Continuity in law with respect to the spatial Hurst index of the solutions to some linear SPDEs

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Thesis submitted to the Faculty of Science in partial fulfillment of the requirements for the degree of
Master of Science Mathematics and Statistics[1]

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[1]The M.Sc. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

In this thesis, we study the linear stochastic heat and wave equations with zero initial conditions, driven by a Gaussian noise, which is fractional in space with Hurst index $H \in (0,1)$, and is either white in time (i.e. fractional in time with index $H_0 = \frac{1}{2}$) or fractional in time with index $H_0 > \frac{1}{2}$. We prove that the solution of each of these equations is continuous in law in the space $C([0,T] \times \mathbb{R})$ of continuous functions, with respect to the index H. This result has already been proved in the recent article [15] for the case $H_0 = \frac{1}{2}$, and we extend it here to the case $H_0 > \frac{1}{2}$.

Dedications

I would like to thank my parents for their unconditional love and continued support throughout the years. I wish to thank all the faculty, staff and friends for making my university life fulfilling and wonderful. In particular I would like to thank Professor Chen Xu for the suggestions he provided during my bachelor study.

Acknowledgement

I must express my sincere gratitude to my supervisor, Professor Raluca Balan, for her persistent support of my study and this research, for her patience and immense knowledge. She sparked my desire in the study of SPDEs and encouraged me going on. She always have a positive attitude, which keeps me engaged and motivated to learn. Throughout the time as her student, she cared so much about my work, and responded to my questions and queries so promptly. This work would not have been possible without her helpful guidance, careful reading and corrections.

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Chapter 1

Introduction

The study of stochastic partial differential equation (SPDE) has gained enormous interest and has been expanding rapidly during the last three decades. SPDEs generalize partial differential equations by introducing random force terms or coefficients, and appear in several different applications, for example, in quantum field theory, statistical mechanics (in physics), systemic risk, and portfolio choice (in finance).

The solution to an SPDE may be viewed in different manners. For example, one can view a solution as a random field (a set of random variables indexed by a multidimensional parameter), which is the approach introduced by John Walsh in his lecture notes [28]. This random field approach extending Itô's construction to higher-dimensions allows us to investigate the probabilistic behavior of the solution, simultaneously in time and space. In the case when the SPDE is an evolution equation, Da Prato and Zabczyk proposed the infinite-dimensional approach in [14], which consists in viewing the solution at a given time as a random element in a function space and thus view the SPDE as a stochastic evolution equation in an infinite dimensional space, mainly Hilbert and Banach spaces. The article [13] explored the links between the random field and the infinite-dimensional approach. In 1999 in [19], Krylov developed an analytic approach to present a theory of solvability of the Cauchy problem for linear and some quasi-linear equations in spaces of summable functions. In this thesis, we will use the random field approach. A gentle introduction to SPDEs using the random field approach can be found in [3] for researchers in mathematics who have a background in probability theory, but may not be familiar with the area of SPDEs. We refer the reader to [12] and [17] for expository lectures of this approach.

The goal of this thesis is to prove the continuity in law with respect to the spatial Hurst index H of the noise of the solution to the linear stochastic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,x)=0, \quad x\in\mathbb{R} \end{cases} \tag{1.0.1}$$

and of the solution to the linear stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t,x), \quad t>0, x\in\mathbb{R} \\ u(0,x)=0, \quad x\in\mathbb{R} \end{cases} \tag{1.0.2}$$

in the case when the noise $\dot{W}$ is fractional in time with index $H_0 \geq \frac{1}{2}$ and fractional in space with index $H \in (0,1)$.

In order to state our results, we need to introduce some notation. Throughout this thesis, we use upper indices h and w to indicate that a quantity is computed for heat equation, respectively wave equation, and we use no upper index when this quantity is computed for either one of the two equations.

Definition 1.0.1. *We say that a process* $\{u^h(t,x); t\geq 0, x\in\mathbb{R}\}$ *is a* **solution** *to equation* (1.0.1) *if*

$$u^h(t,x) = \int_0^t \int_{\mathbb{R}} G^h(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y) \tag{1.0.3}$$

where

$$G^h(t,x) = (2\pi t)^{-\frac{1}{2}} \exp\Big(-\frac{x^2}{2t}\Big) \tag{1.0.4}$$

is the fundamental solution of the heat equation on $\mathbb{R}_+ \times \mathbb{R}$.

Definition 1.0.2. *We say that a process* $\{u^w(t,x); t\geq 0, x\in\mathbb{R}\}$ *is a* **solution** *to equation* (1.0.2) *if*

$$u^w(t,x) = \int_0^t \int_{\mathbb{R}} G^w(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y), \tag{1.0.5}$$

where

$$G^w(t,x) = \frac{1}{2} 1_{\{|x|\leq t\}} \tag{1.0.6}$$

is the fundamental solution of the wave equation on $\mathbb{R}_+ \times \mathbb{R}$.

These solutions exist if the stochastic integrals (which appear in these definitions) are well-defined. We denote by $\mathcal{F}$ the Fourier transform in the space variable. Note that

$$\mathcal{F}G^h(t,\cdot)(\xi) = \exp\Big(-\frac{t|\xi|^2}{2}\Big), \text{ for all } \xi\in\mathbb{R}, \tag{1.0.7}$$

and

$$\mathcal{F}G^w(t,\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \text{ for all } \xi\in\mathbb{R}. \tag{1.0.8}$$

We need to explain the intuitive meaning of the definition of solution given above. First, note that $\dot{W}(t,x) = \frac{\partial^2 W}{\partial t \partial x}(t,x)$ is a formal notation for the space-time derivative of the noise W, which does not exist for any $t > 0$ and $x \in \mathbb{R}$ (exactly as the derivative $B'(t)$ of a Brownian motion $\{B(t)\}_{t\geq 0}$ does not exist at any time $t > 0$). Suppose that in equation (1.0.1) (or equation (1.0.2)), we replace $\dot{W}(t,x)$ by $f(t,x)$, where f is a smooth function. In this case, it is known from the classical theory of partial differential equations that the solution of equation (1.0.1) (or (1.0.2)) is given by $u(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s, x-y) f(s,y) \mathrm{d}y \mathrm{d}s$, where $G = G^h$ for equation (1.0.1), or $G = G^w$ for equation (1.0.2). (This follows using a classical method called the Duhamel's principle.) Replacing f by $\dot{W}$, we obtain a formal integral with respect to $\dot{W}(s,y)\mathrm{d}y\mathrm{d}s$. By the usual convention used in stochastic analysis, this formal integral is interpreted rigorously as the stochastic integral with respect to W, in which $\dot{W}(s,y)\mathrm{d}y\mathrm{d}s$ is replaced by $W(\mathrm{d}s, \mathrm{d}y)$.

Recall that a fractional Brownian motion (fBm) is a zero-mean Gaussian process $(B_t^{(H)})_{t\in\mathbb{R}}$ with covariance

$$E[B_t^{(H)} B_s^{(H)}] = R_H(t,s) := \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

The parameter $H \in (0,1)$ is called the Hurst index. If $H = 1/2$, then $R_H(t,s) = t \wedge s$ and the fBm coincides with the Brownian motion. If $H > 1/2$, the covariance of the fBm can be expressed as follows:

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} \mathrm{d}u \mathrm{d}v,$$

where $\alpha_H = H(2H-1)$. For any $H \in (0,1)$, this covariance admits the spectral representation:

$$R_H(t,s) = c_H \int_{\mathbb{R}} \mathcal{F}1_{[0,t]}(\xi) \overline{\mathcal{F}1_{[0,s]}(\xi)} |\xi|^{1-2H} \mathrm{d}\xi,$$

where $\mathcal{F}1_{[0,t]}(\xi)$ is the Fourier transform of the indicator function $1_{[0,t]}$, given by:

$$\mathcal{F}1_{[0,t]}(\xi) = \int_0^t e^{-i\xi s} ds = \frac{1 - e^{-i\xi t}}{i\xi} \quad \text{for all } \xi \in \mathbb{R},$$

and

$$c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}. \tag{1.0.9}$$

Throughout the thesis, we will use these definitions of the constants α_H and c_H.

A fractional Brownian sheet (fBs) with Hurst indices $H_0, H \in (0,1)$ is a zero-mean Gaussian process $\{W(t,x); t \geq 0, x \in \mathbb{R}\}$ with covariance

$$E[W(t,x)W(s,y)] = R_{H_0}(t,s) R_H(x,y).$$

The noise that we will introduce in Chapter 2 (respectively Chapter 3) corresponds to an fBs with index $H_0 = 1/2$ (respectively $H_0 > 1/2$) in time, and index $H \in (0,1)$ in space.

In Chapter 2, we consider equations (1.0.1) and (1.0.2) driven by a Gaussian noise which is white in time and behaves in space like a fractional Brownian motion with index $H \in (0,1)$. In Section 2.1, we introduce the noise and we define the Wiener integral with respect to this noise. In Section 2.2, we study the problem of existence and uniqueness of solutions, under some conditions, for the linear stochastic heat and wave equations with noise $\dot{W}$ introduced in Section 2.1. In Section 2.3, we focus our attention on the behaviour of the increments of the solution with respect to the time variable and spatial variable. In this section, we follow the method of [15], by using some preliminary results taken from [4]. In Section 2.4, we study the continuity in law, in the space $C([0,T] \times \mathbb{R})$ of continuous functions with respect to the index $H \in (0,1)$, of the solutions to equations introduced in Section 2.2.

In Chapter 3, we continue with the same procedure as in Chapter 2. We examine the linear stochastic heat and wave equations with a Gaussian noise which is fractional in time with index $H_0 \in (\frac{1}{2}, 1)$ and behaves in space like a fractional Brownian motion with index $H \in (0,1)$. We treat the heat and wave equations separately. In Section 3.1, we introduce the noise and we define the Wiener integral with respect to this noise. In Section 3.2, we prove the existence of solution to the linear stochastic heat equation with noise $\dot{W}$ introduced in Section 3.1. The result is a particular case of Theorem 4.2 of [6]. In Section 3.3, we study the problem of existence of solution for the linear stochastic wave equation with noise $\dot{W}$ introduced in Section 3.1, a problem studied in Section 3 of [6]. In Section 3.4, we give some upper bounds for the moments of the increments of the solution to the linear stochastic heat equation (1.0.1) with noise $\dot{W}$ introduced in 3.1. More precisely, we show that:

$$E|u^h(t,x) - u^h(s,y)|^2 \leq C(|t-s|^{2H_0+H-1} + |x-y|^{2(2H_0+H-1)}),$$

for some constant $C > 0$. The space component of this result is a particular case of Theorem 4 of [26], and is obtained under the additional condition

$$2H_0 + H < 2.$$

The time increment component of this result is obtained under no restriction on H_0 and H, and is a particular case of Theorem 2.6 of [25]. This result can also be found in reference [21] for case of the white noise in space (i.e. when $H = 1/2$), and in reference [27], which considers the more general case of a noise $\dot{W}$ which behaves like a bi-fractional Brownian motion (bi-fBm) in time with indices H_0, K such that $H_0 K > 1/2$; in particular, if $K = 1$, we recover our case of a fractional noise in time with index H_0. The proof that we present here is different than the one given in references [21] and [27], following the same lines as the proof given in Chapter

2 for the case $H_0 = 1/2$. In Section 3.5, we make an analysis of the behaviour of the increments in time and space of the solution to linear stochastic wave equation introduced in Section 3.3. Here we proceed as in the proof of Propositions 3.4 and 3.7 in [10] by considering only the upper bounds. More precisely, assuming that

$$2H_0 + 2H < 3,$$

we prove that

$$E|u^w(t,x) - u^w(s,y)|^2 \leq C(|t-s|^{2H_0+2H-1} + |x-y|^{2H_0+2H-1}),$$

for some constant $C > 0$. In Section 3.6, we prove that the solution of each of the equations introduced in Section 3.2 and Section 3.3 is continuous with respect to the index H, by using the convergence in law in the space of continuous functions. This result is new in the literature and is the main contribution of the thesis. The other results presented here are known in the pulished literature.

Appendix A contains some auxiliary results about the Fourier transform in the space variable which are used in the proofs. In Appendix B, we review some classical upper bounds for the moments of the increments of the linear stochastic heat and wave equations with space-time white noise. Appendix C provides an inequality, which is a consequence of the Littlewood-Hardy-Sobolev inequality from analysis, that plays an important role on the thesis. Appendix D contains some useful elementary results. Finally, in Appendix E we review some basic concepts related to the convergence of probability measures on $\mathbb{R}^k$, on a general metric space, on the space $C([0,1])$ of continuous functions on $[0,1]$, and on the space $C([0,1] \times \mathbb{R})$ of continuous functions on $[0,1] \times \mathbb{R}$. The latest of these results are used in the proofs of Theorems 2.4.1 and 3.6.1.

Chapter 2

Equations with white noise in time

In this chapter we review some results related to the linear stochastic heat and wave equations driven by a Gaussian noise which is white in time (i.e. which behaves in time like Brownian motion) and fractional in space with index $H \in (0,1)$.

The results in Sections 2.1 and 2.2 are presented following ideas from Dalang's seminal article [11] and the monograph [22], whereas those presented in Sections 2.3 and 2.4 are taken from the recent preprint [15].

2.1 The noise

In this section, we introduce the noise and we define the Wiener integral with respect to this noise.

Let $H \in (0,1)$ be arbitrary and let $W = \{W([0,t] \times A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R})\}$ be a zero mean Gaussian process defined on a probability space $(\Omega, \mathcal{F}, P)$ with covariance:

$$E[W([0,t] \times A)\ W([0,s] \times B)] = (t \wedge s) c_H \int_{\mathbb{R}} \mathcal{F}1_A(\xi)\overline{\mathcal{F}1_B(\xi)} |\xi|^{1-2H} \mathrm{d}\xi$$
$$=: \langle 1_{[0,t]\times A}\ ,\ 1_{[0,s]\times B} \rangle_{\mathcal{H}}.$$

where $H \in (0,1)$ and the constant c_H is given by (1.0.9). Here we denote by $\mathcal{B}_b(\mathbb{R})$ the set of bounded Borel sets of $\mathbb{R}$.

Remark 2.1.1. If $H > \frac{1}{2}$, by Remark A.6 and the fact that $t \wedge s = \int_0^\infty 1_{[0,t]}(u) 1_{[0,s]}(u) \mathrm{d}u$, we have:

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times B} \rangle_{\mathcal{H}} = (t \wedge s) \alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x) 1_B(y)\, |x-y|^{2H-2}\, \mathrm{d}x\mathrm{d}y \tag{2.1.1}$$
$$= \alpha_H \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[0,t]\times A}(u,x) 1_{[0,s]\times B}(u,y) |x-y|^{2H-2} \mathrm{d}x\mathrm{d}y\mathrm{d}u$$

$$= c_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F}1_{[0,t]\times A}(u,\cdot)(\xi)\overline{\mathcal{F}1_{[0,s]\times B}(u,\cdot)(\xi)}|\xi|^{1-2H} \mathrm{d}\xi \mathrm{d}u,$$

using the fact that $\mathcal{F}1_{[0,t]\times A}(u,\cdot)(\xi) = 1_{[0,t]}(u)\mathcal{F}1_A(\xi)$.

Note that W has the covariance structure of fractional Brownian motion with Hurst index H in space and that of Brownian motion in time. Letting $W(t,x) = W([0,t] \times [0,x])$ for all $t \geq 0$ and $x \in \mathbb{R}$, we see that $\{W(t,x); t \geq 0, x \in \mathbb{R}\}$ is a fractional Brownian sheet of index $H_0 = 1/2$ in time and index H in space.

We set $W(1_{[0,t]\times A}) = W([0,t] \times A)$. Let $\mathcal{E}$ be the set of linear combinations of indicator functions $1_{[0,t]\times A}$ with $t > 0$ and $A \in \mathcal{B}_b(\mathbb{R})$. By linearity, we extend W to $\mathcal{E}$. More precisely, for any $\varphi = \sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}$, with $a_i \in \mathbb{R}$ and $A_i \in \mathcal{B}_b(\mathbb{R})$, we define

$$W(\varphi) = \sum_{i=1}^n a_i W([0,t_i] \times A_i).$$

We endow $\mathcal{E}$ with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := c_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}\, |\xi|^{1-2H} \,\mathrm{d}\xi \mathrm{d}t.$$

We denote by $\|\cdot\|_{\mathcal{H}}$ the corresponding norm, given by: $\|\varphi\|^2_{\mathcal{H}} = \langle \varphi, \varphi \rangle_{\mathcal{H}}$. This means that for any $\varphi \in \mathcal{E}$,

$$\|\varphi\|^2_{\mathcal{H}} = c_H \int_0^\infty \int_{\mathbb{R}} |\mathcal{F}\varphi(t,\cdot)(\xi)|^2 |\xi|^{1-2H} \mathrm{d}\xi \mathrm{d}t.$$

If $H > \frac{1}{2}$, then by relation (2.1.1) and linearity, for any $\varphi, \psi \in \mathcal{E}$,

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t,x)\psi(t,y)\, |x-y|^{2H-2} \,\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

Lemma 2.1.2. *W is an isometry from $\mathcal{E}$ to $L^2(\Omega)$.*

Proof: We consider the following functions

$$\varphi(t,x) = \sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}(t,x) \text{ and } \psi(t,x) = \sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}(t,x).$$

Note that

$$E[\, W(\varphi)W(\psi) \,] = E\Big[\, W\big(\sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}\big) W\big(\sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}\big) \,\Big]$$

$$
\begin{aligned}
&= E\Big[\ \{\sum_{i=1}^{n} a_i W(1_{[0,t_i]\times A_i})\}\{\sum_{j=1}^{m} b_i W(1_{[0,s_j]\times B_j})\}\ \Big] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j E[\ W(1_{[0,t_i]\times A_i}) W(1_{[0,s_j]\times B_j})\ ] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j c_H \int_0^\infty \int_{\mathbb{R}} 1_{[0,t_i]}(t) 1_{[0,s_j]}(t) \mathcal{F}1_{A_i}(\xi)\overline{\mathcal{F}1_{B_j}(\xi)}\,|\xi|^{1-2H}\,\mathrm{d}\xi\mathrm{d}t \\
&= c_H \int_0^\infty \int_{\mathbb{R}} \Big\{\sum_{i=1}^{n} a_i \mathcal{F}1_{[0,t_i]\times A_i}(t,\cdot)(\xi)\Big\}\Big\{\sum_{j=1}^{m} b_j \overline{\mathcal{F}1_{[0,s_j]\times B_j}(t,\cdot)(\xi)}\Big\}|\xi|^{1-2H}\mathrm{d}\xi\mathrm{d}t \\
&= c_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}|\xi|^{1-2H}\mathrm{d}\xi\mathrm{d}t = \langle \varphi,\psi\rangle_{\mathcal{H}}.
\end{aligned}
$$

∎

We now proceed with the construction of the integral with respect to W. For this, let $\mathcal{H}$ be the closure of $\mathcal{E}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. This means that for any $\varphi \in \mathcal{H}$, these exists a sequence $(\varphi_n)_n$ in $\mathcal{E}$ such that $||\varphi_n - \varphi||_{\mathcal{H}} \to 0$. Note that $\{W(\varphi_n)\}_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega)$, since

$$
\begin{aligned}
E|W(\varphi_n) - W(\varphi_m)|^2 &= E|W(\varphi_n - \varphi_m)|^2 = \|\varphi_n - \varphi_m\|^2_{\mathcal{H}} \\
&= \|\varphi_n - \varphi + \varphi - \varphi_m\|^2_{\mathcal{H}} \leq 2(\|\varphi_n - \varphi\|^2_{\mathcal{H}} + \|\varphi - \varphi_m\|^2_{\mathcal{H}}) \\
&\to 0,
\end{aligned}
$$

when $n, m \to \infty$. Since $L^2(\Omega)$ is a Banach space, there exists a random variable $W(\varphi)$ in $L^2(\Omega)$ such that $W(\varphi_n) \to W(\varphi)$ in $L^2(\Omega)$, i.e.

$$E|W(\varphi_n) - W(\varphi)|^2 \to 0.$$

We say that $W(\varphi)$ is the *Wiener integral* of φ with respect to W.
We denote

$$W(\varphi) = \int_0^T \int_{\mathbb{R}} \varphi(t,x) W(\mathrm{d}t,\mathrm{d}x).$$

Note that $W(\varphi)$ has zero mean since convergence in $L^2(\Omega)$ implies convergence in $L^1(\Omega)$ (i.e. $E\,|W(\varphi_n) - W(\varphi)| \to 0$) which in turn implies that $E\Big(W(\varphi_n)\Big) \to E\Big(W(\varphi)\Big)$ since

$$\Big|E\Big(W(\varphi_n)\Big) - E\Big(W(\varphi)\Big)\Big| = \Big|E\Big(W(\varphi_n) - W(\varphi)\Big)\Big| \leq E\,|W(\varphi_n) - W(\varphi)| \to 0.$$

Moreover, $W(\varphi)$ is a normal random variable with mean 0 and variance

$$E|W(\varphi)|^2 = ||\varphi||_{\mathcal{H}}^2.$$

The following theorem is a particular case of Theorem 1.4 of [2], which gives some sufficient conditions for a function φ to be in the Hilbert space $\mathcal{H}$.

Theorem 2.1.3. *(a) Let $H \in (\frac{1}{2}, 1)$ be arbitrary. If $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a function such that*

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(t,x)||\varphi(t,y)||x-y|^{2H-2} \mathrm{d}x\mathrm{d}y\mathrm{d}t < \infty,$$

then $\varphi \in \mathcal{H}$.
(b) Let $H \in (0,1)$ be arbitrary. If $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ is such that $\varphi(t,\cdot) \in L^1(\mathbb{R})$ for all $t \in [0,T]$ and

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}\varphi(t,\cdot)(\xi)|^2 |\xi|^{1-2H} \mathrm{d}\xi\mathrm{d}t < \infty,$$

then $\varphi \in \mathcal{H}$, and in this case

$$||\varphi||_{\mathcal{H}}^2 = c_H \int_0^T \int_{\mathbb{R}} |\mathcal{F}\varphi(t,\cdot)(\xi)|^2 |\xi|^{1-2H} \mathrm{d}\xi\mathrm{d}t.$$

2.2 Existence of solutions

In this section, we study the problem of existence of solution for the linear heat and wave equation with noise W introduced in Section 2.1. We first consider the linear stochastic heat equation (1.0.1) with noise W as in Section 2.1. To show that the solution to this equation exists, we must show that for any $t > 0$ and $x \in \mathbb{R}$, the function $g_{t,x}^h$ belongs to the space $\mathcal{H}$ defined in Section 2.1, where

$$g_{t,x}^h(s,y) = 1_{[0,t]}(s) G^h(t-s, x-y).$$

To do this, we will apply Theorem 2.1.3.(b) with $T = t$, which requires some estimates on the Fourier transform of $g_{t,x}^h$ in the space variable y.

Recall that the Fourier transform in the space variable of the fundamental solution of the heat equation is given by relation (1.0.7). The following result gives an upper and lower bound of the time integral of the square of this Fourier transform.

Lemma 2.2.1. *For any $t > 0$ and $\xi \in \mathbb{R}$,*

$$\frac{1}{2} \cdot \frac{t \wedge 1}{1 + |\xi|^2} \leq \int_0^t |\mathcal{F}G^h(s,\cdot)(\xi)|^2 \mathrm{d}s \leq 2 \cdot \frac{t \vee 1}{1 + |\xi|^2}.$$

Proof: We use the same argument as in Lemma 6.1 of [22]. Note that

$$\begin{aligned}\int_0^t |\mathcal{F}G^h(s,\cdot)(\xi)|^2 \mathrm{d}s &= \int_0^t |\mathrm{e}^{-\frac{s|\xi|^2}{2}}|^2 \mathrm{d}s = -\frac{1}{|\xi|^2}\mathrm{e}^{-s|\xi|^2}\Big|_0^t \\ &= -\frac{1}{|\xi|^2}\mathrm{e}^{-t|\xi|^2} + \frac{1}{|\xi|^2} = \frac{1-\mathrm{e}^{-t|\xi|^2}}{|\xi|^2}.\end{aligned}$$

We will use the following inequalities:

$$1-\mathrm{e}^{-x} \le x, \qquad \text{for any } x \ge 0, \tag{2.2.1}$$

$$1-\mathrm{e}^{-x} \ge \frac{x}{1+x}, \quad \text{for any } x \ge 0. \tag{2.2.2}$$

When $|\xi| > 1$, we use the inequality $1 - e^{-x} \le 1$, for all $x > 0$:

$$\int_0^t |\mathcal{F}G^h(s,\cdot)(\xi)|^2 \mathrm{d}s = \frac{1-\mathrm{e}^{-t|\xi|^2}}{|\xi|^2} \le \frac{1}{|\xi|^2} = \frac{2}{2|\xi|^2} \le \frac{2}{1+|\xi|^2}.$$

When $|\xi| \le 1$, we use inequality (2.2.1):

$$\int_0^t |\mathcal{F}G^h(s,\cdot)(\xi)|^2 \mathrm{d}s = \frac{1-\mathrm{e}^{-t|\xi|^2}}{|\xi|^2} \le t \le \frac{2t}{1+|\xi|^2}.$$

Then we obtain the following upper bound:

$$\int_0^t |\mathcal{F}G^h(s,\cdot)(\xi)|^2 \mathrm{d}s = \frac{1-\mathrm{e}^{-t|\xi|^2}}{|\xi|^2} \le \max\left\{\frac{2}{1+|\xi|^2}, \frac{2t}{1+|\xi|^2}\right\} = (t \vee 1)\frac{2}{1+|\xi|^2}.$$

For the lower bound, we use inequality (2.2.2). We obtain:

$$\int_0^t |\mathcal{F}G^h(s,\cdot)(\xi)|^2 \mathrm{d}s = \frac{1-\mathrm{e}^{-t|\xi|^2}}{|\xi|^2} \ge \frac{t}{1+t|\xi|^2}.$$

If $t|\xi|^2 > 1$, then

$$\frac{t}{1+t|\xi|^2} \ge \frac{t}{t|\xi|^2 + t|\xi|^2} \ge \frac{t}{2t(1+|\xi|^2)} = \frac{1}{2(1+|\xi|^2)}.$$

If $t|\xi|^2 \le 1$, then

$$\frac{t}{1+t|\xi|^2} \ge \frac{t}{2} \ge \frac{t}{2}\cdot\frac{1}{1+|\xi|^2} = \frac{t}{2(1+|\xi|^2)}.$$

We get the lower bound as follows:

$$\int_0^t |\mathcal{F}G^h(s,\cdot)(\xi)|^2 \mathrm{d}s = \frac{1-\mathrm{e}^{-t|\xi|^2}}{|\xi|^2} \ge \frac{1}{2(1+|\xi|^2)} \wedge \frac{t}{2(1+|\xi|^2)} = \frac{1\wedge t}{2(1+|\xi|^2)}.$$

∎

We will use the following elementary result.

Lemma 2.2.2. *For any* $H \in (0,1)$,

$$I = \int_{\mathbb{R}} \frac{1}{1+|\xi|^2} |\xi|^{1-2H} \mathrm{d}\xi < \infty.$$

Proof: We consider separately the integrals $I_1 = \int_{|\xi|\leq 1} \frac{1}{1+|\xi|^2}|\xi|^{1-2H}\mathrm{d}\xi$ and $I_2 = \int_{|\xi|>1} \frac{1}{1+|\xi|^2}|\xi|^{1-2H}\mathrm{d}\xi$. First, we consider I_1:

$$I_1 = \int_{|\xi|\leq 1} \frac{1}{1+|\xi|^2}|\xi|^{1-2H}\mathrm{d}\xi \leq \int_{|\xi|\leq 1} |\xi|^{1-2H}\mathrm{d}\xi = 2\int_0^1 \xi^{1-2H}\mathrm{d}\xi < \infty,$$

since $1-2H+1 = 2-2H > 0$. Next, we consider I_2. We have

$$I_2 = \int_{|\xi|>1} \frac{1}{1+|\xi|^2}|\xi|^{1-2H}\mathrm{d}\xi \leq \int_{|\xi|>1} \frac{|\xi|^{1-2H}}{|\xi|^2}\mathrm{d}\xi = 2\int_1^{+\infty} \xi^{-1-2H}\mathrm{d}\xi < \infty,$$

since $-1-2H+1 = -2H < 0$. ∎

Theorem 2.2.3. *For any* $H \in (0,1)$, *equation* (1.0.1) *with noise* W *as in Section 2.1 has a unique solution.*

Proof: We have to prove that the function $g^h_{t,x}(s,y) = G^h(t-s,x-y)1_{[0,t]}(s)$ belongs to $\mathcal{H}$ so that the Wiener integral which appears on the right hand side of equation (1.0.3) is well defined. By applying Theorem 2.1.3(b) to the function $\varphi = g^h_{t,x}$, it is enough to prove that

$$\int_0^t \int_{\mathbb{R}} |\mathcal{F}g^h_{t,x}(s,\cdot)(\xi)|^2 |\xi|^{1-2H}\mathrm{d}\xi\mathrm{d}s < \infty. \tag{2.2.3}$$

By Lemma A.2,

$$\mathcal{F}g^h_{t,x}(s,\cdot)(\xi) = \mathcal{F}G^h(t-s,x-\cdot)(\xi)1_{[0,t]}(s) = \mathrm{e}^{-i\xi x}\overline{\mathcal{F}G^h(t-s,\cdot)(\xi)}1_{[0,t]}(s).$$

Hence, relation (2.2.3) becomes (since $|\mathrm{e}^{-i\xi x}| = 1$)

$$\int_0^t \int_{\mathbb{R}} |\mathcal{F}G^h(t-s,\cdot)(\xi)|^2 |\xi|^{1-2H}\mathrm{d}\xi\mathrm{d}s < \infty.$$

By the substitution $\bar{s} = t - s$, this is equivalent to:

$$I_t := \int_0^t \int_{\mathbb{R}} |\mathcal{F}G^h(s,\cdot)(\xi)|^2 |\xi|^{1-2H} \mathrm{d}\xi \mathrm{d}s < \infty. \tag{2.2.4}$$

By Fubini's Theorem,

$$I_t = \int_{\mathbb{R}} \left(\int_0^t |\mathcal{F}G^h(s,\cdot)(\xi)|^2 \mathrm{d}s \right) |\xi|^{1-2H} \mathrm{d}\xi.$$

We multiply the inequalities in Lemma 2.2.1 by $|\xi|^{1-2H}$ and then we integrate with respect to ξ. We obtain:

$$c_1(t \wedge 1) \int_{\mathbb{R}} \frac{1}{1+|\xi|^2} |\xi|^{1-2H} \mathrm{d}\xi \leq I_t \leq c_2(t \wedge 1) \int_{\mathbb{R}} \frac{1}{1+|\xi|^2} |\xi|^{1-2H} \mathrm{d}\xi.$$

Finally, note that $\int_{\mathbb{R}} \frac{1}{1+|\xi|^2} |\xi|^{1-2H} \mathrm{d}\xi < \infty$ by Lemma 2.2.2. Hence $I_t < \infty$. ∎

We consider now the linear stochastic wave equation. Recall equation (1.0.8) which was stated in the introduction.

Lemma 2.2.4. *For any $t \geq 0$, $\xi \in \mathbb{R}$, it holds that*

$$\frac{\cos^2(1)}{3} (t \wedge t^3) \frac{1}{1+|\xi|^2} \leq \int_0^t |\mathcal{F}G^w(s,\cdot)(\xi)|^2 \mathrm{d}s \leq 2(t \vee t^3) \frac{1}{1+|\xi|^2}.$$

Proof: If $|\xi| \leq 1$, then $|\sin(x)| \leq |x|$ for all $x > 0$, and consequently, $\sin^2(s|\xi|) \leq s^2|\xi|^2$,

$$\frac{\sin^2(s|\xi|)}{|\xi|^2} \leq \frac{s^2|\xi|^2}{|\xi|^2} = s^2.$$

In this case,

$$\int_0^t |\mathcal{F}G^w(s,\cdot)(\xi)|^2 \mathrm{d}s \leq \int_0^t s^2 \mathrm{d}s = \frac{t^3}{3} \leq \frac{t^3}{3} \cdot \frac{2}{1+|\xi|^2}.$$

If $|\xi| \geq 1$, we use the fact that $|\sin(s\xi)| \leq 1$. Therefore, $\frac{\sin^2(s|\xi|)}{|\xi|^2} \leq \frac{1}{|\xi|^2}$, and

$$\int_0^t |\mathcal{F}G^w(s,\cdot)(\xi)|^2 \mathrm{d}s \leq \frac{1}{|\xi|^2} \int_0^t \mathrm{d}s = t \cdot \frac{1}{|\xi|^2} \leq t \cdot \frac{2}{1+|\xi|^2}.$$

Hence,

$$\int_0^t |\mathcal{F}G^w(s,\cdot)(\xi)|^2 \mathrm{d}s \leq 2 \frac{t \vee t^3}{1+|\xi|^2}.$$

This yield the upper bound.

We now treat the lower bound. Assuming that $t|\xi| \leq 1$, and applying the inequality $\sin x \geq x\cos 1$ for any $x \in [0,1]$, we have:

$$\int_0^t \frac{\sin^2(s|\xi|)}{|\xi|^2}\mathrm{d}s \geq \cos^2(1)\int_0^t s^2\mathrm{d}s = \frac{\cos^2(1)}{3}t^3 \geq \frac{\cos^2(1)}{3}\frac{t^3}{1+|\xi|^2}.$$

Next, we assume that $t|\xi| > 1$. Then

$$\begin{aligned}\int_0^t \frac{\sin^2(s|\xi|)}{|\xi|^2}\mathrm{d}s &= \frac{1}{2|\xi|^2}\int_0^t \left[1-\cos\left(2s|\xi|\right)\right]\mathrm{d}s \\ &= \frac{1}{2|\xi|^2}\left[t - \frac{1}{2|\xi|}\sin\left(2t|\xi|\right)\right].\end{aligned}$$

Then, using the fact that $\sin(2x) \leq x$ if $x > 1$, we have

$$\int_0^t \frac{\sin^2(s|\xi|)}{|\xi|^2}\mathrm{d}s \geq \frac{t}{4|\xi|^2},$$

and since $\frac{1}{|\xi|^2} \geq \frac{1}{1+|\xi|^2}$, we obtain the following:

$$\int_0^t \frac{\sin^2(s|\xi|)}{|\xi|^2}\mathrm{d}s \geq \frac{1}{4}\frac{t}{1+|\xi|^2}.$$

Thus,

$$\int_0^t |\mathcal{F}G^w(s,\cdot)(\xi)|^2\mathrm{d}s \geq \frac{\cos^2(1)}{3}\frac{t^3}{1+|\xi|^2} \wedge \frac{1}{4}\frac{t}{1+|\xi|^2} \geq \frac{\cos^2(1)}{3}\frac{t\wedge t^3}{1+|\xi|^2}.$$

∎

Theorem 2.2.5. *For any $H \in (0,1)$, equation* (1.0.2) *with noise W as in Section 2.1 has unique solution.*

Proof: We use the same argument as for Theorem 2.2.3 using this time the inequalities given by Lemma 2.2.4. ∎

2.3 Moment estimates

In this section, we include some estimates for the moments of the increments of the solutions to equations (1.0.1) and (1.0.2) with noise W given in Section 2.1. We follow very closely the method of [15]. This result is used in the proof of Theorem 2.4.1.

We list below a fact that will be used in the following proofs: if X is a centered normal random variable, then for any $p > 0$,

$$E|X|^p = z_p(E|X|^2)^{p/2}, \tag{2.3.1}$$

where $z_p = E|Z|^p$ and Z is a standard normal random variable. The exact form of the constant z_p is given by Lemma D.1. Note that relation (2.3.1) is proved simply by writing $X = \sigma Z$, where $\sigma^2 = E|X|^2$; then $E|X|^p = \sigma^p E|Z|^p = \sigma^p z_p$.

We begin with some preliminary results taken from [4].

Lemma 2.3.1. *[Lemma 3.5 of [4]] Let G be the fundamental solution of the heat or wave equation. For any $\alpha \in (-1, 1)$ and for any $h \in \mathbb{R}$,*

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}G(t+h, y) - \mathcal{F}G(t, y)|^2 |\xi|^\alpha \mathrm{d}\xi \mathrm{d}t \leq \begin{cases} C_\alpha |h|^{(1-\alpha)/2} & \textit{for heat equation} \\ C'_\alpha T |h|^{1-\alpha} & \textit{for wave equation} \end{cases}$$

where

$$C_\alpha = \int_{\mathbb{R}} (1 - e^{-\eta^2/2})^2 |\eta|^{\alpha-2} \mathrm{d}\eta \quad \textit{and} \quad C'_\alpha = 4 \int_{\mathbb{R}} \min\left(1, |\eta|^2\right) |\eta|^{\alpha-2} \mathrm{d}\eta.$$

Lemma 2.3.2. *[Lemma 3.1 of [4]] Let G be the fundamental solution of the heat or wave equation. The integral*

$$A_T(\alpha) := \int_0^T \int_{\mathbb{R}} |\mathcal{F}G(t, \cdot)(\xi)|^2 |\xi|^\alpha \mathrm{d}\xi \mathrm{d}t$$

converges if and only if $\alpha \in (-1, 1)$. When the integral converges, we have

$$A_T(\alpha) = \begin{cases} \frac{2}{1-\alpha} \, \Gamma\left(\frac{\alpha+1}{2}\right) \, T^{(1-\alpha)/2} & \textit{for heat equation} \\ 2^{1-\alpha} \, \widetilde{C_\alpha} \, \frac{1}{2-\alpha} \, T^{2-\alpha} & \textit{for wave equation} \end{cases}$$

where

$$\widetilde{C_\alpha} = \begin{cases} (1-\alpha)^{-1} \Gamma(\alpha) \sin(\pi\alpha/2) & \textit{if } \alpha \in (0, 1), \\ \alpha^{-1}(1-\alpha)^{-1} \Gamma(1+\alpha) \sin(\pi\alpha/2) & \textit{if } \alpha \in (-1, 0), \\ \pi/2 & \textit{if } \alpha = 0. \end{cases}$$

Lemma 2.3.3. *[Lemma 3.4 of [4]] Let G be the fundamental solution of the heat or wave equation. For any $\alpha \in (-1,1)$ and for any $h \in \mathbb{R}$,*

$$\int_0^T \int_{\mathbb{R}} (1-\cos(\xi h))\ |\mathcal{F}G(t,\cdot)(\xi)|^2\ |\xi|^{\alpha} \mathrm{d}\xi\ \mathrm{d}t \le \begin{cases} \bar{C}_{\alpha}|h|^{1-\alpha} & \textit{for heat equation} \\ \bar{C}_{\alpha}T|h|^{1-\alpha} & \textit{for wave equation} \end{cases}$$

where $\bar{C}_{\alpha} = \int_{\mathbb{R}}(1-\cos\eta)|\eta|^{\alpha-2}\mathrm{d}\eta$.

The following result is essentially contained in Step 1 of the proof of Theorem 2.8 of [15].

Theorem 2.3.4. *Let u be the solution of equation (1.0.1) or equation (1.0.2) with noise W as in Section 2.1. Then, for any $p>0, H\in(0,1), t,t'\in[0,T]$ and $x,x'\in\mathbb{R}$, we have*

$$E|u(t',x)-u(t,x)|^p \le \begin{cases} z_p (C_H^{(1)})^{p/2}\big|t'-t\big|^{pH/2} & \textit{for heat equation} \\ z_p T^{p/2} (C_H^{(2)})^{p/2}\big|t'-t\big|^{pH} & \textit{for wave equation} \end{cases}$$

$$E\big|u(t,x')-u(t,x)\big|^p \le \begin{cases} z_p (C_H^{(3)})^{p/2}\big|x'-x\big|^{pH} & \textit{for heat equation} \\ z_p T^{p/2} (C_H^{(3)})^{p/2}\big|x'-x\big|^{pH} & \textit{for wave equation} \end{cases}$$

where

$$C_H^{(1)} = c_H\left(N_H + \frac{1}{H}\Gamma(1-H)\right), \quad C_H^{(2)} = c_H\left(M_H + 2^{2H}\widetilde{C}_{1-2H}\frac{1}{1+2H}\right),$$

$$C_H^{(3)} = 2c_H\int_{\mathbb{R}}\frac{1-\cos\eta}{|\eta|^{1+2H}}\mathrm{d}\eta = 2c_H\bar{C}_{1-2H},$$

$$N_H = \int_{\mathbb{R}}\frac{(1-\mathrm{e}^{-\frac{\eta^2}{2}})^2}{|\eta|^{1+2H}}\mathrm{d}\eta \quad \textit{and} \quad M_H = 4\int_{\mathbb{R}}\frac{\min(1,|\eta|^2)}{|\eta|^{1+2H}}\mathrm{d}\eta.$$

Proof: Recall that the solution u is of the form:

$$u(t,x) = \int_0^t\int_{\mathbb{R}} G(t-s,x-y)W(\mathrm{d}s,\mathrm{d}y),$$

where $G=G^h$ for heat equation, respectively $G=G^w$ for wave equation.

We first treat the time increments. Note that $u(t,x)$ is a centered normal random variable. Using relation (2.3.1), it is enough to consider only the case $p=2$, since

$$E\big|u(t',x)-u(t,x)\big|^p = z_p(E\big|u(t',x)-u(t,x)\big|^2)^{p/2}.$$

Let $t'\ge t$ and denote $h=t'-t$. Then

$u(t+h,x)-u(t,x)$

$$
\begin{aligned}
&= \int_0^{t+h} \int_{\mathbb{R}} G(t+h-s, x-y) W(\mathrm{d}s, \mathrm{d}y) - \int_0^t \int_{\mathbb{R}} G(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y) \\
&= \int_0^t \int_{\mathbb{R}} \Big(G(t+h-s, x-y) - G(t-s, x-y) \Big) W(\mathrm{d}s, \mathrm{d}y) \\
&\qquad + \int_t^{t+h} \int_{\mathbb{R}} G(t+h-s, x-y) W(\mathrm{d}s, \mathrm{d}y) =: I_1 + I_2. \quad (2.3.2)
\end{aligned}
$$

Notice that I_1 and I_2 are uncorrelated. To see this, note that $I_1 = W(g_1)$ and $I_2 = W(g_2)$, where

$$
\begin{aligned}
g_1(s, y) &= 1_{[0,t]}(s) \Big(G(t+h-s, x-y) - G(t-s, x-y) \Big), \\
g_2(s, y) &= 1_{[t,t+h]}(s) G(t+h-s, x-y).
\end{aligned}
$$

Hence

$$
\begin{aligned}
E[I_1 I_2] = E[W(g_1)W(g_2)] &= \langle g_1, g_2 \rangle_{\mathcal{H}} \\
&= c_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F} g_1(s, \cdot)(\xi) \overline{\mathcal{F} g_2(s, \cdot)(\xi)} |\xi|^{1-2H} \mathrm{d}\xi \mathrm{d}s \\
&= 0
\end{aligned}
$$

because $\mathcal{F} g_1(s, \cdot)(\xi)$ contains $1_{[0,t]}(s)$ and $\mathcal{F} g_2(s, \cdot)(\xi)$ contains $1_{[t,t+h]}(s)$.

Thus, we have:

$$
E \big| u(t+h, x) - u(t, x) \big|^2 = A(t, h) + B(t, h), \quad (2.3.3)
$$

where

$$
\begin{aligned}
A(t, h) &:= E \left| \int_0^t \int_{\mathbb{R}} G(t+h-s, x-y) - G(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y) \right|^2, \\
B(t, h) &:= E \left| \int_t^{t+h} \int_{\mathbb{R}} G(t+h-s, x-y) W(\mathrm{d}s, \mathrm{d}y) \right|^2.
\end{aligned}
$$

We start by estimating $A(t, h)$. Using the change of variable $s' = t - s$, we obtain:

$$
\begin{aligned}
A(t, h) &= E \left| \int_0^t \int_{\mathbb{R}} G(t+h-s, x-y) - G(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y) \right|^2 \\
&= c_H \int_0^t \int_{\mathbb{R}} \big| \mathcal{F} G(s'+h, x-\cdot)(\xi) - \mathcal{F} G(s', x-\cdot)(\xi) \big|^2 \big| \xi \big|^{1-2H} \mathrm{d}\xi \mathrm{d}s'
\end{aligned}
$$

$$= c_H \int_0^t \int_{\mathbb{R}} \left|\mathcal{F}G(s'+h,\cdot)(\xi) - \mathcal{F}G(s',\cdot)(\xi)\right|^2 \left|\xi\right|^{1-2H} \mathrm{d}\xi \mathrm{d}s',$$

where we used the fact that $\mathcal{F}G(s, x-\cdot)(\xi) = e^{-i\xi x}\overline{\mathcal{F}G(s,\cdot)(\xi)}$ from Lemma A.2. Thus, by Lemma 2.3.1 with $\alpha = 1-2H$,

$$A(t,h) \leq \begin{cases} c_H N_H h^H & \text{for heat equation,} \\ c_H M_H T h^{2H} & \text{for wave equation.} \end{cases} \tag{2.3.4}$$

Next, we consider $B(t,h)$. Notice that $B(t,h)$ in fact does not depend on t. Using the change of variable $s' = t+h-s$, we obtain:

$$\begin{aligned} B(t,h) &= E\left|\int_t^{t+h}\int_{\mathbb{R}} G(t+h-s, x-y) W(\mathrm{d}s, \mathrm{d}y)\right|^2 \\ &= c_H \int_t^{t+h}\int_{\mathbb{R}} \left|\mathcal{F}G(t+h-s, x-\cdot)(\xi)\right|^2 \left|\xi\right|^{1-2H} \mathrm{d}\xi \mathrm{d}s \\ &= c_H \int_0^h \int_{\mathbb{R}} \left|\mathcal{F}G(s',\cdot)(\xi)\right|^2 \left|\xi\right|^{1-2H} \mathrm{d}\xi \mathrm{d}s', \end{aligned}$$

using again Lemma A.2 for the last line. Thus, by Lemma 2.3.2,

$$B(t,h) \leq \begin{cases} c_H \ \frac{1}{H}\ \Gamma(1-H)\ h^H & \text{for heat equation,} \\ c_H \ 2^{2H}\widetilde{C}_{1-2H}\ \frac{1}{1+2H}\ h^{1+2H} & \text{for wave equation.} \end{cases} \tag{2.3.5}$$

We now use equality (2.3.3), combined with inequalities (2.3.4) and (2.3.5). For heat equation, we have

$$\begin{aligned} E\left|u^h(t+h,x) - u^h(t,x)\right|^2 &\leq c_H N_H h^H + c_H \frac{1}{H}\Gamma(1-H) h^H \\ &= C_H^{(1)} h^H, \end{aligned}$$

whereas for wave equation,

$$\begin{aligned} E\left|u^w(t+h,x) - u^w(t,x)\right|^2 &\leq c_H M_H T h^{2H} + c_H 2^{2H} \tilde{C}_{1-2H} \frac{1}{1+2H}\ h^{1+2H} \\ &\leq c_H T\Big(M_H + 2^{2H}\tilde{C}_{1-2H}\frac{1}{1+2H}\Big) h^{2H} \\ &= C_H^{(2)} T h^{2H}. \end{aligned}$$

using the fact that $h^{1+2H} \leq T h^{2H}$ since $h \leq T$.

Finally, for heat equation,

$$E\left|u^h(t+h,x) - u^h(t,x)\right|^p = z_p\Big(E\left|u^h(t+h,x) - u^h(t,x)\right|^2\Big)^{p/2} \ \ \leq z_p\Big(C_H^{(1)}\Big)^{p/2} h^{pH/2},$$

and for wave equation,

$$E\big|u^w(t+h,x)-u^w(t,x)\big|^p = z_p\Big(E\big|u^w(t+h,x)-u^w(t,x)\big|^2\Big)^{p/2} \quad \le z_p\Big(C_H^{(2)}T\Big)^{p/2}h^{pH}.$$

We now treat the space increments. By relation (2.3.1), it suffices to consider the case $p=2$, since

$$E\big|u(t,x')-u(t,x)\big|^p = z_p(E\big|u(t,x')-u(t,x)\big|^2)^{p/2}.$$

Let $x'\ge x$ and denote $z=x'-x$. We have

$$u(t,x')-u(t,x) = \int_0^t\int_{\mathbb{R}} G(t-s,x+z-y)W(\mathrm{d}s,\mathrm{d}y) - \int_0^t\int_{\mathbb{R}} G(t-s,x-y)W(\mathrm{d}s,\mathrm{d}y).$$

Then,

$$\begin{aligned}
C(t,z) &:= E\left|u(t,x+z)-u(t,x)\right|^2 \\
&= E\left|\int_0^t\int_{\mathbb{R}} G(t-s,x+z-y)W(\mathrm{d}s,\mathrm{d}y) - G(t-s,x-y)W(\mathrm{d}s,\mathrm{d}y)\right|^2 \\
&= c_H\int_0^t\int_{\mathbb{R}} \left|\mathcal{F}\big(G(t-s,x+z-\cdot)-G(t-s,x-\cdot)\big)(\xi)\right|^2\left|\xi\right|^{1-2H}\mathrm{d}\xi\mathrm{d}s \\
&= c_H\int_0^t\int_{\mathbb{R}} \left|\mathcal{F}\big(G(s',x+z-\cdot)-G(s',x-\cdot)\big)(\xi)\right|^2\left|\xi\right|^{1-2H}\mathrm{d}\xi\mathrm{d}s' \\
&= 2\ c_H\int_0^t\int_{\mathbb{R}} \Big(1-\cos(\xi z)\Big)\left|\mathcal{F}G(s',\cdot)(\xi)\right|^2\left|\xi\right|^{1-2H}\mathrm{d}\xi\mathrm{d}s'.
\end{aligned}$$

For the last equality we used the fact that:

$$\begin{aligned}
&\left|\mathcal{F}G(s',x+z-\cdot)(\xi)-\mathcal{F}G(s',x-\cdot)(\xi)\right|^2 \\
&= \left|\mathrm{e}^{-i\xi(x+z)}\overline{\mathcal{F}G(s',\cdot)(\xi)} - \mathrm{e}^{-i\xi x}\overline{\mathcal{F}G(s',\cdot)(\xi)}\right|^2 = \left|(\mathrm{e}^{-i\xi(x+z)}-\mathrm{e}^{-i\xi x})\overline{\mathcal{F}G(s',\cdot)(\xi)}\right|^2 \\
&= \left|\mathrm{e}^{-i\xi x}(\mathrm{e}^{-i\xi z}-1)\overline{\mathcal{F}G(s',\cdot)(\xi)}\right|^2 = \left|\mathrm{e}^{-i\xi x}\right|^2\left|\mathrm{e}^{-i\xi z}-1\right|^2\left|\mathcal{F}G(s',\cdot)(\xi)\right|^2 \\
&= 2\Big(1-\cos(\xi z)\Big)\left|\mathcal{F}G(s',\cdot)(\xi)\right|^2,
\end{aligned}$$

where we used Lemma A.2, and the fact that for any $a\in\mathbb{R}$,

$$|e^{ia}-1|^2 = (\cos a-1)^2+\sin^2 a = \cos^2 a+1-2\cos a+\sin^2 a = 2(1-\cos a).$$

By Lemma 2.3.3 with $\alpha = 1-2H$, we obtain that

$$C(t,z) \le \begin{cases} 2c_H\overline{C}_{1-2H}|z|^{2H} & \text{for heat equation} \\ 2c_H\overline{C}_{1-2H}T|z|^{2H} & \text{for wave equation,} \end{cases} \tag{2.3.6}$$

where $\overline{C}_{1-2H} = \int_{\mathbb{R}}(1-\cos\eta)|\eta|^{-1-2H}\mathrm{d}\eta$.

Finally, by relation (2.3.1),

$$E\big|u(t,x+z)-u(t,x)\big|^p = z_p\Big(E\big|u(t,x+z)-u(t,x)\big|^2\Big)^{p/2} \le \begin{cases} z_p(C_H^{(3)})^{p/2}|z|^{pH} & \text{for heat equation,} \\ z_p(C_H^{(3)})^{p/2}T^{p/2}|z|^{pH} & \text{for wave equation.} \end{cases}$$

∎

The following lemma gives a bound for the constant N_H and the explicit expression of the constant M_H, appearing in Theorem 2.3.4.

Lemma 2.3.5. *For any $H \in (0,1)$,*

$$N_H \le \frac{1}{4(2-H)} + \frac{1}{H} \text{ and } M_H = 4\Big(\frac{1}{H} + \frac{1}{1-H}\Big).$$

Proof:

$$\begin{aligned} N_H &= \int_{\mathbb{R}} \frac{(1-\mathrm{e}^{-\frac{\eta^2}{2}})^2}{|\eta|^{1+2H}}\mathrm{d}\eta = \int_{|\eta|\le 1} \frac{(1-\mathrm{e}^{-\frac{\eta^2}{2}})^2}{|\eta|^{1+2H}}\mathrm{d}\eta + \int_{|\eta|>1} \frac{(1-\mathrm{e}^{-\frac{\eta^2}{2}})^2}{|\eta|^{1+2H}}\mathrm{d}\eta \\ &\le \int_{|\eta|\le 1} \frac{(-\frac{\eta^2}{2})^2}{|\eta|^{1+2H}}\mathrm{d}\eta + \int_{|\eta|>1} \frac{1}{|\eta|^{1+2H}}\mathrm{d}\eta = 2\cdot\frac{1}{4}\int_0^1 \eta^{3-2H}\mathrm{d}\eta + 2\int_1^\infty \eta^{-1-2H}\mathrm{d}\eta \\ &= \frac{1}{2}\cdot\frac{1}{4-2H} + 2\cdot\frac{1}{2H}, \end{aligned}$$

where for the integral on the set $\{|\eta| \le 1\}$, we used the fact that $1-\mathrm{e}^{-x} \le x$ for any $x > 0$. Similarly,

$$\begin{aligned} M_H &= 4\int_{\mathbb{R}} \frac{\min(1,|\eta|^2)}{|\eta|^{1+2H}}\mathrm{d}\eta = 4\Big(\int_{|\eta|>1} \frac{1}{|\eta|^{1+2H}}\mathrm{d}\eta + \int_{|\eta|\le 1} \frac{|\eta|^2}{|\eta|^{2H+1}}\mathrm{d}\eta\Big) \\ &= 4\Big(\frac{1}{H} + \frac{1}{1-H}\Big). \end{aligned}$$

∎

Lemma 2.3.6. *For any $\alpha \in (-1,1)$,*

$$\bar{C}_\alpha \le 2\Big(\frac{1}{1-\alpha} + \frac{1}{1+\alpha}\Big).$$

Proof: We have:

$$\bar{C}_\alpha = \int_{\mathbb{R}} \frac{1-\cos\eta}{|\eta|^{2-\alpha}}\mathrm{d}\eta \le \left(\int_{|\eta|>1} \frac{1}{|\eta|^{2-\alpha}}\mathrm{d}\eta + \int_{|\eta|\le 1} \frac{\eta^2}{|\eta|^{2-\alpha}}\mathrm{d}\eta \right)$$
$$= 2\int_1^\infty \eta^{\alpha-2}\mathrm{d}\eta + 2\int_0^1 \eta^\alpha \mathrm{d}\eta = 2\Big(\frac{1}{1-\alpha} + \frac{1}{1+\alpha}\Big),$$

where for the integral on the set $\{|\eta| \le 1\}$, we used the fact that $1-\cos(x) \le x^2$ for all $x > 0$. ∎

Remark 2.3.7. Note that for any $H \in (0,1)$,

$$c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \le \frac{\Gamma(3)}{2\pi} = \frac{1}{\pi}.$$

Remark 2.3.8. Taking formally $H = 1/2$ in Theorem 2.3.4, we obtain moment bounds which are consistent with those given in Appendix B for the solutions to equations (1.0.1) and (1.0.2) with space-time white noise $\dot{W}$.

2.4 Continuity in law of the solution with respect to H

In this section, we consider the stochastic heat and wave equations with Gaussian noise which is white in time and behaves in space like a fractional Brownian motion with Hurst index $H \in (0,1)$. We prove that the solution of either one of these equations is continuous in law in the space of continuous functions $C([0,T]\times\mathbb{R})$, with respect to H. Note that by Kolmogorov's continuity criterion (Theorem C.6 of [17]) and the moment estimates given by Theorem 2.3.4, the solution has a modification with sample paths in $C([0,T]\times\mathbb{R})$. We work with this modification.

The following result has been recently proved in reference [15].

Theorem 2.4.1. *Let W_H be Gaussian noise introduced in Section 2.1 which is white in time and fractional in space with $H \in (0,1)$. We denote by*

$$u_H^h(t,x) = \int_0^t \int_{\mathbb{R}} G^h(t-s,x-y) W_H(\mathrm{d}s,\mathrm{d}y),$$

the solution of the stochastic heat equation (1.0.1)*, and by*

$$u_H^w(t,x) = \int_0^t \int_{\mathbb{R}} G^w(t-s,x-y) W_H(\mathrm{d}s,\mathrm{d}y),$$

the solution of the stochastic wave equation (1.0.2). *We fix* $T > 0$ *and we consider the modifications of these processes with sample paths in* $C([0,T] \times \mathbb{R})$*, which we denote also by* u_H^h, u_H^w*, respectively. If* $H_n \to H \in (0,1)$*, then*

$$u_{H_n}^h \xrightarrow{d} u_H^h \text{ in } C([0,T] \times \mathbb{R}),$$

and

$$u_{H_n}^w \xrightarrow{d} u_H^w \text{ in } C([0,T] \times \mathbb{R}).$$

Proof: To simplify the writing, we drop the upper indices h and w from the notation, whenever the calculations are valid for both heat and wave equations. We denote $u_n = u_{H_n}$ and $u = u_H$. We apply Theorem E.4.3 (see also Remark E.4.4). Note that condition (ii) of this theorem clearly holds since $u_n(0,0) = 0$ for all $n \geq 1$.

Step 1: We verify that condition (i) of Theorem E.4.3 holds, i.e. for any $k \geq 1$ and for any $(t_1, x_1), ..., (t_k, x_k) \in [0,T] \times \mathbb{R}$,

$$\Big(u_n(t_1, x_1), ..., u_n(t_k, x_k)\Big) \xrightarrow{\mathrm{d}} \Big(u(t_1, x_1), ..., u(t_k, x_k)\Big).$$

Since $\Big(u_n(t_1, x_1), ..., u_n(t_k, x_k)\Big)$ and $\Big(u(t_1, x_1), ..., u(t_k, x_k)\Big)$ are normal random vectors with zero-mean, by Lemma E.1.5, it suffices to prove that for any $1 \leq i, j \leq k$,

$$\mathrm{Cov}\Big(u_n(t_i, x_i), u_n(t_j, x_j)\Big) \to \mathrm{Cov}\Big(u(t_i, x_i), u(t_j, x_j)\Big).$$

That is, we have to show that for any $(t', x'), (t, x) \in [0,T] \times \mathbb{R}$ with $t' \geq t$, we have:

$$E[u_n(t,x)u_n(t',x')] \to E[u(t,x)u(t',x')]. \tag{2.4.1}$$

We denote by $\mathcal{H}_H$ the Hilbert space associated to W_H. By the isometry property of the Wiener integral with respect to W_{H_n}, we have:

$$\begin{aligned}
&E[u_n(t,x)u_n(t',x')] = \Big\langle 1_{[0,t]} G(t-\cdot, x-\cdot), 1_{[0,t']} G(t'-\cdot, x'-\cdot) \Big\rangle_{\mathcal{H}_{H_n}} \\
&= c_{H_n} \int_0^\infty \int_{\mathbb{R}} 1_{[0,t]}(s) 1_{[0,t']}(s) \mathcal{F}G(t-s, x-\cdot)(\xi) \overline{\mathcal{F}G(t'-s, x'-\cdot)(\xi)} |\xi|^{1-2H_n} \mathrm{d}\xi \mathrm{d}s \\
&= c_{H_n} \int_0^t \int_{\mathbb{R}} e^{-i\xi\cdot x} \overline{\mathcal{F}G(t-s, \cdot)(\xi)} e^{i\xi\cdot x'} \mathcal{F}G(t'-s, \cdot)(\xi) |\xi|^{1-2H_n} \mathrm{d}\xi \mathrm{d}s \\
&= c_{H_n} \int_0^t \int_{\mathbb{R}} e^{-i\xi\cdot(x-x')} \overline{\mathcal{F}G(t-s, \cdot)(\xi)} \mathcal{F}G(t'-s, \cdot)(\xi) |\xi|^{1-2H_n} \mathrm{d}\xi \mathrm{d}s.
\end{aligned}$$

Recalling definition (1.0.9) of c_H, we see that $c_{H_n} \to c_H$, by the continuity of the

Gamma and sin functions. Fix numbers a and b such that $0 < a < H < b < 1$. Since $H_n \to H$, there exists $N \in \mathbb{N}$ such that

$$a \le H_n \le b, \text{ for all } n \ge N.$$

We consider first the heat equation. In this case,

$$E[u_n^h(t,x)u_n^h(t',x')] = c_{H_n}\int_0^t\int_{\mathbb{R}} e^{-i\xi(x-x')}e^{-\frac{(t-s)|\xi|^2}{2}}e^{-\frac{(t'-s)|\xi|^2}{2}}|\xi|^{1-2H_n}\mathrm{d}\xi\mathrm{d}s.$$

As $n \to \infty$, the integrand converges to

$$e^{-i\xi(x-x')}e^{-\frac{(t-s)|\xi|^2}{2}}e^{-\frac{(t'-s)|\xi|^2}{2}}|\xi|^{1-2H}.$$

We show that this integrand is bounded by an integrable function. Let

$$f_n^h(s,\xi) = e^{-i\xi(x-x')}e^{-\frac{(t+t'-2s)|\xi|^2}{2}}|\xi|^{1-2H_n}, \text{ for } s \in [0,t],\ \xi \in \mathbb{R}.$$

We consider two cases.

Case 1: Suppose that $t' > t$,
If $|\xi| \le 1$, using the fact that $e^{-x} \le 1$ for all $x \in \mathbb{R}$, we obtain:

$$|f_n^h(s,\xi)| \le |\xi|^{1-2H_n} \le |\xi|^{1-2b} =: g^h(s,\xi), \text{ for all } n \ge N,$$

since $|\xi|^{2H_n-1} \ge |\xi|^{2b-1}$. Clearly, $\int_0^t\int_{|\xi|\le 1} g^h(s,\xi)\mathrm{d}\xi\mathrm{d}s < \infty$, since $b < 1$.

If $|\xi| > 1$, using the fact that $e^{-x} \le x^{-1}$ for all $x > 0$, we get:

$$\begin{aligned}|f_n^h(s,\xi)| &\le \Big[\frac{(t+t'-2s)|\xi|^2}{2}\Big]^{-1}|\xi|^{1-2H_n} = \frac{2}{t'-t+2(t-s)}|\xi|^{-1-2H_n}\\ &\le \frac{2}{t'-t}|\xi|^{-1-2H_n} \le \frac{2}{t'-t}|\xi|^{-1-2a} =: g^h(s,\xi), \text{ for all } n \ge N,\end{aligned}$$

since $|\xi|^{2H_n+1} \ge |\xi|^{2a+1}$. Clearly, $\int_0^t\int_{|\xi|>1} g^h(s,\xi)\mathrm{d}\xi\mathrm{d}s < \infty$, since $a > 0$.

Case 2: Suppose that $t' = t$,
If $|\xi| \le 1$, using the fact that $e^{-x} \le 1$ for all $x \in \mathbb{R}$, we obtain:

$$|f_n^h(s,\xi)| \le |\xi|^{1-2b} =: g^h(s,\xi) \text{ for all } n \ge N.$$

Clearly, $\int_0^t\int_{|\xi|\le 1} g^h(s,\xi)\mathrm{d}\xi\mathrm{d}s < \infty$.
If $|\xi| > 1$, we have:

$$|f_n^h(s,\xi)| = e^{-(t-s)|\xi|^2}|\xi|^{1-2H_n} \le e^{-(t-s)|\xi|^2}|\xi|^{1-2a} =: g^h(s,\xi), \text{ for all } n \ge N,$$

since $|\xi|^{2H_n-1} \geq |\xi|^{2a-1}$. By applying Lemma 2.3.2 with $\alpha = 1-2a \in (-1,1)$, we see that:

$$\int_0^t \int_{|\xi|>1} g^h(s,\xi)\mathrm{d}\xi\mathrm{d}s = \int_0^t \int_{|\xi|>1} e^{-s|\xi|^2}|\xi|^{1-2a}\mathrm{d}\xi\mathrm{d}s < \infty.$$

It follows that $|f_n^h(s,\xi)| \leq g^h(s,\xi),$ for all $n \geq N, s \in [0,t]$ and $\xi \in \mathbb{R}$ and

$$\int_0^t \int_{\mathbb{R}} g^h(s,\xi)\mathrm{d}\xi\mathrm{d}s < \infty.$$

Therefore, in the case of the heat equation, relation (2.4.1) follows by the Dominated Convergence Theorem.

We consider next the wave equation. In this case,

$$\begin{aligned} E[u_n^w(t,x)u_n^w(t',x')] &= c_{H_n} \int_0^t \int_{\mathbb{R}} e^{-i\xi(x-x')} \frac{\sin\Big((t-s)|\xi|\Big)}{|\xi|} \frac{\sin\Big((t'-s)|\xi|\Big)}{|\xi|} |\xi|^{1-2H_n}\mathrm{d}\xi\mathrm{d}s \\ &= c_{H_n} \int_0^t \int_{\mathbb{R}} e^{-i\xi(x-x')} \frac{\sin\Big((t-s)|\xi|\Big)\sin\Big((t'-s)|\xi|\Big)}{|\xi|^{1+2H_n}}\mathrm{d}\xi\mathrm{d}s. \end{aligned}$$

The integrand in the later integral converges to

$$e^{-i\xi(x-x')} \frac{\sin\Big((t-s)|\xi|\Big)\sin\Big((t'-s)|\xi|\Big)}{|\xi|^{1+2H}}.$$

To apply the Dominated Convergence Theorem, we need to show that the integrand f_n^w is bounded by an integrable function, where

$$f_n^w(s,\xi) = e^{-i\xi(x-x')} \frac{\sin\Big((t-s)|\xi|\Big)\sin\Big((t'-s)|\xi|\Big)}{|\xi|^{1+2H_n}}, \text{ for } s \in [0,T],\ \xi \in \mathbb{R}.$$

If $|\xi| \leq 1$, using the fact that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$, we obtain:

$$\begin{aligned} |f_n^w(s,\xi)| &\leq \frac{(t-s)|\xi|(t'-s)|\xi|}{|\xi|^{1+2H_n}} = (t-s)(t'-s)|\xi|^{1-2H_n} \\ &\leq (t-s)(t'-s)|\xi|^{1-2b} =: g^w(s,\xi), \text{ for all } n \geq N, \end{aligned}$$

since $|\xi|^{2H_n-1} \geq |\xi|^{2b-1}$. Note that $\int_0^t \int_{|\xi|\leq 1} g^w(s,\xi)\mathrm{d}\xi\mathrm{d}s < \infty$, since $b < 1$.

If $|\xi| > 1$, using the fact that $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, we get:

$$|f_n^w(s,\xi)| \leq \frac{1}{|\xi|^{1+2H_n}} = |\xi|^{-1-2H_n} \leq |\xi|^{-1-2a} =: g^w(s,\xi), \text{ for all } n \geq N,$$

since $|\xi|^{2H_n+1} \geq |\xi|^{2a+1}$. Note that $\int_0^t \int_{|\xi|>1} g^w(s,\xi)\mathrm{d}\xi\mathrm{d}s < \infty$, since $a > 0$.

It follows that $|f_n^w(s,\xi)| \leq g^w(s,\xi)$ for all $s \in [0,t]$ and $\xi \in \mathbb{R}$, and

$$\int_0^t \int_{|\xi|\geq 1} g^w(s,\xi)\mathrm{d}\xi\mathrm{d}s < \infty.$$

Hence, in the case of the wave equation, relation (2.4.1) follows again by the Dominated Convergence Theorem.

Step 2: In this step, we verify that condition (iii) of Theorem E.4.3 holds. Let $J = [-M, M]$ be a compact set in $\mathbb{R}$. Recalling the moment estimates given by Thmorem 2.3.4, we see that for any $p > 0$, $t', t \in [0,T]$, $x', x \in \mathbb{R}$,

$$E\Big|u_n(t',x') - u_n(t,x)\Big|^p \leq E\bigg(\Big|u_n(t',x') - u_n(t,x')\Big| + \Big|u_n(t,x') - u_n(t,x)\Big|\bigg)^p$$
$$\leq 2^{p-1}\bigg\{E\Big|u_n(t',x') - u_n(t,x')\Big|^p + E\Big|u_n(t,x') - u_n(t,x)\Big|^p\bigg\}$$
$$\leq \begin{cases} 2^{p-1}z_p\bigg\{\Big(C_{H_n}^{(1)}\Big)^{p/2}|t'-t|^{pH_n/2} + \Big(C_{H_n}^{(3)}\Big)^{p/2}|x'-x|^{pH_n}\bigg\} & \text{for heat equation} \\ 2^{p-1}z_pT^{p/2}\bigg\{\Big(C_{H_n}^{(2)}\Big)^{p/2}|t'-t|^{pH_n} + \Big(C_{H_n}^{(3)}\Big)^{p/2}|x'-x|^{pH_n}\bigg\} & \text{for wave equation,} \end{cases}$$

where for the second inequality, we use the fact that

$$(a+b)^p \leq 2^{p-1}(a^p + b^p), \text{ for all } a, b \in \mathbb{R}.$$

We will use the following inequality: for any $\alpha \geq 1$,

$$x^\alpha + a^\alpha \leq (x+a)^\alpha, \text{ for all } x, a \geq 0. \tag{2.4.2}$$

We examine first the heat equation. For any $x, x' \in J$, $|x'-x| \leq |x'| + |x| \leq 2M$ and $|x'-x|^{pH_n/2} \leq (2M)^{pH_n/2}$. Hence,

$$|x'-x|^{pH_n} \leq (2M)^{pH_n/2}|x'-x|^{pH_n/2}, \text{ for all } x, x' \in J.$$

We use (2.4.2) with $\alpha = pH_n/2$. Note that $pH_n/2 \geq pa/2 \geq 1$ for all $n \geq N$, if $p \geq 2/a$. We obtain the following: for any $n \geq N$ and $p \geq 2/a$,

$$E\Big|u_n^h(t',x') - u_n^h(t,x)\Big|^p \leq 2^{p-1}z_p\Big[C_{H_n}^{(1)} \vee C_{H_n}^{(3)}\Big]^{p/2}\bigg\{|t'-t|^{pH_n/2} + (2M)^{pH_n/2}|x'-x|^{pH_n/2}\bigg\}$$
$$\leq 2^{p-1}z_p\Big[C_{H_n}^{(1)} \vee C_{H_n}^{(3)}\Big]^{p/2}\Big[1 \vee (2M)\Big]^{pH_n/2}\Big(|t'-t|^{pH_n/2} + |x'-x|^{pH_n/2}\Big)$$

$$\le 2^{p-1} z_p A^{p/2} \Big[1 \vee (2M)\Big]^{pb/2} \Big(|t'-t| + |x'-x|\Big)^{pH_n/2}, \tag{2.4.3}$$

where for the last inequality we used relation (2.4.2), the fact that $H_n \le b$ for all $n \ge N$, and

$$C_{H_n}^{(1)} \vee C_{H_n}^{(3)} \le A, \text{ for all } n \ge N, \tag{2.4.4}$$

for some constant A which we specify below.

We show inequality (2.4.4) in following way. Recall the definition of the constant $C_H^{(1)}$ given by Theorem 2.3.4:

$$C_H^{(1)} = c_H \bigg(N_H + \frac{1}{H}\Gamma(1-H) \bigg),$$

where $N_H = \int_{\mathbb{R}} \frac{(1-\mathrm{e}^{-\frac{\eta^2}{2}})^2}{|\eta|^{1+2H}} \mathrm{d}\eta$. By Lemma 2.3.5, we have:

$$N_{H_n} \le \frac{1}{4} \cdot \frac{1}{2-H_n} + \frac{1}{H_n} \le \frac{1}{4} \cdot \frac{1}{2-b} + \frac{1}{a}, \text{ for all } n \ge N.$$

The function $\frac{\Gamma(1-H)}{H}$ is continuous on $[a,b]$, hence it is bounded on $[a,b]$ by a constant $c > 0$. Remark 2.3.7 shows that

$$c_H \le \frac{1}{\pi}, \text{ for all } H \in (0,1). \tag{2.4.5}$$

Hence,

$$C_{H_n}^{(1)} \le \frac{1}{\pi}\Big(\frac{1}{4(2-b)} + \frac{1}{a} + c\Big), \text{ for all } n \ge N. \tag{2.4.6}$$

Next, we treat $C_H^{(3)}$. Recall that

$$C_H^{(3)} = 2c_H \int_{\mathbb{R}} \frac{1-\cos\eta}{|\eta|^{1+2H}} \mathrm{d}\eta = 2c_H \bar{C}_{1-2H},$$

where $\bar{C}_\alpha = \int_{\mathbb{R}} (1-\cos\eta)|\eta|^{\alpha-2} \mathrm{d}\eta$. By Lemma 2.3.6, we obtain

$$C_H^{(3)} \le 2c_H \Big(\frac{1}{H} + \frac{1}{1-H}\Big). \tag{2.4.7}$$

Therefore,

$$C_{H_n}^{(3)} \le \frac{2}{\pi}\Big(\frac{1}{a} + \frac{1}{1-b}\Big), \text{ for all } n \ge N. \tag{2.4.8}$$

Relation (2.4.4) follows from (2.4.6) and (2.4.8).

We return now to relation (2.4.3). It remains to treat the term $\Big(|t'-t|+|x'-x|\Big)^{pH_n/2}$. Note that for any $t,t'\in[0,T]$ and $x,x'\in[-M,M]$,

$$w:=|t'-t|+|x'-x|\le T+2M=:c_0.$$

For all $n\ge N$,

$$w^{pH_n/2}=w^{p(H_n-a)/2}w^{pa/2}\le c_0^{p(H_n-a)/2}w^{pa/2},$$

and

$$c_0^{p(H_n-a)/2}\le\begin{cases}c_0^{p(b-a)/2} & \text{if } c_0\ge 1,\\ 1 & \text{if } c_0<1.\end{cases}$$

Therefore, $c_0^{p(H_n-a)/2}\le c_0^{p(b-a)/2}\vee 1$ and

$$w^{pH_n/2}\le\Big(c_0^{p(b-a)/2}\vee 1\Big)w^{pa/2},\ \text{for all } n\ge N. \tag{2.4.9}$$

Using inequalities (2.4.3) and (2.4.9), we obtain:

$$E\Big|u_n^h(t',x')-u_n^h(t,x)\Big|^p\le 2^{p-1}z_pA^{p/2}[1\vee(2M)]^{pb/2}\Big(c_0^{p(b-a)/2}\vee 1\Big)\Big(|t'-t|+|x'-x|\Big)^{pa/2},$$

for all $t,t'\in[0,T]$, $x,x'\in J$, $n\ge N$ and $p\ge 2/a$. Condition (iii) of Theorem E.4.3 follows with $\delta=\frac{pa}{2}>2$, if we choose $p>\frac{4}{a}$.

Next, we examine the wave equation. In this case, we use inequality (2.4.2) with $\alpha=pH_n$. Note that $pH_n\ge pa\ge 1$ for all $n\ge N$, if $p\ge 1/a$. Hence, for any $n\ge N$ and $p\ge 1/a$,

$$\begin{aligned}E\Big|u_n^w(t',x')-u_n^w(t,x)\Big|^p&\le 2^{p-1}z_pT^{p/2}\Big[C_{H_n}^{(2)}\vee C_{H_n}^{(3)}\Big]^{p/2}\Big\{|t'-t|^{pH_n}+|x'-x|^{pH_n}\Big\}\\&\le 2^{p-1}z_pT^{p/2}B^{p/2}\Big(|t'-t|+|x'-x|\Big)^{pH_n},\end{aligned} \tag{2.4.10}$$

where for the last inequality we use the fact that

$$C_{H_n}^{(2)}\vee C_{H_n}^{(3)}\le B,\ \text{for all } n\ge N, \tag{2.4.11}$$

for some constant $B>0$ which is specified below.

We prove (2.4.11). We treat $C_{H_n}^{(2)}$. Recall the definition of the constant $C_H^{(2)}$ given by Theorem 2.3.4:

$$C_H^{(2)}=c_H\bigg(M_H+2^{2H}\widetilde{C}_{1-2H}\frac{1}{1+2H}\bigg),$$

where $M_H = 4\int_{\mathbb{R}} \frac{\min(1,|\eta|^2)}{|\eta|^{1+2H}} d\eta$, and

$$\widetilde{C}_{1-2H} = \begin{cases} (2H)^{-1}\Gamma(1-2H)\sin\Big(\pi(1-2H)/2\Big) & \text{if } H < \frac{1}{2}, \\ (1-2H)^{-1}(2H)^{-1}\Gamma(2-2H)\sin\Big(\pi(1-2H)/2\Big) & \text{if } H > \frac{1}{2}, \\ \pi/2 & \text{if } H = \frac{1}{2}. \end{cases}$$

Since $0 < a \le H_n \le b < 1$, for all $n \ge N$, by Lemma 2.3.5, we have

$$M_{H_n} = 4\Big(\frac{1}{H_n} + \frac{1}{1-H_n}\Big) \le 4\Big(\frac{1}{a} + \frac{1}{1-b}\Big).$$

It can be proved that the function $H \to \widetilde{C}_{1-2H}$ is continuous on $(0,1)$ (see the proof of Theorem of [15]), hence it is bounded on the interval $[a,b]$. Since $H_n \in [a,b]$, for all $n \ge N$, it follows that there exists a constant $c' > 0$ such that

$$\widetilde{C}_{1-2H_n} \le c', \text{ for all } n \ge N.$$

The other constant in $C_H^{(2)}$ is also bounded since

$$\frac{2^{2H}}{1+2H} \le 2^{2H} \le 4, \text{ for all } H \in (0,1).$$

Using (2.4.5), we obtain:

$$C_{H_n}^{(2)} \le \frac{1}{\pi}\left[4\Big(\frac{1}{a} + \frac{1}{1-b}\Big) + 4c'\right], \text{ for all } n \ge N. \tag{2.4.12}$$

Relation (2.4.11) follows from (2.4.8) and (2.4.12).

We return now to relation (2.4.10). The term $\Big(|t'-t| + |x'-x|\Big)^{pH_n}$ is treated similarly to the term $\Big(|t'-t| + |x'-x|\Big)^{pH_n/2}$ which appears for the heat equation. More precisely, for any $t,t' \in [0,T]$ and $x,x' \in [-M,M]$, we have:

$$w := |t'-t| + |x'-x| \le T + 2M =: c_0.$$

For all $n \ge N$,

$$w^{pH_n} = w^{p(H_n-a)} w^{pa} \le c_0^{p(H_n-a)} w^{pa}.$$

Similarly to (2.4.9), we have:

$$w^{pH_n} \le \Big(c_0^{p(b-a)} \vee 1\Big) w^{pa}, \text{ for all } n \ge N. \tag{2.4.13}$$

Using inequalities (2.4.10) and (2.4.13), we obtain:

$$E\Big|u_n^w(t',x') - u_n^w(t,x)\Big|^p \le 2^{p-1} z_p T^{p/2} B^{p/2} \Big(c_0^{p(b-a)} \vee 1\Big)\Big(|t'-t| + |x'-x|\Big)^{pa}$$

for all $t,t' \in [0,T]$, $x,x' \in J$, $n \ge N$ and $p \ge 1/a$. Condition (iii) of Theorem E.4.3 follows with $\delta = pa > 2$, if we choose $p > \frac{2}{a}$. ∎

Chapter 3

Equations with fractional noise in time

In this chapter, we examine equations (1.0.1) and (1.0.2) with fractional noise in time with index $H_0 > \frac{1}{2}$. We assume that the noise is also fractional in space with index $H \in (0,1)$.

We recall that the covariance of fractional Brownian motion of index $H \in (0,1)$ is given by:

$$R_H(t,s) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big).$$

In particular, if $H > \frac{1}{2}$,

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2}\, \mathrm{d}u\mathrm{d}v,$$

where $\alpha_H = H(2H-1)$. Note that $R_H(t,t) = t^{2H}$.

3.1 The noise

In this section, we consider a zero-mean Gaussian process $W = \{W([0,t] \times A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R})\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$, with covariance:

$$\begin{aligned}
&E\Big[W([0,t]\times A)W([0,s]\times B)\Big] = R_{H_0}(t,s)c_H \int_{\mathbb{R}} \mathcal{F}1_A(\xi)\overline{\mathcal{F}1_B(\xi)}|\xi|^{1-2H}\mathrm{d}\xi \\
&= \bigg(\alpha_{H_0}\int_0^\infty\int_0^\infty 1_{[0,t]}(u)1_{[0,s]}(v)|u-v|^{2H_0-2}\mathrm{d}u\mathrm{d}v\bigg)\bigg(c_H\int_{\mathbb{R}} \mathcal{F}1_A(\xi)\overline{\mathcal{F}1_B(\xi)}|\xi|^{1-2H}\mathrm{d}\xi\bigg) \\
&= \alpha_{H_0}c_H \int_0^\infty\int_0^\infty\int_{\mathbb{R}} \mathcal{F}1_{[0,t]\times A}(u,\cdot)(\xi)\overline{\mathcal{F}1_{[0,s]\times B}(v,\cdot)(\xi)}|\xi|^{1-2H}|u-v|^{2H_0-2}\mathrm{d}\xi\mathrm{d}u\mathrm{d}v \\
&=: \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{H}}
\end{aligned}$$

since $\mathcal{F}1_{[0,t]\times A}(u,\cdot)(\xi) = 1_{[0,t]}(u)\mathcal{F}1_A(\xi)$. Here we use the constant c_H given by (1.0.9).

Remark 3.1.1. Similarly to Remark 2.1.1, if $H > \frac{1}{2}$, then

$$\begin{aligned}
&E\left[W([0,t]\times A)W([0,s]\times B)\right] = R_{H_0}(t,s)\alpha_H \int_A\int_B |x-y|^{2H-2}\mathrm{d}x\mathrm{d}y \\
&= \left(\alpha_{H_0}\int_0^\infty\int_0^\infty 1_{[0,t]}(u)1_{[0,s]}(v)|u-v|^{2H_0-2}\mathrm{d}u\mathrm{d}v\right)\left(\alpha_H\int_A\int_B|x-y|^{2H-2}\mathrm{d}x\mathrm{d}y\right) \\
&= \alpha_{H_0}\alpha_H\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{[0,t]\times A}(u,x)1_{[0,s]\times B}(v,y)|u-v|^{2H_0-2}|x-y|^{2H-2}\mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

Note that the process $\{W(t,x); t\geq 0, x\in\mathbb{R}\}$ defined by $W(t,x) = W([0,t]\times[0,x])$ is a fBs of index H_0 in time and index H in space, since

$$E[W(t,x)W(s,y)] = R_{H_0}(t,s)R_H(x,y).$$

We define $W(1_{[0,t]\times A}) = W([0,t]\times A)$ and we extend this definition by linearity to the set $\mathcal{E}$ of linear combinations of elementary functions $1_{[0,t]\times A}$, with $t\geq 0$, and $A\in\mathcal{B}_b(\mathbb{R})$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. For any $\varphi,\psi\in\mathcal{E}$,

$$\langle\varphi,\psi\rangle_{\mathcal{H}} = \alpha_{H_0}c_H\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)(\xi)}|\xi|^{1-2H}|t-s|^{2H_0-2}\mathrm{d}\xi\mathrm{d}t\mathrm{d}s.$$

Note that if $H > \frac{1}{2}$, then for any $\varphi,\psi\in\mathcal{E}$:

$$\langle\varphi,\psi\rangle_{\mathcal{H}} = \alpha_{H_0}\int_0^\infty\int_0^\infty\left(\alpha_H\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(t,x)\psi(s,y)|x-y|^{2H-2}\mathrm{d}x\mathrm{d}y\right)|t-s|^{2H_0-2}\mathrm{d}t\mathrm{d}s.$$

Lemma 3.1.2. *W is an isometry from $\mathcal{E}$ to $L^2(\Omega)$.*

Proof: We consider the following functions:

$$\varphi(t,x) = \sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}(t,x) \text{ and } \psi(s,x) = \sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}(s,x).$$

Note that

$$\begin{aligned}
&E[\ W(\varphi)W(\psi)\ ] = E\Big[\ W\big(\sum_{i=1}^n a_i 1_{[0,t_i]\times A_i}\big)W\big(\sum_{j=1}^m b_j 1_{[0,s_j]\times B_j}\big)\ \Big] \\
&= E\Big[\ \big\{\sum_{i=1}^n a_i W(1_{[0,t_i]\times A_i})\big\}\big\{\sum_{j=1}^m b_j W(1_{[0,s_j]\times B_j})\big\}\ \Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j E[\ W(1_{[0,t_i]\times A_i})W(1_{[0,s_j]\times B_j})\ ] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \Big(\alpha_{H_0} c_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \mathcal{F}1_{[0,t_i]\times A_i}(u,\cdot)(\xi)\overline{\mathcal{F}1_{[0,s_j]\times B_j}(v,\cdot)(\xi)} \\
&\qquad\qquad |\xi|^{1-2H}|u-v|^{2H_0-2}\mathrm{d}\xi\mathrm{d}u\mathrm{d}v \Big) \\
&= \alpha_{H_0} c_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \Big\{ \sum_{i=1}^{n} a_i \mathcal{F}1_{[0,t_i]\times A_i}(u,\cdot)(\xi) \Big\} \Big\{ \sum_{j=1}^{m} b_j \overline{\mathcal{F}1_{[0,s_i]\times B_j}(v,\cdot)(\xi)} \Big\} \\
&\qquad\qquad |\xi|^{1-2H}|u-v|^{2H_0-2}\mathrm{d}\xi\mathrm{d}u\mathrm{d}v \\
&= \alpha_{H_0} c_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \Big\{ \mathcal{F}\Big(\sum_{i=1}^{n} a_i 1_{[0,t_i]\times A_i}(t,\cdot) \Big)(\xi) \Big\} \Big\{ \overline{\mathcal{F}\Big(\sum_{j=1}^{m} b_i 1_{[0,s_i]\times B_j}(s,\cdot) \Big)(\xi)} \Big\} \\
&\qquad\qquad |\xi|^{1-2H}|u-v|^{2H_0-2}\mathrm{d}\xi\mathrm{d}u\mathrm{d}v \\
&= \alpha_{H_0} c_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(s,\cdot)(\xi)}|\xi|^{1-2H}|u-v|^{2H_0-2}\mathrm{d}\xi\mathrm{d}u\mathrm{d}v \\
&= \langle \varphi, \psi \rangle_{\mathcal{H}}.
\end{aligned}
$$

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Exactly as in Section 2.1, we can extend W from $\mathcal{E}$ to $\mathcal{H}$ using the isometry property. For any $\varphi \in \mathcal{H}$, we denote

$$W(\varphi) = \int_0^\infty \int_{\mathbb{R}} \varphi(t,x) W(\mathrm{d}t, \mathrm{d}x)$$

and we say that $W(\varphi)$ is the *Wiener integral* of φ with respect to W. Note that $W(\varphi)$ is a normal random variable with zero mean and variance $\|\varphi\|_{\mathcal{H}}^2$.

This following result is a particular case of Theorem 2.6(c) of [5].

Theorem 3.1.3. *Let $\varphi : [0,T]\times\mathbb{R} \to \mathbb{R}$ be such that $\varphi(t,\cdot) \in L^1(\mathbb{R})$ for all $t \in [0,T]$ and let*

$$\mathcal{F}\varphi(t,\cdot)(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\varphi(t,x)\mathrm{d}x,\ \xi \in \mathbb{R},$$

be the Fourier transform of $\varphi(t,\cdot)$. Suppose that

$$\int_0^T |\mathcal{F}\varphi(t,\cdot)(\xi)|\,\mathrm{d}t < \infty,\ \textit{for all } \xi \in \mathbb{R}.$$

If

$$I := \alpha_{H_0} c_H \int_0^T \int_0^T \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\varphi(s,\cdot)(\xi)}\,|t-s|^{2H_0-2}\,|\xi|^{1-2H}\,\mathrm{d}\xi\mathrm{d}t\mathrm{d}s < \infty, \tag{3.1.1}$$

then $\varphi \in \mathcal{H}$ *and* $E|W(\varphi)|^2 = \|\varphi\|_{\mathcal{H}}^2 = I$.

3.2 Existence of solution: heat equation

In this section, we consider the linear stochastic heat equation (1.0.1) with noise W as in Section 3.1. The next result is a particular case of Theorem 4.2 of [6].

Theorem 3.2.1. *For any* $H_0 \in (\frac{1}{2}, 1)$ *and* $H \in (0,1)$, *equation* (1.0.1) *with noise* W *as in Section 3.1 has a unique solution.*

Proof: To prove that the solution exists we need to check that $g_{t,x}^h \in \mathcal{H}$, where $g_{t,x}^h(s,y) = G^h(t-s, x-y)1_{[0,t]}(s)$. For this, we apply Theorem 3.1.3. We verify that condition (3.1.1) holds.

Note that, by Lemma A.2

$$\begin{aligned}\mathcal{F}g_{t,x}^h(s,\cdot)(\xi) &= \mathcal{F}G^h(t-s, x-\cdot)(\xi)1_{[0,t]}(s)\\ &= e^{-i\xi x}\overline{\mathcal{F}G^h(t-s,\cdot)(\xi)}1_{[0,t]}(s).\end{aligned}$$

Hence

$$\begin{aligned}I_t &:= \alpha_{H_0} c_H \int_0^t \int_0^t \int_{\mathbb{R}} \mathcal{F}g_{t,x}^h(s,\cdot)(\xi)\overline{\mathcal{F}g_{t,x}^h(r,\cdot)(\xi)}\,|s-r|^{2H_0-2}\,|\xi|^{1-2H}\,\mathrm{d}\xi\mathrm{d}s\mathrm{d}r\\ &= \alpha_{H_0} c_H \int_0^t \int_0^t \int_{\mathbb{R}} \mathcal{F}G^h(t-s,\cdot)(\xi)\overline{\mathcal{F}G^h(t-r,\cdot)(\xi)}\,|s-r|^{2H_0-2}\,|\xi|^{1-2H}\,\mathrm{d}\xi\mathrm{d}s\mathrm{d}r\\ &= c_H \int_{\mathbb{R}} \left(\alpha_{H_0}\int_0^t\int_0^t \mathcal{F}G^h(s',\cdot)(\xi)\overline{\mathcal{F}G^h(r',\cdot)(\xi)}\,|s'-r'|^{2H_0-2}\,\mathrm{d}s'\mathrm{d}r'\right)|\xi|^{1-2H}\,\mathrm{d}\xi,\end{aligned}$$

where we applied the change of variables $s' = t-s$ and $r' = t-r$ in the last equality.

Therefore,

$$I_t = c_H \int_{\mathbb{R}} N_t^h(\xi)|\xi|^{1-2H}\mathrm{d}\xi \tag{3.2.1}$$

where

$$\begin{aligned}N_t^h(\xi) &= \alpha_{H_0}\int_0^t\int_0^t \mathcal{F}G^h(s,\cdot)(\xi)\overline{\mathcal{F}G^h(r,\cdot)(\xi)}\,|s-r|^{2H_0-2}\,\mathrm{d}s\mathrm{d}r\\ &= \alpha_{H_0}\int_0^t\int_0^t \exp\Big(-\frac{s|\xi|^2}{2}\Big)\exp\Big(-\frac{r|\xi|^2}{2}\Big)|r-s|^{2H_0-2}\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Step 1: In this step, we check that $I_t < \infty$ if and only if

$$\int_{\mathbb{R}} \Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0} |\xi|^{1-2H} \mathrm{d}\xi < \infty.$$

We claim that: for any $t > 0,\ \xi \in \mathbb{R}$,

$$\frac{1}{4}(t^{2H_0} \wedge 1)\Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0} \le N_t^h(\xi) \le C'_{H_0}(t^{2H_0} \vee 1)\Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0}, \tag{3.2.2}$$

where $C'_{H_0} = b_{H_0}(4H_0)^{2H_0}$ and b_{H_0} is given by Lemma C.2.

To prove relation (3.2.2), we treat the upper bound first. We consider $|\xi| \le 1$ first. Using Corollary C.4 and the fact that

$$e^{-x} \le 1 \text{ for all } x > 0 \text{ and } 1 \le \frac{2}{1+|\xi|^2},$$

we have:

$$\begin{aligned} N_t^h(\xi) &\le b_{H_0} t^{2H_0-1} \int_0^t \exp(-s|\xi|^2)\mathrm{d}s \le b_{H_0} t^{2H_0} \\ &\le b_{H_0} t^{2H_0} 2^{2H_0} \Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0}. \end{aligned}$$

Next we consider $|\xi| > 1$. In this case, we use Lemma C.2 and the fact that

$$1 - e^{-x} \le 1 \text{ for all } x > 0 \text{ and } \frac{1}{|\xi|^2} \le \frac{2}{1+|\xi|^2},$$

we obtain:

$$\begin{aligned} N_t^h(\xi) &\le b_{H_0} \bigg(\int_0^t \exp\Big(-\frac{r|\xi|^2}{2H_0}\Big)\mathrm{d}r \bigg)^{2H_0} = b_{H_0}\Big(\frac{2H_0}{|\xi|^2}\Big)^{2H_0} \Big[1 - \exp\Big(-\frac{t|\xi|^2}{2H_0}\Big)\Big]^{2H_0} \\ &\le b_{H_0}\Big(\frac{2H_0}{|\xi|^2}\Big)^{2H_0} = b_{H_0}(2H_0)^{2H_0}\Big(\frac{1}{|\xi|^2}\Big)^{2H_0} \le b_{H_0}(2H_0)^{2H_0}\Big(\frac{2}{1+|\xi|^2}\Big)^{2H_0} \\ &= b_{H_0}(4H_0)^{2H_0}\Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0}. \end{aligned}$$

For lower bound, we consider $t|\xi|^2 \le 1$ first. We used the fact that

$$\exp\Big(-\frac{r|\xi|^2}{2}\Big) \ge 1 - \frac{r|\xi|^2}{2} \ge \frac{1}{2} \quad \text{for any } r \in [0,t].$$

Hence,

$$\begin{aligned}
N_t^h(\xi) &= \alpha_{H_0} \int_0^t \int_0^t \exp\Big(-\frac{r|\xi|^2}{2}\Big) \exp\Big(-\frac{s|\xi|^2}{2}\Big) |r-s|^{2H_0-2} \mathrm{d}r\mathrm{d}s \\
&\geq \alpha_{H_0} \Big(\frac{1}{2}\Big)^2 \int_0^t \int_0^t |r-s|^{2H_0-2} \mathrm{d}r\mathrm{d}s \\
&= \frac{1}{4} t^{2H_0} \geq \frac{1}{4} t^{2H_0} \Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0},
\end{aligned}$$

where for the last inequalities we need the fact that $\frac{1}{1+|\xi|^2} \leq 1$.

Next, we consider the case $t|\xi|^2 > 1$. We introduce the notation: for any $T > 0$,

$$\|\varphi\|^2_{\mathcal{H}_0(0,T)} = \alpha_{H_0} \int_0^T \int_0^T \varphi(t)\varphi(s)|t-s|^{2H_0-2} \mathrm{d}t\mathrm{d}s.$$

Using the change of variables $r' = r|\xi|^2/2$ and $s' = s|\xi|^2/2$, we have:

$$\begin{aligned}
N_t^h(\xi) &= \alpha_{H_0} \int_0^t \int_0^t \exp\Big(-\frac{r|\xi|^2}{2}\Big) \exp\Big(-\frac{s|\xi|^2}{2}\Big) |r-s|^{2H_0-2} \mathrm{d}r\mathrm{d}s \\
&= \alpha_{H_0} \int_0^{\frac{t|\xi|^2}{2}} \int_0^{\frac{t|\xi|^2}{2}} \exp(-r') \exp(-s') \left|\frac{2r'}{|\xi|^2} - \frac{2s'}{|\xi|^2}\right|^{2H_0-2} \Big(\frac{2}{|\xi|^2}\Big)^2 \mathrm{d}r'\mathrm{d}s' \\
&= \alpha_{H_0} \frac{2^{2H_0}}{|\xi|^{4H_0}} \int_0^{\frac{t|\xi|^2}{2}} \int_0^{\frac{t|\xi|^2}{2}} \exp(-r) \exp(-s) |r-s|^{2H_0-2} \mathrm{d}r\mathrm{d}s \\
&\geq \alpha_{H_0} \frac{2^{2H_0}}{|\xi|^{4H_0}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \exp(-r) \exp(-s) |r-s|^{2H_0-2} \mathrm{d}r\mathrm{d}s \\
&= \frac{2^{2H_0}}{|\xi|^{4H_0}} \|e^{-t}\|^2_{\mathcal{H}_0(0,\frac{1}{2})} \geq 2^{2H_0} \Big(\frac{1}{2}\Big)^{2H_0+2} \Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0} = \frac{1}{4} \Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0},
\end{aligned}$$

where for the last inequality we used the fact that

$$\frac{1}{|\xi|^2} \geq \frac{1}{1+|\xi|^2} \text{ and } \|e^{-t}\|^2_{\mathcal{H}_0(0,\frac{1}{2})} \geq \Big(\frac{1}{2}\Big)^{2H_0+2} \text{ since } e^{-u} \geq 1-u \geq \frac{1}{2},\ \forall u \in \Big[0, \frac{1}{2}\Big].$$

This finishes the proof of relation (3.2.2).

Coming back to equation (3.2.1) and using relation (3.2.2), we obtain:

$$c_H \frac{1}{4} (t^{2H_0} \wedge 1) \int_{\mathbb{R}} \Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0} |\xi|^{1-2H} \mathrm{d}\xi \leq I_t \leq c_H C'_{H_0} (t^{2H_0} \vee 1) \int_{\mathbb{R}} \Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0} |\xi|^{1-2H} \mathrm{d}\xi.$$

Hence $I_t < \infty$ if and only if $\int_{\mathbb{R}} \Big(\frac{1}{1+|\xi|^2}\Big)^{2H_0} |\xi|^{1-2H} \mathrm{d}\xi < \infty$. This finishes **Step 1** of the proof.

Step 2: In this step, we prove that $I = \int_{\mathbb{R}} \left(\frac{1}{1+|\xi|^2}\right)^{2H_0} |\xi|^{1-2H} d\xi < \infty$ if and only if $2H_0 + H > 1$. Note that this condition is satisfied for any $H_0 \in (\frac{1}{2}, 1)$ and $H \in (0,1)$. We write

$$I = \int_{|\xi|\leq 1} \left(\frac{1}{1+|\xi|^2}\right)^{2H_0} |\xi|^{1-2H} d\xi + \int_{|\xi|>1} \left(\frac{1}{1+|\xi|^2}\right)^{2H_0} |\xi|^{1-2H} d\xi$$
$$:= I^{(1)} + I^{(2)}.$$

We treat separately $I^{(1)}$ and $I^{(2)}$. For $I^{(1)}$, note that $\frac{1}{2} \leq \frac{1}{1+|\xi|^2} \leq 1$, if $|\xi| \leq 1$. Hence,

$$\left(\frac{1}{2}\right)^{2H_0} \int_{|\xi|\leq 1} |\xi|^{1-2H} \, d\xi \leq I^{(1)} \leq \int_{|\xi|\leq 1} |\xi|^{1-2H} \, d\xi.$$

Note that $\int_{|\xi|\leq 1} |\xi|^{1-2H} \, d\xi = 2\int_0^1 \xi^{1-2H} d\xi < \infty$ since $1 - 2H + 1 > 0$.

Next, we treat $I^{(2)}$. Note that $\frac{1}{2}\frac{1}{|\xi|^2} \leq \frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2}$, if $|\xi| > 1$. Hence,

$$\left(\frac{1}{2}\right)^{2H_0} \int_{|\xi|>1} \left(\frac{1}{|\xi|^2}\right)^{2H_0} |\xi|^{1-2H} \, d\xi \leq I^{(2)} \leq \int_{|\xi|>1} \left(\frac{1}{|\xi|^2}\right)^{2H_0} |\xi|^{1-2H} \, d\xi.$$

So $I^{(2)} < \infty$ if and only if $\int_{|\xi|>1} |\xi|^{-4H_0} |\xi|^{1-2H} d\xi < \infty$. This last integral is equal to $2\int_1^\infty \xi^{-4H_0+1-2H} d\xi$ which is finite if and only if $-4H_0 + 1 - 2H + 1 < 0$, which is equivalent to $2H_0 + H > 1$.

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3.3 Existence of solution: wave equation

In this section, we consider the linear stochastic wave equation (1.0.2) with noise W as in Section 3.1. In this case, the existence of the solution was first proved in reference [6], using a different argument than the one that we include here. The proof that we present here relies on inequality (3.3.2) below. For the upper bound, we follow the same argument as in [6]. For the lower bound (which was missing from [6]), we use the argument from the proof of Theorem 4.3 of [1]. We carefully include the explicit form of all the constants appearing in (3.3.2).

Theorem 3.3.1. *For any $H_0 \in (\frac{1}{2}, 1)$ and $H \in (0,1)$, equation (1.0.2) with noise W as in Section 3.1 has a unique solution.*

Proof: To prove that the solution exists we need to check that $g^w_{t,x} \in \mathcal{H}$, where $g^w_{t,x}(s,y) = G^w(t-s, x-y)1_{[0,t]}(s)$. For this, we apply Theorem 3.1.3. We verify that condition (3.1.1) holds. As in the proof of Theorem 3.2.1, we need to find the necessary and sufficient condition for $I_t < \infty$, where

$$I_t = c_H \int_{\mathbb{R}} \left(\alpha_{H_0} \int_0^t \int_0^t \mathcal{F}G^w(s,\cdot)(\xi) \overline{\mathcal{F}G^w(r,\cdot)(\xi)}\, |s-r|^{2H_0-2}\, \mathrm{d}s\mathrm{d}r \right) |\xi|^{1-2H}\, \mathrm{d}\xi.$$

We write:

$$I_t = c_H \int_{\mathbb{R}} N_t^w(\xi)|\xi|^{1-2H}\mathrm{d}\xi \tag{3.3.1}$$

where

$$\begin{aligned} N_t^w(\xi) &= \alpha_{H_0} \int_0^t \int_0^t \mathcal{F}G^w(s,\cdot)(\xi)\overline{\mathcal{F}G^w(r,\cdot)(\xi)}\, |s-r|^{2H_0-2}\, \mathrm{d}s\mathrm{d}r \\ &= \frac{\alpha_{H_0}}{|\xi|^2} \int_0^t \int_0^t \sin(s|\xi|)\sin(r|\xi|)|s-r|^{2H_0-2}\mathrm{d}s\mathrm{d}r. \end{aligned}$$

Step 1: In this step, we check that $I_t < \infty$ if and only if

$$\int_{\mathbb{R}} \Big(\frac{1}{1+|\xi|^2}\Big)^{H_0+1/2} |\xi|^{1-2H}\mathrm{d}\xi < \infty.$$

We claim that: for any $t>0,\ \xi \in \mathbb{R}$,

$$D_{H_0}^{(1)}\Big(t^{2H_0+2} \wedge t\Big)\Big(\frac{1}{1+|\xi|^2}\Big)^{H_0+1/2} \le N_t^w(\xi) \le D_{H_0}^{(2)}\Big(t^{2H_0+2} \vee 1\Big)\Big(\frac{1}{1+|\xi|^2}\Big)^{H_0+1/2}, \tag{3.3.2}$$

where

$$\begin{aligned} D_{H_0}^{(1)} &= \min\left\{\alpha_{H_0}\cos^2(1)\frac{B(2,2H_0-1)}{H_0+1}, c_H 4^{-(2H_0-1)}\Big(\frac{\pi}{2}-\frac{4}{3}\Big)\right\}, \\ D_{H_0}^{(2)} &= \max\left\{\frac{1}{3}b_{H_0}2^{H_0+1/2}, c_H\Big(\frac{100}{9}\cdot\frac{1}{1-H_0}2^{-1}+4\pi\Big)2^{3H_0-1/2}\right\}, \end{aligned}$$

and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function.

For the upper bound, we consider $|\xi| \le 1$ first. In this case, we use Corollary C.4 and the fact that $|\sin x| \le x$, for all $x>0$ and $1 \le \frac{2}{1+|\xi|^2}$, if $|\xi| \le 1$, we have:

$$\begin{aligned} N_t^w(\xi) &\le b_{H_0}t^{2H_0-1}\frac{1}{|\xi|^2}\int_0^t \sin^2(s|\xi|)\mathrm{d}s \le b_{H_0}t^{2H_0-1}\frac{1}{|\xi|^2}\int_0^t (s|\xi|)^2\mathrm{d}s \\ &= b_{H_0}t^{2H_0-1}\frac{1}{3}t^3 = \frac{1}{3}b_{H_0}t^{2H_0+2} \le \frac{1}{3}b_{H_0}t^{2H_0+2}\Big(\frac{2}{1+|\xi|^2}\Big)^{H_0+1/2} \end{aligned}$$

$$= \frac{1}{3} b_{H_0} 2^{H_0+1/2} t^{2H_0+2} \left(\frac{1}{1+|\xi|^2} \right)^{H_0+1/2}. \tag{3.3.3}$$

Next, we consider $|\xi| > 1$. To simplify the notation, we introduce the following functions: for $\lambda > 0$ and $\tau \in \mathbb{R}$, let

$$f_t(\lambda, \tau) = \sin(\tau\lambda t) - \tau \sin(\lambda t), \quad g_t(\lambda, \tau) = \cos(\tau\lambda t) - \cos(\lambda t). \tag{3.3.4}$$

Using the expression of the $\mathcal{H}(0, t|\xi|)$-norm of the sin function given by Lemma A.8, we obtain:

$$\begin{aligned} N_t^w(\xi) &= \frac{c_H}{|\xi|^{2H_0+2}} \int_{\mathbb{R}} \frac{|\tau|^{-(2H_0-1)}}{(\tau^2-1)^2} \Big[f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau) \Big] \mathrm{d}\tau \\ &= N_t^{(1)}(\xi) + N_t^{(2)}(\xi), \end{aligned}$$

where

$$N_t^{(1)}(\xi) = \frac{c_H}{|\xi|^{2H_0+2}} \int_{\{|\tau| \le 1/2\}} \frac{|\tau|^{-(2H_0-1)}}{(\tau^2-1)^2} \Big(f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau) \Big) \mathrm{d}\tau,$$

and

$$N_t^{(2)}(\xi) = \frac{c_H}{|\xi|^{2H_0+2}} \int_{\{|\tau| > 1/2\}} \frac{|\tau|^{-(2H_0-1)}}{(\tau^2-1)^2} \Big(f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau) \Big) \mathrm{d}\tau.$$

We consider $N_t^{(1)}(\xi)$ first. Using the fact that,

$$\begin{aligned} |f_t(\lambda, \tau)| &= |\sin(\tau\lambda t) - \tau \sin(\lambda t)| \le |\sin(\tau\lambda t)| + |\tau \sin(\lambda t)| \le 1 + |\tau| \\ |g_t(\lambda, \tau)| &= |\cos(\tau\lambda t) - \cos(\lambda t)| \le |\cos(\tau\lambda t)| + |\cos(\lambda t)| \le 2, \end{aligned}$$

for any $t > 0$, we have:

$$\begin{aligned} N_t^{(1)}(\xi) &= \frac{c_H}{|\xi|^{2H_0+2}} \int_{\{|\tau| \le 1/2\}} \frac{|\tau|^{-(2H_0-1)}}{(\tau^2-1)^2} \Big(f_t^2(|\xi|, \tau) + g_t^2(|\xi|, \tau) \Big) \mathrm{d}\tau \\ &\le \frac{c_H}{|\xi|^{2H_0+1}} \int_{\{|\tau| \le 1/2\}} \frac{|\tau|^{-(2H_0-1)}}{(\tau^2-1)^2} \Big(\big(1 + |\tau|\big)^2 + 2^2 \Big) \mathrm{d}\tau \\ &\le \frac{c_H}{|\xi|^{2H_0+1}} \int_{\{|\tau| \le 1/2\}} \frac{100}{9} |\tau|^{-(2H_0-1)} \mathrm{d}\tau = \frac{100}{9} \frac{c_H}{1-H_0} \left(\frac{1}{2} \right)^{2-2H_0} \frac{1}{|\xi|^{2H_0+1}} \\ &\le \frac{100}{9} \frac{c_H}{1-H_0} \left(\frac{1}{2} \right)^{2-2H_0} \left(\frac{2}{1+|\xi|^2} \right)^{H_0+1/2} \\ &= \frac{100}{9} \frac{c_H}{1-H_0} 2^{3H_0-3/2} \left(\frac{1}{1+|\xi|^2} \right)^{H_0+1/2}, \end{aligned} \tag{3.3.5}$$

since $\frac{1}{(1-\tau^2)^2}[(1+|\tau|)^2+4] \leq \frac{1}{(3/4)^2}[(3/2)^2+4] = \frac{100}{9}$ if $|\tau| \leq 1/2$ and

$$|\xi|^{2H_0+2} \geq |\xi|^{2H_0+1} \quad \text{and} \quad \frac{1}{|\xi|^2} \leq \frac{2}{1+|\xi|^2} \quad \text{if } |\xi| \geq 1.$$

Now we consider $N_t^{(2)}(\xi)$. By Lemma A.9, for any $\lambda > 0$ and $t > 0$,

$$\frac{2\pi \sin^2 1}{3}\Big(t \wedge t^3\Big)\frac{\lambda^3}{1+\lambda^2} \leq \int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2}\Big[f_t^2(\lambda,\tau)+g_t^2(\lambda,\tau)\Big]\mathrm{d}\tau \leq 4\pi\Big(t \vee t^3\Big)\frac{\lambda^3}{1+\lambda^2}.$$

Using this lemma with $\lambda = |\xi|$ and the fact that $|\tau|^{-(2H_0-1)} \leq (1/2)^{-(2H_0-1)}$ if $|\tau| \geq 1/2$, and $|\xi|^2/(1+|\xi|^2) \leq 1$, we have:

$$\begin{aligned}
N_t^{(2)}(\xi) &= \frac{c_H}{|\xi|^{2H_0+2}} \int_{\{|\tau|>1/2\}} \frac{|\tau|^{-(2H_0-1)}}{(\tau^2-1)^2}\Big(f_t^2(|\xi|,\tau)+g_t^2(|\xi|,\tau)\Big)\mathrm{d}\tau \\
&\leq c_H \frac{1}{2^{-(2H_0-1)}}\frac{1}{|\xi|^{2H_0+2}} \int_{|\tau|>1/2} \frac{1}{(\tau^2-1)^2}\Big(f_t^2(|\xi|,\tau)+g_t^2(|\xi|,\tau)\Big)\mathrm{d}\tau \\
&\leq c_H 2^{2H_0-1}\frac{1}{|\xi|^{2H_0+2}} \int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2}\Big(f_t^2(|\xi|,\tau)+g_t^2(|\xi|,\tau)\Big)\mathrm{d}\tau \\
&\leq c_H 2^{2H_0-1}\frac{1}{|\xi|^{2H_0+2}} 4\pi \frac{|\xi|^3}{1+|\xi|^2}\Big(t \vee t^3\Big) \\
&\leq c_H 2^{2H_0-1} 4\pi \frac{1}{|\xi|^{2H_0+1}}\Big(t \vee t^3\Big) \leq c_H 2^{2H_0-1} 4\pi \left(\frac{2}{1+|\xi|^2}\right)^{H_0+1/2}\Big(t \vee t^3\Big) \\
&= c_H 4\pi 2^{3H_0-1/2}\left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2}\Big(t \vee t^3\Big). \qquad (3.3.6)
\end{aligned}$$

Note that for the last inequality, we used the fact that $\frac{1}{|\xi|^2} \leq \frac{2}{1+|\xi|^2}$. Therefore, taking the sum of (3.3.5) and (3.3.6), we have:

$$\begin{aligned}
N_t^w(\xi) &\leq c_H\left(\frac{100}{9}\frac{1}{1-H_0}2^{3H_0-3/2} + 4\pi 2^{3H_0-1/2}\Big(t \vee t^3\Big)\right)\left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2} \\
&\leq c_H\left(\frac{100}{9}\cdot\frac{1}{1-H_0}2^{-1}+4\pi\right)2^{3H_0-1/2}\Big(t^3 \vee 1\Big)\left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2}, \qquad (3.3.7)
\end{aligned}$$

since if $t > 1$, then $t^3 \geq t \geq 1$ and if $t < 1$, then $t^3 < t < 1$. Hence, $t^3 \vee t \vee 1 = t^3 \vee 1$. Combining the relations (3.3.3) and (3.3.7), we obtain the upper bound in relation (3.3.2), noting that $t^{2H_0+2} \vee t^3 \vee 1 = t^{2H_0+2} \vee 1$.

For the lower bound, suppose that $t|\xi| \leq 1$ first. By using the fact that $\sin x \geq x\cos 1$ for all $x \in [0,1]$ and $\sin x > 0$ for all $x \in [0,1]$, we obtain:

$$\begin{aligned}
N_t^w(\xi) &= \frac{\alpha_{H_0}}{|\xi|^2}\int_0^t\int_0^t \sin(s|\xi|)\sin(r|\xi|)|r-s|^{2H_0-2}\mathrm{d}s\mathrm{d}r \\
&\geq \alpha_{H_0}\cos^2(1)\int_0^t\int_0^t rs|r-s|^{2H_0-2}\mathrm{d}s\mathrm{d}r = \alpha_{H_0}\cos^2(1)\frac{B(2,2H_0-1)}{H_0+1}t^{2H_0+2} \\
&\geq \alpha_{H_0}\cos^2(1)\frac{B(2,2H_0)}{H_0+1}t^{2H_0+2}\left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2}, \qquad (3.3.8)
\end{aligned}$$

where we used Lemma D.2 and the fact that $1 \geq \left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2}$.

Now we consider $t|\xi| > 1$. Let $\rho > 1$ be a constant to be specified later. We know that the integrand of

$$N_t^w(\xi) = \frac{c_H}{|\xi|^{2H_0+2}}\int_{\mathbb{R}} \frac{|\tau|^{-(2H_0-1)}}{(\tau^2-1)^2}\Big[f_t^2(|\xi|,\tau) + g_t^2(|\xi|,\tau)\Big]\mathrm{d}\tau$$

is non-negative. Hence $N_t^w(\xi)$ is bounded below by the integral over the region $|\tau| < \rho$. In that region, $|\tau|^{-(2H_0-1)} \geq \rho^{-(2H_0-1)}$, and therefore,

$$\begin{aligned}
N_t^w(\xi) &\geq \frac{c_H\rho^{-(2H_0-1)}}{|\xi|^{2H_0+2}}\int_{\{|\tau|<\rho\}} \frac{f_t^2(|\xi|,\tau)+g_t^2(|\xi|,\tau)}{(\tau^2-1)^2}\mathrm{d}\tau \\
&= \frac{c_H\rho^{-(2H_0-1)}}{|\xi|^{2H_0+2}}\left(J(t|\xi|) - \int_{\{|\tau|\geq\rho\}} \frac{f_t^2(|\xi|,\tau)+g_t^2(|\xi|,\tau)}{(\tau^2-1)^2}\mathrm{d}\tau\right), \qquad (3.3.9)
\end{aligned}$$

where

$$\begin{aligned}
J(t|\xi|) &:= \int_{\mathbb{R}} \frac{f_t^2(|\xi|,\tau)+g_t^2(|\xi|,\tau)}{(\tau^2-1)^2}\mathrm{d}\tau = \int_0^T \left|\mathcal{F}_{0,t|\xi|}\sin(\tau)\right|^2\mathrm{d}\tau \\
&= 2\pi\int_0^{t|\xi|}\sin^2(x)\mathrm{d}x = \pi\int_0^{t|\xi|}\big(1-\cos(2x)\big)\mathrm{d}x = \pi t|\xi|\left[1 - \frac{\sin(2t|\xi|)}{2t|\xi|}\right],
\end{aligned}$$

using Plancherel theorem for the third equality above. Note that

$$J(t|\xi|) \geq \frac{1}{2}\pi t|\xi|, \qquad (3.3.10)$$

since $t|\xi| \geq 1$ and $\sin(2x) \leq x$, if $x > 1$. To find an upper bound for the second integral in the right-hand side of (3.3.9), we use the fact that for any $\lambda > 0$ and $\tau \in \mathbb{R}$,

$$f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau) \leq 2\lambda t\Big(1+|\tau|\Big)^2. \qquad (3.3.11)$$

To prove (3.3.11), note that $|f_t(\lambda,\tau)| \leq 1+|\tau|$ and $|f_t(\lambda,\tau)| \leq 2|\tau|\lambda t$, where for the second inequality, we used the fact that $|\sin x| \leq |x|$. Taking the product of these two inequalities, we get:

$$f_t^2(\lambda,\tau) \leq 2|\tau|\lambda t(1+|\tau|). \tag{3.3.12}$$

Similarly, $|g_t(\lambda,\tau)| \leq 2$ and $|g_t(\lambda,\tau)| \leq |\cos(\lambda\tau t)-1| + |\cos\lambda t - 1| \leq |\tau|\lambda t + \lambda t = \lambda t(1+|\tau|)$ using the inequality $|1-\cos x| \leq |x|$ for all x. Taking the product of these two inequalities, we get:

$$g_t^2(\lambda,\tau) \leq 2\lambda t(1+|\tau|). \tag{3.3.13}$$

Relation (3.3.11) follows by taking the sum of (3.3.12) and (3.3.13).

Using (3.3.11), we have

$$\int_{\{|\tau|\geq\rho\}} \frac{f_t^2(|\xi|,\tau)+g_t^2(|\xi|,\tau)}{(\tau^2-1)^2}\mathrm{d}\tau \leq \int_{\{|\tau|\geq\rho\}} \frac{2t|\xi|\Big(1+|\tau|\Big)^2}{(\tau^2-1)^2}\mathrm{d}\tau = C_\rho t|\xi|, \tag{3.3.14}$$

where

$$\begin{aligned} C_\rho &= 2\int_{|\tau|\geq\rho} \frac{(1+|\tau|)^2}{(\tau^2-1)^2}\mathrm{d}\tau = 2\int_{|\tau|\geq\rho}\frac{1}{(|\tau|-1)^2}\mathrm{d}\tau = 4\int_\rho^\infty \frac{1}{(\tau-1)^2}\mathrm{d}\tau \\ &= 4\int_{\rho-1}^\infty \frac{1}{x^2}\mathrm{d}x = \frac{4}{\rho-1}. \end{aligned}$$

Combining relations (3.3.9), (3.3.10) and (3.3.14), we get

$$N_t^w(\xi) \geq \frac{c_H\rho^{-(2H_0-1)}}{|\xi|^{2H_0+2}}\Big(\frac{\pi}{2}-C_\rho\Big)t|\xi|.$$

Choose ρ large enough such that $C_\rho < \pi/2$. For instance, when $\rho=4$, $C_\rho = 4/3$. Then we have:

$$N_t^w(\xi) \geq c_H 4^{-(2H_0-1)}\Big(\frac{\pi}{2}-\frac{4}{3}\Big)t\frac{1}{|\xi|^{2H_0+1}} \geq c_H 4^{-(2H_0-1)}\Big(\frac{\pi}{2}-\frac{4}{3}\Big)t\Big(\frac{1}{1+|\xi|^2}\Big)^{H_0+1/2}, \tag{3.3.15}$$

since $\frac{1}{|\xi|^2} \geq \frac{1}{1+|\xi|^2}$.

The lower bound in (3.3.2) follows from relations (3.3.8) and (3.3.15).

Using (3.3.2), we obtain:

$$D_{H_0}^{(1)}\Big(t^{2H_0+2}\wedge t\Big)I \leq I_t \leq D_{H_0}^{(2)}\Big(t^{2H_0+2}\vee 1\Big)I,$$

where $I = \int_{\mathbb{R}} \left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2} |\xi|^{1-2H} \mathrm{d}\xi$. It follows that $I_t < \infty$ if and only if

$$I = \int_{\mathbb{R}} \left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2} |\xi|^{1-2H} \mathrm{d}\xi < \infty.$$

This finishes **Step 1** of the proof.

Step 2: In this step, we prove that $I = \int_{\mathbb{R}} \left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2} |\xi|^{1-2H} \mathrm{d}\xi < \infty$ if and only if $H_0 + H > \frac{1}{2}$. Note that this condition is satisfied for any $H_0 \in (\frac{1}{2}, 1)$ and $H \in (0,1)$. We write

$$\begin{aligned} I &= \int_{|\xi|\leq 1} \left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2} |\xi|^{1-2H} \mathrm{d}\xi + \int_{|\xi|>1} \left(\frac{1}{1+|\xi|^2}\right)^{H_0+1/2} |\xi|^{1-2H} \mathrm{d}\xi \\ &:= I^{(1)} + I^{(2)}. \end{aligned}$$

We treat separately $I^{(1)}$ and $I^{(2)}$. For $I^{(1)}$, note that $\frac{1}{2} \leq \frac{1}{1+|\xi|^2} \leq 1$, if $|\xi| \leq 1$. Hence,

$$\left(\frac{1}{2}\right)^{H_0+1/2} \int_{|\xi|\leq 1} |\xi|^{1-2H} \, \mathrm{d}\xi \leq I^{(1)} \leq \int_{|\xi|\leq 1} |\xi|^{1-2H} \, \mathrm{d}\xi.$$

Note that $\int_{|\xi|\leq 1} |\xi|^{1-2H} \, \mathrm{d}\xi = 2\int_0^1 \xi^{1-2H} \mathrm{d}\xi < \infty$ since $1 - 2H + 1 > 0$.

Next, we treat $I^{(2)}$. Note that $\frac{1}{2}\frac{1}{|\xi|^2} \leq \frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2}$, if $|\xi| > 1$. Hence,

$$\left(\frac{1}{2}\right)^{H_0+1/2} \int_{|\xi|>1} \left(\frac{1}{|\xi|^2}\right)^{H_0+1/2} |\xi|^{1-2H} \, \mathrm{d}\xi \leq I^{(2)} \leq \int_{|\xi|>1} \left(\frac{1}{|\xi|^2}\right)^{H_0+1/2} |\xi|^{1-2H} \, \mathrm{d}\xi.$$

So $I^{(2)} < \infty$ if and only if $\int_{|\xi|>1} |\xi|^{-2(H_0+1/2)} |\xi|^{1-2H} \mathrm{d}\xi < \infty$. This last integral is equal to $2\int_1^\infty \xi^{-2H_0-2H} \mathrm{d}\xi$ which is finite if and only if $-2H_0 - 2H + 1 < 0$, which is equivalent to $H_0 + H > \frac{1}{2}$. ∎

Remark 3.3.2. Note that the upper bound in relation (3.3.2) in the case $|\xi| > 1$ cannot be obtained by applying Lemma C.2 as in the case of the heat equation, since by doing so, we would obtain:

$$\begin{aligned} N_t^w(\xi) &= \frac{\alpha_{H_0}}{|\xi|^2} \int_0^t \int_0^t \sin(s|\xi|) \sin(r|\xi|) |r-s|^{2H_0-2} \mathrm{d}s \mathrm{d}r \\ &= \frac{\alpha_{H_0}}{|\xi|^2} \left| \int_0^t \int_0^t \sin(s|\xi|) \sin(r|\xi|) |r-s|^{2H_0-2} \mathrm{d}s \mathrm{d}r \right| \end{aligned}$$

$$\leq \frac{\alpha_{H_0}}{|\xi|^2}\int_0^t\int_0^t \Big|\sin(s|\xi|)\Big|\Big|\sin(r|\xi|)\Big|(r-s)^{2H_0-2}\mathrm{d}s\mathrm{d}r$$
$$\leq \frac{b_{H_0}}{|\xi|^2}\bigg(\int_0^t \Big|\sin(s|\xi|)\Big|^{1/H_0}\mathrm{d}s\bigg)^{2H_0} \leq b_{H_0}t^{2H_0}\frac{1}{|\xi|^2},$$

which does not give the upper bound in the desired form $C_t|\xi|^{-2H_0-1}$ for some constant C_t.

3.4 Moment estimates: heat equation

In this section, we give some moment estimates for the increments of solution to the linear stochastic heat equation (1.0.1) with noise $\dot{W}$ as in Section 3.1. The result presented in this section is an extension to the case $H_0 > 1/2$ of the result obtained by [15] for the case $H_0 = 1/2$ (given by Theorem 2.3.4). Note that for the space increments, we had to impose the additional condition $2H_0 + H < 2$. Our estimate for the time increments given by relation (3.4.1) below can be derived from Theorem 2.6 of [25], but here we present a different proof than in this reference. On the other hand, the bound given by relation (3.4.2) below for the space increments is a particular case of Theorem 4 of [26].

Theorem 3.4.1. *Let u^h be the solution of the linear stochastic heat equation* (1.0.1) *with noise $\dot{W}$ as in Section 3.1.*
(a) For any $p > 0, H_0 \in (\frac{1}{2}, 1), H \in (0,1), t, t' \in [0,T]$ and $x, x' \in \mathbb{R}$, we have

$$E\Big|u^h(t',x) - u^h(t,x)\Big|^p \leq z_p\Big(C^{(1)}_{H_0,H}\Big)^{p/2}\Big|t'-t\Big|^{p(2H_0+H-1)/2}, \tag{3.4.1}$$

where

$$C^{(1)}_{H_0,H} = 2c_H\bigg(b_{H_0}(2H_0)^{2H_0}N_{H_0,H} + \Gamma(1-H)2^{1-H}R_{H_0,H}\bigg),$$
$$N_{H_0,H} = \int_{\mathbb{R}}\frac{\Big(1-\exp(-\eta^2/2)\Big)^2}{|\eta|^{4H_0+2H-1}}\mathrm{d}\eta,$$
$$R_{H_0,H} = \alpha_{H_0}\int_0^1\int_0^1 |r-s|^{2H_0-2}(r+s)^{H-1}\mathrm{d}r\mathrm{d}s.$$

(b) For any $p > 0$, for any $H_0 \in (\frac{1}{2}, 1), H \in (0,1)$ with

$$2H_0 + H < 2,$$

and for any $t, t' \in [0,T]$ and $x, x' \in \mathbb{R}$, we have

$$E\Big|u^h(t,x') - u^h(t,x)\Big|^p \leq z_p\Big(C^{(3)}_{H_0,H}\Big)^{p/2}\Big|x'-x\Big|^{p(2H_0+H-1)}, \tag{3.4.2}$$

where

$$C^{(3)}_{H_0,H} = 2c_H b_{H_0}(2H_0)^{2H_0}\bar{C}_{3-4H_0-2H},$$

and $\bar{C}_\alpha = \int_{\mathbb{R}}(1-\cos\eta)|\eta|^{\alpha-2}\mathrm{d}\eta$ *is the same as in Lemma 2.3.3.*

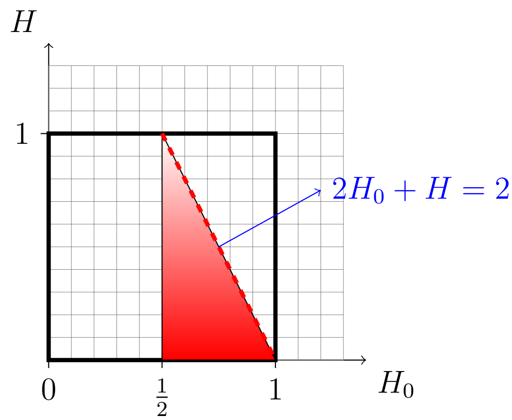


Figure 3.1: The shaded area represents the region $\{H_0 \in (1/2,1), H \in (0,1)$ and $2H_0 + H < 2\}$.

Proof: We first study the time increments. Assume that $t' > t$ and let $h = t' - t$. We use the same decomposition as on the proof of Theorem 2.3.4. Note that decomposition (2.3.2) still holds, but the variables I_1 and I_2 are correlated. We have:

$$E\big|u^h(t+h,x) - u^h(t,x)\big|^2 \le 2\Big(A(t,h) + B(t,h)\Big), \tag{3.4.3}$$

where

$$A(t,h) := E\left|\int_0^t\int_{\mathbb{R}} G^h(t+h-s,x-y) - G^h(t-s,x-y)W(\mathrm{d}s,\mathrm{d}y)\right|^2,$$
$$B(t,h) := E\left|\int_t^{t+h}\int_{\mathbb{R}} G^h(t+h-s,x-y)W(\mathrm{d}s,\mathrm{d}y)\right|^2.$$

We first study $A(t,h)$. Using the change of variables $s' = t-s$ and $r' = t-r$ and the Littlewood-Hardy type inequality given by Lemma C.2, we have:

$$A(t,h) = c_H\int_{\mathbb{R}}\Bigg(\alpha_{H_0}\int_0^t\int_0^t |r-s|^{2H_0-2}\mathcal{F}[G^h(t+h-s,\cdot) - G^h(t-s,\cdot)](\xi)$$
$$\overline{\mathcal{F}[G^h(t+h-r,\cdot) - G^h(t-r,\cdot)](\xi)}\mathrm{d}s\mathrm{d}r\Bigg)|\xi|^{1-2H}\mathrm{d}\xi$$

$$
\begin{aligned}
&= c_H \int_{\mathbb{R}} \Big(\alpha_{H_0} \int_0^t \int_0^t |r' - s'|^{2H_0-2}\, \mathcal{F}[G^h(s'+h,\cdot) - G^h(s',\cdot)](\xi) \\
&\qquad\qquad \overline{\mathcal{F}[G^h(r'+h,\cdot) - G^h(r',\cdot)](\xi)} \mathrm{d}r'\mathrm{d}s' \Big) |\xi|^{1-2H} \mathrm{d}\xi \\
&\le c_H b_{H_0} \int_{\mathbb{R}} \Big(\int_0^t \big|\mathcal{F}G^h(s+h,\cdot)(\xi) - \mathcal{F}G^h(s,\cdot)(\xi)\big|^{\frac{1}{H_0}} \mathrm{d}s \Big)^{2H_0} |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H b_{H_0} \int_{\mathbb{R}} \Big(e^{-\frac{h|\xi|^2}{2}} - 1 \Big)^2 \Big(\int_0^t e^{-\frac{s|\xi|^2}{2H_0}} \mathrm{d}s \Big)^{2H_0} |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H b_{H_0} (2H_0)^{2H_0} \int_{\mathbb{R}} \Big(1 - e^{-\frac{h|\xi|^2}{2}} \Big)^2 |\xi|^{-4H_0} \Big(1 - e^{-\frac{t|\xi|^2}{2H_0}} \Big)^{2H_0} |\xi|^{1-2H} \mathrm{d}\xi \\
&\le c_H b_{H_0} (2H_0)^{2H_0} \int_{\mathbb{R}} \frac{\big(1 - e^{-\frac{h|\xi|^2}{2}}\big)^2}{|\xi|^{4H_0+2H-1}} \mathrm{d}\xi \\
&= c_H b_{H_0} (2H_0)^{2H_0} (\sqrt{h})^{4H_0+2H-2} \int_{\mathbb{R}} \frac{(1 - e^{-\eta^2/2})^2}{|\eta|^{4H_0+2H-1}} \mathrm{d}\eta \\
&= c_H b_{H_0} (2H_0)^{2H_0} N_{H_0,H} h^{2H_0+H-1},
\end{aligned}
\tag{3.4.4}
$$

where for the second last equality, we used the change of variable $\eta = \sqrt{h}\xi$.

Next, we consider $B(t,h)$. Using the change of variables $s' = t+h-s$ and $r' = t+h-r$, we have:

$$
\begin{aligned}
B(t,h) &= c_H \int_{\mathbb{R}} \Big(\alpha_{H_0} \int_t^{t+h} \int_t^{t+h} |r-s|^{2H_0-2} \mathcal{F}G^h(t+h-s, x-\cdot)(\xi) \\
&\qquad\qquad \overline{\mathcal{F}G^h(t+h-r, x-\cdot)(\xi)} \mathrm{d}r\mathrm{d}s \Big) |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H \int_{\mathbb{R}} \Big(\alpha_{H_0} \int_0^h \int_0^h |r'-s'|^{2H_0-2} \mathcal{F}G^h(s',\cdot)(\xi) \overline{\mathcal{F}G^h(r',\cdot)(\xi)} \mathrm{d}r'\mathrm{d}s' \Big) |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H \int_{\mathbb{R}} \Big(\alpha_{H_0} \int_0^h \int_0^h |r-s|^{2H_0-2} \exp\Big(-\frac{s|\xi|^2}{2}\Big) \exp\Big(-\frac{r|\xi|^2}{2}\Big) \mathrm{d}r\mathrm{d}s \Big) |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H \alpha_{H_0} \int_0^h \int_0^h |r-s|^{2H_0-2} \Big(\int_{\mathbb{R}} \exp\Big(-\frac{(s+r)|\xi|^2}{2}\Big) |\xi|^{1-2H} \mathrm{d}\xi \Big) \mathrm{d}r\mathrm{d}s \\
&= c_H \alpha_{H_0} \int_0^h \int_0^h |r-s|^{2H_0-2} \Big(\frac{s+r}{2}\Big)^{H-1} \Gamma\Big(\frac{2-2H}{2}\Big) \mathrm{d}r\mathrm{d}s \\
&= c_H \alpha_{H_0} \Gamma(1-H) 2^{1-H} \int_0^h \int_0^h |r-s|^{2H_0-2} (s+r)^{H-1} \mathrm{d}r\mathrm{d}s,
\end{aligned}
$$

where for the second last equality, we used the following identity: (see equation (34) of [4])

$$\int_{\mathbb{R}} e^{-t|\xi|^2}|\xi|^{\alpha}\mathrm{d}\xi = t^{-(\alpha+1)/2}\Gamma\left(\frac{\alpha+1}{2}\right), \text{ for } \alpha > -1. \tag{3.4.5}$$

Using the change of variables $\bar{r} = r/h$, and $\bar{s} = s/h$, we have:

$$\begin{aligned} B(t,h) &= c_H\alpha_{H_0}\Gamma(1-H)2^{1-H}\int_0^1\int_0^1 |\bar{r}h - \bar{s}h|^{2H_0-2}(\bar{r}h+\bar{s}h)^{H-1}h^2\mathrm{d}r\mathrm{d}s \\ &= c_H\Gamma(1-H)2^{1-H}R_{H_0,H}h^{2H_0+H-1}, \end{aligned} \tag{3.4.6}$$

where

$$R_{H_0,H} = \alpha_{H_0}\int_0^1\int_0^1 |r-s|^{2H_0-2}(r+s)^{H-1}\mathrm{d}r\mathrm{d}s. \tag{3.4.7}$$

Using relations (3.4.3), (3.4.4) and (3.4.6), we obtain:

$$\begin{aligned} E|u^h(t+h,x) - u^h(t,x)|^2 &\le 2c_H\Big\{b_{H_0}(2H_0)^{2H_0}N_{H_0,H} + \Gamma(1-H)2^{1-H}R_{H_0,H}\Big\}h^{2H_0+H-1} \\ &= C^{(1)}_{H_0,H}h^{2H_0+H-1}. \end{aligned}$$

Using relation (2.3.1),

$$\begin{aligned} E|u^h(t+h,x) - u^h(t,x)|^p &= z_p\big(E|u^h(t+h,x)-u^h(t,x)|^2\big)^{p/2} \\ &\le z_p\left(C^{(1)}_{H_0,H}\right)^{p/2} h^{p(2H_0+H-1)/2}. \end{aligned}$$

Now we consider the space increments. Again, we use the same decomposition as on the proof of Theorem 2.3.4. Let $x' = x + z$. Using the change of variables $s' = t - s$ and $r' = t - r$, we have:

$$\begin{aligned} C(t,z) &:= E\big|u^h(t,x+z) - u^h(t,x)\big|^2 \\ &= E\left|\int_0^t\int_{\mathbb{R}} G^h(t-s,x+z-y) - G^h(t-s,x-y)W(\mathrm{d}s,\mathrm{d}y)\right|^2 \\ &= c_H\int_{\mathbb{R}}\Bigg(\alpha_{H_0}\int_0^t\int_0^t |s-r|^{2H_0-2}\mathcal{F}\Big[G^h(t-s,x+z-\cdot) - G^h(t-s,x-\cdot)\Big](\xi) \\ &\qquad \overline{\mathcal{F}[G^h(t-r,x+z-\cdot) - G^h(t-r,x-\cdot)](\xi)}\mathrm{d}s\mathrm{d}r\Bigg)|\xi|^{1-2H}\mathrm{d}\xi \\ &= c_H\int_{\mathbb{R}}\Bigg(\alpha_{H_0}\int_0^t\int_0^t |s'-r'|^{2H_0-2}\mathcal{F}[G^h(s',x+z-\cdot) - G^h(s',x-\cdot)](\xi) \\ &\qquad \overline{\mathcal{F}[G^h(r',x+z-\cdot) - G^h(r',x-\cdot)](\xi)}\mathrm{d}s'\mathrm{d}r'\Bigg)|\xi|^{1-2H}\mathrm{d}\xi \end{aligned}$$

$$= c_H \int_{\mathbb{R}} \Big|e^{-i\xi z} - 1\Big|^2 \Big(\alpha_{H_0} \int_0^t \int_0^t |s-r|^{2H_0-2} \mathcal{F}G^h(s,\cdot)(\xi)\overline{\mathcal{F}G^h(r,\cdot)(\xi)} \mathrm{d}s\mathrm{d}r\Big)|\xi|^{1-2H}\mathrm{d}\xi$$
$$= c_H \int_{\mathbb{R}} 2\Big[1-\cos(\xi z)\Big]\ N_t^h(\xi)\ |\xi|^{1-2H}\mathrm{d}\xi,$$

where $N_t^h(\xi)$ is defined in Theorem 3.2.1 and we used the fact that

$$\mathcal{F}\big[G^h(s, x+z-\cdot) - G^h(s, x-\cdot)\big](\xi) = e^{-i\xi x}(e^{-i\xi z}-1)\mathcal{F}G^h(s,\cdot)(\xi),$$

and

$$\Big|e^{-ia}-1\Big|^2 = 2(1-\cos a).$$

Using the Littlewood-Hardy type inequality given by Lemma C.2, we see that

$$N_t^h(\xi) \le b_{H_0}\Big(\int_0^t \exp\Big(-\frac{s|\xi|^2}{2H_0}\Big)\mathrm{d}s\Big)^{2H_0} = b_{H_0}\Big(\frac{2H_0}{|\xi|^2}\Big)^{2H_0}\Big(1-\exp\Big(-\frac{t|\xi|^2}{2H_0}\Big)\Big)^{2H_0}$$
$$\le b_{H_0}(2H_0)^{2H_0}|\xi|^{-4H_0}.$$

Hence,

$$C(t,z) \le c_H b_{H_0}(2H_0)^{2H_0} 2\int_{\mathbb{R}} \frac{1-\cos(\xi z)}{|\xi|^{4H_0+2H-1}}\mathrm{d}\xi.$$

To evaluate the previous integral, we use the change of variable $\eta = \xi z$. We obtain:

$$C(t,z) \le c_H b_{H_0}(2H_0)^{2H_0} 2|z|^{4H_0+2H-2}\int_{\mathbb{R}} \frac{1-\cos\eta}{|\eta|^{4H_0+2H-1}}\mathrm{d}\eta$$
$$= c_H b_{H_0}(2H_0)^{2H_0} 2\bar{C}_{3-4H_0-2H}|z|^{4H_0+2H-2} = C^{(3)}_{H_0,H}\,|z|^{4H_0+2H-2}\,.$$

Note that $\bar{C}_{3-4H_0-2H} < \infty$ since $3-4H_0-2H \in (-1,1)$. For this, we need the condition $2H_0 + H < 2$.

Using relation (2.3.1),

$$E|u^h(t,x+z) - u^h(t,x)|^p = z_p C(t,z)^{p/2}$$
$$\le z_p (C^{(3)}_{H_0,H})^{p/2}|z|^{p(2H_0+H-1)}.$$

∎

Remark 3.4.2. Note that the upper bound estimates given in Theorem 3.4.1 are sharp, i.e. there exist some matching lower bounds. In [27], the authors gave a sharp estimate for the temporal regularity of the solution u to the linear heat equation with bifractional-colored noise. In [26], the authors studied the sample path regularity of solution u to the fractional-colored stochastic heat equation in time and space separately. We omit the details here.

The following lemmas give some bounds for the constants $N_{H_0,H}$ and $R_{H_0,H}$ appearing in Theorem 3.4.1. Note that $N_{\frac{1}{2},H} = N_H$, where the constant N_H was defined in Theorem 2.3.4.

Lemma 3.4.3. *For any $H_0, H \in \mathbb{R}$ such that $1 < 2H_0 + H < 3$,*

$$N_{H_0,H} \leq \frac{1}{2} \cdot \frac{1}{6 - 4H_0 - 2H} + 2 \cdot \frac{1}{4H_0 + 2H - 2}.$$

Proof: We have:

$$\begin{aligned}
N_{H_0,H} &= \int_{|\eta|\leq 1} \frac{\left(1 - \exp(-\frac{\eta^2}{2})\right)^2}{|\eta|^{4H_0+2H-1}} \mathrm{d}\eta + \int_{|\eta|>1} \frac{\left(1 - \exp(-\frac{\eta^2}{2})\right)^2}{|\eta|^{4H_0+2H-1}} \mathrm{d}\eta \\
&\leq \int_{|\eta|\leq 1} \frac{(\eta^2/2)^2}{|\eta|^{4H_0+2H-1}} \mathrm{d}\eta + \int_{|\eta|>1} \frac{1}{|\eta|^{4H_0+2H-1}} \mathrm{d}\eta \\
&= 2 \cdot \frac{1}{4} \int_0^1 \eta^{4-4H_0-2H+1} \mathrm{d}\eta + 2 \int_1^\infty \eta^{-4H_0-2H+1} \mathrm{d}\eta \\
&= \frac{1}{2} \cdot \frac{1}{6 - 4H_0 - 2H} + 2 \cdot \frac{1}{4H_0 + 2H - 2},
\end{aligned}$$

where for the inequalities, we used the fact that

$$1 - e^{-x} \leq x \text{ if } |x| \leq 1,$$

$$1 - e^{-x} \leq 1 \text{ if } |x| > 1,$$

and we also used the fact that $1 < 2H_0 + H < 3$. ∎

Lemma 3.4.4. *For any $H_0 \in (\frac{1}{2}, 1)$ and $H \in (0, 1)$,*

$$R_{H_0,H} \leq 2^{H-1} b_{H_0} \frac{1}{H}.$$

Proof: By identity (3.4.5),

$$(r + s)^{H-1} = 2^{H-1} \frac{1}{\Gamma(1 - H)} \int_{\mathbb{R}} \exp\left(-\frac{(r + s)|\xi|^2}{2}\right) |\xi|^{1-2H} \mathrm{d}\xi.$$

Then, by applying Corollary C.4 and Fubini's theorem, we have:

$$R_{H_0,H} = 2^{H-1} \frac{1}{\Gamma(1 - H)} \int_{\mathbb{R}} \left(\int_0^1 \int_0^1 |r - s|^{2H_0-2} \exp\left(-\frac{(r + s)|\xi|^2}{2}\right) \mathrm{d}r \mathrm{d}s \right) |\xi|^{1-2H} \mathrm{d}\xi$$

$$\leq 2^{H-1}\frac{1}{\Gamma(1-H)}b_{H_0}\int_{\mathbb{R}}\left(\int_0^1 e^{-s|\xi|^2}\mathrm{d}s\right)|\xi|^{1-2H}\mathrm{d}\xi$$
$$= 2^{H-1}\frac{1}{\Gamma(1-H)}b_{H_0}\int_0^1\left(\int_{\mathbb{R}} e^{-s|\xi|^2}|\xi|^{1-2H}\mathrm{d}\xi\right)\mathrm{d}s.$$

Note that by (3.4.5),

$$\int_{\mathbb{R}} e^{-s|\xi|^2}|\xi|^{1-2H}\mathrm{d}\xi = \Gamma(1-H)s^{H-1}.$$

Hence,

$$R_{H_0,H} \leq 2^{H-1}b_{H_0}\int_0^1 s^{H-1}\mathrm{d}s = 2^{H-1}b_{H_0}\frac{1}{H}.$$

∎

Remark 3.4.5. Taking formally $H_0 = 1/2$ in relations (3.4.1) and (3.4.2), we obtain moment bounds which are consistent with those given by Theorem 2.3.4 for the white noise in time.

3.5 Moment estimates: wave equation

In this section, we give some moment estimates for the increments of the solution to the linear stochastic wave equation (1.0.2) with noise W as in Section 3.1. To obtain these bounds, we need to impose the condition $2H_0+2H<3$. We follow very closely the arguments presented in reference [10], by including the explicit form of all the constants.

We begin with some elementary results which will be used in the proof of Theorem 3.5.4.

Lemma 3.5.1. *[Lemma 3.1 of [10]] For $a, b \in \mathbb{R}$ with $a<b$ and $H_0 \in (\frac{1}{2}, 1)$,*

$$\int_a^b\int_a^b \cos(u)\cos(v)|u-v|^{2H_0-2}\mathrm{d}u\mathrm{d}v$$
$$= \int_0^{b-a}(b-a-v)v^{2H_0-2}\cos(v)\mathrm{d}v + \cos(a+b)\int_0^{b-a}(b-a-v)v^{2H_0-2}\cos(v)\mathrm{d}v, \tag{3.5.1}$$

and

$$\int_a^b\int_a^b \sin(u)\sin(v)|u-v|^{2H_0-2}\mathrm{d}u\mathrm{d}v$$

$$= \int_0^{b-a} (b-a-v)v^{2H_0-2}\cos(v)\mathrm{d}v - \cos(a+b)\int_0^{b-a}(b-a-v)v^{2H_0-2}\cos(v)\mathrm{d}v. \tag{3.5.2}$$

Remark 3.5.2. From Lemma 3.5.1, taking the sum of (3.5.1) and (3.5.2), we conclude that

$$\int_a^b\int_a^b \cos(u)\cos(v)|u-v|^{2H_0-2}\mathrm{d}u\mathrm{d}v \leq 2\int_0^{b-a}(b-a-v)v^{2H_0-2}\cos(v)\mathrm{d}v.$$

Remark 3.5.3. We will use the following formulas: (see relations 3.761-4 and 3.761-9 of [16]): for any $\mu \in (0,1)$,

$$\int_0^\infty x^{u-1}\sin x \ \mathrm{d}x = \frac{\pi}{2\Gamma(1-\mu)\cos(\mu\pi/2)},$$

$$\int_0^\infty x^{u-1}\cos x \ \mathrm{d}x = \frac{\pi}{2\Gamma(1-\mu)\sin(\mu\pi/2)}.$$

In particular, for $\mu = 2H_0 - 1$ with $H_0 \in (\frac{1}{2}, 1)$, we obtain:

$$I_{H_0} := \int_0^\infty x^{2H_0-2}\sin x \ \mathrm{d}x = \frac{\pi}{2\Gamma(2-2H_0)\cos\Big((2H_0-1)\pi/2\Big)}$$

$$I'_{H_0} := \int_0^\infty x^{2H_0-2}\cos x \ \mathrm{d}x = \frac{\pi}{2\Gamma(2-2H_0)\sin\Big((2H_0-1)\pi/2\Big)}.$$

Note that $I_{H_0} = \lim_{R\to\infty} I_{H_0}(R)$ and $I'_{H_0} = \lim_{R\to\infty} I'_{H_0}(R)$, where

$$I_{H_0}(R) = \int_0^R x^{2H_0-2}\sin x \ \mathrm{d}x \quad \text{and} \quad I'_{H_0}(R) = \int_0^R x^{2H_0-2}\cos x \ \mathrm{d}x.$$

From this, it follows that there exists $R_0 > 0$ such that

$$\frac{1}{2}I_{H_0} \leq I_{H_0}(R) \leq \frac{3}{2}I_{H_0}, \quad \text{for all } R > R_0.$$

Note that the function $R \to I_{H_0}(R)$ is continuous. Taking

$$M_{H_0} = \max\Big\{\frac{3}{2}I_{H_0}, \max_{R\in[0,R_0]}\big|I_{H_0}(R)\big|\Big\}, \tag{3.5.3}$$

we conclude that

$$|I_{H_0}(R)| \leq M_{H_0}, \quad \text{for all } R > 0. \tag{3.5.4}$$

Arguing similarly for $I'_{H_0}(R)$, we have:

$$|I'_{H_0}(R)| \le M'_{H_0}, \quad \text{for all } R > 0, \tag{3.5.5}$$

where

$$M'_{H_0} = \max\Big\{\frac{3}{2}I'_{H_0}, \max_{R\in[0,R'_0]}\big|I'_{H_0}(R)\big|\Big\}. \tag{3.5.6}$$

Theorem 3.5.4. *Let u^w be the solution of stochastic linear wave equation* (1.0.2) *with noise W as in Section 3.1. For any $p > 0, H_0 \in (\frac{1}{2}, 1)$, and $H \in (0,1)$ with*

$$2H_0 + 2H < 3, \tag{3.5.7}$$

and for any $t, t' \in [0,T]$ and $x, x' \in [-M, M]$, we have

$$E\Big|u^w(t',x) - u^w(t,x)\Big|^p \le z_p\Big(C^{(2)}_{H_0,H}\Big)^{p/2}T^{p/2}\Big|t'-t\Big|^{p(2H_0+2H-1)/2},$$

$$E\Big|u^w(t,x') - u^w(t,x)\Big|^p \le z_p\Big(C^{(4)}_{H_0,H}\Big)^{p/2}\Big(T^{2H_0+2}\vee 1\Big)^{p/2} \Big(M^{3-2H_0-2H}\vee 1\Big)^{p/2}\Big|x'-x\Big|^{p(2H_0+2H-1)/2},$$

where

$$C^{(2)}_{H_0,H} = 2c_H\Big\{2^{3-2H_0-2H}C^{(5)}_{H_0,H} + C^{(6)}_{H_0,H}\Big\},$$
$$C^{(4)}_{H_0,H} = c_H\Big\{\frac{2^{3-2H}}{1-H} + 4D^{(2)}_{H_0}\Big(\frac{1}{3-2H_0-2H} + \frac{4}{2H_0+2H-1}\Big)\Big\},$$

and

$$C^{(5)}_{H_0,H} = 2M'_{H_0}N'_{H_0,H} + \frac{8H_0}{3-2H_0-2H} + \frac{4}{1+2H} + \frac{(2H_0-1)M_{H_0}}{H_0+H},$$
$$C^{(6)}_{H_0,H} = D^{(2)}_{H_0}\Big(\frac{1}{1-H} + \frac{2}{2H_0+2H-1}\Big).$$

Here M_{H_0}, M'_{H_0} are given by (3.5.3), (3.5.6), *respectively,*

$$N'_{H_0,H} = \int_{\mathbb{R}} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H}}\mathrm{d}\xi,$$

and $D^{(2)}_{H_0}$ is given by relation (3.3.2).

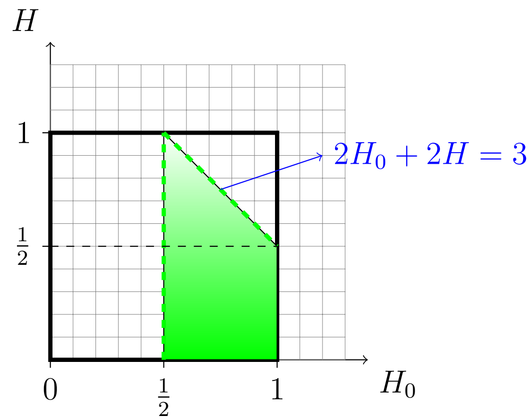


Figure 3.2: The shaded area represents the region $\{H_0 \in (1/2, 1), H \in (0,1)$ and $2H_0 + 2H < 3\}$.

Proof: We proceed as in the proof of Propositions 3.4 and 3.7 in [10] (considering only the upper bounds).

First, we study the time increments. Assume that $t' > t$ and let $h = t' - t$. We use the same decomposition as on the proof of Theorem 2.3.4. Note that decomposition (2.3.2) still holds, but the variables I_1 and I_2 are correlated. We have:

$$E\left|u^h(t+h,x) - u^h(t,x)\right|^2 \leq 2\Big(A(t,h) + B(t,h)\Big), \tag{3.5.8}$$

where

$$A(t,h) := E\left|\int_0^t \int_{\mathbb{R}} G^w(t+h-s, x-y) - G^w(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y)\right|^2,$$

and

$$B(t,h) := E\left|\int_t^{t+h} \int_{\mathbb{R}} G^w(t+h-s, x-y) W(\mathrm{d}s, \mathrm{d}y)\right|^2.$$

We first study $A(t,h)$. Recall that

$$\mathcal{F}G^w(t+h-s, x-\cdot)(\xi) = e^{-i\xi x}\overline{\mathcal{F}G^w(t+h-s,\cdot)(\xi)},$$
$$\mathcal{F}G^w(t-s, x-\cdot)(\xi) = e^{-i\xi x}\overline{\mathcal{F}G^w(t-s,\cdot)(\xi)}.$$

Using the change of variable $s' = t - s$ and $r' = t - r$, and the formula

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right),$$

we have:

$A(t,h)$

$$
\begin{aligned}
&= c_H \int_{\mathbb{R}} \Bigg(\alpha_{H_0} \int_0^t \int_0^t |s-r|^{2H_0-2} \mathcal{F}[G^w(t+h-s,\cdot) - G^w(t-s,\cdot)](\xi) \\
&\qquad \overline{\mathcal{F}[G^w(t+h-r,\cdot) - G^w(t-r,\cdot)](\xi)} \mathrm{d}r \mathrm{d}s \Bigg) |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H \int_{\mathbb{R}} \Bigg(\alpha_{H_0} \int_0^t \int_0^t |s'-r'|^{2H_0-2} \mathcal{F}[G^w(s'+h,\cdot) - G^w(s',\cdot)](\xi) \\
&\qquad \overline{\mathcal{F}[G^w(r'+h,\cdot) - G^w(r',\cdot)](\xi)} \mathrm{d}r' \mathrm{d}s' \Bigg) |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H \int_{\mathbb{R}} \Bigg(\alpha_{H_0} \int_0^t \int_0^t |s-r|^{2H_0-2} \frac{\sin\Big((s+h)|\xi|\Big) - \sin\Big(s|\xi|\Big)}{|\xi|} \cdot \\
&\qquad \frac{\sin\Big((r+h)|\xi|\Big) - \sin\Big(r|\xi|\Big)}{|\xi|} \mathrm{d}r \mathrm{d}s \Bigg) |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H \int_{\mathbb{R}} \Bigg(\alpha_{H_0} \int_0^t \int_0^t |s-r|^{2H_0-2} \frac{4\sin^2(h|\xi|/2)}{|\xi|^2} \cos\Big(\frac{2s+h}{2}|\xi|\Big) \\
&\qquad \cos\Big(\frac{2r+h}{2}|\xi|\Big) \mathrm{d}r \mathrm{d}s \Bigg) |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H 2^{2-2H} \int_{\mathbb{R}} \Bigg(\alpha_{H_0} \int_0^t \int_0^t |s-r|^{2H_0-2} \frac{\sin^2(h|\xi'|)}{|\xi'|^2} \cos\Big((2s+h)|\xi'|\Big) \\
&\qquad \cos\Big((2r+h)|\xi'|\Big) \mathrm{d}r \mathrm{d}s \Bigg) |\xi'|^{1-2H} \mathrm{d}\xi',
\end{aligned}
$$

where for the last line we used $\xi' = \xi/2$. Then we continue by applying the change of variables $s' = (2s+h)|\xi|$ and $r' = (2r+h)|\xi|$, with $\mathrm{d}s = (\frac{1}{2|\xi|})\mathrm{d}s'$ and $\mathrm{d}r = (\frac{1}{2|\xi|})\mathrm{d}r'$. We get:

$$
\begin{aligned}
&A(t,h) \\
&= c_H 2^{2-2H_0-2H} \int_{\mathbb{R}} \frac{\sin^2(h|\xi|)}{|\xi|^{2H_0+2H+1}} \Bigg(\alpha_{H_0} \int_{h|\xi|}^{(2t+h)|\xi|} \int_{h|\xi|}^{(2t+h)|\xi|} |s'-r'|^{2H_0-2} \\
&\qquad \cos(r')\cos(s') \mathrm{d}r' \mathrm{d}s' \Bigg) \mathrm{d}\xi \\
&= c_H 2^{2-2H_0-2H} \int_{\mathbb{R}} \frac{\sin^2(h|\xi|)}{|\xi|^{2H_0+2H+1}} \Big\| \cos(\cdot) 1_{\big[h|\xi|,(2t+h)|\xi|\big]} \Big\|^2_{\mathcal{H}_0} \mathrm{d}\xi, \qquad (3.5.9)
\end{aligned}
$$

where $\|\cdot\|_{\mathcal{H}_0}$ is given by

$$
\|\varphi\|^2_{\mathcal{H}_0} = \alpha_{H_0} \int_0^\infty \int_0^\infty \varphi(t)\varphi(s)|t-s|^{2H_0-2} \mathrm{d}t \mathrm{d}s.
$$

By Remark 3.5.2,

$$\begin{aligned}\Big\| \cos(\cdot)1_{\big[h|\xi|,(2t+h)|\xi|\big]}\Big\|^2_{\mathcal{H}_0} &\le 2\int_0^{2t|\xi|}(2t|\xi|-u)u^{2H_0-2}\cos(u)\mathrm{d}u \\ &= 2\bigg(2t|\xi|\int_0^{2t|\xi|}u^{2H_0-2}\cos(u)\mathrm{d}u - \int_0^{2t|\xi|}u^{2H_0-1}\cos(u)\mathrm{d}u\bigg) \\ &= 2\bigg(2t|\xi|\int_0^{2t|\xi|}u^{2H_0-2}\cos(u)\mathrm{d}u - \Big(2t|\xi|\Big)^{2H_0-1}\sin(2t|\xi|) \\ &\qquad + (2H_0-1)\int_0^{2t|\xi|}u^{2H_0-2}\sin(u)\mathrm{d}u\bigg), \end{aligned} \tag{3.5.10}$$

where for last equation, we used integration by parts. By relations (3.5.9) and (3.5.10), it follows that

$$\begin{aligned} A(t,h) &\le c_H 2^{3-2H_0-2H}\int_{\mathbb{R}}\frac{\sin^2(h|\xi|)}{|\xi|^{2H_0+2H+1}}\bigg(2t|\xi|\int_0^{2t|\xi|}u^{2H_0-2}\cos(u)\mathrm{d}u \\ &\qquad - \Big(2t|\xi|\Big)^{2H_0-1}\sin(2t|\xi|) + (2H_0-1)\int_0^{2t|\xi|}u^{2H_0-2}\sin(u)\mathrm{d}u\bigg)\mathrm{d}\xi \\ &= c_H 2^{3-2H_0-2H}\Big(I_1 - I_2 + I_3\Big), \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}}\frac{\sin^2(h|\xi|)}{|\xi|^{2H_0+2H+1}}\bigg(2t|\xi|\int_0^{2t|\xi|}u^{2H_0-2}\cos(u)\mathrm{d}u\bigg)\mathrm{d}\xi, \\ I_2 &:= \int_{\mathbb{R}}\frac{\sin^2(h|\xi|)}{|\xi|^{2H_0+2H+1}}\Big(2t|\xi|\Big)^{2H_0-1}\sin(2t|\xi|)\mathrm{d}\xi, \\ I_3 &:= (2H_0-1)\int_{\mathbb{R}}\frac{\sin^2(h|\xi|)}{|\xi|^{2H_0+2H+1}}\bigg(\int_0^{2t|\xi|}u^{2H_0-2}\sin(u)\mathrm{d}u\bigg)\mathrm{d}\xi. \end{aligned}$$

Since $A(t,h)\ge 0$, it follows that $I_1 - I_2 + I_3 \ge 0$. Hence, $I_1 - I_2 + I_3 = |I_1 - I_2 + I_3| \le |I_1| + |I_2| + |I_3|$ and

$$A(t,h) \le c_H 2^{3-2H_0-2H}\Big(|I_1| + |I_2| + |I_3|\Big). \tag{3.5.11}$$

We treat separately the three terms. We study I_1 first. Using the substitution $\xi' = h\xi$, we have

$$\begin{aligned} |I_1| &\le 2t\int_{\mathbb{R}}\frac{\sin^2(h|\xi|)}{|\xi|^{2H_0+2H}}\bigg|\int_0^{2t|\xi|}u^{2H_0-2}\cos(u)\mathrm{d}u\bigg|\mathrm{d}\xi \\ &= 2t\int_{\mathbb{R}}\frac{\sin^2(|\xi'|)}{|\xi'/h|^{2H_0+2H}}\bigg|\int_0^{2t|\xi'/h|}u^{2H_0-2}\cos(u)\mathrm{d}u\bigg|h^{-1}\mathrm{d}\xi' \end{aligned}$$

$$
\begin{aligned}
&= 2th^{2H_0+2H-1}\int_{\mathbb{R}}\frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H}}\left|\int_0^{2t|\xi|/h}u^{2H_0-2}\cos(u)\mathrm{d}u\right|\mathrm{d}\xi \\
&\le 2th^{2H_0+2H-1}M'_{H_0}\int_{\mathbb{R}}\frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H}}\mathrm{d}\xi \le 2TM'_{H_0}N'_{H_0,H}h^{2H_0+2H-1}, \qquad (3.5.12)
\end{aligned}
$$

where we used relation (3.5.5) for the first inequality and we recall that

$$
N'_{H_0,H} = \int_{\mathbb{R}}\frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H}}\mathrm{d}\xi.
$$

Next, we treat I_2. By using the substitution $\xi' = h\xi$, we have

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}}\frac{\sin^2(h|\xi|)}{|\xi|^{2H_0+2H+1}}\Big(2t|\xi|\Big)^{2H_0-1}\sin(2t|\xi|)\mathrm{d}\xi \\
&= (2t)^{2H_0-1}\int_{\mathbb{R}}\frac{\sin^2(h|\xi|)}{|\xi|^{2+2H}}\sin(2t|\xi|)\mathrm{d}\xi = (2t)^{2H_0-1}\int_{\mathbb{R}}\frac{\sin^2(|\xi'|)}{|\xi'/h|^{2+2H}}\sin\Big(\frac{2t|\xi'|}{h}\Big)h^{-1}\mathrm{d}\xi' \\
&= (2t)^{2H_0-1}h^{1+2H}\int_{\mathbb{R}}\frac{\sin^2(|\xi|)}{|\xi|^{2+2H}}\sin\Big(\frac{2t|\xi|}{h}\Big)\mathrm{d}\xi = (2t)^{2H_0-1}h^{1+2H}\Big(I'_2 + I''_2\Big), \qquad (3.5.13)
\end{aligned}
$$

where

$$
I'_2 = \int_{|\xi|\le 1}\frac{\sin^2(|\xi|)}{|\xi|^{2+2H}}\sin\Big(\frac{2t|\xi|}{h}\Big)\mathrm{d}\xi \quad\text{and}\quad I''_2 = \int_{|\xi|>1}\frac{\sin^2(|\xi|)}{|\xi|^{2+2H}}\sin\Big(\frac{2t|\xi|}{h}\Big)\mathrm{d}\xi.
$$

We first study I'_2. Note that

$$
\left|\sin\Big(\frac{2t|\xi|}{h}\Big)\right| = \left|\sin\Big(\frac{2t|\xi|}{h}\Big)\right|^{2-2H_0}\left|\sin\Big(\frac{2t|\xi|}{h}\Big)\right|^{2H_0-1} \le \Big(\frac{2t|\xi|}{h}\Big)^{2-2H_0},
$$

since $\sin(x) \le 1$ for all $x \in \mathbb{R}$ and $\sin(x) \le x$ for all $x > 0$. Here we used the fact that $H_0 > 1/2$. Then,

$$
\begin{aligned}
|I'_2| &\le \int_{|\xi|\le 1}\frac{\sin^2(|\xi|)}{|\xi|^{2+2H}}\left|\sin\Big(\frac{2t|\xi|}{h}\Big)\right|\mathrm{d}\xi \le \int_{|\xi|\le 1}\frac{\sin^2(|\xi|)}{|\xi|^{2+2H}}\Big(\frac{2t|\xi|}{h}\Big)^{2-2H_0}\mathrm{d}\xi \\
&= (2t)^{2-2H_0}h^{2H_0-2}\int_{|\xi|\le 1}\frac{\sin^2(|\xi|)}{|\xi|^{2H+2H_0}}\mathrm{d}\xi \le (2t)^{2-2H_0}h^{2H_0-2}\int_{|\xi|\le 1}\frac{|\xi|^2}{|\xi|^{2H+2H_0}}\mathrm{d}\xi \\
&= (2t)^{2-2H_0}h^{2H_0-2}\int_{|\xi|\le 1}|\xi|^{2-2H_0-2H}\mathrm{d}\xi = (2t)^{2-2H_0}h^{2H_0-2}2\int_0^1\xi^{2-2H_0-2H}\mathrm{d}\xi \\
&= 2^{3-2H_0}t^{2-2H_0}\frac{1}{3-2H_0-2H}h^{2H_0-2}, \qquad (3.5.14)
\end{aligned}
$$

where for the last equality, we used condition (3.5.7). For I_2'', we use the fact that $\sin(x) \le 1$ for all $x \in \mathbb{R}$, we obtain

$$\begin{aligned} |I_2''| &\le \int_{|\xi|>1} \frac{\sin^2(|\xi|)}{|\xi|^{2+2H}} \left| \sin\left(\frac{2t|\xi|}{h}\right) \right| \mathrm{d}\xi \le \int_{|\xi|>1} \frac{\sin^2(|\xi|)}{|\xi|^{2+2H}} \mathrm{d}\xi \\ &\le \int_{|\xi|>1} |\xi|^{-2-2H} \mathrm{d}\xi = 2\int_1^{\infty} \xi^{-2-2H} \mathrm{d}\xi = \frac{2}{1+2H}. \end{aligned} \tag{3.5.15}$$

Therefore, using (3.5.13), (3.5.14), and (3.5.15), and the fact that $h \le T$, we obtain:

$$\begin{aligned} |I_2| &\le (2t)^{2H_0-1} h^{1+2H} \Big(|I_2'| + |I_2''| \Big) \\ &\le (2t)^{2H_0-1} h^{1+2H} \Big\{ 2^{3-2H_0} t^{2-2H_0} \frac{1}{3-2H-2H_0} h^{2H_0-2} + \frac{2}{1+2H} \Big\} \\ &= \frac{4t}{3-2H_0-2H} h^{2H_0+2H-1} + \frac{4^{H_0} t^{2H_0-1}}{1+2H} h^{1+2H} \\ &= \frac{4t}{3-2H_0-2H} h^{2H_0+2H-1} + \frac{4^{H_0} t^{2H_0-1}}{1+2H} h^{2-2H_0} h^{2H_0+2H-1} \\ &\le \frac{4T}{3-2H_0-2H} h^{2H_0+2H-1} + \frac{4T^{2H_0-1}}{1+2H} T^{2-2H_0} h^{2H_0+2H-1} \\ &= 4\Big(\frac{1}{3-2H_0-2H} + \frac{1}{1+2H} \Big) T h^{2H_0+2H-1}. \end{aligned} \tag{3.5.16}$$

Now we treat I_3. By using the substitution $\xi' = h\xi$, we have

$$\begin{aligned} I_3 &= (2H_0-1) \int_{\mathbb{R}} \frac{\sin^2(h|\xi|)}{|\xi|^{2H_0+2H+1}} \bigg(\int_0^{2t|\xi|} u^{2H_0-2} \sin(u) \mathrm{d}u \bigg) \mathrm{d}\xi \\ &= (2H_0-1) \int_{\mathbb{R}} \frac{\sin^2(|\xi'|)}{(|\xi'|/h)^{2H_0+2H+1}} \bigg(\int_0^{2t|\xi'|/h} u^{2H_0-2} \sin(u) \mathrm{d}u \bigg) h^{-1} \mathrm{d}\xi' \\ &= (2H_0-1) h^{2H_0+2H} \int_{\mathbb{R}} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H+1}} \bigg(\int_0^{2t|\xi|/h} u^{2H_0-2} \sin(u) \mathrm{d}u \bigg) \mathrm{d}\xi \\ &= (2H_0-1) h^{2H_0+2H} \Big(I_3' + I_3'' \Big), \end{aligned} \tag{3.5.17}$$

where

$$\begin{aligned} I_3' &= \int_{|\xi|\le 1} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H+1}} \bigg(\int_0^{2t|\xi|/h} u^{2H_0-2} \sin(u) \mathrm{d}u \bigg) \mathrm{d}\xi, \\ I_3'' &= \int_{|\xi|>1} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H+1}} \bigg(\int_0^{2t|\xi|/h} u^{2H_0-2} \sin(u) \mathrm{d}u \bigg) \mathrm{d}\xi. \end{aligned}$$

We treat I_3' and I_3'' separately. We study I_3' first. Using the substitution $u' = uh/|\xi|$, we obtain:

$$\begin{aligned} I_3' &= \int_{|\xi|\le 1} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H+1}} \left(\int_0^{2t} \left(\frac{u'|\xi|}{h}\right)^{2H_0-2} \sin\left(\frac{u'|\xi|}{h}\right) \frac{|\xi|}{h} \mathrm{d}u' \right) \mathrm{d}\xi \\ &= h^{1-2H_0} \int_{|\xi|\le 1} \frac{\sin^2(|\xi|)}{|\xi|^{2+2H}} \left(\int_0^{2t} u^{2H_0-2} \sin\left(\frac{u|\xi|}{h}\right) \mathrm{d}u \right) \mathrm{d}\xi. \end{aligned}$$

Therefore, by using the fact that

$$\left|\sin\left(\frac{u|\xi|}{h}\right)\right| = \left|\sin\left(\frac{u|\xi|}{h}\right)\right|^{2-2H_0} \cdot \left|\sin\left(\frac{u|\xi|}{h}\right)\right|^{2H_0-1} \le \left(\frac{u|\xi|}{h}\right)^{2-2H_0},$$

and $\sin(x) \le 1$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} |I_3'| &\le h^{1-2H_0} \int_{|\xi|\le 1} \frac{\sin^2(|\xi|)}{|\xi|^{2+2H}} \left(\int_0^{2t} u^{2H_0-2} \left|\sin\left(\frac{u|\xi|}{h}\right)\right| \mathrm{d}u \right) \mathrm{d}\xi \\ &\le h^{1-2H_0} \int_{|\xi|\le 1} \frac{\sin^2(|\xi|)}{|\xi|^{2+2H}} \left(\int_0^{2t} u^{2H_0-2} \left(\frac{u|\xi|}{h}\right)^{2-2H_0} \mathrm{d}u \right) \mathrm{d}\xi \\ &= 2th^{-1} \int_{|\xi|\le 1} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H}} \mathrm{d}\xi \le 2th^{-1} \int_{|\xi|\le 1} \frac{|\xi|^2}{|\xi|^{2H_0+2H}} \mathrm{d}\xi \\ &= 2th^{-1} \int_{|\xi|\le 1} |\xi|^{2-2H_0-2H} \mathrm{d}\xi = 4th^{-1} \int_0^1 \xi^{2-2H_0-2H} \mathrm{d}\xi \\ &= \frac{4t}{3-2H_0-2H} h^{-1} \le \frac{4T}{3-2H_0-2H} h^{-1}, \end{aligned} \tag{3.5.18}$$

where for the last equality, we used again condition (3.5.7).

For I_3'', by using relation (3.5.4) and the fact that $\sin(x) \le 1$ for all $x > 0$,

$$\begin{aligned} |I_3''| &\le \int_{|\xi|>1} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H+1}} \left| \int_0^{2t|\xi|/h} u^{2H_0-2} \sin(u) \mathrm{d}u \right| \mathrm{d}\xi \\ &\le M_{H_0} \int_{|\xi|>1} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H+1}} \mathrm{d}\xi \le M_{H_0} \int_{|\xi|>1} \frac{1}{|\xi|^{2H_0+2H+1}} \mathrm{d}\xi \\ &= 2M_{H_0} \int_1^\infty \xi^{-2H_0-2H-1} \mathrm{d}\xi = M_{H_0} \cdot \frac{1}{H_0+H}. \end{aligned} \tag{3.5.19}$$

Hence, using (3.5.17), (3.5.18), and (3.5.19), and the fact that $h \le T$, we obtain:

$$\begin{aligned} |I_3| &\le (2H_0-1)h^{2H_0+2H} \Big(|I_3'| + |I_3''| \Big) \\ &\le (2H_0-1)h^{2H_0+2H} \left\{ \frac{4T}{3-2H_0-2H} h^{-1} + M_{H_0} \cdot \frac{1}{H_0+H} \right\} \end{aligned}$$

$$
\begin{aligned}
&= (2H_0-1)\frac{4T}{3-2H_0-2H}h^{2H_0+2H-1} + (2H_0-1)M_{H_0}\cdot\frac{1}{H_0+H}h^{2H_0+2H} \\
&= (2H_0-1)\frac{4T}{3-2H_0-2H}h^{2H_0+2H-1} + (2H_0-1)M_{H_0}\cdot\frac{1}{H_0+H}hh^{2H_0+2H-1} \\
&\leq (2H_0-1)\frac{4T}{3-2H_0-2H}h^{2H_0+2H-1} + (2H_0-1)M_{H_0}\cdot\frac{1}{H_0+H}Th^{2H_0+2H-1} \\
&= (2H_0-1)\Big(\frac{4}{3-2H_0-2H} + \frac{M_{H_0}}{H_0+H}\Big)Th^{2H_0+2H-1}. \qquad (3.5.20)
\end{aligned}
$$

Therefore, combining relations (3.5.11), (3.5.12), (3.5.16) and (3.5.20), we have

$$
\begin{aligned}
&A(t,h) \\
&\leq c_H 2^{3-2H_0-2H}\Big\{2M'_{H_0}N'_{H_0,H} + 4\Big(\frac{1}{3-2H_0-2H} + \frac{1}{1+2H}\Big) \\
&\qquad + (2H_0-1)\Big(\frac{4}{3-2H_0-2H} + \frac{M_{H_0}}{H_0+H}\Big)\Big\}Th^{2H_0+2H-1} \\
&= c_H 2^{3-2H_0-2H} C^{(5)}_{H_0,H} Th^{2H_0+2H-1}, \qquad (3.5.21)
\end{aligned}
$$

where

$$
C^{(5)}_{H_0,H} = 2M'_{H_0}N'_{H_0,H} + \frac{8H_0}{3-2H_0-2H} + \frac{4}{1+2H} + \frac{(2H_0-1)M_{H_0}}{H_0+H}.
$$

Now we treat $B(t,h)$. Using the change of variables $s' = t+h-s$ and $r' = t+h-r$, we obtain:

$$
\begin{aligned}
&B(t,h) \\
&= c_H\int_{\mathbb{R}}\Big(\alpha_{H_0}\int_t^{t+h}\int_t^{t+h}|s-r|^{2H_0-2}\mathcal{F}G^w(t+h-s,\cdot)(\xi)\overline{\mathcal{F}G^w(t+h-r,\cdot)(\xi)}\mathrm{d}r\mathrm{d}s\Big) \\
&\qquad |\xi|^{1-2H}\mathrm{d}\xi \\
&= c_H\int_{\mathbb{R}}\Big(\alpha_{H_0}\int_0^{h}\int_0^{h}|s'-r'|^{2H_0-2}\mathcal{F}G^w(s',\cdot)(\xi)\overline{\mathcal{F}G^w(r',\cdot)(\xi)}\mathrm{d}r'\mathrm{d}s'\Big)|\xi|^{1-2H}\mathrm{d}\xi \\
&= c_H\int_{\mathbb{R}}\Big(\alpha_{H_0}\int_0^{h}\int_0^{h}|s-r|^{2H_0-2}\frac{\sin(s|\xi|)}{|\xi|}\frac{\sin(r|\xi|)}{|\xi|}\mathrm{d}r\mathrm{d}s\Big)|\xi|^{1-2H}\mathrm{d}\xi.
\end{aligned}
$$

We now use the change of variables $\bar{s} = s/h$ and $\bar{r} = r/h$, followed by $\xi' = \xi h$. We obtain:

$$
\begin{aligned}
&B(t,h) \\
&= c_H\int_{\mathbb{R}}\frac{1}{|\xi|^2}\Big(\alpha_{H_0}\int_0^1\int_0^1\Big|h\bar{s}-h\bar{r}\Big|^{2H_0-2}\sin\Big(h\bar{s}|\xi|\Big)\sin\Big(h\bar{r}|\xi|\Big)h^2\mathrm{d}\bar{s}\mathrm{d}\bar{r}\Big)|\xi|^{1-2H}\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&= c_H h^{2H_0} \int_{\mathbb{R}} \frac{1}{|\xi|^2} \Big(\alpha_{H_0} \int_0^1 \int_0^1 |s-r|^{2H_0-2} \sin\Big(hs|\xi|\Big) \sin\Big(hr|\xi|\Big) \mathrm{d}r\mathrm{d}s \Big) |\xi|^{1-2H} \mathrm{d}\xi \\
&= c_H h^{2H_0} \int_{\mathbb{R}} \frac{h^2}{|\xi'|^2} \Big(\alpha_{H_0} \int_0^1 \int_0^1 |s-r|^{2H_0-2} \sin(s|\xi'|) \sin(r|\xi'|) \mathrm{d}r\mathrm{d}s \Big) \Big|\frac{\xi'}{h}\Big|^{1-2H} h^{-1} \mathrm{d}\xi' \\
&= c_H h^{2H_0+2H} \int_{\mathbb{R}} \Big(\alpha_{H_0} \frac{1}{|\xi|^2} \int_0^1 \int_0^1 |s-r|^{2H_0-2} \sin(s|\xi|) \sin(r|\xi|) \mathrm{d}r\mathrm{d}s \Big) |\xi|^{1-2H} \mathrm{d}\xi.
\end{aligned}
$$

Note that the expression appearing in the inner parenthesis above is equal to $N_1^w(\xi)$, where $N_t^w(\xi)$ was defined in the proof of Theorem 3.3.1. By relation (3.3.2), we have

$$
N_1^w(\xi) \leq D_{H_0}^{(2)} \Big(\frac{1}{1+|\xi|^2} \Big)^{H_0+1/2}, \text{ for all } \xi \in \mathbb{R},
$$

where $D_{H_0}^{(2)}$ is given by (3.3.2). Therefore,

$$
B(t,h) \leq c_H D_{H_0}^{(2)} h^{2H_0+2H} \int_{\mathbb{R}} \Big(\frac{1}{1+|\xi|^2} \Big)^{H_0+1/2} |\xi|^{1-2H} \mathrm{d}\xi.
$$

From Step 2 of the proof of Theorem 3.3.1, we know that

$$
\begin{aligned}
\int_{\mathbb{R}} \Big(\frac{1}{1+|\xi|^2} \Big)^{H_0+1/2} |\xi|^{1-2H} \mathrm{d}\xi &\leq \int_{|\xi|\leq 1} |\xi|^{1-2H} \mathrm{d}\xi + \int_{|\xi|>1} \Big(\frac{1}{|\xi|^2} \Big)^{H_0+1/2} |\xi|^{1-2H} \mathrm{d}\xi \\
&= \frac{1}{1-H} + \frac{2}{2H_0+2H-1}.
\end{aligned}
$$

Using the fact that $h \leq T$, we obtain:

$$
B(t,h) \leq c_H C_{H_0,H}^{(6)} h^{2H_0+2H} \leq c_H C_{H_0,H}^{(6)} T h^{2H_0+2H-1}, \tag{3.5.22}
$$

where $C_{H_0,H}^{(6)} = D_{H_0}^{(2)} \Big(\frac{1}{1-H} + \frac{2}{2H_0+2H-1} \Big)$.

Therefore, by relations (3.5.8), (3.5.21) and (3.5.22), we have

$$
\begin{aligned}
E|u^w(t+h,x) - u^w(t,x)|^2 &\leq 2c_H \Big\{ 2^{3-2H_0-2H} C_{H_0,H}^{(5)} + C_{H_0,H}^{(6)} \Big\} T h^{2H_0+2H-1} \\
&= C_{H_0,H}^{(2)} T h^{2H_0+2H-1}.
\end{aligned}
$$

Using relation (2.3.1), it follows that

$$
\begin{aligned}
E|u^w(t+h,x) - u^w(t,x)|^p &= z_p \Big(E|u^w(t+h,x) - u^w(t,x)|^2 \Big)^{p/2} \\
&\leq z_p \Big(C_{H_0,H}^{(2)} \Big)^{p/2} T^{p/2} h^{p(2H_0+2H-1)/2}.
\end{aligned}
$$

Now we consider the space increments. Let $x' = x + z$. We assume that $z > 0$. Using the change of variable $s' = t - s$ and $r' = t - r$, we have

$$
\begin{aligned}
C(t,z) &:= E\big|u^w(t,x+z) - u^w(t,x)\big|^2 \\
&= E\left|\int_0^t \int_{\mathbb{R}} G^w(t-s,x+z-y) - G^w(t-s,x-y)W(\mathrm{d}s,\mathrm{d}y)\right|^2 \\
&= c_H \int_{\mathbb{R}} \Bigg(\alpha_{H_0} \int_0^t \int_0^t |s-r|^{2H_0-2} \mathcal{F}[G^w(t-s,x+z-\cdot) - G^w(t-s,x-\cdot)](\xi) \\
&\qquad \overline{\mathcal{F}[G^w(t-r,x+z-\cdot) - G^w(t-r,x-\cdot)](\xi)}\mathrm{d}s\mathrm{d}r\Bigg)|\xi|^{1-2H}\mathrm{d}\xi \\
&= c_H \int_{\mathbb{R}} |e^{-i\xi z} - 1|^2 \left(\alpha_{H_0} \int_0^t \int_0^t |s-r|^{2H_0-2} \mathcal{F}G^w(t-s,\cdot)(\xi)\overline{\mathcal{F}G^w(t-r,\cdot)(\xi)}\mathrm{d}s\mathrm{d}r\right) \\
&\qquad |\xi|^{1-2H}\mathrm{d}\xi \\
&= c_H \int_{\mathbb{R}} |e^{-i\xi z} - 1|^2 \left(\alpha_{H_0} \int_0^t \int_0^t |s'-r'|^{2H_0-2} \mathcal{F}G^w(s',\cdot)(\xi)\overline{\mathcal{F}G^w(r',\cdot)(\xi)}\mathrm{d}s'\mathrm{d}r'\right) \\
&\qquad |\xi|^{1-2H}\mathrm{d}\xi \\
&= c_H \int_{\mathbb{R}} 2[1-\cos(\xi z)]\left(\alpha_{H_0} \int_0^t \int_0^t |s-r|^{2H_0-2} \frac{\sin(s|\xi|)}{|\xi|}\cdot\frac{\sin(r|\xi|)}{|\xi|}\mathrm{d}s\mathrm{d}r\right)|\xi|^{1-2H}\mathrm{d}\xi \\
&= c_H \int_{\mathbb{R}} 2[1-\cos(\xi z)]N_t^w(\xi)|\xi|^{1-2H}\mathrm{d}\xi =: C'(t,z) + C''(t,z),
\end{aligned}
$$

where $N_t^w(\xi)$ is given in Theorem 3.3.1 by

$$
N_t^w(\xi) = \frac{\alpha_{H_0}}{|\xi|^2} \int_0^t \int_0^t |s-r|^{2H_0-2} \sin(s|\xi|)\sin(r|\xi|)\mathrm{d}s\mathrm{d}r,
$$

and

$$
\begin{aligned}
C'(t,z) &:= c_H \int_{|\xi|\le 1} 2[1-\cos(\xi z)]N_t^w(\xi)|\xi|^{1-2H}\mathrm{d}\xi, \\
C''(t,z) &:= c_H \int_{|\xi|>1} 2[1-\cos(\xi z)]N_t^w(\xi)|\xi|^{1-2H}\mathrm{d}\xi.
\end{aligned}
$$

We treat $C'(t,z)$ and $C''(t,z)$ separately. We study $C'(t,z)$ first. Recall that $x, x' \in [-M, M]$. Hence $|z| \le |x| + |x'| \le 2M$. We use the fact that $1 - \cos(x) \le \frac{1}{2}x^2$ and $\sin(x) \le 1$, for all $x > 0$. We obtain:

$$
C'(t,z) \le c_H \int_{|\xi|\le 1} \xi^2 z^2 N_t^w(\xi)|\xi|^{1-2H}\mathrm{d}\xi
$$

$$= z^2 c_H \int_{|\xi|\le 1} \Big(\alpha_{H_0}\int_0^t\int_0^t \sin(r|\xi|)\sin(s|\xi|)|s-r|^{2H_0-2}\mathrm{d}s\mathrm{d}r\Big)|\xi|^{1-2H}\mathrm{d}\xi$$
$$\le z^2 c_H \int_{|\xi|\le 1} \Big(\alpha_{H_0}\int_0^t\int_0^t |s-r|^{2H_0-2}\mathrm{d}s\mathrm{d}r\Big)|\xi|^{1-2H}\mathrm{d}\xi = z^2 c_H (2t)^{2H_0}\int_{|\xi|\le 1}|\xi|^{1-2H}\mathrm{d}\xi$$
$$= c_H(2t)^{2H_0}\frac{1}{1-H}z^2 \le c_H(2T)^{2H_0}(2M)^{3-2H_0-2H}\frac{1}{1-H}z^{2H_0+2H-1}, \qquad (3.5.23)$$

where for the last inequality, we decompose $z^2 = z^{3-2H_0-2H}\cdot z^{2H_0+2H-1}$.

Now we study $C''(t,z)$. Using the upper bound for $N_t^w(\xi)$ given by relation (3.3.2), followed by the change of variable $w=\xi z$, we have:

$$C''(t,z) \le 2c_H D_{H_0}^{(2)}\Big(t^{2H_0+2}\vee 1\Big)\int_{|\xi|>1}[1-\cos(\xi z)]\Big(\frac{1}{1+|\xi|^2}\Big)^{H_0+1/2}|\xi|^{1-2H}\mathrm{d}\xi$$
$$= 4c_H D_{H_0}^{(2)}\Big(t^{2H_0+2}\vee 1\Big)\int_1^\infty[1-\cos(\xi z)]\Big(\frac{1}{1+\xi^2}\Big)^{H_0+1/2}|\xi|^{1-2H}\mathrm{d}\xi$$
$$= 4c_H D_{H_0}^{(2)}\Big(t^{2H_0+2}\vee 1\Big)\int_z^\infty\Big[1-\cos(w)\Big]\Big(\frac{z^2}{z^2+w^2}\Big)^{H_0+1/2}\Big(\frac{w}{z}\Big)^{1-2H}\frac{1}{z}\mathrm{d}w$$
$$= 4c_H D_{H_0}^{(2)}\Big(t^{2H_0+2}\vee 1\Big)z^{2H_0+2H-1}\int_z^\infty\Big[1-\cos(w)\Big]\Big(\frac{1}{z^2+w^2}\Big)^{H_0+1/2}w^{1-2H}\mathrm{d}w$$
$$\le 4c_H D_{H_0}^{(2)}\Big(t^{2H_0+2}\vee 1\Big)z^{2H_0+2H-1}\int_{\mathbb{R}}\Big[1-\cos(w)\Big]\Big(\frac{1}{z^2+w^2}\Big)^{H_0+1/2}|w|^{1-2H}\mathrm{d}w$$
$$\le 4c_H D_{H_0}^{(2)}\Big(T^{2H_0+2}\vee 1\Big)\Big(\frac{1}{3-2H_0-2H}+\frac{4}{2H_0+2H-1}\Big)z^{2H_0+2H-1}, \qquad (3.5.24)$$

where for last inequality, we use Lemma 3.5.7 below.

Therefore, by relations (3.5.23) and (3.5.24),

$$C(t,z) = C'(t,z)+C''(t,z)$$
$$\le c_H\Big\{\frac{(2T)^{2H_0}(2M)^{3-2H_0-2H}}{1-H} + 4D_{H_0}^{(2)}\Big(T^{2H_0+2}\vee 1\Big)\Big(\frac{1}{3-2H_0-2H}+\frac{4}{2H_0+2H-1}\Big)\Big\} z^{2H_0+2H-1}$$
$$\le C_{H_0,H}^{(4)}\Big(T^{2H_0+2}\vee 1\Big)\Big(M^{3-2H_0-2H}\vee 1\Big)z^{2H_0+2H-1},$$

where

$$C_{H_0,H}^{(4)} = c_H\Big\{\frac{2^{3-2H}}{1-H}+4D_{H_0}^{(2)}\Big(\frac{1}{3-2H_0-2H}+\frac{4}{2H_0+2H-1}\Big)\Big\}.$$

We used the fact that $T^{2H_0} \leq T^{2H_0+2}$ if $T \geq 1$ and $T^{2H_0} \leq 1$ if $T < 1$, and hence $T^{2H_0} \leq T^{2H_0+2} \vee 1$.

Using relation (2.3.1),

$$
\begin{aligned}
&E|u^w(t,x+z) - u^w(t,x)|^p = z_p\Big(C(t,z)\Big)^{p/2} \\
&\leq z_p\Big(C^{(4)}_{H_0,H}\Big)^{p/2}\Big(T^{2H_0+2} \vee 1\Big)^{p/2}\Big(M^{3-2H_0-2H} \vee 1\Big)^{p/2} z^{p(2H_0+2H-1)/2}.
\end{aligned}
$$

∎

Remark 3.5.5. Note that the upper bound estimates given in Theorem 3.5.4 are sharp, i.e. there exist matching lower bounds. In [10], the authors studied the regularity of the solution u to the stochastic wave equation driven by a linear fractional-colored noise, with respect to time variable and space variable. We omit the details here.

The following lemmas were used in the proof of Theorem 3.5.4.

Lemma 3.5.6. *For any* $H_0, H \in \mathbb{R}$ *such that* $1 < 2H_0 + 2H < 3$,

$$
N'_{H_0,H} := \int_{\mathbb{R}} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H}} \mathrm{d}\xi \leq 2\bigg(\frac{1}{3-2H_0-2H} + \frac{1}{2H_0+2H-1}\bigg).
$$

Proof: We split the integral in two regions: $|\xi| \leq 1$ and $|\xi| \geq 1$. When $|\xi| \leq 1$, we use the inequality $|\sin x| \leq x$, and when $|\xi| > 1$, we use the inequality $|\sin x| \leq 1$. We have:

$$
\begin{aligned}
N'_{H_0,H} &= \int_{|\xi|\leq 1} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H}} \mathrm{d}\xi + \int_{|\xi|>1} \frac{\sin^2(|\xi|)}{|\xi|^{2H_0+2H}} \mathrm{d}\xi \\
&\leq \int_{|\xi|\leq 1} \frac{|\xi|^2}{|\xi|^{2H_0+2H}} \mathrm{d}\xi + \int_{|\xi|>1} \frac{1}{|\xi|^{2H_0+2H}} \mathrm{d}\xi \\
&= \int_{|\xi|\leq 1} |\xi|^{2-2H_0-2H} \mathrm{d}\xi + \int_{|\xi|>1} |\xi|^{-2H_0-2H} \mathrm{d}\xi \\
&= 2\bigg(\int_0^1 \xi^{2-2H_0-2H} \mathrm{d}\xi + \int_1^\infty \xi^{-2H_0-2H} \mathrm{d}\xi\bigg) \\
&= 2\bigg(\frac{1}{3-2H_0-2H} + \frac{1}{2H_0+2H-1}\bigg).
\end{aligned}
$$

∎

Lemma 3.5.7. *For any $H_0, H \in \mathbb{R}$ such that $1 < 2H_0 + 2H < 3$ and for any $z \in \mathbb{R}$,*

$$I(z) := \int_{\mathbb{R}} [1-\cos(w)]\Big(\frac{1}{w^2+z^2}\Big)^{H_0+1/2}|w|^{1-2H}\mathrm{d}w \leq \frac{1}{3-2H_0-2H} + \frac{4}{2H_0+2H-1}.$$

Proof: Let $I(z) = I'(z) + I''(z)$, where

$$I'(z) := \int_{|w|\leq 1} [1-\cos(w)]\Big(\frac{1}{w^2+z^2}\Big)^{H_0+1/2}|w|^{1-2H}\mathrm{d}w$$
$$I''(z) := \int_{|w|>1} [1-\cos(w)]\Big(\frac{1}{w^2+z^2}\Big)^{H_0+1/2}|w|^{1-2H}\mathrm{d}w.$$

We treat $I'(z)$ and $I''(z)$ separately. For both $I'(z)$ and $I''(z)$, we will use the fact that $w^2 + z^2 \geq w^2$. We study $I'(z)$ first. Using the fact that $1 - \cos(x) \leq \frac{1}{2}x^2$ for all $x \in \mathbb{R}$ and $2H_0 + 2H < 3$, we have:

$$I'(z) \leq \frac{1}{2}\int_{|w|\leq 1} w^2\Big(\frac{1}{w^2+z^2}\Big)^{H_0+1/2}|w|^{1-2H}\mathrm{d}w \leq \frac{1}{2}\int_{|w|\leq 1} w^2\Big(\frac{1}{w^2}\Big)^{H_0+1/2}|w|^{1-2H}\mathrm{d}w$$
$$= \frac{1}{2}\int_{|w|\leq 1} |w|^{2-2H_0-2H}\mathrm{d}w = \int_0^1 w^{2-2H_0-2H}\mathrm{d}w = \frac{1}{3-2H_0-2H}.$$

We treat $I''(z)$ next. Using the fact that $1-\cos(x) \leq 2$ for all $x \in \mathbb{R}$ and $2H_0+2H > 1$, we obtain:

$$I''(z) \leq 2\int_{|w|>1} \Big(\frac{1}{w^2+z^2}\Big)^{H_0+1/2}|w|^{1-2H}\mathrm{d}w \leq 2\int_{|w|>1} |w|^{-2H_0-2H}\mathrm{d}w$$
$$= 4\int_1^{\infty} w^{-2H_0-2H}\mathrm{d}w = \frac{4}{2H_0+2H-1}.$$

∎

Remark 3.5.8. Taking formally $H_0 = 1/2$ in Theorem 3.5.4, we obtain moment bounds which are consistent with those given by Theorem 2.3.4 for the white noise in time.

3.6 Continuity in law of the solution with respect to H

In this section, we consider equations (1.0.1) and (1.0.2) with noise W as in Section 3.1. We prove that the solution of either one of these equations is continuous in law in the space of continuous functions $C([0,T]\times\mathbb{R})$, with respect to H. Note that by Kolmogorov's continuity criterion (Theorem C.6 of [17]) and the moment estimates given by Theorem 3.4.1 and Theorem 3.5.4, the solution has a modification with sample paths in $C([0,T]\times\mathbb{R})$. We work with this modification.

The results presented in this section are new and constitute the major contribution of this thesis, extending the results obtained in [15] for $H_0=\frac{1}{2}$ (which were presented in Section 2.4) to the case $H_0>\frac{1}{2}$.

Theorem 3.6.1. *Let $W_{H_0,H}$ be Gaussian noise introduced in Section 3.1 which is fractional in time with index $H_0>\frac{1}{2}$ and fractional in space with index $H\in(0,1)$. We denote by*

$$u^h_{H_0,H}(t,x)=\int_0^t\int_{\mathbb{R}}G^h(t-s,x-y)W_{H_0,H}(\mathrm{d}s,\mathrm{d}y),$$

the solution of the stochastic heat equation (1.0.1)*, and by*

$$u^w_{H_0,H}(t,x)=\int_0^t\int_{\mathbb{R}}G^w(t-s,x-y)W_{H_0,H}(\mathrm{d}s,\mathrm{d}y),$$

the solution of the stochastic wave equation (1.0.2)*. We fix $T>0$ and we consider the modifications of these processes with sample paths in $C([0,T]\times\mathbb{R})$, which we denote also by $u^h_{H_0,H}$, $u^w_{H_0,H}$, respectively. Let $(H_n)_{n\geq1}$ be a sequence in $(0,1)$ such that $H_n\to H$.*
(a) If

$$2H_0+H<2,$$

then

$$u^h_{H_0,H_n}\xrightarrow{d}u^h_{H_0,H}\ \text{in } C([0,T]\times\mathbb{R}),$$

(b) If

$$2H_0+2H<3, \tag{3.6.1}$$

then

$$u^w_{H_0,H_n}\xrightarrow{d}u^w_{H_0,H}\ \text{in } C([0,T]\times\mathbb{R}).$$

Proof: We use the same approach in the proof of Theorem 2.4.1. To simplify the writing, we drop the upper indices h and w from the notation, whenever the calculations are valid for both heat and wave equations. We denote $u_n=u_{H_0,H_n}$ and $u=u_{H_0,H}$. We apply Theorem E.4.3 (see also Remark E.4.4). Note that condition (ii) of this theorem clearly holds since $u_n(0,0)=0$ for all $n\geq1$.

Step 1: We verify that condition (i) of Theorem E.4.3 holds. We need to prove that

$$E[u_n(t,x)u_n(t',x')]\to E[u(t,x)u(t',x')]. \tag{3.6.2}$$

In this case, we denote by $\mathcal{H}_{H_0,H}$ the Hilbert space associated to $W_{H_0,H}$ for any $H \in (0,1)$. By the isometry property of Wiener integral, with respect to W_{H_0,H_n}, we have:

$$\begin{aligned}E[u_n(t,x)u_n(t',x')] &= \left\langle 1_{[0,t]}G(t-\cdot,x-\cdot),1_{[0,t']}G(t'-\cdot,x'-\cdot)\right\rangle_{\mathcal{H}_{H_0,H_n}}\\ &= \alpha_{H_0}c_{H_n}\int_0^t\int_0^{t'}\int_{\mathbb{R}}|s-r|^{2H_0-2}\mathcal{F}G(t-s,x-\cdot)(\xi)\overline{\mathcal{F}G(t'-r,x'-\cdot)(\xi)}\\ &\qquad |\xi|^{1-2H_n}\mathrm{d}\xi\mathrm{d}r\mathrm{d}s\\ &= \alpha_{H_0}c_{H_n}\int_0^t\int_0^{t'}\int_{\mathbb{R}}|s-r|^{2H_0-2}e^{-i\xi(x-x')}\overline{\mathcal{F}G(t-s,\cdot)(\xi)}\mathcal{F}G(t'-r,\cdot)(\xi)\\ &\qquad |\xi|^{1-2H_n}\mathrm{d}\xi\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Fix numbers a and b such that $0<a<H<b<1$. Since $H_n \to H$, there exists $N \in \mathbb{N}$ such that

$$a \le H_n \le b, \text{ for all } n \ge N. \tag{3.6.3}$$

We consider first the heat equation. In this case,

$$\begin{aligned}&E[u_n^h(t,x)u_n^h(t',x')]\\ &= c_{H_n}\int_0^t\int_0^{t'}\int_{\mathbb{R}}\alpha_{H_0}|s-r|^{2H_0-2}e^{-i\xi(x-x')}e^{-\frac{(t-s)|\xi|^2}{2}}e^{-\frac{(t'-r)|\xi|^2}{2}}|\xi|^{1-2H_n}\mathrm{d}\xi\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Note that as $n\to\infty$, $c_{H_n}\to c_H$ and the integrand

$$f_n^h(s,r,\xi)=\alpha_{H_0}|s-r|^{2H_0-2}e^{-i\xi(x-x')}e^{-\frac{(t-s)|\xi|^2}{2}}e^{-\frac{(t'-r)|\xi|^2}{2}}|\xi|^{1-2H_n},$$

converges to

$$f^h(s,r,\xi)=\alpha_{H_0}|s-r|^{2H_0-2}e^{-i\xi(x-x')}e^{-\frac{(t-s)|\xi|^2}{2}}e^{-\frac{(t'-r)|\xi|^2}{2}}|\xi|^{1-2H},$$

for all $s\in[0,t],\ r\in[0,t']$ and $\xi\in\mathbb{R}$. We show that f_n is bounded by an integrable function on $[0,t]\times[0,t']\times\mathbb{R}$ for all $n\ge1$.

If $|\xi|\le1$, using the fact that $e^{-x}\le1$ for all $x\in\mathbb{R}$, we obtain:

$$|f_n^h(s,r,\xi)|\le\alpha_{H_0}|s-r|^{2H_0-2}|\xi|^{1-2H_n}\le\alpha_{H_0}|s-r|^{2H_0-2}|\xi|^{1-2b}=:g^h(s,r,\xi),$$

for all $n\ge N$, since $|\xi|^{2H_n-1}\ge|\xi|^{2b-1}$. Clearly, $\int_0^t\int_0^{t'}\int_{|\xi|\le1}g^h(s,r,\xi)\mathrm{d}\xi\mathrm{d}r\mathrm{d}s<\infty$, since $b<1$.

If $|\xi|>1$, $|f_n^h(s,r,\xi)|\le g^h(s,r,\xi)$, where

$$g^h(s,r,\xi)=:\alpha_{H_0}|s-r|^{2H_0-2}e^{-(t-s+t'-r)|\xi|^2/2}|\xi|^{1-2a},$$

since $|\xi|^{1-2H_n} \leq |\xi|^{1-2a}$. We want to prove that $g^h(s,r,\xi)$ is integrable on $[0,t] \times [0,t'] \times \{|\xi|>1\}$. We denote by

$$\langle \varphi, \psi \rangle_{\mathcal{H}_0} = \alpha_{H_0} \int_0^\infty \int_0^\infty |s-r|^{2H_0-2} \varphi(s)\psi(r) \mathrm{d}s\mathrm{d}r,$$

and by $\|\cdot\|_{\mathcal{H}_0}$ the corresponding norm. Note that

$$\begin{aligned}
\|1_{[0,t]} e^{-(t-\cdot)|\xi|^2/2}\|^2_{\mathcal{H}_0} &= \alpha_{H_0} \int_0^t \int_0^t |s-r|^{2H_0-2} e^{-(t-s)|\xi|^2/2} e^{-(t-r)|\xi|^2/2} \mathrm{d}s\mathrm{d}r \\
&\leq b_{H_0} \Big[\int_0^t \exp\Big(-\frac{(t-s)|\xi|^2}{2H_0} \Big) \mathrm{d}s \Big]^{2H_0} = b_{H_0} \Big\{ \frac{2H_0}{|\xi|^2} \Big[1 - \exp\Big(-\frac{t|\xi|^2}{2H_0} \Big) \Big] \Big\}^{2H_0} \\
&\leq b_{H_0} (2H_0)^{2H_0} |\xi|^{-4H_0} =: c_{H_0} |\xi|^{-4H_0},
\end{aligned}$$

where for the first inequality, we apply Lemma C.2. Therefore,

$$\begin{aligned}
&\alpha_{H_0} \int_0^t \int_0^{t'} |s-r|^{2H_0-2} e^{-(t-s)|\xi|^2/2} e^{-(t'-r)|\xi|^2/2} \mathrm{d}s\mathrm{d}r \\
&= \Big\langle 1_{[0,t]} e^{-(t-\cdot)|\xi|^2/2}, 1_{[0,t']} e^{-(t'-\cdot)|\xi|^2/2} \Big\rangle_{\mathcal{H}_0} \leq \|1_{[0,t]} e^{-(t-\cdot)|\xi|^2/2}\|_{\mathcal{H}_0} \cdot \|1_{[0,t']} e^{-(t'-\cdot)|\xi|^2/2}\|_{\mathcal{H}_0} \\
&\leq c_{H_0} |\xi|^{-4H_0},
\end{aligned}$$

where for the first inequality we applied the Cauchy-Schwarz inequality. Hence,

$$\int_0^t \int_0^{t'} \int_{|\xi|>1} g^h(s,r,\xi) \mathrm{d}\xi \mathrm{d}r \mathrm{d}s \leq c_{H_0} \int_{|\xi|>1} |\xi|^{-8H_0} |\xi|^{1-2a} \mathrm{d}\xi < \infty,$$

since $-4H_0 + 1 - 2a + 1 < 0$. This is true since $4H_0 > 2 - 2a$ and $H_0 > \frac{1}{2} > \frac{1-a}{2}$.

Therefore, in the case of the heat equation, relation (3.6.2) follows by the Dominated Convergence Theorem.

We consider next the wave equation. In this case,

$$\begin{aligned}
&E[u_n^w(t,x) u_n^w(t',x')] \\
&= c_{H_n} \int_0^t \int_0^{t'} \int_{\mathbb{R}} \alpha_{H_0} |s-r|^{2H_0-2} e^{-i\xi(x-x')} \frac{\sin\Big((t-s)|\xi|\Big) \sin\Big((t'-r)|\xi|\Big)}{|\xi|^{1+2H_n}} \mathrm{d}\xi \mathrm{d}r \mathrm{d}s.
\end{aligned}$$

Note that as $n \to \infty$, the integrand

$$f_n^w(s,r,\xi) = \alpha_{H_0} |s-r|^{2H_0-2} e^{-i\xi(x-x')} \frac{\sin\Big((t-s)|\xi|\Big) \sin\Big((t'-r)|\xi|\Big)}{|\xi|^{1+2H_n}},$$

converges to

$$f^w(s,r,\xi) = \alpha_{H_0}|s-r|^{2H_0-2}e^{-i\xi(x-x')}\frac{\sin\Big((t-s)|\xi|\Big)\sin\Big((t'-r)|\xi|\Big)}{|\xi|^{1+2H}},$$

for all $s \in [0,t], r \in [0,t']$ and $\xi \in \mathbb{R}$. To apply the Dominated Convergence Theorem, we need to show that $f_n^w(s,r,\xi)$ is bounded by an integrable function on $[0,t]\times[0,t']\times\mathbb{R}$ for all $n \geq 1$.

If $|\xi| \leq 1$, using the fact that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$, we obtain:

$$\begin{aligned}
&|f_n^w(s,r,\xi)| \leq \alpha_{H_0}|s-r|^{2H_0-2}\frac{(t-s)|\xi|\cdot(t'-s)|\xi|}{|\xi|^{1+2H_n}} \\
&= \alpha_{H_0}|s-r|^{2H_0-2}(t-s)(t'-s)|\xi|^{1-2H_n} \leq \alpha_{H_0}|s-r|^{2H_0-2}(t-s)(t'-s)|\xi|^{1-2b} \\
&=: g^w(s,r,\xi), \text{ for all } n \geq N.
\end{aligned}$$

To see that the function g^w is integrable on $[0,t]\times[0,t']\times\{|\xi|\leq 1\}$, we note first that $\int_{|\xi|\leq 1}|\xi|^{1-2b}\mathrm{d}\xi < \infty$. Moreover, by the Cauchy-Schwarz inequality in the space $\mathcal{H}_0$, we have

$$\begin{aligned}
&\alpha_{H_0}\int_0^t\int_0^{t'}|s-r|^{2H_0-2}(t-s)(t'-r)\mathrm{d}r\mathrm{d}s \\
&= \Big\langle (t-\cdot)1_{[0,t]},(t'-\cdot)1_{[0,t']}\Big\rangle_{\mathcal{H}_0} \leq \Big\|(t-\cdot)1_{[0,t]}\Big\|_{\mathcal{H}_0}\cdot\Big\|(t'-\cdot)1_{[0,t']}\Big\|_{\mathcal{H}_0} \leq t^{2+2H_0},
\end{aligned}$$

since

$$\begin{aligned}
\|(t-\cdot)1_{[0,t]}\|_{\mathcal{H}_0}^2 &= \alpha_{H_0}\int_0^t\int_0^t|s-r|^{2H_0-2}(t-s)(t-r)\mathrm{d}r\mathrm{d}s \\
&= \alpha_{H_0}\int_0^t\int_0^t|s-r|^{2H_0-2}sr\ \mathrm{d}r\mathrm{d}s \leq t^2\alpha_{H_0}\int_0^t\int_0^t|s-r|^{2H_0-2}\mathrm{d}r\mathrm{d}s \\
&= t^{2+2H_0}.
\end{aligned}$$

If $|\xi| > 1$, using the fact that $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, we have:

$$|f_n^w(s,r,\xi)| \leq \alpha_{H_0}|s-r|^{2H_0-2}\frac{1}{|\xi|^{1+2H_n}} \leq \alpha_{H_0}|s-r|^{2H_0-2}\frac{1}{|\xi|^{1+2a}} =: g^w(s,r,\xi),$$

for all $n \geq N$. Clearly,

$$\int_0^t\int_0^{t'}\int_{|\xi|>1}g^w(s,r,\xi)\mathrm{d}\xi\mathrm{d}r\mathrm{d}s = \alpha_{H_0}\Big(\int_0^t\int_0^{t'}|s-r|^{2H_0-2}\mathrm{d}r\mathrm{d}s\Big)\Big(\int_{|\xi|>1}|\xi|^{-1-2a}\mathrm{d}\xi\Big)$$

$$< \infty.$$

Therefore, in the case of the wave equation, relation (3.6.2) follows by the Dominated Convergence Theorem.

Step 2 We verify that condition (iii) of Theorem E.4.3 holds. Let $J = [-M, M]$ be a compact set in $\mathbb{R}$. Recalling the moment estimates given by Theorem 3.4.1 and Theorem 3.5.4, we see that for any $p > 0$, $t, t' \in [0, T]$, and $x, x' \in \mathbb{R}$,

$$E\Big|u_n(t',x') - u_n(t,x)\Big|^p \le 2^{p-1}\Big\{E\Big|u_n(t',x') - u_n(t,x')\Big|^p + E\Big|u_n(t,x') - u_n(t,x)\Big|^p\Big\}$$

$$\le \begin{cases} 2^{p-1} z_p \Big\{\Big(C^{(1)}_{H_0,H_n}\Big)^{p/2} |t'-t|^{p(2H_0+H_n-1)/2} + \Big(C^{(3)}_{H_0,H_n}\Big)^{p/2} |x'-x|^{p(2H_0+H_n-1)}\Big\}, \\ \qquad \text{for heat equation} \\ 2^{p-1} z_p \Big(T^{2H_0+2} \vee 1\Big)^{p/2} \Big(M^{3-2H_0-2H_n} \vee 1\Big)^{p/2} \Big\{\Big(C^{(2)}_{H_0,H_n}\Big)^{p/2} |t'-t|^{p(2H_0+2H_n-1)/2} \\ \qquad + \Big(C^{(4)}_{H_0,H_n}\Big)^{p/2} |x'-x|^{p(2H_0+2H_n-1)/2}\Big\}, \\ \qquad \text{for wave equation} \end{cases}$$

Note that for the wave equation, we used the fact that $T \le T^{2H_0+2} \vee 1$, which can be proved considering separately the cases $T > 1$ and $T \le 1$. To apply Theorem 3.5.4, we need

$$2H_0 + 2H_n < 3.$$

Note that due to our hypothesis (3.6.1), $\lim_{n\to\infty}(2H_0 + 2H_n) = 2H_0 + 2H < 3$. Hence there exists $N^* \in \mathbb{N}$ such that $2H_0 + 2H_n < 3$ for all $n \ge N^*$. We assume that N given by (3.6.3) is greater than N^*.

We consider first the heat equation. For any $x, x' \in J$,

$$|x - x'|^{p(2H_0+H_n-1)} \le (2M)^{p(2H_0+H_n-1)/2} |x - x'|^{p(2H_0+H_n-1)/2}.$$

We use (2.4.2) with $\alpha = p(2H_0 + H_n - 1)/2$. Note that $p(2H_0 + H_n - 1)/2 \ge p(2H_0 + a - 1)/2 \ge 1$ for all $n \ge N$, if $p \ge 2/(2H_0 + a - 1)$. We obtain that for all $n \ge N$ and $p \ge 2/(2H_0 + a - 1)$,

$$E|u^h_n(t',x') - u^h_n(t,x)|^p \le 2^{p-1} z_p \Big[C^{(1)}_{H_0,H_n} \vee C^{(3)}_{H_0,H_n}\Big]^{p/2} \Big[1 \vee (2M)\Big]^{p(2H_0+H_n-1)/2}$$
$$\Big(|t'-t|^{p(2H_0+H_n-1)/2} + |x'-x|^{p(2H_0+H_n-1)/2}\Big)$$

$$\le 2^{p-1} z_p A^{p/2} \Big[1 \vee (2M)\Big]^{p(2H_0+b-1)/2} \Big(|t'-t| + |x'-x|\Big)^{p(2H_0+H_n-1)/2}, \qquad (3.6.4)$$

where for the last inequality, we used relation (2.4.2), the fact that $H_n \le b$, for all $n \ge N$ and

$$C^{(1)}_{H_0,H_n} \vee C^{(3)}_{H_0 H_n} \le A, \text{ for all } n \ge N, \tag{3.6.5}$$

for some constant A which we specify below.

We prove (3.6.5) in the following way. Recall that (see Theorem 3.4.1)

$$C^{(1)}_{H_0,H} = 2c_H\Big(b_{H_0}(2H_0)^{2H_0}N_{H_0,H} + \Gamma(1-H)2^{1-H}R_{H_0,H}\Big),$$

where

$$N_{H_0,H} = \int_{\mathbb{R}} \frac{\Big(1-\exp(-\eta^2/2)\Big)^2}{|\eta|^{4H_0+2H-1}}\mathrm{d}\eta,$$

and

$$R_{H_0,H} = \alpha_{H_0}\int_0^1\int_0^1 |r-s|^{2H_0-2}(r+s)^{H-1}\mathrm{d}r\mathrm{d}s.$$

For any $H \in [a,b]$, by Lemma 3.4.3,

$$N_{H_0,H} \le \frac{1}{4}\cdot\frac{1}{3-2H_0-H} + \frac{1}{2H_0+H-1} \le \frac{1}{4}\cdot\frac{1}{3-2H_0-b} + \frac{1}{2H_0+a-1},$$

and by Lemma 3.4.4,

$$\Gamma(1-H)2^{1-H}R_{H_0,H} \le b_{H_0}\frac{\Gamma(1-H)}{H} \le b_{H_0}c,$$

using the fact that $\frac{\Gamma(1-H)}{H}$ is bounded on $[a,b]$ by a constant $c>0$ since it is continuous on $[a,b]$. Recalling that $c_H \le \frac{1}{\pi}$ (see Remark 2.3.7), we have for all $n \ge N$,

$$C^{(1)}_{H_0,H_n} \le \frac{2}{\pi}b_{H_0}\Big[(2H_0)^{2H_0}\Big(\frac{1}{4}\cdot\frac{1}{3-2H_0-b} + \frac{1}{2H_0+a-1}\Big) + c\Big]. \tag{3.6.6}$$

Now we treat $C^{(3)}_{H_0,H_n}$. Recall that

$$C^{(3)}_{H_0,H} = 2c_H b_{H_0}(2H_0)^{2H_0}\bar{C}_{3-4H_0-2H},$$

where $c_H \le \frac{1}{\pi}$ and $\bar{C}_\alpha = \int_{\mathbb{R}}(1-\cos\eta)|\eta|^{\alpha-2}\mathrm{d}\eta \le 2\Big(\frac{1}{1-\alpha} + \frac{1}{1+\alpha}\Big)$ (see Lemma 2.3.6). Hence, for all $H \in (0,1)$,

$$\begin{aligned} C^{(3)}_{H_0,H} &\le 2c_H b_{H_0}(2H_0)^{2H_0}\cdot 2\Big(\frac{1}{4H_0+2H-2} + \frac{1}{4-4H_0-2H}\Big) \\ &\le \frac{2}{\pi}b_{H_0}(2H_0)^{2H_0}\cdot\Big(\frac{1}{2H_0+H-1} + \frac{1}{2-2H_0-H}\Big). \end{aligned}$$

Note that the constant appearing in the upper bound above is positive, due to our condition $2H_0 + H < 2$. We have,

$$C^{(3)}_{H_0,H_n} \leq \frac{2}{\pi} b_{H_0} (2H_0)^{2H_0} \Big(\frac{1}{2H_0 + a - 1} + \frac{1}{2 - 2H_0 - b} \Big), \tag{3.6.7}$$

for all $n > N$. Relation (3.6.5) follows from (3.6.6) and (3.6.7).

We return now to relation (3.6.4). It remains to treat that the term $\Big(|t' - t| + |x' - x|\Big)^{p(2H_0+H_n-1)/2}$. Note that for any $t, t' \in [0,T]$ and $x, x' \in [-M, M]$,

$$w := |t' - t| + |x' - x| \leq T + 2M =: c_0.$$

For all $n \geq N$,

$$w^{p(2H_0+H_n-1)/2} = w^{p(H_n-a)/2} w^{p(2H_0+a-1)/2} \leq c_0^{p(H_n-a)/2} w^{p(2H_0+a-1)/2},$$

and

$$c_0^{p(H_n-a)/2} \leq \begin{cases} c_0^{p(b-a)/2} & \text{if } c_0 \geq 1, \\ 1 & \text{if } c_0 < 1. \end{cases}$$

Therefore, $c_0^{p(H_n-a)/2} \leq c_0^{p(b-a)/2} \vee 1$ and

$$w^{p(2H_0+H_n-1)/2} \leq \Big(c_0^{p(b-a)/2} \vee 1 \Big) w^{p(2H_0+a-1)/2}, \text{ for all } n \geq N. \tag{3.6.8}$$

Using inequalities (3.6.4) and (3.6.8), we obtain:

$$\begin{aligned} &E\Big|u_n^h(t',x') - u_n^h(t,x)\Big|^p \\ &\leq 2^{p-1} z_p A^{p/2} \Big[1 \vee (2M)\Big]^{p(2H_0+b-1)/2} \Big(c_0^{p(b-a)/2} \vee 1 \Big) \Big(|t'-t| + |x'-x| \Big)^{p(2H_0+a-1)/2}, \end{aligned}$$

for all $t, t' \in [0,T]$, $x, x' \in J$, $n \geq N$ and $p \geq 2/(2H_0 + a - 1)$. Condition (iii) of Theorem E.4.3 follows with $\delta = p(2H_0+a-1)/2 > 2$, if we choose $p \geq 4/(2H_0+a-1)$.

Next, we examine the wave equation. In this case, we use inequality (2.4.2) with $\alpha = p(2H_0 + 2H_n - 1)/2$. Note that

$$p(2H_0 + 2H_n - 1)/2 \geq p(2H_0 + 2a - 1)/2 \geq 1$$

for all $n \geq N$, if $p \geq 2/(2H_0+2a-1)$. Hence, for any $n \geq N$ and $p \geq 2/(2H_0+2a-1)$,

$$E|u_n^w(t',x') - u_n^w(t,x)|^p$$

$$
\begin{aligned}
&\leq 2^{p-1} z_p \Big(T^{2H_0+2} \vee 1\Big)^{p/2} \Big(M^{3-2H_0-2H_n} \vee 1\Big)^{p/2} \Big[C^{(2)}_{H_0,H_n} \vee C^{(4)}_{H_0,H_n}\Big]^{p/2} \\
&\qquad\qquad \Big\{|t'-t|^{p(2H_0+2H_n-1)/2} + |x'-x|^{p(2H_0+2H_n-1)/2}\Big\} \\
&\leq 2^{p-1} z_p \Big(T^{2H_0+2} \vee 1\Big)^{p/2} \Big(M^{3-2H_0-2H_n} \vee 1\Big)^{p/2} B^{p/2} \Big(|t'-t| + |x'-x|\Big)^{p(2H_0+2H_n-1)/2},
\end{aligned} \tag{3.6.9}
$$

where for the last inequality, we use the fact that

$$
C^{(2)}_{H_0,H_n} \vee C^{(4)}_{H_0,H_n} \leq B, \text{ for all } n \geq N, \tag{3.6.10}
$$

for some constant B which we specify below.

We prove (3.6.10) in the following way. Recall that (see Theorem 3.5.4)

$$
C^{(2)}_{H_0,H} = 2c_H \Big\{ 2^{3-2H_0-2H} C^{(5)}_{H_0,H} + C^{(6)}_{H_0,H} \Big\},
$$

where

$$
\begin{aligned}
C^{(5)}_{H_0,H} &= 2M'_{H_0} N'_{H_0,H} + \frac{8H_0}{3-2H_0-2H} + \frac{4}{1+2H} + \frac{(2H_0-1)M_{H_0}}{H_0+H} \\
C^{(6)}_{H_0,H} &= D^{(2)}_{H_0} \Big(\frac{1}{1-H} + \frac{2}{2H_0+2H-1}\Big).
\end{aligned}
$$

We first study $C^{(5)}_{H_0,H}$. Using Lemma 3.5.6, we have for all $n \geq N$,

$$
\begin{aligned}
&C^{(5)}_{H_0,H_n} \\
&\leq 4M'_{H_0} \Big(\frac{1}{3-2H_0-2H_n} + \frac{1}{2H_0+2H_n-1}\Big) + \frac{8H_0}{3-2H_0-2H_n} \\
&\qquad + \frac{4}{1+2H_n} + \frac{(2H_0-1)M_{H_0}}{H_0+H_n} \\
&\leq 4M'_{H_0} \Big(\frac{1}{3-2H_0-2b} + \frac{1}{2H_0+2a-1}\Big) + \frac{8H_0}{3-2H_0-2b} \\
&\qquad + \frac{4}{1+2a} + \frac{(2H_0-1)M_{H_0}}{H_0+a}.
\end{aligned} \tag{3.6.11}
$$

We consider $C^{(6)}_{H_0,H_n}$ next: for all $n \geq N$,

$$
C^{(6)}_{H_0,H_n} \leq D^{(2)}_{H_0} \Big(\frac{1}{1-b} + \frac{2}{2H_0+2a-1}\Big). \tag{3.6.12}
$$

Combining relations (3.6.11) and (3.6.12) and recalling that $c_H \leq \frac{1}{\pi}$, we have: for all $n \geq N$,

$$C^{(2)}_{H_0,H_n} \leq \frac{2}{\pi}\Big\{2^{3-2H_0-2a}\Big[4M'_{H_0}\Big(\frac{1}{3-2H_0-2b}+\frac{1}{2H_0+2a-1}\Big)+\frac{8H_0}{3-2H_0-2b} + \frac{4}{1+2a}+\frac{(2H_0-1)M_{H_0}}{H_0+a}\Big]+\Big[D^{(2)}_{H_0}\Big(\frac{1}{1-b}+\frac{2}{2H_0+2a-1}\Big)\Big]\Big\}. \tag{3.6.13}$$

Next we treat $C^{(4)}_{H_0}$. Recall that

$$C^{(4)}_{H_0,H} = c_H\Big\{\frac{2^{3-2H}}{1-H}+4D^{(2)}_{H_0}\Big(\frac{1}{3-2H_0-2H}+\frac{4}{2H_0+2H-1}\Big)\Big\}.$$

Therefore, for any $n \geq N$,

$$C^{(4)}_{H_0,H_n} \leq \frac{1}{\pi}\Big\{\frac{2^{3-2a}}{1-b}+D^{(2)}_{H_0}\Big(\frac{1}{3-2H_0-2b}+\frac{4}{2H_0+2a-1}\Big)\Big\}. \tag{3.6.14}$$

Relation (3.6.10) follows from (3.6.13) and (3.6.14).

We return now to relation (3.6.9). It remains to treat that the term $\Big(|t'-t|+|x'-x|\Big)^{p(2H_0+2H_n-1)/2}$. Note that for any $t,t' \in [0,T]$ and $x,x' \in [-M,M]$,

$$w := |t'-t|+|x'-x| \leq T+2M =: c_0.$$

For all $n \geq N$,

$$w^{p(2H_0+2H_n-1)/2} = w^{p(H_n-a)}w^{p(2H_0+2a-1)/2} \leq c_0^{p(H_n-a)}w^{p(2H_0+2a-1)/2},$$

and

$$c_0^{p(H_n-a)} \leq \begin{cases} c_0^{p(b-a)} & \text{if } c_0 \geq 1, \\ 1 & \text{if } c_0 < 1. \end{cases}$$

Therefore, $c_0^{p(H_n-a)} \leq c_0^{p(b-a)} \vee 1$ and

$$w^{p(2H_0+2H_n-1)/2} \leq \Big(c_0^{p(b-a)} \vee 1\Big)w^{p(2H_0+2a-1)/2}, \text{ for all } n \geq N. \tag{3.6.15}$$

Using inequalities (3.6.9) and (3.6.15), we obtain:

$$E\Big|u^w_n(t',x')-u^w_n(t,x)\Big|^p$$

$$\leq 2^{p-1} z_p \Big(T^{2H_0+2} \vee 1\Big)^{p/2} \Big(M^{3-2H_0-2a} \vee 1\Big)^{p/2} B^{p/2} \Big(c_0^{p(b-a)} \vee 1\Big) \\ \Big(|t'-t| + |x'-x|\Big)^{p(2H_0+2a-1)/2},$$

for all $t, t' \in [0, T]$, $x, x' \in J$, $n \geq N$ and $p \geq 2/(2H_0 + 2a - 1)$. Condition (iii) of Theorem E.4.3 follows with $\delta = p(2H_0 + 2a - 1)/2 > 2$, if we choose $p \geq 4/(2H_0 + 2a - 1)$.

∎

Appendix A

Fourier Transform

In this appendix chapter, we present some results involving Fourier transforms which were used in the thesis.

A.1 Basic properties of Fourier transform

In this section, we review some basic properties of the Fourier transform. In this thesis, the Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-i\xi x} f(x)\mathrm{d}x, \text{ for all } \xi \in \mathbb{R}.$$

Recall that for any $x \in \mathbb{R}$, we have the following Euler formula:

$$\mathrm{e}^{ix} = \cos(x) + i\sin(x).$$

The convolution of two functions $\varphi, \psi \in L^1(\mathbb{R})$ is defined by

$$(\varphi * \psi)(x) = \int_{\mathbb{R}} \varphi(x-y)\psi(y)\mathrm{d}y.$$

Lemma A.1. *For any $\varphi, \psi \in L^1(\mathbb{R})$, $\mathcal{F}(\varphi * \psi)(\xi) = \mathcal{F}\varphi(\xi) \cdot \mathcal{F}\psi(\xi)$.*

Proof: Let $h = \varphi * \psi$. Then

$$\mathcal{F}h(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-i\xi z} h(z)\mathrm{d}z = \int_{\mathbb{R}} \mathrm{e}^{-i\xi z}\left(\int_{\mathbb{R}} \varphi(x)\psi(z-x)\mathrm{d}x\right)\mathrm{d}z.$$

We have $|\varphi(x)\psi(z-x)\mathrm{e}^{-i\xi z}| = |\varphi(x)\psi(z-x)|$ since $|\mathrm{e}^{-i\xi z}| = 1$, and hence

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\varphi(x)\psi(z-x)e^{-i\xi z}|\mathrm{d}x\mathrm{d}z = \Big(\int_{\mathbb{R}} |\varphi(x)|\mathrm{d}x\Big)\Big(\int_{\mathbb{R}} |\psi(z-x)|\mathrm{d}z\Big) < \infty.$$

By Fubini's theorem, $\mathcal{F}h(\xi) = \int_{\mathbb{R}} \varphi(x)\bigg(\int_{\mathbb{R}} \psi(z-x)\mathrm{e}^{-i\xi z}\mathrm{d}z\bigg)\mathrm{d}x$. Using the substitution $y = z - x$, we obtain:

$$\begin{aligned}\mathcal{F}h(\xi) &= \int_{\mathbb{R}} \varphi(x)\bigg(\int_{\mathbb{R}} \psi(y)\mathrm{e}^{-i\xi(x+y)}\mathrm{d}y\bigg)\mathrm{d}x = \int_{\mathbb{R}} \varphi(x)\mathrm{e}^{-i\xi x}\bigg(\int_{\mathbb{R}} \psi(y)\mathrm{e}^{-i\xi y}\mathrm{d}y\bigg)\mathrm{d}x \\ &= \bigg(\int_{\mathbb{R}} \varphi(x)\mathrm{e}^{-i\xi x}\mathrm{d}x\bigg)\bigg(\int_{\mathbb{R}} \psi(y)\mathrm{e}^{-i\xi y}\mathrm{d}y\bigg) = \mathcal{F}\varphi(\xi)\cdot\mathcal{F}\psi(\xi).\end{aligned}$$

∎

Lemma A.2. *If $\varphi \in L^1(\mathbb{R})$ and $\varphi_x(y) = \varphi(x-y)$ be the shifted function , for some $x \in \mathbb{R}$. Then $\mathcal{F}\varphi_x(\xi) = \mathrm{e}^{-i\xi x}\overline{\mathcal{F}\varphi(\xi)}$. (The shifted function is also denoted by $\varphi(x-\cdot)$)*

Proof: Using the substitution $\bar{y} = x - y$, we have:

$$\begin{aligned}\mathcal{F}\varphi_x(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-i\xi y}\varphi_x(y)\ \mathrm{d}y &= \int_{\mathbb{R}} \mathrm{e}^{-i\xi y}\varphi(x-y)\mathrm{d}y = \int_{\mathbb{R}} \mathrm{e}^{-i\xi(x-\bar{y})}\varphi(\bar{y})\mathrm{d}\bar{y} \\ &= \int_{\mathbb{R}} \mathrm{e}^{-i\xi x}\mathrm{e}^{i\xi\bar{y}}\varphi(\bar{y})\mathrm{d}\bar{y} = \mathrm{e}^{-i\xi x}\int_{\mathbb{R}} \mathrm{e}^{i\xi\bar{y}}\varphi(\bar{y})\mathrm{d}\bar{y} \\ &= \mathrm{e}^{-i\xi x}\overline{\mathcal{F}\varphi(\xi)}\ ,\end{aligned}$$

where we recall that the complex conjugate is $\overline{\mathcal{F}\varphi(\xi)} = \int_{\mathbb{R}} \mathrm{e}^{i\xi y}\varphi(y)\ \mathrm{d}y$. ∎

Lemma A.3. *For any $\varphi, \psi \in L^1(\mathbb{R})$, $\mathcal{F}(\varphi * \widetilde{\psi})(\xi) = \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}$, where $\widetilde{\varphi}(x) = \varphi(-x)$.*

Proof: By Lemma A.1, we know that $\mathcal{F}(\varphi * \widetilde{\psi})(\xi) = \mathcal{F}\varphi(\xi)\cdot\mathcal{F}\widetilde{\psi}(\xi)$. Note that

$$\mathcal{F}\widetilde{\psi}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-i\xi x}\widetilde{\psi}(x)\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{-i\xi x}\psi(-x)\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{i\xi x'}\psi(x')\mathrm{d}x' = \overline{\mathcal{F}\psi(\xi)},$$

where for the third equality we use the substitution $x' = -x$ and for the last equality, we used the fact that $\int_R \mathrm{e}^{i\xi x'}\psi(x')\mathrm{d}x' = \overline{\mathcal{F}\psi(\xi)}$ by the definition of the complex conjugate. ∎

A.2 Riesz kernels

In this section, we present some results from the potential theory of Riesz kernels. We denote by $\mathcal{S}(\mathbb{R}^d)$ the set of rapidly decreasing functions on $\mathbb{R}^d$, i.e. infinitely differentiable functions on $\mathbb{R}^d$ which decrease faster than any polynomial.

The following result is taken from page 117 of [23], and gives the Fourier transform (in the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions on $\mathbb{R}^d$) of the Riesz kernel $f(x) = |x|^{-\alpha}$. Here $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$ given by $|x| = (x_1^2 + \ldots + x_d^2)^{1/2}$ if $x = (x_1, \ldots, x_d)$. We denote by $x \cdot y = \sum_{j=1}^d x_j y_j$ the scalar product of $x, y \in \mathbb{R}^d$.

Lemma A.4. *Let $\alpha \in (0, d)$ be arbitrary. For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have:*

$$\int_{\mathbb{R}^d} \varphi(x)|x|^{-\alpha} dx = C_{d,\alpha} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)|\xi|^{-(d-\alpha)} d\xi, \tag{A.2.1}$$

where $\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x}\varphi(x)dx$ and

$$C_{d,\alpha} = \pi^{-d/2} 2^{-\alpha} \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}.$$

In particular, if we take $d = 1$, and $\alpha = 2 - 2H$ in equation (A.2.1) with $H \in (\frac{1}{2}, 1)$, we obtain that for any $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\alpha_H \int_{\mathbb{R}} \varphi(x)|x|^{2H-2} \mathrm{d}x = c_H \int_{\mathbb{R}} \mathcal{F}\varphi(\xi)|\xi|^{1-2H} \mathrm{d}\xi, \tag{A.2.2}$$

where $\alpha_H = H(2H-1)$ and $c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}$. To deduce this expression of the constant c_H, we used the following properties of the Gamma function:

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}, \quad \Gamma(x+1) = x\Gamma(x), \quad \Gamma(x)\Gamma(x+\frac{1}{2}) = 2^{1-2x}\sqrt{\pi}\Gamma(2x).$$

By applying relation (A.2.2), we obtain the following result.

Lemma A.5. *If $H \in (\frac{1}{2}, 1)$, then for any $\varphi, \psi \in \mathcal{S}(\mathbb{R})$,*

$$\alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x)\psi(y)|x-y|^{2H-2} \mathrm{d}x\mathrm{d}y = c_H \int_{\mathbb{R}} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}|\xi|^{1-2H} \mathrm{d}\xi. \tag{A.2.3}$$

Proof: We use the change of variable $z = x - y$. By Fubini's theorem and letting $\widetilde{\psi}(y) = \psi(-y)$, we have:

$$\int_{\mathbb{R}} \varphi(x)\Big(\int_{\mathbb{R}} \psi(y)|x-y|^{2H-2}\mathrm{d}y\Big)\mathrm{d}x = \int_{\mathbb{R}} \varphi(x)\Big(\int_{\mathbb{R}} \psi(x-z)|z|^{2H-2}\mathrm{d}z\Big)\mathrm{d}x$$

$$= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \varphi(x)\psi(x-z)\mathrm{d}x \Big) |z|^{2H-2}\mathrm{d}z = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \varphi(x)\widetilde{\psi}(z-x)\mathrm{d}x \Big) |z|^{2H-2}\mathrm{d}z$$
$$= \int_{\mathbb{R}} (\varphi * \widetilde{\psi})(z)|z|^{2H-2}\mathrm{d}z.$$

By relation (A.2.2), we have:

$$\alpha_H \int_{\mathbb{R}} (\varphi * \widetilde{\psi})(z)|z|^{2H-2}\mathrm{d}z = c_H \int_{\mathbb{R}} \mathcal{F}(\varphi * \widetilde{\psi})(\xi)|\xi|^{1-2H}\mathrm{d}\xi.$$

The conclusion follows by Lemma A.3. ∎

Remark A.6. By Lemma 5.6 of [18], for any $H \in (\frac{1}{2}, 1)$ and for non-negative function $\varphi \in L^1(\mathbb{R})$,

$$\alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x)\varphi(y)|x-y|^{2H-2}\mathrm{d}x\mathrm{d}y = c_H \int_{\mathbb{R}} |\mathcal{F}\varphi(\xi)|^2 \, |\xi|^{1-2H}\mathrm{d}\xi =: \mathcal{E}_H(\varphi) \quad \text{(A.2.4)}$$

where $\mathcal{E}_H(\varphi)$ is called the *energy* of φ.

By relation [5.37] of [18], it can be proved that relation (A.2.3) holds for any $\varphi, \psi \in L^1(\mathbb{R})$ with $\mathcal{E}_H(|\varphi|) < \infty$ and $\mathcal{E}_H(|\psi|) < \infty$. In particular, relation (A.2.3) holds for $\varphi = 1_A$ and $\psi = 1_B$ for any $A, B \in \mathcal{B}_b(\mathbb{R})$.

A.3 Energy of sin function

In this section, we include a result taken from [6] about the energy of the sin function restricted to the interval $[0, T]$, which is used in the proof of Theorem 3.3.1 for the existence of solution of the stochastic wave equation with fractional noise in time.

We begin with the calculation of the Fourier transform of the sin function restricted to the interval $[0, T]$.

Lemma A.7. *For any $T > 0$, and for any $\tau \in \mathbb{R}$,*

$$\Big|\mathcal{F}\Big(1_{[0,T]} \sin\Big)(\tau)\Big|^2 = \frac{\Big(\sin(\tau T) - \tau \sin T\Big)^2 + \Big(\cos(\tau T) - \cos T\Big)^2}{(\tau^2-1)^2}.$$

Proof: Recall the Euler formulas:

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x. \quad \text{(A.3.1)}$$

Then $\sin x = \frac{\exp(ix) - \exp(-ix)}{2i}$. Hence,

$$I := \mathcal{F}\Big(1_{[0,T]} \sin\Big)(\tau) = \int_0^T e^{-i\tau x} \sin x \; \mathrm{d}x = \int_0^T e^{-i\tau x} \Big(\frac{e^{ix} - e^{-ix}}{2i}\Big)\mathrm{d}x$$

$$= \int_0^T e^{-i\tau x}\left(\frac{e^{ix}}{2i}\right)\mathrm{d}x - \int_0^T e^{-i\tau x}\left(\frac{e^{-ix}}{2i}\right)\mathrm{d}x = \frac{e^{(i-i\tau)T}-1}{2(\tau-1)} - \frac{e^{(-i-i\tau)T}-1}{2(\tau+1)}.$$

Using relations (A.3.1), we have:

$$\begin{aligned} e^{iT-i\tau T} &= e^{(T-\tau T)i} = \cos(T-\tau T) + i\sin(T-\tau T) \\ e^{-iT-i\tau T} &= e^{-(T+\tau T)i} = \cos(T+\tau T) - i\sin(T+\tau T). \end{aligned} \tag{A.3.2}$$

We continue to calculate I using relations (A.3.2):

$$\begin{aligned} I &= \frac{\cos(T-\tau T) + i\sin(T-\tau T) - 1}{2(\tau-1)} - \frac{\cos(T+\tau T) - i\sin(T+\tau T) - 1}{2(\tau+1)} \\ &= \left(\frac{\cos(T-\tau T)-1}{2(\tau-1)} - \frac{\cos(T+\tau T)-1}{2(\tau+1)}\right) + i\left(\frac{\sin(T-\tau T)}{2(\tau-1)} + \frac{\sin(T+\tau T)}{2(\tau+1)}\right). \end{aligned}$$

Hence,

$$\begin{aligned} |I|^2 &= \left(\frac{\cos(T-\tau T)-1}{2(\tau-1)} - \frac{\cos(T+\tau T)-1}{2(\tau+1)}\right)^2 + \left(\frac{\sin(T-\tau T)}{2(\tau-1)} + \frac{\sin(T+\tau T)}{2(\tau+1)}\right)^2 \\ &= A + B, \end{aligned}$$

where $A = \Big(\mathrm{Re}(I)\Big)^2$ and $B = \Big(\mathrm{Im}(I)\Big)^2$.

Using elementary trigonometric identities, we see that:

$$\cos(T-\tau T) + \cos(T+\tau T) = 2\cos T\cos(\tau T) \tag{A.3.3}$$

$$\cos(T-\tau T) - \cos(T+\tau T) = 2\sin T\sin(\tau T) \tag{A.3.4}$$

$$\sin(T-\tau T) + \sin(T+\tau T) = 2\sin T\cos(\tau T) \tag{A.3.5}$$

$$\sin(T-\tau T) - \sin(T+\tau T) = -2\cos T\sin(\tau T). \tag{A.3.6}$$

We treat A and B separately. Let us consider A first. Using relations (A.3.3) and (A.3.4), we obtain:

$$\begin{aligned} A &= \left(\frac{\cos(T-\tau T)-1}{2(\tau-1)} - \frac{\cos(T+\tau T)-1}{2(\tau+1)}\right)^2 \\ &= \left[\frac{\Big(\cos(T-\tau T)-1\Big)\Big(\tau+1\Big) - \Big(\cos(T+\tau T)-1\Big)\Big(\tau-1\Big)}{2(\tau-1)(\tau+1)}\right]^2 \\ &= \left(\frac{\tau\sin T\sin(\tau T) + \cos T\cos(\tau T) - 1}{\tau^2-1}\right)^2. \end{aligned}$$

Similarly, using relations (A.3.5) and (A.3.6), we have:

$$B = \left(\frac{\sin(T-\tau T)}{2(\tau-1)} + \frac{\sin(T+\tau T)}{2(\tau+1)}\right)^2 = \left(\frac{\sin(T-\tau T)(\tau+1)+\sin(T+\tau T)(\tau-1)}{2(\tau-1)(\tau+1)}\right)^2$$
$$= \left(\frac{2\tau\sin T\cos(\tau T) - 2\cos T\sin(\tau T)}{2(\tau-1)(\tau+1)}\right)^2 = \left(\frac{\tau\sin T\cos(\tau T)-\cos T\sin(\tau T)}{\tau^2-1}\right)^2.$$

We can simplify the numerator of $A+B$ as follows:

$$\begin{aligned}
&\Big(\tau\sin T\sin(\tau T)+\cos T\cos(\tau T)-1\Big)^2 + \Big(\tau\sin T\cos(\tau T)-\cos T\sin(\tau T)\Big)^2\\
&= \tau^2\sin^2 T\sin^2(\tau T)+\cos^2 T\cos^2(\tau T)+1+2\tau\sin T\sin(\tau T)\cos T\cos(\tau T)\\
&\qquad - 2\tau\sin T\sin(\tau T) - 2\cos T\cos(\tau T) + \tau^2\sin^2 T\cos^2(\tau T)\\
&\qquad + \cos^2 T\sin^2(\tau T) - 2\tau\sin T\sin(\tau T)\cos T\cos(\tau T)\\
&= \tau^2\sin^2 T\Big(\sin^2(\tau T)+\cos^2(\tau T)\Big) + \cos^2 T\Big(\cos^2(\tau T)+\sin^2(\tau T)\Big)\\
&\qquad + \Big(-2\tau\sin T\sin(\tau T) - 2\cos T\cos(\tau T)\Big) + 1\\
&= \tau^2\sin^2 T + \cos^2 T + \sin^2(\tau T) + \cos^2(\tau T) - 2\tau\sin T\sin(\tau T) - 2\cos T\cos(\tau T)\\
&= \Big(\sin(\tau T) - \tau\sin T\Big)^2 + \Big(\cos(\tau T)-\cos T\Big)^2,
\end{aligned}$$

using the fact that $1 = \sin^2(\tau T)+\cos^2(\tau T)$. Therefore,

$$\begin{aligned}
|I|^2 &= A+B\\
&= \frac{\Big(\tau\sin T\sin(\tau T)+\cos T\cos(\tau T)-1\Big)^2 + \Big(\tau\sin T\cos(\tau T)-\cos T\sin(\tau T)\Big)^2}{(\tau^2-1)^2}\\
&= \frac{\Big(\sin(\tau T)-\tau\sin T\Big)^2 + \Big(\cos(\tau T)-\cos T\Big)^2}{(\tau^2-1)^2}.
\end{aligned}$$

∎

For any function $\varphi \in L^1([0,T])$, we define:

$$\|\varphi\|^2_{\mathcal{H}(0,T)} := \alpha_H \int_0^T\int_0^T \varphi(s)\varphi(r)|s-r|^{2H-2}dsdr.$$

Based on Lemma A.7, we obtain the following result.

Lemma A.8. *For any* $H \in (\frac{1}{2}, 1)$ *and for any* $T > 0$,

$$\| \sin(\cdot)\|^2_{\mathcal{H}(0,T)} = c_H \int_0^T \frac{\Big(\sin(\tau T) - \tau \sin(T)\Big)^2 + \Big(\cos(\tau T) - \cos(T)\Big)^2}{(1-\tau^2)^2} |\tau|^{1-2H} \mathrm{d}\tau,$$

where $c_H = \Gamma(2H+1)\sin(\pi H)/(2\pi)$.

Proof: Applying Remark A.6 to $\varphi = \psi = 1_{[0,T]} \sin$, we have

$$\begin{aligned} \| \sin(\cdot)\|^2_{\mathcal{H}(0,T)} &= \alpha_H \int_0^T \int_0^T \sin(s)\sin(r)|s-r|^{2H-2} \mathrm{d}s\mathrm{d}r \\ &= \alpha_H \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[0,T]}(s)\sin(s) 1_{[0,T]}(r)\sin(r)|s-r|^{2H-2}\mathrm{d}s\mathrm{d}r \\ &= c_H \int_{\mathbb{R}} \left|\mathcal{F}\Big(1_{[0,T]}\sin\Big)(\tau)\right|^2 |\tau|^{1-2H}\mathrm{d}\tau \\ &= c_H \int_{\mathbb{R}} \frac{1}{(1-\tau^2)^2}\left[\Big(\sin(\tau T) - \tau\sin(T)\Big)^2 + \Big(\cos(\tau T) - \cos(T)\Big)^2\right] |\tau|^{1-2H}\mathrm{d}\tau, \end{aligned}$$

where we used Lemma A.7 for the last equality. ∎

We conclude this section with a result which was also used in the proof of Theorem 3.3.1. Denote: $f_t(\lambda, \tau) = \sin(\tau\lambda t) - \tau \sin(\lambda t)$, $\quad g_t(\lambda,\tau) = \cos(\tau\lambda t) - \cos(\lambda t)$.

Lemma A.9. *[Lemma 3.6 in [6]] For any* $\lambda > 0$ *and* $t > 0$,

$$c_1\Big(t \wedge t^3\Big)\frac{\lambda^3}{1+\lambda^2} \le \int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2}\Big[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau)\Big]\mathrm{d}\tau \le c_2\Big(t \vee t^3\Big)\frac{\lambda^3}{1+\lambda^2},$$

where $c_1 = 2\pi \sin^2(1)/3$ *and* $c_2 = 4\pi$.

Proof: By Lemma A.7 and Plancherel theorem,

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{(\tau^2-1)^2}\Big[f_t^2(\lambda,\tau) + g_t^2(\lambda,\tau)\Big]\mathrm{d}\tau &= \int_{\mathbb{R}} \left|\mathcal{F}\Big(1_{[0,\lambda t]}\sin\Big)(\tau)\right|^2 \mathrm{d}\tau \\ = 2\pi \int_0^{\lambda t} \sin^2 x \mathrm{d}x = 2\pi\lambda \int_0^t \sin^2(\lambda s)\mathrm{d}s &= 2\pi\lambda^3 \int_0^t \frac{\sin^2(\lambda s)}{\lambda^2}\mathrm{d}s. \end{aligned}$$

The result follows using Lemma 2.2.4 (in which we denote $\lambda = |\xi|$). ∎

Appendix B

Space-Time White Noise

In this chapter, we review some classical upper bounds for the moments of the increments of the solutions to equations (1.0.1) and (1.0.2) driven by a space-time white noise W. The existence of the solutions to these equations was studied in Walsh' lecture notes [28]. These results are not used in the thesis. We include them for the sake of a comparison with the results presented in Section 2.3 for the fractional noise in space; see Remark 2.3.8.

We assume that $W = \{W([0,t] \times A) : t \geq 0, A \in \mathcal{B}_b(\mathbb{R})\}$ is the space time white noise, i.e. W is a zero - mean Gaussian process with covariance:

$$E[W([0,t] \times A)W([0,s] \times B)] = (t \wedge s) \text{ Leb}(A \cap B) = \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{L^2(\mathbb{R}_+\times\mathbb{R})},$$

where Leb denotes the Lebesgue measure.

Let $W(1_{[0,t]\times A}) = W([0,t] \times A)$. By linearity, W can be extended to the set $\mathcal{E}$ of linear combination of functions of the form $1_{[0,t]\times A}$. The space $L^2(\mathbb{R}_+ \times \mathbb{R})$ is the closure of $\mathcal{E}$ with respect to $\langle\cdot,\cdot\rangle_{L^2}$. Hence W can be extended to $L^2(\mathbb{R}_+ \times \mathbb{R})$.

We denote $W(\varphi) = \int_0^\infty \int_{\mathbb{R}} \varphi(t,x)W(\mathrm{d}t, \mathrm{d}x)$, for any $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R})$. The map $\varphi \to W(\varphi)$ is an isometry from $L^2(\mathbb{R}_+ \times \mathbb{R})$ to $L^2(\Omega)$: for any $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R})$

$$E\,|W(\varphi)|^2 = \int_0^\infty \int_{\mathbb{R}} |\varphi(t,x)|^2 \,\mathrm{d}x\mathrm{d}t. \tag{B.0.1}$$

We say that $W(\varphi)$ is the *Wiener integral* with respect to W.

B.1 Linear stochastic heat equation

In this section, we consider equation (1.0.1) with space-time white noise $\dot{W}$. The following lemma is presented also in Section 3.3 of [17].

Lemma B.1.1. *If u^h is the solution of equation* (1.0.1) *with space-time white noise $\dot{W}$, then:*
(a) for any $t', t \in [0,T]$ and $x \in \mathbb{R}$,

$$E\left|u^h(t',x) - u^h(t,x)\right|^2 \leq \sqrt{\frac{2\,|t'-t|}{\pi}}, \tag{B.1.1}$$

(b) for any $t \in [0,T]$ and $x, x' \in \mathbb{R}$,

$$E\left|u^h(t,x') - u^h(t,x)\right|^2 \leq |x'-x|\,. \tag{B.1.2}$$

Proof: (a) Let $t' > t$. Say $t' = t+h$, for some $h > 0$. Arguing as in the proof of Theorem 2.3.4, we write

$$E\big|u^h(t',x) - u^h(t,x)\big|^2 = A(t,h) + B(t,h), \tag{B.1.3}$$

where

$$A(t,h) = E\left|\int_0^t \int_{\mathbb{R}} [G^h(t+h-s, x-y) - G^h(t-s,x-y)] W(\mathrm{d}s, \mathrm{d}y)\right|^2,$$
$$B(t,h) = E\left|\int_t^{t+h} \int_{\mathbb{R}} G^h(t+h-s, x-y) W(\mathrm{d}s, \mathrm{d}y)\right|^2.$$

We first compute $A(t,h)$. Using the isometry property (B.0.1) of W and the change of variable $s' = t+h-s$ and $y' = x-y$, we have:

$$\begin{aligned} A(t,h) &= \int_0^t \int_{\mathbb{R}} |G^h(t+h-s, x-y) - G^h(t-s, x-y)|^2 \mathrm{d}y\mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}} |G^h(s'+h, y') - G^h(s', y')|^2 \mathrm{d}y'\mathrm{d}s' \\ &= J_1 - J_2, \end{aligned}$$

where

$$J_1 := \int_0^\infty \int_{\mathbb{R}} |G^h(s+h,y) - G^h(s,y)|^2 \mathrm{d}y\mathrm{d}s,$$
$$J_2 := \int_t^\infty \int_{\mathbb{R}} |G^h(s+h,y) - G^h(s,y)|^2 \mathrm{d}y\mathrm{d}s.$$

To evaluate J_1, we use the Plancherel theorem as follows:

$$J_1 = \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{R}} \left|\mathcal{F}G^h(s+h,\cdot)(\xi) - \mathcal{F}G^h(s,\cdot)(\xi)\right|^2 \mathrm{d}\xi\mathrm{d}s$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^\infty \int_{\mathbb{R}} \left| e^{-(s+h)|\xi|^2/2} - e^{-s|\xi|^2/2}\right|^2 \mathrm{d}\xi\mathrm{d}s \\
&= \frac{1}{\pi}\int_0^\infty \int_0^{+\infty} \left| e^{-s\xi^2/2}\right|^2 \left|1 - e^{-h\xi^2/2}\right|^2 \mathrm{d}\xi\mathrm{d}s \\
&= \frac{1}{\pi}\int_0^\infty \Big(1 - e^{-h\xi^2/2}\Big)^2 \Big(\int_0^{+\infty} e^{-s\xi^2}\mathrm{d}s\Big)\mathrm{d}\xi \\
&= \frac{1}{\pi}\int_0^\infty \Big(\frac{1 - e^{-h\xi^2/2}}{\xi}\Big)^2 \mathrm{d}\xi.
\end{aligned}
$$

Using the change of variable $z = \sqrt{h}\xi$, we obtain:

$$
\int_0^\infty \Big(\frac{1 - e^{-h\xi^2/2}}{\xi}\Big)^2 \mathrm{d}\xi = \int_0^\infty \Big(\frac{1 - e^{-z^2/2}}{z/\sqrt{h}}\Big)^2 \frac{1}{\sqrt{h}}\mathrm{d}z = \sqrt{h}\sqrt{\pi}(\sqrt{2}-1)
$$

where the last equality follows the identity (see Lemma A.1 of [17]):

$$
\int_0^{+\infty} \frac{(1 - e^{-w^2/2})^2}{w^2}\mathrm{d}w = \sqrt{\pi}(\sqrt{2}-1).
$$

Hence,

$$
A(t,h) \le J_1 = \frac{\sqrt{h}}{\sqrt{\pi}}(\sqrt{2}-1). \tag{B.1.4}
$$

Next, we compute $B(t,h)$. In fact, $B(t,h)$ does not depend on t. By the isometry property (B.0.1) of W,

$$
B(t,h) = \int_t^{t+h}\int_{\mathbb{R}} G^2(t+h-s, x-y)\mathrm{d}y\mathrm{d}s.
$$

Using the change of variable $s' = t+h-s$ and $y' = x-y$, we have:

$$
\begin{aligned}
B(t,h) &= \int_0^h \int_{\mathbb{R}} G^2(s',y')\mathrm{d}y'\mathrm{d}s' = \int_0^h \bigg(\int_{\mathbb{R}} \frac{1}{2\pi s'}\exp\Big(-\frac{|y'|^2}{s'}\Big)\mathrm{d}y'\bigg)\mathrm{d}s' \\
&= \int_0^h \frac{1}{2\pi}\bigg(\int_{\mathbb{R}} \frac{1}{s}\exp\Big(-\frac{|y|^2}{s}\Big)\mathrm{d}y\bigg)\mathrm{d}s = \int_0^h \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{s}}\mathrm{d}s \\
&= \int_0^h \frac{1}{\sqrt{4\pi s}}\mathrm{d}s = \frac{1}{2\sqrt{\pi}}\int_0^h \frac{1}{\sqrt{s}}\mathrm{d}s \\
&= \frac{\sqrt{h}}{\sqrt{\pi}}.
\end{aligned} \tag{B.1.5}
$$

Using relations (B.1.3), (B.1.4) and (B.1.5), it follows that

$$E\big|u^h(t',x)-u^h(t,x)\big|^2 \le \frac{\sqrt{h}}{\sqrt{\pi}}(\sqrt{2}-1)+\frac{\sqrt{h}}{\sqrt{\pi}} = \sqrt{h}\cdot\sqrt{\frac{2}{\pi}}.$$

(b) Assume $x' > x$. Let $x' = x+z$ for some $z \ge 0$. By the isometry property (B.0.1) of W, and the change of variables $s' = t-s$ and $y' = x-y$, followed by Plancharel theorem, we have:

$$\begin{aligned}
C(t,z) &:= E\big|u^h(t,x+z)-u^h(t,x)\big|^2\\
&= E\Big|\int_0^t\int_{\mathbb{R}} G^h(t-s,x+z-y)-G^h(t-s,x-y)W(\mathrm{d}s,\mathrm{d}y)\Big|^2\\
&= \int_0^t\int_{\mathbb{R}}\big|G^h(s',y'+z)-G^h(s',y')\big|^2\,\mathrm{d}y'\mathrm{d}s'\\
&= \frac{1}{2\pi}\int_0^t\int_{\mathbb{R}}\big|\mathcal{F}G^h(s,z+\cdot)(\xi)-\mathcal{F}G^h(s,\cdot)(\xi)\big|^2\,\mathrm{d}\xi\mathrm{d}s\\
&= \frac{1}{2\pi}\int_0^t\int_{\mathbb{R}} 2\Big(1-\cos(\xi z)\Big)\,\big|\mathcal{F}G^h(s,\cdot)(\xi)\big|^2\,\mathrm{d}\xi\mathrm{d}s\\
&= \frac{1}{\pi}\int_0^t\int_{\mathbb{R}}\Big(1-\cos(\xi z)\Big)\exp(-s\xi^2)\mathrm{d}\xi\mathrm{d}s\\
&= \frac{1}{\pi}\int_{\mathbb{R}}\Big(1-\cos(\xi z)\Big)\frac{1}{\xi^2}\Big(1-\exp(-t\xi^2)\Big)\mathrm{d}\xi.
\end{aligned}$$

To continue, let $\xi z = w$. We obtain:

$$\begin{aligned}
C(t,z) &= \frac{1}{\pi}\int_{\mathbb{R}}\Big(1-\cos w\Big)\frac{z^2}{w^2}\Big(1-\exp\Big(-t\frac{z^2}{w^2}\Big)\Big)\frac{1}{z}\mathrm{d}w\\
&= \frac{z}{\pi}\int_{\mathbb{R}}\frac{1-\cos w}{w^2}\Big(1-\exp\Big(-t\frac{z^2}{w^2}\Big)\Big)\mathrm{d}w\\
&= \frac{z}{\pi}\int_{\mathbb{R}}\frac{1-\cos w}{w^2}\mathrm{d}w-\frac{z}{\pi}\int_{\mathbb{R}}\exp\Big(-t\frac{z^2}{w^2}\Big)\frac{1-\cos w}{w^2}\mathrm{d}w\\
&\le z
\end{aligned}$$

since $\int_{\mathbb{R}}\frac{1-\cos(w)}{w^2}\mathrm{d}w = \pi$ from Lemma A.2, p.95 in [17]. ∎

Remark B.1. It can be proved that the solution u^h of the linear stochastic heat equation (1.0.1) with space-time white noise $\dot{W}$ has covariance:

$$E[u^h(t,x)u^h(s,x)] = (2\pi)^{-1/2}\Big((t+s)^{1/2}-|t-s|^{1/2}\Big)$$

(see [24]). That is, for any $x \in \mathbb{R}$, the process $\{u^h(t,x); t \geq 0\}$ coincides (modulo a constant) with a bi-fractional Brownian motion (bi-fBm) with indices $H = K = 1/2$. Recall that a bi-fBm with indices $H \in (0,1)$ and $K \in (0,1]$ is a zero-mean Gaussian process $\{X(t)\}_{t \geq 0}$ with covariance:

$$E[X(t)X(s)] = \frac{1}{2^K}\Big((t^{2H} + s^{2H})^K - |t-s|^{2HK}\Big)$$

(see e.g. relation (4.1) of [24]).

B.2 Linear stochastic wave equation

In this section, we consider equation (1.0.2) with space-time white noise $\dot{W}$. The following result is essentially Proposition 4.35 of [9], but we give here a different proof than the one presented in [9].

Lemma B.2.1. *If u^w is the solution of equation* (1.0.2) *with space-time white noise $\dot{W}$, then:*
(a) for any $t', t \in [0,T]$ and $x \in \mathbb{R}$,

$$E\left|u^w(t',x) - u^w(t,x)\right|^2 \leq \frac{5}{4}T\left|t'-t\right|, \tag{B.2.1}$$

(b) for any $t \in [0,T]$ and $x, x' \in \mathbb{R}$,

$$E\left|u^w(t,x') - u^w(t,x)\right|^2 \leq \frac{1}{8}T\left|x'-x\right|. \tag{B.2.2}$$

Proof: (a) Let $t' > t$. Say $t' = t+h$, for some $h > 0$. Arguing as in the proof of Theorem 2.3.4, we have

$$E\Big|u^w(t',x) - u^w(t,x)\Big|^2 = A(t,h) + B(t,h),$$

where

$$A(t,h) = E\left|\int_0^t \int_{\mathbb{R}} [G^w(t+h-s, x-y) - G^w(t-s,x-y)] W(\mathrm{d}s, \mathrm{d}y)\right|^2,$$
$$B(t,h) = E\left|\int_t^{t+h} \int_{\mathbb{R}} G^w(t+h-s, x-y) W(\mathrm{d}s, \mathrm{d}y)\right|^2.$$

We treat $A(t,h)$ first. Using the change of variable $s' = t-s$ and $x' = x-y$, we obtain:

$$A(t,h) = \int_0^t \int_{\mathbb{R}} |G^w(t+h-s, x-y) - G^w(t-s,x-y)|^2 \mathrm{d}y\mathrm{d}s$$

$$
\begin{aligned}
&= \int_0^t \int_{\mathbb{R}} |G^w(s+h,y) - G^w(s,y)|^2 \mathrm{d}y\mathrm{d}s \\
&= \frac{1}{4}\int_0^t \Big(\int_{\mathbb{R}} 1_{\{s\le |y| \le s+h\}}\mathrm{d}y\Big)\mathrm{d}s = \frac{1}{2}ht \le \frac{1}{2}hT,
\end{aligned}
$$

where we used the fact that the time increment of fundamental solution G^w can be expressed as:

$$
G^w(s+h,y) - G^w(s,y) = \frac{1}{2}\Big(1_{\{|y|\le s+h\}} - 1_{\{|y|\le s\}}\Big) = \frac{1}{2}1_{\{s\le |y|\le s+h\}}.
$$

Next, we treat $B(t,h)$. In fact, $B(t,h)$ does not depend on t. We have:

$$
\begin{aligned}
B(t,h) &= \int_t^{t+h} \int_{\mathbb{R}} |G^w(t+h-s,x-y)|^2 \mathrm{d}y\mathrm{d}s \\
&= \int_0^h \int_{\mathbb{R}} |G^w(s+h,y)|^2 \mathrm{d}y\mathrm{d}s = \frac{1}{4}\int_0^h \Big(\int_{\mathbb{R}} 1_{\{|y|\le s+h\}}\mathrm{d}y\Big)\mathrm{d}s \\
&= \frac{1}{2}\int_0^h (s+h)\mathrm{d}s = \frac{1}{2}\Big(\frac{h^2}{2} + h^2\Big) = \frac{3}{4}h^2 \le \frac{3}{4}hT.
\end{aligned}
$$

(b) Assume $x' > x$. Let $x' = x + z$ for some $z \ge 0$. We have

$$
\begin{aligned}
C(t,z) &= E\Big|u^w(t,x+z) - u^w(t,x)\Big|^2 \\
&= E\Big|\int_0^t \int_{\mathbb{R}} G^w(t-s,x+z-y) - G^w(t-s,x-y)W(\mathrm{d}s,\mathrm{d}y)\Big|^2 \\
&= \int_0^t \int_{\mathbb{R}} |G^w(t-s,x+z-y) - G^w(t-s,x-y)|^2 \,\mathrm{d}y\mathrm{d}s \\
&= \int_0^t \Big(\int_{\mathbb{R}} |G^w(s',y'+z) - G^w(s',y')|^2 \,\mathrm{d}y'\Big)\mathrm{d}s,
\end{aligned}
$$

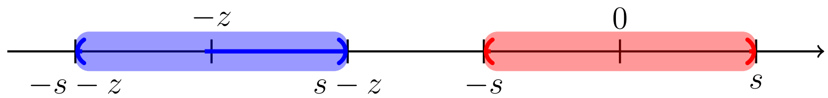
where for the fourth equality, we set $t - s = s'$ and $x - y = y'$. We denote

$$
\begin{aligned}
I_z(s) &= \int_{\mathbb{R}} |G^w(s,y+z) - G^w(s,y)|^2 \,\mathrm{d}y \\
&= \frac{1}{4}\int_{\mathbb{R}} \big|1_{(-s-z,s-z)}(y) - 1_{(-s,s)}(y)\big|^2 \,\mathrm{d}y.
\end{aligned}
$$

We need to consider two cases:

$$
\begin{cases}
s - z < -s & \text{i.e. } z > 2s, \\
s - z > -s & \text{i.e. } z < 2s.
\end{cases}
$$

Case 1: Assume that $s - z < -s$.



In this case, the intervals $(-s-z, s-z)$ and $(-s, s)$ are disjoint and hence

$$\left|1_{(-s-z,s-z)}(y) - 1_{(-s,s)}(y)\right|^2 = 1_{(-s-z,s-z)} + 1_{(-s,s)},$$

and

$$I_z(s) = \frac{1}{4}\Big(\int_{-s-z}^{s-z} \mathrm{d}y + \int_{-s}^{s} \mathrm{d}y\Big) = \frac{1}{4}(2s + 2s) = s.$$

Case 2: Assume that $s - z > -s$.

In this case, the intervals $(-s-z, s-z)$ and $(-s, s)$ are overlapping and hence

$$\begin{aligned}
&\left|1_{(-s-z,s-z)}(y) - 1_{(-s,s)}(y)\right|^2 \\
&= \left|1_{(-s-z,-s)}(y) + 1_{(-s,s-z)}(y) - 1_{(-s,s-z)}(y) - 1_{(s-z,s)}(y)\right|^2 \\
&= 1_{(-s-z,-s)}(y) + 1_{(s-z,s)}(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_z(s) &= \frac{1}{4}\Big(\int_{-s-z}^{-s} \mathrm{d}y + \int_{s-z}^{s} \mathrm{d}y\Big) = \frac{1}{4}\Big\{[-s - (-s-z)] + [s - (s-z)]\Big\} \\
&= \frac{1}{4}(z + z) = \frac{1}{2}z.
\end{aligned}$$

In summary,

$$I_z(s) = \begin{cases} s & \text{if } s \le z/2, \\ z/2 & \text{if } s > z/2. \end{cases}$$

Thus, if $t > \frac{z}{2}$,

$$\begin{aligned}
E\left|u^w(t, x+z) - u^w(t,x)\right|^2 &= \frac{1}{4}\int_0^t I_z(s)\mathrm{d}s = \frac{1}{4}\Big(\int_0^{z/2} s\mathrm{d}s + \int_{z/2}^{t} \frac{1}{2}z\mathrm{d}s\Big) \\
&= \frac{1}{4}\Big\{\frac{s^2}{2}\Big|_0^{z/2} + \frac{1}{2}z\Big(t - \frac{z}{2}\Big)\Big\} = \frac{1}{4}\Big(\frac{1}{2}zt - \frac{z^2}{8}\Big) \\
&= \frac{1}{8}zt - \frac{1}{32}z^2 \le \frac{zt}{8} \le \frac{1}{8}Tz,
\end{aligned}$$

and if $t \leq \frac{z}{2}$,

$$E\left|u^w(t,x+z) - u^w(t,x)\right|^2 = \frac{1}{4}\int_0^t s \ \mathrm{d}z = \frac{t^2}{8} \leq \frac{zt}{16} \leq \frac{1}{16}Tz.$$

∎

Appendix C

Important Inequalities

In this appendix chapter, we give some inequalities which play an important role in the thesis.

The first inequality is a consequence of Littlewood-Hardy-Sobolev inequality from analysis. We recall this result. For any $f \in L^1(\mathbb{R}_+)$ and $\alpha \in (0,1)$, and we define the *fractional integral* of f of order α by:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty |x-y|^{\alpha-1} f(y) \mathrm{d}y. \tag{C.0.1}$$

Theorem C.1. *(Hardy - Littlewood - Sobolev Inequality) [Theorem 1, page 119 of [23]] For any $\alpha \in (0,1)$ and $1 < p < q < \infty$ satisfying $\frac{1}{q} = \frac{1}{p} - \alpha$, we have*

$$\|I^\alpha f\|_{L^q(0,\infty)} \le A_{p,q} \|f\|_{L^p(0,\infty)} \tag{C.0.2}$$

where $A_{p,q} > 0$ is a constant depend on p and q.

The following inequality is used several times in the thesis, for the study of the heat equation. (Usually, the application of this inequality does not yield optimal results for the wave equation). This inequality was proved in reference [20].

Lemma C.2. *For any $H \in (\frac{1}{2}, 1)$ and for any function $\varphi \in L^{1/H}(\mathbb{R}_+)$,*

$$\alpha_H \int_0^\infty \int_0^\infty \varphi(t)\varphi(s)|t-s|^{2H-2} \,\mathrm{d}t\mathrm{d}s \le b_H \Big(\int_0^\infty |\varphi(t)|^{1/H} \mathrm{d}t \Big)^{2H},$$

where $\alpha_H = H(2H-1)$ and $b_H > 0$ is a constant depend on H. More precisely, $b_H = \alpha_H \Gamma(2H-1) A_H$, where A_H is the constant appearing in inequality (C.0.2) with $\alpha = 2H-1$, $p = \frac{1}{H}$, and $q = \frac{1}{1-H}$.

Proof: Using Hölder's inequality with exponents $p = 1/H$ and $q = 1/(1-H)$, we obtain:

$$\begin{aligned}
0 &\le \int_0^\infty \int_0^\infty \varphi(t)\varphi(s)|t-s|^{2H-2}\,\mathrm{d}t\mathrm{d}s = \left|\int_0^\infty \varphi(t)\Big(\int_0^\infty \varphi(s)|t-s|^{2H-2}\,\mathrm{d}s\Big)\mathrm{d}t\right| \\
&\le \int_0^\infty |\varphi(t)|\left|\int_0^\infty \varphi(s)|t-s|^{2H-2}\mathrm{d}s\right|\mathrm{d}t \\
&\le \Big(\int_0^\infty |\varphi(t)|^{1/H}\mathrm{d}t\Big)^H\Big(\int_0^\infty \left|\int_0^\infty \varphi(s)|t-s|^{2H-2}\,\mathrm{d}s\right|^{1/(1-H)}\mathrm{d}t\Big)^{1-H} \\
&= \|\varphi\|_{L^{1/H}(0,\infty)} I,
\end{aligned} \tag{C.0.3}$$

where $I = \Big(\int_0^T \left|\int_0^T \varphi(s)|t-s|^{2H-2}\mathrm{d}s\right|^{1/(1-H)}\mathrm{d}t\Big)^{1-H}$. Note that

$$\begin{aligned}
\|I^{2H-1}\varphi\|_{L^{1/(1-H)}(0,\infty)} &= \Big(\int_0^\infty |I^{2H-1}\varphi(t)|^{1/(1-H)}\mathrm{d}t\Big)^{1-H} \\
&= \frac{1}{\Gamma(2H-1)}\Big(\int_0^\infty \left|\int_0^\infty |t-s|^{2H-2}\varphi(s)\mathrm{d}s\right|^{1/(1-H)}\mathrm{d}t\Big)^{1-H} \\
&= \frac{1}{\Gamma(2H-1)} I.
\end{aligned}$$

By Theorem C.1,

$$\|I^{2H-1}\varphi\|_{L^{1/(1-H)}(0,\infty)} \le A_H\|\varphi\|_{L^{1/H}(0,\infty)},$$

where A_H is the constant appearing in (C.0.2) with $\alpha = 2H-1$, $p = \frac{1}{H}$, and $q = \frac{1}{1-H}$. Hence,

$$I \le \Gamma(2H-1)A_H\|\varphi\|_{L^{1/H}(0,\infty)}. \tag{C.0.4}$$

Therefore, combining (C.0.3) and (C.0.4), we obtain:

$$\begin{aligned}
\alpha_H \int_0^\infty \int_0^\infty \varphi(t)\varphi(s)|t-s|^{2H-2}\,\mathrm{d}t\mathrm{d}s &\le \alpha_H\|\varphi\|_{L^{1/H}(0,\infty)} \cdot \Gamma(2H-1)A_H\|\varphi\|_{L^{1/H}(0,\infty)} \\
&= b_H\|\varphi\|^2_{L^{1/H}(0,\infty)},
\end{aligned}$$

where $b_H = \alpha_H A_H \Gamma(2H-1)$. ∎

Lemma C.3. *For any $\varphi \in L^2(0,T)$ and for any $H \in (\frac{1}{2}, 1)$,*

$$\Big(\int_0^T |\varphi(t)|^{1/H}\,\mathrm{d}t\Big)^{2H} \le T^{2H-1}\int_0^T |\varphi(t)|^2\,\mathrm{d}t.$$

Proof: We apply Hölder's inequality with $p = 2H$ and $q = \frac{2H}{2H-1}$. We obtain:

$$\begin{aligned}\int_0^T |\varphi(t)|^{1/H} \mathrm{d}t &\leq \Big(\int_0^T |\varphi(t)|^{\frac{1}{H}\cdot 2H} \, \mathrm{d}t\Big)^{1/(2H)} \Big(\int_0^T 1 \, \mathrm{d}t\Big)^{\frac{2H-1}{2H}} \\ &= \Big(\int_0^T |\varphi(t)|^2 \, \mathrm{d}t\Big)^{1/(2H)} T^{\frac{2H-1}{2H}}.\end{aligned}$$

Taking power $2H$, we infer that

$$\Big(\int_0^T |\varphi(t)|^{1/H} \, \mathrm{d}t\Big)^{2H} \leq \Big(\int_0^T |\varphi(t)|^2 \, \mathrm{d}t\Big) T^{2H-1}.$$

∎

Corollary C.4. *For any $H \in (\frac{1}{2}, 1)$ and for any $\varphi \in L^2[0,T]$,*

$$\alpha_H \int_0^T \int_0^T \varphi(t)\varphi(s) \, |t-s|^{2H-2} \, \mathrm{d}t \mathrm{d}s \leq b_H T^{2H-1} \int_0^T |\varphi(t)|^2 \, \mathrm{d}t,$$

where $b_H > 0$ is the constant given by Lemma C.2.

Proof: This follows immediately from Lemma C.2 and Lemma C.3. ∎

Appendix D

Useful results

In this appendix chapter, we include some useful results which are used in the thesis.

Lemma D.1. *The p-order moment of a standard normal random variable Z is*

$$z_p = E|Z|^p = 2^{p/2}\pi^{-1/2}\Gamma\Big(\frac{p+1}{2}\Big).$$

Proof: By definition,

$$z_p = E|Z|^p = \int_{\mathbb{R}} |z|^p \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\mathrm{d}z = \frac{2}{\sqrt{2\pi}}\int_0^\infty z^p e^{-z^2/2}\mathrm{d}z.$$

Using the change of variables $y = z^2$, we have:

$$z_p = \frac{2}{\sqrt{2\pi}}\int_0^\infty (\sqrt{y})^p e^{-y/2}\frac{1}{2\sqrt{y}}\mathrm{d}y = \frac{1}{\sqrt{2\pi}}\int_0^\infty y^{(p-1)/2}e^{-y/2}\mathrm{d}y.$$

Applying the change of variables $y/2 = x$, we obtain:

$$\begin{aligned} z_p &= \frac{2}{\sqrt{2\pi}}\int_0^\infty (2x)^{(p-1)/2}e^{-x}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\cdot 2^{(p-1)/2+1}\int_0^\infty x^{(p-1)/2}e^{-x}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\cdot 2^{(p-1)/2+1}\Gamma\Big(\frac{p+1}{2}\Big) = 2^{p/2}\pi^{-1/2}\Gamma\Big(\frac{p+1}{2}\Big). \end{aligned}$$

∎

Lemma D.2. *For any $H \in (\frac{1}{2}, 1)$ and $T > 0$,*

$$\int_0^T\int_0^T ts|t-s|^{2H-2}\mathrm{d}t\mathrm{d}s = \mathrm{B}(2, 2H-1)\frac{1}{H+1}T^{2H+2},$$

where $\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function.

Proof: We split the integral in two regions: $s < t$ and $s \geq t$. By symmetry, we have:

$$\begin{aligned}
&\int_0^T \int_0^T ts|t-s|^{2H-2}\mathrm{d}t\mathrm{d}s \\
&= \int_0^T \int_0^T 1_{\{s<t\}} ts(t-s)^{2H-2}\mathrm{d}t\mathrm{d}s + \int_0^T \int_0^T 1_{\{s\geq t\}} ts(s-t)^{2H-2}\mathrm{d}t\mathrm{d}s \\
&= 2\int_0^T t\Big(\int_0^t s(t-s)^{2H-2}\mathrm{d}s\Big)\mathrm{d}t = 2\int_0^T t\Big(\int_0^1 s't(t-s't)^{2H-2}t\mathrm{d}s'\Big)\mathrm{d}t \\
&= 2\int_0^T t^{2H+1}\Big(\int_0^1 s(1-s)^{2H-2}t\mathrm{d}s\Big)\mathrm{d}t = 2\mathrm{B}(2, 2H-1)\int_0^T t^{2H+1}\mathrm{d}t \\
&= \mathrm{B}(2, 2H-1)\frac{1}{H+1}T^{2H+2},
\end{aligned}$$

where for the third equality, we used the change of variable $s' = s/t$.

∎

Appendix E

Convergence of probability measure

In this appendix chapter, we review some basic concepts related to convergence of probability measures on $\mathbb{R}^k$, on a general metric space, on the space $C([0,1])$ of continuous functions on $[0,1]$, and on the space $C([0,1]\times\mathbb{R})$ of continuous functions on $[0,1]\times\mathbb{R}$. We use references [7] and [8].

E.1 Convergence of probability measures on $\mathbb{R}^k$

In this section, we study the case of classical random vectors with values in $\mathbb{R}^k$.

Definition E.1.1. *Let X be a random vector in $\mathbb{R}^k$. The* **characteristic function** *of X is the function $\varphi : \mathbb{R}^k \to \mathbb{C}$ given by:*

$$\varphi_X(u) = E[e^{iu\cdot X}] = \int_\Omega e^{iu\cdot X}\mathrm{d}P = \int_{\mathbb{R}^k} e^{iu\cdot x}\Big(P\circ X^{-1}\Big)(\mathrm{d}x),$$

for all $u \in \mathbb{R}^k$, where $u\cdot x = u_1x_1 + ... + u_kx_k$ if $u = (u_1, ..., u_k)$, and $x = (x_1, ..., x_k)$.

Remark E.1.2. If $X = (X_1, ..., X_k)$ has a normal distribution with mean $\mu = (\mu_1, ..., \mu_k)$ and covariance matrix Σ, then we write $X \sim N_k(\mu, \Sigma)$. In this case,

$$\varphi_X(u) = \exp\Big\{iu\cdot\mu - \frac{1}{2}u^T\Sigma u\Big\}.$$

Definition E.1.3. *If $\big(X_n\big)_n$ and X are random vectors in $\mathbb{R}^k$, then we say that $\big(X_n\big)_n$* **converge in distribution** *to X if*

$$F_{X_n}(x) = P(X_n \le x) \to F_X(x) = P(X \le x),$$

for all x which is a continuous point of F_X. In this case, we write $X_n \xrightarrow{d} X$.

Theorem E.1.4. *If $\left(X_n\right)_n$ and X are random vectors in $\mathbb{R}^k$, then $X_n \xrightarrow{d} X$ if and only if*

$$\varphi_{X_n}(u) \to \varphi_X(u), \text{ for all } u \in \mathbb{R}^k.$$

If $X_n = (X_n^{(1)}, ..., X_n^{(k)}) \sim N_k(\mu_n, \Sigma_n)$ and $X = (X^{(1)}, ..., X^{(k)}) \sim N_k(\mu, \Sigma)$, then $X_n \xrightarrow{\mathrm{d}} X$ if and only if $\mu_n(i) \to \mu(i)$ for all $i = 1, ..., k$ and $\sigma_n(i,j) \to \sigma(i,j)$, for all $i, j = 1, 2, ..., k$, where $\mu_n = \Big(\mu_n(1), ..., \mu_n(k)\Big)$, $\mu = \Big(\mu(1), ..., \mu(k)\Big)$, $\Sigma_n = \Big(\sigma_n(i,j)\Big)_{1 \le i,j \le k}$ and $\Sigma = \Big(\sigma(i,j)\Big)_{1 \le i,j \le k}$. Consequently, we obtain the following result.

Lemma E.1.5. *Let $X_n = \Big(X_n^{(1)}, ..., X_n^{(k)}\Big)$ and $X = \Big(X^{(1)}, ..., X^{(k)}\Big)$ be zero-mean normal random vectors in $\mathbb{R}^k$. Then $X_n \xrightarrow{d} X$ if and only if*

$$Cov\Big(X_n^{(i)}, X_n^{(j)}\Big) \xrightarrow{d} Cov\Big(X^{(i)}, X^{(j)}\Big), \text{ for all } 1 \le i, j \le k.$$

Definition E.1.6. *Let $\left(\mu_n\right)_n$ and μ be probability measures on $\Big(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)\Big)$. We say that $\left(\mu_n\right)_n$* **converge weakly** *to μ if*

$$\int_{\mathbb{R}^k} h(x)\mu_n(\mathrm{d}x) \to \int_{\mathbb{R}^k} h(x)\mu(\mathrm{d}x),$$

for any function $h : \mathbb{R}^k \to \mathbb{R}$ which is continuous and bounded. In this case, we write $\mu_n \xrightarrow{w} \mu$.

Remark E.1.7. If $(X_n)_n$ and X are random vectors in $\mathbb{R}^k$, $\mu_n = P \circ X_n^{-1}$ and $\mu = P \circ X^{-1}$, then for any measurable function $h : \mathbb{R}^k \to \mathbb{R}$,

$$E[h(X_n)] = \int_\Omega h(X_n) \ \mathrm{d}P = \int_{\mathbb{R}^k} h(x)\Big(P \circ X_n^{-1}\Big)(\mathrm{d}x) = \int_{\mathbb{R}^k} h(x) \ \mu_n(\mathrm{d}x),$$

and

$$E[h(X)] = \int_\Omega h(X) \ \mathrm{d}P = \int_{\mathbb{R}^k} h(x)\Big(P \circ X^{-1}\Big)(\mathrm{d}x) = \int_{\mathbb{R}^k} h(x) \ \mu(\mathrm{d}x).$$

Hence,

$$\mu_n \xrightarrow{\mathrm{w}} \mu \ \text{ if and only if } \ E[h(X_n)] \to E[h(X)],$$

for any function $h : \mathbb{R}^k \to \mathbb{R}$ which is continuous and bounded.

Theorem E.1.8. *If $\left(X_n\right)_n$ and X are random vectors in $\mathbb{R}^k$, $\mu_n = P \circ X_n^{-1}$ and $\mu = P \circ X^{-1}$, then*

$$X_n \xrightarrow{d} X \text{ if and only if } \mu_n \xrightarrow{w} \mu.$$

E.2 Convergence of probability measures on metric space

We start with a general definition for an arbitrary metric space.

Definition E.2.1. *Let* (S, d) *be a metric space and* $\mathcal{S}$ *be its Borel* σ*-field, i.e. the* σ*-field generated by the open sets of* S*. Let* $\big(\Omega, \mathcal{F}, P\big)$ *be a probability space. A* **random element** *in* S *is a function* $X : \Omega \to S$ *which is* $\mathcal{F}/\mathcal{S}$*-measurable. The* **law** *of* X *is the probability measure* $\mu = P \circ X^{-1}$ *on* $(S, \mathcal{S})$ *defined by:*

$$\mu(A) = (P \circ X^{-1})(A) = P(X \in A), \text{ for all } A \in \mathcal{S}.$$

Definition E.2.2. *Let* $\big(\mu_n\big)_n$ *and* μ *be probability measures on* $(S, \mathcal{S})$*. We say that* $\big(\mu_n\big)_n$ **converges weakly** *to* μ *if*

$$\int_S h(x)\mu_n(\mathrm{d}x) \to \int_S h(x)\mu(\mathrm{d}x),$$

for any function $h : S \to \mathbb{R}$ *which is continuous and bounded. In this case, we write* $\mu_n \overset{w}{\to} \mu$*.*

Theorem E.2.3. *(Portmanteau Theorem) Let* $\big(\mu_n\big)_n$ *and* μ *be probability measures on* $(S, \mathcal{S})$*. The following statements are equivalent:*

(i) $\mu_n \overset{w}{\to} \mu$

(ii) $\limsup_{n\to\infty}(F) \leq \mu(F)$*, for any closed set* F *in* S

(iii) $\liminf_{n\to\infty}(G) \geq \mu(G)$*, for any open set* G *in* S

(iv) $\lim_{n\to\infty} \mu_n(A) = \mu(A)$*, for any set* $A \in \mathcal{S}$ *such that* $\mu(\partial A) = 0$*, where* $\partial A = \bar{A} \cap \overline{A^c}$*.*

Definition E.2.4. *Let* (S, d) *be a metric space and* $(X_n)_n$ *and* X *be random elements in* S*. We say that* $(X_n)_n$ **converges in distribution** *to* X *if* $\mu_n \overset{w}{\to} \mu$*, where* μ_n *is the law of* X_n *and* μ *is the law of* X*.*

Remark E.2.5. Note that for any $\mathcal{S}$-measurable function $h : S \to \mathbb{R}$,

$$E[h(X_n)] = \int_\Omega h(X_n) \, \mathrm{d}P = \int_S h(x)\Big(P \circ X_n^{-1}\Big)(\mathrm{d}x) = \int_S h(x) \, \mu_n(\mathrm{d}x),$$

and

$$E[h(X)] = \int_\Omega h(X) \, \mathrm{d}P = \int_S h(x)\Big(P \circ X^{-1}\Big)(\mathrm{d}x) = \int_S h(x) \, \mu(\mathrm{d}x).$$

Hence, $X_n \overset{\mathrm{d}}{\to} X$ if and only if $E[h(X_n)] \to E[h(X)]$, for all function $h : S \to \mathbb{R}$ which is continuous and bounded.

In order to check convergence in distribution of processes, we need to introduce the concept of tightness.

Definition E.2.6. *A sequence $(\mu_n)_n$ of probability measures on S is* **tight** *if for any $\varepsilon > 0$, there exist a compact set $K \subset S$ such that*

$$\mu_n(K) > 1 - \varepsilon, \text{ for all } n \geq 1.$$

E.3 Convergence of probability measures on $C([0,1])$

In this section, we consider the particular case $S = C([0,1])$, equipped with the uniform distance:

$$d(x,y) = \|x - y\| = \sup_{t\in[0,1]} |x(t) - y(t)|, \text{ for all } x, y \in C([0,1]),$$

where $C([0,1])$ is the space of continuous functions $x : [0,1] \to \mathbb{R}$.

A random element in $C([0,1])$ is a function $X : \Omega \to C([0,1])$ which is $\mathcal{F}/\mathcal{C}$-measurable where $\mathcal{C}$ is the Borel σ-field of $C([0,1])$. It can be proved that $\mathcal{C}$ is the σ-field generated by the projection maps $\pi_t : C([0,1]) \to \mathbb{R}$ given by:

$$\pi_t(x) = x(t).$$

A consequence of this is the fact that $X : \Omega \to C([0,1])$ is $\mathcal{F}/\mathcal{C}$-measurable if and only if

$$X_t : \Omega \to \mathbb{R} \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}) - \text{measurable}, \ \forall t \in [0,1].$$

In other words, if $X = (X_t)_{t\in[0,1]}$ is a process defined on a probability space $(\Omega, \mathcal{F}, P)$ which has continuous sample paths, i.e. the map $t \mapsto X_t(\omega)$ is continuous for all $\omega \in \Omega$, then automatically $X : \Omega \to C([0,1])$ is a random element in $C([0,1])$.

Hence, if $(X_n)_n$ and X are processes with continuous sample paths, where $X_n = \{X_n(t)\}_{t\in[0,1]}$ and $X = \{X(t)\}_{t\in[0,1]}$, then $X_n \xrightarrow{\text{d}} X$ if and only if

$$\mu_n \xrightarrow{\text{w}} \mu,$$

where μ_n is the law of X_n and on $C([0,1])$ and μ is the law of X on $C([0,1])$.

The following theorem allows us to prove weak convergence of processes with sample paths in $C([0,1])$.

Theorem E.3.1 (Theorem 8.1 of [7]). *Let $\big(X_n\big)_n$ and X be processes with sample paths in $C([0,1])$. If the following conditions hold:*

(i) $\Big(X_n(t_1), ..., X_n(t_k)\Big) \xrightarrow{d} \Big(X(t_1), ..., X(t_k)\Big)$ *for any* $t_1, ..., t_k \in [0,1]$ *and for any* $k \geq 1$,

(ii) $\big(\mu_n\big)_n$ *is tight, where* μ_n *is the law of* X_n *in* $C([0,1])$,

then $X_n \xrightarrow{d} X$ *in* $C([0,1])$.

The following theorem gives a useful method for checking tightness.

Theorem E.3.2. *Let $\big(X_n\big)_n$ be processes with sample paths in $C([0,1])$. Suppose that the following two conditions are satisfied:*

(i) $\sup\limits_{n\geq 1} E|X_n(0)|^{p'} < \infty,$ *for some* $p' > 0$,

(ii) *there exist* $\gamma \geq 0, \alpha > 1$ *and a non-decreasing continuous function* $F : [0,1] \to \mathbb{R}$ *such that*

$$E\,|X(t_2) - X(t_1)|^{\gamma} \leq \Big(F(t_2) - F(t_1)\Big)^{\alpha} \text{ for all } t_1, t_2 \in [0,1], \tag{E.3.1}$$

for all $t_1, t_2 \in [0,1]$ *and for all* $\lambda > 0$.

Then $(\mu_n)_n$ *is tight, where* μ_n *is the law of* X_n *in* $C([0,1])$.

Combining Theorems E.3.1 and E.3.2, we obtain:

Theorem E.3.3. *Let $(X_n)_n$ and X be processes with sample paths in $C([0,1])$. If the following conditions hold:*

(i) $\Big(X_n(t_1), ..., X_n(t_k)\Big) \xrightarrow{d} \Big(X(t_1), ..., X(t_k)\Big)$ *for any* $t_1, ..., t_k \in [0,1]$ *and for any* $k \geq 1$,

(ii) $\sup\limits_{n\geq 1} E|X_n(0)|^{p'} < \infty,$ *for some* $p' > 0$,

(iii) *there exist* $\gamma \geq 0, \alpha > 1$ *and a non-decreasing continuous function* $F : [0,1] \to \mathbb{R}$ *such that relation* (E.3.1) *holds,*

then $X_n \xrightarrow{d} X$ *in* $C([0,1])$.

E.4 Convergence of probability measures on $C([0,1]\times\mathbb{R})$

In this section we consider the case $S = C([0,1]\times\mathbb{R})$, where $C([0,1]\times\mathbb{R})$ is the space of continuous functions $f : [0,1]\times\mathbb{R}\to\mathbb{R}$.

A collection $X = \Big\{X(t,x)\Big\}_{t\in[0,1],\ x\in\mathbb{R}}$ of random variables $X(t,x) : \Omega \to \mathbb{R}$ is called a **multi-parameter stochastic process** or **random field**.

A random element in $C([0,1]\times\mathbb{R})$ is a function $X : \Omega \to C([0,1]\times\mathbb{R})$ which is $\mathcal{F}/\mathcal{C}([0,1]\times\mathbb{R})$-measurable, where $\mathcal{C}([0,1]\times\mathbb{R})$ is the Borel σ-field on $C([0,1]\times\mathbb{R})$. It can be proved that $\mathcal{C}([0,1]\times\mathbb{R})$ coincides with the σ-field generated by the projection maps $\pi_{t,x} : C([0,1]\times\mathbb{R})\to\mathbb{R}$ given by

$$\pi_{t,x}(f) = f(t,x), \text{ for all } t\in[0,1], x\in\mathbb{R}.$$

Hence any random field $X = \Big\{X(t,x)\Big\}_{t\in[0,1],\ x\in\mathbb{R}}$ with continuous paths is a random element in $C([0,1]\times\mathbb{R})$.

Theorem E.4.1. *Let $X_n = \{X_n(t,x)\}$ and $X = \{X(t,x)\}$ be random fields with sample paths in $C([0,1]\times\mathbb{R})$. If the following conditions hold:*

(i) $\Big(X_n(t_1,x_1),...,X_n(t_k,x_k)\Big) \xrightarrow{d} \Big(X(t_1,x_1),...,X(t_k,x_k)\Big)$ for all $t_1,...,t_k\in[0,1]$, $x_1,...,x_k\in\mathbb{R}$ and $k\geq 1$,

(ii) $\big(\mu_n\big)_n$ is tight where μ_n is the law of X_n on $C([0,1]\times\mathbb{R})$.

Then $X_n \xrightarrow{d} X$ in $C([0,1]\times\mathbb{R})$.

Theorem E.4.2 (Proposition 2.3 of [29])**.** *Let $X_n = \{X_n(t,x)\}_{t\in[0,1],\ x\in\mathbb{R}}, n\geq 1$ be random fields with sample paths in $C([0,1]\times\mathbb{R})$. Suppose that the following two conditions hold:*

(i) $\sup_{n\geq 1} E|X_n(0,0)|^{p'} < \infty$, for some $p' > 0$,

(ii) for any compact set $J\subset\mathbb{R}$, there exist $p>0, \delta>2, C>0$ and $N\in\mathbb{N}$ such that

$$E|X_n(t',x') - X_n(t,x)|^p \leq C\Big(|t'-t| + |x'-x|\Big)^{\delta}, \tag{E.4.1}$$

for all $t',t\in[0,1], x',x\in J$, and $n\geq N$.

Then $(X_n)_n$ is tight in $C([0,1]\times\mathbb{R})$.

Theorem E.4.3. *Let* $X_n = \{X_n(t,x)\}$ *and* $X = \{X(t,x)\}$ *be random fields with sample paths in* $C([0,1]\times\mathbb{R})$. *If the following conditions hold:*

(i) $\Big(X_n(t_1,x_1),...,X_n(t_k,x_k)\Big) \xrightarrow{d} \Big(X(t_1,x_1),...,X(t_k,x_k)\Big)$ *for all* $t_1,...,t_k \in [0,1]$, $x_1,...,x_k \in \mathbb{R}$ *and* $k \geq 1$,

(ii) $\sup_{n\geq 1} E|X_n(0,0)|^{p'} < \infty$, *for some* $p' > 0$,

(iii) *for any compact set* $J \subset \mathbb{R}$, *there exist* $p > 0, \delta > 2, C > 0$ *and* $N \in \mathbb{N}$ *such that relation* (E.4.1) *holds, for all* $t', t \in [0,1], x', x \in J$, *and* $n \geq N$.

Then $X_n \xrightarrow{d} X$ *in* $C([0,1]\times\mathbb{R})$.

Remark E.4.4. Theorem E.4.3 remains valid (with obvious modification) for random fields with sample paths in $C([0,T]\times\mathbb{R})$ for any $T > 0$.

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