# AN ALGEBRA OF PSEUDO-DIFFERENCE OPERATORS

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by

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CONTENTS

## Abstract

The purpose of the present work is to generalize Lax-Nirenberg's theorem and stability criteria for a wide class of pseudo-difference operators whose algebra is based on Calderón and Vaillancourt's boundedness theorem for pseudo-differential operators. In chapter 1, pseudo-difference operators with symbols of Hörmander's class $S^0_{\delta,\delta}$ , $0 \leq \delta < 1$ , are shown to be bounded in $L^2$ by using a method of discrete partition of unity. An example which shows that operators in $S^0_{1,1}$ may not be bounded is also given. In chapter 2, an algebra $\mathbb{S}_\delta$ of pseudo-difference operators with symbols in $S^0_{\delta,\delta}$ is developed. In chapter 3, the Lax-Nirenberg theorem is established for operators in $\mathbb{S}_\delta$ . The two stability criteria are also derived for those operators. Some open problems on the application of operators in $\mathbb{S}_\delta$ are mentioned.

## INTRODUCTION

In 1966 Lax and Nirenberg have established two new stability criteria for difference schemes with variable coefficients. To state these results, we summarize the notations and calculus used in [12] .

Let $P_h$ be a family of difference operators depending on a positive parameter $h$

$$(0.1) \qquad P_h = \sum_{\alpha} p_{\alpha}(x) T_h^{\alpha} \quad ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of integers $\alpha_j$ , and $T_h^{\alpha}$ is the translation operator

$$(T_h^{\alpha} u)(x) = u(x + h\alpha) \quad .$$

The coefficients $p_{\alpha}(x)$ are $m \times m$ matrix functions of $x$ . We denote the scalar product and norm in $C^m$ by $u \cdot v = \sum u_j \bar{v}_j$ and $|u|$, respectively. When no confusion arises, we shall often write $u\bar{v}$ for $u \cdot v$ . The $\ell_2$-absolute value $|p_{\alpha}(x)|$ of the matrix $p_{\alpha}(x)$ is defined as its operator norm in $C^m$ ,

$$|p_{\alpha}(x)| = \sup_{|u|=1} |p_{\alpha}(x) u| \quad .$$

The conjugate transpose of a matrix $p_{\alpha}$ is denoted by $p_{\alpha}^*$ .

The natural domain for difference operators is the space of functions defined on lattice points. Since the boundedness and positivity of difference operators over lattice functions of class $\ell_2$ are equivalent to the same properties for these operators when defined over functions of class $L^2(R^n)$, we let the difference operators $P_h$ act on functions $u(x)$ which are defined for all $x$ in $R^n$ and whose values lie in $C^m$. These functions form a Hilbert space under the norm

$$||u||^2 = \int |u(x)|^2 dx \quad .$$

The corresponding scalar product is denoted by

$$(u,v) = \int u(x)\overline{v(x)}dx \quad .$$

We denote by $||P_h||$ the operator norm,

$$||P_h|| = \sup_u \frac{||P_h u||}{||u||} \quad ,$$

by $P_h^*$ the Hilbert-space adjoint of $P_h$, and by $\operatorname{Re} P_h$ its hermitian part,

$$\operatorname{Re} P_h = \tfrac{1}{2}(P_h + P_h^*) \quad .$$

With the difference operator $P_h = \sum p_\alpha(x)T_h^\alpha$, we associate the symbol

$$p(x,\xi) = \sum_\alpha p_\alpha(x)e^{i\alpha\cdot\xi} \quad ,$$

where

$$\alpha\cdot\xi = \alpha_1\xi_1 + \ldots + \alpha_n\xi_n \quad .$$

In order to express some differentiability conditions on the symbol $p(x,\xi)$ conveniently, we shall assign the $(j,k)$-norms $|p|_{j,k}$ to $p$. We denote by $|p_\alpha|_j$ the j-maximum norm of $p_\alpha(x)$,

$$|p_\alpha|_j = \sup_{0\le|\beta|\le j} \{\sup_x |D_x^\beta\, p_\alpha(x)|\} \quad .$$

Here $|\beta|$ is the length of the multi-index $\beta$ of nonnegative integers,

$$|\beta| = \beta_1 + \ldots + \beta_n \quad .$$

Now we define the $(j,k)$ norm of $p$ as

$$|p|_{j,k} = \sum_\alpha |p_\alpha|_j \langle\alpha\rangle^k \quad ,$$

where $j$ and $k$ are nonnegative integers and $\langle\alpha\rangle$ is given by

$$\langle\alpha\rangle^2 = 1 + \alpha_1^2 + \ldots + \alpha_n^2 \quad .$$

We denote by $C_{j,k}$ the class of symbols $p(x,\xi)$ with finite (j,k)-norm. Clearly the norm of the operator $P_h = \sum p_\alpha(x)T_h^\alpha$ with symbol p satisfies the inequality

$$\| P_h \| \le |p|_{0,0} \quad .$$

The class of bounded difference operators with symbol p belonging to $C_{0,0}$ forms an algebra under either the operator product $P_h Q_h$ or the symbol product $P_h o Q_h$ , the latter being the operator with symbol pq . The following estimates hold

$$\| P_h Q_h - P_h o Q_h \| \le h |p|_{0,1} |q|_{1,0} \quad ,$$

$$\| P_h^* - P_h^\# \| \le h |p|_{1,1} \quad .$$

Here $P_h^\#$ is the difference operator with symbol $p^*(x,\xi)$ .

The basic result of Lax and Nirenberg is stated in the following theorem.

<u>Lax-Nirenberg Theorem</u>. Let $P_h$ be a one-parameter family of difference operators of the form (0.1) whose symbol $p(x,\xi)$ belongs to the class $C_{0,2}$ and $C_{2,0}$ . Suppose further that the symbol $p(x,\xi)$ is a hermitian and nonnegative matrix for every x and $\xi$, then the operator $P_h$ satisfies the inequality

$$\text{Re } P_h \ge -Kh$$

for all h and some constant K independent of h .

As a consequence of this theorem, two new stability criteria for difference schemes have been established. In short, the first of these criteria states that if the norm of the symbol is bounded by 1, $|p| \le 1$, then the operator $P_h$ is stable. The second says that if the numerical range of the symbol is contained in the unit disk of the complex plane, $|u \cdot pu| \le 1$ for $|u| = 1$, then the operator $P_h$ is stable.

In 1969 Vaillancourt [13] has given a simple proof of Lax-Nirenberg's theorem for difference operators and has extended these criteria to a class of pseudo-difference schemes for which Friedrich's calculus for pseudo-differential operators, as developped in [6] holds, thus including in [14] the results of Yamaguti and Nogi [17].

The purpose of the present work is to establish these results for a wide class of pseudo-difference operators whose algebra is derived from Calderón-Vaillancourt's boundedness theorem for pseudo-differential operators [2].

We shall conclude by a few remarks on the application of this class of pseudo-difference operators to some open problems.

# CHAPTER I

## A Class of $L^2$-Continuous Pseudo-Difference Operators.

In this chapter we shall define the class of pseudo-difference operators which will be used subsequently and we shall give a detailed proof of their continuity in $L^2$.

Pseudo-difference operators are a generalization of difference operators. To see this we shall use Fourier transformation to represent a difference operator. Let $\hat{u}(\xi)$ be the Fourier transform of $u(x)$ in $C_o^\infty(R^n)$ ,

$$\hat{u}(\xi) \equiv \int e^{-ix\cdot\xi} u(x)dx,$$

and $u(x)$ be the inverse Fourier transform of $\hat{u}(\xi)$,

$$u(x) \equiv \int e^{ix\cdot\xi} \hat{u}(\xi) đ\xi,$$

where the constant $(2\pi)^{-n}$ is absorbed in $đ\xi$,

$$đ\xi \equiv (2\pi)^{-n} d\xi \equiv (2\pi)^{-n} d\xi_1 \dots d\xi_n ,$$

and both integrations are over the whole $R^n$ space. Then the difference operator $P_h$ can be expressed as:

$$P_h u(x) = \sum p_\alpha(x)\, T_h^\alpha u(x)$$

$$= \sum p_\alpha(x) \int e^{i(x+h\alpha)\cdot\xi}\, \hat{u}(\xi) đ\xi$$

$$= \int e^{ix\cdot\xi} \sum p_\alpha(x) e^{ih\alpha\cdot\xi} \hat{u}(\xi) đ\xi$$

(1.1) $$= \int e^{ix\cdot\xi} p(x,h\xi)\hat{u}(\xi)đ\xi$$

where the matrix $p(x,\xi) = \sum p_\alpha(x)e^{i\alpha\cdot\xi}$ is called the symbol of $P_h$ .

Formula (1.1) will be used to define pseudo-difference operators $P_h$ whose symbols $p(x,\xi)$ are not necessarily of the exponential form $\sum p_\alpha(x)e^{i\alpha\cdot\xi}$ .

We can even consider difference operators

$$P_h u(x) = \sum_\alpha p_\alpha(x, x + h\alpha) T_h^\alpha u(x)$$

$$= \iint e^{i(x-y)\cdot\xi} \sum p_\alpha(x,y) e^{ih\alpha\cdot\xi} u(y) dy đ\xi$$

with symbol

$$p(x,y,\xi) = \sum p_\alpha(x,y)e^{i\alpha\cdot\xi} \quad .$$

In the following it will be as easy to consider double symbols $p(x,y,\xi)$ instead of simple symbols $p(x,\xi)$. The class of simple symbols will follow as a special case.

The pseudo-difference operator $P_h$ associated with the double symbol $p(x,y,\xi)$ will be represented by the formula

(1.2) $$P_h u(x) = \iint e^{i(x-y)\cdot\xi} p(x,y,h\xi) u(y) dy đ\xi \quad .$$

The symbol $p(x,y,\xi)$ is an $m \times m$ matrix function of $x,y$ and $\xi$, and possibly of $h$. The absolute value $|p(x,y,\xi)|$ of $p$ is defined as in the introduction.

Formula (1.2), with $h = 1$, defines a pseudo-differential operator.

We now define a class of symbols which was introduced by Hörmander in [7] in 1966.

Definition 1.1. If $m$, $\rho$ and $\delta$ are real numbers with $\rho \geq 0$ and $\delta \geq 0$, we denote by $S^m_{\rho,\delta}$ the class of matrices $p(x,y,\xi)$ such that

$$|\partial_y^\gamma \partial_x^\beta \partial_\xi^\alpha p(x,y,\xi)| \leq C_{\alpha,\beta,\gamma}(1+|\xi|)^{m-\rho|\alpha| + \delta(|\beta| + |\gamma|)}$$

for all multi-indices $\alpha,\beta,\gamma$ in $Z_+^n$ and all $x,y,\xi$ in $R^n$.

As we are interested in bounded operators in $L^2$, we take $m = 0$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$. Hörmander has shown in [8] that if $\rho > 0$, $\delta < \rho$ and $p$ has compact support in $(x,y)$, then the pseudo-differential operator with symbol $p$ is bounded in $L^2$. On the other hand, he also proved there that this need not be true if $\rho < \delta$. Therefore, we shall consider the borderline case and remove the restriction on the support of $p$. We set

$$S \equiv S^0_{\delta,\delta} \qquad \delta < 1 .$$

The continuity of pseudo-difference operators with symbol $p(x,y,\xi)$ in $S$ can be derived directly from a general theorem of Calderón and Vaillancourt [2] for pseudo-differential operators.

In fact the parameter $h > 0$ does not play any role in the proof; hence, with $h = 1$, the problem reduces to that of a pseudo-differential operator for which the above theorem applies. The proof of this theorem rests on the use of a continuous partition of unity and of a general lemma for almost orthogonal operators in a Hilbert space.

However the use of a discrete partition of unity, originally introduced by Chin-Hung Ching, in 1971, in an unpublished manuscript in the case $\rho = \delta = 0$, and adapted by Vaillancourt in [15] when $0 \leq \rho = \delta < 1$, requires weaker differentiability conditions on the symbols. We shall, therefore, present a detailed proof of the boundedness of pseudo-difference operators of class S, using a discrete partition of unity.

We begin with two lemmas which will be needed in the proof of the boundedness theorem.

The first lemma for almost orthogonal operators, which was stated without proof in [3], will be proved in its full generality, although we need only the discrete form due to Cotlar as reported without proof by Knapp and Stein [9].

Lemma 1.1 Let $A$ denote a bounded operator on a separable Hilbert space $H$ and $A(z)$ be a weakly-measurable, uniformly bounded, operator-valued function on a measure space $Z$ with measure $dz$.

If

$$||A(z_1)\ A^*(z_2)|| \leq h_1(z_1,z_2)^2$$

$$||A^*(z_1)\ A\ (z_2)|| \leq h_2(z_1,z_2)^2$$

and

$$\int h_1(z_1,z)\, h_2(z,z_2)dz$$

is the kernel of a bounded integral operator on $L^2(Z)$ with the norm $M^2$, then

$$\|\int_E A(z)dz\| \leq M ,$$

where $E$ is any subset of finite measure of $Z$ .

<u>Proof</u>: Consider

$$\|[(\int_E A(z)dz)(\int_E A(z)dz)^*]^m\|^{1/m}$$

$$= \|(\int_E A(z_1)dz_1)(\int_E A(z_2)dz_2)^* \ldots (\int_E A(z_{2m-1})dz_{2m-1})(\int_E A(z_{2m})dz_{2m})^*\|^{1/m}$$

$$\leq [\int \ldots \int_{z_i \in E} \|A(z_1)A^*(z_2) \ldots A(z_{2m-1})A^*(z_{2m})\| dz_1 \ldots dz_{2m}]^{1/m} .$$

Let $T_m \equiv \|A(z_1)A^*(z_2) \ldots A(z_{2m-1})A^*(z_{2m})\|$, then

$$T_m \leq \|A(z_1)A^*(z_2)\| \ldots \|A(z_{2m-3})A^*(z_{2m-2})\| \|A(z_{2m-1})\| \cdot \|A^*(z_{2m})\|$$

and

$$T_m \leq \|A(z_1)\| \|A^*(z_2)A(z_3)\| \ldots \|A^*(z_{2m-2})A(z_{2m-1})\| \|A^*(z_{2m})\| ;$$

we multiply the above two inequalities to obtain the estimate

$$T_m^2 \leq C^4 ||A(z_1)A^*(z_2)|| \, ||A^*(z_2)A(z_3)|| \ldots ||A(z_{2m-3})A^*(z_{2m-2})|| \, ||A^*(z_{2m-2}) A(z_{2m-1})||$$

which, after taking square roots and using the hypotheses of the lemma, becomes

$$T_m \leq C^2 h_1(z_1,z_2) \, h_2(z_2,z_3) \, h_1(z_3,z_4) \ldots h_1(z_{2m-3},z_{2m-2}) \, h_2(z_{2m-2},z_{2m-1}) \; .$$

Thus, letting $\chi_E(z)$ be the characteristic function of the set $E$ with finite measure $\mu(E)$, we have

$$||[(\int_E A(z)dz) \, (\int_E A(z)dz)^*]^m||^{1/m}$$

$$\leq |C^2 \int_E \ldots \int h_1(z_1,z_2) \, h_2(z_2,z_3) \ldots h_1(z_{2m-3},z_{2m-2}) \, h_2(z_{2m-2},z_{2m-1}) \, dz_2 \ldots dz_{2m-2} dz_1 \, dz_{2m-1} dz_{2m}|^{1/m}$$

$$= |C^2 \iiint_E h^{(2m-2)}(z_1,z_{2m-1}) dz_1 \, dz_{2m-1} \, dz_{2m} \,|^{1/m}$$

$$= |C^2 \, \mu(E) \iint \chi_E(z_1) \, h^{(2m-2)}(z_1,z_{2m-1}) \, \chi_E(z_{2m-1}) dz_1 \, dz_{2m-1}|^{1/m}$$

$$\leq [C^2 \, \mu^2(E) \, M^{2m-2}]^{1/m} \quad ;$$

the last estimate follows from the fact that $h^{(2m-2)}(z,z')$ is the kernel of the bounded operator $H^{(2m-2)}$ and

$$||H^{(2m-2)}|| \leq M^{2m-2} \quad .$$

Letting m go to infinity, this becomes

$$\lim_{m\to\infty} ||[(\int_E A(z)dz)\ (\int_E A(z)dz^*]^m\ ||^{1/m} \le M^2 \ .$$

Since

$$|| \int_E A(z)dz||^2 = \lim_{m\to\infty} ||[(\int_E A(z)dz)(\int_E A(z)dz)^*]^m||^{1/m} ,$$

we have, after taking square roots, the desired estimate

$$|| \int_E A(z)dz \ || \le M \ .$$

This completes the proof of the first lemma.

The second lemma is the following simple result.

Lemma 1.2 The norm of the operator $Q_h$ associated with the symbol $q(x,y,\xi) \equiv p(a^{-1}x,\ a^{-1}y,\ a\xi)$ is independent of $a$ for $a > 0$, that is,

$$||P_h|| = ||Q_h|| \ .$$

Proof: For any $a > 0$ ,

$$Q_h u(x) = \int \int e^{i(x-y)\cdot\xi} p(a^{-1}x,a^{-1}y,ah\xi)u(y)dy đ\xi$$

$$= \int \int e^{i(a^{-1}x-a^{-1}y)\cdot a\xi} p(a^{-1}x,a^{-1}y,ah\xi)u(y)dy đ\xi \ .$$

Setting $a^{-1}x = x'$, $a^{-1}y = y'$ and $a\xi = \xi'$, this becomes

$$Q_h u(ax') = \iint e^{i(x'-y')\cdot\xi'} p(x',y',h\xi')\, u(ay')dy'đ\xi' ,$$

and dropping the primes, we have

$$Q_h u(ax) = \iint e^{i(x-y)\cdot\xi} p(x,y,h\xi)\, u(ay)dyđ\xi \quad .$$

Setting

$$v(x) \equiv u(ax)$$

and

$$P_h v(x) = \iint e^{i(x-y)\cdot\xi} p(x,y,h\xi)\, v(y)dyđ\xi \quad ,$$

we easily see that

$$Q_h u(ax) = P_h v(x) \quad .$$

Thus we have

$$||Q_h|| = \sup_u \frac{||Q_h u||}{||u||} = \sup_u \frac{|(Q_h u, Q_h u)|^{\frac{1}{2}}}{|(u,u)|^{\frac{1}{2}}} = \sup_u \frac{[\int |Q_h u(ax)|^2\, dax]^{\frac{1}{2}}}{[\int |u(ax)|^2\, dax]^{\frac{1}{2}}}$$

$$= \sup_v \frac{[\int |P_h v(x)|^2\, dx]^{\frac{1}{2}}}{[\int |v(x)|^2\, dx]^{\frac{1}{2}}} = \sup_v \frac{||P_h v||}{||v||} = ||P_h|| \quad .$$

This completes the proof of the second lemma.

We are now ready to prove the continuity of pseudo-differential operators with symbols in S .

Boundedness Theorem 1.1. Let the symbol $p(x,y,\xi)$ be an $m \times m$ matrix of functions $p_{i,j}(x,y,\xi)$ defined on $R^n \times R^n \times R^n$ such that

$$|\partial_{x_n}^{\beta_n} \cdots \partial_{x_1}^{\beta_1} \partial_{\xi_n}^{\alpha_n} \cdots \partial_{\xi_1}^{\alpha_1} p_{i,j}(x,y,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{(|\beta|-|\alpha|)\delta}$$

(1.3)

$$|\partial_{y_n}^{\beta_n} \cdots \partial_{y_1}^{\beta_1} \partial_{\xi_n}^{\alpha_n} \cdots \partial_{\xi_1}^{\alpha_1} p_{i,j}(x,y,\xi)| \le C'_{\alpha,\beta}(1+|\xi|)^{(|\beta|-|\alpha|)\delta}$$

for $0 \le \alpha_j \le 2$, $0 \le \beta_j \le 2+\tau$ , $j = 1,\ldots, n$ and all $x,y,\xi$ ,

where $\tau \equiv$ the least integer $> \delta/(1-\delta)$ . Then the pseudo-differential operator $P$ defined by

$$Pu(x) = \iint e^{i(x-y)\cdot\xi} p(x,y,\xi)\, u(y)\,dy\,d\!\!\!/\xi$$

is bounded in $L^2$ and

$$||P|| \le C\, ||p|| ,$$

where $||p||$ denotes the least value of $C_{\alpha,\beta}$ and $C'_{\alpha,\beta}$ for which the inequalities(1.3) hold.

<u>Proof</u>: For simplicity, we first restrict ourselves to the one-dimensional case; the general n-dimensional case will follow easily.

We choose a non-negative smooth function $q(\xi)$ with compact support, $0 \le q(\xi) \le 1$, such that

$$q(\xi) \begin{cases} > 0 , & \text{for } |\xi| \le 2/3 \\ = 0 , & \text{for } |\xi| > 3/4 . \end{cases}$$

Define $M = M(\sigma) \equiv$ the smallest integer $\ge \frac{1}{2^{1/\sigma}-1}$

where $\sigma \equiv \frac{\delta}{1-\delta}$, and set

$$q_o(\xi) = q\left(\frac{\xi}{\frac{3}{2} M^{\sigma+1}}\right)$$

and

$$q_m(\xi) = q\left(\frac{\xi - m|m|^\sigma}{(\sigma+1)|m|^\sigma}\right) \qquad \text{for} \quad |m| \ge M .$$

Clearly $\text{supp } q_{m+n} \cap \text{supp } q_m = \phi$ for $n > \frac{3}{2}(\sigma+1)$. Letting

$$Q(\xi) = q_o(\xi) + \sum_{|m| \ge M} q_m(\xi)$$

and

$$w_m(\xi) = q_m(\xi) / Q(\xi) ,$$

we define the symbol $p_m(x,y,\xi)$ by

$$p_m(x,y,\xi) \equiv p(x,y,\xi)\, w_m(\xi) \quad .$$

Let $P_m$ be the pseudo-differential operator with the symbol $p_m$ . Since

$$p(x,y,\xi) = p_o(x,y,\xi) + \sum_{|m|\geq M} p_m(x,y,\xi)$$

the operator $P$ has been split as a sum

$$P = P_o + \sum_{|2m|\geq M} P_{2m} + \sum_{|2m+1|\geq M} P_{2m+1} \quad .$$

We remark the following properties of our partition of unity:

(i) $$Q(\xi) = q_o(\xi) + \sum_{|m|\geq M} q_m(\xi) > 0 \quad ,$$

(ii) $$w_m(\xi) \in C_o^{\infty} \quad ,$$

(iii) $\operatorname{supp} w_n(|m|^{\sigma}\xi) \subset [n|\frac{n}{m}|^{\sigma} - \frac{3}{4}(\sigma+1)|\frac{n}{m}|^{\sigma},\ n|\frac{n}{m}|^{\sigma} + \frac{3}{4}(\sigma+1)|\frac{n}{m}|^{\sigma}]$

for $m \neq 0$ , $|n| \geq M$ , and

(iv) distance $[\operatorname{supp} w_m(|m|^{\sigma}\xi),\ \operatorname{supp} w_n(|m|^{\sigma}\xi)] > \frac{1}{4}(|n|-|m|)|\frac{n}{m}|^{\sigma}$

for $|n|-|m| \geq 2(\sigma+1)$ , $|n| > |m| \geq M$ .

To prove (i), we need only show that, for all $m$, the supports of neighboring $q_m(\xi)$ overlap. By construction, $q_o(\xi)$ is positive on $[-M^{\sigma+1}, M^{\sigma+1}]$.

Since

$$\text{supp } q_M(\xi) \supset [M^{\sigma+1} - \tfrac{2}{3}(\sigma+1)M^{\sigma}, M^{\sigma+1} + \tfrac{2}{3}(\sigma+1)M^{\sigma}]$$

and

$$\text{supp } q_{-M}(\xi) \supset [-M^{\sigma+1} - \tfrac{2}{3}(\sigma+1)M^{\sigma}, -M^{\sigma+1} + \tfrac{2}{3}(\sigma+1)M^{\sigma}] ,$$

it is easy to see that

$$\text{supp } q_M(\xi) \cap \text{supp } q_o(\xi) \neq \phi$$

and

$$\text{supp } q_{-M}(\xi) \cap \text{supp } q_o(\xi) \neq \phi .$$

For $|m| \geq M$, we have to verify that

$$(1.4)\quad m|m|^{\sigma} + \tfrac{2}{3}(\sigma+1)|m|^{\sigma} \geq (m+1)|m+1|^{\sigma} - \tfrac{2}{3}(\sigma+1)|m+1|^{\sigma} .$$

If $m \geq M$, this inequality becomes

$$m^{\sigma+1} + \tfrac{2}{3}(\sigma+1)m^{\sigma} \geq (m+1)^{\sigma+1} - \tfrac{2}{3}(\sigma+1)(m+1)^{\sigma} ,$$

$$\tfrac{2}{3}(\sigma+1)[m^{\sigma} + (m+1)^{\sigma}] \geq (m+1)^{\sigma+1} - m^{\sigma+1} ,$$

$$\frac{2}{3}[m^{\sigma} + (m+1)^{\sigma}] \geq (m+\theta)^{\sigma}, \qquad 0 < \theta < 1 ;$$

this holds if

$$\frac{2}{3}[m^{\sigma} + (m+1)^{\sigma}] \geq (m+1)^{\sigma} ,$$

or

$$2^{1/\sigma} m \geq m+1 .$$

Replacing $m$ by $M+k$, $k \geq 0$, we have

$$2^{1/\sigma}(M+k) \geq M+k+1 ,$$

That is

$$M \geq \frac{k(1-2^{1/\sigma})+1}{2^{1/\sigma}-1} .$$

Since the right hand side of this inequality is a decreasing function of $k$, we have

$$M \geq \frac{1}{2^{1/\sigma}-1} ,$$

which holds by our definition of $M$. Hence (1.4) is true for $m \geq M$.

For $m \leq -M$, we have a similar result.

Therefore (1.4) is true for $|m| \geq M$.

This shows that $Q(\xi) > 0$ .

Now (ii) follows from (i). In fact $q_m(\xi) \epsilon C_0^{\infty}$ ,

thus $Q(\xi) \epsilon C^{\infty}$ and, by (i), $Q(\xi) > 0$ ; hence

$$w_m(\xi) = q_m(\xi)/Q(\xi)$$

is clearly in $C_o^\infty$.

To prove (iii), we note that for $m \neq 0$ and $|n| \geq M$,

$$q_n(|m|^\sigma\xi) = q\left(\frac{|m|^\sigma\xi - n|n|^\sigma}{(\sigma+1)|n|^\sigma}\right) = q\left(\frac{\xi - n\left|\frac{n}{m}\right|^\sigma}{(\sigma+1)\left|\frac{n}{m}\right|^\sigma}\right) ;$$

hence (iii) follows from the inclusion relation:

$$\operatorname{supp}\ q_n(|m|^\sigma\xi) \subset [n|\tfrac{n}{m}|^\sigma - \tfrac{3}{4}(\sigma+1)|\tfrac{n}{m}|^\sigma,\ n|\tfrac{n}{m}|^\sigma + \tfrac{3}{4}(\sigma+1)|\tfrac{n}{m}|^\sigma] .$$

To prove (iv), for $\xi \in \operatorname{supp} w_m(|m|^\sigma\xi)$ and $\zeta \in \operatorname{supp} w_n(|m|^\sigma\xi)$,

with $n > |m| \geq M$, we have

$$|\xi - \zeta| > |n|\tfrac{n}{m}|^\sigma - \tfrac{3}{4}(\sigma+1)|\tfrac{n}{m}|^\sigma - [m + \tfrac{3}{4}(\sigma+1)]|$$

$$> [|n| - \tfrac{3}{4}(\sigma+1) - |m| - \tfrac{3}{4}(\sigma+1)]|\tfrac{n}{m}|^\sigma$$

$$> [|n| - |m| - \tfrac{3}{2}(\sigma+1)]|\tfrac{n}{m}|^\sigma .$$

If $|n|-|m| \geq 2(\sigma+1)$, i.e., $(\sigma+1) \leq \frac{1}{2}(|n|-|m|)$, then

$$|\xi - \zeta| > [|n| - |m| - \tfrac{3}{4}(|n|-|m|)]|\tfrac{n}{m}|^\sigma$$

$$= \tfrac{1}{4}(|n|-|m|)|\tfrac{n}{m}|^\sigma .$$

Similarly, for $-n > |m| \geq M$,

$$|\xi - \zeta| > |n|\left|\frac{n}{m}\right|^{\sigma} + \frac{3}{4}(\sigma+1)\left|\frac{n}{m}\right|^{\sigma} - [m - \frac{3}{4}(\sigma+1)]|$$

$$> |n|\left|\frac{n}{m}\right|^{\sigma} - \frac{3}{4}(\sigma+1)\left|\frac{n}{m}\right|^{\sigma} - |m| - \frac{3}{4}(\sigma+1)$$

$$> [|n| - |m| - \frac{3}{2}(\sigma+1)]\left|\frac{n}{m}\right|^{\sigma}$$

$$> \frac{1}{4}(|n| - |m|)\left|\frac{n}{m}\right|^{\sigma},$$

provided $|n| - |m| \geq 2(\sigma+1)$.

Thus for $|n| > |m| \geq M$ and $|n| - |m| \geq 2(\sigma+1)$,

distance $[\text{supp } w_m(|m|^{\sigma}\xi), \text{ supp } w_n(|m|^{\sigma}\xi)] > \frac{1}{4}(|n| - |m|)\left|\frac{n}{m}\right|^{\sigma}$.

This proves (iv).

Now by using properties (i)-(iv), we shall be able to prove the theorem.

We show that $P_0$ is bounded. Since $p_0(x,y,\xi)$ is smooth and has compact support in $\xi$, then

$$\int |\partial_{\xi}^{\alpha} p_0(x,y,\xi)|\, d\xi < C, \qquad \alpha = 0,1,2,$$

and

$$P_0 u(x) = (2\pi)^{-1} \iint e^{i(x-y)\xi} p_0(x,y,\xi) u(y)\, dy\, d\xi$$

$$= (2\pi)^{-1} \iint e^{i(x-y)\xi}(1+|x-y|^2)^{-1}(1-\partial_\xi^2)p_o(x,y,\xi)u(y)dyd\xi \quad .$$

Since $(1 + |y|^2)^{-1}$ is integrable, we have

$$||P_o u|| \leq C\, ||p_o||\, ||u||, \text{ i.e., } ||P_o|| < C\, ||p_o|| \; .$$

Here

$$||p_o|| = \sup_{x,y,\xi} |(1-\partial_\xi^2)p_o(x,y,\xi)| \;\; . \;\; E_o \quad ,$$

where $E_o$ is the measure of the support of $q_o(\xi)$ .
To apply the same argument to $p_m$ with $|m| \geq M$, we note that

$$q_m(|m|^\sigma \xi) = q(\frac{\xi - m}{\sigma + 1}) \quad ,$$

hence $p_m(|m|^{-\sigma}x, |m|^{-\sigma}y, |m|^\sigma \xi)$ has $\xi$-support of length at most $\frac{3}{2}(\sigma+1)$ around $m$ .

Since, by (ii), $w_m(\xi) \in C_o^\infty$ and $p(x,y,\xi) \in S$ , then $p_m(|m|^{-\sigma}x, |m|^{-\sigma}y, |m|^\sigma \xi)$ is $C_o^2$ in $\xi$ ; thus

$$\int |\partial_\xi^\alpha\, p_m(|m|^{-\sigma}x, |m|^{-\sigma}y, |m|^\sigma \xi)| d\xi < C \; , \quad \alpha = 0, 1, 2.$$

In lemma 1.2, let $h\xi = \xi$, i.e., $h = 1$, so that it can be applied to the pseudo-differential operators with symbols $p_m(x,y,\xi)$ and $q_m(x,y,\xi) \equiv p_m(|m|^{-\sigma}x, |m|^{-\sigma}y, |m|^\sigma \xi)$ , thus giving

$$||P_m|| = ||Q_m|| \leq C \quad ,$$

where $C$ depends on $\|p\|$ but not on $m$ for $|m| \geq M(\sigma)$.

Next for $|m|$ , $|n| > M(\sigma)$ , we show that

$$||P_{2n}^* P_{2m}|| \leq \frac{C \; \|p\|^2}{1 + \big||n| - |m|\big|^{2+\varepsilon}} \quad , \quad \varepsilon > 0 \; .$$

Since

$$(P_{2n}^* P_{2m} u, v) = (P_{2m} u, P_{2n} v) = \int P_{2m} u(x) \overline{P_{2n} v}(x) dx$$

$$= (2\pi)^{-2} \int \ldots \int e^{i(x-y)\xi} p_{2m}(x,y,\xi) u(y) dy d\xi e^{-i(x-z)\zeta} \overline{p_{2n}}(x,z,\zeta) \overline{v(z)} dz d\zeta dx$$

$$= (2\pi)^{-2} \int \ldots \int (1+|x-y|^2)^{-1} e^{i(x-y)\xi} (1-\partial_\xi^2) p_{2m}(x,y,\xi) u(y) dy d\xi (1+|x-z|^2)^{-1}$$

$$e^{-i(x-z)\zeta} (1-\partial_\zeta^2) \overline{p_{2n}}(x,z,\zeta) \overline{v}(z) dz d\zeta dx$$

$$= (2\pi)^{-2} \int \ldots \int \frac{1}{1+|x-y|^2} \; \frac{1}{1+|x-z|^2} \; \frac{1}{[i(\xi-\zeta)]^{2+\tau}} e^{ix(\xi-\zeta) + iz\zeta - iy\xi}$$

$$(1-\partial_\xi^2)(1-\partial_\zeta^2)\partial_x^{2+\tau} [p_{2m}(x,y,\xi) u(y) \overline{p_{2n}}(x,z,\zeta) \overline{v}(z)] dy d\xi dz d\zeta dx$$

where $\tau$ is the smallest integer $> \sigma$ .

Hence

$$||P_{2n}^* P_{2m}|| = \sup_{\substack{u,v \\ ||u||=||v||=1}} |(P_{2n}^* P_{2m} u, v)|$$

$$= \sup_{u,v} \left|(2\pi)^{-2}\int\cdots\int \frac{1}{1+|x-y|^2}\,\frac{1}{1+|x-z|^2}\,\frac{1}{[i(\xi-\zeta)]^{2+\tau}}\, e^{ix(\xi-\zeta)+iz\zeta-iy\xi}\right.$$

$$(1-\partial_\xi^2)(1-\partial_\zeta^2)\partial_x^{2+\tau}\quad [p_{2m}(\mu^{-\sigma}x,\ \mu^{-\sigma}y,\ \mu^{\sigma}\xi)u(y)\bar{p}_{2n}(\mu^{-\sigma}x,\mu^{-\sigma}z,\mu^{\sigma}\zeta)\bar{v}(z)]$$

$$\left. dy\,d\xi\,dz\,d\zeta\,dx \right|$$

$$\le (2\pi)^{-2}\int\cdots\int \frac{|u(y)|}{1+|x-y|^2}\,\frac{|v(z)|}{1+|x-z|^2}\,\frac{1}{|\xi-\zeta|^{2+\tau}}\left|(1-\partial_\xi^2)(1-\partial_\zeta^2)\partial_x^{2+\tau}\right.$$

(1.5)

$$\left.[p_{2m}(\mu^{-\sigma}x,\mu^{-\sigma}y,\mu^{\sigma}\xi)\bar{p}_{2n}(\mu^{-\sigma}x,\mu^{-\sigma}z,\mu^{\sigma}\zeta)]\right|dy\,d\xi\,dz\,d\zeta\,dx,$$

where

$$\mu = \min(|2m|\ ,\ \ |2n|)\ .$$

Without loss of generality, we assume $|n| > |m| \ge M(\sigma)$; hence

$$\mu = |2m|\ .$$

Thus, for $0 \le \alpha \le 2$, $0 \le \beta \le 2+\tau$

$$|\partial_\xi^\alpha\, \partial_x^\beta\, p_{2m}(|2m|^{-\sigma}x,|2m|^{-\sigma}y,|2m|^{\sigma}\xi)|$$

$$= |2m|^{-|\beta|\sigma+|\alpha|\sigma}|p_{2m(\beta)}^{(\alpha)}(|2m|^{-\sigma}x,|2m|^{-\sigma}y,|2m|^{\sigma}\xi)|$$

$$\le C_{\alpha,\beta}|2m|^{-|\beta|\sigma+|\alpha|\sigma}(1+|2m|^{\sigma}\xi)^{(|\beta|-|\alpha|)\delta}\ .$$

Because $|2m| > M$ and $\xi \in \operatorname{supp} p_{2m}(|2m|^{-\sigma}x,|2m|^{-\sigma}y,|2m|^{\sigma}\xi)$

which has uniform length in $\xi$ about $2m$, we have

$$|\partial_\xi^\alpha \partial_x^\beta p_{2m}(|2m|^{-\sigma}x, |2m|^{-\sigma}y, |2m|^{\sigma}\xi)|$$

$$\leq C_{\alpha,\beta} |2m|^{-(|\beta|-|\alpha|)\sigma} (|2m|^{\sigma+1})^{(|\beta|-|\alpha|)\delta}$$

$$\leq ||p|| |2m|^{-(|\beta|-|\alpha|)\sigma} |2m|^{(|\beta|-|\alpha|)\frac{\delta}{1-\delta}}$$

$$\leq ||p|| .$$

Similarly,

$$|\partial_\zeta^\alpha \partial_x^\beta p_{2n}(|2m|^{-\sigma}x, |2m|^{-\sigma}z, |2m|^{\sigma}\zeta)|$$

$$\leq C_{\alpha,\beta} |2m|^{-(|\beta|-|\alpha|)\sigma} (|2n|^{\sigma+1})^{(|\beta|-|\alpha|)\delta}$$

$$\leq ||p|| |2m|^{-(|\beta|-|\alpha|)\sigma} |2n|^{(|\beta|-|\alpha|)\sigma}$$

$$= ||p|| \left|\frac{2n}{2m}\right|^{(|\beta|-|\alpha|)\sigma} .$$

Thus inequality (1.5) becomes

$$(1.6) \quad ||P_{2n}^* P_{2m}|| \leq C ||p||^2 \frac{1}{|\xi-\zeta|^{2+\tau}} \left|\frac{2n}{2m}\right|^{(2+\tau)\sigma} .$$

By (iv),

$$|\xi-\zeta| > \frac{1}{4}(|2n|-|2m|)\left|\frac{2n}{2m}\right|^{\sigma} = \frac{1}{2}(|n|-|m|)\left|\frac{n}{m}\right|^{\sigma}$$

provided $|2n|-|2m| \geq 2(\sigma+1)$ .

Also

$$|\xi-\zeta| > \frac{1}{4}(|2n|-|2m|)\left|\frac{2n}{2m}\right|^{\sigma}$$

$$= \frac{1}{4}\ |2m|\left(\left|\frac{2n}{2m}\right|-1\right)\left|\frac{n}{m}\right|^{\sigma}$$

$$= \frac{1}{2}\ |m|\left(\left|\frac{n}{m}\right|-1\right)\left|\frac{n}{m}\right|^{\sigma}$$

$$> \frac{1}{6}\ |m|\ \left|\frac{n}{m}\right|\ \left|\frac{n}{m}\right|^{\sigma}$$

provided $\left|\frac{n}{m}\right| - 1 \geq \frac{1}{3}\left|\frac{n}{m}\right|$ , i.e., $\left|\frac{n}{m}\right| \geq \frac{3}{2}$ .

Thus

$$|\xi-\zeta|^{2+\tau} = |\xi-\zeta|^{(2+\tau-\sigma)+\sigma}$$

$$> \frac{1}{2^{(2+\tau-\sigma)}6^{\sigma}}(|n|-|m|)^{2+\tau-\sigma}|m|^{\sigma}\left|\frac{n}{m}\right|^{(2+\tau+1)\sigma}$$

$$> \frac{1}{6^{2+\tau}}(|n|-|m|)^{2+\tau-\sigma}\left|\frac{n}{m}\right|^{(2+\tau)\sigma}|n|^{\sigma} ,$$

and from (1.6), we have

$$||P_{2n}^{*}\ P_{2m}|| \leq C||p||^{2}\frac{1}{(|n|-|m|)^{2+\tau-\sigma}} \cdot \frac{6^{2+\tau}}{|n|^{\sigma}}$$

$$\leq\ C||p||^{2}\frac{1}{(|n|-|m|)^{2+\varepsilon}} , \qquad \varepsilon = \tau-\sigma > 0 ,$$

provided $|n|>|m|\geq M$, $|2n|-|2m|\geq 2(\sigma+1)$ and $|\frac{n}{m}|\geq\frac{3}{2}$ .

If $|2n|-|2m|<2(\sigma+1)$, then $|\frac{n}{m}|-1<\frac{\sigma+1}{|m|}$ ; thus $|\frac{n}{m}|<1+\frac{\sigma+1}{|m|}<C$ .

If $|\frac{n}{m}|<\frac{3}{2}$ . Then $|\frac{n}{m}|-1<\frac{1}{2}$ ; hence $|n|-|m|<C$ .

Both of these make $||P_{2n}^{*}P_{2m}||\leq C||p||^{2}$ for $|n|-|m|\leq C$ .
Therefore we have shown that

$$||P_{2n}^{*}P_{2m}||\leq C||p||^{2}\frac{1}{1+||n|-|m||^{2+\varepsilon}} .$$

Similarly,

$$||P_{2n}P_{2m}^{*}||\leq C||p||^{2}\frac{1}{1+||n|-|m||^{2+\varepsilon}} .$$

Since $\sum\limits_{n}\left(\frac{1}{1+|n|^{2+\varepsilon}}\right)^{1/2}$ is absolutely convergent, the inequality

$$||\sum_{|2m|\geq M}P_{2m}||\leq C||p||$$

follows by lemma 1.1.

By a similar argument,

$$||\sum_{|2m+1|\geq M}P_{2m+1}||\leq C||p|| .$$

Grouping these inequalities , we have

$$||P|| \le ||P_o|| + ||\sum_{|2m|\ge M} P_{2m}|| + ||\sum_{|2m+1|\ge M} P_{2m+1}|| \le C||p|| \quad ;$$

this establishes the theorem when $x,y,\xi \in R^1$ .

In the n-dimensional case, i.e., $x,y,\xi \in R^n$ , we write

$$q_i(\xi) = q_{i_1 \cdots i_n}(\xi) = q_{i_1}(\xi_1)q_{i_2}(\xi_2)\cdots q_{i_n}(\xi_n) \quad ,$$

$$w_i(\xi) = w_{i_1 \cdots i_n}(\xi) = w_{i_1}(\xi_1)w_{i_2}(\xi_2)\cdots w_{i_n}(\xi_n) \quad ,$$

$$p_i(x,y,\xi) = p_{i_1 \cdots i_n}(x,y,\xi) = p(x,y,\xi)\, w_i(\xi) \quad ,$$

and decompose the symbol $p(x,y,\xi)$ into a sum of $2^n$ series as follows:

$$p(x,y,\xi) = \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} \Big[ \sum_{i_\alpha \equiv k_\alpha (\mathrm{mod} 2)} p_{i_1 \cdots i_n}(x,y,\xi) \Big] \quad ;$$

then the previous arguments can be applied. This completes the proof of the theorem.

<u>Corollary 1.1</u> The pseudo-difference operators with symbols in $S$ are bounded in $L^2$ .

This corollary does not hold when $\rho = 1$ . In fact there are symbols in $S^o_{1,1}$ which yield unbounded operators. Ching [4] has constructed such pseudo-differential operators. We adapt here his result to pseudo-difference operators.

<u>Theorem 1.2</u> Let $\chi(\xi)$ be a $C_0^\infty(R^n)$ function with supp $\chi(\xi) \subset \{ \xi \mid 1 \le |\xi| \le 5 \}$, which is equal to 1 for $2 \le |\xi| \le 4$. Let $\{\eta_k\}$ be a sequence in $R^n$ such that $|\eta_k| = 3\cdot 5^k$. Then the pseudo-difference operator $P_h$ associated with the symbol $p(x,y,\xi) = \sum_{k=1}^{\infty} a_k e^{-i\eta_k(x-y)}\chi(5^{-k}h^{-1}\xi)$ is not bounded from $L^2(R^n)$ to $L^2(K)$ for any $K$ with nonempty interior, if $\sum_1^\infty |a_k|^2$ is divergent.

<u>Proof</u>: Assume to the contrary that, for some constant $C$,

$$(1.7) \qquad ||P_h u||_K \le C||u||_{R^n}$$

Choose $\Psi \not\equiv 0$ such that $\hat{\Psi} \in C_0^\infty(R^n)$ and supp $\hat{\Psi}$ is contained in the unit ball. Define

$$u_m(y) \equiv \sum_1^m b_k e^{i\eta_k y}\Psi(y) \quad ;$$

hence

$$(1.8) \qquad \hat{u}_m(\xi) = \sum_1^m b_k \hat{\Psi}(\xi-\eta_k) \quad .$$

As the terms in (1.8) have disjoint supports, it follows that

$$||\hat{u}_m||^2 = \sum_1^m |b_k|^2$$

and

$$P_h u_m(x) = \iint e^{i(x-y)\cdot\xi} p(x,y,h\xi) u_m(y)\,dy\,d\!\!\bar{}\,\xi$$

$$= \sum_1^m a_k b_k\, e^{-i\eta_k x} \iint e^{ix\cdot\xi - iy\cdot\xi}\chi(5^{-k}\xi) e^{i\eta_k y}\Psi(y)\,dy\,d\!\!\bar{}\,\xi$$

$$= \sum_1^m a_k b_k e^{-i\eta_k x} \int e^{ix\cdot\xi} \chi(5^{-k}\xi)\hat{\Psi}(\xi-\eta_k) đ\xi$$

Since supp $\hat{\Psi} \subset$ unit ball, if $\hat{\Psi}(\xi-\eta_k) \neq 0$, then $|\xi-\eta_k| < 1$

and $\qquad |\xi| < |\eta_k| + 1 = 3\cdot 5^k + 1$, i.e.,

$$|5^{-k}\xi| < 3 + 5^{-k}, \text{ hence } 2 < |5^{-k}\xi| < 4 .$$

Thus $\qquad \chi(5^{-k}\xi) = 1$ and we obtain

$$P_h u_m(x) = \sum_1^m a_k b_k \Psi(x) \quad .$$

Hence

$$||P_h u_m||_K = |\sum_1^m a_k b_k|^2 \int |\Psi(2)|^2 dx \quad .$$

Applying (1.7) to $u_m$ gives

$$|\sum_1^m a_k b_k|^2 \leq C \sum_1^m |b_k|^2 \quad .$$

We then have

$$\sum_1^\infty |a_k|^2 \leq C ,$$

contradicting our hypothesis on $\{a_k\}$ .

The proof is complete.

<u>Corollary 1.2</u> There exist unbounded pseudo-difference operators from $L^2(R^n)$ to $L^2(K)$.

<u>Proof:</u> Define $p(x,y,\xi)$ as in theorem 1.2 with

$$a_k = 1/\sqrt{k}\ , \quad \text{i.e.,}$$

$$(1.9)\quad p(x,y,\xi) = \sum_1^\infty \frac{1}{\sqrt{k}}\, e^{-i\eta_k\cdot(x-y)}\chi(5^{-k}h^{-1}\xi)\ .$$

To prove $p(x,y,\xi) \in S^0_{1,1}$, we observe that the terms in (1.9) have disjoint supports in $\xi$ and hence we have

$$|\partial_y^\gamma\partial_x^\beta\partial_\xi^\alpha p(x,y,\xi)| \le \sup_k |\eta_k^{|\gamma|+|\beta|} 5^{-|\alpha|k} h^{-|\alpha|} \chi^{(\alpha)}(5^{-k}h^{-1}\xi)|$$

$$\le |\xi|^{-|\alpha|} \sup_k |\eta_k^{|\gamma|+|\beta|} (5^{-k}h^{-1}\xi)^{|\alpha|} \chi^{(\alpha)}(5^{-k}h^{-1}\xi)|$$

$$\le C\ |\xi|^{-|\alpha|+|\beta|+|\gamma|} \sup_{|\lambda|\le\alpha} |\eta^{|\lambda|} \chi^{(\lambda)}(\eta)|$$

$$\le C\ (1+|\xi|)^{|\gamma|+|\beta|-|\alpha|}\ .$$

Thus

$$p(x,y,\xi) \in S^0_{1,1}\ .$$

Since the $|a_k|^2$ form a divergent series,

$$\sum_1^\infty |a_k|^2 = \sum_1^\infty \frac{1}{k} = \infty\ ,$$

the corollary follows immediately from the theorem 1.2.

CHAPTER II

## Algebraic Properties of Pseudo-Difference Operators.

In this chapter we establish some algebraic properties of pseudo-difference operators. We denote by $P_h^R$ the operator associated with the symbol $p(y,x,\xi)$, i.e.,

$$P_h^R u(x) = \iint e^{i(x-y)\cdot\xi} p(y,x,h\xi)u(y)dy đ\xi \quad .$$

The superscript R, which stands for "reversor", is a useful notation introduced by Friedrichs [6]. By $P_h^{\#}$, we denote the operator associated with the symbol $p^*(x,y,\xi)$, i.e.,

$$P_h^{\#} u(x) = \iint e^{i(x-y)\cdot\xi} p^*(x,y,h\xi)u(y)dy đ\xi .$$

It is easy to see that the Hilbert adjoint $P_h^*$ of the operator $P_h$ has the symbol $p^*(y,x,\xi)$

$$P_h^* u(x) = \iint e^{i(x-y)\cdot\xi} p^*(y,x,h\xi)u(y)dy đ\xi \quad .$$

If the symbol $p(x,y,\xi)$ belongs to the class $S = S^o_{\delta,\delta}$ then the symbols $p(y,x,\xi)$, $p^*(x,y,\xi)$ and $p^*(y,x,\xi)$ are evidently in $S$ . From the corollary to theorem 1.1, the operators $P_h^R$, $P_h^{\#}$ and $P_h^*$ are bounded in $L^2$. We shall denote by $\mathbb{S}_\delta$ the class of all bounded pseudo-difference operators with symbols belonging to S .

We now define the addition and two different products in $\mathbb{S}_\delta$ .

<u>Definition 2.1</u> Let $P_h$ and $Q_h$ be two operators in $\mathbb{S}_\delta$ with the symbols $p(x,y,\xi)$ and $q(x,y,\xi)$ respectively. The addition $P_h + Q_h$ is defined as:

$$(P_h+Q_h)u(x) \equiv \iint e^{i(x-y)\cdot\xi}[p(x,y,h\xi) + q(x,y,h\xi)]u(y)dy đ\xi ,$$

the operator product $P_hQ_h$ is defined as the composition

$$P_hQ_hu(x) \equiv P_h(Q_hu(x)) ,$$

and the symbol product $P_h \circ Q_h$ is defined by means of the product of the symbols:

$$(P_h \circ Q_h)u(x) \equiv \iint e^{i(x-y)\cdot\xi} p(x,y,h\xi)\, q(x,y,h\xi)u(y)dy đ\xi .$$

The notation $P_h \circ Q_h$ is motivated by the fact that the product of the symbols in the $\xi$-space is a convolution product of the kernels in the x-space, namely

$$\hat{f}(\xi)\, \hat{g}(\xi) = \widehat{f \circ g}(\xi) \equiv \int e^{-ix\cdot\xi} \int f(x-y)g(y)dydx.$$

<u>Algebra Theorem 2.1</u> The class $\mathbb{S}_\delta$ of pseudo-difference operators forms an algebra under the addition and either the operator product or the symbol product. In other words, if $P_h$, $Q_h \in \mathbb{S}_\delta$ , then

$$P_h + Q_h \in \mathbb{S}_\delta \quad ,$$

$$P_h Q_h \in \mathbb{S}_\delta \quad ,$$

and

$$P_h \circ Q_h \in \mathbb{S}_\delta \quad .$$

Proof: It is clear that $P_h + Q_h$ belongs to $\mathbb{S}_\delta$ . To prove $P_hQ_h \in \mathbb{S}_\delta$ , we follow Kumano-go [11]. By definition,

$$P_hQ_hu(x) = P_h(Q_hu(x))$$

$$= \iint e^{i(x-x')\cdot\zeta} p(x,x',h\zeta)(Q_hu(x'))dx' đ\zeta$$

$$= \iint e^{i(x-x')\cdot\zeta} p(x,x',h\zeta) \iint e^{i(x'-y)\cdot\xi} q(x',y,h\xi)u(y)dy đ\xi dx' đ\zeta \ .$$

Changing the order of integration, which can be easily justified, and letting $\zeta = \xi + \eta$ , $đ\zeta = đ\eta$, we obtain

$$P_hQ_hu(x) = \iint e^{i(x-y)\cdot\xi} [\iint e^{i(x-x')\cdot\eta} p(x,x',h(\xi+\eta))q(x',y,h\xi)dx' đ\eta]u(y)dy đ\xi \quad .$$

Now with $x' = x+z$, $dx' = dz$, this becomes

$$P_hQ_hu(x) = \iint e^{i(x-y)\cdot\xi} [\iint e^{-iz\cdot\eta} p(x,x+z,h(\xi+\eta))q(x+z,y,h\xi)dz đ\eta]u(y)dy đ\xi$$

$$\equiv \iint e^{i(x-y)\cdot\xi} a(x,y,h\xi)u(y)dy đ\xi \ ,$$

where the symbol

$$(2.1)\quad a(x,y,\xi) \equiv \iint e^{-iz\cdot\eta} p(x,x+z,\xi+\eta)q(x+z,y,\xi)dz đ\eta \quad .$$

To show $a(x,y,\xi) \in S$, we use the following identity

$$(2.2)\quad (1+ \langle\xi\rangle^{2\delta m}|z|^{2m})^{-1}[1+ \langle\xi\rangle^{2\delta m}(-\Delta_\eta)^m]e^{-iz\cdot\eta} = e^{-iz\cdot\eta}$$

where

$$\Delta_\eta \equiv |\partial_\eta|^2 = \partial^2_{\eta_1} + \partial^2_{\eta_2} + \ldots + \partial^2_{\eta_n} \quad .$$

Setting $m$ to be the smallest integer such that $2m > n$, substituting (2.2) into (2.1) and integrating by parts, we obtain

$$a(x,y,\xi) = \iint e^{-iz\cdot\eta}(1+ \langle\xi\rangle^{2\delta m}|z|^{2m})^{-1}[1+\langle\xi\rangle^{2\delta m}(-\Delta_\eta)^m]\{p(x,x+z,\xi+\eta)q(x+z,y,\xi)\} dz đ\eta$$

$$(2.3)\quad \equiv \iint e^{-iz\cdot\eta} g(x,y,z,\xi,\eta)dz đ\eta$$

where

$$g(x,y,z,\xi,\eta) \equiv (1+ \langle\xi\rangle^{2\delta m}|z|^{2m})^{-1}[1+\langle\xi\rangle^{2\delta m}(-\Delta_\eta)^m]\{p(x,x+z,\xi+\eta)q(x+z,y,\xi)\} \ .$$

Now using the identity

$$|\eta|^{-2\ell}(-\Delta_z)^\ell \, e^{-iz\cdot\eta} = e^{-iz\cdot\eta} \quad ,$$

in (2.3) with $\ell >> m$ and integrating by parts, we have

$$a(x,y,\xi) = \iint e^{-iz\cdot\eta}|\eta|^{-2\ell}(-\Delta_z)^{\ell}\, g(x,y,z,\xi,\eta)dz đ\eta \quad .$$

Set

$$r(x,y,z,\xi,\eta) \equiv e^{-iz\cdot\eta}|\eta|^{-2\ell}(-\Delta_z)^{\ell}\, g(x,y,z,\xi,\eta)$$

and write

$$a(x,y,\xi) = \iint_{|\eta|\geq<\xi>/2} r dz đ\eta + \iint_{<\xi>^{\delta}/2\leq|\eta|\leq<\xi>/2} r dz đ\eta + \iint_{|\eta|\leq<\xi>^{\delta}/2} r dz đ\eta$$

$$\equiv a'(x,y,\xi) + a''(x,y,\xi) + a'''(x,y,\xi) \quad .$$

The growth properties of $p$ and $q$ and the equivalence of the norms $(1+|\xi|)$ and $<\xi>$ give the inequality

$$|a(x,y,\xi)|$$

$$=|\iint e^{-iz\cdot\eta}|\eta|^{-2\ell}(-\Delta_z)^{\ell}\{(1+<\xi>^{2\delta m}|z|^{2m})^{-1}[1+<\xi>^{2\delta m}(-\Delta_\eta)^{m}]$$

$$[p(x,x+z,\xi+\eta)q(x+z,y,\xi)]\}dz đ\eta|$$

$$\leq C\iint|\eta|^{-2\ell}(1+<\xi>^{2\delta m}|z|^{2m})^{-1} <\xi>^{2\delta m}\{<\xi+\eta>^{(2\ell-2m)\delta} + <\xi>^{2\ell\delta}\}dz đ\eta \quad .$$

When $|\eta| \leq <\xi>/2$ , one gets the estimate

(2.4) $$|a(x,y,\xi)| \leq C_1 \iint(1+<\xi>^{2\delta m}|z|^{2m})^{-1}dz đ\eta \quad .$$

By noting that

$$\langle\xi+\eta\rangle \le \langle\xi\rangle + |\eta| \le 3|\eta| \quad \text{for} \quad |\eta| \ge \langle\xi\rangle/2,$$

the above estimate gives

$$|a'(x,y,\xi)| \le C_2 \int_{|\eta|\ge\langle\xi\rangle/2} |\eta|^{-2\ell(1-\delta)}\langle\xi\rangle^{2\delta m} \int (1+\langle\xi\rangle^{2\delta m}|z|^{2m})^{-1} dz \,đ\eta \,.$$

Since $2m > n$, we see that

$$\int (1+\langle\xi\rangle^{2\delta m}|\xi|^{2m})^{-1} d(\langle\xi\rangle^{\delta} z) d\eta \le C \quad .$$

Thus for a large fixed $\ell$, $\ell \gg m$, we have

$$|a'(x,y,\xi)| \le C_3 \int_{|\eta|\ge\langle\xi\rangle/2} |\eta|^{-2\ell(1-\delta)}\langle\xi\rangle^{2\delta m-\delta n} đ\eta$$

$$\le C_4 \langle\xi\rangle^{-2\ell(1-\delta) + n + 2\delta m - \delta n}$$

$$\le C_4 \langle\xi\rangle^{-N} \,, \qquad N > 0 \quad .$$

For $a''(x,y,\xi)$, we observe that for $|\eta| \le \langle\xi\rangle/2$ there exists a constant $C > 0$ such that

$$(2.5) \qquad C^{-1}\langle\xi\rangle \le \langle\xi+\eta\rangle \le C\langle\xi\rangle \,.$$

hence

$$|a''(x,y,\xi)| \le C_5 \iint_{\langle\xi\rangle^{\delta}/2\le|\eta|\le\langle\xi\rangle/2} |\eta|^{-2\ell}(1+\langle\xi\rangle^{2\delta m}|z|^{2m})^{-1}\langle\xi\rangle^{2\ell\delta}dz\,d\!\!\!{}^-\eta$$

$$\le C_6 \int_{\langle\xi\rangle^{\delta}/2\le|\eta|\ \langle\xi\rangle/2} |\eta|^{-2\ell}\langle\xi\rangle^{2\ell\delta-n\delta}d\!\!\!{}^-\eta$$

$$\le C_7\ \langle\xi\rangle^{-2\ell\delta+n\delta+2\ell\delta-n\delta} = C_7 \ .$$

Finally for $a'''(x,y,\xi)$, using (2.4) and (2.5), we have

$$|a'''(x,y,\xi)| \le C_8 \iint_{|\eta|\le\langle\xi\rangle^{\delta}/2} (1+\langle\xi\rangle^{2\delta m}|z|^{2m})^{-1}\ dz\,d\!\!\!{}^-\eta$$

$$\le C_8\ \langle\xi\rangle^{-n\delta} \ .$$

Combining these estimates, we obtain that

$$|\ a(x,y,\xi)| \le C \ .$$

To prove the growth condition for $a(x,y,\xi)$, namely

$$|\partial_y^{\gamma}\partial_x^{\beta}\partial_\xi^{\alpha}\ a(x,y,\xi)| \le C < \xi >^{(|\gamma|+|\beta|-|\alpha|)\delta}$$

for all multi-indices $\alpha,\beta,\gamma$, we simply note that the left hand sides of these inequalities satisfy the estimate

$$|\partial_y^\gamma \partial_x^\beta \partial_\xi^\alpha \ a(x,y,\xi)|$$

$$= |\iint e^{-iz\cdot\eta}|\eta|^{-2\ell}\partial_y^\gamma\partial_x^\beta\partial_\xi^\alpha(-\Delta_z)^\ell\{(1+<\xi>^{2\delta m}|z|^{2m})^{-1}[1+<\xi>^{2\delta m}(-\Delta_\eta)^m]$$

$$[p(x,x+z,\xi+\eta)q(x+z,y,\xi)]\}dz đ\eta|$$

$$\leq C \iint|\eta|^{-2\ell}(1+<\xi>^{2\delta m}|z|^{2m})^{-1}<\xi>^{2\delta m}<\xi+\eta>^{(2\ell-2m+|\gamma|+|\beta|-|\alpha|)\delta}dz đ\eta .$$

Then by the same splitting as above, we obtain

$$|\partial_y^\gamma \partial_x^\beta \partial_\xi^\alpha \ a(x,y,\xi)| \leq C < \xi >^{(|\gamma|+|\beta|-|\alpha|)\delta} .$$

Thus

$$a(x,y,\xi) \in S ,$$

which, by definition, implies that

$$P_h \, Q_h \in \mathbb{S}_\delta .$$

Finally we show that the operator $P_h o Q_h \in \mathbb{S}_\delta$ .

In this case a simple application of Leibnitz' formula to the symbol $p(x,y,\xi)\ q(x,y,\xi)$ of $P_h o Q_h$ easily shows that $pq \in S$, that is, $P_h o Q_h \in \mathbb{S}_\delta$ .

Thus under the addition and either the operator product or the symbol product, $\mathbb{S}_\delta$ forms an algebra.

For further reference, we state as a corollary a result which was derived in the course of the proof of the previous theorem.

Corollary 2.1 Let $p(x,y,\xi)$ and $q(x,y,\xi)$ be two symbols which belong to $S$ and set

$$f(x,y,z,\xi,\eta) \equiv p(x,x+z,\xi+\eta)\, q(x+z,y,\xi) \quad .$$

Then

$$a(x,y,\xi) \quad \equiv \iint e^{-iz\cdot\eta} f(x,y,z,\xi,\eta)\, dz\, đ\eta$$

also belongs to $S$ .

We shall prove two more theorems which hold for the algebra of operators in $\mathbb{S}_\delta$ , namely the Reversor theorem and the Product-Difference theorem.

Reversor Theorem 2.2 If $P_h \in \mathbb{S}_\delta$ , then

$$||P_h - P_h^R|| = ||P_h^\# - P_h^*|| = O(h) \ .$$

Proof : $(P_h - P_h^R)u(x)$

$$= \iint e^{i(x-y)\cdot\xi}[p(x,y,h\xi) - p(y,x,h\xi)]u(y)\,dy\,đ\xi$$

$$= \iint e^{i(x-y)\cdot\xi}\{[p(x,y,h\xi)-p(x,x,h\xi)]+[p(x,x,h\xi)-p(y,x,h\xi)]\} u(y)\,dy\,đ\xi$$

$$= \iint e^{i(x-y)\cdot\xi}\{\sum_j \int_o^1 p_{y_j}(x,x+\mu(y-x),h\xi)d\mu(y_j-x_j)+\sum_j \int_o^1 p_{x_j}(y+\nu(x-y),$$

$$x,h\xi)d\nu(x_j-y_j)\}u(y)dy đ\xi \ .$$

Using the identity

$$[i(x_j-y_j)]^{-1}\partial_{\xi_j} e^{i(x-y)\cdot\xi} = e^{i(x-y)\cdot\xi}$$

and integrating by parts, we obtain

$$(P_h-P_h^R)u(x)$$

$$= \iint e^{i(x-y)\cdot\xi}\sum_j ih\{\int_o^1 p_{y_j\xi_j}(x,x+\mu(y-x),h\xi)d\mu-\int_o^1 p_{x_j\xi_j}(y+\nu(x-y),x,h\xi)d\nu\}$$

$$u(y)dy đ\xi \ .$$

Thus the symbol corresponding to the operator $P_h-P_h^R$ is in $S$ and the following estimate holds:

$$|p(x,y,h\xi)-p(y,x,h\xi)| \le Ch[<h\xi>^{\delta-\delta}+<h\xi>^{\delta-\delta}] = Ch \ .$$

This implies that

$$||P_h - P_h^R|| = O(h) \ .$$

Since the symbol of the operator $P_h^{\#} - P_h^{*}$ satisfies the same estimate,

$$|p^*(x,y,h\xi)-p^*(y,x,h\xi)| = |p(x,y,h\xi)-p(y,x,h\xi)| \le Ch \ ,$$

we have shown that

$$||P_h - P_h^R|| = ||P_h^\# - P_h^*|| = O(h) \ .$$

Product-Difference Theorem 2.3 If $P_h$ and $Q_h$ are in $\mathbb{S}_\delta$, then the difference of the operator product from the symbol product is of order $h$, that is,

$$||P_h\, Q_h - P_h \circ Q_h|| = O(h) \ .$$

Proof: By definition,

$$\begin{aligned} P_h\, Q_h\, u(x) &= P_h(Q_h u(x)) \\ &= \iint e^{i(x-y)\cdot\xi} p(x,y,h\xi) Q_h u(y) dy đ\xi \\ &= \iint e^{i(x-y)\cdot\xi} p(x,y,h\xi) \iint e^{i(y-z)\cdot\zeta} q(y,z,h\zeta) u(z) dz đ\zeta dy đ\xi \\ &= \int\ldots\int e^{i(x-y)\cdot\xi + i(y-z)\cdot\zeta} p(x,y,h\xi) q(y,z,h\zeta) u(z) dz đ\zeta dy đ\xi \ . \end{aligned}$$

and

$$(P_h \circ Q_h) u(x) = \int\ldots\int e^{ix\cdot\xi - iy\cdot\xi + i(y-z)\cdot\zeta} p(y,z,h\zeta) q(y,z,h\zeta) u(z) dz đ\zeta dy đ\xi .$$

Hence

$$(P_h\ Q_h - P_h \circ Q_h)u(x)$$

$$= \int \dots \int e^{i(x-y)\cdot\xi + i(y-z)\cdot\zeta}[p(x,y,h\xi)-p(y,z,h\zeta)]q(y,z,h\zeta)u(z)dz đ\zeta dy đ\xi \ .$$

Splitting the term involving the difference in the following way,

$$p(x,y,h\xi) - p(y,z,h\zeta)$$

$$= [p(x,y,h\xi) - p(x,y,h\zeta)] + [p(x,y,h\zeta) - p(x,z,h\zeta)] + [p(x,z,h\zeta) - p(y,z,h\zeta)]$$

and expanding the last two terms in a Taylor series, we obtain

$$(P_h\ Q_h - P_h \circ Q_h)u(x)$$

$$= \int \dots \int e^{i(x-y)\cdot\xi + i(y-z)\cdot\zeta}\sum_j [(h\xi_j - h\zeta_j)\ \frac{p(x,y,h\xi)-p(x,y,h\zeta)}{h\xi_j - h\zeta_j}]q(y,z,h\zeta)u(z)dz đ\zeta dy đ\xi$$

$$+ \int \dots \int e^{i(x-y)\cdot\xi + i(y-z)\cdot\zeta}\sum_j (y_j - z_j)\int_0^1 p_{y_j}(x,z+\mu(y-z),h\zeta)d\mu\, q(y,z,h\zeta)u(z)dz đ\zeta dy đ\xi$$

$$+ \int \dots \int e^{i(x-y)\cdot\xi + i(y-z)\cdot\zeta}\sum_j (x_j - y_j)\int_0^1 p_{x_j}(y+\mu(x-y),z,h\zeta)d\mu\, q(y,z,h\zeta)u(z)dz đ\zeta dy d\xi \, .$$

The third term vanishes, since the integration with respect to $\xi$ will produce a delta function $\delta(x_j - y_j)$ multiplied by $(x_j - y_j)$ .

In the first and second term, we make use of the following identities respectively,

$$[-i(\xi_j-\zeta_j)]^{-1}\partial_{y_j}\ e^{-iy\cdot(\xi-\zeta)} = e^{-iy\cdot(\xi-\zeta)} ,$$

$$[i(y_j-z_j)]^{-1}\partial_{\zeta_j}\ e^{i(y-z)\cdot\zeta} = e^{i(y-z)\cdot\zeta} ,$$

and integrate by parts to obtain

$$(P_h\ Q_h - P_h o Q_h)u(x)$$

$$= \int\ldots\int e^{i(x-y)\cdot\xi+i(y-z)\cdot\zeta}\sum_j(-i)\{h\partial_{y_j}[\frac{p(x,y,h\xi)-p(x,y,h\zeta)}{h\xi_j - h\zeta_j}\,q(y,z,h\zeta)]$$

$$-\partial_{\zeta_j}[\int_o^1 p_{y_j}(x,z+\mu(y-z),h\zeta)d\mu q(y,z,h\zeta)]\}u(z)dz\bar{d}\zeta dy\bar{d}\xi$$

$$= \int\ldots\int e^{i(x-y)\cdot\xi+i(y-z)\cdot\zeta}(-ih)\sum_j\{\int_o^1 p_{y_j\xi_j}(x,y,h\zeta+\mu h(\xi-\zeta))d\mu q(y,z,h\zeta)$$

$$+ (h\xi_j-h\zeta_j)^{-1}[p(x,y,h\xi)-p(x,y,h\zeta)]q_{y_j}(y,z,h\zeta)-\int_o^1 p_{y_j\zeta_j}(x,z+\mu(y-z),h\zeta)d\mu q(y,z,h\zeta)$$

$$- \int_o^1 p_{y_j}(x,z+\mu(y-z),h\zeta)d\mu q_{\zeta_j}(y,z,h\zeta)\}u(z)dz\bar{d}\zeta dy\bar{d}\xi .$$

Since p and q belong to S, by a simple application of the Algebra theorem and the corollary 2.1, the first, third and fourth terms are bounded. It remains only to estimate the second term:

$$Ch|h\xi-h\zeta|^{-1}|p(x,y,h\xi)-p(x,y,h\zeta)|\ \langle h\zeta\rangle^{\delta} \equiv I .$$

To estimate I, we consider three cases:

(i) If $|\xi| \le \frac{1}{2}|\zeta|$ , then $|\xi-\zeta| \ge \frac{1}{2}|\zeta|$ , and

$$I \le Ch\langle h\zeta\rangle^{-1+\delta}[|p(x,y,h\xi)|+|p(x,y,h\zeta)|] \le Ch .$$

(ii) If $|\xi| \ge \frac{3}{2}|\zeta|$ , then $|\xi-\zeta| \ge \frac{1}{2}|\zeta|$ , and

$I \le Ch$ as in (i) .

(iii) If $\frac{1}{2}|\zeta|<|\xi|<\frac{3}{2}|\zeta|$, then $|\xi-\zeta| < \frac{1}{2}|\zeta|$, and

$$|\zeta+\mu(\xi-\zeta)| \ge ||\zeta|-\mu|\xi-\zeta|| \ge \frac{1}{2}|\zeta|,\ 0<\mu<1 ,$$

thus

$$\begin{aligned} I &\le Ch|p_{\xi}(x,y,h\zeta+\mu h(\xi-\zeta))|\ \langle h\zeta\rangle^{\delta} \\ &\le C'h\ \langle h\zeta+\mu h(\xi-\zeta)\rangle^{-\delta}\langle h\zeta\rangle^{\delta} \\ &\le C''h\ \langle h\zeta\rangle^{-\delta+\delta} \\ &= C''h . \end{aligned}$$

Therefore,

$$|p(x,y,h\xi)q(y,z,h\zeta)-p(y,z,h\zeta)q(y,z,h\zeta)| \le Ch .$$

This implies that

$$\| P_h\ Q_h - P_h \circ Q_h \| = O(h) .$$

## CHAPTER III

The Lax-Nirenberg Theorem and Stability Criteria.

In this chapter we shall show that the Lax-Nirenberg theorem for difference operators established in [12] also holds for pseudo-difference operators $P_h$ with symbols $p(x,y,\xi)$ in $S$. The following lemma relating double symbols in $S$ to simple symbols in $S$ will be used in the proof of Lax-Nirenberg's theorem.

Lemma 3.1 The pseudo-difference operator $P_h$, with double symbol $p(x,y,\xi)$ in $S$, differs by $O(h)$ from the pseudo-difference operator $G_h$, with the contracted simple symbol $g(x,\xi) \equiv p(x,x,\xi)$ in $S$, that is,

$$\| P_h - G_h \| = O(h) \quad .$$

Proof: $P_h u(x)$

$$= \iint e^{i(x-y)\cdot\xi} p(x,y,h\xi)u(y)dy đ\xi$$

$$= \iint e^{i(x-y)\cdot\xi}[p(x,x,h\xi)-(p(x,x,h\xi)-p(x,y,h\xi))]u(y)dy đ\xi$$

$$= \iint e^{ix\cdot\xi} g(x,h\xi)\hat{u}(\xi) đ\xi - \iint e^{i(x-y)\cdot\xi}\int_0^1 \sum_j p_{y_j}(x,y+\mu(x-y),h\xi)d\mu(x_j-y_j)u(y)dy đ\xi \ .$$

Substituting the following identity in the second term

$$[i(x_j-y_j)]^{-1}\partial_{\xi_j} e^{i(x-y)\cdot\xi} = e^{i(x-y)\cdot\xi}$$

and integrating by parts to eliminate $x_j - y_j$ , we have

$$P_h u(x) = G_h u(x) - i\iint e^{i(x-y)\cdot\xi_h} \int_0^1 \sum_j p_{y_j\xi_j}(x, y+\mu(x-y), h\xi)\, d\mu\, u(y)\, dy\, d\xi$$

where $G_h$ is the pseudo-difference operator associated with the simple symbol $g(x,\xi) \equiv p(x,x,\xi)$ in $S$ .
Since

$$p_{y_j\xi_j}(x, y+\mu(x-y), h\xi) \in S ,$$

it follows that

$$\|(P_h - G_h)u\| \le Kh\, \|u\| .$$

This proves the lemma.

Lax-Nirenberg Theorem 3.1 Let $P_h$ be a pseudo-difference operator with symbol $p(x,y,\xi)$ in $S$ .
Suppose $p(x,y,\xi)$ is a hermitian and nonnegative square matrix, then

$$\mathrm{Re}(u, P_h u) \ge -Kh\|u\|^2$$

for some constant $K$ and for all $u$ and $h$ .

Proof: To prove this theorem, we adapt to the class $S$ the simple proof given by Vaillancourt in [13].

We choose a smooth even function $q^2(\sigma)$ in $C_0^\infty(R^n)$ with support in the unit ball, $|\sigma| \le 1$, and integral 1,

$$(3.1) \qquad \int q^2(\sigma) d\sigma = 1 \quad .$$

In order to construct an approximation to the symbol $p$, we use $q^2(\sigma)$ to mollify $p$ in a rather special way to obtain the mollified symbol,

$$(3.2) \quad a(x,y,\xi) = \int p(x,y,\xi+h^{\frac{1}{2}}<\xi>^{\delta}\sigma)q^2(\sigma)d\sigma \quad .$$

To see that $a(x,y,\xi)$ belongs to $S$, we consider

$$\partial_\xi a(x,y,\xi) = \int p_\xi(x,y,\xi+h^{\frac{1}{2}}<\xi>^{\delta}\sigma)(1+h^{\frac{1}{2}}\delta<\xi>^{\delta-2}\xi\cdot\sigma)q^2(\sigma)d\sigma \ .$$

Since $p \in S$, $\delta < 1$ and $|h^{\frac{1}{2}}\sigma| \le 1$, thus, for large $|\xi|$,

$$|\partial_\xi a(x,y,\xi)| \le C <\xi+h^{\frac{1}{2}} <\xi>^{\delta}\sigma>^{-\delta} \le C <\xi>^{-\delta} \quad .$$

Similarly for higher derivatives in $\xi$, we have

$$|\partial_\xi^\alpha a(x,y,\xi)| \le C <\xi>^{-|\alpha|\delta} \quad .$$

Since the derivatives of $a(x,y,\xi)$ in $x$ and $y$ involves only $p(x,y,\xi + h^{\frac{1}{2}} <\xi>^{\delta}\sigma)$, it is clear that, for large $|\xi|$,

$$|\partial_y^\gamma \partial_x^\beta a(x,y,\xi)| \le C<\xi+h^{\frac{1}{2}}<\xi>^{\delta}\sigma>^{(|\beta|+|\gamma|)\delta} \le C<\xi>^{(|\beta|+|\gamma|)\delta} \quad .$$

Thus we have shown that $a(x,y,\xi) \in S$ .

It is convenient to rewrite the right hand side of (3.2) by using the change of variable

$$\tau = \xi + h^{\frac{1}{2}} < \xi >^{\delta} \sigma$$

and the notation

$$\phi(\xi,\tau) = h^{-\frac{n}{4}} <\xi>^{-\frac{n}{2}\delta} q(h^{-\frac{1}{2}} <\xi>^{-\delta}[\tau-\xi]) ,$$

so that the symbol $a(x,y,\xi)$ becomes

$$a(x,y,\xi) = \int p(x,y,\tau) \phi^2 (\xi,\tau) đ\tau .$$

If we denote by $A_h$ the pseudo-difference operator with the symbol $a$ , the Hilbert adjoint $A_h^*$ has the symbol

$$a^{R*}(x,y,\xi) = \int p^*(y,x,\tau) \phi^2 (\xi , \tau) đ\tau .$$

By the assumption that the symbol $p$ is hermitian,

$$p^*(y,x,\tau) = p(y,x,\tau) ,$$

$a^{R*}(x,y,\xi)$ becomes

$$a^R(x,y,\xi) = \int p(y,x,\tau) \phi^2 (\xi , \tau) đ\tau .$$

We assert that

$$(3.3) \qquad \| \operatorname{Re} A_h - \operatorname{Re} P_h \| = O(h) \quad .$$

Let $F_h$ and $G_h$ be the pseudo-difference operators in $\mathbb{S}_\delta$ with symbols $f(x,\xi) \equiv a(x,x,\xi)$ and $g(x,\xi) \equiv p(x,x,\xi)$ respectively. By the lemma 3.1, we have

$$\| A_h - F_h \| = O(h) \quad \text{and} \quad \| P_h - G_h \| = O(h) \quad .$$

Now we make use of (3.2) and (3.1) to obtain

$$(F_h - G_h)\, u(x)$$

$$= \int e^{ix\cdot\xi}[\, f(x,h\xi) - g(x,h\xi)]\, \hat{u}(\xi) đ\xi$$

$$= \int e^{ix\cdot\xi}\int [\, g(x,h\xi+h^{\frac{1}{2}}<h\xi>^{\delta}\sigma) - g(x,h\xi)]q^2(\sigma) đ\sigma \hat{u}(\xi) đ\xi$$

$$= \int e^{ix\cdot\xi}\int h^{\frac{1}{2}}<h\xi>^{\delta} \sum_j g_{\xi_j}(x,h\xi)\sigma_j q^2(\sigma) đ\sigma \hat{u}(\xi) đ\xi$$

$$+ \int e^{ix\cdot\xi}\int h<h\xi>^{2\delta} \int_0^1 \sum_{j,k} g_{\xi_j\xi_k}(x,h\xi+\mu h^{\frac{1}{2}}<h\xi>^{\delta}\sigma)(1-\mu)d\mu$$

$$\sigma_j\sigma_k q^2(\sigma) đ\sigma \hat{u}(\xi) đ\xi \ .$$

The first term vanishes immediately because $q^2(\sigma)$ is even. If we observe that

$$|\int \sigma \cdot \sigma q^2(\sigma) d\sigma| \le 1$$

and

$$|g_{\xi_j \xi_k}(x, h\xi + \mu h^{\frac{1}{2}} \langle h\xi \rangle^{\delta} \sigma)| \le C \langle h\xi + \mu h^{\frac{1}{2}} \langle h\xi \rangle^{\delta} \sigma \rangle^{-2\delta} \le C \langle h\xi \rangle^{-2\delta} ,$$

we see that

$$|(F_h - G_h)u(x)| \le Ch|u(x)| .$$

Therefore

$$\|(A_h - P_h)u\| \le \|(A_h - F_h)u\| + \|(F_h - G_h)u\| + \|(G_h - P_h)u\|$$

$$\le Kh\|u\| ,$$

that is,

$$\|A_h - P_h\| = O(h) .$$

Similarly,

$$\|A_h^* - P_h^*\| = O(h) .$$

These estimates imply that

$$\|\mathrm{Re}A_h - \mathrm{Re}P_h\| = \|\tfrac{1}{2}(A_h - P_h) + \tfrac{1}{2}(A_h^* - P_h^*)\| = O(h) .$$

Thus we have shown (3.3) .

Now we use Friedrichs' trick [6] to define a double symbol $b(\xi',x,\xi)$ ,

$$b(\xi',x,\xi) \equiv \int \phi(\xi',\tau) g(x,\tau) \phi(\xi,\tau) d\tau .$$

Because of the symmetric form of the symbol $b(\xi',x,\xi)$ and the hermitian property of the symbol $g(x,\tau) = p(x,x,\tau)$, we see that

$$b^*(\xi,x,\xi') = b(\xi',x,\xi) .$$

This implies that the symbol $b(\xi',x,\xi)$ generates a symmetric pseudo-difference operator $B_h$ , that is,

$$(B_h u,v) = (u,B_h v) \quad \text{for all } u,v .$$

Furthermore, we claim that

$$B_h \geq 0 . \tag{3.4}$$

To prove this, we use the nonnegativity of $g$ ,

$$g(x,\tau) = p(x,x,\tau) \geq 0 ;$$

thus

$$(B_h u,u)$$

$$= \int \overline{u(z)} B_h u(z) dz$$

$$= \int \overline{u(z)} \iiint e^{iz\cdot\xi' - i\xi'\cdot x + ix\cdot\xi} b(h\xi', x, h\xi)\hat{u}(\xi)d\!\!\!/\xi dx d\!\!\!/\xi' dz$$

$$= \int \overline{u(z)} \iiiint e^{iz\cdot\xi' - i\xi'\cdot x + ix\cdot\xi} \phi(h\xi',\tau)g(x,\tau)\phi(h\xi,\tau)d\!\!\!/\tau\hat{u}(\xi)d\!\!\!/\xi dx d\!\!\!/\xi' dz$$

$$= \iiiint \overline{u(z)e^{i\xi'\cdot x - iz\cdot\xi'}\phi(h\xi',\tau)}dz d\!\!\!/\xi' g(x,\tau)\iint u(y)e^{ix\cdot\xi - i\xi\cdot y}\phi(h\xi,\tau)dy d\!\!\!/\xi \; d\!\!\!/\tau dx$$

$$= \iint \overline{w(x,\tau)}g(x,\tau)w(x,\tau)dx d\!\!\!/\tau \geq 0 \quad ,$$

since

$$w \cdot gw \geq 0 \quad ,$$

where

$$w(x,\tau) = \iint e^{ix\cdot\xi - i\xi\cdot y}u(y)\phi(h\xi,\tau)dy d\!\!\!/\xi \quad .$$

Finally we shall show that

$$(3.5) \qquad \| B_h - \mathrm{Re}A_h \| = O(h) \quad .$$

Consider

$$a(x,x,\xi) + a^R(x,x,\xi') - 2b(\xi',x,\xi)$$

$$= \int [ g(x,\tau)\phi^2(\xi,\tau) + g(x,\tau)\phi^2(\xi',\tau) - 2\phi(\xi',\tau)g(x,\tau)\phi(\xi,\tau)]d\!\!\!/\tau$$

$$= \int g(x,\tau)[\phi(\xi',\tau) - \phi(\xi,\tau)]^2 d\!\!\!/\tau$$

$$= \int g(x,\tau)[\sum_j(\xi_j'-\xi_j)\int_0^1\phi_{\xi_j}(\xi+\mu[\xi'-\xi],\tau)d\mu]^2 d\tau \ .$$

Since

$$\phi(\xi,\tau) = h^{-\frac{n}{4}} \langle\xi\rangle^{-\frac{n}{2}\delta} q(h^{-\frac{1}{2}} \langle\xi\rangle^{-\delta}[\tau-\xi]) \ ,$$

then a simple computation gives

$$\phi_{\xi_j}(\xi,\tau) = h^{-\frac{n}{4}} \langle\xi\rangle^{-\frac{n}{2}\delta}\{-\tfrac{n}{2}\delta \langle\xi\rangle^{-2}\xi_j q(h^{-\frac{1}{2}} \langle\xi\rangle^{-\delta}[\tau-\xi])$$

$$-h^{-\frac{1}{2}} \langle\xi\rangle^{-\delta}[\delta\langle\xi\rangle^{-2}\xi_j(\tau_j-\xi_j)+1]q_{\xi_j}(h^{-\frac{1}{2}}\langle\xi\rangle^{-\delta}[\tau-\xi])\} \ .$$

Set

$$\eta \equiv \xi + \mu(\xi'-\xi) \ , \qquad 0<\mu<1 \ .$$

Since

$$(A_h+A_h^* - 2B_h)u(z)$$

$$= \{(A_h-F_h)+(A_h^*-F_h^*)+(F_h+F_h^*-2B_h)\}u(z)$$

and by lemma 3.1 ,

$$\| A_h-F_h \| = O(h) \quad \text{and} \quad \| A_h^*-F_h^*\| = O(h) \ ,$$

we may neglect these two terms and consider the last one only.

Thus we have

$$(F_h+F_h^* - 2B_h)u(z)$$

$$= \int\ldots\int e^{iz\cdot\xi'-i\xi'\cdot x+ix\cdot\xi-i\xi\cdot y}[a(x,x,h\xi)+a^R(x,x,h\xi')-2b(h\xi',x,h\xi)]$$

$$u(y)dyd\!\!\!/\xi dx d\!\!\!/\xi'$$

$$= \int\ldots\int e^{iz\cdot\xi'-i\xi'\cdot x+ix\cdot\xi-i\xi\cdot y}\int g(x,\tau)\{\sum_j (h\xi_j'-h\xi_j)\int_0^1 h^{-\frac{n}{4}}\langle h\eta\rangle^{-\frac{n}{2}\delta}$$

$$\cdot [-\frac{n}{2}\delta\langle h\eta\rangle^{-2}h\eta_j q(h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}[\tau-h\eta])$$

$$-h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}[\delta\langle h\eta\rangle^{-2}h\eta_j(\tau_j-h\eta_j)+1]$$

$$\cdot\; q_{\xi_j}(h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}[\tau-h\eta])]d\mu\}^2 d\!\!\!/\tau u(y)dyd\!\!\!/\xi dx d\!\!\!/\xi' \; .$$

Substituting the following identity

$$[-i(\xi_j'-\xi_j)]^{-1}\partial_{x_j}e^{-ix\cdot(\xi'-\xi)} = e^{-ix\cdot(\xi'-\xi)}$$

and integrating by parts, we have

$$(F_h+F_h^* - 2B_h)u(z)$$

$$= \int\ldots\int e^{iz\cdot\xi'-i\xi'\cdot x+ix\cdot\xi-i\xi\cdot y}\int\sum_j g_{x_jx_j}(x,\tau)h^2\{\int_0^1 h^{-\frac{n}{4}}\langle h\eta\rangle^{-\frac{n}{2}\delta}[-\frac{n}{2}\delta\langle h\eta\rangle^{-2}h\eta_j$$

$$q(h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}[\tau-h\eta]) - h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}\,\delta\langle h\eta\rangle^{-2}h\eta_j[\tau_j-h\eta_j]q_{\xi_j}(h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}[\tau-h\eta])$$

$$- h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta} q_{\xi_j}(h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}[\tau - h\eta])] d\mu\}^2 đ\tau u(y) dy đ\xi dx đ\xi' \quad .$$

We recall that the support of $q$ is contained in the unit ball. Denote by $\chi(\tau,\eta)$ the characteristic function of the support of the mollifier $q(h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}[\tau-h\eta])$ , then the function

$$V(\eta) \equiv \int \chi(\tau,\eta) đ\tau = h^{-\frac{n}{2}}\langle h\eta\rangle^{-n\delta} \int_{|\tau-h\eta|\le h^{\frac{1}{2}}\langle h\eta\rangle^{\delta}} đ\tau$$

$$= \int_{|\sigma|\le 1} đ\sigma \equiv V \ .$$

is independent of $\eta$ .

Now we can obtain the estimate for the symbol as

$$|a(x,x,h\xi)+a^R(x,x,h\xi')-2b(h\xi',x,h\xi)|$$

$$\le \int \sum_j |g_{x_j x_j}(x,\tau)|h^2 \int_0^1 \chi(\tau,\eta) \{\tfrac{n}{2}\delta\langle h\eta\rangle^{-2}|h\eta_j||q|+h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}\delta\langle h\eta\rangle^{-2}$$

$$|h\eta_j||\tau_j-h\eta_j||q_{\xi_j}| + h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}|q_{\xi_j}|\}^2 d\mu đ\tau$$

$$\le \iint_0^1 \sum_j Ch^2\chi(\tau,\eta)\langle\tau\rangle^{2\delta} \{\tfrac{n}{2}\delta \langle h\eta\rangle^{-1}+h^{-\frac{1}{2}}\delta\langle h\eta\rangle^{-1-\delta}|\tau_j-h\eta_j|+h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}\}^2 d\mu đ\tau$$

$$\le \iint_0^1 h\chi(\tau,\eta)\{C_1 h^{\frac{1}{2}}\langle\tau\rangle^{\delta}\langle h\eta\rangle^{-1} + C_2\langle\tau\rangle^{\delta}\langle h\eta\rangle^{-1-\delta}|\tau-h\eta|+C_3\langle\tau\rangle^{\delta}\langle h\eta\rangle^{-\delta}\}^2 d\mu đ\tau$$

$$\equiv \iint_0^1 h\chi(\tau,\eta)\ \{I + II + III\}^2 d\mu đ\tau \ .$$

Since on the support of $q(h^{-\frac{1}{2}}\langle h\eta\rangle^{-\delta}[\tau-h\eta]$, we have

$$|\tau-h\eta| \leq h^{\frac{1}{2}} \langle h\eta\rangle^{\delta} \quad ,$$

that is

$$|\tau| \leq |h\eta|+h^{\frac{1}{2}} \langle h\eta\rangle^{\delta} \ ,$$

we can bound $\tau$ by

$$|\tau| \leq 2 \max(|h\eta| \ , h^{\frac{1}{2}} \langle h\eta\rangle^{\delta}) \leq 2\langle h\eta\rangle$$

This implies obviously that

$$\langle\tau\rangle \leq C \langle h\eta\rangle \quad ,$$

thus

$$I \equiv C_1 h^{\frac{1}{2}}\langle\tau\rangle^{\delta}\langle h\eta\rangle^{-1} \leq Ch^{\frac{1}{2}}\langle h\eta\rangle^{-1+\delta} \ ,$$

$$II \equiv C_2\langle\tau\rangle^{\delta}\langle h\eta\rangle^{-1-\delta}|\tau-h\eta| \leq Ch^{\frac{1}{2}}\langle h\eta\rangle^{-1-\delta+2\delta} = Ch^{\frac{1}{2}} \langle h\eta\rangle^{-1+\delta} \ ,$$

$$III \equiv C_3\langle\tau\rangle^{\delta}\langle h\eta\rangle^{-\delta} \leq C\langle h\eta\rangle^{\delta-\delta} = C \quad .$$

Since $0 \leq \delta < 1$, the above estimates are bounded for all $\eta$, hence

$$|a(x,x,h\xi) + a^R(x,x,h\xi') - 2b(h\xi',x,h\xi)| \leq C V h \quad .$$

This implies that

$$\| F_h + F_h^* - 2B_h \| = 0(h) \quad .$$

Thus we have shown that

$$\| \mathrm{Re}\, A_h - B_h \| = 0(h) \quad .$$

Now the desired result follows easily from (3.3), (3.4) and (3.5), namely,

$$- \mathrm{Re}(P_h u,u) \leq ([B_h - \mathrm{Re}P_h]u,u)$$

$$\leq [\, \| B_h - \mathrm{Re}A_h \| + \| \mathrm{Re}A_h - \mathrm{Re}P_h \| \,] \, \| u \|^2$$

$$\leq Kh \| u \|^2 \quad .$$

Multiplying across by -1 , this proves the theorem.

After having shown that the Lax-Nirenberg theorem is valid for the pseudo-difference operators in $\mathbb{S}_\delta$, we now extend Lax-Nirenberg's stability criteria [12] to pseudo-difference schemes with symbols in S . The following lemma, which follows from the Reversor theorem, the Product-Difference theorem and Lax-Nirenberg theorem for operators in $\mathbb{S}_\delta$ , will be used to derive the two stability criteria.

<u>Lemma 3.2</u> Let $P_h$ be a pseudo-difference operator with symbol $p(x,y,\xi)$ in S . Suppose $|p(x,y,\xi)| \leq 1$ for all x,y and $\xi$ in $R^n$ . Then the operator $P_h$ is bounded by

$$\| P_h \| \leq 1 + Kh$$

for all $h$ and some constant $K$ .

<u>Proof</u>: Consider

$$\| P_h u \|^2 = (P_h u, P_h u) = (u, P_h^* P_h u) \quad ,$$

From the assumption that $p(x,y,\xi)$ belongs to $S$ , it follows that $p^*(x,y,\xi)$ and $p^*(y,x,\xi)$ are also in $S$ . Hence the corresponding operators $P_h^\#$ and $P_h^*$ , respectively, belong to the algebra $\mathbb{S}_\delta$ . Since

$$\| P_h^* P_h - P_h^\# o P_h \| \leq \| P_h^* P_h - P_h^\# P_h \| + \| P_h^\# P_h - P_h^\# o P_h \|$$

$$\leq \| P_h^* - P_h^\# \| \, \| P_h \| + \| P_h^\# P_h - P_h^\# o P_h \| \quad ,$$

it follows from the boundedness property of $\| P_h \|$ , the Reversor theorem and the Product-Difference theorem that $P_h^* P_h$ differs by $O(h)$ from the symbol product operator $P_h^\# o P_h$ which has the symbol $p^*(x,y,\xi) p(x,y,\xi)$, i.e.,

$$\| P_h^* P_h - P_h^\# o P_h \| = O(h) \quad .$$

Now we define the symbol $q(x,y,\xi)$ as $q \equiv I - p^* p$ , and denote by $Q_h$ the corresponding operator; then

$$\| u \|^2 - \| P_h u \|^2 = (u, [I - P_h^* P_h] u)$$

$$= (u,[(I-P_h^{\#} o P_h) + O(h)]u)$$

$$= (u,Q_h u) + O(h)\|u\|^2 .$$

Since

$$|p^*(x,y,\xi)| = |p(x,y,\xi)| \leq 1 ,$$

the symbol $q(x,y,\xi)$ defined by $q \equiv I-p^*p$ is nonnegative. Because $p^*p$ belongs to $S$, so does $q$. It is evident that $q$ is hermitian, i.e., $q(x,y,\xi) = q^*(x,y,\xi)$. Therefore by the Lax-Nirenberg theorem,

$$\mathrm{Re}(u,Q_h u) \geq -Kh\|u\|^2 .$$

Since $\|u\|^2 - \|P_h u\|^2$ is real, we obtain that

$$\|u\|^2 - \|P_h u\|^2 \geq -Kh\|u\|^2 .$$

Thus we have shown that

$$\|P_h\|^2 \leq 1 + Kh ,$$

which implies

$$\|P_h\| \leq 1 + Kh .$$

<u>Stability Theorem I</u>. Consider the pseudo-difference scheme

(3.6) $\quad u(t+h) = P_h u(t) \; , \quad 0 \le t \le T$

where $P_h$ is a pseudo-difference operator in $\mathbb{S}_\delta$ , which may depend on $t$ as well. Suppose the norm of the symbol $p(t,x,\xi)$ is bounded by 1 ,

$$|p(t,x,\xi)| \le 1 \qquad \text{for all } \; t,x,\xi \; .$$

Then the pseudo-difference scheme is stable in the sense that

$$\| u(T) \| \le M(T) \| u(0) \|$$

for all solutions of (3.6), where $M$ is a function of T only but is independent of $h$ .

<u>Proof</u>: By lemma 3.2 , we have

$$\| P_h \| \le 1+Kh \; ,$$

with $K$ independent of $h$ , therefore

$$\| P_h u(t) \| = \| u(t+h) \| \le (1+Kh) \| u(t) \| \; ,$$

and so inductively,

$$\| u(nh) \| \le (1+Kh)^n \; \| u(0) \| \le e^{Khn} \| u(0) \| \; .$$

Setting $nh = T$ , we get the desired inequality

$$\|u(T)\| \le M(T) \ \|u(0)\|.$$

with

$$M(T) = (1+kh)^n \le e^{KT} ,$$

which is indeed independent of $h$ .

To state the next lemma and the second stability theorem, we need the concept of numerical range.

Definition 3.1 The numerical range of an operator $A$ in a Hilbert space, denoted by $w(A)$ , is the set of complex numbers of the form

$$(Ae,e)$$

where $e$ is any function in the Hilbert space of unit norm, $\|e\| = 1$ . The numerical range of a matrix is defined analogously.

Lemma 3.3 Suppose that the numerical range of the symbol $p(x,y,\xi)$ of class $S$ is contained in the unit disk for all $x,y$ and $\xi$ :

$$|w(p)| \le 1 ,$$

then the numerical range of the operator $P_h$ is contained in a disk of radius $1+Kh$ :

$$|w(P_h)| \le 1+Kh .$$

Here $K$ is a constant independent of $h$ .

Proof: The inequality

$$|w(P_h)| \leq 1+Kh$$

means that, for all vector functions $u \epsilon L^2$ ,

$$|(P_h u,u)| \leq [1+Kh]\|u\|^2 .$$

This can also be expressed by saying that for all complex numbers $z$, $|z| = 1$ , and all $u$ ,

$$\text{Re}\, z\,(P_h u,u) \leq (u,u) + Kh\,\|u\|^2 ,$$

which is the same as

$$\text{(3.7)} \qquad \text{Re}([I-zP_h]u,u) \geq -Kh\,\|u\|^2 .$$

The symbol of $I - zP_h$ is $I - zp$ ; it follows from the hypothesis on the numerical range of $p$ , $|w(p)| \leq 1$ , that $I - zp$ has a nonnegative real part. Therefore, by the Lax-Nirenberg theorem, the real part of the corresponding operator $I - zP_h$ is greater than $-Kh$ ; this proves the inequality (3.7) and hence the lemma.

Stability theorem II. Consider the pseudo-difference scheme

$$u(t+h) = P_h u(t) , \qquad 0 \leq t \leq T$$

where the pseudo-difference operator $P_h$ is independent of $t$ .

Suppose that the symbol $p(x,\xi)$ belongs to $S$ and its numerical range satisfies the inequality

$$|w(p)| \le 1$$

for all $x$ and $\xi$. Then the scheme is stable.

<u>Proof</u>: According to Lemma 3.3, it follows from $|w(p)| \le 1$ that

$$|w(P_h)| \le 1 + Kh \quad .$$

We need the following theorem of Berger [1] :

"If an operator $A$ satisfies $|w(A)| \le r$, then $|w(A^n)| \le r^n$."

Applying this to the operator $A = P_h$ with $r = 1 + Kh$, we have

$$|w(P_h^n)| \le (1 + Kh)^n \le e^{Khn} \quad .$$

We also use the following observation:

"If an operator $B$ satisfies $|w(B)| \le d$, then $\|B\| \le 2d$ ".

Applying this result to $B = P_h^n$ with $d = e^{Khn}$, we conclude that

$$\|P_h^n\| \le 2e^{Khn} \quad .$$

Since $P_h$ is independent of $t$, the operator relating $u(nh)$ to $u(0)$ is $P_h^n$. Denoting $nh$ by $T$ we find therefore from $\| P_h^n \| \le 2e^{Khn}$ that

$$\|u(T)\| \le 2e^{KT} \|u(0)\| ,$$

which is stability as defined in the first stability theorem above.

## CONCLUSION

In this work the main theorems of the algebra of difference operators have been extended to the class of bounded pseudo-difference operators which are defined by symbols of Hörmander's type $\rho,\delta$, for $0 \le \rho = \delta < 1$. Parallel results for pseudo-differential operators of type $\delta,\delta$ are still unknown; in fact for pseudo-differential operators of type $\rho,\delta$, $\delta < \rho$, the Reversor theorem and the Product-Difference theorem are obtained by means of asymptotic expansions with remainder term of lower order, but the remainder of such expansions is not of lower order if $\rho = \delta$.

As an application of the results of this work, one could derive in a simple way and extend the results obtained by Vaillancourt in [16] for symmetrizable difference schemes, thus removing the restriction that the coefficients $a_i(x)$ of the approximated differential operator $\Sigma a_i(x)\partial_{x_i}$ are constant for $|x| > R$ and reducing their differentiability to class $C^{2n}$.

Further applications will be possible when one will have found a way of approximating pseudo-differential operators by consistent pseudo-difference schemes. In [5] Frank announced that such approximations will be considered in a forthcoming paper. Krée has also studied these approximations in [10].

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