial*** *if and only if it is isomorphic to the vector bundle $B \times \mathbb{R}^n \to B$ $(B \times \mathbb{C}^n \to B)$, where the projection map is just projection onto $B$.*

**Remark 1.1.14.** A morphism of vector bundles $\eta : E \to E'$ induces a map $\bar{\eta} : B \to B'$: We define $\eta(b) = b'$, where $\eta$ maps $E_b$ into $E'_{b'}$.

**Definition 1.1.15.** *Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be two (real or complex) vector bundles over $B$ with $E_1$ a submanifold of $E_2$ and $\pi_1 = \pi_2|_{E_1}$. We say that $E_1$ is a* ***subbundle*** *of $E_2$ if for each $b \in B$, $(E_1)_b$ is a linear subspace of $(E_2)_b$.*

The following remark is one instance of the general fact that an isomorphism of two objects carries a substructure of the first object isomorphically onto a substructure of the second.

**Remark 1.1.16.** Let $E_1 \to B$ be a subbundle of $E_2 \to B$, and let $\pi' : E' \to B'$ be another vector bundle which is isomorphic to $E_2$ by the isomorphism $\eta : E_2 \to E'$.

Then $\eta(E_1)$ is a subbundle of $E'$ and $\eta$ restricted to $E_1$ is an isomorphism from $E_1$ to $\eta(E_1)$.

There are many functors which assign to one or more vector spaces another vector space. For example, the functor that assigns to a vector space its dual, or to two vector spaces their tensor product. Analogs of these functors exist for vector bundles; they assign to one or more vector bundles another vector bundle. In order for this construction to work, the functor in question must be "smooth" as defined next.

**Definition 1.1.17.** *[MS74, p. 32] Let $\mathcal{V}$ be the category with finite-dimensional real vector spaces as objects and linear isomorphisms as morphisms and, for $m$ a positive integer, let $\mathcal{V}^m$ be the $m$-fold Cartesian product of this category. A covariant functor $F : \mathcal{V}^m \to \mathcal{V}$ is called **smooth** if the map*

$$\begin{gathered}\mathrm{Mor}(V_1, W_1) \times \cdots \times \mathrm{Mor}(V_m, W_m) \to \mathrm{Mor}(F(V_1, \ldots, V_m), F(W_1, \ldots, W_m))\\ (T_1, \ldots, T_m) \mapsto F(T_1, \ldots, T_m)\end{gathered}$$

*is smooth for all vector spaces $V_1 \cong W_1, \ldots, V_m \cong W_m$. Similarly, a contravariant functor $F' : \mathcal{V}^m \to \mathcal{V}$ is called smooth if the map*

$$\begin{gathered}\mathrm{Mor}(V_1, W_1) \times \cdots \times \mathrm{Mor}(V_m, W_m) \to \mathrm{Mor}(F'(W_1, \ldots, W_m), F'(V_1, \ldots, V_m))\\ (T_1, \ldots, T_m) \mapsto F'(T_1, \ldots, T_m)\end{gathered}$$

*is smooth for all vector spaces $V_1 \cong W_1, \ldots, V_m \cong W_m$.*

**Remark 1.1.18.** As the morphisms in $\mathcal{V}$ are isomorphisms, we can make any contravariant functor $F' : \mathcal{V}^m \to \mathcal{V}$ into a covariant functor $F'_{cov.}$ by defining

$$\begin{aligned}F'_{cov.}(V_1, \ldots, V_m) &= F'(V_1, \ldots, V_m)\\ F'_{cov.}(T_1, \ldots, T_m) &= F'((T_1)^{-1}, \ldots, (T_m)^{-1})\end{aligned}$$

for all vector spaces $V_1 \cong W_1, \ldots, V_m \cong W_m$, and all linear isomorphisms $T_j \in \mathrm{Mor}(V_j, W_j)$, $j = 1, \ldots, m$. Similarly, a covariant functor can be made into a contravariant one. Therefore the distinction between contravariant and covariant functors is not strictly necessary in this context.

**Example 1.1.19.** The following are examples of smooth contravariant functors with $m = 1$:

1. The map that assigns to every vector space $V$ its dual space $V^*$ and to every isomorphism $T$ in $\mathrm{Mor}(V, W)$ its transpose $T^*$ in $\mathrm{Mor}(W^*, V^*)$.

2. The map that assigns to $V$ the vector space of endomorphisms of $V$, $\mathrm{End}(V)$, and to any $T \in \mathrm{Mor}(V, W)$ the isomorphism

$$\mathrm{End}(W) \to \mathrm{End}(V), \quad S \mapsto T^{-1} \circ S \circ T.$$

The following are examples of smooth covariant functors with $m = 1$:

1. For any non-negative integer $k$, the map that assigns to $V$ the $k$-fold tensor product of $V$, $\otimes^k V$, and to any $T \in \mathrm{Mor}(V, W)$ the isomorphism from $\otimes^k(V)$ to $\otimes^k(W)$ determined by the following function defined on decomposable elements $v_1 \otimes \cdots \otimes v_k$ of $\otimes^k(V)$:

$$v_1 \otimes \cdots \otimes v_k \mapsto T(v_1) \otimes \cdots \otimes T(v_k).$$

2. For any non-negative integer $k$, the map that assigns to $V$ the $k$-th symmetric power of $V$, $\mathrm{Sym}^k(V)$, and to any $T \in \mathrm{Mor}(V, W)$ the isomorphism from $\mathrm{Sym}^k(V)$ to $\mathrm{Sym}^k(W)$ determined by the following function defined on decomposable elements $v_1 \odot \cdots \odot v_k$ of $\mathrm{Sym}^k(V)$:

$$v_1 \odot \cdots \odot v_k \mapsto T(v_1) \odot \cdots \odot T(v_k),$$

where $\odot$ represents the symmetric product.

3. For any non-negative integer $k$, the map that assigns to $V$ the $k$-th exterior power of $V$, $\Lambda^k(V)$, and to any $T \in \mathrm{Hom}(V, W)$ the isomorphism from $\Lambda^k(V)$ to $\Lambda^k(W)$ determined by the obvious function defined on decomposable elements $v_1 \wedge \cdots \wedge v_k$ of $\Lambda^k(V)$.

Similarly, the tensor, symmetric and exterior products of $m$ vector spaces define smooth covariant functors $\mathcal{V}^m \to \mathcal{V}$.

**Theorem 1.1.20.** *[MS74, p. 36] Let $F : \mathcal{V}^m \to \mathcal{V}$ be a smooth (covariant or contravariant) functor and let $\pi_1 : E_1 \to B, \ldots, \pi_m : E_m \to B$ be real vector bundles with fibres of ranks $n_1, \ldots, n_m$. Define the set*

$$F(E_1, \ldots, E_m) := \{(b, \xi)|\ b \in B, \xi \in F((E_1)_b, \ldots, (E_m)_b)\}$$

*and the map*

$$\pi_F : F(E_1, \ldots, E_m) \to B, \quad (b, \xi) \mapsto b.$$

*Then $F(E_1, \ldots, E_m)$ is a real vector bundle over $B$ with model fibre $F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_m})$ and projection map $\pi_F$.*

Although we have only stated Theorem 1.1.20 for real vector bundles, an analogous theorem holds for complex vector bundles.

**Example 1.1.21.** Applying this theorem to Example 1.1.19, we see that $E^*$, $\mathrm{End}(E)$, $\otimes^k(E)$, $\mathrm{Sym}^k(E)$ and $\Lambda^k(E)$ are all real vector bundles over $B$. Similarly the tensor, symmetric and exterior products of $k$ distinct vector bundles over the same base are real vector bundles. Moreover, we can apply these vector bundle constructions in succession to obtain the bundle $\Lambda^2(E^*)$, for example, which will be important in defining symplectic vector bundles and $\mathrm{Sym}^2(E^*)$, which will be needed for the definition of a Riemannian structure below.

**Remark 1.1.22.** In the case of the tangent bundle $TM$ of a smooth manifold $M$, these constructions simply yield the familiar vector bundles $T^*M$, $\mathrm{End}(TM)$, $T^k(M)$, $\mathrm{Sym}^k(TM)$, and $\Lambda^k(TM)$.

There are many situations where we study vector bundles which possess additional structure, such as an inner or Hermitian product on the fibres as described in the following definition.

**Definition 1.1.23.** *[MS74, p. 21] A* ***Riemannian structure*** *on a real vector bundle $\pi : E \to B$ is an inner product $g_b$ on each fibre such that $g_b$ varies smoothly with the base point $b \in B$. In other words, a Riemannian structure is a smooth section $g$ of the symmetric product $\mathrm{Sym}^2(E^*)$, satisfying*

$$g_b(v,v) > 0 \text{ for all } v \neq 0 \in E_b, \text{ for all } b \in B.$$

*Here we use $g_b$ to denote $g(b)$. A* ***Hermitian structure*** *on a complex vector bundle $\pi : E \to B$ is a Hermitian inner product $h_b$ on each fibre, such that $h_b$ varies smoothly with $b \in B$. More precisely, a Hermitian inner product is a smooth section $h$ of the complex vector bundle $E^* \otimes_{\mathbb{C}} \overline{E}^*$ satisfying*

$$h_b(\eta,\xi) = \overline{h_b(\xi,\eta)} \text{ for all } \eta,\xi \in E_b \text{ and}$$
$$h_b(\zeta,\zeta) > 0 \text{ for all } \zeta \in E_b, \zeta \neq 0,$$

*for all $b \in B$. Here $E^*$ is the complex dual of $E$ and $\overline{E}^*$ is the space of complex antilinear maps from $E$ to $\mathbb{C}$, i.e. real linear maps $f : E \to \mathbb{C}$ satisfying $f(iv) = -if(v)$ for all $v \in E$.*

**Remark 1.1.24.** If $E \to B$ is a real vector bundle with Riemannian structure $g$ and $f : B' \to B$ is a smooth map from a manifold $B'$ into $B$, then the pullback bundle $f^*E \to B'$ possesses a Riemannian structure $f^*g$ defined by

$$(f^*g)_{b'}(v,w) = g_{f(b')}(\mathrm{pr}_2(v), \mathrm{pr}_2(w)) \text{ and all } v,w \in (f^*E)_{b'},$$

for all $b' \in B'$. Here $\mathrm{pr}_2$ is the projection from

$$(f^*E)_{b'} = \{(b',v) |\, v \in E_{f(b')}\}$$

onto $E_{f(b')}$. Similarly, a Hermitian structure pulls back to a Hermitian structure on the pullback bundle.

**Definition 1.1.25.** *Let $\pi : E \to B$ be a real (complex) vector bundle of rank $n$ and let $U \subset B$ be open. A **smooth frame** on $U$ for $E$ is a set of $n$ smooth sections $X_1, \ldots, X_n$ of $E|_U$ such that for all $b \in U$, the set $\{X_1(b), \ldots, X_n(b)\}$ forms a basis for $E_b$. If $E$ is equipped with a Riemannian (Hermitian) structure, the frame is called **orthonormal** (**unitary** in the complex case) if, in addition, the sections are orthonormal with respect to the Riemannian (Hermitian) structure at each point $b \in B$.*

**Proposition 1.1.26.** *[MS74, p. 23] Let $\pi : E \to B$ be a real (complex) vector bundle with a Riemannian (Hermitian) structure $g$. If $E$ is trivial over the open set $U \subset B$, then there exists a smooth orthonormal (unitary) frame for $E$ on $U$.*

**Proof:** We will prove this for a real vector bundle. The complex case is completely analogous. Let $n$ be the rank of $E$ and let $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a trivialization. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Define the following smooth sections of $E$ over $U$:

$$X_j(b) := \phi^{-1}(b, e_j) \quad \text{for } j = 1, \ldots, n.$$

The vector fields $X_j$ form a smooth frame over $U$ and applying the Gram-Schmidt procedure to this smooth frame yields a smooth orthonormal frame over $U$. ∎

**Corollary 1.1.27.** *Let $\pi : E \to B$ be a rank-$n$ real (complex) vector bundle with Riemannian structure $g$ (Hermitian structure $h$) and let $\{(U_\alpha, \phi_\alpha)\}_\alpha$ be a trivializing atlas for $E$. Then there is a trivializing atlas $\{(U_\alpha, \phi'_\alpha)\}_\alpha$ such that the image of all transition functions $\phi'_{\beta\alpha}$ lies in $\mathrm{O}(n)$ $(\mathrm{U}(n))$.*

**Proof:** We will again prove only the real case as the proof of the complex case is completely analogous. For each trivializing neighbourhood $U_\alpha$ in the trivializing cover of $B$, we use Proposition 1.1.26 to choose a smooth orthonormal frame

$\{(X_1)_\alpha, \ldots, (X_n)_\alpha\}$ for $g$ on $U_\alpha$. We then define the diffeomorphisms

$$\phi'_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n, \quad \sum_{j=1}^{n} \lambda_j((X_j)_\alpha(b)) \mapsto \left(b, \sum_{j=1}^{n} \lambda_j e_j\right)$$

for $b \in U_\alpha$, $\lambda_j \in \mathbb{R}$. Then the maps

$$(\phi'_\alpha)_b : (E_b, g_b) \to (\mathbb{R}^n, g_0)$$

are isomorphisms of inner product spaces. Here $g_0$ is the standard inner product on $\mathbb{R}^n$. It follows that the images of the transition functions $\phi'_{\beta\alpha}$ lie in $\mathrm{O}(n)$. ∎

**Definition 1.1.28.** *Let $E \to B$ be a locally trivial fibre bundle with fibre $F$ and structure group $G$ (* $\mathrm{Diff}(F)$ *for a generic locally trivial fibre bundle and* $\mathrm{GL}(F)$ *for a vector bundle). If there is a trivializing atlas of $E$ such that the images of all transition functions lie in a subgroup $H$ of $G$, then the structure group $G$ is said to* ***reduce*** *to $H$ and the subgroup $H$ is called a* ***reduction of the structure group*** *of the bundle.*

A locally trivial fibre bundle where the fibre $F$ is a vector space and the structure group reduces from $\mathrm{Diff}(F)$ to $\mathrm{GL}(F)$ is a vector bundle. The previous corollary shows that a Riemannian structure on a real vector bundle of rank $n$ allows a reduction of the structure group of the bundle from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n)$. Similarly, a Hermitian metric on a complex vector bundle of rank $n$ allows the structure group of the bundle to be reduced from $\mathrm{GL}(n, \mathbb{C})$ to $U(n)$.

Conversely, if there is a reduction of the structure group of a real vector bundle from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n)$ we may define a Riemannian structure on $E$ by

$$g_b(v_1, v_2) := g_0((\phi_\alpha)_b(v_1), (\phi_\alpha)_b(v_2)) \text{ for } v_1, v_2 \in E_b,\ b \in B.$$

Here $(U_\alpha, \phi_\alpha) \in \{(U_\alpha, \phi_\alpha)\}_\alpha$ is a local trivialization with $b \in U_\alpha$ and the transition functions of the atlas $\{(U_\alpha, \phi_\alpha)\}_\alpha$ have images in $\mathrm{O}(n)$. The definition of $g_b$ does not depend on the choice of $U_\alpha$ containing $b$. Similarly, a reduction of the structure group of a complex vector bundle from $\mathrm{GL}(n,\mathbb{C})$ to $\mathrm{U}(n)$ allows us to define a Hermitian structure on the vector bundle. We will use this type of construction again when we come to symplectic and complex structures on vector bundles.

A rank-$n$ real vector bundle $E$ is called **orientable** if its structure group can be reduced from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{GL}^+(n,\mathbb{R})$, the group of linear isomorphisms of $\mathbb{R}^n$ with positive determinant. This is equivalent to the existence of a smoothly varying choice of orientation for each fibre, or the existence of a nowhere-vanishing section of $\Lambda^n E$. An **orientation** of $E$ is a choice of equivalence class of nowhere-vanishing sections of $\Lambda^n E$ (where two sections are considered equivalent if they differ by a factor of a positive smooth function on the base). Equivalently, an orientation of $E$ can be defined as a choice of smoothly varying orientation of the fibres.

**Remark 1.1.29.** The underlying real vector bundle corresponding to a complex vector bundle is always orientable. This follows from the existence of the natural inclusion

$$\mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}^+(2n,\mathbb{R}), \quad X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

for $X, Y \in M(n,\mathbb{R})$. The image of this map lies in $\mathrm{GL}^+(2n,\mathbb{R})$ since

$$\det \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = |\det(X+iY)|^2.$$

We consider the natural orientation on the underlying real vector bundle of a complex vector bundle to be the one induced by the following ordered $\mathbb{R}$-basis of $\mathbb{C}^n$:

$$((1,0,\ldots,0),\ldots,(0,\ldots,0,1),(i,0,\ldots,0),\ldots,(0,\ldots,0,i))\,. \tag{1.1.1}$$

The following fact will be useful in the section on symplectic vector bundles, where we will need it to prove a similar statement about symplectic vector bundles.

**Proposition 1.1.30.** *[Hus94, p. 30] A vector bundle over a contractible base is trivial.*

## 1.2 Differential graded modules

As we will see in Chapter 4, the coefficient ring for the pearl complex is graded, and therefore this 'complex' does not fit the traditional definition of a 'chain complex of modules'. Rather it can be more accurately described as a differential graded module over a graded ring. The goal of this section is to define these objects as well as their tensor product, and to prove an algebraic Künneth theorem for the tensor product. This is simply a graded counterpart to the familiar algebraic Künneth theorem for chain complexes of modules (see for example [ML63, p. 166]).

### 1.2.1 Graded modules over graded rings

An abelian group $G$ is called a **graded abelian group** if there is a specified family of subgroups $G_j$ indexed by the integers such that $G = \oplus_{j\in\mathbb{Z}} G_j$. By 'graded', we will always mean $\mathbb{Z}$-graded, and therefore we suppress the $\mathbb{Z}$ in our notation. For any $j \in \mathbb{Z}$, the elements of $G_j$ are called **homogeneous** elements and the integer $j$ is called their **degree**. For $g \in G_j$ we write $\deg(g) = j$. Every element of a graded abelian group has a unique expression as a sum of homogeneous elements.

A homomorphism of graded abelian groups $f : G \to H$ is called homogeneous if there exists an integer $s$ such that $f(G_j) \subset H_{j+s}$ for all $j \in \mathbb{Z}$. The integer $s$ is called the **degree** of $f$ and denoted $\deg(f)$. We denote by $(\mathrm{Hom}(G, H))_s$ the group of homogeneous homomorphisms of degree $s$ from $G$ to $H$ and we define $\mathrm{Hom}(G, H)$ to be the graded abelian group $\oplus_{s\in\mathbb{Z}}\mathrm{Hom}_s(G, H)$. We denote by **gr-Ab** the category whose objects are graded abelian groups and whose morphisms are homogeneous maps. Note that, in general, the set of morphisms from $G$ to $H$ in this category is a proper subset of $\mathrm{Hom}(G, H)$, consisting of only those elements of $\mathrm{Hom}(G, H)$ which are homogeneous.

A **graded ring** is a ring $R$ with unity that is graded as an abelian group, i.e. $R = \oplus_{j\in\mathbb{Z}} R_j$ where the $R_j$ are additive subgroups, and satisfies the additional condition $R_k R_l \subset R_{k+l}$. A graded ring is called **graded commutative** if for all $r, s \in R$,

$$rs = (-1)^{\deg(r)\deg(s)} sr.$$

Examples of graded commutative rings include the exterior algebra on a vector space, graded by the degree of $k$-vectors, and rings of polynomials of even degree, graded by the degree of the polynomials. Notice that graded commutativity agrees with ordinary commutativity when the ring is concentrated in even degrees, or when it satisfies $r = -r$ for all $r \in R$. Arbitrary polynomial rings over $\mathbb{Z}_2$, for instance, satisfy the latter condition.

If $R$ is a graded ring, a **graded left $R$-module** $M$ is a left $R$-module $M$ that is graded as an abelian group, i.e. $M = \oplus_{j\in\mathbb{Z}} M_j$, where each $M_j$ is an additive subgroup of $M$, and $R_k M_l \subset M_{k+l}$. Graded right $R$-modules are defined similarly. When $R$ is graded commutative, a graded left $R$-module can be given the structure of a graded right $R$-module via

$$xr := (-1)^{\deg(x)\deg(r)} rx. \tag{1.2.1}$$

The minus sign in this definition is necessary in order for the right $R$-action to satisfy $x(rs) = (xr)s$. Notice that if we consider $\mathbb{Z}$ as a ring concentrated in degree 0, a graded $\mathbb{Z}$-module is simply a graded abelian group. A submodule $N \subset M$ is called a **graded submodule** if $N = \oplus_{j\in\mathbb{Z}} N \cap M_j$. If $N$ is a graded submodule, $M/N$ becomes a graded $R$-module with the grading defined by $(M/N)_j = \pi_j(M_j/N_j)$, where $\pi_j : M/N_j \to M/N$ is the natural projection.

A map of graded left $R$-modules $f : M \to L$ is called **homogeneous** if it is homogeneous as a homomorphism of abelian groups and

$$f(rx) = (-1)^{\deg(f)\deg(r)} r f(x), \tag{1.2.2}$$

for all $r \in R$ and homogeneous $x \in M$. We denote the abelian group of degree-$s$ homogeneous homomorphism from $M$ to $L$ by $(\mathrm{Hom}_R(M, L))_s$ and we define the graded abelian group $\mathrm{Hom}_R(M, L)$ by

$$\mathrm{Hom}_R(M, L) := \oplus_{s\in Z}(\mathrm{Hom}_R(M, L))_s.$$

Here the minus sign in Equation (1.2.2) is necessary for the graded abelian group $\mathrm{Hom_R(M, L)}$ to be a graded $R$-module when $R$ is graded commutative. We denote the category with graded left $R$-modules as objects and homogeneous homomorphisms as morphisms by **gr-$R$-Mod**.

A map of graded right $R$-modules is called **homogeneous** if it is homogeneous as a homomorphism of abelian groups and it is $R$-linear. We account for the discrepancy in the definitions of homogeneous maps of graded left and right $R$-modules as follows: When $R$ is graded commutative and the graded left $R$-modules $M$ and $L$ are given the structure of graded right modules via the right $R$-action defined in Equation (1.2.1) on the previous page, a homogeneous map of graded left $R$-modules $f : M \to L$ induces a map of graded right $R$-modules which is $R$-linear in the usual sense. As we did for graded left $R$-modules, we denote the abelian group of degree-$s$ homogeneous homomorphisms of graded right $R$-modules from $K$ to $N$ by $(\mathrm{Hom}_R(K, N))_s$ and we define the graded abelian group $\mathrm{Hom}_R(K, N)$ by

$$\mathrm{Hom}_R(K, N) := \oplus_{s\in Z}(\mathrm{Hom}_R(K, N))_s.$$

When $R$ is graded-commutative, $\mathrm{Hom}_R(K, N)$ becomes a graded right $R$-module via $(fr)(x) = (-1)^{\deg(r)\deg(x)}f(x)r$. We denote the category with graded right $R$-modules as objects and homogeneous homomorphisms as morphisms by **gr-Mod-$R$**.

Let $M$, $M'$, $N$ and $N'$ be all either graded left or graded right $R$-modules. A homogeneous homomorphism $f : M \to M'$ induces a homogeneous homomorphism of graded abelian groups

$$f^* : \mathrm{Hom}_R(M', N) \to \mathrm{Hom}_R(M, N), \quad \phi \mapsto (-1)^{\deg(f)\deg(\phi)}\phi \circ f.$$

When $R$ is graded commutative, $f^*$ is a homogeneous homomorphism of graded $R$-modules. One can verify easily that $\mathrm{Hom}_R(-, N)$ defines a contravariant functor from

**gr-**$R$**-Mod** or **gr-Mod-**$R$ to **gr-Ab** (or to **gr-**$R$**-Mod** or **gr-Mod-**$R$ in the graded commutative case). Similarly, an element $g \in \mathrm{Hom}_R(N, N')$ induces a homogeneous homomorphism of graded abelian groups

$$g_* : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'), \quad \psi \mapsto g \circ \psi.$$

In this case, it is an easy verification that $\mathrm{Hom}_R(M, -)$ defines a covariant functor from **gr-Mod-**$R$ or **gr-**$R$**-Mod** to the category of graded abelian groups, or of graded $R$-modules when $R$ is graded commutative.

For graded right or left $R$-modules $M$ and $L$, if $f : M \to L$ is homogeneous, then both $\ker(f)$ and $\mathrm{im}(f)$ are graded submodules and there is a homogeneous isomorphism $M/\ker(f) \cong \mathrm{im}(f)$.

If $M$ is a graded right $R$-module and $N$ is a graded left $R$-module, then $M \otimes_R N$ is a graded abelian group, where the grading is defined by setting $(M \otimes_R N)_j$ to be the subgroup of $M \otimes_R N$ generated by all elements of the form $x \otimes y$ where $x \in M_k$ and $y \in N_l$ for $k + l = j$. In the case where $R$ is graded commutative, this grading gives $M \otimes_R N$ the structure of a graded $R$-module, where the $R$-action is defined by

$$r(x \otimes y) = (-1)^{\deg(r)\deg(x)}(xr) \otimes y,$$

for $r \in R$, $x \in M$ and $y \in N$. Here the minus sign is necessary to ensure that $(rs)(x \otimes y) = r(s(x \otimes y))$.

A homogeneous map $f : M \to M'$ of graded right $R$-modules and a homogeneous map $g : N \to N'$ of graded left $R$-modules induce a homogeneous map of abelian groups

$$f \otimes g : M \otimes_R N \to M' \otimes_R N', \quad x \times y \mapsto (-1)^{\deg(x)\deg(g)} f(x) \otimes g(y),$$

with $\deg(f \otimes g) = \deg(f) + \deg(g)$. Here the sign ensures that $f \otimes g(xr \otimes y) = f \otimes g(x \otimes ry)$. When $R$ is graded commutative, $f \otimes g$ is a map of graded $R$-modules. For a fixed graded left $R$-module $M$, $M \otimes_R -$ defines a covariant functor from **gr-**$R$**-Mod** to **gr-Ab**. A morphism $g : N \to N'$ of graded left $R$-modules is mapped to the morphism of graded abelian groups

$$\mathrm{Id}_M \otimes g : M \otimes_R N \to M \otimes_R N'.$$

Similarly, $-\otimes_R$ defines a functor from **gr-Mod-$R$** to **gr-Ab**.

**Remark 1.2.1.** There are two concepts of commutativity for graded rings: There is the usual concept of commutativity when the ring is considered without its grading, and there is the concept of 'graded commutativity' that we have defined. Depending on the context, one notion of commutativity may be more appropriate than the other. When working with graded commutative rings, as we have seen, the usual definitions often must be altered by a sign in order for various conditions of 'compatibility' with the ring multiplication to be fulfilled. We have attempted to explain these compatibility conditions in a few instances. The **Koszul sign convention** is a rule of thumb which always ensures compatibility with graded commutative rings. This rule states that when an object of degree $a$ is commuted with one of degree $b$, a factor of $(-1)^{\deg(a)\deg(b)}$ arises.

For all coefficient rings which are used to define the pearl complex, the two notions of commutativity coincide, and therefore the choice of which concept of commutativity to use is not entirely relevant. However, only when one uses the Koszul sign convention, does the tensor product of two differential graded modules always possess a naturally defined differential. Thus we have chosen to use the Koszul sign convention, as it will allow us to state the Künneth theorem for differential graded modules in full generality.

For $k \in \mathbb{Z}$, the $k$**-suspension** of a graded $R$-module $M$, denoted $s_k M$, is the graded $R$-module whose underlying abelian group is the same as that of $M$, but whose grading is given by

$$(s_k M)_n := M_{n-k}.$$

If $M$ is a graded right $R$-module, we endow $s_k M$ with the right $R$-action induced by the $R$-action on $M$. If $M$ is a graded left $R$-module, the $R$ action on $s_k M$ is defined as follows: Let $s_k : M \to s_k M$ be the homogeneous degree-$k$ map of graded abelian groups induced by the identity map on the underlying non-graded abelian groups. Define

$$rs_k(x) = (-1)^{k|r|} s_k(rx)$$

for $r \in R$ homogeneous. With this $R$-action, $s_k M$ become a graded left $R$-module. In both the left and right cases, the map $s_k : M \to s_k M$ is a homogeneous degree-$k$ map of graded (left or right) $R$-modules. For $N$ another graded $R$-module, a homogeneous homomorphism $f : M \to N$ of degree $k$ induces a degree-zero homomorphism $\bar{f} : s_k M \to N$ defined by $\bar{f} = f \circ (s_k)^{-1}$.

There exist versions of the familiar concepts of free, projective, injective and flat modules in the graded context, as covered in the next definition. By an exact sequence of graded $R$-modules, we mean an exact sequence whose maps are morphisms in the category **gr-$R$-Mod** or **gr-Mod**-$R$. In other words, the maps are homogeneous homomorphisms.

**Definition 1.2.2.** *Let $R$ be a graded ring with unity.*

1. *A graded (left or right) $R$-module $F$ is said to be **gr-free** if it has a basis consisting of homogeneous elements.*

2. *The functor $- \otimes_R M$ from **gr-Mod**-$R$ to **gr-Ab** is always right exact. A graded left $R$-module $L$ is called **gr-flat** if the functor $- \otimes_R L$ is also left exact. Similarly, a graded right $R$-module $N$ is called gr-flat if $N \otimes_R -$ is left exact.*

3. *The functor $\mathrm{Hom}_R(M, -)$ from **gr-$R$-Mod** to **gr-Ab** is always left exact. A graded left $R$-module $P$ is called **gr-projective** if the functor $\mathrm{Hom}_R(P, -)$ is also right exact. Similarly, a graded right $R$-module $N$ is called projective if $\mathrm{Hom}_R(M, -)$ is a right exact functor from **gr-Mod**-$R$ to **gr-Ab**.*

**Proposition 1.2.3.**

1. *If a graded module is gr-free, then it is graded and free, but modules that are graded and free need not be gr-free.*

2. *A graded module is gr-flat if and only if it is graded and flat.*

3. *A graded module is gr-projective if and only if it is graded and projective.*

**Proof:**

1. Since a homogeneous basis of a graded $R$-module is a basis for the underlying module ignoring its grading, it is clear that 'gr-free' implies 'free'. As an example of a graded module over a graded ring which is free but not gr-free, take $R = \mathbb{Z} \times \mathbb{Z}$ and let $F$ be the graded $R$-module with $F_0 = \mathbb{Z} \times \{0\}$, $F_2 = \{0\} \times \mathbb{Z}$ and $F_j = \{0\}$ for $j \neq 0, 2$. The ring $R$ acts on $F$ by componentwise multiplication. Then $F$ is clearly free, but it does not possess a homogeneous basis. [NVO04, p. 21]

2. The proof of "flat implies gr-flat" is straightforward and we omit it.

   To prove that 'gr-flat' implies 'flat', one first proves a graded version of the Govorov-Lazard theorem which says that flat modules are precisely the direct limits of finitely-generated free modules. We can then write a flat module $M$ as $M = \varinjlim F_j$, where the $F_j$ are gr-free graded $R$-modules. As the $F_j$ are also free, an exact sequence of $R$-modules $0 \to A \overset{\alpha}{\to} B$, where $\alpha$ is only $R$-linear but not necessarily homogeneous, induces an exact sequence of graded abelian groups

$$0 \to F_j \otimes_R A \xrightarrow{\mathrm{Id}_{F_j} \otimes \alpha} F_j \otimes_R B$$

   for each $j$. Then using the fact that tensor products commute with direct limits, we obtain an exact sequence

$$0 \to M \otimes_R A \xrightarrow{\mathrm{Id}_F \otimes \alpha} M \otimes_R B.$$

When $R$ is graded commutative, this is an exact sequence of graded $R$-modules. We conclude that $M$ is flat.

3. First assume $P$ is gr-projective. Choose a homogeneous set of generators of $P$ and let $F$ be the gr-free graded $R$-module on this set of generators. Let $\phi : F \to P$ be the surjective homogeneous homomorphism of $R$-modules that takes a generator of $F$ to the corresponding generator of $P$. By gr-projectivity of $P$, $\phi$ lifts to a homogeneous homomorphism $\mu : P \to F$ satisfying $\phi \circ \mu = \mathrm{Id}_P$. Then

$$F = \mathrm{im}(\mu) \oplus \ker(\phi) \cong P \oplus \ker(\phi).$$

Thus $P$ is a direct summand of a gr-free $R$-module. As gr-free implies free, $P$ is a direct summand of a free $R$-module, and thus projective.

Now assume $P$ is projective as an $R$-module ignoring its grading. Consider the following exact diagram:

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, "\psi"] & N \arrow[r] & 0 \end{tikzcd}$$

Here $\psi$ and $f$ are homogeneous homomorphisms of graded $R$-modules with $\deg(\psi) = k$ and $\deg(f) = j$. This diagram induces an exact diagram of graded $R$-modules

$$\begin{tikzcd} & s_j P \arrow[d, "\bar{f}"] & \\ s_k M \arrow[r, "\bar{\psi}"] & N \arrow[r] & 0 \end{tikzcd}$$

where $\bar{f}$ and $\bar{\psi}$ are the degree-zero homomorphisms induced by $f$ and $\psi$. In particular, the maps $\bar{\psi}$ and $\bar{f}$ are $R$-linear, and therefore projectivity of $P$ implies the existence of an $R$-linear map $\bar{\rho} : s_j P \to s_k M$ satisfying $\bar{\psi} \circ \bar{\rho} = \bar{f}$. The map $\bar{\rho}$ can be chosen to be homogeneous as both $\psi$ and $f$ are homogeneous. We define $\rho : P \to M$ by $\rho = (s_k)^{-1} \circ \bar{\rho} \circ (s_j)$. The map $\rho$ satisfies $\psi \circ \rho = f$

and is homogeneous. Thus $P$ is gr-projective. ∎

**Remark 1.2.4.** Together with the fact that free modules are flat, parts 1 and 2 of Proposition 1.2.3 imply that gr-free modules are gr-flat.

### 1.2.2 Differential graded modules

In what follows, $R$ will be a graded ring with unity.

**Definition 1.2.5.** *We define a* ***differential graded (left or right)*** $R$***-module*** *to be a pair* $(K, d)$ *consisting of a (left or right) graded* $R$*-module* $K$ *together with a homogeneous* $R$*-module homomorphism* $d : K \to K$ *of degree* $-1$*, called the* ***differential****, satisfying* $d \circ d = 0$*. We denote the graded submodule* $\ker(d)$ *by* $Z(K)$ *and call its elements* ***cycles****. The graded submodule* $\mathrm{im}(d)$ *is denoted by* $B(K)$ *and its elements are called* ***boundaries****. The homology of a differential graded* $R$*-module is the graded* $R$*-module* $H(K) := \ker(d)/\mathrm{im}(d)$*. A* ***morphism of differential graded*** $R$***-modules*** *is a homogeneous map of the underlying graded* $R$*-modules that commutes with the differential. A morphism* $f : K \to M$ *of differential graded* $R$*-modules induces a morphism of graded modules* $f_* : H(K) \to H(M)$*.*

*A note on terminology:* The concept of a differential graded $R$-module is related to the usual concept of a 'chain complex of $R$-modules,' but is more general. Recall that for a ring $R$ (which in this case is ungraded), a chain complex of $R$-modules is a sequence of $R$-modules $(C_n)_{n\in\mathbb{Z}}$ together with $R$-module homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. A chain complex of $R$-modules $(C_n)_{n\in\mathbb{Z}}$ with differential $d_n$ results in a differential graded $R$-module $(\oplus_{n\in\mathbb{Z}} C_n, \oplus_{n\in\mathbb{Z}} d_n)$, where we consider $R$ as a graded ring concentrated in degree zero. However the converse is not true: If $(K, d)$ is a differential graded $R$-module, then for a given $n \in \mathbb{Z}$, the set of homogeneous elements of degree $n$ is not necessarily an $R$-module. In fact this is only true if $R$ is concentrated in degree zero. Likewise, the $n$th homology of a differential graded $R$-module is not necessarily an $R$-module. However, since the set $K_n$ of homogeneous elements of degree $n$ does form an abelian group for all $n$, and the group homomorphisms $d|_{K_n}$ satisfy $d|_{K_n} \circ d|_{K_{n+1}} = 0$, the groups $K_n$ and boundary maps $d|_{K_n}$ do form a chain complex of abelian groups.

If $(K, d_K)$ is a differential graded right $R$-module and $(L, d_L)$ is a differential graded left $R$-module, the tensor product $K \otimes_R L$ is the complex of abelian groups with differential defined by

$$d(a \otimes b) = d_K a \otimes b + (-1)^{\deg(a)} a \otimes d_L b, \tag{1.2.3}$$

for homogeneous elements $a \in K$ and $b \in L$. When $R$ is graded commutative, $K \otimes_R L$ is a differential graded $R$-module.

The next lemma provides an analogue for differential graded modules of the long exact sequence in homology induced by a short exact sequence of chain complexes.

**Lemma 1.2.6.** *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence of differential graded (left or right) $R$-modules with $\deg(f) = i$ and $\deg(g) = j$. Then there is a homogeneous homomorphism of graded $R$-modules $\partial : H(N) \to H(L)$ of degree $-(1 + i + j)$ such that the following triangle of graded $R$-modules is exact:*

$$\begin{tikzcd} H(L) \arrow[dr, "f_*"'] & & H(N) \arrow[ll, "\partial"'] \\ & H(M) \arrow[ur, "g_*"'] & \end{tikzcd}$$

**Proof:** Consider $L$, $M$ and $N$ as chain complexes of abelian groups. Then the following is a short exact sequence of chain complexes of abelian groups:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & L_n \arrow[r, "f_n"] \arrow[d, "d_L"] & M_{n+i} \arrow[r, "g_{n+i}"] \arrow[d, "d_M"] & N_{n+i+j} \arrow[r] \arrow[d, "d_N"] & 0 \\
0 \arrow[r] & L_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & M_{n-1+i} \arrow[r, "g_{n-1+i}"] \arrow[d] & N_{n-1+i+j} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Therefore we have a long exact sequence of abelian groups in homology

$$\cdots \to H_n(L) \xrightarrow{(f_n)_*} H_{n+i}(M) \xrightarrow{(g_{n+i})_*} H_{n+i+j}(N) \xrightarrow{\partial_{n+i+j}}$$
$$H_{n-1}(L) \xrightarrow{(f_{n-1})_*} H_{n-1+i}(M) \xrightarrow{(g_{n-1+i})_*} H_{n-1+i+j}(N) \to \cdots$$

The maps $f_*$ and $g_*$ in the triangle above are equal to $\oplus_{n\in\mathbb{Z}}(f_n)_*$ and $\oplus_{n\in\mathbb{Z}}(g_n)_*$. We define the map $\partial = \oplus_{n\in\mathbb{Z}}\partial_n$. Clearly $\partial$ is a homogeneous homomorphism of graded abelian groups of degree $-(1+i+j)$ . It remains to check that $\partial$ is a homogeneous homomorphism of graded $R$-modules. We assume our differential graded modules are left $R$-modules. The proof for right $R$-modules is even simpler. Let $r \in R_s$ and $x \in Z(N)_t$. Then $r\partial([x]) = r\partial_t([x]) \in H_{t-(1+i+j)}(L)$. Recall the definition of $\partial_t([x])$: We pull $x \in Z(N)_t$ back to some $y \in M_{t-j}$ via $g$. Then we choose an element $z \in (f_{t-(1+i+j)})^{-1}(d_M(y))$. This $z$ will be a cycle, and its homology class is independent of the choices we make in finding it. We set $\partial_t([x]) = [z]$. As $g$ is a homogeneous homomorphism of graded $R$-modules,

$$g((-1)^{sj}ry) = rg(y) = rx.$$

Also, by homogeneity of $f$, we have

$$f((-1)^{-s(1+i+j)}rz) = (-1)^{-s(j+1)}rf(z)$$

$$= (-1)^{-s(j+1)} r d_M(y)$$
$$= d_M((-1)^{sj} r y)$$

Therefore the map $\partial_{s+t} : H_{s+t}(N) \to H_{s+t-(1+i+j)}(L)$ maps $r[x]$ to $(-1)^{-(1+i+j)} r[z]$. Thus $\partial$ is a homogeneous map of graded left $R$-modules. ∎

### 1.2.3 The Künneth theorem for differential graded modules

In this section, we prove versions for differential graded modules of both the universal coefficient theorem and the Künneth theorem. These proofs are adapted from those presented in [ML63, p. 166-168] for chain complexes of modules.

**Definition 1.2.7.** *Let $(K, d_K)$ be a differential graded right $R$-module and let $(L, d_L)$ be a differential graded left $R$-module. The following is a well-defined degree-zero homogeneous homomorphism of graded abelian groups:*

$$p : H(K) \otimes_R H(L) \to H(K \otimes_R L), \quad [a] \otimes [b] \mapsto [a \otimes b],$$

*for $a \in Z(K)$, $b \in Z(L)$. The homomorphism $p$ is called the* ***external homology product****. In the case that $R$ is commutative, then $p$ is also a homogeneous homomorphism of graded $R$-modules.*

We will need the following algebraic lemma to prove the universal coefficient theorem for differential graded $R$-modules.

**Lemma 1.2.8.** *Suppose we have the following commutative diagram of graded $R$-modules, where the rows and columns are exact and all maps are homogeneous homo-*

*morphisms of graded $R$-modules:*

$$\begin{tikzcd} & 0 & 0 \arrow[d] & & \\ 0 \arrow[r] & A \arrow[u] \arrow[r, "a"] & B \arrow[r, "b"] \arrow[d, "f"] & C \arrow[r] & 0 \\ & D \arrow[u, "c"] \arrow[r, "e"] & E \arrow[d, "g"] & & \\ & & F & & \end{tikzcd}$$

*Then there is a homogeneous isomorphism* $q : C \to \ker(g)/\mathrm{im}(e)$ *of degree* $\deg(f) - \deg(b)$.

**Proof:** The proof is a standard diagram chase and we omit the details. As we will need the explicit form of the isomorphism $q$ in Lemma 1.2.9 we mention it here: For $x \in C$ we choose $y \in b^{-1}(x)$ and define $q(x) = f(y) + \mathrm{im}(e)$. ∎

**Lemma 1.2.9** (Universal coefficient theorem for differential graded modules)**.** *Let* $(K, d_K)$ *be a differential graded right* $R$*-module and let* $L$ *be a gr-flat graded left* $R$*-module. We equip* $L$ *with the trivial differential and* $K \otimes_R L$ *with the tensor product differential defined in Equation* (1.2.3)*, making it into a chain complex of abelian groups. Then the external homology product*

$$p : H(K) \otimes_R L \xrightarrow{\cong} H(K \otimes_R L)$$

*is a degree-zero isomorphism of graded abelian groups. If* $R$ *is graded commutative, then* $p$ *is an isomorphism of graded* $R$*-modules.*

**Proof:** The following diagram of graded $R$-modules commutes, the rows and columns are exact, and all maps are homogeneous:

$$\begin{tikzcd}
 & 0 & 0 \arrow[d] & & \\
0 \arrow[r] & B(K) \arrow[r, "\iota"] \arrow[u] & Z(K) \arrow[r, "\pi"] \arrow[d, "\iota'"] & H(K) \arrow[r] & 0 \\
 & K \arrow[r, "d_K"] \arrow[u, "d_K"] & K \arrow[d, "d_K"] & & \\
 & & K & &
\end{tikzcd}$$

Here the maps $\iota$ and $\iota'$ are just inclusions and $\pi$ is the natural projection. As $L$ is gr-flat, the rows and columns remain exact when we tensor with $L$. Therefore we obtain a diagram of graded $\mathbb{Z}$-modules as in Lemma 1.2.8 (where we consider $\mathbb{Z}$ as a graded ring concentrated in degree zero):

$$\begin{tikzcd}
 & 0 & 0 \arrow[d] & & \\
0 \arrow[r] & B(K) \otimes_R L \arrow[r, "\iota \otimes \mathrm{Id}_L"] \arrow[u] & Z(K) \otimes_R L \arrow[r, "\pi \otimes \mathrm{Id}_L"] \arrow[d, "\iota"] & H(K) \otimes_R L \arrow[r] & 0 \\
 & K \otimes_R L \arrow[r, "d_K \otimes \mathrm{Id}_L"] \arrow[u, "d_K \otimes \mathrm{Id}_L"] & K \otimes_R L \arrow[d, "d_K \otimes \mathrm{Id}_L"] & & \\
 & & K \otimes_R L & &
\end{tikzcd}$$

Lemma 1.2.8 yields an isomorphism of graded $\mathbb{Z}$-modules

$$\begin{aligned} H(K) \otimes_R L &\cong \ker(d_K \otimes \mathrm{Id}_L) / \mathrm{im}(d_K \otimes \mathrm{Id}_L) \\ &= H(K \otimes_R L). \end{aligned}$$

Inspecting the definition of the isomorphism $q$ of Lemma 1.2.8, we see that it is simply the external homology product.

It remains to note that in the case where $R$ is graded commutative, $p$ is homogeneous map of graded $R$-modules and thus is also an isomorphism of graded $R$-modules. ∎

The following describes a version of the homology groups $\mathrm{Tor}_n^R(A,B)$ for a graded ring $R$ and graded $R$-modules $A$ and $B$.

**Definition 1.2.10.** *[FHT01, p. 274] Let $R$ be a graded ring and let $A$ be a graded (left or right) $R$-module. A* ***graded projective resolution*** *of $A$ is an exact sequence of the form*

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0, \tag{1.2.4}$$

*where the $P_j$ are gr-projective (left or right) $R$-modules and all maps are degree-zero morphisms of graded $R$-modules.*

Although for an arbitrary abelian category, one usually only requires the maps in a projective resolution to be morphisms in the category, here we use the simplified definition of [FHT01]. A projective resolution of a graded module in the sense of Definition 1.2.10 can be obtained from a projective resolutions whose maps are morphisms of arbitrary degree by taking appropriate suspensions of the modules.

**Proposition 1.2.11.** *[FHT01, p. 274] Every graded (left or right) $R$-module has a graded projective resolution.*

**Definition 1.2.12.** *Let $A$ be a graded left $R$-module and let $D$ be a graded right $R$-module. Then for a graded projective resolution of $A$ as in diagram* (1.2.4)*, we have a chain complex of graded abelian groups*

$$\cdots \xrightarrow{\mathrm{Id}_D \otimes d_3} D \otimes_R P_2 \xrightarrow{\mathrm{Id}_D \otimes d_2} D \otimes_R P_1 \xrightarrow{\mathrm{Id}_D \otimes d_1} D \otimes_R P_0 \xrightarrow{\mathrm{Id}_D \otimes \epsilon} D \otimes_R A \to 0.$$

*As the maps above are morphisms of graded $\mathbb{Z}$-modules, for $n \geq 1$, we can define the graded abelian groups*

$$\mathrm{Tor}_n^R(D,A) = \ker(\mathrm{Id}_D \otimes d_n)/\mathrm{im}(\mathrm{Id}_D \otimes d_{n+1}),$$

*and we set $\mathrm{Tor}_0^R(D,A) = D \otimes_R P_0/\mathrm{im}(\mathrm{Id}_D \otimes d_1)$. If $R$ is graded commutative, the groups $\mathrm{Tor}_n^R(D,A)$ are graded $R$-modules.*

Equivalently, one can define the $\mathrm{Tor}_n^R(D,A)$ by taking a projective resolution of $D$ and tensoring on the right with $A$.

The proofs of the following three propositions are adaptations of the standard proofs to the graded case. We state the results for graded left $R$-modules, but analogous results hold for graded right $R$-modules.

**Proposition 1.2.13.** *[FHT01, p. 276] Up to an isomorphism in the category* ***gr-Ab*** *(or in the category* ***gr-R-Mod*** *in the graded commutative case),* $\mathrm{Tor}_n^R(D,A)$ *does not depend on the choice of graded projective resolution.*

**Proposition 1.2.14.** *[FHT01, p. 277] A short exact sequence of graded left R-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

*results in a long exact sequence of graded abelian groups (or of graded R-modules in the graded commutative case)*

$$\begin{aligned} \cdots \to \mathrm{Tor}_2^R(D,C) &\xrightarrow{\delta_2} \mathrm{Tor}_1^R(D,A) \xrightarrow{\bar{\alpha}_1} \mathrm{Tor}_1^R(D,B) \\ &\xrightarrow{\bar{\beta}_1} \mathrm{Tor}_1^R(D,C) \xrightarrow{\delta_1} D \otimes_R A \xrightarrow{\bar{\alpha}_0} D \otimes_R B \xrightarrow{\bar{\beta}_0} D \otimes_R C \to 0, \end{aligned}$$

*for any graded right R-module D. Here the connecting homomorphisms $\delta_n$ are of degree zero,* $\deg(\bar{\alpha}_n) = \deg(\alpha)$ *and* $\deg(\bar{\beta}_n) = \deg(\beta)$ *for all n,* $\bar{\alpha_0} = \mathrm{Id}_D \otimes \alpha$ *and* $\bar{\beta}_0 = \mathrm{Id}_D \otimes \beta$.

**Proposition 1.2.15.** *For a graded left R-module B, the following are equivalent:*

1. *$B$ is gr-flat,*
2. $\mathrm{Tor}_1^R(D,B) = 0$ *for all graded right R-modules D,*
3. $\mathrm{Tor}_n^R(D,B) = 0$ *for all graded right R-modules D and all $n \geq 1$.*

**Proof:** The implication (2) implies (1) follows directly from 1.2.14 and the definition of gr-flat. The implication (3) implies (2) is trivial. To see that (1) implies (3), choose a graded projective resolution of $D$:

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} D \to 0.$$

If $B$ is gr-flat, tensoring this resolution with $B$, we obtain another exact sequence

$$\cdots \to P_2 \otimes_R B \xrightarrow{d_2 \otimes \mathrm{Id}_B} P_1 \otimes_R B \xrightarrow{d_1 \otimes \mathrm{Id}_B} P_0 \otimes_R B \xrightarrow{\epsilon \otimes \mathrm{Id}_B} D \otimes_R B \to 0.$$

Exactness of this sequence implies $\mathrm{Tor}^n_R(D, B) = 0$ for all $n \geq 1$. ∎

**Theorem 1.2.16** (The Künneth theorem for differential graded modules)**.** *Let $(K, d_K)$ be a differential graded right $R$-module and $(L, d_L)$ be a differential graded left $R$-module. In addition, assume that $Z(L)$ and $B(L)$ are gr-flat. Then there is a short exact sequence of graded abelian groups*

$$0 \to H(K) \otimes_R H(L) \xrightarrow{p} H(K \otimes_R L) \xrightarrow{\beta} \mathrm{Tor}^R_1(H(K), H(L)) \to 0,$$

*where the map $\beta$ is of degree $-1$. Here $p$ is the external homology product (see Definition 1.2.7). If $R$ is graded commutative, then this is a short exact sequence of graded $R$-modules.*

**Proof:** Consider $Z(L)$ and $L/Z(L) \cong B(L)$ as differential graded left $R$-modules with zero boundary. Then we have a short exact sequence of differential $R$-modules

$$0 \to Z(L) \xrightarrow{\iota} L \xrightarrow{\pi} L/Z(L) \to 0, \tag{1.2.5}$$

where $\iota$ is the inclusion map, and $\pi$ is the natural projection. Since $B(L)$ is flat, $B(L) \cong L/Z(L)$ is also flat. Therefore, by Proposition 1.2.15, $\mathrm{Tor}^R_n(K, L/Z(L)) = 0$ for $n \geq 1$. Applying Proposition 1.2.14, tensoring the short exact sequence (1.2.5) above with $K$, we obtain a short exact sequence of graded differential $\mathbb{Z}$-modules, or of graded $R$-modules in the graded commutative case:

$$0 \to K \otimes_R Z(L) \xrightarrow{\mathrm{Id}_K \otimes \iota} K \otimes_R L \xrightarrow{\mathrm{Id}_K \otimes \pi} K \otimes_R (L/Z(L)) \to 0.$$

We have the following exact triangle in homology for this sequence:

$$\begin{tikzcd}
H(K \otimes_R Z(L)) \arrow[dr, "(\mathrm{Id}_K \otimes \iota)_*"'] & & H(K \otimes_R (L/Z(L))) \arrow[ll, "\partial"'] \\
& H(K \otimes_R L) \arrow[ur, "(\mathrm{Id}_K \otimes \pi)_*"'] &
\end{tikzcd} \tag{1.2.6}$$

Equivalently, we have a short exact sequence

$$0 \to \operatorname{coker}(\partial) \to H(K \otimes_R L) \to \ker(\partial) \to 0.$$

We will prove the theorem by determining the modules $\operatorname{coker}(\partial)$ and $\ker(\partial)$ in the above sequence. Let $d'_L$ be the map from $L/Z(L) \to Z(L)$ induced by $d_L$. The following short exact sequence of $R$-modules describes the homology of $L$:

$$0 \to L/Z(L) \xrightarrow{d'_L} Z(L) \xrightarrow{\pi'} H(L) \to 0,$$

where $\pi'$ is the natural projection. Since $Z(L)$ is flat, $\operatorname{Tor}_1^R(H(K), Z(L)) = 0$ and tensoring the above sequence with $H(K)$, again by Proposition 1.2.14, we obtain the following exact sequence

$$\begin{gathered}
0 \to \operatorname{Tor}_1^R(H(K), H(L)) \xrightarrow{\delta_1} H(K) \otimes_R (L/Z(L)) \to \\
H(K) \otimes_R Z(L) \to H(K) \otimes_R H(L) \to 0.
\end{gathered} \tag{1.2.7}$$

Consider the following diagram involving the second and third non-zero terms in the above sequence:

$$\begin{tikzcd}
H(K) \otimes_R L/Z(L) \arrow[r, "\mathrm{Id}_{H(K)} \otimes d'_L"] \arrow[d, "p"'] & H(K) \otimes_R Z(L) \arrow[d, "p"'] \\
H(K \otimes_R L/Z(L)) \arrow[r, "\partial"] & H(K \otimes_R Z(L))
\end{tikzcd} \tag{1.2.8}$$

Here the map $p$ is the external homology product, which by Lemma 1.2.9 is an isomorphism in this case. This diagram commutes: Any element of $H(K)\otimes_R L/Z(L)$ can be expressed as a sum $\sum_{j=1}^{J}[x_j]\otimes(y_j+Z(L))$ where $x_j\in Z(L)$ and $y_j\in L$ are homogeneous for $j=1,\ldots,J$. We have,

$$\begin{aligned} p\circ(\mathrm{Id}_{H(K)}\otimes d'_L)\left([x_j]\otimes(y_j+Z(L))\right) &= p\left((-1)^{\deg(d_L)\deg([x_j])}[x_j]\otimes d_L y_j\right), \\ &= p\left((-1)^{\deg(x_j)}[x_j]\otimes d_L y_j\right), \\ &= (-1)^{\deg(x_j)}[x_j\otimes d_L y_j]. \end{aligned}$$

To compute $\partial\circ p([x_j]\otimes(y_j+Z(L))$, we simply apply the definition of the connecting homomorphism $\partial$ to $p([x_j]\otimes(y_j+Z(L))=[x_j\otimes(y_j+Z(L))]$. To do so, we first choose $x_j\otimes y_j$ as the required element of $K\otimes_R L$ which maps to $x_j\otimes(y_j+Z(L))$ under $\mathrm{Id}_K\otimes\pi$. Then to verify that $\partial([x_j\otimes(y_j+Z(L)])=(-1)^{\deg(x_j)}[x_j\otimes d_L y_j]$, one simply checks that

$$\mathrm{Id}_K\otimes\iota((-1)^{\deg(x_j)}x_j\otimes d_L y_j)=d_{K\otimes L}(x_j\otimes y_j).$$

By commutativity of the above diagram, and using the fact that $p$ is an isomorphism, we have

$$\ker(\partial)=p\left(\ker\left(\mathrm{Id}_{H(K)}\otimes d'_L\right)\right).$$

Thus, by exactness of the sequence (1.2.7),

$$\ker(\partial)=p\left(\mathrm{im}(\delta_1)\right).$$

Then as $\delta_1$ is an isomorphism onto its image (again by exactness of the sequence (1.2.7)),

$$p\circ\delta_1:\mathrm{Tor}_1^R(H(K),H(L))\to\ker(\partial),$$

is a homogeneous isomorphism of degree zero. We define $\beta$ to be the degree-zero homogeneous homomorphism

$$\beta:H(K\otimes_R L)\to\mathrm{Tor}_1^R(H(K),H(L)),\quad \beta=(p\circ\delta_1)^{-1}\circ(\mathrm{Id}_K\otimes\pi)_*.$$

Notice that this composition is well-defined as exactness of the triangle (1.2.6) implies $\ker(\partial) = \mathrm{im}(\mathrm{Id}_K \otimes \pi)_*$.

To determine $\mathrm{coker}(\partial)$, we see from the commutative diagram (1.2.8) that the map $p$ on the right induces an isomorphism $p'$

$$\begin{tikzcd}
\left(H(K) \otimes_R Z(L)\right)/\mathrm{im}\left(\mathrm{Id}_{H(K)} \otimes d'_L\right) \arrow[r, "p'"] \arrow[d, equal] & H(K \otimes_R Z(L))/\mathrm{im}(\partial) \\
\left(H(K) \otimes_R Z(L)\right)/\left(H(K) \otimes_R B(K)\right) &
\end{tikzcd}$$

and the composition

$$H(K) \otimes_R H(L) \xrightarrow{\cong} \left(H(K) \otimes_R Z(L)\right)/\left(H(K) \otimes_R B(K)\right) \xrightarrow{p'}$$
$$H(K \otimes_R Z(L))/\mathrm{im}(\partial) \xrightarrow{(\mathrm{Id}_K \otimes \iota)'_*} H(K \otimes_R L)$$

is simply the external homology product. Here the map $(\mathrm{Id}_K \otimes \iota)'_*$ is the map induced by $(\mathrm{Id}_K \otimes \iota)_* : H(K \otimes_R Z(L)) \to H(K \otimes_R L)$. Therefore we obtain the short exact sequence of the theorem. It remains to notice that in the case where $R$ is graded commutative, all maps are $R$-linear, and therefore we have a short exact sequence of graded $R$-modules. ∎

As the pearl complex is a differential graded module over a graded Laurent polynomial ring over the field $\mathbb{Z}_2$, we will be most interested in Corollary 1.2.18, which is the version of Theorem 1.2.16 that applies to these particular differential graded modules. To prove this corollary we need the following simple algebraic lemma.

**Lemma 1.2.17.** *Let $\mathbb{K}$ be a field and $\mathbb{K}[t, t^{-1}]$ be the graded Laurent polynomial ring with $\deg(t) = k \neq 0$. Then any graded $\mathbb{K}[t, t^{-1}]$-module is gr-free.*

**Proof:** Let $M$ be a graded $\mathbb{K}[t, t^{-1}]$-module. Then for each $n \in \mathbb{Z}$, the map

$$t^n : M_0 \to M_{kn}, \quad x \mapsto t^n x$$

is an isomorphism of $\mathbb{K}$-vector spaces. From this, it follows that the map

$$\psi : \mathbb{K}[t, t^{-1}] \otimes_{\mathbb{K}} M_0 \to M, \quad p \otimes x \mapsto px$$

is also an isomorphism of $\mathbb{K}$-vector spaces. Thus if we consider $\mathbb{K}[t, t^{-1}] \otimes_{\mathbb{K}} M_0$ as a $\mathbb{K}[t, t^{-1}]$-module in the obvious way, $\psi$ is also an isomorphism of $\mathbb{K}[t, t^{-1}]$-modules. But the module $\mathbb{K}[t, t^{-1}] \otimes_{\mathbb{K}} M_0$ is clearly free: If $\{b_\alpha\}_{\alpha \in A}$ is a $\mathbb{K}$-basis of $M_0$, then $\{1 \otimes b_\alpha\}_{\alpha \in A}$ is a $\mathbb{K}[t, t^{-1}]$-basis of $\mathbb{K}[t, t^{-1}] \otimes_{\mathbb{K}} M_0$. ∎

**Corollary 1.2.18.** *Let* $(K, d_K)$ *and* $(L, d_L)$ *be differential graded modules over* $\mathbb{K}[t, t^{-1}]$, *where* $\deg(t) = 2n \neq 0$. *Then the external homology product*

$$p : H(K) \otimes_R H(L) \to H(K \otimes_R L)$$

*is an isomorphism of* $\mathbb{K}[t, t^{-1}]$*-modules.*

**Proof:** As $L$ is a graded $\mathbb{K}[t, t^{-1}]$-module, the modules $Z(L)$, $B(L)$ and $H(L)$ are also graded $\mathbb{K}[t, t^{-1}]$-modules and by Lemma 1.2.17 are gr-free and thus by Remark 1.2.4 also gr-flat. Therefore Theorem 1.2.16 applies and by Proposition 1.2.15, $\mathrm{Tor}_1^{\mathbb{K}[t,t^{-1}]}(H(K), H(L)) = 0$. Then exactness of

$$0 \longrightarrow H(K) \otimes_{\mathbb{K}[t,t^{-1}]} H(L) \xrightarrow{p} H(K \otimes_{\mathbb{K}[t,t^{-1}]} L) \longrightarrow 0$$

implies $p$ is an isomorphism. ∎

## 1.3 Morse theory

Morse theory is based on the idea that a typical smooth function on a compact manifold encodes significant details about the topology of the manifold. In this section, we will describe the Morse complex, which allows one to compute the homology of a compact manifold using such a function.

### 1.3.1 Morse functions

**Definition 1.3.1.** *Let $M$ be a smooth manifold and let $f \in C^\infty(M, \mathbb{R})$. Recall that a critical point of $f$ is a point $p \in M$ for which $df_p = 0$. We denote the set of critical points of $f$ by* $\mathrm{Crit}(f)$*. For $p \in \mathrm{Crit}(f)$, we may define a bilinear map*

$$\mathrm{Hess}_p(f) : T_pM \times T_pM \to \mathbb{R}$$

*called the **Hessian** of $f$ at $p$ as follows: For $\eta, \xi \in T_pM$, choose extensions $\widetilde{\eta}$ and $\widetilde{\xi}$ of $\eta$ and $\xi$ to smooth vector fields defined on a neighbourhood of $p$. We then set*

$$\mathrm{Hess}_p(f)(\eta, \xi) = \widetilde{\eta}(\widetilde{\xi}(f))(p).$$

*On the right side of this equation we are using the definition of vector fields as derivations on the algebra of smooth functions. A critical point $p$ is called **non-degenerate** if the Hessian of $f$ at $p$ is a non-degenerate bilinear map. As usual, by this we mean that if* $\mathrm{Hess}_p(f)(\eta, \xi) = 0$ *for all $\eta \in T_pM$, then $\xi = 0$. If all critical points of $f$ are non-degenerate, $f$ is called a **Morse function**. Equivalently, $f$ is Morse if the image of the section $df$ of $T^*M$ intersects the image of the zero section of $T^*M$ transversally.*

**Remark 1.3.2.** The Hessian does not depend on the choice of extensions $\widetilde{\eta}$ and $\widetilde{\xi}$: Notice that

$$\mathrm{Hess}_p(f)(\eta, \xi) = \widetilde{\eta}(p)(\widetilde{\xi}(f)) = \eta(\widetilde{\xi}(f)),$$

and thus $\mathrm{Hess}_p(f)$ does not depend on the extension $\widetilde{\eta}$. Moreover, $\mathrm{Hess}_p(f)$ is symmetric, as we have

$$\mathrm{Hess}_p(f)(\xi, \eta) - \mathrm{Hess}_p(f)(\eta, \xi) = [\widetilde{\xi}, \widetilde{\eta}](p)(f) = df_p([\widetilde{\xi}, \widetilde{\eta}]) = 0,$$

as $p$ is a critical point of $f$. Therefore, by symmetry, $\mathrm{Hess}_p(f)$ does not depend on the choice of extension $\widetilde{\xi}$.

For coordinates $(x_1, \dots, x_m)$ around $p$, the matrix of the Hessian of $f$ at $p$ with respect to the standard basis of $T_pM$, $\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_m}\Big|_p$, is the familiar Hessian matrix of second partial derivatives from multivariable calculus. Clearly, $\mathrm{Hess}_p(f)$ is non-degenerate precisely when this matrix, computed in any coordinate system, is non-singular. Therefore at a non-degenerate critical point $p$, the matrix of $\mathrm{Hess}_p(f)$ is non-singular and symmetric, and thus diagonalizable with real eigenvalues. The magnitude of these eigenvalues may depend on the choice of coordinate chart, but by Sylvester's law of inertia, their signs do not. This leads to the following definition.

**Definition 1.3.3.** *The number of negative eigenvalues of the Hessian of $f$ at a non-degenerate critical point $p$ is called the* ***Morse index*** *of $p$, and denoted by $|p|$.*

Geometrically, the Morse index of a non-degenerate critical point indicates the number of independent directions in which the Morse function decreases around the critical point. This is best illustrated by an example.

**Example 1.3.4.** Consider the upright torus in $\mathbb{R}^3$,

$$\mathbb{T}_{\mathrm{up}} = \{ \ ((3+\cos\phi)\cos\theta, \sin\phi, (3+\cos\phi)\sin\theta) \mid \theta, \phi \in [0, 2\pi] \ \} \, .$$

The height function

$$h_{\mathrm{up}} : \mathbb{T}_{\mathrm{up}} \to \mathbb{R}, \quad (x, y, z) \mapsto z,$$

is Morse with four critical points: a critical point $x_0$ of index 0 at the bottom of the torus, two saddle points $x_1$ and $y_1$ of index 1 at the bottom and top of the inner circle respectively, and one critical point $x_2$ of index 2 at the top of the torus (see Figure 1.1).

The following lemma reveals that, in a neighbourhood of a non-degenerate critical point, a smooth function admits a very simple local description depending only on the Morse index of the critical point.

**Lemma 1.3.5** (Morse lemma)**.** *[Mor25] Let $M$ be an $m$-dimensional smooth manifold, $f : M \to \mathbb{R}$ a smooth function and $p \in M$ a non-degenerate critical point of $f$ with $|p| = k$. Then there exists a coordinate chart on a neighbourhood $U$ of $p$, $\phi : U \to \phi(U) \subset \mathbb{R}^m$ such that $\phi(p) = 0$ and*

$$f \circ \phi^{-1}(x_1, \ldots, x_m) = f(p) - x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + x_{k+2}^2 + \cdots x_m^2.$$

We will make use of the following corollary when defining Morse homology for compact manifolds.

**Corollary 1.3.6.** *Non-degenerate critical points are isolated. Moreover, if $M$ is compact and $f : M \to \mathbb{R}$ is Morse, then $f$ has finitely many critical points.*

**Proof:** For $p \in \mathrm{Crit}(f)$, we choose a coordinate chart on a neighbourhood $U$ of $p$ of the form described in Lemma 1.3.5. As a matrix with respect to the standard basis of $T_x\mathbb{R}^m$, we have

$$d(f \circ \phi^{-1})_x = \begin{pmatrix} -2x_1 & \cdots & -2x_k & 2x_{k+1} & \cdots & 2x_m \end{pmatrix}^T,$$

for $x = (x_1, \ldots, x_m) \in \phi(U)$, where $|p| = k$. From this it is clear that the only point in $\phi(U)$ where $d(f \circ \phi^{-1})$ vanishes is the origin. Therefore $p$ is the only point in $U$ at which $df$ vanishes.

To see that a Morse function $f$ on a compact manifold $M$ has only finitely many critical points, first notice that $\mathrm{Crit}(f)$ is closed as

$$\mathrm{Crit}(f) = (df)^{-1}(\iota(M)),$$

where $\iota : M \hookrightarrow T^*M$ is the inclusion of $M$ in $T^*M$ as the zero section. As $df : M \to T^*M$ is smooth and $\iota(M)$ is closed in $T^*M$, $\mathrm{Crit}(f)$ is closed in $M$. The result now follows from the general fact that for a compact topological space, closed subsets consisting of isolated points have finite cardinality. ∎

The following proposition tells us that Morse functions always exist in abundance and clarifies what we mean when we say a 'typical' function is Morse.

**Theorem 1.3.7.** *[Mor96][AD10, p. 11] Let $M$ be a compact manifold. The set of Morse functions on $M$ forms an open dense subset of the space of smooth functions $C^\infty(M, \mathbb{R})$ equipped with the Whitney $C^\infty$-topology. We refer the reader to [Hir94] for a detailed description of the Whitney topologies.*

### 1.3.2 The Morse complex

The definition of the boundary operator in the Morse complex will involve counting certain trajectories of the flow of the gradient vector field of a Morse function.

**Definition 1.3.8.** *Let $(M, g)$ be a Riemannian manifold. Recall that the **gradient** of a function $f \in C^\infty(M, \mathbb{R})$ is the smooth vector field $\nabla f$ on $M$ defined by*

$$df_p = g_p(\nabla f, \,\cdot\,)$$

*for all $p \in M$. Suppose $M$ is compact and let $\Phi : \mathbb{R} \times M \to M$ denote the flow of the negative gradient $-\nabla f$. The **stable manifold** of a non-degenerate critical point $p \in \mathrm{Crit}(f)$ is the subset of $M$ defined by*

$$W^s(p) := \{x \in M | \lim_{t \to \infty} \Phi_t(x) = p\},$$

*and the **unstable manifold** of $p$ is*

$$W^u(p) := \{x \in M | \lim_{t \to -\infty} \Phi_t(x) = p\}.$$

*For critical points $p$ and $q$, we denote by $W(p, q)$ the intersection $W^u(p) \cap W^s(q)$.*

For $x \in M$, we will use the notation $\Phi(x)$ for the path in $M$ given by

$$\mathbb{R} \to M, \quad t \mapsto \Phi_t(x).$$

By a **(parametrized) Morse trajectory** or a **connecting orbit** from $p$ to $q$, we mean a function $\gamma \in C^\infty(\mathbb{R}, M)$ such that $\gamma = \Phi(x)$ for some $x \in M$ and $\lim_{t\to\infty} \gamma(t) = p$, $\lim_{t\to\infty} \gamma(t) = q$. We denote the set of Morse trajectories from $p$ to $q$ by $\mathcal{M}(p, q)$. Note that the elements of $W(p, q)$ are in bijective correspondence with the elements of $\mathcal{M}(p, q)$. The bijection is given by

$$W(p, q) \to \mathcal{M}(p, q), \quad x \mapsto \Phi(x).$$

The following theorem confirms that the stable and unstable manifolds are indeed aptly named.

**Theorem 1.3.9.** *[BH04, p. 94] Let $M$ be a compact smooth manifold of dimension $m$ and let $f \in C^\infty(M, \mathbb{R})$ be Morse. Then for $p \in \mathrm{Crit}(f)$, $W^s(p)$ is a smoothly embedded open ball of dimension $m - |p|$ and $W^u(p)$ is a smoothly embedded open ball of dimension $|p|$.*

In order for the Morse homology to be well-defined, we will require the Riemannian metric and the Morse function to satisfy a genericity condition, as defined next.

**Definition 1.3.10.** *Let $M$ be a smooth manifold. A pair $(f, g)$ consisting of a Morse function $f \in C^\infty(M, \mathbb{R})$ and a Riemannian metric $g$ on $M$ is said to satisfy the **Morse-Smale transversality condition**, or is simply called Morse-Smale, if the stable and unstable manifolds of $f$ intersect transversally. That is, for all $p, q \in \mathrm{Crit}(f)$, and for all $x \in W(p, q)$, we have*

$$T_x M = T_x W^u(p) + T_x W^s(q).$$

*If the pair $(f, g)$ is Morse-Smale, one also calls $f$ a Morse-Smale function on the Riemannian manifold $(M, g)$.*

**Example 1.3.11.** We return to Example 1.3.4, where we considered the height function $h_{\mathrm{up}}$ as Morse function on the upright torus $\mathbb{T}_{\mathrm{up}}$ in $\mathbb{R}^3$. With the metric $g_{\mathrm{up}}$ on

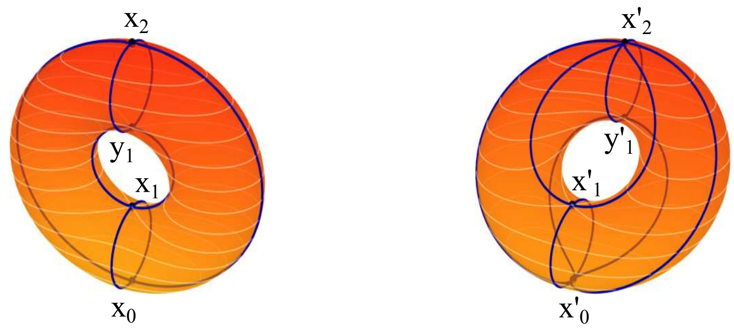



**Figure 1.1:** The height function on the upright torus (left) has four critical points. The subscript indicates the Morse index. Select trajectories of the gradient flow of the height function with respect to the metric $g_{\mathrm{up}}$ are indicated in blue. The unstable manifold of $y_1$ intersects the stable manifold of $x_1$ non-transversally. Tilting the torus (right) results in transversal intersections of all stable and unstable manifolds.[1]

$\mathbb{T}_{\mathrm{up}}$ induced by the standard metric on $\mathbb{R}^3$, the pair $(h_{\mathrm{up}}, g_{\mathrm{up}})$ is *not* Morse-Smale. The left half of Figure 1.1 illustrates select trajectories of the negative gradient of $h_{\mathrm{up}}$ with respect to the metric $g_{\mathrm{up}}$. Consider the intersection of the stable manifold of the critical point $x_1$ with the unstable manifold of $y_1$. This intersection is not transversal: It consists of all points in the blue inner circle excluding $x_1$ and $y_1$. This is a submanifold of $\mathbb{T}_{\mathrm{up}}$ diffeomorphic to the disjoint union of two open intervals.

Tilting the torus slightly with respect to the vertical axis, we obtain another height function $h_{\mathrm{tilt}}$ which is Morse on the tilted torus $\mathbb{T}_{\mathrm{tilt}}$. With the metric $g_{\mathrm{tilt}}$ induced by the standard metric on $\mathbb{R}^3$, the pair $(h_{\mathrm{tilt}}, g_{\mathrm{tilt}})$ *is* Morse-Smale, as illustrated in the right half of Figure 1.1.

Theorem 1.3.7 established the abundance of Morse functions on a compact manifold. The following theorem ensures the abundance of Morse-Smale functions on a compact Riemannian manifold.

**Theorem 1.3.12** (Kupka-Smale theorem)**.** *[Sma61] Let $(M, g)$ be a compact smooth*

[1]Figure by Sam Derbyshire, licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license. Modified and reprinted with permission.

*Riemannian manifold. The set consisting of all Morse-Smale functions on $M$ contains a countable intersection of open dense subsets of $C^\infty(M,\mathbb{R})$. Again, $C^\infty(M,\mathbb{R})$ is equipped with the Whitney $C^\infty$-topology.*

By the Baire category theorem, the intersection of countably many open dense subsets of a complete metric space is dense. Since for compact $M$, $C^\infty(M,\mathbb{R})$ is a complete metric space, the Kupka-Smale theorem implies that the set of Morse-Smale functions is dense in $C^\infty(M,\mathbb{R})$.

In the remainder of this section $M$ will be a compact smooth manifold and $(f,g)$ will be a Morse-Smale pair.

**Proposition 1.3.13.** *[BH04, p. 158, 175] For $p,q \in \mathrm{Crit}(f)$:*

- *If $|p|-|q|<0$, then $W(p,q)=\emptyset$.*
- *If $|p|-|q|=0$, then $W(p,q)=\{p\}$ if $p=q$, and $W(p,q)=\emptyset$ otherwise.*
- *If $|p|-|q|\geq 0$ and $W(p,q)\neq\emptyset$, then $W(p,q)$ is an embedded submanifold of dimension $|p|-|q|$. Moreover if $p$ and $q$ satisfy $|p|-|q|=1$, then $W(p,q)$ has finitely many components.*

This Proposition also implies that the identification of $\mathcal{M}(p,q)$ with $W(p,q)$ induces a manifold structure on $\mathcal{M}(p,q)$.

**Definition 1.3.14.** *For $p,q\in\mathrm{Crit}(f)$, $\mathbb{R}$ acts smoothly on $\mathcal{M}(p,q)$ by*

$$(t\cdot\gamma)(t')=\gamma(t+t') \text{ for all } t,t'\in\mathbb{R},$$

*where $\gamma\in\mathcal{M}(p,q)$. For $p\neq q$, this action is free [AD10, p. 36] and proper, and thus the quotient of $\mathcal{M}(p,q)$ by this action is a smooth manifold. We denote this quotient by $\mathcal{L}(p,q)$ and we call elements of $\mathcal{L}(p,q)$ **unparametrized Morse trajectories** from $p$ to $q$. By Proposition 1.3.13,*

$$\dim\mathcal{L}(p,q)=|p|-|q|-1$$

*for* $|p| - |q| \geq 0$.

**Remark 1.3.15.** For critical points $p$ and $q$ of relative Morse index one, that is $|p| - |q| = 1$, the orbits of the $\mathbb{R}$-action on $\mathcal{M}(p,q)$ are precisely the connected components of $\mathcal{M}(p,q)$. By Proposition 1.3.13, $\mathcal{M}(p,q)$ has finitely many connected components in this case, and hence $\mathcal{L}(p,q)$ is a compact 0-dimensional manifold.

We are now in a position to define the Morse homology. To circumvent issues of orientation, we use coefficients in $\mathbb{Z}_2$ rather than in $\mathbb{Z}$.

**Definition 1.3.16.** *Let* $\dim(M) = m$*, and let* $CM_k(M; f, g)$ *be the* $\mathbb{Z}_2$ *vector space with generators the critical points of* $f$ *of index* $k$ *(for* $k \notin \{0, \ldots, m\}$*, this is simply the* 0*-dimensional vector space). For each* $k \in \{1, \ldots, m\}$*, we define the linear map* $d_k : CM_k(M; f, g) \to CM_{k-1}(M; f, g)$ *by the following map on generators*

$$d_k p = \sum_{\substack{q \\ |p|-|q|=1}} \#_{\mathbb{Z}_2} \mathcal{L}(p,q) q. \tag{1.3.1}$$

*Here* $\#_{\mathbb{Z}_2}\mathcal{L}(p,q)$ *denotes the cardinality of* $\mathcal{L}(p,q)$ *modulo 2. We take* $d_k = 0$ *for* $k \notin \{1, \ldots, m\}$*. By Corollary 1.3.6 and Remark 1.3.15, the sum in* (1.3.1) *is well-defined. The linear map* $d_k$ *is called the* ***Morse boundary operator*** *(or sometimes the Morse-Smale-Witten boundary operator) and the pair* $(CM_*(M; f, g), d_*)$ *is called the* ***Morse chain complex*** *(or the Morse-Smale-Witten chain complex). The homology of this complex is denoted* $HM(M; f, g)$ *and is called the* ***Morse homology*** *of* $M$*.*

**Theorem 1.3.17** (Morse homology theorem)**.** *[BH04, p. 198] The Morse-Smale-Witten chain complex is in fact a chain complex. That is,* $d_k \circ d_{k-1} = 0$ *for all* $k$*. Moreover, the Morse homology of* $M$ *is isomorphic to the singular homology of* $M$ *with coefficients in* $\mathbb{Z}_2$*. In particular,* $HM(M; f, g)$ *is independent of the choice of Morse-Smale pair* $(f, g)$ *and indeed also of the smooth structure on* $M$*.*

**Example 1.3.18.** We return one last time to the example of the torus in $\mathbb{R}^3$. We compute the Morse homology of the tilted torus $\mathbb{T}_{\text{tilt}}$ using the Morse-Smale pair

$(h_{\text{tilt}}, g_{\text{tilt}})$. The Morse complex of $\mathbb{T}_{\text{tilt}}$ has the following form

$$0 \xrightarrow{d_3} \mathbb{Z}_2\langle x_2' \rangle \xrightarrow{d_2} \mathbb{Z}_2\langle x_1', y_1' \rangle \xrightarrow{d_1} \mathbb{Z}_2\langle x_0' \rangle \xrightarrow{d_0} 0.$$

As there are two Morse trajectories connecting the critical point $x_2'$ to both $x_1'$ and $y_1'$, $d_2(x_2') = 0$. Similarly, there are two Morse trajectories joining both $x_1'$ and $y_1'$ to $x_0'$. Thus $d_1(x_1') = d_1(y_1') = 0$. Finally, $d_0(x_0') = 0$ as well. Since the differential is trivial, we recover the usual homology of the torus with coefficients in $\mathbb{Z}_2$

$$HM_0(\mathbb{T}_{\text{tilt}}) \cong \mathbb{Z}_2, \quad HM_1(\mathbb{T}_{\text{tilt}}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad HM_2(\mathbb{T}_{\text{tilt}}) \cong \mathbb{Z}_2.$$

### 1.3.3 The Künneth theorem for Morse homology

In what follows, we will establish a relationship between the Morse homology of a product space and the Morse homology of its factors. Indeed, by the isomorphism of Theorem 1.3.17, the Künneth theorem for Morse homology follows immediately from the Künneth theorem for singular homology with coefficients in the field $\mathbb{Z}_2$. However it is worthwhile to prove the Künneth theorem for Morse homology directly, as the approach used can be adapted to prove the Künneth theorem for Lagrangian quantum homology.

The Morse-Smale pair we will use to compute the Morse homology of a product space will be the product of Morse-Smale pairs for the factor spaces. The following lemma establishes basic facts about the product of Morse-Smale pairs, including the fact that this product is again Morse-Smale.

**Lemma 1.3.19.** *Let $(M_1, g_1)$ and $(M_2, g_2)$ be compact Riemannian manifolds with Morse-Smale functions $f_1 \in C^\infty(M_1, \mathbb{R})$ and $f_2 \in C^\infty(M_2, \mathbb{R})$, and let*

$$p_1 : M_1 \times M_2 \to M_1 \text{ and } p_2 : M_1 \times M_2 \to M_2$$

*be the projection maps. Define the function,*

$$f_1 \oplus f_2 \in C^\infty(M_1 \times M_2, \mathbb{R}), \quad f_1 \oplus f_2 := {p_1}^* f_1 + {p_2}^* f_2,$$

*and the metric $g_1 \oplus g_2$ on $M_1 \times M_2$, $g_1 \oplus g_2 := {p_1}^* g_1 + {p_2}^* g_2$. Then the following statements hold:*

1. *The function $f_1 \oplus f_2$ is Morse and the set of critical points of $f_1 \oplus f_2$ is*

$$\mathrm{Crit}(f_1 \oplus f_2) = \mathrm{Crit}(f_1) \times \mathrm{Crit}(f_2).$$

   *For $(p,q) \in \mathrm{Crit}(f_1 \oplus f_2)$, $|(p,q)| = |p| + |q|$.*

2. *Under the isomorphism $T_{(p,q)}(M_1 \times M_2) \cong T_p M_1 \oplus T_q M_2$,*

$$\nabla_{(p,q)}(f_1 \oplus f_2) = (\nabla_p f_1, \nabla_q f_2)\,.$$

3. *Let $\Phi_1 : \mathbb{R} \times M_1 \to M_1$ and $\Phi_2 : \mathbb{R} \times M_2 \to M_2$ be the flows of $-\nabla f_1$ and $-\nabla f_2$. Then the flow of $-\nabla(f_1 \oplus f_2)$ is given by*

$$\Phi : \mathbb{R} \times (M_1 \times M_2) \to M_1 \times M_2, \quad (t,(x,y)) \mapsto (\Phi_1(x), \Phi_2(y)).$$

4. *The unstable and stable manifolds of $f_1 \oplus f_2$ satisfy*

$$W^u((p,q)) = W^u(p) \times W^u(q) \text{ and } W^s((p,q)) = W^s(p) \times W^s(q),$$

   *for $(p,q) \in \mathrm{Crit}(f_1 \oplus f_2)$. The connecting orbits of $f_1 \oplus f_2$ are given by*

$$\mathcal{M}((p,q),(x,y)) = \mathcal{M}(p,x) \times \mathcal{M}(q,y),$$

   *for $(p,q),(x,y) \in \mathrm{Crit}(f_1 \oplus f_2)$.*

5. *The product pair $(f_1 \oplus f_2, g_1 \oplus g_2)$ is Morse-Smale.*

**Proof:** For $x \in M_1$ and $y \in M_2$, the isomorphism between $T_{(x,y)}(M_1 \times M_2)$ and $T_x M_1 \oplus T_y M_2$ is given by

$$I_{(x,y)} : T_{(x,y)}(M_1 \times M_2) \to T_x M_1 \oplus T_y M_2,\ v \mapsto \left((p_{1*})_{(x,y)} v, (p_{2*})_{(x,y)} v\right),$$

for $v \in T_{(x,y)}(M_1 \times M_2)$. The isomorphism $I_{(x,y)}$ induces isomorphisms

$$(I_{(x,y)})^* : (T_xM_1 \oplus T_yM_2)^* \to T^*_{(x,y)}(M_1 \times M_2),$$

and

$$(I_{(x,y)})^* : \mathrm{Sym}^2((T_xM_1 \oplus T_yM_2)^*) \to \mathrm{Sym}^2 T^*_{(x,y)}(M_1 \times M_2).$$

1. For $x \in M_1$ and $y \in M_2$,

$$\begin{aligned}(d(f_1 \oplus f_2))_{(x,y)} &= (d(p_1^* f_1 + p_2^* f_2))_{(x,y)}\,, = ({p_1}^*)_x (df_1)_x + ({p_2}^*)_y (df_2)_y, \\ &= (I_{(x,y)})^*((df_1)_x, (df_2)_y),\end{aligned}$$

where we have implicitly used the identification $(T_xM_1 \oplus T_yM_2)^* \cong T^*_xM_1 \oplus T^*_yM_2$ in the last line. It follows that $(x, y)$ is a critical point of $f_1 \oplus f_2$ if and only if $x$ is a critical point of $f_1$ and $y$ is a critical point of $f_2$. For $(p, q) \in \mathrm{Crit}(f_1 \oplus f_2)$, define

$$\begin{aligned}&\mathrm{Hess}_p(f_1) \times \mathrm{Hess}_q(f_2) : T_pM_1 \oplus T_qM_2 \times T_pM_1 \oplus T_qM_2 \to \mathbb{R}, \\ &((\xi, \zeta), (\xi', \zeta')) \mapsto \mathrm{Hess}_p(f_1)(\xi, \xi') + \mathrm{Hess}_q(f_2)(\zeta, \zeta'),\end{aligned}$$

for $\xi, \xi' \in T_pM_1$ and $\zeta, \zeta' \in T_qM_2$. Notice that the number of negative eigenvalues of $\mathrm{Hess}_p(f_1) \times \mathrm{Hess}_q(f_2)$ is $|p| + |q|$. Then from

$$\begin{aligned}\mathrm{Hess}_{(p,q)}(f_1 \oplus f_2)(v, w) &= (\mathrm{Hess}_p(f_1) \times \mathrm{Hess}_q(f_2))\,(I_{(p,q)}(v), I_{(p,q)}(w)) \\ &= (I_{(p,q)})^*(\mathrm{Hess}_p(f_1) \times \mathrm{Hess}_q(f_2))(v, w),\end{aligned}$$

for all $v, w \in T_{(p,q)}(M_1 \times M_2)$, it follows that $(p, q)$ is a non-degenerate critical point for $f_1 \oplus f_2$ if and only if $p$ is a non-degenerate critical point of $f_1$ and $q$ is a non-degenerate critical point of $f_2$. Moreover, we see that $|(p, q)| = |p| + |q|$.

2. Consider the inner product

$$(g_1 \times g_2)_{(x,y)} : T_xM_1 \oplus T_yM_2 \times T_xM_1 \oplus T_yM_2 \to \mathbb{R},$$

$$(g_1 \times g_2)_{(x,y)}((\xi, \zeta), (\xi', \zeta')) = (g_1)_x(\xi, \xi') + (g_2)_y(\zeta, \zeta'),$$

for $\xi, \xi' \in T_x M_1$ and $\zeta, \zeta' \in T_y M_2$. Similarly we define

$$(df_1 \times df_2)_{(x,y)} : T_x M_1 \oplus T_y M_2 \to \mathbb{R}.$$

We have,

$$(df_1 \times df_2)_{(x,y)} = (g_1 \times g_2)_{(x,y)}((\nabla_x f_1, \nabla_y f_2),\ \cdot\ ),$$

and therefore

$$\begin{aligned}
(d(f_1 \oplus f_2))_{(x,y)} &= (I_{(x,y)})^*((df_1 \times df_2)_{(x,y)}),\\
&= (I_{(x,y)})^* \left[(g_1 \times g_2)_{(x,y)}((\nabla_x f_1, \nabla_y f_2),\ \cdot\ )\right],\\
&= (I_{(x,y)})^*(g_1 \times g_2)_{(x,y)}({I_{(x,y)}}^{-1}(\nabla_x f_1, \nabla_y f_2),\ \cdot\ ),\\
&= (g_1 \oplus g_2)_{(x,y)}({I_{(x,y)}}^{-1}(\nabla_x f_1, \nabla_y f_2),\ \cdot\ ).
\end{aligned}$$

From this we see that $I_{(x,y)}(\nabla_{(x,y)}(f_1 \oplus f_2)) = (\nabla_x f_1, \nabla_y f_2)$.

3. This follows immediately from

$$\begin{aligned}
I_{((\Phi_1)_t(x),(\Phi_2)_t(y))}\left(\frac{d}{dt}\left(\Phi_1(x), \Phi_2(y)\right)\right) &= \left(\frac{d}{dt}\Phi_1(x), \frac{d}{dt}\Phi_2(y)\right)\\
&= \left(-\nabla_{(\Phi_1)_t(x)} f_1, -\nabla_{(\Phi_2)_t(y)} f_2\right),
\end{aligned}$$

for $x \in M_1$ and $y \in M_2$.

4. From Part 3, we conclude that

$$W^u((p,q)) = W^u(p) \times W^u(q) \text{ and } W^s((p,q)) = W^s(p) \times W^s(q),$$

for $(p,q) \in \mathrm{Crit}(f_1 \oplus f_2)$. This immediately implies

$$\mathcal{M}((p,q),(x,y)) = \mathcal{M}(p,x) \times \mathcal{M}(q,y),$$

5. It follows from Part 4 that

$$I_{(x,y)}(T_{(x,y)}W^u((p,q))) = T_xW^u(p) \times T_yW^u(q)$$

and

$$I_{x'\times y'}(T_{x'\times y'}W^s((p,q))) = T_{x'}W^s(p) \times T_{y'}W^s(q),$$

for $(x,y) \in W^u((p,q))$ and $(x',y') \in W^s((p,q))$.

Then for $(p,q),(p',q') \in \mathrm{Crit}(f_1 \oplus f_2)$ such that $\mathcal{M}((p,q),(p',q')) \neq \emptyset$, we have

$$\begin{aligned} I_{(x,y)}(T_{(x,y)}(M_1 \times M_2)) &= T_xM_1 \oplus T_yM_2 \\ &= (T_xW^u(p) + T_xW^s(p')) \oplus (T_yW^u(q) + T_yW^s(q')) \\ &= I_{(x,y)}(T_{(x,y)}W^u(p,q) + T_{(x,y)}W^s(p',q')), \end{aligned}$$

for $(x,y) \in \mathcal{M}((p,q),(p',q'))$. From this it is clear that $W^u(p,q)$ and $W^s(p',q')$ intersect transversally. ∎

To prove the Künneth theorem for Morse homology, we will apply the algebraic Künneth theorem for tensor products of chain complexes. We will consider the chain complex $CM(M_1; f_1, g_1) \otimes_{\mathbb{Z}_2} CM(M_2; f_2, g_2)$ with the aim of relating the Morse homology of the product space $M_1 \times M_2$ to the homology of this chain complex. Notice that in this case, as we use coefficients in $\mathbb{Z}_2$, the boundary operator for $CM(M_1; f_1, g_1) \otimes_{\mathbb{Z}_2} CM(M_2; f_2, g_2)$ satisfies

$$d(p \otimes q) = dp \otimes q + p \otimes dq.$$

The homology of this chain complex is isomorphic as a graded $\mathbb{Z}_2$ vector space to

$$HM(M_1; f_1, g_1) \otimes_{\mathbb{Z}_2} HM(M_2; f_2, g_2).$$

This follows from the special case of the algebraic Künneth theorem for tensor products of chain complexes over a field.

**Theorem 1.3.20.** *[Algebraic Künneth theorem (special case)][May99, p. 132] Let $X$ and $Y$ be chain complexes over a field $R$. Then*

$$H_*(X) \otimes_R H_*(Y) \cong H_*(X \otimes_R Y).$$

Note that Theorem 1.3.20 is a simple corollary of the Künneth theorem for differential graded modules (Theorem 1.2.16) obtained by taking the graded ring to be a field concentrated in degree zero. The next lemma relates the differential in the chain complex $CM(M_1 \times M_2; f_1 \oplus f_2, g_1 \oplus g_2)$ to the differentials for $CM(M_1; f_1, g_1)$ and $CM(M_2; f_2, g_2)$.

**Lemma 1.3.21.** *Let $(M_1, g_1)$, $(M_2, g_2)$, $f_1$ and $f_2$ be as in Lemma 1.3.19. Define a bilinear map via the following map on pairs of generators*

$$\begin{aligned} \times : CM_*(M_1; f_1, g_1) \times CM_*(M_2; f_2, g_2) \to CM_*(M_1 \times M_2; f_1 \oplus f_2, g_1 \oplus g_2), \\ (p, q) \mapsto (p, q), \end{aligned} \tag{1.3.2}$$

*for $p \in \operatorname{Crit}(f_1)$, $q \in \operatorname{Crit}(f_2)$. Then for $(p, q) \in \operatorname{Crit}(f_1 \oplus f_2)$,*

$$d((p, q)) = dp \times q + p \times dq.$$

**Proof:**

$$\begin{aligned} d((p, q)) &= \sum_{\substack{(x,y) \\ |(p,q)|-|(x,y)|=1}} \#_{\mathbb{Z}_2}\mathcal{L}((p, q), (x, y))(x, y) \\ &= \sum_{\substack{(x,y) \\ |p|+|q|-|x|-|y|=1}} \#_{\mathbb{Z}_2}\mathcal{L}((p, q), (x, y))(x, y). \end{aligned}$$

Here the sum is taken over critical points $(x, y) \in \operatorname{Crit}(f_1 \oplus f_2)$. Suppose $(x, y) \in \operatorname{Crit}(f_1 \oplus f_2)$ satisfies $|p| + |q| - |x| - |y| = 1$ and let $(x_0, y_0) \in \mathcal{M}((p, q), (x, y))$. Then $x_0 \in \mathcal{M}(p, x)$ and $y_0 \in \mathcal{M}(q, y)$.

If $|p| - |x| \geq 2$, then $|q| - |y| \leq -1$, implying $\mathcal{M}(q, y) = \emptyset$ by Theorem 1.3.13. This contradicts the existence of $y_0 \in \mathcal{M}(q, y)$. We conclude that $\mathcal{M}((p, q), (x, y)) = \emptyset$

for $|p| - |x| \geq 2$. Similarly $\mathcal{M}((p,q),(x,y)) = \emptyset$ for $|q| - |y| \geq 2$. We obtain

$$\begin{aligned} d((p,q)) &= \sum_{\substack{x,y \\ |p|-|x|=1 \\ |q|-|y|=0}} \#_{\mathbb{Z}_2}\mathcal{L}((p,q),(x,y))(x,y) + \sum_{\substack{x,y \\ |p|-|x|=0 \\ |q|-|y|=1}} \#_{\mathbb{Z}_2}\mathcal{L}((p,q),(x,y))(x,y) \\ &= \sum_{\substack{x \\ |p|-|x|=1}} \#_{\mathbb{Z}_2}\mathcal{L}((p,q),(x,q))(x,q) + \sum_{\substack{y \\ |q|-|y|=1}} \#_{\mathbb{Z}_2}\mathcal{L}((p,q),(p,y))(p,y) \\ &= \sum_{\substack{x \\ |p|-|x|=1}} \#_{\mathbb{Z}_2}\mathcal{L}(p,x)(x,q) + \sum_{\substack{y \\ |q|-|y|=1}} \#_{\mathbb{Z}_2}\mathcal{L}(q,y)(p,y) \\ &= dp \times q + p \times dq. \end{aligned}$$

Here we have used the fact that from Part 4 of Lemma 1.3.19 that

$$\mathcal{M}((p,q),(x,y)) = \mathcal{M}(p,x) \times \mathcal{M}(q,y).$$

We have also applied the second part of Proposition 1.3.13 for $|q| = |y|$ and $|p| = |x|$. ∎

**Proposition 1.3.22** (Künneth theorem for Morse homology)**.** *Let* $(M_1, g_1)$ *and* $(M_2, g_2)$ *be compact Riemannian manifolds with Morse-Smale functions* $f_1 \in C^\infty(M_1, \mathbb{R})$ *and* $f_2 \in C^\infty(M_2, \mathbb{R})$. *Then the linear map*

$$\times : CM_*(M_1; f_1, g_1) \otimes_{\mathbb{Z}_2} CM_*(M_2; f_2, g_2) \to CM_*(M_1 \times M_2; f_1 \oplus f_2, g_1 \oplus g_2),$$

*which is induced by the bilinear map* $\times$ *defined in Equation* (1.3.2) *of Lemma 1.3.21, induces an isomorphism in homology*

$$HM_*(M_1; f_1, g_1) \otimes_{\mathbb{Z}_2} HM_*(M_2; f_2, g_2) \cong HM_*(M_1 \times M_2; f_1 \oplus f_2, g_1 \oplus g_2).$$

**Proof:** We will show that $\times$ is an isomorphism of chain complexes. The result then follows from Theorem 1.3.20. To do this, we must first check that it commutes with the differential. This amounts to showing that for $p \in \mathrm{Crit}(f_1)$, $q \in \mathrm{Crit}(f_2)$,

$$d((p,q)) = dp \times q + p \times dq, \tag{1.3.3}$$

as we have proved in Lemma 1.3.21. The inverse of $\times$ is given by

$$CM(M_1 \times M_2; f_1 \oplus f_2, g_1 \oplus g_2) \to CM_*(M_1; f_1, g_1) \otimes CM_*(M_2; f_2, g_2),$$
$$(p, q) \mapsto p \otimes q.$$

By the identity in Equation 1.3.3, this map is also a homomorphism of chain complexes. ∎

# Chapter 2

# Linear symplectic geometry

Here we describe the geometry of vector spaces equipped with a non-degenerate skew-symmetric bilinear form. The relationship between linear symplectic geometry and global symplectic geometry is more pronounced than the relationship between the linear and global theory in Riemannian geometry, for example. Many results in global symplectic geometry, such as Darboux's theorem (Theorem 3.2.5) and the contractibility of the spaces of compatible and tame almost complex structures (Theorem 3.3.2), also have linear versions. Even Gromov's celebrated non-squeezing theorem has a linear version, although we do not treat it here.

We also discuss complex structures on symplectic vector spaces, and Lagrangian subspaces of symplectic vector spaces. These are the linear counterparts of two important concepts in the global theory: almost complex structures on symplectic manifolds, and Lagrangian submanifolds of symplectic manifolds.

The last section of this chapter is devoted to the Maslov indices for the linear symplectic group and for the Lagrangian Grassmannian. These Maslov indices are isomorphisms between the fundamental groups of these spaces and the integers. Our description in Chapter 3 of the first Chern class of a symplectic vector bundle relies on the definition of the Maslov index for the linear symplectic group. The Maslov index for the Lagrangian Grassmannian forms the basis for defining the Maslov index of a Lagrangian submanifold, an important ingredient in the description of the pearl complex.

All vector spaces in this chapter will be finite-dimensional.

## 2.1 Symplectic vector spaces

**Definition 2.1.1.** *[MS98, p. 38] A* ***symplectic vector space*** *is a real vector space $V$ equipped with a non-degenerate, skew-symmetric bilinear form*

$$\Omega : V \times V \to \mathbb{R}.$$

*Here non-degeneracy means that if for some $v_1 \in V$, we have $\Omega(v_1, v_2) = 0$ for all $v_2 \in V$, then $v_1 = 0$. We denote this symplectic vector space by $(V, \Omega)$. The bilinear form $\Omega$ is called a* ***bilinear symplectic form****. The set of all bilinear symplectic forms on $V$ is simply the subset of $\Lambda^2 V^*$ consisting of those elements which are non-degenerate. We denote this subset by $\Lambda^2_+ V^*$*[1]*. A* ***symplectic subspace*** *of $(V, \Omega)$ is a linear subspace $U \subset V$ such that $(U, \Omega|_U)$ is again symplectic. If $W$ is another vector space and $R : W \to V$ is a linear isomorphism, then $R$ induces a bilinear symplectic form on $W$, $R^*W$, called the* ***pullback*** *of $\Omega$ and defined by*

$$R^*\Omega(w_1, w_2) := \Omega(Rw_1, Rw_2) \text{ for all } w_1, w_2 \in W.$$

*A linear isomorphism $S : Y \to V$, where $(Y, \Omega')$ is also a symplectic vector space, is called a* ***linear symplectomorphism*** *if $S^*\Omega = \Omega'$. In this case, the symplectic vector spaces $(Y, \Omega')$ and $(V, \Omega)$ are said to be* ***symplectomorphic****. Notice that the map $R$ above is trivially a symplectomorphism from $(W, R^*\Omega)$ to $(V, \Omega)$.*

[1] The set $\Lambda^2_+ V^*$ is not a vector subspace of $\Lambda^2 V^*$ (it does not even contain the zero vector), but it is an open submanifold of $\Lambda^2 V^*$: A choice of basis for $V$ allows us to define the continuous map $\det : \Lambda^2_+ V^* \to \mathbb{R}$ and $\Lambda^2_+ V^* = \det^{-1}(\mathbb{R} \setminus \{0\})$.

**Example 2.1.2.** The most important example of a symplectic vector space is $\mathbb{R}^{2n}$ with the bilinear symplectic form $\Omega_0$ determined by the matrix

$$J_0 = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} \in M(2n, \mathbb{R}).$$

Explicitly, $\Omega_0$ is defined by

$$\Omega_0(v, w) := -v^T J_0 w$$

for all vectors $v, w \in \mathbb{R}^{2n}$.

**Remark 2.1.3.**

1. Symplectic vector spaces are always even-dimensional. This follows easily from the fact that skew-symmetric matrices always have even rank. Conversely, every even-dimensional vector space has a bilinear symplectic form: we can simply take the pullback of $\Omega_0$ by any linear isomorphism from the vector space to $\mathbb{R}^{2n}$.

2. Any symplectic vector space $V$ of dimension $n$ has a canonical orientation determined by the top form $\wedge^{n/2}\Omega \in \Lambda^n(V)$. Indeed, a skew-symmetric bilinear form is non-degenerate if and only if its top power is a volume form.

The following proposition establishes the functoriality of the assignment of the space of bilinear symplectic forms to a vector space. This in turn implies that being symplectomorphic is an equivalence relation on the class of symplectic vector spaces (Corollary 2.1.5) and that the symplectomorphisms of a symplectic vector space form a group under composition (Remark 2.1.6).

**Proposition 2.1.4.** *Let $\mathcal{E}$ be the category whose objects are even-dimensional vector spaces and whose morphisms are linear isomorphisms and let $\mathcal{M}$ be the category whose objects are smooth manifolds and whose morphisms are diffeomorphisms. Then the maps*

$$\mathrm{Obj}(\mathcal{E}) \to \mathrm{Obj}(\mathcal{M}), \quad V \mapsto \Lambda^2_+ V^*,$$

$$\mathrm{Mor}(\mathcal{E}) \to \mathrm{Mor}(\mathcal{M}), \quad T \mapsto T^*,$$

*define a contravariant functor from $\mathcal{E}$ to $\mathcal{M}$.*

**Corollary 2.1.5.** *Being linearly symplectomorphic is an equivalence relation on the class of symplectic vector spaces.*

**Remark 2.1.6.** The set of symplectomorphisms from a symplectic vector space $(V, \Omega)$ to itself forms a group under composition, which we denote by $\mathrm{Sp}(V, \Omega)$.

From elementary linear algebra, we know that real vector spaces are classified by their dimension, and that any choice of bases for two real vector spaces of the same dimension determines an explicit isomorphism between the vector spaces. A similar statement holds for symplectic vector spaces, and relies only on the definition of a symplectic basis and Theorem 2.1.9 which establishes the existence of symplectic bases.

**Definition 2.1.7.** *[MS98, p. 39] Let $(V, \Omega)$ be a symplectic vector space and let*

$$\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$$

*be a basis of $V$. This basis is called a **symplectic basis** if it satisfies*

$$\Omega(v_j, v_k) = 0 = \Omega(w_j, w_k) \text{ and } \Omega(v_j, w_k) = \delta_{jk} \text{ for all } j, k \in \{1, \ldots, n\}.$$

**Example 2.1.8.** Let $\{e_1, \ldots, e_{2n}\}$ be the standard basis for $\mathbb{R}^{2n}$ and define $f_j := e_{n+j}$ for $j = 1, \ldots, n$. Then the set $\{e_j, f_j |\ j = 1, \ldots, n\}$ is a symplectic basis for $(\mathbb{R}^{2n}, \Omega_0)$.

**Theorem 2.1.9.** *[MS98, p. 39] Every symplectic vector space admits a symplectic basis.*

**Remark 2.1.10.** Clearly a linear isomorphism which caries a symplectic basis onto a symplectic basis is a symplectomorphism. Hence this theorem implies that a symplectic vector space is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$ if and only if it is of dimension $2n$. Therefore we can verify properties of arbitrary symplectic vector spaces by considering only the space $(\mathbb{R}^{2n}, \Omega_0)$.

## 2.2 Complex structures

An important feature of symplectic manifolds that has been exploited very successfully in modern symplectic topology is the existence of almost complex structures. As we will see in the following chapter, such a structure allows one to define the concept of a pseudoholomorphic map from a Riemann surface into the manifold[2]. These 'pseudoholomorphic curves' are at the heart of many important constructions in symplectic topology. Examples include Gromov-Witten invariants, Floer homology, and indeed the Lagrangian quantum homology we describe in Chapter 4.

An almost complex structure on a manifold is nothing more than a smooth field of complex structures on the tangent spaces. Here we review the theory of complex structures on vector spaces as well as the interaction between complex structures and bilinear symplectic forms.

**Definition 2.2.1.** *[MS98, p. 61] A* ***complex structure*** *on a real vector space $V$ is an endomorphism $J$ of $V$ satisfying*

$$J^2 = -\mathrm{Id}.$$

*The vector space $V$ equipped with the complex structure $J$ is denoted by $(V, J)$. If $W$ is another vector space with a complex structure $J'$, a linear map $A \in \mathrm{Hom}(V, W)$ is called* ***complex linear*** *if $J' \circ A = A \circ J$. The set of complex linear maps is a vector subspace of $\mathrm{Hom}(V, W)$, which we denote by $\mathrm{Hom}((V, J), (W, J'))$. We denote the space of complex linear endomorphisms of $V$ by $\mathrm{End}(V, J)$. The complex linear isomorphisms of $V$ form a group, which we denote by $\mathrm{GL}(V, J)$.*

**Remark 2.2.2.** Any $n$-dimensional complex vector space, when viewed as a $2n$-dimensional real vector space, has the complex structure given by multiplication by the imaginary unit $i$.

[2] Here the 'pseudo' in 'pseudoholomorphic' refers to the fact that the almost complex structure need not be integrable.

Notice that if $J$ is a complex structure on an $n$-dimensional vector space $V$, we have $(\det(J))^2 = (-1)^n$ and it follows that $n$ must be even. The complex structure $J$ provides $V$ with the structure of an $\frac{n}{2}$-dimensional complex vector space, where we define

$$(a + ib)v := av + bJv.$$

The identity $J^2 = -\mathrm{Id}$ ensures that this scalar multiplication is compatible with the multiplication in $\mathbb{C}$, i.e.

$$z_1(z_2v) = (z_1z_2)v \text{ for all } z_1, z_2 \in C, v \in V.$$

A linear map $A : V \to W$, where $W$ is a vector space with a complex structure $J'$, is complex linear in the sense of Definition 2.2.1 if and only if it induces a linear map between the complex vector spaces determined by the complex structures $J$ and $J'$ on $V$ and $W$. The real vector space $\mathrm{Hom}((V, J), (W, J'))$ also possesses a complex structure $J_{(V,W)}$ defined by

$$J_{(V,W)} : \mathrm{Hom}((V, J), (W, J')) \to \mathrm{Hom}((V, J), (W, J')), \quad A \mapsto J' \circ A.$$

With this complex structure, $\mathrm{Hom}((V, J), (W, J'))$ becomes a complex vector space.

**Example 2.2.3.** The prototypical example of a vector space equipped with a complex structure is $\mathbb{R}^{2n}$ with the complex structure given by left multiplication by $J_0 \in M(2n, \mathbb{R})$. This complex structure endows $\mathbb{R}^{2n}$ with the structure of an $n$-dimensional complex vector space and the map

$$\mathbb{C}^n \to \mathbb{R}^{2n}, \quad a + ib \mapsto (a, b), \quad a, b \in \mathbb{R}^{2n},$$

is a complex linear isomorphism. A matrix $X + iY$ representing an endomorphism of $\mathbb{C}^n$, where $X, Y \in M(n, \mathbb{R})$, induces an endomorphism of $\mathbb{R}^{2n}$ represented by the matrix

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M(2n, \mathbb{R}). \tag{2.2.1}$$

On the other hand, a matrix $A \in M(2n, \mathbb{R})$ of a real linear endomorphism of $\mathbb{R}^{2n}$ represents a complex linear endomorphism of $\mathbb{C}^n$ if and only if it commutes with $J_0$. This condition is equivalent to $A$ being of the form shown in (2.2.1). Therefore we have a bijection

$$\Theta : \mathrm{End}(\mathbb{C}^n) \to \mathrm{End}(\mathbb{R}^{2n}, J_0), \quad X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M(2n, \mathbb{R}).$$

It is not hard to see that the map $\Theta$ is complex linear, where the complex structure on $\mathrm{End}(\mathbb{C}^n)$ is multiplication by $i$ and the complex structure on $\mathrm{End}(\mathbb{R}^{2n}, J_0)$ is $(J_0)_{(\mathbb{R}^{2n},\mathbb{R}^{2n})}$ as defined above. Morever, $\Theta$ preserves matrix multiplication and therefore restricts to an isomorphism from any group contained in $\mathrm{End}(\mathbb{C}^n)$ onto its image. We consider the image under $\Theta$ of the following groups

$$\begin{aligned} \mathrm{GL}(n, \mathbb{C}) &= \{A \in M(n, \mathbb{C}) |\ \det(A) \neq 0\} \\ \mathrm{U}(n) &= \{A \in M(n, \mathbb{C}) |\ A^\dagger A = \mathrm{Id}_n\} \end{aligned}$$

We have

$$\begin{aligned} \Theta(\mathrm{GL}(n, \mathbb{C})) &= \{B \in M(2n, \mathbb{R}) |\ J_0 B = B J_0 \text{ and } \det(B) \neq 0\} \\ &= \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \middle|\ X, Y \in \mathrm{M}(n, \mathbb{R}), (\det X)(\det Y) \neq 0 \right\} \end{aligned}$$

and

$$\begin{aligned} \Theta(\mathrm{U}(n)) &= \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \middle|\ \begin{pmatrix} X^T & Y^T \\ -Y^T & X^T \end{pmatrix} \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = \mathrm{Id}_{2n} \right\} \\ &= \{B \in \Theta(\mathrm{GL}(n, \mathbb{C})) |\ B^T B = \mathrm{Id}_{2n}\} \end{aligned}$$

As the groups $\mathrm{GL}(n, \mathbb{C}) \subset M(n, \mathbb{C})$ and $\Theta(\mathrm{GL}(n, \mathbb{C})) \subset M(2n, \mathbb{R})$ are isomorphic, we will refer to $\Theta(\mathrm{GL}(n, \mathbb{C}))$ as $\mathrm{GL}(n, \mathbb{C})$. Similarly we will refer to $\Theta(U(n))$ as $U(n)$. It should be clear from the context whether we are referring to a subgroup of $M(n,\mathbb{C})$ or to one of $M(2n,\mathbb{R})$.

**Definition 2.2.4.** *If $(V,\Omega)$ is a symplectic vector space, a complex structure $J$ on $V$ is called $\Omega$-tame if*

$$\Omega(v,Jv) > 0 \text{ for all non-zero } v \in V.$$

*$J$ is called $\Omega$-compatible if it is $\Omega$-tame and it is a linear symplectomorphism of the symplectic vector space $(V,\Omega)$, that is*

$$\Omega(Jv,Jw) = \Omega(v,w) \text{ for all } v,w \in V.$$

*We denote the set of all $\Omega$-tame complex structures on $V$ by $\mathcal{J}_\tau(V,\Omega)$ and the set of all $\Omega$-compatible complex structures on $V$ as $\mathcal{J}(V,\Omega)$. Both sets inherit a topology as subsets of* $\mathrm{End}(V)$.

**Remark 2.2.5.** An $\Omega$-tame complex structure $J$ gives rise to an inner product $G_J$ on $V$ defined by

$$G_J(v,w) = \frac{1}{2}(\Omega(v,Jw) - \Omega(Jv,w)) \quad v,w \in V.$$

If $J$ is $\Omega$-compatible, then $G_J$ has the simpler form

$$G_J(v,w) = \Omega(v,Jw).$$

In this case $J$ is skew-adjoint with respect to $G_J$, since

$$G_J(-Jv,w) = \Omega(-Jv,Jw) = \Omega(v,J^2w) = G_J(v,Jw).$$

The standard complex structure $J_0$ on $\mathbb{R}^{2n}$ is compatible with the standard bilinear symplectic form $\Omega_0$ (see Examples 2.1.2 and 2.2.3). The existence of symplectic bases therefore implies the existence of a compatible (and thus tame) complex structure on arbitrary symplectic vector spaces. Therefore $\mathcal{J}(V,\Omega)$ and $\mathcal{J}_\tau(V,\Omega)$ are non-empty, but more can be said of these spaces.

**Proposition 2.2.6.** *Both $\mathcal{J}(V,\Omega)$ and $\mathcal{J}_\tau(V,\Omega)$ are non-empty and, with the subspace topology inherited from* $\mathrm{End}(V)$*, these spaces are also contractible.*

The global counterparts of these results are important for showing that various symplectic invariants defined using a choice of almost complex structure are independent of the choice. We will prove Proposition 2.2.6 in the next section for $\mathcal{J}(V,\Omega)$. We refer the reader to [MS98, p. 65] for a proof of this proposition for $\mathcal{J}_\tau(V,\Omega)$.

## 2.3 The linear symplectic group

By definition, a matrix $\Psi \in M(2n,\mathbb{R})$ represents a linear symplectomorphism of $(\mathbb{R}^{2n},\Omega_0)$ with respect to the standard basis if and only if

$$\Omega_0(\Psi v,\Psi w) = \Omega_0(v,w) \text{ for all } v,w \in \mathbb{R}^{2n}. \tag{2.3.1}$$

Notice that the requirement that $\Psi$ be a linear isomorphism is implicitly satisfied. By definition of $\Omega_0$, the condition in Equation (2.3.1) is satisfied if and only if

$$\Psi^T J_0 \Psi = J_0. \tag{2.3.2}$$

**Definition 2.3.1.** *Matrices satisfying the condition in Equation* (2.3.2) *are called* ***symplectic*** *and the set of such matrices, which by Corollary 2.1.6 is a group under matrix multiplication, is denoted by* $\mathrm{Sp}(2n)$*.*

**Remark 2.3.2.** Notice that the group $\mathrm{Sp}(2n)$ is closed under transposition: Using the fact that $J_0^2 = -\mathrm{Id}$, the condition in Equation (2.3.2) is equivalent to $\Psi^T = -J_0\Psi^{-1}J_0$. We then have

$$(\Psi^T)^T J_0 \Psi^T = \Psi J_0 \Psi^T = \Psi J_0(-J_0\Psi^{-1}J_0) = J_0.$$

The following proposition establishes a relationship between four subgroups of $\mathrm{GL}(2n,\mathbb{R})$: $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{Sp}(2n)$, $\mathrm{O}(2n)$ and $\mathrm{U}(n)$.

**Proposition 2.3.3.** *[MS98, p. 44]*

$$\mathrm{GL}(n,\mathbb{C}) \cap \mathrm{Sp}(2n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{GL}(n,\mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$$

**Proof:** By definition, we have

$$\begin{aligned} \mathrm{O}(2n) &= \{A \in M(2n,\mathbb{R}) |\ A^T A = \mathrm{Id}\}, \\ \mathrm{Sp}(2n) &= \{A \in M(2n,\mathbb{R}) |\ A^T J_0 A = J_0\}, \end{aligned}$$

and as explained in Example 2.2.3,

$$\begin{aligned} \mathrm{GL}(n,\mathbb{C}) &= \{A \in M(2n,\mathbb{R}) |\ \det(A) \neq 0 \text{ and } AJ_0 = J_0 A\}, \\ \mathrm{U}(n) &= \{A \in M(2n,\mathbb{R}) |\ AJ_0 = J_0 A \text{ and } A^T A = \mathrm{Id}\}. \end{aligned}$$

From this it is clear that $\mathrm{GL}(n,\mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$. We now prove the leftmost equality in the statement of the proposition.

$$\begin{aligned} A \in \mathrm{GL}(n,\mathbb{C}) \cap \mathrm{Sp}(2n) &\iff AJ_0 = J_0 A \text{ and } A^T J_0 A = J_0 \\ &\iff A^T A J_0 = J_0 \text{ and } A^T J_0 A = J_0 \\ &\iff A^T A = \mathrm{Id} \text{ and } A^T J_0 A = J_0 \\ &\iff A \in \mathrm{O}(2n) \cap \mathrm{Sp}(2n) \end{aligned}$$

Lastly we prove the centre equality.

$$\begin{aligned} A \in \mathrm{O}(2n) \cap \mathrm{Sp}(2n) &\iff A^T A = \mathrm{Id} \text{ and } A^T J_0 A = J_0 \\ &\iff A^T A = \mathrm{Id} \text{ and } J_0 A = AJ_0 \\ &\iff A \in \mathrm{GL}(n,\mathbb{C}) \cap \mathrm{O}(2n) \end{aligned}$$

∎

If a symplectic matrix is symmetric and positive definite, real powers of the matrix are defined. We see next that these powers are also symplectic, which will be important for showing that $\mathrm{Sp}(2n)$ deformation retracts on $\mathrm{U}(n)$ (Theorem 2.3.6) and that the space of compatible complex structures on a symplectic vector space is contractible (Proposition 2.2.6).

**Proposition 2.3.4.** *[MS98, p. 45] Suppose $P \in \mathrm{Sp}(2n)$ is symmetric and positive definite. Then for any real number $\alpha \in \mathbb{R}$, $P^\alpha \in \mathrm{Sp}(2n)$.*

**Proof:** We choose a basis of $\mathbb{R}^{2n}$ consisting of eigenvectors $v_{\lambda_i}$, $i \in \{1, \dots, 2n\}$, of $P$, with corresponding eigenvalues $\lambda_i$. Then each $v_{\lambda_i}$ is also an eigenvector of $P^\alpha$ with eigenvalue $\lambda_i^\alpha$. It is enough to check that

$$\Omega_0(v_{\lambda_i}, v_{\lambda_j}) = \Omega_0(P^\alpha v_{\lambda_i}, P^\alpha v_{\lambda_j}) \text{ for } i, j \in \{1, \dots, 2n\}.$$

Notice that if $u_\mu, u_{\mu'}$, are eigenvectors of an arbitrary symplectic matrix $\Psi$, corresponding to the eigenvalues $\mu$ and $\mu'$ respectively, then

$$\mu\mu' \neq 1 \implies \Omega_0(u_\mu, u_{\mu'}) = 0$$

since

$$\Omega_0(u_\mu, u_{\mu'}) = \Omega_0(\Psi u_\mu, \Psi u_{\mu'}) = \mu\mu'\Omega_0(u_\mu, u_{\mu'}).$$

We can therefore consider two cases: $\lambda_i\lambda_j \neq 1$, where we have

$$\Omega_0(P^\alpha v_{\lambda_i}, P^\alpha v_{\lambda_j}) = (\lambda_i\lambda_j)^\alpha \Omega_0(v_{\lambda_i}, v_{\lambda_j}) = 0 = \Omega_0(v_{\lambda_i}, v_{\lambda_j})$$

and $\lambda_i\lambda_j = 1$, where we have

$$\Omega_0(P^\alpha v_{\lambda_i}, P^\alpha v_{\lambda_j}) = (\lambda_i\lambda_j)^\alpha \Omega_0(v_{\lambda_i}, v_{\lambda_j}) = \Omega_0(v_{\lambda_i}, v_{\lambda_j}).$$ ∎

**Remark 2.3.5.** This proposition implies that the space $\mathcal{P}$ of symmetric positive-definite symplectic matrices is contractible, since we can define the map

$$G : [0, 1] \times \mathcal{P} \to \mathcal{P}, \quad (t, P) \mapsto P^{(1-t)},$$

which is a deformation retraction of $\mathcal{P}$ onto the identity matrix, $\mathrm{Id} \in \mathrm{Sp}(2n)$.

Having established contractibility of the space $\mathcal{P}$, we can prove contractibility of the space of compatible complex structures $\mathcal{J}(V,\Omega)$ on a symplectic vector space $(V,\Omega)$ (Proposition 2.2.6).

**Proof of Proposition 2.2.6 for $\mathcal{J}_\tau(V,\Omega)$:** By Remark 2.1.10, it is enough to show this for $(\mathbb{R}^{2n},\Omega_0)$. We will show that $\mathcal{J}(\mathbb{R}^{2n},\Omega_0)$ is homeomorphic to the space of symmetric, positive-definite symplectic matrices. This space is contractible by Remark 2.3.5 and is non-empty as it contains the identity. We define a map

$$h : \mathcal{J}(\mathbb{R}^{2n},\Omega_0) \to \mathcal{P}, \quad J \mapsto -J_0 J.$$

To verify that this map is well-defined, we must check that $-J_0J$ is symplectic, symmetric and positive-definite for $J \in (\mathbb{R}^{2n},\Omega_0)$. It's clear that $-J_0J$ is symplectic since both $J_0$ and $J$ are compatible complex structures on $(\mathbb{R}^{2n},\Omega_0)$. Using $J_0^T = -J_0$ and $J^TJ_0J = J_0$, we have

$$(-J_0J)^T = -J^T(J_0)^T = J^TJ_0 = -J^TJ_0J^2 = -J_0J,$$

and we see that $-J_0J$ is symmetric. Positive-definiteness follows from the tameness of $J$:

$$-v^TJ_0Jv = \Omega_0(v,Jv) > 0$$

for all $v \in \mathbb{R}^{2n}, v \neq 0$.

The map $h$ is clearly injective. It is also surjective since for $P \in \mathcal{P}$, $P = h(J_0P)$ and $J_0P \in \mathcal{J}(\mathbb{R}^{2n},\Omega_0)$:

$$(J_0P)^2 = J_0P^TJ_0P = J_0^2 = -\mathrm{Id}.$$

Both $h$ and its inverse are clearly continuous, and we see that $\mathcal{J}(\mathbb{R}^{2n},\Omega_0)$ is homeomorphic to the contractible space $\mathcal{P}$. ∎

**Theorem 2.3.6.** *The group $Sp(2n)$ deformation retracts onto the subgroup $U(n)$.*

**Proof:** We first prove the following polar decomposition result for symplectic matrices: Any $\Psi \in \mathrm{Sp}(2n)$ has a unique decomposition $\Psi = PQ$, where $P$ is symplectic, symmetric and positive-definite and $Q$ is unitary [MS98, p. 45]. Since any $\Psi \in \mathrm{Sp}(2n)$ is nonsingular, we have a unique decomposition

$$\Psi = PQ$$

where $P$ is symmetric and positive-definite and $Q$ is orthogonal. Furthermore, we know $P = (\Psi\Psi^T)^{1/2}$. Using Proposition 2.3.4 along with the fact that the transpose of a symplectic matrix is symplectic, we see that $P$ is symplectic. We also have

$$Q = (\Psi\Psi^T)^{-1/2}\Psi \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{U}(n)$$

The uniqueness of the decomposition of $\Psi$ as the product of a symplectic, symmetric, positive-definite matrix and a unitary one follows from the uniqueness of its decomposition as the product of a symmetric, positive-definite matrix and an orthogonal one.

Using the polar decomposition of symplectic matrices, we can define a deformation retraction of $\mathrm{Sp}(2n)$ onto $\mathrm{U}(n)$,

$$H : [0,1] \times \mathrm{Sp}(2n) \to \mathrm{U}(n), \quad (t, \Psi) \mapsto P^{(1-t)}Q = (\Psi\Psi^T)^{-t/2}\Psi. \qquad \blacksquare$$

The remainder of this section will establish some facts about the topology of $\mathrm{Sp}(2n)$ which are important for the definition of the Maslov index in Section 2.5.

**Corollary 2.3.7.** $\mathrm{Sp}(2n)$ *is path-connected.*

**Proof:** This follows immediately from the fact that $\mathrm{U}(n)$ is path-connected, which we prove. For this we consider $\mathrm{U}(n)$ as a subset of $M(n, \mathbb{C})$. The eigenvalues of a unitary matrix are complex numbers of modulus one, and by the spectral theorem of linear algebra, any $Q \in \mathrm{U}(n)$ can be diagonalized by a unitary matrix:

$$Q = S \begin{pmatrix} e^{i\theta_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{i\theta_n} \end{pmatrix} S^{-1}$$

where $S$ is unitary and $\theta_j \in [0, 2\pi]$. Then

$$\gamma : [0,1] \to \mathrm{U}(n), \quad \gamma_t := S \begin{pmatrix} e^{ti\theta_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{ti\theta_n} \end{pmatrix} S^{-1}$$

is a path from Id to $Q$. ∎

**Proposition 2.3.8.** *[MS98, p. 46] The map* $\det : \mathrm{U}(n) \to S^1$ *induces an isomorphism of fundamental groups. Therefore* $\pi_1(\mathrm{U}(n)) \cong \mathbb{Z}$*.*

**Proof:** $\mathrm{U}(n)$ and $S^1$ are Lie groups and $\det : \mathrm{U}(n) \to S^1$ is a surjective homomorphism with kernel $\mathrm{SU}(n)$. By Corollary 1.1.8 $\det : \mathrm{U}(n) \to S^1$ is a locally trivial fibration with fibre $\mathrm{SU}(n)$. We have the following homotopy long exact sequence for this fibration:

$$\cdots \to \underbrace{\pi_1(\mathrm{SU}(n))}_{=\{0\}} \to \pi_1(\mathrm{U}(n)) \to \pi_1(S^1) \to \underbrace{\pi_0(\mathrm{SU}(n))}_{=\{0\}}.$$

Here we have used the fact that $\mathrm{SU}(n)$ is simply connected (see below). Exactness of this sequence implies

$$\det_* : \pi_1(\mathrm{U}(n)) \to \pi_1(S^1) \cong \mathbb{Z}$$

is an isomorphism.

To prove that $\mathrm{SU}(n)$ is simply connected, we proceed by induction. For $n = 1$, $\mathrm{SU}(n)$ is a single point. For $n > 1$, we assume $\mathrm{SU}(n-1)$ is simply connected and consider the inclusion of $\mathrm{SU}(n-1)$ as a closed subgroup in $\mathrm{SU}(n)$:

$$\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n), \quad A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

By Theorem 1.1.7, this determines a locally trivial fibration

$$\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n) \to \mathrm{SU}(n)/\mathrm{SU}(n-1).$$

The group $\mathrm{SU}(n)$ acts transitively on the unit sphere $S^{2n-1}$ in $\mathbb{C}^n$ and the stabilizer of the vector $(1, 0, \ldots, 0) \in S^{2n-1}$ is the subgroup $\mathrm{SU}(n-1)$. Then by the orbit-stabilizer theorem, $\mathrm{SU}(n)/\mathrm{SU}(n-1) \cong S^{2n-1}$. Using this homeomorphism, we write the homotopy long exact sequence for the above fibration:

$$\cdots \to \pi_2(S^{2n-1}) \to \underbrace{\pi_1(\mathrm{SU}(n-1))}_{=\{0\}} \to \pi_1(\mathrm{SU}(n)) \to \underbrace{\pi_1(S^{2n-1})}_{=\{0\}} \to \cdots .$$

Exactness of this sequence implies that $\pi_1(\mathrm{SU}(n))$ is trivial. ∎

**Corollary 2.3.9.** *Due to the deformation retraction of* $\mathrm{Sp}(2n)$ *onto* $\mathrm{U}(n)$ *established in Corollary 2.3.6, Proposition 2.3.8 implies that* $\pi_1(\mathrm{Sp}(2n))$ *is also isomorphic to the integers.*

## 2.4 Lagrangian subspaces

**Definition 2.4.1.** *A **Lagrangian subspace** of a* $2n$*-dimensional symplectic vector space* $(V, \Omega)$ *is an* $n$*-dimensional subspace* $L$ *on which the bilinear symplectic form* $\Omega$ *vanishes. The set of all Lagrangian subspaces of* $(\mathbb{R}^{2n}, \Omega_0)$ *is called the **La-***

***grangian Grassmannian*** *and denoted by* $\Lambda(n)$*. A* ***frame*** *for a Lagrangian subspace* $L \subset (\mathbb{R}^{2n}, \Omega_0)$ *is a matrix*

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} \in M(2n, n, \mathbb{R}), \quad X, Y \in M(n, \mathbb{R})$$

*such that* $L$ *is the span of the columns of* $Z$*. The frame* $Z$ *is called* ***unitary*** *if* $X + iY \in \mathrm{U}(n)$*.*

We define a topology on $\Lambda(n)$ as follows: Any $L \in \Lambda(n)$ is determined by a basis of $n$ vectors. The set of $n$-tuples of vectors whose span is a Lagrangian subspace of $V$ is a subset of the $n$-fold Cartesian product of $V$, and we equip this subset with the subspace topology. We define an equivalence relation on the subspace by considering two sets of $n$ linearly independent vectors to be equivalent if they span the same Lagrangian subspace of $V$. The set of equivalence classes is in bijective correspondence with $\Lambda(n)$, and the topology on $\Lambda(n)$ is the topology induced from the quotient topology on the set of equivalence classes of bases of Lagrangian subspaces.

**Proposition 2.4.2.** *[MS98, p. 50] The matrix*

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} \in M(2n, n, \mathbb{R})$$

*is the frame for a Lagrangian subspace* $\Lambda \subset (\mathbb{R}^{2n}, \Omega_0)$ *if and only if* $Z$ *is of rank* $n$ *and*

$$X^T Y = Y^T X.$$

*The frame* $Z$ *is unitary if and only if the columns of* $Z$ *form an orthonormal basis for* $\Lambda$*.*

**Remark 2.4.3.** In particular, this proposition implies that every Lagrangian subspace has a unitary frame, since the Gram-Schmidt process can be used to produce an orthonormal basis for the subspace.

**Proof:** Let $X, Y \in M(n, \mathbb{R})$, and define

$$\Lambda := \text{range} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Two vectors

$$\begin{pmatrix} X \\ Y \end{pmatrix} v \text{ and } \begin{pmatrix} X \\ Y \end{pmatrix} w$$

in $\Lambda$ satisfy

$$\begin{aligned} \Omega_0 \left( \begin{pmatrix} X \\ Y \end{pmatrix} v, \begin{pmatrix} X \\ Y \end{pmatrix} w \right) &= v^T \begin{pmatrix} X^T & Y^T \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} w \\ &= v^T(-X^T Y + Y^T X) w \end{aligned}$$

Therefore $\Lambda$ is Lagrangian if and only if it is of rank $n$ and satisfies

$$X^T Y = Y^T X.$$

We also have

$$\begin{aligned} (X + iY)^*(X + iY) &= (X^T - iY^T)(X + iY) \\ &= X^T X + i(X^T Y - Y^T X) + Y^T Y \\ &= \begin{pmatrix} X \\ Y \end{pmatrix}^T \begin{pmatrix} X \\ Y \end{pmatrix}, \end{aligned}$$

and hence the frame is unitary if and only if

$$\begin{pmatrix} X \\ Y \end{pmatrix}^T \begin{pmatrix} X \\ Y \end{pmatrix} = \mathrm{Id}.$$

∎

**Remark 2.4.4.** Let $\Lambda_{\text{hor}}$ be the Lagrangian subspace of $(\mathbb{R}^{2n}, \Omega_0)$ given by

$$\Lambda_{\text{hor}} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) |\; y_i = 0, 1 \leq i \leq n\}.$$

In a neighbourhood of $\Lambda_{\text{hor}}$ every Lagrangian subspace $\Lambda$ is the graph of a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$:

$$\Lambda = \{(v, Av)|\ v \in \mathbb{R}^n\}.$$

In other words, $\left(\begin{smallmatrix} \mathrm{Id} \\ A \end{smallmatrix}\right)$ is a frame for $\Lambda$ and by Proposition 2.4.2, $A$ is symmetric. It is equally clear that every symmetric matrix $A \in M(n, \mathbb{R})$ determines a unique Lagrangian subspace of $(\mathbb{R}^{2n}, \Omega_0)$. Therefore, in a neighbourhood of $\Lambda_{\text{hor}}$, $\Lambda(n)$ is homeomorphic to Euclidean space of dimension $n(n+1)/2$. Indeed, as the following proposition indicates, $\Lambda(n)$ is equipped with a transitive $\mathrm{U}(n)$ action which makes it into a homogeneous space and it follows that every element of $\Lambda(n)$ has a Euclidean neighbourhood of dimension $n(n+1)/2$. In other words, $\Lambda(n)$ is a manifold of dimension $n(n+1)/2$.

**Lemma 2.4.5.**

1. *If* $\Lambda \in \Lambda(n)$ *and* $\Psi \in \mathrm{Sp}(2n)$*, then* $\Psi\Lambda \in \Lambda(n)$.

2. *For any two* $\Lambda, \Lambda' \in \Lambda(n)$*, there is a* $\Psi \in \mathrm{U}(n)$ *with* $\Psi\Lambda = \Lambda'$.

3. $\Lambda(n) \cong \mathrm{U}(n)/\mathrm{O}(n)$.

**Proof:**

1. This is clear from the definition of $\mathrm{Sp}(2n)$.

2. By Remark 2.4.3, for any $\Lambda \in \Lambda(n)$, we can choose a unitary frame of the form $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ and define

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{U}(n) = \mathrm{Sp}(2n) \cap \mathrm{O}(2n).$$

Then

$$\Lambda = \Psi\Lambda_{\text{hor}}$$

Any other $\Lambda' \in \Lambda(n)$ can be written similarly as $\Lambda' = \Psi'\Lambda_{\text{hor}}$ with $\Psi' \in \mathrm{U}(n)$. Then

$$\Lambda' = \Psi'\Psi^{-1}\Lambda.$$

3. Again, by Proposition 2.4.2, any Lagrangian subspace in $\Lambda(n)$ has a unitary frame and this frame is unique up to an orthonormal change of basis for the subspace, in other words, up to right multiplication by elements of $\mathrm{O}(n)$. Right multiplication of unitary frames by elements of $\mathrm{O}(n)$ defines a right action of $\mathrm{O}(n)$ on $\mathrm{U}(n)$. We therefore have an isomorphism between $\Lambda(n)$ and the orbit space of this action, $\mathrm{U}(n)/\mathrm{O}(n)$. Since the action of $\mathrm{O}(n)$ on $\mathrm{U}(n)$ is smooth, free, and proper $\mathrm{U}(n)/\mathrm{O}(n)$ is a topological manifold and possesses a unique smooth structure such that the natural projection $\mathrm{U}(n) \to \mathrm{U}(n)/\mathrm{O}(n)$ is a submersion [Lee13, p. 544]. The topology on $\Lambda(n)$ described at the beginning of this section coincides with the quotient topology on $\mathrm{U}(n)/\mathrm{O}(n)$. Hence $\Lambda(n)$ is also a smooth manifold with this topology. ∎

**Definition 2.4.6.** *We define a map $\tau : \Lambda(n) \to S^1$ as follows:*

$$\Lambda \mapsto \det{}^2(X + iY) \ \textit{where}\ \Lambda = \operatorname{range}\begin{pmatrix} X \\ Y \end{pmatrix} \ \textit{and}\ X + iY \in \mathrm{U}(n).$$

*This map is well-defined since every Lagrangian subspace has a unitary frame which is unique up to multiplication by elements of* $\mathrm{O}(n)$.

**Proposition 2.4.7.** *The map $\tau : \Lambda(n) \to S^1$ induces an isomorphism of fundamental groups and therefore $\pi_1(\Lambda(n)) \cong \mathbb{Z}$, [Arn67].*

**Proof:** We define a subset $S\Lambda(n)$ of $\Lambda(n)$ as follows:

$$S\Lambda(n) = \left\{ \operatorname{range}\begin{pmatrix} X \\ Y \end{pmatrix} \middle|\, X + iY \in \mathrm{SU}(n) \right\}.$$

This allows us to write the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{SO}(n) & \longrightarrow & \mathrm{O}(n) & \xrightarrow{\det} & S^0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{SU}(n) & \longrightarrow & \mathrm{U}(n) & \xrightarrow{\det} & S^1 \\
\big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle z^2} \\
\underbrace{S\Lambda(n)}_{\cong \mathrm{SU}(n)/\mathrm{SO}(n)} & \longrightarrow & \underbrace{\Lambda(n)}_{\cong \mathrm{U}(n)/\mathrm{O}(n)} & \xrightarrow{\tau} & S^1
\end{array}$$

where the columns are locally trivial fibrations by Theorem 1.1.7 and the rows are locally trivial fibrations by Corollary 1.1.8. From the homotopy exact sequence of the left column,

$$\cdots \to \pi_1(\mathrm{SO}(n)) \to \underbrace{\pi_1(\mathrm{SU}(n))}_{=\{0\}} \to \pi_1(S\Lambda(n)) \to \underbrace{\pi_0(\mathrm{SO}(n))}_{=\{0\}},$$

we see that $S\Lambda(n)$ is simply connected, and from the homotopy exact sequence for the bottom row,

$$\cdots \to \underbrace{\pi_1(S\Lambda(n))}_{=\{0\}} \to \pi_1(\Lambda(n)) \to \pi_1(S^1) \to \underbrace{\pi_0(S\Lambda(n))}_{=\{0\}},$$

we have an isomorphism

$$\tau_* : \pi_1(\Lambda(n)) \xrightarrow{\sim} \mathbb{Z}.$$

∎

The following connection between Lagrangian submanifolds and symplectomorphisms of symplectic vector spaces highlights the importance of Lagrangian submanifolds in linear symplectic geometry. Its proof is an easy consequence of the definitions.

**Proposition 2.4.8.** *[MS98, p. 39] Let $(V, \Omega)$ be a symplectic vector space and equip the product $V \times V$ with the bilinear symplectic form*

$$(-\Omega) \oplus \Omega \in \Lambda^2_+((V \times V)^*), \quad (-\Omega) \oplus \Omega((v, w), (v', w')) = -\Omega(v, v') + \Omega(w, w'),$$

*for all $v, v', w, w' \in V$. Then a linear map $\Psi : V \to V$ is a symplectomorphism if and only if the graph of $\Psi$ is a Lagrangian subspace of $(V \times V, (-\Omega) \oplus \Omega)$.*

A global version of this proposition motivates the definition of Lagrangian intersection Floer homology and by extension the Lagrangian quantum homology.

## 2.5 The Maslov index

"The Maslov index" has many different meanings, all of which involve assigning integer invariants in some setting. Here we will define the Maslov index for $\mathrm{Sp}(2n)$ and for the Lagrangian Grassmannian. In a later section we will describe the Maslov index for a Lagrangian submanifold of a symplectic manifold.

### 2.5.1 The Maslov Index for loops in Sp(2n)

As we have seen, the fundamental group of $\mathrm{Sp}(2n)$ is isomorphic to $\mathbb{Z}$. The Maslov index for loops in $\mathrm{Sp}(2n)$ is an explicit isomorphism.

**Remark 2.5.1.** In the following it will be convenient to use the description of the fundamental group of $\mathrm{Sp}(2n)$ and of $\mathrm{U}(n)$ which does not assume a fixed basepoint or require homotopies of loops to fix a basepoint. In general, for a topological group $G$, there are two operations on homotopy classes of loops in the identity element: There is the operation defined by concatenation of paths, $[\gamma] * [\eta] = [\gamma * \eta]$, and there is the operation defined by pointwise multiplication of paths, $[\gamma] \cdot [\eta] = [\gamma \cdot \eta]$. We will use the fact that these two operations determine the same group structure on homotopy classes of loops:

$$[\gamma] * [\eta] = [\gamma] \cdot [\eta].$$

Unlike concatenation of loops, pointwise multiplication of loops is defined even when the loops do not share a basepoint. If $G$ is path-connected, with the operation of pointwise multiplication of loops, the space of all homotopy classes of loops in $G$ (in all basepoints) forms a group, where a homotopy of loops is no longer required to fix basepoints. This group is isomorphic to the ordinary fundamental group of $G$ with respect to a given basepoint. In the remainder of this section we will use this description of the fundamental group that does not require a choice of basepoint and we will understand "homotopic" to mean homotopic without regard to any fixed basepoint.

**Definition 2.5.2.** *We first define a function $\rho : Sp(2n) \to S^1$ by*

$$\rho(\Psi) = \det(X + iY) \ \textit{where} \ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = (\Psi\Psi^T)^{-1/2}\Psi \in U(n)$$

*for $\Psi \in Sp(2n)$. The function $\rho$ simply maps every symplectic matrix to the determinant of the unitary factor in its polar decomposition. The **Maslov index** $\mu$ of a loop $\gamma : S^1 \to Sp(n)$ is the degree of the composition $\rho \circ \gamma : S^1 \to S^1$:*

$$\mu(\gamma) = \deg(\rho \circ \gamma).$$

**Remark 2.5.3.** A map $f : S^1 \to S^1$ can be lifted to $\alpha : [0, 1] \to \mathbb{R}$:

$$f(e^{2\pi i t}) = e^{2\pi i \alpha(t)}$$

and the **degree** of $f$ is

$$\deg(f) = \alpha(1) - \alpha(0).$$

This is equivalent to defining the degree of $f$ to be the integer representing the image of a generator of $\pi_1(S^1)$ under the homomorphism

$$f_* : \pi_1(S^1) \to \pi_1(S^1) \cong \mathbb{Z}.$$

It is clear that $f, g : S^1 \to S^1$ have the same degree if and only if they are homotopic and that the product of two maps satisfies

$$\deg(f \cdot g) = \deg(f) + \deg(g).$$

**Theorem 2.5.4.** *[MS98, p. 48] The Maslov index is the unique function which assigns an integer to loops in* $\mathrm{Sp}(2n)$ *and satisfies the following properties:*

1. *For* $\Psi_1, \Psi_2 : S^1 \to \mathrm{Sp}(2n)$, $\mu(\Psi_1) = \mu(\Psi_2) \iff [\Psi_1] = [\Psi_2]$ *and*

2. $\mu(\Psi_1 \Psi_2) = \mu(\Psi_1) + \mu(\Psi_2)$.

3. *The loop* $\bar{\Psi} : S^1 \to U(1) \subset \mathrm{Sp}(2)$, $t \mapsto e^{2\pi i t}$ *has* $\mu(\bar{\Psi}) = 1$.

4. *If* $m + l = n$, *we can identify* $\mathrm{Sp}(2m) \oplus \mathrm{Sp}(2l)$ *with a subgroup of* $\mathrm{Sp}(2n)$. *Then two loops* $\Psi : S^1 \to \mathrm{Sp}(2m)$, $\Psi' : S^1 \to \mathrm{Sp}(2l)$ *satisfy*

$$\mu(\Psi \oplus \Psi') = \mu(\Psi) + \mu(\Psi').$$

**Remark 2.5.5.** In particular these properties imply that for each $n \in \mathbb{Z}$, the Maslov index defines an isomorphism from $\pi_1(\mathrm{Sp}(2n))$ to $\mathbb{Z}$: Properties 1 and 2 imply that the Maslov index is an injective homomorphism from $\pi_1(\mathrm{Sp}(2n))$ to $\mathbb{Z}$. From Property 3, for $n = 1$, the Maslov index clearly defines a surjection from $\pi_1(\mathrm{Sp}(2))$ to $\mathbb{Z}$, since the loop $\bar{\Psi} : S^1 \to \mathrm{U}(1)$ has $\mu(\bar{\Psi}) = 1$. For $n > 1$, define a constant loop

$$I : S^1 \to \mathrm{Sp}(2(n-1)), \quad t \mapsto \mathrm{Id}.$$

From Property 4,

$$\mu(I \oplus \bar{\Psi}) = \mu(I) + \mu(\bar{\Psi}) = 1$$

and we see that the Maslov index is surjective for all $n$.

**Proof:** We first verify the properties of the Maslov index.

1. If $\Psi_1$ and $\Psi_2$ are homotopic loops in $\mathrm{Sp}(2n)$, we have a homotopy between $\rho \circ \Psi_1$ and $\rho \circ \Psi_2$ and it follows that these maps have the same degree. Therefore

$$\mu(\Psi_1) = \mu(\Psi_2).$$

To see that loops with the same Maslov index are homotopic, assume $\Phi_1, \Phi_2 : S^1 \to \mathrm{Sp}(2n)$ satisfy

$$\deg(\rho \circ \Phi_1) = \deg(\rho \circ \Phi_2).$$

Then $\rho \circ \Phi_1$ and $\rho \circ \Phi_2$ must be homotopic maps from $S^1$ to $S^1$. In other words

$$\det(X_1 + iY_1) \text{ and } \det(X_2 + iY_2)$$

are homotopic, where

$$X_1 + iY_1 = (\Phi_1 \Phi_1^T)^{-1/2} \Phi_1 \text{ and } X_2 + iY_2 = (\Phi_2 \Phi_2^T)^{-1/2} \Phi_2.$$

Since $\det : \mathrm{U}(n) \to S^1$ induces an isomorphism of fundamental groups,

$$(\Phi_1 \Phi_1^T)^{-1/2} \Phi_1 \text{ and } (\Phi_2 \Phi_2^T)^{-1/2} \Phi_2$$

are homotopic maps from $S^1$ to $\mathrm{U}(n)$. But these are simply the images of $\Phi_1$ and $\Phi_2$ in $\mathrm{U}(n)$ under the deformation retraction of $\mathrm{Sp}(2n)$ onto $\mathrm{U}(n)$ from Corollary 2.3.6, which means $\Phi_1$ and $\Phi_2$ are also homotopic.

2. For loops in $\mathrm{U}(n)$, this is clear since the $\det : U(n) \to S^1$ induces an isomorphism of fundamental groups and the degree is an isomorphism from $\pi_1(S^1)$ to $\mathbb{Z}$. For general loops, $\Psi_1$, $\Psi_2$ in $\mathrm{Sp}(2n)$, we compose with the deformation retraction $H$ of Corollary 2.3.6:

$$H(\cdot, \Psi_j) : [0,1] \times [0,1] \to \mathrm{Sp}(2n), \quad j = 1, 2.$$

This yields a homotopy between $\Psi_j$ and $H(1, \Psi_j)$. Using the fact that $H(1, \Psi_j)$ is a path in $\mathrm{U}(n)$, along with the first property of $\mu$, we have

$$\begin{aligned} \mu(\Psi_1 \Psi_2) &= \mu(H(1, \Psi_1) H(1, \Psi_2)) \\ &= \mu(H(1, \Psi_1)) + \mu(H(1, \Psi_2)) = \mu(\Psi_1) + \mu(\Psi_2). \end{aligned}$$

3. This is clear.

4. It is simplest to describe the embedding of $\mathrm{Sp}(2m) \oplus \mathrm{Sp}(2l)$ as a subgroup in $\mathrm{Sp}(2n)$ if we reorder the basis of $\mathbb{R}^{2k}$, $k = m, l, n$, so that the matrix of $-\Omega_0$ can be written as

$$\tilde{J}_0 = \begin{pmatrix} J_2 & 0 & \cdots & \cdots \\ 0 & J_2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_2 \end{pmatrix} \text{ where } J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then if $\Psi : S^1 \to \mathrm{Sp}(2m)$, $\Psi' : S^1 \to \mathrm{Sp}(2l)$ are two matrix loops written with respect to these reordered bases, the direct sum of these loops is simply

$$\Psi \oplus \Psi' : S^1 \to \mathrm{Sp}(2n), \quad \Psi \oplus \Psi' = \begin{pmatrix} \Psi & 0 \\ 0 & \Psi' \end{pmatrix}.$$

Computing the Maslov index,

$$\begin{aligned} \mu(\Psi \oplus \Psi') &= \mu\left(\begin{pmatrix} (\Psi\Psi^T)^{-1/2}\Psi & 0 \\ 0 & (\Psi'\Psi'^T)^{-1/2}\Psi' \end{pmatrix}\right) \\ &= \deg\left(\det\left(\begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} + i\begin{pmatrix} Y & 0 \\ 0 & Y' \end{pmatrix}\right)\right) \end{aligned}$$

where $(\Psi\Psi^T)^{-1/2}\Psi = X + iY$, $(\Psi'\Psi'^T)^{-1/2}\Psi' = X' + iY'$. Using the product property for the degree of maps,

$$\begin{aligned} \mu(\Psi \oplus \Psi') &= \deg(\det(X + iY)\det(X' + iY')) \\ &= \deg(\det(X + iY)) + \deg(\det(X' + iY')) \\ &= \mu(\Psi) + \mu(\Psi') \end{aligned}$$

We now prove uniqueness of the Maslov index. As we verified in Remark 2.5.5, the Maslov index induces an isomorphism from $\pi_1(\mathrm{Sp}(2n))$ to $\mathbb{Z}$, which we will also denote by $\mu$. Now assume we have another function $\chi$ which assigns an integer to loops in $\mathrm{Sp}(2n)$ and satisfies the properties of Theorem 2.5.4. By Remark 2.5.5, $\chi$ also induces an isomorphism from $\pi_1(\mathrm{Sp}(2n))$ to $\mathbb{Z}$. Consider the map $\chi \circ \mu^{-1}$. This is an isomorphism from $\mathbb{Z}$ to $\mathbb{Z}$. We consider its action on $1 \in \mathbb{Z}$,

$$1 \overset{\mu^{-1}}{\longmapsto} [\bar{\Psi}] \overset{\chi}{\longmapsto} 1.$$

Here we are using Property 3 from Theorem 2.5.4 and the loop $\bar{\Psi}$ defined there. From this, it follows that $\chi \circ \mu^{-1} = \mathrm{Id}_{\mathbb{Z}}$, and hence $\mu$ and $\chi$ are equal. ∎

### 2.5.2 The Maslov index for loops in $\Lambda(n)$

By Corollary 2.4.7, the fundamental group of the Lagrangian Grassmannian is isomorphic to $\mathbb{Z}$. As with the Maslov index for loops in $\mathrm{Sp}(2n)$, the Maslov index for loops in $\Lambda(n)$ is an explicit isomorphism between $\pi_1(\Lambda(n))$ and $\mathbb{Z}$.

**Definition 2.5.6.** *The* ***Maslov index*** *is a function which assigns an integer to loops in $\Lambda(n)$ and is defined as follows: For $\gamma : S^1 \to \Lambda(n)$, the Maslov index of $\gamma$ is the degree of the composition $\tau \circ \gamma : S^1 \to S^1$:*

$$\mu(\gamma) = \deg(\tau \circ \gamma),$$

*where $\tau$ is the map of Definition 2.4.6.*

**Remark 2.5.7.** Since we have a locally trivial fibration

$$\mathrm{O}(n) \to \mathrm{U}(n) \xrightarrow{\pi} \mathrm{U}(n)/\mathrm{O}(n) \cong \Lambda(n),$$

we may lift $\gamma$ to a path $u : [0,1] \to \mathrm{U}(n)$:

$$\pi(u(t)) = \gamma(e^{2\pi i t}).$$

To calculate the Maslov index $\mu$ of $\gamma$, we then choose a lift $\alpha$ of $\det \circ u$ over the covering map $\varphi : \mathbb{R} \to S^1$, $t \mapsto e^{\pi i t}$:

$$\alpha : [0,1] \to \mathbb{R}, \quad \det \circ u(t) = e^{\pi i \alpha(t)}.$$

Squaring this equation, we see that $\alpha$ is also a lift of $\det^2 \circ u$ over the standard covering map $\varphi^2 : \mathbb{R} \to S^1$,

$$e^{2\pi i \alpha(t)} = \det{}^2 \circ u(t) = \tau \circ \gamma(t),$$

and we have

$$\deg(\tau \circ \gamma) = \alpha(1) - \alpha(0).$$

**Theorem 2.5.8.** *[MS98, p. 52] The Maslov index satisfies the following properties and it is uniquely determined by Properties 1-4:*

1. *Two loops in $\Lambda(n)$ with the same basepoint are homotopic through a homotopy which fixes the basepoint if and only if they have the same Maslov index.*

2. *For any loops $\Psi : S^1 \to \mathrm{Sp}(2n)$ and $\gamma : S^1 \to \Lambda(n)$, we have*

$$\mu(\Psi\gamma) = 2\mu(\Psi) + \mu(\gamma).$$

3. *For $m + l = n$, we can identify $\Lambda(m) \oplus \Lambda(l)$ with a submanifold of $\Lambda(n)$. We then have*

$$\mu(\gamma' \oplus \gamma'') = \mu(\gamma') + \mu(\gamma'').$$

4. *The constant loop at $\Lambda_0$ has Maslov index 0.*

5. *The loop $\bar{\gamma} : S^1 \to \Lambda(1)$, $\bar{\gamma}(t) = e^{\pi i t}\Lambda_{\mathrm{hor}}$ has Maslov index 1.*

The uniqueness part of this theorem is a result of the following lemma.

**Lemma 2.5.9.** *It follows from the properties of Theorem 2.5.8 that $\mu$ induces an isomorphism from $\pi_1(\Lambda(n))$ to $\mathbb{Z}$.*

**Proof of Lemma 2.5.9**: From Property 1, $\mu$ is well-defined on homotopy classes in $\pi_1(\Lambda(n))$ and the map $\mu$ from $\pi_1(\Lambda(n))$ to $\mathbb{Z}$ induced by $\mu$ is injective. The fact that $\mu$ is a group homomorphism follows from Property 2. First notice that since $\mathrm{Sp}(2n)$ acts transitively on $\Lambda(n)$, every loop in $\Lambda(n)$ based at $\Lambda_0 \in \Lambda(n)$ can be written as $\Psi\gamma_0$, where $\Psi$ is a loop in $\mathrm{Sp}(2n)$ based at the identity and $\gamma_0$ is the constant loop $\gamma_0(t) = \Lambda_0$. If $\gamma_1 = \Psi_1\gamma_0, \gamma_2 = \Psi_2\gamma_0$ are two loops in $\Lambda(n)$ based at $\Lambda_0$, the product of their homotopy classes is given by

$$[\gamma_1] * [\gamma_2] = [(\Psi_1 * \Psi_2)\gamma_0].$$

The action of $\mathrm{Sp}(2n)$ on $\Lambda(n)$ induces a map of fundamental groups,

$$\pi_1(\mathrm{Sp}(2n) \times \Lambda(n)) \to \pi_1(\Lambda(n)).$$

Using the fact that $\pi_1(\mathrm{Sp}(2n) \times \Lambda(n))$ is isomorphic to $\pi_1(\mathrm{Sp}(2n)) \times \pi_1(\Lambda(n))$, we obtain a map

$$h : \pi_1(\mathrm{Sp}(2n)) \times \pi_1(\Lambda(n)) \to \pi_1(\Lambda(n)) \quad ([\Psi], [\gamma]) \mapsto [\Psi\gamma]$$

Since the two group operations on homotopy classes of loops in $\mathrm{Sp}(2n)$ based at the identity – multiplication of loops and concatentation of loops – agree, we have

$$\begin{aligned}
[\gamma_1] * [\gamma_2] = [(\Psi_1 * \Psi_2)\gamma_0] &= h([\Psi_1 * \Psi_2], [\gamma_0]) \\
&= h([\Psi_1\Psi_2], [\gamma_0]) \\
&= [\Psi_1\Psi_2\gamma_0].
\end{aligned}$$

We can use this along with Property 2 to verify that $\mu$ is a homomorphism:

$$\begin{aligned}
\mu([\gamma_1] * [\gamma_2]) &= \mu([\Psi_1\Psi_2\gamma_0]) \\
&= 2\mu(\Psi_1\Psi_2) + \mu(\gamma_0) \\
&= 2\mu(\Psi_1) + 2\mu(\Psi_2) + \mu(\gamma_0) \quad \text{(Theorem 2.5.4, Property 2)}
\end{aligned}$$

Then since $\mu(\gamma_0) = 0$ by Property 4, we have

$$\begin{aligned}\mu([\gamma_1] * [\gamma_2]) &= (2\mu(\Psi_1) + \mu(\gamma_0)) + (2\mu(\Psi_2) + \mu(\gamma_0)) \\ &= \mu(\Psi_1\gamma_0) + \mu(\Psi_2\gamma_0) \\ &= \mu([\gamma_1]) + \mu([\gamma_2]).\end{aligned}$$

It remains to verify that $\mu$ is surjective. This is clear for $n = 1$, since the loop $\bar{\gamma}$ defined in Property 5 has Maslov index 1. For $n > 1$, define the loop $\gamma_0 \oplus \bar{\gamma}$ where $\gamma_0$ is a constant loop in $\Lambda(n-1)$. By Properties 3 and 4, the Maslov index of this loop is 1, and we see that $\mu$ is surjective. ∎

**Proof of Theorem 2.5.8**:

1. If $\gamma_1, \gamma_2 : S^1 \to \Lambda(n)$ are homotopic loops with the same basepoint, the maps

   $$\tau \circ \gamma_1, \tau \circ \gamma_2 : S^1 \to S^1$$

   are homotopic and therefore have the same degree. By the definition of the Maslov index, this means $\gamma_1$ and $\gamma_2$ have the same Maslov index.

   On the other hand, assume $\gamma_1$ and $\gamma_2$ are loops in the same basepoint with the same Maslov index. From Lemma 2.4.5, there is a unitary symplectomorphism $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

   $$\Psi(\gamma_j(0)) = \Psi(\gamma_j(1)) = \Lambda_{\text{hor}}, \quad j = 1, 2.$$

   By Property 2 for the constant loop $\Psi \in \mathrm{U}(n)$, $\gamma_1$ and $\gamma_2$ have the same Maslov index if and only if $\Psi(\gamma_1)$ and $\Psi(\gamma_2)$ do[3]. Clearly $\gamma_1$ and $\gamma_2$ are also homotopic if and only if $\Psi(\gamma_1)$ and $\Psi(\gamma_2)$ are. We may therefore assume without loss of generality that $\gamma_1$ and $\gamma_2$ are loops in $\Lambda_{\text{hor}}$.

[3] Although we use part of Property 2 to prove Property 1 and Property 1 to prove part of Property 2, the proof of Property 1 relies only on Property 2 for the case where $\Psi$ is a loop of unitary matrices. The proof of this case of Property 2 does not rely on Property 1.

As in Remark 2.5.7, we lift $\gamma_1$ and $\gamma_2$ to paths $u_1$ and $u_2$ in $\mathrm{U}(n)$. Since $\gamma_1$ and $\gamma_2$ are loops in $\Lambda_{\mathrm{hor}}$, the paths $u_1$ and $u_2$ can be chosen to start at $\mathrm{Id} \in \mathrm{U}(n)$. The endpoints $u_j(1)$, $j = 1, 2$, of these lifts will be a real unitary matrices. If $\det(u_j(1)) = 1$, up to multiplication by an element of $\mathrm{SO}(n)$, $u_j(1) = \mathrm{Id}$. If $\det(u_j(1)) = -1$, up to multiplication by an element of $\mathrm{SO}(n)$,

$$u_j(1) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

Since $\mathrm{SO}(n)$ is connected, by multiplying $u_j(n)$ by a path in $\mathrm{SO}(n)$, we get a lift of $\gamma_j$ starting in $\mathrm{Id} \in \mathrm{U}(n)$ and ending in the matrix

$$\begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

We have $\mu(\gamma_1) = \mu(\gamma_2)$ and therefore, by Remark 2.5.7,

$$\alpha_1(1) - \alpha_1(0) = \alpha_2(1) - \alpha_2(0) \text{ where } e^{\pi i \alpha_j(t)} = \det(u_j(t)).$$

This implies $u_1(1) = u_2(1)$ and it follows that $u(t) := u_2(t)u_1(t)^{-1}$ defines a loop in $\mathrm{U}(n)$. This loop satisfies

$$\begin{aligned} \deg(\det(u)) &= \deg(\det(u_2)) + \deg(\det(u_1^{-1})) \\ &= \deg(\det(u_2)) - \deg(\det(u_1)) \\ &= \frac{1}{2}(\alpha_2(1) - \alpha_2(0) - \alpha_1(1) + \alpha_1(0)) \\ &= 0. \end{aligned}$$

Hence $\det(u) : S^1 \to S^1$ is contractible and it follows from Proposition 2.3.8 that $u$ is contractible in $\mathrm{U}(n)$, i.e.

$$[u_1] = [u_2] \in \pi_1(\mathrm{U}(n)).$$

Taking the image of a homotopy between $u_1$ and $u_2$ under the quotient map $\mathrm{U}(n) \to \mathrm{U}(n)/\mathrm{O}(n)$, we see that $\gamma_1$ and $\gamma_2$ are homotopic.

2. Suppose we have a loop of unitary matrices $\Psi : S^1 \to \mathrm{U}(n)$, $\Psi = X + iY$, and a loop of Lagrangian subspaces $\gamma : S^1 \to \Lambda(n)$ with a corresponding loop of unitary frames $\left(\begin{smallmatrix} X' \\ Y' \end{smallmatrix}\right)$. Then

$$\begin{aligned}\mu(\Psi\gamma) &= \deg(\tau(\Psi\gamma)) = \deg(\det{}^2((X+iY)(X'+iY')))\\ &= 2\deg(\det(X+iY)\det(X'+iY'))\\ &= 2\deg(\det(X+iY)) + \deg(\det{}^2(X'+iY'))\\ &= 2\mu(\Psi) + \mu(\gamma)\end{aligned}$$

The result for general loops in $\mathrm{Sp}(2n)$ follows using the deformation retraction of Corollary 2.3.6 and the fact that homotopic loops in both $\mathrm{Sp}(2n)$ and $\Lambda(n)$ have the same Maslov index.

3. We embed $\Lambda(m) \oplus \Lambda(l)$ in $\Lambda(n)$ using the following mapping defined for unitary frames:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} \oplus \begin{pmatrix} X'' \\ Y'' \end{pmatrix} \mapsto \begin{pmatrix} X' & 0 \\ 0 & X'' \\ Y' & 0 \\ 0 & Y'' \end{pmatrix}$$

where $X' + iY' \in U(m)$ and $X'' + iY'' \in U(l)$. If the loop $\gamma'$ in $\Lambda(m)$ is represented by $X' + iY'$ and the loop $\gamma''$ in $\Lambda(l)$ is represented by $X'' + iY''$, the Maslov index of the direct sum $\gamma \oplus \gamma'$ satisfies

$$\mu(\gamma' \oplus \gamma'') = \deg\left(\det{}^2\left(\begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix} + i\begin{pmatrix} Y' & 0 \\ 0 & Y'' \end{pmatrix}\right)\right)$$

$$= \deg(\det^2(X' + iY')\det^2(X'' + iY''))$$
$$= \mu(\gamma') + \mu(\gamma'')$$

Properties 4 and 5 are clear from the definition of the Maslov index.

We show that Property 5 is implied by Properties 2, 3, and 4. The path

$$\bar{\gamma} : S^1 \to \Lambda(1), \quad t \mapsto e^{\pi i t}\Lambda_{\mathrm{hor}}$$

corresponds to the path of unitary frames

$$t \mapsto \begin{pmatrix} \cos(\pi t) \\ \sin(\pi t) \end{pmatrix}.$$

Embedding $\Lambda(1) \oplus \Lambda(1)$ in $\Lambda(2)$ as described in the proof of Property 3, we obtain the path $\bar{\gamma} \oplus \bar{\gamma} : S^1 \to \Lambda(2)$ with corresponding path of unitary frames

$$t \mapsto \begin{pmatrix} \cos(\pi t) & 0 \\ 0 & \cos(\pi t) \\ \sin(\pi t) & 0 \\ 0 & \sin(\pi t) \end{pmatrix}.$$

The path $\bar{\gamma} \oplus \bar{\gamma}$ can be expressed as

$$\bar{\gamma} \oplus \bar{\gamma} = \Psi\gamma_0$$

where $\gamma_0$ is the constant loop in $\Lambda_{\mathrm{hor}} \subset \mathbb{R}^4$ and $\Psi$ is the loop in $\mathrm{U}(n)$ defined by

$$\Psi : S^1 \to \mathrm{U}(n), \quad t \mapsto \begin{pmatrix} \cos(\pi t) & 0 \\ 0 & \cos(\pi t) \end{pmatrix} + i \begin{pmatrix} \sin(\pi t) & 0 \\ 0 & \sin(\pi t) \end{pmatrix}.$$

By Properties 2 and 4,

$$\mu(\bar{\gamma}) + \mu(\bar{\gamma}) = \mu(\bar{\gamma} \oplus \bar{\gamma}) = \mu(\Psi\gamma_0) = 2\mu(\Psi) + \mu(\gamma_0) = 2\mu(\Psi)$$

and it follows that $\mu(\bar{\gamma}) = \mu(\Psi)$. We have

$$\det(\Psi(t)) = \cos^2(\pi t) - \sin^2(\pi t) + i(2\cos(\pi t)\sin(\pi t))$$

$$= \cos(2\pi t) + i\sin(2\pi t) = e^{2\pi i t}.$$

Therefore

$$\mu(\Psi) = \deg(\rho \circ \Psi) = \deg(\det \circ \Psi) = 1.$$

and we see that $\mu(\bar{\gamma}) = 1$.

As in the proof of Theorem 2.5.4, the uniqueness of the Maslov index for loops in $\Lambda(n)$ follows from the fact that the Maslov index induces an isomorphism from $\pi_1(\Lambda(n))$ to $\mathbb{Z}$ (Lemma 2.5.9) and that this induced isomorphism maps the generator $\bar{\gamma}$ of $\pi_1(\Lambda(n))$ to $1 \in \mathbb{Z}$. ∎

**Remark 2.5.10.** The form of the statement of Property 1 in Theorem 2.5.8 is convenient for establishing that the Maslov index is an isomorphism from $\pi_1(\Lambda(n))$ to $\mathbb{Z}$. However, the statement is still true for two loops $\gamma_0, \gamma_1 : S^1 \to \Lambda(n)$ with distinct basepoints and an arbitrary homotopy

$$H : [0,1] \times S^1 \to \Lambda(n), \quad H_0 = \gamma_0,\ H_1 = \gamma_1.$$

To see this, choose a lift $\tilde{\eta} : [0,1] \to \mathrm{U}(n)$ of the path

$$\eta : [0,1] \to \Lambda, \quad t \mapsto H(t,0).$$

Define

$$H' : [0,1] \times S^1 \to \Lambda(n), \quad (t,\theta) \mapsto \tilde{\eta}^{-1}(t) H(t,\theta).$$

By Property 2 for loops in $\mathrm{U}(n)$,

$$\mu(H'_0) = \mu(\tilde{\eta}^{-1}(0)\gamma_0) = \mu(\gamma_0), \quad \mu(H'_1) = \mu(\tilde{\eta}^{-1}(1)\gamma_1) = \mu(\gamma_1).$$

The proof of Property 1 then proceeds as before, but for the loops $\tilde{\eta}^{-1}(0)\gamma_0$ and $\tilde{\eta}^{-1}(1)\gamma_1$ and the homotopy $H'$.

# Chapter 3

# Global symplectic geometry

In this chapter, we will introduce the required background material about symplectic manifolds and Lagrangian submanifolds. Symplectic manifolds are manifolds equipped with a smooth field of bilinear symplectic forms on the tangent spaces, just as Riemannian manifolds are manifolds with a smooth field of inner products on the tangent spaces. However we also put an additional condition on the two-form that specifies the bilinear symplectic form on the tangent space at each point of the manifold: We require that this two-form be closed. This requirement leads to possibly the most striking difference between symplectic geometry and Riemannian geometry, namely that all symplectic manifolds of a given dimension locally look alike. This result is know as Darboux's theorem, and in particular it implies that local invariants do not exist in symplectic geometry. This is a significant contrast to the situation in Riemannian geometry, where it is possible to study local invariants such as Riemannian curvature.

A key observation which forms the basis for the theory of $J$-holomorphic curves, and indeed the construction of the pearl complex, is that every symplectic manifold possesses an almost complex structure. With the goal of defining the pearl complex in mind, we give a parallel account of the elements of almost complex geometry we will need. We also attempt to provide a broader view of the role of almost complex structures and the theory of $J$-holomorphic curves in symplectic topology in general.

## 3.1 Symplectic and complex vector bundles

We first introduce symplectic and complex vector bundles as this naturally leads to the definition of symplectic and complex manifolds as those manifolds whose tangent bundles are symplectic or complex vector bundles respectively, and which satisfy an additional condition (closedness of the symplectic form and integrability of the almost complex structure respectively). We will need the concept of a symplectic vector bundle in its own right when we define the Maslov index of a Lagrangian submanifold. In that case, we deal with a symplectic vector bundle which is not the tangent bundle to a symplectic manifold.

**Definition 3.1.1.** *[MS98, p. 69] A* ***symplectic structure*** *on a real vector bundle $\pi : E \to B$ is a smooth section $\omega$ of the bundle $\Lambda^2(E^*)$ such that $\omega(b)$ is non-degenerate for all $b \in B$. We will write $\omega(b)$ as $\omega_b$. In other words, a symplectic structure is a smooth field of bilinear symplectic forms on the fibres over $B$. A real vector bundle $E$ equipped with a symplectic structure $\omega$ is called a* ***symplectic vector bundle*** *and is denoted by $(E, \omega)$. Note that the rank of a symplectic vector bundle must be even. A smooth frame over an open set $U \subset B$,*

$$\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\},$$

*is called a* ***symplectic frame*** *over $U$ if, for each point $b \in U$, the set*

$$\{X_j(b), Y_j(b)|\ j = 1, \ldots, n\}$$

*is a symplectic basis for the symplectic vector space $(E_b, \omega_b)$.*

**Remark 3.1.2.** As we saw in section 1.1, the existence of a Riemannian structure on a vector bundle of rank $n$ is equivalent to a reduction of the structure group of the bundle from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$. Similarly, the existence of a symplectic structure on a vector bundle of rank $2n$ is equivalent to a reduction of the structure group from $\mathrm{GL}(2n,\mathbb{R})$ to $\mathrm{Sp}(2n)$. As in the Riemannian case, this is a consequence of the local existence of symplectic frames on a symplectic vector bundle, which follows from the proof of Proposition 3.1.9.

Most of the basic definitions and facts about symplectic vector bundles in this section are simple generalizations of the versions for symplectic vector spaces.

**Remark 3.1.3.** If $(E,\omega)$ is a symplectic vector bundle with base $B$ and $f : B' \to B$ is a smooth map from a smooth manifold $B'$ into $B$, then the pullback bundle $f^*E \to B'$ possesses a symplectic structure $f^*\omega$ defined by

$$(f^*\omega)_{b'}(v,w) = \omega_{f(b')}(\mathrm{pr}_2 v, \mathrm{pr}_2 w) \text{ for all } b' \in B \text{ and all } v,w \in (f^*E)_{b'}$$

(cf. Remark 1.1.24).

**Definition 3.1.4.** *Two symplectic vector bundles,* $(E,\omega)$ *and* $(E',\omega')$*, over the base spaces* $B$ *and* $B'$ *are* ***symplectomorphic*** *if there is an isomorphism* $\eta : E \to E'$ *of the underlying vector bundles satisfying* $\eta^*\omega' = \omega$*, where*

$$(\eta^*\omega')_b(v,w) := \omega_{\bar{\eta}(b)}(\eta(v),\eta(w)) \textit{ for all } b \in B,\ v,w \in E_b.$$

*Here* $\bar{\eta} : B \to B'$ *is the map induced by* $\eta$ *(see Remark 1.1.14). Notice that if* $\eta$ *is an isomorphism between the underlying vector bundles* $E$ *and* $E'$*, then it is trivially a symplectomorphism from* $(E,\eta^*\omega')$ *to* $(E',\omega')$*.*

**Remark 3.1.5.** It follows from the corresponding linear statement (cf. Proposition 2.1.4) that the map which takes a vector bundle of even rank to its set of symplectic structures is functorial, although in this case we must allow for the possibility that this set is empty, as not every vector bundle of even rank has a symplectic structure (for example the tangent bundles of the spheres $S^4$, $S^8$, $S^{10}$, etc. possess no symplectic structure [Can01, p. 100]). Being symplectomorphic is an equivalence relation on the class of symplectic vector bundles (cf. Corollary 2.1.5) and the set of symplectomorphisms from a symplectic vector bundle to itself forms a group under composition (cf. Corollary 2.1.6).

**Definition 3.1.6.** *A symplectic vector bundle $(E, \omega)$ with base $B$ is* ***symplectically trivial*** *if it is symplectomorphic to the symplectic vector bundle $(B \times \mathbb{R}^{2n}, \omega_0)$ for some integer $n$. Here the projection map for $B \times \mathbb{R}^{2n}$ is simply projection onto $B$ and*

$$(\omega_0)_b((b,v),(b,w)) := \Omega_0(v,w) \text{ for all } b \in B,\ v,w \in \mathbb{R}^{2n}.$$

*A symplectomorphism between $(E, \omega)$ and $(B \times \mathbb{R}^{2n}, \omega_0)$ is called a* ***symplectic trivialization****.*

The following facts about symplectic trivializations will be important in defining the first Chern class of a symplectic vector bundle in Section 3.4 and the Maslov index of a Lagrangian submanifold in Section 4.2.

**Proposition 3.1.7.** *Let $\pi : E \to B$ be a rank-$2n$ symplectic vector bundle with symplectic structure $\omega$. The bundle $(E, \omega)$ is symplectically trivial if and only if it is trivial as a real vector bundle.*

**Proof:** A symplectic trivialization is clearly also a trivialization of the underlying real vector bundle. We prove the opposite implication. Since a vector bundle is trivial if and only if its restriction to each connected component of the base is trivial, we may assume without loss of generality that $B$ is connected.

Let $\phi : E \to B \times \mathbb{R}^{2n}$ be a trivialization of the underlying real vector bundle and define

$$\omega' := (\phi^{-1})^* \omega.$$

Since $\phi$ is a vector bundle isomorphism, it is a symplectomorhism from $(E, \omega)$ to $(B \times \mathbb{R}^{2n}, \omega')$. Therefore it is sufficient to show that $(B \times \mathbb{R}^{2n}, \omega')$ is symplectically trivial.

Define the following smooth matrix field on $B$:

$$A : B \to M(2n, \mathbb{R}), \quad (A(b))_{jk} = -\omega'_b(e_j, e_k).$$

Since at each point $b \in B$, $A(b)$ is skew-symmetric, $iA : B \to M(2n, \mathbb{C})$ is a self-adjoint matrix field. It follows that $iA$ defines a Hermitian structure on the vector bundle $B \times \mathbb{C}^{2n}$:

$$h_b(z_1, z_2) := \bar{z}_1^T iA(b) z_2 \quad \text{for } z_1, z_2 \in \mathbb{C}^{2n}.$$

Applying Proposition 1.1.26, we may choose a unitary frame on $B$,

$$\{X_j + iY_j |\ X_j, Y_j : B \to \mathbb{R}^{2n},\ j = 1, \ldots, 2n\}.$$

At each point $b \in B$ the vector $(X_j + iY_j)(b)$ is an eigenvector of $iA(b)$. Since $iA$ is purely imaginary, if $\lambda$ is an eigenvalue of $iA(b)$, so is $-\lambda$. By non-degeneracy of $\omega$, zero can not be an eigenvalue, and hence $iA$ has $n$ positive eigenvalues at each point $b \in B$. Notice that the eigenvalues $\lambda_j$ of $iA$ are smooth functions since they can be expressed as

$$\lambda_j : B \to \mathbb{R}, \quad b \mapsto (X_j(b) - iY_j(b))^T (iA(b))(X_j(b) + iY_j(b)).$$

It follows from the intermediate value theorem that if $\lambda_j(b) > 0$ for some $b \in B$, then $\lambda_j$ is positive everywhere in $B$. Therefore there are $n$ positive eigenfunctions $\lambda_j$. Reordering the symplectic frame if necessary, we may assume $\{X_j + iY_j |\ j = 1, \ldots, n\}$ consists of eigenvector fields with positive eigenfunctions $\lambda_j$. This allows us to define the following $2n$ sections of $B \times \mathbb{R}^{2n}$:

$$X'_j := \left(\frac{2}{\lambda_j}\right)^{1/2} X_j, \quad Y'_j := \left(\frac{2}{\lambda_j}\right)^{1/2} Y_j.$$

These sections form a symplectic frame for the symplectic vector bundle $(B \times \mathbb{R}^{2n}, \omega')$. Define the following map

$$\rho : B \times \mathbb{R}^{2n} \to B \times \mathbb{R}^{2n},$$

$$\left(b, \sum_{j=1}^{n} (\nu^j X'_j(b) + \xi^j Y'_j(b))\right) \mapsto \left(b, \sum_{j=1}^{n} (\nu^j e_j + \xi^j f_j)\right),$$

where $\nu^j, \xi^j \in \mathbb{R}$, $j \in \{1, \dots, n\}$. The map $\rho$ is a symplectic trivialization of $(B \times \mathbb{R}^{2n}, \omega')$ and therefore $(E, \omega)$ is symplectically trivial. ∎

**Corollary 3.1.8.** *A symplectic vector bundle $(E, \omega)$ over a contractible base $B$ is symplectically trivial.*

In particular, a symplectic vector bundle over the disk is symplectically trivial. This is the base case in the pair of pants induction which is used to prove the following theorem. We refer the reader to [MS98] for the proof.

**Proposition 3.1.9.** *[MS98, p. 72] A symplectic vector bundle $E \to \Sigma$ over a compact oriented surface $\Sigma$ with non-empty boundary admits a symplectic trivialization.*

Next we generalize the concepts of symplectic and Lagrangian subspaces, as well as complex structures on symplectic vector spaces, to symplectic vector bundles.

**Definition 3.1.10.** *A **Lagrangian subbundle** $L$ of a symplectic vector bundle $(E, \omega)$ is a subbundle of $E$ such that for every $b \in B$, $L_b$ is a Lagrangian subspace of $(E_b, \omega_b)$. **Symplectic subbundles** of a symplectic vector bundle are defined analogously.*

**Remark 3.1.11.** The following observations follow directly from the definitions:

- Let $(E, \omega)$ and $(E', \omega')$ be symplectic vector bundles. If $\eta : E \to E'$ is a symplectomorphism and $L$ is a Lagrangian subbundle of $E$, then $\eta(L)$ is a Lagrangian subbundle of $E'$ (see Remark 1.1.16).

- If $f : X \to B$ is a map from a smooth manifold $X$ into the base of the symplectic vector bundle $(E, \omega)$ with Lagrangian subbundle $L$, then $f^*L$ is a Lagrangian subbundle of the pullback bundle $(f^*E, f^*\omega)$ (see Remark 3.1.3).

**Definition 3.1.12.** *A* ***complex structure*** *on a vector bundle $\pi : E \to B$ is a smooth section $J$ of the endomorphism bundle of $E$ satisfying*

$$J_b^2 = -\mathrm{Id}, \text{ for all } b \in B.$$

*A vector bundle $E$ equipped with a complex structure $J$ is called a* ***complex vector bundle*** *and is denoted by $(E, J)$. If $E$ also has a symplectic structure $\omega$, the complex structure $J$ is called* $\omega$***-tame*** *if $J_b$ is an $\omega_b$-tame complex structure on the vector space $E_b$ for every $b$ in $B$. The structure $J$ is called* $\omega$***-compatible*** *if $J_b$ is $\omega_b$-compatible on the vector space $E_b$ for all $b$ in $B$ (cf. Definition 2.2.4). We denote the set of $\omega$-tame and $\omega$-compatible complex structures on the symplectic vector bundle $(E, \omega)$ by $\mathcal{J}_\tau(E, \omega)$ and $\mathcal{J}(E, \omega)$ respectively. These are subsets of the set of all smooth sections of the endomorphism bundle of $E$. We endow the space of smooth sections of $E$ with the $C^\infty$-topology and we give $\mathcal{J}_\tau(E, \omega)$ and $\mathcal{J}(E, \omega)$ the subspace topology.*

*A* ***morphism*** *of complex vector bundles from $(E, J)$ to $(E', J')$ is a morphism $\eta : E \to E'$ of the underlying real vector bundles such that*

$$\eta|_{E_b} \circ J_b = J_{\bar{\eta}(b)} \circ \eta|_{E_b}$$

*for all $b \in B$. Here $\bar{\eta}$ is the map on the base spaces induced by $\eta$.*

**Remark 3.1.13.** This definition of a complex vector bundle does not conflict with Definition 1.1.9. A complex vector bundle $E \to B$ in the sense of Definition 1.1.9 possesses a canonical complex structure $J$ defined in terms of a trivializing atlas $\{(U_\alpha, \phi_\alpha)\}_\alpha$ by

$$J_b(v) := (\phi_\alpha)_b^{-1}(i(\phi_\alpha)_b(v))$$

for all $b \in U_\alpha$ and $v \in E_b$. Because the transition functions in the trivializing atlas are complex linear, this definition does not depend on the choice of $U_\alpha$ containing $b$. Conversely, a complex structure in the sense of Definition 3.1.12 on a real vector bundle $E' \to B'$ of rank $2n$ leads to a reduction of the structure group from $\mathrm{GL}(2n, \mathbb{R})$

to $\mathrm{GL}(n,\mathbb{C})$. This results in a complex vector bundle of rank $n$ in the sense of Definition 1.1.9. Morphisms of complex vector bundles as defined in Definition 3.1.12 are precisely those morphisms of the underlying real vector bundles which are complex linear (cf. the discussion after Remark 2.2.2).

In Remark 1.1.29, we noted that the underlying real vector bundle of a complex vector bundle is always orientable. The equivalence of the two definitions of a complex vector bundle then implies that a real vector bundle equipped with a complex structure is also orientable.

**Remark 3.1.14.** An $\omega$-tame complex structure on $(E,\omega)$ induces a Riemannian structure $g_J$ on $E$ defined by

$$g_J(b)(v,w) = \frac{1}{2}(\omega_b(v,J_b w) - \omega_b(J_b v,w))$$

for all $v,w \in E_b$ and all $b \in B$. If $J$ is $\omega$-compatible, $g_J$ has the form

$$g_J(b)(v,w) = \omega_b(v,J_b w).$$

In the compatible case, it follows from Remark 2.2.5 that $J$ is a $g_J$-skew-adjoint section of $\mathrm{End}(E)$.

Finally we generalize Proposition 2.2.6 about the contractibility of the spaces of $\Omega$-tame and $\Omega$-compatible complex structures on symplectic vector spaces to symplectic vector bundles.

**Theorem 3.1.15.** *Suppose $(E,\omega)$ is a symplectic vector bundle. Then both $\mathcal{J}(E,\omega)$ and $\mathcal{J}_\tau(E,\omega)$ are non-empty and contractible in the $C^\infty$-topology.*

**Proof:** Let $B$ be the base of the symplectic vector bundle $(E,\omega)$. The spaces $\mathcal{J}(E,\omega)$ and $\mathcal{J}_\tau(E,\omega)$ are spaces of sections of the locally trivial fibre bundles with the fibres over a given point $b \in B$ being $\mathcal{J}(E_b,\omega_b)$ and $\mathcal{J}_\tau(E_b,\omega_b)$ respectively. By Proposition 2.2.6 these fibres are contractible. Contractibility of $\mathcal{J}(E_b,\omega_b)$ and

$\mathcal{J}_\tau(E_b, \omega_b)$ then follows from the general fact that spaces of sections of locally trivial fibre bundles with contractible fibre are themselves contractible [Can06, p. 112]. ∎

## 3.2 Symplectic manifolds

Just as a Riemannian metric provides a means of measuring the length of curves in a manifold, a symplectic form provides a means of measuring two-dimensional area. However this analogy between symplectic and Riemannian geometry fails in one important respect: There is no counterpart in Riemannian geometry to the closedness condition on the symplectic manifold. Indeed, it might be more appropriate to compare Riemannian manifolds to *almost* symplectic manifolds, as defined next.

**Definition 3.2.1.** *An **almost symplectic manifold** $(M, \omega)$ is a manifold $M$ equipped with a non-degenerate two-form $\omega$. In other words, the tangent bundle $TM$ of $M$ is a symplectic vector bundle. An almost symplectic manifold $(M, \omega)$ is called a **symplectic manifold** if the form $\omega$ is closed.*

It follows from Remark 2.1.3 that symplectic manifolds are always even-dimensional and possess a natural orientation determined by the volume form $\wedge^n \omega$ where $\dim(M) = 2n$.

**Example 3.2.2.**

1. The most basic example of a symplectic manifold is $\mathbb{R}^{2n}$ equipped with the symplectic form $\omega_0 = \sum_{j=1}^n dx^j \wedge dy^j$.

2. In dimension two, every two-form is trivially closed and the non-degeneracy condition is equivalent to requiring that the symplectic form be a volume form. Therefore every oriented surface is symplectic. Moreover, symplectic surfaces

are completely classified: Two symplectic surfaces are symplectomorphic if and only if they have the same genus and total area [Can01, p. 48].

The study of $J$-holomorphic curves revolves largely around maps from the symplectic surface $(S^2, \omega_{\mathrm{vol}})$ into an arbitrary symplectic manifold. Here $\omega_{\mathrm{vol}}$ is the volume form on the 2-sphere, given locally in cylindrical coordinates by $dh \wedge d\theta$. However we will be mainly interested in maps from the symplectic unit disc in $\mathbb{C}$ satisfying Lagrangian boundary conditions.

3. Cotangent bundles are always symplectic. Recall the definition of the Liouville one-form on a contangent bundle $T^*M$: If $(x_1, \ldots, x_n)$ are local base coordinates and $(p_1, \ldots, p_n)$ are corresponding local fibre coordinates then the Liouville one-form is locally given by
$$\theta_{T^*M} = \sum_{i=1}^{n} p_i dx_i.$$
This definition does not depend on the choice of local coordinates and thus $\theta$ determines a globally defined one-form on $T^*M$. The symplectic form on $T^*M$ is $\omega_{T^*M} := -d\theta_{T^*M}$.

4. Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be symplectic manifolds. Then for $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, $M_1 \times M_2$ is symplectic with symplectic form
$$\lambda_1\omega_1 \oplus \lambda_2\omega_2 := \lambda_1 p_1^*\omega_1 + \lambda_2 p_2^*\omega_2,$$
where $p_j : M_1 \times M_2 \to M_j$, $j = 1, 2$, are the projection maps.

5. We mention also some "non-examples": The sphere $S^n$ is not symplectic for $n \neq 2$. For odd $n$ this is clear. For even $n$, this follows from Stokes' theorem. Suppose $\omega \in \Omega^2(S^{2k})$ is symplectic. As $d\omega = 0$, $\omega$ represents a de Rham homology class in $H^2_{\mathrm{dR}}(S^{2k})$. If $k \neq 1$, $H^2_{\mathrm{dR}}(S^{2k})$ is trivial and thus $\omega$ is exact. Choose $\eta \in \Omega^1(M)$ such that $\omega = d\eta$. Then the integral of the volume form

$\wedge^k\omega$ satisfies

$$0 \neq \int_{S^{2k}} \wedge^k\omega = \int_{S^{2k}} d\eta \wedge \left(\wedge^{k-1}\omega\right) = \int_{S^{2k}} d\left(\eta \wedge \left(\wedge^{k-1}\omega\right)\right),$$
$$= \int_{\partial S^{2k}} \eta \wedge \left(\wedge^{k-1}\omega\right).$$

This last integral must be zero as the boundary of $S^{2k}$ is empty. Thus we have a contradiction and we conclude that $S^{2k}$ is not symplectic for $k \neq 1$. This argument implies more generally that an **exact symplectic manifold**, i.e. a symplectic manifold whose symplectic form is exact, must either be non-compact or have non-empty boundary. Note that Examples 1 and 3 above are exact but non-compact. More specifically, we have a cohomological obstruction for a compact manifold with empty boundary to be symplectic: If $H^2_{\mathrm{dR}}(M)$ is trivial, $M$ cannot be symplectic.

In the next section we will also see how a Kähler potential on a complex manifold determines a symplectic form. This leads to more examples of symplectic manifolds, notably complex projective space $\mathbb{CP}^n$.

In the introduction, we discussed the Arnold conjecture, which in one form places a lower bound on the number of fixed points of a Hamiltonian diffeomorphism. Next we define precisely the concept of a Hamiltonian diffeomorphism, which is also used in some situations to compute the Lagrangian quantum homology.

**Definition 3.2.3.** *Let $(M, \omega)$ be a symplectic manifold. A smooth function $H \in C^\infty(M)$ is called a* ***Hamiltonian****. The function $H$ determines a vector field $X_H$ on $M$, defined by*

$$dH = \iota_{X_H}\omega,$$

*where $\iota_{X_H}$ is the interior product, given by*

$$\iota_{X_H} : \Omega^p(M) \to \Omega^{p-1}(M), \quad \iota_{X_H}(\sigma)(X_1, \ldots, X_{p-1}) = \sigma(X_H, X_1, \ldots, X_{p-1}),$$

*for vector fields $X_1, \ldots, X_{p-1}$ on $M$. A vector field of the form $X_H$ for some Hamiltonian $H$ is called a* ***Hamiltonian vector field****. In other words, a vector field $X$ on $M$ is Hamiltonian precisely when the one-form $\iota_X\omega$ is exact.*

If $M$ is compact, the flow $\rho_t$ of $X_H$ exists for all time. Moreover, at any $t \in \mathbb{R}$, $\rho_t$ is a symplectomorphism of $M$: Using the identity $\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_H}\omega$ [Can01, p. 34],

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_H}\omega = \rho_t^*(d(\iota_{X_H}\omega) + \iota_{X_H}(d\omega)) = 0. \tag{3.2.1}$$

Hence $\rho_t^*\omega$ is constant and equal to $\rho_0^*\omega = \omega$ for all time. The second equality in Equation 3.2.1 comes from Cartan's formula for the Lie derivative. We have also used closedness of $\omega$ and the fact that $\iota_{X_H}\omega$ is exact.

One can also consider time-dependent Hamiltonians on compact symplectic manifolds. A time-dependent Hamiltonian vector field $X_{H_t}$ can also be integrated, but yields a smooth isotopy $\rho_t$ rather than a flow. However, the identity $\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_{H_t}}\omega$ still holds. Therefore Equation 3.2.1 is still valid and again implies that the diffeomorphism $\rho_t$ is a symplectomorphism for all $t$.

**Definition 3.2.4.** *Symplectomorphisms which are the time one map of a Hamiltonian isotopy are called* ***Hamiltonian diffeomorphisms****. The Hamiltonian diffeomorphisms form a normal subgroup of* $\mathrm{Symp}(M, \omega)$ *[MS98, p. 309].*

The existence of symplectic bases for symplectic vector spaces (Theorem 2.1.9) ensures that that the tangent spaces of two symplectic manifolds of the same dimension are symplectomorphic. In other words, a local diffeomorphism between the manifolds can be chosen such that it induces a symplectomorphism on the tangent space at a given point. The following theorem, which is proved using Moser's trick, says that much more is true: Around any two points in two symplectic manifolds of the same dimension, there exist neighbourhoods which are symplectomorphic. That is, there exists a local diffeomorphism that induces a symplectomorphism on the tangent space at *every point* in its domain.

**Theorem 3.2.5** (Darboux's theorem)**.** *[MS98, p. 95] An almost symplectic manifold $(M, \omega)$ is symplectic if and only if around every point $p \in M$ there exists a coordinate neighbourhood $U$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, such that the almost symplectic form has the local expression*

$$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k.$$

Finally we define symplectic and Lagrangian submanifolds. The following chapter is devoted to a detailed description of the latter.

**Definition 3.2.6.** *A **symplectic submanifold** of a symplectic manifold $(M, \omega)$ is a submanifold $N \subset M$ such that $(N, \omega|_N)$ is again symplectic. As the restriction $\omega|_N$ will be closed if $\omega$ is closed, symplectic submanifolds are precisely those submanifolds whose tangent bundles are symplectic subbundles of $TM$. A **Lagrangian submanifold** $L$ of $(M, \omega)$ is an $n$-dimensional submanifold of $M$ whose tangent space at each point $x \in M$ is a Lagrangian subspace of the symplectic vector space $(T_xM, \omega_x)$. That is, a submanifold $L$ is a Lagrangian submanifold of $M$ if and only if $TL$ is a Lagrangian subbundle of $TM|_L$.*

## 3.3 Almost complex structures

In the previous section, we first considered almost symplectic manifolds, manifolds whose tangent bundle is a symplectic vector bundle. We now consider manifolds whose tangent bundle is a complex vector bundle.

**Definition 3.3.1.** *An **almost complex structure** on a manifold $M$ is a complex structure $J$ on the tangent bundle $TM$ of $M$. The pair $(M, J)$ is called an **almost complex manifold**. If $(N, j)$ is another almost complex manifold, a $(J, j)$-**holomorphic map** from $M$ to $N$ is a smooth map $f : M \to N$ such that the tangent map $df : TM \to TN$ is a morphism of complex vector bundles.*

*If $M$ is symplectic with symplectic form $\omega$, **tameness** and **compatibility** of almost complex structures on $(M, \omega)$ are defined as in Definition 3.1.12. That is, an almost complex structure $J$ is $\omega$-tame if, for all $p \in M$,*

$$\omega_p(v, J_p v) > 0$$

*for all non-zero $v \in T_pM$. The almost complex structure is called $\omega$-compatible if it is $\omega$-tame and*

$$\omega_p(J_p v, J_p w) = \omega_p(v, w)$$

*for all $v, w \in T_pM$ and all $p \in M$. The spaces of $\omega$-tame and $\omega$-compatible complex structures on $M$ are denoted by $\mathcal{J}_\tau(M, \omega)$ and $\mathcal{J}(M, \omega)$ respectively. These spaces are again equipped with the $C^\infty$-topology.*

Almost complex manifolds are always even-dimensional and possess a natural orientation (see Remark 1.1.29). By Remark 3.1.14 applied to the symplectic vector bundle $(TM, \omega)$, a tame almost complex structure on the symplectic manifold $(M, \omega)$ induces a Riemannian metric $g_J$ on $M$. We also have a version of Theorem 3.1.15 for almost complex manifolds, which is simply a statement of that theorem applied to the symplectic vector bundle $(TM, \omega)$.

**Theorem 3.3.2.** *[MS98, p. 118], [MS12, p. 17–18] Every almost symplectic manifold has a compatible almost complex structure and the spaces $\mathcal{J}_\tau(M, \omega)$ and $\mathcal{J}(M, \omega)$ are contractible.*

The 'almost' in 'almost complex structure' refers to the fact that a manifold equipped with an almost complex structure need not have the structure of a complex manifold. Recall that a complex manifold is a smooth manifold whose coordinate charts are homeomorphisms from open subsets of the manifold to open sets in $\mathbb{C}^n$ for some $n$, and whose transition maps are biholomorphic. Every complex manifold $M$ possesses the obvious natural complex structure.

**Definition 3.3.3.** *An almost complex structure $J$ on a manifold $M$ is called* ***integrable*** *if $J$ is induced by the structure of a complex manifold on $M$. An integrable almost complex structure is also simply called a complex structure. A symplectic form which is compatible (in the sense of Definition 3.3.1) with an integrable complex structure is called a* ***Kähler form****.*

The Newlander-Nirenberg theorem gives a criterion for the integrability of an arbitrary almost complex structure in terms of the vanishing of the Nijenhuis tensor [NN57].

Almost complex structures, when they exist, are never unique. A given almost complex structure can always be perturbed to produce other almost complex structures. There was once speculation about the possibility that every almost complex structure is homotopic to an integrable one. In 1966, Van de Ven showed that this is not the case when he constructed the first examples of (compact) almost complex manifolds which do not admit a complex structure [VdV66]. Despite this result and more recent progress, much remains unknown about the existence of complex structures. The six-sphere, for example, is the orthogonal complement of $i$ in the unit sphere in the octonions, and inherits an almost complex structure from the octonion multiplication. However, due to the non-associativity of the octonions, this almost complex structure is not integrable. It is not yet known if $S^6$ possesses an almost complex structure which is integrable [Can06, p. 116].

We will not need to concern ourselves with the integrability of our almost complex structures. We mention the relationship between complex and almost complex structures only for general background and to justify the terminology.

**Example 3.3.4.** Examples of almost complex structures include those induced by the structure of a complex manifold on $\mathbb{R}^{2n}$, on the sphere $S^2$, on any orientable surface, or on $\mathbb{CP}^n$. As we hinted, it requires some work to give examples of almost complex structures which are not induced by the structure of a complex manifold. Indeed it is a non-trivial problem to construct examples of symplectic manifolds which are not Kähler (although they do exist – the first compact examples appeared in Thurston's famous two-page paper [Thu76]).

In proving the Künneth theorem in Chapter 4, we will endow a product almost complex manifold with a product almost complex structure, as defined next.

**Example 3.3.5.** Let $(M_1, J_1)$ and $(M_2, J_2)$ be almost complex manifolds, and let

$$p_1 : M_1 \times M_2 \to M_1, \quad p_2 : M_1 \times M_2 \to M_2,$$

be the projection maps. Furthermore, let

$$\iota_s : M_2 \hookrightarrow M_1 \times M_2, \quad y \mapsto (s, y)$$

be the inclusion of $M_2$ in $M_1 \times M_2$ at the point $s \in M_1$, and let $\iota_r$ be the analogous inclusion of $M_1$ in $M_1 \times M_2$ at the point $r \in M_2$. Then $M_1 \times M_2$ has the almost complex structure $J_1 \oplus J_2$ defined by

$$(J_1 \oplus J_2)_x(v) := ({\iota_{p_2(x)}}_*)_{p_1(x)}(J_1)_{p_1(x)}(p_{1*})_x v + ({\iota_{p_1(x)}}_*)_{p_2(x)}(J_2)_{p_2(x)}(p_{2*})_x v,$$

for $x \in M_1 \times M_2$ and $v \in T_x(M_1 \times M_2)$.

In Example 3.3.8 we describe the symplectic structure on $\mathbb{CP}^n$ for arbitrary $n$. The definition of the symplectic form on $\mathbb{CP}^n$ relies on the following recipe for defining Kähler forms on complex manifolds.

**Definition 3.3.6.** *Let $M$ be a complex manifold. A function $\rho \in C^\infty(M)$ is called **strictly plurisubharmonic** if, with respect to any complex coordinate chart $(U, \phi)$ on $M$, the matrix*

$$\frac{\partial^2 \rho \circ \phi^{-1}}{\partial z_j \partial \bar{z}_k}(x)$$

*is positive-definite at every point $x \in \phi(U)$.*

**Proposition 3.3.7.** *[Can01, p. 90] If $\rho \in C^\infty(M)$ is a strictly plurisubharmonic function on the complex manifold $M$, then the two-form*

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho$$

*is Kähler.*

The operators $\partial$ and $\bar{\partial}$ are the Dolbeault operators on complex-valued forms

$$\partial : \Omega^{l,m}(M) \to \Omega^{l+1,m}(M), \quad \bar{\partial} : \Omega^{l,m}(M) \to \Omega^{l,m+1}(M).$$

We refer the reader to [Can01] for the basics of Dolbeault theory. In local complex coordinates $(z_1, \dots, z_n)$ on a complex manifold, an element $f \in \Omega^{l,m}(M)$ has the expression

$$\sum_{\substack{i_1,\cdots,i_l \\ j_1,\cdots,j_m}} f_{i_1\cdots i_l j_1\cdots j_m} dz_{i_1} \wedge \cdots \wedge dz_{i_l} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_m}.$$

The operators $\partial$ and $\bar{\partial}$ are given locally by

$$\partial(f) = \sum_r \sum_{\substack{i_1,\cdots,i_l \\ j_1,\cdots,j_m}} \frac{\partial}{\partial z_r} f_{i_1\cdots i_l j_1\cdots j_m} dz_r \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_l} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_m},$$

$$\bar{\partial}(f) = \sum_s \sum_{\substack{i_1,\cdots,i_l \\ j_1,\cdots,j_m}} \frac{\partial}{\partial \bar{z}_r} f_{i_1\cdots i_l j_1\cdots j_m} d\bar{z}_s \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_l} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_m}.$$

One can make sense of these expressions as complex-valued forms by defining real coordinates

$$x_j = \frac{1}{2}(z_j + \bar{z}_j), \quad y_j = \frac{1}{2}(z_j - \bar{z}_j).$$

The complex-valued forms $dz_j$ and $d\bar{z}_j$ are then given by

$$dz_j = dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j.$$

**Example 3.3.8.** (The Fubini-Study form.) Recall that $\mathbb{CP}^n$ is the set of complex lines through the origin in $\mathbb{C}^{n+1}$. That is,

$$\mathbb{CP}^n = \{[z_0, \dots, z_n] | \, (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}\},$$

where $[z_0, \dots, z_n]$ is the equivalence class of points in $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation $\sim$ defined by

$$(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n) \text{ for all } \lambda \in \mathbb{C} \setminus \{0\},$$

for all $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. We define complex coordinate charts on $\mathbb{CP}^n$ as follows:

$$U_j := \{[z_0, \dots, z_n] \in \mathbb{CP}^n | \ z_j \neq 0\}, \quad j \in \{0 \dots, n\},$$

$$\xi_j^n : U_j \to \mathbb{C}^n, \quad [z_0, \dots, z_n] \mapsto (\tfrac{z_0}{z_j} \dots, \tfrac{z_{j-1}}{z_j}, \tfrac{z_{j+1}}{z_j}, \dots, \tfrac{z_n}{z_j}).$$

To construct the Fubini-Study form, the standard symplectic form on $\mathbb{CP}^n$, we begin by defining a function $\rho$ on $\mathbb{C}^n$,

$$\rho : \mathbb{C}^n \to \mathbb{R}, \quad z \mapsto \ln(|z|^2 + 1).$$

It is not hard to verify that the function $\rho$ is strictly plurisubharmonic, and therefore, by Proposition 3.3.7, the two-form $\omega_{\text{FS}} := \frac{i}{2}\partial\bar{\partial}\rho$ is a Kähler form on $\mathbb{C}^n$. This form is known as the Fubini-Study form on $\mathbb{C}^n$. We define the Fubini-Study form on the coordinate domain $U_j \subset \mathbb{CP}^n$ as

$$\sigma_{\mathbb{CP}^n}|_{U_j} := (\xi_j^n)^* \omega_{\text{FS}}.$$

One can verify by a routine computation that $\sigma_{\mathbb{CP}^n}$ is well-defined on the intersections of coordinate domains, and therefore defines a global Kähler form on $\mathbb{CP}^n$ [Can01, p. 96]. Furthermore, on $\mathbb{CP}^1 \cong S^2$, $\sigma_{\mathbb{CP}^2} = \frac{1}{4}\omega_{\text{std}}$, where $\omega_{\text{std}}$ is the standard symplectic form on $S^2$.

In dimension two, any orientable manifold possesses an almost complex structure and all such almost complex structures are integrable [MS12, p. 2]. In the final section of this chapter we will consider $J$-holomorphic curves, maps defined on such manifolds, known as Riemann surfaces. We formulate our precise definition of a Riemann surface here.

**Definition 3.3.9.** *By a* ***Riemann surface****, we mean a complex manifold* $\Sigma$ *(possibly with boundary) of complex dimension one. When we want to draw attention to the complex structure* $j$ *on* $\Sigma$*, we write* $(\Sigma, j)$*. We will also consider Riemann surfaces equipped with a choice of volume form* $\mathrm{dvol}_\Sigma$ *corresponding to the natural orientation of* $\Sigma$*. In this case, we write the data as a triple* $(\Sigma, j, \mathrm{dvol}_\Sigma)$*. The fact that* $\mathrm{dvol}_\Sigma$ *induces the natural orientation on* $\Sigma$ *means that* $j$ *is* $\mathrm{dvol}_\Sigma$*-tame, and thus* $j$ *and* $\mathrm{dvol}_\Sigma$ *induce a metric* $g_\Sigma$ *on* $\Sigma$*.*

In the remainder of this section we introduce the concept of 'tameness' of symplectic manifolds, not to be confused with the concept of tameness of an almost complex structure with respect to a symplectic form. Tameness is a technical condition that the ambient symplectic manifold of a Lagrangian submanifold must satisfy in order for the Lagrangian quantum homology to be well-defined. Before defining this condition, we first need to recall some concepts from Riemannian geometry.

**Definition 3.3.10.** *Let* $(M, g)$ *be a Riemannian manifold.*

1. *For* $p \in M$*, the* ***injectivity radius at*** $p$*,* $\iota(p)$ *is the largest radius for which the exponential map at* $p$ *is injective:*

$$\iota(p) := \sup\{\rho \in \mathbb{R}_+ \mid \exp_p|_{B_\rho(0)} \textit{ is injective }\}.$$

   *We set* $\iota(p) = \infty$ *if* $\exp_p$ *is injective on all of* $T_pM$*. The* ***injectivity radius of*** $M$ *is the infimum of* $\iota(p)$ *over all points in* $M$*.*

2. *The* ***sectional curvature*** *of* $M$ *is a real-valued function* $K$ *on the* 2*-Grassmannian bundle of* $M$ *(i.e. the locally trivial fibre bundle over* $M$ *whose fibre over a point* $p$ *is the set of planes in* $T_pM$*). For a plane* $\sigma \subset T_pM$ *spanned by* $u, v \in T_pM$*, the sectional curvature is defined by*

$$K_p(\sigma) := \frac{g_p(R_p(u,v)v, u)}{g_p(u,u)g_p(v,v) - (g_p(u,v))^2}.$$

*Here $R$ is the Riemann curvature tensor of $(M, g)$. This definition does not depend on the choice of linearly independent vectors $u, v \in \sigma$.*

**Proposition 3.3.11.** *[Kli95, p. 131] The function $\iota : M \to \mathbb{R}_+ \cup \{\infty\}$ which maps every point to its injectivity radius is continuous.*

**Definition 3.3.12.** *[ALP94, p. 286] A symplectic manifold $(M, \omega)$ is called **geometrically bounded** or **tame** if there exists an almost complex structure $J$ and a complete Riemannian metric $g$ on $M$ such that the following properties are satisfied:*

1. *There exist positive constants $\alpha$ and $\beta$ such that*

$$\omega_p(v, J_p v) \geq \alpha |v|_g^2 \quad \text{and} \quad |\omega_p(v, w)| \leq \beta |v|_g |w|_g,$$

*for all $p \in M$ and all $v, w \in T_p M$.*

2. *The sectional curvature of $(M, g)$ is bounded above and its injectivity radius has a positive lower bound.*

**Remark 3.3.13.** The first condition in Definition 3.3.12 implies that $J$ is $\omega$-tame. If $M$ is compact, then any $\omega$-tame complex structure $J$ will satisfy this first condition for any Riemannian metric on $M$. To see this, first note that any Riemannian metric on a compact manifold is complete. Let $UT(M)$ be the unit tangent bundle of $M$ and define a function

$$f : UT(M) \otimes UT(M) \to \mathbb{R}, \quad (p, v \otimes w) \mapsto \omega_p(v, J_p w).$$

Set

$$\alpha = \min\{f(p, v \otimes v) | \, (p, v) \in UT(M)\},$$
$$\beta = \max\{f(p, v \otimes w) | \, (p, v \otimes w) \in UT(M) \otimes UT(M)\}.$$

These constants are well-defined by compactness of $UT(M)$, and positive by tameness of $J$. Clearly they satisfy the first condition above.

Moreover, if $M$ is compact, the second condition of Definition 3.3.12 is automatically satisfied: By Proposition 3.3.11, the infimum of the set $\mathrm{Im}(\iota) \subset \mathbb{R}_+ \cup \{\infty\}$ is positive, and thus the injectivity radius of $M$ has a positive lower bound. Furthermore, the total space of the 2-Grassmannian bundle over $M$ is also compact, as both the fibre and base of this bundle are compact. Therefore the sectional curvature is also a continuous function on a compact space and is thus bounded above.

**Example 3.3.14.**

1. Euclidean space with the standard symplectic form, $(\mathbb{R}^{2n}, \omega_0)$, is tame. One can take as almost complex structure $J_0$, and as complete Riemannian metric the standard metric $g_0$. Then the constants $\alpha = 1$ and $\beta = 1$ fulfill the first condition of Definition 3.3.12. The injectivity radius of $(\mathbb{R}^{2n}, g_0)$ is infinite and its sectional curvature vanishes everywhere. Hence the second condition of Definition 3.3.12 is also satisfied.

2. [ALP94, p. 286] The cotangent bundle of a closed manifold $M$, equipped with the standard symplectic form for cotangent bundles, is tame. This is also true if $M$ is an open subset of a closed manifold, or if $M$ is the interior of a compact manifold with boundary.

The following technical lemma about the product of tame symplectic manifolds will be needed to prove the quantum Künneth theorem.

**Lemma 3.3.15.** *Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be tame symplectic manifolds. Then the product $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ is also tame.*

**Proof:** Let $J_i$ and $g_i$, $i = 1, 2$, be the complete Riemannian metrics on $M_1$ and $M_2$ satisfying the conditions of Definition 3.3.12. Let $\alpha_i$ and $\beta_i$ be the corresponding constants for the first condition. We show that the product structures $J_1 \oplus J_2$ and $g_1 \oplus g_2$ fulfil these conditions for the product manifold $M_1 \times M_2$.

1. Recall the definition of the product almost complex structure $J_1 \oplus J_2$, (see

Example 3.3.5)

$$(J_1 \oplus J_2)_x(v) := (\iota_{p_2(x)_*})_{p_1(x)}(J_1)_{p_1(x)}(p_{1*})_x v + (\iota_{p_1(x)_*})_{p_2(x)}(J_2)_{p_2(x)}(p_{2*})_x v,$$

for $x \in M_1 \times M_2$ and $v \in T_x(M_1 \times M_2)$. Let $v_1 = (p_{1*})_x v$ and $v_2 = (p_{2*})_x v$.

We have

$$
\begin{aligned}
(\omega_1 \oplus \omega_2)_x(v, (J_1 \oplus J_2)_x v) &= (\omega_1)_{p_1(x)}(v_1, (p_{1*})_x (J_1 \oplus J_2)_x v) \\
&\quad + (\omega_2)_{p_2(x)}(v_2, (p_{2*})_x (J_1 \oplus J_2)_x v), \\
&= (\omega_1)_{p_1(x)}(v_1, (J_1)_{p_1(x)} v_1) + (\omega_2)_{p_2(x)}(v_2, (J_2)_{p_2(x)} v_2), \\
&\geq \alpha_1 |v_1|^2_{g_1} + \alpha_2 |v_2|^2_{g_2}, \\
&\geq \min(\alpha_1, \alpha_2)(|(p_{1*})_x v|^2_{g_1} + |(p_{2*})_x v|^2_{g_2}), \\
&= \min(\alpha_1, \alpha_2) |v|^2_{g_1 \oplus g_2}.
\end{aligned}
$$

Similarly, for $v, w \in T_x(M_1 \times M_2)$, a routine computation shows that

$$
(\omega_1 \oplus \omega_2)_x(v, w) \leq \max(\beta_1, \beta_2) |v|_{g_1 \oplus g_2} |w|_{g_1 \oplus g_2}.
$$

2. A computation shows that the Riemann curvature tensor $R_1 \oplus R_2$ for the metric $g_1 \oplus g_2$ is given by

$$
\begin{aligned}
(R_1 \oplus R_2)_x(u, v)(w) &= (\iota_{p_2(x)_*})_{p_1(x)} \left[ (R_1)_{p_1(x)}(u_1, v_1) w_1 \right] \\
&\quad + (\iota_{p_1(x)_*})_{p_2(x)} \left[ (R_2)_{p_2(x)}(u_2, v_2) p_{2*} w_2 \right],
\end{aligned}
$$

for $x \in M_1 \times M_2$, $u, v, w \in T_x(M_1 \times M_2)$, and

$$
u_i := (p_{i*})_x u, \quad v_i := (p_{i*})_x v, \quad w_i := (p_{i*})_x w,
$$

for $i = 1, 2$. It follows that if $u$ and $v$ are linearly independent, the sectional curvature of the plane $\sigma \subset T_x(M_1 \times M_2)$ spanned by $u$ and $v$ is

$$
\begin{aligned}
(K_1 \oplus K_2)_x(\sigma) &= \frac{(g_1 \oplus g_2)_x((R_1 \oplus R_2)_x(u, v)v, u)}{|u|^2_{g_1 \oplus g_2} |v|^2_{g_1 \oplus g_2} - ((g_1 \oplus g_2)_x(u, v))^2}, \\
&= \frac{(g_1)_{p_1(x)}((R_1)_{p_1(x)}(u_1, v_1)v_1, u_1) + (g_2)_{p_2(x)}((R_2)_{p_2(x)}(u_2, v_2)v_2, u_2)}{(|u_1|^2_{g_1} + |u_2|^2_{g_2})(|v_1|^2_{g_1} + |v_2|^2_{g_2}) - ((g_1)_{p_1(x)}(u_1, v_1) + (g_2)_{p_2(x)}(u_2, v_2))^2}.
\end{aligned}
$$

Consider the denominator in the above expression:

$$
\begin{aligned}
(|u_1|_{g_1}^2 + |u_2|_{g_2}^2)(|v_1|_{g_1}^2 + |v_2|_{g_2}^2) &- ((g_1)_{p_1(x)}(u_1,v_1) + (g_2)_{p_2(x)}(u_2,v_2))^2, \\
&= |u_1|_{g_1}^2|v_1|_{g_1}^2 - ((g_1)_{p_1(x)}(u_1,v_1))^2 + |u_2|_{g_1}^2|v_2|_{g_2}^2 - ((g_2)_{p_2(x)}(u_2,v_2))^2 \\
&\quad + |u_1|_{g_1}^2|v_2|_{g_2}^2 + |u_2|_{g_1}^2|v_1|_{g_2}^2 - 2(g_1)_{p_1(x)}(u_1,v_1)(g_2)_{p_2(x)}(u_2,v_2).
\end{aligned}
$$

By the Cauchy-Schwarz inequality,

$$|u_i|_{g_i}^2|v_i|_{g_i}^2 - ((g_i)_{p_i(x)}(u_i,v_i))^2 \geq 0,$$

for $i = 1, 2$. Also by the Cauchy-Schwarz inequality

$$
\begin{aligned}
|u_1|_{g_1}^2|v_2|_{g_2}^2 &+ |u_2|_{g_1}^2|v_1|_{g_2}^2 - 2(g_1)_{p_1(x)}(u_1,v_1)(g_2)_{p_2(x)}(u_2,v_2) \\
&\geq |u_1|_{g_1}^2|v_2|_{g_2}^2 + |u_2|_{g_1}^2|v_1|_{g_2}^2 - 2|u_1|_{g_1}|v_1|_{g_1}|u_2|_{g_2}|v_2|_{g_2}, \\
&= (|u_1|_{g_1}|v_2|_{g_2} - |u_2|_{g_1}|v_1|_{g_2})^2.
\end{aligned}
$$

Hence the denominator in the expression for the sectional curvature of $\sigma$, $(K_1 \oplus K_2)_x(\sigma)$ is positive and bounded below by

$$|u_i|_{g_i}^2|v_i|_{g_i}^2 - ((g_i)_{p_i(x)}(u_i,v_i))^2 \geq 0$$

for $i = 1, 2$. As $u$ and $v$ are linearly independent, at least one of these lower bounds is strictly positive. We consider two cases: One of the sets $\{u_1, v_1\}$ or $\{u_2, v_2\}$ is linearly dependent, or both are are linearly independent. In the first case, assume without loss of generality that $\{u_1, v_1\}$ is linearly dependent. Then the sectional curvature of $\sigma$ satisfies

$$
\begin{aligned}
(K_1 \oplus K_2)_x(\sigma) \leq \frac{(g_1)_{p_1(x)}((R_1)_{p_1(x)}(u_1,v_1)v_1,u_1)}{|u_2|_{g_2}^2|v_2|_{g_2}^2 - ((g_2)_{p_2(x)}(u_2,v_2))^2} \\
+ \frac{(g_2)_{p_2(x)}((R_2)_{p_2(x)}(u_2,v_2)v_2,u_2)}{|u_2|_{g_2}^2|v_2|_{g_2}^2 - ((g_2)_{p_2(x)}(u_2,v_2))^2}.
\end{aligned}
\tag{3.3.1}
$$

By the symmetry properties of the Riemann tensor, the first term in this sum vanishes: In general, for tangent vectors $t, s \in T_pM$ on a Riemannian manifold $M$, the Riemann curvature satisfies

$$R_p(s,t) = -R_p(t,s).$$

This implies that $R_p(t,s) = 0$ if $t$ and $s$ are linearly dependent. The second term in the inequality (3.3.1) is simply the sectional curvature $(K_2)_{p_2(x)}(\langle u_2, v_2\rangle)$. This quantity is bounded above by assumption.

If both $\{u_1, v_1\}$ and $\{u_2, v_2\}$ span planes, then the sectional curvature of $\sigma$ satisfies

$$\begin{aligned}(K_1 \oplus K_2)_x(\sigma) \leq \frac{(g_1)_{p_1(x)}((R_1)_{p_1(x)}(u_1, v_1)v_1, u_1)}{|u_1|^2_{g_1}|v_1|^2_{g_1} - ((g_1)_{p_1(x)}(u_1, v_1))^2} \\ + \frac{(g_2)_{p_2(x)}((R_2)_{p_2(x)}(u_2, v_2)v_2, u_2)}{|u_2|^2_{g_2}|v_2|^2_{g_2} - ((g_2)_{p_2(x)}(u_2, v_2))^2}, \\ = (K_1)_{p_1(x)}(\langle u_1, v_1\rangle) + (K_2)_{p_2(x)}(\langle u_2, v_2\rangle).\end{aligned}$$

Boundedness of $K_1$ and $K_2$ therefore implies boundedness of $K_1 \oplus K_2$.

Similarly, one can verify using the geodesic equation that the product of geodesics in $M_1$ and $M_2$ is a geodesic in $M_1 \times M_2$. It follows that the exponential map, $\exp_1 \oplus \exp_2$, for $M_1 \times M_2$ is of the form

$$(\exp_1 \oplus \exp_2)_x(v) = ((\exp_1)_{p_1(x)}((p_{1*})_x(v)), (\exp_2)_{p_2(x)}((p_{2*})_x(v))).$$

From this, one sees that the injectivity radius of $M_1 \times M_2$ is the minimum of the injectivity radii of $M_1$ and $M_2$. Therefore, if the injectivity radii of both $M_1$ and $M_2$ have positive lowers bounds, so too will the injectivity radius of $M_1 \times M_2$. ∎

## 3.4 The first Chern class for symplectic vector bundles

In this section, we introduce an integer invariant of symplectic vector bundles over oriented surfaces without boundary, the first Chern number. We show that this invariant also determines a homomorphism, called the first Chern class[1], from the second homotopy group of the base space of an arbitrary symplectic vector bundle to the integers.

The definition of the first Chern number will rely on Proposition 3.1.9 and on the concept of a decomposition of a surface.

**Definition 3.4.1.** *Let $\Sigma$ be a compact oriented surface with empty boundary. A* ***decomposition*** *of $\Sigma$ is a pair of submanifolds, $\Sigma_1$ and $\Sigma_2$, of $\Sigma$, such that*

$$\Sigma = \Sigma_1 \cup \Sigma_2, \quad \partial\Sigma_1 = \partial\Sigma_2.$$

Decompositions of a surface always exist; One can simply cut a disc $D$ out of the surface $\Sigma$ and take the decomposition $D \cup (\Sigma \setminus D)$, for example. This fact, together with Proposition 3.1.9, implies that symplectic vector bundles over compact oriented surfaces without boundary are constructed by gluing together two trivial bundles. The definition of the first Chern number is based on this observation.

**Definition 3.4.2.** *Let $E \to \Sigma$ be a symplectic vector bundle of rank $2n$ over a compact, connected, oriented surface without boundary. Choose a decomposition of $\Sigma$,*

$$\Sigma = \Sigma_1 \cup \Sigma_2.$$

[1]The first Chern class is usually defined for complex vector bundles: If $E \to B$ is a complex vector bundle, the first Chern class is an element of $H^2(B, \mathbb{Z})$. In remark 3.4.9, we will clarify the relationship between the usual first Chern class for complex vector bundles and the first Chern class for symplectic vector bundles that we define here.

*By Proposition 3.1.9, there exist symplectic trivializations*

$$\phi_1 : E|_{\Sigma_1} \to \Sigma_1 \times \mathbb{R}^{2n}, \quad \phi_2 : E|_{\Sigma_2} \to \Sigma_2 \times \mathbb{R}^{2n}.$$

*Let $C = \partial\Sigma_1 = \partial\Sigma_2$. The trivializations $\phi_1$ and $\phi_2$ determine a map*

$$\Psi := \phi_{12} : C \to \mathrm{Sp}(2n), \quad p \mapsto (\phi_1)_p \circ (\phi_2)_p^{-1}.$$

*Moreover, as $\Sigma_1$ and $\Sigma_2$ are compact, the 1-manifold $C$ is a disjoint union of circles. We endow $C$ with the orientation induced by the orientation of $\Sigma_1 \subset \Sigma$. To define this orientation, let $X : C \to T\Sigma|_C$ be a normal vector field on $C$ which points out of $\Sigma_1$. Then a vector $v \in T_pC$ is considered positively oriented if $(X(p), v)$ is a positively oriented basis of $T_p\Sigma$. The* ***first Chern number*** *of $E$ is defined*[2] *to be*

$$c_1(E) = \deg(\rho \circ \Psi),$$

*where $\rho : \mathrm{Sp}(2n) \to S^1$ is the map defined in Definition 2.5.2. More precisely, if for each connected component $C_j$ of $C$, we choose an orientation-preserving diffeomorphism $\eta_j : S^1 \to C_j$, the first Chern number of $E$ is the sum of the Maslov indices of the loops $\Psi|_{C_j} \circ \eta_j$,*

$$c_1(E) = \sum_{j=1}^{l} \mu(\Psi|_{C_j} \circ \eta_j),$$

*where $l$ is the number of connected components of $C$.*

The next proposition will establish the basic properties and the well-definedness of the first Chern number, but first we consider an example.

**Example 3.4.3.** We compute the first Chern number of the tangent bundle of the sphere $S^2$ with the standard symplectic form, locally expressed in spherical coordinates by $\omega_{\text{vol}} = \sin\phi d\phi \wedge d\theta$ except at the poles. As a decomposition of $S^2$, we take

[2] If $C$ has more than one component, the degree of $\rho \circ \Psi$ is defined to be the sum of the degrees of the restrictions of $\rho \circ \Psi$ to the connected components of $C$.

the upper and lower hemispheres $D_+$, and $D_-$. We define the following symplectomorphisms:

$$f_1 : (D_+, \omega_{\mathrm{vol}}) \to (\overline{B}_{\sqrt{2}}(0), \omega_0), \quad (x,y,z) \mapsto \left(\sqrt{\frac{2}{1+z}}x, \sqrt{\frac{2}{1+z}}y\right),$$

$$f_2 : (D_-, \omega_{\mathrm{vol}}) \to (\overline{B}_{\sqrt{2}}(0), \omega_0), \quad (x,y,z) \mapsto \left(\sqrt{\frac{2}{1-z}}x, -\sqrt{\frac{2}{1-z}}y\right),$$

where $\overline{B}_{\sqrt{2}}(0)$ denotes the closed unit ball in $\mathbb{R}^2$ of radius $\sqrt{2}$ and $\omega_0$ is the restriction to $\overline{B}_{\sqrt{2}}(0)$ of the standard symplectic form on $\mathbb{R}^2$. The second map is the Lambert azimuthal equal-area projection onto the plane tangent to the south pole, with a minus sign in the second coordinate to preserve orientation. The first map is the analogous projection onto the plane tangent to the north pole. As these maps are symplectomorphisms, the tangent maps

$$\phi_1 := df_1 : TD_+ \to T\overline{B}_{\sqrt{2}}(0), \quad \phi_2 := df_2 : TD_- \to T\overline{B}_{\sqrt{2}}(0),$$

are symplectic trivializations. In spherical coordinates in a neighbourhood of the equator in $S^2$, and Euclidean coordinates on $\overline{B}_{\sqrt{2}}(0)$, we have

$$\phi_1(\phi,\theta) = \begin{pmatrix} \frac{\cos(\theta)\sin(\phi)}{\sqrt{2-2\cos(\phi)}} & -\sqrt{2-2\cos(\phi)}\sin(\theta) \\ \frac{\sin(\theta)\sin(\phi)}{\sqrt{2-2\cos(\phi)}} & \cos(\theta)\sqrt{2-2\cos(\phi)} \end{pmatrix},$$

$$\phi_2(\phi,\theta) = \begin{pmatrix} -\frac{\cos(\theta)\sin(\phi)}{\sqrt{2\cos(\phi)+2}} & -\sqrt{2\cos(\phi)+2}\sin(\theta) \\ \frac{\sin(\theta)\sin(\phi)}{\sqrt{2\cos(\phi)+2}} & -\cos(\theta)\sqrt{2\cos(\phi)+2} \end{pmatrix}.$$

Given the orientation of the equator in $S^2$ described in Definition 3.4.2, we can take as orientation-preserving diffeomorphism

$$\eta : S^1 \to D_- \cap D_+, \quad \theta \mapsto (0,\theta).$$

The first Chern number of $S^2$ is the Maslov index of the map

$$\phi_{12} \circ \eta : S^1 \to \mathrm{Sp}(2), \quad \theta \mapsto (\phi_1|_{\eta(\theta)}) \circ (\phi_2|_{\eta(\theta)})^{-1} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

We have

$$c_1(TS^2) = \mu(\phi_{12} \circ \eta) = \deg(\det(\cos 2\eta + i \sin 2\eta)) = 2.$$

**Proposition 3.4.4.** *[MS98, p. 74] The first Chern number is well-defined: It does not depend on the decomposition of $\Sigma$ or on the choice of symplectic trivializations $\phi_1$ and $\phi_2$. Moreover, the first Chern number is the unique function which assigns an integer to symplectic vector bundles over compact oriented surfaces with empty boundary, which satisfies the following conditions:*

1. *Two symplectic vector bundles $E'$ and $E$ over $\Sigma$ are isomorphic if and only if they have the same rank and the same Chern number.*
2. *If $\nu : \Sigma' \to \Sigma$ is a smooth map of oriented surfaces and $E \to \Sigma$ is a symplectic vector bundle, then*
$$c_1(\nu^* E) = \deg(\nu) c_1(E).$$
3. *For symplectic vector bundles $E_1 \to \Sigma$, $E_2 \to \Sigma$, of rank $n_1$ and $n_2$,*
$$c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2), \quad c_1(E_1 \otimes E_2) = n_2 c_1(E_1) + n_1 c_1(E_2).$$
4. *We may consider the tangent bundle $T\Sigma$ of an oriented surface $\Sigma$ of genus $g$ as a symplectic vector bundle with symplectic form the volume form on $\Sigma$. Then*
$$c_1(T\Sigma) = 2 - 2g.$$

In the second axiom of the preceding proposition, we have used the following notion of 'degree' for smooth maps.

**Remark 3.4.5.** [Mil97, p. 27] The degree of a map $f : M \to N$ between compact, connected, oriented smooth manifolds $M$ and $N$ of the same dimension is defined to be

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(df_x),$$

where $y$ is any regular value of $f$, and $\operatorname{sign}(df_x) := +1$ if $df_x$ preserves orientation and $\operatorname{sign}(df_x) := -1$ otherwise. The value of $\deg(f)$ does not depend on the choice of regular value $y$.

**Remark 3.4.6.** It follows from Proposition 3.4.4 that a symplectic vector bundle over an oriented surface with empty boundary is symplectically trivial if and only if it has first Chern number zero: Note that if $E \to \Sigma$ is a symplectic vector bundle over the oriented surface $\Sigma$ and $\nu : \Sigma' \to \Sigma$ is a constant map from the oriented surface $\Sigma'$ to $\Sigma$, then the pullback symplectic vector bundle $\nu^* E \to \Sigma'$ is symplectically trivial and since $\deg(\nu) = 0$, $c_1(\nu^* E) = 0$. The statement then follows from Property 1 of Proposition 3.4.4.

The first Chern class for an arbitrary symplectic vector bundle is defined in terms of the first Chern number for symplectic vector bundles over oriented surfaces.

**Proposition 3.4.7.** *For any symplectic vector bundle $E \to B$, the **first Chern class**, also denoted $c_1$, is a map which assigns an integer to homotopy classes of maps from compact oriented surfaces without boundary into $B$. For a smooth map $u : \Sigma \to B$, the first Chern class of $[u]$ is defined by*

$$c_1([u]) := c_1(u^* E). \tag{3.4.1}$$

*In particular, setting $\Sigma = S^2$ and restricting to homotopy classes of based maps, the first Chern class determines a map*

$$c_1|_{\pi_2(B,b_0)} : \pi_2(B, b_0) \to \mathbb{Z}, \quad [v] \mapsto c_1(v^* E), \tag{3.4.2}$$

*where* $v : (S^2, x_0) \to (B, b_0)$*, and* $x_0 \in S^2$ *and* $b_0 \in B$ *are basepoints. The map defined in Equation* (3.4.2) *is a group homomorphism.*

**Proof:** We first show that the map defined in Equation (3.4.1) is well-defined. Let $u, v : \Sigma \to B$ be homotopic maps. From the theory of classifying spaces [MS74], $E$ is isomorphic as a symplectic vector bundle to $f^*E\mathrm{Sp}(2n)$, where $E\mathrm{Sp}(2n) \to B\mathrm{Sp}(2n)$ is the universal symplectic vector bundle and $f : B \to B\mathrm{Sp}(2n)$ is some smooth map. As $f \circ u$ and $f \circ v$ are homotopic, $(f \circ u)^*E\mathrm{Sp}(2n) = u^*E$ is isomorphic as a symplectic vector bundle to $(f \circ v)^*E\mathrm{Sp}(2n) = v^*E$. It then follows from Property 1 of the first Chern number that $c_1(u^*E) = c_1(v^*E)$.

We now show that for $u, v : (S^2, x_0) \to (B, b_0)$, $c_1([u * v]) = c_1([u]) + c_1([v])$, where $u * v$ is the concatenation of the maps $u$ and $v$. We denote by $N$ and $S$ the north and south poles in $S^2$. We may assume without loss of generality that $x_0 = N$. Let $\omega$ be the symplectic structure on $E$. We choose symplectic trivializations

$$\phi_1 : (u^*E|_{D_+}, u^*\omega) \to (D \times \mathbb{R}^{2n}, \omega_0), \quad \phi_2 : (u^*E|_{D_-}, u^*\omega) \to (D \times \mathbb{R}^{2n}, \omega_0),$$
$$\phi_3 : (v^*E|_{D_+}, v^*\omega) \to (D \times \mathbb{R}^{2n}, \omega_0), \quad \phi_4 : (v^*E|_{D_-}, v^*\omega) \to (D \times \mathbb{R}^{2n}, \omega_0),$$

where $2n$ is the rank of $E$, and $D$ denotes the closed unit disc in $\mathbb{R}^2$. We may assume without loss of generality that $(\phi_1)_N = (\phi_3)_N$. Otherwise, we simply define the symplectic trivialization

$$\phi_3' : v^*E|_{D_+} \to D \times \mathbb{R}^{2n} \quad \phi_3' := (\mathrm{Id}_D, (\phi_1)_N \circ (\phi_3)_N^{-1}) \circ \phi_3.$$

Let $\xi : S^2 \to S^2 \vee S^2$ be the map which collapses the great circle through $N$ and $S$, $\{(x, y, z) \in S^2 |\ x = 0\}$, to a point and maps the circle $C_1 = \{(x, y, z) \in S^2 |\ x = -\frac{1}{2}\}$ in $S^2$ to the equator of the upper sphere in $S^2 \vee S^2$ and the circle $C_2 = \{(x, y, z) \in S^2 |\ x = \frac{1}{2}\}$ to the equator of the lower sphere in $S^2 \vee S^2$ (see Figure 3.1). We take the following decomposition of $S^2$:

$$\Sigma_1 = \left\{(x, y, z) \in S^2 \,\middle|\, x \in \left[-1, -\tfrac{1}{2}\right] \cup \left[\tfrac{1}{2}, 1\right]\right\},$$

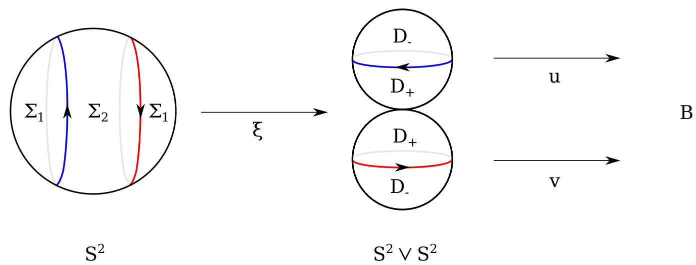



**Figure 3.1:** The map $u * v$. The arrows on the curves $C_1$ and $C_2$ indicate the orientation of these curves induced by a normal vector field pointing out of $\Sigma_1$. The arrows on the images of $C_1$ and $C_2$ under $\xi$ indicate the orientation induced by normal vector fields pointing out of the surfaces $D_+$.

$$\Sigma_2 = \left\{ (x, y, z) \in S^2 \,\middle|\, x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \right\},$$

and we take as symplectic trivializations

$$\psi_1 : (u * v)^* E|_{\Sigma_1} \to \Sigma_1 \times \mathbb{R}^{2n},$$

$$(p, v) \mapsto \begin{cases} (p, (\phi_2)_{\xi(p)}(v)), & p = (x, y, z),\ x \in [\tfrac{1}{2}, 1], \\ \\ (p, (\phi_4)_{\xi(p)}(v)), & p = (x, y, z),\ x \in [-1, -\tfrac{1}{2}], \end{cases}$$

$$\psi_2 : (u * v)^* E|_{\Sigma_2} \to \Sigma_2 \times \mathbb{R}^{2n},$$

$$(p, v) \mapsto \begin{cases} (p, (\phi_1)_{\xi(p)}(v)), & p = (x, y, z),\ x \in [0, \tfrac{1}{2}], \\ \\ (p, (\phi_3)_{\xi(p)}(v)), & p = (x, y, z),\ x \in [-\tfrac{1}{2}, 0]. \end{cases}$$

for $p \in S^2$ and $v \in E_{u*v(p)}$. We define the overlap map

$$\Psi : C_1 \cup C_2 \to \mathrm{Sp}(2n), \quad p \mapsto (\psi_1)_p \circ (\psi_2)_p^{-1}.$$

Taking into account the induced orientations on $C_1$ and $C_2$, we have orientation-preserving diffeomorphisms

$$\eta_1 : S^1 \to C_1, \quad \theta \mapsto (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\cos\theta, -\tfrac{\sqrt{3}}{2}\sin\theta),$$
$$\eta_2 : S^1 \to C_2, \quad \theta \mapsto (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\cos\theta, \tfrac{\sqrt{3}}{2}\sin\theta).$$

Finally we calculate $c_1([u * v])$. As in Example 3.4.3, let $\eta : S^1 \to D_+ \cap D_-$ be the map $\theta \mapsto (0, \theta)$. We have

$$\begin{aligned}
c_1([u * v]) &= \mu(\Psi \circ \eta_1) + \mu(\Psi \circ \eta_2) \\
&= \mu\left((\phi_{21}) \circ \xi|_{C_1} \circ \eta_1\right) + \mu\left((\phi_{43}) \circ \xi|_{C_2} \circ \eta_2\right) \\
&= \deg(\rho \circ \phi_{21} \circ \xi|_{C_1} \circ \eta_1) + \deg(\rho \circ \phi_{43} \circ \xi|_{C_2} \circ \eta_2) \\
&= -\deg(\rho \circ \phi_{21} \circ \eta) - \deg(\rho \circ \phi_{43} \circ \eta) \\
&= -\mu(\phi_{21} \circ \eta) - \mu(\phi_{43} \circ \eta) \\
&= \mu(\phi_{12} \circ \eta) + \mu(\phi_{34} \circ \eta) \\
&= c_1([u]) + c_1([v]).
\end{aligned}$$

Here we have used the fact that $\xi|_{C_1}$ and $\xi|_{C_2}$ are orientation-reversing. The second last equality is an application of the fact that, for a loop $\eta$ in $\mathrm{Sp}(2n)$, $\mu(\eta) = -\mu(\eta')$, where $\eta'$ is the loop defined by $\eta'(p) = (\eta(p))^{-1}$. ∎

**Remark 3.4.8.** For a symplectic vector bundle $E \to B$, the first Chern class of a map $u : \Sigma \to B$ can be viewed as an obstruction to extending $u$ to the compact oriented three-manifold bounded by $\Sigma$. If $u$ can be extended, then it is clear that the maps

$$\gamma|_{C_j} : C_j \to \mathrm{Sp}(2n)$$

are all null-homotopic, where $C_j$ are the connected components of $\partial\Sigma_1 = \partial\Sigma_2$ for a decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$ of $\Sigma$. Hence $\mu(\gamma|_{C_j}) = 0$ for all $j$, and thus $\mu([u]) = 0$.

**Remark 3.4.9.** The definition for the first Chern class we present here is less general than the usual description of the first Chern class as an element of $H^2(B, \mathbb{Z})$ for a complex vector bundle $E \to B$. However, it is possible to recover the usual first Chern class from the map $c_1$ defined in Proposition 3.4.7. First, fix a compatible complex structure on $E$ and let $\bar{c}_1 \in H^2(B, \mathbb{Z})$ be the usual first Chern class of the corresponding complex vector bundle over $B$. Note that, since the space of compatible complex structures on $E$, $\mathcal{J}(E, \omega)$ is contractible, different choices of compatible complex structure determine isomorphic complex vector bundles. Therefore the class $\bar{c}_1$ does not depend on the choice of compatible complex structure.

We begin by showing how $c_1$ can be obtained from $\bar{c}_1$. By the universal coefficient theorem for cohomology, there is an isomorphism

$$H^2(B, \mathbb{Z}) \cong \mathrm{Hom}(H_2(B), \mathbb{Z}) \oplus T_1,$$

where $T_1$ is the torsion part of $H_1(B)$. By projecting onto the first summand, $\bar{c}_1$ determines an element $p_1(\bar{c}_1) \in \mathrm{Hom}(H_2(B), \mathbb{Z})$. For any smooth map $u : \Sigma \to B$, where $\Sigma$ is a compact oriented surface without boundary, the following relationship holds:

$$(p_1(\bar{c}_1))(u_*([\Sigma])) = c_1([u]),$$

where $[\Sigma]$ is the fundamental class of $\Sigma$.

To recover $\bar{c}_1$ from the map $c_1$, we use the fact that, for any smooth manifold $M$, if $k \leq 6$, every element $\kappa \in H_k(M)$ can be represented by a smooth manifold [Tho54]. By this, we mean there is a compact oriented $k$-manifold $N$ without boundary, together with a smooth map $F : N \to M$ such that $f_*([N]) = \kappa$. Thus, for the symplectic vector bundle $E \to B$, every element in $H_2(B)$ is the pushforward of the fundamental class of an oriented surface $\Sigma$ by a smooth map $f : \Sigma \to B$. Not only is the first Chern class well-defined on homotopy classes, it is also well-defined on homology [MS98, p. 80]. In other words, if $g : \Sigma' \to B$ is another smooth map from a compact oriented surface without boundary such that

$$f_*([\Sigma]) = g_*([\Sigma']) \in H_2(B),$$

then $c_1([f]) = c_1([g])$.

It follows that $c_1$ determines a map from all of $H_2(B)$ into $\mathbb{Z}$. Moreover, the map $c_1 : H_2(B) \to \mathbb{Z}$ is a homomorphism. Finally, there is a natural choice of lift (which we do not describe) of this element of $\mathrm{Hom}(H_2(B), \mathbb{Z})$ to $\mathrm{Hom}(H_2(B), \mathbb{Z}) \oplus T_1 \cong H^2(B, \mathbb{Z})$ [MS98, p. 80]. In this way, we obtain an element of $H^2(B, \mathbb{Z})$, which is in fact equal to $\bar{c}_1$ for any choice of compatible complex structure on $E$.

Since the tangent bundle of a symplectic manifold $(M, \omega)$ is a symplectic vector bundle, the first Chern class of the tangent bundle is defined. Often one simply refers to this first Chern class as the first Chern class of $M$.

**Remark 3.4.10.** For a symplectic vector bundle over $B$, we will also denote the element of $\mathrm{Hom}(H_2(B), \mathbb{Z})$ induced by the first Chern class, as explained in Remark 3.4.9, by $c_1$.

### 3.4.1 Monotonicity

When Andreas Floer originally developed Floer homology for Lagrangian intersections and applied it to prove a special case of the Arnold conjecture, he did so under the restriction that the Lagrangian submanifold $L$ in the symplectic manifold $M$ satisfy $\pi_2(M, L) = \{0\}$ [Flo88]. Floer was later able to relax this condition to prove this case of the Arnold conjecture for all symplectic manifolds which are merely monotone [Flo89a], as we define next. A relative version of this monotonicity condition will be needed to define the Lagrangian quantum homology. We will see in Section 4.3 of Chapter 4 that the relative and absolute notions of monotonicity are closely related.

**Definition 3.4.11.** *Let $(M, \omega)$ be a symplectic manifold. The symplectic form $\omega$ determines a homomorphism, which we also denote by $\omega$,*

$$\omega : H_2(M) \to \mathbb{R}, \quad [c] \mapsto \int_c \omega, \tag{3.4.3}$$

*where $c \in Z_2(M)$ is a smooth closed singular two-chain. Note that this homomorphism is well-defined by closedness of $\omega$. A symplectic manifold $M$ is called **(spherically) monotone** if it satisfies*

$$c_1(A) > 0 \Longleftrightarrow \omega(A) > 0 \tag{3.4.4}$$

*for all classes $A$ in the image of the Hurewicz homomorphism*

$$h : \pi_2(M) \to H_2(M).$$

**Remark 3.4.12.** The symplectic form also determines a homomorphism from $\pi_2(M)$, which by the standard abuse of notation we also call $\omega$,

$$\omega : \pi_2(M) \to \mathbb{R}, \quad [u] \mapsto \int_{S^2} u^*\omega, \tag{3.4.5}$$

where $u : S^2 \to M$ is smooth. The homomorphisms defined in Equations (3.4.3) and (3.4.5) are related by

$$\omega(h([u])) = \omega([u]).$$

As with the first Chern class, in some contexts, it is more convenient to think of the homomorphism $\omega$ as a function on homotopy classes rather than on homology classes.

**Example 3.4.13.**

1. Any symplectic manifold $M$ with $\pi_2(M) = \{e\}$ is monotone. In particular, any symplectic surface with positive genus is monotone.

2. Complex projective space with the Fubini-Study form, $(\mathbb{CP}^n, \sigma_{\mathbb{CP}^n})$, is monotone. In Example 3.4.17, we compute the first Chern class of $\mathbb{CP}^n$ and prove this statement. As $\sigma_{\mathbb{CP}^1} = \frac{1}{4}\omega$, where $\omega_{\mathrm{vol}} \stackrel{\text{loc.}}{=} \sin\phi d\phi \wedge d\theta$ is the standard symplectic form on $S^2 \cong \mathbb{CP}^1$, we see that $(S^2, \omega_{\mathrm{vol}})$ is also monotone. This is also easy to see directly.

3. The product symplectic manifold $(\mathbb{CP}^m \times \mathbb{CP}^n, \sigma_{\mathbb{CP}^m} \oplus \sigma_{\mathbb{CP}^n})$ is monotone if and only if $m = n$. We will examine this example in more detail in Remark 3.4.18.

**Remark 3.4.14.** Often those classes in the image of the Hurewicz homomorphism are referred to as 'spherical' classes, hence the term 'spherically monotone'. By the next proposition, a symplectic manifold is spherically monotone if and only if the homomorphisms $c_1$ and $\omega$ are related by a positive proportionality constant on the image of the Hurewicz homomorphism. There is also a more restrictive notion of montonicity, where the class $c_1$ in $H^2(M)$ is required to be a positive multiple of $[\omega]$. Although we will only be interested in spherical monotonicity, we mention that it is possible for a symplectic manifold to be spherically monotone without satisfying this more restrictive monotonicity condition. For example, let $(T^2, \tau)$ be the two-torus with symplectic form locally given by $d\theta_1 \wedge d\theta_2$, where $\theta_1$ and $\theta_2$ are angular coordinates. Then the product $(\mathbb{CP}^n \times T, \sigma_{\mathbb{CP}^n} \oplus \tau)$ is spherically monotone, but is not monotone in this more restrictive sense. More generally, if $(N, \omega_N)$ is a symplectic manifold such that $\omega_N$ and $c_1$ both vanish on $h(\pi_2(N))$, then $(\mathbb{CP}^n \times N, \sigma_{\mathbb{CP}^n} \oplus \omega_N)$ will be spherically monotone but need not be monotone in the more restrictive sense [BC01].

The following reformulation of the monotonicity condition is often useful:

**Proposition 3.4.15.** *The symplectic manifold $(M, \omega)$ is monotone if and only if there exists a positive constant $\nu$ such that*

$$\omega|_{H_2^S(M)} = \nu c_1|_{H_2^S(M)},$$

*where $H_2^S(M) \subset H_2(M)$ is the image of the Hurewicz homomorphism.*

**Proof:** Clearly the existence of such a constant $\nu$ implies $M$ is monotone. Conversely, assume $M$ is monotone. Notice that $c_1|_{H_2^S(M)} \equiv 0$ if and only if $\omega|_{H_2^S(M)} \equiv 0$, and in this case the statement holds for any choice of constant $\nu$. Therefore we may assume that $c_1$ and $\omega$ are not identically zero on $H_2^S(M)$.

Define

$$\omega_M = \inf\{\omega(\alpha) |\, \omega(\alpha) > 0,\ \alpha \in H_2^S(M)\}$$

and

$$N_M = \min\{c_1(\alpha) |\, c_1(\alpha) > 0,\ \alpha \in H_2^S(M)\}.$$

We first notice that $\omega_M > 0$ and therefore there exists $\alpha_0 \in H_2^S(M)$ such that $\omega(\alpha_0) = \omega_M$: Since $\omega(H_2^S(M))$ is a subgroup of $\mathbb{R}$, either $\omega(H_2^S(M))$ is dense in $\mathbb{R}$ or it is of the form $\langle \omega(\alpha_0) \rangle$ for $\alpha_0 \in H_2^S(M)$ such that $\omega(\alpha_0) = \omega_M$. However, if $\omega(H_2^S(M))$ is dense, there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $\omega(\alpha_{n+1}) < \omega(\alpha_n)$ and $\lim_{n\to\infty} \omega(\alpha_n) = 0$. We have,

$$\begin{aligned} \omega(\alpha_n) - \omega(\alpha_{n+1}) > 0 &\Longleftrightarrow \omega(\alpha_n \alpha_{n+1}^{-1}) > 0 \Longleftrightarrow c_1(\alpha_n \alpha_{n+1}^{-1}) > 0 \\ &\Longleftrightarrow c_1(\alpha_n) - c_1(\alpha_{n+1}) > 0 \end{aligned}$$

and we see that $(c_1(\alpha_n))_{n\in\mathbb{N}}$ is a sequence of integers with every term strictly less than the preceding term. However, by the monotonicity of $M$, all $c_1(\alpha_n)$ must be positive. Clearly this is a contradiction.

Therefore we have some $\alpha_0 \in H_2^S(M)$ with $\omega(\alpha_0) = \omega_M$. We will show that $c_1(\alpha_0) = N_M$. Assume to the contrary that there exists $\alpha_1 \in H_2^S(M)$ with $0 < c_1(\alpha_1) < c_1(\alpha_0)$. Notice that $c_1(\alpha_1) < c_1(\alpha_0)$ implies $c_1(\alpha_0\alpha_1^{-1}) > 0$ and therefore $\omega(\alpha_0\alpha_1^{-1}) > 0$. We have

$$\omega(\alpha_0) = \omega(\alpha_0\alpha_1^{-1}\alpha_1) = \omega(\alpha_0\alpha_1^{-1}) + \omega(\alpha_1) > 0,$$

where the inequality on the right follows from $c_1(\alpha_0) > 0$ and the monotonicity of $M$. This implies $\omega(\alpha_0) > \omega(\alpha_1)$, contradicting the minimality of $\omega(\alpha_0)$.

Take $\alpha \in H_2^S(M)$ and choose $m, n \in \mathbb{Z}$ such that

$$\omega(\alpha) = m\omega(\alpha_0) \text{ and } c_1(\alpha) = nc_1(\alpha_0).$$

We show that $m = n$. Clearly this is true for $\omega(\alpha) = 0$. Therefore we assume without loss of generality that $\omega(\alpha) > 0$ and by way of contradiction, we assume $m > n$.

We have

$$\begin{aligned}\omega(\alpha) &= \omega(\alpha_0^l \alpha_0^{-l}\alpha) = \omega(\alpha_0^l) + \omega(\alpha_0^{-l}\alpha)\\ &= m\omega(\alpha_0),\end{aligned}$$

for any $l \in \mathbb{Z}$. It follows that $\omega(\alpha_0^{-l}\alpha) = (m-l)\omega(\alpha_0)$. Similarly, $c_1(\alpha_0^{-l}\alpha) = (n-l)c_1(\alpha_0)$. Setting $l = n$, we see that $\omega(\alpha_0^{-l}\alpha) = (m-l)\omega(\alpha_0) > 0$, but $c_1(\alpha_0^{-l}\alpha) = 0$, contradicting the monotonicity of $M$.

This allows us to define the positive constant $\nu \in \mathbb{R}$ by $\nu = \frac{\omega(\alpha_0)}{c_1(\alpha_0)}$. ∎

**Definition 3.4.16.** *The constant $\nu$ of Proposition 3.4.15 is called the* ***monotonicity constant*** *of the symplectic manifold $M$. It is unique if $c_1|_{H_2^S(M)} \not\equiv 0$. The number $N_M \in \mathbb{Z}$ is called the* ***minimal Chern number*** *of $M$. In the case where $c_1|_{H_2^S(M)} \equiv 0$, we define $N_M = \infty$.*

**Example 3.4.17.** We show that $\mathbb{CP}^n$ with the Fubini-Study form $\sigma_{\mathbb{CP}^n}$ is monotone and compute its monotonicity constant. To establish the monotonicity of $\mathbb{CP}^n$, we first note that $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$ and that a generator of $\pi_2(\mathbb{CP}^n)$ is represented by the map

$$u : S^2 \cong \mathbb{CP}^1 \to \mathbb{CP}^n, \quad [z_0, z_1] \mapsto [z_0, z_1, 0, \dots, 0].$$

It is sufficient to evaluate both $\sigma_{\mathbb{CP}^n}$ and $c_1$ on $[u]$ and to show that they have the same sign. We have,

$$\sigma_{\mathbb{CP}^n}([u]) = \int_{S^2} u^*\sigma_{\mathbb{CP}^n} = \int_{S^2} \sigma_{\mathbb{CP}^1} = \frac{1}{4}\int_{S^2} \omega_{std} = \pi.$$

In order to compute $c_1([u])$ we take the following decomposition of $\mathbb{CP}^1$:

$$\begin{aligned}\Sigma_1 &:= \{[z_0, z_1] \in U_1 |\; |\xi_1^1([z_0, z_1])| \le 1\},\\ \Sigma_2 &:= \{[z_0, z_1] \in U_0 |\; |\xi_0^1([z_0, z_1])| \le 1\}.\end{aligned}$$

We now need to find symplectic trivializations

$$\begin{aligned}\phi_1 &: (u^*T\mathbb{CP}^n|_{\Sigma_1}, u^*\sigma_{\mathbb{CP}^n}) \to (\Sigma_1 \times \mathbb{C}^n, \omega_0),\\ \phi_2 &: (u^*T\mathbb{CP}^n|_{\Sigma_2}, u^*\sigma_{\mathbb{CP}^n}) \to (\Sigma_2 \times \mathbb{C}^n, \omega_0),\end{aligned}$$

where $(\omega_0)_p = \frac{i}{2} dz \wedge d\bar{z}$ for all $p \in \Sigma_1, \Sigma_2$. We will make use of the fact that the following map is a symplectomorphism

$$f^n : (\mathbb{C}^n, \omega_{\mathrm{FS}}) \to (\mathbb{C}^n, \omega_0), \quad z \mapsto \frac{z}{\sqrt{1+|z|^2}}.$$

This can be checked by a routine computation.[3] With this in mind, the three maps defined in the diagram below are isomorphisms of symplectic vector bundles. We simply take the symplectic trivialization $\phi_1$ to be the composition of these three maps.

$$\begin{tikzcd}
(p,v) \arrow[d, mapsto] & (u^*T\mathbb{CP}^n|_{\Sigma_1}, u^*\sigma_{\mathbb{CP}^n}) \arrow[r, "\phi_1"] \arrow[d] & (\Sigma_1 \times \mathbb{C}^n, \omega_0) & ((f^1 \circ \xi_1^1)^{-1}(r), x) \\
(\xi_1^1(p), (d\xi_1^n)_{u(p)}(v)) & (D \times \mathbb{C}^n, (\omega_{\mathrm{FS}})_\iota) \arrow[r] & (D \times \mathbb{C}^n, \omega_0) \arrow[u] & (r,x) \arrow[u, mapsto] \\
 & (q,w) \arrow[r, mapsto] & (f^1(q), df^n_{\iota(q)}(w)) &
\end{tikzcd}$$

for $p \in \Sigma_1, v \in T_{u(p)}\mathbb{CP}^n$, $q, r \in D$, $w, x \in \mathbb{C}^n$. Here $D$ denotes the unit disc in $\mathbb{C}$, and $\iota : \mathbb{C} \hookrightarrow \mathbb{C}^n$ denotes the inclusion of $\mathbb{C}$ as the first coordinate in $\mathbb{C}^n$. The symplectic form $(\omega_{\mathrm{FS}})_\iota$ is defined by $((\omega_{\mathrm{FS}})_\iota)_q(w_1, w_2) = (\omega_{\mathrm{FS}})_{\iota(q)}(w_1, w_2)$, where $\omega_{\mathrm{FS}}$ is the Fubini-Study from on $\mathbb{C}^n$ and $w_1, w_2 \in \mathbb{C}^n \cong T_{\iota(q)}\mathbb{C}^n$. Similarly, we define $\phi_2$

[3]For $n = 1$, the map $f$ has a simple geometric description: It is the composition of the inverse stereographic projection, the Lambert azimuthal equal-area projection and a linear scaling by a factor of $\frac{1}{2}$. The scaling accounts for the fact that the Fubini-Study form on $\mathbb{CP}^1 \cong S^2$ is $\frac{1}{4}\omega_{std}$.

to be the composition of the three maps in the diagram below.

$$\begin{tikzcd}
(p,v) \arrow[d, mapsto] & (u^*T\mathbb{CP}^n|_{\Sigma_2}, u^*\sigma_{\mathbb{CP}^n}) \arrow[r, "\phi_2"] \arrow[d] & (\Sigma_2 \times \mathbb{C}^n, \omega_0) & ((f^1 \circ \xi_0^1)^{-1}(r), x) \\
(\xi_0^1(p), (d\xi_0^n)_{u(p)}(v)) & (D \times \mathbb{C}^n, (\omega_{\mathrm{FS}})_\iota) \arrow[r] & (D \times \mathbb{C}^n, \omega_0) \arrow[u] & (r,x) \arrow[u, mapsto] \\
 & (q,w) \arrow[r, mapsto] & (f^1(q), df^n_{\iota(q)}(w)) &
\end{tikzcd}$$

We endow $\Sigma_1 \cap \Sigma_2$ with the orientation induced by a normal vector field pointing out of $\Sigma_1$. With this orientation, the following map is an orientation-preserving diffeomorphism:

$$\eta : S^1 \to \Sigma_1 \cap \Sigma_2, \quad \theta \mapsto (\xi_1^1)^{-1}(e^{i\theta}).$$

We have,

$$(\phi_1 \circ \phi_2^{-1})|_{\Sigma_1 \cap \Sigma_2} : \Sigma_1 \cap \Sigma_2 \times \mathbb{C}^n \to \Sigma_1 \cap \Sigma_2 \times \mathbb{C}^n \quad (p,v) \mapsto (p, \phi_{12}(v))$$

where

$$\begin{aligned}
\phi_{12} &= (df^n)_{\iota(\xi_1^1(p))} \circ (d\xi_1^n)_{u(p)} \circ \left((d\xi_0^n)_{u(p)}\right)^{-1} \circ \left((df^n)_{\iota(\xi_0^1(p))}\right)^{-1} \\
&= (df^n)_{\iota(\xi_1^1(p))} \circ \left[(df^n)_{\iota(\xi_0^1(p))} \circ (d\xi_0^n)_{u(p)} \circ \left(d\xi_1^n)_{u(p)}\right)^{-1}\right] \\
&= (df^n)_{\iota(\xi_1^1(p))} \circ d(f^n \circ \xi_{01}^n)_{\xi_1^n(u(p))}.
\end{aligned}$$

The last equality is an application of the chain rule. Therefore the overlap map $\psi : S^1 \to \mathrm{Sp}(2n)$ is

$$\psi := \phi_{12} \circ \eta, \quad \theta \mapsto (df^n)_{(e^{i\theta},0,\ldots,0)} \circ d(f^n \circ \xi_{01}^n)_{(e^{i\theta},0,\ldots,0)}.$$

Computing the derivatives, we have

$$\psi(\theta) = \begin{pmatrix} e^{i(2\theta-\pi)} & 0 & \cdots & 0 \\ 0 & e^{i\theta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta} \end{pmatrix}.$$

Using the direct sum property of the Maslov index for loops in $\mathrm{Sp}(2n)$, we can write the Maslov index of $\psi$ as

$$\mu(\psi) = \mu(\psi_1) + (n-1)\mu(\psi_2)$$

where

$$\psi_1, \psi_2 : S^1 \to \mathrm{Sp}(2), \quad \psi_1(\theta) = e^{2i\theta}, \quad \psi_2(\theta) = e^{i\theta}.$$

Clearly $\mu(\psi_1) = 2$ and $\mu(\psi_2) = 1$. Therefore

$$c_1([u]) = \mu(\psi) = n + 1.$$

As both $c_1([u])$ and $\sigma_{\mathbb{CP}^n}([u])$ are positive, it follows that $\mathbb{CP}^n$ is monotone. Moreover, $\mathbb{CP}^n$ has monotonicity constant

$$\nu = \frac{\sigma_{\mathbb{CP}^n}([u])}{c_1([u])} = \frac{\pi}{n+1}.$$

**Remark 3.4.18.** Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be symplectic manifolds and consider the product $(M_1 \times M_2, \lambda_1\omega_1 \oplus \lambda_2\omega_2)$ for $\lambda_1, \lambda_2 \in \mathbb{R}_+$ (see Part 4 of Example 3.2.2). First notice that scaling $\omega_1$ and $\omega_2$ by positive constants does not alter the first Chern class of $M_1$ or $M_2$: A compatible almost complex structure for $\omega_i$, for $i = 1, 2$, will also be compatible for $\lambda_i\omega_i$. Since the first Chern class of a symplectic vector bundle is determined by the first Chern class of the complex vector bundle associated to any compatible almost complex structure (see Remark 3.4.9), the symplectic vector bundles $(TM, \omega_i)$ and $(TM, \lambda_i\omega_i)$ have identical first Chern class.

It also follows from the definition in Equation (3.4.5) of Remark 3.4.5 and from Property 3.4.4 of Proposition 3.4.4 that the homomorphisms

$$\lambda_1\omega_1 \oplus \lambda_2\omega_2 : \pi_2(M_1 \times M_2) \to \mathbb{R} \quad \text{and} \quad c_1 : \pi_2(M_1 \times M_2) \to \mathbb{Z}$$

satisfy

$$\omega_1 \oplus \omega_2([u]) = \lambda_1\omega_1([p_1 \circ u]) + \lambda_2\omega_2([p_2 \circ u]),$$

$$c_1([u]) = c_1([p_1 \circ u]) + c_1([p_2 \circ u]),$$

where $u : S^2 \to M_1 \times M_2$ is a smooth map and $p_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, are the projection maps. Thus if $(M_1, \omega_1)$ and $(M_2, \omega_2)$ are monotone with monotonicity constants $\nu_1$ and $\nu_2$, $(M_1 \times M_2, \lambda_1\omega_1 \oplus \lambda_2\omega_2)$ is monotone if and only if $\lambda_1\nu_1 = \lambda_2\nu_2$. In this case, the monotonicity constant for the product symplectic manifold is simply $\lambda_1\nu_1$.

In particular, $(\mathbb{CP}^m \times \mathbb{CP}^n, \lambda_1\sigma_{\mathbb{CP}^m} \oplus \lambda_2\sigma_{\mathbb{CP}^n})$ is monotone if and only if

$$\frac{\lambda_1}{m+1} = \frac{\lambda_2}{n+1}.$$

Hence $(\mathbb{CP}^m \times \mathbb{CP}^n, \sigma_{\mathbb{CP}^m} \oplus \sigma_{\mathbb{CP}^n})$ is monotone if and only if $m = n$.

## 3.5 J-holomorphic curves

Since they were introduced by Gromov in 1985, $J$-holomorphic curves have played a central role in symplectic topology. They were originally used by Gromov to prove the non-squeezing theorem, which states that if the open ball of radius $R$ in $(\mathbb{R}^{2n}, \omega_0)$, $n \geq 2$, can be embedded symplectically inside the cylinder of radius $r$ in $(\mathbb{R}^{2n}, \omega_0)$, then $R$ must be no larger than $r$ [Gro85].

If the embedding is only required to preserve volume but not the symplectic form, then any open ball can be embedded into a cylinder of any radius, no matter how small, simply by 'squeezing' the ball. Prior to Gromov's result, the rigidity of symplectic embeddings in comparison with volume-preserving embeddings was unknown. The theory of $J$-holomorphic curves is also the basis for the construction of the famed Gromov-Witten invariants, which count $J$-holomorphic curves in a symplectic manifold subject to certain conditions.

**Definition 3.5.1.** *A **$J$-holomorphic curve** (sometimes called a **pseudoholomorphic curve**) is a $(j, J)$-holomorphic map $u : \Sigma \to M$ from a Riemann surface $(\Sigma, j)$*

*into an almost complex manifold $(M, J)$. A $J$-holomorphic curve from $\mathbb{CP}^1$ is called a **$J$-holomorphic sphere**, and a $J$-holomorphic curve from the unit disc in $\mathbb{C}$ is called a **$J$-holomorphic disc**.*

The condition for a curve $u : \Sigma \to M$ to be $J$-holomorphic can be written as

$$J \circ du = du \circ j,$$

or equivalently $du + J \circ du \circ j = 0$. One often expresses this condition in terms of the del-bar operator as $\bar{\partial}_J(u) = 0$. The del-bar operator is the composition of the exterior derivative with the natural projection

$$\Omega^1(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM), \quad \eta \mapsto \frac{1}{2}(\eta + J \circ \eta \circ j).$$

Here $\Omega^1(\Sigma, u^*TM)$ is the space of vector-valued one-forms on $\Sigma$ with values in $u^*TM$ and $\Omega^{0,1}(\Sigma, u^*TM)$ is the subspace of $\Omega^1(\Sigma, u^*TM)$ consisting of complex antilinear one-forms. More precisely,

$$\Omega^{0,1}(\Sigma, u^*TM) := \{\eta \in \Omega^1(\Sigma, u^*TM) |\ \eta \circ j = -J \circ \eta\}.$$

The 'pseudo' in 'pseudoholomorphic curve' refers to the fact that the almost complex structure on the target space of a pseudoholomorphic curve is not required to be integrable. The notion of a pseudoholomorphic curve is related to the traditional notion of a holomorphic curve in $\mathbb{C}^n$ as follows: A map $u : \Sigma \to \mathbb{C}^n$ is a $J_0$-holomorphic curve if and only if, for any conformal coordinate chart $\phi : U \to \Sigma$ on an open subset $U \subset \Sigma$ (i.e. $d\phi : (TU, j) \to (T\mathbb{C}, J_0)$ is a morphism of complex vector bundles), $u \circ \phi^{-1}$ satisfies the Cauchy-Riemann equations.

Although $J$-holomorphic curves can be defined for arbitrary almost complex manifolds, in symplectic geometry, the almost complex structure $J$ is generally a tame or compatible almost complex structure on a symplectic manifold. If $J$ is a tame almost complex structure, the images of embedded $J$-holomorphic curves are precisely the two-dimensional symplectic submanifolds. This follows from the fact that an almost complex structure on a manifold of real dimension two is always integrable. As integrability can fail in higher dimensions, it is not possible to develop a similar theory of pseudoholomorphic maps from complex manifolds of higher dimensions.

Pseudoholomorphic curves play a role in symplectic geometry similar to the role of geodesics in Riemannian geometry. In Riemannian geometry, geodesics are local minima of the length functional on curves connecting two points. In symplectic geometry, when $J$ is a compatible almost complex structure on a symplectic manifold, $J$-holomorphic curves minimize a quantity called the energy among curves in their homology class.

We will be mainly interested in $J$-holomorphic discs in symplectic manifolds which map the boundary of the disc into a Lagrangian submanifold.

**Definition 3.5.2.** *Let $(M, \omega)$ be an almost symplectic manifold and $J$ be an $\omega$-tame almost complex structure. Given a smooth map $u : \Sigma \to M$ from a Riemann surface $(\Sigma, j, \mathrm{dvol}_\Sigma)$ into $M$, the* ***energy*** *of $u$ is defined to be the $L^2$-norm of the vector-valued one-form $du \in \Omega^1(\Sigma, u^*TM) := \Gamma(u^*TM \otimes (T\Sigma)^*)$:*

$$E(u) := \frac{1}{2}\int_\Sigma |du|_J^2 \mathrm{dvol}_\Sigma.$$

*The norm on the space of real linear maps $\mathrm{Hom}(T_z\Sigma, T_{u(z)}M)$ is defined by*

$$|L|_J := |\zeta|^{-1}\sqrt{|L(\zeta)|_J^2 + |L(j\zeta)|_J^2}, \tag{3.5.1}$$

*for $L \in \mathrm{Hom}(T_z\Sigma, T_{u(z)}M)$. Here $\zeta$ is any choice of non-zero vector in $T_z\Sigma$ and the norm $|\zeta|$ is computed using the metric determined by $j$ and $\mathrm{dvol}_\Sigma$. It is not difficult to show that this norm does not depend on the choice of $\zeta$ [MS98, p. 21].*

The energy does not depend on the volume form on $\Sigma$ (as long as the choice induces the same orientation on $\Sigma$). To see this, note that the norm defined in Equation (3.5.1) depends on the volume form on $\Sigma$ since the norm $|\zeta|$ is computed using the metric induced by $\mathrm{dvol}_\Sigma$ and $j$. If $\mathrm{dvol}'_\Sigma = f\mathrm{dvol}_\Sigma$ is another volume form on $\Sigma$, where $f$ is a positive smooth function on $\Sigma$, then the factor of $f(z)$ in $\mathrm{dvol}'_\Sigma(z)$ cancels with the factor $f(z)$ appearing in $|du(z)|^2_J$ computed with respect to $\mathrm{dvol}'_\Sigma$.

Notice also that the energy is always a non-negative quantity and can vanish only for locally constant curves. With the goal of studying Lagrangian submanifolds in view, we state the relative version of the following proposition. The absolute version version is obtained by assuming $\partial\Sigma = \emptyset$.

**Proposition 3.5.3.** *[MS12, p. 21] Let $(M, \omega)$ be a symplectic manifold with $\omega$-tame almost complex structure $J$ and let*

$$u : (\Sigma, \partial\Sigma) \to (M, L)$$

*be a smooth map from a compact Riemann surface.*

1. *(The energy identity.) If $u$ is $J$-holomorphic, then its energy satisfies*

$$E(u) = \int_{u_*(\sigma)} \omega = \int_\Sigma u^*\omega. \tag{3.5.2}$$

   *Here $\sigma \in Z_2(\Sigma, \partial\Sigma)$ is a fundamental cycle for $\Sigma$.*

2. *If $J$ is $\omega$-compatible (and $u$ is not necessarily $J$-holomorphic), then*

$$E(u) = \int_\Sigma |\bar\partial(u)|^2_J \mathrm{dvol}_\Sigma + \int_\Sigma u^*\omega. \tag{3.5.3}$$

**Proof:** The second equality in Equation (3.5.2) is simply the definition of integration over $\Sigma$. As $j$ is integrable, any point in $\Sigma$ has a neighbourhood $U$ with a $(j, J_0)$-holomorphic coordinate chart $\phi : U \to \mathbb{C}$. The coordinates $(s, t)$ on $U$ with respect to such a chart are called conformal coordinates and satisfy

$$j(\partial_s) = \partial_t, \quad j(\partial_t) = -\partial_s.$$

As the integrand in Equation (3.5.2) is independent of the choice of volume form, we may also assume that $\mathrm{dvol}_\Sigma$ is locally given by $\phi^*\omega_0$. This implies that $\partial_s$ and $\partial_t$

are normal vector fields on $U$ with respect to the metric $g_\Sigma$ induced by $\mathrm{dvol}_\Sigma$ and $j$. Choosing $\zeta = \partial_s$ in calculating the norm in Equation (3.5.1), on the open set $U \subset \Sigma$ we have

$$\begin{aligned}
\frac{1}{2}|du|_J^2 \mathrm{dvol}_\Sigma &= \frac{1}{2}\left(|du(\partial_s)|_J^2 + |du(\partial_t)|_J^2\right) ds \wedge dt, \\
&= \frac{1}{2}\left(|\partial_s u)|_J^2 + |\partial_t u)|_J^2\right) ds \wedge dt, \\
&= \frac{1}{2}\left(\omega(\partial_s u, J\partial_s u) + \omega(\partial_t u, J\partial_t u)\right) ds \wedge dt, \\
&= \frac{1}{2}\left[|\partial_s u + J\partial_t u|_J^2 - \left(\omega(\partial_s u, -\partial_t u) + \omega(J\partial_t u, J\partial_s u)\right)\right] ds \wedge dt, \\
&= \frac{1}{2}\left[|\partial_s u + J\partial_t u|_J^2 + \left(\omega(\partial_s u, \partial_t u) + \omega(J\partial_s u, J\partial_t u)\right)\right] ds \wedge dt, \\
&= |\bar{\partial}(u)|_J^2 + \frac{1}{2}\left(\omega(\partial_s u, \partial_t u) + \omega(J\partial_s u, J\partial_t u)\right) ds \wedge dt.
\end{aligned}$$

If $u$ is $J$-holomorphic, then $\bar{\partial}(u) = 0$ and

$$J\partial_s u = \partial_t u, \quad J\partial_t u = -\partial_s u.$$

Therefore

$$\frac{1}{2}|du|_J^2 \mathrm{dvol}_\Sigma = \omega(\partial_s u, \partial_t u) ds \wedge dt = u^*\omega.$$

If $u$ is not necessarily $J$-holomorphic but $J$ is $\omega$-compatible, then

$$\frac{1}{2}|du|_J^2 \mathrm{dvol}_\Sigma = |\bar{\partial}(u)|_J^2 + \omega(\partial_s u, \partial_t u) ds \wedge dt = |\bar{\partial}(u)|_J^2 + u^*\omega. \qquad \blacksquare$$

**Remark 3.5.4.** Together with Stokes' theorem, this proposition implies that the energy of a $J$-holomorphic curve depends only on its homology class. This means that a $J$-holomorphic curve which represents the trivial homology class in $H_2(M, L)$ must have zero energy and is therefore locally constant. In other words, if $\Sigma$ is connected, the curve must be constant.

The following lemma is an easy generalization of the fact that locally constant curves are the only $J$-holomorphic curves representing the trivial homology class.

**Lemma 3.5.5.** *Let $(M, \omega)$ be a symplectic manifold, let $L \subset M$ be a Lagrangian submanifold, and let $J$ be an $\omega$-tame almost complex structure on $M$. A sequence of $J$-holomorphic curves $(u_i)_{i=1}^l$,*

$$u_i : (\Sigma_i, \partial\Sigma_i) \to (M, L)$$

*from compact, connected Riemann surfaces $\Sigma_i$, satisfies*

$$\sum_{i=1}^{l} (u_i)_*([\sigma_i]) = 0 \in H_2(M, L)$$

*if and only if all $u_i$ are constant. Here the $\sigma_i \in Z_2(\Sigma, \partial\Sigma_i)$ are fundamental cycles for the surfaces $\Sigma_i$.*

**Proof:** Define the map

$$u : (\dot\cup_i \Sigma_i, \partial (\dot\cup_i \Sigma_i)) \to (M, L), \quad u|_{\Sigma_i} = u_i.$$

Notice that the cycle $\sigma := \sum_{i=1}^l \sigma_i \in Z_2 (\dot\cup_i \Sigma_i, \partial (\dot\cup_i \Sigma_i))$ is a fundamental cycle for the disjoint union $\dot\cup_i \Sigma_i$. We have,

$$\sum_{i=1}^{l} (u_i)_*([\sigma_i]) = u_*([\sigma]) \in H_2(M, L).$$

Thus $\sum_{i=1}^l (u_i)_*([\sigma_i]) = 0$ if and only if $E(u) = 0$. This in turn holds precisely when $u$ is locally constant, which is the case only when all $u_i$ are constant. ∎

Next, we provide a brief outline, following [MS12], of the approach used to study moduli spaces of $J$-holomorphic curves. This is the same general approach which is used to study the moduli spaces of pearly trajectories we will deal with in Chapter 4. This approach uses the following injectivity condition for $J$-holomorphic curves.

**Definition 3.5.6.** *[BC09a, p. 15] Let $(M, J)$ be an almost complex manifold. A $J$-holomorphic curve $u : \Sigma \to M$ is called **simple** if it is non-constant and the preimage $u^{-1}(p)$ of every regular value $p \in M$ of $u$ contains a single point.*

For Riemann surfaces without boundary, all $J$-holomorphic curves which are not multiply covered are simple [MS12, p. 30]. By a multiply covered curve, we mean a curve $u : \Sigma \to M$ that can be factored as $u = u' \circ \phi$ where $u'$ is a simple $J$-holomorphic curve from the Riemann surface $(\Sigma', j')$ and $\phi : \Sigma \to \Sigma'$ is a holomorphic branched covering map of degree greater than one. If $\Sigma$ has empty boundary, many problems can be reduced to the simple case, which is generally easier to approach, by replacing $u$ by $u'$. For $(M, \omega)$ a symplectic manifold and $L \subset M$ a Lagrangian submanifold, the study of $J$-holomorphic curves $u : (\Sigma, \partial\Sigma) \to (M, L)$, where $\partial\Sigma \neq \emptyset$, is more nuanced. It is possible for such curves to be non-simple without being multiply covered [BC09a, p. 15].

Let $D$ be the unit disc in $\mathbb{C}$ and denote by $\mathcal{M}^*(A, D; J)$ the set of simple $J$-holomorphic discs in $M$ with boundary in $L$ representing the relative homology class $A \in H_2(M, L)$. An important result in the theory of $J$-holomorphic cuves is that for a choice of almost complex $J$ which is in some sense generic (as we explain below), $\mathcal{M}^*(A, D; J)$ is a smooth finite dimensional manifold.

To prove this, elements of $\mathcal{M}^*(A, D; J)$ are described as the zeros of a section of an infinite-dimensional vector bundle. Specifically, one defines $\mathcal{E} \to \mathcal{B}$ to be the vector bundle with base the infinite-dimensional Fréchet manifold

$$\mathcal{B} := C^\infty((D, \partial D), (M, L))$$

and with fibre over $u \in \mathcal{B}$, given by $\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM)$. Then the map

$$\mathcal{S} : \mathcal{B} \to \mathcal{E}, \quad u \mapsto \bar{\partial}(u),$$

defines a section of this vector bundle whose zeros are precisely the $J$-holomorphic curves from $(D, \partial D)$ to $(M, L)$. The set $\mathcal{M}^*(A, D; J)$ is the intersection of the set of zeros with the open subset $\mathcal{B}^* \subset \mathcal{B}$ of simple maps from $(D, \partial D)$ to $(M, L)$ representing the class $A$. The problem of showing that $\mathcal{M}^*(A, D; J)$ is a smooth finite-dimensional manifold can be reduced to showing that $\mathcal{S}$ is transverse to the zero section, as we explain next.

By definition of transversality, to show that the image of $\mathcal{S}$ is transversal to the zero section, we must show that for a $J$-holomorphic curve $u \in \mathcal{M}(A, \Sigma; J)$,

$$T_{(u,0)}\mathcal{E} = T_{(u,0)}\mathcal{S} \oplus T_u\mathcal{B}.$$

In other words,

$$T_{(u,0)}\mathcal{E} = (\mathrm{im}(d\mathcal{S})_u) \oplus T_u\mathcal{B}.$$

A tangent vector at a curve $u \in \mathcal{B}$ is simply a smooth choice of vector in $T_{u(z)}M$ for each $z \in \Sigma$. That is, $T_u\mathcal{B} \cong \Gamma(u^*TM)$. The tangent space to the Fréchet space $\mathcal{E}_u$ at any point in $\mathcal{E}_u$ is naturally isomorphic to $\mathcal{E}_u$. Therefore the differential of the section $\mathcal{S}$ at a $J$-holomorphic curve $u$ is of the form

$$(d\mathcal{S})_u : \Gamma(u^*TM) \to T_{(u,0)}\mathcal{E} \cong T_u\mathcal{B} \oplus T_{(u,0)}\mathcal{E}_u \cong T_u\mathcal{B} \oplus \mathcal{E}_u.$$

We define the **vertical differential**, $D_u$, of the section $\mathcal{S}$ at $u$ to be the composition of $(d\mathcal{S})_u$ with the projection $T_u\mathcal{B} \oplus \mathcal{E}_u \to \mathcal{E}_u$. Proving transversality then amounts to showing that the vertical differential is surjective.

To show surjectivity, one first shows that the operator $D_u$ is a real linear Cauchy-Riemann operator and hence Fredholm. Its Fredholm index is given by the Riemann-Roch theorem as

$$\mathrm{Ind}(D_u) = n + \mu(u^*TM, u^*TL), \tag{3.5.4}$$

where $2n$ is the dimension of $M$ and $\mu$ is a relative version of the first Chern number for symplectic vector bundles over surfaces with boundary, called the Maslov index of a bundle pair. We will describe this Maslov index in detail in the next chapter. The integer $\mu(u^*TM, u^*TL)$ does not depend on the choice of representative $u$ of the homology class $A \in H_2(M, L)$.

Let $\mathcal{J}^l$ be the set of $\omega$-compatible $C^l$ almost complex structures on $M$ and define the **universal moduli space**,

$$\mathcal{M}^*(A, D; \mathcal{J}^l) := \{(u, J) |\ J \in \mathcal{J}^l, u \in \mathcal{M}^*(A, D; J)\}.$$

The universal moduli space is a separable Banach manifold for sufficiently large $l$ and the projection

$$\pi : \mathcal{M}^*(A, D; \mathcal{J}^l) \to \mathcal{J}^l$$

is a Fredholm map, i.e. its differential at every point is Fredholm. Moreover, the differential of $\pi$ at a point $(u, J) \in \mathcal{M}^*(A, D; \mathcal{J}^l)$ has the same Fredholm index as $D_u$ and is surjective precisely when $D_u$ is surjective. The implicit function theorem for Banach manifolds then implies that for a regular value $J \in \mathcal{J}^l$ of $\pi$, the preimage $\pi^{-1}(J) = \mathcal{M}^*(A, D; J)$ is a smooth finite-dimensional manifold whose dimension is the Fredholm index of $\pi$ at any point in $\pi^{-1}(J)$. This dimension is given by Equation (3.5.4). Finally, the Sard-Smale theorem says that the set of regular values of $\pi$ is a second category subset of $\mathcal{J}^l$ for large enough $l$. Hence, for large enough $l$, there is a second category subset, $\mathcal{J}^l_{\mathrm{reg}}$, of $\mathcal{J}^l$ such that for every $J \in \mathcal{J}^l_{\mathrm{reg}}$, the set $\mathcal{M}^*(A, D; J)$ is a smooth finite-dimensional manifold of the prescribed dimension. Finally, an argument of Taubes extends this to the smooth case and proves the following theorem.

**Theorem 3.5.7.** *[MS98, p. 43] There is a second category subset $\mathcal{J}_{\mathrm{reg}}$ of $\mathcal{J}(M, \omega)$, such that for all $J \in \mathcal{J}_{\mathrm{reg}}$, the set of simple $J$-holomorphic curves $\mathcal{M}^*(A, D; J)$ is a smooth finite-dimensional manifold of dimension*

$$n + \mu(u^*TM, u^*TL),$$

*where $u : (D, \partial D) \to (M, L)$ is any representative of the class $A \in H_2(M, L)$ in the image of the Hurewicz homomorphism.*

Elements of the set $\mathcal{J}_{\mathrm{reg}}$ are called **generic** almost complex structures.

# Chapter 4

# Lagrangian geometry

In Definition 3.2.6, we defined a Lagrangian submanifold to be an $n$-dimensional submanifold $L$ of a $2n$-dimensional symplectic manifold $(M, \omega)$ such that the symplectic form $\omega$ vanishes on $L$. In this chapter we first present several examples of Lagrangian submanifolds. Then we define the Maslov index for a Lagrangian submanifold, which is a relative version of the first Chern class. We also discuss the Lagrangian, or 'relative', version of the monotonicity condition introduced in Section 3.4.1, and relate it to the absolute version. Finally we introduce the pearl complex and prove a Künneth theorem for the Lagrangian quantum homology.

## 4.1 Examples of Lagrangian submanifolds

**Example 4.1.1.** The simplest example of a Lagrangian submanifold is $\mathbb{R}^n$ as a submanifold of $(\mathbb{C}^n, \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j) \cong (\mathbb{R}^{2n}, \omega_0)$. Here $\mathbb{R}^n$ is simply the set of all vectors in $\mathbb{C}^n$ whose imaginary component vanishes.

Weinstein's symplectic creed states, "Everything is a Lagrangian submanifold." [Wei77] Elaborating on his creed, Weinstein clarified that the most fruitful approach to many problems in symplectic geometry is to formulate them in terms of Lagrangian submanifolds. The creed also has a more concrete interpretation: Indeed, every smooth manifold is Lagrangian, as we see next.

**Example 4.1.2.** An arbitrary smooth manifold $M$ is an embedded submanifold of its cotangent bundle $T^*M$, as the image of the zero section $M \to T^*M$. If $T^*M$ is equipped with the canonical 2-form $\omega_{T^*M}$ as symplectic form (see Part 3 of Example 3.2.2), then $M$ is Lagrangian.

Although Example 4.1.2 may seem rather trivial, the zero sections of cotangent bundles are an important class of Lagrangians. Such Lagrangian submanifolds provide a local model for arbitrary compact embedded Lagrangian submanifolds. This is the content of the Weinstein tubular neighbourhood theorem, which is a Lagrangian version of the tubular neighbourhood theorem in differential geometry.

**Theorem 4.1.3** (Weinstein tubular neighbourhood theorem)**.** *[Wei71, Can06, p. 97] Let $(M, \omega)$ be a symplectic manifold and let $L \subset M$ be a compact, embedded Lagrangian submanifold with inclusion map $\iota : L \hookrightarrow M$. Let $\omega_{T^*L}$ be the canonical symplectic form on $T^*L$ and $\iota_0 : L \hookrightarrow T^*L$ be the zero section. Then there is a neighbourhood $U$ of $L$ in $M$ and $U_0$ of $L$ in $T^*L$ such that there exists a symplectomorphism*

$$\varphi : (U_0, \omega_{T^*L}|_{U_0}) \to (U, \omega|_U)$$

*satisfying $\varphi \circ \iota_0 = \iota$.*

Sections of $T^*M$ other than the zero section can also give rise to Lagrangian submanifolds.

**Example 4.1.4.** Let $M$ be an arbitrary manifold and let $\alpha \in \Omega^1(M)$. A straightforward computation shows that the image of $\alpha$ is a Lagrangian submanifold of $(T^*M, \omega_{T^*M})$ precisely when $\alpha$ is closed [Can01, p. 15]. However, not every Lagrangian submanifold of $T^*M$ is of this form: The fibre of $T^*M$ over any point $x \in M$, for instance, is also a Lagrangian submanifold of $(T^*M, \omega_{T^*M})$.

Together with the Weinstein tubular neighbourhood theorem, this example characterizes the smooth perturbations of a compact embedded Lagrangian which preserve the property of being Lagrangian.

In symplectic surfaces, Lagrangian submanifolds have a particularly simple description.

**Example 4.1.5.** By skew-symmetry, any symplectic bilinear form on a vector space must vanish on a one-dimensional subspace. Therefore any curve in a symplectic two-manifold is Lagrangian.

Examples of Lagrangians in manifolds of higher dimension can be produced by taking products of curves in symplectic surfaces. The following example confirms that the product of two Lagrangians is again Lagrangian.

**Example 4.1.6.** Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be symplectic manifolds and for $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, let $\lambda_1\omega_1 \oplus \lambda_2\omega_2$ be the symplectic form on $M_1 \times M_2$ defined in Part (4) of Example 3.2.2. If $L_1 \subset M_1$ and $L_2 \subset M_2$ are Lagrangian submanifolds, then it follows easily from the definitions that $L_1 \times L_2$ is a Lagrangian submanifold of $(M_1 \times M_2, \lambda_1\omega_1 \oplus \lambda_2\omega_2)$.

Clearly a similar statement holds for a product of finitely many Lagrangians. The torus, $\mathbb{T}^n$ in $(\mathbb{R}^{2n}, \omega_0)$, is an example of a compact Lagrangian submanifold in $\mathbb{R}^{2n}$ which is a product of curves. Since Langrangian tori of arbitrary radii exist in $\mathbb{R}^{2n}$, it follows from Darboux's theorem (Theorem 3.2.5) that all symplectic manifolds contain embedded Lagrangian tori. The question of the existence of embedded Lagrangians which are not simply a product of one-dimensional curves can be of a more global nature. Gromov's famous Non-Exactness theorem, proved using pseudoholomorphic curves, gives a homological obstruction to the existence of compact Lagrangians in $\mathbb{R}^{2n}$.

**Theorem 4.1.7** (Gromov's non-exactness theorem)**.** *[Gro85] Let $\lambda$ be the Liouville one-form on $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$,*

$$\lambda = \sum_{i=1}^{n} y_i dx_i.$$

*If $L \subset \mathbb{R}^{2n}$ is a compact Lagrangian, then the closed form $\iota^*\lambda \in \Omega^1(L)$ is never exact. Here $\iota : L \hookrightarrow M$ is the inclusion map.*

Gromov's theorem implies, in particular, that no compact manifold with trivial first cohomology can be embedded as a Lagrangian in $\mathbb{R}^{2n}$. Thus $\mathbb{R}^{2n}$ does not contain any embedded Lagrangian spheres for $n > 1$. For $n = 1$, clearly the image of any embedding $S^1 \to \mathbb{R}^2$ is Lagrangian.

In keeping with Weinstein's symplectic creed, the study of the symplectomorphism group of a symplectic manifold can be reformulated as the study of Lagrangians in a product space.

**Proposition 4.1.8.** *For a symplectic manifold $(M, \omega)$, a diffeomorphism $f \in \mathrm{Diff}(M)$ is a symplectomorphism precisely when its graph is a Lagrangian submanifold of $(M \times M, (-\omega) \oplus \omega)$ (cf. Proposition 2.4.8).*

As in the linear case, the proof of this proposition is an easy consequence of the definitions.

The following proposition describes an important class of Lagrangians, **symmetric Lagrangians** which are fixed point sets of antisymplectic involutions.

**Proposition 4.1.9.** *[Oh96] Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold and let $F$ be an antisymplectic involution, i.e. a diffeomorphism of $M$ such that $F^2 = \mathrm{Id}_M$ and $F^*\omega = -\omega$. Then if the fixed point set $\mathrm{Fix}(F)$ of $F$ has any $n$-dimensional connected components, they will be Lagrangian submanifolds of $M$.*

**Proof:** For $p \in \mathrm{Fix}(F)$, the condition $(F^*\omega)_p = -\omega_p$ implies $\omega_p = -\omega_p$. Therefore $\omega$ must vanish on $\mathrm{Fix}(F)$. One just needs to check that the connected components of $\mathrm{Fix}(F)$ are smooth submanifolds. If one endows $M$ with any Riemannian metric $g$, $F$ will be a an isometry with respect to the metric $g + F^*g$. The fact that the connected components of $\mathrm{Fix}(F)$ are smooth submanifolds then follows from the fact that connected components of the fixed point set of an isometry of a Riemannian manifold are smooth (totally geodesic) submanifolds [Kli95, p. 95]. ∎

Our last example is an important application of Proposition 4.1.9

**Example 4.1.10.** Real projective space $\mathbb{RP}^n$ sits inside $\mathbb{CP}^n$ as an embedded submanifold consisting of the set of points $[z_0, \ldots, z_n]$ that can be represented by real coordinates $(z_0, \ldots, z_n) \in \mathbb{R}^{n+1}$. When $\mathbb{CP}^n$ is equipped with the Fubini-Study form, $\mathbb{RP}^n$ is Lagrangian: In coordinates $(z_1, \ldots, z_n)$ on $\mathbb{C}^n$, the Fubini-Study form on $\mathbb{C}^n$ is given by

$$(\omega_{\mathrm{FS}})_z(\partial_j|_z, \bar{\partial}_k|_z) = \frac{i}{2}\left(\frac{(|z|^2+1)\delta_{jk} - \bar{z}_k z_j}{(|z|^2+1)^2}\right),$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We define an antisymplectic involution of $\mathbb{CP}^n$ by

$$\phi : \mathbb{CP}^n \to \mathbb{CP}^n, \quad [z_0, \ldots, z_n] \mapsto [\bar{z}_0, \ldots, \bar{z}_n].$$

Clearly $\phi^2 = \mathrm{Id}_{\mathbb{CP}^n}$. We check that $\phi^*\sigma_{\mathbb{CP}^n} = -\sigma_{\mathbb{CP}^n}$. First notice that it is sufficient to show that $\psi^*\omega_{\mathrm{FS}} = -\omega_{\mathrm{FS}}$ where $\psi : \mathbb{C}^n \to \mathbb{C}^n$ is the involution induced by $\phi$ using any of the coordinate charts $\xi_j^n$, $j \in \{0, \ldots, n\}$. We have

$$\begin{aligned}
(\psi^*\omega_{\mathrm{FS}})_z\left(\partial_j|_z, \bar{\partial}_k|_z\right) &= (\omega_{\mathrm{FS}})_{\psi(z)}\left((\psi_*)_z\partial_j|_z, (\psi_*)_z\bar{\partial}_k|_z\right),\\
&= (\omega_{\mathrm{FS}})_{\bar{z}}(\bar{\partial}_j|_{\bar{z}}, \partial_k|_{\bar{z}}),\\
&= -(\omega_{\mathrm{FS}})_{\bar{z}}(\partial_k|_{\bar{z}}, \bar{\partial}_j|_{\bar{z}}),\\
&= -(\omega_{\mathrm{FS}})_z(\partial_j|_z, \bar{\partial}_k|_z).
\end{aligned}$$

Clearly the fixed point set of $\phi$ is $\mathbb{RP}^n$, and thus $\mathbb{RP}^n$ is Lagrangian in $(\mathbb{CP}^n, \sigma_{\mathbb{CP}^n})$.

## 4.2 The Maslov index for Lagrangian submanifolds

Just as we defined the first Chern class of a symplectic vector bundle in terms of the first Chern number of symplectic vector bundles over surfaces, we define the Maslov index of a Lagrangian submanifold in terms of an invariant for bundles over surfaces. More precisely, in this situation we consider bundle pairs, as defined next.

**Definition 4.2.1.** *A **bundle pair** $(E, L)$ consists of a symplectic vector bundle $(E, \omega)$ over a compact oriented surface $\Sigma$ with non-empty boundary and a Lagrangian subbundle $L \to \partial\Sigma$ of $E|_{\partial\Sigma}$. We define the Maslov index of the bundle pair $(E, L)$ as follows: We choose a symplectic trivialization of $(E, \omega)$*

$$\phi : (E, \omega) \to (\Sigma \times \mathbb{R}^{2n}, \omega_0),$$

*which is guaranteed to exist by Proposition 3.1.9. The boundary of $\partial\Sigma$ is diffeomorphic to a finite disjoint union of circles and we endow each connected component $C_j$ of $\partial\Sigma$ with the orientation induced by the orientation of $\Sigma$, (see Definition 3.4.2). By the first part of Remark 3.1.11, the restriction of $\phi$ to $E|_{\partial\Sigma}$ carries the Lagrangian subbundle $L$ of $(E|_{\partial\Sigma}, \omega|_{\partial\Sigma})$ onto a Lagrangian subbundle of $(\partial\Sigma \times \mathbb{R}^{2n}, \omega_0)$. In other words, at any point $p \in \partial\Sigma$, $\phi_p(L_p)$ is a Lagrangian subspace of $\mathbb{R}^{2n}$. Therefore we have a map*

$$\gamma : \partial\Sigma \to \Lambda(n), \quad p \mapsto \phi_p(L_p) \subset \mathbb{R}^{2n}. \tag{4.2.1}$$

*The **Maslov index of the bundle pair** $(E, L)$ is defined by*

$$\mu(E, L) = \deg(\tau \circ \gamma),$$

*where $\tau$ is the map defined in 2.4.6. More explicitly, if for each connected component $C_j$ of $\partial\Sigma$, we choose an orientation preserving diffeomorphism*

$$\eta_j : S^1 \to C_j,$$

*then the Maslov index of the bundle pair* $(E, L)$ *is the sum of the Maslov indices of the loops* $\gamma|_{C_j} \circ \eta_j$ *in* $\Lambda(n)$*:*

$$\mu(E, L) = \sum_{j=1}^{l} \mu(\gamma_j \circ \eta_j), \tag{4.2.2}$$

*where* $l$ *is the number of connected components of* $\partial\Sigma$*.*

**Proposition 4.2.2.** *[FOOO09, p. 42] The Maslov index of a bundle pair does not depend on the choice of symplectic trivialization used to compute it.*

**Proof:** Let $(E, L)$ be a vector bundle pair over $\Sigma$. Suppose $\phi_1$ and $\phi_2$ are two symplectic trivializations of $E$ and let $\gamma_{\phi_1}$ and $\gamma_{\phi_2}$ be the loops in $\Lambda(n)$ defined as in Equation (4.2.1) for these trivializations. Consider the map

$$\Psi := \phi_{12} : \Sigma \to \mathrm{Sp}(2n), \quad p \mapsto (\phi_1)_p \circ (\phi_2)_p^{-1},$$

where $2n$ is the rank of $E$. We have

$$\gamma_{\phi_1} = \Psi\gamma_{\phi_2}.$$

Restricting to a connected component $C_j \subset \partial\Sigma$, the Maslov indices of the loops $\gamma_{\phi_1}|_{C_j}$ and $\gamma_{\phi_2}|_{C_j}$ in $\Lambda(n)$ are related by

$$\mu(\gamma_{\phi_1}|_{C_j}) = \mu(\Psi|_{C_j}\gamma_{\phi_2}|_{C_j}) = 2\mu(\Psi|_{C_j}) + \mu(\gamma_{\phi_2}|_{C_j}).$$

The second equality is an application of Property (2) of the Maslov index of loops in $\Lambda(n)$ (Theorem 2.5.8). Since the same relationship holds for every connected component of $\Sigma$, we obtain

$$\sum_{j=1}^{l} \mu(\gamma_{\phi_1}|_{C_j}) = 2\sum_{j=1}^{l} \mu(\Psi|_{C_j}) + \sum_{j=1}^{l} \mu(\gamma_{\phi_2}|_{C_j}),$$

where $l$ is the number of connected components, and thus

$$\mu_{\phi_1}(E, L) = 2\deg(\rho \circ \Psi|_{\partial\Sigma}) + \mu_{\phi_2}(E, L).$$

Here $\mu_{\phi_1}$ and $\mu_{\phi_2}$ are the Maslov indices of the bundle pair $(E, L)$ computed with respect to the trivializations $\phi_1$ and $\phi_2$, and $\rho$ is the map which takes a symplectic matrix to the determinant of its unitary factor (see Definition 2.5.2). It remains to notice that since $\Psi|_{\partial\Sigma}$ extends to $\Sigma$, so too does $\rho \circ \Psi|_{\partial\Sigma}$. Hence $\deg(\rho \circ \Psi|_{\partial\Sigma}) = 0$ (see [Mil97, p. 28]) and $\mu_{\phi_1} = \mu_{\phi_2}$. ∎

We refer the reader to Appendix 9 by J. Robbin in [MS12] for an axiomatic description of the Maslov index of a bundle pair.

**Remark 4.2.3.** The Maslov index of the bundle pair $(E, L)$ can be viewed as an obstruction to extending $L$ to a Lagrangian subbundle of $E$ defined on all of $\Sigma$. By this we mean that if $L$ can be extended, $\mu(E, L) = 0$. This is due to the fact that an extension of $L$ results in an extension of the map $\gamma : \partial\Sigma \to \Lambda(n)$ to all of $\Sigma$. Thus $\tau \circ \gamma$ extends to $\Sigma$ and it follows that $\deg(\tau \circ \gamma) = 0$.

We now use the definition of the Maslov index for a bundle pair to define the Maslov index of a Lagrangian submanifold.

**Definition 4.2.4.** *Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold and let $L$ be a Lagrangian submanifold of $M$. The* ***Maslov index*** *of $L$ is a map which assigns an integer to homotopy classes of smooth maps of pairs $(\Sigma, \partial\Sigma) \to (M, L)$, defined by*

$$\mu([u]) := \mu(u^*TM, u^*TL),$$

*for $u : (\Sigma, \partial\Sigma) \to (M, L)$. Note that since the boundary of $\Sigma$ is mapped by $u$ into the Lagrangian submanifold $L$, by the second part of Remark 3.1.11, the pullback $(\partial u)^*TL$ is a Lagrangian subbundle of $(\partial u)^*TM$.*

*More explicitly, for a symplectic trivialization*

$$\phi : (u^*TM, u^*\omega) \to (\sigma \times \mathbb{R}^{2n}, \omega_0),$$

*the map $u$ determines a map*

$$\gamma : \partial\Sigma \to \Lambda(n), \quad z \mapsto \phi_z((u^*TL)_z).$$

*The Maslov index of $u$, $\mu([u])$, is given by $\deg(\tau \circ \gamma)$. In particular, setting $\Sigma = D$ and restricting to homotopy classes of based maps, the Maslov index of a Lagrangian submanifold determines a map*

$$\mu|_{\pi_2(M,L)} : \pi_2(M, L) \to \mathbb{Z}. \tag{4.2.3}$$

We omit the proof of the following theorem as it is very similar to the proof of Proposition 3.4.7.

**Proposition 4.2.5.** *The Maslov index for Lagrangian submanifolds is well-defined: It does not depend on the choice of representative $u$ of the homotopy class*

$$[u] \in [(\Sigma, \partial\Sigma), (M, L)].$$

*Moreover, the map $\mu|_{\pi_2(M,L)}$ is a group homomorphism from $\pi_2(M, L)$ to $\mathbb{Z}$.*

The following simple example provides some geometric intuition for the Maslov index of a Lagrangian submanifold.

**Example 4.2.6.** Consider the unit circle $S^1$ as a Lagrangian submanifold of

$$(\mathbb{R}^2, dx \wedge dy) \cong (\mathbb{C}, \tfrac{i}{2} dz \wedge d\bar{z}).$$

We compute the homomorphism $\mu|_{\pi_2(\mathbb{C},S^1)}$. The relative homotopy group $\pi_2(\mathbb{C}, S^1)$ is isomorphic to $\mathbb{Z}$ and, for $n \in \mathbb{Z}$, a representative $u_n$ of the corresponding homotopy class is given by

$$u_n : (D, \partial D) \to (\mathbb{C}, S^1), \quad z \mapsto \begin{cases} z^n, & n \geq 0, \\ \bar{z}^{-n}, & n < 0, \end{cases}$$

for $z$ in the closed unit disc $D \subset \mathbb{C}$. We compute the Maslov index of each class $[u_n]$ explicitly. As a symplectic trivialization of $u_n^* T\mathbb{C}$, we can simply take the identity map

$$\mathrm{Id} : u_n^* T\mathbb{C} \to D \times \mathbb{C},$$

where the tangent space of $\mathbb{C}$ is identified with $\mathbb{C}$ in the usual way. Under this identification, the tangent space to $S^1$ at the point $u_n(z)$ for $z \in \partial D$ is

$$T_{u_n(z)} S^1 = \begin{cases} z^n e^{i\frac{\pi}{2}} \Lambda_{\mathrm{hor}}, & n \geq 0, \\ \bar{z}^{-n} e^{i\frac{\pi}{2}} \Lambda_{\mathrm{hor}}, & n < 0. \end{cases}$$

For all $n$, the loop in $\Lambda(1)$ determined by $u_n$ is thus

$$\gamma_n : S^1 \to \Lambda(1), \quad e^{2\pi i\theta} \mapsto e^{\pi i(2n\theta + \frac{1}{2})} \Lambda_{\mathrm{hor}},$$

and the Maslov index of this loop is

$$\mu(\gamma_n) = \deg(\tau \circ \gamma_n) = 2n.$$

Therefore the homomorphism $\mu|_{\pi_2(\mathbb{C}, S^1)}$ is given by

$$\mu|_{\pi_2(\mathbb{C}, S^1)} : \pi_2(\mathbb{C}, S^1) \to \mathbb{Z}, \quad [u_n] \mapsto 2n.$$

The following results relate the Maslov index of a Lagrangian submanifold to the first Chern class of the ambient symplectic manifold. We will use Corollary 4.2.9 of Proposition 4.2.7 at the end of this section to show that the Maslov index of a Lagrangian submanifold is well-defined on homology. The other results will be needed in the next section.

**Proposition 4.2.7.** *Let $E \to \Sigma$ be a symplectic vector bundle over a compact oriented surface with empty boundary, let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a decomposition of $\Sigma$ and let $L \to \partial\Sigma_1$ be a Lagrangian subbundle. Then*

$$2c_1(E) = \mu(E|_{\Sigma_1}, L) + \mu(E|_{\Sigma_2}, L).$$

**Proof:** Choose symplectic trivializations $\phi_1$ and $\phi_2$ of $E|_{\Sigma_1}$ and $E|_{\Sigma_2}$, and let

$$\Psi : \partial\Sigma_1 \to \mathrm{Sp}(2n)$$

be the overlap map. As $\Sigma_1$ and $\Sigma_2$ induce opposite orientations on $\partial\Sigma_1 = \partial\Sigma_2$, for clarity, we denote the connected components of $\partial\Sigma_1$ with the orientation induced by $\Sigma_1$ by $C_j$, $j = 1, \ldots, l$, and with the orientation induced by $\Sigma_2$ by $\overline{C}_j$. As in the proof of Proposition 4.2.2, the paths $\gamma_{\phi_1}$ and $\gamma_{\phi_2}$ in the Lagrangian Grassmannian determined by $\phi_1$ and $\phi_2$ satisfy $\gamma_{\phi_1} = \Psi\gamma_{\phi_2}$. Then

$$\begin{aligned}
\mu(E|_{\Sigma_1}, L) &= \deg(\tau \circ \gamma_{\phi_1}) = \sum_{j=1}^{l} \mu(\gamma_{\phi_1}|_{C_j}) = \sum_{j=1}^{l} \mu(\Psi\gamma_{\phi_2}|_{C_j}), \\
&= 2\sum_{j=1}^{l} \mu(\Psi|_{C_j}) + \sum_{j=1}^{l} \mu(\gamma_{\phi_2}|_{C_j}), \quad \text{(Theorem 2.5.8, Property 2)} \\
&= 2\sum_{j=1}^{l} \mu(\Psi|_{C_j}) - \sum_{j=1}^{l} \mu(\gamma_{\phi_2}|_{\overline{C}_j}), \\
&= 2c_1(E) - \mu(E|_{\Sigma_2}, L).
\end{aligned}$$

∎

The first corollary of Proposition 4.2.7 is a version of the proposition for a decomposition of $\Sigma$ by finitely many compact oriented surfaces with boundary.

**Corollary 4.2.8.** *Let $\Sigma$ be a compact oriented surface with empty boundary and let $\Sigma = \cup_{j=1}^{N}\Sigma_j$ be a decomposition of $\Sigma$. In other words, the $\Sigma_j$ are submanifolds of $\Sigma$ with non-empty boundary satisfying*

$$\Sigma_i \cap \Sigma_j = \partial\Sigma_i \cap \partial\Sigma_j,$$

*for all $i, j \in \{1, \ldots, N\}$. Suppose $E \to \Sigma$ is a symplectic vector bundle and $L \to \cup_{j=1}^{N}\partial\Sigma_j$ is a Lagrangian subbundle, then*

$$2c_1(E) = \sum_{j=1}^{N} \mu(E|_{\Sigma_j}, L|_{\Sigma_j}).$$

**Proof:** Note that if we define $\Sigma' = \cup_{j=1}^{N-1}\Sigma_j$, then $\Sigma = \Sigma' \cup \Sigma_N$ is a decomposition of $\Sigma$, and thus by Proposition 4.2.7

$$2c_1(E) = \mu(E|_{\Sigma'}, L|_{\partial\Sigma'}) + \mu(E|_{\Sigma_N}, L|_{\partial\Sigma_N}).$$

Therefore we simply need to show that

$$\mu(E|_{\Sigma'}, L|_{\partial\Sigma'}) = \sum_{j=1}^{N-1} \mu(E|_{\Sigma_j}, L|_{\partial\Sigma_j}).$$

To do this, choose a symplectic trivialization $\phi$ of $E|_{\Sigma'}$, which exists as $\Sigma_N$ has non-empty boundary and thus so must $\Sigma'$. Then $\phi|_{E|_{\Sigma_j}}$ is a symplectic trivialization of $\Sigma_j$ for $j = 1, \ldots, N-1$. Let $\gamma^j$ be the path in the Lagrangian Grassmannian determined by $\phi|_{E|_{\Sigma_j}}$ and let $C_k^j$, $k = 1, \ldots, l_j$, be the connected components of the boundary of $\Sigma_j$. Then

$$\sum_{j=1}^{N-1} \mu(E|_{\Sigma_j}, L|_{\partial\Sigma_j}) = \sum_{j=1}^{N-1} \sum_{k=1}^{l_j} \mu(\gamma^j|_{C_k^j}). \tag{4.2.4}$$

In the terms $\mu(\gamma^j|_{C_k^j})$, $C_k^j$ is understood to have the orientation induced by the orientation of $\Sigma_j$. Any given boundary component $C_k^j$ of one of the surfaces $\Sigma_j$ is either contained in $\partial\Sigma'$ or it is contained in precisely two of the surfaces $\Sigma_1, \ldots, \Sigma_{N-1}$. In the latter case, assume $C_k^j \subset \Sigma_j \cap \Sigma_m$. Then $\Sigma_j$ and $\Sigma_m$ induce opposite orientations on $C_k^j$ and thus

$$\mu(\gamma^j|_{C_k^j}) = -\mu(\gamma^m|_{C_{k'}^m}),$$

where $C_{k'}^m \subset \partial\Sigma_m$ is equal to $C_k^j$ but oppositely oriented. It follows that all terms contained in the sum in Equation (4.2.4) cancel in pairs, except for those corresponding to the connected components $C_1, \ldots, C_l$ of $\partial\Sigma'$. Therefore

$$\sum_{j=1}^{N-1} \mu(E|_{\Sigma_j}, L|_{\partial\Sigma_j}) = \mu(E|_{\Sigma'}, L|_{\partial\Sigma'}).$$

∎

The following corollary is a direct consequence of the previous one.

**Corollary 4.2.9.** *Let $M$ be a symplectic manifold with Lagrangian submanifold $L$, let $\Sigma$ be a compact oriented surface with decomposition $\Sigma = \cup_{j=1}^N \Sigma_j$, and let*

$$u : (\Sigma, \cup_{j=1}^N \partial\Sigma_j) \to (M, L)$$

*be a smooth map. Then*

$$c_1([u]) = \sum_{j=1}^N \mu([u|_{\Sigma_j}]).$$

The following propositions, which are also straightforward consequences of Proposition 4.2.7, will be important in the next section for relating the absolute and relative concepts of monotonicity.

**Proposition 4.2.10.** *[Vit87] Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold and $L$ be a Lagrangian submanifold. Suppose $p_0 \in L$ and $x_0 \in \partial D$ are base points, and*

$$u_1, u_2 : (D, \partial D, x_0) \to (M, L, p_0)$$

*are smooth maps of triples with $u_1|_{\partial D} = u_2|_{\partial D}$. Let $D_+$ and $D_-$ be the upper and lower hemispheres of $S^2$ and define the maps*

$$\mathrm{pr}_\pm : D_\pm \to D \subset \mathbb{R}^2, \quad (x, y, z) \mapsto (x, y),$$

*for $(x, y, z) \in D_\pm \subset S^2 \subset \mathbb{R}^3$. Then if the map of pairs $w : (S^2, x_0) \to (M, p_0)$ defined by*

$$w(x) = \begin{cases} u_1(\mathrm{pr}_+(x)), & x \in D_+, \\ u_2(\mathrm{pr}_-(x)), & x \in D_-. \end{cases} \tag{4.2.5}$$

*is smooth, it satisfies*

$$2c_1([w]) = \mu([u_1]) - \mu([u_2]).$$

**Proof:** This follows from Corollary 4.2.9 and the fact that the map pr_ reverses the orientation of $\partial D$, and thus

$$\mu([u_2 \circ \mathrm{pr}_-]) = -\mu([u_2]). \qquad \blacksquare$$

**Remark 4.2.11.** Even if the map of pairs $w$ defined in Equation (4.2.5) is not smooth, one can choose alternative representatives of the classes $[u_1], [u_2] \in \pi_2(M, L)$ so that this map is smooth.

**Proposition 4.2.12.** *[Oh93a] Let $(M, \omega)$ be a symplectic manifold with Lagrangian submanifold $L$ and let $j : \pi_2(M, p_0) \to \pi_2(M, L, p_0)$ be the canonical homomorphism. Then $2c_1|_{\pi_2(M,p_0)} = \mu \circ j$.*

**Proof:** Let $[w] \in \pi_2(M, p_0)$ where $w : (D, \partial D, x_0) \to (M, p_0, p_0)$. Define maps

$$u_1, u_2 : (D, \partial D, x_0) \to (M, L, p_0), \quad u_1(x) = \iota \circ w(x), \quad u_2(x) = p_0,$$

for all $x \in D$. Then the difference map

$$w' : (S^2, x_0) \to (M, p_0), \quad w'(x) = \begin{cases} u_1(\mathrm{pr}_+(p)), & p \in D_+, \\ u_2(\mathrm{pr}_-(p)), & p \in D_-, \end{cases}$$

clearly represents the same element of $\pi_2(M, p_0)$ as $w$. Note that the representative $w$ of the class $[w]$ can be chosen so that $w'$ is smooth. Applying Corollary 4.2.10,

$$\begin{aligned} 2c_1([w]) &= 2c_1([w']) = \mu([u_1]) - \mu([u_2]) \\ &= \mu([u_1]) = \mu([\iota \circ w]) \\ &= \mu(j([w])), \end{aligned}$$

where we have used the fact that the Maslov index of a constant map is zero. ∎

We mentioned in Remark 3.4.9 that the first Chern class is well-defined on homology. A similar statement holds for the Maslov index.

**Proposition 4.2.13.** *The Maslov index is well-defined on homology. That is, if two maps*

$$u : (\Sigma_1, \partial\Sigma_1) \to (M, L), \quad v : (\Sigma_2, \partial\Sigma_2) \to (M, L),$$

*satisfy*

$$u_*([\Sigma_1]) = v_*([\Sigma_2]) \in H_2(M, L),$$

*then* $\mu([u]) = \mu([v])$.

**Proof:** Choose smooth absolute singular chains $\sigma_i \in C_2(\Sigma_i)$ representing the fundamental classes $[\Sigma_i] \in H_2(\Sigma_i, \partial\Sigma_i)$ for $i = 1, 2$. In other words,

$$[\sigma_i + C_2(\partial\Sigma_i)] = [\Sigma_i].$$

Since $u$ and $v$ represent the same relative homology class in $H_2(M, L)$, there exists $\alpha \in C_3(M)$ and $\beta \in C_2(L)$ such that

$$u \circ \sigma_1 - v \circ \sigma_2 = \partial\alpha + \beta.$$

We assume that the singular chain $\alpha$ represents a smooth (immersed) submanifold of $M$ with boundary. In other words, the following exist:

- A compact oriented 3-manifold $X$ such that the boundary of $X$ has the decomposition

  $$\partial X = \Sigma_1 \cup \overline{\Sigma}_2 \cup \Sigma',$$

  where $\Sigma'$ is an oriented surface with boundary, and $\overline{\Sigma}_2$ is equal to $\Sigma_2$ but inherits the opposite orientation from $X$.

- A map $f : (X, \Sigma') \to (M, L)$ extending $u$ and $v$.

Then by Corollary 4.2.9

$$\mu([u]) - \mu([v]) + \mu([f|_{\Sigma'}]) = 2c_1([f|_{\partial X}]).$$

Since the map $f|_{\partial X}$ extends to $X$ (see Remark 3.4.8), $c_1([f|_{\partial X}]) = 0$. Moreover, since $f$ maps $\Sigma'$ into $L$, by Remark 4.2.3, $\mu(f|_{\Sigma'}) = 0$. Therefore $\mu([u]) = \mu([v])$.

In general, the singular chain $\alpha$ may not represent a smooth immersed submanifold of $M$. However, $\alpha$ can always be chosen such that it represents a submanifold with singularities, and the above proof can be adapted to this case. [MS98, p. 80] ∎

**Remark 4.2.14.** From here on, we will only be interested in the case of maps from the unit disc when considering the Maslov index for a Lagrangian submanifold. We adopt the standard abuse of notation in denoting both the Maslov index homomorphism $\pi_2(M, L) \to \mathbb{Z}$ of Definition 4.2.4 as well as the map defined on elements of $H_2(M, L)$, which was established in Proposition 4.2.13, by $\mu$.

## 4.3 Monotonicity

In defining the Lagrangian quantum homology we will use the following relative version of the monotonicity condition.

**Definition 4.3.1.** *Let $(M, \omega)$ be a symplectic manifold and let $L$ be a Lagrangian submanifold. The symplectic form $\omega$ determines a homomorphism, which we again denote by $\omega$,*

$$\omega : \pi_2(M, L) \to \mathbb{R}, \quad [u] \mapsto \int_D u^*\omega.$$

*Since $\omega$ is a closed form which vanishes on $L$, it also defines a homomorphism*

$$H_2(M, L) \to \mathbb{R}, \quad [c] \mapsto \int_c \omega,$$

*where $c$ is a singular 2-chain in $M$ with boundary in $L$. We also denote this homomorphism by $\omega$. These two homomorphisms are related by*

$$\omega([u]) = \omega(h([u])),$$

*where $h : \pi_2(M, L) \to H_2(M, L)$ is the relative Hurewicz homomorphism. The Lagrangian submanifold $L$ is called **monotone** if the homomorphisms $\mu$ and $\omega$ satisfy*

$$\mu(\sigma) > 0 \Longleftrightarrow \omega(\sigma) > 0 \textit{ for all } \sigma \in \pi_2(M, L). \tag{4.3.1}$$

*Equivalently, regarding $\mu$ and $\omega$ as homomorphisms defined on $H_2(M, L)$, $L$ is called monotone if*

$$\mu(A) > 0 \Longleftrightarrow \omega(A) > 0 \textit{ for all } A \in H_2^D(M, L),$$

*where $H_2^D(M, L)$ is the image of the relative Hurewicz homomorphism.*

The exact argument in the proof of Proposition 3.4.15 also applies to the relative case and thus the condition of Equation (4.3.1) is equivalent to the existence of a postive constant $\tau$ such that $\omega|_{H_2^D(M,L)} = \tau\mu|_{H_2^D(M,L)}$. We have the following relative version of Definition 3.4.16.

**Definition 4.3.2.** *The constant $\tau$ is called the* ***monotonicity constant*** *of the Lagrangian submanifold. It is unique if $\mu|_{H_2^D(M,L)} \not\equiv 0$. We denote the positive generator of $\mu(H_2^D(M, L)) \subseteq \mathbb{Z}$ by $N_L$ and call this number the* ***minimal Maslov number*** *of $L$. In the case where $\mu|_{H_2^D(M,L)} \equiv 0$, we define $N_L = \infty$.*

**Example 4.3.3.**

1. If $L$ is a Lagrangian submanifold of the symplectic manifold $M$ with $\pi_2(M, L) = 0$, then $L$ is monotone. Hence $\mathbb{R}^n$ is a monotone Lagrangian in $(\mathbb{C}^n, \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j)$ (see Example 4.1.1).

2. As in Example 4.2.6, consider the unit circle $S^1$ in $(\mathbb{C}, \frac{i}{2} dz \wedge d\bar{z})$. We showed that the homotopy class in $\pi_2(\mathbb{C}, S^1)$ of the map

$$u_n : (D, \partial D) \to (\mathbb{C}, S^1), \quad z \mapsto \begin{cases} z^n, & n \geq 0, \\ \bar{z}^{-n}, & n < 0, \end{cases}$$

has Maslov index $\mu([u_n]) = 2n$. We also have

$$\begin{aligned} \omega([u_n]) &= \int_D u_n^* \left( \frac{i}{2} dz \wedge d\bar{z} \right) = \int_D u_n^*(r dr \wedge d\theta) \\ &= \int_D r dr \wedge d(n\theta) = n \int_D r dr \wedge d\theta \end{aligned}$$

$$= n\pi.$$

Therefore $\omega = \frac{\pi}{2}\mu$ on $\pi_2(\mathbb{C}, S^1)$ and we see that $S^1$ is monotone in $(\mathbb{C}, \frac{i}{2}dz \wedge d\bar{z})$ with monotonicity constant $\frac{\pi}{2}$.

3. Let $L \subset S^2$ be a closed curve, where $S^2$ has the standard symplectic structure $\omega_{\mathrm{vol}}$. Then $L$ is monotone if and only if it divides $S^2$ into two components of equal area: Suppose $A_1$ and $A_2$ are the areas of the two components of $S^2 \setminus L$. We have $\pi_2(S^2, L) \cong \mathbb{Z} \oplus \mathbb{Z}$ and we may choose generators $[v_1]$ and $[v_2]$ of $\pi_2(S^2, L)$ that are represented by symplectomorphisms[1]

$$v_1 : (D_{A_1}, \partial D_{A_1}) \to (S^2, L), \quad v_2 : (D_{A_2}, \partial D_{A_1}) \to (S^2, L),$$

where $D_{A_j}$ is the closed unit disc in $\mathbb{C}$ of area $A_j$ for $j = 1, 2$. Then the differentials $dv_j$ induce isomorphisms of symplectic vector bundles

$$(TD_{A_j}, \omega_0) \to (TS^2|_{v_j(D_{A_j})}, \omega_{\mathrm{vol}}),$$

which carries $T(\partial D_{A_1})$ onto $TL$. It follows that $\mu([v_j]) = \mu([u_j])$ where

$$u_j : (D_{A_j}, \partial D_{A_j}) \to (\mathbb{C}, \partial D_{A_j}), \quad z \mapsto z.$$

From Example 4.2.6, $\mu([u_j]) = 2$. However, $\omega([v_1]) = A_1$ and $\omega([v_2]) = A_2$. Therefore $L$ is monotone if and only if

$$\frac{\omega([v_1])}{\mu([v_1])} = \frac{\omega([v_2])}{\mu([v_2])},$$

which occurs precisely when $A_1 = A_2$.

[1]This follows from a theorem of Moser, and its generalization by Banyaga to manifolds with boundary, which says that if two diffeomorphic compact oriented manifolds have the same total volume, then there is a diffeomorphism between them which preserves the volume form [Ban74, Mos65].

In Example 4.2.6, we computed the Maslov index of $[u_n] \in \pi_2(\mathbb{C}, S^1)$. We saw that the path in the Lagrangian Grassmannian for each $u_n$ could be written as a time-dependent rotation of the horizontal subspace $\Lambda_{\text{hor}} \subset \mathbb{C}$ and that the Maslov index of $[u_n]$ counted the number of rotations of $\Lambda_{\text{hor}}$, without taking its orientation into account. In other words, the subspaces $e^{2\pi is}\Lambda_{\text{hor}}$ and $e^{2\pi i(s-1/2)}\Lambda_{\text{hor}}$, for $s \in \mathbb{R}$, are considered equivalent in this count. A loop of odd Maslov number in the Lagrangian Grassmannian would correspond to a path starting at $e^{2\pi is}\Lambda_{\text{hor}}$ and ending at $e^{2\pi i(s-1/2)}\Lambda_{\text{hor}}$.

This geometric picture suggests that the parity of the minimal Maslov number of a Lagrangian submanifold is related to the orientability of the Lagrangian submanifold. Indeed this is the case, as we prove in Proposition 4.3.6. To prove this proposition, we will use the concept of a frame bundle and Lemma 4.3.5 which gives a condition for orientability in terms of loops in the frame bundle.

**Definition 4.3.4.** *[KN63, p. 55] Let $E \to B$ be a real vector bundle. The* ***frame bundle*** *associated to the vector bundle $E$, $F(E)$, is defined by*

$$F(E) = \bigcup_{b \in B} \{(v_1, \ldots, v_n) | \ \{v_1, \ldots, v_n\} \text{ is a basis for } E_b\}.$$

*The frame bundle associated to $E$ is a locally trivial fibre bundle over $B$ with the obvious projection.*

We omit the routine proof of the following lemma.

**Lemma 4.3.5.** *Let $E \to B$ be a real vector bundle of rank $n$. Then $E$ is orientable if and only if for any loop in $\gamma : [0,1] \to B$ and any lift $\widetilde{\gamma} : [0,1] \to F(E)$ of $\gamma$, the ordered bases $\widetilde{\gamma}(0)$ and $\widetilde{\gamma}(1)$ determine the same orientation on $E_{\gamma(0)}$.*

**Proposition 4.3.6.** *Let $L$ be a connected Lagrangian submanifold of the $2n$-dimensional symplectic manifold $(M, \omega)$. The following statements hold:*

1. *If $L$ is orientable, then the minimal Maslov number $N_L$ of $L$ is even or infinite.*

*2. If $N_L$ is even or infinite and $\pi_1(M, L)$ is trivial, then $L$ is orientable.*

**Proof:**

1. Assume $N_L$ is not even or infinite. Then we can choose a map

$$u : (D, \partial D) \to (M, L)$$

such that $\mu([u])$ is odd. Take a symplectic trivialization

$$\phi : (u^*TM, u^*\omega) \to (D \times \mathbb{R}^{2n}, \omega_0).$$

As usual, let $\gamma : S^1 \to \Lambda(n)$ be the loop in the Lagrangian Grassmannian determined by this trivialization. Furthermore, let

$$\widetilde{\gamma} : [0,1] \to \mathrm{U}(n), \quad t \mapsto \begin{pmatrix} X(t) & -Y(t) \\ Y(t) & X(t) \end{pmatrix}$$

be a lift of $\gamma$. Write

$$X(t) = \begin{pmatrix} x_1(t) & \cdots & x_n(t) \end{pmatrix}, \quad Y(t) = \begin{pmatrix} y_1(t) & \cdots & y_n(t) \end{pmatrix},$$

for $x_j, y_j : [0,1] \to \mathbb{R}^n$.

Without loss of generality, we can assume $\widetilde{\gamma}(0) = \mathrm{Id}_{2n}$, for otherwise we can compose $\phi$ with the symplectic trivialization

$$\psi : (D \times \mathbb{R}^{2n}, \omega_0) \to (D \times \mathbb{R}^{2n}, \omega_0), \quad (z, v) \mapsto (z, \widetilde{\gamma}^{-1}(0)(v)).$$

The loop in $\Lambda(n)$ determined by the symplectic trivialization $\psi \circ \phi$ of $(u^*TM, u^*\omega)$ has as lift $(\widetilde{\gamma}(0))^{-1}\widetilde{\gamma} : [0,1] \to \mathrm{U}(n)$, which starts at the identity.

Since $\gamma$ has odd Maslov index, $\det(\widetilde{\gamma}(1)) = -1$ (see Remark 2.5.7). Define smooth maps

$$\chi_i : [0,1] \to TL|_{u(S^1)}, \quad \chi_i(t) = \mathrm{pr}_2 \circ \phi^{-1}_{e^{2\pi i t}} \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix},$$

where $\mathrm{pr}_2$ is the projection of

$$u^*TM|_{e^{2\pi it}} = \{(e^{2\pi it}, v) |\ v \in T_{u(e^{2\pi it})}M\}$$

onto the second component. Let $\chi$ be the path in $F(TL)$ given by

$$\chi : [0,1] \to F(TL), \quad t \mapsto (\chi_1(t), \dots, \chi_n(t)).$$

We aim to show that $(\chi_1(0), \dots, \chi_n(0))$ and $(\chi_1(1), \dots, \chi_n(1))$ are oppositely orientated bases of $T_{u(1)}L$. As $\chi$ lifts the loop $u|_{\partial D}$, Lemma 4.3.5 will then imply that $L$ is not orientable. Using the vector space isomorphism

$$T_{u(1)}L \to \phi(u^*TL|_1) \subset \mathbb{R}^{2n}, \quad v \mapsto \phi_1((1,v)),$$

it is enough to show that the bases

$$\left(\begin{pmatrix} x_1(0) \\ y_1(0) \end{pmatrix}, \dots, \begin{pmatrix} x_n(0) \\ y_n(0) \end{pmatrix}\right) \text{ and } \left(\begin{pmatrix} x_1(1) \\ y_1(1) \end{pmatrix}, \dots, \begin{pmatrix} x_n(1) \\ y_n(1) \end{pmatrix}\right)$$

of $\phi(u^*TL|_1)$ have opposite orientations. This is clear since

$$\begin{pmatrix} X(1) \\ Y(1) \end{pmatrix} = \widetilde{\gamma}(1)(\widetilde{\gamma}(0))^{-1} \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix},$$

and $\det(\widetilde{\gamma}(1)(\widetilde{\gamma}(0))^{-1}) = -1$.

2. Now assume that $N_L$ is even. We show that $L$ is orientable, again by applying Lemma 4.3.5. Let $\eta : S^1 \to L$ be a loop in $L$ and let

$$\widetilde{\eta} : [0,1] \to F(TL), \quad \widetilde{\eta} = (\widetilde{\eta}_1, \dots, \widetilde{\eta}_n),$$

be a lift of $\eta$. Using the condition $\pi_1(M, L) = \{0\}$, extend $\eta$ to a map $u : (D, \partial D) \to (M, L)$. Choosing a symplectic trivialization

$$\phi : (u^*TM, u^*\omega) \to (D \times \mathbb{R}^{2n}, \omega_0),$$

we obtain a path in $M(2n,n,\mathbb{R})$,

$$\sigma : [0,1] \to M(2n,n,\mathbb{R}), \quad t \mapsto \begin{pmatrix} \phi_{e^{2\pi it}}(e^{2\pi it}, \widetilde{\eta}_1(t)) & \cdots & \phi_{e^{2\pi it}}(e^{2\pi it}, \widetilde{\eta}_n(t)) \end{pmatrix}.$$

We need to show that $\det(\sigma(0))$ and $\det(\sigma(1))$ have the same sign.

By performing the Gram-Schmidt procedure on the columns of $\sigma$, we obtain a path in $U(n)$,

$$\widetilde{\gamma} : [0,1] \to U(n), \quad t \mapsto \begin{pmatrix} \mathrm{GS}(\sigma(t)) & J_0\mathrm{GS}(\sigma(t)) \end{pmatrix}.$$

For all $t \in [0,1]$, the Lagrangian subspace of $\mathbb{R}^{2n}$, $\phi_{e^{2\pi it}}(u^*TL)$ is simply $\mathrm{Range}(\mathrm{GS}(\sigma(t)))$. Therefore $\widetilde{\gamma}$ is a lift to $U(n)$ of the loop $\gamma$ in the Lagrangian Grassmannian determined by the map $u$ and the symplectic trivialization $\phi$.

Since $\mu([u])$ is even by assumption, $\det(\widetilde{\gamma}(0))$ and $\det(\widetilde{\gamma}(1))$ have the same sign (see Remark 2.5.7). Hence the bases of $\phi_1(u^*TL)$ given by the columns of $\mathrm{GS}(\sigma(0))$ and $\mathrm{GS}(\sigma(1))$ have the same orientation. It remains to notice that the Gram-Schmidt procedure preserves the orientation of a basis, and thus $\sigma(0)$ and $\sigma(1)$ have the same orientation. Lemma 4.3.5 then implies that $L$ is orientable. ∎

The preceding proposition will provide us with a corollary of the Künneth theorem in Section 4.5. Namely, it will imply that two monotone orientable Lagrangians with the same monotonicity constants satisfy the hypotheses of the Künneth theorem.

As we see in the next corollary, the monotonicity of a Lagrangian submanifold is related to the monotonicity of the ambient symplectic manifold.

**Proposition 4.3.7.**

1. *If $L$ is a monotone Lagrangian submanifold of the symplectic manifold $(M,\omega)$, then $M$ is monotone.*

*2. If $L$ is a simply connected Lagrangian submanifold of the monotone symplectic manifold $(M, \omega)$, then $L$ is monotone.*

*3. Suppose $L$ is a monotone Lagrangian submanifold of the (necessarily) monotone symplectic manifold $(M, \omega)$ and $L$ and $M$ have monotonicity constants $\tau$ and $\nu$ respectively. Then if $\mu \not\equiv 0$ and $c_1 \not\equiv 0$, $2\tau = \nu$. In particular, $\tau$ does not depend on $L$, but only on $(M, \omega)$.*

*4. If $L$ is a monotone Lagrangian submanifold of the symplectic manifold $M$ and both $N_L, N_M < \infty$, then $N_L \leq 2N_M$ with equality holding when $L$ is simply connected.*

**Proof:**

1. Let $[w] \in \pi_2(M, p_0)$ where $w : (S^2, x_0) \to (M, p_0)$ and suppose $L$ has monotonicity constant $\tau$. Then

$$\omega([w]) = \omega(j([w])) = \tau\mu(j([w])) = 2\tau c_1([w]),$$

where the last equality is an application of Proposition 4.2.12.

2. Let $[u] \in \pi_2(M, L)$ where $u : (D, \partial D, x_0) \to (M, p_0)$ and suppose $M$ has monotonicity constant $\nu$. Notice that if $L$ is simply connected, then the map $j : \pi_2(M, p_0) \to \pi_2(M, L, p_0)$, which appears in the long exact homotopy sequence of the pair $(M, L)$, is surjective. Therefore $[u] = j([w])$ for some $[w] \in \pi_2(M, p_0)$. We have

$$\begin{aligned} \omega([u]) &= \omega(j([w])) = \omega([w]) = \nu c_1([w]) \\ &= \frac{1}{2}\nu\mu(j[w]) = \frac{1}{2}\nu\mu([u]). \end{aligned}$$

3. This is clear from the preceding proofs.

4. Choose $[w_0] \in \pi_2(M, p_0)$ such that $N_M = c_1([w_0])$. Then by Proposition 4.2.12,

$$\mu(j([w_0])) = 2c_1([w_0]) = 2N_M,$$

and therefore $N_L \leq 2N_M$. Suppose $L$ is simply connected and $N_L = \mu([u_0])$ for $[u_0] \in \pi_2(M, L, p_0)$. We know that $[u_0] = j([w])$ for some $[w] \in \pi_2(M, p_0)$. We have

$$N_L = \mu([u_0]) = \mu(j([w])) = 2c_1([w]) \geq 2N_M,$$

and therefore $N_L = 2N_M$. ∎

**Remark 4.3.8.** As an example of a monotone Lagrangian submanifold $L$ such that the inequality in Part 4 of Corollary 4.3.7 is strict, take $L$ to be the equator on the surface of $(S^2, \omega)$. Then from part 3 of Example 4.3.3, $N_L = 2$, and it follows from Example 3.4.3 that $N_M = 2$ as well.

Finally we show that symmetric Lagrangians in monotone symplectic manifolds are monotone. By the previous proposition, we already know that simply connected symmetric Lagrangians in monotone symplectic manifolds are monotone. However, this also holds for symmetric Lagrangians which are not simply connected.

**Proposition 4.3.9.** *Let $(M, \omega)$ be a $2n$-dimensional monotone symplectic manifold with an antisymplectic involution $F \in \mathrm{Diff}(M)$. If the fixed point set of $F$, $\mathrm{Fix}(F)$, has an $n$-dimensional component, then this component is a monotone Lagrangian submanifold.*

**Proof:** Let $L$ denote the $n$-dimensional component of $\mathrm{Fix}(F)$ and assume $L$ is non-empty. By Proposition 4.1.9, $L$ is a Lagrangian submanifold. We show that $L$ is monotone.

Let $u : (D, \partial D) \to (M, L)$ be a smooth map of pairs. Note that

$$\mu([F \circ u]) = -\mu([u]),$$

for if we have a symplectic trivialization,

$$\phi : (u^*TM, u^*\omega) \to (D \times \mathbb{R}^{2n}, \omega_0),$$

using the fact that $F$ is antisymplectic, we obtain a symplectic trivialization

$$\begin{aligned} \overline{\phi} : ((F \circ u)^*TM, (F \circ u)^*\omega) &\to (D \times \mathbb{R}^{2n}, -\omega_0), \\ (z, v) &\mapsto \left(z, \phi_z(z, (dF)_{F\circ u(z)}(v))\right), \end{aligned}$$

where $z \in D$ and $v \in T_{F\circ u(z)}M$. Since $u$ maps the boundary of $D$ into $L$, and $L$ is fixed by $F$, $\phi$ and $\overline{\phi}$ determine the same loop of Lagrangian subspaces in $\mathbb{R}^{2n}$. However for $\overline{\phi}$, $\mathbb{R}^{2n}$ has the bilinear symplectic form $-\Omega_0$. Hence $\mu([u]) = -\mu([F \circ u])$.

Define

$$w : S^2 \to M, \quad w(p) = \begin{cases} u \circ \mathrm{pr}_+(p), & p \in D_+, \\ F \circ u \circ \mathrm{pr}_-(p), & p \in D_-. \end{cases}$$

We have

$$\begin{aligned} \omega([w]) &= \int_{D_+} (u \circ \mathrm{pr}_+)^*\omega + \int_{D_-} (F \circ u \circ \mathrm{pr}_-)^*\omega, \\ &= \int_D u^*\omega - \int_D (F \circ u)^*\omega, \\ &= \int_D u^*\omega - \int_D u^*F^*\omega, \\ &= 2\int_D u^*\omega = 2\omega([u]). \end{aligned}$$

The minus sign in the second line arises since the map $\mathrm{pr}_-$ reverses orientation. If $M$ has monotonicity constant $\nu$, then applying Proposition 4.2.10,

$$\begin{aligned} \frac{\nu}{2}\mu([u]) &= \frac{\nu}{2}\left[\frac{1}{2}(\mu([u]) - \mu([F \circ u]))\right], \\ &= \frac{\nu}{2}c_1([w]) = \frac{1}{2}\omega([w]), \\ &= \omega([u]). \end{aligned}$$

Therefore $L$ is monotone with monotonicity constant $\tau = \frac{\nu}{2}$. ∎

**Example 4.3.10.** In Example 3.4.17 we showed that $\mathbb{CP}^n$ with the Fubini-Study form is monotone with monotonicity constant $\frac{\pi}{n+1}$. Since $\mathbb{RP}^n$ is a symmetric Lagrangian in $\mathbb{CP}^n$ (see Example 4.1.10), Proposition 4.3.9 implies that $\mathbb{RP}^n$ is monotone with monotonicity constant $\frac{\pi}{2(n+1)}$.

## 4.4 The pearl complex

We are now ready to define the pearl complex. The pearl complex for a Lagrangian submanifold is similar to its Morse complex, but the coefficient ring for the pearl complex is a graded polynomial ring rather than $\mathbb{Z}_2$, and instead of counting Morse trajectories to define the differential, we count 'pearly trajectories.' These are Morse trajectories which can be interrupted by pseudoholomorphic discs with boundary on the Lagrangian. We begin by defining these trajectories precisely.

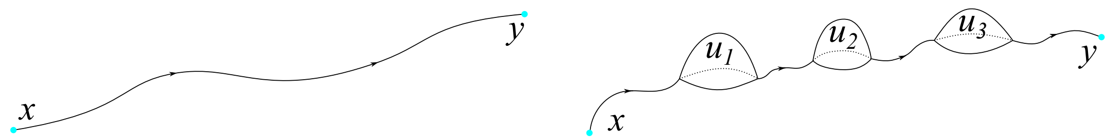
**Definition 4.4.1.** *Let $(M, \omega)$ be a symplectic manifold and let $L \subset M$ be a closed Lagrangian submanifold (i.e. $L$ is compact and has empty boundary). Let $(f, \rho, J)$ be a triple consisting of a Morse function $f \in C^\infty(L)$, a Riemannian metric $\rho$ on $L$, and an almost complex structure $J$ on $M$. Let $x, y \in L$ and $A \in H_2^D(M, L)$.*

- *If $A \neq 0$, a **parametrized pearly trajectory** from $x$ to $y$ of class $A$ is a sequence $(u_i)_{i=1}^l$ of smooth maps*

  $$u_i : (D, \partial D) \to (M, L)$$

  *satisfying*

  1. *$u_i : (D, \partial D) \to (M, L)$ is a non-constant $J$-holomorphic disc.*
  2. *There exists $t' \in [-\infty, 0)$ such that $\Phi_{t'}(u_1(-1)) = x$.*
  3. *For every $1 \leq i \leq l-1$ there exists $t_i \in (0, \infty)$ such that $\Phi_{t_i}(u_i(1)) = u_{i+1}(-1)$.*
  4. *There exists $t'' \in (0, \infty]$ such that $\Phi_{t''}(u_l(1)) = y$.*




**(a)** $A = 0$. **(b)** $A \neq 0$.

**Figure 4.1:** Two pearly trajectories from $x$ to $y$ of class $A$.

5. $h([u_1]) + \cdots + h([u_l]) = A$.

- *If $A = 0$, a* ***parametrized pearly trajectory*** *from $x$ to $y$ of class $A$ is simply a parametrized Morse trajectory from $x$ to $y$.*

*We denote the set of all parametrized pearly trajectories from $x$ to $y$ of class $A$ by $P_{\mathrm{prl}}(x, y; A; f, \rho, J)$. We define an equivalence relation on $P_{\mathrm{prl}}(x, y; A; f, \rho, J)$ as follows:*

- *If $A \neq 0$, two sequences $(u_i)_{i=1}^{l}$ and $(u'_j)_{j=1}^{m}$ are equivalent if $l = m$ and for all $i \in \{1, \ldots, l\}$, there exists a biholomorphic map $\sigma_i \in \mathrm{Aut}(D)$ such that $\sigma_i(1) = 1$, $\sigma_i(-1) = -1$ and $u'_i = u_i \circ \sigma_i$.*

- *If $A = 0$, two pearly trajectories from $x$ to $y$ are equivalent if one is a reparametrization of the other as Morse trajectories.*

*An equivalence class of this relation is called an* ***unparametrized pearly trajectory*** *from $x$ to $y$. We denote the set of unparametrized pearly trajectories from $x$ to $y$ of class $A$ by $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$, or if the triple $(f, \rho, J)$ is clear from the context, simply by $\mathcal{P}_{\mathrm{prl}}(x, y; A)$.*

With some additional assumptions on $M$, $L$, $f$ and $\rho$, for a generic choice of $J$, as defined in the next theorem, the set $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ has the structure of a smooth manifold. Moreover, the dimension of $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ is determined by the Morse indices of $x$ and $y$ as well as the Maslov index of $A$.

**Theorem 4.4.2.** *[BC09a, p. 15] Assume $M$ is connected and tame, that $L \subset M$ is closed, connected and monotone with $N_L \geq 2$, and that $(f, \rho)$ is Morse-Smale. Let $\mathcal{J}$ be the space of $\omega$-compatible almost complex structures $J$ on $M$ for which $(M, g_J)$ is geometrically bounded (i.e. $J$ and $g_J$ satisfy the conditions of Definition 3.3.12). For any points $x, y \in \mathrm{Crit}(f)$ and $A \in H_2^D(M, L)$, define the **virtual dimension** of the space of unparametrized pearly trajectories from $x$ to $y$ of class $A$ by*

$$\delta_{\mathrm{prl}}(x, y; A) = |x| - |y| + \mu(A) - 1.$$

*Then there exists a second category subset $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}$ such that for all $J \in \mathcal{J}_{\mathrm{reg}}$, the following statements hold for all $x, y \in \mathrm{Crit}(f)$ and $A \in H_2^D(M, L)$:*

1. *If $\delta_{\mathrm{prl}}(x, y; A) < -1$, the space $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ is empty.*

2. *If $\delta_{\mathrm{prl}}(x, y; A) = -1$, the space $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ is empty unless $x = y$ and $A = 0$. In the latter case, the space $\mathcal{P}_{\mathrm{prl}}(x, x; 0; f, \rho)$ consists of one element, namely the constant unparametrized Morse trajectory at the point $x$.*

3. *If $\delta_{\mathrm{prl}}(x, y; A) = 0$, the space $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ is either empty or it is a* compact $0$-*dimensional manifold.*

4. *If $\delta_{\mathrm{prl}}(x, y; A) = 1$, $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ is either empty or it is a smooth* $1$-*dimensional manifold.*

*Elements of $\mathcal{J}_{\mathrm{reg}}$ are called **generic** almost complex structures. If the pair $(f, \rho)$ is Morse-Smale and $J$ is generic, we also say that the triple $(f, \rho, J)$ is generic for the pair $(M, L)$.*

Theorem 4.4.2 does not provide any information about the structure of $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ when $\delta_{\mathrm{prl}}(x, y; A) \geq 2$. This is because for $\delta_{\mathrm{prl}}(x, y; A) \leq 1$, one can assume that all $J$-holomorphic discs of the pearly trajectories in $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ are simple (see Definition 3.5.6) and absolutely distinct, meaning there is no disc whose image is contained in the union of the images of the others [BC07, p. 15]. The proof of Theorem 4.4.2 relies on this fact, which is not true in general when $\delta_{\mathrm{prl}}(x, y; A) \geq 2$.

**Remark 4.4.3.** Note that the condition $N_L \geq 2$ implies that the virtual dimension of the space of pearly trajectories from a point $x$ to itself, $\delta_{\mathrm{prl}}(x, x; A)$, cannot vanish for any $A \in H_2^D(M, L)$.

The third statement of the preceding theorem tells us that if $\delta_{\mathrm{prl}}(x, y; A) = 0$, $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ consists of a finite number of points. This fact together with Theorem 4.4.5 allows for the following definition.

**Definition 4.4.4.** *Let $(M, \omega)$, $L$, and $(f, \rho, J)$ be as in Theorem 4.4.2. Let $\Lambda$ be the Laurent polynomial ring $\mathbb{Z}_2[t, t^{-1}]$ with $\deg(t) = -N_L$. If $N_L = \infty$, we take $\Lambda = \mathbb{Z}_2$. The **pearl complex** for $L$ is the $\Lambda$-module*

$$\mathcal{C}(f, \rho, J) := \mathbb{Z}_2\langle \mathrm{Crit}(f) \rangle \otimes_{\mathbb{Z}_2} \Lambda,$$

*where the degree of a critical point $x \in \mathrm{Crit}(f)$ is its Morse index $|x|$. Denote by $\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ the number modulo 2 of elements in $\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)$ and, if $N_L \neq \infty$, let $\bar{\mu} = \frac{\mu}{N_L}$. If $N_L \neq \infty$, the differential for the pearl complex is defined by*

$$d(x) = \sum_{\substack{(y,A) \\ \delta_{\mathrm{prl}}(x,y;A)=0}} \left(\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)\right) y\, t^{\bar{\mu}(A)},$$

*for $x \in \mathrm{Crit}(f)$. The sum is taken over all $y \in \mathrm{Crit}(f)$ and $A \in H_2^D(M, L)$ satisfying $\delta_{\mathrm{prl}}(x, y; A) = 0$. If $N_L = \infty$, the differential is defined by*

$$d(x) = \sum_{\substack{(y,A) \\ \delta_{\mathrm{prl}}(x,y;A)=0}} \left(\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x, y; A; f, \rho, J)\right) y.$$

**Theorem 4.4.5.** *[BC07, p. 5] For $(M, \omega)$, $L \subset M$ and $(f, \rho, J)$ as in Theorem 4.4.2, the pearl complex is in fact a differential graded $\Lambda$-module. That is, $d \circ d = 0$.*

Although in the terminology of Section 1.2, the pearl complex is a differential graded module rather than a complex of modules, we conform to the terminology of Biran and Cornea in referring to it as the pearl 'complex.'

Note that in the case where $N_L = \infty$, the pearl complex for $L$ reduces to the Morse complex for $L$. In this case, $\mu(A) = 0$ for all $A \in H_2^D(M, L)$ and the virtual dimension of the space of pearly trajectories from the critical point $x$ to the critical point $y$ vanishes precisely when $|x| - |y| - 1 = 0$. By monotonicity, $N_L = \infty$ implies $\omega(A) = 0$ for all $A \in H_2^D(M, L)$. However, for a $J$-holomorphic curve $u$, $\omega(h([u]))$ is simply its energy. By the energy identity, the only $J$-holomorphic curves with vanishing energy are constant (see Remark 3.5.4). Therefore if $\delta_{\mathrm{prl}}(x, y; A) = 0$, we must have $A = 0$, and the space $\mathcal{P}_{\mathrm{prl}}(x, y; 0; f, \rho, J)$ is the space of unparametrized Morse trajectories from $x$ to $y$.

**Definition 4.4.6.** *The graded $\Lambda$-module $H(\mathcal{C}(f, \rho, J), d)$ is called the* ***quantum homology*** *of $L$ and is denoted $QH(L; f, \rho, J)$, or simply $QH(L)$ as the following theorem indicates that the quantum homology is independent of the choice of generic triple $(f, \rho, J)$.*

**Theorem 4.4.7.** *[BC07, p. 5] For two choices of generic triples $(f, \rho, J)$ and $(f', \rho', J')$ for the pair $(M, L)$, there is an isomorphism of graded $\Lambda$-modules*

$$QH(L; f, \rho, J) \xrightarrow{\cong} QH(L; f', \rho', J').$$

The quantum homology possesses additional structure which we do not describe here: There is a quantum product which endows $QH(L)$ with the structure of an associative algebra with unity, and $QH(L)$ is also a module over the quantum homology of the ambient symplectic manifold [BC07, BC09a].

Finally we mention that the main motivation for studying the pearl complex is that it provides an intrinsic method of computing the Floer homology of a monotone Lagrangian. We refer the reader to Floer's original articles [Flo88, Flo89a, Flo89b], as well as the article [Oh96] and the book by Audin and Damian [AD10] for the details of Floer homology.

**Theorem 4.4.8.** *[BC07, p. 5] Let $(M, L)$ be a pair consisting of a symplectic manifold $M$ and a Lagrangian $L$ satisfying the conditions of 4.4.2. Let $HF(L)$ denote the Lagrangian intersection Floer homology of $L$ with itself defined over $\Lambda$. There is an isomorphism*

$$HF(L) \cong QH(L),$$

*which is canonical up to a shift in grading.*

To construct the Floer complex of a monotone Lagrangian with itself, one perturbs the Lagrangian by a Hamiltonian diffeomorphism. The complex is generated as a $\Lambda$-module by the intersection points of the Lagrangian and the perturbed Lagrangian. The Floer homology does not depend on the choice of Hamiltonian perturbation. By contrast, the construction of the pearl complex does not rely on a perturbation, and partly for this reason, the quantum homology is often easier to compute than the Floer homology. However, in some situations there is an obvious Hamiltonian diffeomorphism that carries $L$ off of itself. In this case the Floer complex is trivial and the isomorphism of Theorem 4.4.8 implies that the quantum homology vanishes. Such Lagrangians are called **displaceable**.

**Example 4.4.9.** Consider a closed, connected, monotone Lagrangian $L$ in $(\mathbb{R}^{2n}, \omega_0)$. By part 1 of Example 3.3.14, $(\mathbb{R}^{2n}, g_0)$ is geometrically bounded. Define the following (time-independent) Hamiltonian function on $\mathbb{R}^{2n}$,

$$H^\lambda : \mathbb{R}^{2n} \to \mathbb{R}, \quad (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \lambda x_1,$$

for $\lambda \in \mathbb{R} \setminus \{0\}$. The Hamiltonian vector field $X_{H^\lambda}$ determined by $H^\lambda$ and $\omega_0$ is simply $-\lambda \frac{\partial}{\partial y_1}$, and the flow of $X_{H^\lambda}$ is

$$\Phi^\lambda : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad (t, (x_1, \ldots, x_n, y_1, \ldots, y_n)) \mapsto (x_1, \ldots, x_n, y_1 - \lambda t, y_2, \ldots, y_n).$$

Clearly, for any compact subset of $\mathbb{R}^{2n}$, there is a $\lambda \in \mathbb{R} \setminus \{0\}$ for which

$$\Phi_1^\lambda(L) \cap L = \emptyset.$$

Therefore $L$ is displaceable and the quantum homology of $L$ is trivial.

Next we consider a Lagrangian which is not displaceable.

**Example 4.4.10.** We compute the Lagrangian quantum homology of the circle in $(S^2, \omega_{\mathrm{vol}})$. Set

$$L = \{(x, y, z) \in S^2 \subset \mathbb{R}^2 |\ x = 0\}.$$

The circle $L$ divides $S^2$ into two components of equal area and is therefore monotone (see part 3 of Example 4.3.3). Moreover, $L$ is not displaceable by a Hamiltonian diffeomorphism: Every Hamiltonian diffeomorphism is symplectic and the symplectomorphisms of $S^2$ are simply the diffeomorphisms which preserve the area form. Clearly $L$ cannot be displaced by such a diffeomorphism. Therefore it is possible for $L$ to have non-trivial quantum homology. By part 3 of Example 4.3.3, $L$ has minimal Maslov number $N_L = 2$.

We take as Morse function

$$f : L \to \mathbb{R}, \quad (x, y, z) \mapsto z,$$

and as Riemannian metric $\rho$ the metric on $L$ induced by the standard metric on $S^2 \subset \mathbb{R}^3$. The Morse function $f$ has two critical points, one critical point $x_1$ of index 1 at the top of the circle and another $x_0$ of index 0 at the bottom of the circle. We have depicted the negative gradient of $f$ with respect to $\rho$ in Figure 4.2. Clearly the pair $(f, \rho)$ is Morse-Smale. We equip $S^2$ with the standard compatible complex structure $J$ induced by the isomorphism $S^2 \cong \mathbb{CP}^1$. The sphere $S^2$ is compact and therefore $(S^2, g_J)$ is geometrically bounded (see Remark 3.3.13). Moreover, $J$ is generic [Cho04].

The pearl complex for $L$ is

$$\mathbb{Z}_2\langle x_0, x_1\rangle \otimes_{\mathbb{Z}_2} \Lambda = \mathbb{Z}_2\langle x_0, x_1\rangle \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[t, t^{-1}],$$

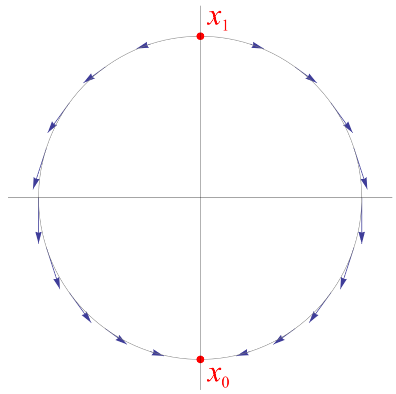



**Figure 4.2:** The vector field $-\nabla f$ on $L = S^1 \subset S^2$.

where $\deg(x_0) = 0$, $\deg(x_1) = 1$, and $\deg(t) = -N_L = -2$.

We compute the boundary operator, which is determined by its value on the critical points $x_0$ and $x_1$.

$d(x_1)$**:** The virtual dimension of the space of pearly trajectories from $x_1$ to $x_0$ for $A \in H_2^D(S^2, L)$ is

$$\delta_{\mathrm{prl}}(x_1, x_0; A) = |x_1| - |x_0| + \mu(A) - 1 = \mu(A).$$

By monotonicity, the only class $A \in H_2^D(M, L)$ with $\mu(A) = 0$ is the trivial class. By definition, the pearly trajectories in $\mathcal{P}_{\mathrm{prl}}(x_1, x_0; 0; f, \rho, J)$ are precisely the Morse trajectories from $x_1$ to $x_0$. There are two of these: one along the right side of the circle, and one along the left side of the circle.

By Remark 4.4.3, there is no $A \in H_2^D(S^2, L)$ satisfying $\delta_{\mathrm{prl}}(x_1, x_1; A) = 0$. Therefore we have

$$\begin{aligned} d(x_1) &= \sum_{\substack{(y,A) \\ \delta_{\mathrm{prl}}(x_1,y;A)=0}} (\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x_1, y; A; f, \rho, J)) y t^{\bar{\mu}(A)}, \\ &= \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x_1, x_0; 0) x_0, \\ &= 0. \end{aligned}$$

**$d(x_0)$:** The virtual dimension of the space of pearly trajectories from $x_0$ to $x_1$ of class $A$ is given by

$$\delta_{\mathrm{prl}}(x_0, x_1; A) = |p_0| - |p_1| + \mu(A) - 1 = \mu(A) - 2,$$

and vanishes when $\mu(A) = 2$. The image of the relative Hurewicz homomorphism for $(S^2, L)$ is

$$H_2^D(S^2, L) = H_2(S^2, L) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The positive generators are $h([u])$ and $h([v])$, where $u, v : (D, \partial D) \to (S^2, L)$ are the maps which carry the disc diffeomorphically onto the right and left hemispheres of $S^2$ respectively and which map the points $-1 \in \partial D$ to $x_0$ and $+1 \in \partial D$ to $x_1$. By part 3 of Example 4.3.3, these generators have Maslov index 2. The elements

$$[(u)] \in \mathcal{P}_{\mathrm{prl}}(x_0, x_1; h([u]); f, \rho, J) \text{ and } [(v)] \in \mathcal{P}_{\mathrm{prl}}(x_0, x_1; h([v]); f, \rho, J)$$

are the only distinct pearly trajectories in these spaces.

We have,

$$\begin{aligned} d(x_0) &= \sum_{\substack{(y,A) \\ \delta_{\mathrm{prl}}(x_0,y;A)=0}} (\#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x_0, y; A; f, \rho, J))yt^{\bar{\mu}(A)}, \\ &= \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x_0, x_1; h([u]))x_1 t + \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}(x_0, x_1; h([v]))x_1 t, \\ &= t + t = 0. \end{aligned}$$

Therefore the differential is trivial, and we obtain

$$QH(L) = \mathbb{Z}_2\langle x_0, x_1\rangle \otimes_{\mathbb{Z}_2} \Lambda \cong H(S^1, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda.$$

Monotone Lagrangians satisfying $QH(L) \cong H(L, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda$ are called **wide**. By contrast, monotone Lagrangians with trivial quantum homology are called **narrow**.

In Example 4.4.9, we saw that all closed monotone Lagrangians in $(\mathbb{R}^{2n}, \omega_0)$ are narrow. Interestingly, all known monotone Lagrangians are either wide or narrow, although it has not yet been proved in general that these are the only two possibilities [BC09b]. Other examples of wide Lagrangians include the Clifford torus and $\mathbb{RP}^n$ in $(\mathbb{CP}^n, \sigma_{\mathbb{CP}^n})$.

## 4.5 The quantum Künneth theorem

In the following we prove a Künneth theorem relating the quantum homology of a product space to the quantum homology of its factors under certain conditions. Since the differential in the pearl complex is defined by taking a sum over classes in the image of the relative Hurwicz homomorphism, we first relate the image of the relative Hurewicz homomorphism for a product pair to the image of the homomorphism for the factors.

**Remark 4.5.1.** Notice that the relative Hurewicz homomorphism is natural in the following sense: If $f : (X, A) \to (Y, B)$ is a map of pairs of topological spaces, then the following diagram commutes:

$$\begin{tikzcd}
\pi_n(X,A) \arrow[r, "f_*"] \arrow[d, "h"'] & \pi_n(Y,B) \arrow[d, "h"'] \\
H_n(X,A) \arrow[r, "f_*"] & H_n(Y,B)
\end{tikzcd}$$

In particular, if $(X_1, A_1)$ and $(X_2, A_2)$ are two pairs of spaces and

$$p_j : (X_1 \times X_2, A_1 \times A_2) \to (X_j, A_j), \quad j = 1, 2,$$

is the projection onto the first or second component, we have

$$h \circ p_{j_*} = p_{j_*} \circ h.$$

As ${p_j}_* : \pi_2(X_1 \times X_2, A_1 \times A_2) \to \pi_2(X_j, A_j)$ is surjective, this implies

$$ {p_j}_* \left( H_2^D(X_1 \times X_2, A_1 \times A_2) \right) = H_2^D(X_j, A_j). $$

In order to relate virtual dimensions for a product space to virtual dimensions for the factors, we also need a relationship between the Maslov index of a product Lagrangian submanifold and the Maslov index of its factors. We establish this relationship in the next proposition, as well as the relationship between the homomorphisms $\omega : H_2^D(M, L) \to \mathbb{R}$ for a product Lagrangian and its factors. This allows us to determine monotonicity for the product in terms of monotonicity for the factors.

**Proposition 4.5.2.** *Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be symplectic manifolds with Lagrangian submanifolds $L_1 \subset M_1$ and $L_2 \subset M_2$. Endow $M_1 \times M_2$ with the symplectic form $\omega_1 \oplus \omega_2$. Then for all $A \in H_2^D(M_1 \times M_2, L_1 \times L_2)$ the Maslov index $\mu$ for $M_1 \times M_2$ and the homomorphism $\omega_1 \oplus \omega_2$ satisfy the following additivity conditions:*

$$ \begin{aligned} \mu(A) &= \mu_1\left(p_{1*}(A)\right) + \mu_2(p_{2*}(A)), \\ \omega_1 \oplus \omega_2(A) &= \omega_1\left(p_{1*}(A)\right) + \omega_2(p_{2*}(A)). \end{aligned} $$

*Here we have used $\mu_1$ and $\mu_2$ to denote the Maslov indices for $L_1$ and $L_2$.*

**Proof:** Additivity for $\omega_1 \oplus \omega_2$ follows immediately from the definitions of the symplectic form $\omega_1 \oplus \omega_2$ and of the associated homomorphism

$$ \omega_1 \oplus \omega_2 : H_2^D(M_1 \times M_2, L_1 \times L_2) \to \mathbb{R}, \quad [c] \mapsto \int_c \omega_1 \oplus \omega_2. $$

We prove additivity for the Maslov index. For $A \in H_2^D(M_1 \times M_2, L_1 \times L_2)$, we may choose a smooth map

$$ u : (D, \partial D) \to (M_1 \times M_2, L_1 \times L_2), $$

such that $A = h([u])$. By Remark 4.5.1,

$$ {p_j}_*(A) = {p_j}_* h([u]) = h {p_j}_*([u]) = h([p_j \circ u]), $$

for $j = 1, 2$. Choose symplectic trivializations

$$\phi_j : {u_j}^* TM_j \to D \times \mathbb{R}^{2n_j},$$

where $\dim(M_j) = 2n_j$. These symplectic trivializations determine loops in the Lagrangian Grassmanians $\Lambda(n_j)$,

$$\gamma_j : S^1 \to \Lambda(n_j), \quad z \mapsto \phi_z\left((u|_{\partial D}^* TL_j)_z\right) \subset \mathbb{R}^{2n_j}.$$

Then by definition of the Maslov index, $\mu_j({p_j}_* A) = \mu(\gamma_j)$.

The trivializations $\phi_1$ and $\phi_2$ also determine a symplectic trivialization

$$\phi : u^* T(M_1 \times M_2) \to D \times \mathbb{R}^{2n_1+2n_2},$$

via the following composition:

$$\begin{array}{ccccc} u^*T(M_1 \times M_2) & \xrightarrow{\cong} & u_1^*TM_1 \oplus u_2^*TM_2 & \longrightarrow & D \times \mathbb{R}^{2n_1+2n_2} \\ (z, v) & \longmapsto & (z, ({p_1}_* v, {p_2}_* v)) & \longmapsto & (z, ((\phi_1)_z({p_1}_* v), (\phi_2)_z({p_2}_* v))) \end{array}$$

where $z \in D$ and $v \in T_{u(z)}(M_1 \times M_2)$.

The path in the Lagrangian Grassmannian $\Lambda(n_1 + n_2) \supseteq \Lambda(n_1) \oplus \Lambda(n_2)$ obtained using this symplectic trizialization is simply $\gamma = \gamma_1 \oplus \gamma_2$. Applying Property 3 of Theorem 2.5.8, we obtain $\mu(\gamma) = \mu(\gamma_1) + \mu(\gamma_2)$. This implies the Proposition. ∎

The following is an immediate corollary of Proposition 4.5.2.

**Corollary 4.5.3.** *1. If the minimal Maslov numbers $N_{L_1}$ and $N_{L_2}$ are both finite, the minimal Maslov number of the product $L_1 \times L_2$ satisfies*

$$N_{L_1 \times L_2} = \gcd(N_{L_1}, N_{L_2}).$$

*2. If $L_1$ and $L_2$ are monotone with monotonicity constants $\tau_1$ and $\tau_2$, then $L_1 \times L_2$ is monotone if and only if $\tau_1 = \tau_2$. In this case $L_1 \times L_2$ has monotonicity constant $\tau_1 = \tau_2$.*

The following lemma is the analogue for the pearl complex of a fact we have already seen for Morse theory. In Morse theory, given two compact Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ equipped with Morse functions $f_1$ and $f_2$, the product manifold $M_1 \times M_2$ can be given the Morse function $f_1 \oplus f_2$ and Riemannian metric $g_1 \oplus g_2$. Then the projection of the flow of the negative gradient of $f_1 \oplus f_2$ with respect to the metric $g_1 \oplus g_2$ onto either $M_1$ or $M_2$ is simply the flow of the negative gradient of the Morse function $f_1$ on $M_1$ or of $f_2$ on $M_2$ (see part 3 of Lemma 1.3.19). There is one technicality we must account for in the case of pearly trajectories that did not arise in the Morse case. Namely, the projection onto a factor of a non-constant holomorphic disc in the product space could be constant.

**Lemma 4.5.4.** *Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be symplectic manifolds and let $L_1 \subset M_1$ and $L_2 \subset M_2$ be Lagrangian submanifolds. Equip $M_1 \times M_2$ with the symplectic form $\omega_1 \oplus \omega_2$. Furthermore suppose $(f_1, \rho_1, J_1)$ and $(f_2, \rho_2, J_2)$ are (not necessarily generic) triples for $(M_1, L_1)$ and $(M_2, L_2)$. Consider the triple $(f_1 \oplus f_2, \rho_1 \oplus \rho_2, J_1 \oplus J_2)$ for $(M_1 \times M_2, L_1 \times L_2)$. Let*

$$(u_i)_{i=1}^l \in P_{prl}\left(r, z; A; f_1 \oplus f_2, \rho_1 \oplus \rho_2, J_1 \oplus J_2\right),$$

*be a parametrized pearly trajectory, where $A \in H_2^D(M_1 \times M_2, L_1 \times L_2)$, $A \neq 0$, and $r, z \in \mathrm{Crit}(f_1 \oplus f_2) = \mathrm{Crit}(f_1) \times \mathrm{Crit}(f_2)$. The following statements hold for $j = 1$ and $j = 2$:*

*(a) If ${p_j}_*(A) \neq 0$, then not all $p_j \circ u_i$ are constant. Furthermore, if $(u_{i_k})_{k=1}^m$ is the subsequence of $(u_i)_{i=1}^l$ consisting of those $u_i$ such that $p_j \circ u_i$ is non-constant, then*

$$(p_j \circ u_{i_k})_{k=1}^m \in P_{prl}(p_j(r), p_j(z); {p_j}_*(A); f_j, \rho_j, J_j).$$

*(b) If ${p_j}_*(A) = 0$, then all $p_j \circ u_i$ are constant and there is a trajectory of $-\nabla f_j$ from $p_j(r)$ to $p_j(z)$ passing (in order) through the points $p_j \circ u_1(D), \ldots, p_j \circ u_l(D)$.*

*In particular, for* $A \neq 0$*, if* $\mathcal{P}_{\mathrm{prl}}\left((p,q),(x,y);A;f_1 \oplus f_2,\rho_1 \oplus \rho_2, J_1 \oplus J_2\right)$ *is non-empty, both* $\mathcal{P}_{\mathrm{prl}}\left(p,x;p_{1*}(A);f_1,\rho_1,J_1\right)$ *and* $\mathcal{P}_{\mathrm{prl}}\left(q,y;p_{2*}(A);f_2,\rho_2,J_2\right)$ *are non-empty.*

**Proof:**

(a) By Remark 4.5.1,

$$p_{j_*}(A) = p_{j_*}\left(\sum_{i=1}^{l} h([u_i])\right) = \sum_{i=1}^{l} h \circ p_{j_*}[u_i] = \sum_{i=1}^{l} h([p_j \circ u_i]).$$

From this it is clear that if $p_j \circ u_i$ is constant for all $i \in \{1,\ldots,l\}$, then $p_{j_*}(A) = 0$. Moreover,

$$p_{j_*}(A) = \sum_{k=1}^{m} h([p_j \circ u_{i_k}]).$$

This verifies Property 5 of Definition 4.4.1 for the sequence $(p_j \circ u_{i_k})_{k=1}^m$. We verify properties 1 and 4 of Definition 4.4.1. Properties 2 and 3 can be verified in a manner similar to Property 4.

1. As the projection $p_j$ is $(J_j, J_1 \oplus J_2)$-holomorphic, $p_j \circ u_i$ is $J_j$-holomorphic. Then by choice of the subsequence $(u_{i_k})_{k=1}^m$, all $p_j \circ u_{i_k}$ are non-constant $J_j$-holomorphic discs.

4. Let $\Phi_1$ be the flow of $-\nabla f_1$ and $\Phi_2$ be the flow of $-\nabla f_2$. Recall that the flow of $-\nabla(f_1 \oplus f_2)$ is $\Phi_1 \times \Phi_2$ (see part 4 of Lemma 1.3.19). As $(u_i)_{i=1}^l$ is a parametrized pearly trajectory, for $i \in \{1,\ldots,l-1\}$, we may choose $t_i \in (0,\infty)$ such that $(\Phi_1 \times \Phi_2)_{t_i}(u_i(1)) = u_{i+1}(-1)$. Note that this implies $\Phi_j(p_j \circ u_i(1)) = p_j \circ u_{i+1}(-1)$. We also choose $t'' \in (0,\infty]$ such that $(\Phi_1 \times \Phi_2)_{t''}(u_l(1)) = z$. Then $(\Phi_j)_{t''}(p_j \circ u_l(1)) = p_j(z)$.

   Define

   $$\bar{t}'' = t'' + t_{l-1} + \cdots + t_{i_{m+1}} + t_{i_m}.$$

   We have,

   $$(\Phi_1)_{\bar{t}''}(p_j \circ u_{i_m}(1)) = (\Phi_1)_{t''+t_{l-1}+\cdots+t_{i_{m+1}}+t_{i_m}}(p_j \circ u_{i_m}(1)),$$

$$= (\Phi_1)_{t''} \circ (\Phi_1)_{t_{l-1}} \circ \cdots \circ (\Phi_1)_{t_{i_m+1}} \circ (\Phi_1)_{t_{i_m}} (p_j \circ u_{i_m}(1)),$$
$$= (\Phi_1)_{t''} \circ (\Phi_1)_{t_{l-1}} \circ \cdots \circ (\Phi_1)_{t_{i_m+1}} (p_j \circ u_{i_m+1}(-1)),$$

Since $p_j \circ u_i$ is constant for $i \in \{i_m+1, i_m+2, \ldots, l\}$, $p_j \circ u_{i_m+1}(1) = p_j \circ u_{i_m+1}(-1)$. Thus $(\Phi_1)_{t_{i_m+1}}(p_j \circ u_{i_m+1}(-1)) = p_j \circ u_{i_m+2}(-1)$. Repeating this step, we obtain

$$(\Phi_1)_{\bar{t}''}(p_j \circ u_{i_m}(1)) = (\Phi_1)_{t''}(p_j \circ u_l(1)) = p_j(z).$$

(b) If $p_{j*}(A) = 0$, Lemma 3.5.5 implies that $p_j \circ u_i$ is constant for all $i = 1, \ldots, l$. Let $t''$ and $t_1, \ldots, t_{l-1}$ be as above and choose $t' \in [-\infty, 0)$ such that $\Phi_{t'}(u_1(-1)) = r$. Then $\Phi_{-t'}(p_j(r)) = p_j \circ u_1(-1) = p_j \circ u_1(D)$ and an argument similar to that used to prove Property 4 in the first part of this proof shows that

$$(\Phi_j)_{-t' + \sum_{i=1}^{k} t_k}(p_j(r)) = p_j(u_{k+1}(D))$$

for all $k \in \{1, \ldots, l-1\}$. For $\bar{t} := -t' + \sum_{i=1}^{l-1} t_i + t''$, it follows that $(\Phi_j)_{\bar{t}}(p_j(r)) = p_j(z)$. ∎

**Remark 4.5.5.** In the preceding lemma, if $A = 0$ then a parametrized pearly trajectory from $r$ to $z$ is simply a parametrized Morse trajectory. We have already seen that the projection onto a factor of a Morse trajectory in a product space is a Morse trajectory in the factor. Therefore, for all $A \in H_2^D(M, L)$, if the set

$$\mathcal{P}_{\mathrm{prl}}\left((p,q),(x,y);A;f_1 \oplus f_2, \rho_1 \oplus \rho_2, J_1 \oplus J_2\right)$$

is non-empty, so are $\mathcal{P}_{\mathrm{prl}}\left(p,x;p_{1*}(A);f_1,\rho_1,J_1\right)$ and $\mathcal{P}_{\mathrm{prl}}\left(q,y;p_{2*}(A);f_2,\rho_2,J_2\right)$.

In the remainder of this section $(M_1, \omega_1)$ and $(M_2, \omega_2)$ will be connected, tame symplectic manifolds with closed monotone Lagrangian submanifolds $L_1 \subset M_1$ and $L_2 \subset M_2$. Furthermore, we will assume that both $N_{L_1}$ and $N_{L_2}$ are finite, that

$N := \gcd(N_{L_1}, N_{L_2}) \geq 2$ and that $\tau_1 = \tau_2$. Then, since the minimal Maslov number of $L_1 \times L_2$ is $N$ (Corollary 4.5.3), and the product of two tame symplectic manifolds is again tame (Lemma 3.3.15), the symplectic manifold $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ and Lagrangian submanifold $L_1 \times L_2 \subset M_1 \times M_2$ satisfy the conditions of Theorem 4.4.2. We will also assume that the triples $(f_1, \rho_1, J_1)$ and $(f_2, \rho_2, J_2)$ are generic for $(M_1, L_1)$ and $(M_2, L_2)$.

The next proposition is the analogue for the pearl complex of Lemma 1.3.21 for the Morse complex. As the two Lagrangian submanifolds whose product we wish to consider might have different minimal Maslov numbers, it is necessary to modify the pearl complexes for these Lagrangian submanifolds by choosing a common grading on their coefficient rings.

Note that by the proof of Lemma 3.3.15 $(M_1 \times M_2, g_{J_1 \oplus J_2})$ is geometrically bounded.

**Lemma 4.5.6.** *Assume the product triple $(f_1 \oplus f_2, \rho_1 \oplus \rho_2, J_1 \oplus J_2)$ is generic. Then we may construct the pearl complex for $L_1 \times L_2$ in $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ using this triple. Define $\Lambda := \mathbb{Z}_2[T, T^{-1}]$, $\Lambda_1 := \mathbb{Z}_2[t_1, t_1^{-1}]$, and $\Lambda_2 := \mathbb{Z}_2[t_2, t_2^{-1}]$ where*

$$\deg(T) = -N := -\gcd(N_1, N_2),\ \deg(t_1) = -N_{L_1},\ \textit{and}\ \deg(t_2) = -N_{L_2}.$$

*Also define the homomorphisms of degree zero*

$$F_j : \Lambda_j \to \Lambda, \quad t_j \mapsto T^{N_{L_j}/N}, \quad j = 1, 2.$$

*The homomorphisms $F_j$ induce homomorphisms of degree zero*

$$F_j : \mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda_j \to \mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda, \quad j = 1, 2.$$

*We will consider the complexes $\mathbb{Z}_2\langle \mathrm{Crit}(f_1)\rangle \otimes_{\mathbb{Z}_2} \Lambda$ and $\mathbb{Z}_2\langle \mathrm{Crit}(f_2)\rangle \otimes_{\mathbb{Z}_2} \Lambda$ with differentials $d'(p) = F_j \circ d(p)$ for $p \in \mathrm{Crit}(f_j)$, $j = 1, 2$. Define a map*

$$\times : (\mathbb{Z}_2\langle \mathrm{Crit}(f_1)\rangle \otimes_{\mathbb{Z}_2} \Lambda) \otimes_\Lambda (\mathbb{Z}_2\langle \mathrm{Crit}(f_2)\rangle \otimes_{\mathbb{Z}_2} \Lambda) \to \mathbb{Z}_2\langle \mathrm{Crit}(f_1 \oplus f_2)\rangle \otimes_{\mathbb{Z}_2} \Lambda$$

*by* $\times(p \otimes q) = (p, q)$ *for* $p \in \mathrm{Crit}(f_1)$ *and* $q \in \mathrm{Crit}(f_2)$*. We also write* $p \times q$ *for* $\times(p \otimes q)$*. Then the map* $\times$ *is a map of differential graded modules. That is,*

$$d(p \times q) = d'p \times q + p \times d'q.$$

**Proof:** By the definition of the differential in the pearl complex $\mathbb{Z}_2\langle \mathrm{Crit}(f_1 \oplus f_2)\rangle \otimes_{\mathbb{Z}_2} \Lambda$, we have

$$d(p \times q) = d(p, q) = \sum_{(x,y),\, A} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}\left((p, q), (x, y); A\right)(x, y)T^{\mu(A)/N},$$

where the sum is taken over all pairs $(x, y) \in \mathrm{Crit}(f_1 \oplus f_2) = \mathrm{Crit}(f_1) \times \mathrm{Crit}(f_2)$ and all elements $A$ of $H_2^D(M_1 \times M_2, L_1 \times L_2; \mathbb{Z})$ satisfying $\delta_{\mathrm{prl}}\left((p, q), (x, y); A\right) = 0$. Then by Proposition 4.5.2,

$$d(p \times q) = \sum_{(x,y),\, A} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}\left((p, q), (x, y); A\right)(x, y)T^{(\mu_1(p_{1*}(A))+\mu_2(p_{2*}(A)))/N}.$$

Now we observe that

$$\begin{aligned}
\delta_{\mathrm{prl}}\left((p, q), (x, y); A\right) &= |(p, q)| - |(x, y)| + \mu(A) - 1 \\
&= |p| + |q| - |x| - |y| + \mu_1\left(p_{1*}(A)\right) + \mu_2\left(p_{2*}(A)\right) - 1 \\
&= \delta_{\mathrm{prl}}(p, x; p_{1*}(A)) + \delta_{\mathrm{prl}}(q, y; p_{2*}(A)) + 1.
\end{aligned}$$

Therefore the condition $\delta_{\mathrm{prl}}((p, q), (x, y); A) = 0$ is equivalent to

$$\delta_{\mathrm{prl}}(p, x; p_{1*}(A)) + \delta_{\mathrm{prl}}(q, y; p_{2*}(A)) = -1.$$

Suppose $\mathcal{P}_{\mathrm{prl}}\left((p, q), (x, y); A\right)$ is non-empty. Then by Lemma 4.5.4 and Remark 4.5.5, this implies that both $\mathcal{P}_{\mathrm{prl}}\left(p, x; p_{1*}(A)\right)$ and $\mathcal{P}_{\mathrm{prl}}\left(q, y; p_{2*}(A)\right)$ are non-empty. By Theorem 4.4.2, $\mathcal{P}_{\mathrm{prl}}\left(p, x; p_{1*}(A)\right)$ is empty for $\delta_{\mathrm{prl}}(p, x; p_{1*}(A)) < -1$ and $\mathcal{P}_{\mathrm{prl}}\left(q, y; p_{2*}(A)\right)$ is empty for $\delta_{\mathrm{prl}}(q, y; p_{2*}(A)) < -1$. Hence, if $\mathcal{P}_{\mathrm{prl}}\left((p, q), (x, y); A\right)$ is non-empty and $\delta_{\mathrm{prl}}(p, x; p_{1*}(A)) + \delta_{\mathrm{prl}}(q, y; p_{2*}(A)) = -1$, then either $\delta_{\mathrm{prl}}(p, x; p_{1*}(A)) = -1$ or

$\delta_{\mathrm{prl}}(q, y; p_{2*}(A)) = -1$. Therefore we may break up the above sum as

$$d(p \times q) = \sum_{\substack{(x,y),\, A \\ \delta_{\mathrm{prl}}(p,x;p_{1*}(A))=0 \\ \delta_{\mathrm{prl}}(q,y;p_{2*}(A))=-1}} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}\left((p,q),(x,y);A\right)(x,y)T^{(\mu_1(p_{1*}(A))+\mu_2(p_{2*}(A)))/N}$$
$$+ \sum_{\substack{(x,y),\, A \\ \delta_{\mathrm{prl}}(p,x;p_{1*}(A))=-1 \\ \delta_{\mathrm{prl}}(q,y;p_{2*}(A))=0}} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}\left((p,q),(x,y);A\right)(x,y)T^{(\mu_1(p_{1*}(A))+\mu_2(p_{2*}(A)))/N}.$$

Now we recall that by Theorem 4.4.2, if $\mathcal{P}_{\mathrm{prl}}\left(p, x; p_{1*}(A)\right)$ is non-empty and $\delta_{\mathrm{prl}}(p, x; p_{1*}(A)) = -1$, then $p_{1*}(A) = 0$, $p = x$, and $\mathcal{P}_{\mathrm{prl}}\left(p, x; p_{1*}(A)\right)$ consists of one trajectory: the constant Morse trajectory at $p$. A similar statement holds for $\mathcal{P}_{\mathrm{prl}}\left(q, y; p_{2*}(A)\right)$. We conclude that the elements of $\mathcal{P}_{\mathrm{prl}}\left((p, q),(x, y); A\right)$ where $\delta_{\mathrm{prl}}(p, x; p_{1*}A) = -1$ are constant in $M_1$. More precisely, if

$$[(u_1, \ldots, u_l)] \in \mathcal{P}_{\mathrm{prl}}\left((p,q),(x,y);A\right),$$

then $p_1 \circ u_j = p = x$ for all $j = 1, \ldots, l$. Likewise, if $\delta_{\mathrm{prl}}(q, y; p_{2*}(A)) = -1$, all elements of $\mathcal{P}_{\mathrm{prl}}\left((p, q),(x, y); A\right)$ with $\delta_{\mathrm{prl}}(q, y; p_{2*}(A)) = -1$ are constant in $M_2$.

Finally we notice that elements of $\mathcal{P}_{\mathrm{prl}}\left((p, q),(x, y); A\right)$ which are constant in $M_2$ are in one-to-one correspondence with elements of $\mathcal{P}_{\mathrm{prl}}\left(p, x; p_{1*}(A)\right)$. Likewise, elements of $\mathcal{P}_{\mathrm{prl}}\left((p, q),(x, y); A\right)$ which are constant in $M_2$ correspond to those of $\mathcal{P}_{\mathrm{prl}}\left(q, y; p_{2*}(A)\right)$. We then have

$$d(p \times q) = \sum_{\substack{x,A \\ \delta_{\mathrm{prl}}(p,x;p_{1*}(A))=0}} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}\left(p,x;p_{1*}(A)\right)(x,q)T^{\mu_1(p_{1*}(A))/N}$$
$$+ \sum_{\substack{y,A \\ \delta_{\mathrm{prl}}(q,y;p_{2*}(A))=0}} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}\left(q,y;p_{2*}(A)\right)(p,y)T^{\mu_2(p_{2*}(A))/N},$$
$$= \left( \sum_{\substack{x,A \\ \delta_{\mathrm{prl}}(p,x;p_{1*}(A))=0}} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathrm{prl}}\left(p,x;p_{1*}(A)\right) x F_1(t_1)^{\mu_1(p_{1*}(A))/N_{L_1}} \right) \times q$$

$$+ p \times \left( \sum_{\substack{y,A \\ \delta_{\mathrm{prl}}(q,y;p_{2*}(A))=0}} \#_{\mathbb{Z}_2} \mathcal{P}_{\mathrm{prl}}\left(q, y; p_{2*}(A)\right) y F_2(t_2)^{\mu_2(p_{2*}(A))/N_{L_2}} \right),$$

$$= (F_1 \circ d(p)) \times q + p \times (F_2 \circ d(q)),$$

$$= d'p \times q + p \times d'q. \qquad \blacksquare$$

With the homomorphism of differential graded modules established in Lemma 4.5.6 and Corollary 1.2.18 of the Künneth theorem for differential graded modules in hand, we are now prepared to prove the quantum Künneth theorem.

**Theorem 4.5.7** (Künneth theorem for Lagrangian quantum homology)**.** *Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be connected, tame symplectic manifolds with closed monotone Lagrangian submanifolds $L_1 \subset M_1$ and $L_2 \subset M_2$. Suppose that both $N_{L_1}$ and $N_{L_2}$ are finite, that $N := \gcd(N_{L_1}, N_{L_2}) \geq 2$ and that $\tau_1 = \tau_2$. Furthermore, assume that the triples $(f_1, \rho_1, J_1)$ and $(f_2, \rho_2, J_2)$ are generic for $(M_1, L_1)$ and $(M_2, L_2)$. Let $\Lambda$, $\Lambda_1$, $\Lambda_2$ and $F_j : \Lambda_j \to \Lambda$, $j = 1, 2$ be as in Lemma 4.5.6. The ring $\Lambda$ becomes a graded $\Lambda_j$-module via the multiplication*

$$s \cdot r = F_j(s) r, \qquad \textit{for } s \in \Lambda_j,\ r \in \Lambda.$$

*Under these assumptions, we have the following relationship between the quantum homology of $L_1$, $L_2$ and $L_1 \times L_2$:*

$$(QH(L_1) \otimes_{\Lambda_1} \Lambda) \otimes_{\Lambda} (QH(L_2) \otimes_{\Lambda_2} \Lambda) \cong QH(L_1 \times L_2).$$

**Proof:** In Lemma 4.5.6 we saw that the map

$$\times : (\mathbb{Z}_2\langle \mathrm{Crit}(f_1) \rangle \otimes_{\mathbb{Z}_2} \Lambda) \otimes_{\Lambda} (\mathbb{Z}_2\langle \mathrm{Crit}(f_2) \rangle \otimes_{\mathbb{Z}_2} \Lambda) \to \mathbb{Z}_2\langle \mathrm{Crit}(f_1 \oplus f_2) \rangle \otimes_{\mathbb{Z}_2} \Lambda$$

is a homomorphism of differential graded $\Lambda$-modules. This map has an obvious inverse and thus it induces an isomorphism in homology

$$H\left((\mathbb{Z}_2\langle \mathrm{Crit}(f_1)\rangle \otimes_{\mathbb{Z}_2} \Lambda) \otimes_\Lambda (\mathbb{Z}_2\langle \mathrm{Crit}(f_2)\rangle \otimes_{\mathbb{Z}_2} \Lambda), d' \otimes \mathrm{Id} + \mathrm{Id} \otimes d'\right) \cong QH(L_1 \times L_2).$$

By Corollary 1.2.18 concerning the tensor product of differential graded modules over Laurent polynomial rings with field coefficients, the homology of

$$(\mathbb{Z}_2\langle \mathrm{Crit}(f_1)\rangle \otimes_{\mathbb{Z}_2} \Lambda) \otimes_\Lambda (\mathbb{Z}_2\langle \mathrm{Crit}(f_2)\rangle \otimes_{\mathbb{Z}_2} \Lambda, d' \otimes \mathrm{Id} + \mathrm{Id} \otimes d')$$

is isomorphic as a $\Lambda$-module to

$$\left(H\left(\mathbb{Z}_2\langle \mathrm{Crit}(f_1)\rangle \otimes_{\mathbb{Z}_2} \Lambda, d'\right)\right) \otimes_\Lambda \left(H\left(\mathbb{Z}_2\langle \mathrm{Crit}(f_2)\rangle \otimes_{\mathbb{Z}_2} \Lambda, d'\right)\right).$$

As $\Lambda$ is a $\Lambda_j$-module for $j = 1, 2$, we have $\Lambda_j \otimes_{\Lambda_j} \Lambda \cong \Lambda$, and thus we have an isomorphism of $\Lambda_j$-modules

$$\mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda_j \otimes_{\Lambda_j} \Lambda \to \mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda,\ ps \otimes r \mapsto pF_j(s)r$$

for $p \in \langle \mathrm{Crit}(f_j)\rangle$, $s \in \Lambda_j$, and $r \in \Lambda$. One checks easily that the above isomorphism is in fact an isomorphism of differential graded $\Lambda$-modules:

$$\left((\mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda_j) \otimes_{\Lambda_j} \Lambda, d \otimes \mathrm{Id}_\Lambda\right) \to \left(\mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda, d'\right).$$

This isomorphism of differential graded $\Lambda$-modules induces an isomorphism of $\Lambda$-modules in homology,

$$H\left((\mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda_j) \otimes_{\Lambda_j} \Lambda, d \otimes \mathrm{Id}_\Lambda\right) \cong H\left(\mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda, d'\right).$$

Finally we note that by the universal coefficient theorem for differential graded modules

$$\begin{aligned} H\left((\mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda_j) \otimes_{\Lambda_j} \Lambda, d \otimes \mathrm{Id}_\Lambda\right) &\cong H\left(\mathbb{Z}_2\langle \mathrm{Crit}(f_j)\rangle \otimes_{\mathbb{Z}_2} \Lambda_j, d\right) \otimes_{\Lambda_j} \Lambda \\ &= QH(L_j) \otimes_{\Lambda_j} \Lambda. \end{aligned}$$

This completes the proof. ∎

**Corollary 4.5.8.** *We have seen in Proposition 4.3.6 that oriented Lagrangians have even minimal Maslov number (when their minimal Maslov number is finite). Therefore the product of two oriented Lagrangians with finite minimal Maslov number has minimal Maslov number at least two. Hence the quantum Künneth theorem holds for the product of oriented Lagrangians satisfying the hypotheses of Lemma 4.5.6, with finiteness the only assumption necessary on their minimal Maslov numbers.*

## 4.6 Genericity

In our statement of the Künneth theorem, we have listed as a hypothesis that the product triple $(f_1 \oplus f_2, \rho_1 \oplus \rho_2, J_1 \oplus J_2)$ be generic. However, we believe this assumption is not necessary; if $(f_1, \rho_1, J_1)$ and $(f_2, \rho_2, J_2)$ are generic, then we suspect it is possible to show that their product will be generic as well. In this section, we examine the genericity condition more closely and outline some details of an approach that could be used to show that the product triple satisfies this condition.

Let $M$ be a connected, tame symplectic manifold and $L$ be a closed, connected, monotone Lagrangian with $N_L \geq 2$. As in Section 4.4, let $\mathcal{J}$ be the set of compatible almost complex structures $J$ on $M$ for which $(M, g_J)$ is geometrically bounded. Let $(f, \rho)$ be a Morse-Smale pair for $L$. In Section 4.4, for simplicity, we took the statement of Theorem 4.4.2 as the definition of the subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}$. However, in the proof of Theorem 4.4.2, the subset $\mathcal{J}_{\text{reg}}$ actually arises as a subset of the set of almost complex structures $J \in \mathcal{J}$ for which a transversality condition holds. The existence of this subset is guaranteed by an argument similar to that outlined at the end of Chapter 3. There we considered the space of simple $J$-holomorphic discs $u : (D, \partial D) \to (M, L)$ of a given class $A \in H_2^D(M, L)$.

The argument can be adapted to spaces of unparametrized pearly trajectories whose $J$-holomorphic discs are simple and absolutely distinct. These are simply sequences of simple, absolutely distinct $J$-holomorphic discs whose images are subject to the conditions detailed in Definition 4.4.1. These conditions can be formulated as restrictions on the image of an evaluation map, as we explain next.

Denote by $Q_{f,\rho}$ the image of the embedding

$$(L \setminus \mathrm{Crit}(f)) \times \mathbb{R}_{>0} \to L \times L, \quad (x,t) \mapsto (x, \Phi_t(x)),$$

where $\Phi$ is the flow of $-\nabla f$. Let $G_{-1,+1}$ be the group of biholomorphic maps of the unit disc $D$ which fix the points $-1$ and $+1$. For $J \in \mathcal{J}$ and $F \in H_2^D(M,L)$, let $\mathcal{M}(F,J)$ be the set of (not necessarily simple) $J$-holomorphic discs of class $F$ with boundary in $L$. There is an obvious action of $G_{-1,+1}$ on $\mathcal{M}(F,J)$ and hence we can consider the quotient $\mathcal{M}(F,J)/G_{-1,+1}$ by this action. For a sequence of non-trivial classes $\mathbf{A} = (A_1, \ldots, A_l)$ in $H_2^D(M,L)$, define the product

$$\mathcal{M}(\mathbf{A},J) := \mathcal{M}(A_1,J)/G_{-1,+1} \times \cdots \times \mathcal{M}(A_l,J)/G_{-1,+1}.$$

Consider the evaluation map

$$\mathrm{ev}_{\mathbf{A}} : \mathcal{M}(\mathbf{A},J) \to L^{\times 2l}, \quad ([u_1], \ldots, [u_l]) \mapsto (u_1(-1), u_1(+1), \ldots, u_l(-1), u_l(+1)).$$

The space of unparametrized pearly trajectories from $x$ to $y$ whose $J$-holomorphic discs represent the sequence of classes $\mathbf{A}$ is simply

$$\mathcal{P}_{\mathrm{prl}}(x,y;\mathbf{A};f,\rho,J) = \mathrm{ev}_{\mathbf{A}}^{-1}(W_x^u \times Q_{f,\rho} \times W_y^s).$$

The space of unparametrized pearly trajectories from $x$ to $y$ of non-trivial class $F \in H_2^D(M,L)$ as defined in Section 4.4 is then given by

$$\mathcal{P}_{\mathrm{prl}}(x,y;F;f,\rho,J) = \dot{\bigcup_{\mathbf{A}}} \mathcal{P}_{\mathrm{prl}}(x,y;\mathbf{A};f,\rho,J),$$

where the union is taken over all sequences $\mathbf{A} = (A_1, \cdots, A_l)$ of non-trivial classes in $H_2^D(M,L)$, of all lengths $l \geq 1$, with $\sum_{i=1}^{l} A_i = F$.

Let $\mathcal{M}^{*,d}(\mathbf{A},J)$ be the subset of $\mathcal{M}(\mathbf{A},J)$ consisting of sequences $([u_1], \ldots, [u_l])$ where the $u_i$, $i = 1, \ldots, l$, are simple and absolutely distinct. By a line of reasoning similar to that outlined at the end of Chapter 3, the following theorem holds. We refer the reader to [MS12] for more details.

**Theorem 4.6.1.** *[BC07, p. 15] For every Morse-Smale pair $(f, \rho)$ on $L$, there exists a second category subset $\mathcal{J}'_{\text{reg}} \subset \mathcal{J}$ such that for every sequence $\mathbf{A}$ of length $l \geq 1$ consisting of non-trivial classes in $H_2^D(M, L)$, and every pair of critical points $x, y \in \text{Crit}(f)$, the image of the map*

$$\text{ev}_{\mathbf{A}}|_{\mathcal{M}^{*,d}(\mathbf{A},J)} \to L^{\times 2l}$$

*intersects $W_x^u \times Q_{f,\rho} \times W_y^s$ transversally. In particular, this implies that the set*

$$\mathcal{P}^{*,d}_{\text{prl}}(x, y; \mathbf{A}; f, \rho, J) := \mathcal{P}_{\text{prl}}(x, y; \mathbf{A}; f, \rho, J) \cap \mathcal{M}^{*,d}(\mathbf{A}, J)$$

*is either empty or a smooth manifold of dimension $\sum_{i=1}^{l} \mu(A_i) + |x| - |y| - 1$.*

The proof of Theorem 4.4.2 in Section 4.4 about the structure of the spaces of pearly trajectories of virtual dimension no greater than one relies on Theorem 4.6.1. One shows that when

$$\sum_{i=1}^{l} \mu(A_i) + |x| - |y| - 1 \leq 1,$$

there is a second category subset of $\mathcal{J}''_{\text{reg}} \subset \mathcal{J}$ such that for all $J \in \mathcal{J}''_{\text{reg}}$, all discs appearing in pearly trajectories from $x$ to $y$ of class $A$ are simple and absolutely distinct [BC07], i.e.

$$\mathcal{P}^{*,d}_{\text{prl}}(x, y; \mathbf{A}; f, \rho, J) = \mathcal{P}_{\text{prl}}(x, y; \mathbf{A}; f, \rho, J).$$

The second category subset $\mathcal{J}_{\text{reg}}$ of Theorem 4.4.2 can then be taken to be the intersection of $\mathcal{J}''_{\text{reg}}$ and the subset $\mathcal{J}'_{\text{reg}}$ of Theorem 4.6.1. As the intersection of two second category subsets, $\mathcal{J}_{\text{reg}}$ is second category.

Using this more direct definition of the set $\mathcal{J}_{\text{reg}}$, we consider the steps necessary to show that a product of generic triples is again generic. Let $(M_1, \omega_1)$ and

$(M_2, \omega_2)$ be symplectic manifolds with Lagrangian submanifolds $L_1$ and $L_2$ satisfying the conditions of the quantum Künneth theorem (Theorem 4.5.7), i.e. $M_1$ and $M_2$ are connected and tame, $L_1$ and $L_2$ are closed, connected and monotone, and

$$\tau_1 = \tau_2, \quad \gcd(N_{L_1}, N_{L_2}) \geq 2,$$

where $\tau_1$ and $\tau_2$ are the monotonicity constants of $L_1$ and $L_2$.

Let $(f_1, \rho_1, J_1)$ and $(f_2, \rho_2, J_2)$ be generic triples for the pairs $(M_1, L_1)$ and $(M_2, L_2)$. We would like to check genericity of $(f_1 \oplus f_2, \rho_1 \oplus \rho_2, J_1 \oplus J_2)$ for the symplectic manifold $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ with Lagrangian $L_1 \times L_2$. Let $\mathcal{J}_{M_i}$, $i = 1, 2$, be the set of $\omega_i$-compatible almost complex structures $J$ on $M_i$ for which $(M_i, g_{J_i})$ is geometrically bounded. Similarly, let $\mathcal{J}_{M_1 \times M_2}$ be the set of $\omega_1 \oplus \omega_2$-compatible almost complex structures $J$ for which $(M_1 \times M_2, g_J)$ is geometrically bounded.

As we already noted in the previous section $(f_1 \oplus f_2, \rho_1 \oplus \rho_2)$ is Morse-Smale. Moreover, $J_1 \oplus J_2 \in \mathcal{J}_{M_1 \times M_2}$. To show genericity for the product triple, it remains to show that for the pair $(f_1 \oplus f_2, \rho_1 \oplus \rho_2)$, $J_1 \oplus J_2$ belongs to the set

$$(\mathcal{J}_{M_1 \times M_2})_{\text{reg}} := (\mathcal{J}_{M_1 \times M_2})'_{\text{reg}} \cap (\mathcal{J}_{M_1 \times M_2})''_{\text{reg}}.$$

It is not hard to show that $J_1 \oplus J_2$ is contained in $(\mathcal{J}_{M_1 \times M_2})'_{\text{reg}}$ under the following assumptions:

**Assumptions.** For a simple $J$-holomorphic curve $u : (D, \partial D) \to (M_1 \times M_2, L_1 \times L_2)$, if the projections $p_i \circ u$, $i = 1, 2$, are non-constant, then they are simple. For a parametrized pearly trajectory $(u_1, \ldots, u_l)$ consisting of absolutely distinct $J$-holomorphic discs in $M_1 \times M_2$, the discs $p_i \circ u_1, \ldots, p_i \circ u_l$ are absolutely distinct for $i = 1, 2$.

We outline the argument using these assumptions here. Let $\mathbf{A} = (A_1, \ldots, A_l)$ be a sequence of non-trivial classes in $H_2^D(M_1 \times M_2, L_1 \times L_2)$ and let $(p, q)$ and $(x, y)$ be critical points of $(f_1 \oplus f_2)$. Suppose

$$(x_1, \ldots, x_{2l}) \in \mathrm{Im}(\mathrm{ev}_{\mathbf{A}}|_{\mathcal{M}^{*,d}(\mathbf{A}, J_1 \oplus J_2)}) \cap (W^u_{(p,q)} \times Q_{f_1 \oplus f_2, \rho_1 \oplus \rho_2} \times W^s_{(x,y)}).$$

Choose $([u_1], \ldots, [u_l]) \in \mathcal{M}^{*,d}(\mathbf{A}, J_1 \oplus J_2)$ such that

$$\mathrm{ev}_{\mathbf{A}}([u_1], \ldots, [u_l]) = (x_1, \ldots, x_{2l}).$$

Assume the sequences $(p_i)_*\mathbf{A} := ((p_i)_*A_1, \ldots, (p_i)_*A_l)$ consist of non-trivial classes for $i = 1, 2$. As in the previous section, the case where some of these projections are trivial must be dealt with separately. We do not treat this case here.

By assumption, the projections $p_i \circ u_1, \ldots p_i \circ u_l$ are simple and absolutely distinct for $i = 1, 2$. As $J_1 \in (\mathcal{J}_{M_1})_{\mathrm{reg}}$, the image of $\mathrm{ev}_{(p_1)_*\mathbf{A}}$ is transverse to $W^u_p \times Q_{f_1,\rho_1} \times W^s_x$. Similarly, the image of $\mathrm{ev}_{(p_2)_*\mathbf{A}}$ is transverse to $W^u_q \times Q_{f_2,\rho_2} \times W^s_y$. There is an obvious isomorphism

$$W^u_{(p,q)} \times Q_{f_1 \oplus f_2, \rho_1 \oplus \rho_2} \times W^s_{(x,y)} \cong W^u_p \times Q_{f_1,\rho_1} \times W^s_x \times W^u_q \times Q_{f_2,\rho_2} \times W^s_y,$$

given by reordering factors. Since $\mathrm{ev}_{\mathbf{A}} = (\mathrm{ev}_{(p_1)_*\mathbf{A}}, \mathrm{ev}_{(p_2)_*\mathbf{A}})$ under this isomorphism, transversality must hold for $\mathrm{ev}_{\mathbf{A}}$ if it holds for $\mathrm{ev}_{(p_1)_*\mathbf{A}}$ and $\mathrm{ev}_{(p_2)_*\mathbf{A}}$. Hence $J_1 \oplus J_2 \in (\mathcal{J}_{M_1 \times M_2})'_{\mathrm{reg}}$.

We have seen the explicit definition of $(\mathcal{J}_{M_1 \times M_2})'_{\mathrm{reg}}$ in terms of the transversality condition, but have not given the explicit definition of $(\mathcal{J}_{M_1 \times M_2})''_{\mathrm{reg}}$ as established by Biran and Cornea in [BC07]. We propose as a direction of future work to show, using the explicit definition of $(\mathcal{J}_{M_1 \times M_2})''_{\mathrm{reg}}$ contained in [BC07], that $J_1 \oplus J_2$ also belongs to the second category subset $(\mathcal{J}_{M_1 \times M_2})''_{\mathrm{reg}}$. Together with the elimination of the assumption stated above about the simplicity of the projections of simple $J$-holomorphic curves in a product space, this would allow for the elimination of the hypothesis about the genericity of the product triple in our statement of the quantum Künneth theorem. Without this hypothesis, our Künneth theorem would be significantly stronger and more easily applicable.

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