

APPLICATION OF THE PERTURBATION METHOD  
TO THE SMALL AND LARGE DEFLECTION  
PROBLEMS OF CLAMPED PLATES

by

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## A B S T R A C T

In the first part of the work, extensive investigations into the application of the perturbation method of successive approximations to the analyses of plate problems are carried out. The rate and pattern of convergence of the method is studied by applying it to analyse small deflection of clamped skewed plates and both small as well as large deflection problems of clamped circular, elliptical and rectangular plates under uniformly distributed loads. Linear and nonlinear results are obtained for each stage of the successive approximation process and for successive increments of the undetermined coefficients in the plate displacement functions. The effects of the assumed displacement functions, plate aspect ratios and additional parameters such as the elastic foundation modulus on the pattern of convergence are also investigated. Based on these results, a discussion is made on the convergence of the method both in small and large deflection range. It is shown that the rate of convergence depends largely on the assumed displacement functions and decreases significantly from the small to the large deflection theory. The range of variation of the linear and nonlinear results in deflection is plotted and compared with available data. It is found that the method yields results which closely agree with existing values by other investigators.

In the second part of the work, the perturbation technique is then used to investigate the effect of a Winkler type elastic support on the bending of clamped circular, elliptical, rectangular and skewed plates. In case of elliptical plates the nonlinear large deflection problem is also examined. Numerical and graphical results are presented. Variation of centre deflection, maximum centre and edge moments and stresses for such plates with elastic support for different aspect ratios, and changes in skew are shown. The influence of elastic support, plate aspect ratio and angle of skew on the design of plates is examined and discussed.

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# CHAPTER I

## INTRODUCTION

Plates are frequently used as components of large structures such as fluid containers (circular, elliptical and rectangular plates), building and bridge floor systems (rectangular and skewed slabs) and aircraft wing panels (skewed plates). In most cases, the deflections of the plates used in the aforementioned structures are usually rather small in comparison with the thickness of the plate, so that for a proper design of such plates, only a small deflection (linear) theory is required. Nevertheless, movable structures such as aircraft and railroad rolling stock belong to a class in which payload is being increased at the expense of dead weight for reasons of economy. This situation has been intensified in recent years through the utilization of materials with thin sections and high elastic strengths. To gain full advantage from these materials in aircraft, railroad and related structures requires efficient design criteria. For example, the large deflection (nonlinear) theory has recently been applied to the design of railroad fluid-containing hopper cars (1) whose side sheets are made of U. S. S. Tenelon stainless steel (70,000 p.s.i. yield strength). The use of a more accurate theory made it possible to employ side sheets

thinner than those that would have been required based on the small deflection analysis. Occasionally environmental and functional phenomena may render structural plates with elastic supports. As a result, the deflections and stresses of structural plates will be reduced. Examples of plates resting on elastic support are plates on springs, slabs on grade, airport runways, mat foundations and ice sheets in frozen lakes and rivers.

Many different methods are now available for the linear analysis of clamped plates subjected to uniform normal loadings. Some of these methods are cited by Timoskenko and Woinowsky-Krieger (2). With the exception of clamped circular and elliptical plates where exact solutions are known, only approximate solutions are available for other plate shapes with similar boundary conditions. For clamped skewed plates, Morley (3) has presented results for a very limited number of skew angles and aspect ratios. Recently, due to the work of Kennedy (4) and (5) deflections as well as principal stresses for such plates have been determined for various skews and aspect ratios both analytically and experimentally.

For the nonlinear analysis, it can be shown that large deflection of elastic plates are governed by two coupled nonlinear partial differential equations first formulated by Von Karman (6). Approximate solutions for such equations

have been obtained, for example, for the case of the clamped rectangular plates by Way (7) using the Rayleigh-Ritz technique and by Levy (8) and (9) who substituted a double Fourier Series solution into the partial differential equations and evaluated the coefficients. The Ritz method has also been used to solve the problem of large deflection of elliptical plates by Weil and Newmark (10). Invariably, the methods of analysis employed are extremely laborious and require considerable computations.

More recently, a number of investigators notably Chien (11), (12), Nash (13) and Huang (14) have adopted a method of successive approximations based on the small perturbation technique and have applied the method quite successfully to obtain large deflection solutions of uniformly loaded circular, elliptical and rectangular plates. In comparison with the Ritz and Fourier Series approach of analysis, the method of successive approximations is simpler to apply and involves much less computational effort. Unfortunately, in all the successive approximation solutions cited above, approximations beyond the third are not considered and invariably solutions are only given from a six-term algebraic assumed displacement functions which are similar to those employed in this thesis. No mention is made of the degree of accuracy of the solution or the pattern of convergence of the method as the number of unknown coefficients in the assumed displacement functions is increased.

The work of this thesis consists of :

- (i) To investigate the rate and pattern of convergence of the method of successive approximations based on the small perturbation technique as applied to linear and nonlinear plate problems. The method is applied to analyse small and large deflection problems of clamped elliptical and rectangular plates under uniform normal loads. Small deflection problem of skewed plates is also studied. Linear and nonlinear results are obtained for each stage of the successive approximation process and for successive increments of the undetermined coefficients in the plate displacement functions. Results so obtained are compared with those by other investigators (2), (3), (4), (7), (8), (9) and (10). The effect of an elastic support on the small deflection problem of uniformly loaded clamped circular, elliptical, rectangular and skewed plates is then studied by this method and the results are compared with those obtained by the variation method of Galerkin (32).
- (ii) To extend the theoretical analysis by this method into the large deflection behaviour of clamped elastically support elliptical plates under uniform normal load.

Throughout the analyses, the successive approximation method (15), (16) and (17) based on the smallness of the central

deflection is used to determine the lateral deflection, the inplane displacements and hence the moments and stresses of the clamped plates. Here the dimensionless ratio of the central deflection to the thickness of the plate,  $W_0$ , is used as the perturbation parameter. Essentially, the perturbation technique is based on the principle that, if a well constructed asymptotic power series should satisfy the differential equation, then each parameter should also satisfy the governing differential equation independently. Following the perturbation procedure, asymptotic power series containing a number of undetermined coefficients are assumed relating first the dimensionless load to the central deflection perturbation parameter  $W_0$ . Power series containing unknown functions of the co-ordinate axes are also assumed relating each of the displacement components  $u$ ,  $v$  and  $w$  in the middle plane of the plate with the same perturbation parameter  $W_0$ . In assuming these functions for displacements, care was taken to ensure that these assumed functions satisfy not only the boundary conditions of the plate but also the condition of symmetry. The assumed displacement functions are next substituted in turn into the governing partial differential equations derived in each step in the successive approximation sequence. By equating equal powers of the co-ordinates, and by solving the set of simultaneous linear equations thus obtained, all the unknown coefficients in the

displacement functions are evaluated. Finally, these displacement functions with their determined parameters are differentiated to yield the required bending and membrane stresses anywhere within the plate boundary.

To facilitate the computational work involved in obtaining the deflection, principal bending and membrane stresses of the plates, all the problems are coded in Fortran for the IBM 360/65. To minimize round-off errors, double precision arithmetic was used throughout the work. The flow chart and a typical program are included in the Appendix B.

## CHAPTER II

### REVIEW OF LITERATURE

In the bending of plate problem when the strain in the middle plane of the plate is negligible and the lateral deflection from its initial plane is small compared with the thickness of the plate, the load and deflection relationship is linear. The analysis of this small deflection behaviour is regarded as linear analysis. However, if the lateral deflection is no longer small, the analysis of the problem must be extended to include the strain developed in the middle plane of the plate. The load and deflection relationship becomes nonlinear. Thus the analysis of this large deflection behaviour is regarded as nonlinear analysis.

#### 2.1 Linear Analysis

The linear small deflection behaviour of circular, elliptical and rectangular plates is well documented since the start of the nineteenth century. For uniformly loaded circular, and elliptical clamped plates, rigorous solutions have been obtained (2). For rectangular plates with uniform lateral load and clamped edges, the problem has been solved by various approximate techniques (2), such as double Fourier series, energy method and finite difference approximations. In case

of skewed plates, the small deflection behaviour has been investigated mainly by the method of finite difference. The first publication was probably due to Brigatti (18) who obtained limited results, by means of finite difference equations, for uniformly loaded rhombic plates simply supported and clamped. In 1941, studies of skewed plates by means of finite difference techniques were also made by Jensen (19). In 1953, Dorman (20) used the energy approach to investigate the bending behaviour of clamped skewed plates. Other outstanding researchers in skewed plates and related structures included Morley (3), Mirsky (21) and Jones (22). More recently, Kennedy and Ng (4), (5) have solved the small deflection problem of uniformly load skewed plates by means of variational techniques and the results were verified by experiments.

## 2.2 Nonlinear Analysis

As opposed to the small deflection analysis of circular, elliptical and rectangular plates, the corresponding nonlinear large deflection analysis of such plates received comparably less attention. However some of the investigations are still quite well known in the technical literature. In case of clamped circular plates under uniform load, in 1924, Timoshenko (23) used the energy method based upon an assumed form of lateral displacement to obtain his approximate solution. In 1925, Nadai

(24) established an approximate solution by considering a circular plate subjected to only approximately uniform pressure. In 1934, Way (25) employed a power series to obtain an exact solution of the same problem. In 1942, McPherson, Ramberg and Levy (26) carried out tests of uniformly loaded circular plates to obtain their experimental results. In 1947, Chien (11) applied the perturbation technique, equivalent to a greatly simplified form of power series method, to obtain his solution of the same circular plate problem.

In case of clamped rectangular plates under uniform load, the problem was solved approximately by Way (7) in 1938 using the Ritz energy method. In 1942, Levy (8) obtained a solution to the same problem by the double Fourier series method. In 1948, Wang (27) solved the problem again by using the method of finite difference. In 1957, Chien and Yeh (12) used the successive approximations of the perturbation technique and obtained a solution to the square plate problem and verified by experiments.

In case of uniformly loaded elliptical plates with built-in edges, the first solution was probably due to Perry (28) in 1950, who employed expansions into Mathieu functions. In 1956, Weil and Newmark (10) solved the same problem based on the Ritz method. In 1959, Nash and Cooley (13) used the successive approximations of the perturbation technique to solve the problem again.

With regard to the influence of elastic support on the behaviour of plates, there are but a few investigations. In 1960, Nash and Ho (29) investigated the elastic support effect on the large deflection of clamped circular plates by means of the perturbation technique. In 1963, Sinha (30) examined the large deflections problems of simply supported rectangular plates on elastic foundations. In 1966, Ng and Kennedy (31) obtained approximate solutions for small and large deflection behaviour of elastically supported clamped rectangular plates. In 1969, Ng (32) obtained results of the influence of elastic support on small deflection of clamped circular, elliptical, rectangular and skewed plates by means of the variational method of Galerkin and verified by the perturbation method.

In the treatments of various kinds of plates, the perturbation method has been used to deal with nonlinear plate problems mainly because it is simpler to apply and requires less computational effort in comparison with the methods of Fourier series and the finite difference. However, in all the successive approximation solutions cited above, no mention is made of the degree of accuracy of the solution or the pattern of convergence as the number of unknown coefficients in the assumed displacement function is increased.

## CHAPTER III

### FORMULATION OF EQUATIONS

In the analysis of small deflection problems of elastic thin plates, the deflections are considered to be of such magnitude that the effect of stretching of the middle plane of the plate on its curvature can be neglected. When the lateral deflection of plate is moderately large, that is, in the neighbourhood of one half the plate thickness or more, the linear theory of thin plates is no longer applicable and the effect of the forces acting in the middle surface must be taken into account.

#### 3.1 Assumptions

Throughout the theoretical work of linear and nonlinear analyses, the following assumptions are made:

- (i) Points which lie on a normal to the mid-plane of the undeflected plate lie on a normal to the mid-plane of the deflected plate.
- (ii) The stresses normal to the mid-plane of the plate, arising from the applied loading, are negligible in comparison with the stresses in the plane of the plate.
- (iii) The deflection,  $w$ , of the plate is small enough so that the first two assumptions still hold, and yet large enough so

that the products of the in-plane forces or their derivatives and the derivatives of  $w$  are of the same order of magnitude as the derivatives of the shear forces.

Also for the linear (small deflection) analysis the forces in the middle plane of the plate is neglected and this gives one additional assumption viz.

(iv) The mid-plane of the plate is a neutral plane i.e. any mid-plane stresses arising from the deflection of the plate (into a non-developable surface) is ignored.

### 3.2 The Governing Partial Differential Equations

The nonlinear large deflection behaviour of thin elastic plates is governed by two coupled nonlinear partial differential equations derived by von Karman. It is often advantageous to express these equations in terms of the displacement components  $u$ ,  $v$ , and  $w$  (parallel to the rectangular coordinate axes  $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$  respectively) of a point in the middle plane of the plate.

(i) Equilibrium of the Plate Element in the  $\bar{x}$  and  $\bar{y}$  Directions.

Consider the equilibrium of a small element cut out from the middle plane of the plate with sides  $d\bar{x}$  and  $d\bar{y}$  as shown in Figure 3-1. Let  $N_{\bar{x}}$ ,  $N_{\bar{y}}$  and  $N_{\bar{xy}}$  be the in-plane forces per

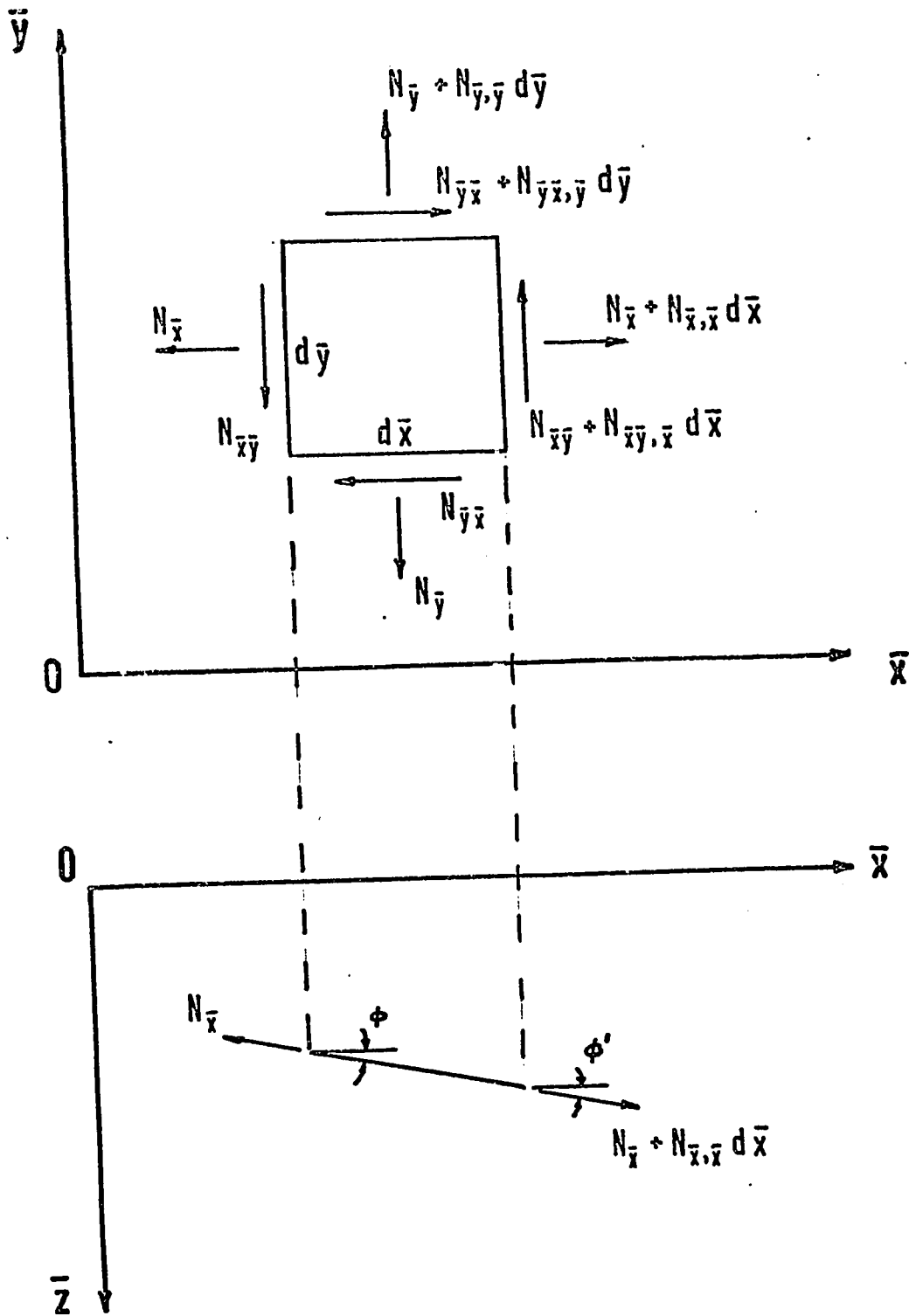


Figure 3-1 In-plane Forces on Plate Element

unit length of the plate.

Neglecting body forces and since  $\phi$  and  $\phi'$  are small ( $\cos \phi = \cos \phi' = 1$ ) the equilibrium of the plate element in the  $\bar{x}$  and  $\bar{y}$  directions yield respectively,

$$N_{\bar{x},\bar{x}} + N_{\bar{xy},\bar{y}} = 0 \dots\dots\dots(3-1)$$

$$N_{\bar{xy},\bar{x}} + N_{\bar{y},\bar{y}} = 0 \dots\dots\dots(3-2)$$

where the comma notation signifies differentiation.

(ii) Equilibrium of the Plate Element in the  $\bar{z}$  Direction.

The equilibrium in the  $\bar{z}$  direction is obtained by considering separately the in-plane forces and lateral loads acting along this direction.

(a) Forces in the  $\bar{z}$  direction due to the in-plane forces:

From Figure 3-1, the net contribution of the downward force by  $N_{\bar{x}}$  and  $(N_{\bar{x}} + N_{\bar{x},\bar{x}} d\bar{x})$  in the plate element is

$$-N_{\bar{x}} d\bar{y} \sin \phi + (N_{\bar{x}} + N_{\bar{x},\bar{x}} d\bar{x}) d\bar{y} \sin \phi'$$

For small  $\phi$  and  $\phi'$ ,

$$\sin \phi \cong \phi \cong w_{,\bar{x}}$$

$$\sin \phi' \cong \phi' \cong \phi + \phi_{,\bar{x}} d\bar{x} = w_{,\bar{x}} + w_{,\bar{xx}} d\bar{x}$$

The above expression becomes:

$$(N_{\bar{x}} w_{,\bar{x}\bar{x}} + N_{\bar{x},\bar{x}} w_{,\bar{x}}) d\bar{x} d\bar{y} \dots\dots\dots(3-3)$$

where higher-order terms are neglected.

Similarly, the net downward contribution of  $N_{\bar{y}}$  and  $(N_{\bar{y}} + N_{\bar{y},\bar{y}} d\bar{y})$  on the plate element is

$$(N_{\bar{y}} w_{,\bar{y}\bar{y}} + N_{\bar{y},\bar{y}} w_{,\bar{y}}) d\bar{x} d\bar{y} \dots\dots\dots(3-4)$$

The net downward contribution for  $N_{\bar{xy}}$  and  $(N_{\bar{xy}} + N_{\bar{xy},\bar{x}} d\bar{x})$  on the plate element is, where higher-order terms are neglected,

$$-N_{\bar{xy}} d\bar{y} w_{,\bar{y}} + (N_{\bar{xy}} + N_{\bar{xy},\bar{x}} d\bar{x}) d\bar{y} (w_{,\bar{y}} + w_{,\bar{xy}} d\bar{x})$$

or 
$$(N_{\bar{xy}} w_{,\bar{xy}} + N_{\bar{xy},\bar{x}} w_{,\bar{y}}) d\bar{x} d\bar{y} \dots\dots\dots(3-5)$$

Similarly, the net downward contribution of  $N_{\bar{yx}}$  and  $(N_{\bar{yx}} + N_{\bar{yx},\bar{y}} d\bar{y})$  is

$$(N_{\bar{yx}} w_{,\bar{xy}} + N_{\bar{yx},\bar{y}} w_{,\bar{x}}) d\bar{x} d\bar{y} \dots\dots\dots(3-6)$$

Since  $N_{\bar{xy}} = N_{\bar{yx}}$ , and making use of equations (3-1) and (3-2), the net downward contribution of all the in-plane forces can be obtained by adding equations (3-3) through (3-6) viz.

$$(N_{\bar{x}} w_{,\bar{x}\bar{x}} + 2N_{\bar{xy}} w_{,\bar{xy}} + N_{\bar{y}} w_{,\bar{y}\bar{y}}) d\bar{x} d\bar{y} \dots\dots\dots(3-7)$$

(b) Forces in the  $\bar{z}$  direction due to lateral loads.

In Figure 3-2, let  $Q_{\bar{x}}$ ,  $Q_{\bar{y}}$  be the shear forces per unit length,  $M_{\bar{x}}$ ,  $M_{\bar{y}}$  and  $M_{\bar{xy}}$  be the bending and twisting moments per unit length,  $q$  the intensity of downward lateral load and  $k$  the foundation modulus the magnitude of which is proportional to the deflection.

Neglecting the body forces of the plate element equilibrium in the downward direction (along the  $\bar{z}$  axis) gives :

$$(Q_{\bar{x},\bar{x}} + Q_{\bar{y},\bar{y}} + q - kw) d\bar{x} d\bar{y} = 0 \dots\dots\dots(3-8)$$

By taking moments of all forces acting about an axis parallel to the  $\bar{y}$  axis and neglecting higher-order terms gives :

$$Q_{\bar{x}} d\bar{x} d\bar{y} - M_{\bar{x},\bar{x}} d\bar{x} d\bar{y} + M_{\bar{xy},\bar{y}} d\bar{x} d\bar{y} = 0$$

or 
$$M_{\bar{x},\bar{x}} - M_{\bar{xy},\bar{y}} - Q_{\bar{x}} = 0 \dots\dots\dots(3-9)$$

Similarly by taking moments about an axis parallel to the  $\bar{x}$  axis, we have

$$M_{\bar{y},\bar{y}} - M_{\bar{xy},\bar{x}} - Q_{\bar{y}} = 0 \dots\dots\dots(3-10)$$

Now making use of the moments curvature relationships (2)

viz.

$$M_{\bar{x}} = -D (w_{,\bar{xx}} + \nu w_{,\bar{yy}})$$

$$M_{\bar{y}} = -D (w_{,\bar{yy}} + \nu w_{,\bar{xx}})$$

$$M_{\bar{xy}} = D (1 - \nu) w_{,\bar{xy}}$$

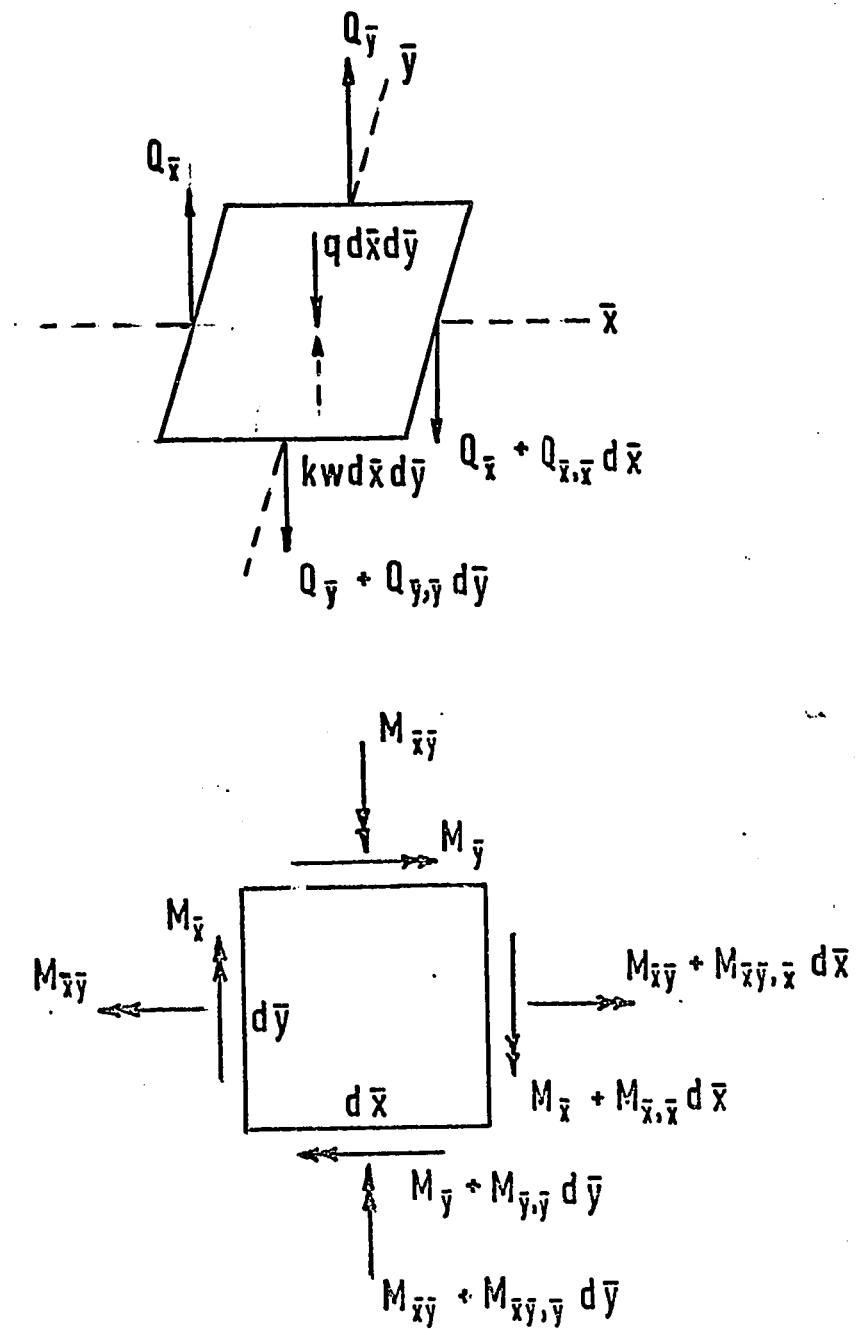


Figure 3-2 Moments and Shear Forces on Plate Element

and substituting into expressions (3-9) and (3-10), we get:

$$Q_{\bar{x}} = -D (w_{,xxx} + w_{,xyy}) \dots\dots\dots(3-11)$$

$$Q_{\bar{y}} = -D (w_{,yyx} + w_{,yxx}) \dots\dots\dots(3-12)$$

Now adding equations (3-7) to (3-8) gives the equilibrium equation in the vertical direction due to the combined action of the lateral and in-plane membrane forces.

$$Q_{\bar{x},\bar{x}} + Q_{\bar{y},\bar{y}} + q - kw + N_{\bar{x}} w_{,xx} + 2N_{\bar{xy}} w_{,xy} + N_{\bar{y}} w_{,yy} = 0 \dots\dots\dots(3-13)$$

Substituting equations (3-11) and (3-12) into equation (3-13) gives :

$$D (w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) = q - kw + N_{\bar{x}} w_{,xx} + 2N_{\bar{xy}} w_{,xy} + N_{\bar{y}} w_{,yy} \dots\dots\dots(3-14)$$

In terms of stresses, equations (3-1), (3-2) and (3-14) can be written as :

$$\sigma_{\bar{x},\bar{x}} + \tau_{\bar{xy},\bar{y}} = 0 \dots\dots\dots(3-15)$$

$$\sigma_{\bar{y},\bar{y}} + \tau_{\bar{xy},\bar{x}} = 0 \dots\dots\dots(3-16)$$

and 
$$D \nabla^2 \nabla^2 w = q - kw + h(\sigma_{\bar{x}} w_{,xx} + \sigma_{\bar{y}} w_{,yy} + 2\tau_{\bar{xy}} w_{,xy}) \dots\dots\dots(3-17)$$

To establish equations (3-15), (3-16) and (3-17) in terms of the displacement  $u$ ,  $v$  and  $w$  of the plate element, it is required to employ the equations of plane strain (33) viz.

$$\left. \begin{aligned} \sigma_{\bar{x}} &= \frac{E}{1-\nu^2} (\epsilon_{\bar{x}} + \nu \epsilon_{\bar{y}}) \\ \sigma_{\bar{y}} &= \frac{E}{1-\nu^2} (\epsilon_{\bar{y}} + \nu \epsilon_{\bar{x}}) \\ \tau_{\bar{xy}} &= \frac{E}{2(1+\nu)} \gamma_{\bar{xy}} \end{aligned} \right\} \dots\dots\dots(3-18)$$

and the equations of compatibility (2) viz.

$$\left. \begin{aligned} \epsilon_{\bar{x}} &= u_{,\bar{x}} + \frac{1}{2} (w_{,\bar{x}})^2 \\ \epsilon_{\bar{y}} &= v_{,\bar{y}} + \frac{1}{2} (w_{,\bar{y}})^2 \\ \gamma_{\bar{xy}} &= u_{,\bar{y}} + v_{,\bar{x}} + w_{,\bar{x}} w_{,\bar{y}} \end{aligned} \right\} \dots\dots\dots(3-19)$$

Substituting equations (3-18) and (3-19) into equations (3-15), (3-16) and (3-17), the three general differential equations in terms of the displacements  $u$ ,  $v$  and  $w$  in rectangular cartesian coordinates governing the large deflection of thin elastic plates are obtained:

$$\begin{aligned}
& u_{,\bar{x}\bar{x}} + w_{,\bar{x}} w_{,\bar{x}\bar{x}} + \nu (v_{,\bar{x}\bar{y}} + w_{,\bar{y}} w_{,\bar{x}\bar{y}}) \\
& + \frac{1}{2} (1 - \nu) (u_{,\bar{y}\bar{y}} + v_{,\bar{x}\bar{y}} + w_{,\bar{x}} w_{,\bar{y}\bar{y}} + w_{,\bar{y}} w_{,\bar{x}\bar{y}}) \\
& = 0 \dots\dots\dots(3-20)
\end{aligned}$$

$$\begin{aligned}
& v_{,\bar{y}\bar{y}} + w_{,\bar{y}} w_{,\bar{y}\bar{y}} + \nu (u_{,\bar{x}\bar{y}} + w_{,\bar{x}} w_{,\bar{x}\bar{y}}) \\
& + \frac{1}{2} (1 - \nu) (v_{,\bar{x}\bar{x}} + u_{,\bar{x}\bar{y}} + w_{,\bar{y}} w_{,\bar{x}\bar{x}} + w_{,\bar{x}} w_{,\bar{x}\bar{y}}) \\
& = 0 \dots\dots\dots(3-21)
\end{aligned}$$

$$\begin{aligned}
D \nabla^2 \nabla^2 w = q - kw + h \left\{ \frac{E}{(1 - \nu^2)} \right. \\
& [u_{,\bar{x}} + \frac{1}{2} (w_{,\bar{x}})^2 + \nu (v_{,\bar{y}} + \frac{1}{2} (w_{,\bar{y}})^2)] w_{,\bar{x}\bar{x}} \\
& + \frac{E}{(1 - \nu^2)} [v_{,\bar{y}} + \frac{1}{2} (w_{,\bar{y}})^2 \\
& + \nu (u_{,\bar{x}} + \frac{1}{2} (w_{,\bar{x}})^2)] w_{,\bar{y}\bar{y}} + \frac{E}{(1 + \nu)} \\
& \left. (u_{,\bar{y}} + v_{,\bar{x}} + w_{,\bar{x}} w_{,\bar{y}}) w_{,\bar{x}\bar{y}} \right\} \dots\dots(3-22)
\end{aligned}$$

To solve the above equation (3-20), (3-21) and (3-22) it is often convenient to have them in dimensionless forms.

To render them non-dimensional the following dimensionless ratios are introduced :

$$R = b/a, \quad x = \bar{x}/a \quad y = \bar{y}/b,$$

$$U = ua/h^2, \quad V = va/h^2 \quad W = w/h$$

$$Q = qb^4/Dh, \quad K = kb^4/D$$

$$\begin{aligned}
& u_{,\bar{x}\bar{x}} + w_{,\bar{x}} w_{,\bar{x}\bar{x}} + \nu (v_{,\bar{x}\bar{y}} + w_{,\bar{y}} w_{,\bar{x}\bar{y}}) \\
& + \frac{1}{2} (1 - \nu) (u_{,\bar{y}\bar{y}} + v_{,\bar{x}\bar{y}} + w_{,\bar{x}} w_{,\bar{y}\bar{y}} + w_{,\bar{y}} w_{,\bar{x}\bar{y}}) \\
& = 0 \dots\dots\dots(3-20)
\end{aligned}$$

$$\begin{aligned}
& v_{,\bar{y}\bar{y}} + w_{,\bar{y}} w_{,\bar{y}\bar{y}} + \nu (u_{,\bar{x}\bar{y}} + w_{,\bar{x}} w_{,\bar{x}\bar{y}}) \\
& + \frac{1}{2} (1 - \nu) (v_{,\bar{x}\bar{x}} + u_{,\bar{x}\bar{y}} + w_{,\bar{y}} w_{,\bar{x}\bar{x}} + w_{,\bar{x}} w_{,\bar{x}\bar{y}}) \\
& = 0 \dots\dots\dots(3-21)
\end{aligned}$$

$$\begin{aligned}
D\nabla^2 \nabla^2 w &= q - kw + h \left\{ \frac{E}{(1 - \nu^2)} \right. \\
& \left[ u_{,\bar{x}} + \frac{1}{2} (w_{,\bar{x}})^2 + \nu (v_{,\bar{y}} + \frac{1}{2} (w_{,\bar{y}})^2) \right] w_{,\bar{x}\bar{x}} \\
& + \frac{E}{(1 - \nu^2)} \left[ v_{,\bar{y}} + \frac{1}{2} (w_{,\bar{y}})^2 \right. \\
& \left. + \nu (u_{,\bar{x}} + \frac{1}{2} (w_{,\bar{x}})^2) \right] w_{,\bar{y}\bar{y}} + \frac{E}{(1 + \nu)} \\
& \left. (u_{,\bar{y}} + v_{,\bar{x}} + w_{,\bar{x}} w_{,\bar{y}}) w_{,\bar{x}\bar{y}} \right\} \dots\dots(3-22)
\end{aligned}$$

To solve the above equation (3-20), (3-21) and (3-22) it is often convenient to have them in dimensionless forms.

To render them non-dimensional the following dimensionless ratios are introduced :

$$\begin{aligned}
R &= b/a, & x &= \bar{x}/a & y &= \bar{y}/b, \\
U &= ua/h^2, & V &= va/h^2 & W &= w/h \\
Q &= qb^4/Dh, & K &= kb^4/D
\end{aligned}$$

With the dimensionless ratios, equations (3-20),  
 (3-21) and (3-22) become:

$$\begin{aligned}
 2R^3 U_{,xxx} + R(1-\nu) U_{,yy} + R^2(1+\nu) V_{,xy} \\
 + 2R^3 W_{,x} W_{,xx} + R(1+\nu) W_{,y} W_{,xy} \\
 + R(1-\nu) W_{,xy} W_{,y} = 0 \dots\dots\dots(3-23)
 \end{aligned}$$

$$\begin{aligned}
 2RV_{,yy} + R^3(1-\nu) V_{,xx} + R^2(1+\nu) U_{,xy} \\
 + 2W_{,y} W_{,yy} + R^2(1+\nu) W_{,x} W_{,xy} \\
 + R^2(1-\nu) W_{,xx} W_{,y} = 0 \dots\dots\dots(3-24)
 \end{aligned}$$

$$\begin{aligned}
 R^4 W_{,xxxx} + 2R^2 W_{,xxyy} + W_{,yyyy} = Q - KW \\
 + 12 \left\{ R^2 U_{,x} + \frac{1}{2} (RW_{,x})^2 \right. \\
 \left. + \nu [RV_{,y} + \frac{1}{2} (W_{,y})^2] \right\} R^2 W_{,xx} \\
 + 12 \left\{ RV_{,y} + \frac{1}{2} (W_{,y})^2 \right. \\
 \left. + \nu [R^2 U_{,x} + \frac{1}{2} (RW_{,x})^2] \right\} W_{,yy} \\
 + 12(1-\nu) (RU_{,y} + R^2 V_{,x} + RW_{,x} W_{,y}) RW_{,xy} \\
 \dots\dots\dots(3-25)
 \end{aligned}$$

Hence the problem of large deflections of plates on  
 elastic foundations is reduced to finding solutions to equations

(3-23), (3-24) and (3-25), satisfying the necessary boundary conditions.

It should be noted that a solution of the linear problem (small deflection) is obtained by solving equation (3-22) when the terms due to the effect of the in-plane forces are neglected. In non-dimensional form the equation becomes:

$$R^4 W_{,xxxx} + 2R^2 W_{,xxyy} + W_{,yyyy} = Q - KW$$

.....(3-26)

Also the solution for plates without the effect of foundation modulus is obtained by solving the equations with  $K = 0$ .

### 3.3 Governing Differential Equation for Small Deflection Problem of Radially Symmetric Loaded Circular Plate.

The governing differential equation of the elastically supported circular plate in terms of the polar coordinates system after some slight modification from (2) is given by:

$$D \frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) \right] \right\} = q - kw \quad \dots(3-27)$$

Now by employing an additional dimensionless ratio,  $\psi = 1 - r^2/b^2$ , equation (3-27) can be transformed into dimensionless form as:

$$16 \frac{d}{d\psi} \left\{ (1 - \psi) \frac{d^2}{d\psi^2} \left[ (1 - \psi) \frac{dw}{d\psi} \right] \right\} = Q - KW$$

.....(3-28)

3.4 Governing Differential Equation for Small Deflection Problem of Skewed Plate.

In investigating the small deflection problem of skewed plates, a coordinate system parallel to the edges of the plate, namely the oblique coordinate system  $\alpha$  and  $\beta$  as shown in Figure 3-3 is used.

By the transformation

$$\left. \begin{aligned} \bar{x} &= \alpha \cos \theta \\ \bar{y} &= \beta + \alpha \sin \theta \end{aligned} \right] \dots\dots\dots(3-29)$$

where  $\theta$  is the skew angle, the following relationship between the rectangular  $(\bar{x}, \bar{y})$  and the oblique  $(\alpha, \beta)$  coordinate systems are obtained :

$$\begin{aligned} w_{,\bar{x}} &= w_{,\alpha} \sec \theta - w_{,\beta} \tan \theta \\ w_{,\bar{xy}} &= w_{,\alpha\beta} \sec \theta - w_{,\beta\beta} \tan \theta \\ w_{,\bar{xx}} &= w_{,\alpha\alpha} \sec^2 \theta - 2w_{,\alpha\beta} \sec \theta \tan \theta + w_{,\beta\beta} \tan^2 \theta \\ w_{,\bar{xxxx}} &= w_{,\alpha\alpha\alpha\alpha} \sec^4 \theta - 4w_{,\alpha\alpha\alpha\beta} \sec^3 \theta \tan \theta \\ &\quad + 6w_{,\alpha\alpha\beta\beta} \sec^2 \theta \tan^2 \theta - 4w_{,\alpha\beta\beta\beta} \sec \theta \tan^3 \theta \\ &\quad + w_{,\beta\beta\beta\beta} \tan^4 \theta \end{aligned}$$

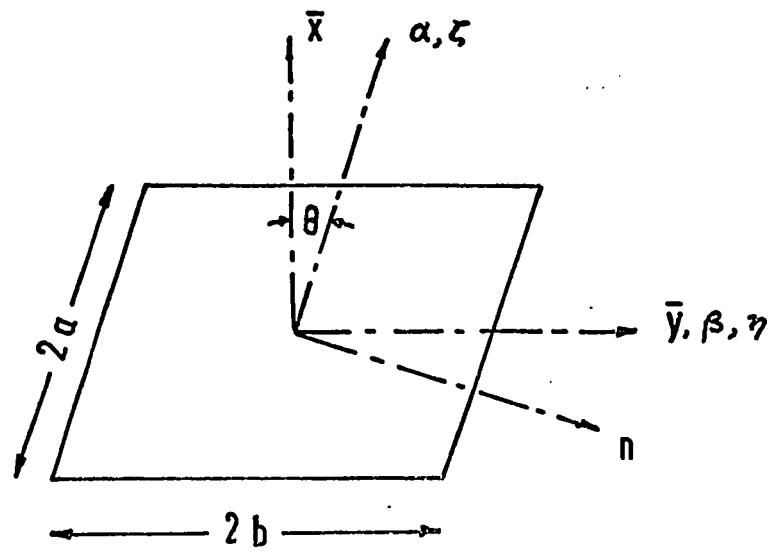


Figure 3-3 The Rectangular and Skewed Coordinate Systems

$$w_{,xyxy} = w_{,\alpha\alpha\beta\beta} \sec^2 \theta - 2w_{,\alpha\beta\beta\alpha} \sec \theta \tan \theta + w_{,\beta\beta\beta\beta} \tan^2 \theta$$

$$w_{,yyyy} = w_{,\beta\beta\beta\beta}$$

Using these transformation relationships and the additional dimensionless quantities  $\zeta = \alpha/a$ ,  $\eta = \beta/b$ , the differential equation governing the small deflection problem with elastic support can be written into the oblique dimensionless form as :

$$R^4 w_{,\zeta\zeta\zeta\zeta} + 2(1 + 2\sin^2 \theta) R^2 w_{,\zeta\zeta\eta\eta} - 4\sin \theta R (R^2 w_{,\zeta\zeta\zeta\eta} + w_{,\zeta\eta\eta\eta}) + w_{,\eta\eta\eta\eta} = \cos^4 \theta (Q - KW) \dots\dots\dots(3-30)$$

Hence the problem of small and large deflections of plates is reduced to finding a solution to the appropriate governing differential equation (3-23), (3-24), (3-25), (3-26), (3-28) and (3-30).

## CHAPTER IV

### METHOD OF SOLUTION

#### 4.1 The Perturbation Procedure

The small parameter perturbation method of successive approximations is now used to obtain approximate solutions to equations (3-23), (3-24), (3-25), (3-26), (3-28) and (3-30) for the linear and nonlinear deflections of clamped plates subjected to lateral uniform pressure with and without the elastic foundation modulus. This method requires the expansion of the displacement components and the dimensionless load quantity in a power series of ascending powers of the dimensionless centre deflection parameter  $W_0$ . Thus let

$$Q = \gamma_1 W_0 + \gamma_3 W_0^3 + \gamma_5 W_0^5 + \dots \quad (4-1)$$

$$W = w_1 W_0 + w_3 W_0^3 + w_5 W_0^5 + \dots \quad (4-2)$$

$$U = s_2 W_0^2 + s_4 W_0^4 + \dots \quad (4-3)$$

$$V = t_2 W_0^2 + t_4 W_0^4 + \dots \quad (4-4)$$

where  $\gamma_1, \gamma_3, \gamma_5 \dots$  are undetermined parameters relating the dimensionless centre deflection  $W_0$  to the dimensionless load  $Q$ ;  $s_2, s_4 \dots, t_2, t_4 \dots$  and  $w_1, w_3, w_5 \dots$  are

functions of the dimensionless coordinates  $(x, y; \zeta, \eta$  and  $\psi$  for rectangular, oblique and polar coordinates respectively) satisfying the boundary conditions of the plate and relating the in-plane and lateral displacements to the same perturbation parameter  $W_0$ .

From the series for  $W$ , equation (4-2) it can be seen that in order the centre deflection be  $W_0$  as defined, it is necessary to require :

$$w_{,1}(0,0) = 1 \quad \text{and} \quad w_{,3}(0,0) = w_{,5}(0,0) = 0$$

.....(4-5)

The prescribed boundary conditions for the clamped rectangular and elliptical plates are respectively :

$$w_{,\bar{x}} = u = w = 0 \quad \text{at} \quad x = \pm 1$$

and

$$w_{,\bar{y}} = v = w = 0 \quad \text{at} \quad y = \pm 1$$

.....(4-6)

$$\left. \begin{aligned} u = v = w = 0 \\ w_{,\bar{x}} = w_{,\bar{y}} = 0 \end{aligned} \right\} \quad \text{at} \quad x^2 + y^2 = 1$$

.....(4-7)

The boundary conditions for the circular plate in polar coordinates are :

$$w_{,r} = w = 0 \quad \text{at} \quad \psi = 0 \quad \text{.....(4-8)}$$

The corresponding boundary conditions in the case of skewed plates can easily be transformed in oblique coordinates as :

$$w_{,\bar{x}} = u \cos \theta = w = 0 \quad \text{at } \zeta = \pm 1$$

and  $w_{,n} = v \cos \theta = w = 0 \quad \text{at } \eta = \pm 1$

$$\dots\dots\dots(4-9)$$

#### 4.2 Solution to the Linear Small Deflection Problem (First Order Approximation)

For the linear small deflection problem, only the first order of  $W_0$  need be considered. Substituting equations (4-1) and (4-2) into equations (3-26), (3-28) and (3-30) and equating coefficients of the terms containing  $W_0$ , the following differential equations governing the small deflection problem of the various types of elastically supported thin plates can be expressed as :

Rectangular Plates :

$$R^4 w_{1,xxxx} + 2R^2 w_{1,xyyy} + w_{1,yyyy} = \gamma_1 - Kw_1$$

$$\dots\dots\dots(4-10)$$

Circular Plates :

$$16 \frac{d}{d\psi} \left\{ (1 - \psi) \frac{d^2}{d\psi^2} \left[ (1 - \psi) \frac{dw_1}{d\psi} \right] \right\}$$

$$= \gamma_1 - Kw_1 \quad \dots\dots\dots(4-11)$$

Skewed Plates :

$$\begin{aligned}
 R^4 w_{1,\zeta\zeta\zeta\zeta} + 2(1 + 2\sin^2\theta) R^2 w_{1,\zeta\zeta\eta\eta} \\
 - 4R \sin\theta (R^2 w_{1,\zeta\zeta\zeta\eta} + w_{1,\zeta\eta\eta\eta}) + w_{1,\eta\eta\eta\eta} \\
 = \cos^4\theta (\gamma_1 - Kw_1) \dots\dots\dots(4-12)
 \end{aligned}$$

To solve the linear small deflection problem of a plate, it is required to assume a deflection function  $w_1$  that satisfies the boundary conditions. Now by substituting the assumed deflection function  $w_1$  for a plate into the appropriate differential equation, (4-10), (4-11) or (4-12) and by equating the corresponding powers of the coordinates, a set of simultaneous equations will result. The solution of these equations defines the value of  $\gamma_1$  in equation (4-1) and all the undetermined coefficients in the assumed deflection function  $w_1$  in equation (4-2) and hence the solution of the linear small deflection problem of the plate.

The deflection functions assumed for the various plate shapes and the associated boundary conditions are outlined as follows :

Rectangular Plates :

$$\begin{aligned}
 w_1 = (1 - x^2)^2 (1 - y^2)^2 \sum_{i,m,n=0}^{\infty} C_i x^{2m} y^{2n} \\
 \text{where } C_0 = 1 \dots\dots\dots(4-13)
 \end{aligned}$$

Boundary Conditions:

$$w_1 = w_{1,\bar{x}} = 0 \quad \text{at } x = \pm 1$$

$$w_1 = w_{1,\bar{y}} = 0 \quad \text{at } y = \pm 1$$

.....(4-13a)

Elliptical Plates:

$$w_1 = (1 - x^2 - y^2)^2 \sum_{i,m,n=0}^{\infty} C_i x^{2m} y^{2n}$$

where  $C_0 = 1$  .....(4-14)

Boundary Conditions:

$$w_1 = w_{1,\bar{x}} = w_{1,\bar{y}} = 0 \quad \text{at } x^2 + y^2 = 1$$

.....(4-14a)

Circular Plates:

$$w_1 = \psi^2 \sum_{i=0}^{\infty} C_i (1 - \psi)^i$$

where  $C_0 = 1$  .....(4-15)

Boundary Conditions:

$$w_1 = w_{1,\psi} = 0 \quad \text{at } \psi = 0 \quad \dots\dots(4-15a)$$

Skewed Plates:

$$w_1 = (1 - \zeta^2)^2 (1 - \eta^2)^2 (1 - \zeta\eta) \sum_{i,m,n=0}^{\infty} C_i \zeta^{2m} \eta^{2n}$$

where  $C_0 = 1$  .....(4-16)

Boundary Conditions:

$$\begin{aligned}
 w_1 = w_{1,\bar{x}} = 0 & \quad \text{at } \zeta = \pm 1 & \dots\dots\dots(4-16a) \\
 w_1 = w_{1,n} = 0 & \quad \text{at } \eta = \pm 1
 \end{aligned}$$

Where the infinite series of the displacement functions are expanded to include an appropriate number of terms, complete expressions of these functions are listed in Appendix A.

4.3 Solution to the Nonlinear Large Deflection Problem (Second and Higher Order Approximations).

Apart from obtaining the linear deflection solution for rectangular, elliptical, circular and skewed plates, the method of successive approximations is also used here to obtain the nonlinear large deflection solution of rectangular and elliptical plates. In case of elliptical plates, the influence of the elastic support on the lateral displacement, and on the distribution of stresses is also investigated.

Now making use of equations (4-2), (4-3) and (4-4) in equations (3-23) and (3-24) yields, after equating powers of  $W_0^2$ , the following differential equations governing the in-plane displacement components of the plate:

$$\begin{aligned}
 2R^3 s_{2,xx} + R(1 - \nu) s_{2,yy} + R^2(1 + \nu) t_{2,xy} \\
 + 2R^3 w_{1,x} w_{1,xx} + R(1 + \nu) w_{1,y} w_{1,xy} \\
 + R(1 - \nu) w_{1,x} w_{1,yy} = 0 \dots\dots\dots(4-17)
 \end{aligned}$$

$$\begin{aligned}
& 2Rt_{2,yy} + R^3 (1 + \nu) t_{2,xx} + R^2 (1 + \nu) s_{2,xy} \\
& + 2w_{1,y} w_{1,yy} + R^2 (1 + \nu) w_{1,x} w_{1,xy} \\
& + R^2 (1 - \nu) w_{1,y} w_{1,xx} = 0 \dots\dots\dots(4-18)
\end{aligned}$$

Substituting the two assumed displacement functions  $s_2$  and  $t_2$  with the boundary conditions and equating powers of  $x$  and  $y$  yield sufficient linear equations to define the undetermined coefficients in the assumed displacement functions  $s_2$  and  $t_2$ , and hence the in-plane displacement components.

The displacement functions assumed for various shapes of plates and the associated boundary conditions are outlined as below:

Rectangular Plates:

$$s_2 = (1 - x^2) (1 - y^2) x \sum_{i,m,n=0}^{\infty} D_i x^{2m} y^{2n} \dots\dots(4-19)$$

$$t_2 = (1 - x^2) (1 - y^2) y \sum_{i,m,n=0}^{\infty} E_i y^{2m} x^{2n} \dots\dots(4-20)$$

where  $D_0 = E_0 = 0$

Boundary Conditions:

$$\begin{aligned}
s_2 = t_2 = 0 \quad \text{at} \quad x = \pm 1 \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad y = \pm 1 \\
\dots\dots\dots(4-20a)
\end{aligned}$$

Elliptical Plates:

$$s_2 = (1 - x^2 - y^2) x \sum_{i,m,n=0}^{\infty} D_i x^{2m} y^{2n} \dots\dots\dots(4-21)$$

$$t_2 = (1 - x^2 - y^2) y \sum_{i,m,n=0}^{\infty} E_i y^{2m} x^{2n} \dots\dots\dots(4-22)$$

where  $D_0 = E_0 = 0$

Boundary conditions:

$$s_2 = t_2 = 0 \quad \text{at} \quad x^2 + y^2 = 1 \quad \dots\dots\dots(4-22a)$$

Where the appropriate complete expressions of the functions are also listed in Appendix A.

Now substituting equations (4-1), (4-2), (4-3) and (4-4) into equations (3-25) and collecting the coefficients of  $w_0^3$  terms, the differential equation governing the first nonlinear term of the lateral displacement expression is obtained, viz.

$$\begin{aligned} R^4 w_{3,xxxx} + 2R^2 w_{3,xyxy} + w_{3,yyyy} &= \gamma_3 - Kw_3 \\ &+ 12 \left\{ (R^4 s_{2,x} w_{1,xx}) + \frac{1}{2} \left[ R^4 w_{1,xx} (w_{1,x})^2 \right] \right. \\ &+ \nu \left[ R^3 w_{1,xx} t_{2,y} + \frac{1}{2} R^2 w_{1,xx} (w_{1,y})^2 \right] \left. \right\} \\ &+ 12 \left\{ R w_{1,yy} t_{2,y} + \frac{1}{2} w_{1,yy} (w_{1,y})^2 \right. \\ &+ \nu \left[ R^2 w_{1,yy} s_{2,x} + \frac{1}{2} R^2 (w_{1,x})^2 w_{1,yy} \right] \left. \right\} \\ &+ 12 (1 - \nu) (R^2 w_{1,xy} s_{2,y} + R^3 w_{1,xy} t_{2,x} \\ &+ R^2 w_{1,y} w_{1,x} w_{1,xy}) \dots\dots\dots(4-23) \end{aligned}$$

By substituting the assumed nonlinear deflection function  $w_3$  satisfying the associated boundary conditions and equating the powers of  $x$  and  $y$  result is a system of simultaneous equations whose solution yields the value of  $\gamma_3$  and all the undetermined coefficients of the assumed nonlinear deflection function  $w_3$ .

The assumed nonlinear deflection function  $w_3$  for plates of various shapes and their associated boundary conditions are outlined as follows:

Rectangular Plates:

$$w_3 = (1 - x^2)^2 (1 - y^2)^2 \sum_{i,m,n=0}^{\infty} F_i x^{2m} y^{2n}$$

where  $F_0 = 0$  .....(4-24)

Boundary Conditions:

$$w_{3,\bar{x}} = w_3 = 0 \quad \text{at } x = \pm 1$$

$$w_{3,\bar{y}} = w_3 = 0 \quad \text{at } y = \pm 1$$

.....(4-24a)

Elliptical Plates:

$$w_3 = (1 - x^2 - y^2)^2 \sum_{i,m,n=0}^{\infty} F_i x^{2m} y^{2n}$$

where  $F_0 = 0$  .....(4-25)

Boundary Conditions :

$$w_{\bar{z},\bar{x}} = w_{\bar{z},\bar{y}} = w_{\bar{z}} = 0 \quad \text{at } x^2 + y^2 = 1 \quad \dots\dots(4-25a)$$

Where the appropriate complete expressions of the functions are again listed in Appendix A.

It should be noted that the assumed displacement functions for both linear and nonlinear problems are chosen because of their simplicity and readiness to suit into another plate shape with the slightest modification.

Following much the same procedure as outlined above, coefficients  $s_4$  and  $t_4$  and subsequently,  $\gamma_5$  and  $w_5$  are obtained.

Thus the foregoing analysis provides a means of determining uniquely the lateral as well as the in-plane displacements anywhere within the plate boundary, hence solutions to linear and nonlinear problems.

#### 4.4 Displacements, Bending Moments and Stresses Relationships.

From the theory of elastic thin plate the bending moments and stresses can be expressed in terms of the displacements (2). In rectangular coordinates  $\bar{x}$  and  $\bar{y}$ , the relation between the displacements  $u$ ,  $v$  and  $w$  and the plate moments and stresses can be expressed as :

Bending Moments :

$$\left. \begin{aligned} M_{\bar{x}} &= -D (w_{,\bar{x}\bar{x}} + \nu w_{,\bar{y}\bar{y}}) \\ M_{\bar{y}} &= -D (w_{,\bar{y}\bar{y}} + \nu w_{,\bar{x}\bar{x}}) \\ M_{\bar{xy}} &= \bar{M}_{\bar{yx}} = D (1 - \nu) w_{,\bar{xy}} \end{aligned} \right] \dots\dots\dots(4-26)$$

Bending Stresses:

$$\left. \begin{aligned} \sigma_{\bar{x}} &= (-6D/h^2) (w_{,\bar{x}\bar{x}} + \nu w_{,\bar{y}\bar{y}}) \\ \sigma_{\bar{y}} &= (-6D/h^2) (w_{,\bar{y}\bar{y}} + \nu w_{,\bar{x}\bar{x}}) \\ \tau_{\bar{x}\bar{y}} &= (6D/h^2) (1 - \nu) w_{,\bar{x}\bar{y}} \end{aligned} \right\} \dots\dots\dots(4-27)$$

Membrane Stresses:

$$\left. \begin{aligned} \sigma'_{\bar{x}} &= \frac{E}{1 - \nu^2} \left\{ u_{,\bar{x}} + \frac{1}{2} (w_{,\bar{x}})^2 + \nu \left[ v_{,\bar{y}} + \frac{1}{2} (w_{,\bar{y}})^2 \right] \right\} \\ \sigma'_{\bar{y}} &= \frac{E}{1 - \nu^2} \left\{ v_{,\bar{y}} + \frac{1}{2} (w_{,\bar{y}})^2 + \nu \left[ u_{,\bar{x}} + \frac{1}{2} (w_{,\bar{x}})^2 \right] \right\} \\ \tau'_{\bar{x}\bar{y}} &= G (u_{,\bar{y}} + v_{,\bar{x}} + w_{,\bar{x}} w_{,\bar{y}}) \end{aligned} \right\} \dots\dots\dots(4-28)$$

Making use of the dimensionless ratios previously outlined and the additional non-dimensional forms from bending moments and stresses, viz.

$$\left. \begin{aligned} M_{\bar{x}} &= \frac{12 (1 - \nu^2) b^2}{Eh^4} M_{\bar{x}} \\ M_{\bar{y}} &= \frac{12 (1 - \nu^2) b^2}{Eh^4} M_{\bar{y}} \\ M_{\bar{x}\bar{y}} &= \frac{6b^2}{Gh^4} M_{\bar{x}\bar{y}} \end{aligned} \right\} \dots\dots\dots(4-29)$$

$$\begin{aligned} \sigma_x &= \frac{(1 - \nu^2) b^2}{Eh^2} \sigma_{\bar{x}} \\ \sigma_y &= \frac{(1 - \nu^2) b^2}{Eh^2} \sigma_{\bar{y}} \\ \tau_{xy} &= \frac{b^2}{Gh^2} \tau_{\bar{xy}} \end{aligned} \quad \left. \begin{array}{l} \dots\dots\dots(4-30) \end{array} \right\}$$

$$\begin{aligned} \sigma'_x &= \frac{(1 - \nu^2) b^2}{Eh^2} \sigma'_{\bar{x}} \\ \sigma'_y &= \frac{(1 - \nu^2) b^2}{Eh^2} \sigma'_{\bar{y}} \\ \tau'_{xy} &= \frac{b^2}{Gh^2} \tau'_{\bar{xy}} \end{aligned} \quad \left. \begin{array}{l} \dots\dots\dots(4-31) \end{array} \right\}$$

Equations (4-26), (4-27) and (4-28) can be shown in the following non-dimensional forms:

$$\begin{aligned} M_x &= -(R^2 W_{,xx} + \nu W_{,yy}) \\ M_y &= -(W_{,yy} + \nu R^2 W_{,xx}) \\ M_{xy} &= RW_{,xy} \end{aligned} \quad \left. \begin{array}{l} \dots\dots\dots(4-32) \end{array} \right\}$$

$$\begin{aligned} \sigma_x &= -\frac{1}{2} (R^2 W_{,xx} + \nu W_{,yy}) \\ \sigma_y &= -\frac{1}{2} (W_{,yy} + \nu R^2 W_{,xx}) \\ \tau_{xy} &= RW_{,xy} \end{aligned} \quad \left. \begin{array}{l} \dots\dots\dots(4-33) \end{array} \right\}$$

$$\begin{aligned}
\sigma'_x &= R^2 U_{,x} + \frac{1}{2} R^2 (W_{,x})^2 + \nu [RV_{,y} + \frac{1}{2} (W_{,y})^2] \\
\sigma'_y &= RV_{,y} + \frac{1}{2} (W_{,y})^2 + \nu [R^2 U_{,x} + \frac{1}{2} (W_{,xx})^2] \\
\tau'_{xy} &= RU_{,y} + R^2 V_{,x} + RW_{,x} W_{,y} \\
&\dots\dots\dots(4-34)
\end{aligned}$$

Relationships between the dimensionless bending moments, stresses and the perturbing parameter  $W_0$ , hence  $Q$ , can be obtained by substituting equations (4-2), (4-3) and (4-4) into equations (4-32), (4-33) and (4-34) as :

$$\begin{aligned}
M_x &= - (R^2 w_{1,xx} + w_{1,yy}) W_0 \\
&\quad - (R^2 w_{3,xx} + \nu w_{3,yy}) W_0^3 + \dots\dots\dots \\
M_y &= - (w_{1,yy} + \nu R^2 w_{1,xx}) W_0 \\
&\quad - (w_{3,yy} + \nu R^2 w_{3,xx}) W_0^3 + \dots\dots\dots \\
M_{xy} &= (R w_{1,xy}) W_0 + (R w_{3,xy}) W_0^3 + \dots\dots\dots \\
&\dots\dots\dots(4-35)
\end{aligned}$$

$$\begin{aligned}
\sigma_x &= - \frac{1}{2} (R^2 w_{1,xx} + \nu w_{1,yy}) W_0 \\
&\quad - \frac{1}{2} (R^2 w_{3,xx} + \nu w_{3,yy}) W_0^3 + \dots\dots\dots \\
\sigma_y &= - \frac{1}{2} (w_{1,yy} + \nu R^2 w_{1,xx}) W_0 \\
&\quad - \frac{1}{2} (w_{3,yy} + \nu R^2 w_{3,xx}) W_0^3 + \dots\dots\dots \\
\tau_{xy} &= (Rw_{1,xy}) W_0 + (Rw_{3,xy}) W_0^3 + \dots\dots\dots \\
&\dots\dots\dots(4-36)
\end{aligned}$$

$$\begin{aligned}
\sigma'_x &= \left\{ R^2 s_{2,x} + \frac{1}{2} R^2 (w_{1,x})^2 + \nu \left[ R t_{2,y} + \frac{1}{2} (w_{1,y})^2 \right] \right\} w_0^2 \\
&\quad + (R^2 w_{1,x} w_{3,x} + \nu w_{1,y} w_{3,y}) w_0^4 + \dots \\
\sigma'_y &= \left\{ R t_{2,y} + \frac{1}{2} (w_{1,y})^2 + \nu \left[ R^2 s_{2,x} + \frac{1}{2} R^2 (w_{1,x})^2 \right] \right\} w_0^2 \\
&\quad + (w_{1,y} w_{3,y} + \nu R^2 w_{1,x} w_{3,x}) w_0^4 + \dots \\
\tau'_{xy} &= (R s_{2,y} + R^2 t_{2,x} + R w_{1,x} w_{1,y}) w_0^2 \\
&\quad + R (w_{1,x} w_{3,y} + w_{3,x} w_{1,y}) w_0^4 + \dots \\
&\quad \dots \dots \dots (4-37)
\end{aligned}$$

To obtain the maximum bending moment, bending and membrane stresses at a point, the following equations are used, viz.

$$\begin{aligned}
M_{\max} &= \frac{M_x + M_y}{2} + \sqrt{\left(\frac{M_x - M_y}{2}\right)^2 + M_{xy}^2} \\
\sigma_{\max} &= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\
\sigma'_{\max} &= \frac{\sigma'_x + \sigma'_y}{2} + \sqrt{\left(\frac{\sigma'_x - \sigma'_y}{2}\right)^2 + \tau'_{xy}{}^2}
\end{aligned}$$

## C H A P T E R V

### RATE OF CONVERGENCE OF THE PERTURBATION

#### METHOD OF SUCCESSIVE APPROXIMATIONS

The method of successive approximations is applied to analyse small and large deflection problems of clamped circular, elliptical, rectangular and skewed plates under uniform pressure. To investigate the rate and pattern of convergence of the method, linear and nonlinear results are obtained for each stage of the successive approximation process and for successive increments of the undetermined coefficients in the plate displacement functions. Furthermore, in order to ascertain qualitatively the effect of additional variable parameters on the pattern of convergence, variables such as the elastic foundation modulus  $K$  is introduced in the linear solution of elliptical plates.

#### 5.1 Linear (Small) Deflection Problem

In order to have a reasonable investigation of the pattern of convergence of the method, assumed small deflection functions listed in Appendix A are used to analyse plates of various shapes. These displacement functions contain up to a maximum of 25 undetermined parameters. Typical results showing the variation of centre deflections with the number of parameters in the assumed lateral deflection function for

rectangular, circular and elliptical plates are tabulated in Tables 5-1 and 5-2 respectively; for elliptical plates with elastic support they are shown in Tables 5-3 and 5-4, while those for skewed plates are shown in Tables 5-5 to 5-7.

It can be observed from Tables 5-1 and 5-2 that, for the rectangular, circular and elliptical plates, the results not only converge rapidly but are also in excellent agreement with those obtained by Timoshenko (2). In the case of rectangular plates at least six terms are required to achieve a reasonable accuracy. However the degree of accuracy is not directly proportional to the number of terms employed in the lateral displacement function. The results reveal no characteristic of monotonic convergence. There is little, if any, tendency that the results will come to constant values as the number of terms in the displacement function increases.

With the circular and elliptical plates a two term solution yields an exact answer. This is to be expected since, unlike the deflection function assumed for rectangular plates, the assumed deflection function for the circular or elliptical plate takes on the exact mathematical equation of a circular or elliptical boundary. Also it can be seen from the results of elliptical plates that the rate of convergence seems to be relatively insensitive with the plate aspect ratio  $R$ . However, with the introduction of the foundation modulus (Tables 5-3 and

T A B L E 5-1

Variation of the Maximum Deflection Coefficient  $\alpha$  with  
the Number of Terms in the Lateral Deflection Function  
 $w_1$  for Rectangular Plates with Built-in Edges

$$w_{\max} = \alpha \frac{qb^4}{D} (10^{-2})$$

| Plate Aspect Ratio R | Number of Terms in Lateral Deflection Function $w_1$ |           |           |           |           |           |           | Timoshenko Ref.(2) |
|----------------------|--|-----------|-----------|-----------|-----------|-----------|-----------|--------------------|
|                      | 3  | 4         | 6         | 8         | 15        | 24        | 25        |                    |
| 1.00                 | 1.8382353  | 1.9453642 | 1.9848554 | 2.0198507 | 2.0259329 | 2.0255942 | 2.0248619 | 2.016              |
| 1.25                 | 1.0838080  | 1.1442166 | 1.1687768 | 1.1898723 | 1.1947186 | 1.1944368 | 1.1941749 |                    |
| 1.50                 | 0.6312883  | 0.6627416 | 0.6777839 | 0.6900133 | 0.6944326 | 0.6942353 | 0.6941993 | 0.695              |
| 1.75                 | 0.3775453  | 0.3933681 | 0.4020698 | 0.4087372 | 0.4122987 | 0.4122081 | 0.4122506 |                    |
| 2.00                 | 0.2344016  | 0.2426151 | 0.2474204 | 0.2508660 | 0.2533260 | 0.2533620 | 0.2533961 | 0.254              |

T A B L E 5-2

Variation of the Maximum Deflection Coefficient  $\beta$  with  
the Number of Terms in the Lateral Deflection Function  
 $w_1$  for Circular and Elliptical Plates with Built-in Edges

$$w_{\max} = \beta \frac{qb^4}{D} (10^{-2})$$

| Plate Aspect<br>Ratio R | Number of Terms in the<br>Lateral Deflection Function $w_1$ |            | Timoshenko<br>Ref. (2) |
|-------------------------|---|------------|------------------------|
|                         | 2   | 25         |                        |
| 1.00                    | 1.56250   | 1.56250    | 1.563                  |
| 1.25                    | 0.92942202  | 0.92942202 | 0.929                  |
| 1.50                    | 0.55096419  | 0.55096419 | 0.551                  |
| 1.75                    | 0.33546495  | 0.33546495 | 0.335                  |
| 2.00                    | 0.21166441  | 0.21166441 | 0.212                  |

T A B L E 5-3

Variation of the Maximum Deflection Coefficient  $\beta$  with the Number of Terms in the Lateral Deflection Function  $w_1$  for Elastically Supported Elliptical Plates with Built-in Edges

$$R = 1.5 \quad K = \frac{kb^4}{D}$$

$$w_{\max} = \beta \frac{ab^4}{D} (10^{-2})$$

| Dimensionless Foundation Modulus K | Number of Terms in Lateral Deflection Function $w_1$ |            |            |            |            |            | Timoshenko Ref. (2) |
|------------------------------------|--|------------|------------|------------|------------|------------|---------------------|
|                                    | 3  | 6          | 8          | 10         | 15         | 25         |                     |
| 0                                  | 0.55096419   | 0.55096419 | 0.55096419 | 0.55096419 | 0.55096419 | 0.55096419 | 0.551               |
| 40                                 | 0.48791282   | 0.48155758 | 0.48141808 | 0.48136592 | 0.48137771 | 0.48137782 | -                   |
| 80                                 | 0.43604628   | 0.42733719 | 0.42692630 | 0.42677169 | 0.42680260 | 0.42680326 | -                   |
| 120                                | 0.39286796   | 0.38379128 | 0.38309435 | 0.38283037 | 0.38287640 | 0.38287808 | -                   |
| 160                                | 0.35652171   | 0.34803753 | 0.34708551 | 0.34672243 | 0.34677666 | 0.34677967 | -                   |
| 200                                | 0.32561367   | 0.31814816 | 0.31698683 | 0.31654083 | 0.31659651 | 0.31660105 | -                   |

T A B L E 5-4

Variation of the Maximum Deflection Coefficient  $\beta$  with the Number of Terms in the Lateral Deflection Function  $w_1$  for Elastically Supported Elliptical Plates with Built-in Edges

$$R = 2.0 \quad K = \frac{kb^4}{D}$$

$$w_{\max} = \beta \frac{qb^4}{D} (10^{-2})$$

| Dimensionless Foundation Modulus K | Number of Terms in Lateral Deflection Function $w_1$ |            |            |            |            |            | Timoshenko Ref. (2) |
|------------------------------------|--|------------|------------|------------|------------|------------|---------------------|
|                                    | 3  | 6          | 8          | 10         | 15         | 25         |                     |
| 0                                  | 0.21186441   | 0.21186441 | 0.21186441 | 0.21186441 | 0.21186441 | 0.21186441 | 0.212               |
| 40                                 | 0.20140821   | 0.20037387 | 0.20026422 | 0.20036043 | 0.20036127 | 0.20036127 | -                   |
| 80                                 | 0.19117770   | 0.19002790 | 0.18999515 | 0.18998198 | 0.18998465 | 0.18998468 | -                   |
| 120                                | 0.18228912   | 0.18066277 | 0.18059953 | 0.18057365 | 0.18057845 | 0.18057854 | -                   |
| 160                                | 0.17446789   | 0.17214458 | 0.17204761 | 0.17200725 | 0.17201408 | 0.17201427 | -                   |
| 200                                | 0.16770753   | 0.16436292 | 0.16423166 | 0.16417611 | 0.16418465 | 0.16418496 | -                   |

T A B L E 5-5

Variation of the Maximum Deflection Coefficient  $\gamma$  with  
the Number of Terms in the Lateral Deflection Function

$w_1$  for Skewed Plates with Built-in Edges

$$R = 1.0 \quad w_{\max} = \gamma \frac{qb^4}{D} (10^{-2})$$

| Skew Angle | Number of Terms in Lateral Deflection Function $w_1$ |        |         |         |         |         |         | Morley Ref. (3) | Kennedy Ref. (4) |
|------------|--|--------|---------|---------|---------|---------|---------|-----------------|------------------|
|            | 3  | 4      | 6       | 8       | 15      | 24      | 25      |                 |                  |
| 15°        | 1.5809   | 1.3847 | 1.7975  | 1.7985  | 1.8359  | 1.8473  | 1.8524  | 1.792           | 1.793            |
| 30°        | 0.8402   | 1.4100 | 1.2278  | 1.0287  | 1.0922  | 1.0961  | 1.1787  | -               | 1.181            |
| 45°        | 0.2961   | 0.6998 | 0.5511  | 0.3771  | 0.4088  | 0.4163  | 0.5574  | -               | 0.508            |
| 60°        | 0.0627   | 0.1712 | 0.1290  | 0.0809  | 0.0879  | 0.0963  | 0.1425  | -               | 0.120            |
| 75°        | 0.00412  | 0.0115 | 0.00865 | 0.00535 | 0.00591 | 0.00672 | 0.00982 | -               | 0.00836          |

T A B L E 5-6

Variation of the Maximum Deflection Coefficient  $\gamma$  with  
the Number of Terms in the Lateral Deflection Function  
 $w_1$  for Skewed Plates with Built-in Edges

$$R = 1.5 \qquad w_{\max} = \gamma \frac{qb^4}{D} (10^{-2})$$

| Skew Angle | Number of Terms in Lateral Deflection Function $w_1$ |         |         |         |         |         |         | Morley Ref.(3) | Kennedy Ref.(4) |
|------------|--|---------|---------|---------|---------|---------|---------|----------------|-----------------|
|            | 3  | 4       | 6       | 8       | 15      | 24      | 25      |                |                 |
| 15°        | 0.5102   | 0.6333  | 0.6154  | 0.6244  | 0.6357  | 0.6390  | 0.6393  | 0.612          | 0.611           |
| 30°        | 0.3103   | 0.4554  | 0.4148  | 0.3750  | 0.4106  | 0.3370  | 0.4360  | 0.405          | 0.411           |
| 45°        | 0.1133   | 0.2253  | 0.1884  | 0.1424  | 0.1650  | 0.1659  | 0.1795  | 0.197          | 0.177           |
| 60°        | 0.02438  | 0.05842 | 0.04564 | 0.03111 | 0.0330  | 0.0431  | 0.0460  | -              | 0.0426          |
| 75°        | 0.00161  | 0.00414 | 0.00314 | 0.00270 | 0.00216 | 0.00315 | 0.00331 | -              | 0.00296         |

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T A B L E 5-7

Variation of the Maximum Deflection Coefficient  $\gamma$  with  
 the Number of Terms in the Lateral Deflection Function  
 $w_1$  for Skewed Plates with Built-in Edges

$R = 20$                        $w_{\max} = \gamma \frac{qb^4}{D} (10^{-2})$

| Skew Angle | Number of Terms in Lateral Deflection Function $w_1$ |          |         |          |          |         |         | Morley Ref. (3) | Kennedy Ref. (4) |
|------------|--|----------|---------|----------|----------|---------|---------|-----------------|------------------|
|            | 3  | 4        | 6       | 8        | 15       | 24      | 25      |                 |                  |
| 15°        | 0.2107   | 0.2274   | 0.2259  | 0.2310   | 0.2344   | 0.2344  | 0.2339  | 0.222           | 0.211            |
| 30°        | 0.1260   | 0.1567   | 0.1506  | 0.1490   | 0.1659   | 0.1751  | 0.1535  | 0.145           | 0.147            |
| 45°        | 0.04863  | 0.07532  | 0.06873 | 0.06050  | 0.04009  | 0.0572  | 0.0684  | 0.0652          | 0.0556           |
| 60°        | 0.01075  | 0.02015  | 0.01726 | 0.01347  | 0.01366  | 0.01556 | 0.0171  | -               | 0.0162           |
| 75°        | 0.000718   | 0.001501 | 0.00123 | 0.000902 | 0.000976 | 0.00117 | 0.00123 | -               | 0.00114          |

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5-4), a two term solution no longer gives an exact answer, but there is almost no appreciable change in result when more than 15 parameters in the lateral displacement function are used. Thus, it can be seen that the rate of convergence decreases as the foundation modulus is included.

In the case of skewed plates, it can be seen from the results of Tables 5-5 to 5-7 that there is again no monotonic convergence, but instead, an oscillating characteristic exists with no guarantee of decrease in amplitudes when the number of terms in the assumed displacement function increases. Here the skewed angles play an important part in the rate of convergence which decreases as the skewed angle increases. It shows that in general six term solutions are rather close to existing results. Here the rate of convergence decreases as the plate aspect ratio increases. This rather irregular behaviour of the rate of convergence of skewed plates may be due to the less refined approximation of the assumed lateral deflection function of the skewed plate as compared with that of the elliptical or the rectangular plate.

## 5.2 Nonlinear (Large) Deflection Problem

The patterns of convergence for the large deflection problem from succeeding approximations are investigated for rectangular, circular and elliptical plates. While only nine parameters are assumed for the in-plane displacement and large deflection

functions for the rectangular plate, fifteen parameters are assumed in corresponding functions on the circular and elliptical plates. The rate and pattern of convergence can best be revealed basically in the linear and nonlinear deflection coefficients  $\gamma_1$  and  $\gamma_2$  of Equation (4-1), tabulated in Tables 5-8, 5-9 for rectangular and elliptical plates respectively. Corresponding results for the large deflection problem obtained from succeeding approximations are shown in Figures 5-1 to 5-6. To exhibit the relative pattern of convergence of the linear and nonlinear analysis, results based on the small deflection theory are shown plotted in the aforementioned figures. From these figures it is quite obvious that, in comparison with the linear results, the rate of convergence of the large deflection results is appreciably slower. For example, in the case of the rectangular plate (  $R = 1.5$  ) Figure 5-2 the deviation between a fourth term solution and a ninth term solution is only 4.2% whereas the corresponding deviation of the large deflection (  $\frac{W_{MAX}}{h} = 1.5$  ) solution is 20%. It is perhaps interesting to note that, again due to the more exact nature of the assumed displacement functions for the elliptical plate, the rate of convergence is more rapid than in the case of rectangular plates. This is well illustrated by the closeness of the deflection curves obtained by taking 9 to 15 parameters in the displacement functions. However, in contrast with the linear

T A B L E 5-8

Linear and Nonlinear Deflection Coefficients

$\gamma_1$  and  $\gamma_3$  for Rectangular Plates

| Aspect<br>Ratio<br>R | Number of Terms in Displacement Functions |            |            |            |            |            |
|----------------------|---|------------|------------|------------|------------|------------|
|                      | 3   |            | 4          |            | 6          |            |
|                      | $\gamma_1$                                | $\gamma_3$ | $\gamma_1$ | $\gamma_3$ | $\gamma_1$ | $\gamma_3$ |
| 1.00                 | 54.4000                                   | -40.5484   | 51.4043    | 23.0277    | 50.3815    | 23.9402    |
| 1.25                 | 92.2673                                   | -68.2327   | 87.3960    | 40.0311    | 85.5595    | 41.3257    |
| 1.50                 | 158.4062                                  | -114.3193  | 150.8884   | 39.1580    | 147.5396   | 74.6610    |
| 1.75                 | 265.3092                                  | -184.0947  | 254.2148   | 85.2396    | 248.7130   | 133.3343   |
| 2.00                 | 426.6182                                  | -283.4899  | 412.1755   | -67.4299   | 404.1704   | 227.8654   |

| Aspect<br>Ratio<br>R | Number of Terms in Displacement Functions |            |            |            |
|----------------------|---|------------|------------|------------|
|                      | 8   |            | 9          |            |
|                      | $\gamma_1$                                | $\gamma_3$ | $\gamma_1$ | $\gamma_3$ |
| 1.00                 | 49.5086                                   | 15.8592    | 49.5267    | 14.5491    |
| 1.25                 | 84.0426                                   | 28.7783    | 85.6837    | 30.4437    |
| 1.50                 | 144.9247                                  | 68.9977    | 144.2164   | 69.1741    |
| 1.75                 | 244.6560                                  | 146.6972   | 243.5170   | 133.3622   |
| 2.00                 | 398.6192                                  | 266.2737   | 397.1186   | 229.0607   |

T A B L E 5-9

Linear and Nonlinear Deflection Coefficients

$\gamma_1$  and  $\gamma_3$  for Elliptical Plates

| Aspect Ratio<br>R | Number of Terms in Displacement Functions |            |            |            |            |            |
|-------------------|---|------------|------------|------------|------------|------------|
|                   | 3   |            | 5          |            | 6          |            |
|                   | $\gamma_1$                                | $\gamma_3$ | $\gamma_1$ | $\gamma_3$ | $\gamma_1$ | $\gamma_3$ |
| 1.00              | 64.0000                                   | -68.8711   | 64.0000    | 31.4054    | 64.0000    | 44.9511    |
| 1.25              | 107.5938                                  | -109.2370  | 107.5938   | 80.7122    | 107.5938   | 74.7713    |
| 1.50              | 181.5000                                  | -175.2995  | 181.5000   | 115.5421   | 181.5000   | 138.0782   |
| 1.75              | 298.0938                                  | -272.8618  | 298.0938   | 186.4661   | 298.0938   | 224.8834   |
| 2.00              | 472.0000                                  | -408.4192  | 472.0000   | 289.7605   | 472.0000   | 351.0374   |

| Aspect Ratio<br>R | Number of Terms in Displacement Functions |            |            |            |            |            |
|-------------------|---|------------|------------|------------|------------|------------|
|                   | 8   |            | 12         |            | 15         |            |
|                   | $\gamma_1$                                | $\gamma_3$ | $\gamma_1$ | $\gamma_3$ | $\gamma_1$ | $\gamma_3$ |
| 1.00              | 64.0000                                   | 36.4983    | 64.0000    | 34.3123    | 64.0000    | 34.9444    |
| 1.25              | 107.5938                                  | 63.6072    | 107.5938   | 65.7409    | 107.5938   | 66.6825    |
| 1.50              | 181.5000                                  | 115.8875   | 181.5000   | 116.6904   | 181.5000   | 117.9107   |
| 1.75              | 298.0938                                  | 193.5308   | 298.0938   | 195.5834   | 298.0938   | 197.0164   |
| 2.00              | 472.0000                                  | 310.1643   | 472.0000   | 313.1533   | 472.0000   | 314.7104   |

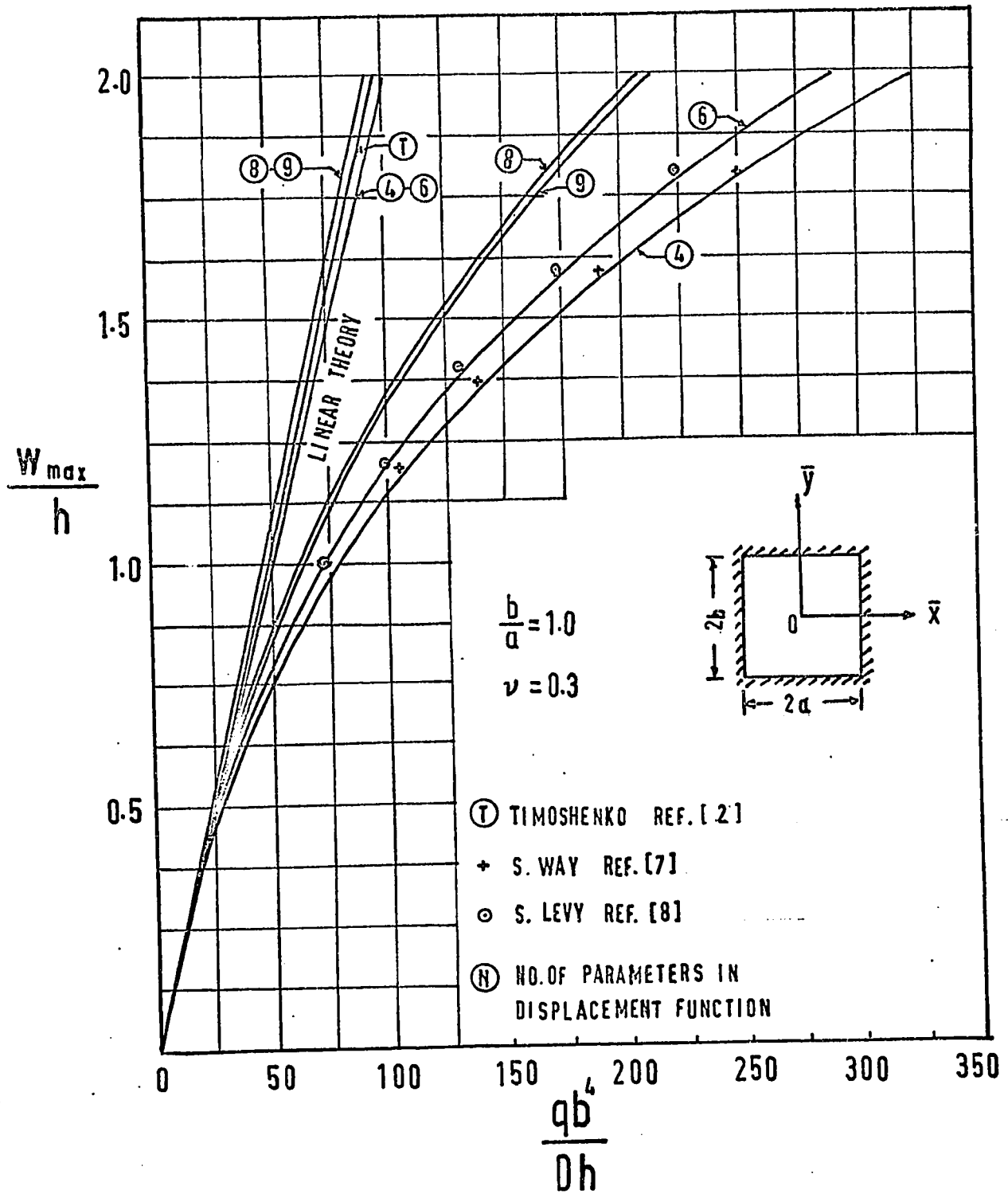


Figure 5-1 Variation of Centre Deflection with Lateral Pressure and Number of Parameters for Square Plate with Built-in Edges

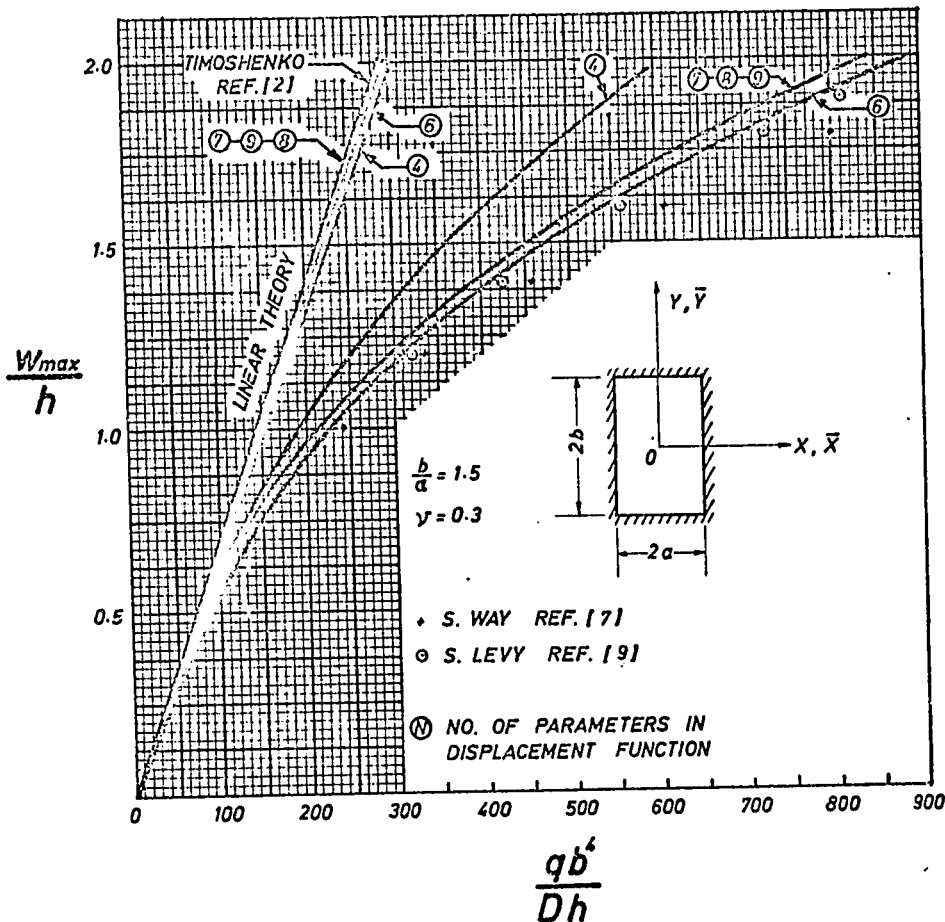


Figure 5-2

Variation of Centre Deflection with Lateral Pressure and Number of Parameters for Rectangular Plate of Aspect Ratio 1.5 with Built-in Edges.

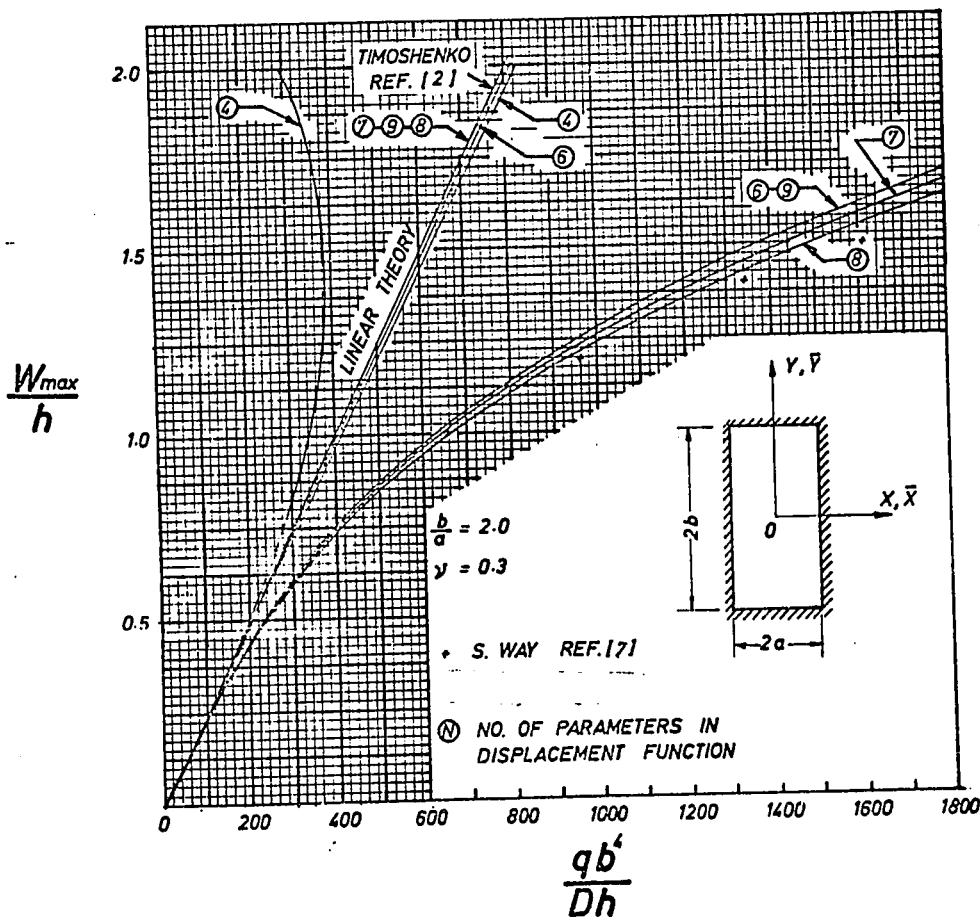


Figure 5-3

Variation of Centre Deflection with Lateral Pressure and Number of Parameters for Rectangular Plate of Aspect Ratio 2 with Built-in Edges.

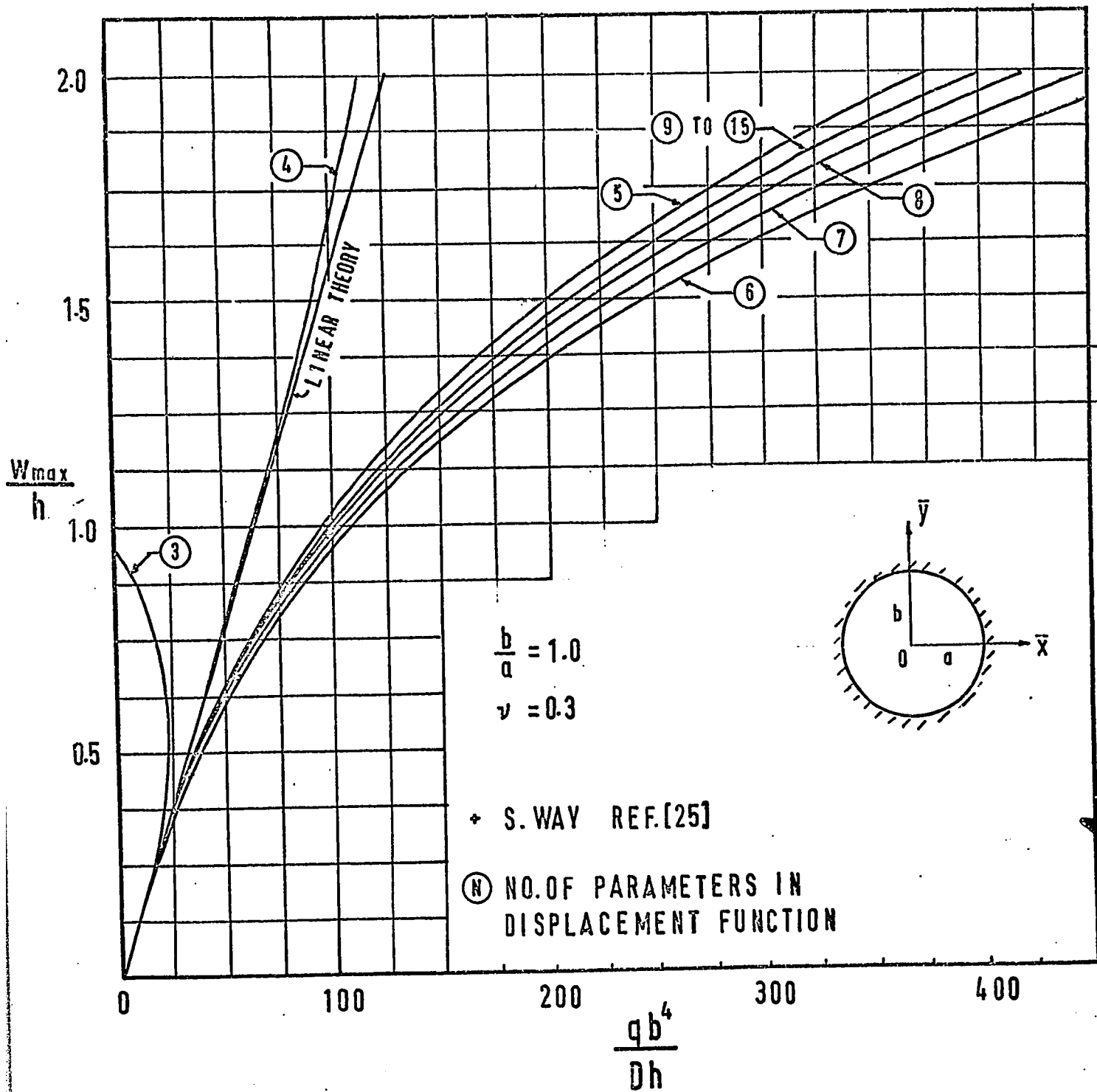


Figure 5-4 Variation of Centre Deflection with Lateral Pressure and Number of Parameters for Circular Plate with Built-in Edges

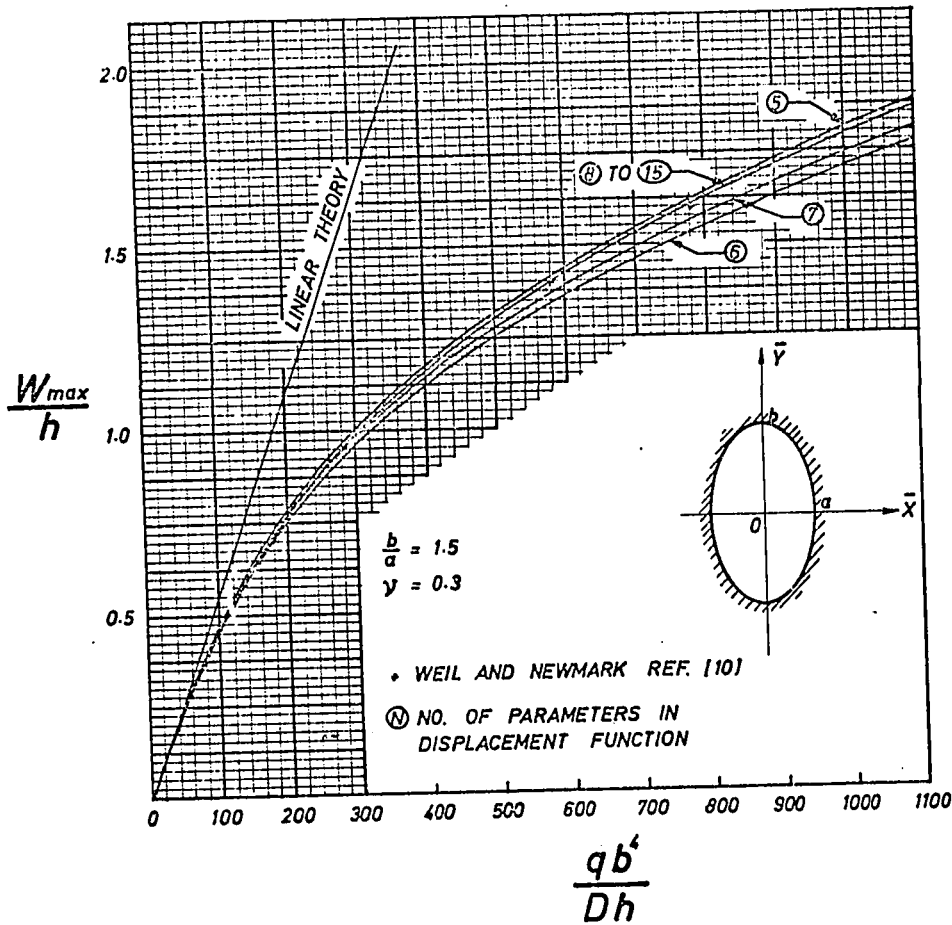


Figure 5-5

Variation of Centre Deflection with Lateral Pressure and Number of Parameters for Elliptical Plate of Aspect Ratio 1.5 with Built-in Edges

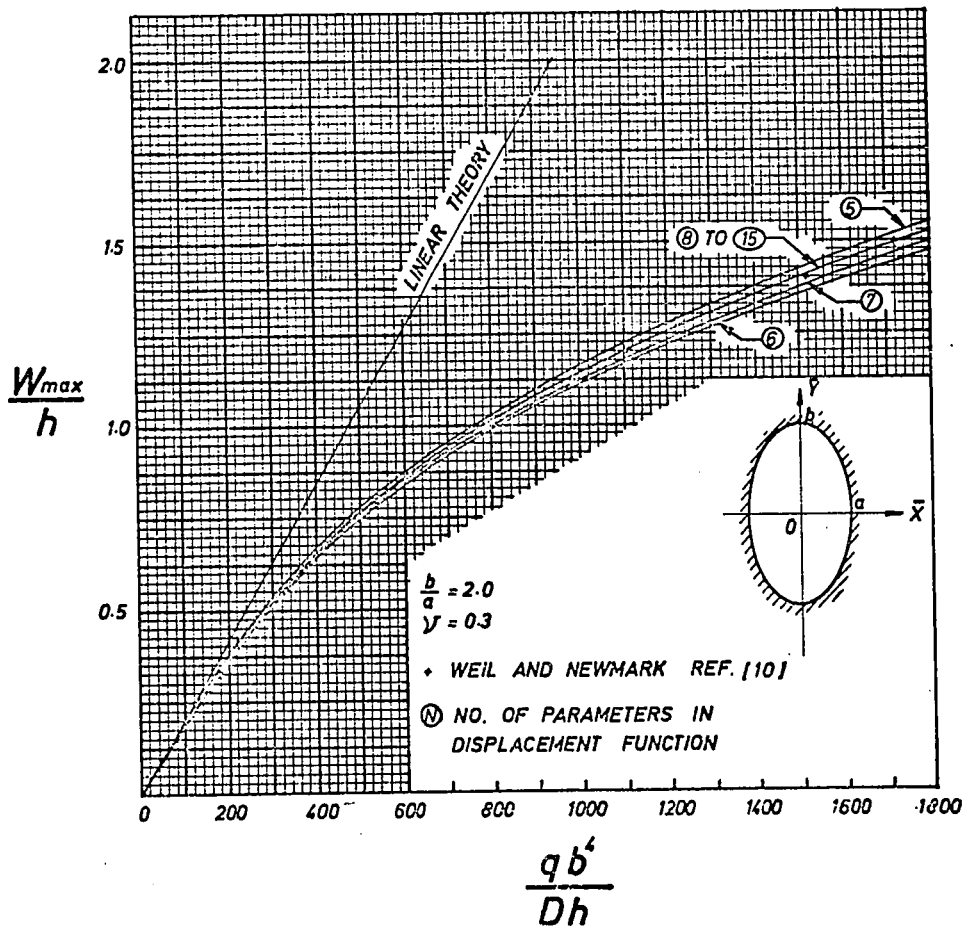


Figure 5-6

Variation of Centre Deflection with Lateral Pressure and Number of Parameters for Elliptical Plate of Aspect Ratio 2 with Built-in Edges.

deflection results, the rate of convergence appears to increase with the plate aspect ratio,  $R$ . For the sake of comparison, existing results for the large deflection problem of clamped circular, elliptical, square and rectangular plates obtained by other investigators are plotted, thus showing the range of variation of the successive approximation results with available data.

#### Summary of Results

From the extensive study of the method, it can be seen that the rate of convergence depends mainly on the closeness of the assumed displacement function to the actual boundary of the plate domain and is more rapid in small deflection than in large deflection. However, the rate of convergence decreases with additional parameters in the plate equation and is not directly proportional to the number of undetermined parameters employed in the assumed displacement function. For the various types of plates investigated, it is found that assumed displacement functions with six undetermined parameters yield results with discrepancy at most up to 7% for skewed plates but less than 4% for other types of plates as compared with available data from other investigators. With the exception of skewed plates, this discrepancy decreases as the number of undetermined parameters employed in the assumed displacement functions increase.

## CHAPTER VI

### INFLUENCE OF ELASTIC LATERAL SUPPORT ON THE LINEAR AND NONLINEAR ANALYSES OF CLAMPED PLATES

The successive approximation fo the small perturbation method is used here to investigate the effect of the foundation modulus on: (a) the linear (small deflection) behaviour of clamped circular, elliptical, rectangular and skewed plates, (b) the nonlinear (large deflection) behaviour of clamped elliptical plates. Results are obtained by employing up to eight parameters in the assumed displacement functions and such results, wherever possible are compared with available data in the technical literature.

#### 6.1 Linear Analysis ---- Deflection and Moments

For circular plates, coefficients of maximum centre deflection, centre and edge moments are tabulated in Tables 6-1 and 6-2. To show the pattern of convergence of the perturbation method, results from three, five, six and eight parameters in the deflection function, together with the corresponding results from the variational method of Galerkin, are tubulated in the aforementioned Tables.

For elliptical and rectangular plates, the variation of the centre deflection, the centre and maximum edge moments with defferent foundation moduli are plotted for various plate aspect

ratios in Figures 6-1, 6-2, 6-6 and 6-7. For skewed plates, the centre deflection, the centre and maximum edge moments are plotted against the various skewed angles for different aspect ratios in Figures 6-3 to 6-5 and 6-8 to 6-10. In these graphs, results are obtained by employing six parameters in the deflection functions and whenever possible are compared with available data.

#### Discussion of Results

From Table 6-1 and Table 6-2, it can be seen that the influence of elastic support in reducing the magnitude of the central deflection, hence the bending moments, of the plate decreases as the foundation modulus increases. Furthermore, the results also show that the elastic support is more effective in reducing the centre moment than the maximum edge moment. This is to be expected since the elastic support reaction is proportional to the lateral deflection and hence most effective at the plate centre. For example, with the foundation modulus  $K$  changing from 0 to 200, the reduction in the centre moment is 75.7% whereas the corresponding reduction in the maximum edge moment is only 54.4%. In the Tables the results agree closely with the corresponding values obtained by the variational method of Galerkin (32).

The effect of elastic support on the centre deflection

T A B L E 6-1

Coefficient  $w$  for the Maximum Deflection  
at Centre for Circular Plates

$$w_{\max} = w \frac{qa^4}{D} (10^{-2})$$

| Dimensionless<br>Foundation<br>Modulus K | Number of Terms in Lateral Deflection Function $w_1$ |         |         |         | Galerkin<br>Ref. (32) |
|--|--|---------|---------|---------|-----------------------|
|  | 3  | 5       | 6       | 8       |                       |
| 0  | 1.56250  | 1.56250 | 1.56250 | 1.56250 | 1.5625                |
| 20                                       | 1.14447  | 1.11818 | 1.11429 | 1.11339 | 1.112                 |
| 80                                       | 0.88668  | 0.86603 | 0.86271 | 0.85881 | 0.858                 |
| 120                                      | 0.71499  | 0.70339 | 0.70137 | 0.69818 | 0.695                 |
| 160                                      | 0.59400  | 0.58974 | 0.58894 | 0.58816 | 0.581                 |
| 200                                      | 0.50500  | 0.50586 | 0.50604 | 0.50015 | 0.498                 |

T A B L E 6-2

Moment Coefficients c and m for Circular Plates

Centre Moment :  $M_{rc} = M_{tc} = cqa^2 (10^{-2})$

Edge Moments :  $M_{re} = mqa^2 (10^{-2})$

$M_{te} = \nu M_{re}$

| Dimensionless<br>Foundation<br>Modulus K | Number of Terms in Lateral Deflection Function $w_1$ |          |        |          |        |          |        |          | Galerkin<br>Ref.(32) |         |
|--|--|----------|--------|----------|--------|----------|--------|----------|----------------------|---------|
|  | 3  |          | 5      |          | 6      |          | 8      |          |                      |         |
|  | c  | m        | c      | m        | c      | m        | c      | m        | c                    | m       |
| 0  | 8.1250   | -12.5000 | 8.1250 | -12.5000 | 8.1250 | -12.5000 | 8.1250 | -12.5000 | 8.125                | -12.500 |
| 40                                       | 5.5648   | -10.5448 | 5.5007 | -9.7715  | 5.4912 | -9.7268  | 4.9525 | -9.6695  | 5.486                | -9.666  |
| 80                                       | 4.0484   | -8.8235  | 4.0203 | -8.2560  | 4.0158 | -8.2144  | 4.0123 | -8.0560  | 4.011                | -8.047  |
| 120                                      | 3.0769   | -7.6923  | 3.0918 | -7.3367  | 3.0738 | -7.2720  | 3.0757 | -7.0096  | 3.076                | -6.993  |
| 160                                      | 2.4174   | -6.8182  | 2.4214 | -6.6471  | 2.4222 | -6.6315  | 2.4323 | -6.2744  | 2.435                | -6.249  |
| 200                                      | 1.9492   | -6.1224  | 1.9473 | -6.1637  | 1.9469 | -6.1677  | 1.9653 | -5.7291  | 1.972                | -5.694  |

for elliptical, rectangular and skewed plates is shown graphically in Figures 6-1 to Figures 6-5. From these graphs it can be seen that introduction of the elastic support reduces the centre deflection of the plates. Furthermore as the aspect ratio of the plate increases the influence of the elastic support on centre deflection decreases. For instance, in the case of circular and elliptical plates, with the dimensionless foundation modulus changing from 0 to 200, there is a decrease in central deflection of 67.5% with aspect ratio equals to 1, however the decrease is only 22.6% with aspect ratio equals to 2. In the case of square and rectangular plates with aspect ratios of 1 and 2, and foundation modulus changing from 0 to 200, the corresponding reduction in central deflection are 74% and 23% respectively. In the case of skewed plates, it can also be observed that, in addition to the aspect ratio, the increase in skewed angle has a rather significant part in the reduction of the effectiveness of the elastic support. With the skewed angle approaching  $60^\circ$ , especially for aspect ratio of 2, the influence of elastic support in reducing the centre deflection is almost insignificant. This reduction in the influence of the elastic support with the increase in skew is probably due to the increased rigidity of the obtuse corners for high angle of skew. In the graphs for skewed plates the values obtained from the variational method of Galerkin (32) are also plotted for comparison.

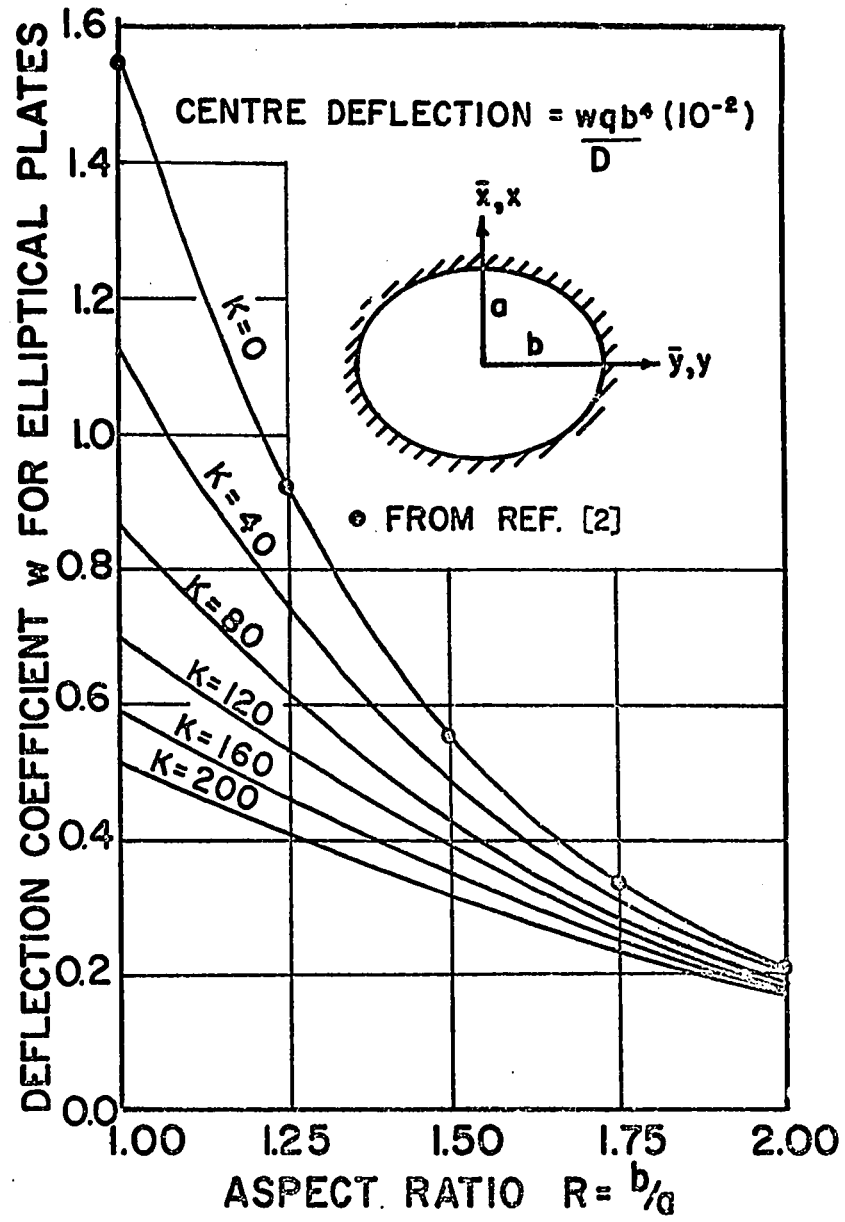


Figure 6-1 Variation of the Maximum Centre Deflection with Elastic Support and Aspect Ratio for Elliptical Plates

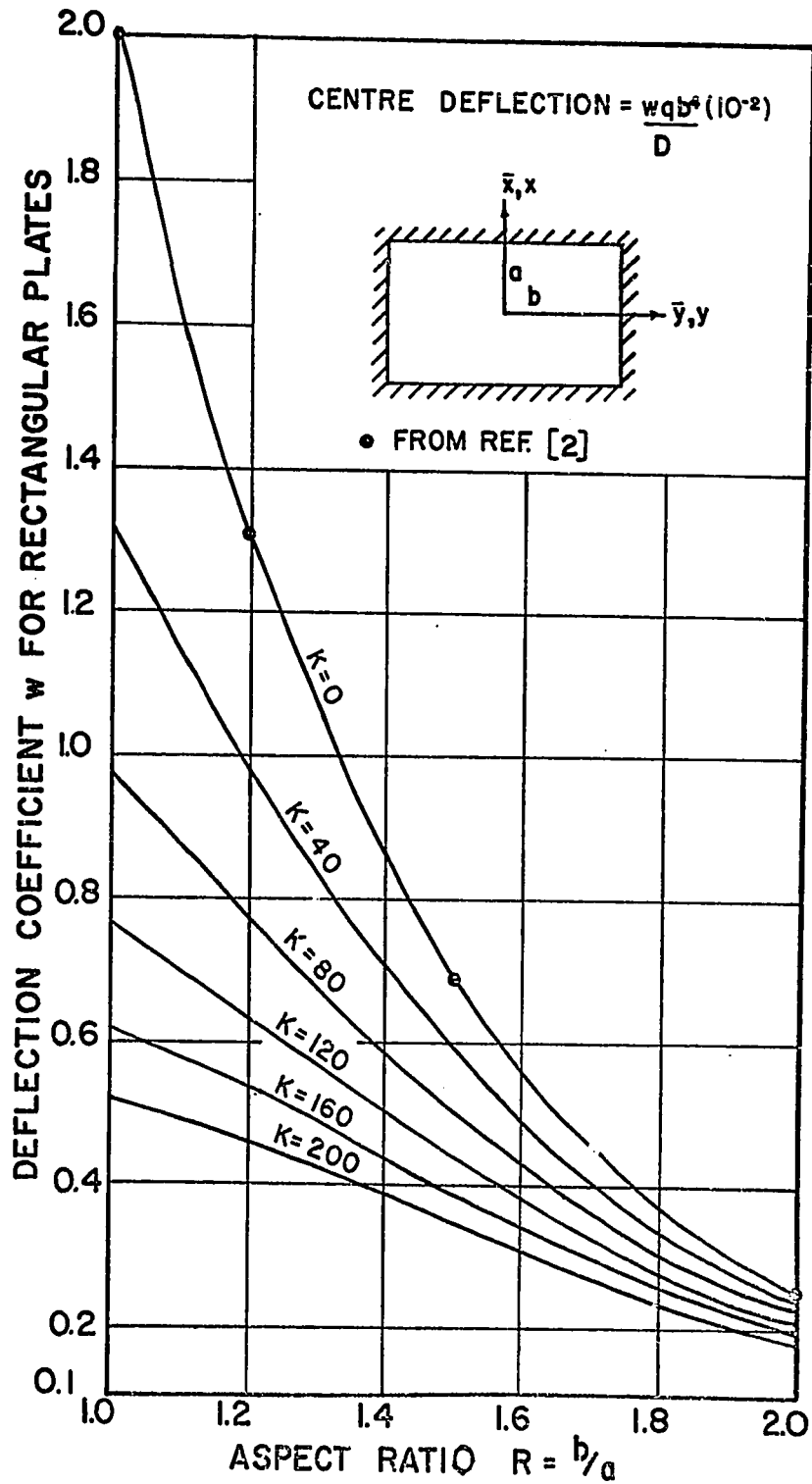


Figure 6-2 Variation of the Maximum Centre Deflection with Elastic Support and Aspect Ratio for Rectangular Plates

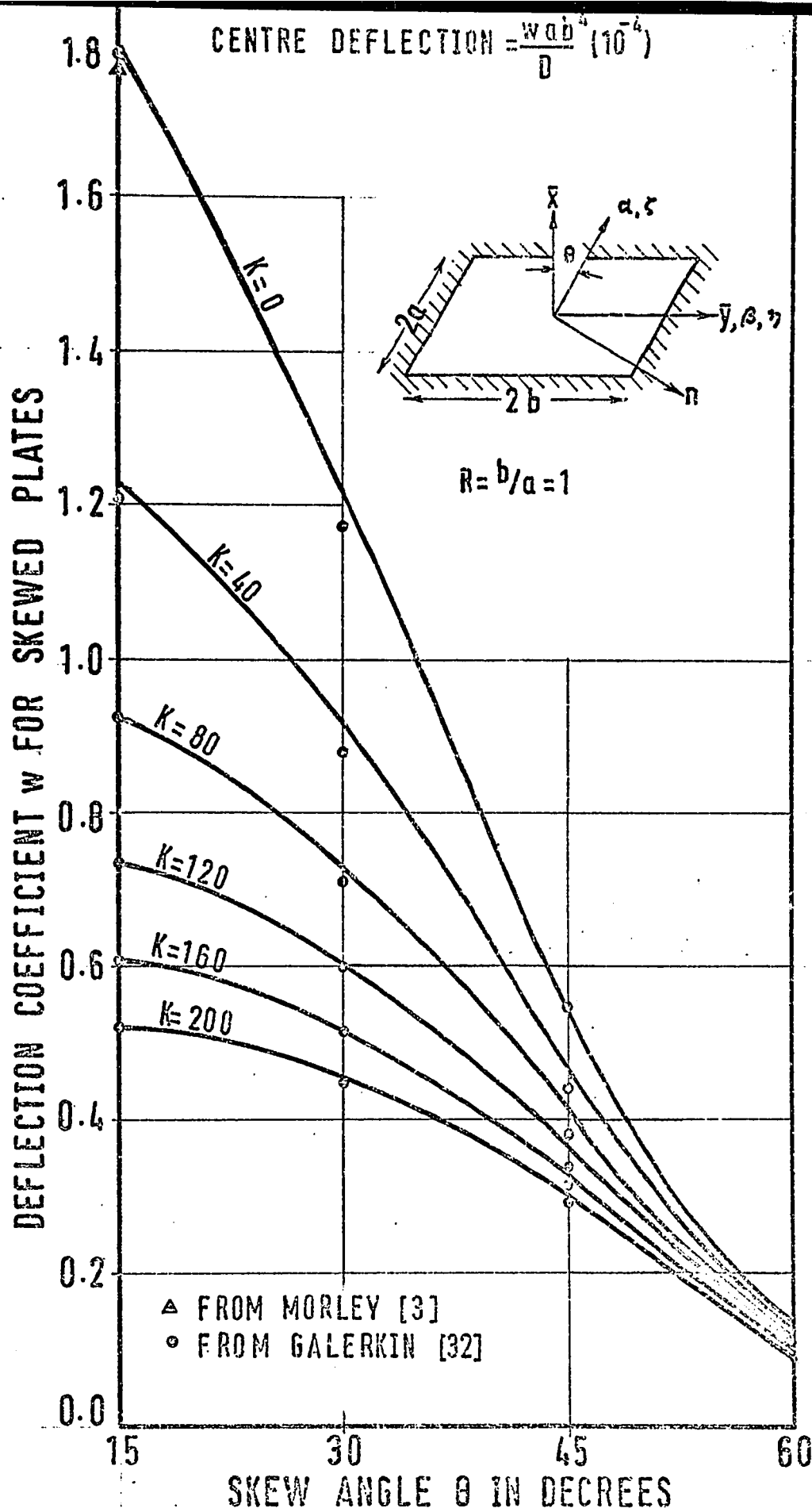


Figure 6-3 Variation of the Maximum Centre Deflection with Elastic Support and Angle of Skew for Skewed Plates,  $R = 1$

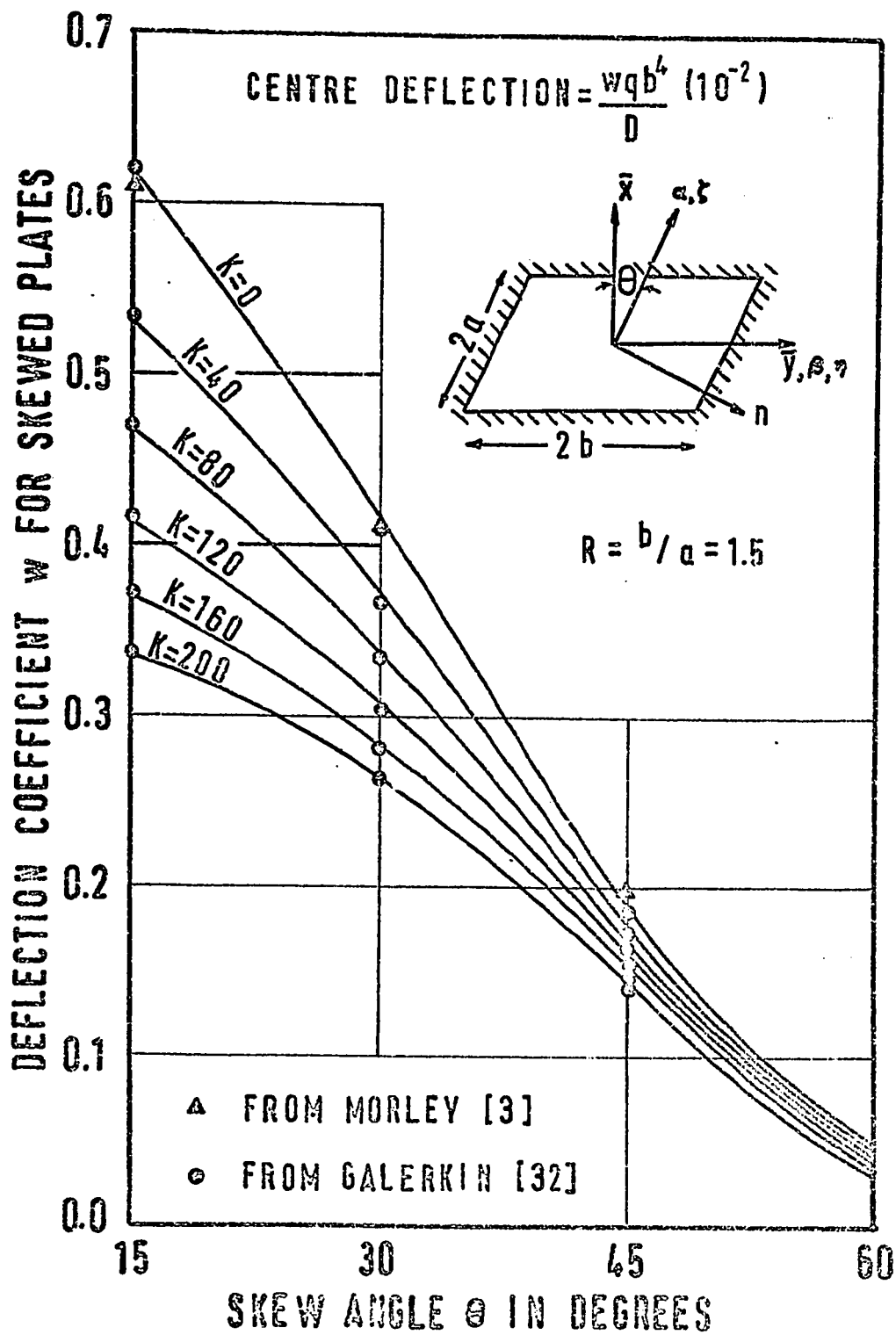


Figure 6-4 Variation of the Maximum Centre Deflection with Elastic Support and Angle of Skew for Skewed Plates,  $R = 1.5$

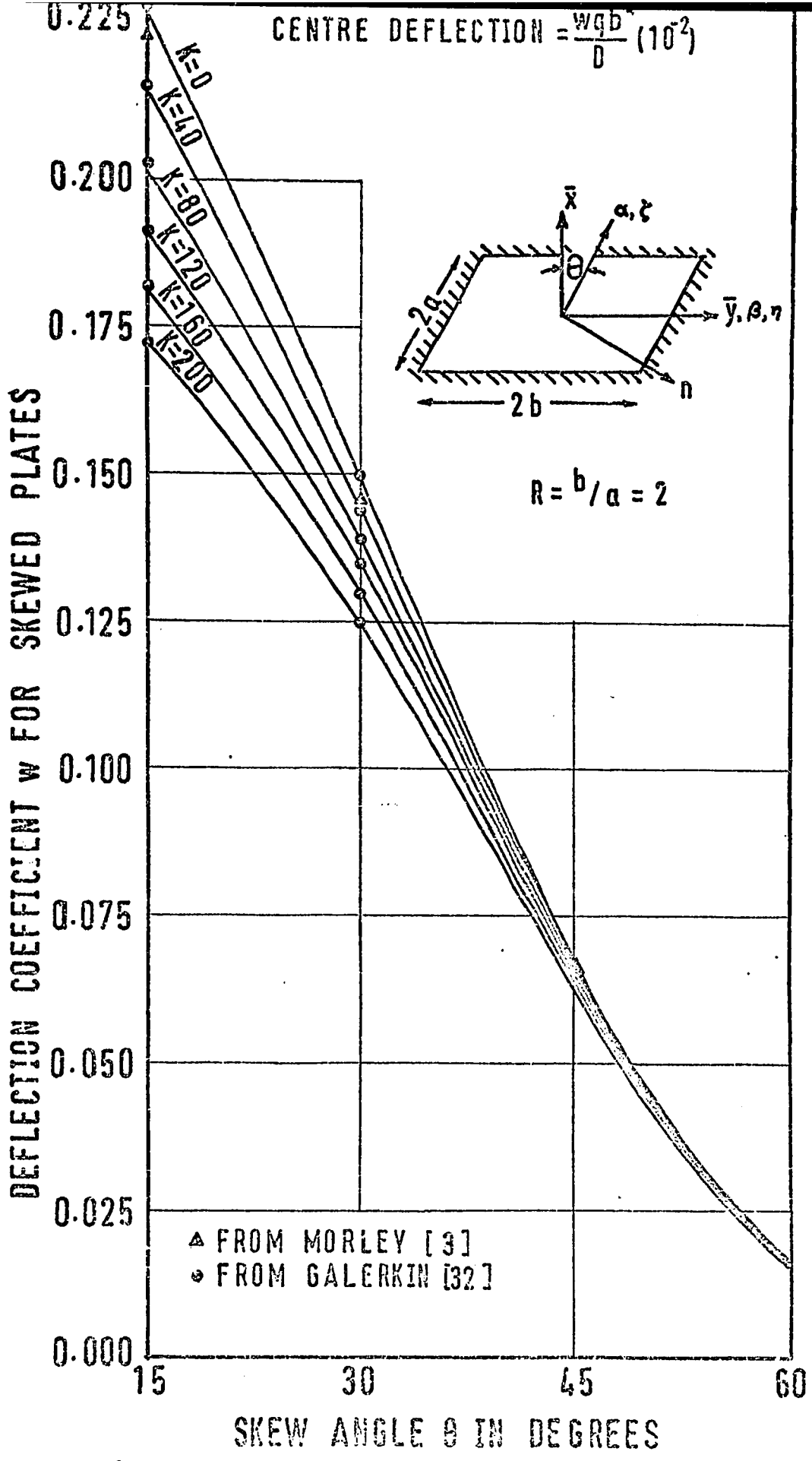


Figure 6-5 Variation of the Maximum Centre Deflection with Elastic Support and Angle of Skew for Skewed Plates,  $R = 2$

The variation of the centre and maximum edge moments with elastic support for circular, elliptical, rectangular and skewed plates are shown in Figures 6-6 through 6-10. As in the case of centre plate deflections, the effectiveness of the elastic support in reducing the centre and edge moments decreases as the aspect ratio of the plate increases. As expected, the centre and maximum edge moments decrease as the influence of elastic support increases, however, as in the case of the circular plate the influence of the elastic support on the reduction of centre moment is more pronounced. For instance, in the case of a square plate, Figure 6-7, for a change of foundation modulus  $K$  from 0 to 200, the centre moment is decreased by 83.6% whereas the corresponding decrease in the maximum edge moment is only 57.5%. In the case of skewed plates, it can be seen from Figures 6-8, 6-9 and 6-10 that the influence of elastic support on centre and maximum edge moments decreases rather rapidly with increase of skew. It is also interesting to note that when the influence of elastic support is up to a certain magnitude, the centre plate moment can actually increase with an increase in plate aspect ratio, Figures 6-6 and 6-7, for a given value of the foundation modulus. This can be due to the fact that the increase in aspect ratio reduces the lateral deflections, thus diminishing the effectiveness of the elastic support to reduce the centre plate moments.

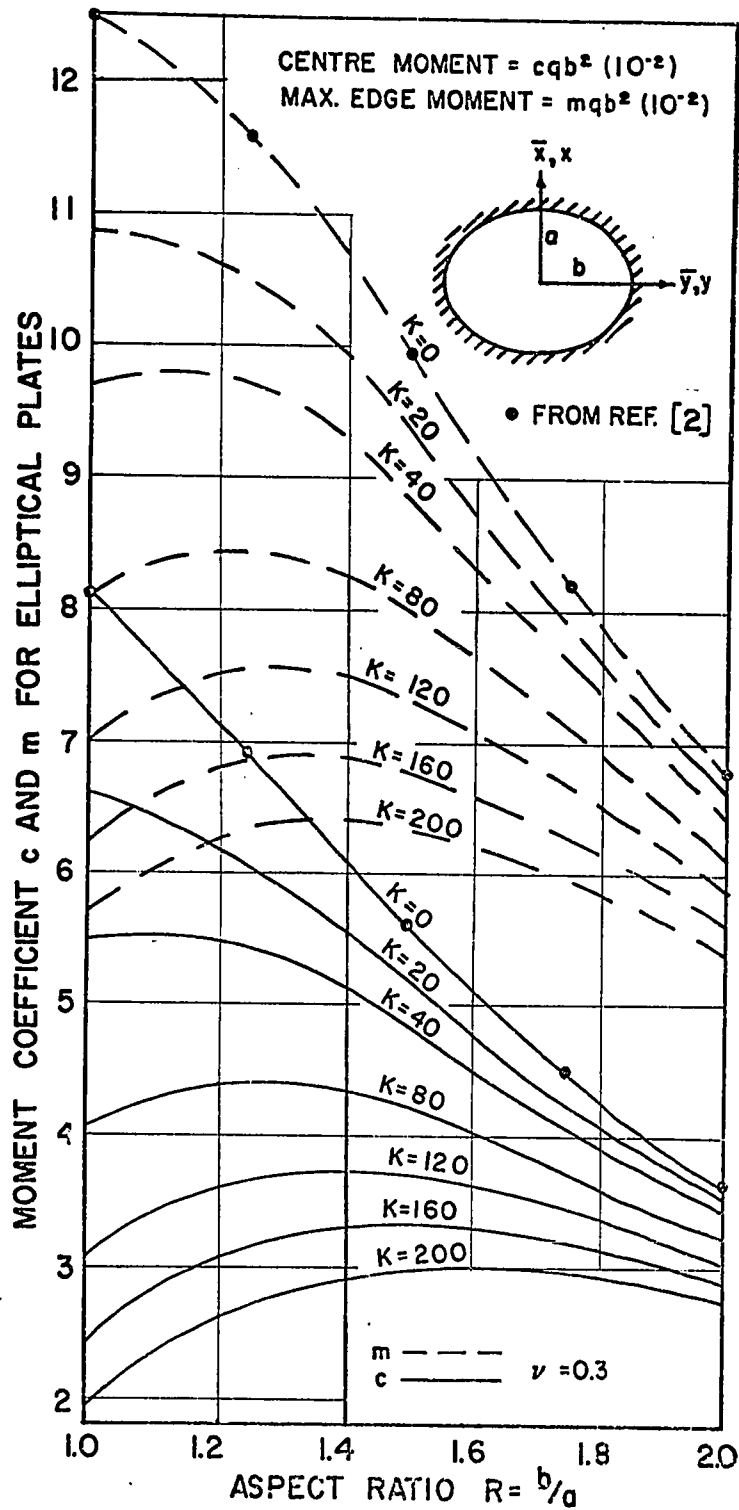


Figure 6-6 Variation of Centre and Maximum Edge Moments with Elastic Support and Aspect Ratio for Elliptical Plates

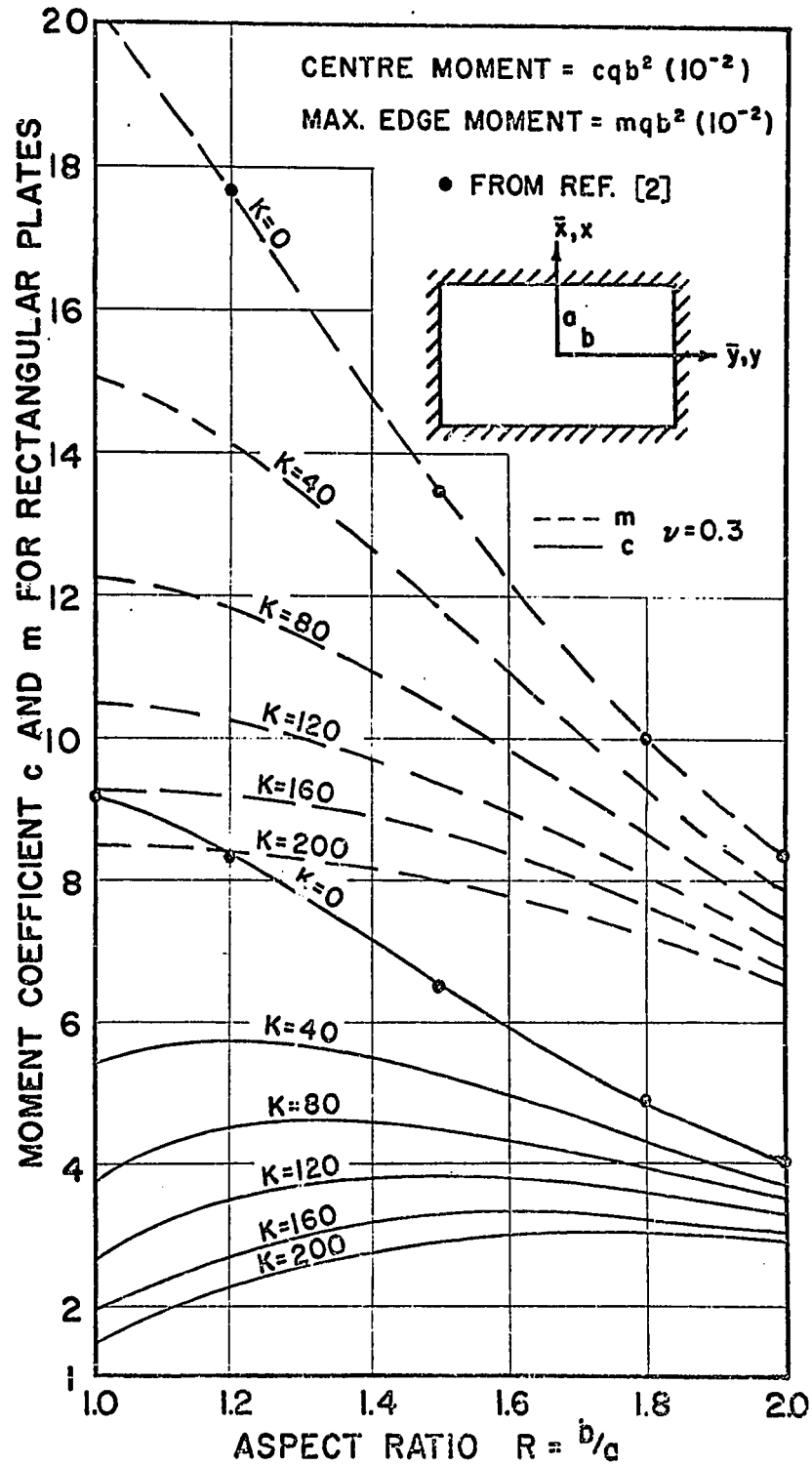


Figure 6-7 Variation of Centre and Maximum Edge Moments with Elastic Support and Aspect Ratio for Rectangular Plate

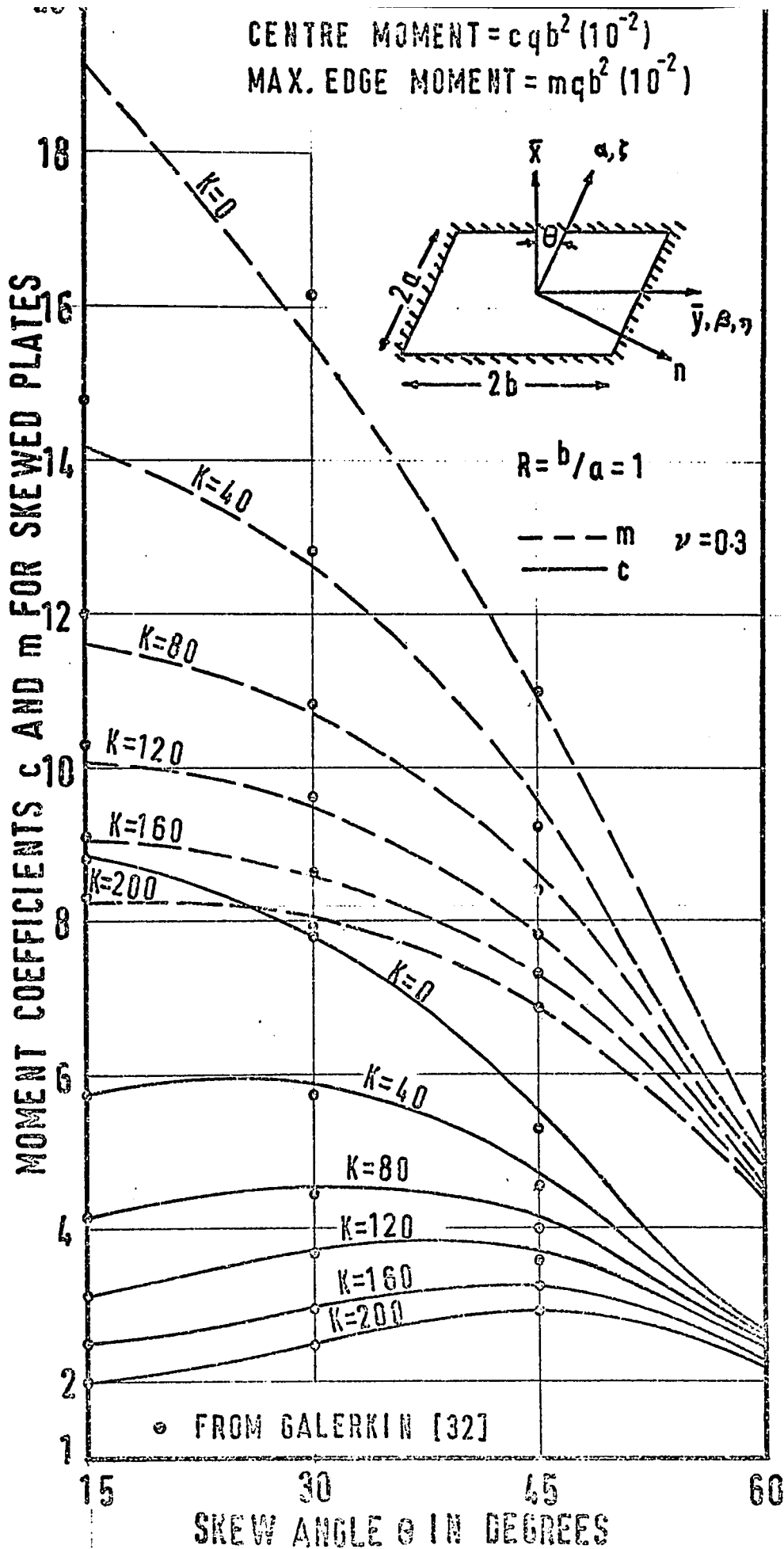


Figure 6-8 Variation of Centre and Maximum Edge Moments with Elastic Support and Angle of Skew for Skewed Plates,  $R = 1$ .

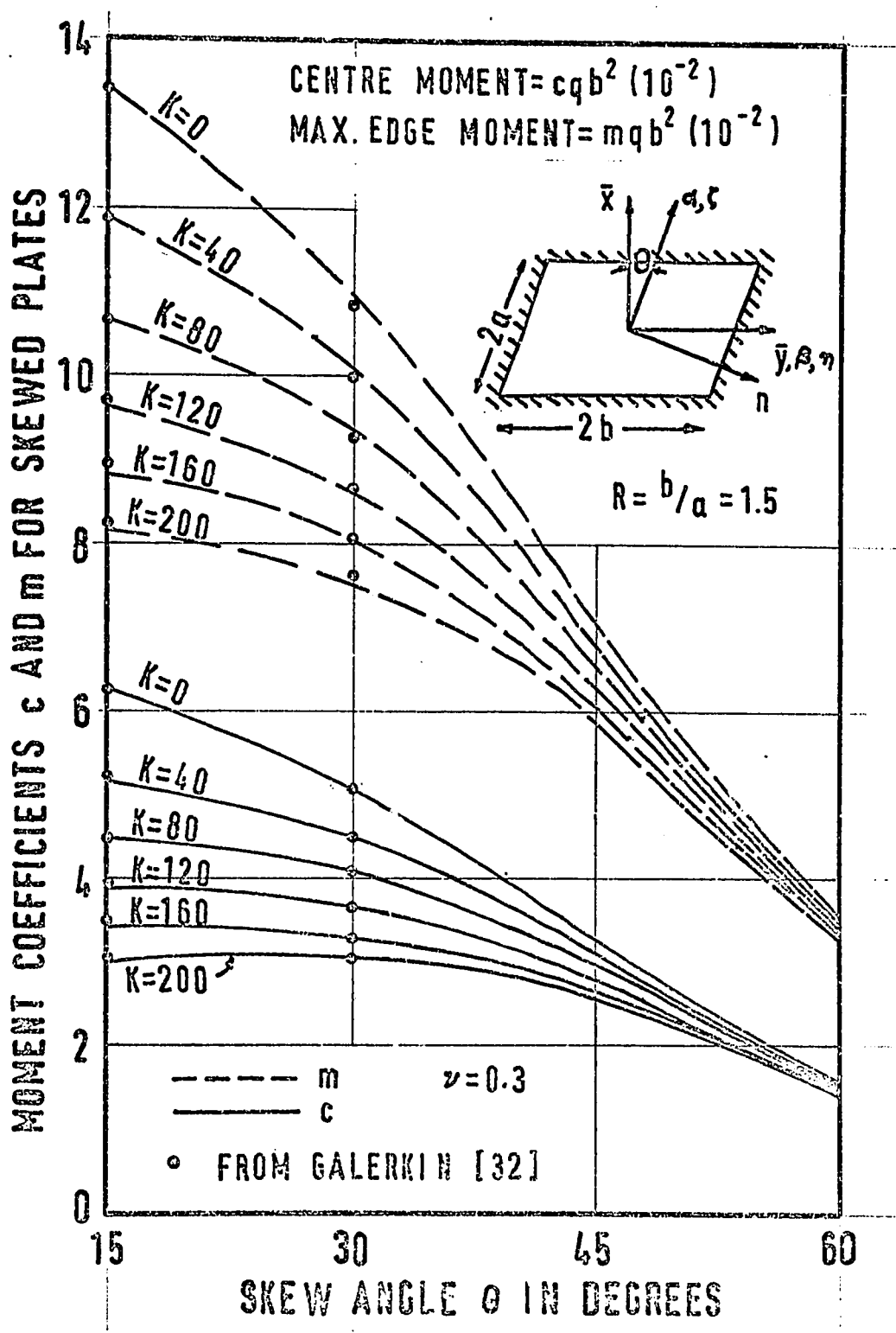


Figure 6-9 Variation of Centre and Maximum Edge Moments with Elastic Support and Angle of Skew for Skewed Plates,  $R = 1.5$

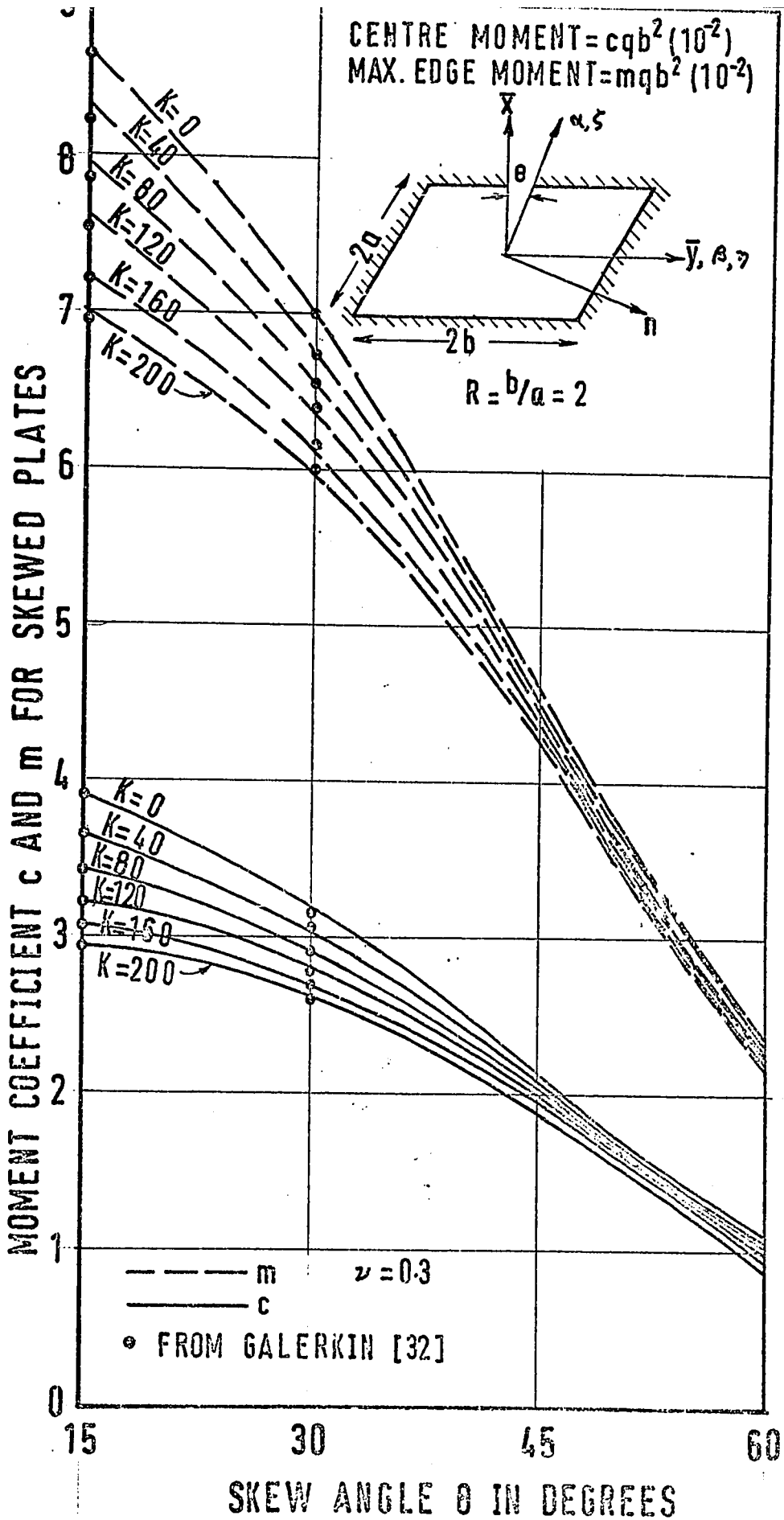


Figure 6-10 Variation of Centre and Maximum Edge Moments with Elastic Support and Angle of Skew for Skewed Plates,  $R = 2$

### Comparison of Results

From the results presented for circular plates (Tables 6-1 and 6-2), it can be seen that they are in extremely good agreement with those obtained from the variational method of Galerkin (32).

For the case of elliptical plates where the assumed deflection function takes on the exact mathematical equation of an elliptical boundary, results for centre deflection and maximum edge moments are in exact agreement with those obtained by Timoshenko (2) for plates with no elastic support. Good agreements are also obtained with the results from Galerkin's method (32) when the influence of elastic support is included.

For the case of rectangular plates, discrepancies in the centre deflection and maximum edge moments do not exceed 0.8% and 1.3% respectively when compared with existing results from Timoshenko (2) and Galerkin (32).

For the case of skewed plates where no quadrant symmetry exists, deviation from existing results (3) and (32) increase although the agreement in general is still good with the maximum discrepancy in the edge moment not exceeding 5%.

### 6.2 Nonlinear Analysis --- Deflection and Total Stresses

Here the influence of elastic support on the large deflection problem of clamped elliptical plates is analysed. Results are obtained by employing eight parameters in the

assumed displacement functions. Plates with aspect ratios of 1.5 and 2.0 are considered. Results showing the variation of the dimensionless centre deflection with the dimensionless load are plotted in Figures 6-11 and 6-12 for different values of foundation modulus  $K$ . Also, the variation of the maximum total dimensionless stresses produced by the effect of both bending and stretching of the plate with the dimensionless load for various values of foundation modulus  $K$  are shown in Figures 6-13 and 6-14. In both cases, results from the linear theory and large deflection theory obtained by Weil and Newmark (10) are plotted for comparison.

#### Discussion and Comparison of Results

From the deflection curves (Figures 6-11 and 6-12), it can be seen that the centre deflection starts to deviate notably from linear theory, when the dimensionless deflection exceeds 0.25. This means that the elementary linear plate theory would not be sufficiently accurate once the ratio of the centre deflection to the thickness of the plate exceeds this value. For plates with no elastic support, the centre deflection results agree very closely with those obtained by Weil and Newmark (10) for both aspect ratios  $R = 1.5$  and  $2$ . No data in the literature have yet been available for large deflections of clamped elliptical plates with elastic support, hence comparison is not possible for deflection values when

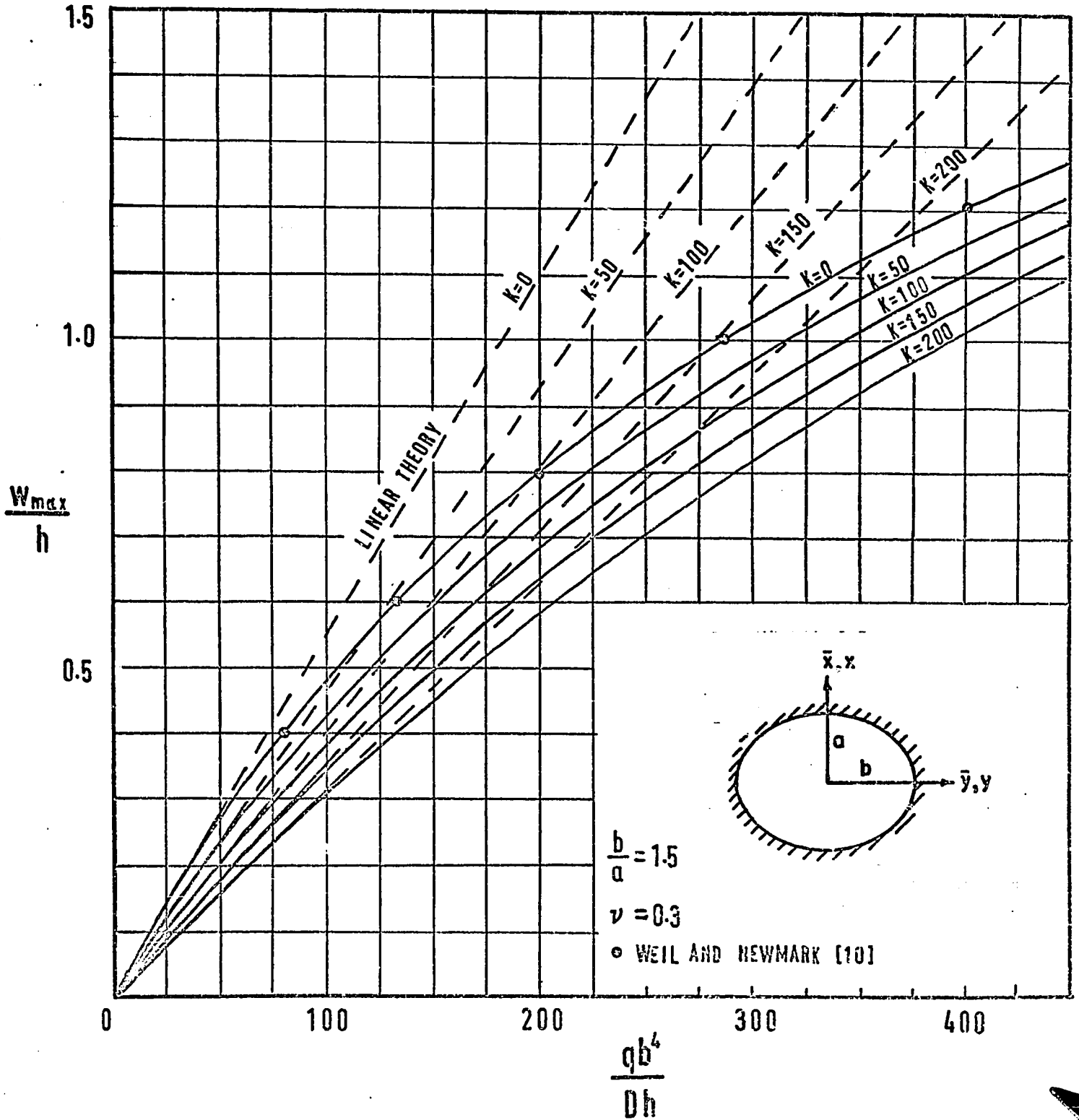


Figure 6-11 Variation of Centre Deflection with Lateral Pressure for Elastically Supported Elliptical Plates,  $R = 1.5$

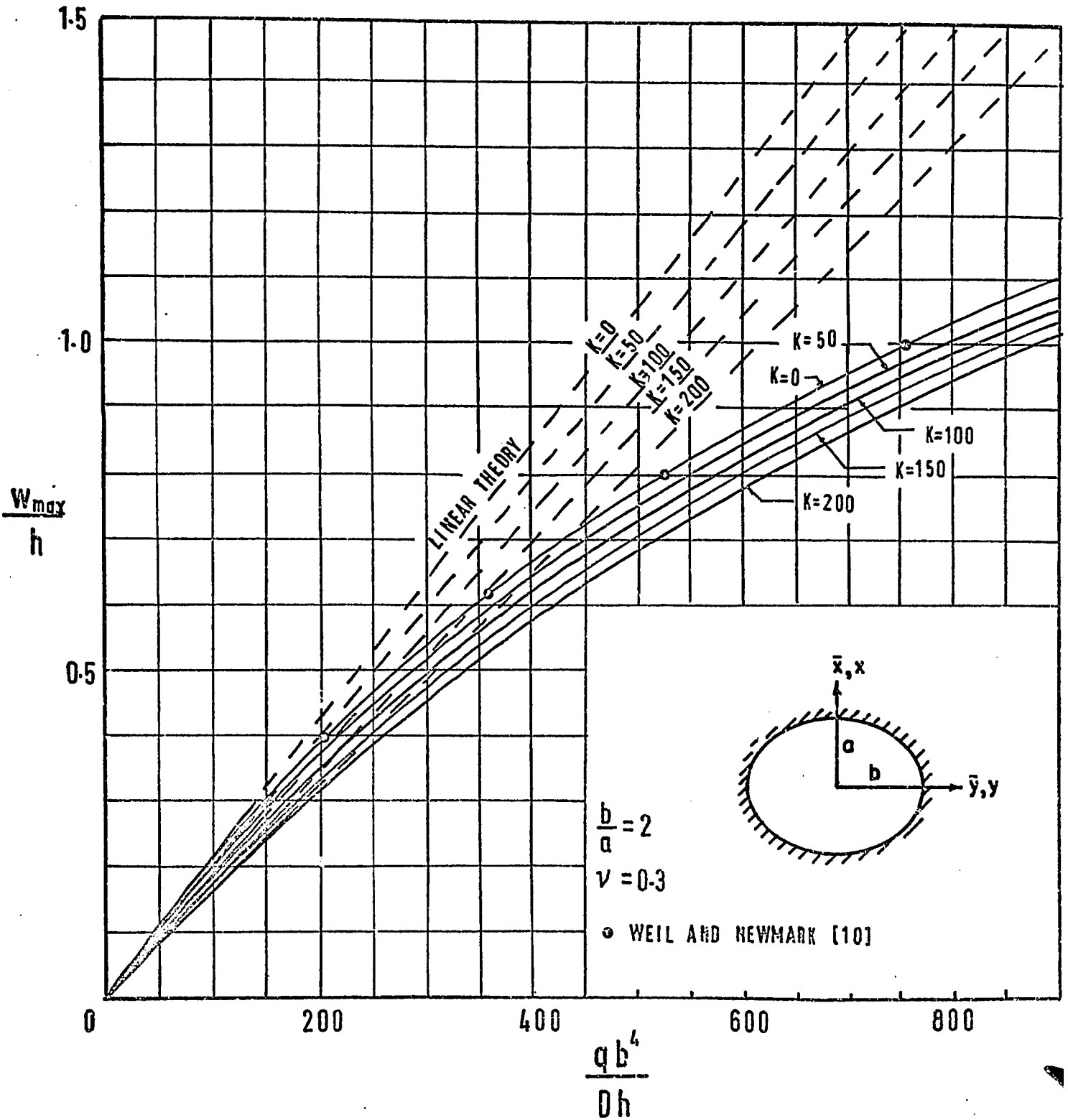


Figure 6-12 Variation of Centre Deflection with Lateral Pressure for Elastically Supported Elliptical Plates,  $R = 2$

foundation modulus  $K > 0$ .

As in the case of small deflection, the influence of the elastic support on centre deflection decreases as the aspect ratios increases. For example, with  $K$  increases from 0 to 200, at  $R = 1.5$  with dimensionless load  $Q = 250$ , there is 24.4% decrease in centre deflection; while at  $R = 2$ , with  $Q = 250$ , there is only 17.8% decrease in centre deflection.

For a given aspect ratio, the centre deflection decreases as the foundation modulus increases. This is to be expected since the effect of the elastic support is to reduce the lateral pressure on the plate. Also, as the foundation modulus increases, the influence of the nonlinear term for deflection decreases. Hence the large deflection curves tend to become relatively more linear as the values of the foundation modulus become higher, though this is also related to the aspect ratio of the plate. For instance such tendency is less pronounced with aspect ratio equals to 2. This is due to the fact, that, as mentioned earlier, the elastic support is less effective in reducing the centre deflection of plate with large aspect ratios.

Figures 6-13 and 6-14 show the relation of total maximum edge stresses,  $C_2^*$ , and centre stresses,  $C_1^*$ , with the dimensionless load,  $Q$ , for various values of the foundation modulus,  $K$ . The total linear edge and centre stresses with no elastic support on the plate ( $K = 0$ ) are plotted in to show the deviation from the

nonlinear theory. Existing data from nonlinear analysis for elliptical plates with no elastic support from Weil and Newmark (10) are also shown. The maximum deviation is less than 7%. It can be noted that since the total edge stress is much higher than the total centre stress, the former generally governs for the design considerations. One point worthy of mention is that the total stress is the sum of two stresses one due to bending and the other due to the stretching of the plate. However, the membrane stresses due to stretching of the plate contribute up to only about 10% of the total stresses. From the figures, it can be seen that the curves of the total edge stresses,  $\sigma_E$ , and centre stresses,  $\sigma_c$ , are essentially linear until they reach respectively a dimensionless magnitude of 5 and 2.5 respectively for  $R = 1.5$ ; 7 and 4.5 respectively for  $R = 2$ . At this magnitude, there corresponds a central dimensionless deflection,  $W_0$ , less than 0.5. Again as in small deflection, the influence of elastic support on the total stresses decreases as the aspect ratio increases. It can also be seen that the relative positions of  $\sigma$ -curves are closer together and tends to straighten out for higher values of foundation modulus  $K$ . This is probably due to the decreasing effect of the elastic support as the foundation modulus increases.

For the same plate aspect ratio, it is to be expected that increase in values of foundation modulus  $K$ , reduces the

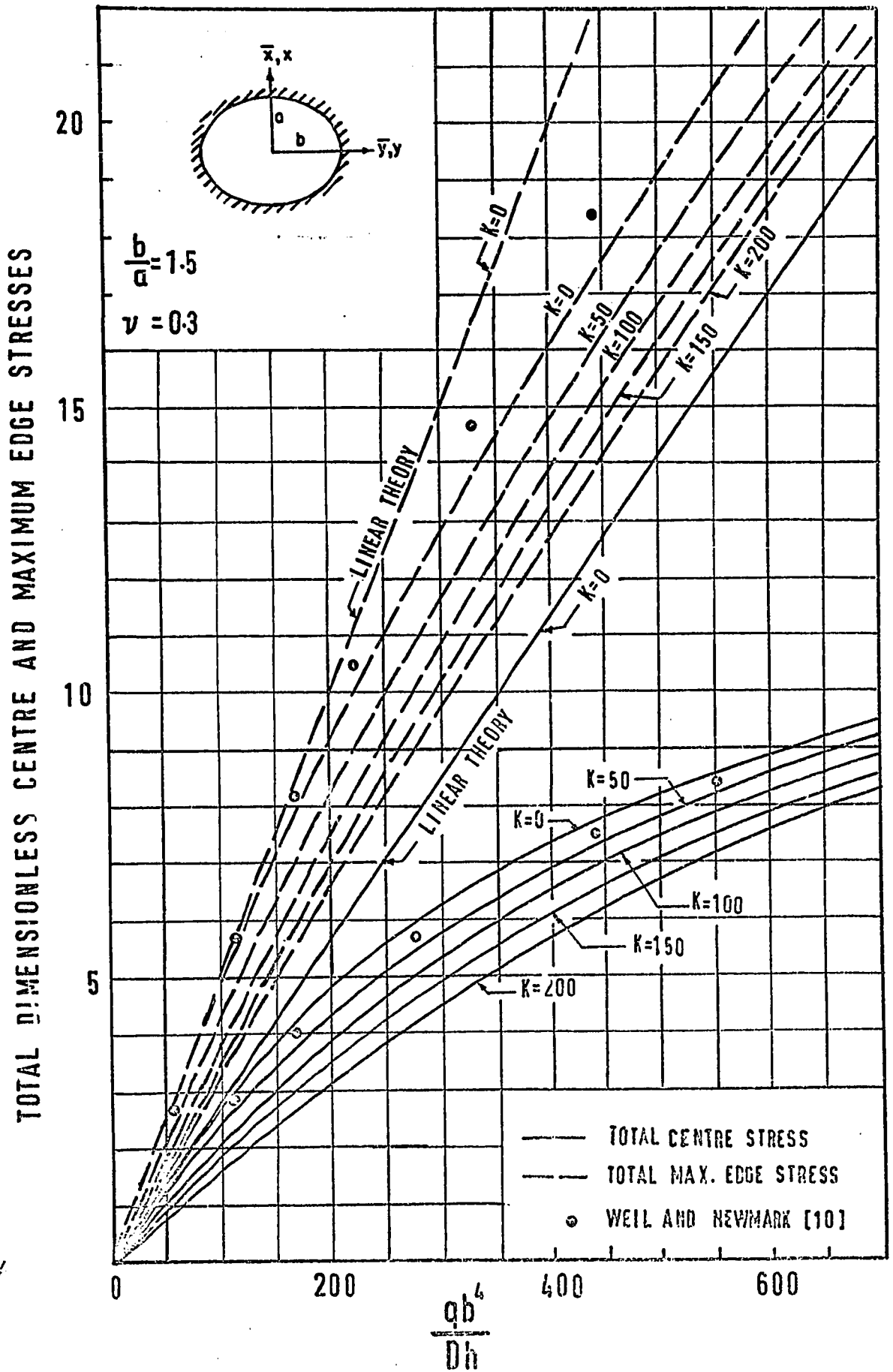


Figure 6-13 Variation of Total Centre and Maximum Edge Stresses with Lateral Pressure for Elastically Supported Elliptical Plates,  $R = 1.5$

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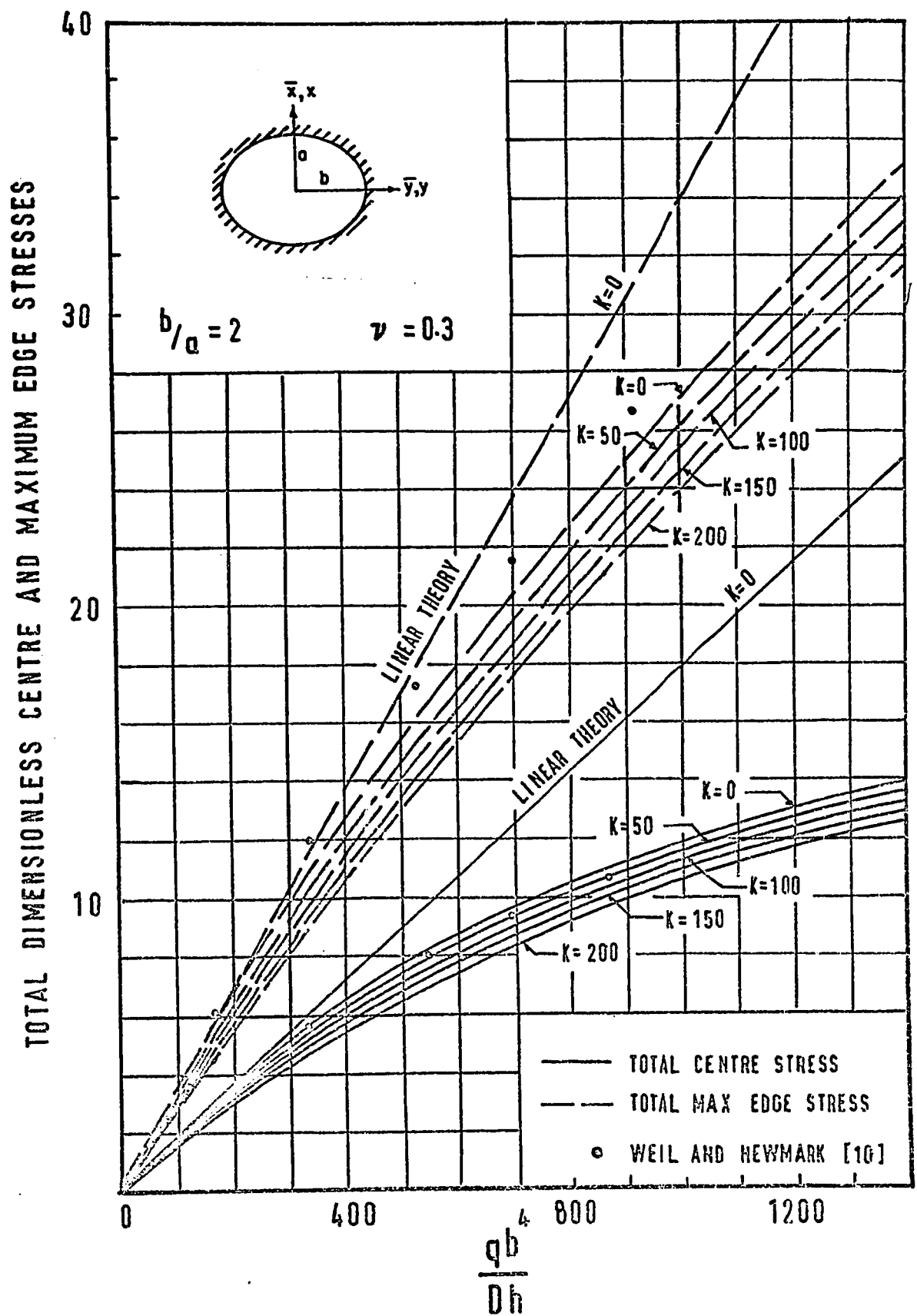


Figure 6-14 Variation of Total Centre and Maximum Edge Stresses with Lateral Pressure for Elastically Supported Elliptical Plates,  $R = 2$

magnitude of the total stresses both at the edge and in the centre. Nevertheless, it should be noted again that the elastic support is more effective in reducing the stress at the centre than it is at the edge. For instance, at  $R = 1.5$  as  $K$  increases from 0 to 200 and  $Q = 250$ , there is a 32.2% decrease of stress in the centre, while the corresponding decrease in the edge stress is only 24.8%. This can be explained by the fact that the elastic support in reducing the lateral pressure of the plate is more effective at the centre than it is at the edge.

### 6.3 Location of Maximum Moments or Stresses

From the analytical solutions, it is found that the maximum moment (or stress) for elliptical and rectangular plates occurs at the end of minor axis and at the middle of the longer edge respectively. The location of this maximum moment (or stress) appears to be invariant with the effect of the elastic support. In case of skewed plates the maximum moment also occurs along the longer edge of the plate but is invariably displaced towards to obtuse corners, the location of this maximum moment varies with the aspect ratio but appears to be relatively insensitive to changes in skew and the effect of elastic support.

## C H A P T E R VII

### CONCLUSIONS

As a result of this analysis on the rate of convergence and the influence of the elastic support on clamped plates by the successive approximations of perturbation, the following conclusions may be drawn :

1. The rate of convergence of the method of successive approximations depends largely on the assumed plate displacement function for the type of plate. Both linear and nonlinear results indicate that the method converges quite rapidly provided suitable displacement functions are chosen to fit closely the mathematical boundary of the plate domain.
2. Invariably, the rate of convergence of the method is more rapid in the case of the small deflection theory than it is with the large deflection theory.
3. Convergence of the method of successive approximation decreases appreciably when additional parameters, such as an elastic foundation modulus, are added to the plate equation.
4. For plates with no elastic support, the perturbation method yields results which closely agree with existing values by other investigators.

5. The maximum centre deflection of the plate decreases with an increase in the foundation modulus.
6. The elastic support is more effective in reducing the centre moment (or stress) than the edge moment (or stress) of the plate.
7. The influence of the elastic support decreases as with an increase in the plate aspect ratio and; in the case of skew plates, the influence of the elastic support also decreases with an increase in the angle of skew.
8. In elliptical plates, the results start to deviate from linear theory as the ratio of maximum centre deflection to the thickness of the plate ( $\frac{w_{max}}{h}$ ) approaches 0.5.
9. In the large deflection of elliptical plates, the membrane stress contributes less than 10% of the total centre or edge stresses.
10. The large deflection of elliptical plates becomes increasing linear as the foundation modulus increases.
11. The maximum resultant stress for elliptical and rectangular plates occurs at the end of the minor axis and at the centre of the longer edge respectively. In skewed plates, it occurs along the longer edge of the plate but is invariably displaced towards the obtuse corners.

A P P E N D I X   A

ASSUMED DISPLACEMENT FUNCTIONS

APPENDIX A

ASSUMED DISPLACEMENT FUNCTIONS

Rectangular Plates:

$$w_1 = (1 - x^2)^2 (1 - y^2)^2 (1 + C_1x^2 + C_2y^2 + C_3x^2y^2 + C_4x^4 + C_5y^4 + \dots + C_{22}x^6y^8 + C_{23}x^8y^6 + C_{24}x^8y^8)$$

$$s_2 = (1 - x^2) (1 - y^2)x(D_1 + D_2x^2 + D_3y^2 + D_4x^2y^2 + D_5x^4 + D_6y^4 + D_7x^2y^4 + D_8x^4y^2 + D_9x^4y^4)$$

$$t_2 = (1 - x^2) (1 - y^2)y(E_1 + E_2y^2 + E_3x^2 + E_4x^2y^2 + E_5y^4 + E_6x^4 + E_7y^2x^4 + E_8y^4x^2 + E_9y^4x^4)$$

$$w_3 = (1 - x^2)^2 (1 - y^2)^2 (F_1x^2 + F_2y^2 + F_3x^2y^2 + F_4x^4 + F_5y^4 + F_6x^2y^4 + F_7x^4y^2 + F_8x^4y^4)$$

Elliptical Plates:

$$w_1 = (1 - x^2 - y^2)^2 (1 + C_1x^2 + C_2y^2 + C_3x^4 + C_4y^4 + C_5x^2y^2 + C_6x^2y^4 + C_7x^4y^2 + \dots + C_{22}x^6y^8 + C_{23}x^8y^6 + C_{24}x^8y^8)$$

$$s_2 = (1 - x^2 - y^2)x(D_1 + D_2x^2 + D_3y^2 + D_4x^4 + D_5y^4 + D_6x^2y^2 + \dots + D_{13}x^4y^4 + D_{14}x^2y^6 + D_{15}x^6y^2)$$

$$t_2 = (1 - x^2 - y^2)y(E_1 + E_2y^2 + E_3x^2 + E_4y^4 + E_5x^4 + E_6y^2x^2 + \dots + E_{13}y^4x^4 + E_{14}y^2x^6 + E_{15}y^6x^2)$$

$$w_3 = (1 - x^2 - y^2)^2 (F_1x^2 + F_2y^2 + F_3x^4 + F_4y^4 + F_5x^2y^2 + \dots + F_{12}x^4y^4 + F_{13}x^2y^6 + F_{14}x^6y^2)$$

Circular Plates:

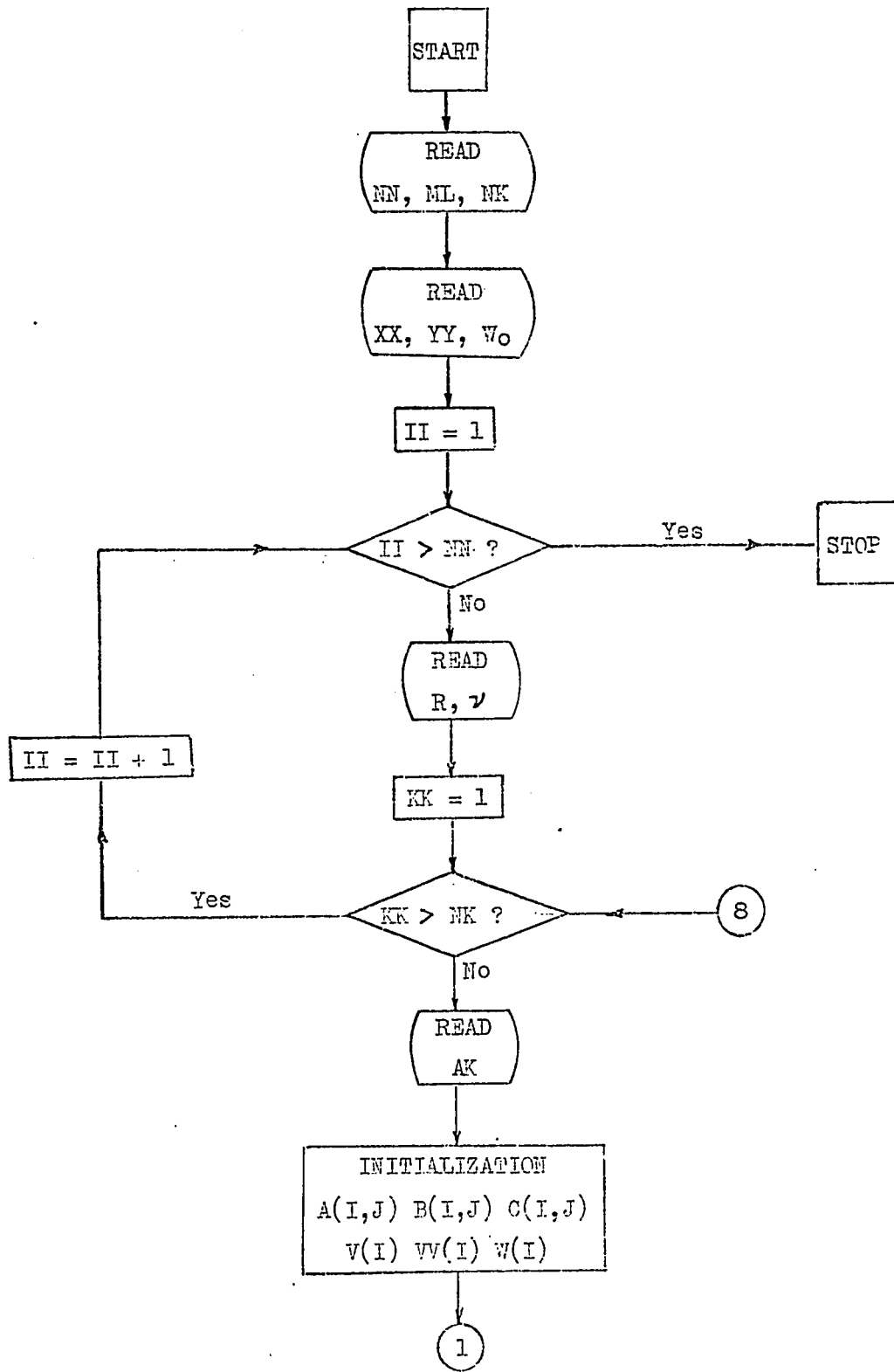
$$w_1 = \psi^2 [1 + c_1(1 - \psi) + c_2(1 - \psi)^2 + c_3(1 - \psi)^3 + c_4(1 - \psi)^4 + c_5(1 - \psi)^5 + c_6(1 - \psi)^6]$$

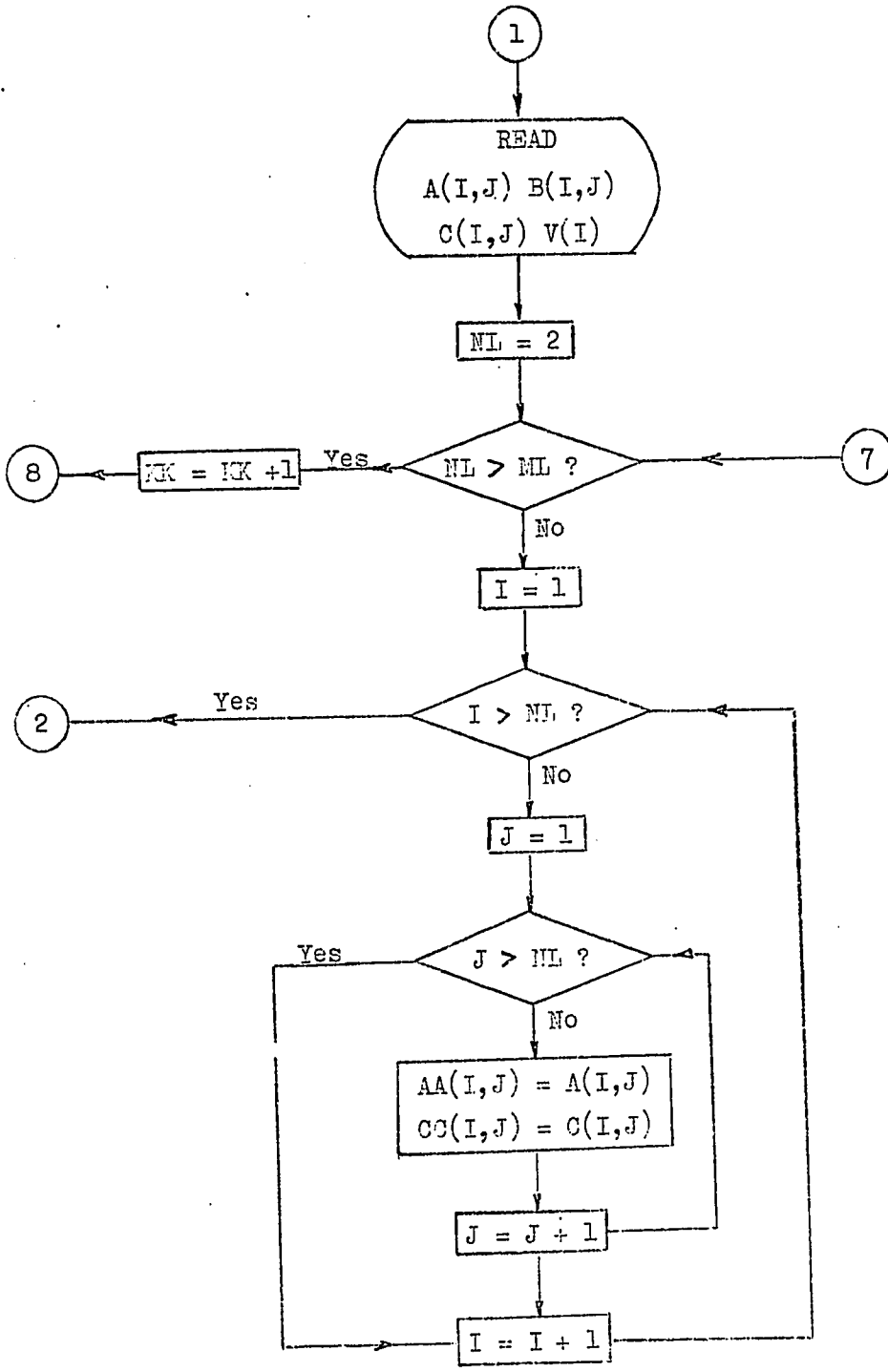
Skewed Plates:

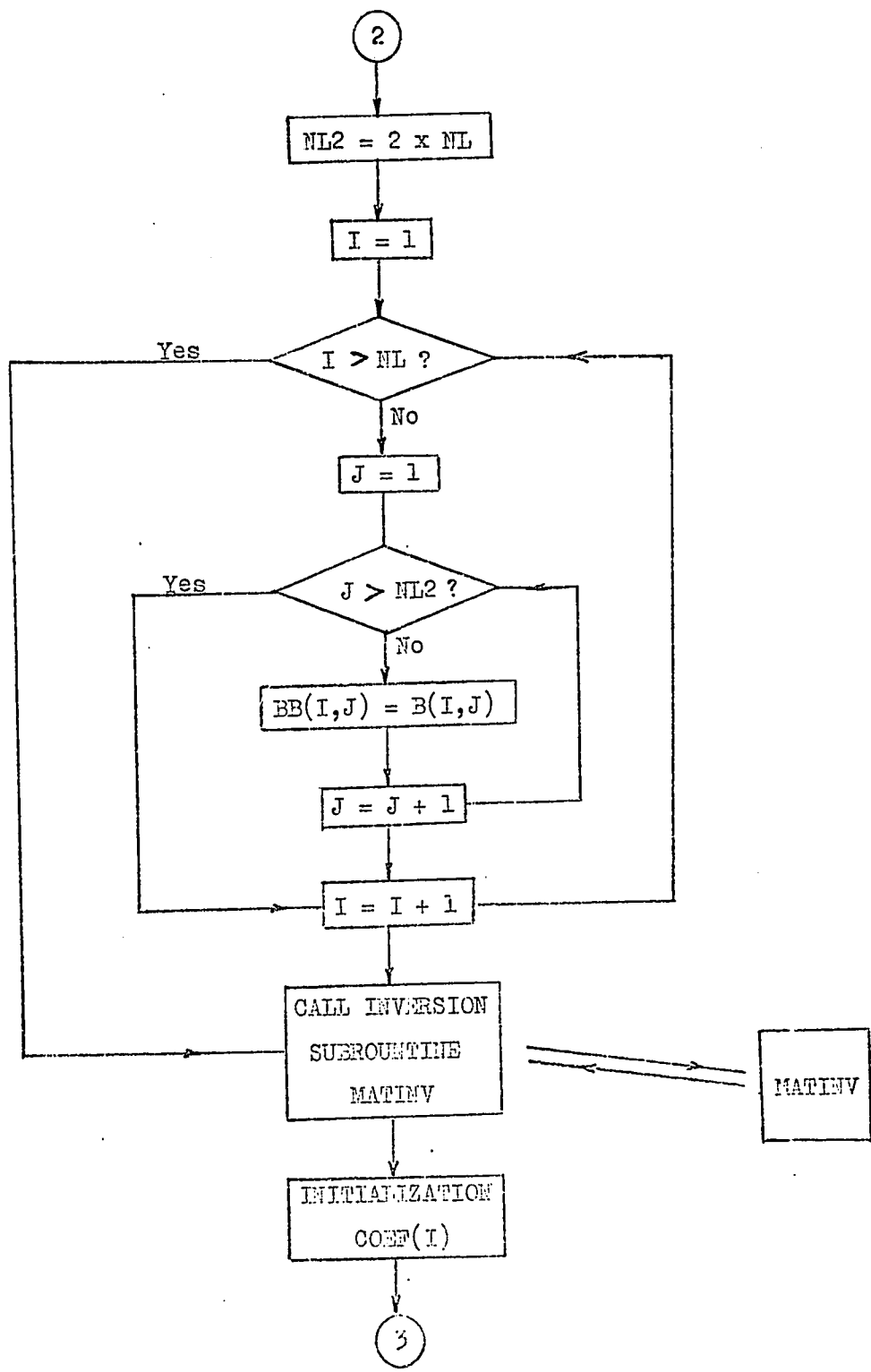
$$w_1 = (1 - \zeta^2)^2 (1 - \eta^2)^2 (1 - \zeta\eta) (1 + c_1\zeta^2 + c_2\eta^2 + c_3\zeta^4 + c_4\eta^4 + c_5\zeta^2\eta^2 + \dots + c_{22}\zeta^6\eta^3 + c_{23}\zeta^8\eta^6 + c_{24}\zeta^8\eta^8)$$

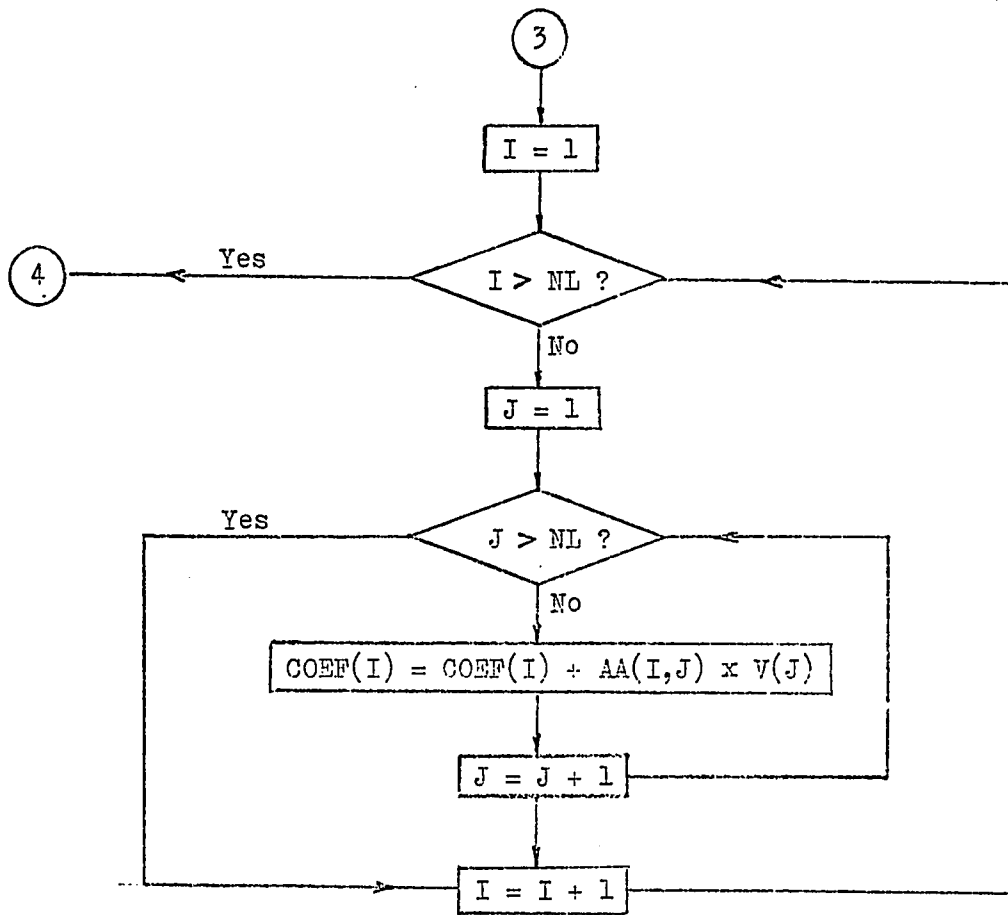
A P P E N D I X   B

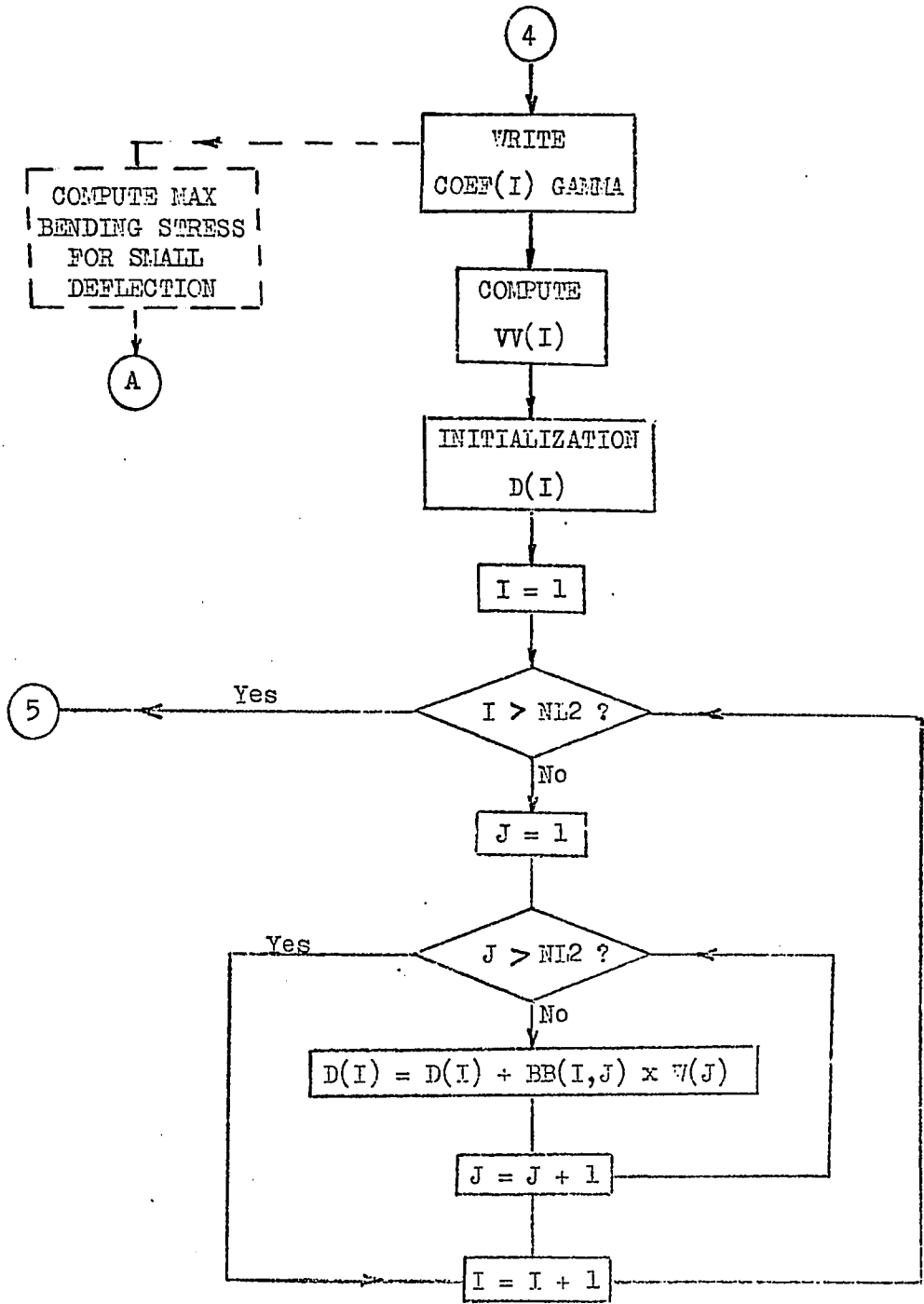
F L O W   C H A R T   A N D   C O M P U T E R   P R O G R A M ' S



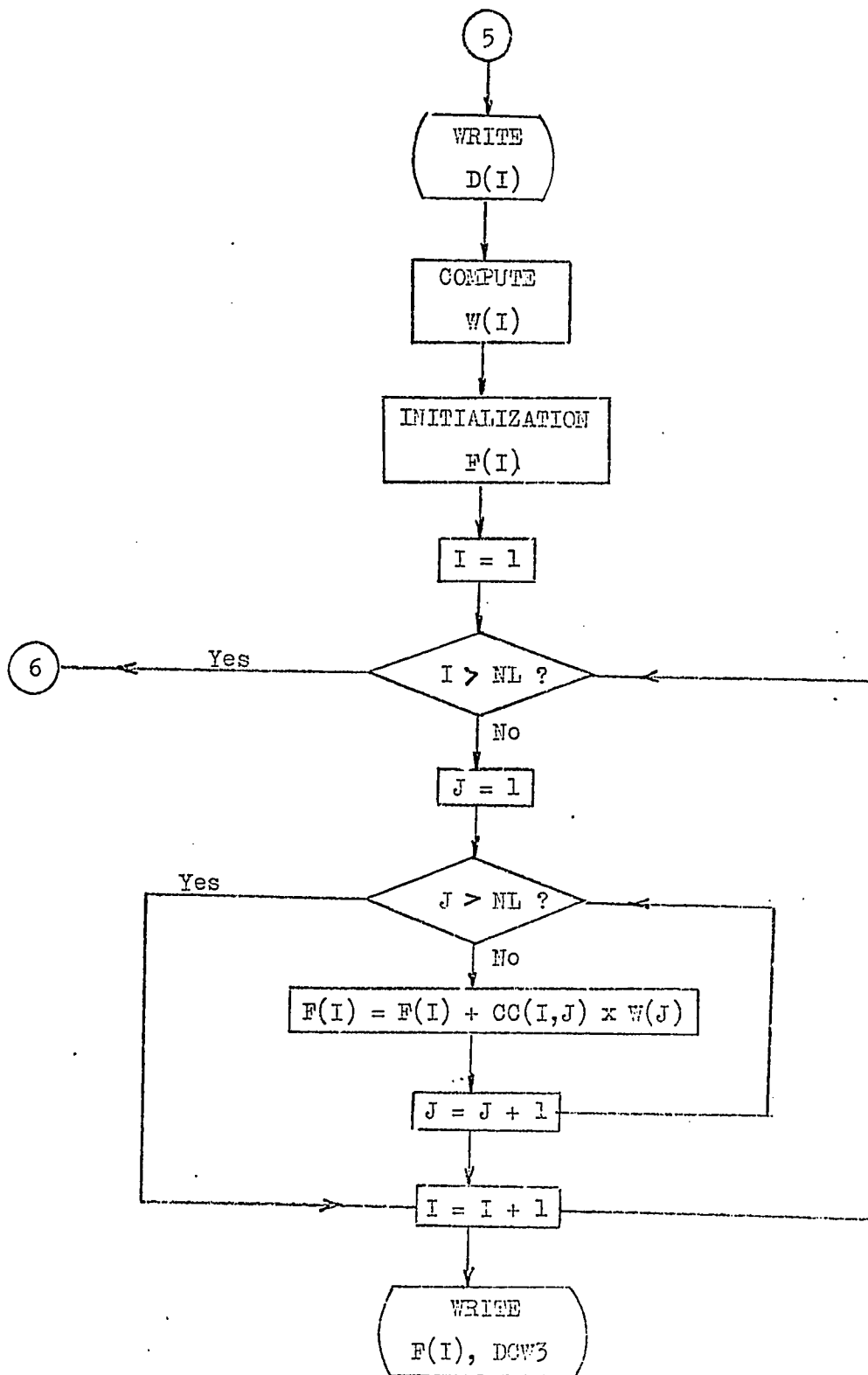




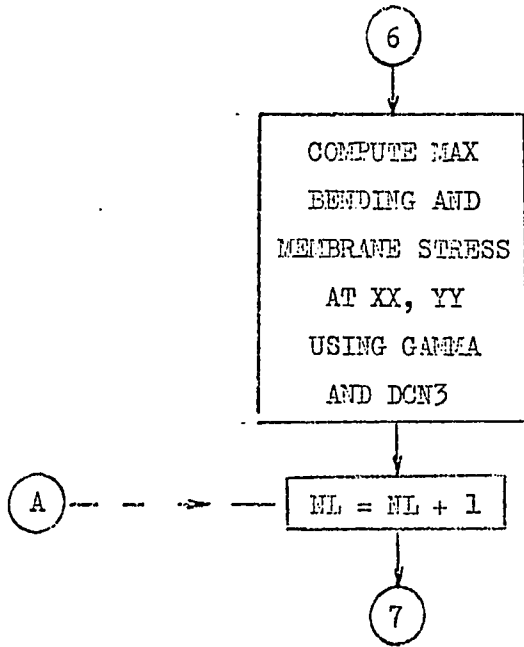




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C ELLIPTICAL PLATE
C ELLIPTICAL WITH K
C B.M. & STRESSES ---LINEAR AND NCN-LINEAR
0001 IMPLICIT REAL*8(A-H,O-Z)
0002 DIMENSION A(20,20),AA(20,20),V(20),COEF(20),
1B(20,20),BB(20,20),VV(20),D(20),
2C(20,20),CC(20,20),W(20),F(20),WO(20),XX(5),YY(5)
0003 READ(1,1)NN,ML,NK
0004 1 FORMAT(8I10)
0005 READ(1,2)(XX(I),YY(I),I=1,3)
0006 READ(1,2)(WO(I),I=1,16)
0007 DO 100 II=1,NN
0008 REAC(1,2)R,ANU
0009 2 FORMAT(8F10.2)
0010 WRITE(3,4)R,ANU
0011 4 FORMAT(1H1,5X,'R = 'F10.3,10X,'ANU = 'F10.3/)
0012 DO 100 KK=1,NK
0013 READ(1,2)AK
0014 WRITE(3,20)AK
0015 20 FORMAT(1HC,10X,'AK = 'F10.2/)
0016 R2=R*R
0017 R3=R2*R
0018 R4=R2*R2
0019 A1=1.-ANU
0020 A2=1.+ANU
0021 ML2=2.*ML
0022 DO 40 I=1,ML
0023 V(I)=0.
0024 W(I)=0.
0025 DO 40 J=1,ML
0026 A(I,J)=0.
0027 40 C(I,J)=0.
0028 DO 400 I=1,ML2
0029 VV(I)=0.
0030 DO 400 J=1,ML2
0031 400 B(I,J)=0.
C
C LHS MATRIX FOR SMALL DEFLECTION COEF.
C
0032 A(1,1)=-(.48.*R4+16.*R2)
0033 A(1,2)=-1.
0034 A(1,3)=-(.16.*R2+48.)
0035 A(1,4)=24.*R4

```

0036 A(1,5)=24.  
 0037 A(1,6)=8.\*R2  
 0038 V(1)=-{24.\*R4+16.\*R2+24.\*AK}  
 0039 A(2,1)=360.\*R4+96.\*R2+24.\*AK  
 0040 A(2,3)=48.\*R2+48.  
 0041 A(2,4)=-{720.\*R4+96.\*R2}  
 0042 A(2,5)=-48.  
 0043 A(2,6)=-{96.\*R2+48.}  
 0044 A(2,7)=360.\*R4  
 0045 A(2,9)=24.  
 0046 A(2,10)=48.\*R2  
 0047 V(2)=2.\*AK  
 0048 A(3,1)=48.\*R4+48.\*R2  
 0049 A(3,3)=24.\*R4+96.\*R2+360.\*AK  
 0050 A(3,4)=-48.\*R4  
 0051 A(3,5)=-{96.\*R2+720.}  
 0052 A(3,6)=-{48.\*R4+96.\*R2}  
 0053 A(3,8)=360.  
 0054 A(3,9)=48.\*R2  
 0055 A(3,10)=24.\*R4  
 0056 V(3)=2.\*AK  
 0057 A(4,1)=-2.\*AK  
 0058 A(4,4)=1680.\*R4+240.\*R2+24.\*AK  
 0059 A(4,5)=24.  
 0060 A(4,6)=120.\*R2+48.  
 0061 A(4,7)=-{3360.\*R4+240.\*R2}  
 0062 A(4,9)=-48.  
 0063 A(4,10)=-{240.\*R2+48.}  
 0064 A(4,11)=1680.\*R4  
 0065 A(4,13)=24.  
 0066 A(4,15)=120.\*R2  
 0067 V(4)=-AK  
 0068 A(5,3)=-2.\*AK  
 0069 A(5,4)=24.\*R4  
 0070 A(5,5)=24.\*R4+240.\*R2+1680.\*AK  
 0071 A(5,6)=48.\*R4+120.\*R2  
 0072 A(5,8)=-{240.\*R2+3360.}  
 0073 A(5,9)=-{48.\*R4+240.\*R2}  
 0074 A(5,10)=-48.\*R4  
 0075 A(5,12)=1680.  
 0076 A(5,13)=24.\*R4  
 0077 A(5,14)=120.\*R2  
 0078 V(5)=-AK

0079 A(6,1)=-2.\*AK  
0080 A(6,3)=-2.\*AK  
0081 A(6,4)=720.\*R4+288.\*R2  
0082 A(6,5)=288.\*R2+720.  
0083 A(6,6)=360.\*R4+576.\*R2+360.\*AK  
0084 A(6,7)=-720.\*R4  
0085 A(6,8)=-720.  
0086 A(6,9)=-576.\*R2+720.  
0087 A(6,10)=-720.\*R4+576.\*R2  
0088 A(6,13)=288.\*R2  
0089 A(6,14)=360.  
0090 A(6,15)=360.\*R4  
0091 V(6)=-2.\*AK  
0092 A(7,1)=AK  
0093 A(7,4)=-2.\*AK  
0094 A(7,7)=5040.\*R4+448.\*R2+24.\*AK  
0095 A(7,9)=24.  
0096 A(7,10)=224.\*R2+48.  
0097 A(7,11)=-10080.\*R4+448.\*R2  
0098 A(7,13)=-48.  
0099 A(7,15)=-448.\*R2+48.  
0100 A(7,17)=24.  
0101 A(7,19)=224.\*R2  
0102 A(8,3)=AK  
0103 A(8,5)=-2.\*AK  
0104 A(8,8)=24.\*R4+448.\*R2+5040.\*AK  
0105 A(8,9)=48.\*R4+224.\*R2  
0106 A(8,10)=24.\*R4  
0107 A(8,12)=-448.\*R2+10080.  
0108 A(8,13)=-48.\*R4  
0109 A(8,14)=-48.\*R4+448.\*R2  
0110 A(8,16)=24.\*R4  
0111 A(8,18)=224.\*R2  
C LHS MATRIX FOR MEMBRANE COEF.  
C  
C  
0112 B(1,1)=-12.\*R3+2.\*R\*A1  
0113 B(1,2)=-2.\*R2\*A2  
0114 B(1,3)=12.\*R3  
0115 B(1,6)=2.\*R2\*A2  
0116 B(1,5)=2.\*R\*A1  
0117 B(2,1)=-2.\*R2\*A2  
0118 B(2,2)=-12.\*R+2.\*R3\*A1

0119 B(2,6)=2.\*R3\*A1  
 0120 B(2,5)=2.\*R2\*A2  
 0121 B(2,4)=12.\*R  
 0122 B(3,3)=- (40.\*R3+2.\*R\*A1)  
 0123 B(3,6)=-4.\*R2\*A2  
 0124 B(3,5)=-2.\*R\*A1  
 0125 B(3,7)=40.\*R3  
 0126 B(3,10)=4.\*R2\*A2  
 0127 B(3,11)=2.\*R\*A1  
 0128 B(4,6)=-2.\*R3\*A1  
 0129 B(4,5)=-4.\*R2\*A2  
 0130 B(4,4)=- (40.\*R+2.\*R3\*A1)  
 0131 B(4,9)=4.\*R2\*A2  
 0132 B(4,8)=40.\*R  
 0133 B(4,12)=2.\*R3\*A1  
 0134 B(5,3)=-12.\*R3  
 0135 B(5,6)=-6.\*R2\*A2  
 0136 B(5,5)=- (12.\*R3+12.\*R\*A1)  
 0137 B(5,5)=- (12.\*R3+12.\*R\*A1)  
 0138 B(5,4)=- (6.\*R2\*A2)  
 0139 B(5,9)=12.\*R\*A1  
 0140 B(5,11)=12.\*R3  
 0141 B(5,12)=6.\*R2\*A2  
 0142 B(6,3)=-6.\*R2\*A2  
 0143 B(6,6)=- (12.\*R+12.\*R3\*A1)  
 0144 B(6,5)=-6.\*R2\*A2  
 0145 B(6,4)=-12.\*R  
 0146 B(6,10)=12.\*R3\*A1  
 0147 B(6,11)=6.\*R2\*A2  
 0148 B(6,12)=12.\*R  
 0149 B(7,7)=- (84.\*R3+2.\*R\*A1)  
 0150 B(7,10)=-6.\*R2\*A2  
 0151 B(7,11)=-2.\*R\*A1  
 0152 B(7,13)=84.\*R3  
 0153 B(7,16)=6.\*R2\*A2  
 0154 B(8,8)=- (84.\*R+2.\*R3\*A1)  
 0155 B(8,9)=-6.\*R2\*A2  
 0156 B(8,12)=- (2.\*R3\*A1)  
 0157 B(8,14)=84.\*R  
 0158 B(8,15)=6.\*R2\*A2  
 0159 B(9,8)=-10.\*R2\*A2  
 0160 B(9,9)=- (12.\*R3+30.\*R\*A1)  
 0161 B(9,11)=-12.\*R3

0162 B(9,12)=-10.\*R2\*A2  
 0163 B(9,15)=30.\*R\*A1  
 0164 B(10,7)=-10.\*R2\*A2  
 0165 B(10,10)=-12.\*R+30.\*R3\*A1  
 0166 B(10,11)=-10.\*R2\*A2  
 0167 B(10,12)=-12.\*R  
 0168 B(10,16)=50.\*R3\*A1  
 0169 B(11,7)=-40.\*R3  
 0170 B(11,9)=-12.\*R\*A1  
 0171 B(11,10)=-12.\*R2\*A2  
 0172 B(11,11)=-40.\*R3+12.\*R\*A1  
 0173 B(11,12)=-12.\*R2\*A2  
 0174 B(12,8)=-40.\*R  
 0175 B(12,9)=-12.\*R2\*A2  
 0176 B(12,10)=-12.\*R3\*A1  
 0177 B(12,11)=-12.\*R2\*A2  
 0178 B(12,12)=-40.\*R+12.\*R3\*A1  
 0179 B(13,13)=-144.\*R3+2.\*R\*A1  
 0180 B(13,16)=-8.\*R2\*A2  
 0181 B(14,14)=-144.\*R+2.\*R3\*A1  
 0182 B(14,15)=-8.\*R2\*A2  
 0183 B(15,14)=-14.\*R2\*A2  
 0184 B(15,15)=-12.\*R3+56.\*R\*A1  
 0185 B(16,13)=-14.\*R2\*A2  
 0186 B(16,16)=-12.\*R+56.\*R3\*A1

C  
 C LHS MATRIX FOR LARGE DEFLECTION COEF  
 C

0187 C(1,1)=-48.\*R4+16.\*R2  
 0188 C(1,2)=-1.  
 0189 C(1,3)=-16.\*R2+48.  
 0190 C(1,4)=24.\*R4  
 0191 C(1,5)=24.  
 0192 C(1,6)=8.\*R2  
 0193 C(2,1)=360.\*R4+96.\*R2+24.\*AK  
 0194 C(2,3)=48.\*R2+48.  
 0195 C(2,4)=-720.\*R4+96.\*R2  
 0196 C(2,5)=-48.  
 0197 C(2,6)=-96.\*R2+48.  
 0198 C(2,7)=360.\*R4  
 0199 C(3,1)=48.\*R4+48.\*R2  
 0200 C(3,3)=24.\*R4+96.\*R2+360.\*AK  
 0201 C(3,4)=-48.\*R4



```

0202 C(3,5)=- (96.*R2+720.)
0203 C(3,6)=- (48.*R4+96.*R2)
0204 C(3,8)=360.
0205 C(4,1)=-2.*AK
0206 C(4,4)=1680.*R4+240.*R2+24.*AK
0207 C(4,5)=24.
0208 C(4,6)=120.*R2+48.
0209 C(4,7)=- (3360.*R4+240.*R2)
0210 C(5,3)=-2.*AK
0211 C(5,4)=24.*R4
0212 C(5,5)=24.*R4+240.*R2+1680.*AK
0213 C(5,6)=48.*R4+120.*R2
0214 C(5,8)=- (240.*R2+3360.)
0215 C(6,1)=-2.*AK
0216 C(6,3)=-2.*AK
0217 C(6,4)=720.*R4+288.*R2
0218 C(6,5)=288.*R2+720.
0219 C(6,6)=360.*R4+576.*R2+360.*AK
0220 C(6,7)=-720.*R4
0221 C(6,8)=-720.
0222 C(7,1)=AK
0223 C(7,4)=-2.*AK
0224 C(7,7)=5040.*R4+448.*R2+24.*AK
0225 C(8,3)=AK
0226 C(8,5)=-2.*AK
0227 C(8,8)=24.*R4+448.*R2+5040.*AK

```

C

```

0228 DO 200 NL=8,ML
0229 DO 211 I=1,NL
0230 DO 211 J=1,NL
0231 AA(I,J)=A(I,J)
0232 211 CC(I,J)=C(I,J)
0233 NL2=2.*NL
0234 DO 201 I=1,NL2
0235 DO 201 J=1,NL2
0236 201 BB(I,J)=B(I,J)
0237 CALL MATINV(AA,NL)
0238 CALL MATINV(BB,NL2)
0239 CALL MATINV(CC,NL)

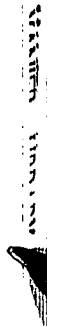
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C

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0240 WRITE(3,1010)
0241 1010 FORMAT(1H,1X,'*****')
0242 WRITE(3,14)NL

```



14 FORMAT(IH0,I2X,I2,2X,'TERM SOLUTION')

DO 30 I=1,10

30 COEF(I)=0.

DO 300 I=1,NL

DO 300 J=1,NL

300 COEF(I)=COEF(I)+AA(I,J)\*V(J)

DO 500 I=1,ML,5

I1=I+1

I2=I+2

I3=I+3

I4=I+4

500 WRITE(3,5) I,COEF(I),I1,COEF(I1),I2,COEF(I2),I3,COEF(I3),I4,COEF(I4)

1)

5 FORMAT(1H,5(1X,'C',I2,D16.8,5X))

GAMMA=COEF(2)

DC=1./GAMMA

WRITE(3,6)DC

6 FORMAT(1H0,5X,'DC ='D16.8)

C

C FORMING RHS OF MEMBRANE COEF

C

P1=-4.+2.\*COEF(1)

P2=4.-8.\*COEF(1)+4.\*COEF(4)

P3=6.\*COEF(1)-12.\*COEF(4)+6.\*COEF(7)

P4=4.-4.\*COEF(1)-4.\*COEF(3)+2.\*COEF(6)

P5=2.\*COEF(1)+4.\*COEF(3)-4.\*COEF(5)-4.\*COEF(6)

P6=8.\*COEF(1)+4.\*COEF(3)-8.\*COEF(4)-8.\*COEF(6)

P7=4.\*COEF(5)+2.\*COEF(6)-4.\*COEF(8)

P8=8.\*COEF(4)-16.\*COEF(7)

PP1=-4.+2.\*COEF(3)

PP2=4.-8.\*COEF(3)+4.\*COEF(5)

PP3=6.\*COEF(3)-12.\*COEF(5)+6.\*COEF(8)

PP4=4.-4.\*COEF(1)-4.\*COEF(3)+2.\*COEF(6)

PP5=8.\*COEF(5)-16.\*COEF(8)

PP6=4.\*COEF(1)+8.\*COEF(3)-8.\*COEF(5)-8.\*COEF(6)

PP7=4.\*COEF(1)+2.\*COEF(3)-4.\*COEF(4)-4.\*COEF(6)

PP8=4.\*COEF(4)+2.\*COEF(6)-4.\*COEF(7)

PX1=-4.+2.\*COEF(1)

PX2=12.-24.\*COEF(1)+12.\*COEF(4)

PX3=4.-4.\*COEF(1)-4.\*COEF(3)+2.\*COEF(6)

PX4=24.\*COEF(1)+12.\*COEF(3)-24.\*COEF(4)-24.\*COEF(6)

PX5=30.\*COEF(1)-60.\*COEF(4)+30.\*COEF(7)

PX6=2.\*COEF(1)+4.\*COEF(3)-4.\*COEF(5)-4.\*COEF(6)

```

0282 PX7=56.*COEF(4)-112.*COEF(7)
0283 PX8=4.*COEF(5)+2.*COEF(6)-4.*COEF(8)
0284 PY1=-4.+2.*COEF(3)
0285 PY2=4.-4.*COEF(1)-4.*COEF(3)+2.*COEF(6)
0286 PY3=12.-24.*COEF(2)+12.*COEF(5)
0287 PY4=12.*COEF(1)+24.*COEF(3)-24.*COEF(5)-24.*COEF(6)
0288 PY5=4.*COEF(1)+2.*COEF(3)-4.*COEF(4)-4.*COEF(6)
0289 PY6=30.*COEF(3)-60.*COEF(5)+30.*COEF(8)
0290 PY7=4.*COEF(4)+2.*COEF(6)-4.*COEF(7)
0291 PY8=56.*COEF(5)-112.*COEF(8)
0292 PXY1=8.-8.*COEF(1)-8.*COEF(3)+4.*COEF(6)
0293 PXY2=8.*COEF(1)+16.*COEF(3)-16.*COEF(5)-16.*COEF(6)
0294 PXY3=16.*COEF(1)+8.*COEF(3)-16.*COEF(4)-16.*COEF(6)
0295 PXY4=24.*COEF(5)+12.*COEF(6)-24.*COEF(8)
0296 PXY5=24.*COEF(4)+12.*COEF(6)-24.*COEF(7)
0297 VV(1)=- (2.*R3*PI*PX1+R*A1*PY1*PI)
0298 VV(2)=- (2.*PY1*PP1+R2*A1*PX1*PP1)
0299 VV(3)=- (2.*R3*(PX1*P2+PX2*PI)+R*A1*(PY1*P2+PY2*PI))
0300 VV(4)=- (2.*(PY1*PP2+PY3*PP1)+R2*A1*(PX1*PP2+PX3*PP1))
0301 VV(5)=- (2.*R3*(P4*PX1+PX3*PI)+R*A2*PXY1*PP1+R*A1*(PY1*P4+PY3*PI))
0302 VV(6)=- (2.*(PY1*PP4+PY2*PP1)+R2*A2*PXY1*PI+R2*A1*(PX1*PP4+PX2*PI))
1))
0303 VV(7)=- (2.*R3*(PX1*P3+PX2*P2+PX5*PI)+R*A1*(PY1*P3+PY2*P2+PY5*PI))
0304 VV(8)=- (2.*(PY1*PP3+PY3*PP2+PY6*PP1)+R2*A1*(PX1*PP3+PX3*PP2+PX6*
3PP1))
0305 VV(9)=- (2.*R3*(PX1*P5+PX3*P4+PX6*PI)+R*A2*(PXY1*PP2+PXY2*PI)+R*A1
5*(PY1*P5+PY3*P4+PY6*PI))
0306 VV(10)=- (2.*(PY1*PP7+PY2*PP4+PY5*PP1)+R2*A2*(PXY1*P2+PXY3*PI)+R2*
6A1*(PX1*PP7+PX2*PP4+PX5*PP1))
0307 VV(11)=- (2.*R3*(PX1*P6+PX2*P4+PX3*P2+PX4*PI)+R*A2*(PXY1*PP4+PXY3*
7PP1)+R*A1*(PY1*P6+PY2*P4+PY3*P2+PY4*PI))
0308 VV(12)=- (2.*(PY1*PP6+PY2*PP2+PY3*PP4+PY4*PP1)+R2*A2*(PXY1*P4+PXY2*
8PI)+R2*A1*(PX1*PP6+PX2*PI+PX3*PP4+PX4*PP1))
0309 VV(13)=- (2.*R3*(PX1*P8+PX2*P3+PX5*P2+PX7*PI)+R*A1*(PY1*P8+PY2*P3+
9PY5*P2+PY7*PI))
0310 VV(14)=- (2.*(PY1*PP5+PY3*PP3+PY6*PP2+PY8*PP1)+R2*A1*(PX1*PP5+PX3*
2PP3+PX6*PP2+PX8*PP1))
0311 VV(15)=- (2.*R3*(PX1*P7+PX3*P5+PX6*P4+PX8*PI)+R*A2*(PXY1*PP3+PXY2
5*PP2+PXY4*PP1)+R*A1*(PY1*P7+PY3*P5+PY6*P4+PY8*PI))
0312 VV(16)=- (2.*(PY1*PP8+PY2*PP7+PY5*PP4+PY7*PP1)+R2*A2*(PXY1*P3
3+PXY3*P2+PXY5*PI)+R2*A1*(PX1*PP8+PX2*PP7+PX5*PP4+PX7*PP1))
WRITE (3,151) (VV(I), I=1, ML2)
151 FORMAT(1H0,5X,5D15.8/(6X,5O16.8))

```

```

0315 DO 31 I=1,20
0316 31 C(I)=0.
0317 DO 301 I=1,NL2
0318 DO 301 J=1,NL2
0319 D(I)=D(I)+BB(I,J)*VV(J)
0320 WRITE(3,16)
0321 16 FORMAT(1H0,5X,'LARGE DEFLECTION')
0322 DO 501 I=1,ML2,5
0323 I1=I+1
0324 I2=I+2
0325 I3=I+3
0326 I4=I+4
0327 501 WRITE(3,15)I,D(I),I1,D(I1),I2,D(I2),I3,D(I3),I4,D(I4)
0328 15 FORMAT(1H,5(1X,'C',I2,D16.8,5X))
C
C FORMING RHS FOR W3
C
0329 SX1=D(1)
0330 SX2=-3.*D(1)+3.*D(3)
0331 SX3=-D(1)+D(5)
0332 SX4=-3.*D(3)-3.*D(5)+3.*D(11)
0333 SX5=-5.*D(3)+5.*D(7)
0334 SX6=-D(5)+D(9)
0335 SX7=-7.*D(7)+7.*D(13)
0336 SX8=-D(9)+D(15)
0337 SY1=-2.*D(1)+2.*D(5)
0338 TX1=-2.*D(2)+2.*D(6)
0339 TY1=D(2)
0340 TY2=-D(2)+D(6)
0341 TY3=-3.*D(2)+3.*D(4)
0342 TY4=-3.*D(6)-3.*D(4)+3.*D(12)
0343 TY5=-D(6)+D(10)
0344 TY6=-5.*D(4)+5.*D(8)
0345 TY7=-D(10)+D(16)
0346 TY8=-7.*D(8)+7.*D(14)
0347 W(1)=12.*(R4*PX1+ANU*(R3*TY1*PX1)+12.*(R*TY1*PY1+ANU*R2*PY1*
1SX1)
W(2)=12.*(R4*(PX1*SX2+PX2*SX1)+5*R4*P1*P1*PX1+ANU*(R3*(PX1*TY2+PX
22*TY1))+12.*(R*(PY1*TY2+PY2*TY1)+ANU*(R2*(PY1*SX2+PY2*SX1))+5*R2*
3P1*P1*PY1))
W(3)=12.*(R4*(PX1*SX3+PX3*SX1)+ANU*(R3*(PX1*TY3+PX3*TY1))+5*R2*
4(PP1*PP1*PX1))+12.*(R*(PY1*TY3+PY3*TY1))+5*PP1*PP1+ANU*(R2*(
5PY1*SX3+PY3*SX1)))
0348
0349

```



```

0371      DO 1000 I=1,3
0372      WRITE(3,1100)
0373      1100 FORMAT(1H,5X,'-----')
0374      X=XX(I)
0375      Y=YY(I)
0376      WRITE(3,1001)X,Y
0377      1001 FURNAT(1H0,5X,'AT',5X,'X='F8.2,3X,'Y='F8.2/)
0378      X2=X*X
0379      X3=X2*X
0380      X4=X3*X
0381      X5=X4*X
0382      X6=X5*X
0383      X7=X6*X
0384      X8=X7*X
0385      X9=X8*X
0386      X10=X9*X
0387      X11=X10*X
0388      Y2=Y*Y
0389      Y3=Y2*Y
0390      Y4=Y3*Y
0391      Y5=Y4*Y
0392      Y6=Y5*Y
0393      Y7=Y6*Y
0394      Y8=Y7*Y
0395      Y9=Y8*Y
0396      Y10=Y9*Y
0397      Y11=Y10*Y
0398      EX1=2.-24.*X2-4.*Y2+24.*X2*Y2+30.*X4+2.*Y4
0399      EX2=-4.*Y2+4.*Y4+12.*X2*Y2
0400      EX3=12.*X2-60.*X4-24.*X2*Y2+60.*X4*Y2+56.*X6+12.*X2*Y4
0401      EX4=-4.*Y4+4.*Y6+12.*X2*Y4
0402      EX5=2.*Y2-24.*X2*Y2-4.*Y4+24.*X2*Y4+30.*X4*Y2+2.*Y6
0403      EX6=30.*X4-112.*X6-60.*X2*Y2+112.*X6*Y2+90.*X8+30.*X4*Y4
0404      EX7=-4.*Y6+4.*Y8+12.*X2*Y6
0405      EY1=-4.*X2+4.*X4+12.*X2*Y2
0406      EY2=2.-4.*X2-24.*Y2+24.*X2*Y2+2.*X4+30.*Y4
0407      EY3=-4.*X4+4.*X6+12.*X4*Y2
0408      EY4=12.*Y2-24.*X2*Y2-60.*Y4+60.*X2*Y4+12.*X4*Y2+56.*Y6
0409      EY5=2.*X2-4.*X4-24.*X2*Y2+24.*X4*Y2+2.*X6+30.*X2*Y4
0410      EY6=-4.*X6+4.*X8+12.*X6*Y2
0411      EY7=30.*Y4-60.*X2*Y4-112.*Y6+112.*X2*Y6+30.*X4*Y4+90.*Y8
0412      EY11=-8.*X*Y+16.*X3*Y+8.*X*Y3
0413      EY2=-8.*X*Y+16.*X*Y3+8.*X*Y3

```

0414 EXY3=-16.\*X3\*Y+24.\*X5\*Y+16.\*X3\*Y3  
 0415 EXY4=-16.\*X\*Y3+24.\*X\*Y5+16.\*X3\*Y3  
 0416 EXY5=4.\*X\*Y-16.\*X3\*Y-16.\*X\*Y3+32.\*X3\*Y3+12.\*X5\*Y+12.\*X\*Y5  
 0417 EXY6=-24.\*X5\*Y+32.\*X7\*Y+24.\*X5\*Y3  
 0418 EXY7=-24.\*X\*Y5+32.\*X\*Y7+24.\*X3\*Y5  
 0419 WX2=-4.\*+4.\*Y2+12.\*X2+COEF(1)\*EX1+COEF(3)\*EX2+COEF(4)\*EX3  
 0420 1+COEF(5)\*EX4+COEF(6)\*EX5+COEF(7)\*EX6+COEF(8)\*EX7  
 WY2=-4.\*+4.\*X2+12.\*Y2+COEF(1)\*EY1+COEF(3)\*EY2+CCEF(4)\*EY3+COEF(5)\*  
 1EY4+COEF(6)\*EY5+COEF(7)\*EY6+COEF(8)\*EY7  
 WXY=8.\*X\*Y+COEF(1)\*EXY1+COEF(3)\*EXY2+COEF(4)\*EXY3+COEF(5)\*EXY4  
 1+COEF(6)\*EXY5+COEF(7)\*EXY6+COEF(8)\*EXY7

C  
 C BM & STRESS VS Q ---- LINEAR  
 C

0422 BMX=-DC\*(R2\*WX2+ANU\*WY2)  
 0423 BMY=-DC\*(WY2+ANU\*R2\*WX2)  
 0424 BMXY=DC\*R\*WXY  
 C425 SIGX=.5\*BMX  
 0426 SIGY=.5\*BMY  
 0427 SIGXY=BMXY  
 C428 WRITE(3,1002)

1002 FORMAT(1H0,5X,'BM & STRESS VS Q ---- LINEAR')  
 WRITE(3,1003)BMX,BMY,BMXY,SIGX,SIGY,SIGXY  
 1003 FORMAT(1H0,5X,'BMX='D16.8,5X,'BMY='D16.8,5X,'BMXY='D16.8/  
 15X,'SIGX='D16.8,5X,'SIGY='D16.8,5X,'SIGXY='D16.8/')

C  
 C BM & STRESS VA W0 ---- LINEAR  
 C

0432 HMX1=- (R2\*WX2+ANU\*WY2)  
 0433 BMY1=- (WY2+ANU\*R\*WX2)  
 0434 BMXY1=R\*WXY  
 0435 SIGX1=.5\*BMX1  
 0436 SIGY1=.5\*BMY1  
 0437 SIGXY1=R\*WXY  
 0438 WRITE(3,1004)  
 0439 1004 FORMAT(1H0,5X,'BM & STRESS VS W0 ---- LINEAR')

1004 WRITE(3,1005)HMX1,BMY1,BMXY1,SIGX1,SIGY1,SIGXY1  
 1005 FORMAT(1H0,5X,'BMX1='D16.8,5X,'BMY1='D16.8,5X,'BMXY1='D16.8/  
 15X,'SIGX1='D16.8,5X,'SIGY1='D16.8,5X,'SIGXY1='D16.8/')

C  
 C Q, BM & STRESS VS W0 ---- NCN-LINEAR  
 C  
 C W3X2=F(1)\*EX1+F(3)\*EX2+F(4)\*EX3+F(5)\*EX4+F(6)\*EX5+F(7)\*EX6  
 0442

```

0443      1+F(8)*EX7
          W3Y2=F(1)*EY1+F(3)*EY2+F(4)*EY3+F(5)*EY4+F(6)*EY5+F(7)*EY6
          1+F(8)*EY7
          W3XY=F(1)*EXY1+F(3)*EXY2+F(4)*EXY3+F(5)*EXY4+F(6)*EXY5+F(7)*EXY6
          1+F(8)*EXY7
          BMX2=-(R2*W3X2+ANU*W3Y2)
          BMY2=-(W3Y2+ANU*W3X2)
          BMXY2=R*W3XY
          WRITE(3,1011)BMX2,BMY2,BMXY2
          1011 FORMAT(1H0,5X,'BMX2='D16.8,5X,'BMY2='D16.8,5X,'BMXY2='D16.8/)
          WRITE(3,1009)
          1009 FORMAT(1H0,5X,'Q, BM, & STRESS VS WO --- NON-LINEAR')
          DO 2000 K=1,16
          MW=W0(K)
          W3=WK**3
          Q=COEF(2)*WW+F(2)*WW**3
          BMX3=BMX1*WW+BMX2*W3
          BMY3=BMY1*WW+BMY2*W3
          BMXY3=BMXY1*WW+BMXY2*W3
          SIGX3=.5*BMX3
          SIGY3=.5*BMY3
          SIGXY3=BMXY3
          1006 FORMAT(1H0,5X,'WC='F6.3,5X,'Q='D16.8/20X,'BMX3='D16.8,5X,'BMY3='
          1D16.8,5X,
          1,'BMXY3='D16.8/20X,'SIGX3='D16.8,5X,'SIGY3='D16.8,5X,'SIGXY3='
          2D16.8)
          BM=DABS((.5*(BMX3-BMY3))*2+BMXY3**2)**.5)
          BMAX=.5*(BMX3+BMY3)+BM
          SIG=CABS((.5*(SIGX3-SIGY3))*2+SIGXY3**2)**.5)
          SIGMAX=.5*(SIGX3+SIGY3)+SIG
          WRITE(3,1012)BMAX,SIGMAX
          1012 FORMAT(1H0,20X,'BMAX='D16.8/20X,'SIGMAX='D16.8/)
          2000 WRITE(3,1006)MW,Q,BMX3,BMY3,BMXY3,SIGX3,SIGY3,SIGXY3
          C
          C      TO COMPUTE MEMBRANE STRESSES
          C
          S2X=D(I)*(1.-3.*X2-Y2)+D(3)*(3.*X2-5.*X4-3.*X2*Y2)
          1+D(5)*(Y2-3.*X2*Y2-Y4)+D(7)*(5.*X4-7.*X6-5.*X4*Y2)
          2+D(9)*(Y4-3.*X2*Y4-Y6)+D(11)*(3.*X2*Y2-5.*X4*Y2-3.*X2*Y4)
          3+D(13)*(7.*X6-9.*X8-7.*X6*Y2)+D(15)*(Y6-3.*X2*Y6-Y8)
          S2Y=D(I)*(-2.*X*Y)+D(3)*(-2.*X3*Y)+D(5)*(2.*X*Y-2.*X3*Y-4.*X*Y3)
          1+D(7)*(-2.*X5*Y)+D(9)*(4.*X*Y3-4.*X3*Y3-6.*X*Y5)
          2+D(11)*(2.*X3*Y-2.*X5*Y-4.*X3*Y3)+D(13)*(-2.*X*Y7)

```

0472 3+D(15)\*(6.\*X\*Y5-6.\*X3\*Y5-8.\*X\*Y7)  
T2X=D(2)\*(-2.\*X\*Y)+D(4)\*(-2.\*X\*Y3)+D(6)\*(2.\*X\*Y-4.\*X3\*Y-2.\*X\*Y3)

1+D(8)\*(-2.\*X\*Y5)+D(10)\*(4.\*X3\*Y-6.\*X5\*Y-4.\*X3\*Y3)  
2+D(12)\*(2.\*X\*Y3-4.\*X3\*Y3-2.\*X\*Y5)+D(14)\*(-2.\*X\*Y7)  
4+D(16)\*(6.\*X5\*Y-8.\*X7\*Y-6.\*X5\*Y3)  
T2Y=D(2)\*(1.-X2-3.\*Y2)+D(4)\*(3.\*Y2-3.\*X2\*Y2-5.\*Y4)

0473 1+D(6)\*(X2-X4-3.\*X2\*Y2)+D(8)\*(5.\*Y4-5.\*X2\*Y4-7.\*Y6)  
2+D(10)\*(X4-X6-3.\*X4\*Y2)+D(12)\*(3.\*X2\*Y2-3.\*X4\*Y2-5.\*X2\*Y4)  
3+D(14)\*(7.\*Y6-7.\*X2\*Y6-9.\*Y8)+D(16)\*(X6-X8-3.\*X6\*Y2)

0474 WX1=-4.\*X+4.\*X\*Y2+4.\*X3  
1+COEF(1)\*(2.\*X-8.\*X3-4.\*X\*Y2+8.\*X3\*Y2+6.\*X5+2.\*X\*Y4)  
2+COEF(3)\*(-4.\*X\*Y2+4.\*X\*Y4+4.\*X3\*Y)  
3+COEF(4)\*(4.\*X3-12.\*X5-8.\*X3\*Y2+12.\*X5\*Y2+8.\*X7+4.\*X3\*Y4)  
4+COEF(5)\*(-4.\*X\*Y4+4.\*X\*Y6+4.\*X3\*Y4)  
5+COEF(6)\*(2.\*X\*Y2-8.\*X3\*Y2-4.\*X\*Y4+8.\*X3\*Y4+6.\*X5\*Y2+2.\*X\*Y6)  
6+COEF(7)\*(6.\*X5-16.\*X7-12.\*X5\*Y2+16.\*X7\*Y2+10.\*X9+6.\*X5\*Y4)  
7+COEF(8)\*(-4.\*X\*Y6+4.\*X\*Y8+4.\*X3\*Y6)

0475 WY1=-4.\*Y+4.\*X2\*Y+4.\*Y3  
1+COEF(1)\*(-4.\*X2\*Y+4.\*X4\*Y+4.\*X2\*Y3)  
2+COEF(3)\*(2.\*Y-4.\*X2\*Y-8.\*Y3+8.\*X2\*Y3+2.\*X4\*Y+6.\*Y5)  
3+COEF(4)\*(-4.\*X4\*Y+4.\*X6\*Y+4.\*X4\*Y3)  
4+COEF(5)\*(4.\*Y3-8.\*X2\*Y3-12.\*Y5+12.\*Y5\*X2+4.\*X4\*Y3+8.\*Y7)  
5+COEF(6)\*(2.\*X2\*Y-4.\*X4\*Y-8.\*X2\*Y3+8.\*X4\*Y3+2.\*X6\*Y+6.\*X2\*Y5)  
6+COEF(7)\*(-4.\*X6\*Y+4.\*X8\*Y+4.\*X6\*Y3)  
7+COEF(8)\*(6.\*Y5-12.\*X2\*Y5-16.\*Y7+16.\*X2\*Y7+6.\*X4\*Y5+10.\*Y9)

0476 WX3=F(1)\*(2.\*X2+6.\*X5+2.\*X\*Y4-8.\*X3-4.\*X\*Y2+8.\*X3\*Y2)  
1+F(3)\*(4.\*X3\*Y2-4.\*X\*Y2+4.\*X\*Y4)  
2+F(4)\*(4.\*X3+8.\*X7+4.\*X3\*Y4-12.\*X5-8.\*X3\*Y2+12.\*X5\*Y2)  
3+F(5)\*(4.\*X3\*Y4-4.\*X\*Y4+4.\*X\*Y6)  
4+F(6)\*(2.\*X\*Y2+6.\*X5\*Y2+2.\*X\*Y6-8.\*X3\*Y2-4.\*X\*Y4+8.\*X3\*Y4)  
5+F(7)\*(6.\*X5+10.\*X9+6.\*X5\*Y4-16.\*X7-12.\*X5\*Y2+16.\*X7\*Y2)  
6+F(8)\*(4.\*X3\*Y6-4.\*X\*Y6+4.\*X\*Y8)

0477 WY3=F(1)\*(4.\*X2\*Y3-4.\*X2\*Y+2.\*X4\*Y)  
1+F(3)\*(2.\*Y+2.\*X4\*Y+6.\*Y5-4.\*X2\*Y-8.\*Y3+8.\*X2\*Y3)  
2+F(4)\*(4.\*X4\*Y3-4.\*X4\*Y+4.\*X6\*Y)  
3+F(5)\*(4.\*Y3+4.\*X4\*Y3+8.\*Y7-8.\*X2\*Y3-12.\*Y5+12.\*X2\*Y5)  
4+F(6)\*(2.\*X2\*Y+2.\*X6\*Y+6.\*X2\*Y5-2.\*X4\*Y-8.\*X2\*Y3+8.\*X4\*Y3)  
5+F(7)\*(4.\*X6\*Y3-4.\*X6\*Y+4.\*X8\*Y)  
6+F(8)\*(6.\*Y5+6.\*X4\*Y5+10.\*Y9-12.\*X2\*Y5-16.\*Y7+16.\*X2\*Y7)

C  
C  
C  
SMX, SMY, SMXY VS W02  
SMX1=(R2\*S2X+.5\*R2\*WX1\*WX1+ANU\*(R\*(2Y+.5\*WY1\*WY1))

```

0479 SMX2=R2*WX1*WX3+ANU*WY1*WY3
0480 SMX3=.5*(R2*WX3*WX3+ANU*WY3*WY3)
0481 SMY1=(R*T2Y+.5*WY1*WY1+ANU*(R2*S2X+.5*R2*WX1*WX1))
0482 SMY2=WY1*WY3+ANU*R2*WX1*WX3
0483 SMY3=.5*(WY3**2+ANU*R2*WX3**2)
0484 SMXY1=R*S2Y+R*T2X+R*WX1*WY1
0485 SMXY2=R*(WX1*WY3+WX3*WY1)
0486 SMXY3=R*WX3*WY3
0487 WRITE(3,1007)SMX1,SMX2,SMX3,SMY1,SMY2,SMY3,SMXY1,SMXY2,SMXY3
0488 1007 FORMAT(1H0,5X,'MEMBRANE STRESS COEF. OF W02',/3(48X,3D16.8/))
0489 DO 2001 K=1,16
0490 W1=W0(K)
0491 W2=W1**2
0492 W4=W1**4
0493 W6=W1**6
0494 SMX=SMX1*W2+SMX2*W4+SMX3*W6
0495 SMY=SMY1*W2+SMY2*W4+SMY3*W6
0496 SMXY=SMXY1*W2+SMXY2*W4+SMXY3*W6
0497 2001 WRITE(3,1008)W1,SMX,SMY,SMXY
0498 1008 FORMAT(1H0,20X,'AC2 =',F6.3,22X,'SMX =',D16.8,5X,'SMY =',D16.8,5X,
1,SMXY =',D16.8)
0499 1000 CONTINUE
0500 200 CONTINUE
0501 100 CONTINUE
0502 STOP
0503 END

```

C SUBROUTINE FOR MATRIX INVERSION

```

0001 SUBROUTINE MATINV(A,N)
0002 IMPLICIT REAL*8(A-H,O-Z)
0003 DIMENSION INDEX(20,2),A(20,20)
0004 DO 108 I=1,N
0005 108 INDEX(I,1)=0
0006 II=0
0007 109 AMAX=-1.
0008 DO 110 I=1,N
0009 IF (INDEX(I,1))110,111,110
0010 DO 112 J=1,N
0011 IF(INDEX(J,1))112,113,112
0012 113 TEMP=DABS(A(I,J))
0013 IF(TEMP-AMAX)112,112,114
0014 114 IROW=I
0015 ICOL=J
0016 AMAX=TEMP
0017 112 CONTINUE
0018 110 CONTINUE
0019 IF(AMAX)225,115,116
0020 INDEX(ICOL,1)=IROW
0021 IF(IROW-ICOL)115,118,115
0022 DO 120 J=1,N
0023 TEMP=A(IROW,J)
0024 A(IROW,J)=A(ICOL,J)
0025 A(ICOL,J)=TEMP
0026 II=II+1
0027 INDEX(II,2)=ICOL
0028 PIVOT=A(ICOL,ICOL)
0029 A(ICOL,ICOL)=1.
0030 PIVOT=1./PIVOT
0031 DO 121 J=1,N
0032 121 A(ICOL,J)=A(ICOL,J)*PIVOT
0033 DO 122 I=1,N
0034 IF(I-ICOL)123,122,123
0035 123 TEMP=A(I,ICOL)
0036 A(I,ICOL)=0.
0037 DO 124 J=1,N
0038 124 A(I,J)=A(I,J)-A(ICOL,J)*TEMP
0039 122 CONTINUE
0040 GO TO 109
0041 125 ICOL=INDEX(II,2)
0042 IROW=INDEX(ICOL,1)

```

```
0043 DO 126 I=1,N
0044 TEMP=A(I,IROW)
0045 A(I,IROW)=A(I,ICOL)
0046 126 A(I,ICOL)=TEMP
0047 II=II-1
0048 225 IF(II)125,127,125
0049 115 WRITE(6,1001)
0050 1001 FORMAT(1H0,2X,11H ZERO PIVOT)
0051 127 CONTINUE
0052 RETURN
0053 END
```

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## N O M E N C L A T U R E

|  |   |
|--|---|
| D  | = flexural rigidity of plate $Eh^3/12(1 - \nu^2)$                                     |
| E  | = modulus of elasticity of plate material   |
| G  | = shear modulus   |
| R  | = plate aspect ratio $b/a$  |
| Q  | = dimensionless uniformly distributed load  |
| U, V, W  | = dimensionless displacement components parallel to $\bar{x}$ , $\bar{y}$ , $\bar{z}$ |
| $W_0$  | = dimensionless lateral displacement at centre of plate                               |
| K  | = dimensionless foundation modulus  |
| $C_i, D_i, E_i, F_i$                                       | = unknown parameters in assumed displacement functions                                |
| N  | = total number of undetermined parameters in plate displacement functions             |
| $Q_{\bar{x}}, Q_{\bar{y}}$                                 | = shearing forces per unit length of plate  |
| $N_{\bar{x}}, N_{\bar{y}}, N_{\bar{xy}}$                   | = in-plane forces per unit length of plate  |
| $M_{\bar{x}}, M_{\bar{y}}, M_{\bar{xy}}$                   | = bending and twisting moments per unit length of plate                               |
| $\bar{M}_{\bar{x}}, \bar{M}_{\bar{y}}, \bar{M}_{\bar{xy}}$ | = dimensionless bending and twisting moments per unit length of plate                 |
| a, b   | = dimensions of plate   |
| h  | = thickness of plate  |
| q  | = intensity of uniformly distributed load   |
| k  | = elastic foundation reaction per unit area per unit deflection                       |

|   |   |
|---|---|
| $\bar{x}, \bar{y}, \bar{z}$                                       | = rectangular cartesian coordinates   |
| $x, y, z$   | = dimensionless cartesian coordinates   |
| $u, v, w$   | = displacement of point on middle plane of plate<br>parallel to $\bar{x}, \bar{y}, \bar{z}$ axes respectively |
| $r$   | = radial direction of circular plate  |
| $n$   | = outwardly drawn normal to oblique side of skewed<br>plate   |
| $\alpha, \beta$   | = oblique coordinates for skewed plate  |
| $\zeta, \eta$   | = dimensionless oblique coordinates for skewed plate  |
| $\theta$  | = skew angle  |
| $\nu$   | = Poisson's ratio   |
| $\gamma_i$  | = undetermined constants  |
| $\nabla$  | = Laplacian operator  |
| $\epsilon_{\bar{x}}, \epsilon_{\bar{y}}, \gamma_{\bar{x}\bar{y}}$ | = longitudinal and shearing strain components   |
| $\sigma_{\bar{x}}, \sigma_{\bar{y}}, \tau_{\bar{x}\bar{y}}$       | = extreme fibre bending and shearing stresses   |
| $\sigma_x, \sigma_y, \tau_{xy}$                                   | = dimensionless bending stresses and shearing stresses  |
| $\sigma'_{\bar{x}}, \sigma'_{\bar{y}}, \tau'_{\bar{x}\bar{y}}$    | = membrane stresses in middle surface of plate  |
| $\sigma'_x, \sigma'_y, \tau'_{xy}$                                | = dimensionless stresses in middle surface of plate   |