APPLICATION OF THE ORTHOGONAL COLLOCATION METHOD TO SOME STATIC AND DYNAMIC PROBLEMS IN STRUCTURAL MECHANICS

BY

UNDRIADI BENGGAWAN

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ABSTRACT

The use of the method of orthogonal collocation for solving boundary value problems in structural and applied mechanics is investigated. Some linear analysis of static problems, free vibrations of isotropic and orthotropic beams and plates with various boundary conditions are presented.

A computer program coded in FORTRAN is written for IBM 360/65.

The results obtained are presented in tabular forms and graphical forms, and whenever possible, are compared with existing solutions based on more tedious and lenghtier methods of analysis. Very good agreements are generally obtained.

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NOMENCLATURE

x, y, z	rectangular Cartesian coordinates
u, v, w	displacements in x,y, and z - directions
σ_x, σ_y	direct stresses
$\tau, \tau_{xy}, \tau_x, \tau_y$	shear stresses
$\varepsilon_x, \varepsilon_y$	direct strains
$\gamma, \gamma_{xy}, \gamma_x, \gamma_y$	shear strains
E_x, E_y, G_{xy}	moduli of elasticity and shear modulus of isotropic material
$E, G,$	modulus of elasticity and shear modulus of isotropic material
ν	Poisson's ratios for isotropic material
ν_x, ν_y	Poisson's ratios for orthotropic material
h	plate thickness
D	flexural rigidity of plates
D_x, D_y, D_{xy}	bending and twisting stiffnesses of orthotropic plates
k	modulus of elastic foundation
q, p	lateral load per unit area
a, b	plate dimensions in x and y - directions
λ	aspect ratio of plate, (a/b)
δ_{ij}	Kronecker delta

ξ, η	dimensionless parameters of x and y directional coordinates, ($\xi = x/a$, $\eta = y/b$)
P_i, P_j, P_n	orthogonal polynomial sets
C_i	constants
λ_n	eigenvalue corresponding to the n^{th} mode of vibration
X_m	functions of x only
Y_m	functions of y only
m,n,i,j	integers
V_x,V_y	shearing forces perpendicular to the plane of the plate
M_x,M_y	bending moments in x and y - directions
N_x,N_y	normal forces in x and y - directions

CHAPTER I

INTRODUCTION

The orthogonal collocation method was first introduced to boundary value problems in structural and applied mechanics by Chan and Ng (4). The types of problem Chan and Ng investigated were static problems such as torsion of rectangular bars, bending of plates, large deflection analysis of rectangular isotropic, orthotropic, and sandwich plates. In this thesis the orthogonal collocation method is further employed to investigate some additional static problems in applied mechanics and to extend the method to the analysis of free vibration of beams (with classical and non classical boundary conditions) and free vibration of rectangular isotropic and orthotropic plates.

The collocation method is one of many methods in the numerical treatment of boundary value problem using error distribution principles. Generally, the most rapidly converging method using error distribution principles is the Galerkin method.

The Galerkin method is not very efficient as it involves tedious and lengthy definite integration procedures. While the collocation method is the simpliest numerical scheme , it does not always yield reliable results since the solution of a problem can fluctuate greatly for arbitrary choices of collocation points.

Orthogonal collocation is a numerical scheme which has the accuracy of the Galerkin method and the simplicity of the collocation method.

In orthogonal collocation method, the most crucial phase of the solution lies in the formulation of the orthogonal polynomial sets, a step which provides the trial function as well as the collocation points. Through the use of a simple computer program, the orthogonal polynomial sets can easily be formulated, and the roots of the polynomials indicating the location of the collocation points are obtained by using standard iteration schemes. Once the orthogonal polynomial sets for the problem is formulated, the rest of the computation is as simple as the collocation method.

The orthogonal collocation method is used to obtain approximate solutions to the governing differential equations of boundary problems in applied mechanics. The numerical method provides convergence to the exact answer as the approximation is refined in successive calculations by using more collocation points, a procedure similar to the technique of using finer grid in the finite difference method or the finite element method.

Furthermore, the first approximation of the orthogonal collocation method which can easily be obtained without resorting to computers would generally give insight to the qualitative behaviour of the structural problem to be solved.

CHAPTER II

Review of Literature

The governing differential equations of boundary value problems in structural and applied mechanics for all but the simpliest of cases are usually rather complex. For this reason, exact solutions to these types of problems can only be obtained for a few simple cases. Because of this, many numerical methods have been developed to solve complex boundary value problems. These numerical methods vary in accuracy, applicability and simplicity.

Every method has its advantages and disadvantages. For a given boundary value problem in applied mechanics, different numerical schemes have to be tried to arrive at a numerical method where the advantages of such a method greatly outweighs the disadvantages.

Collocation methods have been used for more than fifty years to solve differential equations. They were first applied to differential equations by Frazer, Jones and Skan (12) and independently by Lanzos (17). The Lanzos method as adopted by Clenshaw and Norton (5,24) has found many applications. This method capitalizes on the good convergence properties of the expansion of the Tschebysheff polynomials; however , it does not take full advantage of the boundary conditions. For an arbitrary positioning of the collocation points the method may be divergent even for rather simple ordinary differential equations (36) and for partial differential equations such phenomenon probably occurs even more frequently.

In 1967 Villadsen and Stewart (35) developed a new method called Orthogonal Collocation, using orthogonal polynomials as trial functions and the roots as the collocation points over its region. It is a special case of the collocation method and the method of weighted residuals. Such methods are discussed in more detail by Finlayson (11). Since then, many problems have been solved using the orthogonal collacation method, mostly in application to chemical engineering problems.

Serth (27) utilized the method to solve stiff boundary value problems, stiff differential equations are those which are difficult to solve numerically due to accuracy and stability problems associated with eigenvalues of widely different magnitudes. Villadsen and Sorenson (37) used the double collocation method whereby the collocation was extended to both space and time variables to yield algebraic equations. In their work the collocation approach was handled in a manner reminiscent of implicit finite difference approaches in which the values of the dependent variables were solved line by line (or section by section depending on the number of unknown points) into the infinite domains.

Birnbaum and Lapidus (1) illustrated that the infinite domain can be covered with a single orthogonal polynomial and that, in turn, the entire problem can be solved in a single pass. Carey and Finlayson (2) explored the method of orthogonal collocation on finite elements combining the rapid convergence of the orthogonal collocation method with the convenience associated with finite difference methods of locating grid points or elements where the solution is important or has large gradients. The solution of such complex problems as the large deflection analysis of rectangular isotropic, orthotropic, and sandwich plates employing the method of orthogonal collocation were demonstrated by Chan and Ng (4).

A relatively large number of techniques are available for solving boundary value problems in structural and applied mechanics. Many problems of rectangular plates have been studied hitherto by methods of various kinds because of theoretical interest and practical importance.

Problems involving vibrations of thin, elastic plates occur in a wide variety of applications of structural mechanics in all aspects of engineering. An excellent comprehensive summary of existing analytical and experimental information relevant to this problem area has been presented by A.W. Leissa (19).

Various methods have been used by investigators to determine the natural frequencies of free vibration and the associated mode shapes of thin elastic plates (19,32). Some of the most important of these techniques are:

(a) Exact Solutions

In a few cases involving relatively simple plate contours and associated boundary conditions an exact solution of the governing differential equation may be found. This yields frequencies as eigenvalues of the equation and the associated mode shapes are then readily determined. Exact solutions for certain rectangular plates have been found by S. Iguchi as well as H.J.Fletcher, N.Woodfield, and K.Larsen and M.Vet and A.W.Leissa (19).

(b) Energy Methods

In any conservative system the total energy must remain constant. Hence for the vibrating plate the sum of the potential and kinetic energies is constant and if a realistic vibratory configuration (satisfying boundary conditions) is assumed, it is possible to determine the natural frequencies. Since the vibratory configuration employed may not be the true one occurring during motion, the use of this technique leads to values of natural frequencies that are too great. Unfortunately, the error involved is usually difficult to estimate.

Lord Rayleigh in his classic work gives a method for the approximation of the frequencies of dynamical systems. Later W.Ritz produced what is known as the "Rayleigh - Ritz" method for approximating frequencies in vibrating systems.He applied his technique to square plates with all edges free. The Ritz method is one of several possible procedures for obtaining approximate solutions for the frequencies and modes of vibration of thin elastic plates. The convergence and accuracy of the Ritz method have been discussed by various authors including L.Collatz (6). It is known that this method gives upper bounds for the frequencies, that is, the frequencies calculated by Ritz's procedure are always higher than the exact values. Also, the accuracy of the results cannot be estimated with certainty in most cases. In spite of this limitation the method has yielded satisfactory solutions for numerous problems in equilibrium, buckling, and vibration. While the Ritz method is well known, it has not been used as much as might be expected for plate vibration problems. This is probably due, at least in part, to the great amount of computational labour which requires both, setting up and solving the necessary equations. The amount of computations involved depends to a large extent upon the set of functions used to represent the plate deflection problems. For these functions some investigators have taken a series of polynomials while others have used combinations of the characteristic functions which define the normal modes of a vibration. Dana Young (42) used the latter types of functions to obtain three specific plate problems, namely, square plate clamped at all four edges, square plate clamped along two adjacent edges and free along the other two edges, and square plate clamped along one edge and free along the other three edges. Warburton (41) presented frequency formulas for all twenty one types of problems derived by using the Rayleigh method with assumed modes shapes which are the products of vibrating beam eigenfunctions. Later, another set of formulas was published by Janich in 1962 for 18 cases (for fundamental modes only). Again, the Rayleigh technique was utilized, but simple trigonometric functions were chosen to represent the plate deflections. However, these functions do not represent the mode shapes nearly as well as the beam functions. Leissa presented accurate analytical results for the free vibration of rectangular plates of various cases. There are many more authors applying the energy methods for other shapes of plates such as V.G.Sigillito, M.hasegawa, S.T.Odman, N.Aronnzajin and D.Young (42).

(c) Numerical Method

It is worthy to note that several other numerical approaches have been employed by previous investigators. These include: i) point matching techniques by H.D.Conway(7) where an assumed solution satisfies the governing equation at all interior points of the plate but satisfies the boundary only at designated points along the edges; ii) finite difference method: Szilard (28) solved a variety of plate problems using this method. He also has an extensive discussion of the method, and of mean refining the method as applied to plate bending; iii) finite element method: The advancement in the technology in computing facilities makes this method to be widely used to solved many types of problems. Szilard (28) presented this method and many other numerical method for solving plate problems.

More recently laser holography has been applied to determine natural frequencies and associated mode shapes of plates by G.M.Mayer and this approach has been found to be excellent particularly for vibrations at higher modes.

CHAPTER III

3.1 Orthogonal collocation method.

Applications of error distribution principles for numerical treatment of boundary value problems are widely used by many investigators. The expansion coefficients of the trial function used are determined mostly by weighted residual methods. These methods are more attractive due to the simple numerical scheme and compactness of the results, as compared to other principles. Of all the numerical methods that are based on the principles of error distribution, the Galerkin method is generally the most rapidly converging method (12), while the collocation method is definitely the most simple numerical scheme. In the collocation method, it is only necessary to evaluate the residual at the collocation points. This is easily done, so that the collocation method appears to be the simpliest form of weighted residual methods. However, both methods have their drawbacks. The collocation method, though simple in application, is not very reliable since the solution of a problem can fluctuate greatly for arbitrary choices of the collocation points and an equidistant spacing is not generally appropriate. Comparatively, little has been done on setting out definite criteria for selecting the location of collocation points. On the other hand, the Galerkin method is much more cumbersome to use due to tedious finite integrations required before a solution can be obtained.

A numerical scheme which, in the opinion of the writer, has the accuracy of the Galerkin method and the simplicity of the collocation method is the method of orthogonal collocation. In fact, orthogonal collocation is a discrete analogy of Galerkin's method since the formulation of the method is also based on orthogonality though not of the residual function, but of a polynomial which vanishes at the same points. Thus instead of requiring the residual function to be orthogonal to each term of the trial function as is done in most weighted residual methods, the residual here is matched to an orthogonal function at its zeroes. The necessity of integrating the residual is thereby avoided.

The most important is the formulation of the orthogonal polynomial sets, a step which provides the trial functions as well as the collocation points. For a symmetrical boundary value problem in one independent variable, x, in the region $x^2 < 1$, the formulation of the orthogonal polynomial sets can be written as follows :

$$\int_0^1 \bar{w}(x^2)\ P_i(x^2)\ P_n(x^2)\ dx = C_i\ \delta_{in} \qquad 3.1.1$$

where : i and n are positive integers

$\bar{w}(x^2)$ is a weight function

C_i are constants

δ_{in} is the Kronecker delta

$P_i(x^2)$ are polynomials of degree i in x^2

The formulation of the orthogonal polynomials sets will be described in more detail in section 3.2.

For interior collocation, the trial function is chosen such that the boundary conditions are satisfied. The trial function for the solution is then written in the form :

$$\bar{w}(x^2) \sum_{i=0}^{n-1} A_i \; P_i(x^2) \qquad 3.1.2$$

in which $\bar{w}(x^2)$ is usually chosen such that the boundary conditions are satisfied. By adjusting the trial function to satisfy the governing differential equation at n collocation points, which are the roots of the polynomial $P_n(x^2)$, then n equations with n undetermined parameters A_i can be evaluated.

To illustrate the orthogonal collocation method, consider a symmetrical second order boundary value problem in one independent variable, x, in the region $x^2 < 1$.
The differential equation is :

$$L(Y) = 0 \qquad \text{for } x^2 < 1 \qquad 3.1.3$$

and the boundary conditions are :

$$Y = 0 \qquad \text{at } x^2 = 1 \qquad 3.1.4$$

$$\frac{dY}{dx} = 0 \qquad \text{at } x = 0 \qquad 3.1.5$$

For interior collocation, the assumed solution is chosen such that the boundary conditions are satisfied.
A suitable function is :

$$Y = (1 - x^2) \sum_{i=0}^{n-1} A_i \quad P_i(x^2) \qquad 3.1.6$$

where $P_i(x^2)$ are polynomials of degree i in x^2, yet to be specified and the A_i are undetermined constants.

Once Y has been adjusted to satisfy equation 3.1.3 at n collocation points $x_1, x_2 \ldots\ldots x_n$, the residual function L(Y) either vanishes everywhere or contains a polynomial factor $G_n(x^2)$ of degree n in x^2 whose zeroes are the collocation points. Then by analogy with Galerkin's method which specifies that the residual be orthogonal to all the trial functions, the collocation points are selected by specifying that $G_n(x^2)$ be orthogonal to all the functions $(1-x^2)\ P_i(x^2)$ of equation 3.1.6 over the region $x^2 < 1$. Such a specification is automatically satisfied by taking $G_n(x^2)$ and $P_n(x^2)$ from the orthogonal set defined by :

$$\int_0^1 (1-x^2)\ P_i(x^2)\ P_n(x^2)\ dx = C_i\ \delta_{in} \qquad 3.1.7$$

for all positive integers i and n, where C_i is a constant and δ_{in} is a kronecker delta.

The orthogonality relation in equation 3.1.7 ensures that the zeroes of $P_n(x^2)$ are real, distinct and located within the open interval 0,1 . The equation 3.1.7 is the key formula here which provides both the trial functions and the collocation points.

In the Galerkin interior method, the approximate solution of equation 3.1.3 is obtained by setting the differential - equation residual L(Y) orthogonal to all the trial functions. For the assumed solution, equation 3.1.6, this orthogonality relation over the region $x^2 < 1$ becomes

$$\int_0^1 (1-x^2)\, P_i(x^2)\, [\, L(Y)\,]\, dx = 0 \qquad 3.1.8$$

$$(i = 0, 1, \ldots, n-1)$$

The present collocation method, on the other hand, uses the orthogonality relation :

$$\int_0^1 (1-x^2)\, P_i(x^2)\, [(x^2-x_1^2) \ldots\ldots (x^2-x_n^2)]\, dx \qquad 3.1.9$$

$$(i = 0, 1, \ldots, n-1)$$

to define the collocation points, $x_1 \ldots x_n$ where the residual L(Y) is to vanish. The two methods agree if L(Y) is a polynomial of degree $d \le n$ in x^2 .

Although the derivation here is based on one dimensional second order problems, the present method can be easily extended to two dimensional problems and problems involving higher order derivatives. To demonstrate this, boundary value problems such as the static and dynamic analysis of clamped beams and clamped plates will be solved in chapter V using the orthogonal collocation method.

Section 3.3 described the method to develop the orthogonal polynomials used in chapter V. Subsequent steps required by the orthogonal collocation method in solving problems are almost identical as the ordinary collocation method.

Instead of using equal spacing or arbitrary collocation points , the orthogonal collocation method uses the roots of the orthogonal polynomial as collocation points. Therefore unlike the collocation method which forced the residual function to be zero only at the collocation points, the orthogonal collocation method also ensures that the mean of the residual function is zero over the entire region. The extra efforts to develop the orthogonal polynomials used in the orthogonal collocation method compare to the simple collocation method are insignificant considering the enormous advantages in accuracy that results in such a method.

3.2 Formulation of the orthogonal polynomials

Let f(x) be a function for which $\int_a^b \bar{w}(x)\ [f(x)]^2\ dx$ exist and let $[\phi_i(x)]$ be a set of functions for which

$$\int_a^b \bar{w}(x)\ \phi_i(x)\ \phi_j(x)\ dx = C_i\ \delta_{ij} \qquad 3.2.1$$

where δ_{ij} is the krocnecker delta function and C_i the value of the integral for $i = j$.

The weight function $\bar{w}(x)$ is assumed to be integrable (but not necessarily continuous or even defined for all $x \in [a,b]$.)

Then :

1. If equation 3.2.1 holds for every (i,j) the set $[\phi_i(x)]$ is an orthogonal system on [a,b] with respect to the weight function $\bar{w}(x)$.

   The weight function is now assumed to be positive in [a,b].
2. The set of parameters $\bar{a}^*$ for which the weighted L_2 norm

$$L_{2w}(f) = [\int_a^b \bar{w}(x)\{f(x)-L_n(f,x)\}^2dx]^{\frac{1}{2}} \qquad 3.2.2$$

   is minimum is obtained by formula 3.2.3 :

$$\bar{a}^* = \frac{1}{C_i}\int_a^b \bar{w}(x)\ \phi_i(x)\ f(x)\ dx \qquad 3.2.3$$

The parameters $\bar{a}^*$ chosen in accordance with eq. 3.2.3 minimizes the square of distance function weighted by $\bar{w}(x)$

and integrated from a to b . The form of 3.2.2 justifies the name 'best weighted mean' approximation, normally given to the approximation defined by equation 3.2.1 and equation 3.2.3 . Particularly simple expressions for the approximation functions are obtained when $\bar{w}(x) = (x-a)^{\alpha} (x-b)^{\beta}$ and the approximation functions are chosen as algebraic (or trigonometric) polynomials. If $a = 0$ and $b = 1$ the resulting polynomials are called 'shifted' Jacobi polynomials.

The most important relation between orthogonal polynomials is equation 3.2.3 . It defines the polynomials, it illustrates one of their most important geometrical and analytical properties and it may be used to construct the polynomials.

For the specific weight function of Jacobi polynomials :
Let $\bar{w}(x) = x^{\beta} (1-x)^{\alpha}$ where $\alpha \geqslant 0$ and $\beta \geqslant 0$, let $\phi_n(x)$ be polynomials and let the range of orthogonality be [0,1].

The set of approximations $\phi_i(x)$ are then defined as Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$:

$$\int_0^1 x^{\beta}(1-x)^{\alpha} P_n^{(\alpha,\beta)}(x) \; P_j^{(\alpha,\beta)}(x) \, dx = C_n \, \delta_{nj} \qquad 3.2.4$$

The coefficients of an orthogonal polynomial $P_n(x)$, which is normalized such that the coefficients of x is $(-1)^n$, are most conveniently found by a simple recurrence formula

[36 pp 41] .

$$P_n(x) = \gamma_n x^n - \gamma_{n-1} x^{n-1} + \ldots + (-1)^n$$

$$= \sum_{i=0}^{n} (-1)^{n-i} \gamma_i \, x^i \qquad 3.2.5$$

$$\gamma_{k+1} = \frac{n-k}{k+1} \; \frac{n+\alpha+\beta+k+1}{\beta+k+1} \; \gamma_k \qquad 3.2.6$$

where $k = 0,1,2, \ldots, n-1$, $\gamma_0 = 1$ and α and β are both greater than -1. And values of

$$C_n = \int_0^1 \bar{w}(x) \; [P_n(x)]^2 \, dx \qquad 3.2.7$$

or

$$C_n = \frac{(\Gamma(\beta+1))^2 \; n! \; \Gamma(n+\alpha+1)}{\Gamma(n+\beta+1) \; \Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)} \qquad 3.2.8$$

For $n > 0$ and $(\alpha,\beta) > -1$

The Jacobi polynomials defined by equation 3.2.4 and equation 3.2.5 are given explicitly by

$$P_n^{(\alpha,\beta)} \; x^\beta \; (1-x)^\alpha =$$

$$\frac{(-1)^n \; \Gamma(\beta+1)}{\Gamma(n+\beta+1)} \; \frac{d^n}{dx^n} [x^{n+\beta} \; (1-x)^{n+\alpha}] \qquad 3.2.9$$

The proof of this is given by Villadsen [36].

The preceding paragraphs of this section gave an outline of the most important properties of orthogonal polynomials.

In many engineering problems, symmetrical boundary value problems are very often encountered, therefore it is more efficient to solve those problems using polynomials in x^2. The following paragraph studied some useful even degree polynomials as examples of the general theorems and formulas.

The orthogonal relation for Jacobi polynomials is

$$\int_0^1 (1-u)^{\alpha} u^{\beta} P_i(u)\, P_j(u)\, du = C_i\, \delta_{ij} \qquad 3.2.10$$

Substitute $u = x^2$ and $du = 2x\, dx$

$$\int_0^1 (1-x^2)^{\alpha} x^{2\beta+1} P_i(x^2)\, P_j(x^2)\, dx = \frac{C_i}{2}\, \delta_{ij} \qquad 3.2.11$$

Equation 3.2.11 is the recipe for construction of a particu - lar set of even orthogonal polynomials. The physical problem often dictates the form of integral in eq. 3.2.11 that is the exponents α and $2\beta+1$, and the formula allows one to work backwards to α and β, and hence to the appropriate set of polynomials. For $a = 2\beta + 1$ and $C_i^* = \frac{C_i}{2}$ formula 3.2.11 yields

$$\int_0^1 (1-x^2)^{\alpha} x^{a} P_i(x^2)\, P_j(x^2)\, dx = C_i^*\, \delta_{ij} \qquad 3.2.12$$

The orthogonal polynomials are constructed from Rodriques formula (eq.3.2.8).

$$P_n(x) = P_n(u)$$

$$P_n(u) = (-1)^n \frac{u^{\frac{1-a}{2}} (1-u)^{-\alpha} \, \Gamma(\frac{a+1}{2})}{\Gamma(n+\frac{a+1}{2})} \frac{d^n}{du^n} [u^{n+\frac{a-1}{2}} (1-u)^{n+\alpha}]$$

with $a = 2\beta + 1$ 3.2.13

All supplementary information such as the recurrence formula, the formulas for γ_n and the formula for C_i are obtained directly from the preceding paragraphs with proper interpretation of α and β and replacing x with x^2

Villadsen and Stewart (35) showed the orthogonal polynomials and constants where the weight functions are $\overline{w}(x^2) = (1-x^2)$ and $\overline{w}(x) = x(1 - x^2)$ and $\overline{w}(x^2) = x^2 (1 - x^2)$.

In this thesis, while the weight function has to satisfy the boundary conditions of the problem as used by others, the orthogonal polynomials developed here are different from those of other investigators such as Villadsen and Stewart (35). The following section outlines in detail the development of the orthogonal polynomials and the constant used in this thesis.

3.3 Orthogonal polynomials.

From Equation 3.1.1

$$\int_0^1 \bar{w}(x^2)\ P_i(x^2)\ P_n(x^2)\ dx = C_i\ \delta_{in}$$

$$\text{if} \quad i \neq n \qquad \delta_{in} = 0$$

$$\text{if} \quad i = n \qquad \delta_{in} = 1$$

Let $\bar{W}(x^2) = b_1\, x^{d_1} + b_2\, x^{d_2} + \ldots + b_m\, x^{d_m}$ 3.3.1

$$= \sum_{j=1}^{m} b_j\, x^{d_j}$$

and $T_j = \int_0^1 x^{(2j-2)}\ \bar{w}(x^2)\ dx \quad , j = 1,2,\ldots$ 3.3.2

Assumed

$$P_0 = 1$$

$$P_1 = 1 + a_1 x^2$$

$$P_2 = 1 + a_2 x^2 + a_3 x^4$$

$$P_3 = 1 + a_4 x^2 + a_5 x^4 + a_6 x^6$$

$$P_4 = 1 + a_7 x^2 + a_8 x^4 + a_9 x^6 + a_{10} x^8$$

For $i = 0$ and $n = 0$

$$C_0 = \int_0^1 \bar{w}(x^2)\ dx = T_1$$

For $i = 0$ and $n = 1$

$$\int_0^1 \bar{w}(x^2)\ (1)\ (1+a_1 x^2)\ dx = 0$$

$$\int_0^1 \bar{w}(x^2)dx + \int_0^1 \bar{w}(x^2)\ (a_1 x^2)\ dx = 0$$

$$T_1 + a_1 T_2 = 0 \qquad a_1 = -\frac{T_1}{T_2}$$

For $i = 1$ and $\dot{n} = 1$

$$C_1 = \int_0^1 \bar{w}(x^2)\ [(1+a_1x^2)(1+a_1x^2)]\ dx$$

$$= \int_0^1 \bar{w}(x^2)\ dx + 2a_1 \int_0^1 \bar{w}(x^2)\ x^2\ dx + a_1^2 \int_0^1 \bar{w}(x^2)x^4 dx$$

$$= T_1 + 2a_1 T_2 + a_1^2 T_3$$

This can go on to as many terms as required. This type of computation is best achieved by computer programming, because the input to the program is just the coefficients and the powers of the weight function. Once the polynomial is found, the roots can easily be computed using built-in subroutines which usually are readily available. The listing of the program is at the back of this thesis, also the polynomials for different weighing functions, their constants and the roots are tabulated.

CHAPTER IV

Fundamental Theories and Equations of Beams and Plates

4.1 Basic Equations for Free Vibration of Beams.

Generally, the assumptions in the derivation of the basic differential equations governing the free vibration of beams are as follows:

1. Properties of the beams are elastic, homogeneous, isotropic and have uniform cross section throughout the length of the beam.
2. Plane sections that are initially normal to the middle plane remain plane and normal to it.
3. The beam is long in proportion to its depth.
4. The effect of the rotary inertia is negligible and need not be taken into consideration.

In Fig. 4.1 a small cut out section from a loaded beam with length dx is shown. For uniformly distributed load the total load qdx, acts through the center of the element A. By taking moments about A and neglecting the product of dV.dx gives

$$V = \frac{dM}{dx} \qquad 4.1.1$$

Also, by taking the sum of forces in the vertical direction equal to zero:

$$q = -\frac{dV}{dx} \qquad 4.1.2$$

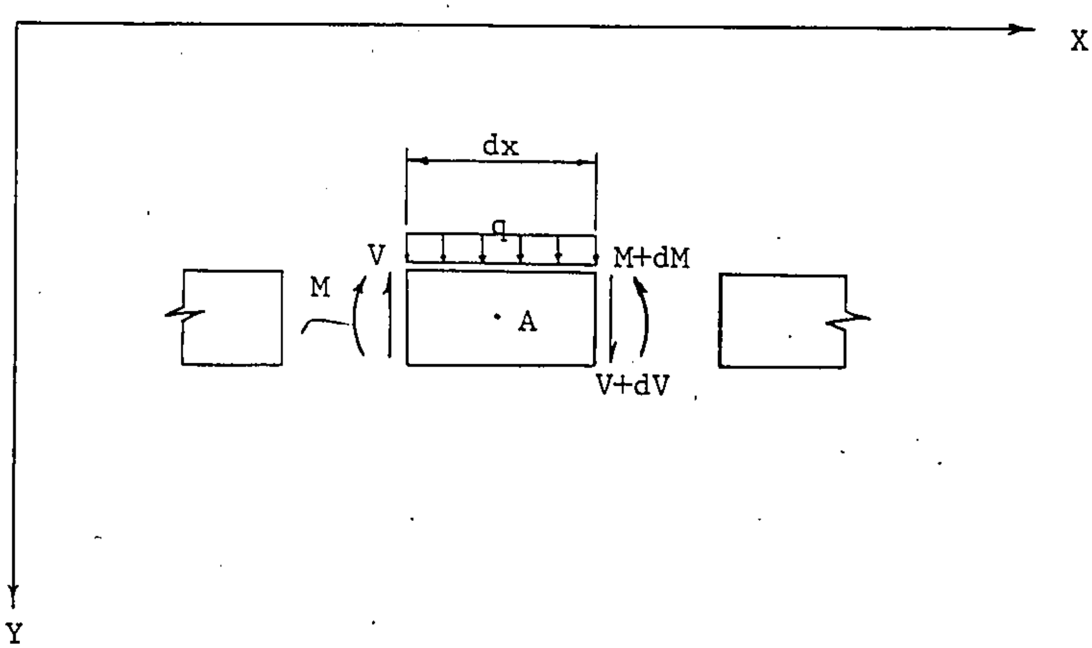


FIGURE 4.1

It can also be shown from elementary strength of material that the moment and the beam curvature are related by

$$M = EI\frac{d^2y}{dx^2} \qquad 4.1.3$$

Provided that M can be expressed as function of x, eq.4.1.1 and eq.4.1.2 yield:

$$V = EI\frac{d^3y}{dx^3} \qquad 4.1.4$$

and the load (or force)

$$q = EI \frac{d^4 y}{dx^4} \qquad 4.1.5$$

Equation 4.1.5 is a differential equation for beams without elastic foundation. For free vibration of beams, using the displacement w(x,t) as a function of time and distance and ρ as mass, the force q in equation 4.1.5 can be substituted and the following equation can be obtained:

$$-\frac{\partial^2 w}{\partial t^2} = \frac{EI}{\rho}\frac{\partial^4 w}{\partial x^4} \qquad 4.1.6$$

Assuming that equation 4.1.6 has a variable separable solution, dependent variable w(x,t) can be represented by the product of W(x) and T(t), where W(x) is the shape function and depends upon x only and T(t) is a function of time. With this, w(x,t) can readily be written as:

$$w(x,t) = W(x)\ T(t) \qquad 4.1.6a$$

Substituting equation 4.1.6a into equation 4.1.6 gives:

$$-W(x)\frac{d^2T(t)}{dt^2} = \frac{EI}{\rho}\ T(t)\ \frac{d^4w(x)}{dx^4} \qquad 4.1.6b$$

Separating the variables we obtain:

$$-\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = \frac{EI}{\rho}\frac{1}{W(x)}\frac{d^4W(x)}{dx^4} = a^4 \qquad 4.1.6c$$

where a^4 is a constant.

It can be shown that the constant a^4 of equation 4.1.6c is a positive quantity. Therefore from equation 4.1.6c the two ordinary homogeneous differential equations are obtained:

$$\frac{d^4W(x)}{dx^4} - \frac{\rho a^4}{EI} W(x) = 0 \qquad 4.1.7$$

and

$$\frac{d^2T(t)}{dt^2} + a^4 T(t) = 0 \qquad 4.1.8$$

It is highly advantageous to work with non-dimensionalized variables of space and displacement. To transform into dimensionless form the variable $\xi = x/L$ is introduced.

Equation 4.1.7 may be written as:

$$\frac{\partial^4W(\xi)}{\partial\xi^4} - k^4W(\xi) = 0 \qquad 4.1.9$$

where now

$$k^4 = \frac{\rho\omega^2L^4}{E\,I} \qquad 4.1.10$$

and $\omega = 2\pi f$ 4.1.11

where f is frequency

4.2 Basic Equations of the Theory of Elasticity for Rectangular Plates.

The basic assumptions.

1. The plates are continuous and homogeneous.
2. The strains are infinitesimal.

The equations will be refered to a rectangular Cartesian system.

4.2.1 The Strain-Displacement relations.

The components of deformation can be expressed in term of displacements. When there are no limitations to the extent of deformation, then the relation between the normal strain ε_x, ε_y, ε_z, and the shear strain γ_{xy}, γ_{xz}, γ_{yz}, are related to the displacement u, v, w in the following manners (21).

$$\varepsilon_x = \sqrt{1+2\frac{\partial u}{\partial x}+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2} - 1$$

$$\varepsilon_y = \sqrt{1+2\frac{\partial v}{\partial y}+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2} - 1 \qquad 4.2.1a$$

$$\varepsilon_z = \sqrt{1+2\frac{\partial w}{\partial z}+\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2} - 1$$

$$\sin\gamma_{xy} = \frac{\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}}{(1+\varepsilon_x)\cdot(1+\varepsilon_y)} \qquad 4.2.1b$$

$$\sin \gamma_{xz} = \frac{\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}+\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial z}}{(1+\varepsilon_x).(1+\varepsilon_z)} \qquad 4.2.1b$$

$$\sin \gamma_{yz} = \frac{\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}+\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial z}}{(1+\varepsilon_y).(1+\varepsilon_z)}$$

In the case of small deflections, when the derivatives of displacements are small compared with unity, these formulas can be simplified, as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad , \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad , \quad \varepsilon_z = \frac{\partial w}{\partial z} \qquad 4.2.2a$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad , \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad 4.2.2b$$

4.2.2 The Stress-Strain Relations According to Generalized Hooke's Law.

For elastic plates in which the components of strain are linear functions of the components of stresses, the generalized Hooke's Law can be written in matrix form as:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \qquad 4.2.3$$

where a_{11}, a_{12} ...a_{66} are the elastic constants or the coefficients of deformation, and $a_{ij} = a_{ji}$:

4.2.3 The Equilibrium Equations.

The following equilibrium equations for the stress components in a continuous body which is in equilibrium must be satisfied.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \qquad 4.2.4$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + f_z = 0$$

where f_x, f_y, f_z are projection of the body force along the directions x, y, z. Eq. 4.2.4 can be transformed into a differential equation of motion by adding inertia terms.

$$\rho \frac{\partial^2 u_x}{\partial t^2}, \quad \rho \frac{\partial^2 u_y}{\partial t^2}, \quad \rho \frac{\partial^2 u_z}{\partial t^2} \qquad 4.2.5$$

where ρ is the material density and t is time.

4.3 Equations for Theory of Rectangular Plates.

Basic assumptions:

1. The plate is considered thin when its thickness h is small compared with its other dimensions. The x-y plane in the undeformed state is called the middle surface, as shown in fig.4.2
2. The plane sections normal to the middle surface before deformation remain plane and normal to the middle surface after deformation.
3. The normal stresses in the transverse direction to the plate can be disregarded in comparison to the normal stresses σ_x and σ_y.
4. There is no straining of the middle surface after bending, and therefore it becomes the neutral surface.
5. Since h is small, and considering (3), the vertical displacement of any point not on the middle surface will be the same as the vertical displacement of the point above it on the middle surface.

4.3.1 The Strain-Displcement Relations.

It follows from second assumption that u and v can be expressed in terms of derivatives of the transverse displacement w, which is a function of x and y only. The expression are as follows:

$$u = -z \frac{\partial w}{\partial x} \quad , \qquad v = -z \frac{\partial w}{\partial y} \qquad 4.3.1$$

Using the above equations the strain-displacement relations in equation 4.2.2a can written as follows:

$$\varepsilon_x = -z \frac{\partial^2 w}{\partial x^2} \quad , \qquad \varepsilon_y = -z \frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y} \qquad 4.3.2$$

4.3.2 The Stress-Strain relations.

Since the transverse stresses $_z$, $_{zy}$, $_{zx}$ are small relative to the in-plane stresses in a plate problems, eq. 4.2.3 reduces to

$$\begin{aligned} \varepsilon_x &= a_{11} \sigma_x + a_{12} \sigma_y + a_{16} \tau_{xy} \\ \varepsilon_y &= a_{12} \sigma_x + a_{22} \sigma_y + a_{26} \tau_{xy} \\ \gamma_{xy} &= a_{16} \sigma_x + a_{26} \sigma_y + a_{66} \tau_{xy} \end{aligned} \qquad 4.3.3$$

Eq. 4.3.3 can be put in matrix form as follows:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \qquad 4.3.4$$

The inversion of eq. 4.3.4 gives the stresses in terms of the strains and result in:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{16} \\ b_{12} & b_{22} & b_{26} \\ b_{16} & b_{26} & b_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \qquad 4.3.5$$

The substitution of eq.4.3.2 into eq.4.3.5 yields:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = -z \begin{bmatrix} b_{11} & b_{12} & b_{16} \\ b_{12} & b_{22} & b_{26} \\ b_{16} & b_{26} & b_{66} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \qquad 4.3.6$$

The shearing stresses τ_{zx} and τ_{zy} can be determined from the equilibrium equations, eq. 4.2.4 and eq.4.3.6

$$\begin{aligned} \tau_{zx} &= \tfrac{1}{2}\left(z^2 - \frac{h^2}{4}\right) b_{11} \frac{\partial^3 w}{\partial x^3} + 3b_{16} \frac{\partial^3 w}{\partial x^2 \partial y} + \\ &\quad (b_{12} + 2b_{66}) \frac{\partial^2 w}{\partial x \partial y^2} + b_{26} \frac{\partial^3 w}{\partial y^3} \\ \tau_{zy} &= \tfrac{1}{2}\left(z^2 - \frac{h^2}{4}\right) b_{16} \frac{\partial^3 w}{\partial x^3} + 3b_{26} \frac{\partial^3 w}{\partial x \partial y^2} + \\ &\quad (b_{12} + 2b_{66}) \frac{\partial^3 w}{\partial x \partial y^2} + b_{22} \frac{\partial^3 w}{\partial y^3} \end{aligned} \qquad 4.3.7$$

4.3.3 The Equilibrium Equations.

When the plate in figure 4.3 is cut with certain surface parallel to the initial middle surface x-y with height equal to the plate thickness h and the bases dx and dy, then the forces and moments per unit length are obtained by integrating the stresses over the plate thickness.

$$M_x = \int_{-h/2}^{h/2} \sigma_x \, z \, dz \quad , \quad M_y = \int_{-h/2}^{h/2} \sigma_y \, z \, dz$$

$$M_{xy} = M_{yx} = \int_{-h/2}^{h/2} \tau_{xy} \, z \, dz \qquad 4.3.8$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{zx} \, dz \quad , \quad Q_y = \int_{-h/2}^{h/2} \tau_{yz} \, dz$$

From this and from eq.4.3.6 and eq. 4.3.7 the expressions for M_x, M_y, M_{xy}, Q_x, and Q_y become:

$$M_x = -D_{11} \frac{\partial^2 w}{\partial x^2} + D_{12} \frac{\partial^2 w}{\partial y^2} + 2D_{16} \frac{\partial^2 w}{\partial x \partial y}$$

$$M_y = -D_{12} \frac{\partial^2 w}{\partial x^2} + D_{22} \frac{\partial^2 w}{\partial y^2} + 2D_{26} \frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xy} = -D_{16} \frac{\partial^2 w}{\partial x^2} + D_{26} \frac{\partial^2 w}{\partial y^2} + 2D_{66} \frac{\partial^2 w}{\partial x \partial y}$$

$$Q_x = -D_{11} \frac{\partial^3 w}{\partial x^3} + 3D_{16} \frac{\partial^3 w}{\partial x^2 \partial y} + (D_{12} + 2D_{66}) \frac{\partial^3 w}{\partial x \partial y^2} + D_{26} \frac{\partial^3 w}{\partial y^3} \qquad 4.3.8$$

$$Q_y = -D_{16} \frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} + 3D_{26} \frac{\partial^3 w}{\partial x \partial y^2} + D_{22} \frac{\partial^3 w}{\partial y^3}$$

where $D_{ij} = \frac{h^3}{12} b_{ij}$. $\qquad$ 4.3.9

The constants D_{ij} are called rigidity; D_{11} , D_{22} are bending rigidities about y and x axes respectively, D_{66} is the twisting rigidity and D_{16} , D_{26} are additional rigidities.

In the case of orthotropic plates equation 4.3.8 can be applied by letting $D_{16} = D_{26} = 0$.
Using relationship such as:

$$\nu_1 = \frac{D_{12}}{D_{22}} \quad , \quad \nu_2 = \frac{D_{12}}{D_{11}} \quad ,$$

$$D_1 = D_{11} = \frac{E_1 h^3}{12(1-\nu_1\nu_2)} \quad , \quad D_3 = D_1\nu_2 + 2D_k \qquad 4.3.10$$

$$D_2 = D_{22} = \frac{E_2 h^3}{12(1-\nu_1\nu_2)} \quad , \quad D_k = D_{66} = \frac{Gh^3}{12}$$

where E_1, E_2, ν_1, ν_2, G are Young's Moduli, Poisson's ratios and shear modulus for the principal directions.
Equation 4.3.3 and equation 4.3.8 becomes:

$$\varepsilon_x = \frac{1}{E_1}(\sigma_x - \nu_1\sigma_y) \quad , \quad \sigma_y = \frac{1}{E_2}(\sigma_y - \nu_2\sigma_x)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad 4.3.11$$

$$M_x = -D_1\left(\frac{\partial^2 w}{\partial x^2} + \nu_2\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D_2\left(\frac{\partial^2 w}{\partial y^2} + \nu_1\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = -D_k\frac{\partial^2 w}{\partial x \partial y} \qquad 4.3.12$$

$$Q_x = -\frac{\partial}{\partial x}\left(D_1\frac{\partial^2 w}{\partial x^2} + D_3\frac{\partial^2 w}{\partial y^2}\right)$$

$$Q_y = -\frac{\partial}{\partial y}\left(D\frac{\partial^2 w}{\partial x^2} + D\frac{\partial^2 w}{\partial y^2}\right) .$$

The above equations for orthotropic plates are valid only if the direction of x and y axes coincide with the principal direction of elasticity.

When the plate is isotropic plate, all the rigidities are reduced to one rigidity only.:

$$D_1 = D_2 = D_3 = D = \frac{E\,h^3}{12(1-\nu^2)} \qquad 4.3.13$$

and

$$E_1 = E_2 = E \quad , \quad \nu_1 = \nu_2 = \nu$$
$$G = \frac{E}{2(1+\nu)} \qquad 4.3.14$$

Therefore for isotropic plates eq. 4.3.12 becomes:

$$M_x = - D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$
$$M_y = - D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)$$
$$M_{xy} = - D (1-\nu) \frac{\partial^2 w}{\partial x \partial y} \qquad 4.3.15$$
$$Q_x = - D \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$
$$Q_y = - D \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

4.4 Equations for Bending Analysis

By considering the equilibrium of the plate element in figure 4.3 and figure 4.4 the following equations results:

$$\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} + q(x,y) = 0 \qquad 4.4.1$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \qquad 4.4.2$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \qquad 4.4.3$$

Substituting eq.4.4.2 and eq.4.4.3 into eq. 4.4.1 and then substituting the values of M_x, M_y and M_{xy} from eq. 4.3.8 will get the governing equation for the bending of an anisotropic plate.

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4 D_{16} \frac{\partial^4 w}{\partial x \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} +$$
$$4 D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} = q(x,y) \qquad 4.4.4$$

When the plate is orthotropic, equations 4.3.10 can be applied to equation 4.4.4 which gives the governing differential equation for bending of an orthotropic plate:

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = q(x,y) \qquad 4.4.5$$

In particular for an isotropic plate with $D = D_1 = D_2 = D_3$, equation 4.4.5 becomes:

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad 4.4.6$$

4.5 Equations for Free Vibration Analysis.

The equations governing the free vibration analysis of anisotropic plates are obtained by writing the equations of motion for the elemental rectangular plate element shown in figure 4.3 and figure 4.4 or by replacing the external force of eq.4.4.4 by inertia term.

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4\ D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4\ D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} + \rho \frac{\partial^2 w}{\partial t^2} = 0 \qquad 4.5.1$$

where ρ is mass density per unit area of the plate.

When free vibrations are assumed, the motion is expressed as:

$$w = W(x,y).\cos \omega t \qquad 4.5.2$$

where ω is the circular frequency in radians/unit time. Substituting eq. 4.5.2 into eq. 4.5.1 yields:

$$D_{11} \frac{\partial^4 W}{\partial x^4} + 4\ D_{16} \frac{\partial^4 W}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 W}{\partial x^2 \partial y^2} + 4\ D_{26} \frac{\partial^4 W}{\partial x \partial y^3} + D_{22} \frac{\partial^4 W}{\partial y^4} - k^4 W = 0 \qquad 4.5.3$$

where k is a parameter of convenience defined as $k^4 = \rho\omega^2$.

For the special case of an orthotropic plate, equation 4.5.3 becomes:

$$D_1 \frac{\partial^4 W}{\partial x^4} + 2D_3 \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 W}{\partial y^4} - k^4 W = 0 \qquad 4.5.4$$

When the plate is isotropic, equation 4.5.4 becomes:

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - \frac{k^4}{D} W = 0 \qquad 4.5.5$$

4.6 Elastic Constant.

When a body possesses symmetry of internal structure, its elastic properties would show the symmetry. If an elastic body possesses symmetry, each point if the body has identical elastic properties (equivalent direction). Specimen made of natural wood, delta wood, plywood have elastic symmetries. The type of symmetry that encountered most in practice are usually homogeneous body with three mutually perpendicular planes of elastic symmetry passing through every point. The body which has that type of symmetry is called orthotropic and the equations of generalized Hooke's Law can be expressed in the following manner:

$$\begin{aligned}
\varepsilon_x &= \frac{1}{E_1}\sigma_x - \frac{\nu_{21}}{E_2}\sigma_y - \frac{\nu_{31}}{E_3}\sigma_z \;, & \gamma_{yz} &= \frac{1}{G_{23}}\tau_{yz} \\
\varepsilon_y &= -\frac{\nu_{12}}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y - \frac{\nu_{32}}{E_2}\sigma_z \;, & \gamma_{xz} &= \frac{1}{G_{13}}\tau_{xz} \qquad 4.6.1 \\
\varepsilon_z &= -\frac{\nu_{13}}{E_1}\sigma_x - \frac{\nu_{23}}{E_2}\sigma_y + \frac{1}{E_3}\sigma_z \;, & \gamma_{xy} &= \frac{1}{G_{12}}\tau_{xy}
\end{aligned}$$

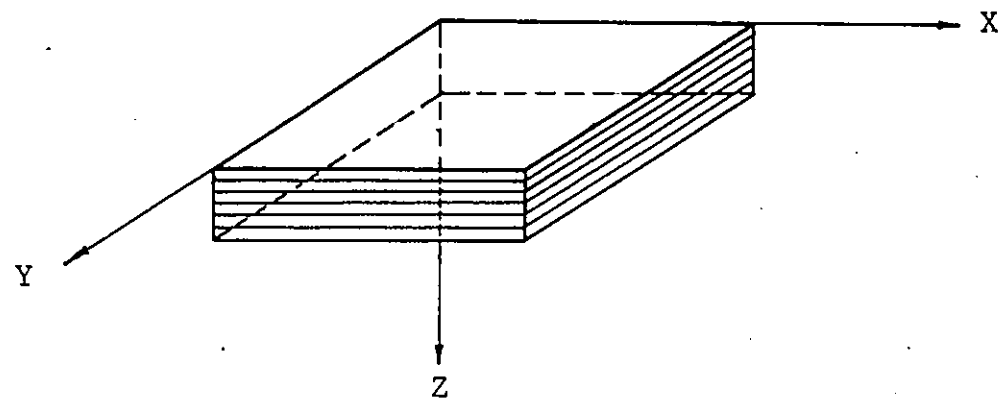
Due to the symmetry of eq. 4.6.1 , the following relations between the Young's moduli and the Poisson's ratios exists:

$$E_1\nu_{21} = E_2\nu_{12} \;,\; E_2\nu_{32} = E_3\nu_{23} \;,\; E_3\nu_{13} = E_1\nu_{31} \qquad 4.6.2$$

The following are some numerical values of the elastic constants for anisotropic plates of non-crystalline nature, namely pinewood, delta wood and plywood.

1. Natural pinewood.

Consider a rectangular plate cut off from a natural pinewood with normal annular rings.



Let the plate, the edges of which are parallel to the annular layers be subjected to generalized plane stress. Therefore,

$$
\begin{aligned}
\varepsilon_x &= \frac{1}{E_1}\sigma_x - \frac{\nu_2}{E_2}\sigma_y \\
\varepsilon_y &= -\frac{\nu_1}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y \qquad 4.6.3 \\
\gamma_{xy} &= \frac{1}{G}\tau_{xy}
\end{aligned}
$$

The numerical values of the elastic constants from equation 4.6.3 for pinewood are given below,

$$E_1 = 10^5 \text{ kg/cm}^2 \qquad E_2 = 0.042 \times 10^5 \text{ kg/cm}^2$$

$$\nu_2 = 0.01 \qquad G = 0.075 \times 10^5 \text{ kg/cm}^2$$

2. Delta wood.

Delta wood is made of wood layers impregnated with resin and compressed. After every ten layers with identical directions of the fibers, comes one layer with fibers perpendicular to the first fibers.

The average elastic constants are:

$$E_1 = 3.05 \times 10^5 \text{ kg/cm}^2 \qquad E_2 = 0.467 \times 10^5 \text{ kg/cm}^2$$

$$\nu_2 = 0.02 \qquad G = 0.220 \times 10^5 \text{ kg/cm}^2$$

3. Birch plywood

Made of odd numbers of wood layers arranged symmetrically with respect to the middle layer and glued with bakelite. The average elastic constants are:

$$E_1 = 1.2 \times 10^5 \text{ kg/cm}^2 \qquad E_2 = 0.6 \times 10^5 \text{ kg/cm}^2$$

$$\nu_2 = 0.036 \qquad G = 0.07 \times 10^5 \text{ kg/cm}^2$$

Anisotropic materials of non-crystalline nature such as pinewood, delta wood and plywood can be considered as homogeneous and orthotropic for the first approximation. The average elastic constants for such materials are generally obtained through various approved testing procedures.

CHAPTER V

Application to Some Problems

5.1 Static and Dynamic Analysis of Clamped Edged Beam by Orthogonal Collocation Method.

There are six cases which involve the possible combinations of clamped, simply supported, and free edge conditions. In static problems only four cases are considered. In what follows, two illustrative examples are presented using the orthogonal collocation method. They are transverse deflections of a clamped edged beam and free vibration of a clamped edged beam. The results for other boundary conditions are also in good agreement with corresponding results obtained by other investigators.

5.1.1 Deflection of a clamped edged beam.

Consider the dimensionless governing differential equation for beams,

$$\frac{d^2}{d\xi^2}\left(EI \frac{d^2y}{d\xi^2} \right) = \frac{qL^4}{16} \qquad 5.1.1$$

The boundary conditions for the beam in figure 5.1. are

$$y = 0 \qquad @ \ \xi = \pm 1$$

$$\frac{dy}{d\xi} = 0 \qquad @ \ \xi = \pm 1$$

The assumed function is :

$$y(\xi) = (1-\xi^2)^2 \sum_{i=0}^{n-1} A_i \, P_i(\xi^2) \qquad 5.1.2$$

which satisfies the boundary condition.

For $n = 1$, $P_0(\xi^2) = 1$ and from table A.4

Collocation point is 0.377964473

The assumed function, i.e. equation 5.1.2 for $n = 1$ become

$$y(\xi) = A_0 \; (1-2\xi^2+\xi^4)$$

Substituting $y(\xi)$ into equation 5.1.1 gives

$$(\; EI \; 24 \; A_0 \;) = \frac{qL^4}{16} \quad \text{or}$$

$$A_0 = \frac{qL^4}{384EI}$$

Substituting A_0 back into the assumed function, the deflection, moments, and hence stresses at any section of the beam can be determined.

Maximum deflection occurs at the center, $\xi = 0$.

$$y(0) = \frac{qL^4}{384EI} \; (1 - 0 - 0)$$

$$y_{max} = \frac{qL^4}{384EI} \qquad \text{(exact answer)}$$

There are many other methods that will yield the same results, this example is for illustration purposes only.

5.1.2. Free vibration of a clamped edged beam

Consider the governing differential equation from equation 4.1.9;

$$\frac{d^4 y(\xi)}{d\xi^4} - k^4 y(\xi) = 0 \qquad 5.1.3$$

The boundary conditions and the assumed function are the same as in the previous case.

For n = 2

$$y(\xi) = (1-\xi^2)^2 |A_0 P_0 \quad A_1 P_1|$$
$$= (1-\xi^2)^2 |A_0 (1) \quad A_1 (1-7\xi^2)| \qquad 5.1.4$$

Collocation points are 0.2505628078 and 0.6947465906 by substitute Eq. 5.1.4 into the governing differential equation and the collocation points,

$$24A_0 + 201.7901A_1 = k^4 (0.8784A_0 + 0.4924A_1) \qquad 5.1.5$$
$$24A_0 - 856.3355A_1 = k^4 (0.2676A_0 - 0.6366A_1) \qquad 5.1.6$$

Putting the equations 5.1.5 and 5.1.6 in matrix form and solving for k, A_0 and A_1

$$\begin{bmatrix} 24 & 201.7901 \\ 24 & -856.3355 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} = k^4 \begin{bmatrix} 0.8784 & 0.4924 \\ 0.2676 & -0.6366 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} \qquad 5.1.7$$

Using a simple computer program, k^4 value from Eq. 5.1.7 is found to be:

$$k^4 = 31.2874$$

From equation 4.1.10 ,

$$k^4 = \frac{\rho\omega^2 (L/2)^4}{EI}$$

Therefore

$$\omega = \frac{(4.7301)^2}{L^2} \left(\frac{EI}{\rho} \right)^{\frac{1}{2}}$$

Comparing it to available result using more complicated method, the error is insignificant, 0.002%. For n = 3 and n = 4 results yield the exact answer.

5.2 Static and Dynamic Analysis of Rectangular Clamped Isotropic and Orthotropic plates.

The derivation of orthogonal collocation polynomials and their roots is based on a one dimensional problem, but the method can be easily extended to two dimensional problems. To demonstrate this, static and dynamic linear analysis of plates will be solved in the following sections using the present method. In order to meet the requirements at the boundary, other orthogonal polynomial sets have to be formulated, since the weight function in the orthogonality relation, equation 3.1.2, must be replaced by a suitable function to meet the various requirements of a particular boundary value problems.

5.2.1 Static analyses of Isotropic rectangular plates

To verify the analogy between the orthogonal collocation method and the Galerkin method, and to prove its validity, the torsion of rectangular barsare considered to serve as an illustrative example by Chan [3] . The results obtained for one term solution agree very favourably with Timoshenko and also identical with the one term solution of the Galerkin method and Ritz method.

To demonstrate the ability and the simplicity of the orthogonal collocation method in handling higher order complex differential equations, the step by step calculation of deflection of an isotropic rectangular plate are shown in the following. For the plate considered here, the effect of the elastic foundation will not be accounted for, though this effect can easily incorporated in the solution. The well known expression of governing differential equation of plates is:

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 dy^2} + \frac{\partial^4 W}{\partial y^4} = \frac{P_z(x,y)}{D}$$

For convenience purposes it is better to work with dimensionless form. See figure 5.2 for the geometry of the plate in Cartesian coordinate system. Introducing non-dimensional parameters, $\xi = x/a$, $\eta = y/b$ and $R = b/a$. Utilizing the chain rule of differentiation, then the dimensionless differential equation can be written as:

$$R^4 \frac{\partial^4 W}{\partial \xi^4} + 2R^2 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 W}{\partial \eta^4} = \frac{P_z(\xi,\eta)}{D} b^4 \qquad 5.2.1$$

For the present method, an admissable solution of equation (5.2.1) for clamped edges all sides can be taken as:

$$W = (1-\xi^2)^2(1-\eta^2)^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} A_{ij}\ P_i(\xi^2)\ P_j(\eta^2) \qquad 5.2.2$$

and the collocation points where the residual is to be set equal to zero are combinations of the roots of $P_n(\xi^2)$ and $P_n(\eta^2)$.

For the particular case of square plate under a uniform distributed load, $P_z(\xi,\eta)=q$, and a one term solution, i.e, n=1, substituting equation (5.2.2) into equation (5.2.1) result in

$$A_{00}|24(1-\eta^2)^2+2(12\xi^2-4)(12\eta^2-4)+24(1-\xi^2)^2| = \frac{q}{D} b^4 \qquad 5.2.3$$

Since there is only one unknown, that is A_{00}, by setting equation (5.2.3) to zero at a single collocation point will yield the solution of this particular case. From table A.4., the roots of P_1 (ξ^2) is $1/\sqrt{7}$, hence the location of the collocation point is $(1/\sqrt{7}, 1/\sqrt{7})$. Substituting the ξ and η coordinate of this collocation point into 5.2.3 gives $A_{00} = 0.02023\frac{qb^4}{D}$ from which the maximum deflection which occur at $\xi=\eta=0$ is $0.02023\frac{qa^4}{D}$. This value has good agreement with Timoshenko's $0.02016\frac{qa^4}{D}$, considering the crudeness of the one term solution used.

To investigate the convergence of the orthogonal collocation method, results are obtained for n=2, n=3 and n=4, i.e, 4 term, 9 term and 16 term solution. These results are shown in table 3a.

As can be seen from the results presented, the convergence is very consistent, and the agreement of these results with those of Timoshenko is excellent.

5.2.2 Dynamic analyses of Isotropic rectangular plates

The free vibration analysis of rectangular plates have been studied until this time by methods of various kinds because of theoretical interest and practical importance. Problems frequently encountered by the structural design engineer is to find the stresses, frequency, modeshapes and deflections in cases of irregular boundary shapes and arbitrary transverse loadings. It is well known that the classical, exact methods of analytical solution cannot be applied to the above problem with any reasonable degree of generality, and that approximate mathematical techniques must be used if realistic answers are to be obtained.

The author introduced the orthogonal collocation method to the free vibration analysis of rectangular plates because from the solution of illustrative examples at the previous section, orthogonal collocation is seen to be an accurate yet simple numerical method for the solution of various boundary problems. As an illustrative example the calculation of the natural frequency of isotropic rectangular plates clamped on all sides is shown below. Consider the governing differential equation for free vibration of isotropic rectangular plates, eq. 4.5.5,

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - \frac{\rho\omega^2}{D} W = 0 \qquad 5.2.4$$

Using dimensionless form, $\xi = x/a$, $\eta = y/b$, and $R = b/a$ and utilizing the chain rule in differentiation, then the dimensionless differential equation can be written as follow,

$$R^4\frac{\partial^4 W}{\partial \xi^4} + 2R^2\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 W}{\partial \eta^4} - k^4 W = 0 \qquad 5.2.5$$

where $\quad k^4 = \frac{\rho}{D}\,\omega^2\, b^4 \qquad 5.2.6$

Since the plate boundary condition is the same as section 5.2.1, the function in eq.5.2.2 can be used as trial function.

For a particular case of square plate and a two term solution, that is n = 2.

$$W = (1-\xi^2)^2(1-\eta^2)^2 |A_{00} + A_{01}(1-7\eta^2) + A_{10}(1-7\xi^2) + A_{11}(1-7\xi^2)(1-7\eta^2)| \qquad 5.2.7$$

The collocation points are combination of the roots of $P_n'(\xi^2)$ and $P_n'(\eta^2)$ in this case $P_2'(\xi^2) = 1-18\xi^2+33\xi^4$ and $P_2'(\eta^2) = 1-18\eta^2+33\eta^4$. The roots are 0.25056280708 and 0.69474659060.

The collocation points are:

(0.25056280708, 0.25056280708)

(0.25056280708, 0.69474659060)

(0.69474659060, 0.25056280708)

(0.69474659060, 0.69474659060)

Then substituting equation 5.2.7 into equation 5.2.5 yields

$$\begin{aligned}
&A_{00}\,|24R^4(1-\eta^2)^2+2R^2(-4+12\xi^2)(-4+12\eta^2)+24(1-\xi^2)^2| + \\
&A_{01}\,|R^4(360-2520\xi^2)(1-\eta^2)^2+2R^2(-18+180\xi^2-210\xi^4)(-4+12\eta^2) \\
&\quad +24(1-9\xi^2+15\xi^4-7\xi^6)| + \\
&A_{10}\,|24R^4(1-9\eta^2+15\eta^4-7\eta^6)+2R^2(-4+12\xi^2)(-18+180\eta^2-210\eta^4) \\
&\quad +(1-\xi^2)^2(360-2520\eta^2)| + \\
&A_{11}\,|R^4(360-2520\xi^2)(1-9\eta^2+15\eta^4-7\eta^6)+2R^2(-18+180\xi^2-210\xi^4) \\
&\quad (-18+180\eta^2-210\eta^4)+(360-2520\eta^2)(1-9\xi^2+15\xi^4-7\xi^6)| = k^4 \{ \\
&A_{00}\,|(1-\xi^2)^2(1-\eta^2)^2| + A_{01}\,|(1-\xi^2)^2(1-\eta^2)^2(1-7\eta^2)| + \\
&A_{10}\,|(1-\xi^2)^2(1-\eta^2)^2(1-7\xi^2)| + A_{11}\,|(1-\xi^2)^2(1-\eta^2)^2(1-7\eta^2)(1-7\xi^2)| \}
\end{aligned} \qquad 5.2.8$$

Substitute collocation points and the aspect ratio into equation 5.2.8 which becomes an algebraic eigenvalue problem in the form $|C||A| = k^4|A|$ 5.2.9

It is clear that the zero vector $|A| = 0$ is a solution of eq.5.2.9 for any value of k^4. A value of k^4 for which eq.5.2.9 has a solution $|A| \neq 0$ is called eigenvalue of the matrix $|C|$. The corresponding solutions $|A| \neq 0$ of eq. 5.2.9 are called eigenvectors of $|C|$. The lowest value of eigenvalues correspond to the natural frequency. For a square plate R = 1, the lowest eigenvalues for the above illustration is

$$k^4 = \frac{\rho\omega^2 b^4}{D} = \frac{\rho\omega^2 a^4}{D} R^4 = 80.8807 \qquad 5.2.10$$

Leissa (19) uses the constant a = the length of the plate while the author choosed a = half the length of the plate. For the purpose of comparison the value at the right hand side of equation 5.2.10 can be multiply by 16. Therefore the natural frequency of a square homogeneous isotropic plate is:

$$\omega a^2 \left(\frac{\rho}{D} \right)^{\frac{1}{2}} = 35.973$$

The result using only 2 terms (i.e. n = 2) has discrepancy less than 0.1%. The results using more terms, also for various aspect ratio are presented in a tabular and graphical forms.

5.2.3 Static Analysis of Orthotropic Rectangular Plates.

Bending analysis of orthotropic rectangular plates with all edges fixed are presented below using 4 collocation points. For the sake of accuracy, all calculations are carried out using double precision arithmetic.

To demonstrate the ability and the simplicity of the orthogonal collocation method in handling higher order complex differential equations, the step by step calculation of bending of an orthotropic rectangular plate are shown in the following. For the plate considered here, the effect of the elastic foundation will not be accounted for, though this effect can be easily incorporated in the solution. The known expression of the governing differential equation of orthotropic plates is:

$$D_{11}\frac{\partial^4 W}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 W}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 W}{\partial y^4} = q \tag{5.2.11}$$

For convenience purposes it is better to work with dimensionless form. Introducing non-dimensional parameters $\xi = x/a$, $\eta = y/b$ and $R = b/a$ and utilizing the chain rule of differentiation, the dimensionless differential equation can be written as:

$$\frac{D_{11}}{D_{22}} R^4 \frac{\partial^4 W}{\partial \xi^4} + 2R^2\left(\frac{D_{12}}{D_{22}} + 2\frac{D_{66}}{D_{22}}\right)\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} +$$

$$\frac{\partial^4 W}{\partial \eta^4} = \frac{q\, b^4}{D_{22}} \tag{5.2.12}$$

For the present method, an admissable solution to equation 5.2.12 for clamped edges all sides can be taken as:

$$W(\xi,\eta) = (1-\xi^2)^2(1-\eta^2)^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} A_{ij} \; P_i(\xi^2) \; P_j(\eta^2) \qquad 5.2.13$$

For a particular case of square plate (R = 1) under a uniformly distributed load q, and a two term solution, that is n = 2, and using the elastic constants :

$$\frac{D_{11}}{D_{22}} = 40 \qquad \frac{D_{12}}{D_{22}} = 0.25 \qquad \frac{D_{66}}{D_{22}} = 0.5 \qquad 5.2.14$$

where

$$\begin{aligned} P_0(\xi^2) &= P_0(\eta^2) = 1.0 \\ P_1(\xi^2) &= 1 - 7\xi^2 \\ P_1(\eta^2) &= 1 - 7\eta^2 \end{aligned} \qquad 5.2.15$$

and the collocation points are:

(0.2505622807, 0.2505622807) , (0.2505622807, 0.6947465906)

(0.6947465906, 0.2505622807) , (0.2505622807, 0.6947465906)

Then substituting eq. 5.2.13 into eq. 5.2.12 yield:

$$\begin{aligned}
&A_{00}\,|40R^4 24(1-\eta^2)^2+1.25R^2 2(-4+12\xi^2)(-4+12\eta^2)+24(1-\xi^2)^2| \; + \\
&A_{01}\,|40R^4(360-2520\xi^2)(1-\eta^2)^2+1.25R^2 2(-18+180\xi^2-210\xi^4)(-4+12\eta^2) \\
&\quad +24(1-9\xi^2+15\xi^4-7\xi^6)| \; + \\
&A_{10}\,|40R^4 24(1-9\eta^2+15\eta^4-7\eta^6)+1.25R^2 2(-4+12\xi^2)(-18+180\eta^2-210\eta^4) \\
&\quad +(1-\xi^2)^2(360-2520\eta^2)| \; + \\
&A_{11}\,|40R^4(360-2520\xi^2)(1-9\eta^2+15\eta^4-7\eta^6)+1.25R^2 2(-18+180\xi^2-210\xi^4) \\
&\quad (-18+180\eta^2-210\eta^4)+(360-2520\eta^2)(1-9\xi^2+15\xi^4-7\xi^6)| \; = \; \frac{qb^4}{D_{22}}
\end{aligned} \qquad 5.2.16$$

Substitute collocation points and result in 4 equation with 4 unknown, which are A_{00} , A_{01} , A_{10} , and A_{11} . This can be put in a matrix form $|C|\ |A| = |I|\ \frac{qb^4}{D_{22}}$. By multiplying both side with the inverse of matrix $|C|$, matrix $|A|$ is computed. Put the A_{ij} back into equation 5.2.13. For maximum deflection of the plate $\xi = 0$ and $\eta = 0$. By substituting this into the equation, $W(\xi,\eta)_{max.}$ can be found in terms of $\frac{qb^4}{D_{22}}$.
The result using only 2 terms has discrepancy less than 1%. The results using larger terms, also for various aspect ratio are presented in a tabular and graphical forms. These quantities represent typical properties of high modulus graphite-epoxy fiber reinforced composite materials.

The solution for plates with other types of boundary conditions are also obtained by the method of orthogonal collocation. Figures 6.6 through 6.9 show the graphs of central deflection, W, and bending moment, M_x, as functions of plate aspect ratio.

CHAPTER VI

DISCUSSIONS AND CONCLUSSIONS

The Collocation method developed here is seen to be an accurate yet simple numerical method for the solution of difficult boundary value problems.

By choosing the collocation points in a well defined manner, very accurate results can be obtained even with a 4 - term (4 collocation points) solution, a feat practically impossible by the conventional collocation method, and with a nine term (9 collocation points) solution, the results are almost identical if not better than those obtained by the collocation least square method which uses a large number of points for a solution.

The Orthogonal Collocation method differs from other weighted residual methods in that the residual method here is not directly orthogonalized, but is matched to an orthogo nal function at its zeroes. The tedious task of integrating the residual is thereby avoided, and the calculations are co respondingly simplified.

The vital part of the solution lies in the formulation of the orthogonal polynomials. Once these polynomials are formulated and their zeroes obtained, the solution of a

problem becomes very straight forwards.
It can be said that in addition to its simplicity in application, the orthogonal collocation method has an accuracy comparable to other weighted residual methods, and as such, can be used as a convenient tool in the numerical treatment of very complex boundary value problems.

Appendix A.

Orthogonal polynomials using trigonometric function.

In the process of searching for orthogonal polynomials the author also tried trigonometric functions as orthogonal polynomials. The orthogonal polynomial trigonometric func - tion are more powerful than the orthogonal polynomial using geometric function due to the ability of the orthogonal polynomial to help the weight function to satisfy boundary condition exactly, which will be illustrated in the following.

Let expression 3.1.2 equal to two separate functions

$$Y = \bar{w}(x^2) \sum_{i=0}^{n-1} A_i \; P_i(x^2) = \bar{w}(x^2) \quad s(x^2) \qquad \text{A.1}$$

Simply supported boundary condition will be used to illustra- te the example.

$$Y = 0 \qquad \text{at} \quad x = \pm\, 1.0 \qquad \text{A.2}$$

$$\frac{d^2Y}{dx^2} = 0 \qquad \text{at} \quad x = \; 1.0 \qquad \text{A.3}$$

Differentiate eq. A.1 with respect to x^2 yields

$$\frac{d^2Y}{dx^2} = \bar{w}\,\frac{d^2s}{dx^2} + 2\,\frac{d\bar{w}}{dx}\,\frac{ds}{dx} + \frac{d^2\bar{w}}{dx^2}\,s \qquad \text{A.4}$$

The weight function $\bar{w}(x^2)$ have to satisfied boundary condi -

tions, in this case :

$$\bar{w}(x^2) = 0 \qquad \text{at} \quad x = \pm 1.0 \tag{A.5}$$

$$\frac{d\,\bar{w}(x^2)}{dx^2} = 0 \qquad \text{at} \quad x = \pm 1.0 \tag{A.6}$$

But as it can be observed from equation A.4, in order Y to satisfy boundary condition A.3 (which is the condition of interior collocation to satisfy boundary conditions exactly), the three groups have to equal zero, that is ;

$$\bar{w}\,\frac{d^2 s}{dx^2} = 0 \quad , \quad \frac{d\bar{w}}{dx}\,\frac{ds}{dx} = 0 \quad , \quad \frac{d^2\bar{w}}{dx^2}\,s = 0 \tag{A.7}$$

Using A.5 and A.6 , now the choices are $\frac{d\bar{w}}{dx} = 0$ at $x = \pm 1$ or $\frac{ds}{dx} = 0$ at $x = \pm 1$. It is not required in the boundary condition considered in this example for $\bar{w}(x^2)$ to satisfy $\frac{d\bar{w}}{dx} = 0$ at $x = \pm 1$. Therefore $\frac{ds}{dx} = 0$ at $x = \pm 1$ is used as a condition to get the orthogonal polynomial.

The selection of geometric polynomials to satisfy the above condition and orthogonality at the same time are restricted only to a few cases of boundary conditions, while the trigonometric polynomials are more flexible but the formulation of the polynomials are considerably longer compared to geometric polynomials due to many integration of trigonometric functions.

Assumed : $P_0 = 1$

$P_1 = 1 + a_1 \cos x\pi$

$P_2 = 1 + a_2 \cos x\pi + a_3 \cos^2 x\pi$

Let $$T_j = \int_0^1 \bar{w}(x^2) \quad [\cos x\pi]^{j-1} \, dx \qquad \text{A.8}$$

for $j = 1, 2, \ldots$

The weight function $\bar{w}(x^2) = x^4 - 6x^2 + 5$ satisfied the boundary conditions A.5 and A.6 .

For j = 1,2,3,4,5 expression A.8 become

$T_1 = 3.2$

$T_2 = 1.056953$

$T_3 = 1.490979$

$T_4 = 0.815991$

$T_5 = -1.911744$

Applying the same technique as section 3.4 yield polynomial as follows ;

$P_0 = 1$

$P_1 = 1 - 3.027570 \cos x\pi$

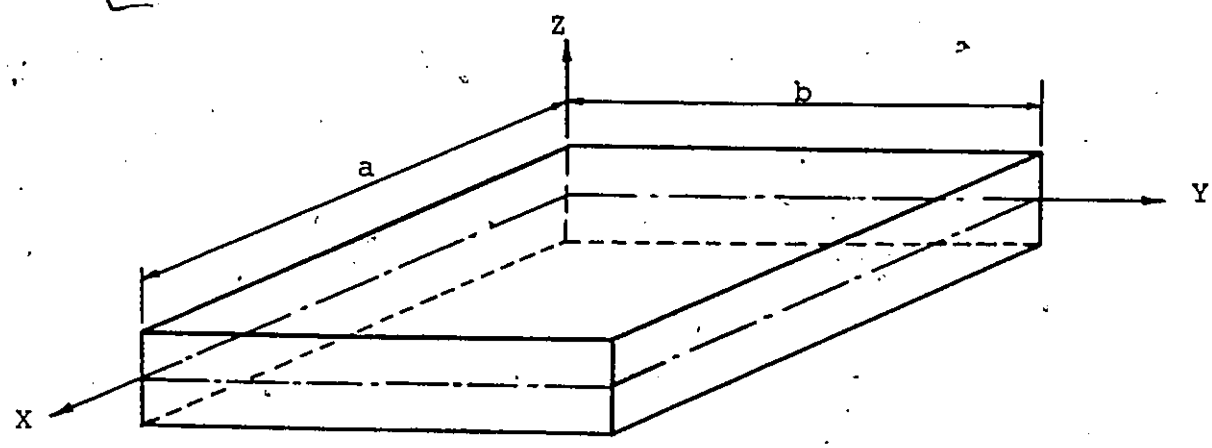
$P_2 = 1 + 0.760921 \cos x\pi - 2.685656 \cos^2 x\pi$

and the roots are ; 0.3928508149 for n = 1

0.2212043140 for n = 2

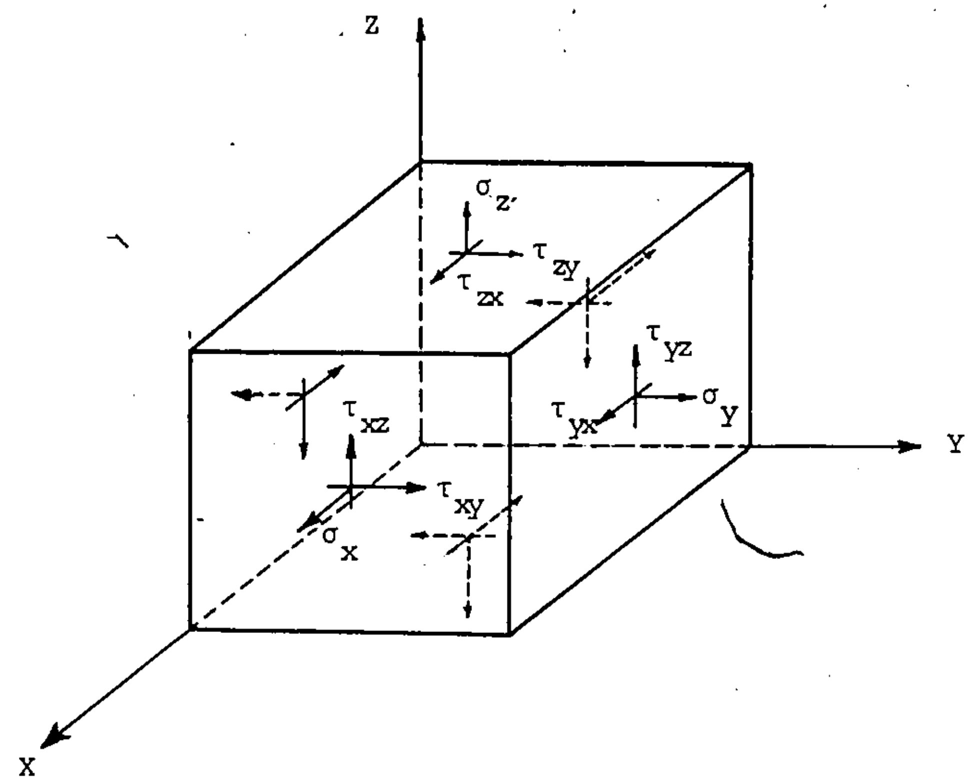
0.6610959681

Using this orthogonal polynomial to solve problems such as free vibration of beams with simply supported edges yields excellent agreements compared with available data in the literature. In formulating the orthogonal polynomials for unsymmetrical problems the variable x^2 shall be substituted by variable u and all the derivatives shall be changed accordingly. (36)



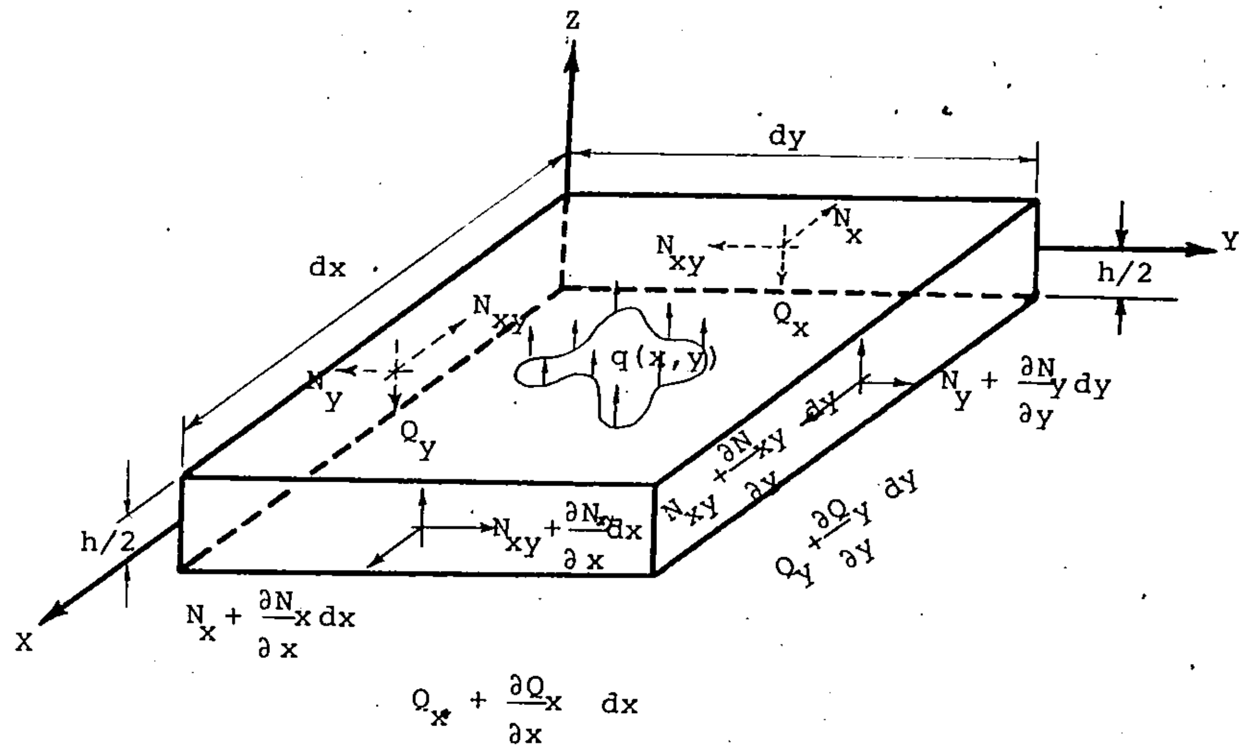
The middle surface of a plate element

Figure 4.1



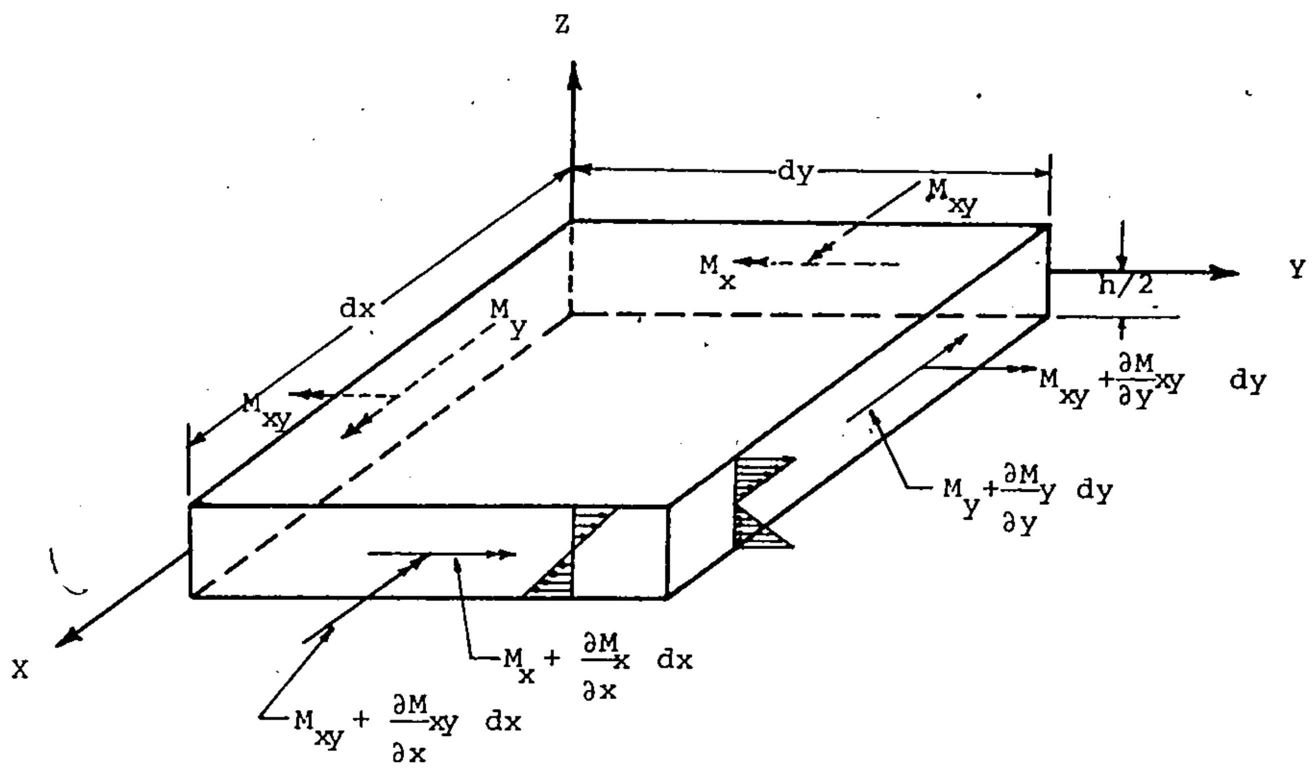
Direction of positive stresses on a plate element

Figure 4.2



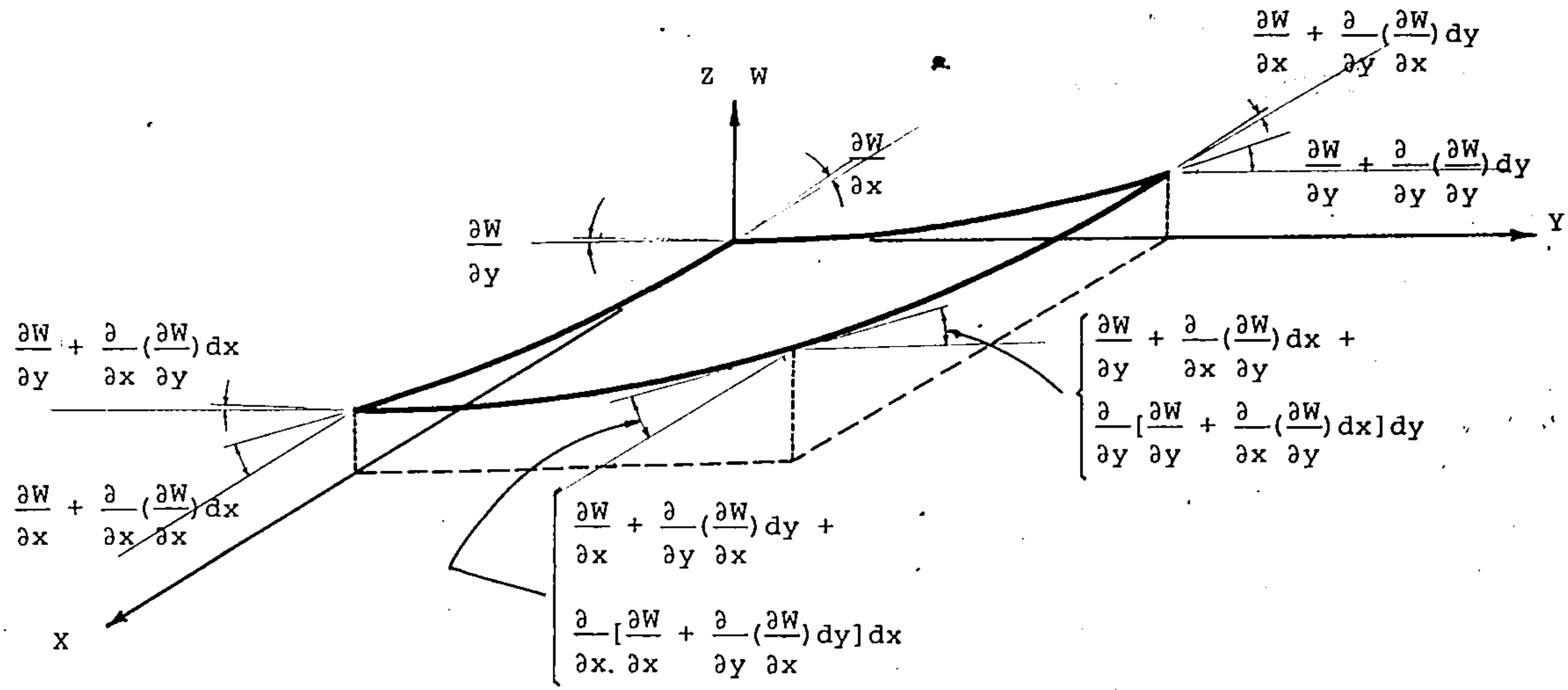
Incremental force resultant on a plate element

Figure 4.3



Incremental moment resultants on a plate element

Figure 4.4



Deformed configuration of the middle

surface of a plate

Figure 4.5

Clamped edged beam

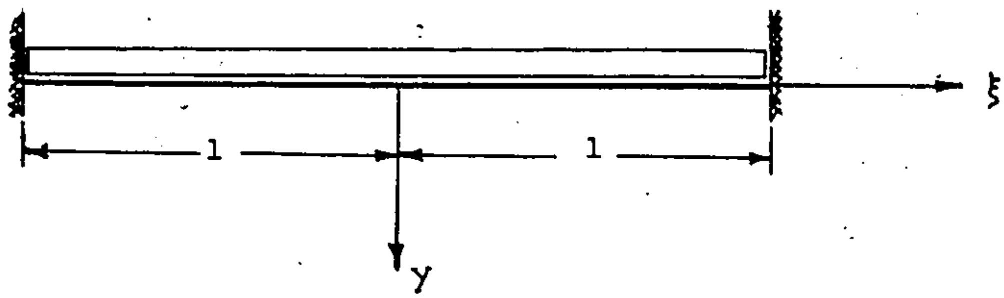


Figure 5.1

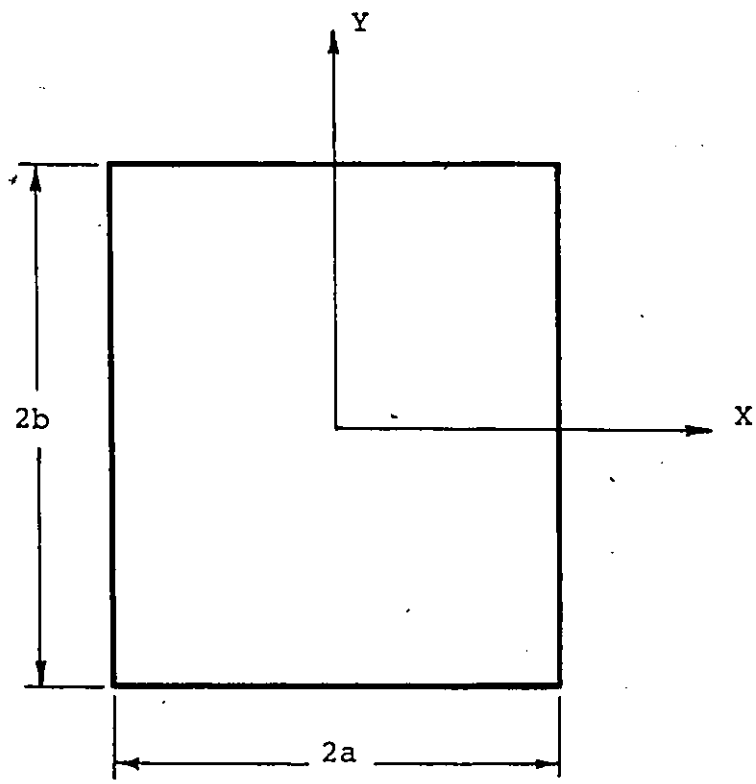


Figure 5.2

The geometry of a plate in Cartesian coordinate system.

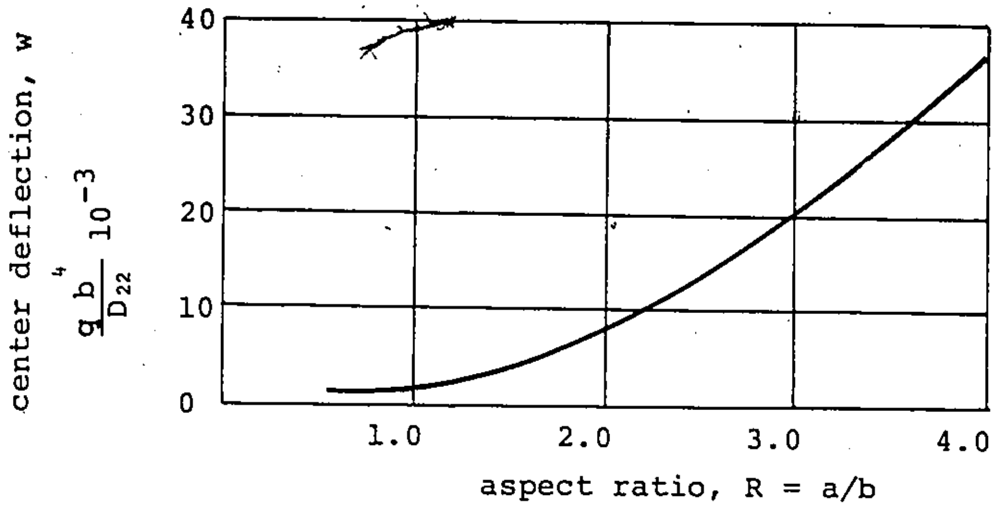


Figure 6.6 Center deflection of a rectangular orthotropic plate clamped all edges

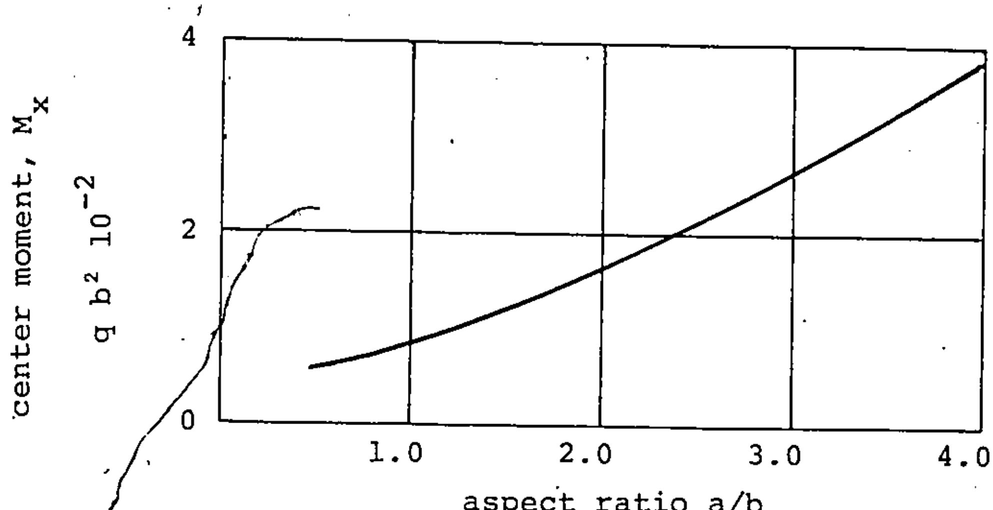


Figure 6.7 Center moment M_x of a rectangular orthotropic plate clamped all edges

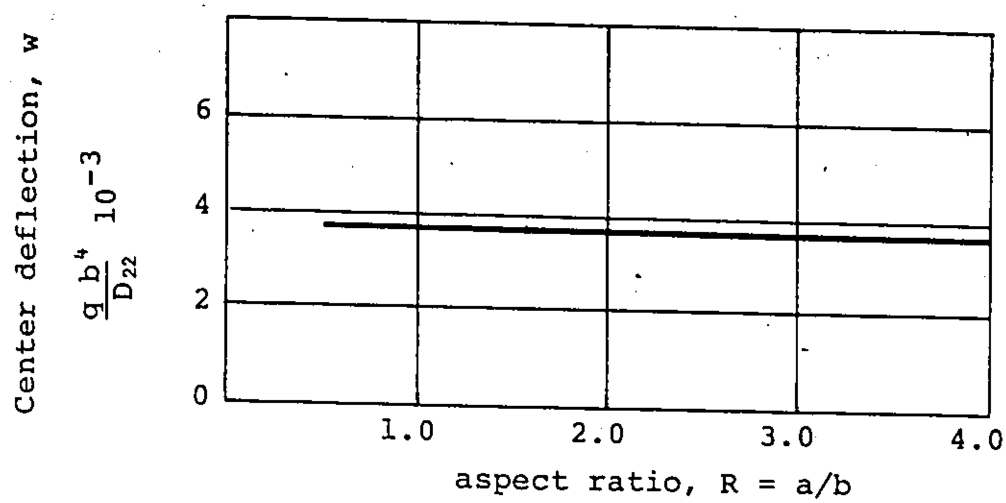


Figure 6.8 Center deflection of a rectangular orthotropic plate with boundary FCFC

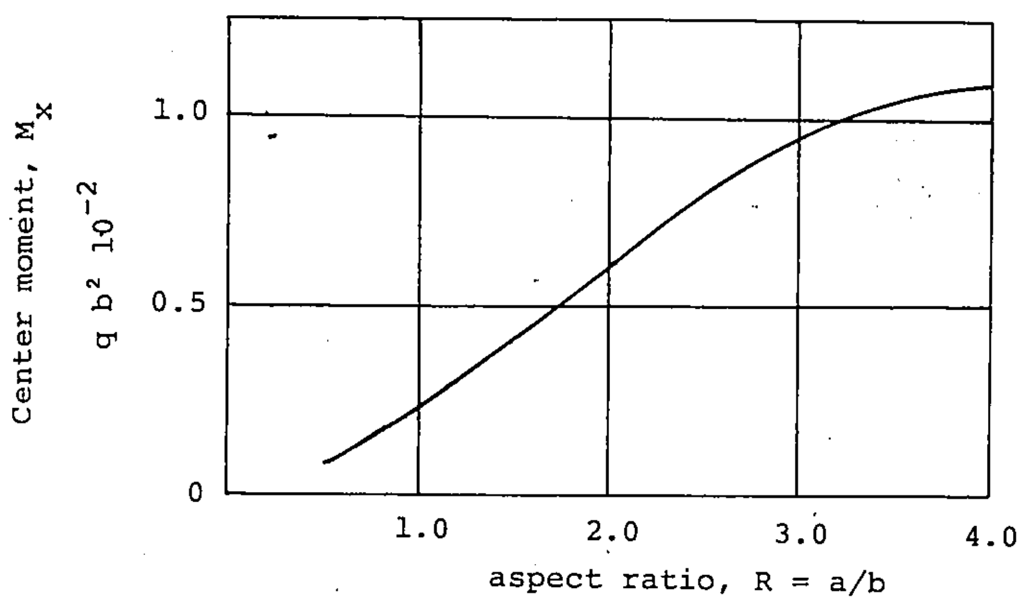


Figure 6.9 Center moment M_x of a rectangular orthotropic plate with boundary FCFC

BOUNDARY FUNCTION
FOR
CLAMPED ** CLAMPED

COEFFICIENTS	X - POWER
1.0	0.0
-2.0	2.0
1.0	4.0

COEFFICIENTS OF ORTHOGONAL POLYNOMIALS

C(1)	=	-0.7000000000000000D 01
C(2)	=	~~-~~0.1800000000000000D 02
C(3)	=	0.3300000000000000D 02
C(4)	=	-0.3300000000000000D 02
C(5)	=	0.1430000000000000D 03
C(6)	=	-0.1430000000000000D 03
C(7)	=	-0.5200000000000000D 03
C(8)	=	0.3900000000000000D 03
C(9)	=	-0.8840000000000000D 03
C(10)	=	0.5998571428563779D 03
ALPHA(1)	=	0.5333333333333333D 00
ALPHA(2)	=	0.7111111111111109D 00
ALPHA(3)	=	0.7501831501831482D 00
ALPHA(4)	=	0.7648926237161255D 00
Alpha(5)	=	0.7719881025927293D 00

TABLE A.1.

```
********************************
*   BOUNDARY   FUNCTION        *
*                              *
*          FOR                 *
*                              *
*   CLAMPED  *  SIMPLY         *
********************************
```

COEFFICIENTS	X - POWER
1.0	2.0
-3.0	4.0
3.0	6.0
-1.0	8.0

```
****************************************************
*                                                  *
*  COEFFICIENTS  OF  ORTHOGONAL  POLYNOMIALS       *
*                                                  *
****************************************************
```

C(1)	=	-0.3666666666666666D 01
C(2)	=	-0.8666666666666666D 01
C(3)	=	0.1299999999999986D 02
C(4)	=	-0.1499999999999429D 02
C(5)	=	0.5099999999999678D 02
C(6)	=	-0.4612485714281991D 02
C(7)	=	-0.2266666666652752D 02
C(8)	=	0.1291999999986653D 03
C(9)	=	-0.2583999999996533D 03
C(10)	=	0.1650888888862808D 03

ALPHA(1)	=	0.5079365079365079D-01
ALPHA(2)	=	0.2083842083842070D-01
ALPHA(3)	=	0.1158929217751501D-01
ALPHA(4)	=	0.7422962524968878D-02
ALPHA(5)	=	0.5172980106053159D-02

TABLE A.2.

```
*******************************
*                             *
*   BOUNDARY    FUNCTION      *
*              FOR            *
*   SIMPLY  **  SIMPLY        *
*******************************
```

COEFFICIENTS	X - POWER
1.0	0.0
-3.0	2.0
3.0	4.0
-1.0	6.0

```
**************************************************
*                                                *
*  COEFFICIENTS  OF  ORTHOGONAL POLYNOMIALS      *
*                                                *
**************************************************
```

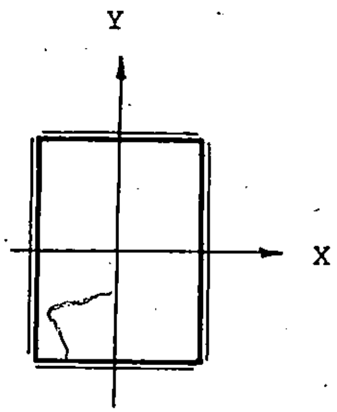
```
   C( 1)   =  -0.9000000000000000D 01
   C( 2)   =  -0.2200000000000000D 02
   C( 3)   =   0.4766666666666697D 02
   C( 4)   =  -0.3900000000000000D 02
   C( 5)   =   0.1950000000000000D 03
   C( 6)   =  -0.2210000000000000D 03
   C( 7)   =  -0.6000000000000000D 02
   C( 8)   =   0.5100000000000000D 03
   C( 9)   =  -0.1292000000000000D 04
   C(10)   =   0.9690000000000000D 03

ALPHA( 1)  =   0.4571428571428571D 00
ALPHA( 2)  =   0.6649350649350660D 00
ALPHA( 3)  =   0.7223985890652727D 00
ALPHA( 4)  =   0.7465937571210510D 00
ALPHA( 5)  =   0.7590786025751299D 00
```

TABLE A.3.

ROOTS OF POLYNOMIALS

N	CLAMPED CLAMPED	SIMPLY SIMPLY	CLAMPED SIMPLY
1	0.3779644730092272	0.3333333333333333	0.5222329678670938
2	0.2505628070857315	0.2260876561655180	0.3852703828805491
	0.6947465906068656	0.6406425159697435	0.7198842953848514
3	0.1886774224907856	0.1731567539180992	0.3071626846160933
	0.5406046373873581	0.5009134932118675	0.5872564900764341
	0.8198459954634863	0.7755355259865654	0.8161152234442589
4	0.1516316642933045	0.1408897472252969	0.2559426475677000
	0.4414329761086064	0.4122676659700963	0.4958599884043239
	0.6920606182569244	0.6532727259932570	0.7047978089790968
	0.8814085756174592	0.8466129750531484	0.8701112991003173

TABLE A.4.

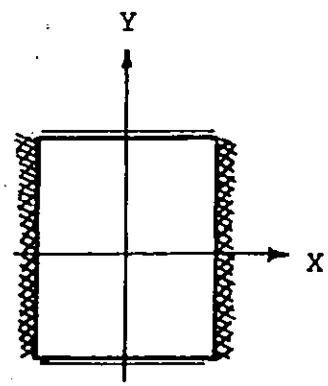


SIMPLY SUPPORTED EDGES

Orthogonal Collocation for N = 2, and
employs EIGZF Subroutine to yield k

$R = {}^{b}/_{a}$	1.0	1.5	2.0	2.5	3.0
k^4	27.5761	75.8670	189.9466	416.3372	814.8557
$\omega a^2 \frac{\rho}{D}$					
ORTH. COLL.	21.0052	15.4847	13.7821	13.0588	12.6869
	21.0052	15.4847	13.7821	13.0588	12.6969
IGUCHI	19.74	14.26	12.34	11.45	10.97
EIGENVECTORS					
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.0162	0.0119	0.0095	0.0081	0.0072
3	0.0162	0.0206	0.0231	0.0246	0.0255
4	0.0002	0.0002	0.0002	0.0002	0.0002

Table 1 : Natural Frequencies of Rectangular Plates and Eigenvectors.



CLAMPED AND SIMPLY SUPPORTED EDGES

Orthogonal Collocation for N = 2 , and employs EIGZF Subroutine to yield k

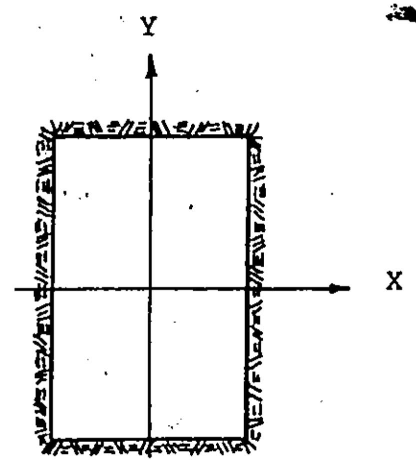
$R = {}^{b}/_{a}$	1.0	1.5	2.0	2.5	3.0
k^4	53.9215	199.6628	568.5379	1324.8823	2680.0741
$\omega a^2\sqrt{\frac{\rho}{D}}$					
ORTHO. COLL.	29.3725	25.1204	23.8440	23.2953	23.0086
IGUCHI	28.95	25.05	23.82	23.27	22.99
LEISSA	28.946	24.047	23.814	23.271	22.985
EIGENVECTORS					
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.0160	0.0112	0.0087	0.0074	0.0067
3	0.0018	0.0136	0.0180	0.0201	0.0212
4	-0.0012	-0.0003	-0.0000	0.0000	0.0000

Table 2 : Natural Frequencies and Eigenvectors of Rectangular Plates with Two Opposite Edges Simply Supported and the Other Edges Clamped.

Aspect Ratio b/a	Coefficients α for maximum deflection.			
	4 terms	[illegible] terms	16 terms	Timoshenko ref (31)
1.0	0.02023	0.02024	0.02024	0.02016
1.25	0.02911	0.029152	0.029153	n a
1.5	0.03501	0.035143	0.03514	0.03520
1.75	0.03833	0.03866	0.03866	n a
2.0	0.03987	0.04054	0.04053	0.04064
4.0	0.03635	0.04306	0.04157	———

$$W_{max} = \alpha q a^4/D$$

Table 3a: Coefficient α for maximum small deflection of clamped isotropic homogeneous rectangular plates.

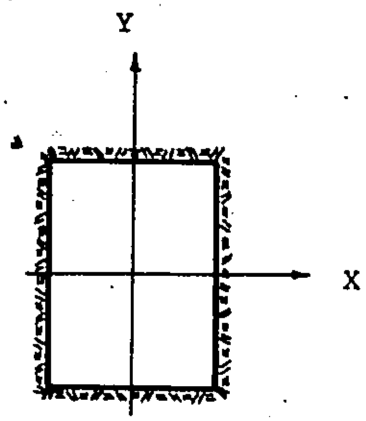


CLAMPED EDGES

Orthogonal Collocation for N=2 ,and employs EIGZF Subroutine to yield k^4

$R = {}^b/a$	1.0	1.5	2.0	2.5	3.0
k^4	80.8807	230.6838	604.0676	1364.9426	2724.3870
$\omega a^2\sqrt{\frac{\rho}{D}}$	35.9735	27.0014	24.5778	23.6449	23.1981
IGUCHI	35.98	27.00	24.57	23.77	23.19
YOUNG	35.99				
EIGENVECTOR					
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.0125	-0.0196	-0.0565	-0.0917	-0.1231
3	0.0125	0.0230	0.0239	0.0237	0.0235
4	-0.0282	-0.0216	-0.0141	-0.0098	-0.0076

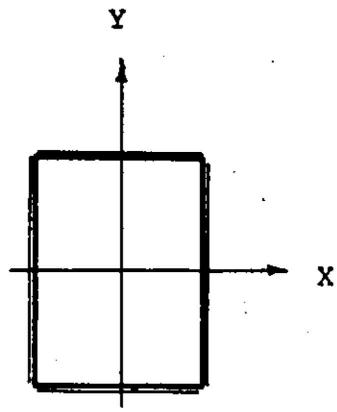
Table 3 : Natural Frequencies of Rectangular Plates and Eigenvectors.



CLAMPED ALL EDGES

R = $^b/a$	n = 2	n = 3	n = 4	IGUCHI
1.0	35.9735	35.9778	35.9832	35.98
1.5	27.0014	27.0072	27.0102	27.00
2.0	24.5778	24.5753	24.5720	24.57
2.5	23.6449	23.6598	23.6972	23.77
3.0	23.1981	23.1998	23.2013	23.19

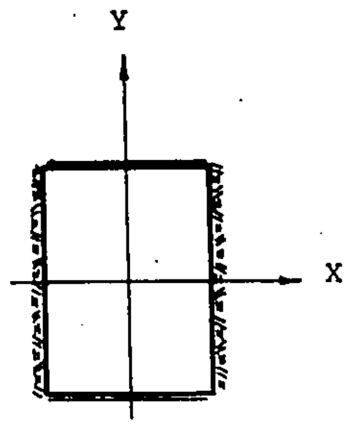
Table 4 : Natural Frequencies ($\omega a^2 \frac{\rho}{D}$) of Isotropic Rectangular Plates Clamped All Edges for n = 2, n = 3 , n = 4 .



SIMPLY SUPPORTED EDGES

R = $^{b}/_{a}$	n = 2	n = 3	n = 4	IGUCHI
1.0	21.0052	20.5319	20.4160	19.74
1.5	15.4847	15.0483	14.8290	14.26
2.0	13.7821	13.2813	12.9370	12.34
2.5	13.0588	12.5018	12.0800	11.45
3.0	12.6869	12.0896	11.6210	10.97

Table 5 : Natural Frequencies ($\omega a^2 \frac{\rho}{D}$) of Isotropic Rectangular Plates Simply Supported All Edges for $n = 2$, $n = 3$, $n = 4$.



SIMPLE-CLAMPED-SIMPLE-CLAMPED

$R = {}^{b}/a$	n = 2	n = 3	n = 4	LEISSA
1.0	29.3725	29.2188	29.0107	28.946
1.5	25.1204	25.0795	25.0603	25.047
2.0	23.8440	23.8231	25.0603	25.047
2.5	23.2953	23.2796	23.2705	23.271
3.0	23.0086	22.9946	22.9838	22.985

Table 6 : Natural Frequencies ($\omega a^2 \frac{\rho}{D}$) of Rectangular Plates with Two Opposite Edges Simply Supported and the Other Two Edges Clamped for n = 2 , n = 3 ,and n = 4 .

COMPUTER PROGRAM

The computer program capable to analyse plate problems using Orthogonal Collocation method for various boundary conditions.

It generates the orthogonal polynomials associated with the boundary conditions given, thus build the shape functions accordingly, than multiply by the boundary function to give the assumed function for the plate which than substitute into appropriate governing differential equation. The roots of the orthogonal polynomial are the collocation points to be substituted into the equations resulting from differentiation. The results after substituting the collocation points were put in a matrix form to be process according to the analyses desired.

The higher accuracy can be achieve by increasing the number of terms used, in this case line number 58 in the main program (NN = 2) can be increase to 3 or 4 . If NN = 3, all the value in line 3 and line 4 have to be changed to 9 in order to acquired enough space storage capacity for the purpose.

Line 71 determine which analyses is desired.

Line 45 is the aspect ratio of the plate.

For orthotropic plates, flexural rigidities are required and can be inputed in lines 28 to 31.
For stability analyses, the types of inplane forces can be specified in lines 39 to 41.

The first input data is an 80 characters title to be printed on the output.

The next input data are coefficients of boundary funtion, power of x and power of y in boundary function. For example :

Rectangular plates clamped all edges.

The boundary function can be taken as

$$(1 - x^2)^2 (1 - y^2)^2$$

The input data will be :

1.0	0.0	0.0
-2.0	2.0	0.0
1.0	4.0	0.0
1.0	0.0	0.0
-2.0	0.0	2.0
1.0	0.0	4.0

Subroutine EIGZF is employs to find the eigenvalues and the eigenvectors. The subroutine is an internal subroutine supplied by IMSL package. The information about the subroutine is accompanied at the end of this thesis.

```
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 NX,NY,NXY
      COMPLEX*16 ALFA(4),Z(4,4),LAMBDA(4),FREQ(4)
      DIMENSION P(4,4),Q(4,4),PD(4),WKAREA(4),BETA(4)
      DIMENSION WK(512),PLATE(80)
      DIMENSION HX10(16),HY10(16),HX01(16),HY01(16),HX00(16),HY00(16)
      DIMENSION HXY10(16),HXY01(16),HXY00(16)
      COMMON /BLK1/HX10,HY10,HXY10,HX01,HY01,HXY01,HX00,HY00,HXY00
      COMMON /BLK2/ND
      COMMON /BLK3/FR1,FR2,FR3,NX,NY,NXY
C *********************************************************************
C                                                                     *
C THIS PROGRAM CAPABLE TO ANALYSE PLATE PROBLEMS                      *
C                                                                     *
C SUCH AS :          STATIC ANALYSIS    !                             *
C                    VIBRATION ANALYSIS !  ISOTROPIC AND ORTHOTROPIC  *
C                    STABILITY ANALYSIS !                             *
C                                                                     *
C THE METHOD USED IS ORTHOGONAL COLLOCATION                           *
C                                                                     *
C THE INPUT ARE :    TITLE OF BOUNDARY CONDITIONS                     *
C                    BOUNDARY FUNCTION                                *
C                                                                     *
C *********************************************************************
C
C FOR ORTHOTROPIC PLATES FLEXURAL RIGIDITIES ARE ALSO REQUIRED
C
      D11=2438.0
      D22=134.8
      D12=32.3
      D66=64.8
      FR1=D11/D22
      FR2=D12/D22
      FR3=D66/D22
C
```

```
C
C FOR STABILITY ANALYSIS  INPLANE FORCES HAVE TO BE SPECIFIED
C
      NX = 1.0
      NY = 0.0
      NXY= 0.0
C
C ASPECT RATIO    R = B/A
C
      R  = 1.0
      R2 = R**2
C
C NBC1 IS THE NUMBER OF X TERMS OF BOUNDARY FUNCTION
C NBC2 IS THE NUMBER OF Y TERMS OF BOUNDARY FUNCTION
C
      NBC1=3
      NBC2=3
      ND=NBC1*NBC2
C
C NN IS THE NUMBER OF TERMS OF ORTHOGONAL COLLOCATION
C NT IS THE SQUARE OF NN
C
      NN = 2
      NT = NN**2
C
C NWHAT is an integer for which to indicate which analysis is desired
C
C          1     STATIC ANALYSIS      !
C          2     VIBRATION ANALYSIS   !  ISOTROPIC PLATES
C          3     BUCKLING ANALYSIS    !
C
C          4     STATIC ANALYSIS      !
C          5     VIBRATION ANALYSIS   !  ORTHOTROPIC PLATES
C          6     STABILITY ANALYSIS   !
C
      NWHAT=1
C
C  TO READ BOUNDARY CONDITIONS OF THE PLATE
C
```

```
C
      READ(5,10)(PLATE(LOV),LOV=1,80)
   10 FORMAT(80A1)
C
C   TO WRITE TITLE OF OUTPUT
C
      CALL TITLE(R,NWHAT,PLATE)
C
      CALL BEN(NN,NT,ND,NBC1,NBC2,R,NWHAT,P,Q)
C
C TO DIRECT TO WHICH ANALYSIS DESIRED
C
      GO TO (111,222,333,444,555,666),NWHAT
C
C STATIC ANALYSIS OF ISOTROPIC PLATES
C
  111 CALL DEFLEC(P,Q,NT,R,WKAREA,PD,NWHAT)
      GO TO 999
C
C VIBRATION ANALYSIS OF ISOTROPIC PLATES
C
  222 CALL EIGEN (P,Q,NT,ALFA,Z,LAMBDA,FREQ,BETA,R)
      GO TO 999
C
C STABILITY ANALYSIS OF ISOTROPIC PLATES
C
  333 CALL BUCKL(P,Q,NT,ALFA,Z,LAMBDA,FREQ,BETA,R)
      GO TO 999
C
C STATIC ANALYSIS OF ORTHOTROPIC PLATES
C
  444 CALL DEFLEC(P,Q,NT,R,WKAREA,PD,NWHAT)
      GO TO 999
C
C VIBRATION ANALYSIS OF ORTHOTROPIC PLATES
C
```

```
C
  555 CALL EIGEN(P,Q,NT,ALFA,Z,LAMBDA,FREQ,BETA,R)
      GO TO 999
C
C STABILITY ANALYSIS OF ORTHOTROPIC PLATES
C
  666 CALL BUCKL(P,Q,NT,ALFA,Z,LAMBDA,FREQ,BETA,R)
  999 STOP
      END
```

```
      SUBROUTINE TITLE(R,NWHAT,PLATE)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION PLATE(80)
      WRITE(6,6)
    6 FORMAT('1',24(/),26X,80('*'),2(/,26X,'*',78X,'*'))
      WRITE(6,7)R
    7 FORMAT(26X,'*',26X,'ASPECT  RATIO    =    ',F4.2,26X,'*')
      WRITE(6,8)
    8 FORMAT(26X,'*',78X,'*',/,26X,'*',78X,'*')
      GO TO(11,22,33,44,55,66),NWHAT
   11 WRITE(6,611)
  611 FORMAT(26X,'*',18X,'STATIC   ANALYSIS   OF   ISOTROPIC   PLATES',
     $17X,'*')
      GO TO 77
   22 WRITE(6,622)
  622 FORMAT(26X,'*',17X,'VIBRATION   ANALYSIS   OF   ISOTROPIC   PLATES
     $',15X,'*')
      GO TO 77
   33 WRITE(6,633)
  633 FORMAT(26X,'*',17X,'STABILITY   ANALYSIS   OF   ISOTROPIC   PLATES
     &',15X,'*')
      WRITE(6,8)
      WRITE(6,6331)NX,NXY,NY
 6331 FORMAT(26X,'*',13X,'NX   =   ',F3.1,10X,'NXY   =   ',F3.1,10X,
     $'NY   =   ',F3.1,13X,'*')
      WRITE(6,8)
      GO TO 77
   44 WRITE(6,644)
  644 FORMAT(26X,'*',17X,'STATIC   ANALYSIS   OF   ORTHOTROPIC   PLATES'
     $,16X,'*')
      WRITE(6,8)
      WRITE(6,6441)FR1,FR2,FR3
 6441 FORMAT(26X,'*',10X,'FR1   =   ',F8.4,5X,'FR2   =   ',F8.4,5X,
     $'FR3   =   ',F8.4,10X,'*')
      WRITE(6,8)
      GO TO 77
```

```
   55 WRITE(6,655)
  655 FORMAT(26X,'*',15X,'VIBRATION   ANALYSIS   OF   ORTHOTROPIC   PLAT
     $ES',15X,'*')
      WRITE(6,8)
      WRITE(6,6441)FR1,FR2,FR3
      WRITE(6,8)
      GO TO 77
   66 WRITE(6,616)
  616 FORMAT(26X,'*',15X,'STABILITY   ANALYSIS   OF   ORTHOTROPIC   PLAT
     $ES',15X,'*')
      WRITE(6,8)
      WRITE(6,6441)FR1,FR2,FR3
      WRITE(6,8)
      WRITE(6,6331)NX,NXY,NY
      WRITE(6,8)
   77 CONTINUE
      WRITE(6,8)
      WRITE(6,9)(PLATE(LOV),LOV=1,80)
    9 FORMAT(26X,80A1)
      WRITE(6,8)
      WRITE(6,10)
   10 FORMAT(26X,80('*'))
      RETURN
      END
```

```
      SUBROUTINE BEN(NN,NT,ND,NBC1,NBC2,R,NWHAT,P,Q)
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 NX,NY,NXY
      DIMENSION POINTX(16),POINTY(16),P(NT,NT),Q(NT,NT)
      DIMENSION BC1(3),BC1X(3),BC1Y(3),BC2(3),BC2X(3),BC2Y(3)
      DIMENSION AC(9),AX(9),AY(9),CS(16),PXS(16),PYS(16)
      DIMENSION ABC(3,16,9),DX2(3,16,9),DXY(3,16,9)
      DIMENSION DY2(3,16,9),DX4(3,16,9),DX2Y2(3,16,9)
      DIMENSION DY4(3,16,9),DY(3,16,9),DX(3,16,9)
      DIMENSION XPT(4),YPT(4),CPX(4,4),CPY(4,4),POW(4,4)
      DIMENSION HX10(16),HY10(16),HX01(16),HY01(16),HX00(16),HY00(16)
      DIMENSION HXY10(16),HXY01(16),HXY00(16)
      COMMON /BLK1/HX10,HY10,HXY10,HX01,HY01,HXY01,HX00,HY00,HXY00
      COMMON /BLK3/FR1,FR2,FR3,NX,NY,NXY
      COMMON /BLK4/AC,AX,AY
      COMMON /BLK5/NC
      COMMON /BLK6/CS,PXS,PYS
      COMMON /BLK7/ABC
C
C     TO READ BOUNDARY FUNCTIONS
C
      READ(5,10)(BC1(I),BC1X(I),BC1Y(I),I=1,NBC1)
      READ(5,10)(BC2(I),BC2X(I),BC2Y(I),I=1,NBC2)
   10 FORMAT(3F10.1)
      CALL ORTPOL(BC1,BC1X,NBC1,XPT,CPX,NN,POW)
      CALL ORTPOL(BC2,BC2Y,NBC2,YPT,CPY,NN,POW)
      CALL COLPTS(XPT,YPT,NN,POINTX,POINTY)
      DO 111 I=1,NBC1
      DO 111 J=1,NBC2
      IJ=NBC2*I-NBC2+J
      AC(IJ)=BC1(I)*BC2(J)
      AX(IJ)=BC1X(I)+BC2X(J)
      AY(IJ)=BC1Y(I)+BC2Y(J)
  111 CONTINUE
C
C     TO WRITE BOUNDARY FUNCTIONS
```

```
C
      WRITE(6,202)
  202 FORMAT('1',4(/),56X,'BOUNDARY FUNCTION',////,30X,'COEFFICIENTS',
     $20X,'POWER OF X',20X,'POWER OF Y',//)
      WRITE(6,20)(AC(I),AX(I),AY(I),I=1,ND)
   20 FORMAT('0',27X,F10.1,21X,F10.1,20X,F10.1)
C
C TO CREATE  P  AND  Q   MATRICES ;  (P)(A) = K(Q)(A)
C
      WRITE(6,201)
  201 FORMAT('1',4(/),56X,'SHAPE FUNCTION',////,30X,'COEFFICIENTS',20X,
     $'POWER OF X',20X,'POWER OF Y',//)
      R4 = R**4
      R2 = R**2
C
C LOOP KC = 1 TO NT     ;   NT IS NUMBER OF TERMS
C
      KC=1
      DO 2000 I=1,NN
      DO 2000 J=1,NN
      NC=I*J
      CALL ARANGE(CPX,CPY,POW,I,J)
C
C TO MULTIPLY BOUNDARY FUNCTION BY SHAPE FUNTION
C
      CALL EXPAND
C
C TO DIFFERENTIATE THE TERM BY TERM ACCORDING TO DIFFERENTIAL EQUATION
C
      CALL DIFF(DX,1)
      CALL DIFF(DY,2)
      CALL DIFF(DX2,3)
      CALL DIFF(DXY,4)
      CALL DIFF(DY2,5)
      CALL DIFF(DX4,6)
      CALL DIFF(DX2Y2,7)
      CALL DIFF(DY4,8)
```

```
C
C TO SUBSTITUTE COLLOCATION POINTS
C
      DO 1000 KR=1,NT
      COOX=POINTX(KR)
      COOY=POINTY(KR)
      CALL SUBSTI(ABC,VW,COOX,COOY)
      CALL SUBSTI(DX,VX,COOX,COOY)
      CALL SUBSTI(DY,VY,COOX,COOY)
      CALL SUBSTI(DX2,VX2,COOX,COOY)
      CALL SUBSTI(DXY,VXY,COOX,COOY)
      CALL SUBSTI(DY2,VY2,COOX,COOY)
      CALL SUBSTI(DX4,VX4,COOX,COOY)
      CALL SUBSTI(DX2Y2,VX2Y2,COOX,COOY)
      CALL SUBSTI(DY4,VY4,COOX,COOY)
C
C TO FORM THE  (P)  AND  (Q)  MATRICES TERM BY TERM
C
      GO TO (1111,2222,3333,4444,5555,6666),NWHAT
 1111 P(KR,KC)=R4*VX4+2*R2*VX2Y2+VY4
      CALL SUB(DX2,VX2,1,0)
      CALL SUB(DY2,VY2,1,0)
      HX10(KC)=VX2
      HY10(KC)=VY2
      CALL SUB(DX2,VX2,0,1)
      CALL SUB(DY2,VY2,0,1)
      HX01(KC)=VX2
      HY01(KC)=VY2
      CALL SUB(DX2,VX2,0,0)
      CALL SUB(DY2,VY2,0,0)
      HX00(KC)=VX2
      HY00(KC)=VY2
      GO TO 1000
 2222 P(KR,KC)=R4*VX4+2*R2*VX2Y2+VY4
      Q(KR,KC)=VW
      GO TO 1000
```

```
3333 P(KR,KC)=R4*VX4+2*R2*VX2Y2+VY4
     Q(KR,KC)=NX*R2*VX2+2*NXY*R*VXY+NY*VY2
     GO TO 1000
4444 P(KR,KC)=FR1*R4*VX4+(FR2+2*FR3)*2*R2*VX2Y2+VY4
     CALL SUB(DX2,VX2,1,0)
     CALL SUB(DY2,VY2,1,0)
     CALL SUB(DXY,VXY,1,0)
     HX10(KC)=VX2
     HY10(KC)=VY2
     HXY10(KC)=VXY
     CALL SUB(DX2,VX2,0,1)
     CALL SUB(DY2,VY2,0,1)
     CALL SUB(DXY,VXY,0,1)
     HX01(KC)=VX2
     HY01(KC)=VY2
     HXY01(KC)=VXY
     CALL SUB(DX2,VX2,0,0)
     CALL SUB(DY2,VY2,0,0)
     CALL SUB(DXY,VXY,0,0)
     HX00(KC)=VX2
     HY00(KC)=VY2
     HXY00(KC)=VXY
     GO TO 1000
5555 P(KR,KC)=FR1*R4*VX4+(FR2+2*FR3)*2*R2*VX2Y2+VY4
     Q(KR,KC)=VW
     GO TO 1000
6666 P(KR,KC)=FR1*R4*VX4+(FR2+2*FR3)*2*R2*VX2Y2+VY4
     Q(KR,KC)=NX*R2*VX2+2*NXY*R*VXY+NY*VY2
1000 CONTINUE
2000 KC=KC+1
     RETURN
     END
```

```
      SUBROUTINE ORTPOL(CB,PB,NBT,CPT,COE,NN,POWER)
      IMPLICIT REAL*8(A-H,O-Z)
      COMPLEX*16 RP1(3),RP2(5),RP3(7),RP4(9)
      DIMENSION CB(NBT),PB(NBT),COE(4,4),POWER(4,4)
      DIMENSION X(9),C(10),ALPHA(5),CPT(NN)
      DIMENSION CP1(3),CP2(5),CP3(7),CP4(9)
C
C     TO INTEGRATE WEIGHT FUNCTION*(X**(2*N))
C
C
C     INTEGRATE WEIGHT FUNCTION*(X**(2*N))
C
      WRITE(6,101)
  101 FORMAT('1',4(/),47X,'***  INTEGRATE WEIGHT FUNCTION  ***',4(/))
      IJ=1
      DO 102 I=1,9
      X(I)=0.0
      DO 103 J=1,NBT
      X(I)=X(I)+CB(J)/(PB(J)+IJ)
  103 CONTINUE
      IJ=IJ+2
      WRITE(6,60)I,X(I)
   60 FORMAT('0',48X,'X(',I1,')  =  ',E24.16)
  102 CONTINUE
      ALPHA(1)=X(1)
      C(1)=-X(1)/X(2)
      ALPHA(2)=X(1)+2*C(1)*X(2)+C(1)*C(1)*X(3)
      VIA1=-(X(2)+C(1)*X(3))/(X(3)+C(1)*X(4))
      C(2)=-X(1)/(X(2)+VIA1*X(3))
      C(3)=VIA1*C(2)
      ALPHA(3)=X(1)+C(2)*C(2)*X(3)+C(3)*C(3)*X(5)+2*C(2)*X(2)+2*C(3)*X(3
     $)+2*C(2)*C(3)*X(4)
      VIA2=-(X(3)+C(2)*X(4)+C(3)*X(5))/(X(4)+C(2)*X(5)+C(3)*X(6))
      VIA3=-(X(2)+C(1)*X(3))/((X(3)+C(1)*X(4))+VIA2*(X(4)+C(1)*X(5)))
```

```
      C(4)=-X(1)/(X(2)+VIA3*X(3)+VIA2*VIA3*X(4))
      C(5)=VIA3*C(4)
      C(6)=VIA2*C(5)
      ALPHA(4)=X(1)+C(4)*C(4)*X(3)+C(5)*C(5)*X(5)+C(6)*C(6)*X(7)+2*C(4)*
     $X(2)+2*C(5)*X(3)+2*C(6)*X(4)+2*C(4)*C(5)*X(4)+2*C(4)*C(6)*X(5)+2*C
     $(5)*C(6)*X(6)
      VIA4=-(X(4)+C(4)*X(5)+C(5)*X(6)+C(6)*X(7))/(X(5)+C(4)*X(6)+C(5)*X(
     $7)+C(6)*X(8))
      VIA5=-(X(3)+C(2)*X(4)+C(3)*X(5))/((X(4)+C(2)*X(5)+C(3)*X(6))+VIA4*
     $(X(5)+C(2)*X(6)+C(3)*X(7)))
      VIA6=-(X(2)+C(1)*X(3))/((X(3)+C(1)*X(4))+VIA5*(X(4)+C(1)*X(5))+VIA
     $4*VIA5*(X(5)+C(1)*X(6)))
      C(7)=-X(1)/(X(2)+VIA6*X(3)+VIA5*VIA6*X(4)+VIA4*VIA5*VIA6*X(5))
      C(8)=VIA6*C(7)
      C(9)=VIA5*C(8)
      C(10)=VIA4*C(9)
      ALPHA(5)=X(1)+C(7)*C(7)*X(3)+C(8)*C(8)*X(5)+C(9)*C(9)*X(7)+C(10)*C
     $(10)*X(9)+2*C(7)*X(2)+2*C(8)*X(3)+2*C(9)*X(4)+2*C(10)*X(5)+2*C(7)*
     $C(8)*X(4)+2*C(7)*C(9)*X(5)+2*C(7)*C(10)*X(6)+2*C(8)*C(9)*X(6)+2*C(
     $8)*C(10)*X(7)+2*C(9)*C(10)*X(8)
      WRITE(6,106)
  106 FORMAT('1',4(/),47X,'***  COEFFICIENT OF POLYNOMIAL  ***',4(/))
      DO 104 IC=1,10
      WRITE(6,70)IC,C(IC)
  104 CONTINUE
      DO 105 IALPHA=1,5
      WRITE(6,80)IALPHA,ALPHA(IALPHA)
  105 CONTINUE
C
C     TO CALCULATE ROOTS OF POLYNOMIALS
C
      CALL ROOTS(C,CPT,NN)
   70 FORMAT('0',48X,'C(',I2,')  =  ',E24.16)
   80 FORMAT('-',44X,'ALPHA(',I2,')  =  ',E24.16)
      NP=4
```

```
      DO 110 IZERO=1,NP
      DO 110 JZERO=1,NP
      COE(IZERO,JZERO)=0.0
      POWER(IZERO,JZERO)=0.0
  110 CONTINUE
      DO 111 IR=1,NP
      COE(IR,1)=1.0
  111 CONTINUE
      COE(2,2)=C(1)
      COE(3,2)=C(2)
      COE(3,3)=C(3)
      COE(4,2)=C(4)
      COE(4,3)=C(5)
      COE(4,4)=C(6)
      POWER(2,2)=2.0
      POWER(3,2)=2.0
      POWER(3,3)=4.0
      POWER(4,2)=2.0
      POWER(4,3)=4.0
      POWER(4,4)=6.0
      RETURN
      END
```

```
      SUBROUTINE COLPTS(XPT,YPT,NN,POINTX,POINTY)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION XPT(NN),YPT(NN),POINTX(16),POINTY(16)
      DO 100 I=1,NN
      DO 100 J=1,NN
      IJ=NN*I-NN+J
      POINTX(IJ)=XPT(I)
      POINTY(IJ)=YPT(J)
  100 CONTINUE
      WRITE(6,120)
  120 FORMAT('1',4(/),57X,'COLLOCATION POINTS',///,42X,'X - COORDINATE',
     $20X,'Y - COORDINATE',///)
      NT=NN**2
      WRITE(6,130)(POINTX(I),POINTY(I),I=1,NT)
  130 FORMAT('-',40X,F14.10,20X,F14.10)
      RETURN
      END
```

```
      SUBROUTINE ROOTS(C,CPT,NN)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION CP1(3),CP2(5),CP3(7),CP4(9),C(10),CPT(NN)
      COMPLEX*16 RP1(3),RP2(5),RP3(7),RP4(9)
      NDEG1=2
      NDEG2=4
      NDEG3=6
      NDEG4=8
      CP1(1)=C(1)
      CP1(2)=0.0
      CP1(3)=1.0
      CP2(1)=C(3)
      CP2(2)=0.0
      CP2(3)=C(2)
      CP2(4)=0.0
      CP2(5)=1.0
      CP3(1)=C(6)
      CP3(2)=0.0
      CP3(3)=C(5)
      CP3(4)=0.0
      CP3(5)=C(4)
      CP3(6)=0.0
      CP3(7)=1.0
      CP4(1)=C(10)
      CP4(2)=0.0
      CP4(3)=C(9)
      CP4(4)=0.0
      CP4(5)=C(8)
      CP4(6)=0.0
      CP4(7)=C(7)
      CP4(8)=0.0
      CP4(9)=1.0
      CALL ZPOLR(CP1,NDEG1,RP1,IER)
      CALL ZPOLR(CP2,NDEG2,RP2,IER)
      CALL ZPOLR(CP3,NDEG3,RP3,IER)
      CALL ZPOLR(CP4,NDEG4,RP4,IER)
```

```
      DO 113 IGO=1,NN
      IG=2*(NN+1-IGO)
      GO TO(113,220,230,240),NN
  220 CPT(IGO)=CDABS(RP2(IG))
      GO TO 113
  230 CPT(IGO)=CDABS(RP3(IG))
      GO TO 113
  240 CPT(IGO)=CDABS(RP4(IG))
  113 CONTINUE
      WRITE(6,46)(CPT(NCOL),NCOL=1,NN)
   46 FORMAT('-',50X,E24.16)
      RETURN
      END
```

```
      SUBROUTINE ARANGE(CPX,CPY,PC,I,J)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION CPX(4,4),CPY(4,4),PC(4,4)
      DIMENSION CS(16),PXS(16),PYS(16)
      COMMON /BLK6/CS,PXS,PYS
      COMMON /BLK5/NC
      DO 200 K=1,J
      DO 200 L=1,I
      IJ=I*K-I+L
      CS(IJ)=CPX(I,L)*CPY(J,K)
      PXS(IJ)=PC(I,L)
      PYS(IJ)=PC(J,K)
  200 CONTINUE
      WRITE(6,20)(CS(M),PXS(M),PYS(M),M=1,NC)
   20 FORMAT('0',27X,F10.1,21X,F10.1,20X,F10.1)
      RETURN
      END
```

```
      SUBROUTINE DIFF(A2,NNN)
C     ABC IS THE MATRIX THAT HAVE TO BE DIFFERENTIATE
C     ND IS NUMBER OF ROW OF ABC
C     NC IS NUMBER OF COLUMN OF ABC
C     A2 IS THE RESULT AFTER DIFFERENTIATION
C     NNN IS A POINTER TO WHAT ORDER DIIFERENTIATION
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ABC(3,16,9),A2(3,16,9)
      COMMON /BLK2/ND
      COMMON /BLK5/NC
      COMMON /BLK7/ABC
      GO TO(10,20,30,40,50,60,70,80),NNN
C     DIFFERENTIATE W.R.T. X
   10 DO 1 I=1,NC
      DO 1 J=1,ND
      A2(1,I,J)=ABC(1,I,J)*ABC(2,I,J)
      A2(2,I,J)=ABC(2,I,J)-1.0
      A2(3,I,J)=ABC(3,I,J)
    1 CONTINUE
      GO TO 999
C     DIFFERENTIATE W.R.T. Y
   20 DO 2 I=1,NC
      DO 2 J=1,ND
      A2(1,I,J)=ABC(1,I,J)*ABC(3,I,J)
      A2(2,I,J)=ABC(2,I,J)
      A2(3,I,J)=ABC(3,I,J)-1.0
    2 CONTINUE
      GO TO 999
C     DIFFERENTIATE W.R.T. X**2
   30 DO 3 I=1,NC
      DO 3 J=1,ND
      A2(1,I,J)=ABC(1,I,J)*ABC(2,I,J)*(ABC(2,I,J)-1.0)
      A2(2,I,J)=ABC(2,I,J)-2.0
      A2(3,I,J)=ABC(3,I,J)
    3 CONTINUE
```

```
      GO TO 999
C     DIFFERENTIATE W.R.T. X*Y
   40 DO 4 I=1,NC
      DO 4 J=1,ND
      A2(1,I,J)=ABC(1,I,J)*ABC(2,I,J)*ABC(3,I,J)
      A2(2,I,J)=ABC(2,I,J)-1.0
      A2(3,I,J)=ABC(3,I,J)-1.0
    4 CONTINUE
      GO TO 999
C     DIFFERENTIATE W.R.T. Y**2
   50 DO 5 I=1,NC
      DO 5 J=1,ND
      A2(1,I,J)=ABC(1,I,J)*ABC(3,I,J)*(ABC(3,I,J)-1.0)
      A2(2,I,J)=ABC(2,I,J)
      A2(3,I,J)=ABC(3,I,J)-2.0
    5 CONTINUE
      GO TO 999
C     DIFFERENTIATE W.R.T. X**4
   60 DO 6 I=1,NC
      DO 6 J=1,ND
      A2(1,I,J)=ABC(1,I,J)*ABC(2,I,J)*(ABC(2,I,J)-1.0)*(ABC(2,I,J)-2.0)*
     $(ABC(2,I,J)-3.0)
      A2(2,I,J)=ABC(2,I,J)-4.0
      A2(3,I,J)=ABC(3,I,J)
    6 CONTINUE
      GO TO 999
C     DIFFERENTIATE W.R.T. (X**2)*(Y**2)
   70 DO 7 I=1,NC
      DO 7 J=1,ND
      A2(1,I,J)=ABC(1,I,J)*ABC(2,I,J)*(ABC(2,I,J)-1.0 )*ABC(3,I,J)*(ABC(
     $3,I,J)-1.0)
      A2(2,I,J)=ABC(2,I,J)-2.0
      A2(3,I,J)=ABC(3,I,J)-2.0
    7 CONTINUE
      GO TO 999
```

```
C     DIFFERENTIATE W.R.T.  Y**4
   80 DO 8 I=1,NC
      DO 8 J=1,ND
      A2(1,I,J)=ABC(1,I,J)*ABC(3,I,J)*(ABC(3,I,J)-1.0)*(ABC(3,I,J)-2.0)*
     $(ABC(3,I,J)-3.0)
      A2(2,I,J)=ABC(2,I,J)
      A2(3,I,J)=ABC(3,I,J)-4.0
    8 CONTINUE
  999 RETURN
      END
```

```
      SUBROUTINE EXPAND
C     CA IS COEFFICIENT OF BOUNDARY FUNCTION
C     PXA IS POWER OF X OF BOUNDARY FUNCTION
C     PYA IS POWER OF Y OF BOUNDARY FUNCTION
C     ND IS NUMBER OF TERM OF BOUNDARY FUNCTION
C     CS IS COEFFICIENT OF ONE OF THE TERM OF SHAPE FUNCTION
C     PXS IS POWER OF X OF SHAPE FUNCTION
C     PYS IS POWER OF Y OF SHAPE FUNCTION
C     NC IS NUMBER OF TERM OF SHAPE FUNCTION
C     ABC IS RESULT OF EXPANSION
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION CA(9),PXA(9),PYA(9)
      DIMENSION CS(16),PXS(16),PYS(16)
      DIMENSION ABC(3,16,9)
      COMMON /BLK2/ND
      COMMON /BLK4/CA,PXA,PYA
      COMMON /BLK5/NC
      COMMON /BLK7/ABC
      COMMON /BLK6/CS,PXS,PYS
      DO 10 I=1,NC
      DO 10 J=1,ND
      ABC(1,I,J)=CS(I)*CA(J)
      ABC(2,I,J)=PXS(I)+PXA(J)
      ABC(3,I,J)=PYS(I)+PYA(J)
   10 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SUBSTI(WXY,VALUE,X,Y)
C     WXY IS MATRIX RESULTED FROM DIFFERENTIATION
C     ND IS NUMBER OF ROW
C     NC IS NUMBER OF COLUMN
C     X IS COORDINATE OF COLLOCATION IN X DIRECTION
C     Y IS COORDINATE OF COLLOCATION IN Y DIRECTION
C     VALUE IS RESULT AFTER SUBSTITUTION
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION WXY(3,16,9)
      COMMON /BLK2/ND
      COMMON /BLK5/NC
      VALUE=0.0
      DO 100 I=1,NC
      DO 100 J=1,ND
      VALUE=VALUE+WXY(1,I,J)*(X**WXY(2,I,J))*(Y**WXY(3,I,J))
  100 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE DEFLEC(P,Q,NT,R,WKAREA,PD,NWHAT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION P(NT,NT),Q(NT,NT),WKAREA(NT)
      DIMENSION PD(NT)
      DIMENSION HX10(16),HY10(16),HX01(16),HY01(16),HX00(16),HY00(16)
      DIMENSION HXY10(16),HXY01(16),HXY00(16)
      COMMON /BLK1/HX10,HY10,HXY10,HX01,HY01,HXY01,HX00,HY00,HXY00
      COMMON /BLK3/FR1,FR2,FR3,NX,NY,NXY
      WRITE(6,210)
  210 FORMAT('1',///,47X,'MATRIX P OF STATIC ANALYSIS',/)
      IF(NT .EQ. 16)GO TO 223
      IF(NT .EQ. 9)GO TO 221
      WRITE(6,220)((P(I,J),J=1,NT),I=1,NT)
  220 FORMAT('0',20X,4E20.8)
      GO TO 225
  221 WRITE(6,222)((P(I,J),J=1,NT),I=1,NT)
  222 FORMAT('0',12X,9E12.5)
      GO TO 225
  223 WRITE(6,224)(I,(P(I,J),J=1,NT),I=1,NT)
  224 FORMAT('0',3X,'ROW ',I2,4X,8E14.7,/,13X,8E14.7)
  225 N=NT
      IA=NT
      IDGT=8
      CALL LINV1F (P,N,IA,Q,IDGT,WKAREA,IER)
      WRITE(6,230)
  230 FORMAT('1',///,52X,'INVERSE OF MATRIX P',/)
      IF(NT .EQ. 16)GO TO 233
      IF(NT .EQ. 9)GO TO 231
      WRITE(6,220)((Q(I,J),J=1,NT),I=1,NT)
      GO TO 235
  231 WRITE(6,222)((Q(I,J),J=1,NT),I=1,NT)
      GO TO 235
  233 WRITE(6,224)(I,(P(I,J),J=1,NT),I=1,NT)
  235 WRITE(6,35)
   35 FORMAT('1',///,55X,'--***COEFFICIENTS***--',//)
```

```
      DO 37 JJ=1,NT
      VAR=0.0
      DO 38 KL=1,NT
      VAR=VAR+Q(JJ,KL)
   38 CONTINUE
      PD(JJ)=VAR
   37 CONTINUE
      WRITE(6,36)(PD(I),I=1,NT)
   36 FORMAT('0',52X,D24.16)
      DEFL=0.0
      DO 19 LI=1,NT
   19 DEFL=DEFL+PD(LI)
      WRITE(6,29)DEFL
   29 FORMAT('-',6(/),40X,'MAXIMUM DEFLECTION    =  ',D24.16)
C
C     TO CALCULATE MOMENTS
C     X1M IS MOMENT X AT X=1.0 AND Y=0.0
C     Y1M IS MOMENT Y AT X=0.0 AND Y=1.0
C     XOM IS MOMENT X AT X=0.0 AND Y=0.0
C     YOM IS MOMENT Y AT X=0.0 AND Y=0.0
C
      R2=R**2
      X1M=0.0
      Y1M=0.0
      XOM=0.0
      YOM=0.0
      DO 112 II=1,NT
      IF(NWHAT .EQ. 1)GO TO 113
      X1M=X1M-(FR1*R2*HX10(II)+FR2*HY10(II))*PD(II)
      Y1M=Y1M-(FR2*R2*HX01(II)+HY01(II))*PD(II)
      XOM=XOM-(FR1*R2*HX00(II)+FR2*HY00(II))*PD(II)
      YOM=YOM-(FR2*R2*HY00(II)+HX00(II))*PD(II)
      GO TO 112
  113 X1M=X1M-(R2*HX10(II)+0.3D 00*HY10(II))*PD(II)
      Y1M=Y1M-(R2*HX01(II)+0.3D 00*HY01(II))*PD(II)
```

```
      XOM=XOM-(R2*HX00(II)+0.3D 00*HY00(II))*PD(II)
      YOM=YOM-(R2*HY00(II)+0.3D 00*HX00(II))*PD(II)
  112 CONTINUE
      X1M=X1M/4.0*R2
      Y1M=Y1M/4.0
      XOM=XOM/4.0*R2
      YOM=YOM/4.0*R2
      WRITE(6,39)X1M,Y1M,XOM,YOM
   39 FORMAT(//,40X,'X=1.0  Y=0.0      MX   =   ',D24.16,//,40X,'X=0.0  Y=1
     $.0      MY   =   ',D24.16,//,40X,'X=0.0  Y=0.0      MX   =   ',D24.16,/
     $/,40X,'X=0.0  Y=0.0      MY   =   ',D24.16)
      RETURN
      END
```

```
      SUBROUTINE SUB(WXY,VAL,N1,N2)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION WXY(3,16,9)
      COMMON /BLK2/ND
      COMMON /BLK5/NC
      VAL=0.0
      DO 400 I=1,NC
      DO 400 J=1,ND
      IF(N1 .EQ. 1)GO TO 401
      IF(N2 .EQ. 1)GO TO 402
      IF(WXY(3,I,J).GT.0.0)GO TO 400
  402 IF(WXY(2,I,J).GT.0.0)GO TO 400
      GO TO 403
  401 IF(WXY(3,I,J).GT.0.0)GO TO 400
  403 VAL=VAL+WXY(1,I,J)
  400 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE EIGEN (P,Q,NT,ALFA,Z,LAMBDA,FREQ,BETA,R)
      IMPLICIT REAL*8(A-H,O-Z)
      COMPLEX*16 ALFA(NT),Z(NT,NT),LAMBDA(NT),FREQ(NT)
      DIMENSION BETA(NT),WK(512)
      DIMENSION P(NT,NT),Q(NT,NT)
      WRITE(6,210)
  210 FORMAT('1',///,47X,'MATRIX P OF EIGENVALUE PROBLEMS',/)
      IF(NT .EQ. 16)GO TO 223
      IF(NT .EQ. 9)GO TO 221
      WRITE(6,220)((P(I,J),J=1,NT),I=1,NT)
  220 FORMAT('0',20X,4E20.8)
      GO TO 225
  221 WRITE(6,222)((P(I,J),J=1,NT),I=1,NT)
  222 FORMAT('0',12X,9E12.5)
      GO TO 225
  223 WRITE(6,224)(I,(P(I,J),J=1,NT),I=1,NT)
  224 FORMAT('0',3X,'ROW ',I2,4X,8E14.7,/,13X,8E14.7)
  225 WRITE(6,230)
  230 FORMAT('1',///,47X,'MATRIX Q OF EIGENVALUE PROBLEMS',/)
      IF(NT .EQ. 16)GO TO 233
      IF(NT .EQ. 9)GO TO 231
      WRITE(6,220)((Q(I,J),J=1,NT),I=1,NT)
      GO TO 235
  231 WRITE(6,222)((Q(I,J),J=1,NT),I=1,NT)
      GO TO 235
  233 WRITE(6,224)(I,(P(I,J),J=1,NT),I=1,NT)
  235 IA=NT
      IB=NT
      IZ=NT
      N =NT
      IJOB=2
      CALL EIGZF(P,IA,Q,IB,N,IJOB,ALFA,BETA,Z,IZ,WK,IER)
```

```
      WRITE(6,691)
  691 FORMAT('-',////,55X,'**** EIGENVALUE ****',////)
      DO 150 I=1,N
      IF(BETA(I))101,102,101
  101 LAMBDA(I)=ALFA(I)/BETA(I)
      FREQ(I)=4.0*(DSQRT(CDABS(LAMBDA(I))))/R**2
      WRITE(6,103)I,LAMBDA(I),I,FREQ(I)
  103 FORMAT('-',15X,'LAMBDA(',I1,') = ',2E16.8,10X,'FREQ  (',I1,') = ',
     $2E16.8)
      GO TO 150
  102 WRITE(6,104)I
  104 FORMAT('0',50X,'LAMBDA(',I1,') = INFINITE')
  150 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE BUCKL (P,Q,NT,ALFA,Z,LAMBDA,FREQ,BETA,R)
      IMPLICIT REAL*8(A-H,O-Z)
      COMPLEX*16 ALFA(NT),Z(NT,NT),LAMBDA(NT),FREQ(NT)
      DIMENSION BETA(NT),WK(512)
      DIMENSION P(NT,NT),Q(NT,NT)
      WRITE(6,210)
  210 FORMAT('1',///,47X,'MATRIX P OF STABILITY PROBLEMS',/)
      IF(NT .EQ. 16)GO TO 223
      IF(NT .EQ. 9)GO TO 221
      WRITE(6,220)((P(I,J),J=1,NT),I=1,NT)
  220 FORMAT('0',20X,4E20.8)
      GO TO 225
  221 WRITE(6,222)((P(I,J),J=1,NT),I=1,NT)
  222 FORMAT('0',12X,9E12.5)
      GO TO 225
  223 WRITE(6,224)(I,(P(I,J),J=1,NT),I=1,NT)
  224 FORMAT('0',3X,'ROW ',I2,4X,8E14.7,/,13X,8E14.7)
  225 WRITE(6,230)
  230 FORMAT('1',///,47X,'MATRIX Q OF STABILITY PROBLEMS',/)
      IF(NT .EQ. 16)GO TO 233
      IF(NT .EQ. 9)GO TO 231
      WRITE(6,220)((Q(I,J),J=1,NT),I=1,NT)
      GO TO 235
  231 WRITE(6,222)((Q(I,J),J=1,NT),I=1,NT)
      GO TO 235
  233 WRITE(6,224)(I,(P(I,J),J=1,NT),I=1,NT)
  235 IA=NT
      IB=NT
      IZ=NT
      N =NT
      IJOB=2
      CALL EIGZF(P,IA,Q,IB,N,IJOB,ALFA,BETA,Z,IZ,WK,IER)
```

```fortran
      WRITE(6,691)
  691 FORMAT('-',////,55X,'**** BUCKLVALUE ****',////)
      DO 150 I=1,N
      IF(BETA(I))101,102,101
  101 LAMBDA(I)=ALFA(I)/BETA(I)
      PHI=3.14159
      FREQ(I)=4.0/(PHI**2)*(CDABS(LAMBDA(I)))/R**2
      WRITE(6,103)I,LAMBDA(I),I,FREQ(I)
  103 FORMAT('-',15X,'LAMBDA(',I1,') = ',2E16.8,10X,'FREQ  (',I1,') = ',
     $2E16.8)
      GO TO 150
  102 WRITE(6,104)I
  104 FORMAT('0',50X,'LAMBDA(',I1,') = INFINITE')
  150 CONTINUE
      RETURN
      END
```

```
C       SUBROUTINE ZPOLR  (A,NDEG,Z,IER)
C
C-ZPOLR---------S/D-----LIBRARY 1------------------------------------------
C
C   FUNCTION            - ZEROS OF A POLYNOMIAL WITH REAL
C                           COEFFICIENTS (LAGUERRE).
C   USAGE               - CALL ZPOLR(A,NDEG,Z,IER)
C   PARAMETERS   A      - REAL VECTOR OF LENGTH NDEG+1 CONTAINING THE
C                           COEFFICIENTS IN ORDER OF DECREASING
C                           POWERS OF THE VARIABLE (INPUT).
C                NDEG   - INTEGER DEGREE OF THE POLYNOMIAL (INPUT).
C                Z      - COMPLEX VECTOR OF LENGTH NDEG CONTAINING
C                           THE COMPUTED ROOTS OF THE POLYNOMIAL
C                           (OUTPUT).
C                IER    - ERROR PARAMETER (OUTPUT)
C                         TERMINAL ERROR
C                           IER = 129, INDICATES THAT THE DEGREE OF THE
C                             POLYNOMIAL IS GREATER THAN 100 OR LESS
C                             THAN 1.
C                           IER = 130, INDICATES THAT THE LEADING
C                             COEFFICIENT IS ZERO. THIS RESULTS IN AT
C                             LEAST ONE ROOT, Z(NDEG), BEING SET TO
C                             POSITIVE MACHINE INFINITY.
C                           IER = 131, INDICATES THAT ZPOLR FOUND
C                             FEWER THAN NDEG ZEROS. IF ONLY M ZEROS
C                             ARE FOUND Z(J),J=M+1,...,NDEG ARE SET TO
C                             POSITIVE MACHINE INFINITY.
C   PRECISION           - SINGLE/DOUBLE
C   REQD. IMSL ROUTINES - SINGLE/UERTST,ZQADC,ZQADR
C                       - DOUBLE/UERTST,VXPADD,ZQADC,ZQADR
C   LANGUAGE            - FORTRAN
C--------------------------------------------------------------------------
```

CALL ZPOLR(A,NDEG,Z,IER)

Purpose

ZPOLR computes the NDEG zeros of the polynomial $P(Z)=A_1*Z^{NDEG}+A_2*Z^{NDEG-1}+ \ldots + A_{NDEG}*Z+A_{NDEG+1}$ where the coefficients, A_i, $i=1,2,\ldots,NDEG+1$, are real. The zeros are stored in the complex array Z with complex conjugate pairs stored contiguously.

Algorithm

ZPOLR uses Laguerre's method. The routine is a modification of B. T. Smith's routine ZERPOL.

ZPOLR iterates toward a zero using Laguerre's method, which is cubically convergent for isolated zeros and linearly convergent for multiple zeros. The maximum length of the step between successive iterates is restricted so that a new iterate lies inside a certain region, about the previous iterate, proved to contain a zero of the polynomial. An iterate is accepted as a zero when the polynomial value at that iterate is smaller than a computed bound for the rounding error in the polynomial value at that iterate. The original polynomial is deflated after each real zero or pair of complex zeros is found, and subsequent zeros are found using the deflated polynomial.

See reference: Smith, B. T., "ZERPOL, a zero finding algorithm for polynomials using Laguerre's method", Department of Computer Science, University of Toronto, May, 1967.

Programming Notes

1. The degree of the polynomial, NDEG, must be greater than or equal to 1 and less than or equal to 100.

2. If the leading coefficient is zero, at least one root, Z(NDEG) is set to positive machine infinity. In this case, ZPOLR does attempt to find the other roots.

3. If the routine finds fewer than NDEG zeros, the remaining roots are set to positive machine infinity (i.e., if only M zeros are found, Z(J), J=M+1,...,NDEG are set to positive machine infinity).

4. If the number of iterations for any one root is greater than 200*NDEG, the remaining roots are set to positive machine infinity and the routine terminates.

Example

```
      DIMENSION          A(4),Z(3)
      COMPLEX            Z
```

Input:

```
NDEG =    3
A    =    (1.0, -3.0, 4.0, -2.0)
CALL ZPOLR(A,NDEG,Z,IER)
```

Output:

```
IER  =    0
Z    =    (1.0, 1.0+i, 1.0-i)  (roots)
```

```
C      SUBROUTINE EIGZF  (A,IA,B,IB,N,IJOB,ALFA,BETA,Z,IZ,WK,IER)
C
C-EIGZF---------S/D-----LIBRARY 1----------------------------------------
C
C   FUNCTION             - TO CALCULATE EIGENVALUES AND (OPTIONALLY)
C                            EIGENVECTORS OF THE SYSTEM A*X=LAMBDA*B*X
C                            WHERE A AND B ARE REAL MATRICES OF ORDER N.
C   USAGE                - CALL EIGZF (A,IA,B,IB,N,IJOB,ALFA,BETA,Z,IZ,
C                            WK,IER)
C   PARAMETERS   A       - THE INPUT REAL GENERAL MATRIX OF ORDER N.
C                            INPUT A IS DESTROYED IF IJOB IS EQUAL
C                            TO 0 OR 1.
C                IA      - THE ROW DIMENSION OF THE MATRIX A IN THE
C                            CALLING PROGRAM. IA MUST BE GREATER THAN
C                            OR EQUAL TO N. (INPUT)
C                B       - THE INPUT REAL GENERAL MATRIX OF ORDER N.
C                            INPUT B IS DESTROYED IF IJOB IS EQUAL
C                            TO 0 OR 1.
C                IB      - THE ROW DIMENSION OF THE MATRIX B IN THE
C                            CALLING PROGRAM. IB MUST BE GREATER THAN
C                            OR EQUAL TO N. (INPUT)
C                N       - THE ORDER OF THE MATRICES A AND B. (INPUT)
C                IJOB    - INPUT OPTION PARAMETER. WHEN
C                            IJOB = 0, COMPUTE EIGENVALUES ONLY .
C                            IJOB = 1, COMPUTE EIGENVALUES AND EIGEN-
C                              VECTORS.
C                            IJOB = 2, COMPUTE EIGENVALUES, EIGENVECTORS
C                              AND PERFORMANCE INDEX.
C                            IJOB = 3, COMPUTE PERFORMANCE INDEX ONLY.
C                            IF THE PERFORMANCE INDEX IS COMPUTED, IT IS
C                            RETURNED IN WK(1). THE ROUTINES HAVE
C                            PERFORMED (WELL, SATISFACTORILY, POORLY) IF
C                            WK(1) IS (LESS THAN 1, BETWEEN 1 AND 100,
C                            GREATER THAN 100).
C                ALFA    - OUTPUT VECTORS OF LENGTH N.
C                BETA        ALFA IS TYPE COMPLEX AND BETA IS TYPE REAL.
C                            IF A AND B WERE SIMULTANEOUSLY REDUCED
C                            TO TRIANGULAR FORM BY UNITARY EQUIVALENCES,
C                            ALFA AND BETA WOULD CONTAIN THE DIAGONAL
C                            ELEMENTS OF THE RESULTING MATRICES. (SEE
C                            MOLER-STEWART REFERENCE).
C                          THE J-TH EIGENVALUE IS THE COMPLEX NUMBER
C                            GIVEN BY ALFA(J)/BETA(J).
C                Z       - THE OUTPUT N BY N COMPLEX MATRIX CONTAINING
C                            THE EIGENVECTORS.
C                            THE EIGENVECTOR IN COLUMN J OF Z CORRES-
C                            PONDS TO THE EIGENVALUE ALFA(J)/BETA(J).
C                            IF IJOB = 0, Z IS NOT USED.
C                IZ      - THE ROW DIMENSION OF THE MATRIX Z IN THE
C                            CALLING PROGRAM. IZ MUST BE GREATER THAN
C                            OR EQUAL TO N IF IJOB IS NOT EQUAL TO ZERO.
C                WK      - WORK AREA, THE LENGTH OF WK DEPENDS
C                            ON THE VALUE OF IJOB AS FOLLOWS,
C                            IJOB = 0, THE LENGTH OF WK IS AT LEAST N.
C                            IJOB = 1, THE LENGTH OF WK IS AT LEAST N.
C                            IJOB = 2, THE LENGTH OF WK IS AT LEAST
C                              2*N*N.
C                            IJOB = 3, THE LENGTH OF WK IS AT LEAST 1.
C                IER     - ERROR PARAMETER
C                          TERMINAL ERROR
```

```
C                                IER = 128+J, INDICATES THAT EQZTF FAILED     EIZF0610
C                                  TO CONVERGE ON EIGENVALUE J. EIGENVALUES   EIZF0620
C                                  J+1,J+2,...,N HAVE BEEN COMPUTED COR-      EIZF0630 110
C                                  RECTLY. EIGENVALUES 1,...,J MAY BE         EIZF0640
C                                  INACCURATE. IF IJOB = 1 OR 2 EIGENVECTORS  EIZF0650
C                                  MAY BE INACCURATE. THE PERFORMANCE INDEX   EIZF0660
C                                  IS SET TO 1000.                            EIZF0670
C                              WARNING ERROR (WITH FIX)                       EIZF0680
C                                IER = 66, INDICATES IJOB IS LESS THAN 0 OR   EIZF0690
C                                  IJOB IS GREATER THAN 3. IJOB IS RESET      EIZF0700
C                                  TO 1.                                      EIZF0710
C                                IER = 67, INDICATES IJOB IS NOT EQUAL TO     EIZF0720
C                                  0, AND IZ IS LESS THAN THE ORDER OF        EIZF0730
C                                  MATRIX A. IJOB IS RESET TO 0.              EIZF0740
C   PRECISION            - SINGLE/DOUBLE                                      EIZF0750
C   REQD. IMSL ROUTINES  - EQZQF,EQZTF,EQZVF,UERTST,VHSH2C,VHSH2R,VHSH3R     EIZF0760
C   LANGUAGE             - FORTRAN                                            EIZF0770
C-----------------------------------------------------------------------------EIZF0780
```

CALL EIGZF(A,IA,B,IB,N,IJOB,ALFA,BETA,Z,IZ,WK,IER)

Purpose

EIGZF computes eigenvalues and (optionally) eigenvectors for the generalized eigenproblem $Ax=\lambda Bx$ where A and B are real N by N matrices. It can also compute a performance index.

The eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_N$ can be calculated from the output by setting λ_I=ALFA(I)/BETA(I) when BETA(I)≠0. If BETA(I)=0 then λ_I is regarded as being infinite. The eigenvectors are returned in the complex matrix Z so that column M of Z contains the eigenvector corresponding to λ_M.

Algorithm

EIGZF calls IMSL routine EQZQF to reduce A to upper Hessenberg form and B to upper triangular form. Then, EQZTF is called to further transform A to quasi-upper triangular form (upper Hessenberg with no two consecutive subdiagonal elements being nonzero) while retaining B in upper triangular form. EQZVF is called to compute ALFA(I) and BETA(I), I=1,2,...,N and, optionally, the associated eigenvectors.

The performance index is defined as follows:

$$P = \max_{1\le j\le N} \frac{||\beta_j A z^j - \alpha_j B z^j||_\infty}{(|\beta_j|\ ||A||_1 ||z^j||_1 + |\alpha_j|\ ||B||_1 ||z||_1)\mathrm{EPS}}$$

where the max is taken over the j eigenvalues $\lambda_j=\alpha_j/\beta_j$ and associated eigenvectors z^j. Here, $\alpha_j \equiv$ALFA(j), $\beta_j \equiv$BETA(j), and z^j denotes column j of Z. EPS specifies the relative precision of floating point arithmetic. When P is less than 1, the performance of the routines is considered to be excellent in the sense that the residuals $\beta Az-\alpha Bz$ are as small as can be expected. When P is between 1 and 100 the performance is good. When P is greater than 100 the performance is considered poor.

See reference: Moler, C. B., and Stewart, G. W., "An algorithm for generalized matrix eigenvalue problems", SIAM J. Numer. Anal. 10, 241-256 (1973).

Programming Notes

1. A and B are preserved when IJOB=2 or 3. In all other cases A and B are destroyed.

2. The eigenvalues are unordered except that complex conjugate pairs of eigenvalues appear consecutively. That is, if λ_M and λ_{M+1} are such a pair then ALFA(M+1)/BETA(M+1) is the complex conjugate of ALFA(M)/BETA(M). ALFA is type COMPLEX and BETA is type REAL. It is not necessarily true that ALFA(M+1) is the conjugate of ALFA(M) for such a pair.

3. The eigenvectors are normalized so that the largest component has absolute value 1.

4. When IJOB=3 (i.e., to compute a performance index only) the eigenvalues, ALFA and BETA, and eigenvectors, Z, are assumed to be input.

5. If parameter IJOB is not in the range 0 to 3, computation continues with IJOB reset to 1 and IER=66 is returned. If IJOB is greater than zero (indicating that eigenvectors are desired) and IZ is less than N, computation continues with IJOB reset to 0 and IER=67 is returned.

Example

DIMENSION A(3,3),B(3,3),ALFA(3),BETA(3),Z(3,3),WK(18)

IA = 3
IB = 3
IZ = 3
N = 3
IJOB = 2

$$A = \begin{bmatrix} 1.0 & 0.5 & 0.0 \\ -10.0 & 2.0 & 0.0 \\ 5.0 & 1.0 & 0.5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0.5 & 0.0 & 0.0 \\ 3.0 & 3.0 & 0.0 \\ 4.0 & 0.5 & 1.0 \end{bmatrix}$$

CALL EIGZF(A,IA,B,IB,N,IJOB,ALFA,BETA,Z,IZ,WK,IER)

Output:

IER = 0
ALFA = (1.2705 + 3.0386i, 0.40869 − 0.97744i, 1.0031 + 0.0000i)
BETA = (1.5246, 0.49043, 2.0061)

$$Z = \begin{bmatrix} -0.25205+0.19169i & -0.25205-0.19169i & 0.0+0.0i \\ -0.08799-0.72598i & -0.08799+0.72598i & 0.0+0.0i \\ 1.00000+0.00000i & 1.00000+0.00000i & 1.0+0.0i \end{bmatrix} \quad \text{(eigenvectors)}$$

WK(1) < 10 (performance index)

The eigenvalues are as follows:

$$\lambda_1 = 0.83333 + 1.99301i$$
$$\lambda_2 = 0.83333 - 1.99301i$$
$$\lambda_3 = 0.50000 + 0.00001i$$

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