Université d'Ottawa · University of Ottawa

DERIVATIONS, INVARIANT FORMS AND THE SECOND HOMOLOGY GROUP OF ORTHOSYMPLECTIC LIE SUPERALGEBRAS

By

Ana Duff, B.Sc., M.Sc.

March 2002

A Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy in Mathematics[1]

[1]The Ph.D. Program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

0-612-72807-2

Acknowledgements

I would like to thank my supervisor, Erhard Neher, for his support and guidance. I would also like to thank the National Science and Engineering Research Council for their support through the Postgraduate Scholarships PGS A and PGS B and the Ontario government for their support through the Ontario Graduate Scholarship in Science and Technology. In addition, I owe my deepest gratitude to my parents, Nataša and Miroslav Kadić, and my husband Paul for their continuous and unconditional love, support and encouragement.

Dedication

To my father, in loving memory.

Abstract

We develop the description of the derivation algebras of orthosymplectic Lie superalgebras over supercommutative, associative superrings containing $\frac{1}{2}$ and determine conditions under which the derivation algebra can be written as a semidirect product of the inner and the outer derivations. We then describe the supersymmetric invariant forms of the elementary orthosymplectic Lie superalgebra and determine the outer derivations which are skew with respect to a given supersymmetric invariant form. Finally, we describe the universal central extension and its centre, the second homology group, of the elementary orthosymplectic Lie superalgebra. The original motivation for this comes from the theory of extended affine Lie algebras. Specialized to the orthosymplectic Lie superalgebras representing the centreless cores of extended affine Lie algebras of type B and D, the above descriptions are the necessary and sufficient building blocks for the construction of an extended affine Lie algebra of type B and D from its centreless core.

Contents

Introduction

In this thesis we study the derivations, invariant forms and the second homology group of certain subalgebras of the orthosymplectic Lie superalgebras. The original motivation for this comes from the theory of extended affine Lie algebras. In order to construct an extended affine Lie algebra one needs to describe its centreless core $\mathcal{K}$. This has been done for the various types of root systems by Allison, Berman, Gao, Krylyuk, Neher and Yoshii. The extended affine Lie algebra of a particular type is then constructed by forming a central extension L_c of $\mathcal{K}$ and adding to it a subalgebra of skew derivations with respect to an invariant symmetric form on L_c (and possibly a 2-cocycle).

The centreless core of an extended affine Lie algebra of type B or D is an orthogonal Lie algebra. To construct the extended affine Lie algebras of these types one needs to know precisely the derivations, invariant forms and the second homology group of this Lie algebra. The results in [Be] and in [ABG] are not specific enough for the purpose of constructing extended affine Lie algebras. The results we develop in this thesis provide us precisely with the descriptions we need for the construction of extended affine Lie algebras of types B and D. Moreover, because of the interest in generalizing the theory of Lie algebras and, in particular the extended affine Lie algebras, to the supersetting, we prove our results in the Lie superalgebra setting but they are new even in the classical, nonsuper, setting.

The Lie superalgebras we consider are generalizations of the orthosymplectic Lie algebras and superalgebras studied by many, including Kac ([K1],[K2]), Scheunert ([S]), Garcia and Neher ([N2], [GN]), all on theory of Lie algebras and superalgebras. Examples of the Lie superalgebras we study include the Lie algebras graded by root systems of type B and D ([BM]), the cores of the extended affine Lie algebras of type B and D ([AG]) and an orthosymplectic Lie superalgebra studied in [KP], [DJKM] and [CW]. The results we develop on the structure of the derivations and supersymmetric invariant forms generalize the results developed by Benkart ([Be]). The description of the second homology group of Lie algebras graded by root systems of type B and D developed by Allison, Benkart and Gao ([ABG]) is a special case of a result we will prove.

We will now explain our program in detail.

Basic definitions: We define the orthosymplectic Lie superalgebra as follows. Let k be a commutative, associative and unital ring containing $\frac{1}{2}$. We let $\mathcal{K}$ be a super-extension of k, i.e., $\mathcal{K} = \mathcal{K}_{\bar{0}} \oplus \mathcal{K}_{\bar{1}}$ is a $\mathbb{Z}_2$-graded algebra, called superalgebra, over k which is supercommutative (i.e., $ab = (-1)^{\alpha\beta}ba$ for all $a \in \mathcal{K}_\alpha$, $b \in \mathcal{K}_\beta$, $\alpha, \beta \in \mathbb{Z}_2$), associative and unital. Let $\mathcal{M}$ be a $\mathcal{K}$-supermodule, i.e., $\mathcal{M} = \mathcal{M}_{\bar{0}} \oplus \mathcal{M}_{\bar{1}}$ is a $\mathbb{Z}_2$-graded $\mathcal{K}$-bimodule with $am = (-1)^{\alpha\beta}ma$ for $a \in \mathcal{K}_\alpha$ and $m \in \mathcal{M}_\beta$. Throughout this thesis, any time we talk about a $\mathbb{Z}_2$-graded structure $Z = Z_{\bar{0}} \oplus Z_{\bar{1}}$ we will call an element $z \in Z$ *homogeneous* if $z \in Z_\alpha$ for some $\alpha \in \mathbb{Z}_2$ and we will say that z is of *degree* α, denoted by $|z| = \alpha$; in this case **we will assume throughout that when $|z|$ occurs in an expression, then it is assumed that z is homogeneous, and that the expression extends to the other elements by linearity.** If $f : \mathcal{M} \to \mathcal{N}$ is an additive map between supermodules then we say that the *degree* of f is $\alpha \in \mathbb{Z}_2$ if $f(\mathcal{M}_\beta) \subset \mathcal{N}_{\alpha+\beta}$ for all $\beta \in \mathbb{Z}_2$.

A $\mathcal{K}$-supermodule L is called a *Lie superalgebra* if it is equipped with a $\mathcal{K}$-bilinear bracket multiplication $[\cdot,\cdot] : L \times L \to L$ such that:

(SL1) for all $x, y \in L$,

$$[x, y] = -(-1)^{|x||y|}[y, x];$$

(SL2) for all $x, y, z \in L$,

$$(-1)^{|x||z|}[[x, y], z] + (-1)^{|y||x|}[[y, z], x] + (-1)^{|z||y|}[[z, x], y] = 0.$$

Note that, using (SL1), (SL2) is equivalent to

$$[x, [y, z]] = [[x, y], z] - (-1)^{|y||z|}[[x, z], y].$$

We let $q : \mathcal{M} \times \mathcal{M} \to K$ be a supersymmetric (i.e., $q(m,n) = (-1)^{|m||n|}q(n,m)$ for all homogeneous $m, n \in \mathcal{M}$) K-bilinear form (q is additive and $q(am,n) = aq(m,n)$, $q(ma,n) = q(m,an)$ for all $a \in K$, $m, n \in \mathcal{M}$) on $\mathcal{M}$ of degree $\bar{0}$, called the quadratic (super)form on $\mathcal{M}$. The *orthosymplectic Lie superalgebra* $\operatorname{osp}(q)$ is defined by

$$\operatorname{osp}(q) = \{x \in \operatorname{End}_K \mathcal{M} \mid q(x(m),n) + (-1)^{\alpha\beta}q(x(n),m) = 0,\ m \in \mathcal{M}_\alpha, n \in \mathcal{M}_\beta\}.$$

The bracket multiplication in $\operatorname{osp}(q)$ is induced from the bracket in $\operatorname{End}_K \mathcal{M}$ which is given by $[x,y](m) = x(y(m)) - (-1)^{|x||y|}y(x(m))$ for all $x, y \in \operatorname{End}_K \mathcal{M}$ and $m \in \mathcal{M}$. If $\mathcal{M} = \mathcal{M}_{\bar{0}}$ then $\operatorname{osp}(q)$ is an orthogonal algebra. If $\mathcal{M} = \mathcal{M}_{\bar{1}}$ then q is a symplectic form. Note that in this case $\operatorname{osp}(q)$ is not a symplectic Lie algebra.

An important subalgebra, in fact an ideal, of $\operatorname{osp}(q)$ is the *elementary orthosymplectic Lie superalgebra* $\operatorname{eosp}(q)$ which is constructed as follows. Given $m \in \mathcal{M}_\alpha$ and $n \in \mathcal{M}_\beta$ we define a map $\mathbf{E}_{m,n} \in \operatorname{End}_K \mathcal{M}$ by setting

$$\mathbf{E}_{m,n}(p) = mq(n,p) - (-1)^{\alpha\beta}nq(m,p),\ p \in \mathcal{M}.$$

We can extend this definition using bilinearity to $\mathbf{E}_{m,n}$ for all $m, n \in \mathcal{M}$. Then $\operatorname{eosp}(q)$ is defined as

$$\operatorname{eosp}(q) = \operatorname{span}_{\mathbb{Z}}\{\mathbf{E}_{m,n} \mid m, n \in \mathcal{M}\},$$

i.e., the additive subgroup generated by $\{\mathbf{E}_{m,n} \mid m, n \in \mathcal{M}\}$. This is indeed an important Lie superalgebra because, among others, the Lie superalgebras graded by (possibly infinite) root systems of types B and D over K ($\frac{1}{6} \in K$) are central extensions of $\operatorname{eosp}(q)$ for a suitable q (see [GN]), including the centreless cores of the extended affine Lie algebras of types B and D ([AG]).

To describe these algebras we need the following notions. Given K-supermodules $\mathcal{M}_1$ and $\mathcal{M}_2$ with K-quadratic forms q_1 and q_2 respectively, we define their orthogonal sum as the K-supermodule $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with the K-quadratic form $q = q_1 \oplus q_2$ where

$$(q_1 \oplus q_2)(m_1 \oplus m_2, n_1 \oplus n_2) = q_1(m_1,n_1) + q_2(m_2,n_2),\ m_1, n_1 \in \mathcal{M}_1,\ m_2, n_2 \in \mathcal{M}_2.$$

An important example of the orthogonal sum is the *hyperbolic superspace* (over K)

$$H(I) = H(I; K) = \bigoplus_{i \in I} (Kh_i \oplus Kh_{-i}),$$

where I is an arbitrary index set. The K-quadratic form q_I on $H(I)$ is given by

$$q_I(h_{\sigma i}, h_{\mu j}) = \delta_{\sigma,-\mu}\delta_{ij}, \; \sigma, \mu = \pm, \; i, j \in I.$$

Note that $H(I)$ is an orthogonal sum of $H(\{i\})$, $i \in I$. If $|I| = 1$ we call $H(I)$ the *hyperbolic superplane* and we denote it by $H(\{\infty\})$. Garcia and Neher ([GN]) proved the following for an arbitrary index set I:

(B_I) Assume that $\frac{1}{6} \in K$ and $|I| \geq 3$. A Lie superalgebra L over K is B_I-graded if and only if there exists a superextension A of K, an A-supermodule $\mathcal{M}$ with an A-quadratic form q and an element $m \in \mathcal{M}$ satisfying $q(m, m) = 1$ such that L is a central extension of the Lie superalgebra $\mathrm{eosp}(q_I \mathbin{\bar\oplus} q)$.

(D_I) Assume that $\frac{1}{6} \in K$ and $|I| \geq 4$. A Lie superalgebra L over K is D_I-graded if and only if there exists a superextension A of K such that L is a central extension of the Lie superalgebra $\mathrm{eosp}(q_I)$.

Here the root systems B_I and D_I are given by

$$\begin{aligned} D_I &= \{\pm\epsilon_i \pm \epsilon_j \mid i, j \in I, \; i \neq j\}; \\ B_I &= \{\pm\epsilon_i \mid i \in I\} \cup D_I \end{aligned}$$

where the ϵ_i form the orthonormal basis of $\bigoplus_{i \in I} \mathbb{R}\epsilon_i$. In [AG], the authors Allison and Gao described the centreless core $\mathcal{K}$ of the extended affine Lie algebra (over $\mathbb{C}$) graded by the root system of type B_n and nullity ν. It can be shown that $\mathcal{K}$ is isomorphic to $\mathrm{eosp}(q_I \mathbin{\bar\oplus} q)$ where $I = \{1, 2, \ldots, n\}$. Here q is an A-quadratic form on $\mathcal{M}$ for $A = \mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]$ the Laurent polynomials over $\mathbb{C}$ in ν variables and a free A-module $\mathcal{M} = \oplus A m_i$ of finite rank satisfying

$$q(m_i, m_j) = \delta_{ij} t^{\tau_j}$$

where $\tau_i \in \mathbb{Z}^\nu$ such that $\tau_1 = 0$ and $\tau_i \not\equiv \tau_j \mod 2\mathbb{Z}^\nu$ for $i \neq j$.

This construction is what motivated this thesis. In Chapter 5 we describe the universal central extensions of certain eosp(q) which in particular cover the case of the centreless core of extended affine Lie algebras of types B and D. In Chapter 4 we describe the invariant supersymmetric forms on this eosp(q) and in Chapter 3 we describe the structure of the derivation algebra of certain orthosymplectic Lie superalgebras including eosp(q).

Using the description of [GN], in order to study the derivations of Lie superalgebras of types B and D we need to study the derivations of the Lie superalgebra eosp($q_I \oplus q$) for an arbitrary quadratic form q. Note that, if $I \neq \emptyset$, eosp($q_I \oplus q$) = eosp($q_{\{\infty\}} \oplus q_{I\setminus\{\infty\}} \oplus q$) where we denote an (arbitrary) element of I by ∞. Hence it suffices to consider Lie superalgebras eosp($q_{\{\infty\}} \oplus q$) for an arbitrary K-quadratic form q (we will denote $q_{\{\infty\}} \oplus q$ by q_∞) however we will do this in a more general setting. We wish to consider all the subalgebras $\mathcal{E}$ of osp(q) which contain eosp(q) for some K-quadratic form q on a K-supermodule $\mathcal{M}$. Given such a superalgebra, we define the Lie superalgebra $\mathcal{E}_\infty$ as follows:

$$\mathcal{E}_\infty = \{(a, m, n, x) \mid a \in A,\ m, n \in \mathcal{M},\ x \in \mathcal{E}\}$$

with multiplication

$$\begin{aligned}
[(a, m, n, x), (a', m', n', x')] = \ & (q(m, n') - q(n, m'), \\
& am' + xm' - ma' - (-1)^{|x'||m|}x'm, \\
& -an' + xn' + na' - (-1)^{|x'||n|}x'n, \\
& \mathbf{E}_{m,n'} + \mathbf{E}_{n,m'} + [x, x'])
\end{aligned}$$

for all $a, a' \in A$, $m, m', n, n' \in \mathcal{M}$ and $x, x' \in \mathcal{E}$. We will show that, for example, eosp(q_∞) $\cong$ (eosp(q))$_\infty$ and osp(q_∞) $\cong$ (osp(q))$_\infty$.

An example of a subalgebra of osp(q) which contains eosp(q) is the Lie superalgebra $\overline{\text{eosp}(q)}$ which is defined as follows. If $\underline{x} = (x_i)_{i\in I}$ is a family in eosp(q) we say that $\underline{x}$ is *summable*, if for every $m \in \mathcal{M}$, $(x_i(m))_{i\in I}$ has only finitely many nonzero terms and in this case we define $x = \sum_{i\in I} x_i \in \mathrm{End}_A \mathcal{M}$ by setting

$$\left(\sum_{i\in I} x_i\right)(m) = \sum_{i\in I_m} x_i(m)$$

for $I_m = \{i \in I \mid x_i(m) \neq 0\}$ (I_m is finite). We say that $\sum_{i\in I} x_i$ is summable to indicate that $(x_i)_{i\in I}$ is summable. The set of all summable $\sum_{i\in I} x_i$, $x_i \in \mathrm{eosp}(q)$ is denoted by $\overline{\mathrm{eosp}(q)}$ and is a subalgebra of $\mathrm{osp}(q)$ which contains $\mathrm{eosp}(q)$ (we discuss summable homomorphisms in greater detail in Section 1.5). We will show that $\overline{\mathrm{eosp}(q_\infty)} \cong (\overline{\mathrm{eosp}(q)})_\infty$.

Another example of a subalgebra of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$ was studied in [KP], [DJKM] and [CW] and this will be explained in the body of the thesis.

Derivations: Our first task is to describe the derivations of $\mathcal{E}_\infty$. However, we need to make it clear whether we are using the proper super-ring as a base ring, keeping in mind the characterization of the Lie superalgebras L graded by root systems of types B and D. Using this characterization we can view $L = \mathrm{eosp}(q)_\infty$ both as a Lie superalgebra over K and as a Lie superalgebra over A where A is a superextension of K. So from now on we assume that A is a superextension of K, all supermodules are over A and all quadratic forms are A-bilinear. Hence we can view $\mathrm{osp}(q)$ and any of its A-subalgebras $\mathcal{E}$ (containing $\mathrm{eosp}(q)$) as subalgebras of both $\mathrm{End}_A \mathcal{M}$ and $\mathrm{End}_K \mathcal{M}$. However, in what follows we will view the Lie superalgebras $\mathcal{E}$ and $\mathcal{E}_\infty$ as subalgebras of $\mathrm{End}_K \mathcal{M}$ even though they consist of A-linear maps.

Let L be a Lie superalgebra over K. A *K-derivation* $d : L \to L$ of L of degree $\alpha \in \mathbb{Z}_2$ is a K-linear map which satisfies

$$d([x,y]) = [d(x), y] + (-1)^{\alpha\beta}[x, d(y)]$$

for all $y \in L$ and $x \in L_\beta$, $\beta \in \mathbb{Z}_2$. The set consisting of all derivations d_α of degree $\alpha \in \mathbb{Z}_2$ is denoted by $(\mathrm{Der}_K L)_\alpha$. The *derivation algebra* of L is defined as

$$\mathrm{Der}_K L = (\mathrm{Der}_K L)_{\bar{0}} \oplus (\mathrm{Der}_K L)_{\bar{1}}$$

and is a Lie superalgebra over K with respect to the usual bracket multiplication in $\mathrm{End}_K \mathcal{M}$ (i.e., $[d, d'](x) = d(d'(x)) - (-1)^{|d||d'|}d'(d(x))$ for all homogeneous $d, d' \in \mathrm{Der}_K L$ and all $x \in L$). Given $x \in L$, the map $\mathbf{ad}(x) : L \to L : y \mapsto [x, y]$ is a derivation, called the inner derivation of L determined by x. We denote the set of all inner derivations (which is an ideal of $\mathrm{Der}_K L$) by $\mathbf{ad}\, L$.

We will show that $L = \mathcal{E}_\infty$ is a *toral 3-graded Lie superalgebra*, i.e., $L = L_+ \oplus L_0 \oplus L_-$ is 3-graded ($[L_\sigma, L_\mu] \subset L_{\sigma+\mu}$, $\sigma, \mu \in \{0, \pm\}$) with a *toral element* $\zeta \in (L_0)_{\bar{0}}$ satisfying $[\zeta, x_\sigma] = \sigma x_\sigma$ for $x_\sigma \in L_\sigma$, $\sigma \in \{0, \pm\}$. Indeed, if we set

$$
\begin{aligned}
L_0 &= \{(a, 0, 0, x) \mid a \in A,\ x \in \mathcal{E}\} \\
L_+ &= \{(0, m, 0, 0) \mid m \in \mathcal{M}\} \\
L_- &= \{(0, 0, m, 0) \mid m \in \mathcal{M}\}
\end{aligned}
$$

then it is clear that L is 3-graded and that $(1, 0, 0, 0)$ is a toral element of L. Then it is well-known that $\mathrm{Der}_K L = \mathbf{ad}\, L + (\mathrm{Der}_K L)_0$ where $(\mathrm{Der}_K L)_0 = \{d \in \mathrm{Der}_K L \mid d(L_\sigma) \in L_\sigma, \sigma = 0, \pm\}$ and $\mathbf{ad}\, L \cap (\mathrm{Der}_K L)_0 = \mathbf{ad}\, L_0$. Therefore, to describe the derivations of L it suffices to describe $(\mathrm{Der}_K L)_0$. We will assume that L is centreless which we prove to be equivalent to $\mathrm{Ann}_A \mathcal{M} = (0)$. We define two sets, $\mathcal{S}_\mathcal{E}$ and $\mathcal{T}_\mathcal{E}$, of K-endomorphisms of $\mathcal{M}$ and show that $\mathcal{S}_\mathcal{E} \oplus \mathcal{T}_\mathcal{E} \subset \mathrm{End}_K \mathcal{M} \oplus \mathrm{End}_K \mathcal{M}$ is a Lie superalgebra with multiplication

$$[S \oplus T, S' \oplus T']^{\sim} = ([S, S'] + [T, T']) \oplus ([S, T'] + [T, S'])$$

where $[\ ,\ ]$ is the usual bracket multiplication in $\mathrm{End}_K \mathcal{M}$. We set $\mathcal{D}_\mathcal{E}$ to be the set of all K-endomorphisms $d_{S,T}$ of L for which there exist $S \in \mathcal{S}_\mathcal{E}$, $T \in \mathcal{T}_\mathcal{E}$ satisfying

$$d_{S,T}(a, m, n, x) = ([S, a \cdot Id] + [T, x], (S+T)(m), (S-T)(n), [T, a \cdot Id] + [S, x]).$$

Using this notation we will prove the following:

Theorem 1. (3.2.13) *Derivations of $\mathcal{E}_\infty$ satisfy the following:*

(1) $\mathcal{D}_\mathcal{E} = (\mathrm{Der}_K \mathcal{E}_\infty)_0$, *in particular, $\mathcal{D}_\mathcal{E}$ is a subalgebra of* $\mathrm{Der}_K \mathcal{E}_\infty$. *Moreover, the map $\phi : \mathcal{S}_\mathcal{E} \oplus \mathcal{T}_\mathcal{E} \to \mathcal{D}_\mathcal{E} : S \oplus T \mapsto d_{S,T}$ is a Lie superalgebra isomorphism.*

(2) $\mathrm{Der}_K\, \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty + \mathcal{D}_\mathcal{E}$.

(3) $\mathbf{ad}\, \mathcal{E}_\infty \cap \mathcal{D}_\mathcal{E} = \{d_{S,T} \in \mathcal{D}_\mathcal{E} \mid S \in \mathcal{E},\ T \in A \cdot Id\} = \mathbf{ad}(\mathcal{E}_\infty)_0$.

(4) *If* $\mathcal{S} = \mathcal{E} \oplus \mathcal{S}_0$ *and* $\mathcal{T} = A \cdot Id \oplus \mathcal{T}_0$ *where* $\mathcal{S}_0 \oplus \mathcal{T}_0$ *is a subalgebra of* $\mathcal{S} \oplus \mathcal{T}$ *then*

$$\mathrm{Der}_K\, \mathcal{E}_\infty \cong \mathbf{ad}\, \mathcal{E}_\infty \rtimes (\mathcal{S}_0 \oplus \mathcal{T}_0).$$

The symbol $\rtimes$ stands for the term *semidirect product*. The item (4) of the above result is important because, using our original motivation to describe the derivations of the (centreless) core of the extended affine Lie algebras of types B and D, it is extremely useful to describe conditions under which the derivation algebra can be written as a semidirect product of the inner and the outer derivations. We need the outer derivations to construct the extended affine Lie algebra from its core.

We make the following definitions. We say that, given a free A-supermodule $\mathcal{M} = \oplus_{i\in I} Am_i$, a quadratic form q on $\mathcal{M}$ is almost diagonizable if for each $i \in I$ there exists $\underset{\cdot}{i} \in I$ such that

$$\begin{aligned} &q(m_i, m_{\underset{\cdot}{i}}) \in A^\times; \text{ and} \\ &q(m_i, m_j) = 0 \text{ for all } j \in I,\ j \neq \underset{\cdot}{i}. \end{aligned}$$

Here $A^\times$ denotes the invertible elements of A. In addition, if q is almost diagonizable with respect to $\{m_i \mid i \in I\}$ where $i = \underset{\cdot}{i}$ for all $i \in I$ then we say that q is *invertibly diagonizable*. For example, q_I is almost diagonizable with respect to the basis $\{h_{\pm i} \mid i \in I\}$ of $H(I)$ and is invertibly diagonizable with respect to the basis $\{h_i \pm h_{-i} \mid i \in I\}$ (since $\frac{1}{2} \in K$).

Using this we have the following:

Theorem 2. (see 3.3.7 for a more detailed version) *Let* $\mathcal{N}$ *and* $\mathcal{R}$ *be free* A*-supermodules with* A*-quadratic forms* $q_\mathcal{N}$ *and* $q_\mathcal{R}$ *respectively such that* $q_\mathcal{R} \equiv 0$ *and* $q_\mathcal{N}$ *is an orthogonal sum of an almost diagonizable* A*-quadratic form and an invertibly* *diagonizable A-quadratic form on a free A-supermodule of rank 2. Let* $\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$ *with the A-quadratic form* $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$ *and let* $\mathcal{E}$ *be a subalgebra of* $\mathrm{osp}(q)$ *containing* $\mathrm{eosp}(q)$*.*

(1) *If* $\mathcal{E} = \mathrm{eosp}(q)$ *then* $\mathrm{Der}_K \mathcal{E}_\infty \cong \mathbf{ad}_{\mathrm{loc}} \mathcal{E}_\infty \rtimes (\mathrm{End}_A \mathcal{R} \oplus \mathrm{Der}_K A)$*. Moreover, if* $\mathcal{M}$ *is free of finite rank then* $\mathrm{Der}_K \mathcal{E}_\infty \cong \mathbf{ad}\, \mathcal{E}_\infty \rtimes (\mathrm{End}_A \mathcal{R} \oplus \mathrm{Der}_K A)$*.*

(2) *If* $\mathcal{E} = \overline{\mathrm{eosp}(q)}$ *then* $\mathrm{Der}_K \mathcal{E}_\infty \cong \mathbf{ad}\, \mathcal{E}_\infty \rtimes (\mathrm{End}_A \mathcal{R} \oplus \mathrm{Der}_K A)$*.*

(3) *If* $\mathcal{E} = \mathrm{osp}(q)$ *then* $\mathrm{Der}_K \mathcal{E}_\infty \cong \mathbf{ad}\, \mathcal{E}_\infty \rtimes \mathrm{Der}_K A$*.*

In (1) above, $\mathbf{ad}_{\mathrm{loc}} \mathcal{E}_\infty$ denotes all *locally inner derivations*. These are the derivations d of $\mathcal{E}_\infty$ such that for each finite set $\{v_1, \ldots, v_n\} \subset \mathcal{E}_\infty$ there exists $v \in \mathcal{E}_\infty$ such that $d(v_i) = [v, v_i]$.

As a consequence to Theorem 2, since the centreless cores L of the extended affine Lie algebras of types B and D are isomorphic to $\mathrm{eosp}(q)$ where q is an invertibly diagonizable A-quadratic form on a free A-supermodule of finite rank and A is the ring of Laurent polynomials over $\mathbb{C}$ in ν variables for some ν,

$$\mathrm{Der}_{\mathbb{C}} L \cong \mathbf{ad}\, L \rtimes \mathrm{Der}_{\mathbb{C}} A.$$

Another result we develop on derivations (see 3.3.9) is a generalization of Benkart's result for Lie algebras of type B and D ([Be, Theorem 3.6]). This result describes the derivations of $\mathrm{eosp}(q_I \oplus q)$ in terms of the inner derivations and the derivations of a certain Jordan superalgebra associated to q.

Supersymmetric invariant forms: We now turn to the topic of supersymmetric invariant forms. Again, the original motivation for this problem comes from the study of extended affine Lie algebras and the fact that one needs a nondegenerate symmetric invariant form on the centreless core of extended affine Lie algebra L in order to be able to contruct L. We wish to describe the supersymmetric invariant forms on the Lie superalgebras graded by root systems of types B and D.

Let L be a Lie superalgebra over K. We say that a K-bilinear map $(\cdot\,|\,\cdot) : L \times L \to K$ of degree $\bar{0}$ is (L-)*invariant* if and only if

$$([x,y]\,|\,z) = (x\,|\,[y,z])$$

for all $x, y, z \in L$. We denote the $K_{\bar{0}}$-module of invariant supersymmetric K-bilinear forms $(\cdot\,|\,\cdot) : L \times L \longrightarrow K$ by $\mathrm{Inv}(L; K)$.

As in the study of derivations, to describe $\mathrm{Inv}(L; K)$ for L a Lie superalgebra over K graded by root systems of types B and D, it suffices to describe the invariant supersymmetric forms on central extensions of $\mathrm{eosp}(q)_\infty$. It is well-known that the $K_{\bar{0}}$-module of supersymmetric invariant forms of a Lie superalgebra L is isomorphic to the $K_{\bar{0}}$-module of supersymmetric invariant forms of $L/Z(L)$ where $Z(L) = \{z \in L \mid [L, z] = 0\}$ denotes the centre of L. Hence it suffices to consider a centreless $\mathrm{eosp}(q_\infty)$. Since $\mathrm{eosp}(q_\infty)$ is a toral 3-graded Lie superalgebra it is natural to first describe the invariant supersymmetric forms on 3-graded Lie superalgebra. In particular, we derive the necessary and sufficient conditions for existence of supersymmetric invariant forms on toral 3-graded Lie superalgebras $L = L_+ \oplus L_0 \oplus L_-$ over K with a toral element ζ and the conditions under which these forms are nondegenerate. We will show the following:

Theorem 3. (4.1.3 and 4.1.8) *Let L be a toral 3-graded Lie superalgebra over K. Then*

$$\mathrm{Inv}(L; K) \cong \mathrm{Inv}(L_0; K).$$

Moreover, if $\zeta \in [L_+, L_-]$ then

$$\mathrm{Inv}([L_+, L_-]; K) \cong \{\varphi \in (\mathrm{Hom}_K([L_+, L_-], K)_{\bar{0}} \mid \varphi([[L_+, L_-], [L_+, L_-]]) = 0\}.$$

In particular, if $L_0 = [L_+, L_-]$ then

$$\mathrm{Inv}(L; K) \cong \mathrm{Inv}(L_0; K) \cong \{\varphi \in (\mathrm{Hom}_K(L_0, K)_{\bar{0}} \mid \varphi([L_0, L_0]) = 0\}$$

where $(\cdot\,|\,\cdot) \in \mathrm{Inv}(L; K)$ is nondegenerate if and only if $(\cdot\,|\,\cdot)|_{L_0 \times L_0}$ is nondegenerate and $\ker(\varphi \circ \mathbf{ad}_{L_\sigma})\,|_{L_{-\sigma}} = 0$. *In addition, if L is a toral 3-graded Lie superalgebra over a* *superextension* A *of* K *with* $L_0 = [L_+, L_-]$ *and* $L_0 = A\zeta \rtimes X$ *for some superextension* A *of* K *and a Lie superalgebra* X *over* A *then*

$$\mathrm{Inv}(L; K) \cong \mathrm{Hom}_K(A, K)_{\bar{0}} \oplus \{\chi \in \mathrm{Hom}_K(X, K)_{\bar{0}} \mid \chi([X, X]) = 0\}.$$

We will show that $\mathrm{eosp}(q_I \oplus q)$ is perfect if $|I| \geq 2$ and so the supersymmetric invariant forms of Lie superalgebras over K (including the EALA) graded by root systems of (not necesarily finite) type B_n, D_n, $n \geq 3, 4$ respectively are isomorphic to $\mathrm{Hom}_K(A, K)_{\bar{0}}$ for some superextension A of K. This is a generalization of Benkart's result (see [Be]) on invariant forms on Lie algebras graded by finite root systems and over fields of characteristic zero for root systems of type B_n and D_n, $n \geq 3, 4$ respectively.

To conclude our study of invariant supersymmetric forms on $L = \mathrm{eosp}(q_\infty)$, using the results we developed on derivations, we describe the derivations of L which are *skew* with respect to a given supersymmetric invariant forms. A homogeneous $d \in \mathrm{Der}_K L$ is skew with respect to an invariant form $(\cdot \mid \cdot)$ if $(d(x) \mid y) = -(-1)^{|d||x|}(x \mid d(y))$ for all $x, y \in L$, x homogeneous. We denote the skew derivations of L by $\mathrm{SDer}_K L$. These are the derivations that we use in constructing the extended affine Lie algebras of type B and D from their cores. In particular, we will show the following:

Corollary 4 (4.3.4): *Let* $\mathcal{N}$ *and* $\mathcal{R}$ *be free* A*-supermodules with* A*-quadratic forms* $q_{\mathcal{N}}$ *and* $q_{\mathcal{R}}$ *respectively such that* $q_{\mathcal{R}} \equiv 0$ *and* $q_{\mathcal{N}}$ *is an orthogonal sum of an almost diagonizable* A*-quadratic form and an invertibly diagonizable* A*-quadratic form on a free* A*-supermodule of rank 2. Let* $\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$ *with the* A*-quadratic form* $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$ *and let* $L = \mathrm{eosp}(q_\infty)$. *Let* $(\cdot \mid \cdot)$ *be an invariant supersymmetric form on* L. *Then*

$$\mathrm{SDer}_K L \cong \mathbf{ad}_{\mathbf{loc}} L \rtimes (\mathrm{End}_A \mathcal{R} \oplus \mathrm{SDer}_K A).$$

Note that if $\mathcal{N}$ above is of finite rank then $\mathbf{ad}_{\mathbf{loc}}\, \mathrm{eosp}(q_\infty) = \mathbf{ad}\, \mathrm{eosp}(q_\infty)$.

Universal central extensions and the second homology group: As the final part of our program of research in this thesis, we consider the second homology group of $\text{eosp}(q_\infty)$. It is the centre of the universal central extension of $\text{eosp}(q_\infty)$ and is used in constructing the central extensions of $\text{eosp}(q_\infty)$.

Let L be a Lie superalgebra over K. A *central extension of* L ([SZ]) is a short exact sequence $0 \longrightarrow C \longrightarrow \tilde{L} \xrightarrow{\pi} L \longrightarrow 0$ of Lie superalgebras where C is a central ideal of L. By abuse of notation, $\tilde{L}$ is also called a central extension. If $\tilde{L}$ is a perfect Lie superalgebra and for any central extension $\varphi : L' \to L$ there exists a unique Lie superalgebra homomorphism $\psi : \tilde{L} \to L'$ such that $\varphi\psi = \pi$, then $\tilde{L}$ is called a *universal central extension of* L. The following construction is due to Neher ([N4]).

Set $\mathcal{L}_L = L \otimes_K L/\mathcal{B}$ where

$$\mathcal{B} = \text{span}_K\{[x,y] \otimes z + (-1)^{|x|(|y|+|z|)}[y,z] \otimes x + (-1)^{|z|(|x|+|y|)}[z,x] \otimes y,$$
$$x \otimes y + (-1)^{|y||x|} y \otimes x \mid x, y, z \in L\}.$$

We let $\langle x, y\rangle = x \otimes y + \mathcal{B}$ for $x, y \in L$ and extend it K-linearly. The map

$$m : \mathcal{L} \to L : \langle x, y\rangle \to [x, y].$$

is a well-defined K-supermodule homomorphism. We can view $\mathcal{L}$ as a superalgebra over K with respect to the multiplication $[l_1, l_2] = \langle m(l_1), m(l_2)\rangle$ and, in this case, the map m is a superalgebra isomorphism. Let

$$\mathrm{H}_2(L) = \left\{ \sum_i \langle X_i, Y_i\rangle \in \mathcal{L} \mid \sum_i [X_i, Y_i] = 0 \right\}.$$

The following theorem justifies to call $\mathrm{H}_2(L)$ the *second homology group of* L.

Theorem 5. (see 5.1.3 for a more detailed version) *Let L be a perfect Lie superalgebra over K and write $\mathcal{L} = \mathcal{L}_L$.*

(1) *The superalgebra $\mathcal{L}$ is a Lie superalgebra over K.*

(2) *The extension $0 \to \mathrm{H}_2(L) \to \mathcal{L} \xrightarrow{m} [L, L] \longrightarrow 0$ is a central extension of the Lie superalgebra $[L, L]$.*

(3) *If L is a perfect Lie superalgebra then $\mathcal{L}$ is the universal central extension of L and $Z(\mathcal{L}) = m^{-1}(Z(L)) \cong \mathrm{H}_2(L/Z(L))$.*

We first consider the toral 3-graded Lie superalgebras $L = L_+ \oplus L_0 \oplus L_-$ with $\zeta \in L_0$ the toral element.

Proposition 6. (see 5.2.1 for a more detailed version) *Let L be a toral 3-graded Lie superalgebra. Then $\mathcal{L}$ is a 3-graded Lie superalgebra where $\mathcal{L}_0 = \langle L_0, L_0\rangle + \langle L_+, L_-\rangle$ and $\mathcal{L}_\sigma = \langle \zeta, L_\sigma\rangle = \langle L_0, L_\sigma\rangle$ for $\sigma = \pm$. If $\zeta \in [L, L]$ then $\mathcal{L}$ is toral. Moreover, the map m preserves the 3-grading and $m|_{\mathcal{L}_\sigma} : \mathcal{L}_\sigma \to L_\sigma$ for $\sigma = \pm$ are K-supermodule isomorphisms. In particular, $\mathrm{H}_2(L) \subset \mathcal{L}_0$.*

We now wish to describe the second homology group of $L = \mathrm{eosp}(q_\infty)$. Consider the K-supermodule $\mathcal{J} = A \oplus \mathcal{M}$ and define the multiplication in $\mathcal{J}$ by

$$(a \oplus m)(b \oplus n) = (ab + q(m, n)) \oplus (an + mb).$$

Then $\mathcal{J}$ is a Jordan superalgebra. Let $\mathfrak{s}$ be the submodule of $\mathcal{J} \otimes_K \mathcal{J}$ spanned by $\{x \otimes y + (-1)^{|x||y|} y \otimes x, xy \otimes z - x \otimes yz - (-1)^{|x||y|} y \otimes xz, a \otimes m \mid x, y, z \in \mathcal{J},\ a \in A,\ m \in \mathcal{M}\}$. Let $\{\mathcal{J}, \mathcal{J}\} := (\mathcal{J} \otimes_K \mathcal{J})/\mathfrak{s}$ and for $x, y \in \mathcal{J}$, let $\{x, y\}$ denote the coset $x \otimes y + \mathfrak{s}$. We define the *full skew-dihedral homology group of $\mathcal{J}$* (see [ABG]) as

$$\mathrm{HF}(\mathcal{J}) = \left\{ \sum_i \{a_i, b_i\} + \sum_k \{m_k, n_k\} \mid a_i, b_i \in A,\ m_k, n_k \in \mathcal{M},\ \sum_k \mathbf{E}_{m_k, n_k} = 0 \right\}.$$

In particular, we have the following:

Theorem 9. (see 5.3.5 for a more detailed version) *Let $\mathcal{N}$ and $\mathcal{P}$ be A-supermodules with quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{P}}$ respectively such that $q_{\mathcal{N}}$ is almost diagonizable with respect to some basis $\{n_i \mid i \in I\}$ and that there exists an $i \in I$ such that $i = \underline{i}$. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ with $q = q_{\mathcal{N}} \oplus q_{\mathcal{P}}$. Let $L = \mathrm{eosp}(q_\infty)$ and assume that $\mathrm{eosp}(q)$ is perfect. Then if we set $\mathcal{J}_{\mathcal{M}} = A \oplus \mathcal{M}$ and $\mathcal{J}_{\mathcal{P}} = A \oplus \mathcal{P}$ we have*

$$\mathrm{HF}(\mathcal{J}_{\mathcal{P}}) \cong \mathrm{HF}(\mathcal{J}_{\mathcal{M}}) \cong \mathrm{H}_2(L).$$

The above is a generalization of the result on the centre of the universal central extension of a Lie algebra over a field and graded by the root system of type B or D proved in [ABG].

This thesis consists of five chapters. Chapter 1 covers the basic notions of superstructures and Lie algebras and superalgebras in particular. Chapter 2 contains the definitions and properties of the orthosymplectic Lie superalgebra $\mathrm{osp}(q)$ and $\mathrm{eosp}(q_\infty)$, the elementary orthosymplectic Lie superalgebra $\mathrm{eosp}(q)$ and $\mathrm{eosp}(q_\infty)$, and certain subalgebras of $\mathrm{osp}(q)$ ($\mathrm{osp}(q_\infty)$) containing $\mathrm{eosp}(q)$ (respectively, $\mathrm{eosp}(q_\infty)$). In Chapter 3 we describe the derivations of subalgebras of $\mathrm{osp}(q_\infty)$ containing $\mathrm{eosp}(q_\infty)$ while in Chapter 4 we describe the supersymmetric invariant forms on $\mathrm{eosp}(q_\infty)$ and determine the derivations which are skew with respect to a given supersymmetric invariant form. Finally, in Chapter 5 we describe the second homology group and the universal central extension of $\mathrm{eosp}(q_\infty)$.

Chapter 1

Preliminaries

1.1 Classical Case

Let R be an associative and unital ring.

Given R-bimodules $\mathcal{M}_1, \cdots, \mathcal{M}_n$ and $\mathcal{N}$, we say that a map $\phi : \mathcal{M}_1 \times \cdots \times \mathcal{M}_n \to \mathcal{N}$ is *R-multilinear* if

$$\begin{aligned}\phi(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_n) +\\ \phi(m_1, \ldots, m_{i-1}, m_i', m_{i+1}, \ldots, m_n) =\\ \phi(m_1, \ldots, m_{i-1}, m_i + m_i', m_{i+1}, \ldots, m_n);\end{aligned} \tag{1.1}$$

$$\begin{aligned}\phi(m_1, \ldots, m_{i-1}, m_i a, m_{i+1}, \ldots, m_n) =\\ \phi(m_1, \ldots, m_i, a m_{i+1}, m_{i+2}, \ldots, m_n); \text{ and}\end{aligned} \tag{1.2}$$

$$\phi(m_1, \ldots, m_{n-1}, m_n a) \;=\; \phi(m_1, \ldots, m_n)a \tag{1.3}$$

for all $a \in R$, $1 \le i \le n-1$ and $m_j, m_j' \in \mathcal{M}_j$, $1 \le j \le n$. The set of all R-multilinear maps $\phi : \mathcal{M}_1 \times \cdots \times \mathcal{M}_n \to \mathcal{N}$ forms an R-bimodule via the action

$$\begin{aligned}(a\phi)(m_1, \ldots, m_n) &:= a\phi(m_1, \ldots, m_n); \text{ and}\\ (\phi a)(m_1, \ldots, m_n) &:= \phi(am_1, m_2, \ldots, m_n)\end{aligned}$$

for all $a \in R$ and $m_i \in \mathcal{M}_i$, $i = 1, \ldots, n$. An R-linear map between R-bimodules $\mathcal{M}$ and $\mathcal{N}$ is called an *R-bimodule homomorphism from $\mathcal{M}$ to $\mathcal{N}$*. A bijective homomorphism is called an *isomorphism.* The set of all R-bimodule homomorphisms from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $\mathrm{Hom}_R(\mathcal{M}, \mathcal{N})$. In the case when $\mathcal{M} = \mathcal{N}$ we write $\mathrm{End}_R(\mathcal{M})$ instead of $\mathrm{Hom}_R(\mathcal{M}, \mathcal{M})$.

If $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are R-bimodules then the direct sum $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$ is also an R-bimodule via the action

$$a(m_1 \oplus \cdots \oplus m_n) := (am_1) \oplus \cdots \oplus (am_n)$$

and

$$(m_1 \oplus \cdots \oplus m_n)a := (m_1 a) \oplus \cdots \oplus (m_n a)$$

for all $a \in R$, $m_i \in \mathcal{M}_i$. Let $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$ and $\mathcal{M}' = \mathcal{M}'_1 \oplus \cdots \oplus \mathcal{M}'_{n'}$ be R-bimodules which can be expressed as direct sums of R-bimodules $\mathcal{M}_i$ and $\mathcal{M}'_i$ respectively. For each $i \in \{1, \ldots, n\}$ we define $\Pi_{\mathcal{M}_i} : \mathcal{M} \to \mathcal{M}_i$ to be the canonical projection of $\mathcal{M}$ onto the submodule $\mathcal{M}_i$ and we let $\iota_{\mathcal{M}_i} : \mathcal{M}_i \to \mathcal{M}$ be the embedding of $\mathcal{M}_i$ into $\mathcal{M}$. Similarly we define $\Pi_{\mathcal{M}'_i} : \mathcal{M}' \to \mathcal{M}'_i$ and $\iota_{\mathcal{M}'_i} : \mathcal{M}'_i \to \mathcal{M}'$. Then we have an isomorphism

$$\varphi \in \mathrm{Hom}_R(\mathcal{M}, \mathcal{M}') \longleftrightarrow (\varphi_{ij})$$

where $\varphi_{ij} = \Pi_{\mathcal{M}'_i} \varphi \iota_{\mathcal{M}_j} \in \mathrm{Hom}_R(\mathcal{M}_j, \mathcal{M}'_i)$, and, in particular, for $\mathrm{End}_R \mathcal{M}$,

$$\varphi \in \mathrm{End}_R(\mathcal{M}) \longleftrightarrow (\varphi_{ij})$$

where $\varphi_{ij} = \Pi_{\mathcal{M}_i} \varphi \iota_{\mathcal{M}_j} \in \mathrm{Hom}_R(\mathcal{M}_j, \mathcal{M}_i)$.

For a bimodule $\mathcal{M}$ we say that a map $\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ is *symmetric* if

$$\phi(m, n) = \phi(n, m)$$

for all $m, n \in \mathcal{M}$. We say that ϕ is *antisymmetric* if

$$\phi(m, n) = -\phi(n, m)$$

for all $m, n \in \mathcal{M}$.

An *algebra over* R (or a *k-algebra*) A is an R-bimodule equipped with an R-bilinear map $\cdot : A \times A \to A$ (note that an algebra as defined here need not be associative). A *subalgebra* of an algebra A is a submodule of A satisfying $A \cdot A \subset A$. If no confusion is possible, we usually abbreviate $ab = a \cdot b$.

Through the remainder of the thesis, k will denote an associative and commutative ring with unity 1. In addition we will assume that $\frac{1}{2} \in k$. In particular, all of the definitions above apply to $R = k$.

Given a left k-module $\mathcal{M}$, we can define a k-bimodule action on $\mathcal{M}$ by setting

$$ma = am \text{ for all } a \in k,\ m \in \mathcal{M}. \tag{1.4}$$

Throughout this section when we talk about a k-module $\mathcal{M}$ we shall assume that $\mathcal{M}$ is a k-bimodule satisfying (1.4).

Remark 1.1.1 The k-module $\mathrm{End}_k(\mathcal{M})$ forms a k-algebra under the multiplication $\circ : \mathrm{End}_k(\mathcal{M}) \times \mathrm{End}_k(\mathcal{M}) \to \mathrm{End}_k(\mathcal{M})$ defined by

$$(f \circ g)(m) := f(g(m))$$

for all $f, g \in \mathrm{End}_k(\mathcal{M})$, $m \in \mathcal{M}$. The algebra $\mathrm{End}_k(\mathcal{M})$ is associative but not, in general, commutative.

Example 1.1.2 Let $\mathcal{G}$ be the associative k-algebra generated by $\{\xi_i\}_{i \in \mathbb{N}}$ subject to the relation

$$\xi_i \xi_j + \xi_j \xi_i = 0 \text{ for all } i, j \in \mathbb{N}.$$

Since $\frac{1}{2} \in k$, we have in particular that $\xi_i^2 = 0$ for all $i \in \mathbb{N}$. It is easily seen that

$$\mathcal{G} = \mathrm{span}_k\{1, \xi_{i_1}\xi_{i_2}\cdots\xi_{i_n} \mid i_j \in \mathbb{N},\ i_1 < i_2 < \ldots < i_n,\ n \in \mathbb{N}\}.$$

The algebra $\mathcal{G}$ is called the *exterior algebra over k on a countable number of generators* ξ_i, $i \in \mathbb{N}$, or simply the *Grassmann algebra.* The Grassmann algebra is associative but not commutative.

An algebra K which is an associative, commutative, unital algebra over k is called an *extension* of k. Throughout this section, K shall denote an extension of k. Note that k is an extension of itself.

Definition 1.1.3 Let A be an algebra over an associative commutative ring R in a variety $\mathcal{V}$ and U an R-bimodule (not necessarily satisfying (1.4)). Suppose that we have a pair of R-bilinear mappings $(a, u) \mapsto au$, $(a, u) \mapsto ua$, $a \in A$, $u \in U$, of $A \otimes U$ into U. Let $X = A \oplus U$ and define multiplication in X by

$$(a + u)(b + v) = ab + av + ub \tag{1.5}$$

for all $a, b \in A$ and $u, v \in U$. Symbolically, this could be written as

$$X = A \oplus U = \left\{ \begin{pmatrix} a & u \\ 0 & a \end{pmatrix} \middle| a \in A,\ u \in U \right\}.$$

Since this product is bilinear, X is an algebra over r. X is called the *split null extension* of A determined by the bilinear mappings of A and U (see [La, Chap. II, Sect. 5]). If X is an algebra in the variety $\mathcal{V}$ then we say that X is a *$\mathcal{V}$-module for* A. If it is clear from the context what $\mathcal{V}$ is, we will write simply that X is an A-module.

If K is an extension of k then a k-module $\mathcal{M}$ is a K-module if there exists a pair of bilinear mappings $(a, m) \mapsto am$, $(a, m) \mapsto ma$, $a \in K$, $m \in \mathcal{M}$, of $K \otimes_k \mathcal{M}$ into $\mathcal{M}$ such that for all $a, b \in K$ and $m \in \mathcal{M}$.

(i) $am = ma$ and

(ii) $(ab)m = a(bm)$.

Quadratic maps. Given two K-modules $\mathcal{M}$ and $\mathcal{N}$, we say that a map $q : \mathcal{M} \to \mathcal{N}$ is *K-quadratic* if

1) $q(am) = a^2 q(m)$ for all $a \in K$, $m \in \mathcal{M}$, and

2) $q(m,n) := q(m+n) - q(m) - q(n)$ for $m,n \in \mathcal{M}$ is a K-bilinear map from $\mathcal{M} \times \mathcal{M}$ to $\mathcal{N}$, called the *polar of* q.

Remark 1.1.4 The polar $q : \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ of a K-quadratic map $q : \mathcal{M} \to \mathcal{N}$ is symmetric.

Quadratic forms. Given a K-module $\mathcal{M}$, a quadratic map $q : \mathcal{M} \to K$ is called a *K-quadratic form.* We define its *radical* by

$$R = \{m \in \mathcal{M} \mid q(m,n) = 0 \text{ for all } n \in \mathcal{M}\}.$$

It is easy to see that R is a submodule of $\mathcal{M}$ and that $q(m) = 0$ for all $m \in R$ (since $\frac{1}{2} \in K$). We say that q is *nondegenerate* if $R = 0$. If we have two K-modules $\mathcal{M}_1$ and $\mathcal{M}_2$ with quadratic forms q_1 and q_2 respectively, we can define the *orthogonal sum of* q_1 *and* q_2, denoted by $q = q_1 \oplus q_2$, as the K-quadratic form on $\mathcal{M}_1 \oplus \mathcal{M}_2$ given by

$$(q_1 \oplus q_2)(m_1 \oplus m_2) = q_1(m_1) + q_2(m_2)$$

for $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$.

Example 1.1.5 For an arbitrary set I, the *hyperbolic K-space (of rank $2|I|$)* is the free K-module of rank $2|I|$,

$$H(I) = H(I,K) = H_+(I,K) \oplus H_-(I,K)$$

where $H_\sigma(I,K) = \oplus_{i \in I} K h_{\sigma i}$ for $\sigma = \pm$. We assign to it a K-quadratic form $q_I : H(I) \to K$ determined by

$$q_I(h_i, h_{-j}) = \delta_{ij} \text{ for } i,j \in I \text{ and } q_I \mid_{H_\pm(I)} = 0.$$

Note that q is nondegenerate. A hyperbolic space of rank 2 is called a *hyperbolic plane.* We note that $H(I)$ is an orthogonal sum of the hyperbolic K-planes $H(\{i\})$, $i \in I$.

Given two K-algebras A and B, a map $\varphi : A \to B$ is called a *K-algebra homomorphism* (or a *K-algebra endomorphism* if $A = B$) if φ is a K-module homomorphism satisfying

$$\varphi(aa') = \varphi(a)\varphi(a') \tag{1.6}$$

for all $a, a' \in A$. If B is an A-module (in the sense of Definition 1.1.3), a map $d : A \to B$ is called a *K-derivation (from A to B)* if d is a K-module homomorphism satisfying

$$d(aa') = d(a) \cdot a' + a \cdot d(a') \tag{1.7}$$

for all $a, a' \in A$. We denote the K-module of all K-derivations from A to B by $\mathrm{Der}_K(A, B)$. We abbreviate $\mathrm{Der}_K(A) = \mathrm{Der}_K(A, A)$.

A K-bilinear map $(\cdot, \cdot) : A \times A \to B$ is called *invariant* if

$$(ab, c) = (a, bc)$$

for all $a, b, c \in A$.

Lie algebras. A *Lie algebra* L over K is a K-module with a multiplication $[\cdot, \cdot] : L \times L \to L$ called the *bracket* which satisfies the following properties for all $x, y, z \in L$:

(L1) $[x, y] = -[y, x]$;

(L2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

The axiom (L2) is called the *Jacobi identity.*

A submodule $\tilde{L}$ of L is called an *ideal* of L if $[\tilde{L}, L] \subset \tilde{L}$.

Example 1.1.6 Let A be an associative algebra over K : then A is a K-module. We can create a Lie algebra structure $A^{(-)}$ on A by defining the bracket of $A^{(-)}$ as

$$[x, y] = xy - yx$$

for all $x, y \in A$. In particular, if $\mathcal{M}$ is a K-module then $\mathrm{End}_K(\mathcal{M})$ is a K-algebra and so $(\mathrm{End}_K \mathcal{M})^{(-)}$ forms a Lie algebra.

Example 1.1.7 Given a K-algebra A, the K-module $\mathrm{Der}_K(A)$ is a Lie subalgebra of $\mathrm{End}_K(A)^{(-)}$.

Example 1.1.8 Given a K-module $\mathcal{M}$ with a K-quadratic form q, we define a K-module $\mathrm{o}(q)$ by

$$\mathrm{o}(q) = \{x \in \mathrm{End}_K \mathcal{M} \mid q(x(m), n) + q(m, x(n)) = 0 \text{ for all } m, n \in \mathcal{M}\}.$$

Since $\frac{1}{2} \in k$, we have that

$$\mathrm{o}(q) = \{x \in \mathrm{End}_K \mathcal{M} \mid q(x(m), m) = 0 \text{ for all } m \in \mathcal{M}\}.$$

It is easy to check that $\mathrm{o}(q)$ is a subalgebra of the Lie algebra $(\mathrm{End}_K \mathcal{M})^{(-)}$, called the *orthogonal Lie algebra of* q. In addition, we define for each $m, n \in \mathcal{M}$ a K-linear map $\mathbf{E}_{m,n} : \mathcal{M} \to \mathcal{M}$ by

$$\mathbf{E}_{m,n}(p) = mq(n, p) - q(m, p)n.$$

It is easy to check that $a\mathbf{E}_{m,n} = \mathbf{E}_{am,n}$ for all $a \in K$, $m, n \in \mathcal{M}$ (for a proof in a more general setting see 2.1.1). Since for all $m, n, p \in \mathcal{M}$,

$$q(\mathbf{E}_{m,n}(p), p) = q(q(n, p)m, p) - q(q(m, p)n, p) = 0$$

we have $\mathbf{E}_{m,n} \in \mathrm{o}(q)$ for all $m, n \in \mathcal{M}$. In addition, we have that for $x \in \mathrm{o}(q)$ and $m, n \in \mathcal{M}$,

$$[x, \mathbf{E}_{m,n}] = \mathbf{E}_{x(m),n} + \mathbf{E}_{m,x(n)}$$

(for a proof of this in more general setting see 2.3) and so the submodule $\mathrm{eo}(q)$ of $\mathrm{o}(q)$ spanned by $\{\mathbf{E}_{m,n} \mid m, n \in \mathcal{M}\}$ is an ideal of $\mathrm{o}(q)$ and is called the *elementary orthogonal Lie algebra (of q)*.

If L is a Lie algebra, we define the *centre of* L by

$$Z(L) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}.$$

Central extensions of Lie algebras. A Lie algebra L is called a *central extension* of the Lie algebra $\tilde{L}$ if there exists a central ideal C of L such that $L/C \cong \tilde{L}$.

Example 1.1.9 ([FLM]) Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ of characteristic 0 and $(\cdot,\cdot)$ an $\mathbb{F}$-bilinear, invariant, symmetric form on $\mathfrak{g}$. Let $K = \mathbb{F}[t^{\pm}]$ be the ring of Laurent polynomials over $\mathbb{F}$. Then the $\mathbb{F}$-vector space

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm}] \oplus \mathbb{F}c$$

becomes a Lie algebra under the bracket operation

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + (x, y)m\delta_{m+n,0}\,c; \; x, y \in \mathfrak{g},\, m, n \in \mathbb{Z};$$
$$[c, \tilde{\mathfrak{g}}] = 0$$

extended $\mathbb{F}$-bilinearly. Moreover, $\tilde{\mathfrak{g}}/\mathbb{F}c \cong \mathfrak{g} \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm}]$ and so $\tilde{\mathfrak{g}}$ is a central extension of the Lie algebra $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm}]$.

Derivations of Lie algebras. Given a Lie algebra L over K, for any element x of L, we can define the map $\mathbf{ad}(x) : L \to L$ by $\mathbf{ad}(x)(y) := [x, y]$. Then from the K-bilinearity of $[\cdot,\cdot]$ and the Jacobi identity, we have that $\mathbf{ad}(x) \in \mathrm{Der}_K(L)$ for all $x \in L$. Moreover, we have a Lie algebra homomorphism $\mathbf{ad} : L \to \mathrm{Der}_K(L)$. The derivations in the image of $\mathbf{ad}$ are called the *inner derivations of* L and are denoted by $\mathbf{ad}(L)$.

Example 1.1.10 The derivations of a finite-dimensional semisimple Lie algebra L over a field of characteristic zero are simply the inner derivations of L.

Example 1.1.11 Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over a field $\mathbb{F}$ of characteristic zero. Let K be an extension of $\mathbb{F}$ and let $L = \mathfrak{g} \otimes_{\mathbb{F}} K$. Then L is a Lie algebra with bracket

$$[g \otimes a, h \otimes b]^{\sim} = [g, h] \otimes ab$$

for $g, h \in \mathfrak{g}$ and $a, b \in K$. It is well known (see for example [K1, Exercises 7.3-7.5], [BM]) that $\mathrm{Der}_{\mathbb{F}}(L) \cong \mathbf{ad}(L) \oplus \mathrm{Der}_{\mathbb{F}} K$ via $\mathrm{Der}_{\mathbb{F}} K \hookrightarrow \mathrm{Der}_{\mathbb{F}} L : \delta \mapsto \delta(g \otimes a) := g \otimes \delta(a)$.

Jordan algebras. A *Jordan algebra* J over K is a K-algebra such that for all $a, b \in J$,

(JA1) $ab = ba$; and

(JA2) $(a^2b)a = a^2(ba)$.

Example 1.1.12 Let U be a K-module with a K-quadratic form q. Define $J = K \oplus U$ to be the K-algebra with multiplication given by

$$(a \oplus m)(b \oplus n) = (ab + q(m, n)) \oplus (an + mb). \tag{1.8}$$

One can easily check that J is a Jordan algebra. This algebra is called *the Jordan algebra associated with the quadratic form q* .

Jordan algebra modules. Let J be a Jordan algebra over K and U a K-module. Suppose we have a pair of bilinear mappings $(a, u) \mapsto au$, $(a, u) \mapsto ua$, $a \in J$, $u \in U$, of $J \otimes U$ into U. Then by definition of modules of the algebras in the variety of Jordan algebras (see 1.1.3), U is a Jordan module for J if the split null extension $X = J \oplus U$ with multiplication

$$(a + u)(b + v) = ab + au + vb \tag{1.9}$$

for $a, b \in J$ and $u, v \in U$ is a Jordan algebra. In fact, this holds if and only if the following are satisfied (see [La, Chap. II, Sect. 5]):

$$\begin{aligned} au &= ua & (1.10)\\ (ua^2)a &= (ua)a^2 & (1.11)\\ 2((ua)b)a + u(a^2b) &= 2(ua)(ab) + (ub)a^2 & (1.12) \end{aligned}$$

for all $a \in J$ and $u \in U$.

Example 1.1.13 Let $\mathcal{M}$ be a K-module. Consider the K-module $U = \text{eo}(q) \oplus \mathcal{M}$. Define the action $\circ$ of $J = K \oplus \mathcal{M}$ on U by setting

$$\begin{aligned} (x + p) \circ (a + m) &= (xa + \mathbf{E}_{p,m}) + (x(m) + pa) & (1.13)\\ &= (a + m) \circ (x + p) \end{aligned}$$

for all $a + m \in J$ and $x + p \in U$. Then U is a Jordan module for J (we will prove this in a more general setting later, see Example 1.3.22).

Jordan pairs. ([L]) A *Jordan pair over k* is a pair $V = (V^+, V^-)$ of k-modules together with a pair (Q^+, Q^-) of k-quadratic maps $Q^\sigma : V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma)$ such that the following identities hold in all scalar extensions V_K of V:

$$\textbf{(JP1)} \quad D^\sigma(x,y)Q^\sigma(x) = Q^\sigma(x)D^{-\sigma}(y,x)$$

$$\textbf{(JP2)} \quad D^\sigma(Q^\sigma(x),y) = D^\sigma(x,Q^\sigma(y)x)$$

$$\textbf{(JP3)} \quad Q^\sigma(Q^\sigma(x)y) = Q^\sigma(x)Q^\sigma(y)Q^\sigma(x)$$

for all $x \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$ where $D = (D^+, D^-)$, $D^\sigma : V^\sigma \times V^{-\sigma} \to \mathrm{Hom}(V^\sigma, V^\sigma)$ is defined by

$$D^\sigma(x,y)z = Q^\sigma(x,z)y =: \{x,y,z\}^\sigma$$

and $Q(\cdot,\cdot)$ is the polar of the quadratic map Q. It is important to note that $D(x,y)z = D(z,y)x$ for all $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$.

Example 1.1.14 ([L]) Given a K-module $\mathcal{M}$ with a K-quadratic form q, we can define a Jordan pair $V = (\mathcal{M}, \mathcal{M})$ by setting $Q(x)y = q(x,y)x - q(x)y$ for all $x, y \in \mathcal{M}$. It is called the *quadratic form pair (associated to q)*.

Tits-Kantor-Koecher algebras of Jordan pairs. ([N1],[Ko]) Let $V = (V^+, V^-)$ be a Jordan pair over K. For $(u,v) \in V$ put

$$\delta(u,v) = (D^+(u,v), -D^-(v,u))$$

and denote by $\mathcal{D}$ the K-span of all maps $\delta(u,v)$. One can show that $\delta(u,v)$ is a derivation on V in the following sense: A pair of maps $D = (D^+, D^-)$, $D^\sigma \in \mathrm{End}_K V^\sigma$, $\sigma = \pm$ is called a *derivation of V* if for all $\sigma = \pm$ and $x, z \in V^\sigma$, $y \in V^{-\sigma}$ we have

$$D^\sigma(\{x,y,z\}^\sigma) = \{D^\sigma(x), y, z\} + \{x, D^{-\sigma}(y), z\} + \{x, y, D^\sigma(z)\}.$$

It is well known that $\delta(u,v)$ is a derivation of V and that

$$[D, \delta(u,v)] = \delta(D^+(u), v) + \delta(u, D^-(v))$$

for all $u \in V^+, v \in V^-$ and so $\mathcal{D}$ forms an ideal of the derivation algebra of V. Hence $\mathcal{D}$ is a Lie algebra called the *inner derivation algebra of V*. We set

$$\mathcal{K} = \mathcal{K}(V) = V^+ \oplus \mathcal{D} \oplus V^-$$

and define the product on $\mathcal{K}$ by

$$\begin{aligned}[][u^+ \oplus a \oplus u^-, v^+ \oplus b \oplus v^-] =& (a^+v^+ - b^+u^+) \oplus ([a,b] + \delta(u^+, v^-) \\ & - \delta(v^+, u^-)) \oplus (a^-v^- - b^-u^-) \end{aligned} \tag{1.14}$$

where $u^\sigma, v^\sigma \in V^\sigma$ and $a = (a^+, a^-)$, $b = (b^+, b^-) \in \mathcal{D}$. We then have that $\mathcal{K}$ is a Lie algebra over K ([M]). It is called the *Tits-Kantor-Koecher algebra* (or *the TKK-algebra) of V.*

Example 1.1.15 ([N1]) Let $\mathcal{M}$ be a K-module and q a K-quadratic form on $\mathcal{M}$. Let $V = (\mathcal{M}, \mathcal{M})$ be the associated quadratic form pair. Then

$$\begin{aligned} D(m,n)(p) &= Q(m,p)(n) = Q(m+p)(n) - Q(m)(n) - Q(p)(n) \\ &= q(m+p,n)(m+p) - q(m+p)n \\ &\quad -q(m,n)m + q(m)n - q(p,n)p + q(p)n \\ &= q(m,n)p + q(p,n)m - q(m,p)n - q(m)n - q(p)n \\ &\quad +q(m)n + q(p)n \\ &= q(m,n)p + \mathbf{E}_{m,n}(p) \end{aligned}$$

for all $m, n, p \in \mathcal{M}$ and so $\mathcal{D} = \mathrm{span}_K\{q(m,n)\cdot Id + \mathbf{E}_{m,n} \mid m, n \in \mathcal{M}\}$. Let $L = L_{-1} \oplus L_0 \oplus L_1$ where

$$\begin{aligned} L_1 &= \{(0, m, 0, 0) \mid m \in \mathcal{M}\} \\ L_0 &= \{(a, 0, 0, x) \mid a \in K, x \in \mathrm{eo}(q)\} \\ L_{-1} &= \{(0, 0, m, 0) \mid m \in \mathcal{M}\} \end{aligned}$$

with multiplication given by

$$\begin{aligned}[][(a,m,n,x),(a',m',n',x')] =& (q(m,n') - q(n,m'), am' + xm' - a'm - x'm, \\ & -an' + xn' + a'n - x'n, \mathbf{E}_{m,n'} + \mathbf{E}_{n,m'} + [x,x'])\end{aligned}$$

for all $a, a' \in K$, $m, m', n, n' \in \mathcal{M}$ and $x, x' \in \mathrm{eo}(q)$. We will show later that L is a Lie algebra isomorphic to $\mathrm{eo}(q \oplus q_{\{\infty\}})$ where $q_{\{\infty\}}$ is the K-quadratic form on the hyperbolic K-plane (see Example 1.1.5). One can see that the TKK-algebra of V is isomorphic to

$$L_1 \oplus [L_1, L_{-1}]/C \oplus L_{-1}$$

where $[L_1, L_{-1}] \subset L_0$ is spanned by commutators

$$[(0, m, 0, 0), (0, 0, n, 0)] = (q(m, n), 0, 0, \mathbf{E}_{m,n})$$

and

$$C = \{l \in [L_1, L_{-1}] \mid [l, L_\sigma] = 0 \text{ for } \sigma = \pm\}.$$

If $\mathrm{Ann}_K \mathcal{M}\{a \in K \mid am = 0 \text{ for all } m \in \mathcal{M}\} = 0$ and there exist $m_i, n_i \in \mathcal{M}$ such that $\sum_i q(m_i, n_i) = 1$ i.e., q represents 1, then $\mathcal{K}(V) = L$ (see Proposition 2.3.10).

1.2 Graded Lie algebras

In this section we review the theory of certain graded Lie algebras, in particular root-graded Lie algebras, 3-graded Lie algebras, and we define toral 3-graded Lie algebras. The reader is referred to [N1] for a more detailed study of those algebras.

We continue to assume that k is a commutative, associative and unital ring containing $\frac{1}{2}$. By a k-module, we mean a k-bimodule satisfying (1.4). We denote by K an extension of k.

Given an abelian group G and an algebra A, we say that A is *G-graded* if there exist submodules A_g, $g \in \mathcal{G}$ of A such that $A = \oplus_{g \in \mathcal{G}} A_g$ and $A_g A_h \in A_{g+h}$ for all $g, h \in \mathcal{G}$.

Root Systems. ([Bo], [N1], [LN]) A subset R of a real vector space X with a scalar product $(\cdot, \cdot)$ is called a *root system (in X)* if R has the following three properties:

(RS1) R generates X as a vector space and $0 \notin R$,

(RS2) for each $\alpha \in R$ we have $s_\alpha(R) = R$ where $s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$ for $x \in X$,

(RS3) for all $\alpha, \beta \in R$, we have $\langle \alpha, \beta \rangle \in \mathbb{Z}$, where for $x \in X$, $\alpha \in R$,

$$\langle x, \alpha \rangle := 2\frac{(x,\alpha)}{(\alpha,\alpha)}.$$

A root system R is *irreducible* if $R \neq 0$ and if R is not an orthogonal sum of two non-empty root systems. Every root system is an orthogonal sum of irreducible root systems which are uniquely determined and are called the *irreducible components of R*. Note that in the finite-dimensional case, these definitions conform with the traditional definition of (finite) root systems. An irreducible root system is isomorphic to a finite root system or to one of the infinite analogues of the root systems of type $A - D$, explained in Example 1.2.1 below.

Example 1.2.1 For a nonempty set I, let X_I be the real vector space with basis $\{\epsilon_i \mid i \in I\}$ and scalar product $(\cdot \mid \cdot)$ satisfying $(\epsilon_i \mid \epsilon_j) = \delta_{ij}$. Assume that ∞ is a symbol not in I and define $I_\infty = I \cup \{\infty\}$. The following are root systems of rank $|I|$ which are irreducible if $|I| \geq 4$ ([N3]):

$$\begin{aligned}
A_I &= \{\epsilon_i - \epsilon_j \mid i, j \in I_\infty, i \neq j\} \subset X_{I_\infty}, \\
B_I &= \{\pm\epsilon_i \pm \epsilon_j \mid i, j \in I, i \neq j\} \cup \{\pm\epsilon_i \mid i \in I\} \subset X_I, \\
C_I &= \{\pm\epsilon_i \pm \epsilon_j \mid i, j \in I, i \neq j\} \cup \{\pm 2\epsilon_i \mid i \in I\} \subset X_I, \\
D_I &= \{\pm\epsilon_i \pm \epsilon_j \mid i, j \in I, i \neq j\} \subset X_I, \\
BC_I &= B_I \cup C_I.
\end{aligned}$$

Example 1.2.2 ([N1]) We call (R, R_1) a *3-graded root system* if R is a root system which has a disjoint decomposition $R = R_1 \dot\cup R_0 \dot\cup R_{-1}$ such that

$$R_{-1} = -R_1$$
$$R_0 = \{\alpha - \beta \mid \alpha, \beta \in R_1, \ (\alpha, \beta) \neq 0, \ \alpha \neq \beta\}$$
$$(R_i + R_j) \cap R \subset R_{i+j} \text{ where} R_k = 0 \text{ if } k \notin \{\pm 1, 0\}.$$

Then one can show that all irreducible reduced root systems of type other than E_8, F_4 or G_2 are 3-graded.

Root-graded Lie algebras. ([N2]) Let R be a root system. A Lie algebra L over K is called *R-graded* (or a *Lie algebra graded by root system R*) if there exist K-submodules L^α, $\alpha \in R \cup \{0\}$ of L such that

(RG1) $L = \oplus_{\alpha \in R \cup \{0\}} L^\alpha$;

(RG2) for all $\alpha, \beta \in R \cup \{0\}$,

$$[L^\alpha, L^\beta] \subset \begin{cases} L^{\alpha+\beta} & \text{if } \alpha + \beta \in R \cup \{0\} \\ \{0\} & \text{if } \alpha + \beta \notin R \cup \{0\} \end{cases} \quad ;$$

(RG3) as a Lie algebra, L is generated by $\cup_{\alpha \in R} L^\alpha$;

(RG4) for every $\alpha \in R$ there exists $0 \neq X_\alpha \in L^\alpha$ such that $H_\alpha := [X_{-\alpha}, X_\alpha]$ operates on L^β ($\beta \in R \cup \{0\}$) by

$$[H_\alpha, z_\beta] = \langle \beta, \alpha \rangle z_\beta \quad (z_\beta \in L^\beta).$$

Remark 1.2.3 Note that (RG3) is equivalent to

$$L^0 = \sum_{\alpha \in R} [L^\alpha, L^{-\alpha}].$$

Remark 1.2.4 The condition (RG4) implies that when K is a field of characteristic zero, the split simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in R} \mathfrak{g}_\alpha)$ whose root system is R relative to the split Cartan subalgebra $\mathfrak{h} = \mathrm{span}_K\{H_\alpha \mid \alpha \in R\} := \mathfrak{g}_0$ is a subalgebra of L and $\mathfrak{g}_\alpha \subset L_\alpha$ for all $\alpha \in R \cup \{0\}$. Here $\mathfrak{g}$ is generated by $\{X_{\pm\alpha} \mid \alpha \in B\}$ where B is a root basis.

Example 1.2.5 Lie algebras over fields of characteristic zero and graded by finite root systems have been classified by Berman and Moody ([BM]) in case of root systems of types A_n, $n \geq 2$, D_n and E, and by Benkart and Zelmanov ([BeZ]) in case of root systems of types A_1, $B_n (n \geq 3)$, $C_n (n \geq 2)$, F_4 and G_2. The classification in all of these cases was arrived to by using the decomposition of an R-graded Lie algebra L

into a direct sum of $\mathfrak{g}$-modules, where $\mathfrak{g}$ is the split simple R-graded Lie algebra, as follows:

$$L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \mathcal{D}$$

where $\mathfrak{a} := A \oplus B$ is an algebra over A, V is the vector space associated with $\mathfrak{g}$ and $\mathcal{D}$ is an algebra of derivations on $\mathfrak{a}$ which map A to A and B to B. Types of both $\mathfrak{a}$ and $\mathcal{D}$ are determined by R.

Example 1.2.6 Lie algebras over rings graded by root systems of types $\neq E_8$, F_4 or G_2 were classified by Neher in [N1]. In particular:

(1) Let k be a ring containing $\frac{1}{6}$. Then a Lie algebra L over k is B_I-graded ($|I| \geq 3$) if and only if there exists an extension K of k and a K-module $\mathcal{M}$ with a K-quadratic form q and an element $m \in \mathcal{M}$ satisfying $q(m) = 1$ such that L is a central extension of the Lie algebra $\mathrm{eo}(q_I \oplus q)$.

(2) Let k be a ring containing $\frac{1}{6}$. Then a Lie algebra L is D_I-graded ($|I| \geq 4$) if and only if there exists an extension K of k such that L is a central extension of the Lie algebra $\mathrm{eo}(q_I)$.

3-graded and toral 3-graded Lie algebras. A Lie algebra L over a ring K is called *3-graded* if it has a decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$ such that

$$[L_i, L_j] \subset L_{i+j} \text{ where } L_{i+j} = 0 \text{ if } i + j \notin \{\pm 1, 0\}. \tag{1.15}$$

Note that we do not assume that $L_0 = [L_1, L_{-1}]$ as was done in [N1]. We say that a 3-graded Lie algebra is *toral* if there exists an element $\zeta \in L_0$ such that

$$[\zeta, x_\sigma] = \sigma x_\sigma \text{ for all } x_\sigma \in L_\sigma,\ \sigma = \pm, 0. \tag{1.16}$$

Example 1.2.7 Let K be an extension of k and $\mathcal{M}$ a K-module with a quadratic form q. Let $\tilde{\mathcal{M}}$ be the orthogonal sum of K-modules $\mathcal{M}$ and a K-hyperbolic plane H with the quadratic form $q_\infty = q \oplus q_H$. Then $\mathrm{eo}(q_\infty)$ is a toral 3-graded Lie algebra where $\zeta = (1, 0, 0, 0)$ (see Example 1.1.15).

Remark 1.2.8 From the Example 1.2.6 we see that a Lie algebra over a ring containing $\frac{1}{6}$ is B_I-graded ($|I| \geq 3$) or D_I graded ($|I| \geq 4$) only if it is a central extension of $\mathrm{eo}(q_\infty)$ for some extension K of k, a K-module $\mathcal{M}$ and a K-quadratic form q on $\mathcal{M}$.

1.3 Superalgebras

In this section we will describe some fundamental concepts of k-superstructures, in particular Lie superalgebras and the superversion of $\mathrm{eo}(q)$. The reader can find more extensive coverage of this material in [S], [K2], [N2] and [N3].

Recall that k is a commutative, associative and unital ring containing $\frac{1}{2}$. We continue to call a k-bimodule satisfying (1.4) a k-module.

Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the field of two elements. A k-module $\mathcal{M}$ is called a *k-supermodule* if $\mathcal{M} = \mathcal{M}_{\bar{0}} \oplus \mathcal{M}_{\bar{1}}$ for some submodules $\mathcal{M}_{\bar{0}}$ and $\mathcal{M}_{\bar{1}}$ of $\mathcal{M}$. We call the elements of $\mathcal{M}_\alpha$, $\alpha \in \mathbb{Z}_2$, *homogeneous elements of degree* α. We denote the degree of a homogeneous element m by $|m|$. The homogeneous elements of degree $\bar{0}$ are called *even* and the homogeneous elements of degree $\bar{1}$ are called *odd*. We will assume throughout that when $|m|$ occurs in an expression, then it is assumed that m is homogeneous, and that the expression extends to the other elements by linearity.

A k-algebra K is called a *superalgebra over* k (or a k-superalgebra) if K is a $\mathbb{Z}_2$-graded algebra, i.e., $K = K_{\bar{0}} \oplus K_{\bar{1}}$ for some k-submodules $K_{\bar{0}}$ and $K_{\bar{1}}$ satisfying $K_\alpha K_\beta \in K_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$. We say that a k-superalgebra K is *supercommutative* if $ab = (-1)^{|a||b|}ba$ for all $a, b \in K$ and *(super)associative* if it is associative as a k-algebra. A k-superalgebra is *unital* if there exists $1 \in K$ such that $1a = a = a1$ for all $a \in K$. Note that $k \cdot 1 \subset K_{\bar{0}}$.

Example 1.3.1 Let R be a $\mathbb{Z}_2$-graded supercommutative ring, i.e. $R = R_{\bar{0}} \oplus R_{\bar{1}}$ and $R_\alpha R_\beta \subset R_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$. Then R is called a *supercommutative superring* and in particular, R is a superalgebra over $R_{\bar{0}}$.

A k-superalgebra K is called a *k-superextension* (or a *superextension of* k) if K is unital, associative and supercommutative. Throughout this section, K shall denote a superextension of k. Note that k is a superextension of itself by setting $k_{\bar{0}} = k$ and $k_{\bar{1}} = (0)$.

Example 1.3.2 An important example of a superextension of k is the *Grassmann algebra* $\mathcal{G}$ (see Example 1.1.2). Fix an order (for example, the natural order) on $\mathbb{N}$. If we write $\xi_\emptyset = 1_{\mathcal{G}}$ and $\xi_I = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}$ for a nonempty set $I = \{i_1, i_2, \cdots, i_n\} \subset \mathbb{N}$ with $i_1 < i_2 < \cdots < i_n$, then

$$\{\xi_I \mid I \text{ is a finite subset of } \mathbb{N}\}$$

is a k-basis of $\mathcal{G}$. Moreover, $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$ where

$$\mathcal{G}_{\bar{0}} = \bigoplus_{|I| \text{ even}} k\xi_I$$

and

$$\mathcal{G}_{\bar{1}} = \bigoplus_{|I| \text{ odd}} k\xi_I,$$

the sums being taken over finite ordered subsets of $\mathbb{N}$, is a $\mathbb{Z}_2$-grading of $\mathcal{G}$ and so $\mathcal{G}$ is a k-superalgebra. Clearly $\mathcal{G}$ is unital with unit $\xi_\emptyset$. By definition, $\mathcal{G}$ is associative. The defining relation $\xi_i\xi_j + \xi_j\xi_i = 0$ implies that $\xi_I\xi_J = (-1)^{|I||J|}\xi_J\xi_I$ and so $\mathcal{G}$ is supercommutative. Hence $\mathcal{G}$ is a superextension of k.

Supermodules over superextensions. In keeping with the Definition 1.1.3, a K-bimodule $\mathcal{M}$ is a *superextension-module for* K or *K-supermodule* for short if $\mathcal{M} = \mathcal{M}_{\bar{0}} \oplus \mathcal{M}_{\bar{1}}$ for some k-submodules $\mathcal{M}_\alpha$, $\alpha \in \mathbb{Z}_2$ such that

$$K_\alpha \mathcal{M}_\beta \subset \mathcal{M}_{\alpha+\beta}$$

for all $\alpha, \beta \in \mathbb{Z}_2$ and the K-bimodule action satisfies

$$am = (-1)^{|a||m|}ma$$

for all $a \in \mathcal{K}$, $m \in \mathcal{M}$. In particular, if $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_n$ are $\mathcal{K}$-supermodules then the direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_n$ is a $\mathcal{K}$-supermodule via the action $a(m_1, m_2, \cdots, m_n) = (am_1, am_2, \cdots am_n)$ for all $a \in \mathcal{K}$, $m_i \in \mathcal{M}_i$, $i = 1, 2, \cdots, n$.

Example 1.3.3 Let I be an arbitrary set. We define the $\mathcal{K}$-supermodule $H(I, \mathcal{K})$ to be the direct sum of free $\mathcal{K}$-supermodules $\mathcal{K}h_{\pm i}$ for $i \in I$ where the $h_{\pm i}$ are of even degree and form a $\mathcal{K}$-basis of $H(I, \mathcal{K})$. In other words

$$H(I) = H(I, \mathcal{K}) = \bigoplus_{i \in I} (\mathcal{K}h_i \oplus \mathcal{K}h_{-i}).$$

where the elements of the basis $\{h_{\pm i}\}_{i \in I}$ are all of even degree. Then we have that $H(I) = H(I)_{\bar{0}} \oplus H(I)_{\bar{1}}$ is a $\mathcal{K}$-supermodule where

$$H(I)_\alpha = \oplus_{i \in I} (\mathcal{K}_\alpha h_i \oplus \mathcal{K}_\alpha h_{-i})$$

for $\alpha \in \mathbb{Z}_2$.

Given a $\mathcal{K}$-supermodule $\mathcal{M}$, a $\mathcal{K}$-submodule $\mathcal{N}$ of $\mathcal{M}$ is a *sub(super)module* of $\mathcal{M}$ if $\mathcal{N} = (\mathcal{M}_{\bar{0}} \cap \mathcal{N}) \oplus (\mathcal{M}_{\bar{1}} \cap \mathcal{N})$. Throughout we will refer to a subsupermodule of the $\mathcal{K}$-supermodule $\mathcal{M}$ as simply a submodule of the $\mathcal{K}$-supermodule $\mathcal{M}$.

Lemma 1.3.4 *Let $\mathcal{M}$ be a supermodule over $\mathcal{K}$ and $\mathcal{N}$ a submodule of $\mathcal{M}$. Then the quotient $\mathcal{M}/\mathcal{N}$ is a $\mathcal{K}$-supermodule with respect to the induced actions $a\bar{m} = \overline{am}$, $\bar{m}a = \overline{ma}$ for $a \in \mathcal{K}$, $m \in \mathcal{M}$.*

Proof: Since $\mathcal{M} = \mathcal{M}_{\bar{0}} \oplus \mathcal{M}_{\bar{1}}$ and $\mathcal{K}_\alpha \mathcal{M}_\beta \subset \mathcal{M}_{\alpha+\beta}$, we have $\mathcal{M}/\mathcal{N} = (\mathcal{M}_{\bar{0}}/\mathcal{N}) \oplus (\mathcal{M}_{\bar{1}}/\mathcal{N})$ with $\mathcal{K}_\alpha(\mathcal{M}_\beta/\mathcal{N}) \subset (\mathcal{K}_\alpha \mathcal{M}_\beta)/\mathcal{N} \subset \mathcal{M}_{\alpha+\beta}/\mathcal{N}$ under the induced actions $a\bar{m} = \overline{am}$, $\bar{m}a = \overline{ma}$ for $a \in \mathcal{K}$, $m \in \mathcal{M}$. Moreover,

$$a\bar{m} = \overline{am} = (-1)^{|a||m|}\overline{ma} = (-1)^{|a||m|}\bar{m}a$$

for all $a \in \mathcal{K}$ and $m \in \mathcal{M}$ and so $\mathcal{M}/\mathcal{N}$ is a $\mathcal{K}$-supermodule. ∎

Let $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_n$ and $\mathcal{N}$ be $\mathcal{K}$-supermodules. A map $\phi : \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n \to \mathcal{N}$ is said to have *degree* α ($\alpha \in \mathbb{Z}_2$) if

$$\phi(m_1, \cdots, m_n) \in \mathcal{N}_{\alpha+|m_1|+\cdots+|m_n|}$$

for all homogeneous $m_i \in \mathcal{M}_i$, $1 \leq i \leq n$.

Remark 1.3.5 If $\mathcal{M}$ and $\mathcal{N}$ are K-supermodules, $\mathrm{Hom}_K(\mathcal{M},\mathcal{N})$ is a K-supermodule with

$$\mathrm{Hom}_K(\mathcal{M},\mathcal{N}) = (\mathrm{Hom}_K(\mathcal{M},\mathcal{N}))_{\bar{0}} \oplus (\mathrm{Hom}_K(\mathcal{M},\mathcal{N}))_{\bar{1}}$$

where $\mathrm{Hom}_K(\mathcal{M},\mathcal{N})_\alpha = \{f \in \mathrm{Hom}_K(\mathcal{M},\mathcal{N}) \mid \text{degree of } f \text{ is } \alpha\}$.

We say that a K-bilinear map $\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ is *supersymmetric* if

$$\phi(m,n) \;=\; (-1)^{|m||n|}\phi(n,m) \tag{1.17}$$

for all $m, n \in \mathcal{M}$ and that is superantisymmetric if

$$\phi(m,n) \;=\; -(-1)^{|m||n|}\phi(n,m) \tag{1.18}$$

for all $m, n \in \mathcal{M}$. Note that if ϕ is supersymmetric, then $\phi|_{\mathcal{M}_{\bar{0}}\times\mathcal{M}_{\bar{0}}}$ is symmetric and $\phi|_{\mathcal{M}_{\bar{1}}\times\mathcal{M}_{\bar{1}}}$ is antisymmetric. The *radical of* a supersymmetric ϕ is defined by

$$\mathrm{Rad}(\phi) = \{m \in \mathcal{M} \mid \phi(m,n) = 0 \text{ for all } n \in \mathcal{M}\}.$$

and ϕ is *nondegenerate* if $\mathrm{Rad}(\phi) = \{0\}$. We have

Proposition 1.3.6 *Let $\mathcal{M}$ be a K-supermodule and $\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ a supersymmetric K-bilinear map. Then*

(i) $\mathrm{Rad}(\phi)$ *is a K-submodule of $\mathcal{M}$;*

(ii) The induced K-quadratic form $\bar{\phi} : \mathcal{M}/\mathrm{Rad}(\phi) \to \mathcal{N} : (\overline{m},\overline{n}) \mapsto \phi(m,n)$ is nondegenerate.

Proof: (i) We need to prove that the homogeneous parts of $r \in \mathrm{Rad}(\phi)$ belong to $\mathrm{Rad}(\phi)$. Let $r = r_{\bar{0}} + r_{\bar{1}} \in \mathrm{Rad}(\phi)$. Then for all homogeneous $m \in \mathcal{M}$,

$$0 = \phi(m,r) = \underbrace{\phi(m,r_{\bar{0}})}_{\in\mathcal{N}_{|m|}} + \underbrace{\phi(m,r_{\bar{1}})}_{\in\mathcal{N}_{|m|+\bar{1}}}$$

and, as a result, $\phi(m, r_{\bar{0}}) = 0 = \phi(m, r_{\bar{1}})$ for all $m \in \mathcal{M}$ and so $r_{\bar{0}}, r_{\bar{1}} \in \mathrm{Rad}(\phi)$.

(ii) For all $m, n \in \mathcal{M}$ and $r, s \in \mathrm{Rad}(\phi)$,

$$\bar{\phi}(\overline{m+r}, \overline{n+s}) = \phi(m+r, n+s) = \phi(m, n) = \bar{\phi}(\bar{m}, \bar{n}),$$

hence $\bar{\phi}$ is well-defined. It inherits from ϕ the $\mathcal{K}$-bilinearity and supersymmetry. Moreover,

$$\bar{\phi}(\bar{m}, \bar{n}) = 0 \ \text{ for all } \bar{n} \in \bar{\mathcal{M}} \Rightarrow \phi(m, n) = 0 \ \text{ for all } n \in \mathcal{M} \Rightarrow m \in \mathrm{Rad}(\phi) \Rightarrow \bar{m} = 0.$$

Hence $\bar{\phi}$ is nondegenerate. ∎

Quadratic maps. A *$\mathcal{K}$-quadratic map* between $\mathcal{K}$-supermodules $\mathcal{M}$ and $\mathcal{N}$ is a $\mathcal{K}$-supersymmetric, $\mathcal{K}$-bilinear map $q : \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ of degree $\bar{0}$. Note that, since $\frac{1}{2} \in \mathcal{K}$, the map q induces a $\mathcal{K}_{\bar{0}}$-quadratic map between $\mathcal{M}_{\bar{0}}$ and $\mathcal{N}_{\bar{0}}$ via $q_{\bar{0}}(m) = \frac{1}{2}q(m, m)$, $m \in \mathcal{M}_{\bar{0}}$.

Quadratic forms. Given a $\mathcal{K}$-supermodule $\mathcal{M}$, a quadratic map q from $\mathcal{M}$ to $\mathcal{K}$ is called a *$\mathcal{K}$-quadratic form on $\mathcal{M}$*. Given two $\mathcal{K}$-supermodules $\mathcal{M}_1$ and $\mathcal{M}_2$ with $\mathcal{K}$-quadratic forms q_1 and q_2 respectively, we define their *orthogonal sum* $q = q_1 \oplus q_2$ to be the $\mathcal{K}$-quadratic form on the $\mathcal{K}$-supermodule $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ given by

$$\begin{aligned} q(m_1 \oplus m_2, n_1 \oplus n_2) &= (q_1 \oplus q_2)(m_1 \oplus m_2, n_1 \oplus n_2) \\ &= q(m_1, n_1) + q(m_2, n_2) \end{aligned}$$

for $m_1, n_1 \in \mathcal{M}_1$, $m_2, n_2 \in \mathcal{M}_2$.

Example 1.3.7 Let I be an arbitrary set. We define the $\mathcal{K}$-quadratic form q_I on the $\mathcal{K}$-supermodule $H(I; \mathcal{K})$ (see Example 1.3.3) by setting

$$q_I(h_{\sigma i}, h_{-\mu j}) = \delta_{\sigma,\mu}\delta_{i,j}$$

for all $i, j \in I$, $\sigma, \mu \in \{\pm\}$ and extending q bilinearly over $\mathcal{K}$. We call $H(I; \mathcal{K})$ the ***hyperbolic $\mathcal{K}$-superspace***. In the case when $|I| = 1$, we call $H(I; \mathcal{K})$ the ***hyperbolic superplane***.

Superalgebras over superextensions. Given a superextension K of k, a K-*superalgebra* A is a K-supermodule with a K-bilinear map $\cdot : A \times A \to A$ of degree $\bar{0}$. A *subalgebra* B *of the superalgebra* A is a K-submodule B of A such that B is closed under $\cdot|_{B\times B}$. A subalgebra B of A is a (left) *ideal* of A if $A \cdot B \subset B$. A unital, associative, supercommutative superalgebra A over K is called a *superextension of* K.

Example 1.3.8 Let $\mathcal{M}$ be a K-supermodule. Then $\mathrm{End}_K(\mathcal{M})$ is a K-superalgebra via $\circ : \mathrm{End}_K(\mathcal{M}) \times \mathrm{End}_K(\mathcal{M}) \to \mathrm{End}_K(\mathcal{M})$:

$$(f \circ g)(m) = f(g(m))$$

for all $m \in \mathcal{M}$.

Given two K-superalgebras A and B, a K-algebra homomorphism $\phi : A \to B$ of degree $\bar{0}$ is called a K-*superalgebra homomorphism.* If, in addition, ϕ is bijective, we say that ϕ is an *isomorphism.*

Definition 1.3.9 Let A be a K-superalgebra in a variety $\mathfrak{V}$ (later $\mathfrak{V}$ will be the variety of Jordan or Lie superalgebras). Let $\mathcal{U}$ be a K-supermodule equipped with a pair of K-bilinear mappings $(a, u) \mapsto au$, $(a, u) \mapsto ua$, $a \in A$, $u \in \mathcal{U}$, of $A \times \mathcal{U}$ into $\mathcal{U}$ of degree 0. Then $X = A \oplus \mathcal{U}$ is a K-supermodule on which we define a multiplication by

$$(a + u)(b + v) = ab + av + ub \tag{1.19}$$

for all $a, b \in A$, $u, v \in \mathcal{U}$. Since this product is K-bilinear, X is a superalgebra over K, called the *split null extension* of $\mathcal{A}$ determined by the bilinear mappings of A and $\mathcal{U}$ (see [J1, Chap. II, Sect., 5] for the classical case). If X is a superalgebra in the variety $\mathfrak{V}$ then we say that $\mathcal{U}$ is a $\mathfrak{V}$-*module* for A. In this case, for each $\alpha \in \mathbb{Z}_2$ let $(\mathrm{Der}_K(A, \mathcal{U}))_\alpha$ be the space of all homogeneous K-module homomorphisms $d \in \mathrm{Hom}_K(A, \mathcal{U})_\alpha$ satisfying for all homogeneous $x \in A$ and all $y \in A$

$$d(xy) = d(x)y + (-1)^{\alpha|x|} x d(y).$$

We define

$$\mathrm{Der}_K(A, \mathcal{U}) = (\mathrm{Der}_K(A, \mathcal{U}))_{\bar{0}} \oplus (\mathrm{Der}_K(A, \mathcal{U}))_{\bar{1}}.$$

Then it is easy to see that $\mathrm{Der}_{K}(A,\mathcal{U})$ is a submodule of $\mathrm{End}_{K}\mathcal{M}$ and hence a K-supermodule. The elements of $\mathrm{Der}_{K}(A,\mathcal{U})$ are called the *K-derivations (from A to $\mathcal{U}$)*. If $A=\mathcal{U}$, we simply write $\mathrm{Der}_{K}A$.

T-envelopes. Let K and T be k-superextensions and $\mathcal{M}$ a K-supermodule. We define the *T-envelope $T(\mathcal{M})$ of $\mathcal{M}$* by

$$T(\mathcal{M})=(T_{\bar{0}}\otimes_k\mathcal{M}_{\bar{0}})\oplus(T_{\bar{1}}\otimes_k\mathcal{M}_{\bar{1}}).$$

If we take $\mathcal{M}=K$, we have that $T(K)$ is an extension of k, i.e., a commutative, unital and associative algebra over k under the multiplication

$$(s\otimes a)(t\otimes b)=st\otimes ab$$

for $s,t\in T$ and $a,b\in K$ such that $|s|=|a|$, $|t|=|b|$.

In general, if $\mathcal{M}$ is a K-supermodule, $T(\mathcal{M})$ is a $T(K)$-module under the action

$$(s\otimes a)\cdot(t\otimes m):=st\otimes am$$

for all $s,t\in T$, $a\in K$ and $m\in\mathcal{M}$ such that $|s|=|a|$, $|t|=|m|$. Moreover, if A is a K-superalgebra, $T(A)$ is a $T(K)$-algebra with multiplication given by

$$(s\otimes x)(t\otimes y):=st\otimes xy$$

for all $s,t\in T$, $x,y\in A$, $|s|=|x|$, $|t|=|y|$.

Let $\mathcal{M}_1,\cdots,\mathcal{M}_n$ and $\mathcal{N}$ be K-supermodules and ϕ a K-module homomorphism from $\mathcal{M}_1\times\cdots\times\mathcal{M}_n$ to $\mathcal{N}$ of degree $\bar{0}$. We define its *T-envelope* $T(\phi):T(\mathcal{M}_1)\times\cdots\times T(\mathcal{M}_n)\to T(\mathcal{N})$ by

$$T(\phi)(t_1\otimes m_1,\cdots,t_n\otimes m_n):=t_1\cdots t_n\otimes\phi(m_1,\cdots,m_n)$$

for all $t_1,\cdots,t_n\in T$ and $m_i\in\mathcal{M}_i$, $i=1,\cdots,n$.

Let $\mathcal{M}$ and $\mathcal{N}$ be K-supermodules and q from $\mathcal{M}$ to $\mathcal{N}$ a K-quadratic map. Let $\mathcal{G}$ be the Grassmann algebra and define the map $\mathcal{G}(q) : \mathcal{G}(\mathcal{M}) \to \mathcal{G}(\mathcal{N})$ by

$$\mathcal{G}(q)(\sum_I \xi_I \otimes m_I) = \frac{1}{2} 1_{\mathcal{G}} \otimes q(m_{\bar{0}}, m_{\bar{0}}) + \sum_{\{I,J\} \subset \mathbb{N}} \xi_I \xi_J \otimes q(m_I, m_J)$$

where the sums are taken over finite and ordered subsets of $\mathbb{N}$ and in the second sum the subsets are distinct (including the possibility $I = \emptyset$). We then have that $\mathcal{G}(q)$ is a $\mathcal{G}(K)$-quadratic map such that the polar of $\mathcal{G}(q)$ is the Grassmann envelope $\mathcal{G}(q)$ of q (for proof see [N3]). $\mathcal{G}(q)$ is called the *Grassmann envelope of* q.

Example 1.3.10 ([N3]) Let A be a K-superalgebra and $\mathcal{G}$ the Grassmann algebra. A $\mathcal{G}$-envelope of A is called the *Grassmann envelope of* A. We will use the Grassmann envelopes to define varieties of superalgebras. For example, suppose A is a K-superalgebra. Then it is easy to show that $\mathcal{G}(A)$ is an associative $\mathcal{G}(K)$-algebra if and only if A is associative. Similarly, A is a supercommutative K-superalgebra if and only if $\mathcal{G}(A)$ is a commutative $\mathcal{G}(K)$-algebra. This is easily seen from the fact that

$$\begin{aligned}
(\xi_I \otimes x)(\xi_J \otimes y) &= \xi_I \xi_J \otimes xy \\
&= ((-1)^{|\xi_I||\xi_J|} \xi_J \xi_I) \otimes ((-1)^{|x||y|} yx) \\
&= \xi_J \xi_I \otimes yx \\
&= (\xi_J \otimes y)(\xi_I \otimes x)
\end{aligned}$$

for all finite ordered subsets $I, J \in \mathbb{N}$ and $x, y \in A$ such that $|\xi_I| = |x|$, $|\xi_J| = |y|$.

In general, we have the following definition:

Definition 1.3.11 Let $\mathcal{V}$ be a homogeneous variety of k-algebras. A K-superalgebra A is called a *$\mathcal{V}$-superalgebra* if $\mathcal{G}(A)$ belongs to $\mathcal{V}$.

Proposition 1.3.12 *Let $\mathcal{V}$ be a homogeneous variety of k-algebras and let A be a $\mathcal{V}$-superalgebra. Then $\mathcal{M}$ is an A-supermodule if and only if $\mathcal{G}(\mathcal{M})$ is a $\mathcal{G}(A)$-module.*

Proof: Using the Definitions 1.3.9 and 1.3.11, an A-module $\mathcal{M}$ is an A-supermodule if and only if $\mathcal{G}(A \oplus \mathcal{M})$ is a $\mathcal{V}$-algebra where the multiplication on $\mathcal{G}(A \oplus \mathcal{M})$ is given by

$$(\xi_I \otimes (a+m))(\xi_J \otimes (b+n)) = \xi_I \xi_J \otimes (ab + an + mb)$$

and extended linearly. Since

$$\begin{aligned}
\mathcal{G}(A \oplus \mathcal{M}) &= \mathcal{G}_{\bar{0}} \otimes (A_{\bar{0}} \oplus \mathcal{M}_{\bar{0}}) + \mathcal{G}_{\bar{0}} \otimes (A_{\bar{1}} \oplus \mathcal{M}_{\bar{1}}) \\
&= ((\mathcal{G}_{\bar{0}} \otimes A_{\bar{0}}) \oplus (\mathcal{G}_{\bar{1}} \otimes A_{\bar{1}})) \oplus ((\mathcal{G}_{\bar{0}} \otimes \mathcal{M}_{\bar{0}}) \oplus (\mathcal{G}_{\bar{1}} \otimes \mathcal{M}_{\bar{1}})) \\
&= \mathcal{G}(A) \oplus \mathcal{G}(\mathcal{M}),
\end{aligned}$$

we have

$$(\xi_I \otimes (a+m))(\xi_J \otimes (b+n)) = ((\xi_I \otimes a) \oplus (\xi_I \otimes m))((\xi_J \otimes b) \oplus (\xi_J \otimes n))$$

and since the typical element of $\mathcal{G}(A) \oplus \mathcal{G}(\mathcal{M})$ looks like $\sum_I (\xi_I \otimes a_I \oplus \xi_I \otimes m_I)$, the assertion now follows. ∎

Lie superalgebras. Let K be a k-superextension and L a K-supermodule. L is called a *Lie superalgebra* if it is equipped with a K-bilinear bracket multiplication $[\cdot,\cdot] : L \times L \to L$ such that:

(SL1) For all $x, y \in L$,

$$[x, y] = -(-1)^{|x||y|}[y, x];$$

(SL2) For all $x, y, z \in L$,

$$(-1)^{|x||z|}[[x, y], z] + (-1)^{|y||x|}[[y, z], x] + (-1)^{|z||y|}[[z, x], y] = 0.$$

The property (SL2) is called the *Jacobi identity* and is equivalent to

$$[[x, y], z] = [x, [y, z]] - (-1)^{|x||y|}[y, [x, z]] \tag{1.20}$$

for all $x, y, z \in L$.

In fact, this definition is consistent with the Definition 1.3.11. We have the well known

Proposition 1.3.13 *Let L be a K-superalgebra. Denote the multiplication in L by $[\cdot,\cdot]$. The Grassmann envelope $\mathcal{G}(L)$ of L is a Lie algebra if and only if L satisfies* (SL1) *and* (SL2).

Proof: $\boxed{\Rightarrow}$ Since $\mathcal{G}$ is free, every $X \in \mathcal{G}(L)$ has a unique representation of the form $\bigoplus_I(\xi_I \otimes x_I)$ where the sum is over finite subsets of $\mathbb{N}$. As a result, $X = 0$ if and only if $x_I = 0$ for all finite $I \subset \mathbb{N}$.

Denote the multiplication in L by $[\cdot,\cdot]$. Let $\xi_I \otimes x, \xi_J \otimes y \in \mathcal{G}(L)$. Then

$$\begin{aligned}
[\xi_I \otimes x, \xi_J \otimes y] + [\xi_J \otimes y, \xi_I \otimes x] = \\
&= \xi_I\xi_J \otimes [x,y] + \xi_J\xi_I \otimes [y,x] \\
&= \xi_I\xi_J \otimes [x,y] + (-1)^{|\xi_I||\xi_J|}\xi_I\xi_J \otimes [y,x] \qquad (1.21)\\
&= \xi_I\xi_J \otimes ([x,y] + (-1)^{|x||y|}[y,x]).
\end{aligned}$$

If L satisfies (SL1) then (1.21) implies that $\mathcal{G}(L)$ satisfies (L1). Conversely, for any $x, y \in L$ we can find disjoint $I, J \subset \mathbb{N}$ such that $|\xi_I| = |x|$ and $|\xi_J| = |y|$ and so, in particular, such that $\xi_I\xi_J \neq 0$. So if $\mathcal{G}(L)$ satisfies (L1), by 1.21 we have that L satisfies (SL1).

Similarly, for $\xi_I \otimes x, \xi_J \otimes y, \xi_K \otimes z \in \mathcal{G}(L)$ we have

$$\begin{aligned}
[\xi_I \otimes x, [\xi_J \otimes y, \xi_K \otimes z]] + [\xi_J \otimes y, [\xi_K \otimes z, \xi_I \otimes x]] + [\xi_K \otimes z, [\xi_I \otimes x, \xi_J \otimes y]] \\
&= \xi_I\xi_J\xi_K \otimes [x,[y,z]] + \xi_J\xi_K\xi_I \otimes [y,[z,x]] \\
&\quad + \xi_K\xi_I\xi_J \otimes [z,[x,y]] \qquad (1.22)\\
&= \xi_I\xi_J\xi_K \otimes ([x,[y,z]] + (-1)^{|x|(|y|+|z|)}[y,[z,x]] \\
&\quad + (-1)^{|z|(|x|+|y|)}[z,[x,y]]).
\end{aligned}$$

Hence if L satisfies (SL2) then, by 1.22, $\mathcal{G}(L)$ satisfies (L2). Conversely, for any $x, y, z \in L$ we can pick disjoint $I, J, K \subset \mathbb{N}$ such that $|\xi_I| = |x|$, $|\xi_J| = |y|$, $|\xi_K| = |z|$ (and so, in particular, such that $\xi_I\xi_J\xi_K \neq 0$). Therefore if $\mathcal{G}(L)$ satisfies (L2) then by 1.22, L satisfies (SL2). Hence L is a Lie superalgebra if and only if $\mathcal{G}(L)$ is a Lie algebra. ∎

Example 1.3.14 Given an associative K-superalgebra A, we can define a Lie superalgebra structure on A, denoted by $A^{(-)}$, by defining the bracket operation via

$$[x, y] = xy - (-1)^{|x||y|}yx$$

for all $x, y \in A$ homogeneous and extending it linearly on all of A. In particular, one can easily check that $\mathrm{Der}_K(A)$ is a subalgebra of $\mathrm{End}_K(A)^{(-)}$.

Example 1.3.15 Let L be a Lie superalgebra over k and K a superextension of k. Then $L \otimes_k K$ is a Lie superalgebra via

$$[x \otimes a, y \otimes b] = (-1)^{|a||y|}[x, y] \otimes ab$$

for all $x, y \in L$, $a, b \in K$. The $\mathbb{Z}_2$-grading is given by

$$(L \otimes_k K)_{\bar{0}} = (L_{\bar{0}} \otimes_k K_{\bar{0}}) \oplus (L_{\bar{1}} \otimes_k K_{\bar{1}});$$
$$(L \otimes_k K)_{\bar{1}} = (L_{\bar{0}} \otimes_k K_{\bar{1}}) \oplus (L_{\bar{1}} \otimes_k K_{\bar{0}}).$$

Example 1.3.16 Let $\mathcal{M}$ be a K-supermodule with a K-quadratic form q. Set

$$\mathrm{osp}(q) = \{x \in \mathrm{End}_K \mathcal{M} \mid q(x(m), n) + (-1)^{|m||n|}q(x(n), m) = 0 \text{ for all } m, n \in \mathcal{M}\}.$$

We define, for homogeneous $m, n, p \in \mathcal{M}$,

$$\mathbf{E}_{m,n}(p) = mq(n, p) - (-1)^{|n||p|}q(m, p)n,$$

and we extend $\mathbf{E} : \mathcal{M} \times \mathcal{M} \to \mathrm{End}_K \mathcal{M}$ linearly. Let

$$\mathrm{eosp}(q) = \mathrm{span}_K\{\mathbf{E}_{m,n} \mid m, n \in \mathcal{M} \text{ homogeneous}\}.$$

We will show in Chapter 2 that $\mathrm{osp}(q)$ is a subalgebra of the Lie superalgebra $\mathrm{End}_K(\mathcal{M})^{(-)}$ and that $\mathrm{eosp}(q)$ is an ideal of $\mathrm{osp}(q)$. We call $\mathrm{osp}(q)$ the *orthosymplectic Lie superalgebra of* q and $\mathrm{eosp}(q)$ the *elementary orthosympletic Lie superalgebra (of q)*. In fact, we will show that the Grassmann envelopes $\mathcal{G}(\mathrm{osp}(q))$ and $\mathcal{G}(\mathrm{eosp}(q))$ are the orthogonal Lie algebra and the elementary orthogonal Lie algebra of $\mathcal{G}(q)$ respectively.

Example 1.3.17 Let L be a Lie superalgebra over K and let $\mathcal{M}$ and $\mathcal{N}$ be submodules of L with $[\mathcal{M},\mathcal{M}] \subset \mathcal{M}$, $[\mathcal{M},\mathcal{N}] \subset \mathcal{N}$ and $[\mathcal{N},\mathcal{N}] \subset \mathcal{M}$. Then $\mathcal{M} \oplus \mathcal{N}$ is a Lie superalgebra under the multiplication

$$[m \oplus n, m' \oplus n']^{\sim} = ([m, m'] + [n, n']) \oplus ([m, n'] + [n, m']) \tag{1.23}$$

where $[\cdot,\cdot]$ denotes the bracket in L. Indeed, since $\mathcal{M}$ and $\mathcal{N}$ are submodules of L, they are themselves K-supermodules and so we can set

$$(\mathcal{M} \oplus \mathcal{N})_\alpha = \mathcal{M}_\alpha \oplus \mathcal{N}_\alpha; \ \alpha \in \mathbb{Z}_2.$$

For all $m \oplus n \in (\mathcal{M} \oplus \mathcal{N})_\alpha, m' \oplus n' \in (\mathcal{M} \oplus \mathcal{N})_\beta$, $\alpha, \beta \in \mathbb{Z}_2$,

$$\begin{aligned} [m \oplus n, m' \oplus n']^{\sim} &= \underbrace{([m, m'] + [n, n'])}_{\in \mathcal{M}_{\alpha+\beta}} \oplus \underbrace{([m, n'] + [n, m'])}_{\in \mathcal{N}_{\alpha+\beta}} \\ &\subset (\mathcal{M} \oplus \mathcal{N})_{\alpha+\beta} \end{aligned}$$

and so $\mathcal{M} \oplus \mathcal{N}$ is a $\mathbb{Z}_2$-graded superalgebra over K. Moreover, using the fact that L is a Lie superalgebra, we have

$$\begin{aligned} [m \oplus n, m' \oplus n']^{\sim} &= ([m, m'] + [n, n']) \oplus ([m, n'] + [n, m']) \\ &= (-1)^{\alpha\beta}([m', m] + [n', n]) \oplus ([m', n] + [n', m]) \\ &= (-1)^{\alpha\beta}[m' \oplus n', m \oplus n]^{\sim} \end{aligned}$$

and

$$\begin{aligned} &[m \oplus n, [m' \oplus n', m'' \oplus n'']^{\sim}]^{\sim} - (-1)^{\alpha\beta}[m' \oplus n', [m \oplus n, m'' \oplus n'']^{\sim}]^{\sim} \\ &= [m \oplus n, ([m', m''] + [n', n'']) \oplus ([m', n''] + [n', m''])^{\sim} \\ &\quad - (-1)^{\alpha\beta}[m' \oplus n', ([m, m''] + [n, n'']) \oplus ([m, n''] + [n, m''])^{\sim} \\ &= ([m, [m', m'']] + [n, [n', n'']] + [n, [m', n'']] + [n, [n', m'']]) \\ &\quad \oplus ([m, [m', n'']] + [m, [n', m''] + [n, [m', n'']] + [n, [n', n''])) \\ &\quad - (-1)^{\alpha\beta}(([m', [m, m'']] + [m', [n', n'']] + [n', [m, n'']] + [n', [n, m'']]) \\ &\quad \oplus ([m', [m, n'']] + [m', [[n, m'']] + [n', [m, m'']] + [n', [n, n'']]) \\ &= ([[m, m'], m''] + [[m, n'], n''] + [[n, m'], n''] + [[n, n'], m'']) \\ &\quad + \oplus([[m, m'], n''] + [[m, n'], m''] + [[n, m'], m''] + [[n, n'], n'']) \\ &= [[m \oplus n, m' \oplus n'], m'' \oplus n'']^{\sim} \end{aligned}$$

and so $[\cdot,\cdot]^{\tilde{}}$ satisfies the Jacobi identity. Therefore $\mathcal{M}\oplus\mathcal{N}$ is a Lie superalgebra over K with the bracket given by (1.23). ■

If L is a Lie superalgebra, we define the *centre of* L by

$$Z(L) = \{x \in L \mid [x,y] = 0 \text{ for all } y \in L\}.$$

We have that $Z(L)$ is an ideal of L. We say that a K-bilinear form $(\cdot,\cdot) : L \times L \to K$ on a Lie superalgebra L over K is a *2-supercocycle* (see [Fuks]) if

(LSC1) $(x,y) = -(-1)^{|x||y|}(y,x)$, and

(LSC2) $([x,y],z) = (x,[y,z]) - (-1)^{|x||y|}(y,[x,z])$

for all homogeneous $x, y, z \in L$.

Homomorphisms of Lie superalgebras. Given two Lie superalgebras L and L' over K, we say that a map $\phi : L \to L'$ is a *Lie superalgebra homomorphism* if ϕ is a K-supermodule homomorphism of degree $\bar{0}$ which satisfies

$$\phi([x,y]) = [\phi(x),\phi(y)]$$

for all $x, y \in L$. We say that a K-homomorphism ϕ is a *Lie superalgebra isomorphism* if ϕ is a bijective map. In this case, we write $L \overset{\phi}{\cong} L'$ or simply $L \cong L'$.

Central extensions of Lie superalgebras. A Lie superalgebra $\tilde{L}$ is called a *central extension* of the Lie superalgebra L if there exists a central ideal C of $\tilde{L}$ (i.e., $C \subset Z(\tilde{L})$) such that $\tilde{L}/C \cong L$.

Example 1.3.18 Let L be a Lie superalgebra over K and $(\cdot,\cdot)$ a 2-supercocycle on L. Consider the K-supermodule $\tilde{L} = L \oplus Kc$ where Kc is a free K-supermodule with even K-basis $\{c\}$. Define the bracket $[\cdot,\cdot]^{\tilde{}}$ on $\tilde{L}$ by

$$[x,y]^{\tilde{}} = [x,y] + (x,y)c$$
$$[x,c]^{\tilde{}} = 0$$

for all $x, y \in L$. Then it is easy to check that $\tilde{L}$ is a Lie superalgebra and so a central extension of L. Indeed, since $Kc \subset Z(\tilde{L})$ we only need to show that $[\cdot,\cdot]^{\sim}{}_{L\times L}$ satisfies (SL1) and (SL2). Let $x, y, z \in L$. Then

$$[x,y]^{\sim} = [x,y] + (x,y)c = -(-1)^{|x||y|}([y,x] + (y,x)c) = -(-1)^{|x||y|}[y,x]^{\sim}$$

and

$$\begin{aligned}[x,[y,z]^{\sim}]^{\sim} - (-1)^{|x||y|}[y,[x,z]^{\sim}]^{\sim} &= [x,[y,z]]^{\sim} - (-1)^{|x||y|}[y,[x,z]]^{\sim} \\ &= [x,[y,z]] + (x,[y,z])c \\ &\quad - (-1)^{|x||y|}[y,[x,z]] - (-1)^{|x||y|}(y,[x,z])c \\ &= [[x,y],z] + ([x,y],z)c \\ &= [[x,y]^{\sim},z]^{\sim}.\end{aligned}$$

Hence $[\cdot,\cdot]^{\sim}$ satisfies (SL1) and (SL2) and so L is a Lie superalgebra.

Example 1.3.19 Let L be a Lie superalgebra over k and $k[t^{\pm}]$ the ring of Laurent polynomials over k. Let $(\cdot,\cdot)$ be a supersymmetric invariant form on L. Set

$$\tilde{L} = (L \otimes k[t^{\pm 1}]) \oplus kc$$

and define the bracket operation by

$$\begin{aligned}&[x \otimes t^m, y \otimes t^n] = [x,y] \otimes t^{m+n} + (x,y)m\delta_{m+n,0}c; \ x, y \in L, m, n \in \mathbb{Z} \\ &[c, \tilde{L}] = 0.\end{aligned}$$

By Example 1.3.15, $L \otimes k[t^{\pm 1}]$ is a Lie superalgebra over k. Let $\overline{(\cdot,\cdot)} : (L \otimes k[t^{\pm 1}]) \otimes (L \otimes k[t^{\pm 1}]) \to k$ be defined by

$$\overline{(x \otimes t^m, y \otimes t^n)} = (x,y)m\delta_{m+n,0}$$

for all $x, y \in L$, $m, n \in \mathbb{Z}$. Then it is easy to check that $\overline{(\cdot,\cdot)}$ is a Lie supercocycle on $L \otimes k[t^{\pm 1}]$ and so by Example 1.3.18, $\tilde{L}$ is a central extension of $L \otimes k[t^{\pm 1}]$. The $\mathbb{Z}_2$-grading of $\tilde{L}$ is given by

$$(\tilde{L})_{\bar{0}} = (L_{\bar{0}} \otimes k[t^{\pm}]) \oplus kc \text{ and } (\tilde{L})_{\bar{1}} = L_{\bar{1}} \otimes k[t^{\pm}].$$

Derivations of Lie superalgebras. In keeping with the definition of derivations of arbitrary K-superalgebras, given a Lie superalgebra L over K, a K-module homomorphism $d : L \to L$ is called a *K-derivation (of L)* if $d = d_{\bar{0}} \oplus d_{\bar{1}}$ where d_α ($\alpha \in \mathbb{Z}_2$) is a K-module homomorphism of degree α satisfying

$$d_\alpha[x, y] = [d_\alpha(x), y] + (-1)^{\alpha|x|}[x, d_\alpha(y)] \tag{1.24}$$

for all $x \in L_\beta, y \in L$. We denote the set of K-derivations of a Lie superalgebra L by $\mathrm{Der}_K(L)$. For any element x of L, we can define the map $\mathbf{ad}(x) : L \to L$ by $\mathbf{ad}(x)(y) := [x, y]$. The K-supermodule $\mathbf{ad}(L)$ forms an ideal of $\mathrm{Der}_K L$ called the *inner derivation algebra of* L, denoted by $\mathbf{ad}\, L$. Further, we define the *locally inner derivations of* L to be the derivations $d \in \mathrm{Der}_K L$ such that for any finite subset $\{x_1, \ldots, x_n\}$ of L there exists $x \in L$ satisfying $d(x_i) = [x, x_i]$ (see [DZ]). We denote the locally inner derivations of L by $\mathbf{ad}_{\mathrm{loc}}\, L$.

Lemma 1.3.20 *Let L be a superalgebra over K. Then $\mathbf{ad}_{\mathrm{loc}}\, L$ is an ideal of* $\mathrm{Der}_K L$.

Proof: Let $d \in \mathrm{Der}_K L$, $d' \in \mathbf{ad}_{\mathrm{loc}}\, L$. Let $\{x_1, \ldots, x_n\}$ be a finite subset of L. Then $\{x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)\}$ is also a finite subset of L and so, since d is locally inner, there exists $x \in L$ such that

$$d'(x_i) = [x, x_i] \text{ and } d'(d(x_i)) = [x, d(x_i)].$$

Note that $|d'| = |x|$. Hence

$$\begin{aligned} [d, d'](x_i) &= d(d'(x_i)) - (-1)^{|d||d'|}d'(d(x_i)) = d([x, x_i]) - (-1)^{|d||d'|}[x, d(x_i)] \\ &= [d(x), x_i] + (-1)^{|d||x|}[x, d(x_i)] - (-1)^{|d||d'|}[x, d(x_i)] \\ &= [d(x), x_i] \end{aligned}$$

and so $[d, d']$ is a locally inner derivation of L. ∎

Example 1.3.21 ([K2, Prop. 2.3.4]) Any finite-dimensional Lie superalgebra L over a field k whose Killing form is nondegenerate has no outer derivations, i.e., $\mathrm{Der}_k(L) = \mathbf{ad}(L)$.

Jordan superalgebras. A *Jordan superalgebra* $J = J_{\bar 0} \oplus J_{\bar 1}$ over K is a K-superalgebra such that its Grassmann envelope $\mathcal{G}(J)$ is a Jordan algebra over $\mathcal{G}(K)$. One can check that this holds if and only if for all $a, b, c, d \in J$,

(JSA1) $ab = (-1)^{|a||b|}ba$

(JSA2) $((ab)c)d + (-1)^{|d||c|}a((bd)c) + (-1)^{|b||a|+|d||c|}b((ad)c) =$

$$(ab)(cd) + (-1)^{|c||b|}(ac)(bd) + (-1)^{|d|(|b|+|c|)}(ad)(bc)$$

(JSA3) $(a^2c)a = a^2(ca)$.

Example 1.3.22 Let $\mathcal{M}$ be a K-supermodule with a K-quadratic form q. Define $J = K \oplus \mathcal{M}$ to be the K-superalgebra with the natural $\mathbb{Z}_2$-grading and multiplication given by

$$(a \oplus m)(b \oplus n) = (ab + q(m, n)) \oplus (an + mb). \tag{1.25}$$

One can easily check that the Grassmann envelope of J is the Jordan algebra $\mathcal{G}(J)$ associated with the $\mathcal{G}(K)$-quadratic form. Consider the K-supermodule $X = \mathrm{eosp}(q) \oplus \mathcal{M}$. Define the action $\circ$ of $J = K \oplus \mathcal{M}$ on X by setting

$$\begin{aligned}(x + p) \circ (a + m) &= (xa + \mathbf{E}_{p,m}) + (x(m) + pa) \\ &= (-1)^{\alpha\beta}(a + m) \circ (x + p)\end{aligned} \tag{1.26}$$

for all $a + m \in J_\alpha$ and $x + p \in X_\beta$ homogeneous. Then, as we will see later (see Example 1.5.14), X is a Jordan supermodule for J.

Jordan superpairs. ([N3]) A *Jordan superpair* over K is a pair $V = (V^+, V^-)$ of K-supermodules together with a pair (Q^+, Q^-) of K-quadratic maps $Q^\sigma : V^\sigma \to \mathrm{Hom}_K(V^{-\sigma}, V^\sigma)$ such that its Grassmann envelope $\mathcal{G}(V)$ together with the $\mathcal{G}(K)$-quadratic maps $(\widetilde{Q}^+, \widetilde{Q}^-)$ is a Jordan pair over $\mathcal{G}(K)$. The maps $\widetilde{Q}^\sigma$ are defined as

$$\widetilde{Q}^\sigma := \widetilde{\ } \circ \mathcal{G}(Q^\sigma) : \mathcal{G}(V^\sigma) \to \mathrm{Hom}_{\mathcal{G}(K)}(\mathcal{G}(V^{-\sigma}), \mathcal{G}(V^\sigma))$$

where $\mathcal{G}(Q^\sigma)$ is the Grassmann envelope of Q^σ for $\sigma = \pm$ and

$$\tilde{}: \mathcal{G}(\mathrm{Hom}_K(V^{-\sigma}, V^\sigma)) \to \mathrm{Hom}_{\mathcal{G}(K)}(\mathcal{G}(V^{-\sigma}), \mathcal{G}(V^\sigma))$$

is defined by

$$\widetilde{\xi_I \otimes \phi_I}(\xi_J \otimes v^{-\sigma}) := \xi_I \xi_J \otimes \phi_I(v^{-\sigma})$$

for all $\xi_I \otimes \phi_I \in \mathcal{G}(\mathrm{Hom}_K(V^{-\sigma}, V^\sigma))$, $\xi_J \otimes v^{-\sigma} \in \mathcal{G}(V^{-\sigma})$, $\sigma = \pm$.

Define a K-trilinear map $\{\cdots\} : V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$ by

$$\{u, v, w\} = (-1)^{|v||w|} Q(u, w) v$$

for $u, w \in V^\sigma$, $v \in V^{-\sigma}$. Then we have

(JSP0) $\{u, v, w\} = (-1)^{|u||v|+|u||w|+|v||w|}\{w, v, u\}$

for all $u, w \in V^\sigma$, $v \in V^{-\sigma}$.

Example 1.3.23 ([N3]) Given a K-supermodule $\mathcal{M}$ with a K-quadratic form q, we can define a Jordan superpair $V = (\mathcal{M}, \mathcal{M})$ by setting

$$Q(m, n)p = (-1)^{|n||p|} q(m, p)n + (-1)^{|n||p|} m q(p, n) - q(m, n)p$$

for all $m, n, p \in \mathcal{M}$. It is called the *quadratic form superpair (associated to q)* and its Grassmann envelope $\mathcal{G}(V)$ is a Jordan pair associated to the Grassmann envelope $\mathcal{G}(Q)$ of Q.

Tits-Kantor-Koecher superalgebras of Jordan superpairs. Let $V = (V^+, V^-)$ be a Jordan superpair over K with respect to the pair (Q^+, Q^-) of K-quadratic maps. We define $D = (D^+, D^-) : (V^+ \times V^-) \times (V^+ \times V^-) \to \mathrm{End}(V^+) \times \mathrm{End}(V^-)$ by

$$D^\sigma(u, v)w = (-1)^{|w||v|} Q^\sigma(u, w)v.$$

It is important to note that $D^\sigma(u, v)w = (-1)^{|u||v|+|u||w|+|v||w|} D^\sigma(w, v)u$ for all $u, w \in V^\sigma$, $v \in V^{-\sigma}$, $\sigma = \pm$. For $(u, v) \in V$ put

$$\delta(u, v) = (D(u, v), -(-1)^{|u||v|} D(v, u)) \in \mathrm{End}_K V^+ \times \mathrm{End}_K V^-$$

and denote by $\mathcal{D}$ the K-span of all maps $\delta(u,v)$. $\mathcal{D}$ is called the *inner derivation superalgebra of* V and the elements the *inner derivations.* We set

$$\mathcal{K} = \mathcal{K}(V) = V^+ \oplus \mathcal{D} \oplus V^-$$

and define the product on $\mathcal{K}$ by

$$\begin{aligned}
[u^+ \oplus a \oplus u^-, v^+ \oplus b \oplus v^-] =& (a^+v^+ - (-1)^{|u^+||b^+|}b^+u^+) \\
&\oplus ([a,b] + \delta(u^+, v^-) - (-1)^{|v^+||u^-|}\delta(v^+, u^-)) \quad (1.27) \\
&\oplus (a^-v^- - (-1)^{|u^-||b^-|}b^-u^-) \quad (1.28)
\end{aligned}$$

where $u^\sigma, v^\sigma \in V^\sigma$ and $a = (a^+, a^-)$, $b = (b^+, b^-) \in \mathcal{D}$. Then $\mathcal{G}(\mathcal{K}(V)) = \mathcal{K}(\mathcal{G}(V))$ (see [Kr]), hence $\mathcal{K}(V)$ is a Lie superalgebra called the *TKK-superalgebra of the Jordan superpair* V where $\mathcal{D}_\gamma = \sum_{\gamma=\alpha+\beta}\delta(V_\alpha^+, V_\beta^-)$ and $\mathcal{K}(V)_\gamma = V_\gamma^+ \oplus \mathcal{D}_\gamma \oplus V_\gamma^-$, $\gamma \in \mathbb{Z}_2$.

Example 1.3.24 ([N3]) Let $\mathcal{M}$ be a K-supermodule and q a K-quadratic form on $\mathcal{M}$. Let $V = (\mathcal{M}, \mathcal{M})$ be the associated quadratic form superpair. Then

$$\begin{aligned}
D(m,n)p &= (-1)^{|n||p|}Q(m,p)n \\
&= q(m,n)p + mq(n,p) - (-1)^{|n||p|}q(m,p)n \\
&= q(m,n)p + \mathbf{E}_{m,n}p
\end{aligned}$$

for all $m, n, p \in \mathcal{M}$ and so $\mathcal{D} = \mathrm{span}_K\{q(m,n)\cdot Id + \mathbf{E}_{m,n} \mid m, n \in \mathcal{M}\}$. Let $L = L_{-1} \oplus L_0 \oplus L_1$ where

$$\begin{aligned}
L_1 &= \{(0, m, 0, 0) \mid m \in \mathcal{M}\} \\
L_0 &= \{(a, 0, 0, x) \mid a \in \mathrm{span}_K\{q(m,n) \mid m, n \in \mathcal{M}\}, x \in \mathrm{eo}(q)\} \\
L_{-1} &= \{(0, 0, m, 0) \mid m \in \mathcal{M}\}
\end{aligned}$$

and multiplication given by

$$\begin{aligned}
[(a,m,n,x),(a',m',n',x')] =& (q(m,n') - q(n,m'), \\
& am' + xm' - (-1)^{|a'||m|}a'm - (-1)^{|x'||m|}x'm, \\
& -an' + xn' + (-1)^{|a'||n|}a'n - (-1)^{|x'||n|}x'n, \\
& \mathbf{E}_{m,n'} + \mathbf{E}_{n,m'} + [x, x']]
\end{aligned}$$

for all $a, a' \in K$, $m, m', n, n' \in \mathcal{M}$, $x, x' \in \mathrm{eosp}(q)$. We will show in Section 2.3 that $L = \mathrm{eosp}(q_{\{\infty\}} \oplus q)$ where $q_{\{\infty\}}$ is the quadratic form on a hyperbolic superplane and so L is a Lie superalgebra. Moreover, one can see that the TKK-superalgebra of V is isomorphic to

$$L_1 \oplus [L_1, L_{-1}]/C \oplus L_{-1}$$

where $[L_1, L_{-1}] \subset L_0$ is spanned by commutators

$$[(0, m, 0, 0), (0, 0, n, 0)] = (q(m, n), 0, 0, \mathbf{E}_{m,n})$$

and

$$C = \{A \in [L_1, L_{-1}] \mid [A, L_\sigma] = 0 \text{ for } \sigma = \pm\}.$$

If $\{a \in K \mid am = 0 \text{ for all } m \in \mathcal{M}\} = 0$ and there exist $m_i, n_i \in \mathcal{M}$ such that $\sum_i q(m_i, n_i) = 1$ then $\mathcal{K}(V) = L$ (see Proposition 2.3.10).

1.4 Graded Lie Superalgebras

In this section we review some basic definitions and facts of root-graded Lie superalgebras and 3-graded Lie superalgebras, as well we define the toral 3-graded Lie superalgebras. The reader is referred to [GN] for a more detailed discussion on these topics.

Let k be a commutative and associative ring with unity and containing $\frac{1}{2}$. We denote by K a superextension of k.

Root-graded Lie superalgebras. ([GN]) Let R be a root system (as defined in Section 1.2). We say that a Lie superalgebra over K is *R-graded* if there exist supermodules L^α, $\alpha \in R \cup \{0\}$ of L, such that

(SG1) $L = \oplus_{\alpha \in R \cup \{0\}} L^\alpha$;

(SG2) for all $\alpha, \beta \in R \cup \{0\}$,

$$[L^\alpha, L^\beta] \subset \begin{cases} L^{\alpha+\beta} & \text{if } \alpha+\beta \in R \cup \{0\} \\ \{0\} & \text{if } \alpha+\beta \notin R \cup \{0\} \end{cases} ;$$

(SG3) as a Lie superalgebra, L is generated by $\cup_{\alpha \in R} L^\alpha$;

(SG4) for every $\alpha \in R$ there exists $0 \neq X_\alpha \in (L^\alpha)_{\bar{0}}$ such that $H_\alpha := [X_{-\alpha}, X_\alpha]$ operates on L^β $(\beta \in R \cup \{0\})$ by

$$[H_\alpha, z_\beta] = \langle \beta, \alpha \rangle z_\beta \quad (z_\beta \in L^\beta)$$

where

$$\langle \beta, \alpha \rangle = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}.$$

Remark 1.4.1 (SRG3) is equivalent to

$$L^0 = \sum_{\delta \in R} [L^\delta, L^{-\delta}].$$

Remark 1.4.2 By definition we have that if L is an R-graded Lie superalgebra, then $L^\alpha = (L^\alpha \cap L_{\bar{0}}) \oplus (L^\alpha \cap L_{\bar{1}})$ for all $\alpha \in R \cup \{0\}$. □

Remark 1.4.3 This is a generalization of R-graded Lie algebras (as defined in Chapter 1.2) in the following sense: Every R-graded Lie algebra is an R-graded Lie superalgebra (by setting $L_{\bar{0}} = L$ and $L_{\bar{1}} = 0$). However, if L is an R-graded Lie superalgebra, $L_{\bar{0}}$ need not be an R-graded Lie algebra ($L_{\bar{0}}$ need not be generated by $(L_\alpha)_{\bar{0}}$; see example below). □

The techniques developed in [N1] to describe root-graded Lie algebras can be adapted to the supercase, see [N2, Section 5.4] and [GN] for part of this adaptation. In particular, given an index set I, one obtains:

(B_I) Assume that $\frac{1}{6} \in K$ and $|I| \geq 3$. A Lie algebra L over K is B_I-graded if and only if there exists a superextension A of K, an A-supermodule $\mathcal{M}$ with an A-quadratic form q and an element $m \in \mathcal{M}$ satisfying $q(m, m) = 1$ such that L is a central extension of the Lie superalgebra $\mathrm{eosp}(q_I \oplus q)$.

(D_I) Assume that $\frac{1}{6} \in K$ and $|I| \geq 4$. A Lie superalgebra L over K is D_I-graded if and only if there exists a superextension A of K such that L is a central extension of the Lie superalgebra $\mathrm{eosp}(q_I)$.

Example 1.4.4 Let $L = \mathrm{eosp}(q_I \oplus q)$ be B_I- or D_I-graded for an index set I and an A-quadratic form q on an A-supermodule $\mathcal{M}$. The root systems B_I and D_I are given by

$$\begin{aligned} D_I &= \{\pm\epsilon_i \pm \epsilon_j \mid i, j \in I,\ i \neq j\}; \\ B_I &= \{\pm\epsilon_i \mid i \in I\} \cup D_I \end{aligned}$$

where the ϵ_i form the orthonormal basis of $\bigoplus_{i\in I} \mathbb{R}\epsilon_i$. If L is D_I-graded then $\mathcal{M} = (0)$. Let $R = B_I$ or D_I. Then

$$\begin{aligned} L_{\sigma(\epsilon_i+\epsilon_j)} &= A\mathbf{E}_{h_{\sigma i},h_{\sigma j}}, \quad i, j \in I,\ i \neq j,\ \sigma = \pm \\ L_{\epsilon_i-\epsilon_j} &= A\mathbf{E}_{h_i,h_{-j}}, \quad i, j \in I,\ i \neq j \\ L_{\sigma\epsilon i} &= A\mathbf{E}_{h_{\sigma i},\mathcal{M}}, \quad i \in I,\ \sigma = \pm \\ L_0 &= \mathrm{eosp}(q) \bigoplus \left(\bigoplus_{i\in I} A\mathbf{E}_{h_i,h_{-i}} \right) \end{aligned}$$

gives $\mathrm{eosp}(q_I \oplus q)$ the R-grading.

Example 1.4.5 Let K be a commutative and unital ring containing $\frac{1}{6}$. We consider K as a superring with $K_{\bar{1}} = (0)$. Let I and J be arbitrary index sets ($|I| \geq 3$ and $J \neq \emptyset$) and let $\mathcal{M} = \oplus_{j\in J} K m_j$ be a free K-supermodule with a basis consisting of odd elements and with a nonzero K-quadratic form $q_{\mathcal{M}}$. Consider $L = \mathrm{eosp}(q_I \oplus q_{\mathcal{M}} \oplus q_0)$ where q_I is defined as in Example 1.3.3 and q_0 is the quadratic form on the free module Kx_0 given by $q_0(x_0, x_0) = 1$. Then by *(B_I)*, L is a B_I-graded Lie superalgebra over K. In particular, $\mathrm{eosp}(q_{\mathcal{M}})(\hookrightarrow \mathrm{eosp}(q_I \oplus q_{\mathcal{M}}))$ is contained in $(L^0)_{\bar{0}}$ but it is not generated by $(L^\alpha)_{\bar{0}}$, $\alpha \in B_I$. Hence $L_{\bar{0}}$ is not a B_I-graded Lie algebra.

3-graded and toral 3-graded Lie superalgebras. A Lie superalgebra L over K is called *3-graded* if it has a decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$ such that

$$[L_i, L_j] \subset L_{i+j} \text{ where } L_k = 0 \text{ if } k \notin \{\pm 1, 0\}.$$

Given a 3-graded Lie superalgebra, we define

$$C(L) = \{x \in L_0 \mid [x, L_\pm] = 0\} \tag{1.29}$$

Remark 1.4.6 Note that if $C(L) = (0)$ then if $x, y \in L_0$ satisfy $\mathbf{ad}\, x|_{L_\pm} = \mathbf{ad}\, y|_{L_\pm}$ then $x = y$.

We say that a 3-graded Lie superalgebra L is *toral 3-graded* if there exists an element $\zeta \in L_0$ such that

$$[\zeta, x_\sigma] = \delta x_\sigma \text{ for all } x_\sigma \in L_\sigma,\ \sigma = 0, \pm 1.$$

One can easily show that the Grassmann envelope $L' = \mathcal{G}(L)$ is a *toral 3-graded Lie algebra over $\mathcal{G}(K)$*, i.e.

$$L' = L'_{-1} \oplus L'_0 \oplus L'_1 \text{ (namely } L'_\alpha = G(L_\alpha)\text{)};$$

$$[L'_\delta, L'_\gamma] \subset L'_{\delta+\gamma},\ \delta, \gamma \in \{0, \pm 1\}. \text{ where } L_{\delta+\gamma} = \{0\} \text{ if } \delta + \gamma \notin \{0, \pm 1\}; \text{ and}$$

$$1_{\mathcal{G}} \otimes \zeta \in L'_0 \text{ satisfies } [1_{\mathcal{G}} \otimes \zeta, \xi_I \otimes x_\delta] = \delta(\xi_I \otimes x_\delta),\ \delta = 0, \pm 1.$$

Example 1.4.7 Every A_1-graded Lie superalgebra over k is a toral 3-graded Lie superalgebra. □

Example 1.4.8 Let $\mathcal{M}$ be a K-supermodule with a K-quadratic form q. Let $\tilde{\mathcal{M}}$ be the direct sum of the K-supermodule $\mathcal{M}$ and a hyperbolic K-superplane $H(\{\infty\})$ with the K-quadratic form $q_\infty = q \oplus q_{\{\infty\}}$. Then $\mathrm{eosp}(q_\infty)$ is 3-graded. In particular, $\zeta = (1, 0, 0, 0)$ and so L is toral. □

Remark 1.4.9 From (B_I) and (D_I) we see that a Lie superalgebra over a superring K containing $\frac{1}{6}$ is B_I-graded ($|I| \geq 3$) or D_I-graded ($|I| \geq 4$) only if it is a central extension of $\mathrm{eosp}(q_\infty)$ for some superextension A of K, an A-supermodule $\mathcal{M}$ and an A-quadratic form q on $\mathcal{M}$. □

1.5 Summable Supermodule Homomorphisms

We assume that A is a superextension of a superring K. As before, when we talk about a submodule $\mathcal{N}$ of a supermodule $\mathcal{M}$, we mean that $\mathcal{N}$ is also a supermodule.

Definition 1.5.1 Let $\mathcal{M}$ and $\mathcal{N}$ be A-supermodules and let $\underline{x} = (x_i)_{i\in I}$ be a family in $\mathrm{Hom}_A(\mathcal{M},\mathcal{N})$. We say that $\underline{x}$ is *summable* if for every $m \in \mathcal{M}$, $(x_i(m))_{i\in I}$ has only finitely many nonzero terms, and in this case, we define

$$\sum_{i\in I} x_i \in \mathrm{Hom}_A(\mathcal{M},\mathcal{N})$$

by setting

$$\left(\sum_{i\in I} x_i\right)(m) = \sum_{i\in I_m} x_i(m)$$

for $I_m = \{i \in I \mid x_i(m) \neq 0\}$ (I_m is finite). We will say that $\sum_{i\in I} x_i$ is summable to indicate that $(x_i)_{i\in I}$ is summable. Also, given a summable $x = \sum_{i\in I} x_i$ we will throughout denote by I_m the finite subset of I such that for $i \in I$, $x_i(m) \neq 0$.

Example 1.5.2 Let I be an index set and $(0_i)_{i\in I}$ be a family in $\mathrm{Hom}_A(\mathcal{M},\mathcal{N})$: $0_i(m) = 0$. Then clearly $(0_i)_{i\in I}$ is summable and $\sum_{i\in I} 0_i = 0$.

Example 1.5.3 Let I be a finite index set and $x_i \in \mathrm{Hom}_A(\mathcal{M},\mathcal{N})$, $i \in I$. Then $(x_i)_{i\in I}$ is summable.

Definition 1.5.4 Given a submodule $X \subset \mathrm{Hom}_A(\mathcal{M},\mathcal{N})$ and a homomorphism $x \in \mathrm{Hom}_A(\mathcal{M},\mathcal{N})$, we say that x is *summable with respect to* X (or X-*summable*) if $x = \sum_{i\in I} x_i$ where $x_i \in X$, $i \in I$ and $\sum_{i\in I} x_i$ is summable. We set, for $\alpha \in \mathbb{Z}_2$,

$$\overline{X}^A_\alpha = \{x \in \mathrm{Hom}_A(\mathcal{M},\mathcal{N})_\alpha \mid x \text{ is } X\text{-summable}\}.$$

Then

$$\overline{X}^A = \overline{X}_{\bar 0} \oplus \overline{X}_{\bar 1} = \{x \in \mathrm{Hom}_A(\mathcal{M},\mathcal{N}) \mid x \text{ is } X\text{-summable}\}.$$

We will abbreviate $\overline{X}^A$ by $\overline{X}$ when there is no possibility for confusion.

Remark 1.5.5 The summability is equivalent to convergence in finite topology in the following sense ([J2]): The basis of open sets in finite topology is the set consisting of $O_{f;\{m_1,\dots,m_n\}} = \{g \in \operatorname{End}_A \mathcal{M} \mid g(m_i) = f(m_i),\ 1 \le i \le n\}$ for some map $f : X \to X$ and $m_1, \dots, m_n \in \mathcal{M}$. To a family $\underline{x} = (x_i)_{i \in I} \subset \operatorname{End}_A \mathcal{M}$ we associate a net

$$\tilde{I} = \{\text{finite subsets of } I\},$$

ordered by inclusion

$$\tilde{x} = \left(\sum_{j \in J} x_j\right)_{J \subset \tilde{I}}, \quad \tilde{x}_J = \sum_{j \in J} x_j.$$

The net $\tilde{x}$ converges to the map $y : \mathcal{M} \to \mathcal{M}$ if and only if for every open neighborhood $O_{y;\{m_1,\dots,m_n\}}$ there exists $J_0 \in \tilde{I}$ such that for all finite $J \supset J_0$ we have $x_J(m_i) = y(m_i)$, $1 \le i \le n$. Note that $O_{y;\{m_1,\dots,m_n\}} = \cap_{i=1}^{n} O_{y;m_i}$. So, convergence is equivalent to: for every $m \in \mathcal{M}$ there exists $J_0 \in \tilde{I}$, depending on m, such that for all finite $J \supset J_0$ we have $y(m) = \tilde{x}_J(m)$ $(= \sum_{i \in J} x_i(m))$. Since $y(m) = \tilde{x}_{J_0}(m)$ this means that $x_j(m) = 0$ for all $j \notin J_0$. Hence convergence in the finite topology is equivalent to summability. It is known ([J3]) that $\operatorname{End}_A \mathcal{M}$ is a topological ring, i.e., multiplication is continuous.

Remark 1.5.6 $X \subset \overline{X}$ (but not equal in general). However we have the following:

Proposition 1.5.7 *Let $\mathcal{M}$ be a finitely generated A-supermodule and let X be a submodule of* $\operatorname{Hom}_A(\mathcal{M}, \mathcal{N})$ *for some A-supermodule $\mathcal{N}$. Then $\overline{X} = X$.*

Proof: Let $x = \sum_{i \in I} x_i \in \overline{X}$. Let $\{m_j \mid 1 \le j \le n\}$ be a generating set for $\mathcal{M}$. Set

$$I_j = \{i \in I \mid x_i(m_j) \neq 0\}.$$

Since x is summable, each I_j, $1 \le j \le n$ is finite. Set $I_0 = \dot{\cup}_{j=1}^{n} I_j$. Then if $i \in I \setminus I_0$, $x_i(m_k) = 0$ for all $1 \le k \le n$ and so $x_i = 0$ for all $i \in I \setminus I_0$. Hence for all $m \in \mathcal{M}$,

$$\left(\sum_{i \in I} x_i\right)(m) = \sum_{i \in I_0} x_i(m) = \left(\sum_{i \in I_0} x_i\right)(m)$$

and so $x = \sum_{i\in I_0} x_i \in X$ since $x_i \in X$ and I_0 is finite. Therefore $\overline{X} = X$. ■

Proposition 1.5.8 *Let $\mathcal{M}$ be an A-supermodule. Suppose that the set $\{m \in \mathcal{M} \mid am = 0$ for $a \in A$ only if $a = 0\}$ is not empty. Then $\overline{A\cdot Id} = A\cdot Id$.*

Proof: Let $x = \sum_{i\in I} a_i Id \in \overline{A\cdot Id}$. Let $m \in \mathcal{M}$ such that $am = 0$ only if $a = 0$. Since x is summable, there exists a finite subset I_m of I such that $a_i m = 0$ for all $i \in I \setminus I_m$ and so $a_i = 0$ for all $i \in I \setminus I_m$. Hence $x = \sum_{i\in I_m} a_i\cdot Id = (\sum_{i\in I_m} a_i)\cdot Id \in A\cdot Id$. ■

Proposition 1.5.9 *Let $\mathcal{M}$ and $\mathcal{N}$ be A-supermodules and let $X \subset \mathrm{Hom}_A(\mathcal{M},\mathcal{N})$. Then*

$$\overline{X}^K = \overline{X}^A$$

and, in particular, $\overline{(\mathrm{Hom}_A(\mathcal{M},\mathcal{N}))}^K = \overline{(\mathrm{Hom}_A(\mathcal{M},\mathcal{N}))}^A$. *Hence,*

$$\overline{(\mathrm{Hom}_A(\mathcal{M},\mathcal{N}))}^K = \mathrm{Hom}_A(\mathcal{M},\mathcal{N}).$$

Proof: Clearly $\overline{X}^A \subset \overline{X}^K$. Let $x = \sum_{i\in I} x_i \in \overline{X}^K$ be homogeneous. Then $\sum_{i\in I} x_i$ is summable where $x_i \in X \subset \mathrm{Hom}_K(\mathcal{M},\mathcal{N})$. Then for all $a \in A$,

$$\begin{aligned} x(am) &= \left(\sum_{i\in I}\right) x_i(am) = \sum_{i\in I_m} x_i(am) = \sum_{i\in I_m} (-1)^{|x_i||a|} a x_i(m) \\ &= (-1)^{|x||a|} a \sum_{i\in I_m} x_i(m) = (-1)^{|x||a|} \left(\sum_{i\in I} x_i\right)(m) \end{aligned}$$

and so $x \in \mathrm{Hom}_A(\mathcal{M},\mathcal{N})$. Hence $\overline{X}^K = \overline{X}^A$. ■

Proposition 1.5.10 *Let X be a submodule of $\mathrm{Hom}_A(\mathcal{M},\mathcal{N})$. Then $\overline{X}$ is a submodule of $\mathrm{Hom}_A(\mathcal{M},\mathcal{N})$.*

Proof: $0 \in X$ so clearly $0 \in \overline{X}$. Let $x = \sum_{i\in I} x_i$, $y = \sum_{j\in J} y_j \in \overline{X}$. Let $K = I\dot\cup J$ and

$$u_k = \begin{cases} x_k & : \ k \in I \\ y_k & : \ k \in J \end{cases}.$$

Then for each $m \in \mathcal{M}$ there exist finite $I_m \subset I$, $J_m \subset J$ such that $x_i(m) = 0$ and $y_j(m) = 0$ for all $i \in I \setminus I_m, j \in J \setminus J_m$ and so $u_k(m) = 0$ for all $k \notin I_m \cup J_m$. Hence $\sum_{k \in K} u_k$ is summable and since $x_i, y_j \in X$, $\sum_{k\in K} u_k \in \overline{X}$. Also,

$$(x+y)(m) = x(m) + y(m) = \sum_{i \in I_m} x_i(m) + \sum_{j \in J_m} y_j(m) = \sum_{k \in K} u_k(m)$$

and so $x + y \in \overline{X}$. Let x be X-summable, say $x = \sum_{i\in I} x_i$, and $a \in A$. Then clearly $\sum_{i \in I} ax_i \in \overline{X}$. Let $m \in \mathcal{M}$ and let $I_m = \{i \in I \mid x_i(m) \neq 0\}$ and $I_{a,m} = \{i \in I \mid ax_i(m) \neq 0\}$. Then $I_{a,m} \subset I_m$ and so

$$a(\sum_{i\in I} x_i)(m) = a \sum_{i \in I_m} x_i(m) = \sum_{i \in I_m} ax_i(m) = \sum_{i \in I_{a,m}} ax_i(m) = (\sum_{i \in I} ax_i)(m).$$

Hence $ax \in \overline{X}$. Similarly, $xa \in \overline{X}$ and so $\overline{X}$ is a submodule of $\mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$ with the induced $\mathbb{Z}_2$-grading. ∎

Proposition 1.5.11 *Let $Y \subset X$ be submodules of* $\mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$. *Then $\overline{Y} \subset \overline{X}$.*

Proof: Let $y = \sum_{j\in J} y_j \in \overline{Y}$. Then $\sum_{j \in J} y_j$ is summable and $y_j \in Y \subset X$ for all $j \in J$. Hence $y = \sum_{j\in J} y_j \in \overline{X}$. ∎

Proposition 1.5.12 *Let X, Y and Z be submodules of* $\mathrm{End}_A \mathcal{M}$ *satisfying $[X, Y] \subset Z$. If $x = \sum_{i \in I} x_i \in \overline{X}$, $y = \sum_{j\in J} y_j \in \overline{Y}$ then $([x_i, y_j])_{(i,j)\in I\times J}$ is Z-summable and*

$$\sum_{(i,j)\in I \times J} [x_i, y_j] = \left[\sum_{i\in I} x_i, \sum_{j \in J} y_j\right].$$

Hence $[\overline{X}, \overline{Y}] \subset \overline{Z}$. In particular, if X is a subalgebra (ideal) of $(\mathrm{End}_A \mathcal{M})^{(-)}$, *then $\overline{X}$ is also a subalgebra (respectively, ideal) of* $(\mathrm{End}_A \mathcal{M})^{(-)}$.

Proof: Let $x = \sum_{i\in I} x_i \in \overline{X}$, $y = \sum_{j \in J} y_j \in \overline{Y}$. Note that if x and y are homogeneous then $|x| = |x_i|$ and $|y| = |y_j|$ for all $i \in I, j \in J$. First we claim that

$$\sum_{(i,j)\in I\times J} [x_i, y_j] \in \overline{Z}.$$

Given $m \in \mathcal{M}$, $x_i(m) \neq 0$ and $y_j(m) \neq 0$ for only finitely many $i \in I$, $j \in J$ which in turn implies that, for all $m \in \mathcal{M}$, $x_i(y_j(m)) - (-1)^{|x||y|} y_j(x_i(m)) \neq 0$ for only

finitely many $(i,j) \in I \times J$, hence $\sum_{(i,j)\in I\times J}[x_i, y_j]$ is summable. Moreover, since $x_i \in X, y_j \in Y$, $[x_i, y_j] \in Z$ and so $\sum_{(i,j)\in I\times J}[x_i, y_j] \in \overline{Z}$.
Now, for all $m \in \mathcal{M}$, let

$$I_m = \{i \in I \mid x_i(m) \neq 0\}, \quad J_m = \{j \in J \mid y_j(m) \neq 0\}$$

and

$$J'_m = (\cup_{j\in J_m} I_{y_j(m)}) \cup I_m, \quad I'_m = (\cup_{i\in I_m} J_{x_i(m)}) \cup J_m.$$

Clearly he set $I_m \times J_m$ is finite. Then

$$\begin{aligned}
[y, z](m) &= y(z(m)) - (-1)^{|y||z|}z(y(m)) \\
&= x\left(\sum_{j\in J_m} y_j(m)\right) - (-1)^{|y||z|}y\left(\sum_{i\in I_m} x_i(m)\right) \\
&= \sum_{j\in J_m}\sum_{i\in I_{y_j(m)}} x_i(y_j(m)) - (-1)^{|y||z|}\sum_{i\in I_m}\sum_{j\in J_{x_i(m)}} y_j(x_i(m)) \\
&= \sum_{(i,j)\in I'_m\times J'_m} (x_i(y_j(m)) - (-1)^{|y||z|}y_j(x_i(m))) \\
&= \left(\sum_{(i,j)\in I\times J} [x_i, y_j]\right)(m)
\end{aligned}$$

and so $[y, z] = \sum_{(i,j)\in I\times J}[x_i, y_j] \in \overline{Z}$. The rest of the assertion now follows. ∎

The following will also be useful later:

Proposition 1.5.13 *Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ where $\mathcal{M}_1$ and $\mathcal{M}_2$ are K-supermodules and let $X \subset \mathrm{End}_A \mathcal{M}$. If $X = X_{11} \oplus X_{12} \oplus X_{22}$ where $X_{ii} = X \cap \mathrm{End}_K \mathcal{M}_i$ for $i = 1, 2$ and $X_{12} = X \cap (\mathrm{Hom}_K(\mathcal{M}_1, \mathcal{M}_2) \oplus \mathrm{Hom}_K(\mathcal{M}_2, \mathcal{M}_1))$ then*

$$\overline{X} = \overline{X_{11}} \oplus \overline{X_{12}} \oplus \overline{X_{22}}.$$

Proof: Let $x \in \overline{X}$. Then $x = x_{11} + x_{12} + x_{22}$ where $x_{ii} = \sum_j y_{ji}$ for some $y_{ji} \in X_{ii}$, $i = 1, 2$ and $x_{12} = \sum_k y'_j$ for some $y'_j \in X_{12}$. Since x is summable, so are x_{11}, x_{12}

and x_{22} and so $x \in \overline{X_{11}} + \overline{X_{12}} + \overline{X_{22}}$. Suppose $x = 0$. Then for all $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$ we have

$$0 = x(m_1) = \underbrace{x_{11}(m_1)}_{\in \mathcal{M}_1} + \underbrace{x_{12}(m_1)}_{\in \mathcal{M}_2} + \underbrace{x_{22}(m_1)}_{=0}$$

and

$$0 = x(m_2) = \underbrace{x_{11}(m_2)}_{=0} + \underbrace{x_{12}(m_2)}_{\in \mathcal{M}_1} + \underbrace{x_{22}(m_2)}_{\in \mathcal{M}_2}.$$

Hence $x_{ij} = 0$ and so we have

$$\overline{X} = \overline{X_{11}} \oplus \overline{X_{12}} \oplus \overline{X_{22}}.$$

∎

Example 1.5.14 Let $\mathcal{M}$ be a K-supermodule with a K-quadratic form q. Define $J = K \oplus \mathcal{M}$ to be the K-superalgebra with the natural $\mathbb{Z}_2$-grading and multiplication given by

$$(a \oplus m)(b \oplus n) = (ab + q(m, n)) \oplus (an + mb). \tag{1.30}$$

Recall from Example 1.3.22 that the Grassmann envelope of J is the Jordan superalgebra $\mathcal{G}(J)$ associated with the $\mathcal{G}(K)$-quadratic form. Let $\mathcal{E}$ be a subalgebra of osp(q) which contains eosp(q). Consider the K-supermodule $X = \mathcal{E} \oplus \mathcal{M}$. Define the action $\circ$ of $J = K \oplus \mathcal{M}$ on X by setting

$$\begin{aligned}(x + p) \circ (a + m) &= (xa + \mathbf{E}_{p,m}) + (x(m) + pa) \qquad (1.31)\\ &= (-1)^{\alpha\beta}(a + m) \circ (x + p)\end{aligned}$$

for all $a + m \in J_\alpha$ and $x + p \in X_\beta$ homogeneous. Then X is a Jordan superalgebra supermodule for J. Indeed, to check this we need to verify that $\mathcal{G}(X)$ is a Jordan module for $\mathcal{G}(J)$. Recall from Chapter 1, Section 1 that a k-module V is a Jordan module for J if it satisfies the following three relations:

$$av = va; \tag{1.32}$$

$$(va^2)a = (va)a^2; \text{ and} \tag{1.33}$$

$$2((va)b)a + v(a^2b) = 2(va)(ab) + (vb)a^2 \tag{1.34}$$

for all $a \in J$ and $v \in V$. We have, given $M, N, P \subset \mathbb{Z}$ and $a+m, b+n \in J, x+p \in X$ homogeneous of degrees M, N and P respectively,

$$\begin{aligned}
(\xi_M \otimes (a+m)) \circ (\xi_P \otimes (x+p)) &= (-1)^{|M||P|} \xi_M \xi_P \otimes (xa + \mathbf{E}_{p,m} + x(m) + pa) \\
&= \xi_P \xi_M \otimes (xa + \mathbf{E}_{p,m} + x(m) + pa) \\
&= (\xi_P \otimes (x+p)) \circ (\xi_M \otimes (a+m))
\end{aligned}$$

and so in that case, $\mathcal{G}(X)$ and $\mathcal{G}(J)$ satisfies (1.32). Note that if M is odd then $\xi_M^2 = 0$ and so (1.33) and (1.34) are satisfied. Now, suppose M is even. Then

$$\begin{aligned}
&((\xi_P \otimes (x+p)) \circ (\xi_M \otimes (a+m))^2) \circ (\xi_M \otimes (a+m)) \\
&\qquad ((\xi_P \otimes (x+p)) \circ (\xi_M \otimes (a+m))) \circ (\xi_M \otimes (a+m))^2 \\
&= (\xi_P \xi_M \otimes (x(a^2 + q(m,m)) + \mathbf{E}_{p,2am} + x(2am) \\
&\qquad + p(a^2 + q(m,m)))) \circ (\xi_M \otimes (a+m)) \\
&\quad - (\xi_P \xi_M \otimes (xa + \mathbf{E}_{p,m} + x(m) + pa)) \\
&\qquad \circ (\xi_M \otimes (a^2 + q(m,m) + 2am)) \\
&= \xi_P \xi_M^2 \otimes ((x(a^2 + q(m,m)) + \mathbf{E}_{p,2am})a + \mathbf{E}_{x(2am)+p(a^2+q(m,m)),m} \\
&\quad + (x(a^2 + q(m,m)))(m) + pq(2am, m) - q(p,m)2am \\
&\qquad + p(a^2 + q(m,m))a) \\
&\quad - \xi_P \xi_M^2 \otimes ((xa + \mathbf{E}_{p,m})(a^2 + q(m,m)) + \mathbf{E}_{x(m)+pa,2am} \\
&\quad + xa(2am) + pq(m, 2am) - q(p, 2am)m \\
&\qquad + (x(m) + pa)(a^2 + q(m,m))) \\
&= 0
\end{aligned}$$

hence (1.33) is satisfied for M even as well. In addition,

$$
\begin{aligned}
2(((&\xi_P \otimes (x+p)) \circ (\xi_M \otimes (a+m)) \circ (\xi_N \otimes (b+n))) \circ (\xi_M \otimes (a+m)) \\
&+ (\xi_P \otimes (x+p)) \circ ((\xi_M \otimes (a+m))^2(\xi_N \otimes (b+n))) \\
&- 2((\xi_P \otimes (x+p)) \circ (\xi_M \otimes (a+m))) \circ ((\xi_M \otimes (a+m)) \\
&\quad \circ ((\xi_M \otimes (a+m))(\xi_N \otimes (b+n))) \\
&- ((\xi_P \otimes (x+p)) \circ (\xi_N \otimes (b+n))) \circ (\xi_M \otimes (a+m))^2 \\
= 2(((&\xi_P\xi_M \otimes (ax + \mathbf{E}_{p,m} + ap + x(m)) \circ (\xi_N \otimes (b+n))) \\
&\quad \circ (\xi_M \otimes (a+m)) \\
&+ (\xi_P \otimes (x+p)) \circ ((\xi_M^2 \otimes (a^2 + q(m,m) + 2am))(\xi_N \otimes (b+n))) \\
&- 2((\xi_P\xi_M \otimes (ax + \mathbf{E}_{p,m} + ap + x(m)) \\
&\quad \circ (\xi_M\xi_N \otimes (ab + q(m,n) + an + bm) \\
&- ((\xi_P\xi_N \otimes (xb + \mathbf{E}_{p,n} + pb + x(n))) \circ (\xi_M^2 \otimes (a^2 + q(m,m) + 2am)) \\
= 2(&\xi_P\xi_M\xi_N \otimes ((ax + \mathbf{E}_{p,m})b + \mathbf{E}_{ap+x(m),n} \\
&\quad + ax(n) + pq(m,n) - q(p,n)m + (ap + x(m))b) \circ (\xi_M \otimes (a+m)) \\
&+ (\xi_P \otimes (x+p)) \circ (\xi_M^2\xi_N \otimes ((a^2 + q(m,m))b + q(2am,n) \\
&\quad + (a^2 + q(m,m))n + 2amb)) \\
&* \begin{cases}
-2(\xi_P\xi_M^2\xi_N \otimes ((ax + \mathbf{E}_{p,m})(ab + q(m,n)) + \mathbf{E}ap + x(m), an + bm \\
\quad +ax(an + bm) + pq(m, an + bm) - q(p,m)(an + bm) \\
\quad +(ap + x(m))(ab + q(m,n)))) \\
-(\xi_P\xi_N\xi_M^2 \otimes ((xb + \mathbf{E}_{p,n})(a^2 + q(m,m)) + \mathbf{E}_{pb+x(n),2am} \\
\quad +xb(2am) + pq(n, 2am) - q(p, 2am)n \\
+(pb + x(n))(a^2 + q(m,m))
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&= 2(\xi_P\xi_M^2\xi_N \otimes (((ax + \mathbf{E}_{p,m})b + \mathbf{E}_{ap+x(m),n})a \\
&\qquad + \mathbf{E}_{x(n)+pq(m,n)-q(p,n)m+(ap+x(m))b,m} \\
&\qquad + ax(bm) + (ap + x(m))q(n,m) - q(ap + x(m), n)m \\
&\qquad + (ax(n) + pq(m,n) - q(p,n)m + (ap + x(m))b)a \\
&\quad + \xi_P\xi_M^2\xi_N \otimes (x((a^2 + q(m,m))b + q(2am,n)) \\
&\qquad + \mathbf{E}_{p,(a^2+q(m,m))n+2amb} \\
&\qquad + x((a^2 + q(m,m))n + 2amb) + p((a^2 + q(m,m))b + q(2am,n)) \\
&\quad + (*) \\
&= 0.
\end{aligned}
$$

Hence $\mathcal{G}(X)$ and $\mathcal{G}(J)$ satisfy (1.34) and so $\mathcal{G}(X)$ is a Jordan module for $\mathcal{G}(J)$. Therefore X is a Jordan supermodule for J.

Chapter 2

Orthosymplectic Lie Superalgebras

Throughout this chapter we will assume that K is an associative supercommutative unital superring containing $\frac{1}{2}$ and A will denote a superextension of K, i.e., an associative supercommutative unital superalgebra over K. We will denote the invertible elements in A by $A^\times$. We will assume throughout that when $|m|$ occurs in an expression for some $m \in \mathcal{M}$ for some supermodule $\mathcal{M}$, then it is assumed that m is homogeneous, and that the expression extends to the other elements of $\mathcal{M}$ by linearity.

2.1 The Elementary Orthosymplectic Lie Superalgebra $\mathrm{eosp}(q)$

Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Recall that the orthosymplectic Lie superalgebra (of q) $\mathrm{osp}(q)$ is defined by

$$\mathrm{osp}(q) = \{x \in \mathrm{End}_A \mathcal{M} \mid q(x(m), n) + (-1)^{|m||n|} q(x(n), m) = 0 \text{ for all } m, n \in \mathcal{M}\}. \tag{2.1}$$

In addition, we defined, for homogeneous $m, n, p \in \mathcal{M}$,

$$\mathbf{E}_{m,n}(p) = mq(n, p) - (-1)^{|n||p|} q(m, p)n.$$

and we extended $\mathbf{E}$ additively to $\mathbf{E} : \mathcal{M} \times \mathcal{M} \to \mathrm{End}_A \mathcal{M}$, i.e.,

$$\mathbf{E}_{m+n,p+r} = \mathbf{E}_{m,p} + \mathbf{E}_{m,r} + \mathbf{E}_{n,p} + \mathbf{E}_{n,r}$$

for all $m, n, p, r \in \mathcal{M}$. The elementary orthosympletic Lie superalgebra (of q) is defined as

$$\mathrm{eosp}(q) = \mathrm{span}_{\mathbf{Z}}\{\mathbf{E}_{m,n} \mid m, n \in \mathcal{M}\}.$$

In this section we will investigate some of the properties and the structure of $\mathrm{eosp}(q)$.

Lemma 2.1.1 *Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Then for all $m, n, p \in \mathcal{M}$,*

(i) $\mathbf{E}_{m,n} \in \mathrm{osp}(q)$, $\quad m, n \in \mathcal{M}$;

(ii) the map $\mathbf{E} : \mathcal{M} \times \mathcal{M} \to \mathrm{End}_A \mathcal{M} : (m, n) \mapsto \mathbf{E}_{m,n}$ is A-bilinear of degree $\bar{0}$,

$$\mathbf{E}_{m,n} = -(-1)^{|m||n|}\mathbf{E}_{n,m} \text{ for all } m, n \in \mathcal{M}, \textit{ and}$$

(iii) we have

$$q(\mathbf{E}_{m,n}(p), r) = (-1)^{(|m|+|n|)(|p|+|r|)} q(\mathbf{E}_{p,r}(m), n). \tag{2.2}$$

Proof: (i) First, for $a \in A$, $m, n, p \in \mathcal{M}$ we have

$$\begin{aligned} \mathbf{E}_{m,n}(pa) &= mq(n, pa) - (-1)^{|n|(|a|+|p|)} q(m, pa)n \\ &= (mq(n, p) - (-1)^{|n||p|} q(m, p)n)a \\ &= (\mathbf{E}_{m,n}(p))a \end{aligned}$$

and so, since obviously $\mathbf{E}_{m,n}(p + p') = \mathbf{E}_{m,n}(p) + \mathbf{E}_{m,n}(p')$ for all $m, n, p, p' \in \mathcal{M}$, $\mathbf{E}_{m,n} \in \mathrm{End}_A(\mathcal{M})$. Since for all $m, n, p, r \in \mathcal{M}$ we have

$$\begin{aligned} q(\mathbf{E}_{m,n}(p), r) &+ (-1)^{|p||r|} q(\mathbf{E}_{m,n}(r), p) = \\ &= q(mq(n, p), r) - (-1)^{|n||p|} q(q(m, p)n, r) \\ &\quad + (-1)^{|p||r|}(q(mq(n, r), p) - (-1)^{|n||r|} q(q(m, r)n, p)) \\ &= (-1)^{(|n|+|p|)|r|} q(m, r)q(n, p) - (-1)^{|n||p|} q(m, p)q(n, r) \\ &\quad + (-1)^{|p||r|+(|n|+|r|)|p|} q(m, p)q(n, r) - (-1)^{(|p|+|n|)|r|} q(m, r)q(n, p) \\ &= 0, \end{aligned}$$

it follows that $\mathbf{E}_{m,n} \in \operatorname{osp}(q)$ for all $m, n \in \mathcal{M}$.

(ii) For $a \in \mathcal{A}$, $m, n \in \mathcal{M}$ we have

$$\begin{aligned}
\mathbf{E}_{ma,n}(p) &= maq(n,p) - (-1)^{|p||n|}q(ma,p)n = mq(an,p) - (-1)^{|n||p|}q(m,ap)n \\
&= mq(an,p) - (-1)^{|n||p|+|a||p|}q(m,p)(an) \\
&\Rightarrow \mathbf{E}_{m,an}(p) = (-1)^{|a||n|}\mathbf{E}_{m,na}(p) \\
&= (-1)^{|a||n|}(mq(na,p) - (-1)^{|p|(|n|+|a|)}q(m,p)(na)) \\
&= (-1)^{|a||n|}(mq(n,ap) - (-1)^{(|n|+|a|)|p|+|a||n|}q(m,pa)n) \\
&= (-1)^{|a||n|}(mq(n,ap) - (-1)^{(|n|+|a|)|p|+|a||n|+|p||a|}q(m,ap)n) \\
&= (-1)^{|a||n|}\mathbf{E}_{m,n}(ap).
\end{aligned}$$

Hence $\mathbf{E} : \mathcal{M} \times \mathcal{M} \to \operatorname{End}_{\mathcal{A}} \mathcal{M} : (m,n) \mapsto \mathbf{E}_{m,n}$ is $\mathcal{A}$-bilinear and clearly of degree $\bar{0}$. In addition, for all $m, n, p \in \mathcal{M}$,

$$\begin{aligned}
\mathbf{E}_{m,n}(p) &= mq(n,p) - (-1)^{|p||n|}q(m,p)n \\
&= (-1)^{|m|(|n|+|p|)}q(n,p)m - (-1)^{|m||n|}nq(m,p) \\
&= -(-1)^{|m||n|}\mathbf{E}_{n,m}(p)
\end{aligned}$$

and so the assertion holds.

(iii) Let $m, n, p, r \in \mathcal{M}$. Then

$$\begin{aligned}
q(\mathbf{E}_{m,n}(p), r) - (-1)^{(|m|+|n|)(|p|+|r|)}q(\mathbf{E}_{p,r}(m), n) &= \\
&= q(mq(n,p) - (-1)^{|n||p|}q(m,p)n, r) \\
&\quad - (-1)^{(|m|+|n|)(|p|+|r|)}q(pq(r,m) - (-1)^{|r||m|}q(p,m)r, n) \\
&= (-1)^{|r|(|n|+|p|)}q(m,r)q(n,p) - (-1)^{|n||p|}q(m,p)q(n,r) \\
&\quad - (-1)^{(|m|+|n|)(|p|+|r|)+|p|(|r|+|m|)}q(r,m)q(p,n) \\
&\quad + (-1)^{(|m|+|n|)(|p|+|r|)+|r||m|}q(p,m)q(r,n)
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{|r|(|n|+|p|)+|n||p|+|r||m|}q(r,m)q(p,n) \\
&\quad - (-1)^{|n||p|+|m||p|+|n||r|}q(p,m)q(r,n) \\
&\quad - (-1)^{(|m|+|n|)(|p|+|r|)+|p|(|r|+|m|)}q(r,m)q(p,n) \\
&\quad + (-1)^{(|m|+|n|)(|p|+|r|)+|r||m|}q(p,m)q(r,n) \\
&= 0.
\end{aligned}$$

∎

In addition,

Proposition 2.1.2 *Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Then*

(i) $\text{osp}(q)$ *is a subalgebra of the Lie superalgebra* $(\text{End}_A \mathcal{M})^{(-)}$*;*

(ii) for all $m, n \in \mathcal{M}$, and $x \in \text{osp}(q)$,

$$[x, \mathbf{E}_{m,n}] = \mathbf{E}_{x(m),n} + (-1)^{|x||m|}\mathbf{E}_{m,x(n)}; \text{ and hence} \tag{2.3}$$

(iii) $\text{eosp}(q)$ *is an ideal of* $\text{osp}(q)$.

Proof: (i) We need to prove first of all that $\text{osp}(q)$ is a K-submodule of $\text{End}_A(\mathcal{M})$, i.e., that $\text{osp}(q)$ contains the homogeneous parts of its elements, that it is closed under addition and that $A \cdot \text{osp}(q) \in \text{osp}(q)$. That $\text{osp}(q)$ is closed under addition follows from the fact that q is bilinear. Let $x = x_{\bar{0}} \oplus x_{\bar{1}} \in \text{osp}(q)$. Then we have for $m, n \in \mathcal{M}$,

$$\begin{aligned}
0 &= q(x(m),n) + (-1)^{|m||n|}q(x(n),m) \\
&= \underbrace{q(x_{\bar{0}}(m),n) + (-1)^{|m||n|}q(x_{\bar{0}}(n),m)}_{\in A_{|m|+|n|}} \\
&\quad + \underbrace{q(x_{\bar{1}}(m),n) + (-1)^{|m||n|}q(x_{\bar{1}}(n),m)}_{\in A_{|m|+|n|+1}}.
\end{aligned}$$

Hence

$$q(x_\alpha(m),n) + (-1)^{|m||n|}q(x_\alpha(n),m) = 0$$

for all $m, n \in \mathcal{M}$, $\alpha \in \mathbb{Z}_2$ and so $x_\alpha \in \operatorname{osp}(q)$ for all $\alpha \in \mathbb{Z}_2$. Let $a \in A$ and $x \in \operatorname{osp}(q)$ be elements. Then for all $m, n \in \mathcal{M}$

$$\begin{aligned} q((ax)(m), n) + (-1)^{|m||n|}q((ax)(n), m) &= q(ax(m), n) + (-1)^{|m||n|}q(ax(n), m) \\ &= aq(x(m), n) + (-1)^{|m||n|}aq(x(n), m) \\ &= 0 \end{aligned}$$

and so $\operatorname{osp}(q)$ is a K-submodule of $\operatorname{End}_A \mathcal{M}$. To show that $\operatorname{osp}(q)$ is a K-subalgebra of $\operatorname{End}_A \mathcal{M}^{(-)}$, we need to show that $[\operatorname{osp}(q), \operatorname{osp}(q)] \in \operatorname{osp}(q)$. Let $x, x' \in \operatorname{osp}(q)$ and $m, n \in \mathcal{M}$. Then

$$\begin{aligned} q([x,x'](m), n) &+ (-1)^{|m||n|}q([x,x'](n), m) = \\ &= q(x(x'(m)), n) - (-1)^{|x||x'|}q(x'(x(m)), n) \\ &\quad + (-1)^{|m||n|}(q(x(x'(n)), m) - (-1)^{|x||x'|}q(x'(x(n)), m) \\ &\overset{(2.1)}{=} q(x(x'(m)), n) - (-1)^{|x||x'|}q(x'(x(m)), n) \\ &\quad - (-1)^{|m||n| + (|x'| + |n|)|m|}q(x(m), x'(n)) \\ &\quad + (-1)^{|m||n| + |x||x'| + (|x| + |n|)|m|}q(x'(m), x(n)) \\ &= 0. \end{aligned}$$

Hence $\operatorname{osp}(q)$ is a K-subalgebra of $\operatorname{End}_A(\mathcal{M})^{(-)}$.

(ii) Let $x \in \operatorname{osp}(q)$, $m, n, p \in \mathcal{M}$ be homogenous. Then

$$\begin{aligned} [x, \mathbf{E}_{m,n}](p) &= x(mq(n,p) - (-1)^{|n||p|}q(m,p)n) \\ &\quad -(-1)^{|x|(|m|+|n|)}(mq(n, x(p)) + (-1)^{|x||m|+|p||n|}q(m, x(p))n) \\ &= x(m)q(n,p) - (-1)^{|n||p|+|x|(|m|+|p|)}q(m,p)x(n) \\ &\quad +(-1)^{|x|(|m|+|n|)+|x||n|}mq(x(n),p) - (-1)^{|x||m|+|n||p|+|x||m|}q(x(m),p)n \\ &= \mathbf{E}_{x(m),n}(p) + (-1)^{|x||m|}\mathbf{E}_{m,x(n)}(p) \end{aligned}$$

(iii) By 2.1.1 (i), $\operatorname{eosp}(q)$ is contained in $\operatorname{osp}(q)$. Moreover,

$$\operatorname{eosp}(q) = \operatorname{eosp}(q)_{\bar 0} \oplus \operatorname{eosp}(q)_{\bar 1}$$

where

$$\operatorname{eosp}(q)_\alpha = \{\mathbf{E}_{m,n} \mid |m| + |n| = \alpha\},\ \alpha \in \mathbb{Z}_2.$$

By Proposition 2.1.1 (ii), we have that $\operatorname{eosp}(q)$ is a K-submodule of $\operatorname{End}_A(\mathcal{M})$. Hence, by (ii), $\operatorname{eosp}(q)$ is an ideal of $\operatorname{osp}(q)$. ∎

Remark 2.1.3 Note that if A is just an extension of k, i.e. $A_{\bar{1}} = (0)$, and $\mathcal{M}$ is an A-supermodule such that $\mathcal{M}_{\bar{1}} = (0)$ with an A-quadratic form q, then $\operatorname{eosp}(q)$ is just $\operatorname{eo}(q)$. If $\mathcal{M}_{\bar{0}} = (0)$ then q is skewsymmetric however $\operatorname{eosp}(q)$ is not a symplectic Lie algebra (it is not a Lie algebra).

Lemma 2.1.4 *Let $\mathcal{M}$ be an A-supermodule with a quadratic form q. Suppose that there exists $m_0 \in \mathcal{M}$ such that $q(m_0, m_0)$ is invertible in A. Then*

$$\mathcal{P} = \{m \in \mathcal{M} \mid q(m, m_0) = 0\}$$

is an A-submodule of $\mathcal{M}$ such that

$$\mathcal{M} = Am_0 \oplus \mathcal{P}$$

and

$$q(Am_0, \mathcal{P}) = 0.$$

Proof: If $am_0 \in \mathcal{P}$ we have $0 = q(m_0, am_0) = aq(m_0, m_0)$ forcing $a = 0$. Moreover, for $m \in \mathcal{M}$ we have

$$m = \underbrace{q(m_0, m)q(m_0, m_0)^{-1}m_0}_{\in Am_0} + \underbrace{(m - q(m_0, m)q(m_0, m_0)^{-1}m_0)}_{\in \mathcal{P}}$$

since

$$\begin{aligned} q(m_0, m - q(m_0, m)q(m_0, m_0)^{-1}m_0) &= q(m_0, m) - q(m_0, m)q(m_0, m_0)^{-1}q(m_0, m_0) \\ &= 0. \end{aligned}$$

∎

Let $\mathcal{G}$ denote the Grassmann algebra. Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Recall that the Grassmann envelopes $\mathcal{G}(\mathcal{M})$ and $\mathcal{G}(A)$ are defined by $\mathcal{G}(\mathcal{M}) = (\mathcal{G}_{\bar{0}} \otimes \mathcal{M}_{\bar{0}}) \oplus (\mathcal{G}_{\bar{1}} \otimes \mathcal{M}_{\bar{1}})$ and $\mathcal{G}(A) = (\mathcal{G}_{\bar{0}} \otimes A_{\bar{0}}) \oplus (\mathcal{G}_{\bar{1}} \otimes A_{\bar{1}})$. One knows that $\mathcal{G}(\mathcal{M})$ is a $\mathcal{G}(A)$-module via $(\xi_I \otimes a) \cdot (\xi_J \otimes m) = \xi_I \xi_J \otimes am$. In addition we had defined the Grassman envelope $\mathcal{G}(q) : \mathcal{G}(\mathcal{M}) \to \mathcal{G}(A)$ where $\mathcal{G}(q)$ is a $\mathcal{G}(A)$-quadratic form on $\mathcal{G}(\mathcal{M})$ (see [N3]) with polar

$$\mathcal{G}(q)(\xi_I \otimes m_I, \xi_J \otimes m_J) = \xi_I \xi_J \otimes q(m_I, m_J)$$

for all $\xi_I \otimes m_I, \xi_J \otimes m_J \in \mathcal{G}(\mathcal{M})$. In fact, we have

Proposition 2.1.5 *Let $\mathcal{M}$ be a A-supermodule with a A-quadratic form q. Then*

$$\mathcal{G}(\mathrm{eosp}(q)) = \mathrm{eo}(\mathcal{G}(q)).$$

Proof: We have

$$\begin{aligned} \mathcal{G}(\mathrm{eosp}(q)) &= (\mathcal{G}_{\bar{0}} \otimes \mathrm{eosp}(q)_{\bar{0}}) \oplus (\mathcal{G}_{\bar{1}} \otimes \mathrm{eosp}(q)_{\bar{1}}) \\ &= \mathrm{span}_A\{\xi_I \otimes \mathbf{E}_{m,n} \mid |m| + |n| = \bar{0} = |I|\} \\ &\quad \oplus \mathrm{span}_A\{\xi_I \otimes \mathbf{E}_{m,n} \mid |m| + |n| = \bar{1} = |I|\} \\ &= \mathrm{span}_A\{\xi_I \otimes \mathbf{E}_{m,n} \mid |m| + |n| = |I|\}. \end{aligned}$$

On the other hand, for any $\xi_I \otimes m$, $\xi_J \otimes n \in \mathcal{G}(\mathcal{M})$, and for all $\xi_P \otimes p \in \mathcal{G}(\mathcal{M})$ we have

$$\begin{aligned} \mathbf{E}_{\xi_I \otimes m, \xi_J \otimes n}(\xi_P \otimes p) &= (\xi_I \otimes m)(\xi_J \xi_P \otimes q(n,p) - (-1)^{|n||p|}(\xi_I \xi_P \otimes q(m,p))(\xi_J \otimes n) \\ &= \xi_I \xi_J \xi_P \otimes mq(n,p) - (-1)^{|n||p|} \xi_I \xi_P \xi_J \otimes q(m,p)n \\ &= \xi_I \xi_J \xi_P \otimes mq(n,p) - (-1)^{|p||n|} \xi_I \xi_J \xi_P \otimes q(m,p)n \\ &= \xi_I \xi_J \xi_P \otimes (m(q(n,p) - (-1)^{|p||n|} q(m,p)n) \\ &= \xi_I \xi_J \xi_P \otimes \mathbf{E}_{m,n} p \\ &= (\xi_I \xi_J \otimes \mathbf{E}_{m,n})(\xi_P \otimes p). \end{aligned}$$

Hence

$$\mathbf{E}_{\xi_I \otimes m, \xi_J \otimes n} = \xi_I \xi_J \otimes \mathbf{E}_{m,n} \tag{2.4}$$

which proves that $\mathrm{eo}(\mathcal{G}(q)) \subset \mathcal{G}(\mathrm{eosp}(q))$. Conversely, for any $\xi_I \otimes \mathbf{E}_{m,n} \in \mathcal{G}(\mathrm{eosp}(q))$ we have that $|I| = |m| + |n|$ and hence $\xi_I = \xi_{I_1}\xi_{I_2}$ for some $I_1, I_2 \subset \mathbb{N}$ satisfying $|I_1| = |m|$, $|I_2| = |n|$. Therefore (2.4) implies the other inclusion. In addition, for all $\xi_I \otimes m, \xi_J \otimes n, \xi_{I'} \otimes m', \xi_{J'} \otimes n' \in \mathcal{G}(\mathcal{M})$, using (2.3) we have that

$$\begin{aligned}
[\mathbf{E}_{\xi_I\otimes m,\xi_J\otimes n}, \mathbf{E}_{\xi_{I'}\otimes m',\xi_{J'}\otimes n'}] &= \mathbf{E}_{\mathbf{E}_{\xi_I\otimes m,\xi_J\otimes n}(\xi_{I'}\otimes m'),\xi_{J'}\otimes n'} \\
&\quad +(-1)^{(|m|+|n|)|m'|}\mathbf{E}_{\xi_{I'}\otimes m',\mathbf{E}_{\xi_I\otimes m,\xi_J\otimes n}(\xi_{J'}\otimes n')} \\
&= \mathbf{E}_{(\xi_I\xi_J\otimes\mathbf{E}_{m,n})(\xi_{I'}\otimes m'),\xi_{J'}\otimes n'} \\
&\quad +(-1)^{(|m|+|n|)|m'|}\mathbf{E}_{\xi_{I'}\otimes m',(\xi_I\xi_J\otimes\mathbf{E}_{m,n})(\xi_{J'}\otimes n'} \\
&= \mathbf{E}_{\xi_I\xi_J\xi_{I'}\otimes\mathbf{E}_{m,n}(m'),\xi_{J'}\otimes n'} + \mathbf{E}_{\xi_{I'}\otimes m',\xi_I\xi_J\xi_{J'}\otimes\mathbf{E}_{m,n}(n')} \\
&= \xi_I\xi_J\xi_{I'}\xi_{J'} \otimes \mathbf{E}_{\mathbf{E}_{m,n}(m'),n'} \\
&\quad +(-1)^{(|m|+|n|)|m'|}\xi_I\xi_J\xi_{I'}\xi_{J'} \otimes \mathbf{E}_{m',\mathbf{E}_{m,n}(n')} \\
&= \xi_I\xi_J\xi_{I'}\xi_{J'} \otimes [\mathbf{E}_{m,n}, \mathbf{E}_{m',n'}] \\
&= [\xi_I\xi_J \otimes \mathbf{E}_{m,n}, \xi_{I'}\xi_{J'} \otimes \mathbf{E}_{m',n'}]
\end{aligned}$$

and so we have $\mathrm{eo}(\mathcal{G}(q)) = \mathcal{G}(\mathrm{eosp}(q))$ as Lie algebras. ∎

Recall that, given A-modules $\mathcal{M}_1, \cdots, \mathcal{M}_n$ and $\mathcal{M} = \oplus_{i=1,\cdots,n}\mathcal{M}_i$, we have the embedding

$$\begin{aligned}
\mathrm{End}_A(\mathcal{M}_i) &\hookrightarrow \mathrm{End}_A\,\mathcal{M} \\
\varphi_i &\mapsto \varphi : (m_1, \cdots, m_n) \mapsto \varphi_i(m_i).
\end{aligned}$$

Also recall that, given two A-supermodules $\mathcal{M}_1$ and $\mathcal{M}_2$ with A-quadratic forms q_1 and q_2 respectively, we define their orthogonal sum $q = q_1 \oplus q_2$ to be the A-quadratic form on the A-supermodule $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ given by

$$\begin{aligned}
q(m_1 \oplus m_2, n_1 \oplus n_2) &= (q_1 \oplus q_2)(m_1 \oplus m_2, n_1 \oplus n_2) \\
&= q(m_1, n_1) + q(m_2, n_2)
\end{aligned}$$

for $m_1, n_1 \in \mathcal{M}_1$ and $m_2, n_2 \in \mathcal{M}_2$. We then have the following:

Proposition 2.1.6 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be A-supermodules with A-quadratic forms q_1 and q_2 respectively. Let $q = q_1 \oplus q_2$. Then*

$$\begin{aligned}
\mathrm{eosp}(q) &= \mathrm{span}_A\{\mathbf{E}_{m_1,n_1} \mid m_1, n_1 \in \mathcal{M}_1\} \\
&\quad \oplus \mathrm{span}_A\{\mathbf{E}_{m_1,n_2} \mid m_1 \in \mathcal{M}_1,\ n_2 \in \mathcal{M}_2\} \\
&\quad \oplus \mathrm{span}_A\{\mathbf{E}_{m_2,n_2} \mid m_2, n_2 \in \mathcal{M}_2\} \\
&= \mathrm{eosp}(q_1) \oplus \mathrm{eosp}(q_1, q_2) \oplus \mathrm{eosp}(q_2)
\end{aligned}$$

where

$$\mathrm{eosp}(q_1, q_2) = \mathrm{span}_A\{\mathbf{E}_{m_1,m_2} \mid m_1 \in \mathcal{M}_1,\ m_2 \in \mathcal{M}_2\}.$$

Proof: Clearly, using bilinearity of $\mathbf{E}$ and Proposition 2.1.1.2,

$$\begin{aligned}
\mathrm{eosp}(q) &= \mathrm{span}_A\{\mathbf{E}_{m_1,n_1} \mid m_1, n_1 \in \mathcal{M}_1\} \\
&\quad + \mathrm{span}_A\{\mathbf{E}_{m_1,n_2} \mid m_1 \in \mathcal{M}_1,\ n_2 \in \mathcal{M}_2\} \\
&\quad + \mathrm{span}_A\{\mathbf{E}_{m_2,n_2} \mid m_2, n_2 \in \mathcal{M}_2\}.
\end{aligned}$$

Let $x = x_{11} + x_{12} + x_{22} \in \mathrm{eosp}(q)$ where $x_{ij} \in \mathrm{span}_A\{\mathbf{E}_{m,n} \mid m \in \mathcal{M}_i,\ n \in \mathcal{M}_j\}$ for $1 \le i \le j \le 2$. Suppose $x = 0$. Then for all $m_1 \in \mathcal{M}_1$ and $m_2 \in \mathcal{M}_2$ we have

$$0 = x(m_1) = \underbrace{x_{11}(m_1)}_{\in \mathcal{M}_1} + \underbrace{x_{12}(m_1)}_{\in \mathcal{M}_2} + \underbrace{x_{22}(m_1)}_{=0}$$

and

$$0 = x(m_2) = \underbrace{x_{11}(m_2)}_{=0} + \underbrace{x_{12}(m_2)}_{\in \mathcal{M}_1} + \underbrace{x_{22}(m_2)}_{\in \mathcal{M}_2}.$$

Hence $x_{ij} = 0$ and so we have

$$\begin{aligned}
\mathrm{eosp}(q) &= \mathrm{span}_A\{\mathbf{E}_{m_1,n_1} \mid m_1, n_1 \in \mathcal{M}_1\} \\
&\quad \oplus \mathrm{span}_A\{\mathbf{E}_{m_1,n_2} \mid m_1 \in \mathcal{M}_1,\ n_2 \in \mathcal{M}_2\} \\
&\quad \oplus \mathrm{span}_A\{\mathbf{E}_{m_2,n_2} \mid m_2, n_2 \in \mathcal{M}_2\}.
\end{aligned}$$

In addition, for $i = 1, 2$, any $x \in \text{span}_A\{\mathbf{E}_{m,n} \mid m, n \in \mathcal{M}_i\}$ satisfies $x(\mathcal{M}) \subset \mathcal{M}_i$ and $x(\mathcal{M}_j) = 0$ for $j \in \{1, 2\} - \{i\}$ and so x can be viewed as an element of $\text{eosp}(q_i)$. Hence

$$\text{eosp}(q) = \text{eosp}(q_1) \oplus \text{eosp}(q_1, q_2) \oplus \text{eosp}(q_2)$$

where

$$\text{eosp}(q_1, q_2) = \text{span}_A\{\mathbf{E}_{m_1,m_2} \in \mid m_1 \in \mathcal{M}_1,\ m_2 \in \mathcal{M}_2\}.$$

■

We shall now consider $\text{eosp}(q)$ for free A-supermodules. Throughout, when we talk of an index set we assume that it is totally ordered. We have the following:

Proposition 2.1.7 *Let A be a superextension of K, I an arbitrary (not necessarily finite) index set and $\mathcal{M} = \oplus_{i \in I} Am_i$ a free A-supermodule with an A-quadratic form q such that the basis $\{m_i\}$ consists of homogeneous elements (not necessarily of degree $\bar{0}$). Then $\{\mathbf{E}_{m_i,m_j} \mid i, j \in I,\ i < j\} \cup \{\mathbf{E}_{m_i,m_i} \mid m_i \in \mathcal{M}_{\bar{1}}\}$ forms a spanning set of $\text{eosp}(q)$ over A, and, if q is nondegenerate, an A-basis.*

Proof: Let $L = \text{eosp}(q)$. We have in general that $\mathbf{E}_{m_i,m_i} = 0$ if and only if for all $p \in \mathcal{M}$,

$$0 = \mathbf{E}_{m_i,m_i}(p) = m_i q(m_i, p) - (-1)^{|m_i|} m_i q(m_i, p) = m_i(1 - (-1)^{|m_i|}) q(m_i, p). \quad (2.5)$$

Then $\mathbf{E}_{m,m} = 0$ for all $m \in \mathcal{M}_{\bar{0}}$ and so by Proposition 2.1.1,

$$L = \text{span}_A(\{\mathbf{E}_{m_i,m_j} \mid i < j\} \cup \{\mathbf{E}_{m_i,m_i} \mid m_i \in \mathcal{M}_{\bar{1}}\}).$$

Assume that q is nondegenerate. Let $x = \sum_{i \leq j} a_{i,j} \mathbf{E}_{m_i,m_j} \in \text{eosp}(q)$ where $a_{ii} = 0$ for $m_i \in \mathcal{M}_{\bar{0}}$. Suppose that $x = 0$. Then for all $m \in \mathcal{M}$,

$$\begin{aligned} x(m_l) &= \sum_{i \leq j} a_{ij} \mathbf{E}_{m_i,m_j}(m) = \sum_{i \leq j} a_{ij}(m_i q(m_j, m) - (-1)^{|m_j||m|} q(m_i, m) m_j) \\ &= \sum_i \Big(\sum_{j \geq i} a_{ij} m_i q(m_j, m) - \sum_{j \leq i} (-1)^{|m_i||m|} a_{ji} q(m_j, m) m_i\Big) \\ &= \sum_i m_i \left(\sum_{j \geq i} (-1)^{|m_i||a_{ij}|} a_{ij} q(m_j, m) - \sum_{j \leq i} (-1)^{|m_i||m_j| + |m_i||a_{ji}|} a_{ji} q(m_j, m) \right). \end{aligned}$$

Since $\mathcal{M}$ is free, for a fixed i we have

$$\begin{aligned}0 &= \sum_{j\geq i}(-1)^{|m_i||a_{ij}|}a_{ij}q(m_j,m) - \sum_{j\leq i}(-1)^{|m_i||m_j|+|m_i||a_{ji}|}a_{ji}q(m_j,m)\\ &= q(\sum_{j>i}(-1)^{|m_i||a_{ij}|}a_{ij}m_j + (-1)^{|m_i||a_{ii}|}(1-(-1)^{|m_i|})a_{ii}m_i\\ &\quad - \sum_{j<i}(-1)^{|m_i||m_j|+|m_i||a_{ji}|}a_{ji}m_j, m)\end{aligned}$$

for all $m \in \mathcal{M}$. Therefore, since q is nondegenerate,

$$\sum_{j>i}(-1)^{|m_i||a_{ij}|}a_{ij}m_j + 2(-1)^{|a_{ii}|}a_{ii}m_i - \sum_{j<i}(-1)^{|m_i||m_j|+|m_i||a_{ji}|}a_{ji}m_j = 0$$

and so using again the fact that $\mathcal{M}$ is free,

$$a_{ij} = 0 \text{ for all } i<j \text{ and } a_{ii} = 0 \text{ for all } m_i \in \mathcal{M}_{\bar{1}}.$$

Hence $\{\mathbf{E}_{m_i,m_j} \mid i,j \in I,\ i<j\} \cup \{\mathbf{E}_{m_i,m_i} \mid m_i \in \mathcal{M}_{\bar{1}}\}$ forms an $\mathcal{A}$-basis of $\mathrm{eosp}(q)$. ∎

In particular, for an arbitrary set I, the hyperbolic $\mathcal{A}$-superspace $H(I)$ is a free $\mathcal{A}$-supermodule with even basis $\{h_{\pm i}\}_{i\in I}$ and a nondegenerate $\mathcal{A}$-quadratic form q_I. Given a total order $<$ on I, define a total order on $-I$ by setting $-i < -j$ if and only if $i<j$. Hence we have:

Corollary 2.1.8 *Let I be an arbitrary set and $H(I)$ a hyperbolic $\mathcal{A}$-superspace. Then $\{\mathbf{E}_{h_i,h_j} \mid i,j \in I,\ i<j\} \cup \{\mathbf{E}_{h_i,h_{-j}} \mid i,j \in I\} \cup \{\mathbf{E}_{h_{-i},h_{-j}} \mid i,j \in I,\ i<j\}$ forms a basis of* $\mathrm{eosp}(q_I)$.

Another example of an $\mathcal{A}$-supermodule $\mathcal{M}$ which satisfies the conditions in Proposition 2.1.7 is the following:

Definition 2.1.9 Let $\mathcal{M}$ be a free $\mathcal{A}$-supermodule with a homogeneous basis $\{m_i \mid i \in I\}$ and an A-quadratic form q. We say that q is *almost diagonizable* if for all $i \in I$ there exists a (necessarily unique) $\underline{i} \in I$ such that $q(m_i, m_{\underline{i}}) \in \mathcal{A}^\times$ and $q(m_i, m_j) = 0$ for all $j \in I$, $j \neq \underline{i}$. In addition, if q is almost diagonizable with respect to $\{m_i \mid i \in I\}$ where $i = \underline{i}$ then we say that q is *invertibly diagonizable*. If q is almost diagonizable with respect to a basis $\{m_i \mid i \in I\}$ consisting of odd elements and $q(m_i, m_{\underline{i}}) \in \{\pm 1\}$ then we say that q is *standard alternating* (see [La]).

Lemma 2.1.10 *Let $\mathcal{M} = \oplus_{i \in I} Am_i$ be a free A-supermodule with the homogeneous basis $\{m_i \mid i \in I\}$ with an almost diagonizable A-quadratic form q.*

(a) If $m_i \in \mathcal{M}_{\bar{0}}$ for all $i \in I$ then q is invertibly diagonizable (with respect to a possibly different basis).

(b) If $m_i \in \mathcal{M}_{\bar{1}}$ for all $i \in I$ then q is standard alternating (with respect to a possibly different basis).

(c) If q is almost diagonizable then q is an orthogonal sum of an invertibly diagonizable form and a standard alternating form (with respect to a possibly different basis).

Proof: (a) If $i \in I$ such that $i \neq \underline{i}$ and $m_i \in \mathcal{M}_{\bar{0}}$ then, using the fact that $\frac{1}{2} \in A$, $Am_i \oplus Am_{\underline{i}} = A(m_i + m_{\underline{i}}) \oplus A(m_i - m_{\underline{i}})$. Moreover, $q(m_i + m_{\underline{i}}, m_i - m_{\underline{i}}) = 0$ and $q(m_i + \sigma m_{\underline{i}}, m_i + \sigma m_{\underline{i}}) = 2\sigma q(m_i, m_{\underline{i}}) \in A^\times$, $\sigma = \pm$, hence we can modify the basis of $\mathcal{M}$ so that q is invertibly diagonizable.
(b) If $\{m_i \mid i \in I\}$ is an odd basis of $\mathcal{M}$ with respect to which q is almost diagonizable, then I is a disjoint union of two index sets J and $\underline{J}$ such that $j \in J \Leftrightarrow \underline{j} \in \underline{J}$. Then $\{m_j \mid j \in J\} \dot{\cup} \{q(m_j, m_{\underline{j}})^{-1} m_j \mid j \in \underline{J}\}$ is an odd basis of $\mathcal{M}$ with respect to which q is standard alternating.
(c) By (a) and (b) we can modify the basis so that q is invertibly diagonizable with respect to $\{m_i \mid i \in I\}_{\bar{0}}$ and standard alternating with respect to $\{m_i \mid i \in I\}_{\bar{1}}$ hence the assertion follows. ∎

Remark 2.1.11 It is clear that an orthogonal sum of two almost diagonizable (invertibly diagonizable, standard alternating) quadratic forms is almost diagonizable (invertibly diagonizable, standard alternating, respectively). It is also clear that an almost diagonizable quadratic form is nondegenerate. □

Example 2.1.12 Let I be an arbitrary set and $H(I)$ a hyperbolic A-superplane. Then q_I is almost diagonizable with respect to the basis $\{h_{\pm i} \mid i \in I\}$ and is invertibly diagonizable with respect to the basis $\{h_i + h_{-i}, h_i - h_{-i} \mid i \in I\}$. □

Example 2.1.13 Let L_c be the core of the extended affine Lie algebra of type B_l ($l \geq 3$) with nullity ν (see [AG]). Then one can show that $L_c/Z(L_c) = \mathrm{eosp}(q_I \oplus q_X)$ where I is an index set of cardinality l, $q_X = q_W \oplus q_0$ is the orthogonal sum of the quadratic form q_W on a free $A_{[\nu]}$-module $W = \bigoplus_{i=1}^{m} A_{[\nu]}w_i$ and the quadratic form q_0 on a free $A_{[\nu]}$-module $A_{[\nu]}x_0$ and $A_{[\nu]} = \mathbb{C}[t_1^{\pm 1}, \cdots, t_\nu^{\pm 1}]$. The quadratic forms q_W and q_0 are given by

$$q_0(x_0, x_0) = 1$$

and

$$q_W(\sum_{i=1}^{m} a_i w_i, \sum_{i=1}^{m} b_i w_i) = \sum_{j=1}^{m} a_j b_j t^{\tau_j}$$

for some fixed $\tau_j \in \mathbb{Z}^\nu$, $j = 1, \cdots, m$. Then it is easily seen that $q_I \oplus q_X$ is invertibly diagonizable with respect to the basis $\{h_i \pm h_{-i} \mid i = 1, \cdots, l\} \cup \{w_j \mid j = 1, \cdots, m\} \cup \{x_0\}$. □

Example 2.1.14 Let I be an arbitrary set and let $\tilde{H}(I) = \tilde{H}(I)_+ \oplus \tilde{H}_-(I)$ be a free left A-module where $\tilde{H}_\sigma = \oplus_{i \in I} A\tilde{h}_{\sigma i}$ for $\sigma = \pm$. Define the right A-module action by setting $\tilde{h}_{\sigma i} a = (-1)^{|a| + \bar{1}} a \tilde{h}_{\sigma i}$ for all $i \in I$, $a \in A$ and $\sigma = \pm$. Then it is easy to see that $\tilde{H}(I)$ is an A-supermodule with the $\mathbb{Z}_2$-grading given by

$$\tilde{H}(I)_\alpha = (\tilde{H}(I)_+)_\alpha \oplus (\tilde{H}(I)_-)_\alpha$$

where

$$(\tilde{H}(I)_\sigma)_{\bar{0}} = \oplus_{i \in I} A_{\bar{1}} \tilde{h}_{\sigma i} \text{ and } (\tilde{H}(I)_\sigma)_{\bar{1}} = \oplus_{i \in I} A_{\bar{0}} \tilde{h}_{\sigma i}.$$

Note that $\{\tilde{h}_i, \tilde{h}_{-i} \mid i \in I\}$ is an odd basis of $\tilde{H}(I)$. Define an A-quadratic form $\tilde{q}_I$ on $\tilde{H}(I)$ using

$$\tilde{q}_I(h_i, h_{-j}) = \delta_{ij} \text{ for all } i, j \in I; \ \tilde{q}_I(\tilde{H}_\sigma(I), \tilde{H}_\sigma(I)) = 0 \text{ for all } \sigma = \pm.$$

Then it is easy to check that $\tilde{q}_I$ is a standard alternating quadratic form with respect to $\{h_{\pm i} \mid i \in I\}$. $\square$

Remark 2.1.15 Let A be a field of characteristic 0 and let $\mathcal{M}$ be a finite dimensional vector space over A with an A-quadratic form on $\mathcal{M}$. Then q is almost diagonizable iff it is nondegenerate. Indeed, it is clear that if q is almost diagonizable that it is nondegenerate. The proof of converse was done by Lang (see [La]): one can show that $\mathcal{M}$ has a homogeneous basis, say $\{n_i\}_{i=1}^k \cup \{r_j\}_{j=1}^l$ where $n_i \in \mathcal{M}_{\bar{0}}$ and $r_j \in \mathcal{M}_{\bar{1}}$. Let $\mathcal{N} = \bigoplus_{i=1}^{k} An_i$ and $\mathcal{R} = \bigoplus_{j=1}^{l} Ar_j$. Since q is supersymmetric, $q(\mathcal{N}, \mathcal{R}) = 0$. Moreover, using the fact that q is nondegenerate, one can readjust the bases of $\mathcal{N}$ and $\mathcal{R}$ so that $q_{\mathcal{N}\times\mathcal{N}}$ is invertibly diagonizable and $q_{\mathcal{R}\times\mathcal{R}}$ is standard alternating. $\square$

Proposition 2.1.16 *Let I be an arbitrary index set and $\mathcal{M}$ a free A-supermodule with a homogeneous basis $\{m_i \mid i \in I\}$ and an almost diagonizable A-quadratic form q_1. Let $\mathcal{N}$ be an A-supermodule with an A-quadratic form q_2 and let $q = q_1 \hat{\otimes} q_2$. Then*

$$\mathrm{span}_A\{\mathbf{E}_{n,m} \mid m \in \mathcal{M}, n \in \mathcal{N}\} = \bigoplus_{i\in I}\{\mathbf{E}_{n,m_i} \mid n \in \mathcal{N}\}.$$

Proof: Let $x \in \mathrm{span}_A\{\mathbf{E}_{m,n} \mid m \in \mathcal{M}, n \in \mathcal{N}\}$. Then $x = \sum_{j\in J}\mathbf{E}_{r_j,p_j}$ for some $p_j \in \mathcal{M}$, $r_j \in \mathcal{N}$, $j \in J$ where J is a finite set. Since m_i, $i \in I$ form a basis of $\mathcal{M}$ we can write $p_j = \sum_{i\in I} a_{ij}m_i$ for some $a_{ij} \in A$. Then by Lemma 2.1.1,

$$\sum_{j\in J}\mathbf{E}_{r_j,p_j} = \sum_{j\in J}\mathbf{E}_{r_j,\sum_{i\in I}a_{ij}m_i} = \sum_{i\in I}\sum_{j\in J}\mathbf{E}_{r_ja_{ij},m_i} = \sum_{i\in I}\mathbf{E}_{\underbrace{\sum_{j\in J} r_ja_{ij}}_{=n_i\in\mathcal{N}},m_i} = \sum_{i\in I}\mathbf{E}_{n_i,m_i}.$$

Suppose $x = \sum_{i\in I_{fin}} \mathbf{E}_{n_i,m_i} = 0$ for some $n_i \in \mathcal{N}$ and $I_{fin} \subset I$ finite. Then for all $k \in I_{fin}$,

$$0 = \sum_{i\in I_{fin}} \mathbf{E}_{n_i,m_i}(m_k) = n_k \underbrace{q(m_k, m_k)}_{\in A^\times}$$

and so $n_i = 0$ for all $i \in I_{fin}$, i.e., $\mathbf{E}_{n_i,m_i} = 0$ for all $i \in I_{fin}$. Hence

$$\mathrm{span}_A\{\mathbf{E}_{n,m} \mid m \in \mathcal{M}, n \in \mathcal{N}\} = \bigoplus_{i\in I}\{\mathbf{E}_{n,m_i} \mid n \in \mathcal{N}\}.$$

∎

In particular, we have that for a hyperbolic A-superspace $H(I)$, $H(I) = (\oplus_{i\in I} Ah_i) \oplus (\oplus Ah_{-i})$ where $q(h_{\sigma i}, h_{\mu j}) = \delta_{-\sigma i,\mu j}$ and so:

Corollary 2.1.17 *Let I be an arbitrary set and $\mathcal{M}$ an A-supermodule with an A-quadratic form q. Let $L = \mathrm{eosp}(q_I \oplus q)$. Then L can be identified with*

$$L = (\bigoplus_{i\in I} L_i) \oplus L_0 \oplus (\bigoplus_{i\in I} L_{-i})$$

where

$$\begin{aligned} L_0 &= \mathrm{eosp}(q_I) \oplus \mathrm{eosp}(q) \\ L_i &\equiv \mathcal{M} \equiv L_{-i} \quad (\text{write } L_{\sigma i} = (\mathcal{M})_{\sigma i}) \end{aligned}$$

with induced multiplication

$$\begin{aligned} [a\mathbf{E}_{h_{\sigma i},h_{\mu j}}, (m)_{\nu k}] &= \nu\sigma\delta_{-\nu k,\mu j}(am)_{\sigma i} - \nu\mu\delta_{-\nu k,\sigma i}(am)_{\mu j} \\ &= -(-1)^{|a||m|}[(m)_{\nu k}, a\mathbf{E}_{h_{\sigma i},h_{\mu j}}] \\ [x,(m)_{\nu k}] &= (xm)_{\nu k} = -(-1)^{|x||m|}[(m)_{\nu k}, x] \\ [(m)_{\sigma i},(n)_{\mu j}] &= \sigma\mu(q(m,n)\mathbf{E}_{h_{\mu j},h_{\sigma i}} - \delta_{-\sigma i,\mu j}\mathbf{E}_{m,n}) \\ &= -(-1)^{|m||n|}[(n)_{\mu j},(m)_{\sigma i}] \end{aligned}$$

for all $a \in A$, $i,j,k \in I$, $m,n \in \mathcal{M}$, $x \in \mathrm{eosp}(q)$, $\sigma,\mu,\nu \in \{\pm\}$.

Proof: By Proposition 2.1.6.

$$\begin{aligned} L &= \mathrm{span}_A\{\mathbf{E}_{h,h'} \mid h,h' \in H(I)\} \\ &\quad \oplus \mathrm{span}_A\{\mathbf{E}_{h,m} \mid h \in H(I),\ m \in \mathcal{M}\} \\ &\quad \oplus \mathrm{span}_A\{\mathbf{E}_{m,n} \mid m,n \in \mathcal{M}\}. \end{aligned}$$

But $H(I)$ and $\mathcal{M}$ satisfy the conditions of Proposition 2.1.16 and so we have

$$\begin{aligned} &\mathrm{span}_A\{\mathbf{E}_{h,m} \mid h \in H(I),\ m \in \mathcal{M}\} \\ &\qquad = \left(\bigoplus_{i\in I}\{\mathbf{E}_{h_i,m} \mid m \in \mathcal{M}\}\right) \bigoplus \left(\bigoplus_{i\in I}\{\mathbf{E}_{h_{-i},m} \mid m \in \mathcal{M}\}\right). \end{aligned}$$

So if we set

$$
\begin{aligned}
L_0 &= \text{span}_A\{\mathbf{E}_{h,h'} \mid h, h' \in H(I)\} \bigoplus \text{span}\{\mathbf{E}_{m,n} \mid m, n \in \mathcal{M}\} \\
L_i &= \{\mathbf{E}_{h_i,m} \mid m \in \mathcal{M}\};\ i \in I \\
L_{-i} &= \{\mathbf{E}_{m,h_{-i}} \mid m \in \mathcal{M}\};\ i \in I
\end{aligned}
$$

then $L = (\bigoplus_{i\in I} L_i) \oplus L_0 \oplus (\bigoplus_{i\in I} L_{-i})$ and we can clearly identify the above with

$$
\begin{aligned}
L_0 &= \text{eosp}(q_I) \oplus \text{eosp}(q) \\
L_i &=: (\mathcal{M})_i \equiv \mathcal{M};\ i \in I \\
L_{-i} &=: (\mathcal{M})_{-i} \equiv \mathcal{M};\ i \in I
\end{aligned}
$$

by identifying $\mathbf{E}_{h_i,m} \equiv m$ and $\mathbf{E}_{m,h_{-i}} \equiv m$, with induced multiplication

$$
\begin{aligned}
[a\mathbf{E}_{h_{\sigma i},h_{\mu j}}, (m)_{\nu k}] &\overset{def'n}{=} [a\mathbf{E}_{h_{\sigma i},h_{\mu j}}, \nu\mathbf{E}_{h_{\nu k},m}] \\
&\overset{(2.3)}{=} \nu\mathbf{E}_{a\mathbf{E}_{h_{\sigma i},h_{\mu j}}(h_{\nu k}),m} + \nu\mathbf{E}_{h_{\nu k},a\mathbf{E}_{h_{\sigma i},h_{\mu j}}(m)} \\
&= \nu(\delta_{-\mu j,\nu k}\mathbf{E}_{h_{\sigma i},am} - \delta_{-\sigma i,\nu k}\mathbf{E}_{h_{\mu j},am}) \\
&\overset{def'n}{=} \nu\sigma\delta_{-\mu j,\nu k}(am)_{\sigma i} - \nu\mu\delta_{-\sigma i,\nu k}(am)_{\mu j};
\end{aligned}
$$

$$
\begin{aligned}
[x, (m)_{\nu k}] &\overset{def'n}{=} [x, \nu\mathbf{E}_{h_{\nu k},m}] = \nu\mathbf{E}_{x(h_{\nu k}),m} + \nu\mathbf{E}_{h_{\nu k},x(m)} \\
&\overset{def'n}{=} (x(m))_{\nu k};\ \text{and}
\end{aligned}
$$

$$
\begin{aligned}
[(m)_{\sigma i}, (n_{\mu j}] &\overset{def'n}{=} [\sigma\mathbf{E}_{h_{\sigma i},m}, \mu\mathbf{E}_{h_{\mu j},n}] \\
&= \sigma\mu\mathbf{E}_{\mathbf{E}_{h_{\sigma i},m}(h_{\mu j}),n} + \sigma\mu\mathbf{E}_{h_{\mu j},\mathbf{E}_{h_{\sigma i},m}(n)} \\
&= -\sigma\mu\delta_{-\sigma i,\mu j}\mathbf{E}_{m,n} + \sigma\mu q(m,n)\mathbf{E}_{h_{\mu j},h_{\sigma i}}
\end{aligned}
$$

for all $a \in A$, $i, j, k \in I$, $m, n \in \mathcal{M}$, $x \in \text{eosp}(q)$, $\sigma, \mu, \nu \in \{\pm\}$. ■

Remark 2.1.18 Another characterization of $\text{eosp}(q_I \oplus q)$ can be developed using the same methods as in [N1] for $\text{eo}(q_I \oplus q)$: Let I be an arbitrary set and $\mathcal{M}$ an

A-supermodule with an A-quadratic form q. Then with respect to the decomposition $H_+(I) \oplus \mathcal{M} \oplus H_-(I)$,

$$A \in \mathrm{eosp}(q_I \oplus q) \Leftrightarrow A = \begin{pmatrix} a & (-m_i^*)_{i\in I} & b \\ (n_i)_{i\in I} & x & (m_i)_{i\in I} \\ c & (-n_i^*)_{i\in I} & -a^{\mathrm{T}} \end{pmatrix}$$

where $a, b, c \in Mat(I, I; A)$, $b = -b^{\mathrm{T}}$, $c = -c^{\mathrm{T}}$, $x \in \mathrm{eosp}(q)$ and $(m_i), (n_i) \in \oplus_{i\in I}\mathcal{M}$ and T denotes the matrix transpose. Here $Mat(I, I; A)$ is the set of $I \times I$ matrices with entries in A and finitely many non-zero rows. $\square$

In later chapters it will be important to know when $\mathrm{eosp}(q)$ is perfect. i.e. when

$$\mathrm{eosp}(q) = [\mathrm{eosp}(q), \mathrm{eosp}(q)].$$

We can establish the following:

Proposition 2.1.19 *Let $\mathcal{N} = \oplus_{i\in I} An_i$ be a free A-supermodule with an almost diagonizable A-quadratic form $q_{\mathcal{N}}$ and let $\mathcal{P}$ be an A-supermodule with the quadratic form $q_{\mathcal{P}}$. Set $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ with $q = q_{\mathcal{N}} \oplus q_{\mathcal{P}}$. Then if $\{n_i \mid i \in I\}_{\bar{1}} \neq \emptyset$ or $|\{n_i \mid i \in I\}_{\bar{0}}| \geq 3$ then* $\mathrm{eosp}(q)$ *is perfect.*

Proof: By Lemma 2.1.10 we can rearrange the basis so that q is an orthogonal sum of an almost diagonizable form and a standard alternating form. Assume that $\{n_i \mid i \in I\}_{\bar{1}} \neq \emptyset$. It is easy to see that $\mathrm{eosp}(q)$ is spanned by the elements of the form $\mathbf{E}_{n_i,n_j}$, $\mathbf{E}_{m,n_i}$ and $\mathbf{E}_{m,m'}$ for $i, j \in I$ such that $n_i, n_j \in \mathcal{N}_{\bar{1}}$ and $m, m' \in \mathcal{N}_{\bar{0}} \oplus \mathcal{P}$. Hence it suffices to show that these elements of $\mathrm{eosp}(q)$ are contained in $[\mathrm{eosp}(q), \mathrm{eosp}(q)]$. Let $i \in I$ so that $n_i \in \mathcal{N}_{\bar{1}}$. Then $\mathbf{E}_{n_i,n_i} = \frac{1}{2}q(n_i, n_{\underline{i}})^{-1}[\mathbf{E}_{n_i,n_i}, \mathbf{E}_{n_{\underline{i}},n_i}]$ and $\mathbf{E}_{n_i,n_{\underline{i}}} = \frac{1}{4}q(n_i, n_{\underline{i}})^{-1}[\mathbf{E}_{n_i,n_i}, \mathbf{E}_{n_{\underline{i}},n_{\underline{i}}}]$. Moreover. for $j \in I$, $j \neq i, \underline{i}$ such that $n_j \in \mathcal{N}_{\bar{1}}$ we have $\mathbf{E}_{n_i,n_j} = q(n_j, n_{\underline{j}})^{-1}[\mathbf{E}_{n_i,n_j}, \mathbf{E}_{n_{\underline{j}},n_j}]$. Hence $\mathbf{E}_{n_i,n_j} \in [\mathrm{eosp}(q), \mathrm{eosp}(q)]$ for all $i, j \in I$ such that $n_i, n_j \in \mathcal{N}_{\bar{1}}$. If $m \in \mathcal{N}_{\bar{0}} \oplus \mathcal{P}$ then for all $i \in I$ such that $n_i \in \mathcal{N}_{\bar{1}}$ we have that $\mathbf{E}_{m,n_i} = q(n_i, n_{\underline{i}})^{-1}[\mathbf{E}_{m,n_i}, \mathbf{E}_{n_{\underline{i}},n_i}] \in [\mathrm{eosp}(q), \mathrm{eosp}(q)]$. In addition, if $m' \in \mathcal{N}_{\bar{0}} \oplus \mathcal{P}$ and $i \in I$ such that $n_i \in \mathcal{N}_{\bar{1}}$ then $\mathbf{E}_{m,m'} = q(n_i, n_{\underline{i}})^{-1}\left([\mathbf{E}_{m,n_i}, \mathbf{E}_{n_{\underline{i}},m'}] - (-1)^{|m|+|m'|}q(m, m')\mathbf{E}_{n_{\underline{i}},n_i}\right) \in [\mathrm{eosp}(q), \mathrm{eosp}(q)]$ and so the assertion follows for the case when $\{n_i \mid i \in I\}_{\bar{1}} = \emptyset$. Now suppose that $\{n_i \mid i \in I\}_{\bar{1}} = \emptyset$ and that $|\{n_i \mid i \in I\}_{\bar{0}}| \geq 3$. Note that $\mathbf{E}_{n_i,n_i} = 0$ for all $i \in I$ and $\mathrm{eosp}(q)$ is spanned by the elements of the form $\mathbf{E}_{n_i,n_j}$ and $\mathbf{E}_{an_i+p,r}$ for all $i,j \in I$, $i \neq j$, $a \in A$ and $p,r \in \mathcal{P}$. Let $i,j \in I$, $i \neq j$. We can take $k \in I$ such that $k \neq i,j$ and obtain

$$\mathbf{E}_{n_i,n_j} = q(n_k,n_k)^{-1}\left([\mathbf{E}_{n_i,n_k}, \mathbf{E}_{n_k,n_j}] - q(n_i,n_j)\mathbf{E}_{n_k,n_k}\right) \in [\mathrm{eosp}(q), \mathrm{eosp}(q)].$$

Moreover, for all $i \in I$, $p,r \in \mathcal{P}$ and $a \in A$ we can take $j \in I$, $j \neq i$ and obtain

$$\mathbf{E}_{an_i+p,r} = q(n_j,n_j)^{-1}\left([\mathbf{E}_{an_i+p,n_j}, \mathbf{E}_{n_j,r}] + q(p,r)\mathbf{E}_{n_j,n_j}\right) \in [\mathrm{eosp}(q), \mathrm{eosp}(q)].$$

Hence $[\mathrm{eosp}(q), \mathrm{eosp}(q)] = \mathrm{eosp}(q)$. ■

Corollary 2.1.20 *Let $\mathcal{M}$ be a free A-module with a quadratic form q which is invertibly diagonizable with respect to a basis $\{m_i \mid i \in I\}$ for some index set I. Then* $\mathrm{eosp}(q)$ *is perfect if and only if $|I| \neq 2$.*

Proof: $\boxed{\Rightarrow}$ Suppose $|I| = 2$. Then since the basis is even, by Proposition 2.1.7, $\mathrm{eosp}(q)$ is a 1-dimensional Lie superalgebra with even basis and so $[\mathrm{eosp}(q), \mathrm{eosp}(q)] = 0 \neq \mathrm{eosp}(q)$.

$\boxed{\Leftarrow}$ Since the basis is even, if $|I| = 1$ we have by Proposition 2.1.7 that $\mathrm{eosp}(q) = 0$ and so $\mathrm{eosp}(q)$ is perfect. If $|I| \geq 3$, the assertion follows from Proposition 2.1.19. ■

Corollary 2.1.21 *Let $\mathcal{M}$ be a free A-supermodule with an A-quadratic form q which is standard alternating. Then* $\mathrm{eosp}(q)$ *is perfect.*

Proof: It follows directly from Proposition 2.1.19. ■

2.2 Subalgebras of $\mathrm{osp}(q)$

We denote by $\mathcal{M}$ an A-supermodule with an A-quadratic form q. In this section we will study the properties of some examples of subalgebras of $\mathrm{osp}(q)$ which contain $\mathrm{eosp}(q)$.

Let $\mathcal{E}$ be a subalgebra of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$. For example, $\mathcal{E} = \mathrm{eosp}(q)$, $\mathcal{E} = \overline{\mathrm{eosp}(q)}$ or $\mathcal{E} = \mathrm{osp}(q)$. We have the following facts:

Proposition 2.2.1 *Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Then* $\mathrm{osp}(q) = \overline{\mathrm{osp}(q)}$ *and* $\overline{\mathrm{eosp}(q)}$ *is an ideal of* $\mathrm{osp}(q)$ *containing* $\mathrm{eosp}(q)$.

Proof: To show that $\mathrm{osp}(q) = \overline{\mathrm{osp}(q)}$ we only need to show that $\overline{\mathrm{osp}(q)} \subset \mathrm{osp}(q)$. Let $x = \sum_{i\in I} x_i \in \overline{\mathrm{osp}(q)}$, i.e., $\sum_{i\in I} x_i$ is summable and $x_i \in \mathrm{osp}(q)$, $i \in I$. Then for all $m, n \in \mathcal{M}$,

$$\begin{aligned}
q(x(m), n) &+ (-1)^{|x||m|} q(m, x(n)) \\
&= q\left(\left(\sum_{i\in I} x_i\right)(m), n\right) + (-1)^{|x||m|} q\left(m, \left(\sum_{i\in I} x_i\right)(n)\right) \\
&= q\left(\sum_{i\in I_m} x_i(m), n\right) + (-1)^{|x||m|} q\left(m, \sum_{I_n} x_i(n)\right) \\
&= \sum_{i\in I_m\cup I_n} (q(x_i(m), n) + (-1)^{|x||m|} q(m, x_i(n)) \\
&= 0
\end{aligned}$$

and so $x \in \mathrm{osp}(q)$. The rest of the assertion now follows from Proposition 1.5.12. ∎

Remark 2.2.2 Recall that, given two A-supermodules $\mathcal{M}_1$ and $\mathcal{M}_2$ with A-quadratic forms q_1 and q_2 respectively, we have for $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $q = q_1 \oplus q_2$,

$$\mathrm{eosp}(q) = \mathrm{eosp}(q_{\mathcal{N}}) \oplus \mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}}) \oplus \mathrm{eosp}(q_{\mathcal{P}})$$

where $\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}}) = \mathrm{span}_A\{\mathbf{E}_{n,p} \mid n \in \mathcal{N},\ p \in \mathcal{P}\}$.

Proposition 2.2.3 *Given two A-supermodules $\mathcal{M}_1$ and $\mathcal{M}_2$ with A-quadratic forms q_1 and q_2 respectively, we have for $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $q = q_1 \oplus q_2$,*

$$\overline{\mathrm{eosp}(q)} = \overline{\mathrm{eosp}(q_1)} \oplus \overline{\mathrm{eosp}(q_1, q_2)} \oplus \overline{\mathrm{eosp}(q_2)}$$

where $\mathrm{eosp}(q_1, q_2) = \mathrm{span}_A\{\mathbf{E}_{m_1,m_2} \mid m_1 \in \mathcal{M}_1,\ m_2 \in \mathcal{M}_2\}$.

Proof: Follows from Propositions 2.1.6 and 1.5.13. ∎

Proposition 2.2.4 *Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ be an A-supermodule where $\mathcal{N}$ and $\mathcal{P}$ are A-supermodules with quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{P}}$ respectively. Assume that $q_{\mathcal{N}}$ is almost diagonizable with respect to a basis $\{n_i \mid i \in I\}$ of $\mathcal{N}$. Then*

$$\mathrm{osp}(q) = \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathrm{osp}(q_{\mathcal{P}}).$$

If, in addition, $q_{\mathcal{P}} \equiv 0$ then

$$\mathrm{osp}(q) = \overline{\mathrm{eosp}(q)} \rtimes \mathrm{End}_A \mathcal{P}.$$

In particular, if $\mathcal{N}$ has finite rank and $\mathcal{P} = (0)$ then

$$\mathrm{osp}(q) = \mathrm{eosp}(q).$$

Proof: By Propositions 2.2.1 and 2.2.3, $\overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \subset \overline{\mathrm{eosp}(q)} \subset \mathrm{osp}(q)$. If $x \in \mathrm{osp}(q_{\mathcal{P}})$ we can extend x to $\mathcal{M}$ by setting $x(n+p) := x(p)$ for $n \in \mathcal{N}$, $p \in \mathcal{P}$. Then for all $n, n' \in \mathcal{N}$ and $p, p' \in \mathcal{P}$,

$$\begin{aligned} q(x(n+p), n'+p') &+ (-1)^{|n+p||n'+p'|} q(x(n'+p'), n+p) \\ &\overset{\mathcal{N}\perp\mathcal{P}}{=} q_{\mathcal{P}}(x(p), p') + (-1)^{|p||p'|} q_{\mathcal{P}}(x(p'), p) = 0 \end{aligned}$$

and so $\mathrm{osp}(q_{\mathcal{P}}) \hookrightarrow \mathrm{osp}(q)$. Conversely, let $x \in \mathrm{osp}(q)$, i.e., $x \in \mathrm{End}_A \mathcal{M}$ such that

$$q(x(m), m') + (-1)^{|m||m'|} q(x(m'), m) = 0 \quad \forall\, m, m' \in \mathcal{M}.$$

Since $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ we can write

$$x = \begin{pmatrix} x_{\mathcal{N},\mathcal{N}} & x_{\mathcal{N},\mathcal{P}} \\ x_{\mathcal{P},\mathcal{N}} & x_{\mathcal{P},\mathcal{P}} \end{pmatrix}$$

where $x_{\mathcal{N},\mathcal{P}} \in \mathrm{Hom}_A(\mathcal{N}, \mathcal{P})$, etc. For all $n \in \mathcal{N}$ and $p \in \mathcal{P}$ we have

$$0 = q(x(n), p) + (-1)^{|n||p|} q(x(p), n) \overset{\mathcal{N}\perp\mathcal{P}}{=} q(x_{\mathcal{N},\mathcal{P}}(n), p) + (-1)^{|n||p|} q(x_{\mathcal{P},\mathcal{N}}(p), n)$$

and so

$$\boxed{q(x_{\mathcal{N},\mathcal{P}}(n), p) = -(-1)^{|n||p|} q(x_{\mathcal{P},\mathcal{N}}(p), n)}. \tag{2.6}$$

Moreover, since $\mathcal{N}$ is free with the homogeneous basis $\{n_i \mid i \in I\}$ we can write

$$\begin{aligned} x_{\mathcal{N},\mathcal{N}}(n_i) &= \sum_{j\in I} a_{ij}n_j, \quad \text{for some } a_{ij} \in A \\ x_{\mathcal{P},\mathcal{N}}(p) &= \sum_{i\in I} b_i(p)n_i, \quad \text{for some } b_i \in \mathrm{Hom}_K(\mathcal{P}, K). \end{aligned}$$

Note that for each $i \in I$ and $p \in \mathcal{P}$, the sets $\{k \in I \mid a_{ik} \neq 0\}$ and $\{k \in I \mid b_k(p) \neq 0\}$ are finite. Then for all $i, k \in I$,

$$\begin{aligned} 0 &= q(x(n_i), n_k) + (-1)^{|n_i||n_k|}q(x(n_k), n_i) \\ &= q(\sum_{j\in I} a_{ij}n_j, n_k) + (-1)^{|n_i||n_k|}q(\sum_{j\in I} a_{kj}n_j, n_i) \\ &= a_{i\underline{k}}q(n_{\underline{k}}, n_k) + (-1)^{|n_i||n_k|}a_{k\underline{i}}q(n_{\underline{i}}, n_i) \\ &\Rightarrow \boxed{a_{i\underline{k}} = -(-1)^{|n_i||n_k|}a_{k\underline{i}}q(n_{\underline{i}}, n_i)q(n_{\underline{k}}, n_k)^{-1} \quad \forall i, k \in I}. \end{aligned} \tag{2.7}$$

Hence in particular, for each $i \in I$, the set $\{k \in I \mid a_{ki} \neq 0\}$ is finite. Let

$$y = \sum_{i\in I} q(n_{\underline{i}}, n_i)^{-1}(\frac{1}{2}\mathbf{E}_{x_{\mathcal{N},\mathcal{N}}(n_i),n_{\underline{i}}} + \mathbf{E}_{x_{\mathcal{N},\mathcal{P}}(n_i),n_{\underline{i}}}).$$

Then for each $k \in I$,

$$\frac{1}{2}\mathbf{E}_{x_{\mathcal{N},\mathcal{N}}(n_i),n_{\underline{i}}}(n_k) = \frac{1}{2}(x_{\mathcal{N},\mathcal{N}}(n_i)q(n_{\underline{i}}, n_k) - (-1)^{|n_i||n_k|}q(x(n_i), n_k)n_{\underline{i}})$$

which is not zero only if $i \in \{k\} \cup \{l \in I \mid a_{l\underline{k}} \neq 0\}$, which is a finite set, and

$$\mathbf{E}_{x_{\mathcal{N},\mathcal{P}}(n_i),n_{\underline{i}}}(n_k) = x_{\mathcal{N},\mathcal{P}}(n_i)q(n_{\underline{i}}, n_k)$$

which is not zero only if $i \in \{k\}$, which is a finite set. Moreover, for each $p \in \mathcal{P}$, $\frac{1}{2}\mathbf{E}_{x_{\mathcal{N},\mathcal{N}}(n_i),n_{\underline{i}}}(p) = 0$ and

$$\mathbf{E}_{x_{\mathcal{N},\mathcal{P}}(n_i),n_{\underline{i}}}(p) = -(-1)^{|n_i||p|}q(x_{\mathcal{N},\mathcal{P}}(n_i), p)n_{\underline{i}} \overset{(2.6)}{=} q(x_{\mathcal{P},\mathcal{N}}(p), n_i) = b_{\underline{i}}(p)q(n_{\underline{i}}, n_i)$$

which is not zero only if $i \in \{k \in I \mid b_k(p) \neq 0\}$ and this is a finite set. Hence $\frac{1}{2}\sum_{i\in I} q(n_{\underline{i}}, n_i)^{-1}\mathbf{E}_{x_{\mathcal{N},\mathcal{N}}(n_i),n_{\underline{i}}} \in \overline{\mathrm{eosp}(q_{\mathcal{N}})}$ and $\sum_{i\in I} q(n_{\underline{i}}, n_i)^{-1}\mathbf{E}_{x_{\mathcal{N},\mathcal{P}}(n_i),n_{\underline{i}}} \in \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})}$

and so $y \in \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})}$. Moreover, for all $k \in I$,

$$\begin{aligned}
y(n_k) &= \frac{1}{2} q(n_{\underline{k}}, n_k)^{-1} x_{\mathcal{N},\mathcal{N}}(n_k) q(n_{\underline{k}}, n_k) - \frac{1}{2} \sum_{i \in I} (-1)^{|n_i||n_k|} q(n_{\underline{i}}, n_i)^{-1} q(x_{\mathcal{N},\mathcal{N}}(n_i), n_k) n_{\underline{i}} \\
&\quad + q(n_{\underline{k}}, n_k)^{-1} x_{\mathcal{N},\mathcal{P}}(n_k) q(n_{\underline{k}}, n_k) \\
&= \frac{1}{2} x_{\mathcal{N},\mathcal{N}}(n_k) - \frac{1}{2} \sum_{i \in I} (-1)^{|n_i||n_k|} q(n_{\underline{i}}, n_i)^{-1} a_{i\underline{k}} q(n_{\underline{k}}, n_k) n_{\underline{i}} + x_{\mathcal{N},\mathcal{P}}(n_k) \\
&\overset{(2.7)}{=} \frac{1}{2} x_{\mathcal{N},\mathcal{N}}(n_k) + \frac{1}{2} \sum_{i \in I} q(n_{\underline{i}}, n_i)^{-1} a_{k\underline{i}} q(n_{\underline{i}}, n_i) q(n_{\underline{k}}, n_k)^{-1} q(n_{\underline{k}}, n_k) n_{\underline{i}} + x_{\mathcal{N},\mathcal{P}}(n_k) \\
&= \frac{1}{2} x_{\mathcal{N},\mathcal{N}}(n_k) + \frac{1}{2} \sum_{i \in I} a_{ki} n_i + x_{\mathcal{N},\mathcal{P}}(n_k) \\
&= (x_{\mathcal{N},\mathcal{N}} \oplus x_{\mathcal{N},\mathcal{P}} \oplus x_{\mathcal{P},\mathcal{N}})(n_k),
\end{aligned}$$

and for all $p \in \mathcal{P}$,

$$\begin{aligned}
y(p) &= -\sum_{i \in I} (-1)^{|n_i||p|} q(n_{\underline{i}}, n_i)^{-1} q(x_{\mathcal{N},\mathcal{P}}(n_i), p) n_{\underline{i}} \overset{(2.6)}{=} \sum_{i \in I} q(n_{\underline{i}}, n_i)^{-1} q(x_{\mathcal{P},\mathcal{N}}(p), n_i) n_{\underline{i}} \\
&= \sum_{i \in I} q(n_{\underline{i}}, n_i)^{-1} b_{\underline{i}}(p) q(n_{\underline{i}}, n_i) n_{\underline{i}} = \sum_{i \in I} b_i(p) n_i \\
&= (x_{\mathcal{N},\mathcal{N}} \oplus x_{\mathcal{N},\mathcal{P}} \oplus x_{\mathcal{P},\mathcal{N}})(p)
\end{aligned}$$

since $x_{\mathcal{P},\mathcal{N}}(\mathcal{N}) = (0)$ and $x_{\mathcal{N},\mathcal{P}}(\mathcal{P}) = (0)$. Hence $x_{\mathcal{N},\mathcal{N}} \oplus x_{\mathcal{N},\mathcal{P}} \oplus x_{\mathcal{P},\mathcal{N}} = y \in \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N},\mathcal{P}})} \subset \mathrm{osp}(q)$ and so $x = y + x_{\mathcal{P},\mathcal{P}}$. Since $x, y \in \mathrm{osp}(q)$, we have that $x_{\mathcal{P},\mathcal{P}} \in \mathrm{osp}(q_{\mathcal{P}})$. Hence

$$x = x_{\mathcal{N},\mathcal{N}} \oplus x_{\mathcal{N},\mathcal{P}} \oplus x_{\mathcal{P},\mathcal{N}} \oplus x_{\mathcal{P},\mathcal{P}} \in \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathrm{osp}(q_{\mathcal{P}}).$$

If $q_{\mathcal{P}} = (0)$ then $\mathrm{osp}(q_{\mathcal{P}}) = \mathrm{End}_A \mathcal{P}$ and $\mathrm{eosp}(q_{\mathcal{P}}) = 0$. Hence, by Proposition 2.2.3, $\overline{\mathrm{eosp}(q)} = \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})}$ and so

$$\mathrm{osp}(q) = \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathrm{End}_A \mathcal{P} = \overline{\mathrm{eosp}(q)} \oplus \mathrm{End}_A \mathcal{P}.$$

Moreover, by Proposition 2.2.1, $\overline{\mathrm{eosp}(q)}$ is an ideal of $\mathrm{osp}(q)$ and clearly $\mathrm{End}_A \mathcal{P}$ is a subalgebra of $\mathrm{osp}(q)$. Hence $\mathrm{osp}(q) = \overline{\mathrm{eosp}(q)} \rtimes \mathrm{End}_A \mathcal{P}$. If $\mathcal{N}$ is of finite rank then by Proposition 1.5.7, $\mathrm{eosp}(q_{\mathcal{N}}) = \overline{\mathrm{eosp}(q_{\mathcal{N}})}$ and so if $\mathcal{P} = (0)$ then $\mathrm{osp}(q) = \mathrm{eosp}(q)$.∎

Another example of a subalgebra of $\mathrm{osp}(q)$ which contains $\mathrm{eosp}(q)$ is the following:

Example 2.2.5 Let $\mathcal{M} = H(I, A) \oplus Ah_0 \oplus \tilde{H}(J, A)$ or $\mathcal{M} = H(I, A) \oplus \tilde{H}(J, A)$ where $I \subset \mathbb{Z}^+$ and $J \subset (\mathbb{Z} + \frac{1}{2})^+$. We assume that $I \cup J$ is well-ordered using the standard well-ordering of $\frac{1}{2}\mathbb{Z}$. Let q be the quadratic form $q = (q_{\mathbb{Z}} \oplus q_0 \oplus \tilde{q}_{\mathbb{Z}+\frac{1}{2}})|_{\mathcal{M}\times\mathcal{M}}$ where $q(h_0, h_0) = 1$. Since q is almost-diagonizable, by Proposition 2.2.4, $\operatorname{osp}(q) = \overline{\operatorname{eosp}(q)}$. Since $\mathcal{M}$ is free we can write the elements of $\operatorname{osp}(q)$ in the form of matrices. With respect to the natural ordering of $\frac{1}{2}\mathbb{Z}$, we index the matrices as follows:

$$\begin{array}{c c} & \begin{array}{ccc} -\frac{1}{2}\mathbb{Z}^+ & 0 & \frac{1}{2}\mathbb{Z}^+ \end{array} \\ \begin{array}{c} -\frac{1}{2}\mathbb{Z}^+ \\ 0 \\ \frac{1}{2}\mathbb{Z}^+ \end{array} & \left(\begin{array}{ccc} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{array} \right) \end{array}.$$

Let $\operatorname{osp}_{\mathrm{fd}}(q)$ be the set of all $(a_{ij}) \in \operatorname{osp}(q)$, $i, j \in I \cup J \cup \{-I\} \cup \{-J\} (\cup \{0\})$ such that (a_{ij}) has only finitely many non-zero diagonals. Note that (a_{ij}) has only finitely many non-zero diagonals if and only if there exists $N \in I \cup J$ such that for all $n \in I \cup J \cup \{-I\} \cup \{-J\} (\cup \{0\})$, $|n| \geq N$ the n^{th} diagonal of (a_{ij}) is zero, i.e., $a_{i,n+i} = 0$ for all $i \in I \cup J \cup \{-I\} \cup \{-J\} (\cup \{0\})$. It is easy to verify that $\operatorname{osp}_{\mathrm{fd}}(q)$ contains $\operatorname{eosp}(q)$. Indeed, let $\{v_i \mid i \in I \cup J \cup \{-I\} \cup \{-J\} (\cup \{0\})\}$ where $v_{\sigma i} = h_{\sigma i}$, $v_{\sigma j} = \tilde{h}_{\sigma j}$ and $v_0 = h_0$ for $i \in I$ and $j \in J$, denote the standard basis of $\mathcal{M}$. The map $\mathbf{E}_{v_k, v_l}$ for $i, j \in I \cup J \cup \{-I\} \cup \{-J\} (\cup \{0\})$ is represented by the matrix (a_{ij}) where $a_{ij} = 0$ for $(i, j) \neq (k, -l), (l, -k)$, $a_{k,-l} = q(v_l, v_{-l})$ and $a_{l,-k} = -(-1)^{|v_l||v_k|} q(v_k, v_{-k})$. This matrix clearly has finitely many diagonals (in fact, at most two) and so is in $\operatorname{osp}_{\mathrm{fd}}(q)$. We need the following result for the description of the derivations of $\operatorname{osp}_{\mathrm{fd}}(q)$.

Proposition 2.2.6 *Let $\mathcal{M} = H(I, A) \oplus Ah_0 \oplus \tilde{H}(J, A)$ or $\mathcal{M} = H(I, A) \oplus \tilde{H}(J, A)$ where $I \subset \mathbb{Z}$ and $J \subset \mathbb{Z} + \frac{1}{2}$ and assume that elements of I and J are invertible in A. Let q be the quadratic form $q = (q_{\mathbb{Z}} \oplus q_0 \oplus \tilde{q}_{\mathbb{Z}+\frac{1}{2}})|_{\mathcal{M}}$ where $q(h_0, h_0) = 1$. Then the normalizer of $\operatorname{osp}_{\mathrm{fd}}(q)$ in $\operatorname{osp}(q)$ is $\operatorname{osp}_{\mathrm{fd}}(q)$, i.e.,*

$$\{x \in \operatorname{osp}(q) \mid [x, \operatorname{osp}_{\mathrm{fd}}(q)] \subset \operatorname{osp}_{\mathrm{fd}}(q)\} = \operatorname{osp}_{\mathrm{fd}}(q).$$

Proof: Since $\operatorname{osp}_{\mathrm{fd}}(q)$ is a Lie superalgebra, the $\boxed{\supset}$ inclusion follows.

$\boxed{\subset}$ Let $\{v_i \mid i \in I \cup J \cup \{-I\} \cup \{-J\} (\cup \{0\})\}$ denote the standard basis of $\mathcal{M}$ (see above). Let $x = \sum_{k\in I\cup J\,(\cup\{0\})} \mathbf{E}_{v_k,v_{-k}}$. Then it is easy to verify that x is eosp(q)-summable and so $x \in$ osp(q). Its matrix representation is given by (b_{ij}) where $b_{k,k} = k$, $k \neq 0$ and $b_{ij} = 0$, $i \neq j$. It is clearly in osp$_{\text{fd}}(q)$. Let $(a_{ij}) \in$ osp(q). Since $\mathcal{M}$ is almost diagonizable, from the proof of Proposition 2.2.4 we have that $a_{i,j} = -a_{-j,-i}q(v_i, v_{-i})q(v_j, v_{-j})$ for all $i, j \in I \cup J \cup \{-I\} \cup \{-J\}\,(\cup\{0\})$. Then if the matrix (a_{ij}) is in the normalizer of osp$_{\text{fd}}(q)$, i.e., $[(a_{ij}), \text{osp}_{\text{fd}}(q)] \subset \text{osp}_{\text{fd}}\, q$, we have in particular that

$$(c_{ij}) = [(a_{ij}), (b_{ij})] \in \text{osp}_{\text{fd}}(q).$$

By definition, since osp(q) is a Lie superalgebra, this holds if and only if (c_{ij}) has only finitely many non-zero diagonals. Consider the n^{th} diagonal of (c_{ij}), i.e., $(c_{i,n+i})_{i\in I\cup J}$:

$$\begin{aligned} c_{i,n+i} &= a_{i,n+i}b_{n+i,n+i} - b_{i,i}a_{i,n+i} \\ &= a_{i,n+i}(\underbrace{b_{n+i,n+i} - b_{i,i}}_{=0 \Leftrightarrow n=0}) \end{aligned}$$

Since $(c_{ij}) \in \text{osp}_{\text{fd}}(q)$, there exists $N \in I \cup J$ $((c_{ij}) \neq 0)$ such that for each $n \in I \cup J$, $|n| \geq N$, the n^{th} diagonal of (c_{ij}) is 0, i.e., $c_{i,n+i} = 0$ for all $i \in I \cup J$. This occurs if and only if for all $n \in I \cup J$, $|n| \geq N > 0$, $a_{i,n+i} = 0$ for all $i \in I \cup J$, i.e., $(a_{ij}) \in \text{osp}_{\text{fd}}(q)$. ∎

Remark 2.2.7 Note that if $A = K = \mathbb{C}$ and if $I = \mathbb{Z}$ and $J = \emptyset$ then osp$_{\text{fd}}(q)$ is isomorphic to $\bar{b}_{\infty}^{\pm[0]}$, a Lie algebra studied by V.G. Kac and others (see for example [KP] and [DJKM]). In addition, if $A = K = \mathbb{C}$ and $I = \mathbb{Z}, J = \mathbb{Z} + \frac{1}{2}$ then osp$_{\text{fd}}(q)$ is isomorphic to $\mathcal{B}$, a Lie superalgebra which was studied recently by S.J. Cheng and W. Wang in [CW].

2.3 The Orthosymplectic Lie Superalgebra $\mathcal{E}_\infty$

Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q.

Definition 2.3.1 Given a subalgebra $\mathcal{E}$ of osp(q) which contains eosp(q), we define the algebra $\mathcal{E}_\infty$ as follows:

$$\mathcal{E}_\infty = \{(a, m, n, x) \mid a \in A,\ m, n \in \mathcal{M},\ x \in \mathcal{E}\}$$

with multiplication

$$\begin{aligned}
[(a,m,n,x),(a',m',n',x')] &= (q(m,n')-q(n,m'), \\
&\quad am'+xm'-ma'-(-1)^{|x'||m|}x'm, \\
&\quad -an'+xn'+na'-(-1)^{|x'||n|}x'n, \\
&\quad \mathbf{E}_{m,n'}+\mathbf{E}_{n,m'}+[x,x'])
\end{aligned} \tag{2.8}$$

for all $a,a' \in \mathcal{A}$, $m,m',n,n' \in \mathcal{M}$, $x,x' \in \mathcal{E}$.

Recall that the centre $Z(L)$ of a Lie superalgebra L is defined by

$$Z(L) = \{x \in L \mid [x,L] = 0\}.$$

In addition, for a 3-graded Lie superalgebra L, we have defined

$$C(L) = \{x \in L_0 \mid [x,L_\pm] = 0\}.$$

Proposition 2.3.2 *Let $\mathcal{E}$ be a subalgebra of* $\mathrm{osp}(q)$ *which contains* $\mathrm{eosp}(q)$*. Then $L = \mathcal{E}_\infty$ is a toral 3-graded Lie superalgebra. Moreover,* $Z(L) = C(L) = (\mathrm{Ann}_A \mathcal{M}, 0, 0, 0)$.

Proof: If we set

$$L_\alpha = \{(a,m,n,x) \mid a \in \mathcal{A}_\alpha,\ m,n \in \mathcal{M}_\alpha,\ x \in \mathcal{E}_\alpha\}$$

then it is clear that L is $\mathbb{Z}_2$-graded $\mathcal{A}$-supermodule. To show that L is a Lie superalgebra we need to verify that (2.8) satisfies (SL1) and (SL2). It is clear that (SL1) holds. To show that (SL2) holds we define the map

$$\begin{aligned}
h : L \times L \times L &\to L \\
(l_1,l_2,l_3) &\mapsto [[l_1,l_2],l_3]-[l_1,[l_2,l_3]]+(-1)^{|l_1||l_2|}[l_2,[l_1,l_3]].
\end{aligned}$$

Then h is K-trilinear and for all homogeneous $l_1,l_2,l_3 \in L$ and π in the symmetric group $\mathcal{S}_3$ we have

$$h(l_1,l_2,l_3) = \sigma h(\pi(l_1),\pi(l_2),\pi(l_3))$$

where $\sigma \in \{\pm\}$ depends on π and the degrees of l_1, l_2 and l_3. Indeed

$$\begin{aligned} h(l_2, l_1, l_3) &= [[l_2, l_1], l_3] - [l_2, [l_1, l_3]] + (-1)^{|l_2||l_1|}[l_1, [l_2, l_3]] \\ &= -(-1)^{|l_1||l_2|}\left([[l_1, l_2], l_3] + (-1)^{|l_1||l_2|}[l_2, [l_1, l_3]] - [l_1, [l_2, l_3]]\right) \\ &= -(-1)^{|l_1||l_2|}h(l_1, l_2, l_3) \end{aligned}$$

and

$$\begin{aligned} h(l_1, l_3, l_2) &= [[l_1, l_3], l_2] - [l_1, [l_3, l_2]] + (-1)^{|l_1||l_3|}[l_2, [l_3, l_1] \\ &= -(-1)^{|l_2||l_3|}\left((-1)^{|l_1||l_2|}[l_2, [l_1, l_3] - [l_1, [l_2, l_3]] + [[l_1, l_2], l_3]\right) \\ &= -(-1)^{|l_2||l_3|}h(l_1, l_2, l_3) \end{aligned}$$

for all $l_1, l_2, l_3 \in L$. Since $\mathcal{S}_3$ is generated by (12) and (23), this implies the assertion.

If we set

$$\begin{aligned} L_0 &= \{(a, 0, 0, x) \mid a \in \mathcal{A},\ x \in \mathcal{E}\} \\ L_+ &= \{(0, m, 0, 0) \mid m \in \mathcal{M}\} \\ L_- &= \{(0, 0, m, 0) \mid m \in \mathcal{M}\} \end{aligned}$$

we have

$$\begin{aligned} h(L_\sigma, L_\sigma, L_\sigma) &= (0) \text{ since } [L_\sigma, L_\sigma] = (0) \text{ and} \\ h(L_\sigma, L_\sigma, L_0) &= (0) \text{ since } [L_\sigma, L_\sigma] = (0) \text{ and } [L_\sigma, L_0] = L_\sigma \end{aligned}$$

In addition, $h(L_\sigma, L_\sigma, L_{-\sigma}) = (0)$ since

$$\begin{aligned} h(m^\sigma, n^\sigma, p^{-\sigma}) = &- [m^\sigma q(n, p) + \mathbf{E}_{n,p}] + (-1)^{|m||n|}[n^\sigma, \sigma q(m, p) + \mathbf{E}_{m,p}] \\ &= mq(n, p) + (-1)^{|n|+|p|)|m|}(nq(p, m) - (-1)^{|n||p|}pq(n, m)) \\ &\quad + (-1)^{|m||n|}(-nq(m, p) - (-1)^{(|m|+|p|)|n|}(mq(p, n) - (-1)^{|m||n|}pq(n, m)) \\ &= 0 \end{aligned}$$

for all $m, n, p \in \mathcal{M}$ and $\sigma = \pm$. Moreover, $h(L_\sigma, L_0, L_{-\sigma}) = (0)$ since

$$\begin{aligned}
h(m^\sigma, a + x, n^{-\sigma}) &= [-\sigma m^\sigma a - (-1)^{|m||x|}x(m)^\sigma, n^{-\sigma}] - [m^\sigma, -\sigma a n^{-\sigma} + x(n)^{-\sigma}] \\
&\quad + (-1)^{|m||a+x|}[a + x, \sigma q(m,n) + \mathbf{E}_{m,n}] \\
&= -q(ma, n) - \sigma(-1)^{|m||x|}q(x(m), n) - \sigma\mathbf{E}_{ma,n} - (-1)^{|m||x|}\mathbf{E}_{x(m),n} \\
&\quad + q(m, an) - \sigma q(m, x(n)) + \sigma\mathbf{E}_{m,an} - \mathbf{E}_{m,x(n)} \\
&\quad + (-1)^{|m||x|}\mathbf{E}_{x(m),n} - (-1)^{|m|(|n|+|x|)}\mathbf{E}_{x(n),m)} \\
&= 0
\end{aligned}$$

for all $m, n \in \mathcal{M}$, $a \in \mathcal{A}$, $x \in \mathcal{E}$ and $\sigma = \pm$ and $h(L_\sigma, L_0, L_0) = (0)$ since

$$\begin{aligned}
h(m^\sigma, a + x, b + y) &= [-\sigma m^\sigma a - (-1)^{|m||x|}x(m)^\sigma, b + y] - [m^\sigma, [x, y]] \\
&\quad + (-1)^{|m||a+x|}[a + x, -\sigma m^\sigma b - (-1)^{|m||y|}y(m)^\sigma] \\
&= m^\sigma ab + \sigma(-1)^{|m||x|}x(m)^\sigma b + \sigma(-1)^{(|m|+|a|)|y|}y(ma)^\sigma \\
&\quad + (-1)^{|m||x|+|y|(|x|+|m|)}y(x(m))^\sigma + (-1)^{(|x|+|y|)|m|}[x, y](m)^\sigma \\
&\quad + (-1)^{|m||a+x|}(-am^\sigma b - \sigma(-1)^{|m||y|}ay(m)^\sigma \\
&\quad - \sigma x(mb)^\sigma - (-1)^{|m||y|}x(y(m))^\sigma \\
&= 0
\end{aligned}$$

for all $a, b \in \mathcal{A}$, $x, y \in \mathcal{E}$, $m \in \mathcal{M}$ and $\sigma = \pm$. Finally, $h(L_0, L_0, L_0) = (0)$ since

$$h(a + x, b + y, c + z) = [[x, y], z] - [x, [y, z]] + (-1)^{|x||y|}[y, [x, z]] = 0.$$

It now follows from the symmetry and the K-trilinearity of h that (2.8) satisfies (SL2). Hence L is a Lie superalgebra and it is clear from (2.8) that L is 3-graded and that $(1, 0, 0, 0)$ is the toral element in L. Now, suppose that $(a_0, m_0, n_0, x_0) \in Z(L)$. Then for all $m \in \mathcal{M}$,

$$0 = [(1, 0, 0, 0), (a_0, m_0, n_0, x_0)] = (0, m_0, -n_0, 0)$$

and so $m_0 = 0 = n_0$. Also, for all $m \in \mathcal{M}$,

$$0 = [(a_0, 0, 0, x_0), (0, m, 0, 0)] = (0, a_0 m + x_0 m, 0, 0)$$

and

$$0 = [(a_0, 0, 0, x_0), (0, 0, m, 0)] = (0, 0, -a_0 m + x_0 m, 0).$$

Hence $a_0 m + x_0 m = 0$ and $-a_0 m + x_0 m = 0$ for all $m \in \mathcal{M}$ and so since $\frac{1}{2} \in A$, $a_0 m = 0$ and $x_0 m = 0$ for all $m \in \mathcal{M}$, i.e., $x_0 = 0$ and $a_0 \in \operatorname{Ann}_A \mathcal{M}$. Coversely, let $a_0 \in \operatorname{Ann}_A \mathcal{M}$. Then for all $(a, m, n, x) \in L$,

$$[(a_0, 0, 0, 0), (a, m, n, x)] = (0, a_0 m, -a_0 n, 0) = 0.$$

Let $a + x \in C(L)$, $a \in A$, $x \in \mathcal{E}$. Then for all $m \in \mathcal{M}$, $\sigma = \pm$,

$$0 = [a + x, m^\sigma] = (\sigma a m + x(m))^\sigma$$

and so $am = 0$ for all $m \in \mathcal{M}$, i.e., $a \in \operatorname{Ann}_A \mathcal{M}$, and $x(m) = 0$ for all $m \in \mathcal{M}$, i.e., $x = 0$. Hence $C(L) \subset (\operatorname{Ann}_A \mathcal{M}, 0, 0, 0)$. The rest is trivial. ∎

Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Let $q_{\{\infty\}}$ be the quadratic form of the hyperbolic superplane $H(\{\infty\}) = Ah_{\{\infty\}} \dotplus Ah_{\{-\infty\}}$. Set

$$q_\infty = q \oplus q_{\{\infty\}}$$

so q_∞ is defined on $\mathcal{M} \oplus H(\{\infty\})$. Let $\mathcal{E}(q_\infty)$ be a subalgebra of $\operatorname{osp}(q_\infty)$ containing $\operatorname{eosp}(q_\infty)$. Set $\mathcal{E}(q)$ to be the Lie superalgebra

$$\mathcal{E}(q) = \{x|_{\mathcal{M}} \mid x \in \mathcal{E}(q_\infty) \text{ such that } x|_{\mathcal{M}} \in \operatorname{End}_A \mathcal{M},\ x(H(\{\infty\})) = (0)\}.$$

Using the Proposition 2.1.6 we have that $\operatorname{eosp}(q_\infty) = \operatorname{eosp}(q) \dotplus \operatorname{eosp}(q, q_{\{\infty\}}) \dotplus \operatorname{eosp}(q_{\{\infty\}})$ and so since $\operatorname{eosp}(q_\infty) \subset \mathcal{E}(q_\infty)$ we have that $\operatorname{eosp}(q) \subset \mathcal{E}(q)$. Moreover, since $\mathcal{E}(q_\infty) \subset \operatorname{osp}(q_\infty)$, for each $x \in \mathcal{E}(q)$ we have, for all $m, n \in \mathcal{M}$,

$$0 = q_\infty(x(m), n) + (-1)^{|m||n|} q_\infty(x(n), m) = q(x(m), n) + (-1)^{|m||n|} q(x(n), m)$$

and so $x \in \operatorname{osp}(q)$. Hence $\mathcal{E}(q) \subset \operatorname{osp}(q)$ and so $\mathcal{E}(q)$ is a subalgebra of $\operatorname{osp}(q)$ containing $\operatorname{eosp}(q)$.

Remark 2.3.3 It is easy to verify, using Propositions 2.1.6, 2.2.3 and 2.2.4, that if $\mathcal{E}(q_\infty)$ is $\mathrm{eosp}(q_\infty)$, $\overline{\mathrm{eosp}(q_\infty)}$ or $\mathrm{osp}(q_\infty)$ then $\mathcal{E}(q)$ is $\mathrm{eosp}(q)$, $\overline{\mathrm{eosp}(q)}$ or $\mathrm{osp}(q)$ respectively.

Proposition 2.3.4 *In the above setting,*

$$\mathcal{E}(q_\infty) \cong \mathcal{E}(q)_\infty.$$

Proof: Since $\mathrm{osp}(q) \overset{\iota}{\hookrightarrow} \mathrm{osp}(q_\infty)$ where $\iota \circ y|_{\mathcal{M}} = y$ and $\iota \circ y|_{H(\{\infty\})} = 0$, we have that $\mathcal{E}(q) \overset{\iota}{\hookrightarrow} \mathcal{E}(q_\infty)$. We will simply write y instead of $\iota \circ y$ when the context is clear.
We claim that

$$\mathcal{E}(q_\infty) = \mathcal{A}\mathbf{E}_{h_\infty,h_{-\infty}} \oplus \{\mathbf{E}_{m,h_\infty} \mid m \in \mathcal{M}\} \oplus \{\mathbf{E}_{m,h_{-\infty}} \mid m \in \mathcal{M}\} \oplus \mathcal{E}(q). \tag{2.9}$$

Indeed, the inclusion " $\supset$ " clearly holds since $\mathrm{eosp}(q_\infty) \subset \mathcal{E}(q_\infty)$ and, by above, $\mathcal{E}(q) \hookrightarrow \mathcal{E}(q_\infty)$. Let $x \in \mathcal{E}(q_\infty)$. Then there exist $a, a', b, b' \in \mathcal{A}$ and $m_0, n_0 \in \mathcal{M}$ such that

$$\begin{aligned} x(h_\infty) &= m_0 \oplus a h_\infty \oplus b h_{-\infty}, \text{ and} \\ x(h_{-\infty}) &= n_0 \oplus a' h_\infty \oplus b' h_{-\infty}. \end{aligned}$$

Since $x \in \mathrm{osp}(q_\infty)$ and $|h_{\pm\infty}| = \bar{0}$,

$$\begin{aligned} 0 &= q(x(h_\infty), h_{-\infty}) + q(x(h_{-\infty}), h_\infty) = a + b', \\ 0 &= q(x(h_\infty), h_\infty) = b, \text{ and} \\ 0 &= q(x(h_{-\infty}), h_{-\infty}) = a'. \end{aligned}$$

Set

$$y = x - \mathbf{E}_{m_0,h_{-\infty}} - \mathbf{E}_{n_0,h_\infty} - a\mathbf{E}_{h_\infty,h_{-\infty}}.$$

Then $y \in \mathcal{E}(q_\infty)$ and $y(H(\{\infty\})) = (0)$. Suppose that

$$y(m) = y'(m) \oplus c(m) h_\infty \oplus d(m) h_{-\infty}$$

for some $y' \in \operatorname{End}_A \mathcal{M}$, $c, d \in \operatorname{Hom}_A(\mathcal{M}, A)$ and for all $m \in \mathcal{M}$. Then since $y \in \mathcal{E}(q_\infty) \subset \operatorname{osp}(q_\infty)$, we have that for all $m \in \mathcal{M}$,

$$\begin{aligned} 0 &= q(y(m), h_\infty) + q(y(h_\infty), m) = d(m); \text{ and} \\ 0 &= q(y(m), h_{-\infty}) + q(y(h_{-\infty}), m) = c(m). \end{aligned}$$

and so $y|_{\mathcal{M}} = y' \in \operatorname{End}_A \mathcal{M}$. Hence $y|_{\mathcal{M}} \in \mathcal{E}(q)$ and therefore

$$\mathcal{E}(q_\infty) = A\mathbf{E}_{h_\infty, h_{-\infty}} + \{\mathbf{E}_{m,h_\infty} \mid m \in \mathcal{M}\} + \{\mathbf{E}_{m,h_{-\infty}} \mid m \in \mathcal{M}\} + \mathcal{E}(q).$$

If $y = a\mathbf{E}_{h_\infty, h_{-\infty}} + \mathbf{E}_{m,h_\infty} + \mathbf{E}_{n,h_{-\infty}} + x \in \mathcal{E}(q_\infty)$ where $a \in A$, $m, n \in \mathcal{M}$ and $x \in \mathcal{E}(q)$ and, if $y = 0$ then

$$\begin{aligned} 0 = y(h_\infty) &= ah_\infty \oplus n, \text{ and} \\ 0 = y(h_{-\infty}) &= -ah_{-\infty} \oplus m \end{aligned}$$

hence $a = 0$ and $\mathbf{E}_{m,h_\infty} = \mathbf{E}_{n,h_{-\infty}} = 0$, and so $x = 0$. Therefore the claim (2.9) is true. Let $\varphi : \mathcal{E}(q_\infty) \to \mathcal{E}(q)_\infty$ be a map defined by $\varphi(a\mathbf{E}_{h_\infty, h_{-\infty}} \oplus \mathbf{E}_{m,h_\infty} \oplus \mathbf{E}_{n,h_{-\infty}} \oplus x) = (a, -m, n, x)$ for all $a \in A$, $m, n \in \mathcal{M}$ and $x \in \mathcal{E}(q)$. Note that $a\mathbf{E}_{h_\infty, h_{-\infty}} = 0$ if and only if $a = 0$, and $\mathbf{E}_{m,h_{\pm\infty}} = 0$ if and only if $m = 0$. Hence φ is well-defined and injective. It is clearly surjective and A-linear, and so it is an A-supermodule homomorphism. It remains to show that φ is a Lie superalgebra isomorphism. Indeed, if $y = a\mathbf{E}_{h_\infty, h_{-\infty}} + \mathbf{E}_{m,h_\infty} + \mathbf{E}_{n,h_{-\infty}} + x$ and $y' = a'\mathbf{E}_{h_\infty, h_{-\infty}} + \mathbf{E}_{m',h_\infty} + \mathbf{E}_{n',h_{-\infty}} + x'$

are elements of $\mathcal{E}(q_\infty)$ then

$$
\begin{aligned}
\varphi([y,y']) &= \varphi([a\mathbf{E}_{h_\infty,h_{-\infty}}, \mathbf{E}_{m',h_\infty} + \mathbf{E}_{n',h_{-\infty}}]) + \varphi([\mathbf{E}_{m,h_\infty} + \mathbf{E}_{n,h_{-\infty}}, a'\mathbf{E}_{h_\infty,h_{-\infty}}]) \\
&\quad + \varphi([\mathbf{E}_{m,h_\infty} + \mathbf{E}_{n,h_{-\infty}}, x']) + \varphi([x, \mathbf{E}_{m',h_\infty} + \mathbf{E}_{n',h_{-\infty}}]) \\
&\quad + \varphi([\mathbf{E}_{m,h_\infty}, \mathbf{E}_{n',h_{-\infty}}]) + \varphi([x,x']) \\
&= \varphi(a\mathbf{E}_{m',h_\infty} - a\mathbf{E}_{n',h_{-\infty}}) + \varphi(-\mathbf{E}_{ma',h_\infty} + \mathbf{E}_{na',h_{-\infty}}) \\
&\quad + \varphi(-q(m,n')\mathbf{E}_{h_\infty,h_{-\infty}} - \mathbf{E}_{m,n'} - (-1)^{|x'||m|}\mathbf{E}_{x'(m),h_\infty}) \\
&\quad + \varphi(q(n,m')\mathbf{E}_{h_\infty,h_{-\infty}} - \mathbf{E}_{n,m'} - (-1)^{|x'||n|}\mathbf{E}_{x'(n),h_{-\infty}}) \\
&\quad + \varphi(\mathbf{E}_{x(m'),h_\infty} + \mathbf{E}_{x(n'),h_{-\infty}}) + \varphi([x,x']) \\
&= (-q(m,n') + q(n,m'), -am' + m'a - x(m) + (-1)^{|x'||m|}x'(m), \\
&\qquad -an' + na' + x(n') - (-1)^{|x'||n|}x'(n), -\mathbf{E}_{m,n'} - \mathbf{E}_{n,m'} + [x,x']) \\
&= [(a,-m,n,x),(a',-m',n',x')] \\
&= [\varphi(y),\varphi(y')].
\end{aligned}
$$

Hence φ is a Lie superalgebra isomorphism. ∎

Corollary 2.3.5 *We have that* $\operatorname{eosp}(q_\infty) \cong \operatorname{eosp}(q)_\infty$, $\overline{\operatorname{eosp}(q_\infty)} \cong \overline{\operatorname{eosp}(q)_\infty}$ *and* $\operatorname{osp}(q_\infty) \cong \operatorname{osp}(q)_\infty$.

We will use the isomorphism of Proposition 2.3.4 throughout the thesis.

We wish to investigate now when $\operatorname{eosp}(q_\infty)$ is perfect. With respect to that we have the following:

Lemma 2.3.6 *Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q and let $\mathcal{E}$ be a subalgebra of* $\operatorname{osp}(q)$ *containing* $\operatorname{eosp}(q)$*. Set $L = \mathcal{E}_\infty$. Then*

(1) $[L_+, L_-] = \operatorname{span}\{q(m,n) \mid m,n \in \mathcal{M}\} \oplus \operatorname{eosp}(q) \subset L_0 = A \oplus \mathcal{E}$*;*

(2) $[L_0, L_0] \subset [\mathcal{E},\mathcal{E}]$*. In particular, if* $\mathcal{E} = \operatorname{eosp}(q)$ *then* $[L_0, L_0] = [L_+, L_-]$*;*

(3) L is perfect if and only if $\mathcal{E} = \operatorname{eosp}(q) + [\mathcal{E},\mathcal{E}]$ *and there exist $m_i, n_i \in \mathcal{M}$ such that $\sum_i q(m_i,n_i) = 1$, i.e., q represents 1. In particular, if* $\mathcal{E} = \operatorname{eosp}(q)$ *then $\mathcal{E}$ is perfect if and only if q represents 1.*

Proof: (1) Let $m, n \in \mathcal{M}$. Then

$$\begin{aligned}\frac{1}{2}[m^+, n^-] - (-1)^{|m||n|}\frac{1}{2}[n^+, m^-] &= \frac{1}{2}q(m,n) - (-1)^{|m||n|}\frac{1}{2}q(n,m) \\ &\quad + \frac{1}{2}E_{m,n} - (-1)^{|m||n|}\frac{1}{2}E_{n,m} \\ &= \frac{1}{2}q(m,n) - \frac{1}{2}q(m,n) + \frac{1}{2}E_{m,n} + \frac{1}{2}E_{m,n} \\ &= E_{m,n}.\end{aligned}$$

Hence $E_{m,n} \in [L^+, L^-]$ for all $m, n \in \mathcal{M}$ and so $\mathrm{eosp}(q) \subset [L^+, L^-]$. Using (2.8) we have, for all $m, n \in \mathcal{M}$,

$$q(m, n) = [m^+, n^-] - E_{m,n}$$

and so $\mathrm{span}\{q(m,n) \mid m, n \in \mathcal{M}\} \in [L_+, L_-]$. Also, using (2.8) it is clear that $[L_+, L_-] \subset \langle q(m,n) \mid m, n \in \mathcal{M}\rangle \oplus \mathrm{eosp}(q)$, hence

$$[L_+, L_-] = \mathrm{span}\{q(m,n) \mid m, n \in \mathcal{M}\} \oplus \mathrm{eosp}(q) \subset L_0 = A \oplus \mathcal{E}.$$

(2) Let $a, b \in A$ and $x, y \in \mathcal{E}$. Then by (2.8),

$$[a + x, b + y] = [x, y] \in [\mathcal{E}, \mathcal{E}].$$

In particular, if $\mathcal{E} = \mathrm{eosp}(q)$ then, by (1), $[L_0, L_0] \subset [\mathcal{E}, \mathcal{E}] \subset [L_+, L_-]$.
(3) Note that $L_\sigma = [(1,0,0,0), L_\sigma]$ for $\sigma = \pm$ and so $L_\sigma \subset [L, L]$. Hence L is perfect if and only if $L_0 \subset [L, L]$. By (2),

$$L_0 \subset [L, L] \cap L_0 = [L_0, L_0] + [L_+, L_-] \subset [\mathcal{E}, \mathcal{E}] + [L_+, L_-] \subset L_0$$

and so $L_0 = [L_+, L_-] + [\mathcal{E}, \mathcal{E}]$. Hence, by (1),

$$L_0 = \mathrm{span}\{q(m,n) \mid m, n \in \mathcal{M}\} \oplus (\mathrm{eosp}(q) + [\mathcal{E}, \mathcal{E}]).$$

It now follows that L is perfect if and only if $\mathcal{E} = \mathrm{eosp}(q) + [\mathcal{E}, \mathcal{E}]$ and $\mathrm{span}\{q(m,n) \mid m, n \in \mathcal{M}\} = A$, i.e., there exist $m_i, n_i \in \mathcal{M}$ such that $\sum_i q(m_i, n_i) = 1$. If $\mathcal{E} = \mathrm{eosp}(q)$ then $\mathrm{eosp}(q) + [\mathcal{E}, \mathcal{E}] = \mathrm{eosp}(q)$ and so the rest of the assertion follows. ∎

We shall now look at when $\mathrm{eosp}(q_\infty)$ is simple.

Lemma 2.3.7 *Let $\mathcal{M}$ be an A-supermodule and let q be a degenerate A-quadratic form on $\mathcal{M}$. Then* $\mathrm{eosp}(q_\infty)$ *is not simple.*

Proof: Let $n \in \mathcal{M}$, $n \neq 0$ such that $q(n, \mathcal{M}) = 0$. Then it is easy to check that the submodule $(0, An, An, \mathbf{E}_{n,\mathcal{M}})$ is a proper ideal of $\mathrm{eosp}(q_\infty)$. ∎

Proposition 2.3.8 *Let A be a field (so $A_{\bar{1}} = (0)$), $\mathcal{N} = \oplus_{i \in N} An_i$ with an invertibly diagonizable A-quadratic form $q_{\mathcal{N}}$ and $\mathcal{R} = \tilde{H}(R; A)$ with the standard alternating form $\tilde{q}_{\mathcal{R}}$. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$ and set $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$. Then* $\mathrm{eosp}(q_\infty)$ *is ($\mathbb{Z}_2$-graded-)simple if $R \neq \emptyset$ or $|N| \geq 3$.*

Proof: Let $L = \mathrm{eosp}(q_\infty)$. Let I be a $\mathbb{Z}_2$-graded ideal of L, hence $I = (I \cap L_{\bar{0}}) \oplus (I \cap L_{\bar{1}})$. First note that if $1 \in I$ then $I = L$. Indeed, suppose first that $R \neq \emptyset$. Let $0 \neq (a, m_+, m_-, x) \in I \cap L_{\bar{1}}$. Then $a = 0$ (since $A_{\bar{1}} = (0)$), $m_+, m_- \in \mathcal{R}$ and $x \in \mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{R}})$. Suppose $m_\mu \neq 0$ for some $\mu \in \{\pm\}$. Then there exists $k \in R$ and $\sigma \in \{\pm\}$ such that $q(m_\mu, \tilde{h}_{\sigma k}) \neq 0$. Hence

$$\begin{aligned}
&[[[[(1, (0, m_+, m_-, x)], \tilde{h}^{-\mu}_{\sigma k}], \tilde{h}^{\mu}_{\sigma k}], \tilde{h}^{-\mu}_{-\sigma k}] - \\
&\quad \frac{1}{4}[[[[[(1, (0, m_+, m_-, x)], \tilde{h}^{-\mu}_{\sigma k}], \tilde{h}^{\mu}_{\sigma k}], \tilde{h}^{-\mu}_{\sigma k}], \mathbf{E}_{\tilde{h}_{-\sigma k}, \tilde{h}_{-\sigma k}}] \\
&= -2\mu\sigma q(m_\mu, \tilde{h}_{\sigma k}) \in I
\end{aligned} \tag{2.10}$$

and so $I = L$. Now suppose that $m_\mu = 0$, $\mu = \pm$ and $x \neq 0$. Then there exists $i \in N$ such that $x(n_i) \neq 0$ and so $[x, n_i^+] = x(n_i)^+ \in (I \cap L_{\bar{1}})^+$ hence using (2.10) we get $I = L$. Hence if $I \cap L_{\bar{1}} \neq (0)$ then $I = L$. Now, let $0 \neq (a, 0, 0, x) \in I \cap L_{\bar{0}}$. If $x = 0$ then $I = L$. Suppose $x \neq 0$. If $x(\mathcal{R}) \neq 0$ then there exists $k \in R$ and $\sigma \in \{\pm\}$ such that $x(\tilde{h}_{\sigma k}) \neq 0$. Therefore $[a + x, \mathbf{E}_{\tilde{h}_{\sigma k}, \tilde{h}_{\sigma k}}] = 2\mathbf{E}_{x(\tilde{h}_{\sigma k}), \tilde{h}_{\sigma k}} \in I \cap L_{\bar{1}}$ and so by (2.10), $I = L$. If $x(\mathcal{R}) = 0$, then there exists $i \in I$ such that $x(n_i) \neq 0$. If $\mathcal{R} \neq (0)$ let $0 \neq g \in \mathcal{R}$. Then $[a + x, \mathbf{E}_{n_i, g}] = \mathbf{E}_{x(n_i), g} \in I \cap L_{\bar{1}}$ and so $I = L$. Now suppose that $\mathcal{R} = (0)$. Assume that $a = 0$. Let $i \in I$ such that $x(n_i) \neq 0$. Then it is easy to check that $x(n_i) \notin An_i$ and so there exists $j \in I$ such that $i \neq j$ and $q(x(n_i), n_j) \neq 0$. Then $[[[x, n_i^+], \mathbf{E}_{n_i, n_j}], n_i^-] = -q(x(n_i), n_j)q(n_i, n_i)$ and so $I = L$. Suppose $a \neq 0$. Then if $k \in I$ is such that $k \neq i, j$, $[[[a + x, n_i^+], \mathbf{E}_{n_i, n_j}], n_i^-], \mathbf{E}_{n_j, n_k}], n_i^+], n_k^-] = aq(n_i, n_i)^2 q(n_j, n_j) q(n_k, n_k)$ and so $I = L$. Finally, suppose $(a, m_+, m_-, x) \in I \cap L_{\bar{0}}$

such that $m_\mu \neq 0$ for some $\mu \in \{\pm\}$. Let $\tilde{m} \in \mathcal{M}$ such that $q(m_\mu, \tilde{m}) \neq 0$. Then

$$[[1,(a,m_+,m_-,x)],\tilde{m}^{-\mu}] = q(m_\mu,\tilde{m}) + \mu\mathbf{E}_{m_\mu,\tilde{m}} \neq 0$$

and so by above $I = L$. Therefore if $I \neq \emptyset$ is a $\mathbb{Z}_2$-graded ideal of L then $I = L$ and so L is simple. ∎

Example 2.3.9 Let A be a field of characteristic zero and $\mathcal{M}$ a finite dimensional A-supermodule. Assume that $\dim\mathcal{M}_{\bar{0}} \geq 3$ or $\dim\mathcal{M}_{\bar{1}} \geq 2$. Let q be a A-quadratic form on $\mathcal{M}$. Then $\mathrm{eosp}(q_\infty)$ is simple if and only if q is nondegenerate. Indeed, if q is nondegenerate, by Remark 2.1.15 one can show that $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ so that $q_{\mathcal{N}\times\mathcal{N}}$ is invertibly diagonizable and $q_{\mathcal{R}\times\mathcal{R}}$ is standard alternating. Hence by Proposition 2.3.8, $\mathrm{eosp}(q_\infty)$ is simple. The converse follows from Lemma 2.3.7. Note that if $\mathcal{N} = H(k/2;A)$ when k is even or $\mathcal{N} = H((k-1)/2;A) \oplus An_0$ where $q(n_0,n_0) = 1$ when k is odd then $L = \mathrm{osp}(2+k,l)$ and, as indeed also proved by Kac ([K2]), L is simple.

The following is worth noting.

Proposition 2.3.10 *Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Assume that $\mathrm{Ann}_A(\mathcal{M}) = 0$ and that there exist $m_i, n_i \in \mathcal{M}$ such that $\sum_i q(m_i,n_i) = 1$. Then $\mathrm{eosp}(q_\infty)$ is the TKK-superalgebra of the associated quadratic form Jordan superpair $V = (\mathcal{M},\mathcal{M})$.*

Proof: By Example 1.3.24, $\mathcal{K}(V)$ can be represented by

$$\mathcal{K}(V) = L_+ \oplus [L_+,L_-]/C \oplus L_-$$

where

$$\begin{aligned}
L_+ &= \{(0,m,0,0) \mid m \in \mathcal{M}\};\\
L_- &= \{(0,0,m,0) \mid m \in \mathcal{M}\};\\
L_0 &= \{(a,0,0,x) \mid a \in A,\ x \in \mathrm{eosp}(q)\};\\
[L_+,L_-]/C = \mathcal{D} &= \mathrm{span}_A\{q(m,n) \mid m,n \in \mathcal{M}\}\cdot Id + \mathrm{eosp}(q) \subset L_0/C;\\
C &= \{A \in [L_+,L_-] \mid [A,L_\sigma] = 0 \text{ for } \sigma = \pm\};
\end{aligned}$$

and the multiplication is given by

$$\begin{aligned}[u^+ \oplus a \oplus u^-, v^+ \oplus b \oplus v^-] =& (a^+v^+ - (-1)^{|u^+||b^+|}b^+u^+) \\ &\oplus ([ab] + \delta(u^+, v^-) - (-1)^{|v^+||u^-|}\delta(v^+, u^-)) \\ &\oplus (a^-v^- - (-1)^{|u^-||b^-|}b^-u^-)\end{aligned} \tag{2.11}$$

where $u^\sigma, v^\sigma \in L_\sigma$, $\sigma = \pm$ and $a = (a^+, a^-)$, $b = (b^+, b^-) \in \mathcal{D}$. Since $\mathrm{eosp}(q_\infty)$ is perfect we have that $[L_+, L_-] = L_0$. In addition, by Proposition 2.3.2, $C = \{(a, 0, 0, 0) \mid a \in \mathrm{Ann}_A(\mathcal{M})\} = (0)$ and so $\mathcal{K}(V) = \mathrm{eosp}(q_\infty)$ with multiplication given by (2.8). ∎

Chapter 3

Derivations

Throughout this chapter k will denote a commutative, associative, unital ring containing $\frac{1}{2}$, K a superextension of k and A a superextension of K. In addition, we will assume throughout that when $|m|$ occurs in an expression for some $m \in \mathcal{M}$ for some supermodule $\mathcal{M}$, then it is assumed that m is homogeneous, and that the expression extends to the other elements by linearity.

3.1 Derivations of toral 3-graded Lie superalgebras

Recall that given a Lie superalgebra L over K, a K-module homomorphism $d : L \to L$ is called a *K-derivation (of L)* if $d = d_{\bar{0}} \oplus d_{\bar{1}}$ where d_α ($\alpha \in \mathbb{Z}_2$) is a K-module homomorphism of degree α satisfying

$$d_\alpha[x, y] = [d_\alpha(x), y] + (-1)^{|\alpha||\beta|}[x, d_\alpha(y)] \tag{3.1}$$

for all $x \in L_\beta, y \in L$. We denote the set of K-derivations of a Lie superalgebra L by $\mathrm{Der}_K(L)$. For any element x of L, we can define the map $\mathbf{ad}(x) : L \to L$ by $\mathbf{ad}(x)(y) := [x, y]$. The K-supermodule $\mathbf{ad}(L)$ forms an ideal of $\mathrm{Der}_K L$ called the *inner derivation algebra of L*, denoted by $\mathbf{ad}\, L$.

In this section we will study the properties of derivations of toral 3-graded Lie superalgebras over K. These will be useful in the sections that follow when we will concentrate on determining the structure of the derivation algebras of orthosymplectic Lie superalgebras.

Recall that a 3-graded Lie superalgebra is a Lie superalgebra L with a decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$ such that

$$[L_i, L_j] \subset L_{i+j} \text{ where } L_k = 0 \text{ if } k \notin \{\pm 1, 0\}.$$

In addition, a toral 3-graded Lie superalgebra is a 3-graded Lie superalgebra L such that there exists an element $\zeta \in L_0$ so that

$$[\zeta, x_\sigma] = \sigma x_\sigma \text{ for all } x_\sigma \in L_\sigma,\ \sigma = 0, \pm 1.$$

Proposition 3.1.1 *Let L be a toral 3-graded Lie superalgebra over A . Then*

$$\mathrm{Der}_K L = \mathbf{ad}\, L + (\mathrm{Der}_K L)_0$$

where

$$\begin{aligned} (\mathrm{Der}_K L)_0 &:= \{d \in \mathrm{Der}_K L \mid d(L_\sigma) \in L_\sigma, \sigma = 0, \pm\} \\ &= \{d \in \mathrm{Der}_K L \mid d(\zeta) \in Z(L)\}. \end{aligned}$$

In particular, $\{d \in \mathrm{Der}_K L \mid d(\zeta) \in Z(L)\}$ is a subalgebra of $\mathrm{Der}_K L$ *and*

$$\mathbf{ad}\, L \cap \{d \in \mathrm{Der}_K L \mid d(\zeta) \in Z(L)\} = \mathbf{ad}\, L_0.$$

Remark 3.1.2 The first assertion of above result is well known (see for example [Fuks]).

Proof: It is clear that $\mathbf{ad}\, L + (\mathrm{Der}_K L)_0 \subset \mathrm{Der}_K L$. For $x \in L$ we will write $x = x_- + x_0 + x_+$ where $x_\mu \in L_\mu$, $\mu = 0, \pm$. Let $d \in \mathrm{Der}_K L$ and set $d' = d + \mathbf{ad}(d(\zeta)_+ - d(\zeta)_-)$. Then for all $x \in L$

$$\begin{aligned} d'(x_+) &= d(x_+) + [d(\zeta)_+ - d(\zeta)_-, x_+] = d[\zeta, x_+] - [d(\zeta)_-, x_+] \\ &= [d(\zeta), x_+] + [\zeta, d(x_+)] - [d(\zeta)_-, x_+] \\ &= \underbrace{[(d(\zeta))_0, x_+]}_{\in L_+} + \underbrace{[\zeta, d(x_+)]}_{\in L_+} \in L_+ \end{aligned}$$

since

$$d(x_+)_- = (d[\zeta, x_+])_- = \underbrace{[d(\zeta), x_+]_-}_{=0} + [\zeta, d(x_+)]_- = -d(x_+)_-$$

and so $d(x_+)_- = 0$. Similarly,

$$d'(x_-) \in L_-.$$

In addition, $[\zeta, x_0] = 0$ implies $0 = [d(\zeta), x_0] + [\zeta, d(x_0)]$ which in turn implies

$$d(x_0)_\pm = \mp[d(\zeta)_\pm, x_0].$$

Hence

$$\begin{aligned} d'(x_0) &= d(x_0) + [d(\zeta)_+ - d(\zeta)_-, x_0] \\ &= d(x_0)_0 - [d(\zeta)_+ - d(\zeta)_-, x_0] + [d(\zeta)_+ - d(\zeta)_-, x_0] \\ &= d(x_0)_0 \in L_0. \end{aligned}$$

Therefore $d' \in (\mathrm{Der}_K L)_0$ and so $\mathrm{Der}_K L = \mathbf{ad}\, L + (\mathrm{Der}_K L)_0$. Now, let $d \in (\mathrm{Der}_K L)_0$ and $x \in L$. Then

$$d(x_+) = d[\zeta, x_+] = [d(\zeta), x_+] + [\zeta, \underbrace{d(x_+)}_{\in L_+}] = [d(\zeta), x_+] + d(x_+)$$

which implies that $[d(\zeta), x_+] = 0$. Similarly, $[d(\zeta), x_-] = 0$. Moreover, since $[\zeta, x_0] = 0$,

$$0 = [d(\zeta), x_0] + [\zeta, \underbrace{d(x_0)}_{\in L_0}]$$

and so $[d(\zeta), x_0] = 0$. Hence $d(\zeta) \in Z(L)$ and so $(\mathrm{Der}_K L)_0 \subset \{d \in \mathrm{Der}_K L \mid d(\zeta) \in Z(L)\}$. Conversely, if $d \in \mathrm{Der}_K L$ is such that $d(\zeta) \subset Z(L)$ then for all $x \in L$,

$$\begin{aligned} d(x_\pm)_- + d(x_\pm)_0 + d(x_\pm)_+ &= d(x_\pm) = \pm d[\zeta, x_\pm] = \pm([d(\zeta), x_\pm] + [\zeta, d(x_\pm)]) \\ &= \pm(d(x_\pm)_+ - d(x_\pm)_-) \Rightarrow d(L_\pm) \subset L_\pm \end{aligned}$$

and

$$[\zeta, x_0] = 0 \Rightarrow [\zeta, d(x_0)] = -[d(\zeta), x_0] = 0 \Rightarrow d(L_0) \subset L_0.$$

Therefore $(\mathrm{Der}_K L)_0 = \{d \in \mathrm{Der}_K L \mid d(\zeta) \in Z(L)\}$. It is easily verified that $(\mathrm{Der}_K L)_0$ is a subalgebra of $\mathrm{Der}_K L$ and so all of the statements of the proposition hold. ∎

3.2 Derivations of orthosymplectic Lie superalgebras

Throughout this section we shall denote by $\mathcal{M}$ an A-supermodule with an A-quadratic form q. We will assume that $\mathrm{Ann}_A \mathcal{M} = 0$.

In this section we will study the K-derivations of $\mathcal{E}_\infty$ where $\mathcal{E}$ is a subalgebra of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$. Recall from Proposition 2.3.1 that $\mathcal{E}_\infty = \{(a, m, n, x) \mid a \in A,\ m, n \in \mathcal{M},\ x \in \mathcal{E}\}$ with multiplication given by (2.8). Since $\mathrm{Ann}_A \mathcal{M} = 0$, by Proposition 2.3.2, $Z(\mathcal{E}_\infty) = (0)$. In addition, we have the superalgebra isomorphism $A \to A \cdot Id|_{\mathcal{M}} : a \mapsto a \cdot Id|_{\mathcal{M}}$. We will sometimes, for ease of notation, write $a \cdot Id$ instead of a for $a \in A$. The meaning of $a \cdot Id$ will be clear from the context.

We make the following definitions which will be instrumental in describing the derivations of $\mathcal{E}_\infty$.

Definition 3.2.1 For each $\alpha \in \mathbb{Z}_2$, let $(\mathcal{S}_\mathcal{E})_\alpha$ be the set of all K-linear endomorphisms S of $\mathcal{M}$ of degree α satisfying

(S1) $[S, a \cdot Id] \in A \cdot Id$ for all $a \in A$;

(S2) $[S, q(m,n) \cdot Id] = (q(S(m), n) + (-1)^{|m||n|} q(S(n), m)) \cdot Id$ for all homogeneous $m, n \in \mathcal{M}$;

(S3) $[S, \mathcal{E}] \subset \mathcal{E}$.

Definition 3.2.2 For each $\alpha \in \mathbb{Z}_2$, let $(\mathcal{T}_\mathcal{E})_\alpha$ be the set of all K-linear endomorphisms T of $\mathcal{M}$ of degree α satisfying

(T1) $[T, a \cdot Id] \in \mathcal{E}$ for all $a \in A$;

(T2) $[T, q(m,n)\cdot Id] = \mathbf{E}_{T(m),n} + (-1)^{|m||n|}\mathbf{E}_{T(n),m}$ for all homogeneous $m, n \in \mathcal{M}$;

(T3) $[T, \mathcal{E}] \subset A\cdot Id$.

Set

$$\mathcal{S}_{\mathcal{E}} = (\mathcal{S}_{\mathcal{E}})_{\bar{0}} \oplus (\mathcal{S}_{\mathcal{E}})_{\bar{1}} \text{ and } \mathcal{T}_{\mathcal{E}} = (\mathcal{T}_{\mathcal{E}})_{\bar{0}} \oplus (\mathcal{T}_{\mathcal{E}})_{\bar{1}}.$$

We see immediately that $\mathcal{S}_{\mathcal{E}}$ and $\mathcal{T}_{\mathcal{E}}$ are submodules of the K-supermodule $\mathrm{End}_K(\mathcal{M})$. We will write $\mathcal{S}$ and $\mathcal{T}$ for $\mathcal{S}_{\mathcal{E}}$ and $\mathcal{T}_{\mathcal{E}}$ respectively when the context is clear. In fact, if we define, for $N \in \{1,2,3\}$ and $\alpha \in \mathbb{Z}_2$,

$$\mathcal{S}_\alpha^{(N)} = \{S \in (\mathrm{End}_K \mathcal{M})_\alpha \mid S \text{ satisfies (SN)}\}$$

and

$$\mathcal{T}_\alpha^{(N)} = \{T \in (\mathrm{End}_K \mathcal{M})_\alpha \mid T \text{ satisfies (TN)}\}$$

and set $\mathcal{S}^{(N)} = \mathcal{S}_{\bar{0}}^{(N)} \oplus \mathcal{S}_{\bar{1}}^{(N)}$, $\mathcal{T}^{(N)} = \mathcal{T}_{\bar{0}}^{(N)} \oplus \mathcal{T}_{\bar{1}}^{(N)}$ then

$$\mathcal{S}_{\mathcal{E}} = \cap_{N=1}^{3}\mathcal{S}^{(N)} \quad \text{and} \quad \mathcal{T}_{\mathcal{E}} = \cap_{N=1}^{3}\mathcal{T}^{(N)}.$$

Remark 3.2.3 Note that the $(\ )^{(N)}$ notation is traditionally used to denote the derived algebras of a Lie algebra however in what follows we will not be using the derived algebras at all so there should be no confusion resulting from the above definition.

Remark 3.2.4 It follows from the definition of $\mathcal{T}_{\mathcal{E}}$ that $\overline{A\cdot Id}^K \subset \mathcal{T}_{\mathcal{E}}$. Indeed, by Proposition 1.5.9, $\overline{A\cdot Id}^K = \overline{A\cdot Id}^A$ and so $[\overline{A\cdot Id}^K, A\cdot Id] = 0 \in \mathcal{E}$,

$$\begin{aligned}
\mathbf{E}_{(\sum_{i\in I} a_i Id)(m),n} + (-1)^{|m||n|}\mathbf{E}_{(\sum_{i\in I} a_i Id)(n),m} &= \mathbf{E}_{(\sum_{i\in I_m\cup I_n} a_i)m,n} + (-1)^{|m||n|}\mathbf{E}_{(\sum_{i\in I_m\cup I_n} a_i)n,m} \\
&= \left(\sum_{i\in I_m\cup I_n} a_i\right)(\mathbf{E}_{m,n} + (-1)^{|m||n|}\mathbf{E}_{n,m}) = 0
\end{aligned}$$

and, using Proposition 1.5.12, for all $x \in \mathcal{E}$,

$$[\sum_{i\in I} a_i\cdot Id, x] = \sum_{i\in I}[a_i\cdot Id, x] = 0 \in A\cdot Id.$$

Hence elements of $\overline{A\cdot Id}^K$ satisfy (T1)-(T3) and so the assertion follows. ∎

We have the following result:

Lemma 3.2.5 *Assume that $\mathcal{M}$ is an A-supermodule with an A-quadratic form q such that q represents 1, i.e., there exist $m_i, n_i \in \mathcal{M}$ such that $\sum_{i=1}^{N} q(m_i, n_i) = 1$. Then*

1) $\mathcal{S}^{(2)} \subset \mathcal{S}^{(1)}$*; and*

2) $\mathcal{T}^{(2)} \subset \mathcal{T}^{(1)}$.

Proof: For all $a \in A$ we have

$$a = \sum_{i=1}^{N} q(m_i, n_i)a = \sum_{i=1}^{N} q(m_i, n_i a).$$

1) Let $S \in \mathrm{End}_K \mathcal{M}$ such that S satisfies (S2). Let $a \in A$. Then if we write $m_i = (m_i)_{\bar{0}} + (m_i)_{\bar{1}}$ and $n_i = (n_i)_{\bar{0}} + (n_i)_{\bar{1}}$,

$$\begin{aligned}
[S, a\cdot Id] &= [S, \sum_{i=1}^{N} q(m_i, n_i a)\cdot Id] = \sum_{i=1}^{N} [S, q(m_i, n_i a)\cdot Id] \\
&= \sum_{i=1}^{N} \left(q(S(m_i), n_i a) + (-1)^{|m_i|(|n_i|+|a|)} q(S(n_i a), m_i)\right)\cdot Id \in A\cdot Id.
\end{aligned}$$

Hence S satisfies (S1).
2) Let $T \in (\mathrm{End}_K \mathcal{M})_a$ such that T satisfies (T2). Let $a \in A$. Then

$$\begin{aligned}
[T, a\cdot Id] &= [T, \sum_{i=1}^{N} q(m_i, n_i a)\cdot Id] = \sum_{i=1}^{N} [T, q(m_i, n_i a)\cdot Id] \\
&= \sum_{i=1}^{N} \left(\mathbf{E}_{T(m_i), n_i a} + (-1)^{|m_i|(|n_i|+|a|)} \mathbf{E}_{T(n_i a), m_i}\right) \in \mathrm{eosp}(q) \subset \mathcal{E}.
\end{aligned}$$

Hence T satisfies (T1). ∎

Remark 3.2.6 If $q \equiv 0$ then (S2) and (T2) hold trivially for all K-linear endomorphisms of $\mathcal{M}$ and so $\mathcal{S}^{(2)} = \mathrm{End}_K \mathcal{M}$ and $\mathcal{T}^{(2)} = \mathrm{End}_K \mathcal{M}$. Hence in this case, $\mathcal{S}_{\mathcal{E}} = \mathcal{S}^{(1)} \cap \mathcal{S}^{(3)}$ and $\mathcal{T}_{\mathcal{E}} = \mathcal{T}^{(1)} \cap \mathcal{T}^{(3)}$.

Proposition 3.2.7 *We have the following properties of $\mathcal{S}$ and $\mathcal{T}$:*

(1) Properties of $\mathcal{S}_{\mathcal{E}}$:

(i) For all $S \in \mathcal{S}^{(1)}$, the map $\Delta : A \to A$ given by $\Delta(a)\cdot Id = [S, a\cdot Id]$ is a K-derivation of A. Moreover, $[\mathrm{End}_A \mathcal{M}, S] \subset \mathrm{End}_A \mathcal{M}$.

(ii) For all $S \in \mathcal{S}^{(2)}$ and for all $m, n \in \mathcal{M}$,

$$[S, \mathbf{E}_{m,n}] = \mathbf{E}_{S(m),n} - (-1)^{|m||n|}\mathbf{E}_{S(n),m}. \tag{3.2}$$

(iii) Let $X = \mathrm{eosp}(q)$, $\overline{\mathrm{eosp}(q)}$ or $\mathrm{osp}(q)$. Then $[\mathcal{S}^{(2)}, X] \subset X$. In particular, if $\mathcal{E} = \mathrm{eosp}(q)$, $\overline{\mathrm{eosp}(q)}$ or $\mathrm{osp}(q)$ then $\mathcal{S}^{(2)} \subset \mathcal{S}^{(3)}$.

(iv) $\mathrm{osp}(q)$ is an ideal of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$ and so

$$\mathrm{osp}(q) \cap \mathcal{S}^{(3)} \lhd \mathcal{S}_{\mathcal{E}}. \tag{3.3}$$

In particular, $\mathcal{E} \lhd \mathcal{S}_{\mathcal{E}}$. Moreover, $\mathrm{osp}(q) \lhd \mathcal{S}_{\mathcal{E}}$ for any ideal $\mathcal{E}$ of $\mathrm{osp}(q)$ which is true in particular for $\mathcal{E} = \mathrm{eosp}(q)$, $\overline{\mathrm{eosp}(q)}$ and $\mathrm{osp}(q)$.

(2) Properties of $\mathcal{T}_{\mathcal{E}}$:

(i) For all $T \in \mathcal{T}^{(1)}$, the map $\varphi : A \to \mathcal{E}$ given by $\varphi(a) = [T, a\cdot Id]$ is a K-derivation from A to $\mathcal{E}$.

(ii) For all $T \in \mathcal{T}^{(2)}$ and for all $m, n \in \mathcal{M}$,

$$[T, \mathbf{E}_{m,n}] = (q(T(m), n) - (-1)^{|m||n|}q(T(n), m))\cdot Id. \tag{3.4}$$

(iii) Suppose that $\mathcal{E} \subset \overline{\mathrm{eosp}(q)}$ and $\overline{A\cdot Id}^K = A\cdot Id$ if $\mathcal{E} \neq \mathrm{eosp}(q)$. Then $\mathcal{T}^{(2)} \subset \mathcal{T}^{(3)}$.

(iv) $[\mathcal{E}, [\mathcal{T}^{(3)}, A\cdot Id]] = (0)$ and $[\mathcal{T}^{(3)}, [\mathcal{E}, \mathcal{E}]] = (0)$.

Remark 3.2.8 We should note the following:

(1) The assertion (1iii) in the above proposition does not hold for general $\mathrm{eosp}(q) \subset \mathcal{E} \subset \mathrm{osp}(q)$, for example see Example 2.2.6.

(2) A sufficient condition for $\overline{A\cdot Id} = A\cdot Id$ to hold was described in Lemma 1.5.8.

Proof: $\boxed{(1i)}$ Let $S \in \operatorname{End}_K \mathcal{M}$ satisfying (S1). Since $\operatorname{Ann}_A \mathcal{M} = 0$, the map $\Delta : A \to A$ is a well-defined K-linear map. For all $a, b \in A$, $p \in \mathcal{M}$,

$$\begin{aligned}
\Delta(ab)p &= [S, ab\cdot Id](p) = S(abp) - (-1)^{|S|(|a|+|b|)}abS(p)\\
&= S(abp) - (-1)^{|S||a|}aS(bp) + (-1)^{|S||a|}aS(bp) - (-1)^{|S|(|a|+|b|)}abS(p)\\
&= [S, a\cdot Id](bp) + (-1)^{|S||a|}a[S, b\cdot Id](p) = \Delta(a)bp + (-1)^{|\Delta||a|}a\Delta(b)p.
\end{aligned}$$

So using the fact that $\operatorname{Ann}_A \mathcal{M} = 0$,

$$\Delta(ab) = \Delta(a)b + (-1)^{|\Delta||a|}a\Delta(b).$$

Hence $\Delta \in \operatorname{Der}_K A$. Let $x \in \operatorname{End}_A \mathcal{M}$. Then for all $a \in A$, $m \in \mathcal{M}$,

$$\begin{aligned}
[x, S](am) &= x(S(am)) - (-1)^{|x||S|}S(x(am))\\
&= (-1)^{|S||a|}x(aS(m)) + x(\Delta(a)m) - (-1)^{|x|(|S|+|a|)}S(a(x(m)))\\
&= (-1)^{|a|(|S|+|x|)}a(x(S(m)) + (-1)^{|x|(|S|+|a|)}\Delta(a)x(m)\\
&\quad -(-1)^{|x|(|S|+|a|)+|S||a|}aS(x(m)) - (-1)^{|x|(|S|+|a|)}\Delta(a)x(m)\\
&= (-1)^{|a|(|x|+|S|)}a[x, S](m)
\end{aligned}$$

and so $[x, S]$ is A-linear.

$\boxed{(1ii)}$ Let $S \in \operatorname{End}_K \mathcal{M}$ satisfying (S2) and let $m, n \in \mathcal{M}$. Then for all $p \in \mathcal{M}$,

$$\begin{aligned}
[S, \mathbf{E}_{m,n}](p) &= S(mq(n,p) - (-1)^{|n||p|}q(m,p)n) - (-1)^{|S|(|m|+|n|)}mq(n, S(p))\\
&\quad + (-1)^{|S|(|m|+|n|)+|n|(|S|+|p|)}q(m, S(p))n\\
&= S(mq(n,p)) - (-1)^{|n||p|}S(q(m,p)n)\\
&\quad - (-1)^{|n||p|+(|p|+|n|)|m|}q(S(p), n)m + (-1)^{(|m|+|n|)|p|}q(S(p), m)n\\
&\overset{(S2)}{=} (-1)^{|m|(|n|+|p|)}[S, q(n,p)\cdot Id](m) + (-1)^{(|m|+|S|)(|n|+|p|)}q(n,p)S(m)\\
&\quad - (-1)^{|n||p|}[S, q(m,p)\cdot Id](n) - (-1)^{|n||p|+|S|(|m|+|p|)}q(m,p)S(n)\\
&\quad - (-1)^{|m|(|p|+|n|)}[S, q(n,p)\cdot Id](m) + (-1)^{|m|(|p|+|n|)}q(S(n), p)m\\
&\quad + (-1)^{|n||p|}[S, q(m,p)\cdot Id](n) - (-1)^{|n||p|}q(S(m), p)n\\
&= S(m)q(n,p) - (-1)^{|m||p|}S(n)q(m,p)\\
&\quad + (-1)^{|m|(|n|+|p|)}q(S(n), p)m - (-1)^{|n||p|}q(S(m), p)n\\
&= (\mathbf{E}_{S(m),n} - (-1)^{|m||n|}\mathbf{E}_{S(n),m})(p).
\end{aligned}$$

$\boxed{(1iii)}$ Let $S \in \mathrm{End}_K \mathcal{M}$ satisfying (S2). Then it follows from (1ii) that $[S, \mathrm{eosp}(q)] \subset \mathrm{eosp}(q)$ and so by Proposition 1.5.12,

$$[S, \overline{\mathrm{eosp}(q)}] \subset \overline{\mathrm{eosp}(q)}.$$

So the assertion for $X = \mathrm{eosp}(q)$ and $\overline{\mathrm{eosp}(q)}$ holds. Now, let $x \in \mathrm{osp}(q)$ and $m, n \in \mathcal{M}$. Then

$$\begin{aligned}
q([S,x](m),n)+&(-1)^{|m||n|}q([S,x](n),m) = \\
&= q(S(x(m)),n) - (-1)^{|S||x|}q(x(S(m)),n) + \\
&\quad + (-1)^{|m||n|}q(S(x(n)),m) - (-1)^{|S||x|+|m||n|}q(x(S(n)),m) \\
&\overset{(S2)}{=} [S, q(x(m),n)\cdot Id] - (-1)^{(|x|+|m|)|n|}q(S(n),x(m)) \\
&\quad - (-1)^{|S||x|}q(x(S(m)),n) + (-1)^{|m||n|}[S, q(x(n),m)\cdot Id] \\
&\quad - (-1)^{|m||n|+(|x|+|n|)|m|}q(S(m),x(n)) - (-1)^{|S||x|+|m||n|}q(x(S(n)),m) \\
&= [S, (q(x(m),n) + (-1)^{|m||n|}q(x(n),m))\cdot Id] \\
&\quad - (-1)^{(|x|+|m|)(|n|+|S|+|n|)}q(x(m),S(n)) - (-1)^{|S||x|+|m||n|}q(x(S(n)),m) \\
&\quad - (-1)^{|S||x|}q(x(S(m)),n) - (-1)^{|m||n|+(|x|+|n|)(|m|+|S|+|m|)}q(x(n),S(m)) \\
&= \ -(-1)^{(|x|+|m|)|S|}(q(x(m),S(n)) + (-1)^{|m|(|S|+|n|)}q(x(S(n)),m)) \\
&\quad - (-1)^{|S||x|}(q(x(S(m)),n) + (-1)^{|n|(|S|+|m|)}q(x(n),S(m)) \\
&= 0.
\end{aligned}$$

Hence $[S, x] \in \mathrm{osp}(q)$. The rest of the assertion now follows from the definition of (S3).

$\boxed{(1iv)}$ Every $x \in \mathrm{osp}(q)$ is A-linear and so $[x, A \cdot Id] = (0)$ and $(q(x(m),n) + (-1)^{|m||n|}q(x(n),m))\cdot Id = 0 = [x, q(m,n)\cdot Id]$. Hence $\mathrm{osp}(q) \subset \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. In addition, if $S \in \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$ and $x \in \mathrm{osp}(q)$ then

$$\begin{aligned}
q([x,S](m),n)+(-1)^{|m||n|}q([x,S](n),m) &= q(x(S(m)),n) - (-1)^{|x||S|}q(S(x(m)),n) \\
&\quad + (-1)^{|m||n|}q(x(S(n)),m) - (-1)^{|m||n|+|x||S|}q(S(x(n)),m) \\
&= -(-1)^{(|S|+|m|)|n|}q(x(n),S(m)) - (-1)^{|x||S|}[S,q(x(m),n)\cdot Id] \\
&\quad + (-1)^{|x||S|+(|x|+|m|)|n|}q(S(n),x(m)) - (-1)^{|S||m|}q(x(m),S(n)) \\
&\quad - (-1)^{|m||n|+|x||S|}[S,q(x(n),m)\cdot Id] + (-1)^{|x|(|S|+|m|)}q(S(m),x(n)) \\
&= 0
\end{aligned}$$

and so $\mathrm{osp}(q)$ is an ideal of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. Now (3.3) follows from the definition of $\mathcal{S}_{\mathcal{E}}$ and from the fact that, if $x \in \mathrm{osp}(q) \cap \mathcal{S}^{(3)}$ then

$$[[x,\mathcal{S}],\mathcal{E}] = \underbrace{[x,\underbrace{[\mathcal{S},\mathcal{E}]}_{\subset\mathcal{E}}]}_{\subset\mathcal{E}} + \underbrace{[\mathcal{S},\underbrace{[x,\mathcal{E}]}_{\subset\mathcal{E}}]}_{\subset\mathcal{E}}.$$

Hence in particular $\mathrm{osp}(q) \lhd \mathcal{S}_{\mathcal{E}}$ whenever $\mathcal{E}$ is an ideal of $\mathrm{osp}(q)$. In particular, $\mathcal{E}$ is an ideal of $\mathrm{osp}(q)$ for $\mathcal{E} = \mathrm{eosp}(q)$ (Proposition 2.1.2), $\mathcal{E} = \overline{\mathrm{eosp}(q)}$ (Proposition 2.2.1), or $\mathcal{E} = \mathrm{osp}(q)$.

$\boxed{(2\ i)}$ Let $T \in \mathrm{End}_K \mathcal{M}$ satisfying (T1). Clearly φ is a K-linear map. Then by an argument similar to the one in (1i), for all $a, b \in A$ and $p \in \mathcal{M}$

$$[T, ab\cdot Id](p) = [T, a\cdot Id](bp) + (-1)^{|T||a|}a[T, b\cdot Id](p).$$

So by (T1) ,

$$\varphi(ab) = \varphi(a)b + (-1)^{|\varphi||a|}a\varphi(b).$$

Hence $\varphi \in \mathrm{Der}_K(A,\mathcal{E})$.

$\boxed{(2\ ii)}$ Let $T \in \mathrm{End}_K \mathcal{M}$ satisfying (T2) and let $m, n \in \mathcal{M}$. Then for all $p \in \mathcal{M}$,

$$\begin{aligned}
[T,\mathbf{E}_{m,n}](p) &= T(mq(n,p) - (-1)^{|m||n|}nq(m,p)) \\
&\quad - (-1)^{|T|(|m|+|n|)}mq(n,T(p)) + (-1)^{|T|(|m|+|n|)+|m||n|}nq(m,T(p)) \\
&= (-1)^{|m|(|n|+|p|)}[T,q(n,p)\cdot Id](m) + (-1)^{|m|(|n|+|p|)+|T|(|n|+|p|)}q(n,p)T(m) \\
&\quad - (-1)^{|n||p|}[T,q(m,p)\cdot Id](n) - (-1)^{|n||p|+|T|(|m|+|p|)}q(m,p)T(n) \\
&\quad - (-1)^{|T|(|m|+|n|)}mq(n,T(p)) + (-1)^{|T|(|m|+|n|)+|m||n|}nq(m,T(p))
\end{aligned}$$

$$\begin{aligned}
&\overset{(T2)}{=} (-1)^{|m|(|n|+|p|)}\mathbf{E}_{T(n),p}(m) + (-1)^{|m|(|n|+|p|)+|n||p|}\mathbf{E}_{T(p),n}(m) \\
&\quad - (-1)^{|n||p|}\mathbf{E}_{T(m),p}(n) - (-1)^{|p|(|n|+|m|)}\mathbf{E}_{T(p),m}(n) \\
&\quad + T(m)q(n,p) - (-1)^{|n||m|}T(n)q(m,p) \\
&\quad - (-1)^{|T|(|m|+|n|)}mq(n,T(p)) + (-1)^{|T|(|m|+|n|)+|m||n|}nq(m,T(p)) \\
&= (-1)^{|m|(|n|+|p|)}T(n)q(p,m) - (-1)^{|m||n|}q(T(n),m)p \\
&\quad + (-1)^{|m|(|n|+|p|)+|n||p|}T(p)q(n,m) - (-1)^{|p|(|m|+|n|)}q(T(p),m)n \\
&\quad - (-1)^{|n||p|}T(m)q(p,n) + q(T(m),n)p \\
&\quad - (-1)^{|p|(|n|+|m|)}T(p)q(m,n) + (-1)^{|p|(|n|+|m|)+|m||n|}q(T(p),n)m \\
&\quad + T(m)q(n,p) - (-1)^{|n||m|}T(n)q(m,p) \\
&\quad - (-1)^{|T|(|m|+|n|)}mq(n,T(p)) + (-1)^{|T|(|m|+|n|)+|m||n|}nq(m,T(p)) \\
&= q(T(m),n)p - (-1)^{|m||n|}q(T(n),m)p
\end{aligned}$$

and so

$$[T,\mathbf{E}_{m,n}] = (q(Tm,n) - (-1)^{|m||n|}q(Tn,m))\cdot Id.$$

$\boxed{(2\ iii)}$ The assertion for $\mathcal{E} = \mathrm{eosp}(q)$ clearly follows from (2ii). Suppose that $\overline{A\cdot Id} = A\cdot Id$. Then by Propositions 1.5.9 and 1.5.12, for any $\mathcal{E} \subset \overline{\mathrm{eosp}(q)}$,

$$[T,\mathcal{E}] \subset [\mathcal{T}_{\mathcal{E}}, \overline{\mathrm{eosp}(q)}^{K}] \subset \overline{[\mathcal{T},\mathrm{eosp}(q)]}^{K} = \overline{[\mathcal{T},\mathrm{eosp}(q)]}^{A} \subset \overline{A\cdot Id} = A\cdot Id.$$

$\boxed{(2\ iv)}$ Let $T \in \mathrm{End}_K \mathcal{M}$ satisfying (T3). Then for all $x,y \in \mathcal{E}$,

$$[T,[x,y]] = \underbrace{[\underbrace{[T,x]}_{\in A\cdot Id},y]}_{=0} - (-1)^{|x||y|}\underbrace{[\underbrace{[T,y]}_{\in A\cdot Id},x]}_{=0} = 0$$

hence $[T,[\mathcal{E},\mathcal{E}]] = (0)$. In addition, for all $x \in \mathcal{E}$ and $a \in A$,

$$[x,[T,a\cdot Id]] = \underbrace{[\underbrace{[x,T]}_{\in A\cdot Id},a\cdot Id]}_{=0} - (-1)^{|T||a|}[\underbrace{[x,a\cdot Id]}_{=0},T] = 0.$$

■

In fact, $\mathcal{S}$ and $\mathcal{T}$ give rise to a Lie superalgebra structure in the following way:

Proposition 3.2.9

(1) For all $N \in \{1,2,3\}$, $\mathcal{S}^{(N)}$ is a subalgebra of $\mathrm{End}_K \mathcal{M}$. *In particular, $\mathcal{S}$ is a subalgebra of* $\mathrm{End}_K \mathcal{M}$.

(2) $[\mathcal{S}^{(1)} \cap \mathcal{S}^{(3)}, \mathcal{T}^{(1)} \cap \mathcal{T}^{(3)}] \subset \mathcal{T}^{(1)} \cap \mathcal{T}^{(3)}$ *and* $[\mathcal{S}^{(2)}, \mathcal{T}^{(2)}] \subset \mathcal{T}^{(2)}$. *In particular,* $[\mathcal{S}, \mathcal{T}] \subset \mathcal{T}$.

(3) $[\mathcal{T}^{(1)} \cap \mathcal{T}^{(3)}, \mathcal{T}^{(1)} \cap \mathcal{T}^{(3)}]$ *and* $[\mathcal{T}^{(2)}, \mathcal{T}^{(2)}]$ *are subalgebras of* $\mathcal{S}^{(1)} \cap \mathcal{S}^{(3)}$ *and* $\mathcal{S}^{(2)}$ *respectively. In particular,* $[\mathcal{T}, \mathcal{T}]$ *is a subalgebra of* $\mathcal{S}$.

(4) $\mathcal{S} \oplus \mathcal{T} \subset \mathrm{End}_K \mathcal{M} \oplus \mathrm{End}_K \mathcal{M}$ *is a Lie superalgebra with multiplication*

$$[S \oplus T, S' \oplus T']^{\sim} = ([S, S'] + [T, T']) \oplus ([S, T'] + [T, S']) \tag{3.5}$$

where $[\ ,\ ]$ *is the usual bracket multiplication in* $(\mathrm{End}_K \mathcal{M})^{(-)}$.

Proof: Let $N \in \{1, 2, 3\}$. Clearly $\mathcal{S}^{(N)}$ and $\mathcal{T}^{(N)}$ are A-supermodules.
(1) Let $S, S' \in \mathcal{S}^{(N)}$ and $T, T' \in \mathcal{T}^{(N)}$. If $N = 1$ then using the Jacobi identity, for all $a \in A$,

$$[[S, S'], a \cdot Id] = \underbrace{[S, \underbrace{[S', a \cdot Id]}_{\in A \cdot Id}]}_{\in A \cdot Id} - (-1)^{|S||S'|} \underbrace{[S', \underbrace{[S, a \cdot Id]}_{\in A \cdot Id}]}_{\in A \cdot Id} \in A \cdot Id$$

and so $\mathcal{S}^{(1)}$ is a subalgebra of $(\mathrm{End}_K \mathcal{M})^{(-)}$. If $N = 2$ then for all $m, n \in \mathcal{M}$,

$$\begin{aligned}
[[S, S'], q(m, n) \cdot Id] &= [S, [S', q(m, n) \cdot Id]] - (-1)^{|S||S'|}[S', [S, q(m, n) \cdot Id]] \\
&= [S, q(S'(m), n) \cdot Id] + (-1)^{|m||n|}[S, q(S'(n), m) \cdot Id] \\
&\quad - (-1)^{|S||S'|}[S', q(S(m), n) \cdot Id] - (-1)^{|S||S'|+|m||n|}[S', q(S(n), m) \cdot Id] \\
&= q(S(S'(m)), n) \cdot Id + (-1)^{(|S'|+|m|)|n|} q(S(n), S'(m)) \cdot Id \\
&\quad + (-1)^{|m||n|} q(S(S'(n)), m) \cdot Id + (-1)^{|S'||m|} q(S(m), S'(n)) \cdot Id \\
&\quad - (-1)^{|S||S'|} q(S'(S(m)), n) \cdot Id - (-1)^{|S||S'|+(|S|+|m|)|n|} q(S'(n), S(m)) \\
&\quad - (-1)^{|S||S'|+|m||n|} q(S'(S(n)), m) \cdot Id - (-1)^{|S|(|S'|+|m|)} q(S'(m), S(n)) \cdot Id \\
&= q([S, S'](m), n) \cdot Id + (-1)^{|m||n|} q([S, S](n), m) \cdot Id
\end{aligned}$$

and so $\mathcal{S}^{(2)}$ is a subalgebra of $(\mathrm{End}_K \mathcal{M})^{(-)}$. If $N = 3$ then

$$[[S,S'],\mathcal{E}] \subset \underbrace{[S,\underbrace{[S',\mathcal{E}]}_{\subset\mathcal{E}}]}_{\subset\mathcal{E}} + \underbrace{[S',\underbrace{[S,\mathcal{E}]}_{\subset\mathcal{E}}]}_{\subset\mathcal{E}} \subset \mathcal{E}$$

hence $\mathcal{S}^{(3)}$ is a subalgebra of $(\mathrm{End}_K \mathcal{M})^{(-)}$. It now follows that $\mathcal{S} = \cap_{N=1}^{3}\mathcal{S}^{(N)}$ is a subalgebra of $(\mathrm{End}_K \mathcal{M})^{(-)}$.

(2) Let $S \in \mathcal{S}^{(1)} \cap \mathcal{S}^{(3)}$, $T \in \mathcal{T}^{(1)} \cap \mathcal{T}^{(3)}$. Then

$$[[S,T],A\cdot Id] \subset \underbrace{[S,\underbrace{[T,A\cdot Id]}_{\subset\mathcal{E}}]}_{\subset\mathcal{E}} + \underbrace{[T,\underbrace{[S,A\cdot Id]}_{A\cdot Id}]}_{\subset\mathcal{E}} \subset \mathcal{E}$$

so $[S,T] \in \mathcal{T}^{(1)}$. In addition,

$$[[S,T],\mathcal{E}] = \underbrace{[S,\underbrace{[T,\mathcal{E}]}_{\subset A\cdot Id}]}_{\subset A\cdot Id} + \underbrace{[T,\underbrace{[S,\mathcal{E}]}_{\subset A\cdot Id}]}_{\subset A\cdot Id}$$

and so $[S,T] \in \mathcal{T}^{(3)}$. For $S \in \mathcal{S}^{(2)}$ and $T \in \mathcal{T}^{(2)}$ and for all $m,n \in \mathcal{M}$ we have

$$\begin{aligned}
[[S,T],\ q(m,n)\cdot Id] &= [S,[T,q(m,n)\cdot Id]] - (-1)^{|S||T|}[T,[S,q(m,n)\cdot Id] \\
&\overset{(S2)}{\underset{(T2)}{=}} [S,\mathbf{E}_{T(m),n} + (-1)^{|m||n|}\mathbf{E}_{T(n),m}] \\
&\quad - (-1)^{|S||T|}[T,q(S(m),n)\cdot Id + (-1)^{|m||n|}q(S(n),m)\cdot Id] \\
&\overset{(3.2)}{\underset{(T2)}{=}} \mathbf{E}_{S(T(m)),n} - (-1)^{(|T|+|m|)|n|}\mathbf{E}_{S(n),T(m)} \\
&\quad + (-1)^{|m||n}\mathbf{E}_{S(T(n)),m} - (-1)^{|m||n|+(|T|+|n|)|m|}\mathbf{E}_{S(m),T(n)} \\
&\quad - (-1)^{|S||T|}\mathbf{E}_{T(S(m)),n} - (-1)^{|S||T|+(|S|+|m|)|n|}\mathbf{E}_{T(n),S(m)} \\
&\quad - (-1)^{|S||T|+|m||n|}\mathbf{E}_{T(S(n)),m} - (-1)^{|S|(|T|+|m|)}\mathbf{E}_{T(m),S(n)} \\
&= \mathbf{E}_{[S,T]m,n} + (-1)^{|m||n|}\mathbf{E}_{[S,T](n),m}
\end{aligned}$$

Hence $[S,T] \in \mathcal{T}^{(2)}$. The rest of (2) now follows.

(3) Let $T,T' \in \mathcal{T}^{(1)} \cap \mathcal{T}^{(3)}$. Then $[T,T'] \in \mathcal{S}^{(1)}$ since for all $a \in A$,

$$[[T,T'],a\cdot Id] = \underbrace{[T,\underbrace{[T',a\cdot Id]}_{\in\mathcal{E}}]}_{\in A\cdot Id} + (-1)^{|T||T'|}\underbrace{[T',\underbrace{[T,a\cdot Id]}_{\in\mathcal{E}}]}_{\in A\cdot Id} \in A\cdot Id.$$

Moreover,

$$[[T,T'],\mathcal{E}] = \underbrace{[T,\underbrace{[T',\mathcal{E}]}_{\subset A\cdot Id}]}_{\subset\mathcal{E}} + \underbrace{[T',\underbrace{[T,\mathcal{E}]}_{\subset A\cdot Id}]}_{\subset\mathcal{E}}$$

and so $[T,T'] \in \mathcal{S}^{(1)} \cap \mathcal{S}^{(3)}$. For $T,T' \in \mathcal{T}^{(2)}$, for all $m,n \in \mathcal{M}$,

$$\begin{aligned}
[[T,T'],q(m,n)\cdot Id] &= [T,[T',q(m,n)\cdot Id]] - (-1)^{|T||T'|}[T',[T,q(m,n)\cdot Id]] \\
&\overset{(T2)}{=} [T,\mathbf{E}_{T'(m),n} + (-1)^{|m||n|}\mathbf{E}_{T'(n),m}] \\
&\quad - (-1)^{|T||T'|}[T',\mathbf{E}_{Tm,n} + (-1)^{|m||n|}\mathbf{E}_{Tn,m}] \\
&\overset{(3.4)}{=} q(T(T'(m)),n)\cdot Id - (-1)^{(|T'|+|m|)|n|}q(T(n),T'(m))\cdot Id \\
&\quad + (-1)^{|m||n|}q(T(T'(n)),m)\cdot Id \\
&\quad - (-1)^{|m||n|+(|T'|+|n|)|m|}q(T(m),T'(n))\cdot Id \\
&\quad - (-1)^{|T||T'|}q(T'(T(m)),n)\cdot Id \\
&\quad + (-1)^{|T||T'|+(|T|+|m|)|n|}q(T'(n),T(m))\cdot Id \\
&\quad - (-1)^{|T||T'|+|m||n|}q(T'(T(n)),m)\cdot Id \\
&\quad + (-1)^{|T||T'|+|m||n|+(|T|+|n|)|m|}q(T'(m),T(n))\cdot Id \\
&= q([T,T'](m),n)\cdot Id + (-1)^{|m||n|}q([T,T'](n),m)\cdot Id
\end{aligned}$$

and so $[T,T'] \in \mathcal{S}^{(2)}$. It now follows that $[\mathcal{T},\mathcal{T}] \subset \mathcal{S}$. Clearly $[\mathcal{T},\mathcal{T}]$ has a $\mathbb{Z}_2$-grading given by

$$[\mathcal{T},\mathcal{T}]_{\bar{0}} = [\mathcal{T}_{\bar{0}},\mathcal{T}_{\bar{0}}] + [\mathcal{T}_{\bar{1}},\mathcal{T}_{\bar{1}}] = [\mathcal{T},\mathcal{T}] \cap \mathcal{S}_{\bar{0}}$$

and

$$[\mathcal{T},\mathcal{T}]_{\bar{1}} = [\mathcal{T}_{\bar{0}},\mathcal{T}_{\bar{1}}] = [\mathcal{T},\mathcal{T}] \cap \mathcal{S}_{\bar{1}}$$

and so it is a submodule of $\mathcal{S}$. Hence we only have to show that $[[\mathcal{T},\mathcal{T}],[\mathcal{T},\mathcal{T}]] \subset \mathcal{T}$. We have

$$[[\mathcal{T},\mathcal{T}],[\mathcal{T},\mathcal{T}]] \subset [\mathcal{T},\underbrace{[\mathcal{T},\underbrace{[\mathcal{T},\mathcal{T}]}_{\subset\mathcal{S}}]}_{\subset\mathcal{T}}] \subset [\mathcal{T},\mathcal{T}]$$

and so $[\mathcal{T},\mathcal{T}]$ is a subalgebra of the Lie superalgebra $\mathcal{S}$.
(4) Follows from (1)-(3) and from Example 1.3.17. ∎

Remark 3.2.10 It is important to note that $\mathcal{S}_{\mathcal{E}} \oplus \mathcal{T}_{\mathcal{E}}$ does not in general naturally imbed in $\mathrm{End}_K \mathcal{M}$. For example, if $q \equiv 0$ then $\mathrm{End}_A \mathcal{M} \subset \mathcal{S}_{\mathcal{E}} \cap \mathcal{T}_{\mathcal{E}}$ and hence $\mathcal{S}_{\mathcal{E}} \cap \mathcal{T}_{\mathcal{E}} \neq 0$.

Recall that we identify $A \cdot Id$ with A via $a \cdot Id \mapsto a$ and we will sometimes write $a \cdot Id$ instead of a for $a \in A$. The meaning should be clear from the context.

Definition 3.2.11 We set $\mathcal{D}_{\mathcal{E}}$ to be the set of all K-endomorphisms $d_{S,T}$ of $\mathcal{E}_\infty$ for which there exist $S \in \mathcal{S}_{\mathcal{E}}$, $T \in \mathcal{T}_{\mathcal{E}}$ satisfying

$$\begin{aligned} d_{S,T}(a,m,n,x) &= ([S,a\cdot Id] + [T,x], (S+T)(m), \\ &\quad (S-T)(n), [T,a\cdot Id] + [S,x]). \end{aligned} \tag{3.6}$$

Note that $\mathcal{D}_{\mathcal{E}}$ is a K-supermodule with the $\mathbb{Z}_2$-grading given by $(\mathcal{D}_{\mathcal{E}})_\alpha = \{d_{S,T} \mid S \in (\mathcal{S}_{\mathcal{E}})_\alpha, T \in (\mathcal{T}_{\mathcal{E}})_\alpha\}$ for $\alpha \in \mathbb{Z}_2$.

Using Proposition 3.2.9 it is easy to see that $\mathcal{D}_{\mathcal{E}}$ is well defined.

Remark 3.2.12 Note that for $(S,T),(S',T') \in \mathcal{S}_{\mathcal{E}} \oplus \mathcal{T}_{\mathcal{E}}$ such that $d_{S,T} = d_{S',T'}$ we have

$$(S+T)(m) = (S'+T')(m)$$
$$(S-T)(m) = (S'-T')(m)$$

for all $m \in \mathcal{M}$. Since $\frac{1}{2} \in K$ it follows that $S = S'$, $T = T'$ and so S and T are uniquely determined by d.

We will often use the following notation:

$$\begin{aligned} a &:= (a,0,0,0) \quad \text{for all } a \in A; \\ m^+ &:= (0,m,0,0) \quad \text{for all } m \in \mathcal{M}; \\ m^- &:= (0,0,m,0) \quad \text{for all } m \in \mathcal{M}; \\ x &:= (0,0,0,x) \quad \text{for all } x \in \mathcal{E}; \end{aligned}$$

the meaning will be clear from the context.

Theorem 3.2.13 *Derivations of $\mathcal{E}_\infty$ satisfy the following:*

(1) $\mathcal{D}_\mathcal{E} = (\mathrm{Der}_K \mathcal{E}_\infty)_0$, *in particular,* $\mathcal{D}_\mathcal{E}$ *is a subalgebra of* $\mathrm{Der}_K \mathcal{E}_\infty$. *Moreover, for all* $(S,T),(S',T') \in \mathcal{S} \oplus \mathcal{T}$,

$$[d_{S,T}, d_{S',T'}] = d_{[S\oplus T, S'\oplus T']}.$$

In other words, the map $\phi : \mathcal{S}_\mathcal{E} \oplus \mathcal{T}_\mathcal{E} \to \mathcal{D}_\mathcal{E} : S \oplus T \mapsto d_{S,T}$ *is a Lie superalgebra isomorphism.*

(2) $\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty + \mathcal{D}_\mathcal{E}$.

(3) $\mathbf{ad}\, \mathcal{E}_\infty \cap \mathcal{D}_\mathcal{E} = \{d_{S,T} \in \mathcal{D}_\mathcal{E} \mid S \in \mathcal{E},\ T \in A\cdot Id\} = \mathbf{ad}(\mathcal{E}_\infty)_0$.

(4) If $\mathcal{S} = \mathcal{E} \oplus \mathcal{S}_0$ *and* $\mathcal{T} = A\cdot Id \oplus \mathcal{T}_0$ *where* $\mathcal{S}_0 \oplus \mathcal{T}_0$ *is a subalgebra of* $\mathcal{S} \oplus \mathcal{T}$ *then*

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty \rtimes (\widehat{\mathcal{S}_0 \oplus \mathcal{T}_0})$$

where $\widehat{\ }: \mathcal{S}_0 \oplus \mathcal{T}_0 \to \mathrm{Der}_K \mathcal{E}_\infty : S \oplus T \mapsto d_{S,T}$ *is a Lie superalgebra monomorphism.*

Remark 3.2.14 We will see some examples where the conditions in 3.2.13.4 are fulfilled in the next section.

Proof:

(1) $\boxed{\subset}$ Let $d = d_{S,T} \in \mathcal{D}_\mathcal{E}$. Clearly $d \in (\mathrm{End}_K \mathcal{E}_\infty)_0$. We have to check that d is indeed a K-derivation. i.e. we have to check that for all $u, v \in L$ homogeneous.

$$d([u,v]) = [d(u), v] + (-1)^{|d||u|}[u, d(v)]$$

We have for all $u = (a, m, n, x), v = (b, p, r, 0) \in \mathcal{E}_\infty$,

$$\begin{aligned}
d[(a,m,n,x),(b,p,r,0)] &= \\
&= d(q(m,r) - q(n,p), ap + xp - mb, -ar + xr + nb, \mathbf{E}_{m,r} + \mathbf{E}_{n,p}) \\
&= ([S,(q(m,r) - q(n,p))\cdot Id] + [T, \mathbf{E}_{m,r} + \mathbf{E}_{n,p}], (S+T)(ap + xp - mb), \\
&\qquad (S-T)(-ar + xr + nb), [T,(q(m,r) - q(n,p))\cdot Id] + [S, \mathbf{E}_{m,r} + \mathbf{E}_{n,p}]).
\end{aligned}$$

On the other hand, using (S2), (T2) and Proposition 3.2.7 we have

$$
\begin{aligned}
&[d(a,m,n,x),(b,p,r,0)] + (-1)^{|d||u|}[(a,m,n,x),d(b,p,r,0)] = \\
&= ([S,a\cdot Id] + [T,x],(S+T)m,(S-T)n,[T,a\cdot Id] + [S,x]),(b,p,r,0)] \\
&\quad + (-1)^{|d||u|}[(a,m,n,x),([S,b\cdot Id],(S+T)p,(S-T)r,[T,b\cdot Id]) \\
&= (q((S+T)m,r) - q((S-T)n,p), \\
&\quad [S,a\cdot Id]p + [T,x]p + [T,a\cdot Id]p + [S,x]p - (-1)^{|v|(|u|+|d|)}b(S+T)m, \\
&\quad - [S,a\cdot Id]r - [T,x]r + [T,a\cdot Id]r + [S,x]r + (-1)^{|v|(|u|+|d|)}b(S-T)n, \\
&\quad \mathbf{E}_{(S+T)m,r} + \mathbf{E}_{(S-T)n,p}) \\
&\quad + (-1)^{|d||u|}(q(m,(S-T)r) - q(n,(S+T)p), \\
&\quad a(S+T)p + x(S+T)p - (-1)^{(|d|+|v|)|u|}[S,b\cdot Id]m - (-1)^{(|d|+\beta)\alpha}[T,b\cdot Id]m, \\
&\quad - a(S-T)r + x(S-T)r + (-1)^{(|d|+|v|)|u|}[S,b\cdot Id]n - (-1)^{(|d|+|v|)|u|}[T,b\cdot Id]n, \\
&\quad \mathbf{E}_{m,(S-T)r} + \mathbf{E}_{n,(S+T)p} + [x,[T,b\cdot Id]]) \\
&= ([S,(q(m,r) - q(n,p))\cdot Id] + [T,\mathbf{E}_{m,r} + \mathbf{E}_{n,p}], \\
&\quad S(ap) + T(xp) + T(ap) + S(xp) - (-1)^{|u||v|}S(bm) - (-1)^{|u||v|}T(bm), \\
&\quad - S(ar) - T(xr) + T(ar) + S(xr) + (-1)^{|u||v|}S(bn) - (-1)^{|u||v|}T(bn), \\
&\quad [S,\mathbf{E}_{m,r} + \mathbf{E}_{n,p}] + [T,(q(m,r) - q(n,p))\cdot Id] \\
&= d[(a,m,n,x),(b,p,r,0)].
\end{aligned}
$$

Now, let $u = (0,0,0,y)$, $v = (a,m,n,x) \in \mathcal{E}_\infty$. Then, using Proposition 3.2.7,

$$
\begin{aligned}
[d(u),v]+(-1)^{|d||u|}[u,d(v)] &= [([T,y],0,0,[S,y]),(a,m,n,x)] \\
&\quad + (-1)^{|d||u|}[(0,0,0,y),([S,a\cdot Id] + [T,x],(S+T)(m), \\
&\qquad\qquad (S-T)(n),[S,x] + [T,a\cdot Id])] \\
&= (0,[T,y](m) + [S,y](m), -[T,y](n) + [S,y](n),[[S,y],x]) \\
&\quad + (-1)^{|d||u|}(0,y((S+T)(m)), \\
&\qquad\qquad y((S-T))(n),[y,[S,x]] + [y,[T,a\cdot Id]]) \\
&= (0,(S+T)(y(m)),(S-T)(y(n)),[S,[y,x]]) \\
&= d(0,y(m),x(n),[y,x]) = d([u,v])
\end{aligned}
$$

Hence $\mathcal{D}_{\mathcal{E}} \subset (\mathrm{Der}_K \mathcal{E}_\infty)_0$.

$\boxed{\supset}$ Let $d = d_{\bar 0} \oplus d_{\bar 1} \in (\mathrm{Der}_K \mathcal{E}_\infty)_0$. Then there exist $M, N \in \mathrm{End}_K \mathcal{M}$, $\varphi_A \in \mathrm{End}_K A$, $\varphi_{\mathcal{E}} \in \mathrm{Hom}_K(A, \mathcal{E})$ and $\psi_A \in \mathrm{Hom}_K(\mathcal{E}, A)$, $\psi_{\mathcal{E}} \in \mathrm{End}_K \mathcal{E}$ such that

$$d(a,m,n,x) = (\varphi_A(a) + \psi_A(x), M(m), N(n), \varphi_{\mathcal{E}}(a) + \psi_{\mathcal{E}}(x)).$$

Since d is a derivation, for all $a, b \in A$ and $x, y \in \mathcal{E}$,

$$\begin{aligned}
(\psi_A([x,y]), 0, 0, \psi_{\mathcal{E}}([x,y])) &= d([(a,0,0,x),(b,0,0,y)] \\
&= [(\varphi_A(a) + \psi_A(x), 0, 0, \varphi_{\mathcal{E}}(a) + \psi_{\mathcal{E}}(x)), (b,0,0,y)] \\
&\quad + (-1)^{|(a,0,0,x)||(b,0,0,y)|}[(a,0,0,x), (\varphi_A(b) + \psi_A(y), 0, 0, \varphi_{\mathcal{E}}(b) + \psi_{\mathcal{E}}(y))) \\
&= (0,0,0, [\varphi_{\mathcal{E}}(a) + \psi_{\mathcal{E}}(x), y] + (-1)^{|(a,0,0,x)||(b,0,0,y)|}[x, \varphi_{\mathcal{E}}(b) + \psi_{\mathcal{E}}(y)])
\end{aligned}$$

which implies that

$$\psi_A[x,y] = 0, \ \varphi_{\mathcal{E}}(A) \subset Z(\mathcal{E}) \text{ and } \psi_{\mathcal{E}}(\mathcal{E}) \subset \mathrm{Der}_K \mathcal{E}.$$

Moreover, for all $a \in A$ and $m \in \mathcal{M}$,

$$\begin{aligned}
(0, M(am + xm), 0, 0) = d(0, am + xm, 0, 0) &= d([(a,0,0,x),(0,m,0,0)]) \\
&= [(\varphi_A(a) + \psi_A(x), 0, 0, \varphi_{\mathcal{E}}(a) + \psi_{\mathcal{E}}(x)), (0,m,0,0)] \\
&\quad + (-1)^{|d||a|}[(a,0,0,x), (0, M(m), 0, 0)] \\
&= (0, (\varphi_A(a) + \psi_A(x) + \varphi_{\mathcal{E}}(a) + \psi_{\mathcal{E}}(x))(m), 0, 0) \\
&\quad + (-1)^{|d||a+x|}(0, aM(m) + xM(m), 0, 0).
\end{aligned}$$

Hence

$$[M, a\cdot Id] = \varphi_A(a)\cdot Id + \varphi_{\mathcal{E}}(a) \text{ and } [M, x] = \psi_A(x)\cdot Id + \psi_{\mathcal{E}}(x)$$

for all $a \in A$ and $x \in \mathcal{E}$. Similarly,

$$[N, a\cdot Id] = \varphi_A(a)\cdot Id - \varphi_{\mathcal{E}}(a) \text{ and } [N, x] = -\psi_A(x)\cdot Id + \psi_{\mathcal{E}}(x)$$

for all $a \in A$ and $x \in \mathcal{E}$. Let $S, T \in \mathrm{End}_K \mathcal{M}$ be given by

$$\begin{aligned} S &= \frac{1}{2}(M+N); \\ T &= \frac{1}{2}(M-N). \end{aligned}$$

Then $[S, a\cdot Id] = \varphi_A(a)\cdot Id \in A\cdot Id$ and $[T, a\cdot Id] = \varphi_{\mathcal{E}}(a) \in \mathcal{E}$. Moreover, $[S,x] = \psi_{\mathcal{E}}(x) \in \mathcal{E}$ and $[T,x] = \psi_A(x)\cdot Id \in A\cdot Id$. Hence S satisfies (S1) and (S3) and T satisfies (T1) and (T3). Moreover, we have that

$$d(a,m,n,x) = ([S,a\cdot Id] + [T,x], (S+T)m, (S-T)n, [S,x] + [T,a\cdot Id])$$

and so for all $m, n \in \mathcal{M}$,

$$\begin{aligned} d(\mathbf{E}_{m,n}) &= d([(0,m,0,0),(0,0,n,0)]) - d(q(m,n),0,0,0) \\ &= [(0,(S+T)(m),(S-T)(n),0),(0,n,0,0)] \\ &\quad +(-1)^{|d||m|}[(0,m,0,0),(0,0,(S-T)(n),0)] \\ &\quad -([S,q(m,n)\cdot Id],0,0,[T,q(m,n)\cdot Id]) \\ &= (q((S+T)(m),n) + (-1)^{|d||m|}q(m,(S-T)(n)) - [S,q(m,n)\cdot Id]),0, \\ &\qquad 0, \mathbf{E}_{(S+T)(m),n} + (-1)^{|d||m|}\mathbf{E}_{m,(S-T)(n)} - [T,q(m,n)\cdot Id]). \end{aligned}$$

Since $\mathbf{E}_{m,n} = -(-1)^{|m||n|}\mathbf{E}_{n,m}$, we have to have

$$\begin{aligned} &q((S+T)(m),n) + (-1)^{|d||m|}q(m,(S-T)(n)) - [S,q(m,n)\cdot Id]) = \\ &\quad -(-1)^{|m||n|}q((S+T)(n),m) - (-1)^{|d||n|+|m||n|}q(n,(S-T)(m)) \\ &\qquad + (-1)^{|m||n|}[S,q(n,m)\cdot Id]) \end{aligned}$$

and so

$$\begin{aligned} &(q((S+T)(m),n) + (-1)^{|d||n|+|m||n|+(|d|+|m|)|n|}q((S-T)(m),n) + \\ &\quad + (-1)^{|d||m|+(|d|+|n|)|m|}q((S-T)(n),m) + (-1)^{|m||n|}q((S+T)(n),m))\cdot Id = \\ &\qquad = 2[S,q(m,n)\cdot Id]. \end{aligned}$$

Therefore we have

$$(q(S(m),n) + (-1)^{|m||n|}q(S(n),m))\cdot Id = [S,q(m,n)\cdot Id]$$

and so S satisfies (S2).

Moreover, we have

$$
\begin{aligned}
\mathbf{E}_{(S+T)(m),n} + (-1)^{|d||m|}\mathbf{E}_{m,(S-T)(n)} - \varphi(q(m,n)) &= \\
&= -(-1)^{|m||n|}\mathbf{E}_{(S+T)(n),m} - (-1)^{|d||n|+|m||n|}\mathbf{E}_{n,(S-T)(m)} \\
&\quad + (-1)^{|m||n|}\varphi(q(n,m))
\end{aligned}
$$

and so

$$
\begin{aligned}
\mathbf{E}_{(S+T)(m),n} &- (-1)^{|d||n|+|m||n|+(|d|+|m|)|n|}\mathbf{E}_{(S-T)(m),n} - \\
&- (-1)^{|d||m|+(|d|+|n|)|m|}\mathbf{E}_{(S-T)(n),m} + (-1)^{|m||n|}\mathbf{E}_{(S+T)(n),m} = \\
&= 2[T, q(m,n)\cdot Id].
\end{aligned}
$$

Therefore, we have

$$
\mathbf{E}_{T(m),n} + (-1)^{|m||n|}\mathbf{E}_{T(n),m} = [T, q(m,n)\cdot Id])
$$

and so T satisfies (T2). Hence $(S,T) \in \mathcal{S} \oplus \mathcal{T}$ and so $d \in \mathcal{D}_{\mathcal{E}}$. Therefore we have

$$
\mathcal{D}_{\mathcal{E}} = (\mathrm{Der}_K\, \mathcal{E}_\infty)_0
$$

and so $\mathcal{D}_{\mathcal{E}}$ is a subalgebra of $\mathrm{Der}_K\, \mathcal{E}_\infty$. In particular we have, for $d = d_{S,T}$, $d' = d'_{S',T'} \in \mathcal{D}_{\mathcal{E}}$

$$
\begin{aligned}
&[d,d'](a,m,n,x) = \\
&\quad d([S', a\cdot Id] + [T', x], (S'+T')(m), (S'-T')(n), [S', x] + [T', a\cdot Id]) \\
&\quad - (-1)^{|d||d'|}d'([S, a\cdot Id] + [T, x], (S+T)(m), (S-T)(n), [S, x] + [T, a\cdot Id]) \\
&= ([S, [S', a\cdot Id] + [T', x]] + [T, [S, x] + [T', a\cdot Id]], (S+T)((S'+T')(m)), \\
&\qquad (S-T)((S-T')(n)), [S, [S', x] + [T', a\cdot Id]] + [T, [S', a\cdot Id] + [T', x]]) \\
&\quad - (-1)^{|d||d'|}([S', [S, a\cdot Id] + [T, x]] + [T', [S, x] + [T, a\cdot Id]], (S'+T')((S+T)(m)), \\
&\qquad (S'-T')((S-T)(n)), [S', [S, x] + [T, a\cdot Id]] + [T', [S, a\cdot Id] + [T, x]]) \\
&= ([[S, S'] + [T, T'], a\cdot Id] + [[S, T'] + [T, S'], x], ([S, S'] + [T, T'] + [S, T'] + [T, S'])(m), \\
&\qquad ([S, S'] + [T, T'] - [S, T'] - [T, S'])(n), [[S, S'] + [T, T'], x] + [[S, T'] + [T, S'], a\cdot Id]) \\
&= d_{[S,S']+[T,T'],[S,T']+[T,S']}(a,m,n,x).
\end{aligned}
$$

Hence

$$[d_{S,T}, d_{S',T'}] = d_{[S\oplus T, S'\oplus T']^{\sim}}. \tag{3.7}$$

Now, let $\phi : \mathcal{S} \oplus \mathcal{T} \longrightarrow \mathcal{D}_{\mathcal{E}}$ be defined by $\phi(S,T) = d_{S,T}$. Clearly this map is well defined and is K-linear since S and T are K-linear.

$\boxed{injective}$ Follows from Remark 3.2.12.

$\boxed{surjective}$ Follows from the definition of $\mathcal{D}_{\mathcal{E}}$.

$\boxed{LSA - homomorphism}$ Let $(S,T),(S',T') \in \mathcal{S} \oplus \mathcal{T}$. Then

$$\begin{aligned} \phi([(S,T),(S',T')]^{\sim}) &= d_{[(S,T),(S',T')]^{\sim}} \\ &\overset{(3.7)}{=} [d_{S,T}, d_{S',T'}] \\ &= [\phi(S,T), \phi(S',T')] \end{aligned}$$

and so ϕ is a Lie superalgebra isomorphism.

(2) Since $\mathcal{E}_\infty$ is a toral 3-graded Lie superalgebra, by Proposition 3.1.1.

$$\mathrm{Der}_K\, \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty + (\mathrm{Der}_K\, \mathcal{E}_\infty)_0$$

and so by (1),

$$\mathrm{Der}_K\, \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty + \mathcal{D}_{\mathcal{E}}.$$

(3) $\boxed{\supset}$ Let $d = d_{S,T} \in \mathcal{D}_{\mathcal{E}}$ such that

$$d(a,m,n,x) = ([S, a\cdot Id] + [T,x], (S+T)(m), (S-T)(n), [S,x] + [T, a\cdot Id])$$

for some $S \in \mathcal{E}$ and $T = b\cdot Id \in A\cdot Id$. Then since S and T are A-linear, and $x \in \mathcal{E}$ is A-linear,

$$[S, a\cdot Id] = 0; \ [T, a\cdot Id] = 0; \text{ and } [T,x] = [b\cdot Id, x] = 0.$$

Hence,

$$d(a,m,n,x) = (0, (S+T)(m), (S-T)(n), [S,x]) = [(b,0,0,S),(a,m,n,x)]$$

and so

$$d = \mathbf{ad}(b, 0, 0, S) \in \mathbf{ad}\,\mathcal{E}_\infty \cap \mathcal{D}_\mathcal{E}.$$

Therefore, $\{d_{S,T} \in \mathcal{D}_\mathcal{E} \mid S \in \mathcal{E}, T \in A\cdot Id\} \subset \mathbf{ad}\,\mathcal{E}_\infty \cap \mathcal{D}_\mathcal{E}$.

$\boxed{\subset}$ Let $d \in \mathbf{ad}\,\mathcal{E}_\infty \cap \mathcal{D}_\mathcal{E}$. Then $d = d_{S,T} = \mathbf{ad}(a_0, m_0, n_0, x_0)$ for some $S \in \mathcal{S}$, $T \in \mathcal{T}$, $(a_0, m_0, n_0, x_0) \in \mathcal{E}_\infty$. So for all $m \in \mathcal{M}$,

$$\begin{aligned} (S+T)(m) &= a_0 m + x_0 m \\ (S-T)(m) &= -a_0 m + x_0 m. \end{aligned}$$

Hence, since $\frac{1}{2} \in K$, $S(m) = x_0(m)$ and $T(m) = a_0 m$ for all $m \in \mathcal{M}$ and so $S = x_0 \in \mathcal{E}$ and $T = a_0\cdot Id \in A\cdot Id$. Therefore,

$$\mathbf{ad}\,\mathcal{E}_\infty \cap \mathcal{D}_\mathcal{E} \subset \{d_{S,T} \in \mathcal{D}_\mathcal{E} \mid S \in \mathcal{E},\ T \in A\cdot Id\}.$$

Hence,

$$\begin{aligned} \mathbf{ad}\,\mathcal{E}_\infty \cap \mathcal{D}_\mathcal{E} &= \{d_{S,T} \in \mathcal{D}_\mathcal{E} \mid S \in \mathcal{E},\ T \in A\cdot Id\} \\ &= \mathbf{ad}(A\cdot Id \oplus \mathcal{E}) = \mathbf{ad}(\mathcal{E}_\infty)_0. \end{aligned}$$

(4) Assume $\mathcal{S} = \mathcal{E} \oplus \mathcal{S}_0$ and $\mathcal{T} = A\cdot Id \oplus \mathcal{T}_0$ where $\mathcal{S}_0 \oplus \mathcal{T}_0$ is a subalgebra of $\mathcal{S} \oplus \mathcal{T}$. Then

$$[\mathcal{S} \oplus \mathcal{T}, \mathcal{E} \oplus A\cdot Id] = ([\mathcal{S}, \mathcal{E}] + [\mathcal{T}, A\cdot Id]) \oplus ([\mathcal{S}, A\cdot Id] + [\mathcal{T}, \mathcal{E}]) \subset \mathcal{E} \oplus A\cdot Id$$

and so $\mathcal{E} \oplus A\cdot Id$ is an ideal of $\mathcal{S} \oplus \mathcal{T}$. Hence

$$\mathcal{S} \oplus \mathcal{T} = (\mathcal{E} \oplus A\cdot Id) \rtimes (\mathcal{S}_0 \oplus \mathcal{T}_0).$$

Therefore, since by (1), $\mathcal{S} \oplus \mathcal{T} \cong \mathcal{D}_\mathcal{E}$ as Lie superalgebras via $S \oplus T \mapsto d_{S,T}$, we have

$$\begin{aligned} \mathcal{D}_\mathcal{E} &= \{d_{S,T} \mid S \in \mathcal{E},\ T \in A\cdot Id\} \rtimes \{d_{S,T} \mid S \in \mathcal{S}_0,\ T \in \mathcal{T}_0\} \\ &= \mathbf{ad}(\mathcal{E} \oplus A\cdot Id) \rtimes \{d_{S,T} \mid S \in \mathcal{S}_0,\ T \in \mathcal{T}_0\} \end{aligned}$$

i.e. $\{d_{S,T} \mid S \in \mathcal{S}_0,\ T \in \mathcal{T}_0\}$ is a subalgebra of the Lie superalgebra $\mathcal{D}_\mathcal{E}$. By (3), we have $\mathbf{ad}\,\mathcal{E}_\infty \cap \mathcal{D}_\mathcal{E} = \mathbf{ad}(\mathcal{E}_\infty)_0 = \mathbf{ad}(\mathcal{E} \oplus A\cdot Id)$ and so by (2),

$$\mathrm{Der}_K\,\mathcal{E}_\infty = \mathbf{ad}\,\mathcal{E}_\infty + \mathcal{D}_\mathcal{E} = \mathbf{ad}\,\mathcal{E}_\infty \oplus \{d_{S,T} \mid S \in \mathcal{S}_0, T \in \mathcal{T}_0\}.$$

Since $\{d_{S,T} \mid S \in \mathcal{S}_0,\ T \in \mathcal{T}_0\}$ is a subalgebra of $\mathcal{D}_\mathcal{E}$ and $\mathcal{D}_\mathcal{E}$ is a subalgebra of $\mathrm{Der}_K\, \mathcal{E}_\infty$, we have

$$\begin{aligned} \mathrm{Der}_K\, \mathcal{E}_\infty &= \mathbf{ad}\, \mathcal{E}_\infty \rtimes \{d_{S,T} \mid S \in \mathcal{S}_0,\ T \in \mathcal{T}_0\} \\ &= \mathbf{ad}\, \mathcal{E}_\infty \rtimes \widehat{\mathcal{S}_0 \oplus \mathcal{T}_0} \end{aligned}$$

where

$$\begin{aligned} \widehat{\ } : \mathcal{S}_0 \oplus \mathcal{T}_0 &\to \mathrm{Der}_K\, \mathcal{E}_\infty \\ S \oplus T &\mapsto d_{S,T} \end{aligned}$$

is a Lie superalgebra monomorphism. ∎

Recall from Section 1.3 that locally inner derivations of a Lie superalgebra L are derivations d of L such that for each finite set $\{v_1, \ldots, v_n\} \subset L$ there exists $v \in L$ such that $d(v_i) = [v, v_i]$.

Proposition 3.2.15 *Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q and let $\mathcal{E} = \mathrm{eosp}(q)$. Then $\{d_{S,T} \mid S \in \overline{\mathrm{eosp}(q)},\ T \in \overline{A\cdot Id}\} \subset \mathbf{ad}_{\mathrm{loc}}\, \mathcal{E}_\infty$.*

Proof: Recall first from Remark 3.2.4 and Proposition 3.2.7 that $\overline{A\cdot Id} \subset \mathcal{T}_\mathcal{E}$ and $\mathrm{osp}(q) \subset \mathcal{S}$. If we consider $S = \sum_{i\in I} y_i \in \overline{\mathrm{eosp}(q)} \subset \mathrm{osp}(q)$ and $T = \sum_{j\in J} a_j\cdot Id$ then for all $(a, m, n, x) \in \mathcal{E}_\infty$

$$\begin{aligned} d_{S,T}(a, m, n, x) &= (0, (S+T)(m), (S-T)(n), [S, x]) \\ &= (0, \sum_{i\in I_m} y_i(m) + (\sum_{j\in J_m} a_j)m, \sum_{i\in I_n} y_i(n) - (\sum_{j\in J_m} a_j)n, [S, x]), \end{aligned}$$

where we use the fact that S and T are A-linear and that $[\overline{A\cdot Id}, \mathcal{E}] = 0$. Let $v_l = (a_l, m_l, n_l, x_l) \in \mathrm{eosp}(q_\infty)$, $l = 1, \ldots, N$. Since $x_l \in \mathrm{eosp}(q)$ we can write

$$x_l = \sum_{k=1}^{K_l} \mathbf{E}_{p_{lk}, r_{lk}}$$

where $p_{lk}, r_{lk} \in \mathcal{M}$. Then using the fact that S is eosp(q)-summable and using Lemma 2.1.2 we get

$$S(m_l) = \left(\sum_{i\in I} y_i\right)(m_l) = \sum_{i\in I_{m_l}} y_i(m_l); \qquad S(n_l) = \left(\sum_{i\in I} y_i\right)(n_l) = \sum_{i\in I_{n_l}} y_i(n_l);$$
$$T(m_l) = \left(\sum_{j\in J} a_j\right)(m_l) = \sum_{j\in J_{m_l}} a_j(m_l); \quad T(n_l) = \left(\sum_{j\in J} a_j\right)(n_l) = \sum_{j\in J_{n_l}} a_j(n_l)$$

and

$$\begin{aligned}[S, x_l] &= [S, \sum_{k=1}^{K_l} \mathbf{E}_{p_{lk}, r_{lk}}] = \sum_{k=1}^{K_l} (\mathbf{E}_{S(p_{lk}), r_{lk}} + (-1)^{|S||p_{lk}|} \mathbf{E}_{p_{lk}, S(r_{lk})}) \\ &= \sum_{k=1}^{K_l} \sum_{i\in I_{p_{lk}} \cup I_{r_{lk}}} (\mathbf{E}_{y_i(p_{lk}), r_{lk}} + (-1)^{|S||p_{lk}|} \mathbf{E}_{p_{lk}, y_i(r_{lk})})\end{aligned}$$

Take

$$J_{v_1,\dots,v_N} = \bigcup_{l=1}^{n} (J_{m_l} \cup J_{n_l}); \text{ and}$$

$$I_{v_1,\dots,v_N} = \bigcup_{l=1}^{n} \left(I_{m_l} \cup J_{n_l} \cup \left(\bigcup_{k=1}^{K_l} (I_{p_{lk}} \cup I_{r_{lk}}) \right) \right).$$

Then $I_{v_1,\dots,v_N}$ and $J_{v_1,\dots,v_N}$ are finite and

$$\left[\left(\sum_{i\in I_{v_1,\dots,v_N}} y_i \right) + \left(\sum_{j\in J_{v_1,\dots,v_N}} a_j \right), v_l \right] = d_{S,T}(v_l)$$

where $\sum_{j\in J_{v_1,\dots,v_N}} a_j \in A \cdot Id$ and $\sum_{i\in I_{v_1,\dots,v_N}} y_i \in \text{eosp}(q)$. Therefore the assertion follows. ∎

In fact, the A-supermodules $\mathcal{S}$ and $\mathcal{T}$ can be described in the following terms: Recall the Jordan superalgebra $J = A \oplus \mathcal{M}$ with the multiplication $\circ$ given by

$$(a \oplus m) \circ (b \oplus n) = (ab + q(m, n)) \oplus (an + mb). \tag{3.8}$$

In addition, given a subalgebra $\mathcal{E}$ of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$, recall the J-supermodule X from Example 1.5.14 where $X = \mathcal{E} \oplus \mathcal{M}$ with action of J on X given by

$$\begin{aligned}(x \oplus p) * (a \oplus m) &= (xa + \mathbf{E}_{p,m}) \oplus (x(m) + pa) \\ &= (-1)^{|a\oplus m||x \oplus p|}(a \oplus m) * (x \oplus p)\end{aligned}$$

for $a \in K$, $m, p \in \mathcal{M}$ and $x \in \mathcal{E}$. Recall that

$$\begin{aligned}\mathrm{Der}_K(J) = \{d \in \mathrm{End}_K(J) \mid d((a \oplus m)\circ(b \oplus n)) &= d(a \oplus m) \circ (b \oplus n) \\ &+ (-1)^{|d||a\oplus m|}(a \oplus m) \circ d(b \oplus n).\end{aligned}$$

Let

$$\mathrm{Der}_*(J) = \{d \in \mathrm{Der}_K(J) \mid d(A) \subset A \text{ and } d(\mathcal{M}) \subset \mathcal{M}\}.$$

Set

$$\begin{aligned}\mathrm{Der}_K(J, X) = \{d \in \mathrm{Hom}_K(J, X) \mid d((a \oplus m)\circ(b \oplus n)) &= d(a \oplus m) * (b \oplus n) \\ &+ (-1)^{|d||a\oplus m|}(a \oplus m) * d(b \oplus n)\}\end{aligned}$$

and

$$\mathrm{Der}_*(J, X) = \{d \in \mathrm{Der}_K(J, X) \mid d(\mathcal{E}) \subset A \text{ and } d(\mathcal{M}) \subset \mathcal{M}\}.$$

Proposition 3.2.16 *In the above setting we have that*

(1) the map $\mathrm{Der}_*(J) \to \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} : d \mapsto d|_{\mathcal{M}}$ *is a Lie superalgebra isomorphism; and*

(2) the map $\mathrm{Der}_*(J, X) \to \mathcal{T}^{(1)} \cap \mathcal{T}^{(2)} : d \mapsto d|_{\mathcal{M}}$ *is a* K*-supermodule isomorphism.*

Proof: (1) Let $d \in \mathrm{End}_K \mathcal{M}$ such that $d(A) \subset A$ and $d(\mathcal{M}) \subset \mathcal{M}$. Then for all homogeneous $a \oplus m, b \oplus n \in A \oplus \mathcal{M}$,

$$d((a \oplus m) \circ (b \oplus n)) = (d(ab) + d(q(m, n)) \oplus (d(an) + d(mb))$$

and

$$\begin{aligned}d(a \oplus m) \circ (b \oplus n) + &(-1)^{|d||a+m|}(a \oplus m) \circ d(b \oplus n) = \\ &= (d(a)b + q(d(m), n) + (-1)^{|d||a+m|}(ad(b) + q(m, d(n))) \\ &\quad \oplus (d(a)n + d(m)b + (-1)^{|d||a+m|}(ad(n) + md(b))).\end{aligned}$$

Hence $d \in \mathrm{Der}_*(J)$ if and only if

$$d(ab) = d(a)b + (-1)^{|d||a|}ad(b) \text{ for all } a, b \in \mathcal{A}; \tag{3.9}$$

$$d(am) = d(a)m + (-1)^{|d||a|}ad(m) \text{ for all } a \in \mathcal{A},\ m \in \mathcal{M}; \text{ and} \tag{3.10}$$

$$\begin{aligned}d(q(m, n)) &= q(d(m), n) + (-1)^{|d||a|}q(m, d(n)) \\ &= q(d(m), n) + (-1)^{|m||n|}q(d(n), m) \text{ for all } m, n \in \mathcal{M}\end{aligned} \tag{3.11}$$

which holds if and only if

$$d|_{\mathcal{A}} \in \mathrm{Der}_K \mathcal{A}; \tag{3.12}$$

$$[d|_{\mathcal{M}}, a \cdot Id] = d(a) \cdot Id|_{\mathcal{M}} \text{ for all } a \in \mathcal{A},\ m \in \mathcal{M}; \text{ and} \tag{3.13}$$

$$\begin{aligned}[d, q(m, n) \cdot Id](p) &= d(q(m, n)p) - (-1)^{|d|(|m|+|n|)}q(m, n)d(p) \\ &\overset{(3.10)}{=} d(q(m, n))p - (-1)^{|d|(|m|+|n|)}q(m, n)d(p) \\ &\quad + (-1)^{|d|(|m|+|n|)}q(m, n)d(p) \\ &\overset{(3.11)}{=} (q(d(m), n) + (-1)^{|m||n|}q(d(n), m))p.\end{aligned} \tag{3.14}$$

Now, (3.13) holds if and only if $d|_{\mathcal{M}}$ satisfies (S1) and (3.14) holds if and only if $d|_{\mathcal{M}}$ satisfies (S2). Moreover, by Proposition 3.2.7.1.1, (3.12) follows from (3.13). Hence we have that $d \in \mathrm{Der}_*(\mathcal{A} \oplus \mathcal{M})$ if and only if $d|_{\mathcal{M}} \in \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. It is obvious that the map is a Lie superalgebra isomorphism.

(2) Let $d \in \mathrm{Hom}_K(\mathcal{A} \oplus \mathcal{M}, \mathcal{E} \oplus \mathcal{M})$ such that $d(\mathcal{A}) \subset \mathcal{E}$ and $d(\mathcal{M}) \subset \mathcal{M}$. Then

$$d((a \oplus m) \circ (b \oplus n)) = (d(ab) + d(q(m, n))) \oplus (d(an) + d(ma))$$

and

$$\begin{aligned}d(a \oplus m) * (b \oplus n) + &(-1)^{|d||a+m|}(a \oplus m) * d(b \oplus n) = \\ &= (d(a)b + \mathbf{E}_{d(m), n} + (-1)^{|a+m||b+n|}(d(b)a + \mathbf{E}_{d(n), m})) \\ &\quad \oplus (d(a)(n) + d(m)b + (-1)^{|a+m||b+n|}d(n)a + d(b)(m)).\end{aligned}$$

Hence $d \in \mathrm{Der}_*(A \oplus \mathcal{M}, \mathcal{E} \oplus \mathcal{M})$ if and only if

$$d(ab) = d(a)b + (-1)^{|d||a|} a d(b) \text{ for all } a, b \in A; \tag{3.15}$$

$$d(am) = d(a)(m) + (-1)^{|d||a|} a d(m) \text{ for all } a \in A,\ m \in \mathcal{M}; \text{ and} \tag{3.16}$$

$$d(q(m,n)) = \mathbf{E}_{d(m),n} + (-1)^{|m||n|}\mathbf{E}_{d(n),m} \text{ for all } m, n \in \mathcal{M} \tag{3.17}$$

which holds if and only if

$$d|_A \in \mathrm{Der}_K(A, \mathcal{E}); \tag{3.18}$$

$$[d, a \cdot Id]|_{\mathcal{M}} = d(a)|_{\mathcal{M}} \cdot Id; \tag{3.19}$$

$$\begin{aligned} [d, q(m,n) \cdot Id](p) &= d(q(m,n)p) - (-1)^{|d|(|m|+|n|)} q(m,n) d(p) \\ &\overset{(3.16)}{=} d(q(m,n))(p) \\ &\overset{(3.17)}{=} (\mathbf{E}_{d(m),n} + (-1)^{|m||n|}\mathbf{E}_{d(n),m})(p). \end{aligned} \tag{3.20}$$

It is clear that (3.19) and (3.20) hold if and only if $d|_{\mathcal{M}}$ satisfies (T1) and (T2) respectively. Moreover, by Proposition 3.2.7.2.1, (3.19) implies (3.18) and so the assertion holds. ∎

3.3 Derivations as a semidirect product of the inner derivations and the outer derivations

Here we will present some examples of subalgebras of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$ where we get a direct splitting of the algebra of derivations into inner and outer derivations.

We let $A^\times$ to be the set of all invertible elements of A. Recall that, given a free A-supermodule $\mathcal{M} = \oplus_{i \in I} A m_i$ with a homogeneous basis $\{\mathcal{M}_i \mid i \in I\}$ and an A-quadratic form q, we say that q is almost diagonizable if for each $i \in I$ there exists $\underset{\cdot}{i} \in I$ such that

$$q(m_i, m_{\underset{\cdot}{i}}) \in A^\times; \text{ and}$$

$$q(m_i, m_j) = 0 \text{ for all } j \in I,\ j \neq \underset{\cdot}{i}.$$

If q is almost diagonizable then for each $i \in I$, $\underset{=}{i}$ is unique, $\underset{=}{i} = i$ and $|m_i| = |m_{\underset{=}{i}}|$. In addition, if q is almost diagonizable with respect to $\{m_i \mid i \in I\}$ where $i = \underset{=}{i}$ then we say that q is *invertibly diagonizable*. Note that if q is invertibly diagonizable with respect to $\{m_i \mid i \in I\}$ then the basis elements m_i are even.

Proposition 3.3.1 *Let $\mathcal{P}$ be an A-supermodule with an A-quadratic form $q_{\mathcal{P}}$ and an element $p \in \mathcal{P}$ such that $\mathrm{Ann}_A(p) = (0)$. Let $\mathcal{N} = \oplus_{i=\pm 1} An_i$ be a free A-supermodule with an invertibly diagonizable A-quadratic form $q_{\mathcal{N}}$. Set $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ with $q = q_{\mathcal{N}} \oplus q_{\mathcal{P}}$ and let $\mathcal{E}$ be a subalgebra of $\mathrm{osp}(q)$ containing $\mathrm{eosp}(q)$. Then*

$$\mathcal{T}_{\mathcal{E}} = \mathcal{T}^{(2)} = A \cdot Id$$

and

$$\mathbf{ad}(\mathcal{E}_\infty) + \{d_{S,0} \in \mathcal{D} \mid S \in \mathcal{S}_{\mathcal{E}}\} = \mathrm{Der}_K \mathcal{E}_\infty = \{d_{S,0} \mid S \in \mathcal{E}\}$$

with

$$\mathbf{ad}\,\mathcal{E} = \mathbf{ad}(\mathcal{E}_\infty) \cap \{d_{S,0} \in \mathcal{D} \mid S \in \mathcal{S}_{\mathcal{E}}\}.$$

Hence, whenever $\mathcal{S}_{\mathcal{E}} = \mathcal{E} \rtimes \mathcal{S}_0$,

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) \rtimes \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathcal{S}_0\}.$$

Proof: Recall that by Remark 3.2.4, $A \cdot Id \subset \mathcal{T} \subset \mathcal{T}^{(2)}$. Let $T \in \mathcal{T}^{(2)}$ and write

$$T = \begin{pmatrix} T_{\mathcal{N},\mathcal{N}} & T_{\mathcal{P},\mathcal{N}} \\ T_{\mathcal{N},\mathcal{P}} & T_{\mathcal{P},\mathcal{P}} \end{pmatrix}$$

with respect to $\begin{pmatrix} \mathcal{N} \\ \mathcal{P} \end{pmatrix}$, so $T_{\mathcal{N},\mathcal{P}} \in \mathrm{Hom}_K(\mathcal{N}, \mathcal{P})$, etc. Let $p \in \mathcal{P}$ and let $i, k \in \{\pm 1\}$, $k \neq i$. Then by (T2),

$$\begin{aligned} 0 &= [T, q(an_i, p) \cdot Id](n_k) = \mathbf{E}_{T(an_i),p}(n_k) + (-1)^{|a||p|}\mathbf{E}_{T(p),an_i}(n_k) \\ &= -q(T(an_i), n_k)p - (-1)^{|a||p|}q(T(p), n_k)an_i. \end{aligned} \tag{3.21}$$

Since $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ and $q(\mathcal{N}, \mathcal{P}) = q(\mathcal{P}, \mathcal{N}) = 0$ we have that if $p \in \mathcal{P}$ is such that $\mathrm{Ann}_A(p) = (0)$, then

$$0 = q(T(an_i), n_k) = q_{\mathcal{N}}(T_{\mathcal{N},\mathcal{N}}(an_i), n_k) \text{ for all } i, k \in I,\ k \neq i.$$

We write $T_{\mathcal{N},\mathcal{N}}(an_i) = t_{i,1}(a)n_1 + t_{i,-1}(a)n_{-1}$ where $t_i \in \mathrm{End}_K A$ for $i \in \{\pm 1\}$ and we obtain

$$0 = t_{i,k}(a)q(n_k, n_k).$$

Since $q(n_k, n_k) \in A^{\times}$, we therefore have that $t_{i,k}(a) = 0$ for $i \neq k$ and so $T_{\mathcal{N},\mathcal{N}}(an_i) = t_{ii}(a)n_i$. Moreover, since $p \in \mathcal{P}$ and $i \in \{\pm 1\}$ were arbitrary in (3.21) we have that

$$0 = q(T(p), n) = q_{\mathcal{N}}(T_{\mathcal{P},\mathcal{N}}(p), n) \text{ for all } n \in \mathcal{N},\ p \in \mathcal{P}$$

and so since $q_{\mathcal{N}}$ is nondegenerate, we have that $T_{\mathcal{P},\mathcal{N}} = 0$. Now, let $i \in \{\pm 1\}$. Then for all $p \in \mathcal{P}$,

$$\begin{aligned} 0 &= [T, q(n_i, p)\cdot Id](n_i) = \mathbf{E}_{T(n_i),p}(n_i) + \mathbf{E}_{T(p),n_i}(n_i) \\ &= -q(T(n_i), n_i)p + T(p)q(n_i, n_i) \\ &= -t_{ii}(1)q(n_i, n_i)p - T(p)q(n_i, n_i) \end{aligned}$$

and so we have that $T(p) = t_{ii}(1)p$, $i \in \{\pm 1\}$. Set $t := t_{ii}(1)$.

Let $i, k \in \{\pm 1\}$, $i \neq k$ and let $a \in A$. Then

$$\begin{aligned} 0 &= [T, q(n_i, an_k)\cdot Id](n_i) = (\mathbf{E}_{T(n_i),an_k} + \mathbf{E}_{T(an_k),n_i})(n_i) \\ &= -q(T(n_i), n_i)an_k + T(an_k)q(n_i, n_i) - q(T(an_k), n_i)n_i \\ &= -tq(n_i, n_i)an_k - t_{kk}(a)n_kq(n_i, n_i - T_{\mathcal{N},\mathcal{P}}(an_k)q(n_i, n_i)). \end{aligned}$$

Since $q(n_i, n_i) \in A^{\times}$, $T_{\mathcal{N},\mathcal{P}} = 0$ and $t_{kk}(a) = ta$. Hence $T = t\cdot Id$ and so $\mathcal{T} = A\cdot Id$. The rest of the assertion now follows from Theorem 3.2.13.3 and 3.2.13.4. ■

Remark 3.3.2 For I and $\mathcal{P}$ different from above the situation seems to be more complicated. For example, if $\mathrm{rank}(\mathcal{N}) = 2$ and $\mathcal{P} = (0)$ then $\mathcal{T} \neq A\cdot Id \rtimes \mathcal{T}_0$ for a submodule $\mathcal{T}_0$.

Recall that if $\mathcal{N}$ and $\mathcal{P}$ are A-supermodules with A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{P}}$ respectively, then for $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ with $q = q_{\mathcal{N}} \oplus q_{\mathcal{P}}$,

$$\mathrm{eosp}(q) = \mathrm{eosp}(q_{\mathcal{N}}) \oplus \mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}}) \oplus \mathrm{eosp}(q_{\mathcal{P}})$$

where $\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}}) = \mathrm{span}_A\{\mathbf{E}_{n,p} \mid n \in \mathcal{N},\ p \in \mathcal{P}\}$ and

$$\overline{\mathrm{eosp}(q)} = \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{P}})}$$

We define

$$\mathcal{S}_{\mathcal{N}}^{\mathcal{P}} = \{S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{N}}^{(2)} \oplus \mathcal{S}_{\mathcal{P}}^{(2)} \mid [S_{\mathcal{N}} \oplus S_{\mathcal{P}}, a] = \Delta(a) \text{ for some } \Delta \in \mathrm{Der}_K A\}.$$

Proposition 3.3.3 *Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ where $\mathcal{N}$ and $\mathcal{P}$ are A-supermodules with A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{P}}$ respectively. Assume that $q_{\mathcal{N}}$ is almost diagonizable with respect to some basis $\{n_i \mid i \in I\}$. Then*

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$$

and

$$\mathcal{S} = \{S \in \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{N}}^{\mathcal{P}} \mid [S, \mathcal{E}] \subset \mathcal{E}\}.$$

Proof: Let $S \in \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. By (S1), $[S, a\cdot Id] = \Delta(a)\cdot Id$ for some $\Delta \in \mathrm{Der}_K A$. Write

$$S = \begin{pmatrix} S_{\mathcal{N},\mathcal{N}} & S_{\mathcal{N},\mathcal{P}} \\ S_{\mathcal{P},\mathcal{N}} & S_{\mathcal{P},\mathcal{P}} \end{pmatrix}$$

where $S_{\mathcal{N},\mathcal{P}} \in \mathrm{Hom}_K(\mathcal{N}, \mathcal{P})$, etc. Then for all $n \in \mathcal{N}$ and $p \in \mathcal{P}$,

$$\begin{aligned} \Delta(a)(n+p) &= [S, a](n+p) = S(a(n+p)) - (-1)^{|S||a|} aS(n+p) \\ &= S_{\mathcal{N},\mathcal{N}}(an) - (-1)^{|S||a|} aS_{\mathcal{N},\mathcal{N}}(n) + S_{\mathcal{N},\mathcal{P}}(an) - (-1)^{|S||a|} aS_{\mathcal{N},\mathcal{P}}(n) \\ &\quad + S_{\mathcal{P},\mathcal{P}}(ap) - (-1)^{|S||a|} aS_{\mathcal{P},\mathcal{P}}(p) + S_{\mathcal{P},\mathcal{N}}(ap) - (-1)^{|S||a|} aS_{\mathcal{P},\mathcal{N}}(p) \end{aligned}$$

and so

$$\begin{aligned} [S_{\mathcal{N},\mathcal{N}} \oplus S_{\mathcal{P},\mathcal{P}}, a] &= \Delta(a), \text{ and} \\ S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}} &\in \mathrm{Hom}_A \mathcal{M}. \end{aligned}$$

In addition, for all $n, n' \in \mathcal{N}$, $p, p' \in \mathcal{P}$,

$$\begin{aligned}[S_{\mathcal{N},\mathcal{N}}, q(n,n')]|_{\mathcal{N}} &= \Delta(q(n,n'))|_{\mathcal{N}} = [S, q(n,n')]|_{\mathcal{N}} \\ &= q(S(n),n') + (-1)^{|n||n'|}q(S(n'),n) \\ &\overset{\mathcal{N}\perp\mathcal{P}}{=} q(S_{\mathcal{N},\mathcal{N}}(n),n') + (-1)^{|n||n'|}q(S_{\mathcal{N},\mathcal{N}}(n'),n)\end{aligned}$$

and, similarly,

$$[S_{\mathcal{P},\mathcal{P}}, q(p,p')]|_{\mathcal{P}} = q(S_{\mathcal{P},\mathcal{P}}(p),p') + (-1)^{|p||p'|}q(S_{\mathcal{P},\mathcal{P}}(p'),p).$$

Hence $S_{\mathcal{N},\mathcal{N}} \oplus S_{\mathcal{P},\mathcal{P}} \in \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$. Moreover,

$$\begin{aligned}&q((S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}}(n+p), n'+p') + (-1)^{|n+p||n'+p'|}q((S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}}(n'+p'), n+p) \\ &= q(S(n+p), n'+p') + (-1)^{|n+p||n'+p'|}q(S(n'+p'), n+p) \\ &\quad - q(S_{\mathcal{N},\mathcal{N}}(n+p), n'+p') - (-1)^{|n+p||n'+p'|}q(S_{\mathcal{N},\mathcal{N}}(n'+p'), n+p) \\ &\quad - q(S_{\mathcal{P},\mathcal{P}}(n+p), n'+p') - (-1)^{|n+p||n'+p'|}q(S_{\mathcal{P},\mathcal{P}}(n'+p'), n+p) \\ &= [S, q(n+p, n'+p')] - \Delta(q(n,n')) - \Delta(q(p,p')) \\ &= 0\end{aligned}$$

and so by Proposition 2.2.4

$$S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}} \in \operatorname{osp}(q) = \overline{\operatorname{eosp}(q_{\mathcal{N}})} \oplus \overline{\operatorname{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \operatorname{osp}(q_{\mathcal{P}}).$$

Hence clearly

$$S_{\mathcal{N},\mathcal{P}} \oplus S_{\mathcal{P},\mathcal{N}} \in \overline{\operatorname{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})}$$

and so

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} \subset \overline{\operatorname{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}.$$

Conversely, to show that $\overline{\operatorname{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{N}}^{\mathcal{P}} \subset \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$, by Proposition 3.2.7.1.4, we only need to show that $\mathcal{S}_{\mathcal{N}}^{\mathcal{P}} \subset \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. Let $S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$ and let $\Delta \in \operatorname{Der}_K A$ such that $[S_{\mathcal{N}} \oplus S_{\mathcal{P}}, a] = \Delta(a)$. Hence clearly $S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}^{(1)}$. Moreover, for all

$n, n' \in \mathcal{N}$, $p, p' \in \mathcal{P}$,

$$\begin{aligned} q((S_{\mathcal{N}} \oplus S_{\mathcal{P}})(n+p), n'+p') &+ (-1)^{|n+p||n'+p'|} q((S_{\mathcal{N}} \oplus S_{\mathcal{P}})(n'+p'), n+p) \\ &= q_{\mathcal{N}}(S_{\mathcal{N}}(n), n') + (-1)^{|n||n'|} q_{\mathcal{N}}(S_{\mathcal{N}}(n'), n) \\ &\quad + q_{\mathcal{P}}(S_{\mathcal{P}}(p), p') + (-1)^{|p||p'|} q_{\mathcal{P}}(S_{\mathcal{P}}(p'), p) \\ &= \Delta(q_{\mathcal{N}}(n, n')) + \Delta(q_{\mathcal{P}}(p, p') \\ &= \Delta(q(n+p, n'+p')) \end{aligned}$$

and so $S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}^{(2)}$. Rest of the assertion now follows. ■

Proposition 3.3.4 *Let $\mathcal{N}$ and $\mathcal{R}$ be free with homogeneous bases $\{n_i \mid i \in I\}$ and $\{r_j \mid j \in J\}$ and A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{R}}$ respectively, where $q_{\mathcal{N}}$ is almost diagonizable with respect to $\{n_i \mid i \in I\}$ and $q_{\mathcal{R}} \equiv 0$. Set $\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$ with the A-quadratic form $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$. For $\Delta \in \operatorname{Der}_K A$ define $\Delta_{\mathcal{M}} \in \operatorname{End}_A \mathcal{M}$ by setting*

$$\begin{aligned} \Delta_{\mathcal{M}}\left(\sum_{i\in I} a_i n_i + \sum_{j\in J} b_j r_j\right) = \sum_{i\in I}\left(\frac{1}{2}\Delta(q(n_i, n_i))q(n_i, n_i)^{-1}a_i + \Delta(a_i)\right) n_i \\ + \sum_{j\in J}\Delta(b_j)r_j. \end{aligned} \tag{3.22}$$

Set $(\operatorname{Der}_K A)_{\mathcal{M}} = \{\Delta_{\mathcal{M}} \mid \Delta \in \operatorname{Der}_K A\}$. Then the map $\operatorname{Der}_K A \to (\operatorname{Der}_K A)_{\mathcal{M}} : \Delta \mapsto \Delta_{\mathcal{M}}$ is a Lie superalgebra isomorphism and

$$\begin{aligned} \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} &= \operatorname{osp}(q) \rtimes (\operatorname{Der}_K A)_{\mathcal{M}} \\ &= \overline{\operatorname{eosp}(q)} \rtimes (\operatorname{End}_A \mathcal{R} \rtimes (\operatorname{Der}_K A)_{\mathcal{M}}). \end{aligned}$$

Hence if $\mathcal{E}$ is a subalgebra of $\operatorname{osp}(q)$ containing $\operatorname{eosp}(q)$ then

$$\mathcal{S}_{\mathcal{E}} = \{S \in \operatorname{osp}(q) \rtimes (\operatorname{Der}_K A)_{\mathcal{M}} \mid [S, \mathcal{E}] \subset \mathcal{E}\}$$

and in particular, if $\mathcal{E} = \operatorname{eosp}(q)$, $\overline{\operatorname{eosp}(q)}$ or $\operatorname{osp}(q)$ then

$$\begin{aligned} \mathcal{S}_{\mathcal{E}} &= \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \operatorname{osp}(q) \rtimes (\operatorname{Der}_K A)_{\mathcal{M}} \\ &= \overline{\operatorname{eosp}(q)} \rtimes (\operatorname{End}_A \mathcal{R} \rtimes (\operatorname{Der}_K A)_{\mathcal{M}}). \end{aligned}$$

Proof: Let $S \in \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. By (S1), $[S, a\cdot Id] = \Delta(a)\cdot Id$ for some $\Delta \in \mathrm{Der}_K A$. Then for each $m \in \mathcal{M}$ and $a \in A$,

$$S(am) = (-1)^{|S||a|} aS(m) + \Delta(a)m.$$

Set $S_0 \in \mathrm{End}_A \mathcal{M}$ by

$$\begin{aligned} S_0(n_i) &= S(n_i) - \frac{1}{2}\Delta(q(n_i, n_{\underline{i}}))q(n_i, n_{\underline{i}})^{-1} n_i \\ S_0(r_j) &= S(r_j) \end{aligned}$$

for $i \in I$, $j \in J$. Then for each $i, k \in I$,

$$\begin{aligned} q(S_0(n_i), n_k) &+ (-1)^{|n_i||n_k|} q(S_0(n_k), n_i) \\ &= q(S(n_i), n_k) - \frac{1}{2}\Delta(q(n_i, n_{\underline{i}}))q(n_i, n_{\underline{i}})^{-1} q(n_i, n_k) \\ &\quad + (-1)^{|n_i||n_k|}(q(S(n_k), n_i) - \frac{1}{2}\Delta(q(n_k, n_{\underline{k}}))q(n_k, n_{\underline{k}})^{-1} q(n_k, n_i)) \\ &= \delta_{i\underline{k}}\left([S, q(n_i, n_{\underline{i}})\cdot Id] - \Delta(q(n_i, n_{\underline{i}}))\right) \\ &\overset{(S2)}{=} 0. \end{aligned}$$

Moreover, for each $i \in I$, $j \in J$,

$$q(S_0(r_j), n_i) + (-1)^{|r_j||n_i|} q(S_0(n_i), r_j) = q(S(r_j), n_i) \overset{(S2)}{=} [S, q(r_j, n_i)\cdot Id] = 0$$

and for all $j, l \in J$,

$$q(S_0(r_j), r_l) + (-1)^{|r_j||r_l|} q(S_0(r_l), r_j) = 0.$$

Hence since $S_0 \in \mathrm{End}_A \mathcal{M}$, $S_0 \in \mathrm{osp}(q)$ and so

$$\begin{aligned} S(an_i + br_j) &= (-1)^{|S||a|} aS(n_i) + (-1)^{|S||b|} bS(r_j) + \Delta(a)n_i + \Delta(b)r_j \\ &= S_0(an_i) + (\frac{1}{2}\Delta(q(n_i, n_{\underline{i}}))q(n_i, n_{\underline{i}})^{-1} a + \Delta(a))n_i \\ &\quad + S_0(br_j) + \Delta(b)r_j \\ &= (S_0 + \Delta_{\mathcal{M}})(an_i + br_j). \end{aligned}$$

Therefore it follows by linearity of S that

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} \subset \mathrm{osp}(q) + (\mathrm{Der}_K A)_{\mathcal{M}}.$$

Conversely, to show that $\operatorname{osp}(q) + (\operatorname{Der}_K A)_{\mathcal{M}} \subset \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$, by Proposition 3.2.7.1.4, we only need to show that $(\operatorname{Der}_K A)_{\mathcal{M}} \subset \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. So let $\Delta \in \operatorname{Der}_K A$. By linearity, it suffices to show that $\Delta_{\mathcal{M}}$ satisfies (S1) and (S2) for elements of $\mathcal{M}$ of the form $an_i + br_j$ for some $a, b \in A$, $i \in I$, $j \in J$. We have

$$\begin{aligned}
[\Delta_{\mathcal{M}}, a \cdot Id](bn_i + cr_j) &= \Delta_{\mathcal{M}}(abn_i + acr_j) - (-1)^{|\Delta||a|} a\Delta_{\mathcal{M}}(bn_i) - (-1)^{|\Delta||a|} a\Delta_{\mathcal{M}}(cr_j) \\
&= (\Delta(q(n_i, n_{\underline{i}}))q(n_i, n_{\underline{i}})^{-1}ab + \Delta(ab))n_i \\
&\quad - (\Delta(q(n_i, n_{\underline{i}}))q(n_i, n_{\underline{i}})^{-1}ab + (-1)^{|\Delta||a|}a\Delta(b))n_i \\
&\quad + (\Delta(ac) - (-1)^{|\Delta||a|}a\Delta(c))r_j \\
&= \Delta(a)(bn_i + cr_j)
\end{aligned}$$

and so $\Delta_{\mathcal{M}} \in \mathcal{S}^{(1)}$. Moreover,

$$\begin{aligned}
&q(\Delta_{\mathcal{M}}(an_i + br_j), cn_k + dr_l) + (-1)^{|an_i+br_j||cn_k+dr_l|} q(\Delta_{\mathcal{M}}(cn_k + dr_l), an_i + br_j) \\
&\quad = \frac{1}{2}\Delta(q(n_i, n_{\underline{i}}))q(n_i, n_{\underline{i}})^{-1}a + \Delta(a))q(n_i, bn_k) \\
&\qquad + (-1)^{(|a|+|n_i|)(|b|+|n_k|)} \frac{1}{2}\Delta(q(n_i, n_{\underline{i}}))q(n_i, n_{\underline{i}})^{-1}b + \Delta(b))q(n_k, an_i) \\
&\quad = \delta_{i\underline{k}}\Delta(q(n_i, n_{\underline{i}}))q(n_i, n_{\underline{i}})^{-1}q(an_i, bn_{\underline{i}}) \\
&\qquad + \delta_{i\underline{k}}((-1)^{|b||m_i|}\Delta(a)bq(n_i, n_{\underline{i}}) + (-1)^{|\Delta||b|+|b||n_i|}a\Delta(b)q(n_i, n_{\underline{i}})) \\
&\quad = \delta_{i\underline{k}}(-1)^{|b||n_i|}(\Delta(q(n_i, n_{\underline{i}}))ab + \Delta(ab)q(n_i, n_{\underline{i}})) \\
&\quad = \delta_{i\underline{k}}(-1)^{|b||n_i|}\Delta(abq(n_i, n_{\underline{i}})) = \delta_{i\underline{k}}\Delta(q(an_i, bn_{\underline{i}}) \\
&\quad = \Delta(q(an_i, bn_{\underline{i}}))
\end{aligned}$$

and so $\Delta_{\mathcal{M}} \in \mathcal{S}^{(2)}$. Hence $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \operatorname{osp}(q) + (\operatorname{Der}_K A)_{\mathcal{M}}$. Since $(\operatorname{Der}_K A)_{\mathcal{M}} \cap \operatorname{End}_A \mathcal{M} = (0)$ and $\operatorname{osp}(q) \in \operatorname{End}_A \mathcal{M}$, $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \operatorname{osp}(q) \oplus (\operatorname{Der}_K A)_{\mathcal{M}}$. It is easy to see that

$$\begin{aligned}
\operatorname{Der}_K A &\rightarrow (\operatorname{Der}_K A)_{\mathcal{M}} \\
\Delta &\mapsto \Delta_{\mathcal{M}}
\end{aligned}$$

is an A-supermodule isomorphism. To see that it is also a Lie superalgebra isomorphism let $\Delta, \Delta' \in \operatorname{Der}_K A$. Then for all $a, b \in A$, $i \in I$ and $j \in J$, if we set

$c = q(n_i, n_i)$,

$$\begin{aligned}
[\Delta_{\mathcal{M}}, \Delta'_{\mathcal{M}}](an_i + br_j) &= \Delta_{\mathcal{M}}((\tfrac{1}{2}\Delta'(c)c^{-1}a + \Delta'(a))n_i + \Delta'(b)r_j) \\
&\quad -(-1)^{|\Delta||\Delta'|}\Delta'_{\mathcal{M}}((\tfrac{1}{2}\Delta(c)c^{-1}a + \Delta(a))n_i + \Delta(b)r_j) \\
&= \tfrac{1}{2}\Delta(c)c^{-1}(\tfrac{1}{2}\Delta'(c)c^{-1}a + \Delta'(a))n_i \\
&\quad +\Delta(\tfrac{1}{2}\Delta'(c)c^{-1}a + \Delta'(a))n_i + \Delta(\Delta'(b))r_j \\
&\quad -(-1)^{|\Delta||\Delta'|}\tfrac{1}{2}\Delta'(c)c^{-1}(\tfrac{1}{2}\Delta(c)c^{-1}a + \Delta(a))n_i \\
&\quad -(-1)^{|\Delta||\Delta'|}\Delta'(\tfrac{1}{2}\Delta(c)c^{-1}a + \Delta(a))n_i - (-1)^{|\Delta||\Delta'|}\Delta'(\Delta(b))r_j \\
&= (\tfrac{1}{4}\Delta(c^{-1})\Delta'(c)c^{-2}a + \tfrac{1}{2}\Delta(c)c^{-1}\Delta'(a) \\
&\quad +\tfrac{1}{2}\Delta(\Delta'(c)c^{-1}a + (-1)^{|\Delta||\Delta'|}\Delta'(c)\Delta(c^{-1})a \\
&\quad +(-1)^{|\Delta||\Delta'|}\tfrac{1}{2}\Delta'(c)c^{-1}\Delta(a) \\
&\quad +\Delta(\Delta'(a)) - (-1)^{|\Delta||\Delta'|}\tfrac{1}{4}\Delta'(c)\Delta(c)c^{-2} - (-1)^{\Delta||\Delta'|}\tfrac{1}{2}\Delta'(c)c^{-1}\Delta(a) \\
&\quad -(-1)^{|\Delta||\Delta'|}\tfrac{1}{2}\Delta'(\Delta(c)c^{-1}a - (-1)^{|\Delta||\Delta'|}\Delta(c)\Delta'(c^{-1})a \\
&\quad -\Delta(c)c^{-1}\Delta'(a) - (-1)^{|\Delta||\Delta'|}\Delta'(\Delta(a))\big)\, n_i \\
&\quad +[\Delta, \Delta'](b)r_j \\
&= \big(\tfrac{1}{2}[\Delta, \Delta'](c)c^{-1}a + [\Delta, \Delta'](a)\big)\, n_i + [\Delta, \Delta'](b)r_j \\
&= [\Delta, \Delta']_{\mathcal{M}}(an_i + br_j).
\end{aligned}$$

Using linearity, we therefore have that $\mathrm{Der}_K A \to (\mathrm{Der}_K A)_{\mathcal{M}} : \Delta \mapsto \Delta_{\mathcal{M}}$ is a Lie superalgebra isomorphism. Hence $(\mathrm{Der}_K A)_{\mathcal{M}}$ is a subalgebra of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$ and so since by Proposition 3.2.7.1.4. $\mathrm{osp}(q)$ is an ideal of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$.

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \mathrm{osp}(q) \rtimes (\mathrm{Der}_K A)_{\mathcal{M}}.$$

By Proposition 2.2.4, $\mathrm{osp}(q) = \overline{\mathrm{eosp}(q)} \rtimes \mathrm{End}_A \mathcal{R}$ and so

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = (\overline{\mathrm{eosp}(q)} \rtimes \mathrm{End}_A \mathcal{R}) \rtimes (\mathrm{Der}_K A)_{\mathcal{M}} = \overline{\mathrm{eosp}(q)} \dotplus (\mathrm{End}_A \mathcal{R} \dotplus (\mathrm{Der}_K A)_{\mathcal{M}})$$

Since $(\mathrm{Der}_K A)_{\mathcal{M}}$ and $\mathrm{End}_A \mathcal{R}$ are subalgebras of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$ to show that $\mathrm{End}_A \mathcal{R} \dotplus (\mathrm{Der}_K A)_{\mathcal{M}}$ is a subalgebra of $\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$, we only need to show that $[(\mathrm{Der}_K A)_{\mathcal{M}}, \mathrm{End}_A \mathcal{R}] \subset (\mathrm{Der}_K A)_{\mathcal{M}} \oplus \mathrm{End}_A \mathcal{R}$. In fact, we will show that

$$[(\mathrm{Der}_K A)_{\mathcal{M}}, \mathrm{End}_A \mathcal{R}] \subset \mathrm{End}_A \mathcal{R}.$$

Indeed, let $\Delta \in \operatorname{Der}_K A$ and $S \in \operatorname{End}_A \mathcal{R}$. Then with respect to $\mathcal{M} = \begin{pmatrix} \mathcal{N} \\ \mathcal{R} \end{pmatrix}$, we have

$$[\Delta_{\mathcal{M}}, S] = \left[\begin{pmatrix} \Delta|_{\mathcal{N}} & 0 \\ 0 & \Delta|_{\mathcal{R}} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & [\Delta|_{\mathcal{R}}, S] \end{pmatrix} \in \operatorname{Hom}_K(\mathcal{M}, \mathcal{R})$$

and so $[(\operatorname{Der}_K A)_{\mathcal{M}}, \operatorname{End}_A \mathcal{R}] \subset \operatorname{Hom}_K(\mathcal{M}, \mathcal{R})$. Moreover, if $S(r_j) = \sum_{i \in J} s_{ji} r_i$ for some $s_{ji} \in A$, $i, j \in J$ then

$$\begin{aligned}
[[\Delta_{\mathcal{M}}, S], a \cdot Id](br_j) &= (-1)^{(|a|+|b|)|S|} \sum_{i \in J} (\Delta(abs_{ji}) - \Delta(ab)s_{ji}) r_i \\
&\quad -(-1)^{(|\Delta|+|S|)|a|+|b||S|} \sum_{i \in J} (a(\Delta(bs_{ji}) - \Delta(b)s_{ji}) r_i \\
&= (-1)^{(|a|+|b|)|S|} \sum_{i \in J} (\Delta(a)bs_{ji} - \Delta(a)bs_{ji}) r_i \\
&= 0
\end{aligned}$$

and so

$$[(\operatorname{Der}_K A)_{\mathcal{M}}, \operatorname{End}_A \mathcal{R}] \subset 0 \oplus \operatorname{End}_A \mathcal{R} = \operatorname{End}_A \mathcal{R}.$$

Hence $[(\operatorname{Der}_K A)_{\mathcal{M}} \oplus \operatorname{End}_A \mathcal{R}, (\operatorname{Der}_K A)_{\mathcal{M}} \oplus \operatorname{End}_A \mathcal{R}] \subset (\operatorname{Der}_K A)_{\mathcal{M}} \oplus \operatorname{End}_A \mathcal{R}$ and so $(\operatorname{Der}_K A)_{\mathcal{M}} \oplus \operatorname{End}_A \mathcal{R}$ is a subalgebra of $\mathcal{S}$. The rest of the assertion now follows from the definition of $\mathcal{S}$ and from Proposition 3.2.7.1.3. ∎

Corollary 3.3.5 *Let $\mathcal{N}$ and $\mathcal{P}$ be A-supermodules with A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{P}}$ respectively where $q_{\mathcal{N}}$ is almost diagonizable with respect to some basis $\{n_i \mid i \in I\}$ of $\mathcal{N}$. Then*

$$\begin{aligned}
\overline{\operatorname{eosp}(q_{\mathcal{N}})} \oplus \overline{\operatorname{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus (\mathcal{S}_{\mathcal{P}}^{(1)} \cap \mathcal{S}_{\mathcal{P}}^{(2)}) &\to \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} \\
x \oplus S &\mapsto x \oplus (\operatorname{ad}_A S)_{\mathcal{N}} \oplus S
\end{aligned}$$

is a Lie superalgebra isomorphism. If $\mathcal{E} = \operatorname{eosp}(q)$, $\overline{\operatorname{eosp}(q)}$ or $\operatorname{osp}(q)$ then

$$\mathcal{S} \cong \overline{\operatorname{eosp}(q_{\mathcal{N}})} \oplus \overline{\operatorname{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{P}}$$

and in particular, if q is the orthogonal sum of an almost diagonizable quadratic form and an invertibly diagonizable A-quadratic form of rank 2, then

$$\mathrm{Der}_K\,\mathcal{E}_\infty \cong \begin{cases} \mathbf{ad}_{\mathrm{loc}}(\mathcal{E}_\infty) + \mathcal{S}_{\mathcal{P}} & : \mathcal{E} = \mathrm{eosp}(q); \\ \mathbf{ad}(\mathcal{E}_\infty) + \mathcal{S}_{\mathcal{P}} & : \mathcal{E} = \overline{\mathrm{eosp}(q)},\ \mathrm{osp}(q). \end{cases}$$

Proof: By Proposition 3.3.3,

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}.$$

By Proposition 3.3.4, $\mathcal{S}_{\mathcal{N}} = \overline{\mathrm{eosp}(q_{\mathcal{N}})} \rtimes (\mathrm{Der}_K\,A)_{\mathcal{N}}$. We claim that

$$\mathcal{S}_{\mathcal{N}}^{\mathcal{P}} = (\overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus 0) \oplus \{([S_{\mathcal{P}}, A\cdot Id])_{\mathcal{N}} \oplus S_{\mathcal{P}} \mid S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}\}.$$

Indeed, if $S_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$ then $S_{\mathcal{N}} = x + \Delta_{\mathcal{N}} \in \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus (\mathrm{Der}_K\,A)_{\mathcal{N}}$ and $[S_{\mathcal{N}}, A\cdot Id] = \Delta = [S_{\mathcal{P}}, A\cdot Id]$. Hence

$$S_{\mathcal{N}} \oplus S_{\mathcal{P}} = (x \oplus 0) + (\Delta_{\mathcal{N}} \oplus S_{\mathcal{P}}) = (x \oplus 0) + (([S_{\mathcal{P}}, A\cdot Id])_{\mathcal{N}} \oplus S_{\mathcal{P}}).$$

Conversely, since $[\overline{\mathrm{eosp}(q_{\mathcal{N}})}, A\cdot Id] = 0$, $\overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus 0 \subset \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$. Moreover, given $S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}$, $[S_{\mathcal{P}}, A\cdot Id] = \Delta \in \mathrm{Der}_K\,A$ and so $\Delta_{\mathcal{N}} \in \mathcal{S}_{\mathcal{N}}$. Moreover, for all $a, b \in A$ and $i \in I$,

$$\begin{aligned} [\Delta_{\mathcal{N}}, a\cdot Id](bn_i) &= (\frac{1}{2}ab\Delta(q(n_i,n_i))q(n_i,n_i)^{-1} + \Delta(ab))n_i \\ &\quad -(-1)^{|\Delta||a|}(\frac{1}{2}ab\Delta(q(n_i,n_i))q(n_i,n_i)^{-1} + a\Delta(b))n_i \\ &= \Delta(a)(bn_i) \end{aligned}$$

and so $[\Delta_{\mathcal{N}}, A\cdot Id] = \Delta$ hence $([S_{\mathcal{P}}, A\cdot Id])_{\mathcal{N}} \oplus S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{N}}^{\mathcal{P}}$. In addition, if $0 = (x \oplus 0) + (([S_{\mathcal{P}}, A\cdot Id])_{\mathcal{N}} \oplus S_{\mathcal{P}}) = (x + ([S_{\mathcal{P}}, A\cdot Id])_{\mathcal{N}}) \oplus S_{\mathcal{P}})$ then $S_{\mathcal{P}} = 0$ hence $([S_{\mathcal{P}}, A\cdot Id])_{\mathcal{N}} = 0$ and so $x = 0$. Therefore

$$\mathcal{S}^{(1)} \cap \mathcal{S}^{(2)} = \overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}})} \oplus \{([S_{\mathcal{P}}, A\cdot Id])_{\mathcal{N}} \oplus S_{\mathcal{P}} \mid S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}\}.$$

It is easy to see that $\mathcal{S}_{\mathcal{P}} \to \{([S_{\mathcal{P}}, A\cdot Id])_{\mathcal{N}} \oplus S_{\mathcal{P}} \mid S_{\mathcal{P}} \in \mathcal{S}_{\mathcal{P}}\} : S_{\mathcal{P}} \mapsto ([S_{\mathcal{P}}, A\cdot Id])_{\mathcal{N}} \oplus S_{\mathcal{P}}$ is a Lie superalgebra isomorphism and so the first assertion of the Corollary holds. The second assertion follows from 3.2.7.1.3. If $\mathcal{E} = \mathrm{eosp}(q)$, $\overline{\mathrm{eosp}(q)}$ or $\mathrm{osp}(q)$ then by Proposition 3.3.1, $\mathcal{S} = \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$. The rest of the Corollary now follows from Theorem 3.2.13 using the fact that $\overline{\mathrm{eosp}(q_{\mathcal{N}}, q_{\mathcal{P}})} \oplus \overline{\mathrm{eosp}(q_{\mathcal{N}})} \subset \overline{\mathrm{eosp}(q)}$ and that, by Proposition 3.2.15, $\overline{\mathrm{eosp}(q)} \subset \mathbf{ad}_{\mathrm{loc}}(\mathrm{eosp}(q))$. ∎

Remark 3.3.6 In [Be], Benkart described the derivations of root-graded Lie algebras over a field $\mathbb{F}$ of characteristic 0. In particular, for a centreless Lie algebra L graded by root systems B or D she proves that

$$\operatorname{Der}_{\mathbb{F}} L = \mathbf{ad}\, L + \operatorname{Der}_*(J)$$

for a certain Jordan algebra J. Since L is isomorphic to $\mathcal{E} = \operatorname{eosp}(q_I \oplus q_{x_0} \oplus q_{\mathcal{P}})$ for some index set I, base point x_0 and an A-module $\mathcal{P}$ where A is an extension of $\mathbb{F}$, $q_I \oplus q_{x_0}$ is almost diagonizable and by the above corollary, $\operatorname{Der}_K \mathcal{E}_\infty \cong \mathbf{ad}(\mathcal{E}_\infty) + \mathcal{S}_{\mathcal{P}}$. However, by Proposition 3.2.16, $\mathcal{S}_{\mathcal{P}} \cong \operatorname{Der}_*(A \oplus \mathcal{P})$ where $A \oplus \mathcal{P}$ is a Jordan algebra and this algebra is isomorphic to the Jordan algebra which appears in Benkart's description of B-graded Lie algebras.

Theorem 3.3.7 *Let $\mathcal{N}$ and $\mathcal{R}$ be free A-supermodules with A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{R}}$ respectively such that $q_{\mathcal{R}} \equiv 0$ and $q_{\mathcal{N}}$ is an orthogonal sum of an almost diagonizable A-quadratic form and an invertibly diagonizable A-quadratic form on a free A-supermodule of rank 2. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$ with the A-quadratic form $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$ and let $\mathcal{E}$ be a subalgebra of $\operatorname{osp}(q)$ containing $\operatorname{eosp}(q)$. For $\Delta \in \operatorname{Der}_K A$ define $\Delta_{\mathcal{M}} \in \operatorname{End}_K \mathcal{M}$ using (3.22). Let the map $\{S \in \operatorname{osp}(q) \oplus (\operatorname{Der}_K A)_{\mathcal{M}} \mid [S, \mathcal{E}] \subset \mathcal{E}\} \to \mathcal{D}_{\mathcal{E}} : S \mapsto d_{S,0}$ be the Lie superalgebra monomorphism given by (3.6). Then $\operatorname{Der}_K A \to (\operatorname{Der}_K A)_{\mathcal{M}} : \Delta \mapsto \Delta_{\mathcal{M}}$ is a Lie superalgebra isomorphism and*

$$\operatorname{Der}_K \mathcal{E}_\infty = \mathbf{ad}\, \mathcal{E}_\infty + \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \operatorname{osp}(q) \oplus (\operatorname{Der}_K A)_{\mathcal{M}} \textit{ such that } [S, \mathcal{E}] \subset \mathcal{E}\}.$$

with

$$\mathbf{ad}\, \mathcal{E}_\infty \cap \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \operatorname{osp}(q) \textit{ such that } [S, \mathcal{E}] \subset \mathcal{E}\} = \mathbf{ad}\, \mathcal{E} = \{d_{S,0} \mid S \in \mathcal{E}\}.$$

Hence, whenever $\mathcal{S} = \mathcal{E} \rtimes \mathcal{S}_0$,

$$\operatorname{Der}_K \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) \rtimes \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathcal{S}_0\}.$$

In particular:

(1) If $\mathcal{E} = \operatorname{eosp}(q)$ then

$$\operatorname{Der}_K \mathcal{E}_\infty = \mathbf{ad}_{\mathrm{loc}}\, \mathcal{E}_\infty \rtimes \{d_{S,0} \in \mathcal{D} \mid S \in \operatorname{End}_A \mathcal{R} \oplus (\operatorname{Der}_K A)_{\mathcal{M}}\}.$$

Moreover, if $\mathcal{M}$ is free of finite rank then

$$\mathrm{Der}_K\,\mathcal{E}_\infty = \mathbf{ad}\,\mathcal{E}_\infty \rtimes \{d_{S,0} \in \mathcal{D} \mid S \in \mathrm{End}_A\,\mathcal{R} \oplus (\mathrm{Der}_K\,A)_{\mathcal{M}}\}.$$

(2) If $\mathcal{E} = \overline{\mathrm{eosp}(q)}$ then

$$\mathrm{Der}_K\,\mathcal{E}_\infty = \mathbf{ad}\,\mathcal{E}_\infty \rtimes \{d_{S,0} \in \mathcal{D} \mid S \in \mathrm{End}_A\,\mathcal{R} \oplus (\mathrm{Der}_K\,A)_{\mathcal{M}}\}.$$

(3) If $\mathcal{E} = \mathrm{osp}(q)$ then

$$\mathrm{Der}_K\,\mathcal{E}_\infty = \mathbf{ad}\,\mathcal{E}_\infty \rtimes \{d_{S,0} \in \mathcal{D} \mid S \in (\mathrm{Der}_K\,A)_{\mathcal{M}}\}.$$

Proof: Recall that by Theorem 3.2.13.2 we have that the map $\mathcal{S}\oplus\mathcal{T} \to \mathcal{D}_{\mathcal{E}} : (S,T) \mapsto d_{S,T}$ given by

$$d_{S,T}(a,m,m',x) = ([S,a\cdot Id] + [T,x], (S+T)(m), (S-T)(m'), [S,x] + [T,a\cdot Id])$$

is a Lie superalgebra isomorphism with the induced bracket operation and that

$$\mathrm{Der}_K\,\mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) + \mathcal{D}_{\mathcal{E}}.$$

By Proposition 3.3.4,

$$\mathcal{S} = \{S \in \mathrm{osp}(q) \rtimes (\mathrm{Der}_K\,A)_{\mathcal{M}} \mid [S,\mathcal{E}] \subset \mathcal{E}\}$$

where the map $\mathrm{Der}_K\,A \to (\mathrm{Der}_K\,A)_{\mathcal{M}} : \Delta \mapsto \Delta_{\mathcal{M}}$ is a Lie superalgebra isomorphism. Moreover, by Proposition 3.3.1, $\mathcal{T} = A\cdot Id$. Hence since by Theorem 3.2.13.3,

$$\mathbf{ad}\,\mathcal{E}_\infty \cap \mathcal{D}_{\mathcal{E}} = \{d_{S,T} \mid S \in \mathcal{E},\ T \in A\cdot Id\},$$

$$\mathrm{Der}_K\,\mathcal{E}_\infty = \mathbf{ad}\,\mathcal{E}_\infty + \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathrm{osp}(q) \oplus (\mathrm{Der}_K\,A)_{\mathcal{M}} \text{ such that } [S,\mathcal{E}] \subset \mathcal{E}\}.$$

The rest of the main assertion now follows from Theorem 3.2.13.3 and 3.2.13.4. and so (3) follows immediately from this and (2) follows from Proposition 2.2.4. Let $\mathcal{E} = \mathrm{eosp}(q)$. By Propositions 3.3.4 and 2.2.4,

$$\mathcal{S} = \mathrm{osp}(q) \rtimes (\mathrm{Der}_K\,A)_{\mathcal{M}} = \overline{\mathrm{eosp}(q)} \rtimes (\mathrm{End}_A\,\mathcal{R} \oplus (\mathrm{Der}_K\,A)_{\mathcal{M}})$$

and so

$$\mathrm{Der}_K\,\mathcal{E}_\infty = \mathbf{ad}\,\mathcal{E}_\infty + (\overline{\mathrm{eosp}(q)} \oplus \mathrm{End}_A\,\mathcal{R} \oplus (\mathrm{Der}_K\,A)_\mathcal{M})_{\mathcal{E}_\infty}.$$

We will show that $\mathbf{ad}_{\mathbf{loc}}(\mathcal{E}_\infty) = \mathbf{ad}(\mathcal{E}_\infty) + \overline{\mathrm{eosp}(q)}$. Indeed, by Proposition 3.2.15, $\overline{\mathrm{eosp}(q)} \in \mathbf{ad}_{\mathbf{loc}}\,\mathrm{eosp}(q_\infty)$. Moreover, since $\mathbf{ad}_{\mathbf{loc}}(\mathcal{E}_\infty) \subset \mathrm{End}_A\,\mathcal{E}_\infty$, from (3.6) we have $\mathbf{ad}_{\mathbf{loc}}(\mathcal{E}_\infty) \cap (\mathrm{End}_A\,\mathcal{R} \oplus (\mathrm{Der}_K\,A)_\mathcal{M})_{\mathcal{E}_\infty} = \mathbf{ad}_{\mathbf{loc}}(\mathcal{E}_\infty) \cap \mathrm{End}_A\,\mathcal{R}$. Suppose that $S_{\mathcal{E}_\infty} \in \mathbf{ad}_{\mathbf{loc}}(\mathrm{eosp}(q_\infty))$ for some $S \in \mathrm{End}_A\,\mathcal{R}$. Then for each $j \in R$ there exists $(a_0, m_0, n_0, x_0) \in \mathrm{eosp}(q_\infty)$ such that

$$\begin{aligned}
S_{\mathcal{E}_\infty}(0, r_j, 0, 0) &= (0, S(r_j), 0, 0) = [(a_0, m_0, n_0, x_0), (0, r_j, 0, 0)] \\
&= (0, a_0 r_j, 0, \mathbf{E}_{n_0, r_j}) \\
S_{\mathcal{E}_\infty}(0, 0, r_j, 0) &= (0, 0, S(r_j), 0) = [(a_0, m_0, n_0, x_0), (0, 0, r_j, 0)] \\
&= (0, 0, -a_0 r_j, \mathbf{E}_{m_0, r_j})
\end{aligned}$$

and so $S(r_j) = a_0 r_j$ and $S(r_j) = -a_0 r_j$ hence $S(r_j) = 0$. Since $j \in R$ was arbitrary, $S = 0$ and so $\mathbf{ad}_{\mathbf{loc}}(\mathcal{E}_\infty) \cap (\mathrm{End}_A\,\mathcal{R})_{\mathcal{E}_\infty} = (0)$. Hence since clearly $\mathbf{ad}\,\mathcal{E}_\infty \subset \mathbf{ad}_{\mathbf{loc}}\,\mathcal{E}_\infty$,

$$\mathbf{ad}_{\mathbf{loc}}(\mathcal{E}_\infty) = \mathbf{ad}(\mathcal{E}_\infty) + (\overline{\mathrm{eosp}(q)})_{\mathcal{E}_\infty}$$

and so since $\mathbf{ad}_{\mathbf{loc}}(\mathcal{E}_\infty)$ is an ideal of $\mathcal{E}_\infty$,

$$\mathrm{Der}_K(\mathcal{E}_\infty) = \mathbf{ad}_{\mathbf{loc}}(\mathcal{E}_\infty) \rtimes (\mathrm{End}_A\,\mathcal{R} \oplus (\mathrm{Der}_K\,A)_\mathcal{M})_{\mathcal{E}_\infty}.$$

Now, if $\mathcal{M}$ is of finite rank then by Proposition 1.5.7, $\overline{\mathrm{eosp}(q)} = \mathrm{eosp}(q)$ and so $\mathbf{ad}_{\mathbf{loc}}\,\mathcal{E}_\infty = \mathbf{ad}\,\mathcal{E}_\infty$ hence the assertion follows. ■

Corollary 3.3.8 *If $\mathcal{M} = H(I, A) \oplus Ah_0 \oplus \tilde{H}(J, A)$ or $\mathcal{M} = H(I, A) \oplus \tilde{H}(J, A)$ where $I \subset \mathbb{Z}$ and $J \subset \mathbb{Z} + \frac{1}{2}$ with the quadratic form $q = (q_\mathbb{Z} \oplus q_0 \oplus \tilde{q}_{\mathbb{Z}+\frac{1}{2}})|_\mathcal{M}$ and $\mathcal{E} = \mathrm{osp}_{\mathrm{fd}}(q)$ where $I \cup J \subset A^\times$ then*

$$\mathrm{Der}_K\,\mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) \rtimes \{d_{S,0} \in \mathcal{D}_\mathcal{E} \mid S \in (\mathrm{Der}_K\,A)_\mathcal{M}\}.$$

In particular, if $A = K$ then all the derivations of $\mathcal{E}_\infty$ are inner.

Proof: By Proposition 3.3.4,

$$\mathcal{S} = \{S \in \mathrm{osp}(q) \oplus (\mathrm{Der}_K A)_{\mathcal{M}} \mid [S, \mathcal{E}] \subset \mathcal{E}\}.$$

Since the elements of $(\mathrm{Der}_K A)_{\mathcal{M}}$ satisfy (S1), by Proposition 3.2.7.1.1, $[(\mathrm{Der}_K A)_{\mathcal{M}}, \mathrm{End}_A \mathcal{M}] \subset \mathrm{End}_A \mathcal{M}$. Hence since the elements of $(\mathrm{Der}_K A)_{\mathcal{M}}$ are diagonal, $[(\mathrm{Der}_K A)_{\mathcal{M}}, \mathcal{E}] \subset \mathcal{E}$. Therefore by Proposition 2.2.6,

$$\begin{aligned} \mathcal{S} &= \{S \in \mathrm{osp}(q) \rtimes (\mathrm{Der}_K A)_{\mathcal{M}} \mid [S, \mathcal{E}] \subset \mathcal{E}\} \\ &= \{S \in \mathrm{osp}(q) \mid [S, \mathcal{E}] \subset \mathcal{E}\} \rtimes (\mathrm{Der}_K A)_{\mathcal{M}} \\ &= \mathrm{osp}_{\mathrm{fd}}(q) \rtimes (\mathrm{Der}_K A)_{\mathcal{M}} \end{aligned}$$

and so

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) + \{d_{S,0} \in \mathcal{D}_\mathcal{E} \mid S \in \mathrm{osp}_{\mathrm{fd}}(q) \rtimes (\mathrm{Der}_K A)_{\mathcal{M}}\}.$$

Hence by Theorem 3.2.13.4,

$$\mathrm{Der}_K \mathcal{E}_\infty = \mathbf{ad}(\mathcal{E}_\infty) \rtimes \{d_{S,0} \in \mathcal{D}_\mathcal{E} \mid S \in (\mathrm{Der}_K A)_{\mathcal{M}}\}.$$

∎The following corollary is a generalization of Benkart's result for Lie algebras of type B and D ([Be, Theorem 3.6]).

Corollary 3.3.9 *Suppose $\mathcal{M} = H(I; A) \oplus \mathcal{P}$ where $|I| \geq 2$ or $|I| \geq 1$ and $\mathrm{Ann}_A(p) = (0)$ for some $p \in \mathcal{P}$. Let $q_\mathcal{P}$ be an A-quadratic form on $\mathcal{P}$ and set $q = q_I \oplus q_\mathcal{P}$. Let $L = \mathrm{eosp}(q_\infty)$ and $\mathcal{E} = \mathrm{eosp}(q)$ (e.g. L is a centreless D_J- or B_J-graded Lie superalgebra for $J = I \dot{\cup} \{\infty\}$, [GN]). Then*

$$\begin{aligned} \mathrm{Der}_K L &\cong \mathbf{ad}_{\mathrm{loc}} L + \{d_{S,0} \in \mathcal{D}_\mathcal{E} \mid S \in \mathcal{S}_\mathcal{P}\} \\ &\cong \mathbf{ad}_{\mathrm{loc}} L + \mathrm{Der}_*(\mathcal{J}) \end{aligned}$$

where $\mathcal{J} = A \oplus \mathcal{P}$ and $\mathrm{Der}_*(\mathcal{J})$ are defined as in Proposition 3.2.16. Moreover,

$$\mathbf{ad}\, L \cap \{d_{S,0} \in \mathcal{D}_\mathcal{E} \mid S \in \mathcal{S}_\mathcal{P}\} = \mathbf{ad}\, \mathrm{eosp}(q_\mathcal{P}).$$

Hence if $\mathcal{S}_\mathcal{P} = \mathrm{eosp}(q_\mathcal{P}) \rtimes \mathcal{S}_0$ then

$$\mathrm{Der}_K L = \mathbf{ad}_{\mathrm{loc}} L \rtimes \{d_{S,0} \in \mathcal{D}_\mathcal{E} \mid S \in \mathcal{S}_0\}.$$

In particular, if $\mathcal{P} = \mathcal{N} \oplus \mathcal{R}$ with the A-quadratic form $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$ where $\mathcal{N}$ and $\mathcal{R}$ are free A-supermodules with A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{R}}$ respectively such that $q_{\mathcal{N}}$ is almost diagonizable and $q_{\mathcal{R}} \equiv 0$ then

$$\mathrm{Der}_K L = \mathbf{ad}_{\mathrm{loc}} L \rtimes \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathrm{End}_A \mathcal{R} \oplus (\mathrm{Der}_K A)_{\mathcal{M}})$$

where $d_{(\mathrm{Der}_K A)_{\mathcal{M}},0}$ is isomorphic (as a Lie superalgebra) to $\mathrm{Der}_K A$. In particular, if L is of finite rank,

$$\mathrm{Der}_K L = \mathbf{ad}\, L \rtimes \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathrm{End}_A \mathcal{R} \oplus (\mathrm{Der}_K A)_{\mathcal{M}}\}.$$

Proof: Since $|I| \geq 1$, $H(I; A)$ contains a two-dimensional invertibly diagonizable free A-supermodule as a direct summand and so by Proposition 3.3.1

$$\mathrm{Der}_K L = \mathbf{ad}\, L + \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathcal{S}_{\mathcal{E}}\}.$$

By Corollary 3.3.5,

$$\mathcal{S} \cong \overline{\mathrm{eosp}(q_I)} \oplus \overline{\mathrm{eosp}(q_I, q_{\mathcal{P}})} \oplus \mathcal{S}_{\mathcal{P}}$$

and so

$$\mathrm{Der}_K L \cong \mathbf{ad}_{\mathrm{loc}} L + \{d_{S,0} \in \mathcal{D}_{\mathcal{E}} \mid S \in \mathcal{S}_{\mathcal{P}}\}.$$

The rest follows from Proposition 3.2.16 and Theorem 3.3.7. ∎

We can apply Theorem 3.3.7 to extended affine Lie algebras of types B and D:

Corollary 3.3.10 *Let $\mathcal{K}$ be the centreless core of an extended affine Lie algebra of type B_l or D_l ($l \geq 3, 4$ respectively) with nullity ν. Then*

$$\mathrm{Der}_{\mathbb{C}} \mathcal{K} \cong \mathbf{ad}\, \mathcal{K} \rtimes \mathrm{Der}_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1}, \dots, t_\nu^{\pm 1}].$$

The last assertion of the following was proved in [K2, Proposition 2.3.4] and [S, Proposition 3.1.2.3] and is a special case of Theorem 3.3.7.

Corollary 3.3.11 *Let K be an algebraically closed field of characteristic 0.*

(1) *Let* $m, n \in \mathbb{Z}^+$, $q = q_{\{1,\dots,m\}} \oplus \tilde{q}_{\{1,\dots,n\}}$ *over* K *and let* $q_0 : Km_0 \times Km_0 \to K$ *be the* K*-quadratic form on a free supermodule* Km_0 *given by* $q(m_0, m_0) = 1$. *Then* $\mathrm{eosp}(q) = \mathrm{osp}(2m, 2n)$ *and* $\mathrm{eosp}(q \oplus q_0) = \mathrm{osp}(2m+1, 2n)$.

(2) *Let* A *be a superextension of* K *and let* $L = \mathrm{osp}(m, 2n)$, $m, n \in \mathbb{Z}^+$. *Then* $Der_K(L \otimes A) \cong \mathbf{ad}\, L \rtimes \mathrm{Der}_K A$. *In particular, for* $A = K$, L *has no outer derivations.*

Chapter 4

Invariant forms

Throughout this chapter k will denote a commutative, associative, unital ring containing $\frac{1}{2}$, K a superextension of k and A a superextension of K. In addition, we will assume throughout that when $|m|$ occurs in an expression for some $m \in \mathcal{M}$ for some supermodule $\mathcal{M}$, then it is assumed that m is homogeneous, and that the expression extends to the other elements by linearity.

4.1 Invariant forms on 3-graded Lie algebras

Let L be a Lie superalgebra over K. We say that a K-bilinear map $(\cdot \mid \cdot) : L \times L \to K$ of degree $\bar{0}$ is $(L\text{-})$*invariant* if and only if $([x, y] \mid z) = (x \mid [y, z])$ for all $x, y, z \in L$. Denote the $K_{\bar{0}}$-module of invariant supersymmetric K-bilinear forms $(\cdot \mid \cdot) : L \times L \longrightarrow K$ by $\mathrm{Inv}(L; K)$.

First we consider the relation between supersymmetric invariant forms of a Lie superalgebra L and its central extensions $\tilde{L}$. Recall that $\tilde{L}$ is a central extension of a perfect Lie superalgebra L if there exists a short exact sequence $0 \to C \to \tilde{L} \xrightarrow{\pi} L \to 0$ with $C \subset Z(\tilde{L})$. We have the following result which is well known in case of Lie algebras (for example, see [BK, Remark (a), p. 315]):

Proposition 4.1.1 *Let L be a Lie superalgebra over K and let $\pi : \tilde{L} \to L$ be a* *covering, i.e., $\tilde{L}$ is perfect and $\pi : \tilde{L} \to L$ is a central extension. Then*

(a) Every $(\cdot \mid \cdot)_L \in \mathrm{Inv}(L; K)$ gives rise to an invariant bilinear form $(\cdot \mid \cdot)_{\tilde{L}}$ on $\tilde{L}$ via

$$(x \mid y)_{\tilde{L}} = (\pi(x) \mid \pi(y))_L \qquad x, y \in \tilde{L}. \tag{4.1}$$

(b) $(Z(\tilde{L}) \mid \tilde{L})_{\tilde{L}} = (0)$ for any $(\cdot \mid \cdot)_{\tilde{L}} \in \mathrm{Inv}(\tilde{L}; K)$ and hence every $(\cdot \mid \cdot)_{\tilde{L}} \in \mathrm{Inv}(\tilde{L}; K)$ gives rise to an invariant bilinear form $(\cdot \mid \cdot)_L$ on L via (4.1).

(c) The maps described in (a) and (b) are inverses of each other, thus

$$\mathrm{Inv}(L; K) \cong \mathrm{Inv}(\tilde{L}; K)$$

(isomorphism of $K_{\bar{0}}$-modules).

Proof: Let $(\cdot \mid \cdot)_L \in \mathrm{Inv}(L; K)$. Define $(\cdot \mid \cdot)_{\tilde{L}}$ using (4.1). Then for $x_i \in \tilde{L}$, $i = 1, 2, 3$,

$$\begin{aligned}([x_1, x_2] \mid x_3)_{\tilde{L}} &= (\pi([x_1, x_2]) \mid \pi(x_3))_L = ([\pi(x_1), \pi(x_2)] \mid \pi(x_3))_L \\ &= (\pi(x_1) \mid [\pi(x_2), \pi(x_3)])_L = (\pi(x_1) \mid \pi([x_2, x_3]))_L \\ &= (x_1 \mid [x_2, x_3])_{\tilde{L}}.\end{aligned}$$

Conversely, given a form $(\cdot \mid \cdot)_{\tilde{L}} \in \mathrm{Inv}(\tilde{L}; K)$ we have, for all $z \in Z(\tilde{L})$, $\tilde{x}, \tilde{y} \in \tilde{L}$,

$$(z \mid [\tilde{x}, \tilde{y}]) = ([z, \tilde{x}] \mid \tilde{y}) = 0$$

and so since $\tilde{L}$ is perfect, $(Z(\tilde{L}) \mid \tilde{L})_{\tilde{L}} = (0)$. Moreover, since π is surjective, $(\cdot \mid \cdot)_{\tilde{L}}$ defines a supersymmetric invariant form on L via (4.1). The rest of the assertion is clear. ∎

We now specialize to toral 3-graded Lie superalgebras over K. Recall that a Lie superalgebra L is *toral 3-graded* if $L = L_+ \oplus L_0 \oplus L_-$ is a 3-graded Lie superalgebra (i.e., $[L_\sigma, L_\mu] \subset L_{\sigma+\mu}$, $\sigma, \mu \in \{0, \pm\}$) with $\zeta \in (L_0)_{\bar{0}}$ such that $[\zeta, x_\sigma] = \sigma x_\sigma$ for $x_\sigma \in L_\sigma$, $\sigma \in \{0, \pm\}$. The following lemma will be instrumental in characterizing the supersymmetric invariant K-bilinear forms on L:

Lemma 4.1.2 *Let $L = L_+ \oplus L_0 \oplus L_-$ be a 3-graded Lie superalgebra. Then for all $x_\sigma, y_\sigma \in L_\sigma$, $\sigma = \pm$,*

$$[[[x_+, x_-], y_+], y_-] - (-1)^{(|x_+|+|x_-|)(|y_+|+|y_-|)}[[[y_+, y_-], x_+], x_-] \in [L_0, L_0].$$

Proof: Using the Jacobi identity and the fact that $[L_\sigma, L_\sigma] = 0$ for $\sigma = \pm$, we have

$$\begin{aligned}
[[[x_+, x_-], y_+], y_-] &= [[x_+, [x_-, y_+]], y_-] \\
&= [x_+, [[x_-, y_+], y_-]] - (-1)^{|x_+|(|x_-|+|y_+|)}[[x_-, y_+], [x_+, y_-]] \\
&= [x_+, [x_-, [y_+, y_-]]] - (-1)^{|x_+|(|x_-|+|y_+|)}[[x_-, y_+], [x_+, y_-]] \\
&= [[x_+, x_-], [y_+, y_-]] - (-1)^{|x_-|(|y_+|+|y_-|)}[[x_+, [y_+, y_-]], x_-] \\
&\quad - (-1)^{|x_+|(|x_-|+|y_+|)}[[x_-, y_+], [x_+, y_-]] \\
&= [[x_+, x_-], [y_+, y_-]] + (-1)^{(|y_+|+|y_-|)(|x_+|+|x_-|)}[[[y_+, y_-], x_+], x_-] \\
&\quad + (-1)^{|y_-|(|x_-|+|y_+|)}[[x_+, y_-], [x_-, y_+]]
\end{aligned}$$

and so the assertion follows. ∎

Theorem 4.1.3 *Let L be a toral 3-graded Lie superalgebra over K such that $L_0 = [L_+, L_-]$. Let*

$$\widetilde{\mathrm{Inv}(L_0; K)} = \{(\cdot \mid \cdot) \in \mathrm{Inv}(L_0; K) \;\mid\; ([[x_0, x_+], x_-] \mid \zeta) = (x_0 \mid [x_+, x_-]), \; x_\sigma \in L_\sigma, \sigma = 0, \pm\}.$$

Then the map

$$\begin{array}{rcl} \mathrm{Inv}(L; K) & \xrightarrow{\psi} & \widetilde{\mathrm{Inv}(L_0; K)} \\ (\cdot \mid \cdot) & \mapsto & (\cdot \mid \cdot)|_{L_0 \times L_0} \end{array}$$

is a $K_{\bar{0}}$-module isomorphism. Moreover, the map

$$\begin{array}{rcl} \widetilde{\mathrm{Inv}(L_0; K)} & \xrightarrow{\phi} & \{\varphi \in (\mathrm{Hom}_K(L_0, K))_{\bar{0}} \mid \varphi([L_0, L_0]) = (0)\} \\ (\cdot \mid \cdot) & \mapsto & \varphi_{(\cdot \mid \cdot)}, \end{array}$$

where $\varphi_{(\cdot|\cdot)}(x_0) = (\zeta \mid x_0)$ for all $(\cdot \mid \cdot) \in \widetilde{\mathrm{Inv}(L_0; K)}$ and $x_0 \in L_0$, is an isomorphism of $K_{\bar{0}}$-modules. Hence, the map

$$\begin{array}{rcl} \mathrm{Inv}(L; K) & \xrightarrow{\phi \circ \psi} & \{\varphi \in (\mathrm{Hom}_K(L_0, K))_{\bar{0}} \mid \varphi([L_0, L_0]) = (0)\} \\ (\cdot \mid \cdot) & \mapsto & \varphi_{(\cdot \mid \cdot)}, \end{array} \tag{4.2}$$

where $\varphi_{(\cdot|\cdot)}(x_0) = (\zeta \mid x_0)$ *for all* $(\cdot|\cdot) \in \mathrm{Inv}(L_0; K)$ *and* $x_0 \in L_0$, *is an isomorphism of* $K_{\bar{0}}$*-modules. In addition, a form* $(\cdot|\cdot) \in \mathrm{Inv}(L; K)$ *is nondegenerate if and only if* $(\cdot|\cdot)|_{L_0 \times L_0}$ *is nondegenerate and* $\ker(\varphi_{(\cdot|\cdot)} \circ \mathbf{ad}_{L_\sigma})|_{L_{-\sigma}} = 0$, $\sigma = \pm$.

Remark 4.1.4 If L is a toral 3-graded Lie superalgebra satisfying $L_0 = [L_+, L_-]$ then L is perfect. This holds in particular when $L = \mathrm{eosp}(q_\infty)$ where q represents 1. However, this is not true for $L = \overline{\mathrm{eosp}(q_\infty)}$ nor for any subalgebra of $\overline{\mathrm{eosp}(q_\infty)}$ which strictly contains $\mathrm{eosp}(q_\infty)$.

Proof of Theorem 4.1.3: Let $(\cdot|\cdot) \in \mathrm{Inv}(L; K)$. Then clearly $(\cdot|\cdot)|_{L_0 \times L_0} \in \mathrm{Inv}(L_0; K)$. By invariance of $(\cdot|\cdot)$, we have

$$([[x_0, x_+], x_-] \mid \zeta) = ([x_0, x_+] \mid x_-) = (x_0 \mid [x_+, x_-])$$

for all $x_\sigma \in L_\sigma$, $\sigma = 0, \pm$ and so $\psi((\cdot|\cdot)) \in \mathrm{Inv}(\widetilde{L_0; K})$. Hence ψ is well-defined and clearly a $K_{\bar{0}}$-module homomorphism. Given $(\cdot|\cdot), (\cdot|\cdot)^{\sim} \in \mathrm{Inv}(L; K)$ such that $\psi((\cdot|\cdot)) = \psi((\cdot|\cdot)^{\sim})$, we have that

$$(x_+ \mid x_-) = (\zeta \mid [x_+, x_-]) = (\zeta \mid [x_+, x_-])^{\sim} = (x_+ \mid x_-)^{\sim}$$

and so since $(L_\sigma \mid L_\mu) = (0)$ for $\sigma \neq -\mu$, it follows that $(\cdot|\cdot) = (\cdot|\cdot)^{\sim}$. Hence ψ is a monomorphism. Let $(\cdot|\cdot)^{\sim} \in \mathrm{Inv}(\widetilde{L_0; K})$. Note that

$$([L_0, L_0] \mid \zeta)^{\sim} = (L_0 \mid [L_0, \zeta])^{\sim} = (0).$$

Define $(\cdot|\cdot) : L \times L \to K$ by setting

$$\begin{aligned}
(\cdot|\cdot)|_{L_0 \times L_0} &= (\cdot|\cdot)^{\sim}, \\
(x_+ \mid x_-) &= (\zeta \mid [x_+, x_-])^{\sim} = (-1)^{|x_+||x_-|}(x_- \mid x_+), \quad x_\sigma \in L_\sigma, \sigma = \pm, \\
(L_\sigma \mid L_\mu) &= (0), \quad \sigma \neq -\mu, \ \sigma, \mu \in \{0, \pm\}.
\end{aligned}$$

Then clearly by definition, $(\cdot|\cdot)$ is supersymmetric. It is invariant on $L_0 \times L_0$. Hence to show that it is invariant on $L \times L$, using bilinearity and the fact that $([L_\sigma, L_\mu] \mid L_\nu) - (L_\sigma \mid [L_\mu, L_\nu]) = (0)$ for all $\sigma + \mu + \nu \neq 0$, it suffices to show that $([x_\sigma, y_\mu] \mid z_\nu) = (x_\sigma \mid [y_\mu, z_\nu])$ for all $\sigma + \mu + \nu = 0$, $(\sigma, \mu, \nu) \neq (0, 0, 0)$. In addition, proving the case for (σ,μ,ν) we prove the case for (ν,μ,σ) by using supersymmetry. Hence the combinations we need to check are: $(1,-1,0)$, $(1,0,-1)$, $(-1,1,0)$.

$$
\begin{aligned}
(1,-1,0) \quad : \quad & ([x_+,y_-] \mid z_0) = (-1)^{|z_0|(|x_+|+|y_-|)}([[z_0,x_+],y_-] \mid \zeta)^{\sim} \\
& = (-1)^{|z_0|(|x_+|+|y_-|)}([z_0,[x_+,y_-]] \mid \zeta)^{\sim} - (-1)^{|z_0||y_-|}([x_+,[z_0,y_-]] \mid \zeta)^{\sim} \\
& = (x_+ \mid [y_-,z_0]). \\
(1,0,-1) \quad : \quad & ([x_+,y_0] \mid z_-) = ([[x_+,y_0],z_-] \mid \zeta)^{\sim} \\
& = ([x_+,[y_0,z_-]] \mid \zeta)^{\sim} - (-1)^{|x_+||y_0|}([y_0,[x_+,z_-]] \mid \zeta)^{\sim} \\
& = (x_+ \mid [y_0,z_-]). \\
(-1,1,0) \quad : \quad & ([x_-,y_+] \mid z_0) = -(-1)^{|z_0|(|x_-|+|y_+|)+|x_-||y_+|}([[z_0,y_+],x_-] \mid \zeta)^{\sim} \\
& = -(-1)^{|z_0|(|x_+|+|y_-|)+|x_-||y_+|}([z_0,y_+] \mid x_-) \\
& = (x_- \mid [y_+,z_0]).
\end{aligned}
$$

Hence $(\cdot\mid\cdot)$ is invariant and so $(\cdot\mid\cdot) \in \mathrm{Inv}(L; K^{\sim})$. Clearly, $\psi((\cdot\mid\cdot)) = (\cdot\mid\cdot)^{\sim}$ and so ψ is epimorphism, hence isomorphism, of $K^{\sim}_{\bar{0}}$-modules. To prove that ϕ is an isomorphism, let $(\cdot\mid\cdot) \in \mathrm{Inv}(L_0; K^{\sim})$ and define the map $\varphi : L_0 \to K^{\sim}$ by setting $\varphi(x_0) = (\zeta \mid x_0)$. Then using the bilinearity of $(\cdot\mid\cdot)$ and the fact that $(\cdot\mid\cdot)$ and ζ are homogeneous of degree $\bar{0}$, we have that $\varphi \in \mathrm{Hom}_{K^{\sim}}(L_0, K^{\sim})_{\bar{0}}$. Also, since

$$([L_0, L_0] \mid \zeta) = (0),$$

$\varphi([L_0, L_0]) = (0)$ and so ϕ is a well-defined $K^{\sim}_{\bar{0}}$-module homomorphism. Given $(\cdot\mid\cdot), (\cdot\mid\cdot)^{\sim} \in \mathrm{Inv}(L_0; K^{\sim})$ such that $\phi_{(\cdot\mid\cdot)} = \phi_{(\cdot\mid\cdot)^{\sim}}$ we have that

$$
\begin{aligned}
(x_0 \mid [y_+,y_-]) &= ([[x_0,y_+],y_-] \mid \zeta) = \phi_{(\cdot\mid\cdot)}([[x_0,y_+],y_-]) \\
&= \phi_{(\cdot\mid\cdot)^{\sim}}([[x_0,y_+],y_-]) = ([[x_0,y_+],y_-] \mid \zeta)^{\sim} \\
&= (x_0 \mid [y_+,y_-])^{\sim}
\end{aligned}
$$

and so, since $L_0 = [L_+, L_-]$, the map ϕ is a monomorphism. For $\varphi \in \mathrm{Hom}_{K^{\sim}}(L_0, K^{\sim})_{\bar{0}}$ such that $\varphi([L_0, L_0]) = (0)$ we can define the map $(\cdot\mid\cdot) : L_0 \times L_0 \to K^{\sim}$ by setting

$$(x_0 \mid [y_+,y_-]) = \varphi([[x_0,y_+],y_-])$$

for all $x_0 \in L_0$, $y_+ \in L_+$, $y_- \in L_-$. Then clearly $(\cdot|\cdot)$ is K-bilinear and of degree $\bar{0}$. Moreover, for $x_0 = \sum_i [x_+^i, x_-^i]$, $y_0 = \sum_j [y_+^j, y_-^j] \in [L_+, L_-]$ we have

$$\begin{aligned}\sum_j \varphi([[x_0, y_+^j], y_-^j]) &= \sum_{i,j} \varphi([[[x_+^i, x_-^i], y_+^j], y_-^j]) \\ &\overset{(4.1.2)}{=} \sum_{i,j} (-1)^{|x_0||y_0|} \varphi([[[y_+^j, y_-^j], x_+^i], x_-^i]) \\ &= \sum_i (-1)^{|x_0||y_0|} \varphi([[y_0, x_+^i], x_-^i])\end{aligned}$$

and so $(x \mid y)$ is well defined and satisfies $(x \mid y) = (-1)^{|x||y|}(y \mid x)$. To check the invariance, let $u_\pm, v_\pm \in L_\pm$ and $x_0, y_0 \in L_0$. Then using the Jacobi identity and the fact that $\varphi([L_0, L_0]) = (0)$ we have

$$\begin{aligned}([x_0, [u_+, u_-]] \mid y_0) - (x_0 \mid [[u_+, u_-], y_0]) &= ([[x_0, u_+], u_-] - (-1)^{|u_+||u_-|}[[x_0, u_-], u_+] \mid y_0) \\ &\quad - (x_0 \mid [u_+, [u_-, y_0]] - (-1)^{|u_+||u_-|}[u_-, [u_+, y_0]]) \\ &= \varphi([[x_0, u_+], [u_-, y_0]]) - (-1)^{|u_-||u_+|}\varphi([[x_0, u_-], [u_+, y_0]]) \\ &\quad - \varphi([[x_0, u_+], [u_-, y_0]]) - (-1)^{|u_+||u_-|}\varphi([[x_0, u_-], [u_+, y_0]]) \\ &= 0.\end{aligned}$$

Hence $(\cdot|\cdot) \in \mathrm{Inv}(L; K)$. Moreover,

$$([[x_0, x_+], x_-] \mid \zeta) = \varphi([[\zeta, [x_0, x_+]], x_-]) = \varphi([[x_0, x_+], x_-]) = (x_0 \mid [x_+, x_-])$$

for all $x_\sigma \in L_\sigma$, $\sigma = 0, \pm$ and so $(\cdot|\cdot) \in \widetilde{\mathrm{Inv}(L_0; K)}$. Since clearly $\phi((\cdot|\cdot)) = \varphi$, the map ϕ is an epimorphism, hence isomorphism, of $K_{\bar{0}}$-modules. It is now clear that the map (4.2) is an isomorphism of $K_{\bar{0}}$-modules. Now suppose $(\cdot|\cdot) \in \mathrm{Inv}(L; K)$ is nondegenerate. Then clearly the form $(\cdot|\cdot)|_{L_0 \times L_0}$ on L_0 is nondegenerate. Moreover, if $x_\sigma \in \ker(\varphi \circ \mathbf{ad}_{L_{-\sigma}})\,|_{L_\sigma}$, $\varphi = \varphi_{(\cdot|\cdot)}$, then for all $y = y_+ + y_0 + y_- \in L$,

$$(x_\sigma \mid y_+ + y_0 + y_-) = (x_\sigma \mid y_{-\sigma}) = \sigma(\zeta \mid [x_\sigma, y_\sigma]) = \sigma\varphi([x_\sigma, y_{-\sigma}]) = 0$$

and so since $(\cdot|\cdot)$ is nondegenerate, $x_\sigma = 0$. Conversely, if we assume that the form $(\cdot|\cdot)_0$ on L_0 is nondegenerate and that $\ker(\varphi \circ \mathbf{ad}_{L_{-\sigma}})|_{L_\sigma} = 0$, then for all

$x = x_+ + x_0 + x_- \in L$ satisfying $(x \mid y) = 0$ for all $y = y_+ + y_0 + y_- \in L$,

$$\begin{aligned} 0 = (x_+ + x_0 + x_- \mid y_+ + y_0 + y_-) &= (x_+ \mid y_-) + (x_0 \mid y_0) + (x_- \mid y_+) \\ &= (\zeta \mid [x_+, y_-])_0 + (x_0 \mid y_0)_0 - (\zeta \mid [x_-, y_+])_0 \\ &= \varphi([x_+, y_-]) + (x_0 \mid y_0) - \varphi([x_-, y_+]) \end{aligned}$$

and so $x = 0$, i.e., $(\cdot \mid \cdot)$ is nondegenerate. ■

Note that the above characterization of supersymmetric invariant forms was arrived at by "shifting" to L_0. It is also possible to derive another characterization by "shifting" to $L_+ \times L_-$ as was done by Benkart in [Be] for root-graded Lie algebras over fields of characteristic 0. In this case we have the following:

Proposition 4.1.5 *Let $L = L_+ \oplus L_0 \oplus L_-$ be a toral 3-graded Lie superalgebra satisfying $L_0 = [L_+, L_-]$. Then there exists a supersymmetric K-bilinear invariant form $(\cdot \mid \cdot) : L \times L \longrightarrow K$ on L if and only if there exists a K-bilinear map $\phi : L_+ \times L_- \longrightarrow K$ of degree $\bar{0}$ satisfying $\phi([x_+, x_0], x_-) = \phi(x_+, [x_0, x_-])$ for all $x_\sigma \in L_\sigma$, $\sigma \in \{0, +, -\}$. In this case, $(\cdot \mid \cdot)$ and ϕ uniquely determine each other via*

$$\begin{cases} (L_\sigma \mid L_\sigma + L_0) = 0, \quad \sigma \in \{+, -\} \\ (x_0 \mid [x_+, x_-]) = \phi([x_0, x_+], x_-), \quad x_\sigma \in L_\sigma, \ \sigma \in \{0, +, -\} \\ (x_+ \mid x_-) = \phi(x_+, x_-), \quad x_+ \in L_+, \ x_- \in L_-. \end{cases} \tag{4.3}$$

Moreover, $(\cdot \mid \cdot)$ is nondegenerate iff ϕ is nondegenerate and $\ker(\mathbf{ad}_{L_0}) \mid_{L_\sigma} = 0$, $\sigma = \pm$.

Remark 4.1.6 If a Lie superalgebra over $K \ni \frac{1}{6}$ is R-graded, where R is a root system not of E_8, F_4 or G_2-type, then L is perfect, 3-graded (see [GN]) and $L_0 = [L_+, L_-]$. Moreover, for $R = A_n$ ($\frac{1}{n+1} \in K$ if n is even), B_n, C_n, D_n and E_7, L is toral. Note that Benkart described the symmetric invariant forms of R-graded Lie algebras over fields of characteristic 0 where R is any finite reduced root system. In this case, if $R = A_n$ ($\frac{1}{n+1} \in K$ if n is even), B_n, C_n, D_n and E_7, L is toral 3-graded and our result also applies, even for Lie superalgebras.

Throughout the rest of this chapter we will use the result of Theorem 4.1.3 rather than that of Proposition 4.1.5.

Proposition 4.1.7 *Let L be a toral 3-graded Lie superalgebra over A for some superextension A of K and assume that $\zeta \in [L_+, L_-]$. Then every $(\cdot \mid \cdot) \in \mathrm{Inv}(L; K)$ induces a form $(\cdot \mid \cdot)_A \in \mathrm{Inv}(A; K)$ via*

$$(a \mid b)_A = (a\zeta \mid b\zeta)$$

and, in particular,

$$(a \mid b)_A = (1 \mid ab)_A = (\zeta \mid ab\zeta) = (a\zeta \mid b\zeta)$$

for all $a, b \in A$.

Proof: Since $\zeta \in [L_+, L_-]$, $\zeta = \sum_i [x_+^i, x_-^i]$ and so

$$(a\zeta \mid b\zeta) = (a\sum_i [x_+^i, x_-^i] \mid b\zeta) = \sum_i (ax_+^i \mid [x_-^i, b\zeta]) = \sum_i (\zeta \mid [ax_+^i, x_-^i b]) = (\zeta \mid ab\zeta).$$

Rest of the assertion now follows. ∎

Consider now a toral 3-graded Lie superalgebra over K with $L_0 = A\zeta \rtimes X$ for some superextension A of K and a Lie superalgebra X over A. Note that $A\zeta$ is a central ideal of L_0. We have the following:

Proposition 4.1.8 *Let L be a toral 3-graded Lie superalgebra over A with $L_0 = [L_+, L_-]$ and $L_0 = A\zeta \rtimes X$ for some superextension A of K and a Lie superalgebra X over A. Then*

$$\begin{aligned}\{\varphi \in \mathrm{Hom}_K(L_0; K) \mid \varphi([L_0, L_0]) = 0\} = {} & \mathrm{Hom}_K(A, K)_{\bar{0}} \\ & \oplus \{\chi \in \mathrm{Hom}_K(X, K)_{\bar{0}} \mid \chi([X, X]) = 0\}\end{aligned}$$

and so the map

$$\begin{aligned}\mathrm{Inv}(L; K) &\to \mathrm{Hom}_K(A, K)_{\bar{0}} \oplus \{\chi \in \mathrm{Hom}_K(X, K)_{\bar{0}} \mid \chi([X, X]) = 0\} \\ (\cdot \mid \cdot) &\mapsto \varphi = \phi \oplus \chi \end{aligned} \tag{4.4}$$

defined by (4.2) is a $K_{\bar{0}}$-module isomorphism where

$$\left.\begin{array}{ll}(L_\pm \mid L_\pm + L_0) = (0); & \\ (A\zeta \mid [X,X]) = (0); & \\ (a\zeta \mid b\zeta) = (\zeta \mid ab\zeta), & a,b \in A; \\ (x_0 \mid [y_+, y_-]) = (\zeta \mid [[x_0, y_+], y_-]), & x_0 \in L_0,\ y_\pm \in L_\pm; \\ (x_+ \mid x_-) = (\zeta \mid [x_+, x_-]), & x_\pm \in L_\pm\end{array}\right\} \qquad (4.5)$$

and $\varphi(x_0) = (\zeta \mid x_0)$ *for all* $x_0 \in L_0$. *Moreover,* $(\cdot \mid \cdot) \in \mathrm{Inv}(L; K)$ *is nondegenerate if and only if* $(\cdot \mid \cdot)|_{L_0 \times L_0}$ *is nondegenerate and* $\ker(\varphi \circ \mathbf{ad}_{L_\sigma})\,|_{L_{-\sigma}} = 0$. *In particular,*

(i) *if* $[X, X] = X$ *then* $\mathrm{Inv}(L; K) \cong \mathrm{Hom}_K(A, K)_{\bar{0}}$;

(ii) *if* $[X, X] = 0$ *then* $\mathrm{Inv}(L; K) \cong \mathrm{Hom}_K(A, K)_{\bar{0}} \oplus \mathrm{Hom}_K(X, K)_{\bar{0}}$.

Proof: By Theorem 4.1.3, the map $\mathrm{Inv}(L; K) \to \{\varphi \in \mathrm{Hom}_K(L_0, K)_{\bar{0}} \mid \varphi([L_0, L_0]) = 0\}$ given by (4.2) is a $K_{\bar{0}}$-module isomorphism. Since $L_0 = A\zeta \rtimes X$, we have $\mathrm{Hom}_K(L_0, K)_{\bar{0}} = \mathrm{Hom}_K(A\zeta, K)_{\bar{0}} \oplus \mathrm{Hom}_K(X, K)_{\bar{0}}$. We can identify A with $A\zeta$ via $a \mapsto a\zeta$. Moreover, $[A\zeta, L_0] = 0$ and so $[L_0, L_0] = [X, X]$ hence by Theorem 4.1.3, $(0) = (\zeta \mid [L_0, L_0]) = (\zeta \mid [X, X])$. The rest of the assertion now follows from Theorem 4.1.3 and Proposition 4.1.7. ■

Remark 4.1.9 For examples of toral 3-graded Lie algebras of type (i) and (ii) above, see Corollary 4.2.7 and Example 4.2.6.

We now turn to the study of the *module of co-invariants* $V_K(L)$ of L (for example, see [BeK]). Given a Lie superalgebra L over K we define $V_K(L) = L \otimes_K L / \langle [xy] \otimes z - x \otimes [yz] \mid x, y, z \in L \rangle$. We write $x \circ y$ for the image of $x \otimes y$ under this canonical homomorphism. Note that if L is finite-dimensional, $V_K(L) \cong \mathrm{Inv}(L; K)$.

In [BK], Berman and Kryliouk studied the universal central extensions of twisted and untwisted Lie algebras over fields of characteristic not 2, extended over commutative rings. These algebras generalize affine Lie algebras. Berman and Kryliouk described the second homology group (for definition see Chapter 5) of this type of algebra L using the module of co-invariants of L (for more background see [G]).

If we consider the toral 3-graded Lie superalgebras over K we can express $V_K(L)$ in terms of L_0. We have the following

Proposition 4.1.10 *Let L be a toral 3-graded Lie superalgebra. Then $V_K(L) = L_0 \circ L_0$ and $\zeta \circ [L_0, L_0] = 0$. Moreover, if $L_0 = [L_+, L_-]$ then $V_K(L) = \zeta \circ L_0$.*

Proof: Since L is toral 3-graded we have for all $x_0 \in L_0$ and $x_\pm, y_\pm \in L_\pm$,

$$(x_0 + x_\pm) \circ y_\pm = (x_0 + x_\pm) \circ (\pm[\zeta, y_\pm]) = \pm[x_0 + x_\pm, \zeta] \circ y_\pm = -x_\pm \circ y_\pm$$

and so since $\frac{1}{2} \in K$, $(L_\pm + L_0) \circ L_\pm = (0)$. Moreover, if $x_0, y_0 \in L_0$ then

$$\zeta \circ [x_0, y_0] = [\zeta, x_0] \circ y_0 = 0$$

and so $\zeta \circ [L_0, L_0] = (0)$. If $L_0 = [L_+, L_-]$ then for all $x_0 \in L_0$, $y_\pm \in L_\pm$,

$$x_0 \circ [y_+, y_-] = [x_0, y_+] \circ y_- = \zeta \circ [[x_0, y_+], y_-]$$

and so since $L_0 = [L_+, L_-] = [[\zeta, L_+], L_-] \subset [[L_0, L_+], L_-] \subset L_0$, the last assertion holds. ∎

In particular we have the following consequence:

Corollary 4.1.11 *Let L be a toral 3-graded Lie superalgebra with $L_0 = [L_+, L_-]$ and $L_0 = A\zeta \rtimes [L_0, L_0]$. Then $V_K(L) = \zeta \circ A\zeta$.*

4.2 Invariant forms of $\mathrm{eosp}(q_\infty)$

Throughout this section A will denote a superextension of K and $\mathcal{M}$ will denote an A-supermodule with an A-quadratic form q. We now consider the Lie superalgebra $\mathrm{eosp}(q_\infty)$. Recall that we can write $\mathrm{eosp}(q_\infty) = L_+ \oplus L_0 \oplus L_-$ where

$$\begin{aligned} L_0 &= \{(a, 0, 0, x) \mid a \in K, x \in \mathrm{eosp}(q)\} \\ L_+ &= \{(0, m, 0, 0) \mid m \in \mathcal{M}\} \\ L_- &= \{(0, 0, m, 0) \mid m \in \mathcal{M}\} \end{aligned}$$

with multiplication given by

$$\begin{aligned}
&[(a,m,n,x),(\tilde{a},\tilde{m},\tilde{n},\tilde{x})] = \\
&\quad = (q(m,\tilde{n}) - q(n,\tilde{m}),\ a\tilde{m} - m\tilde{a} + x(\tilde{m}) - (m)\tilde{x}, \\
&\qquad n\tilde{a} - a\tilde{n} + x(\tilde{n}) - (n)\tilde{x},\ \mathbf{E}_{m,\tilde{n}} + \mathbf{E}_{n,\tilde{m}} + [x,\tilde{x}]).
\end{aligned}$$

We will sometimes abbreviate $L_\pm$ by $\mathcal{M}^\pm$ when there is no possibility of confusion. Hence L is 3-graded and $\zeta = (1,0,0,0)$ satisfies $[a\zeta, x_\sigma] = \sigma a x_\sigma$ for $a \in A$, $x_\sigma \in L_\sigma$, $\sigma \in \{0, \pm\}$, and so L is toral 3-graded. Moreover, $L_0 = A \cdot 1 \rtimes \mathrm{eosp}(q)$ and L can be viewed as a Lie superalgebra over A. Hence we have

Corollary 4.2.1 *Let A be a superextension of K and $\mathcal{M}$ an A-supermodule with an A-quadratic form q. Let $L = \mathrm{eosp}(q_\infty)$ and assume that L is perfect. Then*

$$\mathrm{Inv}(L,K) \cong \mathrm{Hom}_K(A,K)_{\bar{0}} \oplus \{\chi \in \mathrm{Hom}_K(\mathrm{eosp}(q),K)_{\bar{0}} \mid \chi([\mathrm{eosp}(q),\mathrm{eosp}(q)]) = 0\}.$$

via (4.4).

Proof: By Lemma 2.3.6, L perfect implies $L_0 = [L_+, L_-]$. Hence the assertion follows from Proposition 4.1.8. ∎

Corollary 4.2.2 *Let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q such that $\mathrm{eosp}(q)$ is perfect. Let $L = \mathrm{eosp}(q_\infty)$ and assume that L is perfect. Then*

$$\mathrm{Inv}(L;K) \cong \mathrm{Hom}_K(A,K)_{\bar{0}}$$

via (4.4). Moreover, if q is nondegenerate and $(\cdot\,|\,\cdot) \in \mathrm{Inv}(L;K)$, the following are equivalent:

(1) $(\cdot\,|\,\cdot)$ is nondegenerate;

(2) $(\cdot\,|\,\cdot)|_{A\times A}$ is nondegenerate;

(3) $S = \{a \in A \mid \varphi(ab) = 0,\ \text{for all } b \in A\} = 0$ where $\varphi \in \mathrm{Hom}_K(A,K)_{\bar{0}} : a \mapsto (1 \mid a)$, i.e., φ induces a nondegenerate K-bilinear invariant supersymmetric form on A via $(a\,|\,b)_A = \varphi(ab)$.

Proof: Since $[\mathrm{eosp}(q), \mathrm{eosp}(q)] = \mathrm{eosp}(q)$, by Corollary 4.2.1,

$$\mathrm{Inv}(L; K) \xrightarrow{(4.4)} \mathrm{Hom}_K(A, K)_{\bar{0}}$$

is a $K_{\bar{0}}$-module isomorphism. Note that since $\mathrm{eosp}(q)$ is perfect,

$$(A \mid \mathrm{eosp}(q)) \quad = \quad (0) \tag{4.6}$$

for any $(\cdot \mid \cdot) \in \mathrm{Inv}(L; K)$.

$\boxed{(1) \Rightarrow (2)}$ Assume that $(\cdot \mid \cdot)$ is nondegenerate. Let $a \in A$ such that $(a \mid b) = 0$ for all $b \in A$. Then by (4.5) and (4.6), $(a \mid (b, m, n, x)) = 0$ for all $b \in A$, $m, n \in \mathcal{M}$, $x \in \mathrm{eosp}(q)$ and so $a = 0$.

$\boxed{(2) \Rightarrow (3)}$ Assume $(\cdot \mid \cdot) \mid_{A \times A}$ is nondegenerate. Let $a \in S$. Then by (4.5), for all $b \in A$,

$$0 = \varphi(ab) = (1 \mid ab) = (a \mid b)$$

and so $a = 0$.

$\boxed{(3) \Rightarrow (1)}$ Assume $S = 0$. Let $(a, m, n, x) \in L$. Suppose

$$((a, m, n, x) \mid (b, p, r, y)) = 0 \text{ for all } (b, p, r, y) \in L.$$

Then by (4.5) and (4.6), $(a \mid b) = 0$, $(m^+ \mid r^-) = 0$, $(n^- \mid p^+) = 0$, $(x \mid y) = 0$ for all $b \in A$, $p, r \in \mathcal{M}$, $y \in \mathrm{eosp}(q)$. Hence

$$\varphi(ab) = (1 \mid ab) = (a \mid b) = 0 \quad \text{for all } b \in A$$

$$\varphi(q(m, r)b) = (1 \mid q(m, rb)) = (m^+ \mid r^- b) = 0 \quad \text{for all } b \in A, \ r \in \mathcal{M}$$

$$\varphi(q(n, p)b) = (1 \mid q(n, pb)) = (n^- \mid p^+ b) = 0 \quad \text{for all } b \in A, \ p \in \mathcal{M}$$

and so $a = 0$ and since q is nondegenerate, $m = 0 = n$. Moreover, for all $p, r \in \mathcal{M}$, $a \in A$,

$$\begin{aligned} 0 = (x \mid a\mathbf{E}_{p,r}) &= (x \mid [ap^+, r^-]) - (x \mid q(ap, r)) \overset{(4.6)}{=} ([x, ap^+] \mid r^-) \\ &= (1 \mid [[x, ap^+], r^-]) = (1 \mid [x(ap)^+, r^-]) \overset{(4.6)}{=} (-1)^{|a||x|}(1 \mid q(ax(p), r)) \\ &= (-1)^{|a||x|}\varphi(aq(x(p), r)). \end{aligned}$$

Therefore $q(x(p), r) \in S$ for all $p, r \in \mathcal{M}$ and so $q(x(p), r) = 0$ for all $p, r \in \mathcal{M}$. Since q is nondegenerate, $x(p) = 0$ for all $p \in \mathcal{M}$ and so $x = 0$. Therefore $(\cdot \mid \cdot)$ is nondegenerate. ∎

Corollary 4.2.3 *Let $\mathcal{N} = \oplus_{i \in I} An_i$ be a free A-supermodule with an almost diagonizable A-quadratic form $q_{\mathcal{N}}$ and let $\mathcal{P}$ be an arbitrary A-supermodule. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ with $q = q_{\mathcal{N}} \oplus q_{\mathcal{P}}$. Then if $\{n_i \mid i \in I\}_{\bar{1}} \neq \emptyset$ or $|\{n_i \mid i \in I\}_{\bar{0}}| \geq 3$, the map*

$$\mathrm{Inv}(L; K) \to \mathrm{Hom}_{K}(A, K)_{\bar{0}} : (\cdot \mid \cdot) \mapsto \varphi$$

given by (4.4) is a $K_{\bar{0}}$-module isomorphism and

$$\left.\begin{array}{ll}
(L_\sigma \mid L_\sigma + L_0) = 0 & \\
(A \mid \mathrm{eosp}(q)) = 0 & \\
(a \mid b) = (1 \mid ab) = \varphi(ab), & a, b \in A \\
(x \mid \mathbf{E}_{m,n}) = (1 \mid q(x(m), n)) = \varphi(q(x(m), n)), & m, n \in \mathcal{M},\ x \in \mathrm{eosp}(q) \\
(m^+ \mid n^-) = (1 \mid q(m, n)) = \varphi(q(m, n)), & m, n \in \mathcal{M}.
\end{array}\right\} \quad (4.7)$$

Proof: Since $\mathcal{N} \neq (0)$, L is perfect by Lemma 2.3.6. If $\{n_i \mid i \in I\}_{\bar{1}} \neq \emptyset$ or $|\{n_i \mid i \in I\}_{\bar{0}}| \geq 3$ then $\mathrm{eosp}(q)$ is perfect by Proposition 2.1.19. The assertion now follows from Proposition 4.1.8. ∎

Corollary 4.2.4 *Suppose $\mathcal{M} = H(I; A) \oplus \mathcal{N}$ with $q_{\mathcal{N}}$ an A-quadratic form on $\mathcal{N}$ and set $q = q_I \oplus q_{\mathcal{N}}$. Assume that $|I| \geq 2$ or $|I| \geq 1$ and $\mathcal{N}$ contains as a direct summand either an invertibly diagonizable A-supermodule of rank 1 or an almost diagonizable A-supermodule of rank 2. Let L be a central extension of $\mathrm{eosp}(q_\infty)$ (e.g. L is a D_J- or B_J-graded Lie superalgebra for $J = I \dot{\cup} \{\infty\}$, [GN]). Then*

$$\mathrm{Inv}(L; K) \cong \mathrm{Hom}_{K}(A, K)_{\bar{0}}$$

via (4.7).

Proof: It follows from Corollary 4.2.3. ∎

Example 4.2.5 For example, if L is the centreless core of an extended affine Lie algebra of type B_l or D_l ($l \geq 3, 4$ respectively) then the invariant forms of L are isomorphic to $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}], \mathbb{C})$ for some $\nu \in \mathbb{N}$.

We can now also look at an example of $\mathrm{eosp}(q_\infty)$ for which $\mathrm{Inv}(\mathrm{eosp}(q_\infty); K)$ is not isomorphic to $\mathrm{Hom}_K(A, K)_{\bar{0}}$:

Example 4.2.6 *Let $\mathcal{M} = Am \oplus An$ be a free A-supermodule with the A-quadratic form q defined by $q(m, m) = q(n, n) = 1$, $q(m, n) = 0$. Let $L = \mathrm{eosp}(q_\infty)$. Then*

$$\mathrm{Inv}(L; K) \cong \mathrm{Hom}_K(A, K)_{\bar{0}} \oplus \mathrm{Hom}_K(A, K)_{\bar{0}}$$

Proof: Since $L_0 = A \cdot 1 \oplus A \cdot \mathbf{E}_{m,n}$, $[L_0, L_0] = 0$ and $\mathrm{Hom}_K(A(\mathbf{E}_{m,n}), K)_{\bar{0}} \cong \mathrm{Hom}_K(A, K)_{\bar{0}}$, and so by Proposition 4.1.8,

$$\mathrm{Inv}(L; K) \cong \mathrm{Hom}_K(A, K)_{\bar{0}} \oplus \mathrm{Hom}_K(A, K)_{\bar{0}}$$

■

Proposition 4.2.7 *Let $L = \mathrm{eosp}(q_\infty)$, $L = [L, L]$ and assume that* $\mathrm{eosp}(q)$ *is perfect. Then* $\mathrm{V}(L) = 1 \circ A$.

Proof: It follows from Corollary 4.1.11.

4.3 Skew derivations of $\mathrm{eosp}(q_\infty)$

In this section we will describe the skew derivations of a toral 3-graded Lie superalgebra. We continue to assume that A is a superextension of K and that $\mathcal{M}$ is an A-supermodule with an A-quadratic form q. In addition, we assume that $\mathrm{Ann}_A \mathcal{M} = (0)$.

In general, we have the following definition:

Definition 4.3.1 Let $\alpha \in \mathbb{Z}_2$. A K-derivation $d \in (\mathrm{Der}_K L)_\alpha$ of a Lie superalgebra L over K with a K-bilinear supersymmetric invariant form $(\cdot|\cdot)$ is called a *skew derivation* if

$$(d(x) \mid y) = -(-1)^{|d||x|}(x \mid d(y))$$

for all $x, y \in L$. We denote the K-supermodule of all skew derivations of L by $\mathrm{SDer}_K L$.

Lemma 4.3.2 *Let L be a Lie superalgebra equipped with $(\cdot\,|\,\cdot) \in \mathrm{Inv}(L;K)$. Then*

$$\mathbf{ad}_{\mathrm{loc}}\, L \lhd \mathrm{SDer}_K\, L.$$

Proof: By Lemma 1.3.20, $\mathbf{ad}_{\mathrm{loc}}\, L \lhd \mathrm{Der}_K\, L$. Let $d \in \mathbf{ad}_{\mathrm{loc}}\, L$ and $x, y \in L$. Then there exists $z \in L$ such that $d(x) = (\mathbf{ad}\, z)(x)$ and $d(y) = (\mathbf{ad}\, z)(y)$. Hence

$$\begin{aligned}(d(x) \mid y) &= ([z,x] \mid y) = -(-1)^{|z||x|}([x,z] \mid y)\\ &= -(-1)^{|z||x|}(x \mid [z,y]) = -(-1)^{|d||x|}(x \mid d(y)\end{aligned}$$

and so $d \in \mathrm{SDer}_K\, L$. ∎

Theorem 4.3.3 *Let A be a superextension of K and let $\mathcal{M}$ be an A-supermodule with an A-quadratic form q. Set $L = \mathrm{eosp}(q_\infty)$ and assume that L and $\mathrm{eosp}(q)$ are perfect. Recall from Theorem 3.2.13.2 that*

$$\mathrm{Der}_K\, L = \mathbf{ad}\, L + \{d_{S,T} \in \mathcal{D} \mid S \in \mathcal{S},\, T \in \mathcal{T}\}.$$

Then given an invariant form $(\cdot\,|\,\cdot) \in \mathrm{Inv}(L;K)$,

$$\mathrm{SDer}_K\, L = \mathbf{ad}\, L + \{d_{S,T} \in \mathcal{D} \mid (1 \mid [S, A\cdot Id]) = (0) \text{ and } ([T, A\cdot Id] \mid \mathrm{eosp}(q)) = (0)\}.$$

Moreover,

$$\{d_{S,0} \in \mathcal{D} \mid (1 \mid [S, A\cdot Id]) = (0)\} = \{d_{S,0} \in \mathcal{D} \mid (\mathbf{ad}\, S)|_{A\cdot Id} \subset \mathrm{SDer}_K\, A\}.$$

Proof: By Lemma 4.3.2, $\mathbf{ad}\, L \subset \mathbf{ad}_{\mathrm{loc}}\, L \subset \mathrm{SDer}_K\, L$. Recall from Theorem 4.1.3 that $(L \mid L) = (L_0 \mid L_0) + (L_+ \mid L_-)$ and so since $\mathcal{D} \subset (\mathrm{Der}_K\, L)_0$, to check that $d = d_{S,T} \in \mathcal{D}$ is skew, we only need to check that

(a) $(d(a+x) \mid b+y) + (-1)^{|d||a+x|}(a+x \mid d(b+y)) = 0$ for all $a, b \in A$, $x, y \in \mathrm{eosp}(q)$; and

(b) $(d(m^+) \mid n_-) + (-1)^{|d||m|}(m^+ \mid d(n^-)) = 0$ for all $m, n \in \mathcal{M}$.

Recall that since $\operatorname{eosp}(q)$ is perfect, by Proposition 3.2.7.2.4 we have that $[\mathcal{T}, \operatorname{eosp}(q)] = (0)$ and, by Theorem 4.1.3, $(1 \mid \operatorname{eosp}(q)) = (0)$. Let $S \in \mathcal{S}$ and $T \in \mathcal{T}$. Recall (4.7). Then (a) holds if and only if for all $a, b \in A$, $x \in \operatorname{eosp}(q)$ and $m, n \in \mathcal{M}$ we have

$$\begin{aligned}
0 &= (d(a+x) \mid b + \mathbf{E}_{m,n}) + (-1)^{|S+T||a+x|}(a + x \mid d(b + \mathbf{E}_{m,n})) \\
&= ([S, a\cdot Id] \mid b) + ([S, x] + [T, a\cdot Id] \mid \mathbf{E}_{m,n}) \\
&\quad +(-1)^{|S+T||a+x|}((a \mid [S, b\cdot Id]) + (x \mid [S, \mathbf{E}_{m,n}] + [T, b\cdot Id])) \\
&= (1 \mid [S, a\cdot Id]b + (-1)^{|S||a|}a[S, b\cdot Id]) + (1 \mid q([S, x](m), n)) \\
&\quad +(-1)^{|S||x|}(x \mid \mathbf{E}_{S(m),n} + (-1)^{|S||m|}\mathbf{E}_{m,S(n)}) \\
&\quad +([T, a\cdot Id] \mid \mathbf{E}_{m,n}) + (-1)^{|S||x|}(x \mid [T, b\cdot Id]) \\
&= (1 \mid [S, ab\cdot Id]) + ([T, a\cdot Id] \mid \mathbf{E}_{m,n}) + (-1)^{|S||x|}(x \mid [T, b\cdot Id]) \\
&\quad +(1 \mid q(([S, x])m, n) + (-1)^{|S||x|}q(x(S(m)), n) + (-1)^{|S|(|x|+|m|)}q(x(m), S(n))) \\
&\overset{(S2)}{=} (1 \mid [S, ab\cdot Id]) + ([T, a\cdot Id] \mid \mathbf{E}_{m,n}) + (-1)^{|S||x|}(x \mid [T, b\cdot Id]) \\
&\quad (1 \mid [S, q(x(m), n)\cdot Id]).
\end{aligned}$$

Since $a, b \in A$, $m, n \in \mathcal{M}$ and $x \in \operatorname{eosp}(q)$ above are arbitrary, $d_{S,T}$ satisfies (a) if and only if

$$\left.\begin{array}{ll}
(1 \mid [S, a\cdot Id]) = 0 & \text{for all } a \in A\text{; and} \\
([T, a\cdot Id] \mid y) = 0 & \text{for all } a \in A,\ y \in \operatorname{eosp}(q).
\end{array}\right\} \tag{4.8}$$

In addition, if $m, n \in \mathcal{M}$ and S and T satisfy (4.8) then using (4.7) and Proposition 3.2.7.2.4 we have that

$$\begin{aligned}
(d_{S,T}(m^+) \mid n^-) &+ (-1)^{|S+T||m|}(m^+ \mid d_{S,T}(n^-)) \\
&= ((S+T)(m)^+ \mid n^-) + (-1)^{|S+T||m|}(m^+ \mid (S-T)(n)^-) \\
&= (1 \mid q((S+T)(m), n) + (-1)^{|S+T||m|}q(m, (S-T)(n))) \\
&= (1 \mid [S, q(m, n)\cdot Id] + [T, \mathbf{E}_{m,n}]) \\
&= 0,
\end{aligned}$$

i.e., $d_{S,T}$ satisfies (b). It remains to be shown that $\{S \in \mathcal{S} \mid (1 \mid [S, A\cdot Id]) = 0\} = \{S \in \mathcal{S} \mid (\operatorname{ad} S)|_{A\cdot Id} \subset \operatorname{SDer}_K A\}$. Let $S \in \mathcal{S}$ and $a, b \in A$. Then by Proposition

3.2.7.1.1 and (4.7),

$$
\begin{aligned}
((\mathbf{ad}\, S)(a\cdot Id) \mid b) + (-1)^{|S||a|}(a \mid (\mathbf{ad}\, S)(b)) &= (1 \mid [S, a\cdot Id]b + (-1)^{|S||a|}a[S, b\cdot Id]) \\
&= (1 \mid [S, ab\cdot Id])
\end{aligned}
$$

and so $(1 \mid [S, a\cdot Id]) = 0$ if and only if $(\mathbf{ad}\, S)|_{A\cdot Id} \in \mathrm{SDer}_K\, A$. Hence the assertion follows. ∎

In particular, we have that

Corollary 4.3.4 *Let $\mathcal{N}$ and $\mathcal{R}$ be free A-supermodules with A-quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{R}}$ respectively such that $q_{\mathcal{R}} \equiv 0$ and $q_{\mathcal{N}}$ is an orthogonal sum of an almost diagonizable A-quadratic form and an invertibly diagonizable A-quadratic form on a free A-supermodule of rank 2. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$ with the A-quadratic form $q = q_{\mathcal{N}} \oplus q_{\mathcal{R}}$ and let $L = \mathrm{eosp}(q_\infty)$. Let $(\cdot \mid \cdot)$ be an invariant supersymmetric form on L. Then*

$$\mathrm{SDer}_K\, L = \mathbf{ad}_{\mathbf{loc}}\, L \rtimes \{d_{S,0} \in \mathcal{D} \mid S \in \mathrm{End}_A\, \mathcal{R} \oplus (\mathrm{SDer}_K\, A)_{\mathcal{M}}\}.$$

Proof: It follows from Theorems 3.3.7 and 4.3.3. ∎

Chapter 5

The Second Homology Group

Throughout this chapter k will denote a commutative, associative, unital ring containing $\frac{1}{2}$, K a superextension of k and A a superextension of K. In addition, we will assume throughout that when $|m|$ occurs in an expression for some $m \in \mathcal{M}$ for some supermodule $\mathcal{M}$, then it is assumed that m is homogeneous, and that the expression extends to the other elements by linearity.

5.1 Universal Central Extensions of Lie Superalgebras

Many results in this section are due to Neher ([N4]).

Let L be a Lie superalgebra over K. A *central extension of* L ([SZ]) is a short exact sequence

$$0 \longrightarrow C \longrightarrow \tilde{L} \xrightarrow{\pi} L \longrightarrow 0$$

of Lie superalgebras where C is a central ideal of $\tilde{L}$. By abuse of notation, $\tilde{L}$ is also called a central extension. If, in addition, $\tilde{L}$ is a perfect Lie superalgebra then $\tilde{L}$ is called a *covering of* L. If $\tilde{L}$ is a covering and for any central extension $\varphi : L' \to L$ there exists a unique Lie superalgebra homomorphism $\psi : \tilde{L} \to L'$ such that $\varphi\psi = \pi$, then $\tilde{L}$ is called a *universal central extension of* L.

Lemma 5.1.1 *Let L be a Lie superalgebra over K.*

(1) A universal central extension of L is unique up to isomorphism.

(2) If L has a covering then L is perfect. In particular, if L has a universal central extension then L is perfect.

Proof: (1) Let $0 \to C_1 \to \tilde{L} \xrightarrow{\pi} L \to 0$ and $0 \to C_2 \to L' \xrightarrow{\pi'} L \to 0$ be universal central extensions of L. Then there exist unique superalgebra homomorphisms $\phi : \tilde{L} \to L'$, $\varphi : L' \to \tilde{L}$ such that $\pi'\phi = \pi$ and $\pi\varphi = \pi'$ hence $\pi(\varphi\phi) = \pi$ and $\pi'(\phi\varphi) = \pi'$. Note that since $\tilde{L}$ is a universal central extension, it is also a central extension and so the map $\varphi\phi : \tilde{L} \to \tilde{L}$ is the unique homomorphism satisfying $\pi(\varphi\phi) = \pi$. Since clearly $\pi Id|_{\tilde{L}} = \pi$ it therefore follows that $\varphi\phi = Id|_{\tilde{L}}$. Similarly we can argue that $\phi\varphi = Id|_{L'}$ and so the assertion follows.
(2) Suppose that $0 \to C \to \tilde{L} \xrightarrow{\pi} L \to 0$ is a covering of L. Then using the fact that π is surjective, we have

$$L = \pi(\tilde{L}) = \pi([\tilde{L}, \tilde{L}]) = [\pi(\tilde{L}, \pi(\tilde{L})] = [L, L].$$

Since a universal central extension is a covering, the rest of the assertion follows. ■

Lemma 5.1.2 *Let L and L' be Lie superalgebras over K and let $\varphi : L \to L'$ be a Lie superalgebra epimorphism.*

(1) Then $Z(L) \subset \varphi^{-1}(Z(L'))$.

(2) In addition, if φ is a covering then $Z(L) = \varphi^{-1}(Z(L'))$.

Proof: (1) Since φ is a homomorphism, $\varphi(Z(L)) \subset Z(L')$ so by surjectivity of φ, $Z(L) \subset \varphi^{-1}(Z(L'))$.
(2) Let $x \in \varphi^{-1}(Z(L'))$. Then $\varphi([x, L]) = [\varphi(x), \varphi(Z(L))] = 0$ so since φ is a central extension, $[x, L] \subset \ker\varphi \subset Z(L)$. Hence $0 = [[x, L], L] = [x, [L, L]] + [L, [x, L]]$ and so by Lemma 5.1.1.2, $[x, L] = [x, [L, L]] = 0$. The assertion now follows. ■

Set $\mathcal{L}_L = L \otimes_K L/\mathcal{B}$ where

$$\mathcal{B} = \mathrm{span}_K\{[x,y]\otimes z + (-1)^{|x|(|y|+|z|)}[y,z]\otimes x + (-1)^{|z|(|x|+|y|)}[z,x]\otimes y,$$
$$x\otimes y + (-1)^{|y||x|}y\otimes x \mid x,y,z \in L\}.$$

When it is clear which algebra L we are considering, we will simply write $\mathcal{L}$ instead of $\mathcal{L}_L$. In addition, we will always be considering the tensor products over K and will write $\otimes$ instead of $\otimes_K$. We let $\langle x,y\rangle = x\otimes y+\mathcal{B}$ for $x,y \in L$ and extend it K-bilinearly.

We will use extensively the following two basic identities in $\mathcal{L}$:

$$\langle x,y\rangle = -(-1)^{|x||y|}\langle y,x\rangle \tag{5.1}$$
$$\langle [x,y],z\rangle = \langle x,[y,z]\rangle - (-1)^{|x||y|}\langle y,[x,z]\rangle. \tag{5.2}$$

Note that $\mathcal{L}$ is a K-supermodule with the $\mathbb{Z}_2$-grading given by $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ where

$$\mathcal{L}_\mu = \{x \in L_\alpha \otimes L_\beta \mid \alpha+\beta = \mu\}. \tag{5.3}$$

Consider the map

$$\tilde{m} : L\times L \to L : (x,y) \mapsto [x,y].$$

Then $\tilde{m}$ is a K-bilinear map and so it gives rise to the map $m : L\otimes L \to L : x\otimes y \mapsto [x,y]$. It is easy to check that the elements of $\mathcal{B}$ are in the kernel of m and so m induces the map, also called m,

$$m : \mathcal{L} \to L : \langle x,y\rangle \to [x,y]. \tag{5.4}$$

It is easy to see that m is a K-supermodule homomorphism of degree $\bar{0}$. The K-supermodule $\mathcal{L}$ is a superalgebra with respect to the multiplication

$$[l_1,l_2] = \langle m(l_1), m(l_2)\rangle \tag{5.5}$$

where $\langle m(l_1), m(l_2)\rangle$ is understood in the obvious sense: if $l_1 = \sum_i\langle x_i,y_i\rangle$ and $l_2 = \sum_j\langle u_j,v_j\rangle$ then $\langle m(l_1),m(l_2)\rangle = \langle\sum_i[x_i,y_i],\sum_j[u_j,v_j]\rangle = \sum_{i,j}\langle[x_i,y_i],[u_j,v_j]\rangle$. Hence in particular, for $x,y,u,v \in L$ we have

$$[\langle x,y\rangle,\langle u,v\rangle] = \langle [x,y],[u,v]\rangle. \tag{5.6}$$

Since for all $l_1, l_2 \in \mathcal{L}$ we have that

$$m([l_1, l_2]) = m(\langle m(l_1), m(l_2)\rangle) = [m(l_1), m(l_2)],$$

the map m is a superalgebra homomorphism. Let

$$\mathrm{H}_2(L) = \left\{ \sum_i \langle X_i, Y_i\rangle \in \mathcal{L} \mid \sum_i [X_i, Y_i] = 0 \right\}.$$

The following theorem justifies calling $\mathrm{H}_2(L)$ the *second homology group of* L.

Theorem 5.1.3 *Let L be a Lie superalgebra over K and write $\mathcal{L} = \mathcal{L}_L$.*

(1) With respect to the product (5.5), the algebra $\mathcal{L}$ is a Lie superalgebra over K.

(2) If $\chi : L \to L'$ is a Lie superalgebra epimorphism and $\mathcal{L}' = \mathcal{L}_{L'}$ then

$$\psi : \mathcal{L} \to \mathcal{L}' : \langle x, y\rangle \mapsto \langle \varphi(x), \varphi(y)\rangle \tag{5.7}$$

is a well-defined Lie superalgebra epimorphism.

(3) Let L be a perfect Lie superalgebra and set $\bar{\mathcal{L}} = \mathcal{L}_{\bar{L}}$ where $^- : L \to L/Z(L)$ is the canonical map. Then the map

$$\begin{array}{cccc} \chi : & \mathcal{L} & \longrightarrow & \bar{\mathcal{L}} \\ & \langle x, y\rangle & \longmapsto & \langle \bar{x}, \bar{y}\rangle \end{array} \tag{5.8}$$

is a Lie superalgebra isomorphism.

(4) The extension

$$0 \longrightarrow \mathrm{H}_2(L) \longrightarrow \mathcal{L} \xrightarrow{m} [L, L] \longrightarrow 0 \tag{5.9}$$

is a central extension of the Lie superalgebra $[L, L]$.

(5) If L is a perfect Lie superalgebra then $\mathcal{L}$ is the universal central extension of L and $Z(\mathcal{L}) = m^{-1}(Z(L)) \cong \mathrm{H}_2(L/Z(L))$.

Proof: (1) By construction the map $m : \mathcal{L} \to L$ is homogeneous of degree $\bar{0}$. Hence for $l_1, l_2 \in \mathcal{L}$ we have

$$[l_1, l_2] = \langle m(l_1), m(l_2) \rangle \overset{(5.1)}{=} -(-1)^{|l_1||l_2|}\langle m(l_2), m(l_1) \rangle = -(-1)^{|l_1||l_2|}[l_2, l_1].$$

In addition, using the fact that m is a superalgebra homomorphism, we have that for all $l_1, l_2, l_3 \in \mathcal{L}$

$$\begin{aligned}
[[l_1, l_2], l_3] &= [m([l_1, l_2]), l_3] = \langle [m(l_1), m(l_2)], m(l_3) \rangle \\
&\overset{(5.2)}{=} \langle m(l_1), [m(l_2), m(l_3)] \rangle - (-1)^{|l_1||l_2|}\langle m(l_2), [m(l_1), m(l_3)] \rangle \\
&= [l_1, [l_2, l_3]] - (-1)^{|l_1||l_2|}[l_2, [l_1, l_3]].
\end{aligned}$$

Hence by linearity we have that $[l_1, l_2] = (-1)^{|l_1||l_2|}[l_2, l_1]$ for all $l_1, l_2 \in \mathcal{L}$ and that the Jacobi identity holds and so $\mathcal{L}$ is a Lie superalgebra.
(2) Let $\tilde{\chi} : L \times L \to \mathcal{L}' : (x, y) \mapsto \langle \varphi(x), \varphi(y) \rangle$. Then $\tilde{\chi}$ is a K-bilinear map and is surjective since φ is surjective. Hence it gives rise to a surjective K-supermodule homomorphism $\chi : L \otimes L \to \mathcal{L}' : x \otimes y \mapsto \langle \varphi(x), \varphi(y) \rangle$. Moreover, since $\varphi : L \to L'$ is a superalgebra homomorphism of degree $\bar{0}$. we have for all $x, y, z \in L$.

$$\chi(x \otimes y + (-1)^{|x||y|} y \otimes x) = \langle \varphi(x), \varphi(y) \rangle + (-1)^{|\varphi(x)||\varphi(y)|}\langle \varphi(y), \varphi(x) \rangle = 0$$

and

$$\begin{aligned}
&\chi([x, y] \otimes z - x \otimes [y, z] + (-1)^{|x||y|} y \otimes [x, z]) \\
&\quad = \langle [\varphi(x), \varphi(y)], \varphi(z) \rangle - \langle \varphi(x), [\varphi(y), \varphi(z)] \rangle + (-1)^{|\varphi(x)||\varphi(y)|}\langle \varphi(y), [\varphi(x), \varphi(z)] \rangle \\
&\quad = 0.
\end{aligned}$$

Hence χ induces the epimorphism (5.7). In addition, for all $x, y, u, v \in L$.

$$\begin{aligned}
\chi([\langle x, y \rangle, \langle u, v \rangle]) &= \chi(\langle [x, y], [u, v] \rangle) = \langle \varphi([x, y]), \varphi([u, v]) \rangle \\
&= \langle [\varphi(x), \varphi(y)], [\varphi(u), \varphi(v)] \rangle = [\langle \varphi(x), \varphi(y) \rangle, \langle \varphi(u), \varphi(v) \rangle] \\
&= [\chi(\langle x, y \rangle), \chi(\langle u, v \rangle)]
\end{aligned}$$

and so χ is a surjective Lie superalgebra homomorphism.
(3) Using the canonical epimorphism $^{-} : L \to L/Z(L)$ and (2) we have a Lie superalgebra epimorphism (5.8). Let $\tilde{v} : \bar{L} \times \bar{L} \to \mathcal{L} : (\bar{x}, \bar{y}) \mapsto \langle x, y \rangle$. This is a well

defined K-bilinear map. Indeed, for $x, y, u, v \in L$ such that $(\bar{x}, \bar{y}) = (\bar{u}, \bar{v})$ we have that $x = u + c$ and $y = v + d$ for some $c, d \in Z(L)$. Note that

$$\langle [L, L], Z(L) \rangle = (0)$$

since for all $[w, z] \in L$, $u \in Z(L)$,

$$\langle [w, z], u \rangle = \langle w, [z, u] \rangle - (-1)^{|w||z|} \langle z, [w, u] \rangle = 0$$

hence

$$\langle x, y \rangle = \langle u + c, v + d \rangle = \langle u, v \rangle$$

and so $\tilde{v}$ is well defined. Moreover, since $\tilde{v}$ is K-bilinear and clearly surjective, it gives rise to a surjective K-supermodule homomorphism $v : \bar{L} \otimes \bar{L} \to \mathcal{L} : \bar{x} \otimes \bar{y} \mapsto \langle x, y \rangle$. In addition, for $x, y, z \in L$ we have

$$v(\bar{x} \otimes \bar{y} + (-1)^{|\bar{x}||\bar{y}|} \bar{y} \otimes \bar{x}) = \langle x, y \rangle + (-1)^{|x||y|} \langle y, x \rangle = 0$$

and

$$\begin{aligned} &v([\bar{x}, \bar{y}] \otimes \bar{z} - \bar{x} \otimes [\bar{y}, \bar{z}] + (-1)^{|\bar{x}||\bar{y}|} \bar{y} \otimes [\bar{x}, \bar{z}]) \\ &\qquad = \langle [x, y], z \rangle - \langle x, [y, z] \rangle + (-1)^{|x||y|} \langle y, [x, z] \rangle = 0. \end{aligned}$$

Hence v induces the K-supermodule homomorphism $v : \bar{\mathcal{L}} \to \mathcal{L} : \langle \bar{x}, \bar{y} \rangle \mapsto \langle x, y \rangle$. Moreover, for all $x, y, u, v \in L$,

$$\begin{aligned} v([\langle \bar{x}, \bar{y} \rangle, \langle \bar{u}, \bar{v} \rangle]) &= v(\langle [\bar{x}, \bar{y}], [\bar{u}, \bar{v}] \rangle) = v(\langle \overline{[x, y]}, \overline{[u, v]} \rangle) = \langle [x, y], [u, v] \rangle = [\langle x, y \rangle, \langle u, v \rangle] \\ &= [v(\langle \bar{x}, \bar{y} \rangle), v(\langle \bar{u}, \bar{v} \rangle)] \end{aligned}$$

and so v is a surjective K-superalgebra homomorphism. It is easy to check that $\chi \circ v = Id_{\bar{\mathcal{L}}}$ and $v \circ \chi = Id_{\mathcal{L}}$ and so (5.8) is a K-superalgebra isomorphism.

(4) By definition, $\mathrm{H}_2(L) = \ker m$ and so $[\mathrm{H}_2(L), \mathcal{L}] = \langle m(\mathrm{H}_2(L)), [L, L] \rangle = (0)$. Hence (5.9) is a central extension.

(5) Since L is perfect, $\mathcal{L}$ is also perfect. Let $\tilde{L} \xrightarrow{\pi} L$ be a central extension of L. Choose a section s of π in the category of $\mathbb{Z}_2$-graded sets: a function $s : L \to \tilde{L}$

satisfying $s(L_\alpha) \subset \tilde{L}_\alpha$ and $\pi \circ s = Id_L$. Note that in general s is not K-linear but since π is K-linear and of degree $\bar{0}$, we have for $a, b \in K$ and $x, y \in L$,

$$\pi(s(ax+by)) = ax + by = a\pi(s(x)) + b\pi(s(y)) \underset{K\text{-linear}}{\overset{\pi \text{ is}}{=}} \pi(as(x) + bs(y))$$

and so $s(ax+by) = as(x) + bs(y) + c$ for some $c \in \ker\pi \subset Z(\tilde{L})$.
Define $\varphi : L \times L \longrightarrow \tilde{L} : (x, y) \longmapsto [sx, sy]$. Then φ is of degree $\bar{0}$ and for all $a, b \in K$ and $x, y, z \in L$ with $c \in Z(\tilde{L})$ as above,

$$\begin{aligned}\varphi((ax+by), z) &= [s(ax+by), sz] = [as(x) + bs(y) + c, sz] = a[s(x), s(z)] + b[s(y), s(z)] \\ &= a\varphi(x, z) + b\varphi(y, z).\end{aligned}$$

Similarly, $\varphi(x, (ay+bz)) = a\varphi(x, z) + b\varphi(y, z)$ so φ is K-bilinear and hence it gives rise to the K-linear map

$$\varphi : L \otimes_K L \longrightarrow \tilde{L} : x \otimes y \mapsto [sx, sy].$$

Moreover, for all $x, y \in L$,

$$\pi(s([x,y]) - [s(x), s(y)]) = [x,y] - [\pi(s(x)), \pi(s(y))] = [x,y] - [x,y] = 0$$

we get that $s([x,y]) - [s(x), s(y)] \in \ker\pi \subset Z(\tilde{L})$ and so using the Jacobi identity, it is easy to show that

$$\varphi([x,y] \otimes z + (-1)^{|x|(|y|+|z|)}[y,z] \otimes x + (-1)^{|z|(|x|+|y|)}[z,x] \otimes y) = 0.$$

In addition, since $\varphi(x \otimes y) = -(-1)^{|x||y|}\varphi(y \otimes x)$, φ induces a K-linear map $\varphi : \mathcal{L} \longrightarrow \tilde{L} : \langle x, y\rangle \longmapsto [sx, sy]$ of degree $\bar{0}$. Moreover, φ is a Lie superalgebra homomorphism since

$$\begin{aligned}\varphi([\langle x,y\rangle, \langle x',y'\rangle]) &= \varphi\langle [x,y], [x',y']\rangle = [s([x,y]), s([x',y'])] \\ &= [[s(x), s(y)], [s(x'), s(y')]] = [\varphi\langle x,y\rangle, \varphi\langle x',y'\rangle].\end{aligned}$$

Since

$$(\pi \circ \varphi)\langle x,y\rangle = \pi[s(x), s(y)] = [\pi s(x), \pi s(y)] = [x,y] = m\langle x,y\rangle$$

we have that $\pi \circ \varphi = m$.
To show that φ is unique, suppose $\psi : \mathcal{L} \longrightarrow \bar{L}$ is another Lie superalgebra homomorphism satisfying $\pi \circ \psi = m$. Then for $l_i \in \mathcal{L}$ we have $\psi(l_i) - \varphi(l_i) \in Z(\bar{L})$ and hence

$$\psi[l_1, l_2] = [\psi l_i, \psi l_2] = [\varphi l_1, \varphi l_2] = \varphi[l_1, l_2]$$

and so since $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ we have $\psi = \varphi$. Therefore $\mathcal{L}$ is the universal central extension of L. Since $\mathcal{L} \cong \bar{\mathcal{L}}$, we have that $Z(\mathcal{L}) \cong Z(\bar{\mathcal{L}})$. By (4), $\mathrm{H}_2(\bar{L}) \subset Z(\bar{\mathcal{L}})$. Note that since L is perfect, $\bar{L}$ is also perfect. In addition, $Z(\bar{L}) = (0)$. Indeed, let $z \in L$ such that $\bar{z} \in Z(\bar{L})$. Then for all $x \in L$, $Z(L) = [\bar{z}, \bar{x}] = \overline{[z, x]} = [z, x] + Z(L)$ and so $[z, L] \subset Z(L)$. Hence,

$$[[z, L], L] = [z, [L, L]] + [L, [z, L]] = [z, [L, L]] = [z, L]$$

and so $z \in Z(L)$. Therefore, $Z(\bar{L}) = (0)$. Then by Proposition 5.1.2 we have that $Z(\bar{\mathcal{L}}) = \bar{m}^{-1}(Z(\bar{L})) = \bar{m}^{-1}(0) = \mathrm{H}_2(\bar{L})$ and so the assertion follows. ∎

5.2 The Second Homology Group of Toral 3-Graded Lie Superalgebras

We consider in this section toral 3-graded Lie superalgebras L over K, i.e., $L = L_+ \oplus L_0 \oplus L_-$ satisfying with $\zeta \in L_0$ such that $[\zeta, L_\sigma] = \sigma x_\sigma$, $\sigma = 0, \pm$.

Recall from Proposition 5.1.3.2 that the map $m : \mathcal{L} \to [L, L] : \langle x, y\rangle \mapsto [x, y]$ is a central extension of $[L, L]$.

Proposition 5.2.1 *Let L be a toral 3-graded Lie superalgebra. Then*

$$\left.\begin{array}{l}\langle L_\sigma, L_\sigma\rangle = (0), \ \textit{for } \sigma = \pm;\\ \langle \zeta, [L_0, L_0] + [L_+, L_-]\rangle = (0);\\ \langle x_0, x_\sigma\rangle = \sigma\langle \zeta, [x_0, x_\sigma]\rangle, \ \textit{for } \sigma = \pm;\\ \langle x_0, [x_+, x_-]\rangle = \langle [x_0, x_+], x_-\rangle - (-1)^{|x_+||x_-|}\langle [x_0, x_-], x_+\rangle.\end{array}\right\} \quad (5.10)$$

for all $x_\sigma \in L_\sigma$, $\sigma = 0, \pm$ and $\mathcal{L}$ is a 3-graded Lie superalgebra where

$$\begin{aligned} \mathcal{L}_0 &= \langle L_0, L_0\rangle + \langle L_+, L_-\rangle; \\ \mathcal{L}_\sigma &= \langle \zeta, L_\sigma\rangle = \langle L_0, L_\sigma\rangle, \text{ for } \sigma = \pm \end{aligned} \tag{5.11}$$

and $\mathcal{L}$ is toral if $\zeta \in [L, L]$. Moreover, the map m preserves the 3-grading and $m|_{\mathcal{L}_\sigma} : \mathcal{L}_\sigma \to L_\sigma$ for $\sigma = \pm$ are K-supermodule isomorphisms. In particular,

$$\mathrm{H}_2(L) \subset \mathcal{L}_0.$$

Moreover, if $L_0 = [L_+, L_-]$ then

$$\mathcal{L} = \langle \zeta, L_+ \oplus L_-\rangle \oplus \langle L_+, L_-\rangle$$

and

$$\mathrm{H}_2(L) \subset \langle L_+, L_-\rangle.$$

Proof: Let $x_\sigma \in L_\sigma$, $y_\mu \in L_\mu$ for $\sigma, \mu \in \{0, \pm\}$. Then by (5.2),

$$\langle \zeta, [x_\sigma, y_\mu]\rangle = \langle [\zeta, x_\sigma], y_\mu\rangle - (-1)^{|x||y|}\langle [\zeta, y_\mu], x_\sigma\rangle = \sigma\langle x_\sigma, y_\mu\rangle + \mu\langle x_\sigma, y_\mu\rangle. \tag{5.12}$$

Hence for $\sigma = \mu \in \{\pm\}$ we get $\langle x_\sigma, y_\sigma\rangle = -\langle x_\sigma, y_\sigma\rangle$ and so since $\frac{1}{2} \in K$, $\langle L_\sigma, L_\sigma\rangle = (0)$ for $\sigma = \pm$. If $\sigma = \mu = 0$ then $\langle \zeta, [L_0, L_0]\rangle = (0)$ and if $\sigma = +$ and $\mu = -$ then $\langle \zeta, [L_+, L_-]\rangle = (0)$. Moreover, if $\sigma = 0$ and $\mu \in \{\pm\}$ then

$$\langle \zeta, [x_0, x_\mu]\rangle = \mu\langle x_0, x_\mu\rangle.$$

In addition, by (5.2),

$$\langle x_0, [x_+, x_-]\rangle = \langle [x_0, x_+], x_-\rangle - (-1)^{|x_+||x_-|}\langle [x_0, x_-], x_+\rangle$$

and so (5.10) holds. It now follows from (5.10) that $\langle L_0, L_\sigma\rangle \subset \langle \zeta, [L_0, L_\sigma]\rangle = \langle \zeta, L_\sigma\rangle$ for $\sigma \in \{\pm\}$ and so

$$\mathcal{L} = \langle L, L\rangle = \langle \zeta, L_+\rangle + (\langle L_0, L_0\rangle + \langle L_+, L_-\rangle) + \langle \zeta, L_-\rangle.$$

Suppose that $\langle\zeta, x_+\rangle + (\langle x_0, y_0\rangle + \langle y_+, y_-\rangle) + \langle\zeta, x_-\rangle = 0$. Recall the K-superalgebra homomorphism $m : \mathcal{L} \to L : \langle x, y\rangle \mapsto [x, y]$. Then

$$\begin{aligned} 0 &= m(\langle\zeta, x_+\rangle + (\langle x_0, y_0\rangle + \langle y_+, y_-\rangle) + \langle\zeta, x_-\rangle) \\ &= [\zeta, x_+] + ([x_0, y_0] + [y_+, y_-]) + [\zeta, x_-] \\ &= x_+ + ([x_0, y_0] + [y_+, y_-]) - x_- \end{aligned}$$

and so $x_+ = 0$ and $x_- = 0$. Hence $\langle\zeta, x_+\rangle = 0$. $\langle\zeta, x_-\rangle = 0$ and so $(\langle x_0, y_0\rangle + \langle y_+, y_-\rangle) = 0$. Hence $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_0 \oplus \mathcal{L}_-$ where $\mathcal{L}_{0,\pm}$ are as in (5.11). It is easy to check that this gives $\mathcal{L}$ a 3-grading. If $\zeta = \sum_i [x^i, y^i] \in [L, L]$ then for all $\langle\zeta, z_\sigma\rangle \in \mathcal{L}_\sigma$, $\sigma = \pm$,

$$[\sum_i \langle x^i, y^i\rangle, \langle\zeta, z_\sigma\rangle] = \langle\sum_i [x^i, y^i], \sigma z_\sigma\rangle = \sigma\langle\zeta, z_\sigma\rangle.$$

Moreover, for $\langle w, z\rangle \in \mathcal{L}_0 = \langle L_0, L_0\rangle + \langle L_+, L_-\rangle$ we have by (5.10)

$$[\sum_i \langle x^i, y^i\rangle, \langle w, z\rangle] = \langle\zeta, [w, z]\rangle = 0$$

and so $\sum_i \langle x^i, y^i\rangle$ acts as a toral element. Hence if $\zeta \in [L, L]$ then $\mathcal{L}$ is toral 3-graded. For $x_\sigma \in \mathcal{L}_\sigma$, $y_\mu \in \mathcal{L}_\mu$, $\sigma, \mu \in \{0, \pm\}$ we have $m(\langle x_\sigma, y_\mu\rangle) = [x_\sigma, y_\mu] \in L_{\sigma+\mu}$ and so m preserves the 3-grading. Moreover, $m|_{\mathcal{L}_\sigma} : \mathcal{L}_\sigma \to \mathcal{L}_\sigma$ is clearly surjective and it is injective since

$$m(\langle\zeta, x_\sigma\rangle) = [\zeta, x_\sigma] = \sigma x_\sigma = 0 \Rightarrow x_\sigma = 0$$

and so it is a K-supermodule isomorphism. Let $X = \langle\zeta, x_+ \oplus x_-\rangle + \sum_i \langle x^i_+, x^i_-\rangle + \sum_j \langle x^j_0, y^j_0\rangle \in \mathcal{L}$. Then $X \in \mathrm{H}_2(L)$ if and only if

$$\begin{aligned} 0 &= [\zeta, x_+ \oplus x_-] + \sum_i [x^i_+, x^i_-] + [x^j_0, y^j_0] \\ &= x_+ \oplus (\sum_i [x^i_+, x^i_-] + \sum_j [x^j_0, y^j_0]) \oplus (-x_-) \end{aligned}$$

hence if and only if

$$x_+ = x_- = 0$$

and

$$\sum_i [x_+^i, x_-^i] + \sum_j [x_0^j, y_0^j] = 0.$$

Hence

$$X = \sum_i \langle x_+^i, x_-^i \rangle + \sum_j \langle x_0^j, y_0^j \rangle \subset \langle L_+, L_- \rangle + \langle L_0, L_0 \rangle = \mathcal{L}_0$$

where $\sum_i [x_+^i, x_-^i] + \sum_j [x_0^j, y_0^j] = 0$. If $L_0 = [L_+, L_-]$ then $\langle L_0, L_0 \rangle \subset \langle L_+, L_- \rangle$ using (5.1) and (5.2) and so the rest of the assertion follows. ∎

For example, Lie algebras over a ring containing $\frac{1}{6}$ graded by a root system $R \neq E_8, F_4, G_2, A_n$ (if $\frac{1}{n+1} \notin K$) are toral 3-graded and so their $\mathcal{L}$ as well as the second homology group can be described as above.

The following result will be used extensively in the next section.

Proposition 5.2.2 *Let L be a toral 3-graded Lie superalgebra over A where A is a superextension of K and suppose that $\zeta \in [L_+, L_-]$. Then*

$$\langle A\zeta, [L_0, L_0] \rangle = (0)$$

and

$$\langle a\zeta, bc\zeta \rangle + (-1)^{|b||a|} \langle b\zeta, ac\zeta \rangle + (-1)^{|c|(|a|+|b|)} \langle c\zeta, ab\zeta \rangle = 0 \tag{5.13}$$

for all $a, b, c \in A$ where the tensor products are all over K. In particular, $\langle \zeta, A\zeta \rangle = (0)$.

Proof: We have $\langle A\zeta, [L_0, L_0]\rangle \subset \langle [A\zeta, L_0], L_0\rangle = (0)$ and so the first assertion follows. Since $\zeta \in [L_+, L_-]$, we can write $\zeta = \sum_i [x_+^i, x_-^i]$. Let $a, b, c \in A$. Then

$$
\begin{aligned}
&(\langle ab\zeta, c\zeta\rangle - \langle a\zeta, bc\zeta\rangle - (-1)^{|a||b|}\langle b\zeta, ac\zeta\rangle) \\
&\quad = \langle \sum_i [abx_+^i, x_-^i], c\zeta\rangle - \langle a\zeta, \sum_i [bx_+^i, x_-^i c]\rangle - (-1)^{|a||b|}\langle \sum_i [bx_+^i, x_-^i], ac\zeta\rangle \\
&\quad = \sum_i \Big(\langle abx_+^i, x_-^i c\rangle + (-1)^{(|a|+|b|+|x_+^i|)|x_-^i|}\langle x_-^i, abx_+^i c\rangle\Big) \\
&\qquad - \sum_i \Big(\langle abx_+^i, x_-^i c\rangle + (-1)^{(|b|+|x_+^i|)(|x_-^i|+|c|}\langle ax_-^i c. bx_+^i\rangle\Big) \\
&\qquad - \sum_i \Big((-1)^{|a||b|}\langle bx_+^i, x_-^i ac\rangle + (-1)^{|a||b|+(|b|+|x_+^i|)|x_-^i|}\langle x_-^i, bx_+^i ac\rangle\Big) \\
&\quad = 0
\end{aligned}
$$

and so (5.13) holds. For $b = c = 1$, (5.13) implies that $\langle \zeta, A\zeta\rangle = (0)$. ∎

5.3 The Second Homology Group of $\mathrm{eosp}(q_\infty)$

Throughout this section we assume that $\mathcal{M}$ is an A-supermodule with a quadratic form q. Set $L = \mathrm{eosp}(q_\infty)$. Recall that L is toral 3-graded and that $L_0 = A \oplus \mathrm{eosp}(q)$ where $1 \in A$ plays the role of the toral element. Recall from Lemma 2.3.6 that L is perfect if and only if $1 \in [L_+, L_-]$ and so then by Proposition 5.2.2 if L is perfect we have that

$$\langle a, bc\rangle = \langle ab, c\rangle + (-1)^{|b||c|}\langle ac, b\rangle \tag{5.14}$$

for all $a, b, c \in A$. In particular, $\langle 1, A\rangle = (0)$.

Proposition 5.3.1 *Assume that* $\mathrm{eosp}(q)$ *and* L *are perfect.*

(i) Then

$$\langle A, \mathrm{eosp}(q)\rangle = (0); \text{ and} \tag{5.15}$$

(ii) for all $a \in A$ *and* $m, n \in \mathcal{M}$,

$$\langle am^+, n^-\rangle \;=\; (-1)^{|a||m|}\langle m^+, an^-\rangle + \langle a, q(m, n)\rangle. \tag{5.16}$$

Moreover, if there exists $m_0 \in \mathcal{M}_{\bar{0}}$ such that $q(m_0, m_0) \in A^\times$ then for all $m, n \in \mathcal{M}$ we have

$$\begin{aligned}\langle m^-, n^+\rangle &= \langle m^+, n^-\rangle + \langle q(m_0, m_0), q(m, n)q(m_0, m_0)^{-1}\rangle \\ &\quad -2\langle m_0^+, q(m_0, m_0)^{-1}q(m, n)m_0^-\rangle. \end{aligned} \tag{5.17}$$

In particular, for all $m \in \mathcal{M}_{\bar{0}}$, the following holds:

$$\begin{aligned}\langle am^+, m_-\rangle &= \langle m_0^+, aq(m_0, m_0)^{-1}q(m, m)m_0^-\rangle - \frac{1}{2}\langle a, q(m, m)\rangle \\ &\quad -\frac{1}{2}\langle q(m_0, m_0), aq(m, m)q(m_0, m_0)^{-1}\rangle \end{aligned} \tag{5.18}$$

Proof: $\boxed{(i)}$ Since $[L_0, L_0] = [\mathrm{eosp}(q), \mathrm{eosp}(q)] = \mathrm{eosp}(q)$ and $1 \in [L_+, L_-]$ by Proposition 5.2.2 we have $(0) = \langle A, [L_0, L_0]\rangle = \langle A, \mathrm{eosp}(q)\rangle$.
$\boxed{(ii)}$ Let $a \in A$ and $m, n \in \mathcal{M}$. Then

$$\begin{aligned}\langle am^+, n^-\rangle = \langle [a, m^+], n^-\rangle &\overset{(5.2)}{=} (-1)^{|a||m|}\langle m^+, an^-\rangle + \langle a, [m^+, n^-]\rangle \\ &\overset{(5.15)}{=} (-1)^{|a||m|}\langle m^+, an^-\rangle + \langle a, q(m, n)\rangle\end{aligned}$$

By Lemma 2.1.4, we have that $\mathcal{M} = Am_0 \oplus \mathcal{P}$ with $q = q|_{Am_0} \oplus q|_{\mathcal{P}}$. For $a, b \in A$.

$$\begin{aligned}\langle am_0^-, bm_0^+\rangle - \langle am_0^+, bm_0^-\rangle &\overset{(5.1)}{=} -(-1)^{|a||b|}\langle bm_0^+, am_0^-\rangle - \langle am_0^+, bm_0^-\rangle \\ &\overset{(5.16)}{=} -(-1)^{|a||b|}\langle m_0^+, bam_0^-\rangle - (-1)^{|a||b|}\langle b, aq(m_0, m_0)\rangle \\ &\quad -\langle m_0^+, abm_0^-\rangle - \langle a, bq(m_0, m_0)\rangle \\ &\overset{(5.14)}{=} -2\langle m_0^+, abm_0^-\rangle + \langle q(m_0, m_0), ab\rangle.\end{aligned}$$

Hence (5.17) holds for $m, n \in Am_0$. Moreover, for any $p \in \mathcal{P}$,

$$\begin{aligned}\langle p^-, am_0^+\rangle &= \langle [q(m_0, m_0)^{-1}\mathbf{E}_{p,m_0}, m_0^-], am_0^+\rangle \\ &\overset{(5.2)}{=} \langle q(m_0, m_0)^{-1}\mathbf{E}_{p,m_0}, aq(m_0, m_0)\rangle - \langle m_0^-, (pa)^+\rangle \\ &\overset{(5.15)}{=} (-1)^{|a||p|}\langle ap^+, m_0^-\rangle \\ &\overset{(5.16)}{=} \langle p^+, am_0^-\rangle + (-1)^{|a||p|}\langle a, q(p, m_0)\rangle \\ &= \langle p^+, am_0^-\rangle\end{aligned}$$

and so (5.17) holds for $m \in \mathcal{P}$ and $n \in Am_0$. In addition, for all $r \in \mathcal{P}$, we have that

$$\begin{aligned}
\langle p^-, r^+\rangle &= \langle p^-, [q(m_0,m_0)^{-1}\mathbf{E}_{r,m_0}, m_0^+]\rangle \\
&\overset{(5.2)}{=} \langle q(m_0,m_0)^{-1}q(p,r)m_0^-, m_0^+\rangle - \langle \mathbf{E}_{p,m_0}, q(m_0,m_0)^{-1}\mathbf{E}_{r,m_0}\rangle \\
&\underset{(5.15)}{\overset{(5.1)}{=}} -\langle m_0^+, q(m_0,m_0)^{-1}q(p,r)m_0^-\rangle - \langle [p^+, m_0^-], q(m_0,m_0)^{-1}\mathbf{E}_{r,m_0}\rangle \\
&\overset{(5.2)}{=} -\langle m_0^+, q(m_0,m_0)^{-1}q(p,r)m_0^-\rangle + \langle p^+, r^-\rangle + \langle m_0^-, q(m_0,m_0)^{-1}q(p,r)m_0^+\rangle \\
&\underset{(5.16)}{\overset{(5.1)}{=}} \langle p^+, r^-\rangle - 2\langle m_0^+, q(m_0,m_0)^{-1}q(p,r)m_0^-\rangle - \langle q(m_0,m_0)^{-1}q(p,r), q(m_0,m_0)\rangle
\end{aligned}$$

hence (5.17) holds for $m, n \in \mathcal{P}$ and so by linearity (5.17) holds for all $m, n \in \mathcal{M}$. ∎

In what follows, we abbreviate $[x,y]\otimes z + (-1)^{|x|(|y|+|z|)}[y,z]\otimes x + (-1)^{|z|(|x|+|y|)}[z,x]\otimes y$ by $\sum_{\circlearrowright}[x,y]\otimes z$. The set of $\sum_{\circlearrowright}[x_\sigma, y_\mu]\otimes z_\nu$ for $x_\sigma \in L_\sigma$, $y_\mu \in L_\mu$, $z_\nu \in L_\nu$ will be denoted by $\sum_{\circlearrowright}[L_\sigma, L_\mu]\otimes L_\nu$. The following is a technical result needed to simplify the proof of the result which follows it.

Lemma 5.3.2 *Assume that L is perfect i.e., $L_0 = [L_+, L_-]$ (see Lemma 2.3.6). Let $\varphi : L \otimes L \longrightarrow V$ be a K-supermodule homomorphism satisfying*

$$\varphi(x\otimes y) = -(-1)^{|x||y|}\varphi(y\otimes x) \text{ for all } x, y \in L \tag{5.19}$$

and

$$\sum_{\circlearrowright}\varphi([L_+\oplus L_-, L]\otimes L) = 0. \tag{5.20}$$

Then $\sum_{\circlearrowright}\varphi([L,L]\otimes L) = 0$.

Proof: Note that if we set $g : L\times L\times L \to L\otimes L : (x,y,z)\mapsto \sum_{\circlearrowright}[x,y]\otimes z$ then g is a well-defined K-trilinear homomorphism. Hence to show that $\sum_{\circlearrowright}\varphi([L,L]\otimes L) = 0$ we only need to show that $\sum_{\circlearrowright}\varphi([L_+\oplus L_-, L]\otimes L) = 0$ and $\sum_{\circlearrowright}\varphi([L_0, L]\otimes L) = 0$. Assume that $\varphi(x\otimes y) = -(-1)^{|x||y|}\varphi(y\otimes x)$ for all $x, y \in L$ and that $\sum_{\circlearrowright}\varphi([L_+\oplus L_-, L]\otimes L) = 0$. Note that this implies in particular that

$$\varphi([x,y]\otimes z) = \varphi(x\otimes[y,z]) - (-1)^{|x||y|}\varphi(y\otimes[x,z]) \tag{5.21}$$

for all $x \in L_+ \oplus L_-$ and $y, z \in L$. Let $m, n \in \mathcal{M}$ and $y, z \in L$. Then

$$
\begin{aligned}
&\varphi([m^+, n^-] \otimes z - [m^+, n^-] \otimes [y, z] + (-1)^{(|m|+|n|)|y|} y \otimes [[m^+, n^-], z]) = \\
&\overset{(1.20)}{=} \varphi([m^+, [n^-, y]] \otimes z - (-1)^{|m||n|}[n^-, [m^+, y]] \otimes z - [m^+, n^-] \otimes [y, z] \\
&\qquad + (-1)^{(|m|+|n|)|y|} y \otimes [m^+, [n^-, z]] - (-1)^{(|m|+|n|)|y|+|m||n|} y \otimes [n^-, [m^+, z]]) \\
&\overset{(5.21)}{=} \varphi\,(m^+ \otimes [[n^-, y], z] - (-1)^{|m|(|n|+|y|)}[n^-, y] \otimes [m^+, z] \\
&\qquad - (-1)^{|m||n|} n^- \otimes [[m^+, y], z] + (-1)^{|y||n|}[m^+, y] \otimes [n^-, z] \\
&\qquad - m^+ \otimes [n^-, [y, z]] + (-1)^{|m||n|} n^- \otimes [m^+, [y, z]] \\
&\qquad + (-1)^{|x||y|}[y, m^+] \otimes [n^-, z] - (-1)^{|z||m|+|m||n|+(|m|+|n|)|y|}[y, [n^-, z]] \otimes m^+ \\
&\qquad -(-1)^{(|m|+|n|)|y|+|m||n|}[y, n^-] \otimes [m^+, z] \\
&\qquad +(-1)^{(|m|+|n|)|y|+|z||n|}[y, [m^+, z]] \otimes n^+) \\
&= \varphi\,(m^+ \otimes ([[n^-, y], z] + (-1)^{|y||n|}[y, [n^-, z]] - [n^-, [y, z]]) \\
&\qquad + (-1)^{|m||n|} n^- \otimes ([[m^+, y], z] + [m^+, [y, z]] - (-1)^{|y||m|}[y, [m^+, z]] \\
&\qquad - (-1)^{|}m|(|n| + |y|)[n^-, y] \otimes [m^+, z] \\
&\qquad - (-1)^{(|m|+|n|)|y|+|m||n|}[y, n^-] \otimes [m^+, z] \\
&\qquad +(-1)^{|y||n|}[m^+, y] \otimes [n^-, z] + (-1)^{(|m|+|n|)|y|}[y, m^+] \otimes [n^-, z]) \\
&\overset{(1.20)}{=} 0.
\end{aligned}
$$

Since L is perfect, by Lemma 2.3.6 we have that $L_0 = [L_+, L_-]$ and so $\sum_{\circlearrowleft} \varphi([L_0, L] \otimes L) = 0$. ■

Let $\mathcal{J} = A \oplus \mathcal{M}$ be the Jordan superalgebra constructed in Example 1.3.22. Recall that the multiplication in $\mathcal{J}$ is given by

$$(a \oplus m)(b \oplus n) = (ab + q(m, n)) \oplus (an + mb).$$

Let $\mathfrak{s}$ be the submodule of $\mathcal{J} \otimes_K \mathcal{J}$ spanned by the elements

$$
\begin{gathered}
x \otimes y + (-1)^{|x||y|} y \otimes x \\
xy \otimes z - x \otimes yz - (-1)^{|x||y|} y \otimes xz \qquad (5.22) \\
a \otimes m
\end{gathered}
$$

for all $x, y, z \in \mathcal{J}$ and $a \in A$, $m \in \mathcal{M}$. Let

$$\{\mathcal{J}, \mathcal{J}\} := (\mathcal{J} \otimes_K \mathcal{J})/\mathbf{s}$$

and for $x, y \in \mathcal{J}$, let $\{x, y\}$ denote the coset $x \otimes y + \mathbf{s}$. In particular, the elements of $\{J, J\}$ satisfy the following equations:

$$\begin{aligned} \{ab, c\} &= \{a, bc\} + (-1)^{|a||b|}\{b, ac\}, \ a, b, c \in A & (5.23) \\ \{am, n\} &= \{a, q(m, n)\} + (-1)^{|a||m|}\{m, an\}, \ a \in A, \ m, n \in \mathcal{M}. & (5.24) \end{aligned}$$

Lemma 5.3.3 *Let J be the Jordan algebra as above. Then there exists a K-linear map $\rho : \{J, J\} \to \mathrm{eosp}(q)$ such that $\rho(\{m, n\}) = \mathbf{E}_{m,n}$.*

Proof: Define $\tilde{\rho} : J \times J \to \mathrm{eosp}(q)$ by setting $\tilde{\rho}(a \oplus m, b \oplus n) = \mathbf{E}_{m,n}$. This is a well-defined K-bilinear map and so it gives rise to the map, also called $\tilde{\rho}$, $\tilde{\rho} : J \otimes J \to \mathrm{eosp}(q) : (a \oplus m) \otimes (b \oplus n) \mapsto \mathbf{E}_{m,n}$. Since

$$\tilde{\rho}(a \otimes m) = 0.$$

$$\tilde{\rho}\left((a \oplus m) \otimes (b \oplus n) + (-1)^{|a\oplus m||b\oplus n|}(b \oplus n) \otimes (a \oplus n)\right) = \mathbf{E}_{m,n} + (-1)^{|m||n|}\mathbf{E}_{n,m} = 0$$

and

$$\begin{aligned} &\tilde{\rho}(((a \oplus m)(b \oplus n)) \otimes (c \oplus p) - (a \oplus m) \otimes ((b \oplus n)(c \oplus p)) \\ &\qquad - (-1)^{|a\oplus m||b\oplus n|}(b \oplus n) \otimes ((a \oplus m)(c \oplus p)) \\ &\quad = \tilde{\rho}((q(m, n) \oplus (an + mb)) \otimes (c \oplus p) - (a \oplus m) \otimes (q(n, p) \oplus (bp + nc)) \\ &\qquad - (-1)^{|a\oplus m||b\oplus n|}(b \oplus n) \otimes (q(m, p) \oplus (ap + mc) \\ &\quad = \mathbf{E}_{an+mb,p} - \mathbf{E}_{m,bp+nc} - (-1)^{|a\oplus m||b\oplus n|}\mathbf{E}_{n,ap+mc} \\ &\quad = 0 \end{aligned}$$

for all $a, b, c \in A$ and $m, n, p \in \mathcal{M}$ we have $\tilde{\rho}(\mathbf{s}) = (0)$. Hence $\tilde{\rho}$ induces a K-linear map $\rho : \{J, J\} \to \mathrm{eosp}(q)$ such that $\rho(\{m, n\}) = \mathbf{E}_{m,n}$ for all $m, n \in \mathcal{M}$. ∎

We define the *full skew-dihedral homology group of* $\mathcal{J}$ (see [ABG]) as

$$\begin{aligned} \mathrm{HF}(\mathcal{J}) &= \ker\rho \\ &= \left\{ \sum_i \{a_i, b_i\} + \sum_k \{m_k, n_k\} \mid a_i, b_i \in \mathcal{A},\ m_k, n_k \in \mathcal{M},\ \sum_k \mathbf{E}_{m_k,n_k} = 0 \right\}. \end{aligned}$$

Note that for all $a, b, c \in \mathcal{A}$,

$$ab \otimes c + (-1)^{|a|(|b|+|c|)} bc \otimes a + (-1)^{|c|(|a|+|b|)} ca \otimes b \in \mathfrak{s}$$

if and only if

$$ab \otimes c - a \otimes bc - (-1)^{|a||b|} b \otimes ac \in \mathfrak{s}$$

since $a \otimes b + (-1)^{|a||b|} b \otimes a \in \mathfrak{s}$ for all $a, b \in \mathcal{A}$. Also note that if we set $\mathcal{M} = (0)$ then $\mathrm{HF}(\mathcal{J})$ is the *Connes first homology group* $\mathrm{HC}_1(\mathcal{A})$ of an algebra $\mathcal{A}$ which is isomorphic to $\Omega^1_{A/K}/dA$, the *Kähler differentials of the algebra* $\mathcal{A}$ *modulo exact forms* (see [Ks] or [ABG]).

Theorem 5.3.4 *Assume* $\mathrm{eosp}(q)$ *and* L *are perfect. Then there exists a well-defined* K*-supermodule homomorphism* $\varphi : \mathcal{L} \to \{\mathcal{J}, \mathcal{J}\}$ *satisfying*

$$\left.\begin{array}{r} \varphi(\langle L_0 \oplus L_\sigma, L_\sigma\rangle) = (0),\ \sigma = \pm \\ \varphi(\langle m^+, n^-\rangle) = \{m, n\},\ m, n \in \mathcal{M} \\ \varphi(\langle a, b + x\rangle) = \{a, b\},\ a, b \in \mathcal{A},\ x \in \mathrm{eosp}(q) \\ \varphi(\langle x, \mathbf{E}_{m,n}\rangle) = \{x(m), n\} - (-1)^{|m||n|}\{x(n), m\},\ m, n \in \mathcal{M},\ x \in \mathrm{eosp}(q). \end{array}\right\} \tag{5.25}$$

Conversely, there exists a K*-supermodule homomorphism* $\psi : \{\mathcal{J}, \mathcal{J}\} \to \mathcal{L}_0$ *satisfying*

$$\psi(\{a + m, b + n\}) = \langle a, b\rangle + \frac{1}{2}(\langle m^+, n^-\rangle + \langle m^-, n^+\rangle) \tag{5.26}$$

and $\varphi \circ \psi = Id$. *Moreover,* $\psi(\mathrm{HF}(\mathcal{J})) \subset \mathrm{H}_2(L)$ *and* $\varphi(\mathrm{H}_2(L)) = \mathrm{HF}(\mathcal{J})$. *In addition,*

$$\psi \circ \varphi|_{\langle L_0, L_0\rangle + S} = Id \tag{5.27}$$

where $S = \mathrm{span}\{\langle m^+, n^-\rangle + \langle m^-, n^+\rangle \mid m, n \in \mathcal{M}\}$. *Hence if*

$$\mathrm{H}_2(L) \subset \langle L_0, L_0\rangle + S \tag{5.28}$$

then $\psi \circ \varphi|_{\mathrm{H}_2(L)} = Id$ *and* $\mathrm{H}_2(L) \cong \mathrm{HF}(\mathcal{J})$.

Proof: Let $f : L \times L \to \{\mathcal{J}, \mathcal{J}\}$ be the map defined by

$$f(L_0 \oplus L_\sigma, L_\sigma) = f(L_\sigma, L_0 \oplus L_\sigma) = (0), \ \sigma = \pm$$
$$f(m^+, n^-) = f(m^-, n^+) = \{m, n\}, \ m, n \in \mathcal{M}$$
$$f(a, b + x) = f(a + x, b) = \{a, b\}, \ a, b \in \mathcal{A}, \ x \in \mathrm{eosp}(q)$$
$$f(x, \mathbf{E}_{m,n}) = \{x(m), n\} - (-1)^{|m||n|}\{x(n), m\}, \ m, n \in \mathcal{M}, \ x \in \mathrm{eosp}(q).$$

and extended to $L \times L$ bilinearly over K. This is a well-defined map. indeed. to verify this we only need to show that f is well defined on $\mathrm{eosp}(q) \times \mathrm{eosp}(q)$, i.e. that $f(x, y) = 0$ if $x = 0$ or $y = 0$. The first case clearly holds by definition. If $m, n, p, r \in \mathcal{M}$ then

$$\begin{aligned}
f(\mathbf{E}_{p,r}, \mathbf{E}_{m,n}) &= \{\mathbf{E}_{p,r}(m), n\} - (-1)^{|m||n|}\{\mathbf{E}_{p,r}(n), m\} \\
&= \{pq(r, m) - (-1)^{|p||r|}rq(p, m), n\} \\
&\quad -(-1)^{|m||n|}\{pq(r, n) - (-1)^{|p||r|}rq(p, n), m\} \\
&\overset{(5.24)}{=} \{p, q(r, m)n\} + (-1)^{(|r|+|m|)|p|}\{q(r, m), q(p, n)\} \\
&\quad -(-1)^{|p||r|}\{r, q(p, m)n\} - (-1)^{|m||r|}\{q(p, m), q(r, n)\} \\
&\quad -(-1)^{|m||n|}\{p, q(r, n)m\} - (-1)^{(|r|+|n|)|p|+|m||n|}\{q(r, n), q(p, m)\} \\
&\quad +(-1)^{|m||n|+|p||r|}\{r, q(p, n)m\} + (-1)^{|n|(|r|+|m|)}\{q(p, n), q(r, m)\} \\
&\overset{(5.22)}{=} -(-1)^{(|p|+|r|)(|m|+|n|)}\left(\{\mathbf{E}_{m,n}(p), r\} - (-1)^{|p||r|}\{\mathbf{E}_{m,n}(r), p\}\right) \\
&= -(-1)^{(|p|+|r|)(|m|+|n|)} f(\mathbf{E}_{m,n}, \mathbf{E}_{p,r}).
\end{aligned}$$

Therefore if $x = \sum_i \mathbf{E}_{p_i,r_i}$, $y = \sum_j \mathbf{E}_{m_j,n_j} \in \mathrm{eosp}(q)$ such that $y = 0$ then

$$\begin{aligned}
&\sum_j \left(\{x(m_j), n_j\} - (-1)^{|m_j||n_j|}\{x(n_j), m_j\}\right) \\
&\qquad = -(-1)^{|x||y|} \sum_i \left(\{y(p_i), r_i\} - (-1)^{|p_i||r_i|}\{y(r_i), p_i\}\right) = 0
\end{aligned}$$

and so f is a well-defined K-bilinear map and, in particular,

$$f(x, y) = -(-1)^{|x||y|} f(y, x) \tag{5.29}$$

for all $x, y \in \mathrm{eosp}(q)$. Hence there exists a unique K-supermodule homomorphism $\bar{\varphi} : L \otimes L \to \{\mathcal{J}, \mathcal{J}\}$ such that $\bar{\varphi}(l_1 \otimes l_2) = f(l_1, l_2)$ for all $l_1, l_2 \in L$. It follows from

(5.29) that $x\otimes y+(-1)^{|x||y|}y\otimes x$ is in the kernel of $\tilde{\varphi}$ for all $x, y \in \mathrm{eosp}(q)$. By inspection we can see that $a \otimes b + (-1)^{|a||b|}b \otimes a$ and $m^+ \otimes n^- + (-1)^{|m||n|}n^- \otimes m^+$ are also in the kernel of $\tilde{\varphi}$ for all $a, b \in A$ and $m, n \in \mathcal{M}$. In addition, since $\tilde{\varphi}((L_0 \oplus L_\sigma) \otimes L_\sigma) = \tilde{\varphi}(L_\sigma \otimes (L_0 \oplus L_\sigma)) = (0)$ and $\tilde{\varphi}(A \otimes \mathrm{eosp}(q)) = \tilde{\varphi}(\mathrm{eosp}(q) \otimes A) = (0)$, by linearity of $\tilde{\varphi}$ we have that $l_1 \otimes l_2 + (-1)^{|l_1||l_2|}l_2 \otimes l_1$ lies in the kernel of $\tilde{\varphi}$ for all $l_1, l_2 \in L$. Let $g : L \times L \times L \to L \otimes L : (l_1, l_2, l_3) \mapsto \sum_{\circlearrowleft}[l_1, l_2] \otimes l_3$. Then g is K-trilinear and for all homogeneous $l_1, l_2, l_3 \in L$ and π in the symmetric group S_3 we have

$$g(l_1, l_2, l_3) = \sigma g(\pi(l_1), \pi(l_2), \pi(l_3))$$

where $\sigma \in \{\pm\}$ depends on π and the degrees of l_1, l_2 and l_3. Indeed

$$\begin{aligned}
g(l_2, l_1, l_3) &= [l_2, l_1] \otimes l_3 + (-1)^{|l_2|(|l_1|+|l_3|)}[l_1, l_3] \otimes l_2 + (-1)^{|l_3|(|l_1|+|l_2|)}[l_3, l_2] \otimes l_1 \\
&= -(-1)^{|l_1||l_2|}\left([l_1, l_2] \otimes l_3 + (-1)^{|l_3|(|l_1|+|l_2|)}[l_3, l_1] \otimes l_2\right. \\
&\quad \left.+(-1)^{|l_1|(|l_2|+|l_3|)}[l_2, l_3] \otimes l_1\right) \\
&= -(-1)^{|l_1||l_2|}g(l_1, l_2, l_3)
\end{aligned}$$

and

$$\begin{aligned}
g(l_1, l_3, l_2) &= [l_1, l_3] \otimes l_2 + (-1)^{|l_1|(|l_2|+|l_3|)}[l_3, l_2] \otimes l_1 + (-1)^{|l_2|(|l_1|+|l_3|)}[l_2, l_1] \otimes l_3 \\
&= -(-1)^{|l_1||l_3|}\left([l_3, l_1] \otimes l_2 + (-1)^{|l_2|(|l_1|+|l_3|)}[l_2, l_3] \otimes l_2\right. \\
&\quad \left.+(-1)^{|l_3|(|l_1|+|l_2|)}[l_1, l_2] \otimes l_3\right) \\
&= -(-1)^{|l_2||l_3|}g(l_1, l_2, l_3)
\end{aligned}$$

for all $l_1, l_2, l_3 \in L$. Since S_3 is generated by (12) and (23), this implies the assertion. We claim that $\sum_{\circlearrowleft}[L, L]\otimes L$ is in the kernel of $\tilde{\varphi}$. Indeed, by Lemma 5.3.2 to show this it is sufficient to show that $\sum_{\circlearrowleft}\tilde{\varphi}([L_+ \oplus L_-, L] \otimes L) = (0)$ i.e., $\tilde{\varphi}(g(L_\sigma, L, L)) = (0)$ for $\sigma \in \{\pm\}$. We have for $\sigma \in \{\pm\}$,

$$\begin{aligned}
\tilde{\varphi}(g(L_\sigma, L_\sigma, L_\sigma)) &= (0) \text{ since } [L_\sigma, L_\sigma] = (0) \\
\tilde{\varphi}(g(L_\sigma, L_\sigma, L_0)) &= (0) \text{ since } [L_\sigma, L_\sigma] = (0) \text{ and } \tilde{\varphi}(L_\sigma \otimes L_\sigma) = (0) \\
\tilde{\varphi}(g(L_\sigma, L_\sigma, L_{-\sigma})) &= (0) \text{ since } [L_\sigma, L_\sigma] = (0) \text{ and } \tilde{\varphi}(L_\sigma \otimes L_0) = (0) \\
\tilde{\varphi}(g(L_\sigma, L_0, L_0)) &= (0) \text{ since } \tilde{\varphi}(L_\sigma \otimes L_\sigma) = (0).
\end{aligned}$$

Hence we only need to verify that $\tilde{\varphi}(g(L_+, L_0, L_-)) = (0)$. Let $m, r \in \mathcal{M}$ and $a \in A$, $x \in \mathrm{eosp}(q)$. Then

$$\begin{aligned}
&\tilde{\varphi}([m^+, a + x] \otimes r^-) - \tilde{\varphi}(m^+ \otimes [a + x, r^-]) + (-1)^{|m||a+x|}\tilde{\varphi}((a + x) \otimes [m^+, r^-]) \\
&\quad = \tilde{\varphi}\left((-m^+a - (-1)^{|x||m|}x(m)^+) \otimes r^-\right) - \tilde{\varphi}\left(m^+ \otimes (-ar^- + x(r)^-)\right) \\
&\qquad + (-1)^{|m||a+x|}\tilde{\varphi}\left((a + x) \otimes (q(m, r) + \mathbf{E}_{m,r})\right) \\
&\quad = -\{ma, r\} - (-1)^{|x||m|}\{x(m), r\} + \{m, ar\} - \{m, x(r)\} \\
&\qquad + (-1)^{|a||m|}\{a, q(m, r)\} + (-1)^{|x||m|}\{x(m), r\} - (-1)^{|m|(|x|+|r|)}\{x(r), m\} \\
&\quad \overset{(5.23)}{\underset{(5.24)}{=}} 0
\end{aligned}$$

and so $\sum_{\circlearrowleft} \tilde{\varphi}([L, L] \otimes L) = (0)$ hence $\tilde{\varphi}$ induces the map $\varphi : \mathcal{L} \to \{\mathcal{J}, \mathcal{J}\}$ given by (5.25). Consider now the K-bilinear map $g : \mathcal{J} \times \mathcal{J} \to \mathcal{L}_0$ given by

$$f'(a + m, b + n) = \langle a, b\rangle + \frac{1}{2}\left(\langle m^+, n^-\rangle + \langle m^-, n^+\rangle\right).$$

This map is clearly well-defined and so there exists a unique K-supermodule homomorphism $\tilde{\psi} : \mathcal{J} \otimes \mathcal{J} \to \mathcal{L}_0$ such that $\tilde{\psi}(u \otimes v) = f'(u, v)$ for all $u, v \in \mathcal{J}$. Clearly $a \otimes m$, $m \otimes a$ and $a \otimes b + (-1)^{|a||b|}b \otimes a$ are in the kernel of $\tilde{\psi}$ for all $a, b \in A$ and $m \in \mathcal{M}$. Since

$$\frac{1}{2}(\langle m^+, n^-\rangle + \langle m^-, n^+\rangle) = -\frac{1}{2}(-1)^{|m||n|}(\langle n^+, m^-\rangle + \langle n^-, m^+\rangle).$$

we have that $m\otimes n+(-1)^{|m||n|}n\otimes m$ is in the kernel of $\tilde{\psi}$ for all $m,n\in\mathcal{M}$. Moreover, for all $a,b,c\in A$ and $m,n,p\in\mathcal{M}$,

$$\begin{aligned}
&\tilde{\psi}(((a+m)(b+n))\otimes(c+p))-\tilde{\psi}(a+m\otimes((b+n)(c+p)))\\
&\qquad\qquad -(-1)^{|a+m||b+n|}\tilde{\psi}(b+n\otimes((a+m)(c+p)))\\
&\qquad = \tilde{\psi}((ab+q(m,n)+an+mb)\otimes(c+p))\\
&\qquad\qquad -\tilde{\psi}((a+m)\otimes(bc+q(n,p)+bp+nc))\\
&\qquad\qquad -(-1)^{|a+m||b+n|}\tilde{\psi}((b+n)\otimes(ac+q(m,p)+ap+mc))\\
&\qquad = \langle ab,c\rangle+\langle q(m,n),c\rangle+\frac{1}{2}(\langle an^{+}+m^{+}b,p^{-}\rangle+\langle an^{-}+m^{-}b,p^{+}\rangle)\\
&\qquad\qquad -\langle a,bc\rangle-\langle a,q(n,p)\rangle-\frac{1}{2}(\langle m^{+},bp^{-}+n^{-}c\rangle+\langle m^{-},bp^{+}+n^{+}c\rangle\\
&\qquad\qquad -(-1)^{|a||b|}\langle b,ac\rangle-(-1)^{|m||b|}\langle b,q(m,p)\rangle\\
&\qquad\qquad -\frac{1}{2}(-1)^{|n||a+m|}\langle n^{+},ap^{-}+m^{-}c\rangle+\langle n^{-},ap^{+}+m^{+}c\rangle)\\
&\qquad \overset{(5.14)}{\underset{(5.15)}{=}} \frac{1}{2}(\langle[m^{+},n^{-}],c\rangle-\langle m^{+},[n^{-},c]\rangle+(-1)^{|m||n|}\langle n^{-},[m^{+},c]\rangle)\\
&\qquad\qquad -\frac{1}{2}(\langle[m^{-},n^{+}],c\rangle+\langle m^{-},[n^{+},c]\rangle+(-1)^{|m||n|}\langle n^{+},[m^{-},c]\rangle)\\
&\qquad\qquad -\frac{1}{2}(\langle a,[n^{+},p^{-}],c\rangle-\langle[a,n^{+}],p^{-}\rangle-(-1)^{|a||n|}\langle n^{+},[a,p^{-}]\rangle)\\
&\qquad\qquad +\frac{1}{2}(\langle a,[n^{-},p^{+}]\rangle-\langle[a,n^{-}],p^{+}\rangle+(-1)^{|a||n|}\langle n^{-},[a,p^{+}]\rangle)\\
&\qquad\qquad +((-1)^{|m||b|}\langle b,[m^{+},p^{-}]\rangle-\langle m^{+},[b,p^{-}]\rangle+\langle[m^{+},b],p^{-}\rangle)\\
&\qquad\qquad +\frac{1}{2}((-1)^{|b||m|}\langle b,[m^{-},p^{+}]\rangle-\langle m^{-},[b,p^{+}]\rangle+\langle[m^{-},b],p^{+}\rangle)\\
&\qquad \overset{(5.2)}{=} 0.
\end{aligned}$$

Hence $s\in\ker\tilde{\psi}$ and so $\tilde{\psi}$ gives rise to the map $\psi:\{\mathcal{J},\mathcal{J}\}\to\mathcal{L}_0$ given by (5.26). Moreover, for $a,b\in A$ and $m,n\in\mathcal{M}$,

$$\begin{aligned}
\varphi(\psi(\{a+m,b+n\})) &= \varphi(\langle a,b\rangle)+\frac{1}{2}\varphi(\langle m^{+},n^{-}\rangle+\langle m^{-},n^{+}\rangle)\\
&= \{a,b\}+\frac{1}{2}(\{m,n\}+\{m,n\})\\
&= \{a+m,b+n\}
\end{aligned}$$

and so $\varphi \circ \psi = Id$. If $x = \sum_j \{a_j, b_j\} + \sum_k \{p_k, r_k\} \in \mathrm{HF}(\mathcal{J})$ then $\sum_k \mathbf{E}_{p_k, r_k} = 0$ and so

$$\sum_j [a_j, b_j] + \frac{1}{2} \sum_k ([p_k^+, r_k^-] + [p_k^-, r_k^+]) = \sum_k \mathbf{E}_{p_k, r_k} = 0$$

hence $\psi(x) \in \mathrm{H}_2(L)$. If $X = \sum_j \langle a_j, b_j \rangle + \sum_k \langle m_k, n_k \rangle \in \mathrm{H}_2(L)$ then $\sum_j q(m_k, n_k) = 0$ and $\sum_k \mathbf{E}_{m_k, n_k} = 0$. Hence

$$\varphi(X) = \sum_j \{a_j, b_j\} + \frac{1}{2} \sum_k (\{m_k^+, n_k^-\} + \{m_k^-, n_k^+\}) \in \mathrm{HF}(\mathcal{J}).$$

Given $x \in \mathrm{HF}(\mathcal{J})$ we have that $x = \varphi \circ \psi(x) = \varphi(\psi(x)) \in \varphi(\mathrm{H}_2(L))$ and so $\varphi(\mathrm{H}_2(L)) = \mathrm{HF}(\mathcal{J})$. In addition, any element of $\langle L_0, L_0 \rangle$ can be written as a sum of elements of the form $\langle a, b \rangle$ and $\langle x, \mathbf{E}_{m,n} \rangle$ and so since

$$\begin{aligned}
\psi(\varphi(\langle a, b \rangle + \langle x, \mathbf{E}_{m,n} \rangle)) &= \psi(\{a, b\} + \{x(m), n\} - (-1)^{|m||n|}\{x(n), m\}) \\
&= \langle a, b \rangle + \frac{1}{2}\langle x(m)^+, n^- \rangle + \frac{1}{2}\langle x(m)^-, n^+ \rangle \\
&\quad - \frac{1}{2}(-1)^{|m||n|}\langle x(n)^+, m^- \rangle - \frac{1}{2}(-1)^{|m||n|}\langle x(n)^-, m^+ \rangle \\
&= \langle a, b \rangle + \frac{1}{2}\langle [x, m^+], n^- \rangle + \frac{1}{2}\langle [x, m^-], n^+ \rangle \\
&\quad - \frac{1}{2}(-1)^{|m||n|}\langle x(n)^+, m^- \rangle - \frac{1}{2}(-1)^{|m||n|}\langle x(n)^-, m^+ \rangle \\
&\overset{(5.2)}{\underset{(5.15)}{=}} \langle a, b \rangle + \frac{1}{2}\langle x, \mathbf{E}_{m,n} \rangle - \frac{1}{2}(-1)^{|x||m|}\langle m^+, x(n)^- \rangle \\
&\quad + \frac{1}{2}\langle x, \mathbf{E}_{m,n} \rangle - \frac{1}{2}(-1)^{|x||m|}\langle m^-, x(n)^+ \rangle \\
&\quad - \frac{1}{2}(-1)^{|m||n|}\langle x(n)^+, m^- \rangle - \frac{1}{2}(-1)^{|m||n|}\langle x(n)^-, m^+ \rangle \\
&\overset{(5.1)}{=} \langle a, b \rangle + \langle x, \mathbf{E}_{m,n} \rangle
\end{aligned}$$

we have $\psi o \varphi|_{\langle L_0, L_0\rangle} = Id$. Moreover, for all $\sum_j \langle a_j, b_j\rangle + \frac{1}{2}\sum_k \left(\langle m_k^+, n_k^-\rangle + \langle m_k^-, n_k^+\rangle\right) \in$ $\mathrm{H}_2(L)$,

$$\begin{aligned}\psi(\varphi(\sum_j \langle a_j, b_j\rangle + \frac{1}{2}\sum_k (\langle m_k^+, n_k^-\rangle + \langle m_k^-, n_k^+\rangle)))\\ &= \psi(\sum_j \{a_j, b_j\} + \frac{1}{2}\sum_k (\{m_k^+, n_k^-\} + \{m_k^-, n_k^+\}))\\ &= \sum_j \langle a_j, b_j\rangle + \frac{1}{2}\sum_k (\langle m_k^+, n_k^-\rangle + \langle m_k^+, n_k^-\rangle)\end{aligned}$$

and so (5.27) holds. Hence if (5.28) holds, $\psi|_{\mathrm{HF}(\mathcal{J})}$ and $\varphi|_{\mathrm{H}_2(L)}$ are inverses of each other and so $\mathrm{H}_2(L) \cong \mathrm{HF}(\mathcal{J})$. ∎

Theorem 5.3.5 *Let $\mathcal{N}$ and $\mathcal{P}$ be A-supermodules with quadratic forms $q_{\mathcal{N}}$ and $q_{\mathcal{P}}$ respectively such that $q_{\mathcal{N}}$ is almost diagonizable with respect to some basis $\{n_i \mid i \in I\}$ and that there exists an $i \in I$ such that $i = \underline{i}$. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ with $q = q_{\mathcal{N}} \oplus q_{\mathcal{P}}$. Let $L = \mathrm{eosp}(q_\infty)$ and assume that $\mathrm{eosp}(q)$ is perfect. Then*

$$\mathcal{L} = \langle 1, L_+\rangle \oplus \langle L_+, L_-\rangle \oplus \langle 1, L_-\rangle$$

and (5.28) holds. In particular,

$$\mathrm{H}_2(L) = \langle A, A\rangle + \left\{\frac{1}{2}\sum_k (\langle p_k^+, r_k^-\rangle + \langle p_k^-, r_k^+\rangle) \mid p_k, r_k \in \mathcal{P}, \sum_k \mathbf{E}_{p_k, r_k} = 0\right\}. \quad (5.30)$$

Moreover, if $\mathcal{J}_{\mathcal{M}} = A \oplus \mathcal{M}$ and $\mathcal{J}_{\mathcal{P}} = A \oplus \mathcal{P}$ are Jordan superalgebras as defined in Example 1.3.22 then the maps

$$\begin{aligned}\varphi : \{\mathcal{J}_{\mathcal{P}}, \mathcal{J}_{\mathcal{P}}\}_{\mathcal{P}} &\to \{\mathcal{J}_{\mathcal{M}}, \mathcal{J}_{\mathcal{M}}\}_{\mathcal{M}}\\ \{a \oplus p, b \oplus r\}_{\mathcal{P}} &\mapsto \{a \oplus p, b \oplus r\}_{\mathcal{M}}\end{aligned} \quad (5.31)$$

and

$$\begin{aligned}\phi : \{\mathcal{J}_{\mathcal{M}}, \mathcal{J}_{\mathcal{M}}\}_{\mathcal{M}} &\to \{\mathcal{J}_{\mathcal{P}}, \mathcal{J}_{\mathcal{P}}\}_{\mathcal{P}}\\ \{a \oplus bn_i \oplus p, c \oplus dn_j \oplus r\}_{\mathcal{M}} &\mapsto \{a, c\}_{\mathcal{P}} + \{p, r\}_{\mathcal{P}}\\ &\quad + \tfrac{1}{2}\left(\{b, q(n_i, dn_j)\}_{\mathcal{P}} - (-1)^{|bn_i||dn_j|}\{d, q(n_j, bn_i)\}_{\mathcal{P}}\right),\end{aligned} \quad (5.32)$$

extended bilinearly over K where $a,b,c,d \in A$, $p,r \in \mathcal{P}$ and $i,j \in I$, are well-defined K-supermodule homomorphisms satisfying $\varphi \circ \phi_{\{A\oplus\mathcal{P},A\oplus\mathcal{P}\}_{\mathcal{M}}} = Id$, $\phi \circ \varphi = Id$, $\phi(\mathrm{HF}(\mathcal{J}_{\mathcal{M}})) \subset \mathrm{HF}(\mathcal{J}_{\mathcal{P}})$ and $\varphi(\mathrm{HF}(\mathcal{J}_{\mathcal{P}})) \subset \mathrm{HF}(\mathcal{J}_{\mathcal{M}})$. Hence

$$\{\mathcal{J}_{\mathcal{P}},\mathcal{J}_{\mathcal{P}}\}_{\mathcal{P}} \cong \{A\oplus\mathcal{P},A\oplus\mathcal{P}\}_{\mathcal{M}} \tag{5.33}$$

and

$$\mathrm{HF}(\mathcal{J}_{\mathcal{P}}) \cong \mathrm{HF}(\mathcal{J}_{\mathcal{M}}) \tag{5.34}$$

and, in particular,

$$\mathrm{H}_2(L) \cong \mathrm{HF}(\mathcal{J}_{\mathcal{P}}).$$

Proof: The first assertion follows from Proposition 5.2.1. Moreover, by the same result we have that

$$\mathrm{H}_2(L) = \left\{ \sum_k \langle m_k^+, \tilde{m}_k^- \rangle \mid m_k, \tilde{m}_k \in \mathcal{M}, \sum_k [m_k^+, \tilde{m}_k^-]) = 0 \right\}.$$

It is easy to verify that

$$\langle A, A\rangle + \left\{ \frac{1}{2}\sum_k (\langle p_k^+, r_k^-\rangle + \langle p_k^-, r_k^+\rangle) \mid p_k, r_k \in \mathcal{P}, \sum_k \mathbf{E}_{p_k,r_k} = 0 \right\} \subset \mathrm{H}_2(L).$$

Take $i \in I$ such that $i = \underline{i}$, say $i = 0$. By Proposition 5.3.1 we can write every element x of $\langle L_+, L_-\rangle$ in the form

$$x = \sum_{\substack{i,j\in I\\ i<j}} \langle a_{ij} n_i^+, n_j^-\rangle + \sum_{\substack{i\in I\\ |n_i|=\bar{1}}} \langle a_{ii} n_i^+, n_i^-\rangle + \langle a n_0^+, n_0^-\rangle + \sum_{i\in I} \langle n_i^+, p_i^-\rangle + \sum_k \langle r_k^+, s_k^-\rangle + \chi$$

for some $a, a_{ij} \in A$, $p_i, r_k, s_k \in \mathcal{P}$ and $\chi \in \langle A, A\rangle$. Suppose $x \in \mathrm{H}_2(L)$. Then

$$\begin{aligned} 0 &= \sum_{\substack{i,j\in I\\ i<j}} [a_{ij} n_i^+, n_j^-] + \sum_{\substack{i\in I\\ |n_i|=\bar{1}}} [a_{ii} n_i^+, n_i^-] + [a n_0^+, n_0^-] + \sum_{i\in I} [n_i^+, p_i^-] + \sum_k [r_k^+, s_k^-] \\ &= \underbrace{\sum_{\substack{i,j\in I\\ i<j}} a_{ij}\mathbf{E}_{n_i,n_j} + \sum_{\substack{i\in I\\ |n_i|=\bar{1}}} a_{ii}\mathbf{E}_{n_i,n_i}}_{\in \mathrm{eosp}(q_{\mathcal{N}})} + \underbrace{\sum_{i\in I} \mathbf{E}_{n_i,p_i}}_{\in \mathrm{eosp}(q_{\mathcal{N}},q_{\mathcal{P}})} + \underbrace{\sum_k \mathbf{E}_{r_k,s_k}}_{\in \mathrm{eosp}(q_{\mathcal{P}})} + b \end{aligned}$$

for some $b \in A$ and so by Propositions 2.1.6, 2.1.7 and 2.1.16, $a_{ij} = 0$ and $p_i = 0$ for all $i, j \in I$. Hence

$$x = \langle an_0^+, n_0^- \rangle + \sum_k \langle r_k^+, s_k^- \rangle + \chi.$$

Now, since $x \in \mathrm{H}_2(L)$,

$$0 = [an_0^+, n_0^-] + \sum_k [r_k^+, s_k^-] = aq(n_0, n_0) + \sum_k (q(r_k, s_k) + \mathbf{E}_{r_k, s_k})$$

and so $a = -\sum_k q(n_0, n_0)^{-1} q(r_k, s_k)$. By Proposition 5.3.1 we can write

$$\begin{aligned} \langle an_0^+, n_0^- \rangle &= -\sum_k \langle q(n_0, n_0)^{-1} q(r_k, s_k) n_0^+, n_0^- \rangle \\ &\overset{(5.16)}{=} \frac{1}{2} \sum_k \left(\langle r_k^-, s_k^+ \rangle - \langle r_k^+, s_k^- \rangle + \langle q(n_0, n_0), q(r_k, s_k) q(n_0, n_0)^{-1} \rangle \right) \end{aligned}$$

and so

$$x = \frac{1}{2} \sum_k \left(\langle r_k^+, s_k^- \rangle + \langle r_k^-, s_k^+ \rangle \right) + \langle c, d \rangle$$

for some $c, d \in A$. Hence

$$\mathrm{H}_2(L) = \langle A, A \rangle + \left\{ \frac{1}{2} \sum_k (\langle p_k^+, r_k^- \rangle + \langle p_k^-, r_k^+ \rangle) \mid p_k, r_k \in \mathcal{P}, \ \sum_k \mathbf{E}_{p_k, r_k} = 0 \right\}.$$

and so (5.28) and (5.30) hold. Since

$$\mathrm{HF}(\mathcal{J}_{\mathcal{M}}) = \{A, A\}_{\mathcal{M}} + \left\{ \sum_k \{m_k, m_k'\}_{\mathcal{M}} \mid m_k, m_k' \in \mathcal{M}, \ \sum_k \mathbf{E}_{m_k, m_k'} = 0 \right\}$$

we clearly have that

$$\{A, A\}_{\mathcal{M}} + \left\{ \sum_k \{p_k, r_k\}_{\mathcal{M}} \mid p_k, r_k \in \mathcal{P}, \ \sum_k \mathbf{E}_{p_k, r_k} = 0 \right\} \subset \mathrm{HF}(\mathcal{J}_{\mathcal{M}}).$$

We claim that indeed

$$\{A, A\}_{\mathcal{M}} + \left\{ \sum_k \{p_k, r_k\}_{\mathcal{M}} \mid p_k, r_k \in \mathcal{P}, \ \sum_k \mathbf{E}_{p_k, r_k} = 0 \right\} = \mathrm{HF}(\mathcal{J}_{\mathcal{M}}). \tag{5.35}$$

It is sufficient to show that

$$\left\{\sum_k \{m_k, m_k'\}_{\mathcal{M}} \mid m_k, m_k' \in \mathcal{M},\ \sum_k \mathbf{E}_{m_k,m_k'} = 0\right\}$$
$$\subset \{A, A\}_{\mathcal{M}} + \left\{\sum_k \{p_k, r_k\}_{\mathcal{M}} \mid p_k, r_k \in \mathcal{P},\ \sum_k \mathbf{E}_{p_k,r_k} = 0\right\}.$$

First we should note that if $m \in \mathcal{M}_{\bar{0}}$ and $a \in A$ then $\{am, m\}_{\mathcal{M}} = \{a, q(m,m)\}_{\mathcal{M}} + (-1)^{|a||m|}\{m, am\}_{\mathcal{M}}$ and so $\{am, m\}_{\mathcal{M}} \in \{A, A\}_{\mathcal{M}}$. Hence any $x = \sum_k \{m_k, m_k'\}_{\mathcal{M}}$ such that $m_k, m_k' \in \mathcal{M}$, $\sum_k \mathbf{E}_{m_k,m_k'} = 0$ can be written as

$$x = \sum_{i,j\in I, i<j} \{a_{ij}n_i, n_j\}_{\mathcal{M}} + \sum_{i\in I, |n_i|=\bar{1}} \{a_{ii}n_i, n_i\}_{\mathcal{M}} + \sum_{i\in I} \{b_i n_i, p_i\}_{\mathcal{M}} + \sum_j \{r_j, s_j\}_{\mathcal{M}} + \chi$$

for some $a_{ij}, b_i \in A$, $p_i, r_j, s_j \in \mathcal{P}$ and $\chi \in \{A, A\}_{\mathcal{M}}$ where

$$0 = \sum_{i,j\in I, i<j} a_{ij}\mathbf{E}_{n_i,n_j} + \sum_{i\in I, |n_i|=\bar{1}} a_{ii}\mathbf{E}_{n_i,n_i} + \sum_{i\in I} b_i\mathbf{E}_{n_i,p_i} + \sum_j \mathbf{E}_{r_j,s_j}.$$

Hence by Propositions 2.1.6, 2.1.7 and 2.1.16, $a_{ij} = 0$ and $p_i = 0$ for all $i, j \in I$ and so

$$x = \sum_j \{r_j, s_j\}_{\mathcal{M}} + \chi$$

for some $r_j, s_j \in \mathcal{P}$ and $\chi \in \{A, A\}_{\mathcal{M}}$. Therefore (5.35) follows. It is easy to verify that (5.31) is well-defined and that $\varphi(\mathrm{HF}(\mathcal{J}_{\mathcal{P}})) \subset \mathrm{HF}(\mathcal{J}_{\mathcal{M}})$. Consider the map

$$\begin{aligned} f : \mathcal{J}_{\mathcal{M}} \times \mathcal{J}_{\mathcal{M}} &\to \{\mathcal{J}_{\mathcal{P}}, \mathcal{J}_{\mathcal{P}}\}_{\mathcal{P}} \\ (a \oplus bn_i \oplus p, c \oplus bn_j \oplus r) &\mapsto \{a, b\}_{\mathcal{P}} + \{p, r\}_{\mathcal{P}} \\ &\quad + \tfrac{1}{2}\left(\{a, q(n_i, bn_j)\}_{\mathcal{P}} - (-1)^{|an_i||bn_j|}\{b, q(n_j, an_i)\}_{\mathcal{P}}\right) \end{aligned}$$

and extend it bilinearly over K. Then it gives rise to the map $\tilde{\phi} : \mathcal{J}_{\mathcal{M}} \otimes \mathcal{J}_{\mathcal{M}} \to \{\mathcal{J}_{\mathcal{P}}, \mathcal{J}_{\mathcal{P}}\}_{\mathcal{P}}$ where $\tilde{\phi}(x \otimes y) = f(x, y)$. We claim that

$$x \otimes y + (-1)^{|x||y|} y \otimes x \tag{5.36}$$

$$xy \otimes z - x \otimes yz - (-1)^{|x||y|} y \otimes xz \tag{5.37}$$

$$a \otimes m \tag{5.38}$$

are in the kernel of $\tilde{\phi}$ for all $x, y, z \in \mathcal{J}$ and $a \in A$, $m \in \mathcal{M}$. This is obvious for (5.38). It is easy to see that the elements of the form (5.37) are in the kernel of $\tilde{\phi}$ once one verifies that

$$\begin{aligned}\tilde{\phi}(an_i \otimes bn_j) &= \frac{1}{2}\left(\{a, q(n_i, bn_j)\}_{\mathcal{P}} - (-1)^{|an_i||bn_j|}\{b, q(n_j, an_i)\}\right) \\ &= -(-1)^{|an_i||bn_j|}\frac{1}{2}\left(\{b, q(n_j, an_i)\} - (-1)^{|bn_j||an_i|}\{a, q(n_i, bn_j)\}_{\mathcal{P}}\right). \\ &= -(-1)^{|an_i||bn_j|}\tilde{\phi}(bn_j \otimes an_i).\end{aligned}$$

To show that the elements of the form (5.36) lie in the kernel of $\tilde{\phi}$, we need to show it for the following combinations:

$x, y, z \in A$	(follows from relations in $\{\mathcal{J}_{\mathcal{P}}, \mathcal{J}_{\mathcal{P}}\}_{\mathcal{P}}$)
$x, y, z \in \mathcal{M}$	(follows from the fact that $\tilde{\phi}(A, \mathcal{M}) = \tilde{\phi}(\mathcal{M}, A) = (0)$)
$x, y \in A, z \in \mathcal{M}$	(follows from the fact that $\tilde{\phi}(A, \mathcal{M}) = \tilde{\phi}(\mathcal{M}, A) = (0)$)
$x \in A, y, z \in \mathcal{P}$	(follows from relations in $\{\mathcal{J}_{\mathcal{P}}, \mathcal{J}_{\mathcal{P}}\}_{\mathcal{P}}$)
$x \in A, y \in \mathcal{N}, z \in \mathcal{P}$	(follows from $\tilde{\phi}(\mathcal{N}, \mathcal{P}) = (0)$ and $q(\mathcal{N}, \mathcal{P}) = (0)$)
$x \in A, y, z \in \mathcal{N}$.	

Since $\tilde{\phi}$ is bilinear, to show that (5.38) for $x \in A$ and $y, z \in \mathcal{N}$ are in the kernel of $\tilde{\phi}$, we only need to show it for $x = a$, $y = bn_i$ and $z = cn_j$ for all $a, b, c \in A$ and $i, j \in I$. Using (5.23) and (5.24) we have

$$\begin{aligned}&2\tilde{\phi}(abn_i \otimes cn_j - a \otimes bq(n_i, cn_j) - (-1)^{|a||bn_i|}bn_i \otimes acn_j) \\ &\quad = \{ab, q(n_i, cn_j)\} - (-1)^{|abn_i||cn_j|}\{c, q(n_j, abn_i)\} - 2\{a, bq(n_i, cn_j)\} \\ &\qquad - (-1)^{|a||bn_i|}\{b, q(n_i, acn_j)\} + (-1)^{|a||bn_i| + |bn_i||acn_j|}\{ac, q(n_j, bn_i)\} \\ &\quad = \{ab, q(n_i, cn_j)\} - (-1)^{|abn_i||cn_j| + |a||n_j|}\{ca, q(n_j, bn_i)\} \\ &\qquad - (-1)^{|abn_i||cn_j| + |a||bn_i|}\{cq(n_j, bn_i), a\} - 2\{a, bq(n_i, cn_j)\} \\ &\qquad - (-1)^{|a||bn_i|}\{b, q(n_i, acn_j)\} + (-1)^{|bn_i|(|a| + |acn_j|)}\{ac, q(n_j, bn_i)\} \\ &\quad = \{ab, q(n_i, cn_j)\} - \{a, bq(n_i, cn_j)\} - (-1)^{|a||b|}\{ac, q(n_j, bn_i)\} \\ &\qquad - (-1)^{|bn_i||cn_j|}\{ac, q(n_j, bn_i)\} - (-1)^{|bn_i||cn_j|}\{ac, q(n_j, bn_i)\} \\ &\qquad + (-1)^{|bn_i||cn_j|}\{a, cq(n_j, bn_i)\} - \{a, bq(n_i, cn_j)\} \\ &\quad = 0\end{aligned}$$

where we abbreviated $\{\cdot,\cdot\}_{\mathcal{P}}$ by $\{\cdot,\cdot\}$. Hence (5.36), (5.37) and (5.38) are in the kernel of $\tilde{\phi}$ and so $\tilde{\phi}$ induces the map (5.32). It is easy to verify that $\varphi o \phi_{\{A\oplus\mathcal{P},A\oplus\mathcal{P}\}_{\mathcal{M}}} = Id$ and that $\phi \circ \varphi = Id$, and so (5.33) holds. Moreover, using (5.35), $\phi(\mathrm{HF}(\mathcal{J}_{\mathcal{M}})) \subset \mathrm{HF}(\mathcal{J}_{\mathcal{P}})$ and so since $\varphi(\mathrm{HF}(\mathcal{J}_{\mathcal{P}})) \subset \mathrm{HF}(\mathcal{J}_{\mathcal{M}})$ we have that (5.34) holds. Hence $\mathrm{H}_2(L) \cong \mathrm{HF}(\mathcal{J}_{\mathcal{P}})$ by Theorem 5.3.4. ∎

Remark 5.3.6 The center of the universal central extension of a Lie algebra L over a field $\mathbb{F}$ of characteristic zero and graded by finite root system was described in [ABG]. The Lie superalgebra L over A and graded by the root system of type B_I ($|I| \geq 3$) is a central extension of (a centreless) $L' = \mathrm{eosp}(q_I \oplus q_{x_0} \oplus q_{\mathcal{P}})$ for some base point x_0 and an A-supermodule $\mathcal{P}$ where A is a superextension of $\mathbb{F}$ with a zero annihilator on $\mathcal{P}$. Since $q_I \oplus q_{x_0}$ is invertibly diagonizable and $|I| \geq 3$ we have by Proposition 2.1.19 that L is perfect. Moreover, by Theorem 5.1.3, the center $Z(\mathcal{L})$ of the universal central extension of L is isomorphic to $\mathrm{H}_2(L')$. Hence by Theorem 5.3.5, $Z(\mathcal{L}) \cong \mathrm{HF}(\mathcal{J}_{\mathcal{P}})$ which, for the case of the Lie algebra over a field and graded by the finite root system of type B in particular, is precisely what was proved in [ABG]. Similarly, the Lie superalgebra L over A and graded by the root system of type D_I ($|I| \geq 4$) is a central extension of (a centreless) $L' = \mathrm{eosp}(q_I)$ for some superextension A of $\mathbb{F}$. Since q_I is invertibly diagonizable and $|I| \geq 3$ we have by Proposition 2.1.19 that L is perfect. Hence by the same argument as for type B, the center of the universal central extension of L is isomorphic to $\mathrm{HC}_1(A)$ which, for the case of the Lie algebra over a field and graded by the finite root system of type D in particular, is precisely what was proved in [ABG].

Corollary 5.3.7 *Let $\mathcal{M}$ be a A-supermodule with a quadratic form q such that q is almost diagonizable with respect to some basis $\{m_i \mid i \in I\}$ and that there exists an $i \in I$ such that $i = \underline{i}$. Let $L = \mathrm{eosp}(q_\infty)$ and assume that $\mathrm{eosp}(q)$ is perfect. Then*

$$\mathrm{H}_2(L) = \{\mathcal{J}_0, \mathcal{J}_0\}$$

where $\mathcal{J}_0 = A = A \oplus (0)$.

Remark 5.3.8 Corollary 5.3.7 holds in particular when $|\{i \in I \mid i = \underline{i}\}| \geq 3$ or $\{i \in I \mid |m_i| = \bar{1}\} \neq \emptyset$ since in that case $\mathrm{eosp}(q)$ is perfect by Lemma 2.1.19.

Example 5.3.9 If L is the centreless core of an extended affine Lie algebra of type B_l or D_l ($l \geq 3, 4$ respectively) then $\mathrm{H}_2(L) \cong \{\mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}], \mathbb{C}[t_1^{\pm 1}, \ldots, t_\nu^{\pm 1}]\}$ for some $\nu \in \mathbb{N}$.

Remark 5.3.10 In [IK], Iohara and Koga described the universal central extensions of the basic classical Lie superalgebras $\mathfrak{g}_k \otimes A$ over a free K-extension A where K is a commutative ring containing $\frac{1}{2}$. In particular, for the case when $\mathfrak{g}$ is of type $B(m,n)$ (and $\frac{1}{3} \in K$), $C(n)$ and $D(m,n)$ they proved that the universal central extension of $\mathfrak{g}_k \otimes A$ is isomorphic to $(\mathfrak{g}_k \otimes A) \oplus \mathrm{HC}_1(A)$ and that $\mathrm{H}_2(\mathfrak{g}_k \otimes A) = \mathrm{HC}_1(A)$. Note that

$$\begin{aligned}
B(m,n) &\cong \mathrm{eosp}(q_I \oplus q_{x_0} \oplus \tilde{q}_J) \text{ where } |I| = m \geq 0,\ |J| = n \geq 1 \text{ and} \\
&\quad\ x_0 \text{ is a base point} \\
D(m,n) &\cong \mathrm{eosp}(q_I \oplus \tilde{q}_J) \text{ where } |I| = m \geq 2,\ |J| = n \geq 1 \\
C(n) &\cong \mathrm{eosp}(q_{\{\infty\}} \oplus \tilde{q}_J) \text{ where } |J| = n - 1 \geq 2
\end{aligned}$$

and so every one of these algebras but $B(0,n)$ is isomorphic to $\mathrm{eosp}(q_\infty)$ for some almost diagonizable q for which $\mathrm{eosp}(q)$ is perfect and hence by Corollary 5.3.7 we can recover the result of [IK] for these types of Lie superalgebras as [IK].

Bibliography

[AABGP] B.N. Allison, S. Azam, S. Berman, Y. Gao, A. Pianzola, *Extended Affine Lie Algebras and Their Root Systems*, Mem. Amer. Math. Soc., vol. 126 (1997)

[ABG] B. Allison, G. Benkart, Y. Gao, *Central Extensions of Lie Algebras graded by Finite Root Systems*, Math. Ann. 316, 499-527 (2000)

[AG] B.N. Allison, Y. Gao, *The Root System and the Core of an Extended Affine Lie Algebra*, Selecta Math. (N.S.) **7** (2001), no. 2, 149–212

[Be] G. Benkart, *Derivations and invariant forms of Lie algebras graded by finite root systems*, Can. J. Math. Vol. **50** (2) (1998) 225-241

[BeZ] G.M. Benkart and E.I. Zelmanov, *Lie algebras graded by finite root systems and the intersection matrix algebras*, Invent. Math. **126** (1996), 1–45

[BM] S. Berman, R.V. Moody, *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy*, Invent. Math. **108** (1992), 323-347

[BK] S. Berman, Y Krylioúk, *Universal central extensions of twisted and untwisted Lie algebras over commutatie rings*, J. Algebra **173** (1995), 302-347

[Bo] N. Bourbaki, *Groupes et algebres de Lie, Chap. IV, V, VI*, Hermann, Paris (1968)

[CW] S.J. Cheng, W. Wang, *Lie subalgebras of differential operators on super circle*, preprint 2001, posted on http://xxx.lanl.gov/, math.QA/0103092

[DZ] D.Z. Djokovic, K. Zhao, *Generalized Witt Algebras*, Trans. Amer. Math. Soc. **350** (2) (1998), 643-664

[DJKM] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Operator approach to the Kadomstsev-Petviashivili equation. Transformation groups for solition equations III*, J. Phys. Soc. Japan **50** (1981), 3806-3812.

[Fuks] D.B. Fuks, *Cohomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics. Consultants Bureau, New York (1986)

[FLM] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster*, Pure Appl. Math., **134** (1988)

[G] H. Garland, *The arithmetic theory of loop groups*, Inst. Hautes Etudes Sci. Publ. Math., **52** (1980) 5-136

[GN] E. Garcia, E. Neher, *Tits-Kantor-Koecher superalgebras of Jordan superpairs covered by grids*, preprint July 2001

[J1] N. Jacobson, *Structure and representations of Jordan algebras*. American Mathematical Society Colloquium Publications, Vol. XXXIX (1968)

[J2] N. Jacobson, *Basic algebra II*, W. H. Freeman and Company, New York, 2nd Ed. (1989)

[J3] N. Jacobson, *Structure of rings*. American Mathematical Society Colloquium Publications, **37** (1964)

[IK] K. Iohara, Y. Koga, *Central Extensions of Lie Superalgebras*. Comment. Math. Helv. **76** (2001) 110-154

[K1] V.G. Kac, *Infinite Dimensional Lie Algebras*. Third Ed., Cambridge U. Press. Cambridge (1990)

[K2] V.G. Kac, *Lie Superalgebras*, Advances in Mathematics **26**, (1977) 8-96

[Ko] M. Koecher, *Imbedding of Jordan algebras into Lie algebras I*, Amer. J. Math. **89** (1967), 787-816; II, Amer. J. Math. **90** (1968), 476-510

[KP] V.G. Kac, D.H. Peterson, *Spin and wedge representations of infinite-dimensional Lie algebras and groups*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981). 3308-3312

[Kr] S.V. Krutelevich, *Simple Jordan superpairs*, Comm. Algebra **25** (1997), no. 8, 2635-2657.

[Ks] C. Kassel, *Kähler differentials and coverings of complex simple Lie algebra extended over a commutative algebra*, J. Pure and Applied Alg. **34** (1984) 265-275

[L] O. Loos, *Jordan pairs*, Lecture notes in mathematics, Springer-Verlag, Berlin-New York **460** (1975)

[La] S. Lang, *Algebra*, Addison-Wesley Series in Mathematics. 2nd Ed. (1971)

[LN] O. Loos and E. Neher, *Locally finite root systems*. preprint 2001

[M] K. Meyberg, *Lectures on algebras and triple systems*. The University of Virginia, Charlottesville, Va. (1972)

[N1] E. Neher, *Systemes de racines 3-gradues*. C.R. Acad.Sci. Paris Ser. I **310** (1990). 687-690

[N2] E. Neher, *Lie algebras graded by 3-graded root systems*. Amer. J. Math. **118** (1996)

[N3] E. Neher, *Quadratic Jordan superpairs covered by grids*. preprint 2000. posted on the Jordan Theory Preprint Archive. http://mathematik.uibk.ac.at/jordan/

[N4] E. Neher, *On Universal Central Extensions of Lie superalgebras*, unpublished manuscript, 2000

[S] M. Scheunert, *The Theory of Lie Superalgebras*, Lecture Notes in Mathematics, Springer, Berlin **716** (1979)

[SZ] M. Scheunert, R.B. Zhang, *Cohomology of Lie Superalgebras and of their generalizations*, J. Math. Phys. **39** (1998), no. 9, 5024-5061