Explicit Structure and Data for Supercuspidal Representations of SO_5

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Abstract

This thesis presents an explicit classification of the structural data needed for the construction of supercuspidal representations for the split special orthogonal group $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ over a non-archimedean local field F. While general constructions of supercuspidal representations, such as that of Yu, are well established, their application to specific groups requires detailed knowledge of objects that are often not available in an explicit form.

We classify the elliptic maximal tori of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ and describe their embeddings into twisted Levi subgroups, which are themselves classified via both root-theoretic and Lie algebraic methods. We determine the non-regular elements in the Lie algebras of these tori and compute their centralizers, providing a second, independent verification of the twisted Levi subgroups. We further determine the Moy–Prasad filtrations of each toral Lie algebra and study the genericity conditions essential to Yu's construction. We establish precise criteria for so-called generic elements in terms of valuations and eigenvalue data, and determine the admissible sequences in the G-datum. We then develop explicit methods to determine the corresponding points in the Bruhat–Tits building by constructing explicit embeddings of each elliptic torus, which also allows us to compute the reductive quotients associated to relevant twisted Levi subgroup.

We present the results in a series of tables and diagrams that provide the necessary inputs for the construction of supercuspidal representations for ease of reference for practioners in the field. We also include a comparison with the corresponding data for Sp_4.

Dedications

To my parents,
for their enduring pride, trust, and inspiration

To my father,
whose love for mathematics lives on through me.
This thesis is as much his achievement as it is mine.

To my husband,
for his constant support, patience, and unwavering belief in me,
especially in moments when I doubted myself.

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Chapter 1

Introduction

The representation theory of reductive p-adic groups plays a central role in modern number theory and the Langlands program. Within the category of smooth complex representations of $G = \mathbb{G}(F)$, where $\mathbb{G}$ is a connected reductive group over a non-archimedean local field F, the supercuspidal representations occupy a distinguished position: they are the "atoms" from which all irreducible smooth representations are obtained via parabolic induction, and they cannot themselves be constructed in this way from any proper Levi subgroup. As such, the explicit construction and classification of supercuspidal representations has been a central problem in the field, with implications for automorphic forms, harmonic analysis, and the local Langlands correspondence.

In this thesis, we study the special orthogonal group SO_5 over a non-archimedean local field F of odd residual characteristic, focusing primarily on the split form, and explicitly solve the problem of constructing the structural data needed for the creation of irreducible supercuspidal representations.

Specifically, we produce the key components of G-data in Yu's construction of supercuspidal representations. Our inspiration and motivation for this work arose from a paper by Ju-Lee Kim and Jiang-Kang Yu on this question for Sp_4.

Historical context

Historically, the case of GL_n was the first to be understood in detail. Howe [How77] constructed supercuspidal representations of GL_n via compact induction, and this was later systematized in the influential theory of types developed by Bushnell and Kutzko [BK91], who provided a complete classification of all supercuspidal representations of GL_n for arbitrary n. Building on this progress, Yu introduced in 2001 a uniform construction of *tame* supercuspidal representations for arbitrary connected reductive groups [Yu01]. The key innovation is the notion of a *generic cuspidal G-datum* $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$, consisting of a twisted Levi sequence $\vec{\mathbb{G}} = (\mathbb{G}^0 \subset \cdots \subset \mathbb{G}^d = \mathbb{G})$ with compact center, a point y in the Bruhat–Tits building, a sequence of increasing depths, a depth-zero representation ρ, and a sequence of generic quasicharacters. From such a datum, Yu constructs via compact induction an irreducible supercuspidal representation $\pi_G(\Psi)$ of G. Yu's construction has since been widely used to study harmonic analysis on p-adic groups, including distinction problems, character formulas, and explicit realizations of instances of the local Langlands correspondence.

A major advance came with Kim's exhaustion theorem [Kim07, Thm 19.1], which showed that Yu's construction produces *all* supercuspidal representations under some hypotheses. A later proof by Fintzen [Fin21c] simplified these hypotheses to whenever G splits over a tamely ramified extension and the residual characteristic p of F does not divide the order of the Weyl group of G. For $G = \mathrm{SO}_5$, the Weyl group is the dihedral group of order 8, so this condition reduces to $p \neq 2$. These results guarantee that a complete understanding of the supercuspidal spectrum can be achieved by enumerating all admissible G-data. In a work that inspired our thesis, Kim and Yu [KY11] constructed all twisted Levi sequences for $G = \mathrm{Sp}_4$ and developed explicit types in that setting.

Beyond the exhaustion theorem, Kim and Murnaghan [KM03] developed essential tools including the notion of $\mathcal{T}$-good elements, depth functions relative to a torus, and a characterization of the Bruhat–Tits building of a torus as a subset of the building of G; these play a foundational role in our work. At the same time, doubts emerged regarding Yu's original proof of irreducibility: as pointed out by Spice, it relied on a misprinted statement of Gérardin [Gé77, Thm 2.4(b)], and the intertwining results of [Yu01, Prop 14.1 and Thm 14.2] were shown to be false in general. This issue was resolved by Fintzen [Fin21a, Thm 3.1], who gave a new proof that Yu's construction nevertheless yields irreducible supercuspidal representations.

In parallel to these general constructions, there was extensive work on constructing supercuspidal representations of *classical groups*. Notably, Stevens developed explicit models of supercuspidal representations of classical groups, including orthogonal and symplectic groups, by extending the methods of Bushnell and Kutzko [Ste08, BK91]. However, these works do not provide a complete explicit description of the underlying structures in small rank cases such as SO_5. While the abstract construction of supercuspidal representations is well understood, its application to a specific group requires detailed structural information: the conjugacy classes of elliptic maximal tori, the twisted Levi subgroups and how tori sit inside them, the classification of non-regular elements and their centralizers in the Lie algebra, the description of generic elements of a given depth, and the geometry of the Bruhat–Tits building of the group and its subgroups. These objects are essential ingredients in the G-datum and their construction is the subject of this thesis. As this thesis illustrates, their explicit determination is often nontrivial. It has not been fully developed in the literature for many small rank classical groups.

A summary of this thesis

This thesis focuses on the concrete case of the split special orthogonal group denoted $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. This group, while of rank two, exhibits rich structural features that make it an illuminating test case for the general theory. Its root system is of type B_2, giving rise to both long and short roots and a varied collection of twisted Levi subgroups $\mathcal{M} = \mathbb{M}(F)$, which include products of F-forms of $\mathrm{SO}_2 \times \mathrm{SO}_3$ as well as rank-two unitary groups. The classification of its elliptic maximal tori involves a subtle interplay between quadratic and quartic field extensions, and their associated involutions, and must further account for whether -1 is a square in F.

We now describe our main results in more detail.

Chapter 2 provides the necessary background on reductive groups over local fields and introduces the main objects used throughout the thesis, including quadratic and hermitian forms and their role in the structure of G and its Levi subgroups. We also recall the Bruhat–Tits building and Moy–Prasad filtrations, describe the embedding of twisted Levi subgroups in G, and fix notation for the subsequent chapters. In particular, if $\mathbb{T}$ is an algebraic torus defined over F, we write $\mathcal{T} = \mathbb{T}(F)$ for its group of F-points; thus, by a maximal elliptic torus $\mathcal{T}$ of G, we mean the group of F-points of a maximal elliptic algebraic torus $\mathbb{T}$. We apply similar conventions throughout.

In Chapter 3, we determine the elliptic maximal tori of G and of its relevant subgroups, using explicit constructions guided by the work of Morris [Mor91]. Morris established a bijection between conjugacy classes of maximal elliptic tori in a classical group and equivalence classes of triples $(\mathcal{A}, \Phi, f_{\mathcal{A}})$, consisting of a commutative semisimple algebra, an embedding, and a sesquilinear form. We apply this correspondence systematically to produce all conjugacy classes of maximal elliptic tori in each of the F-forms of SO_n for $n \leq 5$, as well as in the rank-two unitary groups $U_2(F)$. The algebras that arise include direct sums of quadratic extensions, biquadratic extensions, and quartic extensions with a unique quadratic subfield. We construct a unified parametrization that describes all conjugacy classes of elliptic maximal tori, treating simultaneously the cases where $-1 \in F^{\times 2}$ and where $-1 \notin F^{\times 2}$, while making explicit the differences between them. In particular, the number of conjugacy classes varies depending on whether -1 is a square in F. We develop explicit algorithms that produce representatives for each class, and we introduce a notation for the tori that encodes the key structural data of each torus and allows the reader to recognize its properties directly from its label. We carry out this parametrization in both the split and non-split settings, although the primary focus of this thesis is on the split group; the broader treatment provides a foundation for future work in the non-split case. We then identify how these tori embed into the twisted Levi subgroups of G.

The maximal elliptic tori are first constructed abstractly as subgroups of $\mathrm{SO}(q)$, where q is isometric to the split five dimensional quadratic form $2\mathbb{H} \oplus \langle 1 \rangle$, where $\mathbb{H}$ represents a hyperbolic plane. To our knowledge, the complete classification of maximal elliptic tori in $\mathrm{SO}_5(F)$ is not available in the literature, although partial results for $\mathrm{SO}_4(F)$ were obtained by Chinner [Chi21].

In Chapter 4, we classify the conjugacy classes of twisted Levi subgroups $\mathcal{M}$ of $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ using a Lie-theoretic approach. Given a maximal elliptic torus $\mathcal{T}$, we determine all one-dimensional subalgebras of its Lie algebra $\mathfrak{t} = \mathrm{Lie}(\mathcal{T}) = \mathfrak{t}(F)$ that consist entirely of non-regular elements, and compute their centralizers in G. Since the centralizer of a non-regular element is a twisted Levi subgroup containing the torus, this yields a complete description of these subgroups. We also observe that the number of conjugacy classes of twisted Levi subgroups remains the same whether $-1 \in F^{\times 2}$ or not.

We verify the completeness of this classification in two independent ways. First, we use the explicit description of twisted Levi subgroups obtained in Chapter 2 from the F-forms of Levi subgroups of SO_5 associated to the short and long roots. Second, we compute the centralizers of non-regular elements across all tori and show that they coincide with these subgroups. This comparison ensures that the classification is complete. We further support these classifications with carefully designed diagrams and tables, which provide a clear and organized representation of the relationships between the tori, their Lie algebras, and the associated twisted Levi subgroups. As part of this analysis, we find that certain elliptic maximal tori do not admit any non-regular elements, and consequently such tori are not embedded in any proper twisted Levi subgroup.

Next, in Chapter 5, we define and study the genericity conditions that are essential inputs to Yu's construction. The notion of a *G-generic element of depth $-r$* in the dual of the center of the Lie algebra of a twisted Levi subgroup is central to defining the allowed quasicharacters $\vec{\phi}$ in the G-datum. We use the Killing form to dualize this notion to the Lie algebra, and give precise definitions and explicit descriptions of G-generic and $\mathcal{M}$-generic elements of the Lie algebra of a torus in Section 5.1. We further relate these to the notion of $\mathcal{T}$-good elements introduced by Kim and Murnaghan [KM03].

We determine the Moy–Prasad filtrations of the Lie algebra of each elliptic torus, and for each torus and each depth r, we identify which elements are G-generic and which are only $\mathcal{M}$-generic for a proper twisted Levi subgroup $\mathcal{M}$. We identify phenomena that play a central role in the explicit construction for $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. In particular, some elliptic maximal tori have no nonzero non-regular elements in their Lie algebra. For such tori, every nonzero element of depth $-r$ is automatically G-generic. At the same time, there exist elliptic maximal tori which contain no G-generic elements at all. This restriction has direct consequences for the construction, as it limits the admissible sequences in Yu's framework: such tori can only appear in sequences of length three. To make these conditions explicit, we state and prove Lemma 5.3.1, which provides precise criteria for genericity in terms of valuations and eigenvalue data, and we present these criteria systematically in tables allowing direct verification in each case.

At this stage of the thesis, in Chapter 6, we turn to the Bruhat–Tits building. For each elliptic torus $\mathcal{T}$, the building $\mathcal{B}(\mathcal{T})$ is a single point, which must be identified as a point in $\mathcal{B}(G)$. Using a characterization due to Kim and Murnaghan [KM03, Thm 2.3.1], we develop systematic algorithms to determine this point explicitly. This requires constructing Witt bases for each torus, that is, finding explicit isometries between the quadratic form preserved by the torus and the standard form $2\mathbb{H} \oplus \langle 1 \rangle$, computing matrix realizations, and analyzing the Moy–Prasad filtration lattices $\mathfrak{g}_{y,r}$ at candidate points y to match the toral filtration with the ambient one. The construction of Witt bases involves a case-by-case analysis depending on the type of binary quadratic forms that arise and on whether -1 is a square in F. A key step in this computation is the identification of the precise quadratic form preserved by the torus, which allows us to locate the corresponding point in the building. For twisted Levi subgroups $\mathcal{M}$, we present two methods to determine the building: one indirect, using the tori they contain, and one direct, applying the same systematic inequalities restricted to the $\mathcal{T}$-good element X of depth r such that $C_G(X) = \mathcal{M}$.

We then determine the vertices y in the building of each twisted Levi subgroup, which may not be vertices of $\mathcal{B}(G)$. We also determine the reductive quotients $\mathcal{M}_{y,0}/\mathcal{M}_{y,0^+}$ at each such vertex by explicitly realizing each twisted Levi subgroup $\mathcal{M}$ inside $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ via matrix models and determining the quotient $\mathcal{M}(\mathcal{O})/\mathcal{M}(\mathcal{P})$, where $\mathcal{O}$ is the ring of integers of F and $\mathcal{P}$ its maximal ideal. This approach yields results that agree with those obtained in [GHY01, §3, §6], where the reductive quotients are derived via lattice-stabilizer arguments. The reductive quotients are necessary to enumerate the cuspidal representations ρ of $\mathcal{M}_{y,0}/\mathcal{M}_{y,0^+}$ needed for the G-datum. These representations can be obtained using Deligne–Lusztig theory [DL76] and we do not include this classification in this thesis.

In the final part of the thesis, Chapter 7, we assemble all the components of the generic cuspidal G-data and describe how they fit into Yu's construction of supercuspidal representations of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. We do not carry out the full construction, but sketch the main steps and instead focus on making all input data explicit.

We gathered the essential tables into Appendix A. In Appendix B, we carry out a detailed comparison with the influential work of Kim–Yu for Sp_4, whose root system is dual to that of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. While the two settings share the same Weyl group and admit analogous constructions, subtle differences arise in the structure of conjugacy classes, twisted Levi subgroups, and their parametrization. By translating their description into our setting, and presenting parallel tables and diagrams for elliptic tori, twisted Levi subgroups, and length-two twisted Levi sequences, we make these similarities and differences explicit. This comparison highlights both the robustness of the Kim–Yu framework and the distinct features that appear in the $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ case.

Contributions and future work

A distinguishing feature of this work is the explicit and systematic treatment of the structure of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. Throughout the thesis, we provide numerous tables and diagrams that give explicit computations for all the elements of the supercuspidal datum: classifications of tori and their parameters, matrix realizations, root data, Moy–Prasad filtration lattices, generic and non-regular elements, Bruhat–Tits building coordinates, and reductive quotients. These computations are carried out independently from multiple directions and serve both to corroborate the results and to make the structure of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ concrete. In particular, our independent enumeration of all non-regular elements and their centralizers in Chapter 4 confirms that the list of twisted Levi subgroups identified in Chapter 2 is complete. The embeddings of tori from lower-rank groups into SO_5 are verified by comparing with the results obtained via Morris' bijection, the tables of generic elements and filtrations are cross-checked to ensure consistency, and the building points computed via our algorithms are validated against filtration data from independent sources. Kim and Murnaghan's work [KM03] plays an essential role throughout, providing the theoretical foundation for the characterization of buildings, the notion of $\mathcal{T}$-good elements, and the relationship between depth and genericity that underpins the construction.

There are several possible directions for future work. One is to extend the explicit analysis carried out here to the non-split form, which we identify in this thesis with $\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, -\varpi, \varepsilon\varpi \rangle)$, where additional complications arise from the underlying quadratic forms. Another direction is to apply the explicit and systematic methods developed in this thesis to other rank two groups, with the aim of obtaining uniform and comparable descriptions of their structural data. It would also be of interest to make the connection with representation theory more explicit, in particular by relating the supercuspidal representations of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ to those of Sp_4, using the fact that Sp_4 is isomorphic to a double cover of SO_5. Finally, the explicit data developed in this thesis could be used to carry out the full construction of supercuspidal representations and to study their properties in concrete terms.

Chapter 2

Preliminaries

In this chapter, we present the general background and foundational material that will be used throughout the thesis. We begin with properties of local non-archimedean fields in Section 2.1, then quadratic forms in Section 2.2 and Hermitian forms in Section 2.3, before delving into some basic structure theory of algebraic groups in Section 2.4, then the Bruhat–Tits building of a p-adic group in Section 2.5 and finally some representation theory in Section 2.6. Readers familiar with these topics are invited to look at the tables and examples to see the notation and basic conventions that will be used in subsequent chapters.

2.1 p-adic fields and basic notation

Throughout this thesis, let F be a non-archimedean local field of odd residual characteristic, and $\overline{F}$, a fixed separable algebraic closure of F. Fix the (discrete) valuation

$$\nu_F : F^\times \to \mathbb{Z}, \qquad \nu_F(\varpi) = 1,$$

for a chosen uniformizer $\varpi \in F$. Set

$$\mathcal{O}_F = \{x \in F : \ \nu_F(x) \geq 0\}, \quad \text{the ring of integers of } F,$$

with maximal ideal $\mathcal{P}_F = (\varpi)$, and residue field $\mathfrak{f} = \mathbb{F}_q = \mathcal{O}_F/\mathcal{P}_F$, with cardinality $q = |\mathbb{F}_q|$ and characteristic p.

Extend the valuation by setting $\nu_F(0) = +\infty$, and write the non-archimedean absolute value of x as $|x| = q^{-\nu_F(x)}$ for $x \in F^\times$. Write $\mathcal{O}_F^\times$ for the unit group of $\mathcal{O}_F$, and $U_F^n := 1 + \mathcal{P}_F^n$ for $n \geq 1$. This gives the nth filtration of the unit group; by convention $U_F^0 = \mathcal{O}_F^\times$. Every $x \in F^\times$ decomposes uniquely as $x = u\varpi^{\nu_F(x)}$ with $u \in \mathcal{O}_F^\times$, so $F^\times \cong \mathcal{O}_F^\times \times \mathbb{Z}$.

For brevity, we drop field subscripts and write $\mathcal{O}$, $\mathcal{P}$, ν, and U^n whenever the base field is clear; subscripts will be restored when working over an extension of F.

The following fundamental result in p-adic analysis allows one to lift approximate roots of polynomials to exact roots; see [Con09, Theorem 2.1].

Lemma 2.1.1 (Hensel's Lemma). *Let $f(x) \in \mathcal{O}_F[x]$ and $a \in \mathcal{O}_F$ satisfy $f(a) \in \mathcal{P}_F$ and $f'(a) \notin \mathcal{P}_F$. Then there exists a unique $a_0 \in \mathcal{O}_F$ such that $f(a_0) = 0$ and $a_0 - a \in \mathcal{P}_F$.*

2.1.1 Finite extensions, ramification, and valuation normalization

Let E/F be a finite separable extension of degree n. Write $\mathcal{O}_E$ for the valuation ring of E, $\mathcal{P}_E$ for its maximal ideal, and $\mathfrak{e} = \mathcal{O}_E/\mathcal{P}_E$ for its residue field. Following [FV02, §2] and [Gou20], the *ramification index* $e(E/F) \in \mathbb{Z}_{\geq 1}$ and *residue degree* (also called *inertial degree*) $f(E/F) \in \mathbb{Z}_{\geq 1}$ are defined by

$$\mathcal{P}_F\mathcal{O}_E = \mathcal{P}_E^{e(E/F)}, \qquad [\mathfrak{e} : \mathfrak{f}] = f(E/F),$$

and they satisfy $[E : F] = e(E/F)\, f(E/F) = n$.

We say that E/F is: *unramified* if $e(E/F) = 1$; *totally ramified* if $f(E/F) = 1$; *partially ramified* if $1 < e(E/F) < n$ and $1 < f(E/F) < n$; and *tamely ramified* if $p \nmid e(E/F)$.

Two maps play a fundamental role in relating E and F: the *trace* and *norm* maps.

Definition 2.1.2. Let E/F be a finite separable field extension. The **trace map** and the **norm map** from E to F are defined by

$$\mathrm{Tr}_{E/F}(x) = \sum_{\sigma : E \hookrightarrow \overline{F}} \sigma(x), \qquad N_{E/F}(x) = \prod_{\sigma : E \hookrightarrow \overline{F}} \sigma(x),$$

where σ ranges over $\mathrm{Hom}_F(E, \overline{F})$.

To normalize the valuation on E we define $\nu_E : E^\times \longrightarrow \mathbb{Q}$ by

$$\nu_E(x) := \frac{1}{[E : F]}\, \nu_F\big(N_{E/F}(x)\big),$$

for all $x \in E^\times$, as in [Gou20, Thm. 5.3.5]. Then $\nu_E|_{F^\times} = \nu_F$ and $\nu_E(E^\times) = \frac{1}{e(E/F)}\,\mathbb{Z}$. We will always normalize our valuations in this way and so may omit the subscript, writing simply ν.

Choose a uniformizer ϖ_E of E; then $\nu_E(\varpi_E) = \frac{1}{e(E/F)}$. It follows that there exists a unit $u \in \mathcal{O}_E^\times$ such that $\varpi = u\, \varpi_E^{e(E/F)}$. In particular, in an unramified extension, we may choose $\varpi_E = \varpi$.

2.1.2 Quadratic extensions

Write $F^{\times 2}$ for the group $\{x^2 \mid x \in F^\times\}$. Let $\varepsilon \in \mathcal{O}_F^\times \smallsetminus \mathcal{O}_F^{\times 2}$; by Hensel's Lemma 2.1.1, this is equivalent to requiring its image in $\mathbb{F}_q^\times$ to be a nonsquare.

When $-1 \notin F^{\times 2}$ (which we often denote $-1^{\boxtimes}$) we choose $\varepsilon = -1$, but when $-1 \in F^{\times}$ (often abbreviated $-1^{\square}$) there is no standard choice. We write i for $\sqrt{-1}$.

The following result follows from the decomposition $F^{\times} \cong \mathcal{O}_F^{\times} \times \mathbb{Z}$, see Neukirch [Neu99, Prop. 5.3], for example.

Proposition 2.1.3. *Let F be a non-archimedean local field with residue field $\mathbb{F}_q$ of odd cardinality q. Let ϖ be a uniformizer of F, and choose a unit $\varepsilon \in \mathcal{O}_F^{\times}$ whose residue $\bar{\varepsilon} \in \mathbb{F}_q^{\times}$ is a nonsquare. Then*

$$F^{\times}/F^{\times 2} \;\cong\; (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}),$$

with representatives $\{\,1,\ \varepsilon,\ \varpi,\ \varepsilon\varpi\,\}$.

Since $\mathrm{char}(F) \neq 2$, every quadratic extension is of the form $E = F(\sqrt{\alpha})$ for some $\alpha \in F^{\times} \setminus F^{\times 2}$, and is uniquely determined by the class of α modulo squares. It follows that there are three quadratic extension fields of F, given by $E = F(\sqrt{\alpha})$ for some $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$. The one corresponding to $\alpha = \varepsilon$ is unramified but the other two are ramified.

The following lemma summarizes facts about norm groups of quadratic extensions from [Chi21, Prop. 2.3.8, Lemma. 2.4.3, Prop. 2.4.4]).

Lemma 2.1.4 (Norms from a quadratic extension)**.** *Let E/F be a quadratic extension of non-archimedean local fields with odd residue characteristic. Then:*

1. *The image of the norm map $N_{E/F} : E^{\times} \to F^{\times}$ is a multiplicative subgroup of $F^{\times}$ containing $F^{\times 2}$.*
2. *If E/F is unramified, $N_{E/F}$ maps $\mathcal{O}_E^{\times}$ surjectively onto $\mathcal{O}_F^{\times}$; in particular, $-1 \in N_{E/F}(E^{\times})$.*
3. *If E/F is ramified, then $-1 \in N_{E/F}(E^{\times})$ if and only if $-1 \in F^{\times 2}$.*
4. *Representatives for the quotient $F^{\times}/N_{E/F}(E^{\times})$ are given by*

$$\{1, \varepsilon\} \quad \text{if } E/F \text{ is ramified}, \qquad \{1, \varpi\} \quad \text{if } E/F \text{ is unramified},$$

where $\varepsilon \in \mathcal{O}_F^{\times}$ is any non-square unit and ϖ a uniformizer of F.

2.1.3 Quartic extensions

We now introduce quartic extensions of F, that is, of degree $n = 4$. Since $n = ef$, we have 3 cases: unramified ($e = 1, f = 4$), partially ramified ($e = 2, f = 2$), and totally ramified ($e = 4, f = 1$). Each one is obtained as a quadratic extension of a quadratic extension.

If $E = F(\sqrt{\varpi'})$, with $\varpi' \in \{\varpi, \varepsilon\varpi\}$, is a ramified quadratic extension, then by Proposition 2.1.3, a set of representatives for $E^\times/E^{\times 2}$ is given by

$$\{1, \varepsilon, \sqrt{\varpi'}, \varepsilon\sqrt{\varpi'}\}.$$

In the unramified case $E = F(\sqrt{\varepsilon})$, however, the classes are represented by

$$\{1, \varepsilon_E, \varpi, \varepsilon_E\varpi\},$$

for some nonsquare $\varepsilon_E \in \mathcal{O}_E^\times$. We make distinguished choices, as follows.

Lemma 2.1.5. *Let $E = F(\sqrt{\varepsilon})$ denote the unramified quadratic extension of F. If $-1 \in F^{\times 2}$, then $\varepsilon_E := \sqrt{\varepsilon}$ is a nonsquare unit. If $-1 \notin F^{\times 2}$, then we may choose $\varepsilon = -1$ and, setting $i = \sqrt{-1}$, there exist $s, t \in \mathcal{O}_F$ such that*

$$\varepsilon_E = s^2 + it^2$$

satisfies $N_{E/F}(\varepsilon_E) = s^4 + t^4 = -1$ and ε_E is a nonsquare unit.

Proof. Note that by Hensel's Lemma 2.1.1, this question reduces to the finite residue field $\mathfrak{e}$ of E, which is a quadratic extension of $\mathfrak{f} = \mathbb{F}_q$ and has order q^2. Let θ be a generator of $\mathfrak{e}^\times$; then θ^{q+1} generates $\mathfrak{f}^\times$. The nonsquares are the odd powers of the generator. Thus ε is (some lift to $\mathcal{O}_F$ of) $\theta^{(q+1)k}$ for some odd k.

The square root $\sqrt{\varepsilon} = \theta^{(q+1)k/2}$ is again a nonsquare if and only if $q \equiv 1 \mod 4$, that is, if and only if $-1 \in F^{\times 2}$, so we may choose $\varepsilon_E = \sqrt{\varepsilon}$ in this case. Otherwise, we choose $\varepsilon = -1 = \theta^{(q+1)k}$ where $k = (q-1)/2$ and set $\varepsilon_E = \theta^k$, so that $N_{\mathfrak{e}/\mathfrak{f}}(\theta^k) = \theta^k\theta^{qk} = \varepsilon = -1$. By [Chi21, Lem. 2.4.7], there exist $s, t \in \mathcal{O}_F$ such that ε_E has the given convenient form. ∎

The classification of quartic extensions up to isomorphism is provided in [Chi21, Section 2.4]. We reproduce the result here, in Tables 2.1 (for the case that -1 is a square) and 2.2 (when it is not). For each quartic extension field E, we identify its ramification over F and give an explicit realization of E and of all of its quadratic subfields E_1 in terms of our preferred set of representatives from Proposition 2.1.3 and Lemma 2.1.5. Note that each quartic field is either biquadratic over F (in which case it is $F(\sqrt{\varepsilon}, \sqrt{\varpi})$) or contains a unique quadratic subfield.

2.2 Quadratic forms and special orthogonal groups

Given F as in Section 2.1, let us in this section classify symmetric bilinear forms and their associated quadratic forms, focusing particularly on those of small dimension. We follow [Sch85, §2, §7], [Kit93, Chapter 1, Chapter 3] and [GHY01, §3,§6] for the classification of quadratic and Hermitian forms.

Ramification	**Quartic field**	**Quadratic subfields**	**Aut(E/F)**
$e(E/F)=1$	$F(\sqrt[4]{\varepsilon})$	$F(\sqrt{\varepsilon})$	Galois, $\mathbb{Z}/4\mathbb{Z}$
$e(E/F)=2$	$F(\sqrt{\varepsilon},\sqrt{\varpi})$	$F(\sqrt{\varepsilon})$, $F(\sqrt{\varpi})$, $F(\sqrt{\varepsilon\varpi})$	Galois, $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
	$F(\sqrt[4]{\varepsilon\varpi^2})$	$F(\sqrt{\varepsilon})$	Galois, $\mathbb{Z}/4\mathbb{Z}$
$e(E/F)=4$	$F(\sqrt[4]{\varpi})$	$F(\sqrt{\varpi})$	Galois, $\mathbb{Z}/4\mathbb{Z}$
	$F(\sqrt[4]{\varepsilon\varpi})$	$F(\sqrt{\varepsilon\varpi})$	
	$F(\sqrt[4]{\varepsilon^2\varpi})$	$F(\sqrt{\varpi})$	
	$F(\sqrt[4]{\varepsilon^3\varpi})$	$F(\sqrt{\varepsilon\varpi})$	

Table 2.1: *Quartic extensions of F up to isomorphism when $-1\in F^{\times 2}$.*

2.2.1 Hyperbolic planes

Let $(,)$ denote a nondegenerate symmetric bilinear form on an F-vector space V, and denote the associated quadratic form by

$$q(x)=(x,x).$$

A *quadratic form* is a map $q:V\longrightarrow F$ satisfying

$$q(av)=a^2q(v),\quad \forall a\in F,\ v\in V.$$

The pair (V,q) constitutes a quadratic space. Since $\mathrm{char}F\neq 2$, the symmetric bilinear form can be uniquely recovered from q by the polarization identity

$$(v,w)=\tfrac{1}{2}\big(q(v+w)-q(v)-q(w)\big).$$

Definition 2.2.1. Two quadratic forms q_1 and q_2 on a vector space V are said to be **isometric** if there exists $g\in\mathrm{GL}(V)$ such that $q_2(x)=q_1(gx)$ for all $x\in V$. In this case we write $q_2\simeq q_1$.

In other words, q_1 and q_2 are isometric if they become equal after a change of basis. Every quadratic form is isometric to a diagonal quadratic form, which we write as

$$q\simeq\langle a_1,a_2,\dots,a_n\rangle\simeq\langle a_1\rangle\oplus\langle a_2\rangle\oplus\cdots\oplus\langle a_n\rangle,$$

where $\langle a_i\rangle$ denotes the one-dimensional form $x\mapsto a_ix^2$. We define the dimension of a quadratic form to be the dimension of its corresponding quadratic space.

Ramification	**Quartic field**	**Quadratic Subfields**	**Aut**(E/F)
$e(E/F)=1$	$E \simeq F(\sqrt{\varepsilon_E})$	$E \simeq F(i)$	Galois, $\mathbb{Z}/4\mathbb{Z}$
$e(E/F)=2$	$F(i,\sqrt{\varpi})$	$F(i),\ F(\sqrt{\varpi}),\ F(i\sqrt{\varpi})$	Galois, $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
	$F(\sqrt{\varepsilon_E\varpi})$	$E \simeq F(i)$	Galois, $\mathbb{Z}/4\mathbb{Z}$
$e(E/F)=4$	$F(\sqrt[4]{\varpi})$	$F(\sqrt{\varpi})$	Non-Galois, $\mathbb{Z}/2\mathbb{Z}$
	$F(\sqrt[4]{-\varpi})$	$F(i\sqrt{\varpi})$	

Table 2.2: *Quartic extensions of F up to isomorphism when $-1 \notin F^{\times 2}$.*

Definition 2.2.2. A **hyperbolic plane** $\mathbb{H}$ is a 2-dimensional quadratic space $(V, q_{\mathbb{H}})$ admitting a basis $\{e, f\}$, called a Witt basis for $\mathbb{H}$, such that

$$q_{\mathbb{H}}(e) = q_{\mathbb{H}}(f) = 0, \qquad (e, f) = 1,$$

where $(,)$ is the symmetric form associated to $q_{\mathbb{H}}$. That is, for all $x_1, x_2 \in F$, we have $q_{\mathbb{H}}(x_1 e + x_2 f) = 2x_1x_2$.

A quadratic space is called *hyperbolic* if it is isometric to a direct sum of hyperbolic planes.

Lemma 2.2.3. *Let q be the diagonal quadratic form $\langle 1, -1\rangle$ on F^2. Then q is isometric to $q_{\mathbb{H}}$, and thus represents a hyperbolic plane.*

Proof. Let $q(x_1, x_2) = x_1^2 - x_2^2$ on F^2, and let $g = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \in \mathrm{GL}_2(F)$. For $v = (x, y) \in F^2$ we have

$$gv = \left(x + \tfrac{1}{2}y,\ x - \tfrac{1}{2}y\right),$$

and hence

$$q(gv) = \left(x + \tfrac{1}{2}y\right)^2 - \left(x - \tfrac{1}{2}y\right)^2 = 2xy = q_{\mathbb{H}}(v).$$

Therefore $q \simeq q_{\mathbb{H}}$. ∎

In what follows, we write briefly $q \simeq \mathbb{H}$ to mean that q is isometric to the hyperbolic plane. Lemma 2.2.4 records a basic but useful fact about hyperbolic planes under scalar multiplication and isometry.

Lemma 2.2.4. *Suppose $q = \langle \lambda, -\lambda \rangle$ with respect to a basis $\{u, w\}$ of V. Then $q \simeq \mathbb{H}$ where the isometry is given by choosing the Witt basis*

$$e := u + w, \qquad f := \tfrac{1}{2}\,\lambda^{-1}(u - w).$$

Proof. We evaluate the associated symmetric bilinear form directly. Since $(u, u) = \lambda$, $(w, w) = -\lambda$ and $(u, w) = 0$, we have $(u \pm w, u \pm w) = 0$ and $(u + w, u - w) = 2\lambda$. It follows that $q(e) = q(f) = 0$ and that $(e, f) = 1$. This change of basis thus defines an isometry $q \simeq \mathbb{H}$. ∎

Combining these lemmas shows that when $q \simeq \mathbb{H}$, then for any $\lambda \in F^\times$ we have $\lambda q \simeq \mathbb{H}$. We caution that not every quadratic form is invariant under scaling. For example, the forms $q_1(x) = x^2$ and $q_2(x) = \varepsilon x^2$ are not isometric since $q_1(F^\times) \neq q_2(F^\times)$.

We will require the explicit isometries in what follows; here is another.

Lemma 2.2.5. *Suppose $\{u, w\}$ is a basis for a quadratic space (q, V) such that for all $v_1, v_2 \in F$ and a fixed constant $\lambda \in F^\times$ we have*

$$q(v_1 u + v_2 w) = 2\lambda v_1 v_2.$$

Then $q \simeq \mathbb{H}$ and a Witt basis is given by

$$e := u, \qquad f := \lambda^{-1} w.$$

Proof. Indeed, we verify immediately that $q(e) = q(f) = 0$ and $(e, f) = 1$ is obtained by a direct application of the polarization identity. ∎

We collect in Table 2.3 several special binary quadratic forms that will arise in Chapters 3 and 6, together with an explicit change of basis that transforms the form to a target standard form. The first three were proven here; the remaining cases, corresponding to $-1 \notin F^{\times 2}$, were derived in [Chi21, §2,§4]. They use the choice of nonsquare unit $\varepsilon_E = s^2 + it^2 \in F(i)$ from Lemma 2.1.5, and one can verify them directly.

Quadratic Form	-1	target form	basis
$p_1^\lambda(v_1\mathbf{u}+v_2\mathbf{w}) = \lambda(v_1^2 - v_2^2)$	any	$\mathbb{H}$	$e = \mathbf{u}+\mathbf{w}$ $f = \frac{1}{2}\lambda^{-1}(\mathbf{u}-\mathbf{w})$
$p_2^\lambda(v_1\mathbf{u}+v_2\mathbf{w}) = 2\lambda v_1 v_2$	any	$\mathbb{H}$	$e = \mathbf{u}$ $f = \lambda^{-1}\mathbf{w}$
$p_3^\lambda(v_1\mathbf{u}+v_2\mathbf{w}) = \lambda\,(v_1^2 + v_2^2)$	$-1^\square$	$\mathbb{H}$	$e = \mathbf{u}+i\mathbf{w}$ $f = \frac{1}{2}\lambda^{-1}(\mathbf{u}-i\mathbf{w})$
$p_4^\lambda(v_1\mathbf{u}+v_2\mathbf{w}) = \lambda(v_1^2 + v_2^2)$	$-1^\boxtimes$	$\langle -\lambda, -\lambda\rangle$	$g_1 = s^2\mathbf{u}+t^2\mathbf{w}$ $g_2 = t^2\mathbf{u}-s^2\mathbf{w}$
$p_5^\lambda(v_1\mathbf{u}+v_2\mathbf{w}) = \lambda(-s^2v_1^2 + 2t^2v_1v_2 + s^2v_2^2)$	$-1^\boxtimes$	$\langle \lambda, \lambda\rangle$	$g_1 = s\mathbf{u}-t^2s^{-1}\mathbf{w}$ $g_2 = s^{-1}\mathbf{w}$
$p_6^\lambda(v_1\mathbf{u}+v_2\mathbf{w}) = \lambda((s^4-t^4)(v_2^2-v_1^2) + 4s^2t^2\,v_1v_2)$	$-1^\boxtimes$	$\mathbb{H}$	$e = (s^2-t^2)\mathbf{u}-(s^2+t^2)\mathbf{w}$ $f = -\frac{s^2+t^2}{2\lambda}\mathbf{u} - \frac{s^2-t^2}{2\lambda}\mathbf{w}$

Table 2.3: *Frequently occurring binary quadratic forms, with respect to a basis* $\mathbf{u}, \mathbf{w}$*, together with an explicit change of basis that transform them to the isometric target form, as reproduced from [Chi21, §2,§4]. Here* $\lambda \in F^\times$ *and* $i = \sqrt{-1}$*. When* $-1 \notin F^{\times 2}$*, indicated by* $-1^\boxtimes$*, then* s, t *are chosen as in Lemma 2.1.5 and when* $-1 \in F^{\times 2}$*, indicated* $-1^\square$*, we set* i *to be a square root of* -1*.*

Remark 2.2.6. In Table 2.3, the forms p_i^λ for $i = 1,2,3$ illustrate several common presentations of a hyperbolic plane: we refer to p_i^λ for $i = 1,3$ as **diagonal forms** (Lemma 2.2.4) and to p_2^λ as the **cross-term form** (Lemma 2.2.5).

In Chapter 6, the valuation of the scaling factors λ in the forms p_i^λ will end up being the crucial ingredient.

2.2.2 Witt Decomposition and Witt cancellation

We introduce Witt's fundamental results, which are the foundation of the algebraic theory of quadratic forms. Proofs can be found in standard references such as [Sch85, Theorem 1.11, Corollary 5.8, Corollary 5.11].

Theorem 2.2.7. ***(Witt's Cancellation Theorem)*** *Let* (V_1, q_1)*,* (V_2, q_2)*, and* (V, q) *be non-degenerate quadratic forms over a field* F *of char* $F \neq 2$*. Suppose*

$$(V_1, q_1) \oplus (V, q) \simeq (V_2, q_2) \oplus (V, q).$$

Then

$$(V_1, q_1) \simeq (V_2, q_2).$$

Definition 2.2.8. A vector $v \in V$ is called **isotropic** if $q(v) = 0$ and **anisotropic** otherwise. If (V, q) admits a nonzero isotropic vector, we say that q is isotropic; otherwise, it is anisotropic.

For example, the hyperbolic plane $\mathbb{H}$ is isotropic, but the quadratic form $q(x) = ax^2$ with $a \neq 0$ is anisotropic.

One application of Witt's cancellation theorem is the following result.

Theorem 2.2.9. ***(Witt Decomposition).*** *Let (V, q) be a non-degenerate quadratic form. Then there is a decomposition*

$$(V, q) = (V_1, q_1) \oplus (V_2, q_2)$$

where $q_1 = q|_{V_1}$ is anisotropic and $q_2 = q|_{V_2}$ is hyperbolic. That is, any non-degenerate quadratic form q admits a unique decomposition

$$q \simeq q_{an} \oplus n\mathbb{H},$$

with q_{an} anisotropic, and $n\mathbb{H}$ denotes a sum of n hyperbolic planes.

We call q_{an} the **anisotropic kernel** of q, and the natural number n the **Witt index** of V which represents half the dimension of any maximal totally isotropic subspace of V. A non-degenerate quadratic form q over F is called **split** if it has maximal Witt index. Equivalently, if $\dim q = 2n$, then $q \simeq n\mathbb{H}$, while if $\dim q = 2n + 1$, then $q \simeq n\mathbb{H} \oplus \langle a \rangle$ for some $a \in F^\times$.

Finding these subspaces q_{an} and $n\mathbb{H}$ is constructive. When $x \in V$ is an isotropic vector with respect to q, then one can find a hyperbolic plane in V such that $V = \mathbb{H} \oplus W$ is an orthogonal decomposition. Repeating this process leads to the Witt decomposition of (q, V) and the following definition.

Definition 2.2.10 (Witt basis)**.** Let (V, q) be a non-degenerate quadratic space over F with associated symmetric bilinear form $(\cdot, \cdot)$. An ordered basis

$$\{e_1, \ldots, e_n, f_1, \ldots, f_n, g_1, \ldots, g_k\}$$

of V is called a **Witt basis** if:

- $(e_i, f_j) = \delta_{ij}$ for all i, j,
- $(e_i, e_j) = (f_i, f_j) = 0$ for all i, j,
- $(g_\ell, e_i) = (g_\ell, f_i) = 0$ for all i, ℓ,
- $(g_i, g_j) = \delta_{ij}$ for all i, j, and
- $\langle g_1, \ldots, g_k \rangle$ is anisotropic.

Note that with respect to a Witt basis, each hyperbolic block appears in cross-term form. For example, in dimension 5 the form $2\mathbb{H} \oplus \langle 1 \rangle$ is represented in coordinates with respect to such a Witt basis by the matrix

$$J = \left(\begin{array}{c|c|c} & I_2 & \\ \hline I_2 & & \\ \hline & & 1 \end{array}\right) \simeq 2\mathbb{H} \oplus \langle 1 \rangle. \tag{2.2.1}$$

Finally, let us introduce an equivalence relation on the set of quadratic forms, whose classes form a group under direct sums.

Definition 2.2.11 (Witt group)**.** Let $\mathcal{Q}(F)$ be the set of isometry classes of non-degenerate quadratic forms over F. Define an equivalence relation $\sim$ on $\mathcal{Q}(F)$ by

$$q_1 \sim q_2 \quad \Longleftrightarrow \quad q_1 \oplus r\,\mathbb{H} \cong q_2 \oplus s\,\mathbb{H} \quad \text{for some } r, s \in \mathbb{N},$$

where $\mathbb{H}$ denotes the hyperbolic plane. The **Witt group** $W(F)$ is the set of equivalence classes

$$W(F) = \mathcal{Q}(F)/\sim,$$

with addition induced by orthogonal sum:

$$[q_1] + [q_2] := [\, q_1 \oplus q_2 \,].$$

This operation is well-defined, commutative, and associative; the zero element is the class of any hyperbolic form (in particular $[\mathbb{H}] = 0$), and the inverse of $[q]$ is $[-q]$.

2.2.3 Invariants of quadratic forms

Quadratic spaces over non-archimedean local fields can be classified by three fundamental invariants: the dimension, the discriminant and the Hasse invariant. We present these and then use them to identify isometries between various pairs of forms required in this thesis. We refer to [Sch85, Chapter 2, Chapter 7] and [Kit93] for the general theory, and to Gan–Hanke–Yu [GHY01] for the formulation and notation adopted here.

Let (V, q) be a non-degenerate quadratic space over F. Its *dimension* is $\dim(V)$. If $q \simeq \langle a_1, \cdots, a_n \rangle$ then its *discriminant* or determinant is the square class of $\delta(q) = \prod a_i \in F^\times/F^{\times 2}$. To define the Hasse invariant we require the Hilbert symbol.

Definition 2.2.12 (Hilbert symbol)**.** For $a, b \in F^\times$, the **Hilbert symbol** $(a, b)_F$ is defined by

$$(a,b)_F = \begin{cases} 1, & \text{if the equation } ax^2 + by^2 = z^2 \text{ has a nontrivial solution in } F^3, \\ -1, & \text{otherwise.} \end{cases}$$

It is symmetric, with values in $\{\pm 1\}$, and bimultiplicative on $F^\times/F^{\times^2}$; that is,

$$(ab, c)_F = (a, c)_F (b, c)_F, \quad (a, bc)_F = (a, b)_F (a, c)_F.$$

The Hilbert symbol for a p-adic field is computed in Table 2.4. Note that the result depends on whether -1 is a square or not.

	1	ε	ϖ	$\varepsilon\varpi$
1	1	1	1	1
ε	1	1	-1	-1
ϖ	1	-1	1	-1
$\varepsilon\varpi$	1	-1	-1	1

(a) *Hilbert symbol when* $-1 \in F^{\times 2}$

	1	ε	ϖ	$\varepsilon\varpi$
1	1	1	1	1
ε	1	1	-1	-1
ϖ	1	-1	-1	1
$\varepsilon\varpi$	1	-1	1	-1

(b) *Hilbert symbol when* $-1 \notin F^{\times 2}$

Table 2.4: *Values of the Hilbert symbol*

The **Hasse invariant** of $q = \langle a_1, a_2, \dots, a_n \rangle$ is given by

$$\epsilon_H(q) = \prod_{1 \le i < j \le n} (a_i, a_j)_F \in \{\pm 1\}.$$

Among its properties, let us record that the Hasse invariant satisfies the *orthogonal-sum identity* over non-archimedean fields (see [Lam05, §V], [Kit93, Theorem 3.4.2]). That is, for any two quadratic forms q, q' over F, the Hasse invariant of their orthogonal sum satisfies

$$\epsilon_H(q \oplus q') = \epsilon_H(q)\, \epsilon_H(q')\, (\delta(q), \delta(q'))_F. \tag{2.2.2}$$

Note that the hyperbolic quadratic space $n\mathbb{H}$ always has Hasse invariant 1.

By [Kit93, Theorem 3.5.2], the isometry class of q is determined by the triple

$$(\dim(V), \delta(q), \epsilon_H(q)).$$

Since δ takes values in $F^\times/F^{\times 2}$, we can assume that $\delta(q) \in \{1, \varepsilon, \varpi, \varepsilon\varpi\}$. Since for each dimension m there are exactly 8 possible pairs (δ, ϵ_H), there are at most eight isometry classes of quadratic forms of dimension m. However, in dimension 1 there are only 4 isometry classes and in dimension 2, there are seven, as one can determine by enumerating all possible diagonal forms and computing their invariants.

We now classify some key forms up to isometry.

Lemma 2.2.13. *Let $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ and $\mu \in F^\times$. Set $E_\alpha = F(\sqrt{\alpha})$. Then $\epsilon_H(\langle \mu, -\mu\alpha \rangle) = 1$ if and only if $\mu \in N_{E_\alpha/F}(E_\alpha^\times)$. In particular, for $\mu_1, \mu_2 \in F^\times$ we have*

$$\langle \mu_1, -\mu_1\alpha \rangle \simeq \langle \mu_2, -\mu_2\alpha \rangle$$

if and only if $\mu_1/\mu_2 \in N_{E_\alpha/F}(E^\times)$.

Proof. Note that since $\mu x^2 - \mu y^2 = 0$ has many nontrivial solutions in $F^\times$, we know $(\mu, -\mu)_F = 1$. Therefore using the bilinearity of the Hilbert symbol, we compute

$$\epsilon_H(\langle \mu, -\mu\alpha \rangle) = (\mu, -\mu)_F (\mu, \alpha)_F = (\mu, \alpha)_F.$$

Now $\mu x^2 + \alpha y^2 = z^2$ can be rewritten $\mu x^2 = z^2 - \alpha y^2$, which expresses μx^2 as the norm of an element of $E = F(\sqrt{\alpha})$. Therefore $(\mu, \alpha)_F = 1$ if and only if $\mu \in N_{E_\alpha/F}(E_\alpha^\times)$.

The two forms $\langle \mu_1, -\mu_1\alpha\rangle$ and $\langle \mu_2, -\mu_2\alpha\rangle$ have the same dimension, and the same discriminant since

$$\delta(\langle \mu_i, -\mu_i\alpha\rangle) = -\mu_i^2\alpha \equiv -\alpha \quad \text{in } F^\times/F^{\times 2}.$$

Hence it suffices to compare their Hasse invariants, which are 1 if $\mu_i \in N_{E_\alpha/F}(E_\alpha^\times)$ and -1 otherwise. Since $|F^\times/N_{E_\alpha/F}(E_\alpha^\times)| = 2$, we obtain the same coset if and only if $\mu_1/\mu_2 \in N_{E_\alpha/F}(E_\alpha^\times)$. ∎

Let us abbreviate $E_{\alpha,\mu} = \langle \mu, -\mu\alpha\rangle$. As a consequence of Lemma 2.2.13, we can compute the invariants of all possible $E_{\alpha,\mu}$, as in Table 2.5..

Extension E_α/F	Quadratic space $E_{\alpha,\mu}$	δ	μ	ϵ_H
unramified	$\langle 1, -\varepsilon\rangle$	$-\varepsilon$	1	1
unramified	$\langle \varpi, -\varepsilon\varpi\rangle$	$-\varepsilon$	ϖ	-1
ramified	$\langle 1, -\varpi\rangle$	$-\varpi$	1	1
ramified	$\langle \varepsilon, -\varepsilon\varpi\rangle$	$-\varpi$	ε	-1
ramified	$\langle 1, -\varepsilon\varpi\rangle$	$-\varepsilon\varpi$	1	1
ramified	$\langle \varepsilon, -\varpi\rangle$	$-\varepsilon\varpi$	ε	-1

Table 2.5: *Two-dimensional anisotropic quadratic spaces $E_{\alpha,\mu} = \langle \mu, -\mu\alpha\rangle$ up to isometry, together with their invariants.*

We present the classification of quadratic forms of dimensions $2 \leq m \leq 5$ in Table 2.6. We infer from this table that in dimension 2 the six classes $E_{\beta,\mu}$ are all the anisotropic forms, up to isometry. In dimension 3, using the invariants $(\delta(q), \epsilon_H(q))$, there are four isotropic forms $\mathbb{H} \oplus \langle \delta\rangle$ and we introduce the notation W_δ to denote the three-dimensional anisotropic quadratic space that has the same discriminant and opposite Hasse invariant of $\mathbb{H} \oplus \langle \delta\rangle$. In dimension 4, $q_{4\mathrm{an}} = \langle 1, -\varepsilon, \varpi, -\varepsilon\varpi\rangle$ is the unique anisotropic class, while for dimensions ≥ 5 all forms are isotropic.

q	δ	ϵ_H
Dimension 2		
$E_{\alpha,1} = \langle 1, -\alpha\rangle$	$-\alpha$	1
$E_{\alpha,\mu} = \langle \mu, -\mu\alpha\rangle$	$-\alpha$	-1
$\mathbb{H} \simeq \langle 1, -1\rangle$	-1	1
Dimension 3		
$\mathbb{H} \oplus \langle \beta\rangle$	$-\beta$	1
$W_1 \simeq \langle \varepsilon, -\varpi, \varepsilon\varpi\rangle$	-1	-1
$W_\varepsilon \simeq \langle 1, \varpi, -\varepsilon\varpi\rangle$	$-\varepsilon$	-1
$W_\varpi \simeq \langle 1, -\varepsilon, \varepsilon\varpi\rangle$	$-\varpi$	-1
$W_{\varepsilon\varpi} \simeq \langle 1, -\varepsilon, \varpi\rangle$	$-\varepsilon\varpi$	-1

q	δ	ϵ_H
Dimension 4		
$2\mathbb{H}$	1	1
$\mathbb{H} \oplus \langle 1, -\alpha\rangle$	α	1
$\mathbb{H} \oplus \langle \mu, -\mu\alpha\rangle$	α	-1
$\langle 1, -\varepsilon, \varpi, -\varepsilon\varpi\rangle \simeq q_{4\text{an}}$	1	-1
Dimension 5		
$2\mathbb{H} \oplus \langle \beta\rangle$	β	1
$\mathbb{H} \oplus W_\beta$	β	-1

Table 2.6: *A list of quadratic forms q up to isometry and their invariants, for each dimension from 2 to 5, using unified parameters so that the expressions are the same regardless of whether or not -1 is a square. Here the parameter α runs over the set $\{\varepsilon, \varpi, \varepsilon\varpi\}$ while β runs over the set $\{1, \varepsilon, \varpi, \varepsilon\varpi\}$, and μ represents the nontrivial coset of $F^\times/N_{F(\sqrt{\alpha})/F)}(F(\sqrt{\alpha})^\times)$.*

2.2.4 q-partners

In this subsection, we introduce the notion of a q-partner, a concept proposed here to streamline the classification of quadratic forms. To the best of the author's knowledge, this terminology and the associated construction have not appeared in the existing literature.

Definition 2.2.14. Let q be a non-degenerate quadratic form over F of dimension at least 2. We say that a quadratic form $q^\triangledown$ is a **q-partner** of q if it has the same dimension and discriminant as q, but opposite Hasse invariant.

It is easy to see that every form of dimension at least 2, except for $\mathbb{H}$, has a q-partner, and moreover, that when it exists it is unique up to isometry. Therefore we may speak of *the* q-partner of q to refer to (the isometry class of) $q^\triangledown$.

We can identify q and $q^\triangledown$ explicitly for dimensions between 2 and 5 from Table 2.6.

Let us record some properties of q-partners.

Theorem 2.2.15. *Let q_{4an} be a 4-dimensional anisotropic quadratic form over F. Let q be any nondegenerate quadratic form with $\dim q \geq 2$, such that $q \neq \mathbb{H}$. Then the following statements hold:*

1. $q^\triangledown \oplus 2\mathbb{H} \simeq q \oplus q_{4\text{an}}$.

2. *In the Witt ring $W(F)$, one has $[q^\triangledown] - [q] = [q_{4\mathrm{an}}] = [q] - [q^\triangledown]$.*

3. *The partner operation is involutive: $(q^\triangledown)^\triangledown = q$.*

4. *If q' is any quadratic form, then $(q \oplus q')^\triangledown \simeq q^\triangledown \oplus q'$.*

5. *If q is 3-dimensional then exactly one of $\{q, q^\triangledown\}$ is anisotropic.*

6. *If $q \simeq 2\mathbb{H} \oplus \langle 1 \rangle$ then $q^\triangledown \simeq \mathbb{H} \oplus W_1$.*

Proof. We begin with some observations. For the hyperbolic plane, $\delta(2\mathbb{H}) = 1$, $\epsilon_H(2\mathbb{H}) = 1$, and $\dim(2\mathbb{H}) = 4$. For the unique anisotropic quaternary form, we have $\delta(q_{4\mathrm{an}}) = 1$, $\epsilon_H(q_{4\mathrm{an}}) = -1$, and $\dim(q_{4\mathrm{an}}) = 4$. By Definition 2.2.14, the forms q and $q^\triangledown$ share the same dimension and discriminant, but have opposite Hasse invariants.

(1) Consider the two forms $q^\triangledown \oplus 2\mathbb{H}$ and $q \oplus q_{4\mathrm{an}}$. They have the same dimension and discriminant, as follows from the observations above. Since $(x, 1)_F = 1$ for all $x \in F^\times$, it follows from the orthogonal sum identity (2.2.2) and the above observations that

$$\epsilon_H(q^\triangledown \oplus 2\mathbb{H}) = \epsilon_H(q^\triangledown) = -\epsilon_H(q), \quad \text{and} \quad \epsilon_H(q \oplus q_{4\mathrm{an}}) = \epsilon_H(q) \cdot (-1) = -\epsilon_H(q).$$

Thus the two forms are isometric, as required.

(2) In the Witt group, the class of $q^\triangledown \oplus 2\mathbb{H}$ is $[q^\triangledown]$, and the Witt class of $q \oplus q_{4an}$ is $[q] + [q_{4an}]$, so (2) follows.

(3) This is immediate from our Definition 2.2.14.

(4) First recall that $\dim q \geq 2$. Let $\dim(q') = m$. Suppose first $m = 1$, so $q' = \langle \alpha \rangle$. The forms $(q \oplus \langle \alpha \rangle)^\triangledown$ and $q^\triangledown \oplus \langle \alpha \rangle$ have the same dimension and discriminant. For the Hasse invariant, since $\epsilon_H(\langle \alpha \rangle) = 1$, we again use the orthogonal-sum identity (2.2.2) to obtain

$$\epsilon_H(q^\triangledown \oplus \langle \alpha \rangle) = \epsilon_H(q^\triangledown)\,(\delta(q^\triangledown), \alpha)_F = (-\epsilon_H(q))\,(\delta(q), \alpha)_F = -\,\epsilon_H(q \oplus \langle \alpha \rangle).$$

Thus $(q \oplus \langle \alpha \rangle)^\triangledown \simeq q^\triangledown \oplus \langle \alpha \rangle$. Since every quadratic form q' can be diagonalized, the general statement follows by induction on $\dim q'$.

(5) In Table 2.6 we observed that the anisotropic 3-dimensional forms are precisely those with Hasse invariant -1; therefore their q-partners have Hasse invariant 1 by definition and are isotropic.

(6) From Table 2.6, the form $2\mathbb{H} \oplus \langle 1 \rangle$ has discriminant 1 and Hasse invariant 1. The only other 5-dimensional form with the same discriminant and opposite Hasse invariant (-1) is $\mathbb{H} \oplus W_1$. Hence it represents the q-partner of q.

∎

The main focus in this thesis is the 5-dimensional split quadratic form $q \simeq 2\mathbb{H} \oplus \langle 1 \rangle$, and we will often therefore encounter its q-partner.

We next wish to identify how the q-partner operation interacts with the Witt group.

Lemma 2.2.16. *Let q_4 be a 4-dimensional quadratic form over F, not isometric to $q_{4\mathrm{an}}$. Then there exists a unique $\lambda \in F^\times/F^{\times 2}$ such that*

$$q_4 \oplus \langle \lambda \rangle \simeq 2\mathbb{H} \oplus \langle 1 \rangle, \quad \textit{and} \quad q_4^{\triangledown} \oplus \langle \lambda \rangle \simeq \mathbb{H} \oplus W_1.$$

Proof. By the classification of 4-dimensional quadratic forms over F, any such form q_4 is isometric to one $2\mathbb{H}$, $q_{4\mathrm{an}}$, or $\mathbb{H} \oplus \langle \mu, -\mu\alpha \rangle$, for some $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ and $\mu \in F^\times$.

If $q_4 \simeq q_{4\mathrm{an}}$, then for any $\lambda \in F^\times$, the form $q_4 \oplus \langle \lambda \rangle$ remains non-split, since its Hasse invariant does not match that of $2\mathbb{H} \oplus \langle 1 \rangle$. Thus no such λ exists in this case.

Assume now that q_4 is not anisotropic. If $q_4 \simeq 2\mathbb{H}$, then $q_4 \oplus \langle \lambda \rangle \simeq 2\mathbb{H} \oplus \langle 1 \rangle$, if and only if $\lambda = 1$ in $F^\times/F^{\times 2}$. If $q_4 \simeq \mathbb{H} \oplus \langle \mu, -\mu\alpha \rangle$, then $q_4 \oplus \langle \lambda \rangle \simeq \mathbb{H} \oplus \langle \mu, -\mu\alpha, \lambda \rangle$. The form is isometric to $2\mathbb{H} \oplus \langle 1 \rangle$ if and only if its discriminant and Hasse invariant are both 1. A direct computation shows that this occurs precisely when $\lambda = \alpha$ in $F^\times/F^{\times 2}$ if μ is a norm, and no such λ exists otherwise. This proves existence and uniqueness of λ.

Finally, applying Theorem 2.2.15(4) to this decomposition, we obtain

$$q_4^{\triangledown} \oplus \langle \lambda \rangle \simeq (2\mathbb{H} \oplus \langle 1 \rangle)^{\triangledown} \simeq \mathbb{H} \oplus W_1.$$

∎

Table 2.7 gives the explicit values of λ from Lemma 2.2.16 for each of the isotropic forms q_4. Some forms are repeated for ease of later reference.

$q_4 \oplus \langle \lambda \rangle = q_5 \simeq 2\mathbb{H} \oplus \langle 1 \rangle$	$q_4^{\triangledown} \oplus \langle \lambda \rangle = q_5^{\triangledown} \simeq \mathbb{H} \oplus W_1$
$(\mathbb{H} \oplus \langle 1, -\varepsilon \rangle) \oplus \boldsymbol{\langle \varepsilon \rangle}$	$(\mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi \rangle) \oplus \boldsymbol{\langle \varepsilon \rangle}$
$(\mathbb{H} \oplus \langle 1, -\varepsilon\varpi \rangle) \oplus \boldsymbol{\langle \varepsilon\varpi \rangle}$	$(\mathbb{H} \oplus \langle \varepsilon, -\varpi \rangle) \oplus \boldsymbol{\langle \varepsilon\varpi \rangle}$
$(\mathbb{H} \oplus \langle 1, -\varpi \rangle) \oplus \boldsymbol{\langle \varpi \rangle}$	$(\mathbb{H} \oplus \langle \varepsilon, -\varepsilon\varpi \rangle) \oplus \boldsymbol{\langle \varpi \rangle}$
$(\mathbb{H} \oplus \langle 1, \varpi \rangle) \oplus \boldsymbol{\langle -\varpi \rangle}$	$(\mathbb{H} \oplus \langle \varepsilon, \varepsilon\varpi \rangle) \oplus \boldsymbol{\langle -\varpi \rangle}$
$(\mathbb{H} \oplus \langle 1, \varepsilon\varpi \rangle) \oplus \boldsymbol{\langle -\varepsilon\varpi \rangle}$	$(\mathbb{H} \oplus \langle \varepsilon, \varpi \rangle) \oplus \boldsymbol{\langle -\varepsilon\varpi \rangle}$
$2\mathbb{H} \oplus \boldsymbol{\langle 1 \rangle}$	$q_{4an} \oplus \boldsymbol{\langle 1 \rangle} \simeq \mathbb{H} \oplus W_1$

Table 2.7: *Table illustrating Lemma 2.2.16: the unique $\lambda \in F^\times/F^{\times 2}$ for which $q_4 \oplus \langle \lambda \rangle$ is split, and the corresponding behavior of the partner form.*

Lemma 2.2.17. *Let $E_{\alpha,\mu} = \langle \mu, -\mu\alpha \rangle$ be a two-dimensional anisotropic quadratic space as listed in Table 2.5. Then, for each such $E_{\alpha,\mu}$, $q_3 \simeq \langle -\mu, \mu\alpha, 1 \rangle$ is the unique (up to isometry)* *ternary form such that*

$$E_{\alpha,\mu} \oplus q_3 \simeq 2\mathbb{H} \oplus \langle 1 \rangle \quad \textit{and} \quad E_{\alpha,\mu} \oplus q_3^{\triangledown} \simeq \mathbb{H} \oplus W_1.$$

Moreover, if $\mu \in N_{F(\sqrt{\alpha})/F}(F(\sqrt{\alpha})^{\times})$, *then* $q_3 \simeq \mathbb{H} \oplus \langle \alpha \rangle$; *whereas otherwise* $q_3 \simeq W_\alpha$.

Proof. Set $q_3 := \langle -\mu, \mu\alpha, 1 \rangle$. Then

$$E_{\alpha,\mu} \oplus q_3 = \langle \mu, -\mu\alpha, -\mu, \mu\alpha, 1 \rangle \simeq \langle \mu, -\mu \rangle \oplus \langle -\mu\alpha, \mu\alpha \rangle \oplus \langle 1 \rangle \simeq 2\mathbb{H} \oplus \langle 1 \rangle,$$

by Lemma 2.2.4. This proves existence. For uniqueness, if q_3' is another ternary form such that $E_{\alpha,\mu} \oplus q_3' \simeq 2\mathbb{H} \oplus \langle 1 \rangle$, then $E_{\alpha,\mu} \oplus q_3' \simeq E_{\alpha,\mu} \oplus q_3$, so $q_3' \simeq q_3$ by Witt cancellation.

Now apply Theorem 2.2.15(4) to the decomposition $q_3 \oplus E_{\alpha,\mu} \simeq 2\mathbb{H} \oplus \langle 1 \rangle$. We obtain $(2\mathbb{H} \oplus \langle 1 \rangle)^{\triangledown} \simeq q_3^{\triangledown} \oplus E_{\alpha,\mu}$. By Theorem 2.2.15(6) we have $(2\mathbb{H} \oplus \langle 1 \rangle)^{\triangledown} \simeq \mathbb{H} \oplus W_1$, yielding the result. ∎

Table 2.8 summarizes this result: for each anisotropic binary form $E_{\alpha,\mu}$, there exists a unique (up to isometry) ternary form q_3 such that

$$E_{\alpha,\mu} \oplus q_3 \simeq 2\mathbb{H} \oplus \langle 1 \rangle,$$

together with the corresponding identity for the q-partner $q_3^{\triangledown}$.

$q_3 \oplus E_{\alpha,\mu} = q_5 \simeq 2\mathbb{H} \oplus \langle 1 \rangle$	$q_3^{\triangledown} \oplus E_{\alpha,\mu} = q_5^{\triangledown} \simeq \mathbb{H} \oplus W_1$
$\mathbb{H} \oplus \langle \varepsilon \rangle \oplus \boldsymbol{E_{\varepsilon,1}}$	$W_\varepsilon \oplus \boldsymbol{E_{\varepsilon,1}}$
$\mathbb{H} \oplus \langle \varpi \rangle \oplus \boldsymbol{E_{\varpi,1}}$	$W_\varpi \oplus \boldsymbol{E_{\varpi,1}}$
$\mathbb{H} \oplus \langle \varepsilon\varpi \rangle \oplus \boldsymbol{E_{\varepsilon\varpi,1}}$	$W_{\varepsilon\varpi} \oplus \boldsymbol{E_{\varepsilon\varpi,1}}$
$W_\varepsilon \oplus \boldsymbol{E_{\varepsilon,\varpi}}$	$\mathbb{H} \oplus \langle \varepsilon \rangle \oplus \boldsymbol{E_{\varepsilon,\varpi}}$
$W_\varpi \oplus \boldsymbol{E_{\varpi,\varepsilon}}$	$\mathbb{H} \oplus \langle \varpi \rangle \oplus \boldsymbol{E_{\varpi,\varepsilon}}$
$W_{\varepsilon\varpi} \oplus \boldsymbol{E_{\varepsilon\varpi,\varepsilon}}$	$\mathbb{H} \oplus \langle \varepsilon\varpi \rangle \oplus \boldsymbol{E_{\varepsilon\varpi,\varepsilon}}$

Table 2.8: *Explicit realization of Lemma 2.2.17: for each anisotropic binary form $E_{\alpha,\mu}$, the unique ternary form q_3 such that $E_{\alpha,\mu} \oplus q_3 \simeq 2\mathbb{H} \oplus \langle 1 \rangle$, together with the corresponding identity for the partner $q_3^{\triangledown}$, distinguishing the cases $q_3 \simeq \mathbb{H} \oplus \langle \alpha \rangle$ and $q_3 \simeq W_\alpha$.*

2.2.5 From quadratic forms to special orthogonal groups

Finally, a quadratic form gives rise to the special orthogonal group that preserves the form.

Definition 2.2.18. Let (V, q) be an n-dimensional non-degenerate quadratic space over F. The associated special orthogonal group is

$$\mathrm{SO}(V) = \{\, A \in \mathrm{GL}(V) \mid q(Av) = q(v)\ \forall\, v \in V,\ \det(A) = 1 \,\}.$$

It is often denoted $\mathrm{SO}(q)$ instead of $\mathrm{SO}(V)$.

If (V_1, q_1) and (V_2, q_2) are isometric nondegenerate quadratic spaces (over possibly different F-vector spaces of the same dimension), then their corresponding special orthogonal groups are *isomorphic*:

$$\mathrm{SO}(q_1) \;\cong\; \mathrm{SO}(q_2).$$

However, in addition, scaling a form by a nonzero scalar does not affect the group:

$$\mathrm{SO}(q) = \mathrm{SO}(\lambda q) \qquad \text{for all } \lambda \in F^\times.$$

In fact, two quadratic forms give isomorphic special orthogonal groups if and only if one is isometric to a scalar multiple of the other [Chi21, §2]. Therefore we can classify the special orthogonal groups corresponding to forms of dimension 2 to 5 up to isomorphism by grouping those quadratic forms from Table 2.6 that differ by a scalar multiple.

q	$\mathrm{SO}(q)$
Dimension 2	
$E_{\varepsilon,1} = \langle 1, -\varepsilon\rangle$	$\mathrm{SO}(E_{\varepsilon,1})$
$E_{\varpi,1} = \langle 1, -\varpi\rangle$	$\mathrm{SO}(E_{\varpi,1})$
$E_{\varepsilon\varpi,1} = \langle 1, -\varepsilon\varpi\rangle$	$\mathrm{SO}(E_{\varepsilon\varpi,1})$
$\mathbb{H} \simeq \langle 1, -1\rangle$	$\mathrm{SO}(\mathbb{H}) \cong F^\times$
Dimension 3	
$\mathbb{H} \oplus \langle 1\rangle$	$\mathrm{SO}(\mathbb{H} \oplus \langle 1\rangle)$
$W_1 \simeq \langle \varepsilon, -\varpi, \varepsilon\varpi\rangle$	$\mathrm{SO}(W_1)$

q	$\mathrm{SO}(q)$
Dimension 4	
$2\mathbb{H}$	$\mathrm{SO}(2\mathbb{H})$
$\mathbb{H} \oplus \langle 1, -\varepsilon\rangle$	$\mathrm{SO}(\mathbb{H} \oplus \langle 1, -\varepsilon\rangle)$
$\mathbb{H} \oplus \langle 1, -\varpi\rangle$	$\mathrm{SO}(\mathbb{H} \oplus \langle 1, -\varpi\rangle)$
$\mathbb{H} \oplus \langle 1, -\varepsilon\varpi\rangle$	$\mathrm{SO}(\mathbb{H} \oplus \langle 1, -\varepsilon\varpi\rangle)$
$q_{4\mathrm{an}} \simeq \langle 1, -\varepsilon, \varpi, -\varepsilon\varpi\rangle$	$\mathrm{SO}(q_{4\mathrm{an}})$
Dimension 5	
$2\mathbb{H} \oplus \langle 1\rangle$	$\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$
$\mathbb{H} \oplus W_1$	$\mathrm{SO}(\mathbb{H} \oplus W_1)$

Table 2.9: *A list of isomorphism classes of special orthogonal groups* $\mathrm{SO}(q)$, *for* $\dim(q) \in \{2, 3, 4, 5\}$.

2.2.6 Embeddings of special orthogonal groups into $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$

Special orthogonal groups arising from orthogonal direct sums admit natural embeddings into $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$, as follows.

Proposition 2.2.19. *Let $(V, q) = (V_1, q_1) \oplus (V_2, q_2)$ be an orthogonal decomposition of quadratic spaces over F. Then: there is a natural embedding*

$$\phi : \mathrm{SO}(q_1) \times \mathrm{SO}(q_2) \hookrightarrow \mathrm{SO}(q)$$

given by the corresponding block-diagonal embedding. In particular, for each isometry

$$q_4 \oplus \langle \lambda\rangle \simeq 2\mathbb{H} \oplus \langle 1\rangle \quad \textit{and} \quad q_3 \oplus E_{\alpha,\mu} \simeq 2\mathbb{H} \oplus \langle 1\rangle,$$

we may obtain explicit embeddings

$$\mathrm{SO}(q_4) \times \mathrm{SO}(\langle \lambda\rangle) \hookrightarrow \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle) \quad \textit{and} \quad \mathrm{SO}(q_3) \times \mathrm{SO}(E_{\alpha,\mu}) \hookrightarrow \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle).$$

Proof. Choosing orthogonal bases of V_1 and V_2, the quadratic form q is represented by a block-diagonal matrix. Thus any pair $(A, B) \in \mathrm{SO}(q_1) \times \mathrm{SO}(q_2)$ acts on $V_1 \oplus V_2$ by

$$\phi(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

which preserves q and has determinant 1, hence lies in $\mathrm{SO}(q)$.

Let $\iota : (V, q) \to (F^5, 2\mathbb{H} \oplus \langle 1 \rangle)$ be an isometry. Explicit choices of isometries are given in Table 2.7 and Table 2.8 to identify the quadratic spaces $q_4 \oplus \langle \lambda \rangle$ and $q_3 \oplus E_{\alpha,\mu}$ with $2\mathbb{H} \oplus \langle 1 \rangle$. Then conjugation by ι transports the embedding ϕ into $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, giving

$$(A, B) \mapsto \iota \circ \phi(A, B) \circ \iota^{-1}.$$

∎

The groups of $\mathrm{SO}(q_3) \times \mathrm{SO}(E_{\alpha,\mu})$ will be examples of (twisted) Levi subgroups of G in Section 2.4.3.

2.3 Hermitian forms over a quadratic extension

Now suppose that E is a quadratic extension of F, and that σ is the nontrivial element of $\mathrm{Gal}(E/F)$. Let V denote an m-dimensional vector space over E. A nondegenerate sesquilinear form

$$(\quad , \quad) : V \times V \longrightarrow E$$

that is linear in the second variable and σ-conjugate linear in the first variable is called **Hermitian** if

$$(y, x) = \sigma\big((x, y)\big), \qquad \text{for all } x, y \in V.$$

In this section, we sketch the essentials of Hermitian forms.

2.3.1 Invariants of Hermitian forms

Given a choice of basis $B = \{b_1, \cdots, b_m\}$ for V, the Hermitian matrix $J = ((b_i, b_j))_{i,j}$ represents the form. Unlike quadratic forms, the isometry class of (V, J) is determined by its rank or dimension $m = \dim_E(V)$ and its discriminant or determinant $\delta(V) := \det(J) \in F^\times / N_{E/F}(E^\times)$. Since $|F^\times / N_{E/F}(E^\times)| = 2$ for a quadratic extension (Lemma 2.1.4), for each fixed quadratic extension E and fixed rank m there are exactly two isometry classes (see [GHY01, §3]).

In this work, we only require the cases $m = 1$ and $m = 2$. When $m = 1$, the forms are given by $(x, y) = \alpha x \sigma(y)$ for some $\alpha \in F^\times / N_{E/F}(E^\times)$ and when $m = 2$ there are two, one split and one anisotropic. We denote the anisotropic one by $\mathbf{C}$ and, by abuse of notation, use the same symbol $\mathbb{H}$ to represent the split one. This will not cause confusion as it is also represented by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has determinant -1. This yields the following lemma.

Lemma 2.3.1. *Let V be a rank-2 Hermitian space over E/F with discriminant $\delta(V)$. Then:*

$$V \simeq \mathbb{H} \iff \delta(V) \equiv -1 \text{ in } F^\times/N_{E/F}(E^\times).$$

Otherwise, $V \simeq \mathbf{C}$.

Example 2.3.2. Let $E = F(\sqrt{\varepsilon})$ be the unramified quadratic extension and consider the form represented by the diagonal matrix $J = \langle 1,\ \varepsilon\varpi\rangle$. By Lemma 2.1.4 we have $\mathcal{O}_F^\times \subset N_{E/F}(E^\times)$, so in particular $-1 \in N_{E/F}(E^\times)$. Thus since $\delta(J) = \varepsilon\varpi \equiv \varpi \notin N_{E/F}(E^\times)$, J is anisotropic.

Example 2.3.3. Let $E = F(\sqrt{\varpi})$ be a ramified quadratic extension. Consider the 2-dimensional E/F-Hermitian form $h \simeq \langle 1, 1\rangle$. Its discriminant satisfies $\delta(h) = 1$ in $F^\times/N_{E/F}(E^\times) = \{1, \varepsilon\}$. Therefore if $-1 \notin F^{\times 2}$, then $\delta(h)$ does not represent the same class as -1 in $F^\times/N_{E/F}(E^\times)$, and h is anisotropic. If $-1 \in F^{\times 2}$, however, then $\delta(h) \equiv -1 \mod N_{E/F}(E^\times)$ and h is split.

2.3.2 From Hermitian forms to unitary groups

In this thesis our interest is in the full unitary group of a Hermitian form, rather than the special unitary group, and it is defined as follows.

Definition 2.3.4. Let E/F be a quadratic field extension of F, and let $V = \bigoplus^n E$ be a E-vector space equipped with a nondegenerate Hermitian form $\langle,\rangle$ relative to the Galois involution on E/F. The associated **unitary group** is

$$\{\, g \in \mathrm{GL}(V) \mid (gv, gw) = (v, w) \text{ for all } v, w \in V \,\}.$$

Equivalently, fixing a basis and letting J denote the matrix of the Hermitian form, we may define

$$U_{E/F}(J) = \{\, g \in \mathrm{GL}_n(E) \mid g^* J g = J \,\}, \qquad \text{where } g^* = \sigma g^{\,t}.$$

When $E = F(\sqrt{\gamma})$ for some $\gamma \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ we may write $U_\gamma(\,\cdot\,)$ to denote $U_{F(\sqrt{\gamma})/F}(\,\cdot\,)$, following the notation of Kim and Yu [KY11].

If two Hermitian spaces (V_1, J_1) and (V_2, J_2) are isometric, then their corresponding unitary groups are isomorphic, and again, if $J_1 = \lambda J_2$ for some $\lambda \in F^\times$ then the corresponding groups are equal.

For example, for any $\alpha \in F^\times$,

$$U_{E/F}(\langle\alpha\rangle) = \{\, z \in E^\times : N_{E/F}(z) = 1 \,\} = E^1,$$

so the unitary group of a one-dimensional Hermitian space over E/F is simply the group of elements of E of norm one.

Lemma 2.3.5. *Let $E_\alpha = F(\sqrt{\alpha})$ be a quadratic field extension of F. Then we have an isomorphism*

$$U_{E_\alpha/F}(\langle 1\rangle) \;\cong\; \mathrm{SO}(E_{\alpha,1}) \;\cong\; E_\alpha^1.$$

Proof. Recall from Section 2.2.3 that $E_{\alpha,1}$ is the quadratic space $(\langle 1, -\alpha\rangle, E)$, where this form is taken with respect to the basis $\{1, \sqrt{\alpha}\}$ of E over F. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F)$. The condition $A \in \mathrm{SO}(q)$ is equivalent to $A^T J A = J$, where $J = \mathrm{diag}(1, -\alpha)$ and $\det(A) = 1$. A direct computation shows that this holds if and only if

$$A = \begin{pmatrix} p & \alpha r \\ r & p \end{pmatrix} \quad \text{with} \quad p^2 - \alpha r^2 = 1.$$

For such a matrix, let $t = p + r\sqrt{\alpha} \in E_\alpha^\times$. Then $N_{E_\alpha/F}(t) = p^2 - \alpha r^2 = 1$, and multiplication by t on E_α has matrix $\begin{pmatrix} p & \alpha r \\ r & p \end{pmatrix}$ in the basis $\{1, \sqrt{\alpha}\}$. Thus every element of $\mathrm{SO}(E_{\alpha,1})$ arises uniquely as multiplication by an element of E_α^1 which is by definition is isomorphic to $U_{E_\alpha/F}(\langle 1\rangle)$. ∎

Now we come to the case when $m = 2$. For any $\gamma \in \{\varepsilon, \varpi, \varepsilon\varpi\} U_\gamma(\mathbb{H})$ is the unitary group on $F(\sqrt{\gamma})^2$ preserving the hyperbolic plane and $U_\gamma(\mathbf{C})$ preserves the anisotropic form, and therefore is compact. In Table 2.10 we give the diagonal forms corresponding to these two cases, which depends on whether or not $F(\sqrt{\gamma})/F$ is ramified, and also on whether or not $-1 \in F^{\times 2}$. A convenient choice of representatives allows us to give a uniform description independent of whether or not $-1 \in F^{\times 2}$.

Unitary Group	**Unramified** $E = F(\sqrt{\varepsilon})$	**Ramified** $E = F(\sqrt{\varpi'})$
Quasi-split $U_\gamma(\mathbb{H})$	$U_\varepsilon(\langle 1, 1\rangle)$	$U_{\varpi'}(\langle 1, -1\rangle)$
Compact $U_\gamma(\mathbf{C})$	$U_\varepsilon(\langle 1, \varpi\rangle)$	$U_{\varpi'}(\langle 1, -\varepsilon\rangle)$

Table 2.10: *Unitary groups U_γ associated to quadratic extensions E/F. The table distinguishes the unramified extension $E = F(\sqrt{\varepsilon})$ and the ramified extension $E = F(\sqrt{\varpi'})$ where $\varpi' \in \{\varpi, \varepsilon\varpi\}$, and lists the corresponding quasi-split and compact forms.*

2.3.3 Embedding unitary groups into orthogonal groups

Unitary groups have natural embeddings into special orthogonal groups, as follows.

Proposition 2.3.6. *Let E be a quadratic extension of F and let $(\cdot,\cdot)$ denote a Hermitian form on E^2. Then:*

(a) for any $x \in E^2$, the formula $q(x) = \mathrm{Tr}_{E/F}((x,x))$ defines a 4-dimensional quadratic form on $E^2 \cong F^4$;

(b) every isomorphism $\eta : E^2 \to F^4$ induces an embedding $\eta : U_{E/F}((\cdot,\cdot)) \to \mathrm{SO}(q)$;

(c) q is isometric to $2\mathbb{H}$ if $(\cdot,\cdot)$ is hyperbolic and to $q_{4\mathrm{an}}$ if not.

Moreover, η extends to an embedding of $U_{E/F}((\cdot,\cdot))$ into $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$ or $\mathrm{SO}(\mathbb{H}\oplus W_1)$, respectively.

Proof. (a) The proof that q is a nondegenerate quadratic form is a special case of a more general result that we will prove in Chapter 3 (Lemma 3.1.1).

(b) Each isomorphism $\eta_B : E^2 \to F^4$ corresponds to making a choice of F-basis B of E^2. Left multiplication by $h \in U_{E/F}((\cdot,\cdot))$ is an F-linear transformation of $E^2 \cong F^4$ and we let $\eta_B(h) \in \mathrm{GL}_4(F)$ denote the matrix of this transformation with respect to B. Since for any $x \in E^2$ we have

$$\mathrm{Tr}_{E/F}((hx,hx)) = \mathrm{Tr}_{E/F}((x,x)),$$

it follows that

$$q(\eta_B(h)\eta_B(x)) = q(\eta_B(x))$$

and so $\eta_B(h)$ preserves the quadratic form q. We have moreover that

$$\det(\eta_B(h)) = \det(h)\sigma(\det(h)) = N_{E/F}(\det(h)),$$

which is equal to 1 since $\sigma(h^T)Jh = J$ implies $\det(\sigma(h))\det(h) = 1$.

(c) Write $(,)_E$ for the Hermitian form on E^2. We explicitly determine the isometry class of the associated symmetric bilinear form

$$(x,y)_F = \mathrm{Tr}_{E/F}((x,y)_E).$$

Choose the basis

$$B = \{(1,0), (\sqrt{\alpha},0), (0,1), (0,\sqrt{\alpha})\}$$

and write $\overline{x} := \sigma(x)$ for short. Let

$$v = (v_1 + v_2\sqrt{\alpha},\ v_3 + v_4\sqrt{\alpha}), \qquad w = (w_1 + w_2\sqrt{\alpha},\ w_3 + w_4\sqrt{\alpha}).$$

We may assume J has the form $\langle 1, -\mu\rangle$, for some $\mu \in F^\times$; then $J \simeq \mathbb{H}$ if $\mu = 1$ and $J \simeq \mathbf{C}$ if $\mu \in F^\times \setminus N_{E/F}(E^\times)$. We compute

$$\begin{aligned}(v,w)_F &= \mathrm{Tr}_{E/F}\Big(\overline{(v_1 + v_2\sqrt{\alpha})}(w_1 + w_2\sqrt{\alpha}) - \mu\,\overline{(v_3 + v_4\sqrt{\alpha})}(w_3 + w_4\sqrt{\alpha})\Big)\\ &= 2(v_1w_1 - \alpha v_2w_2 - \mu(v_3w_3 - \alpha v_4w_4))\end{aligned}$$

so that the associated quadratic form is

$$q = \langle 2, -2\alpha, -2\mu, 2\mu\alpha\rangle$$

When $\mu = 1$ this is isometric to $2\mathbb{H}$ by Lemma 2.2.4. Otherwise, compute $\delta(q) = 1$ and since we may assume $(\alpha,\mu) \in \{(\varepsilon,\varpi), (\varpi,\varepsilon), (\varepsilon\varpi,\varepsilon)\}$, a calculation yields $\epsilon_H(q) = -1$ so that this is the anisotropic form $q_{4\mathrm{an}}$ by Table 2.6.

The final statement follows by composing with the embedding ϕ from Section 2.2.6. ∎

The unitary groups of the Hermitian forms in Proposition 2.3.6 will be examples of (twisted) Levi subgroups of G in Section 2.4.3.

2.4 Algebraic groups

The groups $\mathrm{SO}(q)$ and $U_\gamma(J)$ introduced in Sections 2.2.5 and 2.3.3 are more than abstract groups: they are the F-points of certain algebraic groups defined over F, and this gives them a great deal of additional structure. Let us give a minimal introduction to the parts of the structure theory of algebraic groups that are needed for this thesis.

Following Humphreys [Hum78], a **reductive algebraic group** defined over F is an affine algebraic F-group $\mathbb{G}$ whose unipotent radical is trivial. We denote **a p-adic group** by $G = \mathbb{G}(F)$, that is, the group of F-points of $\mathbb{G}$, for any local non-archimedean field F.

For example, $\mathbb{G}_a$ and $\mathbb{G}_m$ denote the additive and multiplicative groups, respectively, with

$$\mathbb{G}_a(F) = F, \qquad \mathbb{G}_m(F) = F^\times.$$

Typical examples of algebraic groups include the general linear group GL_n, and the special orthogonal group SO_n associated with a nondegenerate quadratic form q on a vector space of dimension n. Their F-points are $\mathrm{GL}_n(F)$ and $\mathrm{SO}(q)$, respectively.

In this chapter, we wish to understand $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ and its twisted Levi subgroups. The structure theory of reductive groups is most effectively expressed in terms of their maximal tori and the associated root data. These provide the framework for describing the characters and cocharacters of a torus, the root and coroot systems, and ultimately the classification of reductive groups. Root data also provide the natural language for describing parabolic subgroups and their Levi factors, which play an essential role in the construction of representations.

Unless stated otherwise, the material in this section follows [Hum75]; see also [Bor91] for background.

2.4.1 Tori

We begin by recalling the definitions.

Definition 2.4.1. A **torus** is an algebraic group $\mathbb{T}$ that is isomorphic to a direct product of one or more copies of the multiplicative group, denoted $\mathbb{G}_m^r$. The integer r is called the **rank** or **absolute rank** of the torus $\mathbb{T}$.

We are only interested of tori that are defined over F. Note that the isomorphism to $\mathbb{G}_m^r$ may only be defined over an extension of F.

Moreover, the following terminology is associated with tori:

1. The torus $\mathbb{T}$ is said to be **split over a field extension** E/F if $\mathbb{T}(E) \cong (E^\times)^r$. The minimal Galois extension of F over which $\mathbb{T}$ is split is called the **splitting field** of $\mathbb{T}$.

2. The F**-rank** of $\mathbb{T}$ is the maximal rank of an F-split subtorus of $\mathbb{T}$. A torus is F**-split** if and only if its F-rank equals its absolute rank, that is, when $\mathbb{T}(F) \cong (F^\times)^r$.

3. The torus $\mathbb{T}$ is said to be F**-anisotropic** (or **elliptic**) if it contains no nontrivial F-split subtorus, equivalently if its F-rank is zero.

All maximal F-split tori of a connected reductive group $\mathbb{G}$ over F are conjugate under $\mathbb{G}(F)$ [Hum75, Thm. 21.3].

Example 2.4.2. We noted in Table 2.9 that $\mathrm{SO}(\mathbb{H}) \cong F^\times$, so this p-adic group corresponds to an F-split torus of rank 1. On the other hand, by Lemma 2.3.5, $\mathrm{SO}(E_{\alpha,1}) = \mathrm{SO}(\langle 1, -\alpha\rangle) \cong U_\alpha(1) \cong E^1$ is (the F-points of) an elliptic torus that splits over the quadratic extension $E = F(\sqrt{\alpha})$.

Definition 2.4.3. Let $\mathbb{G}$ be a connected reductive algebraic group defined over a field F.

1. The **absolute rank** of $\mathbb{G}$ is the rank of a maximal torus $\mathbb{T}$ of $\mathbb{G}$. Since all maximal tori of $\mathbb{G}$ are conjugate over $\overline{F}$, the absolute rank is well defined.

2. The F**-rank** of $\mathbb{G}$ is the maximal rank of an F-split torus contained in $\mathbb{G}$. Equivalently, it is the dimension of a maximal F-split torus in $\mathbb{G}$.

The group $\mathbb{G}$ is said to be **split over** F if its absolute rank equals its F-rank, that is, if it contains a maximal torus that is F-split.

We now make this explicit for $\mathbb{G} = \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$. With respect to a Witt basis, this group is realized as

$$\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle) \;=\; \{\, g \in \mathrm{GL}_5(F) \;:\; g^T J g = J,\ \det(g) = 1 \,\}, \tag{2.4.1}$$

where J is the matrix in (2.2.1). A maximal F-split torus of this group $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$ is given by

$$\mathbb{T}(F) \;=\; \left\{(\mathrm{diag}(a, b, a^{-1}, b^{-1}, 1)) :\ a, b \in F^\times\right\}. \tag{2.4.2}$$

Therefore, this group is split over F in the sense of Definition 2.4.3.

2.4.2 Root Datum $(X^*(\mathbb{T}), \Phi, X_*(\mathbb{T}), \check{\Phi})$

Suppose $\mathbb{G}$ is F-split and let $\mathbb{T}$ be an F-split maximal torus of $\mathbb{G}$. The structure of $\mathbb{G}$ relative to $\mathbb{T}$ is encoded in its root datum. We begin with the definitions of character and cocharacter groups, their natural pairing, and the associated root system.

2.4.2.1 Characters, cocharacters, and the canonical pairing

Definition 2.4.4. Let $\mathbb{T}$ be an F-split maximal torus. Its **character group** and **cocharacter group** are

$$X^*(\mathbb{T}) := \operatorname{Hom}_F(\mathbb{T}, \mathbb{G}_m), \qquad X_*(\mathbb{T}) := \operatorname{Hom}_F(\mathbb{G}_m, \mathbb{T}),$$

free abelian groups of the same rank. For $\chi \in X^*(\mathbb{T})$ and $\lambda \in X_*(\mathbb{T})$, the composition $\chi \circ \lambda : \mathbb{G}_m \to \mathbb{G}_m$ is $t \mapsto t^k$ for a unique $k \in \mathbb{Z}$, and we define the *canonical pairing* by the rule that $\langle\!\langle \chi, \lambda \rangle\!\rangle := k \in \mathbb{Z}$.

If we choose a basis $\{\epsilon_1, \dots, \epsilon_r\}$ of $X^*(\mathbb{T})$ then there is a dual basis $\{e_1, \dots, e_r\}$ of $X_*(\mathbb{T})$ such that the canonical pairing satisfies

$$\left\langle\!\!\left\langle \sum_i m_i \epsilon_i, \sum_j n_j e_j \right\rangle\!\!\right\rangle = \sum_i m_i n_i.$$

Example 2.4.5. Let $\mathbb{T} \subset \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ be the maximal split torus of (2.4.2). Define $\epsilon_1, \epsilon_2 \in X^*(\mathbb{T})$ by $\epsilon_1(\operatorname{diag}(a, b, a^{-1}, b^{-1}, 1)) = a$ and $\epsilon_2(\operatorname{diag}(a, b, a^{-1}, b^{-1}, 1)) = b$. Then $\{\epsilon_1, \epsilon_2\}$ is a $\mathbb{Z}$–basis of $X^*(\mathbb{T})$, and any $\chi \in X^*(\mathbb{T})$ can be written as $m\epsilon_1 + n\epsilon_2$, for $m, n \in \mathbb{Z}$ in the following sense:

$$\chi_{(m,n)} := m\epsilon_1 \oplus n\epsilon_2 \quad \Longrightarrow \quad \chi_{(m,n)}(\operatorname{diag}(a, b, a^{-1}, b^{-1}, 1)) = a^m b^n.$$

Define $e_1, e_2 \in X_*(\mathbb{T})$ by

$$e_1(t) = \operatorname{diag}(t, 1, t^{-1}, 1, 1) \qquad e_2(t) = \operatorname{diag}(1, t, 1, t^{-1}, 1)$$

Then $\{e_1, e_2\}$ is a $\mathbb{Z}$–basis of $X_*(\mathbb{T})$, and for $(m, n) \in \mathbb{Z}^2$,

$$\lambda_{(m,n)} := me_1 \oplus ne_2 \quad \Rightarrow \quad \lambda_{(m,n)}(t) = \operatorname{diag}(t^m, t^n, t^{-m}, t^{-n}, 1).$$

These are dual bases with respect to the canonical pairing since $\epsilon_i(e_j(t)) = t^{\delta_{ij}}$.

2.4.2.2 Root and coroot systems

We begin by defining the absolute root system, which may not be defined over F. For material in this section, see [Mur05, §7] or [Bor91, §21].

Let $\mathfrak{g} = \operatorname{Lie}(\mathbb{G})$ and let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$. For each $\alpha \in X^*(\mathbb{T})$, we define

$$\mathfrak{g}_\alpha \;=\; \{\, X \in \mathfrak{g} \;:\; \operatorname{Ad}(t)X = \alpha(t)X \text{ for all } t \in \mathbb{T} \,\},$$

or equivalently, under the action of the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$,

$$\mathfrak{g}_\alpha \;=\; \{\, Y \in \mathfrak{g} \mid [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{t} \,\}, \tag{2.4.3}$$

where α in (2.4.3) denotes the differential of the character, viewed as a linear map $\mathfrak{t} \to \overline{F}$.

Definition 2.4.6 (Root system)**.** The set of nonzero characters α for which $\mathfrak{g}_\alpha \neq 0$ is called the **root system** of $\mathbb{G}$ with respect to $\mathbb{T}$:

$$\Phi(\mathbb{G}, \mathbb{T}) \;=\; \{\, \alpha \in X^*(\mathbb{T}) \setminus \{0\} \;:\; \mathfrak{g}_\alpha \neq 0 \,\}.$$

This is referred to as the **absolute root system** of $\mathbb{G}$. Its elements are called **roots**. The root system Φ is said to be **reduced** if for every $\alpha \in \Phi$, the only scalar multiples of α lying in Φ are $\pm\alpha$.

When $\mathbb{T}$ is F-split, then these root subalgebras are defined over F. The root system gives us a decomposition of the Lie algebra $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{2.4.4}$$

where for each α, the subalgebra $\mathfrak{g}_\alpha$ is one-dimensional.

Example 2.4.7. Let $\{e_1, e_2, f_1, f_2, g_1\}$ be a Witt basis associated to the form $2\mathbb{H} \oplus \langle 1 \rangle$; then the corresponding matrix form J is as in (2.2.1). The group and Lie algebra are split over F so we may carry out the above construction on the F-points of the Lie algebra. Here,

$$\mathfrak{so}(J) = \{\, X \in M_5(F) : X^{\mathsf{T}} J + JX = 0 \,\}.$$

The matrix J pairs the coordinates $(1,3)$ and $(3,1)$, representing (e_1, f_1), $(2,4)$ and $(4,2)$, representing (e_2, f_2), while the fifth coordinate is self-paired, g_1. Accordingly, we define

$$1' = 3, \quad 2' = 4, \quad 3' = 1, \quad 4' = 2, \quad 5' = 5.$$

Let E_{ij} denote the matrix in M_5 with 1 in the (i,j)-position and 0 elsewhere. The condition $X^{\mathsf{T}} J + JX = 0$ then implies that if $E_{ij} \in \mathfrak{g}$, then the entry $E_{j'i'}$ must appear with the opposite sign. Hence, for $i \neq j$,

$$X_{ij} := E_{ij} - E_{j'i'} \in \mathfrak{g}.$$

Setting $X_{11} = E_{11} - E_{33}$ and $X_{22} = E_{22} - E_{44}$, the Cartan subalgebra is given over F by $\mathfrak{t} = \{aX_{11} + bX_{22} \mid a, b \in F\}$. These elements X_{ij} span $\mathfrak{g}$.

Let ϵ_1, ϵ_2 be the basis of the character group as in Example 2.4.5. Then if we define $\epsilon_3 = -\epsilon_1,\ \epsilon_4 = -\epsilon_2,\ \epsilon_5 = 0$, then the adjoint action of T on $\mathfrak{g}$ satisfies

$$\mathrm{Ad}(t)\, X_{ij} = \frac{\epsilon_i(t)}{\epsilon_j(t)} X_{ij}. \qquad t \in T$$

Thus if $i \neq j$, X_{ij} is a root vector with root $\epsilon_i - \epsilon_j$, and these are precisely the roots in Φ. We record our choice of basis for the positive and negative root spaces in Table 2.11. The notation a_{l_i} and a_{s_i} will be explained in Example 2.4.10. It is clear that the 10-dimensional algebra $\mathfrak{so}(,5)$ is spanned by these eight root vectors together with the two-dimensional diagonal torus.

Root α	Basis vector $\tilde{X}_\alpha$	Basis vector $\tilde{X}_{-\alpha}$
$a_{l_1} := \epsilon_1 - \epsilon_2$	$E_{12} - E_{43}$	$E_{21} - E_{34}$
$a_{l_2} := \epsilon_1 + \epsilon_2$	$E_{14} - E_{23}$	$E_{41} - E_{32}$
$a_{s_1} := \epsilon_1$	$E_{15} - E_{53}$	$E_{51} - E_{35}$
$a_{s_2} := \epsilon_2$	$E_{25} - E_{54}$	$E_{52} - E_{45}$

Table 2.11: *Basis vectors for the Lie algebra root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ of $\mathfrak{so}_5$*

A **simple system** is a subset $\Delta \subset \Phi$ which is a basis of $\mathrm{span}_{\mathbb{R}}\Phi$ such that every root is an integral linear combination of elements of Δ with all coefficients either nonnegative or nonpositive. A simple system Δ determines a set of **positive roots**

$$\Phi^+ \ = \ \{\, \beta \in \Phi \ : \ \beta = \textstyle\sum_{\alpha_i \in \Delta} n_i \alpha_i, \ n_i \geq 0 \,\},$$

and we have $\Phi = \Phi^+ \sqcup (-\Phi^+)$. Every positive root can therefore be written uniquely as

$$\beta = \sum_{i=1}^{\ell} n_i(\beta)\, \alpha_i, \qquad n_i(\beta) \in \mathbb{Z}_{\geq 0}.$$

The *height* of β is $\mathrm{ht}(\beta) := \sum_{i=1}^{\ell} n_i(\beta)$.

Definition 2.4.8. Let Φ be a reduced irreducible root system with positive system Φ^+. The **highest root** θ is the positive root of maximal height:

$$\mathrm{ht}(\theta) = \max\{\mathrm{ht}(\beta) : \beta \in \Phi^+\}.$$

We now recall the definition of the Cartan matrix associated with a simple root system, following [EW06, §11.4]. The Cartan matrix encodes all relations among the simple roots.

Definition 2.4.9. Let Δ be a simple system in a root system Φ. Fix an order on the elements of Δ, say $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, where $r = \mathrm{rank}(\mathbb{G}) = \dim(\mathbb{T})$. The **Cartan matrix** of Φ is the $r \times r$ matrix $A = (a_{ij})$ whose entries are given by

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \qquad 1 \leq i, j \leq r,$$

Here $(\cdot, \cdot)$ denotes a G-invariant inner product on the real vector space spanned by Φ.

Given simple roots $\Delta = \{\alpha_i\}$ and the corresponding Cartan matrix $A = (a_{ij})$, there exists a unique set of cocharacters $\{\alpha_i^\vee\} \subset X_*(\mathbb{T})$ such that $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij} \quad (1 \leq i, j \leq r)$. where $r = \mathrm{rank}(\mathbb{G})$. The set

$$\check{\Phi}(\mathbb{G}, \mathbb{T}) = \{\alpha^\vee : \alpha \in \Phi(\mathbb{G}, \mathbb{T})\} \subset X_*(\mathbb{T})$$

is the coroot system and its elements are **coroots**.

Let us interpret these definitions for $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

Example 2.4.10. Let $\mathbb{T} \subset \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ be the maximal split diagonal torus of (2.4.2). Define ϵ_i and e_i as in Example 2.4.5. Set $a_{s_2} = \epsilon_2$ and $a_{l_1} = \epsilon_1 - \epsilon_2$. Then following Example 2.4.7 we define

$$a_{s_1} := \epsilon_1 = a_{s_2} + a_{l_1}, \qquad\qquad a_{l_2} := \epsilon_1 + \epsilon_2 = 2a_{s_2} + a_{l_1},$$

and then

$$\Phi = \Phi(\mathbb{G}, \mathbb{T}) = \{\pm a_{s_2},\ \pm a_{l_1},\ \pm(a_{s_2} + a_{l_1}),\ \pm(2a_{s_2} + a_{l_1})\},$$

is the root system of SO_5 with respect to $\mathbb{T}$. It is easy to verify that a corresponding simple system is $\Delta = \{a_{s_2}, a_{l_1}\}$.

In Figure 2.1 we draw the root system, which is of type B_2 [EW06, §12.3], with $\{\pm a_{s_1}, \pm a_{s_2}\}$ corresponding to the short roots and $\{\pm a_{l_1}, \pm a_{l_2}\}$ to the long roots. Fixing the simple roots, the positive roots are $\Phi^+ = \{a_{s_2}, a_{l_1}, a_{s_2} + a_{l_1}, 2a_{s_2} + a_{l_1}\}$. Writing each $\beta \in \Phi^+$ as $\beta = n_1(\beta)\, a_{s_2} + n_2(\beta)\, a_{l_1}$, the heights are

$$\mathrm{ht}(a_{s_2}) = 1, \quad \mathrm{ht}(a_{l_1}) = 1, \quad \mathrm{ht}(a_{s_2} + a_{l_1}) = 2, \quad \mathrm{ht}(2a_{s_2} + a_{l_1}) = 3.$$

Thus the highest root is $\theta = 2a_{s_2} + a_{l_1} = \epsilon_1 + \epsilon_2$.

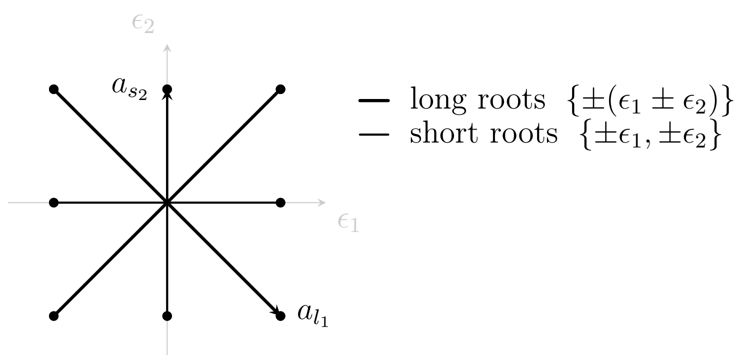

Figure 2.1: *Root system of* $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ *(type B_2) and* $\Delta = \{a_{s_2}, a_{l_1}\}$.

The Cartan matrix for $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ with respect to Δ is $A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, so that

$$\langle a_{s_2}, \check{a}_{s_2} \rangle = 2, \quad \langle a_{s_2}, \check{a}_{l_1} \rangle = -1, \quad \langle a_{l_1}, \check{a}_{s_2} \rangle = -2, \quad \langle a_{l_1}, \check{a}_{l_1} \rangle = 2.$$

Solving the Cartan relations gives the coroots

$$\check{a}_{s_2}(t) = \mathrm{diag}(1, t^2, 1, t^{-2}, 1) = 2e_2(t), \qquad \check{a}_{l_1}(t) = \mathrm{diag}(t, t^{-1}, t^{-1}, t, 1) = (e_1 - e_2)(t).$$

Thus $\check{a}_{s_2} = 2e_2$ and $\check{a}_{l_1} = e_1 - e_2$ in $X_*(\mathbb{T})$. This gives coroot system

$$\check{\Phi} = \{\pm 2e_1, \pm 2e_2, \pm(e_1 - e_2), \pm(e_1 + e_2)\},$$

which is of type C_2.

2.4.2.3 Root subgroups

In (2.4.4) we defined the root subalgebras of $\mathfrak{g}$. The following definition/theorem, which we state without proof, is summarized from [Mur05, §7].

Theorem 2.4.11. *Let $\mathbb{G}$ be F-split and let $\mathbb{T}$ be a fixed maximal F-split torus of $\mathbb{G}$ with Lie algebra $\mathfrak{t}$. For each root $\alpha \in \Phi(\mathbb{G}, \mathbb{T})$, let $\mathfrak{g}_\alpha$ be the corresponding root subalgebra of* $\mathrm{Lie}(\mathbb{G})$*. Then there exists a unique connected, $\mathbb{T}$-stable, unipotent subgroup $\mathbb{U}_\alpha \subset \mathbb{G}$ whose Lie algebra is $\mathfrak{g}_\alpha$. This subgroup is called the* ***root subgroup*** *of $\mathbb{G}$ associated with α. Moreover, there exists an isomorphism*

$$u_\alpha : \mathbb{G}_a \longrightarrow \mathbb{U}_\alpha, \text{ such that } \quad t\, u_\alpha(x)\, t^{-1} = u_\alpha(\alpha(t)x), \qquad t \in \mathbb{T},\ x \in \mathbb{G}_a.$$

Finally, the groups $\mathbb{U}_\alpha$ for $\alpha \in \Phi$, together with $\mathbb{T}$, generate the group $\mathbb{G}$.

When $\mathbb{G}$ is a matrix group, the subgroup $\mathbb{U}_\alpha(F)$ can be identified with the additive group of F via the exponential map, as follows. Fix a basis vector $\tilde{X}_\alpha \in \mathfrak{g}_\alpha$.

Set

$$u_\alpha(t) := \exp(t\widetilde{X}_\alpha), \qquad t \in F.$$

Since $\widetilde{X}_\alpha$ is nilpotent, this exponential series terminates after finitely many terms.

Let us describe these explicitly for $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

Example 2.4.12. Write $\tilde{X}_\alpha$ for the root vector corresponding to $\alpha \in \Phi(\mathrm{SO}_5, \mathbb{T})$ in Table 2.11. Then we compute directly that

$$\exp(t\tilde{X}_\alpha) \;=\; \begin{cases} I + t\tilde{X}_\alpha, & \text{if } \alpha \text{ is a long root,} \\ I + t\tilde{X}_\alpha + \dfrac{t^2}{2}\tilde{X}_\alpha^2, & \text{if } \alpha \text{ is a short root.} \end{cases}$$

We give these expressions explicitly in matrix coordinates in Table 2.12. Then the root subgroup is $\mathbb{U}_\alpha(F) \;=\; \{\, u_\alpha(t) := \exp(t\tilde{X}_\alpha) \;:\; t \in F \,\}$.

Root α	**Root subgroup** U_α	**Root subgroup** $U_{-\alpha}$
$a_{l_1} := \epsilon_1 - \epsilon_2$	$I + t(E_{12} - E_{43})$	$I + t(E_{21} - E_{34})$
$a_{l_2} := \epsilon_1 + \epsilon_2$	$I + t(E_{14} - E_{23})$	$I + t(E_{41} - E_{32})$
$a_{s_1} := \epsilon_1$	$I + t(E_{15} - E_{53}) - \frac{t^2}{2}E_{13}$	$I + t(E_{51} - E_{35}) - \frac{t^2}{2}E_{31}$
$a_{s_2} := \epsilon_2$	$I + t(E_{25} - E_{54}) - \frac{t^2}{2}E_{24}$	$I + t(E_{52} - E_{45}) - \frac{t^2}{2}E_{42}$

Table 2.12: *One-parameter unipotent root subgroups of* $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

2.4.2.4 Regular elements

In this section we write $\mathbb{G}$ for a connected semisimple algebraic group, and let $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$ be its Lie algebra. We use $\mathfrak{g} = \mathfrak{g}(F)$ to represent the F-points of $\mathfrak{g}$. Recall that $X \in \mathfrak{g}$ is semisimple if it is contained in a Cartan subalgebra.

Definition 2.4.13. A semisimple element $X \in \mathfrak{g}$ is called **regular** if its centralizer

$$C_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \mid [Y, X] = 0\}$$

is a Cartan subalgebra, say $\mathfrak{t}$. Equivalently, X is regular if and only if its centralizer in $\mathbb{G}$,

$$C_{\mathbb{G}}(X) = \{g \in \mathbb{G} \mid \mathrm{Ad}_g(X) = X\}, \tag{2.4.5}$$

is a maximal torus $\mathbb{T}$. In this case, the Lie algebra of $\mathbb{T}$ is $\mathrm{Lie}(\mathbb{T}) = \mathfrak{t}$.

We now describe a useful criterion for regularity in terms of the root system.

Theorem 2.4.14. *An element $X \in \mathfrak{t}$ is regular if and only if $\alpha(X) \neq 0$ for all $\alpha \in \Phi(\mathbb{G}, \mathbb{T})$.*

Proof. Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and let $\Phi = \Phi(\mathbb{G}, \mathbb{T})$ be the associated root system. Let $X \in \mathfrak{t}$. If $Y \in \mathfrak{g}_\alpha$, then by (2.4.3) we have $[X, Y] = \alpha(X)Y$. Moreover, since $\mathfrak{t}$ is abelian, we have $[X, H] = 0$ for all $H \in \mathfrak{t}$. Therefore from the direct sum decomposition (2.4.4) it follows that $C_{\mathfrak{g}}(X)$ is the direct sum of $\mathfrak{t}$ and all root spaces $\mathfrak{g}_\alpha$ such that $\alpha(X) = 0$. By Definition 2.4.13, X is regular if and only if $C_{\mathfrak{g}}(X) = \mathfrak{t}$, which gives the result. ∎

Corollary 2.4.15. *An element $X \in \mathfrak{t}$ is non-regular if and only if $\alpha(X) = 0$ for some root $\alpha \in \Phi(\mathbb{G}, \mathbb{T})$.*

Proof. By the proof of Theorem 2.4.14,

$$C_{\mathfrak{g}}(X) = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in \Phi \\ \alpha(X) = 0}} \mathfrak{g}_\alpha$$

which has dimension strictly larger than $\dim \mathfrak{t}$ exactly when $\alpha(X) = 0$ for at least one $\alpha \in \Phi(\mathbb{G}, \mathbb{T})$. ∎

These results are essential in Section 4.1.

2.4.3 Twisted Levi subgroups

Levi subgroups are a class of connected reductive subgroups obtained from subsets of the simple roots.

Definition 2.4.16. Let $\mathbb{G}$ be a reductive group with maximal torus $\mathbb{T}$ and system of simple roots $\Delta \subset X^*(\mathbb{T})$. For a subset $\delta_s \subset \Delta$, let Φ_{δ_s} denote the subroot system generated by δ_s. The associated **standard Levi subgroup** is the group generated by $\mathbb{T}$ and the corresponding root subgroups, written

$$\mathbb{M}_{\delta_s} \;=\; \langle\, \mathbb{T},\, \mathbb{U}_\alpha : \alpha \in \Phi_{\delta_s} \,\rangle.$$

A **Levi subgroup** is a subgroup that is conjugate to a standard Levi subgroup. (See [Hum75, §26.2,§30.2].)

Theorem 2.4.17. *For $\mathbb{G} = \mathrm{SO}_5$, with maximal split torus $\mathbb{T}$ as above and $\Delta = \{a_{s_2}, a_{l_1}\}$, there are three proper standard Levi subgroups:*

$$\mathbb{M}_\varnothing = \mathbb{T}, \qquad \mathbb{M}_{a_{s_2}} = \langle \mathbb{T}, \mathbb{U}_{\pm a_{s_2}} \rangle, \qquad \mathbb{M}_{a_{l_1}} = \langle \mathbb{T}, \mathbb{U}_{\pm a_{l_1}} \rangle.$$

Concretely, in terms of matrices with respect to the form J (2.2.1), we have $\mathbb{M}_\varnothing = \mathbb{T}$ while

$$\mathbb{M}_{a_{s_2}} = \left\{ \begin{pmatrix} * & & & & \\ & * & & * & * \\ & & * & & \\ & * & & * & * \\ & * & & * & * \end{pmatrix} \right\} \cong \mathrm{SO}_3 \times \mathrm{GL}_1 \qquad \mathbb{M}_{a_{l_1}} = \left\{ \begin{pmatrix} * & * & & & \\ * & * & & & \\ & & * & * & \\ & & * & * & \\ & & & & * \end{pmatrix} \right\} \cong \mathrm{GL}_2.$$

where the $*$ entries indicate entries that may be nonzero.

Given a Levi subgroup $\mathbb{M}$ of $\mathbb{G}$ that is defined over F, it may not be $G = \mathbb{G}(F)$-conjugate to a standard Levi subgroup. That is, it is only $\mathbb{G}(E)$-conjugate for some finite field extension E/F. We introduce the following concept as in [HM08, Definition 2.40] and [DeB06, Remark 2.1.3].

Definition 2.4.18. A connected reductive F-subgroup $\mathbb{M} \subset \mathbb{G}$ is a **twisted Levi F-subgroup** if there exists a finite extension E/F such that $\mathbb{M}(E)$ is a Levi subgroup of $\mathbb{G}(E)$.

Those twisted Levi subgroups having compact center are precisely the subgroups needed in the construction of tame supercuspidal representations and will be our primary focus from now on.

Example 2.4.19. An elliptic maximal torus is a twisted Levi subgroup. Indeed, if $\mathbb{T}$ is an elliptic torus in $\mathbb{G}$, then $\mathbb{T}$ becomes E-split over its splitting field E/F, the minimal finite extension of F over which $\mathbb{T}$ is isomorphic to a product of copies of $\mathbb{G}_m$. In particular, an elliptic torus (of a semisimple group) remains compact over F, but becomes split after extension to its splitting field.

To give more examples, we need a lemma.

Lemma 2.4.20. *Let E/F be a quadratic extension and let J be a Hermitian form over E/F of rank m. Then the unitary group $U_{E/F}(J)$ is a twisted form of $\mathrm{GL}_m(F)$, that is, arises as the fixed points of $\mathrm{GL}_m(E)$ under a choice of $\mathrm{Gal}(E/F)$-action.*

Proof. Let $\sigma \in \mathrm{Gal}(E/F)$ denote the nontrivial element of the Galois group. Let J denote the Gram matrix of the Hermitian form. We define an action of σ on $\mathrm{GL}_m(E)$ by

$$\tilde{\sigma}(g) = J^{-1} \left(\sigma(g)^t\right)^{-1} J,$$

where $\sigma(g^t)$ denotes the conjugate transpose of g. It is straightforward to verify that $\tilde{\sigma}^2 = 1$.

The fixed points of this action are given by

$$g = J^{-1}(\sigma(g)^t)^{-1}J \iff \sigma(g)^t J g = J.$$

Hence the group of fixed points is

$$(\mathrm{GL}_m(E))^{\mathrm{Gal}(E/F)} = \{\, g \in \mathrm{GL}_2(E) \mid g^* J g = J \,\} = U_{E/F}(J).$$

Therefore, $U_{E/F}(J)$ is a twisted form of $\mathrm{GL}_m(F)$. ∎

For constructing supercuspidal representations of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, we need to know the possible twisted Levi subgroups with compact center. Recall the notation from Table 2.6.

Proposition 2.4.21. *Let (V, q) be a quadratic space of dimension 5. Up to isomorphism, the possible twisted Levi subgroups with compact center that can occur as subgroups of $\mathrm{SO}(q)$ are as follows:*

1. *Twisted Levi subgroups of type $\mathbb{M}_{a_{s_2}} = \mathrm{GL}_1 \times \mathrm{SO}_3$ have F-points that are isomorphic to*

   $$\mathrm{SO}(\mu, -\mu\alpha) \times \mathrm{SO}(q_3),$$

   where $\langle \mu, -\mu\alpha \rangle$ is a two-dimensional anisotropic quadratic form and q_3 is a three-dimensional quadratic form such that $\langle \mu, -\mu\alpha \rangle \oplus q_3 \simeq q$. We denote this group by ${}^{\alpha}\mathcal{S}_{\mu,s}$ if q_3 is isotropic and ${}^{\alpha}\mathcal{S}_{\mu,c}$ if q_3 is anisotropic.

2. *Twisted Levi subgroups of type $\mathbb{M}_{a_{l_1}} = \mathrm{GL}_2$ have F-points that are isomorphic to ${}^{\alpha}\mathcal{L}_s := U_{E/F}(\mathbb{H})$ if $q \simeq 2\mathbb{H} \oplus \langle 1 \rangle$, and to ${}^{\alpha}\mathcal{L}_c := U_{E/F}(\mathbf{C})$ if $q \simeq \mathbb{H} \oplus W_\delta$.*

Proof. By Theorem 2.4.17, the proper standard Levi subgroups of SO_5 are of type $\mathrm{GL}_1 \times \mathrm{SO}_3$ or GL_2. Hence any twisted Levi subgroup of $\mathrm{SO}(q)$ with compact center is an F-form of one of these types.

For $\mathrm{GL}_1 \times \mathrm{SO}_3$, the compact F-forms of GL_1 are precisely the anisotropic one-dimensional tori, which by Lemma 2.3.5 are of the form $\mathrm{SO}(E_{\alpha,\mu})$. The F-forms of SO_3 are of the form $\mathrm{SO}(q_3)$. Thus the subgroup is isomorphic to $\mathrm{SO}(E_{\alpha,\mu}) \times \mathrm{SO}(q_3)$ for choices such that $E_{\alpha,\mu} \oplus q_3 \simeq q$ (see Table 2.5).

For GL_2, Lemma 2.4.20 shows that $U_{E/F}(J)$, for a rank-2 form J, are twisted F-forms of GL_2. They have center $U_{E/F}(1)$ which is compact. These are either $U_{E/F}(\mathbb{H})$ or $U_{E/F}(\mathbf{C})$. By Proposition 2.3.6, $U_{E/F}(\mathbb{H}) \subset \mathrm{SO}(2\mathbb{H}) \subset \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ and $U_{E/F}(\mathbf{C}) \subset \mathrm{SO}(q_{4\mathrm{an}}) \subset \mathrm{SO}(\mathbb{H} \oplus W_1)$. ∎

Definition 2.4.22. A **Levi sequence** is an increasing chain of Levi subgroups

$$\mathbb{G}^0 \subset \mathbb{G}^1 \subset \cdots \subset \mathbb{G}^d = \mathbb{G}.$$

It is an F-Levi sequence if each of these groups is defined over F. It is a **tamely ramified twisted Levi sequence** if in addition, $Z(\mathbb{G}^0)/Z(\mathbb{G})$ is compact and each subgroup splits over a tamely ramified extension of F.

Remark 2.4.23. Note that $\mathrm{SO}(2\mathbb{H}\oplus\langle 1 \rangle)$ has absolute rank 2. In a Levi (or twisted Levi) sequence, the Levi subgroups must have strictly increasing ranks; therefore, any such sequence in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ can have length at most 3.

Levi subgroups can also be viewed structurally as centralizers of semisimple elements. By [Kim07, Lemma 2.4], if $X \in \mathfrak{g}$ is a semisimple element, then its centralizer $C_{\mathbb{G}}(X)$ (2.4.5) is a (twisted) Levi subgroup, and every (twisted) Levi subgroup arises as the centralizer of an element of the center of its Lie algebra.

We must define parabolic subgroups to define supercuspidal representations in Section 2.6.

Definition 2.4.24. Let $\mathbb{G}$ be a connected reductive group with maximal torus $\mathbb{T}$ and simple root system Δ. For a subset $\delta_s \subset \Delta$, the associated **standard parabolic subgroup** is defined by

$$\mathbb{P}_{\delta_s} \;=\; \langle\, \mathbb{M}_{\delta_s},\, \mathbb{U}_\alpha : \alpha \in \Phi^+ \setminus \Phi_{\delta_s} \,\rangle,$$

where Φ^+ denotes the set of positive roots relative to Δ. Thus, $\mathbb{P}_{\delta_s}$ is generated by the Levi subgroup $\mathbb{M}_{\delta_s}$ together with the root subgroups corresponding to the positive roots outside Φ_{δ_s}. It is called a **Borel subgroup** when $\delta_s = \emptyset$.

Parabolic subgroups admit a **Levi decomposition**

$$\mathbb{P}_{\delta_s} = \mathbb{M}_{\delta_s} \ltimes R_u(\mathbb{P}_{\delta_s}),$$

where $\mathbb{M}_{\delta_s}$ is a Levi subgroup and $R_u(\mathbb{P}_{\delta_s})$ is the unipotent radical, generated by the root subgroups $\mathbb{U}_\alpha$ with $\alpha \in \Phi^+ \setminus \Phi_{\delta_s}$. This decomposition highlights that each Levi subgroup is reductive, whereas proper parabolic subgroups are not.

2.5 The Bruhat–Tits building and Moy–Prasad filtrations

It is now well established that the representation theory of reductive p-adic groups is organized around the Bruhat–Tits building $\mathcal{B}(G) = \mathcal{B}(\mathbb{G}, F)$ of $G = \mathbb{G}(F)$, introduced in

[BT72, BT84]. The building encodes the structure of a natural family of compact open subgroups of G, called the parahoric subgroups $G_{x,0}$ for $x \in \mathcal{B}(\mathbb{G}, F)$, as well as their associated Moy–Prasad filtration subgroups $G_{x,r}$ and G_{x,r^+} [MP94, MP96].

We define an apartment $\mathscr{A}$ and its structure in Section 2.5.1 before specializing to the case that $\mathbb{G}$ is split over F, and particularly the case of $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. We then define filtrations of the torus and of root subgroups and subalgebras, leading to the Moy–Prasad filtration of G and of $\mathfrak{g}$ at a point $x \in \mathcal{B}(G)$, giving an explicit form for $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ and $x \in \mathscr{A}$.

Throughout this section, we refer to [Fin21b, §2.4], [Fin21a, §2], [HM08, §2.5], where these constructions are presented and summarized. Our work follows the descriptions given there.

2.5.1 The apartment

Let $\mathbb{T} \subset \mathbb{G}$ be a maximal F-split torus. Let $\Phi = \Phi(\mathbb{G}, \mathbb{T})$ be the (relative) root system of G with respect to $\mathbb{T}$. The **apartment** attached to $\mathbb{T}$ is the real affine space

$$\mathscr{A} = \mathscr{A}(\mathbb{G}, \mathbb{T}, F) = X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

We usually fix an origin (so treat $\mathscr{A}$ as a vector space) for convenience. For example, for each root $\alpha \in \Phi$ and each $m \in \mathbb{Z}$, define the affine functional

$$\psi_{\alpha,m}(x) \;=\; \alpha(x) - m \qquad (x \in \mathscr{A}),$$

and the associated affine hyperplane (or wall)

$$H_{\alpha,m} = \{\, x \in \mathscr{A} : \psi_{\alpha,m}(x) = 0 \,\} = \{\, x \in \mathscr{A} : \alpha(x) = m \,\}.$$

Set $\mathcal{H} = \bigcup_{\alpha\in\Phi} \bigcup_{m\in\mathbb{Z}} H_{\alpha,m}$, the collection of all affine root hyperplanes. The following definitions are from Humphreys [Hum90, Chapter 4]. The connected components of $\mathscr{A} \setminus \mathcal{H}$ are called **alcoves**. Lower-dimensional pieces of the decomposition (edges, vertices, and so on) are called **facets**. Each alcove is a facet of maximal dimension. When we remove only the linear hyperplanes, the connected components of $\mathscr{A} \setminus \bigcup_{\alpha\in\Phi} H_{\alpha,0}$ are called **Weyl chambers**.

Definition 2.5.1. Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Phi$ be a simple system and let θ be the unique highest root (Definition 2.4.8). Then the **fundamental alcove** in the apartment $\mathscr{A}$ is

$$\mathcal{C}_{\text{fund}} \;=\; \big\{\, x \in \mathscr{A} :\; \alpha_i(x) > 0 \text{ for all } i = 1, \ldots, \ell, \text{ and } \theta(x) < 1 \,\big\}.$$

Its bounding walls are $H_{\alpha_i,0} = \{x : \alpha_i(x) = 0\}$ for $i = 1, \ldots, \ell$ and $H_{\theta,1} = \{x : \theta(x) = 1\}$.

Write $S_{\alpha,m}$ for the affine reflection of $\mathscr{A}$ through $H_{\alpha,m}$.

Definition 2.5.2. The **Weyl group** associated with Φ is the group generated by the linear reflections $W = \langle S_{\alpha,0} : \alpha \in \Phi\rangle$. The **affine Weyl group** is defined as the group generated by all affine reflections $W_{\text{aff}} = \langle S_{\alpha,m} : \alpha \in \Phi, m \in \mathbb{Z}\rangle$.

These groups are related: we have $W_{\text{aff}} \cong \mathbb{Z}\check{\phi} \rtimes W$ where $\mathbb{Z}\check{\phi}$ is the coroot lattice. Let $N_{\mathbb{G}}(\mathbb{T})$ be the normalizer of the torus. In the case that $\mathbb{T}$ is a maximal torus, we have $W \cong N_{\mathbb{G}}(\mathbb{T})/C_{\mathbb{G}}(\mathbb{T})$. Therefore in this case, every $n \in N_{\mathbb{G}}(\mathbb{T})$ can be written as $n = ws$ for some lift $n \in N_{\mathbb{G}}(\mathbb{T})$ representing $w \in W$, and some $s \in \mathbb{T} = C_{\mathbb{G}}(\mathbb{T})$.

As a concrete illustration, consider the split group $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$.

Example 2.5.3. Consider $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$ with maximal F-split torus

$$\mathbb{T}(F) = \Big\{\mathrm{diag}(a, b, a^{-1}, b^{-1}, 1) : a, b \in F^\times\Big\},$$

we have $X_*(\mathbb{T}) \cong \mathbb{Z}^2$, hence $\mathscr{A} \cong \mathbb{R}^2$. The absolute root system is

$$\Phi = \{\pm a_{s_2},\ \pm a_{l_1},\ \pm(a_{s_2} + a_{l_1}),\ \pm(2a_{s_2} + a_{l_1})\}, \qquad \Delta = \{a_{s_2} = \epsilon_2,\ a_{l_1} = \epsilon_1 - \epsilon_2\}.$$

Each root $\alpha \in \Phi$ determines a family of affine hyperplanes

$$H_{\alpha,m} = \{(x_1, x_2) \in \mathbb{R}^2 : \alpha(x_1, x_2) = m\}, \qquad m \in \mathbb{Z},$$

and the collection of all such hyperplanes tessellates $\mathscr{A}$ into facets. Identifying $\mathscr{A}$ with $\mathbb{R}^2$ via the cocharacter lattice and writing $x = (x_1, x_2)$, we have

$$a_{s_2}(x) = x_2, \qquad a_{l_1}(x) = x_1 - x_2, \qquad \theta(x) = x_1 + x_2,$$

as in Example 2.4.10. By Definition 2.4.8, the fundamental alcove is

$$\begin{aligned}\mathcal{C}_{\text{fund}} &= \{\, (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0,\ x_1 - x_2 > 0,\ x_1 + x_2 < 1 \,\} \\ &= \{\, (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0,\ x_2 < x_1 < 1 - x_2 \,\}.\end{aligned}$$

Its three bounding walls are

$$H_{a_{s_2},0} = \{x_2 = 0\}, \qquad H_{a_{l_1},0} = \{x_1 - x_2 = 0\}, \qquad H_{\theta,1} = \{x_1 + x_2 = 1\}.$$

The closed fundamental alcove $\overline{\mathcal{C}}_{\text{fund}}$ is the right–isosceles triangle with vertices

$$(0, 0), \qquad (1, 0), \qquad \Big(\tfrac{1}{2}, \tfrac{1}{2}\Big),$$

Since $\mathbb{T}$ is a maximal torus, it is equal to its own centralizer $C_{\mathbb{G}}(\mathbb{T})$. Its normalizer $N_{\mathbb{G}}(\mathbb{T})$ is generated by $\mathbb{T}$ together with all permutation matrices (up to sign) that preserve $\mathbb{T}$ and the quadratic form and one can show the resulting Weyl group is $W(\Phi) = \langle S_{a_{s_2}}, S_{a_{l_1}}\rangle \cong D_4$. Hence, $|W| = 8$, while W_{aff} is infinite. We have drawn a part of $\mathscr{A}$ with the affine hyperplanes (some are labelled) and the fundamental alcove shaded in grey in Figure 2.2.

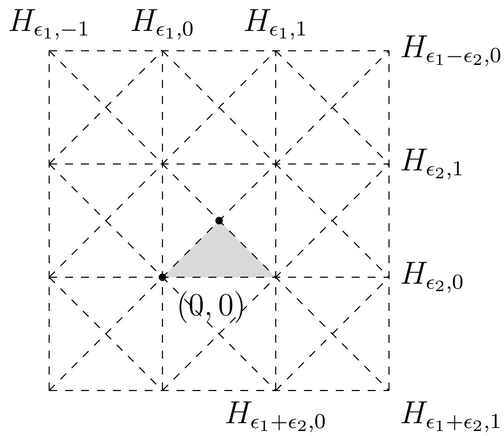

Figure 2.2: *A portion of the apartment of* $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$ *with affine hyperplanes marked with dashed lines. The shaded region is the fundamental alcove.*

2.5.2 Moy–Prasad filtrations

From this point onwards, we assume that $\mathbb{G}$ splits over F, that is, it contains an F-split maximal torus $\mathbb{T}$.

We begin by defining the parahoric subgroup of a torus. Since T is split, it is given by the following standard definition (see [Fin21b, §2.4]).

Definition 2.5.4. The parahoric subgroup of a split maximal torus $T = \mathbb{T}(F)$ is

$$T_0 = \{\, t \in T :\ \nu(\chi(t)) = 0 \text{ for all } \chi \in X^*(\mathbb{T})\}$$

which is a maximal bounded subgroup of T.

In fact if $\mathbb{T} \cong \mathbb{G}_m^n$ then $T_0 \cong (\mathcal{O}^\times)^n$, which we abbreviate as $\mathbb{T}(\mathcal{O}^\times)$.

Then T_0 has a natural decreasing filtration by compact open subgroups T_r for $r \geq 0$ given by $T_r = \{\, t \in T_0 :\ \nu(\chi(t) - 1) \geq r \text{ for all } \chi \in X^*(\mathbb{T}) \,\} = \mathbb{T}(1 + \mathcal{P}^{\lceil r \rceil})$ where $\lceil r \rceil = \min\{n \in \mathbb{Z}_{\geq 0} : n \geq r\}$. We also write

$$T_{r+} = \bigcup_{s>r} T_s$$

for any $r \geq 0$.

If $\mathfrak{t} = \mathbf{t}(F)$ denotes the F-points of the Lie algebra of $\mathbb{T}$, $\mathbf{t}$, then

$$\mathfrak{t}_0 = \{\, X \in \mathfrak{t} :\ \nu(\chi(X)) = 0 \text{ for all } \chi \in X^*(\mathbb{T}) \,\}$$

where again we abuse notation and write χ also for its differential $d\chi : \mathbf{t} \to \mathbb{G}_a$. If $\mathbf{t} \cong \mathbb{G}_a^n$ then $\mathfrak{t}_0 \cong \mathcal{O}^n =: \mathbf{t}(\mathcal{O})$. Then we correspondingly have

$$\mathfrak{t}_r = \{\, X \in \mathfrak{t} :\ \nu(\chi(X)) \geq r \text{ for all } \chi \in X^*(\mathbb{T}) \,\} = \mathbf{t}(\mathcal{P}^{\lceil r \rceil}), \quad \mathfrak{t}_{r+} = \bigcup_{s>r} \mathfrak{t}_s.$$

Example 2.5.5. Let $T = \mathbb{T}(F)$ be the diagonal maximal F-split torus of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, so that $\mathfrak{t} = \mathfrak{t}(F)$ is the maximal split Cartan subalgebra of $\mathfrak{g} = \mathfrak{so}(2\mathbb{H} \oplus \langle 1 \rangle)$, the F-points of the Lie algebra of SO_5. We defined the basis $\{\epsilon_1, \epsilon_2\}$ of $X^*(\mathbb{T})$ in Example 2.4.5. Then, for every positive real number r, we have

$$T_r := \{X \in \mathbb{T}(\mathcal{O}) \mid \nu(\epsilon_i(X) - 1) \geq r \text{ for } i = 1, 2\}$$

and for every $r \in \mathbb{R}$,

$$\mathfrak{t}_r := \{X \in \mathfrak{t} \mid \nu(\epsilon_i(X)) \geq r \text{ for } i = 1, 2\}.$$

Next, we recall from Theorem 2.4.11 that for each root $\alpha \in \Phi$, we have fixed an isomorphism $u_\alpha : \mathbb{G}_a \to \mathbb{U}_\alpha$ associated with a good choice of basis vector $\widetilde{X}_\alpha$ of the root space $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha(F)$. For each $x \in \mathscr{A}(\mathbb{G}, \mathbb{T}, F)$, these give Moy–Prasad filtrations of the root groups and their Lie algebras defined by

$$U_{\alpha,x,r} := u_\alpha\big(\mathcal{P}^{-\lfloor \alpha(x) - r \rfloor}\big), \qquad \forall\, r \geq 0; \qquad \mathfrak{g}_{\alpha,x,r} = \mathfrak{g}_\alpha(F)_{x,r} := \tilde{X}_\alpha\big(\mathcal{P}^{-\lfloor \alpha(x) - r \rfloor}\big), \qquad \forall\, r \in \mathbb{R}.$$

We similarly define $U_{\alpha,x,r^+} := \bigcup_{s>r} U_{\alpha,x,s}$ and $\mathfrak{g}_{\alpha,x,r^+} =:= \bigcup_{s>r} \mathfrak{g}_{\alpha,x,s}$. To rewrite these more explicitly, we use the identity

$$\min_{s>r}\big(-\lfloor t - s \rfloor\big) = 1 - \lceil t - r \rceil, \qquad t \in \mathbb{R},$$

with $t = \alpha(x)$ to deduce

$$U_{\alpha,x,r^+} = u_\alpha\big(\mathcal{P}^{1-\lceil \alpha(x) - r \rceil}\big), \qquad \mathfrak{g}_{\alpha,x,r^+} = \tilde{X}_\alpha\big(\mathcal{P}^{1-\lceil \alpha(x) - r \rceil}\big).$$

In particular

$$U_{\alpha,x,0} := u_\alpha(\mathcal{P}^{-\lfloor \alpha(x) \rfloor}), \quad U_{\alpha,x,0^+} = u_\alpha\big(\mathcal{P}^{1-\lceil \alpha(x) \rceil}\big).$$

Finally, having established the necessary groundwork, we now can define parahoric subgroups and their Moy–Prasad filtrations.

Definition 2.5.6. Let $\mathbb{G}$ and $\mathbb{T}$ as above. For every point $x \in \mathscr{A}(\mathbb{G}, \mathbb{T}, F)$, the associated **parahoric subgroup** $G_{x,0}$ is

$$G_{x,0} = \langle T_0, U_{\alpha,x} : \alpha \in \Phi \rangle.$$

Its pro-unipotent radical is

$$G_{x,0^+} = \langle T_{0^+}, U_{\alpha,x,0+} : \alpha \in \Phi \rangle$$

Note that $G_{x,0}$ and its pro-unipotent radical depend only on the facet of $\mathscr{A}$ containing x. For each $x \in \mathscr{A}$ one can define the Moy–Prasad filtration subgroups (due to Moy and Prasad in [MP96]) of $G_{x,0}$ as follows.

For any real number $r \geq 0$ and $x \in \mathscr{A}(\mathbb{G}, \mathbb{T}, F)$, the **Moy–Prasad filtration** of $G_{x,0}$ of depth r is given by

$$G_{x,r} \ = \ \big\langle\, T_r,\ U_{\alpha,x,r} : \alpha \in \Phi \,\big\rangle, \qquad G_{x,r^+} \ = \ \big\langle\, T_{r^+},\ U_{\alpha,x,r^+} : \alpha \in \Phi \,\big\rangle = \bigcup_{s>r} G_{x,s}.$$

Note that each $G_{x,r}$ is open and compact, and $G_{x,r} \lhd G_{x,0}$. However, in contrast to $G_{x,0}$, the higher-depth subgroups $G_{x,r}$ depend on the point x itself, not just on the facet containing x.

Analogously, we have similar filtrations on the Lie algebra $\mathfrak{g}$ given by

$$\mathfrak{g}_{x,r} \ = \ \mathfrak{t}_r \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_{\alpha,x,r}, \qquad \mathfrak{g}_{x,r^+} \ = \ \mathfrak{t}_{r^+} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_{\alpha,x,r^+}, \quad \forall r \in \mathbb{R}.$$

We define the filtration subspace $\mathfrak{g}^*_{x,r}$ of the dual of the Lie algebra $\mathfrak{g}^*$ by

$$\mathfrak{g}^*_{x,r} \ = \ \{\, X^* \in \mathfrak{g}^* \mid X^*(Y) \in \mathcal{P} \text{ for all } Y \in \mathfrak{g}_{x,(-r)+} \,\}. \tag{2.5.1}$$

Here, for each $Y \in \mathfrak{g}_{x,s}$, the expression $X^*(Y)$ denotes the value of $X^* \in \mathfrak{g}^* = \mathrm{Hom}_F(\mathfrak{g}, F)$ on Y.

Theorem 2.5.7. *Let $\mathbb{G} = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, let $\mathbb{T} \subset \mathbb{G}$ be the diagonal maximal torus given in (2.4.2), and let $\mathscr{A} \simeq \mathbb{R}^2$ be the corresponding apartment, with coordinates $x = (x_1, x_2)$ such that $\epsilon_1(x) = x_1$ and $\epsilon_2(x) = x_2$. Let $\mathcal{P}$ denote the maximal ideal of $\mathcal{O}$ and $U^m := 1 + \mathcal{P}^m$.*

(i) Group filtration. *For $r \geq 0$, the Moy–Prasad subgroup $G_{x,r}$ is*

$$G_{x,r} \ = \ \left\{ \begin{pmatrix} U^{\lceil r\rceil} & \mathcal{P}^{-\lfloor (x_1-x_2)-r\rfloor} & A_r & \mathcal{P}^{-\lfloor (x_1+x_2)-r\rfloor} & \mathcal{P}^{-\lfloor x_1-r\rfloor} \\ \mathcal{P}^{\lceil (x_1-x_2)+r\rceil} & U^{\lceil r\rceil} & \mathcal{P}^{-\lfloor (x_1+x_2)-r\rfloor} & B_r & \mathcal{P}^{-\lfloor x_2-r\rfloor} \\ C_r & \mathcal{P}^{\lceil (x_1+x_2)+r\rceil} & U^{\lceil r\rceil} & \mathcal{P}^{\lceil (x_1-x_2)+r\rceil} & \mathcal{P}^{\lceil x_1+r\rceil} \\ \mathcal{P}^{\lceil (x_1+x_2)+r\rceil} & D_r & \mathcal{P}^{-\lfloor (x_1-x_2)-r\rfloor} & U^{\lceil r\rceil} & \mathcal{P}^{\lceil x_2+r\rceil} \\ \mathcal{P}^{\lceil x_1+r\rceil} & \mathcal{P}^{\lceil x_2+r\rceil} & \mathcal{P}^{-\lfloor x_1-r\rfloor} & \mathcal{P}^{-\lfloor x_2-r\rfloor} & U^{\lceil r\rceil} \end{pmatrix} \cap G \right\},$$

where

$$A_r = \mathcal{P}^{-2\lfloor x_1-r\rfloor}, \qquad B_r = \mathcal{P}^{-2\lfloor x_2-r\rfloor}, \qquad C_r = \mathcal{P}^{2\lceil x_1+r\rceil}, \qquad D_r = \mathcal{P}^{2\lceil x_2+r\rceil}.$$

(ii) Lie algebra filtration. *For $r \in \mathbb{R}$, The associated Moy–Prasad lattice $\mathfrak{g}_{x,r} \subset \mathfrak{g} =$ Lie$(\mathbb{G})(F)$ is*

$$\mathfrak{g}_{x,r} \ = \ \left\{ \begin{pmatrix} \mathcal{P}^{\lceil r\rceil} & \mathcal{P}^{-\lfloor (x_1-x_2)-r\rfloor} & 0 & \mathcal{P}^{-\lfloor (x_1+x_2)-r\rfloor} & \mathcal{P}^{-\lfloor x_1-r\rfloor} \\ \mathcal{P}^{\lceil (x_1-x_2)+r\rceil} & \mathcal{P}^{\lceil r\rceil} & \mathcal{P}^{-\lfloor (x_1+x_2)-r\rfloor} & 0 & \mathcal{P}^{-\lfloor x_2-r\rfloor} \\ 0 & \mathcal{P}^{\lceil (x_1+x_2)+r\rceil} & \mathcal{P}^{\lceil r\rceil} & \mathcal{P}^{\lceil (x_1-x_2)+r\rceil} & \mathcal{P}^{\lceil x_1+r\rceil} \\ \mathcal{P}^{\lceil (x_1+x_2)+r\rceil} & 0 & \mathcal{P}^{-\lfloor (x_1-x_2)-r\rfloor} & \mathcal{P}^{\lceil r\rceil} & \mathcal{P}^{\lceil x_2+r\rceil} \\ \mathcal{P}^{\lceil x_1+r\rceil} & \mathcal{P}^{\lceil x_2+r\rceil} & \mathcal{P}^{-\lfloor x_1-r\rfloor} & \mathcal{P}^{-\lfloor x_2-r\rfloor} & \mathcal{P}^{\lceil r\rceil} \end{pmatrix} \cap \mathfrak{g} \right\}.$$

In particular, setting $r = 0$ yields the parahoric subgroup $G_{x,0}$ and the Lie algebra lattice $\mathfrak{g}_{x,0}$ respectively.

Proof. We work with the split group $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ of type B_2 and the maximal F-split torus $\mathbb{T}$ as above. Via the chosen identification $\mathscr{A} \simeq \mathbb{R}^2$, a point $x = (x_1, x_2)$ satisfies

$$a_{l_1}(x) = x_1 - x_2, \quad a_{s_2}(x) = x_2, \quad (a_{s_2} + a_{l_1})(x) = x_1, \quad (2a_{s_2} + a_{l_1})(x) = x_1 + x_2.$$

The result is then a direct application of the definitions of $G_{x,r}$, $G_{x,0}$, $\mathfrak{g}_{x,r}$ and $\mathfrak{g}_{x,0}$. Using the explicit bases of the root subgroups (Table 2.12) and of the Lie algebra root spaces (Table 2.11), together with the valuation bounds determined by $\alpha(x)$, one reads off the required powers of $\mathcal{P}$ in each matrix entry. The entries denoted A_r, B_r, C_r, D_r arise from the short root subgroups $U_{\pm a_{s_1}}$ and $U_{\pm a_{s_2}}$, through the quadratic terms in the exponential map. In particular, the entries in positions $(1,3), (2,4), (3,1), (4,2)$ come from the $\frac{t^2}{2}$-terms in $u_\alpha(t)$ for short roots in Table 2.12, hence their valuations are doubled relative to the linear root coordinates. ∎

In the non-split case the explicit description of parahoric subgroups and Moy-Prasad filtrations is considerably more subtle; we do not give formulas here, but these filtrations are defined in full generality in [Fin25] and exist for every reductive group over F.

A key feature is the **Moy–Prasad isomorphism**: for every $x \in \mathscr{A}$, for every $r > 0$, we have an isomorphism

$$e : \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+} \xrightarrow{\sim} G_{x,r}/G_{x,r^+} \tag{2.5.2}$$

between these quotients, which are both abelian groups. For $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, if $Y \in \mathfrak{g}_{x,r}$ then the class of $e(Y)$ in $G_{x,r}/G_{x,r+}$ can be represented by the matrix $I + Y$.

We summarize a result from Adler and Roche, which is equivalent to one given by Fintzen in [Fin21c, Lemma 2.2].

Proposition 2.5.8 ([AR00, Proposition 4.1])**.** *If G splits over a tamely ramified extension of F and the residue characteristic p does not divide $|W|$, where W is the Weyl group of G, there there exists a nondegenerate, F-valued, G-invariant, symmetric bilinear form*

$$\kappa : \mathfrak{g} \times \mathfrak{g} \longrightarrow F,$$

such that, under the associated identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ via $X^*(Y) = \kappa(X, Y)$, each Moy–Prasad filtration lattice $\mathfrak{g}_{x,r}$ is identified with its dual $\mathfrak{g}^*_{x,r}$.

For $G = \mathrm{SO}_5$, which satisfies the hypothesis of [AR00, Proposition 4.1] when $p \neq 2$, we may take the trace form

$$\kappa(X, Y) = \mathrm{Tr}(XY).$$

2.5.3 The Bruhat–Tits building $\mathcal{B}(\mathbb{G}, F)$

The Bruhat-Tits building is a geometric structure that glues together all the apartments of G in a consistent and well-defined manner. That is, when defining an apartment for the p-adic group $G = \mathbb{G}(F)$, one must choose a maximal F-split torus of G. Since all such tori are G-conjugate, there are many possible choices. Each choice produces a corresponding apartment, together with parahoric subgroups and Moy-Prasad filtration subgroups defined relative to that torus. To unify these descriptions, Bruhat and Tits introduced the Bruhat–Tits building. As such, a point from the building automatically belongs to some affine apartment.

Having fixed a maximal split torus $\mathbb{T} \subset \mathbb{G}$, and writing $N_G(T)$ for the normalizer of $T = \mathbb{T}(F)$ in $G = \mathbb{G}(F)$, we now recall the definition of the Bruhat–Tits building of G.

Definition 2.5.9. [Rab07, Definition 4.2] **The Bruhat-Tits building** is the set of equivalence classes

$$\mathcal{B}(G) = \mathcal{B}(\mathbb{G}, F) = (G \times \mathscr{A}(\mathbb{G}, \mathbb{T}, F))/ \sim$$

where $(g, x) \sim (h, y)$ if there exist an element $n \in N_G(T)$ such that $n{\cdot}y = x$ and $g^{-1}hn \in G_{x,0}$. Then G acts on $\mathcal{B}(\mathbb{G}, F)$ by the rule : $h \cdot (g, x) = (hg, x)$.

With this identification, for each $g \in G$, the apartment corresponding to ${}^gT$ is $g \cdot \mathscr{A}(\mathbb{G}, \mathbb{T}, F)$ (see [Rab07, §4]). Moreover, the Moy–Prasad filtration is compatible with the action on the building, in the sense that for all $g \in G$, $r \geq 0$, and $x \in \mathcal{B}(\mathbb{G}, F)$, one has ${}^gG_{x,r} = G_{g\cdot x,r}$ and ${}^gG_{x,r^+} = G_{g\cdot x,r^+}$ (see [Fin25, Definition 2.1.4]).

When $\mathbb{G}'$ is a twisted Levi subgroup of $\mathbb{G}$, such as an elliptic torus, then there is an embedding of $\mathcal{B}(\mathbb{G}', F)$ into $\mathcal{B}(\mathbb{G}, F)$ and we'll identify $\mathcal{B}(\mathbb{G}', F)$ with its image (see [KY17, §3]). Let us determine some properties of the buildings of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ and those of its twisted Levi subgroups.

Lemma 2.5.10. *The building $\mathcal{B}(\mathbb{T}, F)$ of an F-torus $\mathbb{T}$ with maximal F-split subtorus $\mathbb{S}$ consists of a single apartment, which is the affine space $X_*(\mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{R}$. In particular, if $\mathbb{S} = \{1\}$, so that $\mathbb{T}$ is elliptic, then $\mathcal{B}(\mathbb{T}, F)$ is a single point.*

Proof. A torus contains a unique maximal F-split subtorus, so the building has only one apartment. The root system is empty. If $\mathbb{T}$ is elliptic, then $\mathbb{T}$ has no nontrivial F-split subtorus, hence $\mathbb{S} = \{1\}$. This gives $X_*(\mathbb{S}) = \{0\}$, yielding $X_*(\mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{R} = \{0\}$. ∎

Recall that in Proposition 2.4.21 we have labeled the various isomorphism classes of twisted Levi subgroups of $G = \mathrm{SO}_5$ by the symbols ${}^{\alpha}\mathcal{S}_{\mu,s}$ and ${}^{\alpha}\mathcal{S}_{\mu,c}$ for those arising from the short roots and ${}^{\alpha}\mathcal{L}_s$ and ${}^{\alpha}\mathcal{L}_c$ for those arising from the long roots, with various parameter choices α, μ.

Lemma 2.5.11. *The buildings $\mathcal{B}({}^{\alpha}\mathcal{S}_{\mu,s})$ and $\mathcal{B}({}^{\alpha}\mathcal{L}_s)$ have one-dimensional apartments, while $\mathcal{B}({}^{\alpha}\mathcal{S}_{\mu,c})$ and $\mathcal{B}({}^{\alpha}\mathcal{L}_c)$ each consist of a single point.*

Proof. The dimension of each apartment of $\mathcal{B}(\mathbb{G}, F)$ is equal to its F-rank. By the notation in Proposition 2.4.21, the maximal F-split torus of each of the groups ${}^{\alpha}\mathcal{S}_{\mu,s}$ and ${}^{\alpha}\mathcal{L}_s$ has F-rank 1 (indicated by the subscript s) whereas the twisted Levi subgroups ${}^{\alpha}\mathcal{S}_{\mu,c}$ and ${}^{\alpha}\mathcal{L}_c$ are anisotropic (so in particular contain no F-split subtori), as indicated by the subscript c. The result follows. ∎

Let us now consider $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

Example 2.5.12. By Example 2.5.3, it follows that each apartment is 2-dimensional. Let $\mathscr{A}$ denote the standard apartment. The affine Weyl group acts simply transitively on the set of alcoves. The elements of G that preserve $\mathscr{A}$ are exacty $N_G(T)$. The action of $N_G(T)$ on $\mathscr{A}$ is given as follows. Write $n \in N_G(T)$ as a product $n = wt$ where w represents the action of the Weyl group and preserves the chosen origin, and $t = \mathrm{diag}(a, b, a^{-1}, b^{-1}, 1) \in T$. By [Tit79, §1], its action on an element $xe_1 + ye_2 \in \mathscr{A}$ (for $x, y \in \mathbb{R}$) is given by

$$t \cdot (xe_1 + ye_2) = (x - \nu(a))e_1 + (y - \nu(b))e_2.$$

Thus the action of T on $\mathscr{A}$ is by the integer lattice, which contains the coroot lattice as a subgroup of index 2. That is, the **fundamental domain** for the action of G on $\mathcal{B}(\mathbb{G}, F)$ is half the size of the fundamental alcove. We illustrate this in Figure 2.3. The additional reflections are indicated by dashed lines.

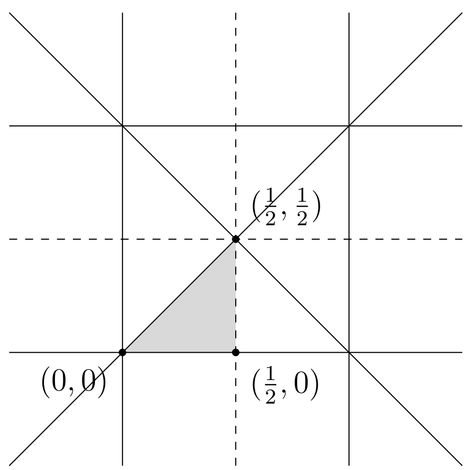

Figure 2.3: *A portion of the apartment of* $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ *with additional reflection hyperplanes induced by the action of G on $\mathcal{B}(G)$ indicated by dashed lines. The shaded region is a fundamental domain for the action, defined by $0 \le x_2 \le x_1 \le \frac{1}{2}$. Coordinates are relative to the choice of origin and basis $\{e_1, e_2\}$ of $X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$.*

For a point $x \in \mathcal{B}(\mathbb{G}, F)$, its stabilizer under this action is denoted G_x. If $\mathbb{G}$ is semisimple and simply connected, then G_x coincides with the parahoric subgroup at x, i.e. $G_x = G_{x,0}$ [Rab07, Lemma 4.5]. For more general reductive groups, $G_{x,0}$ is a subgroup of finite index of G_x. In the case of $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, $G_x \neq G_{x,0}$ when x lies on (some conjugate of) the dashed lines indicated in Figure 2.3 in which case $[G_x : G_{x,0}] = 2$.

The embeddings of the buildings of the maximal elliptic tori and the intermediate twisted Levi subgroups of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ will be described in Chapter 6.

2.6 Representation theory basics

In this section we recall the minimal notions from representation theory needed for the construction of tame supercuspidal representations. We begin with basic definitions of representations, then introduce smoothness and admissibility, quasi-characters, restriction and induction of representation and the concept of depth, as well as the definition of supercuspidal representations. We conclude with the definition of generic quasi-characters realized by generic elements.

2.6.1 Basic Definitions

In this section, G denotes an arbitrary p-adic group. We omit proofs and refer the reader to references such as [CN09] and [Cas18].

Definition 2.6.1. A **representation** (π, V) of a group G is a group homomorphism

$$\pi : G \longrightarrow \mathrm{GL}(V),$$

where V is a complex vector space and $\mathrm{GL}(V)$ is the group of invertible linear operators on V, which may be infinite dimensional. It is called **smooth** if for every $v \in V$ there exists an open compact subgroup $K \subset G$ such that $v \in V^K$, where

$$V^K = \{v \in V : \pi(k)v = v \text{ for all } k \in K\}.$$

It is called **admissible** if it is smooth and $\dim V^K < \infty$ for all open compact subgroups $K \subset G$.

For example, the trivial representation of G is the trivial homomorphism to $\mathrm{GL}(\mathbb{C})$. More generally, the simplest admissible representations are one-dimensional.

Definition 2.6.2. A **quasi-character** of G is a smooth one-dimensional representation

$$\phi : G \longrightarrow \mathbb{C}^\times,$$

that is, a continuous group homomorphism.

Remark 2.6.3. Since the image of a quasi-character is commutative, every quasi-character of a group G is trivial on its commutator subgroup $[G,G]$, and hence factors through the abelianization $G/[G,G]$. For a perfect group such as $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, we have $[G,G] = G$, and therefore G has no nontrivial quasi-characters. On the other hand, if $T \subset G$ is a torus, then it is an abelian group and it will have many nontrivial quasi-characters.

Quasi-characters are special examples of irreducible representations.

Definition 2.6.4. Let (π, V) be a representation of G. A subspace $W \subset V$ is called **G-invariant** if

$$\pi(g)w \in W \quad \text{for all } g \in G,\ w \in W,$$

that is, W is preserved by the action of G. The representation (π, V) is said to be **irreducible** if it has no proper closed nontrivial G-invariant subspaces.

A complex representation that is smooth and irreducible is admissible [Blo05, Proposition 2].

Theorem 2.6.5. (Moy–Prasad, [MP94, Theorem 5.2]) *Let (π, V) be an irreducible admissible representation of G. Then there exists a rational number $r \geq 0$ such that*

$$r = \inf\{s \in \mathbb{R} \mid \exists\, x \in \mathcal{B}(G) \quad \textit{with} \quad V^{G_{x,s+}} \neq 0\}.$$

*The number r is called the **depth** of π.*

The concept of depth refines the idea of smoothness by quantifying how deep inside these compact open subgroups $G_{x,r}$, as x varies over the building, one must go before nonzero fixed vectors appear.

This notion applies both to irreducible admissible representations and to quasi-characters of G, dividing them into those of depth zero and those of positive depth.

Lemma 2.6.6. *Let ϕ be a quasi-character of G. The depth of ϕ, denoted r_ϕ, is the smallest rational number $r \geq 0$ for which there exists $x \in \mathcal{B}(G)$ satisfying $\phi|_{G_{x,r+}} = 1$.*

Proof. Let (ϕ, V) be a quasi-character of depth r. Then there exists $x \in \mathcal{B}(G)$ such that $V^{G_{x,r+}} \neq 0$, so this subspace must be all of V since $\dim(V) = 1$. Thus $G_{x,r+} \subset \ker(\phi)$. Conversely, if $G_{x,r+} \subset \ker(\phi)$ then $V^{G_{x,r+}} \neq 0$. ∎

Example 2.6.7. The trivial quasi-character of G has depth zero, since it is trivial on all filtration subgroups $G_{x,r+}$ with $r \geq 0$.

Fix now an additive quasi-character $\Psi : F \to \mathbb{C}^\times$, and assume that G is such that we may identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the trace form, as in Proposition 2.5.8. Let $\mathfrak{z}$ be the F-points of the Lie algebra of the center of $\mathbb{G}$; that is $\mathfrak{z} = \boldsymbol{\mathfrak{z}}(F)$. and let $y \in \mathcal{B}(\mathbb{G}, F)$. Via the trace pairing, any $X \in \mathfrak{z}_{-r}$ defines an F-linear homomorphism $\mathfrak{g} \to F$ given by

$$Y \longmapsto \mathrm{Tr}(XY).$$

Composing this map with $\Psi : F \to \mathbb{C}^\times$ yields a quasi-character

$$Y \longmapsto \Psi\big(\mathrm{Tr}(XY)\big)$$

of the additive group $\mathfrak{g}$. Since $X \in \mathfrak{z}_{-r}$, one has $\mathrm{Tr}(XY) \in \mathcal{P}_F$ for all $Y \in \mathfrak{g}_{y,r^+}$, and therefore this quasi-character is trivial on $\mathfrak{g}_{y,r^+}$. Consequently, it factors through the quotient

$$\mathfrak{g}_{y,r}/\mathfrak{g}_{y,r^+}.$$

Via the Moy–Prasad isomorphism (2.5.2)

$$e : \mathfrak{g}_{y,r}/\mathfrak{g}_{y,r^+} \xrightarrow{\sim} G_{y,r}/G_{y,r^+},$$

this character descends to a quasicharacter of $G_{y,r}$ which has depth r, by Lemma 2.6.6. Since $X \in \mathfrak{z}$, it is trivial on the intersection of $G_{y,r}$ with $[G, G]$.

Definition 2.6.8. Let ϕ be a quasi-character $\phi : G \to \mathbb{C}^\times$ of depth r. If $X \in \mathfrak{z}_{-r}$ satisfies

$$\phi(e(Y)) = \Psi(\mathrm{Tr}(XY))$$

for all $Y \in \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$, then we say that ϕ is **realized by** X.

Every quasi-character of $G' \subset G$ is realized by some element $X \in \mathfrak{z}'_{-r}/\mathfrak{z}'_{-r^+}$, where $\mathfrak{z}'$ denotes the Lie algebra of the center of G' [Yu01, §9].

Remark 2.6.9. In the literature, one uses the condition $\phi(e(Y)) = \Psi(X^*(Y))$ for some $X^* \in \mathfrak{z}^*_{-r}$ but for $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ and $p \neq 2$ these are equivalent.

2.6.2 Supercuspidal Representations

Irreducible admissible representations of p-adic groups form a vast and intricate class. Among these, the supercuspidal representations occupy a distinguished place: they are, in a sense, the building blocks of all irreducible admissible representations.

To define them, we first need the notion of an induced representation.

Definition 2.6.10 (Induction)**.** Let (π_1, V_1) be a smooth representation of a subgroup $K \subseteq G$. The **induced representation** $\mathrm{Ind}_K^G \pi_1$ is defined on the space

$$\mathrm{Ind}_K^G V_1 = \Big\{ f : G \to V_1 \text{ locally constant } \;\Big|\; f(gk) = \pi_1(k^{-1}) f(g) \text{ for all } g \in G,\, k \in K \Big\},$$

with G acting by left translation:

$$(\mathrm{Ind}_K^G \pi_1)(g)f(x) = f(g^{-1}x), \qquad g, x \in G.$$

If K is open in G, the **compactly induced representation** $\text{c-Ind}_K^G \pi_1$ is the subrepresentation of $\mathrm{Ind}_K^G \pi_1$ consisting of functions with compact support modulo K:

$$\text{c-Ind}_K^G V_1 = \{ f \in \mathrm{Ind}_K^G V_1 \mid \mathrm{supp}(f) \text{ is compact modulo } K \}.$$

If G/K is compact, then $\mathrm{Ind}_K^G \pi_1 = \text{c-Ind}_K^G \pi_1$.

Now suppose $P \subset G$ is a parabolic subgroup. It can be written as a semidirect product $P = MN$, where M is a Levi subgroup and N its unipotent radical. Given a representation σ of M, we can extend it to a representation ρ of P by the rule $\rho_\sigma(mn) = \sigma(m)$.

Definition 2.6.11. An irreducible admissible representation (π, V) of G is said to be **supercuspidal** if it does not occur as a subrepresentation of $\mathrm{Ind}_P^G(\rho_\sigma)$ for any proper parabolic subgroup $P = MN$ of G and any representation σ of M.

Note that if F is a p-adic field, one speaks of supercuspidal representations, whereas if F is a finite field, the corresponding notion is that of **cuspidal** representations.

Example 2.6.12. If $G = T$ is a torus, then it has no roots and hence no proper parabolic subgroups. Therefore, when F is a p-adic field, every irreducible representation of T is supercuspidal. Likewise, when F is a finite field, every irreducible representation of T is cuspidal.

In general, determining whether a given representation is supercuspidal can be subtle, and constructing explicit examples is even more challenging. A key tool for such constructions is Mautner's Theorem, stated below.

Theorem 2.6.13 (Mautner's Theorem [Mau64]). *Let (π, V) be a smooth representation of a subgroup $K \subset G$, with K open and compact modulo the center of G. If the compactly induced representation $\text{c-Ind}_K^G \pi$ is irreducible, then it is supercuspidal (and admissible).*

All known constructions of supercuspidal representations use this result. They construct pairs (K, π) of compact-mod-center open subgroups and irreducible representations of such subgroups K for which the compactly induced representation $\text{c-Ind}_K^G \pi$ is irreducible. The supercuspidal representations of depth zero are constructed from cuspidal representations of certain finite groups of Lie type following the procedure in [MP96]. Supercuspidal representations of positive depth $r > 0$, which are the focus of this thesis, can be constructed using certain data (called Yu data here) following the procedure in [Yu01], as we outline in the next section.

2.7 Construction of tame supercuspidals

Having established the necessary background on local fields, algebraic groups, and their filtrations, we conclude this chapter by recalling the framework for the construction of tame supercuspidal representations.

This construction has a distinguished history, including important work by Howe [How77] and Adler [Adl96]. We follow the algorithm of Jiu-Kang Yu [Yu01] (with corrections and modifications by [Fin21a] and [FKS23]). It is based on a Yu datum; from a Yu datum we can construct a supercuspidal representation. Hakim and Murnaghan determined when two data give isomorphic supercuspidal representations. Ju-Lee Kim [Kim09] showed that for p sufficiently large, all irreducible supercuspidal representations would arise from this construction, and later Fintzen established minimal hypotheses under which this holds (that $\mathbb{G}$ splits over a tamely ramified extension of F, and that p does not divide the order of the Weyl group) [Fin21c].

In brief, the components of a Yu datum are:

1. A sequence of twisted Levi subgroups of $\mathbb{G}$, denoted $\mathbb{G}^0 \subset \mathbb{G}^1 \subset \cdots \subset \mathbb{G}^d \subseteq \mathbb{G}$ such that $Z(\mathbb{G}^0)/Z(\mathbb{G})$ is anisotropic;

2. A vertex of $\mathcal{B}(\mathbb{G}^0, F) \subset \mathcal{B}(\mathbb{G}, F)$;

3. A sequence of quasicharacters ϕ^i of G^i of depth r_i, with $0 < r_0 < r_1 < \cdots < r_d$ that are realized by $\mathbb{G}$**-generic** elements of $\mathfrak{z}^i_{-r_i}$, where $\mathfrak{z}^i$ is the F-points of the Lie algebra of the center of $\mathbb{G}^i$;

4. The ingredients for a depth zero supercuspidal representation of G^0 constructed from the vertex x, as in [MP96].

In this thesis, we produce the key components of the data for the F-split form of $\mathbb{G} = \mathrm{SO}_5$ across several chapters.

Since the center of $\mathbb{G}^i$ is contained in the center of $\mathbb{G}^0$, every twisted Levi subgroup occurring in a valid twisted Levi sequence must have anisotropic center. In Chapter 3 we construct all of the maximal elliptic (or anisotropic) tori of $\mathbb{G}$ up to conjugacy, and in Chapter 4 we construct all the twisted Levi subgroups of $\mathbb{G}$ that have anisotropic center, up to conjugacy.

We define generic elements (or $\mathcal{T}$-good elements) in Chapter 5. These govern the depth and compatibility conditions and determine which twisted Levi sequences will arise as part of the construction; these are the ones we will term *valid twisted Levi sequences*. Knowing all the generic elements is the essential ingredient for producing the sequence of quasi-characters.

In Chapter 6, we develop a method to identify the image of the building of each maximal elliptic torus $\mathcal{T}$ inside the building of $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ by explicitly constructing an embedding of $\mathcal{T}$ into $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, using a Witt basis. Using the embeddings of these tori, we find the buildings of the twisted Levi subgroups and determine their vertices as points in the building of G. From this data we can determine the reductive quotients of the twisted Levi subgroups at these vertices, from which one can generate the 4th part of the datum.

Finally, in Chapter 7, we explain how to assemble all these ingredients and sketch the construction of all the resulting irreducible (positive-depth) supercuspidal representations of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

Chapter 3

Conjugacy classes of elliptic tori of SO_n for $n \leq 5$, and U_2

The primary objective of this chapter is to determine the conjugacy classes of maximal elliptic tori in the split group $\mathbb{G} = \mathrm{SO}_5$, whose group of F-points is

$$G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle).$$

The meticulous calculations in this chapter give us the first essential components in the construction of twisted Levi sequences and, ultimately, in the realization of supercuspidal representations. For completeness and for readers interested in the non-split case, we also classify the elliptic tori in

$$\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, -\varpi, \varepsilon\varpi \rangle).$$

We also classify the elliptic tori in the split and non-split forms of the smaller groups U_2, and SO_3. These groups arise naturally as factors of twisted Levi subgroups of SO_5, and in the final section we provide some independent verification of our detailed calculations by identifying the images of these tori in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ via the embeddings we established in Sections 2.2.6 and 2.3.3.

In Section 3.1, we outline the general algorithm for classifying elliptic tori in classical groups via the algebra–torus correspondence described in [Mor91, §1]. In Section 3.2 we classify the maximal elliptic tori in all forms of SO_3, SO_4, and SO_5, and Section 3.3 is devoted to the case of U_2. We conclude in Section 3.4 by identifying how the maximal elliptic tori of various forms of $\mathrm{SO}_2 \times \mathrm{SO}_3$ and U_2 embed into $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. Section 3.2 extends previous work by Chinner, who classified the maximal elliptic tori in SO_4 [Chi21, Table 4.2]. To our knowledge, the classification of maximal elliptic tori in SO_5 is not available in the literature.

Throughout the chapter, we assume F is a non-archimedean local field of odd residual characteristic. We write $\varepsilon \in \mathcal{O}^\times$ for a fixed non-square unit and $\varpi \in \mathcal{P}$ for a uniformizer. For any finite extension E/F, we similarly write $\varepsilon_E \in \mathcal{O}_E^\times$ and $\varpi_E \in \mathcal{P}_E$ for corresponding elements.

3.1 An algorithm for classifying elliptic tori

In [Mor91, §1], Lawrence Morris presented a general framework for the classification of the conjugacy classes of maximal elliptic tori in classical groups G via an explicit bijection with certain equivalence classes of semisimple commutative subalgebras.

To treat the special orthogonal and unitary groups together, let us define some common notation. When G is a unitary group, we let $\mathcal{K}$ denote the corresponding quadratic extension F, and we set $\sigma_0 \in \mathrm{Gal}(\mathcal{K}/F)$ to be the nontrivial element. When G is a special orthogonal group, then we instead set $\mathcal{K} = F$ and $\sigma_0 = \mathrm{Id}$. Thus in both cases, G acts on an n-dimensional $\mathcal{K}$-vector space V preserving a σ_0-hermitian form $(\cdot, \cdot)$. This data defines an anti-involution $\widetilde{\sigma}$ on $\mathrm{GL}(V)$ and $\mathrm{End}(V)$ such that $G \subset \{x \in \mathrm{GL}(V) | x\widetilde{\sigma}(x) = I\}$. Explicitly, fixing a basis and writing J for the Gram matrix of $(\cdot, \cdot)$, the anti-involution is given by

$$\widetilde{\sigma}(x) = J^{-1}\sigma_0(x)^{\top} J,$$

where σ_0 acts entry-wise. In the orthogonal case this specialises to $\widetilde{\sigma}(x) = J^{-1}x^{\top}J$, with J as in (2.2.1), recovering the condition $x^{\top}Jx = J$ from the definition of $\mathrm{SO}(V)$; in the unitary case it becomes $\widetilde{\sigma}(x) = J^{-1}x^{*}J$, with $x^{*} = \sigma(x)^{\top}$, recovering the condition $g^{*}Jg = J$ from Definition 2.3.4.

3.1.1 The Algebra–Torus Correspondence

Let G be as above and let T be a maximal elliptic torus in $G \subset \mathrm{GL}(V)$. Its centralizer in $\mathrm{End}(V)$,

$$\mathcal{A} = \{\, y \in \mathrm{End}(V) : xy = yx \quad \forall x \in T \,\},$$

is a maximal commutative semisimple F-algebra of dimension $\dim_F(V)$. The involution $\widetilde{\sigma}$ on $\mathrm{End}(V)$ restricts to an involution σ^{*} on $\mathcal{A}$, and Morris observed that we recover T from $\mathcal{A}$ simply as $T = \mathcal{A} \cap G$. Identifying $\mathcal{K}$ with scalar matrices in $\mathcal{A}$, we have $\sigma^{*}|_{\mathcal{K}} = \sigma_0$.

Since every commutative, semisimple F-algebra is isomorphic to a direct product of separable field extensions of F (see [Chi21, Theorem 3.2.17], for example), we can write

$$\mathcal{A} \cong E_1 \oplus \cdots \oplus E_m,$$

where each E_i/F is a finite separable field extension. In [Mor91, §1], Morris proves these are each nontrivially stable under σ^{*} (hence of even degree over F), except that in the odd orthogonal case exactly one factor is $\mathcal{K} = F$. Then it follows that the torus T decomposes as a direct product of subgroups

$$T_i = \{x \in E_i^{\times} \mid x\sigma^{*}(x) = 1\}.$$

By [Mor91, §1.5], the σ_0-sesquilinear form $(\cdot, \cdot)$ on V induces a non-degenerate $\mathcal{A}$-valued σ^{*}-sesquilinear form on V (viewed as an $\mathcal{A}$-module)

$$f_{\mathcal{A}} : V \times V \to \mathcal{A}$$

defined by the relation that

$$(av, w) = \mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(a\, f_{\mathcal{A}}(v, w))$$

for all $a \in \mathcal{A}$ and $v, w \in V$. Conversely, every such form $f_{\mathcal{A}}$ defines a σ_0-sesquilinear form on V, as follows.

Lemma 3.1.1. *If $f_{\mathcal{A}}$ is a nondegenerate σ^*-sesquilinear $\mathcal{A}$-valued form on V, then the $\mathcal{K}$-valued form defined by*

$$(v, w) \longmapsto \mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(f_{\mathcal{A}}(v, w))$$

is a nondegenerate σ_0-sesquilinear form on V.

Proof. We need to show that $\sigma_0(\mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(f_{\mathcal{A}}(v, w))) = \mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(f_{\mathcal{A}}(w, v))$. We first claim that for all $x \in \mathcal{A}$, $\mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(\sigma^*(x)) = \sigma_0(\mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(x))$. Indeed:

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(\sigma^*(x)) &= \sum_{\lambda \in \mathrm{Aut}(\mathcal{A}/\mathcal{K})} \lambda(\sigma^*(x)) && \text{(by definition 2.1.2)} \\
&= \sum_{\lambda \in \mathrm{Aut}(\mathcal{A}/\mathcal{K})} \sigma^*(\lambda(x)) && \text{(since conjugation by } \sigma^* \text{ preserves } \mathrm{Aut}(\mathcal{K}/F)) \\
&= \sigma^*\left(\sum_{\lambda \in \mathrm{Aut}(\mathcal{A}/\mathcal{K})} \lambda(x) \right) && \text{(by the linearity of } \sigma^*) \\
&= \sigma^*(\mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(x)) \\
&= \sigma_0(\mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(x)) && \text{(since } \sigma^*|_{\mathcal{K}} = \sigma_0).
\end{aligned}$$

Now, let $v, w \in V$ be arbitrary. By our claim,

$$\sigma_0(\mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(f_{\mathcal{A}}(v, w))) = \mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(\sigma^*(f_{\mathcal{A}}(v, w))).$$

Since $f_{\mathcal{A}}$ is a σ^*-sesquilinear form, we have $\sigma^*\big(f_{\mathcal{A}}(v, w)\big) = f_{\mathcal{A}}(w, v)$, so that we conclude

$$\sigma_0(\mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(f_{\mathcal{A}}(v, w))) = \mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(f_{\mathcal{A}}(w, v)),$$

as required. Since there exists $a \in \mathcal{A}$ such that $\mathrm{Tr}_{\mathcal{A}/\mathcal{K}}(a) \neq 0$, it follows that if $v \in V$ and $w \in V$ such that $f_{\mathcal{A}}(v, w) \neq 0$ then there is some $a \in \mathcal{A}$ for which $\langle v, aw \rangle \neq 0$, proving nondegeneracy. ∎

Morris then defined a *triple* $(\mathcal{A}, \Phi, f_{\mathcal{A}})$ consisting of a maximal commutative semisimple F-algebra $\mathcal{A}$ of $\mathrm{End}(V)$ arising via an embedding $\Phi : \mathcal{A} \to \mathrm{End}(V)$ and a non-degenerate σ^*-sesquilinear $\mathcal{A}$-form $f_{\mathcal{A}}$. He showed that each such triple defines a maximal elliptic torus in the group G defined by the σ_0-sesquilinear form of Lemma 3.1.1, by sending $\mathcal{A}$ to $\mathcal{A} \cap G$.

Here G is the group of isometries of a fixed σ_0-sesquilinear form $(\cdot, \cdot)$ on V, and the constructions above, the algebra $\mathcal{A}$, the involution σ^*, and the triple $(\mathcal{A}, \Phi, f_{\mathcal{A}})$ apply to any such $G \subset \mathrm{GL}(V)$.

Two triples $(\mathcal{A}, \Phi, f_{\mathcal{A}})$ and $(\mathcal{A}', \Phi', f_{\mathcal{A}'})$ are said to be *equivalent* if there is an F-algebra isomorphism $\mathcal{A} \to \mathcal{A}'$ compatible with Φ and Φ' (that is, which intertwines the $\mathcal{A}$-module structure on V with the $\mathcal{A}'$-module structure on V) and sending $f_{\mathcal{A}}$ to $f_{\mathcal{A}'}$.

Letting A_G denote the set of equivalence classes of triples giving rise to $\langle \cdot, \cdot \rangle$ via Lemma 3.1.1, and T_G the set of conjugacy classes of maximal elliptic tori in G, the main theorem of [Mor91, §1] is the following.

Theorem 3.1.2 ([Mor91, Theorem 1.6]). *With A_G and T_G as above, the map $\Theta : A_G \to T_G$ defined by*

$$\Theta(\mathcal{A}, \Phi, f_{\mathcal{A}}) = \mathcal{A} \cap G$$

induces a bijection. Each maximal elliptic torus arises from a unique equivalence class of triples, and every triple determines a unique conjugacy class of maximal elliptic tori.

A nicely elaborated proof can be found in [Chi21, Theorem 3.4.5].

3.1.2 Algorithm for constructing maximal elliptic tori

We now explain how Theorem 3.1.2 provides a concrete method for constructing maximal elliptic tori in a special orthogonal or unitary group $G \subset \mathrm{GL}(V)$ via equivalence classes of triples. The procedure, as given in [Mor91, §1.11], is as follows. Recall that $n = \dim(V)$.

Proposition 3.1.3 (Construction procedure). *A maximal elliptic torus in G may be constructed as follows:*

1. *Choose an n-dimensional commutative semisimple $\mathcal{K}$-algebra*

   $$\mathcal{A} \cong E_1 \oplus \cdots \oplus E_m,$$

   where each $E_i \supset \mathcal{K}$ is a finite separable field extension of F of even degree, except $E_m = F$ when G is a special orthgonal group and n is odd.

2. *Fix an involution $\sigma^* \in \mathrm{Aut}(\mathcal{A}/F)$ that restricts to a nontrivial involution on each factor E_i (except E_m, in the case of SO_n with n odd), such that $\sigma^*|_{\mathcal{K}} = \sigma_0$.*

3. *Choose $\mu^* \in (\mathcal{A}^\times)^{\sigma^*}$, modulo the group generated by the norm subgroup $N_{\sigma^*}(\mathcal{A}^\times) = \{a\sigma^*(a) : a \in \mathcal{A}^\times\}$ and the automorphisms of $\mathcal{A}$ arising from permuting isomorphic factors. This determines a non-degenerate σ^*-sesquilinear form on the $\mathcal{A}$-module $V = \mathcal{A}$ by the rule $f_{\mathcal{A}}(v, w) = \mu^* \, v \, \sigma^*(w)$ for all $v, w \in V$.*

4. *Define the induced σ_0-sesquilinear form on V, viewed now as a $\mathcal{K}$-vector space, by $(v, w) = \mathrm{Tr}_{\mathcal{A}/\mathcal{K}}\big(f_{\mathcal{A}}(v, w)\big)$.*

Then $T = \{a \in \mathcal{A}^\times : a\sigma^*(a) = 1\}$ is a maximal elliptic torus in the group G that preserves the form $(\cdot, \cdot)$ and its Lie algebra is $\mathfrak{t} = \{a \in \mathcal{A} : a + \sigma^*(a) = 0\}$.

We illustrate this proposition by showing how it applies to the simplest case: where G is an F-form of SO_2.

Example 3.1.4. Let us use Proposition 3.1.3 to produce all maximal elliptic tori in the various forms of SO_2. In this case, $n = 2$ and $\mathcal{K} = F$, so there are only three choices for the algebra $\mathcal{A}$ in (1): $\mathcal{A} = K = F(\sqrt{\beta})$ for some $\beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$. In each case, the involution σ^* of (2) must be the nontrivial element of $\mathrm{Gal}(K/F)$. The parameter μ^* of (3) is any element of the quotient $(\mathcal{A}^\times)^{\sigma^*}/N_{\mathcal{A}/F}(\mathcal{A}^\times) = F^\times/N_{K/F}(K^\times)$. Then the nondegenerate σ^*-sesquilinear form on $V = K$ is defined on $v, w \in K$ by $f_{\mathcal{A}}(v, w) = \mu^* v \sigma^*(w)$. This yields, in part (4), the quadratic form on K, viewed as an F vector space, given on $v \in K$ by $q(v) = \mathrm{Tr}_{K/F}(\mu^* v \sigma^* v)$. Choosing coordinates $v = v_1 + v_2\sqrt{\beta} \in K$ gives the explicit form

$$q(v_1, v_2) = \mathrm{Tr}_{K/F}(\mu^*(v_1^2 - \beta v_2^2) = 2\mu^* v_1^2 - 2\mu^*\beta v_2^2$$

and thus $q = \langle 2\mu^*, -2\mu^*\beta\rangle$. Note that one choice of a set of representatives is given in Lemma 2.1.4(4): $\{1, \varepsilon\}$ if K/F is ramified and $\{1, \varpi\}$ if K/F is unramified. Scaling these by $\frac{1}{2}$ gives another set of representatives, potentially permuting the two cosets. With this scaling, the proposition yields six tori, each one preserving a different anisotropic quadratic form

$$\begin{aligned} \beta = \varepsilon: &\quad T \cong F(\sqrt{\varepsilon})^1 \quad q = \langle 1, -\varepsilon\rangle \text{ and } \langle \varpi, -\varepsilon\varpi\rangle \\ \beta = \varpi: &\quad T \cong F(\sqrt{\varpi})^1 \quad q = \langle 1, -\varpi\rangle \text{ and } \langle \varepsilon, -\varepsilon\varpi\rangle \\ \beta = \varepsilon\varpi: &\quad T \cong F(\sqrt{\varepsilon\varpi})^1 \quad q = \langle 1, -\varepsilon\varpi\rangle \text{ and } \langle \varepsilon, -\varepsilon^2\varpi\rangle. \end{aligned}$$

However, for any μ we have $\mathrm{SO}(\langle 1, -\beta\rangle) = \mathrm{SO}(\langle \mu, -\mu\beta\rangle)$. There are only three F-forms of SO_2, up to isomorphism, and each one contains one maximal elliptic torus. Clearly, since each one is in fact a torus.

The essential point is that (3) gives precisely all the equivalence classes of σ^*-sesquilinear $\mathcal{A}$-valued forms on V ([Mor91, §1.7]), and that (4) sorts the corresponding triples thus produced according to the various possible symmetric (if $\mathcal{K} = F$) or unitary (if $[\mathcal{K} : F] = 2$) forms that arise. Where two quadratic forms have isomorphic special orthogonal groups, the algorithm produces the maximal elliptic tori in each (which must therefore be pairwise isomorphic).

We apply this theorem to the special orthogonal groups with $3 \le n \le 5$ and the unitary group with $n = 2$ in the following sections.

3.2 Conjugacy classes of elliptic tori in SO_n for $n \le 5$

The groups SO_2 are themselves tori, as noted in Example 2.4.2. Thus, in this section, we compute representatives of the conjugacy classes of all maximal elliptic tori of each of the F-forms of SO_n, with $3 \le n \le 5$. That is, we will apply Proposition 3.1.3 with $\mathcal{K} = F$ and $\sigma_0 = \mathrm{Id}$. We will write q for the quadratic form corresponding to the symmetric bilinear form output by this algorithm.

3.2.1 Conjugacy classes of elliptic tori in SO_3

Recall from Table 2.6 that there are exactly eight 3-dimensional non-degenerate quadratic forms (q, V) up to isometry. Consequently, Proposition 3.1.3 will generate outputs that account for all these isometry classes.

The 3-dimensional commutative semisimple algebras $\mathcal{A}$ that meet the requirement of Proposition 3.1.3(1) are of the form $\mathcal{A} \cong K \oplus F$ where K is a quadratic field extension of F; in this case, the involution of Proposition 3.1.3(2) must satisfy that $\sigma^*|_K$ is the nontrivial element of $\mathrm{Gal}(K/F)$. Thus we write $\sigma^* = (\sigma, \mathrm{Id})$ where $\sigma \in \mathrm{Gal}(K/F) \setminus \{\mathrm{Id}\}$. Thus in Proposition 3.1.3(3) we have $\mathcal{A}^{\times\sigma^*} = F^\times \times F^\times$ and $N_{\sigma^*}(\mathcal{A}^\times) = N_{K/F}(K^\times) \times F^{\times 2}$, implying $\mu^* = (\mu, \lambda)$ with μ running over a set of representatives for $F^\times/N_{K/F}(K^\times)$ and λ running over a set of representatives for $F^\times/F^{\times 2}$.

By Lemma 2.1.4, $F^\times/N_{K/F}(K^\times) = \{1, \mu\}$, where we may choose $\mu = \varepsilon$ if K/F is ramified and $\mu = \varpi$ otherwise. Note that scaling our set of coset representatives by any element of $F^\times$ simply permutes the classes.

Thus, for example, $\{\frac{1}{2}, \frac{\mu}{2}\}$ is also a set of representatives for the cosets of $N_{K/F}(K^\times)$ in $F^\times$; this will be a convenient normalization. On the other hand, by Proposition 2.1.3, $F^\times/F^{\times^2}$ is represented by $\{1, \varepsilon, \varpi, \varepsilon\varpi\}$. Thus in total there are eight choices $\mu^* = \left(\frac{\mu}{2}, \lambda\right)$, where $\mu \in F^\times/N_{K/F}(K^\times)$ and $\lambda \in F^\times/F^{\times 2}$, each giving a inequivalent form $f_{\mathcal{A}}$.

Lemma 3.2.1. *Let $\mathcal{A} = K \oplus F$, where $K = F[\sqrt{\alpha}]$ is a quadratic extension of F. Let $\mu^* = (\frac{\mu}{2}, \lambda)$ such that $\mu \in F^\times/N_{K/F}(K^\times)$ and $\lambda \in F^\times/(F^\times)^2$. Then the quadratic form defined by $q(v) = \mathrm{Tr}_{\mathcal{A}/F}(\mu^* v \sigma^*(v))$ for $v \in \mathcal{A}$ is isometric to $\langle \mu, -\mu\alpha, \lambda\rangle$.*

Proof. For $v_1 = a + b\sqrt{\alpha} \in K$, $v_2 \in F$ and $\mu^* = (\frac{\mu}{2}, \lambda)$, we have

$$q(v_1, v_2) = \mathrm{Tr}_{K/F}\left(\frac{\mu}{2} v_1 \sigma(v_1)\right) + \lambda v_2^2.$$

Since $v_1\sigma(v_1) = a^2 - b^2\alpha$ and $a, b, \mu, \alpha \in F$, we have

$$\mathrm{Tr}_{K/F}\left(\frac{\mu}{2} v_1 \sigma(v_1)\right) = \frac{\mu}{2}\mathrm{Tr}_{K/F}(a^2 - b^2\alpha) = \mu(a^2 - b^2\alpha).$$

Thus, with respect to the basis $B = \{(1, 0), (\sqrt{\alpha}, 0), (0, 1)\}$, we have $q = \langle \mu, -\mu\alpha, \lambda\rangle$, as required. ∎

We summarize our steps in Algorithm 1. Since the resulting tori are parametrized by the triples (α, μ, λ), we denote the corresponding torus ${}^{\alpha}\mathcal{T}_{[\mu,\lambda]}$.

Algorithm 1 Generating a quadratic form q from $\mathcal{A} = K \oplus F$, where K is a quadratic extension of F, and its associated elliptic torus in $\mathrm{SO}(q)$.

Input: $(n = 3, \mathcal{K} = F)$
Choose a quadratic extension of F, $K = F(\sqrt{\alpha})$ for $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$, and set $\mathcal{A} = K \oplus F$.
Set $\sigma^* = (\sigma, \mathrm{Id}) \in \mathrm{Aut}(\mathcal{A}/F)$, where the involution $\sigma \in \mathrm{Gal}(K/F)$ is non-trivial.
Choose an element $\mu^* = (\frac{\mu}{2}, \lambda)$ such that $\mu \in F^\times/N_{K/F}(K^\times)$ and $\lambda \in F^\times/F^{\times^2}$.
Compute the associated quadratic form $q = \langle \mu, -\mu\alpha, \lambda \rangle$.
Then $\{a \in \mathcal{A} : a\sigma^*(a) = 1\}$ is a maximal elliptic torus of $\mathrm{SO}(q)$ denoted by ${}^\alpha\mathcal{T}_{[\mu,\lambda]}$.
Output: the pairs $(q, {}^\alpha\mathcal{T}_{[\mu,\lambda]})$

There are 3 choices for K, namely $F(\sqrt{\varepsilon}), F(\sqrt{\varpi}), F(\sqrt{\varepsilon\varpi})$. For each of them, we have two distinct choices of $\mu \in F^\times/N_{K/F}(K^\times)$ and four distinct choices of $\lambda \in F^\times/(F^\times)^2$. Therefore, we have $3 \times 2 \times 4$ classes of elliptic tori, contained in the disjoint union of the groups $G = \mathrm{SO}(V)$, as V runs over the isometry classes of 3-dimensional quadratic forms.

We now want to answer the more precise question: What are the tori in a given group $\mathrm{SO}(q)$? To answer this question, we now need to sort the output of the algorithm by isometry class of quadratic form. Our first observation uses the notion of q-partners, as defined in 2.2.14.

Proposition 3.2.2. *Let $K = F(\sqrt{\alpha})$, where $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ and let $\lambda \in F^\times/F^{\times 2}$. Write $F^\times/N_{K/F}(K^\times) = \{\mu_1, \mu_2\}$. Then exactly one of ${}^\alpha\mathcal{T}_{[\mu_1,\lambda]}$ and ${}^\alpha\mathcal{T}_{[\mu_2,\lambda]}$ preserves an anisotropic form q, while the other preserves an isotropic one.*

Proof. By Lemma 3.2.1, the form associated to the choice (α, μ_1, λ) is $q = \langle \mu_1, -\mu_1\alpha, \lambda \rangle$ while the form associated to the choice (α, μ_2, λ) is $q' = \langle \mu_2, -\mu_2\alpha, \lambda \rangle$. By Lemma 2.2.13, since one of $\{\mu_1, \mu_2\}$ is a norm and the other is not, the forms $q_1 = \langle \mu_1, -\mu_1\alpha \rangle$ and $q_2 = \langle \mu_2, -\mu_2\alpha \rangle$ have opposite Hasse invariants and hence by Definition 2.2.14 are q-partners. Thus by Theorem 2.2.15(4),

$$q^\triangledown = (q_1 \oplus \langle \lambda \rangle)^\triangledown = q_1^\triangledown \oplus \langle \lambda \rangle = q'.$$

By Theorem 2.2.15(5), exactly one of $\{q, q^\triangledown\}$ is anisotropic. ∎

We now list the 24 pairs output by Algorithm 1 in Table 3.1, identifying the isometry class of each of the quadratic forms produced (notation as in Table 2.6). Note that we have made clever choices for the set of representatives of $F^\times/N_{K/F}(K^\times)$ in some cases so that the resulting list is independent of whether or not -1 is a square.

Finally, recall from Table 2.9 that although there are 8 isometry classes of quadratic forms of dimension 3, there are only two distinct isomorphism classes of special orthogonal groups, represented by

$$\mathrm{SO}(\mathbb{H} \oplus \langle 1 \rangle) \qquad \text{and} \qquad \mathrm{SO}(\langle \varepsilon, -\varepsilon\varpi, \varpi \rangle).$$

$\mathcal{A} = K \oplus F$	$\mu \backslash \lambda$	$\lambda = 1$	$\lambda = \varepsilon$	Class of ${}^{\alpha}\mathcal{T}_{[\mu,\lambda]}$
$F(\sqrt{\varepsilon}) \oplus F$	1	$\langle 1, -\varepsilon, 1\rangle \simeq \mathbb{H} \oplus \langle \varepsilon \rangle$	$\langle 1, -\varepsilon, \varepsilon\rangle \simeq \mathbb{H} \oplus \langle 1 \rangle$	${}^{\varepsilon}\mathcal{T}_{[1,\lambda]}$
	ϖ	$\langle \varpi, -\varepsilon\varpi, 1\rangle \simeq W_{\varepsilon}$	$\langle \varpi, -\varepsilon\varpi, \varepsilon\rangle \simeq W_1$	${}^{\varepsilon}\mathcal{T}_{[\varpi,\lambda]}$
$F(\sqrt{\varpi}) \oplus F$	-1	$\langle -1, \varpi, 1\rangle \simeq \mathbb{H} \oplus \langle \varpi \rangle$	$\langle -1, \varpi, \varepsilon\rangle \simeq W_{\varepsilon\varpi}$	${}^{\varpi}\mathcal{T}_{[-1,\lambda]}$
	$-\varepsilon$	$\langle -\varepsilon, \varepsilon\varpi, 1\rangle \simeq W_{\varpi}$	$\langle -\varepsilon, \varepsilon\varpi, \varepsilon\rangle \simeq \mathbb{H} \oplus \langle \varepsilon\varpi \rangle$	${}^{\varpi}\mathcal{T}_{[-\varepsilon,\lambda]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$	-1	$\langle -1, \varepsilon\varpi, 1\rangle \simeq \mathbb{H} \oplus \langle \varepsilon\varpi \rangle$	$\langle -1, \varepsilon\varpi, \varepsilon\rangle \simeq W_{\varpi}$	${}^{\varepsilon\varpi}\mathcal{T}_{[-1,\lambda]}$
	$-\varepsilon$	$\langle -\varepsilon, \varpi, 1\rangle \simeq W_{\varepsilon\varpi}$	$\langle -\varepsilon, \varpi, \varepsilon\rangle \simeq \mathbb{H} \oplus \langle \varpi \rangle$	${}^{\varepsilon\varpi}\mathcal{T}_{[-\varepsilon,\lambda]}$

$\mathcal{A} = K \oplus F$	$\mu \backslash \lambda$	$\lambda = \varpi$	$\lambda = \varepsilon\varpi$	Class of ${}^{\alpha}\mathcal{T}_{[\mu,\lambda]}$
$F(\sqrt{\varepsilon}) \oplus F$	1	$\langle 1, -\varepsilon, \varpi\rangle \simeq W_{\varepsilon\varpi}$	$\langle 1, -\varepsilon, \varepsilon\varpi\rangle \simeq W_{\varpi}$	${}^{\varepsilon}\mathcal{T}_{[1,\lambda]}$
	ϖ	$\langle \varpi, -\varepsilon\varpi, \varpi\rangle \simeq \mathbb{H} \oplus \langle \varepsilon\varpi \rangle$	$\langle \varpi, -\varepsilon\varpi, \varepsilon\varpi\rangle \simeq \mathbb{H} \oplus \langle \varpi \rangle$	${}^{\varepsilon}\mathcal{T}_{[\varpi,\lambda]}$
$F(\sqrt{\varpi}) \oplus F$	1	$\langle 1, -\varpi, \varpi\rangle \simeq \mathbb{H} \oplus \langle 1 \rangle$	$\langle 1, -\varpi, \varepsilon\varpi\rangle \simeq W_{\varepsilon}$	${}^{\varpi}\mathcal{T}_{[1,\lambda]}$
	ε	$\langle \varepsilon, -\varepsilon\varpi, \varpi\rangle \simeq W_1$	$\langle \varepsilon, -\varepsilon\varpi, \varepsilon\varpi\rangle \simeq \mathbb{H} \oplus \langle \varepsilon \rangle$	${}^{\varpi}\mathcal{T}_{[\varepsilon,\lambda]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$	1	$\langle 1, -\varepsilon\varpi, \varpi\rangle \simeq W_{\varepsilon}$	$\langle 1, -\varepsilon\varpi, \varepsilon\varpi\rangle \simeq \mathbb{H} \oplus \langle 1 \rangle$	${}^{\varepsilon\varpi}\mathcal{T}_{[1,\lambda]}$
	ε	$\langle \varepsilon, -\varpi, \varpi\rangle \simeq \mathbb{H} \oplus \langle \varepsilon \rangle$	$\langle \varepsilon, -\varpi, \varepsilon\varpi\rangle \simeq W_1$	${}^{\varepsilon\varpi}\mathcal{T}_{[\varepsilon,\lambda]}$

Table 3.1: *The set of outputs of Algorithm 1 with input $n = 3$, $\mathcal{K} = F$: for each set of parameters (α, μ, λ), we list the resulting quadratic form q (and its isometry class) together with the label of the corresponding conjugacy class of maximal elliptic torus ${}^{\alpha}\mathcal{T}_{[\mu,\lambda]}$ in $\mathrm{SO}(q)$.*

$\mathcal{A}$, **Parameters**	**Class of Torus**
$\mathrm{SO}(\mathbb{H} \oplus \langle 1 \rangle)$	
$F(\sqrt{\varepsilon}) \oplus F$, $(1, \varepsilon)$	${}^{\varepsilon}\mathcal{T}_{[1,\varepsilon]}$
$F(\sqrt{\varpi}) \oplus F$, $(1, \varpi)$	${}^{\varpi}\mathcal{T}_{[1,\varpi]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$, $(1, \varepsilon\varpi)$	${}^{\varepsilon\varpi}\mathcal{T}_{[1,\varepsilon\varpi]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon \rangle)$	
$F(\sqrt{\varepsilon}) \oplus F$, $(1, 1)$	${}^{\varepsilon}\mathcal{T}_{[1,1]}$
$F(\sqrt{\varpi}) \oplus F$, $(\varepsilon, \varepsilon\varpi)$	${}^{\varpi}\mathcal{T}_{[\varepsilon,\varepsilon\varpi]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$, (ε, ϖ)	${}^{\varepsilon\varpi}\mathcal{T}_{[\varepsilon,\varpi]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle \varpi \rangle)$	
$F(\sqrt{\varepsilon}) \oplus F$, $(\varpi, \varepsilon\varpi)$	${}^{\varepsilon}\mathcal{T}_{[\varpi,\varepsilon\varpi]}$
$F(\sqrt{\varpi}) \oplus F$, $(-1, 1)$	${}^{\varpi}\mathcal{T}_{[-1,1]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$, $(-\varepsilon, \varepsilon)$	${}^{\varepsilon\varpi}\mathcal{T}_{[-\varepsilon,\varepsilon]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon\varpi \rangle)$	
$F(\sqrt{\varepsilon}) \oplus F$, (ϖ, ϖ)	${}^{\varepsilon}\mathcal{T}_{[\varpi,\varpi]}$
$F(\sqrt{\varpi}) \oplus F$, $(-\varepsilon, \varepsilon)$	${}^{\varpi}\mathcal{T}_{[-\varepsilon,\varepsilon]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$, $(-1, 1)$	${}^{\varepsilon\varpi}\mathcal{T}_{[-1,1]}$

$\mathcal{A}$, **Parameters**	**Class of Torus**
$\mathrm{SO}(W_1)$	
$F(\sqrt{\varepsilon}) \oplus F$, (ϖ, ε)	${}^{\varepsilon}\mathcal{T}_{[\varpi,\varepsilon]}$
$F(\sqrt{\varpi}) \oplus F$, (ε, ϖ)	${}^{\varpi}\mathcal{T}_{[\varepsilon,\varpi]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$, $(\varepsilon, \varepsilon\varpi)$	${}^{\varepsilon\varpi}\mathcal{T}_{[\varepsilon,\varepsilon\varpi]}$
$\mathrm{SO}(W_{\varepsilon})$	
$F(\sqrt{\varepsilon}) \oplus F$, $(\varpi, 1)$	${}^{\varepsilon}\mathcal{T}_{[\varpi,1]}$
$F(\sqrt{\varpi}) \oplus F$, $(1, \varepsilon\varpi)$	${}^{\varpi}\mathcal{T}_{[1,\varepsilon\varpi]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$, $(1, \varpi)$	${}^{\varepsilon\varpi}\mathcal{T}_{[1,\varpi]}$
$\mathrm{SO}(W_{\varpi})$	
$F(\sqrt{\varepsilon}) \oplus F$, $(1, \varepsilon\varpi)$	${}^{\varepsilon}\mathcal{T}_{[1,\varepsilon\varpi]}$
$F(\sqrt{\varpi}) \oplus F$, $(-\varepsilon, 1)$	${}^{\varpi}\mathcal{T}_{[-\varepsilon,1]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$, $(-1, \varepsilon)$	${}^{\varepsilon\varpi}\mathcal{T}_{[-1,\varepsilon]}$
$\mathrm{SO}(W_{\varepsilon\varpi})$	
$F(\sqrt{\varepsilon}) \oplus F$, $(1, \varpi)$	${}^{\varepsilon}\mathcal{T}_{[1,\varpi]}$
$F(\sqrt{\varpi}) \oplus F$, $(-1, \varepsilon)$	${}^{\varpi}\mathcal{T}_{[-1,\varepsilon]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F$, $(-\varepsilon, 1)$	${}^{\varepsilon\varpi}\mathcal{T}_{[-\varepsilon,1]}$

Table 3.2: *Maximal elliptic tori in the groups $\mathrm{SO}(\mathbb{H} \oplus \langle a \rangle)$ and $\mathrm{SO}(W_a)$, for $a \in F^{\times}/F^{\times^2}$*

Theorem 3.2.3. *Each isomorphism class of F-forms of SO_3 contains exactly three conjugacy classes of maximal elliptic tori.*

Proof. By Proposition 3.1.3 it suffices to count the number of tori in Table 3.1 that correspond to each of these quadratic forms. ∎

The groups $\mathrm{SO}(q)$, with q a 3-dimensional quadratic form, will arise as factors of twisted Levi subgroups of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. For ease of reference later on, we now sort the maximal elliptic tori listed in Table 3.1 according to the quadratic forms they preserve, in Table 3.2.

Note that the isomorphism class of a torus labeled ${}^{\alpha}\mathcal{T}_{[\mu,\lambda]}$ is determined by α alone and we will denote it by ${}^{\alpha}\mathcal{T}$. In this case, ${}^{\alpha}\mathcal{T}$ is the group of norm one elements of $F(\sqrt{\alpha})$. Observe that, as expected, when $\mathrm{SO}(q) \cong \mathrm{SO}(q')$, the corresponding maximal elliptic tori are isomorphic.

3.2.2 Conjugacy classes of elliptic tori in SO_4

By Proposition 3.1.3, the commutative semisimple algebras $\mathcal{A}$ needed for the construction of maximal elliptic tori of SO_4 are 4-dimensional, with each field summand of even degree. Moreover, the algebras $\mathcal{A} \oplus F$ are then precisely the commutative semisimple algebras needed to construct the maximal elliptic tori of SO_5. Thus the work in this section will simplify our exposition in Section 3.2.3, which is our primary case of interest.

The classification of four-dimensional quadratic forms and their associated elliptic tori in SO_4 has been thoroughly established by Trinity Chinner in [Chi21]. Her work provides a complete description using Morris' bijection, forming the basis for further analysis in higher-dimensional cases. Given this detailed account, we summarize her main results here without reproducing the full details.

The 4-dimensional commutative semisimple algebras $\mathcal{A}$ that meet the requirement of Proposition 3.1.3(1) are either of the form $\mathcal{A} \cong K_1 \oplus K_2$ where K_i is a quadratic field extension of F (Section 3.2.2.1), or $\mathcal{A} = E$, where E is a degree 4 field extension of F (Section 3.2.2.2).

3.2.2.1 When $\mathcal{A}$ is a sum of two quadratic extensions

We begin with the first case, where $\mathcal{A} \cong K_1 \oplus K_2$. Let $\sigma_i \in \mathrm{Gal}(K_i/F) \smallsetminus \{\mathrm{Id}\}$ for each $i \in \{1, 2\}$. The involution σ^* of Proposition 3.1.3(2) must satisfy that its restriction to K_i is σ_i, so we write $\sigma^* = (\sigma_1, \sigma_2)$. Thus in the setting of Proposition 3.1.3(3) we have $\mathcal{A}^{\times\sigma^*} = F^\times \times F^\times$ and $N_{\sigma^*}(\mathcal{A}^\times) = N_{K_1/F}(K_1^\times) \times N_{K_2/F}(K_2^\times)$. We must choose $\mu^* = (\mu_1, \mu_2)$ with μ_i running over a set of representatives for $F^\times/N_{K_i/F}(K^\times)$. However, when the factors are equal, the pairs (μ_1, μ_2) and (μ_2, μ_1) are equivalent, via the permutation of these two factors.

In total, up to equivalence, there are three choices for each K_i, hence six choices for $K_1 \oplus K_2$. For three of these choices, namely when $K_1 = K_2$, there are three distinct choices of μ^*, while for the remaining three choices of $K_1 \oplus K_2$, there are four distinct choices of μ^*. Therefore, we obtain 21 classes of elliptic tori from this class of algebras (contained in the disjoint union of the groups $G = \mathrm{SO}(V)$, where V ranges over the isometry classes of 4-dimensional quadratic forms).

As in Lemma 3.2.1, we may normalize our representatives by $\frac{1}{2}$ for convenience, and an identical calculation yields the following lemma.

Lemma 3.2.4. *Let $\mathcal{A} = K_1 \oplus K_2$, where $K_1 = F[\sqrt{\alpha}]$ and $K_1 = F[\sqrt{\beta}]$ are quadratic extensions of F. Let $\mu^* = (\frac{\mu_1}{2}, \frac{\mu_2}{2})$ such that $\mu_i \in F^\times/N_{K_i/F}(K_i^\times)$. Then the quadratic form defined by $q(v) = \mathrm{Tr}_{\mathcal{A}/F}(\mu^* v\sigma^*(v))$ for $v \in \mathcal{A}$ is isometric to $\langle \mu_1, -\mu_1\alpha, \mu_2, -\mu_2\beta\rangle$.*

We summarize these steps in Algorithm 2. Since the tori are parametrized by the quadruple $(\alpha, \beta, \mu_1, \mu_2)$, with fewer choices for the parameters when $\alpha = \beta$, we denote the torus thus produced by ${}^{\alpha}\mathcal{T}^{\beta}_{[\mu_1,\mu_2]}$ if $\alpha \neq \beta$ and by ${}^{\alpha}\mathcal{T}^{2}_{[\mu_1,\mu_2]}$ if $\beta = \alpha$.

Algorithm 2 Generating a quadratic form q from $\mathcal{A} = K_1 \oplus K_2$, where K_i is a quadratic extension of F for $i \in \{1, 2\}$, and the associated maximal elliptic torus in $\mathrm{SO}(q)$.

1: Input:$(n = 4,\ \mathcal{K} = F)$
2: Choose two quadratic extensions of F: $K_1 = F(\sqrt{\alpha})$ and $K_2 = F(\sqrt{\beta})$ for $\alpha, \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$, and set $\mathcal{A} = K_1 \oplus K_2$.
3: Set $\sigma^* = (\sigma_1, \sigma_2) \in \mathrm{Aut}(\mathcal{A}/F)$ where each $\sigma_i \in \mathrm{Gal}(K_i/F)$ is non-trivial.
4: Choose $\mu_i \in F^\times/N_{K_i/F}(K_i^\times)$ for $i \in \{1, 2\}$ and set $\mu^* = (\frac{\mu_1}{2}, \frac{\mu_2}{2})$, with the condition that if $K_1 = K_2$ then we retain only one element of any pair $(\frac{\mu_1}{2}, \frac{\mu_2}{2}) \sim (\frac{\mu_2}{2}, \frac{\mu_1}{2})$.
5: Compute the associated quadratic form $q = \langle \mu_1, -\mu_1\alpha, \mu_2, -\mu_2\beta\rangle$.
6: Then $\{a \in \mathcal{A} : a\sigma(a) = 1\}$ is a maximal elliptic torus of $\mathrm{SO}(q)$, denoted by ${}^{\alpha}\mathcal{T}^{2}_{[\mu_1,\mu_2]}$ when $\alpha = \beta$ and by ${}^{\alpha}\mathcal{T}^{\beta}_{[\mu_1,\mu_2]}$ otherwise.
7: Output: the pairs $(q, {}^{\alpha}\mathcal{T}^{2}_{[\mu_1,\mu_2]})$ or $(q, {}^{\alpha}\mathcal{T}^{\beta}_{[\mu_1,\mu_2]})$

As before, a careful selection of representatives for the various norm class groups allows us to readily identify the isometry class of the quadratic form q, regardless of whether -1 is a square or not. Nevertheless, one must take the square class of -1 into account when removing the duplicate (that is, conjugate) tori that result from permuting the two parameters when the two quadratic extensions are equal. We enumerate the 21 pairs of quadratic forms and tori produced by Algorithm 2 in Table 3.3. To strealine the notation, we use the symbols $-1^{\square}$ to indicate -1 is a square in F, and $-1^{\boxtimes}$ to indicate that -1 is not a square in F.

We next sort the tori into the corresponding special orthogonal groups. As in Section 3.2.1 we make a useful observation.

Proposition 3.2.5. *Let $K_1 = F(\sqrt{\alpha})$ and $K_2 = F(\sqrt{\beta})$, where $\alpha, \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$. Write $F^\times/N_{K_i/F}(K_i^\times) = \{1, \mu_i\}$. For each $\mu \in \{1, \mu_2\}$, if q is the quadratic form produced by Algorithm 2 for the quadruple $(\alpha, \beta, 1, \mu)$ and q' is the one produced for the quadruple $(\alpha, \beta, \mu_1, \mu)$, then $q^{\triangledown} \simeq q'$.*

$\mathcal{A} = K_1 \oplus K_2$	$\mu_1 \backslash \mu_2$	$\mu_2 = 1$	$\mu_2 = \varpi$	$\mathcal{T}$
$F[\sqrt{\varepsilon}] \oplus F[\sqrt{\varepsilon}]$	1	$\langle 1, -\varepsilon, 1, -\varepsilon\rangle \simeq 2\mathbb{H}$	$\langle 1, -\varepsilon, \varpi, -\varepsilon\varpi\rangle \simeq q_{4an}$	${}^{\varepsilon}\mathcal{T}^2_{[1,\mu_2]}$
	ϖ	$\setminus$	$\langle \varpi, -\varepsilon\varpi, \varpi, -\varepsilon\varpi\rangle \simeq 2\mathbb{H}$	${}^{\varepsilon}\mathcal{T}^2_{[\varpi,\mu_2]}$
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varepsilon}]$	1	$\langle 1, -\varpi, 1, -\varepsilon\rangle \simeq \mathbb{H} \oplus \langle \varepsilon, -\varpi\rangle$	$\langle 1, -\varpi, \varpi, -\varepsilon\varpi\rangle \simeq \mathbb{H} \oplus \langle 1, -\varepsilon\varpi\rangle$	${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\mu_2]}$
	ε	$\langle \varepsilon, -\varepsilon\varpi, 1, -\varepsilon\rangle \simeq \mathbb{H} \oplus \langle 1, -\varepsilon\varpi\rangle$	$\langle \varepsilon, -\varepsilon\varpi, \varpi, -\varepsilon\varpi\rangle \simeq \mathbb{H} \oplus \langle \varepsilon, -\varpi\rangle$	${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,\mu_2]}$
$F[\sqrt{\varepsilon\varpi}] \oplus F[\sqrt{\varepsilon}]$	1	$\langle 1, -\varepsilon\varpi, 1, -\varepsilon\rangle \simeq \mathbb{H} \oplus \langle \varepsilon, -\varepsilon\varpi\rangle$	$\langle 1, -\varepsilon\varpi, \varpi, -\varepsilon\varpi\rangle \simeq \mathbb{H} \oplus \langle 1, -\varpi\rangle$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\mu_2]}$
	ε	$\langle \varepsilon, -\varpi, 1, -\varepsilon\rangle \simeq \mathbb{H} \oplus \langle 1, -\varpi\rangle$	$\langle \varepsilon, -\varpi, \varpi, -\varepsilon\varpi\rangle \simeq \mathbb{H} \oplus \langle \varepsilon, -\varepsilon\varpi\rangle$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,\mu_2]}$

$\mathcal{A} = K_1 \oplus K_2$	$\mu_1 \backslash \mu_2$	$\mu_2 = 1$	$\mu_2 = \varepsilon$	$\mathcal{T}$
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varpi}]$	-1	$\langle -1, \varpi, 1, -\varpi\rangle \simeq 2\mathbb{H}$	$\langle -1, -\varpi, \varepsilon, -\varepsilon\varpi\rangle \simeq q_{4an}$ (omit if $-1^{\square}$)	${}^{\varpi}\mathcal{T}^2_{[-1,\mu_2]}$
	$-\varepsilon$	$\langle -\varepsilon, \varepsilon\varpi, 1, -\varpi\rangle \simeq q_{4an}$	$\langle -\varepsilon, \varepsilon\varpi, \varepsilon, -\varepsilon\varpi\rangle \simeq 2\mathbb{H}$ (omit if $-1^{\boxtimes}$)	${}^{\varpi}\mathcal{T}^2_{[-\varepsilon,\mu_2]}$
$F[\sqrt{\varepsilon\varpi}] \oplus F[\sqrt{\varepsilon\varpi}]$	-1	$\langle -1, \varepsilon\varpi, 1, -\varepsilon\varpi\rangle \simeq 2\mathbb{H}$	$\langle -1, \varepsilon\varpi, \varepsilon, -\varpi\rangle \simeq q_{4an}$ (omit if $-1^{\square}$.)	${}^{\varepsilon\varpi}\mathcal{T}^2_{[-1,\mu_2]}$
	$-\varepsilon$	$\langle -\varepsilon, \varpi, 1, -\varepsilon\varpi\rangle \simeq q_{4an}$	$\langle -\varepsilon, \varpi, \varepsilon, -\varpi\rangle \simeq 2\mathbb{H}$ (omit if $-1^{\boxtimes}$.)	${}^{\varepsilon\varpi}\mathcal{T}^2_{[-\varepsilon,\mu_2]}$
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varepsilon\varpi}]$	-1	$\langle -1, \varpi, 1, -\varepsilon\varpi\rangle \simeq \mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi\rangle$	$\langle -1, \varpi, \varepsilon, -\varpi\rangle \simeq \mathbb{H} \oplus \langle 1, -\varepsilon\rangle$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-1,\mu_2]}$
	$-\varepsilon$	$\langle -\varepsilon, \varepsilon\varpi, 1, -\varepsilon\varpi\rangle \simeq \mathbb{H} \oplus \langle 1, -\varepsilon\rangle$	$\langle -\varepsilon, \varepsilon\varpi, \varepsilon, -\varpi\rangle \simeq \mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi\rangle$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-\varepsilon,\mu_2]}$

Table 3.3: *The set of outputs of Algorithm 2 with input $n = 4$, $\mathcal{K} = F$: for each set of parameters $(\alpha, \beta, \mu_1, \mu_2)$, we list the resulting quadratic form q (and its isometry class) together with the label of the corresponding conjugacy class of maximal elliptic torus in* $\mathrm{SO}(q)$*. When $\mu_1 \notin \mu_2 N_{F(\sqrt{\alpha})/F}(F(\alpha)^\times)$, the quadruples $(\alpha, \alpha, \mu_1, \mu_2)$ and $(\alpha, \alpha, \mu_2, \mu_1)$ are equivalent and give rise to the same class of torus. We therefore remove one, as indicated variously by the symbol $\setminus$ and the notes "omit if $-1^{\square}$" and "omit if $-1^{\boxtimes}$".*

Proof. By Lemma 3.2.4, the quadratic form produced from the choice $(\alpha, \beta, 1, \mu)$ is $q = \langle 1, -\alpha, \mu, -\mu\beta\rangle$, while the one produced from $(\alpha, \beta, \mu_1, \mu)$ is $q' = \langle \mu_1, -\mu_1\alpha, \mu, -\mu\beta\rangle$. By Lemma 2.2.13, the forms $q_1 = \langle 1, -\alpha\rangle$ and $q_2 = \langle \mu_1, -\mu_1\alpha\rangle$ have opposite Hasse invariants and hence $q_1^{\triangledown} \simeq q_2$. Thus, by Theorem 2.2.15(4),

$$q^{\triangledown} = (q_1 \oplus \langle \mu, -\mu\beta\rangle)^{\triangledown} = q_1^{\triangledown} \oplus \langle \mu - \mu\beta\rangle = q'.$$

∎

In particular, the effect of varying the parameters μ_1 and μ_2 is to toggle between two quadratic forms, q and $q^{\triangledown}$. In particular, suppose $\mathcal{A} = K \oplus K$ and let $F^\times/N_{K/F}(K^\times) = \{1, \mu\}$ and suppose $(1, \mu)$ gives the quadratic form q. Then $(1, \mu)$ and $(\mu, 1)$ give rise to a single conjugacy class of maximal elliptic torus in $\mathrm{SO}(q)$, whereas it follows from the proposition that $(1, 1)$ and (μ, μ) give rise to two distinct conjugacy classes of maximal elliptic tori in $\mathrm{SO}(q^{\triangledown})$.

We collect the results in Table 3.4. The groups $\mathrm{SO}(q)$ are presented so that q-partners are next to each other. Note that there are five F-isomorphism classes of 4-dimensional special orthogonal groups (as in Table 2.9): the split form, the compact form, and the three forms which in this case may be written $\mathrm{SO}(\mathbb{H} \oplus q) \cong \mathrm{SO}(\mathbb{H} \oplus q^{\triangledown})$ for various 2-dimensional anistropic forms q. We observe from the table that, as expected, isomorphic groups contain isomorphic tori.

$\mathcal{A}$	$\mathcal{T}$
$\mathrm{SO}(2\mathbb{H})$	
$F[\sqrt{\varepsilon}] \oplus F[\sqrt{\varepsilon}]$	${}^{\varepsilon}\mathcal{T}^2_{[1,1]}$ ${}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi]}$
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varpi}]$	${}^{\varpi}\mathcal{T}^2_{[-1,1]}$ ${}^{\varpi}\mathcal{T}^2_{[-\varepsilon,\varepsilon]}$ (omit if $-1^{\boxtimes}$.)
$F[\sqrt{\varepsilon\varpi}] \oplus F[\sqrt{\varepsilon\varpi}]$	${}^{\varepsilon\varpi}\mathcal{T}^2_{[-1,1]}$ ${}^{\varepsilon\varpi}\mathcal{T}^2_{[-\varepsilon,\varepsilon]}$ (omit if $-1^{\boxtimes}$.)
$\mathrm{SO}(\mathbb{H} \oplus \langle 1, -\varepsilon \rangle)$	
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varepsilon\varpi}]$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-1,\varepsilon]}$ ${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-\varepsilon,1]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle 1, -\varpi \rangle)$	
$F[\sqrt{\varepsilon\varpi}] \oplus F[\sqrt{\varepsilon}]$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi]}$ ${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle 1, -\varepsilon\varpi \rangle)$	
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varepsilon}]$	${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi]}$ ${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1]}$

$\mathcal{A}$	$\mathcal{T}$
$\mathrm{SO}(q_{4an})$	
$F[\sqrt{\varepsilon}] \oplus F[\sqrt{\varepsilon}]$	${}^{\varepsilon}\mathcal{T}^2_{[1,\varpi]}$
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varpi}]$	${}^{\varpi}\mathcal{T}^2_{[-1,\varepsilon]}$ (omit if $-1^{\square}$) ${}^{\varpi}\mathcal{T}^2_{[-\varepsilon,1]}$
$F[\sqrt{\varepsilon\varpi}] \oplus F[\sqrt{\varepsilon\varpi}]$	${}^{\varepsilon\varpi}\mathcal{T}^2_{[-1,\varepsilon]}$ (omit if $-1^{\square}$) ${}^{\varepsilon\varpi}\mathcal{T}^2_{[-\varepsilon,1]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi \rangle)$	
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varepsilon\varpi}]$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-1,1]}$ ${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-\varepsilon,\varepsilon]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, -\varepsilon\varpi \rangle)$	
$F[\sqrt{\varepsilon\varpi}] \oplus F[\sqrt{\varepsilon}]$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,1]}$ ${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,\varpi]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, -\varpi \rangle)$	
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varepsilon}]$	${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,1]}$ ${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,\varpi]}$

Table 3.4: *The conjugacy classes of maximal elliptic tori in* $\mathrm{SO}(q)$ *arising from algebras of the form* $\mathcal{A} = K_1 \oplus K_2$*, as* q *varies over all 4-dimensional isometry classes of quadratic forms, from Table 3.3. In some cases, one representative of a pair of conjugate tori must be omitted, depending on whether* $-1 \in F^{\times 2}$*: we write* $-1^{\boxtimes}$ *when* -1 *is not a square in* F*, and* $-1^{\square}$ *when it is.*

3.2.2.2 When $\mathcal{A}$ is a quartic extension

It only remains to consider the case that $\mathcal{A}$ is a quartic extension E of F. Denote by $E_1 = E^{\sigma^*}$ the fixed field of the involution σ^* of Proposition 3.1.3(2), which is a quadratic extension field of F. Thus $\sigma^* \in \mathrm{Gal}(E/E_1) \setminus \{\mathrm{Id}\}$ and it follows that in Proposition 3.1.3(3) we have $\mathcal{A}^{\times\sigma^*} = E_1^\times$ and $N_{\sigma^*}(\mathcal{A}^\times) = N_{E/E_1}(E^\times)$. Therefore μ^* runs over a set of representatives for $E_1^\times / N_{E/E_1}(E^\times)$, which has order two. Given $\mu \in E_1^\times / N_{E/E_1}(E^\times)$, we generate the corresponding form $f_{\mathcal{A}}$ with $\mu^* = \mu/4$, for convenience.

As summarized in Tables 2.1 and 2.2, there are 9 such algebras $\mathcal{A}$ when -1 is a square in F: three are biquadratic extensions and six admit a unique quadratic subfield. When -1 is not a square there are only 7: three are biquadratic and four admit a unique quadratic subfield.

We write $E_1 = F[\sqrt{\alpha}]$ throughout, with $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$. The tori are parameterized by the triple $(E, F[\sqrt{\alpha}], \mu)$. To easily distinguish the cases and for ease of reference, when E is the biquadratic extension of F we denote the associated torus by ${}^{\alpha}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu]}$; but when E has a unique quadratic subfield we write instead ${}^{\alpha}\Upsilon^{E}_{4[\mu]}$. We summarize the preceding discussion in Algorithm 3.

Algorithm 3 Generating a quadratic form q and the associated maximal elliptic torus in $\mathrm{SO}(q)$ from a quartic extension $\mathcal{A}$ of F.

1: Input: $(n = 4,\ \mathcal{K} = F)$
2: Choose a quartic extension field E/F and set $\mathcal{A} = E$.
3: Choose a quadratic subextension $E_1 = F[\sqrt{\alpha}]$ with $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$.
4: Let $\sigma \in \mathrm{Gal}(E/F)$ be the nontrivial involution such that $E^\sigma = E_1$.
5: Choose $\mu \in E_1^\times / N_{E/E_1}(E^\times)$, and set $\mu^* = \frac{\mu}{4}$.
6: Compute the associated quadratic form $q(v) = \mathrm{Tr}_{E/F}\mu^* v\sigma(v)$ for $v \in E$.
7: Then $T = \{a \in E^\times : a\sigma(a) = 1\}$ is a maximal elliptic torus of $\mathrm{SO}(q)$, denoted by $T = {}^{\alpha}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu]}$ if E/F is biquadratic and $T = {}^{\alpha}\Upsilon^{E}_{4[\mu]}$ when E/F has a unique quadratic subfield.
8: Output: the pair (q, T)

This algorithm was executed in [Chi21, §4.4,§4.5], where Chinner computed the quadratic forms and determined their isometry classes. Her results are presented in two comprehensive tables: one corresponding to the case where -1 is a square ([Chi21, Table 4.2]) and another for the case where -1 is not a square ([Chi21, Table 4.3]). As we prove in the following two lemmas, it is possible to make careful parameter choices to unify them.

Lemma 3.2.6. *Suppose $E = F(\sqrt{\varepsilon}, \sqrt{\varpi})$ and $E_1 = F(\sqrt{\varepsilon})$. Then the quadratic form associated to the choice $\mu = \sqrt{-\varepsilon} \in E_1^\times / N_{E/E_1}(E^\times)$ is the split form $2\mathbb{H}$ while the one associated to the other coset yields q_{4an}.*

Proof. Write an arbitrary element $v \in E$ as $v = (v_1 + v_2\sqrt{\varepsilon}) + (v_3 + v_4\sqrt{\varepsilon})\sqrt{\varpi}$; then with $\sigma \in \mathrm{Gal}(E/E_1) \smallsetminus \{1\}$, we compute

$$\begin{aligned} v\sigma(v) &= (v_1 + v_2\sqrt{\varepsilon})^2 - ((v_3 + v_4\sqrt{\varepsilon})^2\varpi \\ &= (v_1^2 + \varepsilon v_2^2 - v_3^2\varpi - v_4^2\varepsilon\varpi) + \sqrt{\varepsilon}(2v_1v_2 - 2v_3v_4\varpi). \end{aligned}$$

Since E/E_1 is ramified, by Lemma 2.1.4, a set of representatives for $E_1^\times / N_{E/E_1}(E^\times)$ is $\{1, \varepsilon_{E_1}\}$ where ε_{E_1} is a nonsquare chosen as in Lemma 2.1.5. For each $\mu \in E_1^\times / N_{E/E_1}(E^\times)$ we must evaluate the form $q(v) = \mathrm{Tr}_{E/F}((\mu/4)v\sigma(v))$; we use our classification in Section 2.2.3.

First let $\mu = \sqrt{-\varepsilon}$. When $-1 \in F^{\times 2}$ we have $\sqrt{-\varepsilon} = \sqrt{\varepsilon}$, and then $q(v) = \varepsilon(2v_1v_2 - 2v_3v_4\varpi) \simeq 2\mathbb{H}$. When $-1 \notin F^{\times 2}$ we have $\varepsilon = -1$ so $\sqrt{-\varepsilon} = 1$, giving $q(v) = v_1^2 + \varepsilon v_2^2 - v_3^2\varpi - v_4^2\varepsilon\varpi$. Since $\varepsilon = -1$ this is isometric with $\langle 1, -1, -\varpi, \varpi\rangle \simeq 2\mathbb{H}$.

Now consider the other coset. When $-1 \in F^{\times 2}$ it is represented by $\mu = 1$, yielding as above $q \simeq \langle 1, \varepsilon, -\varpi, -\varepsilon\varpi\rangle \simeq q_{4an}$. When $-1 \notin F^{\times 2}$, we have $\sqrt{\varepsilon} = i$ and it is represented by $\mu = \varepsilon_{E_1} = s^2 + it^2$ as in Lemma 2.1.5. Evaluating $\mathrm{Tr}_{E/F}((\mu/4)v\sigma(v))$ carefully, we obtain

$$q(v) = s^2(v_1^2 - v_2^2 - v_3^2\varpi + v_4^2\varpi) - t^2(2v_1v_2 - 2v_3v_4\varpi) = p_5(-v_2, v_1) \oplus \varpi p_5(v_3, v_4)$$

where p_5 is the form defined in Table 2.3, which is isometric to $\langle 1, 1\rangle$. Therefore we have $q \simeq \langle 1, 1, \varpi, \varpi\rangle \simeq q_{4an}$ in this case as well. ∎

Given the lemma, it will be convenient in this case to write $E_1^\times/N_{E/E_1}(E^\times) = \{\sqrt{-\varepsilon}, \eta_E\}$ where

$$\eta_E = \begin{cases} 1 & \text{if } -1 \in F^{\times 2}, \\ \varepsilon_{E_1} = s^2 + it^2 & \text{if } -1 \notin F^{\times 2}. \end{cases} \tag{3.2.1}$$

This is the notation used in Table 3.6.

Lemma 3.2.7. *Suppose $E_1 = F(\sqrt{\varepsilon})$ and $E = F\big(\sqrt{\varepsilon_{E_1}\varpi}\big)$. Then the quadratic form associated to the choice $\mu = \sqrt{-\varepsilon} \in E_1^\times/N_{E/E_1}(E^\times)$ is isometric to $\mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi\rangle$ while the one associated to the other coset $\mu = \eta_E$ yields $\mathbb{H} \oplus \langle 1, -\varepsilon\rangle$.*

Proof. We follow the same computation as in the proof of Lemma 3.2.6. Write an arbitrary element $v \in E$ as $v = (v_1 + v_2\sqrt{\varepsilon}) + (v_3 + v_4\sqrt{\varepsilon})\sqrt{\varepsilon_{E_1}\varpi}$; then with $\sigma \in \mathrm{Gal}(E/E_1) \smallsetminus \{1\}$, we have

$$v\sigma(v) = (v_1 + v_2\sqrt{\varepsilon})^2 - (v_3 + v_4\sqrt{\varepsilon})^2\varepsilon_{E_1}\varpi.$$

We must evaluate the form $q(v) = \mathrm{Tr}_{E/F}((\mu/4)v\sigma(v))$, which is more involved in this case since $\sqrt{\varepsilon}$ and ε_{E_1} are related.

First suppose $-1 \in F^{\times 2}$. Then $\varepsilon_{E_1} = \sqrt{\varepsilon}$ so

$$v\sigma(v) = (v_1^2 + v_2^2\varepsilon - 2v_3v_3\varepsilon\varpi) + \sqrt{\varepsilon}(2v_1v_2 - \varpi v_3^2 - \varepsilon\varpi v_4^2).$$

If $\mu = \sqrt{\varepsilon}$ then $q(v) = \varepsilon(2v_1v_2 - \varpi v_3^2 - \varepsilon\varpi v_4^2)$, which is isometric to $\mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi\rangle$ whereas if $\mu = 1$ then $q(v) = v_1^2 + v_2^2\varepsilon - 2v_3v_3\varepsilon\varpi$, which yields $q \simeq \mathbb{H} \oplus \langle 1, -\varepsilon\rangle$.

Now suppose $-1 \notin F^{\times 2}$, so that we have $\varepsilon = -1$, $\sqrt{\varepsilon} = i$ and $\varepsilon_{E_1} = s^2 + it^2$. This gives

$$v\sigma(v) = (v_1^2 - v_2^2 - s^2\varpi v_3^2 + s^2\varpi v_4^2 + 2t^2\varpi v_3v_4) + i(2v_1v_2 - t^2\varpi v_3^2 + t^2\varpi v_4^2 - 2s^2\varpi v_3v_4).$$

Taking $\mu = \sqrt{-\varepsilon} = 1$ yields $q(v) = v_1^2 - v_2^2 + \varpi p_5(v_3, v_4)$, as in Table 2.3, so $q \simeq \langle 1, -1\rangle \oplus \langle \varpi, \varpi\rangle$. Since $\varepsilon = -1$ this is isometric to $\mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi\rangle$. Taking instead $\mu = \varepsilon_{E_1}$ and evaluating $\mathrm{Tr}_{E/F}((\mu/4)v\sigma(v))$ carefully, we obtain $q(v) = p_5(-v_2, v_1) \oplus \varpi p_6(v_3, v_4)$ and Table 2.3 yields $q \simeq \langle 1, 1\rangle \oplus \mathbb{H}$ which is isometric with $\mathbb{H} \oplus \langle 1, -\varepsilon\rangle$, as desired. ∎

We now reproduce what we will need, for both the next section and for further work in Chapter 6, from the tables of [Chi21, §4] as Table 3.5, with the modifications defined by the two lemmas. We record the explicit quadratic form relative to our standard basis because it will be necessary for constructing the appropriate Witt basis in Section 6.2, and this information is also key to the different valuation relations in Section 6.3.3.

We now sort these elliptic tori corresponding to quartic extensions of F according to the group into which they embed in Table 3.6.

Theorem 3.2.8. *There are 6 conjugacy classes of maximal elliptic tori in $\mathrm{SO}(q_{4an})$ if $-1 \in F^{\times 2}$ and 8 otherwise. There are 9 conjugacy classes of maximal elliptic tori in $\mathrm{SO}(2\mathbb{H})$*

$E_1 \subset E$		**First choice of** μ	**Second choice of** μ	$\mathcal{T}$
$F(\sqrt{\alpha}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$				${}^{\alpha}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu]}$
$*F(\sqrt{\varepsilon})$ $\subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	$-1^{\square}$	$\mu = \varepsilon_{E_1} = \sqrt{\varepsilon}$ $\varepsilon(2v_1v_2 - 2\varpi v_3v_4) \simeq 2\mathbb{H}$	$\mu = 1$ $v_1^2 + \varepsilon v_2^2 - \varpi v_3^2 - \varepsilon\varpi v_4^2 \simeq q_{4an}$	${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu]}$
	$-1^{\boxtimes}$	$\mu = 1$ $v_1^2 - v_2^2 - \varpi v_3^2 + \varpi v_4^2 \simeq 2\mathbb{H}$	$\mu = \varepsilon_{E_1} = s^2 + it^2$ $p_5(-v_2, v_1) \oplus \varpi p_5(v_3, v_4) \simeq q_{4an}$	${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu]}$
$F(\sqrt{\varpi})$ $\subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	any	$\mu = \sqrt{\varpi}$ $\varpi(2v_1v_2 - 2\varepsilon v_3v_4) \simeq 2\mathbb{H}$	$\mu = 1$ $v_1^2 + \varpi v_2^2 - \varepsilon v_3^2 - \varepsilon\varpi v_4^2 \simeq q_{4an}$	${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu]}$
$F(\sqrt{\varepsilon\varpi})$ $\subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	any	$\mu = \sqrt{\varepsilon\varpi}$ $\varepsilon\varpi(2v_1v_2 - 2\varepsilon v_3v_4) \simeq 2\mathbb{H}$	$\mu = 1$ $v_1^2 + \varepsilon\varpi v_2^2 - \varepsilon v_3^2 - \varepsilon^2\varpi v_4^2 \simeq q_{4an}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu]}$
$F[\sqrt{\varepsilon}]$ $\subset F[\sqrt{\varepsilon_{E_1}}]$	$-1^{\square}$	$\mu = 1$ $q_1(v) = v_1^2 + \varepsilon v_2^2 - 2\varepsilon v_3v_4$ $\simeq \mathbb{H} \oplus \langle 1, -\varepsilon\rangle$	$\mu = \varpi$ $q_2(v) = \varpi q_1(v)$ $\simeq \mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi\rangle$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[\mu]}$
	$-1^{\boxtimes}$	$\mu = 1$ $q_1(v) = v_1^2 - v_2^2 + p_5(v_3, v_4)$ $\simeq \mathbb{H} \oplus \langle 1, -\varepsilon\rangle$	$\mu = \varpi$ $q_2(v) = \varpi q_1(v)$ $\simeq \mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi\rangle$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[\mu]}$
$*F[\sqrt{\varepsilon}]$ $\subset F[\sqrt{\varepsilon_{E_1}\varpi}]$	$-1^{\square}$	$\mu = 1$ $q_1(v) = v_1^2 + \varepsilon v_2^2 - 2\varepsilon\varpi v_3v_4$ $\simeq \mathbb{H} \oplus \langle 1, -\varepsilon\rangle$	$\mu = \varepsilon_{E_1} = \sqrt{\varepsilon}$ $q_2(v) = \varepsilon(2v_1v_2 - \varpi v_3^2 - \varepsilon\varpi v_4^2)$ $\simeq \mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi\rangle$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\mu]}$
	$-1^{\boxtimes}$	$\mu = \varepsilon_{E_1} = s^2 + it^2$ $q_1(v) = p_5^1(-v_2, v_1) + p_6^{\varpi}(v_3, v_4)$ $\simeq \mathbb{H} \oplus \langle 1, -\varepsilon\rangle$	$\mu = 1$ $q_2(v) = v_1^2 - v_2^2 + p_5^{\varpi}(v_3, v_4)$ $\simeq \mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi\rangle$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\mu]}$
$F(\sqrt{\varpi}) \subset F(\sqrt[4]{\varepsilon^k\varpi})$ $k = 0, 2$ when $-1^{\square}$, $k = 0$ when $-1^{\boxtimes}$		$\mu = 1$ $q_1(v) = v_1^2 + \varpi v_2^2 - 2\varpi v_3v_4$ $\simeq \mathbb{H} \oplus \langle 1, \varpi\rangle$	$\mu = \varepsilon$ $q_2(v) = \varepsilon v_1^2 + \varepsilon\varpi v_2^2 - 2\varepsilon\varpi v_3v_4$ $\simeq \mathbb{H} \oplus \langle \varepsilon, \varepsilon\varpi\rangle$	${}^{\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[\mu]}$
$F(\sqrt{\varepsilon\varpi}) \subset F(\sqrt[4]{\varepsilon^k\varpi})$ $k = 1, 3$ when $-1^{\square}$ $k = 1$ when $-1^{\boxtimes}$		$\mu = 1$ $q_1(v) = v_1^2 + \varepsilon\varpi v_2^2 - 2\varepsilon\varpi v_3v_4$ $\simeq \mathbb{H} \oplus \langle 1, \varepsilon\varpi\rangle$	$\mu = \varepsilon$ $q_2(v) = \varepsilon v_1^2 + \varpi v_2^2 - 2\varpi v_3v_4$ $\simeq \mathbb{H} \oplus \langle \varepsilon, \varpi\rangle$	${}^{\varepsilon\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[\mu]}$

Table 3.5: *The set of outputs of Algorithm 3: for each triple (E, E_1, μ), we give the explicit resulting quadratic form q (where the forms p_5, p_6 are defined in Table 2.3) and its isometry class together with the label of the corresponding conjugacy class of maximal elliptic torus in* $\mathrm{SO}(q)$*. Here, $v = v_1 + v_2\sqrt{\alpha} + (v_3 + v_4\sqrt{\alpha})\sqrt{\theta} \in E$ where: if E is biquadratic then $\theta = \varpi$ if $\alpha = \varepsilon$, and $\theta = \varepsilon$ if $\alpha \in \{\varpi, \varepsilon\varpi\}$; and otherwise θ is the given generator of $E = F(\theta)$. The second column indicates whether -1 is a square in F (denoted $-1^{\square}$) or not (denoted $-1^{\boxtimes}$). When $-1^{\boxtimes}$, we set $\varepsilon = -1$ and $\varepsilon_{E_1} = s^2 + it^2$. Rows marked with $*$ correspond to Lemmas 3.2.6 and 3.2.7; the rest are from [Chi21, Tables 4.2,4.3].*

$\mathcal{A}$	**Class of Torus**
$\mathrm{SO}(2\mathbb{H})$	
$F(\sqrt{\varepsilon}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{-\varepsilon}]}$
$F(\sqrt{\varpi}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi}]}$
$F(\sqrt{\varepsilon\varpi}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi}]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle 1, -\varepsilon \rangle)$	
$F[\sqrt{\varepsilon}] \subset F[\sqrt{\varepsilon_{E_1}}]$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[1]}$
$F[\sqrt{\varepsilon}] \subset F[\sqrt{\varepsilon_{E_1}\varpi}]$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\eta_E]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle 1, \varpi \rangle)$	
$\ast F(\sqrt{\varpi}) \subset F(\sqrt[4]{\varepsilon^k\varpi})$	${}^{\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[1]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle 1, \varepsilon\varpi \rangle)$	
$\ast F(\sqrt{\varepsilon\varpi}) \subset F(\sqrt[4]{\varepsilon^k\varpi})$	${}^{\varepsilon\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[1]}$

$\mathcal{A}$, **Parameters**	**Class of Torus**
$\mathrm{SO}(q_{4an})$	
$F(\sqrt{\varepsilon}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\eta]}$
$F(\sqrt{\varpi}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[1]}$
$F(\sqrt{\varepsilon\varpi}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[1]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle \varpi, -\varepsilon\varpi \rangle)$	
$F[\sqrt{\varepsilon}] \subset F[\sqrt{\varepsilon_{E_1}}]$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[\varpi]}$
$F[\sqrt{\varepsilon}] \subset F[\sqrt{\varepsilon_{E_1}\varpi}]$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\sqrt{-\varepsilon}]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, \varepsilon\varpi \rangle)$	
$\ast F(\sqrt{\varpi}) \subset F(\sqrt[4]{\varepsilon^k\varpi})$	${}^{\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[\varepsilon]}$
$\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, \varpi \rangle)$	
$\ast F(\sqrt{\varepsilon\varpi}) \subset F(\sqrt[4]{\varepsilon^k\varpi})$	${}^{\varepsilon\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[\varepsilon]}$

Table 3.6: *Maximal elliptic tori in* $\mathrm{SO}(q)$ *for all isometry classes of* 4*-dimensional quadratic forms arising from quartic field extensions, grouped by the isometry class of the preserved form* q*. The entries are extracted from Table 3.5. The starred rows represent two conjugacy classes of elliptic tori when* $-1 \in F^{\times 2}$*, and one otherwise, with the parameter* k *defined as in Table 3.5 and* η_E *defined as in* (3.2.1).

if $-1 \in F^{\times 2}$ *and* 7 *otherwise. When* q *is* 4*-dimensional with a* 2*-dimensional anisotropic kernel, then* $\mathrm{SO}(q)$ *has 24 conjugacy classes of maximal elliptic tori if* $-1 \in F^{\times 2}$ *and 20 otherwise. Moreover, the biquadratic extension gives rise to tori in* $\mathrm{SO}(q_{4an})$ *and* $\mathrm{SO}(2\mathbb{H})$*, whereas the quartic extensions with a unique subfield do not.*

Proof. We count these tori from Table 3.4 (when $\mathcal{A}$ is the sum of two quadratic field extensions of F) and Table 3.6 (when $\mathcal{A}$ is a quartic extension of F). The final remark is an observation from the latter table. ∎

3.2.3 Conjugacy classes of elliptic tori in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ and $\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, -\varpi, \varepsilon\varpi \rangle)$

In this section, we finally determine the conjugacy classes of maximal elliptic tori in the two F-forms of SO_5. We do so by leveraging the results of the preceding section.

Namely, suppose we have run either Algorithm 2 or Algorithm 3, with a four-dimensional algebra $\mathcal{A}_4$, equipped with an involution σ_4^*, and a coset representative $\mu_4^* \in (\mathcal{A}_4^\times)^{\sigma_4^*}/N_{\sigma_4^*}(\mathcal{A}_4^\times)$.

Set $\mathcal{A} = \mathcal{A}_4 \oplus F$; this is a 5-dimensional commutative semisimple algebra, equipped with an involution $\sigma^* = (\sigma_4^*, \mathrm{Id}_F)$ satisfying the conditions of Proposition 3.1.3(1) and (2). Choose $\lambda \in F^\times/F^{\times 2}$; then

$$\mu^* = (\mu_4^*, \lambda) \in (\mathcal{A}_4^\times)^{\sigma_4^*}/N_{\sigma_4^*}(\mathcal{A}_4^\times) \times F^\times/F^{\times 2} = (\mathcal{A}^\times)^{\sigma^*}/N_{\sigma^*}(\mathcal{A}^\times).$$

Moreover, every valid set of choices for the construction procedure of Proposition 3.1.3 for $n = 5$ arises in this way.

Note that if $v = (v_4, w) \in \mathcal{A} = \mathcal{A}_4 \oplus F$, then

$$\mathrm{Tr}_{\mathcal{A}/F}(\mu^* v \sigma^*(v)) = \mathrm{Tr}_{\mathcal{A}_4/F}(\mu_4^* v_4 \sigma_4^*(v_4)) + \lambda w^2.$$

Therefore if the quadratic form associated with $(n = 4, \mathcal{A}_4, \mu_4^*)$ is q, then the quadratic form associated to $(n = 5, \mathcal{A} = \mathcal{A}_4 \oplus F, \mu^* = (\mu_4^*, \lambda)$ by Proposition 3.1.3 is $q \oplus \langle \lambda \rangle$.

If the torus of SO_4 produced by this input was labeled $\mathcal{T}_{[\mu_4^*]}$, then we label the corresponding tori of SO_5 by adding λ as an additional subscript, as in $\mathcal{T}_{[\mu_4^*, \lambda]}$.

We now turn to Table 2.7, which identifies the pairs (q_4, λ) such that $q_4 \oplus \langle \lambda \rangle$ is isometric to $2\mathbb{H} \oplus \langle 1 \rangle$ or to $\mathbb{H} \oplus W_1$. Note that the forms q_4 that yield $2\mathbb{H} \oplus \langle 1 \rangle$ are precisely those appearing in the leftmost tables of Tables 3.4 and 3.6. Thus the list of all the conjugacy classes of maximal elliptic tori in the two groups $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle$ and $\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, -\varpi, \varepsilon\varpi \rangle)$ is obtained by adding the appropriate parameter to each of the tori in the corresponding tables. Table 3.7 summarizes the results.

Proposition 3.2.9. *Let F be a non-archimedean local field of odd residual characteristic, and let G be either $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ or $\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, -\varpi, \varepsilon\varpi \rangle)$.*

Let N_G^{ell} be the total number of G-conjugacy classes of elliptic maximal tori of G. Then

$$N^{\mathrm{ell}}_{\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)} = \begin{cases} 21 & \text{if } -1 \in F^{\times 2}, \\ 17 & \text{if } -1 \notin F^{\times 2}, \end{cases} \qquad N^{\mathrm{ell}}_{\mathrm{SO}(\mathbb{H}\oplus\langle \varepsilon, -\varpi, \varepsilon\varpi\rangle)} = \begin{cases} 18 & \text{if } -1 \in F^{\times 2}, \\ 18 & \text{if } -1 \notin F^{\times 2}. \end{cases}$$

Proof. These results follow from Table 3.7. ∎

$\mathcal{A}$	$\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$	$\mathrm{SO}(\mathbb{H} \oplus W_1)$
$F[\sqrt{\varepsilon}] \oplus F[\sqrt{\varepsilon}] \oplus F$	${}^{\varepsilon}\mathcal{T}^2_{[1,1,1]}$ ${}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi,1]}$	${}^{\varepsilon}\mathcal{T}^2_{[1,\varpi,1]}$
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varpi}] \oplus F$	${}^{\varpi}\mathcal{T}^2_{[-1,1,1]}$(omit if $-1^{\boxtimes}$) ${}^{\varpi}\mathcal{T}^2_{[-\varepsilon,\varepsilon,1]}$	${}^{\varpi}\mathcal{T}^2_{[-1,\varepsilon,1]}$ (omit if $-1^{\square}$) ${}^{\varpi}\mathcal{T}^2_{[-\varepsilon,1,1]}$
$F[\sqrt{\varepsilon\varpi}] \oplus F[\sqrt{\varepsilon\varpi}] \oplus F$	${}^{\varepsilon\varpi}\mathcal{T}^2_{[-1,1,1]}$(omit if $-1^{\boxtimes}$) ${}^{\varepsilon\varpi}\mathcal{T}^2_{[-\varepsilon,\varepsilon,1]}$	${}^{\varepsilon\varpi}\mathcal{T}^2_{[-1,\varepsilon,1]}$ (omit if $-1^{\square}$) ${}^{\varepsilon\varpi}\mathcal{T}^2_{[-\varepsilon,1,1]}$
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varepsilon\varpi}] \oplus F$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-1,\varepsilon,\varepsilon]}$ ${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-\varepsilon,1,\varepsilon]}$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-1,1,\varepsilon]}$ ${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-\varepsilon,\varepsilon,\varepsilon]}$
$F[\sqrt{\varepsilon\varpi}] \oplus F[\sqrt{\varepsilon}] \oplus F$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$ ${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varpi]}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,1,\varpi]}$ ${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,\varpi,\varpi]}$
$F[\sqrt{\varpi}] \oplus F[\sqrt{\varepsilon}] \oplus F$	${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varepsilon\varpi]}$ ${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varepsilon\varpi]}$	${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,1,\varepsilon\varpi]}$ ${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,\varpi,\varepsilon\varpi]}$
$F(\sqrt{\varepsilon}) \oplus F \subset F(\sqrt{\varepsilon}, \sqrt{\varpi}) \oplus F$	${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{-\varepsilon},1]}$	${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\eta,1]}$
$F(\sqrt{\varpi}) \oplus F \subset F(\sqrt{\varepsilon}, \sqrt{\varpi}) \oplus F$	${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$	${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[1,1]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F \subset F(\sqrt{\varepsilon}, \sqrt{\varpi}) \oplus F$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[1,1]}$
$F(\sqrt{\varepsilon}) \oplus F \subset F(\sqrt{\varepsilon_{E_1}}) \oplus F$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[1,\varepsilon]}$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[\varpi,\varepsilon]}$
$F(\sqrt{\varepsilon}) \oplus F \subset F(\sqrt{\varepsilon_{E_1}\varpi}) \oplus F$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\eta_E,\varepsilon]}$	${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\sqrt{-\varepsilon},\varepsilon]}$
$*F(\sqrt{\varpi}) \oplus F \subset F(\sqrt[4]{\varepsilon^k\varpi}) \oplus F$	${}^{\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[1,-\varpi]}$	${}^{\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[\varepsilon,-\varpi]}$
$*F(\sqrt{\varepsilon\varpi}) \oplus F \subset F(\sqrt[4]{\varepsilon^k\varpi}) \oplus F$	${}^{\varepsilon\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[1,-\varepsilon\varpi]}$	${}^{\varepsilon\varpi}\Upsilon^{\varepsilon^k\varpi}_{4[\varepsilon,-\varepsilon\varpi]}$

Table 3.7: *Maximal elliptic tori in* $\mathrm{SO}(q)$ *for all isometry classes of* 5*-dimensional quadratic forms isometric to* $2\mathbb{H} \oplus \langle 1 \rangle$ *and* $\mathbb{H} \oplus \langle \varepsilon, -\varpi, \varepsilon\varpi \rangle$*, arising from quadratic and quartic algebras, grouped by the isometry class of the preserved form* q*. The starred rows represent two conjugacy classes of elliptic tori when* $-1 \in F^{\times 2}$ *and one otherwise, with the parameter* k *defined as in Table 3.5 and* η_E *defined as in* (3.2.1)*.*

3.3 Conjugacy classes of elliptic tori in U_2

Let $\beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ and consider the unitary groups in two variables, $U_\beta(\mathbb{H})$ and $U_\beta(\mathbf{C})$, defined with respect to the quadratic extension $K = F(\sqrt{\beta})$ (with notation as in Section 2.3.2). To classify their conjugacy classes of maximal elliptic tori, we apply Proposition 3.1.3 with $n = 2$, $\mathcal{K} = K$ and $\sigma_0 \in \mathrm{Gal}(K/F)$ the non-trivial element. The resulting Hermitian form will be denoted J.

A commutative semisimple algebra $\mathcal{A}$ meeting the conditions of Proposition 3.1.3(1),(2) can take only one of two forms: $\mathcal{A} \simeq E$, where E is a quadratic extension of K equipped with an involution $\sigma^* \in \mathrm{Gal}(E/F)$ with fixed field $E_1 = F(\sqrt{\alpha})$ such that $\sigma^*|_K = \sigma_0$ (which we treat in Section 3.3.1), or $\mathcal{A} \simeq K^2$, equipped $\sigma^* = \sigma_0 \times \sigma_0$ (which we treat in Section 3.3.2). We conclude the chapter in Section 3.4.2 by showing that the embeddings of these unitary groups preserve conjugacy classes of maximal elliptic tori.

3.3.1 Generating a σ_0-Hermitian form J for U_2 when $\mathcal{A}$ is a quartic extension of F

We begin by classifying the relevant quartic extensions.

Lemma 3.3.1. *If E/F is a quartic extension equipped with an involution σ^* extending σ_0, then E is biquadratic over F. In this case there are two choices for σ^*.*

Proof. The fixed field of σ^* must be a quadratic subfield different from K. By our classification of quartic extensions in Tables 2.1 and 2.2, we must have $E = F(\sqrt{\varepsilon}, \sqrt{\varpi})$, the (unique) biquadratic extension, and this yields two options for the fixed field E_1 and thus for σ^*. In particular, since $\sigma^*|_K$ is nontrivial, it agrees with σ_0, as required. ∎

We illustrate the situation as follows, writing $E_1 = F(\sqrt{\alpha})$ with $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\} \smallsetminus \{\beta\}$ for the fixed field of σ^*.

$$\begin{tikzcd}
& E = F(\sqrt{\varepsilon}, \sqrt{\varpi}) \arrow[dl, dash] \arrow[dr, dash, "\sigma^*"] & \\
K = F(\sqrt{\beta}) \arrow[dr, dash, "\sigma_o"'] & & E_1 = F(\sqrt{\alpha}) \arrow[dl, dash] \\
& F &
\end{tikzcd}$$

We next identify the parameter choices in Proposition 3.1.3(3), noting that $\mathcal{A}^{\sigma^*}/N_{\sigma^*}(\mathcal{A}^\times) \cong E_1^\times/N_{E/E_1}(E^\times)$.

Lemma 3.3.2. *If $E_1 = F(\sqrt{\alpha}) \subset E = F(\sqrt{\varepsilon}, \sqrt{\varpi})$ then a set of coset representatives for $E_1^\times/N_{E/E_1}(E^\times)$ is $\{1, \sqrt{\alpha}\}$ unless $-1 \notin F^{\times 2}$ and E_1/F is unramified, in which case we take $\{1, \varepsilon_{E_1}\}$.*

Proof. Suppose first that E_1/F is unramified. Then E/E_1 is ramified, so by Lemma 2.1.4, a set of representatives is $\{1, \varepsilon_{E_1}\}$, where $\varepsilon_{E_1} = s^2 + it^2$ as in Lemma 2.1.5 if $-1 \notin F^{\times 2}$ and $\varepsilon_{E_1} = \sqrt{\varepsilon} = \sqrt{\alpha}$ otherwise. When E_1/F is ramified, E/E_1 is unramified and since $\sqrt{\alpha}$ is a uniformizer of E_1, we may take $E_1^\times/N_{E/E_1}(E^\times) = \{1, \sqrt{\alpha}\}$ by Lemma 2.1.4. ∎

As before, we scale these coset representatives by $\frac{1}{2}$ for convenience (at the expense of potentially permuting the two cases).

We set out the steps of Proposition 3.1.3 as Algorithm 4. We write J for the resulting Hermitian form. Since the tori in these unitary groups are parametrized by the triples $(K = F(\sqrt{\beta}), E_1 = F(\sqrt{\alpha}), \mu)$, we denote the torus produced by ${}^{\alpha}\mathcal{U}^{\varepsilon,\varpi}_{\beta,\mu}$. The superscript (ε, ϖ) records that the associated algebra $\mathcal{A} = F(\sqrt{\varepsilon}, \sqrt{\varpi})$ is a biquadratic field extension; this distinguishes these tori from those arising in Algorithm 5 below, where $\mathcal{A} = K \oplus K$ with K a quadratic extension of F.

Algorithm 4 Generating σ_0– Hermitian Form J corresponding to the associated class of torus embedded in $U_\beta(J)$ from $\mathcal{A} = E = F(\sqrt{\varepsilon}, \sqrt{\varpi})$, where $K = F(\sqrt{\beta})$ is a quadratic extension of F, and producing the associated class of elliptic torus up to conjugacy

1: Input : $(n = 4, \mathcal{K} = K = F[\sqrt{\beta}]$ equipped with $\sigma_0 \in \mathrm{Gal}(K/F)$ non-trivial),
2: Choose $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\} \setminus \{\beta\}$ and set $\mathcal{A} = E = K(\sqrt{\alpha})$.
3: Let $\sigma^* \in \mathrm{Gal}(E/F)$ be the involution with fixed field $E_1 = F(\sqrt{\alpha})$. Then $\sigma^*|_K = \sigma_0$.
4: Choose $\mu \in E_1^\times / N_{E/E_1}(E^\times)$, and set $\mu^* = \frac{\mu}{2}$.
5: Compute the σ_0-Hermitian form J that sends $a \in \mathcal{A}$ to $\mathrm{Tr}_{E/K}\mu^* a\sigma^*(a)$.
6: Then $\{a \in \mathcal{A} : a\sigma^*(a) = 1\}$ is a maximal elliptic torus of $U_\beta(J)$, denoted ${}^{\alpha}\mathcal{U}^{\varepsilon,\varpi}_{\beta,\mu}$
7: Output: the pair $(J, {}^{\alpha}\mathcal{U}^{\varepsilon,\varpi}_{\beta,\mu})$.

This algorithm yields four outputs for each set of inputs. We classify them as follows.

Lemma 3.3.3. *In the setting of Algorithm 4, let $E_1^\times / N_{E/E_1}(E^\times) = \{1, \tau\}$. Recall that $\mathbf{C}$ denotes the anisotropic Hermitian form in dimension 2 over the relevant quadratic extension (Section 2.3.1). Then*

$$ {}^{\alpha}\mathcal{U}^{\varepsilon,\varpi}_{\beta,1} \subset \begin{cases} U_\beta(\mathbb{H}) & \text{if } -1 \notin F^{\times 2} \text{ and } E_1/F \text{ is unramified} \\ U_\beta(\mathbf{C}) & \text{otherwise,} \end{cases} $$

while ${}^{\alpha}\mathcal{U}^{\varepsilon,\varpi}_{\beta,\tau}$ embeds in the opposite group.

Proof. Recall from Lemma 2.3.1 that a Hermitian form J for K/F is split when its discriminant $\delta(J)$ is equivalent to $-1 \in F^\times / N_{K/F}(K^\times)$ and otherwise it is compact. Let $E^{\sigma^*} = E_1 = F(\sqrt{\alpha})$ and $\sigma^*|_K = \sigma_0$. Write $\sigma_0(x) = \overline{x}$ to simplify the notation. We may write an arbitrary $a \in E$ as $a = x + y\sqrt{\alpha}$, with $x, y \in K$, and then $\sigma^*(a) = \overline{x} + \overline{y}\sqrt{\alpha}$. Therefore, $a\sigma^*(a) = x\overline{x} + x\overline{y}\sqrt{\alpha} + y\overline{y}\alpha + y\overline{x}\sqrt{\alpha}$ and we compute

$$ \mathrm{Tr}_{E/K}\Big(\frac{\mu}{2}(a\sigma^*(a))\Big) = \begin{cases} x\overline{x} + y\overline{y}\alpha & \text{if } \mu = 1; \\ s^2(x\overline{x} - y\overline{y}) - t^2(x\overline{y} + \overline{x}y) & \text{if } -1 \notin F^{\times 2},\ \alpha = -1 \text{ and } \mu = s^2 + it^2. \\ x\overline{y}\alpha + y\overline{x}\alpha & \text{with } \mu = \sqrt{\alpha}, \text{ otherwise,} \end{cases} $$

where we have used Lemma 3.3.2 to determine the values of μ. We can read the discriminant of the resulting Hermitian form J from these expressions. For example, $\delta(J) = -1$ when either $\mu = 1$ and $-1 \notin F^{\times 2}$ and $\alpha = -1$, or when $\mu = \sqrt{\alpha}$ and $-1 \in F^{\times 2}$ or E_1/F is ramified; thus in these cases the resulting torus embeds into $U_\beta(\mathbb{H})$.

Now suppose $-1 \notin F^{\times 2}$, $\alpha = -1$ and $\mu = s^2 + it^2$. Then $\delta(J) = -s^4 - t^4 = 1 \in N_{K/F}(K^\times)$. But note that since $\alpha \neq \beta$, the extension K/F is ramified and $\alpha = -1 \notin N_{K/F}(K^\times)$. Thus $\mu = \varepsilon_{E_1}$ yields a torus embedded in the compact form $U_\beta(\mathbf{C})$.

Finally, suppose that either $-1 \in F^{\times 2}$ or E_1/F is ramified, and $\mu = 1$, so that $\delta(J) = \alpha$. These conditions imply that $-\alpha \notin F^{\times 2}$. Note that $-\alpha \in N_{E_1/F}(E_1^\times)$ and since the quadratic extensions K/F and E_1/F are distinct, $-\alpha$ cannot also be a norm of K/F. Therefore $\alpha \notin -1 \cdot N_{K/F}(K^\times)$ and the corresponding torus again embeds into the compact form $U_\beta(\mathbf{C})$. ∎

Note that the exceptional case in Lemma 3.3.3 occurs only when $\alpha = \varepsilon$, and in this case, replace the set of representatives $\{1, \tau\}$ of $E_1/N_{E/E_1}(E^\times)$ with $\{\sqrt{-\varepsilon}, \eta\}$ where η is as in (3.2.1) yields the simpler statement that ${}^{\alpha}\mathcal{U}^{\varepsilon,\varpi}_{\beta,\sqrt{-\varepsilon}} \subset U_\beta(\mathbb{H})$ and ${}^{\alpha}\mathcal{U}^{\varepsilon,\varpi}_{\beta,\eta} \subset U_\beta(\mathbf{C})$. We record the resulting classification in Table 3.8.

$K = F(\sqrt{\beta})$ $E_1 = F(\sqrt{\alpha})$	$a\sigma^*(a)$ $a = x + y\sqrt{\alpha} \in E$ $x, y \in K$	μ	$U_\beta(J)$	${}^{\alpha}\mathcal{U}^{\varepsilon,\varpi}_{\beta,\mu}$
$F(\sqrt{\varepsilon})$ $F(\sqrt{\varpi})$	$x\overline{x} + y\overline{y}\varpi$ $+$ $x\overline{y}\sqrt{\varpi} + y\overline{x}\sqrt{\varpi}$	1	$U_\varepsilon(\mathbf{C})$	${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,1}$
		$\sqrt{\varpi}$	$U_\varepsilon(\mathbb{H})$	${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,\sqrt{\varpi}}$
$F(\sqrt{\varepsilon})$ $F(\sqrt{\varepsilon\varpi})$	$x\overline{x} + y\overline{y}\varepsilon\varpi$ $+$ $x\overline{y}\sqrt{\varepsilon\varpi} + y\overline{x}\sqrt{\varepsilon\varpi}$	1	$U_\varepsilon(\mathbf{C})$	${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,1}$
		$\sqrt{\varepsilon\varpi}$	$U_\varepsilon(\mathbb{H})$	${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,\sqrt{\varepsilon\varpi}}$
$F(\sqrt{\varpi})$ $F(\sqrt{\varepsilon})$	$x\overline{x} + y\overline{y}\varepsilon$ $+$ $x\overline{y}\sqrt{\varepsilon} + y\overline{x}\sqrt{\varepsilon}$	η_E	$U_\varpi(\mathbf{C})$	${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,\eta}$
		$\sqrt{-\varepsilon}$	$U_\varpi(\mathbb{H})$	${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,\sqrt{-\varepsilon}}$
$F(\sqrt{\varpi})$ $F(\sqrt{\varepsilon\varpi})$	$x\overline{x} + y\overline{y}\varepsilon\varpi$ $+$ $x\overline{y}\sqrt{\varepsilon\varpi} + y\overline{x}\sqrt{\varepsilon\varpi}$	1	$U_\varpi(\mathbf{C})$	${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,1}$
		$\sqrt{\varepsilon\varpi}$	$U_\varpi(\mathbb{H})$	${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,\sqrt{\varepsilon\varpi}}$
$F(\sqrt{\varepsilon\varpi})$ $F(\sqrt{\varepsilon})$	$x\overline{x} + y\overline{y}\varepsilon$ $+$ $x\overline{y}\sqrt{\varepsilon} + y\overline{x}\sqrt{\varepsilon}$	η_E	$U_{\varepsilon\varpi}(\mathbf{C})$	${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,\eta}$
		$\sqrt{-\varepsilon}$	$U_{\varepsilon\varpi}(\mathbb{H})$	${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,\sqrt{-\varepsilon}}$
$F(\sqrt{\varepsilon\varpi})$ $F(\sqrt{\varpi})$	$x\overline{x} + y\overline{y}\varpi$ $+$ $x\overline{y}\sqrt{\varpi} + y\overline{x}\sqrt{\varpi}$	1	$U_{\varepsilon\varpi}(\mathbf{C})$	${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,1}$
		$\sqrt{\varpi}$	$U_{\varepsilon\varpi}(\mathbb{H})$	${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,\sqrt{\varpi}}$

Table 3.8: *Outputs of Algorithm 4. We identify the fields $K = F(\sqrt{\beta})$ and $E_1 = F(\sqrt{\alpha})$ in the first column and the choice of parameter $\mu \in E_1^\times/N_{E/E_1}(E^\times)$ in the third. Note that when $\alpha = \varepsilon$ we use the unified parameters $\{\sqrt{-\varepsilon}, \eta_E\} = E_1/N_{E/E_1}(E^\times)$. The fourth column identifies the unitary group into which the resulting torus, whose label is in the last column, embeds, by Lemma 3.3.3 and the comments thereafter. The second column simply recaps a step of the proof of the lemma.*

3.3.2 Constructing a σ_0-Hermitian Form J for U_2 when $\mathcal{A} = K \oplus K$, where K is a Quadratic Extension of F

Finally, we consider the tori in groups U_2 produced from $\mathcal{A} = K \oplus K$, where $K = F(\sqrt{\beta})$, with involution $\sigma^* = (\sigma_0, \sigma_0) \in \mathrm{Aut}(\mathcal{A}/F)$. The parameter μ^* of Proposition 3.1.3(3) is an element of $\mathcal{A}^{\times\sigma^*}/N_{\sigma^*}(\mathcal{A}^\times) = F^\times/N_{K/F}(K^\times) \times F^\times/N_{K/F}(K^\times)$ modulo permutation of the components, which yields three choices for μ^* in total (as seen in Section 3.2.2.1). Writing $\mu^* = (\frac{1}{2}\mu_1, \frac{1}{2}\mu_2)$ and $\sigma_0(x) = \overline{x}$ for $x \in K$, we have for $a = (x, y) \in K \oplus K$ that

$$\mathrm{Tr}_{\mathcal{A}/F}(\mu^* a \sigma^*(a)) = \mu_1 x\overline{x} + \mu_2 y\overline{y}. \tag{3.3.1}$$

Let J be the corresponding Hermitian form and denote the resulting torus in $U_\beta(J)$ by $\mathcal{U}^2_{\beta,\mu_1,\mu_2}$. This is summarized in Algorithm 5.

Algorithm 5 Generating tori in U_2 from $\mathcal{A} = K \oplus K$, where $K = F(\sqrt{\beta})$ is a quadratic extension of F.

1: Input: $(n = 2, \mathcal{K} = F(\sqrt{\beta}))$
2: Set $\mathcal{A} = K \oplus K$ and $\sigma^* = (\sigma_0, \sigma_0) \in \mathrm{Aut}(\mathcal{A}/F)$.
3: Choose a representative $\mu^* = (\mu_1, \mu_2) \in F^\times/N_{K/F}(K^\times) \times F^\times/N_{K/F}(K^\times)$ modulo the equivalence $(\mu_1, \mu_2) \sim (\mu_2, \mu_1)$.
4: Compute the associated Hermitian form J that sends $a = (x, y) \in K^2$ to $\mu_1 x\overline{x} + \mu_2 y\overline{y}$.
5: Then $\{a \in \mathcal{A} : a\sigma^*(a) = 1\}$ is a maximal elliptic torus of $U_\beta(J)$, denoted by $\mathcal{U}^2_{\beta,\mu_1,\mu_2}$
6: Output: the pair $(J, \mathcal{U}^2_{\beta,\mu_1,\mu_2})$.

For each of the three quadratic extension fields K there are three choices of parameters in Algorithm 5, yielding nine classes of tori distributed among six nonisomorphic groups.

Lemma 3.3.4. *In the setting of Algorithm 5, write $F^\times/N_{K/F}(K^\times) = \{1, \tau\}$. For fixed β there are two unitary groups, denoted $U_\beta(J)$ and $U_\beta(\mathbf{C})$. Then if K/F is unramified or $-1 \in F^{\times 2}$, then both $\mathcal{U}^2_{\beta,1,1}$ and $\mathcal{U}^2_{\beta,\tau,\tau}$ embed in $U_\beta(J)$ while $\mathcal{U}^2_{\beta,1,\tau}$ embeds in $U_\beta(\mathbf{C})$. The opposite holds, otherwise.*

Proof. Let $a = (x, y) \in \mathcal{A}$, with $x, y \in K$. When $\mu^* = (\mu_1, \mu_2)$, the discriminant of J is computed from (3.3.1) to be $\mu_1\mu_2$. This lies in $N_{K/F}(K^\times)$ when $\mu_1 = \mu_2$ and is a non-norm when $\mu_1 \neq \mu_2$. Recall that $-1 \in N_{K/F}(K^\times)$ if either K/F is unramified or $-1 \in F^{\times 2}$ yields the result. ∎

We summarize the results in Table 3.9.

For ease of reference, we gather all of the tori of U_2 from Tables 3.8 and 3.9, and sort them according to the unitary groups into which they embed. The result is Table 3.10.

$K = F(\sqrt{\beta})$	μ^*	$U_\alpha(J)$	$\mathcal{T}$
$F(\sqrt{\varepsilon})$	$(1, 1)$	$U_\varepsilon(\mathbb{H})$	$\mathcal{U}^2_{\varepsilon,1,1}$
	(ϖ, ϖ)	$U_\varepsilon(\mathbb{H})$	$\mathcal{U}^2_{\varepsilon,\varpi,\varpi}$
	$(1, \varpi)$	$U_\varepsilon(\mathbf{C})$	$\mathcal{U}^2_{\varepsilon,1,\varpi}$
$F(\sqrt{\varpi'})$	$(1, -1)$	$U_{\varpi'}(\mathbb{H})$	$\mathcal{U}^2_{\varpi',1,-1}$
	$(\varepsilon, -\varepsilon)$ omit if $-1^{\boxtimes}$.	$U_{\varpi'}(\mathbb{H})$	$\mathcal{U}^2_{\varpi',\varepsilon,-\varepsilon}$
	$(1, -\varepsilon)$	$U_{\varpi'}(\mathbf{C})$	$\mathcal{U}^2_{\varpi',1,-\varepsilon}$
	$(-1, \varepsilon)$ omit if $-1^{\square}$.	$U_{\varpi'}(\mathbf{C})$	$\mathcal{U}^2_{\varpi',-1,\varepsilon}$

Table 3.9: *Classification of Hermitian forms from $\mathcal{A} = K \oplus K$ from Algorithm 5 using unified parameters. Here, $\varpi' \in \{\varpi, \varepsilon\varpi\}$.*

Group	Representatives for Conjugacy Classes of Maximal Elliptic Tori
$U_\varepsilon(\mathbb{H})$	$\mathcal{U}^2_{\varepsilon,1,1}$ $\mathcal{U}^2_{\varepsilon,\varpi,\varpi}$ ${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,\sqrt{\varpi}}$ ${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,\sqrt{\varepsilon\varpi}}$
$U_\varepsilon(\mathbf{C})$	$\mathcal{U}^2_{\varepsilon,1,\varpi}$ ${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,1}$ ${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,1}$
$U_\varpi(\mathbb{H})$	$\mathcal{U}^2_{\varpi,1,-1}$, $\mathcal{U}^2_{\varpi,\varepsilon,-\varepsilon}$(omit if $-1^{\boxtimes}$), ${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,\sqrt{-\varepsilon}}$ ${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,\sqrt{\varepsilon\varpi}}$
$U_\varpi(\mathbf{C})$	$\mathcal{U}^2_{\varpi,1,-\varepsilon}$, $\mathcal{U}^2_{\varpi,-1,\varepsilon}$(omit if $-1^{\square}$), ${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,1}$, ${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,\eta_E}$
$U_{\varepsilon\varpi}(\mathbb{H})$	$\mathcal{U}^2_{\varepsilon\varpi,1,-1}$ $\mathcal{U}^2_{\varepsilon\varpi,\varepsilon,-\varepsilon}$(omit if $-1^{\boxtimes}$) ${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,\sqrt{-\varepsilon}}$ ${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,\sqrt{\varpi}}$
$U_{\varepsilon\varpi}(\mathbf{C})$	$\mathcal{U}^2_{\varepsilon\varpi,1,-\varepsilon}$ $\mathcal{U}^2_{\varepsilon\varpi,-1,\varepsilon}$(omit one if $-1^{\square}$), ${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,1}$, ${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,\eta_E}$

Table 3.10: *Representatives for Conjugacy Classes of Maximal Elliptic Tori in U_2*

3.4 Comparison with embeddings into $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$

In this section, we consider the embeddings of various F-forms H of U_2 and of $\mathrm{SO}_2 \times \mathrm{SO}_3$ into $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. Each embedding maps maximal elliptic tori of H to maximal elliptic tori of G. We show this map is injective on conjugacy classes and provide the explicit parametrization. This provides some independent corroboration of the precision and accuracy of our answers.

3.4.1 Embeddings of maximal elliptic tori of $\mathrm{SO}_2 \times \mathrm{SO}_3$

Recall by Proposition 2.4.21, that the first type twisted Levi subgroups in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ is $\mathrm{SO}(q_2) \times \mathrm{SO}(q_3)$. Specifically, by Proposition 2.2.19, is

$$\mathrm{SO}(\langle \mu, -\mu\alpha \rangle) \times \mathrm{SO}(\langle \mu, -\mu\alpha, 1 \rangle)$$

There are exactly two F-isomorphism classes of such subgroups in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, as established by Table 2.8, given by

$$\text{A partially split subgroup:} \quad {}^{\alpha}\mathcal{S}_{1,s} = \phi(\mathrm{SO}(\langle 1, -\alpha \rangle) \times \mathrm{SO}(\mathbb{H} \oplus \langle \alpha \rangle)),$$

$$\text{A compact subgroup:} \quad {}^{\alpha}\mathcal{S}_{\mu,c} = \phi(\mathrm{SO}({}^{\nabla}(\langle 1, -\alpha \rangle)) \times \mathrm{SO}({}^{\nabla}(\mathbb{H} \oplus \langle \alpha \rangle))).$$

Each conjugacy class of maximal elliptic torus of $\mathrm{SO}(\langle \mu, -\mu\alpha \rangle) \times \mathrm{SO}(\langle \mu, -\mu\alpha, 1 \rangle)$ is represented by a torus of the form $T \times \mathrm{SO}(\langle \mu, -\mu\alpha \rangle)$, where T is a maximal elliptic torus of $\mathrm{SO}(\langle \mu, -\mu\alpha, 1 \rangle)$. The image of this torus is a maximal elliptic torus of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

Proposition 3.4.1. *For each elliptic torus $T \times \mathrm{SO}(\langle \mu, -\mu\alpha \rangle) \subset \mathrm{SO}(q_3) \times \mathrm{SO}(q_2)$ we have*

$$\phi({}^{\alpha}\mathcal{T}_{[\mu_1,\lambda]} \times \mathrm{SO}(\langle \mu_2, -\mu_2\beta \rangle)) = \begin{cases} {}^{\alpha}\mathcal{T}^{\beta}_{[\mu_1,\mu_2,\lambda]} & \text{if } \alpha \neq \beta \\ {}^{\alpha}\mathcal{T}^{2}_{[\mu_1,\mu_2,\lambda]} & \text{if } \alpha = \beta. \end{cases}$$

Proof. There are six pairs (q_2, q_3) to consider, as listed in Table 2.6. In $\mathrm{SO}(q_3)$, the elliptic tori (Table 3.2) arise from algebras of the form $K \oplus F = F(\sqrt{\alpha}) \oplus F$, with parameters $\mu^* = (\frac{1}{2}\mu_1, \lambda)$, where $\lambda \in F^\times/F^{\times 2}$ and $\mu_1 \in \{\frac{1}{2}, \frac{1}{2}\tau_1\}$ where $\tau_1 = \varpi$ if $\alpha = \varepsilon$ and $\tau_1 = \varepsilon$ otherwise. The corresponding label is ${}^{\alpha}\mathcal{T}_{[\mu_1,\lambda]}$. By Example 3.1.4, the torus $\mathrm{SO}(\langle \mu_2, -\mu_2\beta \rangle)$ arises from the algebra $E = F(\sqrt{\beta})$ with the parameter $\mu_2 \in \{\frac{1}{2}, \frac{1}{2}\tau_2\}$ where $\tau_2 = \varpi$ if $\beta = \varepsilon$ and $\tau_2 = \varepsilon$ otherwise. Then $K \oplus E$ is one of the algebras input into Algorithm 2, and (μ_1, μ_2) is a corresponding choice of parameter. Finally, to upgrade to a torus in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle$, as in Section 3.2.3, we add a copy of F to the algebra and include a parameter $\lambda \in F^\times/F^{\times 2}$. Thus the conjugacy class of the image $\phi({}^{\alpha}\mathcal{T}_{[\mu_1,\lambda]} \times \mathrm{SO}(\langle \mu_2, -\mu_2\beta \rangle))$ is as given in the statement. ∎

Notice that for fixed $\alpha = \beta$, and $\mu_1, \mu_2 \in N_{F(\sqrt{\alpha})/F}(F(\sqrt{\alpha})^\times$, the tori $\phi({}^{\alpha}\mathcal{T}_{[\mu_1,\lambda]} \times \mathrm{SO}(\langle \mu_2, -\mu_2\alpha \rangle))$ and $\phi({}^{\alpha}\mathcal{T}_{[\mu_2,\lambda]} \times \mathrm{SO}(\langle \mu_1, -\mu_1\alpha \rangle))$ are conjugate in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

Now to specify the conjugacy class of twisted Levi that contains this torus, we use the following process:

$$\phi({}^{\alpha}\mathcal{T}_{[\mu_1,\lambda]} \times \mathrm{SO}(\langle \mu_2, -\mu_2\beta \rangle)) \cong {}^{\alpha}\mathcal{T}^{\beta}_{[\mu_1,\mu_2,\lambda]} \subset \phi(\mathrm{SO}(\langle \mu_1, -\mu_1\alpha, \lambda \rangle) \times \mathrm{SO}(\langle \mu_2, -\mu_2\beta \rangle)) = {}^{\beta}\mathcal{S}_{\mu_2,x},$$

$$\phi({}^{\beta}\mathcal{T}_{[\mu_2,\lambda]} \times \mathrm{SO}(\langle \mu_1, -\mu_1\alpha \rangle)) \cong {}^{\alpha}\mathcal{T}^{\beta}_{[\mu_1,\mu_2,\lambda]} \subset \phi(\mathrm{SO}(\langle \mu_2, -\mu_2\beta, \lambda \rangle) \times \mathrm{SO}(\langle \mu_1, -\mu_1\alpha \rangle)) = {}^{\alpha}\mathcal{S}_{\mu_1,y}.$$

Here x (resp. y) is denoted by s whenever the associated 3 dimensional quadratic form is split and c i.e. compact otherwise.

Theorem 3.4.2. *The isomorphism classes of twisted Levi subgroups of the type $\mathcal{S}$ that embed in the split group* $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ *are:*

$$^{\varepsilon}\mathcal{S}_{1,s}, {}^{\varepsilon}\mathcal{S}_{\varpi,c}, {}^{\varpi}\mathcal{S}_{1,s}, {}^{\varpi}\mathcal{S}_{\varepsilon,c}, {}^{\varepsilon\varpi}\mathcal{S}_{1,s}, {}^{\varepsilon\varpi}\mathcal{S}_{\varepsilon,c}$$

We list all the embeddings of elliptic tori in this type of twisted Levi in Table A.1.

3.4.2 Embeddings of maximal elliptic tori of U_2

Now we work with the second type of twisted Levi in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, which is by Proposition 2.4.21, is $U_{E/F}(J)$ for some rank 2 Hermitian form J. Specifically as established in Proposition 2.3.6, each of the groups $U_\beta(\mathbb{H})$ embeds in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ and the compact groups embed in $\mathrm{SO}(\mathbb{H} \oplus W_1)$. This embedding preserves isomorphism classes of tori, but in fact also preserves conjugacy classes. We denote this embedding by ϕ and the image of U_2 by $\mathcal{L}$.

Proposition 3.4.3. *Let $\beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$, and either $J = \mathbb{H}$ and $q = 2\mathbb{H} \oplus \langle 1 \rangle$ or $J = \mathbf{C}$ and $q = \mathbb{H} \oplus W_1$. Let $\phi : U_\beta(J) \to \mathrm{SO}(q)$ be the embedding from Proposition 2.3.6. Then up to conjugacy in $\mathrm{SO}(q)$ we have*

$$\phi({}^{\alpha}\mathcal{U}^{\varepsilon,\varpi}_{\beta,\mu/2}) \simeq {}^{\alpha}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu,1]}$$
$$\phi(\mathcal{U}^{2}_{\beta,\mu_1/2,\mu_2/2}) \simeq {}^{\beta}\mathcal{T}^{2}_{[\mu_1,\mu_2,1]}.$$

In particular, ϕ maps the conjugacy classes of maximal elliptic tori of $U_\beta(J)$ injectively into conjugacy classes of maximal elliptic tori of $\mathrm{SO}(q)$.

Proof. The conjugacy classes of maximal elliptic tori embedded in $U_\beta(J)$ arise from the two constructions, Algorithms 4 and 5. In each case, taking the trace of K/F of the resulting Hermitian form yields the quadratic form output, on a corresponding set of parameters for Algorithms 3 and 2, respectively. Therefore, up to conjugacy in either of these groups, the algorithms produce the same number of tori. However, due to our choice of scaling, the value of μ^* in Algorithms 4 and 5 is exactly twice the value of μ^* in Algorithms 3 and 2, implying that by our indexing, a parameter $\mu/2$ for the unitary group case corresponds to μ in the case of the 4-dimensional quadratic form.

To further embed into $\mathrm{SO}(q)$, note from Table 2.7 that we simply have $q' \oplus \langle 1 \rangle \simeq q$. That is, in all of these cases, the additional parameter λ that must be added to the labeling of a torus in $\mathrm{SO}(q')$ to produce the corresponding torus in $\mathrm{SO}(q)$ (as discussed in Section 3.2.3) is $\lambda = 1$. ∎

Now to specify the conjugacy class of twisted Levi that contains this torus, we use the following process:

$$\phi(U_\alpha(\mathbb{H}) = {}^{\alpha}\mathcal{L}_s, \qquad \phi(U_\alpha(\mathbf{C}) = {}^{\alpha}\mathcal{L}_c.$$

Since our focus is the split group $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, then only we are concerned with $U_\alpha(\mathbb{H})$ and its associated image ${}^{\alpha}\mathcal{L}_s$.

Theorem 3.4.4. *The isomorphism classes of twisted Levi subgroups of the type $\mathcal{L}$ that embed in the split group* $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ *are:*

$$^{\varepsilon}\mathcal{L}_s, {}^{\varpi}\mathcal{L}_s, {}^{\varepsilon\varpi}\mathcal{L}_s.$$

We list all the embeddings of elliptic tori in this type of twisted Levi in Table A.2. These embeddings are also summarized in Figure 3.1

The sharp-eyed reader will have noticed that some tori of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, as listed in Table 3.7 appear twice in these lists, and others do not appear at all. This mystery will be resolved in Chapter 4, using the classification of non-regular elements in the Lie algebras of these tori in Section 4.1.

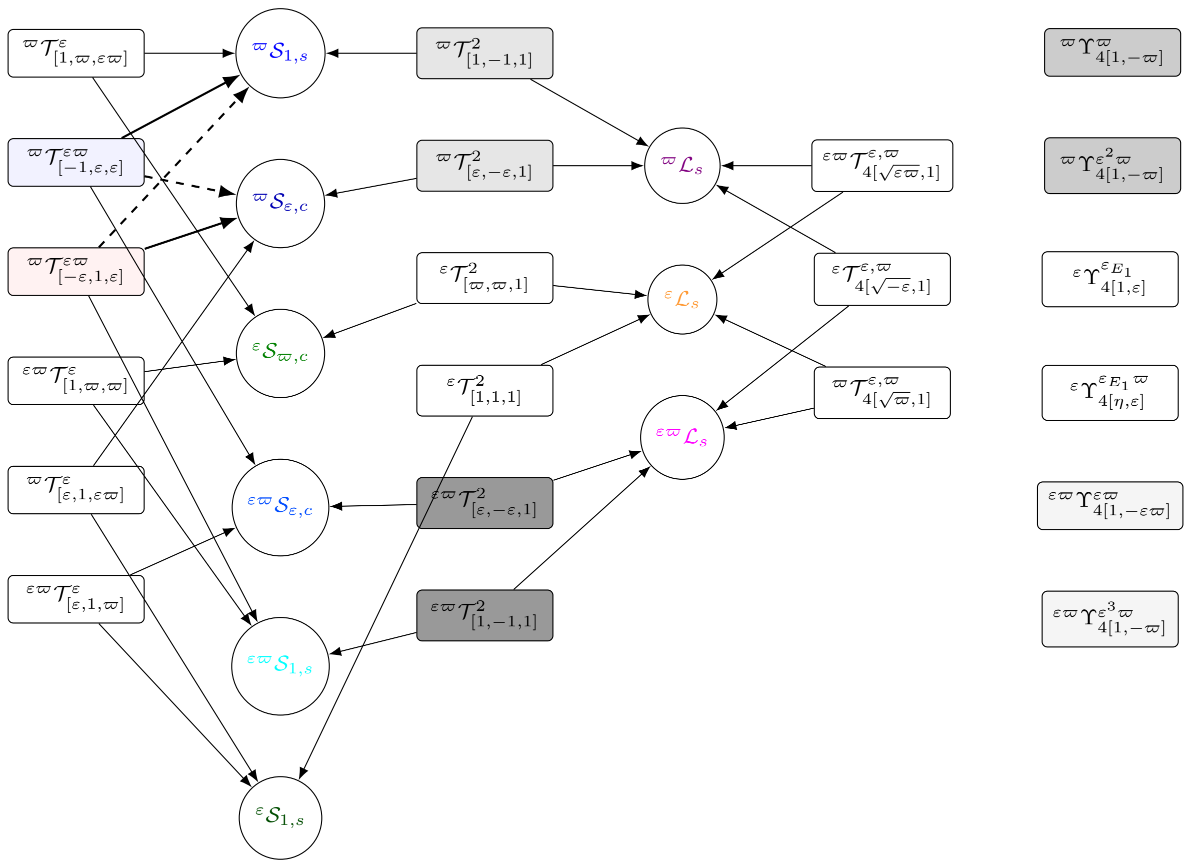

Figure 3.1: *Elliptic maximal tori G^0 and their associated twisted Levi subgroups $G^1 \subset \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. When $-1 \notin F^{\times 2}$, the solid arrow from the lightly shaded tori in the first column is replaced by the dashed arrow, while the remaining twisted Levi subgroups remain unchanged. In the middle column and the last column, tori with the same gray shade merge into a single G-conjugacy class. This diagram is given in a table form in Table A.1 and Table A.2*

Chapter 4

Conjugacy classes of twisted Levi subgroups

In Chapter 2, we identified some twisted Levi subgroups of G that have compact center. In Chapter 3, we showed how each of the maximal elliptic tori of G embed into these various twisted Levi subgroups. In this chapter, we prove that we have indeed identified all conjugacy classes of twisted Levi subgroups of G having compact center, which is the first necessary ingredient for the data for the construction of supercuspidal representations.

Our strategy is as follows. Every twisted Levi subgroup $\mathcal{M}$ is the centralizer of the center $\mathfrak{z}$ of its Lie algebra. We are interested in the case that $\mathcal{M}$ is proper and not itself a torus. Therefore $\dim(\mathfrak{z}) = 1$. Recall that we further require $\mathcal{M}$ to have compact center. Thus $\mathfrak{z}$ will be contained in the Lie algebra $\mathfrak{t}$ of an elliptic maximal torus $\mathcal{T}$. Since its centralizer $\mathcal{M}$ is not equal to $\mathcal{T}$, the elements of $\mathfrak{z}$ will be non-regular.

In Section 4.1 we find, for each of our maximal elliptic tori $\mathcal{T}$ (representing a conjugacy class in G), all of its 1-dimensional subalgebras that consist entirely of non-regular elements, up to conjugacy. In Section 4.2, we prove that every conjugacy class of twisted Levi subgroup of G (that is proper, not a torus, and has compact center) is represented by one of the twisted Levi subgroups

$$^{\varepsilon}\mathcal{S}_{1,s}, {}^{\varepsilon}\mathcal{S}_{\varpi,c}, {}^{\varpi}\mathcal{S}_{1,s}, {}^{\varpi}\mathcal{S}_{\varepsilon,c}, {}^{\varepsilon\varpi}\mathcal{S}_{1,s} {}^{\varepsilon\varpi}\mathcal{S}_{\varepsilon,c}, {}^{\varepsilon}\mathcal{L}_s, {}^{\varpi}\mathcal{L}_s, {}^{\varepsilon\varpi}\mathcal{L}_s.$$

4.1 Non-regular elements of $\mathfrak{t} \subset \mathfrak{so}_5$

We now turn to the Lie algebra of the group, namely $\mathfrak{g} = \mathfrak{so}(2\mathbb{H} \oplus \langle 1 \rangle)$. Our goal in this section is to identify the (conjugacy classes) of non-regular elements of the Lie algebras of each of our maximal elliptic tori.

We write ${}^{\alpha}\mathfrak{t}^{\beta}_{[\mu_1,\mu_2,\lambda]}$ for the Lie algebra of the torus ${}^{\alpha}\mathcal{T}^{\beta}_{[\mu_1,\mu_2,\lambda]}$, and ${}^{\alpha}\mathfrak{u}^{\gamma}_{4[\mu_1,\mu_2]}$ for the Lie algebra of ${}^{\alpha}\Upsilon^{\gamma}_{4[\mu_1,\mu_2]}$, for example. We may omit the parameters μ^* that define the subscript when these are not important, so we may for example simply write ${}^{\alpha}\mathcal{T}^{\beta}$ and ${}^{\alpha}\mathfrak{t}^{\beta}$. These correspond to isomorphism classes of tori in G, so we sometimes refer to this as the isomorphism class.

In this section, we present our tori relative to the basis of the algebra $\mathcal{A}$ we chose in Chapter 4, such that the torus preserves a quadratic form q (that is isometric to $\mathbb{H} \oplus \langle 1 \rangle$). Then the G-conjugacy of the corresponding torus in G is equivalent to $\mathrm{SO}(q)$-conjugacy of the torus as constructed. We adopt this abuse of notation throughout.

Throughout this section, for a given torus $\mathcal{T}$ and $\mathfrak{t}$ its Lie algebra we let E_s denote their splitting field.

Theorem 4.1.1. *Let $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of a maximal torus $\mathcal{T}$ of G. Then the set of eigenvalues of $X \in \mathfrak{t}$ has the form $\{\lambda_1, \lambda_2, -\lambda_1, -\lambda_2, 0\}$ for some $\lambda_1, \lambda_2 \in E_s$. Moreover, X is regular if and only if λ_1 and λ_2 are distinct and non-zero.*

Proof. Since $\mathcal{T}$ splits over E_s, the group $\mathcal{T}(E_s)$ is $G(E_s)$-conjugate to the diagonal split torus, and its Lie algebra is $G(E_s)$-conjugate to the diagonal Cartan subalgebra $\mathfrak{s}(E_s)$. It follows that any element of $\mathfrak{t}$ is conjugate to a diagonal matrix of the form

$$Y = \mathrm{diag}(\lambda_1, \lambda_2, -\lambda_1, -\lambda_2, 0), \qquad \lambda_1, \lambda_2 \in E_s,$$

so its eigenvalues take the stated form.

Now recall from Theorem 2.4.14 that Y is regular if and only if $a(Y) \neq 0$ for every root $a \in \Phi(\mathfrak{g}(E_s), \mathfrak{s}(E_s))$. With our naming convention from Example 2.4.10, evaluating the roots on Y gives

$$\pm a_{s_2}(Y) = \pm\lambda_2, \qquad \pm a_{l_1}(Y) = \pm(\lambda_1 - \lambda_2),$$
$$\pm(a_{s_2} + a_{l_1})(Y) = \pm\lambda_1, \qquad \pm(2a_{s_2} + a_{l_1})(Y) = \pm(\lambda_1 + \lambda_2).$$

Therefore Y is regular if and only if

$$\lambda_1 \neq 0, \quad \lambda_2 \neq 0, \quad \text{and} \quad \lambda_1 \neq \pm\lambda_2,$$

as asserted. ∎

Corollary 4.1.2. *Let $X \in \mathfrak{t} \subset \mathfrak{g}$ with eigenvalues $\{\pm\lambda_1, \pm\lambda_2, 0\}$. Then X is non-regular if and only if*

$$\lambda_1 = 0, \quad \text{or} \quad \lambda_2 = 0, \quad \text{or} \quad \lambda_1 = \pm\lambda_2.$$

Although the set of non-regular elements of $\mathfrak{t}$ is not a subalgebra, it is the union of one-dimensional subalgebras. To find these subalgebras, we must first find the eigenvalues of each of the nonzero elements of $\mathfrak{t}$. Recall from Proposition 3.1.3 that if $\mathcal{T}$ was constructed from an algebra $\mathcal{A}$ and involution σ then $\mathfrak{t} = \{a \in \mathcal{A} \mid a + \sigma(a) = 0\}$. Let us first note that the splitting field of $\mathcal{T}$ may be strictly larger than any of the field extensions of F occurring as summands of $\mathcal{A}$.

Example 4.1.3. Let $E = F(\sqrt[4]{\varpi})$, which arises in the algebra $\mathcal{A}$ defining the tori in the isomorphism class $\mathcal{T} = {}^{\alpha}\Upsilon_4^E$. If $-1 \notin F^{\times 2}$, E/F is not Galois. Since $i \notin F$, its Galois closure is $E_s = F(\sqrt[4]{\varpi}, i)$, which is a splitting field for $\mathcal{T}$.

More generally, we may take E_s to be the Galois closure of the union of the field summand of $\mathcal{A}$. By Theorem 4.1.1, the eigenvalues of the elements of $\mathfrak{t}$ will belong to E_s. We summarize the isomorphism classes of elliptic tori in G, together with their Lie algebras and splitting fields, in Table 4.1.

$\mathcal{T}$	$\mathfrak{t}$	$\mathcal{A} = K_1 \oplus K_2 \oplus F$	E_s	Parameters	$\mathrm{Gal}(E_s/F)$
${}^{\alpha}\mathcal{T}^2$	${}^{\alpha}\mathfrak{t}^2$	$F[\sqrt{\alpha}] \oplus F[\sqrt{\alpha}] \oplus F$	$F[\alpha]$	$\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$	$(\mathbb{Z}/2\mathbb{Z})^2$
${}^{\alpha}\mathcal{T}^{\beta}$	${}^{\alpha}\mathfrak{t}^{\beta}$	$F[\sqrt{\alpha}] \oplus F[\sqrt{\beta}] \oplus F$	$F[\sqrt{\varepsilon}, \sqrt{\varpi}]$	$\alpha, \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$	$(\mathbb{Z}/2\mathbb{Z})^2$
${}^{\alpha}\mathcal{T}_4^{\varepsilon,\varpi}$	${}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi}$	$E_1 = F[\sqrt{\alpha}] \oplus F \subset F[\sqrt{\varepsilon}, \sqrt{\varpi}] \oplus F$	$F[\sqrt{\varepsilon}, \sqrt{\varpi}]$	$\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$	$(\mathbb{Z}/2\mathbb{Z})^2$
$\mathcal{T}$	$\mathfrak{t}$	$\mathcal{A} = E \oplus F$	E_s	Parameters	$\mathrm{Gal}(E_s/F)$
${}^{\alpha}\Upsilon_4^E$	${}^{\alpha}\mathfrak{u}_4^E$	$-1^{\square}$, $E = F[\sqrt[4]{\alpha}]$, $E_1 = F[\sqrt{\alpha}]$	E	$\alpha \in \{\varepsilon, \varepsilon\varpi^2\}$	$\mathbb{Z}/4\mathbb{Z}$
${}^{\alpha}\Upsilon_4^E$	${}^{\alpha}\mathfrak{u}_4^E$	$-1^{\square}$, $E = F[\sqrt[4]{\alpha}]$, $E_1 = F[\sqrt{\alpha}]$	E	$\alpha \in \{\varepsilon^k\varpi \mid k = 0,1,2,3\}$	$\mathbb{Z}/4\mathbb{Z}$
${}^{\alpha}\Upsilon_4^E$	${}^{\alpha}\mathfrak{u}_4^E$	$-1^{\boxtimes}$, $E = F[\sqrt{\gamma}]$, $E_1 = F[i]$	E	$\gamma \in \{\varepsilon_{E_1}, \varepsilon_{E_1}\varpi\}$	$\mathbb{Z}/4\mathbb{Z}$
${}^{\alpha}\Upsilon_4^E$	${}^{\alpha}\mathfrak{u}_4^E$	$-1^{\boxtimes}$, $E = F[\sqrt[4]{\alpha}]$, $E_1 = F[\sqrt{\alpha}]$	$F[\sqrt[4]{\alpha}, i]$	$\alpha \in \{\varepsilon^k\varpi \mid k = 0,1\}$	D_8

Table 4.1: *The isomorphism classes of elliptic tori in G and their associated algebras, together with their splitting fields. We identify $\mathcal{A}^{\sigma} = E_1$ where needed. In the bottom four rows, we put the qualifier $-1^{\square}$ to indicate this corresponds only to the case that $-1 \in F^{\times 2}$ and the qualifier $-1^{\boxtimes}$ when the row corresponds only to the case that $-1 \notin F^{\times 2}$.*

Lemma 4.1.4. *Let $n \geq 1$. Suppose $\mathcal{A} = \bigoplus_{i=1}^{n} E_i \oplus F$ and $\sigma^* = (\sigma_1, \cdots, \sigma_n, \mathrm{Id}_F) \in \mathrm{Aut}(\mathcal{A})$, with each σ_i is a nontrivial involution, and set $\mathfrak{t} = \{a \in \mathcal{A} \mid a + \sigma^*(a) = 0\}$. Then the eigenvalues of an arbitrary element $X = (a_1, \cdots, a_n, 0) \in \mathfrak{t}$ are $\mathrm{Gal}(E_s/F) \cdot \{a_1, \cdots, a_n\}$.*

Proof. The eigenvalues of left multiplication of an element a_i of E_i on E_i as an F-vector space are the roots of its minimal polynomial over F, which are precisely $a_i \in E_i$ and its Galois conjugates under $\mathrm{Gal}(E_s/F)$. ∎

We now apply these ideas to each of the isomorphism classes of maximal elliptic tori to classify their non-regular elements, up to conjugacy, in the following subsections.

4.1.1 Non-regular elements in tori of the forms ${}^{\alpha}\mathfrak{t}^2$ and ${}^{\alpha}\mathfrak{t}^{\beta}$

Let ${}^{\alpha}\mathcal{T}^{\beta}$ be an elliptic torus associated with $\mathcal{A} = K_1 \oplus K_2 \oplus F$, where $K_1 = F(\sqrt{\alpha})$ and $K_2 = F(\sqrt{\beta})$ with $\alpha, \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$, as in Section 3.2.3. Here $\sigma^* = (\sigma_\alpha, \sigma_\beta, \mathrm{Id}_F)$ where $\sigma_\gamma \in \mathrm{Gal}(F(\sqrt{\gamma})/F) \smallsetminus \{\mathrm{Id}\}$. Then the Lie algebra of $\mathcal{T}$ is

$$ {}^{\alpha}\mathfrak{t}^{\beta} = \{(x\sqrt{\alpha}, y\sqrt{\beta}, 0) \mid x, y \in F\}. \tag{4.1.1}$$

and given such an element, its set of eigenvalues is $\{\pm x\sqrt{\alpha}, \pm y\sqrt{\beta}, 0\}$.

Proposition 4.1.5. *In the setting above, if $\alpha \neq \beta$ then the non-regular elements in ${}^{\alpha}\mathfrak{t}^{\beta}$ form the union of the two subalgebras*

$$ {}^{\alpha}\mathfrak{s}^{\beta}_{y=0} = \{(x\sqrt{\alpha}, 0, 0) \mid x \in F\}, \qquad {}^{\alpha}\mathfrak{s}^{\beta}_{x=0} = \{(0, y\sqrt{\beta}, 0) \mid y \in F\}. $$

If $\alpha = \beta$ then the non-regular elements in ${}^{\alpha}\mathfrak{t}^{2}$ form the union of the four subalgebras ${}^{\alpha}\mathfrak{s}^{2}_{y=0}$, ${}^{\alpha}\mathfrak{s}^{2}_{x=0}$ and

$$ {}^{\alpha}\mathfrak{s}^{2}_{y=x} = \{(x\sqrt{\alpha}, x\sqrt{\alpha}, 0) \mid x \in F\}, \quad {}^{\alpha}\mathfrak{s}^{2}_{y=-x} = \{(x\sqrt{\alpha}, -x\sqrt{\alpha}, 0) \mid x \in F\}. $$

Proof. Let $X = (x\sqrt{\alpha}, y\sqrt{\beta}, 0) \in {}^{\alpha}\mathfrak{t}^{\beta}$, whose eigenvalues by Lemma 4.1.4 are $\pm x\sqrt{\alpha}, \pm y\sqrt{\beta}, 0$. By Corollary 4.1.2, X is non-regular if and only if

$$ x\sqrt{\alpha} = 0, \text{ or } y\sqrt{\beta} = 0, \text{ or } x\sqrt{\alpha} - y\sqrt{\beta} = 0, \text{ or } x\sqrt{\alpha} + y\sqrt{\beta} = 0. $$

If $\alpha \neq \beta$, then $\{\sqrt{\alpha}, \sqrt{\beta}\}$ is linearly independent over F. Hence the only non-regular elements are those for which one eigenvalue vanishes, namely $x = 0$ or $y = 0$, which gives ${}^{\alpha}\mathfrak{s}^{\beta}_{y=0}$ and ${}^{\alpha}\mathfrak{s}^{\beta}_{x=0}$, respectively. If $\alpha = \beta$, then the four non-regular conditions become

$$ x = 0, \qquad y = 0, \qquad x = y, \qquad x = -y. $$

These exactly define the four sets ${}^{\alpha}\mathfrak{s}^{2}_{y=0}, {}^{\alpha}\mathfrak{s}^{2}_{x=0}, {}^{\alpha}\mathfrak{s}^{2}_{y=x}, {}^{\alpha}\mathfrak{s}^{\beta}_{y=-x}$. ∎

Each of these sets of non-regular elements define a one-dimensional subalgebra of ${}^{\alpha}\mathfrak{t}^{\beta}$. When $\alpha \neq \beta$, these subalgebras are not isomorphic, hence their embeddings in $\mathfrak{g}$ will be non-conjugate under G. The following proposition determines the G-conjugacy in the case that $\alpha = \beta$.

Proposition 4.1.6. *Let $E_\alpha = F(\sqrt{\alpha})$, where $\alpha \in F^\times \setminus F^{\times 2}$, and let $\mu_1, \mu_2 \in F^\times / N_{E_\alpha/F}(E_\alpha^\times)$. Let q denote the quadratic form on $F^5 \cong E_\alpha \oplus E_\alpha \oplus F$ preserved by the torus ${}^{\alpha}\mathfrak{t}^{2}_{\mu_1,\mu_2,1}$ and set $G' = \mathrm{SO}(q)$. Let ${}^{\alpha}\mathfrak{s}^{2}_{y=0}, {}^{\alpha}\mathfrak{s}^{2}_{x=0}, {}^{\alpha}\mathfrak{s}^{2}_{y=x}, {}^{\alpha}\mathfrak{s}^{2}_{y=-x}$ denote the four F–subalgebras of non-regular elements described in Proposition 4.1.5. Then the following hold:*

1. *${}^{\alpha}\mathfrak{s}^{2}_{y=0}$ and ${}^{\alpha}\mathfrak{s}^{2}_{y=x}$ are not G'-conjugate;*

2. *${}^{\alpha}\mathfrak{s}^{2}_{y=x}$ and ${}^{\alpha}\mathfrak{s}^{2}_{y=-x}$ are G'–conjugate; and*

3. *${}^{\alpha}\mathfrak{s}^{2}_{y=0}$ and ${}^{\alpha}\mathfrak{s}^{2}_{x=0}$ are G'–conjugate if and only if $\mu_1/\mu_2 \in N_{E_\alpha/F}(E_\alpha^\times)$.*

Proof. Choose the basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (\sqrt{\alpha}, 0, 0)$, $e_4 = (0, \sqrt{\alpha}, 0)$ and $e_5 = (0, 0, 1)$ of $E_\alpha \oplus E_\alpha \oplus F \cong F^5$. Then ${}^{\alpha}\mathfrak{t}^{2}_{\mu_1,\mu_2,1}$ preserves the quadratic form $q = \langle \mu_1, \mu_2, -\mu_1\alpha, -\mu_2\alpha, 1\rangle$. Let $G' = \mathrm{SO}(q)$. We note first that the elements of ${}^{\alpha}\mathfrak{s}^{2}_{y=0}$ and ${}^{\alpha}\mathfrak{s}^{2}_{y=x}$

have distinct sets of eigenvalues, so are not isomorphic (hence not G'-conjugate). On the other hand, consider the nonzero representatives $X_3 \in {}^{\alpha}\mathfrak{s}^2_{y=x}$ and $X_4 \in {}^{\alpha}\mathfrak{s}^2_{y=-x}$ defined by

$$X_3e_1 = e_3, \qquad X_3e_2 = e_4, \qquad X_3e_3 = \alpha e_1, \qquad X_3e_4 = \alpha e_2,$$

$$X_4e_1 = e_3, \qquad X_4e_2 = -e_4, \qquad X_4e_3 = \alpha e_1, \qquad X_4e_4 = -\alpha e_2,$$

and $X_3e_5 = X_4e_5 = 0$. If we let $g = \mathrm{diag}(1,-1,1,1,-1)$ then then $g^TJg = J$ and $\det(g) = 1$, so $g \in G'$. Now $g = g^{-1}$, and it is the identity on the span of $\{e_1, e_3, e_4\}$. Since

$$gX_3g^{-1}(e_2) = gX_3(-e_2) = g(-e_4) = -e_4,$$

and

$$gX_3g^{-1}(e_4) = gX_3(e_4) = g(\alpha e_2) = -\alpha e_2$$

we conclude that $gX_3g^{-1} = X_4$ and so in particular $g{}^{\alpha}\mathfrak{s}^2_{y=x}g^{-1} = {}^{\alpha}\mathfrak{s}^2_{y=-x}$.

Now consider the conjugacy of ${}^{\alpha}\mathfrak{s}^2_{y=0}$ and ${}^{\alpha}\mathfrak{s}^2_{x=0}$. Take basis elements $X_1 \in {}^{\alpha}\mathfrak{s}^2_{y=0}$ and $X_2 \in {}^{\alpha}\mathfrak{s}^2_{x=0}$ defined by

$$X_1e_1 = e_3, \qquad X_1e_3 = \alpha e_1, \qquad X_1e_i = 0 \text{ for } i \neq 1,3,$$

$$X_2e_2 = e_4, \qquad X_2e_4 = \alpha e_2, \qquad X_2e_i = 0 \text{ for } i \neq 2,4.$$

Then the restriction of q to $\ker(X_1) = \mathrm{span}\{e_2, e_4, e_5\}$ is given by $\langle \mu_2, -\mu_2\alpha, 1\rangle$ while the restriction of q to $\ker(X_2) = \mathrm{span}\{e_1, e_3, e_5\}$ is $\langle \mu_1, -\mu_1\alpha, 1\rangle$. Note that by the Witt Cancellation Theorem 2.2.7, these are isometric if and only if $\langle \mu_1, -\mu_1\alpha\rangle \simeq \langle \mu_2, -\mu_2\alpha\rangle$, which by Lemma 2.2.13 can happen if and only if $\mu_1/\mu_2 \in N_{E_\alpha/F}$.

Assume first that $\lambda X_2 = gX_1g^{-1}$ for some $g \in G'$ and $\lambda \in F$. Then g defines an isometry from $\ker(X_1)$ onto $\ker(X_2)$, whence $\mu_1/\mu_2 \in N_{E_\alpha/F}$.

Conversely, assume that $\mu_1/\mu_2 \in N_{E_\alpha/F}(E_\alpha^\times)$. Then by Lemma 2.2.13 we may choose an F-linear isometry $h : \mathrm{span}\{e_2, e_4\} \to \mathrm{span}\{e_1, e_3\}$. Define $g \in GL_5(F)$ by

$$g|_{\langle e_2,e_4\rangle} = h, \qquad g|_{\langle e_1,e_3\rangle} = h^{-1}, \qquad g(e_5) = e_5.$$

Since h is an isometry, it follows that g preserves q, and moreover we have $\det(g) = 1$, so $g \in G'$. Since gX_1g^{-1} vanishes on $\langle e_2, e_4, e_5\rangle$ and for $v \in \mathrm{span}\{e_2, e_4\}$ we have by the definition of h that

$$gX_1g^{-1}(v) = h^{-1}X_1h(v) = X_2(v),$$

we conclude that $gX_1g^{-1} = X_2$, so ${}^{\alpha}\mathfrak{s}^2_{y=0}$ and ${}^{\alpha}\mathfrak{s}^2_{x=0}$ are G'-conjugate. ∎

Note that with respect to our parameter choices, when $\alpha = \varepsilon$, then $\mu_1/\mu_2 \in N_{F(\sqrt{\alpha})/F}((\sqrt{\alpha})^\times)$ if and only if $\mu_1 = \mu_2$, whereas when $\alpha \in \{\varpi, \varepsilon\varpi\}$, the answer will depend on whether or not -1 is a square. When $-1 \in F^{\times 2}$, this is a norm, but otherwise, $-1 \notin N_{F(\sqrt{\alpha})/F}((\sqrt{\alpha})^\times)$. We summarize this in the first rows of Table A.7.

4.1.2 Non-regular elements in tori of the form ${}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi}$ and ${}^{\alpha}\mathfrak{u}_4^{E}$

Now suppose $\mathcal{A} = E \oplus F$, where E is a quartic extension of F and E_1 is a quadratic subfield fixed by σ, as in Section 3.2.3.

We begin with the biquadratic extension, whose associated tori have Lie algebras denoted ${}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi}$, for $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$.

Lemma 4.1.7. *Choose $\alpha \neq \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ and let $\mathcal{A} = E \oplus F$ with $E = F(\sqrt{\varepsilon}, \sqrt{\varpi})$ and $E_1 = F(\sqrt{\alpha}) \subset E$. Then the Lie algebra of the associated torus is given by*

$$ {}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi} = \{(x\sqrt{\beta} + y\sqrt{\alpha\beta}, 0) \mid x, y \in F\}. $$

The set of eigenvalues of $(x\sqrt{\beta}+y\sqrt{\alpha\beta}, 0)$ is $\{\pm x\sqrt{\beta} \pm y\sqrt{\alpha\beta}, 0\}$. The non-regular elements of ${}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi}$ form the union of the two non-conjugate subalgebras

$$ {}^{\alpha}\mathfrak{s}_{4,y=0}^{\varepsilon,\varpi} = \{(x\sqrt{\beta}, 0) \mid x \in F\}, \quad {}^{\alpha}\mathfrak{s}_{4,x=0}^{\varepsilon,\varpi} = \{(y\sqrt{\alpha\beta}, 0) \mid y \in F\}. $$

Proof. An arbitrary element of $\mathcal{A}$ is of the form $(a+b\sqrt{\beta}, c)$ for some $a, b \in F(\sqrt{\alpha})$ and $c \in F$. Since σ^* fixes E_1, we deduce that ${}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi} = \{a \in \mathcal{A} \mid a+\sigma^*(a) = 0\} = \{(b\sqrt{\beta}, 0) \mid b \in E_1\} = \{(x\sqrt{\beta}+y\sqrt{\alpha\beta}, 0) \mid x, y \in F\}$. In this case, $E = E_s$ so $\mathrm{Gal}(E_s/F) \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, generated by sign changes on $\sqrt{\alpha}$ and $\sqrt{\beta}$, which gives for each choice of $X = (x\sqrt{\beta}+y\sqrt{\alpha\beta}, 0) \in {}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi}$, the eigenvalues by Lemma 4.1.4

$$ \{\pm\lambda_1 = \pm(x\sqrt{\beta} + y\sqrt{\alpha\beta}), \pm\lambda_2 = \pm(x\sqrt{\beta} - y\sqrt{\alpha\beta}), 0\}. $$

When $X \neq 0$, $\lambda_1, \lambda_2 \neq 0$. Thus by Corollary 4.1.2, X is non-regular if and only if $\lambda_1 \in \pm\lambda_2$. Now $\lambda_1 = \lambda_2$ if and only if $y = 0$ while $\lambda_1 = -\lambda_2$ if and only if $x = 0$. Hence the non-regular elements are exactly those with $x = 0$ or $y = 0$, as required. Note that these subalgebras have distinct sets of eigenvalues, so are not isomorphic. ∎

These results are summarized in Table A.7.

We now examine the isomorphism classes of tori ${}^{\alpha}\Upsilon_4^{E}$ associated to an algebra $\mathcal{A} = E \oplus F$ where E is a quartic extension with a unique subfield. These forms of tori will stand out from the rest: they have no nonzero non-regular elements.

Lemma 4.1.8. *Let ${}^{\alpha}\mathfrak{u}_4^{E}$ be the elliptic torus associated with the algebra $\mathcal{A} = E \oplus F$, with fixed field $E_1 = F(\sqrt{\alpha}) \subset E = F(\zeta)$ where $\zeta \in \{\sqrt[4]{\alpha}, \sqrt{\gamma}\}$ for the ranges of parameters α, γ set out in Table 4.1 (so in particular $\sqrt{\alpha} = i$ when $E = F(\sqrt{\gamma})$). Then we have*

$$ {}^{\alpha}\mathfrak{u}_4^{E} = \{(x + y\sqrt{\alpha})\zeta, 0) \mid x, y \in F\} $$

and every nonzero element of ${}^{\alpha}\mathfrak{u}_4^{E}$ is regular.

Furthermore, given $X = ((x+y\sqrt{\alpha})\zeta, 0) \in {}^{\alpha}\mathfrak{u}_4^{E}$, its set of eigenvalues is $\{\pm\lambda_1, \pm\lambda_2, 0\}$ where *(a) if* $E = F(\sqrt[4]{\alpha})$ *then* $\lambda_1 = (x + y\sqrt{\alpha})\sqrt[4]{\alpha}$ *and* $\lambda_2 = (x - y\sqrt{\alpha})i\sqrt[4]{\alpha}$*, where* $i = \sqrt{-1}$*;*

(b) if $E = F(\sqrt{\gamma})$ *where* $\gamma \in \{\varepsilon_{E_1}, \varepsilon_{E_1}\varpi\}$*, then* $\lambda_1 = (x + yi)\sqrt{\gamma}$ *and* $\lambda_2 = (x - yi)\sqrt{\tau(\gamma)}$ *where* $\tau \in \mathrm{Gal}(E_1/F) \smallsetminus \{1\}$*.*

Proof. Since in each case, $\sigma(\zeta) = -\zeta$, the definition of the torus yields, as previously, ${}^{\alpha}\mathfrak{u}_4^E = \{(b\zeta, 0) \mid b \in E_1\}$. Setting $X = ((x + y\sqrt{\alpha})\zeta, 0) \in {}^{\alpha}\mathfrak{u}_4^E$ we see that $\lambda_1 = (x + y\sqrt{\alpha})\gamma$ is an eigenvalue of X, with σ-conjugate $-\lambda_1$. Now choose $\tau \in \mathrm{Gal}(E_s/F)$ that restricts nontrivially to E_1.

In the first case it must send $\sqrt[4]{\alpha}$ to $\pm i\sqrt[4]{\alpha}$ (as these are the remaining roots of its minimal polynomial), yielding $\lambda_2 = (x - y\sqrt{\alpha})(i\sqrt[4]{\alpha})$. In the second case, τ must send $i \to -i$ and thus $\tau(\varepsilon_{E_1}) = \tau(s^2 + it^2) = s^2 - it^2$.

In either case, if λ_1 or λ_2 is zero, then X is zero. Moreover, either of the conditions $\lambda_1 = \pm\lambda_2$ entail $x = y = 0$, since otherwise λ_1 and λ_2 are linearly independent over F. Thus by Corollary 4.1.2, there are no nonzero non-regular elements in ${}^{\alpha}\mathfrak{u}_4^E$. ∎

This lemma explains why none of the tori of the form ${}^{\alpha}\mathfrak{u}_4^E$ appear in Table A.7: they have no nonzero non-regular elements.

4.1.3 Number of non-regular subalgebras in elliptic tori

Let us summarize the maze of results of the previous section for all the maximal elliptic tori of $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

Theorem 4.1.9. *Let* $G = \mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$*. If* $-1 \in F^{\times 2}$ *then* G *contains a total of* 21 *conjugacy classes of maximal elliptic tori, each one associated to a set of parameters* $(\mathcal{A}, \sigma^*, \mu^*)$*. Of these, there are:*

- 12 *conjugacy classes of tori where* $\mathcal{A} = F(\sqrt{\alpha}) \oplus F(\sqrt{\beta}) \oplus F$*, with* $\alpha, \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$*, such that each conjugacy class contains exactly two non-conjugate nonzero subalgebras of non-regular elements;*

- 3 *conjugacy classes of tori where* $\mathcal{A} = F(\sqrt{\varepsilon}, \sqrt{\varpi}) \oplus F$*, each one containing exactly two non-conjugate nonzero subalgebras of non-regular elements;*

- 6 *conjugacy classes of tori when* $\mathcal{A} = E \oplus F$*, such that* E *is a quartic extension with a unique quadratic subfield, none of which contain any nonzero non-regular elements.*

When $-1 \notin F^{\times 2}$*, then instead* G *contains a total of* 17 *conjugacy classes of maximal elliptic tori, each one associated to a set of parameters* $(\mathcal{A}, \sigma^*, \mu^*)$*. Of these, there are:*

- 6 *conjugacy classes of tori where* $\mathcal{A} = F(\sqrt{\alpha})\oplus F(\sqrt{\beta})\oplus F$*, with* $\alpha, \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$*, such that each one contains exactly two non-conjugate nonzero subalgebras of non-regular* *elements;*

- 2 *conjugacy classes of tori where* $\mathcal{A} = F(\sqrt{\varepsilon}) \oplus F(\sqrt{\varepsilon}) \oplus F$*, such that each one contains exactly two non-conjugate nonzero subalgebras of non-regular elements;*

- 2 *conjugacy classes of tori where* $\mathcal{A} = F(\sqrt{\alpha}) \oplus F(\sqrt{\alpha}) \oplus F$*, with* $\alpha \in \{\varpi, \varepsilon\varpi\}$ *such that each one contains exactly three non-conjugate nonzero subalgebras of non-regular elements;*

- 3 *conjugacy classes of tori where* $\mathcal{A} = F(\sqrt{\varepsilon}, \sqrt{\varpi}) \oplus F$*, each one containing exactly two non-conjugate nonzero subalgebras of non-regular elements;*

- 4 *conjugacy classes of tori when* $\mathcal{A} = E \oplus F$*, such that* E *is a quartic extension with a unique quadratic subfield, none of which contain any nonzero non-regular elements.*

Proof. The total number of conjugacy classes of tori can be counted from Table 3.7, noting that the last two rows correspond to two tori each exactly when $-1 \in F^{\times 2}$. The last four rows of the table correspond to the case that $\mathcal{A} = E \oplus F$, such that E is a quartic extension with a unique quadratic subfield, which have no nonzero nonregular elements by Lemma 4.1.8. There are three tori corresponding to the biquadratic extension, and by Lemma 4.1.7 each contains two subalgebras of non-regular elements that are not G-conjugate. There are 6 tori corresponding to $\mathcal{A} = F(\sqrt{\alpha}) \oplus F(\sqrt{\beta}) \oplus F$, with $\alpha \neq \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$, and by the remarks preceding Proposition 4.1.6, they each contain two non-conjugate lines of non-regular elements.

When $\mathcal{A} = F(\sqrt{\varepsilon}) \oplus F(\sqrt{\varepsilon}) \oplus F$, the parameters $\mu_i \in \{1, \varpi\}$ so the quotient is a norm of $F(\sqrt{\varepsilon})$ if and only if $\mu_1 = \mu_2$. Thus both these tori contain just two lines of non-regular elements, up to conjugacy.

When $\mathcal{A} = F(\sqrt{\varpi'}) \oplus F(\sqrt{\varpi'}) \oplus F$, with $\varpi' \in \{\varpi, \varepsilon\varpi\}$, then we choose parameters $\mu_1 \in \{-1, -\varepsilon\}$ and $\mu_2 \in \{1, \varepsilon\}$. Therefore when $-1 \in F^{\times 2}$, μ_1/μ_2 is a norm of $F(\sqrt{\varpi'})$ iff $\mu_1 = -\mu_2$, which is always the case for the tori appearing in G. Each of these has two lines of non-regular elements, up to conjugacy.

When $-1 \notin F^{\times 2}$, then $\varepsilon = -1$ so μ_1/μ_2 is a norm of $F(\sqrt{\varpi'})$ when $(\mu_1, \mu_2) = (1, 1)$ or $(-1, -1)$, but none of these tori appear in G. Therefore each of the tori of this form in G have exactly three lines of non-regular elements, up to conjugacy. ∎

We summarize this visually in Figure 4.1, using the notation $\sum_{[\mathcal{T}]} \#\mathcal{N}(\mathcal{T})$ for the total number of non-regular subspaces over $G(F)$-conjugacy classes of elliptic maximal tori $\mathcal{T}$ in G. For each such torus, $\mathcal{N}(\mathcal{T})$ denotes the set of F-subalgebras in $\mathfrak{t} \setminus \{0\}$ consisting of non-regular elements, and $\#\mathcal{N}(\mathcal{T})$ is its cardinality.

We group these contributions according to the type of the algebra $\mathcal{A}$ associated with $\mathcal{T}$.

Namely,

$$\#\mathcal{N}_{K\oplus K} = \sum_{\substack{[\mathcal{T}] \\ \mathcal{T} \text{ of } K\oplus K\text{-type}}} \#\mathcal{N}(\mathcal{T}), \qquad \#\mathcal{N}_E = \sum_{\substack{[\mathcal{T}] \\ \mathcal{T} \text{ of } E\text{-type}}} \#\mathcal{N}(\mathcal{T})$$

Thus the total number of non-regular subspaces decomposes as

$$\sum_{[\mathcal{T}]} \#\mathcal{N}(\mathcal{T}) = \#\mathcal{N}_{K\oplus K} + \#\mathcal{N}_E = \begin{cases} 24+6=30 & -1 \in F^{\times 2} \\ 22+6=28 & -1 \notin F^{\times 2} \end{cases}.$$

The counting argument in the case of $G = \mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, -\varpi, \varepsilon\varpi \rangle)$ is the same as in the proof of Theorem 4.1.9, differing only on those tori arising from $\mathcal{A} = F(\sqrt{\varpi'}) \oplus F(\sqrt{\varpi'}) \oplus F$, with $\varpi' \in \{\varpi, \varepsilon\varpi\}$. In that case, there are 27 conjugacy classes of maximal elliptic tori when $-1 \in F^{\times 2}$ and 29 when $-1 \notin F^{\times 2}$.

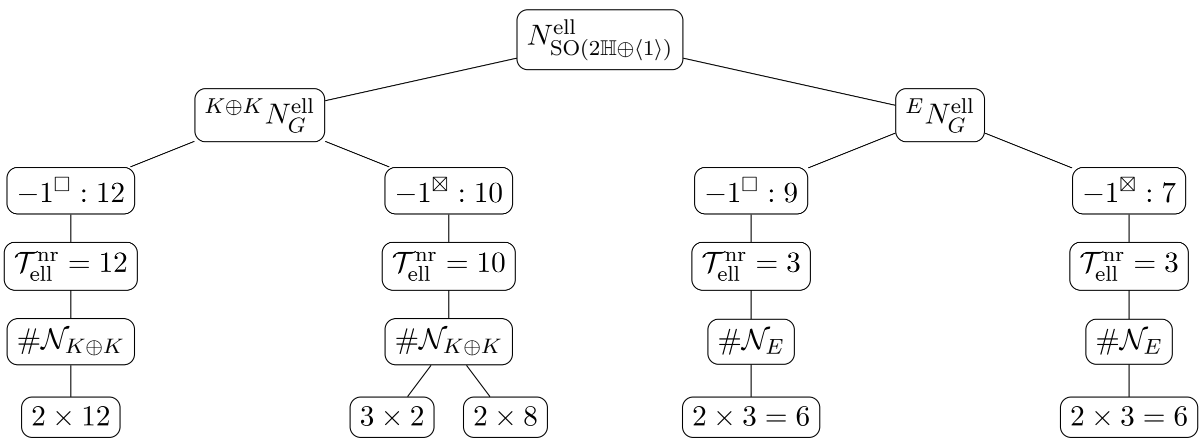

Figure 4.1: *A tree diagram starting from the number $N^{\text{ell}}_{\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)}$ of conjugacy classes of elliptic tori in $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$, branching to the two classes of tori (denoted ${}^{K\oplus K}N^{\text{ell}}_G$ for the sum of quadratic extensions and ${}^{E}N^{\text{ell}}_G$ for the quartic extensions), and then branching by whether or not $-1 \in F^{\times 2}$. Thereafter is a row counting the total number of tori of this form, then the total number of tori containing nonzero nonregular elements $\mathcal{T}^{nr}_{\text{ell}}$, then $\#\mathcal{N}$ is the number of sets of nonregular subalgebras $S_{nr} \subset \mathfrak{t} \setminus \{0\}$. Note, however, that subsets of nonconjugate tori can be conjugate; this is addressed in Section 4.2.*

4.2 Conjugacy classes of twisted Levi subgroups

In this section, we classify the conjugacy classes of twisted Levi subgroups in $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$ by realizing them as centralizers of non-regular elements. We note that if two twisted Levi subgroups $\mathcal{M}, \mathcal{M}'$ are G-conjugate, then each maximal torus of $\mathcal{M}$ is G-conjugate to some maximal torus of $\mathcal{M}'$. However, the conjugacy class of $\mathcal{M}$ is not determined by the conjugacy class of one of its maximal tori $\mathcal{T}$, because there are in general several twisted Levi subgroups containing a given torus. The key observation is the following.

Lemma 4.2.1. *Let $S_{nr_1}, S_{nr_2} \subset \mathfrak{t} \subset \mathfrak{g}$ be G-conjugate subalgebras of non-regular elements of a torus $\mathfrak{t}$. Then $C_G(S_{nr_1})$ is G-conjugate to $C_G(S_{nr_2})$.*

Proof. Since S_{nr_1} and S_{nr_2} are G-conjugate, there exists a fixed element $g \in G$ such that

$$S_{nr_2} = \mathrm{Ad}(g)(S_{nr_1}) = \{\mathrm{Ad}(g)(X) \mid X \in S_{nr_1}\}.$$

Let $Y \in S_{nr_2}$; this it can be written as $Y = \mathrm{Ad}(g)(X)$ for some $X \in S_{nr_1}$. Recall that

$$C_G(S_{nr_i}) = \{h \in G \mid \mathrm{Ad}(h)(X) = X \text{ for all } X \in S_{nr_i}\}.$$

Let $h \in C_G(S_{nr_1})$. Then $\mathrm{Ad}(h)(X') = X'$ for all $X' \in S_{nr_1}$. Then since $\mathrm{Ad}(g^{-1})(Y) = X$ we have

$$\mathrm{Ad}(ghg^{-1})(Y) = \mathrm{Ad}(g)\mathrm{Ad}(h)(X) = \mathrm{Ad}(g)(X) = Y.$$

Thus $ghg^{-1} \in C_G(S_{nr_2})$, so $g\, C_G(S_{nr_1})\, g^{-1} \subset C_G(S_{nr_2})$. The reverse inclusion follows by applying the same argument with g^{-1}. Hence $C_G(S_{nr_2}) = g\, C_G(S_{nr_1})\, g^{-1}$. ∎

Therefore, to exhaust the list of conjugacy classes of twisted Levi subgroups of G (that are proper, are not tori and have compact center), it suffices to enumerate all of the maximal elliptic tori up to conjugacy, then enumerate all of the subalgebras of nonregular elements in each torus up to conjugacy and compute their centralizers. This will produce multiple representatives for each conjugacy class of twisted Levi subgroup $\mathcal{M}$, with multiplicity equal to the number of $\mathcal{M}$-conjugacy classes of maximal elliptic tori contained in $\mathcal{M}$. This multiplicity was computed in Section 4.1.3.

4.2.1 Uniqueness of conjugacy classes of twisted Levi subgroups in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$

The following theorem holds regardless of whether -1 is a square or not.

Theorem 4.2.2. *In $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, each isomorphism class of twisted Levi subgroup corresponds to a unique conjugacy class embedded in the group.*

Proof. The proof is a counting argument. Consider

$$\{(\mathcal{T}, S) \mid \mathcal{T} \text{ elliptic maximal torus of } G,\ S \subset \mathfrak{t} \setminus \{0\} \text{ non-regular } F\text{-subalgebra}\}/G.$$

We count the number of elements in this set in two ways and show they are equal:

$$\sum_{[\mathcal{T}]} \#\,\mathcal{N}(\mathcal{T}) = \sum_{[M]} \#\,\mathrm{Ell}(\mathcal{M}),$$

where $[\mathcal{T}]$ runs over G-conjugacy classes of elliptic maximal tori and $[\mathcal{M}]$ runs over G-conjugacy classes of proper nonabelian twisted Levi subgroups. For each torus, $\#\,\mathcal{N}(\mathcal{T})$ denotes the cardinality of the set of F-subalgebras $S \subset \mathfrak{t} \setminus \{0\}$ consisting of non-regular elements. For each $\mathcal{M}$, $\#\,\mathrm{Ell}(\mathcal{M})$ denotes the number of elliptic tori in each twisted Levi up to conjugacy.

Step 1: Counting non-regular lines for each elliptic torus. In Section 4.1.3, we have proved

$$\sum_{[\mathcal{T}]} \#\mathcal{N}(\mathcal{T}) = \#\mathcal{N}_{K\oplus K} + \#\mathcal{N}_E = \begin{cases} 24+6=30 & -1 \in F^{\times 2} \\ 22+6=28 & -1 \notin F^{\times 2} \end{cases}.$$

Step 2: Counting the number of elliptic tori in each twisted Levi by using the Algorithm method. By using Table A.1 and Table A.2 we record these numbers in the two cases.

Case 1: $-1 \in F^{\times 2}$.

- There are **6** G-conjugacy classes of twisted Levi subgroups of type ${}^{\alpha}\mathcal{S}_{\mu}$. For each such class, we have **3** distinct G-conjugacy classes of elliptic maximal tori, namely of type

$$ {}^{\alpha}\mathfrak{t}^{\beta}_{\mu,-}, \qquad {}^{\alpha}\mathfrak{t}^{\alpha\beta}_{\mu,-}, \qquad {}^{\alpha}\mathfrak{t}^{2}_{\mu,\mu,1}.$$

  Thus these contribute $6\cdot 3 = 18$.

- There are **3** G-conjugacy classes of twisted Levi subgroups of type ${}^{\alpha}\mathcal{L}_s$. For each such class, there are **4** distinct G-conjugacy classes of elliptic maximal tori, namely

$$({}^{\varepsilon}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon},1]}, \quad {}^{\varepsilon\varpi}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon,\varpi},1]}, \quad {}^{\varpi}\mathfrak{t}^{2}_{1,1,1}, \quad {}^{\varpi}\mathfrak{t}^{2}_{\varepsilon,\varepsilon,1}) \text{ for } {}^{\varpi}\mathcal{L}_s.$$

$$({}^{\varepsilon}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon},1]}, \quad {}^{\varpi}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}, \quad {}^{\varepsilon\varpi}\mathfrak{t}^{2}_{1,1,1}, \quad {}^{\varepsilon\varpi}\mathfrak{t}^{2}_{\varepsilon,\varepsilon,1}) \text{ for } {}^{\varepsilon\varpi}\mathcal{L}_s.$$

$$({}^{\varepsilon\varpi}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]}, \quad {}^{\varpi}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}, \quad {}^{\varepsilon}\mathfrak{t}^{2}_{1,1,1}, \quad {}^{\varepsilon}\mathfrak{t}^{2}_{\varpi,\varpi,1}) \text{ for } {}^{\varepsilon}\mathcal{L}_s.$$

  Thus these contribute $3\cdot 4 = 12$.

Adding the contributions gives

$$\sum_{[\mathcal{M}]} \#\,\mathrm{Ell}(M)|_{-1\in F^{\times 2}} = (6\cdot 3) + (3\cdot 4) = 18 + 12 = 30 = \sum_{[\mathcal{T}]} \#\mathcal{N}(\mathcal{T})|_{-1\in F^{\times 2}}.$$

Case 2: $-1 \notin F^{\times 2}$.

- There are **6** G-conjugacy classes of twisted Levi subgroups of type ${}^{\alpha}\mathcal{S}_{\mu}$. For each such class, we have **3** distinct G-conjugacy classes of elliptic maximal tori, namely of type

$$ {}^{\alpha}\mathfrak{t}^{\beta}_{\mu,-}, \qquad {}^{\alpha}\mathfrak{t}^{\alpha\beta}_{\mu,-}, \qquad {}^{\alpha}\mathfrak{t}^{2}_{\mu,\mu,1}.$$

  Thus these contribute $6\cdot 3 = 18$.

- There are **2** G-conjugacy classes of twisted Levi subgroups

$$ {}^{\varpi}\mathcal{L}_s, \quad {}^{\varepsilon\varpi}\mathcal{L}_s,$$

For each such class, there are **3** distinct G-conjugacy classes of elliptic tori namely

$$({}^{\varepsilon}\mathfrak{t}^{\varepsilon,\varpi}_{4[1,1]},\quad {}^{\varepsilon\varpi}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]},\quad {}^{\varpi}\mathfrak{t}^{2}_{1,\varepsilon,1}) \text{ for } {}^{\varpi}\mathcal{L}_s.$$

$$({}^{\varepsilon}\mathfrak{t}^{\varepsilon,\varpi}_{4[1,1]},\quad {}^{\varpi}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]},\quad {}^{\varepsilon\varpi}\mathfrak{t}^{2}_{1,\varepsilon,1}) \text{ for } {}^{\varepsilon\varpi}\mathcal{L}_s.$$

Thus these contribute $2 \cdot 3 = 6$.

- There is **1** G-conjugacy class of twisted Levi subgroup ${}^{\varepsilon}\mathcal{L}_s$, that has exactly **4** classes of elliptic tori namely.

$$({}^{\varepsilon\varpi}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]},\quad {}^{\varpi}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]},\quad {}^{\varepsilon}\mathfrak{t}^{2}_{1,1,1},\quad {}^{\varepsilon}\mathfrak{t}^{2}_{\varpi,\varpi,1}) \text{ for } {}^{\varepsilon}\mathcal{L}_s.$$

  Thus this contributes $1 \cdot 4 = 4$ classes.

Adding the contributions gives

$$\sum_{[\mathcal{M}]} \#\,\mathrm{Ell}(M)|_{-1\notin F^{\times 2}} = (6\cdot 3) + (2\cdot 3) + (1\cdot 4) = 28 = \sum_{[\mathcal{T}]} \#\mathcal{N}(\mathcal{T})|_{-1\notin F^{\times 2}}.$$

∎

To further confirm our work, and for use in Chapter 5, in the following sections we compute the isomorphism class of the centralizers of each of the subalgebras of non-regular elements we found in Section 4.1.

4.2.2 Centralizers of non-regular elements of tori of the forms ${}^{\alpha}\mathfrak{t}^2$ and ${}^{\alpha}\mathfrak{t}^{\beta}$

Throughout this section let $\alpha, \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ and $\mathcal{A} = F(\sqrt{\alpha}) \oplus F(\sqrt{\beta}) \oplus F$. Consider ${}^{\alpha}\mathfrak{t}^{\beta}_{\mu_1,\mu_2,1}$ for some $\mu_1 \in F^{\times}/N_{F(\sqrt{\alpha})/F}(F(\sqrt{\alpha})^{\times})$ and $\mu_2 \in F^{\times}/N_{F(\sqrt{\beta})/F}(F(\sqrt{\beta})^{\times})$. Then ${}^{\alpha}\mathfrak{t}^{\beta}_{\mu_1,\mu_2,1}$ preserves a quadratic form q on $\mathcal{A}$ that is isometric to either $2\mathbb{H} \oplus \langle 1\rangle$ or $\mathbb{H} \oplus W_1$. Set $G' = \mathrm{SO}(q)$.

Proposition 4.2.3. *Let $X = (x\sqrt{\alpha}, 0, 0) \in {}^{\alpha}\mathfrak{s}^{\beta}_{y=0}$, as defined in Proposition 4.1.5, be nonzero. Then*

$$C_{G'}(X) \;\cong\; \mathrm{SO}(E_{\alpha,\mu_1}) \times \mathrm{SO}(\mu_2, -\mu_2\beta, 1) = {}^{\alpha}\mathcal{S}_{\mu_1,x}.$$

Proof. With respect to the basis $e_1 = (1,0,0)$, $e_2 = (\sqrt{\alpha},0,0)$, $e_3 = (0,1,0)$, $e_4 = (0,\sqrt{\beta},0)$, $e_5 = (0,0,1)$, we have $\ker(X) = \mathrm{span}\{e_3,e_4,e_5\}$. Set $W = \mathrm{span}\{e_1,e_2\}$. The Lie algebra ${}^{\alpha}\mathfrak{t}^{\beta}_{\mu_1,\mu_2,\lambda}$ preserves quadratic form $q(a,b,c) = \mathrm{Tr}_{\mathcal{A}/F}(\mu_1 a\bar{a}, \mu_2 b\bar{b}, c^2)$, for $(a,b,c) \in \mathcal{A}$; set $G' = \mathrm{SO}(q)$. We note that the quadratic form q restricts to these subspaces as

$$q|_W = \langle \mu_1, -\mu_1\alpha\rangle, \qquad q|_{W^{\perp}} = \langle \mu_2, -\mu_2\alpha, 1\rangle.$$

Note that $X(W) \subset W$ and $X|_{W^\perp} = 0$, hence $\ker(X) = W^\perp$. If $g \in C_{G'}(X)$, then $g(\ker X) = \ker X$, so $g(W^\perp) = W^\perp$. Since $g \in \mathrm{SO}(q)$, it is an isometry of the quadratic space (V, q). Hence it also preserves the orthogonal complement W of $W^\perp$. Thus g is block diagonal with respect to the orthogonal decomposition $V = W \oplus W^\perp$ and its two blocks lie in $\mathrm{SO}(\langle \mu_1, -\mu_1\alpha\rangle)$ and $\mathrm{SO}(\langle \mu_2, -\mu_2\alpha, 1\rangle)$. Conversely, note that $\mathrm{SO}(W)$ acts trivially on $X|_W \in \mathfrak{so}(W)$ because it is abelian and $\mathrm{SO}(W^\perp)$ acts trivially on $X|_{W^\perp} = 0$. Thus any element of $\mathrm{SO}(\langle \mu_1, -\mu_1\alpha\rangle) \times \mathrm{SO}(\langle \mu_2, -\mu_2\alpha, 1\rangle)$ commutes with X. Therefore

$$C_{G'}(X_1) \cong \mathrm{SO}(\langle \mu_1, -\mu_1\alpha\rangle) \times \mathrm{SO}(\langle \mu_2, -\mu_2\beta, 1\rangle) = {}^{\alpha}\mathcal{S}_{\mu_1,x},$$

where x is s if $\langle \mu_2, -\mu_2\beta, 1\rangle$ is split, and c otherwise. ∎

It follows by symmetry that if $X = (0, y\sqrt{\beta}, 0) \in {}^{\alpha}\mathfrak{s}^{\beta}_{x=0} \subset {}^{\alpha}\mathfrak{t}^{\beta}_{\mu_1,\mu_2,1}$ in nonzero, with ${}^{\alpha}\mathfrak{s}^{\beta}_{x=0}$ as defined in Proposition 4.1.5, then $C_{G'}(X) \cong \mathrm{SO}(E_{\beta,\mu_2}) \times \mathrm{SO}(\mu_1, -\mu_1\alpha, 1) = {}^{\beta}\mathcal{S}_{\mu_2,x}$.

We summarize this in Figure 4.2, where we also include the labels for the associated isomorphism class of twisted Levi subgroup. Note that in the figure we include all tori arising from a sum of two quadratic extensions, even those that do not embed into our preferred forms, as indicated by the variable $\lambda \in F^\times/F^{\times 2}$.

Non-regular elements in ${}^{\alpha}\mathfrak{t}^{\beta}_{[\mu_1,\mu_2,\lambda]} \in \mathrm{SO}(\langle \mu_1, \mu_2, -\mu_1\alpha, -\mu_2\beta, \lambda\rangle)$
$\alpha \neq \beta$

$X_1 = (a\sqrt{\alpha}, 0, 0)$	$X_2 = (0, b\sqrt{\beta}, 0)$
Eigenvalues	**Eigenvalues:**
$(t, 0, -t, 0, 0)$	$(0, s, 0, -s, 0)$
Centralizer $C_G(X_1)$:	**Centralizer** $C_G(X_2)$:
$\mathrm{SO}(E_{\alpha,\mu_1}) \times \mathrm{SO}(\mu_2, -\mu_2\beta, \lambda)$	$\mathrm{SO}(E_{\beta,\mu_2}) \times \mathrm{SO}(\mu_1, -\mu_1\alpha, \lambda)$
${}^{\alpha}\mathcal{S}_{\mu_1,x}$	${}^{\beta}\mathcal{S}_{\mu_2,x}$

Figure 4.2: *Centralizers of non-regular elements in the Lie algebra of the elliptic torus* ${}^{\alpha}\mathfrak{t}^{\beta}_{[\mu_1,\mu_2,\lambda]}$ *when* $\alpha \neq \beta$. *Each branch records the eigenvalue pattern* ($t = a\sqrt{\alpha}, s = b\sqrt{\beta} : a, b \in F^\times$) *of the indicated non-regular element, and by Proposition 4.2.3, has the given centralizer in* $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$. *We use the letter* x *as a unified parameter to represent* s *if* $\langle \mu_2, -\mu_2\beta, 1\rangle$ *is split, and* c *otherwise.*

To understand the centralizer of the subset denoted ${}^{\alpha}\mathfrak{s}^2_{y=-x}$ (which by Proposition 4.1.6 is conjugate to ${}^{\alpha}\mathfrak{s}^2_{y=x}$), we require a basic lemma.

Lemma 4.2.4. *Let* $W = F^4$ *with quadratic form* $q_W = \langle \mu_1, \mu_2, -\mu_1\alpha, -\mu_2\alpha\rangle$, *for some* $\alpha, \mu_1, \mu_2 \in F$ *with* $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ *and let*

$$J_\alpha = \begin{pmatrix} 0 & \alpha I_2 \\ I_2 & 0 \end{pmatrix} \in \mathrm{End}_F(W).$$

Then $J_\alpha^2 = \alpha I_4$, so J_α defines on W the structure of a 2-dimensional $E = F(\sqrt{\alpha})$-vector space by defining $(a + b\sqrt{\alpha}) \cdot w$ to be $(aI + bJ_\alpha)w$. With respect to this structure, J_α lies in the center of the Lie algebra $\mathfrak{u}_\alpha(\mu_1/2, \mu_2/2)$.

Proof. A direct computation gives $J_\alpha^2 = \alpha I_4$, so the map $\phi : F(\sqrt{\alpha}) \to \mathrm{End}_F(W)$ given for all $a, b \in F$ by $\phi(a + b\sqrt{\alpha}) = aI_4 + bJ_\alpha$ is an injective ring homomorphism. If we write vectors in W as

$$w = (x_1, x_2, y_1, y_2), \qquad x_i, y_i \in F,$$

we can identify $w \in W$ with $\phi'(w) = z \in E^2$ by setting $z = (x_1 + y_1\sqrt{\alpha},\ x_2 + y_2\sqrt{\alpha})$. Then $\sqrt{\alpha}z = J_\alpha w$.

Now define the Hermitian form on $(z_1, z_2), (w_1, w_2) \in E^2$ by

$$h((z_1, z_2), (w_1, w_2)) = \tfrac{\mu_1}{2} z_1\sigma(w_1) + \tfrac{\mu_2}{2} z_2\sigma(w_2),$$

where σ is the nontrivial element of $\mathrm{Gal}(E/F)$. Its trace over F is the quadratic form on W defined by

$$q(x_1, x_2, y_1, y_2) = \mathrm{Tr}_{E/F}(h((z_1, z_2), (z_1, z_2))) = \big(\mu_1 x_1^2 - \mu_1\alpha y_1^2 + \mu_2 x_2^2 - \mu_2\alpha y_2^2\big).$$

Thus with respect to our embedding we have $U_\alpha(\mu_1, \mu_2) \subset \mathrm{SO}(q)$.

By definition, $\mathfrak{u}_\alpha(\mu_1, \mu_2)$ consists of the F-linear endomorphisms of W that are E-linear and skew with respect to q. Since E-linearity is equivalent to commuting with multiplication by $\sqrt{\alpha}$, we get

$$\mathfrak{u}_\alpha(\mu_1, \mu_2) = \{Y \in \mathfrak{so}(q) \mid YJ_\alpha = J_\alpha Y\}.$$

In particular, J_α lies in the center of $\mathfrak{u}_\alpha(\mu_1, \mu_2)$. ∎

Proposition 4.2.5. *Let $X = (x\sqrt{\alpha}, -x\sqrt{\alpha}, 0) \in {}^\alpha\mathfrak{s}^2_{y=-x}$, as defined in Proposition 4.1.5, be nonzero. Then*

$$C_{G'}(X) \cong U_\alpha(\mu_1/2, \mu_2/2) = {}^\alpha\mathcal{L}_x.$$

Proof. Set $E = F(\sqrt{\alpha})$. By choosing the basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (\sqrt{\alpha}, 0, 0)$, $e_4 = (0, \sqrt{\beta}, 0)$, $e_5 = (0, 0, 1)$, we have that $W = \mathrm{span}\{e_1, e_2, e_3, e_4\} \cong E^2$ via the isomorphism of ϕ^{-1} where ϕ is as in Lemma 4.2.4. Set $L = \langle e_5\rangle$. Then $V = W \perp L$ with $q|_W = \langle \mu_1, \mu_2, -\mu_1\alpha, -\mu_2\alpha\rangle$ and $J|_L = (1)$. Then we have that $X|_W = J_\alpha$ while $X|_L = 0$. Thus by Lemma 4.2.4, it follows that $U_\alpha(\mu_1/2, \mu_2/2) \subseteq C_{G'}(X)$. Conversely, let $g \in C_{G'}(X)$. Since $gX = Xg$ and $\ker(X) = L$, we deduce as before that g preserves the orthogonal decomposition $V = W \oplus L$. Thus we can write

$$g = \begin{pmatrix} g_W & 0 \\ 0 & c \end{pmatrix}, \qquad g_W \in O(W),\ c \in \{\pm 1\}.$$

The relation $gX = Xg$ implies $g_W J_\alpha = J_\alpha g_W$, so g_W is E-linear, and it preserves the Hermitian form $h = \langle \mu_1/2, \mu_2/2\rangle$ on E^2 under this identification ϕ. Hence $\phi^{-1}(g_W) \in U_\alpha(\mu_1/2, \mu_2/2)$. Since $\det_E(\phi^{-1}(g_W))$ has norm 1, we have that $\det_F(g_W) = 1$, so $g_W \in \mathrm{SO}(W)$, and it follows that $c = 1$. Therefore $C_{G'}(X) \cong U_\alpha(\mu_1/2, \mu_2/2)$. ∎

As before, we summarize this in Figure 4.3, with the labels of the corresponding isomorphism class of twisted Levi subgroup.

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{c}\textbf{Non-regular elements in } {}^{\alpha}\mathfrak{t}^2_{[\mu_1,\mu_2,1]} \in \mathrm{SO}(\langle \mu_1,\mu_2,-\mu_1\alpha,-\mu_2\alpha,\lambda\rangle)\\ \alpha=\beta,\ t=a\sqrt{\alpha} : a\in F^\times\end{array} \arrow[dl] \arrow[d] \arrow[dr] & \\
X_1 = (a\sqrt{\alpha},0,0) \arrow[d] & X_2 = (0,a\sqrt{\alpha},0) \arrow[d] & X_3 = (a\sqrt{\alpha},a\sqrt{\alpha},0) \arrow[d] \\
\begin{array}{c}\textbf{Eigenvalues:}\\ (t,0,-t,0,0)\end{array} \arrow[d] & \begin{array}{c}\textbf{Eigenvalues}\\ (0,t,0,-t,0)\end{array} \arrow[d] & \begin{array}{c}\textbf{Eigenvalues}\\ (t,t,-t,-t,0)\end{array} \arrow[d] \\
\begin{array}{c}\textbf{Centralizer } C_{G'}(X_1):\\ \mathrm{SO}(E_{\alpha,\mu_1})\times\ \mathrm{SO}(\mu_2,-\mu_2\alpha,1)\\ {}^{\alpha}\mathcal{S}_{\mu_1,x}\end{array} & \begin{array}{c}\textbf{Centralizer } C_{G'}(X_2):\\ \mathrm{SO}(E_{\alpha,\mu_2})\times\ \mathrm{SO}(\mu_1,-\mu_1\alpha,1)\\ {}^{\alpha}\mathcal{S}_{\mu_2,x}\end{array} & \begin{array}{c}\textbf{Centralizer } C_{G'}(X_3):\\ U_\alpha(\mu_1/2,\mu_2/2)\\ {}^{\alpha}\mathcal{L}_{x}\end{array}
\end{tikzcd}$$

Figure 4.3: *Centralizers of non-regular elements in* ${}^{\alpha}\mathfrak{t}^2_{[\mu_1,\mu_2,\lambda]}$*, i.e. when* $\alpha=\beta$*. The first two branches follow the same pattern as in Figure 4.2, while the third branch corresponds to the additional non-regular direction that occurs in this case and yields the unitary twisted Levi subgroup* $U_\alpha(\mu_1/2,\mu_2/2)$ *by Proposition 4.2.5. Moreover, the centralizers* $C_{G'}(X_1)$ *and* $C_{G'}(X_2)$ *are* $G'(F)$*-conjugate whenever* $\mu_1/\mu_2 \in F^{\times 2}$*, by Lemma 4.2.1.*

4.2.3 Centralizers of non-regular elements in tori of the form ${}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi}$

Consider a torus arising from a biquadratic extension that embeds into $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle$. Then by Table 3.5 we have ${}^{\alpha}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu,1]}$ for some $\alpha\in\{\varepsilon,\varpi,\varepsilon\varpi\}$ where $\mu=\sqrt{-\varepsilon}$ if $\alpha=\varepsilon$ and $\mu=\sqrt{\alpha}$ otherwise. That is, ${}^{\alpha}\mathcal{T}^{\varepsilon,\varpi}_{4[\mu,1]}$ preserves a quadratic form q that is isometric with $2\mathbb{H}\oplus\langle 1\rangle$; set $G'=\mathrm{SO}(q)$.

Proposition 4.2.6. *Choose* $\beta\in\{\varepsilon,\varpi,\varepsilon\varpi\}\smallsetminus\{\alpha\}$*. Let* $X=(x\sqrt{\beta},0)\in{}^{\alpha}\mathfrak{s}^{\varepsilon,\varpi}_{4,y=0}$ *and* $Y=(y\sqrt{\alpha\beta},0)\in{}^{\alpha}\mathfrak{s}^{\varepsilon,\varpi}_{4,x=0}$ *be non-regular elements, where* $x,y\in F^\times$*. Then their centralizers satisfy*

$$C_{G'}(X)\cong U_\beta(\mathbb{H})={}^{\beta}\mathcal{L}_s \quad \textit{and} \quad C_{G'}(Y)\cong U_{\alpha\beta}(\mathbb{H})={}^{\alpha\beta}\mathcal{L}_s$$

Proof. Set $K=F(\sqrt{\alpha})$ and $E=F(\sqrt{\beta})$. Choose the basis $e_1=(1,0)$, $e_2=(\sqrt{\beta},0)$, $e_3=(\sqrt{\alpha},0)$, $e_4=(\sqrt{\alpha\beta},0)$, $e_5=(0,1)$ of $\mathcal{A}=F(\sqrt{\varepsilon},\sqrt{\varpi})$. Then $W=\mathrm{span}\{e_1,e_2,e_3,e_4\}\cong E(\sqrt{\alpha})\cong E^2$ via the isomorphism sending $ae_1+be_2+ce_3+de_4$ to $(a+b\sqrt{\beta})1+(c+d\sqrt{\beta})\sqrt{\alpha}$. Set $L=\langle e_5\rangle$. Then $V=W\perp L$ with $q\simeq 2\mathbb{H}$ by Table 3.5. Then $X=(x\sqrt{\beta},0)\in{}^{\alpha}\mathfrak{s}^{\varepsilon,\varpi}_{4,y=0}$ acts as xJ_β on W, where β is as defined in Lemma 4.2.4, and $W\cong E^2$ is a $F(\sqrt{\beta})/F$ Hermitian space carrying the split Hermitian form $\mathbb{H}$, such that X lies in the center of its Lie algebra $\mathfrak{u}_\beta(\mathbb{H})$. Permuting e_2 and e_3 allows us to instead view W as a vector space over $E'=F(\sqrt{\alpha\beta})$ and a similar analysis yields that Y lies in the center of a Lie algebra of $\mathfrak{u}_{\alpha\beta}(\mathbb{H})$. Finally, we compute the centralizer as in the proof of Proposition 4.2.5 to conclude the result. ∎

We summarize this in Figure 4.4, including also the label ${}^{\gamma}\mathcal{L}_s$ for the twisted Levi subgroup $U_\gamma(\mathbb{H})$.

Elliptic torus ${}^{\alpha}\mathcal{T}^{\varepsilon,\varpi}_{[\mu,1]}$
$\mathcal{A} = F[\sqrt{\varepsilon}, \sqrt{\varpi}] \supset F[\sqrt{\alpha}]$

$\downarrow$

Non-regular elements in ${}^{\alpha}\mathfrak{t}^{\varepsilon,\varpi}_{[\mu,1]}$

$\downarrow$

$X_\gamma = (a\sqrt{\gamma}, 0)$
$\gamma \in \{\varepsilon, \varpi, \varepsilon\varpi\} \setminus \{\alpha\}$

$\downarrow$

Eigenvalues: $(t, t, -t, -t, 0)$, $t = a\sqrt{\gamma}$

$\downarrow$

Centralizer: $C_G(X_\gamma)$
${}^{\gamma}\mathcal{L}_s = U_\gamma(\mathbb{H})$

Figure 4.4: *Centralizers of non-regular elements in* ${}^{\alpha}\mathfrak{t}^{\varepsilon,\varpi}_{[\mu,1]} \in \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$*, see Lemma 4.1.7. Here* $\gamma \in \{\varepsilon, \varpi, \varepsilon\varpi\} \setminus \{\alpha\}$ *runs over the two square classes distinct from* α*, so each such choice yields a unitary twisted Levi subgroup* $U_\gamma(\mathbb{H})$ *by Proposition 4.2.6.*

Chapter 5

Genericity and $\mathcal{T}$-good elements in $\mathfrak{so}_5$

Our goal in this chapter is to classify the $\mathcal{T}$-good elements of $\mathfrak{g}$, for each maximal elliptic torus $\mathcal{T}$, and use this to identify which twisted Levi sequences $\mathcal{T} \subset \mathcal{M} \subset G$ will be valid inputs for the construction of supercuspidal representations.

In Section 5.1, we give an explicit description of generic elements and relate them to T-good elements using results from the literature. In Section 5.2, We give an explicit description of filtration on $\mathfrak{t}$ for each elliptic torus $\mathcal{T} \subset G$, and determine how the depth depends on the parameters defining the torus.

We then, in Section 5.3, introduce G-generic elements in $\mathfrak{t}$ and relative $\mathcal{M}$-generic elements for twisted Levi subgroups $\mathcal{M}$. A key point is that not every torus admits G-generic elements. Using these results, we determine in Section 5.4 the valid twisted Levi sequences that arise in the construction.

5.1 On genericity and $\mathcal{T}$-good elements

In this section, we define and relate several equivalent formulations of the notion of "genericity“ that is crucial in the construction of supercuspidal representations. To state them carefully, in this section only, let $\mathbb{G}$ be a connected reductive group over F with Lie algebra $\mathfrak{g}$, and let $\mathbb{G}'$ denote a twisted Levi subgroup of $\mathbb{G}$. Let $\mathbb{T}$ denote a maximal F-torus contained in $\mathbb{G}'$, which is thus a maximal torus of $\mathbb{G}$. Let E be a splitting field of $\mathbb{T}$. We write $\Phi(\mathbb{G}, \mathbb{T})$ and $\Phi(\mathbb{G}', \mathbb{T})$ for the corresponding (absolute) root systems. We assume throughout that $\mathbb{G}$ splits over a tamely ramified Galois extension and that p does not divide the order of the Weyl group of $\mathbb{G}$, so that we may identify $\mathfrak{g}$ and $\mathfrak{g}^*$, and $\mathrm{Lie}(Z(\mathbb{G})) = Z(\mathrm{Lie}(\mathbb{G}))$ by [Spr09, Lemma 13.3.3], for example.

Example 5.1.1. When $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, the pairs $(\mathbb{G}'(F), \mathbb{G}(F))$ could be any of

$$\{(\mathcal{T}, G), (\mathcal{T}, \mathcal{M}), (\mathcal{M}, G)\}$$

where $\mathcal{M}$ is one of the twisted Levi subgroups containing a given maximal elliptic torus $\mathbb{T}(F) = \mathcal{T}$ that we found in Chapter 4.

Write $\boldsymbol{\mathfrak{z}}'$ for the Lie algebra of the center of $\mathbb{G}'$ and $\boldsymbol{\mathfrak{z}}'^*$ for its algebraic dual. Since $\boldsymbol{\mathfrak{z}}'$ is contained in the Lie algebra $\boldsymbol{\mathfrak{t}}$ of a maximal torus $\mathbb{T}$ of $\mathbb{G}'$, it inherits a Moy–Prasad filtration. We set $\mathfrak{z}' := \boldsymbol{\mathfrak{z}}(F)$, and $\mathfrak{z}'^* = \boldsymbol{\mathfrak{z}}'^*(F)$. We also have $\mathfrak{z}'_r := \boldsymbol{\mathfrak{z}}(F) \cap \mathfrak{t}_r$ and recall from (2.5.1) that we define

$$\mathfrak{z}'^*_{-r} = \left\{X^* \in \mathfrak{z}'^* \ : \ X^*(Y) \in \mathcal{P}_F \text{ for all } Y \in \mathfrak{z}'_{r+}\right\}.$$

Definition 5.1.2 ([KM03, Definition 2.1.1])**.** For $X \in \mathfrak{t}$, define the *depth of X relative to T* by

$$d_T(X) := \sup\{r \in \mathbb{R} \mid X \in \mathfrak{t}_r\}.$$

That is, if $X \neq 0$ then $r = d_T(X)$ is the unique real number such that

$$X \in \mathfrak{t}_r \smallsetminus \mathfrak{t}_{r+}.$$

The definition of depth on elements of $\mathfrak{t}^*$ is analogous. The following definition is from [Yu01, §8].

Definition 5.1.3 (G-generic element of depth $-r$)**.** Let $r \in \mathbb{R}$ and $X^* \in \mathfrak{z}'^*_{-r}$. For each root $a \in \Phi(\mathbb{G}, \mathbb{T})$, set $H_a = d\check{a}(1)$. If

$$\nu\big(X^*(H_a)\big) = -r \quad \text{for all } a \in \Phi(\mathbb{G}, \mathbb{T}) \setminus \Phi(\mathbb{G}', \mathbb{T}),$$

then X^* is called a G**-generic element of depth** $-r$.

Remark 5.1.4. In Definition 5.1.3, we define $H_a \in \mathfrak{t}$ as the differential of the coroot $\check{a} : \mathbb{G}_m \to \mathbb{T}$ at the identity, that is, $H_a = d\check{a}(1)$. If we let E denote a splitting field of $\mathfrak{t}$ then we have $H_a \in \mathfrak{t}(E)_0$, but in general $H_a \notin \boldsymbol{\mathfrak{t}}(F)$. Since $X^* \in \mathfrak{t}_{-r}$ and $H_a \in \boldsymbol{\mathfrak{t}}(E)_0$, we have from properties of the Moy–Prasad filtration, [AD02, §2.2], $\nu(X^*(H_a)) \geq -r$.

As mentioned in Proposition 2.5.8, for $\mathbb{G} = \mathrm{SO}_5$ we may identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form κ. Let us interpret Definition 5.1.3 in the Lie algebra instead of its dual.

Lemma 5.1.5. *Let $\mathbb{G} = \mathrm{SO}_5$ and let $\mathbb{T}$ denote a maximal torus of $\mathbb{G}$ that is defined over F and split over E. Suppose $X \in \mathfrak{t}$ is identified with the dual element $X^* \in \mathfrak{t}^*$ under κ. For any root $a \in \Phi(\mathbb{G}, \mathbb{T})$ and any $r \in \mathbb{R}$, we have*

$$\nu\big(X^*(H_a)\big) = -r \quad \textit{if and only if} \quad \nu\big(a(X)\big) = -r.$$

Proof. Let $a \in \Phi(\mathbb{G}, \mathbb{T}) \subset X^*(\mathbb{T})$ be a root. Then its differential, also denoted a, is an element of $\mathfrak{t}^*$. By construction, $\kappa(X, Y) = \mathrm{Tr}(XY)$ for all $X, Y \in \mathfrak{g}$. By [Hum78, Corollary

8.2], κ is nondegenerate on $\mathbf{t}$ so there exists a unique $t_a \in \mathbf{t}$ such that

$$X^*(t_a) = \kappa(t_a, X) = a(X) \qquad \text{for all } X \in \mathbf{t} \tag{5.1.1}$$

and thus that $H_a = \frac{2t_a}{\kappa(t_a,t_a)}$. It follows that $X^*(H_a) = \frac{2}{\kappa(t_a,t_a)} X^*(t_a)$. Since $\mathrm{Tr}(H_a H_a) \in \{4, 8\}$ in SO_5, we have $\kappa(t_a, t_a) = \frac{4}{\kappa(H_a,H_a)}$ is a power of 2. Since $p \neq 2$, this has valuation 0. Combining these equalities gives

$$\nu(X^*(H_a)) = \nu\left(\frac{2}{\kappa(t_a, t_a)}\right)\nu(a(X)) = \nu(a(X)),$$

as required. ∎

The preceding lemma, Lemma 5.1.5 together with Definition 5.1.3, allows us to make the following definition of genericity of elements of the Lie algebra.

Definition 5.1.6. An element $X \in \mathfrak{z}'_r$ is said to be G-generic of depth r if for every $a \in \Phi(\mathbb{G}, \mathbb{T}) \smallsetminus \Phi(\mathbb{G}', \mathbb{T})$, we have $\nu(a(X)) = r$. The set of all such elements is denoted $\mathbf{Gen}_{G,G',r}$.

A more convenient notion is that of $\mathcal{T}$-good elements defined in [KM03, Definition 2.1.1].

Definition 5.1.7. Given $r \in \mathbb{R}$, a nonzero element $X \in \mathfrak{t}$ of depth r is called $\mathcal{T}$**–good of depth** r if, for every root $a \in \Phi(\mathbb{G}, \mathbb{T})$, either

$$a(X) = 0 \quad \text{or} \quad \nu(a(X)) = r.$$

To make precise the relationship between $\mathcal{T}$-good elements and the G-generic elements introduced earlier, we need the following lemma.

Lemma 5.1.8. *Let $\mathbb{T} \subset \mathbb{G}' \subset \mathbb{G}$ be as above. Then*

$$\mathfrak{z}' = \{\, X \in \mathfrak{t} \mid a(X) = 0 \textit{ for all } a \in \Phi(\mathbb{G}', \mathbb{T}) \,\}.$$

Proof. Since $\mathbb{T}$ splits over E, the Lie algebra $\mathfrak{g}'$ of G' admits the root space decomposition

$$\mathfrak{g}' = \mathfrak{t} \oplus \bigoplus_{a \in \Phi(\mathbb{G}', \mathbb{T})} \mathfrak{g}_a.$$

Let $X \in \mathfrak{t}$. For each root $a \in \Phi(\mathbb{G}', \mathbb{T})$ and each $Y \in \mathfrak{g}_a$, we have $[X, Y] = a(X)Y$. Thus X commutes with the root space $\mathfrak{g}_a$ if and only if $a(X) = 0$. Since every element of $\mathfrak{t}$ commutes with X, it follows that X lies in the center of $\mathfrak{g}'$ if and only if $a(X) = 0$ for all $a \in \Phi(\mathbb{G}', \mathbb{T})$. Since the Lie algebra of the center of $\mathbb{G}'$ is the center of the Lie algebra of $\mathbb{G}'$, we conclude that

$$\mathfrak{z}' = \{\, X \in \mathbf{t} \mid a(X) = 0 \text{ for all } a \in \Phi(\mathbb{G}', \mathbb{T}) \,\}$$

which implies the statement upon intersecting with $\mathfrak{t}$. ∎

We can now establish the equivalence we needed.

Proposition 5.1.9. *Let $X \in \mathfrak{t}$ be a $\mathcal{T}$-good element of depth r and set $\mathbb{G}' = C_{\mathbb{G}}(X)$ to be its centralizer. Then $X \in \mathbf{Gen}_{G,G',r}$ and conversely, every element of $\mathbf{Gen}_{G,G',r}$ is $\mathcal{T}$-good of depth r.*

Proof. The centralizer of an element X of the Lie algebra of a torus is a (twisted) Levi subgroup, and its root system is precisely

$$\Phi(\mathbb{G}', \mathbb{T}) = \{a \in \Phi(\mathbb{G}, \mathbb{T}) : a(X) = 0\}.$$

In this case, X lies in the Lie algebra of the center of G', that is, in $\mathfrak{z}'$. If X is $\mathcal{T}$-good, then for every $a \in \Phi(\mathbb{G}, \mathbb{T}))$ one has $a(X) = 0$ or $\nu(a(X)) = r$. Therefore it follows that $\nu(a(X)) = r$ for all $a \in \Phi(\mathbb{G}, \mathbb{T}) \smallsetminus \Phi(\mathbb{G}', \mathbb{T})$, so X satisfies Definition 5.1.6.

Conversely, suppose $X \in \mathfrak{z}'_r$ is such that for every $a \in \Phi(\mathbb{G}, \mathbb{T}) \smallsetminus \Phi(\mathbb{G}', \mathbb{T})$, we have $\nu(a(X)) = r$. Since $X \in \mathfrak{z}'$, its centralizer $C_G(X)$ contains G' and thus $a(X) = 0$ for every $\Phi(\mathbb{G}', \mathbb{T})$. Therefore X is $\mathcal{T}$-good of depth r. ∎

Our goal in the remaining sections of this chapter is to determine the sets $\mathbf{Gen}_{G,G',r}$, for pairs (G', G) as in Example 5.1.1, as explicitly as possible.

Remark 5.1.10. The definition of $\mathbf{Gen}_{G,G',r}$ seems to depend on the choice of maximal torus $\mathbb{T}$, but by [Fin21c, Lemma 3.6], it is independent of the choice of maximal torus of $\mathbb{G}'$. In particular, in the following we are free to choose a convenient elliptic maximal torus for explicit computations.

5.2 The Moy–Prasad filtrations of $\mathfrak{t}$

In this section we describe explicitly the Moy–Prasad filtrations of the Lie algebras $\mathfrak{t}$ of each of the maximal elliptic tori in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. The filtration takes the form

$$\mathfrak{t} = \bigcup_{r \in \mathbb{R}} \mathfrak{t}_r$$

although the set of values for which $\mathfrak{t}_r \neq \mathfrak{t}_{r+}$ is discrete. We refer to $\mathfrak{t}_r$ as a subalgebra of $\mathfrak{t}$ but unlike $\mathfrak{t}$, it is only an algebra over $\mathcal{O}_F$.

For the next lemma, recall that we normalize the valuation ν of any finite extension of F so that its restriction to $F^\times$ surjects onto the integers.

Lemma 5.2.1. *Let $\mathcal{T}$ be a maximal elliptic torus of G with Lie algebra $\mathfrak{t}$ and let $r \in \mathbb{R}$. Then $X \in \mathfrak{t}_r$ if and only if $X \in \mathfrak{t}$ and each of the eigenvalues λ of X satisfies $\nu(\lambda) \geq r$.*

Proof. By the proof of Theorem 4.1.1, if $\mathcal{T}$ is an elliptic torus of G that splits over E_s, then $\mathcal{T}(E_s)$ is $G(E_s)$-conjugate to a split torus $S(E_s)$. This conjugation identifies $\mathfrak{t}(E_s)$ with the Lie algebra of a split torus $\mathfrak{s}(E_s) \subset \mathfrak{g}(E_s)$.

By Definition 2.5.5, the Moy–Prasad filtration on the Lie algebra of a maximal split torus $S(E_s) \cong (E_s^\times)^n$ is given by the filtration on the discrete valued field E_s. In this case, $n = 2$ and we note that by Lemma 4.1.4 an element $Y = (Y_1, Y_2) \in E_s^2 \cong \mathfrak{s}(E_s)$ acts on E_s^5 with eigenvalues $\{\pm Y_1, \pm Y_2, 0\}$. Thus for $Y \in \mathfrak{s}(E_s)$, we have that $Y \in \mathfrak{s}(E_s)_r$ if and only if each of its eigenvalues has valuation at least r.

Since $\mathfrak{s}(E_s)$ is conjugate to $\mathfrak{t}(E_s)$, this definition of the filtration passes to $\mathfrak{t}(E_s)$, and hence to $\mathfrak{t}$ by intersection, as required. ∎

In the following subsections, we use our results from Section 4.1 to describe the filtration subalgebras $\mathfrak{t}_r$ as well as the values of r for which there are breaks in the filtration. These are important because an element $X \in \mathfrak{t}_r \setminus \mathfrak{t}_{r+}$ is said to have *depth* r, and these exist only for values r that are breaks in the filtration.

We first consider the tori arising from either a sum of two quadratic, or a biquadratic, extension.

Proposition 5.2.2. *Let $\alpha, \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ and $r \in \mathbb{R}$. If $\mathfrak{t} = {}^\alpha\mathfrak{t}^\beta = \{(x\sqrt{\alpha}, y\sqrt{b}, 0) \mid x, y \in F\}$, then*

$$\mathfrak{t}_r = \left\{ (x\sqrt{\alpha}, y\sqrt{\beta}, 0) \in {}^\alpha\mathfrak{t}^\beta \,\middle|\, \nu(x) \geq r - \frac{1}{2}\nu(\alpha) \text{ and } \nu(y) \geq r - \frac{1}{2}\nu(\beta) \right\}$$

and the jumps in the filtration occur at values $r \in \{\frac{1}{2}\nu(\alpha), \frac{1}{2}\nu(\beta)\} + \mathbb{Z}$. If $\alpha \neq \beta$ and we set $\mathfrak{t} = {}^\alpha\mathfrak{t}_4^{\varepsilon,\varpi} = \{((x + y\sqrt{\alpha})\sqrt{\beta}, 0) \mid x, y \in F\}$ then

$$\mathfrak{t}_r = \left\{ ((x + y\sqrt{\alpha})\sqrt{\beta}, 0) \in {}^\alpha\mathfrak{t}_4^{\varepsilon,\varpi} \,\middle|\, \nu(x) \geq r - \frac{1}{2}\nu(\beta) \text{ and } \nu(y) \geq r - \frac{1}{2}\nu(\alpha\beta) \right\},$$

and the jumps in the filtration occur at values $r \in \{\frac{1}{2}\nu(\beta), \frac{1}{2}\nu(\alpha\beta)\} + \mathbb{Z}$.

Proof. In the first case, set $X = (x\sqrt{\alpha}, y\sqrt{b}, 0) \in {}^\alpha\mathfrak{t}^\beta$. Its eigenvalues, as an operator on $\mathcal{A} = F(\sqrt{\alpha}) \oplus F(\sqrt{\beta}) \oplus F$, are

$$\{\pm x\sqrt{\alpha}, \pm y\sqrt{\beta}, 0\}.$$

As $\nu(x\sqrt{\alpha}) = \nu(x) + \frac{1}{2}\nu(\alpha)$, the description of ${}^\alpha\mathfrak{t}_r^\beta$ now follows from Lemma 5.2.1.

Similarly, in the second case, where $\alpha \neq \beta$, we set $X = ((x + y\sqrt{\alpha})\sqrt{\beta}, 0) \in {}^\alpha\mathfrak{t}_4^{\varepsilon,\varpi}$. By Lemma 4.1.7, its eigenvalues as an operator on $\mathcal{A} = F(\sqrt{\varepsilon}, \sqrt{\varpi}) \oplus F$ are

$$\{\pm x\sqrt{\beta} \pm y\sqrt{\alpha\beta}, 0\}.$$

Thus if we choose x, y satisfying the conditions $\nu(x) \geq r - \frac{1}{2}\nu(\beta), \nu(y) \geq r - \frac{1}{2}\nu(\alpha\beta)$, then all eigenvalues will have valuation at least r. Conversely, suppose the valuations of these eigenvalues are all at least r. If $\nu(\beta) \neq \nu(\alpha\beta)$ then $\nu(x\sqrt{\beta}) \neq \nu(y\sqrt{\alpha\beta})$, since one is an integer and the other a half-integer, and therefore each term must have valuation at least r, individually. If $\nu(\beta) = \nu(\alpha\beta)$ then $\{\beta, \alpha\beta\} = \{\varpi, \varepsilon\varpi\}$, and the nonzero eigenvalues are $\{(\pm x \pm y\sqrt{\varepsilon})\sqrt{\varpi}\}$. Since $\{1, \sqrt{\varepsilon}\} \subset \mathcal{O}_{F(\sqrt{\varepsilon})}$ are linearly independent over the residue field of F, the valuation of these eigenvalues is equal to the minimum of the valuations of x and of y. This gives the description of $\mathfrak{t}_r$.

To determine the breaks in the filtration, note that $x, y \in F$ and therefore $\nu(x), \nu(y) \in \mathbb{Z}$. We will thus have $\mathfrak{t}_r \neq \mathfrak{t}_{r+}$ only if $r \equiv \frac{1}{2}\nu(\gamma) \mod \mathbb{Z}$ for some γ occuring in the description, which gives the result. ∎

We now turn to the tori ${}^{\alpha}\mathfrak{u}_4^E$ arising from a quartic extension E with a unique subfield $E_1 = F(\alpha)$. These are summarized in Table 4.1, and two of their eigenvalues were computed in Lemma 4.1.8. For ease of reference we provide a table in which we give the values of the absolute roots on an arbitrary element $X \in {}^{\alpha}\mathfrak{u}_4^E$ in Table 5.1.

$\chi \in \Phi(G,T)$	$E = F(\sqrt[4]{\alpha})$	$E = F(\sqrt{\gamma})$
$a_{s_2} + a_{l_1} = \epsilon_1$	$(x + y\sqrt{\alpha})\sqrt[4]{\alpha}$	$(x + yi)\sqrt{\gamma}$
$a_{s_2} = \epsilon_2$	$(x - y\sqrt{\alpha})i\sqrt[4]{\alpha}$	$(x - yi)\sqrt{\tau(\gamma)}$
$a_{l_1} = \epsilon_1 - \epsilon_2$	$\sqrt[4]{\alpha}(x(1-i) + y(i+1)\sqrt{\alpha})$	$x(\sqrt{\gamma} - \sqrt{\tau(\gamma)}) + yi(\sqrt{\gamma} + \sqrt{\tau(\gamma)})$
$2a_{s_2} + a_{l_1} = \epsilon_1 + \epsilon_2$	$\sqrt[4]{\alpha}(x((1+i) + y(i-1)\sqrt{\alpha})$	$x(\sqrt{\gamma} + \sqrt{\tau(\gamma)}) + yi(\sqrt{\gamma} - \sqrt{\tau(\gamma)})$

Table 5.1: *The values of the absolute roots on and element $X = (x+y\sqrt{\alpha})\zeta, 0) \in {}^{\alpha}\mathfrak{u}_4^E$. In the second column, $\alpha \in \{\varepsilon, \varepsilon^k\varpi, \varepsilon\varpi^2\}$, where $k = 0,1,2,3$ if $-1 \in F^{\times 2}$, and $k = 0,1$ otherwise, and $\zeta = \sqrt[4]{\alpha}$. In the third column, which arises only when $-1 \notin F^{\times 2}$, $\gamma \in \{\varepsilon_{E_1}, \varepsilon_{E_1}\varpi\}$, $\zeta = \sqrt{\gamma}$, $\sqrt{\alpha} = i$ and τ is the nontrivial element of* $\mathrm{Gal}(F(i)/F)$*.*

We now carefully determine the filtrations on each of these tori.

Proposition 5.2.3. *Let ${}^{\alpha}\mathfrak{u}_4^E$ be the elliptic torus associated with the algebra $\mathcal{A} = E \oplus F$, with fixed field $E_1 = F(\sqrt{\alpha}) \subset E = F(\zeta)$, with α, ζ ranging over sets of parameters from Table 4.1 and enumerated in (a)–(e) below. Then we have*

$$\mathfrak{t} = {}^{\alpha}\mathfrak{u}_4^E = \{((x + y\sqrt{\alpha})\zeta, 0) \mid x, y \in F\}$$

and its Moy–Prasad filtration of depth r is given by

$$\mathfrak{t}_r = \left\{ (x + y\sqrt{\alpha})\zeta, 0) \in \mathfrak{t} \;\middle|\; \nu(x) \geq r - \nu(\zeta) \text{ and } \nu(y) \geq r - \frac{1}{2}\nu(\alpha) - \nu(\zeta) \right\},$$

and in fact if r is such that there exists $X \in \mathfrak{t}_r \setminus \mathfrak{t}_{r+}$, then all of the eigenvalues of X have valuation exactly r. These values r depend on the parameters ζ and $\sqrt{\alpha}$ as follows.

(a) *Suppose* $E = F(\sqrt[4]{\varepsilon})$ *and* $-1 \in F^{\times 2}$. *Then* $\alpha = \varepsilon$ *and* $\zeta = \sqrt[4]{\alpha}$ *and the filtration is indexed by* $\mathbb{Z}$.

(b) *Suppose* $E = F(\sqrt[4]{\varepsilon\varpi^2})$ *and* $-1 \in F^{\times 2}$. *Then* $\alpha = \varepsilon\varpi^2$ *and* $\zeta = \sqrt[4]{\varepsilon\varpi^2}$ *and the filtration is indexed by* $\frac{1}{2} + \mathbb{Z}$.

(c) *Suppose* $E = F(\sqrt[4]{\varepsilon^k\varpi})$ *with* $k \in \{0,1,2,3\}$ *if* $-1 \in F^{\times 2}$ *and* $k \in \{0,1\}$ *if* $-1 \notin F^{\times 2}$. *Then* $\alpha = \varepsilon^k\varpi$ *and* $\zeta = \sqrt[4]{\varepsilon^k\varpi}$ *and the filtration is indexed by* $\frac{1}{4} + \frac{1}{2}\mathbb{Z}$.

(d) *Suppose* $E = F(\sqrt{\varepsilon_{E_1}})$ *and* $-1 \notin F^{\times 2}$. *Then* $\sqrt{\alpha} = i$ *and* $\zeta = \sqrt{\varepsilon_{E_1}}$ *and the filtration is indexed by* $\mathbb{Z}$.

(e) *Suppose* $E = F(\sqrt{\varepsilon_{E_1}\varpi})$ *and* $-1 \notin F^{\times 2}$. *Then* $\sqrt{\alpha} = i$ *and* $\zeta = \sqrt{\varepsilon_{E_1}\varpi}$ *and the filtration is indexed by* $\frac{1}{2} + \mathbb{Z}$.

Proof. If $X = ((x+y\sqrt{\alpha})\zeta, 0) \in \mathfrak{t}_r$ then $\nu((x+y\sqrt{\alpha})\zeta \geq r$. We now consider the possibilities in Table 5.1. In the setting of the second column, if $\nu(\alpha) > 0$ then $\nu((x+y\sqrt{\alpha})\zeta) = \min\{\nu(x\sqrt{\alpha}), \nu(y\sqrt{\alpha^3})\}$ because these terms have distinct valuation. When $\alpha = \varepsilon$, we instead note that $\{1, \sqrt{\varepsilon}\}$ are linearly independent over the residue field of F, so that again $\nu((x + y\sqrt{\varepsilon})\sqrt[4]{\varepsilon}) = \min\{\nu(x\sqrt{\varepsilon}), \nu(y\sqrt{\varepsilon^3})\}$. It then follows that all remaining eigenvalues have valuation at least r. A similar argument holds for the setting of the third column: by construction, $\{1, i, \sqrt{\varepsilon_{E_1}}, \sqrt{\tau(\varepsilon_{E_1})}\}$ is linearly independent over the residue field of F, and so any linear combination will have valuation equal to the minimum valuation of any individual term.

We thus find that $X \in \mathfrak{t}_r$ if and only if each of $x\zeta$ and $y\sqrt{\alpha}\zeta$ have valuation at least r, yielding the description of $\mathfrak{t}_r$ given in the lemma. Moreover, all four eigenvalues will have valuation equal to the minimum of $\nu(x\zeta)$ and $\nu(y\sqrt{\alpha}\zeta)$. The breaks in the filtration therefore occur for $r \in \{\nu(\zeta), \frac{1}{2}\nu(\alpha) + \nu(\zeta)\} + \mathbb{Z}$ and enumerating these values as in (a)–(e) gives our result. ∎

Remark 5.2.4. The indexing $r \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ occurs only in the case $\alpha = \varepsilon^i\varpi$. In this situation the quartic extension is generated by $\sqrt[4]{\alpha}$ with $\nu(\alpha) = 1$, so that $\nu(\sqrt[4]{\alpha}) = \frac{1}{4}$. Suppose $X = ((x + y\sqrt{\alpha})\sqrt[4]{\alpha}, 0)$ with $x, y \in F$. If $r \in \frac{1}{4} + \mathbb{Z}$ then $X \in \mathfrak{t}_r$ if and only if $\nu(x) \geq r - \frac{1}{4}$ and $\nu(y) \geq \lceil r - \frac{3}{4} \rceil = r - \frac{1}{4}$. On the other hand, if $r \in \frac{3}{4} + \mathbb{Z}$ then $X \in \mathfrak{t}_r$ if and only if $\nu(x) \geq \lceil r - \frac{1}{4} \rceil = r + \frac{1}{4}$ and $\nu(y) \geq r - \frac{3}{4}$.

The results are summarized in Table 5.2.

Torus $\mathfrak{t}$	$\mathbf{u} \in \mathfrak{t} : x, y \in F$	x, y such that $\mathbf{u} \in \mathfrak{t}_r$	**Depth** r
${}^{\varepsilon}\mathfrak{t}^2$	$(x\sqrt{\varepsilon}, y\sqrt{\varepsilon}, 0)$	$\nu(x) = \nu(y) \geq r$	$r \in \mathbb{Z}$
${}^{\varpi'}\mathfrak{t}^2$	$(x\sqrt{\varpi'}, y\sqrt{\varpi'}, 0)$	$\nu(x) = \nu(y) \geq r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
${}^{\varpi}\mathfrak{t}^{\varepsilon\varpi}$	$(x\sqrt{\varpi}, y\sqrt{\varepsilon\varpi}, 0)$	$\nu(x) = \nu(y) \geq r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$

Torus $\mathfrak{t}$	$\mathbf{u} \in \mathfrak{t} : x, y \in F$	x, y such that $\mathbf{u} \in \mathfrak{t}_r$	**Depth** r
${}^{\varpi'}\mathfrak{t}^{\varepsilon}$	$(x\sqrt{\varpi'}, y\sqrt{\varepsilon}, 0)$	$\nu(x) = \nu(y) \geq r$	$r \in \mathbb{Z}$
		$\nu(x) \geq r - \frac{1}{2}$, $\nu(y) \geq r + \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
${}^{\varepsilon}\mathfrak{t}_4^{\varepsilon,\varpi}$	$((x + y\sqrt{\varepsilon})\sqrt{\varpi}, 0)$	$\nu(x) = \nu(y) \geq r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
${}^{\varpi'}\mathfrak{t}_4^{\varepsilon,\varpi}$	$((x + y\sqrt{\varpi'})\sqrt{\varepsilon}, 0)$	$\nu(x) = \nu(y) \geq r$	$r \in \mathbb{Z}$
		$\nu(x) \geq r + \frac{1}{2}$, $\nu(y) \geq r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$-1^{\square}$, ${}^{\varepsilon}\mathfrak{u}_4^{\varepsilon_{E_1}\varpi}$	$((x + y\sqrt{\varepsilon})\sqrt[4]{\varepsilon\varpi^2}, 0)$	$\nu(x) = \nu(y) \geq r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$-1^{\boxtimes}$, ${}^{\varepsilon}\mathfrak{u}_4^{\varepsilon_{E_1}\varpi}$	$((x + yi)\sqrt{\varepsilon_{E_1}\varpi}, 0)$	$\nu(x) = \nu(y) \geq r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$-1^{\square}$, ${}^{\varepsilon}\mathfrak{u}_4^{\varepsilon_{E_1}}$	$((x + y\sqrt{\varepsilon})\sqrt[4]{\varepsilon}, 0)$	$\nu(x) = \nu(y) \geq r$	$r \in \mathbb{Z}$
$-1^{\boxtimes}$, ${}^{\varepsilon}\mathfrak{u}_4^{\varepsilon_{E_1}}$	$((x + yi)\sqrt{\varepsilon_{E_1}}, 0)$	$\nu(x) = \nu(y) \geq r$	$r \in \mathbb{Z}$
${}^{\varepsilon^k\varpi}\mathfrak{u}_4^{\varepsilon^k\varpi}$	$(x + y\sqrt{\varepsilon^k\varpi})\sqrt[4]{\varepsilon^k\varpi}$	$\nu(x) = \nu(y) \geq r - \frac{1}{4}$	$r \in \frac{1}{4} + \mathbb{Z}$
		$\nu(x) \geq r + \frac{1}{4}$, $\nu(y) \geq r - \frac{3}{4}$	$r \in \frac{3}{4} + \mathbb{Z}$

Table 5.2: *The Moy–Prasad filtrations of the Lie algebras $\mathfrak{t}_r$ of each of the isomorphism classes of elliptic maximal tori of G, together with the values r that define a distinct step in the filtration.*

5.3 The generic sets in $\mathfrak{so}_5$

5.3.1 The sets $\mathbf{Gen}_{G,\mathcal{T},-r}$

In this section we apply the general theory of G-generic elements to the maximal elliptic tori in $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. We begin by stating an explicit criterion for $X \in \mathbf{Gen}_{G,T,-r}$.

Lemma 5.3.1. *Let $\mathcal{T}$ be a maximal elliptic torus of G and let $r \in \mathbb{R}$. Suppose the eigenvalues of $X \in \mathfrak{t}_{-r}$ are $\{\pm\lambda_1, \pm\lambda_2, 0\}$. Then $X \in \mathbf{Gen}_{G,T,-r}$ if and only if*

$$\nu(\lambda_1) = \nu(\lambda_2) = \nu(\lambda_1 \pm \lambda_2) = -r.$$

Proof. Write the roots of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ as $\{\pm\epsilon_1, \pm\epsilon_2, \pm(\epsilon_1 \pm \epsilon_2)\}$. Then by conjugating by an element of $\mathrm{SO}_5(E)$ we may without loss of generality relabel so that $\lambda_1 = \epsilon_1(X), \lambda_2 = \epsilon_2(X)$.

By Definition 5.1.6, $X \in \mathbf{Gen}_{G,T,-r}$ if and only if $\nu(a(X)) = -r$ for all roots a, and it follows now that this occurs if and only if

$$\nu(\lambda_1) = \nu(\lambda_2) = \nu(\lambda_1 \pm \lambda_2) = -r.$$

∎

Comparing with Lemma 5.2.1, we see that the condition for X to be G-generic of depth $-r$ is more restrictive than the condition for X to have depth r. It is weaker than the condition of being a regular element, which was that none of the eigenvalues of X are zero.

Our first result is that for those tori that admit no nonzero, non-regular elements, every element of depth $-r$ is in fact G-generic of depth $-r$.

Proposition 5.3.2. *Let $\mathfrak{t} = {}^{\alpha}\mathfrak{u}_4^E$ be the Lie algebra of the elliptic torus corresponding to the algebra $\mathcal{A} = E \oplus F$, with a unique subfield as in Table 4.1. Then every nonzero element of $\mathfrak{t}_{-r} \setminus \mathfrak{t}_{-r+}$ is G-generic of depth $-r$.*

Proof. As noted in Proposition 5.2.3, the four nonzero eigenvalues of every nonzero element $X \in \mathfrak{t}_{-r} \setminus \mathfrak{t}_{-r+}$ have valuation exactly $-r$. Furthermore, using Table 5.1 we further deduce that the sum and differences of these eigenvalues must also have valuation exactly $-r$, whence these elements are G-generic. ∎

We now turn to the task of identifying G-generic elements of depth $-r$ for the remaining elliptic tori, beginning with the form ${}^{\alpha}\mathfrak{t}^2$.

Lemma 5.3.3. *Let $\mathfrak{t} = {}^{\alpha}\mathfrak{t}^2$ be the elliptic torus corresponding to the algebra $\mathcal{A} = F(\sqrt{\alpha}) \oplus F(\sqrt{\alpha}) \oplus F$, for $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$. Then*

$$\mathbf{Gen}_{G,T,-r} = \left\{(x\sqrt{\alpha}, y\sqrt{\alpha}, 0) \;\middle|\; \nu(x\sqrt{\alpha}) = \nu(y\sqrt{\alpha}) = -r,\ x \not\equiv \pm y \mod \mathcal{P}_F^{-r-\frac{1}{2}\nu(\alpha)+1}\right\}.$$

Proof. Let $X = (x\sqrt{\alpha}, y\sqrt{\alpha}, 0) \in \mathfrak{t}$. The eigenvalues of X we computed in (4.1.1) to be $\{\pm x\sqrt{\alpha}, \pm y\sqrt{\alpha}\}$, so we apply Lemma 5.3.1 directly. The first condition is the statement that each of these eigenvalues has valuation equal to $-r$, which implies $x, y \in \mathcal{P}_F^{-r-\frac{1}{2}\nu(\alpha)}$. In order for their pairwise differences to have valuation r, their sum and difference cannot have greater valuation. This gives this description of $\mathbf{Gen}_{G,T,-r}$. ∎

Lemma 5.3.4. *Let $\mathfrak{t} = {}^{\varepsilon\varpi}\mathfrak{t}^{\varpi}$, and $\mathfrak{t}' = {}^{\varepsilon}\mathfrak{t}_4^{\varepsilon,\varpi}$. Then*

$$\mathbf{Gen}_{G,\mathcal{T},-r} = \left\{(x\sqrt{\varepsilon\varpi}, y\sqrt{\varpi}, 0) \in \mathfrak{t} \;\middle|\; \nu(x\sqrt{\varepsilon\varpi}) = \nu(y\sqrt{\varpi}) = -r\right\},$$

$$\mathbf{Gen}_{G,\mathcal{T}',-r} = \left\{(x\sqrt{\varpi} + y\sqrt{\varepsilon\varpi}, 0) \in \mathfrak{t}' \;\middle|\; \nu(x\sqrt{\varpi}) = \nu(y\sqrt{\varepsilon\varpi}) = -r\right\}.$$

Proof. Let $X = (x\sqrt{\varepsilon\varpi}, y\sqrt{\varpi}, 0) \in \mathfrak{t}$ for $x, y \in F$. By (4.1.1) its eigenvalues are given by the set $\{\pm x\sqrt{\varepsilon\varpi}, \pm y\sqrt{\varpi}\}$. Hence by Lemma 5.3.1, the condition $\nu(x\sqrt{\varepsilon\varpi}) = \nu(y\sqrt{\varpi}) = -r$ is necessary. If it holds, then since $x\sqrt{\varepsilon\varpi} \pm y\sqrt{\varpi} = (x\sqrt{\varepsilon} \pm y)\sqrt{\varpi}$ we have $\varpi^{r+\frac{1}{2}}(x\sqrt{\varepsilon} \pm y) \in \mathcal{O}_{E_s}$. But since $\{1, \sqrt{\varepsilon}\}$ are linearly independent over the residue field of F, and both $\varpi^{r+\frac{1}{2}}x$ and $\varpi^{r+\frac{1}{2}}y$ lie in $\mathcal{O}_F^{\times}$, this sum cannot lie in $\mathcal{P}_{E_s}$. Therefore $\nu(x\sqrt{\varepsilon\varpi} \pm y\sqrt{\varpi}) = -r$ and so $X \in \mathbf{Gen}_{G,\mathcal{T},-r}$.

Now consider $X = (x\sqrt{\varpi} + y\sqrt{\varepsilon\varpi}, 0) \in \mathfrak{t}'$, with $x, y \in F$. By Lemma 4.1.7, the eigenvalues of X are $\{\pm(x\sqrt{\varpi} \pm y\sqrt{\varepsilon\varpi})\}$. By Lemma 5.3.1, for X to be G-generic of depth $-r$, these eigenvalues must have valuation $-r$, and also their sums and differences. This yields the condition $\nu(x\sqrt{\varpi}) = \nu(y\sqrt{\varepsilon\varpi}) = -r$ since $p \neq 2$, and the argument above shows it is equivalent. ∎

Surprisingly, the remaining tori do not admit *any* G-generic elements.

Theorem 5.3.5. *Let $\mathfrak{t} \in \{{}^{\varepsilon}\mathfrak{t}^{\varpi'}, {}^{\varpi'}\mathfrak{t}_4^{\varepsilon,\varpi}\}$, for $\varpi' \in \{\varpi, \varepsilon\varpi\}$. Then $\mathfrak{t}$ does not contain any G-generic elements of any depth, that is, $\mathbf{Gen}_{G,\mathcal{T},-r} = \emptyset$ for all $r \in \mathbb{R}$.*

Proof. Consider $\mathfrak{t} = {}^{\varepsilon}\mathfrak{t}^{\varpi'}$. By (4.1.1) the eigenvalues of $X = (x\sqrt{\varepsilon}, y\sqrt{\varpi'}, 0) \in \mathfrak{t}$ are $\{\pm x\sqrt{\varepsilon}, \pm y\sqrt{\varpi'}, 0\}$. Then $\nu(x\sqrt{\varepsilon}) \in \mathbb{Z}$ while $\nu(y\sqrt{\varpi'}) \in \frac{1}{2} + \mathbb{Z}$, so there does not exist a real number r for which these valuations can be equal. By Lemma 5.3.1, we conclude that $\mathfrak{t}$ does not contain any G-generic elements of depth $-r$, for any $r \in \mathbb{R}$.

Now suppose $\mathfrak{t} = {}^{\varpi'}\mathfrak{t}_4^{\varepsilon,\varpi}$. By Lemma 4.1.7, the eigenvalues of $X = (x\sqrt{\varepsilon} + y\sqrt{\varepsilon\varpi'}, 0) \in \mathfrak{t}$ are $\{\pm(x\sqrt{\varepsilon} \pm y\sqrt{\varepsilon\varpi'}), 0\}$. If X were G-generic of depth $-r$, then we would have $\nu(x\sqrt{\varepsilon} \pm y\sqrt{\varepsilon\varpi'}) = -r$. Taking their difference yields

$$\nu\big((x\sqrt{\varepsilon} + y\sqrt{\varepsilon\varpi'}) - (x\sqrt{\varepsilon} - y\sqrt{\varepsilon\varpi'})\big) = \nu(2y\sqrt{\varepsilon\varpi'}) = -r,$$

so $\nu(y) + \frac{1}{2}\nu(\varepsilon\varpi') = -r$, hence $\nu(y) = -r - \frac{1}{2}$ and $r \in \frac{1}{2} + \mathbb{Z}$. But then taking the sum instead yields

$$\nu\big((x\sqrt{\varepsilon} + y\sqrt{\varepsilon\varpi'}) + (x\sqrt{\varepsilon} - y\sqrt{\varepsilon\varpi'})\big) = \nu(2x\sqrt{\varepsilon}) \in \mathbb{Z},$$

which is a contradiction. Thus not all eigenvalues have valuation $-r$, and there are no G-generic elements of depth $-r$ in $\mathfrak{t}$, for any $r \in \mathbb{R}$. ∎

This result has implications to the possible choices of twisted Levi sequence, as we'll see in Section 5.4. For the reader's convenience, we put all the results in Table A.8 and Table A.9

5.3.2 The sets $\mathbf{Gen}_{G,\mathcal{M},-r}$, for $\mathcal{M}$ a twisted Levi subgroup of G

In this section, let $\mathcal{M}$ be a proper, nonabelian twisted Levi subgroup of $G \cong \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ with compact centre. These were classified up to conjugacy in Chapter 4. Our goal is to determine, for each pair $\mathcal{T} \subset \mathcal{M}$, the structure of the sets $\mathbf{Gen}_{G,\mathcal{M},-r}$.

We use the notation from Section 4.2. Our first step is to identify the (absolute) root system $\Phi(\mathcal{M}, \mathcal{T})$ relative to our choice of presentation of $\mathcal{T}$, in each case that $\mathcal{M}$ is realized as the centralizer of a non-regular element of the Lie algebra of $\mathcal{T}$. As noted when these twisted Levi subgroups were first identified in Section 2.4.3, the (absolute) root systems of twisted Levi subgroups of the form $\mathcal{S}$ correspond to short roots of the root system of G, and $\mathcal{L}$ to long roots.

Lemma 5.3.6. *Let $\mathcal{M} \subset G$ be a proper nonabelian twisted Levi subgroup with compact centre, and let $\mathcal{T} \subset \mathcal{M}$ be an elliptic maximal torus.*

- *If $\mathcal{M}$ is of type $\mathcal{S}$, then we may assume $\Phi(\mathcal{M},\mathcal{T}) = \{\pm\epsilon_2\}$.*
- *If $\mathcal{M}$ is of type $\mathcal{L}$, then we may assume $\Phi(\mathcal{M},\mathcal{T}) = \{\pm(\epsilon_1-\epsilon_2)\}$.*

Proof. If $\mathcal{M}$ is of type $\mathcal{S}$, we may by Proposition 4.2.3 choose a convenient torus $\mathcal{T}$ and realize $\mathcal{M}$ as the centralizer of a non-regular element $X \in \mathrm{Lie}(\mathcal{T})$ of the form $X = (x\sqrt{\alpha}, 0, 0)$. In this case

$$\Phi(\mathcal{M},\mathcal{T}) = \{a \in \Phi(G,\mathcal{T}) : a(X) = 0\} = \{\pm\epsilon_2\}.$$

If $\mathcal{M}$ is of type $\mathcal{L}$, we may take either $\mathcal{T}$ of the form in Proposition 4.2.5, so that $X = (x\sqrt{\alpha}, x\sqrt{\alpha}, 0)$ or as in Proposition 4.2.6, so that $X = (x\sqrt{\alpha}, 0)$, and realized $\mathcal{M}$ as the centralizer of X. In both cases,

$$\Phi(\mathcal{M},\mathcal{T}) = \{a \in \Phi(G,\mathcal{T}) : a(X) = 0\} = \{\pm(\epsilon_1-\epsilon_2)\}.$$

∎

It is now easy to identify the elements of $\mathbf{Gen}_{G,\mathcal{M},-r}$.

Theorem 5.3.7. *Let $\mathcal{M} \subset G$ be a proper, nonabelian twisted Levi subgroup with compact center, and let $\mathfrak{m} = \mathrm{Lie}(\mathcal{M})$. Then*

$$\mathbf{Gen}_{G,\mathcal{M},-r} = \mathfrak{z}(\mathfrak{m})_{-r} \setminus \mathfrak{z}(\mathfrak{m})_{-r^+}.$$

Equivalently, every element of depth $-r$ in $\mathfrak{z}(\mathfrak{m})$ is G-generic.

Proof. By Remark 5.1.10, the set $\mathbf{Gen}_{G,\mathcal{M},-r}$ is independent of the choice of elliptic maximal torus $\mathcal{T} \subset \mathcal{M}$, so we may fix our choices as in Lemma 5.3.6. Let $\mathfrak{z}(\mathfrak{m})$ denote the center of the Lie algebra of $\mathcal{M}$. Since $\dim \mathfrak{z}(\mathfrak{m}) = 1$, if there exists an element $X \in \mathfrak{z}(\mathfrak{m})_{-r}$ has depth exactly $-r$, then every scalar multiple uX with $u \in \mathcal{O}_F^\times$ also has depth $-r$. Moreover, if X is G-generic, then so is uX, since $\nu(a(X))$, for $a \in \Phi(\mathbb{G},\mathbb{T})$, are unchanged under multiplication by units.

Suppose $X \in \mathfrak{z}(\mathfrak{m})_{-r} \smallsetminus \mathfrak{z}(\mathfrak{m})_{-r^+}$. Then by Lemma 5.2.1 at least one of its eigenvalues must have valuation exactly $-r$, that is, either $\nu(\varepsilon_1(X)) = -r$ or $\nu(\varepsilon_2(X)) = -r$. Since $X \in \mathfrak{z}(\mathfrak{m})$, we have that $a(X) = 0$ for every $a \in \Phi(\mathbb{M},\mathbb{T})$. Together with the description of the root systems in Lemma 5.3.6, this yields that $\nu(a(X)) = -r$ for every $a \in \Phi(\mathbb{G},\mathbb{T}) \smallsetminus \Phi(\mathbb{M},\mathbb{T})$. ∎

Corollary 5.3.8. *Let $\mathcal{M} \subset G$ be a proper nonabelian twisted Levi subgroup with compact centre. Then the values of r for which $\mathbf{Gen}_{G,\mathcal{M},-r}$ is nonempty, and a description of this set in that case, is given explicitly as follows.*

- *If $\mathcal{M} = {}^\alpha\mathcal{S}$, then $\mathbf{Gen}_{G,\mathcal{M},-r} = \{(x\sqrt{\alpha}, 0, 0) : \nu(x) = -r - \frac{1}{2}\nu(\alpha)\}$ and $r \in \frac{1}{2}\nu(\alpha) + \mathbb{Z}$.*
- *If $\mathcal{M} = {}^\alpha\mathcal{L}$, then $\mathbf{Gen}_{G,\mathcal{M},-r} = \left\{(x\sqrt{\alpha}, x\sqrt{\alpha}, 0) \in {}^\alpha\mathfrak{t}^2 : \nu(x) = -r - \frac{1}{2}\nu(\alpha)\right\}$, and $r \in \frac{1}{2}\nu(\alpha) + \mathbb{Z}$.*

Proof. The filtration on $\mathfrak{z}(\mathfrak{m})$ is obtained by intersection with $\mathfrak{t}$, for any choice of $\mathfrak{t}$ containing $\mathfrak{z}(\mathfrak{m})$, and these were determined in Section 5.2. Thus the statement follows from Theorem 5.3.7. Note that in the second case, we could also have chosen $\mathfrak{t} = {}^{\gamma}\mathfrak{t}_4^{\varepsilon,\varpi}$, for some $\gamma \in \{\varepsilon, \varpi, \varepsilon\varpi\} \smallsetminus \{\alpha\}$. This would instead give the equivalent description $\mathbf{Gen}_{G,\mathcal{M},-r} = \{(x\sqrt{\alpha}, 0) \in {}^{\gamma}\mathfrak{t}_4^{\varepsilon,\varpi} : \nu(x) = -r - \frac{1}{2}\nu(\alpha)\}$, which yields the same values for the depths of elements of $\mathfrak{z}(\mathfrak{m})$, as expected. ∎

Table A.10 gives a more explicit summary of all the cases, listing the G-generic elements of the center of the Lie algebra of $\mathcal{M}$, together with their depths.

5.3.3 The sets $\mathbf{Gen}_{\mathcal{M},\mathcal{T},-r}$

We now consider which elements of the Lie algebra of a torus $\mathcal{T}$ of $G \cong \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ will be $\mathcal{M}$-generic, relative to some proper nonabelian twisted Levi subgroup $\mathcal{M}$ containing $\mathcal{T}$.

We first make a simple observation.

Lemma 5.3.9. *Let $G \cong \mathrm{SO}_5$, let $\mathcal{T} \subset G$ be a maximal elliptic torus, and let $\mathcal{M}$ be a twisted Levi subgroup of G containing $\mathcal{T}$. Then $\mathrm{Gen}_{G,\mathcal{T},-r} \subseteq \mathrm{Gen}_{\mathcal{M},\mathcal{T},-r}$.*

Proof. If X is G-generic of depth $-r$, then for every root $a \in \Phi(G, \mathcal{T})$ one has $\nu(a(X)) = -r$. Since $\Phi(\mathcal{M}, \mathcal{T}) \subset \Phi(G, \mathcal{T})$ for every twisted Levi subgroup $\mathcal{M} \supset \mathcal{T}$, it follows immediately that X is $\mathcal{M}$-generic of depth $-r$ for every such $\mathcal{M}$. ∎

By Lemma 5.3.6, $\Phi(\mathcal{M}, \mathcal{T}) = \{\pm a\}$ for a single root a, so the $\mathcal{M}$-genericity condition of Definition 5.1.6 reduces to requiring that this root evaluate on $X \in \mathfrak{t}$ with valuation $-r$. Concretely, since this root vanishes on $\mathfrak{z}(\mathfrak{m}) \subset \mathfrak{t}$, the genericity condition ensures that $X \notin \mathfrak{z}(\mathfrak{m})$) and in some sense measures the deviation of X from the non-regular subspace $\mathfrak{z}(\mathfrak{m})$.

The pairs $(\mathcal{T}, \mathcal{M})$ were found in Section 4.2. We begin with the tori arising from a sum of two quadratic extensions, ${}^{\alpha}\mathcal{T}^{\beta}$ and ${}^{\alpha}\mathcal{T}^{\alpha} := {}^{\alpha}\mathcal{T}^{2}$. These may embed in two kinds of twisted Levi subgroups.

Proposition 5.3.10. *Let $\mathcal{T} = {}^{\alpha}\mathcal{T}^{\beta}$ and let $\mathfrak{t} = {}^{\alpha}\mathfrak{t}^{\beta}$ be its Lie algebra, where $\alpha, \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$.*

1. *The ${}^{\alpha}\mathcal{S}$-generic elements in ${}^{\alpha}\mathfrak{t}^{\beta}$ have depth $-r \in \frac{1}{2}\nu(\beta) + \mathbb{Z}$ and are given by*

$$\mathrm{Gen}_{{}^{\alpha}\mathcal{S},{}^{\alpha}\mathcal{T}^{\beta},-r} = \left\{ (x\sqrt{\alpha}, y\sqrt{\beta}, 0) \in {}^{\alpha}\mathfrak{t}^{\beta} : \begin{array}{l} \nu(y) = -r - \frac{1}{2}\nu(\beta), \\ \nu(x) \geq -r - \frac{1}{2}\nu(\alpha) \end{array} \right\}.$$

2. *The ${}^{\beta}\mathcal{S}$-generic elements in ${}^{\alpha}\mathfrak{t}^{\beta}$ have depth $-r \in \frac{1}{2}\nu(\alpha)+\mathbb{Z}$ and are given by*

$$\mathrm{Gen}_{{}^{\beta}\mathcal{S},{}^{\alpha}\mathcal{T}^{\beta},-r} = \left\{ (x\sqrt{\alpha}, y\sqrt{\beta}, 0) \in {}^{\alpha}\mathfrak{t}^{\beta} : \begin{array}{l} \nu(x) = -r - \frac{1}{2}\nu(\alpha), \\ \nu(y) \geq -r - \frac{1}{2}\nu(\beta) \end{array} \right\}.$$

3. *If $\alpha = \beta$, then the ${}^{\alpha}\mathcal{L}$-generic elements in ${}^{\alpha}\mathfrak{t}^{2}$ have depth $-r \in \frac{1}{2}\mathrm{val}(\alpha)+\mathbb{Z}$ and are given by*

$$\mathrm{Gen}_{{}^{\alpha}\mathcal{L},T,-r} = \left\{ (x\sqrt{\alpha}, y\sqrt{\alpha}, 0) \in {}^{\alpha}\mathfrak{t}^{2} : \begin{array}{l} \nu(x) \geq -r - \frac{1}{2}\nu(\alpha), \\ \nu(y) \geq -r - \frac{1}{2}\nu(\alpha), \\ x \not\equiv y \pmod{\mathcal{P}^{-r+1-\frac{1}{2}\nu(\alpha)}} \end{array} \right\}.$$

In particular, at least one of $\nu(x)+\frac{1}{2}\nu(\alpha)$ and $\nu(y)+\frac{1}{2}\nu(\alpha)$ is equal to $-r$.

Proof. Let $X = (x\sqrt{\alpha}, y\sqrt{\beta}, 0) \in {}^{\alpha}\mathfrak{t}^{\beta}$.

We showed in Proposition 4.2.3 that $\mathcal{M} = {}^{\alpha}\mathcal{S} = C_G(S_{nr_1})$ where $S_{nr_1} = \{(x\sqrt{\alpha}, 0, 0) \in \mathfrak{t} : x \in F\}$. By Lemma 5.3.6, $\Phi(\mathcal{M},\mathcal{T}) = \{\pm\epsilon_2\}$. Thus $X \in \mathrm{Gen}_{{}^{\alpha}\mathcal{S},\mathcal{T},-r}$ if and only if $X \in \mathfrak{t}_{-r}$ and $-r = \nu(\epsilon_2(X)) = \nu(y\sqrt{\beta})$, so $\nu(y) = -r - \frac{1}{2}\nu(\beta)$. The other inequality follows from $X \in \mathfrak{t}_{-r}$.

By interchanging (α, x, ϵ_1) and (β, y, ϵ_2), we obtain from Proposition 4.2.3 that $\mathcal{M} = {}^{\alpha}\mathcal{S} = C_G(S_{nr_2})$ where $S_{nr_2} = \{(0, y\sqrt{\beta}, 0) \in \mathfrak{t} : x \in F\}$ and the second result follows as above.

Now suppose $\alpha = \beta$. The eigenvalues of X are $\{\pm x\sqrt{\alpha}, \pm y\sqrt{\alpha}, 0\}$. By Proposition 4.2.5, ${}^{\alpha}\mathcal{L} = C_G(S_{nr_3})$ where $S_{nr_3} = \{(x\sqrt{\alpha}, x\sqrt{\alpha}, 0) \in \mathfrak{t} : x \in F^{\times}\}$ (which by Proposition 4.1.6 is conjugate to S_{nr_4}). By Lemma 5.3.6, $\Phi({}^{\alpha}\mathcal{L}, T) = \{\pm(\epsilon_1 - \epsilon_2)\}$. Thus $X \in \mathrm{Gen}_{{}^{\alpha}\mathcal{L},{}^{\alpha}\mathfrak{t}^2,-r}$ if and only if $X \in {}^{\alpha}\mathfrak{t}^{2}_{-r}$ and $\nu((\epsilon_1-\epsilon_2)(X)) = -r$. The first condition implies that $\nu(x), \nu(y) \geq -r - \nu(\alpha)$. Since $(\epsilon_1-\epsilon_2)(X) = (x-y)\sqrt{\alpha}$, the second condition implies $x \not\equiv y \pmod{\mathcal{P}^{-r+1-\frac{1}{2}\nu(\alpha)}}$. ∎

We now turn to the tori arising from biquadratic extensions.

Proposition 5.3.11. *Let $\alpha \neq \beta \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ and set $\mathcal{T} = {}^{\alpha}\mathcal{T}_4^{\varepsilon,\varpi}$ with $\mathfrak{t}$ be its Lie algebra. Then*

1. *The ${}^{\beta}\mathcal{L}$-generic elements in ${}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi}$ have depth $-r \in \frac{1}{2}\nu(\alpha\beta)+\mathbb{Z}$ and are given by*

$$\mathrm{Gen}_{{}^{\beta}\mathcal{L},\mathcal{T},-r} = \left\{ (x\sqrt{\beta} + y\sqrt{\alpha\beta}, 0) \in {}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi} : \begin{array}{l} \nu(y) = -r - \frac{1}{2}\nu(\alpha\beta), \\ \nu(x) \geq -r - \frac{1}{2}\nu(\beta) \end{array} \right\},$$

2. The ${}^{\alpha\beta}\mathcal{L}$-generic elements in ${}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi}$ have depth $-r \in \frac{1}{2}\nu(\beta) + \mathbb{Z}$ and are given by

$$\mathrm{Gen}_{{}^{\alpha\beta}\mathcal{L},\mathcal{T},-r} = \left\{ (x\sqrt{\beta} + y\sqrt{\alpha\beta}, 0) \in {}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi} : \begin{array}{l} \nu(y) \geq -r - \frac{1}{2}\nu(\alpha\beta), \\ \nu(x) = -r - \frac{1}{2}\nu(\beta) \end{array} \right\}.$$

Proof. Let $X = (x\sqrt{\beta} + y\sqrt{\alpha\beta}, 0) \in \mathfrak{t} = {}^{\alpha}\mathfrak{t}_4^{\varepsilon,\varpi}$. By Lemma 4.1.7, the nonzero eigenvalues of X are

$$x\sqrt{\beta} + y\sqrt{\alpha\beta}, \quad x\sqrt{\beta} - y\sqrt{\alpha\beta}, \quad -x\sqrt{\beta} - y\sqrt{\alpha\beta}, \quad -x\sqrt{\beta} + y\sqrt{\alpha\beta}.$$

We showed in Proposition 4.2.6 that ${}^{\beta}\mathcal{L} = C_G(S_{nr_1})$ where $S_{nr_1} = \{(x\sqrt{\beta}, 0) \in \mathfrak{t} : x \in F\}$. By Lemma 5.3.6, $\Phi({}^{\beta}\mathcal{L}, \mathcal{T}) = \{\pm(\epsilon_1 - \epsilon_2)\}$. Thus $X \in \mathrm{Gen}_{{}^{\beta}\mathcal{L},\mathcal{T},-r}$ if and only if $X \in \mathfrak{t}_{-r}$ and $\nu((\epsilon_1 - \epsilon_2)(X)) = -r$. Since $(\epsilon_1 - \epsilon_2)(X) = 2y\sqrt{\alpha\beta}$, and $2 \in \mathcal{O}_F^{\times}$, this gives $\nu(y) = -r - \frac{1}{2}\nu(\alpha\beta)$, and $\nu(x) \geq -r - \frac{1}{2}\nu(\beta)$. The case of ${}^{\alpha\beta}\mathcal{L} = C_G(S_{nr_2})$ where $S_{nr_2} = \{(y\sqrt{\alpha\beta}, 0) \in \mathfrak{t} : y \in F\}$ is obtained by interchanging $(\beta, x, \epsilon_1 - \epsilon_2)$ and $(\alpha\beta, y, \epsilon_1 + \epsilon_2)$, yielding the stated description of $\mathrm{Gen}_{{}^{\alpha\beta}\mathcal{L},\mathcal{T},-r}$. ∎

Similarly, Table A.11 summarizes the $\mathcal{M}$-generic elements in $\mathfrak{t}$ corresponding to all the cases discussed.

5.4 Valid twisted Levi sequences

In the definition of a cuspidal Yu-datum, the key item is a sequence of twisted Levi subgroups. By Remark 2.4.23, such a sequence has at most three terms. It must also satisfy additional conditions.

The first condition on this sequence is that the first group has compact center, and this was what guided our classification of twisted Levi subgroups. We also determined all possible inclusion relations among our maximal elliptic tori and twisted Levi subgroups, as illustrated in Figure 3.1.

Another condition arises from the requirement that if $(\mathbb{G}_1, \mathbb{G}_2)$ occurs in this sequence, then there must also exist a character ϕ' of $G_1 = \mathbb{G}_1(F)$ that is $\mathbb{G}_2$-generic of some depth r. By Definition 2.6.8, this is equivalent to the existence of an element $X \in \mathrm{Lie}(G_1)$ that is G_2-generic of depth $-r$.

The results of the previous sections show that genericity is a firm condition: not every element of depth $-r$ is generic, and in some cases no generic elements exist at all. In this section, we list all of the *valid* twisted Levi sequences, that is, twisted Levi sequences

$$G^0 \subset G^1 \subset \cdots \subset G^d = G$$

(with $d \leq 2$) that satisfy both conditions.

Let us begin by summarizing the key observations from Section 5.3 and the impact on the choices of twisted Levi sequences.

Proposition 5.4.1.

- *If $\mathcal{T}$ is an elliptic maximal torus that is not contained in any intermediate twisted Levi subgroup, then every element in $\mathfrak{t}_{-r} \setminus \mathfrak{t}_{-r^+}$ is G-generic of depth $-r$.*

- *If $\mathcal{M}$ is a proper nonabelian twisted Levi subgroup of G, then every element of $\mathfrak{z}(\mathfrak{m})_{-r} \smallsetminus \mathfrak{z}(\mathfrak{m})_{-r+}$ is G-generic of depth r.*

- *If $\mathcal{T}$ is an elliptic maximal torus that admits no G-generic elements of depth $-r$, for any r, then the twisted Levi sequence $\mathcal{T} \subset G$ is not valid. However, in this case for each depth $-r$ occurring as the depth of an element of $\mathfrak{t}$, there exists a twisted Levi subgroup $\mathcal{M}$ containing $\mathcal{T}$ such that $\mathcal{T}$ has $\mathcal{M}$-generic elements of depth $-r$, and then the sequence $\mathcal{T} \subset \mathcal{M} \subset G$ is valid.*

Proof. The first statement is from Proposition 5.3.2 and the second from Theorem 5.3.7. For the last, a careful study of Tables A.8, A.9, A.10, and A.11 reveals that for all tori, except ${}^{\alpha}\Upsilon_4^E$, for each element $X \in \mathfrak{t}$ we can find a root a for which $\nu(a(X)) = -r$ and correspondingly there exists a twisted Levi subgroup $\mathcal{M}$ such that X is $\mathcal{M}$-generic of depth $-r$. ∎

Recall that in Figure 3.1 we had given all twisted Levi sequences in G up to conjugacy. Using Proposition 5.4.1 we can now determine which of these are valid, that is, that arise in the construction of supercuspidal representations.

Corollary 5.4.2. *All twisted Levi sequences are valid, with the exception of those beginning with the maximal elliptic tori of the form ${}^{\varpi'}\mathcal{T}_4^{\varepsilon,\varpi}$ and ${}^{\varpi'}\mathcal{T}^{\varepsilon}$. These only occur in twisted Levi sequences with three terms, that is, of the form $\mathcal{T} \subset \mathcal{M} \subset G$.*

On the other hand, tori of the form ${}^{\alpha}\Upsilon_{4[\mu]}^E$ occur only in twisted Levi sequences with two terms, that is, sequences of the form $\mathcal{T} \subset G$.

We summarize this result in Figure 5.1, showing all the inclusions that are part of a valid twisted Levi sequence. Sequences of length two may begin either with an elliptic maximal torus $\mathcal{T}$ or with a twisted Levi subgroup $\mathcal{M}$ leading to $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. Sequences of length three begin with an elliptic maximal torus $\mathcal{T}$ and pass through a twisted Levi subgroup $\mathcal{M}$ before reaching $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

The figure indicates the tori that admit sequences of length two, and shows that the tori ${}^{\varpi'}\mathcal{T}^{\varepsilon}$ and ${}^{\varpi'}\mathcal{T}^{\varepsilon,\varpi}$ do not occur in sequences of length two. It also shows that all tori participate in sequences of length three, except for the quartic torus with a unique subfield (shown in dark blue).

Note that the arithmetic of F comes into play: the torus ${}^{\varpi'}\mathcal{T}^2$ appears in three sequences of length three when $-1 \notin F^{\times 2}$, and in only two such sequences otherwise.

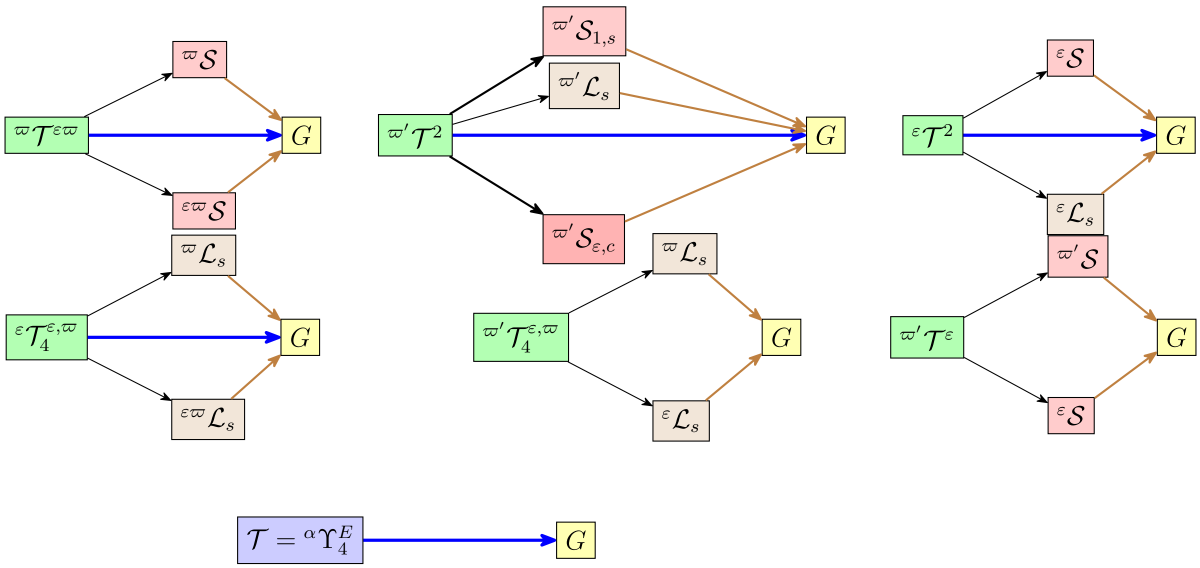

Figure 5.1: *Valid twisted Levi sequences for the construction of supercuspidal representations of* $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. *We use colors to distinguish twisted Levi subgroups of type* M_{short} *(pink) and* M_{long} *(brown). The blue line shows the sequence of length two from* $\mathcal{T}$ *to* G*, the brown line shows the sequence of length two from* $\mathcal{M}$ *to* G*, and the black with brown together show the sequence of length three.*

Chapter 6

Determining the Bruhat–Tits building of the tori of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$

In Chapter 3, we classified the conjugacy classes of elliptic maximal tori in $G = \mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$, but abstractly: each one is naturally embedded in a group G isomorphic to $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$. Our work in Chapters 4 and 5 depended only on the isomorphism class of each torus. That is, until now, we have not required an explicit embedding of our tori or twisted Levi sequences into the group $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$. However, to complete the work of constructing supercuspidal representations, we need to make a choice of embedding for each $\mathcal{T}$ and then identify the point in the Bruhat–Tits building of G that is the image of the building of $\mathcal{T}$.

To set the stage, recall that the initial subgroup G^0 of a valid twisted Levi sequence in G can either be a maximal elliptic torus or an intermediate twisted Levi subgroup with compact center. By Lemma 2.5.10, if G^0 is a maximal elliptic torus, then it is compact, and its building consists of a single point. On the other hand, by Corollary 2.5.11, if G^0 is a non-toral twisted Levi subgroup, it may be either compact or partially split. In such cases, the building is either a point or its apartments are affine lines, which contain among their points the buildings corresponding to the tori in G^0. Note that a vertex in G^0 need not to be vertex in G. We begin with the case where G^0 is a torus, that is, compact.

In Section 6.1, we show how to apply a result of Kim and Murnaghan to find the point in the building of G corresponding to a torus or a twisted Levi subgroup using the Moy–Prasad filtration. With this in mind, in Section 6.2, we construct, for each elliptic torus in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$, an explicit Witt basis, which allows us to realize the torus inside the standard form of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$. In Section 6.3, we construct explicit matrix realizations of the tori using the Witt bases obtained earlier. In Section 6.4, we combine these results to develop algorithms that determine the building of each elliptic torus in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$. In Section 6.5, we present two methods to determine the building of a twisted Levi subgroup: from the maximal elliptic tori it contains, and more directly, from the Lie algebra of its center. Finally, in Section 6.6.2, we determine the admissible points y in the building of G

that arise for each twisted Levi sequence in the construction of supercuspidal representations, as well as the finite groups of Lie type associated to a given sequence and point y.

6.1 Determining $\mathcal{B}(T)$

The goal of this section is to present a useful characterization, due to Kim and Murnaghan [KM03], of the building of a torus $\mathcal{T}$ as the set of points in the building of G for which $\mathfrak{g}_{x,r} \cap \mathfrak{t} = \mathfrak{t}_r$ (Proposition 6.1.3, below).

We start with the following definition, which generalizes Definition 5.1.2 to one that is related to a choice of point $x \in \mathcal{B}(G)$.

Definition 6.1.1. [KM03, §2.1] For $x \in \mathcal{B}(G)$, $r \in \mathbb{R}$, and $\Gamma \in \mathfrak{g}$, define the **depth of Γ in the x-filtration** by

$$d(x,\Gamma) := \sup\{r \in \mathbb{R} \mid \Gamma \in \mathfrak{g}_{x,r}\}.$$

That is, if $\Gamma \neq 0$ then $d(x,\Gamma)$ is the unique real number such that $\Gamma \in \mathfrak{g}_{x,r} \setminus \mathfrak{g}_{x,r^+}$.

Then in [KM03, Lemma 2.1.5], Kim and Murnaghan note that if $\Gamma \in \mathfrak{t}$ is a nonzero element of the Lie algebra of a torus T, then its depth $d_T(\Gamma)$ with respect to that torus, as defined in Definition 5.1.2, satisfies

$$d_T(\Gamma) := \sup_{x \in \mathcal{B}(C_G(\Gamma))} d(x,\Gamma).$$

Theorem 6.1.2 (Kim–Murnaghan [KM03, Thm. 2.3.1])**.** *Let $\mathcal{T}$ be a maximal torus and let $\Gamma \in \mathfrak{g}$ be a non-zero $\mathcal{T}$-good semisimple element. Then*

$$\{x \in \mathcal{B}(G) \mid d(x,\Gamma) = d(\Gamma)\} = \mathcal{B}\big(C_G(\Gamma)\big).$$

In the case of $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$, we have established the abundance of $\mathcal{T}$-good elements: in Section 5.3, we found all of the G-generic elements of the Lie algebra $\mathfrak{t}$ of a maximal elliptic torus ($\mathbf{Gen}_{G,\mathcal{T}}$) and of the Lie algebra $\mathfrak{z}(\mathfrak{m})$ of the center of a nonabelian proper twisted Levi subgroup $\mathcal{M}$ with compact center ($\mathbf{Gen}_{G,\mathcal{M}}$). Since all of these elements are $\mathcal{T}$-good by Proposition 5.1.9, and their centralizers are $\mathcal{T}$ and $\mathcal{M}$, respectively, the theorem applies to each of them and gives us a characterization of the buildings of these subgroups [KM03], which we state and prove explicitly for $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$.

Proposition 6.1.3. *Let $\mathcal{T}$ be a maximal elliptic torus in $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$, $\mathfrak{t}$ be the Lie algebra of $\mathcal{T}$. Then*

$$\mathcal{B}(\mathcal{T}) = \{x\} \quad \textit{if and only if} \quad \forall r \in \mathbb{R},\ \mathfrak{t}_r = \mathfrak{t} \cap \mathfrak{g}_{x,r},$$

which is equivalent to requiring the inclusion $\mathfrak{t}_r \subset \mathfrak{g}_{x,r}$ for all r.

Proof. Note that by [Yu01, Lemma 8.2], for any point $x \in \mathcal{B}(G)$ we have $\mathfrak{t} \cap \mathfrak{g}_{x,r} \subset \mathfrak{t}_r$. Therefore, given a point $x \in \mathcal{B}(G)$, it suffices to prove the inclusion $\mathfrak{t}_r \subset \mathfrak{g}_{x,r}$ for every $r \in \mathbb{R}$ to conclude equality.

Suppose now that $\mathcal{B}(T) = \{x\}$. Then the property that $\mathfrak{t}_r = \mathfrak{t} \cap \mathfrak{g}_{x,r}$ is the key compatibility property of Moy–Prasad filtrations under subgroup embeddings (see [Adl96, Proposition 1.9.1]), proving one implication.

To prove the reverse implication, let $x \in \mathcal{B}(G)$ satisfy $\mathfrak{t} \cap \mathfrak{g}_{x,r} = \mathfrak{t}_r$ for all $r \in \mathbb{R}$.

We begin by supposing that Γ is a $\mathcal{T}$-good element of depth r, which exists for every tamely ramified torus in SO_5 by [Adl96, Proposition 2.2.5]. Set $\mathcal{M} = C_G(\Gamma)$, its centralizer. Then $\Gamma \in \mathfrak{t}_r \smallsetminus \mathfrak{t}_{r+}$. By hypothesis, this implies $\Gamma \in \mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r+}$. Therefore by Definition 6.1.1 we have $d(x, \Gamma) = r = d_T(\Gamma)$, which implies by Theorem 6.1.2, that $x \in \mathcal{B}(C_G(\Gamma))$.

If in the preceding step we could choose Γ to be regular then $C_G(\Gamma) = \mathcal{T}$ and we are done. Otherwise, $M = C_G(\Gamma) \subsetneq G$ is a proper twisted Levi subgroup and we proceed as follows. We wish to choose a $\mathcal{T}$-good element $\Gamma' \in \mathfrak{t}_r \smallsetminus \mathfrak{t}_{r+}$ for some $r \in \mathbb{R}$, where now $\mathcal{T}$-good is defined relative to the root system of the smaller subgroup M, which yields a larger class of $\mathcal{T}$-good elements. In particular, since by [Adl96, Proposition 2.2.5] every coset of $\mathfrak{t}_r/\mathfrak{t}_{r+}$ contains a $\mathcal{T}$-good element, we may choose a coset that does not contain an element of the one-dimensional subspace $\mathfrak{z}(\mathfrak{m})$ of $\mathfrak{t}$. Doing so ensures that $C_M(\Gamma') = \mathcal{T}$. As before, our hypothesis allows us to conclude that $d(x, \Gamma') = r$, and therefore by Theorem 6.1.2, that $x \in \mathcal{B}(C_G(\Gamma')) = \mathcal{B}(\mathcal{T})$, as required. ∎

6.2 Witt bases for elliptic tori in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$

We established the complete list of conjugacy classes of maximal elliptic tori in the split group $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ in Section 3.2.3. Each such torus $\mathcal{T}$ was constructed from an algebra $\mathcal{A}$ and embeds, with respect to a basis B of $\mathcal{A}$, into an orthogonal group $\mathrm{SO}(q)$, where the preserved quadratic form q is isometric to $2\mathbb{H} \oplus \langle 1 \rangle$. In order to embed $\mathcal{T}$ into $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, we must represent the torus with respect to a Witt basis for $\mathcal{A}$ instead. Recall from Definition 2.2.10 that a Witt basis

$$B^* = \{e_1, e_2, f_1, f_2, g_1\} \subset \mathcal{A}$$

for $2\mathbb{H} \oplus \langle 1 \rangle$ satisfies

$$(e_i, e_i) = (f_i, f_i) = 0, \qquad (e_i, f_i) = 1, \qquad (g_1, g_1) = 1,$$

with all other undetermined pairings equal to zero.

6.2.1 Quadratic forms preserved by maximal elliptic tori

To produce the Witt bases, we need to describe the forms q preserved by each torus, and we will need to identify its isometry with $2\mathbb{H} \oplus \langle 1 \rangle$ explicitly. For the convenience of the reader, we reproduce Table 2.3 here, as Table 6.1, which lists the 2-dimensional quadratic forms that arise. Note that we refer to forms of the type p_1^λ and p_3^λ (for some $\lambda \in F^\times/F^{\times 2}$) as *diagonal* forms, and those of type p_2^λ as *cross-term* forms.

When we classified the maximal elliptic tori of G in Chapter 3 we listed them in Table 3.7 without specifying the explicit associated quadratic forms. To obtain these forms, we must return to Tables 3.3 and 3.5, which gave the explicit forms for SO_4, and insert the appropriate parameters from Table 3.7. Doing so gives the first three columns of Table 6.2 (in the case that -1 is a square) and Table 6.3 (when -1 is not a square).

In the final column of these tables, we identify the form q as a sum of forms p_i^λ from Table 6.1. Note that only the forms p_i^λ for $i = 1, 2, 3$ arise in the case $-1 \in F^{\times 2}$, while the forms p_i^λ for $i = 1, 2, 4, 5, 6$ occur when $-1 \notin F^{\times 2}$. This structure is made transparent by the color coding used in the tables. When the result is $p_i^{\lambda_1} \oplus p_j^{\lambda_2}$, we order the forms so that $\nu(\lambda_1) \geq \nu(\lambda_2)$. As we will explain in Section 6.4, this will help ensure that the resulting point in the building corresponds to a point in the fundamental domain.

We illustrate this process explicitly in several examples in the following sections, in the course of determining a corresponding Witt basis. We choose to construct the Witt basis according to whether -1 is a square in F, as this makes the distinction between the two cases transparent.

Quadratic Form	-1	Target form	Witt basis
$p_1^\lambda(v_1\mathbf{u} + v_2\mathbf{w}) = \lambda(v_1^2 - v_2^2)$	any	$\mathbb{H}$	$e = (\mathbf{u} + \mathbf{w})$ $f = \frac{1}{2}\lambda^{-1}(\mathbf{u} - \mathbf{w})$
$p_2^\lambda(v_1\mathbf{u} + v_2\mathbf{w}) = 2\lambda v_1 v_2$	any	$\mathbb{H}$	$e = \mathbf{u}$ $f = \lambda^{-1}\mathbf{w}$
$p_3^\lambda(v_1\mathbf{u} + v_2\mathbf{w}) = \lambda\,(v_1^2 + v_2^2)$	$-1^{\square}$	$\mathbb{H}$	$e = (\mathbf{u} + i\mathbf{w})$ $f = \frac{1}{2}\lambda^{-1}(\mathbf{u} - i\mathbf{w})$
$p_4^\lambda(v_1\mathbf{u} + v_2\mathbf{w}) = \lambda(v_1^2 + v_2^2)$	$-1^{\boxtimes}$	$\langle -\lambda, -\lambda\rangle$	$g_1 = s^2\mathbf{u} + t^2\mathbf{w}$ $g_2 = t^2\mathbf{u} - s^2\mathbf{w}$
$p_5^\lambda(v_1\mathbf{u} + v_2\mathbf{w}) =$ $\lambda(-s^2v_1^2 + 2t^2 v_1 v_2 + s^2 v_2^2)$	$-1^{\boxtimes}$	$\langle \lambda, \lambda\rangle$	$g_1 = s\mathbf{u} - t^2 s^{-1}\mathbf{w}$ $g_2 = s^{-1}\mathbf{w}$
$p_6^\lambda(v_1\mathbf{u} + v_2\mathbf{w}) =$ $\lambda((s^4 - t^4)(v_2^2 - v_1^2) + 4s^2t^2\, v_1 v_2)$	$-1^{\boxtimes}$	$\mathbb{H}$	$e = (s^2 - t^2)\mathbf{u} - (s^2 + t^2)\mathbf{w}$ $f = -\dfrac{s^2 + t^2}{2\lambda}\mathbf{u} - \dfrac{s^2 - t^2}{2\lambda}\mathbf{w}$

Table 6.1: *Frequently occurring binary quadratic forms, with respect to a basis* $\{\mathbf{u}, \mathbf{w}\}$. *For each one, we list a different quadratic form in its isometry class and the corresponding change of basis, reproduced from Section 2.2. Here* $\lambda \in F^\times/F^{\times 2}$ *and* $i = \sqrt{-1}$. *When* $-1 \notin F^{\times 2}$, *indicated by* $-1^{\boxtimes}$, *then* s, t *satisfy* $s^2 + it^2 = \varepsilon_{E_1}$ *and* $s^4 + t^4 = -1$.

6.2.2 Witt basis for elliptic tori in Table 6.2.

When $-1 \in F^{\times 2}$, so that $i=\sqrt{-1} \in F$, all the forms p_j^{λ} that arise are hyperbolic, that is, each is isometric to $\mathbb{H}$. Hence, a direct application of Table 6.1 yields a Witt basis for each elliptic torus.

We illustrate how to read Table 6.2 through two representative examples, one from a sum of quadratic extensions and another from a quartic extension.

Example 6.2.1. Consider the elliptic torus ${}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi,1]}$, which arises from the algebra $\mathcal{A} = F(\varepsilon) \oplus F(\varepsilon) \oplus F$. With respect to the basis

$$B = \{u_1, u_2, u_3, u_4, u_5\} = \{(1,0,0), (\sqrt{\varepsilon},0,0), (0,1,0), (0,\sqrt{\varepsilon},0), (0,0,1)\},$$

of $\mathcal{A}$, the quadratic form preserved is

$$q(v_1u_1 + v_2u_2 + v_3u_3 + v_4u_4 + v_5u_5) = \varpi v_1^2 - \varepsilon\varpi v_2^2 + \varpi v_3^2 - \varepsilon\varpi v_4^2 + v_5^2$$

where each $v_i \in F$. Using Table 6.1 we see that this quadratic form decomposes as $p_3^{\varpi}(v_1,v_3) + p_3^{-\varepsilon\varpi}(v_2,v_4) + v_5^2$, which we record in the last column of Table 6.2.

We now write down a Witt basis B^* of $\mathcal{A}$ using the last column of Table 6.1, which defines which linear combinations of the vectors of B to use. We obtain the Witt basis $B^* = \{e_1, e_2, f_1, f_2, g_1\}$ where

$$e_1 = (1, i, 0), \quad e_2 = (-\sqrt{\varepsilon}, -i\sqrt{\varepsilon}, 0),$$
$$f_1 = \left(\tfrac{1}{2\varpi}, -\tfrac{i}{2\varpi}, 0\right), \quad f_2 = \left(\tfrac{\sqrt{\varepsilon^{-1}}}{2\varpi}, -\tfrac{i\sqrt{\varepsilon^{-1}}}{2\varpi}, 0\right), \quad g_1 = (0,0,1).$$

In general, for every torus $\mathcal{T}$ such that with respect to some permutation $\{u_1, v_1, u_2, v_2, g\}$ of our fixed basis of $\mathcal{A}$ we obtained a quadratic form $q = p_{i_1}^{w_1} \oplus p_{i_2}^{w_2} \oplus \langle 1 \rangle$, with $i_1, i_2 \in \{1,3\}$ the resulting Witt basis takes the form

$$e_1 = u_1 + \gamma_{i_1}v_1, \quad f_1 = \frac{u_1 - \gamma_{i_1}v_1}{2w_1}, \qquad e_2 = u_2 + \gamma_{i_2}v_2, \quad f_2 = \frac{u_2 - \gamma_{i_2}v_2}{2w_2}, \qquad g \tag{6.2.1}$$

where $\gamma_{i_j} = 1$ if $i_j = 1$ and $\gamma_{i_j} = i$ if $i_j = 3$. These Witt bases are recorded in the first two columns of Table A.3.

Example 6.2.2. Consider the elliptic torus ${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon},1]}$, which acts on the algebra $\mathcal{A} = F(\sqrt{\varepsilon},\sqrt{\varpi}) \oplus F$. With respect to the basis

$$B = \{(1,0), (\sqrt{\varepsilon},0), (\sqrt{\varpi},0), (\sqrt{\varepsilon\varpi},0), (0,1)\} = \{u_1, \sqrt{\varepsilon}\,u_1, u_2, \sqrt{\varepsilon}\,u_2, u_3\},$$

of $\mathcal{A}$ the associated quadratic form preserved is

$$q(v_1u_1 + v_2\sqrt{\varepsilon}\,u_1 + v_3u_2 + v_4\sqrt{\varepsilon}\,u_2 + v_5u_3) = \varepsilon(2v_1v_2 - 2\varpi v_3v_4) + v_5^2,$$

where each $v_i \in F$. Using Table 6.1, the form decomposes as a sum of two hyperbolic planes $p_2^{\varepsilon}(v_1u_1 + v_2\sqrt{\varepsilon}u_1)$ and $p_2^{-\varepsilon\varpi}(v_3u_2 + v_4\sqrt{\varepsilon}u_2)$ and anisotropic line $q(u_3) = 1 = g_1$. Here, $\nu(\varepsilon\varpi) > \nu(\varepsilon)$ so we order the two hyperbolic planes so that the decomposition is given as $p_2^{-\varepsilon\varpi}(v_3u_2 + v_4\sqrt{\varepsilon}u_2) + p_2^{\varepsilon}(v_1u_1 + v_2\sqrt{\varepsilon}u_1) + v_5^2$, and record this in the last column of Table 6.2.

Applying now the change of basis from Table 6.1, we obtain the Witt basis

$$e_1 = (\sqrt{\varpi}, 0), \quad e_2 = (1, 0),$$
$$f_1 = (-\sqrt{\varepsilon\varpi}^{-1}, 0), \quad f_2 = (\sqrt{\varepsilon^{-1}}, 0), \quad g_1 = (0, 1).$$

All remaining tori in Table 6.2 are treated in the same way as these two examples: we order the hyperbolic pairs so that the parameters have non-increasing valuation, and for each hyperbolic component of the quadratic form, we apply the appropriate change of basis to obtain the corresponding Witt pairs. The chosen Witt bases for all tori in this setting are collected in Tables A.3, A.5 , A.6.

6.2.3 Witt basis for elliptic tori in Table 6.3

We now construct Witt bases for the conjugacy classes of elliptic tori in the case $-1 \notin F^{\times 2}$. Throughout this subsection we set $\varepsilon = -1$ and write $\sqrt{\varepsilon} = i$. When E/F is unramified and -1 is not a square, recall that we choose $\varepsilon_E = s^2 + it^2 \in \mathcal{O}_E^{\times}$ as a nonsquare unit, with $s, t \in F$ and $s^4 + it^4$. There are four fewer conjugacy classes of elliptic tori, compared to the case $-1 \in F^{\times 2}$.

When the form q preserved by $\mathcal{T}$ is a sum of forms from the set $\{p_1^{\lambda}, p_2^{\lambda}, p_3^{\lambda} | \lambda \in F^{\times}/F^{\times 2}\}$, the approach is the same as in the previous section. However, the construction of a Witt basis $\{e_1, e_2, f_1, f_2, g_1\}$ requires additional care in six cases, where the quadratic form involves the types p_i^{λ} for $i \in \{4, 5, 6\}$ from Table 6.1. The key difference is that the forms p_4^{λ} and p_5^{λ} are not hyperbolic; rather, they are isometric to $\langle \lambda, \lambda \rangle$, which is not isometric to $\mathbb{H}$ when $-1 \notin F^{\times 2}$. Instead, we must combine forms and make a sequence of transformations to obtain the result.

For example, when $-1 \notin F^{\times 2}$, we have the isometry

$$\langle \lambda, \lambda, \lambda, \lambda \rangle \simeq 2\mathbb{H}.$$

The form on the left is $p_4^{\lambda} \oplus p_4^{\lambda}$. Choosing the change of basis in Table 6.1 on the *first* of these two factors yields a new diagonal form $\langle -\lambda, -\lambda, \lambda, \lambda \rangle$. Reordering the basis yields the form $p_1^{\lambda} \oplus p_1^{\lambda}$ which by the change of basis in Table 6.1 is isometric to $2\mathbb{H}$.

Thus these cases require additional work to construct Witt bases for the elliptic tori admitting these forms, which are exactly the tori

$${}^{\varepsilon}\mathcal{T}^{2}_{[1,1,1]}, \ {}^{\varepsilon}\mathcal{T}^{2}_{[\varpi,\varpi,1]}, \ {}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[\varepsilon,-1,\varepsilon]}, \ {}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}, \ {}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[1,\varepsilon]}, \ {}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\eta,\varepsilon]}.$$

$\mathcal{T}$	$v \in \mathcal{T}$	**Quadratic form** q	**Decomposition**
${}^{\varepsilon}\mathcal{T}^{2}_{[1,1,1]}$	$(v_1 + v_2\sqrt{\varepsilon}, v_3 + v_4\sqrt{\varepsilon}, v_5)$	$v_1^2 - \varepsilon v_2^2 + v_3^2 - \varepsilon v_4^2 + v_5^2$	$p_3^{1}(v_1, v_3)+ p_3^{-\varepsilon}(v_2, v_4) +v_5^2$
${}^{\varepsilon}\mathcal{T}^{2}_{[\varpi,\varpi,1]}$		$\varpi v_1^2 - \varepsilon\varpi v_2^2 + \varpi v_3^2 - \varepsilon\varpi v_4^2 + v_5^2$	$p_3^{\varpi}(v_1, v_3)+ p_3^{-\varepsilon\varpi}(v_2, v_4) +v_5^2$
${}^{\varpi}\mathcal{T}^{2}_{[1,1,1]}$	$(v_1 + v_2\sqrt{\varpi}, v_3 + v_4\sqrt{\varpi}, v_5)$	$v_1^2 - \varpi v_2^2 + v_3^2 - \varpi v_4^2 + v_5^2$	$p_3^{-\varpi}(v_2, v_4)+ p_3^{1}(v_1, v_3)+v_5^2$
${}^{\varpi}\mathcal{T}^{2}_{[\varepsilon,\varepsilon,1]}$		$\varepsilon v_1^2 - \varepsilon\varpi v_2^2 + \varepsilon v_3^2 - \varepsilon\varpi v_4^2 + v_5^2$	$p_3^{-\varepsilon\varpi}(v_2, v_4)+p_3^{\varepsilon}(v_1, v_3)+v_5^2$
${}^{\varepsilon\varpi}\mathcal{T}^{2}_{[1,1,1]}$	$(v_1 + v_2\sqrt{\varepsilon\varpi}, v_3 + v_4\sqrt{\varepsilon\varpi}, v_5)$	$v_1^2 - \varepsilon\varpi v_2^2 + v_3^2 - \varepsilon\varpi v_4^2 + v_5^2$	$p_3^{-\varepsilon\varpi}(v_2, v_4)+ p_3^{1}(v_1, v_3)+v_5^2$
${}^{\varpi}\mathcal{T}^{2}_{[\varepsilon,\varepsilon,1]}$		$\varepsilon v_1^2 - \varepsilon^2\varpi v_2^2 + \varepsilon v_3^2 - \varepsilon^2\varpi v_4^2 + v_5^2$	$p_3^{-\varepsilon^2\varpi}(v_2, v_4)+ p_3^{\varepsilon}(v_1, v_3)+v_5^2$
${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[1,\varepsilon,\varepsilon]}$	$(v_1 + v_2\sqrt{\varpi}, v_3 + v_4\sqrt{\varepsilon\varpi}, v_5)$	$v_1^2 - \varpi v_2^2 + \varepsilon v_3^2 - \varepsilon^2\varpi v_4^2 + \varepsilon v_5^2$	$p_3^{-\varpi}(v_2, v_4)+, p_3^{\varepsilon}(v_3, v_5)+v_1^2$
${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[\varepsilon,1,\varepsilon]}$		$\varepsilon v_1^2 - \varepsilon\varpi v_2^2 + v_3^2 - \varepsilon\varpi v_4^2 + \varepsilon v_5^2$	$p_3^{-\varepsilon\varpi}(v_2, v_4)+p_3^{\varepsilon}(v_1, v_5)+v_3^2$
${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$	$(v_1 + v_2\sqrt{\varepsilon\varpi}, v_3 + v_4\sqrt{\varepsilon}, v_5)$	$v_1^2 - \varepsilon\varpi v_2^2 + \varpi v_3^2 - \varepsilon\varpi v_4^2 + \varpi v_5^2$	$p_3^{-\varepsilon\varpi}(v_2, v_4)+p_3^{\varpi}(v_3, v_5)+v_1^2$
${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varpi]}$		$\varepsilon v_1^2 - \varepsilon^2\varpi v_2^2 + v_3^2 - \varepsilon v_4^2 + \varpi v_5^2$	$p_1^{\varpi}(v_2, v_5)+ p_1^{\varepsilon}(v_1, v_4)+v_3^2$
${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varepsilon\varpi]}$	$(v_1 + v_2\sqrt{\varpi}, v_3 + v_4\sqrt{\varepsilon}, v_5)$	$v_1^2 - \varpi v_2^2 + \varpi v_3^2 - \varepsilon\varpi v_4^2 + \varepsilon\varpi v_5^2$	$p_1^{\varpi}(v_2, v_3)+ p_1^{\varepsilon\varpi}(v_4, v_5)+v_1^2$
${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varepsilon\varpi]}$		$\varepsilon v_1^2 - \varepsilon\varpi v_2^2 + v_3^2 - \varepsilon v_4^2 + \varepsilon\varpi v_5^2$	$p_1^{\varepsilon\varpi}(v_2, v_5), p_1^{\varepsilon}(v_1, v_4)+v_3^2$
${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon},1]}$	$(v_1 + v_2\sqrt{\varepsilon} + (v_3 + v_4\sqrt{\varepsilon})\sqrt{\varpi}, v_5)$	$\varepsilon(2v_1v_2 - 2\varpi v_3v_4) + v_5^2$	$p_2^{-\varepsilon\varpi}(v_3, v_4)+p_2^{\varepsilon}(v_1, v_2) +v_5^2$
${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$	$(v_1 + v_2\sqrt{\varpi} + (v_3 + v_4\sqrt{\varpi})\sqrt{\varepsilon}, v_5)$	$\varpi(2v_1v_2 - 2\varepsilon v_3v_4) + v_5^2$	$p_2^{-\varepsilon\varpi}(v_3, v_4)+p_2^{\varpi}(v_1, v_2)+v_5^2$
${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]}$	$(v_1 + v_2\sqrt{\varepsilon\varpi} + (v_3 + v_4\sqrt{\varepsilon\varpi})\sqrt{\varepsilon}, v_5)$	$\varepsilon\varpi(2v_1v_2 - 2\varepsilon v_3v_4) + v_5^2$	$p_2^{\varepsilon\varpi}(v_1, v_2)+p_2^{-\varepsilon^2\varpi}(v_3, v_4)+v_5^2$
${}^{\varepsilon}\Upsilon^{\varepsilon E_1}_{4[1,\varepsilon]}$	$(v_1 + v_2\sqrt{\varepsilon} + (v_3 + v_4\sqrt{\varepsilon})\sqrt[4]{\varepsilon}, v_5)$	$v_1^2 + \varepsilon v_2^2 - 2\varepsilon v_3v_4 + \varepsilon v_5^2$	$p_3^{\varepsilon}(v_2, v_5)+p_2^{-\varepsilon}(v_3, v_4)+v_1^2$
${}^{\varepsilon}\Upsilon^{\varepsilon E_1\varpi}_{4[1,\varepsilon]}$	$(v_1 + v_2\sqrt{\varepsilon} + (v_3 + v_4\sqrt{\varepsilon})\sqrt[4]{\varepsilon\varpi^2}, v_5)$	$v_1^2 + \varepsilon v_2^2 - 2\varepsilon\varpi v_3v_4 + \varepsilon v_5^2$	$p_2^{-\varepsilon\varpi}(v_3, v_4)+ p_3^{\varepsilon}(v_2, v_5)+v_1^2$
${}^{\varpi}\Upsilon^{\varpi}_{4[1,\varpi]}$	$(v_1 + v_2\sqrt{\varpi} + (v_3 + v_4\sqrt{\varpi})\sqrt[4]{\varpi}, v_5)$	$v_1^2 + \varpi v_2^2 - 2\varpi v_3v_4 + \varpi v_5^2$	$p_3^{\varpi}(v_2, v_5)+p_2^{-\varpi}(v_3, v_4)+v_1^2$
${}^{\varpi}\Upsilon^{\varepsilon^2\varpi}_{4[1,\varpi]}$	$(v_1 + v_2\sqrt{\varpi} + (v_3 + v_4\sqrt{\varpi})\sqrt[4]{\varepsilon^2\varpi}, v_5)$	$v_1^2 + \varpi v_2^2 - 2\varepsilon\varpi v_3v_4 + \varpi v_5^2$	$p_3^{\varpi}(v_2, v_5)+p_2^{-\varepsilon\varpi}(v_3, v_4)+v_1^2$
${}^{\varepsilon\varpi}\Upsilon^{\varepsilon\varpi}_{4[1,\varepsilon\varpi]}$	$(v_1 + v_2\sqrt{\varepsilon\varpi} + (v_3 + v_4\sqrt{\varepsilon\varpi})\sqrt[4]{\varepsilon\varpi}, v_5)$	$v_1^2 + \varepsilon\varpi v_2^2 - 2\varepsilon\varpi v_3v_4 + \varepsilon\varpi v_5^2$	$p_3^{\varepsilon\varpi}(v_2, v_5)+p_2^{-\varepsilon\varpi}(v_3, v_4)+v_1^2$
${}^{\varepsilon\varpi}\Upsilon^{\varepsilon^3\varpi}_{4[1,\varepsilon\varpi]}$	$(v_1 + v_2\sqrt{\varepsilon\varpi} + (v_3 + v_4\sqrt{\varepsilon\varpi})\sqrt[4]{\varepsilon^3\varpi}, v_5)$	$v_1^2 + \varepsilon\varpi v_2^2 - 2\varepsilon^2\varpi v_3v_4 + \varepsilon\varpi v_5^2$	$p_3^{\varepsilon\varpi}(v_2, v_5)+p_2^{-\varepsilon^2\varpi}(v_3, v_4)+v_1^2$

Table 6.2: *Elliptic tori in* $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$ *when* $-1 \in F^{\times 2}$. *For each elliptic torus, we give the explicit quadratic form and then the decomposition of the quadratic form by the forms displayed in Table 6.1. We use the blue color as a code for the diagonal forms* p_1^{λ} *and* p_3^{λ}*, and the red color to code the cross-term forms* p_2^{λ}*.*

These cases are indicated by $*$ in Table 6.3. We illustrate the steps in constructing the appropriate Witt basis for three of these tori, the other three being quite similar.

Example 6.2.3. Consider the elliptic torus ${}^{\varepsilon}\mathcal{T}^{2}_{[\varpi,\varpi,1]}$. With respect to the basis

$$B = \{(1, 0, 0), (i, 0, 0), (0, 1, 0), (0, i, 0), (0, 0, 1)\} = \{u_1, u_2, u_3, u_4, u_5\},$$

the preserved quadratic form is

$$q(v_1u_1 + v_2u_2 + v_3u_3 + v_4u_4 + v_5u_5) = \varpi v_1^2 + \varpi v_2^2 + \varpi v_3^2 + \varpi v_4^2 + v_5^2.$$

Using Table 6.1, the form decomposes into two binary forms of type p_4^{ϖ}. We first consider the subspace $\mathrm{span}\{u_1, u_3\}$, on which

$$q(v_1u_1 + v_3u_3) = \varpi v_1^2 + \varpi v_3^2.$$

$\mathcal{T}$	$v \in \mathcal{T}$	**Quadratic form** q	**Form decomposition**
${}^{\varepsilon}\mathcal{T}^2_{[1,1,1]}$	$(v_1 + v_2 i,\ v_3 + v_4 i,\ v_5)$	$v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2$	$*\quad p_4^1(v_1, v_3) + p_4^1(v_2, v_4) + v_5^2$
${}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi,1]}$		$\varpi v_1^2 + \varpi v_2^2 + \varpi v_3^2 + \varpi v_4^2 + v_5^2$	$*\quad p_4^{\varpi}(v_1, v_3) + p_4^{\varpi}(v_2, v_4) + v_5^2$
${}^{\varpi}\mathcal{T}^2_{[1,\varepsilon,1]}$	$(v_1 + v_2\sqrt{\varpi},\ v_3 + v_4\sqrt{\varpi},\ v_5)$	$v_1^2 - \varpi v_2^2 - v_3^2 + \varpi v_4^2 + v_5^2$	$p_1^{\varpi}(v_2, v_4) + p_1^1(v_1, v_3) + v_5^2$
${}^{\varepsilon\varpi}\mathcal{T}^2_{[1,\varepsilon,1]}$	$(v_1 + v_2 i\sqrt{\varpi},\ v_3 + v_4 i\sqrt{\varpi},\ v_5)$	$v_1^2 + \varpi v_2^2 - v_3^2 - \varpi v_4^2 + v_5^2$	$p_1^{\varpi}(v_2, v_4) + p_1^1(v_1, v_3) + v_5^2$
${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[\varepsilon,\varepsilon,\varepsilon]}$	$(v_1 + v_2\sqrt{\varpi},\ v_3 + v_4 i\sqrt{\varpi},\ v_5)$	$-v_1^2 + \varpi v_2^2 - v_3^2 - \varpi v_4^2 - v_5^2$	$*\quad p_1^{\varpi}(v_2, v_4) + p_4^{-1}(v_1, v_3) - v_5^2$
${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[1,1,\varepsilon]}$		$v_1^2 - \varpi v_2^2 + v_3^2 + \varpi v_4^2 - v_5^2$	$p_1^{-\varpi}(v_2, v_4) + p_1^1(v_3, v_5) + v_1^2$
${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$	$(v_1 + v_2 i\sqrt{\varpi},\ v_3 + v_4 i,\ v_5)$	$v_1^2 + \varpi v_2^2 + \varpi v_3^2 + \varpi v_4^2 + \varpi v_5^2$	$*\quad p_4^{\varpi}(v_2, v_4) + p_4^{\varpi}(v_3, v_5) + v_1^2$
${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[-1,1,\varpi]}$		$-v_1^2 - \varpi v_2^2 + v_3^2 + v_4^2 + \varpi v_5^2$	$p_1^{-\varpi}(v_2, v_5) + p_1^{-1}(v_1, v_4) + v_3^2$
${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varepsilon\varpi]}$	$(v_1 + v_2\sqrt{\varpi},\ v_3 + v_4 i,\ v_5)$	$v_1^2 - \varpi v_2^2 + \varpi v_3^2 + \varpi v_4^2 - \varpi v_5^2$	$p_1^{-\varpi}(v_2, v_4) + p_1^{\varpi}(v_3, v_5) + v_1^2$
${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varepsilon\varpi]}$		$-v_1^2 + \varpi v_2^2 + v_3^2 + v_4^2 - \varpi v_5^2$	$p_1^{\varpi}(v_2, v_5) + p_1^1(v_1, v_4) + v_3^2$
${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[1,1]}$	$(v_1 + v_2 i + (v_3 + v_4 i)\sqrt{\varpi},\ v_5)$	$v_1^2 - v_2^2 - \varpi v_3^2 + \varpi v_4^2 + v_5^2$	$p_1^{-\varpi}(v_3, v_4) + p_1^1(v_1, v_2) + v_5^2$
${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$	$(v_1 + v_2\sqrt{\varpi} + (v_3 + v_4\sqrt{\varpi})i,\ v_5)$	$\varpi(2v_1v_2 + 2v_3v_4) + v_5^2$	$p_2^{\varpi}(v_1, v_2) + p_2^{\varpi}(v_3 v_4) + v_5^2$
${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[i\sqrt{\varpi},1]}$	$(v_1 + v_2 i\sqrt{\varpi} + (v_3 + v_4 i\sqrt{\varpi})i,\ v_5)$	$-\varpi(2v_1v_2 + 2v_3v_4) + v_5^2$	$p_2^{-\varpi}(v_1, v_2),\quad p_2^{-\varpi}(v_3, v_4) + v_5^2$
${}^{\varepsilon}\Upsilon^{\varepsilon E_1}_{4[1,\varepsilon]}$	$(v_1 + v_2\sqrt{\varepsilon} + (v_3 + v_4\sqrt{\varepsilon})\sqrt[4]{\varepsilon},\ v_5)$	$v_1^2 - v_2^2 + p_5^1(v_3, v_4) - v_5^2$	$*\quad p_1^1(v_1, v_2) + p_5^1(v_3, v_4) - v_5^2$
${}^{\varepsilon}\Upsilon^{\varepsilon E_1 \varpi}_{4[\varepsilon_{E_1},\varepsilon]}$	$(v_1 + v_2\sqrt{\varepsilon} + (v_3 + v_4\sqrt{\varepsilon})\sqrt[4]{\varepsilon\varpi^2},\ v_5)$	$p_5^1(v_2, -v_1) + p_6^{\varpi}(v_3, v_4) - v_5^2$	$*\quad p_6^{\varpi}(v_3, v_4) + p_5^1(v_2, -v_1) - v_5^2$
${}^{\varpi}\Upsilon^{\varpi}_{4[1,\varepsilon\varpi]}$	$(v_1 + v_2\sqrt{\varpi} + (v_3 + v_4\sqrt{\varpi})\sqrt[4]{\varpi},\ v_5)$	$v_1^2 + \varpi v_2^2 - 2\varpi v_3v_4 - \varpi v_5^2$	$p_1^{\varpi}(v_2, v_5) + p_2^{-\varpi}(v_3, v_4) + v_1^2$
${}^{\varepsilon\varpi}\Upsilon^{\varepsilon\varpi}_{4[1,\varpi]}$	$(v_1 + v_2 i\sqrt{\varpi} + (v_3 + v_4 i\sqrt{\varpi})\sqrt[4]{-\varpi},\ v_5)$	$v_1^2 - \varpi v_2^2 + 2\varpi v_3v_4 + \varpi v_5^2$	$p_1^{-\varpi}(v_2, v_5) + p_2^{\varpi}(v_3, v_4) + v_1^2$

Table 6.3: *Elliptic tori in* $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$ *when* $-1 \notin F^{\times 2}$*. For each elliptic torus, we give the explicit quadratic form, followed by its decomposition using the forms listed in Table 6.1. We use blue to denote the diagonal forms* p_1^{λ} *and* p_6^{λ}*, and red to denote the cross-term forms* p_2^{λ}*. The remaining cases correspond to forms that are isometric to* $\langle 1, 1\rangle$ *and require a change of basis to show that the full form is isometric to* $2\mathbb{H}\oplus\langle 1\rangle$*.*

Applying the change of basis corresponding to $p_4^{\varpi}(v_1u_1 + v_3u_3)$ yields the vectors

$$h_1 = s^2u_1 + t^2u_3, \qquad h_2 = t^2u_1 - s^2u_3.$$

The vectors h_1 and h_2 form a basis of $\mathrm{span}\{u_1, u_3\}$, with respect to which the restriction of q is isometric to $\langle -\varpi, -\varpi\rangle$. By choosing the new ordered basis

$$\tilde{B} = \{h_1, u_2, h_2, u_4, u_5\} = \{(s^2, t^2, 0), (i, 0, 0), (t^2, -s^2, 0), (0, i, 0), (0, 0, 1)\}.$$

Our form is now given by

$$q \simeq \langle -\varpi, \varpi, -\varpi, \varpi, 1\rangle. \tag{6.2.2}$$

Finally, we consider the subspaces $\mathrm{span}\{h_1, u_2\}$ and $\mathrm{span}\{h_2, u_4\}$ which each carry the form $p_1^{-\varpi}$. Applying the corresponding change of basis from Table 6.1 yields the Witt basis vectors

$$e_1 = h_1 + u_2 = (s^2 + i, t^2, 0), \qquad f_1 = \frac{u_2 - h_1}{2\varpi} = \left(\frac{i - s^2}{2\varpi}, \frac{-t^2}{2\varpi}, 0\right),$$

$$e_2 = h_2 + u_4 = (t^2, i - s^2, 0), \qquad f_2 = \frac{u_4 - h_2}{2\varpi} = \left(\frac{-t^2}{2\varpi}, \frac{i + s^2}{2\varpi}, 0\right),$$

together with $g = (0, 0, 1)$ since $q(g) = 1$.

A similar construction to the elliptic tori ${}^{\varepsilon}\mathcal{T}^{2}_{[1,1,1]}$, ${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[\varepsilon,\varepsilon,\varepsilon]}$, and ${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$, where in all of these case the key is to first transform one of the factors p_4^{λ} to the isometric form $\langle -\lambda, -\lambda\rangle$ and then combine with another term to produce a form p_1^{λ}, to which a second change of basis yields our required Witt basis.

Now we compute the Witt basese for the two remaining starred cases from Table 6.3.

Example 6.2.4. Consider the elliptic torus ${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[1,\varepsilon]}$. With respect to the basis

$$B = \{(1,0),(i,0),(\sqrt{\varepsilon_{E_1}},0),(i\sqrt{\varepsilon_{E_1}},0),(0,1)\} = \{u_1,u_2,u_3,u_4,u_5\},$$

the preserved quadratic form, using Table 3.5 is

$$q(v_1u_1+v_2u_2+v_3u_3+v_4u_4+v_5u_5) = v_1^2 - v_2^2 + p_5^1(v_3u_3+v_4u_4) - v_5^2.$$

We first consider the subspace $\operatorname{span}\{u_3,u_4\}$. Since

$$p_5^1(v_3u_3+v_4u_4) \simeq \langle 1,1\rangle,$$

applying the corresponding change of basis from Table 6.1 yields the vectors

$$h_1 = s\,u_3 - t^2s^{-1}u_4, \qquad h_2 = s^{-1}u_4,$$

which form a basis of $\operatorname{span}\{u_3,u_4\}$. If we now choose the ordered basis $\tilde{B} = \{u_1,u_2,h_1,h_2,u_5\}$, then with respect to this basis, q is given by $\langle 1,-1,1,1,-1\rangle$. Finally, we consider the subspaces $\operatorname{span}\{u_1,u_2\}$ and $\operatorname{span}\{h_2,u_5\}$. Applying the change of basis corresponding to p_1^1 to each gives the Witt basis

$$e_1 = u_1+u_2 = (1+i,0), \qquad f_1 = \frac{u_1-u_2}{2} = \left(\frac{1-i}{2},0\right),$$

$$e_2 = h_2+u_5 = \left(s^{-1}i\sqrt{\varepsilon_{E_1}},1\right), \qquad f_2 = \frac{h_2-u_5}{2} = \left(\frac{s^{-1}i\sqrt{\varepsilon_{E_1}}}{2},\frac{-1}{2}\right),$$

together with $g_1 = h_1 = \left(s\sqrt{\varepsilon_{E_1}} - t^2s^{-1}i\sqrt{\varepsilon_{E_1}},0\right)$.

In the final case, we require the use of the most complex form, p_6.

Example 6.2.5. Consider the elliptic torus ${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\eta,\varepsilon]}$. With respect to the basis

$$B = \{(1,0),(i,0),(\sqrt{\varepsilon_{E_1}\varpi},0),(i\sqrt{\varepsilon_{E_1}\varpi},0),(0,1)\} = \{u_1,u_2,u_3,u_4,u_5\},$$

the preserved quadratic form is, again as precomputed in Table 3.5, is

$$q(v_1u_1+v_2u_2+v_3u_3+v_4u_4+v_5u_5) = p_5^1(v_2u_2-v_1u_1)+p_6^{\varpi}(v_3u_3+v_4u_4)-v_5^2 \simeq \langle 1,1,\varpi,-\varpi,-1\rangle.$$

Note the additional change of sign and order needed to match our labeling. We first consider $\mathrm{span}\{u_1, u_2\}$. Then we have $p_5^1(v_2u_2 - v_1u_1) \simeq \langle 1, 1\rangle$ via the change of basis in Table 6.1 given here by

$$h_1 = su_2 + t^2s^{-1}u_1, \qquad h_2 = -s^{-1}u_1.$$

Next, we consider $\mathrm{span}\{u_3, u_4\}$. Then we have $p_6^{\varpi}(v_3u_3 + v_4u_4) \simeq \mathbb{H}$ via the change of basis

$$e_1 = (s^2 - t^2)u_3 - (s^2 + t^2)u_4, \quad f_1 = -\frac{s^2 + t^2}{2\varpi}u_3 - \frac{s^2 - t^2}{2\varpi}u_4.$$

Thus, our next intermediate basis is

$$\tilde{B} = \{h_1, h_2, e_1, f_1, u_5\}.$$

Finally, applying p_1^1 to $\mathrm{span}\{h_2, u_5\}$ gives

$$e_2 = h_2 + u_5 = (-s^{-1}, 1), \qquad f_2 = \frac{h_2 - u_5}{2} = \left(\frac{-s^{-1}}{2}, \frac{-1}{2}\right),$$

with $g_1 = h_1 = (si + t^2s^{-1}, 0)$.

The chosen Witt bases for all tori in this setting are collected in Tables A.4, A.5 and A.6.

6.3 Explicit matrices for tori and their root coordinates

In this section, we use the embedding defined by the Witt bases of the previous section to give an explicit embedding of the Lie algebra of each torus into $\mathfrak{so}_5$, viewed as a Lie algebra of 5×5 matrices as described in Chapter 2. In addition to giving a very satisfactory explicit description of these tori, this description will allow us to apply Proposition 6.1.3 in the next section to find the associated point in $\mathcal{B}(G)$.

6.3.1 Overview of the strategy

Fix the maximal diagonal split torus $S \subset G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$ with Lie algebra $\mathfrak{s}$. Let Φ denote the associated root system. Fix also a maximal elliptic torus $\mathcal{T} \subset G$ constructed from an algebra $\mathcal{A}$, an involution $\sigma \in \mathrm{Aut}(\mathcal{A}/F)$ and a parameter $\mu \in (\mathcal{A}^\times)^\sigma$, as in Chapter 3.

Let $\mathfrak{t}$ be its Lie algebra, and choose an element $v \in \mathfrak{t}$. With respect to the root space decomposition of $\mathfrak{g} = \mathfrak{so}_5$, the element v admits a unique decomposition

$$v = v_0 + \sum_{\alpha \in \Phi} v_\alpha, \qquad v_0 \in \mathfrak{s},\ v_\alpha \in \mathfrak{g}_\alpha,$$

where each root subspace $\mathfrak{g}_\alpha$ is one-dimensional, as recalled in Section 2.4.2.3. Fixing basis vectors $\tilde{X}_\alpha \in \mathfrak{g}_\alpha$ as in Table 2.11, we may write

$$v = a_1(E_{11} - E_{33}) + a_2(E_{22} - E_{44}) + \sum_{\alpha \in \Phi} c_\alpha \tilde{X}_\alpha,$$

with coefficients $a_1, a_2 \in F$ and $c_\alpha \in F$ for each $\alpha \in \Phi$.

To determine these coefficients, we compute the matrix M_v corresponding to left multiplication by the element $v \in \mathfrak{t} \subset \mathcal{A}$ on $V = \mathcal{A}$ with respect to the Witt basis

$$B^* = \{b_1 = e_1,\, b_2 = e_2,\, b_3 = f_1,\, b_4 = f_2,\, b_5 = g\}$$

constructed in Section 6.2. The entries of M_v, given by the coordinates of $v \cdot b_i$ for $b_i \in B^*$, express v explicitly in terms of the root coordinates relative to the chosen bases. Let c_{b_i,b_j} be the coordinates defined by

$$v \cdot b_i = \sum_{j=1}^{5} c_{b_i,b_j} b_j. \tag{6.3.1}$$

Then the matrix representing v is given by

$$M_v = (c_{b_i,b_j})_{i,j} = \sum_{i,j} c_{b_i,b_j}\, E_{ji}.$$

More precisely, let $(B^*)^{\mathfrak{d}}$ denote the dual basis to B^* with respect to the symmetric bilinear form $\langle , \rangle$. Then

$$(B^*)^{\mathfrak{d}} = \{b_1^{\mathfrak{d}} = f_1,\, b_2^{\mathfrak{d}} = f_2,\, b_3^{\mathfrak{d}} = e_1,\, b_4^{\mathfrak{d}} = e_2,\, b_5^{\mathfrak{d}} = g\}.$$

It has the property that

- $\langle b_j, b_j^{\mathfrak{d}} \rangle = 1$; and
- $\langle b_j, b \rangle = 0$ for all $b \in B^* \setminus \{b_j^{\mathfrak{d}}\}$.

Lemma 6.3.1. *With the notation as above, and c_{b_i,b_j} as in (6.3.1), we have*

$$c_{b_i,b_j} = \langle v \cdot b_i, b_j^{\mathfrak{d}} \rangle = \mathrm{Tr}_{\mathcal{A}/F}(\mu\, v\, b_i\, \sigma(b_j^{\mathfrak{d}})) \tag{6.3.2}$$

Proof. First, since

$$v \cdot b_i = \sum_k c_{b_i,b_k} b_k,$$

pairing both sides with $b_j^{\mathfrak{d}}$ gives

$$\langle v \cdot b_i, b_j^{\mathfrak{d}} \rangle = \left\langle \sum_k c_{b_i,b_k} b_k, b_j^{\mathfrak{d}} \right\rangle = \sum_k c_{b_i,b_k} \langle b_k, b_j^{\mathfrak{d}} \rangle = c_{b_i,b_j},$$

since $\langle b_k, b_j^{\mathfrak{d}} \rangle = 0$ unless $k = j$, in which case it equals 1. This proves the first equality.

For the second equality, recall from Proposition 3.1.3 that the bilinear form preserved by $\mathcal{T}$ is given on $v, w \in \mathcal{A}$ by

$$\langle v, w \rangle = \mathrm{Tr}_{\mathcal{A}/F}(\mu\, v\, \sigma(w)).$$

Applying this to $v \cdot b_i$ and $b_j^{\mathfrak{d}}$ yields $\langle v \cdot b_i, b_j^{\mathfrak{d}} \rangle = \mathrm{Tr}_{\mathcal{A}/F}(\mu\, v\, b_i\, \sigma(b_j^{\mathfrak{d}}))$, as required. ∎

With Lemma 6.3.1 in hand, let us now derive several identities among the 25 entries of the matrix $M_v = (c_{b_i,b_j})$. These structural constraints must reduce the number of independent coefficients to ten, which is $\dim(\mathfrak{so}_5)$, from which all remaining entries are then determined.

Lemma 6.3.2. *The matrix coefficients defined in* (6.3.1)*satisfy that for all* $1 \le i, j \le 5$, *we have*

$$c_{b_i,b_j} \;=\; -\, c_{b_j^{\mathfrak{d}},\, b_i^{\mathfrak{d}}}. \tag{6.3.3}$$

Consequently,

$$c_{e_1,f_1} = c_{e_2,f_2} = c_{f_1,e_1} = c_{f_2,e_2} = c_{g,g} = 0.$$

Proof. This follows directly from the matrix realization of $\mathfrak{so}_5$, however, it is a valuable consistency check to confirm it directly from our definition. Since $c_{b_i,b_j} \in F$, this element is invariant under the automorphism σ of $\mathcal{A}$. Thus by Lemma 6.3.1, we have

$$c_{b_i,b_j} = \sigma\big(\mathrm{Tr}_{\mathcal{A}/F}(\mu\, v\, b_i\, \sigma(b_j^{\mathfrak{d}}))\big) = \mathrm{Tr}_{\mathcal{A}/F}\big(\sigma(\mu)\, \sigma(v)\, \sigma(b_i)\, b_j^{\mathfrak{d}}\big).$$

Here we also used the fact that σ is an automorphism of the (commutative) algebra $\mathcal{A}$. Since $v \in \mathfrak{t} = \{X \in \mathcal{A} \mid X + \sigma(X) = 0\}$, we have $\sigma(v) = -v$, and because $\mu \in (\mathcal{A}^\times)^\sigma$, the above expression becomes

$$c_{b_i,b_j} = \mathrm{Tr}_{\mathcal{A}/F}\big(\mu\,(-v)\,\sigma(b_i)\, b_j^{\mathfrak{d}}\big) = -\,\mathrm{Tr}_{\mathcal{A}/F}\big(\mu\, v\, b_j^{\mathfrak{d}}\, \sigma(b_i)\big) = -c_{b_j^{\mathfrak{d}},\, b_i^{\mathfrak{d}}}.$$

Taking $b_i = b$ and $b_j = b_i^{\mathfrak{d}}$ in Lemma 6.3.2 gives $c_{b,b^{\mathfrak{d}}} = -\,c_{(b^{\mathfrak{d}})^{\mathfrak{d}},\, b^{\mathfrak{d}}} = -\,c_{b,b^{\mathfrak{d}}}$, so $c_{b,b^{\mathfrak{d}}} = 0$ since $\mathrm{char}(F) \neq 2$. Applying this to the pairs of Witt duals $e_1^{\mathfrak{d}} = f_1$, $e_2^{\mathfrak{d}} = f_2$, and $g^{\mathfrak{d}} = g$, yields that the corresponding coefficients must be zero. ∎

Through Lemma 6.3.2, we may visualize the matrix M_v of an arbitrary element as in (6.3.4), where we have colour-coded matching pairs (which correspond to multiples of the eight distinct root vectors, see Section 2.4.2.3).

$$M_v = \left[\begin{array}{cc|cc|c}
c_{e_1,e_1} & c^{a_{l_1}}_{f_1,f_2} & 0 & -c^{a_{l_2}}_{f_1,e_2} & c^{a_{s_1}}_{f_1,g} \\
c^{-a_{l_1}}_{e_1,e_2} & c_{e_2,e_2} & c^{a_{l_2}}_{f_1,e_2} & 0 & c^{a_{s_2}}_{f_2,g} \\
\hline
0 & -c^{-a_{l_2}}_{e_1,f_2} & -c_{e_1,e_1} & -c^{-a_{l_1}}_{e_1,e_2} & -c^{-a_{s_1}}_{e_1,g} \\
c^{-a_{l_2}}_{e_1,f_2} & 0 & -c^{a_{l_1}}_{f_1,f_2} & -c_{e_2,e_2} & -c^{-a_{s_2}}_{e_2,g} \\
\hline
c^{-a_{s_1}}_{e_1,g} & c^{-a_{s_2}}_{e_2,g} & -c^{a_{s_1}}_{f_1,g} & -c^{a_{s_2}}_{f_2,g} & 0
\end{array}\right] \tag{6.3.4}$$

With this notation in hand, in the following subsections we will for each torus give explicit relations on the valuations of the entries of the matrix representing $v \in \mathfrak{t}_r$. The key will be to express the proportionalities between the valuations of root coefficients in terms of the parameters (w_1, w_2) appearing in the forms $q = p_i^{w_1} \oplus p_j^{w_2} \oplus \langle 1 \rangle$ that arose in Tables 6.2, 6.3. Accordingly, we treat the tori in classes, depending on these forms: in Section 6.3.2 we treat the case that $i, j \in \{1, 3\}$; in Section 6.3.3 we treat the case that $i = j = 2$; in Section 6.3.4 we treat the case that $i \in \{1, 3\}$ and $j = 2$, and finally in Section 6.3.5 we treat the special ones in Table 6.3.

The robust output in all cases is the resulting valuation data.

6.3.2 Elliptic tori corresponding to two diagonal hyperbolic forms

Suppose in this section that $\mathcal{T}$ is a torus constructed from a datum $(\mathcal{A}, \sigma, \mu)$ such that with respect to the natural basis chosen in Tables 6.2 , 6.3, the resulting quadratic form q has the form decomposition

$$q = p_{i_1}^{w_1} \oplus p_{i_2}^{w_2} \oplus \langle 1 \rangle \quad \text{such that } i_1, i_2 \in \{1, 3\} \tag{6.3.5}$$

with respect to some fixed ordered basis $\{u_1, v_1, u_2, v_2, g\}$ of $\mathcal{A}$. For ease of identification: these are the tori whose entire entry in the last column of these tables is in blue. All but one corresponds to an algebra $\mathcal{A}$ of the form $\mathcal{A} = K_1 \oplus K_2 \oplus F$.

Let $B^* = \{e_1, e_2, f_1, f_2, g\}$ be the associated Witt basis. As show in Table A.3 and Table A.4, it takes the form

$$e_1 = u_1 + \gamma_{i_1} v_1, \quad f_1 = \frac{u_1 - \gamma_{i_1} v_1}{2w_1}, \qquad e_2 = u_2 + \gamma_{i_2} v_2, \quad f_2 = \frac{u_2 - \gamma_{i_2} v_2}{2w_2}, \qquad g$$

where w_1 and w_2 are obtained from the q as in (6.3.5), and $\gamma_{i_j} = 1$ if $i_j = 1$ and $\gamma_{i_j} = i$ if $i_j = 3$. Note that only $i_j = 1$ can occur is $-1 \notin F^{\times 2}$, so in all cases $\gamma_{i_j} \in F^{\times}$ so is fixed by σ.

Lemma 6.3.3. *Suppose that $\mathcal{A} = K_1 \oplus K_2 \oplus F$ is a sum of two quadratic extensions and that (6.3.5) holds. Then the following relations hold:*

$$e_1\sigma(e_2) = \pm 4w_1w_2\, f_1\sigma(f_2), \qquad e_1\sigma(f_2) = \pm\frac{w_1}{w_2}\, f_1\sigma(e_2), \qquad e_i\sigma(g) = \pm 2w_i\, f_i\sigma(g).$$

Moreover, the entries of $b\sigma(b^{\mathfrak{d}})$ lie in F for all $b \in B^*$.

Proof. Suppose that $\mathcal{A} = K_1 \oplus K_2 \oplus F$ for some quadratic extensions K_1, K_2 of F. We have, for $i_1, i_2 \in \{1, 3\}$ as above, that $q = p_{i_1}^{w_1} \oplus p_{i_2}^{w_2} \oplus \langle 1 \rangle = \langle w_1, -\gamma_{i_1}^2 w_1, w_2, -\gamma_{i_2}^2 w_2, 1 \rangle$ with respect to the ordered basis $\{u_1, v_1, u_2, v_2, g\}$ of $\mathcal{A}$. Perusing these rows in Tables 6.2, 6.3 (omitting those indicated by a $*$ in the latter table), we see that in the permutation of the basis of $\mathcal{A}$ to construct the diagonal forms $p_{i_k}^{w_k}$, we chose for each p_{i_k} vectors from *distinct* summands of $\mathcal{A}$, meaning in particular that $u_i v_i = 0$ (and thus $u_i\sigma(v_i) = 0$) for $i \in \{1, 2\}$.

In most cases, it is also true that $u_1 v_2 = 0$ and $u_2 v_1 = 0$ because they come from distinct summands, which means

$$u_i\sigma(v_j) = 0 \quad \forall i, j \in \{1, 2\}. \tag{6.3.6}$$

There are exactly five exceptions to (6.3.6) and for all of these we have $\{u_2, v_1\} \in K_2$ but no other pairs from the set $\{u_1, u_2, v_1, v_2\}$ lie in the same summand; thus all other pairwise products are zero. When $-1 \in F^{\times 2}$ these exceptions are ${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[1,\varepsilon,\varepsilon]}$, ${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$ and ${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varepsilon\varpi]}$ and when $-1 \notin F^{\times 2}$ they are ${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[1,1,\varepsilon]}$ and ${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varepsilon\varpi]}$. The calculations are simpler in these exceptional cases and are omitted.

Therefore assuming (6.3.6), we may calculate

$$e_1\sigma(e_2) = (u_1 + \gamma_{i_1} v_1)\sigma(u_2 + \gamma_{i_2} v_2) = u_1\sigma(u_2) + \gamma_{i_1}\gamma_{i_2} v_1\sigma(v_2)$$

whereas

$$f_1\sigma(f_2) = \left(\frac{u_1 - \gamma_{i_1} v_1}{2w_1}\right)\sigma\left(\frac{u_2 - \gamma_{i_2} v_2}{2w_2}\right) = \frac{1}{4w_1w_2}(u_1\sigma(u_2) + \gamma_{i_1}\gamma_{i_2} v_1\sigma(v_2)),$$

yielding the first relation. We also have

$$e_1\sigma(f_2) = (u_1 + \gamma_{i_1} v_1)\sigma\left(\frac{u_2 - \gamma_{i_2} v_2}{2w_2}\right) = \frac{1}{2w_2}(u_1\sigma(u_2) - \gamma_{i_1}\gamma_{i_2} v_1\sigma(v_2))$$

and

$$f_1\sigma e_2 = \left(\frac{u_1 - \gamma_{i_1} v_1}{2w_1}\right)\sigma(u_2 + \gamma_{i_2} v_2) = \frac{1}{2w_1}(u_1\sigma(u_2) - \gamma_{i_1}\gamma_{i_2} v_1\sigma(v_2)),$$

yielding the second. Finally, note that if g is a basis for the factor F of $\mathcal{A}$ (meaning v_5^2 is the final term of the expression in the blue rows of Tables 6.2, and 6.3, then $e_i\sigma(g) = 0$ and $f_i\sigma(g) = 0$ for all i. Otherwise, we have $g \in K_j$ for some $j = 1, 2$, and exactly one of the vectors $\{u_1, u_2, v_1, v_2\}$ is also an element of K_j. Therefore in each pair $\{u_i, v_i\}$ there can be at most one basis vector that lies in the same summand as g, meaning that for all j, at most one of $u_j\sigma(g)$ or $v_j\sigma(g)$ is nonzero. From this observation we conclude that

$$e_j\sigma(g) = (u_j + \gamma_{i_j} v_j)\sigma(g) = (\pm 2w_j)\left(\frac{u_j - \gamma_{i_j} v_j}{2w_j}\right)\sigma(g) = (\pm 2w_j) f_j\sigma(g)$$

where the sign is positive if $u_j\sigma(g) \neq 0$ and is negative if $v_j\sigma(g) \neq 0$. For the last point

$$e_1\sigma(f_1) = (u_1 + \gamma_{i_1} v_1)\sigma\left(\frac{u_1 - \gamma_{i_1} v_1}{2w_1}\right) = \frac{1}{2w_1}(u_1\sigma(u_1) - \gamma_{i_1}\gamma_{i_1} v_1\sigma(v_1)) = \frac{1}{2w_1}(u_1^2 - \gamma_{i_1}\gamma_{i_1} v_1^2).$$

Hence the entries lie in F. replacing e_1, f_1 by e_2, f_2, we get the same result. By (6.3.3), we conclude it for $b\sigma(b^{\mathfrak{d}})$ for every $b \in B^*$. ∎

Now consider the one remaining case, in Table 6.3 assuming $-1 \notin F^{\times 2}$, of the torus ${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[1,1]}$. Here, $\mathcal{A} = F(\sqrt{\varepsilon}, \sqrt{\varpi}) \oplus F$ and the form is

$$q(v_1 + v_2 i + (v_3 + v_4 i)\sqrt{\varpi}, v_5) = p_1^{-\varpi}(v_3, v_4) + p_1^1(v_1, v_2) + v_5^2.$$

In this case, $i = \sqrt{\varepsilon}$ and $\sigma(v_1 + v_2 i + (v_3 + v_4 i)\sqrt{\varpi}, v_5) = (v_1 + v_2 i - (v_3 + v_4 i)\sqrt{\varpi}, v_5)$. The ordered basis is $\{u_1 = (\sqrt{\varpi}, 0), u_2 = (1, 0), v_1 = (i\sqrt{\varpi}, 0), v_2 = (i, 0), g = (0, 1)\}$.

Lemma 6.3.4. *In the case of* ${}^{\varepsilon}\mathcal{T}_{4[1,1]}^{\varepsilon,\varpi}$ *when* -1 *is not a square, we have* $w_1 = -\varpi$, $w_2 = 1$ *and*

$$e_1\sigma(e_2) = 4w_1w_2\, f_1\sigma(f_2), \quad e_1\sigma(f_2) = \frac{-w_1}{w_2}\, f_1\sigma(e_2), \quad \textit{and} \quad e_i\sigma(g) = f_i\sigma(g) = 0$$

for all $i \in \{1,2\}$. *Moreover,* $c_{b,b} = 0$ *for all* $b \in B^*$.

Proof. Since g lies in a different summand of $\mathcal{A}$ than all other vectors, we have $e_i\sigma(g) = f_i\sigma(g) = 0$ for all $i \in \{1,2\}$. We compute

$$e_1\sigma(e_2) = (u_1+v_1)\sigma(u_2+v_2) = (\sqrt{\varpi}+i\sqrt{\varpi},0)\sigma(1+i,0) = (1+i)\sigma(1+i)(\sqrt{\varpi},0) = 2(\sqrt{\varpi},0)$$

whereas

$$\begin{aligned} f_1\sigma(f_2) &= \left(\frac{u_1-v_1}{2w_1}\right)\sigma\left(\frac{u_2-v_2}{2w_2}\right) = \frac{1}{4w_1w_2}\left(\sqrt{\varpi}-i\sqrt{\varpi},0\right)\sigma\left(1-i,0\right) \\ &= \frac{(1-i)\sigma(1-i)}{4w_1w_2}(\sqrt{\varpi},0) = \frac{2}{4w_1w_2}(\sqrt{\varpi},0), \end{aligned}$$

yielding the first relation. On the other hand,

$$e_1\sigma(f_2) = (\sqrt{\varpi}+i\sqrt{\varpi},0)\sigma\left(\frac{1}{2w_2}(1-i,0)\right) = \frac{(1+i)^2}{2w_2}(\sqrt{\varpi},0)$$

whereas

$$f_1\sigma(e_2) = \frac{1}{2w_1}\left(\sqrt{\varpi}-i\sqrt{\varpi},0\right)\sigma(1+i,0) = \frac{(1-i)^2}{2w_1}(\sqrt{\varpi},0)$$

so that in this case we have $e_1\sigma(f_2) = \frac{w_2}{w_1}\cdot\frac{(1-i)^2}{(1+i)^2}f_1\sigma(e_2)$. Since

$$\frac{(1-i)^2}{(1+i)^2} = \frac{1}{4}(1-i)^4 = \frac{1}{4}(-2i)^2 = -1,$$

we have the second relation. Since

$$e_1\sigma(f_1) = (\sqrt{\varpi}+i\sqrt{\varpi},0)\left(\frac{-i\sqrt{\varpi}+\sqrt{\varpi}}{2\varpi},0\right) = \left(\frac{\varpi(1+i)(1-i)}{2\varpi},0\right) = (1,0) \in (F,F),$$

$$e_2\sigma(f_2) = (1+i,0)\left(\frac{1-i}{2},0\right) = \left(\frac{(1+i)(1-i)}{2},0\right) = (1,0) \in (F,F),$$

then $c_{e_1,e_1} = c_{e_2,e_2} = 0$. Consequently, by (6.3.3), we have $c_{f_1,f_1} = c_{f_2,f_2} = 0$. ∎

We can now use Lemma 6.3.1 to deduce relations among the valuations of the matrix entries of M_v, for any $v \in \mathfrak{t}$.

Proposition 6.3.5. *Suppose $\mathcal{T}$ is a torus such that the quadratic form q obtained in Tables 6.2 and 6.3 has the form $p_{i_1}^{w_1} \oplus p_{i_2}^{w_2} \oplus \langle 1 \rangle$, for $i_1, i_2 \in \{1, 3\}$. Then for any $v \in \mathfrak{t}$, the coordinates of M_v with respect to B^* satisfy the relations*

$$\begin{aligned}
\nu(c_{e_1,f_2}^{-a_{l_2}}) &= \nu(w_1 w_2) + \nu(c_{f_1,e_2}^{a_{l_2}}), \\
\nu(c_{e_1,e_2}^{-a_{l_1}}) &= \nu(w_1/w_2) + \nu(c_{f_1,f_2}^{a_{l_1}}), \\
\nu(c_{e_i,g}^{-a_{s_i}}) &= \nu(w_i) + \nu(c_{f_i,g}^{a_{s_i}}).
\end{aligned} \tag{6.3.7}$$

In particular, the constants depend only on the valuations of the scales w_1, w_2. Moreover $c_{b,b} = 0$ for all $b \in B^*$.

Proof. Let $v \in \mathfrak{t}$. By Lemma 6.3.1 its coordinates are given by $c_{x,y} = \mathrm{Tr}_{\mathcal{A}/F}(\mu^* \, v \, x \, \sigma(y^{\mathfrak{d}}))$. Lemmas 6.3.3 and 6.3.4 give us the relations

$$e_1\sigma(e_2) = \pm 4 w_1 w_2 \, f_1 \sigma(f_2), \quad e_1 \sigma(f_2) = \pm \frac{w_1}{w_2} \, f_1 \sigma(e_2), \quad e_i \sigma(g) = \pm 2 w_i \, f_i \sigma(g)$$

up to sign, where some equalities may be trivial with both terms equal to 0. Note that $w_1, w_2 \in F$. Therefore we may compute

$$c_{e_1,f_2}^{-a_{l_2}} = \mathrm{Tr}\big(\mu^* v \, e_1 \, \sigma(e_2)\big) = 4 w_1 w_2 \, \mathrm{Tr}\big(\mu^* v \, f_1 \, \sigma(f_2)\big) = 4 w_1 w_2 c_{f_1,e_2}^{a_{l_2}}.$$

Similarly,

$$c_{e_1,e_2}^{-a_{l_1}} = \mathrm{Tr}\big(\mu^* v \, e_1 \, \sigma(f_2)\big) = \pm \frac{w_1}{w_2} \, \mathrm{Tr}\big(\mu^* v \, f_1 \, \sigma(e_2)\big) = \pm (w_1/w_2) c_{f_1,f_2}^{a_{l_1}}.$$

and,

$$c_{e_i,g}^{-a_{s_i}} = \mathrm{Tr}\big(w^* v \, e_i \, \sigma(g)\big) = \pm 2 w_i \, \mathrm{Tr}\big(\mu^* v \, f_i \, \sigma(g)\big) = \pm \, 2 w_i \, c_{f_i,g}^{a_{s_i}}.$$

We note that $2, 4 \in \mathcal{O}_F^\times$. Therefore, taking the valuation of each of these terms yields the relations in the statement of the proposition. Finally, $c_{b,b} = \mathrm{Tr}(\mu^* v b \sigma(b^{\mathfrak{d}})) = 0$, since μ^* and $b\sigma(b^{\mathfrak{d}})$ lie in F for each $b \in B^*$. ∎

We describe the resulting matrix M_v graphically in (6.3.8).

$$M_v = \left[\begin{array}{cc|cc|c}
0 & -c_{f_1,f_2}^{a_{l_1}} & 0 & -c_{f_1,e_2}^{a_{l_2}} & -c_{f_1,g}^{a_{s_1}} \\
(w_1/w_2) \, c_{f_1,f_2}^{a_{l_1}} & 0 & c_{f_1,e_2}^{a_{l_2}} & 0 & -c_{f_2,g}^{a_{s_2}} \\
\hline
0 & -4 \, w_1 w_2 \, c_{f_1,e_2}^{a_{l_2}} & 0 & (-w_1/w_2) \, c_{f_1,f_2}^{a_{l_1}} & -(\pm 2 w_1) \, c_{f_1,g}^{a_{s_1}} \\
4 \, w_1 w_2 \, c_{f_1,e_2}^{a_{l_2}} & 0 & c_{f_1,f_2}^{a_{l_1}} & 0 & -(\pm 2 w_2) \, c_{f_2,g}^{a_{s_2}} \\
\hline
(\pm 2 w_1) \, c_{f_1,g}^{a_{s_1}} & (\pm 2 w_2) \, c_{f_2,g}^{a_{s_2}} & c_{f_1,g}^{a_{s_1}} & c_{f_2,g}^{a_{s_2}} & 0
\end{array}\right] \tag{6.3.8}$$

Having established the relations among the coefficients, it remains to determine the valuation of each independent coefficient for a $\mathcal{T}$-good element v of depth r. These valuations are computed explicitly using the chosen Witt bases and the identity (6.3.2). The results of these computations are summarized in Table 6.4.

$\mathfrak{t}$	$\mathcal{T}$-good v	$\nu(c^{a_{l_1}}_{f_1,f_2})$	$\nu(c^{a_{l_2}}_{f_1,e_2})$	$\nu(c^{a_{s_1}}_{f_1,g})$	$\nu(c^{a_{s_2}}_{f_2,g})$	$\nu(\frac{w_1}{w_2})$	$\nu(w_1 w_2)$	$\nu(w_1)$	$\nu(w_2)$
${}^{\varepsilon}\mathfrak{t}^2_{1,1,1}$	$(a\sqrt{\varepsilon}, b\sqrt{\varepsilon}, 0)$	r	r			0	0	0	0
${}^{\varepsilon}\mathfrak{t}^2_{\varpi,\varpi,1}$	$(a\sqrt{\varepsilon}, b\sqrt{\varepsilon}, 0)$	r	$r-1$			0	2	1	1
${}^{\varpi'}\mathfrak{t}^2$	$(a\sqrt{\varpi'}, b\sqrt{\varpi'}, 0)$	$r-\frac{1}{2}$	$r-\frac{1}{2}$			1	1	1	0
${}^{\varpi}\mathfrak{t}^{\varepsilon\varpi}$	$(a\sqrt{\varpi}, b\sqrt{\varepsilon\varpi}, 0)$	$r-\frac{1}{2}$	$r-\frac{1}{2}$	$r-\frac{1}{2}$		1	1	1	0
${}^{\varepsilon}\mathfrak{t}^{\varepsilon,\varpi}_{4[1,1]}$, $-1^{\boxtimes}$	$(a\sqrt{\varpi} + bi\sqrt{\varpi}, 0)$	$r-\frac{1}{2}$	$r-\frac{1}{2}$			1	1	1	0
${}^{\varpi'}\mathfrak{t}^{\varepsilon}_{1,\varpi,\lambda}$	$(a\sqrt{\varpi'}, 0, 0)$			$r-\frac{1}{2}$		0	2	1	1
	$(0, b\sqrt{\varepsilon}, 0)$	r	$r-1$			0	2	1	1
${}^{\varpi'}\mathfrak{t}^{\varepsilon}_{\varepsilon,1,\lambda}$	$(a\sqrt{\varpi'}, 0, 0)$	$r-\frac{1}{2}$	$r-\frac{1}{2}$			1	1	1	0
	$(0, b\sqrt{\varepsilon}, 0)$				r	1	1	1	0

Table 6.4: *Valuations of structure constants associated to each elliptic torus admitting quadratic form $q \simeq p_i^{w_1} \oplus p_j^{w_2} \oplus \langle 1 \rangle$, $i, j \in \{1, 3\}$, computed using the Witt bases given in Tables A.3 and A.4 and equation (6.3.2). Here $\nu(c)$ denotes the valuation of the root coefficient in the matrix of a $\mathcal{T}$-good element of depth r. Gray cells indicate that the corresponding coefficient $c^{\alpha} = 0$.*

6.3.3 Elliptic tori corresponding to two cross-term forms

Suppose in this section that $\mathcal{T}$ is a torus constructed from a datum $(\mathcal{A}, \sigma, \mu)$ such that with respect to the natural basis chosen in Tables 6.2 and 6.3, the resulting quadratic form q has the form decomposition

$$q = p_2^{w_1} \oplus p_2^{w_2} \oplus \langle 1 \rangle$$

with respect to a permutation $\{u_1, v_1, u_2, v_2, g\}$ of the natural basis.

For ease of identification: these are the five tori whose entire entry in the last column of these tables is in red, and they are precisely the tori arising from a biquadratic field extension (except for ${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{-\varepsilon},1]}$ in the case that $-1 \notin F^{\times^2}$, which was treated in Section 6.3.2). In each of these five cases, if the fixed field is $F(\sqrt{\alpha})$ for some $\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ then $\mu^* = (\sqrt{\alpha}, 1)$.

Let $B^* = \{e_1, e_2, f_1, f_2, g\}$ be the associated Witt basis from Table A.5.

We record a more general lemma that applies to the form p_2^{λ}.

Lemma 6.3.6. *Let E be a quartic extension of F and suppose $u \in E$ and $\kappa \in \{\varepsilon, \varpi, \varepsilon\varpi\}$ such that $\{u, \sqrt{\kappa} u\}$ is a basis for the quadratic space p_2^{λ} for some $\lambda \in F^{\times}$. Then the elements of its Witt basis satisfy $e = \lambda\, \kappa^{-1/2} f$.*

Proof. By Table 6.1, the corresponding Witt basis is $e = u$ and $f = \lambda^{-1}\sqrt{\kappa}u$. The stated relation follows directly. ∎

Proposition 6.3.7. *Let E/F be biquadratic and let $\sigma \in \mathrm{Gal}(E/F)$ have quadratic fixed field $E_1 = F(\sqrt{\alpha})$. Set $\mathcal{A} = E \oplus F$ and take $\mu^* = (\sqrt{\alpha}, 1)$. Choose a Witt basis $B^* = \{e_1, e_2, f_1, f_2, g\}$ as in Table A.5. Then for all $v \in \mathfrak{t}$ one has the following valuation identities:*

$$\nu(c_{e_1,f_2}^{-a_{l_2}}) = \nu(w_1 w_2 \alpha^{-1}) + \nu(c_{f_1,e_2}^{a_{l_2}}), \qquad \nu(c_{e_1,e_2}^{-a_{l_1}}) = \nu(w_1/w_2) + \nu(c_{f_1,f_2}^{a_{l_1}}).$$

$$c_{e_i,g} = 0 = c_{f_i,g} \quad (i = 1, 2), \qquad c_{b,b} = 0 \quad \textit{for all } b \in B^*.$$

Proof. By Lemma 6.3.1, $c_{x,y} = \mathrm{Tr}_{\mathcal{A}/F}\big(\mu^*\, v\, x\, \sigma(y^{\mathfrak{d}})\big)$ with $\sigma|_{E_1} = \mathrm{Id}$. Then $v = (v_E, 0)$ for some $v_E \in E$ so we can consider only the first coordinate in each vector and parameter. Using Lemma 6.3.6 we compute

$$\begin{aligned} c_{e_1,f_2}^{-a_{l_2}} &= \mathrm{Tr}_{\mathcal{A}/F}\big(\sqrt{\alpha}\, v\, e_1\, \sigma(e_2)\big) \\ &= \mathrm{Tr}_{\mathcal{A}/F}\big(\sqrt{\alpha}\, v\, w_1 \alpha^{-1/2} f_1\, \sigma(w_2 \alpha^{-1/2} f_2)\big) \\ &= w_1 w_2\, \alpha^{-1}\, \mathrm{Tr}_{\mathcal{A}/F}\big(\sqrt{\alpha}\, v\, f_1\, \sigma(f_2)\big) \\ &= w_1 w_2\, \alpha^{-1}\, c_{f_1,e_2} \end{aligned}$$

$$\begin{aligned} c_{e_1,e_2}^{-a_{l_1}} &= \mathrm{Tr}_{\mathcal{A}/F}\big(\sqrt{\alpha}\, v\, e_1\, \sigma(f_2)\big) \\ &= w_1\, \alpha^{-1/2}\, \mathrm{Tr}_{\mathcal{A}/F}\big(\sqrt{\alpha}\, v\, f_1\, \sigma\big(w_2^{-1} \alpha^{1/2} e_2\big)\big) \\ &= \frac{w_1}{w_2}\, \mathrm{Tr}_{\mathcal{A}/F}\big(\sqrt{\alpha}\, v\, f_1\, \sigma(e_2)\big) = \frac{w_1}{w_2}\, c_{f_1,f_2}. \end{aligned}$$

Hence the valuation relations follow. Next, note that since $g = (0, 1)$, $\sigma(g) = g \in F$ and multiplication in $\mathcal{A} = E \oplus F$ is componentwise, so for any $x \in E$ we have $x \cdot \sigma(g) = (x, 0)\cdot(0, 1) = (0, 0)$. Therefore $c_{x,g} = \mathrm{Tr}_{\mathcal{A}/F}(\mu\, v\, x\, \sigma(g)) = 0$, and in particular $c_{e_i,g} = c_{f_i,g} = 0$ for $i = 1, 2$. Finally, again by Lemma 6.3.6, we compute

$$\begin{aligned} c_{e_i,e_i} &= \mathrm{Tr}_{\mathcal{A}/F}\big(\sqrt{\alpha}\, v\, e_i\, \sigma(f_i)\big) \\ &= \mathrm{Tr}_{\mathcal{A}/F}\big(\sqrt{\alpha}\, v\, w_i \alpha^{-1/2} f_i\, \sigma(f_i)\big) \\ &= w_i\, f_i\, \sigma(f_i) \mathrm{Tr}_{\mathcal{A}/F}\big(v\big) = 0 \end{aligned}$$

since w_i and $f_i\, \sigma(f_i)$ both in F, and $\mathrm{Tr}_{\mathcal{A}/F}(v) = 0$. Similarly, one has $c_{f_i,f_i} = 0$. ∎

We describe the resulting matrix M_v graphically in (6.3.9).

$$M_v = \left[\begin{array}{cc|cc|c} 0 & -c^{a_{l_1}}_{f_1,f_2} & 0 & -c^{a_{l_2}}_{f_1,e_2} & 0 \\ (w_1/w_2)\, c^{a_{l_1}}_{f_1,f_2} & 0 & c^{a_{l_2}}_{f_1,e_2} & 0 & 0 \\ \hline 0 & -\, w_1 w_2 \alpha^{-1}\, c^{a_{l_2}}_{f_1,e_2} & 0 & (-w_1/w_2)\, c^{a_{l_1}}_{f_1,f_2} & 0 \\ w_1 w_2 \alpha^{-1}\, c^{a_{l_2}1}_{f_1,e_2} & 0 & c^{a_{l_1}}_{f_1,f_2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{6.3.9}$$

As in the previous cases, we compute the valuations of the independent coefficients for $\mathcal{T}$-good elements of depth r. The results for the biquadratic case are recorded in Table 6.5.

${}^{\alpha}\mathfrak{t}^{\varepsilon,\varpi}_{4}$	$\mathcal{T}$-good v	$\nu(c^{a_{l_1}}_{f_1,f_2})$	$\nu(c^{a_{l_2}}_{f_1,e_2})$	$\nu(c^{a_{s_1}}_{f_1,g})$	$\nu(c^{a_{s_2}}_{f_2,g})$	$\nu(\frac{w_1}{w_2})$	$\nu(w_1 w_2 a^{-1})$
${}^{\varepsilon}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon},1]}$, $-1^{\square}$	$(a\sqrt{\varpi}+b\sqrt{\varepsilon\varpi},0)$	$r-\frac{1}{2}$	$r-\frac{1}{2}$			1	1
${}^{\varpi'}\mathfrak{t}^{\varepsilon,\varpi}_{4[\sqrt{\varpi'},1]}$	$(a\sqrt{\varepsilon\varpi'},0)$		$r-\frac{1}{2}$			0	1
	$(b\sqrt{\varepsilon},0)$	r				0	1

Table 6.5: *Valuations of structure constants associated to each elliptic torus admitting quadratic form $q \simeq p_2^{w_1} \oplus p_2^{w_2} \oplus \langle 1\rangle$, computed using the Witt bases given in Table A.5 and equation (6.3.2). Here $\nu(c)$ denotes the valuation of the root coefficient in the matrix of a $\mathcal{T}$-good element of depth r. Gray cells indicate that the corresponding coefficient $c^{\alpha}=0$.*

6.3.4 Elliptic tori corresponding to a diagonal and a cross-term form

Suppose in this section that $\mathcal{T}$ is a torus constructed from a datum $(\mathcal{A},\sigma,\mu)$ such that with respect to the natural basis chosen in Tables 6.2 , 6.3, the resulting quadratic form q has the form decomposition

$$q = p_i^{w_1} \oplus p_2^{w_2} \oplus \langle 1\rangle \quad \text{such that } i \in \{1,3\}.$$

For ease of identification: these are the tori whose entry in the last column of these tables is a mix of blue and red.

The only elliptic tori that fall under this case are those for which $\mathcal{A} = E \oplus F$ where E/F is quartic with a unique subfield $F(\sqrt{\alpha})$ and these are denoted ${}^{\alpha}\Upsilon^{E}_{4[\mu,\lambda]}$. However, when -1 is not a square, not all of these tori lie in this category (as noted by $*$ in Table 6.3) and they will be treated separately in the next section.

First, we prove the following relation.

Lemma 6.3.8. *Let $\{e_1, e_2, f_1, f_2, g\}$ be a Witt basis of $\mathcal{A} = E \oplus F$ that takes the form*

$$e_1 = (\sqrt{\alpha}, \eta), \quad f_1 = \left(\pm\frac{\sqrt{\alpha}}{2w_1}, \eta'\right), \quad e_2 = (r, 0), \quad f_2 = (z, 0),$$

where $w_1 \in F^\times$ obtained from the q as $p_i^{w_1}$ with $i \in \{1, 3\}$ and $\sigma(\sqrt{\alpha}) = \sqrt{\alpha} \in E$. Here, $\eta, \eta' \in F$ and $r, z \in E$. Then

$$e_1\sigma(f_2) = \pm 2w_1\, f_1\sigma(f_2), \qquad e_2\sigma(e_1) = \pm 2w_1\, e_2\sigma(f_1).$$

Proof. Since $f_2 = (z, 0)$, $\sigma(f_2) = (\sigma(z), 0)$, so $e_1\sigma(f_2) = (\sqrt{\alpha}\sigma(z), 0)$ and $f_1\sigma(f_2) = \left(\pm\frac{\sqrt{\alpha}}{2w_1}\sigma(z), 0\right)$, giving $e_1\sigma(f_2) = \pm 2w_1\, f_1\sigma(f_2)$.

Likewise, $e_2 = (r, 0)$ gives $e_2\sigma(e_1) = (r\sigma(\sqrt{\alpha}), 0)$ and $e_2\sigma(f_1) = \left(r\sigma\left(\pm\frac{\sqrt{\alpha}}{2w_1}\right), 0\right)$, and since $\sigma(\sqrt{\alpha}) = \sqrt{\alpha}$, we obtain $e_2\sigma(e_1) = \pm 2w_1\, e_2\sigma(f_1)$. ∎

We now address most of the tori that fall under this case (with the exception of ${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[1,\varepsilon]}$, which is treated in Proposition 6.3.10 instead).

Proposition 6.3.9. *Suppose $\mathcal{T}$ is an elliptic torus associated to the algebra*

$$\mathcal{A} = E \oplus F, \qquad E = F(\sqrt[4]{\alpha}),$$

with unique quadratic subfield $E_1 \subset E$, as described above. Fix a coset representative $\mu^* = (1, \lambda)$ and an element $v \in \mathfrak{t}$. Choose a Witt basis B^* as in Table A.6. Then the coordinates of M_v with respect to B^* satisfy the relations.

$$\nu\big(c^{-a_{l_1}}_{e_1,e_2}\big) = \nu(w_1) + \nu\big(c^{a_{l_2}}_{f_1,e_2}\big), \qquad \nu\big(c^{-a_{l_2}}_{e_1,f_2}\big) = \nu(w_1) + \nu\big(c^{a_{l_1}}_{e_2,e_1}\big). \tag{6.3.10}$$

$$c^{\pm a_{s_1}}_{x,g} = 0 \qquad \textit{for all } x \in \{e_1, f_1\},$$
$$c_{b,b} = 0 \qquad \textit{for all } b \in B^*.$$

Proof. Now, as $g = (1, 0)$ in our choice for B^*, $\sigma(e_1) = e_1, \sigma(f_1) = f_1$ (as they belong the fixed subfield), and $\sigma(v) = -v$. Write $\mu^* = (1, \lambda)$. Moreover, in this setting we have all the Witt basis that belong to this criteria satisfy the hypothesis of Lemma 6.3.8. Therefore we compute

$$c_{e_1,e_2} = \mathrm{Tr}(\mu^* v\, e_1\, \sigma(f_2)) = \pm 2w_1 \mathrm{Tr}(\mu^* v\, f_1\, \sigma(f_2)) = \pm 2w_1\, c_{f_1,e_2}.$$

and

$$c_{e_1,f_2} = -c_{e_2,f_1} = -\mathrm{Tr}(\mu^* v\, e_2\, \sigma(e_1)) = \pm 2w_1 \mathrm{Tr}(\mu^* v\, e_2\, \sigma(f_1)) = \pm 2w_1\, c_{e_2,e_1}.$$

The valuation relation follows.

For $x \in \{e_1, f_1\}$, since x and $g = (1, 0)$ are σ-fixed but $\sigma(v) = -v$ we have

$$\sigma\big(\mu\, v\, x\, \sigma(g^{\mathfrak{d}})\big) = \sigma(\mu\, v\, x) = -\mu\, v\, x.$$

This implies $\mathrm{Tr}_{E/E_1}\big(\mu\, v\, x\, \sigma(g^{\eth})\big) = 0$. Therefore the coefficient

$$c_{x,g}^{\pm a_{s_1}} = \mathrm{Tr}_{E/F}\big(\mu\, v\, x\, \sigma(g^{\eth})\big) = \mathrm{Tr}_{E_1/F} \circ \mathrm{Tr}_{E/E_1}\big(\mu\, v\, x\, \sigma(g^{\eth})\big) = 0.$$

Consequently, $c_{x,g}^{\pm a_{s_1}} = 0$ for all $x \in \{e_1, f_1\}$. The diagonal entries vanish in these cases. For any basis vector b,

$$c_{b,b} = \mathrm{Tr}_{\mathcal{A}/F}\big(\mu^* v\, b\sigma(b)\big).$$

Since $b\sigma(b) \in \mathcal{A}^{\sigma}$ and $\sigma(v) = -v$, the element $\mu^* v\, b\sigma(b)$ is σ-anti-invariant. Hence its trace to F is zero. Therefore

$$c_{b,b} = 0$$

for all b, so every diagonal entry is zero.

∎

Consequently, the quartic matrix M_v in (6.3.11) is determined by only six coefficients; all other entries follow from the proportionalities recorded in the general matrix 6.3.4.

$$M_v = \left[\begin{array}{cc|cc|c} 0 & c_{e_2,e_1}^{a_{l_1}} & 0 & c_{f_1,e_2}^{a_{l_2}} & 0 \\ -\pm 2w_1 c_{f_1,e_2}^{a_{l_2}} & 0 & -c_{f_1,e_2}^{a_{l_2}} & 0 & -c_{f_2,g}^{a_{s_2}} \\ \hline 0 & \pm 2w_1\, c_{e_2,e_1}^{a_{l_1}} & 0 & \pm 2w_1 c_{f_1,e_2}^{a_{l_2}} & 0 \\ -\pm 2w_1\, c_{e_2,e_1}^{a_{l_1}} & 0 & -c_{e_2,e_1}^{a_{l_1}} & 0 & -c_{e_2,g}^{-a_{s_2}} \\ \hline 0 & c_{e_2,g}^{-a_{a_2}} & 0 & c_{f_2,g}^{a_{s_2}} & 0 \end{array}\right] \tag{6.3.11}$$

Now finally consider the torus ${}^{\varepsilon}\Upsilon_{4[1,\varepsilon]}^{\varepsilon_{E_1}\varpi}$. By Table 6.2 its form is $q = p_2^{-\varepsilon\varpi}(v_3, v_4) + p_3^{\varepsilon}(v_2, v_5) + v_1^2$ so $w_1 = -\varepsilon\varpi$ and $w_2 = \varepsilon$. This order was chosen, as mentioned earlier, to ensure that $\nu(w_1) \geq \nu(w_2)$. Hence for this case we have different scenario.

Proposition 6.3.10. *For the elliptic torus ${}^{\varepsilon}\Upsilon_{4[1,\varepsilon]}^{\varepsilon_{E_1}\varpi}$, in Table 6.2 with associated algebra $\mathcal{A} = E \oplus F$, $\quad E = F(\sqrt[4]{\varepsilon\varpi^2})$, with unique quadratic subfield $E_1 = F(\sqrt{\varepsilon}) \subset E$, and the parameter $\mu^* = (1, \varepsilon)$, choose a Witt basis B^* as in Table A.6.*

Then the coefficients of the matrix M_v corresponding to an arbitrary element $v \in {}^{\alpha}\mathfrak{u}_{4[1,\varepsilon]}^{\varepsilon_{E_1}\varpi}$ satisfy the relations

$$\nu(c_{e_1,f_2}^{-a_{l_2}}) = \nu(w_2) + \nu(c_{e_1,e_2}^{-a_{l_1}}), \qquad \nu(c_{f_1,f_2}^{a_{l_1}}) = \nu(w_2) + \nu(c_{f_1,e_2}^{a_{l_2}}), \quad \textit{and}$$
$$c_{x,g}^{\pm a_{s_2}} = 0 \qquad \forall\, x \in \{e_2, f_2\}.$$

Proof. We only have to permute the order of the two hyperbolic planes p_2 and p_3. Hence the proof follows the same argument as that of Proposition 6.3.9, with the roles of the two hyperbolic planes interchanged. ∎

We now determine the valuations of the independent coefficients for a $\mathcal{T}$-good element v of depth r, building on the relations established above. These computations use the chosen Witt bases together with equation (6.3.2), and the results are collected in Table 6.6.

${}^{\alpha}\mathfrak{u}_4^E$	$\mathcal{T}$-good v	$\nu(c^{a_{l_1}}_{f_1,f_2})$	$\nu(c^{a_{l_2}}_{f_1,e_2})$	$\nu(c^{a_{s_2}}_{f_2,g})$	$\nu(c^{-a_{s_2}}_{e_2,g})$	$\nu(w_1)$
${}^{\varepsilon}\mathfrak{t}^{\varepsilon_{E_1}}_{4[1,\varepsilon]}$, $-1^{\square}$	$(a\sqrt[4]{\varepsilon}+b\sqrt[4]{\varepsilon}^3,0)$	r	r	r	r	0
${}^{\varepsilon^k\varpi}\mathfrak{u}^{\varepsilon^k\varpi}_{4[1,\lambda]}$	$(a\sqrt[4]{\varepsilon^k\varpi}+b\sqrt[4]{\varepsilon^k\varpi}^3,0)$	$r-\frac{1}{4}$	$r-\frac{1}{4}$	$r-\frac{1}{4}$	$r+\frac{3}{4}$	1
${}^{\alpha}\mathfrak{u}_4^E$	$\mathcal{T}$-good v	$\nu(c^{\pm a_{l_1}}_{f_1,f_2})$	$\nu(c^{\pm a_{l_2}}_{f_1,e_2})$	$\nu(c^{-a_{s_1}}_{e_1,g})$	$\nu(c^{a_{s_1}}_{f_1,g})$	$\nu(w_2)$
${}^{\varepsilon}\mathfrak{t}^{\varepsilon_{E_1}\varpi}_{4[1,\varepsilon]}$, $-1^{\square}$	$(a\sqrt[4]{\varepsilon\varpi^2}+b\sqrt[4]{\varepsilon\varpi^2}^3,0)$	$r+\frac{1}{2}$	$r+\frac{1}{2}$	$r+\frac{1}{2}$	$r-\frac{1}{2}$	0

Table 6.6: *Valuations of structure constants associated to each elliptic torus admitting quadratic form $q \simeq p_i^{w_1} \oplus p_2^{w_2} \oplus \langle 1\rangle$, $i \in \{1,3\}$ for the first two, and $q \simeq p_i^{w_1} \oplus p_2^{w_2} \oplus \langle 1\rangle$, $i \in \{1,3\}$ for the last one. We compute the valuation using the Witt bases given in Tables A.6 and equation (6.3.2). Here $\nu(c)$ denotes the valuation of the root coefficient in the matrix of a $\mathcal{T}$-good element of depth r.*

6.3.5 The special elliptic tori in Table 6.3

Here we deal with the final outstanding cases: the special cases marked by asterisks in Table 6.3, which are explicitly

$$ {}^{\varepsilon}\mathcal{T}^2_{[1,1,1]},\ {}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi,1]},\ {}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[\varepsilon,-1,\varepsilon]},\ {}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]},\ {}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[1,\varepsilon]},\ {}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\eta,\varepsilon]} $$

in the case that $-1 \notin F^{\times 2}$. So in this section $i = \sqrt{-1} = \sqrt{\varepsilon} \notin F$. Therefore, we are going to deal with the cases individually, starting with ${}^{\varepsilon}\mathcal{T}^2_{[1,1,1]}$ and ${}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi,1]}$.

Example 6.3.11. Consider the tori ${}^{\varepsilon}\mathcal{T}^2_{[1,1,1]}$ and ${}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi,1]}$ (which we also saw in Example 6.2.3). Here $\mathcal{A} = F(i) \oplus F(i) \oplus F$ and $\sigma = (\sigma_1, \sigma_2, \mathrm{Id})$ where $\sigma_k(i) = -\sqrt{\varepsilon}$ for $k = 1, 2$. Take $\mu^* = (\lambda, \lambda, 1)$, for $\lambda = \{1, \varpi\}$; then note that from Table 6.3 we have $w_1 = w_2 = \lambda$. The Witt basis computed in Table A.4 is

$$ e_1 = (i + s^2, t^2, 0), \quad e_2 = (t^2, i - s^2, 0) $$
$$ f_1 = (\tfrac{i-s^2}{2\lambda}, \tfrac{-t^2}{2\lambda}, 0), \quad f_2 = (\tfrac{-t^2}{2\lambda}, \tfrac{i+s^2}{2\lambda}, 0) $$
$$ g = (0, 0, 1) $$

with $\varepsilon_{E_1} = s^2 + it^2$ as before. Recall that these were obtained by transforming $p_4 \oplus p_4$ to $p_1 \oplus p_1$ and then to $2\mathbb{H}$. Given $v = (ai,\, bi,\, 0) \in \mathfrak{t}$, a direct component-wise multiplication in $\mathcal{A}$, noting that $\sigma(g) = g$ and $\sigma(v) = -v$ gives

$$ c^{-a_{l_2}}_{e_1,f_2} = 2\lambda t^2(b - a), \qquad c^{a_{l_2}}_{f_1,e_2} = \frac{t^2}{2\lambda}(a - b), $$

$$c_{e_1,e_2}^{-a_{l_1}} = t^2(a+b), \qquad c_{f_1,f_2}^{a_{l_1}} = -t^2(a+b),$$
$$c_{e_1,e_1} = 2as^2 = -c_{f_1,f_1}, \qquad c_{e_2,e_2} = -2bs^2 = -c_{f_2,f_2}.$$
$$c_{e_1,g}^{-a_{s_1}} = 0, \qquad c_{e_2,g}^{-a_{s_2}} = 0,$$

Note that for $v \in t_r$, $\nu(c_{b,b}) \geq r$

Importantly, we note that for these two tori, even though the methods are different, the coefficient valuation identities (6.3.7) of Proposition 6.3.5 hold, just as all other tori arising from a sum of two quadratic extensions.

Example 6.3.12. Consider the elliptic torus ${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[\varepsilon,-1,\varepsilon]} = {}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-1,-1,-1]}$; the case of ${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$ is analogous. Here,

$$\mathcal{A} = F(\sqrt{\varpi}) \oplus F(i\sqrt{\varpi}) \oplus F, \qquad \sigma = (\sigma_1, \sigma_2, \mathrm{Id}),$$

where $\sigma_1(\sqrt{\varpi}) = -\sqrt{\varpi}$ and $\sigma_2(i\sqrt{\varpi}) = -\, i\sqrt{\varpi}$ and the parameter is $\mu^* = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. Our form gives the scaling factors $w_1 = \varpi$ and $w_2 = -1$ and we have the following Witt basis as in Table A.4:

$$e_1 = (\sqrt{\varpi}, i\sqrt{\varpi}, 0), \qquad e_2 = (t^2, -s^2, 1),$$
$$f_1 = (\tfrac{\sqrt{\varpi}^{-1}}{2}, -\tfrac{i\sqrt{\varpi}^{-1}}{2}, 0), \qquad f_2 = (\tfrac{t^2}{2}, -\tfrac{s^2}{2}, -\tfrac{1}{2}),$$
$$g_1 = (s^2, t^2, 0).$$

Here, we recall that we first transformed to $p_1^{\varpi} \oplus p_1^1$, with $w_1 = \varpi$, $w_2 = 1$, before transforming to $2\mathbb{H}$. For $v = (a\sqrt{\varpi},\, b\, i\sqrt{\varpi},\, 0) \in \mathfrak{t}$, to compute the coefficients of M_v, note that $\sigma(e_2) = e_2$ and $\sigma(f_2) = f_2$ since their coordinates lie in F. A direct component-wise multiplication in $\mathcal{A}$ gives

$$c_{e_1,f_2}^{-a_{l_2}} = \mathrm{Tr}_{\mathcal{A}/F}\big(\mu^* v e_1 e_2\big) = -\varpi(at^2 + bs^2), \qquad c_{f_1,e_2}^{a_{l_2}} = \mathrm{Tr}_{\mathcal{A}/F}\big(\mu^* v f_1 f_2\big) = \frac{bs^2 - at^2}{4},$$
$$c_{e_1,e_2}^{-a_{l_1}} = \mathrm{Tr}_{\mathcal{A}/F}\big(\mu^* v e_1 f_2\big) = -\frac{\varpi}{2}(at^2 + bs^2), \qquad c_{f_1,f_2}^{a_{l_1}} = \mathrm{Tr}_{\mathcal{A}/F}\big(\mu^* v f_1 e_2\big) = \frac{-at^2 + bs^2}{2},$$
$$c_{e_1,g}^{-a_{s_1}} = \mathrm{Tr}_{\mathcal{A}/F}\big(\mu^* v e_1 g\big) = as^2\varpi = 2\varpi\, c_{f_1,g}^{a_{s_1}}, \qquad c_{e_2,g}^{-a_{s_2}} = 0 = c_{f_2,g}^{a_{s_2}}.$$

Here $\mathcal{A}^{\sigma} = F \oplus F \oplus F$, and each component of $\mu^* v b \sigma(b)$ lies in the σ-anti-invariant subspace of a quadratic extension, hence, the diagonal entries vanish. We infer from the computation above that for this torus, the coefficients satisfy the relations

$$\begin{aligned} \nu(c_{e_1,f_2}^{-a_{l_2}}) &= \nu(w_1 w_2) + \nu(c_{f_1,e_2}^{a_{l_2}}), \\ \nu(c_{e_1,e_2}^{-\alpha_{l_1}}) &= \nu(w_1/w_2) + \nu(c_{f_1,f_2}^{a_{l_1}}). \end{aligned} \tag{6.3.12}$$

Similar relations also applies to ${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$. Hence, the valuation identity (6.3.7) holds for these two tori as well.

Example 6.3.13. Consider the elliptic torus ${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[1,\varepsilon]}$ with

$$F(i) \oplus F \subset F(\sqrt{\varepsilon_{E_1}}) \oplus F, \qquad \sigma = (\sigma_1, \mathrm{Id}),$$

where $\varepsilon_{E_1} = s^2 + it^2$ so $\sigma_1(\sqrt{\varepsilon_{E_1}}) = -\sqrt{\varepsilon_{E_1}}$.

We have $\mu^* = (1, -1)$ and our Witt basis, as in Table A.4, is given by

$$\begin{gathered} e_1 = (1+i, 0), \quad e_2 = (s^{-1} i\sqrt{\varepsilon_{E_1}}, 1) \\ f_1 = (\tfrac{1}{2} - \tfrac{i}{2}, 0), \quad f_2 = (\tfrac{s^{-1} i\sqrt{\varepsilon_{E_1}}}{2}, \tfrac{-1}{2}) \\ g = ((s - t^2 s^{-1} i)\sqrt{\varepsilon_{E_1}}, 0) \end{gathered}$$

In this case, the scaling factors from Table 6.3 are $w_1 = w_2 = 1$. Given

$$v = (x\sqrt{(s^2+it^2)} + yi\sqrt{(s^2+it^2)},\, 0) \in \mathfrak{t}$$

we compute

$$\begin{gathered} c^{-a_{l_2}}_{e_1,f_2} = \frac{1}{s}\big((s^2+t^2)x + (s^2-t^2)y\big), \qquad c^{a_{l_2}}_{f_1,e_2} = -\frac{1}{4s}\big((s^2-t^2)x - (s^2+t^2)y\big), \\ c^{-a_{l_1}}_{e_1,e_2} = \frac{1}{2s}\big((s^2+t^2)x + (s^2-t^2)y\big), \qquad c^{a_{l_1}}_{f_1,f_2} = -\frac{1}{2s}\big((s^2-t^2)x - (s^2+t^2)y\big), \\ c^{a_{-s_1}}_{e_1,g} = \frac{x-y}{2s}, \qquad c^{a_{s_1}}_{f_1,g} = \frac{x+y}{4s}. \\ c_{b,b} = 0, \forall\ b \in B^*. \end{gathered}$$

In spite of the technical computations, we can see that the coefficients satisfy the valuation identity (6.3.7).

The case of ${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\eta,\varepsilon]}$ is similar but requires more computations; the coefficients also satisfy (6.3.7).

Corollary 6.3.14. *Suppose $\mathcal{T}$ is one of the special tori, in Table A.4, then for any $v \in \mathfrak{t}$, the coordinates of M_v with respect to B^* satisfy the relations*

$$\begin{aligned} \nu(c^{-\alpha_{l_2}}_{e_1,f_2}) &= \nu(w_1 w_2) + \nu(c^{\alpha_{l_2}}_{f_1,e_2}), \\ \nu(c^{-\alpha_{l_1}}_{e_1,e_2}) &= \nu(w_1/w_2) + \nu(c^{\alpha_{l_1}}_{f_1,f_2}), \\ \nu(c^{-\alpha_{s_i}}_{e_i,g}) &= \nu(w_i) + \nu(c^{\alpha_{s_i}}_{f_i,g}). \end{aligned}$$

In particular, the constants depend only on the valuations of the scales w_1, w_2.

$\mathfrak{t}$	$\mathcal{T}$-good v	$\nu(c^{\alpha_{l_1}}_{f_1,f_2})$	$\nu(c^{\alpha_{l_2}}_{f_1,e_2})$	$\nu(c^{\alpha_{s_1}}_{f_1,g})$	$\nu(c^{\alpha_{s_2}}_{f_2,g})$	$\nu(\frac{w_1}{w_2})$	$\nu(w_1 w_2)$	$\nu(w_1)$	$\nu(w_2)$
$*^{\varepsilon}\mathfrak{t}^2_{1,1,1}$	$(ai, bi, 0)$	r	r			0	0	0	0
$*^{\varepsilon}\mathfrak{t}^2_{\varpi,\varpi,1}$	$(ai, bi, 0)$	r	$r-1$			0	2	1	1
$*^{\varpi}\mathfrak{t}^{\varepsilon\varpi}_{\varepsilon,-1,\varepsilon}$	$(a\sqrt{\varpi}, bi\sqrt{\varpi}, 0)$	$r-\frac{1}{2}$	$r-\frac{1}{2}$	$r-\frac{1}{2}$		1	1	1	0
$*^{\varepsilon\varpi}\mathfrak{t}^{\varepsilon}_{1,\varpi,\varpi}$	$(ai\sqrt{\varpi}, 0, 0)$			$r-\frac{1}{2}$		0	2	1	1
	$(0, b\sqrt{\varepsilon}, 0)$	r	$r-1$			0	2	1	1
$*^{\varepsilon}\mathfrak{u}^{\varepsilon_{E_1}\varpi}_{4[\eta,\varepsilon]}$	$(a\sqrt{\varpi\varepsilon_{E_1}} + bi\sqrt{\varpi\varepsilon_{E_1}}, 0)$	$r-\frac{1}{2}$	$r-\frac{1}{2}$			1	1	1	0
$*^{\varepsilon}\mathfrak{u}^{\varepsilon_{E_1}}_{4[1,\varepsilon]}$	$(a\sqrt{\varepsilon_{E_1}} + bi\sqrt{\varepsilon_{E_1}}, 0)$	r	r	r		1	1	1	0

Table 6.7: *Valuations of structure constants associated to each "special" elliptic torus, computed using the Witt bases given in Table A.4 and equation (6.3.2). Here $\nu(c)$ denotes the valuation of the root coefficient in the matrix of a $\mathcal{T}$-good element of depth r. Gray cells indicate that the corresponding coefficient $c^{\alpha} = 0$.*

6.4 Computing $\mathcal{B}(\mathcal{T})$

In this Section, we determine the point of $\mathcal{B}(G)$ that corresponds to $\mathcal{B}(\mathcal{T})$, for each maximal elliptic torus $\mathcal{T}$, using Proposition 6.1.3, which says that $x \in \mathcal{B}(\mathcal{T})$ is the unique point for which $\mathfrak{t}_r \subset \mathfrak{g}_{x,r}$, for all $r \in \mathbb{R}$. Due to our careful choices, this point x will be found not only in the standard apartment $\mathscr{A}$, but in fact in the fundamental alcove.

We begin by relating this condition to the coefficients of the matrix M_v of an element $v \in \mathfrak{t}_r$. Recall that our positive roots are given by

$$\Phi^+ = \{a_{s_1} = \epsilon_1, \qquad a_{s_2} = \epsilon_2, \qquad a_{l_1} = \epsilon_1 - \epsilon_2, \qquad a_{l_2} = \epsilon_1 + \epsilon_2\}.$$

Theorem 6.4.1. *Let $\mathcal{T}$ be a maximal elliptic torus in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, with Lie algebra $\mathfrak{t}$ and fixed Witt basis $B^* = \{e_1, e_2, f_1, f_2, g\}$. Write $c^{\alpha}_{b_i,b_j}(v)$ for the coefficients of the matrix M_v of $v \in \mathfrak{t}$ with respect to B^*, where the superscript α redundantly denotes the corresponding root space, for ease of reference.*

Then a point $x \in \mathscr{A}$ satisfies $\mathcal{B}(T) = \{x\}$ if and only if for every $r \in \mathbb{R}$ and $v \in \mathfrak{t}_r \setminus \mathfrak{t}_{r+}$, x satisfies the inequalities

$$-\nu(c^{a_{l_1}}_{f_1,f_2}(v)) + r \leq (\epsilon_1 - \epsilon_2)(x) \leq \nu(c^{-a_{l_1}}_{e_1,e_2}(v)) - r,$$

$$-\nu(c^{a_{l_2}}_{f_1,e_2}(v)) + r \leq (\epsilon_1 + \epsilon_2)(x) \leq \nu(c^{-a_{l_2}}_{e_1,f_2}(v)) - r,$$

$$-\nu(c^{a_{s_1}}_{f_1,g}(v)) + r \leq \epsilon_1(x) \leq \nu(c^{-a_{s_1}}_{e_1,g}(v)) - r,$$

$$-\nu(c^{a_{s_2}}_{f_2,g}(v)) + r \leq \epsilon_2(x) \leq \nu(c^{-a_{s_2}}_{e_2,g}(v)) - r$$

and $\nu(c_{b,b}(v)) \geq r$ for each $b \in B^*$.

Proof. Suppose $\mathcal{B}(\mathcal{T}) \subset \mathscr{A}$. To satisfy $\mathfrak{t}_r \subset \mathfrak{g}_{x,r}$ for some $x \in \mathscr{A}$, the matrix M_v, with $v \in \mathfrak{t}_r$, must lie in $\mathfrak{g}_{x,r}$, which was described explicitly in Theorem 2.5.7(ii). This gives a lower bound on the valuation of each matrix entry. On the diagonal it is the condition that $\nu(c_{b,b}(v)) \geq r$. For each positive root α we have

$$\nu(c^{\alpha}(v)) \geq -\lfloor \alpha(x) - r \rfloor$$

so that $\alpha(x) - r \geq -\nu(c^{\alpha}(v))$ and for each negative root α takes the form

$$\nu(c^{-\alpha}(v)) \geq \lceil \alpha(x) + r \rceil$$

so that $\alpha(x) + r \leq \nu(c^{-\alpha}(v))$. This is equivalent to

$$-\nu(c^{\alpha}(v)) + r \leq \alpha(x) \leq \nu(c^{-\alpha}(v)) - r.$$

Writing these inequalities for each of the positive roots α, using the notation of our matrix entries, yields the desired inequalities.

Conversely, if all of these inequalities hold for all r, then we conclude by Proposition 6.1.3 that $\mathfrak{t}_r \subset \mathfrak{g}_{x,r}$. In particular, we may conclude $\mathcal{B}(T) \subset \mathscr{A}$. ∎

Note that the diagonal components to not impose a constraint on x, but are necessary to allow us to conclude that $x \in \mathscr{A}$, as hypothesized.

Before continuing, let us make a few observations and sketch the strategy.

If $X \in \mathfrak{t}_r \setminus \mathfrak{t}_{r^+}$ is G-generic of depth r, then by Proposition 5.1.9, X is a $\mathcal{T}$-good element of depth r, and furthermore, its centralizer is exactly $\mathcal{T}$. Therefore by Theorem 6.1.2, it suffices to verify the inequalities of Theorem 6.4.1 for this single element $v = X$.

However, when no G-generic elements with centralizer equal to $\mathcal{T}$ exist (that is, for each of the tori of Theorem 5.3.5), then the $\mathcal{T}$–good elements have centralizers that are strictly larger than $\mathcal{T}$, so considering the relations for a single $\mathcal{T}$-good element is not enough. In this case, there are $\mathcal{T}$-good elements X' and X'' with $C_G(X') = \mathcal{M}'$ and $C_G(X'') = \mathcal{M}''$, so that $x \in \mathcal{B}(\mathcal{M}') \cap \mathcal{B}(\mathcal{M}'')$. Furthermore, the $\mathcal{T}$–good elements occur in two depth families $r_1 + \mathbb{Z}$ and $r_2\mathbb{Z}$ where $r_1 - r_2 \in \frac{1}{2} + \mathbb{Z}$. We'll see that each depth family controls a different collection of root coefficients in the matrix M_v. Thus, imposing $\mathfrak{t}_r \subset \mathfrak{g}_{x,r}$ for $r \in \mathbb{Z}$ yields one set of inequalities, while imposing it for $r \in \frac{1}{2} + \mathbb{Z}$ yields the complementary set, allowing us determine the unique point x such that $\mathcal{B}(\mathcal{T}) = \{x\}$.

6.4.1 Buildings of elliptic tori with two diagonal hyperbolic forms, including special cases

We begin with the tori whose coefficients follow the valuation identity (6.3.7). These are all the tori covered in Section 6.3.2 as well as all the special tori in Section 6.3.5; equivalently, there are all tori in Tables A.3 and A.4.

Theorem 6.4.2. *Assume $\mathcal{T}$ is as above, with Witt scales w_1, w_2 as given in Tables A.3 and A.4. Then $\mathcal{B}(\mathcal{T}) = \{x\}$ where*

$$x = \left(\frac{\nu(w_1)}{2}, \frac{\nu(w_2)}{2}\right).$$

Proof. Set

$$S_{l_1} := \nu\left(\frac{w_1}{w_2}\right), \quad S_{l_2} := \nu(w_1 w_2), \quad S_{s_1} := \nu(w_1), \quad S_{s_2} := \nu(w_2).$$

Let $v \in \mathfrak{t}_r$. By Proposition 6.3.5, and Corollary 6.3.14, the diagonal coefficients of the corresponding matrix M_v are 0, and so satisfy $\nu(c_{b,b}(v)) \geq r$. By Theorem 6.4.1, it remains to have four sets of inequalities. Using Proposition 6.3.5, and Corollary 6.3.14, these inequalities reduce to the following system:

$$\begin{aligned}
-\nu\big(c^{a_{l_1}}_{f_1,f_2}\big) + r \leq (\epsilon_1 - \epsilon_2)(x) \leq S_{l_1} + \nu\big(c^{a_{l_1}}_{f_1,f_2}\big) - r,\\
-\nu\big(c^{a_{l_2}}_{f_1,e_2}\big) + r \leq (\epsilon_1 + \epsilon_2)(x) \leq S_{l_2} + \nu\big(c^{a_{l_2}}_{f_1,e_2}\big) - r,\\
-\nu\big(c^{a_{s_1}}_{f_1,g}\big) + r \leq \epsilon_1(x) \leq S_{s_1} + \nu\big(c^{a_{s_1}}_{f_1,g}\big) - r,\\
-\nu\big(c^{a_{s_2}}_{f_2,g}\big) + r \leq \epsilon_2(x) \leq S_{s_2} + \nu\big(c^{a_{s_2}}_{f_2,g}\big) - r.
\end{aligned} \tag{6.4.1}$$

We first note that if

$$\nu(c^\alpha) = r - \tfrac{S_\alpha}{2}, \tag{6.4.2}$$

then

$$-\nu(c^\alpha) + r = S_\alpha + \nu(c^\alpha) - r$$

and so the corresponding inequality holds if and only if

$$\alpha(x) = \tfrac{S_\alpha}{2}. \tag{6.4.3}$$

Furthermore, since $\nu(w_1/w_2) = \nu(w_1) - \nu(w_2)$ and $\nu(w_1 w_2) = \nu(w_1) + \nu(w_2)$, if (6.4.3) holds for at least two roots α, then it holds for all.

We summarized the valuations of these coefficients in Table 6.4 for the tori appearing in Section 6.3.2. In all cases, the given coefficient is either zero (meaning the corresponding inequality is trivially satisfied) or satisfies the condition (6.4.2).

For the first five rows, corresponding to the tori

$${}^{\varepsilon}\mathfrak{t}^2_{1,1,1},\ {}^{\varepsilon}\mathfrak{t}^2_{\varpi,\varpi,1},\ {}^{\varpi'}\mathfrak{t}^2,\ {}^{\varpi}\mathfrak{t}^{\varepsilon\varpi},\ {}^{\varepsilon}\mathfrak{t}^{\varepsilon,\varpi}_{4[1,1]},$$

we have G-generic elements and we have

$$\nu\big(c^{a_{l_1}}_{f_1,f_2}\big) = r - \tfrac{S_{l_1}}{2} = r - \frac{1}{2}\nu(w_1/w_2), \quad \nu\big(c^{a_{l_2}}_{f_1,e_2}\big) = r - \tfrac{S_{l_2}}{2} = r - \frac{1}{2}\nu(w_1 w_2).$$

Therefore the point x must satisfy the equalities (6.4.3), specifically,

$$(\epsilon_1 - \epsilon_2)(x) = \frac{1}{2}\nu(w_1/w_2), \qquad (\epsilon_1 + \epsilon_2)(x) = \frac{1}{2}\nu(w_1 w_2), \tag{6.4.4}$$

which yields $x = \left(\frac{1}{2}\nu(w_1), \frac{1}{2}\nu(w_2)\right)$ as the unique solution to the system of inequalities. By Theorem 6.4.1, we conclude $\mathcal{B}(\mathcal{T}) = \{x\} \subset \mathscr{A}$.

For the remaining tori in Table 6.4, namely ${}^{\varpi'}\mathfrak{t}^{\varepsilon}_{1,\varpi,\lambda}$ and ${}^{\varpi'}\mathfrak{t}^{\varepsilon}_{\varepsilon,1,\lambda}$, each of the two choices of a $\mathcal{T}$-good element yields at least one root α for which the coefficient satisfies (6.4.2). (These arise from different depths r.) Together they yield at least two distinct positive roots for which (6.4.3) hold, and this again implies

$$\epsilon_1(x) = \frac{\nu(w_1)}{2}, \qquad \epsilon_2(x) = \frac{\nu(w_2)}{2}.$$

Since any element of $\mathfrak{t}_r$ can be written as a sum of a $\mathcal{T}$-good element of depth r and an element of greater depth, we conclude that the inclusion $M_v \in \mathfrak{g}_{x,r}$ holds for all $v \in \mathfrak{t}_r$, as required.

Finally, we repeat these arguments with the special tori of Section 6.3.5, whose coefficient valuations are listed in Table 6.7. Once again, every coefficient satisfies (6.4.2) for at least two roots, whence (6.4.3) holds for two independent roots, giving the point $x = (\nu(w_1)/2, \nu(w_2)/2)$, as required. ∎

Consequently, for tori arising from this case, the unique point $x \in \mathscr{A}$ is specified entirely by the values of w_1 and w_2. We record these points in the final columns of Table A.3 and Table A.4.

Remark 6.4.3. As we have $\epsilon_i(x) = \frac{\nu(w_i)}{2}$, we intend for x to lie in the chosen fundamental domain (Figure 2.3). To do so, it should satisfy $0 \leq x_2 \leq x_1 \leq \frac{1}{2}$, which happens if and only if

$$0 \leq \nu(w_2) \leq \nu(w_1) \leq 1.$$

This explains and justifies our choice and ordering of the scales: it was to yield x to lie in our favourite fundamental domain.

6.4.2 Building of tori admitting two cross-term forms

We now study the tori admitting two cross-term forms, as described in Section 6.3.3.

Theorem 6.4.4. *Assume $\mathcal{T} = {}^{\alpha}\mathcal{T}_4^{\varepsilon,\varpi}$ is an elliptic torus treated in Section 6.3.3 and let w_1, w_2 be the Witt scales recorded in Table A.5. Then $\mathcal{B}(\mathcal{T}) = \{x\}$ where*

$$\epsilon_1(x) = \frac{2\nu(w_1) + \nu(\alpha^{-1})}{4} \qquad \epsilon_2(x) = \frac{2\nu(w_2) + \nu(\alpha^{-1})}{4} \tag{6.4.5}$$

Proof. Set

$$S_1 := \nu\left(\frac{w_1}{w_2}\right), \qquad S_2 := \nu(w_1 w_2 \alpha^{-1}).$$

Our argument is very similar to that of Theorem 6.4.2. Considering Proposition 6.3.7 and Table 6.5, for any $v \in \mathfrak{t}_r$, many coefficients are zero. Thus the inequalities of Theorem 6.4.1 are automatically satisfied. In fact it suffices to prove that the point x in (6.4.5) satisfies

$$\begin{aligned} -\nu\big(c^{a_{l_1}}_{f_1,f_2}\big) + r \le (\epsilon_1 - \epsilon_2)(x) \le S_1 + \nu\big(c^{a_{l_1}}_{f_1,f_2}\big) - r, \\ -\nu\big(c^{a_{l_2}}_{f_1,e_2}\big) + r \le (\epsilon_1 + \epsilon_2)(x) \le S_2 + \nu\big(c^{a_{l_2}}_{f_1,e_2}\big) - r. \end{aligned} \tag{6.4.6}$$

As in the previous lemma, we note from Table 6.5 that, for possibly different $\mathcal{T}$-good elements, $\nu(c^{a_{l_i}}) = r - \frac{S_i}{2}$ for $i = 1, 2$. Therefore (6.4.6) admits the unique solution

$$a_{l_i}(x) = \frac{S_i}{2}.$$

Hence

$$(\epsilon_1 - \epsilon_2)(x) = \frac{S_1}{2}, \qquad (\epsilon_1 + \epsilon_2)(x) = \frac{S_2}{2}$$

which yields solution

$$\epsilon_1(x) = \frac{2\nu(w_1) + \nu(\alpha^{-1})}{4}, \qquad \epsilon_2(x) = \frac{2\nu(w_2) + \nu(\alpha^{-1})}{4}.$$

It follows that $\mathfrak{t}_r \subset \mathfrak{g}_{x,r}$ for every r. ∎

Thus in this situation, the biquadratic case, the building of the torus is completely determined by the choice of the fixed quadratic field together with the valuation of w_1 and w_2. We record these points in the final column of Table A.5.

Remark 6.4.5. Note that the point x found in Theorem 6.4.4 lies in the fundamental domain (see Figure 2.3). In the case of the torus ${}^{\varepsilon}\mathcal{T}_4^{\varepsilon,\varpi}$, this was due to our choice of the ordering of the basis. Namely, when $\alpha = \varepsilon$, to satisfy $a_{s_2}(x) = \frac{1}{4}(2\nu(w_2) + \nu(\alpha^{-1})) \ge 0$, $a_{l_1}(x) = \frac{1}{2}(\nu(w_1) - \nu(w_2)) \ge 0$, and $a_{s_1}(x) = \frac{1}{4}(2\nu(w_1) + \nu(\alpha^{-1})) \le \frac{1}{2}$, one needs to choose

$$\nu(w_1) \ge \nu(w_2),$$

exactly as in the previous case.

6.4.3 Building of tori admitting a diagonal and a cross-term form

Finally, we address the remaining tori. These were the tori that are quartic with a unique subfield and were treated in Section 6.3.4. We treat them individually.

Lemma 6.4.6. *Let $\mathcal{T}$ be an elliptic torus associated to a quartic extension of F with a fixed quadratic subfield as above. Fix a Witt basis $B^* = \{e_1, e_2, f_1, f_2, g\}$ as in Table A.6, and an element $v \in \mathfrak{t}_r \setminus \mathfrak{t}_{r^+}$. Set $S_1 := \nu(w_1)$. Then $M_v \in \mathfrak{g}_{x,r}$ if and only if x satisfies*

$$-\nu\left(c^{a_{l_1}}_{f_1,f_2}\right) + r \;\le\; (\epsilon_1 - \epsilon_2)(x) \;\le\; S_1 + \nu\left(c^{a_{l_2}}_{f_1,e_2}\right) - r,$$

$$-\nu\left(c_{f_1,e_2}^{a_{l_2}}\right) + r \ \leq\ (\epsilon_1+\epsilon_2)(x)\ \leq\ S_1 + \nu\left(c_{e_2,e_1}^{a_{l_1}}\right) - r,$$
$$-\nu\left(c_{f_2,g}^{a_{s_2}}\right) + r \ \leq\ \epsilon_2(x)\ \leq\ \nu\left(c_{e_2,g}^{-a_{s_2}}\right) - r.$$

Proof. By Theorem 6.4.1, the condition $M_v \in \mathfrak{g}_{x,r}$ reduces to verifying the valuation of the diagonal elements of M_v and a set of inequalities. In this case, by Proposition 6.3.9, most inequalities are satisfied and these are the remaining conditions to be verified. ∎

Proposition 6.4.7. *Let $\mathcal{T}$ be a torus of type ${}^{\varepsilon^i\varpi}\Upsilon_4^{\varepsilon^i\varpi}$ as in Table A.6. Then*

$$\mathcal{B}(\mathcal{T}) = \left\{\left(\tfrac{1}{2}, \tfrac{1}{4}\right)\right\}.$$

Proof. We apply Lemma 6.4.6. By Table 6.6, for these tori we have $S_1 = 1$ and

$$\nu\!\left(c^{a_{l_1}}\right) = \nu\!\left(c^{a_{l_2}}\right) = \nu\!\left(c^{a_{s_2}}\right) = r - \tfrac{1}{4}, \quad \nu\!\left(c_{e_2,g}^{-a_{s_2}}\right) = r + \tfrac{3}{4}.$$

Substituting these values into the inequalities of Lemma 6.4.6 and writing $x_1 = \epsilon_1(x)$ and $x_2 = \epsilon_2(x)$ yields

$$\frac{1}{4} \leq x_1 - x_2 \leq \frac{3}{4}, \quad \frac{1}{4} \leq x_1 + x_2 \leq \frac{3}{4}, \quad \frac{1}{4} \leq x_2 \leq \frac{3}{4}.$$

Evidently $x = (1/2, 1/4)$ satisfies these inequalities, and thus should be the unique solution since the point $\mathcal{B}(\mathcal{T})$ is unique. To confirm this, we solve the system as follows. From the first two inequalities,

$$\frac{1}{4} + x_2 \leq x_1 \leq \frac{3}{4} + x_2, \qquad \frac{1}{4} - x_2 \leq x_1 \leq \frac{3}{4} - x_2.$$

Hence

$$\max\left(\tfrac{1}{4} + x_2, \tfrac{1}{4} - x_2\right) \leq x_1 \leq \min\left(\tfrac{3}{4} + x_2, \tfrac{3}{4} - x_2\right).$$

Since $x_2 \geq \frac{1}{4}$, we obtain

$$\frac{1}{4} + x_2 \leq x_1 \leq \frac{3}{4} - x_2.$$

For this interval to be nonempty, we must have

$$\frac{1}{4} + x_2 \leq \frac{3}{4} - x_2 \iff x_2 \leq \frac{1}{4}.$$

Combining with $\frac{1}{4} \leq x_2 \leq \frac{3}{4}$, we obtain $x_2 = \frac{1}{4}$. Substituting back gives $x_1 = \frac{1}{2}$. Therefore

$$\mathcal{B}(\mathcal{T}) = \{(\tfrac{1}{2}, \tfrac{1}{4})\}.$$

∎

Proposition 6.4.8. *Let $\mathcal{T}$ be a torus of type ${}^{\varepsilon}\Upsilon_4^{\varepsilon E_1}$ with $-1^{\square}$, as in Table A.6. Then*

$$\mathcal{B}(\mathcal{T}) = \{(0,0)\}.$$

Proof. We apply Lemma 6.4.6. By Table 6.6, all relevant valuations are equal to r, and $S_1 = 0$. Substituting into the inequalities yields $x = (0,0)$ immediately. ∎

Now, the remaining torus is $({}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\eta,\varepsilon]}, -1 \in F^{\times 2})$ which requires separate consideration, as the hyperbolic planes permuted.

Lemma 6.4.9. *Let $\mathcal{T} = {}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[1,\varepsilon]}$ be the elliptic torus associated to a quartic extension of F with a fixed quadratic subfield $F(\sqrt{\alpha})$. Fix a Witt basis $B^* = \{e_1, e_2, f_1, f_2, g\}$ as in Table A.6, and an element $v \in \mathfrak{t}_r \setminus \mathfrak{t}_{r^+}$. Set $S_2 := \nu(w_2)$. Then $M_v \in \mathfrak{g}_{x,r}$ if and only if x satisfies*

$$-\nu\left(c^{a_{l_1}}_{f_1,f_2}\right) + r \ \le\ (\epsilon_1 - \epsilon_2)(x)\ \le\ -S_2 + \nu\left(c^{-a_{l_2}}_{e_1,f_2}\right) - r,$$
$$-\nu\left(c^{a_{l_2}}_{f_1,e_2}\right) + r \ \le\ (\epsilon_1 + \epsilon_2)(x)\ \le\ S_2 + \nu\left(c^{-a_{l_1}}_{e_1,e_2}\right) - r,$$
$$-\nu\left(c^{a_{s_2}}_{f_1,g}\right) + r \ \le\ \epsilon_1(x)\ \le\ \nu\left(c^{-a_{s_2}}_{e_1,g}\right) - r.$$

Proof. By Theorem 6.4.1, the condition $M_v \in \mathfrak{g}_{x,r}$ reduces to verifying the valuation of the diagonal elements of M_v and a set of inequalities. In this case, by Proposition 6.3.10, most inequalities are satisfied and these are the remaining conditions to be verified. ∎

Proposition 6.4.10. *Let $\mathcal{T} = {}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[1,\varepsilon]}$ as above. Then*

$$\mathcal{B}(\mathcal{T}) = \left\{\left(\tfrac{1}{2}, 0\right)\right\}.$$

Proof. We apply Lemma 6.4.6. By Table 6.6, for these tori we have $S_2 = 0$ and

$$\nu\!\left(c^{\pm a_{l_1}}\right) = \nu\!\left(c^{\pm a_{l_2}}\right) = \nu\!\left(c^{-a_{s_1}}\right) = r + \tfrac{1}{2}, \quad \nu\!\left(c^{a_{s_1}}_{f_1,g}\right) = r - \tfrac{1}{2}.$$

Substituting these values into the inequalities of Lemma 6.4.6 and writing $x_1 = \epsilon_1(x)$ and $x_2 = \epsilon_2(x)$ yields

$$-\frac{1}{2} \le x_1 - x_2 \le \frac{1}{2}, \quad -\frac{1}{2} \le x_1 + x_2 \le \frac{1}{2}, \quad \frac{1}{2} \le x_1 \le \frac{1}{2}.$$

Evidently $x = (1/2, 0)$ satisfies these inequalities, and thus should be the unique solution since the point $\mathcal{B}(\mathcal{T})$ is unique. To confirm this, we solve the system as follows. clearly,

$$x_1 = \frac{1}{2}$$

Therefore, by substitution in the two other inequalities, we have

$$0 \le x_2 \le 1, \qquad \text{and} \qquad -1 \le x_2 \le 0$$

Hence $x_2 = 0$, and

$$\mathcal{B}(\mathcal{T}) = \{(\tfrac{1}{2}, 0)\}.$$

∎

Remark 6.4.11. Once again the point we found, by our choice of basis, is in the fundamental domain.

We now illustrate the results of this section in Figure 6.1.

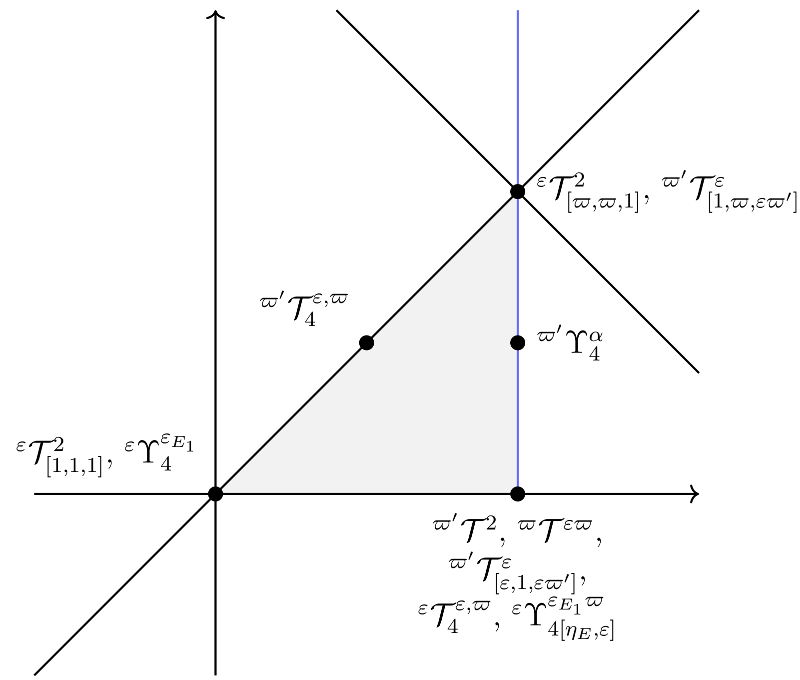

Figure 6.1: *We identify the points corresponding to each of the conjugacy classes of maximal elliptic tori in the fundamental domain of the action of G on $\mathcal{B}(G)$. For brevity, we omit subscripts when all parameters yield the same point, and write ϖ' for the choice of ϖ or $\varepsilon\varpi$.*

6.5 Building of intermediate twisted Levi subgroups of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$

We now turn to the buildings of the proper twisted Levi subgroups that contain these tori. More precisely, we identify the intersection of their embeddings into $\mathcal{B}(G)$ with the standard apartment $\mathscr{A}$. By our choices of embeddings of these twisted Levi subgroups into $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$, they contain maximal split tori that are contained in the maximal split torus of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. Therefore the standard apartment for each twisted Levi subgroup, if it is nontrivial, embeds as an affine line in the standard apartment $\mathscr{A}$.

Given the work in Section 6.4, we have two complementary methods to arrive at the answer:

Indirect approach. For a given twisted Levi subgroup $\mathcal{M}$, we consider all elliptic tori $\mathcal{T}$ contained in it. Since $\mathcal{B}(\mathcal{T}) \subset \mathcal{B}(\mathcal{M})$, these points are all contained in $\mathcal{B}(\mathcal{M})$. When $\mathcal{M}$ is compact, this uniquely identifies $\mathcal{B}(\mathcal{M})$. When $\mathcal{M}$ is non-compact, connecting the points corresponding to its tori gives an affine line which is the intersection of $\mathcal{B}(\mathcal{M})$ with $\mathscr{A}$, and particularly, with the portion contained in $\mathcal{C}_{\mathrm{fund}}$.

Direct approach. For each twisted Levi subgroup $\mathcal{M}$, choose a maximal elliptic torus $\mathcal{T}$ contained in $\mathcal{M}$, and then choose a $\mathcal{T}$-good element of depth r, $X \in \mathfrak{t}_r \setminus \mathfrak{t}_{r^+}$, such that $C_G(X) = \mathcal{M}$. Then by Theorem 6.1.2, it suffices to prove that $X \in \mathfrak{g}_{x,r}$ to deduce that $x \in \mathcal{B}(\mathcal{M})$. This leads to a system of inequalities from Theorem 6.4.1 that we can solve for x.

In this section, we illustrate these approaches to determining the buildings of our twisted Levi subgroups through some representative examples.

Example 6.5.1. The subgroup ${}^{\varepsilon}\mathcal{S}_{1,s}$. *Indirect approach.* The subgroup ${}^{\varepsilon}\mathcal{S}_{1,s}$ has split rank one, so the apartments in its building are one-dimensional. The elliptic tori contained in ${}^{\varepsilon}\mathcal{S}_{1,s}$ are ${}^{\varepsilon}\mathcal{T}^2_{[1,1,1]}$, ${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varepsilon\varpi]}$, and ${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varpi]}$. As recorded in Tables A.1 and illustrated in Figure 6.1, the images of their buildings in the standard apartment are $(0,0)$ for ${}^{\varepsilon}\mathcal{T}^2_{[1,1,1]}$ and $(\frac{1}{2},0)$ for the other two tori. The image of $\mathcal{B}({}^{\varepsilon}\mathcal{S}_{1,s})$ in the standard apartment is therefore the unique line passing through these points, and hence is the affine line joining $(0,0)$ and $(\frac{1}{2},0)$.

Direct approach. Choose an elliptic torus that is embedded in the subgroup that has no G-generic elements. In this case we may choose ${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varepsilon\varpi]} \subset {}^{\varepsilon}\mathcal{S}_{1,s}$, and a non-regular element $X = (0, a\sqrt{\varepsilon}, 0) \in \mathfrak{t}_r \setminus \mathfrak{t}_{r^+}$ where $C_G(X) = {}^{\varepsilon}\mathcal{S}_{1,s}$. By Theorem 6.1.2,

$$\{x \in \mathcal{B}(G) \mid d(x, X) = d(X)\} = \mathcal{B}({}^{\varepsilon}\mathcal{S}_{1,s}).$$

We have only $\nu(c^{a_{s_2}}_{f_2,g}) = r - \frac{S_{s_2}}{2}$ in Table 6.4. Hence, $a_{s_2}(x) = \nu(w_2)/2 = 0$ in (6.4.1). Therefore the set of points $x \in \mathscr{A}$ for which $X \in \mathfrak{g}_{x,r}$ is

$$\mathcal{B}({}^{\varepsilon}\mathcal{S}_{1,s}) \cap \mathscr{A} = \{(x_1, 0) \in \mathscr{A}\}$$

which recovers the same affine line.

The same arguments apply to the other split twisted Levi subgroups ${}^{\varpi}\mathcal{S}_{1,s}$, and ${}^{\varepsilon\varpi}\mathcal{S}_{1,s}$, yielding one-dimensional intersections with $\mathscr{A}$. However, as these two twisted Levi subgroups are ramified, we instead obtain (for $\varpi' \in \{\varpi, \varepsilon\varpi\}$) that

$$\mathcal{B}({}^{\varpi'}\mathcal{S}_{1,s}) \cap \mathscr{A} = \left\{(\frac{1}{2}, x_2) \in \mathscr{A}\right\},$$

which is the affine line joining $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0)$. Note that this coincides with joining the two points corresponding to the buildings of the maximal elliptic tori contained in these twisted Levi subgroups.

Finally, for the compact twisted Levi subgroups

$${}^{\varepsilon}\mathcal{S}_{\varpi,c}, \qquad {}^{\varpi}\mathcal{S}_{\varepsilon,c}, \qquad {}^{\varepsilon\varpi}\mathcal{S}_{\varepsilon,c},$$

both approaches show that the building reduces to a single point: $(\frac{1}{2}, \frac{1}{2})$ in the unramified case and $(\frac{1}{2}, 0)$ in the ramified cases. This is illustrated in Figure 6.2.

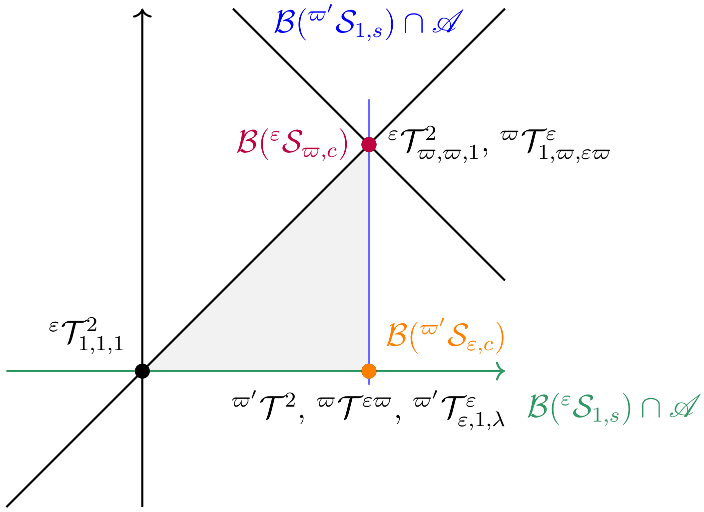

Figure 6.2: *The buildings of the six twisted Levi subgroups of type $\mathcal{S}$ in the standard apartment $\mathscr{A}$ of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. Here, $\varpi' \in \{\varpi, \varepsilon\varpi\}$. The building $\mathcal{B}({}^{\varepsilon}\mathcal{S}_{1,s})$ is represented by the green segment, $\mathcal{B}({}^{\varpi'}\mathcal{S}_{1,s})$ by the light blue segment, $\mathcal{B}({}^{\varepsilon}\mathcal{S}_{\varpi,c})$ by the purple point, and $\mathcal{B}({}^{\varpi'}\mathcal{S}_{\varepsilon,c})$ by the orange point. We consider only those elliptic tori that embed into the given twisted Levi subgroup.*

We observe that the building of a twisted Levi subgroup corresponding to a short root is perpendicular to that short root, so the buildings are oriented vertically and horizontally. When we consider a twisted Levi subgroup corresponding to a long root, the building should embed diagonally.

Example 6.5.2. Consider the subgroup ${}^{\varepsilon\varpi}\mathcal{L}_s$. Choose an elliptic torus that is embedded in the subgroup that has no G-generic elements. In this case we may choose $\mathcal{T} = {}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$, and a non-regular element $X = (a\sqrt{\varepsilon\varpi}, 0) \in \mathfrak{t}_r \setminus \mathfrak{t}_{r^+}$ where $C_G(X) = {}^{\varepsilon\varpi}\mathcal{L}_s$. By Theorem 6.1.2,

$$\{x \in \mathcal{B}(G) \mid d(x, X) = d(X)\} = \mathcal{B}({}^{\varepsilon\varpi}\mathcal{L}_s).$$

We have in Table 6.5, that only $\nu(c^{a_{l_2}}_{f_1,e_2}) = \nu(w_1 w_2 \alpha^{-1})$, hence $a_{l_2}(x) = \nu((w_1 w_2 \alpha^{-1})/2) = 1/2$ in (6.4.6). This yields

$$\mathcal{B}({}^{\varepsilon\varpi}\mathcal{L}_s) \cap \mathscr{A} = \{(x_1, x_2) \in \mathscr{A} : x_1 + x_2 = \tfrac{1}{2}\},$$

which gives the line segment between $(\frac{1}{2}, 0)$ and $(\frac{1}{4}, \frac{1}{4})$. This agrees with the indirect approach, since by Table A.2, we have the following elliptic tori embedded in ${}^{\varepsilon\varpi}\mathcal{L}_s$

$${}^{\varepsilon\varpi}\mathcal{T}^2_{[1,-1,1]}, {}^{\varepsilon\varpi}\mathcal{T}^2_{[\varepsilon,-\varepsilon,1]}, {}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{-\varepsilon},1]}, {}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$$

Similar calculations show that $\mathcal{B}({}^{\varpi}\mathcal{L}_s)$ intersects $\mathscr{A}$ in the same line. For the unramified case, performing the calculation with a $\mathcal{T}$-good nonregular element X whose centralizer is

${}^{\varepsilon}\mathcal{L}_s$, yields

$$\mathcal{B}({}^{\varepsilon}\mathcal{L}_s) \cap \mathscr{A} = \{(x_1, x_2) \in \mathrm{A} : x_1 - x_2 = 0\}$$

which is the affine line joining $(0,0)$ and $(\frac{1}{2}, \frac{1}{2})$. We obtain the same result by the indirect method since this group contains the tori

$${}^{\varepsilon}\mathcal{T}^2_{[1,1,1]},\ {}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi,1]},\ {}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]},\ {}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]}$$

by Table A.2.

We illustrate these embeddings in Figure 6.3 .

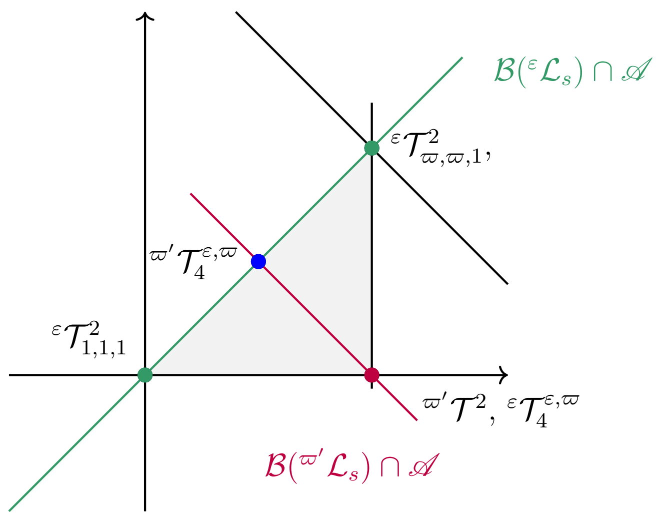

Figure 6.3: *The buildings of the three twisted Levi subgroups of type M^{Long} in the standard apartment $\mathscr{A}$ of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. Here, $\varpi' \in \{\varpi, \varepsilon\varpi\}$. The building $\mathcal{B}({}^{\varepsilon}\mathcal{L}_s)$ is represented by the green segment, $\mathcal{B}({}^{\varpi'}\mathcal{L}_s)$ by the red segment. We consider only those elliptic tori that embed into the given twisted Levi subgroup.*

6.6 Vertices y and reductive quotients

As summarized in Section 2.7, two more ingredients of the construction of supercuspidal representations are a vertex y in the building of the smallest twisted Levi subgroup G^0 of the sequence, and the group $G^0_{y,0}/G^0_{y,0+}$, called the reductive quotient. In this section, we explain how to identify both.

6.6.1 Valid points y for the construction

When $G^0 = \mathcal{T}$ is a maximal elliptic torus, or a compact nonabelian twisted Levi subgroup $\mathcal{M}_c$ then there is a unique choice for y, because it must lie in the building of G^0. Therefore these points have been determined.

However, when G^0 is a proper nonabelian noncompact twisted Levi subgroup $\mathcal{M}_s$, one must choose a vertex

$$y \in \mathcal{B}(\mathcal{M}_s).$$

We emphasize that this vertex is taken in the Bruhat–Tits building of $\mathcal{M}_s$ and need not be a vertex of the building of the ambient group G.

One way to identify the vertices is as follows.

Lemma 6.6.1. *Let* $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$*. The buildings of the least ramified maximal elliptic tori contained in* $\mathcal{M}_s$ *correspond to vertices of* $\mathcal{B}(\mathbb{M}_s, F)$*.*

Proof. The affine root hyperplanes defining the simplicial structure of $\mathcal{B}(\mathbb{M}_s, F)$ occur at integer multiples of the affine roots in $\Phi(\mathbb{M}_s, T')$, where T' is a maximal split torus of $\mathbb{M}_s$. Suppose $\mathcal{M}_s$ contains a maximal elliptic torus $\mathbb{T}$ that splits over an unramified extension E of F. Then $\mathscr{A}(\mathbb{M}_s, \mathbb{T}, E)$ will be an apartment of $\mathcal{B}(\mathbb{M}_s, E)$, and its intersection with the standard apartment $\mathscr{A}$ is thus a hyperplane of $\mathcal{B}(\mathbb{M}_s, F) \cap \mathscr{A}$, that is, a vertex. Since $\mathcal{B}(\mathbb{T}, F) \cap \mathscr{A}$ is contained in this intersection, we deduce the result.

On the other hand, when $\mathcal{M}_s$ is ramified, it will not contain any maximal elliptic tori that split over an unramified extension. In that case, consider a maximal elliptic torus $\mathbb{T}$ that is partially split over an unramified extension E. Then since $\mathcal{M}_s$ has split rank 1 over E and over F, we deduce again that $\mathscr{A}(\mathbb{M}_s, \mathbb{T}, E)$ will be an apartment of $\mathcal{B}(\mathbb{M}_s, E)$, and again the result follows. ∎

In contrast, for ramified tori the affine roots take fractional values because they split over a ramified extension, so the associated hyperplanes occur at fractional translates, and the distinguished point lies in the interior of a positive-dimensional facet of both $\mathcal{B}(G)$ and $\mathcal{B}(\mathcal{M})$.

Thus the process we use is that among all maximal tori contained in $\mathcal{M}_s$, we identify all maximal tori $\mathcal{T} \subset \mathcal{M}_s$ that are the least ramified; their corresponding points are the vertices of $\mathcal{B}(\mathcal{M}_s)$ lying in $\mathscr{A}$. Doing this for each of the twisted Levi subgroups of the families $\mathcal{S}_s$ and $\mathcal{L}_s$ yields Table 6.8.

Note that only ${}^{\varepsilon}\mathcal{L}_s \cong U_\varepsilon(\mathbb{H})$ yields two vertices. This is because in the building of the unramified unitary group, the two vertices are not conjugate, whereas for $\mathcal{S}_s \cong \mathrm{SO}(\mathbb{H} \oplus \langle \alpha \rangle)$ and ${}^{\varpi'}\mathcal{L}_s \cong U_{\varpi'}(\mathbb{H})$ all vertices of their corresponding buildings are conjugate.

In Figure 6.2, the vertices of the twisted Levi subgroups of the form $\mathcal{S}_s$ are exactly the vertices of G that they meet. This is also true for ${}^{\varepsilon}\mathcal{L}_s$ in Figure 6.3, but the buildings of the ramified groups ${}^{\varpi'}\mathcal{L}_s$ do not meet any vertices of $\mathcal{B}(G)$, the single conjugacy class of vertices in $\mathcal{B}({}^{\varpi'}\mathcal{L}_s)$ occurs at a nonvertex of $\mathcal{B}(G)$.

Twisted Levi $\mathcal{M}$	$\mathcal{T}^{\text{unram}}$	$\mathcal{B}(\mathcal{T}^{\text{unram}}) = \{y\}$
${}^{\varepsilon}\mathcal{S}_{1,s}$	${}^{\varepsilon}\mathcal{T}^{2}_{[1,1,1]}$	$(0,0)$
${}^{\varpi}\mathcal{S}_{1,s}$	${}^{\varepsilon}\mathcal{T}^{\varpi}_{[\varpi,1,\varepsilon\varpi]}$	$(\frac{1}{2},\frac{1}{2})$
${}^{\varepsilon\varpi}\mathcal{S}_{1,s}$	${}^{\varepsilon}\mathcal{T}^{\varepsilon\varpi}_{[\varpi,1,\varpi]}$	$(\frac{1}{2},\frac{1}{2})$
${}^{\varepsilon}\mathcal{L}_{s}$	${}^{\varepsilon}\mathcal{T}^{2}_{[1,1,1]}$, ${}^{\varepsilon}\mathcal{T}^{2}_{[\varpi,\varpi,1]}$	$(0,0)$, $(\frac{1}{2},\frac{1}{2})$
${}^{\varpi}\mathcal{L}_{s}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]}$	$(\frac{1}{4},\frac{1}{4})$
${}^{\varepsilon\varpi}\mathcal{L}_{s}$	${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$	$(\frac{1}{4},\frac{1}{4})$

Table 6.8: *Vertices in the buildings of the twisted Levi subgroups of split rank* 1*, up to conjugacy, obtained as the buildings associated to the most unramified elliptic tori contained in each twisted Levi subgroup.*

6.6.2 Reductive quotients $\mathcal{M}_{y,0}/\mathcal{M}_{y,0^+}$

In the construction of irreducible supercuspidal representations, one ingredient is a cuspidal representation of the group

$$G^0_{y,0}/G^0_{y,0^+},$$

where G^0 is the first group of the twisted Levi sequence and y is a vertex of $\mathcal{B}(G^0)$.

If G^0 is a maximal elliptic torus, however, this quotient is abelian and as noted in [Nev13], one can omit this part of the datum. Therefore it remains, for Levi sequences beginning with a nonabelian twisted Levi subgroup $\mathcal{M}$, to identify the corresponding reductive quotients $\mathcal{M}_{y,0}/\mathcal{M}_{y,0^+}$, where y is a vertex in the building of $\mathcal{M}$ determined in the preceding section.

Since y is a vertex of $\mathcal{B}(\mathcal{M})$, $\mathcal{M}_{y,0}$ is a maximal parahoric subgroup, and the structure of these reductive quotients is determined in the literature via the Bruhat–Tits theory of parahoric subgroups. In particular, the quotient $\mathcal{M}_{y,0}/\mathcal{M}_{y,0^+}$ can be identified with the $\mathbb{F}_q$-points of a connected reductive group defined over the residue field.

These quotients were explicitly determined for special orthogonal groups and unitary groups by Gan, Hanke and Yu in [GHY01, §3,§6] via lattice-stabilizer arguments. For example, when $y \in \mathcal{B}(G)$ is the hyperspecial vertex $(0,0)$, then

$$G_{y,0}/G_{y,0+} \cong \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)(\mathbb{F}_q),$$

which is the corresponding split special orthogonal group over the residue field. On the other hand, at the non-hyperspecial vertex $x = (\frac{1}{2},\frac{1}{2})$, the reductive quotient is

$$G_{x,0}/G_{x,0+} \cong \mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{SL}_2(\mathbb{F}_q).$$

We record the classification for all the proper twisted Levi subgroups in the last column of Table 6.9.

Let us now demonstrate, in support of these results, how we can explicitly and directly compute $\mathfrak{m}_{x,0}/\mathfrak{m}_{x,0+}$, which is the corresponding Lie algebra over $\mathbb{F}_q$. We compute $\mathfrak{m}_{y,0}/\mathfrak{m}_{y,0^+}$

Twisted Levi $\mathcal{M}$	$\mathfrak{m}_{\mathbf{y},\mathbf{0}}/\mathfrak{m}_{\mathbf{y},\mathbf{0}^+}$	$\mathbf{M}_{\mathbf{y}}/\mathbf{M}_{\mathbf{y},\mathbf{0}^+}$
${}^{\varepsilon}\mathcal{S}_{1,s} \cong \mathrm{SO}(1,-\varepsilon) \times \mathrm{SO}(\mathbb{H} \oplus \varepsilon)$	$\mathfrak{so}_2(\mathbb{F}_q) \oplus \mathfrak{so}_3(\mathbb{F}_q)$	$\mathrm{SO}(1,-\varepsilon)(\mathbb{F}_q) \times \mathrm{SO}(\mathbb{H} \oplus \langle\varepsilon\rangle)(\mathbb{F}_q)$
${}^{\varpi}\mathcal{S}_{1,s} \cong \mathrm{SO}(1,-\varpi) \times \mathrm{SO}(\mathbb{H} \oplus \varpi)$	$\mathfrak{so}(\mathbb{H} \oplus \langle 1\rangle)(\mathbb{F}_q)$	$O(\mathbb{H})(\mathbb{F}_q)$
${}^{\varepsilon\varpi}\mathcal{S}_{1,s} \cong \mathrm{SO}(1,-\varepsilon\varpi) \times \mathrm{SO}(\mathbb{H} \oplus \varepsilon\varpi)$	$\mathfrak{so}(\mathbb{H} \oplus \langle 1\rangle)(\mathbb{F}_q)$	$O(\mathbb{H})(\mathbb{F}_q)$
${}^{\varepsilon}\mathcal{S}_{\varpi,c} \cong \mathrm{SO}(\varpi,-\varepsilon\varpi) \times \mathrm{SO}(-\varpi,\varepsilon\varpi,1)$	$\mathfrak{so}_2(\mathbb{F}_q) \oplus \mathfrak{so}_2(\mathbb{F}_q)$	$\mathrm{SO}(1,-\varepsilon)(\mathbb{F}_q) \times \mathrm{O}(1,-\varepsilon)(\mathbb{F}_q)$
${}^{\varpi}\mathcal{S}_{\varepsilon,c} \cong \mathrm{SO}(\varepsilon,-\varepsilon\varpi) \times \mathrm{SO}(1,-\varepsilon,\varepsilon\varpi)$	$\mathfrak{so}_2(\mathbb{F}_q)$	$\mathrm{O}(1,-\varepsilon)(\mathbb{F}_q)$
${}^{\varepsilon\varpi}\mathcal{S}_{\varepsilon,c} \cong \mathrm{SO}(\varepsilon,-\varpi) \times \mathrm{SO}(1,-\varepsilon,\varpi)$	$\mathfrak{so}_2(\mathbb{F}_q)$	$\mathrm{O}(1,-\varepsilon)(\mathbb{F}_q)$
${}^{\varepsilon}\mathcal{L}_s \cong U_{\varepsilon}(\mathbb{H})$	$\mathfrak{sl}_2(\mathbb{F}_q) \oplus \mathbb{F}_q$	$U_{\varepsilon}(1,1)(\mathbb{F}_q)$
${}^{\varpi}\mathcal{L}_s \cong U_{\varpi}(\mathbb{H})$	$\mathfrak{sl}_2(\mathbb{F}_q)$	$\mathrm{Sp}_2(\mathbb{F}_q)$
${}^{\varepsilon\varpi}\mathcal{L}_s \cong U_{\varepsilon\varpi}(\mathbb{H})$	$\mathfrak{sl}_2(\mathbb{F}_q)$	$\mathrm{Sp}_2(\mathbb{F}_q)$

Table 6.9: *We give the (possibly disconnected) reductive quotients $\mathcal{M}_y/\mathcal{M}_{y,0^+}$ for each of the proper nonabelian twisted Levi subgroups of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1\rangle)$. Here, y denotes a vertex of $\mathcal{B}(\mathcal{M},F)$, and in the case $U_{\varepsilon}(\mathbb{H})$ although there are two conjugacy classes of vertices, both yield the same reductive quotient up to isomorphism.*

using Moy–Prasad filtrations and an explicit realization of $\mathfrak{m}$ inside $\mathfrak{g}$ via a suitable Witt basis.

The procedure consists of the following steps, given $\mathcal{M}$ and a vertex $y \in \mathcal{B}(\mathcal{M})$, and an embedding of $\mathcal{B}(\mathcal{M}) \subset \mathcal{B}(G)$.

1. **Moy–Prasad filtrations.** The point $y \in \mathcal{B}(G)$ determines a Moy–Prasad filtration on the Lie algebra $\mathfrak{g}$,
$$\mathfrak{g}_{y,0} \supset \mathfrak{g}_{y,0^+},$$
and hence the quotient
$$\mathfrak{g}_{y,0}/\mathfrak{g}_{y,0^+},$$
viewed as a Lie algebra over the residue field $\mathbb{F}_q$.

2. **Induced filtrations.** Through our embedding, we obtain the Moy–Prasad filtration on $\mathfrak{m}$ by intersection with those of $\mathfrak{g}$ as
$$\mathfrak{m}_{y,0} := \mathfrak{m} \cap \mathfrak{g}_{y,0}, \qquad \mathfrak{m}_{y,0^+} := \mathfrak{m} \cap \mathfrak{g}_{y,0^+}.$$

3. **Lie algebra quotient.** Form the Lie algebra quotient
$$\mathfrak{m}_{y,0}/\mathfrak{m}_{y,0^+},$$
which in fact embeds as a Lie subalgebra of $\mathfrak{g}_{y,0}/\mathfrak{g}_{y,0^+}$.

4. **Identification of the Lie type.** Compute the dimension of $\mathfrak{m}_{y,0}/\mathfrak{m}_{y,0+}$, and determine its structure as a Lie algebra (by for example constructing an explicit basis and computing its structure constants) in order to identify the isomorphism type of $\mathfrak{m}_{y,0}/\mathfrak{m}_{y,0+}$.

We illustrate this with the example of $\mathcal{M} = {}^{\varepsilon}\mathcal{L}_s$.

Example 6.6.2. Let F be a non-archimedean local field with ring of integers $\mathcal{O}$, maximal ideal $\mathcal{P}$, and residue field $\mathbb{F}_q = \mathcal{O}/\mathcal{P}$. Let $\varepsilon \in \mathcal{O}^\times$ be a nonsquare unit, and consider the Lie algebra

$$\mathfrak{u}_\varepsilon(1,1) = \left\{ \begin{pmatrix} u_1 + u_2\sqrt{\varepsilon} & v\sqrt{\varepsilon} \\ z\sqrt{\varepsilon} & -u_1 + u_2\sqrt{\varepsilon} \end{pmatrix} : u_1, u_2, v, z \in F \right\}.$$

embedded into $\mathfrak{so}(2\mathbb{H}) \subset \mathfrak{so}(2\mathbb{H} \oplus \langle 1 \rangle)$ via the Witt basis $\{e_1 = (0, \sqrt{\varepsilon}), e_2 = (1,0), f_1 = (-\sqrt{\varepsilon}^{-1}, 0), f_2 = (0,1)\}$ to give for each $X \in \mathfrak{u}_\varepsilon(1,1)$

$$\iota(X) = \begin{pmatrix} -u_1 & z & 0 & u_2 \\ \varepsilon v & u_1 & -u_2 & 0 \\ 0 & -\varepsilon u_2 & u_1 & -\varepsilon v \\ \varepsilon u_2 & 0 & -z & -u_1 \end{pmatrix}.$$

Let $\mathfrak{m} = \iota(\mathfrak{u}_\varepsilon(\mathbb{H})) \subset \mathfrak{g} = \mathfrak{so}_5$, and let $x = (0,0)$ be a point in the standard apartment corresponding to a vertex of $\mathcal{B}(\mathcal{M})$. From the general description of the Moy–Prasad lattices $\mathfrak{g}_{x,r}$ at $x = (0,0)$, one computes $\mathfrak{g}_{x,0}$ and $\mathfrak{g}_{x,0+}$, and then restricts to the upper-left 4×4 block corresponding to $\mathfrak{so}(2\mathbb{H})$. Therefore, we have

$$\mathfrak{m}_{x,0} = \{\, \iota(X) : u_1, u_2, v, z \in \mathcal{O} \,\}, \qquad \mathfrak{m}_{x,0+} = \{\, \iota(X) : u_1, u_2, v, z \in \mathcal{P} \,\}.$$

Hence

$$\mathfrak{m}_{x,0}/\mathfrak{m}_{x,0+} \;\cong\; (\mathcal{O}u_1 \oplus \mathcal{O}u_2 \oplus \mathcal{O}v \oplus \mathcal{O}z)\big/(\mathcal{P}u_1 \oplus \mathcal{P}u_2 \oplus \mathcal{P}v \oplus \mathcal{P}z) \;\cong\; \mathbb{F}_q^4.$$

Reducing the defining relation for $\mathfrak{u}_\varepsilon(1,1)$ modulo $\mathcal{P}$ preserves the involution $\overline{\sqrt{\varepsilon}} = -\sqrt{\varepsilon}$. Thus the reduced Lie algebra is

$$\mathfrak{m}_{x,0}/\mathfrak{m}_{x,0+} \cong \left\{ \begin{pmatrix} \alpha & \beta\sqrt{\varepsilon} \\ \gamma\sqrt{\varepsilon} & -\overline{\alpha} \end{pmatrix} : \alpha \in \mathbb{F}_q(\sqrt{\varepsilon}),\ \beta, \gamma \in \mathbb{F}_q \right\}.$$

Writing $\alpha = u_1 + u_2\sqrt{\varepsilon}$, one decomposes this algebra as

$$\underbrace{\left\{ \begin{pmatrix} u_2\sqrt{\varepsilon} & 0 \\ 0 & u_2\sqrt{\varepsilon} \end{pmatrix} \right\}}_{\text{1-dimensional center}} \oplus \underbrace{\left\{ \begin{pmatrix} u_1 & v\sqrt{\varepsilon} \\ z\sqrt{\varepsilon} & -u_1 \end{pmatrix} \right\}}_{\text{3-dimensional algebra}}.$$

A direct computation shows

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H,$$

for

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & \sqrt{\varepsilon} \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ \sqrt{\varepsilon} & 0 \end{pmatrix},$$

so the 3-dimensional subalgebra is isomorphic to $\mathfrak{sl}_2(\mathbb{F}_q)$(after scaling). Thus

$$\mathfrak{m}_{x,0}/\mathfrak{m}_{x,0+} \cong \mathfrak{sl}_2(\mathbb{F}_q) \oplus \mathbb{F}_q.$$

Doing this for each of the twisted Levi subgroups gives the reductive quotients of the Lie algebras that are recorded in the second column of Table 6.9.

Chapter 7

Construction of supercuspidal representations

In this chapter, we briefly recap how the ingredients computed in this thesis can be used to construct all of the irreducible supercuspidal representations of $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

7.1 G-datum for the construction

The construction relies on the notion of a *generic cuspidal G-datum*. Such a datum consists of five interrelated components satisfying specific axioms. We recall the definition below.

Definition 7.1.1. A 5-tuple $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ is called a **(generic cuspidal) G-datum** if

(D1) $\mathbb{G}$ is a sequence of twisted Levi subgroups

$$\vec{\mathbb{G}} = (\mathbb{G}^0 \subset \mathbb{G}^1 \subset \ldots \subset \mathbb{G}^d \subseteq \mathbb{G})$$

in $\mathbb{G}$, where each $\mathbb{G}^i$ is defined over F and splits over a tamely ramified Galois extension E of F, and $Z(\mathbb{G}^0)/Z(\mathbb{G})$ is F-anisotropic;

(D2) y is a vertex in $\mathcal{B}(\mathbb{G}, F)$;

(D3) $\vec{r} = (0 < r_0 < r_1 < \ldots < r_d)$ is a sequence of real numbers;

(D4) ρ is an irreducible representation of $K^0 = N_G(G_{y,0})$ such that $\rho|_{G^0_{y,0+}}$ is 1-isotypic and $\rho|_{G^0_{y,0}}$ contains a cuspidal representation of the reductive quotient $G_{y,0}/G_{y,0+}$;

(D5) $\vec{\phi} = (\phi^0, \phi^1, \ldots, \phi^d)$ is a sequence of quasicharacters, where ϕ^i is a G^{i+1}-generic character of G^i (relative to y) of depth r_i for $0 \leq i \leq d$.

Remark 7.1.2. In Axiom (D4), the group $N_G(G_{y,0})$ is explicitly identified in [Yu01, Lemma

3.3] as $G_{[y]}$, which is the stabilizer of the image of y in the *reduced building* of G. Note also that the restriction of a representation (ρ, V) to the subgroup $G_{y,0+}$ is called 1-isotypic when $\rho(h) = \mathrm{Id}_V$ for all $h \in G_{y,0+}$, meaning it decomposes as a direct sum of copies of the trivial representation. Thus, this hypothesis ensures that ρ factors through the reductive quotient group.

We briefly outline the construction in Section 7.1.1, below.

Ju-Lee Kim in [Kim07] proved that for p sufficiently large and $\mathrm{char}(F) = 0$, all irreducible supercuspidal representations of G arise from this construction. Later, Fintzen in [Fin21c] extended this to cover all p that do not divide the order of the Weyl group, as long as G splits over a tamely ramified extension of F (which may have positive characteristic).

Furthermore, Hakim and Murnaghan in [HM08] determined when two data Ψ, Ψ' would give isomorphic supercuspidal representations. While it is an involved definition, we note that the datum depends on the choices of the twisted Levi subgroups only up to conjugacy.

Remark 7.1.3. Fintzen provides a proof that $\pi_G(\Psi)$ is irreducible and thus supercuspidal in [Fin21a, Theorem 3.1], as an alternative proof of the original proof of Yu in [Yu01], which was based on a misprinted statement. In [FKS23], Fintzen, Kaletha and Spice introduce the FKS twist of this construction, in which the inducing representation $\kappa_G(\Psi)$ is multiplied by a very particular quasi-character (taking values only in $\{\pm 1\}$). This correction modifies the precise parametrization of supercuspidal representations but does not impact the set of supercuspidal representations that are produced.

7.1.1 A brief overview of the construction

The goal of this section is to outline, in a step-by-step manner, the structural components of Yu's construction clarifying how each element of the datum contributes to the formation of the final representation. We do not, however, reproduce the construction in full detail. The following outline follows the original reference [Yu01] and the expositions in [Bou20, §4.1, 4.2.1, 4.3.1] and [Fin25, §3.7]. For proofs, structural details of the Heisenberg–Weil extension, and explicit calculations of the inflation process, the reader is referred to [Bou20, §4.1–4.3] and the references therein, in particular Yu [Yu01, §15], Hakim–Murnaghan [HM08]. We also point the reader to modern survey articles such that [Fin25, §3.8].

Given a generic cuspidal G-datum

$$\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi}), \qquad \vec{\mathbb{G}} = (\mathbb{G}^0 \subsetneq \mathbb{G}^1 \subsetneq \cdots \subsetneq \mathbb{G}^d \subset \mathbb{G}),$$

that splits over a tamely ramified Galois extension E, set $s_i = r_i/2$ for each i.

1. **Define compact-mod-center subgroups.** Define recursively

$$K^0 = G^0_y \quad K^0_+ = G^0_{y,+},$$

$$K^{i+1} = K^i\, G^{i+1}_{y,s_i} \quad K^{i+1}_+ = K^i_+\, G^{i+1}_{y,s_i+} \quad (0 \le i \le d-1).$$

Each K^i is an open subgroup of G that is compact modulo the center of G^i.

2. **Define special subgroups.** For each $0 \le i \le d-1$, one defines certain subgroups J^{i+1} and J^{i+1}_+ that satisfy $K^{i+1} = K^i J^{i+1}$ and $K^{i+1}_+ = K^i J^{i+1}_+$ such that $K^i \cap J^{i+1} \subset G^i_{y,r_i}$. These subgroups, defined by passing to the splitting field E/F, have the additional property if $J^{i+1} \neq (J^{i+1})_+$, the quotient group $J^{i+1}/(J^{i+1})_+$ carries a natural symplectic form induced by composing the commutator (which lies in G^i_{y,r_i}) with ϕ^i.

3. **Construct the representations $\phi^{i'}$ of K^{i+1} for $0 \le i \le d-2$.** Start from the generic character ϕ^i of G^i_{y,r_i}, which is trivial on G^i_{y,r_i^+}. Thus $\phi^i|_{G^i_{y,s_i+}}$ can be viewed as a character of $G^i_{y,s_i^+}/G^i_{y,r_i^+}$ that can be extended to a character $\overline{\phi^i}$ of G^{i+1}_{y,s_i^+}.

   If $J^{i+1} \neq J^{i+1}_+$ then one uses ϕ^i and $\overline{\phi^i}$ to construct a *Heisenberg–Weil representation* of the semidirect product $K^i \ltimes J^{i+1}$, which is a finite-dimensional representation (ω_i, V_i), whose dimension related to the size of the quotient $J^{i+1}/(J^{i+1})_+$. One then uses this to define a representation $\widehat{\phi^{i'}}$ of $K^i J^{i+1}_+ = K^{i+1}$. (See [Fin25, §3.8] for more details.) When $J^{i+1} = J^{i+1}_+$ then ϕ^i and $\overline{\phi^i}$ together define a character of K^{i+1}, which we also denote $\widehat{\phi^{i'}}$.

4. **Inflate the intermediate representations to K^d.** Each ϕ_i' constructed above is successively *inflated* from K^{i+1} to larger groups K^j with $j > i+1$, eventually to K^d. Inflation here means extending a representation by letting the additional factors (in the product decomposition of K^d) act trivially. Denote the inflated representation on K^d by κ^i. For consistency, one sets

   $$\kappa^i = \inf{}^{K^d}_{K^{i+1}} \phi^{i'} \quad (0 \le i \le d-2),\ \kappa^{-1} = \inf{}^{K^d}_{K^0}\, \rho, \quad \kappa^{d-1} = \phi^{d-1'},\ \kappa^d = \phi^d|_{K^d}.$$

   Each κ^i remains an irreducible representation of K^d.

5. **Form $\kappa_G(\Psi)$.** The next step is to form the tensor product

   $$\kappa_G(\Psi) = \kappa^{-1} \otimes \kappa^0 \otimes \cdots \otimes \kappa^d,$$

   which is a finite-dimensional representation of K^d encoding all depth and genericity data from the sequence.

6. **Compact induction to obtain the supercuspidal representation.** Finally, define

   $$\pi_G(\Psi) = c\text{-}\mathrm{Ind}^G_{K^d} \kappa_G(\Psi),$$

   which yields an irreducible, depth-r_d representation of G, which is then supercuspidal by Mautner's Theroem 2.6.13.

7.2 Interpretation for $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$

We now summarize the results of our thesis by interpreting the components of a generic cuspidal G-datum

$$\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$$

in the specific setting where $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. This discussion clarifies how each ingredient of the datum appears in the construction of tame supercuspidal representations for G.

(D1) Twisted Levi sequence. Let us recap our analysis from Section 5.4.

As observed in Section 2.4.3, the group SO_5 has rank 2. Consequently, any twisted Levi sequence has length at most 3. There are three possible cases:

- **Length** 1 **(trivial sequence).** The sequence consists only of $\mathbb{G}$ itself.
- **Length** 2**.** The sequence begins with either
  - an elliptic torus $\mathbb{T}$, or
  - a proper twisted Levi subgroup $\mathbb{M} \subset \mathbb{G}$.
- **Length** 3 **(maximal length).** These sequences involve an elliptic torus followed by an intermediate twisted Levi subgroup before reaching $\mathbb{G}$.

The actual realization of a twisted Levi sequence is governed by condition (D5), which imposes G^{i+1}-genericity on the associated characters. This condition determines which of the above possibilities occurs. More precisely:

- Certain elliptic tori appear only in length 2 sequences because they are not contained in any proper twisted Levi subgroup of $\mathbb{G}$.
- Conversely, not every elliptic torus gives rise to a sequence of length 2. Only those whose Lie algebras contain G-generic elements will do.
- Elliptic tori failing this condition instead appear in sequences of length 3, where the presence of an intermediate twisted Levi subgroup is forced by the behavior of generic characters.

(D2) The point y in the building. We recap our analysis from Section 6.6.1.

If G^0 is an elliptic torus, then $\mathcal{B}(\mathbb{G}^0, F)$ consists of a single point $\{y\}$, which serves as the vertex y in the $\mathcal{B}(G^0)$. These were determined in Section 6.4. If instead G^0 is an intermediate twisted Levi subgroup $\mathcal{M}$ then we found the vertices of its building in Section 6.6, by choosing a maximally unramified torus satisfying $\mathcal{T} \subset \mathcal{M} \subset G$. Such a point y need not be a vertex of $\mathcal{B}(\mathbb{G}, F)$ itself. Section 6.6.

(D3) The depth sequence. We recap the results from Tables A.8, A.9, A.10 and A.11 .

For the trivial Levi sequence $G = \mathrm{SO}(\vec{2}\mathbb{H} \oplus \langle 1 \rangle)$, we have noted that this group does not have any nontrivial characters, so there is no depth sequence in this case.

For nontrivial sequences, the values of the depths $r_0, \ldots, r_{d-1}$ depend on the choice of the respective G^{i+1}-generic elements, whose explicit description was developed in Section 5.3.1 and Section 5.3.2. These values have been recorded in the tables in the Appendix, for the convenience of the reader.

(D4) The representation of the reductive quotient. We recap the results from Section 6.6.2.

When $G^0 = \mathcal{T}$ is an elliptic torus in G, we have

$$K^0 = \mathcal{T},$$

Let ρ be an irreducible depth zero representation of $\mathcal{T}$. Since $\mathcal{T}$ is abelian, ρ is one-dimensional, and ρ is 1-isotypic on $\mathcal{T}_{0^+}$, and hence $\rho|_{\mathcal{T}_{0^+}} = 1$. Moreover, because $\mathcal{T}$ has no roots, it admits no nontrivial parabolic subgroups, and every irreducible representation of $\mathcal{T}$ is automatically supercuspidal. Thus we may take ρ to be the trivial character of $\mathcal{T}$, that is, replace the pair (ρ, ϕ^0) with $(1, \rho\phi^0)$ (see [Nev13, Remark 3.2]).

When G^0 is an intermediate twisted Levi subgroup (for example M with $T \subset M \subset G$), one must instead choose a nontrivial irreducible cuspidal representation ρ of the reductive quotient $M_{y,0}/M_{y,0^+}$ satisfying condition (D4). The possible reductive quotients $M_{y,0}/M_{y,0^+}$ are listed in Section 6.6.2. The cuspidal representations σ of these groups are given by Deligne–Lusztig theory.

Since G is semisimple, the normalizer $N_G(G_{y,0})$ is equal to G_y, the stabilizer of y in $\mathcal{B}(G)$. This was discussed in Section 2.5.3. Therefore when $G_y = G_{y,0}$ we may take $\rho = \sigma$. When $y \in \{(\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\}$, however, we have $[G_y : G_{y,0}] = 2$ and one must first choose an extension of σ to create a representation ρ of G_y.

(D5) Generic quasi-characters. We recap the results of Chapter 5.

The sequence $\vec{\phi} = (\phi^0, \ldots, \phi^d)$ of quasi-characters are characters chosen so that they are realized by G^{i+1}-generic elements of $\mathfrak{z}^i$, which we denoted $\mathbf{Gen}_{G^{i+1},G^i}$. Given such an element X of depth $-r_i$, we showed in Section 2.6 that it defines a quasi-character of G^i_{y,r_i} that is trivial on G^i_{y,r_i+}. Thus, for each such X one may choose any character ϕ^i of G^i that has this restriction to $G^i_{y,r}$. We determined the sets $\mathbf{Gen}_{G,\mathcal{T}}$, $\mathbf{Gen}_{G,\mathcal{M}}$ and $\mathbf{Gen}_{\mathcal{M},\mathcal{T}}$ for all possible pairs (G^{i+1}, G^i) that could occur as a valid twisted Levi sequence in Figure 5.1.

This completes our work.

Appendix A

Extra tables

In this appendix, we have gathered all the key tables summarizing the conjugacy classes of elliptic tori and twisted Levi subgroups.

A.1 Embeddings of maximal elliptic tori of $\mathrm{SO}_2 \times \mathrm{SO}_3$ in $\mathcal{S}_{1,s}$ and $\mathcal{S}_{\mu,c}$, the twisted Levi (short) of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$

$\mathcal{A}$	Product torus	Class in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$
$\phi(\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon \rangle) \times \mathrm{SO}(1, -\varepsilon)) \cong {}^{\varepsilon}\mathcal{S}_{1,s}$		
$F(\sqrt{\varepsilon}) \oplus F(\sqrt{\varepsilon}) \oplus F$	${}^{\varepsilon}\mathcal{T}^{2}_{[1,1]} \times {}^{\varepsilon}\mathcal{T}_{[1]}$	${}^{\varepsilon}\mathcal{T}^{2}_{[1,1,1]}$
$F(\sqrt{\varpi}) \oplus F(\sqrt{\varepsilon}) \oplus F$	${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,\varepsilon\varpi]} \times {}^{\varepsilon}\mathcal{T}_{[1]}$	${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varepsilon\varpi]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F(\sqrt{\varepsilon}) \oplus F$	${}^{\varepsilon\varpi}\mathcal{T}_{[\varepsilon,\varpi]} \times {}^{\varepsilon}\mathcal{T}_{[1]}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varpi]}$
$\phi(\mathrm{SO}(\mathbb{H} \oplus \langle \varpi \rangle) \times \mathrm{SO}(1, -\varpi)) \cong {}^{\varpi}\mathcal{S}_{1,s}$		
$F(\sqrt{\varepsilon}) \oplus F(\sqrt{\varpi}) \oplus F$	${}^{\varepsilon}\mathcal{T}_{[\varpi,\varepsilon\varpi]} \times {}^{\varpi}\mathcal{T}_{[1]}$	${}^{\varepsilon}\mathcal{T}^{\varpi}_{[\varpi,1,\varepsilon\varpi]}$
$F(\sqrt{\varpi}) \oplus F(\sqrt{\varpi}) \oplus F$	${}^{\varpi}\mathcal{T}_{[-1,1]} \times {}^{\varpi}\mathcal{T}_{[1]}$	${}^{\varpi}\mathcal{T}^{2}_{[-1,1,1]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F(\sqrt{\varpi}) \oplus F$	${}^{\varepsilon\varpi}\mathcal{T}_{[-\varepsilon,\varepsilon]} \times {}^{\varpi}\mathcal{T}_{[1]}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varpi}_{[-\varepsilon,1,\varepsilon]}*$
$\phi(\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon\varpi \rangle) \times \mathrm{SO}(1, -\varepsilon\varpi)) \cong {}^{\varepsilon\varpi}\mathcal{S}_{1,s}$		
$F(\sqrt{\varepsilon}) \oplus F(\sqrt{\varepsilon\varpi}) \oplus F$	${}^{\varepsilon}\mathcal{T}_{[\varpi,\varpi]} \times {}^{\varepsilon\varpi}\mathcal{T}_{[1]}$	${}^{\varepsilon}\mathcal{T}^{\varepsilon\varpi}_{[\varpi,1,\varpi]}$
$F(\sqrt{\varpi}) \oplus F(\sqrt{\varepsilon\varpi}) \oplus F$	${}^{\varpi}\mathcal{T}_{[-\varepsilon,\varepsilon]} \times {}^{\varepsilon\varpi}\mathcal{T}_{[1]}$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-\varepsilon,1,\varepsilon]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F(\sqrt{\varepsilon\varpi}) \oplus F$	${}^{\varepsilon\varpi}\mathcal{T}_{[-1,1]} \times {}^{\varepsilon\varpi}\mathcal{T}_{[1]}$	${}^{\varepsilon\varpi}\mathcal{T}^{2}_{[-1,1,1]}$
$\phi(\mathrm{SO}(W_{\varepsilon}) \times \mathrm{SO}(\varpi, -\varepsilon\varpi)) \cong {}^{\varepsilon}\mathcal{S}_{\varpi,c}$		
$F(\sqrt{\varepsilon}) \oplus F(\sqrt{\varepsilon}) \oplus F$	${}^{\varepsilon}\mathcal{T}_{[\varpi,1]} \times {}^{\varepsilon}\mathcal{T}_{[\varpi]}$	${}^{\varepsilon}\mathcal{T}^{2}_{[\varpi,\varpi,1]}$
$F(\sqrt{\varpi}) \oplus F(\sqrt{\varepsilon}) \oplus F$	${}^{\varpi}\mathcal{T}_{[1,\varepsilon\varpi]} \times {}^{\varepsilon}\mathcal{T}_{[\varpi]}$	${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varepsilon\varpi]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F(\sqrt{\varepsilon}) \oplus F$	${}^{\varepsilon\varpi}\mathcal{T}_{[1,\varpi]} \times {}^{\varepsilon}\mathcal{T}_{[\varpi]}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$
$\phi(\mathrm{SO}(W_{\varpi}) \times \mathrm{SO}(\varepsilon, -\varepsilon\varpi)) \cong {}^{\varpi}\mathcal{S}_{\varepsilon,c}$		
$F(\sqrt{\varepsilon}) \oplus F(\sqrt{\varpi}) \oplus F$	${}^{\varepsilon}\mathcal{T}_{[1,\varepsilon\varpi]} \times {}^{\varpi}\mathcal{T}_{[\varepsilon]}$	${}^{\varepsilon}\mathcal{T}^{\varpi}_{[1,\varepsilon,\varepsilon\varpi]}$
$F(\sqrt{\varpi}) \oplus F(\sqrt{\varpi}) \oplus F$	${}^{\varpi}\mathcal{T}_{[-\varepsilon,1]} \times {}^{\varpi}\mathcal{T}_{[\varepsilon]}$	${}^{\varpi}\mathcal{T}^{2}_{[-\varepsilon,\varepsilon,1]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F(\sqrt{\varpi}) \oplus F$	${}^{\varepsilon\varpi}\mathcal{T}_{[-1,\varepsilon]} \times {}^{\varpi}\mathcal{T}_{[\varepsilon]}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varpi}_{[-1,\varepsilon,\varepsilon]}*$
$\phi(\mathrm{SO}(W_{\varepsilon\varpi}) \times \mathrm{SO}(\varepsilon, -\varpi)) \cong {}^{\varepsilon\varpi}\mathcal{S}_{\varepsilon,c}$		
$F(\sqrt{\varepsilon}) \oplus F(\sqrt{\varepsilon\varpi}) \oplus F$	${}^{\varepsilon}\mathcal{T}_{[1,\varpi]} \times {}^{\varepsilon\varpi}\mathcal{T}_{[\varepsilon]}$	${}^{\varepsilon}\mathcal{T}^{\varepsilon\varpi}_{[1,\varepsilon,\varpi]}$
$F(\sqrt{\varpi}) \oplus F(\sqrt{\varepsilon\varpi}) \oplus F$	${}^{\varpi}\mathcal{T}_{[-1,\varepsilon]} \times {}^{\varepsilon\varpi}\mathcal{T}_{[\varepsilon]}$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[-1,\varepsilon,\varepsilon]}$
$F(\sqrt{\varepsilon\varpi}) \oplus F(\sqrt{\varepsilon\varpi}) \oplus F$	${}^{\varepsilon\varpi}\mathcal{T}_{[-\varepsilon,1]} \times {}^{\varepsilon\varpi}\mathcal{T}_{[\varepsilon]}$	${}^{\varepsilon\varpi}\mathcal{T}^{2}_{[-\varepsilon,\varepsilon,1]}$

Table A.1: *Maximal elliptic tori arranged according to the algebra $\mathcal{A}$ and their realization inside the twisted Levi $\mathrm{SO}_3 \times \mathrm{SO}_2$. The second column records the product torus, ${}^{\alpha}\mathcal{T}_{[\mu_1,\lambda]} \times \mathrm{SO}(\mu_2, -\mu_2\beta)$, before applying the isomorphism ϕ of Proposition 3.4.1, and the third column records its image, ${}^{\alpha}\mathcal{T}^{\beta}_{[\mu_1,\mu_2,\lambda]}$, namely the corresponding class of elliptic tori in $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. We write ${}^{\alpha}\mathcal{T}_{[\mu]}$ for the torus $\mathrm{SO}(\mu, -\mu\alpha)$. The starred torus ${}^{\varepsilon\varpi}\mathcal{T}^{\varpi}_{[-\varepsilon,1,\varepsilon]}$ is the same as the torus ${}^{\varepsilon\varpi}\mathcal{T}^{\varpi}_{[\varepsilon,-1,\varepsilon]}$ when $-1^{\square}$ and torus ${}^{\varepsilon\varpi}\mathcal{T}^{\varpi}_{[1,-\varepsilon,\varepsilon]}$ when $-1^{\boxtimes}$. While, the torus ${}^{\varepsilon\varpi}\mathcal{T}^{\varpi}_{[-1,\varepsilon,\varepsilon]}$ is the same as the torus ${}^{\varepsilon\varpi}\mathcal{T}^{\varpi}_{[1,-\varepsilon,\varepsilon]}$ when $-1^{\square}$, and the torus ${}^{\varepsilon\varpi}\mathcal{T}^{\varpi}_{[\varepsilon,-1,\varepsilon]}$ otherwise. Moreover, tori shaded with the same color represent the same elliptic torus class when $-1^{\boxtimes}$. Figure 3.1 demonstrates this embedding.*

A.2 Embeddings of maximal elliptic tori of U_2 in $\mathcal{L}_s$, the twisted Levi(long) of $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

$\mathcal{U}_{\varepsilon,*}$ in $U_\varepsilon(\mathbb{H})$ over $K = F(\sqrt{\varepsilon})$	$\phi(\mathcal{U}_{\varepsilon,*})$ in ${}^{\varepsilon}\mathcal{L}_s$	Splitting field
$\mathcal{U}^2_{\varepsilon,1/2,1/2}$	${}^{\varepsilon}\mathcal{T}^2_{[1,1,1]}$	$K = F(\sqrt{\varepsilon})$
$\mathcal{U}^2_{\varepsilon,\varpi/2,\varpi/2}$	${}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi,1]}$	$K = F(\sqrt{\varepsilon})$
${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,\sqrt{\varpi}/2}$	${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$	$K(\sqrt{\varpi})$
${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon,\sqrt{\varepsilon\varpi}/2}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]}$	$K(\sqrt{\varepsilon\varpi})$
$\mathcal{U}_{\varpi,*}$ in $U_\varpi(\mathbb{H})$ over $K = F(\sqrt{\varpi})$	$\phi(\mathcal{T}^{\varpi})$ in ${}^{\varpi}\mathcal{L}_s$	Splitting field
$\mathcal{U}^2_{\varpi,1/2,-1/2}$	${}^{\varpi}\mathcal{T}^2_{[1,-1,1]}$	$K = F(\sqrt{\varpi})$
$\mathcal{U}^2_{\varpi,\varepsilon/2,-\varepsilon/2}$, omit if $-1^{\boxtimes}$	${}^{\varpi}\mathcal{T}^2_{[\varepsilon,-\varepsilon,1]}$, omit if $-1^{\boxtimes}$	$K = F(\sqrt{\varpi})$
${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,\sqrt{-\varepsilon}/2}$	${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{-\varepsilon},1]}$	$K(\sqrt{\varepsilon})$
${}^{\varepsilon\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varpi,\sqrt{\varepsilon\varpi}/2}$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]}$	$K(\sqrt{\varepsilon})$
$\mathcal{U}^2_{\varepsilon\varpi,*,*}$ in $U_{\varepsilon\varpi}(1,-1)$ over $K = F(\sqrt{\varepsilon\varpi})$	$\phi(\mathcal{T}^{\varepsilon\varpi})$ in ${}^{\varepsilon\varpi}\mathcal{L}_s$	Splitting field
$\mathcal{U}^2_{\varepsilon\varpi,1,-1}$	${}^{\varepsilon\varpi}\mathcal{T}^2_{[1,-1,1]}$	$K = F(\sqrt{\varepsilon\varpi})$
$\mathcal{U}^2_{\varepsilon\varpi,\varepsilon/2,-\varepsilon/2}$, omit if $-1^{\boxtimes}$	${}^{\varepsilon\varpi}\mathcal{T}^2_{[\varepsilon,-\varepsilon,1]}$, omit if $-1^{\boxtimes}$	$K = F(\sqrt{\varepsilon\varpi})$
${}^{\varepsilon}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,\sqrt{-\varepsilon}/2}$	${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{-\varepsilon},1]}$	$K(\sqrt{\varepsilon})$
${}^{\varpi}\mathcal{U}^{\varepsilon,\varpi}_{\varepsilon\varpi,\sqrt{\varpi}/2}$	${}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$	$K(\sqrt{\varepsilon})$

Table A.2: *Maximal elliptic tori in the twisted Levi $U_\alpha(2)$ ($\alpha \in \{\varepsilon, \varpi, \varepsilon\varpi\}$), together with their images inside SO_5 under the maps ϕ given in Proposition 3.4.2. Each row lists the unitary torus, the corresponding elliptic torus in SO_5, and the extension field over which the torus splits.*

A.3 Witt basis and building of elliptic tori tables .

$\mathcal{T}$	Witt Basis	w_1, w_2	$\mathcal{B}(\mathcal{T}) = \left(\frac{1}{2}\nu(w_1), \frac{1}{2}\nu(w_2)\right)$
${}^{\varepsilon}\mathcal{T}^2_{[1,1,1]}$	$e_1 = (1, i, 0)$, $e_2 = (-\sqrt{\varepsilon}, -i\sqrt{\varepsilon}, 0)$ $f_1 = (\frac{1}{2}, -\frac{i}{2}, 0)$, $f_2 = (\frac{\sqrt{\varepsilon^{-1}}}{2}, \frac{-i\sqrt{\varepsilon^{-1}}}{2}, 0)$ $g_1 = (0, 0, 1)$	$1, -\varepsilon$	$(0, 0)$
${}^{\varepsilon}\mathcal{T}^2_{[\varpi,\varpi,1]}$	$e_1 = (1, i, 0)$, $e_2 = (-\sqrt{\varepsilon}, -i\sqrt{\varepsilon}, 0)$ $f_1 = (\frac{1}{2\varpi}, -\frac{i}{2\varpi}, 0)$, $f_2 = (\frac{\sqrt{\varepsilon}}{2\varepsilon\varpi}, \frac{-i\sqrt{\varepsilon}}{2\varepsilon\varpi}, 0)$ $g_1 = (0, 0, 1)$	$\varpi, -\varepsilon\varpi$	$(\frac{1}{2}, \frac{1}{2})$
${}^{\varpi}\mathcal{T}^2_{[1,-1,1]}$	$e_1 = (-\sqrt{\varpi}, -i\sqrt{\varpi}, 0)$, $e_2 = (1, i, 0)$ $f_1 = (\frac{\sqrt{\varpi^{-1}}}{2}, \frac{-i\sqrt{\varpi^{-1}}}{2}, 0)$, $f_2 = (\frac{1}{2}, -\frac{i}{2}, 0)$ $g_1 = (0, 0, 1)$	$-\varpi, 1$	$(\frac{1}{2}, 0)$
${}^{\varpi}\mathcal{T}^2_{[\varepsilon,-\varepsilon,1]}$	$e_1 = (-\sqrt{\varpi}, -i\sqrt{\varpi}, 0)$, $e_2 = (1, i, 0)$ $f_1 = (\frac{\sqrt{\varpi}^{-1}}{2\varepsilon}, \frac{-i\sqrt{\varpi}^{-1}}{2\varepsilon}, 0)$, $f_2 = (\frac{1}{2\varepsilon}, -\frac{i}{2\varepsilon}, 0)$ $g_1 = (0, 0, 1)$	$-\varepsilon\varpi, \varepsilon$	$(\frac{1}{2}, 0)$

$\mathcal{T}$	Witt Basis	w_1, w_2	$\mathcal{B}(\mathcal{T}) = \left(\frac{1}{2}\nu(w_1), \frac{1}{2}\nu(w_2)\right)$
${}^{\varepsilon\varpi}\mathcal{T}^2_{[1,-1,1]}$	$e_1 = (-\sqrt{\varepsilon\varpi}, -i\sqrt{\varepsilon\varpi}, 0), \quad e_2 = (1, i, 0)$ $f_1 = (\frac{\sqrt{\varepsilon\varpi}^{-1}}{2}, \frac{-i\sqrt{\varepsilon\varpi}^{-1}}{2}, 0), \quad f_2 = (\frac{1}{2}, -\frac{i}{2}, 0)$ $g_1 = (0, 0, 1)$	$-\varepsilon\varpi, 1$	$(\frac{1}{2}, 0)$
${}^{\varepsilon\varpi}\mathcal{T}^2_{[\varepsilon,-\varepsilon,1]}$	$e_1 = (-\sqrt{\varepsilon\varpi}, -i\sqrt{\varepsilon\varpi}, 0), \quad e_2 = (1, i, 0)$ $f_1 = (\frac{\sqrt{\varepsilon\varpi}^{-1}}{2\varepsilon}, \frac{-i\sqrt{\varepsilon\varpi}^{-1}}{2\varepsilon}, 0), \quad f_2 = (\frac{1}{2\varepsilon}, -\frac{i}{2\varepsilon}, 0)$ $g_1 = (0, 0, 1)$	$-\varepsilon^2\varpi, \varepsilon$	$(\frac{1}{2}, 0)$
${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[1,-\varepsilon,\varepsilon]}$	$e_1 = (-\sqrt{\varpi}, -\frac{i\sqrt{\varepsilon\varpi}}{\varepsilon}, 0), \quad e_2 = (0, 1, i)$ $f_1 = (\frac{\sqrt{\varpi}^{-1}}{2}, -\frac{i\sqrt{\varepsilon\varpi}^{-1}}{2}, 0), \quad f_2 = (0, \frac{1}{2\varepsilon}, -\frac{i}{2\varepsilon})$ $g_1 = (1, 0, 0)$	$-\varpi, \varepsilon$	$(\frac{1}{2}, 0)$
${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[\varepsilon,-1,\varepsilon]}$	$e_1 = (-\sqrt{\varpi}, -i\sqrt{\varepsilon\varpi}, 0), \quad e_2 = (1, 0, i)$ $f_1 = (\frac{\sqrt{\varpi}^{-1}}{2\varepsilon}, -\frac{i\sqrt{\varepsilon\varpi}^{-1}}{2}, 0), \quad f_2 = (\frac{1}{2\varepsilon}, 0, -\frac{i}{2\varepsilon})$ $g_1 = (0, 1, 0)$	$-\varepsilon\varpi, \varepsilon$	$(\frac{1}{2}, 0)$
${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varepsilon\varpi]}$	$e_1 = (\sqrt{\varpi}, 1, 0), \quad e_2 = (0, \sqrt{\varepsilon}, 1)$ $f_1 = (-\frac{\sqrt{\varpi}^{-1}}{2}, \frac{1}{2\varpi}, 0), \quad f_2 = (0, -\frac{\sqrt{\varepsilon}^{-1}}{2\varpi}, \frac{1}{2\varepsilon\varpi})$ $g_1 = (1, 0, 0)$	$\varpi, \varepsilon\varpi$	$(\frac{1}{2}, \frac{1}{2})$
${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varepsilon\varpi]}$	$e_1 = (\sqrt{\varpi}, 0, 1), \quad e_2 = (1, \sqrt{\varepsilon}, 0)$ $f_1 = (-\frac{\sqrt{\varpi}^{-1}}{2\varepsilon}, 0, \frac{1}{2\varepsilon\varpi}), \quad f_2 = (\frac{1}{2\varepsilon}, -\frac{\sqrt{\varepsilon}^{-1}}{2}, 0)$ $g_1 = (0, 1, 0)$	$-\varepsilon\varpi, \varepsilon$	$(\frac{1}{2}, 0)$
${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$	$e_1 = (-\sqrt{\varepsilon\varpi}, -i\sqrt{\varepsilon}, 0), \quad e_2 = (0, 1, i)$ $f_1 = (\frac{\sqrt{\varepsilon\varpi}^{-1}}{2}, -\frac{i\sqrt{\varepsilon}^{-1}}{2\varpi}, 0), \quad f_2 = (0, \frac{1}{2\varpi}, -\frac{i}{2\varpi})$ $g_1 = (1, 0, 0)$	$-\varepsilon\varpi, \varpi$	$(\frac{1}{2}, \frac{1}{2})$
${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varpi]}$	$e_1 = (\frac{\sqrt{\varepsilon\varpi}}{\varepsilon}, 0, 1), \quad e_2 = (1, \sqrt{\varepsilon}, 0)$ $f_1 = (-\frac{\sqrt{\varepsilon\varpi}^{-1}}{2}, 0, \frac{1}{2\varpi}), \quad f_2 = (\frac{1}{2\varepsilon}, -\frac{\sqrt{\varepsilon}^{-1}}{2}, 0)$ $g_1 = (0, 1, 0)$	ϖ, ε	$(\frac{1}{2}, 0)$

Table A.3: *Witt bases and building for each conjugacy class of elliptic tori of* $\mathrm{SO}\langle 2\mathbb{H} \oplus \langle 1\rangle\rangle$ *admitting two diagonal forms when* $-1 \in F^{\times 2}$*, i.e* $p_i^{w_1}, p_j^{w_2}$ *for* $i, j \in \{1, 3\}$ *in Table 6.2. This table contains basically the sum of quadratic tori. It is associated with Section 6.2.2, Section 6.3.2 and Section 6.4.1.*

T_i	Witt Basis	w_1, w_2	$\mathcal{B}(\mathcal{T}) = \{(\frac{1}{2}\nu(w_1), \frac{1}{2}\nu(w_2))\}$
$*\,{}^{\varepsilon}\mathcal{T}^2_{[1,1,1]}$	$e_1 = (i + s^2, t^2, 0), \quad e_2 = (t^2, i - s^2, 0)$ $f_1 = (\frac{i-s^2}{2}, \frac{-t^2}{2}, 0), \quad f_2 = (\frac{-t^2}{2}, \frac{i+s^2}{2}, 0)$ $g_1 = (0, 0, 1)$	$1, -1$	$(0, 0)$

T_i	Witt Basis	w_1, w_2	$\mathcal{B}(\mathcal{T}) = \{(\frac{1}{2}\nu(w_1), \frac{1}{2}\nu(w_2))\}$
$* \, {}^{\varepsilon}\mathcal{T}^{2}_{[\varpi,\varpi,1]}$	$e_1 = (i+s^2, t^2, 0), \quad e_2 = (t^2, i-s^2, 0)$ $f_1 = (\frac{i-s^2}{2\varpi}, \frac{-t^2}{2\varpi}, 0), \quad f_2 = (\frac{-t^2}{2\varpi}, \frac{i+s^2}{2\varpi}, 0)$ $g_1 = (0,0,1)$	$\varpi, -\varpi$	$(\frac{1}{2}, \frac{1}{2})$
${}^{\varpi}\mathcal{T}^{2}_{[1,-1,1]}$	$e_1 = (\sqrt{\varpi}, \sqrt{\varpi}, 0), \quad e_2 = (1,1,0)$ $f_1 = (\frac{-\sqrt{\varpi}^{-1}}{2}, \frac{\sqrt{\varpi}^{-1}}{2}, 0), \quad f_2 = (\frac{1}{2}, -\frac{1}{2}, 0)$ $g_1 = (0,0,1)$	$\varpi, 1$	$(\frac{1}{2}, 0)$
${}^{\varepsilon\varpi}\mathcal{T}^{2}_{[1,-1,1]}$	$e_1 = (i\sqrt{\varpi}, i\sqrt{\varpi}, 0), \quad e_2 = (1,1,0)$ $f_1 = (\frac{i\sqrt{\varpi}^{-1}}{2}, \frac{-i\sqrt{\varpi}^{-1}}{2}, 0), \quad f_2 = (\frac{1}{2}, -\frac{1}{2}, 0)$ $g_1 = (0,0,1)$	$\varpi, 1$	$(\frac{1}{2}, 0)$
${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[1,-\varepsilon,\varepsilon]}$	$e_1 = (\sqrt{\varpi}, i\sqrt{\varpi}, 0), \quad e_2 = (0,1,1)$ $f_1 = (-\frac{\sqrt{\varpi}^{-1}}{2}, \frac{i\sqrt{\varpi}^{-1}}{2}, 0), \quad f_2 = (0, \frac{1}{2}, -\frac{1}{2})$ $g_1 = (1,0,0)$	$\varpi, 1$	$(\frac{1}{2}, 0)$
$* \, {}^{\varpi}\mathcal{T}^{\varepsilon\varpi}_{[\varepsilon,-1,\varepsilon]}$	$e_1 = (\sqrt{\varpi}, i\sqrt{\varpi}, 0), \quad e_2 = (t^2, -s^2, 1)$ $f_1 = (\frac{\sqrt{\varpi}^{-1}}{2}, -\frac{i\sqrt{\varpi}^{-1}}{2}, 0), \quad f_2 = (\frac{t^2}{2}, \frac{-s^2}{2}, \frac{-1}{2})$ $g_1 = (s^2, t^2, 0)$	$\varpi, 1$	$(\frac{1}{2}, 0)$
${}^{\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varepsilon\varpi]}$	$e_1 = (\sqrt{\varpi}, i, 0), \quad e_2 = (0,1,1)$ $f_1 = (-\frac{\sqrt{\varpi}^{-1}}{2}, \frac{i}{2\varpi}, 0), \quad f_2 = (0, \frac{1}{2\varpi}, -\frac{1}{2\varpi})$ $g_1 = (1,0,0)$	ϖ, ϖ	$(\frac{1}{2}, \frac{1}{2})$
${}^{\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varepsilon\varpi]}$	$e_1 = (\sqrt{\varpi}, 0, 1), \quad e_2 = (1, i, 0)$ $f_1 = (\frac{\sqrt{\varpi}^{-1}}{2}, 0, -\frac{1}{2\varpi}), \quad f_2 = (-\frac{1}{2}, \frac{i}{2}, 0)$ $g_1 = (0,1,0)$	$\varpi, 1$	$(\frac{1}{2}, 0)$
$* \, {}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[1,\varpi,\varpi]}$	$e_1 = (i\sqrt{\varpi}, s^2, t^2), \quad e_2 = (0, i+t^2, -s^2)$ $f_1 = (\frac{i\sqrt{\varpi}^{-1}}{2}, \frac{-s^2}{2\varpi}, \frac{-t^2}{2\varpi}), \quad f_2 = (0, \frac{i-t^2}{2\varpi}, \frac{s^2}{2\varpi})$ $g_1 = (1,0,0)$	ϖ, ϖ	$(\frac{1}{2}, \frac{1}{2})$
${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}_{[\varepsilon,1,\varpi]}$	$e_1 = (i\sqrt{\varpi}, 0, 1), \quad e_2 = (1, i, 0)$ $f_1 = (-\frac{i\sqrt{\varpi}^{-1}}{2}, 0, \frac{1}{2\varpi}), \quad f_2 = (-\frac{1}{2}, \frac{i}{2}, 0)$ $g_1 = (0,1,0)$	$\varpi, 1$	$(\frac{1}{2}, 0)$
${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{-\varepsilon},1]}$	$e_1 = (\sqrt{\varpi} + i\sqrt{\varpi}, 0), \quad e_2 = (1+i, 0)$ $f_1 = (\frac{i\sqrt{\varpi} - \sqrt{\varpi}}{2\varpi}, 0), \quad f_2 = (\frac{1-i}{2}, 0)$ $g_1 = (0,1)$	$\varpi, 1$	$(\frac{1}{2}, 0)$
$* \, {}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\eta,\varepsilon]}$	$e_1 = (\tilde{a}\sqrt{\varepsilon_{E_1}\varpi} - ai\sqrt{\varepsilon_{E_1}\varpi}, 0), \quad e_2 = (-s^{-1}, 1)$ $f_1 = (\frac{-a}{2\varpi}\sqrt{\varepsilon_{E_1}\varpi} - \frac{\tilde{a}}{2\varpi}i\sqrt{\varepsilon_{E_1}\varpi}, 0), \quad f_2 = (\frac{-s^{-1}}{2}, -\frac{1}{2})$ $g_1 = (si + t^2 s^{-1}, 0)$	$\varpi, 1$	$(\frac{1}{2}, 0)$

T_i	Witt Basis	w_1, w_2	$\mathcal{B}(\mathcal{T}) = \{(\frac{1}{2}\nu(w_1), \frac{1}{2}\nu(w_2))\}$
$*\,{}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[1,\varepsilon]}$	$e_1 = (1+i, 0), \quad e_2 = (s^{-1}i\sqrt{\varepsilon_{E_1}}, 1)$ $f_1 = (\frac{1}{2} - \frac{i}{2}, 0), \quad f_2 = (\frac{s^{-1}i\sqrt{\varepsilon_{E_1}}}{2}, \frac{-1}{2})$ $g_1 = ((s - t^2s^{-1}i)\sqrt{\varepsilon_{E_1}}, 0)$	$1, 1$	$(0, 0)$

Table A.4: *Witt basis and building of for each conjugacy class of the elliptic tori of* $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ *admitting two diagonal forms when* $-1 \notin F^{\times 2}$*, i.e.* $p_1^{w_1}, p_1^{w_2}$ *and the tori indicated by* $*$ *in Table 6.3. Note that* $\varepsilon_E = s^2 + it^2$ *for some* $s, t \in F^\times$ *such that* $N_{E/F}(\varepsilon_E) = s^4 + t^4 = -1$*. For the torus* ${}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}\varpi}_{4[\eta,\varepsilon]}$*,* $a = s^2 + t^2$ *and* $\tilde{a} = s^2 - t^2$*. This table is associated with Section 6.2.3 , Section 6.3.2 and Section 6.4.1.*

${}^{\alpha}\mathcal{T}^{\varepsilon,\varpi}_4$	Witt Basis	w_1, w_2	$\mathcal{B}(\mathcal{T}) = \{(\frac{2\nu(w_1)+\nu(\alpha^{-1})}{4}, \frac{2\nu(w_2)+\nu(\alpha^{-1})}{4})\}$
$-1^{\square}, {}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{-\varepsilon},1]}$	$e_1 = (\sqrt{\varpi}, 0), \quad e_2 = (1, 0)$ $f_1 = (-\sqrt{\varepsilon\varpi}^{-1}, 0), \quad f_2 = (\sqrt{\varepsilon^{-1}}, 0)$ $g_1 = (0, 1)$	$-\varepsilon\varpi, \varepsilon$	$(\frac{1}{2}, 0)$
$-1^{\square}, {}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$	$e_1 = (1, 0), \quad e_2 = (\sqrt{\varepsilon}, 0)$ $f_1 = (\sqrt{\varpi}^{-1}, 0), \quad f_2 = (-\sqrt{\varepsilon\varpi}^{-1}, 0)$ $g_1 = (0, 1)$	$\varpi, -\varepsilon\varpi$	$(\frac{1}{4}, \frac{1}{4})$
$-1^{\square}, {}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]}$	$e_1 = (1, 0), \quad e_2 = (\sqrt{\varepsilon}^{-1}, 0)$ $f_1 = (\sqrt{\varepsilon\varpi}^{-1}, 0), \quad f_2 = (-\sqrt{\varpi}^{-1}, 0)$ $g_1 = (0, 1)$	$\varepsilon\varpi, -\varpi$	$(\frac{1}{4}, \frac{1}{4})$
$-1^{\boxtimes}, {}^{\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varpi},1]}$	$e_1 = (i, 0), \quad e_2 = (1, 0)$ $f_1 = (-i\sqrt{\varpi}^{-1}, 0), \quad f_2 = (\sqrt{\varpi}^{-1}, 0)$ $g_1 = (0, 1)$	$-\varpi, \varpi$	$(\frac{1}{4}, \frac{1}{4})$
$-1^{\boxtimes}, {}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon,\varpi}_{4[\sqrt{\varepsilon\varpi},1]}$	$e_1 = (1, 0), \quad e_2 = (i, 0)$ $f_1 = (-i\sqrt{\varpi}^{-1}, 0), \quad f_2 = (\sqrt{\varpi}^{-1}, 0)$ $g_1 = (0, 1)$	$-\varpi, \varpi$	$(\frac{1}{4}, \frac{1}{4})$

Table A.5: *Witt bases and building of each conjugacy class of the elliptic tori of* $\mathrm{SO}\langle 2\mathbb{H} \oplus \langle 1 \rangle\rangle$ *admitting two cross-term forms. That is the quadratic form decomposes as* $p_2^{w_1}, p_2^{w_2}$ *in Tables 6.2 and 6.3. Here each* w_i *is obtained via* $e_i = w_i\, \alpha^{-1/2} f_i$*. This table is associated with the following sections 6.2.2, 6.2.3, 6.3.3 and 6.4.2.*

${}^{\alpha}\Upsilon^{E}_4$	Witt Basis	w_1, w_2	$\mathcal{B}(\mathcal{T}) = \{(x_1, , x_2)\}$
$-1^{\square}, {}^{\varepsilon}\Upsilon^{\varepsilon_{E_1}}_{4[1,\varepsilon]}$	$e_1 = (\sqrt{\varepsilon}, i), \quad e_2 = (\sqrt[4]{\varepsilon}, 0)$ $f_1 = (\frac{\sqrt{\varepsilon}^{-1}}{2}, -\frac{i}{2\varepsilon}), \quad f_2 = (-\sqrt[4]{\varepsilon}^{-1}, 0)$ $g_1 = (1, 0)$	$\varepsilon, -\varepsilon$	$(0, 0)$

${}^{\alpha}\Upsilon_4^E$	Witt Basis	w_1, w_2	$\mathcal{B}(\mathcal{T}) = \{(x_1,, x_2)\}$
$-1^{\square}, {}^{\varpi}\Upsilon^{\varpi}_{4[1,-\varpi]}$	$e_1 = (\sqrt{\varpi}, i), \quad e_2 = (\sqrt[4]{\varpi}, 0)$ $f_1 = (\frac{\sqrt{\varpi}^{-1}}{2}, -\frac{i}{2\varpi}), \quad f_2 = (-\sqrt[4]{\varpi}^{-1}, 0)$ $g_1 = (1, 0)$	$\varpi, -\varpi$	$(\frac{1}{2}, \frac{1}{4})$
$-1^{\square}, {}^{\varpi}\Upsilon^{\varepsilon^2\varpi}_{4[1,-\varpi]}$	$e_1 = (\sqrt{\varpi}, i), \quad e_2 = (\sqrt[4]{\varepsilon^2\varpi}, 0)$ $f_1 = (\frac{\sqrt{\varpi}^{-1}}{2}, -\frac{i}{2\varpi}), \quad f_2 = (-\sqrt[4]{\varepsilon^2\varpi}^{-1}, 0)$ $g_1 = (1, 0)$	$-\varepsilon\varpi, -\varpi$	$(\frac{1}{2}, \frac{1}{4})$
$-1^{\square}, {}^{\varepsilon\varpi}\Upsilon^{\varepsilon\varpi}_{4[1,-\varepsilon\varpi]}$	$e_1 = (\sqrt{\varepsilon\varpi}, i), \quad e_2 = (\sqrt[4]{\varepsilon\varpi}, 0)$ $f_1 = (\frac{\sqrt{\varepsilon\varpi}^{-1}}{2}, -\frac{i}{2\varepsilon\varpi}), \quad f_2 = (-\sqrt[4]{\varepsilon\varpi}^{-1}, 0)$ $g_1 = (1, 0)$	$-\varepsilon\varpi, \varepsilon\varpi$	$(\frac{1}{2}, \frac{1}{4})$
$-1^{\square}, {}^{\varepsilon\varpi}\Upsilon^{\varepsilon^3\varpi}_{4[1,-\varepsilon\varpi]}$	$e_1 = (\sqrt{\varepsilon\varpi}, i), \quad e_2 = (\sqrt[4]{\varepsilon^3\varpi}, 0)$ $f_1 = (\frac{\sqrt{\varepsilon\varpi}^{-1}}{2}, -\frac{i}{2\varepsilon\varpi}), \quad f_2 = (-\sqrt[4]{\varepsilon^3\varpi}^{-1}, 0)$ $g_1 = (1, 0)$	$-\varepsilon^2\varpi, \varepsilon\varpi$	$(\frac{1}{2}, \frac{1}{4})$
$-1^{\boxtimes}, {}^{\varpi}\Upsilon^{\varpi}_{4[1,-\varpi]}$	$e_1 = (\sqrt{\varpi}, 1), \quad e_2 = (\sqrt[4]{\varpi}, 0)$ $f_1 = (\frac{\sqrt{\varpi}^{-1}}{2}, -\frac{1}{2\varpi}), \quad f_2 = (-\sqrt[4]{\varpi}^{-1}, 0)$ $g_1 = (1, 0)$	$\varpi, -\varpi$	$(\frac{1}{2}, \frac{1}{4})$
$-1^{\boxtimes}, {}^{\varepsilon\varpi}\Upsilon^{\varepsilon\varpi}_{4[1,-\varepsilon\varpi]}$	$e_1 = (\sqrt{\varepsilon\varpi}, 1), \quad e_2 = (\sqrt[4]{\varepsilon\varpi}, 0)$ $f_1 = (\frac{\sqrt{\varepsilon\varpi}^{-1}}{2}, -\frac{1}{2\varepsilon\varpi}), \quad f_2 = (-\sqrt[4]{\varepsilon\varpi}^{-1}, 0)$ $g_1 = (1, 0)$	$-\varpi, \varpi$	$(\frac{1}{2}, \frac{1}{4})$
$-1^{\square}, {}^{\varepsilon}\Upsilon^{\varepsilon E_1 \varpi}_{4[\eta,\varepsilon]}$	$e_1 = (\sqrt[4]{\varepsilon\varpi^2}, 0), \quad e_2 = (\sqrt{\varepsilon}, i)$ $f_1 = (-\sqrt[4]{\varepsilon\varpi^2}^{-1}, 0), \quad f_2 = (\frac{\sqrt{\varepsilon}^{-1}}{2}, -\frac{i}{2\varepsilon})$ $g_1 = (1, 0)$	$-\varepsilon\varpi, \varepsilon$	$(\frac{1}{2}, 0)$

Table A.6: *Witt bases and Building of each conjugacy class of the elliptic tori of* $\mathrm{SO}\langle 2\mathbb{H} \oplus \langle 1 \rangle\rangle$ *admitting a diagonal and a cross-term forms, that is the quadratic form decomposes as* $p_i^{w_1}, p_2^{w_2}$ *for* $i \in \{1, 3\}$ *in Table 6.2, and* $i = 1$ *in Table 6.3. For the last one, the quadratic form decomposes as* $p_2^{-\varepsilon\varpi}$ *and* p_3^{ε}*. This table is associated with the following sections 6.2.2, 6.2.3, 6.3.4 and 6.4.3.*

A.4 Non-regular elements

In Section (4.1) we determined the non-regular elements of each of our tori.

Torus $\mathfrak{t}$	$\mathbf{u} \in \mathfrak{t} : x, y \in F$	**Eigenvalues**	**Non-Regular Elements**
${}^{\varepsilon}\mathfrak{t}^2_{[1,1,1]}$	$(x\sqrt{\varepsilon}, y\sqrt{\varepsilon})$	$\lambda_1 = x\sqrt{\varepsilon}, \quad \lambda_2 = y\sqrt{\varepsilon}$	${}^{\varepsilon}\mathfrak{s}^2_{y=0} = \{(x\sqrt{\varepsilon}, 0) \mid x \in F\}$ ${}^{\varepsilon}\mathfrak{s}^2_{x=0} = \{(0, y\sqrt{\varepsilon}) \mid y \in F\}$ ${}^{\varepsilon}\mathfrak{s}^2_{y=x} = \{(x\sqrt{\varepsilon}, x\sqrt{\varepsilon}) \mid x \in F\}$ ${}^{\varepsilon}\mathfrak{s}^2_{y=-x} = \{(x\sqrt{\varepsilon}, -x\sqrt{\varepsilon}) \mid x \in F\}$
${}^{\varpi'}\mathfrak{t}^2_{[1,-1,1]}$	$(x\sqrt{\varpi'}, y\sqrt{\varpi'})$	$\lambda_1 = x\sqrt{\varpi'}, \quad \lambda_2 = y\sqrt{\varpi'}$	${}^{\varpi'}\mathfrak{s}^2_{y=0} = \{(x\sqrt{\varpi'}, 0) \mid x \in F\}$ ${}^{\varpi'}\mathfrak{s}^2_{x=0} = \{(0, y\sqrt{\varpi'}) \mid y \in F\}$ ${}^{\varpi'}\mathfrak{s}^2_{y=x} = \{(x\sqrt{\varpi'}, x\sqrt{\varpi'}) \mid x \in F\}$ ${}^{\varpi'}\mathfrak{s}^2_{y=-x} = \{(x\sqrt{\varpi'}, -x\sqrt{\varpi'}) \mid x \in F\}$
${}^{\varpi}\mathfrak{t}^{\varepsilon\varpi}$	$(x\sqrt{\varpi}, b\sqrt{\varepsilon\varpi})$	$\lambda_1 = x\sqrt{\varpi}, \quad \lambda_2 = y\sqrt{\varepsilon\varpi}$	${}^{\varpi}\mathfrak{s}^{\varepsilon\varpi}_{y=0} = \{(x\sqrt{\varpi}, 0), x \in F\}$ ${}^{\varpi}\mathfrak{s}^{\varepsilon\varpi}_{x=0} = \{(0, y\sqrt{\varepsilon\varpi}), b \in F\}$
${}^{\varepsilon}\mathfrak{t}^{\varpi'}$	$(x\sqrt{\varepsilon}, y\sqrt{\varpi'})$	$\lambda_1 = x\sqrt{\varepsilon}, \quad \lambda_2 = y\sqrt{\varpi'}$	${}^{\varepsilon}\mathfrak{s}^{\varpi'}_{y=0} = \{(x\sqrt{\varepsilon}, 0), x \in F\}$ ${}^{\varepsilon}\mathfrak{s}^{\varpi'}_{x=0} = \{(0, y\sqrt{\varpi'}), y \in F\}$
${}^{\varepsilon}\mathfrak{t}^{\varepsilon,\varpi}_4$	$(x + y\sqrt{\varepsilon})\sqrt{\varpi}$	$\lambda_1 = (x + y\sqrt{\varepsilon})\sqrt{\varpi}$ $\lambda_2 = (x - y\sqrt{\varepsilon})\sqrt{\varpi}$	${}^{\varepsilon}\mathfrak{s}^{\varepsilon,\varpi}_{4,y=0} = \{x\sqrt{\varpi} \mid x \in F\}$ ${}^{\varepsilon}\mathfrak{s}^{\varepsilon,\varpi}_{4,x=0} = \{y\sqrt{\varepsilon\varpi} \mid y \in F\}$
${}^{\varpi}\mathfrak{t}^{\varepsilon,\varpi}_4$	$(x + y\sqrt{\varpi})\sqrt{\varepsilon}$	$\lambda_1 = (x + y\sqrt{\varpi})\sqrt{\varepsilon}$ $\lambda_2 = (x - y\sqrt{\varpi})\sqrt{\varepsilon}$	${}^{\varpi}\mathfrak{s}^{\varepsilon,\varpi}_{4,y=0} = \{x\sqrt{\varepsilon} \mid x \in F\}$ ${}^{\varpi}\mathfrak{s}^{\varepsilon,\varpi}_{4,x=0} = \{y\sqrt{\varepsilon\varpi} \mid y \in F\}$
${}^{\varepsilon\varpi}\mathfrak{t}^{\varepsilon,\varpi}_4$	$(x + y\sqrt{\varepsilon\varpi})\sqrt{\varepsilon}$	$\lambda_1 = (x + y\sqrt{\varepsilon\varpi})\sqrt{\varepsilon}$ $\lambda_2 = (x - y\sqrt{\varepsilon\varpi})\sqrt{\varepsilon}$	${}^{\varepsilon\varpi}\mathfrak{s}^{\varepsilon,\varpi}_{4,y=0} = \{x\sqrt{\varepsilon}, x \in F\}$ ${}^{\varepsilon\varpi}\mathfrak{s}^{\varepsilon,\varpi}_{4,x=0} = \{y\sqrt{\varpi}, y \in F\}$

Table A.7: *List of all conjugacy classes of maximal elliptic tori in $\mathfrak{g}' = \mathfrak{so}(q)$ together with conjugacy classes of non-regular elements. The Third column represents the eigenvalues; the remaining eigenvalues are determined by symmetry, namely $\lambda_3 = -\lambda_1$ and $\lambda_4 = -\lambda_2$ (with the fifth eigenvalue equal to 0). For the first case ${}^{\varepsilon}\mathfrak{t}^2$, Proposition 4.1.6 implies that there are always two G'-conjugacy classes of non-regular elements. For the second case ${}^{\varpi'}\mathfrak{t}^2$, it implies that there are two classes when -1 is a square and three classes otherwise, since ${}^{\varpi'}\mathfrak{s}^2_{y=0} \nsim_{G'} {}^{\varpi'}\mathfrak{s}^2_{x=0}$ in the latter case. In particular, this depends on the parameters μ_i. The remaining cases are obtained by direct computation and yield exactly two conjugacy classes of non-regular elements.*

A.5 Generic elements

In this section, we enumerate the G-generic elements of $\mathfrak{t}$ and of the Lie algebras of each of our twisted Levi subgroups M, as well as the M-generic elements of T. Note that these sets depend only on the isomorphism class of the groups, independent of their embeddings via conjugacy classes.

Torus $\mathfrak{t}$	$\mathfrak{u}$	**Eigenvalues** λ_1, λ_2	**Conditions on** $\mathbf{Gen}_{G,T,-r}$	**Depth** r
$-1^{\square}, {}^{\varepsilon}\mathfrak{u}_4^{\varepsilon_{E_1}\varpi}$	$(a+b\sqrt{\varepsilon})\sqrt[4]{\varepsilon\varpi^2}$	$\lambda_1 = (a+b\sqrt{\varepsilon})\sqrt[4]{\varepsilon\varpi^2}$ $\lambda_2 = (a-b\sqrt{\varepsilon})i\sqrt[4]{\varepsilon\varpi^2}$	$\nu(a) = \nu(b) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$-1^{\boxtimes}, {}^{\varepsilon}\mathfrak{u}_4^{\varepsilon_{E_1}\varpi}$	$(a+bi)\sqrt{\varepsilon_{E_1}\varpi}$	$\lambda_1 = (a+bi)\sqrt{\varepsilon_{E_1}\varpi}$ $\lambda_2 = (a-bi)\sqrt{\tau(\varepsilon_{E_1})\varpi}$	$\nu(a) = \nu(b) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$-1^{\square}, {}^{\varepsilon}\mathfrak{u}_4^{\varepsilon}$	$(a+b\sqrt{\varepsilon})\sqrt[4]{\varepsilon}$	$\lambda_1 = (a+b\sqrt{\varepsilon})\sqrt[4]{\varepsilon}$ $\lambda_2 = (a-b\sqrt{\varepsilon})i\sqrt[4]{\varepsilon}$	$\nu(a) = \nu(b) = -r$	$r \in \mathbb{Z}$
$-1^{\boxtimes}, {}^{\varepsilon}\mathfrak{u}_4^{\varepsilon}$	$(a+bi)\sqrt{\varepsilon_{E_1}}$	$\lambda_1 = (a+bi)\sqrt{\varepsilon_{E_1}}$ $\lambda_2 = (a-bi)\sqrt{\tau(\varepsilon_{E_1})}$	$\nu(a) = \nu(b) = -r$	$r \in \mathbb{Z}$
${}^{\varepsilon^k\varpi}\mathfrak{u}_4^{\varepsilon^k\varpi}$	$(a+b\sqrt{\varepsilon^k\varpi})\sqrt[4]{\varepsilon^k\varpi}$	$\lambda_1 = (a+b\sqrt{\varepsilon^k\varpi})\sqrt[4]{\varepsilon^k\varpi}$ $\lambda_2 = (a-b\sqrt{\varepsilon^k\varpi})i\sqrt[4]{\varepsilon^k\varpi}$	$\nu(a) = \nu(b) = -r - \frac{1}{4}$	$r \in \frac{3}{4} + \mathbb{Z}$
			$\nu(a) = -r + \frac{1}{4}$, $\nu(b) = -r - \frac{3}{4}$	$r \in \frac{1}{4} + \mathbb{Z}$

Table A.8: *The G-generic elements of depth $-r$ of the Lie algebras of quartic elliptic tori with unique subfield, ${}^{\alpha}\mathfrak{u}_4^E$.*

Torus $\mathfrak{t}$	$\mathfrak{u}$	**Eigenvalues** λ_1, λ_2	**Conditions on** $\mathbf{Gen}_{G,T,-r}$	**Depth** r
${}^{\varepsilon}\mathfrak{t}^2$	$(a\sqrt{\varepsilon}, b\sqrt{\varepsilon})$	$\lambda_1 = a\sqrt{\varepsilon}$ $\lambda_2 = b\sqrt{\varepsilon}$	$\nu(a) = \nu(b) = -r$ $a \not\equiv \pm b \pmod{\mathcal{P}^{-r+1}}$	$r \in \mathbb{Z}$
${}^{\varpi'}\mathfrak{t}^2$	$(a\sqrt{\varpi'}, b\sqrt{\varpi'})$	$\lambda_1 = a\sqrt{\varpi'}$ $\lambda_2 = b\sqrt{\varpi'}$	$\nu(a) = \nu(b) = -r - \frac{1}{2}$ $a \not\equiv \pm b \pmod{\mathcal{P}^{-r+\frac{1}{2}}}$	$r \in \frac{1}{2} + \mathbb{Z}$
${}^{\varepsilon}\mathfrak{t}^{\varpi'}$	$(a\sqrt{\varepsilon}, b\sqrt{\varpi'})$	$\lambda_1 = a\sqrt{\varepsilon}$ $\lambda_2 = b\sqrt{\varpi'}$	No G-generic elements	
${}^{\varpi}\mathfrak{t}^{\varepsilon\varpi}$	$(a\sqrt{\varpi}, b\sqrt{\varepsilon\varpi})$	$\lambda_1 = a\sqrt{\varpi}$ $\lambda_2 = b\sqrt{\varepsilon\varpi}$	$\nu(a) = \nu(b) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
${}^{\varepsilon}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a+b\sqrt{\varepsilon})\sqrt{\varpi}$	$\lambda_1 = (a+b\sqrt{\varepsilon})\sqrt{\varpi}$ $\lambda_2 = (a-b\sqrt{\varepsilon})\sqrt{\varpi}$	$\nu(a) = \nu(b) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
${}^{\varpi}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a+b\sqrt{\varpi})\sqrt{\varepsilon}$	$\lambda_1 = (a+b\sqrt{\varpi})\sqrt{\varepsilon}$ $\lambda_2 = (a-b\sqrt{\varpi})\sqrt{\varepsilon}$	No G-generic elements	
${}^{\varepsilon\varpi}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a+b\sqrt{\varepsilon\varpi})\sqrt{\varepsilon}$	$\lambda_1 = (a+b\sqrt{\varepsilon\varpi})\sqrt{\varepsilon}$ $\lambda_2 = (a-b\sqrt{\varepsilon\varpi})\sqrt{\varepsilon}$	No G-generic elements	

Table A.9: *The G-generic elements of depth $-r$ of the $\mathfrak{t} \setminus \{0\}$ that contain non-regular elements.*

Torus $\mathfrak{t}$	X : Non-regular elt	$\mathcal{M} : C_G(X)$	**Conditions on** $X \in \mathbf{Gen}_{G,\mathcal{M},-r}$	**Depth** r
$^{\varepsilon}\mathfrak{t}^2$	$(a\sqrt{\varepsilon}, 0, 0)$	$^{\varepsilon}\mathcal{S}$	$\nu(a) = -r$	$r \in \mathbb{Z}$
$^{\varepsilon}\mathfrak{t}^2$	$(a\sqrt{\varepsilon}, a\sqrt{\varepsilon}, 0)$	$^{\varepsilon}\mathcal{L}$	$\nu(a) = -r$	$r \in \mathbb{Z}$
$^{\varpi'}\mathfrak{t}^2$	$(a\sqrt{\varpi'}, 0, 0)$	$^{\varpi'}\mathcal{S}$	$\nu(a) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varpi'}\mathfrak{t}^2$	$(a\sqrt{\varpi'}, a\sqrt{\varpi'}, 0)$	$^{\varpi'}\mathcal{L}$	$\nu(a) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varepsilon}\mathfrak{t}^{\varpi'}$	$(0, a\sqrt{\varpi'}, 0)$	$^{\varpi'}\mathcal{S}$	$\nu(a) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varepsilon}\mathfrak{t}^{\varpi'}$	$(a\sqrt{\varepsilon}, 0, 0)$	$^{\varepsilon}\mathcal{S}$	$\nu(a) = -r$	$r \in \mathbb{Z}$
$^{\varpi}\mathfrak{t}^{\varepsilon\varpi}$	$(0, a\sqrt{\varepsilon\varpi}, 0)$	$^{\varepsilon\varpi}\mathcal{S}$	$\nu(a) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varpi}\mathfrak{t}^{\varepsilon\varpi}$	$(a\sqrt{\varpi}, 0, 0)$	$^{\varpi}\mathcal{S}$	$\nu(a) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varepsilon}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varepsilon\varpi}, 0)$	$^{\varepsilon\varpi}\mathcal{L}$	$\nu(a) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varepsilon}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varpi}, 0)$	$^{\varpi}\mathcal{L}$	$\nu(a) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varpi}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varepsilon}, 0)$	$^{\varepsilon}\mathcal{L}$	$\nu(a) = -r$	$r \in \mathbb{Z}$
$^{\varpi}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varepsilon\varpi}, 0)$	$^{\varepsilon\varpi}\mathcal{L}$	$\nu(a) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varepsilon\varpi}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varpi}, 0)$	$^{\varpi}\mathcal{L}$	$\nu(a) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varepsilon\varpi}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varepsilon}, 0)$	$^{\varepsilon}\mathcal{L}$	$\nu(a) = -r$	$r \in \mathbb{Z}$

Table A.10: *The G-generic elements of depth $-r$ of the center of the intermediate twisted Levis $\mathfrak{m}$.*

$\mathfrak{t}$	$\mathbf{u}$	$\mathcal{M} = C_G(X)$	$\mathbf{Gen}_{\mathcal{M},T,-r}$	**Depth** r
$^{\varepsilon}\mathfrak{t}^2$	$(a\sqrt{\varepsilon}, b\sqrt{\varepsilon}, 0)$	$^{\varepsilon}\mathcal{S} = C_G(a\sqrt{\varepsilon}, 0, 0)$	$\nu(a) \geq -r, \quad \nu(b) = -r$	$r \in \mathbb{Z}$
$^{\varepsilon}\mathfrak{t}^2$	$(a\sqrt{\varepsilon}, b\sqrt{\varepsilon}, 0)$	$^{\varepsilon}\mathcal{L} = C_G(a\sqrt{\varepsilon}, a\sqrt{\varepsilon}, 0)$	$\nu(a) \geq -r, \quad \nu(b) \geq -r$ $a \not\equiv b \pmod{\mathcal{P}^{-r+1}}$	$r \in \mathbb{Z}$
$^{\varpi'}\mathfrak{t}^2$	$(a\sqrt{\varpi'}, b\sqrt{\varpi'}, 0)$	$^{\varpi'}\mathcal{S} = C_G(a\sqrt{\varpi'}, 0, 0)$	$\nu(a) \geq -r - \frac{1}{2}, \quad \nu(b) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varpi'}\mathfrak{t}^2$	$(a\sqrt{\varpi'}, b\sqrt{\varpi'}, 0)$	$^{\varpi'}\mathcal{L} = C_G(a\sqrt{\varpi'}, a\sqrt{\varpi'}, 0)$	$\nu(a) \geq -r - \frac{1}{2}, \quad \nu(b) \geq -r - \frac{1}{2}$ $a \not\equiv b \pmod{\mathcal{P}^{-r+\frac{1}{2}}}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varpi'}\mathfrak{t}^{\varepsilon}$	$(a\sqrt{\varpi'}, b\sqrt{\varepsilon}, 0)$	$^{\varpi}\mathcal{S} = C_G(a\sqrt{\varpi'}, 0, 0)$	$\nu(a) \geq -r, \quad \nu(b) = -r$	$r \in \mathbb{Z}$
$^{\varpi'}\mathfrak{t}^{\varepsilon}$	$(a\sqrt{\varpi'}, b\sqrt{\varepsilon}, 0)$	$^{\varepsilon}\mathcal{S} = C_G(0, b\sqrt{\varepsilon}, 0)$	$\nu(a) = -r - \frac{1}{2}, \quad \nu(b) \geq -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varpi}\mathfrak{t}^{\varepsilon\varpi}$	$(a\sqrt{\varpi}, b\sqrt{\varepsilon\varpi}, 0)$	$^{\varepsilon\varpi}\mathcal{S} = C_G(0, b\sqrt{\varepsilon\varpi}, 0)$	$\nu(a) = -r - \frac{1}{2}, \quad \nu(b) \geq -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varpi}\mathfrak{t}^{\varepsilon\varpi}$	$(a\sqrt{\varpi}, b\sqrt{\varepsilon\varpi}, 0)$	$^{\varpi}\mathcal{S} = C_G(a\sqrt{\varpi}, 0, 0)$	$\nu(a) \geq -r - \frac{1}{2}, \quad \nu(b) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varepsilon}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varpi} + b\sqrt{\varepsilon\varpi}, 0)$	$^{\varepsilon\varpi}\mathcal{L} = C_G(b\sqrt{\varepsilon\varpi}, 0)$	$\nu(a) = -r - \frac{1}{2}, \quad \nu(b) \geq -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varepsilon}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varpi} + b\sqrt{\varepsilon\varpi}, 0)$	$^{\varpi}\mathcal{L} = C_G(a\sqrt{\varpi}, 0)$	$\nu(a) \geq -r - \frac{1}{2}, \quad \nu(b) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varpi'}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varepsilon} + b\sqrt{\varepsilon\varpi'}, 0)$	$^{\varepsilon}\mathcal{L} = C_G(a\sqrt{\varepsilon}, 0)$	$\nu(a) \geq -r + \frac{1}{2}, \quad \nu(b) = -r - \frac{1}{2}$	$r \in \frac{1}{2} + \mathbb{Z}$
$^{\varpi'}\mathfrak{t}_4^{\varepsilon,\varpi}$	$(a\sqrt{\varepsilon} + b\sqrt{\varepsilon\varpi'}, 0)$	$^{\varepsilon\varpi'}\mathcal{L} = C_G(b\sqrt{\varepsilon\varpi'}, 0)$	$\nu(a) = -r, \quad \nu(b) \geq -r$	$r \in \mathbb{Z}$

Table A.11: *The M-generic elements of depth $-r$ of the Lie algebra of elliptic tori $\mathfrak{t} \setminus \{0\}$.*

Appendix B

Comparison with the case of $\mathrm{Sp}_4(F)$

In [KY11], Ju-Lee Kim and J.K. Yu gave an explicit classification of the twisted Levi sequences for the construction of all *Kim-Yu types* for irreducible representations of the group $\mathrm{Sp}_4(F)$, and this was the inspiration for this thesis about $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$.

The group Sp_4 and SO_5 share the dual root systems C_2 and B_2 respectively which are isomorphic. The group Sp_4 is simply connected and is a double cover of SO_5, so the corresponding p-adic groups are strongly related.

In this appendix, we compare our results.

B.1 Root systems and alcoves

	Sp_4 (Kim–Yu)	$\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ (this work)
Root system	C_2	B_2
Simple roots	$\alpha_1 = \epsilon_1 - \epsilon_2$ (short), $\alpha_2 = 2\epsilon_2$ (long)	$a_{l_1} = \epsilon_1 - \epsilon_2$ (long), $a_{s_2} = \epsilon_2$ (short)
Highest root θ	$2\epsilon_1 = 2\alpha_1 + \alpha_2$	$\epsilon_1 + \epsilon_2 = a_{l_1} + 2a_{s_2}$
Weyl group	D_4 (dihedral, order 8)	D_4 (same)

The fundamental alcoves differ:

$$\mathcal{C}_{C_2} :\ 0 \leq x_2 \leq x_1 \leq \tfrac{1}{2} \qquad \text{vs.} \qquad \mathcal{C}_{B_2} :\ 0 \leq x_2 \leq x_1,\ x_1 + x_2 \leq 1.$$

The fundamental alcove for the affine Weyl group of B_2 is the triangle with vertices $(0,0)$, $(1,0)$, $(1/2,1/2)$, while for C_2 it is $(0,0)$, $(1/2,0)$, $(1/2,1/2)$. That said, as noted in Section 2.5.3, the fundamental domain for the action of $G = \mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ on $\mathcal{B}(G)$, which is contained in the fundamental alcove, can be identified with the fundamental alcove of $\mathcal{B}(\mathrm{Sp}_4(F))$, as we record in Table B.5.

B.2 Kac coordinates to Cartesian coordinates

In [KY11], building points are given in **Kac coordinates** (c_0, c_1, c_2). For Sp_4 (type $\widetilde{C}_2$), the affine simple roots evaluate on a point $x = (x_1, x_2)$ in the apartment as

$$\alpha_0(x) = 1 - 2x_1, \qquad \alpha_1(x) = x_1 - x_2, \qquad \alpha_2(x) = 2x_2.$$

These numbers are the normalized affine coordinates of x; they satisfy

$$\alpha_0(x) + 2\alpha_1(x) + \alpha_2(x) = 1.$$

The Kac coordinates are obtained from these normalized coordinates by clearing denominators. More precisely, if

$$a_i = \alpha_i(x), \qquad i = 0, 1, 2,$$

then the Kac coordinates of x are the smallest integral triple $(c_0, c_1, c_2) = N(a_0, a_1, a_2)$, where N is chosen so that $c_0, c_1, c_2 \in \mathbb{Z}_{\geq 0}$ and $\gcd(c_0, c_1, c_2) = 1$. Equivalently,

$$N = c_0 + 2c_1 + c_2.$$

Conversely, given Kac coordinates (c_0, c_1, c_2), let $N = c_0 + 2c_1 + c_2$. Then the corresponding Cartesian point is

$$x_1 = \frac{2c_1 + c_2}{2N}, \qquad x_2 = \frac{c_2}{2N}. \tag{B.2.1}$$

Thus the Kac coordinates determine the position of the point in $\mathcal{C}_{\text{fund}}$, while a Cartesian point determines the Kac coordinates only after clearing denominators. Therefore we can easily determine their positions in $\mathcal{C}_{\text{fund}}$.

Kac coord. (c_0, c_1, c_2)	N	**Cartesian** (x_1, x_2)	**Location in alcove**
$(1, 0, 0)$	1	$(0,\ 0)$	Vertex
$(1, 1, 1)$	4	$(3/8,\ 1/8)$	Interior
$(1, 0, 1)$	2	$(1/4,\ 1/4)$	Wall $\alpha_1 = 0$ $(x_1 = x_2)$
$(0, 1, 0)$	2	$(1/2,\ 0)$	Vertex P_1
$(2, 1, 0)$	4	$(1/4,\ 0)$	Wall $\alpha_2 = 0$ $(x_2 = 0)$
$(0, 1, 2)$	4	$(1/2,\ 1/4)$	Wall $\alpha_0 = 0$ $(x_1 = 1/2)$

Table B.1: *Translation of Kac coordinates from [KY11] to Cartesian coordinates via formula* (B.2.1).

B.3 Correspondence of isomorphism classes of elliptic tori

Both Sp_4 and $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ have the same abstract set of isomorphism classes of elliptic maximal tori, since these are parameterized, via the algorithm due to Morris (described for special orthogonal groups and unitary groups in Chapter 3) by 4-dimensional algebras over F. In Table B.2 we make the correspondence explicit, by identifying the labels of tori used in [KY11] with the labels of corresponding tori used here.

Kim–Yu label	**Algebra $\mathcal{A}$**	**This work (label)**	**Splitting field**
$T[8]$	$F(\sqrt{\varepsilon}) \oplus F(\sqrt{\varepsilon})$	${}^{\varepsilon}\mathcal{T}^2$	$F(\sqrt{\varepsilon})$ (unramified)
$T[7]$, $\varpi' = \varpi$	$F(\sqrt{\varpi}) \oplus F(\sqrt{\varpi})$	${}^{\varpi}\mathcal{T}^2$	$F(\sqrt{\varpi})$ (ramified)
$T[7]$, $\varpi' = \varepsilon\varpi$	$F(\sqrt{\varepsilon\varpi}) \oplus F(\sqrt{\varepsilon\varpi})$	${}^{\varepsilon\varpi}\mathcal{T}^2$	$F(\sqrt{\varepsilon\varpi})$ (ramified)
$T[5]$, $(\varpi, \varepsilon\varpi)$	$F(\sqrt{\varpi}) \oplus F(\sqrt{\varepsilon\varpi})$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}$	$F(\sqrt{\varepsilon}, \sqrt{\varpi})$
$T[6]$, (ϖ, ε)	$F(\sqrt{\varpi}) \oplus F(\sqrt{\varepsilon})$	${}^{\varpi}\mathcal{T}^{\varepsilon}$	$F(\sqrt{\varepsilon}, \sqrt{\varpi})$
$T[6]$, $(\varepsilon\varpi, \varepsilon)$	$F(\sqrt{\varepsilon\varpi}) \oplus F(\sqrt{\varepsilon})$	${}^{\varepsilon\varpi}\mathcal{T}^{\varepsilon}$	$F(\sqrt{\varepsilon}, \sqrt{\varpi})$
$T[9]$, $E_1 = F(\sqrt{\varepsilon})$	$F(\sqrt{\varepsilon}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	${}^{\varepsilon}\mathfrak{t}_4^{\varepsilon,\varpi}$	$F(\sqrt{\varepsilon}, \sqrt{\varpi})$
$T[10]$, $E_1 = F(\sqrt{\varpi})$	$F(\sqrt{\varpi}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	${}^{\varpi}\mathcal{T}_4^{\varepsilon,\varpi}$	$F(\sqrt{\varepsilon}, \sqrt{\varpi})$
$T[10]$, $E_1 = F(\sqrt{\varepsilon\varpi})$	$F(\sqrt{\varepsilon\varpi}) \subset F(\sqrt{\varepsilon}, \sqrt{\varpi})$	${}^{\varepsilon\varpi}\mathcal{T}_4^{\varepsilon,\varpi}$	$F(\sqrt{\varepsilon}, \sqrt{\varpi})$
$T[4]$, $E_1 = F(\sqrt{\varepsilon})$	$F(\sqrt{\varepsilon}) \subset F(\sqrt[4]{\alpha})$	${}^{\varepsilon}\Upsilon_4^{E}$	$F(\sqrt[4]{\varepsilon})$
$T[3]$, $E_1 = F(\sqrt{\varepsilon})$	$F(\sqrt{\varepsilon}) \subset F(\sqrt[4]{\varepsilon\varpi^2})$	${}^{\varepsilon}\Upsilon_4^{\varepsilon\varpi^2}$	$F(\sqrt[4]{\alpha})$
$T[1], T[2]$, $E_1 = F(\sqrt{\varepsilon^k\varpi})$	$F(\sqrt{\varepsilon^k\varpi}) \subset F(\sqrt[4]{\varepsilon^k\varpi})$	${}^{\varepsilon^k\varpi}\Upsilon_4^{\varepsilon^k\varpi}$	$F(\sqrt[4]{\varepsilon^k\varpi})$

Table B.2: *Correspondence of 12 isomorphism classes of elliptic maximal tori between [KY11] and the classification in this work. Here ϖ' denotes a uniformizer, ε a nonsquare unit.*

B.4 Conjugacy classes within each isomorphism class

While the isomorphism classes are abstractly the same, the number of *conjugacy classes* within each isomorphism class differs between $\mathrm{Sp}_4(F)$ and $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$. Abstractly, this is to be expected since although the groups have isomorphic Lie algebras, their group structures are not the same. Explicitly, this occurs in part because the parameter μ^* from the construction lives in different quotient groups, and for SO_5 there is an additional parameter $\lambda \in F^\times/F^{\times 2}$ arising from the fifth coordinate. Moreover, the various conjugacy classes of tori are distributed among the two groups $\mathrm{SO}(2\mathbb{H} \oplus \langle 1 \rangle)$ and $\mathrm{SO}(\mathbb{H} \oplus \langle \varepsilon, -\varpi, \varepsilon\varpi \rangle)$, and in some sense the correct comparison is of the union of the two. In Table B.3 we compare the numbers of conjugacy classes explicitly.

B.5 Comparison of twisted Levi subgroups

The nonabelian proper Levi subgroups of Sp_4 are isomorphic to $\mathrm{GL}_1 \times \mathrm{SL}_2$ (which is a double cover of $\mathrm{GL}_1 \times \mathrm{SO}_3$) or to GL_2, although we must swap the roles of the long and the short roots. Thus we can compare the isomorphism classes of the F-forms of these groups directly.

In [KY11], Kim and Yu show that Sp_4 has 12 classes of twisted Levi subgroups with compact center up to conjugacy when -1 is a square, and 10 if -1 is not a square.

Isomorphism class	**Kim–Yu** (Sp_4)	**This work** (SO_5)	
	# conj. classes	$\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$	$\mathrm{SO}(\mathbb{H}\oplus\langle\varepsilon,-\varpi,\varepsilon\varpi\rangle)$
$^{\varepsilon}\mathcal{T}^2$ ($T[8]$)	3	2	1
$^{\varpi'}\mathcal{T}^2$ ($T[7]$)	3 each if $-1^{\square}$, otherwise 1 each	2 each (omit 1 if $-1^{\boxtimes}$)	2 each(omit 1 if $-1^{\square}$)
$^{\varpi}\mathcal{T}^{\varepsilon\varpi}$ ($T[5]$)	4 if $-1^{\square}$, otherwise 1	2	2
$^{\varpi'}\mathcal{T}^{\varepsilon}$ ($T[6]$)	4 each	2 each	2 each
$^{\varpi'}\mathcal{T}_4^{\varepsilon,\varpi}$ ($T[10]$)	2 each	1 each	1 each
$^{\varepsilon}\mathcal{T}_4^{\varepsilon,\varpi}$ ($T[9]$)	2	1	1
$^{\varepsilon}\Upsilon_4^{E}$ ($T[4]$)	2	1	1
$^{\varepsilon}\Upsilon_4^{\varepsilon\varpi^2}$ ($T[3]$)	2	1	1
$^{\varpi}\Upsilon_4^{\varepsilon^k\varpi}$ ($T[1]$)	4 $-1^{\square}$, otherwise 2	2 $-1^{\square}$, otherwise 1	2 $-1^{\square}$, otherwise 1
$^{\varepsilon\varpi}\Upsilon_4^{\varepsilon^k\varpi}$ ($T[2]$)	4 $-1^{\square}$, otherwise 2	2 $-1^{\square}$, otherwise 1	2 $-1^{\square}$, otherwise 1
Total	39 when $-1^{\square}$, 28 otherwise	**21** when $-1^{\square}$, 17 otherwise	**18** in both cases

Table B.3: *Conjugacy class counts per isomorphism class. The* $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$ *counts are from Proposition 3.2.9; the* Sp_4 *counts are from the Galois cohomology classification in [KY11, Table 1.3.3].*

In constrast, in our classification of $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$, the twisted Levi subgroups with compact center form 9 conjugacy classes in two types. As explained in Sections 2.2.6 and 2.3.3, different F-forms of the two Levi subgroups may embed in different F-forms of SO_5. Thus to make a full comparison, we have computed the twisted Levi subgroups in the group $\mathrm{SO}(\mathbb{H}\oplus\langle\varepsilon,-\varpi,\varepsilon\varpi\rangle)$ and give the correspondence in Table B.4.

Kim–Yu (Sp_4)	**This work** ($\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$)	$\mathrm{SO}(\mathbb{H}\oplus\langle\varepsilon,-\varpi,\varepsilon\varpi\rangle)$
$U_{\varepsilon,1}\times SL_2$	$^{\varepsilon}\mathcal{S}_{1,s}$	$^{\varepsilon}\mathcal{S}_{\varpi,s}$
$U_{\varepsilon,\varpi}\times SL_2$	$^{\varepsilon}\mathcal{S}_{\varpi,c}$	$^{\varepsilon}\mathcal{S}_{1,c}$
$U_{\varpi,1}\times SL_2$	$^{\varpi}\mathcal{S}_{1,s}$	$^{\varpi}\mathcal{S}_{\varepsilon,s}$
$U_{\varpi,\varepsilon}\times SL_2$	$^{\varpi}\mathcal{S}_{\varepsilon,c}$	$^{\varpi}\mathcal{S}_{1,c}$
$U_{\varepsilon\varpi,1}\times SL_2$	$^{\varepsilon\varpi}\mathcal{S}_{1,s}$	$^{\varepsilon\varpi}\mathcal{S}_{\varepsilon,s}$
$U_{\varepsilon\varpi,\varepsilon}\times SL_2$	$^{\varepsilon\varpi}\mathcal{S}_{\varepsilon,c}$	$^{\varepsilon\varpi}\mathcal{S}_{1,c}$
$U_{\varepsilon}(1,1)$	$^{\varepsilon}\mathcal{L}_s$	
$U_{\varpi}(1,1)$	$^{\varpi}\mathcal{L}_s$	
$U_{\varepsilon\varpi}(1,1)$	$^{\varepsilon\varpi}\mathcal{L}_s$	
$U_{\varepsilon}(2)$		$^{\varepsilon}\mathcal{L}_c$
$U_{\varpi}(2)$		$^{\varpi}\mathcal{L}_c$
$U_{\varepsilon\varpi}(2)$		$^{\varepsilon\varpi}\mathcal{L}_c$

Table B.4: *Twisted Levi subgroups of* Sp_4 *(12 conjugacy classes if* $-1^{\square}$*, 10 otherwise.Both in two types) compared with those of* $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$ *and* $\mathrm{SO}(\mathbb{H}\oplus\langle\varepsilon,-\varpi,\varepsilon\varpi\rangle)$*(9 conjugacy classes in 2 types).*

B.6 Comparison of buildings of twisted Levi subgroups

We have drawn the embeddings of the buildings of the twisted Levi subgroups of $\mathrm{Sp}_4(F)$ in the fundamental alcove of $\mathcal{B}(\mathrm{Sp}_4(F))$ in Figure B.1, for comparison with the case of $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$, which was given in Figures 6.2 and 6.3. Note that to compare the fundamental alcoves of $\mathcal{B}(\mathrm{Sp}_4(F))$ and $\mathcal{B}(\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle))$, one needs to perform a change of coordinates that swaps diagonal lines for vertical/horizontal ones.

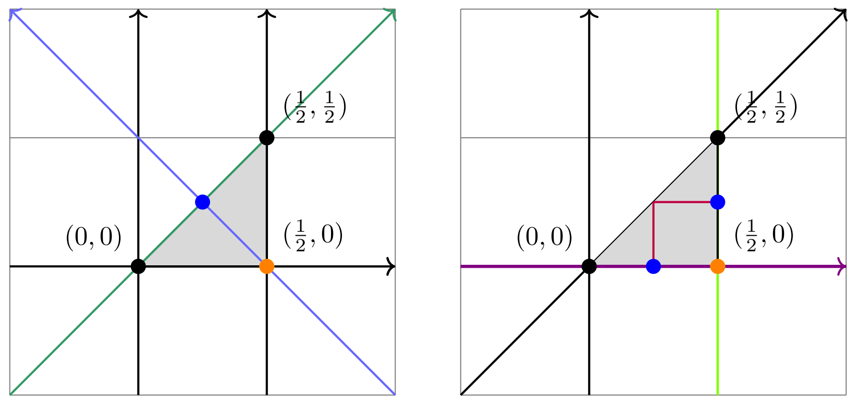

Figure B.1: *On the left are the embeddings of the buildings of type $\mathcal{S}$ inside the fundamental alcove $\mathcal{C}_{\text{fund}}$ of Sp_4. The building $\mathcal{B}(U_\varepsilon(1,1))$ is represented by the green segment, $\mathcal{B}(U_{\varpi'}(1,1))$ by the light blue segment, $\mathcal{B}(U_\varepsilon(2))$ by the orange point, and $\mathcal{B}(U_{\varpi'}(2))$ by the blue point. On the right, the buildings of twisted Levi subgroups of type $\mathcal{L}$. The building $\mathcal{B}(U_{\varepsilon,1}\times SL_2)$ is represented by the horizontal purple segment, $\mathcal{B}(U_{\varepsilon,\varpi}\times SL_2)$ by the vertical green segment, $\mathcal{B}(U_{\varpi',1}\times SL_2)$ and $\mathcal{B}(U_{\varpi',\varepsilon}\times SL_2)$ by the red segments.*

B.7 Comparison of twisted Levi sequences

For the purposes of comparison, we shall only consider the twisted Levi sequences corresponding to pairs $G^0\subset G$ such that there exists an element $X\in\mathrm{Lie}(Z(G^0))$ that is G-generic of depth $-r$ for some $r\in\mathbb{R}$.

These sequences for $\mathrm{Sp}_4(F)$ were provided in [KY11, Table 3.12]. In Table B.5 we match the Kim–Yu data with ours.

We have grouped together all tori in a given isomorphism class when they are attached to the same vertex. In some cases, as noted above, there are more conjugacy classes in $\mathrm{Sp}_4(F)$ and so there are no corresponding sequences in $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$ (such as for $T[4]$ and $T[8]$). Moreover, there are no compact unitary subgroups in $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$, so these twisted Levi sequences in $\mathrm{Sp}_4(F)$ have no analogue in $\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)$.

For the torus identified in [KY11] as $T[9]$ and in our thesis as ${}^{\varepsilon}\mathcal{T}_4^{\varepsilon,\varpi}$, arising from a biquadratic extension, we have found that this torus does admit G-generic elements, and hence can occur in a two-step twisted Levi sequence, but this sequence was omitted in [KY11, Table 3.12].

Kim–Yu G^0	**Depth** r_0	**Vertices in $\mathcal{B}(G_0)$** y_{Sp_4}	**This work** G^0	**Vertices in $\mathcal{B}(G_0)$** $y_{\mathrm{SO}(2\mathbb{H}\oplus\langle 1\rangle)}$	**Kac coordinates** (c_0, c_1, c_2)
$T[1]$	$\frac{1}{4}\mathbb{Z}_+$	$(3/8,\ 1/8)$	${}^{\varpi}\Upsilon_4^{\varepsilon^k\varpi}$	$(1/2, 1/4)$	$(1, 1, 1)$
$T[2]$	$\frac{1}{4}\mathbb{Z}_+$	$(3/8,\ 1/8)$	${}^{\varepsilon\varpi}\Upsilon_4^{\varepsilon^k\varpi}$	$(1/2,\ 1/4)$	$(1, 1, 1)$
$T[3]$	$\frac{1}{2}\mathbb{Z}_+$	$(1/4,\ 1/4)$	${}^{\varepsilon}\Upsilon_4^{\varepsilon\varpi^2}$	$(1/2,\ 0)$	$(1, 0, 1)$
$T[4], c_1$	$\mathbb{Z}_+$	$(0,\ 0)$	${}^{\varepsilon}\Upsilon_4^{E}$	$(0,\ 0)$	$(1, 0, 0)$
$T[4], c_2$	$\mathbb{Z}_+$	$(1/2,\ 1/2)$			$(0, 0, 1)$
$T[5]$	$\frac{1}{2}\mathbb{Z}_+$	$(1/4, 1/4)$	${}^{\varpi}\mathcal{T}^{\varepsilon\varpi}$	$(1/2,\ 0)$	$(1, 0, 1)$
$T[7]$	$\frac{1}{2}\mathbb{Z}_+$	$(1/4, 1/4)$	${}^{\varpi'}\mathcal{T}^{2}$	$(1/2,\ 0)$	$(1, 0, 1)$
$T[8], c_1$	$\mathbb{Z}_+$	$(0, 0)$	${}^{\varepsilon}\mathcal{T}^2_{1,1,1}$	$(0, 0)$	$(1, 0, 0)$
$T[8], c_2$	$\mathbb{Z}_+$	$(1/2, 0)$	${}^{\varepsilon}\mathcal{T}^2_{\varpi,\varpi,1}$	$(1/2, 1/2)$	$(0, 1, 0)$
$T[8], c_3$	$\mathbb{Z}_+$	$(1/2, 1/2)$			$(0, 0, 1)$
$T[9]$?	$\frac{1}{2}\mathbb{Z}_+$		${}^{\varepsilon}\mathcal{T}^{\varepsilon,\varpi}$	$(1/2,\ 0)$	$(1, 0, 1)$
$U_\varepsilon(2)$	$\mathbb{Z}_+$	$(1/2,\ 0)$			
$U_{\varpi'}(2)$	$\frac{1}{2}\mathbb{Z}_+$	$(1/4, 1/4)$			
$U_\varepsilon(1,1)$	$\mathbb{Z}_+$	$(0, 0)$ or $(1/2, 1/2)$	${}^{\varepsilon}\mathcal{L}_s$	$(0, 0)$ or $(1/2, 1/2)$	
$U_{\varpi'}(1,1)$	$\frac{1}{2}\mathbb{Z}_+$	$(1/2, 0)$	${}^{\varpi'}\mathcal{L}_s$	$(1/4, 1/4)$	
$U_{\varpi',1} \times SL_2$	$\frac{1}{2}\mathbb{Z}_+$	$(1/4,\ 0)$	${}^{\varpi'}\mathcal{S}_{1,s}$	$(1/2, 1/2)$	
$U_{\varpi',a} \times SL_2$	$\frac{1}{2}\mathbb{Z}_+$	$(1/2, 1/4)$	${}^{\varpi'}\mathcal{S}_{1,c}$	$(1/2, 0)$	
$U_{\varepsilon,1} \times SL_2$	$\mathbb{Z}_+$	$(0, 0), (1/2, 0)$	${}^{\varepsilon}\mathcal{S}_{1,s}$	$(0, 0)$	
$U_{\varepsilon,\varpi} \times SL_2$	$\mathbb{Z}_+$	$(1/2, 1/2)$ or $(1/2, 0)$	${}^{\varepsilon'}\mathcal{S}_{1,c}$	$(1/2, 1/2)$	

Table B.5: *Comparison of the $d = 1$ case. For each Kim–Yu entry, we give the Cartesian coordinates for the vertex via (B.2.1), and the corresponding torus or twisted Levi from our work with its vertex/vertices. Note that the Cartesian coordinates live in* different *alcoves (C_2 vs. B_2) and are not directly comparable; the comparison is at the level of the abstract torus and depth.*

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