"COMPLEX STRUCTURES ON VECTOR SPACES"

A thesis submitted

by

LOUIS CULUMOVIĆ

to

the School of Graduate Studies of

the University of Ottawa

in partial fulfillment of the requirements

for the degree of

Master of Science

in the subject of

Mathematics

August, 1974

CONTENTS

ACKNOWLEDGEMENT

I wish to thank Dr. H. G. Helfenstein for the generous assistance he has given me as my thesis advisor. I wish also to acknowledge his personnal warmth and friendship which he extended towards me and which made it a pleasure for me to work on this thesis.

ABSTRACT

It is well-known that the set $\mathcal{J}_{2n}$ of all complex structures on a finite-dimensional real vectorspace is a homogenous space G/H , where G is a Lie group and H a closed subgroup. Instead of the standard procedure for introducing a smooth manifold structure on $\mathcal{J}_{2n}$, we present in Chapter III a different ad hoc atlas. Some aspects of $\mathcal{J}_{2n}$ are then investigated. Among them is the study of complex structures on the quaternions, in particular, those structures induced in various ways by the ring structure.

In Chapter IV, the set $\mathcal{K}_{2n}$ of quaternion structures is defined and developed in a manner very analogous to that of $\mathcal{J}_{2n}$.

Chapter I is a short chapter which motivates the definition of $\mathcal{J}_{2n}$, and Chapter II develops the necessary algebraic preliminaries.

Chapter I - INTRODUCTION

1. *Definition*: the 4-tuple $(V,+,\cdot,F)$, or just simply $(V,+,\cdot)$ over F, where V is a nonempty set, $+:V\times V \to V$ and $\cdot:F\times V \to V$ are functions, and F is a skew-field (ie. possibly non-commutative), is called a *vector space* if:

   I. $(V,+)$ is an Abelian group;

   II. $\forall \alpha,\beta \in F$, $\forall v,w \in V$:

   1) $\alpha\cdot(v+w) = \alpha\cdot v + \alpha\cdot w$,
   2) $(\alpha+\beta)\cdot v = \alpha\cdot v + \beta\cdot v$,
   3) $\alpha\cdot(\beta\cdot v) = (\alpha\beta)\cdot v$,
   4) $1\cdot v = v$ (where 1 is the unit of F).

2. *Theorem*: Let $(V,+,\cdot_C)$ be an n-dimensional vector space over C $(\cdot_C : C\times V \to V)$. Define function $\cdot_R = \cdot_C|R\times V$. Then $(V,+,\cdot_R)$ is a 2n-dimensional vector space over R.

   Proof: Clearly $(V,+,\cdot_R)$ is a vector space. To determine its dimension, let $\mathcal{B}_C = \{x_1,\dots,x_n\} \subset V$ be a basis of $(V,+,\cdot_C)$. Then $\mathcal{B}_R = \{x_1,\dots,x_n,(i\cdot_C x_1),\dots,(i\cdot_C x_n)\} \subset V$ is a basis of $(V,+,\cdot_R)$:

   1) $$\sum_{k=1}^{n} \alpha_k\cdot_R x_k + \sum_{k=1}^{n} \beta_k\cdot_R(i\cdot_C x_k) = 0 \to$$

   $$\sum_{k=1}^{n} (\alpha_k+i\beta_k)\cdot_C x_k = 0 \to \alpha_k+i\beta_k = 0 \quad k=1,\dots,n \to$$

   $$\alpha_k=\beta_k=0 \quad k=1,\dots,n \to \mathcal{B}_R \text{ is linearly independent;}$$

   2) $$x\in V \to x = \sum_{k=1}^{n}(\alpha_k+i\beta_k)\cdot_C x_k = \sum_{k=1}^{n}\alpha_k\cdot_R x_k + \sum_{k=1}^{n}\beta_k\cdot_R(i\cdot_C x_k).$$

3. Let $(V,+,\cdot_R)$ be the real vector space obtained in such a manner from the complex vector space $(V,+,\cdot_C)$. Define map

$f: V \to V$ where $f(x) = i \cdot_C x$. Then $f(x+y) = i \cdot_C (x+y) = i \cdot_C x + i \cdot_C y = f(x) + f(y)$ and $f((\alpha+i\beta) \cdot_C x) = i \cdot_C ((\alpha+i\beta) \cdot_C x) = (\alpha+i\beta) \cdot_C f(x)$ imply that f is a linear endomorphism of $(V,+,\cdot_C)$. Clearly, any linear endomorphism f of the complex vector space $(V,+,\cdot_C)$ is also a linear endomorphism of the real linear space $(V,+,\cdot_R)$.

f satisfies $f^2(x) = i \cdot_C (i \cdot_C x) = -x$, (thus $f^2 = -id_V$), and $(\alpha+i\beta) \cdot_C x = \alpha \cdot_C x + \beta \cdot_C (i \cdot_C x) = \alpha \cdot_R x + \beta \cdot_R f(x)$.

4. We now turn the question around: Given $(V,+,\cdot_R)$ over R, is it possible to find an extension $\cdot_J : C \times V \to V$ of $\cdot_R : R \times V \to V$ such that $(V,+,\cdot_J)$ is a complex vector space?

Clearly, necessary conditions are the existence of a linear endomorphism J of $(V,+,\cdot_R)$ such that $J^2 = -id_V$ and the definition of the extension $\cdot_J : C \times V \to V$ being

$$(\alpha+i\beta) \cdot_J x = \alpha \cdot_R x + \beta \cdot_R J(x).$$

We show that these are also sufficient conditions by showing that $(V,+,\cdot_J)$ is a complex vector space and that $J(x) = i \cdot_J x$ $\forall x \in V$.

We call a linear endomorphism J which satisfies $J^2 = -I$ a <u>complex structure</u> on the real vector space $(V,+,\cdot_R)$.

5. <u>Theorem</u>: Let J and $(V,+,\cdot_J)$ be defined as above. Then $J(x) = i \cdot_J x$ and $(V,+,\cdot_J)$ is a complex vector space.

<u>proof</u>: It is clear that $J(x) = i \cdot_J x$ $\forall x \in V$.
To show that $(V,+,\cdot_J)$ is a complex vector space we check the axioms. Let $\alpha+i\beta,\ \gamma+i\delta \in C$, $x,y \in V$:

1) $$\begin{aligned}(\alpha+i\beta) \cdot_J (x+y) &= \alpha \cdot_R (x+y) + \beta \cdot_R J(x+y) \\ &= (\alpha \cdot_R x + \beta \cdot_R J(x)) + (\alpha \cdot_R y + \beta \cdot_R J(y)) \\ &= (\alpha+i\beta) \cdot_J x + (\alpha+i\beta) \cdot_J y \ ;\end{aligned}$$

2) $$\begin{aligned}((\alpha+i\beta)+(\gamma+i\delta)) \cdot_J x &= (\alpha+\gamma) \cdot_R x + (\beta+\delta) \cdot_R J(x) \\ &= (\alpha \cdot_R x + \beta \cdot_R J(x)) + (\gamma \cdot_R x + \delta \cdot_R J(x)) \\ &= (\alpha+i\beta) \cdot_J x + (\gamma+i\delta) \cdot_J x \ ;\end{aligned}$$

3) $$\begin{aligned}(\alpha+i\beta) \cdot_J ((\gamma+i\delta) \cdot_J x) &= \alpha \cdot_R (\gamma \cdot_R x + \delta \cdot_R J(x)) + \beta \cdot_R J(\gamma \cdot_R x + \delta \cdot_R J(x)) \\ &= (\alpha\gamma - \beta\delta) \cdot_R x + (\alpha\delta + \beta\gamma) \cdot_R J(x) \\ &= ((\alpha+i\beta)(\gamma+i\delta)) \cdot_J x \ ;\end{aligned}$$

4) $1 \cdot_J x = 1 \cdot_R x = x$. ¶

Thus for a given real vector space $(V,+,\cdot_R)$, there is a 1-1 correspondance between the set $\{(V,+,\cdot_J)\}$ of complex vector spaces **on V** and the set $\{J \mid J^2=-I\}$ of linear endomorphisms of $(V,+,\cdot_R)$.

6. Theorem: If $\{x_1,\ldots,x_n\}$ is a basis of $(V,+,\cdot_J)$, then $\{x_1,\ldots,x_n,J(x_1),\ldots,J(x_n)\}$ is a basis of $(V,+,\cdot_R)$:

proof: follows from the proof of theorem I.2 . ¶

Corollary: Only even-dimensional vector spaces can have complex structures.

From now on we will consider complex structures only on the specific vector spaces R^{2n}.

7. Definition: GL(n;R(or C)) = the general linear group of real (or complex) non-singular matrices of order n, where the group operation is matrix multiplication.

Since a complex structure J is a non-singular $(J^{-1}=-J)$ linear endomorphism, we can consider it as an element of GL(2n;R).

8. Definition: $\mathbf{J}_{2n} = \{J \in GL(2n;R) \mid J^2=-I_{2n}\} \subset GL(2n;R)$.

$\mathbf{J}_{2n} \neq \phi$ because $J_0 = \left[\begin{array}{c|c} O_n & I_n \\ \hline -I_n & O_n \end{array}\right] \in \mathbf{J}_{2n}$. We call J_0 the canonical complex structure.

$\mathbf{J}_{2n}$ is not a subgroup of GL(2n;R) because (for example) $I_{2n}, J_0^2 \notin \mathbf{J}_{2n}$.

Chapter II - (GROUP) ACTIONS AND EQUIVARIANCES

0. Now that the set $\mathcal{J}_{2n}$ has been defined, we take this chapter to build up some of the machinery of actions and equivariances. The important part of this theory for the next chapter is that concerning homogenous spaces.

The first part of the chapter is developed in the setting of general categories. For the rest of the chapter we restrict ourselves to the category of sets and maps because the theorems that we prove are not true in general for arbitrary categories. It is true that if one considers categories with underlying sets and maps, one can also develop a theory not too different from ours, e.g. for groups and homomorphisms, topological spaces and continuous maps, or smooth manifolds and differentiable maps, but additional theorems and conditions would be needed. Since our purposes are best served with the study of the category of sets and maps, we develop only this theory, and we prove other specific results when we need them.

1. Definitions: Let $\mathcal{C}$ be a category. We denote by obj($\mathcal{C}$) the class of objects in $\mathcal{C}$ and by hom(X,Y) the set of morphisms $f:X \to Y$ in $\mathcal{C}$.

2. Definition: Let $f \in \hom(X,Y)$. Then f is called an equivalence in $\mathcal{C}$ if $\exists\, g \in \hom(Y,X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$ (ie. f has a 2-sided inverse g in $\mathcal{C}$).

3. Definition: Let $X \in \text{obj}(\mathcal{C})$. Then
$\text{Aut } X = \{ f \in \hom(X,X) \mid f \text{ is an equivalence} \}$ ($\neq \phi$).

Aut X becomes a group under composition in $\mathcal{C}$. This group is sometimes called the automorphism group of X in $\mathcal{C}$ or the group of symmetries of X in $\mathcal{C}$.

If $\mathcal{C}$ is the category of sets and maps, then Aut X consists of the set of all bijections (or permutations) of X.

Although generally it is not a group, we will sometimes use the notation $\mathrm{Aut}(X,Y) = \{f \varepsilon \hom(X,Y) \mid f \text{ is an equivalence}\}$.

4. Definition: Let G be a group. Let $\mathcal{C}$ be a category, and let $X \varepsilon \mathrm{obj}(\mathcal{C})$. Then a (left) (group) action ϕ of G on X in $\mathcal{C}$ is a group homomorphism

$$\phi : G \to \mathrm{Aut}\ X.$$

ϕ is also called a representation of G by automorphisms of X in $\mathcal{C}$. (X,ϕ) is called a G-object wrt ϕ, and G is called a transformation group acting on X via ϕ.
We will denote $\phi(g):X \to X$ by ϕ_g.

5. If $\mathcal{C}$ is the category of sets and functions, then an action has the following equivalent definition:

An action ϕ is a map $\phi:G\times X \to X$ satisfying:
1) $\phi(g_1g_2,x) = \phi(g_1,\phi(g_2,x)) \ \forall g_1,g_2 \varepsilon G,\ \forall x \varepsilon X;$
2) $\phi(e,x) = x \ \forall x \varepsilon X$, where e is the neutral element of G.

6. Definition: Let $(G,\cdot)$ be a group. Then we define the opposite group of G to be the group $(G,\circ)$ (which we will also denote by G^o) where $\circ:G\times G \to G$ is defined as

$$g_1 \circ g_2 = g_2 \cdot g_1 \ \forall\ g_1, g_2 \ \varepsilon\ G.$$

G^o is isomorphic to G by the isomorphism $x \mapsto x^{-1}$.

7. Definition: Let G be a group. Let $\mathcal{C}$ be a category, and let $X \varepsilon \mathrm{obj}(\mathcal{C})$. Then a right (group) action ϕ of G on X in $\mathcal{C}$ is a group homomorphism

$$\phi : G^o \to \mathrm{Aut}\ X.$$

Again, if $\mathcal{C}$ is sets and maps, a right action can be defined equivalently as a map $\phi:X\times G \to X$ satisfying:
1) $\phi(x,g_1g_2) = \phi(\phi(x,g_1),g_2) \ \forall g_1,g_2 \varepsilon G,\ \forall x \varepsilon X;$
2) $\phi(x,e) = x \ \forall x \varepsilon X$, where e is the neutral element of G.

8. Note that the only reason why our definition of a left action appears more natural than that of a right action is because

of our convention of writing the function symbol to the left of the element symbol. (ie. we write fx, not xf.)

9. Example: Let $\mathcal{C}$ be the category of sets and maps. Let G be any group. Then the set $G \in \mathrm{obj}(\mathcal{C})$. We define $L:G \to \mathrm{Aut}\ G$ by $L(g) = L_g : G \to G$ where $L_g(x) = gx$ (ie. left translation by g). Then L_g is a bijection $\forall g \in G$. Also, L is a group homomorphism because $L_{g_1 g_2}(x) = g_1 g_2 x = g_1 L_{g_2}(x) = (L_{g_1} L_{g_2})(x)$ $\forall x \in G$, $\forall g_1, g_2 \in G$. Thus L is a (left) action of G on the set G in $\mathcal{C}$.

In a similar way, we define $R:G \to \mathrm{Aut}\ G$ where $R_g(x) = xg$. Then R is a right action of G on the set G in $\mathcal{C}$.

Note that L and R would not be actions if one took the category of groups and homomorphisms. (The L_g's and R_g's would not be group isomorphisms.)

10. Example: Let $\mathcal{C}$ be the category of Lie groups and smooth (C^∞) homomorphisms, and let R^n have the (additive) Lie group structure. Define an action (which depends on the choice of basis for R^n) $\phi: GL(n;R) \to \mathrm{Aut}\ R^n$ where $\phi_A(\underline{x}) = A\underline{x}$. Then ϕ is an action of $GL(n;R)$ on the Lie group R^n in $\mathcal{C}$. (ϕ is also an action if $\mathcal{C}$ is sets and maps, topological vector spaces and continuous linear operators, differentiable manifolds and smooth maps, etc.)

11. Definition: Let $\mathcal{C}$ be a category, let $X, X' \in \mathrm{obj}(\mathcal{C})$, let G, G' be two groups, let $\phi: G \to \mathrm{Aut}\ X$, $\phi': G' \to \mathrm{Aut}\ X'$ be two actions, and let $\rho: G \to G'$ be a group homomorphism. Then a ρ-equivariant morphism Φ is a $\Phi \in \mathrm{hom}(X,X')$ such that $\forall g \in G$, the diagram

$$\begin{tikzcd}
X \arrow[r, "\Phi"] \arrow[d, "\phi_g"'] & X' \arrow[d, "\phi'_{\rho(g)}"] \\
X \arrow[r, "\Phi"'] & X'
\end{tikzcd}$$

commutes.

In particular, if $G = G'$ and $\rho = id_G$, then we call $\Phi \in \hom(X,X')$ an <u>equivariance</u>. (ie. Φ is "compatible" with the G-actions on X and X'.)

Clearly the composition of equivariances is also an equivariance.

12. In case $\mathcal{C}$ is sets and maps, then $\Phi \in \hom(X,X')$ is a ρ-equivariant morphism if

$$\begin{tikzcd} G\times X \arrow[r, "\rho\times\Phi"] \arrow[d, "\phi"'] & G'\times X' \arrow[d, "\phi'"] \\ X \arrow[r, "\Phi"'] & X' \end{tikzcd}$$

commutes.

13. <u>Example</u>: Let $\mathcal{C}$ be sets and maps, let our group be the real line $(R,+)$, and take as objects $R \in obj(\mathcal{C})$. Define the R-actions as $\phi: R \to Aut\ R$ where $\phi_g(x) = g + x$ and $\phi': R \to Aut\ R$ where $\phi'_g(x) = e^g x$. Then $\Phi: R \to R$ where $\Phi(x) = e^x$ is an equivariance.

$$\begin{tikzcd} R \arrow[r, "e^x"] \arrow[d, "g+x"'] & R \arrow[d, "e^g x"] \\ R \arrow[r, "e^x"'] & R \end{tikzcd}$$

14. <u>Theorem</u>: Let $\mathcal{C}$ be any category, and let G be any group. Then all G-objects in $\mathcal{C}$ together with all equivariant morphisms in $\mathcal{C}$ form a category which we will call <u>$\mathcal{C}(G)$</u>. (The composition is the same as in $\mathcal{C}$.)

<u>proof</u>: Since $1_X \in \hom_{\mathcal{C}}(X,X)$ commutes with every morphism in $\hom_{\mathcal{C}}(X,X)$, 1_X is an equivariant morphism and so $1_X \in \hom_{\mathcal{C}(G)}((X,\phi),(X,\phi))$. The other conditions follow trivially. ¶

We cannot in general define a similar category where the morphisms are ρ-equivariances because in general composition of ρ-equivariances is not ~~a~~ ρ-equivariance and in general the identities 1_X are not ρ-equivariances.

15. **Theorem**: Let $\Phi \varepsilon \hom_{\mathcal{C}(G)}((X,\phi),(X',\phi'))$. Then $\Phi \varepsilon \mathrm{Aut}_{\mathcal{C}(G)}((X,\phi),(X',\phi'))$ *iff* $\Phi \varepsilon \mathrm{Aut}_{\mathcal{C}}(X,X')$.

proof: One direction is trivial. In the other direction, let $\Phi \varepsilon \mathrm{Aut}_{\mathcal{C}}(X,X')$. Then $\exists$ $\Psi \varepsilon \mathrm{Aut}_{\mathcal{C}}(X',X)$ such that $\Phi\Psi = 1_{X'}$ and $\Psi\Phi = 1_X$. But $\Phi \varepsilon \hom_{\mathcal{C}(G)}((X,\phi),(X',\phi'))$ means $\Phi\phi_g = \phi'_g\Phi$ $\forall g \varepsilon G$. Thus $\forall g \varepsilon G$ we have

$$\Phi\phi_g=\phi'_g\Phi \rightarrow \Psi\Phi\phi_g\Psi=\Psi\phi'_g\Phi\Psi \rightarrow 1_X\phi_g\Psi=\Psi\phi'_g 1_{X'} \rightarrow \phi_g\Psi=\Psi\phi'_g .$$

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\phi_g"'] & X' \arrow[d, "\phi'_g"] \\ X \arrow[r, "\Phi"'] & X' \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "\Psi"] \arrow[d, "\phi'_g"'] & X \arrow[d, "\phi_g"] \\ X' \arrow[r, "\Psi"'] & X \end{tikzcd}$$

Thus Ψ is an equivariance and so $\Phi \varepsilon \mathrm{Aut}_{\mathcal{C}(G)}((X,\phi),(X',\phi'))$. ¶

16. **Definition**: Let (X,ϕ) and (X',ϕ') be two G-objects in $\mathcal{C}$. Then (X,ϕ) and (X',ϕ') are called *isomorphic* G-objects if $\exists$ $\Phi \varepsilon \mathrm{Aut}_{\mathcal{C}(G)}((X,\phi),(X',\phi'))$ (ie. Φ is an equivariant equivalence).

We see that here "isomorphic" has its usual meaning, namely, that isomorphic G-objects are indistinguishable as G-objects.

17. **Definitions**: Let X be a G-object in $\mathcal{C}$ via $\phi:G \rightarrow \mathrm{Aut}\, X$. Then we define $K =$ *ker* ϕ $= \{g\varepsilon G \mid \phi_g=1_X\} \subset G$. ker ϕ is a normal subgroup of G $(gkg^{-1} \varepsilon K \ \forall g\varepsilon G, \forall k\varepsilon K)$.

We call the action ϕ *effective* if ker ϕ is trivial. (ie. if there is a faithful representation of G by automorphisms of X in $\mathcal{C}$.)

18. **Theorem**: Let G act on X in $\mathcal{C}$ via $\phi:G \rightarrow \mathrm{Aut}\, X$. Let $K = \ker \phi$. Then there is a natural way of inducing an effective action on X, called "dividing the kernel out". Define $\psi:G/K \rightarrow \mathrm{Aut}\, X$ by $\psi(gK) = \phi(g)$. Then ψ is an effective action of G/K on X in $\mathcal{C}$.

proof:

ψ is well-defined: $g_1K=g_2K \rightarrow g_1^{-1}g_2 \varepsilon K \rightarrow \phi(g_1^{-1}g_2)=1_X \rightarrow \phi(g_1)=\phi(g_2)$.

ψ is a homomorphism: $\psi(g_1Kg_2K)=\psi(g_1g_2K)=\phi(g_1g_2)=\phi(g_1)\phi(g_2)=\psi(g_1K)\psi(g_2K)$

ψ is effective: $g_1K \neq g_2K \rightarrow g_1^{-1}g_2 \notin K \rightarrow \phi(g_1^{-1}g_2) \neq 1_X \rightarrow \phi(g_1) \neq \phi(g_2)$. ¶

19. For the rest of this chapter, we restrict ourselves to the category of sets and maps.

20. <u>Definition</u>: Let X be a G-object in $\mathcal{C}$ via ϕ, and let $x_0 \varepsilon X$. Then define $O(x_0) = \{x\varepsilon X \mid \exists g\varepsilon G \text{ st } \phi_g(x_0)=x\} = \bigcup_{g\varepsilon G} \{\phi_g(x_0)\} \subset X$ to be the <u>orbit</u> of x_0 under ϕ.

Clearly, the orbits in X are equivalence classes under the equivalence relation: $x_1 \sim x_2$ <u>if</u> $\exists g\varepsilon G$ such that $\phi_g(x_1)=x_2$.
The set of all orbits of X we will denote by $X/_\phi G$ (ie. the <u>orbit space</u>) .

21. <u>Theorem</u>: Let ϕ be an action of G on X, and let $O(x_0)$ be an orbit in X. Then $\psi: G \rightarrow \text{Aut}\, O(x_0)$ where $\psi_g = \phi_g|O(x_0)$ is an action of G on $O(x_0)$.

<u>proof</u>: We show $\psi_{g_0}(O(x_0)) = O(x_0) \ \forall g_0 \varepsilon G$.

Let $g_0 \varepsilon G$ and let $x \varepsilon O(x_0)$. Then $\exists g\varepsilon G$ such that $x=\phi_g(x_0)$.
Then $\psi_{g_0}(x)=\phi_{g_0}(x)=\phi_{g_0}\phi_g(x_0)=\phi_{g_0 g}(x_0)\varepsilon O(x_0) \rightarrow \psi_{g_0}(O(x_0)) \subset O(x_0)$,
and $\psi_{g_0}(\phi_{g_0^{-1}g}(x_0)) = x \rightarrow \psi_{g_0}(O(x_0)) \supset O(x_0)$. ¶

22. <u>Definition</u>: Let ϕ be an action of G on X, and let $x_0 \varepsilon X$.
Then we define $G_{x_0} = \{g\varepsilon G \mid \phi_g(x_0)=x_0\} \subset G$ to be the <u>isotropy subgroup</u> of G at x_0 , or the <u>stabilizer</u> of x_0 under ϕ.

Clearly, G_{x_0} is a subgroup of G.

We call the action ϕ <u>free</u> if G_{x_0} is trivial $\forall x_0 \varepsilon X$.

23. <u>Theorem</u>: Let ϕ be an action of G on X. Then

$$\ker \phi = \bigcap_{x_0 \varepsilon X} G_{x_0} .$$

<u>proof</u>: $g\varepsilon \ker \phi \leftrightarrow \phi_g = 1_X \leftrightarrow g\varepsilon G_{x_0} \ \forall x_0 \varepsilon X \leftrightarrow g\varepsilon \bigcap_{x_0\varepsilon X} G_{x_0}$. ¶

<u>Corollary</u>: Every free action is effective.

24. *Theorem*: Let G act on X via ϕ, and let $x, y \in X$ lie in a common orbit (then $\exists\, g_0 \in G$ st $y=\phi_{g_0}(x)$ and $x=\phi_{g_0^{-1}}(y)$). Then $G_y = g_0 G_x g_0^{-1}$ (ie. G_x and G_y are conjugate subgroups of G).

proof: $g \in G_x \to \phi_{g_0 g g_0^{-1}}(y)=\phi_{g_0 g}(x)=\phi_{g_0}(x)=y \to g_0 g g_0^{-1} \in G_y$ and so $g_0 G_x g_0^{-1} \subset G_y$. $g_0^{-1} G_y g_0 \subset G_x$ follows similarly. ¶

25. *Definition*: Let G act on X via ϕ. Then ϕ is called a *transitive* action (and X is called a *homogenous* G-set) if for any two points $x, y \in X$, $\exists\, g \in G$ such that $y = \phi_g(x)$. (ie. if there is exactly one orbit in X.)

26. *Theorem*: Let G be a group, and $H \subset G$ a subgroup. Let G/H be the *set* of all left cosets (ie. $G/H = \{gH \mid g \in G\}$, which in general is not a group). Then $\phi : G \to \operatorname{Aut} G/H$ where $\phi_g(\gamma H) = g\gamma H$ is a transitive action.

proof: Clearly every ϕ_g is a bijection. Since $\phi_{g_1 g_2}(\gamma H) = g_1 g_2 \gamma H = g_1 \phi_{g_2}(\gamma H) = \phi_{g_1}\phi_{g_2}(\gamma H)$, ϕ is a homomorphism. And since $\phi_{g_2 g_1^{-1}}(g_1 H) = g_2 H$, ϕ is transitive. ¶

27. *Theorem*: Conversely, let X be a homogenous (left) G-set via ϕ. Then $\exists$ subgroup $H \subset G$ such that X is isomorphic (as a G-set) to the left coset space G/H, where the action ϕ' on G/H is defined by $\phi'_g(\gamma H) = g\gamma H$.

proof: Take any $x_0 \in X$. Then $G_{x_0} = \{g \in G \mid \phi_g(x_0)=x_0\}$ is the isotropy subgroup of G at x_0. Define map $f : X \to G/G_{x_0}$ by $f(x) = gG_{x_0}$ where g is such that $x = \phi_g(x_0)$. This is possible because X is homogenous. Let $x, y \in X$. Then $\exists\, g_1, g_2 \in G$ st $x = \phi_{g_1}(x_0)$, $y = \phi_{g_2}(x_0)$ (ie. $f(x) = g_1 G_{x_0}$, $f(y) = g_2 G_{x_0}$). Then f is well-defined and 1-1 because $x=y \leftrightarrow \phi_{g_1}(x_0)=\phi_{g_2}(x_0)$

$\leftrightarrow x_0=\phi_{g_1^{-1} g_2}(x_0) \leftrightarrow g_1^{-1} g_2 \in G_{x_0} \leftrightarrow g_1 G_{x_0}=g_2 G_{x_0} \leftrightarrow f(x)=f(y)$.

f is onto because $f(\phi_g(x_0)) = gG_{x_0}$. Thus f is a bijection. To show that f is an equivariance, we will show that $\forall\, g\varepsilon G$ the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_g"'] & G/G_{x_0} \arrow[d, "\phi'_g"] \\ X \arrow[r, "f"'] & G/G_{x_0} \end{tikzcd}$$

commutes.

We have $(f\phi_g f^{-1})(\gamma G_{x_0}) = f\phi_g(\phi_\gamma(x_0)) = f(\phi_{g\gamma}(x_0)) = g\gamma G_{x_0}$ $= \phi'_g(\gamma G_{x_0})$ and so $f\phi_g = \phi'_g f$. ¶

The action $\phi'_g(\gamma H) = g\gamma H$ on G/H we will call the <u>canonical action</u> on G/H .

In the above two theorems, we have an analogous result if we consider right cosets and right actions.

28. <u>Definition</u>: Let G be a group and $H \subset G$ a subgroup. Then define $N(H) = \{g\varepsilon G \mid gHg^{-1}=H\}$ to be the <u>normalizer</u> of H in G.

N(H) is the largest subgroup of G in which H is normal.

29. <u>Theorem</u>: Let G be a group and $H \subset G$ a subgroup. Thus G/H is a (left) G-set with the canonical action $\phi_g(\gamma H) = g\gamma H$. Then as <u>groups</u>, $\mathrm{Aut}_{\mathcal{C}(G)}(G/H,\phi)$ is isomorphic to $N(H)/H$.

<u>proof</u>: Define action $\Phi: N(H) \to \mathrm{Aut}_{\mathcal{C}(G)}(G/H,\phi)$ where

$$\Phi_n(gH) = gn^{-1}H .$$

Φ_n is well-defined because $g_1H = g_2H \to g_1Hn^{-1} = g_2Hn^{-1}$ $\to g_1n^{-1}H = g_2n^{-1}H \to \Phi_n(g_1H) = \Phi_n(g_2H)$.

Φ_n is a bijection because $\Phi_n^{-1}(gH) = gnH$ is the inverse.

Φ_n is an equivariance because $\Phi_n\phi_g(\gamma H) = \Phi_n(g\gamma H)$ $= g\gamma n^{-1}H = \phi_g(\gamma n^{-1}H) = \phi_g\Phi_n(\gamma H)$.

Φ is a homomorphism because $\Phi_{n_1n_2}(gH) = gn_2^{-1}n_1^{-1}H$ $= \Phi_{n_1}(gn_2^{-1}H) = \Phi_{n_1}\Phi_{n_2}(gH)$.

$\ker \Phi = \{n\varepsilon N(H) \mid \Phi_n = 1_{G/H}\}$. But because $\Phi_n = 1_{G/H}$ $\leftrightarrow \Phi_n(\gamma H) = \gamma H \;\forall \gamma\varepsilon G \leftrightarrow \gamma n^{-1}H = \gamma H \;\forall \gamma\varepsilon G \leftrightarrow n^{-1}H = H \leftrightarrow n\varepsilon H$, we have $\ker \Phi = H$. Thus by theorem II.18, there is an induced effective action $\Psi: N(H)/H \to \text{Aut}_{\mathcal{C}(G)}(G/H,\phi)$ where $\Psi_{nH}(gH) = gn^{-1}H$. To show Ψ is a (group) isomorphism it remains to show that Ψ is onto. Thus let $\alpha \varepsilon \text{Aut}_{\mathcal{C}(G)}(G/H,\phi)$. Then call $\alpha(H) = nH$.

Since α is an equivariance, $\alpha\phi_g(H) = \phi_g\alpha(H) \to \alpha(gH) = g\alpha(H) = gnH$, and so α is completely determined by its value $\alpha(H)=nH$.

Since α is well-defined, $hH=H \to hnH=nH \to n^{-1}hnH=H \to n^{-1}hn\varepsilon H$.

Thus $n^{-1}Hn \subset H$.

Since α is bijective, α has an inverse α^{-1} which is equivariant (by II.15). Thus α^{-1} is also completely determined by its value $\alpha^{-1}(H) = n'H$. And since α^{-1} is well-defined we also have $n'^{-1}Hn' \subset H$. n and n' are thus related:

$gH=1_{G/H}(gH)=\alpha\alpha^{-1}(gH)=\alpha(gn'H)=gn'nH \to H=n'nH \to n'n=h_0\varepsilon H \to n'=h_0n^{-1}$.

Thus $n'^{-1}Hn' \subset H \to nh_0^{-1}Hh_0n^{-1} \subset H \to nh_0^{-1}hh_0n^{-1} \varepsilon H \;\; \forall h\varepsilon H$

$\to nh_0^{-1}(h_0hh_0^{-1})h_0n^{-1} \varepsilon H \;\; \forall h\varepsilon H \to nhn^{-1} \varepsilon H \;\; \forall h\varepsilon H$

$\to nHn^{-1} \subset H \to H \subset n^{-1}Hn$.

Thus $H = n^{-1}Hn$ and so $n \varepsilon N(H)$.

Thus $\alpha = \Psi_{n^{-1}H}$ and so Ψ is an onto map. ¶

30. Corollary: If (X,ψ) is a homogenous G-set, then $\text{Aut}_{\mathcal{C}(G)}(X,\psi)$ is isomorphic to $N(H)/H$, where $H = G_{x_0}$ for any $x_0 \varepsilon X$.

31. Theorem: Let $\phi: G \to \text{Aut } X$ be an action, let $x_0\varepsilon X$, and let G_{x_0} be the isotropy subgroup at x_0 . Thus by II.21 , $\psi: G \to \text{Aut } O(x_0)$ where $\psi_g = \phi_g | O(x_0)$ is also an action. Then $\ker\psi$ is the largest normal subgroup of G contained in G_{x_0} .

proof: Clearly $\ker\psi$ is normal in G and $\ker\psi \subset G_{x_0}$.

From II.23, $\ker\psi = \bigcap_{x\varepsilon O(x_0)} G_x = \bigcap_{g\varepsilon G} G_{\phi_g(x_0)} \underset{.24)}{\overset{(\text{II}}{=}} \bigcap_{g\varepsilon G} gG_{x_0}g^{-1}$.

Take any $M \subset G_{x_0}$ which is normal in G . Then $gMg^{-1} = M \;\; \forall g\varepsilon G$, and so

$M = \bigcap_{g\varepsilon G} gMg^{-1} \subset \bigcap_{g\varepsilon G} gG_{x_0}g^{-1} = \ker\psi$. ¶

32. **Corollary**: If $\phi : G \to \mathrm{Aut}\ X$ is a transitive action, then for any $x_0 \in X$, $\ker \phi$ is the largest normal subgroup of G contained in G_{x_0}.

33. **Corollary**: If $\phi : G \to \mathrm{Aut}\ G/H$ is the canonical action, then $\ker \phi = \bigcap_{g\in G} gHg^{-1}$ is the largest normal subgroup of G contained in H.

proof: $G_{eH} = \{g\in G \mid \phi_g(eH)=eH\} = \{g\in G \mid gH=H\} = H$. ¶

34. **Theorem**: Let H be a subgroup of G. Then $N = \{h\in H \mid ghg^{-1}\in H\ \forall g\in G\} \subset H \subset G$ is the largest normal subgroup of G contained in H.

proof:

1) N is a subgroup of H:
$n\in N \to gng^{-1}\in H\ \forall g\in G \to gn^{-1}g^{-1}\in H\ \forall g\in G \to n^{-1}\in N$.
$n_1, n_2\in N \to gn_1g^{-1}\in H,\ gn_2g^{-1}\in H\ \forall g\in G \to gn_1n_2g^{-1}\in H\ \forall g\in G \to n_1n_2\in N$.

2) N is normal in G:
$n\in N \to g_0ng_0^{-1}\in H\ \forall g_0\in G \to g(g_0ng_0^{-1})g^{-1}\in H\ \forall g_0, g\in G$
$\to g_0ng_0^{-1}\in N\ \forall g_0\in G$. Thus $gNg^{-1}\subset N\ \forall g\in G$ and so $gNg^{-1} = N\ \forall g\in G$.

3) Let $M\subset H$ be normal in G. Then $gMg^{-1} = M\ \forall g\in G$.
Then $n\in M \to gng^{-1}\in M\subset H\ \forall g\in G \to n\in N$, and so $M\subset N$. ¶

Chapter III - THE SET OF COMPLEX STRUCTURES

0. There are natural bijections between the set $M_n(R)$ (or $M_n(C)$) of all n-square real (or complex) matrices and the set R^{n^2} (or C^{n^2}) of n^2-tuples. Any such natural bijection can be used to give the set $M_n(R)$ a topology, and a smooth (C^∞) manifold structure whose complete atlas is generated by a single chart. Thus every subset of $M_n(R)$ can be topologized by the induced topology, and every open subset of $M_n(R)$ can be given a manifold structure which would make it a regular submanifold of $M_n(R)$.

Since the determinant function $\det: M_n(R) \to R$ is continuous (it is even smooth), we have that $GL(n;R)$ is an open subset of $M_n(R)$. Thus $GL(n;R)$ has a natural topology and manifold structure. Since the operations of matrix multiplication and inversion are smooth, we have that $GL(n;R)$ is an n^2-dimensional Lie group. Since $\mathcal{J}_{2n}$ is a subset of $GL(2n;R)$ (and of $M_{2n}(R)$), the natural topology to give it is the induced topology of $M_{2n}(R)$. But $\mathcal{J}_{2n}$ is not an open subset of $M_{2n}(R)$ (though it is a closed subset). Thus we will attempt to give $\mathcal{J}_{2n}$ a "natural" manifold structure in a different way. By defining the transitive action on $\mathcal{J}_{2n}$, we automatically ensure the existence of such a (unique) manifold structure (cf. Warner , p. 120) .

1. Define action $\phi: GL(2n;R) \to \text{Aut}\, \mathcal{J}_{2n}$ where $\phi_g(J) = gJg^{-1}$. Since $(gJg^{-1})^2 = -I_{2n}$ $\forall J \in \mathcal{J}_{2n}$, we have $\phi_g(\mathcal{J}_{2n}) \subset \mathcal{J}_{2n}$. Clearly every ϕ_g is a homeomorphism. Also, ϕ is a homomorphism because $\phi_{g_1 g_2}(J) = g_1 g_2 J g_2^{-1} g_1^{-1} = g_1 \phi_{g_2}(J) g_1^{-1} = \phi_{g_1}\phi_{g_2}(J)$. Thus ϕ is a (left) action of $GL(2n;R)$ on the set and topological space $\mathcal{J}_{2n}$.

2. <u>Theorem</u>: The action ϕ of $GL(2n;R)$ on the set $\mathcal{J}_{2n}$ is transitive.

<u>proof</u>: Let $J, J' \varepsilon \mathcal{J}_{2n}$. Then by I.6, $\exists$ bases of R^{2n} $\{x_1,\ldots,x_n,Jx_1,\ldots,Jx_n\}$ and $\{x'_1,\ldots,x'_n,J'x'_1,\ldots,J'x'_n\}$.

Then define $g\varepsilon GL(2n;R)$ by $g(x_k)=x'_k$ and $g(Jx_k)=J'x'_k$ $k=1,\ldots,n$.

Then

$$
\begin{aligned}
gJg^{-1}(x) &= gJg^{-1}(a_1x'_1+\ldots+a_nx'_n+b_1J'x'_1+\ldots+b_nJ'x'_n)\\
&= a_1\cdot gJg^{-1}(x'_1)+\ldots+a_n\cdot gJg^{-1}(x'_n)+b_1\cdot gJg^{-1}(J'x'_1)+\ldots+b_n\cdot gJg^{-1}(J'x'_n)\\
&= a_1\cdot gJ(x_1)+\ldots+a_n\cdot gJ(x_n)+b_1\cdot gJ(Jx_1)+\ldots+b_n\cdot gJ(Jx_n)\\
&= a_1\cdot J'x'_1+\ldots+a_n\cdot J'x'_n+b_1\cdot J'^2x'_1+\ldots+b_n\cdot J'^2x'_n\\
&= J'(x)\ .
\end{aligned}
$$

Thus $\phi_g(J) = J'$ and so ϕ is transitive. ¶

3. <u>Corollary</u>: $\mathcal{J}_{2n} = \{gJ_0g^{-1} \mid g\varepsilon GL(2n;R)\}$, where $J_0 = \left[\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right]$ is the canonical complex structure.

4. Thus by II.27, $\exists$ subgroup $H \subset GL(2n;R)$ such that $\mathcal{J}_{2n} \approx GL(2n;R)/H$. We take $H = G_{J_0}$, the isotropy subgroup at the canonical complex structure $J_0 = \left[\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right]$.

$G_{J_0} = \{g\varepsilon GL(2n;R) \mid \phi_g(J_0)=J_0\} = \{g\varepsilon GL(2n;R) \mid gJ_0=J_0g\}$

(ie. the centralizer of J_0 in $GL(2n;R)$). To obtain a more explicit characterization of G_{J_0}, we have the following

<u>lemma</u>: $G_{J_0} = \left\{ \left[\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right] \varepsilon\ GL(2n;R) \right\}$.

<u>proof</u>:

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \varepsilon\ G_{J_0} \leftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}
$$

$$
\leftrightarrow \begin{bmatrix} -B & A \\ -D & C \end{bmatrix} = \begin{bmatrix} C & D \\ -A & -B \end{bmatrix} \leftrightarrow \begin{cases} A = D \\ B = -C \end{cases} .
$$

¶

G_{J_0} is not a normal subgroup of $GL(2n;R)$ because, for example,

$$
\begin{bmatrix} 0 & 2I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} 0 & 2I_n \\ -I_n & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -2I_n & 0 \\ 0 & -I_n \end{bmatrix}\begin{bmatrix} 0 & -I_n \\ \frac{1}{2}I_n & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2I_n \\ -\frac{1}{2}I_n & 0 \end{bmatrix} \not\in G_{J_0} .
$$

5. **Theorem:** $GL(n;C)$ is (group) isomorphic to $G_{J_0} \subset GL(2n;R)$.

proof: Define $e: GL(n;C) \to G_{J_0}$, where $e(A+iB) = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$.

Since $A+iB$ is invertible $\leftrightarrow$ $\exists$ unique C, D st $(A+iB)(C+iD)=I_n$

$\leftrightarrow$ $\exists$ unique C, D st $AC-BD=I_n$ and $AD+BC=O_n$

$\leftrightarrow$ $\exists$ unique C, D st

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}\begin{bmatrix} C & D \\ -D & C \end{bmatrix} = \begin{bmatrix} AC-BD & AD+BC \\ -BC-AD & -BD+AC \end{bmatrix} = \begin{bmatrix} I_n & O \\ O & I_n \end{bmatrix} = I_{2n}$$

$\leftrightarrow$ $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ is invertible, it is clear that e is a bijection.

e is a homomorphism because

$$e((A+iB)(C+iD)) = e((AC-BD)+i(AD+BC)) = \begin{bmatrix} AC-BD & AD+BC \\ -AD-BC & AC-BD \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ -B & A \end{bmatrix}\begin{bmatrix} C & D \\ -D & C \end{bmatrix} = e(A+iB)e(C+iD) .$$ ¶

(Thus $GL(n;C)$ may be considered as a subgroup of $GL(2n;R)$.)

6. G_{J_0} and $GL(n;C)$ are even more strongly related:

Theorem: $\det\begin{bmatrix} A & B \\ -B & A \end{bmatrix} = |\det(A+iB)|^2 \quad \forall\, A, B \in M_n(R)$.

proof: Since the determinant is a continuous function and since the set of matrices in $M_n(C)$ whose eigenvalues are all distinct is dense in $M_n(C)$, it is sufficient to prove the theorem only for our dense subset. Furthermore, since every such matrix $A+iB$ is similar to a matrix in diagonal form and since $\det(A+iB) = \det((S+iT)(A+iB)(S+iT)^{-1})$ and $\det\begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \det(\begin{bmatrix} S & T \\ -T & S \end{bmatrix}\begin{bmatrix} A & B \\ -B & A \end{bmatrix}\begin{bmatrix} S & T \\ -T & S \end{bmatrix}^{-1})$, it is sufficient to prove the theorem only for diagonal matrices $A+iB$.

$$\text{Thus } \det\left[\begin{array}{ccc|ccc} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ \hline -b_1 & & & a_1 & & \\ & \ddots & & & \ddots & \\ & & -b_n & & & a_n \end{array}\right] = (a_1^2+(-1)^{2n}b_1^2)\dots(a_n^2+(-1)^2b_n^2)$$

$$= |a_1+ib_1|^2\dots|a_n+ib_n|^2 = |(a_1+ib_1)\dots(a_n+ib_n)|^2 = \left|\det\begin{bmatrix} a_1+ib_1 & & \\ & \ddots & \\ & & a_n+ib_n \end{bmatrix}\right|^2 .$$

7. <u>Corollary</u>: If $\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \varepsilon\ GL(2n;R)$, then $\det \begin{bmatrix} A & B \\ -B & A \end{bmatrix} > 0$.

8. From now on we will use the simplified notation G/H instead of $GL(2n;R)/G_{J_0}$.

9. The specific G-isomorphism $f:\mathcal{J}_{2n} \to G/H$ is given by $f(gJ_0g^{-1}) = gH$. If we give G/H the quotient topology, then f and f^{-1} are continuous functions, and so $\mathcal{J}_{2n}$ and G/H are homeomorphic as well.

There is a well-known general procedure (see Warner) to introduce a manifold structure on a homogenous space G/H if G is a Lie group and H is a closed subgroup; we however prefer to use an ad hoc method for our special case, as follows:

10. The set $V = \{ \begin{bmatrix} P \\ Q \end{bmatrix} \mid P\varepsilon M_n(R),\ Q\varepsilon GL(n;R) \}$ can be considered as an open subset of R^{2n^2} .

Define $\psi: V \to G/H$ where $\psi(\begin{bmatrix} P \\ Q \end{bmatrix}) = \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} H$.

ψ is 1-1 because $\psi(\begin{bmatrix} P \\ Q \end{bmatrix}) = \psi(\begin{bmatrix} P' \\ Q' \end{bmatrix}) \to \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} H = \begin{bmatrix} I_n & P' \\ O & Q' \end{bmatrix} H$

$$\to \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} = \begin{bmatrix} I_n & P' \\ O & Q' \end{bmatrix} \cdot h = \begin{bmatrix} I_n & P' \\ O & Q' \end{bmatrix} \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \begin{bmatrix} A-P'B & B+P'A \\ -Q'B & Q'A \end{bmatrix}$$

$$\to \begin{cases} O = -Q'B \\ I_n = A-P'B \end{cases} \to B = O \text{ (because } \det Q' \neq 0 \text{) and } A = I_n$$

$$\to \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} P' \\ Q' \end{bmatrix} .$$

11. <u>lemma</u>: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H = \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} H \ (\varepsilon G/H)$ <u>iff</u> $\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} -C \\ A \end{bmatrix}$ with $\det Q \neq 0$.

<u>proof</u>: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H = \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} H \leftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} = \begin{bmatrix} A & AP+BQ \\ C & CP+DQ \end{bmatrix} \varepsilon\ H$

$$\leftrightarrow \begin{bmatrix} -C \\ A \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \leftrightarrow \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} -C \\ A \end{bmatrix} . \quad ¶$$

12. We show that $\psi(V)$ is an open neighbourhood of the coset H homeomorphic to V (via ψ^{-1}) :

If we let $P = O$, $Q = I_n$ then we see that the coset $H \epsilon \psi(V)$. To show that $\psi(V)$ is an open subset in G/H we prove the

lemma: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H \epsilon \psi(V)$ iff $\det\begin{bmatrix} A & -C \\ C & A \end{bmatrix} \neq 0$.

proof: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H \epsilon \psi(V) \leftrightarrow \exists\ P, Q, \det Q \neq 0$ st $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H = \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} H$

$\overset{\text{(III}}{\underset{\text{.ii)}}{\leftrightarrow}} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} -C \\ A \end{bmatrix}$ with $\det Q \neq 0$

$\leftrightarrow \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} A & -C \\ C & A \end{bmatrix}$ with $\det Q \neq 0 \leftrightarrow \det\begin{bmatrix} A & -C \\ C & A \end{bmatrix} \neq 0$. ¶

Thus $\psi(V)$ is an open neighbourhood of H in G/H .

In general $\psi(V) \neq G/H$; for example,

$$\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}^{-1} = 1 \quad \text{but} \quad \det\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} = 0 .$$

It is not too difficult to see that ψ and ψ^{-1} are continuous: we omit the details.

13. Thus $(\psi(V), \psi^{-1})$ is a chart around $H \epsilon G/H$. Since the ϕ_g's are homeomorphisms, it is now clear how one would define a chart around any point in G/H : For any $gH \epsilon G/H$, $\phi_g \circ \psi(V)$ is an open nbd of gH . Since $\phi_g, \phi_g{}^{-1}, \psi, \psi^{-1}$ are continuous, both $\phi_g \circ \psi$ and $(\phi_g \circ \psi)^{-1} = \psi^{-1} \circ \phi_g{}^{-1}$ are also continuous. Thus $(\phi_g \circ \psi(V),\ (\phi_g \circ \psi)^{-1})$ is a chart around $gH \epsilon G/H$. Thus $\mathcal{J}_{2n}$ (through G/H) becomes a homogenous manifold of dimension $2n^2$.

14. One can generalize the lemma in III.12 :

lemma: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H \epsilon \phi_{\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}} \circ \psi(V)$ iff $\det\begin{bmatrix} AW+BY & -CW-DY \\ CW+DY & AW+BY \end{bmatrix} \neq 0$.

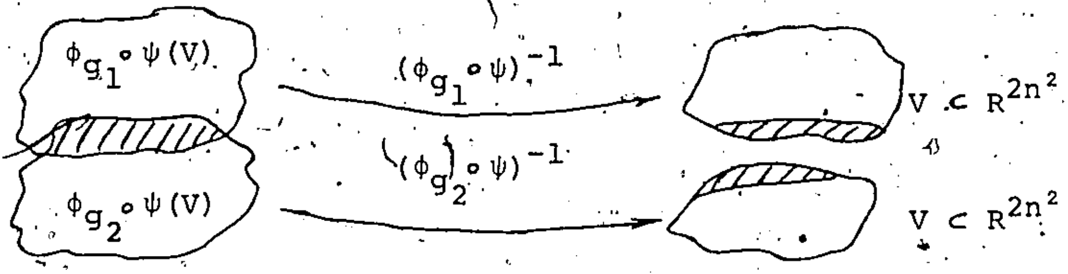
proof: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H \epsilon \phi_{\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}} \circ \psi(V) \leftrightarrow \exists\ P, Q, \det Q \neq 0$ st

$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} H \overset{\text{(III}}{\underset{\text{.ii)}}{\leftrightarrow}} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} -CW-DY \\ AW+BY \end{bmatrix}$ with $\det Q \neq 0$

$$\leftrightarrow \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} AW+BY & -CW-DY \\ CW+DY & AW+BY \end{bmatrix} \quad \text{with} \quad \det Q \neq 0$$

$$\leftrightarrow \det \begin{bmatrix} AW+BY & -CW-DY \\ CW+DY & AW+BY \end{bmatrix} \neq 0 \ . \quad ¶$$

15. We show that we have a smooth (C^∞) manifold by showing that the coordinate transformations are smooth.



Let $g = g_2^{-1} g_1$. Then $(\phi_{g_2} \circ \psi)^{-1} \circ (\phi_{g_1} \circ \psi) = \psi^{-1} \circ \phi_g \circ \psi$ is a map

$$\psi^{-1} \circ \phi_g \circ \psi : (\phi_{g_1} \circ \psi)^{-1} (\phi_{g_1} \circ \psi(V) \cap \phi_{g_2} \circ \psi(V)) \to (\phi_{g_2} \circ \psi)^{-1} (\phi_{g_1} \circ \psi(V) \cap \phi_{g_2} \circ \psi(V)) .$$

Take any $\begin{bmatrix} P \\ Q \end{bmatrix}$ in the domain of $\psi^{-1} \circ \phi_g \circ \psi$ above, denote g by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, and denote $\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} \right)^{-1}$ by $\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$.

Then $\begin{bmatrix} P \\ Q \end{bmatrix} \overset{\psi}{\mapsto} \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} H \overset{\phi_g}{\mapsto} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} H \overset{\psi^{-1}}{\mapsto} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} \begin{bmatrix} -C' \\ A' \end{bmatrix}$.

And because matrix multiplication and matrix inversion are smooth operations (they are even real analytic), we have that $\psi \circ \phi_g \circ \psi^{-1}$ is smooth.

Thus $\mathcal{J}_{2n}$ (through G/H) with the complete smooth atlas generated by the charts $\{(\phi_g \circ \psi(V), (\phi_g \circ \psi)^{-1}) \mid g \varepsilon G\}$ becomes a smooth (C^∞) homogenous manifold of dimension $2n^2$.

16. We have defined a smooth manifold structure on G/H in a certain way. What distinguishes this structure from other ones are two nice properties which, in fact, characterize our structure. We identify $M_{2n}(R)$ with R^{4n^2} and define $e: G \to M_{2n}(R)$ to be the inclusion map. (ie. (G,e) is the chart for G)

The first property is that the projection $\pi: G \to G/H$ is smooth, as is easily seen from: (for fixed $g = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$)

$$R^{4n^2} = M_{2n}(R) \xrightarrow{e^{-1}} G \xrightarrow{\pi} G/H \xrightarrow{(\phi_g\circ\psi)^{-1}} V \subset R^{2n^2}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \mapsto \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \mapsto \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H \overset{III}{\underset{.11}{\mapsto}} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} -CW-DY \\ AW+BY \end{bmatrix} .$$

The second property is that for each point in G/H there exists a <u>local</u> smooth function τ whose range is in G and is such that $\pi\circ\tau = id$:

$$R^{2n^2} \supset V \xrightarrow{\phi_g\circ\psi} G/H \xrightarrow{\tau_g} G \xrightarrow{e} M_{2n}(R) = R^{4n^2}$$

$$\begin{bmatrix} P \\ Q \end{bmatrix} \mapsto \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} H \mapsto \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix} \mapsto \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \begin{bmatrix} I_n & P \\ O & Q \end{bmatrix}$$

where $\tau_g : \phi_g\circ\psi(V) \to G$ and $\pi\circ\tau_g = id$ on $\phi_g\circ\psi(V)$.

17. Conversely, suppose we have any other complete smooth atlas defining a manifold structure on G/H that satisfies the two properties above. Denote this new structure by $(G/H)'$.

$$\begin{tikzcd} & G \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\ G/H \arrow[rr, "id"'] & & (G/H)' \end{tikzcd}$$

Then the identity map and its inverse are smooth at a nbd of every point and so are themselves smooth. Thus the identity map is a diffeomorphism and so we have $(G/H)' = G/H$.

18. Now that a smooth homogenous manifold structure has been defined on G/H (and on J_{2n}), we examine more closely its consequences.

19. In the case that $n = 1$, lemma III.12 tells us that $\psi(V) = G/H$, and so the manifold structure on J_{2n} can be generated by only one chart. This also means that J_{2n} is homeomorphic to a real plane with a straight line removed, ie. the topological sum $R^2 \vee R^2$.

    However, for higher n it is in general not true that the manifold structure can be specified by one of our charts, as the counterexample in III.12 shows.

20. <u>Theorem</u>: Let ϕ be the action defined in III.1 . Then $\ker \phi = \{ kI_{2n} \mid k \in R \setminus \{0\} \}$.

<u>proof</u>: By II.33 and II.34, $\ker \phi = \{h \in H \mid ghg^{-1} \in H \ \forall g \in G\}$. Thus all matrices of the form kI_{2n}, $k \neq 0$ belong to $\ker \phi$. To show that there are no others we take specific cases of $g \in G$ to eliminate the others.

We will denote by square partitioning $g=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, $g^{-1}=\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, $h=\begin{bmatrix} C & D \\ -D & C \end{bmatrix}$. Take case $A_2=A_3=O$, $A_4=I$, $A_1=2I$. Then $B_2=B_3=O$, $B_4=I$, $B_1=\frac{1}{2}I$. Then $ghg^{-1}=\begin{bmatrix} 2I & O \\ O & I \end{bmatrix}\begin{bmatrix} C & D \\ -D & C \end{bmatrix}\begin{bmatrix} \frac{1}{2}I & O \\ O & I \end{bmatrix}=\begin{bmatrix} 2C & 2D \\ -D & C \end{bmatrix}\begin{bmatrix} \frac{1}{2}I & O \\ O & I \end{bmatrix}=$ $\begin{bmatrix} C & 2D \\ -\frac{1}{2}D & C \end{bmatrix} \in H \rightarrow 2D=\frac{1}{2}D \rightarrow D=O$. Now take the simplified case $D=O$, $A_2=A_3=O=B_2=B_3$, $A_4=I=B_4$, $B_1=A_1^{-1}$. Then $ghg^{-1} =$ $\begin{bmatrix} A_1 & O \\ O & I \end{bmatrix}\begin{bmatrix} C & O \\ O & C \end{bmatrix}\begin{bmatrix} A_1^{-1} & O \\ O & I \end{bmatrix} = \begin{bmatrix} A_1CA_1^{-1} & O \\ O & C \end{bmatrix} \in H \rightarrow A_1CA_1^{-1} = C \ \forall A_1 \in GL(n;R)$.

For $1 \leq s,t \leq n$ $s \neq t$ define $\mathbb{1}_{st}$ to be the identity matrix except for the s,t th entry which we set $= 1$. Then $\mathbb{1}_{st}^{-1}$ is the identity matrix which has -1 in the s,t th entry.

Then $\mathbb{1}_{st}C\mathbb{1}_{st}^{-1} = \mathbb{1}_{st}\begin{bmatrix} c_{11} \cdots c_{1n} \\ \vdots \\ c_{n1} \cdots c_{nn} \end{bmatrix}\mathbb{1}_{st}^{-1} = \begin{bmatrix} c_{11} & \cdots c_{1n} \\ \vdots & \\ c_{s1}+c_{t1} & \cdots c_{sn}+c_{tn} \\ \vdots & \\ c_{n1} & \cdots c_{nn} \end{bmatrix}\mathbb{1}_{st}^{-1}$

$$= \begin{bmatrix} c_{11} & \cdots c_{1t}-c_{1s} \cdots & c_{1n} \\ \vdots & & \\ c_{s1}+c_{t1} & \cdots c_{st}+c_{tt} \cdots & c_{sn}+c_{tn} \\ & -c_{ss}-c_{ts} & \\ \vdots & & \\ c_{n1} & \cdots c_{nt}-c_{ns} \cdots & c_{nn} \end{bmatrix} = C$$ implies, if one takes all choices of s,t , that $c_{ij} = 0$ for $i \neq j$, and that $c_{ii} = c_{jj}$. ¶

Thus $\ker \phi = \{g \in G \mid gJ=Jg \ \forall J \in \mathcal{J}_{2n}\} = \{kI_{2n} \mid k \in R \setminus \{0\} \}$.

22. <u>lemma</u>: Let H be a subgroup of (any group) G. Then $\{g\varepsilon G \mid gHg^{-1}=H\} = \{g\varepsilon G \mid gHg^{-1} \subset H\}$ iff

$$g \ \varepsilon\ \{g\varepsilon G \mid gHg^{-1} \subset H\} \rightarrow g^{-1} \ \varepsilon\ \{g\varepsilon G \mid gHg^{-1} \subset H\}.$$

<u>proof</u>: trivial. ¶

23. <u>lemma</u>: If $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ has an inverse, it is of the form $\begin{bmatrix} C & D \\ -D & C \end{bmatrix}$; and if $\begin{bmatrix} A & B \\ B & -A \end{bmatrix}$ has an inverse, it is of the form $\begin{bmatrix} C & D \\ D & -C \end{bmatrix}$.

<u>proof</u>: The first part follows from III.4 or III.5. The second part follows from the observation, that

if $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}^{-1} = \begin{bmatrix} C & D \\ -D & C \end{bmatrix}$, then $\begin{bmatrix} A & B \\ B & -A \end{bmatrix}^{-1} = \begin{bmatrix} C & -D \\ -D & -C \end{bmatrix}$. ¶

24. <u>Theorem</u>: $N(H) = \{g\varepsilon G \mid gHg^{-1}=H\} = \{\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \varepsilon G\} \cup \{\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \varepsilon G\}$.

<u>proof</u>: Take $g\varepsilon G$. Then denote $g = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ and $g^{-1} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$. We will denote an $h\varepsilon H$ by $h = \begin{bmatrix} C & D \\ -D & C \end{bmatrix}$.

Then $g\varepsilon\{g\varepsilon G \mid gHg^{-1} \subset H\} \leftrightarrow ghg^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} C & D \\ -D & C \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$

$$= \begin{bmatrix} (A_1CB_1+A_2CB_3)+(A_1DB_3-A_2DB_1) & (A_1CB_2+A_2CB_4)+(A_1DB_4-A_2DB_2) \\ (A_3CB_1+A_4CB_3)+(A_3DB_3-A_4DB_1) & (A_3CB_2+A_4CB_4)+(A_3DB_4-A_4DB_2) \end{bmatrix} \varepsilon H \quad \forall h\varepsilon H$$

$$\leftrightarrow \begin{cases} (A_1CB_1+A_2CB_3)+(A_1DB_3-A_2DB_1) = (A_3CB_2+A_4CB_4)+(A_3DB_4-A_4DB_2) \\ -(A_3CB_1+A_4CB_3)-(A_3DB_3-A_4DB_1) = (A_1CB_2+A_2CB_4)+(A_1DB_4-A_2DB_2) \end{cases} \forall h\varepsilon H.$$

If we take $C = I_n$, then all the terms containing C cancel and only terms containing D remain. If we now take all possible sums of these equations where D has exactly one non-zero entry in each, and if we proceed in exactly the same way for the remaining terms containing C only, then we have that the last two equations are equivalent to the following:

$$\left.\begin{aligned} A_1CB_1+A_2CB_3 &= A_3CB_2+A_4CB_4 \\ -A_3CB_1-A_4CB_3 &= A_1CB_2+A_2CB_4 \\ A_1DB_3-A_2DB_1 &= A_3DB_4-A_4DB_2 \\ -A_3DB_3+A_4DB_1 &= A_1DB_4-A_2DB_2 \end{aligned}\right\} \text{ for } \underline{\text{every}}\ C \text{ and } D.$$

We now denote $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ by $\begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix}$ and $\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ by $\begin{bmatrix} w_{ij} & x_{ij} \\ y_{ij} & z_{ij} \end{bmatrix}$.

Define 1_{st} to be the matrix whose s,t th entry is 1 and 0 elsewhere. If we let C and D equal to 1_{st} for all values of s and t , then the four above equations are equivalent to:

$$\left.\begin{array}{ll} 1. & a_{is}w_{tj}+b_{is}y_{tj} = c_{is}x_{tj}+d_{is}z_{tj} \\ 2. & -c_{is}w_{tj}-d_{is}y_{tj} = a_{is}x_{tj}+b_{is}z_{tj} \\ 3. & a_{is}y_{tj}-b_{is}w_{tj} = c_{is}z_{tj}-d_{is}x_{tj} \\ 4. & -c_{is}y_{tj}+d_{is}w_{tj} = a_{is}z_{tj}-b_{is}x_{tj} \end{array}\right\} \quad \forall\ 1 \leq i,j,s,t \leq n \ . \qquad (*)$$

Then
$$\left.\begin{array}{l} 1.+4. \rightarrow (x_{tj}+y_{tj})(b_{is}-c_{is}) = (z_{tj}-w_{tj})(a_{is}+d_{is}) \\ 2.-3. \rightarrow (x_{tj}+y_{tj})(a_{is}+d_{is}) = -(z_{tj}-w_{tj})(b_{is}-c_{is}) \end{array}\right\} \forall 1 \leq i,j,s,t \leq n.$$

<u>case</u> $\exists\ t,j$ st $z_{tj}-w_{tj} \neq 0$

  <u>case</u> $x_{tj}+y_{tj}=0$: $a_{is}+d_{is}=0$, $b_{is}-c_{is}=0$ $\forall i,s$

  <u>case</u> $x_{tj}+y_{tj} \neq 0$: $$\left.\begin{array}{l} \frac{x+y}{z-w}(b-c)=(a+d) \\ \frac{x+y}{z-w}(a+d)=-(b-c) \end{array}\right\} \rightarrow a_{is}+d_{is}=0 \ \& \ b_{is}-c_{is}=0 \ \forall i,s$$

<u>case</u> $z_{tj}-w_{tj}=0$ $\forall t,j$

  <u>case</u> $\exists\ t,j$ st $x_{tj}+y_{tj} \neq 0$: $a_{is}+d_{is}=0$ & $b_{is}-c_{is}=0$ $\forall i,s$

  <u>case</u> $x_{tj}+y_{tj}=0$ $\forall t,j$:III.23 → $a_{is}-d_{is}=0$ & $b_{is}+c_{is}=0$ $\forall i,s$.

Thus any $g \varepsilon \{g \varepsilon G \mid gHg^{-1} \subset H\}$ can only have the form $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ or $\begin{bmatrix} A & B \\ B & -A \end{bmatrix}$. And by verifying the equations (*) we have that any non-singular matrix of the form $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ or $\begin{bmatrix} A & B \\ B & -A \end{bmatrix}$ is in our set.

But from III.22 and III.23 we have that N(H) is made up precisely of these matrices. ¶

25. <u>Corollary</u>: N(H)/H is a cyclic group of order 2 .

<u>proof</u>: This follows easily from the fact that $\begin{bmatrix} A & B \\ B & -A \end{bmatrix}\begin{bmatrix} C & D \\ D & -C \end{bmatrix} \varepsilon H$. ¶

26. <u>Corollary</u>: From II.30 it now follows that there are exactly two bijective equivariances from $(\mathbf{J}_{2n},\phi) \rightarrow (\mathbf{J}_{2n},\phi)$. (In fact there are exactly two equivariances as well, because we have: every equivariance on a homogenous G-set is bijective <u>iff</u> $\{g \varepsilon G \mid gHg^{-1}=H\} = \{g \varepsilon G \mid gHg^{-1} \subset H\}$.)

27. We can find the equivariances explicitly by substituting in the functions defined in II.29. As coset representatives we choose

$n_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ and $n_2 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

The first equivariance $\Psi_{n_1H}(gH) = gH$ is the identity map.

The second one is $\Psi_{n_2H}(gH) = gn_2^{-1}H = gn_2H$. But this has a nice interpretation in $\mathcal{J}_{2n}$: $\Psi_{n_2H}(gJ_0g^{-1}) = gn_2J_0n_2^{-1}g^{-1} = -gJ_0g^{-1}$.

Thus the second equivariance (in $\mathcal{J}_{2n}$) is the map which assigns to each matrix (or linear endomorphism) its inverse.

28. One of the topological properties of $\mathcal{J}_{2n}$ is that $\mathcal{J}_{2n}$ is not compact because, for example, the sequence
$\left\{\begin{bmatrix} 0 & kI_n \\ -\frac{1}{k}I_n & 0 \end{bmatrix}\right\}_{k=1}^{\infty} \subset \mathcal{J}_{2n}$ is unbounded.

29. Another topological property of $\mathcal{J}_{2n}$ is that it has exactly two (path) connected components. To show this we will use the well-known facts that $GL(n;C)$ is connected and that $GL(n;R)$ has precisely two connected components.

30. Define an equivalence relation $\sim$ on $\mathcal{J}_{2n}$ by
$J_1 \sim J_2$ <u>if</u> $\exists\, S$ $\det S > 0$ st $J_1 = SJ_2S^{-1}$.

31. <u>lemma</u>: If $J_1 \sim J_2$ and if $J_1 = TJ_2T^{-1}$, then $\det T > 0$.
<u>proof</u>: $J_1 \sim J_2 \rightarrow \exists\, S$ $\det S > 0$ st $J_1 = SJ_2S^{-1}$.
Also $J_2 \in \mathcal{J}_{2n} \xrightarrow[.3]{III} \exists\, V$ st $J_2 = VJ_0V^{-1}$.
Thus $SJ_2S^{-1} = TJ_2T^{-1} \rightarrow VJ_0V^{-1} = (S^{-1}T)VJ_0V^{-1}(S^{-1}T)^{-1}$
$\rightarrow J_0 = (V^{-1}S^{-1}TV)J_0(V^{-1}S^{-1}TV)^{-1} \rightarrow V^{-1}S^{-1}TV \in H$
$\xrightarrow[.7]{III} \det(V^{-1}S^{-1}TV) > 0 \rightarrow \det(S^{-1}T) > 0 \rightarrow \det(ST) > 0$.
And since $\det S > 0$, also $\det T > 0$. ¶

32. <u>Corollary</u>: If $J_1 \not\sim J_2$ then $\exists\, S$ $\det S < 0$ st $J_1 = SJ_2S^{-1}$.

33. <u>Corollary</u>: If $J_1 \not\sim J_2$ and $J_1 \not\sim J_3$, then $J_2 \sim J_3$.

proof: $J_1 = SJ_2S^{-1} = TJ_3T^{-1}$ where det S , det T < 0 .
Then $J_2 = (S^{-1}T)J_3(S^{-1}T)^{-1}$ where $\det(S^{-1}T) > 0$. ¶

34. Thus there are precisely two equivalence classes in $\mathcal{J}_{2n}$. If one takes $J_1 \sim J_2$ then $J_1 = SJ_2S^{-1}$ det $S > 0$. Then, because GL(2n;R) has two connected components which consist of those matrices of positive and negative determinant, there exists a path $p: [0,1] \to GL^+(2n;R)$ from I to S . Then $p': [0,1] \to$ equiv. class in $\mathcal{J}_{2n}$ defined by $p'(t) = p(t)J_2p(t)^{-1}$ is a path from J_2 to J_1 .

And because elements in distinct classes cannot be joined by a path, therefore the connected components of $\mathcal{J}_{2n}$ are precisely its two equivalence classes under $\sim$.

35. It is well-known that the set H of quaternions can by considered in a natural way as a complex vectorspace, whereby the complex structure is induced by the ring operations.

Since according to our general theory R^4 can carry an 8-dimensional manifold of complex structures, it seems worthwhile to investigate which of them can be considered "compatible" with the algebra operations.

36. We first state a useful lemma:

<u>lemma</u>: If J is a real square matrix, then any two of the following imply the third.

1) J is orthogonal: $JJ^{\tau} = I$;
2) J is a complex structure: $JJ = -I$;
3) J is skew-symmetric: $J^{\tau} = -J$.

37. Let H denote the algebra of quaternions. Then as a real vector space H is isomorphic to R^4 .

In the following whenever we use matrices to represent linear endomorphisms we will do so with respect to the canonical basis $\mathcal{B}$ of H . Also, our quaternion norm N will be defined wrt the canonical scalar product $xy = \Sigma x_iy_i$.

The simplest way in which one can consider a complex structure "compatible" with H is the following:
Let $q_0 \varepsilon H$. Then define R-linear endomorphisms ${}_{q_0}J(x) = q_0 \cdot_H x$ and $J_{q_0}(x) = x \cdot_H q$ $\forall x \varepsilon H$. Then

38. <u>Definition</u>: We say that the complex structure J can be <u>simply represented by a quaternion</u> if $\exists q_0 \varepsilon H$ st either $J = {}_{q_0}J$ or $J = J_{q_0}$.

39. Since ${}_{q_0}J^2(x) = q_0{}^2 x$ and $J_{q_0}{}^2(x) = x q_0{}^2$, ${}_{q_0}J$ and J_{q_0} are complex structures <u>iff</u> $q_0{}^2 = (-1,0,0,0)$.
And since $q_0{}^2 = (a,b,c,d)^2 = (a^2-b^2-c^2-d^2, 2ab, 2ac, 2ad)$, this is true <u>iff</u> $a=0$, $b^2+c^2+d^2=1$.

A simple calculation shows that the representation wrt $\mathcal{B}$ of such complex structures is

$${}_{q_0}J = \begin{bmatrix} 0 & -b & -c & -d \\ b & 0 & -d & c \\ c & d & 0 & -b \\ d & -c & b & 0 \end{bmatrix} \text{ and } J_{q_0} = \begin{bmatrix} 0 & -b & -c & -d \\ b & 0 & d & -c \\ c & -d & 0 & b \\ d & c & -b & 0 \end{bmatrix}, \text{ where } b^2+c^2+d^2=1 .$$

Thus ${}_{q_0}J$ and J_{q_0} are orthogonal and skew-symmetric wrt $\mathcal{B}$.

40. There is also a converse to the above. If we take any orthogonal (wrt $\mathcal{B}$) complex structure, then by III.36 it is also skew-symmetric. But

41. <u>lemma</u>: Every 4×4 orthogonal skewsymmetric matrix J has the form of one of the above two matrices.

<u>proof</u>:

J has the form $J = \begin{bmatrix} 0 & -b & -c & -d \\ b & 0 & x & y \\ c & -x & 0 & z \\ d & -y & -z & 0 \end{bmatrix}$, J orthogonal.

By applying algebraic manipulations and by distinguishing cases, one obtains the solutions for x,y,z : either $x=d$, $y=-c$, $z=b$ or $x=-d$, $y=c$, $z=-b$. ¶

42. Summarizing the above, we have the

<u>theorem</u>: If J is a complex structure on H, then
J can be (uniquely) simply represented by a quaternion
<u>iff</u> J is orthogonal (wrt $\mathcal{B}$)
<u>iff</u> J is skew-symmetric (wrt $\mathcal{B}$).

43. <u>Corollary</u>: The set of matrices which are complex structures simply representable by a quaternion is precisely the set of matrices which are orthogonal <u>and</u> skew-symmetric.

The set of such complex structures forms a submanifold of the manifold of all complex structures which is homeomorphic to two copies of the sphere S^2.

44. <u>Example</u>: Take the canonical complex structure

$$J = J_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \quad \text{Then } J_0 = J_{(0,0,-1,0)}. \text{ ie. } J_0(x) = x \cdot_H (-j).$$

45. There are other ways in which one can try to represent complex structures by quaternions.

One way is by $J(x) = p \cdot_H x + x \cdot_H q$, where $p=(p_0,p_1,p_2,p_3)$ and $q=(q_0,q_1,q_2,q_3)$ are constant. Then the representation wrt $\mathcal{B}$ is

$$J = \begin{bmatrix} p_0+q_0 & -p_1-q_1 & -p_2-q_2 & -p_3-q_3 \\ p_1+q_1 & p_0+q_0 & -p_3+q_3 & p_2-q_2 \\ p_2+q_2 & p_3-q_3 & p_0+q_0 & -p_1+q_1 \\ p_3+q_3 & -p_2+q_2 & p_1-q_1 & p_0+q_0 \end{bmatrix}.$$

If we take the special case $x=(1,0,0,0)$, then $J^2(x) = (p+q)^2 - qp + pq = (-1,0,0,0)$

$$\rightarrow \quad (p+q)^2 = ((p_0+q_0)^2-(p_1+q_1)^2-(p_2+q_2)^2-(p_3+q_3)^2, 2(p_0+q_0)(p_1+q_1), 2(p_0+q_0)(p_2+q_2), 2(p_0+q_0)(p_3+q_3))$$

$$= (-1, 2p_3q_2-2p_2q_3, 2p_1q_3-2p_3q_1, 2p_2q_1-2p_1q_2)$$

$$\rightarrow \begin{cases} (p_0+q_0)^2-(p_1+q_1)^2-(p_2+q_2)^2-(p_3+q_3)^2 = -1 \\ (p_0+q_0)(p_1+q_1) = p_3q_2-p_2q_3 \\ (p_0+q_0)(p_2+q_2) = p_1q_3-p_3q_1 \\ (p_0+q_0)(p_3+q_3) = p_2q_1-p_1q_2 \end{cases}.$$

If we compare the result after choosing $x=(0,1,0,0)$, $x=(0,0,1,0)$, and $x=(0,0,0,1)$ then we see that the RHS of the last three above equations is zero. Thus we obtain two results: first, that p and q commute; and second, that $p_0+q_0=0$. The second result tells us that J is a skew-symmetric matrix and thus orthogonal, and thus simply representable by a quaternion. Thus we do not get any more complex structures in this way.

46. Nor do we get any more if we try $J(x) = p\cdot_H x\cdot_H q$. If we let $N(x) = N((x_0,x_1,x_2,x_3)) = \sqrt{(\sum_{i=0}^{3} x_i^2)}$ be the quaternion norm, and if we take the norm of both sides of $J^2(x) = p^2xq^2 = -x$, then we obtain $N(pq) = 1$.

Thus $N(J(x)) = N(pxq) = N(x)$, and so the representation of J wrt an orthonormal basis is orthogonal. Thus J is simply representable by a quaternion.

Chapter IV - THE SET OF QUATERNION STRUCTURES

0. Very much in the same vein as for complex structures, one has similar results if one studies the ways to make a quaternion vector space from a complex vector space. Since the quaternions are not commutative multiplicatively we can have two types of quaternion vector spaces, where the scalar multiplication is $a \cdot x$ or is $x \cdot a$. For our purposes it does not matter which one we pick, so we arbitrarily choose multiplication by the scalar on the left.

1. Similarly as in Chapter I, if $(V,+,\cdot_H)$ is an n-dimensional vector space over the quaternions H , then $(V,+,\cdot_C)$, where $\cdot_C = \cdot_H|C\times V$, is a 2n-dimensional vector space over H .

   Define $f:V\to V$ where $f(x) = j\cdot_H x$. (Using j gives the simplest formulae, but similar consequences could be obtained by multiplication by k .) Then $f(x+y) = f(x) + f(y)$ and $f((\alpha+i\beta)\cdot_C x) = (\alpha-i\beta)\cdot_C f(x)$ imply that f is a <u>conjugate</u> linear endomorphism of the vector space $(V,+,\cdot_C)$.

   Note that f satisfies $f^2 = -id_V$, and that the relation between $\cdot_H$ and $\cdot_C$ is

$$(\alpha+i\beta+j\gamma+k\delta)\cdot_H x = (\alpha+i\beta)\cdot_C x + (\gamma+i\delta)\cdot_C f(x) \ .$$

2. <u>Theorem</u>: Conversely, given a conjugate linear endomorphism K on $(V,+,\cdot_C)$ such that $K^2 = -id_V$, then we obtain a quaternion vector space $(V,+,\cdot_K)$ where the extension $\cdot_K$ of $\cdot_C$ is defined by $(\alpha+i\beta+j\gamma+k\delta)\cdot_K x = (\alpha+i\beta)\cdot_C x + (\gamma+i\delta)\cdot_C K(x)$.

   <u>proof</u>: We need to check the axioms:

   1) $(\alpha+i\beta+j\gamma+k\delta)\cdot_K(x+y) = (\alpha+i\beta+j\gamma+k\delta)\cdot_K x + (\alpha+i\beta+j\gamma+k\delta)\cdot_K y$;

   2) $((\alpha_1+i\beta_1+j\gamma_1+k\delta_1)+(\alpha_2+i\beta_2+j\gamma_2+k\delta_2))\cdot_K x$

$$= ((\alpha_1+i\beta_1)+(\alpha_2+i\beta_2))\cdot_C x + ((\gamma_1+i\delta_1)+(\gamma_2+i\delta_2))\cdot_C J(x)$$

$$= (\alpha_1+i\beta_1+j\gamma_1+k\delta_1)\cdot_K x + (\alpha_2+i\beta_2+j\gamma_2+k\delta_2)\cdot_K x \ ;$$

3) $$\begin{aligned}&(\alpha_1+i\beta_1+j\gamma_1+k\delta_1)\cdot_K((\alpha_2+i\beta_2+j\gamma_2+k\delta_2)\cdot_K x)\\
&= (\alpha_1+i\beta_1)\cdot_C((\alpha_2+i\beta_2)\cdot_C x+(\gamma_2+i\delta_2)\cdot_C K(x))\\
&\quad + (\gamma_1+i\delta_1)\cdot_C K((\alpha_2+i\beta_2)\cdot_C x+(\gamma_2+i\delta_2)\cdot_C K(x))\\
&= ((\alpha_1+i\beta_1)(\alpha_2+i\beta_2)-(\gamma_1+i\delta_1)(\gamma_2-i\delta_2))\cdot_C x\\
&\quad + ((\alpha_1+i\beta_1)(\gamma_2+i\delta_2)+(\gamma_1+i\delta_1)(\alpha_2-i\beta_2))\cdot_C K(x)\\
&= ((\alpha_1+i\beta_1)(\alpha_2+i\beta_2)+(j\gamma_1+k\delta_1)(j\gamma_2+k\delta_2)+\\
&\quad (\alpha_1+i\beta_1)(\gamma_2+i\delta_2)j+(\gamma_1+i\delta_1)(\alpha_2-i\beta_2)j)\cdot_K x\\
&= ((\alpha_1+i\beta_1+j\gamma_1+k\delta_1)(\alpha_2+i\beta_2+j\gamma_2+k\delta_2))\cdot_K x \;;\end{aligned}$$

4) $1\cdot_K x = 1\cdot_H x = x$, ¶

3. Definition: Such a conjugate linear endomorphism K as described in the previous theorem we will call a quaternion structure on the complex vector space $(V,+,\cdot_C)$.

Thus a quaternion structure is a conjugate linear endomorphism K such that $K^2 = -I$.

Note in all the above that if we had considered quaternion vector spaces with scalar multiplication on the right, then everything would still be true, with only a minor adjustment required in the definition of the extension $\cdot_K$. Namely,

$$x\cdot_K(\alpha+i\beta+j\gamma+k\delta) = x\cdot_C(\alpha+i\beta) + K(x)\cdot_C(\gamma-i\delta) .$$

Thus a quaternion structure can be considered independent of whether we have a right or left vector space.

4. The interesting fact about quaternion structures is that, though one expects that the fact that K is conjugate linear (instead of linear) will cause problems, actually almost everything which works for complex structures also works for quaternion structures.

5. One can define a matrix representation (wrt a base) of a conjugate linear endomorphism in entirely the analogous way as one does for linear endomorphisms. A slight problem is that the composition of two conjugate linear endomorphisms is a linear endomorphism, and that the composition of a conjugate linear endomorphism with a linear endomorphism is a conjugate linear endomorphism.

Since a given matrix K can denote either a linear endomorphism or a conjugate linear endomorphism, we will denote the associated endomorphism by K and K_C, respectively.

Then the matrix representation of the composition of different types of endomorphisms is: $m(J\circ K) = JK \qquad m(J_C\circ K) = (J\overline{K})_C$
$m(J\circ K_C) = (JK)_C \qquad m(J_C\circ K_C) = J\overline{K}$.

6. Definition: $\mathbb{K}_{2n} = \{K_C \varepsilon GL(2n;C) \mid K_C^2 = -I_{2n}\} \subset GL(2n;C)$.

7. Define action $\phi : GL(2n;C) \to \operatorname{Aut} \mathbb{K}_{2n}$ where $\phi_g(K_C) = gK_Cg^{-1}$.
Then results analogous to III.1,2,3 hold.
In III.4 however, the explicit characterization of $G_{(K_0)_C}$ becomes

$$G_{K_0} = \{\begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix} \varepsilon\ GL(2n;C)\} .$$

In III.5 the isomorphism $e : GL(n;H) \to G_{K_0}$ is given by

$$e(A+Bj) = \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix} .$$

Lemma III.11 becomes

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} H = \begin{bmatrix} I_n & P \\ 0 & Q \end{bmatrix} H \quad (\varepsilon G/H) \quad \underline{\text{iff}} \quad \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} -\overline{C} \\ \overline{A} \end{bmatrix} \text{ with } \det Q \neq 0 .$$

The change in this lemma, though, prevents us from making our next result stronger:

8. Since matrix multiplication, inversion, and conjugation are smooth operations, we have that $\mathbb{K}_{2n}$ can be given a smooth real manifold structure of dimension $4n^2$, but unfortunately in this wav one can not make a complex manifold out of $\mathbb{K}_{2n}$, though one would have a much stronger result than one has now. Also, complex manifolds, as all nice things, are rare.

9. ker ϕ is the same as in III.20 but $N(H)$ is probably larger than in III.24 .

$\mathbb{K}_{2n}$ is not compact, but $\mathbb{K}_{2n}$ <u>is</u> connected (in contrast to $\mathbb{J}_{2n}$) .

BIBLIOGRAPHY

1. Bredon, Glen E., Introduction to Compact Transformation Groups, Academic Press, New York, 1972.
2. Chevalley, Claude, Theory of Lie Groups, Princeton University Press, Princeton, 1946.
3. Kobayashi, Shoshichi, & Katsumi Nomizu, Foundations of Differential Geometry (vol. I & II), Interscience Publishers (John Wiley & Sons), New York, 1963 (& 1969).
4. Warner, Frank W., Foundations of Differentiable Manifolds and Lie Groups, Scott, Foresman and Company, Glenview, Illinois, 1971.