

**ON THE MECHANICAL CHARACTERIZATION OF A
CLASS OF VISCOELASTIC MATERIALS USING A
DYNAMIC SYSTEM APPROACH**

By

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**A thesis submitted to the University of Ottawa
in partial fulfillment of the requirement for the degree of Master of
Applied Science in Mechanical Engineering**

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ABSTRACT

The present research aims at the use of the dynamic system approach for the characterization of the mechanical response of linear viscoelastic materials.

The new and basic idea of the research is to consider the viscoelastic material as a dynamic system. Starting with this concept, the research is carried out according to the following steps:

1. Developing a method for the identification of a general dynamic system. Discrete-time system analysis is introduced to achieve this goal.

2. Extending the method mentioned under (1) above to characterize the creep (or relaxation) function of the viscoelastic material from the measurements of the strain (or stress) output and the corresponding rate of the stress (or strain) input signals.

3. Extending the method further to characterize the creep (or relaxation) function directly from the measurements of the strain (or stress) output and the corresponding stress (or strain) input signals.

In this part, the linear viscoelastic material is considered as a dynamic system. From this point of view, a dynamic system identification method is developed for the determination of the creep or relaxation function of the material from dynamic experimental measurements. First, the relation between the creep or relaxation function and the transfer function of the system is established by assuming a model of rational functions of polynomials for the transfer function. Second, a discrete-time system analysis method is introduced to identify the order and parameters of the proposed model from the discrete-time series of both the input and output

signals.

The numerical examples dealt with in the thesis show that the proposed procedure is reasonable and the models are accurate and efficient.

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LIST OF SYMBOLS

a, b_m ($m=1, 2, \dots, p$)	parameters of a frequency response function
a_i, b_i ($i=1, 2, \dots, p$)	parameters of a transfer function
A_m ($m=1, 2, \dots, p$)	constants corresponding to roots ξ_m
B_m ($m=1, 2, \dots, p$)	constants corresponding to roots λ_m
$C(t)$	creep function
$f(t)$	inverse Fourier transform of $H(i\omega)$
$h(t)$	characteristic function of a continuous system
$h_d(k)$ ($k=0, 1, 2, \dots$)	characteristic series of a discrete-time system
$H_d(z)$	transfer function of a discrete-time system
$H(i\omega)$	Fourier transform of the system characteristic function
$H(s)$	Laplace transform of the system characteristic function
p, q	order of a transfer function
$R(t)$	relaxation function
t	time parameter
ΔT	time interval of sampling
$x(t)$	input to a dynamic system
$\{x_i\}$ ($i=0, 1, 2, \dots$)	discrete-time series of an input
$X(i\omega)$	Fourier transform of $x(t)$
$X(s)$	Laplace transform of an input signal
$y(t)$	output of a dynamic system

$\{y_i\}$ ($i=0,1,2,\dots$)	discrete-time series of an output
$Y(i\omega)$	Fourier transform of $y(t)$
$Y(s)$	Laplace transform of an output signal
α, β_m ($m=1,2,\dots,p$)	parameters of a discrete-time system
α'_m ($m=1,2,\dots,p$)	parameters of the modified discrete-time system
$\varepsilon(t)$	time-dependent strain
λ_m ($m=1,2,\dots,p$)	roots of the characteristic equation of a discrete-time system
ξ_m ($m=1,2,\dots,p$)	roots of the characteristic equation of a continuous system
$\sigma(t)$	time-dependent stress

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CHAPTER 1

Introduction

With the development of modern industrial materials, such as high polymers and polymeric base composite materials, and their increasing applications in engineering, the study of the mechanical properties of such classes of materials is attracting more and more attention from material scientists and practical engineers, and the subject may be judged to be one of the most active research domains in the field of engineering science.

As it is well-known in the classical material science, the theory of elasticity plays an extremely important role, because most of widely-used engineering materials such as metals can be considered as elastic bodies in a part or whole of their application ranges. Elasticity is a concept which is based on the following physical phenomenon: The stress in an elastic body, that is induced by loading, is directly proportional to the deformation or strain in the body and it is independent of the strain rate. Thus, for a perfectly elastic body, a constant loading will produce a constant strain, and the deformation in the body will disappear instantaneously upon the removal of the load. The mathematical statement of such material property is given by the well-known Hooke's law. The constant proportional coefficient between the stress and the corresponding strain is the Young's Modulus of the material. Because of the direct proportional relation between the stress and the strain, ideal elastic materials have a capacity to store mechanical energy without any dissipation of such energy.

Another classical mechanical property concept, which is related to the present subject, is the concept of viscosity. The theory of viscosity mainly deals with the flow of Newtonian viscous fluids, for which, the rate of strain is assumed to be directly proportional to the imposed shear stress. The mathematical statement of such material property is Newton's Law for viscous fluids. Because of the mentioned direct proportional relation between the strain rate and the stress, the Newtonian viscous fluid has no capacity of storing mechanical energy [Gurtin and Herrera, 1965].

Elasticity and viscosity are two idealized concepts which are used to describe two types of totally different physical phenomena. But in the real world, there exist materials which possess properties characteristic of both elastic and viscous materials. This special phenomenon is termed "viscoelasticity". Thus, materials with such combined response characteristics are called "viscoelastic" materials.

The basic approaches of characterizing the response behaviour of viscoelastic materials are through the nature of their response to a suddenly applied constant stress or to a suddenly applied constant strain in the material[Gross, 1953; Lee, 1962 and Haddad, 1988, 1994]. Corresponding to these two different methods of characterization, two fundamental concepts in the theory of viscoelasticity are defined, respectively, i.e., the concepts of creep and relaxation of the viscoelastic material.

- **Creep Response**

Unlike a Hookean elastic material, which when subjected to a suddenly applied constant loading responds instantaneously with a corresponding state of deformation which remains

constant, and also unlike a Newtonian viscous fluid which responds to a suddenly applied state of constant shear stress by a corresponding steady flow process, a viscoelastic material, when subjected to a suddenly applied constant stress will experience an instantaneous deformation followed by a "flow" process as time advances. This distinct property is termed "creep". The ratio of the occurring time-dependent strain to the magnitude of the constant stress input is a measure of the time-dependent creep response of the viscoelastic material and is referred to as the "creep function" of this material.

- **Stress Relaxation**

An alternative approach to the mechanical characterization of a viscoelastic material is through its stress-relaxation nature. That is when a constant strain input is applied into the material, the occurring stress in the material decays gradually. The ratio of the occurring time-dependent stress and the magnitude of the constant strain input is a measure of the time-dependent stress-relaxation response of the material and is termed as the "stress-relaxation function" of the considered viscoelastic material.

As discussed above, viscoelastic materials possess a particular property, that is, when they are subjected to a constant loading, the resulting response does not hold constant, but it varies with time. If we consider an arbitrary loading, the response of a viscoelastic material is not only dependent on the total stress level at every instant in time, but is also dependent on the previous history of loading. In other words, the material response is not only determined by the current state of stress, but is also determined by all past states of stress. Generally, the

viscoelastic material is said to have a memory for all past states of loading.

Both creep and relaxation functions form the basis for the analysis of the mechanical response of viscoelastic materials. The basic material law, i.e., the constitutive law which governs the mechanical behaviour of a viscoelastic material, is constructed with the inclusion of either one or both of these material functions which are related to each other [Gross, 1953]. Thus, the determination of such functions for a viscoelastic material is a fundamental step for any further analysis concerning viscoelastic materials.

1.1 Subject of the Present Research

The creep, or the relaxation, function of a viscoelastic material is a time-dependent function. The characterization of the two functions has always been a main research topic in the subject of viscoelasticity. The construction of a precise model for describing the response of the material, in terms of either one or both of these functions, and the establishment of an efficient and accurate method for determining the parameters of the model from experimental data have been basic tasks in this research field.

To achieve this objective, experimental and analytical methods have been developed, and they can be classified into the following three categories.

1. Quasi-static Methods

These are the methods directly based on the definition of the creep or the relaxation function. By these methods, one usually conducts a series of quasi-static creep or relaxation experiments, in which, respectively, a constant stress or constant strain is applied to the

specimen, and the time-dependent response of the material is measured. The pertaining creep, or relaxation, function is then determined from the experimental data.

Quasi-static methods are simple, but they have a vital shortcoming that they usually require very long periods of time so that the creep or relaxation properties of the material are to be fully demonstrated. This requirement of a very long time scale usually constitutes a main obstacle for using the quasi-static method in the identification of the viscoelastic response of the materials.

2. Time-temperature Superposition Methods

To overcome the inconvenience of long time testing periods in the quasi-static methods, the so-called time-temperature superposition (TTS) method[e.g., Aklonis, 1972] is developed. The basis of this method is the fact that the time and temperature effects on a viscoelastic material are interrelated. At lower temperature, the creep, or the relaxation, experiment requires a long period of time, while at a higher temperature, it takes a relatively shorter period of time. Thus, in the TTS method, the experiments are conducted at an elevated temperature for a relatively short period of time, then the measurements are transformed, via a reference temperature, to obtain the required creep, or relaxation, properties at the required lower temperature (usually at the room temperature).

It is obvious that, in the TTS method, a set of complex temperature control facilities would be needed. Another difficulty related to these methods stems from the fact that nonlinearities in the response of materials could be introduced in the viscoelastic material at high temperatures.

3. Dynamic Methods

Dynamic methods are based on the results of dynamic experiments performed on the viscoelastic material. By these methods, the experiments are conducted not by applying a constant loading as in the above mentioned methods, but by employing sinusoidal loading. The analysis of the experimental data are usually carried out in frequency domain to obtain certain types of response spectra, then, the creep, or relaxation, functions are obtained from the analysis of these spectra.

The most distinct advantage of the dynamic methods over the quasi-static and TTS methods is that the dynamic experiments can be carried out in relatively short periods of time and without the need of complex experimental facilities. Therefore, dynamic methods are recently attracting more and more attention from researchers.

The subject of the present research is to use of a investigate the dynamic method approach for the characterization of viscoelastic materials and to develop a new and efficient method for the characterization of the creep and relaxation functions from the results of dynamic experiments.

1.2 Outline of the Present Research

The present research adopts a dynamic method approach to the characterization of the mechanical response of linear viscoelastic materials.

The new and basic idea of the research is to consider the linear viscoelastic materials as a dynamic system. Starting with this new concept, the research has been carried out according to the following steps:

a) Developing a method for the identification of a general dynamic system. Discrete-time system analysis has been introduced to achieve this goal.

b) Extending the method mentioned under (a) above to the characterization of the creep (or the relaxation) function of the viscoelastic material from the measurements of the rate of the stress (or the strain) input and the strain (or the stress) output signals.

c) Extending further the above referred to method to the characterization of the creep (or the relaxation) function directly from the measurements of the stress (or the strain) input and the strain (or the stress) output signals.

Thus, the main objective of the present research is to establish, through the research steps outlined above, a new systematic and efficient method for the determination of the creep (or the relaxation) function of the linear viscoelastic material from dynamic experimental measurements.

CHAPTER 2

Theoretical Basis of the Research

In order to have a basis for the following research, it is necessary to give a brief description of the fundamental concepts and basic theory of viscoelasticity. In this chapter, we discuss the creep and relaxation phenomena, the corresponding creep and relaxation functions and their relation. Then, we introduce the basic material law, or constitutive law of viscoelastic materials.

2.1 Creep

Creep is a concept describing the time-dependent characteristic of strain response of a viscoelastic material when subjected to a constant stress.

Under the application of constant stress, the creep strain can be generally divided into three components Fig.2.1[see, e.g., Gross, 1953; Christensen, 1971; Flügge, 1975].

- Instantaneous component ε_e

This part of response of viscoelastic material is somehow similar to the response of an elastic material. It is directly proportional to the stress and time-independent. With the removal of stress, it disappears at the same time .

- Delayed component $\varepsilon_d(t)$

With the stress remaining constant inside the viscoelastic material, this part of response

increases with time and its rate of increase decreases steadily with the passage of time. This part of response is also elastic, in the sense that when stress is removed it also disappears. But the whole process of recovery does not occur at the same time when stress is removed. It requires time. Thus it is called delayed elastic response or elastic after-effect.

- Viscous flow $\varepsilon_v(t)$

It is an irreversible part of the strain. Usually it increases linearly with time.

The total strain is expressed as the sum of the previously mentioned three components, i.e.,

$$\varepsilon(t) = \varepsilon_e + \varepsilon_d(t) + \varepsilon_v(t) \quad (2.1)$$

The corresponding response relations for the three components of strain are expressed, respectively, as

$$\begin{aligned} \varepsilon_e &= C_0 \sigma_0 \\ \varepsilon_d &= \varphi(t) \sigma_0 \\ \varepsilon_v &= \frac{t}{\eta} \sigma_0 \end{aligned} \quad (2.2)$$

where C_0 , $\varphi(t)$ and η are proportional coefficients related to, respectively, instantaneous, delayed and viscous components, and termed as instantaneous elastic compliance, time-increasing function and Newtonian viscosity coefficient. Eq.(2.1) can, thus, be expressed, in view of Eq.(2.2) as

$$\begin{aligned}\varepsilon(t) &= \left[C_0 + \frac{t}{\eta} + \varphi(t) \right] \sigma_0 \\ &= C(t) \sigma_0\end{aligned}\quad t \geq 0 \quad (2.3)$$

where σ_0 is the constant stress in the material and $C(t)$ is known as the creep function.

Fig.2.1 shows a typical creep response of a viscoelastic material. The specimen is subjected to a constant stress σ_0 which is applied rapidly from zero to σ_0 at $t=0$ in a very short time interval. This results in a strain which increases very rapidly and reaches a value of ε_r in an almost vertical line at $t=0$. This is the instantaneous response. Then, with the passage of time, the strain rate slows down gradually tending asymptotically to a constant value.

It should be pointed out that for solid materials such as rigid polymers at ordinary temperature, the viscous coefficient η is very large. Actually in these cases, there are no viscous flow, and $\varepsilon_v(t)$ may be neglected.

Consider a multi-stage loading programme in Fig.2.2 in which incremental stresses $\Delta\sigma_1$, $\Delta\sigma_2$, $\Delta\sigma_3$ etc. are added at time τ_1 , τ_2 , τ_3 etc., respectively. The total creep at time t is then given, in view of Eq.(2.3), by

$$\begin{aligned}\varepsilon(t) &= \Delta\sigma_1 \left[C_0 + \frac{t-\tau_1}{\eta} + \varphi(t-\tau_1) \right] + \Delta\sigma_2 \left[C_0 + \frac{t-\tau_2}{\eta} + \varphi(t-\tau_2) \right] \\ &+ \Delta\sigma_3 \left[C_0 + \frac{t-\tau_3}{\eta} + \varphi(t-\tau_3) \right] + \dots\end{aligned}\quad (2.4)$$

As the increment of time, corresponding to $\Delta\sigma$, decreases, the response relation Eq.(2.4) can be written in a general form as

$$\varepsilon(t) = \int_{-\infty}^t \frac{d\sigma(\tau)}{d\tau} \left[C_0 + \frac{t-\tau}{\eta} + \varphi(t-\tau) \right] d\tau \quad (2.5)$$

This is the classical Boltzmann Superposition Principle[Ferry,1970; Markovitz, 1977]. Boltzmann proposed [Ward, 1972], (1) that creep in a specimen is a function of the entire loading history, and (2) that each loading step makes an independent contribution to the final deformation and that the final deformation can be obtained by the simple addition of each contribution. Eq.(2.5) is also called the "**Hereditary Integral**". This integral equation is the basic material law constructed in terms of the creep function $C(t)$ for viscoelastic materials.

Eq.(2.5) is suitable for any kind of load such as continuous and discontinuous input load histories. Let us consider the response of viscoelastic material to a loading situation in which a unit stress is applied at $t=0$ and unloaded at $t=t_1$, as shown in Fig.2.3.

If we introduce a unit step function $u(t)$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad (2.6)$$

the derivative of the unit step function is the Delta function, i.e.,

$$\frac{du(t)}{dt} = \delta(t) \quad (2.7)$$

with the properties

$$\delta(t) = \begin{cases} \infty & ; \quad t = 0 \\ 0 & ; \quad t \neq 0 \end{cases} \quad (2.8)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (2.9)$$

$$\int_{-\infty}^{\infty} f(t) \delta(t-\tau) dt = f(\tau) \quad (2.10)$$

then the load can be expressed as

$$\sigma(t) = \sigma_0 u(t) - \sigma_0 u(t-t_1) \quad (2.11)$$

By substituting it into Eq.(2.5), the strain responses are:

for $t < 0$

$$\varepsilon(t) = 0 \quad t < 0 \quad (2.12)$$

for $0 < t < t_1$

$$\begin{aligned} \varepsilon(t) &= \int_{-\infty}^t [C_0 + \frac{t-\tau}{\eta} + \varphi(t-\tau)] \frac{d\sigma(\tau)}{d\tau} d\tau \\ &= \int_{-\infty}^t [C_0 + \frac{t-\tau}{\eta} + \varphi(t-\tau)] \sigma_0 \delta(\tau) d\tau \\ &= [C_0 + \frac{t-\tau}{\eta} + \varphi(t)] \sigma_0 \end{aligned} \quad (2.13)$$

for $t > t_1$

$$\begin{aligned}
 \varepsilon(t) &= \int_{-\infty}^t \left[C_0 + \frac{t-\tau}{\eta} + \varphi(t-\tau) \right] \frac{d\sigma(\tau)}{d\tau} d\tau \\
 &= \int_{-\infty}^t \left[C_0 + \frac{t-\tau}{\eta} + \varphi(t-\tau) \right] \sigma_0 [\delta(\tau) - \delta(\tau-t_1)] d\tau \\
 &= \sigma_0 \frac{t_1}{\eta} + \sigma_0 [\varphi(t) - \varphi(t-t_1)]
 \end{aligned} \tag{2.14}$$

The responses corresponding to equations (2.13) and (2.14) are shown in Fig.2.4

2.2 Relaxation

Relaxation is a concept describing the time-dependent characteristic of stress response of a viscoelastic material when subjected to a constant strain. As shown in Fig.2.5. The stress response is monotonically decreasing with time [Day, 1970; Conway, 1992].

The stress can be expressed as

$$\begin{aligned}
 \sigma(t) &= [R_0 - \psi(t)] \varepsilon_0 \\
 &= R(t) \varepsilon_0
 \end{aligned} \tag{2.15}$$

where R_0 , and $\psi(t)$ are proportional coefficients, and termed as static elastic modulus and time-decay function respectively. $R(t)$ is the relaxation function of the viscoelastic materials. The limiting value of the relaxation function is determined by viscous flow. If there is a viscous flow, the relaxation function decays to zero at sufficiently long time [Hopkins and Hannming,

1957]. On the other hand, in the absence of viscous flow, the relaxation function decays to a nonzero value, which is usually called an equilibrium or "relaxed modulus".

As in the case of creep, we can consider a several-stage straining program in which incremental strains $\Delta\varepsilon_1, \Delta\varepsilon_2, \Delta\varepsilon_3$ etc., are added at times τ_1, τ_2, τ_3 etc., respectively. The total stress at time t is the sum of contributions of each incremental straining. The limiting process will lead to the general expression of the relation between the strain and the stresses as follows.

$$\sigma(t) = \int_{-\infty}^t \frac{d\varepsilon(\tau)}{d\tau} [R_0 - \psi(t-\tau)] \quad (2.16)$$

If we denote

$$R(t) = [R_0 - \psi(t)] \quad (2.17)$$

then, equation (2.16) can be written as

$$\begin{aligned} \sigma(t) &= R(t)\varepsilon_0 + \int_0^t R(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau \\ &= R(0)\varepsilon(t) + \int_0^t \varepsilon(\tau) \frac{dR(t-\tau)}{d(t-\tau)} d\tau \end{aligned} \quad (2.18)$$

Equations (2.16) and (2.18), showing that the stress at any time t is a function of the history of strain input and not just its current value, are the constitutive equations of viscoelastic materials in terms of their relaxation functions.

2.3 The Relation Between the Creep and Relaxation Functions

Equations (2.5) and (2.16) are both the constitutive equations of a viscoelastic materials: one is in terms of creep function and the other is in terms of relaxation function. Obviously, creep function and relaxation function of a certain material are not independent of each other.

In fact, if for a viscoelastic material during an experiment, we have measured the stress $\sigma(t)$ and the corresponding strain $\varepsilon(t)$, Then the equations (2.5) and (2.16) must hold simultaneously for the given $\sigma(t)$ and $\varepsilon(t)$.

To establish a simple relation between the creep and relaxation functions, we need to go into the Fourier Spectrum Domain. First, we note that the creep and relaxation functions are defined for the region of $t \geq 0$. For $t < 0$, $C(t) \equiv 0$, and $R(t) \equiv 0$. Thus if introducing unit step function $u(t)$, we can write

$$\begin{aligned} C(t) &= \left[C_0 + \frac{t}{\eta} + \varphi(t) \right] u(t) & -\infty < t < +\infty \\ R(t) &= [R_0 - \psi(t)] u(t) & -\infty < t < +\infty \end{aligned} \quad (2.19)$$

The time-rate of the creep and relaxation functions can be expressed as

$$\begin{aligned} \dot{C}(t) &= \left[C_0 + \frac{t}{\eta} + \varphi(t) \right] \delta(t) + \left[\frac{1}{\eta} + \dot{\varphi}(t) \right] u(t) \\ \dot{R}(t) &= [R_0 - \psi(t)] \delta(t) - \dot{\psi}(t) u(t) \end{aligned} \quad (2.20)$$

Denoting the Fourier transforms of the rate of creep and relaxation functions by $J(i\omega)$ and $E(i\omega)$, respectively, we have[e.g., Bracewell, 1978; Christensen, 1972]

$$\begin{aligned}
J(i\omega) &= \int_{-\infty}^{\infty} \dot{C}(t) e^{-i\omega t} dt \\
E(i\omega) &= \int_{-\infty}^{\infty} \dot{R}(t) e^{-i\omega t} dt
\end{aligned}
\tag{2.21}$$

Usually $J(i\omega)$ is called complex compliance, $E(i\omega)$ the complex modulus.

Now let us return to Eq.(2.5) and Eq.(2.16). As pointed out before, Eq.(2.5) and Eq.(2.16) must hold simultaneously for the given $\sigma(t)$ and $\varepsilon(t)$. If we denote the Fourier transforms of $\sigma(t)$ and $\varepsilon(t)$ by $\sigma^*(i\omega)$ and $\varepsilon^*(i\omega)$, and take the Fourier transform of Eq.(2.5) and Eq.(2.16), we can have

$$\begin{aligned}
\varepsilon^*(i\omega) &= J(i\omega) \sigma^*(i\omega) \\
\sigma^*(i\omega) &= E(i\omega) \varepsilon^*(i\omega)
\end{aligned}
\tag{2.22}$$

Eq.(2.22) gives the following relation

$$J(i\omega) \cdot E(i\omega) = 1
\tag{2.23}$$

Therefore, the creep function and relaxation function are actually not independent characteristics of a viscoelastic material. If one of them is known, the other can readily be obtained.

Here, it is necessary to investigate further the physical meaning of the complex compliance and the complex modulus:

Let us examine the situation when an oscillating stress with frequency ω is applied to a viscoelastic material, that is

$$\sigma = \sigma_0 e^{i\omega t} \quad (2.24)$$

Substituting the above input into the creep constitutive equation, one gets,

$$\begin{aligned} \varepsilon(t) &= \int_{-\infty}^t \frac{d(\sigma_0 e^{i\omega\tau})}{d\tau} \left[C_0 + \frac{t-\tau}{\eta} + \varphi(t-\tau) \right] d\tau \\ &= \sigma_0 e^{i\omega t} i\omega \int_0^{\infty} e^{-i\omega\tau} \left[C_0 + \frac{\tau}{\eta} + \varphi(\tau) \right] d\tau \\ &= \sigma_0 e^{i\omega t} \left[C_0 + \frac{1}{i\omega\eta} + i\omega \int_0^{\infty} \varphi(\tau) e^{-i\omega\tau} d\tau \right] \end{aligned} \quad (2.25)$$

But from Eq.(2.20) and (2.21), one gets

$$\begin{aligned} J(i\omega) &= \int_{-\infty}^{\infty} \left[C_0 + \frac{t}{\eta} + \varphi(t) \right] \delta(t) e^{-i\omega t} dt \\ &+ \int_{-\infty}^{\infty} \left[\frac{1}{\eta} + \dot{\varphi}(t) \right] u(t) e^{-i\omega t} dt \\ &= C_0 + \int_0^{\infty} \left[\frac{1}{\eta} + \dot{\varphi}(t) \right] e^{-i\omega t} dt \\ &= C_0 + \frac{1}{i\omega\eta} + i\omega \int_0^{\infty} \varphi(t) e^{-i\omega t} dt \end{aligned} \quad (2.26)$$

Thus, equation (2.25) can be written as:

$$\varepsilon(t) = J(i\omega) \sigma_0 e^{i\omega t} \quad (2.27)$$

Equation (2.27) states that the application of an oscillating stress produces an oscillating strain response of the same frequency. The amplitude of the response is

$$\varepsilon_0 = J(i\omega) \sigma_0 \quad (2.28)$$

Similarly, if we apply a sinusoidal strain to a viscoelastic material, we can obtain a sinusoidal stress with the same frequency, and the amplitude of the stress is

$$\sigma_0 = E(i\omega) \varepsilon_0 \quad (2.29)$$

Then if we combine equations (2.28) and (2.29), we can also arrive at the relation Eq.(2.23).

CHAPTER 3

Review on the Characterization of Linear Viscoelastic Materials

As pointed out before, the determination of the creep function (or relaxation function) is the most important step for the analysis of the response behaviour of viscoelastic materials. The development of analytical methods for this purpose has always been an important part of frame of the theory of viscoelasticity, and it is still a very active research subject in viscoelasticity. According to the procedures by which viscoelastic experiments are conducted, the methods developed for the characterization of the creep (or relaxation) can be classified into the following three categories.

3.1 Quasi-Static Methods

These methods are based on the experimental data obtained from direct simple creep and relaxation tests on the viscoelastic material. By definition, the creep function of a viscoelastic material is the ratio of the time-dependent strain response to the applied constant stress. Meantime, the relaxation function is the ratio of the time-dependent stress response to the applied constant strain [Lee, 1960a and 1962; Lee and Rogers, 1963].

In a quasi-static creep experiment, the specimen is loaded with a constant stress, and,

then, the time-dependent strain response is measured. The creep function at different instants of time can be obtained by dividing the measured strain response (usually a series of values at different time instants) by the constant stress input as

$$C_0^* , C_1^* , C_2^* , \dots , C_n^* \quad (3.1)$$

In a quasi-static relaxation experiment, the specimen is strained to a certain level which is maintained constant during the experiment while the time dependent stress response is measured. By dividing the measured stress response at different instants of time by the magnitude of the constant strain input, we obtain a series of values for the relaxation function at different instants of time, as

$$R_0^* , R_1^* , R_2^* , \dots , R_n^* \quad (3.2)$$

It should be pointed out, however, that we call the above mentioned methods quasi-static because from the static property of loading these experiments are static ones. But, on the other hand, the response of the materials are changing with time. Due to the involvement of the time-scale in the whole process, however, the experiment is not truly static.

The discrete series of the measurements of the creep function or the relaxation function at different time instants is, of course, a description of the properties of the viscoelastic material. For further theoretical analysis, however, a theoretical model has to be established from those measurements.

The typical classical method for the analysis of the quasi-static response of a linear

viscoelastic material is reported in [Odeh and Tadjbakhsh, 1965; Gradowczyk and Moavenzadeh, 1969]. The equations (3.1) and (3.2) give, respectively, the experimental values of the creep function and the relaxation function at the time instants $t_0=0, t_1, t_2, \dots, t_n$. In the above reference, the series of $\{C^*\}$ and $\{R^*\}$ are considered as the "averages", in some statistical sense, of different sets of experiments. Then, two functions $C^*(t)$ and $R^*(t)$ can be chosen to satisfy

$$\begin{aligned} C_0^* &= C^*(t_0) ; \dots ; C_n^* = C^*(t_n) \\ R_0^* &= R^*(t_0) ; \dots ; R_n^* = R^*(t_n) \end{aligned} \quad (3.3)$$

and $C^*(t)$ and $R^*(t)$ are called approximate creep and relaxation functions. Then, the exact creep and relaxation functions $C(t)$ and $R(t)$ can be assumed as:

$$\begin{aligned} C(t) &= C^*(t) [1 + \mu_c(t)] \\ R(t) &= R^*(t) [1 + \mu_r(t)] \end{aligned} \quad (3.4)$$

where $\mu_c(t)$ and $\mu_r(t)$ are the corresponding relative error functions. From the basic theory of viscoelasticity[Gross, 1953 and Schultz, 1974], the relaxation and creep functions must satisfy

$$R(t) C(0) = 1 + \int_0^t C(\tau) \frac{\partial R(t - \tau)}{\partial \tau} d\tau \quad (3.5)$$

Therefore only one equation can be used to determine the error functions $\mu_c(t)$ and $\mu_r(t)$. To avoid this indetermination, it is assumed that

$$\mu_c(t) = \gamma \mu_r(t) = \gamma \mu(t) \quad (3.6)$$

where γ is an arbitrary real constant. Substituting Eq.(3.6) into Eq.(3.5) yields a nonlinear integral equation for the error function $\mu(t)$

$$C(0) R^*(t) \mu(t) = 1 - C(0) R^*(t) + \int_0^t R(\tau) \frac{\partial C(t - \tau)}{\partial \tau} d\tau \quad (3.7)$$

For the purpose of computation, it is convenient to write this equation as

$$C(0) R^*(t_m) \mu(t_m) = 1 - C(0) R^*(t_m) + \sum_{i=1}^m \int_{t_{i-1}}^{t_i} R(\tau) \frac{\partial C(t_m - \tau)}{\partial \tau} d\tau \quad (3.8)$$

Applying the mean value theorem of integral to Eq.(3.8) gives

$$C_0 R^*(t_m) \mu_m = 1 - C_0 R^*(t_m) + \sum_{i=1}^m R(t_{i-1} + h_i \zeta) (C_{m,i} - C_{m,i-1}) \quad (3.9)$$

where

$$\begin{aligned} h_i &= t_i - t_{i-1} & 0 < \zeta < 1 \\ C_0 &= C(t_0) \\ C_{m,i} &= C(t_m - t_i) \\ \mu_m &= \mu(t_m) \end{aligned} \quad (3.10)$$

To solve equation (3.9) to obtain $\mu(t)$, ζ has to be determined. But we can set $\zeta=0.5$ to obtain an approximation of the error function $\mu(t)$ in the discrete series form:

$$\mu_m = \frac{\Gamma_m}{\Lambda_m} \quad (3.11)$$

where

$$\begin{aligned} \Gamma_m = & 1 + \frac{1}{2}C_0(R_{m-1} - R_m^*) - \frac{1}{2}C_1(R_{m-1} + R_m^*) \\ & + \sum_{i=2}^{m-1} R_{i-\frac{1}{2}} (C_{m,i} - C_{m,i-1}) \\ & + R_{\frac{1}{2}} (C_{m-1} - C_m^*) \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} \Lambda_m = & \frac{1}{2}R_m^* (C_0 + C_1) + \gamma R_{\frac{1}{2}} C_m^* \\ R_i = & R_i^* (1 + \mu_i) \\ C_i = & C_i^* (1 + \gamma \mu_i) \\ R_{i-\frac{1}{2}} = & \frac{1}{2} (R_i + R_{i-1}) \end{aligned} \quad (3.13)$$

The first two values of μ_m are

$$\begin{aligned} \mu_0 = & -(\gamma + \frac{1}{2\gamma}) + [(\gamma + \frac{1}{2\gamma})^2 - (\frac{1}{\gamma}) (1 - \frac{1}{C_0^* R_0^*})]^{1/2} \\ \mu_1 = & (\frac{B}{2A}) [-1 + (1 - \frac{4AC}{B})^{1/2}] \end{aligned} \quad (3.14)$$

where

$$\begin{aligned}
 A &= \gamma C_1^* R_1^* \\
 B &= R_1^* (C_0 + C_1) + \gamma C_1^* (R_0 + R_1^*) \\
 C &= 2 C_0 R_1^* - 2 - (R_0 + R_1^*) (C_0 - C_1^*)
 \end{aligned} \tag{3.15}$$

In this way, the error function can be evaluated. And the experimental data $\{C_i^*\}$, $\{R_i^*\}$ can be modified. Based on these modified experimental data, we can establish the model of creep and relaxation functions. Generally, the creep and relaxation functions may be represented, respectively, by [see, e.g., Gradowczyk and Moavenzadeh, 1969]

$$\begin{aligned}
 C(t) &= \sum_{i=0}^m a_i f_i(t) \\
 R(t) &= \sum_{i=1}^m b_i g_i(t)
 \end{aligned} \tag{3.16}$$

where $\{f_i(t)\}$ and $\{g_i(t)\}$ are suitable sets of functions, and $\{a_i\}$ and $\{b_i\}$ are unknown real coefficients, which can be determined by the following condition (setting $m = n$)

$$\begin{aligned}
 C(t_i) &= C_i^* [1 + \gamma \mu(t_i)] \\
 R(t_i) &= R_i^* [1 + \mu(t_i)]
 \end{aligned} \quad (i = 0, 1, \dots, n) \tag{3.17}$$

at instants t_0, t_1, \dots, t_n . In the general case, $m=n$, the functions $\{f_i(t)\}$ and $\{g_i(t)\}$ must satisfy, respectively, the following two conditions

$$\Delta_f = \begin{vmatrix} f_0(t_0) & f_1(t_0) & \dots & f_m(t_0) \\ f_0(t_1) & f_1(t_1) & \dots & f_m(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(t_n) & f_1(t_n) & \dots & f_m(t_n) \end{vmatrix} \neq 0 \quad (3.18)$$

and

$$\Delta_g = \begin{vmatrix} g_0(t_0) & g_1(t_0) & \dots & g_m(t_0) \\ g_0(t_1) & g_1(t_1) & \dots & g_m(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ g_0(t_n) & g_1(t_n) & \dots & g_m(t_n) \end{vmatrix} \neq 0 \quad (3.19)$$

Equation (3.16) is in a general form. In practice, the relaxation and creep function are usually written as [see, e.g., Gradowczyk and Moavenzadeh, 1969]

$$\begin{aligned} R(t) &= R_\infty + R'(t) \\ C(t) &= C_0 + C'(t) \end{aligned} \quad (3.20)$$

where R_∞ and C_0 are, respectively, the relaxation function at $t = \infty$ and the creep function at $t = 0$, and $R'(t)$, $C'(t)$ are called, in the same order, transient parts of $R(t)$ and $C(t)$.

For the transient parts $R'(t)$, $C'(t)$, the most widely used representations are, respectively,

$$\begin{aligned} R'(t) &= \sum_{j=0}^n X_j e^{-\frac{t}{\tau_j}} \\ C'(t) &= \sum_{j=0}^n Y_j (1 - e^{-\frac{t}{\tau_j}}) \end{aligned} \quad (3.21)$$

In each equation of Eq.(3.21), there are $2(n+1)$ unknowns $\{X_j, \tau_j^r\}$ or $\{Y_j, \tau_j^c\}$. But there are only $n+1$ equations (for t_0, t_1, \dots, t_n). To avoid this indetermination, it is suggested to take

$$\tau_j = \frac{l_j}{\beta} \quad (3.22)$$

where $l_j = 10^{(j-\alpha)}$

That is, l_j are taken a decade apart. α is the exponent of the initial time decade and $\beta=1/2$.

Through the above presentation, we find that the classical methods are really coarse. The very fundamental problem is that the characteristic times $\{\tau_j\}$ in the model equation (3.21) are arbitrarily set through Eq.(3.22), and the number of terms in Eq.(3.21) are taken to be the same as the number of experimental measurement points in time-scale. Generally speaking, it is sometimes unrealistic.

To overcome this problem, the recent work is reported in [Haddad and Tanary, 1989], in which the so-called differential approximation of the creep and relaxation functions is introduced. By this method, in the creep case, for instance, the rate of the creep function, denoted by $H(t)$ is assumed to satisfy an N -th order differential equation with constant coefficients, i.e.,

$$k_0 H(t) + k_1 H(t)^{(1)} + k_2 H(t)^{(2)} + \dots + k_{N-1} H(t)^{(N-1)} + H(t)^{(N)} = 0 \quad (3.23)$$

where the superscript represent the corresponding order differential of the creep function with respect to time and k_0, k_1, \dots, k_{N-1} are constants to be determined by differential approximation

using the creep experimental data. In the relaxation case, the same assumptions can be made.

Eq.(3.23) is a homogeneous differential equation with constant coefficients which has the following characteristic equation

$$k_0 + k_1 \psi + k_2 \psi^2 + \dots + k_{N-1} \psi^{N-1} + \psi^N = 0 \quad (3.24)$$

Denote the roots of the N -th order algebraic equation (3.24) by

$$\psi_1, \psi_2, \dots, \psi_N \quad (3.25)$$

The general solution of the rate of creep function $H(t)$ can be written as

$$H(t) = \sum_{i=1}^N A_i \exp(\psi_i t) \quad (3.26)$$

where $A_i (i = 1, 2, \dots, N)$ are constants.

To define the governing differential equation (3.23) for the creep function, the coefficients $\{k_i, i = 0, 1, \dots, N-1\}$ and order N have to be determined. To do so, Haddad[Haddad, 1981, 1984 and 1992; Haddad and Tanary, 1987 and 1989] provided the following method, which is based on the definition equation of creep function, that is,

$$\begin{aligned} \varepsilon(t) &= C(t) \sigma_0 \\ &= C_0 \sigma_0 + \sigma_0 \int_0^t H(\tau) d\tau \\ &= \varepsilon_0 + \sigma_0 \int_0^t H(\tau) d\tau \end{aligned} \quad (3.27)$$

where ε_0 is the instantaneous strain response in creep.

Through creep experiments, the experimental strain $\varepsilon_i(t)$ ($i = 1, 2, \dots, n$) can be obtained. Then the first step is to use the following function

$$\begin{aligned} \varepsilon_i(t) &= \varepsilon_i(0) + l_i \theta_i(t) \\ (i &= 1, 2, \dots, n) \end{aligned} \quad (3.28)$$

to fit the experimental results. For a linear viscoelastic material, the function $\theta_i(t)$ usually expressed by the following form

$$\theta_i(t) = t^{m_i} \quad (3.29)$$

where m_i ($i = 1, 2, \dots, n$) are constants.

The second step is to approximate the rate of the creep function $H(t)$ by

$$H(t) \approx \left. \frac{d}{dt} \theta_i(t) \right]_0^T \quad (3.30)$$

$$(i = 1, 2, \dots, n)$$

where T represents the total time of the creep experiment.

Denote

$$\chi_i = \frac{d}{dt} \theta_i(t) \quad (3.31)$$

Because it is assumed that the sought creep function $H(t)$ satisfies the differential equation (3.23), the unknown coefficients $\{k_i\}$ and the order N in the latter equation can be determined

by requiring the following expression

$$\sum_{i=1}^N \int_0^T (k_0 \chi_i + k_1 \chi_i^{(1)} + \dots + k_{N-1} \chi_i^{(N-1)} + \chi_i^{(N)})^2 dt \quad (3.32)$$

$$(i = 1, 2, \dots, N)$$

be a minimum with respect to all possible choices of the coefficients $\{k_i\}$ and order N .

After the coefficients $\{k_i\}$ and N of the equation (3.23) are determined, the roots $\{\psi_i\}$ of the characteristic equation (3.24) can be numerically obtained. Substituting the general solution of $H(t)$ by Eq.(3.26) into Eq.(3.27), we have

$$\varepsilon(t) = \varepsilon_0 + \sigma_0 \sum_{l=1}^N B_l [\exp(\psi_l t) - 1] \quad (3.33)$$

where $B_l = A_l / \psi_l$ ($l = 1, 2, \dots, N$)

Finally, the coefficients $\{B_l\}$, equation (3.33) are determined with the inclusion of the experimental creep data within the specific range of strain input and the extent of time considered.

$$\varepsilon_i(t) = \varepsilon_0 + \sigma_0 \sum_{l=1}^N B_l [\exp(\psi_l t) - 1] + e_i(t) \quad (3.34)$$

where $e_i(t)$ are error functions. By requiring the following

$$\sum_{i=1}^n \alpha_i \int_0^T [e_i(t)]^2 dt \quad (3.35)$$

to be minimum, the constants B_i can be determined. In the Eq.(3.35), α_i represent some suitable weighting factors.

In the relaxation case, the above procedure can be exactly followed to obtain the relaxation function from the quasi-static relaxation experimental measurements.

Practical numerical examples were given in the referred to reference [Haddad and Tanary, 1987 and 1989]. The results show that the differential approximation method is a reasonable method and the model is accurate and efficient. It is obvious that this method gave a significant improvement to the classical method on the following fact: in the classical method presented earlier in this section, the order N and the parameters $\{\tau_i\}$ of the method (Eq.(3.21)) are actually determined arbitrarily, while in the differential approximation method, the pertaining parameters of the model (Eq.(3.26)) are determined by differential approximation with the inclusion of the viscoelastic data.

3.2 The Time-Temperature Superposition Method

In the quasi-static method, the experimental data are obtained from direct simple creep and relaxation tests. By "simple" here, one means that

a) the experiments are conducted directly from the definition of creep or relaxation function as mentioned previously in Chapter 2, sections 2.1 and 2.2.

b) the experiments are conducted at constant temperatures.

As pointed out in section 3.1, to obtain enough experimental data which can represent the characteristics of creep or relaxation of a certain viscoelastic material, one usually requires very long period of time (at least weeks or months) to finish a creep or relaxation experiment. This requirement of very long time often makes carrying a quasi-static experiment an unrealistic task.

To overcome the fundamental problem mentioned above, and to obtain the creep or relaxation data in a relatively short period of time, the time-temperature superposition method is usually employed. This method has been developed on the basis of a particular property of viscoelastic materials, that is, the " time-temperature equivalence " property.

Due to the temperature influence on the viscoelastic response of materials, the creep and relaxation functions are not only functions of time, but also functions of temperature [Plazek, 1965]. Thus in general terms, the creep and relaxation functions may be expressed respectively as follows

$$\begin{aligned} C &= C(T, t) \\ R &= R(T, t) \end{aligned} \tag{3.36}$$

where "T" represents the temperature.

For a large class of linear viscoelastic materials, the effect of temperature on the creep or the relaxation function is as shown in Fig.3.1, in which one can observe that temperature change causes a shift of the creep or relaxation function along the time axis.

The special property of the viscoelastic materials shown in Fig.3.1 can be expressed mathematically as

$$\begin{aligned}
C(T, t) &= C(T_0, \zeta) \\
R(T, t) &= R(T_0, \zeta) \\
\zeta &= \frac{t}{a_T(T)} \\
\log \zeta &= \log t - \log(a_T(T))
\end{aligned}
\tag{3.37}$$

here t is the actual time of observation measured from first application of load, T and T_0 are absolute base temperatures where T is the test temperature and T_0 is a selected reference temperature and ζ is referred to as the "reduced time parameter". As indicated by Eq.(3.37), the reduced time ξ is related to the real time t by a factor $a_T(T)$, which is called the "temperature shift factor". Equation (3.37) means that the effect of changing temperature is the same as applying a multiplication factor to the time scale, or an additive factor to the *log* time-scale. This concept is referred to as the "time-temperature superposition principle". The materials with such properties are termed as thermorheologically simple materials[see, e.g., Schwarzl and Staverman, 1952; Haddad, 1988].

The implication of the time-temperature superposition principle is significant. It states that the creep function or relaxation function at a temperature can be shifted along the time axis by a factor to obtain the creep function or relaxation function at another temperature. Usually, at a high temperature, the creep and relaxation process can proceed at a faster rate. Thus, by employing the time-temperature superposition principle, one can determine the corresponding creep or relaxation behaviour at room temperature, which would, otherwise, require a much longer time to be established by simple quasi-static experiment.

Equation (3.37) is widely used in the practice and was proven to be successful.

Aklonis[1972], however, argued that when shifting experimental data, one additional conversion is necessary. By shifting creep and relaxation curves horizontally (along the time axis), we compensate for a change in the time scale which is brought about by changing temperature. In addition to the change of the time scale, there is another inherent change in the relaxation or creep function, which is also caused by a change in temperature, that is, there must be a vertical shift of the response curves due to the temperature variation. In addition, he concluded that the vertical shift is necessary by the fact that volume of a viscoelastic material is a function of temperature. Therefore the density is obviously the parameter that must be used in the correction. These consideration leads to the following equations concerning the creep and relaxation functions respectively, [see, Aklonis, 1972]

$$\frac{C(T_1, t)}{\rho(T_1) T_1} = \frac{C(T_2, \frac{t}{a_T})}{\rho(T_2) T_2} \quad (3.38)$$

$$\frac{R(T_1, t)}{\rho(T_1) T_1} = \frac{R(T_2, \frac{t}{a_T})}{\rho(T_2) T_2}$$

where ρ is the material density.

In (3.38), division by the temperature accounts for the changes in the values of the material function $C(T,t)$ or $R(T,t)$ due to the inherent dependence of these function on temperature, while the division by the density accounts for the change of matter per unit volume with temperature variation.

If one chooses a reference temperature T_0 , then, the creep or the relaxation function at

the reference temperature can be obtained from the creep or relaxation function at other temperatures by

$$\begin{aligned}
 C(T_0, t) &= \frac{\rho(T_0) T_0}{\rho(T) T} C(T, \frac{t}{a_T}) \\
 R(T_0, t) &= \frac{\rho(T_0) T_0}{\rho(T) T} R(T, \frac{t}{a_T})
 \end{aligned}
 \tag{3.39}$$

The complete creep or the relaxation function at the reference temperature is referred to as the "Master Curve". It is obvious that to obtain a master curve, from equation (3.39), the creep or the relaxation experiments can be carried out at different temperatures. The corresponding viscoelastic experimental curves from these tests can then be shifted horizontally and vertically to the reference temperature T_0 , and combined to form the required master curve. After one obtains the master curve, the methods described in the previous section, such as the differential approximation method [see, e.g., Haddad and Tanary, 1987 and 1989] can be used to establish the model of the creep or the relaxation function.

To use the time-temperature superposition principle, the shift factor has to be determined first. It is now common practice to reduce most relaxation or creep data to the temperature T_g , the glass transition temperature[Aklonis, 1972] of the material. Then the shift factor is given by

$$\log a_T = \frac{-C_1 (T - T_g)}{C_2 + (T - T_g)}
 \tag{3.40}$$

This equation is called the "WLF equation" and it is due to Williams, Landel and Ferry[see, e.g., Aklonis, 1972]. The constants C_1 and C_2 in equation (3.40) were originally

"universal constants" and were given, respectively, the values

$$\begin{aligned} C_1 &= 17.4 \\ C_2 &= 51.6 \end{aligned} \tag{3.41}$$

But later, they have been shown to vary slightly from one linear viscoelastic material to another. Fig.3.2 gives the variation of $\log a_T$ versus the temperature difference $T-T_g$.

It should, however, be pointed out that, although the time-temperature superposition principle has been proven to be successful and generally applicable to most viscoelastic materials, Plazek[1965] has shown that it is not quantitatively correct except in limited time-temperature ranges.

3.3 Dynamic Methods

The quasi-static and time-temperature superposition methods described in the foregoing section are both time-domain methods, in which the time behaviour of creep (or relaxation) process is measured and directly analyzed in the time domain. Another alternative is the dynamic method, which mainly deals with the properties of viscoelastic materials in the frequency domain.

In the dynamic method, the acquisition of experimental data is different from the above mentioned two methods. Instead of applying a step loading to the viscoelastic specimen, the dynamic method applies a sinusoidal strain (or stress) to the specimen, and measures simultaneously the stress (or strain) response of the viscoelastic material [Szyszkowski and

Glockner, 1985].

For a linear viscoelastic material, when the input is sinusoidal, the output signal will also be sinusoidal, but the phase is shifted with respect to the input, as shown in Fig.3.3,

In a dynamic relaxation experiment, let the strain input be

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \quad (3.42)$$

The stress response will be

$$\sigma(t) = \sigma_0 \sin(\omega t + \delta) \quad (3.43)$$

where ω and δ are, respectively, the frequency and the phase shift.

Expanding the stress response, Eq.(3.43), one has

$$\sigma(t) = (\sigma_0 \cos\delta) \sin(\omega t) + (\sigma_0 \sin\delta) \cos(\omega t) \quad (3.44)$$

Thus we see that the stress can be considered to consist of two components: one component of magnitude $\sigma_0 \cos\delta$ in phase with the strain, and the other component is of magnitude $\sigma_0 \sin\delta$ and it is 90° out of phase with the strain.

The stress-strain relationship Eq.(3.44) can therefore be defined by a modulus E_1 in phase with the strain and by a modulus E_2 which is 90° out of phase with the strain, i.e.,

$$\sigma(t) = \varepsilon_0 E_1 \sin(\omega t) + \varepsilon_0 E_2 \cos(\omega t) \quad (3.45)$$

where E_1 and E_2 are defined, respectively, as

$$E_1 = \frac{\sigma_0}{\epsilon_0} \cos\delta$$
$$E_2 = \frac{\sigma_0}{\epsilon_0} \sin\delta$$
(3.46)

With reference to equations (3.45) and (3.46), E_1 is referred to as "the storage modulus", and E_2 is termed as "the loss modulus". The modulus E_1 is associated with the energy storage and release in the periodic deformation, while E_2 is associated with the dissipation or loss of energy as heat. The ratio of the loss modulus to the storage modulus is called "the loss tangent", which is expressed as

$$\tan\delta = \frac{E_2}{E_1}$$
(3.47)

If we change the frequency of strain input, the storage modulus, loss modulus and loss tangent will all change, i.e., E_1 , E_2 and δ are all the functions of frequency. Typical variations of E_1 , E_2 and δ with frequency are shown in Fig.3.4.

Similarly, in a dynamic creep experiment, let the stress input be

$$\sigma(t) = \sigma_0 \sin(\omega t)$$
(3.48)

the resulting strain response is then

$$\begin{aligned}
\varepsilon(t) &= \varepsilon_0 \sin(\omega t - \delta) \\
&= (\varepsilon_0 \cos\delta) \sin(\omega t) - (\varepsilon_0 \sin\delta) \cos(\omega t)
\end{aligned}
\tag{3.49}$$

Thus one can define a storage compliance J_1 and a loss compliance J_2 as

$$\begin{aligned}
J_1 &= \frac{\varepsilon_0}{\sigma_0} \cos\delta \\
J_2 &= \frac{\varepsilon_0}{\sigma_0} \sin\delta \\
\tan\delta &= \frac{J_2}{J_1}
\end{aligned}
\tag{3.50}$$

The storage compliance and loss compliance are also functions of frequency.

Equations (3.46) and (3.50) immediately suggest complex representation for both the modulus and compliance. For the complex modulus, we have:

$$\begin{aligned}
E(i\omega) &= \frac{\sigma_0}{\varepsilon_0} e^{i\delta} \\
&= \frac{\sigma_0}{\varepsilon_0} (\cos\delta + i \sin\delta) \\
&= E_1 + i E_2
\end{aligned}
\tag{3.51}$$

For the complex compliance, we have

$$\begin{aligned}
J(i\omega) &= \frac{\varepsilon_0}{\sigma_0} e^{-i\delta} \\
&= \frac{\varepsilon_0}{\sigma_0} (\cos\delta - i \sin\delta) \\
&= J_1 - i J_2
\end{aligned}
\tag{3.52}$$

From equations (3.51) and (3.52), one sees that the product of $E(i\omega)$ and $J(i\omega)$ are unity, however, their individual components are not reciprocally related, and given by[see, for instance, Lee, 1960b and Ferry, 1970;]

$$\begin{aligned}
 J_1 &= \frac{1}{E_1 + \tan^2\delta} \\
 J_2 &= \frac{1}{E_2 + (\tan^2\delta)^{-1}} \\
 E_1 &= \frac{1}{J_1 + \tan^2\delta} \\
 E_2 &= \frac{1}{J_2 + (\tan^2\delta)^{-1}}
 \end{aligned}
 \tag{3.53}$$

Therefore, in a dynamic experiment, we apply a sinusoidal strain (or stress) to a specimen, and measure the stress (or strain) simultaneously. Then we can obtain the storage modulus (or compliance) and loss modulus (or compliance) of the linear viscoelastic material at the given frequency. By changing the frequency, one obtains the values of modulus (or compliance) at other levels of frequency. Thus, in an experiment of N levels of frequency, one can obtain the following sets for the components of the material complex compliance and modulus [Struik, 1987 and Orbey and Ded, 1991],

$$\begin{aligned}
& J_1(\omega_1), J_1(\omega_2), \dots, J_1(\omega_N) \\
& J_2(\omega_1), J_2(\omega_2), \dots, J_2(\omega_N) \\
& E_1(\omega_1), E_1(\omega_2), \dots, E_1(\omega_N) \\
& E_2(\omega_1), E_2(\omega_2), \dots, E_2(\omega_N)
\end{aligned} \tag{3.53}$$

Combining the above measurements, one can determine two discrete series of the corresponding complex modulus and complex compliance

$$\begin{aligned}
& E(i\omega_1), E(i\omega_2), \dots, E(i\omega_N) \\
& J(i\omega_1), J(i\omega_2), \dots, J(i\omega_N)
\end{aligned} \tag{3.55}$$

From the foregoing dynamic measurements, one needs to determine the creep and relaxation functions. In chapter 2, the relation between the complex modulus and the rate of relaxation function has been determined by equation (2.21), and the relation between the complex compliance and the rate of creep function have been established by equation (2.21). They are related with each other by Fourier transform pairs. For the relationship between the creep function and the complex compliance, we have

$$\begin{aligned}
J(i\omega) &= \int_{-\infty}^{\infty} \dot{C}(t) e^{-i\omega t} dt \\
\dot{C}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} J(i\omega) e^{i\omega t} d\omega
\end{aligned} \tag{3.56}$$

For the relationship between the relaxation function and the complex modulus, we have

$$E(i\omega) = \int_{-\infty}^{\infty} \dot{R}(t) e^{-i\omega t} dt$$

$$\dot{R}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(i\omega) e^{i\omega t} d\omega$$
(3.57)

If the inverse Fourier transform is denoted by $F^{-1}(\cdot)$, the creep and relaxation functions are, then, given, respectively, by

$$C(t) = \int_0^t F^{-1} [J(i\omega)] dt + C(0)$$

$$R(t) = \int_0^t F^{-1} [E(i\omega)] dt + R(0)$$
(3.58)

where $C(0)$ and $R(0)$ are, respectively, the initial values of the creep and the relaxation functions at $t=0$.

It is obvious that in order to obtain the corresponding creep and the relaxation functions from dynamic measurements, the integrals in Eq.(3.58) have to be evaluated. During the early period of development of viscoelasticity theory, the application of computers was not as widely used as today. The evaluation of integrals such as those given in Eq.(3.58) was considered to be a difficult problem. Thus Ninomiya and Ferry[see, Roscoe, 1950 and Ferry, 1970] devised two approximate formulae to fulfil the transformation of dynamic parameters to creep and relaxation functions. They are, respectively,

$$\begin{aligned}
C(t) &= J_1(\omega) - 0.40 j_2(0.40\omega) + 0.014 J_2(10\omega) \Big|_{\omega=\frac{1}{t}} \\
R(t) &= E_1(\omega) + 0.40 E_2(0.40\omega) - 0.014 E_2(10\omega) \Big|_{\omega=\frac{1}{t}}
\end{aligned}
\tag{3.59}$$

Through these two equations, the creep and the relaxation functions are related to complex compliance and complex modulus, respectively, at three different frequency points, that is:

$$\begin{aligned}
\omega_1 &= \frac{1}{t} \\
\omega_2 &= \frac{0.4}{t} \\
\omega_3 &= \frac{10}{t}
\end{aligned}
\tag{3.60}$$

Equations (3.59) were widely used in practice. But from Eq.(3.56) we know that the creep function is dependent on the distribution of the complex compliance in the whole frequency range. Similarly, the relaxation function is dependent on the distribution of the complex modulus in the whole frequency range. Indeed, Ninomiya and Ferry expressions (3.59) are very coarse. To overcome this problem, Gibson, Hwang and Sheppard[1990] employed the Fast Fourier Transformation (FFT) technique. The later technique is based on the Discrete Fourier Transform (DFT)[e.g., Brigham, 1974]. The DFT pairs corresponding to equations (3.56) and (3.57) are as follows

$$\begin{aligned}
J\left(\frac{n}{N \cdot \Delta T}\right) &= \Delta T \sum_{k=0}^{N-1} \dot{C}(k \cdot \Delta T) e^{\frac{i 2\pi n k}{N}} \\
\dot{C}(k \cdot \Delta T) &= \Delta f \sum_{n=0}^{N-1} J\left(\frac{n}{N \cdot \Delta T}\right) e^{\frac{i 2\pi n k}{N}} \\
E\left(\frac{n}{N \cdot \Delta T}\right) &= \Delta T \sum_{k=0}^{N-1} \dot{R}(k \cdot \Delta T) e^{\frac{-i 2\pi n k}{N}} \\
\dot{R}(k \cdot \Delta T) &= \Delta f \sum_{n=0}^{N-1} E\left(\frac{n}{N \cdot \Delta T}\right) e^{\frac{i 2\pi n k}{N}}
\end{aligned} \tag{3.61}$$

where $n = 0, 1, \dots, N-1$

N = number of samples

$k = 0, 1, \dots, N-1$

ΔT = time interval

Δf = frequency interval, must equal to $1/(N \cdot \Delta T)$

It should be pointed out that in equations (3.55) and (3.56), the frequency ω is in radian per second, while in equation (3.61), the frequency is in Hz.

FFT and inverse FFT can be used very effectively in the calculation of Discrete Fourier Transform and the inverse Discrete Fourier Transform. From the dynamical experiments, one can obtain the discrete series of complex modulus and complex compliance (equation (3.55)), therefore, equation (3.61) can be used to obtain the rate of both the creep and relaxation functions. Then, through equation (3.58), one can finally obtain the creep and relaxation functions by numerical integration.

3.4 Concluding Remarks

In this chapter, we have classified the characterization methods of viscoelastical response into three categories, i.e., quasi-static, time-temperature superposition, and dynamic methods. A brief review on the three characterization methods have been presented.

From the description of these methods, it is clear that the dynamic experimental method is an efficient one to conduct, as the quasi-static method requires a very long period of time, and the time-temperature superposition method would require complex temperature control facilities.

But the described dynamic analysis method of linear viscoelasticity serves only as an alternative way to obtain the creep and relaxation data. It first calculates the complex modulus and complex compliance from the measured oscillating strain or stress experiment, then converts the complex modulus and complex compliance to time domain obtaining the creep and relaxation data. Finally, one establishes the models of creep and relaxation functions from those data. The whole analysis process could be time-consuming and not very accurate.

The following chapters of this thesis are based on the measured oscillating strain and stress from dynamical experiments, where one develops a new method to establish the models of creep and relaxation functions directly from the measured oscillating strain and stress.

CHAPTER 4

A New Method for the Identification of a Dynamic System

As it is well known, a fundamental task of most physical sciences is the establishment of a mathematical model, which is essential for the analysis, prediction and even control of the physical process. Basically, models can be obtained through the following two approaches, i.e.,

1. physical reasoning from the observations of the behaviour of the physical process.
2. mathematical modelling as based on the analysis of the experimental data pertaining to the system.

The first approach is mainly based on the analysis of the basic properties, basic principles of the system. The second approach, however, does not concern itself about what kind of physical system really is. It analyzes and establishes a model from the experimental input and output signals of the system [Kolsky, 1967].

In this chapter, adopting the second approach, we develop a new method for the identification of dynamic system [Yu and Haddad, 1994a and 1994b]. This method will act, then, as an important basis for further development of the characterization procedure of the mechanical response of viscoelastic materials.

4.1 Introduction

A system is actually a very abstract and general concept. A specimen in a vibration test

is a system. An airplane, for example, can also be considered as a system, but larger and more complicated. It is convenient to visualize a system schematically by a box, as shown in Fig.4.1 where $x(t)$ and $y(t)$ represent, respectively, the input and the output pertaining to the system.

Symbolically, the dependence of the system output on the input can be expressed as

$$y(t) = \Omega[x(t)] \quad (4.1)$$

where the symbol $\Omega[\cdot]$ represents the system, and it is referred to as an operator, or a functional.

In the present analysis, we assume the system $\Omega[\cdot]$ is linear. Mathematically, a linear system is one which satisfies the following criterion. If

$$\begin{aligned} y_1(t) &= \Omega[x_1(t)] \\ y_2(t) &= \Omega[x_2(t)] \end{aligned} \quad (4.2)$$

and α_1 and α_2 are constants, then

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = \Omega[\alpha_1 x_1(t) + \alpha_2 x_2(t)] \quad (4.3)$$

Usually, we are interested in obtaining explicit relationships between the input and output variables of a system. Let the input $x(t)$ be an arbitrary continuous function of time in the time interval $T_1 \leq t \leq T_2$ and zero elsewhere, as shown in Fig.4.2. We can approximate $x(t)$ by a staircase function $\hat{x}(t)$ [Rodger, 1983 and Smith, 1989], that is

$$\begin{aligned}
\hat{x}(t) &= \sum_{n=N_1}^{N_2} x(n \cdot \Delta\lambda) U\left(\frac{t - n \cdot \Delta\lambda}{\Delta\lambda}\right) \\
&= \sum_{n=N_1}^{N_2} x(n \cdot \Delta\lambda) \frac{1}{\Delta\lambda} U\left(\frac{t - n \cdot \Delta\lambda}{\Delta\lambda}\right) \cdot \Delta\lambda \quad ; \quad T_1 \leq t \leq T_2
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
T_1 &= \left(N_1 - \frac{1}{2}\right) \Delta\lambda \\
T_2 &= \left(N_2 + \frac{1}{2}\right) \Delta\lambda
\end{aligned} \tag{4.5}$$

In the equation (4.4), we introduced the unit pulse function, which is defined as:

$$U(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

From unit pulse function $U(t)$, we can construct another function $\delta_\varepsilon(t)$

$$\begin{aligned}
\delta_\varepsilon(t) &= \frac{1}{2\varepsilon} U\left(\frac{t}{2\varepsilon}\right) \\
&= \begin{cases} \frac{1}{2\varepsilon} & |t| < \varepsilon \\ 0 & |t| > \varepsilon \end{cases}
\end{aligned} \tag{4.7}$$

It is obvious that if we let $\varepsilon \rightarrow 0$, we obtain the following $\delta(t)$ function, that is

$$\delta(t) = \lim_{\epsilon \rightarrow 0} \delta_\epsilon(t) \quad (4.8)$$

From Eq.(4.7), we have

$$\frac{1}{\Delta\lambda} U\left(\frac{t - n \cdot \Delta\lambda}{\Delta\lambda}\right) = \delta_{\frac{1}{2}\Delta\lambda}(t - n \cdot \Delta\lambda) \quad (4.9)$$

In terms of the $\delta_\epsilon(t)$ function of equation (4.7), Eq.(4.4) can be expressed as:

$$\hat{x}(t) = \sum_{n=N_1}^{N_2} x(n \cdot \Delta\lambda) \delta_{\frac{1}{2}\Delta\lambda}(t - n \cdot \Delta\lambda) \cdot \Delta\lambda \quad (4.10)$$

Now consider the output of system $\Omega[\cdot]$ corresponding to the input of Eq.(4.10), i.e.

$$\begin{aligned} \hat{y}(t) &= \Omega[\hat{x}(t)] \\ &= \Omega\left[\sum_{n=N_1}^{N_2} x(n \cdot \Delta\lambda) \delta_{\frac{1}{2}\Delta\lambda}(t - n \cdot \Delta\lambda) \cdot \Delta\lambda\right] \end{aligned} \quad (4.11)$$

Because $\Omega[\cdot]$ is a linear system, from Eq.(4.3), one has

$$\hat{y}(t) = \sum_{n=N_1}^{N_2} x(n \cdot \Delta\lambda) \Omega[\delta_{\frac{1}{2}\Delta\lambda}(t - n \cdot \Delta\lambda)] \cdot \Delta\lambda \quad (4.12)$$

By denoting the unit pulse response of the system by $\hat{h}(t)$, that is,

$$\hat{h}(t) = \Omega[\delta_{\frac{1}{2}\Delta\lambda}(t)] \quad (4.13)$$

Eq.(4.12) becomes

$$\hat{y}(t) = \sum_{n=N_1}^{N_2} x(n \cdot \Delta\lambda) \hat{h}(t - n \cdot \Delta\lambda) \cdot \Delta\lambda \quad (4.14)$$

We know that a characteristic function of a system, the impulse response, is the response to a unit impulse applied at $t=0$ with all initial conditions of the system equal to zero, that is

$$h(t) = \Omega[\delta(t)] \quad (4.15)$$

Hence, from equations (4.8) and (4.13), it is obvious that

$$h(t) = \lim_{\Delta\lambda \rightarrow 0} \hat{h}(t) \quad (4.16)$$

Returning to Eq.(4.14), and taking the limit as $\Delta\lambda \rightarrow 0$ and $n \cdot \Delta\lambda \rightarrow \lambda$, we recognize that Eq.(4.14) becomes

$$\begin{aligned} y(t) &= \lim_{\Delta\lambda \rightarrow 0} \hat{y}(t) \\ &= \int_{T_1}^{T_2} x(\lambda) h(t-\lambda) d\lambda \quad T_1 \leq t \leq T_2 \end{aligned} \quad (4.17)$$

Assuming that the input may have been present since the infinite past and may last indefinitely into the future, we have, in the limit as $T_1 \rightarrow -\infty$, and $T_2 \rightarrow +\infty$

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \quad (4.18)$$

Making the substitution $\xi = t - \lambda$, we obtain the equivalent result

$$y(t) = \int_{-\infty}^{\infty} x(t-\xi)h(\xi)d\xi \quad (4.19)$$

Equations (4.18) and (4.19) are the integral representations of a linear system in terms of the impulse response function of the system.

It should be pointed out that for a causal system, the impulse response function has the following properties

$$h(t) = 0 \quad t < 0 \quad (4.20)$$

Almost every physical system is a causal system, because the output of a system can not be dependent on the future input. Therefore, for a causal system, we have

$$y(t) = \int_0^{\infty} x(t-\xi)h(\xi)d\xi \quad (4.21)$$

From equations (4.18), (4.19) or (4.21), if the input $x(t)$ and the system characteristic function $h(t)$ are known, the output of the system can be easily obtained by integration. Now, our problem is an inverse problem, that is:

If the output $y(t)$ and input $x(t)$ have been obtained from experiments, we would like to determine the system, that is, to establish the characteristic function $h(t)$ of the system. It is the so-called system identification problem [Ljung, 1987]. This chapter will develop a new method for such a problem.

4.2 Theory

Equations (4.18) and (4.19) are the integral representation of the relationship between the input and output of a system in the time domain. Here we would like to obtain a representation of the relationship in frequency domain. Therefore, Fourier transform and its inverse [Bracewell, 1978], i.e.,

$$G(i\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt \quad (4.22)$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega)e^{i\omega t} d\omega$$

are employed. Here $g(t)$ is a time function, $G(i\omega)$ is the Fourier transform of $g(t)$, and $i = \sqrt{-1}$

Denoting by $Y(i\omega)$ and $X(i\omega)$ the Fourier transforms of the output $y(t)$ and input $x(t)$ respectively, and taking Fourier transform of Eq.(4.19) (or Eq.(4.18)), one has

Let $\phi = t - \xi$, we have

$$\begin{aligned}
Y(i\omega) &= \int_{-\infty}^{\infty} y(t)e^{-i\omega t} dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t-\xi)h(\xi)d\xi \right) e^{-i\omega t} dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\xi)e^{-i\omega\xi} x(t-\xi)e^{-i\omega(t-\xi)} d\xi \right) dt
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
Y(i\omega) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(\phi)e^{-i\omega\phi} d\phi \right) h(\xi)e^{-i\omega\xi} d\xi \\
&= \int_{-\infty}^{\infty} X(\phi)e^{-i\omega\phi} d\phi \cdot \int_{-\infty}^{\infty} h(\xi)e^{-i\omega\xi} d\xi
\end{aligned} \tag{4.24}$$

In the above equation, the first integral is the Fourier transform of the input $x(t)$ and the second integral is the Fourier transform of the impulse response function of the system, i.e.,

$$H(i\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt \tag{4.25}$$

in which, $H(i\omega)$ is usually called the frequency response function of a system

Thus, we have the following relation in frequency domain

$$Y(i\omega) = H(i\omega) X(i\omega) \tag{4.26}$$

Eq.(4.26) is obviously an algebraic equation. Thus, in time domain, the input and output relation is governed by an integral equation, while in the frequency domain, the input and output

relation is governed by an algebraic equation. It is why sometimes the analysis in frequency domain is more convenient.

Because the system characteristic function $h(t)$ is unidentified, the frequency response function $H(i\omega)$ is unknown. To model the behaviour of the system, we have to invoke the approximation theory[see, e.g., Dunham, 1930]. According to the theory of approximation, any continuous function can be approximated by a rational function with any desired accuracy. Therefore we assume, in the frequency domain, that the frequency response function $H(i\omega)$ can be expressed by

$$\begin{aligned}
 H(i\omega) &= \frac{a}{(i\omega)^p + b_1(i\omega)^{p-1} + \dots + b_{p-1}(i\omega) + b_p} \\
 &= \frac{a}{\sum_{k=0}^p b_k(i\omega)^{p-k}}
 \end{aligned}
 \tag{4.27}$$

where $a, b_0, b_1, b_2, \dots, b_p$ and p are constant parameters with $b_0=1$, and p is the order of the denominator polynomial.

The rational function of Eq.(4.27) for the frequency response function is the basic model we assumed for the system. From this model, and with reference to Eq.(4.26), we have

$$Y(i\omega) = \frac{a}{\sum_{k=0}^p b_k(i\omega)^{p-k}} X(i\omega)
 \tag{4.28}$$

Rearranging the above equation, one further has

$$\sum_{k=0}^p b_k (i\omega)^{p-k} Y(i\omega) = a X(i\omega) \quad (4.29)$$

By the differentiation theorem of Fourier transform[e.g., Bracewell, 1978], i.e.

$$\frac{d^n g(t)}{dt^n} \leftrightarrow (i\omega)^n G(i\omega) \quad (4.30)$$

and taking the inverse Fourier transform of Eq.(4.29), one obtains the model of the system in the time domain

$$\sum_{k=0}^p b_k y^{(p-k)}(t) = a x(t) \quad (4.31)$$

where superscript " $(p-k)$ " represents the $(p-k)$ -th differentiation order of $y(t)$ with respect to the time t . Explicitly, we have the following p -th order differential equation

$$\frac{d^p}{dt^p} y(t) + b_1 \frac{d^{p-1}}{dt^{p-1}} y(t) + \dots + b_p y(t) = a x(t) \quad (4.32)$$

The differential equation model Eq.(4.32) in the time domain is the very interesting result from the corresponding frequency model. In view of the forgoing analysis, one can consider that using a rational function in frequency domain to approximate the system frequency response function means using a differential equation in the time domain to model the behaviour of the system.

Assume the p -th order equation

$$\xi^p + b_1 \xi^{p-1} + \dots + b_{p-1} \xi + b_p = 0 \quad (4.33)$$

has roots: $\xi_1, \xi_2, \dots, \xi_p$. Eq.(4.27) can, then, be written as

$$H(i\omega) = \frac{a}{(i\omega - \xi_1)(i\omega - \xi_2) \dots (i\omega - \xi_p)} \quad (4.34)$$

Further, the above equation can be expressed in a partial fraction form as

$$\begin{aligned} H(i\omega) &= \frac{A_1}{i\omega - \xi_1} + \frac{A_2}{i\omega - \xi_2} + \dots + \frac{A_p}{i\omega - \xi_p} \\ &= \sum_{m=1}^p \frac{A_m}{i\omega - \xi_m} \end{aligned} \quad (4.35)$$

Where A_m ($m = 1, 2, \dots, p$) are constants corresponding to root ξ_m , and are calculated by the following procedures.

Multiplying $H(i\omega)$ by $(i\omega - \xi_m)$, we have

$$\begin{aligned} H(i\omega)(i\omega - \xi_m) &= A_1 \frac{i\omega - \xi_m}{i\omega - \xi_1} + \dots + A_{m-1} \frac{i\omega - \xi_m}{i\omega - \xi_{m-1}} \\ &\quad + A_m + A_{m+1} \frac{i\omega - \xi_m}{i\omega - \xi_{m+1}} + \dots + A_p \frac{i\omega - \xi_m}{i\omega - \xi_p} \end{aligned} \quad (4.36)$$

From Eq.(4.36), it is obvious that

$$A_m = \lim_{i\omega \rightarrow \xi_m} H(i\omega)(i\omega - \xi_m) \quad (4.37)$$

$$(m = 1, 2, \dots, p)$$

Substituting Eq.(4.34) into Eq.(4.37) we obtain the formula to determine the constants A_m ($m = 1, 2, \dots, p$), i.e.,

$$A_m = \lim_{i\omega \rightarrow \xi_m} \frac{a (i\omega - \xi_m)}{(i\omega - \xi_1) (i\omega - \xi_2) \dots (i\omega - \xi_p)}$$

$$= \frac{a}{\prod_{\substack{k=1 \\ k \neq m}}^p (\xi_m - \xi_k)} \quad (4.38)$$

The system characteristic function $h(t)$ (or the impulse response function) and frequency response function $H(i\omega)$ are a Fourier transform pair. Taking the inverse Fourier transform of Eq.(4.35), one has

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(i\omega) e^{i\omega t} d\omega$$

$$= \frac{1}{2\pi} \sum_{m=1}^p \int_{-\infty}^{\infty} \frac{A_m}{i\omega - \xi_m} e^{i\omega t} d\omega \quad (4.39)$$

$$= \sum_{m=1}^p A_m e^{\xi_m t} u(t)$$

where $u(t)$ is the unit step function defined by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Thus, the system characteristic function can be written as

$$h(t) = \begin{cases} \sum_{m=1}^p A_m e^{\xi_m t} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (4.40)$$

Eq.(4.40) indicates that the system is a causal system. This is exactly a result which we want.

In terms of Eq.(4.40), the input-output relationship can be expressed as

$$\begin{aligned} y(t) &= \int_0^{\infty} x(t-\tau)h(\tau)d\tau \\ &= \sum_{m=1}^p A_m \int_0^{\infty} x(t-\tau)e^{\xi_m \tau}d\tau \end{aligned} \quad (4.41)$$

On the basis of approximation of the frequency response function by a rational function, which actually models the system by a differential equation in the time domain, we finally arrived at the integral representation of the input and output relation Eq.(4.41). Examining Eq.(4.41), we find that using the model Eq.(4.27), the identification of the system is reduced to the determination of the parameters A_m, ξ_m ($m = 1, 2, \dots, p$) and the order p from the input $x(t)$ and the output $y(t)$.

In practice, the input $x(t)$ and the output $y(t)$ signals are usually given in the form of discrete-time signals, that is, the time-continuous signals $x(t)$ and $y(t)$ are sampled into discrete series. Let ΔT represent the sampling interval. Thus from experimental measurements, we can obtain two discrete-time series in the form

$$\left. \begin{array}{l} x_i = x(\Delta T \cdot i) \\ y_i = y(\Delta T \cdot i) \end{array} \right\} \quad (i = 0, 1, 2, \dots) \quad (4.42)$$

To analyze the above two discrete-time series and develop a method to determine the parameters in Eq.(4.41), we introduce the concept of discrete-time system [see, e.g., Cadzow, 1970 and 1973; Freeman, 1965; Pandit and Wu, 1983 and Pandit, 1991]. In doing this, one assumes that the relation between $\{x_i\}$ and $\{y_i\}$ is governed by a discrete-time system. That is

$$y_i + \beta_1 y_{i-1} + \dots + \beta_{p-1} y_{i-p+1} + \beta_p y_{i-p} = \alpha x_i \quad (i = 0, 1, 2, \dots) \quad (4.43)$$

where $\alpha, \beta_1, \beta_2, \dots, \beta_p$ and p are constant parameters. Denoting:

$$\begin{aligned} Dy_i &= y_{i-1} \\ \psi(D) &= 1 + \beta_1 D + \dots + \beta_p D^p \end{aligned} \quad (4.44)$$

where " D " is called "single-step delay operator", Eq.(4.43) can, thus, be written as:

$$\psi(D)y_i = \alpha x_i \quad (4.45)$$

Denoting the z-transforms of $\{x_i\}$ and $\{y_i\}$ by $X(z)$ and $Y(z)$, that is[see Cadzow, 1973]

$$X(z) = \sum_{i=0}^{\infty} X_i z^{-i} \quad (4.46)$$

$$Y(z) = \sum_{i=0}^{\infty} y_i z^{-i}$$

where z is a complex variable. Taking the z-transform of Eq.(4.45), and using the fact that

$$\begin{aligned} \sum_{i=0}^{\infty} D y_i z^{-i} &= \sum_{i=0}^{\infty} y_{i-1} z^{-i} \\ &= z^{-1} Y(z) \end{aligned} \quad (4.47)$$

one has

$$(1 + \beta_1 z^{-1} + \dots + \beta_p z^{-p}) Y(z) = \alpha X(z) \quad (4.48)$$

Thus, it follows that

$$Y(z) = H_d(z) X(z) \quad (4.49)$$

in which, $H_d(z)$ is the transfer function of the discrete-time system, and from Eq.(4.48), it is obvious that

$$H_d(z) = \frac{\alpha}{1 + \beta_1 z^{-1} + \beta_2 z^{-2} + \dots + \beta_p z^{-p}} \quad (4.50)$$

By denoting the roots of the characteristic equation of $H_d(z)$, i.e.,

$$1 + \beta_1 \lambda^{-1} + \dots + \beta_p \lambda^{-p} = 0 \quad (4.51)$$

by $\lambda_1, \lambda_2, \dots, \lambda_p$, the transfer function of discrete-time system Eq.(4.50) can be written as:

$$\begin{aligned} H_d(z) &= \frac{\alpha}{(1-\lambda_1 z^{-1}) \dots (1-\lambda_p z^{-1})} \\ &= \frac{B_1}{1-\lambda_1 z^{-1}} + \dots + \frac{B_p}{1-\lambda_p z^{-1}} \\ &= \sum_{m=1}^p \frac{B_m}{1-\lambda_m z^{-1}} \end{aligned} \quad (4.52)$$

where B_m , corresponding to root λ_m , is calculated by

$$\begin{aligned} B_m &= \lim_{z \rightarrow \lambda_m} H_d(z) \cdot (1 - \lambda_m z^{-1}) \\ &= \frac{\alpha}{\prod_{\substack{k=1 \\ k \neq m}}^p (1 - \lambda_k \lambda_m^{-1})} \quad (m = 1, 2, \dots, p) \end{aligned} \quad (4.53)$$

Taking the inverse z-transform of the transfer function $H_d(z)$, one obtains the system characteristic series of the discrete-time system in the form

$$h_d(k) = \sum_{m=1}^p B_m \lambda_m^k u[k] ; \quad (k = 0, 1, 2, \dots) \quad (4.54)$$

where

$$u[k] = \begin{cases} 0 & k < 0 \\ 1 & k = 0, 1, 2, 3, \dots \end{cases} \quad (4.55)$$

is a discrete-time unit step function. Thus the discrete-time system is also a causal system

By the convolution property of the z-transform, and taking the inverse z-transform of Eq.(4.49), the relation between the input and output of the discrete-time system in the discrete-time domain can be expressed as

$$y_i = \sum_{k=0}^{\infty} h_d(k) x_{i-k} \quad (4.56)$$

($i = 0, 1, 2, \dots$)

Here we have to establish a relation between the system characteristic function $h(t)$ of a continuous-time system and the system characteristic series $h_d(k)$ of the corresponding discrete-time system. Therefore we approximate the integral representation Eq.(4.19) by

$$\begin{aligned} y(\Delta T \cdot i) &= \sum_{k=0}^{\infty} h(k \cdot \Delta T) x[\Delta T \cdot i - \Delta T \cdot k] \cdot \Delta T \\ &= \sum_{k=0}^{\infty} h(k \cdot \Delta T) x[\Delta T \cdot (i-k)] \cdot \Delta T \end{aligned} \quad (4.57)$$

($i = 0, 1, 2, \dots$)

From Eq.(4.42), one has

$$y_i = \sum_{k=0}^{\infty} h(k \cdot \Delta T) x_{i-k} \cdot \Delta T \quad (4.58)$$

($i = 0, 1, 2, \dots$)

By comparing equations (4.56) and (4.58), the following equation holds approximately

$$h(k \cdot \Delta T)_{\Delta T} = h_d(k) \quad (4.59)$$

Meantime, from equations (4.40) and (4.54), one has

$$\sum_{m=1}^p A_m e^{\xi_m k \Delta T} \cdot \Delta T = \sum_{m=1}^p B_m \lambda_m^k \quad (4.60)$$

In order that Eq.(4.60) holds, we can establish the following relation

$$\begin{aligned} \Delta T A_m &= B_m \\ e^{\xi_m \Delta T} &= \lambda_m \end{aligned} \quad (m = 1, 2, \dots, p) \quad (4.61)$$

In terms of the parameters of the discrete-time system, the parameters of the continuous time system can be expressed as

$$\begin{aligned} A_m &= \frac{1}{\Delta T} B_m \\ \xi_m &= \frac{1}{\Delta T} \ln \lambda_m \end{aligned} \quad (m = 1, 2, \dots, p) \quad (4.62)$$

Eq.(4.62) is the relation which determines the parameters of the model equation (4.27). If the parameters $(\alpha, p, \beta_m, m = 1, 2, \dots, p)$ of the discrete-time system are determined from the discrete-time series of the input signal $\{x_i\}$ and of the corresponding output signal $\{y_i\}$, then, the parameters $(a, p, b_m, m = 1, 2, \dots, p)$ for the continuous model equation (4.27) can be

determined from Eq.(4.62).

In what follows, we discuss the method to determine the order p and the parameters α and β_m ($m = 1, 2, \dots, p$) of the discrete-time system.

Choose arbitrarily an order p and parameters $\hat{\alpha}, \hat{\beta}_m$ ($m = 1, 2, \dots, p$) of a discrete system. Then, with reference to Eq.(4.43), one writes that

$$y_i + \hat{\beta}_1 y_{i-1} + \dots + \hat{\beta}_p y_{i-p} = \hat{\alpha} x_i + e_i \quad (i = 1, 2, \dots, N) \quad (4.63)$$

where e_i ($i = 1, 2, \dots, N$) is the error in choosing the values of $\hat{\alpha}, \hat{\beta}_m$ ($m = 1, 2, \dots, p$) and the order p .

The error e_i can be expressed as:

$$\begin{aligned} e_i &= y_i - (-y_{i-1}, \dots, -y_{i-p}, x_i) \begin{Bmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p \\ \hat{\alpha} \end{Bmatrix} \\ &= y_i - \{W_i\}^T \{\hat{\beta}\}, \quad (i = 1, 2, \dots, N) \end{aligned} \quad (4.64)$$

where the vectors in Eq.(4.64) are expressed by

$$\begin{aligned} \{W_i\}^T &= (-y_{i-1}, \dots, -y_{i-p}, x_i) \\ \{\hat{\beta}\}^T &= (\hat{\beta}_1, \dots, \hat{\beta}_p, \hat{\alpha}) \end{aligned} \quad (4.65)$$

By minimizing the sum of the error squared, i.e.

$$\begin{aligned}
e^2 &= \frac{1}{N} \sum_{i=p+1}^N e_i^2 \\
&= \frac{1}{N} \sum_{i=p+1}^N (y_i - \{W_i\}^T \{\hat{\beta}\}) (y_i - \{\hat{\beta}\}^T \{W_i\})
\end{aligned} \tag{4.66}$$

one has

$$\left(\frac{1}{N} \sum_{i=p+1}^N \{W_i\} \{W_i\}^T \right) \{\hat{\beta}\} = \frac{1}{N} \sum_{i=p+1}^N y_i \{W_i\} \tag{4.67}$$

Thus, for every choice of an order p , a corresponding $\{\beta\}$ can be determined by Eq.(4.67). Then, from Eq.(4.66), the error corresponding to the choice of the order p can be calculated. Since e^2 is a function of the order p , the choice of the order of the discrete-time system can be made by requiring that it will result in a minimum e^2 .

4.3 Numerical Examples

To test the analytical model developed in the previous section, a number of numerical illustrations will be carried out. The formalism of these illustrations is outlined as follows:

1. For a given system, calculate the response of the system under certain dynamic loading by a numerical method (Runge-Kutta Method[see, e.g., Morris, 1983] is employed). Then, two discrete-time series (One is the input to the system and the other is its response) are obtained.
2. Assume that no other knowledge about the system is given except the two discrete-time series mentioned by Eq.(4.42). Apply the Dynamic System

Identification Method developed before to the two discrete-time series. First, determine the parameters of the discrete-time system (DTS) function, then, secondly, determine the corresponding continuous system function.

Example (1)

Consider the system

$$\ddot{y} + 25\dot{y} + 100y = 100 \sin(t^{1.5}) \quad (4.68)$$

with the input $x(t)$ is

$$x(t) = 100 \sin(0.5 t^{1.5}) \quad (4.69)$$

The input and response of the system are plotted in Figures 4.3 and 4.4 respectively, with $\Delta T = 0.01$.

The parameters of this system are listed in Tables 1 and 2 below

Table 1

p	b ₁	b ₂	a
2	25.0	100.0	1

Table 2

A ₁	A ₂	ξ ₁	ξ ₂
0.06666	-0.06667	-5.0	-20.0

The errors for different DTS's are shown in Table 3.

Table 3

Order	First	Second	Third	Fourth
Error	0.254506E-02	0.971892E-05	0.234843E-04	0.877769E-03

It is clear from Table 3 that, the DTS of second order is the system with minimum error. The parameters of the pertaining second order DTS are listed in the Tables 4 and 5.

Table 4

Parameter	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\alpha}$
Value	-0.175611E+01	0.764908E+00	0.996642E-04

Table 5

Parameter	λ_1	λ_2
Value	0.95601	0.80010

where λ_1 and λ_2 are two roots of the characteristic equation of the corresponding DTS.

From Eq.(4.53), the parameters B_1 and B_2 for the discrete-time system can be calculated as follows:

$$B_1 = \frac{\alpha}{1 - \lambda_2 \lambda_1^{-1}} = 6.111 \times 10^{-4} \quad (4.70)$$

$$B_2 = \frac{\alpha}{1 - \lambda_1 \lambda_2^{-1}} = -5.111 \times 10^{-4}$$

Then from Eq.(4.61), The parameters for the continuous system can be easily obtained as

$$A_1 = \frac{1}{\Delta T} B_1 = 0.06111$$

$$A_2 = \frac{1}{\Delta T} B_2 = -0.5111$$

(4.71)

$$\xi_1 = \ln \lambda_1 \cdot \frac{1}{\Delta T} = -4.4987$$

$$\xi_2 = \ln \lambda_2 \cdot \frac{1}{\Delta T} = -22.302$$

The estimated and exact values of parameters of the corresponding continuous system are listed in Table 6.

Table 6

Parameter	A_1	A_2	ξ_1	ξ_2
Exact	0.06667	-0.06667	-5.0	-20.0
Estimated	0.06111	-0.05111	-4.4987	-22.302

Figures 4.5, 4.6, and 4.7 show the exact and estimated responses of the system using discrete time system of *1st*, *2nd* and *3rd* order DTS respectively.

Fig.4.8 shows the exact and estimated system characteristic functions $h(t)$ for the *2nd* order DTS.

In the following example, the same procedure in example (1) is followed.

Example (4)

Consider the system

$$\ddot{y} + 5.5\dot{y} + 2.5y = 100 \sin(t^{1.5}) \tag{4.72}$$

with the input $x(t)$ is

$$x(t) = 100 \sin(0.5 t^{1.5}) \tag{4.73}$$

The input and response of the system are plotted in Figures 4.9 and 4.10, respectively, with $\Delta T = 0.01$.

The parameters of this system are listed in Table 7 and 8 below

Table 7

p	b ₁	b ₂	a
2	5.5	2.5	1

Table 8

A_1	A_2	ζ_1	ζ_2
-0.22222	0.22222	-5.0	-0.5

Meantime, the errors for different DTS's are shown in Table 9.

Table 9

Order	First	Second	Third
Error	0.731543E+01	0.654397E-04	0.197870E-02

It is clear that from Table 9, the DTS of second order is the system with minimum error.

The parameters of the obtained second order DTS are listed in the Tables 10 and 11.

Table 10

Parameter	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\alpha}$
Value	-0.194887E+01	0.949087E+00	0.945613E-04

Table 11

Parameter	λ_1	λ_2
Value	0.99522	0.95364

where λ_1 and λ_2 are two roots of the characteristic equation of the corresponding DTS.

The estimated and exact values of parameters of the corresponding continuous system are listed in Table 12.

Table 12

Parameter	A_1	A_2	ξ_1	ξ_2
Exact	-0.2222	0.2222	-5.0	-0.5
Estimated	-0.216877	0.226333	-4.7469	-0.479146

Fig.4.11 shows the exact and estimated responses of the system from the obtained 2nd discrete time system.

Fig.4.12 shows the exact and estimated system characteristic functions $h(t)$ for the obtained 2nd order DTS.

CHAPTER 5

Dynamic System Identification Method for the Characterization of Viscoelastic Materials --- Method I

5.1 Some Basic Considerations

A linear viscoelastic material can be considered as a dynamic system. Therefore, the dynamic system identification method developed in the Chapter 4 can be applied to the characterization of the mechanical response of a viscoelastic material.

For linear viscoelastic behaviour, the general expressions of the constitutive relations between strain and stress are [Christensen, 1971]

$$\begin{aligned} \text{For stress relaxation} \quad \sigma(t) &= \int_{-\infty}^{\infty} \frac{d\varepsilon(\tau)}{d\tau} R(t - \tau) d\tau \\ \text{For creep} \quad \varepsilon(t) &= \int_{-\infty}^{\infty} \frac{d\sigma(\tau)}{d\tau} C(t - \tau) d\tau \end{aligned} \tag{5.1}$$

where $\sigma(t)$ is the time-dependent stress,

$\varepsilon(t)$ is the time-dependent strain,

$R(t)$ is the relaxation function of the material,

$C(t)$ is the creep function of the material.

Here it should be pointed out that by the convention of the research work in viscoelasticity, the loading is usually considered to begin at the time t equal to zero, i.e. usually we assume that

$$\begin{aligned}\varepsilon(t) &= 0 \\ \sigma(t) &= 0\end{aligned}\quad t < 0 \quad (5.2)$$

Further, both the relaxation function $R(t)$ and creep function $C(t)$ are usually defined for $t \geq 0$, whereby for $t < 0$, one has

$$\begin{aligned}R(t) &= 0 \\ C(t) &= 0\end{aligned}\quad t < 0 \quad (5.3)$$

Therefore, Equation (5.1), can be written, respectively, as

$$\sigma(t) = \int_0^{\infty} \frac{d\varepsilon(\tau)}{d\tau} R(t - \tau) d\tau \quad (5.4)$$

$$\varepsilon(t) = \int_0^{\infty} \frac{d\sigma(\tau)}{d\tau} C(t - \tau) d\tau \quad (5.5)$$

Mathematically, equations (5.4) and (5.5) have the same structure and both can be written in the same form. If in the situation of Eq.(5.4), we write

$$\begin{aligned}
 x(t) &= \frac{d(\varepsilon(t))}{dt} \\
 y(t) &= \sigma(t) \\
 h(t) &= R(t)
 \end{aligned}
 \tag{5.6}$$

and in the situation of Eq.(5.5), we write

$$\begin{aligned}
 x(t) &= \frac{d(\sigma(t))}{dt} \\
 y(t) &= \varepsilon(t) \\
 h(t) &= C(t)
 \end{aligned}
 \tag{5.7}$$

Then, both equations (5.4) and (5.5) can be written in the following general form

$$y(t) = \int_0^{\infty} x(\tau)h(t-\tau)d\tau
 \tag{5.8}$$

For simplicity, we shall refer to the situation of Eq.(5.6) as "dynamic relaxation experiment", and to the situation of Eq.(5.7) as "dynamic creep experiment". Thus, in the later analysis, Eq.(5.8) represents the dynamic relaxation experiment, and in this case $y(t)$ is the stress response, $x(t)$ is the rate of strain loading and $h(t)$ is the relaxation function. Similarly, Eq.(5.8) represents also the dynamic creep experiment, and in this case $y(t)$ is the strain response, $x(t)$ is the rate of loading stress and $h(t)$ is the creep function.

Thus, from Eq.(5.8), we can conclude that if one considers the viscoelastic material specimen as a dynamic system, then, the characterization of its rheological response would be

a process of identification of the corresponding system from dynamic measurements. The method developed in the previous chapter can, thus, be modified to apply to our present problem.

5.2 Theoretical Model

In the development of the dynamic system identification method in Chapter 4, we first transformed the problem into frequency domain by Fourier transform, assumed a rational function model for the frequency response function of the system, then, a discrete-time system method was introduced to analyze the input and output discrete-time series and to establish a procedure for the estimation of the order and parameters of the assumed model.

In this chapter, the same idea of Chapter 4 is employed. But in the analysis of viscoelastic properties, almost every function of time, such as strain, stress, creep and relaxation functions are usually considered to begin at time t equal to zero. Therefore Laplace transform may be more suitable for the analysis of the viscoelastic problem.

The Laplace transform pair

$$L(s) = \int_0^{\infty} l(t) e^{-st} dt$$

$$l(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} L(s) e^{st} ds$$
(5.9)

will then be employed in the analysis of this Chapter. Here $l(t)$ is a time function. $L(s)$ is the Laplace transform of $l(t)$. r is a real constant.

Denoting the Laplace transforms of $y(t)$, $x(t)$ and $h(t)$ of the equation (5.8) by $Y(s)$, $X(s)$

and $H(s)$, respectively, and taking Laplace transform of Eq.(5.8), we obtain the following relationship between input and output

$$Y(s) = H(s) X(s) \quad (5.10)$$

where the theorem of Laplace transform of the convolution of two signals has been used [e.g., Fodor, 1965].

In the dynamic relaxation experiment cases, $H(s)$ is

$$H(s) = \int_0^{\infty} R(t) e^{-st} dt \quad (5.11)$$

and in the dynamic creep experiment cases, $H(s)$ is

$$H(s) = \int_0^{\infty} C(t) e^{-st} dt \quad (5.12)$$

Before a model is assumed for $H(s)$, we have to analyze the special behaviour of the relaxation and creep functions at $t=0$. We know that both $R(t)$ and $C(t)$ are not continuous functions at $t=0$. Each is equal to zero at $t=0^-$, and is equal to a finite value at $t=0^+$.

From the initial value theorem of Laplace transform[see, Fodor, 1965], we have

$$\begin{aligned} \lim_{s \rightarrow \infty} s H(s) &= R(0^+) \quad ; \quad \text{for the relaxation case} \\ \lim_{s \rightarrow \infty} s H(s) &= C(0^+) \quad ; \quad \text{for the creep case} \end{aligned} \quad (5.13)$$

Because each of $R(O^+)$ and $C(O^+)$ is not equal to zero, we can assume the following rational function for $H(s)$

$$\begin{aligned}
 H(s) &= \frac{Q(s)}{P(s)} \\
 &= \frac{b_1 s^{p-1} + b_2 s^{p-2} + \dots + b_p}{s^p + a_1 s^{p-1} + \dots + a_p}
 \end{aligned}
 \tag{5.14}$$

where,

$$\begin{aligned}
 Q(s) &= b_1 s^{p-1} + b_2 s^{p-2} + \dots + b_{p-1} s + b_p \\
 P(s) &= s^p + a_1 s^{p-1} + \dots + a_{p-1} s + a_p
 \end{aligned}
 \tag{5.15}$$

in which, $b_1, b_2, \dots, b_p, a_1, a_2, \dots, a_p$ are constant parameters, and p is the order of the polynomial expressed above. To satisfy the condition in Eq.(5.13), the order of polynomial $Q(s)$ has to be one less than the order of $P(s)$.

On the basis of the assumption of Eq.(5.14), we have

$$Y(s) = \frac{Q(s)}{P(s)} X(s)
 \tag{5.16}$$

which can be rewritten as

$$P(s) Y(s) = Q(s) X(s)
 \tag{5.17}$$

Taking the inverse Laplace transform of Eq.(5.17), we obtain the corresponding time-

domain model

$$\begin{aligned} & \frac{d^p}{dt^p}y(t) + a_1 \frac{d^{p-1}}{dt^{p-1}}y(t) + \dots + a_p y(t) \\ & = b_1 \frac{d^{p-1}}{dt^{p-1}}x(t) + b_2 \frac{d^{p-2}}{dt^{p-2}}x(t) + \dots + b_p x(t) \end{aligned} \quad (5.18)$$

Assuming the characteristic equation of Eq.(5.14), i.e.,

$$\xi^p + a_1 \xi^{p-1} + \dots + a_{p-1} \xi + a_p = 0 \quad (5.19)$$

has the roots

$$\xi_1, \xi_2, \dots, \xi_p \quad (5.20)$$

then, we can write the transfer function $H(s)$ in the following partial fraction form

$$\begin{aligned} H(s) &= \frac{b_1 s^{p-1} + b_2 s^{p-2} + \dots + b_p}{(s - \xi_1)(s - \xi_2) \dots (s - \xi_p)} \\ &= \frac{A_1}{s - \xi_1} + \frac{A_2}{s - \xi_2} + \dots + \frac{A_p}{s - \xi_p} \\ &= \sum_{m=1}^p \frac{A_m}{s - \xi_m} \end{aligned} \quad (5.21)$$

where A_m ($m = 1, 2, \dots, p$) can be calculated by

$$\begin{aligned}
A_m &= \lim_{s \rightarrow \xi_m} H(s) (s - \xi_m) \\
&= \frac{b_1 \xi_m^{p-1} + b_2 \xi_m^{p-2} + \dots + b_{p-1} \xi_m + b_p}{\prod_{\substack{k=1 \\ k \neq m}}^p (\xi_m - \xi_k)}
\end{aligned} \tag{5.22}$$

By taking the inverse Laplace transform of Eq.(5.21) and noting that the inverse Laplace transform of $1/(s-\xi)$ is $e^{\xi t}$, we can obtain the time-domain model for $h(t)$ as

$$h(t) = \sum_{m=1}^p A_m e^{\xi_m t} \tag{5.23}$$

5.3 Determination of Parameters

In the previous section, we established the models for the characterization of the response behaviour of a linear viscoelastic material. This section will discuss the determination of the pertaining parameters from the experimental measurements [Partl, Tinic and Rosli, 1982; Partl and Rosli, 1985]. Assume in the dynamic relaxation experiment, we obtain two discrete-time series of stress response and the rate of loading strain, i.e.,

$$\begin{aligned}
&\dot{\epsilon}(t_0), \dot{\epsilon}(t_1), \dots, \dot{\epsilon}(t_{N-1}), \dot{\epsilon}(t_N) \\
&\sigma(t_0), \sigma(t_1), \dots, \sigma(t_{N-1}), \sigma(t_N)
\end{aligned} \tag{5.24}$$

or, in the dynamic creep experiment, we obtain two discrete-time series of strain response and

the rate of stress loading, i.e.,

$$\begin{aligned} & \dot{\sigma}(t_0), \dot{\sigma}(t_1), \dots, \dot{\sigma}(t_{N-1}), \dot{\sigma}(t_N) \\ & \varepsilon(t_0), \varepsilon(t_1), \dots, \varepsilon(t_{N-1}), \varepsilon(t_N) \end{aligned} \quad (5.25)$$

where "·" represents the derivative with respect to time.

From Eq.(5.6) or Eq.(5.7), we can obtain two general input and output discrete-time series

$$\begin{aligned} & x(t_0), x(t_1), \dots, x(t_{N-1}), x(t_N) \\ & y(t_0), y(t_1), \dots, y(t_{N-1}), y(t_N) \end{aligned} \quad (5.26)$$

Now we have to analyze these two discrete-time series Eq.(5.26) to determine the parameters A_m, ξ_m ($m = 1, 2, \dots, p$) in Eq.(5.23). In doing so, corresponding to the continuous-time differential model Eq.(5.18), we introduce the following discrete-time system

$$\begin{aligned} & y_k + \beta_1 y_{k-1} + \dots + \beta_{p-1} y_{k-p+1} + \beta_p y_{k-p} \\ & = \alpha_0 x_k + \alpha_1 x_{k-1} + \dots + \alpha_{p-1} x_{k-p+1} \end{aligned} \quad (5.27)$$

$(k = 0, 1, 2, \dots)$

where $\alpha_0, \alpha_1, \dots, \alpha_p$ and $\beta_1, \beta_2, \dots, \beta_p$ and p are constant parameters. Denoting

$$Dy_k = y_{k-1} \quad (5.28)$$

and

$$\begin{aligned}\varphi(D) &= 1 + \beta_1 D + \dots + \beta_p D^p \\ \theta(D) &= \alpha_0 + \alpha_1 D + \dots + \alpha_{p-1} D^{p-1}\end{aligned}\tag{5.29}$$

Eq.(5.27) can be written as

$$\varphi(D) y_k = \theta(D) x_k\tag{5.30}$$

Representing the z-transform of $\{y_k\}$ and $\{x_k\}$ by $Y(z)$ and $X(z)$ respectively, and taking z-transform of Eq.(5.30), we have

$$\varphi(z^{-1}) Y(z) = \theta(z^{-1}) X(z)\tag{5.31}$$

or

$$\begin{aligned}Y(z) &= \frac{\theta(z^{-1})}{\varphi(z^{-1})} X(z) \\ &= H_d(z) X(z)\end{aligned}\tag{5.32}$$

where $H_d(z)$ is called the "transfer function" of the discrete-time system

$$\begin{aligned}
H_d(z) &= \frac{\theta(z^{-1})}{\varphi(z^{-1})} \\
&= \frac{\alpha_0 + \alpha_1 z^{-1} + \dots + \alpha_{p-1} z^{-(p-1)}}{1 + \beta_1 z^{-1} + \dots + \beta_{p-1} z^{-(p-1)} + \beta_p z^{-p}}
\end{aligned} \tag{5.33}$$

Assume that the equation

$$\varphi(\lambda^{-1}) = 1 + \beta_1 \lambda^{-1} + \dots + \beta_p \lambda^{-p} = 0 \tag{5.34}$$

has roots $\lambda_1, \lambda_2, \dots, \lambda_p$. Then, Eq.(5.33) can be written as

$$\begin{aligned}
H_d(z) &= \frac{\alpha_0 + \alpha_1 z^{-1} + \dots + \alpha_{p-1} z^{-p+1}}{(1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1}) \dots (1 - \lambda_p z^{-1})} \\
&= \frac{B_1}{1 - \lambda_1 z^{-1}} + \frac{B_2}{1 - \lambda_2 z^{-1}} + \dots + \frac{B_p}{1 - \lambda_p z^{-1}} \\
&= \sum_{m=1}^p \frac{B_m}{1 - \lambda_m z^{-1}}
\end{aligned} \tag{5.35}$$

where B_m ($m = 1, 2, \dots, p$) are calculated by

$$\begin{aligned}
B_m &= \lim_{z \rightarrow \lambda_m} H_d(z) (1 - \lambda_m z^{-1}) \\
&= \frac{\alpha_0 + \alpha_1 \lambda_m^{-1} + \dots + \alpha_{p-1} \lambda_m^{-p+1}}{\prod_{\substack{k=1 \\ k \neq m}}^p (1 - \lambda_k \lambda_m^{-1})}
\end{aligned} \tag{5.36}$$

Taking the inverse z-transform of Eq.(5.35), one obtains the system characteristic series or weighting sequence of the discrete-time system:

$$h_d(i) = \sum_{m=1}^p B_m \lambda_m^i u(i) \quad (5.37)$$

$$(i = 0, 1, 2, \dots)$$

As discussed in Chapter 4, a relation between the system characteristic function $h(t)$, Eq.(5.23), of a continuous-time system and the system characteristic series $h_d(i)$, Eq.(5.37) of the corresponding discrete-time system can be established. Finally we can also obtain the following equations

$$A_m = \frac{1}{\Delta T} B_m \quad (m = 1, 2, \dots, p) \quad (5.38)$$

$$\xi_m = \frac{1}{\Delta T} \ln \lambda_m$$

To determine the parameters B_m, λ_m ($m = 1, 2, \dots, p$) for the discrete-time system, the parameters $\alpha_0, \alpha_1, \dots, \alpha_p, \beta_1, \beta_2, \dots, \beta_p$ in the model Eq.(5.27) has to be determined first. Choose arbitrarily an order p and parameters $\hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_p, \hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_p$ of a discrete-time system. Substituting them into Eq.(5.27), we have

$$y_i + \hat{\beta}_1 y_{i-1} + \dots + \hat{\beta}_p y_{i-p} \quad (5.39)$$

$$= \hat{\alpha}_0 x_i + \hat{\alpha}_1 x_{i-1} + \dots + \hat{\alpha}_{p-1} x_{i-p+1} + e_i$$

$$(i = 1, 2, \dots, N)$$

where e_i ($i = 1, 2, \dots, N$) is the error in choosing the values of the parameters $\hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_p,$

$\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_p$ and the order p .

Then, the error $e_i (i = 1, 2, \dots, N)$ can be expressed as:

$$\begin{aligned}
 e_i &= y_i - (-y_{i-1}, \dots, -y_{i-p}, x_i, \dots, x_{i-p+1}) \begin{Bmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p \\ \hat{\alpha}_0 \\ \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_{p-1} \end{Bmatrix} \\
 &= y_i - \{w_i\}^T \{\hat{\beta}\} \\
 &\quad (i = 1, 2, \dots, N)
 \end{aligned} \tag{5.40}$$

where

$$\begin{aligned}
 \{w_i\}^T &= (-y_{i-1}, \dots, -y_{i-p}, x_i, x_{i-1}, \dots, x_{i-p+1}) \\
 \{\hat{\beta}\}^T &= (\hat{\beta}_1, \dots, \hat{\beta}_p, \hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_{p-1})
 \end{aligned} \tag{5.41}$$

and " T " represents the transpose of a matrix.

By minimizing the sum of the error square, i.e.,

$$\begin{aligned}
 e^2 &= \frac{1}{N} \sum_{i=p+1}^N e_i^2 \\
 &= \frac{1}{N} \sum_{i=p+1}^N (y_i - \{w_i\}^T \{\hat{\beta}\}) (y_i - \{w_i\}^T \{\hat{\beta}\})
 \end{aligned} \tag{5.42}$$

one has

$$\left(\frac{1}{N} \sum_{i=p+1}^N \{w_i\}^T \{w_i\} \right) \{\hat{\beta}\} = \frac{1}{N} \sum_{i=p+1}^N y_i \{w_i\} \quad (5.43)$$

Thus, for every choice of an order p , a corresponding $\{\hat{\beta}\}$ can be determined by Eq.(5.43). Then, from Eq.(5.42), the pertaining sum of square error can be calculated. The choice of the order p of the discrete-time system can be made by requiring that it results in a minimum sum of error square.

5.4 Numerical Examples

To test the analytical model developed in the previous section, numerical illustrations will be carried out.

Example 1

Consider the first order system:

$$\dot{y} + 0.2y = x(t) \quad (5.44)$$

By using the notations of Eq.(5.18), the parameters of this system are listed in Table 1.

Table 1

p	b_1	a_1
1	0.2	1.0

Then, the parameters A_m and ξ_m , which are defined in Eq.(5.21) can be calculated from equations (5.19) and (5.22), and listed in Table 2.

Table 2

A_1	ξ_1
1.0	-0.2

Let the input $x(t)$ be

$$x(t) = 100 \sin(t^{1.5}) \quad (5.45)$$

By employing the Runge-Kutta method [Morris, 1983], with $\Delta T=0.01$, we can obtain two discrete-time series of input and output, which are shown in Figures 5.1 and 5.2.

Using discrete-time systems (DTS), Eq.(5.27), of different orders to model the system. The errors for different DTS's are given in Table 3 below.

Table 3

Order	First	Second	Third
Error	0.526948E+00	0.205991E+2	0.206344E+01

From Table 3, the DTS of first order is the system with minimum error, therefore, we choose the first order of DTS to model the system . The parameters of this DTS are listed in Table 4 below.

Table 4

Parameter	$\hat{\beta}_1$	$\hat{\alpha}_0$	λ_1
Value	-0.998250E+00	0.997577E-02	0.99825

where $\hat{\beta}_1$, $\hat{\alpha}_0$ are parameters of the discrete-time system defined in Eq.(5.27) and λ_1 is the root of the characteristic equation (5.34) of the DTS. Because this DTS is of first order, its transfer function, Eq.(5.33), is written as

$$H_d(z) = \frac{B_1}{1 - \lambda_1 z^{-1}} \quad (5.46)$$

where B_1 is calculated according to Eq.(5.36) as

$$B_1 = \alpha = 0.997577E-02$$

Then, according to Eq.(5.38), the parameters of the corresponding continuous-time system can be calculated as

$$\begin{aligned} A_1 &= \left(\frac{1}{\Delta T}\right) B_1 = 0.997577 \\ \xi_1 &= \left(\frac{1}{\Delta T}\right) \ln \lambda_1 = -0.1852 \end{aligned} \quad (5.47)$$

For comparison, we list the exact and estimated values in Table 5

Table 5

parameter	A_1	ξ_1
Exact	1.0	-0.2
Estimated	0.997577	-0.1852

Thus, the estimated system characteristic function is written as

$$\begin{aligned}
 h(t) &= A_1 e^{\xi_1 t} \\
 &= 0.997577 e^{-0.1852t}
 \end{aligned}
 \tag{5.48}$$

Fig.5.3 shows the exact and the estimated responses given by the *1st* order DTS.

Fig.5.4 shows the exact and the estimated system characteristic function $h(t)$ obtained from the *1st* order DTS.

In the remaining examples given below, the same process we carried out in Example 1 is followed.

Example (2)

Consider the system

$$\dot{y} + 5y = 100 \sin(0.5 t^{1.5})
 \tag{5.49}$$

where the input $x(t)$ is

$$x(t) = 100 \sin(0.5 t^{1.5})
 \tag{5.50}$$

The corresponding input and output discrete-time series are plotted in Fig.5.5 and Fig.5.6 respectively, with $\Delta T = 0.01$.

The parameters of this system are listed in Table 6 below

Table 6

p	b_1	a	A_1	ξ_1
1	0.5	1	1.0	-5.0

The errors for discrete-time systems of different orders are listed in Table 7 below

Table 7

Order	First	Second	Third
Error	0.852409E-02	0.253536E+00	0.634591E-1

From Table 7, the DTS of first order is the system with minimum error, the parameters of which are given in Table 8.

Table 8

Parameter	$\hat{\beta}$	$\hat{\alpha}$	λ
Value	-0.952681E+00	0.951930E-02	0.95268

The estimated and exact values of parameters of continuous system model are listed in Table 9.

Table 9

Parameter	A_1	ξ_1
Exact	1.0	-0.5
Estimated	0.95193	-4.8476

Fig.5.7 shows the exact and the estimated response given by the 1st order DTS.

Fig.5.8 shows the exact and the estimated system characteristic function $h(t)$ obtained for the first order DTS.

CHAPTER 6

Dynamic System Identification Method for the Characterization of Viscoelastic Materials --- Method II

6.1 Introduction

In Chapter 5, we developed a dynamic system identification method for the characterization of linear viscoelastic properties by considering the linear viscoelastic material as a dynamic system. A relation was established between the creep or relaxation function and the transfer function of the dynamic system. In the frequency domain, an analytical model was first assumed for the transfer function of the system, then, a discrete-time system analysis was introduced to determine the order and parameters of the proposed model from dynamic experiment measurements. The method requires two discrete-time series, i.e., the time-rate of the input signal and the corresponding output signal.

In the present chapter, the basic idea of Chapter 5 as summarized above, is extended to directly treat the input signal itself (stress in a dynamic creep experiment or strain in a dynamic relaxation experiment) rather than the time-rate of the input signal as considered previously in Chapter 5. Laplace transform method is employed and the relation between the creep or relaxation function and the transfer function of the dynamical system is first established. In this context, a model of rational function of polynomials for the transfer function is assumed. A

discrete-time system analysis method is, then, introduced to identify the order and parameters of the pertaining model from the discrete-time series of both the input and output signals.

6.2 Theory

The mechanical response of a linear viscoelastic material is conventionally characterized by the concepts of creep and relaxation functions. For the convenience of the analysis in this Chapter, the creep and relaxation functions can be represented in Fig.6.1 and 6.2 with some modification of definitions.

With reference to Fig.6.1, in a creep experiment, C_0 represents the instantaneous response of the material per unit stress input, while, $C_c(t)$ represents the time-dependent delayed part of the response. Thus, the creep function $C(t)$ is expressed as

$$C(t) = C_0 + C_c(t) \quad ; \quad t \geq 0 \quad (6.1)$$

Meantime, in a relaxation experiment, Fig.6.2, R_0 represents the instantaneous response of the material per unit strain input, and $R_c(t)$ represents the time-dependent part of the response. Thus, the relaxation function $R(t)$ is written as

$$R(t) = R_0 - R_c(t) \quad ; \quad t \geq 0 \quad (6.2)$$

Taking Laplace transform of the creep function $C(t)$ as expressed in Eq.(6.1), one has

$$\begin{aligned}
C(s) &= \int_0^{\infty} C(t) e^{-st} dt \\
&= \int_0^{\infty} (C_0 + C_c(t)) e^{-st} dt \\
&= C_0 \frac{1}{s} + C_c(s)
\end{aligned} \tag{6.3}$$

where $C_c(s)$ is the Laplace transform of $C_c(t)$

Similarly, the Laplace transform of the relaxation function $R(t)$, Eq.(6.2), is

$$\begin{aligned}
R(s) &= \int_0^{\infty} R(t) e^{-st} dt \\
&= \int_0^{\infty} (R_0 - R_c(t)) e^{-st} dt \\
&= R_0 \frac{1}{s} - R_c(s)
\end{aligned} \tag{6.4}$$

where $R_c(s)$ is the Laplace transform of $R_c(t)$.

Since

$$\begin{aligned}
C_c(t) \Big|_{t=0} &= 0 \\
R_c(t) \Big|_{t=0} &= 0
\end{aligned} \tag{6.5}$$

we have [see, e.g., Fodor, 1965],

$$\begin{aligned}
C_r(t)|_{t=0} &= \lim_{s \rightarrow \infty} s C_r(s) = 0 \\
R_r(t)|_{t=0} &= \lim_{s \rightarrow \infty} s R_r(s) = 0
\end{aligned}
\tag{6.6}$$

Thus,

$$\begin{aligned}
\lim_{s \rightarrow \infty} s C(s) &= C_0 + \lim_{s \rightarrow \infty} s C_r(s) \\
&= C_0 \\
\lim_{s \rightarrow \infty} s R(s) &= R_0 - \lim_{s \rightarrow \infty} s R_r(s) \\
&= R_0
\end{aligned}
\tag{6.7}$$

In a dynamical creep experiment, the stress input is a function of the time, and the resulting strain in the material is given by,

$$\begin{aligned}
\varepsilon(t) &= \int_{-\infty}^t C(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau \\
&= \int_0^{\infty} C(\tau) \frac{d\sigma(t-\tau)}{d(t-\tau)} d\tau
\end{aligned}
\tag{6.8}$$

Similarly, in a dynamic relaxation experiment, the strain input is a function of time, and the occurring stress in the material is given by

$$\begin{aligned}
\sigma(t) &= \int_{-\infty}^t R(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau \\
&= \int_0^{\infty} R(\tau) \frac{d\varepsilon(t-\tau)}{d(t-\tau)} d\tau
\end{aligned}
\tag{6.9}$$

Taking Laplace transform of Eq.(6.8), one finds that

$$\begin{aligned}
\varepsilon(s) &= \int_0^{\infty} \varepsilon(t) e^{-st} dt \\
&= \int_0^{\infty} \int_0^{\infty} C(\tau) \frac{d\sigma(t-\tau)}{d(t-\tau)} e^{-st} d\tau dt \\
&= \int_0^{\infty} C(\tau) e^{-s\tau} \left[\int_{-\tau}^{\infty} \frac{d\sigma(\phi)}{d(\phi)} e^{-s\phi} d\phi \right] d\tau
\end{aligned}
\tag{6.10}$$

where $\phi = t - \tau$.

Since $\sigma(t) = 0$ for $t < 0$, one has

$$\begin{aligned}
\int_{-\tau}^{\infty} \frac{d\sigma(\phi)}{d(\phi)} e^{-s\phi} d\phi &= \int_0^{\infty} \frac{d\sigma(\phi)}{d\phi} e^{-s\phi} d\phi \\
&= s \sigma(s) - \sigma(-0) \\
&= s \sigma(s)
\end{aligned}
\tag{6.11}$$

where $\sigma(s)$ is the Laplace transform of the time-dependent stress input $\sigma(t)$. Thus, by combining equations (6.10) and (6.11), the former equation can be written as

$$\begin{aligned}\varepsilon(s) &= \int_0^{\infty} C(\tau)e^{-s\tau}d\tau \int_0^{\infty} \frac{d\sigma(\phi)}{d\phi} e^{-s\phi}d\phi \\ &= C(s) s \sigma(s)\end{aligned}\tag{6.12}$$

Substituting Eq.(6.3) into Eq.(6.12), one has

$$\begin{aligned}\varepsilon(s) &= [C_0 + sC_c(s)] \sigma(s) \\ &= H_c(s) \sigma(s)\end{aligned}\tag{6.13}$$

Similarly, for a dynamic relaxation experiment, by taking Laplace transform of Eq.(6.9), one can write that

$$\begin{aligned}\sigma(s) &= [R_0 - sR_r(s)]\varepsilon(s) \\ &= H_r(s) \varepsilon(s)\end{aligned}\tag{6.14}$$

From a dynamic system point of view, the viscoelastic material is considered as a dynamic system, thus, the function $H_c(s)=[C_0+sC_c(s)]$ in Eq.(6.13) is the transfer function in a dynamic creep experiment. Similarly, the function $H_r(s)=[R_0-sR_r(s)]$ in Eq.(6.14) is the transfer function in a dynamic relaxation experiment.

At this point, it is necessary to make an assumption concerning the functions $C_c(s)$ and $R_r(s)$ in equations (6.13) and (6.14). Both of the two continuous functions may be approximated by rational functions of polynomials, i.e.,

$$sC_c(s) = \frac{Q_c^q(s)}{P_c^p(s)} \quad (6.15)$$

$$sR_c(s) = \frac{Q_r^q(s)}{P_r^p(s)}$$

where $Q_c^q(s)$ and $Q_r^q(s)$ are two q -th order polynomials, and $P_c^p(s)$ and $P_r^p(s)$ are two p -th order polynomials. Combining equation (6.13) and (6.15), the transfer function in a dynamical creep experiment can be expressed as

$$\begin{aligned} H_c(s) &= C_0 + C_c(s)s \\ &= \frac{C_0 P_c^p(s) + Q_c^q(s)}{P_c^p(s)} \end{aligned} \quad (6.16)$$

With reference to Eq.(6.6), since $sC_c(s)$ and $sR_c(s)$ tend to 0 when $s \rightarrow \infty$, then, in Eq.(6.15), the order q of the polynomial must be smaller than the order p , that is $q \leq p-1$.

Similarly, in a dynamical relaxation experiment, the transfer function $H_r(s)$ can be written as,

$$\begin{aligned} H_r(s) &= R_0 - R_c(s)s \\ &= \frac{R_0 P_r^p(s) - Q_r^q(s)}{P_r^p(s)} \end{aligned} \quad (6.17)$$

Mathematically, Eq.(6.13), for a dynamical creep experiment, and Eq.(6.14), for a dynamical relaxation experiment, can be written in the following general form

$$Y(s) = H(s) X(s) \quad (6.18)$$

In Eq.(6.18), $Y(s)$ represent the Laplace transform of the strain response in a dynamical creep experiment, or Laplace transform of the stress response in a dynamical relaxation experiment. Meantime, $X(s)$ represent the Laplace transform of the stress input in a dynamical creep experiment, or Laplace transform of the strain input in a dynamical relaxation experiment. In this equation, $H(s)$ is a transfer function which can be modelled in the following rational function form

$$H(s) = \frac{Q^p(s)}{P^p(s)} \quad (6.19)$$

where,

$$\begin{aligned} Q^p(s) &= b_0 s^p + b_1 s^{p-1} + \dots + b_{p-1} s + b_p \\ P^p(s) &= s^p + a_1 s^{p-1} + \dots + a_{p-1} s + a_p \end{aligned} \quad (6.20)$$

in which, $b_0, b_1, \dots, b_p, a_1, a_2, \dots, a_p$ are constant parameters, and p is the order of the both polynomials $Q^p(s)$ and $P^p(s)$, because from Eq.(6.16) and Eq.(6.17), the numerator polynomial $Q(s)$ and the denominator polynomial $P(s)$ must have the same order.

With reference to Eq.(6.20), since the two polynomials $Q^p(s)$ and $P^p(s)$ have the same order, the transfer function $H(s)$, can be written as

$$H(s) = b_0 + \frac{N^q(s)}{P^p(s)} \quad (6.21)$$

where $N^q(s)$ is a q -th order polynomial with $q \leq p-1$ and b_0 is a constant.

Assume that the p -th order algebraic equation

$$s^p + a_1 s^{p-1} + \dots + a_{p-1} s + a_p = 0 \quad (6.22)$$

has the roots

$$\xi_1, \xi_2, \dots, \xi_p \quad (6.23)$$

then, the transfer function $H(s)$ can be further written as

$$H(s) = b_0 + \sum_{i=1}^p \frac{A_i}{s - \xi_i} \quad (6.24)$$

where A_i ($i = 1, 2, \dots, p$) are calculated by

$$\begin{aligned} A_i &= \lim_{s \rightarrow \xi_i} (s - \xi_i) \frac{N^q(s)}{P^p(s)} \\ &= \frac{N^q(\xi_i)}{\frac{d}{ds}(P^p(s))|_{s=\xi_i}} \end{aligned} \quad (6.25)$$

$$(i = 1, 2, \dots, p)$$

or, alternatively,

$$A_i = \frac{N^q(\xi_i)}{\prod_{\substack{m=1 \\ m \neq i}}^p (\xi_m - \xi_i)} \quad (6.26)$$

$$(i = 1, 2, \dots, p)$$

In a dynamical creep experiment, one has, with reference to Eq.(6.12),

$$H(s) = s C(s) \quad (6.27)$$

Thus, by combining equations (6.24) and (6.27), it follows that

$$\begin{aligned} C(s) &= \frac{1}{s} H(s) \\ &= \frac{1}{s} b_0 + \sum_{i=1}^p \frac{A_i}{s (s - \xi_i)} \\ &= \frac{1}{s} b_0 + \sum_{i=1}^p \frac{A_i}{\xi_i} \left(\frac{1}{s - \xi_i} - \frac{1}{s} \right) \end{aligned} \quad (6.28)$$

Taking the inverse Laplace transform of Eq.(6.28), one has

$$C(t) = b_0 + \sum_{i=1}^p \frac{A_i}{\xi_i} (e^{\xi_i t} - 1) \quad ; \quad t \geq 0 \quad (6.29)$$

Comparing Eq.(6.29) with Eq.(6.1), it follows that, in a dynamical creep experiment,

$$\begin{aligned}
C_0 &= b_0 \\
C_c(t) &= \sum_{i=1}^p \frac{A_i}{\xi_i} (e^{\xi_i t} - 1)
\end{aligned} \tag{6.30}$$

Following the same procedure for creep, in a dynamical relaxation experiment, we have

$$R(t) = b_0 + \sum_{i=1}^p \frac{A_i}{\xi_i} (e^{\xi_i t} - 1) \quad ; \quad t \geq 0 \tag{6.31}$$

Comparing Eq.(6.31) with Eq.(6.2), then,

$$\begin{aligned}
R_0 &= b_0 \\
R_c(t) &= \sum_{i=1}^p \frac{A_i}{\xi_i} (1 - e^{\xi_i t}) \quad ; \quad (i = 1, 2, \dots, p)
\end{aligned} \tag{6.32}$$

Denoting the inverse Laplace transform of the transfer function $H(s)$ by $h(t)$, then,

$$\begin{aligned}
h(t) &= \int_0^{\infty} H(s) e^{st} ds \\
&= b_0 \delta(t) + \sum_{i=1}^p A_i e^{\xi_i t}
\end{aligned} \tag{6.33}$$

where $\delta(t)$ is the Delta function defined by

$$\delta(t) = \begin{cases} \infty & , \quad t = 0 \\ 0 & , \quad t \neq 0 \end{cases} \quad (6.34)$$

Recalling Eq.(6.19), if the transfer function $H(s)$ of a system can be expressed by this equation, then the relationship between the output $y(t)$ and the input $x(t)$ of the system is governed by the following differential equation

$$\begin{aligned} & \frac{d^p}{dt^p}y(t) + a_1 \frac{d^{p-1}}{dt^{p-1}}y(t) + \dots + a_p y(t) \\ & = b_0 \frac{d^p}{dt^p}x(t) + b_1 \frac{d^{p-1}}{dt^{p-1}}x(t) + \dots + b_p x(t) \end{aligned} \quad (6.35)$$

The problem now is to determine the parameters in the model equations (6.20) from the measurements of input and output which are usually given in the form of discrete-time signals. To analyze these two discrete-time series, we introduce the concept of the discrete-time system [see, Cadzow, 1973],

$$\begin{aligned} & y_k + \beta_1 y_{k-1} + \dots + \beta_{p-1} y_{k-p+1} + \beta_p y_{k-p} \\ & = \alpha_0 x_k + \alpha_1 x_{k-1} + \dots + \alpha_{p-1} x_{k-p+1} + \alpha_p x_{k-p} \end{aligned} \quad (6.36)$$

($k = 0, 1, 2, \dots$)

where $\alpha_0, \alpha_1, \dots, \alpha_p$ and $\beta_0, \beta_1, \dots, \beta_p$ are constant parameters.

Taking the z-transform of Eq.(6.36), one has

$$Y(z) = H_d(z) X(z) \quad (6.37)$$

where

$$\begin{aligned} H_d(z) &= \frac{\alpha_0 + \alpha_1 z^{-1} + \dots + \alpha_{p-1} z^{-(p-1)} + \alpha_p z^{-p}}{1 + \beta_1 z^{-1} + \dots + \beta_{p-1} z^{-(p-1)} + \beta_p z^{-p}} \\ &= \frac{\theta^p(z^{-1})}{\varphi^p(z^{-1})} \end{aligned} \quad (6.38)$$

$$\theta^p(z^{-1}) = \alpha_0 + \alpha_p z^{-1} + \dots + \alpha_{p-1} z^{-(p-1)} + \alpha_p z^{-p}$$

$$\varphi^p(z^{-1}) = 1 + \beta_1 z^{-1} + \dots + \beta_{p-1} z^{-(p-1)} + \beta_p z^{-p}$$

where $H_d(z)$ is the transfer function for the discrete-time system Eq.(6.36).

In order to express Eq.(6.38) in a similar form of Eq.(6.21), we apply some mathematical operations to Eq.(6.38), and, hence, one can show that

$$\begin{aligned}
H_d(z) &= \frac{\alpha_0 + \alpha_1 z^{-1} + \dots + \alpha_{p-1} z^{-(p-1)}}{\varphi^p(z^{-1})} + \frac{\alpha_p z^{-p}}{\varphi^p(z^{-1})} \\
&= \frac{\alpha_0 + \alpha_1 z^{-1} + \dots + \alpha_{p-1} z^{-(p-1)}}{\varphi^p(z^{-1})} + \frac{\alpha_p}{\beta_p} \frac{\beta_p z^{-p}}{\varphi^p(z^{-1})} \\
&= \frac{\alpha_0 + \alpha_1 z^{-1} + \dots + \alpha_{p-1} z^{-(p-1)}}{\varphi^p(z^{-1})} \\
&\quad + \frac{\alpha_p}{\beta_p} \frac{\varphi^p(z^{-1}) - (1 + \beta_1 z^{-1} + \dots + \beta_{p-1} z^{-(p-1)})}{\varphi^p(z^{-1})} \\
&= \frac{\alpha_p}{\beta_p} + \frac{\gamma^{p-1}(z^{-1})}{\varphi^p(z^{-1})}
\end{aligned} \tag{6.39}$$

where

$$\begin{aligned}
\gamma^{p-1}(z^{-1}) &= \alpha_0 + \alpha_1 z^{-1} + \dots + \alpha_{p-1} z^{-(p-1)} \\
&\quad - \frac{\alpha_p}{\beta_p} (1 + \beta_1 z^{-1} + \dots + \beta_{p-1} z^{-(p-1)}) \\
&= \left(\alpha_0 - \frac{\alpha_p}{\beta_p}\right) + \left(\alpha_1 - \alpha_p \frac{\beta_1}{\beta_p}\right) z^{-1} + \dots + \left(\alpha_{p-1} - \alpha_p \frac{\beta_{p-1}}{\beta_p}\right) z^{-(p-1)}
\end{aligned} \tag{6.40}$$

is a polynomial of $(p-1)$ th order.

Assuming the equation

$$\varphi^p(z^{-1}) = 1 + \beta_1 z^{-1} + \dots + \beta_p z^{-p} = 0 \tag{6.41}$$

has roots $\lambda_1, \lambda_2, \dots, \lambda_p$. Then, it follows that

$$\begin{aligned}
H_d(z) &= \frac{\alpha_p}{\beta_p} + \frac{\gamma^{p-1}(z^{-1})}{(1-\lambda_1 z^{-1})\dots(1-\lambda_m z^{-1})} \\
&= \frac{\alpha_p}{\beta_p} + \sum_{i=1}^p \frac{B_i}{1-\lambda_i z^{-1}}
\end{aligned} \tag{6.42}$$

where B_i ($i = 1, 2, \dots, p$) are calculated by

$$\begin{aligned}
B_i &= \frac{\gamma^{p-1}(\lambda_i^{-1})}{\prod_{\substack{m=1 \\ m \neq i}}^p (1-\lambda_i^{-1}\lambda_m)} \\
&(i = 1, 2, \dots, p)
\end{aligned} \tag{6.43}$$

Taking the inverse z-transform of Eq.(6.42), one obtains the system characteristic series or weighting sequence of the discrete-time system as

$$\begin{aligned}
h_d(k) &= \frac{\alpha_p}{\beta_p} \delta_0(k) + \sum_{m=1}^p B_m \lambda_m^k u(k) \\
&(k = 0, 1, 2, \dots)
\end{aligned} \tag{6.44}$$

where $\delta_0(k)$ is the Kronecker δ -function[Cadzow, 1973], defined as

$$\delta_0(k) = \begin{cases} 1 & , \quad k = 0 \\ 0 & , \quad k \neq 0 \end{cases} \tag{6.45}$$

and $u(k)$ is the discrete-time unit step function, i.e.,

$$u(k) = \begin{cases} 0 & , \quad k < 0 \\ 1 & , \quad k = 0, 1, 2, \dots \end{cases} \quad (6.46)$$

By using a weighting sequence of Eq.(6.44), the relation between the input and output series can be written as

$$y_i = \sum_{k=0}^{\infty} h_d(k) x_{i-k} \quad ; \quad (i = 0, 1, 2, \dots) \quad (6.47)$$

Substituting Eq.(6.44) into Eq.(6.47), it follows that

$$\begin{aligned} y_i &= \sum_{k=0}^{\infty} \left(\frac{\alpha_p}{\beta_p} \delta_0(k) + \sum_{m=1}^p B_m \lambda_m^k \right) x_{i-k} \\ &= \sum_{k=0}^{\infty} \frac{\alpha_p}{\beta_p} \delta_0(k) x_{i-k} + \sum_{k=0}^{\infty} \sum_{m=1}^p B_m \lambda_m^k x_{i-k} \\ &= \frac{\alpha_p}{\beta_p} x_i + \sum_{m=1}^p \sum_{k=0}^{\infty} B_m \lambda_m^k x_{i-k} \\ &\quad (i = 0, 1, 2, \dots) \end{aligned} \quad (6.48)$$

For a continuous system, one has

$$y(t) = \int_0^{\infty} h(\tau) x(t-\tau) d\tau \quad (6.49)$$

By substituting Eq.(6.33) into Eq.(6.49), the latter equation becomes

$$\begin{aligned}
y(t) &= \int_0^{\infty} h(\tau) x(t-\tau) d\tau \\
&= \int_0^{\infty} (b_0 \delta(\tau) + \sum_{m=1}^p A_m e^{\xi_m \tau}) x(t-\tau) d\tau \\
&= b_0 x(t) + \sum_{m=1}^p \int_0^{\infty} A_m e^{\xi_m \tau} x(t-\tau) d\tau
\end{aligned} \tag{6.50}$$

Eq.(6.50) can be approximated further by the following expression

$$\begin{aligned}
y (\Delta T \cdot i) &= b_0 x (\Delta T \cdot i) \\
&+ \sum_{m=1}^p \sum_{k=0}^{\infty} A_m e^{(\xi_m \Delta T k)} x ((i-k) \cdot \Delta T) \cdot \Delta T \\
&(i = 0, 1, 2, \dots)
\end{aligned} \tag{6.51}$$

which can be also written in the following form

$$\begin{aligned}
y_i &= b_0 x_i + \sum_{m=1}^p A_m \cdot \Delta T \sum_{k=0}^{\infty} (e^{\xi_m \Delta T})^k x_{i-k} \\
&(i = 0, 1, 2, \dots)
\end{aligned} \tag{6.52}$$

Comparing equations (6.48) and (6.52), one can finally establish the following relations between the parameters of the continuous and discrete-time systems

$$\begin{aligned}
b_0 &= \frac{\alpha_p}{\beta_p} \\
\Delta T A_m &= B_m \\
e^{\xi_m \Delta T} &= \lambda_m \quad (m = 1, 2, \dots, p)
\end{aligned} \tag{6.53}$$

6.3 Practical Considerations

For a viscoelastic material, the creep and relaxation functions are not continuous functions at $t=0$. There are jumps in the two functions at $t=0$. For the creep function, the jump is equal to C_0 , while, for the relaxation function the jump is equal to R_0 . The discontinuous characteristic of the creep or the relaxation function at $t=0$ represents the so-called instantaneous response of the material to the input,[see, Gross, 1953 and Christensen, 1971]. Due to this characteristic, the corresponding dynamic system, which is represented by Eq.(6.33), has a singular property at $t=0$, i.e., it goes to infinity when $t=0$. This singular property will cause significant difficulties in practical numerical calculations. It is, thus, necessary to modify the formulism presented in Section 6.2 so that it can be implemented numerically in a practical manner.

By definition, C_0 and R_0 are the instantaneous responses of the viscoelastic material to a sudden loading of constant strain or constant stress, respectively. Thus the constants C_0 and R_0 are easily determinable through simple creep or relaxation experiments. Accordingly, it is assumed, for the purpose of the present section, that the instantaneous responses C_0 and R_0 are already known before the dynamical experiment begins. This means that b_0 in Eq.(6.21) is known.

Because the transfer function $H(s)$ can be written in the form of Eq.(6.21), Eq.(6.18) can be expressed as

$$\begin{aligned} Y(s) &= H(s) X(s) \\ &= b_0 X(s) + \frac{N^q(s)}{P^p(s)} X(s) \end{aligned} \tag{6.54}$$

If we introduce another function $Y'(s)$ where

$$Y'(s) = Y(s) - b_0 X(s) \quad (6.55)$$

Eq.(6.54) can, thus, be written as

$$Y'(s) = \frac{N^q(s)}{P^p(s)} X(s) \quad (6.56)$$

where $N^q(s)$ is a q -th order polynomial, with $q \leq p-1$, which can be generally written as

$$N^q(s) = b'_1 s^{p-1} + \dots + b'_{p-1} s + b'_p \quad (6.57)$$

where b'_1, b'_2, \dots, b'_p are constants

If we denote

$$y'(t) = y(t) - b_0 x(t) \quad (6.58)$$

and taking the inverse Laplace transform of Eq.(6.56), we can have the following governing differential equation

$$\frac{d^p}{dt^p} y'(t) + a_1 \frac{d^{p-1}}{dt^{p-1}} y'(t) + \dots + a_p y'(t) = b'_1 \frac{d^{p-1}}{dt^{p-1}} x(t) + \dots + b'_p x(t) \quad (6.59)$$

The new system represented by Eq.(6.59), is called the modified system of the original system Eq.(6.39) and $y'(t)$ is called the modified output of the modified system.

In view of Eq.(6.59), the modified system has the following transfer function

$$\begin{aligned}
 H'(s) &= \frac{b'_1 s^{p-1} + \dots + b'_p}{s^p + a_1 s^{p-1} + \dots + a_p} \\
 &= \sum_{i=1}^p \frac{A_i}{s - \xi_i}
 \end{aligned}
 \tag{6.60}$$

The system function $h'(t)$ of the modified system, which is the inverse Laplace transform of $H'(s)$, is:

$$h'(t) = \sum_{i=1}^p A_i e^{\xi_i t}
 \tag{6.61}$$

Corresponding to Eq.(6.58), the modified discrete-time output signals can be calculated by

$$y'_k = y_k - b_0 x_k \quad ; \quad k = 0, 1, 2, \dots
 \tag{6.62}$$

Corresponding to the continuous modified system of Eq.(6.59), we have the following modified discrete-time system

$$\begin{aligned}
 y'_k + \beta_1 y'_{k-1} + \dots + \beta_{p-1} y'_{k-p+1} + \beta_p y'_{k-p} \\
 = \alpha'_1 x_k + \alpha'_2 x_{k-1} + \dots + \alpha'_p x_{k-p+1}
 \end{aligned}
 \tag{6.63}$$

The transfer function of the modified discrete-time system of Eq.(6.63) is

$$\begin{aligned}
H_d(z) &= \frac{\alpha'_1 + \alpha'_2 z^{-1} + \dots + \alpha'_p z^{-(p-1)}}{1 + \beta_1 z^{-1} + \dots + \beta_{p-1} z^{-(p-1)} + \beta_p z^{-p}} \\
&= \sum_{i=1}^p \frac{B_i}{1 - \lambda_i z^{-1}}
\end{aligned} \tag{6.64}$$

By taking the inverse z -transform of Eq.(6.64), the system characteristic series of the modified discrete-time system is

$$h'_d(k) = \sum_{m=1}^p B_m \lambda_m^k \quad ; \quad k = 0, 1, 2, \dots \tag{6.65}$$

With reference to Eq.(6.53), the parameters of the continuous and discrete-time systems are related to each other by

$$\begin{aligned}
\Delta T A_m &= B_m \\
e^{\xi_m \Delta T} &= \lambda_m
\end{aligned} \tag{6.66}$$

Next, we discuss the method for determining the order p and the parameters α'_m and β_m ($m = 1, 2, \dots, p$) of the discrete-time system.

Choosing an arbitrary order p and parameters $\hat{\alpha}'_m, \hat{\beta}_m, (m = 1, 2, \dots, p)$, then, with reference to Eq.(6.63), we have

$$\begin{aligned}
& y'_i + \hat{\beta}_1 y'_{i-1} + \dots + \hat{\beta}_p y'_{i-p} \\
& = \hat{\alpha}'_1 x_i + \dots + \hat{\alpha}'_p + e_i \\
& (i = 1, 2, \dots, N)
\end{aligned} \tag{6.67}$$

where e_i ($i = 1, 2, \dots, N$) is the error in choosing the values of the parameters $\hat{\alpha}'_m, \hat{\beta}_m$ ($m = 1, 2, \dots, p$), and the order p .

Then, the error e_i ($i = 1, 2, \dots, N$) can be expressed as:

$$\begin{aligned}
e_i & = y'_i - (-y'_{i-1}, \dots, -y'_{i-p}, x_i, \dots, x_{i-p+1}) \begin{Bmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p \\ \hat{\alpha}'_1 \\ \vdots \\ \hat{\alpha}'_p \end{Bmatrix} \\
& = y'_i - \{w_i\}^T \{\hat{\beta}\}
\end{aligned} \tag{6.68}$$

where

$$\begin{aligned}
\{w_i\}^T & = (-y'_{i-1}, \dots, -y'_{i-p}, x_i, \dots, x_{i-p+1}) \\
\{\hat{\beta}\}^T & = (\hat{\beta}_1, \dots, \hat{\beta}_p, \hat{\alpha}'_1, \dots, \hat{\alpha}'_p)
\end{aligned} \tag{6.69}$$

where "T" represents the transpose of a matrix.

By minimizing the sum of the square errors, i.e.,

$$\begin{aligned}
e^2 & = \frac{1}{N} \sum_{i=p+1}^N e_i^2 \\
& = \frac{1}{N} \sum_{i=p+1}^N (y'_i - \{w_i\}^T \{\hat{\beta}\}) (y'_i - \{w_i\}^T \{\hat{\beta}\})
\end{aligned} \tag{6.70}$$

one has

$$\left(\frac{1}{N} \sum_{i=p+1}^N \{w_i\}^T \{w_i\} \right) \{\hat{\beta}\} = \frac{1}{N} \sum_{i=p+1}^N y_i' \{w_i\} \quad (6.71)$$

Thus, for every choice of an order p , a corresponding $\{\beta\}$ can be determined by Eq.(6.69). Then, from Eq.(6.70), the pertaining sum of square errors can be calculated. The choice of the order of the discrete-time system can be made by requiring that it results in a minimum square error e^2 by employing equation (6.70).

6.4 Numerical Examples

To test the model presented in the previous sections, several numerical examples are given below. In all examples, the following system is considered

$$\ddot{y} + a_1 \dot{y} + a_2 y = b_0 \ddot{x} + b_1 \dot{x} + b_2 x \quad (6.72)$$

the input $x(t)$ is assumed to be given by:

$$x(t) = \sin [(\beta t^{\gamma-1} + \omega_0)t] \quad (6.73)$$

where β , ω_0 and γ are parameters of an input.

It is obvious that the frequency of the input signal is

$$\omega(t) = \beta t^{\gamma-1} + \omega_0 \quad (6.74)$$

Example 1.

The parameters in equations (6.72) and (6.74) are assumed as shown, respectively, in Table 1 and 2 below

Table 1

a_1	a_2	b_0	b_1	b_2
51.000	50.000	2.000	113.000	160.000

Table 2

β	γ	ω_0
0.5	2.0	30.0

Solving equation (6.72) by using the Runge-Kutta numerical method [see, e.g., Morris, 1983], we obtain two discrete-time series $\{x_k; k = 0, 1, 2, \dots, N\}$ and $\{y_k; k = 0, 1, 2, \dots, N\}$ which are shown, respectively in Fig.6.3 and Fig.6.4 with $\Delta T = 0.01$. Here, the parameter N represents the numbers of discrete points.

In order to implement the method, we assume that the parameter b_0 , which corresponds to the instantaneous response of the viscoelastic material, to be known ($b_0 = 2.000$). Through Eq.(6.62), we obtain the modified output $\{y'_k, k = 0, 1, 2, \dots, N\}$, as shown in Fig.6.5.

Using discrete-time systems (DTS) of different orders of Eq.(6.63) to model the modified system, the errors for different orders of DTS are shown in Table 3.

Table 3

Order	First	Second	Third	Fourth
Error	0.704957E+00	0.124800E-04	0.510821E-04	0.900288E-04

Based on the results shown in Table 3, we choose the second order DTS, with minimum error to model the modified system. The parameters characterizing Eq.(6.63) are determined as shown in Table 4

Table 4

parameters	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\alpha}_1'$	$\hat{\alpha}_2'$
Value	-1.59012	0.594102	0.109973	0.105197

The roots of the characteristic equation Eq.(6.41) of the discrete-time system of the second order can be determined and given in Table 5.

Table 5

Roots	λ_1	λ_2
Value	0.99004	0.60008

Accordingly, the pertaining parameters B_i ($i = 1, 2$) corresponding to Eq.(6.64) can be determined and are shown in Table 6.

Table 6

B_1	B_2
0.00943659	0.10053623

The parameters of the creep model Eq.(6.29) can be finally identified from Eq.(6.66) and are given in Table 7.

Table 7

Parameter	A_1	A_2	ξ_1	ξ_2
Estimated	0.94366	10.05362	-1.00065	-51.069823
Exact	1.00000	10.00000	-1.00000	-50.00000

Fig.6.6 shows the exact modified output $y'(t)$ and the estimated modified output $y'(t)$.

Fig.6.7 shows the exact and estimated creep functions.

Example 2

The parameters in Eq.(6.72) and Eq.(6.73) are assumed, respectively, in Tables 8 and 9 below

Table 8

a_1	a_2	b_0	b_1	b_2
60.0000	500.0000	2.00000	115.0000	830.0000

Table 9

β	γ	ω_0
0.1	2.0	30.0

Then discrete-time series $\{y_k; k = 0, 1, 2, \dots, N\}$, and $\{x_k; k = 0, 1, 2, \dots, N\}$ are obtained by the Runge-Kutta numerical method for solving equation (6.72) with $\Delta T = 0.01$, which are shown, respectively, in Fig.6.8 and Fig.6.9. The modified output $y'(t)$ can be obtained through Eq.(6.62), and is shown in Fig.6.10.

Using discrete-time systems (DTS) of different orders of Eq.(6.63) to model the modified system, the errors for different orders of DTS are shown in Table 10

Table 10

Order	First	Second	Third
Error	0.302092E+00	0.186518E-05	0.482273E-05

Based on the results shown in Table 10, we choose the second order of DTS, with minimum error to model the system. The parameters characterizing Eq.(6.63) are determined and are shown in Table 11.

Table 11

Table 11

parameters	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\alpha}_1'$	$\hat{\alpha}_2'$
Value	-1.503406	0.5417537	-0.0509786	0.0365676

The roots of the characteristic equation (6.41) of the chosen discrete system can be determined and are given in Table 12.

Table 12

Roots	λ_1	λ_2
Value	0.90436	0.59905

Then, the parameters B_i ($i = 1, 2$) of the system corresponding to Eq.(6.64) can be determined and are given in Table 13.

Table 13

B_1	B_2
-0.03123	-0.019747

The parameters of the relaxation model Eq.(6.31) can be finally identified and are shown in Table 14

Table 14

	A_1	A_2	ξ_1	ξ_2
Estimated	-3.123	-1.9747	-10.05304	-51.2413
Exact	-3.000	-2.000	-10.000	-50.000

Fig.6.11 shows the exact modified output $y'(t)$ and the estimated modified output $\hat{y}'(t)$.

Fig 6.12 shows the exact and estimated relaxation functions.

CHAPTER 7

Conclusions

In the development of viscoelasticity theory, the characterization of the properties of viscoelastic materials has been always an active and an important research field. There are two aspects in the research, i.e.,

- developing more efficient experimental methods to obtain practical viscoelastic data
- constructing more efficient and accurate theoretical models to describe the viscoelastic response behaviour under different mechanical and/or physical input conditions.

The experimental methods developed so far can be classified into three groups, i.e.,

- quasi-static methods
- time-temperature superposition methods
- dynamic experimental methods

Quasi-static experiments are usually carried out at room temperature. The specimen is loaded with a suddenly applied constant stress or strain and the corresponding response is recorded. The data that are required to determine the relaxation, or creep, function of the material can directly be obtained from the experimental measurements. This method is simple, but the required long period of experimental time makes it impractical.

In the time-temperature superposition method, the experiment is carried out at an elevated temperature. The viscoelastic experiment here requires much less time to complete by

comparison with the time required at room temperature. The measurements taken at a high temperature can then be transformed to a reference temperature. Then, the data obtained from transformation can be used to determine the viscoelastic properties at room temperature. However, the requirement of a set of complex temperature control facilities makes this method a very expensive one. In addition, at high temperatures nonlinearity in the viscoelastic response of materials may be introduced.

Recently, dynamic methods are attracting more and more attention from researchers, because it has distinct advantages over quasi-static and time-temperature superposition methods in the way that the dynamic experiments are relatively easy to conduct with no requirements on the time-scale or any complex experimental instruments. In dynamic experiments, the specimen is loaded with an oscillating stress or strain, and the dynamic response of the material is measured. From the dynamic experimental measurements, we obtain two oscillating discrete-time series of input and output are obtained.

However, from dynamic measurements, the relaxation and creep data of the material can not be obtained directly. The common procedure to obtain the relaxation and creep data from dynamic measurements is : First, one calculates the complex modulus and complex compliance from the measured oscillating strain or stress. Secondly, one converts the complex modulus and complex compliance to the time domain in order to obtain the creep or relaxation data, then, those data are used by existing analytical methods to establish the required models. The whole process could be very time consuming and sometimes inaccurate. In addition, the existing modelling methods for viscoelastic properties are generally very coarse and are difficult to use.

This thesis is mainly concerned with the theoretical methods by which the viscoelastic

models can be obtained from experimental measurements. As the author assumes that the dynamic experimental methods are efficient procedures to carry out viscoelastic experiments, the analysis of this thesis is based on dynamic experimental measurements. In developing an appropriate method to establish a model of relaxation, or creep, function directly from the dynamical measurements of input and output signals, the following major contributions have been made:

1. A new method for dynamic system identification has been developed.

The basic idea of this method is: First, establishing a rational function of polynomials model for the frequency response function of a dynamic system. Then, a discrete-time system analysis method is introduced to identify the order and parameters of the model directly from the input and output measurements of the system.

Several numerical examples show that the proposed method is efficient and accurate.

2. A characterization method for the mechanical response of linear viscoelastic materials from the measurements of output signal and the rate of input signal has been developed.

To develop this method, the basic idea of the above dynamic system identification method has been directly applied.

3. Finally, a characterization method for the mechanical response of linear viscoelastic materials from the measurements of output and input signals has been developed.

To develop this method, the basic idea of the dynamic system identification method has also been used. But in this case, there occurs a singularity problem. In theory, this problem

causes no difficulties, but in the practical calculation, there will be significant difficulties. To overcome the problem, a modified system is introduced. The numerical examples given show that the proposed method is reasonable and the presented models are accurate and efficient.

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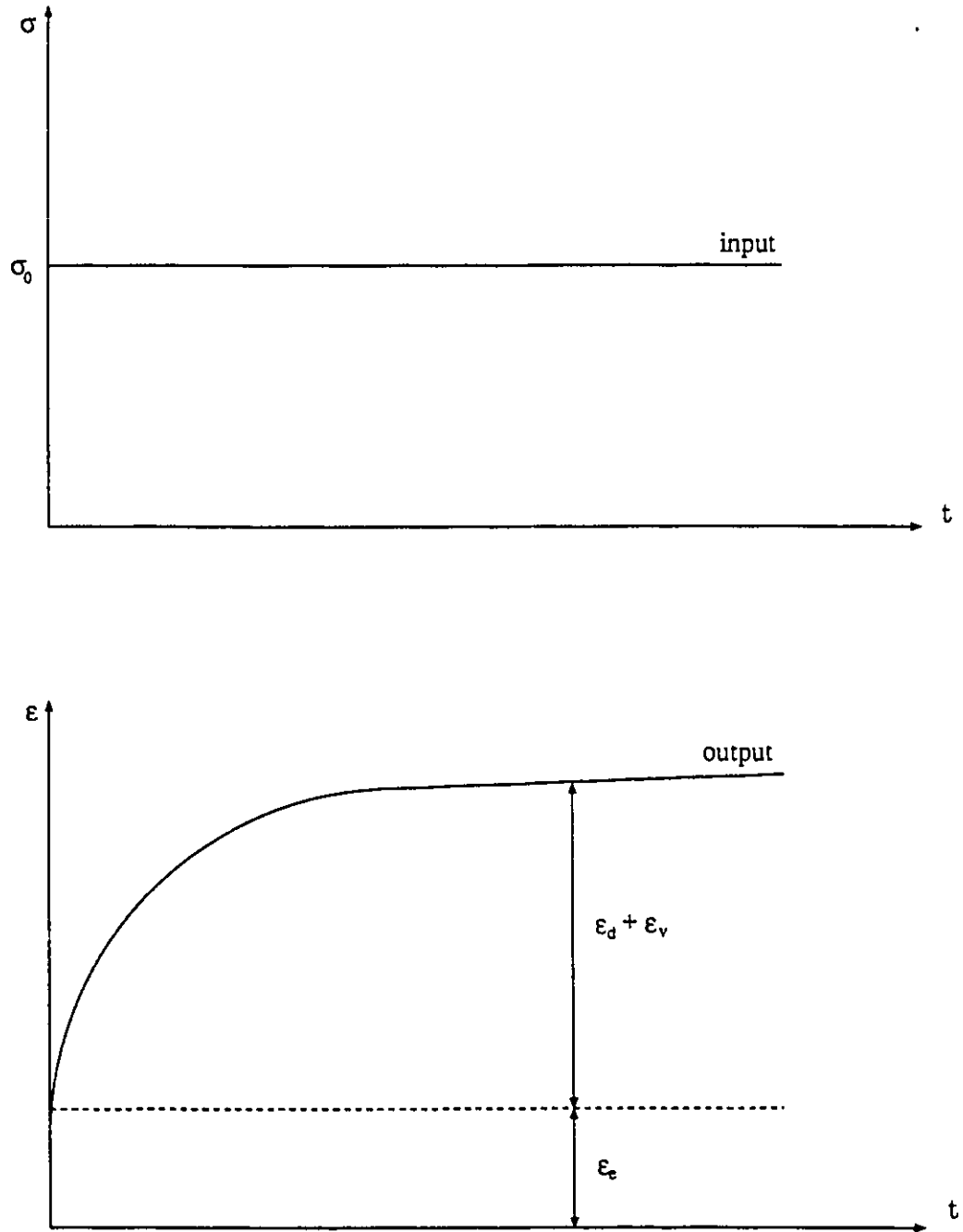


Fig.2.1 Creep response of a linear viscoelastic material under constant level of stress

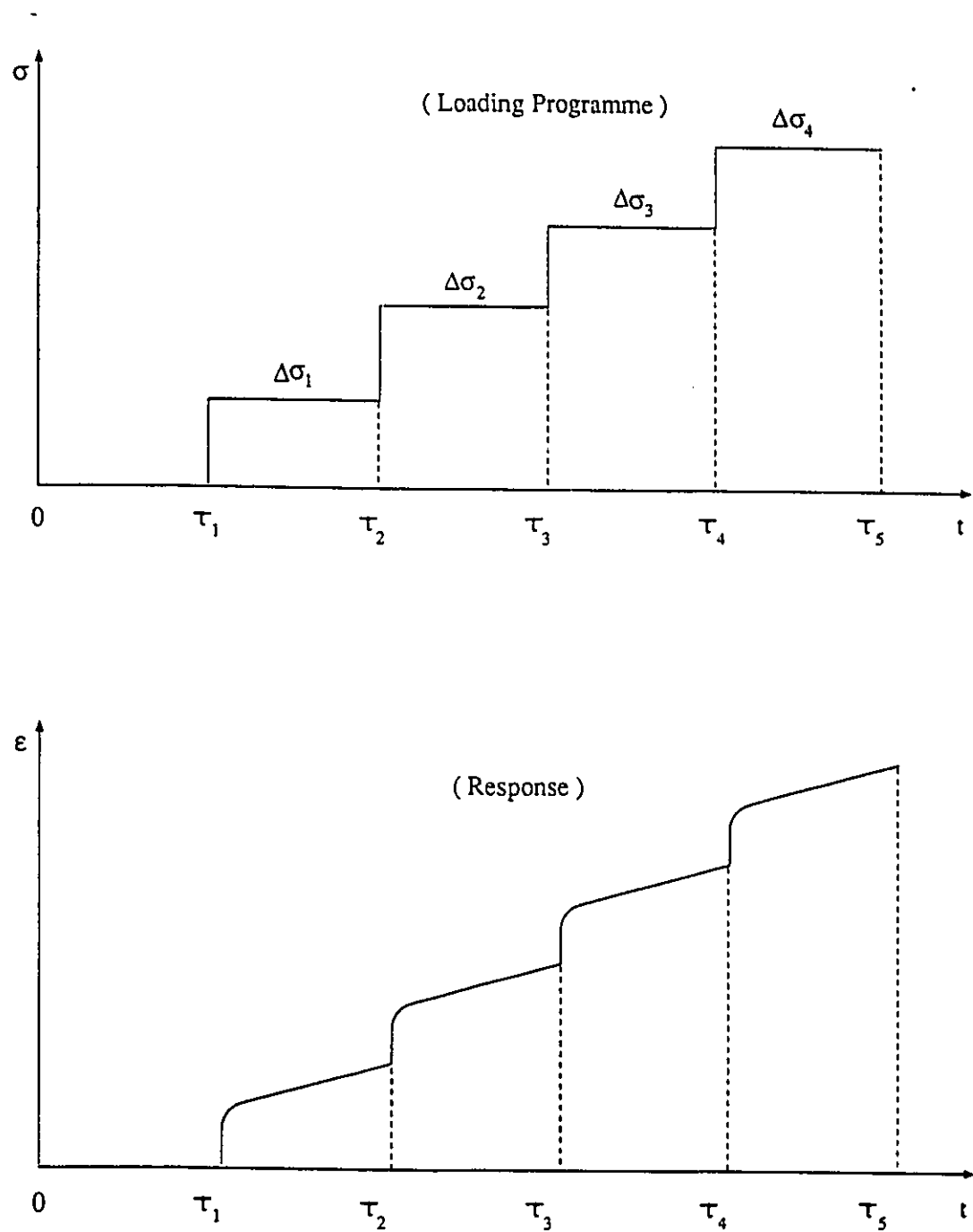


Fig.2.2 Creep behaviour of a linear viscoelastic material under an increasing tensile loading [see ward, 1983]

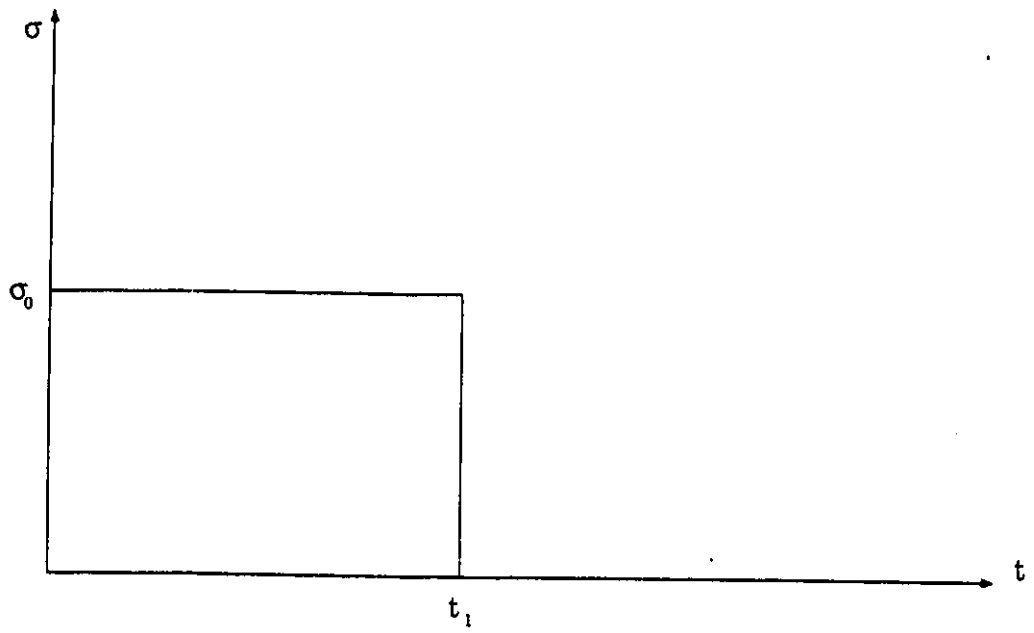


Fig.2.3 An input loading situation in a creep experiment

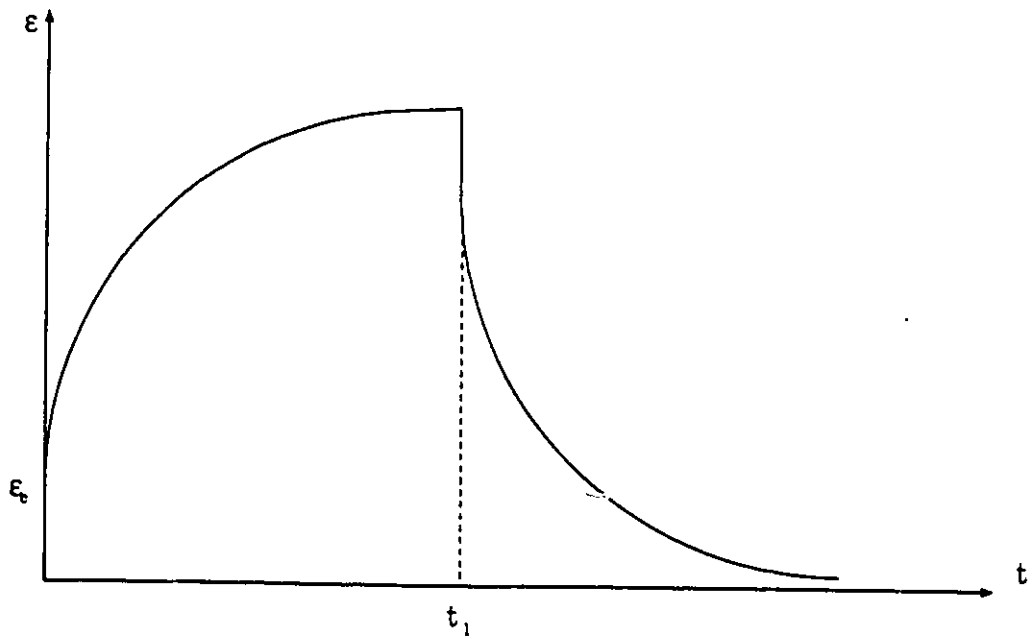


Fig.2.4 The response of a linear viscoelastic material to the loading situation shown in Fig.2.3 [see Ward, 1983]

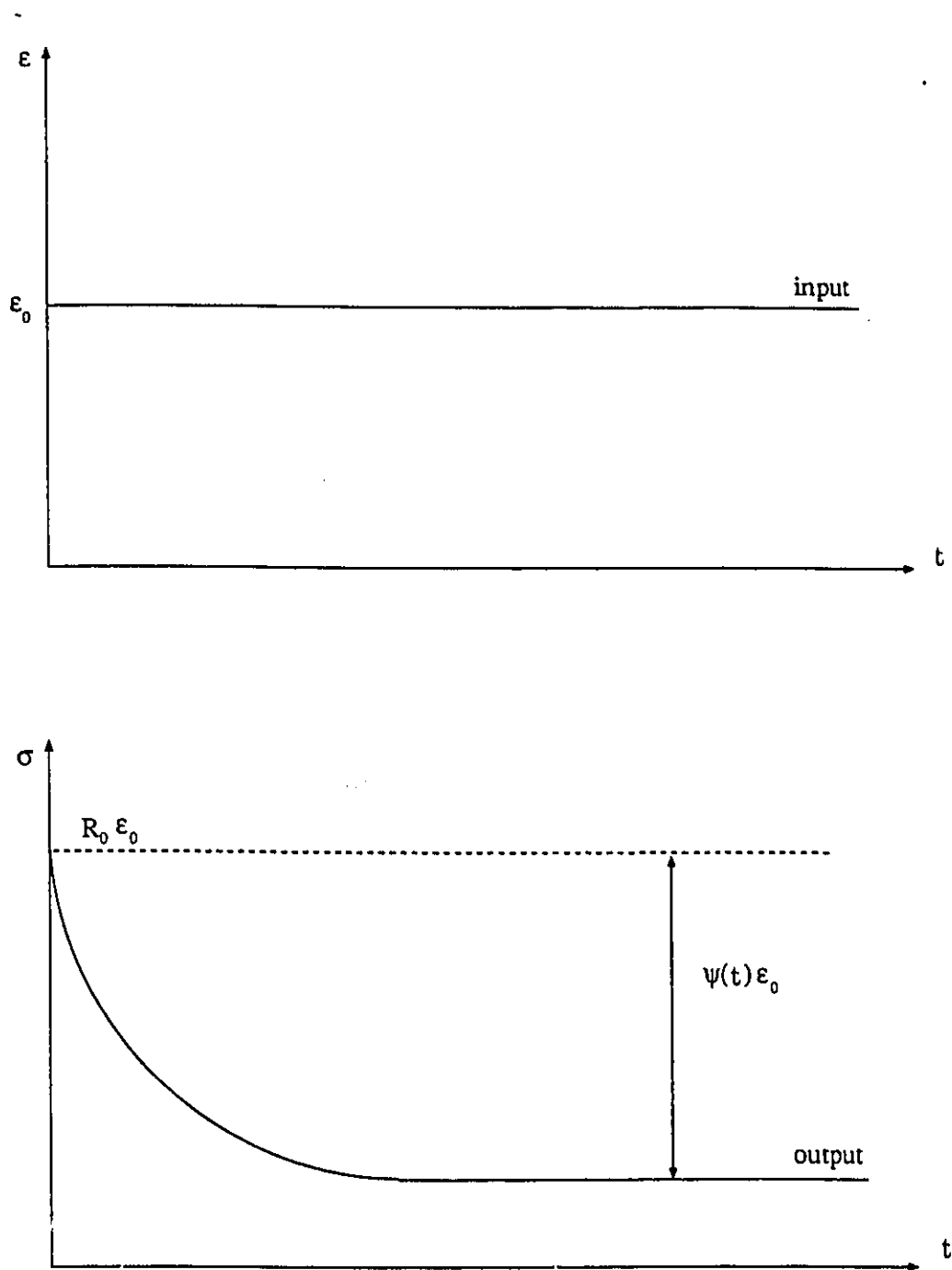


Fig.2.5 Relaxation response of a linear viscoelastic material

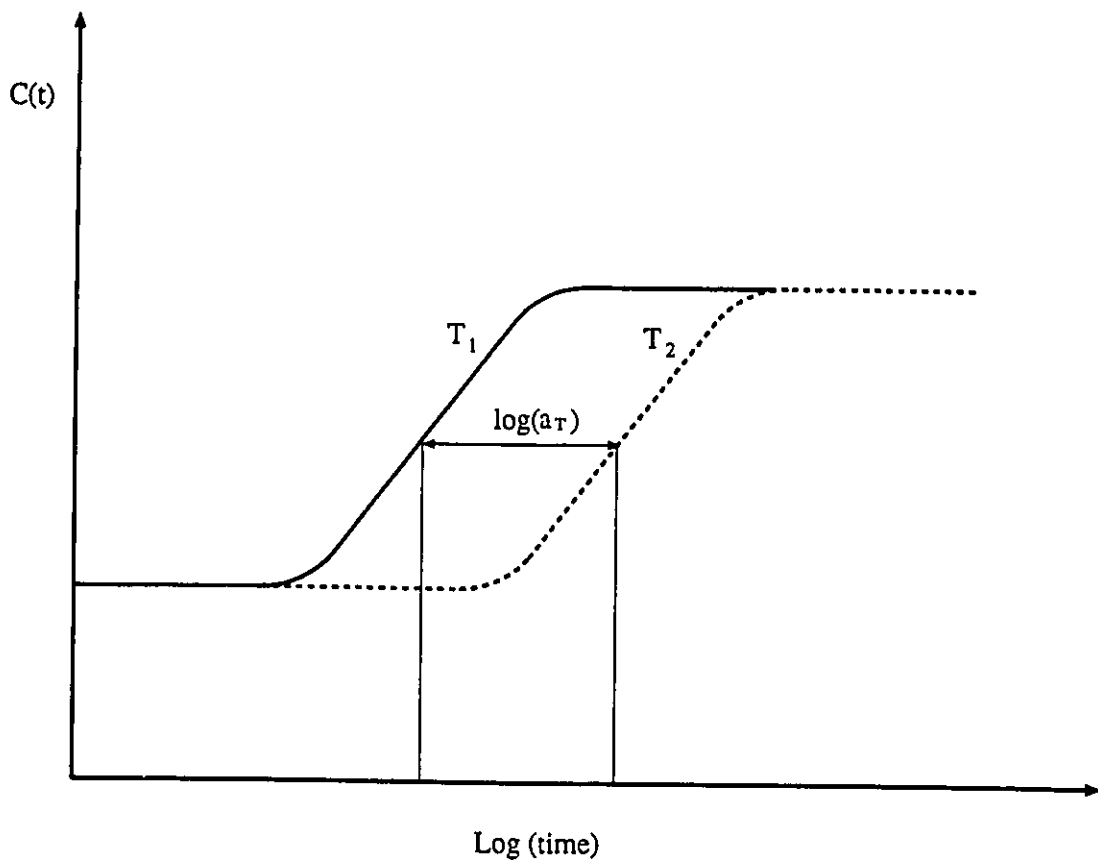


Fig.3.1 Effect of temperature on the creep function [see Aklonis, 1972]

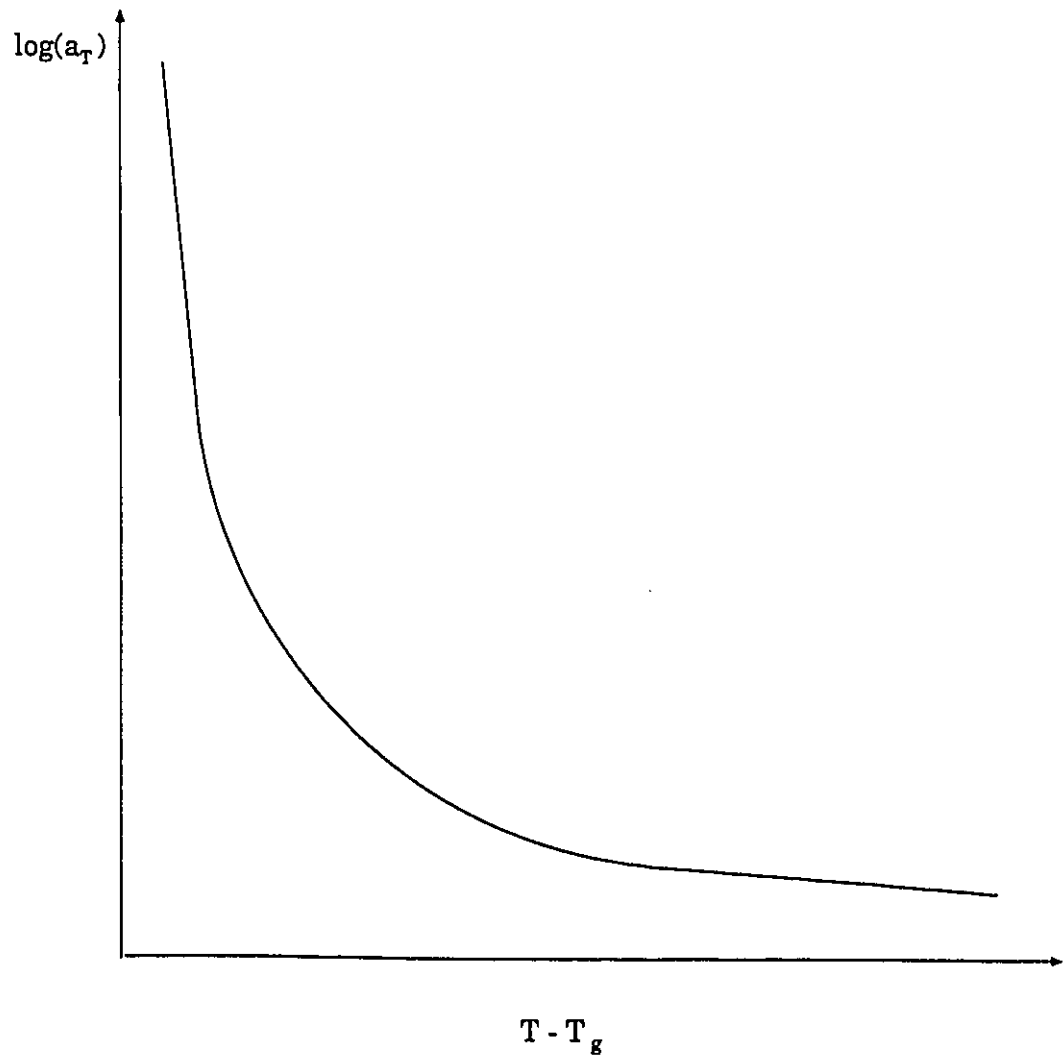
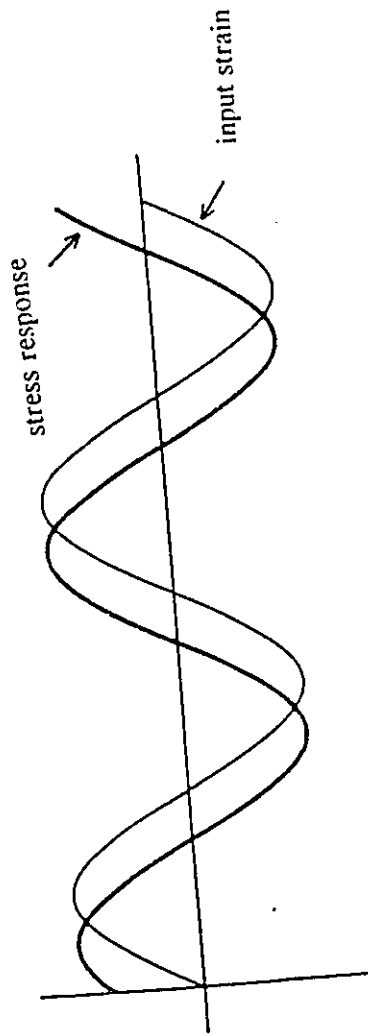
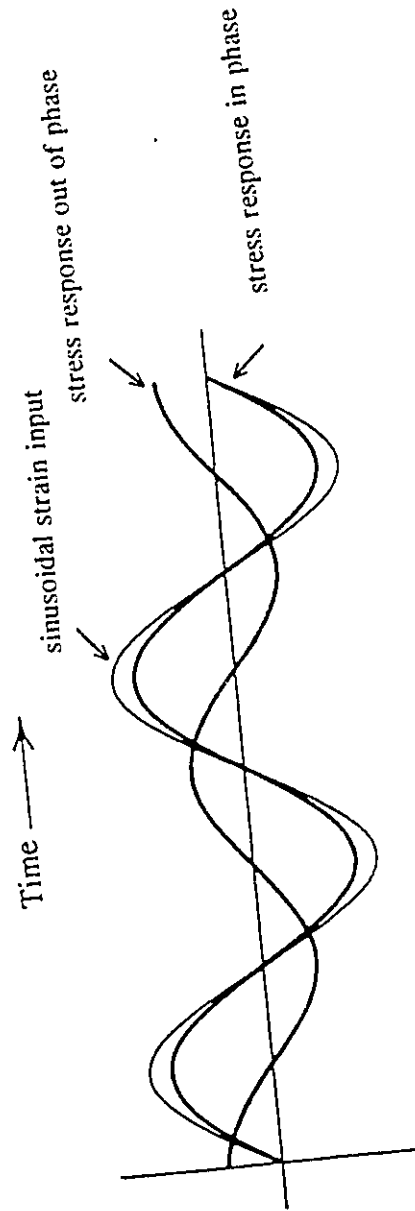


Fig.3.2 The variation of $\log a_T$ versus the temperature difference $T - T_g$
[see Aklonis, 1972]



(a) Stress response is out-of-phase with the sinusoidal strain input



(b) Decomposition of stress response in (a) into two components, one in-phase and one out of phase, with the strain input

Fig.3.3. Sinusoidal input and response [see Ward, 1983]

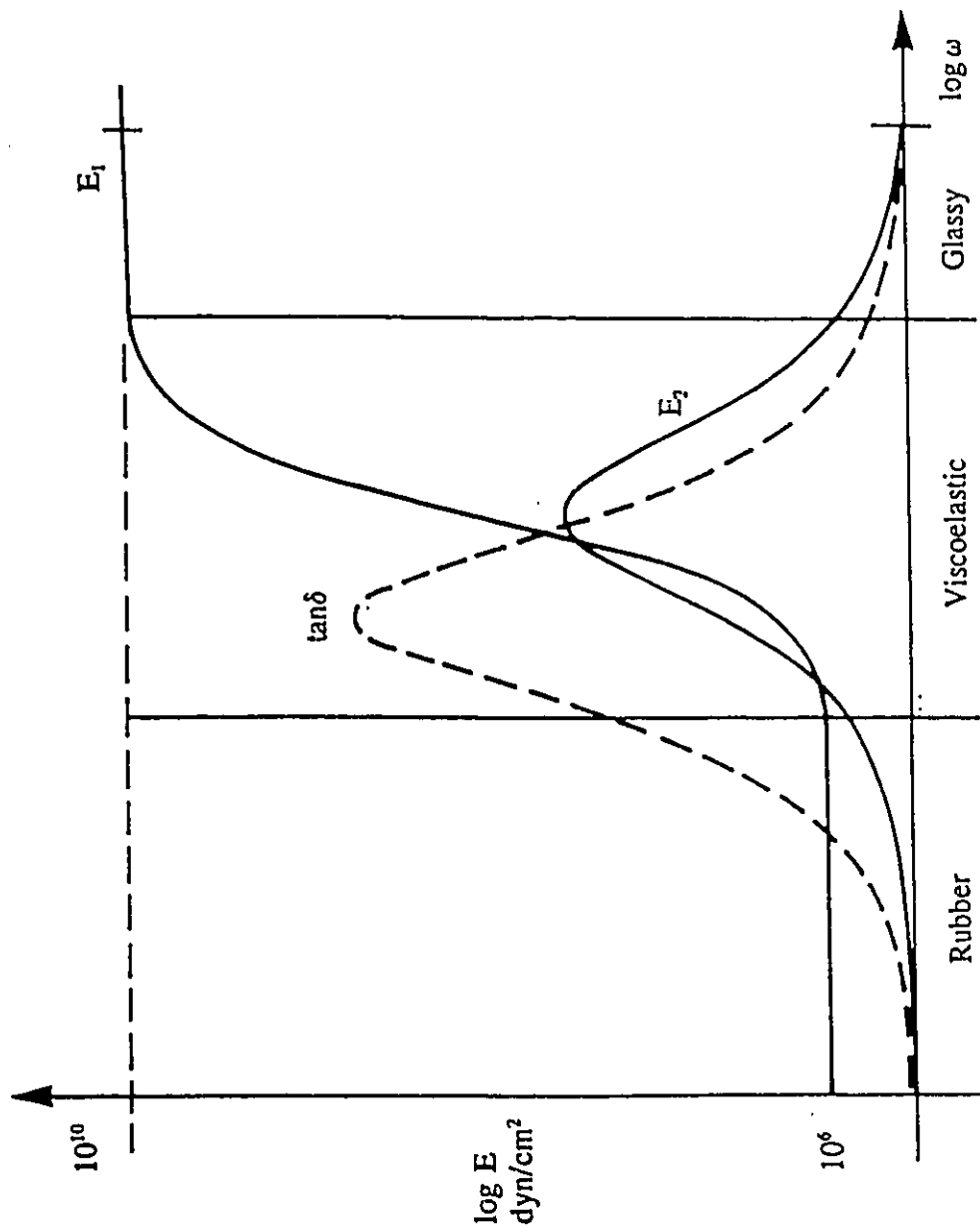


Fig. 3.4 E_1 , E_2 and δ versus frequency [see Ward, 1983]

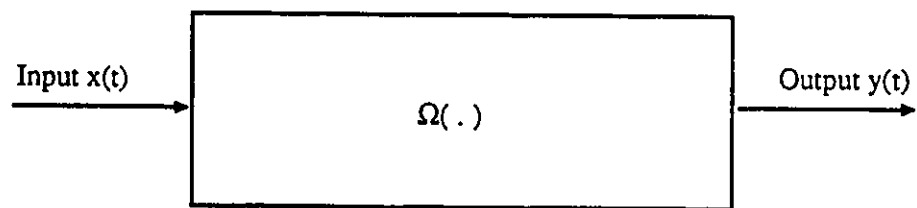
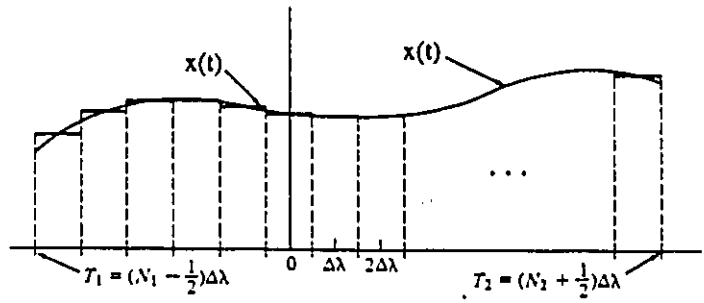
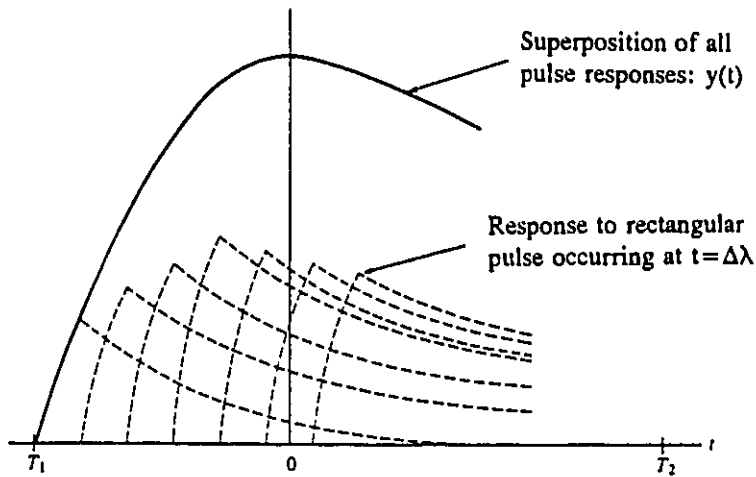


Fig.4.1 A symbolic representation of a mechanical system



(a) Input to a defined, linear system and a staircase approximation



(b) Output of a defined, linear system due to the staircase approximate input

Fig.4.2 Input and output signals for a defined, linear system that is used to derive the superposition integral [see Rodger, 1983]

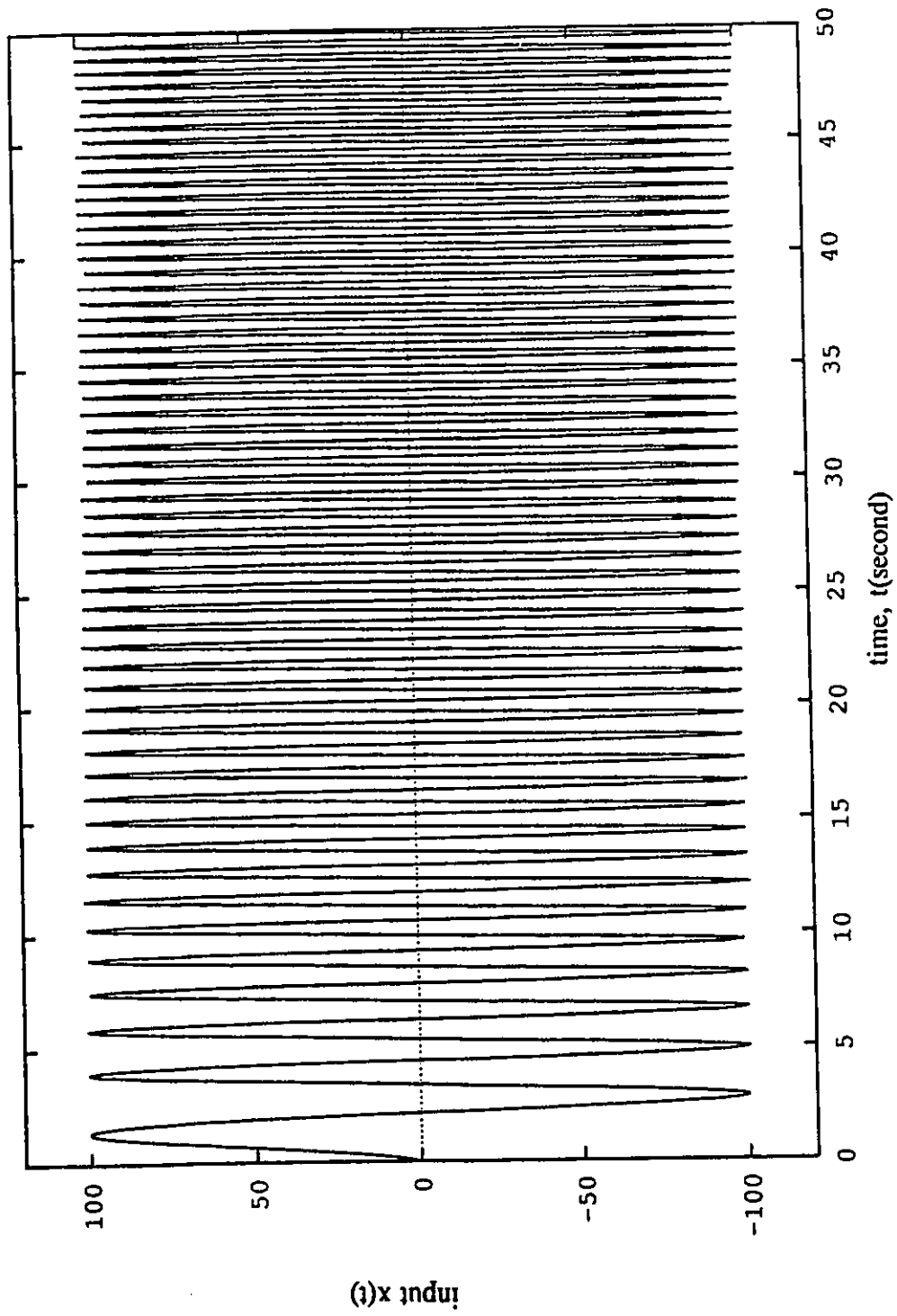


Fig 4.3 Input $x(t) = 100 \sin(t^{1.5})$ with $\Delta T = 0.01$.

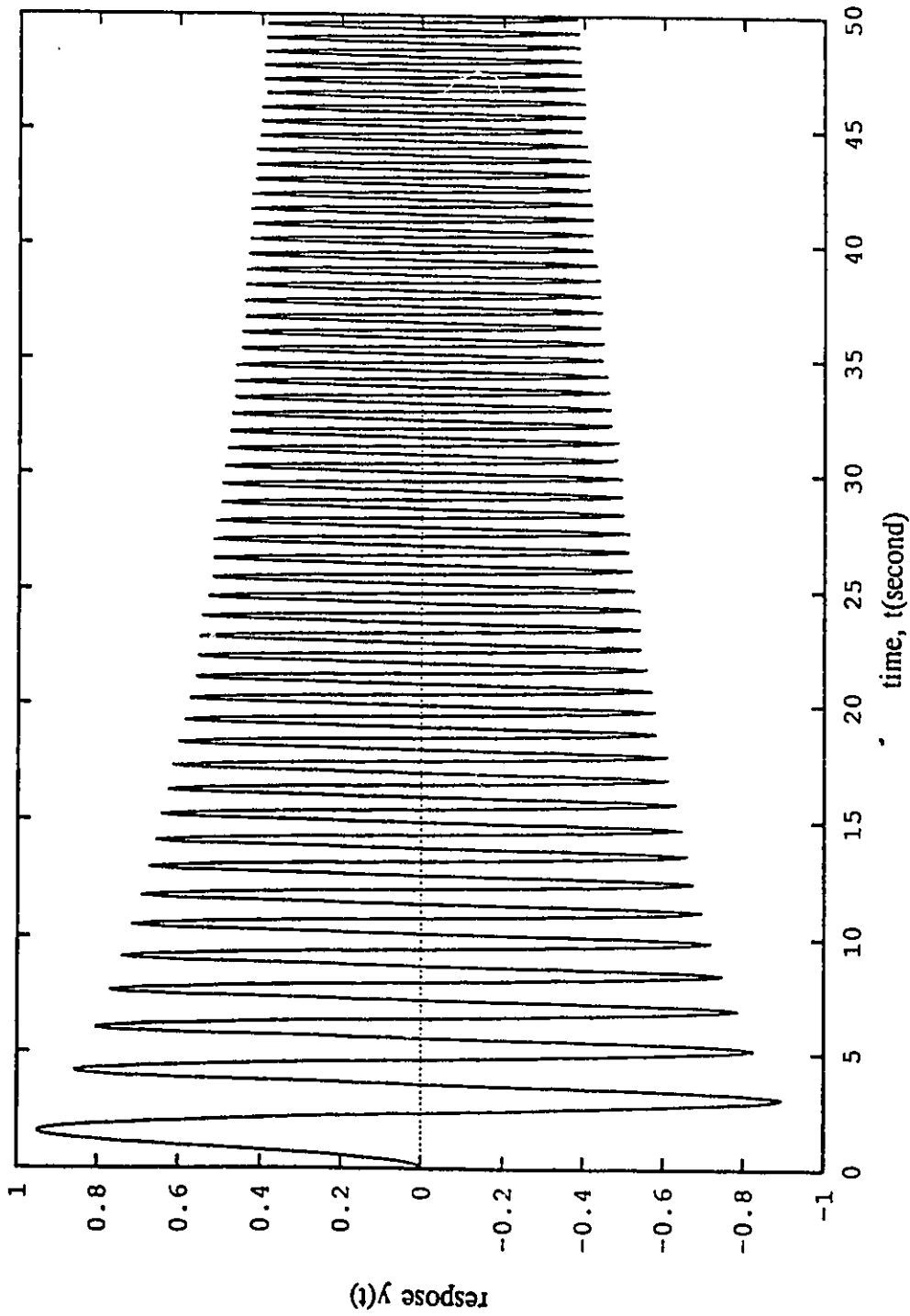


Fig.4.4 Output $y(t)$ corresponding to the input $x(t)$ of Fig.4.3.

Second order system $\ddot{y} + 25\dot{y} + 100y = x(t)$ with parameters $a = 1.0$,

$b_1 = 25$, $b_2 = 100$ and $p = 2$.

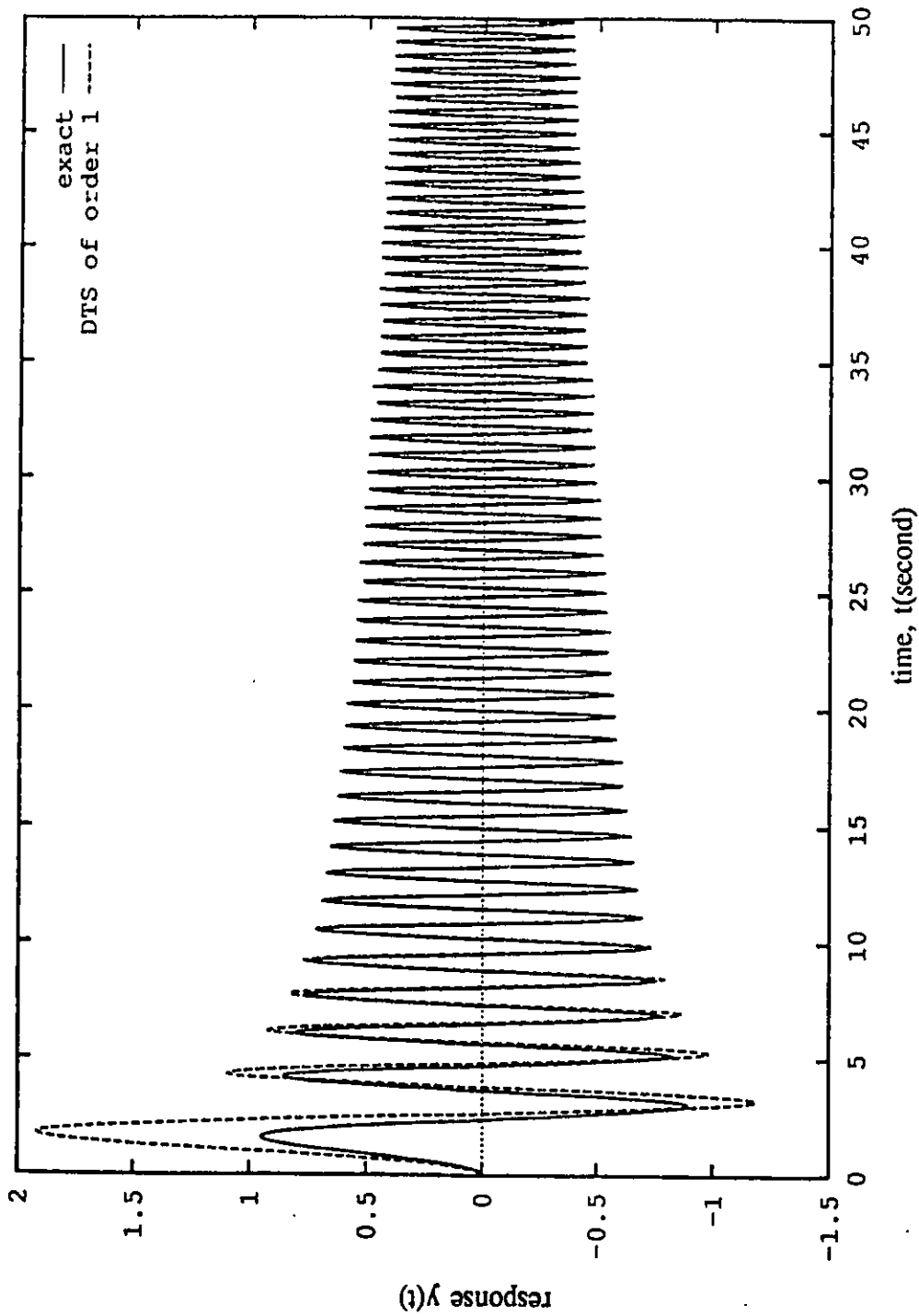


Fig.4.5 The exact response and the estimated response $y(t)$ from the 1st order DTS.

Second order system $\ddot{y} + 25\dot{y} + 100y = x(t)$ with parameters $a = 1.0$,

$b_1 = 25$, $b_2 = 100$, $p = 2$, and input $x(t)$ of Fig.4.3.

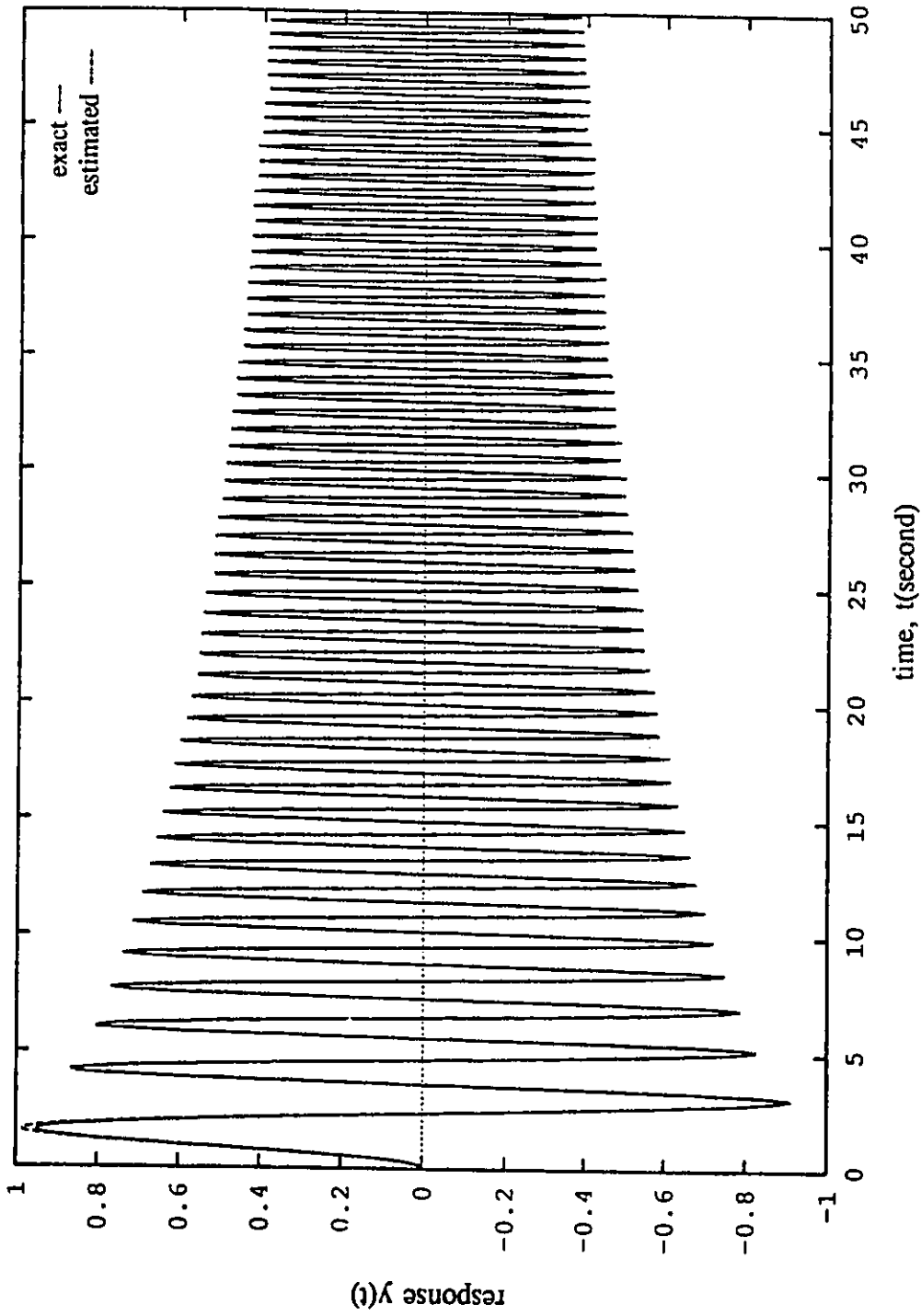


Fig.4.6 The exact response and the estimated response $y(t)$ from the 2nd order DTS.

Second order system $\ddot{y} + 2.5\dot{y} + 100y = x(t)$ with parameters $a = 1.0$,
 $b_1 = 25$, $b_2 = 100$, $p = 2$, and input $x(t)$ of Fig.4.3.

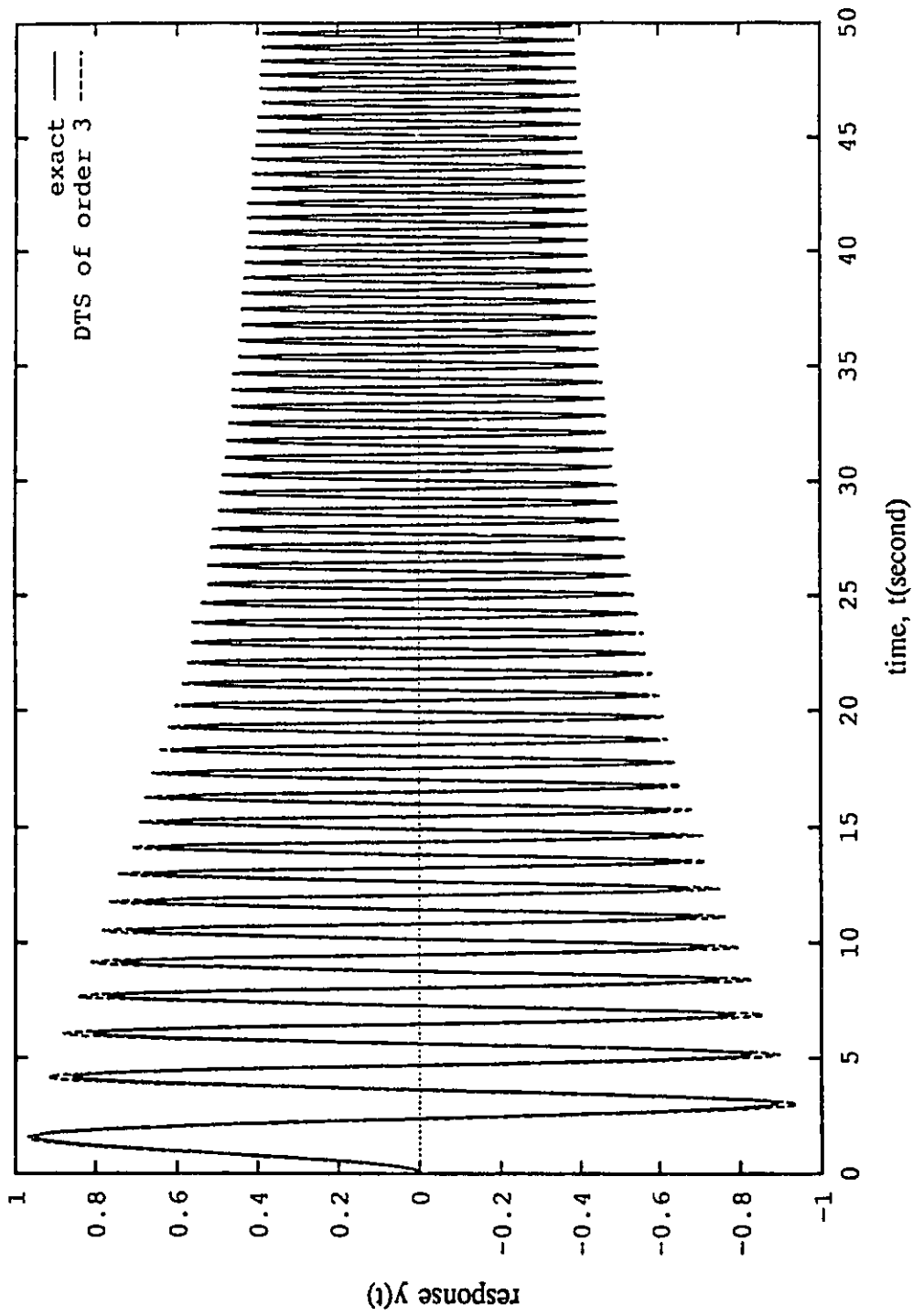


Fig.4.7 The exact response and the estimated response $y(t)$ from the 3rd order DTS.

Second order system $\ddot{y} + 25\dot{y} + 100y = x(t)$ with parameters $a = 1.0$,

$b_1 = 25$, $b_2 = 100$, $p = 2$, and input $x(t)$ of Fig.4.3.

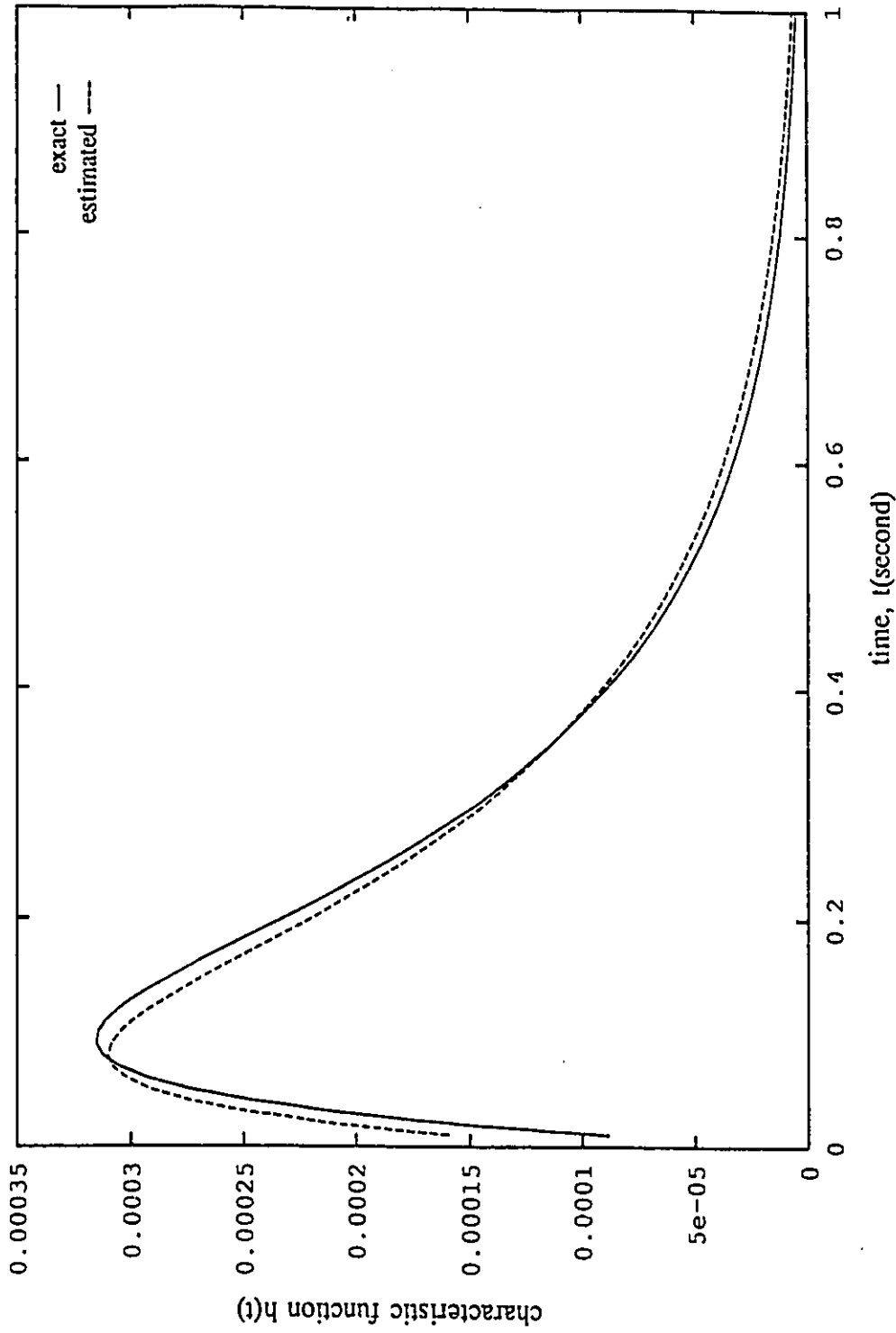


Fig.4.8 The exact and the estimated system characteristic functions $h(t)$ from the 2nd order DTS.

Second order system $\ddot{y} + 25\dot{y} + 100y = x(t)$ with parameters
 $a = 1.0$, $b_1 = 25$, $b_2 = 100$, $p = 2$, and input $x(t)$ of Fig.4.3.

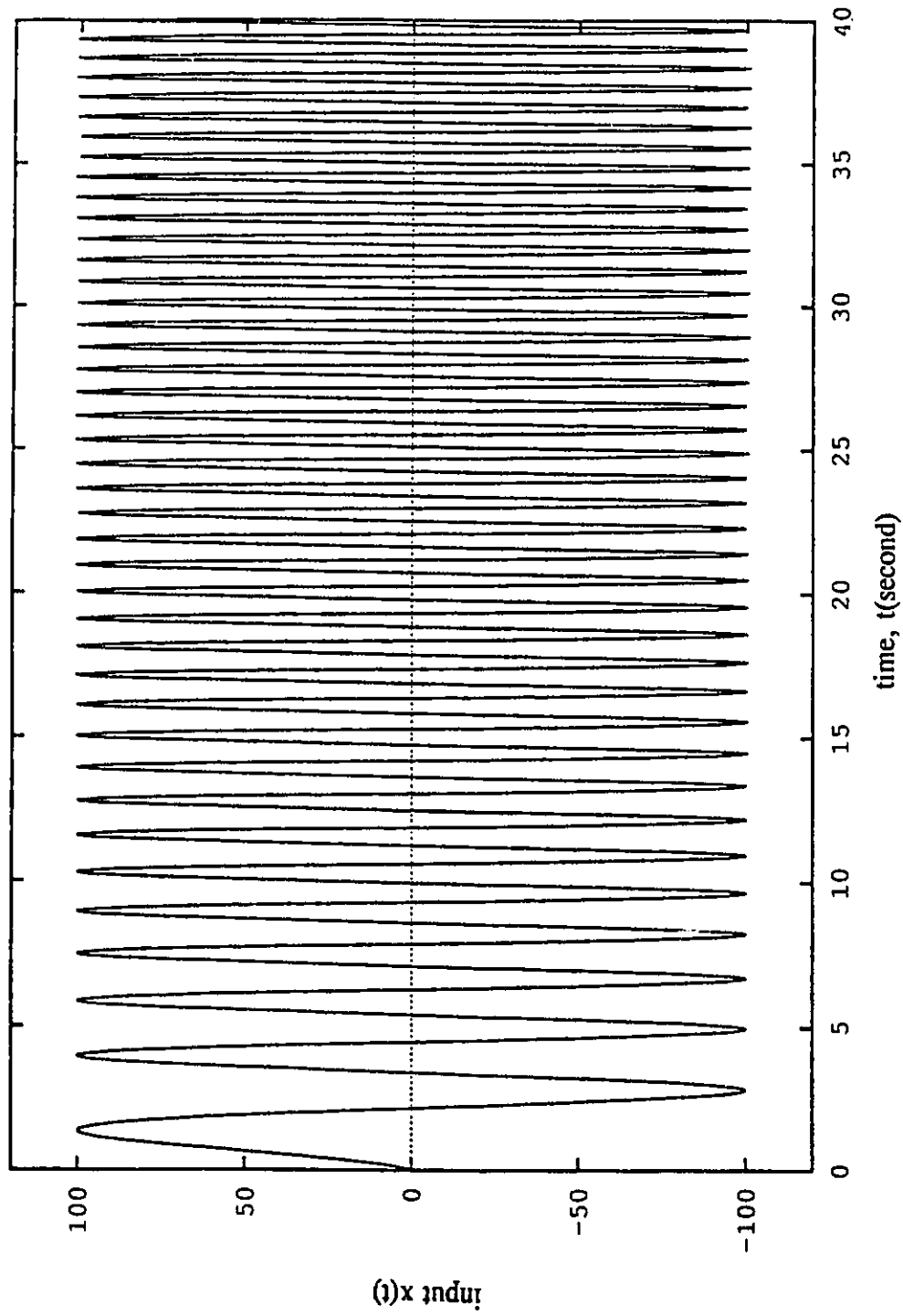


Fig.4.9 Input $x(t) = 100 \sin(t^{1.5})$ with $\Delta T = 0.01$.

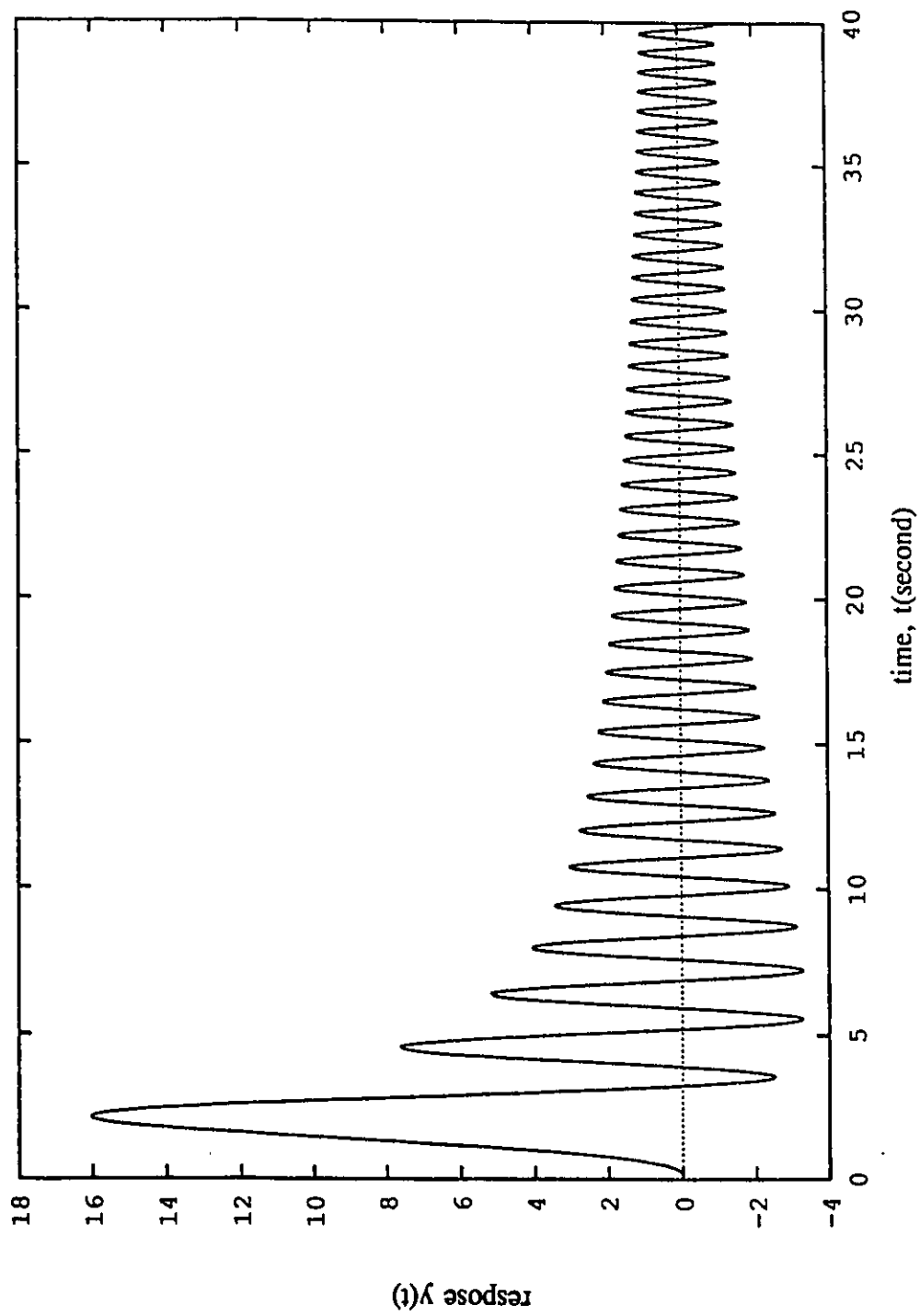


Fig.4.10 Output $y(t)$ corresponding to the input $x(t)$ of Fig.4.9.

Second order system $\ddot{y} + 5.5\dot{y} + 2.5y = x(t)$ with parameters $a = 1.0$,
 $b_1 = 5.5$, $b_2 = 2.5$ and $p = 2$.

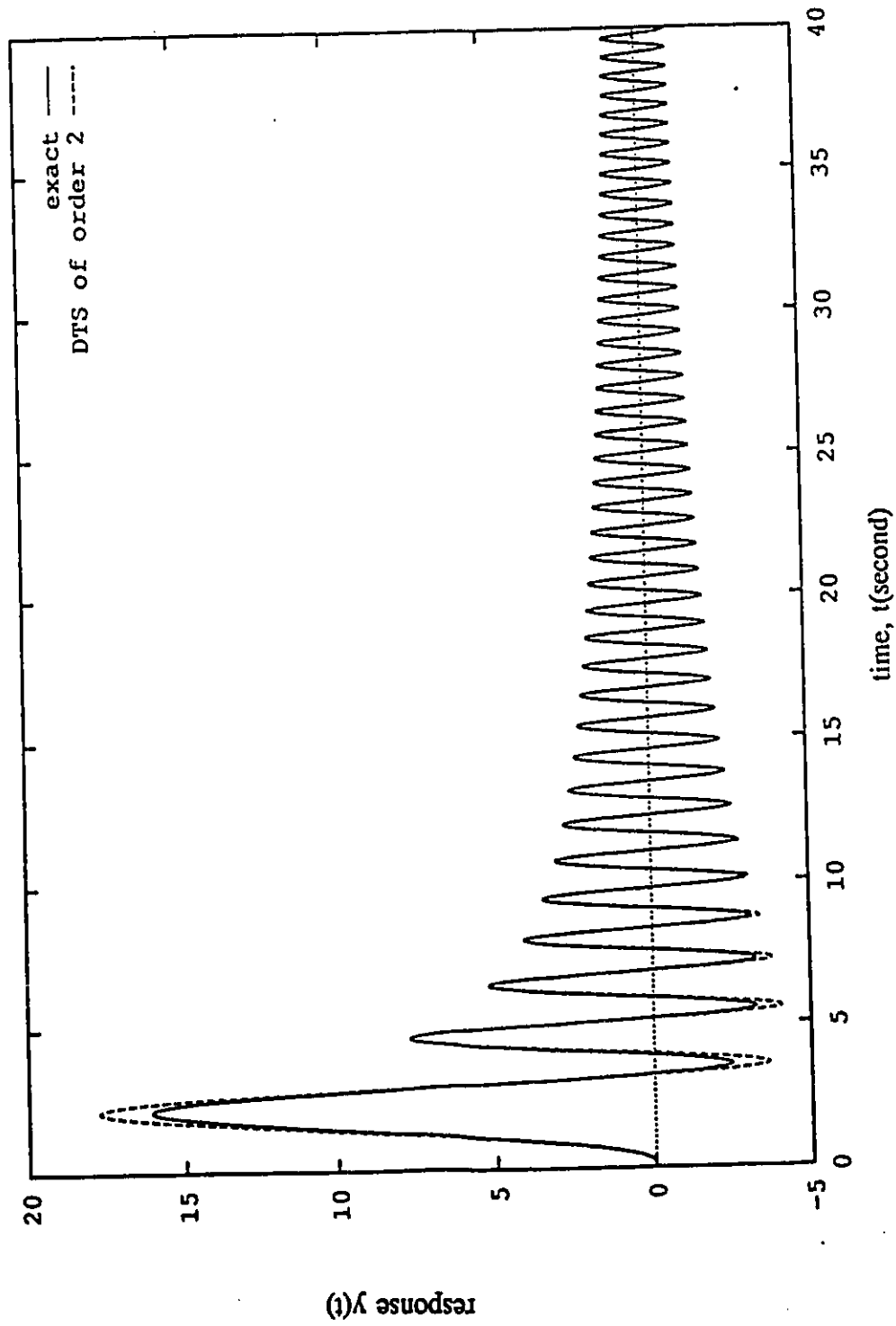


Fig.4.11 The exact response and the estimated response $y(t)$ from the 2nd order DTS. Second order system $\ddot{y} + 5.5\dot{y} + 2.5y = x(t)$ with parameters $a = 1.0$, $b_1 = 5.5$, $b_2 = 2.5$, $p = 2$, and input $x(t)$ of Fig.4.9.

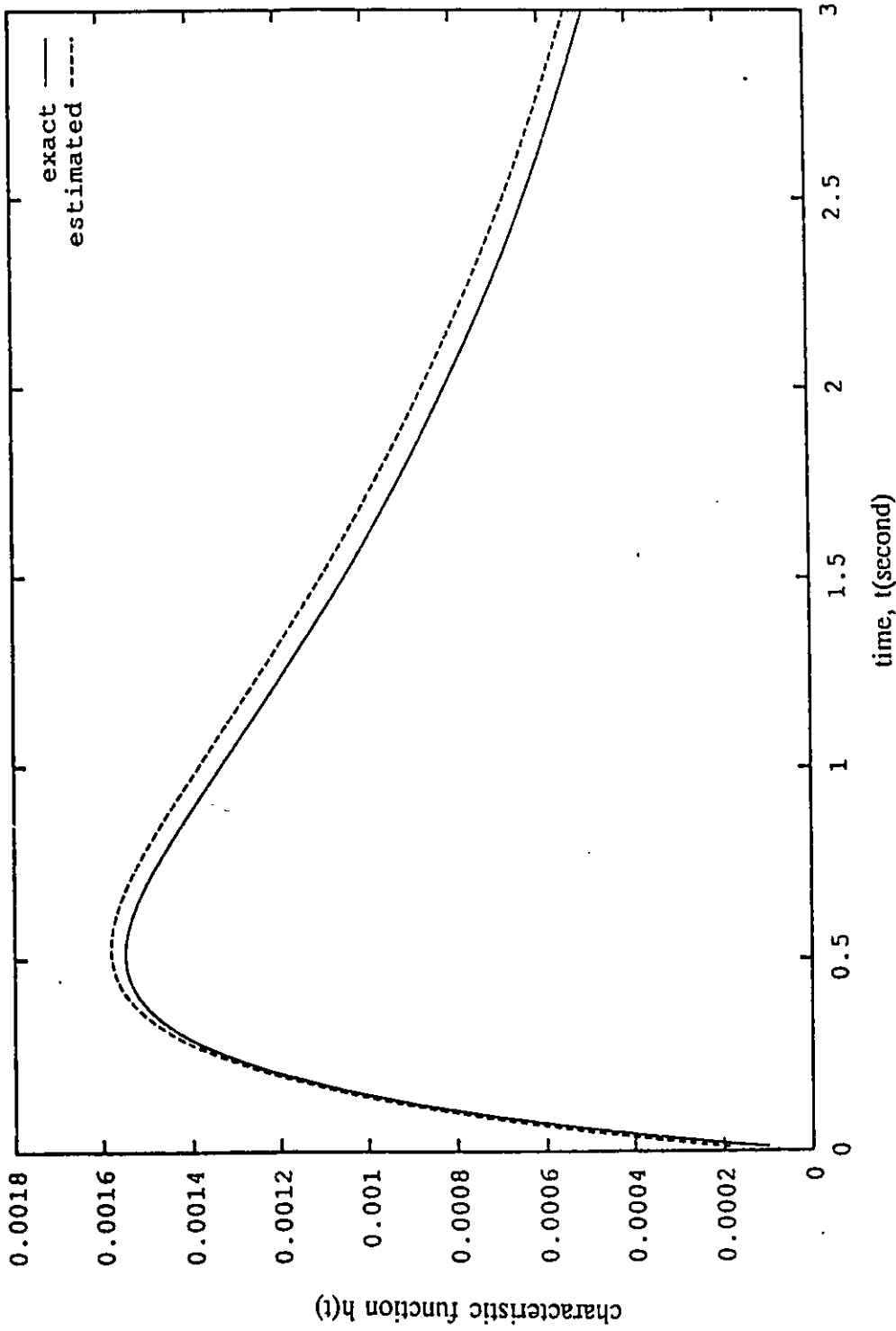


Fig.4.12 The exact and the estimated system characteristic functions $h(t)$ from the 2nd order DTS.

Second order system $\ddot{y} + 5.5\dot{y} + 2.5y = x(t)$ with parameters $a = 1.0$, $b_1 = 5.5$, $b_2 = 2.5$, $p = 2$, and input $x(t)$ of Fig.4.9.

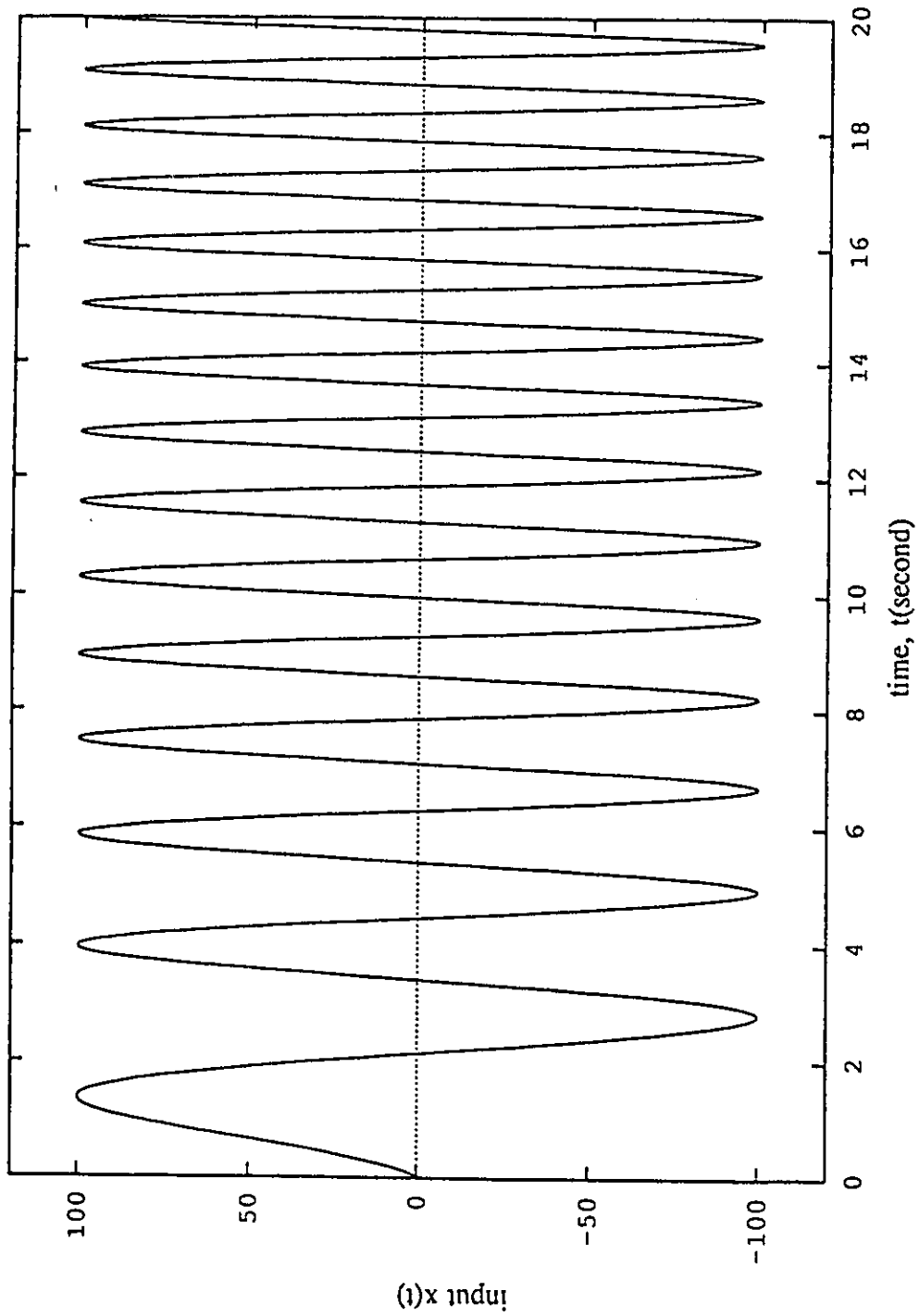


Fig.5.1 Input $x(t) = 100 \sin(t^{1.5})$ with $\Delta T = 0.01$.

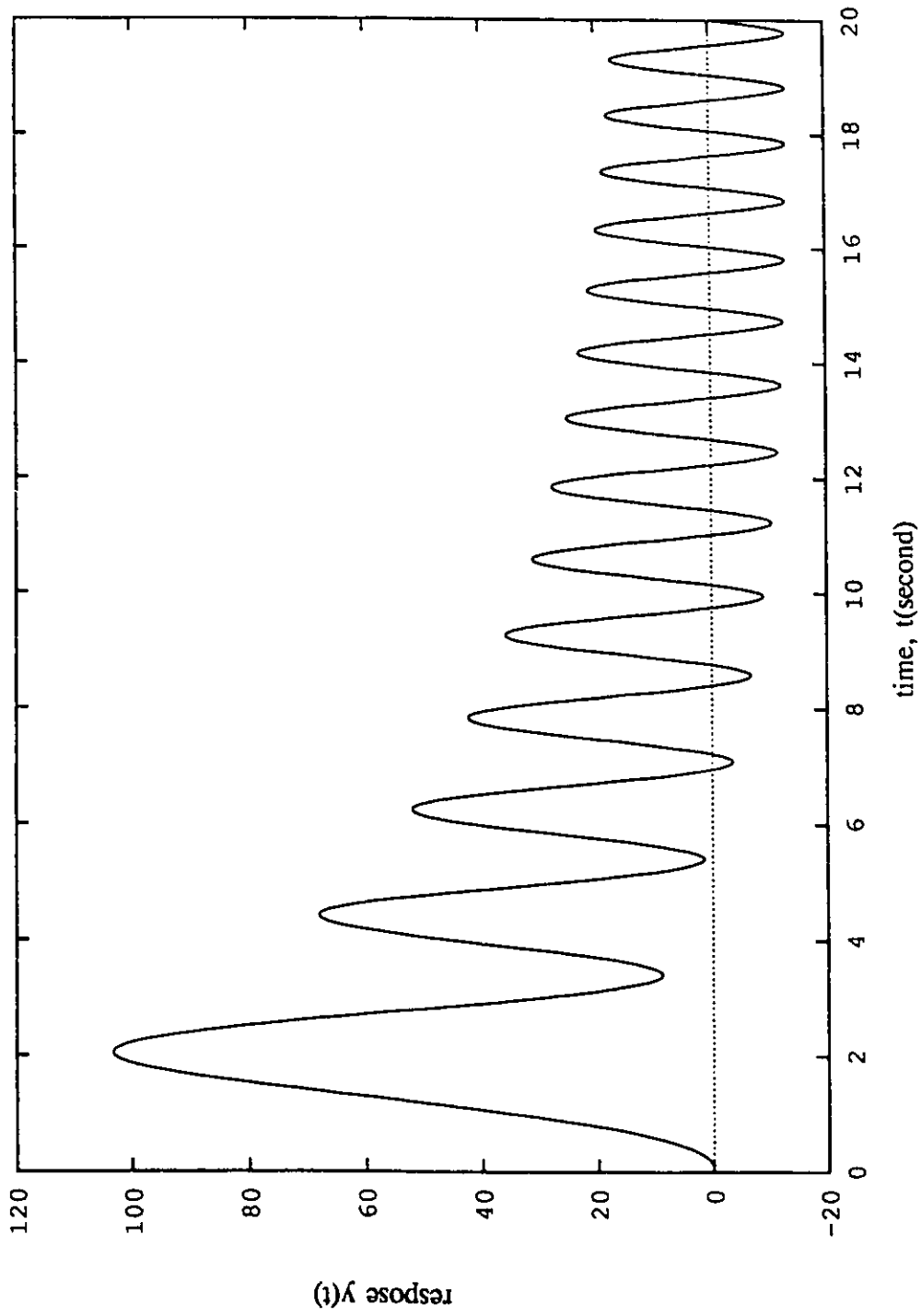


Fig.5.2 Output $y(t)$ corresponding to the input $x(t)$ of Fig.5.1.
 First order system $\dot{y} + 0.2y = x(t)$ with parameters $a = 1.0$, $b_1 = 0.2$
 and $p = 1$.

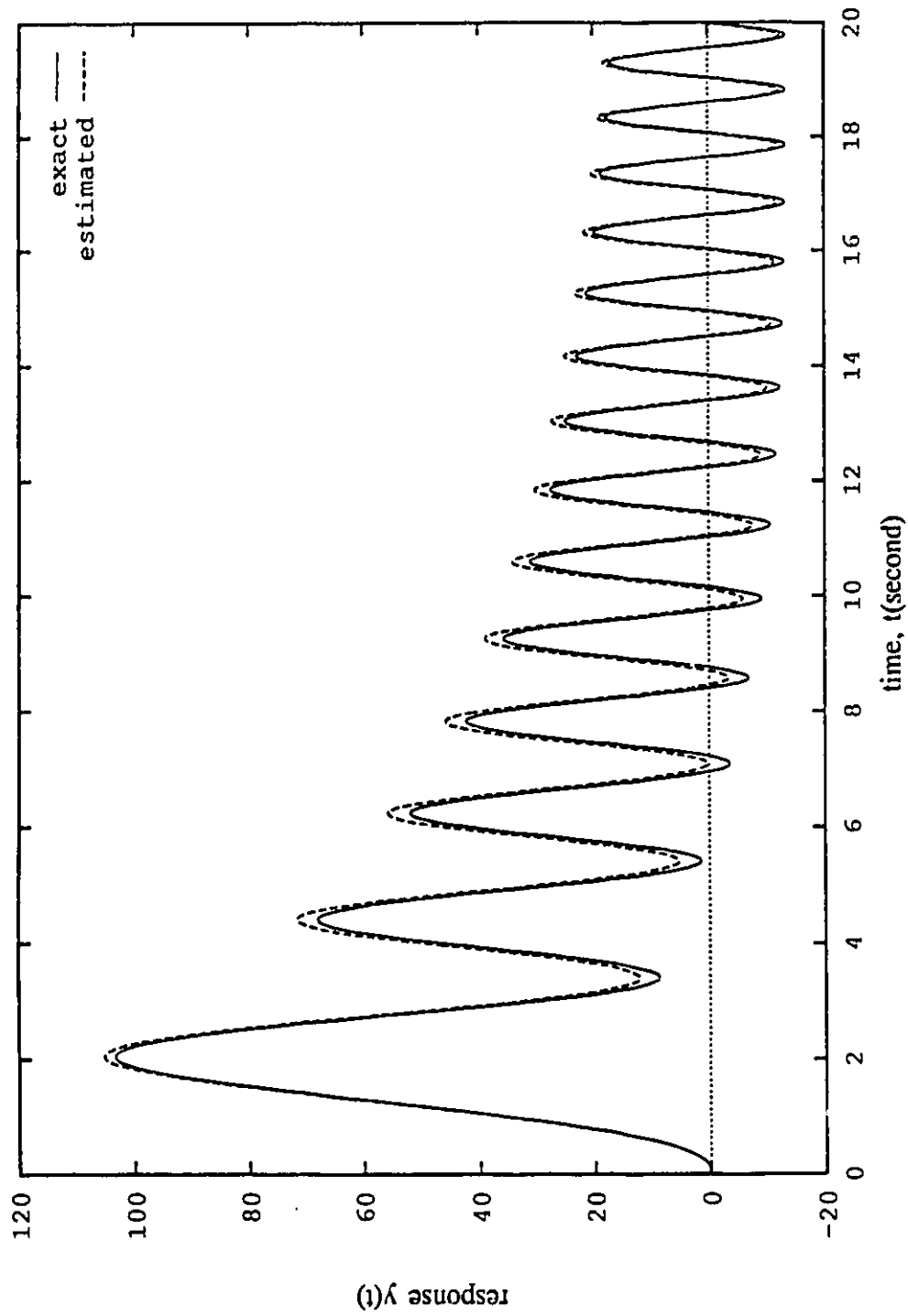


Fig.5.3 The exact and the estimated responses $y(t)$ from the 1st order DTS.

First order system $\dot{y} + 0.2y = x(t)$ with parameters $a = 1.0$, $b_1 = 0.2$ and $p = 1$, and input $x(t)$ of Fig.5.1.

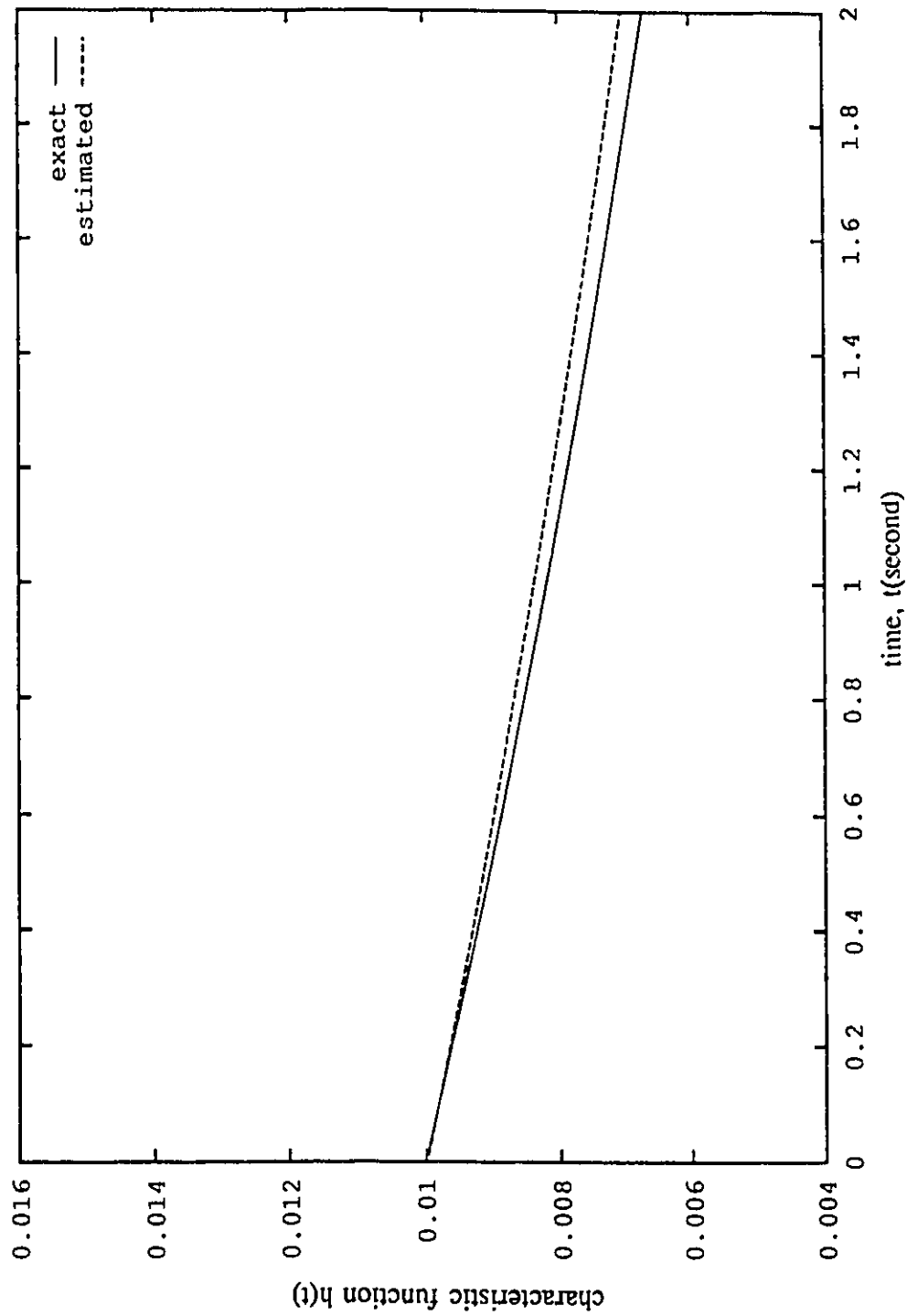


Fig. 5.4 The exact and the estimated system characteristic functions $h(t)$ from the 1st order DTS.

First order system $\dot{y} + 0.2y = x(t)$ with parameters $a = 1.0$, $b_1 = 0.2$ and $p = 1$, and input $x(t)$ of Fig. 5.1.

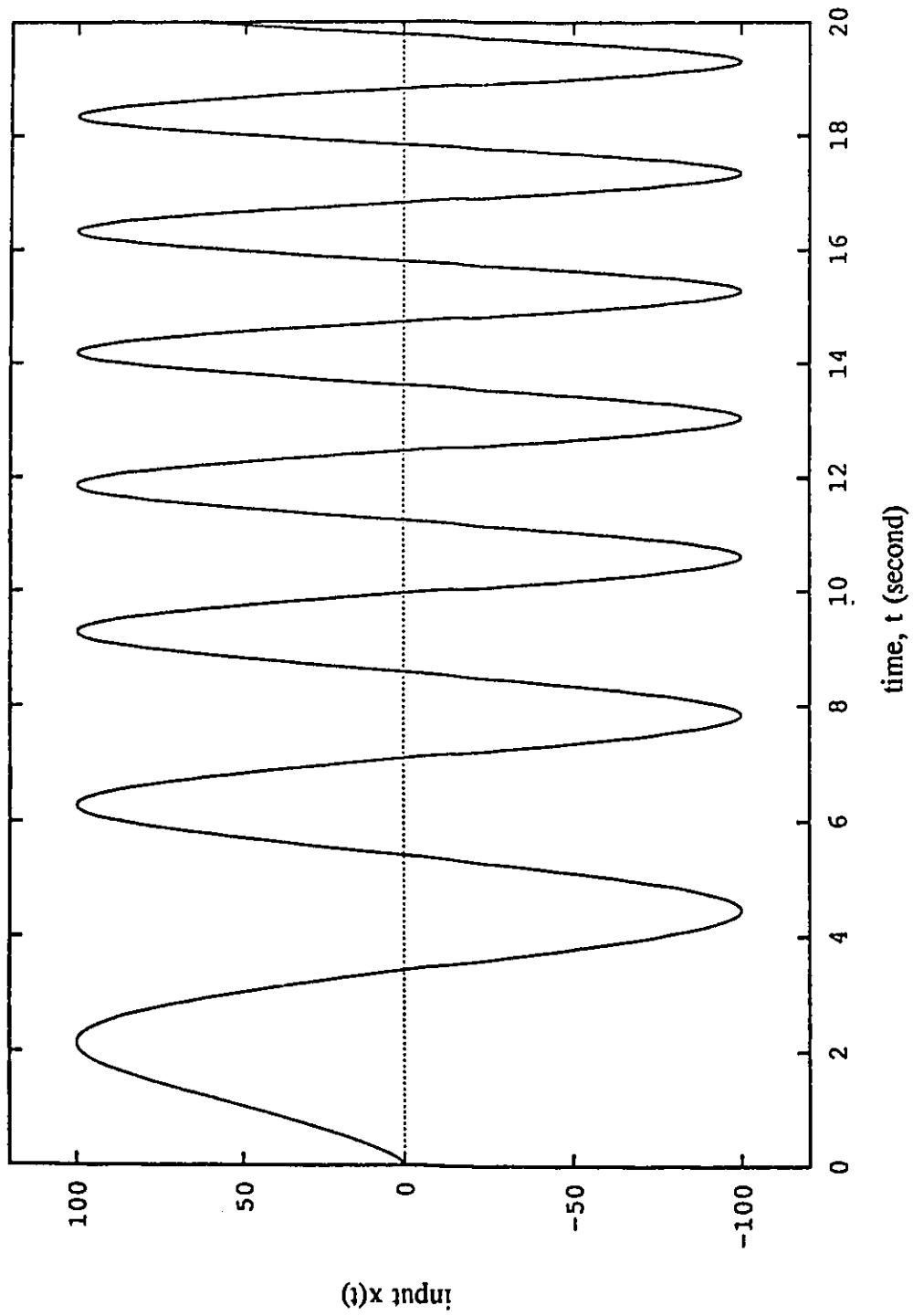


Fig.5.5 Input $x(t) = 100 \sin(t^{1.5})$ with $\Delta T = 0.01$.

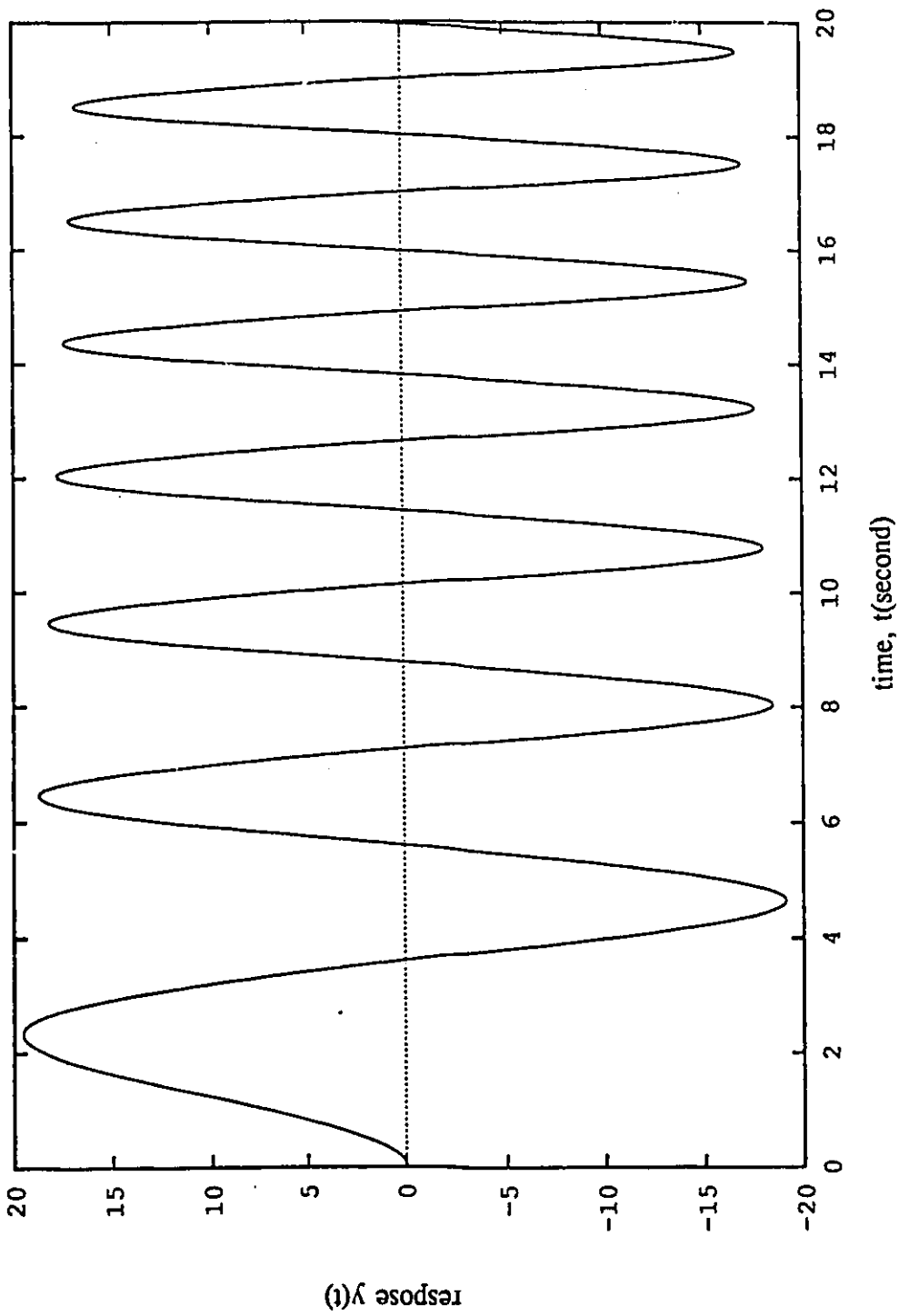


Fig.5.6 Output $y(t)$ corresponding to the input $x(t)$ of Fig.5.5.

First order system $\dot{y} + 5y = x(t)$ with parameters $a = 1.0$, $b_1 = 5$
and $p = 1$.

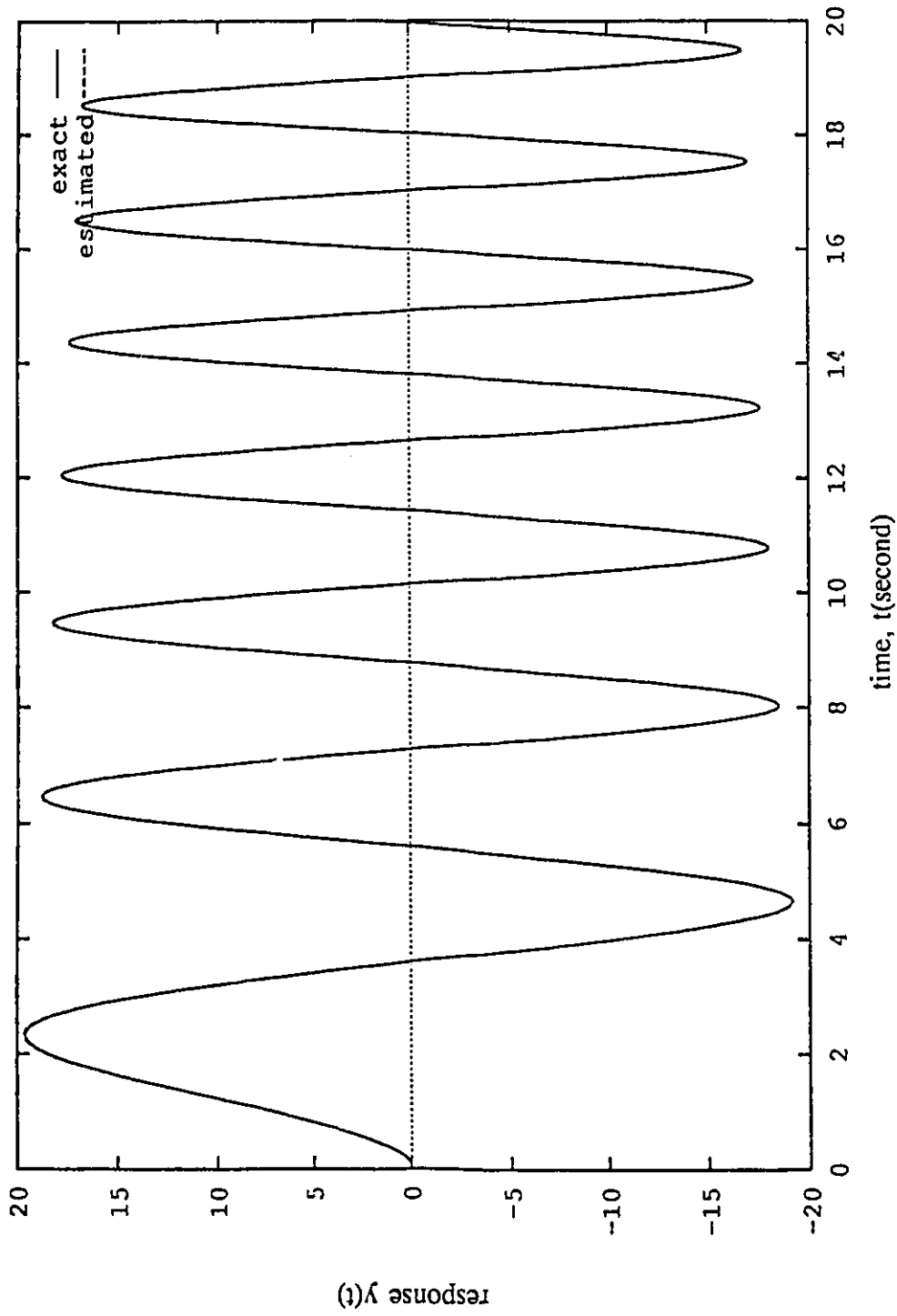


Fig.5.7 The exact and the estimated responses $y(t)$ from the 1st order DTS.

First order system $\dot{y} + 5y = x(t)$ with parameters $a = 1.0$, $b_1 = 5$ and $p = 1$, and input $x(t)$ of Fig.5.5.

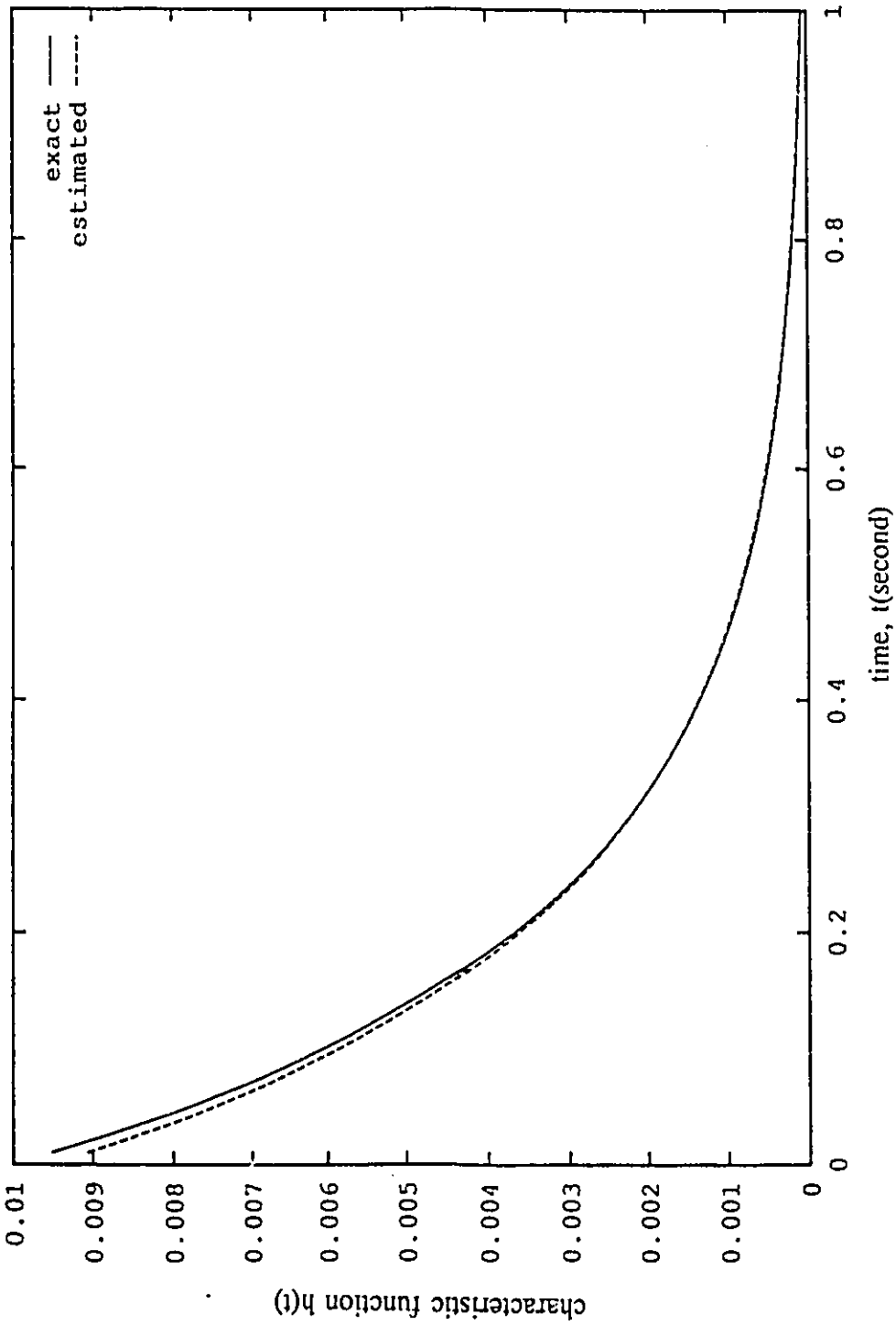


Fig.5.8 The exact and the estimated system characteristic functions $h(t)$ from the 1st order DTS.
 First order system $\dot{y} + 5y = x(t)$ with parameters $a = 1.0, b_1 = 5$ and $p = 1$, and input $x(t)$ of Fig.4.6.

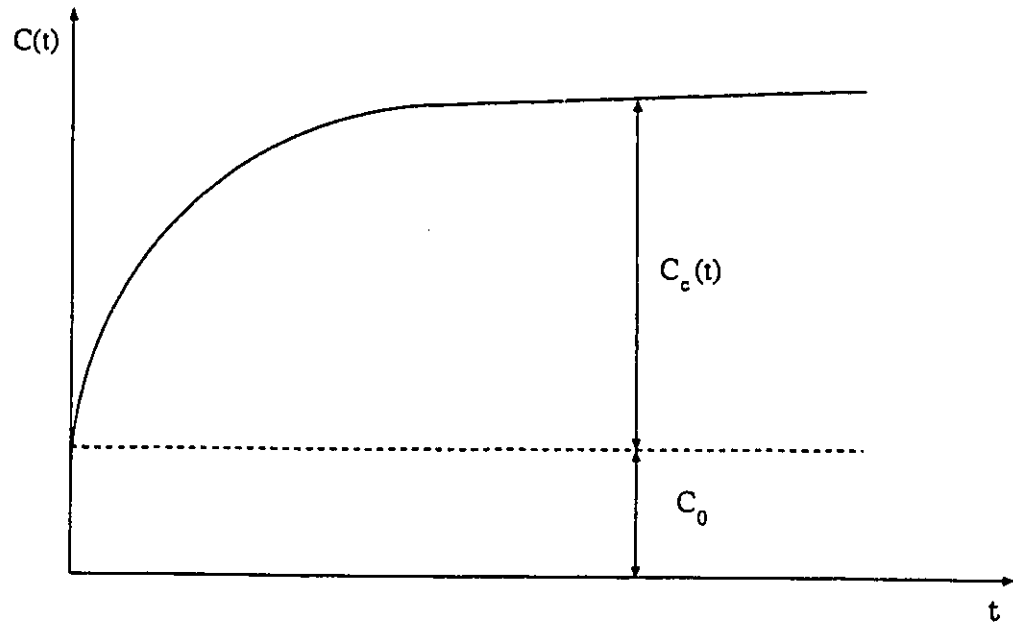


Fig.6.1 Creep function $C(t)$

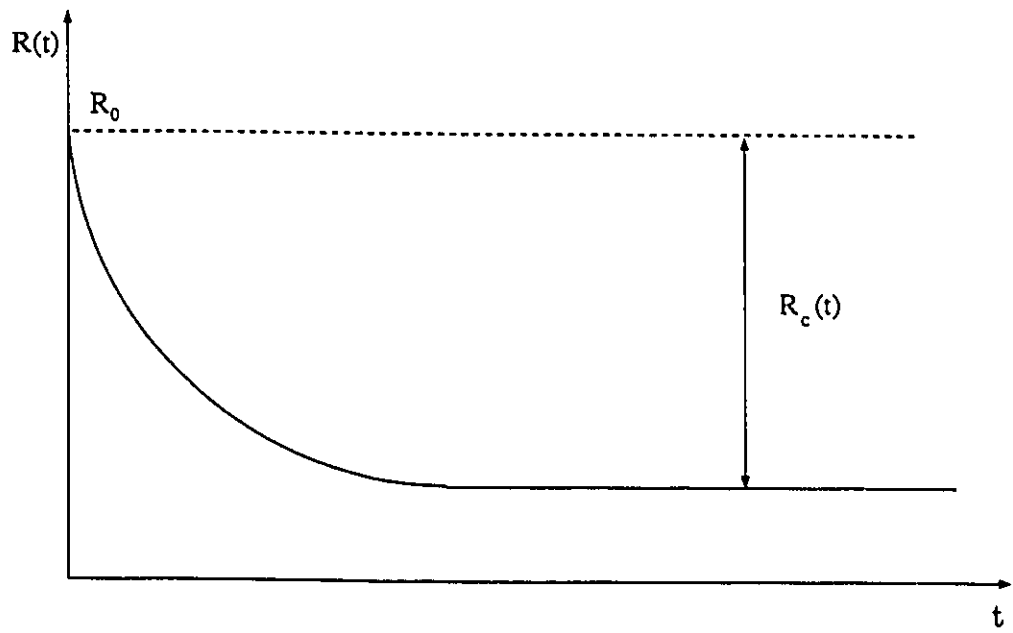


Fig.6.2 Relaxation function $R(t)$

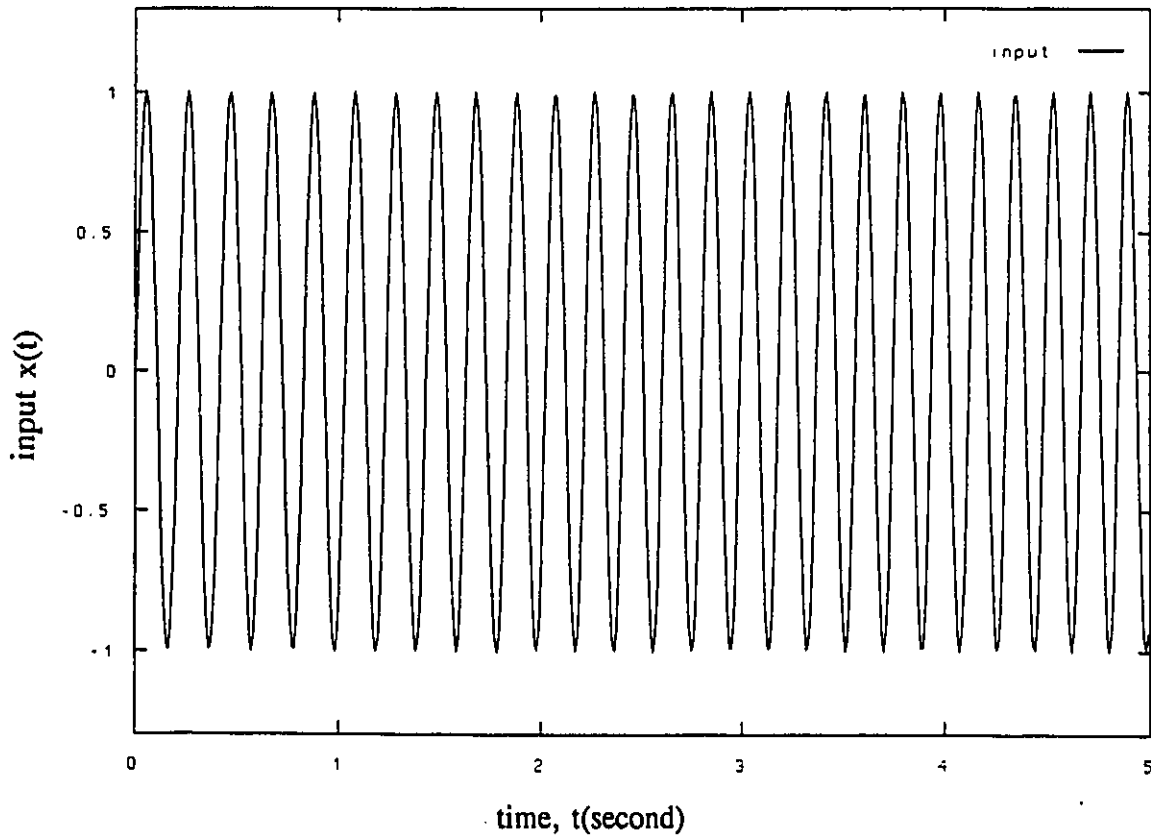


Fig.6.3 Input $x(t)=\sin[\beta t^{-\gamma} + \omega_0 t]$ in which $\beta=0.5$, $\gamma=2.0$ and $\omega_0=30.0$ to system $\ddot{y}+a_1\dot{y}+a_2y = b_0\ddot{x}+b_1\dot{x}+b_2x$ with $a_1=51.0$, $a_2=50.0$, $b_0=2.0$, $b_1=113.0$ and $b_2=160.0$.

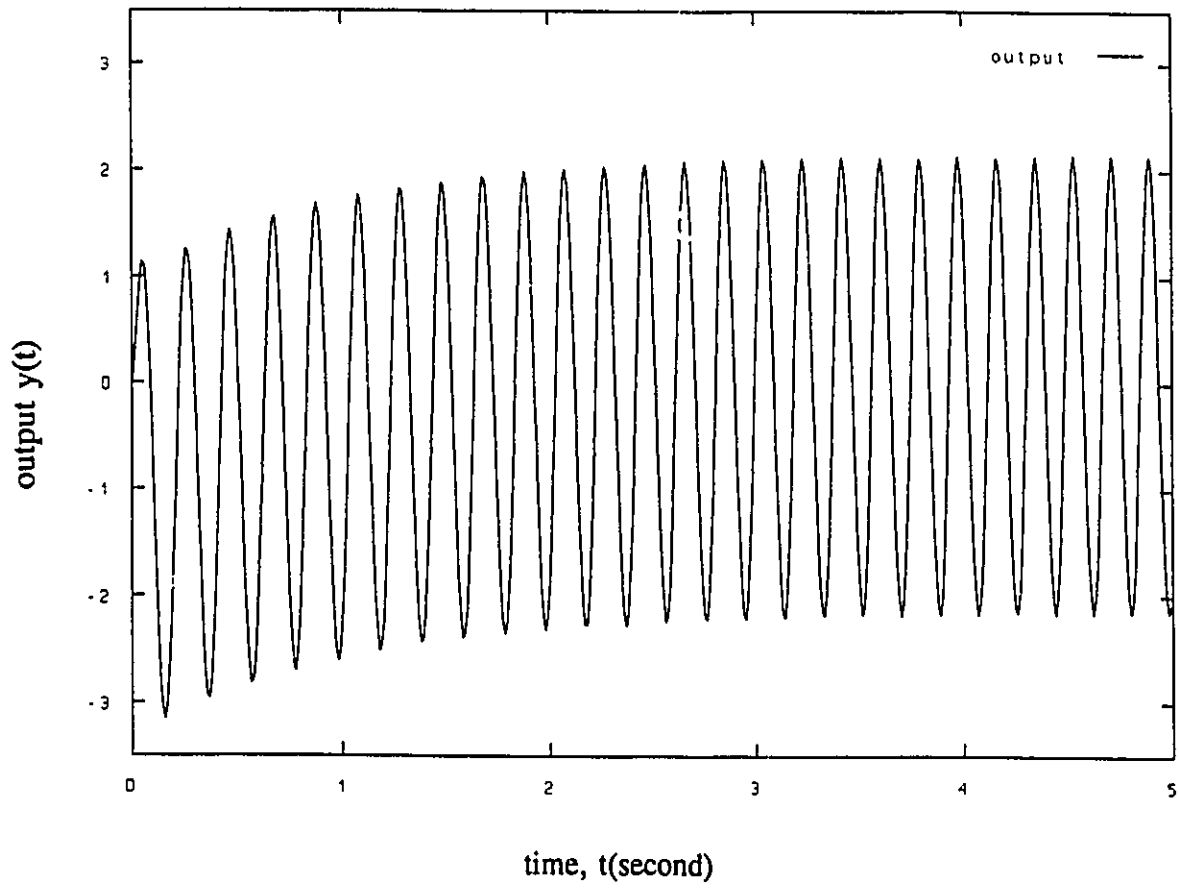


Fig.6.4 Output $y(t)$ corresponding to the input shown in Fig.6.3.

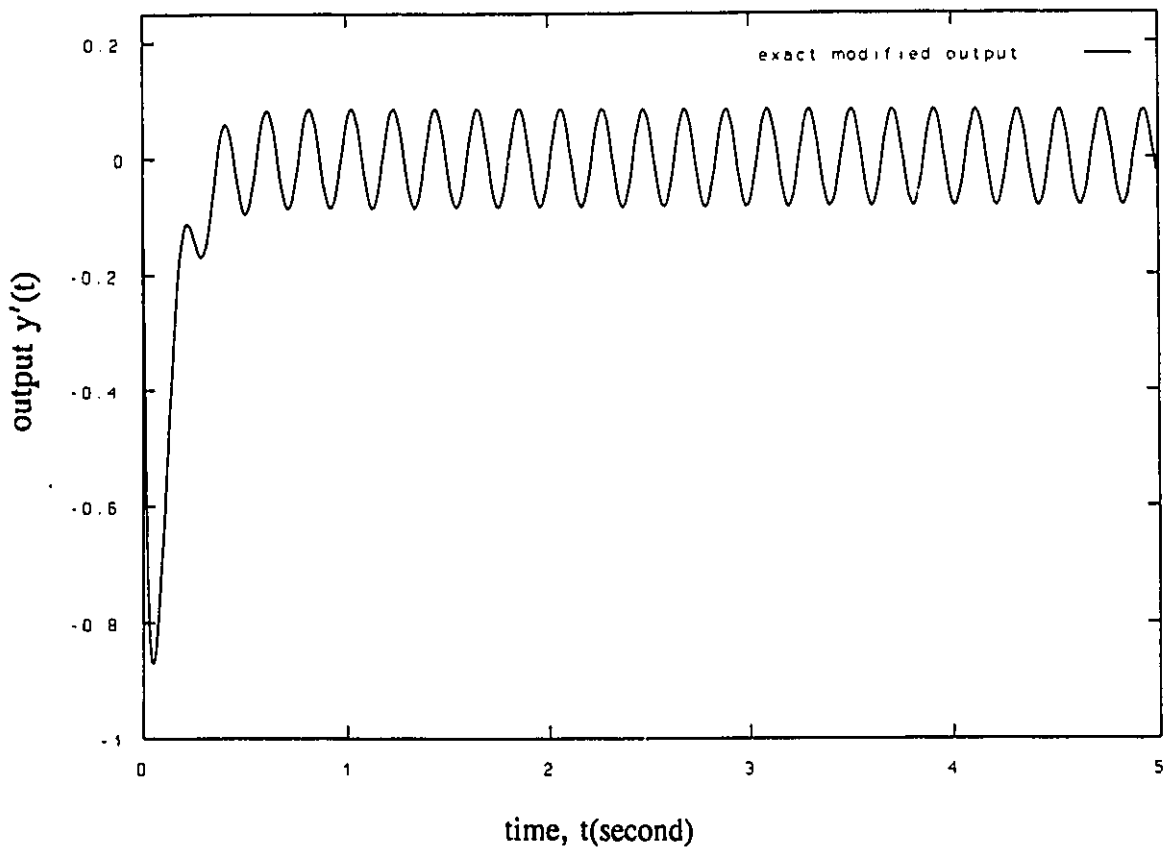


Fig.6.5 Exact modified output $y'(t)$ corresponding to the input shown in Fig.6.3.

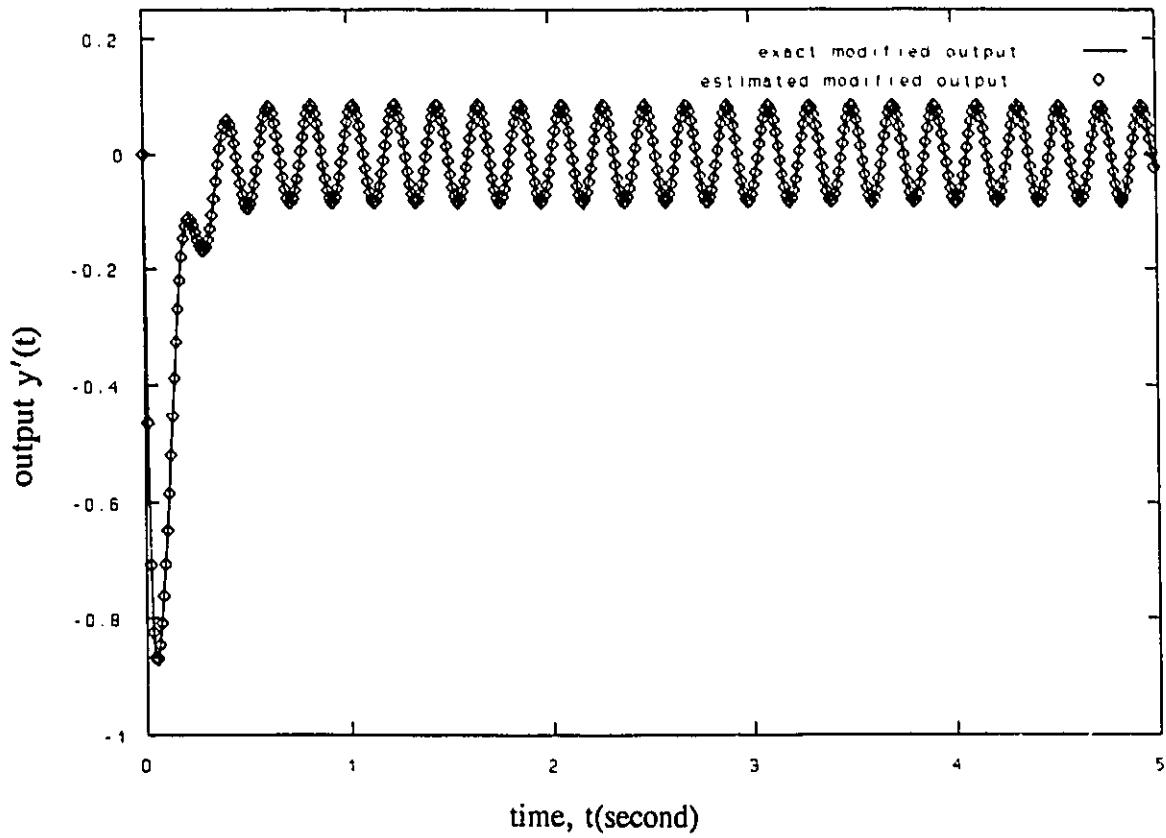


Fig.6.6 Exact and estimated modified outputs $y'(t)$ corresponding to the input shown in Fig.6.3.

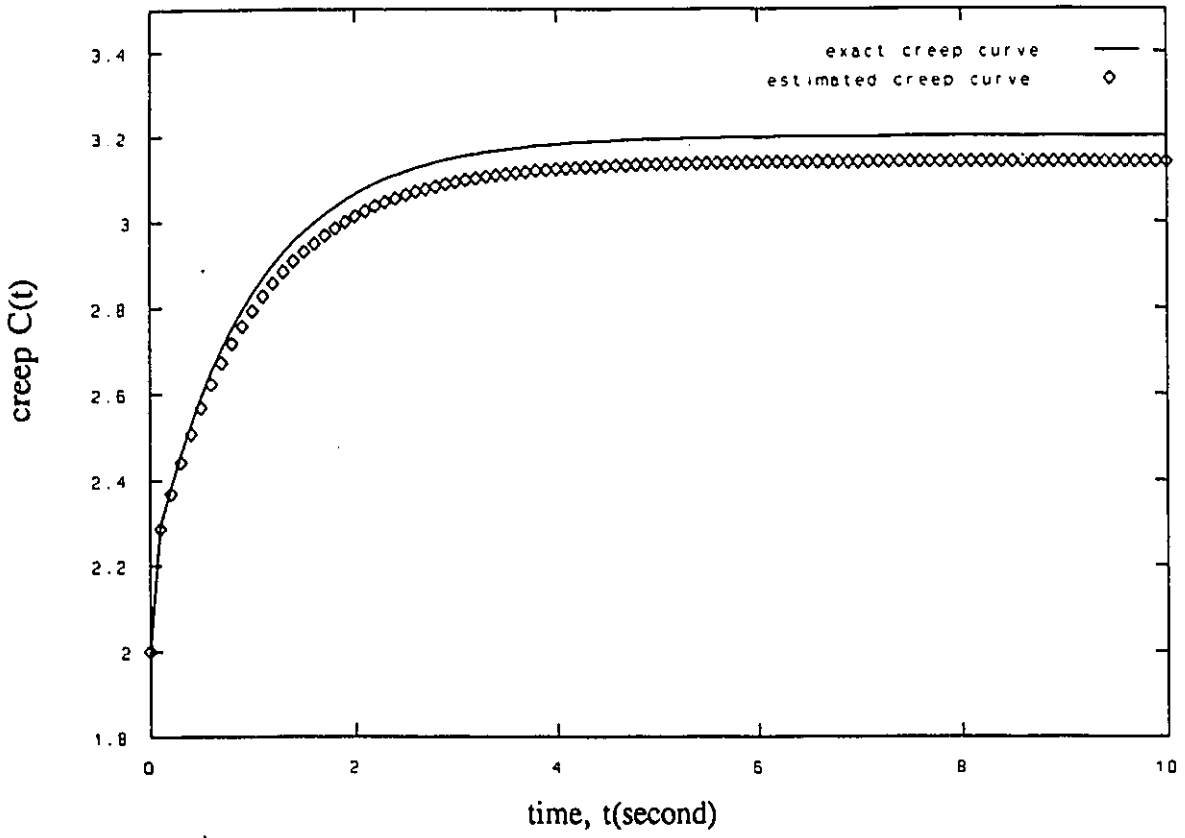


Fig.6.7 Exact and estimated creep functions.

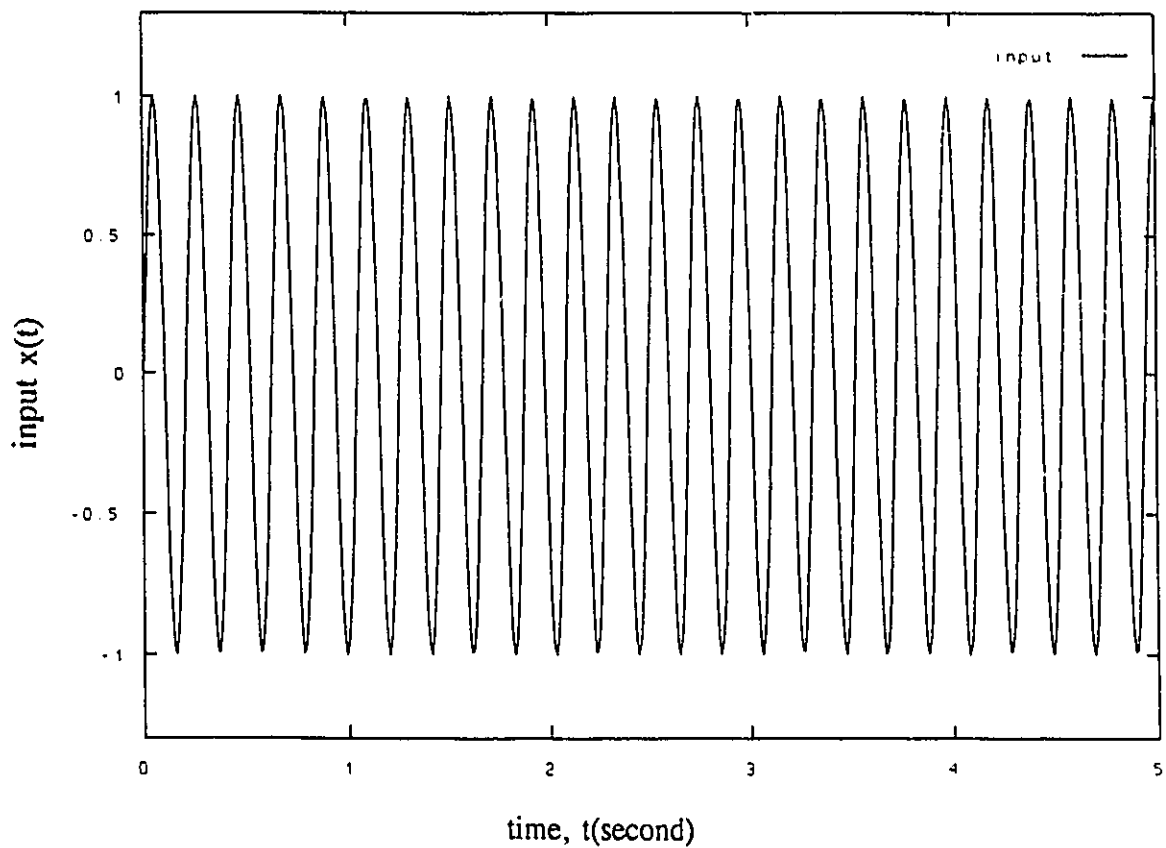


Fig.6.8 Input $x(t)=\sin[(\beta t^{\gamma-1} + \omega_0)t]$ with $\beta=0.1$, $\gamma=2.0$ and $\omega_0=30.0$ to system $\ddot{y}+a_1\dot{y}+a_2y = b_0\ddot{x}+b_1\dot{x}+b_2x$ with $a_1=60.0$, $a_2=500.0$, $b_0=2.0$, $b_1=115.0$ and $b_2=830.0$.

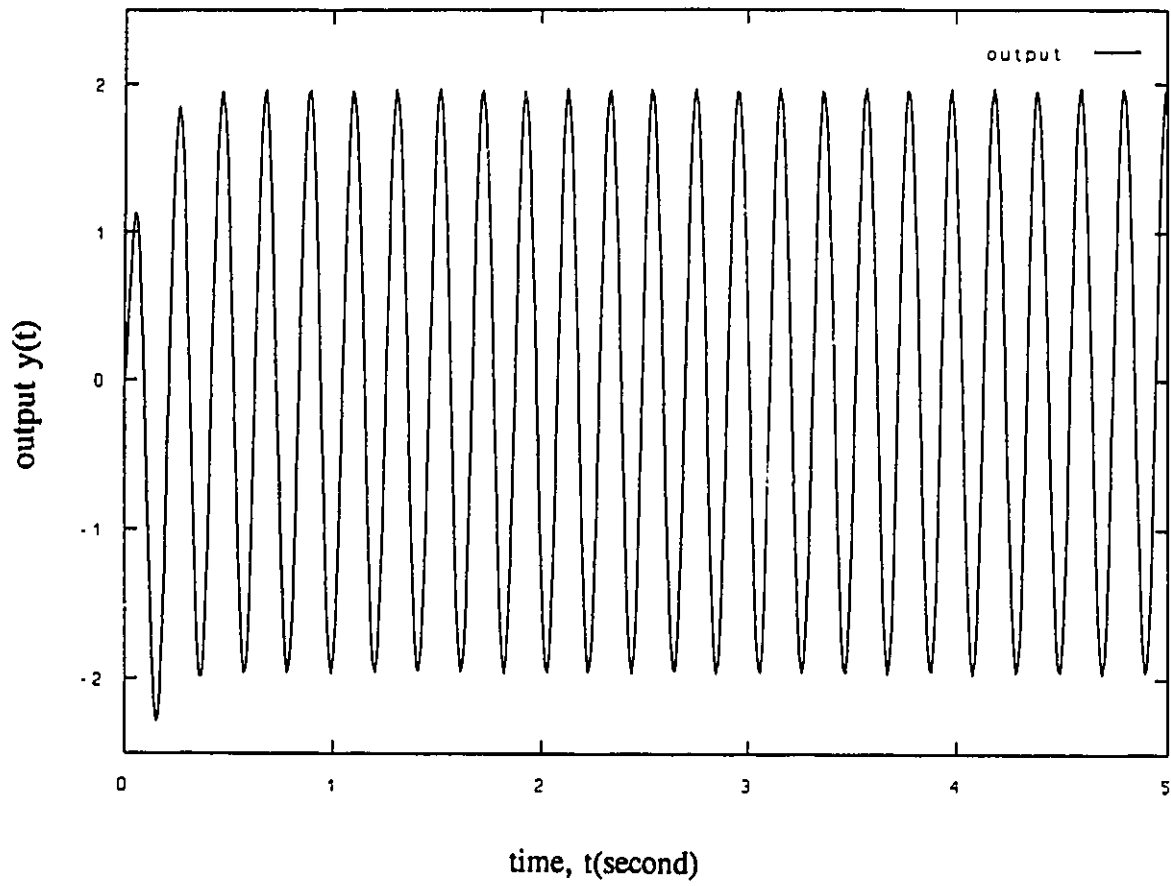


Fig.6.9 Output $y(t)$ corresponding to the input shown in Fig.6.8.

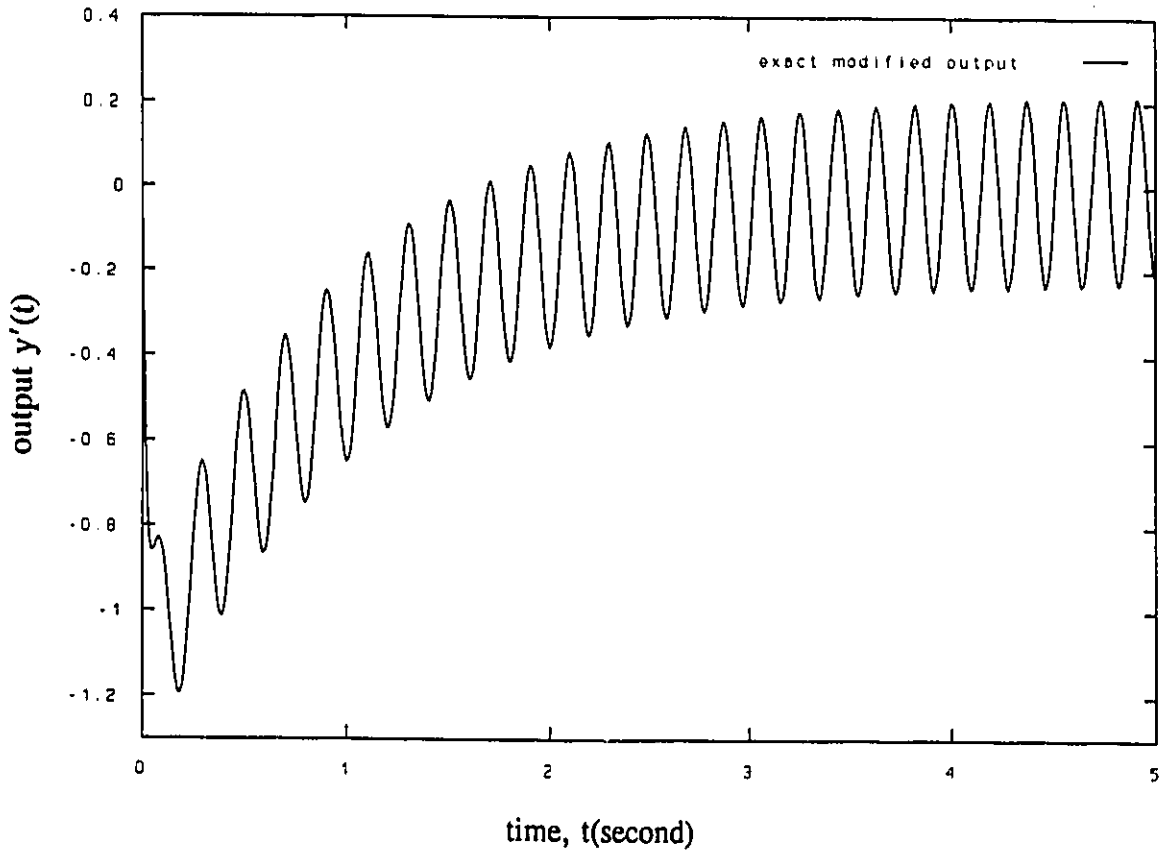


Fig.6.10 Exact modified output $y'(t)$ corresponding to the input given in Fig.6.8.

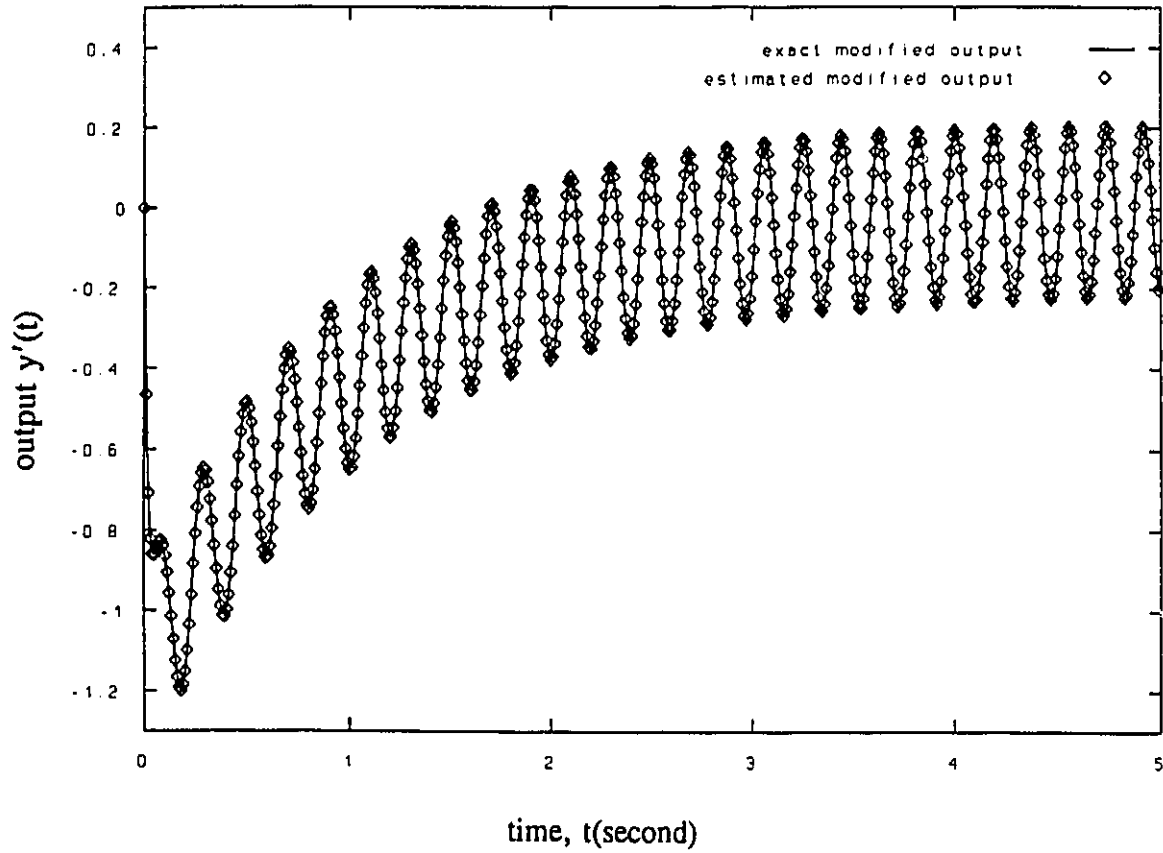


Fig.6.11 Exact and estimated modified outputs $y'(t)$ corresponding to the input shown in Fig.6.8.

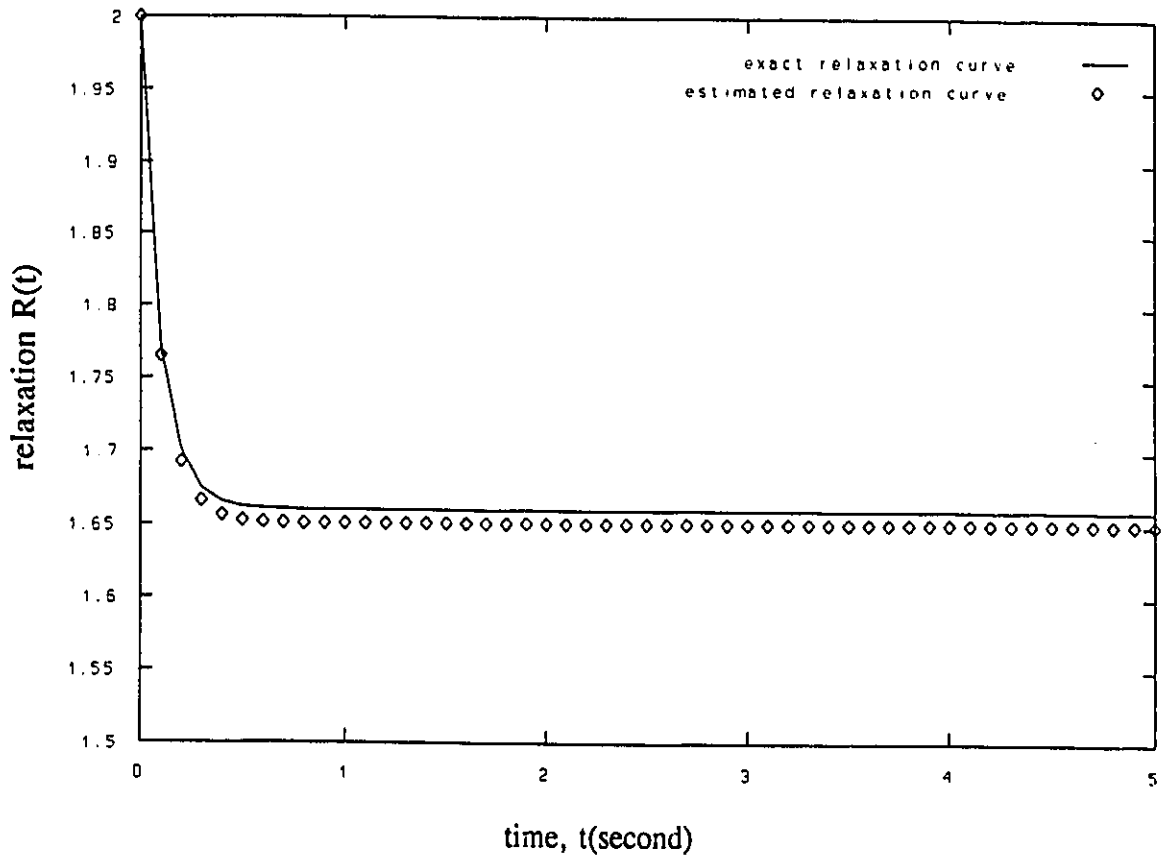


Fig.6.12 Exact and estimated relaxation functions.