On the Existence of a Second Hamilton Cycle in Hamiltonian Graphs With Symmetry

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Abstract

In 1975, Sheehan conjectured that every simple 4-regular hamiltonian graph has a second Hamilton cycle. If Sheehan's Conjecture holds, then the result can be extended to all simple d-regular hamiltonian graphs with $d \geq 3$.

First, we survey some previous results which verify the existence of a second Hamilton cycle if d is large enough. We will then demonstrate some techniques for finding a second Hamilton cycle that will be used throughout this paper. Finally, we use these techniques and show that for certain 4-regular Hamiltonian graphs whose automorphism group is large enough, a second Hamilton cycle exists.

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I would also like to humbly thank both Carleton University and the University of Ottawa for their financial assistance.

Dedication

This work is dedicated to the memory of Paul Bestfather, who passed away earlier this year. It is unlikely that I would be here today without his guidance and wisdom. May he be remembered always.

Contents

Chapter 1

Introduction

1.1 Definitions

For terms not defined here, the reader is referred to [1].

A *graph* consists of a set V of objects, called *vertices*, and a set E of *edges*. Since we will only consider *simple, undirected* graphs, we have $E \subseteq \binom{V}{2}$ — that is, E is a subset of the set of unordered pairs of vertices. We insist that V be non-empty, and that V and E are disjoint. We say that the *order of a graph* is $|V|$. If E is empty, then we call the graph X *edgeless*.

Let $X = (V, E)$ be a graph. Let $u, v \in V$, and suppose $e = \{u, v\} \in E$ is an edge. For convenience, we will write uv instead of $\{u, v\}$. We call u and v *adjacent*, denoted $u \sim_X v$, if uv is an edge in X. When it is understood to which graph we are referring, we may write $u \sim v$ instead. Conversely, we say that the edge uv is *incident to* u and to v. We also say that u and v are *neighbours in* X, that u and v are the *ends* of edge uv, and that u and v are *joined* by the edge uv. If $vw \in E$ for some $w \in V$, then we sometimes say that edges uv and vw are *adjacent.*

A *path* P in X is a sequence $(v_0, v_1, \ldots, v_k)$ of distinct vertices such that $v_i \sim v_{i+1}$, for each $0 \leq i \leq k-1$. The *initial vertex* of P is v_0, and the *terminal vertex* of P

is v_k. If v_0 is the initial vertex of P, then we may call P a *v_0-path*; if we also have that v_k is the terminal vertex of P, then we may call P a v_0v_k-path. The *length* of P is one less than the number of vertices in it: in this case, k. We may call a path of length k a *k-path*. At times, it will be useful to perform a *path concatenation* on two or more paths: suppose $P_1 = (v_0, v_1, \dots, v_k)$ and $P_2 = (v_k, v_{k+1}, \dots, v_{k+l})$. Their concatenation, denoted P_1P_2, is $(v_0, v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l})$. Note that we require the terminal vertex of P_1 to be the same as the initial vertex of P_2. To concatenate more than two paths $P_1, P_2, \dots P_t$, this process is performed recursively on $P_1P_2 \dots P_{t-1}$ and P_t.

A *walk* W in X is a sequence $(v_0, v_1, \dots, v_k)$ of adjacent vertices of X that need not be distinct. In all other respects, a walk behaves as a path (and the terms are defined analogously). Additionally, a if X has a v_0v_k-walk, then X also has a v_0v_k-path ([1]).

A *cycle* C in X is a sequence of vertices $(v_0, v_1, \dots, v_k)$ that are pairwise distinct except that $v_0 = v_k$. In addition, we require that $v_i \sim v_{i+1}$ for each $0 \leq i \leq k-1$. The *length* of a cycle is given by the number of distinct vertices (or alternatively, edges) it contains: in this case, it is k. If k is even, then we say that C is an *even cycle*; analogously, if k is odd, we call C *odd*. A *Hamilton cycle* in a graph $X = (V, E)$ is a cycle of length $|V|$. If X has a Hamilton cycle, we call X *hamiltonian*. Furthermore, we call X *uniquely hamiltonian* if X has exactly one Hamilton cycle.

We call the graph X *bipartite* (with bipartition (A, B)), and denote it by $X[A, B]$, if we can partition $V(X)$ into disjoint sets A and B such that $X[A]$ and $X[B]$ are both edgeless.

Suppose $Y = (V', E')$ is a graph. We say that Y is a *subgraph* of $X = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We call Y a *spanning subgraph of* X if $V' = V$. For a subset $U \subseteq V$, the *subgraph of* X *induced by* U, denoted $X[U]$, has vertex set U along with all edges of E that are disjoint with $V \setminus U$. We can also take disjoint sets $A, B \subseteq V$ and induce a bipartite graph $X[A, B]$ by including all edges of E that have exactly one end in A and one end in B.

Let $X = (V, E)$ and $X' = (V', E')$. We can take the union of these two graphs by letting $X \cup X' = (V \cup V', E \cup E')$. For a set of edges $E'' \subseteq \{uv : u, v \in V(X), u \neq v\} \setminus E(X)$, we also define $X + E'' = (V, E \cup E'')$. Similarly, if instead $E'' \subseteq E$, then we can define $X - E'' = (V, E \setminus E'')$. If Y is a subgraph of X, then we can write $X - Y$ to mean the same thing as $X - E(Y)$.

A graph X is *connected* if, for any two vertices $u, v \in V(X)$, there exists a walk in X from u to v. In general, we are most interested in connected graphs, for only they can admit a Hamilton cycle. However, we sometimes investigate subgraphs which may not be connected.

The *degree* of a vertex is the number of edges incident to it. For $v \in V$, this number is denoted by $\deg(v)$. The *neighbourhood* of v in X is the set of vertices adjacent to v, and is denoted $N_X(v)$. We can also define the neighbourhood of a set U of vertices similarly: $N_X(U)$ is the set of all vertices adjacent to a vertex in U that are not themselves in U. If every vertex of X has the same degree d, then we say that X is *regular of degree d*, or simply *d-regular*. A d-regular spanning subgraph of X is called a *d-factor*.

A *matching* M in X is a subset of $E(X)$ such that every vertex of V is incident with at most one edge of M. A *maximum matching* is one of greatest size among all matchings in X. If $|M| = \frac{|V|}{2}$, then M is a *perfect matching*. Note that if X is the vertex-disjoint union of even cycles (that is, if X is 2-regular), then a perfect matching can be made by choosing alternating edges in each cycle. If a vertex $v \in V(X)$ is incident to an edge of M, then we say that M *saturates* v. Note that a perfect matching is similar to a 1-factor, except that the former is a set of edges in X, whereas the latter is a subgraph of X.

We present a statement of Hall's Marriage Theorem, which gives a necessary and sufficient condition for a bipartite graph to have a perfect matching. The corollary that follows will be useful in many proofs throughout this thesis.

Theorem 1.1.1. (Hall's Marriage Theorem [4]) *Let $X[A, B]$ be a bipartite graph. Then X has a matching that saturates every vertex of A if and only if, for any subset $S \subseteq A$, we have that $|S| \leq |N(S)|$.*

Corollary 1.1.2. *Let $X[A, B]$ be a bipartite k-regular graph, for $k \geq 1$. Then X has a perfect matching.*

Proof: Let $S \subseteq A$. Assume, for the sake of finding a contradiction, that $|N(S)| < |S|$. Let X' be the bipartite subgraph of X induced by the bipartition $[S, N(S)]$, and consider the degrees of the vertices of $N(S)$ in X'. Since each vertex of S has degree k (both in X and in X'), the sum of the degrees of vertices in $N(S)$ must be $k|S|$. However, since $|N(S)| < |S|$, this degree sum implies that the average degree of vertices in $N(S)$ is greater than k. Since no vertex of X' can have degree greater than k, this is a contradiction.

Therefore, we have $|S| \leq |N(S)|$. Then X has a matching M that saturates each vertex of A by Theorem 1.1.1. Suppose, however, that M does not saturate every vertex of B: then $|B| > |A|$. Since X is bipartite, the number of edges with one end in A is equal to the number of edges with one end in B. Since X is k-regular, this number is $k|A| = k|B|$. Therefore, we have $|A| = |B|$, a contradiction. Hence M is a perfect matching in X. ▮

An *automorphism* of a graph $X = (V, E)$ is a bijection $\sigma : V \to V$ which preserves adjacency and non-adjacency. That is, for any $u, v \in V$, we have $u \sim v \Leftrightarrow \sigma(u) \sim \sigma(v)$. Further, we call vertices u and v *similar* in X if there exists an automorphism of X that maps u to v. If all vertices of V are pairwise similar, then X is *vertex-transitive.* The set of all automorphisms of X forms a group; this *automorphism group* is denoted $\mathrm{Aut}(X)$. (For more information on groups, see *e.g.* [7].)

Definition 1.1.3. A *circulant graph* (usually referred to simply as a *circulant*) of order n has vertex set $\mathbb{Z}_n$. Its edges are given by a non-empty connection set $L \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ as follows: vertices $i, j \in \mathbb{Z}_n$ are adjacent if and only if there exists $l \in L$ such that $i - j \equiv l \pmod{n}$ or $j - i \equiv l \pmod{n}$. For example, a cycle on n vertices is isomorphic to a circulant of order n with connection set $L = \{1\}$. We denote circulants by $\mathcal{C}(n; L)$ and the values in L are referred to as *edge lengths*.

One additional important property of circulants is that if we let $\rho = (0\ 1\ 2 \dots\ n-1)$, then ρ is always an automorphism of $\mathcal{C}(n; L)$. This is because for vertices $i, j \in \mathbb{Z}_n$, there exists $l \in L$ such that $i - j \equiv l \pmod{n}$ or $j - i \equiv l \pmod{n}$ if and only if, for the same $l \in L$, we have $(i + 1) - (j + 1) \equiv l \pmod{n}$ or $(j + 1) - (i + 1) \equiv l \pmod{n}$. (Note that $i + 1 = \rho(i)$ and $j + 1 = \rho(j)$.)

Definition 1.1.4. A *bicirculant graph* (or simply *bicirculant*) of order $2n$ has vertex set $A \cup B$ in which A and B are disjoint copies of $\mathbb{Z}_n$. To disambiguate these two sets, let $0_A, 1_A, \dots, (n-1)_A$ represent the vertices of A and $0_B, 1_B, \dots, (n-1)_B$ represent the vertices of B. The edges of a bicirculant are given by a non-empty difference set $D \subseteq \{0, \dots, n-1\}$, whose values are called *edge differences*. Vertices a_A, b_B are adjacent if and only if there exists $d \in D$ such that $b_B - a_A \equiv d \pmod{n}$. We call such a bicirculant $\mathcal{B}(2n; D)$.

1.2 History of the Problem

Problems involving Hamilton cycles seem to be ubiquitous in discussions of graph theory. This is probably because it is not obvious how to succinctly characterize hamiltonian graphs, so there are many unanswered questions. It is known that the problem of determining whether a graph is hamiltonian is NP-complete.

On the subject of finding a *second* Hamilton cycle in a graph which is already known to have one, there is much yet to be explored. The earliest result of interest to us is attributed to Cedric Smith, although it was first published in 1946 in a paper of William Tutte's [16]. It concerns the number of Hamilton cycles in a cubic graph, asserting that no cubic graph can have exactly one.

It is unclear whether the problem of a second Hamilton cycle was uninteresting or merely resistant to proof (though we humbly submit that it was the latter!). It was almost 30 years later when John Sheehan conjectured [11] that every d-regular hamiltonian graph, for $d \geq 3$, has a second Hamilton cycle. Smith's result confirmed the conjecture when $d = 3$, but not much more was known at that point.

It was not much longer before Andrew Thomason produced a lemma [13] which not only gives an elegant proof of Smith's result, but extends it to any regular graph of odd degree. Due to the following theorem of Peteresen's [8], the problem for even values of d can be reduced to the problem for $d - 2$ (see Corollary 1.2.2 below). This implies that Sheehan's conjecture will hold if the case when $d = 4$ is verified.

Theorem 1.2.1. (Petersen [8]) *Let X be a $2k$-regular graph. Then X can be decomposed into edge-disjoint 2-factors.*

Our proof will require the notion of an *Euler tour*. An Euler tour in a graph X is a sequence of adjacent vertices such that each edge of X is used exactly once (inasmuch as an edge is implied to be used to get from one vertex to its successor). Furthermore, an Euler tour starts and ends with the same vertex. It should be noted that a connected graph always has an Euler tour as long as every vertex in that graph has even degree (see [1]).

Proof (of Theorem 1.2.1): Let $X = (V, E)$ be a $2k$-regular graph of order n. Without loss of generality, assume X is connected, since we can apply this argument to every component of X. Since every vertex of $V(X)$ has even degree, there exists an Euler tour $C = (v_0, v_1, \ldots, v_0)$.

We construct a bipartite auxiliary graph $X'[A, B]$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$. Vertices a_i, b_j are adjacent in X' if and only if v_i immediately precedes v_j in C.

Now, as every v_i precedes another vertex exactly k times in C, we have that $a_0, \ldots, a_{n-1}$ each have degree k. Similarly, every v_i follows another vertex exactly k times in C, so $b_0, \ldots, b_{n-1}$ each have degree k as well. Therefore, X' is a k-regular bipartite graph. Repeated application of Corollary 1.1.2 implies that X' can be decomposed into edge-disjoint 1-factors.

Let F be a 1-factor of X'. The edges of X' are in a natural one-to-one correspondence with the edges of X, since each edge in $E(X')$ is given by an edge of $E(C) = E(X)$ and vice-versa. Each a_i and b_i are incident to one edge of F, implying that each v_i is incident to two edges in the subgraph of X corresponding to F. Therefore, F corresponds to a 2-factor in X. Therefore, the edge-disjoint 1-factors comprising X' correspond to a decomposition of X into 2-factors. ∎

Corollary 1.2.2. [1] *Let X be a $2k$-regular hamiltonian graph, for some $k \geq 3$, with a Hamilton cycle C. Suppose that every 4-regular hamiltonian graph has a second Hamilton cycle. Then X has a second Hamilton cycle.*

Proof: Let $k \geq 3$, and $X = (V, E)$ be a $2k$-regular graph with a Hamilton cycle C. Since C is a 2-factor of X, the graph $X' = (V, E \setminus E(C))$ is $2k-2$-regular. Therefore, we may apply Theorem 1.2.1 in order to find a decomposition of X' into 2-factors. Let F be one of these 2-factors. Then $F \cup C$ is a 4-regular hamiltonian graph. By our hypothesis, it has a second Hamilton cycle. Therefore, X also has a second Hamilton cycle. ∎

In 1997, Carsten Thomassen [14] introduced the notion of red-independent green-dominating sets, which changed the complexion of the problem. In a subsequent paper [15], he used his idea to prove Sheehan's Conjecture for $d \geq 300$. He combined the idea of red-independent green-dominating sets with the use of the probabilistic method, which hitherto had not been applied to the problem.

The bound on d was reduced several times, until 2007, when Haxell, Seamone, and Verstraete [5] improved the bound to $d \geq 23$, where it has remained since.

We conclude by noting that although these results of Smith and Thomason hold for multigraphs, Sheehan's Conjecture does not. A result of Fleischner ([3]) shows that not only do there exist infinitely many 4-regular uniquely hamiltonian multigraphs, but that such graphs need not have loops.

Chapter 2

Tools and Technical Lemmas

2.1 Probability Theory

The definitions and lemmas in this section will be useful when we introduce the technique known as the probabilistic method in Chapter 3.4.

Definition 2.1.1. [12] A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ in which Ω is called the *sample space*; we have that $\mathcal{F}$, the *set of events*, is a collection of subsets of Ω; and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a *probability measure*. For our purposes, we will insist that $\mathcal{F}$ is what we call a *σ-field*: that is, $\mathcal{F}$ is closed under taking complements and countable union, and $\Omega \in \mathcal{F}$.

Furthermore, for $\mathbb{P}$ to be a probability measure on $\mathcal{F}$, it must satisfy the following properties:

- $\mathbb{P}(\Omega) = 1$;
- Let $A_i \in \mathcal{F}$ for $i \geq 1$ be a collection of pairwise-disjoint subsets of Ω. Then

$$\mathbb{P}(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} \mathbb{P}(A_i).$$

This property is known as *countable additivity*.

Singletons in $\mathcal{F}$ are called the *elementary events*. Events A and B are *independent* if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

First, we will set up the probability space that we will be using throughout later proofs. The following lemma defines this probability space and demonstrates that it is indeed a probability space.

Lemma 2.1.2. *Let V be a ground set and $p \in (0,1)$. Define a triple $(\Omega, \mathcal{F}, \mathbb{P})$ as follows:*

- $\Omega = 2^V$;
- $\mathcal{F} = 2^\Omega$;
- $\mathbb{P}$ *is defined so that for any elementary event* $\{S\}$ *in* $\mathcal{F}$, *we have* $\mathbb{P}(\{S\}) = p^{|S|}(1-p)^{|V|-|S|}$, *and* $\mathbb{P}$ *satisfies the countable additivity property.*

Then $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Proof: In order to show that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, we need to show that $\mathbb{P}$ truly is a probability measure. According to Definition 2.1.1 we must show that the probability of any event lies between 0 and 1, and that $\mathbb{P}(\Omega) = 1$. Now,

$$\begin{aligned}
\mathbb{P}(\Omega) &= \mathbb{P}(\bigcup_{k=0}^{|V|}\{k\text{-element subsets of } V\}) \\
&= \sum_{k=0}^{|V|} \mathbb{P}(\{k\text{-element subsets of } V\}) \\
&= \sum_{k=0}^{|V|} \binom{|V|}{k} p^k (1-p)^{|V|-k} \\
&= (p + 1 - p)^{|V|} \\
&= 1.
\end{aligned}$$

Since $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}$ has the countable additivity property, it must be the case that all probabilities lie between 0 and 1. It should be clear that all probabilities are at least 0, since the probability of any event is the sum of the probabilities of its aggregate elementary events, which all have probability at least 0. However, suppose there exists some event $S \in \mathcal{F}$ with $\mathbb{P}(S) > 1$. Then, due to countable additivity, $\mathbb{P}(\Omega) = \mathbb{P}(S \cup \overline{S}) > 1 + \mathbb{P}(\overline{S}) > 1$, a contradiction.

Therefore, $\mathbb{P}$ is a probability measure. Hence $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. ∎

There are several times at which it will be convenient to set up the above probability space using the set of vertices of a graph as a ground set. The following definition of "bad events" will also be used.

Definition 2.1.3. Let $X = (V, E)$ be a d-regular graph and suppose C is a Hamilton cycle of X. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space as defined in Lemma 2.1.2. Colour the edges of C red and all other edges of X green. Define the following events, referred to collectively as *bad events*:

- For each $e \in E(C)$, define the *edge-event* $A_e = \{S \subseteq V \mid \text{both ends of } e \text{ are in } S\}$;
- For each $v \in V$, define the *vertex-event* $B_v = \{S \subseteq V \mid v \notin S, v \text{ has a neighbour in } S \text{ via a red edge}, v \text{ has no neighbours in } S \text{ via a green edge}\}$;
- Define $D = \{\emptyset\}$.

Having now defined our bad events, we wish to compute their probabilities under $\mathbb{P}$.

Lemma 2.1.4. *Let X be a d-regular hamiltonian graph with a Hamilton cycle C, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space; let each be defined as above in Definition 2.1.3.*

Then, for any $e \in E(C)$, $v \in V$, we have the following probabilities:

1. $\mathbb{P}(A_e) = p^2$*;*

2. $\mathbb{P}(B_v) = (1-(1-p)^2)(1-p)^{d-1}$*;*

3. $\mathbb{P}(D) = (1-p)^n$*.*

Proof:

1. Let $e = xy \in E(C)$. Then, by enumerating all subsets of V which contain x and y, and using countable additivity, we have

$$\begin{aligned}\mathbb{P}(\{S \mid x, y \in S\}) &= \sum_{i=0}^{n-2} \mathbb{P}(\{S \mid x, y \in S, |S| = i+2\})\\ &= \sum_{i=0}^{n-2} \binom{n-2}{i} p^{i+2}(1-p)^{n-2-i}\\ &= p^2 \cdot 1^{n-2}\\ &= p^2.\end{aligned}$$

2. Let $v \in V(X)$. Let $Q = \{u \mid u = v \text{ or } u \text{ is adjacent to } v \text{ via a green edge}\}$ and R be the pair of vertices adjacent to v via a red edge. Note that $|Q| = d-1$ and $|R| = 2$. Then

$$\begin{aligned}\mathbb{P}(B_v) &= \mathbb{P}(\{S \mid Q \cap S = \emptyset \text{ and } R \cap S \neq \emptyset\})\\ &= \sum_{i=1}^{2} \sum_{j=0}^{n-d-1} \mathbb{P}(\{S \mid S \text{ has } i \text{ elements of } R \text{ and } j \text{ elements of } V \setminus (Q \cup R)\})\\ &= \sum_{i=1}^{2} \sum_{j=0}^{n-d-1} \binom{2}{i}\binom{n-d-1}{j} p^{i+j}(1-p)^{n-i-j}\\ &= \left(\sum_{i=1}^{2} \binom{2}{i} p^i (1-p)^{2-i}\right)\left(\sum_{j=0}^{n-d-1} \binom{n-d-1}{j} p^j (1-p)^{n-d-1-j}(1-p)^{d-1}\right)\end{aligned}$$

Applying the Binomial Theorem, we get

$$\mathbb{P}(B_v) = (1-(1-p)^2) \cdot 1 \cdot (1-p)^{d-1} = (1-(1-p)^2)(1-p)^{d-1}.$$

3. Since D is an elementary event, we can compute $\mathbb{P}(D) = (1-p)^n$ from the definition of $\mathbb{P}$ in Lemma 2.1.2. ∎

Finally, we use the following lemma to determine which of these bad events are independent with each other.

Lemma 2.1.5. *Let X be a d-regular hamiltonian graph with a Hamilton cycle C, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space; let each be defined as above in Definition 2.1.3.*

Let A_{uv}, A_{wx}, B_y, B_z be bad events, for some $u, v, w, x, y, z \in V(X)$. Then the following hold.

1. *If u, v, w, and x are pairwise distinct, then A_{uv} and A_{wx} are independent events;*
2. *If u and v are distinct from y, and neither is in $N_X(y)$, then A_{uv} and B_y are independent events; and*
3. *If $y \neq z$, $y \not\sim z$, and $N_X(y) \cap N_X(z) = \emptyset$, then B_y and B_z are independent events.*

Proof:

1. Suppose u, v, w, and x are pairwise distinct. We show that A_{uv} and A_{wx} are independent:

$$\begin{aligned}
\mathbb{P}(A_{uv} \cap A_{wx}) &= \mathbb{P}(\{S \mid u, v, w, x \in S\}) \\
&= \sum_{i=0}^{n-4} \mathbb{P}(\{S \mid |S| = i+4 \text{ and } u, v, w, x \in S\}) \\
&= \sum_{i=0}^{n-4} \binom{n-4}{i} p^{i+4} (1-p)^{n-4-i}
\end{aligned}$$

Applying the Binomial Theorem, we get

$$\mathbb{P}(A_{uv} \cap A_{wx}) = p^4 \cdot 1$$

$$= \mathbb{P}(A_{uv})\mathbb{P}(A_{wx}). \qquad \text{(By Lemma 2.1.4)}$$

2. Now suppose that $u, v \neq y$ and $u, v \notin N_X(y)$. We show that A_{uv} and B_y are independent. Let $Q = \{a \in V \mid a = y \text{ or } a \text{ is adjacent to } y \text{ via a green edge}\}$ and R be the pair of vertices adjacent to y via a red edge. Note that $|Q| = d-1$ and $|R| = 2$.

$$\begin{aligned}
&\mathbb{P}(A_{uv} \cap B_y)\\
&= \mathbb{P}(\{S \mid u, v \in S, Q \cap S = \emptyset, R \cap S \neq \emptyset\})\\
&= \sum_{i=1}^{2} \sum_{j=0}^{n-d-3} \mathbb{P}(\{S \mid u, v \in S, S \text{ has } i \text{ elements of } R\\
&\qquad\qquad \text{and } j \text{ elements of } V \setminus (Q \cup R \cup \{u, v\})\})\\
&= \sum_{i=1}^{2} \sum_{j=0}^{n-d-3} \binom{2}{i}\binom{n-d-3}{j} p^{i+j+2}(1-p)^{n-i-j-2}\\
&= p^2(1-p)^{d-1}\left(\sum_{i=1}^{2}\binom{2}{i}p^i(1-p)^{2-i}\right)\left(\sum_{j=0}^{n-d-3}\binom{n-d-3}{j}p^j(1-p)^{n-d-3-j}\right)
\end{aligned}$$

Applying the Binomial Theorem, we get

$$\begin{aligned}
\mathbb{P}(A_{uv} \cap B_y) &= p^2(1-p)^{d-1}(1-(1-p)^2) \cdot 1\\
&= \mathbb{P}(A_{uv})\mathbb{P}(B_y). \qquad \text{(By Lemma 2.1.4)}
\end{aligned}$$

3. Finally, suppose $y \neq z$, $y \not\sim z$, and $N_X(y) \cap N_X(z) = \emptyset$. We show that B_y and B_z are independent. Let Q, R be defined as above for y, with Q' and R' defined analogously for z. Since Q, Q', R, and R' are pairwise disjoint, we have $|Q \cup Q' \cup R \cup R'| = 2d + 2$. Then

$$\begin{aligned}
\mathbb{P}(B_y \cap B_z) &= \mathbb{P}(\{S \mid (Q \cup Q') \cap S = \emptyset, R \cap S \neq \emptyset, R' \cap S \neq \emptyset\})\\
&= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=0}^{n-2d-2} \mathbb{P}(\{S \mid S \text{ has } i \text{ elements of } R,\ j \text{ elements of } R',
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\text{and } k \text{ elements of } V \setminus (Q \cup Q' \cup R \cup R')\}) \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=0}^{n-2d-2}\binom{2}{i}\binom{2}{j}\binom{n-2d-2}{k}p^{i+j+k}(1-p)^{n-i-j-k} \\
&= (1-p)^{2d-2}\left(\sum_{i=1}^{2}\binom{2}{i}p^{i}(1-p)^{2-i}\right)\left(\sum_{j=1}^{2}\binom{2}{j}p^{j}(1-p)^{2-j}\right)\cdot \\
&\quad \left(\sum_{k=0}^{n-2d-2}\binom{n-2d-2}{k}p^{k}(1-p)^{n-2d-2-k}\right)
\end{aligned}
$$

Applying the Binomial Theorem yields

$$
\begin{aligned}
\mathbb{P}(B_y \cap B_z) &= (1-p)^{2d-2}(1-(1-p)^2)(1-(1-p)^2) \\
&= \mathbb{P}(B_y)\mathbb{P}(B_z). \qquad \text{(By Lemma 2.1.4)}
\end{aligned}
$$

∎

2.2 Circulants

Since the graphs in which we are interested will have a lot of symmetry, the situation will arise quite often where we have a circulant graph as a subgraph. The following five lemmas work together to show when a circulant has a perfect matching (if the order is even) or a Hamilton cycle (if the order is odd), which will be used in Chapter 4.

Lemma 2.2.1. *Let $X = \mathcal{C}(m; L)$ be a circulant graph of order $m \geq 2$. Suppose $m = 2^k m'$, where $k \geq 1$ and m' is odd. If there exists $l \in L$ such that $2^k \nmid l$, then X has a perfect matching.*

Proof: Let F_l denote the spanning subgraph of X that contains exactly the edges of length l. If $l = \frac{m}{2}$, then F_l is a 1-factor, whose edge set is a perfect matching in X. If not, then F_l is a 2-factor. In fact, we note that F_l is a disjoint union of cycles of length $\frac{m}{\mathrm{GCD}(m,l)}$, which is odd if and only if 2^k divides l. Since 2^k does not divide l, these are even cycles, from which a perfect matching for X can be extracted. ∎

Lemma 2.2.2. [10] *Let $X = \mathcal{C}(m; L)$ be a circulant graph of order $m \geq 2$ and degree $2c$. Suppose $m = 2^k m'$, where $k \geq 1$ and m' is odd. If $2c > m' - 1$, then X has a perfect matching.*

Proof: Suppose 2^k divides each edge length in L. Then $L \subseteq \{\alpha 2^k \colon 1 \leq \alpha \leq \frac{m'-1}{2}\}$, so $c = |L| \leq \frac{m'-1}{2}$. But $c > \frac{m'-1}{2}$ was given, a contradiction. Therefore, one of the edge lengths $l \in L$ must not be divisible by 2^k. By Lemma 2.2.1, there is a perfect matching in X. ∎

Lemma 2.2.3 ([10]). *Let $X = \mathcal{C}(m; L)$ be a circulant graph of order m. Suppose $L = \{l_1, l_2, \ldots, l_c\}$. Then X is connected if and only if GCD$(m, l_1, l_2, \ldots, l_c) = 1$.*

Proof: Let X be a circulant graph have vertex set $\mathbb{Z}_m$. Suppose X is connected. Then there exists a 01-path P. Suppose P utilizes α_1 edges of length l_1, α_2 edges of length l_2, and so on, up to α_c edges of length l_c. Then $0 + \alpha_1 l_1 + \alpha_2 l_2 + \ldots + \alpha_c l_c \equiv 1 \pmod{m}$. This implies that $\alpha_1 l_1 + \alpha_2 l_2 + \ldots + \alpha_c l_c + tm = 1$ for some $t \in \mathbb{Z}$. If $d := \mathrm{GCD}(m, l_1, l_2, \ldots, l_c) > 1$, then we should be able to factor out d from both sides of the preceding equation. We cannot; therefore, $\mathrm{GCD}(m, l_1, l_2, \ldots, l_c) = 1$.

Conversely, suppose $\mathrm{GCD}(m, l_1, l_2, \ldots, l_c) = 1$. Then there exist integers $t, \alpha_1, \alpha_2, \ldots, \alpha_c$ such that $1 = \alpha_1 l_1 + \alpha_2 l_2 + \ldots + \alpha_c l_c + tm$ (see *e.g.* [9]). Therefore, $1 \equiv \alpha_1 l_1 + \alpha_2 l_2 + \ldots + \alpha_c l_c \pmod{m}$. This equality implies that there is a 01-walk W in X. To show that X is connected, let $i, j \in \mathbb{Z}_m$. Since $\rho = (0\ 1\ 2\ \ldots\ m-1)$ is an automorphism of X, we can get an $i(i+1)$-walk by applying ρ^i to W. By concatenating the walks $\rho^i(W)\rho^{i+1}(W)\ldots\rho^{j-1}(W)$, we can get an ij-walk in X. This demonstrates that X is connected. ∎

The following result has been previously proven by several authors (e.g. [6]), but we present an original proof that is more suitable in this context.

Lemma 2.2.4. *Let X be a connected circulant graph of order $m \geq 3$. Then X is hamiltonian.*

Proof: Let X have vertex set $\mathbb{Z}_m$ and connection set $L \subseteq \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$. We give a construction for a Hamilton cycle in X using induction on $|L|$.

If $|L| = 1$, then let $L = \{l_1\}$. By Lemma 2.2.3, $\text{GCD}(l_1, m) = 1$, so l_1 is in fact a generator of $\mathbb{Z}_m$. Hence $(0, l_1, 2l_1, \ldots, (m-1)l_1, 0)$ is a Hamilton cycle.

Our induction hypothesis is as follows: for some fixed $k \geq 1$, if $L = \{l_1, l_2, \ldots, l_k\}$ generates a connected circulant X, then X is hamiltonian.

If $|L| = k + 1 \geq 2$, then let $L = \{l_1, l_2, \ldots, l_{k+1}\}$. Let $d = \text{GCD}(l_1, l_2, \ldots, l_k, m)$. If $d = 1$, then Lemma 2.2.3 implies that the circulant with edge lengths in $L - \{l_{k+1}\}$ is connected, and our induction hypothesis gives a Hamilton cycle in that circulant, and consequently, in X. Otherwise, without loss of generality, we may assume that $d < \frac{m}{2}$: if $\frac{m}{2} \in L$, then we may assume that $l_{k+1} = \frac{m}{2}$. Let $S = \{\frac{l_1}{d}, \frac{l_2}{d}, \ldots, \frac{l_k}{d}\}$, and let Y be a circulant with vertex set $\mathbb{Z}_{\frac{m}{d}}$ and edge length set S. We may apply Lemma 2.2.3 to get that Y is connected.

Since Y is a connected circulant with k edge lengths and Y has order $\frac{m}{d} \geq 3$, we may apply the induction hypothesis to find a Hamilton cycle C for Y. This corresponds to a cycle C_0 in X once we multiply each of its vertices by d. This cycle C_0 includes precisely every vertex of X that is congruent to 0 modulo d. Since X is a circulant, it admits $\rho = (0\ 1\ 2\ \ldots\ m-1)$ as an automorphism, so $C_1 = \rho^{l_{k+1}}(C_0)$, $C_2 = \rho^{2l_{k+1}}(C_0)$, $\ldots, C_{d-1} = \rho^{(d-1)l_{k+1}}(C_0)$ are cycles in X. We note that l_{k+1} and d are coprime, for if not, then suppose some $e > 1$ divides them both. But then e

divides $l_1, \dots, l_k, l_{k+1}$, and m. By Lemma 2.2.3, this means that X is not connected, a contradiction. Therefore, each cycle C_i contains all the vertices of X which are congruent to il_{k+1} modulo d . This fact implies that the cycles are pairwise vertex-disjoint and every vertex of X is in some C_i.

We will now show that these vertex-disjoint cycles can be used to construct a Hamilton cycle in X. For each $0 \leq i \leq d-1$, let $C_i = (v_0^i,\ v_1^i, \dots,\ v_{\frac{m}{d}-1}^i, v_0^i)$. Specifically, we arbitrarily label successive vertices of C_0 as v_0^0, v_1^0, and so on; then we let $v_0^i = \rho^{il_{k+1}}(v_0^0)$, $v_1^i = \rho^{il_{k+1}}(v_1^0)$, and so on. Note that because $v_j^{i+1} = \rho^{l_{k+1}}(v_j^i)$ and two vertices that have difference l_{k+1} are adjacent, this labeling gives that $v_j^{i+1} \sim v_j^i$, for every $0 \leq i \leq d-1$ and $0 \leq j \leq \frac{m}{d} - 1$.

We will combine these cycles iteratively. We first join C_1 to C_0 by taking the edges of each of them, discarding $v_0^0 v_1^0$ and $v_0^1 v_1^1$, and including the edges $v_0^0 v_0^1$ and $v_1^0 v_1^1$. The new cycle, then, is $(v_0^0,\ v_0^1,\ v_{\frac{m}{d}-1}^1, \dots,\ v_1^1,\ v_1^0,\ v_2^0, \dots,\ v_{\frac{m}{d}-1}^0,\ v_0^0)$. Call this new cycle C_1'. Observe that $v_1^1 v_2^1$ is an edge of C_1'.

When $1 \leq i \leq d-2$, we may join C_{i+1} to C_i' in a very similar fashion. We need only ensure that we are not discarding (or adding) edges that were discarded in the previous iteration. However, this is easy to avoid, since C_i has at least 3 edges to choose from. Hence, we may take the edges of C_i' and C_{i+1}, discard $v_i^i v_{i+1}^i$ and $v_i^{i+1} v_{i+1}^{i+1}$, and include edges $v_i^i v_i^{i+1}$ and $v_{i+1}^i v_{i+1}^{i+1}$. (Here, vertex subscripts are evaluated modulo $\frac{m}{d}$.) Call this resulting cycle C_{i+1}'. Note that C_{i+1}' visits all vertices of C_i' and includes all vertices of C_{i+1} as well. In addition, we have that $v_{i+1}^{i+1} v_{i+2}^{i+1}$ is an edge in C_{i+1}', and thus is eligible to be discarded in the next iteration. Therefore, at the end of this process, we shall have a cycle C_{d-1}' which visits all vertices of $C_0, \dots, C_{d-1}$; that is, all the vertices of X.

Therefore, C_{d-1}' is a Hamilton cycle in X, and the result holds by induction. ∎

Lemma 2.2.5. [10] *Let X be a circulant graph of order $m \geq 3$ and degree $2c$. Denote* *by p the smallest prime divisor of m. If $2c \geq \frac{m}{p}$, then X is hamiltonian.*

Proof: Let $L = \{l_1, \ldots, l_c\}$ be the set of edge lengths in X. Let $d = \mathrm{GCD}(m, l_1, \ldots, l_c)$, and suppose $d > 1$. Then $L \subseteq \{\alpha d\colon\ 1 \leq \alpha \leq \frac{m}{2d} - 1\}$. (Here, we cannot have $\frac{m}{2}$ as an edge length, for if we did, the degree of X would be odd.) Hence $c = |L| \leq \frac{m}{2d} - 1$. However, $c \geq \frac{m}{2p} \geq \frac{m}{2d}$ was given, a contradiction. Therefore, $d = 1$, so X is connected by Lemma 2.2.3, and hence X is hamiltonian by Lemma 2.2.4. ∎

2.3 Bicirculants

As with circulants, we often find bicirculants as subgraphs of our graphs. There are times when it is useful to know when we can find a Hamilton cycle in such a subgraph.

Proposition 2.3.1. *Let $n \geq 2$ be an integer, and suppose $X = \mathcal{B}(2n; D)$ is a bicirculant graph with (non-empty) difference set D. Further, suppose that $V(X)$ is partitioned into disjoint copies of $\mathbb{Z}_n$, denoted by A and B. Then the following hold.*

1. *Let $\rho = (0_A\ 0_B\ 1_A\ 1_B \ldots (n-1)_A\ (n-1)_B)$ be a permutation on $A \cup B$. Then X admits ρ^2 as an automorphism.*
2. *The bicirculant X admits as an automorphism $\sigma = (0_A\ (n-1)_B)(1_A\ (n-2)_B) \ldots ((n-1)_A\ 0_B)$.*
3. *X is vertex-transitive.*

Proof:

1. Let $a \in A$ and $b \in B$. Suppose $a \sim b$. Then there exists $d \in D$ such that $b - a \equiv d \pmod{n}$. This implies that $\rho^2(b) - \rho^2(a) \equiv (b+1) - (a+1) \pmod{n} \equiv d \pmod{n}$; hence, ρ^2 preserves adjacency.

On the other hand, suppose $a \not\sim b$. Then, for all $d \in D$, we have $b - a \not\equiv d \pmod{n}$. We deduce that $\rho^2(b) - \rho^2(a) \not\equiv d \pmod{n}$ for any $d \in D$, so ρ^2 preserves non-adjacency. Therefore, ρ^2 is an automorphism of X.

2. Let $a \in A$ and $b \in B$. Then $\sigma = (0_A\ (n-1)_B)(1_A\ (n-2)_B)\ldots$ $((n-1)_A\ 0_B)$ is an automorphism of X because

$$\begin{aligned}\sigma(a) - \sigma(b) &= (n - a - 1) - (n - b - 1)\\ &= b - a\end{aligned}$$

Hence $\sigma(b) \sim \sigma(a) \Leftrightarrow a \sim b$, so σ is an automorphism of X.

3. Let ρ and σ be defined as above. The action of ρ^2 implies that all vertices of A are similar to each other, and all vertices of B are similar to each other. The action of σ implies that each vertex of A is similar to a vertex in B (and vice versa). By transitivity, we get that all vertices of A are similar to all vertices of B, so the graph X is vertex-transitive.

∎

Lemma 2.3.2. [10] *Let* $X = \mathcal{B}(2m; \{a, b\})$ *be a bicirculant graph, for* $m \geq 2$ *and* $0 \leq a < b \leq m - 1$. *If GCD*$(m, b-a) = 1$, *then* X *is hamiltonian.*

Proof: Since m and $b - a$ are coprime, we have that their lowest common multiple is $m \cdot (b - a)$. Therefore, the sequence $0, b-a, 2(b-a), \ldots, (m-1)(b-a)$ represent different residue classes modulo m. Similarly, the same sequence with b added to each value also represents the set of all residue classes modulo m. Therefore, we can construct a cycle of length $2m$ by taking

$$C = (0_A, b_B, (b-a)_A, (2b-a)_B, (2b-2a)_A, \ldots, 0_A).$$

∎

Lemma 2.3.3. [10] *Let* $X = \mathcal{B}(2m; D)$ *be a bicirculant graph of degree* $|D| = d$*, for* $m \geq 2$*. Denote by* p *the smallest prime divisor of* m*. If* $d > \frac{m}{p}$*, then* X *is hamiltonian.*

Proof: Let $p_1, \ldots, p_k$ be the distinct prime divisors of m, and suppose, without loss of generality, that $p = p_1$. Since $D \subseteq \{0, 1, \ldots, m-1\}$ and $|D| > \frac{m}{p_1} \geq 1$, we can find distinct $a, b \in D$ such that $|b - a| < p_1$. Then $|b - a| < p_i$, and consequently $a \not\equiv b \pmod{p_i}$, for all $i = 1, \ldots, k$. It follows that $\mathrm{GCD}(m, b - a) = 1$, so Lemma 2.3.2 implies that X is hamiltonian. ∎

2.4 Second Hamilton Cycles

These next few lemmas can be used in a multitude of situations to obtain a second Hamilton cycle from the original one. The following technique is commonly referred to as a *cycle exchange.*

Lemma 2.4.1. [1] *Let* X *be a graph containing a cycle* $C = (v_0, v_1, \ldots, v_{k-1}, v_0)$*. Suppose that* $v_0 \sim v_j$ *and* $v_1 \sim v_{j+1}$*, for some* $j \in \{2, 3, \ldots k-2\}$*. Then* X *contains another cycle,* C'*, which has the same length as* C *and uses all the edges of* C *except for* v_0v_1 *and* v_jv_{j+1}*.*

Proof: The cycle C' is given by

$$C' = (v_0, v_j, v_{j-1}, \ldots, v_1, v_{j+1}, v_{j+2}, \ldots, v_{k-1}, v_0).$$

∎

The following lemma is an improvement on a lemma from [10].

Lemma 2.4.2. *Let X be a graph containing an even cycle $C = (v_0, v_1, \dots, v_{2k-1}, v_0)$. Let $1 \leq s, t \leq k$ be such that $s + t \leq k - 1$. Suppose that we have that $v_0 \sim v_{2s}$, $v_{2t} \sim v_{2(s+t)}$, $v_1 \sim v_{2t+1}$, and $v_{2s+1} \sim v_{2(s+t)+1}$ in X.*

Then X has another cycle, C', which has the same length as C, such that C' contains all edges of the form $v_{2i+1}v_{2(i+1)}$, for $0 \leq i \leq k-1$.

Proof:

Case (1) $s < t$. Then a second cycle C' is given by

$$\begin{aligned} C' = (&v_0, v_{2s}, v_{2s-1}, \dots, v_1, v_{2t+1}, v_{2t+2}, \dots, v_{2(t+s)}, \\ &v_{2t}, v_{2t-1}, \dots, v_{2s+1}, v_{2(s+t)+1}, v_{2(s+t)+2}, \dots, v_0). \end{aligned}$$

The edges of C which are not present in C' are precisely the following: v_0v_1, $v_{2s}v_{2s+1}$, $v_{2t}v_{2t+1}$, and $v_{2(s+t)}v_{2(s+t)+1}$. Therefore, all edges of the form $v_{2i+1}v_{2(i+1)}$, for $0 \leq i \leq k-1$, are in C'.

Case (2) $s = t$. Since $v_0 \sim v_{2s}$ and $v_1 \sim v_{2s+1}$, we may apply Lemma 2.4.1 to obtain a second cycle C' with the same length as C. The only edges of C that are not present in C' are v_0v_1 and $v_{2s}v_{2s+1}$. Therefore, all edges of the form $v_{2i+1}v_{2(i+1)}$, for $0 \leq i \leq k-1$, are in C'.

Case (3) $s > t$. Then a second cycle C' is given by

$$\begin{aligned} C' = (&v_0, v_{2s}, v_{2s-1}, \dots, v_{2t+1}, v_1, v_2, \dots, v_{2t}, \\ &v_{2(s+t)}, v_{2(s+t)-1}, \dots, v_{2s+1}, v_{2(s+t)+1}, v_{2(s+t)+2}, \dots, v_0). \end{aligned}$$

The edges of C which are not present in C' are precisely the following: v_0v_1, $v_{2s}v_{2s+1}$, $v_{2t}v_{2t+1}$, and $v_{2(s+t)}v_{2(s+t)+1}$. Therefore, all edges of the form $v_{2i+1}v_{2(i+1)}$, for $0 \leq i \leq k-1$, are in C'.

∎

Lemma 2.4.3. *Let X be a hamiltonian graph of order αm, where $\alpha \geq 3$. Let $C = (u_0, u_1, \ldots, u_{\alpha m-1}, u_0)$ be a Hamilton cycle of X. Let $\rho = (u_0\ u_1\ u_2 \ldots u_{\alpha m-1}\ u_0)$ be a permutation on $V(X)$, and suppose $\rho^\alpha \in Aut(X)$. Define $V_i = \{u_j : j \equiv i \ (mod\ \alpha)\}$, for $0 \leq i \leq \alpha - 1$. Denote by X_i the induced subgraph $X[V_i]$.*

If, for some $i \in \mathbb{Z}_\alpha$, we have that X_i and X_{i+1} are not edgeless, then X has a second Hamilton cycle.

Proof: Without loss of generality, we may assume that X_0 and X_1 are not edgeless. Colour the edges of C red and all other edges of X green. We construct an auxiliary graph X' in the following manner:

- $V(X') = V_0 \cup V_1$;
- Two vertices of X' are adjacent if the corresponding vertices were adjacent in X. The edge colour is the same as in X;
- Add a red edge joining $u_{\alpha i+1}$ to $u_{\alpha(i+1)}$, for each $0 \leq i \leq m-1$. (These edges correspond to red paths $(u_{\alpha i+1}, u_{\alpha i+2}, \ldots, u_{\alpha(i+1)})$.)

Now, we will relabel the vertices of X' by changing $u_{\alpha i}$ to v_{2i} and $u_{\alpha i+1}$ to v_{2i+1}, for every $0 \leq i \leq m-1$.

Note that $X'[V_0]$ and $X'[V_1]$ are (non-edgeless) circulants such that there exist $s, t \in \{0, 1, \ldots, \lfloor \frac{m}{2} \rfloor\}$ so that $v_{2i} \sim v_{2(i+s)}$ and $v_{2i+1} \sim v_{2(i+t)+1}$, for every i. Furthermore, X' contains a red Hamilton cycle $(v_0, v_1, \ldots, v_{2m-1}, v_0)$.

If $s = t = \lfloor \frac{m}{2} \rfloor$, then since we have $v_0 \sim v_{2s}$ and $v_1 \sim v_{2s+1}$, we may apply Lemma 2.4.1 to find a second Hamilton cycle C'. On the other hand, if we do not have $s = t = \lfloor \frac{m}{2} \rfloor$, then since $s + t \leq m - 1$, we may instead apply Lemma 2.4.2 to obtain a new Hamilton cycle C'. In either case, the new cycle C' contains all edges of the form $v_{2i+1}v_{2(i+1)}$ for $i \in \mathbb{Z}_m$.

C' corresponds to a Hamilton cycle in X as follows. Every edge of the form $v_{2i+1}v_{2(i+1)}$ corresponds to the red path $(u_{\alpha i+1}, u_{\alpha i+2}, \ldots , u_{\alpha(i+1)})$. All the other edges of C' correspond precisely to an edge that is in X. Furthermore, at least one of these edges is guaranteed to be green because C' is different from C, a cycle that uses all the red edges in X'.

Therefore, X has a second Hamilton cycle. ▮

Chapter 3

Methods for Finding a Second Hamilton Cycle

There are three large ideas that will be used to find a second Hamilton cycle throughout this thesis. The first is known as the Lollipop Lemma, published by Andrew Thomason [13] in 1978. It uses the concept of path exchanges, and is essentially an extension of an earlier theorem of Cedric Smith's (published by Tutte [16] in 1946), which, as we will see, states that every edge of a cubic graph is in an even number of Hamilton cycles. Smith's result, though only a special case of the Lollipop Lemma, will prove to be a powerful tool in its own right. Finally, we make use of a result of Carsten Thomassen's [14], whose notion of red-independent green-dominating sets will set the tone for subsequent chapters.

3.1 The Lollipop Lemma

Definition 3.1.1. Let X be a graph. Suppose $P_1 = (v_1, v_2, \dots, v_m)$ is a path in X. Further, suppose there exists a path $P_2 = (v_1, v_2, \dots, v_i, v_m, v_{m-1}, \dots, v_{i+1})$ for some $i \in \{1, \dots, m-2\}$ in X. We say that (P_1, P_2) is a *path exchange*, or P_2 *is related to*

P_1 *by path exchange.* Note that v_m and v_i must be neighbours in X.

We can quickly prove a useful fact about this relation.

Lemma 3.1.2. *The path exchange relation is symmetric, but not reflexive.*

Proof: Suppose (P_1, P_2) is a path exchange, with $P_1 = (v_1, v_2, \dots, v_m)$ and $P_2 = (v_1, v_2, \dots, v_i, v_m, v_{m-1}, \dots, v_{i+1})$.

Now, v_{i+1} is adjacent to v_i, so the path $(v_1, v_2, \dots, v_m)$ is related to P_2 by path exchange. Hence (P_2, P_1) is also a path exchange, so the relation is symmetric.

On the other hand, (P_1, P_1) is not a path exchange. By Definition 3.1.1, we would require $P_1 = (v_1, v_2, \dots, v_j, v_{j+k})$ for some $k \geq 2$, which is not possible. Therefore, the path exchange relation is not reflexive. ∎

Now we are ready to present Thomason's powerful result, for which we offer an alternative proof.

Theorem 3.1.3. (Lollipop Lemma [13]) *Suppose* $X = (V, E)$ *is a graph, and let* $x \in V(X)$. *Then the number of longest* x*-paths which terminate at a vertex of even degree is even.*

Proof: Let $\mathcal{P}$ be the set of longest x-paths, and let $P \in \mathcal{P}$. Denote by v the terminal vertex of P. Now let $\mathcal{F}_P \subset \mathcal{P}$ be the set of paths related to P by path exchange.

Then $|\mathcal{F}_P| = \deg(v) - 1$, and $\displaystyle\sum_{P \in \mathcal{P}} |\mathcal{F}_P|$ counts the number of ordered pairs of paths in $\mathcal{P}$ that are related by path exchange. By Lemma 3.1.2, this sum must be even since, for example, if (P_1, P_2) is counted, then so is (P_2, P_1), and $P_1 \neq P_2$.

Once again fixing path P with terminal vertex v, if v has odd degree, then $|\mathcal{F}_P|$ is even, so its contribution to $\displaystyle\sum_{P \in \mathcal{P}} |\mathcal{F}_P|$ does not change the parity of the sum. On the other hand, if v is of even degree, then its contribution to the sum does change the parity. As the sum is even, there must be an even number of such paths. Hence the number of x-paths whose terminal vertex has even degree is even. ▮

While this seems like a potentially useful result, it may not be immediately clear how to wield it. Our general strategy will go like this. Suppose X is a hamiltonian graph. We will fix a special vertex x, and try to find a subgraph H whose vertices all have odd degree, except that at least one neighbour y of x should have even degree in H. In H, the special vertex x may have even or odd degree. As long as there is a Hamilton xy-path in H, the Lollipop Lemma will guarantee that we can find another. Typically, the vertex y will be adjacent to x in the given Hamilton cycle, so if we keep all of the cycle edges of X except for xy, we will have a Hamilton xy-path in H, as desired. We take the second Hamilton xy-path that the Lollipop Lemma produces and add the edge xy to get a second Hamilton cycle.

In fact, this strategy is employed to easily prove Smith's Theorem below.

3.2 Smith's Theorem

Theorem 3.2.1. (Smith [16]) *In a cubic graph X, every edge lies in an even number of Hamilton cycles.*

Proof: Suppose X is a cubic graph, and let $e = xy \in E(X)$. We may assume that e is in at least one Hamilton cycle, otherwise the statement of the theorem is true for e. Let C be a Hamilton cycle of X containing e.

Now, in $X \setminus e$, every vertex has odd degree except for x and y, which each have degree 2. Further, a Hamilton xy-path exists via the other edges of C. Theorem 3.1.3 implies that there are, in fact, an even number of Hamilton xy-paths in $X \setminus e$, which, when we include edge e, become distinct Hamilton cycles. In fact, all Hamilton cycles which contain e must be included in this number because every Hamilton cycle containing e contains a Hamilton xy-path. Hence, e is in an even number of Hamilton cycles, as desired. ∎

Now, because we are able to use the result of Theorem 3.1.3 to prove Smith's Theorem, we actually never used the fact that X is cubic. We merely needed the fact that X has odd degree at least 3. This gives rise to the following result.

Corollary 3.2.2. *Suppose X is a graph in which every vertex has odd degree at least 3. Then every edge of X lies in an even number of Hamilton cycles.*

Smith's Theorem is of particular use when we are dealing with hamiltonian graphs. Suppose a graph X has a Hamilton cycle C, and that we can find a 1-factor F of $X - C$. Then $C \cup F$ is a hamiltonian cubic graph, so it must have at least one more Hamilton cycle. We state this formally, since this will be the statement of Smith's Theorem that we use in practice.

Corollary 3.2.3. *Let X be a graph with a Hamilton cycle C. If $X - C$ contains a 1-factor, then X possesses a second Hamilton cycle.*

Proof: Let C be a Hamilton cycle of X, and F be a 1-factor of $X - C$. Then $C \cup F$ is a cubic hamiltonian graph, so by Theorem 3.2.1, X has a second Hamilton cycle. ∎

Corollary 3.2.3 cannot be used in graphs that do not have a 1-factor, particularly graphs of odd order. However, we will see later that a great number of graphs in our proofs will have perfect matchings that are easy to find, so Smith's Theorem is very useful indeed.

3.3 Red-Independent Green-Dominating Sets

First, we must formally introduce the notion of a *red-independent green-dominating* set. We then present Thomassen's proof [14] that the existence of such a set in a hamiltonian graph guarantees a second Hamilton cycle.

Definition 3.3.1. Let $X = (V, E)$ be a hamiltonian graph with a Hamilton cycle C. Colour the edges of C red and all other edges green. Let $S \subseteq V$. Then S is a *red-independent green-dominating* set of X if the following hold:

- $X[S]$ contains no red edges, and
- For all $v \notin S$, there is a green edge joining v to a vertex in S.

Lemma 3.3.2. [14] *Let X be a graph and $S \subset V(X)$. Suppose $X - S$ has exactly $|S|$ components, each of which is a path whose initial and terminal vertices have odd degree in X.*

Then every edge incident with a vertex in S lies in an even number of Hamilton cycles of X.

Proof: Let $v \in S$ and $uv \in E(X)$. Any Hamilton path of X that begins at v must alternate between a vertex of S and a complete traversal of a path (an entire component) in $X-S$. Therefore, any Hamilton v-path must terminate at an endpoint of one of the paths in $X - S$, since there are equally as many such paths as there are vertices in S.

Suppose C is a Hamilton cycle in X containing uv. Theorem 3.1.3 implies that in $X - uv$, there are an even number of Hamilton paths starting at v and ending at a vertex of even degree. However, the observation above tells us that the terminal vertex must be u, since it is the only eligible vertex (endpoint of a path in $X - S$) of even degree. Each of these Hamilton paths, together with the edge uv, gives a Hamilton cycle in X (one of these cycles is C, of course). As there are an even number of such

Hamilton paths, there are an even number of Hamilton cycles containing uv in X. ∎

Theorem 3.3.3. (Thomassen [14]) *Let $X = (V, E)$ be a hamiltonian graph with a Hamilton cycle C. Colour the edges of C red and all other edges green. If X admits a red-independent green-dominating set, then X has a second Hamilton cycle.*

Proof: Let $S \subset V$ be a red-independent green-dominating set in X. Define $T = N_C(S)$.

Construct a spanning subgraph X' of X by taking every edge of C, and for every vertex u in T, include one green edge joining u to a vertex of S. This is possible because S is green-dominating and $T \cap S$ is empty. Now, in X', vertices of T each have degree 3, and vertices of $V \setminus (S \cup T)$ each have degree 2. Note that $X' - S$ has $|S|$ components, and each of them is a path with both endpoints in T, implying that the ends of these paths have odd degree in X'. Therefore, we may apply Lemma 3.3.2 to find that there is a Hamilton cycle in X' which is different from C. This gives a second Hamilton cycle in X. ∎

This result is so powerful that we will almost always colour edges of our Hamilton cycle red and the other edges green. This means that when we apply Theorem 3.2.1 (Smith's Theorem), we will seek a *green* perfect matching to accompany our red Hamilton cycle. When applying Theorem 3.1.3 directly, we will need to preserve all but one of the *red* edges so that we have at least one Hamilton path in the graph.

3.4 The Probabilistic Method

One powerful method for finding a second Hamilton cycle is the probabilistic method. Although we will not be using it in later chapters, it merits mention as the main method for the best known results at the time of this writing. (For basic information on probability spaces, please see Definition 2.1.1.)

Suppose we have a collection of pairwise independent events in a probability space, each with probability less than 1. Then the probability of the complement of each of these events is positive, and in general, the probability that none of these events occur (that is, the probability of the intersection of their complements) is also going to be positive. However, the Lovász Local Lemma [2], published in 1975, says that we can conclude the same thing if the events are "sufficiently independent" from each other. This lemma is the basic tool used by the best known results, and we present its statement here.

Theorem 3.4.1. (Lovász Local Lemma [2], [1]) *Let* $A_1, A_2, \dots, A_n$ *be events in a probability space, and for each* A_i *let* $J_i \subset \{1, 2, \dots, n\} \setminus \{i\}$ *be the set of values* j *such that* A_j *is not known to be independent with* A_i. *Suppose, for each event* A_i, *there exists a real number* $0 < p_i < 1$ *such that*

$$\mathbb{P}(A_i) < p_i \prod_{j \in J_i} (1 - p_j).$$

Then

$$\mathbb{P}(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n) \geq \prod_{i=1}^{n} (1 - p_i) > 0.$$

In 1998, Thomassen [15] used Theorem 3.4.1 to prove that for $d > 72$, every d-regular hamiltonian graph has a second Hamilton cycle. In 2007, this threshold was improved to $d > 22$ by Haxell, Seamone, and Verstraete [5]. In both cases, the setup is similar: the sample set Ω contains all possible subsets of the vertex set of the graph, and the event set $\mathcal{F}$ is 2^{Ω}. We determine which events contain vertex subsets that are not red-independent green-dominating. Theorem 3.4.1 will assure us, with an appropriate definition of the probability measure $\mathbb{P}$, that with positive probability, there is a red-independent green-dominating set (that is, the probability of the event consisting of all red-independent green-dominating sets is positive). Using Theorem 3.3.3, a second Hamilton cycle is assured.

One thing we cautiously note is that in both of these results, the existence of a red-independent green-dominating set is sought after. However, this condition is needlessly strong. We first present a slightly weaker sufficient condition for the existence of a second Hamilton cycle. We then present a variation on Thomassen's proof using this new sufficient condition to get a better requirement for d.

Definition 3.4.2. Let $X = (V, E)$ be a graph with a Hamilton cycle C. Colour the edges of C red and all other edges green. Suppose X has a red-independent set S. Let Y be a subgraph of X induced by the set of vertices in S and their neighbours via red edges. If S is green-dominating in Y, then we call S *weakly green-dominating* in X.

Theorem 3.4.3. *Let $X = (V, E)$ be a graph with a Hamilton cycle C. Colour the edges of C red and all other edges green. If X admits a set S that is red-independent weakly green-dominating, then X has a second Hamilton cycle.*

Proof: The proof is *identical* to that of Theorem 3.3.3. Note that we never use the fact that S is green-dominating in X. We only call upon that fact to find green edges joining vertices in $N_C(S)$ to vertices of S. ∎

This theorem may seem a trivial improvement, but we shall now use it to easily improve upon Thomassen's original proof which showed that every d-regular hamiltonian graph has a second Hamilton cycle if $d > 72$.

Theorem 3.4.4. *Let X be a d-regular hamiltonian graph. If $d > 64$, then X has a second Hamilton cycle.*

Proof: Let $X = (V, E)$ be a d-regular graph of order n and suppose C is a Hamilton cycle of X. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space defined on V as in Lemma 2.1.2.

We wish to compute the probability that a randomly chosen set of vertices S is red-independent weakly green-dominating in X. To that end, we will characterize the sets are not red-independent weakly green-dominating; these vertex sets are the elements of the "bad events" from Definition 2.1.3. We will then use the Lovász Local Lemma to determine when it is possible that some vertex set is not in any bad event.

Lemma 2.1.5 implies that these bad events are independent if the sets of vertices involved in the definition of these events are disjoint. Specifically, for an edge-event A_e, this set of vertices is just e; for a vertex-event B_v, it is $\{v\} \cup N_X(v)$; and for event D, it is V. We need to count, for each bad event, how many other bad events there are with which it may not be independent.

Let $e \in E(C)$. The event A_e may depend on edge-events of red edges that share an end with e; that is, adjacent red edges: there are 2 of these. In addition, A_e may depend on any vertex-event of vertices in the neighbourhood of either of its ends: this number may be as great as $2d$.

Now, let $v \in V(X)$. The event B_v may depend on edge-events of red edges that are incident with a vertex in $N_X(v)$ or with v itself. For each of v's $d-2$ neighbours via a green edge, there are two such incident red edges. For the neighbours of v in C, there are 4 such red edges, and these include the two red edges incident with v. Thus, B_v depends on at most $2d$ edge-events. As for vertex-events, each vertex in $N_X(v)$ is in the neighbourhood of $d-1$ vertices other than v, and v is in the neighbourhood of each of its d neighbours. Therefore, B_v depends on $d(d-1)+d = d^2$ vertex-events.

Finally, D depends‘ on every vertex- and edge-event.

In order to apply Theorem 3.4.1, we must select a parameter $x_e \in (0,1)$ for each event A_e; a $y_v \in (0,1)$ for each event B_v; and some $z \in (0,1)$ for D. For convenience, we choose a single parameter x for all the edge-events and a single parameter y for all the vertex-events. The probabilities of individual events (which are on the left-hand side of each inequality below) are computed using Lemma 2.1.4.

Hence, to apply Theorem 3.4.1, we must solve the following system of inequalities.

$$\mathbb{P}(A_e) = p^2 < x(1-x)^2(1-y)^{2d}(1-z) \tag{3.4.1}$$

$$\mathbb{P}(B_v) = (1-(1-p)^2)(1-p)^{d-1} < y(1-x)^{2d}(1-y)^{d^2}(1-z) \tag{3.4.2}$$

$$\mathbb{P}(D) = (1-p)^n < z(1-x)^n(1-y)^n \tag{3.4.3}$$

First, we set $z = c^n$ so that Inequality 3.4.3 becomes

$$1 - p < c(1-x)(1-y). \tag{3.4.4}$$

Now, we find a solution to the system of inequalities 3.4.1, 3.4.2, and 3.4.4 as follows. Fix $d \in [50, 75]$ and define $U = \{(p, x, y, c) : p, x, y, c \in \{k \cdot 10^{-3} : k = 1, \dots, 10^3\}\}$ as the search space. Find a solution on U using an exhaustive search. Once a solution is found, define a new search space U', which includes values for p, x, y, c up to four decimal places, in the vicinity of the solution. This process was repeated a few times, increasing the number of decimal places accordingly, until it was evident that the inequalities could not be solved for any lower values of d.

This process gave a solution of $p = 0.2335, x = 0.064, y = 0.0002087, z = 0.8195^n$ when $d = 66$. For these parameters, and assuming $|V| = 67$, Inequality 3.4.1 becomes $0.2335^2 \approx 0.05452 < 0.05455 \approx 0.064(1-0.064)^2(1-0.0002087)^{132}(1-0.8195^{67})$. As the left-hand side does not depend on $|V|$ and the right-hand side is increasing in $|V|$, this inequality is satisfied for any $|V| \geq 67$.

Similarly, Inequality 3.4.2 becomes $(1-(1-0.2335)^2)(1-0.2335)^{65} \approx 9.8447 \cdot 10^{-9} < 1.3586 \cdot 10^{-8} \approx 0.0002087(1-0.064)^{132}(1-0.0002087)^{66^2}(1-0.8195^{67})$. Again, the right-hand side is increasing in $|V|$ and the left-hand side is constant, so this inequality is satisfied for any $|V| \geq 67$.

Finally, Inequality 3.4.4 becomes $1 - 0.2335 = 0.7665 < 0.7669 \approx 0.8195(1 - 0.064)(1 - 0.0002087)$. Hence the system of inequalities is satisfied.

Due to Theorem 1.2.1, this solution for $d = 66$ implies that any d-regular hamiltonian graph for even $d \geq 66$ has a second Hamilton cycle. Together with Corollary 3.2.2, this gives the result for any $d > 64$.

I have not been able to find a solution to the above inequalities using this search method when $d \leq 64$. ∎

In their paper [5], Haxell *et al.* conclude by giving an infinite family of 4-regular hamiltonian graphs which do not have a red-independent green-dominating set. Each of these graphs is a circulant of order n, with $n \not\equiv 0 \pmod 3$, having edge length set $\{1, 2\}$. Relative to cycle C generated by the edges of length 1, there cannot be a red-independent green-dominating set S. This is because, for example, if $i \in S$, then vertex $i - 1$ would need either $i + 1$ or $i - 3$ — its only neighbours via a green edge — in S. But if $i + 1$ were in S, then S would not be red-independent. We deduce that S must contain exactly every third vertex of C, an impossibility since 3 does not divide n.

This result is of major interest to us because circulants are vertex-transitive and, as we shall see early on in the next Chapter, vertex-transitive hamiltonian graphs of degree at least 3 have a second Hamilton cycle. However, this result also presents us with some difficulty, for it means that we will not be able to prove Sheehan's Conjecture using only the notion of red-independent green-dominating sets. It is important, then, to have many tools at our disposal.

Chapter 4

Graphs With a Large Automorphism Group

4.1 Introduction

The purpose of this chapter is primarily to determine how large the automorphism group of a hamiltonian graph needs to be before a second Hamilton cycle is assured. As we will see, in graphs with very large automorphism group (*i.e.* if the group has size at least the order of the graph), it is very easy to find a second Hamilton cycle. Therefore, we will try to find the smallest requirement on the size of the automorphism group that still enables us to find a second Hamilton cycle. Starting with large groups, we shall prove that a second Hamilton cycle exists for successively smaller groups.

In this chapter, our graphs X will always have order n and a Hamilton cycle C. Hence we shall assume that $V(X) = \{u_0, u_1, \ldots, u_{n-1}\}$ and that $C = (u_0, u_1, \ldots, u_{n-1}, u_0)$. We shall colour the edges of C red and all other edges of X green. Finally, recall the special permutations $\rho = (u_0\ u_1 \ldots u_{n-1})$ and $\tau = (u_0)(u_1\ u_{n-1})(u_2\ u_{n-2})\ldots$, which act on $V(X)$.

Lemma 4.1.1. *Let X be a d-regular hamiltonian graph for $d \geq 3$. If $\rho \in Aut(X)$,* *then X has a second Hamilton cycle.*

Proof: Let $C = (u_0, u_1, \dots , u_{n-1}, u_0)$. Since $\rho \in \mathrm{Aut}(X)$ and the degree $d \geq 3$, there exist edges u_0u_i and u_1u_{i+1} for some $i \in \mathbb{Z}_n$. Lemma 2.4.1 implies that X has a second Hamilton cycle. ▮

Lemma 4.1.2. [10] *Let X be a d-regular hamiltonian graph for $d \geq 3$. If $\rho^2 \in$ Aut(X), then G has a second Hamilton cycle.*

Proof: If n is odd, then in fact $\langle \rho^2 \rangle = \langle \rho \rangle$, so we may apply Theorem 4.1.1 to get a second Hamilton cycle. On the other hand, if n is even, then let $X_i = X[\{u_j \in V : j \equiv i \pmod 2\}]$ be induced subgraphs for $i = 0, 1$.

Consider the green edges of X. If there is a green edge from X_0 to X_1, say $u_{2i}u_{2j+1}$, then the action of ρ^2 implies that $u_{2(i+k)}u_{2(j+k)+1}$ is an edge in X for every $k \in \mathbb{Z}_m$. These edges then form a green perfect matching, from which we can get a 1-factor. In this case, Theorem 3.2.3 implies that X has a second Hamilton cycle. On the other hand, if X_0 and X_1 are not joined by any green edges, then they are both not edgeless. Because ρ^2 is an automorphism of X, we may apply Lemma 2.4.3 to find a second Hamilton cycle in X. ▮

Theorem 4.1.3. [10] *Let X be a d-regular hamiltonian graph of order n, for $d \geq 3$, with $|$Aut(X)$| \geq n$. Then X has a second Hamilton cycle.*

Proof: Suppose that X is uniquely hamiltonian, and let its unique Hamilton cycle be $C = (u_0, u_1, \dots , u_{n-1}, u_0)$. Since any automorphism of X must preserve C, we have that $\mathrm{Aut}(X) \leq \mathrm{Aut}(C)$, which is a subgroup of the dihedral group. The dihedral group is generated by $\rho = (u_0\ u_1 \dots u_{n-1})$ and $\tau = (u_0)(u_1\ u_{n-1})(u_2\ u_{n-2}) \dots$ and has size $2n$.

Since $|\mathrm{Aut}(X)| \geq n$, we get that either $\mathrm{Aut}(X) \geq \langle\rho\rangle$ or $\mathrm{Aut}(X) \geq \langle\rho^2, \tau\rangle$. In either case, we have that $\rho^2 \in\mathrm{Aut}(X)$, so Lemma 4.1.2 yields a second Hamilton cycle, contradicting the fact that X is uniquely hamiltonian. Therefore, X has a second Hamilton cycle. ▮

Having now established that ρ and ρ^2 cannot be automorphisms of a uniquely hamiltonian graph, we shall turn our attention to larger powers of ρ. Excluding these small powers of ρ necessarily lowers the size of $\mathrm{Aut}(X)$. In the following chapters, we will see how far we can push down the size of $\mathrm{Aut}(X)$ before it becomes too difficult to find a second Hamilton cycle.

4.2 Lowering the Bound on $|\mathrm{Aut}(X)|$

Having verified that there exists no uniquely hamiltonian graph X of order n whose automorphism group has size at least n, we would now like to lower the bound on $|\mathrm{Aut}(X)|$. If X is uniquely hamiltonian, then $\mathrm{Aut}(X)$ is a subgroup of the dihedral group generated by $\langle\rho, \tau\rangle$. (Recall that for Hamilton cycle $C = (u_0, u_1, \ldots, u_{n-1}, u_0)$, we have permutations $\rho = (u_0\ u_1 \ldots u_{n-1})$ and $\tau = (u_0)(u_1\ u_{n-1})(u_2\ u_{n-2})\ldots.)$ In order for $\mathrm{Aut}(X)$ to be smaller, it might not have ρ or ρ^2 as generators, or perhaps τ may not be in $\mathrm{Aut}(X)$. If ρ^3, ρ^4, or ρ^5 are generators of $\mathrm{Aut}(X)$, then n, the order of X, must be a multiple of 3, 4, or 5, respectively; otherwise, there is a second Hamilton cycle in X, as the following lemma shows.

Lemma 4.2.1. [10] *Let X be a hamiltonian graph of order n, and suppose that n is not divisible by any of 3, 4, and 5. If $|Aut(X)| \geq \frac{2n}{5}$, then X has a second Hamilton cycle.*

Proof: Suppose that X is a uniquely hamiltonian graph with a Hamilton cycle $C = (u_0, u_1, \ldots, u_{n-1}, u_0)$. Let ρ and τ be defined as usual (*i.e.* as in the paragraph preceding this lemma). Let β be the smallest positive integer such that $\rho^\beta \in$Aut(X). Since $|\langle\rho\rangle| = n$ and $\langle\rho^\beta\rangle$ is a subgroup of $\langle\rho\rangle$ of order $\frac{n}{\beta}$, we must have $\beta \mid n$.

By Lemmas 4.1.1 and 4.1.2, we have $\beta \geq 3$. On the other hand, since $|\text{Aut}(X)|$ is either $|\langle\rho^\beta, \tau\rangle| = \frac{2n}{\beta}$ or $|\langle\rho^\beta\rangle| = \frac{n}{\beta}$, and $|\text{Aut}(X)| \geq \frac{2n}{5}$, we must have $\beta \leq 5$.

Thus $3 \leq \beta \leq 5$ and $\beta \mid n$, contradicting the assumption that n is not divisible by any of 3, 4, and 5. We conclude that X has a second Hamilton cycle. ∎

We remark that the minimum size of the automorphism group in the above lemma, $\frac{2n}{5}$, was selected because then we must have some power of ρ that is no greater than 5. It is possible to increase the minimum power of ρ, decreasing the size of Aut(X) accordingly, but the scenarios we would encounter become more and more complex. Therefore, we have selected $\frac{2n}{5}$ as a reasonably interesting, but solvable, lower bound.

We have seen what happens when n is not a multiple of any of 3, 4, and 5. Now, the following lemma (modified from [10]) states what Aut(X) looks like when n is a multiple of 3, 4, or 5.

Lemma 4.2.2. [10] *Let X be a d-regular graph of order $n = \alpha m$, for $d \geq 3$ and $\alpha \in \{3, 4, 5\}$. Suppose X admits a unique Hamilton cycle C and $|Aut(X)| = 2m$. Then the vertices of X can be relabelled so that $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ and, for permutations $\rho' = (v_0\ v_1\ \ldots v_{n-1})$ and $\tau' = (v_0)(v_1\ v_{n-1})(v_2\ v_{n-2})\ldots$, we have the following.*

1. *$Aut(X) = \langle\rho'^\alpha, \tau'\rangle$ if $\alpha = 3$ or 5; and*
2. *$Aut(X) = \langle\rho'^\alpha, \tau'\rangle$ or $Aut(X) = \langle\rho'^\alpha, \rho'\tau'\rangle$ if $\alpha = 4$.*

Proof: Let $C = (u_0, u_1, \ldots, u_{n-1}, v_0)$, and define permutations $\rho = (u_0\ u_1 \ldots u_{n-1})$ and $\tau = (u_0)(u_1\ u_{n-1})(u_2\ u_{n-2})\ldots$ as usual. Since X is uniquely hamiltonian, any automorphism must preserve its Hamilton cycle C, so $\mathrm{Aut}(X)$ is a subgroup of the dihedral group $\langle \rho, \tau \rangle$.

Let $\beta > 0$ be minimum such that $\rho^\beta \in \mathrm{Aut}(X)$. If $\beta = 1$ or 2, then we know X is not uniquely hamiltonian by Lemma 4.1.1 or 4.1.2, respectively. If $\beta \geq 3$, we need $\frac{n}{\beta}$ to divide $2m$, since $\frac{n}{\beta} = |\langle \rho^\beta \rangle|$ divides $|\mathrm{Aut}(X)| = 2m$. Since $n = \alpha m$, it follows that $\alpha | 2\beta$.

We wish to show that $\alpha = \beta$. To that end, we will show that $\alpha \leq \beta$ and $\beta \leq \alpha$. If α is 3 or 5, then since 3 and 5 are prime and $\alpha \mid 2\beta$, we in fact have that $\alpha | \beta$, so $\alpha \leq \beta$. If $\alpha = 4$, then we could have $\beta \equiv 0$ or 2 (mod 4). In this case, either $\beta = 2$, a contradiction, or $\beta \geq \alpha$. Hence $\alpha \leq \beta$ in all cases.

However, suppose $\beta > \alpha$. Then for any i, we have that $|\langle \rho^\beta, \rho^i \tau \rangle| = 2\frac{\alpha}{\beta} m < 2m$. As $|\mathrm{Aut}(X)|$ can be no bigger than the maximum of $|\langle \rho^\beta, \rho^i \tau \rangle|$ over all i, we have a contradiction. It must be the case that $\beta \leq \alpha$. Therefore (using the result of the previous paragraph), we have $\alpha = \beta$, as desired. Furthermore, there must be some element $\rho^i \tau \in \mathrm{Aut}(X)$ for some $i \geq 0$; otherwise, the automorphism group would be smaller than $2m$. Therefore, we deduce that $\mathrm{Aut}(X) = \langle \rho^\alpha, \rho^i \tau \rangle$.

We must now select a suitable permutation s by which to relabel the vertices and obtain the permutations ρ' and τ' mentioned in the statement of this lemma. Our selection of s is dependent on the parities of n and i (from the above paragraph). Once we select s, we will set $v_j = s(u_j)$ for each $j \in \mathbb{Z}_n$ and let $\rho = s\rho' s^{-1}$ and $\tau = s\tau' s^{-1}$. We must show in each case that $\rho' = \rho$ and $\tau' = \rho^i \tau$ (or that $\rho'\tau' = \rho^i \tau$), so that $\mathrm{Aut}(X) = \langle \rho'^\alpha, \tau' \rangle$ (or $\langle \rho'^\alpha, \rho'\tau' \rangle$).

Case (I) If $n - i$ is even, we let $s = \rho^{\frac{n+i}{2}}$. Then C is preserved under s, as $s(C) = (s(u_0), s(u_1), \ldots, s(u_{n-1}), s(u_0)) = (u_{\frac{n+i}{2}}, u_{\frac{n+i}{2}+1}, \ldots, u_{\frac{n+i}{2}-1}, u_{\frac{n+i}{2}}) = C$.

By our choice of s, we get $\rho' = \rho^{\frac{n-i}{2}} \rho \rho^{-\frac{n-i}{2}} = \rho$. Hence $\rho'(v_j) = \rho'(s(u_j)) = \rho \cdot \rho^{\frac{n+i}{2}}(u_j) = \rho^{\frac{n+i}{2}} \cdot \rho(u_j) = s(u_{j+1}) = v_{j+1}$, for any $j \in \mathbb{Z}_n$.

Additionally, we have

$$\begin{aligned}
\tau' &= \rho^{\frac{n+i}{2}}\tau\rho^{-\frac{n+i}{2}} \\
&= \rho^{\frac{n+i}{2}}\tau\rho^{\frac{-n+i}{2}} \\
&= \rho^{\frac{n+i}{2}} \cdot \rho^{\frac{n+i}{2}}\tau\rho^{\frac{n+i}{2}} \cdot \rho^{-\frac{n+i}{2}} && \text{(since } \rho\tau\rho = \tau\text{)} \\
&= \rho^{n+i}\tau \\
&= \rho^{i}\tau.
\end{aligned}$$

Hence, for any $j \in \mathbb{Z}_n$, we have

$$\begin{aligned}
\tau'(v_j) &= \rho^{i}\tau s(u_j) \\
&= \rho^{i}\tau\rho^{\frac{n+i}{2}}(u_j) \\
&= \rho^{i-\frac{n+i}{2}}\tau(u_j) && \text{(since } \rho\tau\rho = \tau\text{)} \\
&= \rho^{\frac{i-n}{2}}(u_{n-j}) \\
&= \rho^{\frac{i-n}{2}+n}(u_{n-j}) \\
&= \rho^{\frac{i+n}{2}}(u_{n-j}) = s(u_{n-j}) = v_{n-j}.
\end{aligned}$$

Therefore, the permutations ρ' and τ' act as they should on C.

Case (II) If i is even and n is odd, then we should let $s = \rho^{\frac{i}{2}}$. As above, the cycle C is preserved under s because s is a rotation of $\frac{i}{2}$. We again get that $\rho' = s\rho s^{-1} = \rho$, and by a similar computation to the one in Case (I), we find that $\rho'(v_j) = v_{j+1}$ for any $j \in \mathbb{Z}_n$.

We also have

$$\begin{aligned}
\tau' &= \rho^{\frac{i}{2}}\tau\rho^{-\frac{i}{2}} \\
&= \rho^{\frac{i}{2}} \cdot \rho^{\frac{i}{2}}\tau\rho^{\frac{i}{2}} \cdot \rho^{-\frac{i}{2}} \\
&= \rho^{i}\tau.
\end{aligned}$$

We determine that $\tau'(v_j) = v_{n-j}$ in a manner analogous to that of Case (I). Therefore, the permutations ρ' and τ' act as they should on C.

Case (III) Finally, suppose that n is even and there is no even i such that $\rho^i\tau \in$Aut(X). Then we should let $s = \rho^{\frac{i-1}{2}}$. As above, the cycle C is preserved under this rotation.

We get that $\rho' = s\rho s^{-1} = \rho$ as usual, and that $\rho'(v_j) = v_{j+1}$ for any $j \in \mathbb{Z}_n$.

Now,

$$\begin{aligned}\tau' &= \rho^{\frac{i-1}{2}}\tau\rho^{-\frac{i-1}{2}}\\ &= \rho^{\frac{i-1}{2}} \cdot \rho^{\frac{i-1}{2}}\tau\rho^{\frac{i-1}{2}} \cdot \rho^{-\frac{i-1}{2}}\\ &= \rho^{i-1}\tau.\end{aligned}$$

We can show that $\tau'(v_j) = v_{n-j}$ in a manner analogous to that of Case (I). Therefore, the permutations ρ' and τ' act as they should on C.

Note that in this case, we cannot have $\alpha \in \{3, 5\}$, for if we did, then since $\rho^{i+\alpha}\tau \in$Aut(X), we have a contradiction since we assumed there was no $\rho^{2k}\tau$ in Aut(X) for any k. Therefore, we have $\alpha = 4$ and, in this case, we have $\rho'\tau' = \rho^i\tau$ in Aut(X).

Therefore, the automorphisms ρ' and τ' are as we claimed, and Aut$(X) = \langle \rho'^{\alpha}, \tau' \rangle$ when $\alpha \in \{3, 4, 5\}$, or possibly Aut$(X) = \langle \rho'^{\alpha}, \rho'\tau' \rangle$ when $\alpha = 4$. ∎

4.3 Graphs of Order $n = 3m$

Let X be a graph of order $n = 3m$ for some $m \geq 2$. We set up the following assumptions and notation on our graph X:

- X is d-regular for $d \geq 4$, d even;
- We assume that X has a Hamilton cycle $C = (u_0, u_1, \ldots, u_{3m-1}, u_0)$;

- We assume that $\rho^3, \tau \in$Aut(X) for $\rho = (u_0\ u_1 \dots u_{n-1})$ and $\tau = (u_0)(u_1\ u_{n-1})(u_2\ u_{n-2})\dots$;
- Colour the edges of C red and the remaining edges green;
- Define $X_i = X[V_i]$ for $i = 0, 1, 2$, where $V_i = \{u_j : j \equiv i \pmod 3\}$.

We can partition all possibilities for X based on the number of X_0, X_1, and X_2 that have green edges. Note that, due to the action of ρ^3, any time we have green edges in $X[V_i, V_j]$ for $i \neq j$, we in fact have a perfect matching in $X[V_i, V_j]$. Due to the action of τ, the subgraphs X_1 and X_2 are isomorphic.

Case (0) X_0, X_1, and X_2 are edgeless (Figure 4.1, left);

Case (1) Exactly one of X_0, X_1, and X_2 contains green edges. Since X_1 and X_2 are isomorphic to each other, it must be the case that X_0 contains green edges (Figure 4.1, right);

Case (2) At least two of X_0, X_1, and X_2 contain green edges. If one of them does not contain green edges, then it must be X_0 because X_1 and X_2 are isomorphic (Figure 4.2).

We will show in Lemma 4.3.1 that a second Hamilton cycle can easily be found in Cases (0) and (2) if X is 4-regular. Case (1) is more resistant and we will insist upon additional conditions on the degree.

Lemma 4.3.1. *Let X be a d-regular hamiltonian graph of order $3m$, for $m \geq 2$ and even $d \geq 4$. Suppose $|Aut(X)| = 2m$, and that X falls into one of Cases (0) or (2) above. Then X has a second Hamilton cycle.*

Proof: Suppose X is uniquely hamiltonian. If $|\text{Aut}(X)| > 2m$, then we have ρ or ρ^2 in Aut(X). In this case, either Lemma 4.1.1 or 4.1.2 gives a second Hamilton cycle for X, a contradiction. We conclude that $|\text{Aut}(X)| = 2m$. Without loss of generality, by Lemma 4.2.2, we have $\text{Aut}(X) = \langle \rho^3, \tau \rangle$.

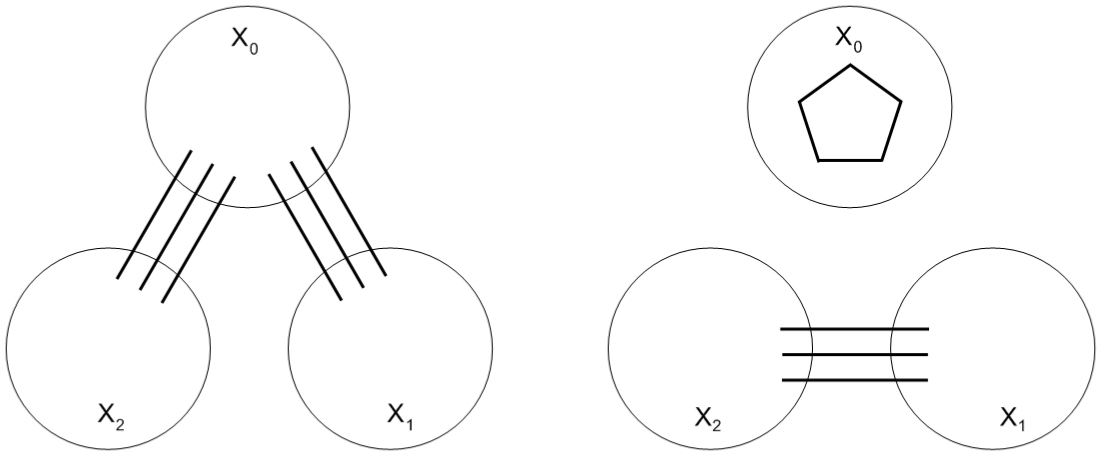

Figure 4.1: Cases (0) and (1).

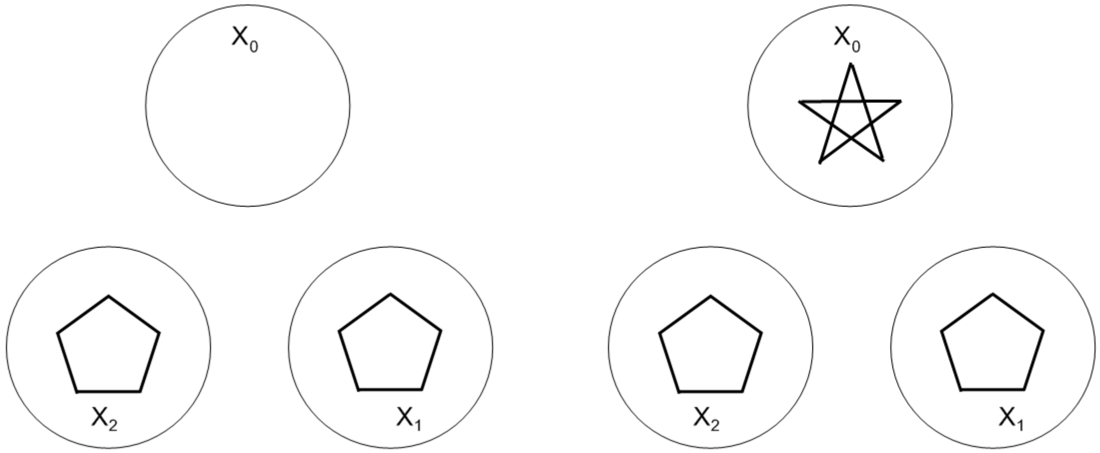

Figure 4.2: There are two possibilities in Case (2).

Case (0) X_0, X_1, and X_2 are edgeless. Since X_0 must have green edges leaving it, such green edges have their other end in either X_1 or X_2. Because the action of τ assures us that X_1 and X_2 are isomorphic to each other, X_0 must have some green edges with ends in X_1, and other green edges with ends in X_2.

Due to the action of ρ^3, all vertices in V_1 are similar to each other, and all vertices in V_2 are similar to each other. Therefore, every vertex in V_1 and V_2 is incident to a green edge with an end in V_0. Hence V_0 forms a red-independent green-dominating set, and Theorem 3.3.3 implies that X admits a second Hamilton cycle, a contradiction.

Case (2) X_1 and X_2 have green edges. Since $\rho^3 \in$Aut(X), Lemma 2.4.3 immediately implies that X has a second Hamilton cycle, a contradiction.

Therefore, in either case, we have a contradiction with the fact that X is uniquely hamiltonian. X must admit a second Hamilton cycle. ▮

Theorem 4.3.2. *Let X be a d-regular hamiltonian graph of order $3m$, for some $m \geq 2$ and even $d \geq 4$. Suppose $|Aut(X)| \geq 2m$. Define p to be the smallest prime divisor of m. Then X admits a second Hamilton cycle if one of the following holds:*

(I) *$d > m' + 1$, where $m = 2^k m'$ for $k \geq 1$, and m' is odd; or*

(II) *$d \geq \frac{m}{p} + 2$.*

Proof: Let X be a graph satisfying the assumptions of this theorem, and suppose X is uniquely hamiltonian. Colour the edges of C red and the remaining edges green. Then X falls into one of the Cases (0), (1), or (2) from the beginning of this section. If X falls into either Case (0) or (2), then Lemma 4.3.1 assures us that X has a second Hamilton cycle, a contradiction. Therefore, we assume that X is such that X_0 is the only non-edgeless subgraph from among X_0, X_1, and X_2.

As usual, let $C = (u_0, u_1, \ldots, u_{3m-1}, u_0)$ be the unique Hamilton cycle of X. Colour the edges of C red and all remaining edges green. Let $\rho = (u_0\ u_1 \ldots u_{3m-1})$ and $\tau = (u_0)(u_1\ u_{3m-1})(u_2\ u_{3m-2})\ldots$. By Lemma 4.2.2, we may assume that Aut$(X) = \langle \rho^3, \tau \rangle$.

We may assume that $X[V_0, V_1]$ and $X[V_0, V_2]$ are devoid of green edges. If this is not the case, then the actions of ρ^3 and τ imply that every vertex of V_1 and V_2 is incident to a green edge with the other end in V_0. Therefore, V_0 is a red-independent green-dominating set, so Theorem 3.3.3 yields a second Hamilton cycle, a contradiction.

We may also assume that $m \geq 3$, for if $m = 2$, then as the degree of each vertex is at least 4, there must be at least one green edge in $X[V_0, V_1]$ or $X[V_0, V_2]$, a contradiction.

So it must be the case that there are green edges in $X[V_1, V_2]$. In fact, $X[V_1, V_2]$ contains a green spanning $(d-2)$-regular bipartite graph. Corollary 1.1.2 implies that $X[V_1, V_2]$ has a (green) perfect matching, the edges of which form a 1-factor, which we shall call F, in that subgraph.

We proceed by using the additional conditions in the statement of the theorem. Let $2c = d-2$, so that c represents the number of distinct edge lengths in the circulant X_0.

Case (I) $d > m' + 1$, where $m = 2^k m'$ for $k \geq 1$, and m' is odd. Then, since $2c \geq m' - 1$, we have that X_0 contains a green 1-factor by Lemma 2.2.2. This 1-factor combined with F gives a green 1-factor for X, so applying Corollary 3.2.3 yields a second Hamilton cycle.

Case (II) $d \geq \frac{m}{p} + 2$, where p is the smallest prime factor of m. Since $m \geq 3$ and $2c \geq \frac{m}{p}$, we may apply Lemma 2.2.5 to get that X_0 contains a Hamilton cycle. If m is even, then this cycle is even, so we can take every other edge of the cycle as edges of a 1-factor F' for X_0. Then $F \cup F'$ is a green 1-factor for X, so we may apply Corollary 3.2.3 to get a second Hamilton cycle in X, a contradiction.

On the other hand, if m is odd, we can extract a maximum matching M in X_0 of size $\frac{m-1}{2}$ that saturates every vertex except for u_0. Let $X' = C \cup F + M - u_0u_1$. Observe that every vertex of X' has odd degree except for u_1, which has degree 2. Furthermore, $C - u_0u_1$ is a Hamilton u_0u_1-path in X'. Therefore, by Theorem 3.1.3, there exists another Hamilton u_0u_1-path in X'. This path, together with the edge u_0u_1, gives a second Hamilton cycle in X, contradicting the fact that X is uniquely hamiltonian.

Therefore, we conclude that X admits a second Hamilton cycle. ∎

Further advances have been made in an attempt to show that there are no uniquely hamiltonian graphs in Case (1). We refer the reader to Chapter 5 for more details.

4.4 Graphs of Order $n = 4m$

Let X be a graph of order $n = 4m$ for some $m \geq 2$. We set up the following assumptions and notation on our graph X:

- X is d-regular for $d \geq 4$, d even;
- We assume that X has a Hamilton cycle $C = (u_0, u_1, \dots, u_{4m-1}, u_0)$;
- We assume that $\langle \rho^4, \tau \rangle \leq \text{Aut}(X)$ (Case (1) below) or $\langle \rho^4, \rho\tau \rangle \leq \text{Aut}(X)$ (Case (2) below) for $\rho = (u_0\ u_1 \dots u_{n-1})$ and $\tau = (u_0)(u_1\ u_{n-1})(u_2\ u_{n-2}) \dots$;
- Colour the edges of C red and the remaining edges green;
- Define $X_i = X[V_i]$ for $i = 0, 1, 2, 3$, where $V_i = \{u_j : j \equiv i \pmod 4\}$.

Note that if $\langle \rho^4, \tau \rangle \leq \text{Aut}(X)$, then recall that τ fixes V_0 and V_2 setwise, and swaps V_1 and V_3. Therefore, we get the following cases:

(1.1) $X[V_0, V_1]$, $X[V_0, V_2]$, $X[V_1, V_2]$, and $X[V_1, V_3]$ contain no green edges (Figure 4.3, left);

(1.2) $X[V_0, V_1]$ or $X[V_1, V_2]$ contain green edges (Figure 4.4);

(1.3) $X[V_0, V_2]$ and $X[V_1, V_3]$ contain green edges (Figure 4.3, right);

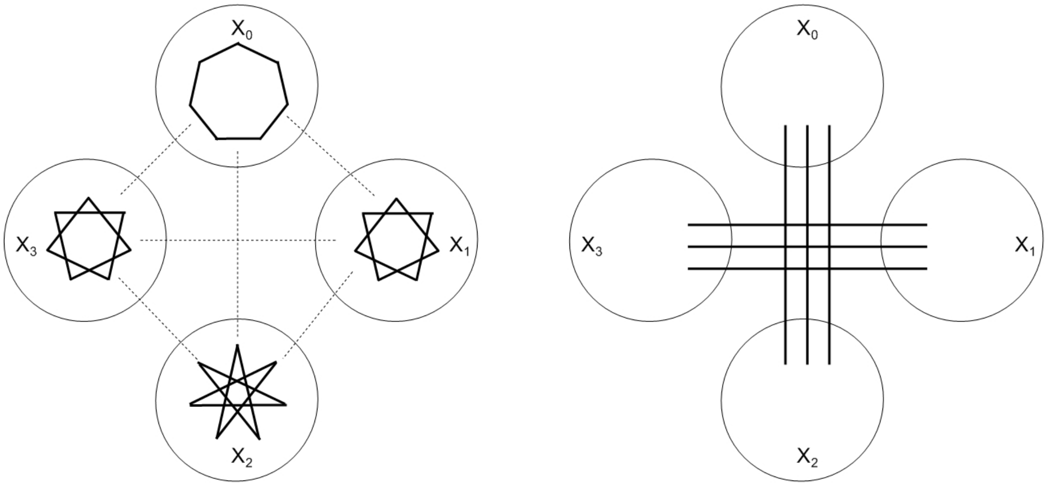

Figure 4.3: Cases (1.1) and (1.3). (Dashed lines indicate edges that we know do not exist.)

(1.4) $X[V_1, V_3]$ contains green edges (but $X[V_0, V_1]$, $X[V_0, V_2]$, and $X[V_1, V_2]$ do not) (Figure 4.5, left);

(1.5) $X[V_0, V_2]$ contains green edges (but $X[V_1, V_3]$, $X[V_0, V_1]$, and $X[V_1, V_2]$ do not) (Figure 4.5, right).

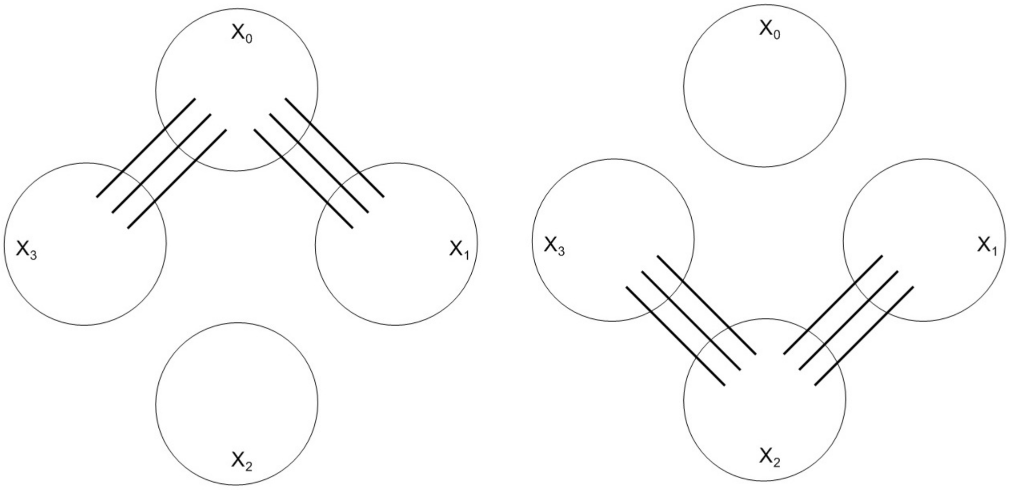

Figure 4.4: Case (1.2) can look one of two ways.

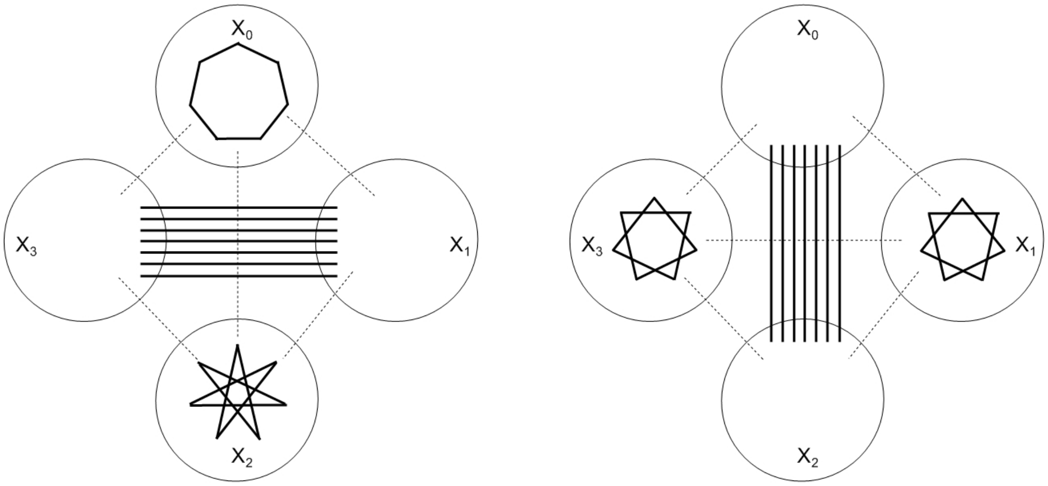

Figure 4.5: Cases (1.4) and (1.5). (Dashed lines indicate edges that we know do not exist.)

Otherwise, $\langle \rho^4, \rho\tau \rangle \leq \mathrm{Aut}(X)$. Observe that $\rho\tau = (u_0\ u_1)(u_{4m-1}\ u_2)\ldots$, hence $\rho\tau$ transposes V_0 with V_1, and V_2 with V_3, so we get the following additional cases:

(2.1) $X[V_0, V_1]$, $X[V_0, V_2]$, and $X[V_0, V_3]$ contain no green edges (Figure 4.6, left);

(2.2) $X[V_0, V_1]$ contains green edges (Figure 4.6, right);

(2.3) $X[V_0, V_2]$ contains green edges (Figure 4.7, left);

(2.4) $X[V_0, V_3]$ contains green edges (Figure 4.7, right).

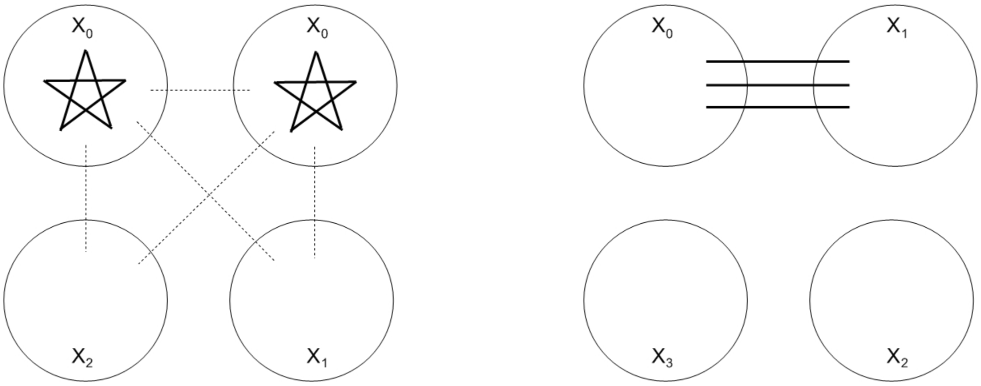

Figure 4.6: Cases (2.1) and (2.2).

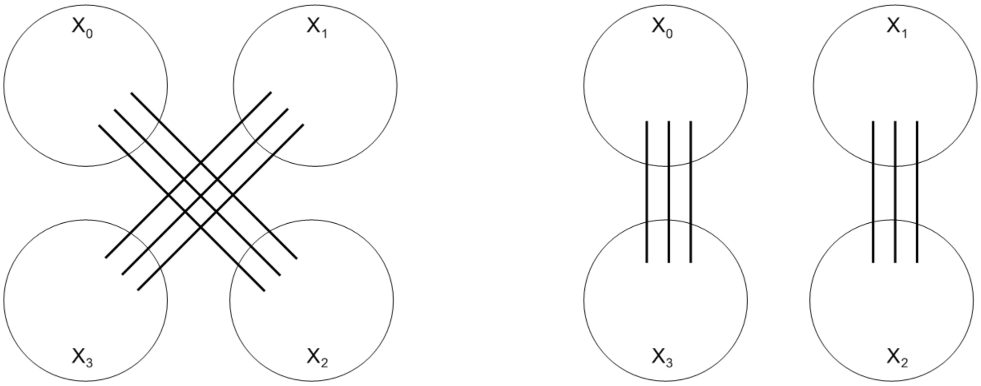

Figure 4.7: Cases (2.3) and (2.4).

We shall see that in most of the above cases, we find a second Hamilton cycle if X is 4-regular. The next two lemmas break up the cases based on the structure of $\mathrm{Aut}(X)$.

Lemma 4.4.1. *Let X be a d-regular graph of order $4m$ with a Hamilton cycle $C = (u_0, u_1, u_2, \ldots, u_{4m-1}, u_0)$ for $m \geq 2$ and even $d \geq 4$, and suppose that $\langle \rho^4, \tau \rangle \leq$ Aut(X). If X falls into one of cases (1.1), (1.2), (1.3), or (1.5) above, then X has a second Hamilton cycle.*

Proof: We begin by colouring the edges of the Hamilton cycle C red, and the remaining edges green.

Case (1.1) $X[V_0, V_2]$, $X[V_0, V_1]$, $X[V_1, V_2]$, and $X[V_1, V_3]$ contain no green edges.

Since τ is an automorphism of X, the subgraph $X[V_0, V_3]$ does not contain green edges either. Therefore, there are no green edges leaving V_0 or V_1. However, X is 4-regular, so X_0 and X_1 must each have green edges. Since $\rho^4 \in \mathrm{Aut}(X)$, we may apply Lemma 2.4.3 to get a second Hamilton cycle.

Case (1.2) $X[V_0, V_1]$ or $X[V_1, V_2]$ contain green edges.

The action of τ implies that $X[V_0, V_3]$ or $X[V_2, V_3]$, respectively, contain green edges, too. Applying ρ^4 to the green edges in either $X[V_0, V_1]$ and $X[V_0, V_3]$ or $X[V_1, V_2]$ and $X[V_2, V_3]$ yields a set of edges that saturates every vertex of V_1 and V_3. That is, every vertex of V_1 and V_3 has a green edge joining it to a vertex of V_0 or V_2. Then $\{V_0, V_2\}$ is a red-independent green-dominating set for X, so Theorem 3.3.3 assures us that there is a second Hamilton cycle.

Case (1.3) $X[V_0, V_2]$ and $X[V_1, V_3]$ contain green edges.

Applying ρ^4 to an edge in $X[V_0, V_2]$ yields a set of green edges M in that subgraph that saturates V_0 and V_2. Similarly, applying ρ^4 to an edge in $X[V_1, V_3]$ yields a set of green edges M' in that subgraph that saturates V_1 and V_3. Then

$M \cup M'$ is the edge set of a green 1-factor in X, so Theorem 3.2.3 implies that X has a second Hamilton cycle.

Case (1.5) $X[V_0, V_2]$ contains green edges, while $X[V_1, V_3]$, $X[V_0, V_1]$, and $X[V_1, V_2]$ are edgeless.

It follows that every green edge of X either has its ends in the same V_i, or has one end in V_0 and the other in V_2. Since X is 4-regular and $\tau \in$Aut(X), we conclude that X_1 and X_3 are isomorphic (non-empty) circulants; more precisely, there exists $j \in \{1, 2, \ldots, \lfloor \frac{m-1}{2} \rfloor\}$ such that $u_{4i+1} \sim u_{4(i+j)+1}$ and $u_{4i+3} \sim u_{4(i+j)+3}$ for all $i \in \mathbb{Z}_m$.

As usual, since $\rho^4 \in \text{Aut}(X)$, there exists $a \in \mathbb{Z}_m$ such that $u_{4i} \sim u_{4(i+a)+2}$, for all $i \in \mathbb{Z}_m$, and these edges form a perfect matching in $X[V_0, V_2]$.

Suppose that $a > \lfloor \frac{m-1}{2} \rfloor$. In this case, we shall relabel all vertices of X by setting $v_i := u_{i-2}$. Note that as subscripts are evaluated modulo $4m$, we should let $v_0 := u_{4m-2}$ and $v_1 := u_{4m-1}$. After relabeling, we have that $v_{4i-2} \sim v_{4(i+a)}$ — that is, $v_{4i} \sim v_{4(i+m-1-a)+2}$ for all $i \in \mathbb{Z}_m$. Since $a > \lfloor \frac{m-1}{2} \rfloor$, we now have that $m - 1 - a \leq \lfloor \frac{m-1}{2} \rfloor$, so let $b = m - 1 - a$.

On the other hand, if $a \leq \lfloor \frac{m-1}{2} \rfloor$, then we should simply let $b = a$. Relabel the vertices of X by $v_i := u_i$.

We now split the analysis into two cases, depending on which of j and b is larger. In both cases, we now have $v_{4i} \sim v_{4(i+b)+2}$, $v_{4i+1} \sim v_{4(i+j)+1}$, and $v_{4i+3} \sim v_{4(i+j)+3}$ for all $i \in \mathbb{Z}_m$.

Case (A): $b \geq j$. Now consider that $v_{4m-1} \sim v_{4j-1}$, $v_{4b+1} \sim v_{4(j+b)+1}$, $v_0 \sim v_{4b+2}$, and $v_{4j} \sim v_{4(j+b)+2}$. We shall, once again, relabel the vertices by letting $w_i := v_{i-1}$. Let $s = 2j$ and $t = 2b + 1$. Now we have $w_0 \sim w_{2s}$, $w_{2t} \sim w_{2(s+t)}$, $w_1 \sim w_{2t+1}$, and $w_{2s+1} \sim w_{2(s+t)+1}$. Because $j \leq b \leq \lfloor \frac{m-1}{2} \rfloor$, we have that $s < t \leq m$. Hence we may apply Lemma 2.4.2, which yields a second Hamilton cycle.

Case (B): $b < j$. Similarly, consider that $v_0 \sim v_{4b+2}$, $v_{4j} \sim v_{4(j+b)+2}$, $v_1 \sim v_{4j+1}$, and $v_{4b+3} \sim v_{4(j+b)+3}$. Now, letting $s = 2b + 1$ and $t = 2j$, we actually have $v_0 \sim v_{2s}$, $v_{2t} \sim v_{2(s+t)}$, $v_1 \sim v_{2t+1}$, and $v_{2s+1} \sim v_{2(s+t)+1}$. Because $b < j \leq \lfloor \frac{m-1}{2} \rfloor$, we have that $s < t \leq m$. Hence we may apply Lemma 2.4.2, which yields a second Hamilton cycle.

∎

Lemma 4.4.2. *Let X be a d-regular graph of order $4m$ with a Hamilton cycle $C = (u_0, u_1, u_2, \ldots, u_{4m-1}, u_0)$ for $m \geq 2$ and even $d \geq 4$, and suppose that $\langle \rho^4, \rho\tau \rangle \leq$ Aut(X). Then X has a second Hamilton cycle.*

Proof: Colour the edges of C red and the remaining edges green. Because $\langle \rho^4, \rho\tau \rangle \leq \mathrm{Aut}(X)$, the graph X falls into one of cases (2.1) through (2.4) on page 50.

Case (2.1) $X[V_0, V_1]$, $X[V_0, V_2]$, and $X[V_0, V_3]$ contain no green edges.

Note that $\rho\tau = (V_0V_1)(V_2V_3)$ setwise, so this implies that V_1 also has no green edges leaving it. However, since X has degree at least 4, X_0 and X_1 must each have green edges. Since $\rho^4 \in \mathrm{Aut}(X)$, we may apply Lemma 2.4.3 to get a second Hamilton cycle.

Case (2.2) $X[V_0, V_1]$ contains green edges.

Suppose there are no green edges leaving $X[V_0 \cup V_1]$. Then X_2 and X_3 are not edgeless, or $X[V_2, V_3]$ contains green edges. In the former case, since $\rho^4 \in \mathrm{Aut}(X)$, Lemma 2.4.3 produces a second Hamilton cycle. In the latter case, the action of ρ^4 implies that there is a set of edges that forms a green perfect matching in $X[V_0, V_1]$, and another green perfect matching in $X[V_2, V_3]$. Then Theorem 3.2.3 yields a second Hamilton cycle.

On the other hand, suppose there are green edges leaving $X[V_0 \cup V_1]$. Then V_0 has green edges joining it to either V_2 or V_3, and V_1 has green edges joining it to the other one of V_2 and V_3, thanks to the automorphism $\rho\tau$. Either way, the action of ρ^4 produces a green perfect matching in X, as above. Then Theorem 3.2.3 implies the existence of a second Hamilton cycle.

Case (2.3) $X[V_0, V_2]$ contains green edges.

Due to the action of $\rho\tau$, we know that $X[V_1, V_3]$ also has green edges, and we proceed as we did in Case (1.3) of Lemma 4.4.1.

Case (2.4) $X[V_0, V_3]$ contains green edges.

Due to the action of $\rho\tau$, we know that $X[V_1, V_2]$ contains green edges as well. Again, the proof is completed as in Case (1.3) of Lemma 4.4.1.

∎

The following theorem is an improvement on [10].

Theorem 4.4.3. *Let X be a d-regular graph of order $4m$, for some $m \geq 2$ and even $d \geq 4$. Let $C = (u_0, u_1, u_2, \ldots, u_{4m-1}, u_0)$ be a Hamilton cycle in X. Suppose that $|Aut(X)| = 2m$. Define p to be the smallest prime divisor of m. Then X admits a second Hamilton cycle if one of the following holds:*

(I) *$d > m' + 1$, where $m = 2^k m'$ for some $k \geq 1$ and odd m';*

(II) *$d > \frac{m}{p} + 2$.*

Proof: Let X be a graph satisfying the assumptions of the theorem, and suppose X is uniquely hamiltonian. Colour the edges of C red and the remaining edges green.

Then Lemma 4.2.2 implies that either $\mathrm{Aut}(X) = \langle \rho^4, \tau \rangle$ or $\mathrm{Aut}(X) = \langle \rho^4, \rho\tau \rangle$. The case breakdown on pages 47–50 describe all possible cases for X that need to be investigated. Lemma 4.4.1 yields a second Hamilton cycle in Cases (1.1), (1.2), (1.3), and (1.5); Lemma 4.4.2 does the same in Cases (2.1), (2.2), (2.3), and (2.4). In each of these cases, we obtain a contradiction. Therefore, what remains to be examined is Case (1.4).

Recall that in this case, we have that $\mathrm{Aut}(X) = \langle \rho^4, \tau \rangle$ and that $X[V_1, V_3]$ contains green edges, while $X[V_0, V_1]$, $X[V_0, V_2]$, and $X[V_1, V_2]$ do not. Because of the action of τ, there are no green edges leaving V_0 or V_2, so these subgraphs are circulants of degree $d-2$; similarly, since there are no edges leaving V_1 or V_3 except in $X[V_1, V_3]$, we have that $X[V_1, V_3]$ is a bicirculant of degree $d-2$.

Let $d-2 = 2c$ be the degree of the circulants X_0 and X_2. Note that because the degree $2c$ is at least 2, we must have that the order of X_0 is at least 3. Hence $m \geq 3$. We split our analysis into cases depending on which of our hypotheses is true.

(I) $d > m' + 1$, where $m = 2^k m'$ for some $k \geq 1$ and odd m'. Since $2c > m' - 1$, Lemma 2.2.2 tells us that each of X_0 and X_2 has a perfect matching. Since the bicirculant $X[V_1 \cup V_3]$ also has a perfect matching, Corollary 3.2.3 implies that X has a second Hamilton cycle, a contradiction.

(II) $d > \frac{m}{p} + 2$. Since $2c > \frac{m}{p}$ and $m \geq 3$, Lemma 2.2.5 implies that X_0 and X_2 are hamiltonian. If m is even, then taking every other edge in each of the Hamilton cycles of X_0 and X_2 yields the edge set of a green 1-factor F in $X[V_0 \cup V_2]$. Since $X[V_1, V_3]$ is a bicirculant, it is a regular bipartite graph, so Corollary 1.1.2 gives a green perfect matching in $X[V_1, V_3]$ which forms the edge set of a green 1-factor F' in $X[V_1, V_3]$. Then $F \cup F'$ is a green 1-factor in X, so Corollary 3.2.3 yields a Hamilton cycle in X, a contradiction.

However, suppose m is odd. If we take every other edge of the Hamiton cycle in X_0 such that we do not choose either edge incident to u_0, we can get a maximum matching M_0 that saturates every vertex of V_0 except for u_0. Similarly, use a Hamilton cycle in X_2 to get a maximum matching M_2 that saturates every vertex of V_2 except u_{4m-2}.

Since $2c > \frac{m}{p}$, Lemma 2.3.3 produces a Hamilton cycle C_1 in the bicirculant $X[V_1 \cup V_3]$. Using the edges of C_1, we can construct a perfect matching M_1 in $X[V_1 \cup V_3]$. However, we wish to construct yet another matching, M_3, which does not saturate the vertices u_{4m-1} and u_{4m-3}. Find an M_1-alternating $u_{4m-1}u_{4m-3}$-path P which starts and ends on an edge of M_1. This is possible because, owing to the fact that $X[V_1 \cup V_3]$ is bipartite, any $u_{4m-1}u_{4m-3}$-path has odd length in this subgraph. We create the matching M_3 by taking edges of the symmetric difference $M_1 \oplus E(P)$. Now, M_3 saturates every vertex of $X[V_1 \cup V_3]$ except for u_{4m-1} and u_{4m-3}.

Now, define a spanning subgraph X' of X by taking all the edges of $(E(C) \cup M_0 \cup M_2 \cup M_3) \setminus \{u_0u_1\}$. This gives a graph in which every vertex has degree 3 except $u_{4m-1}, u_{4m-2}, u_{4m-3}$, and u_1, each of which have degree 2, and u_0, which has degree 1. There is a red Hamilton u_0u_1-path in X'. Theorem 3.1.3 tells us, then, that the number of Hamilton paths beginning at u_0 and terminating at a vertex of even degree is even, hence there must be at least one more. However, any such path must begin with subpath $(u_0, u_{4m-1}, u_{4m-2}, u_{4m-3})$. This is because the only neighbour of u_0 in X' is u_{4m-1}, and each subsequent vertex in this sequence consequently has only one unvisited neighbour. Hence we really have another Hamilton u_0u_1-path. Together with the edge u_0u_1, this gives a second Hamilton cycle in X, a contradiction.

Therefore, in all cases, we obtain a contradiction. Therefore, X is not uniquely hamiltonian but instead admits a second Hamilton cycle. ∎

4.5 Graphs of Order $n = 5m$

Let X be a graph of order $n = 5m$ for some $m \geq 2$. We set up the following assumptions and notation on our graph X:

- X is d-regular for $d \geq 4$, d even;
- We assume that X has a Hamilton cycle $C = (u_0, u_1, \dots , u_{5m-1}, u_0)$;
- We assume that $\langle \rho^5, \tau \rangle \leq \mathrm{Aut}(X)$ for $\rho = (u_0\ u_1 \dots u_{n-1})$ and $\tau = (u_0)(u_1\ u_{n-1})(u_2\ u_{n-2}) \dots$;
- Colour the edges of C red and the remaining edges green;
- Define $X_i = X[V_i]$ for $i = 0, 1, 2, 3, 4$, where $V_i = \{u_j : j \equiv i \pmod 5\}$.

Note that $\tau = (V_0)(V_1V_4)(V_2V_3)$ setwise. Then we get the following cases:

(1) $X[V_0, V_1]$ and $X[V_0, V_2]$ contain green edges (Figure 4.8);

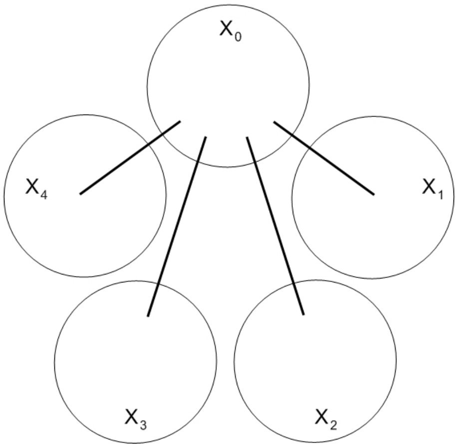

Figure 4.8: Case (1).

(2) $X[V_0, V_1]$ contains green edges but $X[V_0, V_2]$ does not. We note that the vertices of V_1 must have other incident green edges than the ones in $X[V_0, V_1]$. Then we get the following subcases:

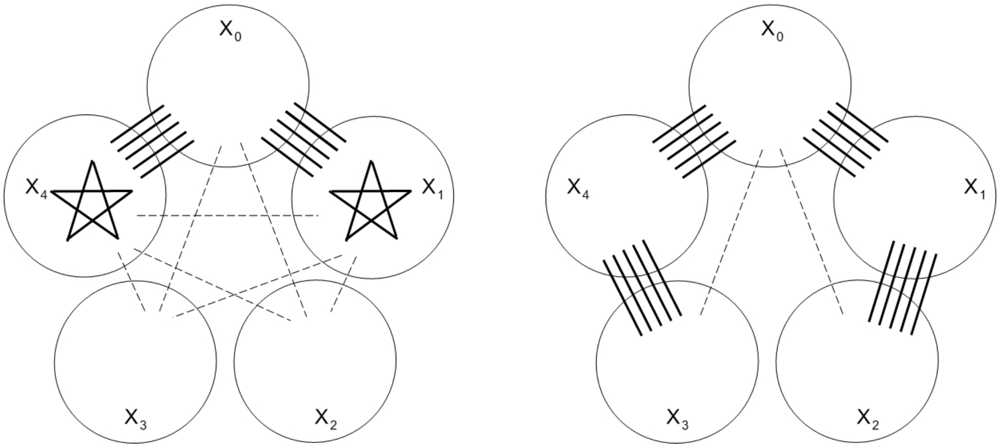

Figure 4.9: Cases (2a) and (2b). (As usual, dashed lines indicate edges that we know do not exist.)

- **(2a)** None of $X[V_1, V_2]$,$X[V_1, V_3]$, and $X[V_1, V_4]$ have green edges. Hence X_1 is not edgeless (Figure 4.9, left);
- **(2b)** $X[V_1, V_2]$ contains green edges (Figure 4.9, right);
- **(2c)** $X[V_1, V_3]$ contains green edges (Figure 4.10, left);
- **(2d)** $X[V_1, V_4]$ contains green edges, while $X[V_1, V_2]$ and $X[V_1, V_3]$ do not (Figure 4.10, right);

(3) $X[V_0, V_2]$ contains green edges but $X[V_0, V_1]$ does not. We note that the vertices of V_2 must have other incident green edges than the ones in $X[V_0, V_2]$. In this case, we get four subcases:

- **(3a)** None of $X[V_1, V_2]$,$X[V_2, V_3]$, and $X[V_2, V_4]$ have green edges. Hence X_2 is not edgeless (Figure 4.11, left);
- **(3b)** $X[V_1, V_2]$ contains green edges (Figure 4.11, right);
- **(3c)** $X[V_2, V_4]$ contains green edges (Figure 4.12, left);

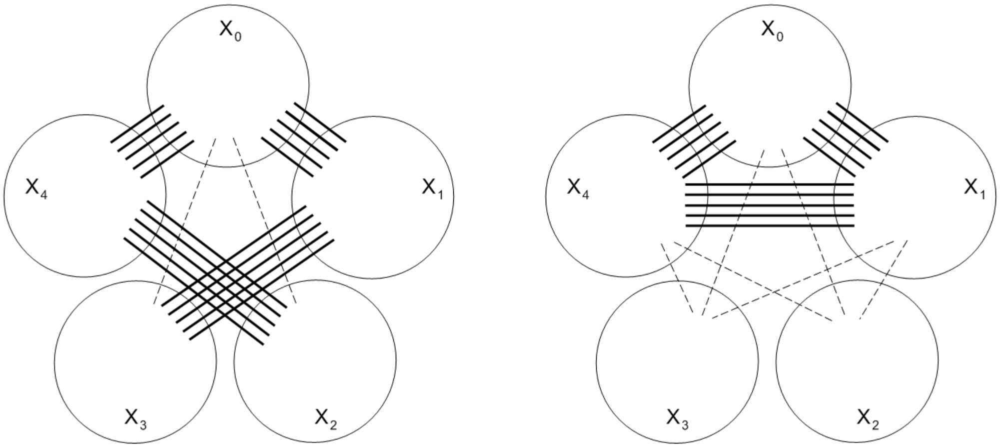

Figure 4.10: Cases (2c) and (2d). (Dashed lines indicate edges that we know do not exist.)

- **(3d)** $X[V_2, V_3]$ contains green edges, while $X[V_1, V_2]$ and $X[V_2, V_4]$ do not (Figure 4.12, right);

(4) None of $X[V_0, V_1]$, $X[V_0, V_2]$, $X[V_0, V_3]$, and $X[V_0, V_4]$ contain green edges; hence X_0 is not edgeless. There are four possible subcases:

- **(4a)** X_2 contains green edges (Figure 4.13, left);
- **(4b)** $X[V_2, V_4]$ contains green edges (Figure 4.13, right);
- **(4c)** $X[V_1, V_2]$ contains green edges but X_2 and $X[V_2, V_4]$ do not (Figure 4.14, left);
- **(4d)** $X[V_2, V_3]$ contains green edges but X_2, $X[V_2, V_4]$, and $X[V_1, V_2]$ do not (Figure 4.14, right).

Recall (from the $3m$ and $4m$ cases) that the actions of ρ^5 and τ are going to imply the existence of many more edges. The automorphism ρ^5 implies that whenever we have a green edge in $X[V_i, V_j]$ for $i \neq j$, we actually have a green perfect matching in that subgraph. The automorphism τ implies that any perfect matching in $X[V_i, V_j]$

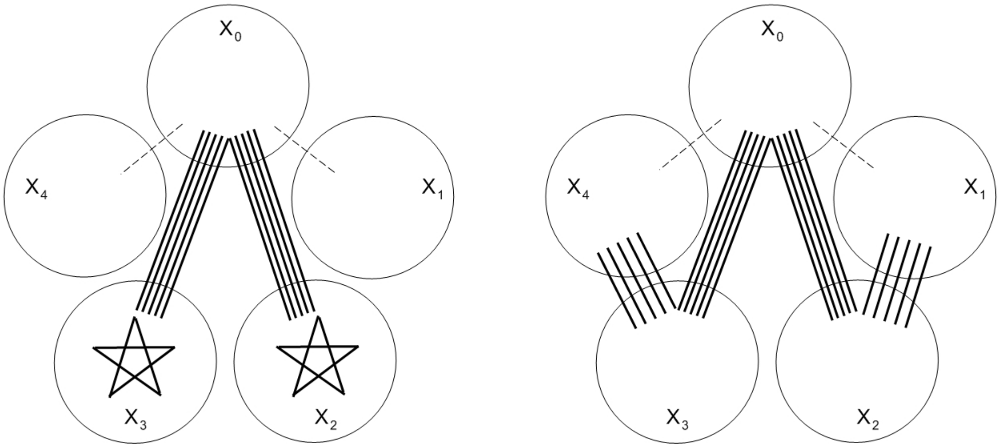

Figure 4.11: Cases (3a) and (3b). (Dashed lines indicate edges that we know do not exist.)

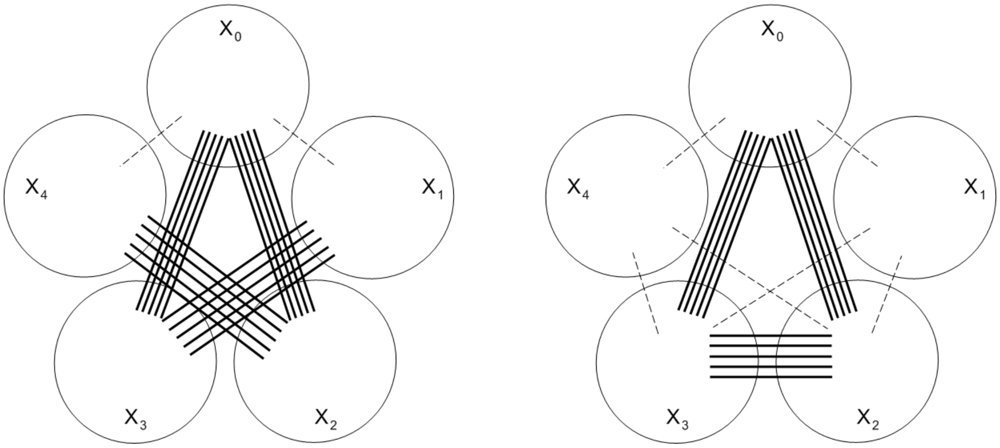

Figure 4.12: Cases (3c) and (3d). (Dashed lines indicate edges that we know do not exist.)

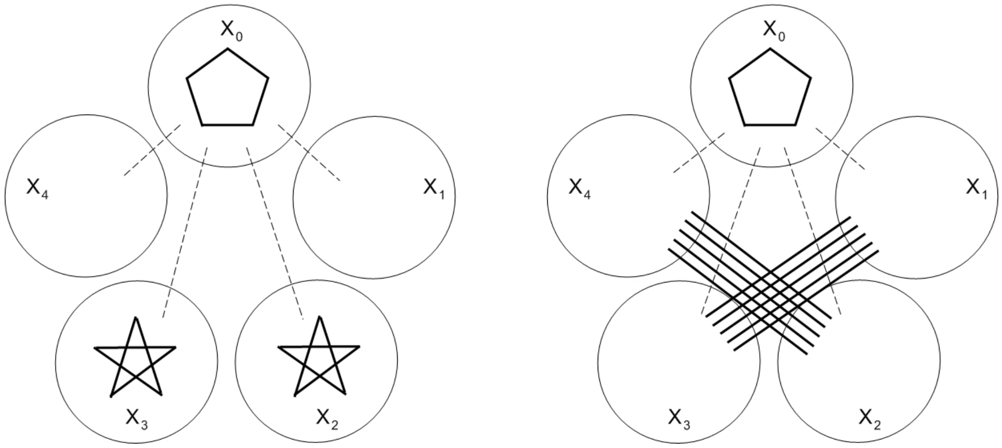

Figure 4.13: Cases (4a) and (4b).

is going to give us a perfect matching in $X[V_{5-i}, V_{5-j}]$. The next lemma tells us when we can easily find a second Hamilton cycle if X is 4-regular. For a couple of subcases, we will insist on additional conditions on the degree; these cases are covered by Theorem 4.5.2

Lemma 4.5.1. *Let X be a d-regular graph of order $5m$ with a Hamilton cycle $C = (u_0, u_1, u_2, \ldots, u_{5m-1}, u_0)$ for $m \geq 2$ and even $d \geq 4$, and suppose that $\langle \rho^5, \tau \rangle \leq$ Aut(X). Suppose X falls into any case from pages 57–59 except possibly Case (4c) or (4d). Then X has a second Hamilton cycle.*

Proof:

Case (1) $X[V_0, V_1]$ and $X[V_0, V_2]$ contain green edges.

> The action of τ implies that $X[V_0, V_4]$ and $X[V_0, V_3]$ contain green edges, too. The action of ρ^5 implies that every vertex of V_1, V_2, V_3, and V_4 is adjacent via a green edge to a neighbour in V_0. Therefore, V_0 is a red-independent green-dominating set for X. Theorem 3.3.3 gives a second Hamilton cycle.

Case (2a) None of $X[V_0, V_2]$, $X[V_1, V_2]$,$X[V_1, V_3]$, and $X[V_1, V_4]$ have green edges,

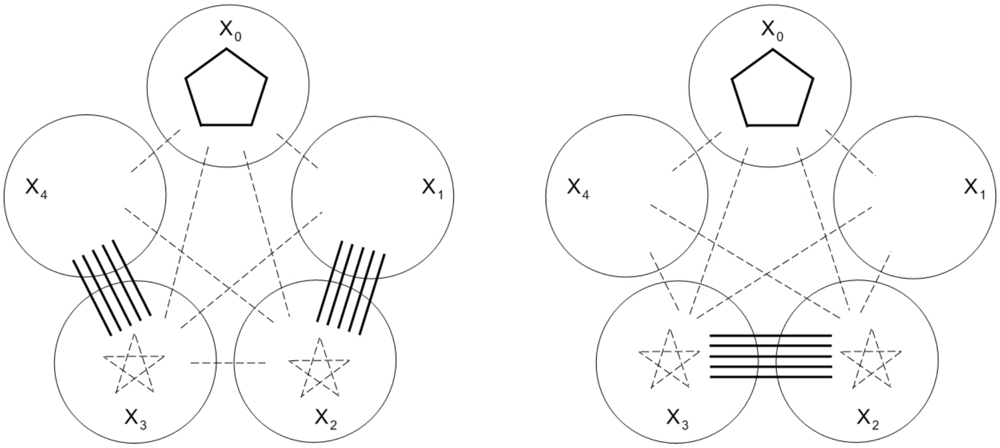

Figure 4.14: Cases (4c) and (4d). (Dashed lines indicate edges that we know do not exist.)

while $X[V_0, V_1]$ does have green edges. Hence X_1 is not edgeless.

We are given that $X[V_0, V_2]$ and $X[V_1, V_2]$ have no green edges, and τ applied to $X[V_1, V_3]$ gives us that $X[V_2, V_4]$ has no green edges. Thus, X_2 or $X[V_2, V_3]$ must have green edges. In the former case, the action of τ implies that X_3 also has green edges, hence X_2 and X_3 are not edgeless. Since $\rho^5 \in \text{Aut}(X)$, Lemma 2.4.3 gives a second Hamilton cycle.

On the other hand, if $X[V_2, V_3]$ contains green edges, then we claim that $V_0 \cup V_2$ is a red-independent green-dominating set. Here we are using the fact that $\tau \in \text{Aut}(X)$ implies that $X[V_0, V_4]$ contains green edges, and the action of ρ^5 implies that there is a green perfect matching in $X[V_i, V_j]$ whenever there are green edges between parts V_i and V_j. Hence every vertex in V_1 and V_4 is adjacent via a green edge to a neighbour in V_0, and every vertex in V_3 is adjacent via a green edge to a neighbour in V_2. Therefore, we may apply Theorem 3.3.3 to obtain a second Hamilton cycle.

Case (2b) $X[V_0, V_1]$ and $X[V_1, V_2]$ contain green edges, and $X[V_0, V_2]$ is edgeless.

The action of ρ^5 implies that there are green perfect matchings in $X[V_1, V_0]$ and $X[V_1, V_2]$, so every vertex of V_0 and V_2 is adjacent via a green edge to a neighbour in V_1. Similarly, the action of τ implies that there is a green perfect matching in $X[V_3, V_4]$, so every vertex of V_3 is adjacent via a green edge to a neighbour in V_4. Therefore, $V_1 \cup V_4$ is a red-independent green-dominating set for X. Once again, Theorem 3.3.3 gives a second Hamilton cycle.

Case (2c) $X[V_0, V_1]$ and $X[V_1, V_3]$ contain green edges, and $X[V_0, V_2]$ is edgeless.

This case is similar to Case (2b). Now we have green perfect matchings in $X[V_0, V_1]$, $X[V_1, V_3]$, and (due to the action of τ) $X[V_2, V_4]$. Therefore, $V_1 \cup V_4$ is again a red-independent green-dominating set for X, so Theorem 3.3.3 gives a second Hamilton cycle.

Case (2d) $X[V_0, V_1]$ and $X[V_1, V_4]$ contain green edges, while $X[V_0, V_2]$, $X[V_1, V_2]$, and $X[V_1, V_3]$ have no green edges.

The action of τ and ρ^5 imply that there are green perfect matchings in $X[V_0, V_1]$, $X[V_1, V_4]$, and $X[V_0, V_4]$. Since $X[V_1, V_2]$ and $X[V_1, V_3]$ have no green edges, we have that $X[V_3, V_4]$ and $X[V_2, V_4]$ also have no green edges. Either X_2 and X_3 are not edgeless, or there is a green perfect matching in $X[V_2, V_3]$. In the former case, since $\rho^5 \in$Aut(X), Lemma 2.4.3 gives a second Hamilton cycle in X. In the latter case, we have that $V_0 \cup V_2$ is a red-independent green-dominating set. Then Theorem 3.3.3 gives a second Hamilton cycle.

Case (3a) X_2 and $X[V_0, V_2]$ contain green edges but $X[V_0, V_1]$ does not.

Since X_2 is a not edgeless, the action of τ implies that X_3 is also not edgeless. Since $\rho^5 \in$Aut(X), we may apply Lemma 2.4.3 to obtain a second Hamilton cycle.

Case (3b) $X[V_0, V_2]$ and $X[V_2, V_1]$ contain green edges, and $X[V_0, V_1]$ is edgeless.

The action of ρ^5 implies that there are green perfect matchings in $X[V_0, V_2]$ and $X[V_1, V_2]$. The action of τ implies that there are also green perfect matchings in $X[V_0, V_3]$ and $X[V_3, V_4]$. Therefore, $V_1 \cup V_3$ forms a red-independent green-dominating set for X. Hence there is a second Hamilton cycle, by Theorem 3.3.3.

Case (3c) $X[V_0, V_2]$ and $X[V_2, V_4]$ contain green edges, and $X[V_0, V_1]$ is edgeless.

Suppose $u_2 \sim u_{5i+4}$. The action of τ gives that $u_{5(m-1)+3} \sim u_{5(m-i-1)+1}$. Then the action of ρ^5 tells us that $u_1 \sim u_{5i+3}$ (since $\rho^{5(i+1)}(u_{5(m-i-1)+1}) = u_1$ and $\rho^{5(i+1)}(u_{5(m-1)+3}) = u_{5i+3}$). We may now apply Lemma 2.4.1, performing a cycle exchange to get a second Hamilton cycle in X.

Case (3d) $X[V_0, V_2]$ and $X[V_2, V_3]$ contain green edges, while $X[V_0, V_1]$, $X[V_1, V_2]$, $X[V_1, V_3]$, and $X[V_2, V_4]$ have no green edges.

If there are green edges in $X[V_1, V_4]$, then the actions of ρ^5 and τ give green perfect matchings in $X[V_1, V_4]$, $X[V_3, V_2]$, and $X[V_3, V_0]$. In this case, $V_1 \cup V_3$ forms a red-independent green-dominating set for X, so Theorem 3.3.3 gives a second Hamilton cycle.

Otherwise, since $X[V_1, V_4]$ has no green edges, X_1 and X_4 are (non-edgeless) circulants. These circulants are isomorphic with isomorphism τ. More precisely, we have $u_{5i+1} \sim u_{5(i+l)+1}$ and $u_{5i+4} \sim u_{5(i+l)+4}$, for each $i \in \mathbb{Z}_m$ and some $1 \leq l < \frac{m}{2}$.

The actions of τ and ρ^5 imply that there are green perfect matchings in $X[V_0, V_2]$, $X[V_2, V_3]$, and $X[V_0, V_3]$; in fact, we get that each of these is a bicirculant graph, due to the action of ρ^5 in particular. Suppose $X[V_0, V_2]$ has edge difference $r \in \mathbb{Z}_m$, so that $u_{5i} \sim u_{5(i+r)+2}$, for all $i \in \mathbb{Z}_m$. Because of τ, we now get that $X[V_0, V_3]$ has edge difference $t = m-1-r$, so that $u_{5i} \sim u_{5(i+t)+3}$ for all $i \in \mathbb{Z}_m$.

We will now construct an auxiliary graph in preparation for applying Lemma 2.4.2. Obtain an auxiliary graph X' from X by identifying vertices u_{5i+2} and u_{5i+3}, for each $i \in \mathbb{Z}_m$, and removing any resulting loops and parallel edges. Colour the edges of $E(X')$ the same way that the corresponding edges are coloured in X.

We now label the vertices of X' in a manner dependent on which of t and $m-1-l$ is larger.

Case (3d.i) $t \leq m-l-1$. Let $V' = \{(v_0, v_1, \dots, v_{4m-1}\}$ be the vertex set of X', which corresponds to $V(X)$ in the following way. For each $i \in \mathbb{Z}_m$,

- Let $v_{4i} \in V'$ correspond to u_{5i} from $V(X)$;
- Let $v_{4i+1} \in V'$ correspond to u_{5i+1} from $V(X)$;
- Let $v_{4i+2} \in V'$ correspond to the vertex obtained by identifying u_{5i+2} with u_{5i+3} in X; and
- Let $v_{4i+3} \in V'$ correspond to u_{5i+4} from $V(X)$.

Now, we have (in particular) that $v_0 \sim v_{4t+2}$, $v_{4l} \sim v_{4(l+t)+2}$, $v_1 \sim v_{4l+1}$, and $v_{4t+3} \sim v_{4(l+t)+3}$. If we define $j = 2t+1$ and $k = 2l$, then we can see that in fact $v_0 \sim v_{2j}$, $v_{2k} \sim v_{2(j+k)}$, $v_1 \sim v_{2k+1}$, and $v_{2j+1} \sim v_{2(j+k)+1}$. Note that X' contains a Hamilton cycle corresponding to C: namely, $(v_0, v_1, \dots, v_{4m-1}, v_0)$. Finally, we have that $j + k = 2t + 2l + 1 \leq 2m - 1$ by our assumption.

Therefore, we may apply Lemma 2.4.2 to obtain a Hamilton cycle C' that contains all edges of the form $v_{2i+1}v_{2(i+1)}$. In particular, the cycle C' contains all edges of the form $v_{4i+1}v_{4i+2}$. Each of these corresponds to a red 2-path $(u_{5i+1}, u_{5i+2}, u_{5i+3})$ in X. Therefore, there is a corresponding Hamilton cycle in X which includes all edges corresponding to the edges of C', together with all red edges of the form $u_{5i+2}u_{5i+3}$. This cycle is different from C because it uses green edges of X', which correspond to green edges in X.

Case (3d.ii) $t > m - l - 1$. Then, since $t = m - 1 - r$, we find that $r < l$. Let $V' = \{v_0, v_1, \ldots, v_{4m-1}\}$ be the vertex set of X', which corresponds to $V(X)$ in the following way. For each $i \in \mathbb{Z}_m$,

- Let $v_{4i} \in V'$ correspond to $u_{5(i-1)+4}$ from $V(X)$;
- Let $v_{4i+1} \in V'$ correspond to u_{5i} from $V(X)$;
- Let $v_{4i+2} \in V'$ correspond to u_{5i+1} from $V(X)$; and
- Let $v_{4i+3} \in V'$ correspond to the vertex obtained by identifying u_{5i+2} with u_{5i+3} in X.

Now, we have that $v_0 \sim v_{4l}$, $v_{4r+2} \sim v_{4(l+r)+2}$, $v_1 \sim v_{4r+3}$, and $v_{4l+1} \sim v_{4(l+r)+3}$. If we define $j = 2l$ and $k = 2r + 1$, then we can see that in fact $v_0 \sim v_{2j}$, $v_{2k} \sim v_{2(j+k)}$, $v_1 \sim v_{2k+1}$, and $v_{2j+1} \sim v_{2(j+k)+1}$. Note that X' contains a Hamilton cycle corresponding to C: namely, $(v_0, v_1, \ldots, v_{4m-1})$. We also have that $j + k = 2l + 2r + 1 \leq 4l - 1 \leq 2m - 1$.

Therefore, we may apply Lemma 2.4.2 to obtain a Hamilton cycle C' which contains all edges of the form $v_{2i+1}v_{2(i+1)}$. In particular, the cycle C' contains all edges of the form $v_{4i+3}v_{4(i+1)}$. Each of these corresponds to a red 2-path $(u_{5i+2}, u_{5i+3}, u_{5i+4})$ in X. Therefore, there is a corresponding Hamilton cycle in X which includes all edges coresponding to the edges of C', together with all red edges of the form $u_{5i+2}u_{5i+3}$. This cycle is different from C because it uses green edges of X', which correspond to green edges in X.

Therefore, in either case (Case (3d.i) or Case (3d.ii)), we can find a second Hamilton cycle in X.

Case (4a) None of $X[V_0, V_1]$, $X[V_0, V_2]$, $X[V_0, V_3]$, and $X[V_0, V_4]$ contain green edges; hence X_0 is not edgeless. In addition, we have that X_2 is not edgeless.

Because of the action of τ, we have that X_3 is isomorphic to X_2, so X_3 also contains edges. Then, since $\rho^5 \in \mathrm{Aut}(X)$, Lemma 2.4.3 produces a second Hamilton cycle.

Case (4b) None of $X[V_0, V_1]$, $X[V_0, V_2]$, $X[V_0, V_3]$, and $X[V_0, V_4]$ contain green edges; hence X_0 is not edgeless. In addition, we have that $X[V_2, V_4]$ contains green edges.

This case is identical to Case (3c), since we only used the fact that $X[V_2, V_4]$ contains green edges.

▮

Theorem 4.5.2. *Let X be a d-regular graph of order $5m$, for some $m \geq 2$ and even $d \geq 4$. Let $C = (u_0, u_1, u_2, \ldots, u_{5m-1}, u_0)$ be a Hamilton cycle in X. Suppose that $|Aut(X)| = 2m$. Define p to be the smallest prime divisor of m. Then X admits a second Hamilton cycle if one of the following holds:*

(I) *$d > m' + 1$, where $m = 2^k m'$ for some $k \geq 1$ and odd m'; or*

(II) *$d \geq \frac{m}{p} + 2$ and m is odd.*

Proof: Let X be a graph satisfying the assumptions of the theorem, and suppose X is uniquely hamiltonian. Colour the edges of C red and the remaining edges green. We may then apply Lemma 4.2.2 to find that $\mathrm{Aut}(X) = \langle \rho^5, \tau \rangle$. Then X falls into one of the cases listed on pages 57–59. Unless X falls into either Case (4c) or (4d), Lemma 4.5.1 shows that X has a second Hamilton cycle, a contradiction. Therefore, we assume that X satisfies the conditions of either Case (4c) or (4d). Note that in either case, since there are no edges leaving V_0, we have that X_0 is a (non-edgeless) circulant. In both cases, we will attempt to find a perfect matching in $X[V_1 \cup V_2 \cup V_3 \cup V_4]$.

Case (4c) $X[V_1, V_2]$ contains green edges but X_2, $X[V_2, V_4]$, $X[V_0, V_1]$, $X[V_0, V_2]$, $X[V_0, V_3]$, and $X[V_0, V_4]$ do not.

Applying ρ^5 to a green edge of $X[V_1, V_2]$ yields a set of edges M_1 saturating every vertex in V_1 and V_2. Applying τ to this set of edges gives a set of edges M_2, disjoint from M_1, that saturates every vertex in V_3 and V_4. Therefore, we have that $M := M_1 \cup M_2$ is a perfect matching in $X[V_1 \cup V_2 \cup V_3 \cup V_4]$.

Case (4d) $X[V_2, V_3]$ contains green edges but X_2, $X[V_2, V_4]$, $X[V_1, V_2]$, $X[V_0, V_1]$, $X[V_0, V_2]$, $X[V_0, V_3]$, and $X[V_0, V_4]$ do not.

We have that either $X[V_1, V_4]$ contains green edges, or it doesn't. If it does, then ρ^5 applied to an edge in $X[V_1, V_4]$, along with an edge in $X[V_2, V_3]$, will produce a perfect matching M in $X[V_1 \cup V_2 \cup V_3 \cup V_4]$, as desired.

Otherwise, if $X[V_1, V_4]$ is edgeless, then there must be green edges in X_1 and X_4. Since X_0 contains green edges and $\rho^5 \in$Aut(X), we may apply Lemma 2.4.3 to find a second Hamilton cycle in X, a contradiction.

We now have a green perfect matching M in $X[V_1 \cup V_2 \cup V_3 \cup V_4]$. We proceed depending on which condition in the hypothesis is true.

(I) $d > m' + 1$, where $m = 2^k m'$ for some $k \geq 1$ and odd m'.

Then X_0 is a circulant of degree $2c = d - 2 > m' - 1$, so we may apply Lemma 2.2.2 to obtain a (necessarily green) perfect matching M' in X_0. Therefore, $M \cup M'$ is a green perfect matching in X, so we invoke Corollary 3.2.3 to obtain a second Hamilton cycle in X, a contradiction.

(II) $d \geq \frac{m}{p} + 2$ and m is odd. (p is the smallest prime divisor of m.)

Then X_0 is a circulant of degree $2c = d - 2 \geq \frac{m}{p}$, so we may apply Lemma 2.2.5 to find a Hamilton cycle C' for X_0. Since m is odd, the cycle C' has odd length.

Therefore, we may use C' to find a maximum matching M' in X_0 that saturates every vertex except for u_0.

Now, the subgraph $X' = (C - u_0u_1) + (M \cup M')$ has vertex u_0 of degree one, u_1 of degree two, and all other vertices of degree 3. Note that X' also contains a Hamilton u_0u_1-path. Therefore, we can apply Theorem 3.1.3 to obtain a second Hamilton u_0u_1-path, which, augmented by the edge u_0u_1, gives a second Hamilton cycle in X, a contradiction.

Since all cases lead to a contradiction, we conclude that X admits a second Hamilton cycle. ∎

4.6 Summary of Results

Corollary 4.6.1. [10] *Let X be a d-regular vertex-transitive hamiltonian graph of order n, for $d \geq 3$. Then X has a second Hamilton cycle.*

Proof: Since X is vertex-transitive, its automorphism group has size at least n. We may apply Theorem 4.1.3 to show that X has a second Hamilton cycle. ∎

The following is an improvement on [10] and aggregates the important results from Sections 4.1–4.5.

Theorem 4.6.2. *Let X be a d-regular hamiltonian graph of order n with $|Aut(X)| \geq \frac{2n}{5}$. Let p denote the smallest prime divisor of m. Then X has a second Hamilton cycle if one of the following conditions is satisfied:*

1. *$n \not\equiv 0 \pmod{\alpha}$ for all $\alpha \in \{3, 4, 5\}$, and $d \geq 3$;*
2. *$n = \alpha \cdot 2^k m'$, where $\alpha \in \{3, 4, 5\}, k \geq 1, m'$ odd; and $d > m' + 1$; or*

3. $n = \alpha m$, where $\alpha \in \{3,4,5\}, m \geq 2$; and $d > \frac{m}{p} + 2$.

Proof: We have in all cases that $d \geq 3$. If d is odd, then Corollary 3.2.2 implies that X has a second Hamilton cycle. On the other hand, suppose d is even. Then Lemma 4.2.1 finds a second Hamilton cycle in the case where n is not divisible by any of 3, 4, and 5. If n is divisible by $\alpha \in \{3,4,5\}$, then Theorem 4.3.2 completes the proof when $\alpha = 3$; Theorem 4.4.3 completes the proof when $\alpha = 4$; and Theorem 4.5.2 completes the proof when $\alpha = 5$. ▮

Corollary 4.6.3. [10] *Let X be a d-regular hamiltonian graph of order n with $|Aut(X)| \geq \frac{2n}{5}$. Suppose $n = \alpha m$ for some $\alpha \in \{3,4,5\}$. If m is an odd prime or a power of 2, then X admits a second Hamilton cycle.*

Proof: If m is an odd prime or a power of 2, then X satisfies one of the conditions of Theorem 4.6.2. We conclude that X admits a second Hamilton cycle. ▮

4.7 Looking Further

A natural question to ask is whether further results are forthcoming. In order to reduce the size of the automorphism group further, we either need to increase the smallest positive power of ρ that is an automorphism, or remove τ. In order to have ρ^6, ρ^7, or ρ^8 as the smallest positive power of ρ, one has to investigate graphs of order $6m$, $7m$, $8m$, respectively, and so on. However, a cursory glance at the uniquely hamiltonian 4-regular graphs of order $6m$ that have automorphism group size $2m$ reveals that there may be at least fifty different cases to check. Though it is certainly possible to check so many cases, it may not be worthwhile, as it seems that there might not be any more insight to be gained on the original problem by this approach.

A curious thing happens when the graph X has order $n = 6m$, for some $m \geq 2$, and $\mathrm{Aut}(X) = \langle \rho^3 \rangle$. In this case, we have $|\mathrm{Aut}(X)| = \frac{n}{3}$, which is smaller than any of the automorphism groups investigated so far. Note that such graphs are very similar to the graphs discussed in Section 4.3, except that the automorphism we call τ is absent. We will set up the notation for X before we give a result about it.

Let X be a graph of order $n = 6m$ for some $m \geq 2$. We set up the following assumptions and notation on our graph X:

- X is d-regular for $d \geq 4$, d even;
- We assume that X has a Hamilton cycle $C = (u_0, u_1, \ldots, u_{6m-1}, u_0)$;
- We assume that $\langle \rho^3, \rangle \leq \mathrm{Aut}(X)$ for $\rho = (u_0\ u_1 \ldots u_{n-1})$ and $\tau = (u_0)(u_1\ u_{n-1})(u_2\ u_{n-2})\ldots$;
- Colour the edges of C red and the remaining edges green;
- Define $X_i = X[V_i]$ for $i = 0, 1, 2$, where $V_i = \{u_j : j \equiv i \pmod 3\}$.

Lemma 4.7.1. *Let X be a d-regular graph of order $6m$, for some $m \geq 2$ and even $d \geq 4$. Suppose Aut(X) $= \langle \rho^3 \rangle$ so that, in particular, $|Aut(X)| = 2m$. Let $C = (u_0, u_1, \ldots, u_{6m-1})$ be a Hamilton cycle of X. Then X has a second Hamilton cycle.*

Proof: Colour the edges of C red and the remaining edges green. We proceed by splitting our analysis into three cases:

(0) All of X_0, X_1, or X_2 are edgeless.

In this case, we must have green edges going from one of V_0, V_1, V_2 to each of the others. If this is not the case, then one of these parts is isolated in the subgraph of only green edges, contradicting the fact that X is d-regular ($d \geq 4$) and X_0, X_1, and X_2 are edgeless. Suppose, without loss of generality, that this set that has green edges going to the other two parts is V_0. Then V_0 is a red-independent green-dominating set, so Theorem 3.3.3 yields the existence of a second Hamilton cycle in X.

(1) Exactly one of X_0, X_1, and X_2 is non-edgeless.

Without loss of generality, suppose X_0 is a circulant and X_1 and X_2 have green edges joining them. The action of ρ^3 implies that $X[V_1 \cup V_2]$ (including red edges) is a bicirculant. Proposition 2.3.1 shows that $(u_1\ u_{6m-1})(u_4\ u_{6m-4}) \ldots (u_{6m-2}\ u_2)$ is an automorphism of $X[V_1 \cup V_2]$. We note that $(u_0)(u_3\ u_{6m-3})(u_6\ u_{6m-6}) \ldots$ is an automorphism on X_0. If we extend these two automorphisms to a permutation on all of $V(X)$, we get $(u_0)(u_1\ u_{6m-1})(u_2\ u_{6m-2}) \ldots$, which is the permutation τ. This τ preserves C in addition to all the green edges of X, so it is in fact an automorphism of X. However, we assumed that $\tau \notin \mathrm{Aut}(X)$, a contradiction.

(2) At least two of X_0, X_1, and X_2 are non-edgeless.

Without loss of generality, suppose X_0 and X_1 are non-edgeless. Since $\rho^3 \in \mathrm{Aut}(X)$, we may apply Lemma 2.4.3 to obtain a second Hamilton cycle in X.

Therefore, X has a second Hamilton cycle. ∎

This result appears a lot less powerful when one realizes that all of the work that would have been done in Case (1) is deflected to Theorem 4.3.2, which attempts to find a second Hamilton cycle in such graphs of order $3m$. Of course, Theorem 4.3.2 sometimes insists on a higher degree than four. In the following chapter, we describe some *ad hoc* constructions which can solve a fair number of cases for graphs of order $3m$ and degree 4.

Chapter 5

An Algorithmic Approach

5.1 Motivation

In this chapter, it is our goal to tackle some of the cases from Section 4.3 that could not be solved except by imposing an additional constraint on the degree of the graph. Recall that these cases considered graphs of order $3m$, for some $m \geq 3$. It would be ideal if we could produce a method to find a second Hamilton cycle in these more difficult cases. In this chapter, we present some constructions for finding a second Hamilton cycle in certain cases, though not in all remaining cases. For the cases that cannot be solved by these constructions, we apply a computer search to verify the existence of a second Hamilton cycle, as long as the order of the graph is sufficiently small.

5.2 Results

We make use of the results of Section 4.3 together with some specific constructions. This enables us to apply a computer search on the remaining cases, achieving the following result.

Theorem 5.2.1. *Let X be a 4-regular hamiltonian graph of order $3m$, for some $2 \leq m \leq 25$. Suppose $|Aut(X)| = 2m$. Then X has a second Hamilton cycle.*

Proof: For additional notation on the graph X, please see pages 42–43. Assume X is uniquely hamiltonian, so that Lemma 4.2.2 tells us that $\rho^3, \tau \in$Aut(X). Then Lemma 4.3.1 gives a Hamilton cycle when exactly zero, two, or three of X_0, X_1, and X_2 are not edgeless, a contradiction. When exactly one of them is not edgeless, we apply Lemma 5.2.3 (below) or verify using Algorithms 5.2.4 and 5.2.5 (to follow), again contradicting the fact that X is uniquely hamiltonian.

Therefore, the graph X admits a second Hamilton cycle. ∎

Before presenting the constructions, it will be useful to define precisely what kinds of graphs we are interested in. The following definition will be used quite often throughout this chapter.

Definition 5.2.2. For any $m \geq 3$, let $a, b \in \{1, \ldots, m-1\}$, $a \neq b$, and $c \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor\}$. We define $X(m, a, b, c) = (V, E)$, a 4-regular hamiltonian graph of order $3m$, as follows. The vertex set $V = \mathbb{Z}_{3m}$ may sometimes be denoted by $\{u_0, u_1, \ldots, u_{3m-1}\}$. The set E of edges is defined in the following way. For all $0 \leq i \leq m-1$, we have $u_{3i+1} \sim u_{3(i+a)+2}$, $u_{3i+1} \sim u_{3(i+b)+2}$, and $u_{3i} \sim u_{3(i+c)}$ — these are represented by green edges of X. The red edges of X are those found in the Hamilton cycle $(u_0, u_1, \ldots, u_{3m-1}, u_0)$.

As usual, recall that we partition the vertex set into three parts $V_i = \{u_j : j \equiv i \pmod 3\}$, and $X_i = X[V_i]$, for $i = 0, 1, 2$.

Observe that for $\rho = (u_0\ u_1 \ldots u_{3m-1})$ and $\tau = (u_0)(u_1\ u_{3m-1})(u_2\ u_{3m-2})\ldots$, the permutations ρ^3 and τ are in Aut($X(m, a, b, c)$).

The following lemma gives a list of sufficient conditions for the existence of a second Hamilton cycle in a graph of the form $X(m, a, b, c)$. Applying these constructions will allow us to skip over several cases when we perform a computer search.

Lemma 5.2.3. *Let $X(m, a, b, c)$ be a graph as defined in Definition 5.2.2. Further, define $d = GCD(a + 1, m)$ and $e = GCD(c, m)$.*

Then X has a second Hamilton cycle if any of the following are true:

(A) $ad \equiv 0$ *(mod m) and $a \neq m - 1$;*

(B) $d > 1$, $b = m - 1$, *and* $ad \equiv 1$ *(mod e);*

(C) $a = c - 1$;

(D) $c \leq a \leq m - c - 1$; *or*

(E) $d = 1$.

Proof:

(A) Assume $ad \equiv 0$ (mod m) and $a \neq m - 1$. Define $Q_0 = (u_{3m-1}, u_0, u_1, u_{3a+2})$. Note Q_0 is actually a path because $3a + 2 \not\equiv 3m - 1$ (mod $3m$), owing to the fact that $a \neq m - 1$. For $1 \leq i \leq \frac{m}{d} - 2$, obtain a path Q_i from Q_0 by adding $3i(a + 1)$ to all subscripts and evaluating modulo $3m$. Define a path $Q_{\frac{m}{d}-1}$ slightly differently: obtain $Q_{\frac{m}{d}-1}$ from Q_0 by adding $3(\frac{m}{d} - 1)(a + 1)$ to the subscripts, but *do not include the fourth vertex*. That is, $Q_0, \ldots, Q_{\frac{m}{d}-2}$ are paths of length 3, but $Q_{\frac{m}{d}-1}$ is a path of length only 2. Note that, for $0 \leq i \leq \frac{m}{d} - 2$, the last vertex of Q_i is the same as the first vertex of Q_{i+1}. Finally, define P_0 to be the concatenation of these paths $Q_0, \ldots, Q_{\frac{m}{d}-1}$.

We would first like to show that P_0 is a path. Our motivation for doing so is that P_0 will be the initial path of a second Hamilton cycle in X. Since P_0 is the concatenation of several smaller paths, we should show that $Q_0, \ldots, Q_{\frac{m}{d}-1}$ are pairwise vertex-disjoint, except that the last vertex of Q_i is the same as the first vertex of Q_{i+1}, for $0 \leq i \leq \frac{m}{d} - 2$. Since each path Q_i is just a "translation" of Q_0 by $3i(a + 1)$ and ρ^3 is an automorphism of X, we just need to show that no two of these paths are the same. We can do this by showing that none of the Q_i contain the same vertex of X_0. That is, we shall show that $u_0, u_{3(a+1)}, u_{3(2(a+1))}, \ldots, u_{3(\frac{m}{d}-1)(a+1)}$ are pairwise distinct.

It suffices to show that the subscripts $0, 3(a+1), 3(2(a+1)), \ldots, 3(\frac{m}{d}-1)(a+1)$ are distinct residues modulo $3m$, or simply that $0, a+1, 2(a+1), \ldots, (\frac{m}{d}-1)(a+1)$ are distinct residues modulo m. Suppose, seeking a contradiction, that for some distinct $0 \leq i < j \leq \frac{m}{d} - 1$, we have $i(a+1) \equiv j(a+1) \pmod{m}$. Then $(j-i)(a+1) \equiv 0 \pmod{m}$; that is, $(j-i)\frac{a+1}{d} \equiv 0 \pmod{\frac{m}{d}}$.

But note that $\mathrm{GCD}(a+1, m) = d$ implies $\mathrm{GCD}(\frac{a+1}{d}, \frac{m}{d}) = 1$, so $\frac{a+1}{d}$ cannot be a zero divisor in $\mathbb{Z}_{\frac{m}{d}}$. This shows that $i \equiv j \pmod{\frac{m}{d}}$: a contradiction. Therefore, P_0 is a path, as desired.

Having now established that P_0 is a path, we may define other paths P_1 through P_{d-1} as follows. For $1 \leq i \leq d-1$, let P_i be obtained from P_0 by increasing all subscripts of vertices by $3((\frac{m}{d} - 1)(a+1) + 1)$. Now, since $\rho^3 \in \mathrm{Aut}(X)$, all of these $P_0, \ldots, P_{d-1}$ are paths, but we would like to show they are pairwise disjoint.

Now, examining the structure of P_i, we note that a vertex $v_{3j} \in V_0$ is in P_i if and only if v_{3j-1} and v_{3j+1} are adjacent to v_{3j} in P_i. Therefore, we need only verify that each vertex of X_0 is in at most one of $P_0, \ldots, P_{d-1}$. As before, we will consider just the subscripts of these vertices.

First, notice that $0, 3(a+1), 3(2(a+1)), \ldots, 3(\frac{m}{d}-1)(a+1)$ represent the vertices of X_0 in P_0. These subscripts are all congruent to 0 mod $3d$ because $3(a+1) \equiv 0 \pmod{3d}$.

In general, when we look at the vertices of X_0 in P_i, for $1 \leq i \leq d-1$, the subscripts are congruent to $3i$ modulo $3d$ because they were obtained by adding $3((\frac{m}{d} - 1)(a+1) + 1) \equiv 3 \pmod{3d}$ to subscripts of P_{i-1}. So the subscripts of vertices of X_0 on different paths are always different, hence the paths $P_0, \ldots, P_{d-1}$ are pairwise vertex-disjoint.

Finally, take $P_0, \ldots, P_{d-1}$, together with red edges joining the last vertex of P_i

and the first vertex of P_{i+1}, for each $0 \leq i \leq d-2$, and a red edge joining the last vertex of P_{d-1} to the first vertex of P_0. We claim that joining the paths in this way yields a second Hamilton cycle C' in X.

In order to show that the last vertex of P_i and the first vertex of P_{i+1}, for $0 \leq i \leq d-2$, are adjacent, we need only check that it holds for P_0 and P_1. Indeed, the last vertex of P_0 is $u_{3(\frac{m}{d}-1)(a+1)+1}$ and the first vertex of P_1 is $u_{(3m-1)+3((\frac{m}{d}-1)(a+1)+1)} = u_{3((\frac{m}{d}-1)(a+1)+1)-1} = u_{3(\frac{m}{d}-1)(a+1)+2}$, which are adjacent via a red edge.

Finally, it suffices to check that the last vertex of P_{d-1} has subscript $3(m-1)+1$. It can be computed by adding $d-1$ times $3((\frac{m}{d}-1)(a+1)+1)$ to $3(\frac{m}{d}-1)(a+1)+1$ and reducing modulo $3m$.

$$\begin{aligned}
&3(d-1)((\frac{m}{d}-1)(a+1)+1)+3(\frac{m}{d}-1)(a+1)+1\\
\equiv\ &3(\frac{m}{d}-1)(a+1)(d-1+1)+3(d-1)+1 \pmod{3m}\\
\equiv\ &3(m-d)(a+1)+3(d-1)+1 \pmod{3m}\\
\equiv &-3d(a+1-1)-3+1 \pmod{3m}\\
\equiv &-3+1 \pmod{3m} \qquad (\text{since } ad \equiv 0 \pmod{m} \Leftrightarrow 3ad \equiv 0 \pmod{3m})\\
\equiv\ &3(m-1)+1 \pmod{3m}, \text{ as desired.}
\end{aligned}$$

This proves that C' is a cycle. Since each path P_i contains $3(\frac{m}{d})$ distinct vertices, and C' contains d of these paths, C' contains $3m$ vertices. Therefore, C' is a Hamilton cycle of X.

(B) Assume $d > 1$, $b = m-1$, and $ad \equiv 1 \pmod{e}$. Observe that in the green circulant X_0, vertices u_{3i} and u_{3j} are in the same connected component if and only if $i - j = k_1c + k_2m = e(k_1\frac{c}{e} + k_2\frac{m}{e})$ for some integers k_1 and k_2. Since $e = \text{GCD}(c, m)$, we know that the equation $k_1\frac{c}{e} + k_2\frac{m}{e} = 1$ has an integer solution for k_1 and k_2. Thus, the above condition is equivalent to $i - j = ke$ for some integer k.

We conclude that u_{3i} and u_{3j} lie in the same connected component of X_0 if and only if $i \equiv j \pmod{e}$. By assumption, we have $m - ad + 1 \equiv 0 \pmod{e}$; hence, there exists a $u_3(m - ad + 1)u_0$-path P in X_0.

For all $0 \leq i \leq 3m - 1$, define *R-paths* R_{3i} of X as:

$$R_{3i} = \begin{cases} (u_{3(i-1)+2}, u_{3i+1}) & \text{if } u_{3i} \in V(P) \\ (u_{3(i-1)+2}, u_{3i}, u_{3i+1}) & \text{if } u_{3i} \notin V(P) \end{cases}$$

That is, each of these is either a (red) 2-path which travels through X_0 if vertex u_{3i} has not already been used in P; otherwise it uses a green edge of difference $m - 1$ to skip that vertex and still end up in the same place.

Now, define paths $Q_0, \ldots, Q_{\frac{m}{d}-1}$ as follows:

$$\begin{aligned} &Q_0 = R_0(u_1, u_{3a+2}), \\ &Q_1 = R_{3(a+1)}(u_{3(a+1)+1}, u_{3(2a+1)+2}), \\ &\ldots \\ &Q_i = R_{3i(a+1)}(u_{3i(a+1)+1}, u_{3(i(a+1)+a)+2}), \\ &\ldots \\ &Q_{\frac{m}{d}-1} = R_{3(\frac{m}{d}-1)(a+1)}(u_{3(\frac{m}{d}-1)(a+1)+1}, u_{3(\frac{m}{d}-1)(a+1)+2}). \end{aligned}$$

Note that $Q_{\frac{m}{d}-1}$ differs from the other Q_i in that its last edge is a *red* edge instead of a green one. We now verify that each Q_i is a path. For $0 \leq i \leq \frac{m}{d} - 1$, the last vertex of $R_{3i(a+1)}$ is $u_{3i(a+1)+1}$ from the definition of the R-paths. From the construction of Q_i, this is what we needed in order for Q_i to be well-defined.

We will define paths $P_0, \ldots, P_d$ in the following ways:

- $P_0 = Q_0 Q_1 \ldots Q_{\frac{m}{d}-1}$ (we verify that this concatenation is well-defined in the paragraphs below);
- For $1 \leq i \leq d - 2$, we obtain P_i from P_0 by adding $3i((\frac{m}{d} - 1)(a + 1) + 1)$ to the subscripts of the R-paths, as well as the vertices explicitly stated in the definition of each Q_i;

- P_{d-1} is obtained from P_0 by adding $3(d-1)((\frac{m}{d}-1)(a+1)+1)$ to all subscripts of vertices and R-subpaths, except that we do not include the last vertex. We compute the subscript of the terminal vertex of P_{d-1} by adding $3(d-1)((\frac{m}{d}-1)(a+1)+1)$ to $3(\frac{m}{d}-1)(a+1)+1$ (the subscript of the second-last vertex in P_0), evaluating modulo 3m:

$$\begin{aligned}
&3(\frac{m}{d}-1)(a+1)+1+3(d-1)((\frac{m}{d}-1)(a+1)+1) \\
&\equiv 3((\frac{m}{d}-1)(a+1)+1)-2+3(d-1)((\frac{m}{d}-1)(a+1)+1) \pmod{3m} \\
&\equiv 3d((\frac{m}{d}-1)(a+1)+1)-2 \pmod{3m} \\
&\equiv 3((m-d)(a+1)+d)-2 \pmod{3m} \\
&\equiv 3(m-ad+am)-2 \pmod{3m} \\
&\equiv 3(m-ad-1)+1 \pmod{3m}
\end{aligned}$$

- $P_d = (u_{3(m-ad-1)+1}, u_{3(m-ad-1)})P(u_0, u_{3(m-1)+2})$.

We claim that $C' = P_0P_1 \dots P_d$ is a second Hamilton cycle in X. To prove this, we will first show that P_0 is a path and not merely a walk. Since $P_1, \dots, P_{d-1}$ are (essentially) "translations" of P_0, this will show that they are paths, too. It will also be seen that P_d is a path. Then we will have to show that these paths are pairwise disjoint except that the last vertex of each P_i is the first vertex of P_{i+1}, for each $i \in \mathbb{Z}_{d+1}$. Finally, we will show that their concatenation is a cycle of length $3m$.

To show that P_0 is a path, we note that it is the concatenation of the paths $Q_0, Q_1, \dots, Q_{\frac{m}{d}-1}$, which are "translations" of each other by some $3i$, except that $Q_{\frac{m}{d}-1}$ differs in its last vertex. Since ρ^3 is an automorphism of X, we need only show that the first vertices of the Q_i are pairwise distinct, and also show that the terminal vertex of $Q_{\frac{m}{d}-1}$ is not among these.

Showing that the initial vertices of the Q_i are pairwise distinct amounts to showing that the initial vertices of the corresponding R_{3i} are pairwise distinct. Therefore, we must show that values of the form $3(i(a+1)+2)$ are distinct modulo $3m$, for $0 \leq i \leq \frac{m}{d}-1$. Adding 1 to each value and dividing by 3 reduces this task to showing that 0, $a+1$, $2(a+1)$, $\ldots$, $(\frac{m}{d}-1)(a+1)$, and $(\frac{m}{d}-1)(a+1)+1$ are distinct modulo m.

First, suppose that for $0 \leq i < j \leq \frac{m}{d}-1$, we have that $i(a+1) \equiv j(a+1) \pmod{m}$. As in Case (A) of this proof, we note that this supposition implies that $(j-i)(\frac{a+1}{d}) \equiv 0 \pmod{\frac{m}{d}}$. However, since $\mathrm{GCD}(\frac{a+1}{d}, \frac{m}{d}) = 1$, this is a contradiction since $\frac{a+1}{d}$ is not a zero divisor modulo $\frac{m}{d}$.

On the other hand, suppose that for some $0 \leq i \leq \frac{m}{d}-1$, we have that $i(a+1) \equiv (\frac{m}{d}-1)(a+1)+1 \pmod{m}$. Note that $m(\frac{a+1}{d}) \equiv 0 \pmod{m}$, so we may eliminate this term from the equation. Then $(i+1)(a+1) \equiv 1 \pmod{m}$, so $a+1$ is invertible in $\mathbb{Z}_m$. If $a+1$ is invertible in $\mathbb{Z}_m$, then $d=1$, which contradicts our hypothesis that $d>1$. This completes the proof of our claim that P_0 is a path. Therefore, each of $P_1, \ldots, P_{d-1}$ is a path as well.

Now, P_d is also a path: its internal vertices form a path P in X_0, so we need only check that its initial and terminal vertices are not the same, and that they are not in V_0. However, its initial vertex is in V_1 and its terminal vertex is in V_2.

We must now show that $P_0, \ldots, P_d$ are vertex-disjoint except for the ends of consecutive paths: note that for each $0 \leq i \leq d-1$, the last vertex of P_i coincides with the first vertex of P_{i+1}. To show that $P_0, \ldots, P_{d-1}$ are pairwise internally vertex-disjoint, recall that these paths are close to being images of each other under some power of ρ^3, so they have similar structures. We will show that each R-path is in at most one of these paths. Since each internal vertex of a path P_i (for $0 \leq i \leq d-1$) lies in an R-path within P_i, this fact will show that any two paths P_i and P_j are internally vertex-disjoint.

Notice that the R-paths in P_0 have subscripts $0, 3(a+1), 3(2(a+1)), \ldots, 3(\frac{m}{d}-1)(a+1)$. These are all congruent to 0 modulo $a+1$. In fact, when looking at P_i, we see that we have added $3i((\frac{m}{d}-1)(a+1)+1)$ to all subscripts of the R-paths, which is congruent to $3i$ modulo $3(a+1)$. Since i only ranges between 0 and $d-1$ (which is at most $a-2$), we have that each P_i has R-paths of a different residue modulo $3(a+1)$. Therefore, a single R-path cannot be in both P_i and P_j, for $i \neq j$.

To show that P_d is also pairwise vertex-disjoint with $P_0, \ldots, P_{d-1}$ (except that the terminal vertex of P_{d-1} and the initial vertex of P_d are the same), note that its vertices which are in V_0 are exactly the ones missed by $P_0, \ldots, P_{d-1}$. We must still check that $u_{3(m-1)+2}$ is the initial vertex of P_0 but does not occur in any of $P_1, \ldots, P_{d-1}$. We do this by confirming that R_0 is found only in P_0.

Finally, we have just shown that C' is a cycle, since it is a well-defined concatenation of several internally-disjoint paths, and its initial and terminal vertices are the same. We determine the length of C' by adding the lengths of its component paths. P_0 has length $3(\frac{m}{d})$, minus the number of vertices of P that it skips. The rest of $P_1, P_2, \ldots, P_{d-1}$ are similar, except that P_{d-1} is shorter by 1. Hence $P_0 P_1 \ldots P_{d-1}$ has length $3m-1$ minus the number of vertices of P. Finally, P_d has length 1 plus the number of vertices of P. Therefore, the total length of C' is $3m$, as desired.

Therefore, C' is a second Hamilton cycle in X.

(C) Assume $a = c-1$. We claim that the following is a Hamilton cycle in X (see Figure 5.1):

$$\begin{aligned}
&(u_0, u_{3c}, u_{3c+1}, u_{3c+2},\\
&u_{3(1)+1}, u_{3(1)+2}, \ldots, u_{3a+2},\\
&u_1, u_2, u_{3(1)},\\
&u_{3(c+1)}, u_{3(c+1)+1}, \ldots, u_0).
\end{aligned}$$

Here, we need only note that $u_{3c+2} \sim u_4$, $3a+2 = 3c-1$, and $u_3 \sim u_{3(c+1)}$.

(D) Assume $c \leq a \leq m-c-1$. We claim that the following is a Hamilton cycle in X (see Figure 5.2):

$$(u_0, u_{3c}, u_{3(c-1)+2}, u_{3(c-1)+1}, \ldots, u_1,$$

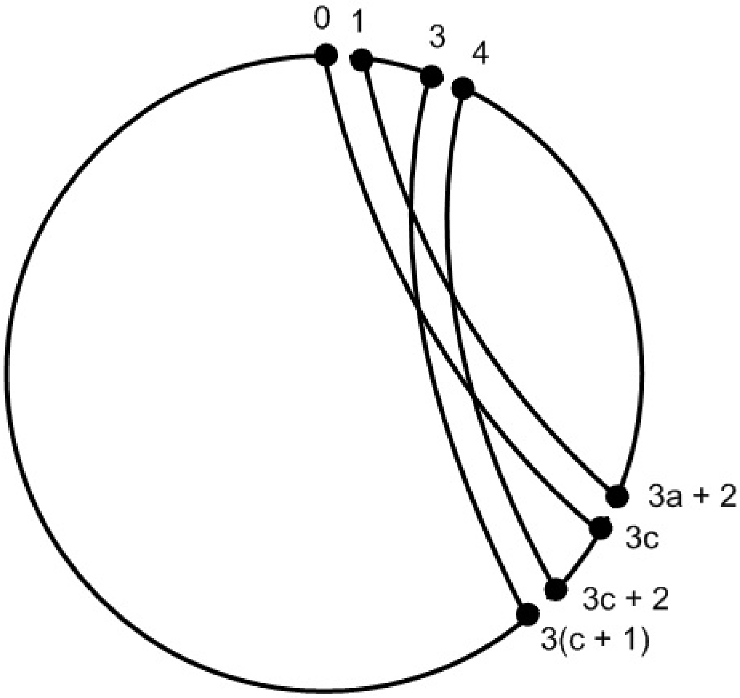

Figure 5.1: An illustration of the construction in (C). The large circle represents the original (red) Hamilton cycle. The curved chords represent newly included green edges, which produce a different Hamilton cycle when four red edges are deleted. Here, a label of i represents the vertex u_i.

$$u_{3a+2}, u_{3a+1}, \dots, u_{3c+1},$$

$$u_{3(a+c)+2}, u_{3(a+c)+1}, \dots, u_{3(a+1)},$$

$$u_{3(a+c+1)}, u_{3(a+c+1)+1}, \dots, u_0).$$

This claim will require us to show that $3c < 3a+2 < 3(a+c)+2 < 3(a+c+1) < 3m$. These are the subscripts of the second ends of the green edges in our proposed cycle. All the other edges are coloured red. We also need $3(a+c+1) \leq 3m$ so that we can proceed to u_0 from $u_{3(a+c+1)}$ along red edges.

By our hypothesis, we have $c \leq a$, so $3c < 3a+2$. Next, we have $3a+2 < 3(a+c)+2$ because $c > 0$. Clearly $3(a+c)+2 < 3(a+c+1)$; these two vertices are adjacent in the red Hamilton cycle. Finally, our hypothesis $a \leq m-c-1$ implies

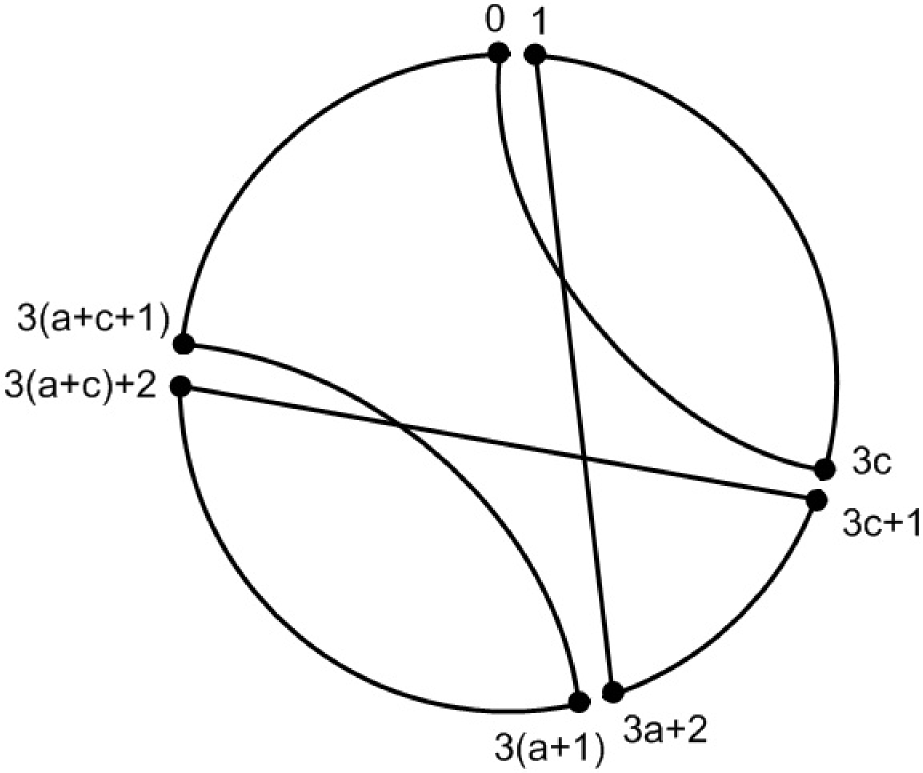

Figure 5.2: An illustration of the construction in (D). The large circle represents the original (red) Hamilton cycle. The chords represent newly included green edges, which produce a different Hamilton cycle when four red edges are deleted. Here, a label of i represents the vertex u_i.

that $3(a+c+1) \leq 3m$. We conclude that we indeed have found a second Hamilton cycle in X.

Now, our hypothesis that $m-c-1 \geq a$ implies that $3m \geq 3(a+c+1)$. Clearly $3(a+c+1) > 3(a+c)+2$; these two vertices are adjacent in C. Next, $3(a+c)+2 > 3a+2$ because $c > 0$. Finally, since our hypothesis assures that $a \geq c$, we have that $3a+2 > 3c$.

(E) Assume $d = 1$. Construct an auxiliary graph X' which contains all the vertices of X_1 and X_2, as well as the green edges between them of difference a. In addition, for each red 2-path in X of the form $(u_{3(i-1)+2}, u_{3i}, u_{3i+1})$, for all $0 \leq i \leq m-1$, add an edge to X' joining $u_{3(i-1)+2}$ to u_{3i+1}.

Now, X' is a bicirculant $\mathcal{B}(2m; \{a, m-1\})$. Lemma 2.3.2 shows that X' is a cycle of length $2m$ because $\mathrm{GCD}(m, m-1-a) = 1$. By converting the edges $u_{3(i-1)+2}u_{3i+1}$ back into their original 2-paths in X, we can see that X' corresponds to a Hamilton cycle in X which is different from C. ∎

We can use constructions from Lemma 5.2.3 in a computer search in order to limit the number of cases we actually have to check. The computer search, based on Algorithms 5.2.4 and 5.2.5 (pages 86–87), can find a second Hamilton cycle for all the remaining cases — graphs of the form $X(m, a, b, c)$ — for $m \leq 25$ in a very short amount of time (under a minute with a current-day typical computer).

The algorithm performs a naive and recursive depth-first search of all possible paths in the graph $X(m, a, b, c)$ that start with a green edge, terminating when it either finds a Hamilton cycle or it has searched every single possibility. Our search tree consists of nodes which correspond to paths in X. The search tree is extended by taking a node whose corresponding path can be extended by one edge in X, and adding a leaf that contains this larger path.

We select a 1-path consisting of a green edge to be the root node of our search tree. Since a second Hamilton cycle of X has to have at least one green edge, this will not limit our search. Furthermore, not all the green edges used can be within X_0 because if they are, then all the vertices of X_0 would be visited before several vertices of X_1 and X_2. Therefore, we in fact use a green edge from $X[V_1 \cup V_2]$, and extend our search tree from there. (In practice, our choice of green edge in $X[V_1 \cup V_2]$ — as there are two different edge differences — does not appear to matter; a second Hamilton cycle can always be found. We suspect that this is because X has more than two Hamilton cycles.) It is easier for humans to discern a pattern if a small number of green edges are used, so we bias our search to include as many red edges as possible.

Though not much work has been done in optimizing the algorithm, we note that an alternative selection of the root node of the search tree can make a big difference in runtime. For example, there are many solutions that include a long subpath of just red edges. If this path is chosen as the root node, the problem effectively becomes much smaller and faster to solve. Of course, not all graphs have such a solution, so this cannot be a universal heuristic.

As the ultimate goal of the algorithm has been to identify more constructions that can be applied to many situations, varying the search method has been useful in dredging up different kinds of solutions to the same problem. As mentioned earlier, the easiest solutions (for the purposes of identifying a pattern) are the ones which use as few green edges as possible: note that a few of the constructions in Lemma 5.2.3 use only four green edges. Such solutions are also most quickly found by our computer search because we have conducted our search in such a red-edge-biased manner.

Algorithm 5.2.4.

```
main function:
input  : An integer $m \geq 3$
output: A second Hamilton cycle for each graph $X(m, a, b, c)$ with $a < b$, or
        an error message if none was found
foreach $a, b, c$ which define a graph as in Definition 5.2.2
do
    if $a, b,$ and $c$ do not satisfy any of the conditions of Lemma 5.2.3 then
        Generate the graph $X = X(m, a, b, c)$;
        // initialize the search:
        $v_0 \leftarrow 1$;
        $v_1 \leftarrow 3a + 2$;
        output findHamCycle $(X, (v_0, v_1))$;

end
```

Algorithm 5.2.5.

```
findHamCycle function:
input  : A graph $X = X(m, a, b, c)$ and an initial path $P = (v_0, \dots, v_k)$
output: A Hamilton cycle containing $P$, or null if none was found

if $k = 3m - 1$ and $v_k \sim v_0$ then
    return $P(v_k, v_0)$; // Recall the notation for concatenation
else if $k = 3m - 1$ and $v_k \not\sim v_0$ then
    return null;
if $v_k + 1 \notin V(P)$ then $secondCycle \leftarrow$ findHamCycle($X, P(v_k, v_k + 1)$);
if $secondCycle =$ null and $v_k - 1 \notin V(P)$ then
    $secondCycle \leftarrow$ findHamCycle($X, P(v_k, v_k - 1)$);
if $secondCycle =$ null and $v_k \equiv 0$ (mod 3) then
    if $v_k + 3c \notin V(P)$ then
        $secondCycle \leftarrow$ findHamCycle($X, P(v_k, v_k + 3c)$);
    if $secondCycle =$ null and $v_k - 3c \notin V(P)$ then
        $secondCycle \leftarrow$ findHamCycle($X, P(v_k, v_k - 3c)$);
else if $secondCycle =$ null and $v_k \equiv 1$ (mod 3) then
    if $v_k + 3a + 1 \notin V(P)$ then
        $secondCycle \leftarrow$ findHamCycle($X, P(v_k, v_k + 3a + 1)$);
    if $secondCycle =$ null and $v_k + 3b + 1 \notin V(P)$ then
        $secondCycle \leftarrow$ findHamCycle($X, P(v_k, v_k + 3b + 1)$);
else if $secondCycle =$ null and $v_k \equiv 2$ (mod 3) then
    if $v_k - 3a - 1 \notin V(P)$ then
        $secondCycle \leftarrow$ findHamCycle($X, P(v_k, v_k - 3a - 1)$);
    if $secondCycle =$ null and $v_k - 3b - 1 \notin V(P)$ then
        $secondCycle \leftarrow$ findHamCycle($X, P(v_k, v_k - 3b - 1)$);
return $secondCycle$;
```

One might wonder how many cases there are left to check. A large number of cases are eliminated by Corollary 4.6.3; these are solved if m is a power of 2 or an odd prime. Many other cases are covered through an application of the techniques used in Theorem 4.3.2, which we summarize in the following proposition.

Proposition 5.2.6. *Let $X = X(m, a, b, c)$ be a graph as defined in Definition 5.2.2. Then X has a second Hamilton cycle if any of the following are true:*

(A) $m = 2^k m'$, *with* $k \geq 1$ *and* m' *odd; and* $2^k \nmid c$;

(B) *GCD*$(c, m) = 1$.

Proof:

(A) We apply Lemma 2.2.1 to obtain a green perfect matching in X_0. Since $X[V_1, V_2]$ is a bicirculant formed by its green edges, it has a green perfect matching as well. Taking the union of these perfect matchings gives the edge set of a 1-factor F for X. Corollary 3.2.3 tells us that X admits a second Hamilton cycle.

(B) If GCD$(c, m) = 1$, then X_0 is connected by Lemma 2.2.3 and, therefore, is hamiltonian by Lemma 2.2.4. We proceed as in Case (II) of Theorem 4.3.2 to determine that X has a second Hamilton cycle.

∎

The following table(s) show, for each $m \leq 80$, the proportion of cases that can be solved by Proposition 5.2.6 (the second column); the proportion of cases that can be solved with the constructions in Lemma 5.2.3 (the third column); and the proportion of cases that cannot be solved by either (the fourth column). The last column shows how many possible triples (a, b, c) there are for the given m. The rows that correspond to m that is a power of 2 or an odd prime have been omitted, as these are completely solved by Corollary 4.6.3.

Table 5.1: Number of Solved and Unsolved Cases by Order

n	m	Proposition 5.2.6 (% of total)	Lemma 5.2.3 (% of total)	Unsolved (% of total)	Total
18	6	80.0	20.0	0.0	20
27	9	96.4	3.6	0.0	84
30	10	84.7	15.3	0.0	144
36	12	92.7	7.3	0.0	275
42	14	85.9	13.2	0.9	468
45	15	91.8	8.1	0.2	546
54	18	82.0	15.9	2.1	1088
60	20	93.1	6.2	0.6	1539
63	21	93.7	5.8	0.5	1710
66	22	86.7	11.4	1.9	2100
72	24	96.7	2.7	0.6	2783
75	25	99.3	0.6	0.0	3036
78	26	86.8	10.9	2.3	3600
81	27	96.3	3.1	0.6	3900
84	28	93.4	5.6	1.0	4563
90	30	81.2	16.6	2.2	5684
99	33	94.9	4.3	0.8	7440
102	34	87.0	10.3	2.7	8448
105	35	97.8	1.9	0.3	8976
108	36	91.7	6.6	1.7	10115
114	38	87.1	10.1	2.8	11988
117	39	95.2	4.0	0.9	12654
120	40	96.9	2.3	0.8	14079
126	42	81.6	15.2	3.2	16400
132	44	93.6	5.1	1.3	18963
135	45	92.0	6.6	1.3	19866

Table 5.2: Number of Solved and Unsolved Cases by Order, continued

n	m	Proposition 5.2.6 (% of total)	Lemma 5.2.3 (% of total)	Unsolved (% of total)	Total
138	46	87.2	9.8	3.1	21780
144	48	98.5	1.0	0.5	24863
147	49	99.8	0.2	0.0	25944
150	50	85.0	11.2	3.8	28224
153	51	95.5	3.6	0.9	29400
156	52	93.6	4.9	1.5	31875
162	54	82.5	12.9	4.6	35828
165	55	98.5	1.3	0.2	37206
168	56	96.9	2.3	0.8	40095
171	57	95.6	3.4	1.0	41580
174	58	87.2	9.5	3.3	44688
180	60	90.9	7.1	2.0	49619
186	62	87.3	9.4	3.3	54900
189	63	93.7	4.8	1.5	56730
195	65	98.7	1.1	0.2	62496
198	66	82.2	13.8	4.0	66560
204	68	93.7	4.7	1.6	72963
207	69	95.7	3.3	1.0	75174
210	70	84.1	12.4	3.5	79764
216	72	96.0	2.9	1.1	86975
222	74	87.3	9.2	3.5	94608
225	75	92.1	5.9	2.0	97236
228	76	93.7	4.7	1.6	102675
231	77	99.3	0.6	0.1	105450
234	78	82.2	13.4	4.4	111188
240	80	98.5	1.1	0.5	120159

5.3 Concluding Remarks

While it would be nice to improve the aforementioned algorithms in order to verify larger and larger cases, that may be missing the point somewhat. While the constructions in Lemma 5.2.3 are used to improve the computer search, the computer search is, in turn, meant to reveal patterns in the Hamilton cycles it finds. We hope that these patterns can give rise to more constructions, until every case can be covered, rendering a more in-depth computer search unnecessary.

We conclude on a speculative note, which may be of some interest for future research. A random algorithm was also applied, which is similar to the deterministic algorithms presented except that leaves are added into the search tree by extending a path of X with a random edge. In a few tests, this increased the runtime, on average, by a factor of at least 10^3. This is unusual at first blush, but understandable. The graphs $X(m, a, b, c)$ are highly structured, so it is easy to imagine that at least one of the Hamilton cycles beyond the first would have a pattern that can be exploited by a deterministic algorithm. Interestingly, though the runtime of the random algorithm is much slower than the deterministic algorithm, we note that the runtime is still much smaller than expected if we assume there are only two Hamilton cycles in the graph. In reality, there may be a number of Hamilton cycles that is exponential in the order of the graph. If we can determine for a graph X what the average time to find a Hamilton cycle is with a random search, then we may be able to estimate more precisely the number of Hamilton cycles in X without performing an exhaustive search.

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