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SUPERINVOLUTIONS OF ASSOCIATIVE SUPERALGEBRAS.

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Abstract

In the first part of this thesis, we study the existence of superinvolutions of both kinds on finite dimensional central simple superalgebras. We study the group of automorphisms of a central simple associative superalgebra with superinvolution, and then study the group of automorphisms for two examples in detail.

In the second part, we give the definition of projective supermodules; it is used to study *separable* superalgebras over super-commutative super-rings. We prove that a central separable superalgebra over a given field is a finite dimensional central simple superalgebra over the field and vice versa. We also present the fundamental theory of separable superalgebras, with the attendant theory of the Brauer-Wall group of supercommutative super-ring. Finally, we give the definition of superinvolutions of central separable superalgebras, and then extend two results on involutions of central separable algebras proved by D.J. Saltman.

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Dedication

I dedicate this work to my parents.

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Introduction

An associative superalgebra is simply a $\mathbf{Z}_2$-graded associative algebra, $\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{1}}$ the elements of $\mathcal{A}_{\bar{0}}$ are said to be *even*, elements of $\mathcal{A}_{\bar{1}}$, *odd.* A *superinvolution* of an associative superalgebra $\mathcal{A}$ is a graded linear map $*$ from $\mathcal{A}$ into $\mathcal{A}$ such that

$$a^{**} = a \quad and \quad (a_\alpha b_\beta)^* = (-1)^{\alpha\beta} b_\beta^* a_\alpha^*, \quad a_\alpha \in \mathcal{A}_\alpha \ , \ b_\beta \in \mathcal{A}_\beta.$$

Our main purpose in the first part of this thesis is to study the existence of superinvolutions of both kinds on finite dimensional central simple associative superalgebras. We extend the *Skolem-Noether* Theorem to the supercase, and use it in our work to study proper analogues of the *Albert-Riehm* results on the existence of involutions.

In Chapter 1, we introduce some basic results about finite dimensional central simple algebras, finite dimensional central simple superalgebras, and *Azumaya* superalgebras.

If $\mathcal{A}$ is a finite dimensional central simple superalgebra over a field K, and $*$ is a superinvolution on $\mathcal{A}$, then $*$ is called a *superinvolution of the first kind* if the restriction $*|_K = id_K$, and $*$ is called a *superinvolution of the second kind* if $\sigma = *|_K$ is a Galois automorphism of order 2 on K.
In Chapter 2, we prove that over a field of characteristic not 2, a finite dimensional central simple superalgebra $\mathcal{A} = M_n(\mathcal{D})$, where $\mathcal{D}$ is a division superalgebra with non-trivial grading, does not have a superinvolution of the first kind. Also we prove for a finite dimensional central simple superalgebra $\mathcal{A} = M_n(\mathcal{D})$ over any field K, where $\mathcal{D}$ is a division superalgebra with non-trivial grading whose center is even, has a superinvolution of the second kind if and only if the corestriction of A $(\mathrm{Tr}(A))$

splits. However, if the center of $\mathcal{D}$ is not even, it is of the form $K[u]$, $u \in \mathcal{A}_{\bar{1}}$, $u^2 = \lambda \in K$. If the characteristic of K is not 2, then such an $\mathcal{A}$ has a superinvolution of the second kind (say $*$) if and only if the corestriction of A splits and $\lambda^* = -\lambda$.

In Chapter 3, we use the classical Skolem-Noether Theorem to study the automorphisms of a central simple superalgebra A with a superinvolution, over fields of characteristic not 2. We prove for A, with a superinvolution $*$, that the group of inner automorphisms of $(A, *)$ is

$$\mathrm{inAut}(A, *) \cong G/\sim = \{[b_{\bar{0}}] \mid b_{\bar{0}} \in G\},$$

where $G = \{b_{\bar{0}} \in A_{\bar{0}} \mid b_{\bar{0}} b_{\bar{0}}^* = \alpha \in K^\times\}$. We use this and the analogous fact from the algebra case (see [27, pages 593, 595]), to study in detail the group of automorphisms of $A = M_{2p}(D)$, and $A = M_{p+q}(D)$, where $D = \mathbf{H}$ or $\mathbf{R}$, where $\mathbf{H}$ is the Hamiltonian Quaternion algebra and $\mathbf{R}$ is the field of real numbers, with a given superinvolution.

The main purpose of the second part of this thesis is to study the *separable* superalgebras over supercommutative super-rings, and to show that a central separable superalgebra over a given field (or Azumaya superalgebra), is a finite dimensional central simple superalgebra over the field and vice versa.
In Chapter 4, we give the definition of projective supermodules, and we extend the Generalized Nakayama Lemma to the super case. The most important results of this chapter are the Morita Theorems (section 3) and the theorem of invariance of rank for projective supermodules over supercommutative super-rings with no idempotents other than 0 and 1 (section 4).

In Chapter 5, we present the fundamental theory of non-supercommutative separable superalgebras, with the attendant theory of the Brauer-Wall group of supercommutative super-ring. In the last section, we give the definition of superinvolutions on Azumaya superalgebras over supercommutative super-rings, and then we extend two results on involutions of central separable algebras proved by David J. Saltman [23, theorem 4.2, 4.4].

Chapter 1

Background

The first part of this chapter contains basic results about finite dimensional simple algebras. A large part of the results was developed by Albert, Artin, Brauer, and Noether. The proof of these results can be found in many algebra texts such as [16], [8], and [25]. In the second part of this chapter we introduce some basic results about finite dimensional central simple superalgebras obtained by C. T. C. Wall [26] and M. Racine [21]. In the final part of this chapter we recall some basic results about *Azumaya* algebras, the proof of which can be found in [18], [6], and [23].

1.1 Simple Algebras and Involutions

Definition 1.1.1 *Let R be a ring. An R-module $M \neq 0$ is called* simple *if it has no submodules other than $\{0\}$ and M. The ring R is called* simple *if it has no two-sided ideals other than (0) and R.*

Theorem 1.1.2 (Schur's lemma) If M is a simple R-module, then the endomorphism ring $A = \mathrm{End_R(M)}$ is a division ring.

Definition 1.1.3 *A ring R is said to be* artinian *if any non-empty set of right ideals has a minimal element.*

Theorem 1.1.4 (Wedderburn-Artin) Let R be a simple artinian ring. Then R is isomorphic to $M_n(D)$, the ring of all $n \times n$ matrices over the division ring D. Moreover

n and D are unique up to isomorphism. Conversely, for any division ring D, $M_n(D)$ is a simple Artinian ring.

Corollary 1.1.5 Let K be a field and R a finite-dimensional simple K-algebra. Then $R \cong M_n(D)$ for a suitable division ring D.

Definition 1.1.6 *A finite dimensional K-algebra A is called* central *if K is exactly the center of A. For any subset B of A the* centralizer *of B in A is*

$$Z_A(B) = \{a \in A \mid ab = ba \ \forall b \in B\}.$$

In this section our main subject is finite dimensional central simple algebras. We will use the abbreviation c.s. algebra for this concept.

Theorem 1.1.7 (i) If A and B are finite dimensional K-algebras and $A' \subset A$, $B' \subset B$ subalgebras, then

$$Z_{A \otimes_K B}(A' \otimes_K B') = Z_A(A') \otimes_K Z_B(B').$$

In particular, $A \otimes_K B$ is central if A and B are central.
(ii) If A is a c.s. algebra and B is simple, then $A \otimes_K B$ is simple.
(iii) $A \otimes_K B$ is a c.s. algebra if A and B are.

Definition 1.1.8 *If R is a ring we define a ring R° by $R^\circ = R$ as additive abelian groups while the product $\circ$ in R° is defined by $a \circ b = ba$. The ring R° is called the* opposite ring *of R.*

It is obvious that R° is in fact a ring. Moreover, R is central or simple or a division ring exactly when R° is. Clearly $R^{\circ\circ} = R$, and $R = R^\circ$ if and only if R is commutative. In general, R and R° are not isomorphic. If the K-algebra A is a c.s. algebra, then A° is a c.s. algebra.

Theorem 1.1.9 Let A be a c.s. algebra over K. Then $A \otimes_K A^\circ \cong M_n(K)$ where $n = dim_K(A)$. $\square$

Definition 1.1.10 *Two c.s. algebras A and B over K are called* similar $(A \sim B)$ *if for some integers m and n, $A \otimes_K M_m(K) \cong B \otimes_K M_n(K)$.*

Similarity is obviously an equivalence relation. The set of similarity classes will be denoted by $\mathrm{Br}(K)$. If $[A]$ denotes the class of A we define a product in $\mathrm{Br}(K)$ by $[A][B] = [A \otimes_K B]$, one can easily check that this product is well-defined. The class of the matrix algebras $M_n(K)$ is a neutral element for this product. By the previous theorem every element in $\mathrm{Br}(K)$ has an inverse, namely the similarity class of the opposite algebra. Therefore, $\mathrm{Br}(K)$ is a group which is called the *Brauer group* of K.

Lemma 1.1.11 If K is an algebraically closed field, then $\mathrm{Br}(K) = \{1\}$.

If A is a K-algebra and α an invertible element of A, then $x \mapsto \alpha x \alpha^{-1}$ is an automorphism of A. Automorphisms of this kind are called *inner automorphisms.*

Theorem 1.1.12 (Skolem-Noether) Let A be a c.s. algebra over K and B a simple K-algebra. Let $\sigma, \tau : B \to A$ be two algebra homomorphisms. Then there exists an inner automorphism φ such that $\tau = \varphi\sigma$.

Corollary 1.1.13 Every automorphism φ of a c.s. algebra A is an inner automorphism.

Definition 1.1.14 *Let R be a ring. A map $* : R \to R$ is called an* involution *if*

$$(x + y)^* = x^* + y^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x$$

for all x, y in R. The pair $(R, *)$ is called a ring with involution.

Remark. If $(A, *)$ is a finite dimensional K-algebra with involution $*$, then $K^* = K$, and therefore we distinguish two possibilities:
(i) $*|_K$ is the identity, that is $*$ is K-linear. In this case $*$ is said to be of the *first kind.*
(ii) $*|_K$ is not the identity, that is $*|_K$ is a non-trivial automorphism σ of K. In this case $*$ is σ-linear, that is $(\lambda x)^* = \sigma(\lambda) x^*$ for all $x \in A$, $\lambda \in K$. The involution is said to be of the *second kind.* If k denotes the fixed field of σ, we get the separable

quadratic extension K/k.

We now give a theorem which allows us to divide involutions of the first kind into two separate classes (if $\mathrm{Char}(K) \neq 2$). If $(A, *)$ is a finite K-algebra with involution $*$ we define

$$\mathcal{H}(A, *) = \{x \in A \mid x^* = x\} \quad \text{(symmetric elements)}$$

$$\mathcal{S}(A, *) = \{x \in A \mid x^* = -x\} \quad \text{(skew symmetric elements)}$$

Theorem 1.1.15 Let A be a c.s. algebra of dimension n^2 over K and $*$ an involution.
(i) If $\mathrm{char}(K) \neq 2$ then $A = \mathcal{H}(A, *) \oplus \mathcal{S}(A, *)$ as vector spaces.
(ii) If $*$ is of the second kind, then

$$\dim_k(\mathcal{H}(A, *)) = \dim_k(\mathcal{S}(A, *)) = n^2,$$

where k is the fixed field of $\sigma = *|_K$.
(iii) If $*$ is of the first kind, then either $\dim_K(\mathcal{H}(A, *)) = \frac{1}{2}n(n+1)$ or $= \frac{1}{2}n(n-1)$.
If $char(K) = 2$ one has always $\dim_K(\mathcal{H}(A, *)) = \frac{1}{2}n(n+1)$. □

Let A be a c.s. algebra of dimension n^2 over K, and $*$ an involution of the first kind. Let L be a splitting field of A. Extend $*$ to an L-linear map (say J) on $A_L = A \otimes_K L \cong M_n(L)$. The involution $*$ is called of *orthogonal type* if there exist a diagonal matrix $H = \mathrm{diag}\{\gamma_1, ..., \gamma_n\}$, where $\gamma_i \neq 0 \in L$ such that

$$X^J = HX^tH^{-1} \;\; \forall\, X \in M_n(L).$$

The involution $*$ is called of *symplectic type* (and in this case n is even) if there exists a matrix $S = \mathrm{diag}\{Q, ..., Q\}$, where $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, such that

$$X^J = SX^tS^{-1}, \;\; \forall\, X \in M_n(L).$$

Examples.
(i) If $A = M_n(K)$, where K is a field, then the transposition is an involution (of the

first kind).
(ii) However, there may exist essentially different involutions on $M_n(K)$. For example

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \rightarrow \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}$$

is the symplectic involution on $M_2(K)$.
(iii) If R is a ring with involution $*$, then $(\alpha_{ij}) \rightarrow (\alpha_{ij}^*)^t$ is an involution on $M_n(R)$.
(iv) Let K/k be a Galois extension with Galois group $G = \{1, \sigma\}$, then $(\alpha_{ij}) \rightarrow (\alpha_{ij}^\sigma)^t$ is an involution of the second kind on $M_n(K)$.

The next theorem gives us the condition on a c.s. algebra A over K to have an involution of the first kind.

Theorem 1.1.16 (Albert) *Let A be a c.s. algebra over K. Then A admits an involution of the first kind if and only if $A \cong A^\circ$, or equivalently A is of order 2 in the Brauer group.*

Let K/k be a separable quadratic extension with non-trivial automorphism σ. We will give a condition on a c.s. algebra A over K to have an involution of the second kind extending σ. We first define the *conjugate* algebra A^σ. We set $A^\sigma = A$ as a ring, but the product by a scalar in A^σ is defined by $\lambda \cdot a = \sigma(\lambda)a$. Of course, A^σ is a c.s. algebra over K.
Now we consider the canonical map

$$\tau : A \otimes_K A^\sigma \rightarrow A \otimes_K A^\sigma, \quad a \otimes b \mapsto b \otimes a.$$

This is obviously a ring automorphism of order 2. Moreover τ is σ-linear; in particular, it is k-linear. We define

$$\mathrm{Tr}(A) = \{x \in A \otimes_K A^\sigma \mid \tau(x) = x\}.$$

Obviously, $\mathrm{Tr}(A)$ is a k-algebra.

Definition 1.1.17 *The k-algebra* $\mathrm{Tr}(A)$ *is called the* corestriction *of the algebra A.*

Lemma 1.1.18 There is a *canonical* K-algebra isomorphism

$$\mathrm{Tr}(A) \otimes_k K \cong A \otimes_K A^\sigma.$$

Corollary 1.1.19 If A is a c.s. algebra of K then $\mathrm{Tr}(A)$ is a c.s. algebra over k.

We can formulate the existence criterion for unitary involutions:

Theorem 1.1.20 (Albert, Riehm) Let K/k be a separable quadratic extension and A a c.s. algebra over K. Then A admits a K/k-involution of the second kind if and only if $\mathrm{Tr}(A)$ splits.

1.2 Central Simple Superalgebras

In this section we start with some examples of central simple associative superalgebras, and then we give some basic results about central simple superalgebras obtained by C. T. C. Wall [26] and M. Racine [21].

First we start with the following definitions.

Definition 1.2.1 *An* associative super-ring *is a* $\mathbf{Z}_2$*-graded associative ring. An associative superalgebra* $\mathcal{A}$ *is a* $\mathbf{Z}_2$*-graded* K*-algebra (where* K *is a field)*

$$\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{1}}$$

which is associative and such that $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$.

Definition 1.2.2 *If* $\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{1}}$ *is an associative superalgebra then the elements in* $\mathcal{A}_{\bar{0}} \cup \mathcal{A}_{\bar{1}}$ *are called the* homogeneous elements *of* $\mathcal{A}$*. The elements of* $\mathcal{A}_{\bar{0}}$ *are said to be* even *and those of* $\mathcal{A}_{\bar{1}}$, odd.

Lemma 1.2.3 If $R = R_{\bar{0}} + R_{\bar{1}}$ is a unital associative super-ring then $1 \in R_{\bar{0}}$.

Proof Assume that $1 = e_{\bar{0}} + e_{\bar{1}}$ where $e_{\bar{0}} \in R_{\bar{0}}$ and $e_{\bar{1}} \in R_{\bar{1}}$. Then $e_{\bar{0}} = 1 \cdot e_{\bar{0}} = e_{\bar{0}}.1$ which implies that $e_{\bar{0}}^2 = e_{\bar{0}}$ and $e_{\bar{0}}e_{\bar{1}} = e_{\bar{1}}e_{\bar{0}} = 0$. Therefore $e_{\bar{1}} = 1 \cdot e_{\bar{1}} = e_{\bar{0}}e_{\bar{1}} + e_{\bar{1}}^2 = e_{\bar{1}}^2$ in $R_{\bar{0}}$ and so $e_{\bar{1}} = 0$, since $e_{\bar{1}} \in R_{\bar{0}} \cap R_{\bar{1}} = \{0\}$. $\square$

Definition 1.2.4 *A unital super-ring R is said to be a* division super-ring *if all nonzero homogeneous elements are invertible.*

Examples:

(1) Let $\mathbf{R}$ be the field of real numbers and let $\lambda, \mu \in \mathbf{R}^{\times}$ such that $\lambda, \mu < 0$, then the quaternion algebra $\mathcal{A} = \mathbf{R}1 + \mathbf{R}u + \mathbf{R}v + \mathbf{R}uv$, where $u^2 = \lambda 1$, $v^2 = \mu 1$, and $uv = -vu$, is a division superalgebra whose center is even. The grading on $\mathcal{A}$ is $\mathcal{A}_{\bar{0}} = \mathbf{R}1 + \mathbf{R}uv$, $\mathcal{A}_{\bar{1}} = \mathbf{R}u + \mathbf{R}v$.

(2) If K is any field then $\mathcal{D} = K1 + Ku$, $u^2 = \lambda \in K^{\times}$, is a division superalgebra whose center is not even. The grading on $\mathcal{D}$ is $\mathcal{D}_{\bar{0}} = K1$, $\mathcal{D}_{\bar{1}} = Ku$.

Definition 1.2.5 *The opposite super-ring R° of the super-ring R is defined to be $R^{\circ} = R$ as an additive group, with the multiplication given by*

$$b_\beta \circ c_\gamma := (-1)^{\beta\gamma} c_\gamma b_\beta \qquad b_\beta \in R_\beta \ , \ c_\gamma \in R_\gamma .$$

Definition 1.2.6 *If $R = R_{\bar{0}} + R_{\bar{1}}$ is an associative super-ring, a (right) R-*supermodule *M is a right R-module with a grading $M = M_{\bar{0}} + M_{\bar{1}}$ as $R_{\bar{0}}$-modules such that $m_\alpha r_\beta \in M_{\alpha+\beta}$ for any $m_\alpha \in M_\alpha$, $r_\beta \in R_\beta$, $\alpha, \beta \in \mathbf{Z}_2$. An R-supermodule M is* simple *if $MR \neq \{0\}$ and M has no proper subsupermodule.*

Definition 1.2.7 *An R-*supermodule homomorphism *from M to N, where M and N are R-supermodules, is an $R_{\bar{0}}$-module homomorphism $h_\gamma : M \to N$, $\gamma \in \mathbf{Z}_2$, such that $h_\gamma M_\alpha \subseteq N_{\alpha+\gamma}$ and*

$$h_\gamma(m_\alpha r_\beta) = (h_\gamma m_\alpha) r_\beta, \ \ \forall\, m_\alpha \in M_\alpha,\ r_\beta \in R_\beta,\ \alpha, \beta \in \mathbf{Z}_2.$$

Definition 1.2.8 *A* (right) super-ideal *$I = I_{\bar{0}} + I_{\bar{1}}$ is a (right) subsupermodule of the super-ring R considered as a (right) R-supermodule. An associative super-ring is* simple *if it has no non-trivial two-sided super-ideal.*

Definition 1.2.9 *A* superinvolution *of an associative superalgebra $\mathcal{A}$ is a graded additive map $* : \mathcal{A} \to \mathcal{A}$ such that*

$$a^{**} = a \quad \text{and} \quad (a_\alpha b_\beta)^* = (-1)^{\alpha\beta} b_\beta^* a_\alpha^* .$$

If $\mathcal{A}$ is of characteristic 2, this is nothing more than an involution respecting the grading. An associative superalgebra $\mathcal{A}$ over a field K is *central* if $\hat{Z}(\mathcal{A}) = K$, where

$$\hat{Z}(\mathcal{A}) = \{a_\alpha \in \mathcal{A}_\alpha \mid a_\alpha b_\beta = (-1)^{\alpha\beta} b_\beta a_\alpha \ \forall \ b_\beta \in \mathcal{A}_\beta,\ \alpha, \beta \in \mathbf{Z}_2\}.$$

If $\hat{Z}(\mathcal{A}) = \mathcal{A}$, then we say that $\mathcal{A}$ is *supercommutative.*

If $\mathcal{A}$ is a central associative superalgebra over K and $*$ a superinvolution on $\mathcal{A}$ then we say that the superinvolution $*$ is of the *first kind* if $*|_K = id_K$ and it is of the *second kind* if $\sigma = *|_K$ is a Galois automorphism on K of order 2.

Examples of associative superalgebras.

(i) Let K be a field of characteristic not 2, then the quaternion algebra $A = K1 + Ku + Kv + Kuv$, where $u^2 = \lambda$, $v^2 = \mu$, and $uv = -vu$. In this case the grading on A is $A_{\bar{0}} = K1 + Kuv$, $A_{\bar{1}} = Ku + Kv$.

(ii) The algebra of $p+q \times p+q$ matrices $M_{p+q}(D)$, where D is a division algebra, can be viewed as an associative superalgebra by taking the diagonal components $M_p(D)$ and $M_q(D)$ as the even part and the off-diagonal components as the odd part; this is an example of simple associative superalgebra.

(iii) A *superspace* over K is a left K-vector space which is $\mathbf{Z}_2$-graded $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The associative algebra $\mathrm{End}_K(V) = [\mathrm{End}_K(V)]_{\bar{0}} + [\mathrm{End}_K(V)]_{\bar{1}}$, where $[\mathrm{End}_K(V)]_\alpha := \{a \in End_K(V) \mid v_\beta a \in v_{\alpha+\beta}\}$, is an associative superalgebra.

An associative super-ring is (right) *artinian* if it satisfies the descending chain condition on right super-ideals. A *superspace* over an associative division superalgebra $D = D_{\bar{0}} + D_{\bar{1}}$ is a left D-supermodule V such that $V = V_{\bar{0}} \oplus V_{\bar{1}}$ as a $D_{\bar{0}}$ (left) vector space. Let $\dim_{D_{\bar{0}}} V_{\bar{0}} = p$ and $\dim_{D_{\bar{0}}} V_{\bar{1}} = q$. If $p + q < \infty$ then we say that V is *finite dimensional.* If $D_{\bar{1}} \neq \{0\}$ then for any $0 \neq d_{\bar{1}} \in D_{\bar{1}}$, $d_{\bar{1}} V_{\bar{0}} \subset V_{\bar{1}}$ and $d_{\bar{1}} V_{\bar{1}} \subset V_{\bar{0}}$. Then, $p = q$ and $\mathrm{End}_D(V) \cong M_p(D)$. If $D_{\bar{1}} = \{0\}$ then $\mathrm{End}_D(V) \cong M_{p+q}(D)$ as in Example(ii). Thus the grading of $\mathrm{End}_D(V)$ is induced by the grading of D if $D_{\bar{1}} \neq \{0\}$ and by a partition of $\dim_D(V) = n = p + q$ if $D = D_{\bar{0}}$.

An associative super-ring is simple if it has no non-trivial graded ideal.

Theorem 1.2.10 If $A = A_{\bar{0}} + A_{\bar{1}}$ is an artinian simple associative super-ring then, as a super-ring, $A \cong End_D(V)$, V a finite dimensional superspace over an associative division superalgebra D.

We recall the Division Superalgebra Theorem which was proved in [21].

Theorem 1.2.11 (Division Superalgebra Theorem) If $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$ is a central division superalgebra over a field K, then exactly one of the following holds, where throughout $\mathcal{C}$ denotes a central division algebra over K.

(i) $\mathcal{D} = \mathcal{D}_{\bar{0}} = \mathcal{C}$ i.e., $\mathcal{D}_{\bar{1}} = 0$.

(ii) $\mathcal{D} = \mathcal{C} \otimes_K K[u]$, $u^2 = \lambda \in K^{\times}$, the multiplicative group of K, $\mathcal{D}_{\bar{0}} = \mathcal{C} \otimes K$; $\mathcal{D}_{\bar{1}} = \mathcal{C} \otimes Ku$, where K in this case is a field of characteristic not 2.

(iii) $\mathcal{D} = \mathcal{C}$, $\mathcal{D}_{\bar{0}} = C_{\mathcal{C}}(u)$, the centralizer of u in $\mathcal{C}$, $\mathcal{D}_{\bar{1}} = \{d \in \mathcal{C} \mid du = u^{\sigma}d\}$, for some quadratic Galois extension $K[u] \subseteq \mathcal{C}$ with Galois automorphism σ.

(iv) $\mathcal{D} = M_2(\mathcal{C}) \cong \mathcal{C} \otimes_K M_2(K)$; $\mathcal{D}_{\bar{0}} = \mathcal{C} \otimes K[u]$, $\mathcal{D}_{\bar{1}} = \mathcal{C} \otimes K[u]w$

where

$u = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$, $w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(K)$, $\lambda \notin K^2$, $Char(K) \neq 2$,

$u = \begin{pmatrix} 0 & 1 \\ \lambda & 1 \end{pmatrix}$, $w = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in M_2(K)$, $\lambda \notin \{\alpha + \alpha^2 \mid \alpha \in K\}$, $Char(K) = 2$

and $K[u]$ does not embed in $\mathcal{C}$.

(v) $\mathcal{D} = \mathcal{C} + \mathcal{C}v$, $\mathcal{D}_{\bar{0}} = \mathcal{C}$, $\mathcal{D}_{\bar{1}} = \mathcal{C}v$, $v^2 = d \in \mathcal{C}^{\times}$, $va = a^{\phi}v$ $(a^{\phi} = vav^{-1})$ $\forall a \in \mathcal{C}$, where ϕ is an outer automorphism of $\mathcal{C}$ over K such that $\phi^2 = \psi_d$, where $\psi_d(a) = dad^{-1}, \forall a \in \mathcal{C}$, and $d^{\phi} = d$.

This last case can occur only if $\mathcal{C}$ is infinite dimensional over its center K. □

Note that in theorem 1.2.11, $Z(\mathcal{D}) \cap \mathcal{D}_{\bar{1}} \neq 0$ if $\mathcal{D}$ is of type (ii). Hence, following Wall [26], we call it a *division superalgebra of odd type.* If $\mathcal{D}$ is of type (i), (iii) or (iv), $Z(\mathcal{D}) \cap \mathcal{D}_{\bar{1}} = 0$. Therefore, we call it a *division superalgebra of even type.*

Also we should recall that a central simple superalgebra over a field K is a finite dimensional central simple superalgebra over K.

Proposition 1.2.12 Let $A = M_n(D)$ be a finite dimensional central simple superalgebra over field K, where D is a division superalgebra with non-trivial grading, then

A has a superinvolution if and only if D has a superinvolution. The superinvolutions on A and D are of the same kind.

Proposition 1.2.13 Let $D = D_{\bar{0}} \otimes_K K[u]$, $u^2 = \lambda \in K^\times$, $D_{\bar{1}} = D \otimes Ku$. If D has a superinvolution, then we can choose u such that $u^* = u$ and $\lambda^* = -\lambda$. If the characteristic is not 2, this implies that $*|_{D_{\bar{0}}}$ is of the second kind. Conversely, if $^-$ is an involution of $D_{\bar{0}}$ and $\lambda \neq 0$ is an element of K the center of $D_{\bar{0}}$ such that $\bar{\lambda} = -\lambda$, the superalgebra $D = D_{\bar{0}} \otimes_K K[u]$, $u^2 = \lambda$, has a superinvolution $*$ extending $^-$ given by

$$(a + bu)^* := \bar{a} + \bar{b}u.$$

□

If D is a division superalgebra as in cases (iii) and (iv) of the Division Superalgebra Theorem(1.2.11) and $0 \neq v \in D_{\bar{1}}$ then for all $a \in D_{\bar{0}}$, $va = a^\phi v$, where $a \mapsto a^\phi := vav^{-1}$ is an automorphism of $D_{\bar{0}}$.

Proposition 1.2.14 Let $D = D_{\bar{0}} + D_{\bar{1}}$ be a division superalgebra as in cases (iii) and (iv). If D has a superinvolution $*$ then $D_{\bar{1}}$ contains a $0 \neq v = v^*$. Moreover,

$$d^* = -d, \quad \text{where } d = v^2, \quad \text{and} \tag{1}$$

$$b^*\phi = b^{\phi^{-1}*} \qquad \forall b \in D_{\bar{0}}. \tag{2}$$

Conversely, if $*$ is an involution of $D_{\bar{0}}$, satisfying (1) and (2), then

$$(a + bv)^* := a^* + b^{*\phi}v$$

extends $*$ to a superinvolution of D.

Remark. For $(D, *)$ as above. One checks that if $*$ satisfies (1) and (2), then $*\phi$ is an involution of $D_{\bar{0}}$ which extends to a superinvolution of D via

$$a + bv^{-1} \mapsto a^{*\phi} + b^*v^{-1}.$$

In order to obtain more information on central simple associative superalgebras $(\mathcal{A}, *)$ we first start with elementary results for super-rings. The next lemma is a version of a standard result for rings with involution. We say that $(\mathcal{A}, *)$ is *simple* if the only $*$-stable super-ideals of $\mathcal{A}$ are $\mathcal{A}$ and (0).

Lemma 1.2.15 If $\mathcal{A}$ is an associative super-ring with superinvolution $*$ such that $(\mathcal{A}, *)$ is simple then either $\mathcal{A}$ is simple as a super-ring or $\mathcal{A} = \mathcal{B} \oplus \mathcal{B}^*$, with $\mathcal{B}$ a simple super-ring. □

In the second case $\mathcal{B}^*$ is isomorphic to the opposite super-ring $\mathcal{B}^\circ$ of $\mathcal{B}$. We will consider a super-ring $\mathcal{A}$ with non-zero odd part. To avoid double indices, we will write $\mathcal{A} = A + B$, where $A = \mathcal{A}_{\bar{0}}$ is the even part and $B = \mathcal{A}_{\bar{1}}$ the odd part.

Theorem 1.2.16 Let $\mathcal{A} = A + B$ be an associative super-ring with $B \neq \{0\}$, and $*$ a superinvolution of $\mathcal{A}$. If $(\mathcal{A}, *)$ is simple, then either $(A, *|A)$ is simple, or

$$A = A_1 \oplus A_2, \qquad B = B_1 \oplus B_2,$$

where $(A_i, *|_{A_i})$ are simple and B_i are irreducible A-bimodules with

$$B_1^* = B_2 \quad \text{and} \quad B_2^* = B_1, \tag{3}$$

such that

$$A_1B_1 = B_1 = B_1A_2, \qquad A_2B_2 = B_2 = B_2A_1, \tag{4}$$

$$B_1B_2 = A_1, \qquad B_2B_1 = A_2, \tag{5}$$

$$\{0\} = A_2B_1 = A_1B_2 = B_1A_1 = B_2A_2 = B_1B_1 = B_2B_2. \tag{6}$$

□

Remark. If $\mathcal{A} = A_1 \oplus A_2 + B_1 \oplus B_2$ with A_i $*$-stable, B_i irreducible A-bimodules satisfying (3), (4), (5) and (6) then there is no proper $*$-stable ideal I of A with $I \cap BIB \neq \{0\}$.

We will need more information on the superinvolutions of $\mathcal{A}$ when the grading is not inherited from that of $\mathcal{D}$, that is, $\mathcal{D} = \mathcal{D}_{\bar{0}}$, and $\mathcal{A}$ is finite dimensional. If $\mathcal{A} = M_{p+q}(\mathcal{D})$, $\mathcal{A}_{\bar{0}} = M_p(\mathcal{D}) \oplus M_q(\mathcal{D})$, $p, q > 0$, then we are in one or the other of the situation, described in Theorem 1.2.16. We consider each case in turn using the notation of Theorem 1.2.16..

Proposition 1.2.17 If $\mathcal{A} = M_{p+q}(\mathcal{D})$, $p, q > 0$, is a superalgebra with $\mathcal{A}_{\bar{0}} = A = M_p(\mathcal{D}) \oplus M_q(\mathcal{D})$ and $(A, *|_A)$ is simple, then $p = q$, $M_p(\mathcal{D})$ has an involution $\tilde{}$ and $(\mathcal{A}, *)$ is isomorphic to $(M_{2p}(\mathcal{D}), *)$ with the superinvolution $*$ on $M_{2p}(\mathcal{D})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ \tilde{c} & \tilde{a} \end{pmatrix}, \tag{7}$$

for $a, b, c, d \in M_p(\mathcal{D})$. Conversely, if $M_p(\mathcal{D})$ has an involution $\tilde{}$ then (7) defines a superinvolution on the simple superalgebra $M_{2p}(\mathcal{D})$. □

Proposition 1.2.18 If $\mathcal{A} = M_{p+q}(\mathcal{D})$, $p, q > 0$, is a superalgebra with $A = A_1 \oplus A_2$, $A_1 = M_p(\mathcal{D})$, $A_2 = M_q(\mathcal{D})$, and $(A, *|_A)$ is not simple then $(A_1, *|_{A_1})$ and $(A_2, *|_{A_2})$ are of the same kind. If $\mathcal{A}$ is finite dimensional over a field of characteristic not 2 and $*$ is of the first kind then one $(A_i, *|_{A_i})$ is orthogonal type and the other of symplectic type. □

1.3 Azumaya Algebras

In this section we shall set out results about projective modules and separable algebras. The proof of these results can be found in [6] and [18]. In what follows R will always be a commutative ring with 1. All rings have a unit element, all modules are unitary.

Definition 1.3.1 *An R-module M is called* projective *if it is a direct summand of a free R-module.*

Equivalently, every epimorphism $N \to M \to 0$ splits. Another characterization of a projective R-module M is that M has projective basis $\{x_i, f_i\}$; i.e., x_i is in M, f_i in $M^* = Hom_R(M, R)$ and for each x in M, $x = \sum f_i(x)x_i$ (almost all $f_i(x)$ are zero).

If M is finitely generated and projective, then it admits a finite projective basis.
If R is commutative then an R-module M is called *faithfully projective* if it is finitely generated, projective and faithful (faithful means that its annihilator in R, $annih_R(M)$, is zero).

For any R-module M, consider the subset $\mathcal{T}_R(M)$ of R consisting of elements of the form $\sum f_i(x_i)$ where the f_i are from $Hom_R(M, R)$ and the x_i are from M, then $\mathcal{T}_R(M)$ is a two-sided ideal of R, called the *trace ideal* of M.

Lemma 1.3.2 (Generalized Nakayama Lemma) Let R be a commutative ring and M a finitely generated R-module. An ideal I of R has the property that $IM = M$ if and only if $I + annih_R(M) = R$.

Corollary 1.3.3 If M is a finitely generated, projective R-module, where R is a commutative ring, then $\mathcal{T}_R(M) \oplus annih_R(M) = R$.

Corollary 1.3.4 Suppose M is faithfully projective over R. Then $\mathcal{T}_R(M) = R$.

Proposition 1.3.5 Let R be a commutative ring and M and N, R-modules. Then
(1) $M \otimes_R N$ is faithfully projective over R if both M and N are.
(2) There is an isomorphism

$$\psi : \mathrm{End}_R(M) \otimes_R \mathrm{End}_R(N) \to \mathrm{End}_R(M \otimes_R N)$$

given by sending $f \otimes g$ to $\psi(f \otimes g)$, where $\psi(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

For an R-algebra A, we shall let A° denote the opposite algebra of A. The *enveloping algebra* is defined by $A^e = A \otimes_R A^\circ$.
A^e has the following useful property: an (A,A)-bimodule M whose induced left and right R-actions coincide (i.e., $a(mb) = (am)b$ and $rm = mr\ \forall a, b \in A, m \in M, r \in R$) is characterized as a left A^e-module, with action $(a \otimes b)m = amb$. In particular, A is a left A^e-module. There is an epimorphism of left A^e-modules

$$\mu : A^e \to A$$

satisfying $\mu(a \otimes b) = ab$.

Definition 1.3.6 *A is said to be R-*separable *if μ splits as an A^e-homomorphism, or, equivalently, if A is A^e-projective.*

If A is R-separable then μ is split by an A^e-map $\psi : A \to A^e$ $(\mu\psi = 1)$. Thus there exists an element $e = \psi(1)$ in A^e satisfying $\mu(e) = 1$ and $(1 \otimes a)e = (a \otimes 1)e$ for all $a \in A$. Conversely, if such an e exists, then A is R-separable. Such an e called a *separability idempotent.*

Proposition 1.3.7 Let A be a separable R-algebra which is projective as an R-module. Then A is finitely generated as an R-module.

Corollary 1.3.8 Let K be a field and A a separable K-algebra. Then $\dim_K(A)$ is finite and $L \otimes_K A$ is semisimple for every field extension L of K.

Remark. Let K be a field. If A is a finite dimensional K-algebra, semisimplicity of $L \otimes_K A$, for every field extension L of K, is equivalent to A being K-separable.

Definition 1.3.9 *An R-algebra A that is faithful, separable over R and has R as its center, will be called a* central separable R-algebra *(or* Azumaya algebra*).*

Remark. If $A = Hom_R(P, P)$, where P is a faithfully projective R-module, then A is a central separable R-algebra.

Proposition 1.3.10 Let R be a field and let A be an R-algebra. Then A is a central simple R-algebra if and only if A is a central separable R-algebra.

For any R-algebra A we have seén that A is naturally a left A^e-module. This structure induces an R-algebra homomorphism $\phi : A^e \to \mathrm{Hom}_R(A, A)$ defined by $\phi(a \otimes b)(x) = axb$ for all $a, b, x \in A$. The next theorem states, in the case that A is a central separable R-algebra, this algebra homomorphism ϕ is an isomorphism.

Theorem 1.3.11 Let A be an R-algebra. The following conditions are equivalent:
(1) A is a central separable R-algebra.
(2) A is A^e-projective and A is R-separable.
(3) A is faithfully projective as an R-module, and the R-algebra map ϕ is an isomorphism.

Corollary 1.3.12 Let A be a central separable R-algebra. Then there is a one-to-one correspondence between ideals I of R and ideals U of A, given by $I \to IA$ and $U \to U \cap R$.

For A,B central separable R-algebras, write $A \sim B$ if there exist faithfully projective R-modules P, Q such that $A \otimes_R \mathrm{End}_R(P) \cong B \otimes_R \mathrm{End}_R(Q)$ (R-algebra isomorphism). Then $\sim$ is an equivalence relation. Let $B(R)$ denote the set of equivalence classes of central separable R-algebras. Write $[A]$ for the class of A in $B(R)$. Define a product on $B(R)$ by $[A][B] = [A \otimes_R B]$; this is a well-defined operation. It is easy to check that $B(R)$ is an abelian group, with $[A]^{-1} = [A^\circ]$ and $[R] = 1$. $B(R)$ is called the *Brauer group* of R.

Finally, we recall two results proved by David J. Saltman [23].

Suppose R is a commutative ring, σ an automorphism of order 2, P a faithfully projective R-module, and I a rank one R projective module. A morphism $e : P \otimes P \to I$ is called a *bilinear I form* on P, a morphism $e : P \otimes P^\sigma \to I$ is called a *σ bilinear I form* on P. Such a form induces a map $e^* : P \to \mathrm{Hom}_R(P, I)$ (or $e^* : P \to \mathrm{Hom}_R(P^\sigma, I)$) given by $e^*(p)(q) = e(p,q)$. In a similar manner, we define e_* by $e_*(p)(q) = e(q,p)$. The morphism e is nondegenerate if e^*, e_* are isomorphisms. The next result shows that the existence of involutions on $\mathrm{End}_R(P)$ is equivalent to the existence of forms on P.

Theorem 1.3.13 Let $A = \mathrm{End}_R(P)$, where P is a faithfully projective R-module.
(a) A has an involution of the first kind if and only if there is a rank one R-projective I, a nondegenerate bilinear I form e on P and a δ in R such that $\delta^2 = 1$ and $e(x,y) = \delta e(y,x)$ for all x, y in P.

(b) Let σ be an automorphism of R of order 2. Then A has an involution of the second kind extending σ if and only if there is a rank one R-projective I with a σ-linear automorphism of order 2 (also called σ) a σ-bilinear I form e on P and an element δ in R such that $\sigma(\delta)\delta = 1$ and $\sigma(e(x,y)) = \delta e(y,x)$ for all x, y in P. $\square$

We call a commutative ring *local* if it contains a unique maximal ideal, and we call it *semilocal* if it is a finite direct product of local rings. Also we call a commutative ring R *connected* if the only idempotents in R are 0 and 1.

Theorem 1.3.14 Suppose R is a connected semilocal ring, and A is Azumaya over R. If R/S is a Galois extension with Galois group $\{1, \sigma\}$, then $\mathrm{Tr}(A) \sim 1$ if and only if A has an involution of the second kind extending σ.

Chapter 2

Superalgebras with superinvolution

In this chapter we will develop the theory of superinvolutions on finite dimensional central simple superalgebras. First we generalize the Skolem-Noether Theorem to the superalgebra case since we use it in this work.

2.1 Superinvolutions of the First Kind

In this section we show which kinds of superalgebras do not have a superinvolution of the first kind.

Before we generalize the Skolem-Noether Theorem, we need the following definitions.

Definition 2.1.1 *Let $\mathcal{A} = M_n(\mathcal{D})$ be a central simple superalgebra over field K. Then $\mathcal{A}$ is of* even type *if $Z(\mathcal{A}) \cap \mathcal{A}_{\bar{1}} = \{0\}$. $\mathcal{A}$ is of the* odd type *if $Z(\mathcal{A}) \cap \mathcal{A}_{\bar{1}} \neq \{0\}$.*

Definition 2.1.2 *Let $\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{1}}$, $\mathcal{B} = \mathcal{B}_{\bar{0}} + \mathcal{B}_{\bar{1}}$ be associative superalgebras. Then the graded tensor product*

$$\mathcal{A}\hat{\otimes}_K\mathcal{B} = [(\mathcal{A}_{\bar{0}} \otimes \mathcal{B}_{\bar{0}}) \oplus (\mathcal{A}_{\bar{1}} \otimes \mathcal{B}_{\bar{1}})] \oplus [(\mathcal{A}_{\bar{0}} \otimes \mathcal{B}_{\bar{1}}) \oplus (\mathcal{A}_{\bar{1}} \otimes \mathcal{B}_{\bar{0}})]$$

where the multiplication on $\mathcal{A}\hat{\otimes}_K\mathcal{B}$ is induced by

$$(a_\alpha \otimes b_\beta)(c_\gamma \otimes d_\delta) = (-1)^{\beta\gamma} a_\alpha c_\gamma \otimes b_\beta d_\delta, \quad a_\alpha \in \mathcal{A}_\alpha,\ c_\gamma \in \mathcal{A}_\gamma,\ b_\beta \in \mathcal{B}_\beta,\ d_\delta \in \mathcal{B}_\delta.$$

If $\mathcal{A}$ and $\mathcal{B}$ are associative superalgebras, then $\mathcal{A}\hat{\otimes}_K\mathcal{B}$ is an associative superalgebra.

Definition 2.1.3 *Let $\mathcal{A}$ be any K-superalgebra. For any $a_\alpha \in \mathcal{A}_\alpha$, we define $a_\alpha^\nu = (-1)^\alpha a_\alpha$.*

The map ν is a superalgebra automorphism since $(a_\alpha b_\beta)^\nu = (-1)^{\alpha+\beta} a_\alpha b_\beta = a_\alpha^\nu b_\beta^\nu$ for any $a_\alpha \in \mathcal{A}_\alpha$ and $b_\beta \in \mathcal{A}_\beta$. If Char$(K) \neq 2$ and $\mathcal{A}_{\bar{1}} \neq \{0\}$ then ν has order 2, otherwise $\nu = id_{\mathcal{A}}$.

Lemma 2.1.4 If $\mathcal{A}$ is a central simple superalgebra over field K of even type. Then ν is an inner automorphism. If $\mathcal{A}$ is of odd type and $Char(K) \neq 2$ then ν is not inner.

Proof If $\mathcal{A} = M_{p+q}(\mathcal{C})$ where $\mathcal{C}$ is a finite dimensional central simple division algebra over K. Then $a_\alpha^\nu = u a_\alpha u^{-1}$ where $u = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. If $\mathcal{A} = M_n(\mathcal{D})$ where $\mathcal{D}$ is a finite dimensional division superalgebra with non-trivial grading of even type over field K of characteristic not 2,then $a_\alpha^\nu = u I_n a_\alpha u^{-1} I_n$ where u as defined in type (iii) or (iv) in the Division Superalgebra Theorem. If Char$(K) = 2$ then $\nu = id_{\mathcal{A}}$, and hence let $u = I_n$.
If $\mathcal{A}$ is of odd type and $char(K) \neq 2$ then $\mathcal{A} = M_n(\mathcal{D})$ where $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{0}} u$, $u^2 = \lambda \in K^\times$, is a division superalgebra of odd type. If b_β is an invertible element in $\mathcal{A}_\beta$ such that $a_\alpha^\nu = b_\beta a_\alpha b_\beta^{-1}$ for all $a_\alpha \in \mathcal{A}_\alpha$ then $u^\nu = -u = b_\beta u b_\beta^{-1} = u$ since $u \in Z(\mathcal{A})$, a contradiction. □

Theorem 2.1.5 (Skolem-Noether Theorem) Let B be a central simple superalgebra over the field K, and let A be a simple subsuperalgebra over K. Then any superalgebra homomorphism f of A into B can be extended to an inner automorphism of B if B is even. If B is odd then f or $f\nu$ can be extended to an inner automorphism but not both of them, where $\nu : B \to B$ such that $a_\alpha^\nu = (-1)^\alpha a_\alpha$.

Proof Let $E = B^\circ \hat{\otimes}_K A$, where B° is the opposite superalgebra of B, then by [16, Theorem 2.3 (2)] E is a simple superalgebra over K. Using the homomorphism f of A into B, we make B into a right E-supermodule in two ways. In the first way, the action is $x_\gamma.(b_\beta \otimes a_\alpha) = (-1)^{\beta\gamma} b_\beta x_\gamma a_\alpha$ and the second action is $x_\gamma.(b_\beta \otimes a_\alpha) = (-1)^{\beta\gamma} b_\beta x_\gamma a_\alpha^f$ where f is the given superalgebra homomorphism of A into B. Then B is a right

E-supermodule under these two actions. By [21, proposition 4], these supermodules are isomorphic. Hence there exists an isomorphism s_δ such that
$s_\delta(x_\gamma.(b_\beta \otimes a_\alpha)) = s_\delta(x_\gamma).(b_\beta \otimes a_\alpha^f)$ therefore

$$(-1)^{\beta\gamma} s_\delta(b_\beta x_\gamma a_\alpha) = (-1)^{\beta(\gamma+\delta)} b_\beta(s_\delta(x_\gamma))a_\alpha^f$$

and so

$$s_\delta(b_\beta x_\gamma a_\alpha) = (-1)^{\beta\delta} b_\beta(s_\delta(x_\gamma))a_\alpha^f. \tag{8}$$

For $b_\beta = 1,\ s_\delta(x_\gamma a_\alpha) = s_\delta(x_\gamma)a_\alpha^f$. Letting $x_\gamma = 1$, we have

$$s_\delta(a_\alpha) = s_\delta(1)a_\alpha^f. \tag{9}$$

For $a_\alpha = 1$, (8) gives you $s_\delta(b_\beta x_\gamma) = (-1)^{\beta\delta} b_\beta s_\delta(x_\gamma)$ and for $x_\gamma = 1$, we have $s_\delta(b_\beta) = (-1)^{\beta\delta} b_\beta s_\delta(1)$, but from (9),

$$(s_\delta(1))b_\beta^f = (-1)^{\beta\delta} b_\beta s_\delta(1)$$

and therefore we have

$$b_\beta^f = (-1)^{\beta\delta}(s_\delta(1))^{-1} b_\beta s_\delta(1).$$

For $\delta = \bar{0}$

$$b_\beta^f = (s_\delta(1))^{-1} b_\beta s_\delta(1).$$

For $\delta = \bar{1}$

$$b_\beta^{f\nu} = (s_\delta(1))^{-1} b_\beta s_\delta(1).$$

For B even, ν is an inner automorphism but not for odd B. Therefore f can be extended to an inner automorphism on B if it is even. If B is odd then f or $f\nu$ is inner but not both of them. □

We now turn to another classic result in the theory of superalgebras; it is usually called the double super-centralizer theorem. We recall the notation: if $\mathcal{A} \subset R$ then $\hat{C}_R(\mathcal{A}) = \mathcal{C}_{\bar{0}} + \mathcal{C}_{\bar{1}}$ where

$$\mathcal{C}_\alpha = \{r_\alpha \in R_\alpha \mid r_\alpha x_\gamma = (-1)^{\alpha\gamma} x_\gamma r_\alpha \ \forall x_\gamma \in A_\gamma\}$$

Theorem 2.1.6 Let R be a central simple superalgebra over field K, and let $A \subset R$ be simple subsuperalgebra over K and containing it. Then
(1) $\hat{C}_R(A)$ is simple, and
(2) $\hat{C}_R(\hat{C}_R(A)) = A$.

Proof Let $E = End_K A$ and let

$$T_{a_\alpha} : A \to A, \quad x_\gamma T_{a_\alpha} = x_\gamma a_\alpha.$$

Then $A_r = \{T_{a_\alpha} \mid a_\alpha \in A_\alpha \ (\alpha = \overline{0}, \overline{1})\} \cong A \ \ (a_\alpha \mapsto T_{a_\alpha})$. Thus, A is imbedded in E as A_r.
Claim :
(1) $\hat{C}_E(A_r) = A_l$, where

$$A_l = \{L_{x_\delta} \mid x_\delta \in A_\delta \ ; \ a_\alpha L_{x_\delta} = (-1)^{\alpha\delta} x_\delta a_\alpha\} \subseteq E.$$

(2) A_l is simple.
Proof of the Claim :
(1) ($\supseteq$)

$$\begin{aligned}(a_\alpha T_{y_\beta}) L_{x_\delta} = (a_\alpha y_\beta) L_{x_\delta} &= (-1)^{\delta(\alpha+\beta)} x_\delta (a_\alpha y_\beta) \\ &= (-1)^{\delta(\alpha+\beta)} (x_\delta a_\alpha) y_\beta \\ &= (-1)^{\delta\beta} (a_\alpha L_{x_\delta}) y_\beta \\ &= (-1)^{\delta\beta} (a_\alpha L_{x_\delta}) T_{y_\beta}.\end{aligned}$$

Therefore, $T_{y_\beta} L_{x_\delta} = (-1)^{\delta\beta} L_{x_\delta} T_{y_\beta}$ which implies that $L_{x_\delta} \in (\hat{C}_E(A_r))_\delta$ so

$$A_l \subseteq \hat{C}_E(A_r)$$

($\subseteq$) let $f_\delta \in (\hat{C}_E(A_r))_\delta$ then it easy to check that $f_\delta = L_{1.f_\delta} \in (A_l)_\delta$. Therefore, $\hat{C}_E(A_r) = A_l$.

(2) Since A° is simple, it is enough to show that $A_l \cong A^\circ$ to prove that A_l is simple.

Let the map ψ from A° to A_l be defined by $x_\delta^\psi = L_{x_\delta}$. Then

$$\begin{aligned} a_\alpha(x_\delta \circ y_\gamma)^\psi = a_\alpha L_{x_\delta \circ y_\gamma} &= (-1)^{\alpha(\delta+\gamma)} x_\delta \circ y_\gamma a_\alpha \\ &= (-1)^{\delta\gamma}(-1)^{\alpha(\delta+\gamma)} y_\gamma x_\delta a_\alpha \\ &= (-1)^{\delta\gamma+\alpha\gamma} y_\gamma (a_\alpha L_{x_\delta}) \\ &= (a_\alpha L_{x_\delta}) L_{y_\gamma} \\ &= a_\alpha x_\delta^\psi y_\gamma^\psi. \end{aligned}$$

Therefore, ψ is an onto superalgebra homomorphism and if $x_\delta \neq y_\gamma$ then $1 \cdot L_{x_\delta} \neq 1 \cdot L_{y_\gamma}$ which implies that ψ is 1-1 and so $A^\circ \cong A_l$. Therefore, A_l is simple.

Now, we show that $\hat{C}_E(\hat{C}_E(A_r)) = \hat{C}_E(A_l) = A_r$.
Let $T_{a_\alpha} \in (A_r)_\alpha$ then $\forall y_\beta, x_\delta,\ \ y_\beta L_{x_\delta} T_{a_\alpha} = (-1)^{\alpha\delta} y_\beta T_{a_\alpha} L_{x_\delta}$. Therefore,

$$T_{a_\alpha} \in (\hat{C}_E(A_l))_\alpha$$

and hence, $A_r \subseteq \hat{C}_E(A_l)$.
Conversely, let $g_\beta \in (\hat{C}_E(A_l))_\beta$ then it is easy to check that $g_\beta = T_{1.g_\beta} \in (A_r)_\beta$.

For the general case let $E = R \hat{\otimes}_K \mathrm{End}_K A$, then E is a central simple superalgebra over K. Since $A \subset \mathrm{End}_K A$ (considering A as A_r) and $A \subset R$, $A \otimes 1$, $1 \otimes A$ are finite dimensional simple subsuperalgebras over K in E. Also, $\phi : A \otimes 1 \to 1 \otimes A$ $(a_\alpha \otimes 1 \mapsto 1 \otimes a_\alpha)$ is a superalgebra isomorphism, so by the Noether-Skolem Theorem, ϕ can be extended to an inner automorphism on E (if E is even) or ϕ or $\phi\nu$ can be extended to an inner automorphism on E but not both of them (if E is odd). Therefore in all cases $\hat{C}_E(A \otimes 1) \approx \hat{C}_E(1 \otimes A)$, and $\hat{C}_E(\hat{C}_E(A \otimes 1)) \approx \hat{C}_E(\hat{C}_E(1 \otimes A))$. But

$$\hat{C}_E(A \otimes 1) = \hat{C}_R(A) \hat{\otimes}_K \mathrm{End}_K A$$

and

$$\hat{C}_E(1 \otimes A) = R \hat{\otimes}_K \hat{C}_{\mathrm{End}_K A}(A) \cong R \hat{\otimes}_K A_l \cong R \hat{\otimes}_K A^\circ.$$

Hence

$$\hat{C}_E(\hat{C}_E(A \otimes 1)) = \hat{C}_R(\hat{C}_R(A)) \hat{\otimes}_K K \cong \hat{C}_R(\hat{C}_R(A))$$

and

$$\begin{aligned}\hat{C}_E(\hat{C}_E(1\otimes A)) &= K\hat{\otimes}_K\hat{C}_{\mathrm{End}_K A}(\hat{C}_{\mathrm{End}_K A}(A))\\ &= K\hat{\otimes}_K\hat{C}_{\mathrm{End}_K A}(A^\circ)\\ &\cong K\hat{\otimes}_K A_r\\ &\cong K\otimes A\\ &\cong A.\end{aligned}$$

Therefore,

$$A\cong\hat{C}_R(\hat{C}_R(A))$$

but $A\subseteq\hat{C}_R(\hat{C}_R(A))$ so, $A=\hat{C}_R(\hat{C}_R(A))$.
Now, we have seen that

$$\hat{C}_E(1\otimes A)\cong R\hat{\otimes}_K A^\circ\cong\hat{C}_E(A\otimes 1)\cong\hat{C}_R(A)\hat{\otimes}_K\mathrm{End}_K A,$$

but $R\hat{\otimes}_K A^\circ$ is simple. Thus $\hat{C}_E(A\otimes 1)$ is simple. In consequence $\hat{C}_R(A)$ is simple.□

Now, by using [16, Theorem 2.3(3)], we shall define the similarity of finite dimensional central simple superalgebras over a given field K.

Definition 2.1.7 *Two finite dimensional central simple superalgebras $\mathcal{A}$ and $\mathcal{B}$ over a given field K are called* similar *($\mathcal{A}\sim\mathcal{B}$), if there exist graded K-vector spaces $V=V_{\bar 0}\oplus V_{\bar 1}$, $W=W_{\bar 0}\oplus W_{\bar 1}$, such that $\mathcal{A}\hat{\otimes}_K\mathrm{End}V\cong\mathcal{B}\hat{\otimes}_K\mathrm{End}W$ as a K-superalgebras.*

It can easily be seen that similarity is an equivalence relation on finite dimensional central simple superalgebras over a field K. The equivalence class of a finite dimensional central simple superalgebra A is denoted by $< A >$. By using [16, Theorem 2.3(3)] the operation $< A >< B > = < A\hat{\otimes}B >$ is well-defined, and makes the set of similarity classes of finite dimensional central simple superalgebras over K into a commutative group, BW(K). (the Brauer-Wall group of K). [16, Proposition 4.13].

Lemma 2.1.8 (1) Let $\mathcal{D}=\mathcal{D}_{\bar 0}+\mathcal{D}_{\bar 1}$ be a central division superalgebra of odd type over field K of characteristic not equal to 2, then $\mathcal{D}$ does not admit a superinvolution of the first kind.

(2) If $\mathcal{D}$ is a central division superalgebra with non-trivial grading of even type over any field K of characteristic not 2, then $\mathcal{D}$ does not admit a superinvolution of the first kind.

Proof (1) Let $*$ be a superinvolution of $\mathcal{D}$, then $(u^2)^* = -(u^*)^2 \Longrightarrow \lambda^* = -\lambda$, where u and λ are as defined in type(ii) of the Division Superalgebra Theorem 1.2.11. Thus, $*$ is a superinvolution of the second kind.

(2) For $\mathcal{D}$ of even type we use a proof by contradiction.

Assume that $\mathcal{D}$ admits a superinvolution $*$ of the first kind and let $\sim\, = * \mid_{\mathcal{D}_{\bar{0}}}$. By [21, Proposition 10] $\mathcal{D}_{\bar{1}}$ contains $v \neq 0$ such that $v^* = v$. Then, let $\phi : \mathcal{D} \longrightarrow \mathcal{D}$ be defined by $x^\phi = vxv^{-1}$. If $\sim$ is an involution of the first kind on $\mathcal{D}_{\bar{0}}$ then $\sim\phi$ is an involution of the second kind on $\mathcal{D}_{\bar{0}}$ and vice versa.

Assume that $\sim$ is of the first kind. We have $\mathcal{H}(K(u), \sim\phi) = \{x \in K(u) \mid \tilde{x}^\phi = x\} = K$, where u is as defined in the cases (iii) and (iv) in the Division Superalgebra Theorem 1.2.11. Then $(\mathcal{D}_{\bar{0}} \otimes_K K(u))e$ is a $(\sim \otimes 1)$-stable proper ideal in $\mathcal{D}_{\bar{0}} \otimes_K K(u)$, where $e = \frac{1-z}{2}$, and $z = \frac{u \otimes u}{\lambda}$, where $\lambda = u^2 \in K^\times$. Therefore, $(\mathcal{D} \otimes_K K(u), * \otimes 1)$ is as in [21, Theorem 12]. Now $\mathcal{D} \otimes_K K(u) \cong M_n(\mathcal{C}')$, where $\mathcal{C}'$ is a central simple division algebra over $K(u)$. The grading on $M_n(\mathcal{C}')$ is not inherited from $\mathcal{C}'$, because if the grading were inherited from $\mathcal{C}'$ then $Z(M_n(\mathcal{C}'))$ would be a field and equal to $Z(\mathcal{D}_{\bar{0}} \otimes_K K(u)) = K(u) \otimes_K K(u)$, a contradiction. Thus, $\mathcal{D} \otimes_K K(u) \cong M_{p+q}(\mathcal{C}')$, where $n = p + q$, but

$$\dim_{K(u)}(\mathcal{D}_{\bar{0}} \otimes_K K(u)) = \dim_{K(u)}(\mathcal{D}_{\bar{1}} \otimes_K K(u)).$$

Hence $p = q$, and therefore $\mathcal{D} \otimes_K K(u) \cong M_{2p}(\mathcal{C}')$, which implies that $\mathcal{D}_{\bar{0}} \otimes_K K(u) \cong M_p(\mathcal{C}') \oplus M_p(\mathcal{C}')$. By [21, proposition 14], $* \otimes 1$ restricts to an orthogonal involution on one of the summands of $\mathcal{D}_{\bar{0}} \otimes_K K(u)$, and to a symplectic involution on the other summand. Thus,

$$\begin{aligned}\dim_{K(u)}\mathcal{H}(\mathcal{D}_{\bar{0}}\otimes_K K(u),\sim\otimes 1) &= \dim_{K(u)}\mathcal{H}(M_p(\mathcal{C}'),\sim\otimes 1)\\ &\quad+\dim_{K(u)}\mathcal{S}(M_p(\mathcal{C}'),\sim\otimes 1)\\ &= \dim_{K(u)} M_p(\mathcal{C}')\\ &= \dim_{K(u)}\mathcal{D}_{\bar{0}},\end{aligned}$$

where $\mathcal{S}(M_p(\mathcal{C}'),\sim\otimes 1)=\{x\in M_p(\mathcal{C}')\mid x^{\sim\otimes 1}=-x\}$. But this is impossible since

$$\begin{aligned}\dim_{K(u)}\mathcal{H}(\mathcal{D}_{\bar{0}}\otimes_K K(u),\sim\otimes 1) &= \dim_{K(u)}(\mathcal{H}(\mathcal{D}_{\bar{0}},\sim)\otimes_K K(u))\\ &= \dim_K\mathcal{H}(\mathcal{D}_{\bar{0}},\sim)\\ &= 2\dim_{K(u)}\mathcal{H}(\mathcal{D}_{\bar{0}},\sim).\end{aligned}$$

Now, if $\sim$ is of the second kind then

$$a+bv^{-1}\mapsto \tilde{a}^{\phi}+\tilde{b}v^{-1}\quad a,b\in\mathcal{D}_{\bar{0}}$$

is another superinvolution on $\mathcal{D}$ whose restriction to $\mathcal{D}_{\bar{0}}$ is of the first kind and will lead to the same contradiction above. □

Moreover, we will find an example of a superalgebra $\mathcal{A}$ of order 2 in the $BW(K)$ (i.e., $\mathcal{A}\cong\mathcal{A}^{\circ}$) which does not have a superinvolution of the first kind.

Example : Let K be any field of characteristic not equal to 2 such that $i=\sqrt{-1}\in K$. Let $a,b\in K$, where a not in K^2 then the K-superalgebra $A=(a,b)$ has two generators u,v with defining relations :
$u^2=a$, $v^2=b$, $uv=-vu$. Therefore, the quaternion algebra $A=K\oplus Ku\oplus Kv\oplus Kuv$ is a division superalgebra with basis $\{1,u,v,uv\}$. ($A_{\bar{0}}=K\oplus Ku, A_{\bar{1}}=Kv\oplus Kuv$). By Lemma 2.1.8, A does not have a superinvolution of the first kind but it is of order 2 in BW(K). To see this, define the K-linear map $*:A\longrightarrow A$ as follows : $\alpha^*=\alpha$ $\forall\alpha\in K$; $u^*=u$; $v^*=iv$; $(uv)^*=ivu=-iuv$ then $*$ is a K-antiautomorphism on A, which is of order 2 in BW(K). □

2.2 Existence of a Superinvolution of the Second Kind

In this section we investigate the existence of superinvolutions of the second kind. We will do that case by case on the division superalgebra $\mathcal{D}$.

Let K/k be a separable quadratic extension. We want to find conditions for $\mathcal{D}$ to admit a K/k-superinvolution of the second kind. A necessary condition is the existence of a K/k-graded anti-automorphism on $\mathcal{D}$ (say J) (i.e., J is a K/k-anti-automorphism respecting the grading on $\mathcal{D}$). We define the conjugate division superalgebra $\overline{\mathcal{D}}$. We set $\overline{\mathcal{D}} = \mathcal{D}$ as a super-ring, but the multiplication by a scalar on $\overline{\mathcal{D}}$ is defined by $\lambda d = \lambda^J d$. Therefore, a K/k-graded anti-automorphism J is simply an isomorphism $\mathcal{D} \cong \overline{\mathcal{D}}^{\circ}$ ($\circ$: the opposite). Fix such a J. Then $x^{J^2} = x$, $\forall x \in \mathcal{D}_{\bar{0}}$. For if not, we can define another graded anti-automorphism I on $\mathcal{D}$ such that $x^{I^2} = x$, $\forall x \in \mathcal{D}_{\bar{0}}$. To show this, assume that $J^2 \mid_{\mathcal{D}_{\bar{0}}} \neq id_{\mathcal{D}_{\bar{0}}}$ but since $J \mid_{\mathcal{D}_{\bar{0}}} : \mathcal{D}_{\bar{0}} \approx \overline{\mathcal{D}}_{\bar{0}}^{\circ}$, so $x^{J^2} = \gamma x \gamma^{-1}$, where $\gamma \in \mathcal{D}_{\bar{0}}$, and $\gamma\gamma^J = 1$ [25, theorem 8.2]. Let $\alpha = (1+\gamma)^{-1}$ ($\gamma \neq -1$). A trivial computation shows that $x^I = \alpha x^J \alpha^{-1}$, $\forall x \in \mathcal{D}$, is another graded anti-automorphism on $\mathcal{D}$, and $x^{I^2} = x$, $\forall x \in \mathcal{D}_{\bar{0}}$ [25, Theorem 8.2]. Therefore we may fix a K/k-graded anti-automorphism J on $\mathcal{D}$ such that $x^{J^2} = x$, $\forall x \in \mathcal{D}_{\bar{0}}$.

Theorem 2.2.1 If $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$ is a central division superalgebra of odd type over the field K of characteristic not equal to 2 then $\mathcal{D}$ admits a superinvolution $*$ of the second kind if and only if $\mathcal{D} \approx \overline{\mathcal{D}}^{\circ}$, and $\mathcal{D} = \mathcal{D}_{\bar{0}} \otimes_K K(u)$, $u^2 = \lambda \in K$, $\lambda^{\sigma} = -\lambda$, where σ is the Galois automorphism on K/k.

Proof. If $*$ is a K/k-superinvolution of the second kind on $\mathcal{D}$ then $* : \mathcal{D} \to \overline{\mathcal{D}}^{\circ}$ is an isomorphism, so, by [21, Proposition 9], $\mathcal{D} = \mathcal{D}_{\bar{0}} \otimes_K K(u)$, where $u^2 = \lambda \in K$ and $u^* = u$. This implies that $\lambda^* = \lambda^{\sigma} = -\lambda$.

Now, if $\mathcal{D}$ is isomorphic to $\overline{\mathcal{D}}^{\circ}$, we may fix a K/k-graded anti-automorphism J on $\mathcal{D}$ such that $x^{J^2} = x$, $\forall x \in \mathcal{D}_{\bar{0}}$. However, $u^2 = \lambda$ and $\lambda^J = \lambda^{\sigma} = -\lambda$, and therefore $(u^2)^J = -(u^J)^2 = -\lambda = -u^2$. This implies that $u^J = u$ or $u^J = -u$. If $u^J = -u$ then replace u by λu. Then in all cases we have $u^J = u$, and therefore $* : \mathcal{D} \longrightarrow \mathcal{D}$ defined by $(a + bu)^* = a^J + b^J u$, $\forall a, b \in \mathcal{D}_{\bar{0}}$, is a superinvolution of the second kind on $\mathcal{D}$. □

The following example shows that the condition $\mathcal{D}$ isomorphic to $\overline{\mathcal{D}}^{\circ}$ under J, where J is a K/k-antiautomorphism on $\mathcal{D}$, is not enough to say that $\mathcal{D}$ admits a K/k-superinvolution.

Example Let $K = \mathbf{Q}[i](\sqrt{2})$ and $\mathcal{D} = K \oplus Ku$, $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $u^2 = 1$. Define $J : \mathcal{D} \longrightarrow \mathcal{D}$ as follows: $u \mapsto iu$ and $\alpha + \beta\sqrt{2} \mapsto \alpha - \beta\sqrt{2}$, $\forall \alpha, \beta \in \mathbf{Q}[i]$. Then a computation shows that $\mathcal{D} \overset{J}{\approx} \overline{\mathcal{D}}^{\circ}$ but $\mathcal{D}$ does not admit a superinvolution of the second kind. For if $\mathcal{D}$ has one (say $*$) then $\exists\ u' = \gamma u \in \mathcal{D}_{\bar{1}}$ such that $(a + bu')^* = a^J + b^J u'$. Then $*J$ is an automorphism on $\mathcal{D}$, and therefore, by the Skolem-Noether Theorem, $*J$ or $\nu * J$ is an inner automorphism, but not both of them (since $\mathcal{D}$ is odd)
Case(1) : $*J$ is inner . Then $\exists v_\alpha \in \mathcal{D}_\alpha$ such that $\forall a, b \in K$ $(a + bu')^{*J} = v_\alpha(a + bu')v_\alpha^{-1} = a + bu' = a + b(u')^J$; therefore, $(u')^J = u' = (\gamma u)^J = \gamma^J u^J = \gamma^J iu = \gamma u$. Then, $\gamma^J i = \gamma$, if $\gamma = x + y\sqrt{2} \in \mathbf{Q}[i](\sqrt{2})$. In that case, $(x - y\sqrt{2})i = x + y\sqrt{2}$ so $x(1 - i) = -y\sqrt{2}(1 + i)$, and therefore $\sqrt{2} = \frac{-x(1-i)}{y(1+i)} \in \mathbf{Q}[i]$ ($y \neq 0$, because $y = 0$ implies that $\gamma^J = \gamma \Rightarrow i = 1$), a contradiction.
Case(2) : $\nu * J$ is inner. Then $\exists v_\alpha \in \mathcal{D}_\alpha$ such that $\forall a, b \in K$ $(a + bu')^{\nu * J} = v_\alpha(a + bu')v_\alpha^{-1} = a + bu' = a - b(u')^J$; therefore, $-(u')^J = u' = -(\gamma u)^J = -\gamma^J u^J = -\gamma^J iu = \gamma u$ and so $-\gamma^J i = \gamma$, if $\gamma = x + y\sqrt{2} \in \mathbf{Q}[i](\sqrt{2})$. In that case, $(-x + y\sqrt{2})i = x + y\sqrt{2}$ so $x(1 + i) = y\sqrt{2}(-1 + i)$, and therefore $\sqrt{2} = \frac{x(1+i)}{y(-1+i)} \in \mathbf{Q}[i]$ ($y \neq 0$, because $y = 0$ implies that $\gamma^J = \gamma \Rightarrow i = -1$), a contradiction. $\square$

Lemma 2.2.2 If an even division superalgebra $\mathcal{D}$ with non-trivial grading admits a K/k-graded antiautomorphism J such that $x^{J^2} = x$, $\forall x \in \mathcal{D}_{\bar{0}}$, then J^2 is an inner automorphism, and $x^{J^2} = bxb^{-1}$ for $b \in Z(\mathcal{D}_{\bar{0}})$.

Proof Since $\mathcal{D}$ is a central simple algebra over K, the map J^2 is an inner automorphism on $\mathcal{D}$. Hence $x^{J^2} = bxb^{-1}$, for a suitable invertible element $b = b_{\bar{0}} + b_{\bar{1}} \in \mathcal{D}$. Now, if Char$(K) \neq 2$ then for $u \in \mathcal{D}$ as defined in cases(iii) and (iv) in the Division Superalgebra Theorem 1.2.11, $u^{J^2} = u = (b_{\bar{0}}+b_{\bar{1}})u(b_{\bar{0}}+b_{\bar{1}})^{-1} = u(b_{\bar{0}}-b_{\bar{1}})(b_{\bar{0}}+b_{\bar{1}})^{-1} \Leftrightarrow (b_{\bar{0}} - b_{\bar{1}})(b_{\bar{0}} + b_{\bar{1}})^{-1} = 1 \Leftrightarrow b_{\bar{0}} - b_{\bar{1}} = b_{\bar{0}} + b_{\bar{1}} \Leftrightarrow b_{\bar{1}} = 0 \Leftrightarrow b \in \mathcal{D}_{\bar{0}}$. If Char$(K) = 2$ then $u^{J^2} = u = (b_{\bar{0}} + b_{\bar{1}})u(b_{\bar{0}} + b_{\bar{1}})^{-1} = (ub_{\bar{0}} + (u + 1)b_{\bar{1}})(b_{\bar{0}} + b_{\bar{1}})^{-1} = u$ implies that

$(u(b_{\bar{0}}+b_{\bar{1}})+b_{\bar{1}})(b_{\bar{0}}+b_{\bar{1}})^{-1} = u$ and so $u+b_{\bar{1}}(b_{\bar{0}}+b_{\bar{1}})^{-1} = u$. Therefore $b_{\bar{1}}(b_{\bar{0}}+b_{\bar{1}})^{-1} = 0$ so $b_{\bar{1}} = 0$. Therefore $b = b_{\bar{0}} \in \mathcal{D}_{\bar{0}}$. But $x^{J^2} = x\ \forall x \in \mathcal{D}_{\bar{0}}$, so $b \in Z(\mathcal{D}_{\bar{0}})$. $\square$

Lemma 2.2.3 Let $\mathcal{D}$ be as in the lemma above. If $\mathcal{D}^\circ \overset{J}{\approx} \overline{\mathcal{D}}$ and $\mathcal{D}^\circ \overset{I}{\approx} \overline{\mathcal{D}}$ such that $J^2 |_{\mathcal{D}_{\bar{0}}} = I^2 |_{\mathcal{D}_{\bar{0}}} = id_{\mathcal{D}_{\bar{0}}}$, where I and J are K/k-graded antiautomorphisms on $\mathcal{D}$. Then $\exists\ a_\alpha \in \mathcal{D}_\alpha$ such that $x^I = a_\alpha x^J a_\alpha^{-1}$, $\forall x \in \mathcal{D}$.

Proof (i) (Char$(K) \neq 2$). Since $u^J \in Z(\mathcal{D}_{\bar{0}})$ and $(u^J)^2 = (u^2)^J \in K$, we have $u^J = \gamma u$, where $\gamma \in K^\times$. So, $u^{J^2} = \gamma\gamma^J u = u$; therefore $\gamma\gamma^J = 1$. By Hilbert's theorem 90, $\exists\ \mu \in K^\times$ such that $\mu(\mu^{-1})^J = \mu\mu^{-J} = \gamma$. Now $(\mu^{-J}u)^J = \mu^{-J}u$, so if we replace u by $\mu^{-J}u$, we have $u^J = u$. However, $u^I = \beta u$ and $u^{I^2} = \beta\beta^I u = u$ implies $\beta\beta^I = 1$ for some $\beta \in K^\times$. By the Skolem-Noether Theorem, $\exists\ a = a_{\bar{0}} + a_{\bar{1}} \in \mathcal{D}$ such that $x^I = ax^J a^{-1}$, $\forall x \in \mathcal{D}$. Now

$$\begin{aligned} u^I = \beta u = au^J a^{-1} &= aua^{-1} \\ &= (a_{\bar{0}} + a_{\bar{1}})u(a_{\bar{0}} + a_{\bar{1}})^{-1} \\ &= u(a_{\bar{0}} - a_{\bar{1}})(a_{\bar{0}} + a_{\bar{1}})^{-1} \end{aligned}$$

$\Longrightarrow \beta = (a_{\bar{0}} - a_{\bar{1}})(a_{\bar{0}} + a_{\bar{1}})^{-1} \Longrightarrow 1 = \beta^I\beta = \beta^I(a_{\bar{0}} - a_{\bar{1}})(a_{\bar{0}} + a_{\bar{1}})^{-1} \Longrightarrow \beta^I(a_{\bar{0}} - a_{\bar{1}}) = a_{\bar{0}} + a_{\bar{1}} \Longrightarrow (\beta^I - 1)a_{\bar{0}} = 0$ and $(\beta^I + 1)a_{\bar{1}} = 0$.
For $(\beta^I - 1)a_{\bar{0}} = 0$, we get that $\beta^I = 1$ or $a_{\bar{0}} = 0 \Leftrightarrow a_{\bar{1}} = 0$ or $a_{\bar{0}} = 0$. For $(\beta^I + 1)a_{\bar{1}} = 0$, we get that $\beta^I = -1$ or $a_{\bar{1}} = 0 \Leftrightarrow a_{\bar{0}} = 0$ or $a_{\bar{1}} = 0$. So in all cases we have that $a = a_\alpha \in \mathcal{D}_\alpha$.
(ii) (Char$(K) = 2$). Since $K = k(\lambda)$ such that $\lambda^J = \lambda + 1$, $\lambda^I = \lambda + 1$ and $u^J \in Z(\mathcal{D}_{\bar{0}}) = K(u)$, we have $u^J = \alpha + \beta u$. However, $v^J = dv \in \mathcal{D}_{\bar{0}}v = \mathcal{D}_{\bar{1}}$ and so $v^J u^J (v^J)^{-1} = \alpha + \beta vuv^{-1} = \alpha + \beta(1 + u) = \alpha + \beta + \beta u = \beta + u^J$. But also $v^J u^J (v^J)^{-1} = (v^{-1}uv)^J = (1+u)^J = 1+u^J$; therefore $\beta = 1$ and so $u^J = \alpha + u$. Since $u^{J^2} = (\alpha + u)^J = \alpha^J + \alpha + u = u$, we have $\alpha^J + \alpha = 0$ and so $\alpha^J = \alpha \in k$. If $\alpha = 0$ then $u^J = u$; if not then replace u by $\frac{1}{\alpha}u$ we get $u^J = 1 + u$ and since $\lambda^J = 1 + \lambda$ we have $(\lambda + u)^J = \lambda + u$. Therefore, we may assume that $u^J = u$ and $u^I = \gamma + u$ for

some $\gamma \in k$. Now,

$$\begin{aligned} u^I = \gamma + u = au^J a^{-1} &= a(1+u)a^{-1} \\ &= (a_{\bar{0}} + a_{\bar{1}})(1+u)(a_{\bar{0}} + a_{\bar{1}})^{-1} \\ &= 1 + (a_{\bar{0}} + a_{\bar{1}})u(a_{\bar{0}} + a_{\bar{1}})^{-1} \\ &= 1 + (ua_{\bar{0}} + (u+1)a_{\bar{1}})(a_{\bar{0}} + a_{\bar{1}})^{-1} \\ &= 1 + (u(a_{\bar{0}} + a_{\bar{1}}) + a_{\bar{1}})(a_{\bar{0}} + a_{\bar{1}})^{-1} \\ &= 1 + u + a_{\bar{1}}(a_{\bar{0}} + a_{\bar{1}})^{-1}. \end{aligned}$$

However, $(\gamma+1) = a_{\bar{1}}(a_{\bar{0}}+a_{\bar{1}})^{-1}$, and therefore $(\gamma+1)(a_{\bar{0}}+a_{\bar{1}}) = a_{\bar{1}}$ and so $(\gamma+1)a_{\bar{0}} = \gamma a_{\bar{1}}$ which implies that $\gamma a_{\bar{1}} = 0$ so $\gamma = 0$ or $a_{\bar{1}} = 0$ but if $\gamma = 0$ then $a_{\bar{0}} = 0$. Therefore, we have $a = a_\alpha \in \mathcal{D}_\alpha$. □

Lemma 2.2.4 Let $b \in \mathcal{D}_{\bar{0}}$ be as in Lemma 2.2.2. Then
(i) $bb^J = b^J b \in k^\times$.
(ii) bb^J does not depend on the choice of J and b.

Proof (i) the equation $x_\alpha^{J^2} = bx_\alpha b^{-1}$ implies

$$\begin{aligned} x_\alpha^{J^3} &= (x_\alpha^{J^2})^J = (bx_\alpha b^{-1})^J = b^{-J} x_\alpha^J b^J \\ &= (x_\alpha^J)^{J^2} \\ &= bx_\alpha^J b^{-1} \end{aligned}$$

$\Rightarrow bx_\alpha^J b^{-1} = b^{-J} x_\alpha^J b^J \Rightarrow x_\alpha^J = b^{-1} b^{-J} x_\alpha^J b^J b$. Hence $b^J b \in K$. In fact $b^J b \in k$ since $(b^J b)^J = b^J b$. Therefore, $(b^J b)^J bb^J = b(b^J b)^J b^J = b(b^J b)b^J = (bb^J)(bb^J) \Rightarrow b^J b = bb^J$.
(ii) Let I be another K/k-graded antiautomorphism on $\mathcal{D}$ such that $x^{I^2} = x$, $\forall x \in \mathcal{D}_{\bar{0}}$. Then, by Lemma 2.2.3, $\exists\ a_\alpha \in \mathcal{D}_\alpha$ such that $x^I = a_\alpha x^J a_\alpha^{-1}\ \forall x \in \mathcal{D}$.
For $\alpha = \bar{0}$:

$$\begin{aligned} x_\beta^{I^2} &= a_{\bar{0}}(a_{\bar{0}} x_\beta^J a_{\bar{0}}^{-1})^J a_{\bar{0}}^{-1} \\ &= a_{\bar{0}} a_{\bar{0}}^{-J} b x_\beta b^{-1} a_{\bar{0}}^J a_{\bar{0}}^{-1}. \end{aligned}$$

Claim : $(a_{\bar{0}}a_{\bar{0}}^{-J}b)(a_{\bar{0}}a_{\bar{0}}^{-J}b)^I = bb^J$.
Proof of the claim :

$$\begin{aligned}(a_{\bar{0}}a_{\bar{0}}^{-J}b)(a_{\bar{0}}a_{\bar{0}}^{-J}b)^I &= a_{\bar{0}}a_{\bar{0}}^{-J}ba_{\bar{0}}(a_{\bar{0}}a_{\bar{0}}^{-J}b)^J a_{\bar{0}}^{-1} \\ &= a_{\bar{0}}a_{\bar{0}}^{-J}ba_{\bar{0}}\underbrace{b^Jb}_{\in k}a_{\bar{0}}^{-1}b^{-1}a_{\bar{0}}^Ja_{\bar{0}}^{-1} \\ &= b^Jb.\end{aligned}$$

For $\alpha = \bar{1}$:

$$\begin{aligned}x_\beta^{I^2} &= a_{\bar{1}}(a_{\bar{1}}x_\beta^Ja_{\bar{1}}^{-1})^Ja_{\bar{1}}^{-1} \\ &= (-1)^{(1+\beta)}(-1)^\beta a_{\bar{1}}(a_{\bar{1}}^{-1})^Jbx_\beta b^{-1}a_{\bar{1}}^Ja_{\bar{1}}^{-1}. \\ &= -a_{\bar{1}}(a_{\bar{1}}^{-1})^Jbx_\beta b^{-1}a_{\bar{1}}^Ja_{\bar{1}}^{-1}.\end{aligned}$$

Since $(a_{\bar{1}}a_{\bar{1}}^{-1})^J = 1 = -(a_{\bar{1}}^{-1})^Ja_{\bar{1}}^J = 1$, we have $(a_{\bar{1}}^{-1})^J = -(a_{\bar{1}}^J)^{-1}$ and therefore, $x_\beta^{I^2} = a_{\bar{1}}(a_{\bar{1}}^J)^{-1}bx_\beta b^{-1}a_{\bar{1}}^Ja_{\bar{1}}^{-1}$.
Claim : $(a_{\bar{1}}(a_{\bar{1}}^J)^{-1}b)(a_{\bar{1}}(a_{\bar{1}}^J)^{-1}b)^I = bb^J$.
Proof of the claim :

$$\begin{aligned}(a_{\bar{1}}(a_{\bar{1}}^J)^{-1}b)(a_{\bar{1}}(a_{\bar{1}}^J)^{-1}b)^I &= a_{\bar{1}}(a_{\bar{1}}^J)^{-1}ba_{\bar{1}}(a_{\bar{1}}(a_{\bar{1}}^J)^{-1}b)^Ja_{\bar{1}}^{-1} \\ &= a_{\bar{1}}(a_{\bar{1}}^J)^{-1}ba_{\bar{1}}(-1)b^J(-1)ba_{\bar{1}}^{-1}b^{-1}a_{\bar{1}}^Ja_{\bar{1}}^{-1} \\ &= a_{\bar{1}}(a_{\bar{1}}^J)^{-1}ba_{\bar{1}}\underbrace{b^Jb}_{\in k}a_{\bar{1}}^{-1}b^{-1}a_{\bar{1}}^Ja_{\bar{1}}^{-1} \\ &= b^Jb \\ &= bb^J.\end{aligned}$$

□

Theorem 2.2.5 In the situation described above. The even division superalgebra $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$ with non-trivial grading admits a K/k-superinvolution if and only if

$$bb^J \in N(K^\times) := \{\alpha\alpha^J \mid \alpha \in K^\times\},$$

where b as defined in the lemma above.

Proof ($\Longrightarrow$) Choose $b = 1$.
($\Longleftarrow$) We can write $bb^J = \alpha\alpha^J$, where $\alpha \in K$, and replacing b by $b\alpha^{-1}$ we may assume $bb^J = 1$. If $b = -1$ we are finished, since J is a superinvolution of the second kind. Otherwise we show that the map

$$I : \mathcal{D} \longrightarrow \mathcal{D}$$

$$x \mapsto (1+b)^{-1}x^J(1+b)$$

is a K/k-superinvolution.
Notice that $\gamma^I = \gamma^J$, $\forall\gamma \in K$. Moreover,

$$\begin{aligned}
(x_\alpha y_\beta)^I &= (1+b)^{-1}(x_\alpha y_\beta)^J(1+b) \\
&= (-1)^{\alpha\beta}(1+b)^{-1}y_\beta^J x_\alpha^J(1+b) \\
&= (-1)^{\alpha\beta}(1+b)^{-1}y_\beta^J(1+b)(1+b)^{-1}x_\alpha^J(1+b) \\
&= (-1)^{\alpha\beta}y_\beta^I x_\alpha^I.
\end{aligned}$$

and

$$\begin{aligned}
x_\beta^{I^2} &= (1+b)^{-1}((1+b)^{-1}x_\beta^J(1+b))^J(1+b) \\
&= (1+b)^{-1}(1+b)^J b x_\beta b^{-1}(1+b)^{-J}(1+b) \\
&= (1+b)^{-1}(1+b)x_\beta b^{-1}(1+b^J)^{-1}(1+b) \\
&= x_\beta((1+b^J)b)^{-1}(1+b) \\
&= x_\beta(1+b)^{-1}(1+b) \\
&= x_\beta.
\end{aligned}$$

Therefore I is a K/k-superinvolution on $\mathcal{D}$. □

2.3 The Corestriction.

We generalize the method due to Riehm which improves on Albert's original approach to superalgebra case, to formulate the existence criterion of superinvolutions of the second kind. First we have to start with the corestriction of central simple superalgebras.

Let $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$ be a central division superalgebra over K with non-trivial grading. Let K/k be a separable quadratic extension with non-trivial automorphism. Let $\overline{\mathcal{D}}$ be the conjugate K-division superalgebra as defined above. Consider the canonical map

$$\nu : \overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D} \longrightarrow \overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}$$
$$(a_\alpha \otimes b_\beta)^\nu = (-1)^{\alpha\beta} b_\beta \otimes a_\alpha.$$

Clearly ν respects the grading and

$$\begin{aligned} ((a_\alpha \otimes b_\beta)(c_\gamma \otimes d_\delta))^\nu &= (-1)^{\beta\gamma}(a_\alpha c_\gamma \otimes b_\beta d_\delta)^\nu \\ &= (-1)^{\beta\gamma}(-1)^{(\alpha+\gamma)(\beta+\delta)} b_\beta d_\delta \otimes a_\alpha c_\gamma \\ &= (-1)^{\alpha\beta+\alpha\delta+\gamma\delta} b_\beta d_\delta \otimes a_\alpha c_\gamma \end{aligned} \tag{10}$$

and

$$\begin{aligned} (a_\alpha \otimes b_\beta)^\nu (c_\gamma \otimes d_\delta)^\nu &= (-1)^{\alpha\beta}(b_\beta \otimes a_\alpha)(-1)^{\gamma\delta}(d_\delta \otimes c_\gamma) \\ &= (-1)^{\alpha\beta+\alpha\delta+\gamma\delta} b_\beta d_\delta \otimes a_\alpha c_\gamma \end{aligned} \tag{11}$$

so, from (10) and (11), ν is a graded algebra homomorphism of order 2. Moreover, ν is semilinear, since

$$\begin{aligned} (\lambda(a_\alpha \otimes b_\beta))^\nu &= (\lambda.a_\alpha \otimes b_\beta)^\nu \\ &= (-1)^{\alpha\beta} b_\beta \otimes \lambda.a_\alpha \\ &= (-1)^{\alpha\beta} b_\beta \otimes \lambda^J a_\alpha \\ &= \lambda^J (a_\alpha \otimes b_\beta)^\nu. \end{aligned}$$

In particular ν is k-linear. Define

$$\mathrm{Tr}(\mathcal{D}) = \{x \in \overline{\mathcal{D}}\hat{\otimes}\mathcal{D} \mid x^\nu = x\}$$

obviously, $\mathrm{Tr}(\mathcal{D})$ is a k-space. However, $\mathrm{Tr}(\mathcal{D})$ is also closed with respect to multiplication since $(xy)^\nu = x^\nu y^\nu = xy$, $\forall x, y \in \mathrm{Tr}(\mathcal{D})$. Therefore, it is a k-superalgebra.

Definition 2.3.1 *The k-superalgebra* $\mathrm{Tr}(\mathcal{D})$ *is called the* Corestriction *of the superalgebra $\mathcal{D}$.*

Lemma 2.3.2 There is a canonical K-superalgebra isomorphism

$$K\hat{\otimes}_k\mathrm{Tr}(\mathcal{D}) \approx \overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}.$$

Proof Consider the canonical map

$$\psi : K \times \mathrm{Tr}(\mathcal{D}) \longrightarrow \overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}$$

$$(\alpha, x) \mapsto \alpha x.$$

ψ is a k-bilinear and multiplicative. It induces a k-superalgebra homomorphism

$$\phi : K\hat{\otimes}_k\mathrm{Tr}(\mathcal{D}) \longrightarrow \overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}.$$

ϕ is K-linear. To prove that ϕ is bijective choose $\lambda \in K$ such that $K = k + k\lambda$ and $\lambda^J = -\lambda$ (if $\mathrm{Char}(K) \neq 2$), $\lambda^J = 1 + \lambda$ (if $\mathrm{Char}(K) = 2$). Now, every $x \in \overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}$ can be written in the form $x = y + \lambda z$ with $y, z \in \mathrm{Tr}(\mathcal{D})$: $x = \frac{1}{2}(x + x^\nu) + \lambda(\frac{1}{2\lambda}(x - x^\nu))$ (if $\mathrm{Char}(K) \neq 2$) and $x = (\lambda^J x + \lambda x^\nu) + \lambda(x^\nu + x)$ (if $\mathrm{Char}(K) = 2$). □

Corollary 2.3.3 If $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$ is a central simple division superalgebra over K with non-trivial grading, then $\mathrm{Tr}(\mathcal{D})$ is a central simple superalgebra over k. □

Definition 2.3.4 *For $\gamma \in k^\times$ a 4-dimensional k-superalgebra with a trivial grading $(K/k,\gamma)$ is defined by $(K/k,\gamma)= K \oplus eK$, $exe^{-1} = \overline{x}$ for $x \in K$, $e^2 = \gamma$ where $\overline{\alpha + \beta\lambda} = \alpha - \beta\lambda$ $\forall\alpha, \beta \in k$ (if Char$(k) \neq 2$) and $\overline{\alpha + \beta\lambda} = \alpha + \beta(\lambda + 1)$ $\forall\alpha, \beta \in k$ (if Char$(k) = 2$).*

This is a central simple superalgebra (with trivial grading). It splits if and only if $\gamma \in N(K^\times)$. See [25, section 11].

Lemma 2.3.5 Let J be a K/k-graded antiautomorphism on $\mathcal{D}$, where $\mathcal{D}$ is an even division superalgebra with non-trivial grading, such that $x^{J^2} = bxb^{-1}$ then the opposite superalgebra $(\mathrm{Tr}(\mathcal{D}))^\circ \sim (K/k, (b^J b)^{-1})$.

Proof Using the graded isomorphism $J : \mathcal{D}^\circ \approx \overline{\mathcal{D}}$, we get a graded isomorphism $\mathrm{End}_K\mathcal{D} = \mathcal{D}^\circ\hat{\otimes}_K\mathcal{D} \cong \overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}$. Since $\mathrm{Tr}(\mathcal{D}) \subseteq \overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}$ we can consider $\mathcal{D}$ as a $\mathrm{Tr}(\mathcal{D})$-supermodule.

Claim : $T^\circ \sim \mathrm{End}_{T^\circ}\mathcal{D}$ (where $T = \mathrm{Tr}(\mathcal{D})$).

Proof of the claim : Let $T \approx \mathcal{M}_n(\mathcal{C})$, where $\mathcal{C}^\circ$ is the commuting super-ring of T on I, a minimal right superideal in T. Now, $\mathcal{D} = M^r$, where M is an irreducible T-supermodule. So, by [21, Proposition 4] $I \approx M$ since $\mathrm{End}_{T^\circ}I \approx \mathcal{C}^\circ$ we have $\mathrm{End}_{T^\circ}M \approx \mathrm{End}_{T^\circ}I \approx \mathcal{C}^\circ$ and therefore $\mathrm{End}_{T^\circ}\mathcal{D} \approx \mathrm{End}_{T^\circ}M^r \approx \mathcal{M}_r(\mathcal{C}^\circ)$. Therefore, $\mathrm{End}_{T^\circ}\mathcal{D} \sim T^\circ$. Now, $\dim_k \mathrm{End}_{T^\circ}\mathcal{D} = r^2d^2$, where $d^2 = \dim_k\mathcal{C} = \dim_k\mathcal{C}^\circ$. But

$$\begin{aligned} n^2d^2 = \dim_k T^\circ &= \dim_K(\overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D})^\circ \\ &= \dim_K \overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D} \\ &= (\frac{1}{2}\dim_k M^r)^2 \\ &= \frac{1}{4}n^2r^2d^4. \end{aligned}$$

Therefore, $\mathrm{End}_{T^\circ}\mathcal{D}$ has dimension 4 over k. For $\gamma \in K$ multiplication by γ is a T-supermodule endomorphism on $\mathcal{D}$. This gives an inclusion $K \subset \mathrm{End}_{T^\circ}\mathcal{D}$. Now, consider the map

$$e : \mathcal{D} \longrightarrow \mathcal{D} \quad (x \mapsto x^J(b^J)^{-1}).$$

Case(1) $\mathrm{Char}(K) \neq 2$. Then $e \in \mathrm{End}_{T^\circ}\mathcal{D}$ since

(i) for $a, d \in \mathcal{D}_{\bar{0}}$ we have

$$\begin{aligned} (x_\gamma.(d\otimes a))^e = (d^Jx_\gamma a)^e &= (d^Jx_\gamma a)^J(b^J)^{-1} \\ &= a^Jx_\gamma^Jbdb^{-1}(b^J)^{-1} \\ &= a^Jx_\gamma^Jbd(b^Jb)^{-1} \\ &= a^Jx_\gamma^J(b^J)^{-1}d \\ &= a^Jx_\gamma^e d \\ &= x_\gamma^e(a\otimes d). \end{aligned}$$

Therefore, $(x_\gamma.(a\otimes d + d\otimes a))^e = x_\gamma^e(a\otimes d + d\otimes a)$ and $(x_\gamma(a\otimes a))^e = x_\gamma^e(a\otimes a)$ $\forall a, d \in \mathcal{D}_{\bar{0}}$.

(ii) for $a, d \in \mathcal{D}_{\bar{1}}$ and for any $\delta \in K$ we have

$$\begin{aligned}
(x_\gamma.(\delta(d \otimes a)))^e &= (x_\gamma.((\delta.d) \otimes a))^e \\
&= (-1)^\gamma((\delta^J d)^J x_\gamma a)^e \\
&= (-1)^\gamma(d^J x_\gamma a\delta)^e \\
&= (-1)^\gamma(d^J x_\gamma a\delta)^J (b^J)^{-1} \\
&= (-1)^\gamma(-1)^{\gamma+1}(-1)^\gamma \delta^J a^J x_\gamma^J bdb^{-1}(b^J)^{-1} \\
&= (-1)^{\gamma+1}\delta^J a^J x_\gamma^J (b^J)^{-1} d \\
&= (-1)^{\gamma+1}\delta^J a^J x_\gamma^e d \\
&= (-1)(-1)^\gamma(\delta a)^J x_\gamma^e d \\
&= (-1)(-1)^\gamma(\delta^J.a)^J x_\gamma^e d \\
&= -x_\gamma^e.((\delta^J.a) \otimes d) \\
&= -x_\gamma^e.(\delta^J(a \otimes d))
\end{aligned}$$

so for $\delta = \lambda$ where $\lambda^J = -\lambda$ we get $(x_\gamma.(\lambda(a \otimes a)))^e = x_\gamma^e.(\lambda(a \otimes a))$ and $(x_\gamma(\lambda(a \otimes d + d \otimes a)))^e = x_\gamma^e(\lambda(a \otimes d + d \otimes a))$ $\forall a, d \in \mathcal{D}_{\bar{1}}$.
(iii) for $a \in \mathcal{D}_{\bar{0}}$ and $d \in \mathcal{D}_{\bar{1}}$ and for any $\delta \in K$

$$\begin{aligned}
(x_\gamma.(\delta(d \otimes a)))^e &= (x_\gamma.((\delta.d) \otimes a))^e \\
&= (-1)^\gamma((\delta^J d)^J x_\gamma a)^e \\
&= (-1)^\gamma(d^J x_\gamma a\delta)^e \\
&= (-1)^\gamma(d^J x_\gamma a\delta)^J (b^J)^{-1} \\
&= (-1)^\gamma(-1)^\gamma \delta^J a^J x_\gamma^J bdb^{-1}(b^J)^{-1} \\
&= \delta^J a^J x_\gamma^J (b^J)^{-1} d \\
&= \delta^J a^J x_\gamma^e d \\
&= (\delta a)^J x_\gamma^e d \\
&= (\delta^J.a)^J x_\gamma^e d \\
&= x_\gamma^e.((\delta^J.a) \otimes d) \\
&= x_\gamma^e.(\delta^J(a \otimes d))
\end{aligned}$$

and

$$\begin{aligned}
(x_\gamma.(\delta(a\otimes d)))^e &= (x_\gamma.((\delta.a)\otimes d))^e \\
&= ((\delta^J a)^J x_\gamma d)^e \\
&= (a^J x_\gamma d\delta)^e \\
&= (a^J x_\gamma d\delta)^J (b^J)^{-1} \\
&= (-1)^\gamma \delta^J d^J x_\gamma^J bab^{-1}(b^J)^{-1} \\
&= (-1)^\gamma \delta^J d^J x_\gamma^J (b^J)^{-1} a \\
&= (-1)^\gamma (\delta d)^J x_\gamma^e a \\
&= (-1)^\gamma (\delta^J.d)^J x_\gamma^e a \\
&= x_\gamma^e.((\delta^J.d)\otimes a) \\
&= x_\gamma^e.(\delta^J(d\otimes a))
\end{aligned}$$

$\Longrightarrow (x_\gamma.((a\otimes d)+(d\otimes a)))^e = x_\gamma^e((a\otimes d)+(d\otimes a))$ and $(x_\gamma.(\lambda(a\otimes d)+\lambda^J(d\otimes a)))^e = x_\gamma^e(\lambda(a\otimes d)+\lambda^J(d\otimes a))$ $\forall a\in\mathcal{D}_{\bar 0}, d\in\mathcal{D}_{\bar 1}$. Therefore, $e\in \mathrm{End}_{T^\circ}\mathcal{D}$ and $x^{e^2}=(x^e)^e = (x^J(b^J)^{-1})^e = (x^J(b^J)^{-1})^J(b^J)^{-1} = b^{-1}bxb^{-1}(b^J)^{-1} = b^{-1}(b^J)^{-1}x = (b^Jb)^{-1}x$ $\forall x\in\mathcal{D}$. Moreover, $x^{e\delta} = \delta x^e = \delta x^J(b^J)^{-1} = (\delta^J b^{-1}x)^J = (b^{-1}(\delta^J x))^J = (\delta^J x)^J(b^J)^{-1} = (\delta^J x)^e = x^{\delta^J e}$ so $e\delta = \delta^J e$ $\forall\delta\in K$.

Case(2) Char$(K)=2$. Then for any a_α and b_β in $\mathcal{D}$, we have

$$\begin{aligned}
(x_\gamma.(a_\alpha\otimes b_\beta))^e = (a_\alpha^J x_\gamma b_\beta)^e &= (a_\alpha^J x_\gamma b_\beta)^J(b^J)^{-1} \\
&= b_\beta^J x_\gamma^J b a_\alpha b^{-1}(b^J)^{-1} \\
&= b_\beta^J x_\gamma^J b a_\alpha (b^J b)^{-1} \\
&= b_\beta^J x_\gamma^J (b^J)^{-1} a_\alpha \\
&= b_\beta^J x_\gamma^e a_\alpha \\
&= x_\gamma^e(b_\beta \otimes' a_\alpha).
\end{aligned}$$

Therefore, $e\in\mathrm{End}_{T^\circ}\mathcal{D}$ and $x^{e^2} = (x^e)^e = (x^J(b^J)^{-1})^e = (x^J(b^J)^{-1})^J(b^J)^{-1} = b^{-1}bxb^{-1}(b^J)^{-1} = b^{-1}(b^J)^{-1}x = (b^Jb)^{-1}x$ $\forall x\in\mathcal{D}$, also $x^{e\delta} = \delta x^e = \delta x^J(b^J)^{-1} = (\delta^J b^{-1}x)^J = (b^{-1}(\delta^J x))^J = (\delta^J x)^J(b^J)^{-1} = (\delta^J x)^e = x^{\delta^J e}$ so $e\delta = \delta^J e$ $\forall\delta\in K$. Therefore, we have $(K/k,(b^Jb)^{-1}) \approx \mathrm{End}_{T^\circ}\mathcal{D} \sim T^\circ$. □

Theorem 2.3.6 (1) If $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$ is an odd central division superalgebra over the field K of characteristic not equal to 2 then $\mathcal{D}$ admits a K/k-superinvolution (of the second kind) if and only if $\mathrm{Tr}(\mathcal{D})$ splits, and $\mathcal{D} = \mathcal{D}_{\bar{0}}\hat{\otimes}_K K(u),\ u^2 = \lambda \in K,\ \lambda^\sigma = -\lambda$ where σ is the Galois automorphism on K/k.
(2) If $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$ is an even central division superalgebra over the field K with non-trivial grading, then $\mathcal{D}$ admits a K/k-superinvolution (of the second kind) if and only if $\mathrm{Tr}(\mathcal{D})$ splits.

Proof (1) ($\Longrightarrow$) If $\mathcal{D}$ admits a K/k-superinvolution (say $*$) then $\mathcal{D}^\circ \overset{*}{\approx} \overline{\mathcal{D}}$ so $\overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}$ splits, and hence $\mathrm{Tr}(\mathcal{D})$ splits, and so by [21, proposition 9] we can choose u such that $u^* = u$ and $\lambda^* = \lambda^\sigma = -\lambda$ where $u^2 = \lambda$.
($\Longleftarrow$) If $\mathrm{Tr}(\mathcal{D})$ splits then $\overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}$ splits, so $\mathcal{D}$ admits a K/k-antiautomorphism J such that $x^{J^2} = x\ \forall x \in \mathcal{D}_{\bar{0}}$, therefore, by Theorem 2.2.1, $\mathcal{D}$ admits a K/k -superinvolution.
(2) ($\Longrightarrow$) If $\mathcal{D}$ admits a K/k-superinvolution (of the second kind) (say J) then by Theorem 2.2.5, $bb^J \in N(K^\times)$, which implies that $(bb^J)^{-1}$ is also in $N(K^\times)$. Therefore, $(K/k,(bb^J)^{-1})$ splits, and hence $\mathrm{Tr}(\mathcal{D})$ splits also, since $(\mathrm{Tr}(\mathcal{D}))^\circ$ splits.
($\Longleftarrow$) If $\mathrm{Tr}(\mathcal{D})$ splits then $\overline{\mathcal{D}}\hat{\otimes}_K\mathcal{D}$ splits, therefore $\mathcal{D}$ admits a K/k-antiautomorphism J such that $x^{J^2} = x\ \forall x \in \mathcal{D}_{\bar{0}}$. Since $K \sim (\mathrm{Tr}(\mathcal{D}))^\circ \sim (K/k,(bb^J)^{-1})$ we get that $(bb^J)^{-1} \in N(K^\times)$, and hence $bb^J \in N(K^\times)$ so again , by Theorem 2.2.5, $\mathcal{D}$ admits a K/k-superinvolution. □

Now, let $\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{1}}$, where $\mathcal{A}_{\bar{1}} \neq 0$, be a central simple superalgebra over K. Then by [21, Theorem3], $\mathcal{A} \cong \mathcal{M}_n(\mathcal{D})$, where $\mathcal{D}$ is an associative division superalgebra over K. Therefore, if $\mathcal{A}$ has a superinvolution then by [21, Theorem 7] $\mathcal{D}$ has.

Lemma 2.3.7 If K is a field of characteristic not equal to 2 and $\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{1}}$, where $\mathcal{A}_{\bar{1}} \neq 0$, is a K-superalgebra with a superinvolution of the first kind, then $\mathcal{A} \cong \mathcal{M}_n(\mathcal{D})$ and $\mathcal{D}$ has the trivial grading.

Proof by [21, Theorem 7] $\mathcal{D}$ has a superinvolution of the first kind. Therefore, $\mathcal{D}$ must be as in case (i) of the Division superalgebra theorem, so $\mathcal{D}$ has a trivial grading.
□

Lemma 2.3.8 For a non-trivial division superalgebra $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$, $\mathcal{A} \cong \mathcal{M}_n(\mathcal{D})$ has a superinvolution if and only if $\mathcal{D}$ has.

Proof ($\Longrightarrow$) If $\mathcal{A}$ has a superinvolution then by [21, Theorem7] $\mathcal{D}$ has.
($\Longleftarrow$) If $\mathcal{D}$ has a superinvolution (say $*$) then the map

$$(t \otimes *) : \mathcal{M}_n(K) \otimes \mathcal{D} \longrightarrow \mathcal{M}_n(K) \otimes \mathcal{D}$$

$$(\Sigma a \otimes d_\delta)^{t \otimes *} \mapsto \Sigma a^t \otimes d_\delta^*$$

where t is the transpose; is a superinvolution on $\mathcal{A}$. □

Corollary 2.3.9 (1) for a division superalgebra $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$ of odd type. $\mathcal{A} \cong \mathcal{M}_n(\mathcal{D})$ has a superinvolution of the second kind if and only if $\mathrm{Tr}(\mathcal{D})$ splits, and $\mathcal{D} = \mathcal{D}_{\bar{0}} \otimes_K K(u)$ where $u^2 = \lambda \in K$ and $\lambda^\sigma = -\lambda$, where σ is the Galois automorphism on K/k.
(2) for a non-trivial division superalgebra $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}}$ of even type. $\mathcal{A} \cong \mathcal{M}_n(\mathcal{D})$ has a superinvolution of the second kind if and only if $\mathrm{Tr}(\mathcal{D})$ splits. □

Chapter 3

Group of Automorphisms

In this chapter all superalgebras are central simple over a given field K of characteristic not equal to 2. Using the classical theorem of Skolem-Noether we study the automorphisms of a simple superalgebra $\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{1}}$ with a superinvolution $*$ (i.e., the automorphisms of the superalgebra $\mathcal{A}$ which commute with the given superinvolution). In this chapter $\psi_u : \mathcal{A} \to \mathcal{A}$ is an inner automorphism on $\mathcal{A}$, conjugation by u.

3.1 Superalgebras of odd and even types

In this section $\mathrm{Aut}(\mathcal{A}, *)$, where $\mathcal{A}$ is a superalgebra of any type, means the set of all automorphisms on $\mathcal{A}$ commuting with a given superinvolution $*$ and $\mathrm{inAut}(\mathcal{A}, *)$ means the set of all inner automorphisms on $\mathcal{A}$ commuting with $*$.

Lemma 3.1.1 Let $\mathcal{A} = \mathcal{A}_{\bar{0}} + \mathcal{A}_{\bar{0}}v$ be a superalgebra of odd type over a field K with a superinvolution $*$. Then

$$\mathrm{Aut}(\mathcal{A}, *) = \mathrm{inAut}(\mathcal{A}, *) \cup \nu.\mathrm{inAut}(\mathcal{A}, *),$$

where ν is the automorphism on $\mathcal{A}$ defined by $a_\alpha^\nu = (-1)^\alpha a_\alpha$.

Proof If $\phi \in \mathrm{Aut}(\mathcal{A}, *)$ then by Skolem-Noether theorem ϕ or $\phi\nu$ is inner but not both of them, so, $\phi \in \mathrm{inAut}(\mathcal{A}, *)$ or $\phi\nu \in \mathrm{inAut}(\mathcal{A}, *)$. $\square$

Now, let any $\phi \in \text{inAut}(\mathcal{A}, *)$. Then $\phi = \psi_{c_\gamma}$ for some invertible $c_\gamma \in \mathcal{A}_\gamma$. But $\phi \mid_{\mathcal{A}_{\bar{0}}}$ is inner , so $\phi \mid_{\mathcal{A}_{\bar{0}}} = \psi_{b_{\bar{0}}}$. Therefore, $c_\gamma(b_{\bar{0}}^{-1} a b_{\bar{0}})c_\gamma^{-1} = a$, $\forall a \in \mathcal{A}_{\bar{0}}$, and $c_\gamma(b_{\bar{0}}^{-1} v b_{\bar{0}})c_\gamma^{-1} = v$ which implies that $\psi_{c_\gamma b_{\bar{0}}^{-1}} = id_{\mathcal{A}} \Rightarrow \psi_{c_\gamma} = \psi_{b_{\bar{0}}}$.
Now, $a_\alpha^{*\psi_{b_{\bar{0}}}} = a_\alpha^{\psi_{b_{\bar{0}}}*} \Leftrightarrow b_{\bar{0}}^* b_{\bar{0}} = \lambda \in K$ and $\psi_{c_{\bar{0}}} = \psi_{b_{\bar{0}}} \Leftrightarrow c_{\bar{0}} = \alpha b_{\bar{0}}$ for some $\alpha \in K$. Let $G = \{b_{\bar{0}} \in \mathcal{A}_{\bar{0}} \mid b_{\bar{0}}^* = \alpha b_{\bar{0}}^{-1}$ for some $\alpha \in K^\times\}$. Let $\sim$ be an equivalence relation on G defined as follows, $a_{\bar{0}} \sim b_{\bar{0}} \Leftrightarrow b_{\bar{0}} a_{\bar{0}}^{-1} = \alpha \in K^\times$. Then $G/\sim = \{[b_{\bar{0}}] \mid b_{\bar{0}} \in G\}$, where $[b_{\bar{0}}] = \{\alpha b_{\bar{0}} \mid \alpha \in K^\times\}$, is a group with a well defined operation $[b_{\bar{0}}][c_{\bar{0}}] = [c_{\bar{0}} b_{\bar{0}}]$.

Theorem 3.1.2 Let $\mathcal{A}$ be a superalgebra of odd type over a field K with a superinvolution $*$, then $\text{inAut}(\mathcal{A}, *) \cong G/\sim$.

Proof Let $\delta : \text{inAut}(\mathcal{A}, *) \to G/\sim$ such that $\delta(\psi_{b_{\bar{0}}}) = [b_{\bar{0}}]$ then $\delta(\psi_{b_{\bar{0}}}\psi_{c_{\bar{0}}}) = \delta(\psi_{c_{\bar{0}} b_{\bar{0}}}) = [c_{\bar{0}} b_{\bar{0}}] = [b_{\bar{0}}][c_{\bar{0}}] = \delta(\psi_{b_{\bar{0}}})\delta(\psi_{c_{\bar{0}}})$. So, δ is a group homomorphism where δ is onto and $ker\delta = id_{\mathcal{A}} = \psi_{\bar{1}}$. Therefore, $\text{inAut}(\mathcal{A}, *) \cong G/\sim$. □

Let $\mathcal{A} = M_n(\mathcal{D})$ be a superalgebra over K, where $\mathcal{D}$ is a non-trivial grading division superalgebra of even type. Then $\mathcal{A}$ is of even type and $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}} = C_{\mathcal{D}}(u) + C_{\mathcal{D}}(u)v$ where $uv = -vu$ and $\hat{Z}(\mathcal{A}) = Z(\mathcal{A}) = K$, $Z(\mathcal{A}_{\bar{0}}) = K(u)$.
Let $*$ be a superinvolution on $\mathcal{A}$, and let $\phi \in \text{Aut}(\mathcal{A}, *)$. Then by the Skolem-Noether theorem $a^\phi = c_\gamma a c_\gamma^{-1}$, $\forall a \in \mathcal{A}$. Since $a^{\phi *} = a^{*\phi}$, $\forall a \in \mathcal{A}$, we have $c_\gamma c_\gamma^* = \lambda \in K$. If $\phi \mid_{\mathcal{A}_{\bar{0}}} = \psi_{c_\gamma} \mid_{\mathcal{A}_{\bar{0}}}$ is inner then $\phi \mid_{\mathcal{A}_{\bar{0}}} = \psi_{b_{\bar{0}}}$ which implies that $a^\phi = b_{\bar{0}} a b_{\bar{0}}^{-1}$, $\forall a \in \mathcal{A}_{\bar{0}}$, and therefore, $\psi_{b_{\bar{0}}^{-1} c_\gamma} = id_{\mathcal{A}_{\bar{0}}}$. Let $x_\gamma = b_{\bar{0}}^{-1} c_\gamma$, where $\gamma = \bar{1}$. Then $x_\gamma a x_\gamma^{-1} = a$ $\forall a \in \mathcal{A}_{\bar{0}}$. Since $\mathcal{A}_{\bar{1}} = \mathcal{A}_{\bar{0}} x_\gamma$, we have x_γ centralizes $\mathcal{A}_{\bar{1}}$, so $x_\gamma \in Z(\mathcal{A}) = K$. This is a contradiction and therefore, $\gamma = \bar{0}$. So that, for any $\phi \in \text{Aut}(\mathcal{A}, *)$ such that $\phi \mid_{\mathcal{A}_{\bar{0}}}$ is inner, $\phi = \psi_{b_{\bar{0}}}$, for some invertible element $b_{\bar{0}} \in \mathcal{A}_{\bar{0}}$, and hence $\text{inAut}(\mathcal{A}, *) = \{\phi \in Aut(\mathcal{A}, *) \mid \phi = \psi_{b_{\bar{0}}},\ b_{\bar{0}} \in \mathcal{A}_{\bar{0}}\}$.

Corollary 3.1.3 Let $\mathcal{A} = M_n(\mathcal{D})$, where $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}} = C_{\mathcal{D}}(u) + C_{\mathcal{D}}(u)v$ and $uv = -vu$, be a superalgebra with a superinvolution $*$. If $\psi_v * = *\psi_v$ then $\text{Aut}(\mathcal{A}, *) = \text{inAut}(\mathcal{A}, *) \cup \psi_v.\text{inAut}(\mathcal{A}, *)$. If not then $\text{Aut}(\mathcal{A}, *) = \text{inAut}(\mathcal{A}, *)$.

Proof If $\psi_v * = *\psi_v$ then for any $\phi \in \text{Aut}(\mathcal{A}, *)$ such that $u^\phi = -u$ we have $u^{\phi\psi_v} = -u^{\psi_v} = u$ which implies that $\phi\psi_v \mid_{\mathcal{A}_{\bar{0}}}$ is inner and so $\phi\psi_v = \psi_{b_{\bar{0}}}$ where $b_{\bar{0}}$ is an invertible element in $\mathcal{A}_{\bar{0}}$. Therefore, $\phi = \psi_{b_{\bar{0}}}\psi_v$. Hence,

$$\mathrm{Aut}(\mathcal{A},*) = \mathrm{inAut}(\mathcal{A},*) \cup \psi_v.\mathrm{inAut}(\mathcal{A},*).$$

But, if $\psi_v * \neq * \psi_v$ then $\mathrm{Aut}(\mathcal{A},*) = \mathrm{inAut}(\mathcal{A},*)$. □

Theorem 3.1.4 Let $\mathcal{A} = M_n(\mathcal{D})$, where $\mathcal{D} = \mathcal{D}_{\bar{0}} + \mathcal{D}_{\bar{1}} = C_{\mathcal{D}}(u) + C_{\mathcal{D}}(u)v$ and $uv = -vu$, be a superalgebra with a superinvolution $*$. Then $\mathrm{inAut}(\mathcal{A},*) \cong G/\sim$.

Proof Let $\delta : \mathrm{inAut}(\mathcal{A},*) \to G/\sim$ such that $\delta(\psi_{b_{\bar{0}}}) = [b_{\bar{0}}]$. Then $\delta(\psi_{b_{\bar{0}}}\psi_{c_{\bar{0}}}) = \delta(\psi_{c_{\bar{0}}b_{\bar{0}}}) = [c_{\bar{0}}b_{\bar{0}}] = [b_{\bar{0}}][c_{\bar{0}}] = \delta(\psi_{b_{\bar{0}}})\delta(\psi_{c_{\bar{0}}})$. So, δ is a group homomorphism where $ker\delta = id_{\mathcal{A}} = \psi_{\bar{1}}$, and δ is onto. Therefore, $\mathrm{inAut}(\mathcal{A},*) \cong G/\sim$. □

We end this section by this example to show that uu^* may equal λ with $\lambda \neq \alpha\alpha^*$ for any $\alpha \in K$,

Example Let $k = \mathbf{Q}(e)$, $K = k(\sqrt{2})$ and $u = \sqrt{e}i$, $v = \sqrt[4]{2}j$ where $i^2 = -1$, $j^2 = -1$ and $ij = -ji$. Then $\mathcal{A} = (K + Ku) \oplus (Kv + Kuv)$ is a quaternion division superalgebra over field K. Let $*$ be a superinvolution on $\mathcal{A}$ defined by $(a + b\sqrt{2})^* = \overline{a + b\sqrt{2}} = a - b\sqrt{2}$, $\forall a, b \in k$, $u^* = u$, $v^* = v$, then $(uv)^* = vu = -uv$. Now, $\psi_u \in \mathrm{Aut}(\mathcal{A},*)$, since $uu^* = -e \in K$. But $uu^* \neq \alpha\alpha^*$, where $\alpha \in K$ because if $-e = (a + b\sqrt{2})(a + b\sqrt{2})^*$ where $a, b \in k$ then

$$-e = (a + b\sqrt{2})(a - b\sqrt{2}) = a^2 - 2b^2 = (\frac{p_1(e)}{q_1(e)})^2 - 2(\frac{p_2(e)}{q_2(e)})^2$$

Where $p_i(e), q_i(e) \in \mathbf{Q}[e]$ and $q_i(e) \neq 0$. Hence,

$$-e(q_1(e))^2(q_2(e))^2 = (p_1(e)q_2(e))^2 - 2(p_2(e)q_1(e))^2.$$

But the highest power of e in the left hand side is odd, and the highest power of e in the right hand side is even, since $\sqrt{2} \notin \mathbf{Q}$, a contradiction. □

3.2 Even Superalgebras Continued.

Let $\mathcal{A} = M_{2p}(\mathcal{D}_{\bar{0}})$, where $\mathcal{D}_{\bar{0}}$ is a division algebra over K and let $*$ be any superinvolution on $\mathcal{A}$. If $\phi = \psi_{M_\alpha} \in \mathrm{Aut}(\mathcal{A}, *)$ then $M_\alpha P_\gamma^* M_\alpha^{-1} = (M_\alpha P_\gamma M_\alpha^{-1})^*$, $\forall P_\gamma \in \mathcal{A}_\gamma$, so,

$$\begin{aligned} M_\alpha P_\gamma^* M_\alpha^{-1} &= (-1)^{\alpha(\alpha+\gamma)}(-1)^{\alpha\gamma}(M_\alpha^{-1})^* P_\gamma^* M_\alpha^* \\ &= (-1)^{\alpha}(M_\alpha^{-1})^* P_\gamma^* M_\alpha^* \\ &= (M_\alpha^*)^{-1} P_\gamma^* M_\alpha^*, \end{aligned}$$

which implies that $M_\alpha^* M_\alpha P_\gamma^* M_\alpha^{-1}(M_\alpha^*)^{-1} = P_\gamma^*$, $\forall P_\gamma$, and therefore, $M_\alpha^* M_\alpha = \lambda I_{2p}$ where $\lambda \in K^\times$. If $*$ is a superinvolution of the first kind then $\alpha = \bar{0}$ and therefore, $\mathrm{Aut}(\mathcal{A}, *) = \{\psi_M \mid M$ is an invertible element in $\mathcal{A}_{\bar{0}}$, with $MM^* = \lambda I_{2p}\}$. Moreover, if $*$ is a superinvolution of the second kind then $\mathrm{Aut}(\mathcal{A}, *) = \{\psi_M \mid M$ is an invertible element in $\mathcal{A}_{\bar{0}}$, with $MM^* = \lambda I_{2p}\} \cup \{\psi_M \mid M$ is an invertible element in $\mathcal{A}_{\bar{1}}$, with $MM^* = \gamma I_{2p}\}$.

Theorem 3.2.1 Let $\mathcal{A} = M_{2p}(\mathcal{D}_{\bar{0}})$ where $\mathcal{D}_{\bar{0}}$ is a division algebra over K and let $*$ be any superinvolution on $\mathcal{A}$. If $*$ is a superinvolution of the first kind then $Aut(\mathcal{A}, *) \cong G/\sim$ where $G/\sim$ is the group defined as in Theorem 3.1.2. If $*$ is a superinvolution of the second kind then $\mathrm{Aut}(\mathcal{A}, *) \cong G/\sim$ where $G = \{b_{\bar{0}} \in \mathcal{A}_{\bar{0}} \mid b_{\bar{0}} b_{\bar{0}}^* = \alpha \in K^\times\} \cup \{b_{\bar{1}} \in \mathcal{A}_{\bar{1}} \mid b_{\bar{1}} b_{\bar{1}}^* = \lambda \in K^\times\}$ and the relation $\sim$ on G is defined by $a_\alpha \sim b_\alpha \Leftrightarrow b_\alpha = \gamma a_\alpha$ for some $\gamma \in K^\times$.

Proof If $*$ is a superinvolution of the second kind then it is easy to check that $G/\sim$ is a group under the well-defined operation $[a_\alpha][b_\beta] = [b_\beta a_\alpha]$, so, in all cases

$$\delta : \mathrm{Aut}(\mathcal{A}, *) \to G/\sim\ ,\ (\psi_{a_\alpha} \mapsto [a_\alpha])$$

is a group isomorphism and therefore $\mathrm{Aut}(\mathcal{A}, *) \cong G/\sim$. □

Let $\mathcal{A} = M_{p+q}(\mathcal{D}_{\bar{0}})$, where $\mathcal{D}_{\bar{0}}$ is a division algebra over K and $(p \neq q)$. Let $*$ be any superinvolution on $\mathcal{A}$. If $\phi = \psi_M \in \mathrm{Aut}(\mathcal{A}, *)$, where M is an invertible element in $\mathcal{A}_{\bar{0}}$, then, $\forall P_\gamma \in \mathcal{A}_\gamma$ we have,

$$\begin{aligned} MP_\gamma^* M^{-1} &= (MP_\gamma M^{-1})^* \\ &= (M^{-1})^* P_\gamma^* M^* \\ &= (M^*)^{-1} P_\gamma^* M^*. \end{aligned}$$

Therefore, $M^*MP_\gamma^*M^{-1}(M^*)^{-1} = P_\gamma^*$, $\forall P_\gamma$, and hence, $M^*M = \lambda I_{p+q}$ where $\lambda \in K^\times$. Similarly, $\mathrm{Aut}(\mathcal{A}, *) = \{\psi_M \mid M$ is an invertible element in $\mathcal{A}_{\bar{0}}$, where $MM^* = \lambda I_{p+q}\} \cong G/\sim$, where $G = \{b_{\bar{0}} \in \mathcal{A}_{\bar{0}} \mid b_{\bar{0}} b_{\bar{0}}^* = \alpha \in K^\times\}$.

3.3 Examples of Even Superalgebras

In this section we study in details the group of automorphisms of superalgebras $\mathcal{A} = M_{2p}(\mathcal{D})$, and $\mathcal{A} = M_{p+q}(\mathcal{D})$, where $\mathcal{D} = \mathbf{H}$ or $\mathbf{R}$, where $\mathbf{H}$ is the hamiltonian quaternion algebra and $\mathbf{R}$ is the field of real numbers.

Let $\mathcal{A} = M_{2p}(\mathcal{D})$, where $\mathcal{D} = \mathbf{H}$ or $\mathbf{R}$, and let $*$ be a superinvolution of the first kind on $\mathcal{A}$ defined by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \tilde{d} & -\tilde{b} \\ \tilde{c} & \tilde{a} \end{pmatrix}$$

where $\tilde{}$ is the involution on $M_p(\mathcal{D})$ induced by $*$. Then $H = \mathrm{Aut}(\mathcal{A}, *) = \{\psi_M \mid M$ is an invertible element in $\mathcal{A}_{\bar{0}}$, where $MM^* = \alpha I_{2p}$, and $\alpha \in \mathbf{R}\}$.

If $\alpha > 0$ then replacing M by $\frac{1}{\sqrt{\alpha}}M$ we get $MM^* = I_{2p}$.

If $\alpha < 0$ then replacing M by $\frac{1}{\sqrt{-\alpha}}M$ we get $MM^* = -I_{2p}$.

So $H = \{\psi_M \mid MM^* = I_{2p}$ or $-I_{2p}\}$.

Theorem 3.3.1 In the situation described above. Let

$$G = \mathrm{Aut}(\mathcal{A}_{\bar{0}}, * |_{\mathcal{A}_{\bar{0}}}).$$

Then $H/\mathbf{R}^\times \cong G_0$ where G_0 is the connected component of the identity in G.

Proof By [27, page 593], $G = G_0 \cup \psi_N G_0$, where $N = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ and where G_0 is the connected component of the identity. Let

$$\phi : H \to G_0$$

$$(\psi_M \mapsto \psi_M \mid_{\mathcal{A}_{\bar{0}}}).$$

Then ϕ is onto because if $g_0 \in G_0$ then $g_0 = \psi_M \mid_{\mathcal{A}_{\bar{0}}}$ for some invertible element $M \in \mathcal{A}_{\bar{0}}$ since G_0 is the connected component of the identity. It is then easy to check that $MM^* = \alpha I_{2p}$ $(\alpha \in \mathbf{R})$ since $\psi_M * = *\psi_M$.

If $\alpha > 0$ then replacing M by $\frac{1}{\sqrt{\alpha}}M$ we get $MM^* = I_{2p}$.

If $\alpha < 0$ then replacing M by $\frac{1}{\sqrt{-\alpha}}M$ we get $MM^* = -I_{2p}$. Therefore, $\psi_M \in H$ and $\phi(\psi_M) = g_0$. Also it is easy to check that $\ker\phi = \{\psi_M \mid M = \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_p \end{pmatrix}$, where $\lambda, \gamma \in \mathbf{R}\}$. But $\begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_p \end{pmatrix}\begin{pmatrix} \gamma I_p & 0 \\ 0 & \lambda I_p \end{pmatrix} = \begin{pmatrix} \lambda\gamma I_p & 0 \\ 0 & \gamma\lambda I_p \end{pmatrix} = I_{2p}$ or $-I_{2p}$. So, $\gamma = \frac{1}{\lambda}$ or $\frac{-1}{\lambda}$. Therefore, $\ker\phi = \{\psi_M \mid M = \begin{pmatrix} \lambda I_p & 0 \\ 0 & \frac{1}{\lambda} I_p \end{pmatrix}$, or $M = \begin{pmatrix} \lambda I_p & 0 \\ 0 & \frac{-1}{\lambda} I_p \end{pmatrix}$, where $\lambda \in \mathbf{R}^\times\} \cong \mathbf{R}^\times$. Therefore, $G_0 \cong H/\ker\phi \cong H/\mathbf{R}^\times$. □

Now, let $\mathcal{A} = M_{p+q}(\mathcal{D})$, where $\mathcal{D} = \mathbf{H}$ or $\mathbf{R}$, and let $*$ be a superinvolution of the first kind on $\mathcal{A}$ defined by:

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} & -N\bar{z}^t M^{-1} \\ M\bar{y}^t N^{-1} & \tilde{w} \end{pmatrix}$$

where $\tilde{x} = N\bar{x}^t N^{-1}$ $(\bar{N}^t = -N \in M_p(\mathcal{D}))$ and $\tilde{w} = M\bar{w}^t M^{-1}$ $(\bar{M}^t = M \in M_q(\mathcal{D}))$, and $^-$ is the standard involution on $\mathcal{D}$.

Since $*$ is a superinvolution of the first kind we have $H = \mathrm{Aut}(\mathcal{A}, *) = \{\psi_M \mid M$ is an invertible element in $\mathcal{A}_{\bar{0}}$, where $MM^* = \alpha I_{p+q}$, and $\alpha \in \mathbf{R}\}$. Similarly, $H = \{\psi_M \mid MM^* = I_{p+q}$ or $-I_{p+q}\}$.

We will start with the case $p \neq q$

Theorem 3.3.2 In the situation described above, let

$$G = \mathrm{Aut}(\mathcal{A}_{\bar{0}}, * \mid_{\mathcal{A}_{\bar{0}}}).$$

Then $H/\ker\phi \cong ImH \leq G$ where

$$\phi : H \to G \quad (\psi_N \mapsto \psi_N \mid_{\mathcal{A}_{\bar{0}}}).$$

Proof Since $\mathcal{A}_{\bar{0}}$ is semi-simple and $p \neq q$ we get $G = \mathrm{Aut}(\mathcal{A}_{\bar{0}}, * |_{\mathcal{A}_{\bar{0}}}) = \{\psi_M \mid M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A, B are invertible matrices $\}$. If $\psi_M \in G$ then

$$\psi_M * |_{\mathcal{A}_{\bar{0}}} = * |_{\mathcal{A}_{\bar{0}}} \psi_M \quad \Leftrightarrow \quad MM^* = \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_q \end{pmatrix}, \ (\lambda, \gamma \in \mathbf{R}),$$

$$\Leftrightarrow \quad MM^* \in V = \{I_{p+q}, -I_{p+q}, \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}\}.$$

So $G = \{\psi_M \mid MM^* \in V\}$. Therefore,

$$\phi : H \to G \quad (\psi_N \mapsto \psi_N |_{\mathcal{A}_{\bar{0}}})$$

is a group homomorphism and $\ker\phi = \{\psi_N \mid \psi_N |_{\mathcal{A}_{\bar{0}}} = id_{\mathcal{A}_{\bar{0}}}\} = \{id_{\mathcal{A}}, \psi_M, \ M = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}\}$. To see this let $N = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ such that $\psi_N \in \ker\phi$ then $NN^* = I_{p+q}$ or $-I_{p+q}$ (since $\psi_N \in H$), and

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

if and only if $A = \lambda I_p, \ B = \gamma I_q$ ($\lambda, \ \gamma \in \mathbf{R}$), since

$$NN^* = \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_q \end{pmatrix} \begin{pmatrix} \lambda I_p & 0 \\ 0 & \gamma I_q \end{pmatrix} = \begin{pmatrix} \lambda^2 I_p & 0 \\ 0 & \gamma^2 I_q \end{pmatrix} \in \{I_{p+q}, -I_{p+q}\}$$

we get $NN^* = I_{p+q}$. Therefore, $\lambda^2 = \gamma^2 = 1 \Rightarrow \lambda, \gamma \in \{1, -1\}$ which implies that $\ker\phi = \{\psi_{I_{p+q}}, \psi_{-I_{p+q}}, \psi_N, \psi_M\}$, where $N = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$ and $M = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and hence $\ker\phi = \{id_{\mathcal{A}}, \psi_M\}$ since $\psi_{I_{p+q}} = \psi_{-I_{p+q}}$ and $\psi_N = \psi_M$. Therefore, $H/\ker\phi \cong ImH \leq G$. $\square$

Lemma 3.3.3 As in the theorem above, if $p \neq q$ then $G = ImH \cup \psi_M |_{\mathcal{A}_{\bar{0}}} ImH$ for a fixed invertible matrix $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ such that

$$MM^* \in \{\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}\}.$$

Proof $\psi_M \mid_{\mathcal{A}_{\bar{0}}} \notin ImH$ since $MM^* \in \{\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}\}$. Let $N = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ such that $NN^* \in \{\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}\}$ then

$$\begin{aligned}
(NM^{-1})(NM^{-1})^* &= NM^{-1}(M^{-1})^*N^* \\
&= \begin{cases} N\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}N^* & : \; MM^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \\ N\begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}N^* & : \; MM^* = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \end{cases} \\
&= \begin{cases} \begin{pmatrix} X\tilde{X} & 0 \\ 0 & -Y\tilde{Y} \end{pmatrix} & : \; MM^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \\ \begin{pmatrix} -X\tilde{X} & 0 \\ 0 & Y\tilde{Y} \end{pmatrix} & : \; MM^* = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \end{cases}
\end{aligned}$$

Therefore, $(NM^{-1})(NM^{-1})^* \in \{I_{p+q}, -I_{p+q}\}$ and so

$$\psi_{NM^{-1}} = \psi_{M^{-1}}\psi_N \in ImH \cong H/\ker\phi.$$

Now, $\psi_N = \psi_{NM^{-1}M} = \psi_M\psi_{M^{-1}}\psi_N = \psi_M \underbrace{\psi_{NM^{-1}}} \in \psi_M ImH$. Therefore,

$$G = ImH \cup \psi_M \mid_{\mathcal{A}_{\bar{0}}} ImH.$$

□

For the case $p = q$, $G = \mathrm{Aut}(\mathcal{A}_{\bar{0}}, * \mid_{\mathcal{A}_{\bar{0}}}) = G_0 \cup \psi_N G_0$, where $N = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ (it is easy to check that $\psi_N * \mid_{\mathcal{A}_{\bar{0}}} = * \mid_{\mathcal{A}_{\bar{0}}} \psi_N$), and where $G_0 = \{\psi_M \mid MM^* \in V\}$ is the connected component of the identity in G. Now, let $\phi : H \to G_0$ ($\psi_N \mapsto \psi_N \mid_{\mathcal{A}_{\bar{0}}}$), where $H = \{\psi_M \mid MM^* = I_{p+q}$ or $-I_{p+q}\}$, then ϕ is a group homomorphism and $\ker\phi = \{id_{\mathcal{A}}, \psi_{N'}\}$ where $N' = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}$. Therefore, $H/\ker\phi \cong ImH \leq G_0$.

Theorem 3.3.4 In the situation described above, if $p = q$ then

$$G_0 = ImH \cup \psi_M \mid_{\mathcal{A}_{\bar{0}}} ImH,$$

for a fixed invertible matrix $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ such that

$$MM^{*} \in \{\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}\}.$$

Where G_0 is the connected component of the identity in G.

Proof $\psi_M \mid_{\mathcal{A}_{\bar{0}}} \notin ImH$ since $MM^* \in \{\begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix}\}$. Let $N = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ such that $NN^* \in \{\begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}, \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix}\}$ then

$$\begin{aligned}
(NM^{-1})(NM^{-1})^* &= NM^{-1}(M^{-1})^*N^* \\
&= \begin{cases} N\begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}N^* & : \quad MM^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix} \\ N\begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix}N^* & : \quad MM^* = \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix} \end{cases} \\
&= \begin{cases} \begin{pmatrix} X\tilde{X} & 0 \\ 0 & -Y\tilde{Y} \end{pmatrix} & : \quad MM^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix} \\ \begin{pmatrix} -X\tilde{X} & 0 \\ 0 & Y\tilde{Y} \end{pmatrix} & : \quad MM^* = \begin{pmatrix} -I_p & 0 \\ 0 & I_p \end{pmatrix} \end{cases}.
\end{aligned}$$

Therefore, $(NM^{-1})(NM^{-1})^* \in \{I_{2p}, -I_{2p}\}$ and so,

$$\psi_{NM^{-1}} = \psi_{M^{-1}}\psi_N \in ImH \cong H/\ker\phi.$$

Now, $\psi_N = \psi_{NM^{-1}M} = \psi_M\psi_{M^{-1}}\psi_N = \psi_M \underbrace{\psi_{NM^{-1}}} \in \psi_M ImH$. Therefore,

$$G_0 = ImH \cup \psi_M \mid_{\mathcal{A}_{\bar{0}}} ImH.$$

□

Chapter 4

Projective Supermodules

The purpose of this chapter is to present material from the theory of super-rings which we will need in our consideration of separable superalgebras.

We begin with some fundamental definitions and conventions. Throughout we will be considering associative super-rings possessing an identity element 1. We will always suppose that subsuper-rings contain the identity of the overring and that a super-ring homomorphism from a super-ring R to a super-ring S takes the identity of R to the identity of S. Finally, all supermodules will be unitary. In what follows, R-supermodule means right R-supermodule.

Suppose $R = R_{\bar{0}} + R_{\bar{1}}$ is a supercommutative super-ring. Then, for any $r_{\bar{1}} \in R_{\bar{1}}$, $r_{\bar{1}}^2 = -r_{\bar{1}}^2$ and $2r_{\bar{1}}^2 = 0$. If R has no 2-*torsion* (i.e., for any $x \in R$, $2x = 0$ if and only if $x = 0$), then $r_{\bar{1}}^2 = 0$. We will assume that if R is supercommutative then $r_{\bar{1}}^2 = 0$, $\forall r_{\bar{1}} \in R_{\bar{1}}$. This assures, for example, that the natural lie superalgebra structure on a supercommutative super-ring is trivial.

By an R-superalgebra we mean a super-ring A along with a super-ring homomorphism θ of R into $\hat{Z}(A)$ (the super-center of A). This induces a natural R-supermodule structure on A by defining

$$a_\beta r_\alpha = a_\beta \theta(r_\alpha) = (-1)^{\alpha\beta}\theta(r_\alpha)a_\beta \quad \forall\ r_\alpha \in R_\alpha,\ a_\beta \in A_\beta. \tag{12}$$

Conversely, if A is a super-ring which is also an R-supermodule satisfying (12), then $r \mapsto 1 \cdot r$ is a super-ring homomorphism of R into the super-center of A and so A is an R-superalgebra.

We describe an R-superalgebra as finitely generated or faithful if it is finitely generated or faithful when considered as an R-supermodule. Thus A is a faithful R-superalgebra if and only if the homomorphism θ from R into the super-center of A is one to one. In this event $R \cong 1 \cdot R$, so we identify R with $1 \cdot R$.

Let R be any super-ring (ring). Suppose $\{M_i\}_{i\in I}$ is a family of right R-supermodules (R-modules), where I is some (not necessarily finite) indexing set. Let $\bigoplus\sum_{i\in I} M_i$ denote the set of all functions from I into $\bigcup_i M_i$ such that $f(i)$ is in M_i and $f(i) = 0$ for all but finite set of indices i of I. Then $\bigoplus\sum_{i\in I} M_i$ has a natural structure as an R-supermodule (R-module) given by

$$(f+g)(i) = f(i) + g(i)$$

and

$$(f.r)(i) = f(i).r \quad \forall\, i \in I,\ r \in R,\ f, g \in \bigoplus\sum_{i\in I} M_i.$$

We call this R-supermodule (R-module) $\bigoplus\sum_{i\in I} M_i$ the *direct sum* of the family $\{M_i\}_{i\in I}$. In the event $I = \{1, 2, ..., n\}$, we sometimes write $\bigoplus\sum_{i\in I} M_i$ as $M_1 \oplus ... \oplus M_n$.

Finally, we speak of a subsupermodule N of a right R-supermodule M as being a *direct summand* of M if M contains a subsupermodule L such that $N + L = M$ and $N \cap L = \{0\}$. In this case, it is apparent that $M \cong N \oplus L$.

4.1 Progenerator Supermodules

Let $R = R_{\bar{0}} + R_{\bar{1}}$ be any super-ring, then for any indexing set I, we denote by R^I the supermodule $\bigoplus\sum_{i\in I} R_i$ where each R_i equals R; hence, $R^I = R^I_{\bar{0}} + R^I_{\bar{1}}$ as an R-supermodule. If $I = \{1, 2, ..., n\}$, we write R^n for R^I.

Definition 4.1.1 *Let $R = R_{\bar{0}} + R_{\bar{1}}$ be any super-ring with $R_{\bar{1}} \neq 0$, then a right (left) R-supermodule M is called* free R-supermodule *if M is isomorphic, as a right (left) R-supermodule to R^I, for some indexing set I. Moreover, if $R_{\bar{1}} = \{0\}$ then a right (left) R-supermodule $M = M_{\bar{0}} + M_{\bar{1}}$ is called* free R-supermodule *if M_α is isomorphic, as a right (left) R-module, to R^{I_α} for some indexing set I_α, where $\alpha = \bar{0}, \bar{1}$.*

One can easily check that M satisfies the usual universal conditions for a free object ([11, Example 2, page 43]).

Define b_i in R^I by $b_i(j) = \delta_{i,j}$, $i, j \in I$, where $\delta_{i,j}$ equals 1 if $i = j$, and 0 if $i \neq j$. Then we see that

(1) every element of R^I has a representation of the form $\sum_{i \in I'} b_i.r_i$, where $r_i \in R$ and I' is some finite subset of I; and,

(2) for any finite subset I' of I, $\sum_{i \in I'} b_i.r_i = 0$ implies $r_i = 0$, $\forall i \in I'$.

Lemma 4.1.2 Let R be any super-ring. If $R_{\bar{1}} = \{0\}$ then a right R-supermodule M is free if and only if there exists $\{m_{i,\bar{0}}\}_{i \in I_{\bar{0}}} \subseteq M_{\bar{0}}$, $\{m_{i,\bar{1}}\}_{i \in I_{\bar{1}}} \subseteq M_{\bar{1}}$ ($I_{\bar{0}}$, $I_{\bar{1}}$ some indexing sets) satisfying (1) and (2) above. If $R_{\bar{1}} \neq 0$ then M is free if and only if for $\delta = \bar{0}$ or $\delta = \bar{1}$, there exists $\{m_{i,\delta}\}_{i \in I} \subseteq M_\delta$ (I some indexing set) satisfying (1) and (2).

If $R_{\bar{1}} = \{0\}$ then the sets $\{m_{i,\bar{0}}\}_{i \in I_{\bar{0}}}$ and $\{m_{i,\bar{1}}\}_{i \in I_{\bar{1}}}$ are called *bases* for $M_{\bar{0}}$ and $M_{\bar{1}}$ (resp.). If $R_{\bar{1}} \neq 0$ then the set $\{m_{i,\delta}\}_{i \in I}$ is called a *basis* for M.

Proof Case(1) $R_{\bar{1}} = 0$. Then by Definition 4.1.1, $M_{\bar{0}}$, $M_{\bar{1}}$ are free R-modules, so by [6, Lemma 1.1.1] each $M_{\bar{0}}$ and $M_{\bar{1}}$ has a basis satisfying (1) and (2) above.
Case(2) $R_{\bar{1}} \neq 0$. Suppose M is free over R then there exists an R-supermodule isomorphism $\psi_\delta : R^I \to M$; put $m_{i,\delta} = \psi_\delta(b_i)$. Then for any $m_\alpha \in M_\alpha$ we have

$$\begin{aligned} m_\alpha = \psi_\delta(f_{\alpha+\delta}) &= \psi_\delta(\sum_{i \in I'} b_i.r_{i,\alpha+\delta}) \\ &= \sum_{i \in I'} m_{i,\delta}.r_{i,\alpha+\delta}. \end{aligned}$$

If $\sum_{i \in I'} m_{i,\delta}.r_{i,\alpha} = 0$, then $\psi_\delta(\sum_{i \in I'} b_i.r_{i,\alpha}) = 0$ and, hence, $\sum_{i \in I'} b_i.r_{i,\alpha} = 0$. Therefore, $r_{i,\alpha} = 0$, $\forall i \in I'$.

Conversely, if M is an R-supermodule and $\{m_{i,\delta}\}_{i\in I} \subseteq M_\delta$ (I some indexing set) satisfying (1) and (2), then the map $\psi_\delta : R^I \to M$ defined by

$$\begin{aligned} \psi_\delta(f_\alpha) &= \psi_\delta(\sum_{i\in I'} b_i.f_\alpha(i)) \\ &= \sum_{i\in I'} m_{i,\delta}.f_\alpha(i) \end{aligned}$$

is an R-supermodule isomorphism, so M is R-free. □

Note that in Lemma 4.1.2 if R is a super-ring where $R_{\bar{1}} \neq 0$ and M is a free R-supermodule then $M_{\bar{0}}.R_{\bar{1}} = M_{\bar{1}}$ if $\delta = \bar{0}$ or $M_{\bar{1}}.R_{\bar{1}} = M_{\bar{0}}$ if $\delta = \bar{1}$.

We speak of a subsupermodule N of an R-supermodule M as being a *direct summand* of M if M contains a subsupermodule L such that $N + L = M$ and $N \cap L = \{0\}$. In this case, we have $M \cong N \oplus L$.

Definition 4.1.3 *If R is a super-ring with $R_{\bar{1}} \neq \{0\}$, an R-supermodule M is called an R-*projective *(or* projective as a supermodule over R*) if M is isomorphic, as an R-supermodule, to a direct summand of some free R-supermodule. If $R_{\bar{1}} = \{0\}$ then M is called R-*projective *if $M_{\bar{0}}$ and $M_{\bar{1}}$ are R-projective as R-modules.*

Note that in Definition 4.1.3, if R is a super-ring with $R_{\bar{1}} \neq \{0\}$, and $M \overset{\psi_{\bar{0}}}{\cong} N$, where $\psi_{\bar{0}}$ is an R-supermodule isomorphism, without loss of generality, we may assume that $R^n \overset{\delta_{\bar{0}}}{\cong} N \oplus L$ as R-supermodules. Then, for any $m_{\bar{1}}$ in $M_{\bar{1}}$, we have that $\psi_{\bar{0}}(m_{\bar{1}}) = \delta_{\bar{0}}(b_1)r_1 + ... + \delta_{\bar{0}}(b_n)r_n$, where each r_i is in $R_{\bar{1}}$, and b_i in $R^n_{\bar{0}}$ defined as before (after Definition 4.1.1). Now for each i, $\delta_{\bar{0}}(b_i) = b'_i + b''_i$, where $b'_i \in N_{\bar{0}}$, and $b''_i \in L_{\bar{0}}$; therefore, $\psi_{\bar{0}}(m_{\bar{1}}) = b'_1 r_1 + ... + b'_n r_n$. However, for any i, $b'_i = \psi_{\bar{0}}(m'_i)$, where m'_i is in $M_{\bar{0}}$. Hence, $\psi_{\bar{0}}(m_{\bar{1}}) = \psi_{\bar{0}}(m'_1 r_1 + ... + m'_n r_n)$. This implies that $m_{\bar{1}} = m'_1 r_1 + ... + m'_n r_n$, and $M_{\bar{1}} = M_{\bar{0}} R_{\bar{1}}$. Similarly, if $M \overset{\psi_{\bar{1}}}{\cong} N$ then $M_{\bar{0}} = M_{\bar{1}} R_{\bar{1}}$.

Throughout this chapter we will assume (unless otherwise specified) that if $R_{\bar{1}} \neq \{0\}$ then an R-supermodule M satisfies at least one of the two conditions
(1) $M_{\bar{0}}.R_{\bar{1}} = M_{\bar{1}}$

(2) $M_{\bar{1}}.R_{\bar{1}} = M_{\bar{0}}$.

When R is a super-commutative super-ring, then by the next theorem it can be shown that if a free R-supermodule M has a finite basis consisting of n elements, then any other basis consists of n elements. This integer n is called the rank of M.

Theorem 4.1.4 For any super-commutative super-ring R, $R^n \cong R^m$ implies that $n = m$.

Proof If $R_{\bar{1}} = \{0\}$ then R is a commutative ring and so, by [10, Theorem 4.3], $n = m$.
If $R_{\bar{1}} \neq \{0\}$ then we will show that it is impossible to have a set $\{c_{i,\bar{1}}\}_{i\in I} \subseteq R_{\bar{1}}^I$ such that $\{c_{i,\bar{1}}\}_{i\in I}$ is a basis for R^I, for any finite set $I = \{1, 2, ..., n\}$. Suppose by contradiction that we have such a set and let, for some $i \in I$, $0 \neq c_{i,\bar{1}} \in R_{\bar{1}}^I$. Then $c_{i,\bar{1}} = (\alpha_1, \alpha_2, ..., \alpha_n)$, where each $\alpha_j \in R_{\bar{1}}$. Since $c_{i,\bar{1}} \neq 0$ there exists $\alpha_j \in \{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that $\alpha_j \neq 0$. We may renumber $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ so that $\alpha_1 \neq 0$. Consider $\alpha_1\alpha_2$, if it is zero just ignore α_2 and multiply α_1 by α_3 to get $\alpha_1\alpha_3$, but if $\alpha_1\alpha_2 \neq 0$, multiply this element by α_3 to get $\alpha_1\alpha_2\alpha_3$. Again multiply the new element by α_4, if this multiplication is not zero otherwise ignore α_4 and multiply it by α_5 and so on. At the end we will have an element of the form $\alpha_1\alpha_{i_1}\alpha_{i_2}...\alpha_{i_k} \neq 0$, where $1 < i_1 < i_2 < ... < i_k \leq n$. Now, for any $1 \leq j \leq n$, $\alpha_1\alpha_{i_1}\alpha_{i_2}...\alpha_{i_k}\alpha_j = 0$, we have a non-zero element $\alpha_1\alpha_{i_1}\alpha_{i_2}...\alpha_{i_k}$ in R such that $c_{i,\bar{1}}(\alpha_1\alpha_{i_1}\alpha_{i_2}...\alpha_{i_k}) = 0$, a contradiction. Therefore it is impossible to form a basis from $R_{\bar{1}}^I$. So if $R^n \cong R^m$ then $R_{\bar{0}}^n \cong R_{\bar{0}}^m$, but again $R_{\bar{0}}$ is a commutative ring so by [10] $n = m$. □

Lemma 4.1.5 If R is a super-ring with $R_{\bar{1}} \neq \{0\}$, then any R-supermodule M is an R-supermodule homomorphic image of a free R-supermodule.

Proof We have assumed that $M_{\bar{0}}R_{\bar{1}} = M_{\bar{1}}$. Let $I = \{m_{\bar{0}} \mid m_{\bar{0}} \in M_{\bar{0}}\}$. We use I as an indexing set. Consider the free R-supermodule R^I and let $\varphi : R^I \to M$ be a map defined by $\varphi(b_{m_{\bar{0}}}) = m_{\bar{0}}$. Then it is easy to see that φ is an onto R-supermodule homomorphism, since $M_{\bar{0}}R_{\bar{1}} = M_{\bar{1}}$. □

Lemma 4.1.6 Let R be a super-ring with $R_{\bar{1}} \neq \{0\}$, and let R^I be a free R-supermodule. Then for any diagram of R-supermodules and R-supermodule homomorphisms

$$\begin{array}{c} R^I \\ \downarrow \mu \\ N \xrightarrow{\alpha} N' \to 0, \end{array}$$

with the bottom row exact, there exists an R-supermodule homomorphism $\beta : R^I \to N$ with $\alpha\beta = \mu$.

Proof For each $i \in I$ choose an element $u \in N$ such that $\alpha(u) = \mu(b_i)$ and let β be the R-supermodule homomorphism of R^I into N such that $\beta(b_i) = u$. Then $\alpha(\beta(b_i)) = \mu(b_i)$ for all $i \in I$ so $\alpha\beta = \mu$. □

The next proposition follows from Lemma 4.1.5, Lemma 4.1.6 and [10, proposition 3.10].

Proposition 4.1.7 If R is a super-ring with $R_{\bar{1}} \neq \{0\}$, then the following properties of an R-supermodule M are equivalent:
(a) M is isomorphic, as an R-supermodule, to a direct summand of some free R-supermodule.
(b) Every short exact sequence of R-supermodules

$$0 \to L \to N \xrightarrow{\varphi} M \to 0$$

splits; that is, there exists an R-supermodule homomorphism $\rho : M \to N$ such that $\varphi\rho = id_M$.
(c) For any diagram of R-supermodules and R-supermodule homomorphisms

$$\begin{array}{c} M \\ \downarrow \mu \\ N \xrightarrow{\alpha} N' \to 0, \end{array}$$

with the bottom row exact, there exists an R-supermodule homomorphism $\beta : M \to N$ with $\alpha\beta = \mu$.

□

We isolate another equivalent condition for projectivity in the following lemma, which we call the Dual Basis Lemma.

Lemma 4.1.8 Let R be any super-ring, and let M be an R-supermodule. If $R_{\bar{1}} = \{0\}$ then M is projective as an R-supermodule if and only if there exist index sets I and J along with $\{m_{i,\bar{0}}\}_{i\in I} \subseteq M_{\bar{0}}$, $\{f_{i,\bar{0}}\}_{i\in I} \subseteq (Hom_R(M,R))_{\bar{0}}$, $\{m_{j,\bar{1}}\}_{j\in J} \subseteq M_{\bar{1}}$ and $\{f_{j,\bar{1}}\}_{j\in J} \subseteq (Hom_R(M,R))_{\bar{1}}$ such that
(a) for every $m_{\bar{0}} \in M_{\bar{0}}$ and for every $m_{\bar{1}} \in M_{\bar{1}}$, $f_{i,\bar{0}}(m_{\bar{0}}) = 0$ and $f_{j,\bar{1}}(m_{\bar{1}}) = 0$ for all but a finite subset of $i \in I$ and $j \in J$ respectively, and
(b) for every $m_{\bar{0}} \in M_{\bar{0}}$ and for every $m_{\bar{1}} \in M_{\bar{1}}$,

$$m_{\bar{0}} = \sum_{i\in I} m_{i,\bar{0}} f_{i,\bar{0}}(m_{\bar{0}}) \quad \text{and} \quad m_{\bar{1}} = \sum_{j\in J} m_{j,\bar{1}} f_{j,\bar{1}}(m_{\bar{1}}).$$

Moreover, I, J are a finite sets if and only if $M_{\bar{0}}$ and $M_{\bar{1}}$ are finitely generated.
If $R_{\bar{1}} \neq \{0\}$ then M is projective as an R-supermodule if and only if there exist an index set I, $\{m_{i,\gamma}\}_{i\in I} \subseteq M_\gamma$ and $\{f_{i,\gamma}\}_{i\in I} \subseteq (Hom_R(M,R))_\gamma$ such that
(a) for every $m_\alpha \in M_\alpha$, $f_{i,\gamma}(m_\alpha) = 0$ for all but a finite subset of $i \in I$ and,
(b) for every $m_\alpha \in M_\alpha$ where $\alpha = \bar{0}$ or $\bar{1}$,

$$m_\alpha = \sum_{i\in I} m_{i,\gamma} f_{i,\gamma}(m_\alpha).$$

Moreover, I is a finite set if and only if M is finitely generated.

Proof Case $R_{\bar{1}} = \{0\}$, use [6, Lemma 1.1.3].
Case $R_{\bar{1}} \neq \{0\}$: Suppose $R^I = N \oplus L$, where N, L are R-subsupermodules of R^I

and I is an indexing set. Suppose further $M \overset{\phi_\gamma}{\cong} N$, where ϕ_γ is an R-supermodule isomorphism. Now, define $\pi_\gamma : R^I \to M$ by $\pi_\gamma(n+l) = \phi_\gamma^{-1}(n)$, for $n \in N$ and $l \in L$. Then it is easy to check that π_γ is an R-supermodule homomorphism and $\pi_\gamma\phi_\gamma = id_M$. Let $\delta_i : R^I \to R$ be the map defined by $\delta_i(f) = f(i)$, for every $f \in R^I$. Then for any $f \in R^I$, $f = \sum_{i\in I} b_i\delta_i(f)$. Thus $\{\delta_i, b_i\}$ forms a dual basis for R^I. Set $m_{i,\gamma} = \pi_\gamma(b_i)$ and $f_{i,\gamma} = \delta_i\phi_\gamma$. Then for any $m_\alpha \in M_\alpha$, $f_{i,\gamma}(m_\alpha) = 0$ for almost all $i \in I$ and

$$\begin{aligned}
\sum_{i\in I} m_{i,\gamma} f_{i,\gamma}(m_\alpha) &= \sum_{i\in I} \pi_\gamma(b_i).\delta_i\phi_\gamma(m_\alpha) \\
&= \pi_\gamma(\sum_{i\in I} b_i.\delta_i\phi_\gamma(m_\alpha)) \\
&= \pi_\gamma\phi_\gamma(m_\alpha) \\
&= m_\alpha.
\end{aligned}$$

Thus $\{f_{i,\gamma}, m_{i,\gamma}\}_{i\in I}$ forms a dual basis for M over R.
Conversely, suppose $\{f_{i,\gamma}, m_{i,\gamma}\}_{i\in I}$ forms a dual basis for M over R. Then define the map $\phi_\gamma : M \to R^I$ by $\phi_\gamma(m_\alpha)(i) = f_{i,\gamma}(m_\alpha)$, and the map $\pi_\gamma : R^I \to M$ by $\pi_\gamma(f) = \sum_{i\in I} m_{i,\gamma} f(i)$. Then it is easy to check that ϕ_γ, π_γ are R-supermodule homomorphisms. Now,

$$\begin{aligned}
\pi_\gamma\phi_\gamma(m_\alpha) &= \pi_\gamma(\sum_{i\in I} b_i.f_{i,\gamma}(m_\alpha)) \\
&= \sum_{i\in I} m_{i,\gamma}.f_{i,\gamma}(m_\alpha) \\
&= m_\alpha.
\end{aligned}$$

Thus, ϕ_γ is a one-to-one R-supermodule homomorphism from M into R^I, so M is isomorphic to a direct summand of R^I and therefore M is R-projective as an R-supermodule. □

Corollary 4.1.9 Let R by any super-ring. If M is a left R-projective as an R-supermodule then M is a right R°-projective as R°-supermodule and vice versa.

Proof Let $\{f_{i,\gamma}, m_{i,\gamma}\}_{i\in I}$ be a dual basis for M. Then for each i, let $F_{i,\gamma}(m_\alpha) = (-1)^{\alpha\gamma} m_\alpha^{f_{i,\gamma}}$, then $F_{i,\gamma} \in (\mathrm{Hom}_{R^\circ}(M, R^\circ))_\gamma$ and hence one can easily check that $\{F_{i,\gamma}, (-1)^\gamma m_{i,\gamma}\}$ forms a dual basis for M as an R°-supermodule. □

Corollary 4.1.10 Let R by any super-ring, with $R_{\bar{1}} \neq \{0\}$. If M is an R-supermodule which is R-projective as an R-module, and $M_{\bar{0}}R_{\bar{1}} = M_{\bar{1}}$ (or $M_{\bar{1}}R_{\bar{1}} = M_{\bar{0}}$) then M is R-projective as an R-supermodule.

Proof By [6, Lemma 1.3], let $\{m_i, f_i\}_{i\in I}$ be a dual basis for M over R. We will show that $\{m_{i,\bar{0}}, f_{i,\bar{0}}\}_{i\in I} \cup \{m_{j,\bar{1}}, f_{j,\bar{1}}\}_{j\in J}$ yields a dual basis for M, as an R-supermodule. We may assume that $M_{\bar{0}}R_{\bar{1}} = M_{\bar{1}}$ then, without loss of generality, for each j we will assume that $m_{j,\bar{1}} = m'_{j,\bar{0}}r_{j,\bar{1}}$. Therefore for any m_α in M_α, we have

$$\begin{aligned} m_\alpha &= \sum_{i,j} m_{i,\bar{0}}f_{i,\bar{0}}(m_\alpha) + m_{j,\bar{1}}f_{j,\bar{1}}(m_\alpha) \\ &= \sum_{i,j} m_{i,\bar{0}}f_{i,\bar{0}}(m_\alpha) + m'_{j,\bar{0}}r_{j,\bar{1}}f_{j,\bar{1}}(m_\alpha) \\ &= \sum_{i,j} m_{i,\bar{0}}f_{i,\bar{0}}(m_\alpha) + m'_{j,\bar{0}}(r_{j,\bar{1}}f_{j,\bar{1}})(m_\alpha). \end{aligned}$$

One easily checks that for each j, $(r_{j,\bar{1}}f_{j,\bar{1}})$ is in $(\mathrm{Hom}_R(M,R))_{\bar{0}}$. Therefore

$$\{m_{i,\bar{0}}, f_{i,\bar{0}}\}_{i\in I} \cup \{m'_{j,\bar{0}}, (r_{j,\bar{1}}f_{j,\bar{1}})\}_{j\in J}$$

form a dual basis for M over R, and hence, by Lemma 4.1.8, M is R-projective as an R-supermodule. □

Suppose R and S are (possibly non-super-commutative) super-rings and $\theta : R \to S$ is a super-ring homomorphism. Then S becomes an R-supermodule under the operation $s.r = s.\theta(r)$. This naturally induces an R-supermodule structure on any S-supermodule.

As a typical application of the Dual Basis Lemma we prove

Proposition 4.1.11 (Transitivity of Projective Supermodules) Let R and S be super-rings and $\theta : R \to S$ a super-ring homomorphism such that S is projective as a supermodule when considered as an R-supermodule. Then, any projective S-supermodule M is R-projective, as an R-supermodule. Furthermore, if M is finitely generated over S and S is finitely generated over R, M will be finitely generated over R.

Proof. Case $R_{\bar{1}} = \{0\}$

(a) If $S_{\bar{1}} = \{0\}$ then use [6, Lemma 1.4].

(b) If $S_{\bar{1}} \neq \{0\}$ then we may assume that $M_{\bar{0}}S_{\bar{1}} = M_{\bar{1}}$ and so by the Dual Basis Lemma 4.1.8 there exist $\{m_{i,\bar{0}}\}_{i\in I} \subseteq M_{\bar{0}}$ and $\{f_{i,\bar{0}}\}_{i\in I} \subseteq (\mathrm{Hom}_S(M,S))_{\bar{0}}$ with $f_{i,\bar{0}}(m_\alpha) = 0$ for almost all $i \in I$ and $\sum_{i\in I} m_{i,\bar{0}}.f_{i,\bar{0}}(m_\alpha) = m_\alpha$ for every $m_\alpha \in M_\alpha$. Similarly there exist $\{g_{j,\bar{0}}\}_{j\in J_{\bar{0}}} \subseteq \mathrm{Hom}_R(S_{\bar{0}}, R)$ and $\{s_{j,\bar{0}}\}_{j\in J_{\bar{0}}} \subseteq S_{\bar{0}}$ with $g_{j,\bar{0}}(s_{\bar{0}}) = 0$, for almost all j in $J_{\bar{0}}$, and $\sum_{j\in J_{\bar{0}}} s_{j,\bar{0}}.g_{j,\bar{0}}(s_{\bar{0}}) = s_{\bar{0}}$, for every $s_{\bar{0}} \in S_{\bar{0}}$; and $\{g_{j,\bar{1}}\}_{j\in J_{\bar{1}}} \subseteq \mathrm{Hom}_R(S_{\bar{1}}, R)$ and $\{s_{j,\bar{1}}\}_{j\in J_{\bar{1}}} \subseteq S_{\bar{1}}$ with $g_{j,\bar{1}}(s_{\bar{1}}) = 0$, for almost all j in $J_{\bar{1}}$, and $\sum_{j\in J_{\bar{1}}} s_{j,\bar{1}}.g_{j,\bar{1}}(s_{\bar{1}}) = s_{\bar{1}}$ for every $s_{\bar{1}} \in S_{\bar{1}}$. Then, $g_{j,\bar{0}}f_{i,\bar{0}}$ is in $\mathrm{Hom}_R(M_{\bar{0}}, R)$ and $m_{i,\bar{0}}s_{j,\bar{0}}$ is in $M_{\bar{0}}$ with $g_{j,\bar{0}}f_{i,\bar{0}}(m_{\bar{0}}) = 0$, for almost all (i,j) in $I \times J_{\bar{0}}$, and

$$\begin{aligned}\sum_{i,j} m_{i,\bar{0}}s_{j,\bar{0}}g_{j,\bar{0}}f_{i,\bar{0}}(m_{\bar{0}}) &= \sum_i m_{i,\bar{0}}(\sum_j s_{j,\bar{0}}g_{j,\bar{0}}(f_{i,\bar{0}}(m_{\bar{0}}))) \\ &= \sum_i m_{i,\bar{0}}f_{i,\bar{0}}(m_{\bar{0}}) \\ &= m_{\bar{0}}.\end{aligned}$$

Moreover, $g_{j,\bar{1}}f_{i,\bar{0}}$ is in $\mathrm{Hom}_R(M_{\bar{1}}, R)$ and $m_{i,\bar{0}}s_{j,\bar{1}}$ is in $M_{\bar{1}}$ with $g_{j,\bar{0}}f_{i,\bar{0}}(m_{\bar{1}}) = 0$, for almost all (i,j) in $I \times J_{\bar{1}}$, and

$$\begin{aligned}\sum_{i,j} m_{i,\bar{0}}s_{j,\bar{1}}g_{j,\bar{1}}f_{i,\bar{0}}(m_{\bar{1}}) &= \sum_i m_{i,\bar{0}}(\sum_j s_{j,\bar{1}}g_{j,\bar{1}}(f_{i,\bar{0}}(m_{\bar{1}}))) \\ &= \sum_i m_{i,\bar{0}}f_{i,\bar{0}}(m_{\bar{1}}) \\ &= m_{\bar{1}}.\end{aligned}$$

Thus M is R-projective as an R-supermodule.

Case $R_{\bar{1}} \neq \{0\}$ then $S_{\bar{1}} \neq \{0\}$. We may again assume that $M_{\bar{0}}S_{\bar{1}} = M_{\bar{1}}$ and $S_{\bar{0}}R_{\bar{1}} = S_{\bar{1}}$ and so by the Dual Basis Lemma (4.1.8), there exist $\{m_{i,\bar{0}}\}_{i\in I} \subseteq M_{\bar{0}}$ and $\{f_{i,\bar{0}}\}_{i\in I} \subseteq (\mathrm{Hom}_S(M,S))_{\bar{0}}$ with $f_{i,\bar{0}}(m_\alpha) = 0$, for almost all $i \in I$, and $\sum_{i\in I} m_{i,\bar{0}}.f_{i,\bar{0}}(m_\alpha) = m_\alpha$ for every $m_\alpha \in M_\alpha$. Similarly there exist $\{s_{j,\bar{0}}\}_{j\in J} \subseteq S_{\bar{0}}$ and $\{g_{j,\bar{0}}\}_{j\in J} \subseteq (\mathrm{Hom}_R(S,R))_{\bar{0}}$ with $g_{j,\bar{0}}(s_\alpha) = 0$, for almost all $j \in J$, and $\sum_{j\in J} s_{j,\bar{0}}.g_{j,\bar{0}}(s_\alpha) = s_\alpha$ for every $s_\alpha \in S_\alpha$. Then $g_{j,\bar{0}}f_{i,\bar{0}}$ is in $(\mathrm{Hom}_R(M_{\bar{0}}, R))_{\bar{0}}$ and

$m_{i,\bar{0}}s_{j,\bar{0}}$ is in $M_{\bar{0}}$, with $g_{j,\bar{0}}f_{i,\bar{0}}(m_\alpha) = 0$, for almost all (i,j) in $I \times J$, and

$$\begin{aligned}
\sum_{i,j} m_{i,\bar{0}}s_{j,\bar{0}}g_{j,\bar{0}}f_{i,\bar{0}}(m_\alpha) &= \sum_i m_{i,\bar{0}}(\sum_j s_{j,\bar{0}}g_{j,\bar{0}}(f_{i,\bar{0}}(m_\alpha))) \\
&= \sum_i m_{i,\bar{0}}f_{i,\bar{0}}(m_\alpha) \\
&= m_\alpha.
\end{aligned}$$

Thus M is also R-projective as an R-supermodule. The conclusion concerning finite generation is obvious. □

For any R-supermodule M, consider the subset $\mathcal{T}_R(M)$ of R consisting of the elements of the form $\sum_{i\in I} f_i(m_i)$ where $f_i \in \mathrm{Hom}_R(M,R)$ and the $m_i \in M$. Since $\mathrm{Hom}_R(M,R)$ is a left R-supermodule under the operation $(r.f)(m) = rf(m)$, we see that $r(\sum_i f_i(m_i)) = \sum_i rf_i(m_i) = \sum_i (r.f)_i(m_i) \in \mathcal{T}_R(M)$ and $(\sum_i f_i(m_i))r = \sum_i f_i(m_i)r = \sum_i f_i(m_ir)$ is in $\mathcal{T}_R(M)$. It follows that $\mathcal{T}_R(M)$ is a two-sided super-ideal of R, called the *trace super-ideal* of M. We call an R-supermodule M an *R-generator* if $\mathcal{T}_R(M) = R$. Therefore, M is an R-generator if and only if there exist $f_{1,\bar{0}}, ..., f_{n,\bar{0}}$ in $(\mathrm{Hom}_R(M,R))_{\bar{0}}$ and $m_{1,\bar{0}}, ..., m_{n,\bar{0}}$ in $M_{\bar{0}}$; and $f_{1,\bar{1}}, ..., f_{k,\bar{1}}$ in $(\mathrm{Hom}_R(M,R))_{\bar{1}}$ and $m_{1,\bar{1}}, ..., m_{k,\bar{1}}$ in $M_{\bar{1}}$ with

$$\sum_{i=1}^{n} f_{i,\bar{0}}(m_{i,\bar{0}}) + \sum_{j=1}^{k} f_{j,\bar{1}}(m_{j,\bar{1}}) = 1.$$

Corollary 4.1.12 Let R and S be super-rings and $\theta : R \to S$ a super-ring homomorphism such that S is a generator, when considered as an R-supermodule. Then any S-generator supermodule M is an R-generator. □

We call an R-supermodule M an *R-progenerator* if M is finitely generated projective and generator over R. We observe that as a result of the transitivity, finite generation, and being a generator, we have

Proposition 4.1.13 (Transitivity of Progenerators) Let R and S be super-rings and $\theta : R \to S$ a super-ring homomorphism such that S is an R-progenerator when considered as an R-supermodule. Then any S-progenerator supermodule M is an R-progenerator. $\square$

It seems an opportune moment to prove Nakayama's Lemma for the super-case. For any super-ideal I of a super-ring R and any R-supermodule M, MI means the set of all finite sums of the form $\sum m_i\alpha_i$ where α_i comes from I and m_i comes from M. Define the *annihilator of* M *in* R by $annih_R(M) = (annih_R(M))_{\bar{0}} + (annih_R(M))_{\bar{1}}$ where $(annih_R(M))_\gamma = \{r_\gamma \in R_\gamma \mid m_\alpha r_\gamma = 0,\ \forall m_\alpha \in M_\alpha\}$

Lemma 4.1.14 (Generalized Nakayama Lemma) Let R be a supercommutative super-ring and M a finitely generated supermodule over R. A super-ideal I of R has the property that $MI = M$ if and only if $I + annih_R(M) = R$.

Proof If $I + annih_R(M) = R$, we can write $1 = \alpha + \beta$ for some α in $I_{\bar{0}}$, β in $(annih_R(M))_{\bar{0}}$, so that for every m in M, $m = m1 = m\alpha + m\beta$. Thus, $MI = M$. Conversely, let $m_{1,\bar{0}}, ..., m_{k,\bar{0}}, m_{k+1,\bar{1}}, ..., m_{n,\bar{1}}$ be elements of M such that $M = m_{1,\bar{0}}R + ... + m_{k,\bar{0}}R + m_{k+1,\bar{1}}R + ... + m_{n,\bar{1}}R$. We proceed by induction on n. Set $M_i = m_iR + m_{i+1}R + ... + m_nR$, $i = 1, 2, ..., n$, where $m_j = \begin{cases} m_{j,\bar{0}} & : \quad 1 \le j \le k \\ m_{j,\bar{1}} & : \quad k+1 \le j \le n \end{cases}$, and $M_{n+1} = 0$. We will show that for every $i = 1, 2, ..., n+1$, there exists α_i in $I_{\bar{0}}$ with $M(1-\alpha_i) \subset M_i$. Clearly we may choose $\alpha_1 = 0$. Suppose we have α_i in $I_{\bar{0}}$ with $M(1-\alpha_i) \subset M_i$. Then $M(1-\alpha_i) = MI(1-\alpha_i) = M(1-\alpha_i)I \subset M_iI$. Then,

$$m_i(1-\alpha_i) = \sum_{j=i}^{n} m_j\alpha_{ij}, \quad \alpha_{ij} \in I.$$

Since m_i are homogeneous elements, we may assume that α_{ii} in $I_{\bar{0}}$. Hence, $m_i(1 - \alpha_i - \alpha_{ii}) \in M_{i+1}$. Then $M(1-\alpha_i-\alpha_{ii})(1-\alpha_i) = M(1-\alpha_i)(1-\alpha_i-\alpha_{ii}) \subset M_i(1-\alpha_i-\alpha_{ii}) \subset M_{i+1}$, showing $M[1-(2\alpha_i+\alpha_{ii}-\alpha_i^2-\alpha_i\alpha_{ii})] \subset M_{i+1}$. We complete the induction step by setting $\alpha_{i+1} = 2\alpha_i + \alpha_{ii} - \alpha_i^2 - \alpha_i\alpha_{ii}$. Then there exists α_{n+1} in $I_{\bar{0}}$ such that $1 - \alpha_{n+1}$ is in $(annih_R(M))_{\bar{0}}$, finishing the proof. $\square$

Corollary 4.1.15 Let M be a finitely generated supermodule over a super-ring R, where R is super-commutative. If $MI = M$ for every maximal super-ideal I of R, then $M = 0$.

Proof $M = 0$ if and only if $annih_R(M) = R$, but if $annih_R(M) \neq R$, there is some maximal super-ideal I with $annih_R(M) \subset I$, whence $annih_R(M) + I \neq R$, a contradiction. □

It is often very difficult to tell whether a finitely generated, projective supermodule is a generator. However, in the case of a super-commutative super-ring, there is an easy criterion.

Proposition 4.1.16 Let R be a super-commutative super-ring and M a finitely generated, projective R-supermodule. Then $\mathcal{T}_R(M) \oplus annih_R(M) = R$.

Proof If $R_{\bar{1}} = 0$ then $M_{\bar{0}}$ and $M_{\bar{1}}$ are finitely generated, projective R-modules. We will show that M is finitely generated, projective as an R-module, and therefore, by [6, Proposition 1.1.9], $\mathcal{T}_R(M) \oplus annih_R(M) = R$. Since $M_{\bar{0}}$ and $M_{\bar{1}}$ are projective R-modules we have $R^{I_0} \cong M_{\bar{0}} \oplus L$ for some indexing set I_0; similarly $R^{I_1} \cong M_{\bar{1}} \oplus N$ for some indexing set I_1. Now,

$$\begin{aligned} R^{I_0 \cup I_1} &\cong R^{I_0} \times R^{I_1} \\ &\cong (M_{\bar{0}} \oplus L) \times (M_{\bar{1}} \oplus N) \\ &\cong (M_{\bar{0}} \times M_{\bar{1}}) \oplus [(M_{\bar{0}} \times N) \oplus (L \times M_{\bar{1}}) \oplus (L \times N)] \end{aligned}$$

and therefore $M = M_{\bar{0}} \times M_{\bar{1}}$ is projective as an R-module. Moreover, M is finitely generated since $M_{\bar{0}}$ and $M_{\bar{1}}$ are finitely generated.
If $R_{\bar{1}} \neq \{0\}$ then there exists a dual basis

$$\{f_{1,\gamma}, ..., f_{n,\gamma}\} \in (\mathrm{Hom}_R(M,R))_\gamma \quad \text{and} \quad \{m_{1,\gamma}, ..., m_{n,\gamma}\} \in M_\gamma$$

such that $m = \sum_{i=1}^n m_{i,\gamma} f_{i,\gamma}(m)$ for every $m \in M$. However, $f_{i,\gamma}(m)$ is in $\mathcal{T}_R(M)$, so $M\mathcal{T}_R(M) = M$. Thus by Lemma 4.1.14, $\mathcal{T}_R(M) + annih_R(M) = R$. We will show that $annih_R(M).\mathcal{T}_R(M) = \{0\}$. Let $r_\delta \in (annih_R(M))_\delta$, $f_\gamma \in (\mathrm{Hom}_R(M,R))_\gamma$, for any $m_\beta \in M_\beta$ we have

$$(r_\delta f_\gamma)(m_\beta) = r_\delta(f_\gamma(m_\beta)) = (-1)^{\delta(\gamma+\beta)} f_\gamma(m_\beta) r_\delta = (-1)^{\delta(\gamma+\beta)} f_\gamma(m_\beta r_\delta) = 0.$$

Thus $annih_R(M).\mathcal{T}_R(M) = \{0\}$. It follows that for every z in $\mathcal{T}_R(M) \cap annih_R(M)$,

$$z = z.1 = z\alpha + z\xi = 0 + 0 = 0,$$

where $1 = \alpha + \xi$ with $\alpha \in (\mathcal{T}_R(M))_{\bar{0}}$ and $\xi \in (annih_R(M))_{\bar{0}}$. Thus $annih_R(M) \cap \mathcal{T}_R(M) = \{0\}$ and hence $\mathcal{T}_R(M) \oplus annih_R(M) = R$. □

Corollary 4.1.17 Let R be a super-commutative super-ring, and an R-supermodule M be a finitely generated, projective as a module over R. Then $\mathcal{T}_R(M) \oplus annih_R(M) = R$. □

Corollary 4.1.18 If R is super-commutative super-ring, an R-supermodule M is an R-progenerator if and only if M is finitely generated, projective and faithful. □

Corollary 4.1.19 If R is super-commutative with no idempotents but 0 and 1, every finitely generated, projective non-zero R-supermodule is faithful and therefore a progenerator. □

4.2 Categories and Functors of Supermodules

In this section we will consider the categories whose objects are supermodules. We therefore restrict our attention in to the collection of all supermodules over some super-ring. For a super-ring R, we will let $R - \mathcal{SM}$ denote the category of left R-supermodules, that is the collection of all left R-supermodules along with, for any R-supermodules M and N, the set $\mathrm{Hom}_R(M, N)$ of all R-supermodule homomorphisms from M to N. Similarly, $\mathcal{SM} - R$ will denote the category of right R-supermodules and R-homomorphisms.

For any super-ring R, we let R° denote the super-ring whose underlying abelian group is the same as that of R but whose multiplication $\circ$ is given by $a_\alpha \circ b_\beta = (-1)^{\alpha\beta} b_\beta a_\alpha$ where $b_\beta a_\alpha$ is the product in R, R° is called the *opposite super-ring* of R. It is clear that a left R-supermodule M can be given the structure of a right R°-supermodule by defining $m_\alpha r_\beta = (-1)^{\alpha\beta} r_\beta m_\alpha$ where $r_\beta \in R_\beta$, $m_\alpha \in M_\alpha$. Thus, in particular, if R is super-commutative, $R = R^{\circ}$ and any one sided R-supermodule can be considered as both a left and right R-supermodule. Consequently, when R is super-commutative, $R - \mathcal{SM} = \mathcal{SM} - R$.

By a *(covariant) functor* from a category of supermodules $\mathfrak{C}$ to a category of supermodules $\mathfrak{D}$, we mean a correspondence $\mathfrak{I}$ which associates to every supermodule M in $\mathfrak{C}$ a supermodule $\mathfrak{I}(M)$ in $\mathfrak{D}$ and which associates to every homomorphism $f_{\delta} : M \to N$ in $\mathfrak{C}$ a homomorphism $\mathfrak{I}(f_{\delta}) : \mathfrak{I}(M) \to \mathfrak{I}(N)$ such that for every M in $\mathfrak{C}$
(a) $\mathfrak{I}(id_M) = id_{\mathfrak{I}(M)}$ and
(b) if f_{α}, g_{β} are homomorphisms in $\mathfrak{C}$ such that $f_{\alpha} g_{\beta}$ is defined then $\mathfrak{I}(f_{\alpha} g_{\beta}) = \mathfrak{I}(f_{\alpha})\mathfrak{I}(g_{\beta})$.

A functor $\mathfrak{I}$ from a category $\mathfrak{C}$ to a category $\mathfrak{D}$ is called *left exact* if for every exact sequence of supermodules and maps

$$0 \to L \to M \to N \to 0$$

in $\mathfrak{C}$,

$$0 \to \mathfrak{I}(L) \to \mathfrak{I}(M) \to \mathfrak{I}(N)$$

is exact in $\mathfrak{D}$. Similarly $\mathfrak{I}$ is called *right exact* if for every sequence of supermodules and maps

$$0 \to L \to M \to N \to 0$$

in $\mathfrak{C}$,

$$\mathfrak{I}(L) \to \mathfrak{I}(M) \to \mathfrak{I}(N) \to 0$$

is exact in $\mathfrak{D}$. A functor that is both left and right exact is called exact.

The two functors we will be must concerned with are the tensor product ($\hat{\otimes}$) and the set of all supermodule homomorphisms from some fixed supermodule to another. Now, we have to recall the following :
(1) For any super-ring R and any right R-supermodule M, left R-supermodule N, $M \hat{\otimes}_R N \cong N \hat{\otimes}_{R^{\circ}} M$ under the map $m_{\alpha} \otimes n_{\beta} \mapsto (-1)^{\alpha\beta} n_{\beta} \otimes m_{\alpha}$. Therefore if R is

super-commutative, $M\hat{\otimes}_R N \cong N\hat{\otimes}_R M$ as R-supermodules.

(2) For any super-ring R and any right R-supermodules M, left R-supermodules N, $M\hat{\otimes}_R N$ is a right R-supermodule under the operation $(m_\alpha \otimes n_\beta)r_\delta = (-1)^{\delta\beta} m_\alpha r_\delta \otimes n_\beta$, since $(m_\alpha \otimes n_\beta)(r_\delta s_\gamma) = (-1)^{(\delta+\gamma)\beta} m_\alpha (r_\delta s_\gamma) \otimes n_\beta = (-1)^{\delta\beta}(m_\alpha r_\delta \otimes n_\beta)s_\gamma = ((m_\alpha \otimes n_\beta)r_\delta)s_\gamma$.

(3) For any right R-supermodules M, M', left R-supermodules N, N', and for any $f_\alpha \in \mathrm{Hom}_R(M, M')$, $g_\beta \in \mathrm{Hom}_R(N, N')$, there is a well-defined R-supermodule homomorphism $f_\alpha \otimes g_\beta$ from $M\hat{\otimes}_R N$ to $M'\hat{\otimes}_R N'$, given by $f_\alpha \otimes g_\beta(m_\epsilon \otimes n_\mu) = (-1)^{\beta(\epsilon+\mu)} f_\alpha(m_\epsilon) \otimes (n_\mu)^{g_\beta}$, since

$$\begin{aligned}
f_\alpha \otimes g_\beta((m_\epsilon \otimes n_\mu)r_\delta) &= (-1)^{\delta\mu} f_\alpha \otimes g_\beta(m_\epsilon r_\delta \otimes n_\mu) \\
&= (-1)^{\delta\mu+\beta(\epsilon+\delta+\mu)} f_\alpha(m_\epsilon r_\delta) \otimes (n_\mu)^{g_\beta} \\
&= (-1)^{\delta\mu+\beta(\epsilon+\delta+\mu)} f_\alpha(m_\epsilon) r_\delta \otimes (n_\mu)^{g_\beta} \\
&= (-1)^{\beta(\epsilon+\mu)} (f_\alpha(m_\epsilon) \otimes (n_\mu)^{g_\beta}) r_\delta \\
&= (f_\alpha \otimes g_\beta(m_\epsilon \otimes n_\mu)) r_\delta.
\end{aligned}$$

(4) For any exact sequence $0 \to N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \to 0$ of left R-supermodules

$$M\hat{\otimes}_R N_1 \xrightarrow{id_M \otimes \alpha} M\hat{\otimes}_R N_2 \xrightarrow{id_M \otimes \beta} N_3 \to 0$$

is exact for any right R-supermodule M. Thus $M\hat{\otimes}_R(\ )$ is a right exact functor. Similarly, $(\ )\hat{\otimes}_R N$ is a right exact functor. If tensoring by a supermodule is exact, that is, it is left exact as well as right exact, we say the supermodule is *flat*.

(5) It follows easily from the characterization of projective supermodules as direct-summands of free supermodules that every projective R-supermodule is flat.

In addition to these properties, we need a few more results on the behavior of supermodules under $\hat{\otimes}$.

Proposition 4.2.1 Let R be a supercommutative super-ring and let A and B be R-superalgebras. If M is a right A-supermodule, then $M\hat{\otimes}_R B$ is projective (resp., a generator) over $A\hat{\otimes}_R B$ if M is projective (resp., a generator) over A.

Proof Case (1) $R_{\bar{1}} \neq \{0\}$. Then $A_{\bar{1}},\ B_{\bar{1}} \neq \{0\}$. Assuming M to be A-projective, there exists a dual basis $\{f_{i,\delta} \in (\mathrm{Hom}_A(M,A))_\delta,\ m_{i,\delta} \in M_\delta\}$. One easily verifies that $\{f_{i,\delta} \otimes id_B,\ m_{i,\delta} \otimes 1\}$ forms a dual basis for $M\hat{\otimes}_R B$ over $A\hat{\otimes}_R B$. Similarly if M is an A-generator, there exist $f_i \in \mathrm{Hom}_A(M,A)$ and $m_i \in M$ with $\sum f_i(m_i) = 1$. Then $\sum (f_i \otimes id_B)(m_i \otimes 1) = 1 \otimes 1$, so $M\hat{\otimes}_R B$ is an $A\hat{\otimes}_R B$-generator.

Case (2) $R_{\bar{1}} = \{0\}$.

(a) If $A_{\bar{1}} \neq \{0\},\ B_{\bar{1}} \neq \{0\}$ then by the same argument used in case (1) we can easily prove the proposition.

(b) If $A_{\bar{1}} \neq \{0\},\ B_{\bar{1}} = \{0\}$, then there exists a dual basis $\{f_{i,\bar{0}} \in \mathrm{Hom}_A(M,A),\ m_{i,\bar{0}} \in M_{\bar{0}}\}$ for M over A. One easily verifies that $\{f_{i,\bar{0}} \otimes id_B,\ m_{i,\bar{0}} \otimes 1\}$ forms a dual basis for $M\hat{\otimes}_R B$ over $A\hat{\otimes}_R B$. Similarly if M is an A-generator, there exist $f_i \in \mathrm{Hom}_A(M,A)$ and $m_i \in M$ with $\sum f_i(m_i) = 1$. Then $\sum (f_i \otimes id_B)(m_i \otimes 1) = 1 \otimes 1$, so $M\hat{\otimes}_R B$ is an $A\hat{\otimes}_R B$-generator.

(c) If $A_{\bar{1}} = \{0\},\ B_{\bar{1}} \neq \{0\}$ then $M_{\bar{0}},\ M_{\bar{1}}$ are A-projective as modules over A. Let $\{f_{i,\bar{0}} \in \mathrm{Hom}_A(M,A),\ m_{i,\bar{0}} \in M_{\bar{0}}\}_{i\in I}$, $\{g_{j,\bar{1}} \in \mathrm{Hom}_A(M,A),\ m_{j,\bar{1}} \in M_{\bar{1}}\}_{j\in J}$ be dual bases for $M_{\bar{0}},\ M_{\bar{1}}$ over A (resp.). We are looking for a dual basis for $M\hat{\otimes}_R B$ over $A\hat{\otimes}_R B$. We may assume that $(M\hat{\otimes}_R B)_{\bar{0}}.(A\hat{\otimes}_R B)_{\bar{1}} = (M\hat{\otimes}_R B)_{\bar{1}}$. Now, $\{f_{i,\bar{0}} \otimes id_B,\ m_{i,\bar{0}} \otimes 1\}_{i\in I}$ forms a dual bases for $M_{\bar{0}}\hat{\otimes}_R B$ over $A\hat{\otimes}_R B$, since for any $m_{\bar{0}} \otimes b_\alpha\ \in M_{\bar{0}}\hat{\otimes}_R B$ we have

$$\begin{aligned} m_{\bar{0}} \otimes b_\alpha &= (\sum_{i\in I} m_{i,\bar{0}} f_{i,\bar{0}}(m_{\bar{0}})) \otimes b_\alpha \\ &= \sum_{i\in I} (m_{i,\bar{0}} f_{i,\bar{0}}(m_{\bar{0}}) \otimes b_\alpha) \\ &= \sum_{i\in I} (m_{i,\bar{0}} \otimes 1)(f_{i,\bar{0}}(m_{\bar{0}}) \otimes b_\alpha) \\ &= \sum_{i\in I} (m_{i,\bar{0}} \otimes 1)(f_{i,\bar{0}} \otimes id_B)(m_{\bar{0}} \otimes b_\alpha). \end{aligned}$$

We now find a dual basis for $M_{\bar{1}}\hat{\otimes}_R B$ over $A\hat{\otimes}_R B$. For any j in J we have $m_{j,\bar{1}} \otimes 1$ in $(M\hat{\otimes}_R B)_{\bar{1}}$, therefore,

$$\sum_{k=1}^{n_j} (m_{j,k,\bar{1}} \otimes b_{k,\bar{1}}).(1 \otimes b'_{k,\bar{1}}) = m_{j,\bar{1}} \otimes 1,$$

since $(M\hat{\otimes}_R B)_{\bar{0}}.(A\hat{\otimes}_R B)_{\bar{1}} = (M\hat{\otimes}_R B)_{\bar{1}}$. Now, for any $m_{\bar{1}} \otimes b_\alpha \in M_{\bar{1}}\hat{\otimes}_R B$ we have

$$\begin{aligned}
m_{\bar{1}} \otimes b_\alpha &= (\sum_{j\in J} m_{j,\bar{1}} g_{j,\bar{1}}(m_{\bar{1}})) \otimes b_\alpha \\
&= \sum_{j\in J} (m_{j,\bar{1}} g_{j,\bar{1}}(m_{\bar{1}}) \otimes b_\alpha) \\
&= \sum_{j\in J} (m_{j,\bar{1}} \otimes 1)(g_{j,\bar{1}}(m_{\bar{1}}) \otimes b_\alpha) \\
&= \sum_{j\in J} [\sum_{k=1}^{n_j} (m_{j,k,\bar{1}} \otimes b_{k,\bar{1}}).(1 \otimes b'_{k,\bar{1}})](g_{j,\bar{1}}(m_{\bar{1}}) \otimes b_\alpha) \\
&= \sum_{j,k} (m_{j,k,\bar{1}} \otimes b_{k,\bar{1}})(g_{j,\bar{1}}(m_{\bar{1}}) \otimes b'_{k,\bar{1}} b_\alpha) \\
&= \sum_{j,k} (m_{j,k,\bar{1}} \otimes b_{k,\bar{1}})(-g_{j,\bar{1}} \otimes b'_{k,\bar{1}} id_B)(m_{\bar{1}} \otimes b_\alpha).
\end{aligned}$$

Therefore, $\{m_{j,k,\bar{1}} \otimes b_{k,\bar{1}},\ -g_{j,\bar{1}} \otimes b'_{k,\bar{1}}.id_B\}_{j\in J,\ k=1,..,n_j}$ forms a dual basis for $M_{\bar{1}}\hat{\otimes}_R B$ over $A\hat{\otimes}_R B$, and hence, $\{m_{i,\bar{0}}\otimes 1,\ m_{j,k,\bar{1}}\otimes b_{k,\bar{1}},\ f_{i,\bar{0}}\otimes id_B,\ -g_{j,\bar{1}}\otimes b'_{k,\bar{1}}.id_B\}_{i\in I\ j\in J,\ k=1,..,n_j}$ forms a dual basis for $M\hat{\otimes}_R B$ over $A\hat{\otimes}_R B$. Similarly, if M is an A-generator, there exist $f_{i,\bar{0}} \in \mathrm{Hom}_A(M, A)$ and $m_{i,\bar{0}} \in M_{\bar{0}}$ and $g_{j,\bar{1}} \in \mathrm{Hom}_A(M, A)$ and $m_{j,\bar{1}} \in M_{\bar{1}}$ with $\sum f_{i,\bar{0}}(m_{i,\bar{0}}) + \sum g_{j,\bar{1}}(m_{j,\bar{1}}) = 1$. Then

$$\sum_i (f_{i,\bar{0}} \otimes id_B)(m_{i,\bar{0}} \otimes 1) + \sum_j \sum_{k=1}^{n_j} (-g_{j,\bar{1}} \otimes b_{k,\bar{1}}.id_B)(m_{j,\bar{1}} \otimes b'_{k,\bar{1}}) = 1 \otimes 1$$

where $\sum_{k=1}^{n_j} (m_{j,k,\bar{1}} \otimes b_{k,\bar{1}}).(1 \otimes b'_{k,\bar{1}}) = m_{j,\bar{1}} \otimes 1$. So $M\hat{\otimes}_R B$ is an $A\hat{\otimes}_R B$-generator.□

Corollary 4.2.2 If M is projective (resp., a generator) over a super-commutative super-ring R, then for any R-superalgebra B, $M\hat{\otimes}_R B$ is projective (resp., a generator) over B.

Proof In the proposition above let $A = R$. □

Proposition 4.2.3 Let R be a super-ring, M a right R-supermodule, N a left R-supermodule. Then,

(a) $M\hat{\otimes}_R N$ is finitely generated over R if both M and N are.

(b) $M\hat{\otimes}_R N$ is R-projective if both M and N are.

(c) $M\hat{\otimes}_R B$ is an R-generator if both M and N are.

(d) $M\hat{\otimes}_R B$ is an R-progenerator if both M and N are.

Proof (a) If M and N are finitely generated over R then it is easy to show that $M\hat{\otimes}_R N$ is finitely generated over R.

(b) Case($R_{\bar{1}} \neq \{0\}$).

Let $\{m_{i,\gamma}, f_{i,\gamma}\}_{i\in I}$ be a dual basis for M over R, and $\{n_{j,\delta}, g_{j,\delta}\}_{j\in J}$ be a dual basis for N over R. Then for any m_α in M_α and n_β in N_β we have $m_\alpha = \sum_i m_{i,\gamma} f_{i,\gamma}(m_\alpha)$, and $n_\beta = \sum_j (n_\beta)^{g_{j,\delta}} n_{j,\delta}$ and so

$$\begin{aligned}
m_\alpha \otimes n_\beta &= \sum_{i,j} m_{i,\gamma} f_{i,\gamma}(m_\alpha) \otimes (n_\beta)^{g_{j,\delta}} n_{j,\delta} \\
&= (-1)^{\delta(\alpha+\gamma+\beta+\delta)} \sum_{i,j} (m_{i,\gamma} \otimes n_{j,\delta})(f_{i,\gamma}(m_\alpha).(n_\beta)^{g_{j,\delta}}) \\
&= (-1)^{\delta(\gamma+\delta)} \sum_{i,j} (m_{i,\gamma} \otimes n_{j,\delta})(f_{i,\gamma} \otimes g_{j,\delta})(m_\alpha \otimes n_\beta) \\
&= \sum_{i,j} (-1)^{\delta(\gamma+\delta)} (m_{i,\gamma} \otimes n_{j,\delta})(f_{i,\gamma} \otimes g_{j,\delta})(m_\alpha \otimes n_\beta)
\end{aligned}$$

Therefore, $\{f_{i,\gamma} \otimes g_{j,\delta},\ (-1)^{\delta(\gamma+\delta)}(m_{i,\gamma} \otimes n_{j,\delta})\}$ form a dual basis for $M\hat{\otimes}_R N$ over R.

Case($R_{\bar{1}} = \{0\}$).

Then $M_{\bar{0}}$, $M_{\bar{1}}$, $N_{\bar{0}}$, $N_{\bar{1}}$ are projective modules over R. Let $\{f_{i,\bar{0}}, m_{i,\bar{0}}\}_{i\in I}$, $\{f_{j,\bar{1}}, m_{j,\bar{1}}\}_{j\in J}$ be dual bases for $M_{\bar{0}}$, $M_{\bar{1}}$ over R (resp.), and $\{g_{k,\bar{0}}, n_{k,\bar{0}}\}_{k\in K}$, $\{g_{l,\bar{1}}, n_{l,\bar{1}}\}_{l\in L}$ be dual bases for $N_{\bar{0}}$, $N_{\bar{1}}$ over R (resp.). Then for any $m_{\bar{0}}$, $m_{\bar{1}}$ in M and $n_{\bar{0}}$, $n_{\bar{1}}$ in N we have

$$\begin{aligned}
m_{\bar{0}} \otimes n_{\bar{0}} &= (\sum_{i\in I} m_{i,\bar{0}} f_{i,\bar{0}}(m_{\bar{0}})) \otimes (\sum_{k\in K} (n_{\bar{0}})^{g_{k,\bar{0}}} n_{k,\bar{0}}) \\
&= \sum_{i,k} m_{i,\bar{0}} f_{i,\bar{0}}(m_{\bar{0}}) \otimes (n_{\bar{0}})^{g_{k,\bar{0}}} n_{k,\bar{0}} \\
&= \sum_{i,k} (m_{i,\bar{0}} \otimes n_{k,\bar{0}})(f_{i,\bar{0}}(m_{\bar{0}})(n_{\bar{0}})^{g_{k,\bar{0}}}) \\
&= \sum_{i,k} (m_{i,\bar{0}} \otimes n_{k,\bar{0}})(f_{i,\bar{0}} \otimes g_{k,\bar{0}})(m_{\bar{0}} \otimes n_{\bar{0}}),
\end{aligned}$$

$$\begin{aligned}
m_{\bar{1}} \otimes n_{\bar{1}} &= (\sum_{j\in J} m_{j,\bar{1}} f_{j,\bar{1}}(m_{\bar{1}})) \otimes (\sum_{l\in L} (n_{\bar{1}})^{g_{l,\bar{1}}} n_{l,\bar{1}}) \\
&= \sum_{j,l} m_{j,\bar{1}} f_{j,\bar{1}}(m_{\bar{1}}) \otimes (n_{\bar{1}})^{g_{l,\bar{1}}} n_{l,\bar{1}} \\
&= \sum_{j,l} (m_{j,\bar{1}} \otimes n_{l,\bar{1}})(f_{j,\bar{1}}(m_{\bar{1}})(n_{\bar{1}})^{g_{l,\bar{1}}}) \\
&= \sum_{j,l} (m_{j,\bar{1}} \otimes n_{l,\bar{1}})(f_{j,\bar{1}} \otimes g_{l,\bar{1}})(m_{\bar{1}} \otimes n_{\bar{1}}),
\end{aligned}$$

$$\begin{aligned}
m_{\bar{0}} \otimes n_{\bar{1}} &= (\sum_{i\in I} m_{i,\bar{0}} f_{i,\bar{0}}(m_{\bar{0}})) \otimes (\sum_{l\in L} (n_{\bar{1}})^{g_{l,\bar{1}}} n_{l,\bar{1}}) \\
&= \sum_{i,l} m_{i,\bar{0}} f_{i,\bar{0}}(m_{\bar{0}}) \otimes (n_{\bar{1}})^{g_{l,\bar{1}}} n_{l,\bar{1}} \\
&= \sum_{i,l} (m_{i,\bar{0}} \otimes n_{l,\bar{1}})(f_{i,\bar{0}}(m_{\bar{0}})(n_{\bar{1}})^{g_{l,\bar{1}}}) \\
&= \sum_{i,l} (m_{i,\bar{0}} \otimes n_{l,\bar{1}})(-f_{i,\bar{0}} \otimes g_{l,\bar{1}})(m_{\bar{0}} \otimes n_{\bar{1}}),
\end{aligned}$$

and

$$\begin{aligned}
m_{\bar{1}} \otimes n_{\bar{0}} &= (\sum_{j\in J} m_{j,\bar{1}} f_{j,\bar{1}}(m_{\bar{1}})) \otimes (\sum_{k\in K} (n_{\bar{0}})^{g_{k,\bar{0}}} n_{k,\bar{0}}) \\
&= \sum_{j,k} m_{j,\bar{1}} f_{j,\bar{1}}(m_{\bar{1}}) \otimes (n_{\bar{0}})^{g_{k,\bar{0}}} n_{k,\bar{0}} \\
&= \sum_{j,k} (m_{j,\bar{1}} \otimes n_{k,\bar{0}})(f_{j,\bar{1}}(m_{\bar{1}})(n_{\bar{0}})^{g_{k,\bar{0}}}) \\
&= \sum_{j,k} (m_{j,\bar{1}} \otimes n_{k,\bar{0}})(f_{j,\bar{1}} \otimes g_{k,\bar{0}})(m_{\bar{1}} \otimes n_{\bar{0}}).
\end{aligned}$$

Therefore, $\{f_{i,\bar{0}} \otimes g_{k,\bar{0}},\ f_{j,\bar{1}} \otimes g_{l,\bar{1}},\ m_{i,\bar{0}} \otimes n_{k,\bar{0}},\ m_{j,\bar{1}} \otimes n_{l,\bar{1}}\}$ forms a dual basis for $M_{\bar{0}} \otimes N_{\bar{0}} \oplus M_{\bar{1}} \hat{\otimes} N_{\bar{1}}$ as a module over R. Similarly $\{-f_{i,\bar{0}} \otimes g_{l,\bar{1}},\ f_{j,\bar{1}} \otimes g_{k,\bar{0}},\ m_{i,\bar{0}} \otimes n_{l,\bar{1}},\ m_{j,\bar{1}} \otimes n_{k,\bar{0}}\}$ forms a dual basis for $M_{\bar{0}} \hat{\otimes} N_{\bar{1}} \oplus M_{\bar{1}} \hat{\otimes} N_{\bar{0}}$ as a module over R.

(c) Let $f_{i,\bar{0}}, m_{i,\bar{0}},\ f_{j,\bar{1}}, m_{j,\bar{1}}$ and $g_{k,\bar{0}}, n_{k,\bar{0}},\ g_{l,\bar{1}}, n_{l,\bar{1}}$ such that

$$\sum f_{i,\bar{0}}(m_{i,\bar{0}}) + \sum f_{j,\bar{1}}(m_{j,\bar{1}}) = 1 \quad \text{and} \quad \sum (n_{k,\bar{0}})^{g_{k,\bar{0}}} + \sum (n_{l,\bar{1}})^{g_{l,\bar{1}}} = 1.$$

Then $[\sum f_{i,\bar{0}}(m_{i,\bar{0}}) + \sum f_{j,\bar{1}}(m_{j,\bar{1}})][\sum (n_{k,\bar{0}})^{g_{k,\bar{0}}} + \sum (n_{l,\bar{1}})^{g_{l,\bar{1}}}] = 1$ and therefore,

$$\begin{aligned}\sum_{i,k}(f_{i,\bar{0}} \otimes g_{k,\bar{0}})(m_{i,\bar{0}} \otimes n_{k,\bar{0}}) \ &+ \ \sum_{i,l}(-f_{i,\bar{0}} \otimes g_{l,\bar{1}})(m_{i,\bar{0}} \otimes n_{l,\bar{1}}) \\ &+ \ \sum_{j,k}(f_{j,\bar{1}} \otimes g_{k,\bar{0}})(m_{j,\bar{1}} \otimes n_{k,\bar{0}}) \\ &+ \ \sum_{j,l}(f_{j,\bar{1}} \otimes g_{l,\bar{1}})(m_{j,\bar{1}} \otimes n_{l,\bar{1}}) = 1.\end{aligned}$$

Therefore, $M\hat{\otimes}_R N$ is an R-generator.
(d) Use (a), (b) and (c). $\square$

For any R-supermodules M, N and N', and any R-supermodule homomorphism $f_\delta \in \mathrm{Hom}_R(N, N')$, we have a mapping

$$\mathrm{Hom}_R(id_M, f_\delta) : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N')$$

given by $\mathrm{Hom}_R(id_M, f_\delta)(g_\gamma) = f_\delta g_\gamma$. With this in mind it is easy to check that $Hom_R(M, \ )$ can be viewed as a functor from $\mathcal{SM} - R$ to $\mathcal{SM} - R$.
The following fundamental properties of Hom are easily verified.
(1) $\mathrm{Hom}_R(M, \ )$ is a left exact functor which by Proposition 4.1.7 is exact if and only if M is R-projective.
(2) For R-supermodules $M_1, ..., M_n$, $N_1, ..., N_k$, we have

$$\mathrm{Hom}_R(\oplus \sum_{i=1}^{n} M_i, \oplus \sum_{j=1}^{k} N_j) \cong \oplus \sum_{i,j}^{n,k} \mathrm{Hom}_R(M_i, N_j)$$

where the map is given by $f \to g$ with $g((i,j))(m_i) = \pi_j f(0, ..., 0, m_i, 0, ..., 0)$ where m_i is in the i-th spot of the n-tuple and π_j is the projection of N onto N_j.
(3) $\mathrm{Hom}_R(R, M) \cong M$ under the map $f \to f(1)$.

We next give some isomorphisms involving the relationship between Hom the supermodule tensor product (i.e., $\hat{\otimes}$).

Theorem 4.2.4 (Hom-Tensor Relation) Let R be a super-commutative superring and let A and B be R-superalgebras. Let M and M' be A-supermodules with M

finitely generated and projective A-supermodule (A-module) and let N and N' be B-supermodules with N finitely generated and projective B-supermodule (B-module). Then for any the mapping

$$\psi : \mathrm{Hom}_A(M, M')\hat{\otimes}_R\mathrm{Hom}_B(N, N') \to \mathrm{Hom}_{A\hat{\otimes}_R B}(M\hat{\otimes}_R N, M'\hat{\otimes}_R N')$$

induced by $\psi(f_\alpha \otimes g_\beta)(m_\mu \otimes n_\epsilon) = (-1)^{\beta\mu} f_\alpha(m_\mu) \otimes g_\beta(n_\epsilon)$ is an R-supermodule isomorphism. If $M = M'$ and $N = N'$, then ψ is a super-ring homomorphism.

Proof That ψ is a well-defined R-supermodule homomorphism which preserves multiplication when $M = M'$ and $N = N'$ is easily checked. Therefore we need only show that ψ is one-to-one and onto.

(a) $A_{\bar{1}}, B_{\bar{1}} \neq \{0\}$.

Case 1: Suppose $M = A$ and $N = B$. Then Property (3) above yields that ψ is an isomorphism.

Case 2: Suppose $M = A^m$ and $N = B^n$. Then by the distributivity of Hom and $\hat{\otimes}$ over finite direct sums, we have

$$\begin{aligned} \mathrm{Hom}_A(A^m, M')\hat{\otimes}_R\mathrm{Hom}_B(B^n, N') &\cong (\mathrm{Hom}_A(A, M'))^m\hat{\otimes}_R(\mathrm{Hom}_B(B, N'))^n \\ &\cong M'^m\hat{\otimes}_R N'^n \\ &\cong \oplus\sum M'\hat{\otimes}_R N' \quad (mn\text{-times}) \\ &\cong \oplus\sum_{mn} \mathrm{Hom}_{A\hat{\otimes}_R B}(A\hat{\otimes}_R B, M'\hat{\otimes}_R N') \\ &\cong \mathrm{Hom}_{A\hat{\otimes}_R B}(A^m\hat{\otimes}_R B^n, M'\hat{\otimes}_R N'), \end{aligned}$$

where the composition of these isomorphisms is ψ.

General case: Suppose M and N are finitely generated and projective. Then there exist integers m, n, an A-supermodule (A-module) L and B-supermodule (B-module) K such that $M \oplus L \cong A^m$ and $N \oplus K \cong B^n$. It follows that there is an R-supermodule (R-module) H of $\mathrm{Hom}_A(A^m, M')\hat{\otimes}_R\mathrm{Hom}_B(B^n, N')$ such that

$$\mathrm{Hom}_A(M, M')\hat{\otimes}_R\mathrm{Hom}_B(N, N') \oplus H \cong \mathrm{Hom}_A(A^m, M')\hat{\otimes}_R\mathrm{Hom}_B(B^n, N')$$

and an R-supermodule (R-module) H' of $\mathrm{Hom}_{A\hat{\otimes}_R B}(A^m\hat{\otimes}_R B^n, M'\hat{\otimes}_R N')$ such that

$$\mathrm{Hom}_{A\hat{\otimes}_R B}(M\hat{\otimes}_R N, M'\hat{\otimes}_R N') \oplus H' \cong \mathrm{Hom}_{A\hat{\otimes}_R B}(A^m\hat{\otimes}_R B^n, M'\hat{\otimes}_R N')$$

in such a fashion that the isomorphism established in case 2 maps H into H' and equals ψ on $\mathrm{Hom}_A(M, M')\hat{\otimes}_R\mathrm{Hom}_B(N, N')$. Therefore ψ is one to one and onto.

(b) $A_{\bar{1}} = \{0\}$ and $B_{\bar{1}} \neq \{0\}$.
Case 1: Suppose $M = A$ and $N = B$. Then Property (3) above yields that ψ is an isomorphism.
Case 2: Suppose $M = A^{m_0} \oplus A^{m_1}$ and $N = B^n$, let $m = m_0 + m_1$. Then by the distributivity of Hom and $\hat{\otimes}$ over finite direct sums, we have

$$\begin{aligned}
\mathrm{Hom}_A(A^{m_0} \oplus A^{m_1}, M')\hat{\otimes}_R\mathrm{Hom}_B(B^n, N') &\cong (\mathrm{Hom}_A(A, M'))^m\hat{\otimes}_R(\mathrm{Hom}_B(B, N'))^n \\
&\cong (M')^m\hat{\otimes}_R(N')^n \\
&\cong \oplus\sum M'\hat{\otimes}_R N' \quad (mn\text{-times}). \\
&\cong \oplus\sum_{mn} \mathrm{Hom}_{A\hat{\otimes}_R B}(A\hat{\otimes}_R B, M'\hat{\otimes}_R N').
\end{aligned}$$

However,

$$\oplus\sum_{mn} \mathrm{Hom}_{A\hat{\otimes}_R B}(A\hat{\otimes}_R B, M'\hat{\otimes}_R N') \cong \mathrm{Hom}_{A\hat{\otimes}_R B}((A^{m_0} \oplus A^{m_1})\hat{\otimes}_R B^n, M'\hat{\otimes}_R N').$$

Therefore,

$$\mathrm{Hom}_A(A^{m_0} \oplus A^{m_1}, M')\hat{\otimes}_R\mathrm{Hom}_B(B^n, N') \cong \mathrm{Hom}_{A\hat{\otimes}_R B}((A^{m_0} \oplus A^{m_1})\hat{\otimes}_R B^n, M'\hat{\otimes}_R N').$$

where the composition of these isomorphisms is ψ.

General case: Suppose M and N are finitely generated and projective. Then there exist integers m_0, m_1, n, an A-modules $L_{\bar{0}}$, $L_{\bar{1}}$ and B-supermodule (B-module) K such that $M_{\bar{0}} \oplus L_{\bar{0}} \cong A^{m_0}$, $M_{\bar{1}} \oplus L_{\bar{1}} \cong A^{m_1}$ and $N \oplus K \cong B^n$. It follows that there is an R-supermodule (R-module) H of $\mathrm{Hom}_A(A^{m_0} \oplus A^{m_1}, M')\hat{\otimes}_R\mathrm{Hom}_B(B^n, N')$ such that

$$\mathrm{Hom}_A(M, M')\hat{\otimes}_R\mathrm{Hom}_B(N, N') \oplus H \cong \mathrm{Hom}_A(A^{m_0} \oplus A^{m_1}, M')\hat{\otimes}_R\mathrm{Hom}_B(B^n, N')$$

and an R-supermodule (R-module) H' of $\mathrm{Hom}_{A\hat{\otimes}_R B}((A^{m_0} \oplus A^{m_1})\hat{\otimes}_R B^n, M'\hat{\otimes}_R N')$ such that

$$\mathrm{Hom}_{A\hat{\otimes}_R B}(M\hat{\otimes}_R N, M'\hat{\otimes}_R N') \oplus H' \cong \mathrm{Hom}_{A\hat{\otimes}_R B}((A^{m_0} \oplus A^{m_1})\hat{\otimes}_R B^n, M'\hat{\otimes}_R N')$$

in such a fashion that the isomorphism established in Case 2 maps H into H' and equals ψ on $\mathrm{Hom}_A(M, M')\hat{\otimes}_R \mathrm{Hom}_B(N, N')$. Therefore ψ is one to one and onto. □

Corollary 4.2.5 For any R-superalgebra A and any finitely generated, projective R-supermodule N, we have $A\hat{\otimes}_R \mathrm{Hom}_R(N, N') \cong \mathrm{Hom}_A(A\hat{\otimes}_R N, A\hat{\otimes}_R N')$ for any R-supermodule N'.

Proof Set $M = M' = A$ and $B = R$. □

Corollary 4.2.6 If the R-supermodules M and N are finitely generated, projective as R-supermodules (or as R-modules), then

$$\mathrm{Hom}_R(M, M)\hat{\otimes}_R \mathrm{Hom}_R(N, N) \cong \mathrm{Hom}_R(M\hat{\otimes}_R N, M\hat{\otimes}_R N)$$

as R-superalgebras.

Proof Set $A = B = R$ and $M = M'$, $N = N'$. □

Theorem 4.2.7 Let A and B be super-rings. Suppose L is a finitely generated, projective A-supermodule, let M be a right A, left B supermodule and N right B-supermodule, then the map

$$\psi : N\hat{\otimes}_B \mathrm{Hom}_A(L, M) \to \mathrm{Hom}_A(L, N\hat{\otimes}_B M)$$

induced by $\psi(n_\alpha \otimes f_\beta)(l_\gamma) = n_\alpha \otimes f_\beta(l_\gamma)$ is an additive group isomorphism.

Proof The proof is analogous to that of Hom-Tensor Relation (Theorem 4.2.4). □

4.3 The Morita Theorems

Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories of supermodules and suppose we have two functors $\mathfrak{J}$ and $\mathfrak{J}'$ from $\mathfrak{C}$ to $\mathfrak{D}$. The two functors $\mathfrak{J}$ and $\mathfrak{J}'$ are *naturally equivalent* if for every supermodule M in $\mathfrak{C}$ there is an isomorphism φ_M in $Hom_{\mathfrak{D}}(\mathfrak{J}(M), \mathfrak{J}'(M))$ such that for every pair of supermodules M and N in $\mathfrak{C}$ and any $f \in \mathrm{Hom}_{\mathfrak{C}}(M, N)$, the diagram

$$\begin{array}{ccc} \mathfrak{J}(M) & \xrightarrow{\mathfrak{J}(f)} & \mathfrak{J}(N) \\ {\scriptstyle \varphi_M}\downarrow & & \downarrow{\scriptstyle \varphi_N} \\ \mathfrak{J}'(M) & \xrightarrow{\mathfrak{J}'(f)} & \mathfrak{J}'(N) \end{array}$$

commutes. We denote by $id_{\mathfrak{C}}$ the identity functor on the category $\mathfrak{C}$ defined by $id_{\mathfrak{C}}(M) = M$ and $id_{\mathfrak{C}}(f) = f$, for supermodules M and maps f. Then the two categories $\mathfrak{C}$ and $\mathfrak{D}$ are equivalent ($\mathfrak{C} \sim \mathfrak{D}$) if there is a functor $\mathfrak{J} : \mathfrak{C} \to \mathfrak{D}$ and a functor $\mathfrak{L} : \mathfrak{D} \to \mathfrak{C}$ such that $\mathfrak{J} \circ \mathfrak{L}$ is naturally equivalent to $id_{\mathfrak{D}}$ and $\mathfrak{L} \circ \mathfrak{J}$ is naturally equivalent to $id_{\mathfrak{C}}$. $\mathfrak{J}$ and $\mathfrak{L}$ are then referred to as inverse equivalences.

In this section we show that $R - \mathcal{SM}$ and $S - \mathcal{SM}$ are equivalent categories when S is chosen as the endomorphism super-ring of some R-progenerator. This equivalence and it's consequences are known as the Morita theorems and we speak of R and S as being Morita equivalent.

The only result of a general categorical nature that we need is the following, where the proof is in [6, Proposition 3.1], since to prove it in the supermodule case is the same as in the module case.

Proposition 4.3.1 Let $\mathfrak{C}$ and $\mathfrak{D}$ be equivalent categories of supermodules under the inverse equivalences $\mathfrak{J} : \mathfrak{C} \to \mathfrak{D}$ and $\mathfrak{L} : \mathfrak{D} \to \mathfrak{C}$. Then for any objects L and L' in $\mathfrak{C}$, the homomorphism from $\mathrm{Hom}_{\mathfrak{C}}(L, L')$ to $\mathrm{Hom}_{\mathfrak{D}}(\mathfrak{J}(L), \mathfrak{J}(L'))$ given by $g \mapsto \mathfrak{J}(g)$ is one-to-one and onto. $\square$

For any super-ring R and any right R-supermodule M, set $M^* = \mathrm{Hom}_R(M, R)$, and $S = \mathrm{Hom}_R(M, M)$. Since R is a left and right R-supermodule, M^* is a left R-supermodule under the operation $(r_\alpha f_\beta)(m_\gamma) = r_\alpha(f_\beta(m_\gamma))$. Moreover, M is a left S-supermodule with $s_\gamma m_\alpha = s_\gamma(m_\alpha)$. Under these operations M is a right R left S supermodule. Whence, M^* becomes a right S-supermodule under the operation $(f_\alpha s_\delta)(m_\beta) = f_\alpha(s_\delta(m_\beta))$. So M^* is a left R, right S supermodule. It follows that we can form $M\hat{\otimes}_R M^*$ and $M^*\hat{\otimes}_S M$. Moreover, $M\hat{\otimes}_R M^*$ is a left S right S-supermodule by virtue of M being a left S right R supermodule and M^* being a right S left R supermodule. (i.e., $M\hat{\otimes}_R M^*$ is a left S right S-supermodule under the operations $s_\delta(m_\alpha \otimes f_\gamma) = (s_\delta m_\alpha) \otimes f_\gamma$ and $(m_\alpha \otimes f_\gamma)s_\delta = m_\alpha \otimes (f_\gamma s_\delta)$. Similarly, $M^*\hat{\otimes}_S M$ is a left R right R supermodule under the operations $r_\delta(f_\alpha \otimes m_\gamma) = (r_\delta f_\alpha) \otimes m_\gamma$ and $(f_\alpha \otimes m_\gamma)r_\delta = f_\alpha \otimes (m_\gamma r_\delta)$.

Let $\theta_R : M\hat{\otimes}_R M^* \to S$ be the map given by $\theta_R(m_\alpha \otimes f_\beta)(n_\delta) = m_\alpha f_\beta(n_\delta)$, $\forall m_\alpha \in M_\alpha$, $f_\beta \in M^*_\beta$, $n_\delta \in M_\delta$. Then θ_R is both a left and right S-supermodule homomorphism, since

$$\begin{aligned} \theta_R(s_\delta(m_\alpha \otimes f_\beta))(n_\mu) &= \theta_R((s_\delta m_\alpha) \otimes f_\beta)(n_\mu) \\ &= (s_\delta m_\alpha) f_\beta(n_\mu) \\ &= s_\delta(m_\alpha f_\beta(n_\mu)) \\ &= s_\delta \theta_R(m_\alpha \otimes f_\beta)(n_\mu) \end{aligned}$$

and

$$\begin{aligned} \theta_R((m_\alpha \otimes f_\beta)s_\delta)(n_\mu) &= \theta_R(m_\alpha \otimes (f_\beta s_\delta))(n_\mu) \\ &= m_\alpha f_\beta(s_\delta(n_\mu)) \\ &= \theta_R(m_\alpha \otimes f_\beta)s_\delta(n_\mu). \end{aligned}$$

Let $\theta_S : M^*\hat{\otimes}_S M \to R$ be the map given by $\theta_S(f_\alpha \otimes m_\beta) = f_\alpha(m_\beta)$. Then θ_S is a left and right R-supermodule homomorphism whose image is the trace super-ideal $\mathcal{T}_R(M)$, since

$$\begin{aligned} \theta_S(r_\delta(f_\alpha \otimes m_\beta)) &= \theta_S((r_\delta f_\alpha) \otimes m_\beta) \\ &= (r_\delta f_\alpha)(m_\beta) \\ &= r_\delta(f_\alpha(m_\beta)) \\ &= r_\delta \theta_S(f_\alpha \otimes m_\beta) \end{aligned}$$

and

$$\begin{aligned}\theta_S((f_\alpha \otimes m_\beta)r_\delta) &= \theta_S(f_\alpha \otimes (m_\beta r_\delta))\\ &= f_\alpha(m_\beta r_\delta)\\ &= f_\alpha(m_\beta)r_\delta\\ &= \theta_S(f_\alpha \otimes m_\beta)r_\delta.\end{aligned}$$

Lemma 4.3.2 (i) θ_R is onto if and only if M is finitely generated and R-projective, as a supermodule over R. If θ_R is onto then it is one to one.
(ii) θ_S is onto if and only if M is an R-generator. If θ_S is onto then it is one-to-one.

Proof We start with the case $R_{\bar{1}} = \{0\}$.
(a) Suppose θ_R is onto. Then there exists $\sum_{i=1}^n m_{i,\bar{0}} \otimes f_{i,\bar{0}} + \sum_{i=1}^n m_{i,\bar{1}} \otimes f_{i,\bar{1}}$ in $M\hat{\otimes}_R M^*$ such that $\theta_R(\sum_{i=1}^n m_{i,\bar{0}} \otimes f_{i,\bar{0}} + \sum_{i=1}^n m_{i,\bar{1}} \otimes f_{i,\bar{1}})$ is the identity map from M to M, that is, for any $m_\alpha \in M_\alpha$,

$$\sum_{i=1}^n m_{i,\bar{0}} f_{i,\bar{0}}(m_\alpha) + \sum_{i=1}^n m_{i,\bar{1}} f_{i,\bar{1}}(m_\alpha) = m_\alpha.$$

For $\alpha = \bar{0}$, we have $\sum_{i=1}^n m_{i,\bar{0}} f_{i,\bar{0}}(m_\alpha) = m_\alpha$, since $R_{\bar{1}} = 0$.
Similarly, for $\alpha = \bar{1}$, we have $\sum_{i=1}^n m_{i,\bar{1}} f_{i,\bar{1}}(m_\alpha) = m_\alpha$. Thus, the set $f_{1,\bar{0}}, ..., f_{n,\bar{0}}$, $m_{1,\bar{0}}, ..., m_{n,\bar{0}}$, $f_{1,\bar{1}}, ..., f_{n,\bar{1}}$, $m_{1,\bar{1}}, ..., m_{n,\bar{1}}$ forms a finite dual basis for M. By the Dual Basis Lemma 4.1.8, M is finitely generated and projective as an R-supermodule. Conversely, suppose there exists a dual basis $f_{1,\bar{0}}, ..., f_{n,\bar{0}} \in (M^*)_{\bar{0}}$, $m_{1,\bar{0}}, ..., m_{n,\bar{0}} \in M_{\bar{0}}$ and $f_{1,\bar{1}}, ..., f_{n,\bar{1}} \in (M^*)_{\bar{1}}$, $m_{1,\bar{1}}, ..., m_{n,\bar{1}} \in M_{\bar{1}}$ such that $\sum_{i=1}^n m_{i,\bar{0}} f_{i,\bar{0}}(m_{\bar{0}}) = m_{\bar{0}}$ and $\sum_{i=1}^n m_{i,\bar{1}} f_{i,\bar{1}}(m_{\bar{1}}) = m_{\bar{1}}$, for any $m_{\bar{0}} \in M_{\bar{0}}$ and $m_{\bar{1}} \in M_{\bar{1}}$. Then for any $g_\alpha \in (Hom_R(M,M))_\alpha$, $\theta_R(\sum_{i=1}^n m_{i,\bar{0}} \otimes f_{i,\bar{0}} g_\alpha + \sum_{i=1}^n m_{i,\bar{1}} \otimes f_{i,\bar{1}} g_\alpha) = g_\alpha$. Thus, θ_R is onto.

Now, given that θ_R is onto, we know M possesses a dual basis $f_{1,\bar{0}}, ..., f_{n,\bar{0}} \in (M^*)_{\bar{0}}$, $m_{1,\bar{0}}, ..., m_{n,\bar{0}} \in M_{\bar{0}}$ and $f_{1,\bar{1}}, ..., f_{n,\bar{1}} \in (M^*)_{\bar{1}}$, $m_{1,\bar{1}}, ..., m_{n,\bar{1}} \in M_{\bar{1}}$. Suppose $z = \sum_j n_j \otimes h_j$ is an element of $M\hat{\otimes}_R M^*$ with $\theta_R(z) = 0$, that is $\sum_j n_j h_j(m) = 0$ for any

$m \in M$. Then

$$\begin{aligned}
z = \sum_j n_j \otimes h_j &= \sum_j (\sum_{i=1}^n m_{i,\bar{0}} f_{i,\bar{0}}(n_j) + \sum_{i=1}^n m_{i,\bar{1}} f_{i,\bar{1}}(n_j)) \otimes h_j \\
&= \sum_{i,j} (m_{i,\bar{0}} f_{i,\bar{0}}(n_j) + m_{i,\bar{1}} f_{i,\bar{1}}(n_j)) \otimes h_j \\
&= \sum_{i,j} m_{i,\bar{0}} f_{i,\bar{0}}(n_j) \otimes h_j + m_{i,\bar{1}} f_{i,\bar{1}}(n_j) \otimes h_j \\
&= \sum_{i,j} m_{i,\bar{0}} \otimes f_{i,\bar{0}}(n_j) h_j + m_{i,\bar{1}} \otimes f_{i,\bar{1}}(n_j) h_j \\
&= \sum_i m_{i,\bar{0}} \otimes (\sum_j f_{i,\bar{0}}(n_j) h_j) + m_{i,\bar{1}} (\sum_j f_{i,\bar{1}}(n_j) h_j) \\
&= 0.
\end{aligned}$$

Since for any i, $m \in M$ and for any $\alpha = \bar{0}$ or $\bar{1}$,

$$\begin{aligned}
(\sum_j f_{i,\alpha}(n_j) h_j)(m) &= \sum_j f_{i,\alpha}(n_j) h_j(m) \\
&= f_{i,\alpha}(\sum_j n_j h_j(m)) \\
&= f_{i,\alpha}(0) = 0.
\end{aligned}$$

This proves (a).

(b) Because the image of θ_S equals $\mathcal{T}_R(M)$, the trace super-ideal of M, it is clear that θ_S is onto if and only if $\mathcal{T}_R(M) = R$, that is, if and only if M is an R-generator.

Suppose θ_S is onto and $\sum_j h_j \otimes n_j \in M^* \hat{\otimes}_S M$ with $\theta_S(\sum_j h_j \otimes n_j) = 0$; that is, $\sum_j h_j(n_j) = 0$. Since θ_S is onto, there exists $f_1, ..., f_n \in M^*$, $m_1, ..., m_n \in M$ with

$\sum_i f_i(m_i) = 1$. Hence

$$\begin{aligned}
\sum_j h_j \otimes n_j &= \sum_j h_j \otimes n_j(\sum_{i=1}^n f_i(m_i)) \\
&= \sum_{i,j} h_j \otimes \theta_R(n_j \otimes f_i)(m_i) \\
&= \sum_i (\sum_j h_j \theta_R(n_j \otimes f_i)(m_i)) \\
&= 0.
\end{aligned}$$

Since, for any i, and for every $m \in M$,

$$\begin{aligned}
\sum_j h_j \theta_R(n_j \otimes f_i)(m) &= \sum_j h_j(n_j(f_i(m))) \\
&= (\sum_j h_j(n_j)).f_i(m) \\
&= 0.f_i(m) = 0.
\end{aligned}$$

Now, for the case $R_{\bar{1}} \neq \{0\}$. We may assume that $M_{\bar{0}}.R_{\bar{1}} = M_{\bar{1}}$
(a) Suppose θ_R is onto. Then there exists $\sum_{i=1}^n m_{i,\bar{0}} \otimes f_{i,\bar{0}} + \sum_{i=1}^n m_{i,\bar{1}} \otimes f_{i,\bar{1}}$ in $M \hat{\otimes}_R M^*$ such that $\theta_R(\sum_{i=1}^n m_{i,\bar{0}} \otimes f_{i,\bar{0}} + \sum_{i=1}^n m_{i,\bar{1}} \otimes f_{i,\bar{1}})$ is the identity map from M to M that is, for any $m_\alpha \in M_\alpha$,

$$\sum_{i=1}^n m_{i,\bar{0}} f_{i,\bar{0}}(m_\alpha) + \sum_{i=1}^n m_{i,\bar{1}} f_{i,\bar{1}}(m_\alpha) = m_\alpha.$$

Since $M_{\bar{0}}.R_{\bar{1}} = M_{\bar{1}}$, we have for any $i \in \{1, 2, ..., n\}$, $m_{i,\bar{1}} = n_{i1,\bar{0}} r_{i1,\bar{1}} + ... + n_{ik_i,\bar{0}} r_{ik_i,\bar{1}}$, where $n_{i1,\bar{0}}, ..., n_{ik_i,\bar{0}}$ in $M_{\bar{0}}$ and $r_{i1,\bar{1}}, ..., r_{ik_i,\bar{1}}$ in $R_{\bar{1}}$. Therefore, for any $m_\alpha \in M_\alpha$,

$$\begin{aligned}
\sum_{i=1}^n [m_{i,\bar{0}} f_{i,\bar{0}}(m_\alpha) + (\sum_{j=1}^{k_i} n_{ij,\bar{0}} r_{ij,\bar{1}}) f_{i\bar{1}}(m_\alpha)] &= \sum_{i,j} [m_{i,\bar{0}} f_{i,\bar{0}}(m_\alpha) + n_{ij,\bar{0}}(r_{ij,\bar{1}} f_{i,\bar{1}})(m_\alpha)] \\
&= m_\alpha.
\end{aligned}$$

For any $i \in \{1, 2, ..., n\}$, let $g_{ij,\bar{0}} = r_{ij,\bar{1}} f_{i,\bar{1}}$ where $j = 1, 2, .., k_i$. Then it easy to check that $g_{ij,\bar{0}} \in (M^*)_{\bar{0}}$, and hence for any $m_\alpha \in M_\alpha$ we have

$$\sum_{i=1}^n [m_{i,\bar{0}} f_{i,\bar{0}} + \sum_{j=1}^{k_i} n_{ij,\bar{0}} g_{ij,\bar{0}}](m_\alpha) = m_\alpha.$$

So, by the Dual Basis Lemma 4.1.8, M is finitely generated and projective. Conversely, supposing there exists a dual basis $f_{1,\bar{0}},...,f_{n,\bar{0}} \in (M^*)_{\bar{0}}$, $m_{1,\bar{0}},...,m_{n,\bar{0}} \in M_{\bar{0}}$, we see that $\theta_R(\sum_{i=1}^n m_{i,\bar{0}} \otimes f_{i,\bar{0}}g_\alpha) = g_\alpha$ for any $g_\alpha \in (Hom_R(M,M))_\alpha$. Thus, θ_R is onto.

Now, given that θ_R is onto, we know M possesses a dual basis $f_{1,\bar{0}},...,f_{n,\bar{0}} \in (M^*)_{\bar{0}}$, $m_{1,\bar{0}},...,m_{n,\bar{0}} \in M_{\bar{0}}$. Suppose $z = \sum_j n_j \otimes h_j \in M\hat{\otimes}_R M^*$ with $\theta_R(z) = 0$, that is $\sum_j n_j h_j(m) = 0$ for any $m \in M$. Then,

$$\begin{aligned} z = \sum_j n_j \otimes h_j &= \sum_j (\sum_{i=1}^n m_{i,\bar{0}} f_{i,\bar{0}}(n_j)) \otimes h_j \\ &= \sum_{i,j} m_{i,\bar{0}} f_{i,\bar{0}}(n_j) \otimes h_j \\ &= \sum_{i,j} m_{i,\bar{0}} \otimes f_{i,\bar{0}}(n_j) h_j \\ &= \sum_i m_{i,\bar{0}} \otimes (\sum_j f_{i,\bar{0}}(n_j) h_j) \\ &= 0. \end{aligned}$$

Since for any i, $m \in M$,

$$\begin{aligned} (\sum_j f_{i,\bar{0}}(n_j) h_j)(m) &= \sum_j f_{i,\bar{0}}(n_j) h_j(m) \\ &= f_{i,\bar{0}}(\sum_j n_j h_j(m)) \\ &= f_{i,\bar{0}}(0) = 0. \end{aligned}$$

This proves (a).

(b) Use the same proof as in case one. $\square$

Corollary 4.3.3 An R-supermodule M is an R-progenerator as a module over R if and only if θ_R, θ_S are supermodule isomorphisms, where $S = \mathrm{Hom}_R(M, M)$. $\square$

For any right R-supermodule M, we have seen that M is a right R, left S supermodule and $M^* = \mathrm{Hom}_R(M, R)$ is a left R, right S supermodule, where $S =$

$\mathrm{Hom}_R(M,M)$. Therefore, for any left R-supermodule L, $M\hat{\otimes}_R L$ has a structure of a left S-supermodule. While for any left S-supermodule N, $M^*\hat{\otimes}_S N$ has a structure of a left R-supermodule. Thus we have functors

$$M\hat{\otimes}_R(\;) : R-\mathcal{SM} \to S-\mathcal{SM} \quad \text{and}$$

$$M^*\hat{\otimes}_S(\;) : S-\mathcal{SM} \to R-\mathcal{SM}.$$

The following is the crucial proposition.

Proposition 4.3.4 Let R be any super-ring and let M be a right R-progenerator as an R-supermodule (R-module). Set $S = \mathrm{Hom}_R(M,M)$ and $M^* = \mathrm{Hom}_R(M,R)$. Then,

$$M\hat{\otimes}_R(\;) : R-\mathcal{SM} \to S-\mathcal{SM} \quad \text{and}$$

$$M^*\hat{\otimes}_S(\;) : S-\mathcal{SM} \to R-\mathcal{SM}$$

are inverse equivalences, establishing $R-\mathcal{SM} \sim S-\mathcal{SM}$.

Proof We must show that each of the two ways of composing the above functors gives a functor naturally equivalent to the identity functor.
Let L be any object of $R-\mathcal{SM}$. Then by the basic properties of tensor product and Lemma 4.3.2(ii) (Corollary 4.3.3), we have $M^*\hat{\otimes}_S(M\hat{\otimes}_R L) \cong (M^*\hat{\otimes}_S M)\hat{\otimes}_R L \cong R\hat{\otimes}_R L \cong L$. This isomorphism allows one to verify that $M\hat{\otimes}_R(\;)$ followed by $M^*\hat{\otimes}_S(\;)$ is naturally equivalent to the identity functor on $R-\mathcal{SM}$.
Similarly Lemma 4.3.2(i) (Corollary 4.3.3) yields that for any left S-supermodule N, $M\hat{\otimes}_R(M^*\hat{\otimes}_S N) \cong (M\hat{\otimes}_R M^*)\hat{\otimes}_S N \cong S\hat{\otimes}_S N \cong N$. Again this gives us that $M^*\hat{\otimes}_S(\;)$ followed by $M\hat{\otimes}_R(\;)$ is naturally equivalent to the identity on $S-\mathcal{SM}$. Thus $M\hat{\otimes}_R(\;)$ and $M^*\hat{\otimes}_S(\;)$ are inverse equivalences. □

Corollary 4.3.5 In the setting of the proposition, we have
(a) $R \cong \mathrm{Hom}_S(M,M)$ (as super-rings) under the mapping which associates to an element $r_\alpha \in R_\alpha$ the endomorphism of M induced by scalar multiplication by r_α.
(b) $M^* \cong \mathrm{Hom}_S(M,S)$ (as right S-supermodules) under the mapping which associates to an element $f_\alpha \in (M^*)_\alpha$ the homomorphism $\theta_R((\;)\otimes f_\alpha)$ from M to S.
(c) $M \cong \mathrm{Hom}_R(M^*,R) = M^{**}$ (as right R-supermodules) under the mapping which

associates to an element $m_\alpha \in M_\alpha$ the element in M^{**} which takes f_γ in $(M^*)_\gamma$ to $f_\gamma(m_\alpha)$.
(d) $S \cong \mathrm{Hom}_R(M^*, M^*)$ (as super-rings) under the mapping which associates an element s_α of S_α to the homomorphism from M^* to M^* given by $f_\gamma \mapsto f_\gamma s_\alpha$.
(e) The left R-supermodule M^* is an R-progenerator as a left R-module.
(f) The left S-supermodule M is an S-progenerator as a left S-module.
(g) The right S-supermodule M^* is an S-progenerator as a right S-module.

Proof Under the equivalence of $R - \mathcal{SM}$ and $S - \mathcal{SM}$, the left R-supermodule R corresponds to the left S-supermodule $M\hat{\otimes}_R R \cong M$, and the left R-supermodule M^* corresponds to the left S-supermodule $M\hat{\otimes}_R M^* \cong S$. Therefore, we can apply Proposition 4.3.1 to obtain
(a) $R \cong \mathrm{Hom}_R(R, R) \cong \mathrm{Hom}_S(M, M)$
(b) $M^* \cong \mathrm{Hom}_R(R, M^*) \cong \mathrm{Hom}_S(M, S)$
(c) $M \cong \mathrm{Hom}_S(S, M) \cong \mathrm{Hom}_R(M^*, R) = M^{**}$
(d) $S \cong \mathrm{Hom}_S(S, S) \cong \mathrm{Hom}_R(M^*, M^*)$.
One can easily check that the composite isomorphisms are in each case the maps described in the statement of the corollary.
To obtain parts (e) and (f) we need to show that $\mathrm{Hom}_R(M^*, R) \cong \mathrm{Hom}_{R^\circ}(M^*, R^\circ)$ as right R-supermodules, $\mathrm{Hom}_S(M, S) \cong \mathrm{Hom}_{S^\circ}(M, S^\circ)$ as right S-supermodules, and $\mathrm{Hom}_R(M, R) \cong \mathrm{Hom}_{R^\circ}(M, R^\circ)$ as left R-supermodules. Since the proofs are the same it is enough to prove one of them. So, let ψ be the map from $\mathrm{Hom}_S(M, S)$ to $\mathrm{Hom}_{S^\circ}(M, S^\circ)$, such that for any f_α in $(\mathrm{Hom}_S(M, S))_\alpha$, $\psi(f_\alpha)$ is in $(\mathrm{Hom}_{S^\circ}(M, S^\circ))_\alpha$ where $\psi(f_\alpha)(m_\gamma) = (-1)^{\alpha\gamma} m_\alpha^{f_\gamma}$ for any m_γ in M_γ. One can easily check that $\psi(f_\alpha) \in (\mathrm{Hom}_{S^\circ}(M, S^\circ))_\alpha$ for any f_α in $(\mathrm{Hom}_S(M, S))_\alpha$. Now we will show that for any s_δ in S_δ, $\psi(f_\alpha.s_\delta) \in (\mathrm{Hom}_{S^\circ}(M, S^\circ))_{\alpha+\delta}$ where for any m_γ in M_γ,

$$\begin{aligned}
\psi(f_\alpha.s_\delta)(m_\gamma) &= (-1)^{\gamma(\alpha+\delta)} m_\gamma^{f_\alpha.s_\delta} \\
&= (-1)^{\gamma(\alpha+\delta)} m_\gamma^{f_\alpha}.s_\delta.
\end{aligned}$$

Then for any s'_β in S_β,

$$\begin{aligned}
\psi(f_\alpha.s_\delta)(m_\gamma \circ s'_\beta)) &= (-1)^{(\gamma+\beta)(\alpha+\delta)}(m_\gamma \circ s'_\beta)^{f_\alpha}.s_\delta \\
&= (-1)^{\gamma\beta+(\gamma+\beta)(\alpha+\delta)} s'_\beta m_\gamma^{f_\alpha}.s_\delta \\
&= (-1)^{\gamma\beta+(\beta)(\alpha+\delta)} s'_\beta(\psi(f_\alpha.s_\delta))(m_\gamma) \\
&= (\psi(f_\alpha.s_\delta))(m_\gamma) \circ s'_\beta.
\end{aligned}$$

Therefore ψ is an S-supermodule homomorphism, $\ker(\psi) = 0$, and if g_α is any element in $(\mathrm{Hom}_{S^\circ}(M, S^\circ))_\alpha$, we let $\overline{g_\alpha}$ be defined by $m_\beta^{\overline{g_\alpha}} = (-1)^{\alpha\beta} g_\alpha(m_\beta)$, then it is easy to check that $\overline{g_\alpha} \in \mathrm{Hom}_S(M, S)$, and $\psi(\overline{g_\alpha}) = g_\alpha$, hence ψ is one to one.
(e) Now, since we have that $S \cong S^\circ$ as a left and right S-supermodules, $R \cong R^\circ$ as a left and right R-supermodules and $M \cong \mathrm{Hom}_R(M^*, R) \cong \mathrm{Hom}_{R^\circ}(M^*, R^\circ)$, $M^* \cong Hom_R(M, R) \cong \mathrm{Hom}_{R^\circ}(M, R^\circ)$. Therefore,

$$\begin{aligned}
M\hat{\otimes}_R M^* &\cong \mathrm{Hom}_R(M^*, R)\hat{\otimes}_R M^* \\
&\cong M^*\hat{\otimes}_{R^\circ}\mathrm{Hom}_R(M^*, R) \\
&\cong M^*\hat{\otimes}_{R^\circ}\mathrm{Hom}_{R^\circ}(M^*, R^\circ) \\
&\cong \mathrm{Hom}_{R^\circ}(M^*, M^*) \\
&\cong S^\circ
\end{aligned}$$

and

$$\begin{aligned}
M^*\hat{\otimes}_S M &\cong M^*\hat{\otimes}_S\mathrm{Hom}_R(M^*, R) \\
&\cong \mathrm{Hom}_{R^\circ}(M^*, R^\circ)\hat{\otimes}_{S^\circ} M^* \\
&\cong \mathrm{Hom}_{S^\circ}(M^*, M^*) \\
&\cong R^\circ.
\end{aligned}$$

Hence by Lemma 4.3.2, M^* is a right R°-progenerator as an R°-module, so M^* is a left R-progenerator as an R-module.
(f) Similarly, since $M^* \cong \mathrm{Hom}_S(M, S) \cong \mathrm{Hom}_{S^\circ}(M, S^\circ)$ we have

$$M\hat{\otimes}_{S^\circ} M^* \cong M\hat{\otimes}_{S^\circ}\mathrm{Hom}_{S^\circ}(M, S^\circ) \cong R^\circ \quad \text{and}$$

$$M^*\hat{\otimes}_{R^\circ}M \cong \mathrm{Hom}_{S^\circ}(M,S^\circ)\hat{\otimes}_{R^\circ}M \cong \mathrm{Hom}_{R^\circ}(M,M) \cong S^\circ.$$

Hence by lemma 4.3.2, M is a right S°-progenerator as an S°-module, and so M is a left S-progenerator as an S-module.

(g) Since M is a left S-progenerator as an S-module, M is a right S°-progenerator, we can apply (e) to the S°-module M to conclude that $\mathrm{Hom}_{S^\circ}(M,S^\circ)$ is an S°-progenerator as a module over S°. But $M^* \cong \mathrm{Hom}_S(M,S) \cong \mathrm{Hom}_{S^\circ}(M,S^\circ)$, therefore M^* is a left S°-progenerator as a module over S° and hence M^* is a right S-progenerator. □

4.4 Local Super-rings

In this section we prove that the concept of the rank of a free supermodule can be extended to finitely generated, projective supermodules over super-commutative super-rings containing no idempotents but 0 and 1. To do this we must introduce the important technique of localization.

We call a super-commutative super-ring a *local super-ring* if it contains a unique maximal super-ideal. Clearly a field is a local super-ring. We will soon show that there is associated to every prime super-ideal P of a super-commutative super-ring R a local super-ring, denoted R_P.

In the next proposition we prove that a finitely generated projective supermodule over a local super-ring is free.

Proposition 4.4.1 Let R be a local super-ring, where $R_{\bar{1}} \neq 0$, with maximal super-ideal I and let M be a finitely generated, projective R-supermodule. Suppose $\{m_{1,\bar{0}}+MI,...,m_{n,\bar{0}}+MI\}$ is a free basis for M/MI over R/I. Then $\{m_{1,\bar{0}},...,m_{n,\bar{0}}\}$ is a free basis for M over R. If $R_{\bar{1}}=0$ and $\{m_{1,\bar{0}}+MI,...,m_{n,\bar{0}}+MI\}$ is a free basis for $M_{\bar{0}}/M_{\bar{0}}I$ over R/I and $\{m_{1,\bar{1}}+MI,...,m_{k,\bar{1}}+MI\}$ is a free basis for $M_{\bar{1}}/M_{\bar{1}}I$ over R/I. Then $\{m_{1,\bar{0}},...,m_{n,\bar{0}}\}$ is a free basis for $M_{\bar{0}}$ over R and $\{m_{1,\bar{1}},...,m_{k,\bar{1}}\}$ is a free basis for $M_{\bar{1}}$ over R.

Proof The case $R_{\bar{1}} = \{0\}$ follows from [6, Proposition 1.4.1].
When $R_{\bar{1}} \neq \{0\}$ we may assume that $M_{\bar{0}}R_{\bar{1}} = M_{\bar{1}}$. Let $\phi : R^n \to M$ be defined by $\phi((r_1, ..., r_n)) = \sum_{i=1}^{n} m_{i,\bar{0}}r_i$. Since $\{m_{1,\bar{0}} + MI, ..., m_{n,\bar{0}} + MI\}$ generates M/MI, we have $(m_{1,\bar{0}}R + ... + m_{n,\bar{0}}R) + MI = M$. Therefore, by applying Corollary 4.1.15 to the supermodule $M/(m_{1,\bar{0}}R + ... + m_{n,\bar{0}}R)$, we have $m_{1,\bar{0}}R + ... + m_{n,\bar{0}}R = M$. Thus, ϕ is an onto R-supermodule homomorphism.
Now suppose $(r_1, ..., r_n)$ is in the kernel of ϕ. Then $\sum_{i=1}^{n} m_{i,\bar{0}}r_i = 0$, so $\sum_{i=1}^{n}(m_{i,\bar{0}} + MI)r_i = 0$ in M/MI. Thus r_i is in I since $m_{i,\bar{0}} + MI$ are free over R/I, so $\ker(\phi) \subset R^n I$. Since M is a projective supermodule over R, the exact sequence

$$0 \to \ker(\phi) \to R^n \xrightarrow{\phi} M \to 0$$

splits, so there exists a supermodule L of R^n with $\ker(\phi) \oplus L = R^n$. Now, $\ker(\phi) = R^n I \cap \ker(\phi) = (\ker(\phi) \oplus L)I \cap \ker(\phi) = ker(\phi)I$. But $\ker(\phi)$, being a direct summand of R^n, is finitely generated over R and so by Corollary 4.1.15 equals (0). Thus ϕ is an isomorphism. $\square$

We call a subset W of $R_{\bar{0}}$, where $R = R_{\bar{0}} + R_{\bar{1}}$ is a super-commutative super-ring, multiplicatively closed if
(1) $1 \in W$
(2) $w, w' \in W$ implies that $ww' \in W$.

In what follows, multiplicatively closed sets will consist either of all the non-negative powers of an element of $R_{\bar{0}}$ or of all the elements of $R_{\bar{0}}$ not contained in some proper prime super-ideal.

Suppose M is any R-supermodule and W is any multiplicatively closed subset of $R_{\bar{0}}$. We introduce a relation $\approx$ into the set $M \times W$ by defining $(m_\alpha, w_1) \approx (m'_\beta, w_2)$ if and only if there exists $w \in W$ with $(m_\alpha w_2 - m'_\beta w_1)w = 0$, where m_α, m'_β are in M. One sees that $\approx$ is an equivalence relation. We denote the equivalence class containing (m_α, w) by $\frac{m_\alpha}{w}$, and the collection of all equivalence classes by M_W (i.e. $M_W = (M_W)_{\bar{0}} + (M_W)_{\bar{1}}$ where $(M_W)_\alpha = \{\frac{m_\alpha}{w} \mid m_\alpha \in M_\alpha,\ w \in W\}$). Define addition and R-scalar multiplication in M_W by $\frac{m_\alpha}{w_1} + \frac{n_\beta}{w_2} = \frac{m_\alpha w_2 + n_\beta w_1}{w_1 w_2}$ and $(\frac{m_\alpha}{w})r_\gamma = \frac{m_\alpha r_\gamma}{w}$. One can easily check that these definitions are independent of the choice of representatives of the equivalence classes and therefore M_W forms an R-supermodule. M_W is called a *supermodule of quotients.*

If, in addition, M is an R-superalgebra, M_W can be checked to be an R-superalgebra under the multiplication $\frac{m_\alpha}{w_1}\frac{n_\beta}{w_2} = \frac{m_\alpha n_\beta}{w_1 w_2}$. In particular, R_W is called a *super-ring of quotients* of R. If we take W to be $R_{\bar{0}} - P_{\bar{0}}$, where P is a prime super-ideal of R, we denote R_W by R_P, and observe that the super-ideal $PR_P = \{\frac{r}{s} \mid r \in P,\ s \in R_{\bar{0}} - P_{\bar{0}}\}$ of R_P has the property that every element in $(R_P)_\gamma - (PR_P)_\gamma$ is a unit of R_P. Therefore every proper super-ideal of R_P is contained in PR_P, whence PR_P is the unique maximal super-ideal of R_P. Thus R_P is a local super-ring, called the *localization of R at the prime super-ideal P.*

We now state some fundamental properties of super-rings and supermodules of quotients without proof since their proof is the same as in [6, Lemma 1.4.2, 1.4.3].

Lemma 4.4.2 Let W be any multiplicatively closed subset of $R_{\bar{0}}$. Then for any R-supermodule M, M_W is an R_W-supermodule and the mapping from M_W to $M\hat{\otimes}_R R_W$ given by $\frac{m_\alpha}{w} \to m_\alpha \otimes \frac{1}{w}$ is an R_W-supermodule isomorphism. □

Lemma 4.4.3 R_W is a flat R-supermodule. □

Proposition 4.4.4 Let M be an R-supermodule such that $M_W = 0$ for every maximal super-ideal W of R. Then $M = \{0\}$.

Proof Let m_α be any element of M_α. It sufficies to show that $m_\alpha = 0$. For each maximal super-ideal W of R, $\frac{m_\alpha}{1} = 0$ in M_W, so there exists an element a_W in $R_{\bar{0}} - W_{\bar{0}}$ with $m_\alpha a_W = 0$. Thus the annihilator super-ideal of m_α in R is contained in no maximal super-ideal and so equals R. Hence $M = \{0\}$. □

We would like to generalize the concept of the rank of free supermodule to projective supermodules. Suppose M is a finitely generated, projective R-supermodule, where R is any super-commutative super-ring, where $R_{\bar{1}} \neq \{0\}$. For any prime super-ideal P of R, we can form the R_P-supermodule $M\hat{\otimes}_R R_P \cong M_P$. By Corollary 4.2.2, and Proposition 4.4.1, we know that $M\hat{\otimes}_R R_P$ is an R_P-free supermodule, so there exists a unique non-negative integer n_p with $M\hat{\otimes}_R R_P \cong (R_P)^{n_p}$. If $R_{\bar{1}} = 0$ then $M_{\bar{0}} \otimes_R R_P \cong (R_P)^{n_p^0}$ and $M_{\bar{1}} \otimes_R R_P \cong (R_P)^{n_P^1}$, let $n_p = n_P^0 + n_P^1$. We call n_p the P-rank of M and denote it $rank_P(M)$. For a non-negative integer n, we call n the *rank* of M and write $rank_R(M) = n$ if and only if all the P ranks are equal to n.

Proposition 4.4.5 Let R be a super-commutative super-ring and M a finitely generated, projective R-supermodule such that $rank_R(M)$ is defined. Then for any super-commutative R-superalgebra S, $rank_S(M\hat{\otimes}_R S)$ is defined and equals to $rank_R(M)$.

Proof Let B be any prime super-ideal of S. It is clear that $P = \{r \in R \mid 1.r \in B\}$ is a prime super-ideal of R. Moreover one checks that S_B is naturally an R_P-superalgebra since $1.(R_{\bar{0}} - P_{\bar{0}}) \subset S_{\bar{0}} - B_{\bar{0}}$. Therefore, for $R_{\bar{1}} \neq \{0\}$, we have

$$\begin{aligned}
(M\hat{\otimes}_R S)\hat{\otimes}_S S_B &\cong M\hat{\otimes}_R(S\hat{\otimes}_S S_B) \\
&\cong M\hat{\otimes}_R S_B \\
&\cong M\hat{\otimes}_R(R_P\hat{\otimes}_{R_P} S_B) \\
&\cong (M\hat{\otimes}_R R_P)\hat{\otimes}_{R_P} S_B \\
&\cong (R_P)^n\hat{\otimes}_{R_P} S_B \cong (S_B)^n
\end{aligned}$$

where $n = rank_R(M)$. If $R_{\bar{1}} = \{0\}$ then

$$\begin{aligned}
((M_{\bar{0}} + M_{\bar{1}})\hat{\otimes}_R S)\hat{\otimes}_S S_B &\cong M\hat{\otimes}_R(S\hat{\otimes}_S S_B) \\
&\cong M\hat{\otimes}_R S_B \\
&\cong M\hat{\otimes}_R(R_P\hat{\otimes}_{R_P} S_B) \\
&\cong (M\hat{\otimes}_R R_P)\hat{\otimes}_{R_P} S_B \\
&\cong [(R_P)^{n_P^0} \oplus (R_P)^{n_P^1}]\hat{\otimes}_{R_P} S_B \cong [(S_B)^{n_P^0} \oplus (S_B)^{n_P^1}]
\end{aligned}$$

where $n = n_P^0 + n_P^1 = rank_R(M)$, therefore if $S_{\bar{1}} \neq 0$ then $(S_B)^{n_P^0} \oplus (S_B)^{n_P^1} \cong (S_B)^n$, but if $S_{\bar{1}} = 0$ then $M_{\bar{0}} \otimes_R S_B \cong (S_B)^{n_P^0}$ and $M_{\bar{1}} \otimes_R S_B \cong (S_B)^{n_P^1}$. Since B is arbitrary, this proves the proposition. □

Now we will prove that when $R_{\bar{0}}$ contains no idempotents other than 0 and 1, the rank of any finitely generated, projective R-supermodule is defined.

For any super-commutative super-ring R, we call the collection of all prime super-ideals of R which do not equal R, the *spectrum of* R and denote it $Spec(R)$. We introduce a topology, the Zariski topology, into $Spec(R)$ as follows: for any subset L of R, let $h(L) = \{P \in Spec(R) \mid P \supset L\}$. If we designate the subsets $h(L)$ of $Spec(R)$ as closed sets, we have the following lemma and proposition, whose proofs are found in [6, page 28]

Lemma 4.4.6 The subsets $h(L)$ form a topology for $Spec(R)$. □

Proposition 4.4.7 The topological space $Spec(R)$ is connected if and only if 0 and 1 are the only idempotents in $R_{\bar{0}}$. □

We return to the consideration of super-rings of quotients.

Lemma 4.4.8 Let W and W' be multiplicatively closed subsets of $R_{\bar{0}}$ with $W \subset W'$. Then the map from R_W to $R_{W'}$ given by $\frac{r}{w} \to \frac{r}{w}$ is a super-ring homomorphism which induces in $R_{W'}$ the structure of an R_W superalgebra. Moreover, $R_{W'} \cong (R_W)_{W'R_W}$ where $W'R_W = \{\frac{w'}{w} \in R_W \mid w' \in W'\}$ is a multiplicatively closed subset of R_W.

Proof Since $W \subset W'$, the map $\frac{r}{w} \to \frac{r}{w}$ is well-defined. To show the isomorphism between $R_{W'}$ and $(R_W)_{W'R_W}$, one can define a map which takes $\frac{r_\alpha}{w'}$ of $R_{W'}$ to $\frac{r_\alpha}{1}/\frac{w'}{1}$ in $(R_W)_{W'R_W}$ and show it is well-defined and an isomorphism. □

Lemma 4.4.9 If M is a finitely generated R-supermodule and W is a multiplicatively closed subset of $R_{\bar{0}}$, then $M_W = \{0\}$ if and only if there exists an element $w \in W$ with $Mw = \{0\}$.

Proof Clearly if $Mw = \{0\}$, then any element $\frac{m_\alpha}{w'}$ of M_W equals zero.
Conversely, suppose $M_W = \{0\}$ and $M = m_{1,\alpha_1}R + ... + m_{n,\alpha_n}R$. Since $\frac{m_{i,\alpha_i}}{1} = 0$ in M_W, there exists w_i in W with $m_{i,\alpha_i}w_i = 0$. Then $w = w_1.w_2 \dots w_n$ has the property that $m_{i,\alpha_i}w = 0$ for every i, so $Mw = \{0\}$. □

We have seen that for a finitely generated, projective R-supermodule M, M_P is a free R_P-supermodule for any prime super-ideal P. Our next proposition shows that we can take a multiplicatively closed subset considerably smaller than $R_{\bar{0}} - P_{\bar{0}}$ and obtain the same result.

For any element a in $R_{\bar{0}}$, we let $R_{(a)}$ denote the super-ring of quotients of R determined by the multiplicatively closed subset of R consisting of the non-negative powers of a.

Proposition 4.4.10 Let M be a finitely generated, projective R-supermodule and P any prime super-ideal of R. If $R_{\bar{1}} \neq 0$ then there exists $a \in R_{\bar{0}} - P_{\bar{0}}$ with $M\hat{\otimes}_R R_{(a)} \cong M_{(a)}$ free as an $R_{(a)}$-supermodule. If $R_{\bar{1}} = 0$ then there exists $a \in R_{\bar{0}} - P_{\bar{0}}$ and $a' \in R_{\bar{0}} - P_{\bar{0}}$ with $M_{\bar{1}} \otimes_R R_{(a')} \cong (M_{\bar{1}})_{(a')}$ free as an $R_{(a')}$-module.

Proof If $R_{\bar{1}} = 0$ then the proof follows immediately from [6, Proposition 1.4.10]. If $R_{\bar{1}} \neq 0$ then we know that M_P is a free supermodule of finite rank. Let $\{\frac{m_{1,\bar{0}}}{a_1}, ..., \frac{m_{n,\bar{0}}}{a_n}\}$ (under the assumption $M_{\bar{0}}R_{\bar{1}} = M_{\bar{1}}$) forms a basis for M_P over R_P. Since $\frac{1}{a_1}, ..., \frac{1}{a_n}$ are units in R_P, it follows that $\{\frac{m_{1,\bar{0}}}{1}, ..., \frac{m_{n,\bar{0}}}{1}\}$ also forms a basis for M_P over R_P. Define a R-supermodule homomorphism $\phi : R^n \to M$ by $\phi((r_1, ..., r_n)) = \sum_{i=1}^n m_{i,\bar{0}} r_i$. This gives rise to an exact sequence

$$0 \to \ker(\phi) \to R^n \xrightarrow{\phi} M \to \operatorname{coker}(\phi) \to 0.$$

Where $\operatorname{coker}(\phi) \cong M/Im(\phi)$, since R_P is R-flat, we obtain the exact sequence

$$0 \to \ker(\phi)_P \to (R^n)_P \xrightarrow{\phi} M_P \to \operatorname{coker}(\phi)_P \to 0.$$

Because $\{\frac{m_{1,\bar{0}}}{1}, ..., \frac{m_{n,\bar{0}}}{1}\}$ is a basis, $\ker(\phi)_P = \operatorname{coker}(\phi)_P = 0$. Now $\operatorname{coker}(\phi)$ is finitely generated, because M is finitely generated. By Lemma 4.4.9, there exists $b \in R_{\bar{0}} - P_{\bar{0}}$ with $(\operatorname{coker}(\phi))b = 0$, so $\operatorname{coker}(\phi)_{(b)} = 0$. Thus we obtain the exact sequence

$$0 \to \ker(\phi)_{(b)} \to (R^n)_{(b)} \xrightarrow{\phi} M_{(b)} \to 0.$$

Next we see that $\ker(\phi)_{(b)}$ must be finitely generated over $R_{(b)}$ since the preceding exact sequence splits, $M_{(b)}$ being $R_{(b)}$-projective supermodule by Corollary 4.2.2. But by Lemma 4.4.8, R_P is an $R_{(b)}$-superalgebra and $R_{(b)} \cong (R_{(b)})_{(R_{\bar{0}}-P_{\bar{0}})R_{(b)}}$, so we have

$$\begin{aligned}\ker(\phi)_{(b)}\hat{\otimes}_{R_{(b)}}R_P &\cong (\ker(\phi)\hat{\otimes}_R R_{(b)})\hat{\otimes}_{R_{(b)}}R_P \\ &\cong \ker(\phi)\hat{\otimes}_R(R_{(b)}\hat{\otimes}_{R_{(b)}}R_P) \\ &\cong \ker(\phi)\hat{\otimes}_R R_P \cong \ker(\phi)_P = \{0\}.\end{aligned}$$

So again by Lemma 4.4.9, there exists $\frac{\mu}{b^k}$ in $(R_{\bar{0}} - P_{\bar{0}})(R_{(b)})_{\bar{0}}$ with $(\ker(\phi)_{(b)})\frac{\mu}{b^k} = 0$. Since $\frac{1}{b^k}$ is a unit in $R_{(b)}$, this equivalent to $(\ker(\phi)_{(b)})\mu = 0$. Since $[R_{(b)}]_{(\mu)}$ is isomorphic to $R_{(b\mu)}$, we have

$$\{0\} = [\ker(\phi)_{(b)}]_{(\mu)} \cong (\ker(\phi)\hat{\otimes}_R R_{(b)})\hat{\otimes}_{R_{(b)}}[R_{(b)}]_{(\mu)} \cong \ker(\phi)\hat{\otimes}_R R_{(b\mu)} \cong \ker(\phi)_{(b\mu)}.$$

Since $(\text{coker}(\phi))b = 0$ we also have $\text{coker}(\phi)_{(b\mu)} = 0$. Thus we obtain the exact sequence

$$0 \to [R^n]_{(b\mu)} \to M_{(b\mu)} \to 0,$$

proving that for $a = b\mu$, $M_{(a)}$ is an $R_{(a)}$-free supermodule. □

Corollary 4.4.11 Let M be a finitely generated, projective R-supermodule. The map from $Spec(R)$ to $\{0, 1, 2, 3, ...\}$ given by $P \to rank_P(M)$ is continuous when the non-negative integers are given the discrete topology.

Proof Suppose a is any element of R such that $M_{(a)}$ is free of rank m over $R_{(a)}$. Then for any prime super-ideal P of R not containing a, we have that R_P is an $R_{(a)}$-superalgebra and

$$\begin{aligned}M\hat{\otimes}_R R_P &\cong (M\hat{\otimes}_R R_{(a)})\hat{\otimes}_{R_{(a)}}R_P \\ &\cong (R_{(a)})^m\hat{\otimes}_{R_{(a)}}R_P \\ &\cong (R_P)^m,\end{aligned}$$

so $rank_P(M) = m$.

Now let P be any prime super-ideal and suppose $rank_P(M) = n$. If $R_{\bar{1}} \neq \{0\}$ then by the preceding proposition, there exists an a in $R_{\bar{0}} - P_{\bar{0}}$ with $M_{(a)}$ free of finite rank over $R_{(a)}$. By the preceding paragraph, this rank must equal n. If $R_{\bar{1}} = \{0\}$ then by the preceding proposition, there exist a, a' in $R_{\bar{0}} - P_{\bar{0}}$ with $(M_{\bar{0}})_{(a)}$ free of finite rank

equal to n_1 over $R_{(a)}$, and $(M_{\bar{1}})_{(a')}$ free of finite rank equal to n_2 over $R_{(a')}$, again by the preceding paragraph, $n_1 + n_2 = n$.
For $R_{\bar{1}} \neq \{0\}$, consider $Spec(R) - h(a)$. Since $h(a)$ is closed, $Spec(R) - h(a)$ is an open subset of $Spec(R)$ containing P. But for $R_{\bar{1}} = 0$, consider $Spec(R) - (h(a) \cup h(a'))$. Since $h(a) \cup h(a')$ is closed, $Spec(R) - (h(a) \cup h(a'))$ is an open subset of $Spec(R)$ containing P. Again by the preceding paragraph, if q in $Spec(R) - h(a)$ $(Spec(R) - (h(a) \cup h(a')))$, then $rank_q(M) = n = rank_P(M)$, so the preimage of n is open in $Spec(R)$. Therefore the map is continuous. □

From now on, we say that a super-commutative super-ring R is *connected* if and only if the only idempotents of R is 0 and 1.

We can now deduce the following theorem as an immediate corollary of this result and Proposition 4.4.7, recalling that the continuous image of a connected space is connected.

Theorem 4.4.12 Let R be a super-commutative super-ring with no idempotents but 0 and 1. Then for any finitely generated, projective R-supermodule, $rank_R(M)$ is defined; that is there exists a non-negative integer n such that $rank_P(M) = n$ for every prime super-ideal P of R. □

Chapter 5

Azumaya Superalgebras

In this chapter we present the fundamental theory of non-commutative separable superalgebras with the theory of the Brauer-Wall group of supercommutative super-ring. This theory successfully generalizes the classical theory of central simple superalgebras over fields and the Brauer-Wall group of a field. Throughout this chapter A will denote a not necessarily supercommutative R-superalgebra. The symbol $\hat{\otimes}$ will always mean tensoring with respect to R.

5.1 Definitions and Basic Properties

This section begins with the fundamental definition of separability. The fact that a superalgebra over a field is separable in the present sense if and only if it is finite dimensional and separable in the classical sense (see Section 2) is the crucial connection with the classical theory.

Let $R = R_{\bar{0}} + R_{\bar{1}}$ be any supercommutative super-ring, then for any R-superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$, we form the enveloping superalgebra $A^e = A^\circ \hat{\otimes} A$ (A° : the opposite superalgebra of A). A has a right A^e-supermodule structure induced by $a_\alpha(c_\gamma \otimes d_\delta) = (-1)^{\alpha\gamma} c_\gamma a_\alpha d_\delta$. We show now that,

$$\mu : A^e \to A, \quad (a_\alpha \otimes b_\beta \mapsto a_\alpha b_\beta)$$

is a left A^e-supermodule homomorphism.

$$\begin{aligned}\mu((a_\alpha \otimes b_\beta).(c_\gamma \otimes d_\delta)) &= (-1)^{\beta\gamma}\mu(a_\alpha c_\gamma \otimes b_\beta d_\delta)\\ &= (-1)^{\beta\gamma+\alpha\gamma} c_\gamma a_\alpha b_\beta d_\delta\end{aligned}$$

and

$$\begin{aligned}\mu(a_\alpha \otimes b_\beta).(c_\gamma \otimes d_\delta) &= a_\alpha b_\beta(c_\gamma \otimes d_\delta)\\ &= (-1)^{\gamma(\alpha+\beta)} c_\gamma a_\alpha b_\beta d_\delta.\end{aligned}$$

Thus, μ is a left A^e-supermodule homomorphism, and if A is a supercommutative super-ring then μ is a super-ring homomorphism. Let $J = \ker(\mu)$ we obtain the exact sequence of right A^e-supermodules :

$$0 \hookrightarrow J \hookrightarrow A^e \xrightarrow{\mu} A \to 0$$

Now, it is easy to check that J is a right super-ideal of A^e generated by all elements of the form $a_\alpha \otimes 1 - 1 \otimes a_\alpha$.

Proposition 5.1.1 The following conditions on an R-superalgebra A are equivalent.
(i) A is projective as a right A^e-supermodule under the μ-structure.
(ii) $0 \hookrightarrow J \hookrightarrow A^e \xrightarrow{\mu} A \to 0$ splits as a sequence of right A^e-supermodules.
(iii) $(A^e)_{\bar{0}}$ contains an element e such that $\mu(e) = 1$ and $eJ = 0$.

Proof The equivalence of (i) and (ii) is clear.
(ii)$\Rightarrow$(iii) If the sequence splits then there exists a left A^e-homomorphism ψ from A to A^e such that $\mu\psi$ is the identity on A. Denoting $\psi(1)$ by e, we see that $\mu(e) = 1$ and $eJ = 0$.
(To see that $eJ = 0$, let $x \in J$ then $\mu(ex) = \mu(e)x = 1.x = \mu(x) = 0$ and so $\psi(0) = \psi(1.x) = \psi(1).x = e.x = 0$)
(iii)$\Rightarrow$(ii) If $(A^e)_{\bar{0}}$ contains an element e such that $\mu(e) = 1$ and $eJ = 0$ then the map $\psi : A \to A^e$ defined by $\psi(a_\alpha) = e.(1 \otimes a_\alpha) = e.(a_\alpha \otimes 1)$ is A^e-homomorphism from A

into A^e since

$$\begin{aligned}
\psi(c_\gamma(a_\alpha \otimes b_\beta)) &= (-1)^{\alpha\gamma}\psi(a_\alpha c_\gamma b_\beta) \\
&= (-1)^{\alpha\gamma}e.(1 \otimes a_\alpha c_\gamma b_\beta) \\
&= (-1)^{\alpha\gamma}e.((1 \otimes a_\alpha c_\gamma)(1 \otimes b_\beta)) \\
&= (-1)^{\alpha\gamma}e.((a_\alpha c_\gamma \otimes 1)(1 \otimes b_\beta)) \\
&= e.((c_\gamma^{op} a_\alpha \otimes 1)(1 \otimes b_\beta)) \\
&= e.(c_\gamma \otimes 1)(a_\alpha \otimes b_\beta) \\
&= \psi(c_\gamma)(a_\alpha \otimes b_\beta).
\end{aligned}$$

Moreover, $\mu\psi$ is the identity on A, since

$$\begin{aligned}
\mu\psi(a_\alpha) &= \mu(e.(a_\alpha \otimes 1)) \\
&= \mu(e)(a_\alpha \otimes 1) \\
&= 1.(a_\alpha \otimes 1) \\
&= a_\alpha.
\end{aligned}$$

Therefore, ψ is a splitting map. □

Definition 5.1.2 (definition of separability) *An R-superalgebra A is called* separable *if it satisfies the equivalent conditions of the proposition.*

We should remark that the element e described in condition (iii) is necessarily an idempotent, since $e^2 - e = e.(e - 1 \otimes 1) \in eJ = 0$.

If M is a right A^e-supermodule, then M can be regarded as a two sided A/R-supermodule (i.e., M is an A-supermodule such that $R \in \hat{C}(M)$) by defining :

(1) $a_\alpha m_\beta = (-1)^{\alpha\beta} m_\beta(a_\alpha \otimes 1)$

(2) $m_\beta a_\alpha = m_\beta(1 \otimes a_\alpha)$.

Conversely, any two sided A/R-supermodule can be considered as a right A^e-supermodule (under the action : $m_\gamma(a_\alpha \otimes b_\beta) = (-1)^{\alpha\gamma} a_\alpha m_\gamma b_\beta$).

Let M_α^A denote the set $\{m_\alpha \in M_\alpha \mid a_\beta m_\alpha = (-1)^{\alpha\beta} m_\alpha a_\beta \ \forall a_\beta \in A_\beta,\ \beta = 0, 1\}$. Then $M^A = M_{\bar{0}}^A + M_{\bar{1}}^A$ is an R-subsupermodule of M. One can check that $(\ )^A$ is a covariant functor from the category of two sided A/R-supermodules to the category of R-subsupermodules.

Lemma 5.1.3 For any two sided A/R-supermodule M, $\mathrm{Hom}_{A^e}(A, M) \cong M^A$ as R-supermodules under the correspondence $f \mapsto f(1)$ where $f \in \mathrm{Hom}_{A^e}(A, M)$. For any other two sided A/R-supermodule N and any $g \in \mathrm{Hom}_{A^e}(M, N)$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{A^e}(A, M) & \xrightarrow{g\circ(\ )} & \mathrm{Hom}_{A^e}(A, N) \\ \downarrow & & \downarrow \\ M^A & \xrightarrow{g|_{M^A}} & N^A \end{array}$$

commutes.

Proof If $f_\gamma \in (\mathrm{Hom}_{A^e}(A, M))_\gamma$ then

$$\begin{aligned} a_\alpha(f_\gamma(1)) &= (-1)^{\alpha\gamma} f_\gamma(1)(a_\alpha \otimes 1) \\ &= (-1)^{\alpha\gamma} f_\gamma(1.(a_\alpha \otimes 1)) \\ &= (-1)^{\alpha\gamma} f_\gamma(a_\alpha) \\ &= (-1)^{\alpha\gamma} f_\gamma(1.(1 \otimes a_\alpha)) \\ &= (-1)^{\alpha\gamma} f_\gamma(1)(1 \otimes a_\alpha) \\ &= (-1)^{\alpha\gamma} f_\gamma(1) a_\alpha. \end{aligned}$$

Therefore, $f_\gamma(1) \in M^A_\gamma$. So $f(1) \in M^A$, $\forall f \in \mathrm{Hom}_{A^e}(A, M)$. Conversely, for any $m_\alpha \in M^A_\alpha$ the function $f_\alpha : A \to M$, defined by $f_\alpha(a_\beta) = m_\alpha a_\beta$, is an element of $(\mathrm{Hom}_{A^e}(A, M))_\alpha$. To see this, we have

$$\begin{aligned} f_\alpha(a_\beta(x_\gamma \otimes y_\delta)) &= m_\alpha(a_\beta(x_\gamma \otimes y_\delta)) \\ &= (-1)^{\gamma\beta} m_\alpha x_\gamma a_\beta y_\delta \end{aligned}$$

and also we have

$$\begin{aligned} f_\alpha(a_\beta)(x_\gamma \otimes y_\delta) &= (m_\alpha a_\beta)(x_\gamma \otimes y_\delta) \\ &= (-1)^{\gamma(\alpha+\beta)} x_\gamma m_\alpha a_\beta y_\delta \\ &= (-1)^{\gamma\beta} m_\alpha x_\gamma a_\beta y_\delta. \end{aligned}$$

Therefore the mapping $f \mapsto f(1)$ is one to one and onto M^A. The rest follows immediately. □

REMARK : In categorical terms the lemma establishes that the left exact covariant functors $\mathrm{Hom}_{A^e}(A, ..)$ and $(..)^A$ are naturally equivalent.

Corollary 5.1.4 $\mathrm{Hom}_{A^e}(A, A) \cong \hat{Z}(A)$, the super-center of A, under the correspondence $f \mapsto f(1)$.

Proof Set $A = M$ in the lemma and note that $A^A = \hat{Z}(A)$. □

Corollary 5.1.5 Let $(J : 0)$ denote the left annihilator of J in A^e. Then

$$\mathrm{Hom}_{A^e}(A, A^e) \cong (J : 0)$$

and, if A is R-separable, $\mu(J : 0) = \hat{Z}(A)$.

Proof Setting $M = A^e$ in the lemma, we have

$$\begin{aligned}
(A^e_\gamma)^A &= \{x_\gamma \in A^e_\gamma \mid a_\alpha x_\gamma = (-1)^{\alpha\gamma} x_\gamma a_\alpha \ \forall a_\alpha \in A_\alpha\} \\
&= \{x_\gamma \in A^e_\gamma \mid (-1)^{\alpha\gamma} x_\gamma (a_\alpha \otimes 1) = (-1)^{\alpha\gamma} x_\gamma (1 \otimes a_\alpha) \ \forall a_\alpha \in A_\alpha\} \\
&= \{x_\gamma \in A^e_\gamma \mid x_\gamma[(a_\alpha \otimes 1) - (1 \otimes a_\alpha)] = 0 \ \forall a_\alpha \in A_\alpha\} \\
&= (J : 0)_\gamma.
\end{aligned}$$

So, $\mathrm{Hom}_{A^e}(A, A^e) \cong (A^e)^A \cong (J : 0)$. Also, $A^e \xrightarrow{\mu} A \to 0$ exact implies that

$$\mathrm{Hom}_{A^e}(A, A^e) \xrightarrow{\mu \circ (..)} \mathrm{Hom}_{A^e}(A, A) \to 0$$

is exact if A is A^e-projective. ($\mathrm{Hom}_{A^e}(A, ..)$ is an exact functor if and only if A is A^e-projective.) Then applying the lemma yields $\mu(J : 0) = \hat{Z}(A)$. □

Corollary 5.1.6 An R-superalgebra A is separable if and only if $(..)^A$ is a right exact functor; that is, if and only if $M \xrightarrow{f_\delta} N \to 0$ exact implies $M^A \xrightarrow{f_\delta|} N^A \to 0$ exact, where $f_\delta \mid$ is the restriction of f_δ to M^A, for every pair of two-sided A/R-supermodules M and N and every A^e-epimorphism f_δ.

Proof We know that $Hom_{A^e}(A, ..)$ is right exact if and only if A is A^e-projective, that is, if and only if A is R-separable. But the lemma yields that $(..)^A$ is right exact if and only if $Hom_{A^e}(A, ..)$ is. $\square$

We now need to investigate the behavior of separable superalgebras under certain formal operations, for example the taking of graded tensor product. The next five propositions set out the crucial properties and in each case we use the separability criterion of Corollary 5.1.6 as a tool with proof.

Proposition 5.1.7 Let S_i be super-commutative R-superalgebras (i=1,2). Let A_i be separable S_i-superalgebras (i=1,2). Assume that $A_1\hat{\otimes}_R A_2 \neq \{0\}$ and $S_1\hat{\otimes}_R S_2 \neq \{0\}$. Then $A_1\hat{\otimes}_R A_2$ is a separable $S_1\hat{\otimes}_R S_2$-superalgebra under the operation induced by $(s_\gamma \otimes t_\delta).(a_\alpha \otimes b_\beta) = (-1)^{\delta\alpha} s_\gamma a_\alpha \otimes t_\delta b_\beta$.

Proof It is easy to check that $(s_\gamma \otimes t_\delta).(a_\alpha \otimes b_\beta) = (-1)^{(\delta+\gamma)(\alpha+\beta)}(a_\alpha \otimes b_\beta).(s_\gamma \otimes t_\delta)$. So $A_1\hat{\otimes}A_2$ is $S_1\hat{\otimes}S_2$-superalgebra. Now, let $M \xrightarrow{f_\zeta} N \to 0$ be an exact sequence of two-sided $A_1\hat{\otimes}A_2/S_1\hat{\otimes}S_2$-supermodules. Then the natural map $A_1 \mapsto A_1\hat{\otimes}A_2$ given by $a \mapsto a \otimes 1$ endows M and N with a structure as two-sided A_1/S_1-supermodules. Therefore

$$M^{A_1} \xrightarrow{f_\zeta|} N^{A_1} \to 0$$

is exact by Corollary 5.1.6 since A_1 is S_1-separable. Now, the natural image of A_2 in $A_1\hat{\otimes}A_2$ super-commute with the image of A_1, therefore M^{A_1} and N^{A_1} are two-sided A_2/S_2-supermodules, under the operation induced from $1 \otimes A_2$, so again applying Corollary 5.1.6, we have

$$(M^{A_1})^{A_2} \xrightarrow{f_\zeta|} (N^{A_1})^{A_2} \to 0$$

is exact by the S_2-separability of A_2. But $(M^{A_1})^{A_2} = M^{A_1\hat{\otimes}A_2}$ and $(N^{A_1})^{A_2} = N^{A_1\hat{\otimes}A_2}$, implying that $A_1\hat{\otimes}A_2$ is $S_1\hat{\otimes}S_2$-separable. $\square$

Corollary 5.1.8 Let A be a separable R-superalgebra and S any super-commutative R-superalgebra. Then $S\hat{\otimes}_R A$ is an S-separable.

Proof Set $A = A_2,\ R = S_2,\ S = S_1 = A_1$ in the proposition. $\square$

We can also prove a converse to the proposition.

Proposition 5.1.9 Let S_i be super-commutative R-superalgebras (i=1,2). Let A_i be any S_i-superalgebra (i=1,2) such that $A_1\hat{\otimes}_R A_2$ is separable over $S_1\hat{\otimes}_R S_2$. Then, if A_1 is faithful as an R-supermodule and contains R as an R-direct summand, then A_2 is separable over S_2.

Proof Let M be any two-sided A_2/S_2-supermodule, then

$$A_1\hat{\otimes}_R M$$

is a two-sided $A_1\hat{\otimes}_R A_2/S_1\hat{\otimes}_R S_2$-supermodule. Since R is an R-direct summand of A_1, we have

$$A_1\hat{\otimes}_R M \cong (L\hat{\otimes}_R M) \oplus (R\hat{\otimes}_R M)$$

as two-sided A_2/S_2-supermodules where L is an R-subsupermodule of A_1. Therefore $M \cong R\hat{\otimes}_R M$ can be identified with a direct summand of $A_1\hat{\otimes}_R M$. Moreover, since the projection of $A_1\hat{\otimes}_R M$ onto M is a two-sided A_2/S_2-homomorphism, it is easily seen that under this map, $(A_1\hat{\otimes}_R M)^{A_1\hat{\otimes}_R A_2}$ is projected onto M^{A_2}. Now consider the commutative diagram

$$\begin{array}{ccccc}
A_1\hat{\otimes}_R M & \xrightarrow{1\otimes f_\delta} & A_1\hat{\otimes}_R N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
M & \xrightarrow{f_\delta} & N & \longrightarrow & 0
\end{array}$$

where the vertical maps are the projections and the rows are exact. This gives rise to the commutative diagram

$$\begin{array}{ccccc}
(A_1\hat{\otimes}_R M)^{A_1\hat{\otimes} A_2} & \xrightarrow{1\otimes f_\delta|} & (A_1\hat{\otimes}_R N)^{A_1\hat{\otimes} A_2} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
M^{A_2} & \xrightarrow{f_\delta|} & N^{A_2} & &
\end{array}$$

where the vertical maps are still projections and the top row is exact since $A_1\hat{\otimes}_R A_2$ is $S_1\hat{\otimes}_R S_2$-separable. it follows that $M^{A_2} \xrightarrow{f_\delta|} N^{A_2} \to 0$ is exact, so A_2 is S_2-separable. $\square$

Corollary 5.1.10 If A_i is an R-superalgebra (i=1,2), with $A_1\hat{\otimes}_R A_2$ separable over R, and A_1 contains R as an R-direct summand, then A_2 is R-separable.

Proof Take $S_1 = S_2 = R$ in the proposition. $\square$

Corollary 5.1.11 If A is an R-superalgebra and S is any super-commutative R-superalgebra containing R as an R-direct summand, then A is R-separable if $S\hat{\otimes}_R A$ is S-separable. Moreover, if $S\otimes 1$ is the super-center of $S\hat{\otimes}_R A$, i.e., $S\otimes 1 = \hat{Z}(S\hat{\otimes}_R A)$, then $R.1$ is the super-center of A.

Proof To obtain the first conclusion, set $A_2 = A$, $S_2 = R$ and $A_1 = S_1 = S$. To see the second, we have by the proof of the proposition that $S \otimes 1 = \hat{Z}(S\hat{\otimes}_R A) = (S\hat{\otimes}_R A)^{S\hat{\otimes}_R A}$ is projected onto $A^A = \hat{Z}(A)$. But $S \otimes 1$ is projected onto $R \otimes 1 \cong R$.$\square$

Remark : If $I = I_{\bar{0}} + I_{\bar{1}}$ is any superideal of R such that $I.1 = 0$ then A is naturally an R/I-superalgebra. Moreover, $A^\circ\hat{\otimes}_R A = A^\circ\hat{\otimes}_{R/I} A$. So A is R-separable if and only if A is R/I-separable.

Proposition 5.1.12 Let A be a separable R-superalgebra and let $\mathcal{U}$ be a two-sided superideal of A. Then $A/\mathcal{U}$ is a separable R-superalgebra (and hence, by the preceding remark, $A/\mathcal{U}$ is separable $R.1/(R.1 \cap \mathcal{U})$-superalgebra). Furthermore, $\hat{Z}(A/\mathcal{U}) = \frac{\hat{Z}(A)+\mathcal{U}}{\mathcal{U}}$.

Proof It is easy to see that every two-sided $(A/\mathcal{U})/R$-supermodule is naturally a two-sided A/R-supermodule under the action $m_\gamma a_\alpha = m_\gamma(a_\alpha + \mathcal{U})$ and $a_\alpha m_\gamma = (a_\alpha + \mathcal{U})m_\gamma$. Under this arrangement, $M^A = M^{A/\mathcal{U}}$, so $A/\mathcal{U}$ is R-separable. Now, $A \to A/\mathcal{U} \to 0$ is an exact sequence of two-sided A/R-supermodules, so separability of A over R implies

$$A^A \to (A/\mathcal{U})^A \to 0$$

is exact, but $A^A = \hat{Z}(A)$ and $(A/\mathcal{U})^A = \hat{Z}(A/\mathcal{U})$, so $\hat{Z}(A/\mathcal{U})$ is the image of $\hat{Z}(A)$ under the natural map, that is $\hat{Z}(A/\mathcal{U}) = \frac{\hat{Z}(A)+\mathcal{U}}{\mathcal{U}}$. $\square$

Proposition 5.1.13 (Transitivity of Separability) Let S be a supercommutative separable R-superalgebra and let A be S-separable. Then A is naturally an R-superalgebra and is R-separable. If, on the other hand, A is given to be R-separable and S is any R-subsuperalgebra of $\hat{Z}(A)$ then A is separable over S.

Proof Any two-sided A/R-supermodule M is a two-sided S/R-supermodule. Moreover, for $x_\delta \in (M^S)_\delta$, $a_\alpha \in A_\alpha$, $s_\gamma \in S_\gamma$ $s_\gamma(a_\alpha x_\delta) = (-1)^{\gamma(\alpha+\delta)}(a_\alpha x_\delta)s_\gamma$. So, $a_\alpha x_\delta \in (M^S)_{\alpha+\delta}$. Similarly, for $x_\delta a_\alpha$. It follows that M^S is a two-sided A/S-supermodule, with $(M^S)^A = M^A$ (as is easy checked). Therefore, the separability of A over S, and the separability of S over R gives, by corollary 5.1.6, that $M \xrightarrow{f_\beta} N \to 0$ exact implies that $(M^S)^A = M^A \xrightarrow{f_\beta|} (N^S)^A = N^A \to 0$ is exact. Hence A is R-separable. To prove the final statement of the proposition, consider M to be any A/S-supermodule then M is A/R-supermodule, so $M \xrightarrow{f_\beta} N \to 0$ is an exact sequence of A/S-supermodules implies that $M^A \xrightarrow{f_\beta|} N^A \to 0$ is exact, so A is S-separable. □

Proposition 5.1.14 Let A_i be any R_i-superalgebra, where i=1,2, and where R_i is a super-commutative super-ring (i=1,2). Then $A_1 \oplus A_2$ is $(R_1 \oplus R_2)$-separable if and only if both A_1 and A_2 are separable over R_1 and R_2 respectively.

Proof ($\Leftarrow$) Let $M \xrightarrow{f_\beta} N \to 0$ be a right exact sequence of two-sided $A_1 \oplus A_2/R_1 \oplus R_2$-supermodules. Then $M^{A_1 \oplus A_2} = (M^{A_1})^{A_2}$, so by R_1-separability of A_1,

$$M^{A_1} \xrightarrow{f_\beta|} N^{A_1} \to 0$$

is a right exact sequence of two-side A_2/R_2-supermodules, again by R_2-separability of A_2

$$(M^{A_1})^{A_2} \xrightarrow{f_\beta|} (N^{A_1})^{A_2} \to 0$$

is exact, so $A_1 \oplus A_2$ is $(R_1 \oplus R_2)$-separable.
($\Rightarrow$) Use corollary 5.1.6 and proposition 5.1.12. □

5.2 Separable Superalgebras over Fields

We devote this section to proving the connection between the present and the classical definitions of separability in the case that the coefficient super-ring R is a field. We begin by extending a general result of Villamayor and Zelinsky to the superalgebra case to show that A is R-separable if and only if A is classically separable and the dimension of A as a vector space over R is finite. But first we start with a definition.

Definition 5.2.1 *Let R be a field. An R-superalgebra A is called* classically separable *if the Jacobson radical of $A\hat{\otimes}_R K$ is zero (i.e., $A\hat{\otimes}_R K$ is semi-simple) for every field extension K of R.*

Proposition 5.2.2 Let A be a separable R-superalgebra which is projective as an R-supermodule. Then A is finitely generated as an R-supermodule.

Proof We start with the case $R_{\bar{1}} \neq \{0\}$.
Let $\{f_i^\gamma,\ a_i^\gamma\}$ be a dual basis for A° over R, where A° is a right R-supermodule and $a_i^\gamma \in A_\gamma^\circ$ and $f_i^\gamma \in (\mathrm{Hom}(A^\circ, R))_\gamma$. (Clearly A° is projective as an R-supermodule if A is.) Then, for any $a_\alpha \in A_\alpha$, $a_\alpha = \sum_i a_i^\gamma f_i^\gamma(a_\alpha)$, where $f_i^\gamma(a_\alpha) = 0$ for all but finitely many i. If we identify $R\hat{\otimes}_R A$ with A, then $f_i^\gamma \otimes 1 \in \mathrm{Hom}_A(A^e, A)_\gamma$, so $\{f_i^\gamma \otimes 1, a_i^\gamma \otimes 1\}$ forms a dual basis for $A^e = A^\circ \hat{\otimes}_R A$ as a projective right A-supermodule, that is,

$$u_\delta = \sum_i (a_i^\gamma \otimes 1).(f_i^\gamma \otimes 1)(u_\delta),$$

for any $u_\delta \in A_\delta^e$, implies that, for $u_\delta = e.(a_\delta \otimes 1)$,

$$\begin{aligned}\mu(e.(a_\delta \otimes 1)) = 1.(a_\delta \otimes 1) = a_\delta &= \mu(\sum_i (a_i^\gamma \otimes 1).(f_i^\gamma \otimes 1)(e.(a_\delta \otimes 1))) \\ &= \sum_i a_i^\gamma.(f_i^\gamma \otimes 1)(e.(a_\delta \otimes 1)). \qquad (13)\end{aligned}$$

Since $e.(a_\delta \otimes 1 - 1 \otimes a_\delta) = 0$, we have $(f_i^\gamma \otimes 1)(e.(a_\delta \otimes 1)) = (f_i^\gamma \otimes 1)(e).(1 \otimes a_\delta)$. So, the set of subscripts i for which $(f_i^\gamma \otimes 1)(e.(a_\delta \otimes 1)) \neq 0$ is contained in the finite set of subscripts for which $(f_i^\gamma \otimes 1)(e) \neq 0$ (independent of a_δ). Therefore, the summation in (13) may be taken over a fixed finite set. Writing $e = \sum_j (x_j \otimes y_j) + \sum_k (x_k' \otimes y_k')$, where $x_j, y_j \in A_{\bar{0}}$, $x_k', y_k' \in A_{\bar{1}}$, we have, from (13),

$$\begin{aligned} a_\delta &= \sum_{i,j,k} a_i^\gamma.(f_i^\gamma \otimes 1)(x_j a_\delta \otimes y_j + (-1)^\delta x_k' a_\delta \otimes y_k') \\ &= \sum_{i,j,k} a_i^\gamma.(f_i^\gamma(x_j a_\delta) \otimes y_j + (-1)^\delta f_i^\gamma(x_k' a_\delta) \otimes y_k') \\ &= \sum_{i,j,k} (-1)^{\delta\gamma+\gamma} f_i^\gamma(x_j a_\delta) a_i^\gamma y_j + (-1)^{\gamma\delta+\delta} f_i^\gamma(x_k' a_\delta) a_i^\gamma y_k' \\ &= \sum_{i,j,k} (-1)^{\delta\gamma}[(-1)^\gamma f_i^\gamma(x_j a_\delta) a_i^\gamma y_j + (-1)^\delta f_i^\gamma(x_k' a_\delta) a_i^\gamma y_k']. \end{aligned}$$

Therefore the union of the finite sets $\{a_i^\gamma y_j\} \cup \{a_i^\gamma y_k'\}$ generates A° (and therefore A) over R.

For the case $R_{\bar{1}} = \{0\}$, let $\{f_{i,\bar{0}},\ a_{i,\bar{0}}\}$, $\{f_{j,\bar{1}},\ a_{j,\bar{1}}\}$ be a dual basis for $(A^\circ)_{\bar{0}}$, $(A^\circ)_{\bar{1}}$ over R (resp.). If we identify $R\hat{\otimes}_R A$ with A, then $f_{i,\bar{0}} \otimes 1 \in Hom_A(A^e, A)_{\bar{0}}$ for all i and $f_{j,\bar{1}} \otimes 1 \in Hom_A(A^e, A)_{\bar{1}}$ for all j, so $\{f_{i,\bar{0}} \otimes 1, a_{i,\bar{0}} \otimes 1\} \cup \{f_{j,\bar{1}} \otimes 1, a_{j,\bar{1}} \otimes 1\}$ forms a dual basis for $A^e = A^\circ\hat{\otimes}_R A$ as a projective right A-module, that is,

$$u_\delta = \sum_i (a_{i,\bar{0}} \otimes 1).(f_{i,\bar{0}} \otimes 1)(u_\delta) + \sum_j (a_{j,\bar{1}} \otimes 1).(f_{j,\bar{1}} \otimes 1)(u_\delta),$$

for any $u_\delta \in A^e_\delta$, implies that, for $u_\delta = e.(a_\delta \otimes 1)$,

$$\begin{aligned}
\mu(e.(a_\delta \otimes 1)) = 1.(a_\delta \otimes 1) = a_\delta &= \mu(\sum_i (a_{i,\bar{0}} \otimes 1).(f_{i,\bar{0}} \otimes 1)(e.(a_\delta \otimes 1))) \\
&+ \mu(\sum_j (a_{j,\bar{1}} \otimes 1).(f_{j,\bar{1}} \otimes 1)(e.(a_\delta \otimes 1))) \\
&= \sum_i a_{i,\bar{0}}.(f_{i,\bar{0}} \otimes 1)(e.(a_\delta \otimes 1)) \\
&+ \sum_j a_{j,\bar{1}}.(f_{j,\bar{1}} \otimes 1)(e.(a_\delta \otimes 1)). \qquad (14)
\end{aligned}$$

Since $e.(a_\delta \otimes 1 - 1 \otimes a_\delta) = 0$, we have $(f_{i,\bar{0}} \otimes 1)(e.(a_\delta \otimes 1)) = (f_{i,\bar{0}} \otimes 1)(e).(1 \otimes a_\delta)$, and $(f_{j,\bar{1}} \otimes 1)(e.(a_\delta \otimes 1)) = (f_{j,\bar{1}} \otimes 1)(e).(1 \otimes a_\delta)$. So, the set of subscripts i for which $(f_{i,\bar{0}} \otimes 1)(e.(a_\delta \otimes 1)) \neq 0$ is contained in the finite set of subscripts for which $(f_{i,\bar{0}} \otimes 1)(e) \neq 0$ (independent of a_δ), similarly the set of subscripts j for which $(f_{j,\bar{1}} \otimes 1)(e.(a_\delta \otimes 1)) \neq 0$ is contained in the finite set of subscripts for which $(f_{j,\bar{1}} \otimes 1)(e) \neq 0$ (independent of a_δ). Therefore, the summation in (14) may be taken over a fixed finite set. Writing

$e = \sum_h (x_h \otimes y_h) + \sum_k (x'_k \otimes y'_k)$ where $x_h, y_h \in A_{\bar{0}}$, $x'_k, y'_k \in A_{\bar{1}}$, we have, from (14),

$$\begin{aligned} a_\delta &= \sum_{i,h,k} a_{i,\bar{0}}.(f_{i,\bar{0}} \otimes 1)(x_h a_\delta \otimes y_h + (-1)^\delta x'_k a_\delta \otimes y'_k) \\ &+ \sum_{j,h,k} a_{i,\bar{1}}.(f_{j,\bar{1}} \otimes 1)(x_h a_\delta \otimes y_h + (-1)^\delta x'_k a_\delta \otimes y'_k) \\ &= \sum_{i,h,k} a_{i,\bar{0}}.(f_{i,\bar{0}}(x_h a_\delta) \otimes y_h + (-1)^\delta f_{i,\bar{0}}(x'_k a_\delta) \otimes y'_k) \\ &+ \sum_{j,h,k} a_{j,\bar{1}}.(f_{j,\bar{1}}(x_h a_\delta) \otimes y_h + (-1)^\delta f_{j,\bar{1}}(x'_k a_\delta) \otimes y'_k) \\ &= \sum_{i,h,k} f_{i,\bar{0}}(x_h a_\delta) a_{i,\bar{0}} y_h + (-1)^\delta f_{i,\bar{0}}(x'_k a_\delta) a_{i,\bar{0}} y'_k \\ &+ \sum_{j,h,k} (-)^{\delta+1} f_{j,\bar{1}}(x_h a_\delta) a_{j,\bar{1}} y_h + f_{j,\bar{1}}(x'_k a_\delta) a_{j,\bar{1}} y'_k. \end{aligned}$$

Therefore the union of finite sets $\{a_{i,\bar{0}} y_h\} \cup \{a_{i,\bar{0}} y'_k\} \cup \{a_{j,\bar{1}} y_h\} \cup \{a_{j,\bar{1}} y'_k\}$ generates A° (and therefore A) over R. □

Corollary 5.2.3 Let A be a separable R-superalgebra where R is a field. Then the dimension of A as a vector space over R is finite. □

Proposition 5.2.4 Let A be a separable R superalgebra. Any left A-supermodule M which is R-projective, under its induced of R-supermodule structure, is A-projective.

Proof We start with case $R_{\bar{1}} \neq \{0\}$.

Let $0 \to L \to N \xrightarrow{\eta_\gamma} M \to 0$ be any exact sequence of left A-supermodules. We need to show that this sequence splits over A. We know that this sequence splits over R, so there exists an R-supermodule homomorphism ψ_γ from M to N with $\psi_\gamma \eta_\gamma = id_M$. Since both N and M are left A-supermodules, $\mathrm{Hom}_R(M, N)$ can be given the structure of right $A^e = A^\circ \hat{\otimes}_R A$-supermodule under the operation induced by

$$m_\epsilon^{f_\gamma.(a_\alpha \otimes b_\beta)} = (-1)^{(\epsilon+\gamma)(\alpha+\beta)} a_\alpha (b_\beta m_\epsilon)^{f_\gamma}$$

$\forall a_\alpha, b_\beta \in A$, $f_\gamma \in Hom_R(M, N)$, $m_\epsilon \in M_\epsilon$. To see this :

$$\begin{aligned} (r_\delta m_\epsilon)^{f_\gamma.(a_\alpha \otimes b_\beta)} &= (-1)^{(\delta+\epsilon+\gamma)(\alpha+\beta)} a_\alpha (b_\beta r_\delta m_\epsilon)^{f_\gamma} \\ &= (-1)^{(\epsilon+\gamma)(\alpha+\beta)} r_\delta a_\alpha (b_\beta m_\epsilon)^{f_\gamma} \\ &= r_\delta (m_\epsilon^{f_\gamma.(a_\alpha \otimes b_\beta)}) \end{aligned}$$

and,

$$\begin{aligned} m_\epsilon^{f_\gamma.((a_\alpha\otimes b_\beta)(c_\delta\otimes d_\mu))} &= (-1)^{\delta\beta} m_\epsilon^{f_\gamma.(a_\alpha\circ c_\delta\otimes b_\beta d_\mu)} \\ &= (-1)^{\delta\beta+(\epsilon+\gamma)(\alpha+\delta+\beta+\mu)}(a_\alpha\circ c_\delta)(b_\beta d_\mu m_\epsilon)^{f_\gamma} \end{aligned}$$

On the other hand

$$\begin{aligned} m_\epsilon^{(f_\gamma.(a_\alpha\otimes b_\beta))(c_\delta\otimes d_\mu)} &= (-1)^{(\alpha+\beta+\epsilon+\gamma)(\delta+\mu)} c_\delta(d_\mu m_\epsilon)^{f_\gamma.(a_\alpha\otimes b_\beta)} \\ &= (-1)^{(\alpha+\beta+\epsilon+\gamma)(\delta+\mu)}(-1)^{(\epsilon+\gamma+\mu)(\alpha+\beta)} c_\delta a_\alpha (b_\beta d_\mu m_\epsilon)^{f_\gamma} \\ &= (-1)^{\delta\beta+(\epsilon+\gamma)(\alpha+\delta+\beta+\mu)}(a_\alpha\circ c_\delta)(b_\beta d_\mu m_\epsilon)^{f_\gamma}. \end{aligned}$$

Therefore $\mathrm{Hom}_R(M,N)$ is a right A^e-supermodule. Now, A is R-separable, so let e be a separability idempotent for A. Let $\psi'_\gamma = \psi_\gamma.e$, that is if $e=\sum x_i\otimes y_i$ then

$$\begin{aligned} (m_\delta^{\psi'_\gamma})^{\eta_\gamma} &= (\sum_i m_\delta^{\psi_\gamma(x_i\otimes y_i)})^{\eta_\gamma} \\ &= (\sum_i x_i(y_i m_\delta)^{\psi_\gamma})^{\eta_\gamma} \\ &= \sum_i x_i(y_i m_\delta)^{\psi_\gamma\eta_\gamma} \\ &= \sum_i x_i y_i m_\delta \\ &= \mu(e).m_\delta \\ &= 1.m_\delta \\ &= m_\delta. \end{aligned}$$

Since $e.J=\{0\}$, we have $\psi'_\gamma(a_\alpha\otimes 1 - 1\otimes a_\alpha) = \psi_\gamma.e(a_\alpha\otimes 1-1\otimes a_\alpha)=0,\ \forall a_\alpha\in A_\alpha$, so

$$m_\delta^{\psi'_\gamma(a_\alpha\otimes 1)} = m_\delta^{\psi'_\gamma(1\otimes a_\alpha)},$$

which implies that $(-1)^{(\delta+\gamma)\alpha}a_\alpha(m_\delta)^{\psi'_\gamma} = (-1)^{(\delta+\gamma)\alpha}(a_\alpha m_\delta)^{\psi'_\gamma}$ and therefore

$$(a_\alpha m_\delta)^{\psi'_\gamma} = a_\alpha(m_\delta)^{\psi'_\gamma}.$$

So $\psi'_\gamma\in\mathrm{Hom}_A(M,N)$ and hence, the sequence splits over A.
For the case $R_{\bar{1}}=0$, let $0\to L\to N\overset{\eta_\gamma}{\to} M\to 0$ be any exact sequence of left

A-supermodules. We need to show that this sequence A-splits. But

$$0 \to L_{\bar{0}} \to N_{\bar{0}} \overset{\eta_\gamma}{\to} M_\gamma \to 0 \quad and 0 \to L_{\bar{1}} \to N_{\bar{1}} \overset{\eta_\gamma}{\to} M_{\gamma+\bar{1}} \to 0$$

are right exact sequences of left R-modules, since $M_{\bar{0}}$, $M_{\bar{1}}$ are R-projective, these sequences are R-split. Therefore there exist R-module homomorphisms g_γ from M_γ to $N_{\bar{0}}$ and h_γ from $M_{\gamma+\bar{1}}$ to $N_{\bar{1}}$ (resp.) with $g_\gamma\eta_\gamma = id_{M_\gamma}$ and $h_\gamma\eta_\gamma = id_{M_{\gamma+\bar{1}}}$. One can easily check that $\psi_\gamma = g_\gamma|_{M_\gamma} \oplus h_\gamma|_{M_{\gamma+\bar{1}}} \in \mathrm{Hom}_R(M, N)$ and $\psi_\gamma\eta_\gamma = id_M$. Again as in the first case $\mathrm{Hom}_R(M, N)$ can be given the structure of right $A^e = A^\circ \hat{\otimes}_R A$-supermodule under the operation induced by

$$m_\epsilon^{f_\gamma.(a_\alpha \otimes b_\beta)} = (-1)^{(\epsilon+\gamma)(\alpha+\beta)} a_\alpha (b_\beta m_\epsilon)^{f_\gamma}$$

$\forall a_\alpha, b_\beta \in A,\ f_\gamma \in \mathrm{Hom}_R(M, N),\ m_\epsilon \in M_\epsilon$.
Now, A is R-separable, let e be a separability idempotent for A, and let $\psi'_\gamma = \psi_\gamma.e$. Then as we shown in the first case $\psi'_\gamma\eta_\gamma = id_M$, $\psi'_\gamma \in \mathrm{Hom}_A(M, N)$, and hence the sequence splits over A. □

Corollary 5.2.5 If A is a separable R-superalgebra where R is a field, then A is classically separable as an R-superalgebra.

Proof Let K be any field extension of R. By the corollary we know that $K\hat{\otimes}_R A$ is K-separable. But every $K\hat{\otimes}_R A$-supermodule is K-projective (since K is a field and every supermodule over a field is free) so by the proposition above every $K\hat{\otimes}_R A$-supermodule is projective. Hence the Jacobson radical of $K\hat{\otimes}_R A$ is zero by the next lemma. □

Lemma 5.2.6 Let $R = R_{\bar{0}} + R_{\bar{1}}$ be an artinian super-ring with 1 such that every R-supermodule is R-projective then the Jacobson radical of R is zero.

Proof Let $I =$ the largest direct sum of minimal right super-ideals of R (since R is artinian it has a minimal right super-ideal).

Claim : $I = R$.
Proof of the claim : Let $\mu : R \to R/I \quad (a \mapsto a + I)$, then $ker\mu = I$. So,

$0 \to I \hookrightarrow R \xrightarrow{\mu} R/I \to 0$ is an exact sequence of right R-supermodules, so it splits, and therefore there exists $\psi : R/I \to R$ such that $\mu\psi = id_{R/I}$. Let $I' = \psi(R/I)$, then $R = I \oplus I'$ where I' is a right super-ideal of R generated by one element. If $I' \neq 0$ let $I'' \subsetneq I'$ be a maximal right super-ideal of I'. Then $I' = I'' \oplus S$, for some $S \cong I'/I''$. Since $S \neq 0$ is a minimal right super-ideal of R, $S \subset I$, a contradiction. Therefore $I' = 0$ and $R = I$.

Since R is artinian we have $R = I_1 \oplus I_2 \oplus .... \oplus I_n$ of minimal right super-ideals of R, therefore the Jacobson radical of R is zero. □

Examples :

(1) Let $R = R_{\bar{0}} + R_{\bar{1}}$ be a super-commutative super-ring. Then $A = M_n(R) = M_n(R_{\bar{0}}) \oplus M_n(R_{\bar{1}})$, is R-separable.

Proof Let e_{ij} denote the matrix having 1 in the (i,j)-th spot and 0 elsewhere. Then it is easy to check that

$$e = \sum_{1 \leq i \leq n} e_{ij} \otimes e_{ji}$$

for a fixed j between 1 and n is a separability idempotent for R. □

(2) Let K be any field of characteristic not equal to 2 and $A = M_n(K) \oplus M_n(K)v$ where $v^2 \in K^\times$. Then $\hat{Z}(A) = K$ and A is K-separable.

Proof Let e_{ij} denote the matrix having 1 in the (i,j)-th spot and 0 elsewhere, and let

$$e = \frac{1}{2}[\sum_{1 \leq i \leq n} e_{ij} \otimes e_{ji} + \frac{1}{v^2} \sum_{1 \leq i \leq n} ve_{ij} \otimes ve_{ji}].$$

Then $\mu(e) = 1$ and

$$\begin{aligned} e(1 \otimes e_{kl} - e_{kl} \otimes 1) &= (e_{kj} \otimes e_{jl} - e_{kj} \otimes e_{jl}) + \frac{1}{v^2}(ve_{kj} \otimes e_{jl} - ve_{kj} \otimes e_{jl}) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

and also,

$$\begin{aligned} e(1 \otimes ve_{kl} - ve_{kl} \otimes 1) &= (e_{kj} \otimes ve_{jl} - ve_{kj} \otimes e_{jl}) + \frac{1}{v^2}(ve_{kj} \otimes v^2 e_{jl} - v^2 e_{kj} \otimes ve_{jl}) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Therefore $eJ = 0$ where $J = ker\mu$ and hence A is K-separable. □

(3) In example (2), if K is a field of characteristic equal to 2 and $A = M_n(K) \oplus M_n(K)v$, then $\hat{Z}(A) = Z(A) = K + Kv$, where $v^2 = 0$ since $\hat{Z}(A)$ is a super-commutative super-ring. Therefore A is not classically separable since the Jacobson radical of A itself not equal to zero. □

(4) If A is classically separable finite dimensional over any field R of any characteristic then A is R-separable.

Proof Since $\overline{R}\hat{\otimes}_R A$, where $\overline{R}$ is an algebraic closure of R, is a finite dimensional over $\overline{R}$ having zero radical, we have $\overline{R}\hat{\otimes}_R A$, is a semi-simple artinian super-ring and therefore by [21, theorem 4]

$$\overline{R}\hat{\otimes}_R A \cong \bigoplus_{i=1}^{k} M_{n_i}(D_i)$$

where each D_i is a finite dimensional division super-ring over $\overline{R}$, so each D_i is either of the form $\overline{R} + \overline{R}v_i$ or $D_i = \overline{R}$ since $\overline{R}$ is algebraically closed field. Therefore, by examples (2) and (3) each $M_{n_i}(D_i)$ is $\overline{R}$-separable, so $\overline{R}\hat{\otimes}_R A$ is $\underbrace{\overline{R} \oplus \overline{R} \oplus ... \oplus \overline{R}}_{k\ times}$-separable. Now, $\underbrace{\overline{R} \oplus \overline{R} \oplus ... \oplus \overline{R}}_{k\ times}$ is $\overline{R}$-separable (easy to check) and so by proposition 5.1.13 $\overline{R}\hat{\otimes}_R A$ is $\overline{R}$-separable, and hence from corollary 5.1.11 A is R-separable. □

Theorem 5.2.7 A superalgebra A over any field R is separable if and only if A is classically separable over R and the dimension of A over R is finite. □

5.3 Central Separable Superalgebras

We have seen that if A is a separable R-superalgebra, then A is separable over its center. We now turn our attention to this type of superalgebras.
An R-superalgebra A is called *central* if A is faithful as an R-supermodule and $R.1$ coincides with $\hat{Z}(A)$. When dealing with faithful superalgebras, we identify R with $R.1$ and therefore consider R as a subsuper-ring of $\hat{Z}(A)$. We call A a *central separable R-superalgebra (or Azumaya superalgebra)* if A is both central and separable.

Proposition 5.3.1 Let R be any field and A be an R-superalgebra. Then A is central simple R-superalgebra finite dimensional over R if and only if A is central separable R-superalgebra.

Proof Assume that A is a central separable R-superalgebra then, by Theorem 5.2.7, A is semi-simple finite dimensional over R, and since $\hat{Z}(A) = R$, A can have only one simple component. Therefore A is central simple R-superalgebra.
Conversely, if A is a finite dimensional central simple R-superalgebra then by [16, Theorem 2.3(2)] $K\hat{\otimes}_R A$ is a finite dimensional central simple K-superalgebra, for every field extension K of R. Therefore A is classically separable and so by Theorem 5.2.7 A is a central separable R-superalgebra. □

For any R-superalgebra A we have seen that A is naturally a right A^e-supermodule. This induces an R-superalgebra homomorphism ϕ from A^e to $\mathrm{Hom}_R(A, A)$ by associating to any element $x_\alpha \otimes y_\beta$ of A^e the element $(x_\alpha \otimes y_\beta)^\phi$, where for any $a_\gamma \in A_\gamma$,

$$a_\gamma^{(x_\alpha \otimes y_\beta)^\phi} = a_\gamma.(x_\alpha \otimes y_\beta) = (-1)^{\alpha\gamma} x_\alpha a_\gamma y_\beta.$$

We now set out to prove that, in the case A is a central separable R-superalgebra, this superalgebra homomorphism ϕ is an isomorphism.

Lemma 5.3.2 Let A be a central separable R-superalgebra. Then R is an R-direct summand of A.

Proof Let e be a separability idempotent for A (in A^e). Consider e^ϕ in $\mathrm{Hom}_R(A, A)$. Since ϕ is a super-ring homomorphism e^ϕ is an idempotent. We will show that

$A = Im(e^{\phi}) \oplus \ker(e^{\phi})$. Let $a_\alpha \in A_\alpha$, then $a_\alpha^{e^\phi} \in Im(e^{\phi})$ and $a_\alpha - a_\alpha^{e^\phi} \in \ker(e^{\phi})$, and If $b_\beta \in Im(e^{\phi}) \cap \ker(e^{\phi})$ then $b_\beta = c_\beta^{e^\phi}$ for some $c_\beta \in A_\beta$, and so $b_\beta^{e^\phi} = c_\beta^{e^\phi} = 0 = b_\beta$ which implies that

$$A = Im(e^{\phi}) \oplus \ker(e^{\phi}),$$

as R-subsupermodules of A. Now, since $e.J = 0$, for any $a_\alpha \in A_\alpha$ and $b_\beta \in A_\beta$ we have

$$\begin{aligned}
(b_\beta^{e^\phi})a_\alpha = (b_\beta.e)a_\alpha &= (b_\beta.e).(1 \otimes a_\alpha) \\
&= b_\beta.(e(a_\alpha \otimes 1)) \\
&= (b_\beta.e).(a_\alpha \otimes 1) \\
&= (-1)^{\alpha\beta} a_\alpha (b_\beta.e) \\
&= (-1)^{\alpha\beta} a_\alpha b_\beta^{e^\phi}.
\end{aligned}$$

Therefore, $b_\beta^{e^\phi} \in \hat{Z}(A) = R$. Hence $Im(e^{\phi}) = R$, since $R \subseteq Im(e^{\phi})$. □

Corollary 5.3.3 Let A be a central separable R-superalgebra. If I is any two-sided super-ideal of R, then $IA \cap R = I$.

Proof By Lemma 5.3.2 $A = R \oplus L$, for some R-subsupermodule L of A. Then

$$\begin{aligned}
IA \cap R = I(R \oplus L) \cap R &= (I \oplus IL) \cap R \\
&= I \quad (\text{since } IL \subseteq L).
\end{aligned}$$

□

Proposition 5.3.4 Let A and B be central separable R-superalgebras. Then $A\hat{\otimes}_R B$ is a central separable R-superalgebra.

Proof Since R is an R-direct summand of both A and B, $R\hat{\otimes}_R R \cong R$ is a subsuperring of $A\hat{\otimes}_R B$, so $A\hat{\otimes}_R B \neq \{0\}$. Therefore $A\hat{\otimes}_R B$ is R-separable by Proposition 5.1.7. Now, by the Hom-tensor relation, we have

$$\begin{aligned}
\mathrm{Hom}_{A^e}(A, A)\hat{\otimes}_R \mathrm{Hom}_{B^e}(B, B) &\cong \mathrm{Hom}_{A^e\hat{\otimes}_R B^e}(A\hat{\otimes}_R B, A\hat{\otimes}_R B) \\
&= \mathrm{Hom}_{(A\hat{\otimes}_R B)^e}(A\hat{\otimes}_R B, A\hat{\otimes}_R B) \\
&\cong (A\hat{\otimes}_R B)^{A\hat{\otimes}_R B}.
\end{aligned}$$

But, by Corollary 5.1.4, this is equivalent to

$$A^A\hat{\otimes}_R B^B \cong (A\hat{\otimes}_R B)^{A\hat{\otimes}_R B}$$

or

$$R \cong R\hat{\otimes}_R R \cong \hat{Z}(A\hat{\otimes}_R B).$$

Therefore $A\hat{\otimes}_R B$ is central. $\square$

The next theorem generalizes the result proved by F. DeMeyer to the supercase, see [6, Theorem 3.4].

Theorem 5.3.5 The following statements about an R-superalgebra A are equivalent

(i) A is central separable over R.

(ii) A is an A^e-progenerator and A is R-central.

(iii) A is an R-progenerator as an R-module and the map ϕ from A^e to $\mathrm{Hom}_R(A, A)$ is an isomorphism.

Proof It is clear that (ii) implies (i).
The equivalence of (ii) and (iii) follows from the Morita Theorems. Indeed, if A is R-central we have by Corollary 5.1.4 that $\mathrm{Hom}_{A^e}(A, A) \cong R$; thus, A is an R-progenerator as a module over R if A is an A^e-progenerator as a supermodule over A^e. Moreover, A^e would then be isomorphic by Corollary 4.3.5 to $\mathrm{Hom}_R(A, A)$ under right multiplication, which is precisely the map ϕ, so (ii) implies (iii). The reverse process takes us from (iii) to (ii).

The only remaining implication is (ii) from (i). By definition A is A^e-projective and it is obvious that 1 generates A over A^e, so A is finitely generated over A^e. It remains to show that A is an A^e-generator. To accomplish this we must show that

$$A^*\hat{\otimes}_{R=\mathrm{Hom}_{A^e}(A,A)} A = \mathrm{Hom}_{A^e}(A, A^e)\hat{\otimes}_R A,$$

where $A^* = \mathrm{Hom}_{A^e}(A, A^e)$, is isomorphic to A^e under the map $f \otimes a \mapsto f(a)$. But, by Corollary 5.1.5, A^* is isomorphic to $J : 0$ under map $f \mapsto f(1)$. Therefore, we have to prove that

$$(J : 0)\hat{\otimes} A \cong A^e$$

under $b \otimes a \mapsto b(a \otimes 1) = b(1 \otimes a)$. But this last is easily seen to be equivalent to $(J : 0)A^e = A^e$. To prove this result, we need the following lemma, which we will later improve in Corollary 5.3.9.

Lemma 5.3.6 For every maximal two-sided super-ideal M of a central separable R-superalgebra A, there exists a super-ideal I of R with $IA = M$.

Proof Let $I = M \cap R$ and set $\overline{M} = IA$. Clearly $\overline{M} \subseteq M$ and we wish to show equality. By Proposition 5.1.12, A/M is a central separable superalgebra over R/I. However, A/M is simple (hence, A/M is a division super-ring) since M is maximal, so by the next lemma, its center R/I is a field. However, since $\overline{M} \cap R = I$, by Corollary 5.3.3, we have, also, by Proposition 5.1.12, that $A/\overline{M}$ is a central separable superalgebra over R/I, and so is simple. Therefore, $M/\overline{M}$ is 0, and so $M = \overline{M} = IA$. □

Lemma 5.3.7 Let $R = R_{\bar{0}} + R_{\bar{1}}$ be any super-commutative super-ring. Then for any super-ideal I of R such that R/I is a division super-ring we have $I_{\bar{1}} = R_{\bar{1}}$ and $I_{\bar{0}}$ is a maximal ideal in $R_{\bar{0}}$ and hence R/I is a field.

Proof For any $0 \neq r_{\bar{1}} \in R_{\bar{1}}$, we have $(r_{\bar{1}} + I)^2 = r_{\bar{1}}^2 + I = I$ and so $r_{\bar{1}} \in I_{\bar{1}}$. This implies that $R_{\bar{1}} = I_{\bar{1}}$. Therefore $I = I_{\bar{0}} + R_{\bar{1}}$ and hence $R/I \approx R_{\bar{0}}/I$ which is a field.□

We now complete the proof of Theorem 5.3.5 by assuming $(J : 0)A^e$ is not all of A^e and proceeding to a contradiction. If $(J : 0)A^e$ is a proper super-ideal of A^e then it is contained in some maximal two-sided super-ideal M of A^e. Then A^e is a central separable R-superalgebra by Proposition 5.3.4. Therefore, by Lemma 5.3.6, $M = IA^e$ for some proper super-ideal I of R, so

$$(J : 0)A^e \subseteq IA^e.$$

Applying the multiplication map μ, we get

$$\mu((J : 0)).A^e \subseteq \mu(IA^e) = IA.$$

By Corollary 5.1.5, $\mu((J : 0)) = R$, so $R.A^e = A \subseteq IA$. But $A = IA$ implies $A^e = IA^e = M$, a contradiction. Therefore $(J : 0)A^e = A^e$ and A is an A^e-generator.
□

Corollary 5.3.8 If A is a central separable left R-superalgebra, then for any R-supermodule M, $(M\hat{\otimes}_R A)^A \cong M$ as R-supermodules under the map $m\otimes 1 \leftrightarrow m$, and for any two-sided A/R-supermodule N, $N^A\hat{\otimes}A \cong N$ as two-sided A/R-supermodules under the map $\sum_i n_i \otimes a_i \mapsto \sum_i n_i a_i$.

Proof Since A is an R-progenerator and $Hom_R(A, A) \cong A^e$, we have by the Morita Theorems that

$$(M\hat{\otimes}_R A)\hat{\otimes}_{A^e}\mathrm{Hom}_{A^e}(A, A^e) \cong M.$$

But by the hom-tensor relation:

$$\begin{aligned}(M\hat{\otimes}_R A)\hat{\otimes}_{A^e}\mathrm{Hom}_{A^e}(A, A^e) &\cong \mathrm{Hom}_{A^e}(A, (M\hat{\otimes}_R A)\hat{\otimes}_{A^e} A^e)\\ &\cong \mathrm{Hom}_{A^e}(A, M\hat{\otimes}_R A)\\ &\cong (M\hat{\otimes}_R A)^A.\end{aligned}$$

Therefore $M \cong (M\hat{\otimes}_R A)^A$.
Similarly,

$$\begin{aligned}N &\cong A\hat{\otimes}_R(N\hat{\otimes}_{A^e}\mathrm{Hom}_{A^e}(A, A^e))\\ &\cong A\hat{\otimes}_R\mathrm{Hom}_{A^e}(A, N\hat{\otimes}_{A^e} A^e)\\ &\cong A\hat{\otimes}_R\mathrm{Hom}_{A^e}(A, N)\\ &\cong A\hat{\otimes}_R N^A.\end{aligned}$$

□

Again, the next corollary generalizes the result proved by F. DeMeyer to the super-case, see [6, Corollary 3.7].

Corollary 5.3.9 Let A be a central separable R-superalgebra. Then there is a one to one correspondence between super-ideals I of R and two-sided super-ideals $\mathcal{U}$ of A given by $I \to IA$ and $\mathcal{U} \to \mathcal{U} \cap R$.

Proof By Corollary 5.3.8,

$$I \cong (I\hat{\otimes}_R A)^A \cong (1 \otimes IA)^A \cong (IA)^A = IA \cap R$$

and

$$\mathcal{U} \cong \mathcal{U}^A \hat{\otimes}_R A = (\mathcal{U} \cap R) \hat{\otimes}_R A \cong (\mathcal{U} \cap R)A.$$

This establishes that the correspondence is one-to-one. □

Theorem 5.3.10 An R-superalgebra A is R-separable if and only if A is separable over $\hat{Z}(A)$ and $\hat{Z}(A)$ is a separable R-superalgebra.

Proof Let $C = \hat{Z}(A)$. By transitivity of separability we have that if A is separable over C and C is separable over R then A is R-separable.

Conversely, assuming A is R-separable, we know that A is separable when considered as a superalgebra over its center C. It remains to show that C is R-separable. We will show that C is $C\hat{\otimes}_R C$-projective as a $C\hat{\otimes}_R C$-supermodule, which implies that C is R-separable. Now, since A and A° are central separable over C, A^e is central separable over $C\hat{\otimes}_R C$ (see Proposition 5.1.7) and therefore, by Theorem 5.3.5, A^e is $C\hat{\otimes}_R C$-projective as a module over $C\hat{\otimes}_R C$. But A is projective over A^e, so A is projective as $C\hat{\otimes}_R C$-module. Now, one can easily check that any $C\hat{\otimes}_R C$ direct summand of A is $C\hat{\otimes}_R C$-projective as a module over $C\hat{\otimes}_R C$. However, any C-direct summand of A is a $C\hat{\otimes}_R C$-direct summand. But C is a C-direct summand of A. Hence C is $C\hat{\otimes}_R C$-projective as a module over $C\hat{\otimes}_R C$, since $C_{\bar{0}}.(C\hat{\otimes}_R C)_{\bar{1}} = C_{\bar{1}}$, C is $C\hat{\otimes}_R C$-projective as a supermodule over $C\hat{\otimes}_R C$, which implies that C is R-separable.
□

5.4 The Commutator Theorems

In this section we will prove two striking theorems on the representation of a central separable superalgebras as the tensor product of subsuperalgebras.

To begin, we remark that when A is a central separable R-superalgebra, we have seen that A is an R-progenerator as a module over R, and if $R_{\bar{1}}A_{\bar{0}} = A_{\bar{1}}$ or $R_{\bar{1}}A_{\bar{1}} = A_{\bar{0}}$ or if $R_{\bar{1}} = \{0\}$ then A is an R-progenerator as a supermodule over R. Also we have seen that $Hom_R(A, A)$, being isomorphic to $A^\circ\hat{\otimes}_R A$, is a central separable R-superalgebra.

We have also seen that $Hom_R(E, E)$ is a central separable R-superalgebra when E is a finitely generated, free R-supermodule. The first proposition generalizes both of these facts.

Proposition 5.4.1 Let an R-supermodule E which is R-progenerator as a module over R. Then $A = Hom_R(E, E)$ is a central separable R-superalgebra.

Proof We will show that A satisfies the condition (iii) of Theorem 5.3.5. By Corollary 4.3.3, $E \hat{\otimes}_R E^* \cong \mathrm{Hom}_R(E, E) = A$. But the R-supermodule E^* is finitely generated and projective as a module over R by the Morita Theorems, so $A \cong E \hat{\otimes}_R E^*$ is finitely generated and projective as a module over R. Moreover, it is clear that A is R-faithful since E is R-faithful. Hence A is an R-progenerator by Corollary 4.1.17. Finally we must show that the map ϕ from A^e to $Hom_R(A, A)$ is an isomorphism. By Corollary 4.3.5, we have $A^\circ \cong \mathrm{Hom}_{R^\circ}(E^*, E^*) \cong \mathrm{Hom}_R(E^*, E^*)$. Therefore, using Corollary 4.2.6, we have

$$\begin{aligned} A^\circ \hat{\otimes}_R A &\cong \mathrm{Hom}_R(E^*, E^*) \hat{\otimes}_R \mathrm{Hom}_R(E, E) \\ &\cong \mathrm{Hom}_R(E^* \hat{\otimes}_R E, E^* \hat{\otimes}_R E) \\ &\cong \mathrm{Hom}_R(A, A). \end{aligned}$$

One can easily check that the composition of these isomorphisms is indeed ϕ. □

Corollary 5.4.2 Let A be an R-superalgebra which is a progenerator as an R-module. Then R is an R-direct summand of any subsuperalgebra B of A.

Proof By the proposition above $\mathrm{Hom}_R(A, A)$ is a central separable R-superalgebra. A can be considered a subsuperalgebra of $\mathrm{Hom}_R(A, A)$ by identifying it with set of right multiplication by elements of A. Under this identification we have $R \subseteq B \subseteq A \subseteq \mathrm{Hom}_R(A, A)$. But by Lemma 5.3.2, R is an R-direct summand of $\mathrm{Hom}_R(A, A)$, that is, there exists an R-subsupermodule L of $\mathrm{Hom}_R(A, A)$ with $R \cap L = (0)$ and $R + L = \mathrm{Hom}_R(A, A)$. Then $R \cap (B \cap L) = (0)$ and $R + (B \cap L) = B$, so R is an R-direct summand of B. □

Consider now the following general situation. Let A be any R-superalgebra and

B any subsuperalgebra of A. Then A is naturally a two-sided B/R-supermodule and, as before, we let $A^B = (A^B)_{\bar{0}} + (A^B)_{\bar{1}}$ where $(A^B)_\alpha = \{a_\alpha \in A_\alpha \mid a_\alpha b_\beta = (-)^{\alpha\beta} b_\beta a_\alpha,\ \forall\ b_\beta \in B_\beta\}$. In this setting A^B is seen to be an R-subsuperalgebra of A which supercommutes with B.

Now the proof of the next Theorem in the algebra case is the same as the proof in the subsuperalgebra case, so we can see its proof in [6, Theorem 2.4.3].

Theorem 5.4.3 Let A be a central separable R-superalgebra. Suppose B is any separable subsuperalgebra of A containing R. Set $C = A^B$. Then C is a separable subsuperalgebra of A and $A^C = B$. If B is also central, so is C and the R-superalgebra map $B\hat{\otimes}_R C \to A$ given by $b_\beta \otimes c_\gamma \mapsto b_\beta c_\gamma$ is an isomorphism. □

While the previous theorem states that the central separable subsuperalgebras of a central separable superalgebra A occurs in pairs, each member of a pair being the commutator subsuperalgebra of the other and whose $\hat{\otimes}_R$ is isomorphic to A, the next theorem tells us that any representation of A as a tensor product of subsuperalgebras can only occur in this way.

Theorem 5.4.4 Let A be a central separable R-superalgebra. Suppose B and C are subalgebras such that the map from $B\hat{\otimes}_R C$ to A given by $b_\beta \otimes c_\gamma \mapsto b_\beta c_\gamma$ is an R-superalgebra isomorphism. Then B and C are central separable R-superalgebras with $A^B = C$ and $A^C = B$.

Proof We know by Corollary 5.4.2 that R is an R-direct summand of both B and C, so B and C are R-separable by Corollary 5.1.10. Since elements of B supercommute with elements of C (i.e., $(b_\beta \otimes 1)(1 \otimes c_\gamma) = (-1)^{\beta\gamma}(1 \otimes c_\gamma)(b_\beta \otimes 1)$), the supercenter of B is contained in the supercenter of A (i.e., is contained in $\hat{Z}(A) = R$), so B is central as well as separable. Similarly for C. Finally, by Corollary 5.3.8, $A^B = (B\hat{\otimes}_R C)^B = C$ and $A^C = (B\hat{\otimes}_R C)^C = B$. □

5.5 The Brauer-Wall Group

We devote this section to the definition and fundamental properties of the Brauer-Wall group of a super-commutative super-ring.

For any super-commutative super-ring R, we proceed to construct the Brauer-Wall group of R. For A, B central separable R-superalgebras, write $A \sim B$ if and only if there exist R-supermodules E_1, E_2, where E_1, E_2 are R-progenerators as modules over R, such that

$$A\hat{\otimes}_R\mathrm{Hom}_R(E_1, E_1) \cong B\hat{\otimes}_R\mathrm{Hom}_R(E_2, E_2).$$

One can easily check that this relation is an equivalence relation.

Definition 5.5.1 (Definition of the Brauer-Wall group) *Let $BW(R)$ denote the set of equivalence classes of central separable R-superalgebras, and $[A]$ denote the class containing A. We define a binary operation in $BW(R)$ by $[A][B] = [A\hat{\otimes}_R B]$.*

If $A' \in [A]$ and $B' \in [B]$, by definition there are R-supermodules E_1, E_1', E_2, E_2', where E_1, E_1', E_2, E_2' are R-progenerators as R-modules, such that $A\hat{\otimes}_R\mathrm{Hom}_R(E_1, E_1) \cong A'\hat{\otimes}_R\mathrm{Hom}_R(E_1', E_1')$, and $B\hat{\otimes}_R\mathrm{Hom}_R(E_2, E_2) \cong B'\hat{\otimes}_R\mathrm{Hom}_R(E_2', E_2')$, from which we obtain

$$\begin{aligned}
A\hat{\otimes}B\hat{\otimes}\mathrm{Hom}_R(E_1, E_1)\hat{\otimes}\mathrm{Hom}_R(E_2, E_2) &\cong A\hat{\otimes}\mathrm{Hom}_R(E_1, E_1)\hat{\otimes}B\hat{\otimes}Hom_R(E_2, E_2) \\
&\cong A'\hat{\otimes}B'\hat{\otimes}\mathrm{Hom}_R(E_1', E_1')\hat{\otimes}\mathrm{Hom}_R(E_2', E_2') \\
&\cong A'\hat{\otimes}\mathrm{Hom}_R(E_1', E_1')\hat{\otimes}B'\hat{\otimes}\mathrm{Hom}_R(E_2', E_2').
\end{aligned}$$

Thus the operation is well-defined. Obviously it is commutative, associative, and possesses an identity $[R]$. Finally by Theorem 5.3.5 if A is a central separable R-superalgebra then A is an R-progenerator as a module over R and $A^\circ\hat{\otimes}_R A$ is isomorphic to $\mathrm{Hom}_R(A, A)$, whence $[A^\circ][A] = [\mathrm{Hom}_R(A, A)] = [\mathrm{Hom}_R(R, R)] = [R]$, proving that $[A^\circ] = [A]^{-1}$. Thus $\mathrm{BW}(R)$ forms an abelian group, known as the Brauer-Wall group of R.

We next establish the basic functorial properties of the Brauer-Wall group.

Lemma 5.5.2 Let A be a central separable R-superalgebra. Then for any supercommutative R-superalgebra S, $S\hat{\otimes}_R A$ is a central separable S-superalgebra.

Proof We know that $S\hat{\otimes}_R A$ contains a copy of S since R is an R-direct summand of A. Thus $S\hat{\otimes}_R A \neq 0$ and is therefore separable over S by Corollary 5.1.8. Moreover, by the hom-tensor relation 4.2.5,

$$S\hat{\otimes}R \cong S\hat{\otimes}\mathrm{Hom}_{A^e}(A, A) \cong \mathrm{Hom}_{S\hat{\otimes}A^e}(S\hat{\otimes}A, S\hat{\otimes}A) \cong \mathrm{Hom}_{(S\hat{\otimes}A)^o\hat{\otimes}(S\hat{\otimes}A)}(S\hat{\otimes}A, S\hat{\otimes}A)$$

so by Corollary 5.1.4 we have S is the super-center of $S\hat{\otimes}_R A$ (i.e. $S = \hat{Z}(S\hat{\otimes}_R A)$).□

Let $f : R \to S$ be any super-ring homomorphism of R to the supercommutative super-ring S. Then f endows S with a structure as an R-superalgebra. By the Lemma above, if A is central separable over R then $S\hat{\otimes}_R A$ is central separable over S. Further, if the R-supermodule E is an R-progenerator as a module over R, then $S\hat{\otimes}_R E$ is an S-progenerator as a module over S, and, by Corollary 4.2.5, we have $S\hat{\otimes}_R \mathrm{Hom}_R(E, E) \cong \mathrm{Hom}_S(S\hat{\otimes}E, S\hat{\otimes}E)$. It follows that the map $B(f) : \mathrm{BW}(R) \to \mathrm{BW}(S)$ given by $B(f)([A]) = [S\hat{\otimes}_R A]$ is well-defined. Also, for central separable R-superalgebras A, B, we have $(S\hat{\otimes}_R A)\hat{\otimes}_S(S\hat{\otimes}_R B) \cong (A\hat{\otimes}_R B)\hat{\otimes}_R S$, so $B(f)$ is a group homomorphism.

Suppose $g : S \to T$ is also a super-ring homomorphism from S the super-commutative super-ring T. Then T is an S-superalgebra via g and R-superalgebra via gf in such a way that for any central separable R-superalgebra A, $T\hat{\otimes}_S(S\hat{\otimes}_R A) \cong T\hat{\otimes}_R A$, so $B(g)B(f) = B(gf)$. Thus we proved

Proposition 5.5.3 $B(\ )$ is a covariant functor from the category of supercommutative super-rings (and super-ring homomorphisms) to the category of abelian groups (and group homomorphisms). □

It is clear that from the definition of the equivalence "$\sim$" that any R-supermodule E which is an R-progenerator as a module over R, $\mathrm{Hom}_R(E, E)$ is equivalent to R. The next proposition shows that conversely $A \sim R$ implies that $A \cong \mathrm{Hom}_R(E, E)$ where E is an R-progenerator as a module over R.

Proposition 5.5.4 For every central separable R-superalgebra A, $A \sim R$ if and only if $A \cong \mathrm{Hom}_R(E, E)$, where the R-supermodule E is an R-progenerator as a module over R.

Proof If $A \cong \mathrm{Hom}_R(E, E)$ then $A \sim R$. Conversely, if $A \sim R$ then there exist R-supermodules E_1 and E_2, where E_1 and E_2 are R-progenerators as modules over R, such that $A\hat{\otimes}\mathrm{Hom}_R(E_1, E_1) \cong R\hat{\otimes}\mathrm{Hom}_R(E_2, E_2) \cong \mathrm{Hom}_R(E_2, E_2)$. Setting $B = \mathrm{Hom}_R(E_1, E_1)$, we can identify A and B via this isomorphism with subsuperalgebras of $\mathrm{Hom}_R(E_2, E_2)$. E_2 can be viewed as a right B-supermodule under the operation $e_\alpha b_\beta = e_\alpha(1 \otimes b_\beta)$, for $e_\alpha \in E_2$ and $b_\beta \in B$. Since $B = \mathrm{Hom}_R(E_1, E_1)$, we have, by the Morita theorems, that the right R-supermodule $M = E_2\hat{\otimes}_B E_1^*$ corresponds to E_2 under the category isomorphism between the category of right R-supermodules and the category of right B-supermodules, since

$$\begin{aligned} M\hat{\otimes}_R E_1 &\cong E_2\hat{\otimes}_B E_1^*\hat{\otimes}_R E_1 \\ &\cong E_2\hat{\otimes}_B \mathrm{Hom}_R(E_1, E_1) \\ &\cong E_2\hat{\otimes}_B B \cong E_2. \end{aligned}$$

So, by Proposition 4.3.1, $\mathrm{Hom}_R(M, M) \cong \mathrm{Hom}_B(E_2, E_2)$ as R-superalgebras. But if we consider E_2 as a right R-supermodule then $\mathrm{Hom}_R(E_2, E_2) \cong A^\circ\hat{\otimes}_R B^\circ$ and therefore

$$\mathrm{Hom}_B(E_2, E_2) = [\mathrm{Hom}_R(E_2, E_2)]^{B^\circ} \cong A^\circ.$$

So that $A^\circ \cong \mathrm{Hom}_R(M, M)$, and hence $A \cong \mathrm{Hom}_R(M, M)$ when we consider M as a left R-supermodule. Now, we will verify that M is an R-progenerator as a module over R.

Since E_2 is R-projective as a module over R, it is B-projective as a module over B by [6, proposition 2.2.3]. Because projectivity is preserved under the category isomorphism, M is R-projective as a module over R. In addition, E_1^* and E_2 are both finitely generated over R, so $M = E_2\hat{\otimes}_B E_1^*$ is finitely generated over R. Finally because $\mathrm{Hom}_R(M, M) \cong A$ is a faithful R-superalgebra, M must be a faithful R-supermodule. Therefore by Corollary 4.1.17 M is an R-progenerator as a module over R. □

Corollary 5.5.5 Let A and B be central separable R-superalgebras. Then $A \sim B$ if and only if $B^{\circ}\hat{\otimes}A \cong \mathrm{Hom}_R(E, E)$, where the R-supermodule E is an R-progenerator as a module over R.

Proof $A \sim B$ if and only if $[A] = [B]$ if and only if $[B]^{-1}[A] = [R]$ if and only if $[B^{\circ}\hat{\otimes}A] = [R]$. Now apply the proposition above. $\square$

5.6 Superinvolutions

On a superalgebra A, a map $J : A \to A$ is called a superinvolution if J^2 is the identity and J is an antiautomorphism. More explicitly, $a_\alpha^{J^2} = a_\alpha$, $(a_\alpha + b_\beta)^J = a_\alpha^J + b_\beta^J$ and $(a_\alpha b_\beta)^J = (-1)^{\alpha\beta} b_\beta^J a_\alpha^J$ for all a_α, b_β in A. Let $C = \hat{Z}(A)$ (the super-center of A) then J must preserve C. If J is the identity on C, J is a superinvolution of the first kind. If not, J induces an automorphism of C of order 2, and J is said to be of the second kind. Two superinvolutions, J, J' which agree on C are said to be of the same kind. In the first theorem we try to find the conditions on a central separable R-superalgebra A to have a superinvolution of the second kind, if R is a connected semilocal super-ring. In the second one we try to find the conditions on $A = \mathrm{Hom}_R(P, P)$ to have a superinvolution of any kind, where P is an R-progenerator as a supermodule over R, if R is a connected super-ring.

Let R/S be a Galois extension of super-commutative super-rings with Galois group $G = \{1, \sigma\}$. Let A be any central separable R-superalgebra, we define A^σ as follows, set $A^\sigma = A$ as a super-ring, but the product by a scalar . on A^σ is defined by $\lambda.a = \lambda^\sigma a$. Then one easily check that A^σ is a central separable R-superalgebra. Now let $\tau : A^\sigma\hat{\otimes}_R A \to A^\sigma\hat{\otimes}_R A$, be defined by $\tau(a_\alpha \otimes b_\beta) = (-1)^{\alpha\beta} b_\beta \otimes a_\alpha$, then τ is a σ-linear automorphism. In particular τ is S-linear. Define the *corestriction*

$$\mathrm{Tr}(A) = \{x \in A^\sigma\hat{\otimes}_R A \mid \tau(x) = x\}.$$

Obviously, $\mathrm{Tr}(A)$ is an S-superalgebra. But by [23, Lemma 2.1] $\mathrm{Tr}(A)$ is an S-progenerator as an S-supermodule, if A is an R-progenerator as an R-supermodule. Moreover if A is central separable over R then by [23, page 529] $\mathrm{Tr}(A)$ is central separable over S.

Our next example involves assuming that R, the base super-ring, is semilocal (i.e., a finite direct sum of local super-rings). We will use the fact, from [22], that if A, B are central separable R-algebras, $A \sim B$, and the rank of A equals rank of B, then $A \cong B$, which is also true in the superalgebra case (i.e., if A, B are central separable R-superalgebras, $A \sim B$, and the rank of A equals rank of B, then $A \cong B$). Let M be the Jacobson radical of R. Then $\bar{A} = A/MA$ is a finite direct sum of simple superalgebras. We call $\bar{A}$ is *perfect* if every simple subsuperalgebra of $\bar{A}$ admits a superinvolution of the second kind.

Theorem 5.6.1 Suppose R is a connected semilocal super-ring, and A is a central separable R-superalgebra. Suppose R/S is a Galois extension with Galois group $\{1, \sigma\}$. Then A has a superinvolution of the second kind extending σ if and only if $\mathrm{Tr}(A) \sim 1$ and $\bar{A}$ is perfect.

Proof Suppose A has a superinvolution, $*$, extending σ. Then it is easy to check that $\bar{A}$ is perfect. Also $*$ induces an isomorphism $A^\sigma \cong A^\circ$, so there is an isomorphism

$$\varphi : A^\sigma \hat{\otimes}_R A \to \mathrm{Hom}_R(A, A)$$

given by $x_\gamma(a_\alpha \otimes b_\beta)^\varphi = (-1)^{\alpha\gamma} a^*_\alpha x_\gamma b_\beta$. Set $A' = A'_{\bar{0}} + A'_{\bar{1}}$, where $A'_\alpha = \{a_\alpha \in A_\alpha \mid a^*_\alpha = a_\alpha\}$. Since $*$ is σ-linear R-supermodule automorphism of A, the S-supermodule A' is an S-progenerator as a module over S. φ induces an isomorphism $\mathrm{Tr}(A) \cong \mathrm{Hom}_S(A', A')$, hence $\mathrm{Tr}(A) \sim 1$.

Conversely, since R is a connected semilocal super-ring, one easily sees that S is a connected semilocal super-ring also. Let $\mathrm{Tr}(A) \cong \mathrm{Hom}_S(P, P)$. In other words let $\tau : A^\sigma \hat{\otimes}_R A \to A^\sigma \hat{\otimes}_R A$, given by $(a_\alpha \otimes b_\beta)^\tau = (-1)^{\alpha\beta} b_\beta \otimes a_\alpha$, be a σ-linear automorphism. Then $\mathrm{Tr}(A)$ is the fixed super-ring of $A^\sigma \hat{\otimes}_R A$ under τ. Say $\mathrm{Tr}(A) \cong \mathrm{Hom}_S(P, P)$, where P is an S-progenerator as a supermodule over S. Then $A^\sigma \hat{\otimes}_R A \cong R \hat{\otimes}_R \mathrm{Hom}_S(P, P) \cong \mathrm{Hom}_R(R \hat{\otimes}_S P, R \hat{\otimes}_S P)$ and if $\varphi = \sigma \otimes 1 : R \hat{\otimes}_S P \to R \hat{\otimes}_S P$, then $(x_\gamma.(a_\alpha \otimes b_\beta))^\varphi = x^\varphi_\gamma.(a_\alpha \otimes b_\beta)^\tau$, for all x_γ in $R \hat{\otimes}_S P$ and $a_\alpha \otimes b_\beta$ in $A^\sigma \hat{\otimes}_R A$.

Since R is connected, $rank_R(A) = rank_R(A^\sigma)$, but $A^\sigma \hat{\otimes}_R A \cong \mathrm{Hom}_R(R \hat{\otimes}_S P, R \hat{\otimes}_S P)$, therefore $A^\sigma \hat{\otimes}_R (A^\circ \hat{\otimes}_R A) \cong A^\sigma \hat{\otimes}_R \mathrm{Hom}_R(A, A) \cong \mathrm{Hom}_R(R \hat{\otimes}_S P, R \hat{\otimes}_S P) \hat{\otimes}_R A^\circ$. So by

[22, Corollary 3.3], $A^\sigma \cong A^\circ$, which implies that $\mathrm{Hom}_R(A, A) \cong \mathrm{Hom}_R(R\hat{\otimes}_S P, R\hat{\otimes}_S P)$, but the R-rank of A equals the R-rank of $R\hat{\otimes}_S P$. So again by [22, Corollary 3.3], $A \cong R\hat{\otimes}_S P$. In other words, A has a σ-linear antiautomorphism J such that for all $a_\alpha, x_\gamma, b_\beta$ in A, setting $x_\gamma.(a_\alpha \otimes b_\beta) = (-1)^{\alpha\gamma} a_\alpha^J x_\gamma b_\beta$ yields the isomorphism $A^\sigma \hat{\otimes}_R A \cong \mathrm{Hom}_R(A, A) \cong A^\circ \hat{\otimes}_R A$, and the map $\varphi = \sigma \otimes 1 : A(\cong R\hat{\otimes}_S P) \to A$ satisfies $\varphi^2 = 1$ and $(x_\gamma.(a_\alpha \otimes b_\beta))^\varphi = x_\gamma^\varphi.(a_\alpha \otimes b_\beta)^\tau$. Therefore $(-1)^{\alpha\gamma}(a_\alpha^J x_\gamma b_\beta)^\varphi = (-1)^{\alpha\beta} x_\gamma^\varphi.(b_\beta \otimes a_\alpha) = (-1)^{\beta(\alpha+\gamma)} b_\beta^J x_\gamma^\varphi a_\alpha$. ($\varphi$ respects the grading). For $w = 1^\varphi \in A_{\bar{0}}$ we have $ww^J = w^J w = 1$ and $w a_\alpha w^{-1} = a_\alpha^{J^2}$, and

$$\varphi^2 = 1, \quad (a_\alpha^J x_\gamma b_\beta)^\varphi = (-1)^{\alpha\gamma}(-1)^{\beta(\alpha+\gamma)} b_\beta^J x_\gamma^\varphi a_\alpha. \tag{15}$$

Lemma 5.6.2 Let A be a central separable R-superalgebra, with J and φ satisfying (15). Then A has a superinvolution agreeing with J on R if φ fixes a unit of A_α.

Proof If u_α is a unit in A_α such that $u_\alpha^\varphi = u_\alpha$ then $u_\alpha = (1.u_\alpha)^\varphi = u_\alpha^J.w$, so $(u_\alpha^J)^{-1} u_\alpha = w$, but $(u_\alpha^J)^{-1} = (-1)^\alpha (u_\alpha^{-1})^J$, therefore $w = (-1)^\alpha (u_\alpha^{-1})^J u_\alpha$, implying that $x_\gamma^{J'} = u_\alpha^{-1} x_\gamma^J u_\alpha$ is a superinvolution since J' is an antiautomorphism on A and

$$\begin{aligned}
(x_\gamma^{J'})^{J'} &= u_\alpha^{-1}(u_\alpha^{-1} x_\gamma^J u_\alpha)^J u_\alpha = (-1)^\alpha u_\alpha^{-1}(u_\alpha^J x_\gamma^{J^2}(u_\alpha^{-1})^J) u_\alpha \\
&= u_\alpha^{-1} u_\alpha^J (w x_\gamma w^{-1}) w \\
&= x_\gamma \qquad \text{since } u_\alpha^{-1} u_\alpha^J = w^{-1}.
\end{aligned}$$

□

Continuing proof of the theorem: Let M be the Jacobson radical of R. Then $\bar{A} = A/MA$ is a finite direct sum of simple superalgebras. On $\bar{A}$, φ and J induce maps $\bar{\varphi}$ and $\bar{J}$ satisfying (15). Every preimage of a unit $\bar{u}_\alpha$ of $\bar{A}$ is a unit u_α of A_α. Thus we can change J by conjugation with a unit u_α, to make $\bar{J}$ any desired antiautomorphism of $\bar{A}$ of the same kind. In fact, if J' is defined by $x_\gamma^{J'} = u_\alpha^{-1} x_\gamma^J u_\alpha$, we can find a corresponding φ' so that J', φ' satisfy (15). Specifically if L is the inverse map to J, we can set $x_\gamma^{\varphi'} = u_\alpha^{-1} x_\gamma^\varphi u_\alpha^L$, to show that we have $(x_\gamma^{\varphi'})^{\varphi'} = u_\alpha^{-1}(u_\alpha^{-1} x_\gamma^\varphi u_\alpha^L)^\varphi u_\alpha^L$ $= (-1)^\alpha u_\alpha^{-1}(u_\alpha^{LJ} x_\gamma z_\alpha) u_\alpha^L$, where $z_\alpha^J = u_\alpha^{-1}$, and hence $z_\alpha = z_\alpha^{JL} = (u_\alpha^{-1})^L$, so that $(x_\gamma^{\varphi'})^{\varphi'} = (-1)^\alpha x_\gamma (u_\alpha^{-1})^L u_\alpha^L = x_\gamma$, since $(-1)^\alpha (u_\alpha^L)^{-1} = (u_\alpha^{-1})^L$. It suffices to find $\bar{u}_\alpha$ of $\bar{A}$ such that $(\bar{u}_\alpha)^{\bar{\varphi}} + \bar{u}_\alpha$ is a unit, for if u_α is a preimage of $\bar{u}_\alpha$, then $(u_\alpha)^\varphi + u_\alpha$ will be a φ fixed unit of A_α. Since $\bar{A}$ is perfect, it suffices to prove:

Lemma 5.6.3 Let $\bar{A}$ be a finite dimensional central simple superalgebra over a field F with a superinvolution J of the second kind and any associated φ to J then there is an element $\bar{a}_\alpha$ in $\bar{A}_\alpha$ such that $(\bar{a}_\alpha)^\varphi + \bar{a}_\alpha$ is a unit.

Proof The element $w = 1^\varphi$ is central since J is a superinvolution. If $w \neq -1$, then $\bar{a}_{\bar{0}} = 1$ will do. If $w = -1$, then $(\bar{a}_\alpha)^\varphi = (\bar{a}_\alpha)^J w = -(\bar{a}_\alpha)^J$. Since J is of order 2 on F, there is f in F such that $f - f^J \neq 0$, so again take $\bar{a}_{\bar{0}} = f - f^J$. □

Suppose R is a local super-commutative super-ring, σ an automorphism of R of order 2, P is an R-progenerator as a supermodule over R, and I a rank one R-projective supermodule. A morphism $e : P\hat{\otimes}_R P \to I$ is called a *bilinear I form on P*, a morphism $e : P^\sigma\hat{\otimes}_R P \to I$ is called a *σ bilinear I form on P*. The image $e(p_\alpha \otimes q_\beta)$ is often written as $e(p_\alpha, q_\beta)$ and in either case, e can be thought of as a map $e : P \times P \to I$. Such a form induces a map $e^* : P \to \mathrm{Hom}_R(P, I)$ ($P^\sigma \to \mathrm{Hom}_R(P, I)$) given by $e^*(p_\alpha)(q_\beta) = e(p_\alpha, q_\beta)$. In a similar manner, we define $e_* : P \to \mathrm{Hom}_R(P, I)$ ($P \to \mathrm{Hom}_R(P^\sigma, I)$) given by $e_*(q_\beta)(p_\alpha) = e(p_\alpha, q_\beta)$. If e^* and e_* are isomorphisms then we say e is *nondegenerate.* Our final result shows that the existence of superinvolutions on $\mathrm{Hom}_R(P, P)$, where $\mathrm{Hom}_R(P, P)$ is an R-progenerator as a supermodule over R, is equivalent to the existence of forms on P.

Theorem 5.6.4 Let R be a connected super-ring, and $A = \mathrm{Hom}_R(P, P)$ be a central separable R-superalgebra such that A is an R-progenerator as a supermodule over R, then

(a) A has a superinvolution of the first kind if and only if there is a rank one R-projective I, a nondegenerate bilinear I form e on P and a δ in $R_{\bar{0}}$ such that $\delta^2 = 1$ and $e(x_\alpha, y_\beta) = (-1)^{\alpha\beta}\delta e(y_\beta, x_\alpha)$ for all x_α, y_β in P.

(b) Let σ be an automorphism of R of order 2. Then A has a superinvolution of the second kind extending σ if and only if there is a rank one R-projective I with a σ-linear automorphism of order 2 (also called σ) a σ-bilinear I form e on P and an element δ in $R_{\bar{0}}$ such that $\sigma(\delta)\delta = 1$ and $\sigma(e(x_\alpha, y_\beta)) = (-1)^{\alpha\beta}\delta e(y_\beta, x_\alpha)$ for all x_α, y_β in P.

As a first step we prove the following lemma:

Lemma 5.6.5 Suppose Q is a right $A^e = A^\circ \hat{\otimes}_R A$-supermodule, then $Q = M \oplus I$, where M is the R-subsupermodule of Q generated by all elements of the form $(a_\alpha \otimes 1 - 1 \otimes a_\alpha)q_\beta$, where a_α in A_α and q_β in Q_β. If Q is R-projective as a supermodule over R then $rank_R(A).rank_R(I) = rank_R(Q)$.

Proof Consider the well-known split exact sequence of A^e-supermodules

$$0 \to J \to A^e \xrightarrow{\mu} A \to 0$$

where $\mu(a_\alpha \otimes b_\beta) = a_\alpha b_\beta$ and J is a right super-ideal of A^e generated by all elements of the form $a_\alpha \otimes 1 - 1 \otimes a_\alpha$ where $a_\alpha \in A_\alpha$. Suppose Q is a right A^e-supermodule. Tensoring by Q over A^e yields a split exact sequence of R-supermodules

$$0 \to Q\hat{\otimes}_{A^e} J \to Q\hat{\otimes}_{A^e} A^e \xrightarrow{1\otimes\mu} Q\hat{\otimes}_{A^e} A \to 0$$

of course, $Q\hat{\otimes}_{A^e} A^e \cong Q$ under the map $a_\alpha \otimes z_\beta \to q_\alpha z_\beta$. Under this isomorphism $Q\hat{\otimes}_{A^e} J$ is mapped onto M defined above. Thus $Q \cong M \oplus I$, where $I \cong Q\hat{\otimes}_{A^e} A$. But $I\hat{\otimes}_R A \cong Q\hat{\otimes}_{A^e}(A\hat{\otimes}_R A) \cong Q\hat{\otimes}_{A^e}(A^\circ\hat{\otimes}_R A)$ (as an R-supermodules), therefore $I\hat{\otimes}_R A \cong Q\hat{\otimes}_{A^e} A^e \cong Q$. □

We will handle cases (a) and (b) simultaneously by letting σ be an automorphism of R of order either one or two. Suppose e is an I form on P as in either (a) or (b). Let $a_\alpha \in A_\alpha$. Then for fixed $y_\gamma \in P_\gamma$, $e(x_\mu a_\alpha, y_\gamma) \in I_{\alpha+\gamma+\mu}$ and so there is a unique $y'_{\gamma+\alpha}$ in $P_{\gamma+\alpha}$ such that $e(x_\mu a_\alpha, y_\gamma) = e(x_\mu, y'_{\gamma+\alpha})$. Define $a^*_\alpha \in A_\alpha$ by $(-1)^{\alpha\gamma} y_\gamma a^*_\alpha = y'_{\gamma+\alpha}$. Then for all y_γ

$$\begin{aligned}
\delta e(x_\mu a_\alpha, y_\gamma) &= (-1)^{\alpha\gamma}\delta e(x_\mu, y_\gamma a^*_\alpha) \\
&= (-1)^{\alpha\gamma}(-1)^{\mu(\alpha+\gamma)}\sigma(e(y_\gamma a^*_\alpha, x_\mu)) \\
&= (-1)^{\mu\gamma+\alpha\gamma}\sigma(e(y_\gamma, x_\mu a^{**}_\alpha)) \\
&= \delta e(x_\mu a^{**}_\alpha, y_\gamma),
\end{aligned}$$

and

$$\begin{aligned}
e(x_\mu a_\alpha b_\beta, y_\gamma) &= (-1)^{(\beta+\alpha)\gamma} e(x_\mu, y_\gamma(a_\alpha b_\beta)^*) \\
&= (-1)^{(\beta+\alpha)\gamma+\alpha\beta} e(x_\mu, y_\gamma b^*_\beta a^*_\alpha).
\end{aligned}$$

Therefore $a_\alpha^{**} = a_\alpha$ for all $a_\alpha \in A_\alpha$, and $e(x_\mu, y_\gamma(-1)^{\alpha\beta} b_\beta^* a_\alpha^*) = e(x_\mu, y_\gamma(a_\alpha b_\beta)^*)$, which implies that $(a_\alpha b_\beta)^* = (-1)^{\alpha\beta} b_\beta^* a_\alpha^*$. So, $*$ is the required superinvolution.

Conversely, suppose A has a superinvolution $*$ extending σ on R (σ is of order 1 or 2). View $*$ as an isomorphism: $A^\circ \cong A^\sigma$. Then $* \otimes 1 : A^\circ \hat{\otimes}_R A \cong A^\sigma \hat{\otimes}_R A \cong \mathrm{Hom}_R(P^\sigma \hat{\otimes}_R P, P^\sigma \hat{\otimes}_R P)$. Therefore, $P^\sigma \hat{\otimes}_R P$ becomes an $A^\circ \hat{\otimes}_R A$-supermodule under the action (from the right) $(p_\delta \otimes q_\gamma).(a_\alpha \otimes b_\beta) = (-1)^{\alpha\gamma} p_\delta a_\alpha^* \otimes q_\gamma b_\beta$. According to the lemma above $P^\sigma \hat{\otimes}_R P \cong M \oplus I$, where I has rank one and M is the R-subsupermodule generated by $b_\beta(a_\alpha \otimes 1 - 1 \otimes a_\alpha)$ for all b_β in $P^\sigma \hat{\otimes}_R P$. Let $e : P^\sigma \hat{\otimes}_R P \to I$ be the projection on I. Since $e(M) = 0$,

$$\begin{aligned} e[(x_\gamma \otimes y_\beta)(a_\alpha^* \otimes 1 - 1 \otimes a_\alpha^*)] &= e[(-1)^{\alpha\beta} x_\gamma a_\alpha \otimes y_\beta - x_\gamma \otimes y_\beta a_\alpha^*] \\ &= 0. \end{aligned}$$

Therefore, $e(x_\gamma a_\alpha, y_\beta) = (-1)^{\alpha\beta} e(x_\gamma, y_\beta a_\alpha^*)$, and $e(P^\sigma \hat{\otimes}_R P) = I$. We need to show now that if σ is of order 2, I has the property required in (b). Recall that $\tau : P^\sigma \hat{\otimes}_R P \to P^\sigma \hat{\otimes}_R P$, $\tau(p_\alpha \otimes q_\beta) = (-1)^{\alpha\beta} q_\beta \otimes p_\alpha$ is a σ-linear automorphism and for $a_\alpha \otimes b_\beta$ in $A^\circ \hat{\otimes}_R A$,

$$\begin{aligned} \tau((p_\gamma \otimes q_\delta).(a_\alpha \otimes b_\beta)) &= (-1)^{\alpha\delta} \tau((p_\gamma a_\alpha^* \otimes q_\delta b_\beta)) \\ &= (-1)^{\alpha\delta + (\delta+\beta)(\alpha+\gamma)} q_\delta b_\beta \otimes p_\gamma a_\alpha^* \\ &= (-1)^{\gamma\delta + \beta\alpha} (q_\delta \otimes p_\gamma)(b_\beta^* \otimes a_\alpha^*) \\ &= (-1)^{\alpha\beta} \tau((p_\gamma \otimes q_\delta))(b_\beta^* \otimes a_\alpha^*). \end{aligned}$$

Thus, $\tau(M) = M$, and τ induces a σ-linear automorphism on $I \cong P^\sigma \hat{\otimes}_R P / M$ which we also call σ.

We now show e_* is an isomorphism. Let $\{p_{i,\gamma}, f_{i,\gamma}\}$ be a finite dual basis for P. So if x_δ in P_δ, $x_\delta = \sum_i x_\delta^{f_{i,\gamma}} . p_{i,\gamma}$. Suppose $f_\beta \in \mathrm{Hom}_R(P^\sigma, I)$ we wish to find x_β in P_β such that $e_*(x_\beta) = f_\beta$. Let

$$\begin{aligned} a_{i,\gamma+\beta} = p_{i,\gamma}^{f_\beta} &= \Big(\sum_j (p_{i,\gamma})^{f_{j,\gamma}} . p_{j,\gamma}\Big)^{f_\beta} \\ &= \sum_j \sigma((p_{i,\gamma})^{f_{j,\gamma}}) a_{j,\gamma+\beta}. \end{aligned}$$

Since $e : P^\sigma \hat{\otimes}_R P \to I$ is onto, the $e(p_{j,\gamma}, p_{k,\gamma})$ for all j and k, generate I. Write $a_{i,\gamma+\beta} = \sum_{j,k} r_{ijk,\gamma+\beta} e(p_{j,\gamma}, p_{k,\gamma})$ then

$$\begin{aligned} a_{m,\gamma+\beta} &= \sum_i \sigma((p_{m,\gamma})^{f_{i,\gamma}}) a_{i,\gamma+\beta} \\ &= \sum_{i,j,k} \sigma((p_{m,\gamma})^{f_{i,\gamma}}) r_{ijk,\gamma+\beta} e(p_{j,\gamma}, p_{k,\gamma}) \\ &= \sum_{i,j,k} r_{ijk,\gamma+\beta} e((p_{m,\gamma})^{f_{i,\gamma}} p_{j,\gamma}, p_{k,\gamma}). \end{aligned}$$

Define $c_{ij} \in \mathrm{Hom}_R(P,P)$ by $x_\alpha c_{ij} = x_\alpha^{f_{i,\gamma}}.p_{j,\gamma}$. Then

$$e((p_{m,\gamma})^{f_{i,\gamma}} p_{j,\gamma}, p_{k,\gamma}) = e(p_{m,\gamma} c_{ij}, p_{k,\gamma}) = e(p_{m,\gamma}, p_{k,\gamma} c_{ij}^*).$$

Thus setting $x_\beta = \sum_{i,j,k} (-1)^{\gamma(\gamma+\beta)} r_{ijk,\gamma+\beta} p_{k,\gamma} c_{ij}^*$, then

$$\begin{aligned} e(p_{m,\gamma}, x_\beta) &= \sum_{i,j,k} (-1)^{\gamma(\gamma+\beta)} e(p_{m,\gamma}, r_{ijk,\gamma+\beta} p_{k,\gamma} c_{ij}^*) \\ &= \sum_{i,j,k} r_{ijk,\gamma+\beta} e(p_{m,\gamma} c_{ij}, p_{k,\gamma}) \\ &= a_{m,\gamma+\beta} \quad \text{for all } m, \end{aligned}$$

implying $e_*(x_\beta) = f_\beta$. Thus e_* is onto, and, by considering ranks, e_* is an isomorphism. Similarly, e^* is an isomorphism and we have proved the nondegeneracy of e. For p_α in P_α define $p_\alpha \theta$ by $e(p_\alpha, x_\gamma) = (-1)^{\alpha\gamma} \sigma(e(x_\gamma, p_\alpha \theta))$, for all x_γ in P_γ. Then for r_δ in R_δ,

$$\begin{aligned} e(r_\delta p_\alpha, x_\gamma) &= (-1)^{\gamma(\alpha+\delta)} \sigma(e(x_\gamma, (r_\delta p_\alpha)\theta)) \\ &= (-1)^{\alpha\delta} e(p_\alpha r_\delta, x_\gamma) \\ &= (-1)^{\alpha\delta+\gamma\delta} e(p_\alpha, x_\gamma \sigma(r_\delta)) \\ &= (-1)^{\alpha\delta+\gamma\delta} (-1)^{\alpha(\gamma+\delta)} \sigma(e(x_\gamma \sigma(r_\delta), p_\alpha \theta)) \\ &= (-1)^{\gamma(\alpha+\delta)} \sigma(e(x_\gamma, r_\delta(p_\alpha \theta))). \end{aligned}$$

Thus, $(r_\delta p_\alpha)\theta = r_\delta(p_\alpha \theta)$, which implies that $\theta \in \mathrm{Hom}_R(P,P) = A$. Now, $e(p_\alpha, q_\beta) = (-1)^{\alpha\beta} \sigma(e(q_\beta, p_\alpha \theta)) = (-1)^{\alpha\beta} \sigma(e(q_\beta \theta^*, p_\alpha)) = e(p_\alpha, q_\beta \theta^* \theta)$. Thus $\theta^* \theta = 1$. Similarly,

$\theta\theta^* = 1$, since $e(p_\alpha, q_\beta\theta^*\theta) = (-1)^{\alpha\beta}\sigma(e(q_\beta\theta^*\theta, p_\alpha\theta)) = (-1)^{\alpha\beta}\sigma(e(q_\beta\theta^*, p_\alpha\theta\theta^*))$, so $\theta\theta^* = 1$. Furthermore, for $a_\alpha \in A_\alpha$,

$$\begin{aligned}
e(x_\gamma a_\alpha, y_\beta) &= (-1)^{\beta(\alpha+\gamma)}\sigma(e(y_\beta, x_\gamma a_\alpha\theta)) \\
&= (-1)^{\beta(\alpha+\gamma)}(-1)^{\alpha\gamma}\sigma(e(y_\beta(a_\alpha\theta)^*, x_\gamma)) \\
&= (-1)^{\beta(\alpha+\gamma)}(-1)^{\alpha\gamma}\sigma(e(y_\beta\theta^* a_\alpha^*, x_\gamma)) \\
&= (-1)^{\beta\alpha}e(x_\gamma, y_\beta\theta^* a_\alpha^*\theta) \\
&= e(x_\gamma\theta^* a_\alpha\theta, y_\beta).
\end{aligned}$$

We deduce that $a_\alpha = \theta^* a_\alpha \theta$ for all a_α in A_α,hence $\theta \in R_{\bar{0}}$ and we set $\delta = \theta$. $\square$

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