Converse Theorems of Borcherds Products

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Abstract

In his paper [4], Borcherds introduced a theta lift which allowed him to lift classical modular forms with poles at cusps to automorphic forms on the orthogonal group $\mathrm{O}(2, l)$. The resulting automorphic forms, called Borcherds products, possess an infinite product expansion and have their singularities located along certain arithmetic divisors, the so-called Heegner divisors. Mainly based on the work of Bruinier, we study the question whether every automorphic form having its divisor along the Heegner divisors can be realized as a Borcherds product.

Résumé

Dans son papier [4], Borcherds a introduit un theta lift qui lui a permi de construire des formes automorphes pour le groupe orthogonal $\mathrm{O}(2, l)$ à partir de formes modulaires pour $\mathrm{SL}_2(\mathbb{Z})$ avec poles aux points paraboliques. Les formes automorphes ainsi construites, appelées produits de Borcherds, se développent en produits infinis et ont leurs singularités situées sur certains diviseurs arithmétiques, appelés diviseurs de Heegner. En se basant principalement sur les travaux de Bruinier, on étudie la question si toute forme automorphe ayant comme diviseur une combinaison linéaire des diviseurs de Heegner est un produit de Borcherds.

Dedications

To my parents.

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Contents

Introduction

Borcherds found a multiplicative lifting from classical vector–valued modular forms to meromorphic modular forms for the indefinite orthogonal group of signature $(2, l), l \geq 2$ by integrating the input functions against a suitable theta-function. This lifting yields meromorphic modular forms with some interesting properties, namely, their zeros and poles along certain arithmetically defined divisors, the so-called Heegner divisors (0.0.5), and they possess absolutely convergent infinite product expansions (called 'Borcherds product expansions'). The aim of this thesis is to provide a reference for the converse of Borcherds results. By this we mean the following: Assume that F is a meromorphic modular form for some orthogonal group whose divisor is given by Heegner divisors. Is there a weakly holomorphic form f whose Borcherds lift is F? We treat 3 different examples, for which the answer to the above question positive.

In a nutshell, "Borcherds products" or the "Borcherds lift" is a method for constructing modular forms on the orthogonal groups $\mathrm{O}(2, n)$. More precisely it is a map

$$f \mapsto \Psi_f,$$

where f is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ and a modular form on some finite index subgroup of $\mathrm{O}(2, n)$. The form Ψ_{f_1} is called a borcherds product. The lifting is multiplicative, in the sense that

$$\Psi_{f_1+f_2} = \Psi_{f_1} \times \Psi_{f_2}.$$

If the constant term of f is $a(0)$ then Ψ_f has weight $a(0)/2$. Moreover the poles and zeros of Ψ_f can be described n terms of the Fourier coefficients of f. Briefly, a modular form of weight k is a holomorphic function on the upper half plane $\mathcal{H} \to \mathbb{C}$ that satisfies the transformation law

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \text{ for } \tau \in \mathbb{H} \text{ and } \gamma \in \mathrm{SL}_2(\mathbb{Z}),$$

where the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is given by fractionnal transformations. In addition, f has a Fourier expansion of the form

$$f(\tau) = \sum_{n=-N}^{\infty} a(n)q^n, \qquad q = \exp(2\pi i\tau).$$

We give an elementary example of a modular form and its Borcherds lift. Consider the function

$$f(\tau) = 12 + 24q + 24q^4 + 24q^9 + \cdots .$$

Then f is a weight 24 modular form for the group $\mathrm{SL}_2(\mathbb{Z})$. Its Borcherds lift is the modular discriminant form defined by

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

a weight 12 modular form.

Special cases of such infinite product formulas seem to appear much earlier in the literature as the following example (Theorem 0.0.1) discovered by D. Zagier many years ago.

Let

$$\mathcal{H} = \{\tau = x + iy \in \mathbb{C} |\ y > 0\},$$

be the upper-half plane and let $j : \mathcal{H} \to \mathbb{C}$ be the classical ellipltic invariant, which when expressed as a Fourier series is given by

$$j(q) = \sum_{n=-1}^{\infty} c(n)q^n = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

here $q := \exp(2\pi i\tau)$.

Theorem 0.0.1 (Zagier)**.** *For* $0 < |p|, |q| < 1$ *one has the equality*

$$j(p) - j(q) = (p^{-1} - q^{-1}) \prod_{k,l=1}^{\infty} (1 - p^k q^l)^{c(kl)}.$$

This equality yields an infinite sequence of algebraic identities between the integer numbers $c(k),\ k \geqslant 1$. The first Non-trivial identity is

$$c(4) = c(3) + \frac{c(1)^2 - c(1)}{2}, \quad 20245856256 = 864299970 + \frac{196884^2 - 196884}{2}.$$

First we define the input functions. Let L be an even lattice of signature $(2, l)$, and denote by $L^\vee$ the dual of L. To the quadratic module $(L^\vee/L, q)$ we attach the Weil representation ρ as defined by 1.2.1.

By a holomorphic modular form of weight κ, we mean a function $f : \mathcal{H} \to \mathbb{C}[L^\vee/L]$ that satisfies:

(i) $f(M \cdot \tau) = \phi(\tau)^{2\kappa}\rho_L(M, \phi)f(\tau)$ for all (M, ϕ) in the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$,

(ii) f is holomorphic on $\mathcal{H}$,

(iii) f is holomorphic at ∞.

If we drop condition (iii), we get the so-called a weakly holomorphic modular form. Such a function f has Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in L^\vee/L} \sum_{\substack{n \in \mathbb{Z}-q(\gamma) \\ n \gg -\infty}} c(\gamma, n)\mathfrak{e}_\gamma(n\tau), \tag{0.0.1}$$

here $\mathfrak{e}_\gamma(z) := \exp(2\pi i z)\mathfrak{e}_\gamma$ with $\{\mathfrak{e}_\gamma,\ \gamma \in L^\vee/L\}$ the standard $\mathbb{C}$-basis of $\mathbb{C}[L^\vee/L]$. Note that if f is holomorphic at ∞, then the second sum is taken over $n \geqslant 0$. We denote the space of holomorphic modular forms with weight k and representation ρ_L by M_{k,ρ_L}, the space of those that are weakly holomorphic by $M^!_{k,\rho_L}$, and the space of cusp forms by S_{k,ρ_L}.

We now describe the image of f under the Borcherds lift. Let V denote the real vector space $L \otimes \mathbb{R}$ and $V_\mathbb{C} = V \otimes \mathbb{C}$ its complexification. Denote by $\mathbb{P}V(\mathbb{C})$ the projective space

$$\mathbb{P}V(\mathbb{C}) = (V(\mathbb{C}) \setminus \{0\})/\mathbb{C}^\times,$$

and let

$$V(\mathbb{C}) \to P(V(\mathbb{C})), \qquad Z \to [Z]$$

be the canonical projection. The subset

$$\mathcal{K} := \{[Z] \in \mathbb{P}V(\mathbb{C}); (Z, Z) = 0, (Z, \bar{Z}) > 0\} \tag{0.0.2}$$

is a complex projective manifold with two connected components, which are permuted by complex conjugation. We choose one component and denote it by $\mathcal{K}^+$. The action of $\mathrm{O}(V)$ on V induces an action on $\mathcal{K}$. Let $\mathrm{O}(V)^+$ be the stabilizer of $\mathcal{K}^+$, then this is exactly the connected component of the identity in $\mathrm{O}(V)$.

Let $\ell \in L$ be a primitive isotropic vector, that is a vector of norm 0, and let $\ell' \in L^\vee$ such that $(\ell, \ell') = 1$. Consider the sub-lattice $K = L \cap \ell^\perp \cap \ell'^\perp$. This is a Lorentzian lattice and $V = (K \otimes \mathbb{R}) + \mathbb{R}\ell + \mathbb{R}\ell'$. We have an isomorphism

$$w : \{Z = X + iY \in K \otimes \mathbb{C}; Y^2 > 0\} \to \mathcal{K}, \tag{0.0.3}$$

given by

$$w(Z) := [Z + \ell' - (q(Z) + q(\ell'))\ell].$$

Let $\mathbb{H}$ be the pre-image of $\mathcal{K}^+$ under this isomorphism. The action of $\mathrm{O}^+(V)$ on $\mathcal{K}^+$ induces an action on $\mathbb{H}$. Let $\mathrm{O}_d(L)$ be the discriminant kernel, that is the kernel of the action of $\mathrm{O}(L) = \{g \in \mathrm{O}(V) |\ gL = L\}$ on the discriminant group $L^\vee/L$ and put

$$\Gamma = \mathrm{O}_d(L) \cap \mathrm{O}^+(V).$$

Then, by an automorphic form on Γ of weight k and with character χ we mean a function $F : \mathbb{H} \to \mathbb{C}$, satisfying

$$F(\sigma Z) = \chi(\sigma) j(\sigma, Z)^k F(Z) \text{ for all } \sigma \in \Gamma, \tag{0.0.4}$$

here $j(\sigma, Z)$ is an automorphic factor.

Next we describe the divisors of the Borcherds products. For $\beta \in L^\vee/L$ and $m \in q(\beta) + \mathbb{Z}$ with $m < 0$. The divisor

$$H(\beta, m) = \sum_{\substack{\lambda \in \beta + L^\vee/L \\ q(\beta) = m}} \lambda^\perp$$

defines a Γ-invariant divisor on $\mathbb{H}$. Following Borcherds we call it the Heegner divisor of index (β, m).

Theorem 0.0.2 (Borcherds)**.** *Let L be an even lattice of signature $(2, l)$ with $l \geq 3$, and $\ell \in L$ a primitive isotropic vector. Let $\ell' \in L^\vee, K = L \cap \ell^\perp \cap \ell'^\perp$. Moreover, assume that K also contains an isotropic vector. Let f be a weakly holomorphic modular form of weight $k = 1 - l/2$ whose Fourier coefficients $c(\gamma, n)$ (as in 0.0.1) are integral for $n < 0$. Suppose that $c(0, 0) \in 2\mathbb{Z}$. Then there is a meromorphic function Ψ on $\mathbb{H}$ with the following properties:*

i) Ψ is a meromorphic modular form of weight $c(0, 0)/2$ for the orthogonal group Γ with some character.

ii) The divisor of $\Psi(Z)$ on $\mathbb{H}$ is given by

$$\mathrm{Div}(\Psi) = \frac{1}{2} \sum_{\beta \in L^\vee/L} \sum_{\substack{m \in \mathbb{Z} + q(\beta) \\ m < 0}} c(\beta, m) H(\beta, m). \tag{0.0.5}$$

(The multiplicities of $H(\beta, m)$ are 2, if $2\beta = 0$ in $L^\vee/L$, and 1, if $2\beta \neq 0$ in $L^\vee/L$. Note that $c(\beta, m) = c(-\beta, m)$ and $H(\beta, m) = H(-\beta, m)$.)

iii) Let $W \subset \mathbb{H}$ be a Weyl chamber with respect to f and $m_0 = \min\{n \in \mathbb{Q}; c(\gamma, n) \neq 0\}$. On the set of $Z \in \mathbb{H}$, which satisfy $q(Y) > |m_0|$, and which belong to the complement of the set of poles of $\Psi(Z)$, the function $\Psi(Z)$ has the normally convergent Borcherds product expansion

$$\Psi(Z) = Ce((\rho_f(W), Z)) \prod_{\substack{\lambda \in K^\vee \\ (\lambda, W) > 0}} \prod_{\substack{\delta \in L_0^\vee/L \\ p(\delta) = \lambda + K}} \left(1 - e\left((\delta, \ell') + (\lambda, Z)\right)\right)^{c(\delta, q(\lambda))}.$$

Here $e(z) := \exp(2\pi i z)$ and C is a constant of absolute value 1 whereas $\rho_f(W) \in K \otimes \mathbb{R}$ denotes an explicit vector depending on W and f. The projection p and the sub-lattice L_0 are as defined by 2.2.8.

The automorphic form in Theorem 0.0.2 is obtained by integrating a weakly modular form f against the Siegel theta function $\Theta_L(\tau, v)$ of the lattice L. Let $\mathrm{Gr}(L)$ be the Grassmanian of L, that is

$$\mathrm{Gr}(L) = \{v \subset L \otimes \mathbb{R};\ \dim v = 2, v \text{ is oriented }, q|_v \text{ is positive definite }\}.$$

If $v \in \mathrm{Gr}(L)$, we write $v^\perp$ for the orthogonal complement of v in V so that $V = v \oplus v^\perp$. This is defined as follows. For any vector $x \in V$, let x_v (resp. $x_{v^\perp}$) be the orthogonal projection of x to v (resp. $v^\perp$). The Siegel theta function of L is defined by

$$\Theta_L(\tau, v) = \sum_{\gamma \in L^\vee/L} \mathfrak{e}_\gamma \sum_{\lambda \in \gamma + L} e(\tau q(\lambda_v) + \bar{\tau} q(\lambda_{v^\perp})). \tag{0.0.6}$$

Let $f : \mathcal{H} \to \mathbb{C}[L^\vee/L]$–valued modular form with weight $1 - \frac{l}{2}$ then its theta integral is defined by

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} < f(\tau), \Theta(\tau, v) > y \frac{dxdy}{y^2},$$

where as usual $\tau = x + iy$. This integral diverges wildly and in order to make sense of it, it has to be regularized as follows. Let $\mathcal{F} = \{\tau;\ |\tau| \geq 1, \Re(\tau) \leq 1/2\}$ be the standard fundamental domain, and $\mathcal{F}_u$ be the truncated fundamental domain

$$\mathcal{F}_u = \{\tau = x + iy \in \mathcal{F}; y \leq u\} \qquad (u \in \mathbb{R}_{>0}).$$

Suppose that the limit $\lim_{u\to\infty} \int_{\mathcal{F}_u} < f(\tau), \Theta_L(\tau, v) > \frac{dxdy}{y^{1+s}}$ exists for $\Re(s) \gg 0$ and can be continued to a meromorphic function defined for all complex s. Then the regularized theta integral of F is defined to be the constant term of the Laurent expansion of this limit at $s = 0$, denoted $\mathcal{C}_{s=0}[-]$.

$$\int^{\mathrm{reg}} < f(\tau), \Theta(\tau, v) > y \frac{dxdy}{y^2} := \mathcal{C}_{s=0}\left[\lim_{u\to\infty} \int_{\mathcal{F}_u} < f(\tau), \Theta_L(\tau, v) > \frac{dxdy}{y^{1+s}}\right]. \tag{0.0.7}$$

With the above notation, define a function on $\mathbb{H}$ by

$$\Phi(Z, f) = \int^{\mathrm{reg}} < f(\tau), \Theta(\tau, Z) > y \frac{dxdy}{y^2}.$$

Note that $\mathrm{Gr}(L)$ and $\mathbb{H}$ are isomorphic; thus the use of the variable $Z = X + iY \in \mathbb{H}$.

Then up to an additive constant, there is a unique function $\Psi(Z, f)$ defined on $\mathbb{H}$ such that

$$\Phi(Z, f) = c(0,0) \log |Y|^2 = -2 \log |\Psi(Z, f)|.$$

This works because the second Cousin problem is universally solvable on $\mathbb{H}$. The resulting function Ψ satisfies the properties of Theorem 0.0.2 and is called a Borcherds lift of f.

Now consider the question whether there is a converse to the above construction, namely, assume that Ψ is a meromorphic modular form for the group Γ whose divisor is given by 0.0.5. Is there a weakly holomorphic form f whose Borcherds lift is Ψ?

It turns out that any modular form, whose divisor is a linear combination of Heegner divisors, is given by the regularized theta lifting of a Harmonic Maass form with singularity at ∞. Such Maass form can be constructed using Eisenstein series, see [8]. One might expect that this Maass form actually has to be weakly holomorphic. However, this seems to be a non trivial question.

Here we mean by a Maass Harmonic form of weight k a smooth function $f : \mathcal{H} \to \mathbb{C}[L^\vee/L]$ that satisfies:

(i) it satisfies the transformation law $f(M\tau) = \phi(\tau)^{2k} \rho_L(M, \phi) f(\tau)$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$;

(ii) it is annihilated by the hyperbolic Laplacian in weight k (*i.e.* $\Delta_k f = 0$);

(iii) there is a $C > 0$ such that for any cusp $s \in \mathbb{Q} \cup \infty$ of $\mathrm{Mp}_2(\mathbb{Z})$ and $(\delta, \phi) \in \mathrm{Mp}(\mathbb{Z})$ with $\delta\infty = s$ the function $f_s(\tau) = \phi(\tau)^{-2k} \rho_L^{-1}(\delta, \phi) f(\delta\tau)$ satisfies $f_s(\tau) = O(e^{Cy})$ as $y \to \infty$.

Thus, any weakly holomorphic modular form is a harmonic Maass form. A harmonic Maass form $f \in H_{k,L}$ of weight k has a Fourier expansion of the form [11]

$$f(\tau) = \sum_{\substack{\mu \in L^\vee/L \\ m \in q(\mu) + \mathbb{Z}}} c^+(m, \mu) q^m \mathfrak{e}_\mu + \sum_{\substack{\mu \in L^\vee/L \\ m \in q(\mu) + \mathbb{Z} \\ m < 0}} c^-(m, \mu) \Gamma(1 - k, -4\pi m y) q^m \mathfrak{e}_\mu, \qquad (0.0.8)$$

where $\Gamma(s, t)$ is the incomplete Gamma function.

To answer the question, we exploit the relation between Maass forms and weakly holomorphic modular forms. It is given the differential operator

$$\xi_k(f) = 2iy^k \overline{\frac{\partial}{\partial \bar{\tau}} f},$$

which takes a weight k harmonic Maass form f to a weakly holomorphic form in weight $2-k$ transforming with the dual of ρ_L. Its kernel is exactly the space of weakly holomorphic modular forms in weight k.

We let $H_{k,L}$ be the subspace of those harmonic Maass forms of weight k with representation ρ_L for $\mathrm{Mp}_2(\mathbb{Z})$ for which $\xi_k(f)$ is a cusp form. We have the exact sequence

$$0 \to M^!_{k,L} \to H_{k,L} \to S_{2-k,L^-} \to 0,$$

where L^- denotes the same lattice L but equipped with the quadratic form $-q$. Thus in order to answer the above question it suffices to pick a suitable Maass form f whose regularized theta integral is Ψ and show that $\xi_k(f)=0$; a process which we now describe.

Let $\mathcal{H}^{1,1}(\Gamma\backslash\mathbb{H})$ be the space of square integrable harmonic $(1,1)$-forms on $\Gamma\backslash\mathbb{H}$. We define a linear map $\Lambda : S_{\kappa,L} \to \mathcal{H}^{1,1}(X_\Gamma)$ as follows. If $g \in S_{\kappa,L}$, we can pick a harmonic Maass form $f \in H_{2-\kappa,L^-}$ with vanishing constant term $c^+(0,0)$ such that $\xi(f)=g$, and define

$$\Lambda(Z,g) = \partial\partial^c\Phi(Z,f). \tag{0.0.9}$$

We define a subspace of $H_{2-\kappa,L^-}$ by

$$N_{2-\kappa,L^-} = \{f \in H_{2-\kappa,L^-} :\ \mathrm{Div}(f) = 0 \in \mathrm{Div}(\Gamma\backslash\mathbb{H}) \otimes \mathbb{C}\}. \tag{0.0.10}$$

Denote by $S^+_{\kappa,L}$, the orthogonal complement of $\xi(N_{2-\kappa,L^-})$ with respect to the Petersson scalar product and restrict Λ to $S^+_{\kappa,L}$. Then we have

Theorem 0.0.3 (4.2 [9])**.** *We keep the same notation as above. Suppose that $l \geq 2$ and that l is greater than the Witt rank of V. The following are equivalent:*

i) The map $\Lambda : S^+_{\kappa,L} \to \mathcal{H}^{1,1}(\Gamma\backslash\mathbb{H})$ is injective.

ii) Every meromorphic modular form Ψ with respect to Γ whose divisor is a linear combination of special divisors as in 0.0.5 is (up to a non-zero constant factor) the Borcherds lift $\Psi(z,f)$ of a weakly holomorphic modular form $f \in M^!_{2-\kappa,L^-}$ with integral principal part.

Using this criterion, Bruinier proved that if the lattice L splits a hyperbolic plane, then every meromorphic modular form Ψ with respect to Γ whose divisor is a linear combination of special divisors $H(m,\mu)$ is (up to a non-zero constant factor) the Borcherds lift $\Psi(Z,f)$ of some $f \in M^!_{1-l/2,L_0^-}$ thanks to the following theorem.

Theorem 0.0.4 (5.3 [9])**.** *Assume that $L \cong D \oplus U(N) \oplus U$ for some positive definite even lattice D of dimension $l-2 \geq 1$ and some positive integer N. Then the map $\Lambda : S^+_{\kappa,L} \to \mathcal{H}^{1,1}(\Gamma\backslash\mathbb{H})$ is injective.*

The same is true for lattices of prime level.

Theorem 0.0.5 (Theorem 1.4 [9]). *Let L be an even lattice of prime level p and signature $(l, 2)$. Assume that $l \geq 3$ and that the Witt rank of L is 2. Then there exists a sublattice $L_0 \subset L$ of level p such that every meromorphic modular form F with respect to Γ whose divisor is a linear combination of special divisors $Z(m, \mu)$ is (up to a non-zero constant factor) the Borcherds lift $\Psi(z, f)$ of some $f \in M^!_{1-l/2, L_0^-}$.*

One might ask what happen if the level of L is a product of two primes or more generally a square free integer. Is the map Λ still injective or can a converse hold in this case? To the best of our knowledge, this is still an open question.

Chapter 1

Preliminaries

In this chapter we recall some basic definitions and properties of lattices and vector–valued modular forms. We give references to places in the literature where similar definitions, theorems and calculations can be found.

1.1 Lattices

A lattice is a free $\mathbb{Z}$–module of finite rank, equipped with a symmetric $\mathbb{Z}$–valued bilinear form $(,)$. A lattice L is called even if the norm of every vector is an integer, that is

$$q(v) = \frac{1}{2}(v, v) \in \mathbb{Z}$$

for all $v \in L$; otherwise it is called odd. The property of being even or odd is known as the type of the lattice. If the bilinear form is non–degenerate we say that the lattice is non–degenerate. Unless otherwise stated all lattices in this dissertation can be assumed to be non–degenerate. The signature of the lattice L is the signature of the vector space $L \otimes \mathbb{R}$ equipped with the natural $\mathbb{R}$–valued bilinear form. The signature is denoted by (b^+, b^-) where b^+ is the dimension of the maximal positive definite subspace of $L \otimes \mathbb{R}$ and b^- is the dimension of the maximal negative definite subspace. If the signature is $(1, b^-)$ we call the lattice Lorentzian. We also define

$$\operatorname{sgn}(L) = b^+ - b^-$$

if the lattice L has signature (b^+, b^-).

Example 1.1.1. *Many important lattices can be found in Chapter* 4 *of [14]. These include the Leech lattice* (Λ_{24})*, Barnes–Wall lattice* (BW_{16})*, Coxeter–Todd lattice* (K_{12})*, and the* E_8 *lattice.*

Definition 1.1.2. *The dual lattice of a lattice L is*

$$L^\vee = \{u \in L \otimes \mathbb{Q} : (u,v) \in \mathbb{Z} \text{ for all } v \in L\}.$$

The dual lattice comes naturally equipped with a $\mathbb{Q}$–valued bilinear form. Note that $L \subset L^\vee$, if $L = L^\vee$ then L said to be *unimodular*.

Example 1.1.3. *An important example of an unimodular lattice which will appear later in this thesis is the lattice $\mathbb{Z}^2$ with the quadratic form $q((a,b)) = ab$. This is up to an isometry the unique unimodular even lattice of signature $(1,1)$. We will call any lattice that is isomorphic to U a Hyperbolic Plane.*

If L is a lattice equipped with a quadratic form q, and N is a non-zero integer, we write $L(N)$ for the lattice given by L as a $\mathbb{Z}$-module, but equipped with the rescaled quadratic form $N \cdot q$. Note that $L(N)^\vee = \frac{1}{N}L^\vee$.

The *discriminant form* of L is the finite abelian group $A = L^\vee/L$ [14]. The order of this group is the *determinant* of the lattice L. For a prime integer p, the *p–rank* of L is the order of the Sylow p–subgroup of A. A lattice L is called *elementary* if the discriminant form is an elementary abelian group, that is a product of copies of $\mathbb{Z}_p$ for some prime p.

A tuple (A,q) consisting of a finite abelian group A (i.e. a $\mathbb{Z}$-module) and a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form is called a discriminant form (See [19]). We will often abuse the notation and write A instead of (A,q).

Definition 1.1.4. *Let (A,q) and (A',q') be discriminant forms.*

i) (A,q) and (A',q') are said to be isometric if there is an $\mathbb{Z}$-linear bijective map $\alpha : A \to A'$ such that

$$q'(\alpha(x)) = q(x) \text{ for all } x \in A.$$

The map α is called an isometry.

ii) The level of a discriminant form A is defined to be

$$N := min\{n \in \mathbb{N} : nq(\lambda) = 0 + \mathbb{Z},\ \forall \lambda \in A\}.$$

An example of a discriminant form is the quotient $L^\vee/L$ together with the quadratic form q modulo $\mathbb{Z}$. Conversely every discriminant form can be obtained this way.

Proposition 1.1.5. *If A is any discriminant form then there exists an even, non–degenerate lattice L such that*

$$A = L^\vee/L.$$

Proof. This is a well known result. See for example [19] or the paper of Wall [23] □

If A is a discriminant form and $H \subset A$ is a subgroup, we let

$$H^{\perp} = \{a \in A | (a,h) = 0 \text{ for all } h \in H\}$$

be the orthogonal complement of H. If H is a totally isotropic subgroup, that is, we have $Q(h) = 0$ for all $h \in H$, then $B = H^{\perp}/H$ together with the induced quadratic form q defines a discriminant form. We have $|A| = |B||H|^2$ and $\operatorname{sgn}(A) = \operatorname{sgn}(B)$.

Many of the modular forms we use take values in the vector space $\mathbb{C}[L^{\vee}/L]$. The standard basis elements for this vector space are denoted by $\mathfrak{e}_{\gamma}$. The discriminant form comes naturally equipped with a $\mathbb{Q}/\mathbb{Z}$–valued bilinear form in the following manner: Extend the $\mathbb{Z}$–valued valued bilinear form $q : L \to \mathbb{Z}$ to $L^{\vee} \subset L \otimes \mathbb{Q}$, then reduce it modulo $\mathbb{Z}$. If L is an even lattice then $q(x)$ is a well–defined $\mathbb{Q}/\mathbb{Z}$–valued quadratic form on $L^{\vee}/L$. So, for an even lattice L, every element of $L^{\vee}/L$ has a well defined norm in $\mathbb{Q}/2\mathbb{Z}$ given by $2q(x)$ for any representative x of the coset. If L is an even lattice the signature is determined modulo 8 by Milgram's formula (cf. [17]):

$$\sum_{\gamma \in L^{\vee}/L} e(\gamma^2/2) = \sqrt{|L^{\vee}/L|}\; e(\operatorname{sgn}(L)/8), \tag{1.1.1}$$

where

$$e(x) = \exp(2\pi i x).$$

The sum on the left hand side is finite. The *level* of a lattice L is the smallest positive integer N such that $Nq(\gamma)$ is integral for all $\gamma \in L^{\vee}$.

We define an inner product on $\mathbb{C}[L^{\vee}/L]$ that is linear in the first argument, anti–linear in the second and such that

$$\langle \mathfrak{e}_{\beta}, \mathfrak{e}_{\gamma} \rangle = \delta_{\beta,\gamma} = \begin{cases} 1 & \text{if } \beta = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

For any $n \in \mathbb{Z}$ and $A = L^{\vee}/L$ we define A^n to be the set of n–th powers of elements of A and A_n to be the set of the elements of order n. It is clear that A^n and A_n are orthogonal to each other. We define A^{n*} to be the elements $\beta \in A$ such that $(\beta, \gamma) \equiv nq(\gamma) \pmod 1$ for all $\gamma \in A_n$. Clearly, when A^{n*} is non–empty, A^{n*} is a coset of A^n. See [6] for more about these sets.

Definition 1.1.6. *Let L be a lattice and $L^{\vee}$ its dual lattice. A primitive vector of $L^{\vee}$ is one that is not an integer multiple of any smaller vector. In other words λ is primitive if and only if $\mathbb{Q}\lambda \cap L^{\vee} = \mathbb{Z}\lambda$.*

A *root* of a Lorentzian lattice L is any vector in $L^\vee$ of negative norm for which the reflection in the plane orthogonal to it is an automorphism of the lattice L. The *reflection* in a plane orthogonal to a vector v is denoted σ_v, and defined by $\sigma_v(x) = x - 2\frac{(v,x)}{v^2}v$, where v^2 is shorthand for (v,v).

If L is a positive definite lattice then its *theta function* is

$$\theta_L(\tau) = \sum_{\gamma\in L} q^{\gamma^2/2}.$$

If we regard q as $e^{2\pi i\tau}$, for τ in the upper half plane, then as a consequence of the Poisson summation formula, the theta function is a modular form of weight $\dim(L)/2$ for the group $\Gamma_0(N)$, where N is the level of the lattice L. In the case of an indefinite lattice the above sum will not converge, so we have to define the theta function in a different way. We will see how to do this in the next section, cf. [2].

Example 1.1.7. *The theta function for the Leech lattice, that is the unique even unimodular lattice in the* 24*–dimensional Euclidian space for which the length of every non zero vector is at least* 2*, is*

$$\theta_\Lambda(\tau) = 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + \cdots,$$

and is a modular form of weight 12 *for the group* $\mathrm{SL}_2(\mathbb{Z})$*, see [14].*

1.2 Vector–Valued Modular Forms

In this section we recall the definition of a vector–valued modular form [5]. When dealing with theta functions of lattices, half–integral weight modular forms naturally occur. This means keeping careful track of exactly which sign the square root should take. One of the easiest ways to do this is to use a double cover of the modular group. If we wanted to deal with more general weights we could use the universal covering group instead, but we don't need this here. $\mathrm{SL}_2(\mathbb{R})$ has a non–trivial double cover $\mathrm{Mp}_2(\mathbb{R})$ (the *metaplectic group*), realized by the two choices of holomorphic square roots of $\tau \to c\tau + d$ for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Elements of this group are pairs

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \pm\sqrt{c\tau + d} \right).$$

The matrix is in $\mathrm{SL}_2(\mathbb{R})$ and τ is a formal complex variable in the upper half-plane $\mathcal{H}$ (in other words, $\Im(\tau) > 0$). The multiplication of two elements of $\mathrm{Mp}_2(\mathbb{R})$ is given by

$$(\alpha, \phi_1(\tau))(\beta, \phi_2(\tau)) = (\alpha\beta, \phi_1(\beta(\tau))\phi_2(\tau)).$$

The matrices act on the upper half plane as Möbius transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : \tau \longmapsto \frac{a\tau + b}{c\tau + d}.$$

The group $\mathrm{Mp}_2(\mathbb{Z})$ is defined to be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. It is generated by two elements T and S, where

$$T = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 1 \right), \qquad S = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \sqrt{\tau} \right).$$

One has the relations $S^2 = (ST)^3 = Z$, $Z^4 = 1$, where

$$Z = \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, i \right)$$

is the standard generator of the centre of $\mathrm{Mp}_2(\mathbb{Z})$.

Throughout let L be an even lattice of signature (b^+, b^-). There is a unitary representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L^\vee/L]$. If we denote the standard basis of $\mathbb{C}[L^\vee/L]$ by $(\mathfrak{e}_\gamma)_{\gamma \in L^\vee/L}$, then ρ_L can be defined by the action of the generators $S, T \in \mathrm{Mp}_2(\mathbb{Z})$ as follows

$$\begin{aligned} \rho_L(T) : \mathfrak{e}_\gamma &\longmapsto e\left(\tfrac{1}{2}(\gamma,\gamma)\right) \mathfrak{e}_\gamma, \\ \rho_L(S) : \mathfrak{e}_\gamma &\longmapsto \frac{e(-\operatorname{sgn}(L)/8)}{\sqrt{|L^\vee/L|}} \sum_{\delta \in L^\vee/L} e(-(\gamma,\delta)) \mathfrak{e}_\delta. \end{aligned} \tag{1.2.1}$$

This is the Weil representation attached to the quadratic module $(L^\vee/L, q)$. It factors through the finite group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ + b^-$ is even, and through a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ + b^-$ is odd, where N is the level of L,see [11]. Note that

$$\rho_L(Z)\mathfrak{e}_\gamma = i^{b^- - b^+} \mathfrak{e}_{-\gamma}.$$

Let $\beta, \gamma \in L^\vee/L$, then the *$\beta, \gamma$–coefficient* of the Weil representation, denoted by $\rho_{\beta\gamma}$, is given by

$$\rho_{\beta\gamma}(M) = \langle \rho_L(M)\mathfrak{e}_\gamma, \mathfrak{e}_\beta \rangle,$$

where $M \in \mathrm{Mp}_2(\mathbb{Z})$. There is a formula for the value of the β, γ–coefficient of the Weil representation for a general element of $\mathrm{Mp}_2(\mathbb{Z})$.

Proposition 1.2.1. (Shintani [22]) *Let $\beta, \gamma \in L^\vee/L$ and*

$$M = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \sqrt{c\tau + d} \right)$$

be an element of $\mathrm{Mp}_2(\mathbb{Z})$*. The* β,γ*–coefficient of the Weil representation is given by*

$$\delta_{\beta,a\gamma}\sqrt{i}^{\;\mathrm{sgn}(L)(\mathrm{sgn}(d)-1)}e\left(\frac{1}{2}ab\beta^2\right)$$

if $c=0$*, and by*

$$\frac{\sqrt{i}^{\;-\mathrm{sgn}(L)\,\mathrm{sgn}(c)}}{|c|^{\dim(L)/2}\sqrt{|L^\vee/L|}}\sum_{r\in L/cL}e\left(\frac{a(\beta+r)^2-2(\beta+r,\gamma)+d\gamma^2}{2c}\right)$$

if $c\neq 0$*.*

Notice that in the above formulas we are taking the square root of i according to the branch chosen by the element of $\mathrm{Mp}_2(\mathbb{Z})$.

Let $\kappa\in\frac{1}{2}\mathbb{Z}$ and f be a $\mathbb{C}[L^\vee/L]$-valued function on $\mathcal{H}$. For $(M,\phi)\in\mathrm{Mp}_2(\mathbb{Z})$ define the Peterson slash operator $(f|_\kappa(M,\phi)):\mathcal{H}\to\mathbb{C}$ by

$$(f|_\kappa(M,\phi))(\tau)=\phi(\tau)^{-2\kappa}\rho_L(M,\phi)^{-1}f(M\tau). \tag{1.2.2}$$

One can check that

$$(f|_\kappa(M_1,\phi_1))|_k(M_2,\phi_2)=f|_k(M_1,\phi_1)(M_2,\phi_2).$$

Thus $((M,\phi),f)\mapsto f|_\kappa(M,\phi)$ defines an action of $\mathrm{Mp}_2(\mathbb{Z})$ on functions $f:\mathcal{H}\to\mathbb{C}[L^\vee/L]$.

We denote by ρ_L^* the dual representation of ρ_L. If we think of $\rho_L(M,\phi)$ as a complex matrix, then $\rho_L^*(M,\phi)$ is just the complex conjugate of $\rho_L(M,\phi)$. We also define a dual action of $\mathrm{Mp}_2(\mathbb{Z})$ on functions $f:\mathcal{H}\to\mathbb{C}[L^\vee/L]$, by

$$(f|_\kappa^*(M,\phi))(\tau)=\phi(\tau)^{-2\kappa}\rho_L^*(M,\phi)^{-1}f(M\tau). \tag{1.2.3}$$

Assume that $f:\mathcal{H}\to\mathbb{C}[L^\vee/L]$ is a holomorphic function which is invariant under the $|_\kappa^*$-operation of $T\in\mathrm{Mp}_2(\mathbb{Z})$. We denote the components of f by f_γ, so that $f=\sum_{\gamma\in L^\vee/L}f_\gamma\mathfrak{e}_\gamma$. The invariance of f under T implies that the functions $e(q(\gamma)\tau)f_\gamma$ are periodic of period 1. Thus f has a Fourier expansion

$$f(\tau)=\sum_{\gamma\in L^\vee/L}\sum_{n\in\mathbb{Z}-q(\gamma)}c(\gamma,n)e(n\tau)\mathfrak{e}_\gamma$$

with Fourier coefficients $c(\gamma,n)=\displaystyle\int_0^1 f_\gamma(\tau)e(-n\tau)dx$, where we have written $\tau=x+iy$. Using the abbreviation $\mathfrak{e}_\gamma(\tau):=e(\tau)\mathfrak{e}_\gamma$ we may write

$$f(\tau)=\sum_{\gamma\in L^\vee/L}\sum_{n\in\mathbb{Z}-q(\gamma)}c(\gamma,n)\mathfrak{e}_\gamma(n\tau).$$

By means of the scalar product in $\mathbb{C}[L^\vee/L]$ the coefficients can be expressed by

$$c(\gamma,n)=\int_0^1 <f(\tau),\mathfrak{e}_\gamma(n\bar{\tau})>dx. \tag{1.2.4}$$

Definition 1.2.2. *Let* $\kappa \in \frac{1}{2}\mathbb{Z}$. *A function* $f : \mathcal{H} \to \mathbb{C}[L^\vee/L]$ *is called vector–valued modular form of weight* κ *of type* ρ_L^* *if*

i) $f|_\kappa^*(M,\phi) = f$ *for all* $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$,

ii) f *is holomorphic on* $\mathcal{H}$,

iii) f *is holomorphic at the cusp* ∞.

Condition (iii) means that f *has Fourier expansion of the form*

$$f(\tau) = \sum_{\gamma \in L^\vee/L} \sum_{\substack{n \in \mathbb{Z} - q(\gamma) \\ n \geqslant 0}} c(\gamma, n)\mathfrak{e}_\gamma(n\tau).$$

Moreover, if $c(\gamma, 0) = 0$ *for all* γ, *then* f *is called a cusp form. The* $\mathbb{C}$*-vector space of modular forms of weight* κ *with respect to* ρ_L^* *and* $\mathrm{Mp}_2(\mathbb{Z})$ *is denoted by* $M_{\kappa,L}$ *and the space of cusp forms by* $S_{\kappa,L}$.

1.3 Maass–Poincaré Series

In this section we introduce a certain type of Maass-Poincaré series, that were defined by Bruinier in [7]. Here again we assume that L is an even lattice of signature $(b^+, b^-) = (2, l)\,, (1, l-1)$ or $(0, l-2)$ with $l \geq 3$. Let $k = 1 - l/2$ and $\kappa = 1 + l/2$.

Put

$$\widetilde{\Gamma}_\infty := \langle T \rangle = \left\{ \left(\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, 1 \right)\,;\ n \in \mathbb{Z} \right\}.$$

Let $M_{r,s}(z)$ and $W_{r,s}(z)$ be the Whittaker functions as defined in [1, chapter 13]. These are two linearly independent solutions to the Whittaker differential equation

$$\frac{d^2 f}{dz^2} + \left(-\frac{1}{4} + \frac{r}{z} - \frac{s^2 - 1/4}{z^2} \right) f = 0.$$

The functions $M_{r,s}(z)$ and $W_{r,s}(z)$ are related by the following relation from [1, chapter 13]:

$$W_{r,s}(z) = \frac{\Gamma(-2s)}{\Gamma(1/2 - r - s)} M_{r,s}(z) + \frac{\Gamma(2s)}{\Gamma(1/2 - r + s)} M_{r,-s}(z),$$

where

$$\Gamma(s) = \int_0^\infty z^{s-1} e^{-z}\, dz$$

is the usual Gamma function defined for all complex numbers except non-positive integers. Hereafter, we give some asymptotic properties of the two functions, which can be found in [1, chapter 13]. As $z \to 0$ we have

$$M_{r,s}(z) \sim z^{s+1/2},$$

when $s \notin -\frac{1}{2}\mathbb{Z}_{\geq 0}$ and

$$W_{r,s}(z) \sim \frac{2s}{-r+s+1/2} z^{-s+1/2},$$

when $s \geq 1/2$. For $y \in \mathbb{R}$ and $y \to \infty$ we have

$$M_{r,s}(y) = \frac{\Gamma(1+2s)}{\Gamma(-r+s+1/2)} e^{y/2} y^{-r}(1+\mathrm{O}(y^{-1})).$$

For $s \in \mathbb{C}$ and $y > 0$ we put

$$\mathcal{M}_s(y) = y^{-k/2} M_{-k/2, s-1/2}(y). \tag{1.3.1}$$

The function $\mathcal{M}_s$ is holomorphic in s and satisfies the identity

$$\mathcal{M}_{k/2}(y) = y^{-k/2} M_{-k/2,k/2-1/2}(y) = e^{y/2}.$$

For $\beta \in L^\vee/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. The function defined by

$$\tau \mapsto \mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)$$

is invariant under T and using the above Whittaker differential equation it can be easily checked that $\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)$ is an eigenfunction of the hyperbolic laplacian Δ_k, defined by equation (3.1.1), with the eigenvalue

$$s(1-s) + \frac{k(k-2)}{4}$$

see [7].

Definition 1.3.1. *With the above notation, the Poincaré series $F_{\beta,m}$ of index (β, m) is given by*

$$F^L_{\beta,m}(\tau, s) = \frac{1}{2\Gamma(2s)} \sum_{(M,\phi)\in\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})} [\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)]|_k(M,\phi), \tag{1.3.2}$$

where $\tau = x + iy \in \mathcal{H}$ and $s = \sigma + it$ with $\sigma > 1$. If it is clear from the context, which lattice L the series 1.3.2 refers to, we just write $F_{\beta,m}$.

The series $F_{\beta,m}$ converges normally for $\tau \in \mathcal{H}$ and $\sigma > 1$. See[20, Section 7.4]

Since the action of Δ_k and the slash operator commute (See[20, Section 7.4]), then $F_{\beta,m}$ is an eigenfunction of Δ_k with the same eigenvalue as $\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)$. The asymptotic behavior of the $M_{r,s}$ implies that $F_{\beta,m}(\tau,s)$ increases exponentially as $e^{2\pi|m|y}$ as $y \to \infty$. Moreover the invariance of $F_{\beta,m}$ under $Z \in \mathrm{Mp}_2(\mathbb{Z})$ implies that $F_{\beta,m} = F_{-\beta,m}$. More properties can be found in [7, Section 1.3].

Definition 1.3.2. *A function $f : \mathcal{H} \to \mathbb{C}[L^\vee/L]$ is called a weakly holomorphic modular form of weight k, with respect to ρ_L and $\mathrm{Mp}_2(\mathbb{Z})$, if*

i) $f|_k(M,\phi) = f$ for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$,

ii) f is holomorphic on $\mathcal{H}$,

iii) f is meromorphic at ∞, i.e. f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in L^\vee/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n \gg -\infty}} c(\gamma,n)\mathfrak{e}_\gamma(n\tau).$$

The space of these weakly holomorphic modular forms is denoted by $M^!_{k,L}$. The Fourier polynomial

$$\sum_{\gamma \in L^\vee/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n < 0}} c(\gamma,n)\mathfrak{e}_\gamma(n\tau)$$

is called the principal part of f.

A weakly holomorphic modular form can be recovered from its principal part by means of Maass–Poincaré series. Indeed we have the following proposition, which will allow us later to compute the Fourier expansion of the theta lift of a vector–valued modular function.

Proposition 1.3.3 ([7] Proposition 1.12)**.** *Let $f(\tau)$ be a weakly holomorphic modular form of weight k and denote its Fourier coefficients by $c(\gamma,n)$ $(\gamma \in L^\vee/L, n \in \mathbb{Z} + q(\gamma))$. Then*

$$f(\tau) = \frac{1}{2} \sum_{\gamma \in L^\vee/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n < 0}} c(\gamma,n)F_{\gamma,n}(\tau, 1-k/2).$$

Proof. See [7] □

1.4 Automorphic forms on orthogonal groups

In this section we define automorphic forms for the orthogonal groups. To this end, let L denote a lattice of signature $(2, l)$ with $l \geq 3$. Let V denote the real vector space $L \otimes \mathbb{R}$ and $V_{\mathbb{C}} = V \otimes \mathbb{C}$ its complexification. The Grassmanian $\mathrm{Gr}(L)$ of L is

$$\mathrm{Gr}(L) = \{v \subset L \otimes \mathbb{R};\, \dim v = 2, q|_v \text{ is positive definite } \}$$

We introduce a complex structure on $\mathrm{Gr}(L)$ as follows. Denote by $\mathbb{P}V(\mathbb{C})$ the projective space

$$\mathbb{P}V(\mathbb{C}) = (V(\mathbb{C}) \setminus \{0\})/\mathbb{C}^{\times},$$

and let

$$V(\mathbb{C}) \to P(V(\mathbb{C})), \qquad Z \to [Z]$$

be the canonical projection. The cone model $\mathcal{K}$ is a subset of $\mathbb{P}V(\mathbb{C})$ defined by

$$\mathcal{K} := \{[Z] \in \mathbb{P}V(\mathbb{C}); (Z, Z) = 0, (Z, \bar{Z}) > 0\}. \tag{1.4.1}$$

It is a complex projective manifold with two connected components $\mathcal{K}^+$ and $\mathcal{K}^-$, permuted by complex conjugation. If we write $Z \in V(\mathbb{C})$ in the form $Z = X + iY$ with $X, Y \in V$, then from the definition of $\mathcal{K}$, $[Z] \in \mathcal{K}$ is equivalent to

$$(X, Y) = 0 \text{ and } X^2 = Y^2 = 0. \tag{1.4.2}$$

In other words, the real and the imaginary part of Z span a two-dimensional positive vector subspace of V. Conversely, for a given $v \in \mathrm{Gr}(L)$ we may choose an orthogonal basis X, Y as in (1.4.2) and obtain a unique $[Z] = [X + iY] \in \mathcal{K}^+$. This proves the following proposition.

Proposition 1.4.1 (Lemma 2.17 [12]). *The assignment* $[Z] \mapsto \mathbb{R}X + \mathbb{R}Y$ *defines a real analytic isomorphism* $\mathcal{K} \to \mathrm{Gr}(L)$.

The orthogonal group $\mathrm{O}(V)$ acts on $\mathcal{K}$ as $g[Z] = [gZ]$ for $g \in \mathrm{O}(V)$.

Definition 1.4.2. *We define*

$$\mathrm{O}(V)^+ := \{g \in \mathrm{O}(V) |\, g\mathcal{K}^+ = \mathcal{K}^+\}$$

to be the stabilizer of the component $\mathcal{K}^+$.

The subgroup $\mathrm{O}^+(V)$ is exactly the connected component of the identity in $\mathrm{O}(V)$. It stabilizes the other component $\mathcal{K}^-$ as well (so the choice in the definition is irrelevant), while $\mathrm{O}(V) \setminus \mathrm{O}^+(V)$ interchanges them.

As one can see the cone model has the advantage that it comes naturally equipped with a complex structure. However, we are more interested in an analogue of the upper-half plane, the standard model for the hermitian symmetric space for $\mathrm{SL}_2(\mathbb{R})$. An analogue to the upper-half plane is the tube model realization defined as follows. For that, let $\ell \in L$ be a primitive isotropic vector, that is a vector of norm 0, and let $\ell' \in L^\vee$ such that $(\ell, \ell') = 1$. Consider the sub-lattice $K = L \cap \ell^\perp \cap \ell'^\perp$. This is a Lorentzian lattice and $V = (K \otimes \mathbb{R}) + \mathbb{R}\ell + \mathbb{R}\ell'$.

Proposition 1.4.3 (Lemma 2.18 [12]). *The map*

$$w : \{Z = X + iY \in K \otimes \mathbb{C};\, Y^2 > 0\} \to \mathcal{K} \tag{1.4.3}$$

given by

$$w(Z) := [Z + \ell' - (q(Z) + q(\ell'))\ell]$$

is an isomorphism.

Proof. It can be easily checked that if $Z = X + iY \in K \otimes \mathbb{C}$ with $Y^2 > 0$ then $w(Z) \in \mathcal{K}$. Conversely assume that $[Z'] \in K$, $Z' = X' + iY'$. The subspace $\ell^\perp$ has signature $(1, l-1)$ while X', Y' span a two dimensional positive definite subspace of V. It follows that either X' or Y' is not in $\ell^\perp$ which implies that $(Z', \ell) \neq 0$. Thus $[Z'] \in \mathbb{P}V(\mathbb{C})$ has a unique representative of the form $Z + a\ell + \ell'$. The condition $(Z', Z') = 0$ implies $a = -q(Z) - q(\ell')$, and therefore $[Z'] = [Z + \ell' - (q(Z) + q(\ell'))\ell]$. Moreover, the condition $(Z', Z') > 0$ yields $Y^2 > 0$. $\square$

Let $\mathbb{H}$ be the pre-image of $\mathcal{K}^+$ under the above isomorphism. Then $\mathbb{H}$ is the tube model realization of $\mathrm{Gr}(L)$ and can be viewed as an analogue of the upper-half plane. Note that this construction depends on the choice of the vectors ℓ and ℓ'.

Now we define a function on $\mathrm{O}^+(V) \times \mathbb{H}$ by setting $j(\sigma, Z) := (\sigma(w(Z)), \ell)$. Then j is holomorphic, does not vanish on $\mathbb{H}$ and satisfies the cocycle relation

$$j(\sigma_1\sigma_2, Z) = j(\sigma_1, \sigma_2 Z) j(\sigma_2, Z).$$

Hence j is an automorphy factor for $\mathrm{O}^+(V)$ (see [12]).

Now we have all we need to define automorphic forms for subgroups of finite index in $\mathrm{O}^+(V)$.

Definition 1.4.4. *Let Γ be a subgroup of finite index in $\mathrm{O}^+(V)$, k an integer and $\chi : \Gamma \to S^1$ a unitary character on Γ. A holomorphic automorphic form on Γ of weight k and with character χ is a holomorphic function $F : \mathbb{H} \to \mathbb{C}$, satisfying*

$$F(\sigma Z) = \chi(\sigma) j(\sigma, Z)^k F(Z) \text{ for all } \sigma \in \Gamma. \tag{1.4.4}$$

Meromorphic automorphic forms are defined similarly

For the rest of this thesis we take Γ to be the intersection

$$\Gamma = \mathrm{O}_d(L) \cap \mathrm{O}^+(V),$$

where $\mathrm{O}_d(L)$ is the discriminant kernel, this is the kernel of the action of $\mathrm{O}(L) = \{g \in \mathrm{O}(V) | \, gL = L\}$ on the discriminant group $L^\vee/L$.

Next, we define the so-called Eichler transformations, which we use to expand automorphic forms as Fourier series.

Definition 1.4.5. *Let $u \in V = L \otimes \mathbb{R}$ be an isotropic vector and $v \in V$ be orthogonal to u. The Eichler transformation $E(u, v)$ is an element of $\mathrm{O}^+(V)$, defined by*

$$E(u, v)(a) = a - (a, u)v + (a, v)u - q(v)(a, u)u \tag{1.4.5}$$

for $a \in V$.

Define a set $\tilde{\mathcal{K}}^+$ by

$$\tilde{\mathcal{K}}^+ = \{W \in V(\mathbb{C}) \setminus \{0\} | \, [W] \in \mathcal{K}^+\},$$

Then a meromorphic modular form on $\mathbb{H}$ of weight k can be identified with a meromorphic function F on $\tilde{\mathcal{K}}^+$ if F satisfies the two conditions:

i) F is homogeneous of degree $-k$, *i.e.* $F(tZ) = t^{-k}F(Z)$ for any $t \in \mathbb{C} \setminus \{0\}$,

ii) F is invariant under Γ, *i.e.* $F(\sigma Z) = \chi(\sigma)F(Z)$ for any $\sigma \in \Gamma$.

Some people still use this as a definition of modular forms and it turns out that it is more convenient for our purpose (see [7]). Let F be an automorphic form for Γ, with a unitary character χ. Let $\nu \in K = L \cap \ell^\perp \cap \ell'^\perp$. Since $\ell, \nu \in L$, the Eichler transformation $E(\ell, \nu)$ is an element of Γ. It acts on $\mathbb{H}$ as $Z \mapsto Z - \nu$. In view of the above identification, this implies that any holomorphic modular form F of weight k satisfies the following: There exists a vector $\varrho \in K \otimes \mathbb{Q}$ (which is unique modulo K) such that

$$F(Z + \nu) = e((\varrho, \nu))F(Z)$$

for all $\nu \in K$. Thus F has a Fourier expansion of the form

$$F(Z) = \sum_{\lambda \in \varrho + K^\vee} a(\lambda) e\left((\lambda, Z)\right).$$

Chapter 2

Borcherds Products

2.1 Product formulas

Motivated by generalized Kac-Moody algebras, the Monster group and vertex operator algebras, R. Borcherds found a multiplicative correspondence between classical modular forms with poles at cusps and meromorphic modular forms for an orthogonal group of signature $(2, l)$. This was achieved by integrating the classical modular forms against a suitable theta-function. The resulting automorphic froms have their zeros and poles along certain arithmetical divisors called Heegner divisors and possess absolutely convergent infinite product expansions called Borcherds' products. Product formulas seem to appear much earlier in the literature. I start with an example discovered by D. Zagier many years ago.

Let $j : \mathcal{H} \to \mathbb{C}$ be the classical ellipltic invariant, which when expressed as a Fourier series is given by

$$j(q) = \sum_{n=-1}^{\infty} c(n)q^n = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n \quad ,$$

where $q := \exp(2\pi i\tau)$ for $\tau \in \mathcal{H}$.

Theorem 2.1.1. *For* $0 < |p|, |q| < 1$ *one has the equality*

$$j(p) - j(q) = (p^{-1} - q^{-1}) \prod_{k,l=1}^{\infty} (1 - p^k q^l)^{c(kl)}.$$

This equality yields an infinite sequence of algebraic identities between the integer numbers $c(k),\ k \geqslant 1$. The first non-trivial identity is

$$c(4) = c(3) + \frac{c(1)^2 - c(1)}{2}, \quad 20245856256 = 864299970 + \frac{196884^2 - 196884}{2}.$$

The second product formula is one of Borcherds' theorems.

Theorem 2.1.2 (Theorem 10.1 [4]). *Let L be an even unimodular lattice of signature $(1, l+1)$ where $l = 8, 16, \ldots$ and $v_0 \in L \otimes \mathbb{R}$ be a vector of negative norm. Let $F(\tau) = \sum_{n \gg -\infty}^{\infty} c(n)q^n \in \mathbb{Z}((q))$ be a meromorphic modular form of weight $(-l/2)$ for the group $\mathrm{SL}_2(\mathbb{Z})$ with poles only at the cusp. Then there is a unique vector $\rho \in L$ such that for $t \gg 1$, the function defined for $v \in L \otimes \mathbb{C}$with $|q(v - it\, v_0)| \ll 1$, by the formula*

$$\Psi(v) = e^{2\pi i(\rho,v)} \prod_{\substack{\gamma \in L \\ (\gamma,v_0)>0}} \left(1 - e^{2\pi i(\gamma,v)}\right)^{c(-(\gamma,\gamma)/2)}$$

can be analytically continued to a meromorphic modular form of weight $c(0)/2$ for the group $\mathrm{O}(2, 2+l)^+$. In particular, the analytic continuation of Ψ satisfies the equation

$$\Psi(2v/(v,v)) = \pm\, ((v,v)/2)^{c(0)/2}\, \Psi(v).$$

The proofs of the above product formulas, as in most of the product formulas, involve integrals, infinite series and non-holomorphic functions.

2.2 Theta correspondence

The theta correspondence or theta lift is a correspondence which relate modular forms for a group G_1 to modular forms for a second group G_2 for certain pairs of real Lie groups (G_1, G_2) (the so-called reductive dual pairs). For us G_1 will be the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ and G_2 the orthogonal subgroup $\mathrm{O}^+(V)$. Then the theta correspondence transforms a weakly holomorphic modular forms for the group $\mathrm{Mp}_2(\mathbb{Z})$ to an automorphic form for $\mathrm{O}^+(V)$. This is achieved by integrating the first one against the Siegel theta function attached to the lattice L.

2.2.1 Siegel theta function

We keep the same notation as in the previous chapter, except that L denotes an even lattice of any signature (b^+, b^-). If $v \in \mathrm{Gr}(L)$, we write $v^\perp$ for the orthogonal complement of v in V so that $V = v \oplus v^\perp$. For any vector $x \in V$ let x_v resp. $x_{v^\perp}$ be the orthogonal projection of x to v resp. $v^\perp$. Then $q(x) = \frac{1}{2}x^2 = q(x_v) + q(x_{v^\perp})$.

The Siegel theta function attached to L is defined by

$$\theta_L(\tau, v) = \sum_{\lambda \in L} e(\tau q(\lambda_v) + \bar{\tau} q(\lambda_{v^\perp})) \qquad (\tau \in \mathcal{H}, v \in \mathrm{Gr}(L)). \tag{2.2.1}$$

Following Borcherds, Bruinier introduced a more general theta function. Let $r, t \in V, \gamma \in L^\vee/L, \tau \in \mathcal{H}$ and $v \in \mathrm{Gr}(L)$. Define

$$\theta_\gamma(\tau, v; r, t) = \sum_{\lambda \in \gamma + L} e(\tau q((\lambda + t)_v) + \bar{\tau} q((\lambda + t)_{v^\perp}) - (\lambda + t/2, r)) \tag{2.2.2}$$

and

$$\Theta_L(\tau, v; r, t) = \sum_{\gamma \in L^\vee/L} \mathfrak{e}_\gamma \theta_\gamma(\tau, v; r, t). \tag{2.2.3}$$

If r and t are both 0 then one can simply write $\theta_\gamma(\tau, v)$ resp. $\Theta_L(\tau, v)$. The theta function $\Theta_L(\tau, v)$ is invariant under the action of the discriminant kernel on $v \in \mathrm{Gr}(L)$, and it transforms under the action of $\mathrm{Mp}_2(\mathbb{Z})$ on $\tau \in \mathcal{H}$ as

Theorem 2.2.1 ([5] Theorem 4.1)**.** *If* $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ *with* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *then* $\Theta_L(\tau, v; r, t)$ *satisfies*

$$\Theta_L(M\tau, v; ar + bt, cr + dt) = \phi(\tau)^{b^+} \overline{\phi(\tau)}^{b^-} \rho_L(M, \phi) \Theta_L(\tau, v; r, t).$$

To establish the above transformation law, it suffices to check it for $T = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 1 \right)$ and $S = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \sqrt{\tau} \right)$; the standard generators of $\mathrm{Mp}_2(\mathbb{Z})$. This is straightforward for the first generator T. For S, it uses the Poisson summation formula

$$\sqrt{|L^\vee/L|} \sum_{\lambda \in L} f(\lambda) = \sum_{\lambda \in L^\vee} \hat{f}(\lambda),$$

where $\hat{f}$ is the Fourier transform of f, see [5].

If F is a $\mathbb{C}[L^\vee/L]$–valued modular form transforming under $\mathrm{Mp}_2(\mathbb{Z})$ with weight $\frac{b^- - b^+}{2}$ then its theta integral is defined by

$$\int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}} < F(\tau), \Theta(\tau, v) > y^{b^+/2} \frac{dxdy}{2},$$

where $\tau = x + iy$. This integral is divergent and in order to make sense of it, it has to be regularized as follows. Let $\mathcal{F} = \{\tau;\ |\tau| \geq 1, \Re(\tau) \leq 1/2\}$ be the standard fundamental domain, and $\mathcal{F}_u$ be the truncated fundamental domain

$$\mathcal{F}_u = \{\tau = x + iy \in \mathcal{F}; y \leq u\} \qquad (u \in \mathbb{R}_{>0}).$$

Suppose that the limit $\lim_{u \to \infty} \int_{\mathcal{F}_u} < F(\tau), \Theta_L(\tau, v) > y^{b^+/2 - s} \frac{dxdy}{y^2}$ exists for $\Re(s) \gg 0$ and can be continued to a meromorphic function defined for all complex s. Then the regularized theta integral of F is defined to be the constant term of the Laurent expansion of this limit at $s = 0$, denoted $\mathcal{C}_{s=0}[-]$.

$$\int_{\mathcal{F}}^{\mathrm{reg}} < F(\tau), \Theta(\tau, v) > y^{\frac{b^+}{y^2}} \frac{dxdy}{y^2} := \mathcal{C}_{s=0}\left[\lim_{u\to\infty}\int_{\mathcal{F}_u} < F(\tau), \Theta_L(\tau, v) > y^{\frac{b^+}{2}-s}\frac{dxdy}{y^2}\right]. \tag{2.2.4}$$

2.2.2 Regularized theta integral

Let L be an even lattice of signature $(b^+, b^-) = (2, l), (1, l-1)$, or $(0, l-2)$ with $l \geq 3$. Put $k = 1 - l/2$ and

$$\sigma_0 = \max\{1, b^+/2 - k/2\}.$$

Let $\beta \in L^\vee/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. Define a function $\Phi^L_{\beta,m}(v, s)$ by a regularized theta integral, in the sense of the previous section:

$$\Phi^L_{\beta,m}(v, s) = \int_{\mathcal{F}} < F^L_{\beta,m}(\tau, s), \Theta_L(\tau, v) > y^{b^+/2}\frac{dxdy}{y^2}, \tag{2.2.5}$$

where $F^L_{\beta,m}(\tau, s)$ denotes the Maass-Poincaré series of index (β, m) defined in (1.3.2) and $\Theta_L(\tau, v)$ the Siegel theta function (2.2.3). If it is clear from the context, to which lattice L the function (2.2.5) refers, we simply write $\Phi_{\beta,m}(v, s)$. According to Theorem 2.2.1, the integrand

$$< F^L_{\beta,m}(\tau, s), \Theta_L(\tau, v) > y^{b^+/2}$$

is $\mathrm{Mp}_2(\mathbb{Z})$-invariant. Because $\Theta_L(\tau, v)$ is invariant under the action of $\mathrm{O}_d(L)$, the function $\Phi^L_{\beta,m}(v, s)$ is formally invariant under $\mathrm{O}_d(L)$ too.

Borcherds showed that $\Phi_{\beta,m}(v, s)$ is a real analytic function on $\mathrm{Gr}(L)$ with singularities along a locally finite set of codimension b^+ sub-Grassmannians. To make this more precise, denote by $\lambda^\perp$ the set

$$\lambda^\perp = \{v \subset V;\, \dim v = b^+, v \perp \lambda, v \text{ is positive definite}\},$$

and define a subset $H(\beta, m)$ of the Grassmannian $\mathrm{Gr}(L)$ by

$$H(\beta, m) = \bigcup_{\substack{\lambda\in\beta+L\\ q(\lambda)=m}} \lambda^\perp. \tag{2.2.6}$$

Proposition 2.2.2 (Proposition 2.7 [7]). *Let $v \in \mathrm{Gr}(L)\setminus H(\beta, m)$ and $s = \sigma + it \in \mathbb{C}$ with $\sigma > \sigma_0$. Then the regularized theta integral of $\Phi_{\beta,m}(v, s)$ in (2.2.5) converges and defines a holomorphic function in s. It can be continued holomorphically to*

$$\{s \in \mathbb{C}; \sigma > 1, s \neq b^+/2 - k/2\},$$

and has a pole of first order at $s = b^+/2 - k/2$.

The computations are achieved by means the Fourier expansions, and the asymptotic behavior of the Coefficients of the $F_{\beta,m}$.

Definition 2.2.3. *For* $v \in \mathrm{Gr}(L) \setminus H(\beta, m)$ *we define*

$$\Phi_{\beta,m}(v) = \mathcal{C}_{s=1-k/2}[\Phi_{\beta,m}(v, s)].$$

As noted by Bruinier, if $b^+ = 0$ or $b^+ = 1$, then $\Phi_{\beta,m}(v, s)$ is holomorphic at $s = 1 - k/2$ and one may write $\Phi_{\beta,m}(v) = \Phi_{\beta,m}(v, 1 - k/2)$.

2.2.3 Singularities of $\Phi_{\beta,m}$

We give the type of singularities of $\Phi_{\beta,m}$ as a function in v. We will say that a function f has singularities of type g at a point if $f - g$ can be continued on to a real analytic function near the point.

Theorem 2.2.4 (Theorem 2.11 [7]). *i) For fixed* $s \in \{s \in \mathbb{C};\ \Re(s) > 1, s \neq b^+/2 - k/2\}$ *the function* $\Phi_{\beta,m}(v, s)$ *is real analytic in* v *on* $\mathrm{Gr}(L) \setminus H(\beta, m)$.

ii) Let $U \subset \mathrm{Gr}(L)$ *be an open subset with compact closure and denote by* $S(\beta, m, U)$ *the finite set*

$$S(\beta, m, U) = \{\lambda \in \beta + L |\ q(\lambda) = m,\ \exists v \in \mathrm{Gr}(L) \text{ with } v \perp \lambda\}.$$

Then function $\Phi_{\beta,m}(v)$ *is real analytic on* $\mathrm{Gr}(L) \setminus H(\beta, m)$. *On* U *it has a singularity of type*

$$-2 \sum_{\lambda \in S(\beta,m,U)} \log q(\lambda_v),$$

if $b^+ = 2$, *and of type*

$$-4\sqrt{2}\pi \sum_{\lambda \in S(\beta,m,U)} |\lambda_v|,$$

if $b^+ = 1$.

2.2.4 Fourier expansion

The trick in computing the Fourier expansion of $\Phi^L_{\beta,m}$ is to reduce L to a smaller sub-lattice K, and consider the attached function $\Phi^K_{\beta,m}$ (See [4, Theorem 7.1]). For this reason, we suppose that L is a Lorentzian lattice. Then $\Phi^L_{\beta,m}$ can be written as the sum of two functions $\xi^L_{\beta,m}, \psi_{\beta,m} : \mathrm{Gr}(L) \to \mathbb{R}$, where $\xi^L_{\beta,m}$ is real analytic on the whole $\mathrm{Gr}(L)$ whereas $\psi^L_{\beta,m}$ is the restriction of a piecewise linear function with the only singularities on $H(\beta, m)$ ([7, Theorem 3.4]).

On other hand, we already have seen that the function $\Phi^L_{\beta,m}$ is real analytic on $\mathrm{Gr}(L) \setminus H(\beta, m)$. The later set is not connected. We call its components Weyl chambers of index (β, m).

Because of the discussion above we can define what is called a Weyl vector of index (β, m).

Definition 2.2.5. *Let W be a Weyl chamber of index (β, m). Then the Weyl vector index (β, m), denoted $\rho_{\beta,m}(W)$, is the unique vector in V with the property*

$$\psi^L_{\beta,m}(v) = 8\sqrt{2}\pi(v, \rho_{\beta,m}(W)).$$

The scalar product here is to be understood as follows: We identity $\mathrm{Gr}(L)$ with the set

$$V' = \{v \in V;\, v^2 = 1, (v, \ell) > 0\},$$

via

$$V' \to \mathrm{Gr}(L), \quad v' \to \mathbb{R}v'.$$

The factor of $8\sqrt{2}\pi$ is put in to give the Weyl vector good integrality properties, see [7, 3.4].

The Fourier expansion of $\psi^L_{\beta,m}$ can be worked out explicitly ([7, Theorem 3.4]). Using its Fourier expansion, Bruinier showed that $\psi^L_{\beta,m}$ is piecewise linear.

Now, suppose L is of signature $(2, l)$ with $l \geq 3$. Let $\ell \in L$ be a primitive isotropic vector and pick a vector $\ell' \in L^\vee$ with $(\ell, \ell') = 1$. Then $K = L \cap \ell^\perp \cap \ell'^\perp$ is a Lorentzian lattice. Assume that K contains primitive isotropic vector. Put $\kappa = 1 - \frac{l}{2}$ and $k = 1 + \frac{l}{2}$. As in the Lorentzian case $\Phi^L_{\beta,m}$ can be written as the sum of two functions $\xi^L_{\beta,m}$ and $\psi_{\beta,m}$, which we will describe later.

For $\kappa \in \frac{1}{2}\mathbb{Z}$, define a function $\mathcal{V}_\kappa$ in two variables $A, B \in \mathbb{R}$ and $\kappa > 0$, by

$$\mathcal{V}_\kappa(A, B) = \int_0^\infty \Gamma(\kappa - 1, A^2 y) e^{-B^2 y - 1/y} y^{-3/2} dy, \tag{2.2.7}$$

and let $\Gamma(s, t)$ be the usual incomplete Γ-function(cf.[1] p. 81), defined by

$$\Gamma(s, t) = \int_t^\infty e^{-u} u^{s-1}.$$

Let $\zeta \in L$ be a lattice vector with $(\zeta, \ell) = N$. It can be uniquely represented as

$$\zeta = \zeta_K + N\ell' + a\ell$$

with $\zeta_K \in\in K^\vee$ and $a \in \mathbb{Q}$. Then L can be written as

$$L = K \oplus \mathbb{Z}\zeta\mathbb{Z}\ell,$$

see [7, Proposition 2.2]. Consider the sub-lattice

$$L_0 = \{\lambda \in L^\vee,\ (\lambda, \ell) \equiv 0 (\operatorname{mod} N)\}$$

of L. Obviously, $L_0^\vee$ contains L as a sub-lattice. There is a projection from $L_0^\vee$ to $K^\vee$ given by

$$p(\lambda) = \lambda_K - \frac{(\lambda, \ell)}{N}\zeta_K.$$

This map has the property that $p(L) = K$. Consequently, p induces a surjective map L

$$L_0^\vee/L \to K^\vee/K \tag{2.2.8}$$

which will also be denoted by p. Note that

$$L_0^\vee/L = \{\ \in L^\vee/L,\ (\lambda, \ell) \equiv 0 (\operatorname{mod} N)\}.$$

Theorem 2.2.6 ([7] theorem 3.9)**.** *Let $v \in Gr(L) \setminus H(\beta, m)$ with $\ell_v^2 < \frac{1}{2|m|}$. Then $\Phi_{\beta,m}^L(v)$ is equal to*

$$\begin{aligned}
&\frac{1}{\sqrt{2}|\ell_v|}\Phi_{\beta,m}^K(w, 1-k/2) + C_{\beta,m} + b(0,0)\log(\ell_v^2) \\
&- 2\sum_{\substack{\lambda \in \pm p(\beta)+K \\ q(\lambda)=m}} \log(1 - e((\pm\beta, \ell') + (\lambda, \mu) + i|\lambda_w|/|\ell_v|)) \\
&- 2\sum_{\substack{\lambda \in K^\vee \setminus \{0\} \\ q(\lambda) \geq 0}} \sum_{\substack{\delta \in L_0^\vee/L \\ p(\delta)=\lambda+K}} b(\delta, q(\lambda)) \log(1 - e((\delta, \ell') + (\lambda, \mu) + i|\lambda_w|/|\ell_v|)) \\
&+ \frac{2}{\sqrt{pi}} \sum_{\substack{\lambda \in K^\vee \setminus \{0\} \\ q(\lambda) \geq 0}} \sum_{\substack{\delta \in L_0^\vee/L \\ p(\delta)=\lambda+K}} b(\delta, q(\lambda)) \sum_{n \geq 1} \frac{1}{n} e(n(\delta, \ell') + n(\lambda, \mu)) \\
&\times \mathcal{V}_{2-k}(\pi n|\lambda|/|\ell_v|, \pi n|\lambda_w|/|\ell_v|),
\end{aligned} \tag{2.2.9}$$

Here $b(\gamma, n) = b(\gamma, n, 1-k/2)$ denote the Fourier coefficients of the Poincaré series $F_{\beta,m}^L(\tau, 1-k/2)$, and $b'(0,0)$ means the derivative of $b(0,0,s)$ at $s = 1-k/2$. And $C_{\beta,m}$ is an explicit constant.

We can also give the Fourier expansion of $\Phi_{\beta,m}^L$ in the coordinate $Z = X + iY \in \mathbb{H}$, if we consider $\Phi_{\beta,m}^L$ as a function on the tube domain $\mathbb{H}$. But we need to define a more general notion of Weyl chambers as follows. Let $\beta \in L^\vee/L$ and $m \in \mathbb{Z}$ with

$m < 0$. Assume that $(\beta, \ell) \equiv 0(\operatorname{mod} N)$. Let W be a Weyl chamber of $\operatorname{Gr}(K)$ of index $(p(\beta), m)$. Then we also call the subset

$$\{Z = X + iY \in \mathbb{H}; \quad Y/|Y| \in W\} \subset \mathbb{H}$$

a Weyl chamber of index (β, m) and denote it by W. For the corresponding Weyl vector in $K \otimes \mathbb{R}$ we write $\rho_{\beta,m}(W)$.

If $(\beta, \ell) \neq 0(\operatorname{mod} N)$, then by a Weyl chamber of index (β, m) we simply mean $\mathbb{H}$ and put $\rho_{\beta,m}(W) = 0$.

Let $W \subset \mathbb{H}$ be a Weyl chamber of index (β, m). Suppose that $Z \in W$ with $q(Y) > |m|$. Then in the Z coordinate, Theorem 2.2.6 becomes

$$\begin{aligned}
&\frac{|Y|}{\sqrt{2}}\Phi^K_{\beta,m}(Y/|Y|, 1 - k/2) + C_{\beta,m} - b(0,0)log(Y^2) \\
&- 4 \sum_{\substack{\lambda \in \pm p(\beta) + K \\ (\lambda, W) > 0 \\ q(\lambda) = m}} \log |1 - e((\pm\beta, \ell') + (\lambda, Z))| \\
&- 4 \sum_{\substack{\lambda \in K^\vee \\ (\lambda, W) > 0 \\ q(\lambda) \geq 0}} \sum_{\substack{\delta \in L_0^\vee / L \\ p(\delta) = \lambda + K}} b(\delta, q(\lambda)) log|1 - e((\delta, \ell') + (\lambda, Z))| \\
&+ 2\sqrt{\pi} \sum_{\substack{\lambda \in K^\vee \\ q(\lambda) < 0}} \sum_{\substack{\delta \in L_0^\vee / L \\ p(\delta) = \lambda + K}} b(\delta, q(\lambda)) \sum_{n \geq 1} \frac{1}{n} e(n(\delta, \ell') + n(\lambda, X)) \\
&\times \mathcal{V}_{2-k}(\pi n |\lambda| |Y|, \pi n (\lambda, Y)).
\end{aligned} \tag{2.2.10}$$

The part of major interest for us in the above expansion is the one encoding the singularities of $\Phi^L_{\beta,m}$. So we isolate what seems to be an analytic term. Define a function $\xi^L_{\beta,m} : \mathbb{H} \to \mathbb{R}$ by

Definition 2.2.7.

$$\begin{aligned}
\xi^L_{\beta,m}(Z) = \frac{|Y|}{\sqrt{2}} \xi^K_{\beta,m}(Y/|Y|) - b(0,0)\log(Y^2) + 2\sqrt{\pi} \sum_{\substack{\lambda \in K^\vee \\ q(\lambda) < 0}} \sum_{\substack{\delta \in L_0^\vee / L \\ p(\delta) = \lambda + K}} b(\delta, q(\lambda)) \\
\sum_{n \geq 1} \frac{1}{n} e(n(\delta, \ell') + n(\lambda, X)) \times \mathcal{V}_{2-k}(\pi n |\lambda| |Y|, \pi n (\lambda, Y)).
\end{aligned} \tag{2.2.11}$$

Using the boundedness of the coefficients $b(\gamma, n)$ for $n <$ of the Maass–Poincaré series, and the asymptotic behavior of the function $\mathcal{V}_{2-k}$, it can be shown that $\xi^L_{\beta,m}(Z)$ is real analytic on $\mathbb{H}$.

Definition 2.2.8. *Define a function* $\psi^L_{\beta,m} : \mathbb{H} \setminus H(\beta,m) \to \mathbb{R}$ *by*

$$\psi^L_{\beta,m}(Z) = \Phi^L_{\beta,m}(Z) - \xi^L_{\beta,m}(Z).$$

Where $\xi^K_{\beta,m}$ *is given by* $\xi^K_{\beta,m} = \begin{cases} 0, & \textit{if } (\beta,\ell) \neq 0 (\operatorname{mod} N), \\ \xi^K_{p(\beta),m}, & \textit{if } (\beta,\ell) \equiv 0 (\operatorname{mod} N). \end{cases}$

Let $W \subset \mathbb{H}$ be a Weyl chamber of index (β,m), substitute $\psi^K_{\beta,m}$ by $8\sqrt{2}\pi(v, \rho_{\beta,m}(W))$ for $\rho_{\beta,m}(W)$ the corresponding Weyl vector. Then for $Z \in W$ with $q(Y) > |m|$ the Fourier expansion of $\psi^L_{\beta,m}$ can be written

$$\begin{aligned} \psi^L_{\beta,m}(Z) =& C_{\beta,m} - b(0,0)\log(Y^2) \\ & - 4 \sum_{\substack{\lambda \in \pm p(\beta)+K \\ (\lambda,W)>0 \\ q(\lambda)=m}} \log|1 - e((\pm\beta,\ell') + (\lambda,Z))| \\ & - 4 \sum_{\substack{\lambda \in K^\vee \\ (\lambda,W)>0 \\ q(\lambda)\geq 0}} \sum_{\substack{\delta \in L_0^\vee/L \\ p(\delta)=\lambda+K}} b(\delta, q(\lambda)) log|1 - e((\delta,\ell') + (\lambda,Z))|. \end{aligned} \tag{2.2.12}$$

Which we "exponentiate" to get the following infinite product.

Definition 2.2.9. *For* $Z \in \mathbb{H}$ *with* $q(Y) > |m|$, *define*

$$\begin{aligned} \Psi_{\beta,m}(Z) = e\left((\rho_{\beta,m}(W), Z)\right) & \prod_{\substack{\lambda \in \pm p(\beta)+K \\ (\lambda,W)>0 \\ q(\lambda)=m}} (1 - e((\pm\beta,\ell') + (\lambda,Z))) \\ \times \prod_{\substack{\lambda \in K^\vee \\ (\lambda,W)>0 \\ q(\lambda)\geq 0}} & \prod_{\substack{\delta \in L_0^\vee/L \\ p(\delta)=\lambda+K}} (1 - e((\delta,\ell') + (\lambda,Z)))^{b(\delta,q(\lambda))}. \end{aligned} \tag{2.2.13}$$

The above product converges normally for $Z \in \mathbb{H}$ with $q(Y) > |m|$ ([7, Lemma 3.14]). Moreover it has an analytic continuation to $\mathbb{H}$ with divisor $H(\beta,m)$ ([7, Theorem 3.15]), and satisfies

$$\log|\Psi^L_{\beta,m}(Z)| = -\frac{1}{4}(\psi^L_{\beta,m}(Z) - C_{\beta,m})$$

on $\mathbb{H} \setminus H(\beta,m)$.

2.3 Borcherds infinite products

We arrive to the main theorem of Borcherds.

Theorem 2.3.1 (Borcherds)**.** *Let* L *be an even lattice of signature* $(2, l)$ *with* $l \geq 3$, *and* $\ell \in L$ *a primitive isotropic vector. Let* $\ell' \in L^\vee, K = L \cap \ell^\perp \cap \ell'^\perp$. *Moreover, assume that* K *also contains an isotropic vector. Let* f *be a nearly holomorphic modular form of weight* $k = 1 - l/2$ *whose Fourier coefficients* $c(\gamma, n)$ *are integral for* $n < 0$. *Then*

$$\Psi(Z) = \prod_{\beta \in L^\vee/L} \prod_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} \psi_{\beta,m}(Z)^{c(\beta,m)/2}$$

is a meromorphic function on $\mathbb{H}$ *with the following properties:*

i) It is a meromorphic modular form of (rational) weight $c(0,0)/2$ *for the orthogonal group* Γ *with some multiplier system* χ *of finite order. If* $c(0,0) \in 2\mathbb{Z}$, *then* χ *is a character.*

ii) The divisor of $\Psi(Z)$ *on* $\mathbb{H}$ *is given by*

$$(\Psi) = \frac{1}{2} \sum_{\beta \in L^\vee/L} \sum_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} c(\beta, m) H(\beta, m).$$

(The multiplicities of $H(\beta, m)$ *are* 2, *if* $2\beta = 0$ *in* $L^\vee/L$, *and* 1, *if* $2\beta \neq 0$ *in* $L^\vee/L$. *Note that* $c(\beta, m) = c(-\beta, m)$ *and* $H(\beta, m) = H(-\beta, m)$.*)*

iii) Let $W \subset \mathbb{H}$ *be a Weyl chamber with respect to* f *and* $m_0 = \min\{n \in \mathbb{Q}; c(\gamma, n) \neq 0\}$. *On the set of* $Z \in \mathbb{H}$, *which satisfy* $q(Y) > |m_0|$, *and which belong to the complement of the set of poles of* $\Psi(Z)$, *the function* $\Psi(Z)$ *has the normally convergent Borcherds product expansion*

$$\Psi(Z) = Ce((\rho_f(W), Z)) \prod_{\substack{\lambda \in K^\vee \\ (\lambda, W)>0}} \prod_{\substack{\delta \in L_0^\vee/L \\ p(\delta)=\lambda+K}} \left(1 - e\left((\delta, \ell') + (\lambda, Z)\right)\right)^{c(\delta, q(\lambda))}.$$

Here C is a constant of absolute value 1, and $\rho_f(W) \in K \otimes \mathbb{R}$ denotes a vector depending on W and f. It is given by a linear combination of Weyl vector attached to Weyl chambers and can be computed explicitly.

Example 2.3.2. *Consider the Jacobi Theta function*

$$f(\tau) = 12 + 24q + 24q^4 + 24q^9 + \cdots$$

then f *is a weight* 24 *modular form for the group* $\mathrm{SL}_2(\mathbb{Z})$*. Its Borchereds lift is the modular discriminant form*

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

a weight 12 *modular form.*

Chapter 3

Converse Theorems

In this chapter, we will discuss the converse to Borcherds theorem, namely, we study cases when modular forms for the orthogonal group whose zeros and poles are supported on special divisors are the Borcherds lifts of weakly holomorphic modular forms.

3.1 Lattices that splits a hyperbolic planes

We keep the same notation as in previous chapters. In particular, let L is a lattice of signature $(l, 2)$ and let $\ell \in L$ be a primitive isotropic vector and $\ell' \in L^\vee$ with $(\ell, \ell') = 1$. Put $K = L \cap \ell^\perp \cap \ell'^\perp$. Recall that we dispose of a generalized version of the upper-half plane $\mathbb{H}$. This is one of the two connected components of

$$\{Z = X + iY \in K \otimes \mathbb{C}| \; q(Y) > 0\}.$$

With a suitable choice of basis, it can be written as

$$\mathbb{H} = \{Z = (z_1, z_2, \ldots, z_l) \in K \otimes \mathbb{C};\; y_1 > 0, y_1y_2 - y_3^2 - y_4^2 - \cdots - y_l^2\},$$

where $X = (x_1, x_2, \ldots, x_l)$ and $Y = (y_1, y_2, \ldots, y_l)$. In this basis we have [7, Section 4.1]

$$q(Y) = y_1y_2 - y_3^2 - y_4^2 - \cdots - y_l^2.$$

For $\mu = 1, \ldots, l$, define the differential operators

$$\partial_\mu := \frac{\partial}{\partial z_\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\mu} - i\frac{\partial}{\partial y_\mu}\right) \quad , \quad \partial_\mu^c := \frac{\partial}{\partial \bar{z_\mu}} = \frac{1}{2}\left(\frac{\partial}{\partial x_\mu} + i\frac{\partial}{\partial y_\mu}\right).$$

We start by defining a family of Maass forms called Harmonic Maass forms. These were introduced by Bruinier and Funke in [11], and will serve as input functions for the theta integral instead of the weakly holomorphic forms.

For $k \in \frac{1}{2}\mathbb{Z}$ and $\tau = x + iy \in \mathcal{H}$ with $x, y \in \mathbb{R}$, the weight k Hyperbolic Laplacian operator Δ_k is defined by

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{3.1.1}$$

Definition 3.1.1. *A smooth function $f : \mathcal{H} \to \mathbb{C}[L^\vee / L]$ is called a harmonic Maass form of weight k with representation ρ_L for* $\mathrm{Mp}_2(\mathbb{Z})$*, if*

(i) it satisfies the transformation law $f(M\tau) = \phi(\tau)^{2k} \rho_L(M, \phi) f(\tau)$ for all $(M, \phi) \in$ $\mathrm{Mp}_2(\mathbb{Z})$ *;*

(ii) it satisfies $\Delta_k f = 0$;

(iii) there is a $C > 0$ such that for any cusp $s \in \mathbb{Q} \cup \infty$ of $\mathrm{Mp}_2(\mathbb{Z})$ *and $(\delta, \phi) \in$* $\mathrm{Mp}(\mathbb{Z})$ *with $\delta\infty = s$ the function $f_s(\tau) = \phi(\tau)^{-2k} \rho_L^{-1}(\delta, \phi) f(\delta\tau)$ satisfies $f_s(\tau) = O(e^{Cy})$ as $y \to \infty$.*

If we compare the definition of a harmonic Maass form with the definition of a weakly holomorphic modular form, we see that we simply replaced the condition that f is holomorphic on $\mathbb{H}$ with the weaker condition that f is annihilated by Δ_k, and the meromorphicity at ∞ with the corresponding growth condition. In particular, any weakly holomorphic modular form is a harmonic Maass form. The second condition implies that f is actually real analytic.

Harmonic Maass forms are related to weakly holomorphic modular forms via the differential operator

$$\xi_k(f) = 2iy^k \overline{\frac{\partial}{\partial \bar{\tau}} f},$$

which takes a weight k harmonic Maass form f to a weakly holomorphic form in weight $2 - k$ transforming with the dual of ρ_L. Its kernel is the space of weakly holomorphic modular forms in weight k.

We let $H_{k,L}$ be the subspace of those harmonic Maass forms of weight k with representation ρ_L for $\mathrm{Mp}_2(\mathbb{Z})$ for which $\xi_k(f)$ is a cusp form. We have the exact sequence

$$0 \to M^!_{k,L} \to H_{k,L} \to S_{2-k,L^-} \to 0,$$

where L^- denotes the same lattice L but equipped with the quadratic form $-q$. This is corollary 3.8 in [11].

We shall describe the Fourier expansion of a harmonic Maass in $H_{k,L}$. Define the incomplete Gamma-function $\Gamma(s, t)$ by

$$\Gamma(s, t) := \int_t^\infty e^{-x} x^{s-1} dx.$$

A harmonic Maass form $f \in H_{k,L}$ of weight k has a Fourier expansion of the form (see [11])

$$f(\tau) = \sum_{\substack{\mu \in L^\vee/L \\ m \in q(\mu)+\mathbb{Z}}} c^+(m,\mu) q^m \mathfrak{e}_\mu + \sum_{\substack{\mu \in L^\vee/L \\ m \in q(\mu)+\mathbb{Z} \\ m<0}} c^-(m,\mu)\Gamma(1-k, -4\pi m y) q^m \mathfrak{e}_\mu, \qquad (3.1.2)$$

where $\tau = x + iy$ as above. We refer to the finite sum

$$P_f = \sum_{\substack{\mu \in L^\vee/L \\ m \in q(\mu)+\mathbb{Z} \\ m<0}} c^+(m,\mu) q^m \mathfrak{e}_\mu$$

as the principal part of f at ∞. Moreover, if all the coefficients of P_f are integers then we say that f has integral principal part. For $f \in H_{k,L}$ as in 3.1.2, the Fourier expansion of the cusp form $\xi_k(f)$ is given by (see [11, Lemma 3.1]),

$$\xi_k(f)(\tau) = \sum_{\substack{\mu \in L^\vee/L \\ m \in q(\mu)+\mathbb{Z} \\ m>0}} c^-(m,\mu)(-4\pi m)^{1-k} q^m \mathfrak{e}_\mu. \qquad (3.1.3)$$

3.1.1 A lifting into the space of $(1,1)$–differential forms

Let L be an even lattice of signature $(l,2)$. We recall that we realize the Grassmanian $\mathrm{Gr}(L)$ as the tube domain $\mathbb{H}$. This is a generalized version of the upper-half plane. The action of the orthogonal group $\mathrm{O}(V)$ on $V(\mathbb{C}) = L \otimes \mathbb{C}$ induces an action of $\Gamma \subset \mathrm{O}(V)$ on $\mathbb{H}$. This gives rise to the modular variety $X_\Gamma = \Gamma \backslash \mathbb{H}$. This is a quasi-projective algebraic variety by the theory of Baily-Borel, see [18] for an introduction to the subject. Recall that a meromorphic modular form of weight $k \in \mathbb{Z}$ for Γ is a meromorphic function Ψ on $\mathbb{H}$ which satisfies $\Psi(\gamma z) = j(\gamma, z)^k \Psi(z)$ for all $\gamma \in \Gamma$ and which is meromorphic at the boundary (Definition. 1.4.4).

We want to be able to recognize among all the these meromorphic modular forms for Γ those that are Borcherds product as described by Theorem 2.3.1. In other words if F is a meromorphic modular form for the group Γ whose zeros and poles are supported on Heegner divisors, that is,

$$\mathrm{Div}(F) = \frac{1}{2} \sum_{\mu} \sum_{m>0} c(-m,\mu) H(m,\mu), \qquad (3.1.4)$$

where $c(n,\mu) = c(n,-\mu)$ for $n < 0$ and all $\mu \in L^\vee/L$, then one would like to know if there is any weakly holomorphic form f whose Borcherds lift $\Psi(z,f)$ is F up to an additive constant.

We fix some notation. Let $\kappa = 1 + l/2$. Let $\mathcal{H}^{1,1}(X_\Gamma)$ be the space of square integrable harmonic differential forms of Hodge type $(1,1)$ on X_Γ. We use the regularized theta lift to construct a linear map $\Lambda : S_{\kappa,L} \to \mathcal{H}^{1,1}(X_\Gamma)$. The properties the map Λ shall give us a criterion to answer the above question. For $\tau = x + iy$ and $v \in \mathrm{Gr}(L)$, recall the definition of the theta function $\Theta_L(\tau, z)$ function in 2.2.3

$$\Theta_L(\tau, v) := \sum_{\lambda \in L^\vee/L} e(q(\lambda_v)\tau + q(\lambda_{v^\perp})\bar{\tau})\mathfrak{e}_\lambda.$$

Let $f \in H_{2-\kappa,L^-}$ and denote the Fourier coefficients of f by $c^\pm(m,\mu)$ as in (3.1.2). We consider the regularized theta integral

$$\Phi(v, f) = \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathcal{H}}^{reg} \langle f(\tau), \overline{\Theta_L(\tau, v)}\rangle \, \frac{dx\,dy}{y^2}. \tag{3.1.5}$$

Then the differential form $\partial\partial^c\Phi(Z, f)$ is closed, Γ-invariant, harmonic and extends to a smooth square integrable harmonic $(1,1)$-form. See [7, Theorem 5.5] and [11, Section 5.2].

We define our map $\Lambda : S_{\kappa,L} \to \mathcal{H}^{1,1}(X_\Gamma)$ as follows. If $g \in S_{\kappa,L}$, we can pick a harmonic Maass form $f \in H_{2-\kappa,L^-}$ with vanishing constant term $c^+(0,0)$ such that $\xi_k(f) = g$, and define

$$\Lambda(Z, g) = \partial\partial^c\Phi(Z, f). \tag{3.1.6}$$

This is a well defined linear map. Clearly it is linear and if $g = \xi_k(f) = 0$ then f is a weakly holomorphic modular form, and by [11, Theorem 2.12] we have $\partial\partial^c\Phi(Z, f) = -i\frac{c^+(0,0)}{2}\partial\partial^c \log(q(Y)) = 0$. Hence Λ is well defined. Moreover its Fourier expansion can be computed explicitly thanks to the following theorem.

Theorem 3.1.2 ([7], Theorem 5.9)**.** *With the same notation as above, in particular $\mathcal{V}_\kappa$ as in 2.2.7. The map $\Lambda : S_{\kappa,L} \to \mathcal{H}^{1,1}(X_\Gamma)$ has the following properties:*

(i) If $g \in S_{\kappa,L}$ with Fourier expansion $g = \sum_\mu \sum_m b(m,\mu)q^m\mathfrak{e}_\mu$, then the Fourier expansion of $\Lambda(g, v)$ is given by

$$\begin{aligned}\Lambda(v, g) = \Lambda_0(y, g) - 2^{2-\kappa}\pi^{1/2-\kappa} \sum_{\substack{\lambda \in K^\vee \\ q(\lambda) > 0}} |\lambda|^{-n} \sum_{d|\lambda} d^{n-1} \sum_{\substack{\delta \in L^\vee/L \\ \delta|L\cap\ell^\perp = \lambda/d + K}} e(d(\delta, \ell'))\\ \times b(q(\lambda)/d^2, \delta)\partial\partial^c\mathcal{V}_\kappa\left(\pi|\lambda||y|, \pi(\lambda, y)\right) e((\lambda, x)).\end{aligned}$$

Here $|\lambda| = |(\lambda,\lambda)|^{1/2}$ and the sum $\sum_{d|\lambda}$ runs through all positive integers d such that $\lambda/d \in K^\vee$. Moreover, the 0-th coefficient $\Lambda_0(y, g)$ is a certain $(1,1)$-form which is independent of x.

(ii) If $f \in H_{2-\kappa,L^-}$ *with Fourier coefficients* $c^{\pm}(m,h)$ *such that* $\xi(f) = g$*, then* $\Lambda(v, \xi(f))$ *is a square integrable harmonic representative for the Chern class in* $H^2(X_\Gamma, \mathbb{C})$ *of the divisor* $2\mathcal{Z}(f)$*.*

(iii) The map Λ *is equivariant with respect to* $\mathrm{O}(L)^+$ *i.e. for* $\gamma \in \mathrm{O}(L)^+$ *we have*

$$\Lambda(v, \gamma.g) = \Lambda(\gamma v, g).$$

Here $g \mapsto \gamma.g$ *denotes the action of* $\mathrm{O}(L)^+$ *on* $S_{\kappa,L}$ *via its action on* $\mathbb{C}[L^\vee/L]$ *through* $\mathrm{O}(L)^+ \to \mathrm{Aut}(L^\vee/L)$*.*

Proof. Note that $\partial\partial^c\Phi(Z) = \partial\partial^c\xi(Z)$. Then the first assertion follows from the Fourier expression of $\xi(Z)$ given by definition 2.2.7. For the second assertion it suffices to prove the result for any Siegel domain of $\mathbb{H}$, this is the analogue to the fundamental domain for the upper-half plane [7, Definition 4.9]. The proof uses the asymptotic behavior of both the coefficients $b(m,\delta)$ and the function $\mathcal{V}_\kappa(A,B)$. See [7, Theorem 5.5]. The third statement follows from the corresponding equivariance property of the Siegel theta function (2.2.3). □

Let $\mathrm{Div}(X_\Gamma)$ be the group of divisors of X_Γ. We define a subspace of $H_{2-\kappa,L^-}$ by

$$N_{2-\kappa,L^-} = \{f \in H_{2-\kappa,L^-} : \mathrm{Div}(f) = 0 \in \mathrm{Div}(X_\Gamma) \otimes \mathbb{C}\}. \tag{3.1.7}$$

It is an immediate consequence of Theorem 3.1.2 that the lifting Λ vanishes identically on $\xi(N_{2-\kappa,L^-})$. We denote $S^+_{\kappa,L}$ the orthogonal complement of $\xi(N_{2-\kappa,L^-})$ with respect to the Petersson scalar product. The spaces $N_{2-\kappa,L^-}$ and $S^+_{\kappa,L}$ are stable under the action of $\mathrm{O}(L)^+$. If the lattice L splits a hyperbolic plane over $\mathbb{Z}$ (*i.e.* $L = D \oplus U$ for some smaller lattice D), then $N_{2-\kappa,L^-} = 0$, but in general it can be non-zero. The constant term $c^+(0,0)$ automatically vanishes for any $f \in N_{2-\kappa,L^-}$.

We write Λ^+ for the restriction of Λ to $S^+_{\kappa,L}$. Then the question raised above regarding the surjectivity of the Borcherds lift is equivalent to the injectivity of Λ^+, as in the theorem bellow. To prove the theorem we shall need the following three lemmas.

Lemma 3.1.3 (Lemma 6.6 [10])**.** *Let* $U \subset \mathbb{C}$ *be a convex domain. Let* D *be an analytic divisor on* U*, and let* $f : U \setminus D \to \mathbb{R}$ *be a* $\mathcal{C}^2$*-function with a logarithmic singularity along* D*. If* f *is pluriharmonic (i.e.* $\partial\partial^c f = 0$*), then there exists a meromorphic function* F *on* U *such that*

$$f = \log|F|.$$

Moreover, F *has divisor* D*.*

Proof. The existence question is a question whether the second Cousin problem is solvable or not. But the second Cousin problem is solvable provided that

$H^1(U, \mathcal{O}^\times) = 0$. This is guaranteed by combining the hypothesis on U which implies $H^1(U, \mathcal{O}_U) = H^2(U, \mathbb{Z}) = 0$ and the long exact cohomology sequence

$$H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^\times) \to 2\pi i H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O}).$$

Here $\mathcal{O}$ is the sheaf of holomorphic functions on U and $\mathcal{O}^\times$ is the sheaf of holomorphic functions that vanish nowhere on U. For the details of the proof see the reference, for a further reading on the Cousin problems see for example [15]. □

Lemma 3.1.4 (Lemma 13.1 [5])**.** *Suppose that F is a meromorphic function on $\mathbb{H}$ such that $-4\log|F(Z)|$ is up to a constant equal to $\Phi(Z, f)$. Then $F(Z)$ is an automorphic form of weight 0 for Γ with some character of finite order.*

Proof. Since $\Phi(Z, f)$ is Γ-invariant, it follows from the definition of F that $|F(\sigma Z)/F(Z)| = 1$ is constant for any $\sigma \in \Gamma$. Thus $F(\sigma(Z)) = \chi(\sigma)F(Z)$ for some complex number $\chi(\sigma)$ of absolute value 1. It is obvious that χ is multiplicative, hence a character. □

Note that Borcherds used the lemmas to prove the existence of his automorphic forms rather than directly defining them by means of the Fourier expansion, which is the approach adopted in this dissertation.

Lemma 3.1.5 (Theorem 4.23 [7])**.** *Let F be a meromorphic modular form of weight r with respect to Γ, whose divisor is a linear combination of Heegner divisors*

$$\mathrm{Div}(F) = \frac{1}{2} \sum_{\beta \in L^\vee/L} \sum_{\substack{m \in q(\beta)+\mathbb{Z} \\ m<0}} c(\beta, m) H(\beta, m).$$

Define $f(Z) = \log(|F(Z)| q(Y)^{r/2})$ (with $Y = \Im(Z)$) and

$$\Phi(Z) = -\frac{1}{8} \sum_{\beta \in L^\vee/L} \sum_{\substack{m \in q(\beta)+\mathbb{Z} \\ m<0}} c(\beta, m) \Phi_{\beta,m}(Z).$$

Then $f - \Phi$ is constant.

Proof. We only give a sketch of the proof. First we want to prove that $f - \Phi$ is L^2-integrable on X_Γ. But this reduces to showing that $f - \Phi$ is L^2-integrable on every admissible Siegel domain S_t for $\mathbb{H}$, that is

$$\int_{S_t} |f(Z) - \Phi(Z)|^2 \frac{dX\,dY}{q(Y)^l} < \infty.$$

Here $dX = dx_1 \cdots dx_l$ and $dY = dy_1 \cdots dy_l$. Define two functions ξ and Ψ by

$$\xi(Z) = -\frac{1}{8} \sum_{\beta \in L^\vee / L} \sum_{\substack{m \in q(\beta)+\mathbb{Z} \\ m<0}} c(\beta, m) \xi^L_{\beta,m}(Z)$$

and

$$\Psi(Z) = \prod_{\beta \in L^\vee / L} \prod_{\substack{m \in \mathbb{Z}+q(\beta) \\ m<0}} \psi_{\beta,m}(Z)^{c(\beta,m)/2}.$$

Then F/Ψ is holomorphic nowhere vanishing on $\mathbb{H}$. Using the K-periodicity of $|F|$ (*i.e.* $|F(Z+k)| = |F(Z)|$ for every $k \in K$) with the maximum modulus principle it can be shown that there is $\lambda_1 \in K \otimes \mathbb{R}$ such that

$$\log(|F/\Psi|) < (\lambda_1, Y) + A,$$

for some constant A. Using the Fourier expansion of $\xi^K_{\beta,m}$ it can be shown that there is $\lambda_2 \in K \otimes \mathbb{R}$ such that

$$|\xi(Z)| \leq B \log(q(Y)) + (\lambda_2, Y) + C,$$

is true on S_t for some non-negative constants B and C. The two inequalities yields on S_t,

$$\begin{aligned} |f(Z) - \Phi(Z)| &= |\log(|F/\Psi|) - \xi(Z) + \frac{r}{2}\log(q(Y))| \\ &\leq |\log(|F/\Psi|)| + |\xi(Z)| + |\frac{r}{2}\log(q(Y))| \\ &\ll |(\lambda_1, Y)| + |(\lambda_2, Y)| + (B + \frac{|r|}{2}) \log\left(q(Y)\right). \end{aligned}$$

The right hand side is L^2-integrable and so is the left hand side. Next we have

$$\Delta(f - \Phi) = c,$$

where Δ is the Laplace operator and c a real constant. So $f - \Phi$ is an element of $L^2(X_\Gamma)$ and satisfies $\Delta(f - \Phi) = c$. But since X_Γ is a complete Riemann manifold of finite volume $f - \Phi$ is constant. The complete details can be found in [7, Theorem 4.23, Section 4.2, Section 4.3]. □

Theorem 3.1.6 (Theorem 5.11 [7])**.** . *Suppose that $l \geq 2$ and that l is greater than the Witt rank of V (i.e. the dimension of a maximal totally isotropic subspace of V). The following are equivalent:*

i) The map $\Lambda^+ : S^+_{\kappa,L} \to \mathcal{H}^{1,1}(X_\Gamma)$ is injective.

ii) Every meromorphic modular form F with respect to Γ whose divisor is a linear combination of special divisors as in (3.1.4) *is (up to a non-zero constant factor) the Borcherds lift $\Psi(z, f)$ of a weakly holomorphic modular form $f \in M^!_{2-\kappa,L^-}$ with integral principal part.*

Proof. $(i \Rightarrow ii)$: Let F be as in (ii). Since there is always Borcherds products for Γ in any weight, we may assume that F has weight 0. By lemma 3.1.5 there exists an $f \in H_{2-\kappa,L^-}$ with vanishing constant term $c^+(0,0)$ and integral principal part such that $\Phi(z,f)$ is equal to $-4\log|F(z)|$ up to a constant. Then $\Lambda(\xi(f)) = \partial\partial^c\Phi(z,f) = 0$, and (i) implies that $\xi(f) \in (S^+_{\kappa,L})^\perp = \xi(N_{2-\kappa,L^-})$. Hence, there exists $h \in N_{2-\kappa,L^-}$ with integral principal part such that $\xi(f) = \xi(h)$. Consequently, $\eta := f - h \in \ker\xi = M^!_{2-\kappa,L^-}$, and satisfies $Z(\eta) = Z(f) = \mathrm{Div}(F)$. The Borcherds lift of η is equal to F up to a constant factor.

Conversely, let $g \in S^+_{\kappa,L}$ such that $\Lambda^+(g) = 0$. Let $f \in H_{2-\kappa,L^-}$ with vanishing constant term such that $\xi(f) = g$. Since $\partial\partial^c\Phi(z,f) = \Lambda^+(g) = 0$, then by Lemma 3.1.3 and Lemma 3.1.4 there exists a meromorphic modular form F of weight 0 transforming under a character of finite order such that $-4\log|F| = \Phi(z,f)$. Moreover, by virtue of the last equation, the divisor of F is supported on the Heegner divisors, and therefore (ii) implies that there exists $h \in M^!_{2-\kappa,L^-}$ such that $\Phi(z,h) = \Phi(z,f)$. Consequently, $f - h \in N_{2-\kappa,L^-}$ and $g = \xi(f-h)$ (because $\xi(h) = 0$). Since $g \in \xi(N_{2-\kappa,L^-})^\perp$, we get $g = 0$. □

3.1.2 Up and Down Maps and Cyclic Isotropic Subgroups

In this subsection we review the machinery introduced by Scheithauer and Bruinier, which allowed the latter to prove the injectivity of the above map in the case when the lattice splits a hyperbolic plane.

Let $f \in M_{k,A}$ and denote by f_μ for $\mu \in A$ the components of f with respect to the standard basis of $\mathbb{C}[A]$. Let $S \subset A$ be a subset. We say that f is supported on S if $f_\mu = 0$ for all $\mu \notin S$. Let $H \subset A$ be a totally isotropic subgroup. Put $B = H^\perp/H$. Then $|A| = |B||H|^2$ and there is an injection $M_{k,B} \to M_{k,A}$, described by the following result due to Scheithauer.

Proposition 3.1.7 ([21]Theorem 4.1). *Let $g = \sum_{\nu\in B} \mathfrak{e}_\nu g_\nu \in M_{k,B}$. Then the $\mathbb{C}[A]$–valued function*

$$g\uparrow^A_H = \sum_{\mu\in H^\perp} \mathfrak{e}_\mu \, g_{\mu+H}$$

belongs to $M_{k,A}$. It is supported on $H^\perp$.

Proof. It suffices to check the transformation behavior of $g\uparrow^A_H = \sum_{\mu\in A} \mathfrak{e}_\mu \, (g\uparrow^A_H)_\mu$ under the generators T and S of $\mathrm{Mp}_2(\mathbb{Z})$. This is clear for T. By means of 1.2.1 we have

$$g_{\mu+H}|_S(\tau) = \frac{e(\mathrm{sgn}(B)/8)}{\sqrt{|B|}} \sum_{\delta\in B} e\left((\mu+H,\delta)\right) g_\delta(\tau).$$

If $\mu \in H^\perp$ then $\sum_{\gamma \in H} e\left((\mu, \gamma)\right) = |H|$ and we get

$$\begin{aligned}
(g{\uparrow}_H^A)_\mu|_S(\tau) &= g_{\mu+H}|_S(\tau) \\
&= \frac{e(\operatorname{sgn}(B)/8)}{\sqrt{|B|}} \sum_{\delta \in B} e\left((\mu + H, \delta)\right) g_\delta(\tau) \\
&= \frac{e(\operatorname{sgn}(B)/8)}{|H|\sqrt{|B|}} \cdot |H| \sum_{\delta \in B} e\left((\mu + H, \delta)\right) g_\delta(\tau) \\
&= \frac{e(\operatorname{sgn}(B)/8)}{\sqrt{|B| \cdot |H|^2}} \cdot \sum_{\delta \in H^\perp} e\left((\mu, \delta)\right) g_\delta(\tau) \sum_{\gamma \in H} e\left((\mu, \gamma)\right) \\
&= \frac{e(\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\delta \in A} e\left((\mu, \delta)\right) g_\delta(\tau).
\end{aligned}$$

If $\mu \notin H^\perp$ then $(g{\uparrow}_H^A)_\mu|_S(\tau) = 0$ and

$$\begin{aligned}
\sum_{\delta \in A} e\left((\mu, \delta)\right) g_\delta(\tau) &= \sum_{\delta \in H^\perp} e\left((\mu, \delta)\right) g_\delta(\tau) \\
&= \sum_{\delta \in H^\perp/H} \sum_{\gamma \in \delta + H} e\left((\mu, \gamma)\right) g_\gamma(\tau) \\
&= \sum_{\delta \in H^\perp/H} \sum_{\gamma \in H} e\left((\mu, \delta + \gamma)\right) g_{\delta+\gamma}(\tau) \\
&= \sum_{\delta \in B} e\left((\mu + H, \delta)\right) g_\delta(\tau) \sum_{\gamma \in H} e\left((\mu, \gamma)\right) \\
&= 0.
\end{aligned}$$

The last equality by orthogonality of characters. ☐

There is a map from $M_{k,A}$ to $M_{k,B}$ as well.

Proposition 3.1.8 (Lemma 5.7 [7]). *Let $f = \sum_{\mu \in A} \mathfrak{e}_\mu f_\mu M_{k,A}$. Then the $\mathbb{C}[B]$–valued function*

$$f \downarrow_H^A = \sum_{\mu \in H^\perp} f_\mu \mathfrak{e}_{\mu+H}$$

belongs to $M_{k,B}$.

Proof. This can be proved the same way as Proposition 3.1.7. ☐

Proposition 3.1.9 (Proposition 3.3 [9]). *Let $f \in M_{k,A}$, and assume that f is supported on $H^\perp$. Then $f_{\mu+\mu'} = f_\mu$ for all $\mu \in A, \mu' \in H$, and*

$$f = \frac{1}{|H|} f \downarrow_H^A \uparrow_H^A . \tag{3.1.8}$$

Proof. Using (1.2.1), we have

$$(f_\mu)|_S(\tau) = \frac{e(\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\nu \in A} e((\mu,\nu)) f_\nu(\tau)$$

for all $\mu \in A$. Since f is supported on $H^\perp$, we have

$$(f_\mu)|_S(\tau) = \frac{e(\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\nu \in H^\perp} e((\mu,\nu)) f_\nu(\tau).$$

Therefore, we get for all $\mu' \in H$

$$\begin{aligned}(f_{\mu+\mu'})|_S(\tau) &= \frac{e(\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\nu \in H^\perp} e((\mu+\mu',\nu)) f_\nu(\tau) \\ &= \frac{e(\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\nu \in H^\perp} e((\mu,\nu)) f_\nu(\tau) \\ &= (f_\mu)|_S(\tau).\end{aligned}$$

This implies that $f_{\mu+\mu'}(-1/\tau) = f_\mu(-1/\tau)$ for all $\tau \in \mathcal{H}$, which proves that $f_{\mu+\mu'} = f_\mu$. Write $f = \sum_{\mu \in H^\perp} f_\mu \mathfrak{e}_\mu$ then

$$\begin{aligned} f \downarrow_H^A \uparrow_H^A &= (\sum_{\mu \in H^\perp} f_\mu \mathfrak{e}_{\mu+H}) \uparrow_H^A \\ &= (\sum_{\mu \in H^\perp/H} \sum_{h \in H} f_{\mu+h} \mathfrak{e}_{\mu+H}) \uparrow_H^A \\ &= |H| (\sum_{\mu \in H^\perp/H} f_\mu \mathfrak{e}_{\mu+H}) \uparrow_H^A \\ &= |H| \sum_{\mu \in H^\perp} f_{\mu+H} \mathfrak{e}_\mu \\ &= |H| f. \end{aligned}$$

□

Lemma 3.1.10. *Let $f \in M_{k,A}$. If $G \subset A$ is any subgroup, then for all $\mu \in A$ we have*

$$\frac{1}{|G|} \sum_{\mu' \in G} f_{\mu+\mu'}(-1/\tau) = \tau^k \frac{e(-\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\nu \in G^\perp} e(-(\mu,\nu)) f_\nu(\tau).$$

Proof. Using (1.2.1), we have

$$\begin{aligned}\frac{1}{|G|}\sum_{\mu'\in G} f_{\mu+\mu'}(-1/\tau) &= \tau^k \frac{e(-\operatorname{sgn}(A)/8)}{|G|\sqrt{|A|}} \sum_{\mu'\in G}\sum_{\nu\in A} e(-(\mu+\mu',\nu))f_\nu(\tau)\\ &= \tau^k \frac{e(-\operatorname{sgn}(A)/8)}{|G|\sqrt{|A|}} \sum_{\nu\in A} e(-(\mu,\nu))f_\nu(\tau)\sum_{\mu'\in G} e(-(\mu',\nu))\\ &= \tau^k \frac{e(-\operatorname{sgn}(A)/8)}{|G|\sqrt{|A|}} \sum_{\nu\in G^\perp} e(-(\mu,\nu))f_\nu(\tau)\sum_{\mu'\in G} e(-(\mu',\nu))\\ &= \tau^k \frac{e(-\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\nu\in G^\perp} e(-(\mu,\nu))f_\nu(\tau).\end{aligned}$$

This is because the sum $\sum_{\mu'\in G} e(-(\mu',\nu))$ is zero if $\nu \notin G^\perp$ and is wqual to $|G|$ if $\nu \in G^\perp$. □

Lemma 3.1.11. *Let $C_1,\dots,C_m \subset A$ be subsets. For every subset $S \subset \{1,\dots,m\}$ put $C(S) = \bigcap_{i\in S} C_i$. Then we have*

$$\chi_{C_1\cup\cdots\cup C_m} = \sum_{\emptyset\neq S\subset\{1,\dots,m\}} (-1)^{|S|+1}\chi_{C(S)}.$$

Here $\chi_C : A \to \{0,1\}$ is the characteristic function of $C \subset A$.

Proof. This is a generalization of $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$. It can be proved by induction on m. □

The following theorem is a generalization of proposition 3.1.9.

Theorem 3.1.12 (Theorem 3.6 [9])**.** *Let $H_1,\dots,H_m \subset A$ be isotropic subgroups of prime order $p_i = |H_i|$ with $p_i \neq p_j$ for $i \neq j$. For a subset $S \subset \{1,\dots,m\}$ let $H_S := \sum_{i\in S} H_i$. If $f \in M_{k,A}$ is supported on $H_1^\perp \cup\cdots\cup H_m^\perp$, then*

$$f = \sum_{\emptyset\neq S\subset\{1,\dots,m\}} (-1)^{|S|+1}\frac{1}{|H_S|} f\downarrow^A_{H_S}\uparrow^A_{H_S}.$$

Proof. Notice that for all $S \subset \{1,\dots,m\}$, the subgroup $H_S \subset A$ is isotropic. We prove the statement by induction on m. For $m = 1$ it is Proposition 3.1.9.

Now assume that $m > 1$. By means of 1.2.1 we see that

$$f_\mu(-1/\tau) = \tau^k \frac{e(-\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\nu\in H_1^\perp\cup\cdots\cup H_m^\perp} e(-(\mu,\nu))f_\nu(\tau)$$

for $\mu \in A$. Notice that for all $S \subset \{1,\dots,m\}$ we have $\bigcap_{i\in S} H_i^\perp = H_S^\perp$ so, using Lemma 3.1.11 with $C_i = H_i^\perp$ we get

$$f_\mu(-1/\tau) = \tau^k \frac{e(-\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\emptyset\neq S\subset\{1,\dots,m\}} (-1)^{|S|+1} \sum_{\nu\in H_S^\perp} e(-(\mu,\nu)) f_\nu(\tau).$$

Using Lemma 3.1.10, this becomes

$$f_\mu(\tau) = \sum_{\emptyset\neq S\subset\{1,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} \sum_{\mu'\in H_S} f_{\mu+\mu'}(\tau).$$

Now we use the transformation behavior (1.2.1) under T. Since $f_\mu(\tau+a) = e(aq(\mu))f_\mu(\tau)$ for $a \in \mathbb{Z}$ and since H_S is isotropic, we find

$$f_\mu(\tau) = \sum_{\emptyset\neq S\subset\{1,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} \sum_{\mu'\in H_S} e(a(\mu,\mu')) f_{\mu+\mu'}(\tau).$$

If we sum over a modulo the level of A, then the sum on the right hand side is zero unless $\mu' \perp \mu$. We obtain,

$$f_\mu(\tau) = \sum_{\emptyset\neq S\subset\{1,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} \sum_{\substack{\mu'\in H_S\\ \mu'\perp\mu}} f_{\mu+\mu'}(\tau).$$

Now assume that $\mu \in H_1^\perp$ and $\mu \notin H_i^\perp$ for $i = 2,\dots,m$. Then for $i \in \{2,\dots,m\}$ there is a $\mu_i \in H_i$ with $(\mu_i,\mu) \notin \mathbb{Z}$. Then $(\mu_i,\mu) \equiv \frac{r}{p_i} \pmod{\mathbb{Z}}$ for some $r \in (\mathbb{Z}/p\mathbb{Z})^\times$ and so μ_i is a generator of H_i. Without loss of generality, we may assume that $(\mu_i,\mu) \equiv \frac{1}{p_i} \pmod{\mathbb{Z}}$. Let $\mu' \in H_S$ and write $\mu' = \mu_1 + a_2\mu_2 + \cdots + a_m\mu_m$ with $a_i \in \mathbb{Z}$ and $\mu_i \in H_i$. Then we see that $\mu' \perp \mu$ if and only if $p_i \mid a_i$ for all $i = 2,\dots,m$. Hence $\mu' \perp \mu$ if and only if $\mu' \in H_1$ and we obtain

$$\begin{aligned} f_\mu(\tau) &= \sum_{\emptyset\neq S\subset\{1,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} \sum_{\mu'\in H_S\cap H_1} f_{\mu+\mu'}(\tau) \\ &= \sum_{\emptyset\neq S\subset\{2,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} f_\mu(\tau) + \sum_{1\in S\subset\{1,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} \sum_{\mu'\in H_1} f_{\mu+\mu'}(\tau), \end{aligned}$$

and therefore

$$\sum_{S\subset\{2,\dots,m\}} (-1)^{|S|} \frac{1}{|H_S|} f_\mu(\tau) = \sum_{1\in S\subset\{1,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} \sum_{\mu'\in H_1} f_{\mu+\mu'}(\tau).$$

In the sum on the left hand side the case $S = \emptyset$ is included and corresponds to $H_S = \{0\}$. Using the bijection between the subsets of $\{2, \dots, m\}$ and the subsets of $\{1, \dots, m\}$ containing 1 given by $S \mapsto \{1\} \cup S$ we get $p_1 |H_S| = |H_{\{1\} \cup S}|$. Therefore

$$f_\mu(\tau) = \frac{1}{p_1} \sum_{\mu' \in H_1} f_{\mu+\mu'}(\tau). \tag{3.1.9}$$

Since $\mu \in H_1^\perp$ and $\mu \notin H_i^\perp$ for $i = 2, \dots, m$, we also have for every $\mu_1 \in H_1$ that $\mu + \mu_1 \in H_1^\perp$ and $\mu + \mu_1 \notin H_i^\perp$ for $i = 2, \dots, m$. This is because $\mu_1 \perp H_i$ for $i = 1, \dots, m$. Therefore, (3.1.9) implies for such μ and $\mu_1 \in H_1$ that

$$f_{\mu+\mu_1}(\tau) = f_\mu(\tau). \tag{3.1.10}$$

Consequently,

$$\tilde{f} = f - \frac{1}{p_1} f \downarrow_{H_1}^A \uparrow_{H_1}^A \in M_{k,A} \tag{3.1.11}$$

is supported on $H_2^\perp \cup \dots \cup H_m^\perp$. By induction, we have

$$\tilde{f} = \sum_{\emptyset \neq S \subset \{2,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} \tilde{f} \downarrow_{H_S}^A \uparrow_{H_S}^A .$$

If we insert $\tilde{f}$ in 3.1.11, we obtain

$$\begin{aligned} f = \frac{1}{p_1} f \downarrow_{H_1}^A \uparrow_{H_1}^A &+ \sum_{\emptyset \neq S \subset \{2,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} f \downarrow_{H_S}^A \uparrow_{H_S}^A \\ &+ \sum_{\emptyset \neq S \subset \{2,\dots,m\}} (-1)^{|S|} \frac{1}{p_1 |H_S|} f \downarrow_{H_1}^A \uparrow_{H_1}^A \downarrow_{H_S}^A \uparrow_{H_S}^A . \end{aligned}$$

Using the identity

$$\frac{1}{p_1 |H_S|} f \downarrow_{H_1}^A \uparrow_{H_1}^A \downarrow_{H_S}^A \uparrow_{H_S}^A = \frac{1}{|H_{\{1\} \cup S}|} f \downarrow_{H_{\{1\} \cup S}}^A \uparrow_{H_{\{1\} \cup S}}^A .$$

We get

$$f = \sum_{\emptyset \neq S \subset \{1,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} f \downarrow_{H_S}^A \uparrow_{H_S}^A ,$$

which completes the proof of the theorem. □

Now we specialize the above theory to cyclic isotropic subgroups. To this end, let us fix some notations. For d a positive integer we denote by $\Omega(d)$ the number of prime factors of d counted with multiplicities. Let $e \in A$ be an isotropic element of order $N \in \mathbb{Z}_{>0}$ so, $(e, A) = \frac{1}{N}\mathbb{Z}$. Following Bruinier, we define the content of λ with respect to e as

$$\mathrm{cont}_e(\lambda) := \gcd(N(e, \lambda), N). \tag{3.1.12}$$

For any divisor $d \mid N$, we consider the isotropic subgroup

$$I_d = \langle \frac{N}{d} e \rangle \subset A \tag{3.1.13}$$

of order d. Its orthogonal complement is given by

$$I_d^{\perp} = \{\lambda \in A : \quad d \mid N(e, \lambda) \in \mathbb{Z}/N\mathbb{Z}\} = \{\lambda \in A : \ d \mid \mathrm{cont}_e(\lambda)\}.$$

We put $A(d) = I_d^{\perp}/I_d$. Then $|A| = d^2|A(d)|$.

Proposition 3.1.13 (Proposition 3.7 [9]). *Assume that* $f = \sum_{\lambda \in A} f_\lambda \mathfrak{e}_\lambda \in M_{k,A}$ *is supported on*

$$\bigcup_{\substack{p \mid N \\ p \ prime}} I_p^{\perp},$$

that is, $f_\lambda = 0$ *for all* $\lambda \in A$ *with* $\mathrm{cont}_e(\lambda) = 1$. *Then there exist modular forms* $f_d \in M_{k,A(d)}$ *such that*

$$f = \sum_{1 < d \mid N} f_d \uparrow_{I_d}^{A}.$$

Proof. This is an application of theorem 3.1.12 as follows. Put

$$f_d := \frac{1}{d}\mu(d) f \downarrow_{I_d}^{A},$$

where μ denotes the Mœbius function. Let $p_1, \ldots, p_m$ be the distinct prime divisors of N. Then $H_i := I_{p_i} \subset A$ is an isotropic subgroup of prime order p_i. For $S \subset \{1, \ldots, m\}$ we have

$$H_S = \sum_{i \in S} H_i = \sum_{i \in S} I_{p_i} = I_d,$$

where $d = |H_S| = \prod_{i \in S} p_i$. As S runs through the non-empty subsets of $\{1, \ldots, m\}$, the quantity $d = |H_S|$ runs through the square-free non-trivial divisors of N. Moreover, we have $(-1)^{|S|} = \mu(d)$. This proves the proposition. □

Lemma 3.1.14. *Let* $n \in \mathbb{Z}_{\geq 0}$. *Assume that* $f = \sum_{\lambda \in A} f_\lambda \mathfrak{e}_\lambda \in M_{k,A}$ *is given by*

$$f = \sum_{\substack{d \mid N \\ \Omega(d) \geq n}} f_d \uparrow_{I_d}^{A}$$

for some $f_d \in M_{k,A(d)}$. *Let* $d' \mid N$ *with* $\Omega(d') = n$, *and let* $\mu \notin I_d^{\perp}$ *for all* $d \mid N$, $d \neq d'$ *with* $\Omega(d) \geq n$. *Then* $f_\mu = (f_{d'} \uparrow_{I_{d'}}^{A})_\mu$ *and* $f_{\mu+\mu'} = f_\mu$ *for all* $\mu' \in I_{d'}$.

Proof. Since each f_d is supported on $I_d^\perp$ it follows that $\mu \notin I_d^\perp$ implies $(f_d)_\mu = 0$ for every d different from d'. The second part follows from Proposition 3.1.9. $\square$

Define the height of λ with respect to e to be $\operatorname{height}_e(\lambda) := \Omega(\operatorname{cont}_e(\lambda))$. The following Theorem is a refinement of Proposition 3.1.13.

Theorem 3.1.15 (Theorem 3.10 [9]). *Let $n \in \mathbb{Z}_{\geq 0}$. Assume that $f = \sum_{\lambda \in A} f_\lambda \mathfrak{e}_\lambda \in M_{k,A}$ is supported on*

$$\bigcup_{\substack{d \mid N \\ \Omega(d)=n}} I_d^\perp, \tag{3.1.14}$$

that is, $f_\lambda = 0$ for all $\lambda \in A$ unless $\operatorname{height}_e(\lambda) \geq n$. Then there exist modular forms $f_d \in M_{k,A(d)}$ such that

$$f = \sum_{\substack{d \mid N \\ \Omega(d) \geq n}} f_d \uparrow_{I_d}^A . \tag{3.1.15}$$

Proof. We prove the proposition by induction on n. For $n = 0$ there is nothing to show. For $n = 1$ the assertion follows from Proposition 3.1.13.

Now assume that $n > 1$ and suppose that f is as in the Theorem. Because if $d \mid d' \mid N$ we have $I_{d'}^\perp \subset I_d^\perp$ it follows that f is supported also on

$$\bigcup_{\substack{d \mid N \\ \Omega(d)=n-1}} I_d^\perp .$$

By induction, there exist modular forms $f_d \in M_{k,A(d)}$ such that

$$f = \sum_{\substack{d \mid N \\ \Omega(d) \geq n-1}} f_d \uparrow_{I_d}^A . \tag{3.1.16}$$

We claim that for every $d \mid N$ with $\Omega(d) = n-1$, the modular form $f_d \in M_{k,A(d)}$ is actually supported on

$$\bigcup_{\substack{p \mid N/d \\ p \text{ prime}}} H_p^\perp, \tag{3.1.17}$$

where $H_p := H_p(d) = \langle \frac{N}{dp} e + I_d \rangle$ is a subgroup of $A(d)$ or order p. In other words f_d satisfies the hypothesis of Proposition 3.1.13 for the discriminant form $A(d)$. So, let d be a divisor of N with $\Omega(d) = n-1$. Put $e' := e + I_d$. We need to show that $(f_d)_\lambda = 0$ for all $\lambda \in A(d)$ with $\operatorname{cont}_{e'}(\lambda) = 1$. Let $\lambda \in A(d)$ with $\operatorname{cont}_{e'}(\lambda) = 1$. Then, if $\mu \in I_d^\perp$ with $\mu \mapsto \lambda$ under the projection $I_d^\perp \to A(d)$, we have $\mathrm{cont}_e(\mu) = d$. Thus $\mu \notin I_{d'}^\perp$ for all $d' \mid N$ with $\Omega(d') \geq n$ and $d' \neq d$. Therefore, by Lemma 3.1.14 we have

$$(f_d)_\lambda = (f_d \uparrow_{I_d}^A)_\mu = f_\mu.$$

Since $\mathrm{height}_e(\lambda) = n - 1$, the assumption on f implies that $f_\mu = 0$, and therefore $(f_d)_\lambda = 0$. Therefore, f_d is supported on the set in 3.1.17. But now Proposition 3.1.13 says that there exist modular forms $f_{d,p} \in M_{k,{H_p}^\perp/H_p}$ such that

$$f_d = \sum_{\substack{p \mid N/d \\ p \text{ prime}}} f_{d,p} \uparrow_{H_p}^{A(d)} .$$

Note that $H_p^\perp/H_p \cong A(dp)$ and

$$f_d \uparrow_{I_d}^A = \sum_{\substack{p \mid N/d \\ p \text{ prime}}} f_{d,p} \uparrow_{I_{dp}}^A .$$

If we insert this into (3.1.16), we get the result for n. □

3.1.3 Lattices that split a hyperbolic plane over $\mathbb{Z}$

Now that we have all the necessary ingredients we can prove the injectivity of the lift Λ as in (3.1.6) for lattices that split a hyperbolic plane. Recall that we mean by a lattice that split a hyperbolic plane, a lattice of the form $L = D \oplus U$ for some smaller lattice D. But these are exactly lattices of the form $L = D \oplus U(N) \oplus U$ where D is positive definite and N is the level of the lattice [9, Lemma 5.1]. So from now on we assume that the lattice L is of the form $L \cong D \oplus U(N) \oplus U$ for some positive definite even lattice D.

Let $\ell, \ell' \in L$ be primitive isotropic with $(\ell, \ell') = 1$ so that $U = \mathbb{Z}\ell \oplus \mathbb{Z}\ell'$. Let $K = L \cap \ell^\perp \cap \ell'^\perp$. Then $K \cong D \oplus U(N)$ and $L = K \oplus \mathbb{Z}\ell \oplus \mathbb{Z}\ell'$. Put

$$A = L^\vee/L \cong D^\vee/D \oplus U(N)^\vee/U(N).$$

We have $U(N)^\vee/U(N) \cong (\mathbb{Z}/N\mathbb{Z})^2$ and for each $r \in (\mathbb{Z}/N\mathbb{Z})^\times$ there is an automorphism $\varphi_r^U \in \mathrm{Aut}(U(N)^\vee/U(N))$ acting on $U(N)^\vee/U(N)$ as $(a,b) \mapsto (ra, r^{-1}b)$, where r^{-1} denotes the inverse of r modulo N.

We define the two congruence subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N} \right\}.$$

Lemma 3.1.16. *For* $r \in (\mathbb{Z}/N\mathbb{Z})^\times$ *there exists* $\varphi_r \in \mathrm{O}(L)^+$ *whose image under*

$$\mathrm{O}(L)^+ \longrightarrow \mathrm{Aut}(A)$$

restricts to the identity on $D^\vee/D$ *and to* φ_r^U *on* $U(N)^\vee/U(N)$. *The transformation* φ_r *is uniquely determined up to multiplication by elements of* Γ.

Proof. Because we can extend any automorphism φ of $U^\vee/U \oplus U(N)^\vee/U(N) \cong U(N)^\vee/U(N)$ to an element of $\mathrm{Aut}(A)$ by taking the restricting on $D^\vee/D$ to be the identity. It suffices to prove the assertion for $L = U(N) \oplus U$. The latter lattice can be identified with the following subset of 2 by 2 integral matrices,

$$L \cong \left\{ X = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z} \right\}$$

with the quadratic form given by the determinant. The dual of L is

$$L^\vee \cong \left\{ X = \begin{pmatrix} a & \frac{1}{N}b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z} \right\}$$

The group $\Gamma_0(N) \times \Gamma_0(N)$ acts on L by $(\gamma_1, \gamma_2).X = \gamma_1 X \gamma_2^{-1}$ preserving the quadratic form while its subgroup $\Gamma_1(N) \times \Gamma_1(N)$ acts trivially on $L^\vee/L$. Thus we get a homomorphism $\Gamma_0(N) \times \Gamma_0(N) \to \mathrm{O}(L)^+$ with the subgroup $\Gamma_1(N) \times \Gamma_1(N)$ being mapped to $\Gamma = O(L)^+ \cap \mathrm{O}_d(L)$. Hence we obtain a homomorphism

$$(\Gamma_0(N) \times \Gamma_0(N))/(\Gamma_1(N) \times \Gamma_1(N)) \longrightarrow \mathrm{Aut}(L^\vee/L).$$

Since the left hand side is isomorphic to $((\mathbb{Z}/N\mathbb{Z})^\times)^2$, it follows that φ_r^U is in the image of this map. ☐

Thus the lemma defines an action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $\mathcal{H}^{1,1}(X_\Gamma)$ via the transformations $\varphi_r \in \mathrm{O}(L)^+$ for $r \in (\mathbb{Z}/N\mathbb{Z})^\times$ and a second action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $S_{\kappa,A}$ via the image of φ_r under $\mathrm{O}(L)^+ \longrightarrow \mathrm{Aut}(A)$. We will abuse notation and note it φ_r^U as its restriction to $U(N)^\vee/U(N)$. On an other hand, the map Λ is equivariant with respect to this action, by Theorem 3.1.2. Consequently, the action of $(\mathbb{Z}/N\mathbb{Z})^\times$ preserves the kernel of Λ, and we obtain a decomposition of the kernel into isotypical components with respect to the characters of $(\mathbb{Z}/N\mathbb{Z})^\times$. Therefore, if $g \in \ker \Lambda$, then we may assume without loss of generality that g is contained in the χ-isotypical component of $S_{\kappa,A}$ for some character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, that is,

$$\varphi_r^U \cdot g = \chi(r) g, \quad r \in (\mathbb{Z}/N\mathbb{Z})^\times. \tag{3.1.18}$$

Remark 1. *Let* $\mu \in A$ *and* $e = \ell'/N \in U(N)^\vee \subset L^\vee$. *Write* $\mu = \mu_D + \mu_N$ *with* $\mu_D \in D^\vee/D$ *and* $\mu_N = (a, b) \in U(N)^\vee/U(N)$ *then we have* $(e, \mu) = \frac{a}{N} + \mathbb{Z}$ *and* $(e, \varphi_r^U \cdot \mu) = \frac{ra}{N} + \mathbb{Z}$ *for any* $r \in (\mathbb{Z}/N\mathbb{Z})^\times$. *Thus if* $(e, \mu) = \frac{r}{N} + \mathbb{Z}$ *with* $r \in (\mathbb{Z}/N\mathbb{Z})^\times$, *then using the action of* $(\mathbb{Z}/N\mathbb{Z})^\times$ *on* A *and* (3.1.18) *we may assume that* $(e, \mu) = \frac{1}{N} + \mathbb{Z}$.

Theorem 3.1.17. *[9, Theorem 5.3] Assume that $L \cong D \oplus U(N) \oplus U$ for some positive definite lattice D of dimension $l-2$. Then the map $\Lambda : S_{\kappa,L} \to \mathcal{H}^{1,1}(X_\Gamma)$ is injective.*

Proof. We put $A = L^\vee/L$. Let $g = \sum_{\mu\in A}\sum_m b(m,\mu)q^m\mathfrak{e}_\mu \in S_{\kappa,A}$ be an element in the kernel of Λ. We denote by g_μ the components of g with respect to the standard basis $(\mathfrak{e}_\mu)_{\mu\in A}$ of $\mathbb{C}[A]$. We have to show that $g = 0$. We use the first part of Theorem 3.1.2 to prove that $\lambda(v,g) = 0$ implies $g = 0$. The assumption $\lambda(v,g) = 0$ implies that

$$\sum_{d|\lambda} d^{n-1} b(q(\lambda)/d^2, \lambda/d) = 0 \tag{3.1.19}$$

for all $\lambda \in K^\vee$ with $q(\lambda) > 0$. By an easy inductive argument we find that

$$b(q(\lambda),\lambda) = 0 \tag{3.1.20}$$

for all $\lambda \in K^\vee$ with $q(\lambda) > 0$. We shall prove that $b(q(\lambda),\lambda) = 0$ for all $\lambda \in L^\vee$ with $q(\lambda) > 0$. Put $e := \frac{\ell'}{N} \in U(N)^\vee \subset L^\vee$. This is a primitive isotropic vector of $U(N)^\vee$ whose image in A has order N. For $d \mid N$ we define the subgroups

$$I_d = \langle \frac{\ell'}{d} + L\rangle \subset I_N = \langle e + L\rangle \subset A.$$

We prove that all components g_μ vanish by induction on the number of prime divisors of the content

$$\mathrm{cont}_e(\mu) = \gcd(N(e,\mu), N)$$

of μ with respect to e.

Case 1. $\Omega(\mathrm{cont}_e(\mu)) = 0$: Let $\mu \in A$ with $\mathrm{cont}_e(\mu) = 1$, that is, $(e,\mu) = \frac{r}{N} + \mathbb{Z}$ with $r \in (\mathbb{Z}/N\mathbb{Z})^\times$. In view of Remark 1, we may assume that $(e,\mu) = \frac{1}{N} + \mathbb{Z}$. Let $\delta \in L^\vee$ such that $\delta \mapsto \mu$ under the projection $L^\vee \to A$ with $(\ell,\delta) = a$ and $(\ell',\delta) = b$. Put $\lambda = \delta - a\ell - b\ell' \in K^\vee$. Then $\lambda \in \delta + K$ and $(\lambda, e) = \frac{1}{N}$. Moreover for any $m \in \mathbb{Z}$ we have

$$\lambda + mNe \in \mu, \qquad q(\lambda + aNe) = q(\lambda) + m.$$

But (3.1.20) yields that $b(q(\lambda),\lambda) = 0$ for all $\lambda \in L^\vee$ with $\Omega(\mathrm{cont}_e(\lambda)) = 0$.

Case 2. $\Omega(\mathrm{cont}_e(\mu)) = n > 0$. Assume that $g_\mu = 0$ for all $\mu \in A$ with $\Omega(\mathrm{cont}_e(\mu)) < n$. This means that g is supported on

$$\bigcup_{\substack{d|N \\ \Omega(d)=n}} I_d^\perp = \bigcup_{\substack{d|N \\ \Omega(d)=n}} \{\mu \in A : \ d \mid \mathrm{cont}_e(\mu)\}.$$

By Theorem 3.1.15 there exist cusp forms $g_d \in S_{k,I_d^\perp/I_d}$ such that

$$g = \sum_{\substack{d|N \\ \Omega(d)\geq n}} g_d \uparrow_{I_d}^A . \tag{3.1.21}$$

Let $\mu \in A$ with $\Omega(\mathrm{cont}_e(\mu)) = n$ and put $d' = \mathrm{cont}_e(\mu)$. There exists $r \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $(e,\mu) = \frac{rd'}{N} + \mathbb{Z}$. As in Remark 1, we may assume that $(e,\mu) = \frac{d'}{N} + \mathbb{Z}$. Then Remark 3.1.14 applied to g in (3.1.21) implies that

$$g_\mu = (g_{d'} \uparrow_{I_{d'}}^A)_\mu$$

and $g_{\mu+\mu'} = g_\mu$ for all $\mu' \in I_{d'}$. Since $(e,\mu) = \frac{d'}{N} + \mathbb{Z}$, there exists $\lambda \in \delta + K$ as in the previous case, such that $(\lambda, e) = \frac{d'}{N}$ with $\bar{\delta} = \mu$. Moreover, for any $m \in \mathbb{Z}$ we have

$$\lambda + \frac{mN}{d'}e \in \mu + I_{d'}, \qquad q(\lambda + \frac{mN}{d'}e) = q(\lambda) + m.$$

But now (3.1.20) implies that $b(q(\lambda),\lambda) = 0$ for all $\lambda \in L^\vee$ with $\Omega(\mathrm{cont}_e(\lambda)) = n$. This completes the proof. $\square$

We state the main theorem of this section.

Theorem 3.1.18 (Bruinier). *Assume that $L \cong D \oplus U(N) \oplus U$ for some positive definite even lattice D of dimension $n-2 \geq 1$ and some positive integer N. Then every meromorphic modular form F with respect to Γ whose divisor is a linear combination of special divisors $Z(m,\mu)$ is (up to a non-zero constant factor) the Borcherds lift $\Psi(z,f)$ of a weakly holomorphic modular form $f \in M^!_{1-n/2,L^-}$.*

Proof. The assertion follows from Theorem 3.1.6 using Theorem 3.1.17. $\square$

Remark 2. *To illustrate the assumption that L split a hyperbolic plane over $\mathbb{Z}$, we consider the possible cases for even lattices of prime level p and signature $(n,2)$ with $n \equiv 2 \pmod 8$.*

Let L be a an even lattice of prime level p and signature $(n,2)$ with $n \equiv 2 \pmod 8$. Then $L^\vee/L = \mathbb{F}_p^r$ with $1 \leq r \leq n+2$. It is a consequence of [14, Chapter 15, Theorem 11 and Theorem 19] that the following cases can occur:

If $p \equiv 3 \pmod 4$ then either $r = n+2$ and $L \cong II_{n,2}(p)$, or $r \leq n$ and $L \cong \tilde{L} \oplus II_{1,1}$ for some even lattice $\tilde{L}$ of signature $(n-1,1)$.

If $p \equiv 1 \pmod 4$ then either $r = n+2$ and $L \cong II_{n,2}(p)$, or $r = n+1$ and $L \cong II_{n-1,1}(p) \oplus \mathcal{O}_p$ where $\mathcal{O}_p$ is the even lattice of signature $(1,1)$ and determinant $-p$, or $r \leq n$ and $L \cong \tilde{L} \oplus II_{1,1}$ for some even lattice $\tilde{L}$ of signature $(n-1,1)$.

3.2 Lattices of prime level

In this section we establish a converse theorem for lattices of prime level. These lattices do not necessarily split a hyperbolic plane over $\mathbb{Z}$.

Definition 3.2.1. *For an isotropic vector $u \in V$ and $v \in u^\perp$ we define the Eichler element $E(u,v) \in \mathrm{O}(V)^+$ by*

$$E(u,v)(a) = a - (a,u)v + (a,v)u - q(v)(a,u)u \tag{3.2.1}$$

for $a \in V$. Moreover, if $u, v \in L$, then $E(u,v) \in \Gamma$

Lemma 3.2.2. *Let L be an even lattice of level N and let $u \in L$ be an isotropic vector such that $(u,L) = N\mathbb{Z}$. If $v \in L^\vee \cap u^\perp$, then $E(u,v) \in \mathrm{O}(L)^+$.*

Proof. Since L has level N, we have $NL^\vee \subset L$. Hence the assertion follows immediately from the definition (3.2.1). □

The following proposition is vital to the proof of the converse theorem.

Proposition 3.2.3 (Proposition 6.4 [9]). *Let L be an even lattice of prime level p and signature $(n,2)$ with $n \geq 4$. Let $g = \sum_{\mu\in A}\sum_m b(m,\mu)q^m\mathfrak{e}_\mu \in S_{\kappa,L}$ be an element in the kernel of Λ. Let $u \in L$ be primitive isotropic, and assume $(u,L) = p\mathbb{Z}$. Then for every $v, \lambda \in L^\vee \cap u^\perp$ we have*

$$b(q(\lambda), E(u,v)\lambda) = b(q(\lambda), \lambda + (\lambda,v)u) = b(q(\lambda),\lambda).$$

We shall use the Weil bound for the coefficients of cusp forms in the sequel of the proof of the proposition.

Lemma 3.2.4 (Weil). *If $g = \sum_{m,\mu} a(m)q^m$ is a cusp form of weight κ for the full modular group, then as $m \to \infty$,*

$$|a(m)| \ll_\varepsilon m^{\kappa - \frac{1}{2} + \varepsilon},$$

for any $\varepsilon > 0$.

Proof. (of) Proposition 3.2.3]) Put $A = L^\vee/L$. If $(v,\lambda) \in \mathbb{Z}$, then there is nothing to show. So we assume that $(v,\lambda) \in r/p + \mathbb{Z}$ with $r \in (\mathbb{Z}/p\mathbb{Z})^\times$.

We consider the Eichler transformation $E := E(u,v)$, which belongs to $\mathrm{O}(L)^+$ according to Lemma 3.2.2. We have $E(\lambda) = \lambda + (\lambda,v)u$ and the element E generates a subgroup $G \subset \mathrm{Aut}(A)$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. The group G acts on $S_{\kappa,A}$ and on $\mathcal{H}^{1,1}(X_\Gamma)$, and in view of the third part of Theorem 3.1.2, the map Λ is equivariant with respect to these actions. Consequently, the action of G preserves the kernel of

Λ, and we obtain a direct sum decomposition of the kernel into isotypical components K_s with respect to the characters of G (*i.e.* $\chi_s : G \to \mathbb{C}, a \mapsto \chi_s(a) = e(as/p)$ for $s \in \mathbb{Z}/p\mathbb{Z}$).

For $s \in \mathbb{Z}/p\mathbb{Z}$, the χ_s-isotypical component or simply the s-isotypical component K_s of $\ker \Lambda$ is given by

$$K_s = \{g \in \ker \Lambda | \; E.g = e(s/p)g\}.$$

For $g \in \ker \Lambda$, its s-isotypical component $g_s \in K_s$ is given by

$$g_s = \sum_{a \; (p)} e(-as/p) E^a.g. \tag{3.2.2}$$

If we write $g_s = \sum_{\mu \in A} \sum_m b_s(m, \mu) q^m \mathfrak{e}_\mu$, we have

$$b_s(q(\lambda), \lambda + (\lambda, v)u) = e(s/p) \cdot b_s(q(\lambda), \lambda). \tag{3.2.3}$$

It suffices to show that $b_s(q(\lambda), \lambda) = 0$ for all $s \in (\mathbb{Z}/p\mathbb{Z})^\times$.

Let $K = (L \cap u^\perp)/\mathbb{Z}u$. We write the image of λ in $K^\vee \cong (L^\vee \cap u^\perp)/\mathbb{Z}\frac{u}{p}$ as $d'\lambda_0$, with a primitive vector $\lambda_0 \in K^\vee$ and $d' \in \mathbb{Z}_{>0}$. Since $(\lambda, v) \notin \mathbb{Z}$, the number d' is coprime to p. We choose an auxiliary prime q coprime to pd', and we put $\lambda_1 = qd'\lambda_0 \in K^\vee$. We use the Fourier expansion of Λ given by Theorem 3.1.2 with $\ell = u$, to deduce that the λ_1-th Fourier coefficient of $\Lambda(g_s)$ vanishes, that is,

$$\sum_{d | qd'} d^{l-1} \sum_{a \; (p)} e(ad/p) b_s(q(\lambda_1/d), \lambda_1/d + a\frac{u}{p}) = 0,$$

or equivalently,

$$\sum_{d | qd'} d^{l-1} \sum_{a \; (p)} e\big(a(\lambda_1, v)\big) b_s\big(q(\lambda_1/d), \lambda_1/d + a(\lambda_1/d, v)u\big) = 0.$$

Using (3.2.3) and the fact that $(\lambda_1, v) \equiv qr/p \pmod{\mathbb{Z}}$, we find

$$\sum_{d | qd'} d^{l-1} \sum_{a \; (p)} e\big(a(qr + s)/p\big) b_s\big(q(\lambda_1/d), \lambda_1/d\big) = 0.$$

If $s \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $qr \equiv -s \pmod{p}$, we obtain

$$\sum_{d | qd'} d^{l-1} b_s\big(q(\lambda_1/d), \lambda_1/d\big) = 0.$$

If we split the sum over the divisors of qd' into a sum over the divisors coprime to q and a sum over the divisors divisible by q, we obtain

$$\sum_{d|d'} d^{1-l} b_s\big(q(qd\lambda_0), qd\lambda_0\big) + q^{l-1} \sum_{d|d'} d^{1-l} b_s\big(q(d\lambda_0), d\lambda_0\big) = 0. \tag{3.2.4}$$

By Dirichlet's theorem, there are infinitely many primes q satisfying $qr \equiv -s \pmod{p}$. If q goes to infinity, then the Weil bound for the coefficients of the cusp form g_s of weight $\kappa = 1+l/2$ implies that for any $\varepsilon > 0$ we have $b_s(q(q\lambda), q\lambda) = O(q^{\kappa-1/2+\varepsilon})$. Employing (3.2.4), we obtain

$$\sum_{d|d'} d^{1-l} b_s\big(q(d\lambda_0), d\lambda_0\big) = -q^{1-l} \sum_{d|d'} d^{1-l} b_s\big(q(qd\lambda_0), qd\lambda_0\big) = O(q^{3/2-l/2+\varepsilon}).$$

By our assumption $l > 3$, the right hand side goes to zero as $q \to \infty$, and therefore

$$\sum_{d|d'} d^{1-l} b_s\big(q(d\lambda_0), d\lambda_0\big) = 0.$$

An inductive argument now shows that $b_s\big(q(d\lambda_0), d\lambda_0\big) = 0$ for all $s \in (\mathbb{Z}/p\mathbb{Z})^\times$ and all $d \mid d'$. This proves the assertion. $\square$

Next we shall show that the coefficients of $g \in \ker \Lambda$ have some "nice" properties under the action of $\mathrm{Aut}(A)$. Let L be an even lattice of prime level p and signature $(l, 2)$ with $l \geq 4$. We also assume that L has Witt rank 2, which is automatically fulfilled if $l > 4$. It turns out that one can break down the lattice L to smaller latticse as follows: Let $\ell \in L$ be a primitive isotropic vector such that $(\ell, L) = p\mathbb{Z}$. Let $\ell' \in L^\vee$ such that $(\ell', \ell) = 1$, and put $\tilde{\ell} = p(\ell' - q(\ell')\ell)$. Then $\tilde{\ell}$ is isotropic and satisfies $(\tilde{\ell}, \ell) = p$. Since L has level p, we have $pL^\vee \subset L$ and $pq(\ell') \in \mathbb{Z}$. Consequently, $\tilde{\ell}$ belongs to L. Furthermore, we have $L = K \oplus \mathbb{Z}\tilde{\ell} \oplus \mathbb{Z}\ell$, where $K = L \cap \ell^\perp \cap \tilde{\ell}^\perp$. In fact $z \in L$ and it can be easily checked that $\tilde{z} = z - (z, \ell/p)\tilde{\ell} - (z, \ell')\ell + (z, \ell/p)(\tilde{\ell}, \ell')\ell$ is an element of K from which we can deduce the splitting. In particular, $L \cong K \oplus U(p)$.

Henceforth we assume that $L = D \oplus M$, where D is a positive definite even lattice of level p and rank $l - 2$, and $M \cong U(p) \oplus U(p)$. Using the map $(a, b, c, d) \to \begin{pmatrix} a & c \\ -d & b \end{pmatrix}$. The lattice M can be identified with the lattice of integral 2×2 matrices with the quadratic form $q(X) = p\det(X)$. The group $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ acts on M by orthogonal transformations via $(\gamma_1, \gamma_2).X = \gamma_1 X \gamma_2^{-1}$. The action induces a homomorphism $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{O}(M)^+$ whose kernel is $\{\pm 1\}$. Following Bruinier, we define what we call a normal form.

Definition 3.2.5. *An element $\mu \in A \cong D^\vee/D \oplus M^\vee/M$ is said to have a* normal form *if $\mu_D = 0$ in $D^\vee/D$ and*

$$\mu_M = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \text{if } \mu = 0, \\ \begin{bmatrix} 1/p & 0 \\ 0 & q(\mu) \end{bmatrix}, & \text{if } \mu \neq 0. \end{cases}$$

Let $L^\vee \to A \cong D^\vee/D \oplus M^\vee/M, \lambda \mapsto \bar{\lambda}$ be the canonical projection. We write $\lambda \in L$ as $\lambda = \lambda_D + \lambda_M$ with $\lambda_D \in D$ and $\lambda_M \in M$. Let $g = \sum_{\mu \in A} \sum_m b(m,\mu) q^m \mathfrak{e}_\mu \in S_{\kappa,L}$ be an element in the kernel of Λ. The coefficients of g have the following nice behavior.

Proposition 3.2.6 (Proposition 6.5 [9])**.** *For every $\mu \in L^\vee$ there exists $\gamma \in \mathrm{O}(L)^+$ such that $\gamma\bar{\mu}$ has normal form and*

$$b(q(\mu),\mu) = b(q(\mu),\gamma\mu).$$

The group $\mathrm{O}(L)^+$ acts transitively on the $\mathrm{Aut}(A)$*-orbits of A*

Proof. 1. We first show that there exists $\gamma \in \mathrm{O}(L)^+$ such that $\gamma\bar{\mu} \in M^\vee/M$ and such that

$$b(q(\mu),\mu) = b(q(\mu),\gamma\mu).$$

1.1. First assume that $\bar{\mu}_D \neq 0$ and $\bar{\mu}_M \neq 0$. By the elementary divisor theorem for $\mathrm{SL}_2(\mathbb{Z})$, there is a basis of M in which $\mu_M \in M$ has the form

$$\mu_M = \frac{1}{p}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

with $a \in \mathbb{Z}_{>0}$, a divides d and $(a,p) = (d,p) = 1$. The hypothesis $\bar{\mu}_M \neq 0$ means that $\mu_D \notin D$. Write $\mu_D = t\mu_{D0}$ for some $t \in \mathbb{Z}$ and a primitive $\mu_{D0} \in D^\vee$. Consequently $p\mu_{D0}$ is primitive in D. So, there is $\nu_0 \in D^\vee$ such that $(\nu_0, p\mu_{D0}) = 1$. Because $\mu_D \notin D$ and L has level p it follows that $(t,p) = 1$. Put $\nu_D = -t^{-1}a^{-1}\nu_0$ (x^{-1} is the residue of the inverse of x modulo p). Then $\nu \in D^\vee$ satisfies

$$(\nu_D, \mu_D)a \equiv \frac{-1}{p} \pmod{\mathbb{Z}}. \tag{3.2.5}$$

Define two vectors

$$u = pa\mu_D - \begin{pmatrix} 0 & 1 \\ pa^2 q(\mu_D) & 2pq(\mu_D) \end{pmatrix}, \text{ and } v = \nu_D - \begin{pmatrix} 0 & 0 \\ a(\mu_D,\nu_D) & 0 \end{pmatrix}.$$

Since p is the level of L, then $p\mu_D \in L$ and $pq(\mu_D) \in \mathbb{Z}$ so, $u \in L$. Similarly $pv \in L$ so, $v \in L^\vee$. Clearly u is primitive isotropic. Using the identity $(x,y) = q(x+y) - q(x) - q(y)$ it can be easily checked that $(u,\mu) = (u,v) = 0$. By lemma 3.2.2 the Eichler element $E := E(u,v)$ belongs to $\mathrm{O}(L)^+$ and we have

$$\begin{aligned} E\mu &= \mu + (\mu,v)u \\ &= \mu + (\nu_D,\mu_D)u \\ &\equiv \mu_M + (\nu_D,\mu_D)u_M \pmod{L}. \end{aligned}$$

The last equality follows because $u_D = pa\mu_D$ and $pa(\nu_D, \mu_D) \equiv -1 \pmod{p\mathbb{Z}}$. The claim follows from Proposition 3.2.3.

1.2. Now assume that $\bar{\mu}_D \neq 0$ and $\bar{\mu}_M = 0$. Since $\mu_D \notin D$ we can find $v \in D^\vee$ such that $(\mu_D, v) \notin \mathbb{Z}$. Put $u = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then $\nu := E(u,v)\mu = \mu + (\mu_D, v)u$ has the property that $\bar{\nu}_D \neq 0$ and $\bar{\nu}_M \neq 0$. By Proposition 3.2.3, we have $b(q(\mu), \mu) = b(q(\nu), \nu)$. We continue as in case 1.1.

2. We may now assume that $\bar{\mu}_D = 0$. By the elementary divisor theorem there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma.\bar{\mu}$ has normal form. Since $(\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}))/\{\pm 1\} \subset \mathrm{O}(M)^+ \subset \mathrm{O}(L)^+$ it suffices to show that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ we have

$$b(q(\mu_M), \gamma.\mu_M) = b(q(\mu_M), \mu_M).$$

We only need to prove this for the generators $(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], 1)$, $(\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right], 1)$, $(1, \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right])$, $(1, \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right])$. We illustrate the argument with the first generator γ_1 as it is the same for the other generators. If we write $\mu_M = \frac{1}{p}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $u = \left[\begin{smallmatrix} c & d \\ 0 & 0 \end{smallmatrix}\right]$, we have

$$\gamma_1.\mu_M = \frac{1}{p}\begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} = \mu_M + \frac{1}{p}u.$$

If $p \mid (c,d)$, then $\gamma_1.\mu_M \equiv \mu_M (\operatorname{mod} L)$ and so there is nothing to show. If $p \nmid (c,d)$ we choose $\alpha, \beta \in \mathbb{Z}$ such that $\alpha d - \beta c \equiv 1 \pmod{p}$ and put

$$v = \frac{1}{p}\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}.$$

Then $(\mu, u) = (v, u) = 0$ and $E(u,v)\mu = \mu + (\mu, v)u = \mu + \frac{u}{p} = \gamma_1.\mu_M$. Hence $b(q(\mu_M), \gamma_1.\mu_M) = b(q(\mu_M), \mu_M)$ by Proposition 3.2.3. $\square$

As a consequence, we get the corollary .

Corollary 3.2.7. *Let* $g = \sum_{\mu \in A} \sum_m b(m, \mu) q^m \mathfrak{e}_\mu \in S_{\kappa,L}$ *be an element in the kernel of* Λ.

i) For every $\lambda \in L^\vee$ *and for every* $\gamma \in \mathrm{Aut}(L^\vee/L)$ *we have* $b(q(\lambda), \lambda) = b(q(\lambda), \gamma\lambda)$.

ii) If g *is actually contained in* $S^+_{\kappa,L}$, *then* g *is invariant under* $\mathrm{Aut}(L^\vee/L)$.

Before proving the corollary we shall recall a pairing introduced by Bruinier and Funke in their paper [11]. For $\kappa \in \frac{1}{2}\mathbb{Z}$, they defined a bilinear pairing between the spaces S_{κ,L^-} and $H_{2-\kappa,L}$ by means of the Peterson inner product, by setting

$$\{g, f\} = \big(g,\, \xi_{2-\kappa}(f)\big)_{\kappa,L^-} \tag{3.2.6}$$

for $g \in M_{\kappa,L^-}$ and $f \in H_{2-\kappa,L}$. The pairing $\{g,f\}$ can be explicitly evaluated in terms of the Fourier coefficients of g and the principal part of f.

Theorem 3.2.8 (Proposition 3.5 [11])**.** *Let* $g = \sum_{\mu\in A}\sum_m b(m,\mu)q^m\mathfrak{e}_\mu \in S_{\kappa,L}$*, and* $f \in H_{k,L}$ *with Fourier expansion as in* (3.1.2)*. Then the pairing* (3.2.6) *of* g *and* f *is equal to*

$$\{g,f\} = \sum_{\mu\in L^\vee/L}\sum_{m<0} c^+(m,\mu)b(-m,\mu). \tag{3.2.7}$$

Proof. [of the corollary] i) In every $\mathrm{Aut}(A)$-orbit of A there exists a unique element in normal form. Hence the corollary directly follows from Proposition 3.2.3 and Proposition 3.2.6.

ii) Let $\mu \in A$ and $m \in \mathbb{Z}+q(\mu)$ be positive. If there does not exist any $\lambda \in \mu + L$ for which $q(\lambda) = m$, then $Z(m,\mu) = 0 \in \mathrm{Div}(X_\Gamma)$. Hence the harmonic Maass form $f_{m,\mu} \in H_{2-\kappa,L^-}$ with principal part $\frac{1}{2}q^{-m}(\mathfrak{e}_\mu + \mathfrak{e}_{-\mu})$ belongs to $N_{2-\kappa,L^-}$. Since $g \in S^+_{\kappa,L}$, by 3.2.8 we have

$$b(m,\mu) = \{f_{m,\mu}, g\} = (\xi(f_{m,\mu}), g) = 0.$$

Moreover, by the second part of Proposition 3.2.6 for every $\gamma \in \mathrm{Aut}(A)$ there does not exists any $\lambda \in \gamma\mu + L$ for which $q(\lambda) = m$. Consequently, we have $b(m,\gamma\mu) = 0$ as well. Combining this with (i) we find that g is invariant under $\mathrm{Aut}(L^\vee/L)$.

□

Let r be the dimension of the $\mathbb{F}_p$-vector space A. The splitting $L = D \oplus M$ implies that $r \geq 4$. Since L has level p, the quotient $L/pL^\vee$ is an $\mathbb{F}_p$-vector space of dimension $l + 2 - r$. The quadratic form q on L induces a non-degenerate $\mathbb{F}_p$-valued quadratic form on $L/pL^\vee$.

Definition 3.2.9. *We say* $L/pL^\vee$ *represents* $0 \in \mathbb{F}_p$ *non-trivially if there is a* $\lambda_0 \in L$ *such that* $m_0 := q(\lambda_0) \in p\mathbb{Z}$ *and* $\lambda_0 \notin pL^\vee$*.*

Lemma 3.2.10. *If* $L/pL^\vee$ *represents* $0 \in \mathbb{F}_p$ *non-trivially, then for any* $m \in p\mathbb{Z}$ *there exists* $\lambda \in L$ *such that* $q(\lambda) = m$ *and* $\lambda/p \notin L^\vee$*. Moreover, such a* λ *can be chosen primitively in* $L^\vee$*.*

Proof. The assumption means that there is $\lambda_0 \in L$ such that $m_0 := q(\lambda_0) \in p\mathbb{Z}$ and $\lambda_0/p \notin L^\vee$. Write $\lambda_0 = \lambda_{0D} + \lambda_{0M}$ with $\lambda_{0D} \in D$ and $\lambda_{0M} \in M$. By the Elementary divisor theorem and using the action defined earlier of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{O}(L)^+$ on M we may assume that λ_{0M} is diagonal. *i.e.*

$$\lambda_0 = \lambda_{0D} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $a, b \in \mathbb{Z}$. Then, for $t \in \mathbb{Z}$, the vector

$$\lambda = \lambda_{0D} + \begin{pmatrix} a & 1 \\ t & b \end{pmatrix}$$

in L represents $m_0 - pt$ since

$$q(\lambda) = q(\lambda_{0D}) + pab - pt = q(\lambda_{0D}) + q(\lambda_{0M}) - pt = m_0 - pt.$$

Moreover λ is primitive in $L^\vee$. $\square$

Theorem 3.2.11 (Theorem 6.9 [9])**.** *If* $g \in S^+_{\kappa,L}$ *and* $\Lambda(g) = 0$*, then* $g = 0$*.*

Proof. [Sketch of the proof] 1. Consider the case in which $L/pL^\vee$ represents $0 \in \mathbb{F}_p$ non-trivially, in which for any $m \in p\mathbb{Z}$ there exists $\lambda \in L$ which is primitive in $L^\vee$ such that $q(\lambda) = m$. Using the action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{O}(L)^+$, we may assume that λ is contained in K and primitive in $K^\vee$. Then for any such m, the vanishing of the λ-coefficient of $\Lambda(g)$ yields

$$b(m, 0) = b(m, \ell/p). \tag{3.2.8}$$

Next we shall show that for any $m \in \frac{1}{p}\mathbb{Z}$ and any $\mu \in A \setminus \{0\}$ the Fourier coefficient $b(m, \mu)$ of g vanishes.

1.1. If there is no $\lambda \in L^\vee$ such that $q(\lambda) = m$ and $\bar{\lambda} = \mu$, then $Z(m, \mu) = 0 \in \mathrm{Div}(X_\Gamma)$. Since $g \in S^+_{\kappa,L}$, this implies $b(m, \mu) = 0$ (see proof of Corollary 3.2.7).

1.2. If there exists $\lambda \in L^\vee$ such that $q(\lambda) = m$, then $\bar{\lambda} \neq 0$ implies that $\lambda/p \notin L^\vee$. Moreover, using the action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{O}(L)^+$, we may assume that λ is primitive in $K^\vee$. Now the vanishing of the $p\lambda$-th coefficient in the Fourier expansion of $\Lambda(v, g)$ yields

$$0 = b(q(p\lambda), 0) - b(q(p\lambda), \ell/p) + p^n b(q(\lambda), \lambda). \tag{3.2.9}$$

If we combine this with (3.2.8), we obtain $b(q(\lambda), \lambda) = 0$. Thus $g_\mu = 0$ for all $\mu \in A \setminus \{0\}$ and $g = \sum_m b(m, 0) q^m \mathfrak{e}_0$. But using (1.2.1) we have

$$g(-1/\tau) = t^k \frac{e(-\operatorname{sgn}(A)/8)}{\sqrt{|A|}} g(\tau).$$

This yields $g = 0$.

2. Now we consider the case in which $L/pL^\vee$ does not represent 0 non-trivially *i.e.* there exists no $\lambda_0 \in L$ such that $p \mid q(\lambda_0)$ and $\lambda_0/p \notin L^\vee$. Then, for any $\lambda \in L$ with $p \mid q(\lambda)$, we have $\lambda/p \in L^\vee$. This implies that for any $m \in \frac{1}{p}\mathbb{Z}$ we have

$$Z(p^2 m, 0) = \sum_{\substack{\lambda \in L \\ q(\lambda) = p^2 m}} \lambda^\perp = \sum_{\substack{\lambda \in L^\vee \\ q(\lambda) = m}} \lambda^\perp = \sum_{\mu \in L^\vee/L} \sum_{\substack{\lambda \in \mu + L \\ q(\lambda) = m}} \lambda^\perp = \sum_{\mu \in A} Z(m, \mu).$$

Since $g \in S^+_{\kappa,L}$, the corresponding relation on the level of the Fourier coefficients is (see proof of Corollary 3.2.7)

$$b(p^2m, 0) = \sum_{\mu\in A} b(m, \mu). \tag{3.2.10}$$

For every $m \in \frac{1}{p}\mathbb{Z}$ with $\mathrm{ord}_p(m) \le 0$. Define a number $B(m)$

$$B(m) = \begin{cases} b(m,\mu), & \text{if there exists } \mu \in A \setminus \{0\} \text{ such that } q(\mu) \equiv m \pmod{\mathbb{Z}}, \\ 0, & \text{otherwise.} \end{cases}$$

This definition is independent of the choice of μ, because of Corollary 3.2.7.

Using 3.2.10 it can be shown by induction that for every $r \in \mathbb{Z}_{\ge 0}$ there are integers $C_m(r) \ge p^{rl}$ and $C'_m(r) \ge 0$ such that

$$B(p^{2r}m) = C_m(r)B(m), \tag{3.2.11}$$

$$b(p^{2r}m, 0) = C'_m(r)B(m). \tag{3.2.12}$$

A comparison of (3.2.11) with the Weil bound 3.2.4 for cusp forms of weight $\kappa = 1+l/2$ implies that for any $\varepsilon > 0$ we have

$$B(p^{2r}m) \ll_\varepsilon p^{2r(1/4+l/4+\varepsilon)}, \quad r \to \infty,$$

which gives

$$B(m) \ll_\varepsilon p^{2r(1/4-\frac{7}{4}l+\varepsilon)}, \quad r \to \infty.$$

Since $l > 1$, this implies that $B(m) = 0$ for all $m \in \frac{1}{p}\mathbb{Z}$ with $\mathrm{ord}_p(m) \le 0$. But using (3.2.11) we get $B(m) = 0$ for all $m \in \mathbb{Z}$. Hence $g = 0$. ☐

Now we can state the converse theorem for lattices of prime level.

Theorem 3.2.12 (Theorem 1.4 [9]). *Let L be an even lattice of prime level p and signature $(l, 2)$. Assume that $l \ge 3$ and that the Witt rank of L is 2. Then there exists a sublattice $L_0 \subset L$ of level p such that every meromorphic modular form F with respect to Γ whose divisor is a linear combination of special divisors $Z(m,\mu)$ is (up to a non-zero constant factor) the Borcherds lift $\Psi(z, f)$ of some $f \in M^!_{1-n/2,L_0^-}$.*

Proof. The assertion follows from Theorem 3.1.6 by means of Theorem 3.2.11. ☐

3.3 Lorentzian Lattices

In his Ph.D dissertation [3], Barnard considered the analogous question in the case of orthogonal groups of signature $(1, l)$ in connection with Lorentzian reflection groups.

For $\delta \in L^\vee/L$ define the δ-Heegner divisor

$$H(\delta) = \bigcup_{\substack{\lambda\in L^\vee \\ \lambda\in\delta+L \\ \delta^2=\lambda^2}} \lambda^\perp.$$

In the definition of the Maass–Poincaré series in section 1.3, we specialize to $s = 1$ and define the function F_δ, for $\delta \in L^\vee/L, m = \mathbb{Z} + q(\delta)$ with $m < 0$, by

$$F_\delta(\tau) = y^{\frac{n+1}{2}} F_{\delta,m}(\tau, 1).$$

Let $P(v)$ be a piecewise linear function that is invariant under the discriminant kernel and having singularities given by a linear combination of Heegner divisors

$$(P) = \sum_{\delta\in L^\vee/L} c(\delta)H(\delta),$$

and define a real analytic modular form by $F_P(\tau) = \sum_{\delta\in L^\vee/L} c(\delta)F_\delta(\tau)$. Let Φ_P be its singular theta transformation.

The function $P - \Phi_P$ is real analytic on $\mathrm{Gr}(L)$, invariant under the discriminant kernel and an eigenfunction of the hyperbolic Laplacian with eigenvalue $-n$ ([3], lemma 3.3.1). Using the Fourier expression of the theta lift Φ_P and the action of the Laplacian, one can show that The difference $P - \Phi_P$ decreases faster than $1/y$ at the cusp represented by z ([3] theorem 3.3.4).

We want to prove that $P = \Phi_P$ butt we need some information about the location of the eigenvalues of the Laplacian for the orthogonal group $\mathrm{O}(1, n)$.

Set $\omega = \frac{n-1}{2}$ and write the eigenvalues of the Laplacian in the form $s^2 - \omega^2$. Because the eigenvalues for the Laplacian are negative reals, this implies that s lies in the set

$$i\mathbb{R} \cup [-\omega, \omega].$$

The continuous spectrum of the Laplacian lies in $i\mathbb{R}$ and it is conjectured that the cuspidal part also lies in this region (these are the Ramanujan-Selberg conjectures). The other eigenvalues that can occur are called exceptional eigenvalues.

Conjecture 3.3.1 (Conjecture 3.4.1 [3]). *For the group* $\mathrm{O}(1, n)$ *the exceptional eigenvalues are given by*

$$s \in \{\omega, \omega - 1, \ldots, -\omega\}.$$

See [13, conjecture 1.6] , for computations supporting the conjecture.

Proposition 3.3.2. *Assume Conjecture 3.3. For* $n > 7$ *the function* $P - \Phi_P$ *is the zero function.*

Proof. The difference $P(v_1) - \Phi_P(v_1)$ is an eigenfunction of the Laplacian with eigenvalue equal to $-n = s^2 - (\frac{n-1}{2})^2$ for some $s \in i\mathbb{R} \cup \{\omega, \omega - 1, \ldots, -\omega\}$. But for $n \geq 6$ we have $\omega^2 - s^2 = n < (\frac{n-1}{2})^2 = \omega^2$. Thus $s \notin i\mathbb{R}$ and $s \in \{\omega, \omega - 1, \ldots, -\omega\}$. This amounts to solving in integers $(\frac{n-1}{2})^2 - n = (\frac{k}{2})^2$ or equivalently $(n-3)^2 - k^2 = 8$ which gives $n = 0$ or 6. Hence for $n \geq 7$ the eigenvalue $-n$ does not occur in the (conjectured) spectrum and so $P - \Phi_P$ is the zero function. This proves the converse theorem assuming the conjecture. $\square$

Appendix A

A.1 Lifting scalar valued modular forms

Let L be an even lattice of signature $(l, 2)$ and prime level p. Then l is even, and $A = L^\vee/L$ is an $\mathbb{F}_p$-vector space, whose rank we denote by r. Clearly $0 \leq r \leq 2 + n$. The quadratic Dirichlet character associated to A is defined by

$$\chi_A(x) = \left(\frac{x}{p}\right)^r, \qquad x \in \mathbb{Z}. \tag{A.1.1}$$

As usual we also view χ_A as a character on $\Gamma_0(p)$ by putting $\chi_A(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) = \chi_A(d)$. For $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$ we define the Petersson slash operator in integral weight k on functions on $\mathcal{H}$ by

$$(g \mid_k M)(\tau) = \det(M)^{k/2}(c\tau + d)^{-k} g(M\tau). \tag{A.1.2}$$

So scalar matrices act trivially. We denote by $W_p = \left[\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right]$ the Fricke involution on the space $M_k(p, \chi_A)$ of scalar valued modular forms of weight k for the group $\Gamma_0(p)$ with character χ_A. Recall that the Hecke operator U_p acts on $g = \sum_l a(l)q^l \in M_k(p, \chi_A)$ by

$$g \mid U_p = \sum_l a(pl)q^l. \tag{A.1.3}$$

The V_p-operator acts by

$$g \mid V_p = p^{-k/2} g \mid_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \sum_l a(l) q^{pl}. \tag{A.1.4}$$

The restriction of ρ_A to $\Gamma_0(p)$ acts on the vector $\mathfrak{e}_0 \in \mathcal{C}[A]$ by multiplication with $\chi_A(M)$. Hence, if $g \in M_k(p, \chi_A)$, then

$$\vec{g} = \sum_{\gamma \in \Gamma_0(p) \backslash \mathrm{SL}_2(\mathbb{Z})} (g \mid_k \gamma) \rho_A^{-1}(\gamma) \mathfrak{e}_0 \tag{A.1.5}$$

belongs to $M_{k,A}$. It is invariant under the action of $\operatorname{Aut}(A)$. We have the following result:

Proposition A.1.1. *Let* $g = \sum_l a(l)q^l \in M_k(p, \chi_A)$ *and write* $g \mid W_p = \sum_l \tilde{a}(l)q^l$. *Denote the Fourier expansion of* $\vec{g}$ *by*

$$\vec{g} = \sum_{\mu \in A} \sum_{m \in Q(\mu)+\mathbb{Z}} \vec{a}(m,\mu) q^m \mathfrak{e}_\mu.$$

For $\mu \in A$ *and* $m \in Q(\mu) + \mathbb{Z}$ *we have*

$$\vec{a}(m,\mu) = \begin{cases} p^{1-k/2-r/2} e(\operatorname{sgn}(A)/8) \tilde{a}(pm), & \text{if } \mu \neq 0, \\ a(m) + p^{1-k/2-r/2} e(\operatorname{sgn}(A)/8) \tilde{a}(pm), & \text{if } \mu = 0. \end{cases}$$

Proof. A system of representatives for $\Gamma_0(p) \backslash \mathrm{SL}_2(\mathbb{Z})$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}, \quad j = 1, \dots p.$$

This implies that

$$\vec{g} = g\mathfrak{e}_0 + \sum_{j\ (p)} g \mid W_p \mid \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \cdot \rho_A \begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} \rho_A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathfrak{e}_0.$$

A computation shows that the sum over j on the right hand side is equal to

$$p^{1-k/2} \frac{e(\operatorname{sgn}(A)/8)}{\sqrt{|A|}} \sum_{\mu \in A} \sum_{\substack{m \in \mathbb{Z} \\ m \equiv pQ(\mu)\ (p)}} \tilde{a}(m) q^{m/p} \mathfrak{e}_\mu.$$

This proves the proposition. □

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