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Bounded Cocycles:  
von Neumann Algebras and  
Amenability

An M.Sc. Thesis  
by Teresa Bates

submitted to the School of Graduate Studies and Research  
in partial fulfillment of the requirements for  
the Master's degree in Mathematics\*

University of Ottawa  
Ottawa, Ontario  
Canada  
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# Abstract

In a 1993 preprint Guyan Robertson proved that every uniformly bounded representation of a discrete group on a finite von Neumann algebra is similar to a unitary representation. We have since discovered that this result was first proved in a paper of Vasilescu and Zsidó, published in 1974 [VZ]. In this thesis we generalise this result for discrete groupoids, proving that every uniformly bounded cocycle into a finite von Neumann algebra is cohomologous to a unitary cocycle. The corresponding result for cocycles into finite dimensional algebras was proved in [Zim3]. We also derive some equivalent definitions of amenability of group actions and provide a new proof of a result of Zimmer regarding uniformly bounded cocycles on amenable  $C^*$ -spaces. We develop some machinery in order to prove these results. This is the theory of  $G$ -flows in which we explore the actions of groupoids on Borel fields of sets. Our development of this theory follows that of the usual theory of flows from topological dynamics.

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# Chapter 1

## Introduction

In a 1993 preprint Guyan Robertson proved the following theorem.

**Theorem 1** (*[Rob, Theorem 1.1]*) *Let  $\pi$  be a uniformly bounded representation of a discrete group  $\Gamma$  such that  $M = \pi(\Gamma)''$  is a finite von Neumann algebra. Then  $\pi$  is similar to a unitary representation. In fact, there is a positive, invertible element  $a \in M$  such that  $\pi(g) = a\rho(g)a^{-1}$  where  $\rho$  is a unitary representation of  $\Gamma$  into  $M$ .*

We have since learned that a slightly more general version of this result was proved in a 1974 paper of Vasilescu and Zsidó [VZ].

The following related result for amenable groups is due to Dixmier.

**Theorem 2** (*Dixmier, e.g. [Pau, Theorem 8.3]*) *Let  $G$  be an amenable group and let  $\rho : G \rightarrow \mathcal{B}(\mathcal{H})$  be a strongly continuous homomorphism with  $\rho(e) = 1$ , such that*

$$\|\rho\| := \sup\{\|\rho(g)\|; g \in G\} < \infty.$$

*Then there exists an invertible  $S \in \mathcal{B}(\mathcal{H})$  with  $\|S\| \cdot \|S^{-1}\| \leq \|\rho\|^2$  such that  $S^{-1}\rho(g)S$  is a unitary representation of  $G$ .*

The proofs of both of the above-mentioned theorems exploit fixed point properties. In the case of the representation into a finite von Neumann algebra  $M$ , the existence of a finite trace on  $M$  enables us to use the Ryll-Nardzewski fixed point theorem. The amenability of the group  $G$  provides the required fixed point property for the proof of Dixmier's theorem.

A corresponding result does not hold in general for group representations  $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$ . Kuntze and Stein gave an example of a uniformly bounded representation of  $SL(2, \mathbb{R})$  which is not similar to a unitary representation in [KuS, Theorem 5].

In this thesis we generalise the above-mentioned results for uniformly bounded cocycles. Recall that a Borel function  $\alpha : S \times G \rightarrow H$ , where  $G$  and  $H$  are topological groups and  $(S, \mu)$  is a standard Borel  $G$ -space is called a cocycle if for all  $g$  and  $h$  in  $G$ ,  $\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$  for  $\mu$ -a.c.  $s \in S$ . We may view cocycles as a generalisation of group representations in the following manner. Let  $G$  be a locally compact group, and let  $\pi : G \rightarrow \mathcal{B}(\mathcal{H})$  be a continuous representation of  $G$ . Suppose  $S$  is a  $G$ -space. Then we can define a cocycle  $\alpha_\pi : S \times G \rightarrow \mathcal{B}(\mathcal{H})$  by  $\alpha_\pi(s, g) = \pi(g)$ .

In order to provide proofs of our results, we generalise the fixed point properties used in the proofs of the original results. Our proof of our generalisation of theorem 1 contains the proof of a version of the Ryll-Nardzewski fixed point theorem for Borel fields of compact sets in von Neumann algebras. We also generalise Day's fixed point theorem to amenable actions.

Our proofs of these results require the development of some new machinery. This is the theory of  $\mathcal{G}$ -flows which is a generalisation of the usual theory of flows from Topological Dynamics as discussed in Chapter 4.

We conclude this discussion with a brief synopsis of the remaining chapters of this thesis.

In chapter 2 we review some basic facts from the theory of von Neumann algebras. We include here only those facts which are necessary for an understanding of our proofs in later chapters.

In chapter 3 we review the basic theory of Borel spaces and give an overview of the theory of Borel fields. We prove that if  $E$  is a separable Banach space, then for a Borel field  $\{A_s\}$  of compact subsets of  $E_1^*$ , the set

$$A = \{ \lambda \in L^\infty(S, E^*) \mid \lambda(s) \in A_s \text{ } \mu\text{-a.e.} \}$$

is a compact subset of  $L_1^\infty(S, E)$ . We also provide a new proof of Zimmer's corresponding result for Borel fields of compact, convex sets.

In chapter 4 we review the basic definitions and theory of Borel group actions and ergodicity. We also review the theory of flows as developed in topological dynamics and give a very brief introduction to the theory of groupoids.

In chapter 5 we define and discuss the theory of cocycles. We prove that every cocycle on a free, transitive  $G$ -space is equivalent to the trivial cocycle. We also prove our result for uniformly bounded cocycles into finite von Neumann algebras in some special cases.

In chapter 6 we prove one of our main results, namely:

**Theorem 3** *Let  $G$  be a countable discrete group, and let  $(S, \mu)$  be a standard Borel  $G$ -space. Let  $\alpha : S \times G \rightarrow GL(M)$  be a uniformly bounded cocycle into a finite von Neumann algebra  $M$  with separable predual. Then  $\alpha$  is cohomologous to a unitary cocycle.*

This is the generalisation of Theorem 1.

In chapter 7 we develop the theory of  $\mathcal{G}$ -flows. Our development follows that of the usual theory of flows from topological dynamics.

In chapter 8 we discuss amenability of groups and of group actions. In the first section we review the theory of amenability for locally compact groups. In the second section we give Zimmer's original definition of amenability for actions as stated in [Zim4]. We develop and discuss several equivalent definitions of amenability of actions. Chief amongst these is the following.

**Definition 1** *Let  $G$  be a countable discrete group and let  $(S, \mu)$  be a standard Borel  $G$ -space. Then  $S$  is an amenable  $G$ -space if for every uniformly bounded cocycle  $\alpha : S \times G \rightarrow B(E)$ , where  $E$  is a separable Banach space, and for every  $\alpha$ -equivariant Borel field  $\{K_s\} \subset E_1^*$  of compact, convex subsets, there exists an  $\alpha$ -equivariant section in  $\{K_s\}$ .*

We use this definition to provide a new proof of the following result of Zimmer which is a generalisation of Theorem 2.

**Theorem 4** *Let  $\mathcal{H}$  be a separable Hilbert space. If  $G$  is a countable discrete group and  $S$  is an amenable ergodic  $G$ -space, then every uniformly bounded cocycle  $\alpha : S \times G \rightarrow GL(\mathcal{H})$  is equivalent to a unitary cocycle.*

# Chapter 2

## von Neumann Algebras

In this chapter we recall some facts from the theory of von Neumann algebras. We shall require these facts in the later chapters of this thesis. For more information and a fuller discussion see, for example, [Dix], [Sak], [Tak] and [Mur].

### 2.1 Topologies on $\mathcal{B}(\mathcal{H})$

The *strong operator topology* on  $\mathcal{B}(\mathcal{H})$  is the topology which has a base of neighbourhoods at the operator  $T_0$  consisting of sets of the form

$$V(T_0; x_1, \dots, x_m; \epsilon) = \{T \in \mathcal{B}(\mathcal{H}); \|(T - T_0)x_j\| < \epsilon, j = 1, \dots, m\}.$$

Here  $\epsilon > 0$  and  $x_1, \dots, x_m \in \mathcal{H}$ . Thus a net  $\{T_j\}$  of operators is strong operator convergent to  $T_0$  if and only if  $\{\|(T_j - T_0)x\|\}$  converges to 0 for each  $x \in \mathcal{H}$ .

The *weak operator topology* on  $\mathcal{B}(\mathcal{H})$  is the locally convex topology having a base of convex open neighbourhoods at the operator  $T_0$  given by the family of sets

$$V(T_0; w_1, \dots, w_m; \epsilon) = \{T \in \mathcal{B}(\mathcal{H}); |\langle (T - T_0)w_i, w_j \rangle| < \epsilon, i, j = 1, \dots, m\}.$$

Here  $\epsilon > 0$  and  $w_1, \dots, w_m \in \mathcal{H}$ . The net of operators  $\{T_j\}$  in  $\mathcal{B}(\mathcal{H})$  converges to  $T_0$  in the weak operator topology if and only if  $\langle T_j x, y \rangle$  converges to  $\langle T_0 x, y \rangle$  for all  $x$  and  $y$  in  $\mathcal{H}$ .

The weak operator topology is strictly coarser than the strong operator topology. However, we have the following well-known result for convex subsets of  $\mathcal{B}(\mathcal{H})$ .

**Theorem 2.1.1** (See, for example, [KR, Theorem 5.1.2]) *The weak and strong operator closures of a convex subset  $\mathcal{K}$  of  $\mathcal{B}(\mathcal{H})$  coincide.*

**Definition 2.1.2** (Compare [Mur, p 126]) *The ultraweak or  $\sigma$ -weak topology on  $\mathcal{B}(\mathcal{H})$  is the Hausdorff locally convex topology on  $\mathcal{B}(\mathcal{H})$  generated by the seminorms*

$$\begin{aligned} \mathcal{B}(\mathcal{H}) &\rightarrow \mathbb{R}^+ \\ u &\mapsto |\operatorname{tr}(uv)|. \end{aligned}$$

Here  $v \in L^1(\mathcal{H})$ , the set of trace-class operators on  $\mathcal{H}$ , and for  $w \in L^1(\mathcal{H})$ , we define  $\operatorname{tr}(w) = \sum_\lambda \langle v(e_\lambda), e_\lambda \rangle$  where  $\{e_\lambda\}$  is an orthonormal basis of  $\mathcal{H}$ . Recall that  $L^1(\mathcal{H})$  is an ideal in  $\mathcal{B}(\mathcal{H})$ .

**Remark** ([Mur, p 126]) The ultraweak topology is just the weak\*-topology on  $\mathcal{B}(\mathcal{H})$ . Hence, by the Banach-Alaoglu theorem, the closed unit ball of  $\mathcal{B}(\mathcal{H})$  is ultraweakly compact.

The operations of addition and scalar multiplication are both weak operator and ultraweakly continuous, as is the involution operation.

The weak operator topology is coarser than the ultraweak topology. However we have the following result for the unit ball of  $\mathcal{B}(\mathcal{H})$ .

**Theorem 2.1.3** (For example, [Mur, Theorem 4.2.4]) *If  $\mathcal{H}$  is a Hilbert space, then the relative weak operator and ultraweak topologies on the closed unit ball of  $\mathcal{B}(\mathcal{H})$  coincide, and hence the ball is compact in the weak operator topology.*

## 2.2 Some Basic Definitions and Properties

Let  $\mathcal{H}$  be a Hilbert space and let  $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$  be an associative \*-algebra of operators. We say that  $\mathcal{A}$  is a *von Neumann algebra* if  $\mathcal{A}$  is closed with respect to the weak operator topology on  $\mathcal{B}(\mathcal{H})$ . All non-zero von Neumann algebras contain a unique identity element. Recall that every von Neumann algebra is a  $C^*$ -algebra.

Let  $\mathcal{H}$  be a Hilbert space and let  $S \subset \mathcal{B}(\mathcal{H})$  be a non-empty subset. The *commutant*  $S'$  of  $S$  in  $\mathcal{B}(\mathcal{H})$  is the collection of all operators in  $\mathcal{B}(\mathcal{H})$  which commute with all the operators in  $S$ . It is always a weak-operator closed algebra. If  $S$  is self-adjoint, then  $S'$  is a von Neumann algebra. The *bicommutant*  $S''$  is the commutant of the algebra  $S'$ .

We have the following characterisation of von Neumann algebras. For a proof see [Mur].

**Theorem 2.2.1** (*von Neumann's Double Commutant Theorem. For example, [Mur, Theorem 4.1.5]*) *Let  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  be a  $*$ -subalgebra with  $1_{\mathcal{H}} \in \mathcal{M}$ . Then the following are equivalent:*

- (i)  $\mathcal{M}$  is strong operator closed.
- (ii)  $\mathcal{M}$  is weak operator closed.
- (iii)  $\mathcal{M} = \mathcal{M}''$ .

The von Neumann algebra *generated* by a set  $S$  is the smallest von Neumann algebra containing all the elements of that set. It is the intersection of all von Neumann algebras containing  $S$ . We shall denote this algebra by  $S''$ .

**Remark** If the set  $S$  is self-adjoint, then the bicommutant of  $S$  and the von Neumann algebra generated by  $S$  coincide.

A von Neumann algebra has an essentially unique *predual*, that is a Banach space  $\mathcal{A}_*$  such that  $(\mathcal{A}_*)^* = \mathcal{A}$ . It is isomorphic to the space of ultraweakly continuous linear forms on  $\mathcal{A}$ . The weak\*-topology  $\sigma(\mathcal{A}, \mathcal{A}_*)$  is called the *weak topology* on  $\mathcal{A}$ . A *factor* is a von Neumann algebra which has trivial centre. That is, one for which only scalar multiples of the identity commute with every operator in the algebra. Finite dimensional examples of such algebras include all the full complex matrix algebras. A more interesting example of such an algebra will be given shortly.

A positive linear functional  $\phi$  is called *normal* if whenever  $(p_i)_{i \in I}$  is an increasing net of projections in  $\mathcal{A}$  we have

$$\phi(\sup_{i \in I} \{p_i\}) = \sup_{i \in I} \phi(p_i).$$

A positive linear functional  $\tau$  on  $\mathcal{A}$  is said to be *tracial* if it satisfies  $\tau(ab) = \tau(ba)$  for all  $a, b \in \mathcal{A}$ . In this thesis we shall be concerned with von Neumann algebras on which there exists a *finite normal trace*, that is, a normal tracial functional such that  $\tau(a) < \infty$  for all  $a \in \mathcal{A}^+$ . Such von Neumann algebras are said to be *finite*. If an infinite dimensional factor is finite, then we call it a *II<sub>1</sub>-factor*. An equivalent definition of finiteness for a von Neumann algebra is the following.  $\mathcal{A}$  is a finite von Neumann algebra if for every partial isometry  $v \in \mathcal{A}$  we have  $v^*v = vv^*$ . In particular, if  $u$  is an operator in a finite von Neumann algebra  $\mathcal{A}$  which satisfies  $u^*u = 1$ , then  $u$  must be unitary. If a von Neumann algebra is finite, we may clearly assume that the trace satisfies  $\tau(1) = 1$ .

**Example 2.2.2** (Compare [Pop, Example II.2.2]) Consider the matrix algebras

$$0 \subseteq \mathbb{C} \subseteq M_2(\mathbb{C}) \subseteq M_4(\mathbb{C}) \subseteq \dots \subseteq M_{2^k}(\mathbb{C}) \subseteq \dots$$

where the inclusion is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

Let

$$R = \bigcup_{k \in \mathbb{N}} M_{2^k}(\mathbb{C}).$$

We define a norm on  $R$  by  $\|a\| = \|a\|_{M_{2^k}}$  where  $k$  is the smallest integer such that  $a \in M_{2^k}$ . The completion of  $R$  in this norm is a  $C^*$ -algebra  $R'$ .

There is a trace  $\tau$  on  $R'$  defined by

$$\tau(A) = \text{Tr}(A)$$

where  $\text{Tr}$  is the usual matrix trace on  $M_{2^k}$  and  $k$  is the smallest integer such that  $A \in M_{2^k}$ .

Consider the GNS-construction  $(\pi_\tau, \mathcal{H}_\tau, \xi_\tau)$  for  $R'$  with respect to  $\tau$ . We can identify  $R'$  and  $\pi_\tau(R')$  since  $R'$  is simple. The weak closure  $R_0$  of  $R'$  in  $\mathcal{B}(\mathcal{H}_\tau)$  is a von Neumann algebra which is a  $\text{II}_1$ -factor. We call  $R_0$  the *hyperfinite  $\text{II}_1$ -factor*.

The  $\|\cdot\|_2$ -norm on a finite von Neumann algebra  $\mathcal{M}$  with trace  $\tau$  is defined by

$$\|a\|_2 = \tau(a^*a)^{1/2}.$$

Let  $r > 0$ . Let  $\mathcal{M}_r$  denote the norm-closed ball in  $\mathcal{M}$  of radius  $r$ . We have the following result for finite von Neumann algebras.

**Proposition 2.2.3** *(For example [Sak, Corollary 1.8.10] and [Dix, Proposition I.4.4]) Let  $\mathcal{M}$  be a finite von Neumann algebra and let  $f : \mathcal{M} \rightarrow \mathbb{C}$  be a linear form on  $\mathcal{M}$ . Then  $f$  is ultraweakly continuous on  $\mathcal{M}_r$  if and only if  $f$  is continuous with respect to the  $\|\cdot\|_2$ -norm on  $\mathcal{M}_r$ .*

## 2.3 Tensor Products

Let  $\mathcal{M}$  and  $\mathcal{N}$  be von Neumann algebras, with preduals  $\mathcal{M}_*$  and  $\mathcal{N}_*$  respectively. Form the  $C^*$ -tensor product  $\mathcal{M} \otimes_{\alpha_0} \mathcal{N}$  of  $\mathcal{M}$  and  $\mathcal{N}$  with respect to the minimal  $C^*$ -norm,  $\alpha_0$ . Let  $\alpha_0^*$  denote the dual norm,

$$\alpha_0^*(f) = \sup_{\alpha_0(x) \leq 1} |\langle x, f \rangle|$$

where  $f$  lies in the algebraic tensor product  $\mathcal{M}^* \odot \mathcal{N}^*$ . Since the preduals of  $\mathcal{M}$  and  $\mathcal{N}$  are contained in their respective duals, we may form the algebraic tensor product  $\mathcal{M}_* \odot \mathcal{N}_*$  and consider the norm  $\alpha_0^*$  on this algebra. We form the completion  $\mathcal{M}_* \otimes_{\alpha_0^*} \mathcal{N}_*$  in this norm, and note that it is a closed subspace of  $(\mathcal{M} \otimes_{\alpha_0} \mathcal{N})^*$ . The spaces  $\mathcal{M} \otimes_{\alpha_0} \mathcal{N}$ ,  $\mathcal{M}_* \odot \mathcal{N}_*$ , and hence  $\mathcal{M}_* \otimes_{\alpha_0^*} \mathcal{N}_*$ , are invariant under left and right multiplication by  $x \in \mathcal{M} \otimes_{\alpha_0} \mathcal{N}$ . It follows that the polar  $\mathcal{I}$  of  $\mathcal{M}_* \otimes_{\alpha_0^*} \mathcal{N}_*$  in  $(\mathcal{M} \otimes_{\alpha_0} \mathcal{N})^{**}$  is a two sided ideal such that

$$(\mathcal{M}_* \otimes_{\alpha_0^*} \mathcal{N}_*)^* \cong (\mathcal{M} \otimes_{\alpha_0} \mathcal{N})^{**} / \mathcal{I}.$$

**Definition 2.3.1** (Compare [Sak, Definition 1.22.10]) *The von Neumann algebra  $(\mathcal{M}_* \otimes_{\alpha_0^*} \mathcal{N}_*)^*$ , denoted by  $\mathcal{M} \bar{\otimes} \mathcal{N}$ , is called the von Neumann tensor product or the tensor product of the von Neumann algebras  $\mathcal{M}$  and  $\mathcal{N}$ .*

We have the following results which shall be of use in the later chapters of this thesis.

**Proposition 2.3.2** (For example, [Sak, Proposition 1.22.12]) *Let  $L^\infty(S, \mu)$  be the commutative von Neumann algebra of all essentially bounded  $\mu$ -measurable functions on a measure space  $S$ . Let  $\mathcal{M}$  be a von Neumann algebra.*

Then

$$L^1(S, \mu) \otimes_{\alpha_0} \mathcal{M}_* \cong L^1(S, \mu) \otimes_{\gamma} \mathcal{M}_* \cong L^1(S, \mu, \mathcal{M}_*),$$

where  $L^1(S, \mu, \mathcal{M}_*)$  is the Banach space of all  $\mathcal{M}_*$ -valued  $\mu$ -integrable functions on  $S$  and  $\gamma$  is the maximal cross-norm on the algebraic tensor product  $L^1(S, \mu) \odot \mathcal{M}_*$ .

**Theorem 2.3.3** (For example, [Sak, Theorem 1.22.19]) Let  $\mathcal{M}$  be a von Neumann algebra with separable predual  $\mathcal{M}_*$ , and let  $L^\infty(S, \mu, \mathcal{M})$  be the Banach space of all  $\mathcal{M}$ -valued, essentially bounded weakly- $*$   $\mu$ -measurable functions on the measure space  $S$ . Then  $L^\infty(S, \mu, \mathcal{M})$  is a von Neumann algebra under pointwise multiplication and its predual is  $L^1(S, \mu, \mathcal{M}_*)$ .

Further, the mapping  $f \otimes a \mapsto f(t)a$  for  $f \in L^\infty(S, \mu)$  and  $a \in \mathcal{M}$  extends uniquely to a  $*$ -isomorphism of the von Neumann algebra  $L^\infty(S, \mu) \bar{\otimes} \mathcal{M}$  onto  $L^\infty(S, \mu, \mathcal{M})$ .

## 2.4 Conditional Expectations

**Definition 2.4.1** Let  $M$  be a von Neumann algebra and  $N \subset M$  a von Neumann subalgebra of  $M$ . A linear map  $E : M \rightarrow N$  satisfying the properties

- (i)  $E(M_+) \subset N_+$ , that is  $E$  is positive
- (ii)  $E(b) = b$  for all  $b \in N$
- (iii)  $E(b_1 x b_2) = b_1 E(x) b_2$  for all  $b_1, b_2 \in N, x \in M$

is called a conditional expectation.

**Remark** Such a mapping is a projection of norm 1 from  $M$  onto  $N$ .

**Proof.** We follow the proof found in [KR2, p 834]. Condition (ii) above implies that  $E$  is a projection. Let  $\mathcal{H}$  be a Hilbert space such that  $M$  has a faithful representation as  $M \subset \mathcal{B}(\mathcal{H})$ . Since  $S^*S \leq \|S\|^2 \cdot 1$  we have  $E(S^*S) \leq \|S\|^2 E(1) = \|S\|^2 \cdot 1$ . Also, by positivity and linearity of  $E$ ,

$$0 \leq E((E(S) - S)^*(E(S) - S)) = E(S^*S) - E(S)^*E(S).$$

Hence for  $x \in \mathcal{H}$

$$\begin{aligned} \|E(S)x\|^2 &= \langle E(S)^*E(S)x, x \rangle \\ &\leq \langle E(S^*S)x, x \rangle \\ &\leq \|S\|^2 \|x\|^2. \end{aligned}$$

Thus  $\|E\| \leq 1$ . Condition (ii) implies that  $\|E\| \geq 1$  and the result follows.

□

In fact, we have the following result.

**Theorem 2.4.2** (*Tomiyama, for example [Str, Theorem II.9.1]*) *Every projection of norm 1 from a von Neumann algebra  $\mathcal{M}$  onto a von Neumann subalgebra  $\mathcal{N}$  is a conditional expectation.*

# Chapter 3

## Borel Spaces

### 3.1 Basic Definitions and Results

We begin by recalling some basic measure-theoretical definitions. For further details see [Arv], [Coh] and [Zim1]. Let  $S$  be a set. A  $\sigma$ -algebra of subsets of  $S$  is a collection  $\mathcal{A}$  of subsets of  $S$  satisfying the following axioms.

- (a)  $S \in \mathcal{A}$ .
- (b)  $\mathcal{A}$  is closed under set complementation.
- (c)  $\mathcal{A}$  is closed under the taking of countable set unions.

The subsets of  $S$  which are contained in  $\mathcal{A}$  are called the *measurable subsets* of  $S$ . We call the pair  $(S, \mathcal{A})$  a *measurable space*.

A *Borel space* is a set  $S$  together with a  $\sigma$ -algebra  $\mathcal{B}$  of subsets of  $S$ , called the *Borel (measurable) sets* of  $S$ . In particular, if  $S$  is a topological space, then  $S$  becomes a Borel space under the  $\sigma$ -algebra generated by the open

(or equivalently, the closed) subsets of  $S$ . A *subspace* of a Borel space  $S$  is a subset  $X \subseteq S$  with the Borel structure induced by the Borel structure on  $S$ . That is,  $E \subset X$  is a Borel set if and only if there exists a Borel subset  $B \subset S$  such that  $E = B \cap X$ .

Let  $S$  and  $T$  be Borel spaces. A function  $f : S \rightarrow T$  is a *Borel function* or a *measurable function* if, for every Borel subset  $A \subset T$ ,  $f^{-1}(A)$  is a Borel subset of  $S$ . Recall that finite sums, limits, and compositions of Borel functions into  $\mathbb{R}$  or  $\mathbb{C}$  are all Borel functions. Let  $X$  and  $Y$  be two Borel spaces. A mapping  $\phi : X \rightarrow Y$  is said to be a *Borel isomorphism* if it is one to one, onto, and both  $\phi$  and  $\phi^{-1}$  are Borel functions.

**Remark** We have the following equivalent definition of measurability for functions into separable Banach spaces [Din]. A function  $f : S \rightarrow \mathbb{C}$  is measurable if for every open set  $U \subset \mathbb{C}$ ,  $f^{-1}(U)$  is a measurable subset of  $S$ . A function  $f : S \rightarrow E$  where  $E$  is a separable Banach space is said to be measurable if for all  $\phi \in E^*$ , the function  $\phi \circ f : S \rightarrow \mathbb{C}$  is measurable. We now prove the equivalence of these definitions in this case.

Clearly the usual definition of measurability implies the above condition, since compositions of measurable functions are measurable. Suppose the above condition holds. Let  $B(a; r)$  denote the closed ball in  $E$  of radius  $r$ , centred at  $a$ . Let  $\{\phi_n\}$  be a countable dense subset of  $E^*$ . Since  $E$  is separable, there exists a sequence  $\{\phi'_n\}$  in  $E^*$  such that for every  $x \in E$

$$\|x\| = \sup_n \frac{|\phi'_n(x)|}{\|\phi_n\|}.$$

Thus

$$\begin{aligned} f^{-1}(B(a; r)) &= \{t; \sup_n \{|\phi'_n(f(t)) - \phi'_n(a)| \leq r\|\phi_n\|\}\} \\ &= \bigcap_{n=1}^{\infty} \{t; |\phi'_n(f(t)) - \phi'_n(a)| \leq r\|\phi_n\|\}. \end{aligned}$$

Since  $\phi'_n \circ f$  is measurable for all  $n$ , the above sets are all measurable. It follows that  $f$  is also measurable.  $\square$

We now discuss some Borel structures on the spaces which we shall consider in later chapters of this thesis.

**Examples 3.1.1** (i) Let  $\mathcal{M}$  be a finite von Neumann algebra with separable predual  $\mathcal{M}_*$ , and let  $GL(\mathcal{M})$  denote the group of invertible elements of  $\mathcal{M}$ . We denote by  $\mathcal{B}(\mathcal{M})$  the Banach space of all  $\sigma(\mathcal{M}, \mathcal{M}_*)$ -continuous linear maps from  $\mathcal{M}$  into itself.

For every  $\phi \in \mathcal{M}_*$  and every  $x \in \mathcal{M}$ , we define a bounded linear form  $\phi(.x)$  on  $\mathcal{B}(\mathcal{M})$  by

$$\phi(.x)(T) = \phi(Tx) \text{ where } T \in \mathcal{B}(\mathcal{M}).$$

Let  $\mathcal{B}(\mathcal{M})_*$  denote the norm-closed linear subspace

$$\mathcal{B}(\mathcal{M})_* = \overline{\text{span}}\{\phi(.x); \phi \in \mathcal{M}_*, x \in \mathcal{M}\}$$

of  $\mathcal{B}(\mathcal{M})^*$ .

For  $x \in \mathcal{M}$ , let  $\text{adx}$  denote the element of  $\mathcal{B}(\mathcal{M})$  defined by

$$\text{adx}(y) = xyx^* \text{ for all } y \in \mathcal{M}.$$

**Lemma 3.1.2** *If  $\mathcal{M}$  is endowed with the strong topology and  $\mathcal{B}(\mathcal{M})$  is endowed with the  $\sigma(\mathcal{B}(\mathcal{M}), \mathcal{B}(\mathcal{M})_*)$ -topology, then the map  $\text{ad} : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{M})$  defined above is continuous on bounded subsets of  $\mathcal{M}$ .*

**Proof.**

Let  $r \geq 0$  and let  $(x_m)_{m \geq 1} \subset \mathcal{M}_r$  be a sequence which converges strongly to  $x \in \mathcal{M}_r$ . Since  $\mathcal{M}$  is finite we also have  $x_m^* \rightarrow x^*$  strongly, by [Sak, Theorem 2.5.6]. Let  $\phi(\cdot, y) \in \mathcal{B}(\mathcal{M})_*$ . Then we have

$$\begin{aligned}
|\phi(\text{ad}x_m(y)) - \phi(\text{ad}x(y))| &= |\phi(x_myx_m^*) - \phi(xyx^*)| \\
&\leq \phi((x_m - x)^*(x_m - x))^{1/2} \phi(x_my^*yx_m^*)^{1/2} \\
&\quad + \phi(y^*x^*xy)^{1/2} \phi((x_m - x)(x_m - x)^*)^{1/2} \\
&\leq \|\phi\| r^2 (\phi((x_m - x)^*(x_m - x))^{1/2} \\
&\quad + \phi((x_m - x)(x_m - x)^*)^{1/2}) \\
&\rightarrow 0.
\end{aligned}$$

Continuity of  $\text{ad}$  on  $\mathcal{M}_r$  follows by the definition of the  $\sigma(\mathcal{B}(\mathcal{M}), \mathcal{B}(\mathcal{M})_*)$ -topology. □

- (ii) (See, for example, [Zim4]) Let  $E$  be a separable Banach space and denote by  $\text{Homeo}(E_1^*)$  the group of homeomorphisms of the unit ball  $E_1^*$  of the dual space  $E^*$ . We shall take the Borel structure on this space to be that induced by the topology of uniform convergence.
- (iii) Let  $E$  be a separable Banach space and let  $\text{End}(E)$  denote the Banach space of bounded linear maps from  $E$  to  $E$ , endowed with the strong

operator topology. For every  $r > 0$ , the ball

$$End(E)_r = \{T \in End(E); \|T\| \leq r\}$$

with the strong operator topology is metrisable as a complete separable metric space. We take the Borel structure on  $End(E)$  induced by the strong operator topology.

Let  $Iso(E) \subset End(E)$  denote the group of isometric isomorphisms of  $E$ . Then, by [Zim4, Lemma 1.1],  $Iso(E)$  is a Borel subset of  $End(E)_1$ . Note that for every  $\xi \in E$ , the map  $T \mapsto T\xi, End(E) \rightarrow E$  is Borel.

The  $\sigma(E^*, E)$ -topology on the dual space  $E^*$  of  $E$  is the locally convex topology induced by the family of semi-norms:

$$p_\xi : E^* \rightarrow \mathbb{R} \text{ where } \xi \in E$$

defined by

$$p_\xi(\cdot) = |\langle \lambda\xi, \cdot \rangle|.$$

On  $End(E^*)$  consider the locally convex topology defined by the semi-norms

$$\begin{aligned} \tilde{p}_\xi & : End(E^*) \rightarrow \mathbb{R} \text{ where } \xi \in E \\ \tilde{p}_\xi(\cdot) & = \sup_{\phi \in E^*_1} |\langle \xi, \cdot\phi \rangle|. \end{aligned} \tag{3.1}$$

Then the transpose map

$$\begin{aligned} End(E) & \rightarrow End(E^*) \\ T & \mapsto T^* \end{aligned}$$

is continuous if  $End(E)$  and  $End(E^*)$  are respectively endowed with the strong operator topology and the topology defined by 3.1. Indeed,

$$\begin{aligned}\tilde{p}_\xi(T^*) &= \sup_{\phi \in \tilde{E}_1^*} |\langle \xi, T^* \phi \rangle| \\ &= \sup_{\phi \in \tilde{E}_1^*} |\langle T\xi, \phi \rangle| \\ &\leq \|T\xi\|.\end{aligned}$$

**Definition 3.1.3** Let  $S$  be a set, and let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of  $S$ . A function  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is said to be a measure if it is countably additive and satisfies  $\mu(\emptyset) = 0$ . The triple  $(S, \mathcal{A}, \mu)$  is then called a measure space, or a Borel space if the  $\sigma$ -algebra  $\mathcal{A}$  is the collection of Borel subsets of  $S$ .

A subset  $A \in \mathcal{A}$  is said to be null if  $\mu(A) = 0$  and conull if  $\mu(S \setminus A) = 0$ .

A measure  $\mu$  is said to be finite if  $\mu(S) < \infty$  and  $\sigma$ -finite if  $S$  can be written as a countable union of disjoint sets of finite measure under  $\mu$ . The Borel space  $(S, \mu)$  is then said to be a finite or  $\sigma$ -finite Borel space, respectively.

A positive Borel measure on the measure space  $X$  is said to be regular if

$$\mu(E) = \sup\{\mu(K) \mid K \subset E\} = \inf\{\mu(G) \mid E \subset G\}$$

for every Borel subset  $E \subset X$  where  $K$  ranges over the compact subsets of  $E$  and  $G$  ranges over the open supersets of  $E$ .

A measure  $\mu$  is said to be non-atomic if  $\mu(\{x\}) = 0$  for every  $x \in X$ . The measure space  $(S, \mu)$  is said to be a probability space if  $\mu$  is a positive measure which satisfies  $\mu(S) = 1$ . The measure  $\mu$  is then called a probability measure.

Two functions  $f$  and  $g : S \rightarrow E$  are said to be equal  $\mu$ -almost everywhere, abbreviated  $\mu$ -a.e., if the set of points at which they differ has measure zero.

We have the following theorem for probability measures on metric spaces. For a proof see [Par, Theorem II.1.2].

**Theorem 3.1.4** *(For example, [Par, Theorem II.1.2]) Let  $X$  be a metric space and  $\mu$  a probability measure on  $X$ . Then  $\mu$  is regular.*

**Definition 3.1.5** *Let  $\mu$  and  $\nu$  be measures on the measurable space  $(S, \mathcal{A})$ . Then  $\mu$  is said to be absolutely continuous with respect to  $\nu$  if each  $A \in \mathcal{A}$  which satisfies  $\nu(A) = 0$  also satisfies  $\mu(A) = 0$ . The measures  $\mu$  and  $\nu$  lie in the same measure class if each is absolutely continuous with respect to the other.*

The following result, which is usually called the Radon-Nikodym theorem, will be of importance in Chapter 5 where we prove some special cases of our main result. For a proof see, for example, [Coh].

**Theorem 3.1.6** *(For example, [Coh, Theorem 4.2.3]) Let  $(X, \mathcal{A})$  be a measurable space, and let  $\mu$  be a  $\sigma$ -finite measure and  $\nu$  a finite measure on  $(X, \mathcal{A})$ . If  $\nu$  is absolutely continuous with respect to  $\mu$ , then there is a function,  $g \in L^1(X, \mathcal{A}, \mu, \mathbb{R}^+)$  that satisfies  $\nu(A) = \int_A g d\mu$  for each  $A \in \mathcal{A}$ .*

The function  $g$  of Theorem 3.1.6 is called the *Radon Nikodym derivative* of  $\nu$  with respect to  $\mu$ . It shall form the main ingredient of one of our principal examples in Chapter 5.

We now discuss the topology with which we shall work on the space of positive measures on a metric space  $X$ .

**Theorem 3.1.7** (For example, [Ash, Theorem 4.5.1]) Let  $X$  be a metric space,  $\mathcal{B}$  a  $\sigma$ -algebra of Borel subsets of  $X$  and  $\mu_1, \mu_2, \dots, \mu$  finite Borel measures on  $(X, \mathcal{B})$ . Then the following conditions are equivalent:

- (a)  $\int_X f d\mu_n \rightarrow \int_X f d\mu$  for all bounded continuous functions  $f : X \rightarrow \mathbb{R}$ .
- (b)  $\int_X f d\mu_n \rightarrow \int_X f d\mu$  for all bounded measurable functions  $f : X \rightarrow \mathbb{R}$  such that  $f$  is continuous  $\mu$ -a.e..
- (c)  $\liminf_{n \rightarrow \infty} \mu_n(A) \geq \mu(A)$  for every open set  $A \subset X$  and  $\mu_n(X) \rightarrow \mu(X)$ .
- (d)  $\limsup_{n \rightarrow \infty} \mu_n(A) \leq \mu(A)$  for every closed set  $A \subset X$  and  $\mu_n(X) \rightarrow \mu(X)$ .

**Definition 3.1.8** ([Ash, p 198]) The topology of convergence as described in Theorem 3.1.7 is called the weak or vague topology on the space of finite measures on  $X$ .

**Remark** ([Ash, p 198]) The equivalences in Theorem 3.1.7 hold when the sequences are replaced by nets.

## 3.2 Special Borel Spaces

Let  $\mathcal{F}$  be a family of Borel subsets of a space  $S$ . The family  $\mathcal{F}$  is said to be *separating* if for every pair of distinct points  $x, y$  in  $S$ , there exists a set  $A \in \mathcal{F}$  such that  $x \in A$ , but  $y \notin A$ . A Borel space  $S$  is said to be *countably separated* if there exists a countable family  $\mathcal{F}$  of Borel subsets of  $S$  which is separating for  $S$ . A Borel space  $S$  is *countably generated* if there

exists a countable family of Borel subsets of  $S$  which is both separating, and generates the Borel structure on  $S$ .

**Example 3.2.1**  $[0, 1]$  is countably separated.

Consider the countable family of Borel subsets

$$\{[0, r] \mid r \in \mathbb{Q} \cap [0, 1]\}$$

of  $[0, 1]$ . Let  $x, y$  be elements of  $[0, 1]$  with  $x \neq y$ . We may assume  $x < y$ . By the density of the rationals in  $\mathbb{R}$  there exists  $q \in \mathbb{Q}$  with  $x \leq q < y$ . Then  $x \in [0, q]$ , but  $y \notin [0, q]$ . It follows that  $[0, 1]$  is countably separated. The family of sets described above also generate the topology on  $[0, 1]$ , and hence, the Borel structure. It follows that  $[0, 1]$  is countably generated.

Our proof of the following Proposition follows that found in [Zim1] and [Coh].

**Proposition 3.2.2** *(For example, [Zim1, Proposition A.1])*

- (i) *A Borel space  $X$  is countably separated if and only if there exists an injective Borel map  $X \rightarrow [0, 1]$ .*
- (ii) *A Borel space  $X$  is countably generated if and only if it is Borel isomorphic to a subset of  $[0, 1]$ .*

**Proof.**

(i) Suppose  $X$  is countably separated. Let  $\{A_i\}$  be a countable separating family. Let  $\Omega = \prod_1^\infty \{0, 1\}$ . Define  $f : X \rightarrow \Omega$  by  $f(a)_i = \chi_{A_i}(a)$  where  $\chi_A$  is the characteristic function of  $A$ . Then  $f$  is injective since  $\{A_i\}$  is separating.

It is well known that  $\Omega$  is homeomorphic to the Cantor set in  $[0, 1]$ . Let  $\phi : \Omega \rightarrow K$  be a homeomorphism. Then  $\phi \circ f : X \rightarrow [0, 1]$  is an injective

map. The map  $\phi$  is Borel since it is a homeomorphism. We claim that  $f$  is Borel. It then follows that  $\phi \circ f$  is an injective Borel map from  $X$  into  $[0, 1]$  as required.

It remains to establish our claim. We follow the proof of [Coh, Lemma 8.6.3]. By [Coh, Proposition 8.1.5], the sets

$$E_k = \{\{n_j\} \in \Omega; n_k = 1\}$$

generate the Borel structure on  $\Omega$ . Further, for each  $i \in \mathbb{N}$ , we have  $A_i = f^{-1}(E_i)$ . The measurability of  $f$  follows by [Coh, Proposition 2.6.2].

Suppose that there exists an injective Borel map  $f$  from  $X$  into  $[0, 1]$ . By Example 3.2.1,  $[0, 1]$  is countably separated. Consider the subsets  $f^{-1}[0, r]$  of  $X$  where  $r \in \mathbb{Q}$ . These form a countable family of Borel subsets of  $X$ . Let  $x$  and  $y$  be elements of  $X$  with  $x \neq y$ , and assume, relabelling if necessary, that  $f(x) < f(y)$ . Then, there exists a set  $[0, q] \subset [0, 1]$  with  $q \in \mathbb{Q}$  such that  $f(x) \in [0, q]$  and  $f(y) \notin [0, q]$ . The set  $f^{-1}[0, q]$  separates  $x$  and  $y$ . It follows that  $X$  is countably separated.

(ii) Suppose  $X$  is countably generated. Since  $\Omega$  is a Borel subset of  $[0, 1]$  ([Coh, Proposition 1.4.6]), it is enough to prove that  $X$  is Borel isomorphic to a subset of  $\Omega$ . But this follows from [Coh, Corollary 8.6.4].

The converse is easy to prove, since it is clear that every subspace of a countably generated space is, itself, countably generated.  $\square$

**Definition 3.2.3** (Compare [Zim1, Definition A.2]) *A Borel space is called standard if it is isomorphic to a Borel subset of a complete separable metric space.*

**Example 3.2.4** ([Zim4]) The Borel structure induced by the strong operator topology on the group  $Iso(E)$  discussed in example 3.1.1(ii) and (iii) is standard.

We have the following Theorem for standard Borel spaces.

**Theorem 3.2.5** (For example, [Zim1, Theorem A.9]) *Every standard Borel space is either finite, isomorphic to  $\mathbb{Z}$ , or isomorphic to  $[0, 1]$ .*

Thus it follows, by Proposition 3.2.2, that every standard Borel space is countably generated.

Let  $E$  be a separable Banach space, and let  $S$  be a Borel space. The space  $L^1(S, E)$  is the collection of functions

$$L^1(S, E) = \{f : S \rightarrow E \mid f \text{ measurable and } \int \|f(s)\| d\mu < \infty\}$$

where functions which agree on a conull set are identified. Recall [Edw, p 587] that this space is a separable Banach space when endowed with the norm

$$\|f\|_1 = \int \|f(s)\| d\mu.$$

The space  $L^\infty(S, E)$  is the collection of functions defined by

$$L^\infty(S, E) = \{f : S \rightarrow E \mid f \text{ measurable and } \text{esssup}\|f(s)\| < \infty\}.$$

We again identify functions which agree on a conull set. This space is a Banach space when endowed with the supremum norm [Edw, p 578].

**Definition 3.2.6** *A simple function  $F : S \rightarrow E$  is a function of the form  $F(s) = \sum_{i=1}^{\infty} x_i \chi_{A_i}(s)$  where  $x_i \in E$  and  $\{A_i\}$  is a countable partition of  $S$ .*

**Lemma 3.2.7** (For example [Zim6, p 292]) *Let  $E$  be a separable Banach space and let  $S$  be a standard Borel space. Then the simple Borel functions are dense in  $L^\infty(S, E)$ .*

**Proof.**

Our proof is based on the proof of [Edw, Theorem 8.15.2]. Let  $(\xi_n)$  be a dense sequence in  $E$ . Let  $f \in L^\infty(S, E)$ . We construct a sequence  $(g_k)$  of simple Borel functions such that  $g_k \rightarrow f$  in  $L^\infty(S, E)$ . We define

$$A_{kn} = \{t \in S : \|f(t) - \xi_n\| < \frac{1}{k}\}.$$

By [Edw, Lemma 2, section 8.15],  $A_{kn}$  is measurable. Also,  $S = \cup_n A_{kn}$  for each  $k$ . Let

$$B_{k1} = A_{k1}$$

and

$$B_{kn} = A_{kn} \setminus \cup \{B_{kj} : j < n\} \text{ for } n > 1.$$

Then for a given  $k$ , the  $B_{kn}$  are measurable sets which partition  $S$ . We define

$$g_k = \sum_{n=1}^{\infty} \xi_n \chi_{B_{kn}}.$$

We claim that for each  $k$ ,  $g_k$  is a Borel simple function.

Fix  $k$ . Then consider the sequence  $(g_{ki})$  of functions defined by

$$g_{ki} = \sum_{n=1}^i \xi_n \chi_{B_{kn}}.$$

Note that

$$\lim_i g_{ki} = g_k$$

pointwise on  $S$ . It follows, by [Edw, Theorem 8.15.1], that  $g_k$  is measurable.

Moreover,

$$\|f(t) - g_k(t)\| < \frac{1}{k}$$

for all  $t \in S$  and all  $k$ . It follows that  $g_k \rightarrow f$  uniformly, establishing our result.  $\square$

**Definition 3.2.8** *Let  $S$  be a set and let  $f : S \rightarrow X$  be a surjective function. A section for  $f$  is a function  $g : X \rightarrow S$  such that  $f \circ g = id_X$ .*

The following result about the existence of Borel sections is due to Kallman. It shall be of use in Chapter 5 when we prove some special cases of our main result.

**Theorem 3.2.9** *(For example, [Zim1, Corollary A.8]) If  $H \subset G$  is a closed subgroup of a locally compact second countable group, then there is a Borel section  $G/H \rightarrow G$ .*

### 3.3 Borel Fields

We give a brief overview of the basic theory of Borel fields. For more details see [Azo] and [Nie].

Let  $E$  be a metric space, and let  $S$  be a standard Borel space. A family  $\{F_s\}_{s \in S}$  of Borel subsets of  $E$  is said to be a *Borel field of subsets of  $E$*  if the set  $F = \{(\lambda, s) | \lambda \in F_s\}$  is a Borel subset of  $E \times S$ .

Let  $X$  be a complete separable metric space. Let  $\mathcal{C}(X)$  denote the collection of non-empty closed subsets of  $X$ . We introduce a Borel structure on

$\mathcal{C}(X)$ , turning it into a standard Borel space as follows. Our discussion is based on that found in [Azo]. Form a metrisable compactification  $(Y, d)$  of  $X$ . We define a metric  $\rho$  on  $\mathcal{C}(Y)$  by

$$\rho(S_1, S_2) = \max\left\{ \sup_{y_1 \in S_1} \{d(y_1, S_2)\}, \sup_{y_2 \in S_2} \{d(y_2, S_1)\} \right\}.$$

Then  $\mathcal{C}(Y)$  is a compact space when endowed with this metric. The induced Borel structure is standard. Let  $j : \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$  be the injective map sending each set  $S \in \mathcal{C}(X)$  to its closure in  $Y$ . The map  $j$  induces a Borel structure on  $\mathcal{C}(X)$  which is standard and independent of the choice of  $Y$ . We call this structure the *Hausdorff Borel structure* on  $\mathcal{C}(X)$ .

Given an open subset  $U$  of  $X$ , we write  $\langle U \rangle$  for the subset  $\{S \in \mathcal{C}(X) \mid S \cap X \neq \emptyset\}$  of  $\mathcal{C}(X)$ . By [Azo, Proposition 2.1], the collection  $\{\langle U \rangle \mid U \text{ open in } X\}$  generates the Hausdorff Borel structure on  $\mathcal{C}(X)$ .

Let  $Y$  be standard Borel space, and  $\Phi : Y \rightarrow \mathcal{C}(X)$ . A *selector* of  $\Phi$  is a function  $\phi : Y \rightarrow X$  satisfying  $\phi(y) \in \Phi(y)$  for all  $y \in Y$ . A *dense sequence of selectors* for  $\Phi$  is a sequence  $\{\phi_n\}_{n=1}^{\infty}$  of selectors for  $\Phi$  such that  $\{\phi_n(y)\}_{n=1}^{\infty}$  is dense in  $\Phi(y)$  for all  $y \in Y$ . A family of Borel subsets  $\{F_s\}_{s \in S}$  of  $X$  where  $S$  is a standard Borel space is a Borel field of sets if and only if there is a Borel map  $\Phi : S \rightarrow \mathcal{C}(X)$  such that  $\Phi(s) = F_s$  for all  $s \in S$ .

Our proofs of the following Proposition and its Corollary follow those found in [Azo].

**Proposition 3.3.1** (For example, see [Azo, Proposition 2.2]) *Let  $X$  be a complete, separable metric space. Then there is a sequence  $\{\psi_n\}_{n=1}^{\infty}$  of Borel functions from  $\mathcal{C}(X)$  into  $X$  such that  $\{\psi_n(S)\}_{n=1}^{\infty}$  is a dense subset of  $S$  for each  $S \in \mathcal{C}(X)$ .*

**Proof.**

Let  $Y = \{x_k\}_{k=1}^{\infty}$  be a dense subset of  $X$ . Let  $r > 0$ . Define a function

$$\eta_1 : \mathcal{C}(X) \rightarrow X$$

by  $\eta_1(S) = x_k$  where  $k$  is the smallest integer for which  $S \cap B(x_k, \frac{r}{2}) \neq \emptyset$ .

We inductively define a sequence of functions  $\eta_n : \mathcal{C}(X) \rightarrow X$  by setting  $\eta_{n+1}(S) = x_k$  where  $x_k$  is the element of  $Y$  of smallest index satisfying  $x_k \in B(\eta_n(S), \frac{r}{2^n})$  and  $B(x_k, \frac{r}{2^{n+1}}) \cap S \neq \emptyset$ .

The sequence  $\{\eta_n\}_{n=1}^{\infty}$  converges to a Borel function  $\psi : \mathcal{C}(X) \rightarrow X$  such that  $\psi(S) \in S$  for every  $S \in \mathcal{C}(X)$ . Note that  $\psi(S)$  is within  $2r$  of  $x_1$ , whenever the ball of radius  $r/2$  about  $x_1$  intersects  $S$ .

We repeat this construction of  $\psi$  for each sequence obtained from  $\{x_k\}_{k=1}^{\infty}$  by interchanging  $x_1$  and some other  $x_j$ , and for each positive  $r \in \mathbb{Q}$ , to obtain a sequence  $\{\psi_n\}_{n=1}^{\infty}$  of functions having the required properties.

□

**Corollary 3.3.2** *(For example, [Azo, Corollary 2.9]) Let  $Y$  be a standard Borel space, and  $X$  a complete, separable metric space. A map from  $Y$  into  $\mathcal{C}(X)$  is Borel if and only if it has a dense sequence of Borel selectors.*

**Proof.**

If  $\Phi : Y \rightarrow \mathcal{C}(X)$  is Borel, then the maps  $\phi_n = \psi_n \circ \Phi$  where the  $\psi_n$  are those from the above Proposition form a dense sequence of Borel selectors for  $\Phi$ .

Suppose that  $\{\phi_n\}_{n=1}^\infty$  is a dense sequence of Borel selectors for  $\Phi$ . Let  $V \subset X$  be open. Then  $\Phi^{-1}(V) = \bigcup_{n=1}^\infty \phi_n^{-1}(V)$  is a Borel subset of  $Y$ . The result follows.  $\square$

Let  $E$  be a separable Banach space. We denote by  $E_1^*$  the unit ball in the dual  $E^*$  of  $E$ . Since  $E$  is separable, this ball is metrisable in the  $\sigma(E^*, E)$ -topology [Con, Theorem V.5.1]. In addition, by the Banach-Alaoglu Theorem (for example, [Con, Theorem V.3.1]),  $E_1^*$  is a compact, convex set with respect to this topology.

Recall that the closed unit ball of  $L^\infty(S, E^*)$  is a compact metrisable space with the  $\sigma(L^\infty(S, E^*), L^1(S, E))$ -topology. We denote this ball by  $L_1^\infty(S, E^*)$ .

**Proposition 3.3.3** *Let  $\{A_s\}_{s \in S}$  be a Borel field of compact subsets of  $E_1^*$ . The set*

$$A = \{\lambda \in L^\infty(S, E^*) \mid \lambda(s) \in A_s \text{ } \mu\text{-a.e.}\}$$

*is a compact subset of  $L_1^\infty(S, E^*)$ .*

**Proof.**

Let  $\tilde{\tau}$  denote the  $\sigma(L^\infty(S, E^*), L^1(S, E))$ -topology on  $L^\infty(S, E_1^*)^* = L_1^\infty(S, E^*)$ . With the  $\tilde{\tau}$ -topology,  $L_1^\infty(S, E^*)$  is a compact metrisable space, and hence is separable. Thus it is enough to prove that  $A$  is  $\tilde{\tau}$ -closed.

Let  $(\xi_n)_{n \geq 1}$  be a countable dense sequence in  $E$ . For  $n, m \geq 1$ , we define a neighbourhood  $V_{n,m}$  of 0 in  $E_1^*$  with the  $\sigma(E^*, E)$ -topology by

$$V_{n,m} = \{\phi \in E_1^*; |\langle \phi, \xi_k \rangle| < \frac{1}{m}, 1 \leq k \leq n\}.$$

The sets  $\{V_{n,m}\}_{n,m \geq 1}$  form a local basis of 0 in  $E_1^*$ , endowed with the  $\sigma(E^*, E)$ -topology.

Let  $\lambda$  be a cluster point of  $A$  in the  $\tilde{\tau}$ -topology. Then there exists a sequence  $(\lambda_p)_{p \geq 1}$  such that  $\lambda_p$  converges to  $\lambda$  in the  $\tilde{\tau}$ -topology. Thus for every  $n \geq 1$  we have

$$\int_S \langle \lambda_p(s) - \lambda(s), \xi_j \rangle d\mu(s) \rightarrow 0 \text{ for } 1 \leq j \leq n \text{ as } p \rightarrow \infty.$$

But, by [Coh, Propositions 3.1.2, 3.1.4], this implies that there exists a subsequence  $(\lambda_{p_k})$  of  $(\lambda_p)$  and  $S_n \subset S$ ,  $\mu(S_n) = 1$  such that

$$\langle \lambda_{p_k}(s) - \lambda(s), \xi_j \rangle \rightarrow 0 \text{ as } k \rightarrow \infty \text{ for } 1 \leq j \leq n.$$

Let  $S_0$  denote the conull set  $\bigcap_{n \geq 1} S_n$ . Then for all  $n, m \geq 1$ , there exists  $p$  such that  $\lambda_p(s) - \lambda(s) \in V_{n,m}$  for all  $s \in S_0$ .

But  $\lambda_p(s) \in A_s$   $\mu$ -a.e.. Since  $A_s$  is compact, it follows that  $\lambda(s) \in A_s$   $\mu$ -a.e., and hence that  $\lambda \in A$ , establishing our result.  $\square$

We shall also make use of the following results due to Zimmer in Chapter 7. For a proof of the first result see [Zim4]. We provide a new proof of the second result.

**Lemma 3.3.4** (*Compare [Zim4, Lemma 1.7]*) *If  $\{A_s\}_{s \in S}$  is a Borel field of compact convex sets in  $E_1^*$ , then there is a countable collection of Borel functions  $a_i : S \rightarrow E_1^*$  such that, for all  $s$  in a conull Borel set,  $A_s = \overline{\{a_i(s) | i = 1, 2, \dots\}}$ . Conversely, given Borel functions  $a_i : S \rightarrow E_1^*$ , then  $s \mapsto \overline{\{a_i(s) | i = 1, 2, \dots\}}$  defines a Borel field of compact convex sets if each of these sets is convex.*

Let  $\{A_s\}_{s \in S}$  be a Borel field of compact convex subsets of  $E_1^*$ . Let

$$B = \{\lambda : S \rightarrow E_1^* \mid \lambda(s) \in A_s \text{ for } \mu\text{-a.e. } s \in S\}.$$

**Proposition 3.3.5** (Compare [Zim4, Proposition 2.2]) *The set  $B$  defined above is a closed convex subset of  $L_1^\infty(S, E^*)$ .*

**Proof.**

The set  $B$  is compact by Proposition 3.3.3. It is easy to see that  $B$  is convex, since all the sets  $A_s$  are convex.  $\square$

**Proposition 3.3.6** *Let  $E$  be a separable Banach space. Let  $K = \{f : S \rightarrow E; f = \sum q_i \chi_{A_i}\}$  where the  $q_i$  are positive numbers and  $\{A_i\}$  is a finite partition of  $S$ . Let  $B$  be a compact, convex subset of  $L_1^\infty(S, E^*)$  such that for all  $f_1, \dots, f_n \in K$  which satisfy  $\sum_{i=1}^n f_i(t) = 1$  for all  $t \in S$ , and all  $b_1, \dots, b_n \in B$  we have*

$$\sum_{i=1}^n f_i(t) b_i(t) \in B.$$

*Then there exists a Borel field of compact, convex sets  $\{B_t\}_{t \in S}$  such that*

$$B = \{\lambda \in L^\infty(S, E^*); \lambda(t) \in B_t \text{ } \mu\text{-a.e.}\}.$$

**Proof.**

Recall that  $L_1^\infty(S, E^*)$ , when endowed with the  $\sigma(L^\infty(S, E^*), L^1(S, E))$ -topology, is a compact, separable, metrisable space. Thus  $B$  is a compact, separable, metrisable space.

Let  $(b_n)_{n \geq 1}$  be a countable dense subset of  $B$ . For each  $t \in [0, 1]$ , let

$$B_t = \overline{\text{co}}\{b_n(t) | n = 1, 2, \dots\}$$

where the closure is taken in the  $\sigma(E^*, E)$ -topology. Since  $(b_n)$  is countable, it follows, by Lemma 3.3.4, that  $\{B_t\}$  is a Borel field of compact, convex subsets of  $E_1^*$ .

Let

$$\tilde{B} = \{\lambda \in L^\infty(S, E^*); \lambda(t) \in B_t \text{ } \mu\text{-a.e.}\}.$$

Then  $\tilde{B} \subset L_1^\infty(S, E^*)$ . By construction, we have  $B \subset \tilde{B}$ . Also, by [Zim4, Lemma 2.5], the functions of the form

$$t \mapsto \sum_{i=1}^n f_i(t) a_i(t)$$

where  $f_i \in K$  and for all  $t \in S$ ,  $\sum f_i(t) = 1$  are dense in  $\tilde{B}$ . It follows that  $\tilde{B} \subset B$ . Our result follows.  $\square$

# Chapter 4

## Group Actions

### 4.1 Introduction

We begin by recalling some basic definitions. An *action* of a group  $G$  on a set  $S$  is a map

$$S \times G \rightarrow S$$

satisfying the following conditions:

- i) For all  $g$  and  $h$  in  $G$  and for all  $s \in S$

$$s(gh) = (sg)h.$$

- ii) If  $e$  denotes the identity of the group, then for all  $s \in S$  we have

$$se = s.$$

An action is said to be *transitive* if for all  $s, t \in S$  there exists  $g \in G$  such that  $sg = t$ . An action is *free* if for all  $s \in S$  the *stabiliser* of  $s$

$$\text{Stab}_G(s) = \{g \in G | sg = s\} = \{e\}.$$

The *orbits* of an action are the sets

$$\text{Orbit}_G(s) = \{sg | g \in G\}.$$

The orbits of an action form a partition of the set  $S$ . It is clear that an action is transitive if and only if there is exactly one orbit. The orbits are invariant under the action of  $G$ . Thus the restriction of an action of a group  $G$  on a set  $S$  to an action on one of its orbits is transitive.

In this chapter we shall consider actions of a second countable locally compact group  $G$  on a standard Borel probability space  $S$  such that the action map

$$\begin{aligned} \alpha &: S \times G \rightarrow S \\ (s, g) &\mapsto sg \end{aligned}$$

is Borel. Such a space  $(S, \mu)$  is called a *standard Borel  $G$ -space*.

## 4.2 Ergodicity

**Definition 4.2.1** Let  $\mu$  be a  $\sigma$ -finite measure on  $S$ . Then  $\mu$  is said to be *quasi-invariant* under the action of  $G$  if the measures  $\mu$  and  $\mu \circ g$  are in the same measure class for all  $g \in G$ .

**Remark** Every  $\sigma$ -finite measure is in the same measure class as a probability measure.

**Proof.**

Let  $\mu$  be a  $\sigma$ -finite measure on a Borel space  $S$ . Choose a strictly positive  $L^1$ -function  $f$  such that  $\int f d\mu = 1$ . The measure  $f d\mu$  is a probability measure in the same measure class as  $\mu$ .  $\square$

**Definition 4.2.2** [Zim1, Definition 2.1.1] *The action of  $G$  on  $(S, \mu)$  with  $\mu$  quasi-invariant is called ergodic if every  $G$ -invariant measurable set is either null or conull.*

*If there is a conull orbit, then the action is said to be essentially transitive. If all orbits are of measure zero, then the action is called properly ergodic.*

**Examples 4.2.3**

- 1) Let  $H \subset G$  be a closed subgroup. Then, by [Mac, Theorem 1.1], there is a unique invariant measure class on  $G/H$ . The action of  $G$  on  $G/H$  defined by

$$[k]g = [kg]$$

is transitive, and hence ergodic.

- 2) [Zim1, Example 2.1.4] Let  $S = \{x \in \mathbb{C} \mid |z| = 1\}$ , and let

$$T : S \rightarrow S$$

be defined by

$$Tz = e^{i\alpha} z$$

where  $\alpha/2\pi$  is irrational. Then  $T$  generates an action of  $\mathbb{Z}$  on  $S$  by

$$n \mapsto T^n z.$$

Let  $\mu$  be Lebesgue measure on  $S$ . Since it is invariant under rotation, this action preserves  $\mu$ . The action is not essentially transitive since all orbits are countable and the measure is non-atomic. We claim that this action is ergodic. Suppose  $A \subset S$  is invariant. Let  $\chi_A(z) = \sum_n a_n z^n$  be the  $L^2$ -Fourier expansion of its characteristic function  $\chi_A$ . Then, by invariance:

$$\chi_A(z) = \chi_A(e^{i\alpha} z) = \sum_n a_n e^{in\alpha} z^n.$$

It follows, by the uniqueness of Fourier expansions, that  $a_n e^{in\alpha} = a_n$  for all  $n$ , and hence that  $a_n = 0$  for  $n \neq 0$  in view of the irrationality of  $\alpha/2\pi$ . Thus  $\chi_A$  is constant and the action is ergodic.

Our proof of the next proposition follows that found in [Zim1].

**Proposition 4.2.4** *(For example, see Proposition 2.1.7 [Zim1]) Suppose that  $S$  is a second countable topological space, that  $G$  acts continuously on  $S$ , and that a quasi-invariant measure  $\mu$  is positive on open sets. If the action is properly ergodic, then for almost every  $s \in S$ ,  $\text{Orbit}_G(s)$  is a dense null set.*

**Proof.**

Suppose that  $W \subset S$  is a non-empty open set. Then  $\cup_{g \in G} W.g$  is an open invariant set for the action. By assumption,  $\mu(\cup_{g \in G} W.g) > 0$ , and the action is ergodic. It follows that  $\mu(S - \cup_{g \in G} W.g) = 0$ . Thus if  $\{W_i\}$  is a countable base for the topology on  $S$ ,  $K = \cap_i (\cup_{g \in G} W_i.g)$  will be a conull invariant set. Now, for every point  $s \in K$ , the orbit intersects all the  $W_i$ 's, and hence must be dense in  $S$ .  $\square$

**Definition 4.2.5** (*Compare Definition 2.1.9 [Zim1]*) *Let  $S$  be a countably separated Borel  $G$ -space. The action of  $G$  on  $S$  is said to be smooth if the quotient Borel structure on  $S/G$  is countably separated. Here  $S/G$  denotes the set of equivalence classes defined by the orbits of the action.*

**Proposition 4.2.6** (*For example, Proposition 2.1.10 [Zim1]*) *Suppose  $G$  acts smoothly on  $S$ . Then every quasi-invariant ergodic measure  $\mu$  on  $S$  is supported on an orbit.*

**Proof.**

We follow the proof found in [Zim1]. Let  $\mathcal{J} = \{A_i\}$  be a sequence of Borel sets separating points in  $S/G$ . We can assume that  $\mathcal{J}$  is closed under taking complements (if not, add the complements).

Let  $p : S \rightarrow S/G$  be the canonical surjection, and let  $\nu = p_*(\mu)$ . By ergodicity of  $\mu$ , for every  $A \in \mathcal{J}$ ,  $\nu(A) = 0$  or  $1$ . Let

$$B = \cap \{A \in \mathcal{J} \mid \nu(A) = 1\}.$$

Then  $\nu(B) = 1$  and it is enough to prove that  $B$  consists of a single point.

Suppose  $B$  contained two points. Then these could be separated by an element of  $\mathcal{J}$ . But then either this set or its complement would have measure 1, contradicting the definition of  $B$ .  $\square$

**Definition 4.2.7** *Let  $S$  be an ergodic  $G$ -space, and  $Y$  a countably separated space. A function*

$$f : S \rightarrow Y$$

*is said to be  $G$ -invariant if for all  $g \in G$  and almost every  $s \in S$  we have*

$$f(sg) = f(s).$$

**Proposition 4.2.8** *(For example, Proposition 2.1.11 [Zim1]) Suppose  $S$  is an ergodic  $G$ -space and that  $Y$  is a countably separated space. If  $f : S \rightarrow Y$  is a  $G$ -invariant Borel function, then  $f$  is essentially constant.*

**Proof.**

Our proof is the one found in [Zim1]. If  $\mu$  is a quasi-invariant ergodic measure on  $S$ , then we claim that it is enough to show that  $f_*(\mu)$  is supported on a point. The proof of Proposition 4.2.6 shows this.

$\square$

**Definition 4.2.9** *A subset  $A$  of a topological space  $(X, \mathcal{T})$  is said to be locally closed if  $A$  is an open subset of  $\bar{A}^{\mathcal{T}}$  with the subspace topology. Equiv-*

alently,  $A$  is locally closed if  $A$  is the intersection of a closed and an open subset of  $(X, \mathcal{T})$ .

The following result gives some equivalent conditions for the smoothness of an action.

**Theorem 4.2.10** (For example, Theorem 2.1.14 [Zim1]) *Suppose  $G$  acts continuously on a complete separable metrisable space. Then the following conditions are equivalent.*

- i) All orbits are locally closed.*
- ii) The action is smooth.*
- iii) For every  $s \in S$ , the natural map  $G/\text{Stab}_G(s) \rightarrow \text{Orbit}_G(s)$  is a homeomorphism, where  $\text{Orbit}_G(s)$  has the relative topology as a subspace of  $S$ .*

**Proof.**

(i)  $\Rightarrow$  (ii) We follow the proof of Proposition 2.1.12 [Zim1].

Let  $\rho : S \rightarrow S/G$  be the natural map. Since  $\rho$  is open (see, for example, [HeR, Theorem 5.17]) and  $S$  is second countable, so is  $S/G$ . Thus to show that  $S/G$  is countably separated, it is enough to show that  $S/G$  is a  $T_0$ -space.

Suppose  $x$  and  $y$  are elements of  $S$  such that  $\rho(x)$  and  $\rho(y)$  are not separated by an open set. Then  $yG \subset \overline{xG}$ , and similarly  $xG \subset \overline{yG}$ . It follows that  $yG$  is dense in  $\overline{xG}$ . But  $xG$  is locally closed, and hence  $xG$  is open in  $\overline{xG}$ . It follows that  $xG \cap yG \neq \emptyset$ , and we must have  $\rho(x) = \rho(y)$ .

The equivalence of (i) and (iii) follows from the following lemma, the proof of which is found in [Zim1, Lemma 2.1.15].

**Lemma 4.2.11** *(For example [Zim1, Lemma 2.1.5]) Suppose  $G$  acts continuously on a complete separable metrisable space  $S$ . Suppose  $s \in S$  has a dense orbit. Then  $\text{Orbit}_G(s)$  is open if and only if  $G/\text{Stab}_G(s) \rightarrow \text{Orbit}_G(s)$  is a homeomorphism.*

The implication (ii)  $\Rightarrow$  (i) is the most difficult to prove. For a proof see [Zim1], Chapter 2: Lemmas 2.1.16 - 2.1.18, and the discussion preceding Lemma 2.1.16.  $\square$

**Corollary 4.2.12** *(For example, [Zim1, Corollary 2.1.21]) Every continuous action of a compact group on a separable metrisable space is smooth.*

**Proof.**

Our proof is based on that found in [Zim1]. Let  $X$  be a separable, metrisable space, and let  $G$  be a compact group acting continuously on  $S$ . Let  $x \in X$ . We consider the orbit  $Gx$ . This is a compact subset of  $X$  since  $G$  acts continuously. It is closed, and hence locally closed, since  $X$  is Hausdorff. It follows, by the equivalence of statements (i) and (ii) in the above Theorem, that the action of  $G$  on  $X$  is smooth.  $\square$

**Theorem 4.2.13** *(For example, [Zim1, Theorem 2.1.19]) Let  $S$  be a countably separated Borel  $G$ -space. Then there is a compact metric space  $X$  on*

which  $G$  acts continuously and an injective Borel  $G$ -map  $S \rightarrow X$ .

**Proof.**

Our proof follows that found in [Zim1]. Let  $\{A_i\}$  be a countable separating set for  $S$ , and let  $\chi_i$  denote the characteristic function of  $A_i$ .

Let  $B$  be the unit ball in  $L^\infty(G)$ . Then, by the Banach-Alaoglu Theorem, and Theorem 9 [Gil],  $B$  is a compact metric space in the weak\*-topology. The action of  $G$  on  $B$  is induced by the action of right translation on  $L^\infty(G)$ . That is, for  $f \in L^\infty(G)$ , we have  $f.g(h) = f(hg)$ . Let  $X = \prod_{i=1}^\infty B$ . Then, by Tychonoff's Theorem,  $X$  is a compact metric  $G$ -space where  $G$  acts by componentwise translation as described above.

Define a map

$$\phi : S \rightarrow X$$

by

$$\phi(s) = (\phi_i(s))$$

where  $\phi_i(s) \in B$  is given by  $[\phi_i(s)](g) = \chi_i(sg)$ . We claim that  $\phi$  is a  $G$ -map. For,

$$\begin{aligned} [\phi_i(sg)](h) &= \chi_i(sgh) \\ &= \chi_i(s(gh)) \\ &= [\phi_i(s)](gh) \end{aligned}$$

for all  $g$  and  $h$  in  $G$  and all  $s \in S$ .

To prove that  $\phi$  is Borel, it is enough to prove that each  $\phi_i$  is Borel. For this it suffices to prove that for all  $f \in L^1(G)$

$$s \mapsto \int f(g)[\phi_i(s)](g)dg = \int f(g)\chi_i(sg)dg$$

is Borel. But this follows from Fubini's Theorem.

We claim that  $\phi$  is injective. Suppose  $s, t \in S$  are such that  $\phi(s) = \phi(t)$ . Then for each  $i$ ,  $\chi_i(sg) = \chi_i(tg)$  for almost every  $g \in G$ . It follows that for some  $g_0 \in G$ ,  $\chi_i(sg_0) = \chi_i(tg_0)$  for all  $i$ , and since  $\{A_i\}$  separates points, we must have  $sg_0 = tg_0$ , and hence  $s = t$ .  $\square$

**Corollary 4.2.14** *(For example, [Zim1, Corollary 2.1.20]) Let  $S$  be a countably separated Borel  $G$ -space. Then orbits are Borel sets and stabilisers of points are closed subgroups.*

**Proof.**

By theorem 4.2.13, there exists a compact metric space  $X$  on which  $G$  acts continuously and an injective Borel  $G$ -map  $\phi : S \rightarrow X$ . Let  $x \in X$ . Since  $G$  is  $\sigma$ -compact, it follows, by continuity of the action, that  $Gx$  is  $\sigma$ -compact. Hence  $Gx$  is a countable union of closed sets, and it follows that  $Gx$  is Borel. Now for  $s \in S$ , we have  $Gs = \phi^{-1}(G\phi(s))$ . Our result follows.

Let  $s \in S$ . Suppose that  $g$  is a cluster point of  $\text{Stab}_G(\phi(s))$ . Then there exists a sequence  $\{g_n\} \subset \text{Stab}_G(\phi(s))$  such that  $g_n \rightarrow g$ . Now, by continuity of the action, we have  $g_n\phi(s) \rightarrow g\phi(s)$ . But  $g_n\phi(s) = \phi(s)$  for all  $n$ , and hence we must have  $g\phi(s) = \phi(s)$ , that is  $g \in \text{Stab}_G(\phi(s))$ . Our result follows since  $\phi$  is an injective  $G$ -map.  $\square$

**Remark** If  $G$  acts transitively on  $S$ , then the action is equivalent to an action of  $G$  on  $G/G_0$ , where  $G_0 = \text{Stab}_G(x)$  for some  $x \in S$ . The above corollary implies that  $G_0$  is a closed subgroup of  $G$ . It follows that the usual quotient map  $q : G \rightarrow G/G_0$  is continuous and open, and that the quotient space  $G/G_0$  is Hausdorff and locally compact. (For a discussion see [HeR, Section 5]).

**Definition 4.2.15** *If  $(S, \mu)$  and  $(S', \mu')$  are standard Borel  $G$ -spaces, then they are said to be equivalent if there exist conull  $G$ -invariant Borel sets  $S_0 \subset S$  and  $S'_0 \subset S'$  and a measure class preserving Borel isomorphism  $\phi : S_0 \rightarrow S'_0$  such that  $\phi(sg) = \phi(s)g$  for all  $s \in S_0$  and all  $g \in G$ .*

Thus, by a similar argument to that discussed above, with  $S_0$  denoting the orbit of measure 1, we see that every essentially transitive  $G$ -space is equivalent to  $G/G_0$  for some closed subgroup  $G_0$  of  $G$ .

**Corollary 4.2.16** *(For example, [Zim1, Corollary 2.1.21]) Every action of a compact group on a countably separated Borel space is smooth.*

**Proof.**

We follow the proof found in [Zim1]. Apply Corollary 4.2.12 to the continuous action of  $G$  on the separable metrisable space of theorem 4.2.13.  $\square$

## 4.3 Flows

In this section we briefly describe the basic theory of flows for discrete groups  $G$ . We shall generalise this theory in Chapter 7. For some fuller discussions and for the development of the theory for continuous groups, see, for example, [Gla1], [Gla2], [Nam], and [Phe].

**Definition 4.3.1** *A flow  $(G, X)$  is a pair consisting of a countable discrete group  $G$  and a compact Hausdorff space  $X$  such that the group  $G$  acts continuously on  $X$ . The pair  $(G, Y)$  is said to be a subflow of  $(G, X)$  if  $Y$  is a non-empty closed subspace of  $X$  which is invariant under the action of  $G$ .*

**Definition 4.3.2** *A flow  $(G, X)$  is said to be minimal if it has no proper subflows.*

**Remark** A simple Zorn's lemma argument shows that every flow has a minimal subflow.

If  $(G, X)$  is a flow and  $x \in X$ , then the orbit of  $x$  is the set  $O(x) = \{xg \mid g \in G\}$ .

**Proposition 4.3.3** *(For example, [Gla1, p 2]) A flow  $(G, X)$  is a minimal flow if and only if  $X = \overline{O(x)}$  for every  $x \in X$ .*

**Proof.**

Suppose  $(G, X)$  is minimal. Then since  $X$  is invariant under the action of  $G$  we must have  $\overline{O(x)} \subseteq X$  for all  $x \in X$ . Further for each  $x \in X$ ,  $\overline{O(x)}$

is a subset of  $X$  which is invariant under the action of  $G$ . It follows, by minimality of  $X$ , that  $X = \overline{O(x)}$  for all  $x \in X$ .

The converse is clear. □

**Definition 4.3.4** *Two points  $x$  and  $y \in X$  are said to be proximal if there exists a sequence  $\{g_i\} \subset G$  and  $z \in X$  such that*

$$\lim xg_i = z = \lim yg_i.$$

*Points  $x$  and  $y$  are said to be distal if we have either  $x = y$  or  $x$  and  $y$  are not proximal. The flow  $(G, X)$  is said to be proximal if every pair of points in  $X$  is proximal and distal if every pair of points in  $X$  is distal.*

**Remark** The above definitions are given in terms of sequences since the group  $G$  we are working with is discrete. See [Fur] for further discussion of these concepts in the case where  $G$  is an abelian discrete group. The corresponding definitions for continuous groups are the same, except that in this case we need to work with nets rather than sequences. See [Gla1] for a discussion.

**Remark** Clearly a flow  $(G, X)$  is both proximal and distal if and only if the space  $X$  consists of exactly one point.

**Example 4.3.5** (Example II.5.2 [Gla1])

Let  $Y = \{e^{2\pi i\theta} | 0 \leq \theta < 1\}$ , and let  $t : Y \rightarrow Y$  be the homeomorphism defined by  $t(e^{2\pi i\theta}) = e^{2\pi i\theta^2}$ . Let  $s : Y \rightarrow Y$  be the homeomorphism defined

by  $s(e^{2\pi i\theta}) = e^{2\pi i(\theta+\beta)}$  and let  $G$  be the group of homeomorphisms generated by  $s$  and  $t$ . If  $\beta$  is irrational, then  $(G, Y)$  is minimal. Further, every point of  $Y$  is clearly proximal to 1 and hence the flow  $(G, Y)$  is proximal.

**Proposition 4.3.6** *(For example, [Gla1, p 20]) Every proximal flow  $(G, X)$  has a unique minimal sub-flow.*

**Proof.**

Suppose  $(G, A)$  and  $(G, B)$  are two minimal sub-flows of  $(G, X)$ . By Proposition 4.3.3,  $A = \overline{O(a)}$  for every  $a \in A$  and  $B = \overline{O(b)}$  for every  $b \in B$ . Fix  $a \in A$ . Then, by proximality of  $(G, X)$ , for every  $b \in B$  there exists a sequence  $\{g_i\} \subset G$  such that  $\lim bg_i = \lim ag_i$ . It follows that  $B \subseteq \overline{O(a)} = A$ , and, by minimality of  $(G, A)$ , we must have  $A = B$ .  $\square$

Let  $Y$  be a compact space. Let  $M(Y)$  denote the space of all probability measures on the set  $Y$ . The space  $M(Y)$  can be considered as a convex subset of the space  $C(Y)^*$  via the mapping

$$\mu(f) = \int_Y f d\mu$$

for  $\mu \in M(Y)$  and  $f \in C(Y)$ . With the weak\*-topology induced from  $C(Y)^*$ ,  $M(Y)$  is a compact Hausdorff space by [Par, Theorem II.6.4]. This topology corresponds to the vague topology on  $M(Y)$ .

We have the following results for  $M(Y)$ . See [Par] for proofs.

**Theorem 4.3.7** *(Compare [Par, Theorem II.6.2]) The space  $M(Y)$  can be metrised as a separable metric space if and only if  $Y$  is separable.*

**Theorem 4.3.8** (Compare [Par, Theorem II.6.3]) Let  $Y$  be a separable metric space, and let  $E$  be a countable dense subset of  $Y$ . Then the set of all measures whose supports are the finite subsets of  $E$  forms a countable dense subset of  $M(Y)$ .

**Theorem 4.3.9** (For example, [Par, Theorem II.6.4]) The space  $M(Y)$  is a compact metric space if and only if  $Y$  is a compact metric space.

**Theorem 4.3.10** (For example, [Par, Theorem II.6.5]) Let  $Y$  be a separable metric space. Then  $M(Y)$  is topologically complete if and only if  $Y$  is so.

**Proposition 4.3.11** (For example, [Con, Theorem V.8.4]) The set of extreme points of  $M(Y)$  is

$$\{\delta_x | x \in Y\}.$$

**Proof.**

Our proof is based on that found in [Con]. Let  $x \in Y$ . We show that  $\delta_x$  is an extreme point of  $M(Y)$ . Suppose  $\mu_1, \mu_2 \in M(Y)$  are such that  $\delta_x = \frac{1}{2}(\mu_1 + \mu_2)$ . We show that the support of  $\mu_1$  and  $\mu_2$  is  $\{x\}$ . Since  $\mu_1$  and  $\mu_2$  are regular measures, it is enough to show that for all compact subsets  $D \subset Y \setminus \{x\}$  we have  $\mu_1(D) = \mu_2(D) = 0$ . Suppose  $D$  is such a subset but  $\mu_1(D) > 0$ . Then since  $\mu_1$  and  $\mu_2$  are positive measures we must have  $\delta_x(D) > 0$ . But this is impossible. It follows that we must have  $\mu_1 = \mu_2 = \delta_x$ .

Suppose that  $\mu$  is an extreme point of  $M(Y)$ . Let  $K$  be the support of  $\mu$ . Fix  $x_0 \in K$ , and suppose that there exists  $x \in K$  with  $x \neq x_0$ . Let  $U$  and  $V$  be open subsets of  $Y$  such that  $x_0 \in U$ ,  $x \in V$  and  $\bar{U} \cap \bar{V} = \emptyset$ .

Then  $0 < \mu(U \cap K) = r < 1$ . Define measures  $\mu_1, \mu_2$  for each Borel subset  $B \subset Y$  by  $\mu_1(B) = r^{-1}\mu((U \cap K) \cap B)$  and  $\mu_2(B) = (1 - r)^{-1}\mu((K \setminus (U \cap K)) \cap B)$ . Then  $\mu_1$  and  $\mu_2$  are non-equal probability measures which satisfy  $\mu = r\mu_1 + (1 - r)\mu_2$ , contradicting  $\mu \in \text{ex}(M(Y))$ . It follows that  $\mu = \delta_y$  for some  $y \in Y$ .  $\square$

**Remark** The above proposition, together with the Krein-Milman Theorem, shows that the set of atomic measures is dense in  $M(Y)$  with the vague topology.

Given a flow  $(G, X)$ , we can view the pair  $(G, M(X))$  as a flow. The action of  $G$  on  $M(X)$  is defined as follows. Let  $\mu \in M(X)$  and  $g \in G$ . We define the measure  $g\mu$  by

$$\int_X f d(g\mu) = (g\mu)f = \mu(f \cdot g) = \int_X f \cdot g d\mu.$$

Further, each  $g \in G$  acts affinely on  $M(X)$ .

**Definition 4.3.12** A flow  $(G, X)$  is said to be strongly proximal if the flow  $(G, M(X))$  is proximal.

**Remark** We can identify the space  $X$  with the subset  $\{\delta_x | x \in X\}$  of  $M(X)$  so that  $(G, X)$  can be regarded as a subflow of  $(G, M(X))$ . Under this identification it becomes clear that every strongly proximal flow is proximal.

We now give an example of a proximal flow which is not strongly proximal.

**Example 4.3.13** ([Gla2, p 162]) Consider the compact group  $X = \{0, 1\}^{\mathbb{Z}}$ . Let  $\tau$  be the shift homeomorphism on  $X$ . That is,

$$(\tau x)(n) = x(n - 1) \text{ for } x \in X.$$

Let the map  $\phi : a \mapsto \bar{a}$  for  $a \in \{0, 1\}$  be defined by  $\bar{1} = 0$  and  $\bar{0} = 1$ . Define  $\sigma x = x$  if  $x(0) = 0$  and  $\sigma x = x'$  if  $x(0) = 1$ . Here  $x'$  is the two sided sequence defined by  $x'(n) = x(n)$  for  $n \neq 1$  and  $x'(1) = \overline{x(1)}$ . Finally, define  $\rho x = x$  if  $x(0) = 1$  and  $\rho x = x'$  if  $x(0) = 0$ . Let  $T$  be the group of homeomorphisms of  $X$  generated by  $\tau$ ,  $\sigma$  and  $\rho$ .

The flow  $(T, X)$  is proximal since, given  $x$  and  $y$  in  $X$ , we can apply the homeomorphisms  $\tau$ ,  $\sigma$  and  $\rho$  alternately to both  $x$  and  $y$  in such a way that an arbitrary given block of  $x$  will become identical to the corresponding block of  $y$ , but this block of  $y$  will remain unchanged.

Finally, the flow  $(T, X)$  is measure preserving. It preserves the product measure on  $X$  obtained from the measure  $\mu(0) = \mu(1) = \frac{1}{2}$  on  $\{0, 1\}$ . It follows that the flow  $(T, X)$  is not strongly proximal.

We prove an analogue of the following Lemma for our generalised flows in Chapter 7. Our proof here follows that found in [Pie].

**Lemma 4.3.14** (For example, [Pie, Lemma 5.1]) *The flow  $(G, X)$  is strongly proximal if and only if given  $\mu \in M(X)$  there exists a sequence  $\{g_i\} \subset G$  such that  $\lim g_i \mu = x$  for some  $x \in X$ .*

**Proof.**

Suppose that  $(G, X)$  is strongly proximal. if  $\mu \in M(X)$  and  $\eta \in X$ , there exists a sequence  $(g_i) \in G$  such that  $\lim_i g_i\mu = \lim_i g_i\eta$ . It suffices to put  $x = \lim_i g_i\eta$ . The result follows under the identification described earlier.

Conversely, let  $\mu, \nu \in M(X)$ . Let  $\sigma = \frac{1}{2}(\mu + \nu) \in M(X)$ . By our assumption, there exist a sequence  $(g_i) \subset G$  and  $x \in X$  such that  $\lim g_i\sigma = x$ . By compactness of  $M(X)$ , we may assume that  $\lim g_i\mu = \mu_1$  and  $\lim g_i\nu = \nu_1$  for some  $\mu_1, \nu_1 \in M(X)$ . Then we must have  $x = \frac{1}{2}(\mu_1 + \nu_1)$ . The result follows since  $x$  is an extreme point of  $M(X)$ .  $\square$

### 4.3.1 Affine Flows

**Definition 4.3.15** *A flow  $(G, X)$  is an affine flow if the space  $X$  is a compact, convex subset of a locally convex Hausdorff linear topological space and  $G$  is a group of affine transformations of  $X$ . The affine flow  $(G, X)$  is said to be irreducible if it has no proper affine subflows.*

**Remark** Again, a simple Zorn's lemma argument shows that every affine flow has an irreducible affine subflow.

The barycentre of a compact, convex set will be of great use in what follows. Our proofs of the following propositions imitate those found in Chapter 1 of [Phe].

**Proposition 4.3.16** (compare Proposition III.2.1 [Gla1]) *Let  $E$  be a locally convex topological vector space. Let  $Q$  be a compact, convex subset of  $E$ . There exists a unique map (the barycentre map)  $\beta : M(Q) \rightarrow Q$  satisfying the following conditions.*

(a)  $\beta$  is onto.

(b)  $\beta$  is affine.

(c)  $\beta$  is weak\*-continuous. That is, if  $\mu_\alpha \rightarrow \mu$  vaguely, then we have  $\beta(\mu_\alpha) \rightarrow \beta(\mu)$ .

(d) If  $f$  is a continuous real valued affine function on  $Q$  then

$$f(\beta(\mu)) = \int_Q f d\mu \text{ for } \mu \in M(Q).$$

**Proof.**

If  $\mu = \sum_{i=1}^n c_i \delta_{a_i}$  is an atomic measure, where  $a_i \in Q$ ,  $0 \leq c_i \leq 1$ , and  $\sum c_i = 1$ , then properties (b) and (d) above imply that we must have  $\beta(\mu) = \sum_{i=1}^n c_i a_i$ . Since the set of atomic measures is dense in  $M(Q)$  this defines the barycentre map  $\beta : M(Q) \rightarrow Q$  on a dense subset of  $M(Q)$ . We can extend this map continuously to  $M(Q)$ . Our proof follows [Phe, Proposition 1.1]. Let  $\mu \in M(Q)$ . If  $f \in E^*$  then, by restriction, we have  $f \in C(Q)$ . For each  $f \in E^*$ , let  $H_f = \{y | f(y) = \mu(f) = \int_Q f d\mu\}$ . These are closed hyperplanes. We wish to show that their intersection is non-empty. Since  $Q$  is compact, it is enough to show, by the finite intersection property, that for

every finite subset  $\{f_1, \dots, f_n\} \subset E^*$ ,  $\bigcap_{i=1}^n H_{f_i} \cap Q \neq \emptyset$ . Define a continuous linear map  $T : E \rightarrow \mathbb{R}^n$  by

$$Ty = (f_1(y), \dots, f_n(y)).$$

Then  $TQ$  is compact and convex. It is enough to show that

$p = (\mu(f_1), \mu(f_2), \dots, \mu(f_n)) \in TQ$ . For then,  $p = Ty$  where  $y \in \bigcap_{i=1}^n H_{f_i} \cap Q$ . Suppose  $p \notin TQ$ . Then there exists a linear functional on  $\mathbb{R}^n$  which strictly separates  $p$  and  $TQ$ . If we represent this functional by  $a = (a_1, a_2, \dots, a_n)$ , then we have  $\langle a, p \rangle > \sup\{\langle a, Ty \rangle \mid y \in Q\}$ . Defining  $g \in E^*$  by  $g = \sum a_i f_i$  this last assertion becomes

$$\int_Q g d\mu > \sup_{x \in Q} \{g(x)\}.$$

But this is impossible since  $\mu(Q) = 1$ . This establishes our claim, and assertion (d) above.

The map  $\beta$  thus obtained is onto. For if  $x \in Q$ , then  $\beta(\delta_x) = x$ . Affineness of  $\beta$  follows from its definition.

We show that  $\beta$  is weak\*-continuous. Let  $\{\mu_\alpha\} \subset M(Q)$  be a net which converges to  $\mu \in M(Q)$ . Let  $x_\alpha$  and  $x$  denote their respective barycentres. Since  $X$  is compact, it is enough to show that every convergent subnet  $\{x_\beta\}$  of  $\{x_\alpha\}$  converges to  $x$ . Suppose  $x_\beta \rightarrow y$ . Then  $\mu_\beta \rightarrow \mu$  and hence  $f(x_\beta) = \mu_\beta(f) \rightarrow \mu(f) = f(x)$  for each  $f \in E^*$ . Since  $E^*$  separates points of  $X$ , we must have  $y = x$ . □

**Remark** ([Zim1, p 61]) If  $T : E \rightarrow E$  is an affine transformation such that  $T(Q) \subseteq Q$ , then  $T$  commutes with  $\beta$ . That is,

$$\beta(T_*\mu) = T(\beta(\mu))$$

for all  $\mu \in M(Q)$ .

**Proposition 4.3.17** (compare Proposition III.2.2 [Gla1])

- (1) Let  $X$  be a closed subset of  $Q$  such that  $\overline{\text{co}}(X) = Q$ , then  $\text{ex}(Q) \subset X$ .
- (2) If  $x \in \text{ex}(Q)$  then  $\delta_x$ , the point mass at  $x$ , is the unique measure  $\mu$  in  $M(Q)$  such that  $\beta(\mu) = x$ .

**Proof.**

We follow the proofs found in [Phe, Chapter 1].

(2) Suppose  $x \in \text{ex}(Q)$  and  $\mu \in M(Q)$  satisfies  $\beta(\mu) = x$ . Since  $\mu$  is a regular measure, to show that  $\mu$  is supported by  $\{x\}$  it suffices to show that for each compact subset  $D \subset Q \setminus \{x\}$  we have  $\mu(D) = 0$ . Suppose  $\mu(D) > 0$  for some such  $D$ . By compactness of  $D$ , there is a point  $y \in D$  such that  $\mu(U \cap Q) > 0$  for every neighbourhood  $U$  of  $y$ . Choose  $U$  to be a closed convex neighbourhood of  $y$  such that  $K = U \cap Q \subset Q \setminus \{x\}$ . Then  $K$  is a compact convex set of measure  $0 < r = \mu(K) < 1$ . We can define Borel measures  $\mu_1$  and  $\mu_2$  on  $Q$  by  $\mu_1(B) = r^{-1}\mu(B \cap K)$  and  $\mu_2(B) = (1 - r)^{-1}\mu(B \cap (Q \setminus K))$  for each Borel subset  $B \subset Q$ . Let  $x_i = \beta(\mu_i)$ . Since  $\mu_1(K) = 1$ , we have  $x_1 \in K$ , and hence  $x_1 \neq x$ . Further,  $\mu = r\mu_1 + (1 - r)\mu_2$ , and hence  $x = \beta(\mu) = rx_1 + (1 - r)x_2$  contradicting  $x \in \text{ex}(Q)$ .

(1) Suppose  $x \in ex(Q)$ . Then, by Proposition 4.3.16, there exists a measure  $\mu$  on  $X$  such that  $\beta(\mu) = x$ . By part (2), we must have  $\mu = \delta_x$ . It follows that  $x \in X$ .  $\square$

The following theorem is due to Furstenberg. The proof we give here is the one found in [Gla1].

**Theorem 4.3.18** (compare Theorem III.2.9 [Gla1]) *Let  $(G, Q)$  be an irreducible affine flow. Then  $(T, Q)$  is a strongly proximal flow. If  $X = \overline{ex}(Q)$ , then  $X$  is the unique minimal set of  $Q$ . Thus  $(G, X)$  is a minimal strongly proximal flow.*

**Proof.**

Since the action of  $G$  is affine, the set  $X = ex(Q)$  is an invariant subset of  $Q$ . Hence  $(G, X)$  is a flow. Let  $x \in Q$ . Then, by irreducibility of  $Q$ , we have  $\overline{co}(\overline{O}(x)) = Q$ . Hence, by Proposition 4.3.17, we must have  $X \subset \overline{O}(x)$ . It follows, by Proposition 4.3.3, that  $X$  is the unique minimal set in  $(G, Q)$ .

Let  $\mu \in M(Q)$  and let  $\beta : M(Q) \rightarrow Q$  be the barycentre map. Then  $\beta(\overline{co}(\overline{O}(\mu)))$  is a closed convex invariant subset of  $Q$ . It follows, by irreducibility of  $(G, Q)$ , that  $Q = \beta(\overline{co}(\overline{O}(\mu)))$ . Note, by weak\*-continuity of  $\beta$ , and compactness of  $\overline{co}(\overline{O}(\mu))$ , we have  $\beta(\overline{co}(\overline{O}(\mu))) = \overline{co}(\beta(\overline{O}(\mu)))$  and hence, by Proposition 4.3.17, we have  $X \subset \beta(\overline{O}(\mu))$ .

If  $x \in ex(Q) \subset X$ , then, by Proposition 4.3.17(2),  $\beta^{-1}(x) = \{\delta_x\}$ . Thus  $\delta_x \in \overline{O}(\mu)$ . It follows, by Lemma 4.3.14, that  $(G, Q)$  is strongly proximal.  $\square$

### 4.3.2 Fixed Point Theorems

We shall use the second of these theorems in the next chapter when we prove some special cases of our main result. For a proof of the first theorem, see [Gla1]. There are many different versions of the proof of the second theorem in the literature. The one we give here is found in [NAs]. For other proofs see [Nam] and [Gla1].

**Theorem 4.3.19** (*Hahn, for example Theorem III.5.1 [Gla1]*) *Let  $(G, Q)$  be an affine flow. If the action of  $G$  on  $Q$  is distal, then  $G$  has a fixed point in  $Q$ .*

**Theorem 4.3.20** (*Ryll Nardzewski, for example, [NAs]*) *Let  $Q$  be a nonempty, weakly compact, convex subset of a Banach space  $E$ , and let  $S$  be a semigroup of weakly continuous affine maps of  $Q$  into itself which is distal in the norm topology on  $E$ . Then there is a common fixed point of  $S$  in  $Q$ .*

**Proof.**

We claim that every finite subset of  $S$  has a common fixed point. Then we can use the finite intersection property to prove our result as follows. We denote by  $\mathcal{F}$  the set of all nonempty finite subsets of  $S$ , and for each  $F \in \mathcal{F}$  let  $Q_F = \{x \in Q | T(x) = x \text{ for all } T \text{ in } F\}$ . Then, since each  $T$  in  $S$  is weakly continuous and affine,  $Q_F$  is a convex, weakly compact set.

The collection  $\{Q_F | F \in \mathcal{F}\}$  satisfies the finite intersection property, and the result follows by compactness of  $Q$ .

It remains to establish our claim. Let  $F = \{T_1, \dots, T_n\} \subseteq \mathcal{S}$ . Put  $T_0 = (T_1 + \dots + T_n)/n$ . Then  $T_0$  is a weakly continuous affine map  $T_0 : Q \rightarrow Q$ . By the Markov-Kakutani fixed point theorem (for example, [Con, Theorem V.10.1]), there exists  $x_0 \in Q$  such that  $T_0(x_0) = x_0$ . We show that  $T_k(x_0) = x_0$  for  $k = 1, \dots, n$ .

Suppose that  $T_k(x_0) \neq x_0$  for some  $k$ . Then, renumbering, we may assume that there is an integer  $m$  such that  $T_k(x_0) \neq x_0$  for  $1 \leq k \leq m$  and  $T_k(x_0) = x_0$  for  $m < k \leq n$ . Let  $T'_0 = (T_1 + \dots + T_m)/m$ . Then

$$\begin{aligned} x_0 &= T_0(x_0) \\ &= \frac{1}{n}[T_1(x_0) + \dots + T_m(x_0)] + \frac{n-m}{n}x_0. \end{aligned}$$

Hence,

$$\begin{aligned} T'_0(x_0) &= \frac{1}{m}[T_1(x_0) + \dots + T_m(x_0)] \\ &= \frac{n}{m} \frac{1}{n}[T_1(x_0) + \dots + T_m(x_0)] \\ &= \frac{n}{m}[x_0 - \frac{n-m}{n}x_0] \\ &= x_0. \end{aligned}$$

Thus we may assume that  $T_k(x_0) \neq x_0$  for all  $k \neq 0$  but that  $T_0(x_0) = x_0$ .

Since  $\mathcal{S}$  is distal, there exists  $\epsilon > 0$  such that for every  $t \in \mathcal{S}$  and  $1 \leq k \leq n$ ,

$$\|T(T_k(x_0)) - T(x_0)\| > \epsilon.$$

Let  $\mathcal{S}_1$  be the semigroup generated by  $\{T_1, \dots, T_n\}$ . So  $\mathcal{S}_1 \subset \mathcal{S}$  and  $\mathcal{S}_1 = \{T_{l_1} \cdots T_{l_m} | m \geq 1, 1 \leq l_j \leq n\}$ . Then  $\mathcal{S}_1$  is a countable subsemigroup of  $\mathcal{S}$ .

Put  $\mathcal{K} = \overline{\text{co}}\{T(x_0) | T \in \mathcal{S}_1\}$ . Then  $K$  is a weakly compact convex subset of  $Q$  which is separable. By [Con, Lemma V.10.2], there is a closed convex subset  $C$  of  $K$  such that  $C \neq K$  and  $\text{diam}(K \setminus C) \leq \epsilon$ . Since  $C \neq K$  there exists  $S \in \mathcal{S}_1$  such that  $S(x_0) \in K \setminus C$ . Hence

$$S(x_0) = ST_0(x_0) = \frac{1}{n}[ST_1(x_0) + \cdots + ST_n(x_0)] \in K \setminus C.$$

Since  $C$  is convex, there exists  $1 \leq k \leq n$  such that  $ST_k(x_0) \in K \setminus C$ . But this implies that  $\|S(T_k(x_0)) - S(x_0)\| \leq \text{diam}(K \setminus C) \leq \epsilon$ , contradicting the norm distality of  $\mathcal{S}$ . Our claim follows.  $\square$

## 4.4 Groupoids

In this section we give the definition of a groupoid, together with some examples of groupoids. For more details of the theory of groupoids see [Ren].

**Definition 4.4.1** *A groupoid is a set  $\mathcal{G}$  together with a subset  $\mathcal{G}^2$  of  $\mathcal{G} \times \mathcal{G}$  called the set of composable pairs and a product map  $(g, h) \mapsto gh : \mathcal{G}^2 \rightarrow \mathcal{G}$  and an inverse map  $g \mapsto g^{-1} : \mathcal{G} \rightarrow \mathcal{G}$  which satisfy the following axioms:*

(1) *If  $(g, h)$  and  $(h, j)$  are in  $\mathcal{G}^2$ , then  $(gh, j)$  and  $(g, hj)$  are also in  $\mathcal{G}^2$ . In addition,*

$$(gh)j = g(hj).$$

(2) *For every  $g \in \mathcal{G}$ , we always have  $(g, g^{-1})$  and  $(g^{-1}, g)$  in  $\mathcal{G}^2$ . Moreover, whenever  $(g, h)$  and  $(j, g)$  are in  $\mathcal{G}^2$ , we have*

$$g^{-1}(gh) = h \text{ and } (jg)g^{-1} = j.$$

**Examples 4.4.2** (1) Every group is a groupoid.

(2) All disjoint unions of groups are groupoids. Elements of the union are composable if and only if they were originally elements of the same group. The groupoid product is then the original group product.

(3) Consider the set of homotopy classes of paths mapping the interval  $[0, 1]$  into a topological space  $X$ . If  $[f]$  is the path homotopy class of  $f$ , then this set becomes a groupoid under the product operation defined by

$$[f] \cdot [g] = [f \cdot g]$$

where if  $f$  and  $g$  are two paths satisfying  $g(0) = f(1)$ , we define

$$\begin{cases} f(2s) & \text{for } s \in [0, \frac{1}{2}] \\ g(2s - 1) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

and the inverse operation  $[f] \mapsto [f^{-1}]$  where  $f^{-1}(s) = f(s - 1)$  for  $s \in [0, 1]$ .

(4) If  $G$  is a group of transformations of the set  $S$ , then the set  $\mathcal{G} = S \times G$  has the following natural groupoid structure:

$$\begin{aligned} \mathcal{G}^2 &= \{((s, g), (t, h)) \mid t = sg\} \\ (s, g)(sg, h) &= (s, gh) \text{ and } (s, g)^{-1} = (sg, g^{-1}). \end{aligned}$$

**Definition 4.4.3** Let  $\mathcal{G}$  and  $\mathcal{H}$  be groupoids. A function  $f : \mathcal{G} \rightarrow \mathcal{H}$  is a groupoid homomorphism if the following conditions are satisfied:

(i) If  $(g, h) \in \mathcal{G}^2$ , then  $(f(g), f(h)) \in \mathcal{H}^2$  and then

$$(ii) f(gh) = f(g)f(h).$$

We shall see some examples of groupoid homomorphisms in the next chapter.

# Chapter 5

## Cocycles

### 5.1 Definitions and Examples

**Definition 5.1.1** *Let  $G$  be a second countable locally compact group and let  $H$  be a topological group. Let  $(S, \mu)$  be a  $G$ -space. A cocycle*

$$\alpha : S \times G \rightarrow H$$

*is a Borel function which satisfies the following equality for all  $g$  and  $h$  in  $G$ :  $\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$  for almost all  $s \in S$ . We shall refer to the above equality as the cocycle identity. If a cocycle satisfies this identity for all  $s \in S$ , then it is said to be a strict cocycle.*

**Remark** Thus a cocycle is a groupoid homomorphism from the groupoid  $S \times G$  to the group  $H$ .

**Remarks** 1) (Compare [Zim1, p 67]) If  $G$  is a countable discrete group, and  $H$  is a Borel group, then for every cocycle  $\alpha : S \times G \rightarrow H$  there is

a strict cocycle  $\alpha' : S \times G \rightarrow H$  such that  $\alpha = \alpha'$  almost everywhere.

**Proof.**

We follow the proof found in [Zim1]. For all  $g$  and  $h$  in  $G$  there exists a conull subset  $S_{g,h} \subset S$  such that  $\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$  for  $s \in S_{g,h}$ . Let  $S_0 = \bigcap_{g,h \in G} S_{g,h}$ . Then  $S_0$  is a conull subset of  $S$  and for all  $g$  and  $h$  in  $G$  we have  $\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$  for all  $s \in S_0$ .

Define a function

$$\alpha' : S \times G \rightarrow H$$

by

$$\alpha'(s, g) = \begin{cases} \alpha(s, g) & \text{for } s \in S_0 \\ e_H & \text{otherwise} \end{cases}$$

Then  $\alpha'$  is a strict cocycle which is equal to  $\alpha$  almost everywhere.

□

- 2) Mackey proved (e.g. Proposition 4.2.15 [Zim1]) that if  $\alpha$  is a cocycle from a transitive  $G$ -space into a second countable group, then there is a strict cocycle  $\alpha'$  such that for each  $g \in G$ ,  $\alpha(s, g) = \alpha'(s, g)$  for almost every  $s \in S$ .
- 3) (e.g. Theorem B.9 [Zim1]): If  $S$  is a standard Borel  $G$ -space with quasi-invariant measure  $\mu$  and  $H$  is a second countable topological group, then for each cocycle  $\alpha : S \times G \rightarrow H$  there exists a strict cocycle  $\beta : S \times G \rightarrow H$  such that for all  $g \in G$ ,  $\beta(s, g) = \alpha(s, g)$  for almost all  $s \in S$ .

**Examples 5.1.2** 1) (Compare [Zim1, Example 4.2.6]) Let  $\pi : G \rightarrow H$  be a group homomorphism, and let  $S$  be a  $G$ -space. Define a map

$$\alpha_\pi : S \times G \rightarrow H$$

by

$$\alpha_\pi(s, g) = \pi(g).$$

Then, for all  $g$  and  $h$  in  $G$ , we have

$$\begin{aligned} \alpha_\pi(s, gh) &= \pi(gh) \\ &= \pi(g)\pi(h) \\ &= \alpha_\pi(s, g)\alpha_\pi(sg, h) \end{aligned}$$

for every  $s \in S$ , and hence  $\alpha_\pi$  is a strict cocycle. The cocycles of this form are precisely those which are independent of  $s$ .

2) (Compare [Zim1, Example 4.2.4]) Let  $(S, \mu)$  be a  $G$ -space with quasi-invariant measure  $\mu$ . Then, for all  $g \in G$ , the measures  $\mu$  and  $\mu \circ g$  have the same null sets, and hence, for all  $g \in G$ , we can form the Radon-Nikodym derivative of  $\mu \circ g$  with respect to  $\mu$ . Let

$$r_\mu : S \times G \rightarrow \mathbb{R}^+$$

be defined by the Radon-Nikodym derivative

$$r_\mu(s, g) = \frac{d\mu(sg)}{d\mu(s)} = \frac{d\mu \circ g}{d\mu}(s).$$

Then for every  $g \in G$  and every  $s \in S$ ,

$$r_\mu(s, g)r_\mu(sg, h) = \frac{d\mu(sg)}{d\mu(s)} \frac{d\mu(sgh)}{d\mu(sg)} = \frac{d\mu(sgh)}{d\mu(s)}$$

$$= r_\mu(s, gh).$$

Thus  $r_\mu$  is a cocycle, called the *Radon-Nikodym cocycle* of the  $G$ -space  $(S, \mu)$ .

We now derive some elementary consequences of the cocycle identity.

- 1) Let  $S$  be a  $G$ -space, let  $\alpha : S \times G \rightarrow H$  be a cocycle and let  $e_G$  and  $e_H$  denote the identity elements of  $G$  and  $H$  respectively. Then for almost every  $s \in S$  we have:

$$\alpha(s, e_G) = \alpha(s, e_G^2) = \alpha(s, e_G)\alpha(s, e_G)$$

and it follows that  $\alpha(s, e_G) = e_H$  for almost every  $s \in S$ .

- 2) Let  $g \in G$ . Then for almost every  $s \in S$  we have

$$\alpha(s, g)\alpha(sg, g^{-1}) = \alpha(s, gg^{-1}) = e_H$$

for almost every  $s \in S$ . That is, for all  $g \in G$  and almost every  $s \in S$ ,  $\alpha(s, g)^{-1} = \alpha(sg, g^{-1})$ .

## 5.2 Equivalence of Cocycles

**Definition 5.2.1** Let  $\alpha : S \times G \rightarrow H$  and  $\beta : S \times G \rightarrow H$  be cocycles. We say that  $\alpha$  and  $\beta$  are equivalent or cohomologous, denoted by  $\alpha \sim \beta$ , if there is a Borel function  $\phi : S \rightarrow H$  such that for each  $g \in G$

$$\beta(s, g) = \phi(s)\alpha(s, g)\phi(sg)^{-1}$$

for almost all  $s \in S$ .

If  $\alpha$  and  $\beta$  are strict cocycles and the above equation holds for all  $s \in S$ , then  $\alpha$  and  $\beta$  are said to be strictly equivalent.

**Definition 5.2.2** Let  $E$  be a Banach space. A cocycle  $\alpha : S \times G \rightarrow \text{End}(E)$  is said to be uniformly bounded if there exists a constant  $C > 0$  such that  $\|\alpha(s, g)\| \leq C$  for all  $g \in G$  and almost every  $s \in S$ .

### Examples 5.2.3

- 1) Suppose  $\pi : G \rightarrow H$  and  $\rho : G \rightarrow H$  are conjugate homomorphisms. That is, there exists  $h \in H$  such that for all  $g \in G$

$$h\pi(g)h^{-1} = \rho(g).$$

Then the map

$$\phi : S \rightarrow H$$

defined by

$$\phi(s) = h$$

for all  $s \in S$  is a Borel map implementing the strict equivalence of  $\alpha_\pi$  and  $\alpha_\rho$ .

- 2) (Compare example 4.2.4 [Zim1]) Suppose  $\mu$  and  $\nu$  are equivalent quasi-invariant measures on a  $G$ -space  $S$ . Then the Radon-Nikodym derivative  $\phi : S \rightarrow \mathbb{R}^+$  is a positive function such that  $d\mu = \phi d\nu$ . It follows that

$$r_\mu(s, g) = \frac{\phi(sg)d\nu(sg)}{\phi(s)d\nu(s)}.$$

That is,

$$\begin{aligned}\phi(s)r_\mu(s, g)\phi(sg)^{-1} &= \frac{d\nu(sg)}{d\nu(s)} \\ &= r_\nu(s, g)\end{aligned}$$

and hence  $r_\mu \sim r_\nu$ .

In fact, every cocycle  $r : S \times G \rightarrow \mathbb{R}^+$  such that  $r \sim r_\mu$  for some measure  $\mu$ , is the Radon Nikodym cocycle of a measure equivalent to  $\mu$ . Suppose that  $\phi : S \rightarrow \mathbb{R}^+$  is a Borel function implementing the equivalence of  $r$  and  $r_\mu$ . Then  $r = r_{\phi\mu}$ .

- 3) Suppose that  $(S, \mu)$  is a free, transitive  $G$ -space, and let  $H$  be a second countable group. Then every cocycle  $\alpha : S \times G \rightarrow H$  is cohomologous to the trivial cocycle

$$1 : S \times G \rightarrow H$$

defined by

$$1(s, g) = e_H$$

for every  $s \in S$  and  $g \in G$ .

**Proof.**

Since the action of  $G$  on  $S$  is transitive and free, it is equivalent to an action of  $G$  on itself. Thus we need only consider a cocycle  $\alpha : G \times G \rightarrow H$ . Define a map

$$\phi : G \rightarrow H$$

by

$$\phi(s) = \alpha(e_G, s) \text{ for every } s \in G.$$

Then,

$$\begin{aligned}
\phi(s)\alpha(s, g)\phi(sg)^{-1} &= \alpha(e, s)\alpha(s, g)\alpha(e, sg)^{-1} \\
&= \alpha(e, s)\alpha(s, g)\alpha(sg, (sg)^{-1}) \\
&= \alpha(e, s)\alpha(s, g(sg)^{-1}) \\
&= \alpha(e, s)\alpha(s, s^{-1}) \\
&= \alpha(e, ss^{-1}) = e_H
\end{aligned}$$

for every  $g \in G$  and almost every  $s \in G$ .  $\square$

**Remark** The result of the previous example does not hold, in general, for essentially free actions. Let  $0 < \lambda \leq 1$  and consider the probability measure  $\mu_\lambda$  defined on  $\{0, 1\}$  by

$$\begin{aligned}
\mu_\lambda(0) &= \frac{1}{1+\lambda} \\
\mu_\lambda(1) &= \frac{\lambda}{1+\lambda}.
\end{aligned}$$

Consider the Cantor set

$$X = \prod_{i=1}^{\infty} \{0, 1\}$$

with product measure

$$\nu_\lambda = \prod_{i=1}^{\infty} \mu_i$$

where  $\mu_i = \mu_\lambda$  for all  $i$ . The group  $X$  is compact by Tychonoff's Theorem. The Borel structure on  $X$  is that induced by the product topology. Consider the action of  $\mathbb{Z}$  on  $X$  induced by the odometer transformation. For a description see [Wei]. This is an essentially free action of  $\mathbb{Z}$ . It is also clearly not essentially transitive since  $\mathbb{Z}$  is countable and the measure  $\nu_\lambda$  is not atomic.

Let  $\tau_{\nu_\lambda} : X \times G \rightarrow \mathbb{R}_+^*$  denote the Radon-Nikodym cocycle. We claim that  $\tau_{\nu_\lambda} \not\sim 1$  for  $\lambda \neq 1$ . Now the only invariant measure on  $X$  is the product measure,  $\nu_1$ . For, suppose  $\theta'$  is an invariant measure on  $X$ . Then, the cylinder sets  $A = \{0\} \times \prod_{i=2}^\infty \{0, 1\}$  and  $B = \{1\} \times \prod_{i=2}^\infty \{0, 1\}$  must have equal measure. Since  $X = A \cup B$ , we must have  $\theta'(A) = \theta'(B) = \frac{1}{2}$ . Fixing the  $j$ th co-ordinate for each  $j \in \mathbb{N}$  we see that the cylinder sets  $\prod_{i=1}^{j-1} \{0, 1\} \times \{0\} \times \prod_{i=j+1}^\infty \{0, 1\}$  and  $\prod_{i=1}^{j-1} \{0, 1\} \times \{1\} \times \prod_{i=j+1}^\infty \{0, 1\}$  must also be of equal measure  $\frac{1}{2}$ . But these sets generate the topology, and hence the Borel structure on  $X$ . It follows that  $\theta' = \nu_1$ , establishing our claim. Further, since

$$\prod_{n \geq 1} \frac{1}{2} \left[ \left( \frac{2}{1+\lambda} \right)^{1/2} + \left( \frac{2\lambda}{1+\lambda} \right)^{1/2} \right] = 0$$

it follows, by Kakutani's criterion [HeS, Theorem 22.36], that for  $\lambda \neq 1$ , the measures  $\nu_\lambda$  and  $\nu_1$  are singular. Thus there is no invariant measure in the same measure class as  $\nu_\lambda$  for  $\lambda \neq 1$ , establishing our claim.

If  $G$  acts transitively on a standard Borel space  $S$ , the action is equivalent to an action of  $G$  on  $G/G_0$  for some closed subgroup  $G_0$  of  $G$ . (See Corollary 4.2.14 and the discussion following.) If  $\alpha : G/G_0 \times G \rightarrow H$  is a strict cocycle, then we can associate with it a continuous group homomorphism  $\sigma_\alpha : G_0 \rightarrow H$  defined by  $\sigma_\alpha(g) = \alpha([e], g)$  for  $g \in G_0$ . Here  $[e]$  denotes the coset of  $G/G_0$  with representative  $e$ , the identity element of  $G$ . There is a one-to-one correspondence between strict equivalence classes of strict cocycles and conjugacy classes of homomorphisms [Zim1, Proposition 4.2.13].

We have the following result for cocycles on transitive Borel  $G$ -spaces.

**Proposition 5.2.4** *Let  $G$  be a countable discrete group. Suppose that  $\alpha : G/G_0 \times G \rightarrow GL(\mathcal{M})$  is cocycle into a finite von Neumann algebra such that the associated homomorphism  $\sigma_\alpha$  is uniformly bounded. Then  $\alpha$  is cohomologous to a unitary cocycle.*

**Proof.**

We may assume that  $\alpha$  is a strict cocycle. (See the remark following Definition 5.1.1. ) Let  $M = \alpha(G/G_0 \times G)''$ . Since  $\alpha$  is uniformly bounded, there exists  $C > 0$  such that  $\|\alpha([s], g)\| \leq C$  for all  $[s] \in G/G_0$  and for all  $g \in G$ . It follows that

$$C^{-2}.1 \leq \alpha([s], g)^* \alpha([s], g) \leq C^2.1$$

for all  $[s] \in G/G_0$  and for all  $g \in G$ . Consider the group of linear mappings:

$$\mathcal{F} = \{T_g : g \in G_0\}$$

where for each  $g \in G$ ,

$$T_g : M \rightarrow M$$

is defined by

$$T_g x = \alpha([e], g)^* x \alpha([e], g)$$

for  $x \in M$ . Let  $K$  be the  $\sigma$ -weakly closed convex hull

$$K = \overline{\text{co}}\{\alpha([e], g)^* \alpha([e], g) : g \in G_0\}.$$

Then  $K$  is a  $\sigma$ -weakly compact, convex subset of  $M$  and

$$C^{-2}.1 \leq x \leq C^2.1$$

for all  $x \in K$ . Consider  $M$  as a locally convex space with the strong topology  $s(M, M_*)$  [Sak]. The  $\sigma$ -weak topology  $\sigma(M, M_*)$  is the corresponding weak topology and  $K$  is weakly compact. Further,  $K$  is invariant under  $\mathcal{F}$ .

We claim that the group  $\mathcal{F}$  of affine mappings is distal. That is, given  $u, v \in K$ ,  $u \neq v$  we have

$$0 \notin \overline{\{T_g u - T_g v : g \in G_0\}}^{s(M, M_*)}. \quad (5.1)$$

Since  $M$  is finite, there exists a normal trace  $\tau$  on  $M$  such that  $\tau((u - v)^2) > 0$ . Since  $\tau$  is a trace,

$$\tau((u - v)^2) = \tau(z T_g(u - v)) \quad (5.2)$$

where  $z = \alpha([e], g)^{-1}(u - v)(\alpha([e], g)^*)^{-1}$  and

$$\|z\| \leq \|\alpha([e], g)^{-1}\| \|u - v\| \|(\alpha([e], g)^*)^{-1}\| \leq 2C^4.$$

By the Cauchy-Schwarz Inequality applied to 5.2, we obtain

$$\tau(T_g(u - v)^* T_g(u - v))^{1/2} \geq \frac{\tau((u - v)^2)}{\tau(z^* z)^{1/2}} \geq \frac{\tau((u - v)^2)}{2C^4} > 0$$

for all  $g \in G_0$ . This proves 5.1.

Thus the Ryll-Nardzewski fixed point theorem applies and there exists an element  $b \in K$  such that

$$\alpha([e], g)^* b \alpha([e], g) = b$$

for all  $g \in G_0$ . Now  $b$  is positive and invertible in  $M$ . Put  $a = b^{1/2}$ .

Let  $\gamma : G/G_0 \rightarrow G$  be a Borel section of the canonical projection with  $\gamma([e]) = e$ . Then for  $[s] \in G/G_0$  and  $g \in G$ ,  $\gamma([s])g$  and  $\gamma([sg])$  are equal

when projected to  $G/G_0$ , and so we have  $\gamma([s])g\gamma([sg])^{-1} \in G_0$ . Let

$$\beta : G/G_0 \times G \rightarrow GL(M)$$

be defined by

$$\beta([s], g) = a\alpha([e], \gamma([s])g\gamma([sg])^{-1})a^{-1}.$$

Then  $\beta$  is a Borel cocycle. Further, for every  $g \in G$  and every  $[s] \in G/G_0$

$$\begin{aligned} \beta([s], g)^* \beta([s], g) &= a^{-1}\alpha([e], \gamma([s])g\gamma([sg])^{-1})^* a\alpha([e], \gamma([s])g\gamma([sg])^{-1})a^{-1} \\ &= a^{-1}aaa^{-1} = e_{GL(M)} \end{aligned}$$

and hence, since  $M$  is finite, we have  $\beta([s], g)\beta([s], g)^* = e_{GL(M)}$ . Thus  $\beta$  is unitary, and if we define

$$\phi : G/G_0 \rightarrow GL(M)$$

by

$$\phi([s]) = \alpha([e], \gamma([s]))^{-1}a^{-1}\beta([e], \gamma(s)),$$

then  $\phi$  is Borel and

$$\phi([s])\beta([s], g)\phi([s]g)^{-1} = \alpha([s], g)$$

for every  $[s] \in G/G_0$  and every  $g \in G$ , establishing our result.  $\square$

We prove the above result for more general actions of discrete groups in the later chapters of this thesis. The above proof is a generalisation of that given in [Rob] and [VZ] for the corresponding result for uniformly bounded group representations. We now obtain a corresponding result for essentially

transitive actions of continuous groups. Note that we now restrict ourselves to strict cocycles, since the group of invertible elements of a von Neumann algebra is not, in general, second countable.

**Proposition 5.2.5** *Let  $G$  be a second countable locally compact group. Let  $(S, \mu)$  be an essentially transitive standard Borel  $G$ -space. Suppose  $\alpha : S \times G \rightarrow GL(M)$  is a uniformly bounded strict cocycle into a finite von Neumann algebra. Then  $\alpha$  is cohomologous to a unitary cocycle.*

**Proof.**

Let  $\theta_i$  be the orbit of full measure. Then  $\mu(S \setminus \theta_i) = 0$ . Since  $G$  acts transitively on its orbits, there exists a closed subgroup  $H$  of  $G$  and an equivalence of actions

$$\phi : \theta_i \rightarrow G/H.$$

Define

$$\tilde{\phi} : S \rightarrow G/H$$

by

$$\tilde{\phi}(s) = \begin{cases} \phi(s) & \text{if } s \in \theta_i \\ [e] & \text{otherwise} \end{cases}.$$

Define a cocycle  $\alpha' : G/H \times G \rightarrow GL(M)$  by

$$\alpha'([s], g) = \alpha(\tilde{\phi}^{-1}([s]), g).$$

Then  $\alpha'$  is a strict cocycle. It follows, by the same argument as in the proof of Proposition 5.2.4, that  $\alpha'$  is equivalent to a unitary cocycle  $\beta' : G/H \times G \rightarrow GL(M)$ . Let  $\psi' : G/H \rightarrow GL(M)$  be a Borel map implementing the equivalence.

Define a cocycle  $\beta : S \times G \rightarrow GL(M)$  by  $\beta(s, g) = \beta'(\tilde{\phi}(s), g)$ . Then  $\beta$  is a unitary cocycle. Consider the map  $\psi : S \times G \rightarrow GL(M)$ , defined by  $\psi(s) = \psi'(\phi(s))$ . Then  $\psi$  is a Borel map implementing the equivalence of  $\alpha$  and  $\beta$ .  $\square$

**Corollary 5.2.6** *Let  $G$  be a second countable locally compact group. Let  $(S, \mu)$  be a standard Borel  $G$ -space such that  $\mu$  is supported by a countable number of  $G$ -orbits. Suppose  $\alpha : S \times G \rightarrow GL(M)$  is a uniformly bounded strict cocycle into a finite von Neumann algebra. Then  $\alpha$  is cohomologous to a unitary cocycle.*

**Proof.**

By a similar argument to that in the proof of the above proposition, we may assume that the action has a non-zero, countable number of orbits. Then the action of  $G$  on  $S$  is equivalent to an action of  $G$  on the space  $\bigcup G/G_i$  where the union taken is the disjoint union, and the  $G_i$ 's are closed subgroups of  $G$ . For each  $i$ , the cocycle  $\alpha_i$  obtained by restriction of  $\alpha$  to  $G/G_i$  is Borel, since each orbit is Borel. After rescaling  $\mu$  to be a probability measure on  $G/G_i$ , it follows, from the above proposition, that there is a unitary cocycle,  $\beta_i : G/G_i \times G \rightarrow GL(M)$  which is equivalent to  $\alpha_i$ . Let  $\phi_i : G/G_i \rightarrow GL(M)$  be a Borel map implementing this equivalence.

Define a map

$$\beta : \bigcup G/G_i \times G \rightarrow GL(M)$$

by  $\beta([s]_i, g) = \beta_i([s]_i, g)$ , where  $[s]_i$  denotes the coset in  $G/G_i$  with representative  $s$ . Then, since the orbits are disjoint Borel sets on which the cocycle

identity holds,  $\beta$  is a Borel cocycle. Further,  $\beta$  is unitary since each  $\beta_i$  is unitary. The map

$$\phi : \cup G/G_i \times G \rightarrow GL(M)$$

defined by

$$\phi([s]_i) = \phi_i([s]_i) \text{ for } [s]_i \in G/G_i$$

is a Borel map implementing the equivalence of  $\alpha$  and  $\beta$ .  $\square$

The proof of the following related result uses the Ryll-Nardzewski theorem directly and is also a generalisation of that given for uniformly bounded group representations in [Rob] and [VZ].

**Proposition 5.2.7** *Let  $M$  be a finite von Neumann algebra, let  $G$  be a countable discrete group, and let  $(S, \mu)$  be a standard Borel  $G$ -space such that the Radon-Nikodym derivative  $f(s) = \frac{d\mu(sg)}{d\mu(s)}$  is an  $L^2$ -function for all  $g \in G$ . Let  $\alpha : S \times G \rightarrow M$  be a uniformly bounded cocycle. Then  $\alpha$  is cohomologous to a unitary cocycle.*

**Proof.**

Since  $G$  is countable, we may assume that our cocycle  $\alpha$  is strict. Let  $T : S \times G \rightarrow \text{End}(M)$  be the strict cocycle defined by

$$T(s, g)x = \alpha(s, g)x\alpha(s, g)^* \text{ for } x \in M.$$

Let  $\tilde{M} = M \otimes L^\infty(S, \mu)$ . We will identify  $\tilde{M}$  with  $\int_S^\oplus M_s d\mu(s)$  where  $M_s \cong M$  for all  $s \in S$ . ([Tak, Corollary 8.3])

Let  $x$  be a measurable field of operators on  $M$ , and let  $\tilde{T} : G \rightarrow \text{End}(\tilde{M})$  be defined by

$$\tilde{T}(g)x(s) = T(s, g)x(sg) \text{ for all } s \in S.$$

Then the operators  $\{\tilde{T}(g) : g \in G\}$  form a group.

For all  $s \in S$ , let  $K_s$  denote the  $\sigma(M_s, M_s^*)$ -weak closure of  $\text{co}\{\alpha(s, h)\alpha(s, h)^* : h \in G\}$ . Then for all  $g \in G$  we have

$$T(s, g)K_{sg} = K_s \text{ for all } s \in S.$$

We claim that  $K = \{x \in L^\infty(S, \mu) \otimes M \mid x(s) \in K_s \text{ a.e.}\}$  is a  $\sigma(\tilde{M}, \tilde{M}_*)$ -compact convex subset of  $\tilde{M}$ . We first prove that  $\{K_s\}$  is a Borel field of compact, convex subsets. We may assume that our cocycle  $\alpha$  is uniformly bounded in norm by 1. We define a countable family of functions  $\phi_n : S \rightarrow E_1^*$  by

$$\phi_n(s) = \sum_{i=1}^{k_n} \lambda_i \alpha(s, g_i) \alpha(s, g_i)^*$$

where the  $\lambda_i \in \mathbb{Q}$  vary over all possible rational convex combinations and the  $g_i \in G$  vary over all possible finite subsets of group elements. These functions are Borel since they are finite linear combinations of Borel functions. Moreover, for all  $s \in S$  we have

$$K_s = \overline{\{\phi_n(s) \mid n = 1, 2, \dots\}}.$$

It follows, by Lemma 3.3.4, that  $\{K_s\}$  is a Borel field of compact, convex sets. Our claim follows by Proposition 3.3.5.

Further, for all  $g \in G$ ,  $\tilde{T}(g)K = K$ . We claim that the group  $\{\tilde{T}(g) \mid g \in G\}$  of operators is distal. It is enough to prove that given  $g \in G$ , and  $x, y$  in  $K$  such that  $x \neq y$  we have

$$0 \notin \{\tilde{T}(g)x - \tilde{T}(g)y\}^{-\sigma(\tilde{M}, \tilde{M}_*)}.$$

Let  $\tau$  be a normal tracial state on  $M$ . Then  $\tilde{\tau} = \mu \otimes \tau$  is a normal tracial state on  $\tilde{M}$ . Let  $z$  be defined by  $z(s) = \tau(s, g)(\alpha(s, g)^*)^{-1}(x-y)(sg)\alpha(s, g)^{-1}$ . Then  $z \in L^2(S, \mu) \otimes M$  and we have

$$\begin{aligned}
\tilde{\tau}(zT(g)(x-y)) &= \int_S \tau(z(s)T(g)(x-y)(s))d\mu(s) \\
&= \int_S \tau(\tau(s, g)(\alpha(s, g)^*)^{-1}(x-y)(sg) \\
&\quad \alpha(s, g)^{-1}(\alpha(s, g)(x-y)(sg)\alpha(s, g)^*))d\mu(s) \\
&= \int_S \tau((\alpha(s, g)^*)^{-1}(x-y)^2(sg)\alpha(s, g)^*)\frac{d\mu(sg)}{d\mu(s)} \\
&= \tau((x-y)^2(sg))d\mu(sg) \\
&= \tilde{\tau}((x-y)^2).
\end{aligned}$$

It follows that

$$\tilde{\tau}(T(g)(x-y)^*T(g)(x-y))^{1/2} \geq \frac{\tilde{\tau}((x-y)^2)}{\tilde{\tau}(z^*z)^{1/2}} \geq \frac{\tilde{\tau}((x-y)^2)}{2C^4} > 0,$$

establishing the norm-distality of our group of operators.

Thus the Ryll-Nardzewski fixed point theorem applies and there exists an element  $b \in K$  such that

$$T(g)b = b$$

for all  $g \in G_0$ . Now  $b$  is positive and invertible in  $M$ . Put  $a = b^{1/2}$ .

Let

$$\beta : S \times G \rightarrow GL(M)$$

be defined by

$$\beta(s, g) = a\alpha(s, g)a^{-1}.$$

Then  $\beta$  is a Borel cocycle. Further, for every  $g \in G$  and almost every  $s \in S$ ,

$$\begin{aligned}\beta(s, g)^* \beta(s, g) &= a^{-1} \alpha(s, g)^* a \alpha(s, g) a^{-1} = a^{-1} T(g) a a a^{-1} \\ &= a^{-1} a a a^{-1} = 1_M\end{aligned}$$

and hence, since  $M$  is finite, we have  $\beta(s, g) \beta(s, g)^* = 1_M$ . Thus  $\beta$  is unitary, and if we define

$$\phi : S \rightarrow GL(M)$$

by

$$\phi(s) = a^{-1},$$

then  $\phi$  is Borel and

$$\phi(s) \beta(s, g) \phi(sg)^{-1} = \alpha(s, g)$$

for every  $g \in G$  and almost every  $s \in S$ , establishing our result.  $\square$

## Chapter 6

# Uniformly Bounded Cocycles into Finite von Neumann Algebras

In this chapter we prove the first of our main results.

### 6.1 Our First Main Result

**Theorem 6.1.1** *Let  $G$  be a countable discrete group, and let  $(S, \mu)$  be a standard Borel  $G$ -space with quasi-invariant measure  $\mu$ . Let  $M$  be a finite von Neumann algebra with separable predual, and let  $GL(M)$  denote the group of invertible elements of  $M$ , endowed with the Borel structure induced by the strong topology. If  $\alpha : S \times G \rightarrow GL(M)$  is a uniformly bounded cocycle, then  $\alpha$  is cohomologous to a unitary cocycle.*

**Proof.**

Since  $G$  is discrete, we may assume, by the remark following Definition 5.1.1, that  $\alpha$  is a strict cocycle. Let  $T : S \times G \rightarrow \mathcal{B}(M)$  be the strict cocycle defined by

$$T(s, g)\xi = \alpha(s, g)\xi\alpha(s, g)^* \text{ for } \xi \in M.$$

Then  $T$  is a Borel map by Lemma 3.1.2.

Denote by  $\tilde{M} = M \otimes L^\infty(S, \mu)$  and let  $\mathcal{B}(\tilde{M})$  be defined as in chapter 3. Let  $\tilde{T} : G \rightarrow \mathcal{B}(\tilde{M})$  be defined by:

If  $x : s \mapsto x_s \in \tilde{M}$ , then

$$\tilde{T}(g)x(s) = T(s, g)x(sg).$$

Since  $\alpha$  is uniformly bounded, one checks easily that  $\tilde{T}(g) \in \mathcal{B}(\tilde{\mathcal{M}})$ .

Further for all  $g$  and  $h$  in  $G$  and almost every  $s \in S$ , we have

$$\begin{aligned} \tilde{T}(g)\tilde{T}(h)x(s) &= T(s, g)\tilde{T}(h)x(sg) \\ &= T(s, g)T(sg, h)x(sgh) \\ &= T(s, gh)x(sgh) \\ &= \tilde{T}(gh)x(s) \end{aligned}$$

and  $\tilde{T}(e) = id_{\tilde{M}}$ . Therefore  $\tilde{T}$  is  $\sigma(\tilde{M}, \tilde{M}_*)$ -continuous representation of  $G$  in  $\mathcal{B}(M)$ .

For all  $s \in S$ , let  $K_s$  denote the  $\sigma(M, M_*)$ -closure of

$$co\{\alpha(s, h)\alpha(s, h)^* | h \in G\}.$$

We claim that  $\{K_s\}$  is a Borel field of compact, convex sets, equivariant under the action of  $T$ .

Let  $C > 0$  be a constant such that for all  $g \in G$  and all  $s \in S$   $\|\alpha(s, g)\| \leq C$ . Then for all  $s \in S$  and for all  $\xi \in K_s$  we must have  $\|\xi\| \leq C^2$ . For each  $s \in S$ ,  $K_s$  is a weakly closed, bounded subset of  $M$ , and hence is weakly compact. Let  $\phi_i : S \rightarrow M$  be the functions  $s \mapsto \sum \lambda_i \alpha(s, g_i) \alpha(s, g_i)^*$  where  $\sum \lambda_i \alpha(s, g_i) \alpha(s, g_i)^*$  is a finite rational convex combination of generating elements of  $K_s$ . Then for every  $s \in S$  we have  $K_s = \overline{\{\phi_i(s) | i = 1, 2, \dots\}}$ . It follows, by Lemma 3.3.4, that  $s \mapsto K_s$  is a Borel field of weakly compact, convex sets. We show that this field is equivariant under  $T$ . Let  $g \in G$ . Then for every  $s \in S$  we have

$$\begin{aligned} T(s, g) \alpha(sg, h) \alpha(sg, h)^* &= \alpha(s, g) \alpha(sg, h) \alpha(sg, h)^* \alpha(s, g)^* \\ &= \alpha(s, gh) \alpha(s, gh)^* \\ &\in K_s. \end{aligned}$$

Further, for  $\alpha(s, h) \alpha(s, h)^* \in K_s$  we have

$$\begin{aligned} T(s, g) \alpha(sg, g^{-1}h) \alpha(sg, g^{-1}h)^* &= \alpha(s, g) \alpha(sg, g^{-1}h) \alpha(sg, g^{-1}h)^* \alpha(s, g)^* \\ &= \alpha(s, h) \alpha(s, h)^*. \end{aligned}$$

Hence, for all  $g \in G$ ,  $T(s, g)K_{sg} = K_s$  for every  $s \in S$ , and  $(T, \{K_s\})$  is an affine  $\mathcal{G}$ -flow.

Since  $\alpha$  is uniformly bounded, there exists a constant  $C > 0$  such that  $K_s \subset M_C$  for all  $s \in S$ . Then

$$K = \{ \lambda \in \tilde{M}; \lambda(s) \in K_s \text{ } \mu\text{-a.e.} \}$$

is a subset of  $\tilde{M}_C$ . Since  $\tilde{M}_C$  is  $\sigma(\tilde{M}, \tilde{M}_*)$ -compact, it follows, by an argument similar to that found in the proof of [Zim4, Proposition 2.2], that  $K$  is a  $\sigma(\tilde{M}, \tilde{M}_*)$ -compact, convex subset of  $\tilde{M}_C$ . Since  $\{K_s\}$  is  $T(G)$ -equivariant,  $(\tilde{T}, K)$  is an affine flow in the sense of Chapter 4.

Let  $x$  and  $y$  be distinct elements of  $K$ . Then there is a subset  $S_0 \subset S$  of positive measures such that  $x(s) \neq y(s)$  for every  $s \in S_0$ . Since  $M$  is a finite von Neumann algebra, there is a normal tracial state,  $\tau$ , on  $M$  such that

$$\tau((x(s) - y(s))^2) > 0, \forall s \in S_0.$$

Let  $\|\cdot\|_2$  be the corresponding tracial norm. Let  $g \in G$ ,  $z \in M$  be defined by  $z(s) = (\alpha(s, g)^*)^{-1}(x(sg) - y(sg))^* \alpha(s, g)^{-1}$ . Then  $\tau(z(s)(\tilde{T}(g)x(s) - \tilde{T}(g)y(s))) = \tau((x(sg) - y(sg))^2)$  and

$$\begin{aligned} \|z(s)\| &\leq \|(\alpha(s, g)^*)^{-1}\| \|x(sg) - y(sg)\| \|\alpha(s, g)^{-1}\| \\ &\leq 2C^4. \end{aligned}$$

Further, by the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \tau(\tilde{T}(g)(x - y)^*(s)\tilde{T}(g)(x - y)(s))^{\frac{1}{2}} &\geq \frac{\tau((x - y)^2(sg))}{\tau(z^*z)^{1/2}} \\ &\geq \frac{\tau((x - y)^2(sg))}{2C^4} > 0 \quad (6.1) \end{aligned}$$

for all  $s \in Sg^{-1}$ . Note that  $\mu(S_0g^{-1}) > 0$  since  $\mu$  is quasi-invariant and  $\mu(S_0) > 0$ .

Let  $(\tilde{T}, Q)$  be an irreducible subflow of  $(\tilde{T}, K)$ . Let  $X = \overline{\text{ex}}(Q)$  where the closure is taken in the  $\sigma(\tilde{M}, \tilde{M}_*)$ -topology. Then  $(\tilde{T}, X)$  is a minimal strongly proximal flow by Proposition 4.3.17.

Let  $\epsilon > 0$ . Let  $B = \{f \in \tilde{M}_C \mid \|f(s)\|_2 \leq \epsilon \text{ for } \mu\text{-a.e. } s \in S\}$ . This is a bounded convex set which is  $\|\cdot\|_2$ -norm closed, strong operator closed, and hence, by Proposition 3.3.3 and since the strong and weak topologies coincide on bounded convex sets,  $B$  is  $\sigma(\tilde{M}, \tilde{M}_*)$ -closed.

Let  $\{\phi_n\}$  be a  $\tilde{\tau}$ -dense sequence in  $\tilde{M}_C$ . Then  $\{\phi_n + B\}$  is a countable cover of  $X$ . By Baire's theorem, there exists a  $\sigma(\tilde{M}, \tilde{M}_*)$ -open neighbourhood  $W$  of some  $\phi_n$  such that  $W \cap X \subset (\phi_n + B) \cap X$ .

Since  $(\tilde{T}, X)$  is strongly proximal, and hence, proximal, we have that for each  $y, z$  in  $X$  there exists a sequence  $\{g_i\} \subset G$  such that for some  $i$

$$\tilde{T}(g_i)y, \tilde{T}(g_i)z \in X \cap W \subset \phi_n + B.$$

This implies that  $\tilde{T}(g_i)(y - z) \in B$ . That is,

$$\|\tilde{T}(g_i)(x(s) - y(s))\|_2 \leq \epsilon \text{ for } \mu\text{-a.e. } s \in S.$$

Since  $\epsilon$  is arbitrary, this implies that

$$\lim_{i \rightarrow \infty} \tilde{T}(g_i)(y - z) = 0$$

in the  $\tilde{\tau}$ -topology. But this implies that

$$\sup_s \|\tilde{T}(g_i)(y - z)(s)\|_2 \rightarrow 0,$$

contradicting 6.1 unless  $X$ , and hence also  $Q$ , is trivial. Thus, there exists a Borel function  $\phi : S \rightarrow M$  such that for all  $g \in G$

$$T(s, g)\phi(sg) = \phi(s) \text{ } \mu\text{-a.e.}$$

and  $\phi(s) \in K_s$  for  $\mu$ -a.e.  $s \in S$ .

Define  $\beta : S \times G \rightarrow M$  by

$$\beta(s, g) = \phi(s)^{-1/2} \alpha(s, g) \phi(sg)^{1/2}.$$

Then  $\beta$  is a cocycle, and

$$\begin{aligned} \beta(s, g) \beta(s, g)^* &= \phi(s)^{-1/2} \alpha(s, g) \phi(sg) \alpha(s, g)^* \phi(s)^{-1/2} \\ &= \phi(s)^{-1/2} \phi(s) \phi(s)^{-1/2} \\ &= 1 \end{aligned}$$

for almost every  $s \in S$ . It follows, by finiteness of  $M$ , that  $\beta(s, g)^* \beta(s, g) = 1$ .

Thus  $\beta$  is a unitary cocycle which is equivalent to  $\alpha$ , establishing our result.

□

# Chapter 7

## $\mathcal{G}$ -Flows

### 7.1 $\mathcal{G}$ -Flows on Borel Fields of Compact Sets

**Definition 7.1.1** *Let  $(S, \mu)$  be a standard Borel space, and let  $G$  be a countable discrete group. If  $(S, \mu)$  is a  $G$ -space, let  $\mathcal{G}$  denote the groupoid  $S \times G$ . Let  $Y$  be a compact Hausdorff space. Let  $\text{Homeo}(Y)$  be the group of homeomorphisms of  $Y$  with the topology of uniform convergence and the corresponding Borel structure. Let  $\alpha : S \times G \rightarrow \text{Homeo}(Y)$  be a Borel cocycle. If  $s \mapsto Y_s$  is a Borel field of compact subsets of  $Y$  which satisfies*

$$\alpha(s, g)Y_{sg} = Y_s \text{ for all } g \in G \text{ and } \mu\text{-a.e. } s \in S$$

*then  $\{Y_s\}_{s \in S}$  is said to be an equivariant field of compact sets for  $\alpha$ . Let  $\{Y_s\}_{s \in S}$  be a Borel field of compact subsets of  $Y$  which is equivariant for  $\alpha$ . Then we call the pair  $(\alpha, \{Y_s\})$  a  $\mathcal{G}$ -flow.*

*Let  $E$  be a separable Banach space, let  $G$  be a countable discrete group and let  $(S, \mu)$  be a standard Borel  $G$ -space. Let  $\{X_s\}_{s \in S}$  be a Borel field of*

$\sigma(E^*, E)$ -compact subsets of  $E_1^*$ . Let  $\alpha : S \times G \rightarrow Iso(E)$  be a Borel cocycle such that  $\{X_s\}$  is equivariant for  $\alpha^*$ . Then we call the pair  $(\alpha^*, \{X_s\})$  a Banach  $G$ -flow.

Let  $(\alpha, \{X_s\})$  be a  $G$ -flow. If  $(\alpha, \{Q_s\})$  is a  $G$ -flow such that for  $\mu$ -a.e.  $s \in S$ ,  $Q_s \subset X_s$ , then  $(\alpha, \{Q_s\})$  is called a sub  $G$ -flow of  $(\alpha, \{X_s\})$ . We say that a  $G$ -flow  $(\alpha, \{X_s\})$  is minimal if it has no proper sub  $G$ -flows.

**Remark** If  $S$  is a one point space, then  $\alpha$  is a continuous group homomorphism since  $G$  is discrete and the pair  $(\{\alpha(s, g)\}, \{Y_s\})$  is a flow in the sense of chapter 4.

**Example 7.1.2** Let  $Y$  be a compact Hausdorff space, and let  $s \mapsto Y_s$  be a Borel field of compact subsets of  $Y$ . Then  $C(Y)$  is a separable Banach space. We show that  $s \mapsto M(Y_s)$  is a Borel field of compact subsets of  $M(Y) \subset C(Y)_1^*$ . First note that for all  $s \in S$  we can consider  $M(Y_s)$  as a subset of  $M(Y)$  by considering the measures in  $M(Y_s)$  to be the restrictions of measures in  $M(Y)$  having supports contained in  $Y_s$ . Further, we claim that  $M(Y_s)$  is a closed, and hence compact, subset of  $M(Y)$  for all  $s \in S$ .

Let  $\mu$  be a cluster point of  $M(Y_s)$ . Let  $\{\mu_n\}_{n \in \mathbb{N}}$  be a sequence in  $M(Y_s)$  such that  $\mu_n \rightarrow \mu$  in the vague topology on  $M(Y)$ . By the definition of the vague topology,  $\mu$  is a probability measure, and hence is regular, by Theorem 3.1.4. Suppose  $\mu \notin M(Y_s)$ . Denote by  $\text{supp } \mu$  the support of  $\mu$  in  $Y$ . Let  $K \subset \text{supp } \mu \setminus Y_s$  be a compact set such that  $\mu(K) > 0$ . Then  $K \cap Y_s = \emptyset$  and both  $K$  and  $Y_s$  are closed in  $Y$ . Thus Urysohn's Lemma applies and there exists a continuous function  $f : Y \rightarrow [0, 1]$  such that  $f(Y_s) = 0$  and

$f(K) = 1$ . But then for all  $n$  we have

$$\int_Y f d\mu_n = 0$$

and

$$\int_Y f d\mu \geq \int_K 1 d\mu = \mu(K) > 0,$$

contradicting  $\mu_n \rightarrow \mu$ . Our claim follows.

Let  $\{\phi_i\}$  be a sequence of Borel selectors for  $\{Y_s\}_{s \in S}$ . Consider the sequence of functions  $\psi_k$  defined by all possible convex combinations

$$s \mapsto \sum_{j=1}^{n_k} \lambda_j \delta_{\phi_i(s)}$$

where  $\lambda_j \in \mathbb{Q}$ . These functions are Borel and satisfy

$$Q_s = \overline{\{\psi_j(s) | j = 1, 2, \dots\}}$$

for  $\mu$ -a.e.  $s \in S$ . Thus, by lemma 3.3.4,  $\{M(Y_s)\}_{s \in S}$  is a Borel field of compact convex subsets of  $M(Y)$ .

Let

$$\alpha : S \times G \rightarrow \text{Homeo}(Y_1)$$

be a Borel cocycle with respect to which  $(\alpha, \{Y_s\})$  is a  $\mathcal{G}$ -flow. Then the cocycle

$$\alpha : S \times G \rightarrow \text{Iso}(C(Y))$$

defined by

$$\alpha(s, g)f(y) = f(\alpha(s, g)^{-1}y)$$

satisfies

$$\alpha^*(s, g)M(Y_{sg}) = M(Y_s) \text{ for all } g \in G \text{ and } \mu\text{-a.e. } s \in S.$$

Thus the pair  $(\alpha^*, \{M(Y_s)\})$  is a Banach  $\mathcal{G}$ -flow.

**Remark** (Compare [Zim4, p 358]) Let  $(\alpha^*, \{A_s\})$  be a Banach  $\mathcal{G}$ -flow. Then

$$A = \{\lambda \in L^\infty(S, E^*); \lambda(s) \in A_s \text{ } \mu\text{-a.e.}\}$$

is a  $\sigma(L^\infty(S, E^*), L^1(S, E))$ -compact, convex subset of  $L^\infty(S, E^*)$  by proposition 3.3.3.

Further, the mapping,

$$T : G \rightarrow Iso(L^1(S, E))$$

defined by

$$T(g)f(s) = r(s, g)\alpha(s, g)f(sg)$$

is a continuous representation of  $G$ .

Also, we have an induced adjoint action

$$T^* : G \rightarrow Homeo(L_1^\infty(S, E^*))$$

given by

$$T^*(g) = (T(g^{-1}))^*.$$

This map is defined by

$$(T^*(g)\lambda)(s) = \alpha^*(s, g)\lambda(sg).$$

Since  $\{A_s\}$  is  $\alpha^*$ -equivariant, it follows that the set  $A$  defined above is  $T^*$ -invariant, and hence, that  $(T^*, A)$  is an affine flow in the sense of chapter 4.

## 7.2 Proximal and Distal $\mathcal{G}$ -Flows

**Definition 7.2.1** Let  $\tau$  be a locally convex topology on  $Y$ . A  $\mathcal{G}$ -flow  $(\alpha, \{X_s\})$  is said to be  $\tau$ -proximal if for all Borel fields of vectors  $x : s \mapsto x_s$  and  $y : s \mapsto y_s$  in  $\{X_s\}$  there exists a sequence  $\{g_i\} \subset G$  and a Borel field of vectors  $z : s \mapsto z_s$  in  $\{X_s\}$  such that

$$\lim_{\dagger} \alpha(s, g_i)x(sg_i) = z(s) = \lim_{\dagger} \alpha(s, g_i)y(sg_i) \text{ } \mu\text{-a.e.}$$

where convergence is with respect to the topology  $\tau$ .

A  $\mathcal{G}$ -flow  $(\alpha, \{X_s\})$  is  $\tau$ -distal if for all Borel fields of vectors  $x : s \mapsto x_s$  and  $y : s \mapsto y_s$  in  $\{X_s\}$  we have either  $x_s = y_s$  for almost every  $s \in S$  or, for all sequences  $g = \{g_i\} \subset G$  there exists a Borel subset  $S_g \subset S$  of positive measure such that for all  $s \in S_g$ ,

$$\lim_{\dagger} \alpha(s, g_i)x(sg_i) \neq \lim_{\dagger} \alpha(s, g_i)y(sg_i).$$

**Remark** We work with sequences again here since the groups we shall consider in this chapter are all countable discrete groups. The corresponding definitions for continuous groups are the same, except that in this case we need to work with nets rather than sequences.

**Remark** For a given topology,  $\tau$  on  $E^n$ , a  $\mathcal{G}$ -flow  $(\alpha, \{X_s\})$  which is both  $\tau$ -proximal and  $\tau$ -distal must clearly be trivial. That is, if we have  $x : s \mapsto x_s$  and  $y : s \mapsto y_s$  in  $s \mapsto X_s$  then, we must have  $x_s = y_s$  for almost every  $s \in S$ .

### 7.3 Affine $\mathcal{G}$ -flows

**Definition 7.3.1** An affine  $\mathcal{G}$ -flow is a  $\mathcal{G}$ -flow  $(\alpha, \{K_s\})$  satisfying the additional condition that for each  $s \in S$ ,  $K_s$  is a compact, convex set.

An affine Banach  $\mathcal{G}$ -flow is a Banach  $\mathcal{G}$ -flow  $(\alpha, \{K_s\})$  which is affine in the sense of the above definition.

An affine  $\mathcal{G}$ -flow is said to be irreducible if it has no proper affine sub  $\mathcal{G}$ -flows.

**Example 7.3.2** If  $(\alpha, \{Y_s\})$  is a Banach  $\mathcal{G}$ -flow, then  $(\alpha^*, \{M(Y_s)\})$  as described in Example 7.1.2 is an affine Banach  $\mathcal{G}$ -flow.

**Lemma 7.3.3** Every affine Banach  $\mathcal{G}$ -flow  $(\alpha^*, \{K_s\})$  has an irreducible sub  $\mathcal{G}$ -flow.

**Proof.**

If  $\{X_s\}$  is a Borel field of compact convex subsets of  $E^n$ , then, by Proposition 3.3.5, the set

$$X = \{\lambda \in L^\infty(S, E^n) \mid \lambda(s) \in X_s \text{ } \mu\text{-a.e.}\}$$

is a compact convex subset of  $L^\infty(S, E^n)$ . Let  $\mathcal{T}$  be the group of affine mappings defined from  $\alpha^*$  as in the remark following example 7.1.2. Then  $(\mathcal{T}, X)$  is an affine flow in the sense of chapter 4.

Let  $\mathcal{S}_+$  be the set of measurable simple functions

$$f = \sum_{i=1}^n q_i \chi_{A_i}$$

where the  $q_i$  are positive rational numbers which satisfy  $\sum_{i=1}^n q_i = 1$  and the  $A_i$  are measurable subsets of  $S$ . We will say that a compact, convex subset  $B \subset L_1^\infty(S, E^n)$  has property (\*) if for all  $f_1, \dots, f_n \in S_+$  which satisfy  $\sum_{i=1}^n f_i(s) = 1$  for all  $s \in S$  and for all  $a_1, \dots, a_n \in B$ , the map

$$s \mapsto \sum_{i=1}^n f_i(s) a_i(s)$$

is in  $B$ .

Since  $S$  is a standard Borel space, we may conclude, by [Zim4, Lemma 2.5], that  $A$  has property (\*).

Let  $(B_i)_{i \in I}$  be a chain, ordered by reverse inclusion, of compact, convex  $\mathcal{T}$ -invariant subsets of  $A$  which have property (\*). Then  $\bigcap_{i \in I} B_i$  is a compact, convex,  $\mathcal{T}$ -invariant subset of  $A$  which has property (\*) and which forms a lower bound for  $(B_i)_{i \in I}$ . It follows, by Zorn's Lemma, that there exists a minimal compact, convex subset  $Q$  of  $A$  which is  $\mathcal{T}$ -invariant and which has property (\*).

By proposition 3.3.6, we can construct a Borel field  $\{Q_s\}$  of compact, convex subsets of  $E^n$  such that

$$Q = \{\lambda \in L_1^\infty(S, E^n) \mid \lambda(s) \in Q_s \text{ } \mu\text{-a.e.}\}.$$

The Borel field  $\{Q_s\}$  is equivariant under  $\alpha^*$  since  $Q$  is  $\mathcal{T}$ -invariant. Irreducibility of  $(\alpha^*, \{Q_s\})$  follows from the irreducibility of  $(\mathcal{T}, Q)$ .

□

**Remark** The proof of the above Lemma shows that if  $E$  is a separable Banach space, and  $\mathcal{T}$  is a group of affine mappings which can be defined in

terms of the cocycle  $\alpha^* : S \times G \rightarrow \text{Homeo}(L_1^\infty(S, E^*))$ , then, given an affine flow  $(T, Q)$  in  $L^\infty(S, E^*)$  in the sense of Chapter 4 which has property (\*), we can construct an affine  $\mathcal{G}$ -flow  $(\alpha^*, \{Q_s\})$  such that

$$Q = \{\lambda \in L^\infty(S, E) \mid \lambda(s) \in Q_s \text{ } \mu\text{-a.e.}\}.$$

## 7.4 Strongly Proximal $\mathcal{G}$ -Flows

**Definition 7.4.1** *A  $\mathcal{G}$ -flow  $(\alpha, \{X_s\})$  is said to be strongly proximal if the  $\mathcal{G}$ -flow  $(\alpha^*, \{M(X_s)\})$  is proximal in the vague topology.*

**Remark** Let  $(\alpha, \{X_s\})$  be a  $\mathcal{G}$ -flow. We can identify  $(\alpha, \{X_s\})$  with a sub  $\mathcal{G}$ -flow of  $(\alpha^*, \{M(X_s)\})$  via the identification of  $s \mapsto x_s \in \{X_s\}$  with  $s \mapsto \delta_{x_s} \in \{M(X_s)\}$ .

Thus, if  $(\alpha, \{X_s\})$  is a strongly proximal  $\mathcal{G}$ -flow, it is also a proximal  $\mathcal{G}$ -flow.

We denote by  $M(X)$  the Borel field of measure spaces  $s \mapsto M(X_s)$ . The elements of this space are Borel fields of measures  $s \mapsto \mu_s$  where  $\mu_s \in M(X_s)$  for  $\mu$ -a.e.  $s \in S$ .

**Lemma 7.4.2** *A separable  $\mathcal{G}$ -flow  $(\alpha, \{X_s\})$  is strongly proximal if and only if for every Borel field of measures,  $\mu : s \mapsto \mu_s \in \{M(X_s)\}$  there exists a Borel field  $y : s \mapsto y_s$  of vectors in  $s \mapsto X_s$  and a sequence  $\{g_i\} \subset G$  such that  $\lim \alpha^*(s, g_i)\mu_{s g_i} = \delta_{y_s}$  in the vague topology for almost every  $s \in S$ .*

**Proof.**

Suppose  $(\alpha, \{X_s\})$  is strongly proximal. If  $s \mapsto \mu_s \in \{M(X_s)\}$  and  $s \mapsto x_s \in \{X_s\}$  there exists a sequence  $(g_i) \subset G$  such that

$$\begin{aligned} \liminf_i \alpha^*(s, g_i) \mu_{s g_i} &= \liminf_i \alpha^*(s, g_i) \delta_{x_{s g_i}} \\ &:= \liminf_i \delta_{\alpha^*(s, g_i) x_{s g_i}} \end{aligned}$$

for  $\mu$ -a.e.  $s \in S$ . The result follows with  $y_s = \liminf_i \alpha^*(s, g_i) x_{s g_i}$ .

Conversely, let  $\mu : s \mapsto \mu_s$  and  $\nu : s \mapsto \nu_s$  be two Borel fields of measures in  $\{M(X_s)\}$ . Let  $\theta = \frac{1}{2}(\mu + \nu)$ . Then  $\theta$  is a Borel field of measures in  $\{M(X_s)\}$ , and there exists  $z \in X$  and a sequence  $\{g_i\} \subset G$  such that

$$\alpha^*(s, g_i) \theta_{s g_i} = \delta_{z_s} \text{ for all } s \text{ in some conull subset } S_0 \subset S.$$

Suppose that for some  $s \in S_0$  we have  $\lim \alpha^*(s, g_i) \nu_{s g_i} = \nu_s^1$  and  $\lim \alpha^*(s, g_i) \mu_{s g_i} = \mu_s^1$ . Then  $\frac{1}{2}(\nu_s^1 + \mu_s^1) = \delta_{z_s}$  and it follows, since  $\delta_{z_s}$  is an extreme point of  $M(X_s)$ , that  $\nu_s^1 = \mu_s^1 = \delta_{z_s}$ . Strong proximality of  $(\alpha, \{X_s\})$  follows.  $\square$

# Chapter 8

## Amenability

### 8.1 Amenable Groups

**Definition 8.1.1** (Compare [Zim1, Definition 4.1.1].) *A locally compact group  $G$  is said to be amenable if, for every continuous  $G$ -action on a compact metrisable space  $X$ , there is a  $G$ -invariant probability measure on  $X$ , that is, a  $G$ -fixed point in  $M(X)$ .*

**Examples 8.1.2** (1) If  $G$  is abelian, then  $G$  is amenable by the Kakutani-Markov fixed point theorem. For a proof see [Zim1, Theorem 4.1.2].

(2) All compact groups are amenable. See [Zim1, Proposition 4.1.6] for a proof.

(3) If  $G$  is amenable and  $H \subset G$  is a closed subgroup, then  $H$  is amenable. For a proof see [Zim1, Proposition 4.1.6].

(4) (See, for example, [Zim1, Corollary 4.1.7]) Solvable groups are amenable.

(5) (Compare [Zim1, Example 4.1.10]) Let  $F_2$  be the free group on 2 generators. Then  $F_2$  is not amenable. For, let  $\phi : [0, 1] \rightarrow [0, 1]$  be defined by  $\phi(x) = x^2$ . Then for  $a < 1$ ,  $\phi([0, a]) = [0, a^2]$ , and  $a^n \rightarrow 0$ . It follows that all invariant measures are supported on  $\{0, 1\}$ . We can consider  $\phi$  to be a map on the circle. Let  $\psi$  denote a rotation of the circle by an angle which is not a rational multiple of  $2\pi$ . Then there are no probability measures on the circle which are invariant under both  $\phi$  and  $\psi$ . The maps  $\phi$  and  $\psi$  generate an  $F_2$  action. It follows that  $F_2$  is not amenable.

There are many equivalent definitions of amenability to be found in the literature. The following is due to Day, and is often referred to as Day's fixed point theorem.

**Proposition 8.1.3** *(For example, [Zim1, Proposition 4.1.4]) A locally compact group  $G$  is amenable if and only if there is a fixed point in every compact affine  $G$ -space.*

The following result for uniformly bounded representations of amenable groups is due to Dixmier. The proof which appears here was suggested by Guyan Robertson in [Rob].

**Theorem 8.1.4** *(For example, [Pau, Theorem 8.9]) Let  $G$  be an amenable group and let  $\rho : G \rightarrow \mathcal{B}(\mathcal{H})$  be a strongly continuous homomorphism with  $\rho(e) = 1$ , such that*

$$\|\rho\| := \sup\{\|\rho(g)\| \mid g \in G\} < \infty.$$

Then there exists an invertible  $S \in \mathcal{B}(\mathcal{H})$  with

$$\|S\| \cdot \|S^{-1}\| \leq \|\rho\|^2$$

such that  $S^{-1}\rho(g)S$  is a unitary representation of  $G$ .

**Proof.**

Since  $\rho$  is uniformly bounded, there exists a constant  $C$  such that  $\|\rho(g)\| \leq C$  for all  $g \in G$ . This implies that for all  $g \in G$

$$C^{-2}.1 \leq \rho(g)^* \rho(g) \leq C^2.1.$$

Let  $M = \rho(G)''$ . Define a group  $\{T_g : g \in G\}$  of linear mappings on  $M$  by

$$T_g x = \rho(g)^* x \rho(g) \text{ for } x \in M.$$

Let  $K$  be the  $\sigma$ -weakly closed convex hull

$$\overline{\text{co}}\{\rho(g)^* \rho(g) : g \in G\}.$$

Then  $K$  is a  $\sigma$ -weakly compact convex subset of  $M$ . Thus Day's fixed point theorem applies and there exists  $b \in K$  such that

$$T_g b = b \text{ for all } g \in G.$$

Now  $b$  is a positive, invertible element of  $M$ . Consider the operator  $b^{1/2}\rho(g)b^{-1/2}$ .

Then

$$\begin{aligned} (b^{1/2}\rho(g)b^{-1/2})^*(b^{1/2}\rho(g)b^{-1/2}) &= b^{-1/2}\rho(g)^*b\rho(g)b^{-1/2} \\ &= b^{-1/2}bb^{-1/2} \\ &= 1 \end{aligned}$$

and, similarly  $(b^{1/2}\rho(g)b^{-1/2})(b^{1/2}\rho(g)b^{-1/2})^* = 1$ . The result follows with  $S = b^{1/2}$ .  $\square$

**Definition 8.1.5** *Let  $G$  be a locally compact group. Let  $\lambda$  denote the left regular representation on  $G$ . A mean  $m$  on  $L^\infty(G)$  is a continuous linear functional on  $L^\infty(G)$  which satisfies  $m(1) = 1 = \|m\|$ . The mean  $m$  is said to be left invariant if  $m(\lambda(g)f) = m(f)$  for all  $g \in G$  and all  $f \in L^\infty(G)$ . A group  $G$  has the fixed point property if for every affine flow  $(G, Q)$  there exists a  $G$ -fixed point in  $Q$ .*

We generalise the following result for amenable actions in the next section.

**Theorem 8.1.6** *(For example, [Gla1, Theorem III.9.1]) Let  $G$  be a locally compact topological group. The following are equivalent.*

- (1)  $G$  has a left invariant mean on  $L^\infty(G)$ .
- (2)  $G$  is amenable.
- (3)  $G$  has the fixed point property.
- (4) Every minimal strongly proximal flow for  $G$  is trivial.

**Proof.**

See [Gla1] for a proof of the equivalence of (1) and (2). The equivalence of (2) and (3) is given by Day's fixed point theorem. Our proof of the remaining implications follows that found in [Gla1].

((3)  $\Rightarrow$  (4)) Let  $(G, X)$  be a minimal strongly proximal flow. Then  $M(X)$  contains a fixed point. It follows, by strong proximality, that this point must lie in  $X$ . Minimality of  $X$  implies that  $X$  must be trivial.

((4)  $\Rightarrow$  (2)) Let  $(G, X)$  be a flow. Consider the affine flow  $(G, M(X))$ . Let  $(G, Q)$  be an irreducible subflow. It follows, by Theorem 4.3.18, that  $(G, Q)$  is strongly proximal. Let  $X = \overline{ex(Q)}$ . Then, by Theorem 4.3.18,  $(G, X)$  is a minimal strongly proximal flow. By assumption,  $X$ , and hence  $Q$ , must be trivial. Thus the unique point of  $Q$  is an invariant measure for  $(G, X)$ .  $\square$

## 8.2 Amenable Actions

In this section we develop some equivalent conditions for amenability of an action of a locally compact group  $G$  on a standard ergodic Borel  $G$ -space,  $(S, \mu)$ . This notion was introduced in [Zim4] where the definition given was the following.

Suppose  $E$  is a separable Banach space, and that  $\alpha : S \times G \rightarrow Iso(E)$  is a cocycle. Let  $\mathcal{G}$  denote the groupoid  $S \times G$ . Let  $(\alpha^*, \{A_s\})$  be an affine Banach  $\mathcal{G}$ -flow in  $E_1^*$ .

**Definition 8.2.1** (compare [Zim4]) *The action of  $G$  on  $(S, \mu)$  is called amenable if every affine Banach  $\mathcal{G}$ -flow  $(\alpha^*, \{A_s\})$  has an  $\alpha$ -invariant section. That is, if there exists  $\phi \in L_1^\infty(S, E^*)$  such that  $\phi(s) \in A_s$  for  $\mu$ -a.e.  $s \in S$  and for all  $g \in G$ ,  $\alpha^*(s, g)\phi(sg) = \phi(s)$  for almost all  $s$ .*

**Examples 8.2.2** (1) ([Zim4, Theorem 2.1]) Let  $G$  be an amenable group and  $(S, \mu)$  an ergodic  $G$ -space. Then  $S$  is an amenable  $G$ -space.

(2) ([Zim4, Corollary 1.6]) The trivial  $G$ -space  $\{e\}$  is amenable if and only if  $G$  is an amenable group.

(3) ([Zim4, Theorem 1.9]) Let  $H \subset G$  be a closed subgroup. Then  $G/H$  is an amenable  $G$ -space if and only if  $H$  is amenable.

**Remark** Note, in particular, that for all groups  $G$ , the trivial subgroup  $\{e\}$  is a closed, amenable subgroup. Thus every group has an amenable action, namely the transitive action on itself.

**Proposition 8.2.3** *The  $G$ -space  $S$  is amenable if and only if every irreducible affine Banach  $G$ -flow is trivial.*

**Proof.**

Suppose that  $S$  is an amenable  $G$ -space. Let  $\alpha : S \times G \rightarrow Iso(E)$  be a cocycle, and consider an irreducible affine Banach  $G$ -flow  $(\alpha^*, \{A_s\})$ . It follows, by the amenability of  $S$ , that there exists an  $\alpha$ -invariant section  $\phi$ . But then,  $(\alpha^*, \{B_s\})$  where  $B_s = \{\phi(s)\}$  is an affine Banach  $G$ -flow such that  $B_s \subseteq A_s$  for almost every  $s \in S$ . It follows that  $B_s = A_s$  for almost every  $s \in S$ , and hence that  $(\alpha^*, \{A_s\})$  is trivial.

Conversely, let  $(\alpha^*, \{A_s\})$  be an affine Banach  $G$ -flow. Then, by lemma 7.3.3, there exists an irreducible sub affine  $G$ -flow  $(\alpha^*, \{B_s\})$  of  $(\alpha^*, \{A_s\})$ . It follows, by our assumption, that  $(\alpha^*, \{B_s\})$  is trivial. That is, there exists

a Borel function  $\phi \in L_1^\infty(S, E^*)$  such that  $\phi(s) = B_s \subset A_s$  for almost every  $s \in S$ , satisfying

$$\alpha^*(s, g)\phi(sg) = \phi(s) \text{ for almost every } s \in S.$$

Thus  $S$  is an amenable  $G$ -space.  $\square$

The initial definition of amenability of actions given in this section is a generalisation of the definition of amenability for groups given by Day's fixed point theorem. In [Zim5], Zimmer generalises the invariant mean definition of amenability for groups given in [Gla1], proving the following result for discrete groups. This result is generalised for second countable locally compact groups in [AEG].

**Proposition 8.2.4** (*[Zim5, Proposition 4.1]*) *If  $G$  is a countable discrete group acting ergodically on  $(S, \mu)$  then  $S$  is an amenable  $G$ -space if and only if there is a conditional expectation*

$$\sigma : L^\infty(S \times G) \rightarrow L^\infty(S)$$

such that

$$\sigma(f.g) = \sigma(f).g$$

where  $(f.g)(s, h) = f(sg, hg)$  and  $(\phi.g)(s) = \phi(sg)$ .

If  $\lambda \in L^\infty(S, E^*)$  and  $f \in L^1(S, E)$ , define

$$\langle \lambda, f \rangle = \int_S \langle \lambda(s), f(s) \rangle d\mu(s).$$

Some of the techniques used in the proof of the following theorem are taken from [Zim6]. We consider the Borel structures on  $End(E)$  and  $End(E^*)$  introduced in example 3.1.1(iii).

**Theorem 8.2.5** *Let  $G$  be a countable discrete group and let  $(S, \mu)$  be a standard Borel  $G$ -space. Then the following conditions are equivalent:*

(i)  *$S$  is an amenable  $G$ -space.*

(ii) *For every separable Banach space  $E$  and for every uniformly bounded cocycle  $\alpha : S \times G \rightarrow \text{End}(E)$  and every equivariant Borel field  $\{K_s\}_{s \in S} \subset E_1^*$  of compact convex sets for  $\alpha^* : S \times G \rightarrow \text{End}(E^*)$  there exists a Borel function  $\phi : S \rightarrow E^*$  such that*

$$\alpha^*(s, g)\phi(sg) = \phi(s) \text{ for all } g \in G \text{ and } \mu\text{-a.e. } s \in S.$$

**Proof.**

Suppose that  $S$  is an amenable  $G$ -space. Let  $\alpha : S \times G \rightarrow \text{End}(E)$  be a uniformly bounded cocycle and let  $M > 0$  be a constant such that  $\|\alpha(s, g)\| \leq M$  for all  $g \in G$  and  $\mu$ -a.e.  $s \in S$ . Let

$$K = \{\lambda \in L^\infty(S, E^*) \mid \lambda(s) \in K_s \text{ } \mu\text{-a.e.}\}.$$

Fix  $k \in K$  and for all  $\xi \in E$  define a Borel map

$$\psi_\xi^k : S \times G \rightarrow \mathbb{C}$$

by

$$\psi_\xi^k(s, g) = \langle \xi, \alpha^*(s, g^{-1})k(sg^{-1}) \rangle.$$

Then

$$\begin{aligned} \sup_{s \in S} |\langle \xi, \alpha^*(s, g^{-1})k(sg^{-1}) \rangle| &\leq \|\xi\| \|\alpha^*(s, g^{-1})k(sg^{-1})\| \\ &\leq M \|k\|_\infty \|\xi\| \end{aligned}$$

and  $\psi_\xi^k \in L^\infty(S \times G)$ .

Let  $P : L^\infty(S \times G) \rightarrow L^\infty(S)$  be a  $G$ -equivariant conditional expectation. Such a  $P$  exists by the amenability of  $S$  and Proposition 8.2.4. Define a map

$$u : E \rightarrow L^\infty(S)$$

by

$$u(\xi) = P(\langle \xi, \alpha^*(s, g^{-1})k(sg^{-1}) \rangle).$$

Then  $u$  is a linear map which is clearly bounded, and hence continuous. By [Edw, Theorem 8.17.2], there exists a measurable map  $a : S \rightarrow E^*$  such that

$$u(\xi)(s) = \langle \xi, a(s) \rangle \mu\text{-a.e.}$$

Moreover, since  $E$  is a separable Banach space, the same theorem implies that  $\|u\| = \|a\|_\infty$  and that  $a$  is unique up to  $\mu$ -a.e. equality.

We claim (compare [Zim6, Lemma]) that for all  $\theta \in L^\infty(S, E)$  we have

$$P(\langle \theta(s), \alpha^*(s, g^{-1})k(sg^{-1}) \rangle) = \langle \theta(s), a(s) \rangle \mu\text{-a.e.}$$

Suppose  $\theta$  is a simple function. That is,

$$\theta(s) = \sum_1^\infty \theta_i \chi_{A_i}(s)$$

where  $\{A_i\}$  is a partition of  $S$  and  $\theta_i \in E$ . Fix  $j \in \{1, 2, \dots, n\}$ . Then for  $\mu$ -a.e.  $s \in A_j$  we have

$$\begin{aligned} P(\langle \theta(s), \alpha^*(s, g^{-1})k(sg^{-1}) \rangle)(s) &= P(\langle \theta(s), \alpha^*(s, g^{-1})k(sg^{-1}) \rangle \chi_{A_j \times G})(s) \\ &= P(\langle \theta_j, \alpha^*(s, g^{-1})k(sg^{-1}) \rangle)(s) \\ &= \langle \theta_j, a(s) \rangle \\ &= \langle \theta(s), a(s) \rangle. \end{aligned}$$

Since  $j$  was chosen arbitrarily, we have

$$P(\langle \theta(s), \alpha^*(s, g^{-1})k(sg^{-1}) \rangle)(s) = \langle \theta(s), a(s) \rangle \text{ for } \mu\text{-a.c. } s \in S$$

for simple  $\theta$ .

Suppose  $\theta$  is an arbitrary function. Then there exists a sequence  $(\theta_n)$  of simple functions such that  $\|\theta_n - \theta\|_\infty \rightarrow 0$ . Then

$$\begin{aligned} |\langle \theta_n(s), \alpha^*(s, g^{-1})k(sg^{-1}) \rangle - \langle \theta(s), \alpha^*(s, g^{-1})k(sg^{-1}) \rangle| \\ \leq \|\theta_n(s) - \theta(s)\| \|\alpha^*(s, g^{-1})k(sg^{-1})\| \\ \leq M\|k\|_\infty \|\theta_n(s) - \theta(s)\| \\ \leq M\|k\|_\infty \|\theta_n - \theta\|_\infty \\ \rightarrow 0. \end{aligned}$$

Let  $F(s, g) = \alpha^*(s, g^{-1})k(sg^{-1})$ . Hence  $P(\langle \theta_n(\cdot), F(\cdot, \cdot) \rangle) \rightarrow P(\langle \theta(\cdot), F(\cdot, \cdot) \rangle)$  in  $L^\infty(S)$ . Our claim follows since the continuity of the pairing  $\langle \cdot, \cdot \rangle$  implies that  $\|\langle \theta(\cdot), a(\cdot) \rangle - \langle \theta_n(\cdot), a(\cdot) \rangle\|_\infty \rightarrow 0$ .

Now for  $h \in G$

$$\begin{aligned} \psi_\xi^k.h(s, g) &= \psi_\xi^k(sh, gh) \\ &= \langle \xi, \alpha^*(sh, (gh)^{-1})k(sg^{-1}) \rangle \\ &= \langle \xi, \alpha^*(sh, h^{-1})\alpha^*(s, g^{-1})k(sg^{-1}) \rangle \\ &= \langle \alpha(s, g)\xi, \alpha^*(s, g^{-1})k(sg^{-1}) \rangle. \end{aligned}$$

Consider the Borel mapping  $g : S \rightarrow E$  defined by

$$g(s) = \alpha(s, h)\xi.$$

Then

$$\|g(s)\| = \|\alpha(s, h)\xi\| \leq \|\alpha(s, h)\| \|\xi\| \leq M \|\xi\|$$

and  $g \in L^\infty(S, E)$ . It follows, by the above claim, that

$$\begin{aligned} P(\psi_\xi^k \cdot h)(s) &= P(\langle g(s), \alpha^*(s, g^{-1})k(sg^{-1}) \rangle) \\ &= \langle g(s), a(s) \rangle \\ &= \langle \alpha(s, h)\xi, a(s) \rangle \\ &= \langle \xi, \alpha^*(sh, h^{-1})a(s) \rangle. \end{aligned}$$

Also,

$$P(\psi_\xi^k)(sh) = \langle \xi, a(sh) \rangle.$$

It follows, by equivariance of  $P$ , that

$$\langle \xi, \alpha^*(sh, h^{-1})a(s) \rangle = \langle \xi, a(sh) \rangle.$$

Hence, by separability of  $E$ , we must have

$$\alpha^*(sh, h^{-1})a(s) = a(sh) \text{ } \mu\text{-a.e.}$$

It remains to show that  $a(s) \in K_s$  for  $\mu$ -a.e.  $s \in S$ . Let  $\{\theta_n\}$  be a countable dense subset of  $E^*$ . For each  $n$ , consider  $\theta_n$  as a linear functional on  $E^*$ . Consider the hyperplanes in  $E^*$  defined by  $\theta_n(f) = q$  for  $q$  rational. These separate all convex compact subsets of  $E_1^*$  from points in  $E_1^*$ . Therefore, it is enough to prove that for all  $n$  and  $q$ ,  $\theta_n(K_s) \geq q$  implies that  $\theta_n(a(s)) \geq q$  for  $\mu$ -a.e.  $s \in S$ .

Given  $\theta_n$  and  $q$ , let

$$S_0 = \{s \in S \mid \theta_n(A_s) \geq q\}.$$

Then  $S_0$  is measurable by Lemma 3.3.4. Suppose  $\mu(S_0) > 0$ . Then, by positivity of  $P$ ,

$$\begin{aligned} P(\langle \theta_n, \alpha(s, g^{-1})k(sg^{-1}) \rangle \chi_{S_0 \times G}) &\geq P(q \cdot \chi_{S_0 \times G}) \\ &= q \chi_{S_0}. \end{aligned}$$

Thus  $\langle \theta_n, a(s) \rangle \cdot \chi_{S_0} \geq q \cdot \chi_{S_0}$ . Hence  $\theta_n(a(s)) \geq q$  for  $\mu$ -a.e.  $s \in S_0$ . The result follows since  $\theta_n$  and  $q$  were chosen arbitrarily.

The converse is easy to verify since the original definition of amenability of actions given in this chapter is just a special case of condition (ii) above.  $\square$

We now provide a new proof of a result of Zimmer, [Zim5, Corollary 4.2].

**Corollary 8.2.6** *Let  $\mathcal{H}$  be a separable Hilbert space. If  $G$  is a countable discrete group and  $S$  is an amenable ergodic  $G$ -space, then every cocycle  $\alpha : S \times G \rightarrow GL(\mathcal{H})$  for which there exists a constant  $C > 0$  such that  $\|\alpha(s, g)\| \leq C$  for all  $g \in G$  and  $\mu$ -a.e.  $s \in S$  is equivalent to a unitary cocycle.*

**Proof.**

Let  $\alpha : S \times G \rightarrow GL(\mathcal{H})$  be a uniformly bounded cocycle. Consider the sets

$$K_s = \overline{\text{co}}\{\alpha(s, g)\alpha(s, g)^* | g \in G\}$$

where the closure is taken in the weak topology. Then the sets  $\{K_s\}$  form a Borel field of compact, convex sets. Define a cocycle  $T : S \times G \rightarrow \text{End}(\mathcal{H})$

where for each  $g \in G$  and each  $s \in S$ ,  $T(s, g)x(s) = \alpha(s, g)x(sg)\alpha(s, g)^*$ . Then, for almost every  $s \in S$ ,  $T(s, g)K_{sg} = K_s$  and hence  $(T, \{K_s\})$  is an affine  $G$ -flow. Applying Theorem 8.2.5 to the cocycle  $T$  and the Borel field  $\{K_s\}$  we obtain an equivariant point  $\phi : S \rightarrow \mathcal{H}^*$  for  $T$ .

Define a cocycle,  $\beta : S \times G \rightarrow GL(\mathcal{H})$  by

$$\beta(s, g) = (\phi(s))^{-1/2}\alpha(s, g)(\phi(sg))^{1/2}.$$

Then  $\beta$  is a unitary cocycle which is equivalent to  $\alpha$ . □

**Remark** For discrete groups, theorem 8.1.4 is a special case of the above result. We apply the above corollary with the  $G$ -space  $S = \{e\}$  and the uniformly bounded cocycle  $\alpha_\pi$ .

**Lemma 8.2.7** *Let  $\{X_s\}_{s \in S}$  be a Borel field of compact subsets of  $E_1^*$  for some separable Banach space  $E$ . Let  $s \mapsto \mu_s$  be a Borel field of measures in the Borel field of compact convex sets  $s \mapsto M(X_s)$ . Consider the barycentre map  $\beta : M(E_1^*) \rightarrow E_1^*$ . Then  $s \mapsto \beta(\mu_s)$  is a Borel field of vectors in  $\{X_s\}_{s \in S}$ .*

**Proof.**

Since  $s \mapsto \mu_s$  is a Borel field of measures in  $M(X_s)$ , the map  $f : S \rightarrow M(E_1^*)$  defined by  $f(s) = \mu_s$  is Borel. The barycentre map  $\beta : M(E_1^*) \rightarrow E_1^*$  is Borel since it is weak\*-continuous. Thus  $\beta \circ f$  is a Borel map. It follows that  $s \mapsto \beta(\mu_s)$  is a Borel field of vectors in  $\{X_s\}_{s \in S}$ . □

**Definition 8.2.8** *If  $(\alpha, \{Y_s\})$  is a  $\mathcal{G}$ -flow, then a function  $\nu : S \rightarrow M(Y)$  is called a  $\mathcal{G}$ -invariant field of probability measures for the  $\mathcal{G}$ -flow  $(\alpha, \{Y_s\})$  if it satisfies the following conditions:*

- (i)  $\nu$  is a Borel map.
- (ii)  $\nu_s \in M(Y_s)$   $\mu$ -a.e.
- (iii) For all  $g \in G$ ,  $\alpha^*(s, g)\nu_{sg} = \nu_s$   $\mu$ -a.e.. That is, for all  $f \in C(Y)$  we have

$$\langle \alpha^*(s, g)\nu_{sg}, f \rangle = \langle \nu_s, f \rangle.$$

**Theorem 8.2.9** *The  $G$ -space  $(S, \mu)$  is amenable if and only if every  $\mathcal{G}$ -flow has a  $\mathcal{G}$ -invariant field of probability measures.*

**Proof.**

Suppose  $(S, \mu)$  be an amenable  $G$ -space. Let  $Y$  be a compact Hausdorff space and let  $(\alpha, \{Y_s\})$  be a  $\mathcal{G}$ -flow in  $Y$ . Let  $E = C(Y)$  with the  $\|\cdot\|_\infty$ -norm. Then  $E$  is a separable Banach space. Let  $\alpha^* : S \times G \rightarrow Iso(C(Y))$  be the Borel cocycle given by

$$\alpha^*(s, g)f(y) = f(\alpha(s, g)^{-1}y)$$

for  $f \in C(Y)$  and  $y \in Y$ . Then  $(\alpha^*, \{M(Y_s)\})$  is an affine Banach  $\mathcal{G}$ -flow. The result follows by amenability of  $\mathcal{G}$ .

Conversely, let  $E$  be a separable Banach space, and let  $\alpha : S \times G \rightarrow Iso(E)$  be a Borel cocycle. Let  $\{A_s\}_{s \in S}$  be a Borel field of compact convex subsets of

$E_1^*$  such that for all  $g \in G$ ,  $\alpha^*(s, g)A_{sg} = A_s$   $\mu$ -a.e.. Then  $(\alpha^*, \{A_s\})$  is a  $\mathcal{G}$ -flow in  $E_1^*$ . By our assumption, there exists a  $\mathcal{G}$ -invariant field of probability measures  $s \mapsto \nu_s$  on  $\{A_s\}$ .

Let  $\beta : M(E_1^*) \rightarrow E_1^*$  be the barycentre map. As  $\alpha^* : S \times G \rightarrow \text{Homeo}(E_1^*)$ , the weak\*-continuity of  $\beta$  implies that

$$\beta \circ \alpha^*(s, g) = \alpha^*(s, g) \circ \beta.$$

Thus

$$\beta(\nu_s) = \alpha^*(s, g)\beta(\nu_{sg})$$

and hence, by lemma 8.2.7,  $s \mapsto \beta(\nu_s)$  is an  $\alpha^*$ -equivariant section of  $\{A_s\}$ . The amenability of  $\mathcal{G}$  follows.  $\square$

**Corollary 8.2.10** *If  $S$  is an amenable  $G$ -space then every strongly proximal Banach  $\mathcal{G}$ -flow  $(\alpha, \{X_s\})$  has an  $\alpha$ -equivariant section.*

**Proof.**

Let  $(\alpha, \{X_s\})$  be a strongly proximal Banach  $\mathcal{G}$ -flow and suppose that  $S$  is an amenable  $G$ -space. Then there exists an  $\alpha^*$ -invariant section  $\phi : S \rightarrow M(X)$ . By strong proximality of the  $\mathcal{G}$ -flow, there exists a sequence  $g_i \subset G$  and a Borel field of vectors  $s \mapsto x(s)$  where  $x(s) \in X_s$  for almost every  $s \in S$  such that

$$\lim \alpha^*(s, g_i)\phi(sg_i) = \delta_{x(s)} \text{ for almost every } s \in S.$$

But,

$$\lim \alpha^*(s, g_i)\phi(sg_i) = \phi(s) \text{ for almost every } s \in S,$$

by the equivariance of  $\phi$ . It follows that  $\phi(s) = \delta_{x_s}$  for almost every  $s \in S$ . Let  $\beta : M(E_1^*) \rightarrow E_1^*$  be the barycentre map. Since  $\alpha^* : S \times G \rightarrow \text{Homeo}(E_1^*)$  and  $\beta$  is weak\*-continuous, we have  $\alpha^* \circ \beta = \beta \circ \alpha^*$ . Thus, by lemma 8.2.7,  $s \mapsto \beta(\delta_{x_s}) = s \mapsto x_s$  is an  $\alpha$ -equivariant section for  $(\alpha, \{X_s\})$ .

□

# Bibliography

- [AEG] Adams, S., Elliott, G.A., and Giordano, T., *Amenable actions of groups*, Trans. Amer. Math. Soc. **344** (1994), 803–822.
- [Arv] Arveson, W., “An Invitation to  $C^*$ -Algebra”, Springer, New York, 1976.
- [Ash] Ash, R., B., “Measure, Integration and Functional Analysis”, Academic Press, New York, 1972.
- [Azo] Azoff, E.A., *Borel maps on sets of von Neumann algebras*, J. Operator Theory **9** (1983), 319–340.
- [Coh] Cohn, D.L., “Measure Theory”, Birkhäuser, Boston, 1980.
- [Con] Conway, J.B., “A Course in Functional Analysis”, Springer, New York, 1990.
- [Din] Dinculeanu, N., “Vector Measures”, Pergamon Press, Oxford, 1967.
- [Dix] Dixmier, J., “von Neumann Algebras”, North-Holland, Amsterdam, 1981.

- [Edw] Edwards, R.E., "Functional Analysis", Holt, Rinehart and Winston, New York, 1965.
- [EG] Elliott, G.A., and Giordano, T., *Amenable actions of discrete groups*, Ergod. Th. and Dynam. Sys. **13** (1993), 289–318.
- [Fur] Furstenberg, H., "Recurrence in Ergodic Theory and Combinatorial Number Theory", Princeton University Press, Princeton, 1981.
- [Gil] Giles, J.,R., "Convex Analysis with Applications in the Differentiation of Convex Functions", Research Notes in Mathematics, 58, Pitman, Boston, 1992.
- [Gla1] Glasner, S., "Proximal Flows", Lecture notes 517, Springer, 1976.
- [Gla2] Glasner, S., *Compressibility properties in topological dynamics*, Amer. J. Math., **97** 1 (1975), 148–171.
- [HeR] Hewitt, E., and Ross, K.A., "Abstract Harmonic Analysis I", Springer-Verlag, New York, 1979.
- [HeS] Hewitt, E., and Stromberg, K., "Real and Abstract Analysis", Springer, New York, 1965.
- [Jac] Jacobson, N., "Basic Algebra I", W.H. Freeman and Co., New York, 1985.
- [KR] Kadison, R.V., and Ringrose, J.R., "Fundamentals of the Theory of Operator Algebras : Volume I", Academic Press, New York, 1983.

- [KR2] Kadison, R.V., and Ringrose, J.R., "Fundamentals of the Theory of Operator Algebras : Volume II", Academic Press, New York, 1986.
- [KuS] Kuntze, R.A., and Stein, E.M., *Uniformly bounded representations and harmonic analysis of the  $2 \times 2$  real unimodular group*, Amer. J. of Math. **82** (1960), 1-62.
- [Mac] Mackey, G.W., *Representations of locally compact groups I*, Ann. Math. **13** (1952), 101-139.
- [Mur] Murphy, G.J., "C\*-algebras and Operator Theory", Academic Press, Boston, 1990.
- [Nam] Namioka, I., *Affine flows and distal points*, Math. Z. **184** (1983), 259-269.
- [NAs] Namioka, I., and Asplund, E., *A geometric Proof of Ryll-Nardzewski's fixed point theorem*, Bull. Amer. Math. Soc., **73** (1967), 443-445.
- [Nie] Nielsen, O.A., "Direct Integral Theory", Lecture Notes in Pure and Applied Mathematics, 61, Marcel Dekker, New York, 1980.
- [Pau] Paulsen, V.I., "Completely Bounded Maps and Dilations", Pitman Research Notes in Mathematics Series 146, Longman, 1986.
- [Par] Parthasarathy, K.R., "Probability Measures on Metric Spaces", Academic Press, New York, 1967.

- [Phe] Phelps, R.R., "Lectures on Choquet's Theorem", D. van Nostrand, Princeton, 1966.
- [Pie] Pier, J.P., "Amenable Locally Compact Groups", Wiley, New York, 1984.
- [Pop] Popa, S., "Lecture notes on type  $II_1$ -factors", in preparation.
- [Ren] Renault, J.N., "A Groupoid Approach to  $C^*$ -algebras", Lecture Notes 793, Springer, New York, 1980.
- [Rob] Robertson, A.G., *Uniformly bounded group representations into a finite von Neumann algebra*, preprint, University of Newcastle, 1993.
- [Sak] Sakai, S., " $C^*$ -algebras and  $W^*$ -algebras", Springer Verlag, Berlin - Heidelberg - New York, 1971.
- [Str] Stratila, S., "Modular Theory in Operator Algebras", Abacus Press, Kent, 1981.
- [Tak] Takesaki, M., "Operator Algebras vol I", Springer Verlag, New York, 1979.
- [VZ] Vasilescu, F.H., and Zsidó, L., *Uniformly bounded groups in finite  $W^*$ -algebras*, Acta Sci. Math. Szeged **36** (1974), 189–192.
- [Wei] Weiss, B., *Orbit equivalence of nonsingular actions*, Théorie Ergodique, Monag. de L'Enseignement Math **29**, (1981), 77–107.
- [Zim1] Zimmer, R.J., "Ergodic Theory and Semisimple Groups", Birkhäuser, Basel, 1984.

- [Zim2] Zimmer, R.J., *Extensions of ergodic group actions*, Illinois J. Math. **20** (1976), 373–409.
- [Zim3] Zimmer, R.J., *Compactness conditions on cocycles of ergodic transformation groups*, J. London Math. Soc. **2** 15 (1977), 155–163.
- [Zim4] Zimmer, R.J., *Amenable ergodic group actions and an application to Poisson boundaries of random walks*, J. Funct. Anal. **27** (1978), 350–372.
- [Zim5] Zimmer, R.J., *Hyperfiniteness factors and amenable ergodic actions*, Invent. Math. **41** (1977), 21–31.
- [Zim6] Zimmer, R.J., *The von Neumann algebra of an ergodic group action*, Proc. Amer. Math. Soc. **66** 2 (1977), 289–283.