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# FINITE ELEMENT FORMULATION FOR ANALYSIS OF PIPES BASED ON THIN SHELL THEORY

by

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A thesis submitted to the University of Ottawa
in partial fulfillment of the requirements for
Masters of Applied Science in Civil Engineering

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Your file Votre référence
ISBN: 978-0-494-50935-7
Our file Notre référence
ISBN: 978-0-494-50935-7

# ABSTRACT

A general solution for the stress-deformation analysis of pipes subjected to general loading conditions is developed. The solution is based on the assumptions of thin-walled shell theory and is limited to straight pipes of prismatic cross-section made of linearly elastic isotropic material subjected to general loading.

The principle of stationary potential energy is used in conjunction with general Fourier series expansion for the displacement fields to formulate the equilibrium conditions and boundary conditions. The equilibrium equations for each Fourier mode are observed to be uncoupled from other modes, a feature that is exploited in formulating a general closed form solution for the displacement fields.

The analytical solution developed is then successfully adopted to solve two pipe problems. Comparisons with established finite element solutions demonstrate the ability of the model to accurately capture complex shell behaviour with a remarkably small number of degrees of freedoms. However, the algebraic manipulations in the analytical solution are found quite tedious for hand calculations, even for simple problems.

In order to remedy this limitation and to make the solution scheme amenable to more complex problems, a finite element solution is formulated based on the analytical solution developed. The expressions for the displacement fields obtained are used to formulate a series of exact shape functions which relate the intermediate displacements within a finite element to the nodal displacements. Using the exact shape functions, the principle of stationary potential energy is adopted to formulate the stiffness matrices and associated energy equivalent load vectors for pipe finite elements. The new finite element is tested for a variety of practical loading conditions including point loads, gravity loads, internal and external pressure, twisting deformation, axial deformation, transverse deformations, and combinations thereof. Results are found in excellent agreement with those based on shell finite elements in ABAQUS.

The pipe element developed is shown to be free of discretization errors and is thus computationally efficient. The element can very accurately predict the displacements and stresses of a pipe with only a very few elements and a few Fourier modes.

# ACKNOWLEDGEMENTS

I would like to extend a great deal of thanks to Dr. Magdi Mohareb for his guidance and support. His help has enabled me to overcome a number of obstacles in writing this thesis. Also, I am sincerely grateful for the financial support that I have received from the National Research Council of Canada (NSERC) and the University of Ottawa over the course of my studies.

My deepest appreciation goes to my wife Kathryn for her endless encouragement, support and patience. She has been my inspiration.

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## LIST OF SYMBOLS

| | |
|---|---|
| $\alpha$ | Total number of terms included in the Fourier expansion |
| $\{\Delta_0\}_{8\times1}$ | Nodal displacement vector corresponding to Fourier mode $n=0$ |
| $\{\Delta_{n,1}\}_{8\times1}$ | Nodal displacement vector corresponding for Fourier mode $n \geq 1$ corresponding to displacements $a_n$, $d_n$ and $f_n$ |
| $\{\Delta_{n,2}\}_{8\times1}$ | Nodal displacement vector corresponding for Fourier mode $n \geq 1$ corresponding to displacements $b_n$, $c_n$ and $g_n$ |
| $\varepsilon_\phi$ | Total normal strain in the tangential direction |
| $\bar{\varepsilon}_\phi$ | Direct (membrane) strain in the tangential direction |
| $\varepsilon_z$ | Total normal strain in the axial direction |
| $\bar{\varepsilon}_z$ | Direct (membrane) strain in the axial direction |
| $\gamma_{z\phi}$ | Total shear strain |
| $\bar{\gamma}_{z\phi}$ | Direct (membrane) shear strain |
| $\gamma_s$ | Density of steel pipe |
| $\gamma_w$ | Density of water |
| $\kappa_\phi$ | Curvature in the tangential direction |
| $\kappa_z$ | Curvature in the axial direction |
| $\kappa_{z\phi}$ | Shear curvature |

| | |
|---|---|
| $\sigma_\phi$ | Normal stress in the tangential direction |
| $\sigma_z$ | Normal stress in the axial direction |
| $\sigma_\xi$ | Normal stress in the radial direction |
| $\tau_{z\phi}$ | Shear stress |
| $\pi$ | Total potential energy |
| $\mu$ | Poisson's Ratio |
| $\phi$ | Polar coordinate in the tangential direction |
| $\xi$ | Material coordinate in the radial direction. The range of $\xi$ is $-t/2$ to $+t/2$ |
| $a_n$, $d_n$, $f_n$, $b_n$, $c_n$, $g_n$ | Functions of $z$ only for mode $n$ of the Fourier expansion of the displacements |
| $\ell$ | Length of the pipe or pipe element |
| $n$ | Fourier mode under consideration |
| $M_n$, $m_n$ | Roots to the determinant of the operator matrix of the field equations $\left(M_n = m_n^2\right)$ |
| $p_1, p_2, p_3, p_4, p_5$ | Coefficients of the 4th order (quartic) polynomial equation resulting from the determinant of the operator matrix of the field equations |
| $q_u(z,\phi)$ | Applied tractions in the longitudinal direction |
| $q_v(z,\phi)$ | Applied tractions in the tangential direction |
| $q_w(z,\phi)$ | Applied tractions in the radial direction |

| | |
|---|---|
| $r$ | Radius of the middle surface of the pipe cross section |
| $t$ | Pipe wall thickness |
| $u(z,\phi)$ | Total displacement in the longitudinal direction |
| $v(z,\phi)$ | Total displacement in the tangential direction |
| $w(z,\phi)$ | Total displacement in the radial direction |
| $z$ | Coordinate in the longitudinal direction |
| $A$ | Area of the pipe middle surface $(A = 2\pi r\ell)$ |
| $\overline{A}_n, \overline{D}_n, \overline{F}_n,$ $\overline{B}_n, \overline{C}_n, \overline{G}_n$ | Integration constants in the assumed displacement field corresponding to Fourier mode $n$ |
| $\overline{\overline{A}}_n, \overline{\overline{D}}_n$ | Constants relating $\overline{A}_n$ and $\overline{D}_n$, respectively, to $\overline{F}_n$ |
| $\overline{\overline{B}}_n, \overline{\overline{C}}_n$ | Constants relating $\overline{B}_n$ and $\overline{C}_n$, respectively, to $\overline{G}_n$ |
| $C_{1,2\ldots17}$ | Constants based on the geometric and material properties of the pipe as well as the Fourier mode $n$ under consideration |
| $E$ | Young's Modulus |
| $G$ | Shear Modulus |
| $\left[K_{j,0}\right]_{8\times8}$ | Contribution to the total stiffness matrix for mode $n=0$ of term $j$ (for $j$ = 1 to 8) of the total internal strain energy equation (from Table 2.1) |
| $\left[K_0\right]_{8\times8}$ | Total stiffness matrix for mode $n=0$ |
| $\left[K_{j,n,1}\right]_{8\times8}$ | Contribution to the total stiffness matrix corresponding to nodal displacements $\left\{\Delta_{n,1}\right\}_{8\times1}$ for mode $n \geq 1$ of term $j$ (for $j$ = 1 to 8) of the total internal strain energy equation (from Table 2.1). |

| | |
|---|---|
| $\left[K_{n,1}\right]_{8\times8}$ | Total stiffness matrix corresponding to nodal displacements $\left\{\Delta_{n,1}\right\}_{8\times1}$ for mode $n\geq1$ |
| $\left[K_{j,n,2}\right]_{8\times8}$ | Contribution to the total stiffness matrix corresponding to nodal displacements $\left\{\Delta_{n,2}\right\}_{8\times1}$ for mode $n\geq1$ of term $j$ (for $j$ = 1 to 8) of the total internal strain energy equation (from Table 2.1). |
| $\left[K_{n,2}\right]_{8\times8}$ | Total stiffness matrix corresponding to nodal displacements $\left\{\Delta_{n,2}\right\}_{8\times1}$ for mode $n\geq1$ |
| $\left[L_0\right]_{8\times8}$ | Matrix relating the vector of integration constants $\left\{\bar{A}_0\right\}_{8\times1}$ to nodal displacements $\left\{\Delta_0\right\}_{8\times1}$ for mode $n=0$ |
| $\left[L_{1,1}\right]_{8\times8}$ | Matrix relating the vector of integration constants $\left\{\bar{F}_1\right\}_{8\times1}$ to nodal displacements $\left\{\Delta_{1,1}\right\}_{8\times1}$ for mode $n=1$ |
| $\left[L_{n,1}\right]_{8\times8}$ | Matrix relating the vector of integration constants $\left\{\bar{F}_n\right\}_{8\times1}$ to nodal displacements $\left\{\Delta_{n,1}\right\}_{8\times1}$ for mode $n\geq2$ |
| $\left[L_{1,2}\right]_{8\times8}$ | Matrix relating the vector of integration constants $\left\{\bar{G}_1\right\}_{8\times1}$ to nodal displacements $\left\{\Delta_{1,2}\right\}_{8\times1}$ for mode $n=1$ |
| $\left[L_{n,2}\right]_{8\times8}$ | Matrix relating the vector of integration constants $\left\{\bar{G}_n\right\}_{8\times1}$ to nodal displacements $\left\{\Delta_{n,2}\right\}_{8\times1}$ for mode $n\geq2$ |
| $\left\langle N_{X,n}(z)\right\rangle_{1\times8}^{\mathrm{T}}$ | Exact shape function vectors for the displacements corresponding to Fourier mode $n$ |
| $\left\{Q_n\right\}_{8\times1}$ | Load potential vector |
| $U$ | Internal strain energy |

| | |
|---|---|
| $W$ | Load potential |
| $V$ | Volume of the pipe element $(V = At = 2\pi r\ell t)$ |
| $[X_0]_{3\times3}$ | Field equation operator matrix mode $n = 0$ |
| $[X_n]_{3\times3}$ | Field equation operator matrix mode $n \geq 1$ |
| $[Y_n]_{3\times3}$ | Field equation operator matrix evaluated with the assumed displacement field for mode $n \geq 1$ |

# CHAPTER 1: INTRODUCTION

Pipes are commonly used to convey oil, natural gas and other commercial products. The ability to accurately model the displacements and stresses in a pipeline is required to produce an economical design. As the scale of pipeline projects can be quite large, the economic savings resulting from an efficient analysis can be significant. However, the model must be comprehensive enough to cope with the wide variety of loading conditions possible. Pipelines can be subjected to biaxial bending moments and shears, torsion as well as internal and external pressure in any combination. Failure to account for any of these loading conditions can have severe consequences.

Traditional shell finite element models are capable of accurately modeling pipe behaviour for any type of loading. However, the effort and resources required to create and execute such comprehensive models make them impractical as design tools. There is a need to develop a simple finite element model that can accurately predict displacements and stresses. Simple elements have been developed in the past but have been hampered with restrictive assumptions regarding the displacement field and associated strains.

In the current study, a simple yet accurate element is established which can model general loading conditions. The element is not restricted by the assumptions made with previous elements and is based on the exact solution of the equilibrium equations.

## 1.1 Previous Work

Much of the previous computational research conducted in the field of stresses and displacements of pipes has been concerned with the deformation of pipe bends subjected to bending moments in the plane of the bend. However, the majority of the results can be applied to straight pipes by assuming the radius of the pipe bend to be infinitely large. Recent studies have also investigated the effects of out-of-plane bending and even general loading on pipe deformation.

The sign convention adopted in most previous studies, and in the present study, for the positive direction of the longitudinal displacement, $u$, tangential displacement, $v$, and radial displacement, $w$, is shown in Figure 1.1.

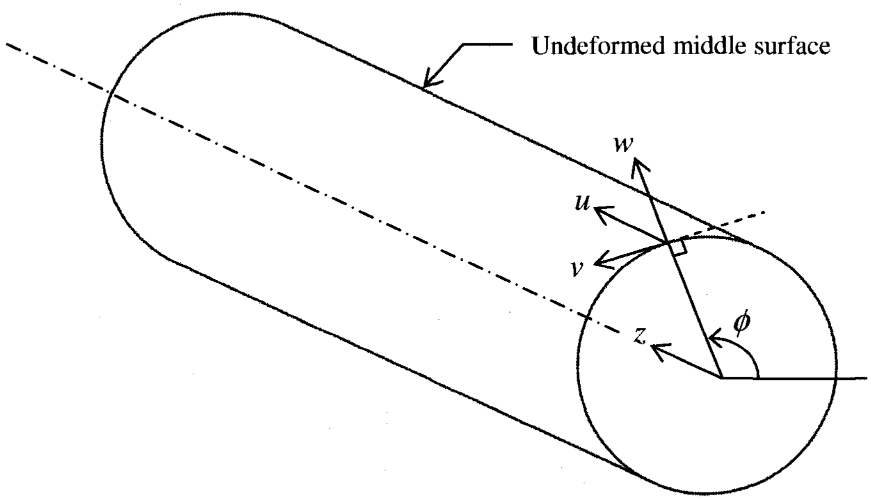


**Figure 1.1** – Sign convention for displacements $u$, $v$, and $w$

In the following, a summary of important work on the analysis of pipelines is presented.

### *von Karman (1911)*

The pioneering work in the study of pipe deformation was conducted by von Karman (1911) and much of the research carried out in the past century is based on his theory. Von Karman's research was related to the response of pipe elbows when subjected to bending moments in the plane of elbow bend. He realized that deformation of the pipe cross section, specifically ovalization, induces additional strains not included in the traditional curved beam theory. The additional strains due to ovalization include a circumferential strain, $\varepsilon_\phi$, and a longitudinal strain, $\varepsilon_z$, that is in addition to the longitudinal strain captured by the traditional beam theory. In his analysis, von Karman included a number of simplifying assumptions:

1. Cross sections which are plane and normal to the neutral axis before deformation remain plane and normal to the neutral axis after deformation.

2. Longitudinal strains, $\varepsilon_z$, are constant throughout the pipe wall thickness (only membrane longitudinal strains are present).

3. Circumferential strains, $\varepsilon_\phi$, vanish at the pipe wall middle surface. All circumferential strains are a result of transverse bending of the pipe wall.

4. The elbow bend radius is much larger than the pipe middle surface radius.

5. Poisson's ratio is disregarded.

In simple terms, Assumption 1 states that warping of the pipe cross section is not accounted for, while Assumption 3 states that pipe is inextensible in the radial direction. Much of the research that followed von Karman adopted the same two assumptions. In contrast, these assumptions are not made in the present study in which both warping and pipe extensibility are represented.

By minimizing the total potential energy of the system, von Karman concluded that for an elbow subjected to in-plane bending, the radial displacement, $w$, of the pipe cross section is given as:

$$w = \sum_{n=1}^{N} c_n \sin 2n\phi \tag{1.1}$$

in which von Karman determined the parameters $c_n$ by Ritz analysis. In Eq. 1.1, the term $N$ is the total number of modes to be included in the analysis. Von Karman gave suggestions on the value of $N$ based on the geometry of the problem.

### *Wood (1958)*

Wood (1958) extended von Karman's work by including the effects of internal pressure. He adopted the same restrictive assumptions as von Karman, including neglecting warping effects and assuming the pipe as inextensible. Wood considered a long, uniformly pressurized circular elbow subjected to a uniform in-plane bending moment. Under the influence of internal pressure and in-plane bending, Wood postulated that the total displacement of a point on the pipe middle surface would be the superposition of three displacement components: St. Venant beam displacement, ovalization displacement and pressurization displacement. He then expanded the tangential displacement, $v$, and radial displacement, $w$, due to ovalization as Fourier series:

$$v(z,\phi)=\sum_{n=1}^{\infty}A_n(z)\sin n\phi \tag{1.2a}$$

$$w(z,\phi)=\sum_{n=1}^{\infty}nA_n(z)\cos n\phi \tag{1.2b}$$

The coefficients $A_n$ were determined by minimizing the total potential energy of the system. In his work, Wood assumed that $A_n = 0$ for $n \geq 4$, thus taking only the first three Fourier modes. Note that mode zero was not included in the Fourier expansion as it was accounted for directly in the calculation of the pressurization displacement.

### *Ohtsubo and Watanabe (1978)*

Ohtsubo and Watanabe (1978) developed a finite element based on ring elements. In the development of their element, they relaxed some of von Karman's assumptions and accounted both for shear deformation and warping of the pipe cross section. However, the inextensibility assumption for the pipe cross section was retained. While the element was developed to model pipe elbows, the authors noted that it can be made equally suitable for straight pipes by assuming an infinitely long bend radius. In addition, both the in-plane bending and out of plane bending were considered. Strain expressions were developed based on the Love-Kirchhoff thin shell theory, which states that "straight fibres perpendicular to the middle surface before deformation remain straight and perpendicular to the middle surface after deformation" (Ohtsubo and Watanabe, 1978). The same Love-Kirchhoff model is also followed in the current study.

The displacements in the longitudinal, tangential and radial directions, $u$, $v$ and $w$ respectively, were expanded in Fourier series. Separate Fourier series were developed based on whether the element was subjected to in-plane or our-of-plane bending.

In-plane bending:

(1.3a) $$u(z,\phi)=\sum_{n=0,2,4...} a_n(z)\cos n\phi+\sum_{n=1,3,5...} b_n(z)\sin n\phi$$

(1.3b) $$v(z,\phi)=\sum_{n=1,3,5...} c_n(z)\cos n\phi+\sum_{n=2,4,6...} d_n(z)\sin n\phi$$

(1.3c) $$w(z,\phi)=\sum_{n=0,2,4...} e_n(z)\cos n\phi+\sum_{n=1,3,5...} f_n(z)\sin n\phi$$

Out-of-plane bending:

(1.4a) $$u(z,\phi)=\sum_{n=1,3,5...} a_n(z)\cos n\phi+\sum_{n=2,4,6...} b_n(z)\sin n\phi$$

(1.4b) $$v(z,\phi)=\sum_{n=0,2,4...} c_n(z)\cos n\phi+\sum_{n=1,3,5...} d_n(z)\sin n\phi$$

(1.4c) $$w(z,\phi)=\sum_{n=1,3,5...} e_n(z)\cos n\phi+\sum_{n=2,4,6...} f_n(z)\sin n\phi$$

In Eqs. 1.3 and 1.4 the coefficients $a_n(z)$, $b_n(z)$, $c_n(z)$, $d_n(z)$, $e_n(z)$ and $f_n(z)$ are functions of the longitudinal coordinate, $z$. The functions were interpolated between nodes with second order Hermite polynomials. The total potential energy expression was formulated and minimized with respect to the nodal displacement parameters in order to recover the discretized form of the equilibrium equations. Results were presented for a number of specialized cases, including: (1) uniform in-plane bending; (2) neglecting the effects of cross section warping; and (3) neglecting the effects of shear deformation.

### *Bathe and Almeida (1980)*

Bathe and Almeida (1980) developed a linear finite element which also relaxed a number of von Karman's assumptions. The von Karman formulation was based on the assumption that the bending moments and ovalization were constant along the length of the elbow. In Bathe and Almeida's formulation, however, the ovalization was assumed to vary along the length of the elbow in the form of a cubic polynomial. This assumed ovalization distribution allowed for the fact that no ovalization may be present at the ends of the elbow while being present in the intermediate section. In addition, von Karman assumptions 2, 4 and 5 were relaxed, permitting the element to: (1) capture the varying longitudinal strains through the pipe wall thickness; (2) model elbows with a small bend radius relative to the pipe radius; and (3) capture the effects of Poisson's ratio. The element does, however, retain von Karman assumptions 1 and 3 and thus the warping of the cross section is neglected and the pipe wall is assumed inextensible in the radial direction.

The element had four nodes and postulated the displacement of a point on the cross as a combination of centreline displacement (beam displacement) and cross section deformation. The centreline displacement was expressed in terms of the nodal displacements and interpolation functions that vary cubically with respect to the longitudinal coordinate. Displacements in the tangential and radial directions, representing the deformation of the cross section, were described with Fourier series expansions similar to those used by Ohtsubo and Watanabe (1978). The difference is that Bathe and Almeida base their interpolation functions on four nodes within the element rather than two. Cubic polynomials were used for the interpolation functions. The formulation of the Fourier series expansion allowed for both in-plane and out-of-plane bending moments to be considered. Equilibrium equations were established and solved to determine the unknown displacement parameters.

Bathe and Almeida continued their work to provide enhancements to the element. Subsequent improvements included developing interaction effects between elbow elements and straight pipe segments (Bathe and Almeida, 1982) and developing the non-linear capabilities for the element (Bathe and Almeida, 1983).

### *Militello and Huespe (1988)*

Further improvements to the Bathe and Almeida element were made by Militello and Huespe (1988) who relaxed the assumption that plane sections remain plane. The element formulation was similar to that presented by Bathe and Almeida in that it used four nodes, assumed the pipe inextensible in the radial direction and expanded the tangential and radial displacements with Fourier series. The contribution of Militello and Huespe was to also expand the longitudinal displacement in a Fourier series to account for warping of the cross section. The coefficients of the longitudinal displacement Fourier series were interpolated with third-order Lagrangian polynomials. Their formulation allowed the number of Fourier terms to be specified by the analyst.

### *Abo-Elkhier (1990)*

Abo-Elkhier (1990) also improved upon the Bathe and Almeida element by presenting a formulation consistent with thin shell theory. The same assumptions employed by Bathe and Almeida (1980) were used, including pipe inextensibility in the radial direction and neglecting the effects of warping. Abo-Elkhier improved the element by including additional strain terms that Bathe and Almeida did not take into consideration. By developing a formulation consistent with thin shell theory, Abo-Elkhier incorporated all direct membrane strains as well as the strains induced by bending of the pipe wall. The Fourier series expansions and interpolation functions were consistent with those used by Bathe and Almeida.

### *Yan et al. (1999)*

A significant improvement to the elements developed by Bathe and Almeida (1980) and Militello and Huespe (1988) was made by Yan et al. (1999) by relaxing the assumption of pipe inextensibility in the radial direction. The remainder of the formulation is consistent with that presented by Militello and Huespe. Cross section deformations, in the form of warping and ovalization, were added to the beam displacement to give the total displacement of the pipe. The longitudinal, tangential and radial displacements were all expressed as Fourier series in the circumferential direction with the interpolation functions being the same as those derived by Militello and Huespe, namely a third-order Lagrangian polynomial for the longitudinal displacement and third-order Hermite polynomials for the tangential and radial displacements. The importance of accounting for pipe extensibility was highlighted as it allows the effects of internal pressure to be included. The internal pressure in the elbow causes direct tangential stresses due to radial expansion, stiffens the elbow which in turn reduces ovalization due to bending and creates a non-uniform stress distribution in the elbow. All of these effects were captured in the new element. The authors also extended their work to determine the plastic limit load of a pipe elbow.

### *Millard and Roche (1984)*

Several variations of the elbow bending problem have also been examined. Millard and Roche (1984) considered the propagation of ovalization from an elbow subjected to in-plane bending along an adjacent straight section of pipe. For a straight pipe subjected to end displacements $v(\phi) = -(a_0/2)\sin 2n\phi$ and $w(\phi) = a_0 \cos 2n\phi$, they expressed the longitudinal, tangential and radial displacements, *u*, *v*, and *w*, as Fourier series:

(1.5a) $$u(z,\phi) = \sum_{n=1}^{5} b_n(z)\cos 2n\phi$$

(1.5b) $$v(z,\phi) = \sum_{n=2}^{5} -\frac{a_n(z)}{2n}\sin 2n\phi$$

(1.5c) $$w(z,\phi) = \sum_{n=1}^{5} a_n(z)\cos 2n\phi$$

A unique feature of the element is that conventional interpolation functions for the coefficients of the Fourier series were not postulated. Rather, the coefficient functions $a_n(z)$ and $b_n(z)$ were assumed to take the form:

(1.6a) $$a_n(z) = \alpha_n e^{\lambda_n z}$$

(1.6b) $$b_n(z) = \beta_n e^{\lambda_n z}$$

The equilibrium equations were established by minimizing the total potential energy of the system. The result was two coupled, fourth order differential equations which were solved for $\lambda_n$. The coefficients $\alpha_n$ and $\beta_n$ were determined from the boundary conditions. The solution procedure is similar to the one used in the current study, with the exception being the loading conditions and assumed strains. Millard and Roche subjected their element to a prescribed displacement at one end and investigated the propagation of the displacement along the length of the element. In contrast, the element in the current study can be subjected to any general loading condition. Also, Millard and Roche neglected the strains due to shear curvature, $\kappa_{\phi z}$, and the tangential hoop strain, $\varepsilon_\phi$ (i.e. pipe inextensibility was assumed). Both of these strains are included in the current study.

The propagation of ovalization due to Fourier modes $2n = 2$ through $2n = 10$ were compared and they discovered that in the straight section of the pipe mode 2 ovalization dominated and the higher modes decayed quickly. The effects of further simplifying assumptions were investigated. These are: (1) neglecting the warping of the cross section; (2) neglecting shear deformation; and (3) neglecting the variation of the bending curvature, $\kappa_{zz}$, in the longitudinal direction. The results based on simplified displacement fields were compared to those based the full formulation. Millard and Roche's solution was only presented analytically and was not implemented in a finite element context.

### *Berton (1992)*

Berton (1992) extended the analytical work of Millard and Roche by investigating the propagation of ovalization of an elbow element subject to in-plane and out-of-plane bending as well as torsional loading along an adjacent straight pipe. The assumptions and displacement functions were identical to those proposed by Millard and Roche. The solution also followed the same procedure but accounts for out-of-plane bending and torsional loads. Only mode 2 ovalization was considered.

### *Karamanos et al. (2002)*

Karamanos (2002) broadened the previous work to include buckling instability of pipes. His work included non-linear elastic effects and investigated the relationship between ovalization and buckling of pipes subjected to a uniform bending moment. This work was followed up by Karamanos et al. (2003) who included the effects of plasticity and internal pressure. They further investigated the influence of the type of loading on the failure mode.

### *Madureira and de Melo (2000, 2004); Fonesca, et al. (2002)*

Simplified analytical models to describe pipe deformation have also been developed. Madureira and de Melo (2000, 2004) and Fonesca, et al. (2002) developed elements based on a semi-membrane formulation. Strains due to longitudinal curvature, $\kappa_z$, and shear curvature, $\kappa_{\phi z}$, were neglected resulting in a much simplified set of equilibrium equations. Warping of the cross section was considered but the pipe was assumed to be inextensible in the radial direction. Fourier series expansions were used to describe the displacement in the longitudinal, tangential and radial directions. The coefficients of the Fourier series were solved exactly for given loading and boundary conditions. The solution procedure was similar to that presented by Millard and Roche (1984).

The solution presented in the current study follows the general procedure used by Millard and Roche (1984) and Berton (1992). However, some of the limiting assumptions in the previous work are relaxed in the current study. In addition, the solution obtained is implemented in a finite element context.

## 1.2 Terms Included in Fourier Series Expansion

In much of the previous work outline in Section 1.1, Fourier series expansions are used to describe the longitudinal, tangential and radial displacements, $u$, $v$ and $w$, in terms of the longitudinal variable $z$ and the tangential variable $\phi$. The general form of the expansion is:

$$\text{(1.7a)} \quad u(z,\phi)=\sum_{n=0}^{\alpha_s} s_u(z)\cos n\phi+\sum_{n=0}^{\alpha_t} t_u(z)\sin n\phi$$

$$\text{(1.7b)} \quad v(z,\phi)=\sum_{n=0}^{\alpha_s} s_v(z)\cos n\phi+\sum_{n=0}^{\alpha_t} t_v(z)\sin n\phi$$

$$\text{(1.7c)} \quad w(z,\phi)=\sum_{n=0}^{\alpha_s} s_w(z)\cos n\phi+\sum_{n=0}^{\alpha_t} t_w(z)\sin n\phi$$

In Eq. 1.7 $\alpha_s$ and $\alpha_t$ represent the number of terms included in the expansion for $\cos n\phi$ and $\sin n\phi$, respectively, while $s_u(z)$, $s_v(z)$, $s_w(z)$, $t_u(z)$, $t_v(z)$ and $t_w(z)$ are functions of $z$ only, which are to be determined. As outlined in Section 1.1, functions $s_u(z)$, $s_v(z)$, $s_w(z)$, $t_u(z)$, $t_v(z)$ and $t_w(z)$ were either solved exactly or approximated by polynomial interpolation functions.

Considerable disparity exists among the previous works with regards to the number of terms included and the postulated shape functions which express $s_u(z)$, $s_v(z)$, $s_w(z)$, $t_u(z)$, $t_v(z)$ and $t_w(z)$. In some work, only even or odd terms are included while other work includes all terms. In addition, a wide variety exists in the number of terms included in the expansion. Some work includes only two terms while others include up to sixteen terms.

The number and type of terms included is dependent on many factors such as loading conditions, assumed displacements and the amount of accuracy desired. In many studies, only bending in the plane of the pipe elbow was considered. For such loading cases only the even or odd terms are required for the Fourier expansion, depending on which displacement is being defined. Conversely, for out-of-plane bending the opposite terms will be required. Both odd and even terms are required to describe the deformation resulting from combined in-plane and out-of-plane loading or any other general loading condition. For certain loading cases, however, only one particular Fourier mode may be required to completely describe the displacement. For example, as will be shown in Chapters 4 and 6, the displacement of a cantilevered pipe subjected to only its own self weight can be completely described with only Fourier mode $n = 1$.

Assumptions regarding the displacement field of the pipe also have a significant impact on which terms are to be included in the expansion. Much of the early work disregards warping of the cross-section, assuming instead that the cross-section remains plane during deformation. As such, a Fourier series expansion is not required to describe the longitudinal displacement of the pipe. In these formulations Fourier series expansions were required only for the tangential and longitudinal displacements.

Table 1.1 contains a summary of the terms included in the Fourier expansions in the previous work for the longitudinal, transverse and radial displacements of the pipe. Also included in the table are the loading and displacement assumptions employed in each of the works. Note that previous works present elastic solutions with no geometric or material non-linear effects.

**Table 1.1** – Number of terms $n$ included in the Fourier series expansions of displacements $u$, $v$ and $w$

| Source | Notes | Longitudinal displacement, $u$ | | Tangential displacement, $v$ | | Radial displacement, $w$ | |
|---|---|---|---|---|---|---|---|
| | | $\cos n\phi$ | $\sin n\phi$ | $\cos n\phi$ | $\sin n\phi$ | $\cos n\phi$ | $\sin n\phi$ |
| Wood, 1958 | 1,6 | none | none | 1,2,3 | none | 0,2,3 | none |
| Ohtsubo and Watanabe, 1978 | 6 | 0,2,4... | 1,3,5... | 1,3,5... | 2,4,6... | 0,2,4... | 1,3,5... |
| | 7 | 1,3,5... | 2,4,6... | 0,2,4... | 1,3,5... | 1,3,5... | 2,4,6... |
| Bathe and Almeida, 1982 | 1,2,8 | none | none | 0,2,4,6 | 0,2,4,6 | 0,2,4,6 | 0,2,4,6 |
| Millard and Roche, 1984 | 6 | 2 | none | none | 2 | 2 | none |
| Militello and Huespe, 1988 | 3,6 | 0,1,4,6,8 | 0,1,4,6,8 | 0,2,4,6 | 0,2,4,6 | 0,2,4,6 | 0,2,4,6 |
| Abo-Elkhier, 1989 | 1 | none | none | 0,2,4,6 | 0,2,4,6 | 0,2,4,6 | 0,2,4,6 |
| Yan, Jospin and Nguyen, 1999 | | 2,3,4 | 2,3,4 | 2,4,6 | 2,4,6 | 2,4,6 | 2,4,6 |
| Karamanos et. al, 2003 | 6 | 2,4,6...,16 | 3,5,7...,15 | 3,5,7...,15 | 1,2,4...,16 | 0,2,...,16 | 1,3,5...,15 |
| Fonesca, Melo and Oliveira, 2002 | 4,5,6 | 0,1,2,4,6,8 | none | none | 0,1,2,4,6,8 | 0,1,2,4,6,8 | none |
| Melo and Madureira, 2004 | 5 | 1,2,3,4,5 | none | none | 1,2,3,4,5 | 1,2,3,4,5 | none |

Notes for Table 1.1:

1. Warping of the cross section is neglected; plane sections remain plane.
2. The number of terms included depends on the pipe and bend geometry.
3. Terms $n = 0$ and $n = 1$ for the longitudinal displacement are accounted for in the beam theory.
4. The solution was also implemented with the odd terms included and the results were compared.
5. Semi-membrane formulation was used in which longitudinal curvature and curvature due to shear were neglected.
6. Only in-plane bending was considered.
7. Only out-of-plane bending was considered.
8. Both in-plane and out-of-plane bending were considered.

For a pipe subjected to a general loading condition, the accuracy of the solution increases as the number of Fourier modes included in the analysis increases. The contribution of each Fourier mode to the total displacement is further examined with examples in Chapters 4 and 6.

## 1.3 Strain and Curvature Terms

In the previous work, as well as in the current study, the equilibrium equations are established by minimizing the total potential energy of the system. The internal strain energy of a pipe is a function of its total strain in the longitudinal and tangential direction and its shear strain. Each of these strains consists of direct (or membrane) strains plus strains due to curvature. The direct and curvature strains are in turn functions of the longitudinal, tangential and radial displacements, $u$, $v$ and $w$.

The previous works, as well as textbooks on thin shell theory (Wang, 1953; Flugge, 1973; Saada, 1974), all agree on the expressions for the direct strain. However, different sources provide different expressions for the strains due to curvature. Table 1.2 contains a comparison of the expressions for strains due to curvature in all three directions from different sources.

Note that the sign convention from some sources in Table 1.2 has been modified to match assumed positive direction of the tangential displacement, $v$, given in Figure 1.1.

**Table 1.2** – Expressions for strains due to curvature

| Source | Tangential curvature $\kappa_\phi(z,\phi)$ | Longitudinal curvature $\kappa_z(z,\phi)$ | Shear curvature $\kappa_{z\phi}(z,\phi)$ |
|---|---|---|---|
| Wang (1953) | $\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-\frac{\partial v}{\partial\phi}\right)$ | $\frac{\partial^2 w}{\partial z^2}$ | $\frac{2}{r}\left(\frac{\partial^2 w}{\partial\phi\partial z}-\frac{1}{2}\frac{\partial v}{\partial z}\right)$ |
| Flugge (1973) | $\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-w\right)$ | $\frac{\partial^2 w}{\partial z^2}$ | $\frac{1}{r}\left(\frac{\partial^2 w}{\partial\phi\partial z}+\frac{1}{2r}\frac{\partial u}{\partial\phi}-\frac{1}{2}\frac{\partial v}{\partial z}\right)$ |
| Saada (1974) | $\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-\frac{\partial v}{\partial\phi}\right)$ | $\frac{\partial^2 w}{\partial z^2}$ | $\frac{2}{r}\left(\frac{\partial^2 w}{\partial\phi\partial z}-\frac{1}{2}\frac{\partial v}{\partial z}\right)$ |
| Ohtsubo and Watanabe (1978) | $\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-\frac{\partial v}{\partial\phi}\right)$ | $\frac{\partial^2 w}{\partial z^2}$ | $\frac{2}{r}\left(\frac{\partial^2 w}{\partial\phi\partial z}-\frac{\partial v}{\partial z}\right)$ |
| Millard and Roche (1984) | $\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-\frac{\partial v}{\partial\phi}\right)$ | $\frac{\partial^2 w}{\partial z^2}$ | $\frac{2}{r}\left(\frac{\partial^2 w}{\partial\phi\partial z}-\frac{\partial v}{\partial z}\right)$ |
| Niordson (1985) | $\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-\frac{2\partial v}{\partial\phi}-w\right)$ | $\frac{\partial^2 w}{\partial z^2}$ | $\frac{2}{r}\left(\frac{\partial^2 w}{\partial\phi\partial z}-\frac{\partial v}{\partial z}\right)$ |
| Abo-Elkhier (1990) | $\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-\frac{\partial v}{\partial\phi}\right)$ | $\frac{\partial^2 w}{\partial z^2}$ | $\frac{2}{r}\left(\frac{\partial^2 w}{\partial\phi\partial z}-\frac{\partial v}{\partial z}\right)$ |
| Berton (1992) | $\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-\frac{\partial v}{\partial\phi}\right)$ | $\frac{\partial^2 w}{\partial z^2}$ | $\frac{2}{r}\left(\frac{\partial^2 w}{\partial\phi\partial z}-\frac{\partial v}{\partial z}\right)$ |
| Madureira and Melo (2000, 2004) | $\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-\frac{\partial v}{\partial\phi}\right)$ | 0 * | 0 * |

* Semi-membrane formulation was used which does not account for longitudinal and shear curvatures

The variation in the terms for strains due to curvature is a result of the assumptions made while deriving the expressions. For consistency with most recently published work, the strains due to curvature in the current study are assumed to be the same as those first presented by Ohtsubo and Watanabe (1978).

## 1.4 Outline and Objective

The objective of the current study is to develop a new finite element that accurately predicts pipeline deformations under general loading conditions. To accomplish this, the strain-displacement relationships for the pipe are established in Chapter 2 within a defined set of assumptions. The stationarity condition is then invoked on the total potential energy of the pipe, which is defined in terms of the strain-displacement relationships, to generate the equilibrium equations and boundary conditions. The displacement field for the pipe is ascertained in Chapter 3 by solving the equilibrium equations within the constraints of the boundary conditions. Chapter 4 presents two examples to verify the displacement field determined in Chapter 3. The results from the current method are compared to both the Euler-Bernoulli beam theory solution and the results from a traditional shell finite element analysis using the commercial software program ABAQUS.

In Chapter 5, the displacement field solution is implemented in a finite element context. The displacement at any point on the pipe middle surface is related to the nodal displacements through the development of exact shape functions. Several examples are presented in Chapter 6 to show the effectiveness of the element. The results are compared to the results of a shell finite element analysis conducted in ABAQUS. Some conclusions and recommendations are discussed in Chapter 7.

# CHAPTER 2: FORMULATION OF EQUILIBRIUM CONDITIONS

The exact displacement fields for a pipe are developed using the principle of stationary potential energy. The unknown displacement fields are expressed as Fourier series, which are used to develop expressions for the strains within a set of assumptions. From these strains and corresponding stresses, the internal strain energy of the pipe is formulated. The internal strain energy is combined with the potential energy from externally applied tractions to give the total potential energy of the system.

Equilibrium is found by evoking the stationarity conditions for the total potential energy functional with respect to the unknown functions. The result is a set of equilibrium equations and boundary conditions. These equations are subsequently solved for the unknown functions in the Fourier series, allowing the displacement of the pipe to be exactly expressed within the limitations of the assumptions of the formulation.

## 2.1 Element Geometry

The pipe element under consideration is shown in Figure 2.1 both in its undeformed and deformed configurations. The geometry of the element is described in a cylindrical coordinate system $(z,\phi,\xi)$. The $z$ direction and corresponding displacement $u$ are orientated along the longitudinal axis of the element. The $\phi$ direction and corresponding displacement $v$ are orientated in a direction tangent to the cross section. Finally, the $\xi$ direction and corresponding displacement $w$ are orientated in a radial direction away from the center of the pipe cross section, with $\xi$ measured from the middle surface of the pipe.

Figure 2.2 shows the pipe cross section, both undeformed and deformed, with thickness $t$ and middle surface radius $r$. As loads are applied to the pipe, point $A$ on the undeformed middle surface moves to $A'$ on the deformed middle surface. The total displacement $\Delta$ of the point $A$ is composed of the tangential displacement $v_A$ and the radial displacement $w_A$. Note that the displacement $v_A$ is tangent to the undeformed middle surface and the displacement $w_A$ is in the radial direction of the undeformed middle surface.

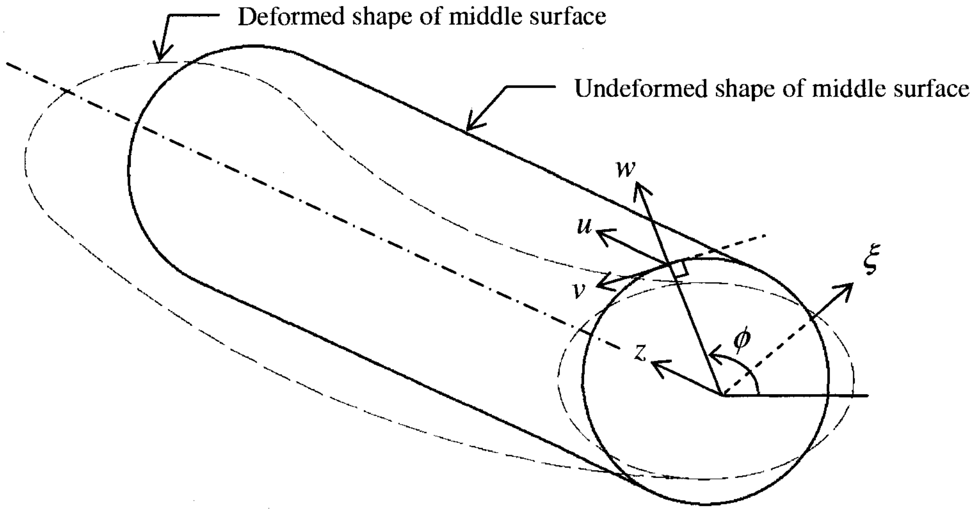


**Figure 2.1** – Pipe element deformation and coordinate system

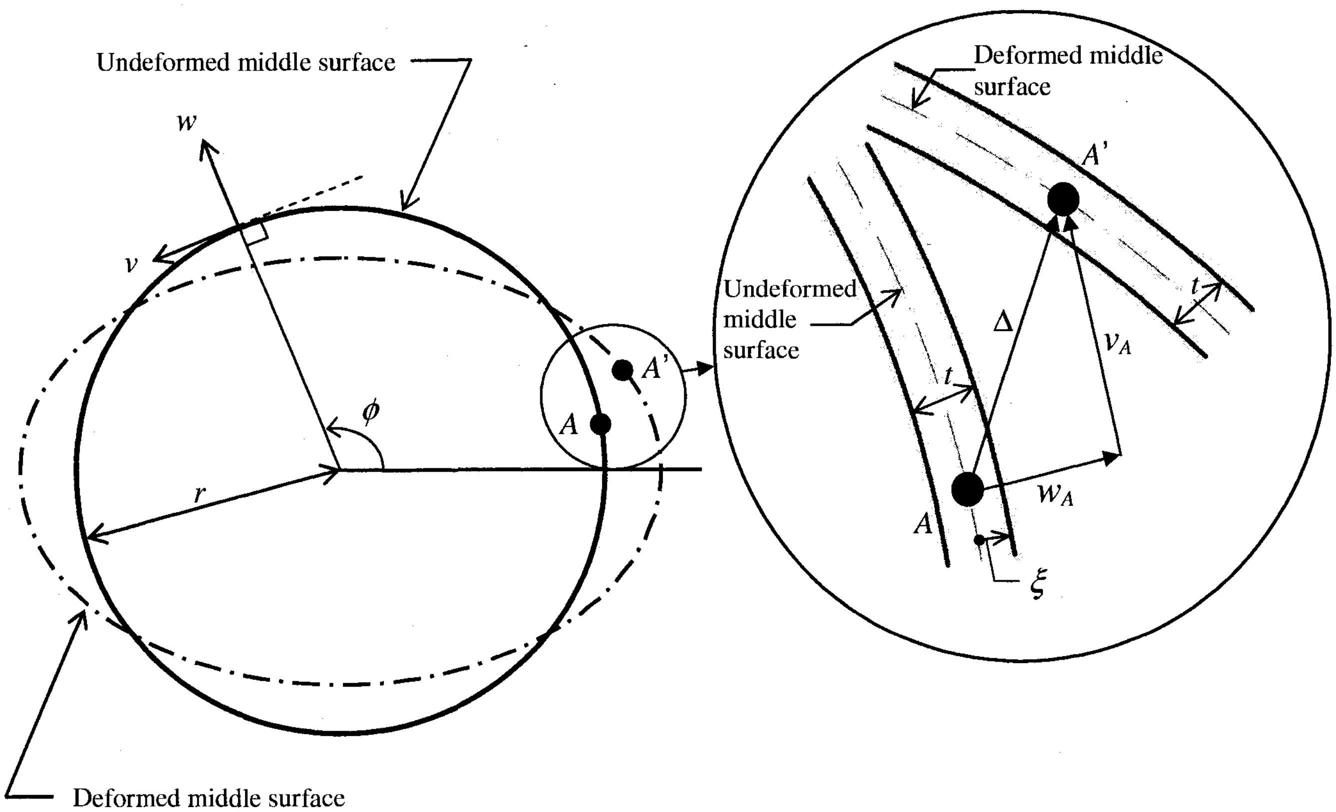


**Figure 2.2** – Pipe cross section deformation

The stresses acting on a small piece of the pipe are shown in Figure 2.3. Normal stresses in the $z$, $\phi$ and $\xi$ directions, as well as shear stresses, are present in the element.

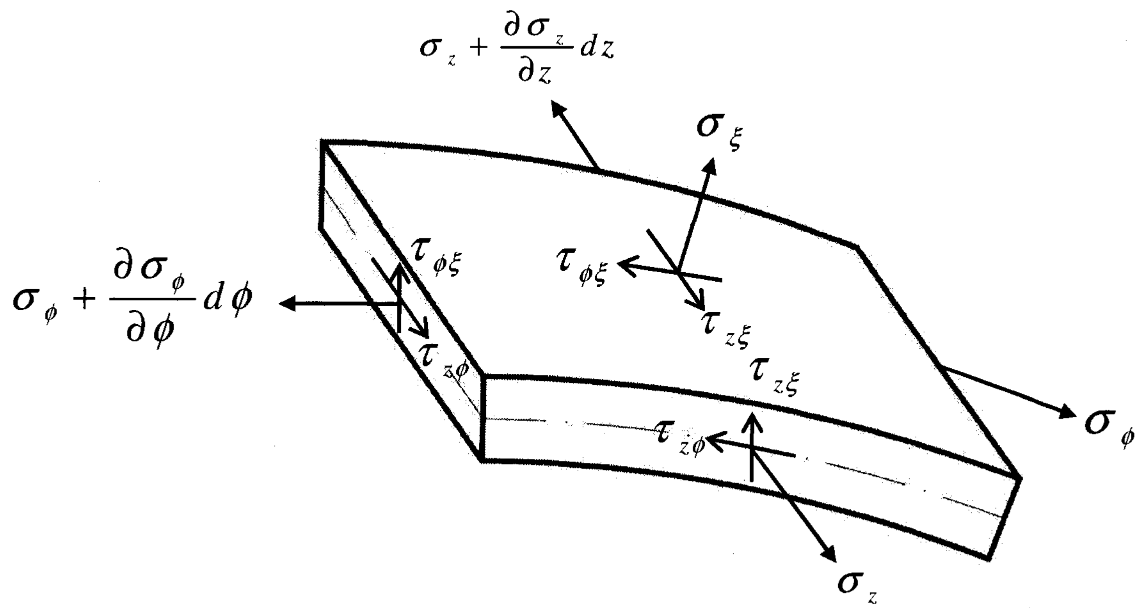


**Figure 2.3** – Stresses on a pipe element with middle surface area $dA = rdzd\phi$

## 2.2 Assumptions

Various assumptions have been made in the development of the element, as outlined below:

1. The pipe element is initially straight with no bends.

2. The pipe wall thickness $t$ and middle surface radius $r$ are constants with respect to tangential coordinate, $\phi$, and longitudinal coordinate, $z$.

3. The pipe is modeled as a thin-walled member, meaning that the ratio of pipe wall thickness to cross section radius is small $(r \gg t)$.

4. Normal stresses in the radial direction throughout the pipe wall thickness are negligible $(\sigma_\xi = 0)$.

5. Transverse shear stresses in the radial direction are negligible $\left(\tau_{\xi z} = \tau_{\xi\phi} = 0\right)$.

6. The Kirchhoff-Love hypothesis is assumed: Straight fibres initially perpendicular to the middle surface before deformation remain straight and perpendicular to the middle surface after deformation.

7. Pipe material is linearly elastic and isotropic.

8. All strains are small.

## 2.3 Total Potential Energy

The displacement field for the pipe is developed by using the principle of stationarity of total potential energy. The total potential energy of the system, $\pi$, is the difference between the internal strain energy in the pipe, $U$, and the work done by externally applied loads, $W$:

$$\pi = U - W$$

$$\begin{aligned} \pi &= \frac{1}{2}\int_V \left\{\varepsilon_z\sigma_z + \varepsilon_\phi\sigma_\phi + \gamma_{z\phi}\tau_{z\phi}\right\}dV \\ &\quad - \int_S \left\{q_u(z,\phi)u(z,\phi) + q_v(z,\phi)v(z,\phi) + q_w(z,\phi)w(z,\phi)\right\}dS \end{aligned} \tag{2.1}$$

In Eq. 2.1 the functions $q_u(z,\phi)$, $q_v(z,\phi)$ and $q_w(z,\phi)$ are the tractions acting on the pipe middle surface in the *u*, *v* and *w* directions respectively. Note that $\sigma_\xi \approx \tau_{\xi z} \approx \tau_{\xi\phi} \approx 0$ as a result of applying Assumptions 4 and 5 in Section 2.2. Thus, terms involving these three stress components are not included in Eq. 2.1.

### *2.3.1 Stress – Strain Relationships*

Applying Hooke's Generalized Law and Assumptions 4 and 5 gives the well known stress-strain relationships in the longitudinal and tangential directions as well as the shear stress-strain relationship:

$$\sigma_z(z,\phi,\xi) = \frac{E}{1-\mu^2}\left[\varepsilon_z(z,\phi,\xi) + \mu\varepsilon_\phi(z,\phi,\xi)\right] \tag{2.2a}$$

(2.2b) $$\sigma_\phi(z,\phi,\xi)=\frac{E}{1-\mu^2}\left[\varepsilon_\phi(z,\phi,\xi)+\mu\varepsilon_z(z,\phi,\xi)\right]$$

(2.2c) $$\tau_{z\phi}(z,\phi,\xi)=\frac{E}{2(1+\mu)}\gamma_{z\phi}(z,\phi,\xi)=G\gamma_{z\phi}(z,\phi,\xi)$$

In Eq. 2.2, $\mu$ is Poisson's Ratio, $E$ is Young's Modulus and $G$ is the shear modulus of the pipe material. Stresses $\sigma_z$, $\sigma_\phi$ and $\tau_{z\phi}$ are defined in Figure 2.3.

The total strain at an infinitesimal element of pipe at a distance $\xi$ from the mid-surface consists of two parts: the direct or membrane strain and strain due to curvature. Based on Assumption 6 in Section 2.2, the total strain is given by:

(2.3a) $$\varepsilon_z(z,\phi,\xi)=\bar{\varepsilon}_z(z,\phi)+\xi\kappa_z(z,\phi)$$

(2.3b) $$\varepsilon_\phi(z,\phi,\xi)=\bar{\varepsilon}_\phi(z,\phi)+\xi\kappa_\phi(z,\phi)$$

(2.3c) $$\gamma_{z\phi}(z,\phi,\xi)=\bar{\gamma}_{z\phi}(z,\phi)+\xi\kappa_{z\phi}(z,\phi)$$

In Eqs. 2.3, $\xi$ is the material coordinate measured from the middle surface of the pipe (see Figure 2.2). The range of $\xi$ is $-t/2$ to $t/2$.

### *2.3.2 Strains in Terms of Mid-surface Displacements*

Using the principles of tensor analysis, membrane strains $\bar{\varepsilon}_z$, $\bar{\varepsilon}_\phi$ and $\bar{\gamma}_{z\phi}$ and bending curvatures $\kappa_z$, $\kappa_\phi$ and $\kappa_{z\phi}$ can be expressed in terms of the mid-surface displacements $u$, $v$ and $w$ and their derivatives (Ohtsubo and Watanabe, 1978) as:

(2.4a) $$\bar{\varepsilon}_z(z,\phi)=\frac{\partial u}{\partial z}$$

(2.4c) $$\bar{\gamma}_{z\phi}(z,\phi)\approx\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial u}{\partial \phi}$$

(2.4b) $$\bar{\varepsilon}_\phi(z,\phi)\approx\frac{1}{r}\left(w+\frac{\partial v}{\partial \phi}\right)$$

(2.4d) $$\kappa_z(z,\phi)=\frac{\partial^2 w}{\partial z^2}$$

$$(2.4e)\qquad \kappa_\phi(z,\phi)=\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial\phi^2}-\frac{\partial v}{\partial\phi}\right) \qquad\qquad (2.4f)\qquad \kappa_{z\phi}(z,\phi)=\frac{2}{r}\left(\frac{\partial^2 w}{\partial z\partial\phi}-\frac{\partial v}{\partial z}\right)$$

Several variations of Eq. 2.4 appear in the literature, as discussed in Chapter 1. The expressions presented above and used in the current study correspond to those used most often in recent publications.

### 2.3.3 *Internal Strain Energy in Terms of Strains*

The total internal strain energy of the pipe, $U$, in Eq. 2.1 is the volume integral of the product of the stresses and the corresponding strains. Eq. 2.1 is expanded by inserting the expressions for stress in Eq. 2.2 as well as the expressions for strain in Eq. 2.3. For simplicity, expand each term of internal strain energy in Eq. 2.1 individually. After the substitutions the first term becomes:

$$(2.5)\qquad U_{\mathrm{I}}=\frac{1}{2}\int_V \varepsilon_z\sigma_z dV=\frac{1}{2}\int_A\int_{\xi=-t/2}^{\xi=t/2}(\bar{\varepsilon}_z+\xi\kappa_z)\left\{\frac{E}{1-\mu^2}\left[\bar{\varepsilon}_z+\xi\kappa_z+\mu\left(\bar{\varepsilon}_\phi+\xi\kappa_\phi\right)\right]\right\}d\xi dA$$

After performing the integration across the thickness along coordinate $\xi$. Eq. 2.5 becomes:

$$(2.6)\qquad U_{\mathrm{I}}=\frac{1}{2}\int_A\left[\frac{Et}{1-\mu^2}\left(\bar{\varepsilon}_z^{\,2}+\mu\bar{\varepsilon}_z\bar{\varepsilon}_\phi\right)+\frac{Et^3}{12\left(1-\mu^2\right)}\left(\kappa_z^{\,2}+\mu\kappa_z\kappa_\phi\right)\right]dA$$

Perform a similar expansion of the second and third terms of Eq. 2.1 and integration with respect to $\xi$ to obtain:

$$(2.7)\qquad U_{\mathrm{II}}=\frac{1}{2}\int_V \varepsilon_\phi\sigma_\phi dV=\frac{1}{2}\int_A\left[\frac{Et}{1-\mu^2}\left(\bar{\varepsilon}_\phi^{\,2}+\mu\bar{\varepsilon}_z\bar{\varepsilon}_\phi\right)+\frac{Et^3}{12\left(1-\mu^2\right)}\left(\kappa_\phi^{\,2}+\mu\kappa_z\kappa_\phi\right)\right]dA$$

$$(2.8)\qquad \begin{aligned} U_{\mathrm{III}}&=\frac{1}{2}\int_V \gamma_{z\phi}\tau_{z\phi} dV=\frac{1}{2}\int_A\left[\frac{Et}{2(1+\mu)}\gamma_{z\phi}^{\,2}+\frac{Et^3}{24(1+\mu)}\kappa_{z\phi}^{\,2}\right]dA\\ &=\frac{1}{2}\int_A\left[Gt\gamma_{z\phi}^{\,2}+\frac{Gt^3}{12}\kappa_{z\phi}^{\,2}\right]dA \end{aligned}$$

Thus the total internal strain energy, $U$, in Eq. 2.1 is written as the sum of Eqs. 2.6 through 2.8:

$$U=\frac{1}{2}\int_A\left[\frac{Et}{1-\mu^2}\left(\bar{\varepsilon}_z^{\,2}+\bar{\varepsilon}_\phi^{\,2}+2\mu\bar{\varepsilon}_z\bar{\varepsilon}_\phi\right)+Gt\gamma_{z\phi}^{\,2}+\frac{Et^3}{12\left(1-\mu^2\right)}\left(\kappa_z^{\,2}+\kappa_\phi^{\,2}+2\mu\kappa_z\kappa_\phi\right)+\frac{Gt^3}{12}\kappa_{z\phi}^{\,2}\right]dA \tag{2.9}$$

In Eq. 2.9, $dA = rd\phi dz$ is an element of area of the middle surface of the pipe.

### 2.3.4 *Internal Strain Energy in Terms of Mid-surface Displacements*

The internal strain energy of the pipe is expressed in terms of the mid-surface displacement by substituting the strain and curvature expressions from Eq. 2.4 into Eq. 2.9. Performing the required substitutions, expanding the squares and converting the area integral into a double integral over the circumference of the middle surface and length of the pipe gives the final form of the internal strain energy:

$$\begin{aligned}U=\frac{1}{2}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\Bigg\langle&\frac{Et}{1-\mu^2}\left\{\left(\frac{\partial u}{\partial z}\right)^2+\frac{1}{r^2}\left[w^2+2w\frac{\partial v}{\partial \phi}+\left(\frac{\partial v}{\partial \phi}\right)^2\right]+2\mu\left(\frac{\partial u}{\partial z}\right)\left(\frac{1}{r}\right)\left(w+\frac{\partial v}{\partial \phi}\right)\right\}\\&+Gt\left[\left(\frac{\partial v}{\partial z}\right)^2+\frac{2}{r}\frac{\partial v}{\partial z}\frac{\partial u}{\partial \phi}+\frac{1}{r^2}\left(\frac{\partial u}{\partial \phi}\right)^2\right]+\frac{Et^3}{12\left(1-\mu^2\right)}\left\{\left(\frac{\partial^2 w}{\partial z^2}\right)^2\right.\\&+\frac{1}{r^4}\left[\left(\frac{\partial^2 w}{\partial \phi^2}\right)^2-2\left(\frac{\partial^2 w}{\partial \phi^2}\right)\left(\frac{\partial v}{\partial \phi}\right)+\left(\frac{\partial v}{\partial \phi}\right)^2\right]\left.+\frac{2\mu}{r^2}\left(\frac{\partial^2 w}{\partial z^2}\right)\left(\frac{\partial^2 w}{\partial \phi^2}-\frac{\partial v}{\partial \phi}\right)\right\}\\&+\frac{Gt^3}{3r^2}\left[\left(\frac{\partial^2 w}{\partial z\partial \phi}\right)^2-2\left(\frac{\partial^2 w}{\partial z\partial \phi}\right)\left(\frac{\partial v}{\partial z}\right)+\left(\frac{\partial v}{\partial z}\right)^2\right]\Bigg\rangle rd\phi dz\end{aligned} \tag{2.10}$$

## 2.4 <u>Assumed Functions for Mid-surface Displacements</u>

The exact equations for the deformation of a pipe section under general loads are unknown. In the present work, the mid-surface displacements $u$, $v$, and $w$ are expressed as Fourier expansions:

(2.11a) $$u(z,\phi)=a_0(z)+\sum_{n=1}^{\alpha_a} a_n(z)\cos n\phi+\sum_{n=1}^{\alpha_b} b_n(z)\sin n\phi$$

(2.11b) $$v(z,\phi)=c_0(z)+\sum_{n=1}^{\alpha_c} c_n(z)\cos n\phi+\sum_{n=1}^{\alpha_d} d_n(z)\sin n\phi$$

(2.11c) $$w(z,\phi)=f_0(z)+\sum_{n=1}^{\alpha_f} f_n(z)\cos n\phi+\sum_{n=1}^{\alpha_g} g_n(z)\sin n\phi$$

In Eq. 2.11 $\alpha_a,\alpha_b,\ldots$ are the number of terms included in the Fourier expansion. As the number of terms included approaches the limits $\alpha_a,\alpha_b,\ldots\to\infty$, Eq. 2.11 approach the exact displacements of the mid-surface within the limitations of the assumptions outlined in Section 2.2. In practicality, the number of terms included must be truncated and an appropriate value for $\alpha_a,\alpha_b,\ldots$ assumed.

The number of terms does not necessarily need to be identical for each Fourier expansion. Different values could be used for each $\alpha_a,\alpha_b,\ldots$ if desired. However, using the same number of terms in each expansion leads to simplifications in handling the equations developed in the following sections. Therefore, all subsequent formulations will use the same number of terms for each Fourier expansion and the notation $\alpha_a,\alpha_b,\ldots=\alpha$ will be adopted.

The following partial derivatives will also be required to formulate the internal strain energy expression in Eq. 2.10:

(2.12a) $$\frac{\partial u(z,\phi)}{\partial z}=a_0'(z)+\sum_{n=1}^{\alpha} a_n'(z)\cos n\phi+\sum_{n=1}^{\alpha} b_n'(z)\sin n\phi$$

(2.12b) $$\frac{\partial u(z,\phi)}{\partial \phi}=\sum_{n=1}^{\alpha} -na_n(z)\sin n\phi+\sum_{n=1}^{\alpha} nb_n(z)\cos n\phi$$

(2.12c) $$\frac{\partial v(z,\phi)}{\partial z}=c_0'(z)+\sum_{n=1}^{\alpha} c_n'(z)\cos n\phi+\sum_{n=1}^{\alpha} d_n'(z)\sin n\phi$$

(2.12d) $$\frac{\partial v(z,\phi)}{\partial \phi}=\sum_{n=1}^{\alpha}-nc_n(z)\sin n\phi+\sum_{n=1}^{\alpha}nd_n(z)\cos n\phi$$

(2.12e) $$\frac{\partial^2 w(z,\phi)}{\partial z^2}=f_0''(z)+\sum_{n=1}^{\alpha}f_n''(z)\cos n\phi+\sum_{n=1}^{\alpha}g_n''(z)\sin n\phi$$

(2.12f) $$\frac{\partial^2 w(z,\phi)}{\partial \phi^2}=\sum_{n=1}^{\alpha}-n^2 f_n(z)\cos n\phi+\sum_{n=1}^{\alpha}-n^2 g_n(z)\sin n\phi$$

(2.12g) $$\frac{\partial^2 w(z,\phi)}{\partial z\partial \phi}=\sum_{n=1}^{\alpha}-nf_n'(z)\sin n\phi+\sum_{n=1}^{\alpha}ng_n'(z)\cos n\phi$$

In Eq. 2.12 above the notation $a_n'(z)=\dfrac{da(z)}{dz}$ and $f_n''(z)=\dfrac{d^2 f(z)}{dz^2}$ , etc. is used.

### *2.4.1 Physical Significance of Fourier Modes*

Each mode $n$ in the Fourier Series in Eq. 2.11 corresponds to a different type of deformation. As shown in Appendix 1, mode $n=0$ represents axi-symmetric deformation. An example of axi-symmetric displacement is the uniform expansion of the cross section of a pipe subjected to internal pressure.

Fourier mode $n=1$ represents a displacement similar to that predicted by traditional Bernoulli beam theory. The displacement from the current method with $n=1$ will not, however, be identical to the displacement obtained from Bernoulli beam theory. The difference between the two methods is a result of the assumptions made in their development.

The higher Fourier modes, $n\geq 2$, correspond to deformation of the pipe cross section, such as ovalization and warping. These deformations are not captured by the traditional Euler-Bernoulli beam theory.

## 2.5 Internal Strain Energy in Terms of Assumed Displacement Functions

The total internal strain energy of the pipe is converted into a function of $z$ only by substituting the Fourier expansions for $u$, $v$ and $w$ and their derivatives from Eqs. 2.11 and 2.12 into the internal strain energy equation of Eq. 2.10. The squares and product terms are then expanded. Finally, the integration with respect to $\phi$ is performed explicitly as the only functions containing $\phi$ are $\cos n\phi$ and $\sin n\phi$. The resulting expression is a function of $z$ only.

The general expression for the integral of the product of two generic Fourier functions, $q_1(\phi)$ and $q_2(\phi)$, with respect to $\phi$, as developed in Appendix 2, is as follows:

Given:

(2.13a) $$q_1(z,\phi) = x_0(z) + \sum_{n=1}^{\alpha} x_n(z)\cos n\phi + \sum_{n=1}^{\alpha} y_n(z)\sin n\phi$$

(2.13b) $$q_2(z,\phi) = u_0(z) + \sum_{n=1}^{\alpha} u_n(z)\cos n\phi + \sum_{n=1}^{\alpha} w_n(z)\sin n\phi$$

It is demonstrated that:

(2.14a) $$\int_{\phi=0}^{2\pi} q_1(z,\phi)q_2(z,\phi)\,d\phi = 2\pi x_0(z)u_0(z) + \sum_{n=1}^{\alpha}\pi x_n(z)u_n(z) + \sum_{n=1}^{\alpha}\pi y_n(z)w_n(z)$$

and

(2.14b) $$\int_{\phi=0}^{2\pi}\left[q_1(z,\phi)\right]^2 d\phi = 2\pi\left[x_0(z)\right]^2 + \sum_{n=1}^{\alpha}\pi\left[x_n(z)\right]^2 + \sum_{n=1}^{\alpha}\pi\left[y_n(z)\right]^2$$

Eq. 2.14 is used to expand each term of the internal strain energy in Eq. 2.10 independently and perform the integrations with respect to $\phi$.

Before performing the substitutions, the following constants are defined for simplicity (note that constants $C_{10}$ through $C_{12}$ and $C_{14}$ through $C_{17}$ are dependent on the Fourier mode $n$ and will be used in Chapter 3):

(2.15a) $$C_1 = \frac{Ert\pi}{1-\mu^2}$$

(2.15b) $$C_2 = \frac{Et\pi}{r\left(1-\mu^2\right)}$$

(2.15c) $$C_3 = \frac{Et\mu\pi}{1-\mu^2}$$

(2.15d) $$C_4 = Grt\pi$$

(2.15e) $$C_5 = Gt\pi$$

(2.15f) $$C_6 = \frac{Ert^3\pi}{12\left(1-\mu^2\right)}$$

(2.15g) $$C_7 = \frac{Et^3\pi}{12r^3\left(1-\mu^2\right)}$$

(2.15h) $$C_8 = \frac{Et^3\mu\pi}{12r\left(1-\mu^2\right)}$$

(2.15i) $$C_9 = \frac{Gt^3\pi}{3r}$$

(2.15j) $$C_{10,n} = \frac{C_4 n^2}{r^2}$$

(2.15k) $$C_{11,n} = \left(C_3 + C_5\right)n$$

(2.15l) $$C_{12,n} = \left(C_2 + C_7\right)n^2$$

(2.15m) $$C_{13} = C_4 + C_9$$

(2.15n) $$C_{14,n} = C_2 n + C_7 n^3$$

(2.15o) $$C_{15,n} = \left(C_8 + C_9\right)n$$

(2.15p) $$C_{16,n} = C_2 + C_7 n^4$$

(2.15q) $$C_{17,n} = \left(2C_8 + C_9\right)n^2$$

Each term of Eq. 2.10, as well as the corresponding term after performing the integration with respect to $\phi$ using the identities in Eq. 2.14, is summarized in Table 2.1 below.

**Table 2.1** – Expansion of internal strain energy terms

| | Term in Eq. 2.14 | Term after integration with respect to $\phi$ |
|---|---|---|
| 1 | $\frac{rEt}{2\left(1-\mu^2\right)}\int\limits_{z=0}^{l}\int\limits_{\phi=0}^{2\pi}\left(\frac{\partial u}{\partial z}\right)^2 d\phi dz$ | $\frac{C_1}{2}\int\limits_{z=0}^{l}\left\{2\left(a_0'(z)\right)^2+\sum\limits_{n=1}^{\alpha}\left(a_n'(z)\right)^2+\sum\limits_{n=1}^{\alpha}\left(b_n'(z)\right)^2\right\}dz$ |
| 2 | $\frac{Et}{2r\left(1-\mu^2\right)}\int\limits_{z=0}^{l}\int\limits_{\phi=0}^{2\pi} w^2+2w\frac{\partial v}{\partial \phi}+\left(\frac{\partial v}{\partial \phi}\right)^2 d\phi dz$ | $\frac{C_2}{2}\int\limits_{z=0}^{l}\left\{2\left(f_0(z)\right)^2+\sum\limits_{n=1}^{\alpha}\left(f_n(z)\right)^2+\sum\limits_{n=1}^{\alpha}\left(g_n(z)\right)^2 + 2\left[\sum\limits_{n=1}^{\alpha}nd_n(z)f_n(z)+\sum\limits_{n=1}^{\alpha}-nc_n(z)g_n(z)\right] + \sum\limits_{n=1}^{\alpha}n^2\left(c_n(z)\right)^2+\sum\limits_{n=1}^{\alpha}n^2\left(d_n(z)\right)^2\right\}dz$ |

| | | |
|---|---|---|
| 3 | $\frac{Et\mu}{\left(1-\mu^2\right)}\int_{z=0}^{l}\int_{\phi=0}^{2\pi}\left(\frac{\partial u}{\partial z}\right)w+\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial \phi}\right)d\phi dz$ | $C_3\int_{z=0}^{l}\left\{2a_0'(z)f_0(z)+\sum_{n=1}^{\alpha}a_n'(z)f_n(z)+\sum_{n=1}^{\alpha}b_n'(z)g_n(z)+\sum_{n=1}^{\alpha}-nb_n'(z)c_n(z)+\sum_{n=1}^{\alpha}na_n'(z)d_n(z)\right\}dz$ |
| 4 | $\frac{rGt}{2}\int_{z=0}^{l}\int_{\phi=0}^{2\pi}\left(\frac{\partial v}{\partial z}\right)^2+\frac{2}{r}\frac{\partial v}{\partial z}\frac{\partial u}{\partial \phi}+\frac{1}{r^2}\left(\frac{\partial u}{\partial \phi}\right)^2 d\phi dz$ | $\frac{C_4}{2}\int_{z=0}^{l}\left\{2\left(c_0'(z)\right)^2+\sum_{n=1}^{\alpha}\left(c_n'(z)\right)^2+\sum_{n=1}^{\alpha}\left(d_n'(z)\right)^2+\frac{2}{r}\left[\sum_{n=1}^{\alpha}nb_n(z)c_n'(z)+\sum_{n=1}^{\alpha}-na_n(z)d_n'(z)\right]+\frac{1}{r^2}\left[\sum_{n=1}^{\alpha}n^2\left(a_n(z)\right)^2+\sum_{n=1}^{\alpha}n^2\left(b_n(z)\right)^2\right]\right\}dz$ |
| 5 | $\frac{rEt^3}{24\left(1-\mu^2\right)}\int_{z=0}^{l}\int_{\phi=0}^{2\pi}\left(\frac{\partial^2 w}{\partial z^2}\right)^2 d\phi dz$ | $\frac{C_6}{2}\int_{z=0}^{l}\left\{2\left(f_0''(z)\right)^2+\sum_{n=1}^{\alpha}\left(f_n''(z)\right)^2+\sum_{n=1}^{\alpha}\left(g_n''(z)\right)^2\right\}dz$ |
| 6 | $\frac{Et^3}{24r^3\left(1-\mu^2\right)}\int_{z=0}^{l}\int_{\phi=0}^{2\pi}\left[\left(\frac{\partial^2 w}{\partial \phi^2}\right)^2-2\frac{\partial^2 w}{\partial \phi^2}\frac{\partial v}{\partial \phi}+\left(\frac{\partial v}{\partial \phi}\right)^2\right]d\phi dz$ | $\frac{C_7}{2}\int_{z=0}^{l}\left\{\sum_{n=1}^{\alpha}n^4\left(f_n(z)\right)^2+\sum_{n=1}^{\alpha}n^4\left(g_n(z)\right)^2-2\left[\sum_{n=1}^{\alpha}-n^3d_n(z)f_n(z)+\sum_{n=1}^{\alpha}n^3c_n(z)g_n(z)\right]+\sum_{n=1}^{\alpha}n^2\left(c_n(z)\right)^2+\sum_{n=1}^{\alpha}n^2\left(d_n(z)\right)^2\right\}dz$ |
| 7 | $\frac{Et^3\mu}{12r\left(1-\mu^2\right)}\int_{z=0}^{l}\int_{\phi=0}^{2\pi}\left(\frac{\partial^2 w}{\partial z^2}\right)\left(\frac{\partial^2 w}{\partial \phi^2}-\frac{\partial v}{\partial \phi}\right)d\phi dz$ | $C_8\int_{z=0}^{l}\left\{\sum_{n=1}^{\alpha}-n^2f_n''(z)f_n(z)+\sum_{n=1}^{\alpha}-n^2g_n''(z)g_n(z)-\sum_{n=1}^{\alpha}-nc_n(z)g_n''(z)-\sum_{n=1}^{\alpha}nd_n(z)f_n''(z)\right\}dz$ |
| 8 | $\frac{Gt^3}{6r}\int_{z=0}^{l}\int_{\phi=0}^{2\pi}\left(\frac{\partial^2 w}{\partial z\partial \phi}\right)^2-2\frac{\partial^2 w}{\partial z\partial \phi}\frac{\partial v}{\partial z}+\left(\frac{\partial v}{\partial z}\right)^2 d\phi dz$ | $\frac{C_9}{2}\int_{z=0}^{l}\left\{\sum_{n=1}^{\alpha}n^2\left(f_n'(z)\right)^2+\sum_{n=1}^{\alpha}n^2\left(g_n'(z)\right)^2-2\left[\sum_{n=1}^{\alpha}-nd_n'(z)f_n'(z)+\sum_{n=1}^{\alpha}nc_n'(z)g_n'(z)\right]+2\left(c_0'(z)\right)^2+\sum_{n=1}^{\alpha}\left(c_n'(z)\right)^2+\sum_{n=1}^{\alpha}\left(d_n'(z)\right)^2\right\}dz$ |

## 2.6 Load Potential in Terms of Assumed Displacement Functions

An expansion similar to that used for the internal strain energy is developed to express the load potential terms from Eq. 2.1 in terms of the Fourier expansions from Eq. 2.11. The load potential $W$ can be expressed as $W = W_u + W_v + W_w$ in which:

(2.16a) $$W_u = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_u(z,\phi)\left[a_0(z)+\sum_{n=1}^{\alpha}a_n(z)\cos n\phi+\sum_{n=1}^{\alpha}b_n(z)\sin n\phi\right]rd\phi dz$$

(2.16b) $$W_v = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_v(z,\phi)\left[c_0(z)+\sum_{n=1}^{\alpha}c_n(z)\cos n\phi+\sum_{n=1}^{\alpha}d_n(z)\sin n\phi\right]rd\phi dz$$

(2.16c) $$W_w = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_w(z,\phi)\left[f_0(z)+\sum_{n=1}^{\alpha}f_n(z)\cos n\phi+\sum_{n=1}^{\alpha}g_n(z)\sin n\phi\right]rd\phi dz$$

In the case of the load potentials the integrations with respect to $\phi$ cannot be performed explicitly because the load functions $q(z,\phi)$ are generally dependent on $\phi$. The integrals will depend on the specific form of $q(z,\phi)$.

## 2.7 Variation of Total Potential Energy

The equilibrium conditions are obtained by evoking the stationarity condition of the total potential energy functional:

(2.17) $$\begin{aligned}\delta\pi = {} & \frac{\partial\pi}{\partial a_0}\delta a_0+\frac{\partial\pi}{\partial a_0'}\delta a_0'+\frac{\partial\pi}{\partial c_0}\delta c_0+\frac{\partial\pi}{\partial c_0'}\delta c_0'+\frac{\partial\pi}{\partial f_0}\delta f_0+\frac{\partial\pi}{\partial f_0'}\delta f_0'+\frac{\partial\pi}{\partial f_0''}\delta f_0'' \\ & +\sum_{n=1}^{\alpha}\left(\frac{\partial\pi}{\partial a_n}\delta a_n+\frac{\partial\pi}{\partial a_n'}\delta a_n'+\frac{\partial\pi}{\partial b_n}\delta b_n+\frac{\partial\pi}{\partial b_n'}\delta b_n'+\frac{\partial\pi}{\partial c_n}\delta c_n+\frac{\partial\pi}{\partial c_n'}\delta c_n'+\frac{\partial\pi}{\partial d_n}\delta d_n\right. \\ & \left.+\frac{\partial\pi}{\partial d_n'}\delta d_n'+\frac{\partial\pi}{\partial f_n}\delta f_n+\frac{\partial\pi}{\partial f_n'}\delta f_n'+\frac{\partial\pi}{\partial f_n''}\delta f_n''+\frac{\partial\pi}{\partial g_n}\delta g_n+\frac{\partial\pi}{\partial g_n'}\delta g_n'+\frac{\partial\pi}{\partial g_n''}\delta g_n''\right)=0\end{aligned}$$

The equilibrium equations and corresponding boundary terms are obtained by applying Eq. 2.17 to the sum of the eight terms in Table 2.1 minus the sum of the load potential terms in

Eq. 2.16. Details of the procedure, including performing integration by parts and grouping similar terms, are outlined in Appendix 3.

As also shown in Appendix 3, the field equations for each Fourier mode $n$ are independent from every other Fourier mode $m$ for $n \neq m$. Given that the functions $\delta a_n$, $\delta c_n$ and $\delta f_n$ for $n = 0,1,\ldots,\alpha$ as well as $\delta b_n$, $\delta d_n$ and $\delta g_n$ for $n = 1,2,\ldots,\alpha$ are arbitrary, the coefficients of each of these functions must vanish. The equilibrium equations and boundary conditions can be written independently for each mode $n$.

## *2.7.1 Equilibrium Equations*

The resulting field equations relating to functions $a_0(z)$, $c_0(z)$ and $f_0(z)$ for mode $n = 0$ are:

(2.18a) $$2C_1 a_0'' + 2C_3 f_0' + \int_{\phi=0}^{2\pi} q_u(z,\phi)\, r d\phi = 0$$

(2.18b) $$2(C_4 + C_9) c_0'' + \int_{\phi=0}^{2\pi} q_v(z,\phi)\, r d\phi = 0$$

(2.18c) $$2C_3 a_0' + 2C_2 f_0 + 2C_6 f_0'''' - \int_{\phi=0}^{2\pi} q_w(z,\phi)\, r d\phi = 0$$

For modes $n \geq 1$ the field equations relating to functions $a_n(z)$, $c_n(z)$ and $f_n(z)$ are:

(2.18d) $$C_1 a_n'' - \frac{C_4 n^2}{r^2} a_n + (C_3 + C_5) n d_n' + C_3 f_n' + \int_{\phi=0}^{2\pi} q_u(z,\phi)\cos(n\phi)\, r d\phi = 0$$

(2.18e) $$\begin{aligned} &(C_3 + C_5) n a_n' + (C_2 + C_7) n^2 d_n - (C_4 + C_9) d_n'' + (C_2 n + C_7 n^3) f_n \\ &- (C_8 + C_9) n f_n'' - \int_{\phi=0}^{2\pi} q_v(z,\phi)\sin(n\phi)\, r d\phi = 0 \end{aligned}$$

$$C_3a_n' + \left(C_2n + C_7n^3\right)d_n - \left(C_8 + C_9\right)nd_n'' + \left(C_2 + C_7n^4\right)f_n - \left(2C_8 + C_9\right)n^2 f_n''$$
$$+C_6 f_n'''' - \int_{\phi=0}^{2\pi} q_w\left(z,\phi\right)\sum_{n=1}^{\alpha}\cos\left(n\phi\right) r d\phi = 0 \tag{2.18f}$$

Similarly, the field equations relating to functions $b_n\left(z\right)$, $d_n\left(z\right)$ and $g_n\left(z\right)$ are for modes $n \geq 1$:

$$C_1 b_n'' - \frac{C_4 n^2}{r^2} b_n - \left(C_3 + C_5\right)nc_n' + C_3 g_n' + \int_{\phi=0}^{2\pi} q_u\left(z,\phi\right)\sin\left(n\phi\right) r d\phi = 0 \tag{2.18g}$$

$$\left(C_3 + C_5\right)nb_n' - \left(C_2 + C_7\right)n^2 c_n + \left(C_4 + C_9\right)c_n'' + \left(C_2 n + C_7 n^3\right) g_n$$
$$-\left(C_8 + C_9\right)ng_n'' + \int_{\phi=0}^{2\pi} q_v\left(z,\phi\right)\cos\left(n\phi\right) r d\phi = 0 \tag{2.18h}$$

$$C_3 b_n' - \left(C_2 n + C_7 n^3\right)c_n + \left(C_8 + C_9\right)nc_n'' + \left(C_2 + C_7 n^4\right) g_n - \left(2C_8 + C_9\right)n^2 g_n''$$
$$+C_6 g_n'''' - \int_{\phi=0}^{2\pi} q_w(z,\phi)\sin\left(n\phi\right) r d\phi = 0 \tag{2.18i}$$

The equilibrium equations in Eq. 2.18 are ordinary, linear differential equations. For Fourier mode $n = 0$, the differential equations are 2$^{\text{nd}}$ order with respect to the functions $a_0\left(z\right)$ and $c_0\left(z\right)$ and 4$^{\text{th}}$ order with respect to the function $f_0\left(z\right)$. For the cases of $n \geq 1$, the differential equations are 2$^{\text{nd}}$ order with respect to the functions $a_n\left(z\right)$, $b_n\left(z\right)$, $c_n\left(z\right)$ and $d_n\left(z\right)$ and 4$^{\text{th}}$ order with respect to the functions $f_n\left(z\right)$ and $g_n\left(z\right)$.

Multiple functions of $z$, $a_n\left(z\right), b_n\left(z\right)\ldots$, appear in each of the differential equations, indicating that the equations are coupled. As a result, the individual functions of a given mode $n$ cannot be solved independently. The solution procedure must consider all of the functions simultaneously. It is of interest to note, however, that the field equations for $a_n\left(z\right)$, $d_n\left(z\right)$ and $f_n\left(z\right)$ are completely uncoupled from the field equations for $b_n\left(z\right)$, $c_n\left(z\right)$ and $g_n\left(z\right)$.

Also, it can also be noted from Eq. 2.18 that the equations for Fourier mode $n$, $a_n(z)$, $b_n(z)$ ..., are uncoupled from the equations for Fourier mode $m$, $a_m(z)$, $b_m(z)$ ..., for $n \neq m$. The fact that the equations for different Fourier modes are uncoupled considerably simplifies the solution of resulting equations.

### 2.7.2 *Boundary Conditions*

The boundary terms arising from integration by parts provide the possible boundary conditions of the problem. For mode $n=0$ these are:

(2.19a) $$\left[\left(2C_1 a_0' + 2C_3 f_0\right)\delta a_0\right]_{z=0}^{l} = 0$$

(2.19b) $$\left[\left(2\left(C_4 + C_9\right)c_0'\right)\delta c_0\right]_{z=0}^{l} = 0$$

(2.19c) $$\left[\left(-2C_6 f_0'''\right)\delta f_0\right]_{z=0}^{l} = 0$$

(2.19d) $$\left[\left(2C_6 f_0''\right)\delta f_0'\right]_{z=0}^{l} = 0$$

For modes $n \geq 1$ the boundary conditions relating to functions $a_n(z)$, $c_n(z)$ and $f_n(z)$ are:

(2.19e) $$\left[\left(C_1 a_n' + C_3 n d_n + C_3 f_n\right)\delta a_n\right]_{z=0}^{l} = 0$$

(2.19f) $$\left[\left(-C_5 n a_n + \left(C_4 + C_9\right)d_n' + C_9 n f_n'\right)\delta d_n\right]_{z=0}^{l} = 0$$

(2.19g) $$\left[\left(\left(C_8 + C_9\right)n d_n' + \left(C_8 + C_9\right)n^2 f_n' - C_6 f_n'''\right)\delta f_n\right]_{z=0}^{l} = 0$$

(2.19h) $$\left[\left(-C_8 n d_n - C_8 n^2 f_n + C_6 f_n''\right)\delta f_n'\right]_{z=0}^{l} = 0$$

Similarly, the boundary conditions relating to functions $b_n(z)$, $d_n(z)$ and $g_n(z)$ are for modes $n \geq 1$:

(2.19i) $$\left[\left(C_1 b_n' - C_3 n c_n + C_3 g_n\right)\delta b_n\right]_{z=0}^{l} = 0$$

(2.19j) $$\left[\left(C_5 n b_n + \left(C_4 + C_9\right) c_n' - C_9 n g_n'\right)\delta c_n\right]_{z=0}^{l} = 0$$

(2.19k) $$\left[\left(-\left(C_8 + C_9\right) n c_n' + \left(C_8 + C_9\right) n^2 g_n' - C_6 g_n'''\right)\delta g_n\right]_{z=0}^{l} = 0$$

(2.19l) $$\left[\left(C_8 n c_n - C_8 n^2 g_n + C_6 g_n''\right)\delta g_n'\right]_{z=0}^{l} = 0$$

The number of boundary terms present in Eq. 2.19 is directly related to the order of the differential equations in Eq. 2.18. For the Fourier mode $n = 0$, the total order of the differential equations with respect to the functions $a_0(z)$, $c_0(z)$ and $f_0(z)$ is eight, resulting in a total of eight integration constants. These eight constants are determined from the 8 boundary conditions in Eqs. 2.19a through 2.19d.

For the cases of $n \geq 1$, the total order of the differential equations, for each mode *n*, with respect to the functions $a_n(z)$, $b_n(z)$, $c_n(z), d_n(z)$, $f_n(z)$ and $g_n(z)$ is sixteen. Correspondingly, the solution will result in a total of sixteen integration constants to be determined from the sixteen boundary conditions in Eqs. 2.19e through 2.19l.

## 2.8 Expressing Equilibrium Equations in Matrix Form

To facilitate the solution of the equilibrium conditions in Eq. 2.18, they are rewritten in matrix form. For Fourier mode $n = 0$, the equilibrium conditions are:

(2.20) $$\left[\mathbf{X_0}\right]_{3\times3} \begin{Bmatrix} a_0(z) \\ f_0(z) \\ c_0(z) \end{Bmatrix}_{3\times1} = \left\{\mathbf{Q_0}\right\}_{3\times1}$$

where

$$(2.21) \qquad [\mathbf{X_0}]_{3\times3} = \left[\begin{array}{c|c|c} 2C_1D^2 & 2C_3D & 0 \\ \hline 2C_3D & 2C_2+2C_6D^4 & 0 \\ \hline 0 & 0 & 2(C_4+C_9)D^2 \end{array}\right]$$

in which the notation $D^n = \dfrac{\partial^n}{\partial z^n} f(z)$ for $(n = 1,2,4)$ is used and

$$(2.22) \qquad \{\mathbf{Q_0}\}_{3\times1} = \begin{Bmatrix} -\int_{\phi=0}^{2\pi} q_u(\phi,z)\,rd\phi \\ \int_{\phi=0}^{2\pi} q_w(\phi,z)\,rd\phi \\ -\int_{\phi=0}^{2\pi} q_v(\phi,z)\,rd\phi \end{Bmatrix}$$

The matrix form of the equilibrium conditions for each Fourier mode $n \geq 1$ is:

$$(2.23) \qquad \left[\begin{array}{c|c} \mathbf{X}_{1,n} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{X}_{2,n} \end{array}\right]_{6\times6} \begin{Bmatrix} a_n(z) \\ d_n(z) \\ f_n(z) \\ b_n(z) \\ c_n(z) \\ g_n(z) \end{Bmatrix}_{6\times1} = \{\mathbf{Q}_n\}_{6\times1}$$

in which

$$(2.24) \qquad [\mathbf{X}_{1,n}]_{3\times3} = \left[\begin{array}{c|c|c} C_1D^2 - \frac{1}{r^2}C_4n^2 & (C_3+C_5)nD & C_3D \\ \hline (C_3+C_5)nD & \begin{matrix}(C_2+C_7)n^2 \\ -(C_4+C_9)D^2\end{matrix} & \begin{matrix}C_2n+C_7n^3 - \\ (C_8+C_9)nD^2\end{matrix} \\ \hline C_3D & \begin{matrix}C_2n+C_7n^3 - \\ (C_8+C_9)nD^2\end{matrix} & \begin{matrix}C_2+C_7n^4+C_6D^4 - \\ (2C_8+C_9)n^2D^2\end{matrix} \end{array}\right]$$

(2.25) $$\left[\mathbf{X}_{2,n}\right]_{3\times3} = \left[\begin{array}{c|c|c} C_1D^2 - \dfrac{1}{r^2}C_4n^2 & -(C_3+C_5)nD & C_3D \\ \hline (C_3+C_5)nD & \begin{array}{c} -(C_2+C_7)n^2 \\ +(C_4+C_9)D^2 \end{array} & \begin{array}{c} C_2n+C_7n^3- \\ (C_8+C_9)nD^2 \end{array} \\ \hline C_3D & \begin{array}{c} -C_2n-C_7n^3+ \\ (C_8+C_9)nD^2 \end{array} & \begin{array}{c} C_2+C_7n^4+C_6D^4- \\ (2C_8+C_9)n^2D^2 \end{array} \end{array}\right]$$

(2.26) $$\left\{\mathbf{Q}_n\right\}_{6\times1} = \left\{\begin{array}{l} -\int_{\phi=0}^{2\pi} q_u(\phi,z)\cos(n\phi)\,rd\phi \\ \int_{\phi=0}^{2\pi} q_v(\phi,z)\sin(n\phi)\,rd\phi \\ \int_{\phi=0}^{2\pi} q_w(\phi,z)\cos(n\phi)\,rd\phi \\ -\int_{\phi=0}^{2\pi} q_u(\phi,z)\sin(n\phi)\,rd\phi \\ -\int_{\phi=0}^{2\pi} q_v(\phi,z)\cos(n\phi)\,rd\phi \\ \int_{\phi=0}^{2\pi} q_w(\phi,z)\sin(n\phi)\,rd\phi \end{array}\right\}$$

By inspection of Eqs. 2.24 and 2.25, it is apparent that the matrices $\mathbf{X}_{1,n}$ and $\mathbf{X}_{2,n}$ are similar. In fact, the two can be equated by multiplying the second column of $\mathbf{X}_{2,n}$ by -1. Correspondingly $c_n(z)$ must be multiplied by -1 for the modified matrix operator $\mathbf{X}_{2,n}$ to be identical to $\mathbf{X}_{1,n}$. Eq. 2.23 can now be written as:

(2.27) $$\left[\begin{array}{c|c} \mathbf{X}_n & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{X}_n \end{array}\right]_{6\times6} \left\{\begin{array}{c} a_n(z) \\ d_n(z) \\ f_n(z) \\ b_n(z) \\ -c_n(z) \\ g_n(z) \end{array}\right\}_{6\times1} = \left\{\mathbf{Q}_n\right\}_{6\times1}$$

In Eq. 2.27 $\mathbf{X}_n$ is the same as $\mathbf{X}_{1,n}$ defined in Eq. 2.24.

It is emphasized that Eq. 2.27 is in fact a series of matrix equations for each Fourier mode ($n \geq 1$). The equations for mode $n$ are completely uncoupled from those of mode $m \neq n$.

# CHAPTER 3: SOLUTION OF EQUILIBRIUM EQUATIONS

The unknowns in the equilibrium equations from Chapter 2 (Eqs. 2.20 and 2.27) are the functions $a_n(z)\ldots g_n(z)$. With the solution of these unknown functions, the displacement field of the pipe element in Eq. 2.11 will be completely defined. The functions are solved by assuming their form and inserting into Eqs. 2.20 and 2.27 for each Fourier mode $n$. The system of equations for mode $n=0$ is different from that of the general solution of modes $n\geq 1$ and will be handled separately.

## 3.1 Solution for Axi-Symmetric Displacement Contribution: $n = 0$

The axi-symmetric displacement components of the pipe cross section, represented in Eq. 2.20 with the terms corresponding to $n=0$, must be considered separately from the general solution. The case of $n=0$ involves a smaller system of equations than the general solution for $n\geq 1$. Expand Eq. 2.20 with Eqs. 2.21 and 2.22 to obtain the equilibrium equations for $n=0$:

$$\begin{bmatrix} 2C_1D^2 & 2C_3D & 0 \\ 2C_3D & 2C_2+2C_6D^4 & 0 \\ 0 & 0 & 2(C_4+C_9)D^2 \end{bmatrix} \begin{Bmatrix} a_0(z) \\ f_0(z) \\ c_0(z) \end{Bmatrix} = \begin{Bmatrix} \int_{\phi=0}^{2\pi} q_u(\phi,z)\,rd\phi \\ \int_{\phi=0}^{2\pi} q_w(\phi,z)\,rd\phi \\ \int_{\phi=0}^{2\pi} q_v(\phi,z)\,rd\phi \end{Bmatrix} \tag{3.1}$$

### *3.1.1 Solution of Homogeneous Part of Equilibrium Equations*

The unknown functions $a_0(z)$, $f_0(z)$ and $c_0(z)$ are determined by considering the homogeneous portion of Eq. 3.1. The homogeneous equation can be simplified by separating it into two systems. Such a separation can be performed because the first two equations of 3.1 involve only $a_0(z)$ and $f_0(z)$ while the third equation involves only $c_0(z)$. The simplified homogeneous equations are:

$$(3.2) \quad \left[\begin{array}{c|c} 2C_1D^2 & 2C_3D \\ \hline 2C_3D & 2C_2+2C_6D^4 \end{array}\right]\begin{Bmatrix} a_0(z) \\ f_0(z) \end{Bmatrix} = 0$$

$$(3.3) \quad 2(C_4+C_9)c_0''(z)=0$$

The solution to Eq. 3.3, a second order linear differential equation, is:

$$(3.4) \quad c_0(z)=\bar{A}_{0,7}+\bar{A}_{0,8}z$$

In Eq. 3.4 $\bar{A}_{0,7}$ and $\bar{A}_{0,8}$ are constants determined from the boundary conditions.

The solution to Eq. 3.2 is more involved. Assume the solution takes the form $\langle a_0(z) \quad f_0(z)\rangle_{1\times2}^{\mathrm{T}} = \langle \bar{A}_0 e^{m_0 z} \quad \bar{F}_0 e^{m_0 z}\rangle_{1\times2}^{\mathrm{T}}$ and substitute into Eq. 3.2:

$$(3.5) \quad \begin{bmatrix} 2C_1{m_0}^2 & 2C_3m_0 \\ 2C_3m_0 & 2C_2+2C_6{m_0}^4 \end{bmatrix}\begin{Bmatrix} \bar{A}_0 \\ \bar{F}_0 \end{Bmatrix} e^{m_0 z} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For the relation in Eq. 3.5 to be true, either the constants $\bar{A}_0$ and $\bar{F}_0$. should vanish or the determinant of the matrix on the left hand side must be zero. Setting $\bar{A}_0=\bar{F}_0=0$ leads to the trivial solution that is of no interest. Therefore, the determinant of the matrix must vanish:

$$(3.6) \quad \begin{vmatrix} 2C_1{m_0}^2 & 2C_3m_0 \\ 2C_3m_0 & 2C_2+2C_6{m_0}^4 \end{vmatrix} = 0$$

Expand Eq. 3.6 and solve for $m_0$:

$$(3.7) \quad \left(C_1C_2-C_3^2\right){m_0}^2+C_1C_6{m_0}^6=0$$

Eq. 3.7 can be simplified to

$$(3.8) \quad {m_0}^2\left[\left(C_1C_2-C_3^2\right)+C_1C_6{m_0}^4\right]=0$$

The roots of Eq. 3.8 are:

(3.9a) $m_{0,1} = 0$

(3.9b) $m_{0,2} = 0$

(3.9c) $m_{0,3} = \sqrt[4]{-\frac{C_1C_2 - C_3^2}{C_1C_6}}$

(3.9d) $m_{0,4} = -\sqrt[4]{-\frac{C_1C_2 - C_3^2}{C_1C_6}}$

(3.9e) $m_{0,5} = \sqrt[4]{-\frac{C_1C_2 - C_3^2}{C_1C_6}}\,i$

(3.9f) $m_{0,6} = -\sqrt[4]{-\frac{C_1C_2 - C_3^2}{C_1C_6}}\,i$

Hence, the solution to the field equations becomes:

(3.10a) $a_0(z) = \bar{A}_{0,1} + \bar{A}_{0,2}z + \sum_{i=3}^{6} \bar{A}_{0,i}e^{m_{0,i}z}$

(3.10b) $f_0(z) = \bar{F}_{0,1} + \bar{F}_{0,2}z + \sum_{i=3}^{6} \bar{F}_{0,i}e^{m_{0,i}z}$

Note that by inserting the values for the constants $C_1$, $C_2$, $C_3$ and $C_6$ from Eq. 2.15, the non-zero roots of Eq. 3.8 become:

(3.11) $m_{0,i}^4 = -\frac{12(1-\mu^2)}{t^2r^2}$ for $i = 3,4,5,6$

The expression for the roots $m_{0,i}$ in Eq. 3.11 matches exactly the roots of the equilibrium equation for the homogeneous case derived by Flugge (1973: 272). Note, however, that the displacement field assumed by Flugge does not include the terms $\bar{A}_{0,1}$, $\bar{A}_{0,2}z$, $\bar{F}_{0,1}$ and $\bar{F}_{0,2}z$ from Eq. 3.10 which correspond to roots $m_{0,1} = m_{0,2} = 0$.

The displacement field in Eq. 3.10 is valid only if the four roots $m_{0,i}$ for $i$ = 3 to 6 are distinct. The roots will be distinct if the right hand side of Eq. 3.11 is non-zero. Given that the condition $\mu \neq 1$ will always hold for practical problems, it follows that the right hand side of Eq. 3.11 will be non-zero and the solution in Eq. 3.10 is always valid.

### *3.1.2 Reducing Integration Constants for Fourier Mode $n = 0$*

In Eqs. 3.4 and 3.10, fourteen integration constants appear in the solution, namely $\bar{A}_{0,i}$ for $i$ = 1 to 8 and $\bar{F}_{0,i}$ for $i$ = 1 to 6, but only eight boundary conditions are provided in Eq. 2.19. However, because the determinant of the equilibrium equation matrix in Eq. 3.5 was set to zero the equations are linearly dependent. As a result, the integration constants $\bar{A}_{0,i}$ and $\bar{F}_{0,i}$ are inter-dependent and one set of constants, $\bar{F}_{0,i}$, can be written in terms of the other, $\bar{A}_{0,i}$. The details of relating the sets of constants are presented in Appendix 4.

The solution to the homogeneous part of the field equations in Eq. 3.1, given in Eqs. 3.4 and 3.10, is written in matrix form:

$$\text{(3.12a)} \quad \begin{Bmatrix} a_0(z) \\ c_0(z) \\ f_0(z) \end{Bmatrix} = \begin{bmatrix} \langle e_0(z) \rangle_{1\times 8}^{\mathrm{T}} \\ \langle \bar{\bar{C}}_0(z) \rangle_{1\times 8}^{\mathrm{T}} \\ \langle \bar{\bar{F}}_0(z) \rangle_{1\times 8}^{\mathrm{T}} \end{bmatrix} \{\bar{A}_0\}_{8\times 1}$$

$$\text{(3.12b)} \quad \begin{Bmatrix} a_0(z) \\ c_0(z) \\ f_0(z) \end{Bmatrix} = \begin{bmatrix} 1 & z & e^{m_{0,3}z} & e^{m_{0,4}z} & e^{m_{0,5}z} & e^{m_{0,6}z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \\ 0 & \bar{\bar{F}}_{0,1} & \bar{\bar{F}}_{0,3}e^{m_{0,3}z} & \bar{\bar{F}}_{0,4}e^{m_{0,4}z} & \bar{\bar{F}}_{0,5}e^{m_{0,5}z} & \bar{\bar{F}}_{0,6}e^{m_{0,6}z} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{A}_{0,1} \\ \bar{A}_{0,2} \\ \bar{A}_{0,3} \\ \bar{A}_{0,4} \\ \bar{A}_{0,5} \\ \bar{A}_{0,6} \\ \bar{A}_{0,7} \\ \bar{A}_{0,8} \end{Bmatrix}$$

in which

(3.13a) $\bar{\bar{F}}_{0,1} = -\dfrac{C_3}{C_2}$

(3.13b) $\bar{\bar{F}}_{0,i} = -\dfrac{C_1 m_{0,i}}{C_3}$ for $i = 3$ to 6

The values contained in vectors $\left\langle \bar{\bar{F}}_0(z) \right\rangle^{\mathrm{T}}_{1\times 8}$ and $\left\langle \bar{\bar{C}}_0(z) \right\rangle^{\mathrm{T}}_{1\times 8}$ are functions of the constants $C_i$, which are known and defined in Eq. 2.15. Thus, the only unknown values are the elements of the vector $\left\{ \bar{A}_0 \right\}_{8\times 1}$, which represent the integration constants. The system now has only eight independent integration constants: $\bar{A}_{0,i}$ for $i$ = 1 to 8. These eight constants are determined from the boundary conditions in Eq. 2.19.

## 3.2 Solution for General Displacements: $n \geq 1$

The equilibrium equations for each Fourier mode $n \geq 1$ are given in Eq. 2.27. Assume the solution takes the form:

$$\text{(3.14)} \quad \left\langle a_n(z) \quad d_n(z) \quad f_n(z) \quad b_n(z) \quad -c_n(z) \quad g_n(z) \right\rangle^{\mathrm{T}}_{1\times 6} = \left\langle \bar{A}_n \quad \bar{D}_n \quad \bar{F}_n \quad \bar{B}_n \quad -\bar{C}_n \quad \bar{G}_n \right\rangle^{\mathrm{T}}_{1\times 6} e^{m_n z}$$

Substitute the assumed solution vector into Eq. 2.27 to obtain:

$$\text{(3.15)} \quad \begin{bmatrix} \mathbf{Y_n} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y_n} \end{bmatrix}_{6\times 6} \begin{Bmatrix} \bar{A}_n e^{m_n z} \\ \bar{D}_n e^{m_n z} \\ \bar{F}_n e^{m_n z} \\ \bar{B}_n e^{m_n z} \\ -\bar{C}_n e^{m_n z} \\ \bar{G}_n e^{m_n z} \end{Bmatrix}_{6\times 1} = \left\{ \mathbf{Q}_n \right\}_{6\times 1}$$

in which

$$
(3.16)\quad [\mathbf{Y_n}]_{3\times3} = \left[\begin{array}{c|c|c} C_1 m_n^{\ 2} - C_{10,n} & C_{11,n} m_n & C_3 m_n \\ \hline C_{11,n} m_n & C_{12,n} - C_{13} m_n^{\ 2} & C_{14,n} - C_{15,n} m_n^{\ 2} \\ \hline C_3 m_n & C_{14,n} - C_{15,n} m_n^{\ 2} & C_6 m_n^{\ 4} - C_{17,n} m_n^{\ 2} + C_{16,n} \end{array}\right]
$$

and the constants $C_{i,n}$ are as defined in Eq 2.15.

### 3.2.1 Solution of Homogeneous Part of Equilibrium Equations

The homogeneous part of the equilibrium equations corresponds to solving Eq. 3.15 with the right hand side equal to zero:

$$
(3.17)\quad \left[\begin{array}{c|c} \mathbf{Y_n} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Y_n} \end{array}\right]_{6\times6} \begin{Bmatrix} \bar{A}_n e^{m_n z} \\ \bar{D}_n e^{m_n z} \\ \bar{F}_n e^{m_n z} \\ \bar{B}_n e^{m_n z} \\ -\bar{C}_n e^{m_n z} \\ \bar{G}_n e^{m_n z} \end{Bmatrix}_{6\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{6\times1}
$$

Eq. 3.17 can be split into two similar sets of equations:

$$
(3.18a)\quad [\mathbf{Y_n}]_{3\times3} \begin{Bmatrix} \bar{A}_n e^{m_n z} \\ \bar{D}_n e^{m_n z} \\ \bar{F}_n e^{m_n z} \end{Bmatrix}_{3\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}_{3\times1}
$$

$$
(3.18b)\quad [\mathbf{Y_n}]_{3\times3} \begin{Bmatrix} \bar{B}_n e^{m_n z} \\ -\bar{C}_n e^{m_n z} \\ \bar{G}_n e^{m_n z} \end{Bmatrix}_{3\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}_{3\times1}
$$

Eqs. 3.18a and 3.18b are similar, but they are not identical. Coefficients $\bar{A}_n$, $\bar{D}_n$ and $\bar{F}_n$ will generally be different from coefficients $\bar{B}_n$, $\bar{C}_n$ and $\bar{G}_n$ due to the difference in boundary conditions and the loading conditions of a particular problem. However, the solution procedure for the homogeneous case considered presently is identical. Eq. 3.18a will be solved for the unknown coefficients $\bar{A}_n$, $\bar{D}_n$ and $\bar{F}_n$ while keeping in mind that the same solution procedure can be used to determine $\bar{B}_n$, $\bar{C}_n$ and $\bar{G}_n$.

Hence, the problem simplifies to:

$$(3.19)\qquad [\mathbf{Y_n}]_{3\times3}\begin{Bmatrix}\bar{A}_n e^{m_n z}\\ \bar{D}_n e^{m_n z}\\ \bar{F}_n e^{m_n z}\end{Bmatrix}_{3\times1}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}_{3\times1}$$

For the left hand side of Eq. 3.19 to vanish, either all of the constants $\bar{A}_n$, $\bar{D}_n$, and $\bar{F}_n$ should vanish or the determinant of the matrix $\mathbf{Y_n}$ must be zero. Setting $\bar{A}_n = \bar{D}_n = \bar{F}_n = 0$ leads to a trivial solution that is of no interest. Therefore the determinant of $\mathbf{Y_n}$ must vanish:

$$(3.20)\qquad \begin{vmatrix} C_1 m_n^{\,2} - C_{10,n} & C_{11,n} m_n & C_3 m_n \\ C_{11,n} m_n & C_{12,n} - C_{13} m_n^{\,2} & C_{14,n} - C_{15,n} m_n^{\,2} \\ C_3 m_n & C_{14,n} - C_{15,n} m_n^{\,2} & C_6 m_n^{\,4} - C_{17,n} m_n^{\,2} + C_{16,n} \end{vmatrix} = 0$$

Expanding Eq. 3.20 leads to an 8th order polynomial in $m_n$ (see Eq. A5.9 in Appendix 5).

$$(3.21)\qquad \begin{aligned}&\{[-C_1C_6C_{13}]m_n^{\,8}\\ &+\left[C_1\left(C_6C_{12,n}+C_{13}C_{17,n}-C_{15,n}^{\,2}\right)+C_6\left(C_{10,n}C_{13}-C_{11,n}^{\,2}\right)\right]m_n^{\,6}\\ &+\Big[C_1\left(-C_{12,n}C_{17,n}-C_{13}C_{16,n}+2C_{14,n}C_{15,n}\right)-C_{10,n}\left(C_6C_{12,n}+C_{13}C_{17,n}-C_{15,n}^{\,2}\right)+C_{11,n}^{\,2}C_{17,n}\\ &\qquad -2C_3C_{11,n}C_{15,n}+C_3^{\,2}C_{13}\Big]m_n^{\,4}\\ &+\Big[C_1\left(C_{12,n}C_{16,n}-C_{14,n}^{\,2}\right)+C_{10,n}\left(C_{12,n}C_{17,n}+C_{13}C_{16,n}-2C_{14,n}C_{15,n}\right)+2C_3C_{11,n}C_{14,n}\\ &\qquad -C_{11,n}^{\,2}C_{16,n}-C_3^{\,2}C_{12,n}\Big]m_n^{\,2}\\ &-C_{10,n}\left(C_{12,n}C_{16,n}-C_{14,n}^{\,2}\right)\}=0\end{aligned}$$

The 8th order polynomial in Eq. 3.21 is converted to a 4th order polynomial with the substitution $M_n = m_n^2$. The determinant is now a 4th order (quartic) polynomial of the form

$$(3.22)\qquad p_{1,n}M_n^{\,4}+p_{2,n}M_n^{\,3}+p_{3,n}M_n^{\,2}+p_{4,n}M_n+p_{5,n}=0$$

in which

(3.23a) $p_{1,n} = -C_1 C_6 C_{13}$

(3.23b) $p_{2,n} = C_1\left(C_6 C_{12,n} + C_{13} C_{17,n} - C_{15,n}{}^2\right) + C_6\left(C_{10,n} C_{13} - C_{11,n}{}^2\right)$

(3.23c) $$\begin{aligned} p_{3,n} &= C_1\left(-C_{12,n} C_{17,n} - C_{13} C_{16,n} + 2C_{14,n} C_{15,n}\right) - C_{10,n}\left(C_6 C_{12,n} + C_{13} C_{17,n} - C_{15,n}{}^2\right) \\ &\quad + C_{11,n}{}^2 C_{17,n} - 2C_3 C_{11,n} C_{15,n} + C_3^2 C_{13,n} \end{aligned}$$

(3.23d) $$\begin{aligned} p_{4,n} &= C_1\left(C_{12,n} C_{16,n} - C_{14,n}{}^2\right) + C_{10,n}\left(C_{12,n} C_{17,n} + C_{13} C_{16,n} - 2C_{14,n} C_{15,n}\right) \\ &\quad + 2C_3 C_{11,n} C_{14,n} - C_{11,n}{}^2 C_{16,n} - C_3^2 C_{12,n} \end{aligned}$$

(3.23e) $p_{5,n} = -C_{10,n}\left(C_{12,n} C_{16,n} - C_{14,n}{}^2\right)$

The quartic equation in Eq. 3.22 is solved by Ferrari's Method (see Appendix 6 for details) to give the four roots $M_n$. The eight values of $m_n$ are then computed from the relation $m_n = \pm\sqrt{M_n}$. Assuming that eight distinct roots are obtained, the solution of the homogeneous part of the equilibrium equations becomes (with $m_{n,i} \neq m_{n,j}$ for $i \neq j$):

(3.24a) $a_n(z) = \sum_{i=1}^{8} \bar{A}_{n,i} e^{m_{n,i} z}$

(3.24b) $d_n(z) = \sum_{i=1}^{8} \bar{D}_{n,i} e^{m_{n,i} z}$

(3.24c) $f_n(z) = \sum_{i=1}^{8} \bar{F}_{n,i} e^{m_{n,i} z}$

### *3.2.2 Solution of Homogeneous Part of Equilibrium Equation for the Case of* $n = 1$

The solution field presented in Eq. 3.24 is valid only when four distinct roots to the quartic equation in Eq. 3.22 are obtained. Repeated roots in Eq. 3.22 lead to $m_{n,i} = m_{n,j}$ for $i \neq j$. The solution for this case is different from the general solution presented in Eq. 3.24. Multiple roots are encountered in the case of $n = 1$ in which four equal roots are present.

For $n=1$, $p_{4,1}=p_{5,1}=0$ in Eq. 3.23 (see Appendix 7) which results in Eq. 3.22 simplifying to:

$$(3.25) \quad p_{1,1}M_1^{\ 4}+p_{2,1}M_1^{\ 3}+p_{3,1}M_1^{\ 2}=M_1^{\ 2}\left(p_{1,1}M_1^{\ 2}+p_{2,1}M_1+p_{3,1}\right)=0$$

Two of the roots of Eq. 3.25 are $M_{1,1}=M_{1,2}=0$ leading to $m_{1,1}=m_{1,2}=m_{1,3}=m_{1,4}=0$. The solution to the equilibrium equations becomes:

$$(3.26a) \quad a_1(z)=\bar{A}_{1,1}+\bar{A}_{1,2}z+\bar{A}_{1,3}z^2+\bar{A}_{1,4}z^3+\sum_{i=5}^{8}\bar{A}_{1,i}e^{m_{1,i}z}$$

$$(3.26b) \quad d_1(z)=\bar{D}_{1,1}+\bar{D}_{1,2}z+\bar{D}_{1,3}z^2+\bar{D}_{1,4}z^3+\sum_{i=5}^{8}\bar{D}_{1,i}e^{m_{1,i}z}$$

$$(3.26c) \quad f_1(z)=\bar{F}_{1,1}+\bar{F}_{1,2}z+\bar{F}_{1,3}z^2+\bar{F}_{1,4}z^3+\sum_{i=5}^{8}\bar{F}_{1,i}e^{m_{1,i}z}$$

The displacement field in Eq. 3.26 is valid only when two distinct roots to the quadratic portion of Eq. 3.25 exist. The solution to the quadratic is:

$$(3.27) \quad M_{1,(3,4)}=\frac{-p_{2,1}\pm\sqrt{p_{2,1}^2-4p_{1,1}p_{3,1}}}{2p_{1,1}}$$

By inspection of Eq. 3.27 the requirement for $M_{1,3}\neq M_{1,4}$ reduces to $p_{2,1}^2\neq 4p_{1,1}p_{3,1}$. Use the definitions of $p_{1,1}$, $p_{2,1}$ and $p_{3,1}$ from Eq. 3.23 to obtain:

$$(3.28) \quad \begin{aligned} p_{2,1}^2 &= \left[C_1\left(C_6C_{12,1}+C_{13}C_{17,1}-C_{15,1}^{\ 2}\right)+C_6\left(C_{10,1}C_{13,1}-C_{11,1}^{\ 2}\right)\right]^2 \\ &= C_1^{\ 2}\left[\left(C_6C_{12,1}\right)^2+\left(C_{13}C_{17,1}\right)^2+C_{15,1}^{\ 4}+2\left(C_6C_{12,1}C_{13}C_{17,1}-C_6C_{12,1}C_{15,1}^{\ 2}-C_{13}C_{17,1}C_{15,1}^{\ 2}\right)\right] \\ &+2C_1C_6\left[C_6C_{12,1}C_{10,1}C_{13}-C_6C_{12,1}C_{11,1}^{\ 2}+C_{13,1}C_{17,1}C_{10,1}C_{13}-C_{13}C_{17,1}C_{11,1}^{\ 2}\right. \\ &\left.-C_{15,1}^{\ 2}C_{10,1}C_{13}+C_{15,1}^{\ 2}C_{11,1}^{\ 2}\right]+C_6^{\ 2}\left[\left(C_{10,1}C_{13}\right)-2C_{10,1}C_{13}C_{11,1}^{\ 2}+C_{11,1}^{\ 4}\right] \end{aligned}$$

$$4p_{1,1}p_{3,1} = 4\left[-C_1C_6C_{13}\right]\left[C_1\left(-C_{12,1}C_{17,1} - C_{13}C_{16,1} + 2C_{14,1}C_{15,1}\right) - C_{10,1}\left(C_6C_{12,1} + C_{13}C_{17,1} - C_{15,1}^{\,2}\right) + C_{11,1}^{\,2}C_{17,1} - 2C_3C_{11,1}C_{15,1} + C_3^{\,2}C_{13}\right] \tag{3.29}$$

By inspection of Eqs. 3.28 and 3.29, it is clear that the condition $p_{2,1}^2 \neq 4p_{1,1}p_{3,1}$ is always satisfied, leading to $M_{1,3} \neq M_{1,4}$. Therefore, the four roots are distinct and the displacement field given in Eq. 3.26 is valid.

Use the relation $m_1 = \pm\sqrt{M_1}$ and Eq. 3.31 to obtain the four roots of $m_{1,i}$ for $i$ = 5 to 8:

$$m_{1,5} = \sqrt{\frac{-p_{2,1} + \sqrt{p_{2,1}^2 - 4p_{1,1}p_{3,1}}}{2p_{1,1}}} \tag{3.30a}$$

$$m_{1,6} = -\sqrt{\frac{-p_{2,1} + \sqrt{p_{2,1}^2 - 4p_{1,1}p_{3,1}}}{2p_{1,1}}} \tag{3.30b}$$

$$m_{1,7} = \sqrt{\frac{-p_{2,1} - \sqrt{p_{2,1}^2 - 4p_{1,1}p_{3,1}}}{2p_{1,1}}} \tag{3.30c}$$

$$m_{1,8} = -\sqrt{\frac{-p_{2,1} - \sqrt{p_{2,1}^2 - 4p_{1,1}p_{3,1}}}{2p_{1,1}}} \tag{3.30d}$$

### *3.2.3 Reducing Integration Constants for Fourier Mode* $n = 1$

In Eq. 3.26, twenty-four integration constants appear in the solution for the displacements $a_1(z)$, $d_1(z)$ and $f_1(z)$, namely $\overline{A}_{1,i}, \overline{D}_{1,i}$ and $\overline{F}_{1,i}$ for $i$ = 1 to 8, but only eight boundary conditions are provided in Eq. 2.19. However, as in Section 3.1.2 for $n = 0$, the equations in the displacement field of Eq. 3.26 are linearly dependent and the integration constants are inter-dependent. By relating the integration constants, as shown in Appendix 8, Eq. 3.26 is written in matrix form as:

$$(3.31)\quad \begin{Bmatrix} a_1(z) \\ d_1(z) \\ f_1(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{\bar{A}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \\ \left\langle \bar{\bar{D}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \\ \left\langle e_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \end{bmatrix} \left\{ \bar{F}_1 \right\}_{8\times 1}$$

in which

$$(3.32a)\quad \left\langle \bar{\bar{A}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} = \Big\langle 0 \mid \bar{\bar{A}}_{1,1.1} \mid \bar{\bar{A}}_{1,2} z \mid \left( \bar{\bar{A}}_{1,1.2} + \bar{\bar{A}}_{1,3} z^2 \right) \mid \bar{\bar{A}}_{1,5} e^{m_{1,5} z} \mid \bar{\bar{A}}_{1,6} e^{m_{1,6} z} \mid \bar{\bar{A}}_{1,7} e^{m_{1,7} z} \mid \bar{\bar{A}}_{1,8} e^{m_{1,8} z} \Big\rangle_{1\times 8}$$

$$(3.32b)\quad \left\langle \bar{\bar{D}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} = \Big\langle \bar{\bar{D}}_{1,1.1} \mid \bar{\bar{D}}_{1,2.1} z \mid \left( \bar{\bar{D}}_{1,1.2} + \bar{\bar{D}}_{1,3} z^2 \right) \mid \left( \bar{\bar{D}}_{1,2.2} z + \bar{\bar{D}}_{1,4} z^3 \right) \mid \bar{\bar{D}}_{1,5} e^{m_{1,5} z} \mid \bar{\bar{D}}_{1,6} e^{m_{1,6} z} \mid \bar{\bar{D}}_{1,7} e^{m_{1,7} z} \mid \bar{\bar{D}}_{1,8} e^{m_{1,8} z} \Big\rangle_{1\times 8}$$

$$(3.32c)\quad \left\langle e_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} = \left\langle 1 \quad z \quad z^2 \quad z^3 \quad e^{m_{1,5} z} \quad e^{m_{1,6} z} \quad e^{m_{1,7} z} \quad e^{m_{1,8} z} \right\rangle_{1\times 8}$$

$$(3.32d)\quad \left\langle \bar{F}_1 \right\rangle_{1\times 8}^{\mathrm{T}} = \left\langle \bar{F}_{1,1} \quad \bar{F}_{1,2} \quad \bar{F}_{1,3} \quad \bar{F}_{1,4} \quad \bar{F}_{1,5} \quad \bar{F}_{1,6} \quad \bar{F}_{1,7} \quad \bar{F}_{1,8} \right\rangle_{1\times 8}$$

In Eq. 3.32, the values of $\bar{\bar{A}}_{1,i}$ and $\bar{\bar{D}}_{1,i}$ are:

$$(3.33a)\quad \bar{\bar{A}}_{1,1.1} = \frac{C_3}{C_{10,1}} - \frac{C_{11,1} C_{14,1}}{C_{10,1} C_{12,1}}$$

$$(3.33b)\quad \bar{\bar{A}}_{1,1.2} = \frac{6C_1}{C_{10,1}^2} \left[ -\frac{C_{11,1} C_{14,1}}{C_{12,1}} + C_3 \right] + \frac{6C_{11,1}}{C_{10,1} C_{12,1}} \left[ \frac{C_{11,1}}{C_{10,1}} \left( \frac{C_{11,1} C_{14,1}}{C_{12,1}} - C_3 \right) - \frac{C_{13} C_{14,1}}{C_{12,1}} + C_{15,1} \right]$$

$$(3.33c)\quad \bar{\bar{A}}_{1,2} = \frac{2}{C_{10,1}} \left( -\frac{C_{11,1} C_{14,1}}{C_{12,1}} + C_3 \right)$$

$$(3.33d)\quad \bar{\bar{A}}_{1,3} = \frac{3}{C_{10,1}} \left( -\frac{C_{11,1} C_{14,1}}{C_{12,1}} + C_3 \right)$$

(3.33e) $$\bar{\bar{A}}_{1,i}=\frac{-\frac{\left(C_{11,1}m_{1,i}\right)}{\left(C_1m_{1,i}^{\;2}-C_{10}\right)}\left[\frac{\left(C_{11,1}m_{1,i}\right)\left(C_3m_{1,i}\right)}{\left(C_1m_{1,i}^{\;2}-C_{10,1}\right)}-\left(C_{14,1}-C_{15,1}m_{1,i}^{\;2}\right)\right]}{\left(C_{12,1}-C_{13}m_{1,i}^{\;2}\right)-\frac{\left(C_{11,1}m_{1,i}\right)^2}{\left(C_1m_{1,i}^{\;2}-C_{10,1}\right)}}-\frac{\left(C_3m_{1,i}\right)}{\left(C_1m_{1,i}^{\;2}-C_{10,1}\right)}$$

for $i$ = 5 to 8

(3.34a) $$\bar{\bar{D}}_{1,1.1}=\bar{\bar{D}}_{1,2.1}=\bar{\bar{D}}_{1,3}=\bar{\bar{D}}_{1,4}=-\frac{C_{14,1}}{C_{12,1}}$$

(3.34b) $$\bar{\bar{D}}_{1,1.2}=\frac{2}{C_{12,1}}\left[\frac{C_{11,1}}{C_{10,1}}\left(\frac{C_{11,1}C_{14,1}}{C_{12,1}}-C_3\right)-\frac{C_{13}C_{14,1}}{C_{12,1}}+C_{15,1}\right]$$

(3.34c) $$\bar{\bar{D}}_{1,2.2}=\frac{6}{C_{12,1}}\left[\frac{C_{11,1}}{C_{10,1}}\left(\frac{C_{11,1}C_{14,1}}{C_{12,1}}-C_3\right)-\frac{C_{13}C_{14,1}}{C_{12,1}}+C_{15,1}\right]$$

(3.34d) $$\bar{\bar{D}}_{1,i}=\frac{\frac{\left(C_{11,1}m_{1,i}\right)\left(C_3m_{1,i}\right)}{\left(C_1m_{1,i}^{\;2}-C_{10,1}\right)}-\left(C_{14,1}-C_{15,1}m_{1,i}^{\;2}\right)}{\left(C_{12,1}-C_{13}m_{1,i}^{\;2}\right)-\frac{\left(C_{11,1}m_{1,i}\right)^2}{\left(C_1m_{1,i}^{\;2}-C_{10,1}\right)}}$$

for $i$ = 5 to 8

The values contained in vectors $\left\langle \bar{\bar{A}}_1(z)\right\rangle^{\mathrm{T}}_{1\times8}$ and $\left\langle \bar{\bar{D}}_1(z)\right\rangle^{\mathrm{T}}_{1\times8}$ are functions of constants $C_{i,1}$, which are known and defined in Eq. 2.15. The only unknown values are the elements of vector $\left\{\bar{F}_1\right\}_{8\times1}$, which represent the integration constants. The system now contains only eight sets of independent integration constants: $\bar{F}_{1,i}$ for $i$ = 1 to 8. These eight constants are determined from the boundary conditions in Eq. 2.19.

Similar to Eq. 3.26, displacements $b_1(z)$, $c_1(z)$ and $g_1(z)$ can be written as:

(3.35a) $$b_1(z)=\bar{B}_{1,1}+\bar{B}_{1,2}z+\bar{B}_{1,3}z^2+\bar{B}_{1,4}z^3+\sum_{i=5}^{8}\bar{B}_{1,i}e^{m_{1,i}z}$$

(3.35b) $c_1(z) = \bar{C}_{1,1} + \bar{C}_{1,2}z + \bar{C}_{1,3}z^2 + \bar{C}_{1,4}z^3 + \sum_{i=5}^{8} \bar{C}_{1,i}e^{m_{1,i}z}$

(3.35c) $g_1(z) = \bar{G}_{1,1} + \bar{G}_{1,2}z + \bar{G}_{1,3}z^2 + \bar{G}_{1,4}z^3 + \sum_{i=5}^{8} \bar{G}_{1,i}e^{m_{1,i}z}$

By following the procedure to relate coefficients $\bar{A}_{1,i}$ and $\bar{D}_{1,i}$ to $\bar{F}_{1,i}$, outlined in Appendix 8, the coefficients $\bar{B}_{1,i}$ and $\bar{C}_{1,i}$ can be related to $\bar{G}_{1,i}$. In fact, coefficients $\bar{B}_{1,i}$ and $\bar{C}_{1,i}$ are related to $\bar{G}_{1,i}$ through vectors $\langle \bar{\bar{A}}_1(z) \rangle_{1\times 8}^{\mathrm{T}}$ and $-\langle \bar{\bar{D}}_1(z) \rangle_{1\times 8}^{\mathrm{T}}$, respectively. The vectors are identical to those used to relate $\bar{A}_{1,i}$ and $\bar{D}_{1,i}$ to $\bar{F}_{1,i}$ because the relationship between integration constants is independent of loading and boundary conditions. In other words, the relationships between $\bar{A}_{1,i}$ and $\bar{F}_{1,i}$ is the same as the relationship between $\bar{B}_{1,i}$ and $\bar{G}_{1,i}$ because the relationships do not depend on loading or boundary conditions. Recall that from Eq. 3.18 that $\bar{A}_{1,i}$ and $\bar{B}_{1,i}$ are different only because of loading and boundary conditions; otherwise the formulations of $\bar{A}_{1,i}$ and $\bar{B}_{1,i}$ are identical.

Note, however, the negative sign in front of the vector $\langle \bar{\bar{D}}_1(z) \rangle_{1\times 8}^{\mathrm{T}}$ relating $\bar{C}_{1,i}$ to $\bar{G}_{1,i}$. The negative sign is required because the displacement $c_1(z)$ was multiplied by -1 in Eq. 2.23 in order to make the matrices $\mathbf{X}_{1,1}$ and $\mathbf{X}_{2,1}$ identical. The equations in these matrices provide the basis for the method used to determine $\langle \bar{\bar{D}}_1(z) \rangle_{1\times 8}^{\mathrm{T}}$ in Appendix 8. As a result, the equations in Appendix 8 also need to be multiplied by -1 for consistency, resulting in $\bar{C}_{1,i}$ being related to $\bar{G}_{1,i}$ through $-\langle \bar{\bar{D}}_1(z) \rangle_{1\times 8}^{\mathrm{T}}$.

Therefore, displacements $b_1(z)$, $c_1(z)$ and $g_1(z)$ are written as:

$$(3.36)\quad \begin{Bmatrix} b_1(z) \\ c_1(z) \\ g_1(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{\bar{A}}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} \\ -\left\langle \bar{\bar{D}}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle e_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} \end{bmatrix} \left\{ \bar{G}_1 \right\}_{8\times 1}$$

where vectors $\left\langle \bar{\bar{A}}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8}$, $\left\langle \bar{\bar{D}}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8}$ and $\left\langle e_1(z) \right\rangle^{\mathrm{T}}_{1\times 8}$ were defined in Eq. 3.32 and vector $\left\{ \bar{G}_1 \right\}_{8\times 1}$ contains the unknown integration constants.

### *3.2.4 Reducing Integration Constants for Fourier Modes $n \geq 2$*

For the general case of $n \geq 2$, the eight roots of Eq. 3.21 are distinct and the equilibrium equations given in Eq. 3.24 are valid. As in the previous case, twenty-four integration constants appear in the solution for displacements $a_n(z)$, $d_n(z)$ and $f_n(z)$ for each mode $n$, namely $\bar{A}_{n,i}$, $\bar{D}_{n,i}$ and $\bar{F}_{n,i}$ for $i$ = 1 to 8, but only eight boundary conditions are provided in Eq. 2.19. However, the equations in the displacement field of Eq. 3.24 are linearly dependent and the integration constants are inter-related. By relating the integration constants, as shown in Appendix 9, Eq. 3.24 is written in matrix form as:

$$(3.37)\quad \begin{Bmatrix} a_n(z) \\ d_n(z) \\ f_n(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{\bar{A}}_n(z) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle \bar{\bar{D}}_n(z) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle e_n(z) \right\rangle^{\mathrm{T}}_{1\times 8} \end{bmatrix} \left\{ \bar{F}_n \right\}_{8\times 1}$$

in which:

$$(3.38a)\quad \left\langle \bar{\bar{A}}_n(z) \right\rangle^{\mathrm{T}}_{1\times 8} = \Big\langle \bar{\bar{A}}_{n,1} e^{m_{n,1}z} \quad \bar{\bar{A}}_{n,2} e^{m_{n,2}z} \quad \bar{\bar{A}}_{n,3} e^{m_{n,3}z} \quad \bar{\bar{A}}_{n,4} e^{m_{n,4}z} \quad \bar{\bar{A}}_{n,5} e^{m_{n,5}z} \quad \bar{\bar{A}}_{n,6} e^{m_{n,6}z} \quad \bar{\bar{A}}_{n,7} e^{m_{n,7}z} \quad \bar{\bar{A}}_{n,8} e^{m_{n,8}z} \Big\rangle_{1\times 8}$$

(3.38b) $$\left\langle \bar{\bar{D}}_n(z) \right\rangle^T_{1\times 8} = \left\langle \bar{\bar{D}}_{n,1} e^{m_{n,1}z} \quad \bar{\bar{D}}_{n,2} e^{m_{n,2}z} \quad \bar{\bar{D}}_{n,3} e^{m_{n,3}z} \quad \bar{\bar{D}}_{n,4} e^{m_{n,4}z} \right.$$
$$\left. \bar{\bar{D}}_{n,5} e^{m_{n,5}z} \quad \bar{\bar{D}}_{n,6} e^{m_{n,6}z} \quad \bar{\bar{D}}_{n,7} e^{m_{n,7}z} \quad \bar{\bar{D}}_{n,8} e^{m_{n,8}z} \right\rangle_{1\times 8}$$

(3.38c) $$\left\langle e_n(z) \right\rangle^T_{1\times 8} = \left\langle e^{m_{n,1}z} \quad e^{m_{n,2}z} \quad e^{m_{n,3}z} \quad e^{m_{n,4}z} \quad e^{m_{n,5}z} \quad e^{m_{n,6}z} \quad e^{m_{n,7}z} \quad e^{m_{n,8}z} \right\rangle_{1\times 8}$$

(3.38d) $$\left\langle \bar{F}_n \right\rangle^T_{1\times 8} = \left\langle \bar{F}_{n,1} \quad \bar{F}_{n,2} \quad \bar{F}_{n,3} \quad \bar{F}_{n,4} \quad \bar{F}_{n,5} \quad \bar{F}_{n,6} \quad \bar{F}_{n,7} \quad \bar{F}_{n,8} \right\rangle_{1\times 8}$$

The values of $\bar{\bar{A}}_{n,i}$ and $\bar{\bar{D}}_{n,i}$ in Eqs. 3.38a and 3.38b are:

(3.39a) $$\bar{\bar{A}}_{n,i} = \frac{-\dfrac{\left(C_{11,n} m_{n,i}\right)}{\left(C_1 m_{n,i}^{\ 2} - C_{10,n}\right)} \left[ \dfrac{\left(C_{11,n} m_{n,i}\right)\left(C_3 m_{n,i}\right)}{\left(C_1 m_{n,i}^{\ 2} - C_{10,n}\right)} - \left(C_{14,n} - C_{15,n} m_{n,i}^{\ 2}\right) \right]}{\left(C_{12,n} - C_{13} m_{n,i}^{\ 2}\right) - \dfrac{\left(C_{11,n} m_{n,i}\right)^2}{\left(C_1 m_{n,i}^{\ 2} - C_{10,n}\right)}} - \frac{\left(C_3 m_{n,i}\right)}{\left(C_1 m_{n,i}^{\ 2} - C_{10,n}\right)}$$

(3.39b) $$\bar{\bar{D}}_{n,i} = \frac{\dfrac{\left(C_{11,n} m_{n,i}\right)\left(C_3 m_{n,i}\right)}{\left(C_1 m_{n,i}^{\ 2} - C_{10,n}\right)} - \left(C_{14,n} - C_{15,n} m_{n,i}^{\ 2}\right)}{\left(C_{12,n} - C_{13} m_{n,i}^{\ 2}\right) - \dfrac{\left(C_{11,n} m_{n,i}\right)^2}{\left(C_1 m_{n,i}^{\ 2} - C_{10,n}\right)}}$$

For a given mode $n$ and a given index $i$ the values contained in vectors $\left\langle \bar{\bar{A}}_n(z) \right\rangle^T_{1\times 8}$ and $\left\langle \bar{\bar{D}}_n(z) \right\rangle^T_{1\times 8}$ are functions of the constants $C_{i,n}$, which are known and defined in Eq. 2.15. The only unknown values, representing the integration constants, are the elements of the vector $\left\{ \bar{F}_n \right\}_{8\times 1}$. The system now contains only eight independent integration constants for each mode $n$: $\bar{F}_{n,i}$ for $i$ = 1 to 8. These eight constants are determined from the boundary conditions in Eq. 2.19.

Similar to Eq. 3.24, displacements $b_n(z)$, $c_n(z)$ and $g_n(z)$ can be written as:

(3.40a) $b_n(z)=\sum_{i=1}^{8}\overline{B}_{n,i}e^{m_{n,i}z}$

(3.40b) $c_n(z)=\sum_{i=1}^{8}\overline{C}_{n,i}e^{m_{n,i}z}$

(3.40c) $g_n(z)=\sum_{i=1}^{8}\overline{G}_{n,i}e^{m_{n,i}z}$

As for the case of $n=1$, the vectors $\left\langle \overline{\overline{A}}_n(z) \right\rangle_{1\times8}^{\mathrm{T}}$ and $\left\langle \overline{\overline{D}}_n(z) \right\rangle_{1\times8}^{\mathrm{T}}$ can also be used to relate coefficients $\overline{B}_{n,i}$ and $\overline{C}_{n,i}$, respectively, to $\overline{G}_{n,i}$. Hence, displacements $b_n(z)$, $c_n(z)$ and $g_n(z)$ are written as:

(3.41) $$\begin{Bmatrix} b_n(z) \\ c_n(z) \\ g_n(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \overline{\overline{A}}_n(z) \right\rangle_{1\times8}^{\mathrm{T}} \\ -\left\langle \overline{\overline{D}}_n(z) \right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle e_n(z) \right\rangle_{1\times8}^{\mathrm{T}} \end{bmatrix} \left\{ \overline{G}_n \right\}_{8\times1}$$

where vectors $\left\langle \overline{\overline{A}}_n(z) \right\rangle_{1\times8}^{\mathrm{T}}$, $\left\langle \overline{\overline{D}}_n(z) \right\rangle_{1\times8}^{\mathrm{T}}$ and $\left\langle e_n(z) \right\rangle_{1\times8}^{\mathrm{T}}$ were defined in Eq. 3.38 and vector $\left\{ \overline{G}_n \right\}_{8\times1}$ contains the unknown integration constants.

# Chapter 4 – Verification of Analytical Solution

The objective of this Chapter is to provide a step by step solution of a simple pipe problem based on the analytical formulation developed in Chapter 3. The solution is compared to solutions based on the Euler-Bernoulli beam theory and established shell finite element analysis.

## 4.1 Example 1: Cantilever Beam Under Concentrated Load

The pipe under consideration is a cantilevered beam with a point load applied to the top of the pipe at the free end, as shown in Figure 4.1. The pipe has a span $\ell$, mid-surface radius $r$ and wall thickness $t$. It is required to find the vertical and longitudinal displacements of Points $A$, $B$ and $C$ which are located on the pipe cross section shown in Figure 4.1 at the free end of the pipe.

The geometric and material properties of the pipe are: $\ell$ = 4 $m$, $t$ = 6 $mm$, $r$ = 200 $mm$,

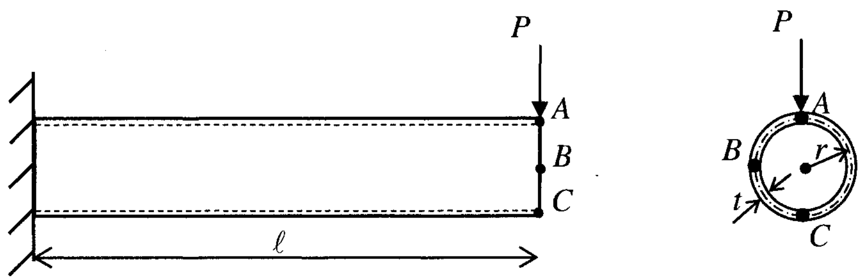


**Figure 4.1** – Pipe geometry and cross section at free end.

$\mu = 0.3$, $E = 200{,}000\,MPa$ and $G = E/2(1+\mu) = 76{,}925\,MPa$. The applied point load is $P$ = 16 $kN$.

### *4.1.1 The Euler-Bernoulli Beam Theory Solution*

The Euler-Bernoulli beam theory does not capture cross section deformation and thus gives only the vertical displacement of the beam centreline. Therefore, the vertical displacements of Points $A$, $B$ and $C$ will all be equal.

For the cantilevered beam under consideration the Euler-Bernoulli beam theory problem includes four degrees of freedom. The degrees of freedom are the rotations and vertical displacements at each end of the beam. The centreline vertical displacement of the free end, $\Delta_{EB}$, is governed by the moment of inertia, *I*, of the pipe cross section:

$$(4.1) \qquad I = \frac{\pi\left[(2r+t)^4-(2r-t)^4\right]}{64} = \frac{\pi\left[(406mm)^4-(394mm)^4\right]}{64} = 150.83\times10^6\ mm^4$$

Hence, the vertical displacement is given as:

$$(4.2) \qquad \Delta_{EB} = \frac{P\ell^3}{3EI} = \frac{(16{,}000N)(4000mm)^3}{3(200{,}000MPa)(150.83\times10^6 mm^4)} = 11.315\,mm$$

The longitudinal displacement at the top fibre at the free end of the cantilever (Point *A*) is given as $r\sin\theta$, where *r* is the pipe cross section radius and $\theta$ is the angle of rotation at the end of the cantilever. The Euler-Bernoulli beam theory gives the angle of rotation as $\theta = P\ell^2/2EI$. For the current problem, the resulting longitudinal displacement of Point *A* is 0.849 *mm.*

#### 4.1.1.1 Displacement due to Shear Deformation

The vertical displacement given by the Euler-Bernoulli beam theory does not account for any deformations due to shear. However, including the effects of shear deformation provides a more accurate estimate of the vertical displacement. The vertical displacement due to shear for a cantilever beam subject to a point load at the tip is:

$$(4.3) \qquad \Delta_{Shear} = \frac{P\ell}{GA_r}$$

where Belvins (1984) gives the value of $A_r$ for a thin walled pipe section as:

$$(4.4) \qquad A_r = \frac{2(1+\mu)}{4+3\mu}A$$

For the current problem $A_r = \dfrac{2(1+\mu)}{4+3\mu}(2\pi rt) = 4000\,mm^2$. Hence the shear contribution of vertical displacement of the free end of the cantilever is:

$$(4.5) \qquad \Delta_{Shear} = \frac{P\ell}{GA_r} = \frac{(16{,}000\,N)(4000\,mm)}{(76{,}925\,MPa)(4000\,mm^2)} = 0.208\,mm$$

The total vertical displacement of the free end from Euler-Bernoulli beam theory and shear deformations is $\Delta_{Total} = 11.523\,mm$.

### *4.1.2 Shell Finite Element Analysis*

A more accurate representation of the deflected shape of the pipe is given by a shell finite element analysis. Such an analysis captures both centreline deflection as well as cross section deformation.

The finite element analysis was conducted using the commercial finite element analysis program ABAQUS. The model was created using the quadrilateral shell element S4R which is internal to the program. The S4R element is a two dimensional element, with user-specified thickness, that is arbitrarily oriented in a three-dimensional space.

The S4R element contains four nodes, each of which has five degrees of freedom. The degrees of freedom include three translations (one in each direction of the global axes) and two rotations. The rotations actually represent the changes of two independent components of the unit vector that is normal to the shell mid-surface. The change of the third component of the unit vector is dependent on the changes to the first two components and is derived from the condition that the length of the vector is unity.

The S4R element contains both in-plane and out-of-plane stiffness, allowing for membrane and bending characteristics, respectively. The membrane strains on the reference surface, which follow a finite strain formulation, are obtained from the derivatives of the position vector for the points on the reference surface in the deformed configuration with respect to their position on the surface. The bending strains are obtained from the derivatives of the vector normal to the mid-surface of the element. A result of the finite strain formulation is that the shell element thickness in the deformed configuration is different than the thickness in the undeformed configuration. However, in the formulation of the S4R element, the strain normal to the reference surface is assumed to be constant over the thickness of the shell (Hibbit et al., 2001).

Both the displacement and position vectors are interpolated with the same functions, which have $C^0$ continuity. The components of the vector normal to the reference surface (i.e. the rotational degrees of freedom) are interpolated independently from the translational degrees of freedom.

Transverse shear strain in the element is the change in the projection of a vector initially normal to the reference surface onto the tangent to the reference surface of the shell. The transverse shear acts as a penalty function to impose a constraint that approximately enforces the Kirchhoff assumption that a line normal to the mid-surface before deformation remains normal to the mid-surface after deformation. In contrast, the Kirchhoff assumption is enforced in the current study in a strict sense (Section 2.2). The transverse shear strains of the S4R element are evaluated at the mid-points of the element edges.

Five points are used for the through-thickness integration, which is conducted numerically using Simpson's Rule.

The material characteristics used in the ABAQUS model are the same as those shown in Figure 4.1. Namely, the material is assumed to be linear elastic with Young's Modulus $E = 200,000\, MPa$ and Poisson's Ratio $\mu = 0.3$.

The finite element mesh for the model contains 102 elements along the length of the pipe and 32 divisions around its circumference, corresponding to an aspect ratio for each element of 1:4, for a total of 3296 nodes, 3264 shell elements and 16,480 degrees of freedom.

The results of the ABAQUS shell finite element analysis, including both the vertical and longitudinal displacements of Points *A, B* and *C* in Figure 4.1, are given in Table 4.1 below.

**Table 4.1** – Displacements of Points $A$, $B$ and $C$ as determined by ABAQUS shell finite element analysis

| Location | Vertical Displacement $\Delta_{Vert,FEA}$ | Longitudinal Displacement $\Delta_{Long,FEA}$ |
|---|---|---|
| Point $A$ | 16.755 $mm$ | 1.142 $mm$ |
| Point $B$ | 11.392 $mm$ | -0.132 $mm$ |
| Point $C$ | 9.599 $mm$ | -0.742 $mm$ |

### 4.1.3 *Methodology Developed in the Current Study*

The displacements are now determined by the method presented in Chapter 3. The unknowns sought are the integration constants $\bar{F}_{n,i}$ for each mode $n$. The starting point for the solution is the equation of the variation of total potential energy, Eq A3.7, from Appendix 3. From this equation the relevant load potential and boundary terms are retained and used to solve for the integration constants. It is clear from Eq. A3.7 that for any two distinct modes, $n$ and $m \neq n$, the field equations are always uncoupled. For a large class of problems, including the present problem, the boundary conditions are also uncoupled.

The solution to the field equations is implemented in Borland C++. Details of the program implementation are included in Appendix 10.

#### 4.1.3.1 Load Potential Terms

The starting point of the solution is to formulate the expression for the load potential, $W$, as given in Eq. 2.16.

$$(4.6) \qquad W = W_w = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_w(\phi,z)\left[f_0(z)+\sum_{n=1}^{\alpha} f_n(z)\cos n\phi+\sum_{n=1}^{\alpha} g_n(z)\sin n\phi\right] r\,d\phi\,dz$$

Note that Eq. 4.6 contains only terms relating to the radial direction, $w$. For the loading condition shown in Figure 4.1, the relation $q_u(\phi,z)=q_v(\phi,z)=0$ holds because the point load at point $A$ is applied vertically. The load is entirely in the radial direction with no components in either the longitudinal or tangential directions. Thus in Eq. 2.16 $W_u = W_v = 0$.

The load function $q_w(\phi, z)$ is expressed using the Dirac Delta Function both in the z and $\phi$ directions to represent the applied point load. In general, the Dirac Delta Function has the following properties:

(4.7a) $\delta(x-a) = \begin{cases} \infty, x = a \\ 0, x \neq a \end{cases}$ where $a$ is any value within the domain of $x$

(4.7b) $\int_{x=b}^{c} \delta(x-a)\,dx = 1$ where $b \leq a \leq c$

(4.7c) $\int_{x=b}^{c} \delta(x-a)\,h(x)\,dx = h(a)$ where $h(x)$ is a continuous function of $x$ on the domain $b < x < c$

In the current example the point load is applied at $z = \ell$ and $\phi = \pi/2$. Hence the load function is:

(4.8) $$q_w(\phi, z) = -P\delta(z-\ell)\,\delta\left(\phi - \pi/2\right)$$

Note that in Eq. 4.8 the negative sign is due to the fact that $P$ is applied in the opposite direction of positive $w$, as defined in Figures 2.1 and 2.2. Substitute Eq. 4.8 into Eq. 4.6:

(4.9) $$W_w = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} -P\delta(z-\ell)\,\delta\left(\phi-\pi/2\right)\left[f_0(z) + \sum_{n=1}^{\alpha} f_n(z)\cos n\phi + \sum_{n=1}^{\alpha} g_n(z)\sin n\phi\right] r\,d\phi\,dz$$

In Eq. 4.9, functions $f_0(z)$, $f_n(z)$, $g_n(z)$, $\cos n\phi$ and $\sin n\phi$ are continuous. Evaluating the integrals for the Dirac Delta functions in $z$ and $\phi$ and using the relation in Eq. 4.7c, gives:

(4.10) $$W_w = -P\left[f_0(\ell) + \sum_{n=1}^{\alpha} f_n(\ell)\cos\left(n\pi/2\right) + \sum_{n=1}^{\alpha} g_n(\ell)\sin\left(n\pi/2\right)\right]$$

Take the variation of Eq. 4.10 to obtain the variation of the work done by the applied load:

(4.11) $$\delta W_w = -P\left[\delta f_0(\ell) + \sum_{n=1}^{\alpha}\cos\left(n\pi/2\right)\delta f_n(\ell) + \sum_{n=1}^{\alpha}\sin\left(n\pi/2\right)\delta g_n(\ell)\right]$$

Eq. 4.11 is simplified by noting that $\cos(n\pi/2)=0$ when $n$ is odd and $\sin(n\pi/2)=0$ when $n$ is even:

$$(4.12)\qquad \delta W_w = -P\left[\delta f_0(\ell) + \sum_{n=2,4,6...}^{\alpha} \cos\left(\frac{n\pi}{2}\right)\delta f_n(\ell) + \sum_{n=1,3,5...}^{\alpha} \sin\left(\frac{n\pi}{2}\right)\delta g_n(\ell)\right]$$

Note that the load potential for a point load applied at the tip of the cantilever does not contribute any terms to the field equations in Eq. A3.7. The field equations remain unchanged while the load potential term in Eq. 4.12 is inserted into Eq. A3.7.

#### 4.1.3.2 Boundary Conditions

The load potential in Eq. 4.12 is substituted into the boundary load potential terms in Eq. A3.7 to form the boundary conditions of the problem. For mode $n=0$ the boundary conditions become:

$$(4.14a)\qquad \left\{2C_1 a_0'(0) + 2C_3 f_0(0)\right\}\delta a_0(0) = 0$$

$$(4.14b)\qquad \left\{2(C_4 + C_9)c_0'(0)\right\}\delta c_0(0) = 0$$

$$(4.14c)\qquad \left\{-2C_6 f_0'''(0)\right\}\delta f_0(0) = 0$$

$$(4.14d)\qquad \left\{2C_6 f_0''(0)\right\}\delta f_0'(0) = 0$$

$$(4.14e)\qquad \left\{2C_1 a_0'(\ell) + 2C_3 f_0(\ell)\right\}\delta a_0(\ell) = 0$$

$$(4.14f)\qquad \left\{2(C_4 + C_9)c_0'(\ell)\right\}\delta c_0(\ell) = 0$$

$$(4.14g)\qquad \left\{-2C_6 f_0'''(\ell) + P\right\}\delta f_0(\ell) = 0$$

$$(4.14h)\qquad \left\{2C_6 f_0''(\ell)\right\}\delta f_0'(\ell) = 0$$

Similarly, the boundary conditions for modes $n \geq 1$ become:

$$(4.15a)\qquad \left\{C_1 a_n'(0) + nC_3 d_n(0) + C_3 f_n(0)\right\}\delta a_n(0) = 0$$

$$(4.15b)\qquad \left\{C_1 b_n'(0) - nC_3 c_n(0) + C_3 g_n(0)\right\}\delta b_n(0) = 0$$

$$(4.15c)\qquad \left\{C_5 n b_n(0) + (C_4 + C_9)c_n'(0) - C_9 n g_n'(0)\right\}\delta c_n(0) = 0$$

(4.15d) $\left\{-C_5 n a_n(0)+(C_4+C_9)d_n'(0)+C_9 n f_n'(0)\right\}\delta d_n(0)=0$

(4.15e) $\left\{-C_6 f_n'''(0)+n(C_8+C_9)d_n'(0)+n^2(C_8+C_9)f_n'(0)\right\}\delta f_n(0)=0$

(4.15f) $\left\{C_6 f_n''(0)-nC_8 d_n(0)-n^2 C_8 f_n(0)\right\}\delta f_n'(0)=0$

(4.15g) $\left\{-C_6 g_n'''(0)-n(C_8+C_9)c_n'(0)+n^2(C_8+C_9)g_n'(0)\right\}\delta g_n(0)=0$

(4.15h) $\left\{C_6 g_n''(0)+nC_8 c_n(0)-n^2 C_8 g_n(0)\right\}\delta g_n'(0)=0$

(4.15i) $\left\{C_1 a_n'(\ell)+nC_3 d_n(\ell)+C_3 f_n(\ell)\right\}\delta a_n(\ell)=0$

(4.15j) $\left\{C_1 b_n'(\ell)-nC_3 c_n(\ell)+C_3 g_n(\ell)\right\}\delta b_n(\ell)=0$

(4.15k) $\left\{C_5 n b_n(\ell)+(C_4+C_9)c_n'(\ell)-C_9 n g_n'(\ell)\right\}\delta c_n(\ell)=0$

(4.15l) $\left\{-C_5 n a_n(\ell)+(C_4+C_9)d_n'(\ell)+C_9 n f_n'(\ell)\right\}\delta d_n(\ell)=0$

(4.15m.i) $\left\{-C_6 f_n'''(\ell)+n(C_8+C_9)d_n'(\ell)+n^2(C_8+C_9)f_n'(\ell)\right\}\delta f_n(\ell)=0$
for $n = 1, 3, 5, \ldots$

(4.15m.ii) $\left\{-C_6 f_n'''(\ell)+n(C_8+C_9)d_n'(\ell)+n^2(C_8+C_9)f_n'(\ell)+P\cos\left(n\pi/2\right)\right\}\delta f_n(\ell)=0$
for $n = 2, 4, 6, \ldots$

(4.15n) $\left\{C_6 f_n''(\ell)-nC_8 d_n(\ell)-n^2 C_8 f_n(\ell)\right\}\delta f_n'(\ell)=0$

(4.15o.i) $\left\{-C_6 g_n'''(\ell)-n(C_8+C_9)c_n'(\ell)+n^2(C_8+C_9)g_n'(\ell)+P\sin\left(n\pi/2\right)\right\}\delta g_n(\ell)=0$
for $n = 1, 3, 5, \ldots$

(4.15o.ii) $\left\{-C_6 g_n'''(\ell)-n(C_8+C_9)c_n'(\ell)+n^2(C_8+C_9)g_n'(\ell)\right\}\delta g_n(\ell)=0$
for $n = 2, 4, 6, \ldots$

(4.15p) $\left\{C_6 g_n''(\ell)+nC_8 c_n(\ell)-n^2 C_8 g_n(\ell)\right\}\delta g_n'(\ell)=0$

Note that the load potential terms from equation 4.12 are included in Eqs. 4.14g, 4.15m.ii and 4.15o.i. These equations represent the boundary conditions at the free end of the pipe that are affected by the load potential.

#### 4.1.3.3 Determining Required Constants

Numerous constants appear in the field equations and boundary conditions that are dependent on the geometric and material properties of the pipe. The constants, from Eq. 2.15, are evaluated for the current problem:

(4.16a) $$C_1 = \frac{Ert\pi}{1-\mu^2} = 828,550\ kN$$

(4.16b) $$C_2 = \frac{Et\pi}{r\left(1-\mu^2\right)} = 2.0714\times10^7\ kN/m^2$$

(4.16c) $$C_3 = \frac{Et\mu\pi}{1-\mu^2} = 1.2428\times10^6\ kN/m$$

(4.16d) $$C_4 = Grt\pi = 290,280\ kN$$

(4.16e) $$C_5 = Gt\pi = 1.4514\times10^6\ kN/m$$

(4.16f) $$C_6 = \frac{Ert^3\pi}{12\left(1-\mu^2\right)} = 2.4857\ kN\cdot m^2$$

(4.16g) $$C_7 = \frac{Et^3\pi}{12r^3\left(1-\mu^2\right)} = 1553.5\ kN/m^2$$

(4.16h) $$C_8 = \frac{Et^3\mu\pi}{12r\left(1-\mu^2\right)} = 18.642\ kN$$

(4.16i) $$C_9 = \frac{Gt^3\pi}{3r} = 87.185\ kN$$

(4.16j) $$C_{13} = C_4 + C_9 = 290,370\ kN$$

The remaining constants, $C_{10,n}$, $C_{11,n}$, $C_{12,n}$, $C_{14,n}$, $C_{15,n}$, $C_{16,n}$ and $C_{17,n}$ in Eq. 2.15, are dependent on the Fourier mode $n$ under consideration. They will be evaluated as required in subsequent sections of this example problem.

#### 4.1.3.4 Displacement Fields for Fourier Mode $n = 0$

The general form of the solution for mode $n = 0$, from Eqs. 3.4 and 3.10, is:

(4.17a) $$a_0(z) = \overline{A}_{0,1} + \overline{A}_{0,2}z + \sum_{i=3}^{6} \overline{A}_{0,i}e^{m_{0,i}z}$$

(4.17b) $$c_0(z) = \overline{A}_{0,7} + \overline{A}_{0,8}z$$

(4.17c) $$f_0(z) = \overline{F}_{0,1} + \overline{F}_{0,2}z + \sum_{i=3}^{6} \overline{F}_{0,i}e^{m_{0,i}z}$$

The values of $m_{0,i}$ are determined by solving the 4$^{th}$ order polynomial in Eq. 3.8:

(4.18a) $$m_{0,3} = \sqrt[4]{\frac{C_3^{\ 2} - C_1 C_2}{C_1 C_6}} = 37.107 + 37.107i\ m^{-1}$$

(4.18b) $$m_{0,4} = -\sqrt[4]{\frac{C_3^{\ 2} - C_1 C_2}{C_1 C_6}} = -37.107 - 37.107i\ m^{-1}$$

(4.18c) $$m_{0,5} = \sqrt[4]{\frac{C_3^{\ 2} - C_1 C_2}{C_1 C_6}}i = -37.107 + 37.107i\ m^{-1}$$

(4.18d) $$m_{0,6} = -\sqrt[4]{\frac{C_3^{\ 2} - C_1 C_2}{C_1 C_6}}i = 37.107 - 37.107i\ m^{-1}$$

As in Eq. 3.12, the displacements $a_0(z)$, $c_0(z)$ and $f_0(z)$ can be written in terms of the integration constants $\bar{A}_{0,i}$.

(4.19a) $$\begin{Bmatrix} a_0(z) \\ c_0(z) \\ f_0(z) \end{Bmatrix} = \begin{bmatrix} \langle e_0(z) \rangle_{1\times 8}^{\mathrm{T}} \\ \langle \bar{\bar{C}}_0(z) \rangle_{1\times 8}^{\mathrm{T}} \\ \langle \bar{\bar{F}}_0(z) \rangle_{1\times 8}^{\mathrm{T}} \end{bmatrix} \{\bar{A}_0\}_{8\times 1}$$

(4.19b) $$\begin{Bmatrix} a_0(z) \\ c_0(z) \\ f_0(z) \end{Bmatrix} = \begin{bmatrix} 1 & z & e^{m_{0,3}z} & e^{m_{0,4}z} & e^{m_{0,5}z} & e^{m_{0,6}z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \\ 0 & \bar{\bar{F}}_{0,1} & \bar{\bar{F}}_{0,3}e^{m_{0,3}z} & \bar{\bar{F}}_{0,4}e^{m_{0,4}z} & \bar{\bar{F}}_{0,5}e^{m_{0,5}z} & \bar{\bar{F}}_{0,6}e^{m_{0,6}z} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{A}_{0,1} \\ \bar{A}_{0,2} \\ \bar{A}_{0,3} \\ \bar{A}_{0,4} \\ \bar{A}_{0,5} \\ \bar{A}_{0,6} \\ \bar{A}_{0,7} \\ \bar{A}_{0,8} \end{Bmatrix}$$

Integration constants $\bar{F}_{0,i}$ were related to constants $\bar{A}_{0,i}$ through the coefficients $\bar{\bar{F}}_{0,i}$. Recall that the coefficients $\bar{\bar{F}}_{0,i}$ were defined in Eq. 3.13 as:

(4.20a) $$\bar{\bar{F}}_{0,1} = -\frac{C_3}{C_2} = -0.0600\ m$$

(4.20b) $\bar{\bar{F}}_{0,2} = 0$

(4.20c) $\bar{\bar{F}}_{0,3} = -\dfrac{C_1 m_{0,3}}{C_3} = -24.738 - 24.738i$

(4.20d) $\bar{\bar{F}}_{0,4} = -\dfrac{C_1 m_{0,4}}{C_3} = 24.738 + 24.738i$

(4.20e) $\bar{\bar{F}}_{0,5} = -\dfrac{C_1 m_{0,5}}{C_3} = 24.738 - 24.738i$

(4.20f) $\bar{\bar{F}}_{0,6} = -\dfrac{C_1 m_{0,6}}{C_3} = -24.738 + 24.738i$

The integration constants $\bar{A}_{0,i}$ are now determined from the boundary conditions in Eq. 4.14. At the fixed end, all displacements are zero. Hence, the boundary conditions at $z = 0$ are:

(4.21a) $a_0(0) = \langle \psi_{0,1} \rangle^{\mathrm{T}}_{1\times 8} \{\bar{A}_0\}_{8\times 1} = 0$

where $\langle \psi_{0,1} \rangle^{\mathrm{T}}_{1\times 8} = \langle e_0(0) \rangle^{\mathrm{T}}_{1\times 8} = \langle 1 \mid 0 \mid 1 \mid 1 \mid 1 \mid 1 \mid 0 \mid 0 \rangle_{1\times 8}$

(4.21b) $c_0(0) = \langle \psi_{0,2} \rangle^{\mathrm{T}}_{1\times 8} \{\bar{A}_0\}_{8\times 1} = 0$

where $\langle \psi_{0,2} \rangle^{\mathrm{T}}_{1\times 8} = \left\langle \bar{\bar{C}}_0(0) \right\rangle^{\mathrm{T}}_{1\times 8} = \langle 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 1 \mid 0 \rangle_{1\times 8}$

(4.21c) $f_0(0) = \langle \psi_{0,3} \rangle^{\mathrm{T}}_{1\times 8} \{\bar{A}_0\}_{8\times 1} = 0$

where $\langle \psi_{0,3} \rangle^{\mathrm{T}}_{1\times 8} = \left\langle \bar{\bar{F}}_0(0) \right\rangle^{\mathrm{T}}_{1\times 8} = \left\langle 0 \mid \bar{\bar{F}}_{0,1} \mid \bar{\bar{F}}_{0,3} \mid \bar{\bar{F}}_{0,4} \mid \bar{\bar{F}}_{0,5} \mid \bar{\bar{F}}_{0,6} \mid 0 \mid 0 \right\rangle_{1\times 8}$

(4.21d) $f_0'(0) = \langle \psi_{0,4} \rangle^{\mathrm{T}}_{1\times 8} \{\bar{A}_0\}_{8\times 1} = 0$

where

$$\langle \psi_{0,4} \rangle^{\mathrm{T}}_{1\times 8} = \left\langle \bar{\bar{F}}_0'(0) \right\rangle^{\mathrm{T}}_{1\times 8} = \left\langle 0 \mid 0 \mid m_{0,3}\bar{\bar{F}}_{0,3} \mid m_{0,4}\bar{\bar{F}}_{0,4} \mid m_{0,5}\bar{\bar{F}}_{0,5} \mid m_{0,6}\bar{\bar{F}}_{0,6} \mid 0 \mid 0 \right\rangle_{1\times 8}$$

At the free end of the pipe, none of the displacements are specified. Therefore, the natural boundary conditions given in Eq. 4.14 must be applied. From Eq. 4.14e, noting that $\delta a_0(\ell) \neq 0$, the first boundary condition becomes:

(4.22) $$2C_1 a_0'(\ell) + 2C_3 f_0(\ell) = 0$$

Now substitute the vector expressions for $a_0'(\ell)$ and $f_0(\ell)$ from Eq. 4.19 into Eq. 4.22. The resulting boundary equation is written in vector form as:

(4.23a) $$\langle \psi_{0,5} \rangle_{1\times 8}^{\mathrm{T}} \{\bar{A}_0\}_{8\times 1} = 0$$

where

(4.23b) $$\{\psi_{0,5}\}_{8\times 1} = \begin{Bmatrix} 0 \\ 2C_1 + 2C_3 \bar{\bar{F}}_{0,1} \\ \left(2C_1 m_{0,3} + 2C_3 \bar{\bar{F}}_{0,3}\right) e^{m_{0,3}\ell} \\ \left(2C_1 m_{0,4} + 2C_3 \bar{\bar{F}}_{0,4}\right) e^{m_{0,4}\ell} \\ \left(2C_1 m_{0,5} + 2C_3 \bar{\bar{F}}_{0,5}\right) e^{m_{0,5}\ell} \\ \left(2C_1 m_{0,6} + 2C_3 \bar{\bar{F}}_{0,6}\right) e^{m_{0,6}\ell} \\ 0 \\ 0 \end{Bmatrix}_{8\times 1}$$

Similarly, from Eq. 4.14f, with $\delta c_0(\ell) \neq 0$, the boundary equation is:

(4.24a) $$2(C_4 + C_9) c_0'(\ell) = 0$$

(4.24b) $$\langle \psi_{0,6} \rangle_{1\times 8}^{\mathrm{T}} \{\bar{A}_0\}_{8\times 1} = 0$$

where

(4.24c) $$\left\{\psi_{0,6}\right\}_{8\times1}=\left\{\begin{matrix}0\\0\\0\\0\\0\\0\\0\\2\left(C_4+C_9\right)\end{matrix}\right\}_{8\times1}$$

The boundary condition from Eq. 4.14g, with $\delta f_0\left(\ell\right)\neq0$, must include the load potential term from Eq. 4.12:

(4.25a) $$-2C_6 f_0'''\left(\ell\right)+P=0$$

(4.25b) $$\left\langle\psi_{0,7}\right\rangle_{1\times8}^{\mathrm{T}}\left\{\bar{A}_0\right\}_{8\times1}=-P$$

where

(4.25c) $$\left\{\psi_{0,7}\right\}_{8\times1}=\left\{\begin{matrix}0\\0\\-2C_6 m_{0,3}{}^3\bar{\bar{F}}_{0,3}e^{m_{0,3}\ell}\\-2C_6 m_{0,4}{}^3\bar{\bar{F}}_{0,4}e^{m_{0,4}\ell}\\-2C_6 m_{0,5}{}^3\bar{\bar{F}}_{0,5}e^{m_{0,5}\ell}\\-2C_6 m_{0,6}{}^3\bar{\bar{F}}_{0,6}e^{m_{0,6}\ell}\\0\\0\end{matrix}\right\}_{8\times1}$$

The final boundary condition is obtained from Eq. 4.14h, noting that $\delta f_0'\left(\ell\right)\neq0$:

(4.26a) $$2C_6 f_0''\left(\ell\right)=0$$

(4.26b) $$\left\langle\psi_{0,8}\right\rangle_{1\times8}^{\mathrm{T}}\left\{\bar{A}_0\right\}_{8\times1}=0$$

where

$$(4.26c) \qquad \left\{\psi_{0,8}\right\}_{8\times1} = \begin{Bmatrix} 0 \\ 0 \\ 2C_6 m_{0,3}{}^2 \overline{\overline{F}}_{0,3} e^{m_{0,3}\ell} \\ 2C_6 m_{0,4}{}^2 \overline{\overline{F}}_{0,4} e^{m_{0,4}\ell} \\ 2C_6 m_{0,5}{}^2 \overline{\overline{F}}_{0,5} e^{m_{0,5}\ell} \\ 2C_6 m_{0,6}{}^2 \overline{\overline{F}}_{0,6} e^{m_{0,6}\ell} \\ 0 \\ 0 \end{Bmatrix}_{8\times1}$$

The boundary equations developed in Eqs. 4.21 through 4.26 can be written in a consolidated matrix form:

$$(4.27) \qquad \left[\Psi_0\right]_{8\times8} \left\{\overline{A}_0\right\}_{8\times1} = \left\{Q_0\right\}_{8\times1}$$

where

$$\left[\Psi_0\right]_{8\times8} = \begin{bmatrix} \left\langle\psi_{0,1}\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle\psi_{0,2}\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle\psi_{0,3}\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle\psi_{0,4}\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle\psi_{0,5}\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle\psi_{0,6}\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle\psi_{0,7}\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle\psi_{0,8}\right\rangle_{1\times8}^{\mathrm{T}} \end{bmatrix}_{8\times8} \quad \text{and} \quad \left\{Q_0\right\}_{8\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P \\ 0 \end{Bmatrix}$$

The components of the vectors $\left\langle\psi_{0,i}\right\rangle_{1\times8}^{\mathrm{T}}$ are defined in Eqs. 4.21 through 4.26. Hence, the only unknowns in Eq. 4.27 are the elements of the vector $\left\{\overline{A}_0\right\}_{8\times1}$. Solving the equation for these unknowns gives:

$$(4.28) \qquad \{\bar{A}_0\} = \begin{Bmatrix} \bar{A}_{0,1} \\ \bar{A}_{0,2} \\ \bar{A}_{0,3} \\ \bar{A}_{0,4} \\ \bar{A}_{0,5} \\ \bar{A}_{0,6} \\ \bar{A}_{0,7} \\ \bar{A}_{0,8} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -2.2191\times10^{-73} + 1.5586\times10^{-71} i\, m \\ 1.5258\times10^{-23} - 5.2897\times10^{-38} i\, m \\ 1.5258\times10^{-23} + 5.2897\times10^{-38} i\, m \\ -2.2191\times10^{-73} - 1.5586\times10^{-71} i\, m \\ 0 \\ 0 \end{Bmatrix}$$

With vector $\{\bar{A}_0\}_{8\times1}$ now completely defined displacements $a_0(z)$, $c_0(z)$ and $f_0(z)$ at the tip of the pipe, with $z = \ell$, can be calculated from Eqs. 3.12:

$$(4.29a) \qquad a_0(\ell) = \langle e_0(\ell) \rangle_{1\times8} \{\bar{A}_0\}_{8\times1} = 0.000637\, mm$$

$$(4.29b) \qquad c_0(\ell) = \langle \bar{\bar{C}}_0(\ell) \rangle_{1\times8} \{\bar{A}_0\}_{8\times1} = 0\, mm$$

$$(4.29c) \qquad f_0(\ell) = \langle \bar{\bar{F}}_0(\ell) \rangle_{1\times8} \{\bar{A}_0\}_{8\times1} = -0.0315\, mm$$

Note that displacement contributions $u_0(\ell) = a_0(\ell)$, $v_0(\ell) = c_0(\ell)$ and $w_0(\ell) = f_0(\ell)$ are equal at all locations on the pipe mid-surface. The fact that the displacement contributions are independent of the value of $\phi$ is expected as $u_0(z)$, $v_0(z)$ and $w_0(z)$ for mode $n = 0$ represent the axi-symmetric component of the longitudinal, tangential and radial displacements, respectively, of the pipe.

#### 4.1.3.5 Displacement Fields for Fourier Mode $n = 1$

The general form of the solution for mode $n = 1$, from Eqs. 3.26 and 3.35, is:

$$(4.30a) \qquad a_1(z) = \bar{A}_{1,1} + \bar{A}_{1,2} z + \bar{A}_{1,3} z^2 + \bar{A}_{1,4} z^3 + \sum_{i=5}^{8} \bar{A}_{1,i} e^{m_{1,i} z}$$

$$(4.30b) \qquad d_1(z) = \bar{D}_{1,1} + \bar{D}_{1,2} z + \bar{D}_{1,3} z^2 + \bar{D}_{1,4} z^3 + \sum_{i=5}^{8} \bar{D}_{1,i} e^{m_{1,i} z}$$

$$(4.30c) \qquad f_1(z) = \bar{F}_{1,1} + \bar{F}_{1,2} z + \bar{F}_{1,3} z^2 + \bar{F}_{1,4} z^3 + \sum_{i=5}^{8} \bar{F}_{1,i} e^{m_{1,i} z}$$

(4.30d) $$b_1(z)=\bar{B}_{1,1}+\bar{B}_{1,2}z+\bar{B}_{1,3}z^2+\bar{B}_{1,4}z^3+\sum_{i=5}^{8}\bar{B}_{1,i}e^{m_{1,i}z}$$

(4.30e) $$c_1(z)=\bar{C}_{1,1}+\bar{C}_{1,2}z+\bar{C}_{1,3}z^2+\bar{C}_{1,4}z^3+\sum_{i=5}^{8}\bar{C}_{1,i}e^{m_{1,i}z}$$

(4.30f) $$g_1(z)=\bar{G}_{1,1}+\bar{G}_{1,2}z+\bar{G}_{1,3}z^2+\bar{G}_{1,4}z^3+\sum_{i=5}^{8}\bar{G}_{1,i}e^{m_{1,i}z}$$

The values of the constants dependent on $n$ in Eq. 2.15, evaluated at $n=1$, are:

(4.31a) $$C_{10,1}=\frac{C_4}{r^2}=7.2571\times10^6\ kN/m^2$$

(4.31b) $$C_{11,1}=C_3+C_5=2.6942\times10^6\ kN/m$$

(4.31c) $$C_{12,1}=C_2+C_7=2.0715\times10^7\ kN/m^2$$

(4.31d) $$C_{14,1}=C_2+C_7=2.0715\times10^7\ kN/m^2$$

(4.31e) $$C_{15,1}=C_8+C_9=105.73\ kN$$

(4.31f) $$C_{16,1}=C_2+C_7=2.0715\times10^7\ kN/m^2$$

(4.31g) $$C_{17,1}=2C_8+C_9=124.37\ kN$$

The numerical values of the coefficients of the polynomial in Eq. 3.25, from Eq. 3.23, are:

(4.32a) $$\begin{aligned}p_{1,1}&=-C_1C_6C_{13}\\&=-5.9802\times10^{11}\ kN^3\cdot m^2\end{aligned}$$

(4.32b) $$\begin{aligned}p_{2,1}&=C_1\left(C_6C_{12,1}+C_{13}C_{17,1}-C_{15,1}{}^2\right)+C_6\left(C_{10,1}C_{13}-C_{11,1}{}^2\right)\\&=5.9770\times10^{13}\ kN^3\end{aligned}$$

(4.32c) $$\begin{aligned}p_{3,1}&=C_1\left(-C_{12,1}C_{17,1}-C_{13}C_{16,1}+2C_{14,1}C_{15,1}\right)-C_{10,1}\left(C_6C_{12,1}+C_{13}C_{17,1}-C_{15,1}{}^2\right)\\&\quad+C_{11,1}{}^2C_{17,1}-2C_3C_{11,1}C_{15,1}+C_3{}^2C_{13}\\&=-4.5343\times10^{18}\ kN^3/m^2\end{aligned}$$

For $n=1$, the polynomial was shown to have four repeated zero roots and four distinct roots. These are:

(4.33a) $$m_{1,1} = m_{1,2} = m_{1,3} = m_{1,4} = 0$$

The distinct roots are determined from Eq. 3.30:

(4.33b) $$m_{1,5} = \sqrt{\frac{-p_{2,1} + \sqrt{p_{2,1}^2 - 4p_{1,1}p_{3,1}}}{2p_{1,1}}} = 37.440 - 36.767i\ m^{-1}$$

(4.33c) $$m_{1,6} = -\sqrt{\frac{-p_{2,1} + \sqrt{p_{2,1}^2 - 4p_{1,1}p_{3,1}}}{2p_{1,1}}} = 37.440 + 36.767i\ m^{-1}$$

(4.33d) $$m_{1,7} = \sqrt{\frac{-p_{2,1} - \sqrt{p_{2,1}^2 - 4p_{1,1}p_{3,1}}}{2p_{1,1}}} = -37.440 + 36.767i\ m^{-1}$$

(4.33e) $$m_{1,8} = -\sqrt{\frac{-p_{2,1} - \sqrt{p_{2,1}^2 - 4p_{1,1}p_{3,1}}}{2p_{1,1}}} = -37.440 - 36.767i\ m^{-1}$$

As in Eq. 3.36, the displacements $b_1(z)$, $c_1(z)$ and $g_1(z)$ can be written in terms of the integration constants $\bar{G}_{1,i}$ as:

(4.34) $$\begin{Bmatrix} b_1(z) \\ c_1(z) \\ g_1(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{\bar{A}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \\ -\left\langle \bar{\bar{D}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \\ \left\langle e_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \end{bmatrix} \left\{ \bar{G}_1 \right\}_{8\times 1}$$

in which

(4.35a) $$\left\langle \bar{\bar{A}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} = \Big\langle 0 \;\Big|\; \bar{\bar{A}}_{1,1.1} \;\Big|\; \bar{\bar{A}}_{1,2} z \;\Big|\; \left( \bar{\bar{A}}_{1,1.2} + \bar{\bar{A}}_{1,3} z^2 \right) \;\Big|\; \bar{\bar{A}}_{1,5} e^{m_{1,5} z} \;\Big|\; \bar{\bar{A}}_{1,6} e^{m_{1,6} z} \;\Big|\; \bar{\bar{A}}_{1,7} e^{m_{1,7} z} \;\Big|\; \bar{\bar{A}}_{1,8} e^{m_{1,8} z} \Big\rangle_{1\times 8}$$

(4.35b) $$\left\langle \bar{\bar{D}}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} = \left\langle \bar{\bar{D}}_{1,1.1} \;\middle|\; \bar{\bar{D}}_{1,2.1} z \;\middle|\; \left( \bar{\bar{D}}_{1,1.2} + \bar{\bar{D}}_{1,3} z^2 \right) \;\middle|\; \left( \bar{\bar{D}}_{1,2.2} z + \bar{\bar{D}}_{1,4} z^3 \right) \;\middle|\; \bar{\bar{D}}_{1,5} e^{m_{1,5} z} \;\middle|\; \bar{\bar{D}}_{1,6} e^{m_{1,6} z} \;\middle|\; \bar{\bar{D}}_{1,7} e^{m_{1,7} z} \;\middle|\; \bar{\bar{D}}_{1,8} e^{m_{1,8} z} \right\rangle_{1\times 8}$$

(4.35c) $$\left\langle e_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} = \left\langle 1 \quad z \quad z^2 \quad z^3 \quad e^{m_{1,5} z} \quad e^{m_{1,6} z} \quad e^{m_{1,7} z} \quad e^{m_{1,8} z} \right\rangle_{1\times 8}$$

(4.35d) $$\left\langle \bar{G}_1 \right\rangle^{\mathrm{T}}_{1\times 8} = \left\langle \bar{G}_{1,1} \quad \bar{G}_{1,2} \quad \bar{G}_{1,3} \quad \bar{G}_{1,4} \quad \bar{G}_{1,5} \quad \bar{G}_{1,6} \quad \bar{G}_{1,7} \quad \bar{G}_{1,8} \right\rangle_{1\times 8}$$

In Eq. 4.35, recall that the coefficients $\bar{\bar{A}}_{1,i}$ and $\bar{\bar{D}}_{1,i}$, which relate the integration constants $\bar{A}_{1,i}$ and $\bar{D}_{1,i}$ to $\bar{G}_{1,i}$, were defined in Eqs. 3.33 and 3.34 as:

(4.36a) $$\bar{\bar{A}}_{1,1.1} = \frac{C_3}{C_{10,1}} - \frac{C_{11,1} C_{14,1}}{C_{10,1} C_{12,1}} = -0.200\, m$$

(4.36b) $$\bar{\bar{A}}_{1,1.2} = \frac{6C_1}{C_{10,1}^2} \left[ -\frac{C_{11,1} C_{14,1}}{C_{12,1}} + C_3 \right] + \frac{6C_{11,1}}{C_{10,1} C_{12,1}} \left[ \frac{C_{11,1}}{C_{10,1}} \left( \frac{C_{11,1} C_{14,1}}{C_{12,1}} - C_3 \right) - \frac{C_{13} C_{14,1}}{C_{12,1}} + C_{15,1} \right]$$
$$= -0.1100\, m^3$$

(4.36c) $$\bar{\bar{A}}_{1,2} = \frac{2}{C_{10,1}} \left( -\frac{C_{11,1} C_{14,1}}{C_{12,1}} + C_3 \right) = -0.400\, m$$

(4.36d) $$\bar{\bar{A}}_{1,3} = \frac{3}{C_{10,1}} \left( -\frac{C_{11,1} C_{14,1}}{C_{12,1}} + C_3 \right) = -0.600\, m$$

(4.36e) $$\bar{\bar{A}}_{1,5} = \frac{ -\dfrac{(C_{11,1} m_{1,5})}{(C_1 m_{1,5}^2 - C_{10,1})} \left[ \dfrac{(C_{11,1} m_{1,5})(C_3 m_{1,5})}{(C_1 m_{1,5}^2 - C_{10,1})} - (C_{14,1} - C_{15,1} m_{1,5}^2) \right] }{ (C_{12,1} - C_{13} m_{1,5}^2) - \dfrac{(C_{11,1} m_{1,5})^2}{(C_1 m_{1,5}^2 - C_{10,1})} } - \frac{(C_3 m_{1,5})}{(C_1 m_{1,5}^2 - C_{10,1})}$$
$$= -0.019412 - 0.021001 i$$

(4.36f) $$\bar{\bar{A}}_{1,6}=\frac{-\dfrac{\left(C_{11,1}m_{1,6}\right)}{\left(C_1m_{1,6}{}^2-C_{10,1}\right)}\left[\dfrac{\left(C_{11,1}m_{1,6}\right)\left(C_3m_{1,6}\right)}{\left(C_1m_{1,6}{}^2-C_{10,1}\right)}-\left(C_{14,1}-C_{15,1}m_{1,6}{}^2\right)\right]}{\left(C_{12,1}-C_{13}m_{1,6}{}^2\right)-\dfrac{\left(C_{11,1}m_{1,6}\right)^2}{\left(C_1m_{1,6}{}^2-C_{10,1}\right)}}-\frac{\left(C_3m_{1,6}\right)}{\left(C_1m_{1,6}{}^2-C_{10,1}\right)}$$

$$=-0.019412+0.021001i$$

(4.36g) $$\bar{\bar{A}}_{1,7}=\frac{-\dfrac{\left(C_{11,1}m_{1,7}\right)}{\left(C_1m_{1,7}{}^2-C_{10,1}\right)}\left[\dfrac{\left(C_{11,1}m_{1,7}\right)\left(C_3m_{1,7}\right)}{\left(C_1m_{1,7}{}^2-C_{10,1}\right)}-\left(C_{14,1}-C_{15,1}m_{1,7}{}^2\right)\right]}{\left(C_{12,1}-C_{13}m_{1,7}{}^2\right)-\dfrac{\left(C_{11,1}m_{1,7}\right)^2}{\left(C_1m_{1,7}{}^2-C_{10,1}\right)}}-\frac{\left(C_3m_{1,7}\right)}{\left(C_1m_{1,7}{}^2-C_{10,1}\right)}$$

$$=0.019412+0.021001i$$

(4.36h) $$\bar{\bar{A}}_{1,8}=\frac{-\dfrac{\left(C_{11,1}m_{1,8}\right)}{\left(C_1m_{1,8}{}^2-C_{10,1}\right)}\left[\dfrac{\left(C_{11,1}m_{1,8}\right)\left(C_3m_{1,8}\right)}{\left(C_1m_{1,8}{}^2-C_{10,1}\right)}-\left(C_{14,1}-C_{15,1}m_{1,8}{}^2\right)\right]}{\left(C_{12,1}-C_{13}m_{1,8}{}^2\right)-\dfrac{\left(C_{11,1}m_{1,8}\right)^2}{\left(C_1m_{1,8}{}^2-C_{10,1}\right)}}-\frac{\left(C_3m_{1,8}\right)}{\left(C_1m_{1,8}{}^2-C_{10,1}\right)}$$

$$=0.019412-0.021001i$$

(4.36i) $$\bar{\bar{D}}_{1,1.1}=-\frac{C_{14,1}}{C_{12,1}}=-1$$

(4.36j) $$\bar{\bar{D}}_{1,1.2}=\frac{2}{C_{12,1}}\left[\frac{C_{11,1}}{C_{10,1}}\left(\frac{C_{11,1}C_{14,1}}{C_{12,1}}-C_3\right)-\frac{C_{13}C_{14,1}}{C_{12,1}}+C_{15,1}\right]=0.024\ m^2$$

(4.36k) $$\bar{\bar{D}}_{1,2.1}=-\frac{C_{14,1}}{C_{12,1}}=-1$$

(4.36l) $$\bar{\bar{D}}_{1,2.2}=\frac{6}{C_{12,1}}\left[\frac{C_{11,1}}{C_{10,1}}\left(\frac{C_{11,1}C_{14,1}}{C_{12,1}}-C_3\right)-\frac{C_{13}C_{14,1}}{C_{12,1}}+C_{15,1}\right]=0.072\ m^2$$

(4.36m) $$\bar{\bar{D}}_{1,3}=-\frac{C_{14,1}}{C_{12,1}}=-1$$

(4.36n) $$\bar{\bar{D}}_{1,4}=-\frac{C_{14,1}}{C_{12,1}}=-1$$

(4.36o) $$\bar{\bar{D}}_{1,5}=\frac{\dfrac{(C_{11,1}m_{1,5})(C_3m_{1,5})}{(C_1m_{1,5}{}^2-C_{10,1})}-(C_{14,1}-C_{15,1}m_{1,5}{}^2)}{(C_{12,1}-C_{13}m_{1,5}{}^2)-\dfrac{(C_{11,1}m_{1,5})^2}{(C_1m_{1,5}{}^2-C_{10,1})}}=-0.00028135+0.020852i$$

(4.36p) $$\bar{\bar{D}}_{1,6}=\frac{\dfrac{(C_{11,1}m_{1,6})(C_3m_{1,6})}{(C_1m_{1,6}{}^2-C_{10,1})}-(C_{14,1}-C_{15,1}m_{1,6}{}^2)}{(C_{12,1}-C_{13}m_{1,6}{}^2)-\dfrac{(C_{11,1}m_{1,6})^2}{(C_1m_{1,6}{}^2-C_{10,1})}}=-0.00028135-0.020852i$$

(4.36q) $$\bar{\bar{D}}_{1,7}=\frac{\dfrac{(C_{11,1}m_{1,7})(C_3m_{1,7})}{(C_1m_{1,7}{}^2-C_{10,1})}-(C_{14,1}-C_{15,1}m_{1,7}{}^2)}{(C_{12,1}-C_{13}m_{1,7}{}^2)-\dfrac{(C_{11,1}m_{1,7})^2}{(C_1m_{1,7}{}^2-C_{10,1})}}=0.00028135-0.020852i$$

(4.36r) $$\bar{\bar{D}}_{1,8}=\frac{\dfrac{(C_{11,1}m_{1,8})(C_3m_{1,8})}{(C_1m_{1,8}{}^2-C_{10,1})}-(C_{14,1}-C_{15,1}m_{1,8}{}^2)}{(C_{12,1}-C_{13}m_{1,8}{}^2)-\dfrac{(C_{11,1}m_{1,8})^2}{(C_1m_{1,8}{}^2-C_{10,1})}}=0.00028135+0.020852i$$

The integration constants, $\{\bar{G}_1\}_{8\times1}$, can be now determined from the boundary conditions in Eq. 4.15. At the fixed end, all displacements are zero. Hence, the boundary conditions at $z=0$ are:

(4.37a) $$b_1(0)=\langle\psi_{1,1}\rangle_{1\times8}^{\mathrm{T}}\{\bar{G}_1\}_{8\times1}=0$$

where $\langle\psi_{1,1}\rangle_{1\times8}^{\mathrm{T}}=\langle\bar{\bar{A}}_1(0)\rangle_{1\times8}^{\mathrm{T}}=\langle 0 \mid \bar{\bar{A}}_{1,1.1} \mid 0 \mid \bar{\bar{A}}_{1,1.2} \mid \bar{\bar{A}}_{1,5} \mid \bar{\bar{A}}_{1,6} \mid \bar{\bar{A}}_{1,7} \mid \bar{\bar{A}}_{1,8}\rangle_{1\times8}$

(4.37b) $$c_1(0)=\langle\psi_{1,2}\rangle_{1\times8}^{\mathrm{T}}\{\bar{G}_1\}_{8\times1}=0$$

where

$$\langle\psi_{1,2}\rangle_{1\times8}^{\mathrm{T}}=-\langle\bar{\bar{D}}_1(0)\rangle_{1\times8}^{\mathrm{T}}=\langle -\bar{\bar{D}}_{1,1.1} \mid 0 \mid -\bar{\bar{D}}_{1,1.2} \mid 0 \mid -\bar{\bar{D}}_{1,5} \mid -\bar{\bar{D}}_{1,6} \mid -\bar{\bar{D}}_{1,7} \mid -\bar{\bar{D}}_{1,8}\rangle_{1\times8}$$

(4.37c) $$g_1(0) = \langle \psi_{1,3} \rangle^{\mathrm{T}}_{1\times 8} \{\bar{G}_1\}_{8\times 1} = 0$$

where $\langle \psi_{1,3} \rangle = \langle e_1(0) \rangle^{\mathrm{T}}_{1\times 8} = \langle 1 \mid 0 \mid 0 \mid 0 \mid 1 \mid 1 \mid 1 \mid 1 \rangle_{1\times 8}$

(4.37d) $$g_1'(0) = \langle \psi_{1,4} \rangle^{\mathrm{T}}_{1\times 8} \{\bar{G}_1\}_{8\times 1} = 0$$

where $\langle \psi_{1,4} \rangle^{\mathrm{T}}_{1\times 8} = \left\langle e_1'(0) \right\rangle^{\mathrm{T}}_{1\times 8} = \langle 0 \mid 1 \mid 0 \mid 0 \mid m_{1,5} \mid m_{1,6} \mid m_{1,7} \mid m_{1,8} \rangle_{1\times 8}$

At the free end of the pipe, none of the displacements are specified. Therefore, the natural boundary conditions given in Eq. 4.15 must be applied. From Eq. 4.15j, noting that $\delta b_1(\ell) \neq 0$, the first boundary condition becomes:

(4.38) $$C_1 b_n'(\ell) + C_3 g_n(\ell) - C_3 c_n(\ell) = 0$$

Now substitute the vector expressions for $b_1'(\ell), c_1(\ell)$ and $g_1(\ell)$ from Eq. 4.34 into Eq. 4.38. The resulting boundary equation is written in vector form as:

(4.39a) $$\langle \psi_{1,5} \rangle^{\mathrm{T}}_{1\times 8} \{\bar{G}_1\}_{8\times 1} = 0$$

where

(4.39b) $$\{\psi_{1,5}\}_{8\times 1} = \left\{ \begin{array}{c} C_3\left(1+\bar{\bar{D}}_{1,1.1}\right) \\ C_3\ell\left(1+\bar{\bar{D}}_{1,2.1}\right) \\ C_1\bar{\bar{A}}_{1,2} + C_3\left(\ell^2 + \bar{\bar{D}}_{1,1.2} + \bar{\bar{D}}_{1,3}\ell^2\right) \\ 2C_1\bar{\bar{A}}_{1,3}\ell + C_3\ell\left(\ell^2 + \bar{\bar{D}}_{1,2.2} + \bar{\bar{D}}_{1,4}\ell^2\right) \\ \left[C_1 m_{1,5}\bar{\bar{A}}_{1,5} + C_3\left(1+\bar{\bar{D}}_{1,5}\right)\right]e^{m_{1,5}\ell} \\ \left[C_1 m_{1,6}\bar{\bar{A}}_{1,6} + C_3\left(1+\bar{\bar{D}}_{1,6}\right)\right]e^{m_{1,6}\ell} \\ \left[C_1 m_{1,7}\bar{\bar{A}}_{1,7} + C_3\left(1+\bar{\bar{D}}_{1,7}\right)\right]e^{m_{1,7}\ell} \\ \left[C_1 m_{1,8}\bar{\bar{A}}_{1,8} + C_3\left(1+\bar{\bar{D}}_{1,8}\right)\right]e^{m_{1,8}\ell} \end{array} \right\}_{8\times 1}$$

Similarly, from Eq. 4.15k, with $\delta c_1(\ell) \neq 0$, the boundary equation is:

(4.40a) $$C_5 b_1(\ell) + (C_4 + C_9) c_1'(\ell) - C_9 g_1'(\ell) = 0$$

(4.40b) $$\langle \psi_{1,6} \rangle_{1\times8}^{\mathrm{T}} \{\bar{G}_1\}_{8\times1} = 0$$

where

(4.40c) $$\{\psi_{1,6}\}_{8\times1} = \begin{Bmatrix} 0 \\ C_5 \bar{\bar{A}}_{1,1.1} - (C_4 + C_9)\bar{\bar{D}}_{1,2.1} - C_9 \\ C_5 \bar{\bar{A}}_{1,2}\ell - 2(C_4 + C_9)\bar{\bar{D}}_{1,3}\ell - 2C_9\ell \\ C_5\left(\bar{\bar{A}}_{1,1.2} + \bar{\bar{A}}_{1,3}\ell^2\right) - (C_4 + C_9)\left(\bar{\bar{D}}_{1,2.2} + 3\bar{\bar{D}}_{1,4}\ell^2\right) - 3C_9\ell^2 \\ \left[C_5 \bar{\bar{A}}_{1,5} - (C_4 + C_9) m_{1,5} \bar{\bar{D}}_{1,5} - C_9 m_{1,5}\right] e^{m_{1,5}\ell} \\ \left[C_5 \bar{\bar{A}}_{1,6} - (C_4 + C_9) m_{1,6} \bar{\bar{D}}_{1,6} - C_9 m_{1,6}\right] e^{m_{1,6}\ell} \\ \left[C_5 \bar{\bar{A}}_{1,7} - (C_4 + C_9) m_{1,7} \bar{\bar{D}}_{1,7} - C_9 m_{1,7}\right] e^{m_{1,7}\ell} \\ \left[C_5 \bar{\bar{A}}_{1,8} - (C_4 + C_9) m_{1,8} \bar{\bar{D}}_{1,8} - C_9 m_{1,8}\right] e^{m_{1,8}\ell} \end{Bmatrix}_{8\times1}$$

The boundary condition from Eq. 4.15o, with $\delta g_1(\ell) \neq 0$, must include the load potential term since $n = 1$. Therefore, use Eq. 4.15o.i:

(4.41a) $$-C_6 g_1'''(\ell) - (C_8 + C_9) c_1'(\ell) + (C_8 + C_9) g_1'(\ell) + P\sin\left(\pi/2\right) = 0$$

(4.41b) $$\langle \psi_{1,7} \rangle_{1\times8}^{\mathrm{T}} \{\bar{G}_1\}_{8\times1} = -P$$

where

$$(4.41c)\qquad \{\psi_{1,7}\}_{8\times1} = \begin{Bmatrix} 0 \\ (C_8+C_9)\left(\bar{\bar{D}}_{1,2.1}+1\right) \\ 2(C_8+C_9)\left(\bar{\bar{D}}_{1,3}+1\right)\ell \\ -6C_6+(C_8+C_9)\left(3\bar{\bar{D}}_{1,4}\ell^2+\bar{\bar{D}}_{1,2.2}+3\ell^2\right) \\ \left[-C_6 m_{1,5}{}^3+(C_8+C_9)m_{1,5}\left(\bar{\bar{D}}_{1,5}+1\right)\right]e^{m_{1,5}\ell} \\ \left[-C_6 m_{1,6}{}^3+(C_8+C_9)m_{1,6}\left(\bar{\bar{D}}_{1,6}+1\right)\right]e^{m_{1,6}\ell} \\ \left[-C_6 m_{1,7}{}^3+(C_8+C_9)m_{1,7}\left(\bar{\bar{D}}_{1,7}+1\right)\right]e^{m_{1,7}\ell} \\ \left[-C_6 m_{1,8}{}^3+(C_8+C_9)m_{1,8}\left(\bar{\bar{D}}_{1,8}+1\right)\right]e^{m_{1,8}\ell} \end{Bmatrix}_{8\times1}$$

The final boundary condition is obtained from Eq. 4.15p, noting that $\delta g_1'(\ell)\neq 0$:

$$(4.42a)\qquad C_6 g_1''(\ell)+C_8 c_1(\ell)-C_8 g_1(\ell)=0$$

$$(4.42b)\qquad \langle\psi_{1,8}\rangle_{1\times8}^{\mathrm{T}}\{\bar{G}_1\}_{8\times1}=0$$

where

$$(4.42c)\qquad \{\psi_{1,8}\}_{8\times1} = \begin{Bmatrix} -C_8\left(\bar{\bar{D}}_{1,1.1}+1\right) \\ -C_8\left(\bar{\bar{D}}_{1,2.1}+1\right)\ell \\ 2C_6-C_8\left(\bar{\bar{D}}_{1,1.2}+\bar{\bar{D}}_{1,3}\ell^2+\ell^2\right) \\ 6C_6\ell-C_8\left(\bar{\bar{D}}_{1,2.2}+\bar{\bar{D}}_{1,4}\ell^2+\ell^2\right)\ell \\ \left[C_6 m_{1,5}{}^2-C_8\left(\bar{\bar{D}}_{1,5}+1\right)\right]e^{m_{1,5}\ell} \\ \left[C_6 m_{1,6}{}^2-C_8\left(\bar{\bar{D}}_{1,6}+1\right)\right]e^{m_{1,6}\ell} \\ \left[C_6 m_{1,7}{}^2-C_8\left(\bar{\bar{D}}_{1,7}+1\right)\right]e^{m_{1,7}\ell} \\ \left[C_6 m_{1,8}{}^2-C_8\left(\bar{\bar{D}}_{1,8}+1\right)\right]e^{m_{1,8}\ell} \end{Bmatrix}_{8\times1}$$

The boundary equations developed in Eqs. 4.37 through 4.42 can be written in a consolidated matrix form:

$$[\Psi_{1,1}]_{8\times8}\{\bar{G}_1\}_{8\times1}=\{Q_{1,1}\}_{8\times1} \tag{4.43}$$

where

$$[\Psi_{1,1}]_{8\times8}=\begin{bmatrix}\langle\psi_{1,1}\rangle^{\mathrm{T}}_{1\times8}\\ \langle\psi_{1,2}\rangle^{\mathrm{T}}_{1\times8}\\ \langle\psi_{1,3}\rangle^{\mathrm{T}}_{1\times8}\\ \langle\psi_{1,4}\rangle^{\mathrm{T}}_{1\times8}\\ \langle\psi_{1,5}\rangle^{\mathrm{T}}_{1\times8}\\ \langle\psi_{1,6}\rangle^{\mathrm{T}}_{1\times8}\\ \langle\psi_{1,7}\rangle^{\mathrm{T}}_{1\times8}\\ \langle\psi_{1,8}\rangle^{\mathrm{T}}_{1\times8}\end{bmatrix}_{8\times8} \quad \text{and} \quad \{Q_{1,1}\}_{8\times1}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\\-P\\0\end{Bmatrix}$$

The components of the vectors $\langle\psi_{1,i}\rangle^{\mathrm{T}}_{1\times8}$ are defined in Eqs. 4.37 through 4.42. Hence, the only unknowns in Eq. 4.43 are the elements of the vector $\{\bar{G}_1\}_{8\times1}$. Solving the equation for these unknowns gives:

$$\{\bar{G}_1\}_{8\times1}=\begin{Bmatrix}\bar{G}_{1,1}\\ \bar{G}_{1,2}\\ \bar{G}_{1,3}\\ \bar{G}_{1,4}\\ \bar{G}_{1,5}\\ \bar{G}_{1,6}\\ \bar{G}_{1,7}\\ \bar{G}_{1,8}\end{Bmatrix}_{8\times1}=\begin{Bmatrix}-2.4916\times10^{-5}\,m\\ -4.3542\times10^{-5}\\ -0.0010610\,m^{-1}\\ 8.8413\times10^{-5}\,m^{-2}\\ 2.3549\times10^{-70}-1.6208\times10^{-70}i\,m\\ 2.3549\times10^{-70}+1.6208\times10^{-70}i\,m\\ 1.2458\times10^{-5}-1.3278\times10^{-5}i\,m\\ 1.2458\times10^{-5}+1.3278\times10^{-5}i\,m\end{Bmatrix}_{8\times1} \tag{4.44}$$

With the vector $\{\bar{G}_1\}_{8\times1}$ now completely defined the displacements $b_1(\ell)$, $c_1(\ell)$ and $g_1(\ell)$ at the tip of the pipe can be calculated from Eq. 4.34:

(4.45a) $$b_1(\ell)=\left\langle \bar{\bar{A}}_1(\ell)\right\rangle_{1\times8}\{\bar{G}_1\}_{8\times1}=0.849\,mm$$

(4.45b) $$c_1(\ell)=-\left\langle \bar{\bar{D}}_1(\ell)\right\rangle_{1\times8}\{\bar{G}_1\}_{8\times1}=-11.516\,mm$$

(4.45c) $$g_1(\ell)=\left\langle \bar{e}_1(\ell)\right\rangle_{1\times8}\{\bar{G}_1\}_{8\times1}=-11.579\,mm$$

*4.1.3.5.1 Determining Functions $a_1(z)$, $d_1(z)$ and $f_1(z)$*

As shown previously, functions $a_1(z)$, $d_1(z)$ and $f_1(z)$ are inter-related but uncoupled from functions $b_1(z)$, $c_1(z)$ and $g_1(z)$. For mode $n$ = 1 the variation of the load potential term contains only $\delta g_1(\ell)$. As such, the boundary conditions in Eq. 4.15 relating to functions $a_1(z)$, $d_1(z)$ and $f_1(z)$ become:

(4.46a) $$a_1(0)=0$$

(4.46b) $$d_1(0)=0$$

(4.46c) $$f_1(0)=0$$

(4.46d) $$f_1'(0)=0$$

(4.46e) $$\delta a_1(\ell)\neq0\rightarrow C_1a_1'(\ell)+C_3d_1(\ell)+C_3f_1(\ell)=0$$

(4.46f) $$\delta d_1(\ell)\neq0\rightarrow -C_5a_1(\ell)+(C_4+C_9)d_1'(\ell)+C_9f_1'(\ell)=0$$

(4.46g) $$\delta f_1(\ell)\neq0\rightarrow -C_6f_1'''(\ell)+(C_8+C_9)d_1'(\ell)+(C_8+C_9)f_1'(\ell)=0$$

(4.46h) $$\delta f_1'(\ell)\neq0\rightarrow C_6f_1''(\ell)-C_8d_1(\ell)-C_8f_1(\ell)=0$$

The only solution to the system of equations in Eq. 4.46 is the trivial one of $a_1(z)=d_1(z)=f_1(z)=0$. In contrast, functions $b_1(z)$, $c_1(z)$ and $g_1(z)$ are non-zero.

This condition can also be realised by considering Eq. 4.43. If Eq. 4.43 had been developed with the boundary conditions in corresponding to displacements $a_1(z)$, $d_1(z)$ and $f_1(z)$, it would have the following form:

$$[\Psi_{1,2}]_{8\times8}\{\bar{F}_1\}_{8\times1} = \{Q_{1,2}\}_{8\times1} \tag{4.47}$$

In Eq. 4.47, matrix $[\Psi_{1,2}]_{8\times8}$ is similar to matrix $[\Psi_{1,1}]_{8\times8}$ defined in Eq. 4.43. The only difference between the matrices is that the sign on the coefficients $\bar{\bar{D}}_{1,i}$ will be opposite. However, vector $\{Q_{1,2}\}_{8\times1} = \{\mathbf{0}\}_{8\times1}$ and is different from vector $\{Q_{1,1}\}_{8\times1} \neq \{\mathbf{0}\}_{8\times1}$ in Eq. 4.43. Thus, vector $\{\bar{F}_1\}_{8\times1}$ must vanish for the relation $[\Psi_{1,2}]_{8\times8}\{\bar{F}_1\}_{8\times1} = \{\mathbf{0}\}_{8\times1}$ to be realised in view of the fact that matrix $[\Psi_{1,2}]_{8\times8}$ is non-singular. Given that $\{\bar{F}_1\}_{8\times1} = \{\mathbf{0}\}_{8\times1}$, it follows that the displacements $a_1(z)$, $d_1(z)$ and $f_1(z)$ are all zero.

#### *4.1.3.5.2 Total Displacement Contributions for Fourier Mode $n=1$*

Given that the displacements $a_1(z)$, $d_1(z)$ and $f_1(z)$ are all zero, the displacement contribution of any point on the pipe mid-surface at the free end, $z=\ell$, from mode $n=1$ is found by substituting Eq. 4.45 into Eq. 2.11:

$$u_1(\ell,\phi) = b_1(\ell)\sin\phi = (0.849mm)\sin\phi \tag{4.48a}$$

$$v_1(\ell,\phi) = c_1(\ell)\cos\phi = (-11.516mm)\cos\phi \tag{4.48b}$$

$$w_1(\ell,\phi) = g_1(\ell)\sin\phi = (-11.579mm)\sin\phi \tag{4.48c}$$

The first mode ($n=1$) contributions to the vertical and longitudinal displacement at Point *A*, *B* and *C* on Figure 4.1 are found by evaluating Eq. 4.48 at the values of $\phi=\pi/2$, $\phi=\pi$ and $\phi=3\pi/2$, respectively. Eq. 4.48a gives the longitudinal displacement at all three points while Eq. 4.48b gives the vertical displacement at Point *B* and Eq. 4.48c gives the vertical displacement at Points *A* and *C*. The displacements are summarized in Table 4.2 below.

**Table 4.2** – Displacement contributions of Fourier mode $n=1$ for Points *A*, *B* and *C*

| Location | Vertical Displacement $\Delta_{Vert,n=1}$ | Longitudinal Displacement $\Delta_{Long,n=1}$ |
|---|---|---|
| Point *A* | 11.579 *mm* | 0.849 *mm* |
| Point *B* | 11.516 *mm* | 0 *mm* |
| Point *C* | 11.579 *mm* | -0.849 *mm* |

In Table 4.2, downward vertical displacements are assumed positive. Also, longitudinal displacements are assumed positive when oriented in the positive *z*-direction. Hence, a positive longitudinal displacement represents elongation of the pipe fibres (as is the case for the longitudinal fibre passing through Point *A*) while a negative longitudinal displacement represents contraction of the pipe fibres (such as is the case of a longitudinal fibre passing through Point *C*).

The vertical displacement of Point *B* ascertained from mode $n=1$ is comparable to that obtained from the Euler-Bernoulli beam theory. The Euler-Bernoulli theory gives a displacement of 11.315 *mm* while the displacement predicted by the methodology in the current study is 11.516 *mm*. As expected, the Euler-Bernoulli beam theory provides a "stiffer" model, resulting in a lower vertical displacement.

The Euler-Bernoulli beam describes the displacement of the pipe-centreline at the free end of the pipe $(z=\ell)$ as a 3rd order polynomial function with respect to the pipe length, $\ell$. The method in the current study, for mode $n=1$, uses a 3rd order polynomial plus four exponential terms with respect to $\ell$, as seen in Eq. 4.35 evaluated at $z=\ell$. If the exponential terms are ignored, then the resulting 3rd order polynomial can be more readily compared to the Euler-Bernoulli solution. Dropping the exponential terms from Eq. 4.35c gives the vertical displacement of Point *B* as $\Delta_B = g(\ell) = \bar{G}_{1,1} + \bar{G}_{1,2}\ell + \bar{G}_{1,3}\ell^2 + \bar{G}_{1,4}\ell^3$, in which the values of $\bar{G}_{1,i}$ are the integration constants in Eq. 4.44. The resulting vertical displacement of all three points *A*, *B* and *C* is 11.516 *mm*, which is nearly identical to the value predicted by the Euler-Bernoulli beam theory.

A similar comparison can be made with the longitudinal displacement at Point *A*. The displacements predicted by both the Euler-Bernoulli beam theory and current method for mode $n=1$ are nearly identical at 0.849 *mm*.

It is of interest to note that the contribution of the exponential terms becomes more significant as the pipe span, $\ell$, reduces. In this case, the present solution yields a significantly more flexible representation of the stiffness and the predicted displacements are larger than those based on the Euler-Bernoulli beam.

### 4.1.3.6 Displacement Fields for Fourier Mode $n$ = 2

The general form of the solution for $n=2$, from Eq. 3.24, is:

(4.49a) $$a_2(z)=\sum_{i=1}^{8}\bar{A}_{2,i}e^{m_{2,i}z}$$

(4.49b) $$b_2(z)=\sum_{i=1}^{8}\bar{B}_{2,i}e^{m_{2,i}z}$$

(4.49c) $$c_2(z)=\sum_{i=1}^{8}\bar{C}_{2,i}e^{m_{2,i}z}$$

(4.49d) $$d_2(z)=\sum_{i=1}^{8}\bar{D}_{2,i}e^{m_{2,i}z}$$

(4.49e) $$f_2(z)=\sum_{i=1}^{8}\bar{F}_{2,i}e^{m_{2,i}z}$$

(4.49f) $$g_2(z)=\sum_{i=1}^{8}\bar{G}_{2,i}e^{m_{2,i}z}$$

Similar to mode $n=1$, functions $a_2(z)$, $d_2(z)$ and $f_2(z)$ are inter-related but are uncoupled from functions $b_2(z)$, $c_2(z)$ and $g_2(z)$. For mode $n$ = 2 the variation of the load potential term contains only $\delta f_2(\ell)$. Therefore, functions $b_2(z)$, $c_2(z)$ and $g_2(z)$ will all vanish while the functions $a_2(z)$, $d_2(z)$ and $f_2(z)$ will be non-zero. The three non-zero functions are determined below.

The values of the constants dependent on $n$ in Eq. 2.15, evaluated at $n=2$, are:

(4.50a) $$C_{10,2}=\frac{4C_4}{r^2}=2.9028\times10^7\ kN/m^2$$

(4.50b) $$C_{11,2}=2(C_3+C_5)=5.3885\times10^6\ kN/m$$

(4.50c) $$C_{12,2}=4(C_2+C_7)=8.2861\times10^7\ kN/m^2$$

(4.50d) $$C_{14,2}=2C_2+8C_7=4.1440\times10^7\ kN/m^2$$

(4.50e) $$C_{15,2}=2(C_8+C_9)=211.45kN$$

(4.50f) $$C_{16,2}=2C_2+16C_7=2.0739\times10^7\ kN/m^2$$

(4.50g) $$C_{17,2}=4(2C_8+C_9)=497.48\ kN$$

The numerical values of the polynomial in Eq. 3.22, from Eq. 3.23, are:

(4.51a)
$$\begin{aligned} p_{1,2} &= -C_1 C_6 C_{13} \\ &= -5.9802 \times 10^{11}\ kN^3 \cdot m^2 \end{aligned}$$

(4.51b)
$$\begin{aligned} p_{2,2} &= C_1\left(C_6 C_{12,2} + C_{13} C_{17,2} - C_{15,2}{}^2\right) + C_6\left(C_{10,2} C_{13} - C_{11,2}{}^2\right) \\ &= 2.3908 \times 10^{14}\ kN^3 \end{aligned}$$

(4.51c)
$$\begin{aligned} p_{3,2} &= C_1\left(-C_{12,2} C_{17,2} - C_{13} C_{16,2} + 2C_{14,2} C_{15,2}\right) - C_{10,2}\left(C_6 C_{12,2} + C_{13} C_{17,2} - C_{15,2}{}^2\right) \\ &\quad + C_{11,2}{}^2 C_{17,2} - 2C_3 C_{11,2} C_{15,2} + C_3{}^2 C_{13} \\ &= -4.5591 \times 10^{18}\ kN^3/m^2 \end{aligned}$$

(4.51d)
$$\begin{aligned} p_{4,2} &= C_1\left(C_{12,2} C_{16,2} - C_{14,2}{}^2\right) + C_{10,2}\left(C_{12,2} C_{17,2} + C_{13} C_{16,2} - 2C_{14,2} C_{15,2}\right) + 2C_3 C_{11,2} C_{14,2} \\ &\quad - C_{11,2}{}^2 C_{16,2} - C_3{}^2 C_{12,2} \\ &= 1.3447 \times 10^{18}\ kN/m^4 \end{aligned}$$

(4.51e)
$$\begin{aligned} p_{5,2} &= -C_{10,2}\left(C_{12,2} C_{16,2} - C_{14,2}{}^2\right) \\ &= -3.3628 \times 10^{19}\ kN/m^6 \end{aligned}$$

Solving Eq. 3.22 using Ferrari's Method (see Appendix 6) gives the eight roots of the polynomial:

(4.52a) $m_{2,1} = 38.476 + 35.787i\ m^{-1}$ (4.52e) $m_{2,5} = 1.1965 + 1.1333i\ m^{-1}$

(4.52b) $m_{2,2} = -38.476 - 35.787i\ m^{-1}$ (4.52f) $m_{2,6} = -1.1965 - 1.1333i\ m^{-1}$

(4.52c) $m_{2,3} = 1.1965 - 1.1333i\ m^{-1}$ (4.52g) $m_{2,7} = 38.476 - 35.787i\ m^{-1}$

(4.52d) $m_{2,4} = -1.1965 + 1.1333i\ m^{-1}$ (4.52h) $m_{2,8} = -38.476 + 35.787i\ m^{-1}$

As in Eq. 3.37, the displacements $a_2(z)$, $d_2(z)$ and $f_2(z)$ can be written in terms of integration constants $\overline{F}_{2,i}$:

$$\text{(4.53)} \qquad \begin{Bmatrix} a_2(z) \\ d_2(z) \\ f_2(z) \end{Bmatrix}_{3\times1} = \begin{bmatrix} \left\langle \bar{\bar{A}}_2(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle \bar{\bar{D}}_2(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle e_2(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix}_{3\times8} \left\{ \bar{F}_2 \right\}_{8\times1}$$

in which

$$\text{(4.54a)} \qquad \left\langle \bar{\bar{A}}_2(z) \right\rangle^{\mathrm{T}}_{1\times8} = \Big\langle \bar{\bar{A}}_{2,1} e^{m_{2,1}z} \quad \bar{\bar{A}}_{2,2} e^{m_{2,2}z} \quad \bar{\bar{A}}_{2,3} e^{m_{2,3}z} \quad \bar{\bar{A}}_{2,4} e^{m_{2,4}z} \quad \bar{\bar{A}}_{2,5} e^{m_{n,5}z} \quad \bar{\bar{A}}_{2,6} e^{m_{n,6}z} \quad \bar{\bar{A}}_{2,7} e^{m_{2,7}z} \quad \bar{\bar{A}}_{2,8} e^{m_{2,8}z} \Big\rangle_{1\times8}$$

$$\text{(4.54b)} \qquad \left\langle \bar{\bar{D}}_2(z) \right\rangle^{\mathrm{T}}_{1\times8} = \Big\langle \bar{\bar{D}}_{2,1} e^{m_{2,1}z} \quad \bar{\bar{D}}_{2,2} e^{m_{2,2}z} \quad \bar{\bar{D}}_{2,3} e^{m_{2,3}z} \quad \bar{\bar{D}}_{2,4} e^{m_{2,4}z} \quad \bar{\bar{D}}_{2,5} e^{m_{2,5}z} \quad \bar{\bar{D}}_{2,6} e^{m_{2,6}z} \quad \bar{\bar{D}}_{2,7} e^{m_{2,7}z} \quad \bar{\bar{D}}_{2,8} e^{m_{2,8}z} \Big\rangle_{1\times8}$$

$$\text{(4.54c)} \qquad \left\langle e_2(z) \right\rangle^{\mathrm{T}}_{1\times8} = \Big\langle e^{m_{2,1}z} \quad e^{m_{2,2}z} \quad e^{m_{2,3}z} \quad e^{m_{2,4}z} \quad e^{m_{2,5}z} \quad e^{m_{2,6}z} \quad e^{m_{2,7}z} \quad e^{m_{2,8}z} \Big\rangle_{1\times8}$$

$$\text{(4.54d)} \qquad \left\langle \bar{F}_2 \right\rangle^{\mathrm{T}}_{1\times8} = \Big\langle \bar{F}_{2,1} \quad \bar{F}_{2,2} \quad \bar{F}_{2,3} \quad \bar{F}_{2,4} \quad \bar{F}_{2,5} \quad \bar{F}_{2,6} \quad \bar{F}_{2,7} \quad \bar{F}_{2,8} \Big\rangle_{1\times8}$$

Recall that integration constants $\bar{A}_{2,i}$ and $\bar{D}_{2,i}$ in Eq. 4.54 are related to constants $\bar{F}_{2,i}$ through the coefficients $\bar{\bar{A}}_{2,i}$ and $\bar{\bar{D}}_{2,i}$, respectively, as defined in Eq. 3.39:

$$\text{(4.55a)} \qquad \bar{\bar{A}}_{2,i} = \frac{-\dfrac{\left(C_{11,2}m_{2,i}\right)}{\left(C_1 m_{2,i}^{\ 2} - C_{10,2}\right)} \left[ \dfrac{\left(C_{11,2}m_{2,i}\right)\left(C_3 m_{2,i}\right)}{\left(C_1 m_{2,i}^{\ 2} - C_{10,2}\right)} - \left(C_{14,2} - C_{15,2} m_{2,i}^{\ 2}\right) \right]}{\left(C_{12,2} - C_{13} m_{2,i}^{\ 2}\right) - \dfrac{\left(C_{11,2}m_{2,i}\right)^2}{\left(C_1 m_{2,i}^{\ 2} - C_{10,2}\right)}} - \frac{\left(C_3 m_{2,i}\right)}{\left(C_1 m_{2,i}^{\ 2} - C_{10,2}\right)}$$

(4.55b) $$\bar{D}_{2,i} = \frac{\dfrac{\left(C_{11,2}m_{2,i}\right)\left(C_3 m_{2,i}\right)}{\left(C_1 m_{2,i}^{\ 2} - C_{10,2}\right)} - \left(C_{14,2} - C_{15,2}m_{2,i}^{\ 2}\right)}{\left(C_{12,2} - C_{13}m_{2,i}^{\ 2}\right) - \dfrac{\left(C_{11,2}m_{2,i}\right)^2}{\left(C_1 m_{2,i}^{\ 2} - C_{10,2}\right)}}$$

Evaluating Eq. 4.55 for $i$ = 1 to 8 gives:

(4.56a) $\bar{\bar{A}}_{2,1} = -0.017102 + 0.023451i$

(4.56b) $\bar{\bar{A}}_{2,2} = 0.017102 - 0.023451i$

(4.56c) $\bar{\bar{A}}_{2,3} = -0.056353 + 0.060468i$

(4.56d) $\bar{\bar{A}}_{2,4} = 0.056353 - 0.060468i$

(4.56e) $\bar{\bar{A}}_{2,5} = -0.056353 - 0.060468i$

(4.56f) $\bar{\bar{A}}_{2,6} = 0.056353 + 0.060468i$

(4.56g) $\bar{\bar{A}}_{2,7} = -0.017102 - 0.023451i$

(4.56h) $\bar{\bar{A}}_{2,8} = 0.017102 + 0.023451i$

(4.56i) $\bar{\bar{D}}_{2,1} = -6.7955\times10^{-5} - 0.041679i$

(4.56j) $\bar{\bar{D}}_{2,2} = -6.7955\times10^{-5} - 0.041679i$

(4.56k) $\bar{\bar{D}}_{2,3} = -0.50048 - 0.0041107i$

(4.56l) $\bar{\bar{D}}_{2,4} = -0.50048 - 0.0041107i$

(4.56m) $\bar{\bar{D}}_{2,5} = -0.50048 + 0.0041107i$

(4.56n) $\bar{\bar{D}}_{2,6} = -0.50048 + 0.0041107i$

(4.56o) $\bar{\bar{D}}_{2,7} = -6.7955\times10^{-5} + 0.041679i$

(4.56p) $\bar{\bar{D}}_{2,8} = -6.7955\times10^{-5} + 0.041679i$

The integration constants $\left\{\bar{F}_2\right\}_{8\times1}$ can now determined from the boundary conditions in Eq. 4.15. At the fixed end, all displacements are zero. Hence, the boundary conditions at $z = 0$ are:

(4.57a) $$a_2(0) = \left\langle \psi_{2,1} \right\rangle_{1\times8}^{\mathrm{T}} \left\{\bar{F}_2\right\}_{8\times1} = 0$$

where $\left\langle \psi_{2,1} \right\rangle_{1\times8}^{\mathrm{T}} = \left\langle \bar{\bar{A}}_2(0) \right\rangle_{1\times8}^{\mathrm{T}} = \left\langle \bar{\bar{A}}_{2,1} \mid \bar{\bar{A}}_{2,2} \mid \bar{\bar{A}}_{2,3} \mid \bar{\bar{A}}_{2,4} \mid \bar{\bar{A}}_{2,5} \mid \bar{\bar{A}}_{2,6} \mid \bar{\bar{A}}_{2,7} \mid \bar{\bar{A}}_{2,8} \right\rangle_{1\times8}$

(4.57b) $$d_2(0) = \left\langle \psi_{2,2} \right\rangle_{1\times8}^{\mathrm{T}} \left\{\bar{F}_2\right\}_{8\times1} = 0$$

where

$$\left\langle \psi_{2,2} \right\rangle_{1\times 8}^{T} = \left\langle \bar{\bar{D}}_2(0) \right\rangle_{1\times 8}^{T} = \left\langle \bar{\bar{D}}_{2,1} \mid \bar{\bar{D}}_{2,2} \mid \bar{\bar{D}}_{2,3} \mid \bar{\bar{D}}_{2,4} \mid \bar{\bar{D}}_{2,5} \mid \bar{\bar{D}}_{2,6} \mid \bar{\bar{D}}_{2,7} \mid \bar{\bar{D}}_{2,8} \right\rangle_{1\times 8}$$

(4.57c) $$f_2(0) = \left\langle \psi_{2,3} \right\rangle_{1\times 8}^{T} \left\{ \bar{F}_2 \right\}_{8\times 1} = 0$$

where $\left\langle \psi_{2,3} \right\rangle_{1\times 8}^{T} = \left\langle e_2(0) \right\rangle_{1\times 8}^{T} = \langle 1 \mid 1 \mid 1 \mid 1 \mid 1 \mid 1 \mid 1 \mid 1 \rangle_{1\times 8}$

(4.57d) $$f_2'(0) = \left\langle \psi_{2,4} \right\rangle_{1\times 8}^{T} \left\{ \bar{F}_2 \right\}_{8\times 1} = 0$$

where

$$\left\langle \psi_{2,4} \right\rangle_{1\times 8}^{T} = \left\langle e_2'(0) \right\rangle_{1\times 8}^{T} = \left\langle m_{2,1} \mid m_{2,2} \mid m_{2,3} \mid m_{2,4} \mid m_{2,5} \mid m_{2,6} \mid m_{2,7} \mid m_{2,8} \right\rangle_{1\times 8}$$

At the free end of the pipe, none of the displacements are specified. Therefore, the natural boundary conditions given in Eq. 4.15 must be applied. From Eq. 4.15i, noting that $\delta a_2(\ell) \neq 0$, the first boundary condition becomes:

(4.58) $$C_1 a_2'(\ell) + 2C_3 d_n(\ell) + C_3 f_n(\ell) = 0$$

Now substitute the vector expressions for $a_2'(\ell), d_2(\ell)$ and $f_2(\ell)$ from Eq. 4.53 into Eq. 4.58. The resulting boundary equation is written in vector form as:

(4.59a) $$\left\langle \psi_{2,5} \right\rangle_{1\times 8}^{T} \left\{ \bar{F}_2 \right\}_{8\times 1} = 0$$

where

(4.59b)
$$\{\psi_{2,5}\}_{8\times1} = \begin{Bmatrix} \left[C_1 m_{2,1}\overline{\overline{A}}_{2,1} + C_3\left(1+2\overline{\overline{D}}_{2,1}\right)\right]e^{m_{2,1}\ell} \\ \left[C_1 m_{2,2}\overline{\overline{A}}_{2,2} + C_3\left(1+2\overline{\overline{D}}_{2,2}\right)\right]e^{m_{2,2}\ell} \\ \left[C_1 m_{2,3}\overline{\overline{A}}_{2,3} + C_3\left(1+2\overline{\overline{D}}_{2,3}\right)\right]e^{m_{2,3}\ell} \\ \left[C_1 m_{2,4}\overline{\overline{A}}_{2,4} + C_3\left(1+2\overline{\overline{D}}_{2,4}\right)\right]e^{m_{2,4}\ell} \\ \left[C_1 m_{2,5}\overline{\overline{A}}_{2,5} + C_3\left(1+2\overline{\overline{D}}_{2,5}\right)\right]e^{m_{2,5}\ell} \\ \left[C_1 m_{2,6}\overline{\overline{A}}_{2,6} + C_3\left(1+2\overline{\overline{D}}_{2,6}\right)\right]e^{m_{2,6}\ell} \\ \left[C_1 m_{2,7}\overline{\overline{A}}_{2,7} + C_3\left(1+2\overline{\overline{D}}_{2,7}\right)\right]e^{m_{2,7}\ell} \\ \left[C_1 m_{2,8}\overline{\overline{A}}_{2,8} + C_3\left(1+2\overline{\overline{D}}_{2,8}\right)\right]e^{m_{2,8}\ell} \end{Bmatrix}_{8\times1}$$

Similarly, from Eq. 4.151, with $\delta d_2(\ell) \neq 0$, the boundary equation is:

(4.60a)
$$-2C_5 a_n(\ell) + (C_4 + C_9) d_n'(\ell) + 2C_9 f_n'(\ell) = 0$$

(4.60b)
$$\langle \psi_{2,6} \rangle_{1\times8}^{\mathrm{T}} \{\overline{F}_2\}_{8\times1} = 0$$

where

(4.60c)
$$\{\psi_{2,6}\}_{8\times1} = \begin{Bmatrix} \left[-2C_5\overline{\overline{A}}_{2,1} + (C_4+C_9)m_{2,1}\overline{\overline{D}}_{2,1} + 2C_9 m_{2,1}\right]e^{m_{2,1}\ell} \\ \left[-2C_5\overline{\overline{A}}_{2,2} + (C_4+C_9)m_{2,2}\overline{\overline{D}}_{2,2} + 2C_9 m_{2,2}\right]e^{m_{2,2}\ell} \\ \left[-2C_5\overline{\overline{A}}_{2,3} + (C_4+C_9)m_{2,3}\overline{\overline{D}}_{2,3} + 2C_9 m_{2,3}\right]e^{m_{2,3}\ell} \\ \left[-2C_5\overline{\overline{A}}_{2,4} + (C_4+C_9)m_{2,4}\overline{\overline{D}}_{2,4} + 2C_9 m_{2,4}\right]e^{m_{2,4}\ell} \\ \left[-2C_5\overline{\overline{A}}_{2,5} + (C_4+C_9)m_{2,5}\overline{\overline{D}}_{2,5} + 2C_9 m_{2,5}\right]e^{m_{2,5}\ell} \\ \left[-2C_5\overline{\overline{A}}_{2,6} + (C_4+C_9)m_{2,6}\overline{\overline{D}}_{2,6} + 2C_9 m_{2,6}\right]e^{m_{2,6}\ell} \\ \left[-2C_5\overline{\overline{A}}_{2,7} + (C_4+C_9)m_{2,7}\overline{\overline{D}}_{2,7} + 2C_9 m_{2,7}\right]e^{m_{2,7}\ell} \\ \left[-2C_5\overline{\overline{A}}_{2,8} + (C_4+C_9)m_{2,8}\overline{\overline{D}}_{2,8} + 2C_9 m_{2,8}\right]e^{m_{2,8}\ell} \end{Bmatrix}_{8\times1}$$

The boundary condition from Eq. 4.15m, with $\delta f_2(\ell) \neq 0$, must include the load potential term since $n = 2$. Therefore, use Eq. 4.15m.ii:

(4.61a) $$-C_6 f_n'''(\ell) + 2(C_8 + C_9) d_n'(\ell) + 4(C_8 + C_9) f_n'(\ell) + P\cos\left(2\pi/2\right) = 0$$

(4.61b) $$\langle \psi_{2,7} \rangle^{\mathrm{T}}_{1\times 8} \{\bar{F}_2\}_{8\times 1} = P$$

where

(4.61c) $$\{\psi_{2,7}\}_{8\times 1} = \begin{Bmatrix} \left[-C_6 m_{2,1}{}^3 + (C_8 + C_9) m_{2,1}\left(2\bar{\bar{D}}_{2,1} + 4\right)\right] e^{m_{2,1}\ell} \\ \left[-C_6 m_{2,2}{}^3 + (C_8 + C_9) m_{2,2}\left(2\bar{\bar{D}}_{2,2} + 4\right)\right] e^{m_{2,2}\ell} \\ \left[-C_6 m_{2,3}{}^3 + (C_8 + C_9) m_{2,3}\left(2\bar{\bar{D}}_{2,3} + 4\right)\right] e^{m_{2,3}\ell} \\ \left[-C_6 m_{2,4}{}^3 + (C_8 + C_9) m_{2,4}\left(2\bar{\bar{D}}_{2,4} + 4\right)\right] e^{m_{2,4}\ell} \\ \left[-C_6 m_{2,5}{}^3 + (C_8 + C_9) m_{2,5}\left(2\bar{\bar{D}}_{2,5} + 4\right)\right] e^{m_{2,5}\ell} \\ \left[-C_6 m_{2,6}{}^3 + (C_8 + C_9) m_{2,6}\left(2\bar{\bar{D}}_{2,6} + 4\right)\right] e^{m_{2,6}\ell} \\ \left[-C_6 m_{2,7}{}^3 + (C_8 + C_9) m_{2,7}\left(2\bar{\bar{D}}_{2,7} + 4\right)\right] e^{m_{2,7}\ell} \\ \left[-C_6 m_{2,8}{}^3 + (C_8 + C_9) m_{2,8}\left(2\bar{\bar{D}}_{2,8} + 4\right)\right] e^{m_{2,8}\ell} \end{Bmatrix}_{8\times 1}$$

The final boundary condition is obtained from Eq. 4.15n, noting that $\delta f_2'(\ell) \neq 0$:

(4.62a) $$C_6 f_n''(\ell) - 2C_8 d_n(\ell) - 4C_8 f_n(\ell) = 0$$

(4.62b) $$\langle \psi_{2,8} \rangle^{\mathrm{T}}_{1\times 8} \{\bar{F}_2\}_{8\times 1} = 0$$

where

$$(4.62c)\qquad \{\psi_{2,8}\}_{8\times1}=\begin{Bmatrix}\left[C_6 m_{2,1}{}^2-C_8\left(2\bar{\bar{D}}_{2,1}+4\right)\right]e^{m_{2,1}\ell}\\ \left[C_6 m_{2,2}{}^2-C_8\left(2\bar{\bar{D}}_{2,2}+4\right)\right]e^{m_{2,2}\ell}\\ \left[C_6 m_{2,3}{}^2-C_8\left(2\bar{\bar{D}}_{2,3}+4\right)\right]e^{m_{2,3}\ell}\\ \left[C_6 m_{2,4}{}^2-C_8\left(2\bar{\bar{D}}_{2,4}+4\right)\right]e^{m_{2,4}\ell}\\ \left[C_6 m_{2,5}{}^2-C_8\left(2\bar{\bar{D}}_{2,5}+4\right)\right]e^{m_{2,5}\ell}\\ \left[C_6 m_{2,6}{}^2-C_8\left(2\bar{\bar{D}}_{2,6}+4\right)\right]e^{m_{2,6}\ell}\\ \left[C_6 m_{2,7}{}^2-C_8\left(2\bar{\bar{D}}_{2,7}+4\right)\right]e^{m_{2,7}\ell}\\ \left[C_6 m_{2,8}{}^2-C_8\left(2\bar{\bar{D}}_{2,8}+4\right)\right]e^{m_{2,8}\ell}\end{Bmatrix}_{8\times1}$$

Similar to mode $n=1$, the boundary equations developed in Eqs. 4.57 through 4.62 can be written in a consolidated matrix form:

$$(4.63)\qquad [\Psi_2]_{8\times8}\{\bar{F}_2\}_{8\times1}=\{Q_2\}_{8\times1}$$

where

$$[\Psi_2]_{8\times8}=\begin{bmatrix}\langle\psi_{2,1}\rangle_{1\times8}^{\mathrm{T}}\\ \langle\psi_{2,2}\rangle_{1\times8}^{\mathrm{T}}\\ \langle\psi_{2,3}\rangle_{1\times8}^{\mathrm{T}}\\ \langle\psi_{2,4}\rangle_{1\times8}^{\mathrm{T}}\\ \langle\psi_{2,5}\rangle_{1\times8}^{\mathrm{T}}\\ \langle\psi_{2,6}\rangle_{1\times8}^{\mathrm{T}}\\ \langle\psi_{2,7}\rangle_{1\times8}^{\mathrm{T}}\\ \langle\psi_{2,8}\rangle_{1\times8}^{\mathrm{T}}\end{bmatrix}_{8\times8}\qquad\text{and}\qquad \{Q_2\}_{8\times1}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\\P\\0\end{Bmatrix}$$

The components of the vectors $\langle \psi_{2,i} \rangle_{1\times 8}^{\mathrm{T}}$ are defined in Eqs. 4.57 through 4.62. Hence, the only unknowns in Eq. 4.63 are the elements of the vector $\{\overline{F}_2\}_{8\times 1}$. Solving the equation for these unknowns gives:

$$\text{(4.64)} \qquad \{\overline{F}_2\}_{8\times 1} = \begin{Bmatrix} \overline{F}_{2,1} \\ \overline{F}_{2,2} \\ \overline{F}_{2,3} \\ \overline{F}_{2,4} \\ \overline{F}_{2,5} \\ \overline{F}_{2,6} \\ \overline{F}_{2,7} \\ \overline{F}_{2,8} \end{Bmatrix}_{8\times 1} = \begin{Bmatrix} -1.9167\times 10^{-6} + 1.0962\times 10^{-5} i\, m \\ 1.7255\times 10^{-6} + 1.4494\times 10^{-5} i\, m \\ -1.9167\times 10^{-6} - 1.0962\times 10^{-5} i\, m \\ 1.7255\times 10^{-6} - 1.4494\times 10^{-5} i\, m \\ 2.3495\times 10^{-72} + 5.4222\times 10^{-72} i\, m \\ 1.9117\times 10^{-7} + 2.1546\times 10^{-7} i\, m \\ 2.3495\times 10^{-72} - 5.4222\times 10^{-72} i\, m \\ 1.9117\times 10^{-7} - 2.1546\times 10^{-7} i\, m \end{Bmatrix}_{8\times 1}$$

With vector $\{\overline{F}_2\}_{8\times 1}$ now completely defined, displacement contributions $a_2(\ell)$, $d_2(\ell)$ and $f_2(\ell)$, corresponding to the second mode $n=2$, at the tip of the pipe can be calculated from Eq. 4.53:

$$\text{(4.65a)} \qquad a_2(\ell) = \langle \overline{\overline{A}}_2(\ell) \rangle_{1\times 8} \{\overline{F}_2\}_{8\times 1} = -0.152 mm$$

$$\text{(4.65b)} \qquad d_2(\ell) = \langle \overline{\overline{D}}_2(\ell) \rangle_{1\times 8} \{\overline{F}_2\}_{8\times 1} = -1.335 mm$$

$$\text{(4.65c)} \qquad f_2(\ell) = \langle \overline{e}_2(\ell) \rangle_{1\times 8} \{\overline{F}_2\}_{8\times 1} = 2.747 mm$$

The displacement contribution of any point on the pipe mid-surface at the free end, $z = \ell$, for mode $n = 2$ is found by substituting Eq. 4.65 into Eq. 2.11:

$$\text{(4.66a)} \qquad u_2(\ell,\phi) = a_2(\ell)\cos 2\phi = (-0.152 mm)\cos 2\phi$$

$$\text{(4.66b)} \qquad v_2(\ell,\phi) = d_2(\ell)\sin 2\phi = (-1.335 mm)\sin 2\phi$$

$$\text{(4.66c)} \qquad w_2(\ell,\phi) = f_2(\ell)\cos 2\phi = (2.747 mm)\cos 2\phi$$

The vertical and longitudinal displacement contributions at Point *A*, *B* and *C* on Figure 4.1 are found by evaluating Eq. 4.66 at the values of $\phi = \pi/2$, $\phi = \pi$ and $\phi = 3\pi/2$ respectively.

Eq. 4.66a gives the longitudinal displacement at all three points while Eq. 4.66b gives the vertical displacement at Point $B$ and Eq. 4.66c gives the vertical displacement at Points $A$ and $C$. The displacements are summarized in Table 4.3 below (recall that a positive vertical displacement indicates that the point moves downward).

**Table 4.3** – Displacement contributions of Fourier mode $n = 2$ for Points $A$, $B$ and $C$

| Location | Vertical Displacement $\Delta_{Vert,n=2}$ | Longitudinal Displacement $\Delta_{Long,n=2}$ |
|---|---|---|
| Point $A$ | 2.747 $mm$ | 0.152 $mm$ |
| Point $B$ | 0 $mm$ | -0.152 $mm$ |
| Point $C$ | -2.747 $mm$ | 0.152 $mm$ |

The displacement contributions for mode $n=2$ represent deformation of the pipe cross section. The difference in the vertical displacements at Points $A$ and $C$, namely that Point $A$ moves downward while Point $C$ moves upward, is evidence of ovalization of the cross section. Similarly, the fibres at Points $A$ and $C$ elongate longitudinally while the fibres at Point $B$ contract, indicating warping of the cross section.

#### 4.1.3.7 Displacement Fields for Fourier Modes $n > 2$

The procedure developed in Section 4.3.6 to calculate the displacements for mode $n=2$ can also be used to determine the displacements for the higher modes. Applying this procedure for modes 3 through 6 gives the vertical and longitudinal displacement contributions at Points $A$, $B$ and $C$ at the free end of the pipe. Note that for each mode $n$ only one of the two sets of orthogonal displacement functions will be non-zero. If $a_n(z)$, $d_n(z)$ and $f_n(z)$ are non-zero then $b_n(z)$, $c_n(z)$ and $g_n(z)$ will be zero, and vice-versa. The displacement contributions are summarized in Tables 4.4 through 4.7 below (recall that a positive vertical displacement indicates that the point moves downward).

**Table 4.4** – Displacement contributions of Fourier mode $n = 3$ for Points $A$, $B$ and $C$

| Location | $\phi$ | $u_3(\ell,\phi) = b_3(\ell)\sin 3\phi$ | $v_3(\ell,\phi) = c_3(\ell)\cos 3\phi$ | $w_3(\ell,\phi) = g_3(\ell)\sin 3\phi$ | Vertical Displacement $\Delta_{Vert,n=3}$ | Longitudinal Displacement $\Delta_{Long,n=3}$ |
|---|---|---|---|---|---|---|
| Point $A$ | $\pi/2$ | 0.0537 $mm$ | 0 $mm$ | -0.972 $mm$ | 0.972 $mm$ | 0.0537 $mm$ |
| Point $B$ | $\pi$ | 0 $mm$ | -0.303 $mm$ | 0 $mm$ | -0.303 $mm$ | 0 $mm$ |
| Point $C$ | $3\pi/2$ | -0.0537 $mm$ | 0 $mm$ | 0.972 $mm$ | 0.972 $mm$ | -0.0537 $mm$ |

**Table 4.5** – Displacement contributions of Fourier mode $n = 4$ for Points $A$, $B$ and $C$

| Location | $\phi$ | $u_4(\ell,\phi) = a_4(\ell)\cos 4\phi$ | $v_4(\ell,\phi) = d_4(\ell)\sin 4\phi$ | $w_4(\ell,\phi) = f_4(\ell)\cos 4\phi$ | Vertical Displacement $\Delta_{Vert,n=4}$ | Longitudinal Displacement $\Delta_{Long,n=4}$ |
|---|---|---|---|---|---|---|
| Point $A$ | $\pi/2$ | 0.0254 $mm$ | 0 $mm$ | -0.503 $mm$ | 0.503 $mm$ | 0.0254 $mm$ |
| Point $B$ | $\pi$ | 0.0254 $mm$ | 0 $mm$ | -0.503 $mm$ | 0 $mm$ | 0.0254 $mm$ |
| Point $C$ | $3\pi/2$ | 0.0254 $mm$ | 0 $mm$ | -0.503 $mm$ | -0.503 $mm$ | 0.0254 $mm$ |

**Table 4.6** – Displacement contributions of Fourier mode $n = 5$ for Points $A$, $B$ and $C$

| Location | $\phi$ | $u_5(\ell,\phi) = b_5(\ell)\sin 5\phi$ | $v_5(\ell,\phi) = c_5(\ell)\cos 5\phi$ | $w_5(\ell,\phi) = g_5(\ell)\sin 5\phi$ | Vertical Displacement $\Delta_{Vert,n=5}$ | Longitudinal Displacement $\Delta_{Long,n=5}$ |
|---|---|---|---|---|---|---|
| Point $A$ | $\pi/2$ | 0.0133 $mm$ | 0 $mm$ | -0.297 $mm$ | 0.297 $mm$ | 0.0133 $mm$ |
| Point $B$ | $\pi$ | 0 $mm$ | 0.0522 $mm$ | 0 $mm$ | 0.0522 $mm$ | 0 $mm$ |
| Point $C$ | $3\pi/2$ | -0.0133 $mm$ | 0 $mm$ | 0.297 $mm$ | 0.297 $mm$ | -0.0133 $mm$ |

**Table 4.7** – Displacement contributions of Fourier mode $n = 6$ for Points $A$, $B$ and $C$

| Location | $\phi$ | $u_6(\ell,\phi) = a_6(\ell)\cos 6\phi$ | $v_6(\ell,\phi) = d_6(\ell)\sin 6\phi$ | $w_6(\ell,\phi) = f_6(\ell)\cos 6\phi$ | Vertical Displacement $\Delta_{Vert,n=6}$ | Longitudinal Displacement $\Delta_{Long,n=6}$ |
|---|---|---|---|---|---|---|
| Point $A$ | $\pi/2$ | 0.0074 $mm$ | 0 $mm$ | -0.187 $mm$ | 0.187 $mm$ | 0.0074 $mm$ |
| Point $B$ | $\pi$ | -0.0074 $mm$ | 0 $mm$ | 0.187 $mm$ | 0 $mm$ | -0.0074 $mm$ |
| Point $C$ | $3\pi/2$ | 0.0074 $mm$ | 0 $mm$ | -0.187 $mm$ | -0.187 $mm$ | 0.0074 $mm$ |

#### 4.1.3.8 Total Displacement

The total displacements of Points $A$, $B$ and $C$ at the free end of the pipe are the sum the values in Eq. 4.29 and Tables 4.2 through 4.7. The vertical and longitudinal displacement contributions for each mode $n$ at Point $A$, as well as the total cumulative displacement, are summarized in Table 4.8.

**Table 4.8** – Contribution of mode $n$ to total displacement for Point $A$

| Mode $n$ | Vertical Displacement ($mm$) | | Longitudinal Displacement ($mm$) | |
|---|---|---|---|---|
| | Mode $n$ | Cumulative | Mode $n$ | Cumulative |
| 0 | 0.032 | 0.032 | 0.001 | 0.001 |
| 1 | 11.579 | 11.611 | 0.849 | 0.850 |
| 2 | 2.747 | 14.358 | 0.152 | 1.002 |
| 3 | 0.972 | 15.330 | 0.0537 | 1.055 |
| 4 | 0.503 | 15.833 | 0.0254 | 1.081 |
| 5 | 0.297 | 16.130 | 0.0133 | 1.094 |
| 6 | 0.187 | 16.317 | 0.0074 | 1.101 |
| Total | - | 16.317 | - | 1.101 |

It is apparent from Table 4.8 that as the number of Fourier modes included in the analysis increases, the predicted vertical and longitudinal displacements of Point $A$ also increases. Also, as the mode $n$ under consideration becomes higher, the displacement contribution for the mode decreases, indicating that the solution is converging to the true displacement value. The convergence of the solution is from below, meaning that the vertical and longitudinal displacements at Point $A$ determined from the method in the current study are always lower than the "true" displacement.

Both the vertical and longitudinal displacements compare well with the values from the ABAQUS shell finite element analysis of 16.755 mm and 1.142 mm, respectively, given in Table 4.1.

The vertical and longitudinal displacement contributions for each mode $n$ at Point $B$, as well as the total cumulative displacement, are summarized in Table 4.9.

**Table 4.9**– Contribution of mode $n$ to total displacement for Point $B$

| Mode $n$ | Vertical Displacement (mm) | | Longitudinal Displacement (mm) | |
|---|---|---|---|---|
| | Mode $n$ | Cumulative | Mode $n$ | Cumulative |
| 0 | 0 | 0 | 0.001 | 0.001 |
| 1 | 11.516 | 11.516 | 0 | 0.001 |
| 2 | 0 | 11.516 | -0.152 | -0.151 |
| 3 | -0.303 | 11.213 | 0 | -0.151 |
| 4 | 0 | 11.213 | 0.0254 | -0.126 |
| 5 | 0.0522 | 11.265 | 0 | -0.126 |
| 6 | 0 | 11.265 | -0.0074 | -0.133 |
| Total | - | 11.265 | - | -0.133 |

The vertical and longitudinal displacement contributions for each mode $n$ at Point $C$, as well as the total cumulative displacement, are summarized in Table 4.10.

**Table 4.10** – Contribution of mode $n$ to total displacement for Point $C$

| Mode $n$ | Vertical Displacement (mm) | | Longitudinal Displacement (mm) | |
|---|---|---|---|---|
| | Mode $n$ | Cumulative | Mode $n$ | Cumulative |
| 0 | -0.032 | -0.032 | 0.001 | 0.001 |
| 1 | 11.579 | 11.548 | -0.849 | -0.848 |
| 2 | -2.747 | 8.801 | 0.152 | -0.696 |
| 3 | 0.972 | 9.773 | -0.0537 | -0.750 |
| 4 | -0.503 | 9.270 | 0.0254 | -0.725 |
| 5 | 0.297 | 9.567 | -0.0133 | -0.738 |
| 6 | -0.187 | 9.380 | 0.0074 | -0.731 |
| Total | - | 9.380 | - | -0.731 |

Unlike Point $A$, the vertical and longitudinal displacements of Point $C$ oscillate as the number of Fourier modes included in the analysis increases, as seen in Table 4.10. The contributions from the odd Fourier modes $(n = 1,3,5)$ increase the displacement while the contributions from the Fourier modes $(n = 2,4,6)$ decrease the displacement. As the number of modes included in the analysis increases, the cumulative displacement converges to the displacement sought. However, the true displacement is not always higher or lower than the displacement predicted by the method on the current study. The current solution oscillates above and below the true displacement at Point $C$ as the total number of Fourier modes included in the solution increases.

Both the vertical and longitudinal displacements for Points $B$ and $C$ provide excellent correlation with the ABAQUS shell finite element analysis results in Table 4.1.

## 4.2 Example 2: Cantilever Beam Under Self Weight

As a second example of the solution developed in Chapter 3, it is required to find the deformed shape of an initially horizontal cantilevered pipe, as shown in Figure 4.2, under its self weight. The pipe has the same geometric and material properties as the previous problem, namely span $\ell = 4\,m$, thickness $t = 6\,mm$, mid-surface radius $r = 200\,mm$, Poisson's ratio $\mu = 0.3$, Young modulus $E = 200{,}000\,MPa$ and density of pipe steel $\gamma_s = 78\,kN/m^3$.

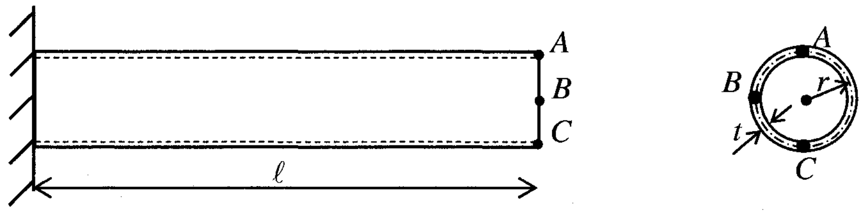


**Figure 4.2** – Pipe under self weight

In the previous example, loads were applied to only the end of the pipe. As a result, only the homogeneous version of the equilibrium equations needed to be solved. In the current example, tractions are applied along the pipe mid-surface and the particular solution is required in addition to the homogeneous solution.

The tractions $q_u(z,\phi)$, $q_v(z,\phi)$ and $q_w(z,\phi)$ associated with the self weight are determined by considering an infinitesimal piece of pipe of volume $t(rd\phi)(dz)$ as shown in Figure 4.3. The load due to self weight acting on the piece of pipe is given as $q = t\gamma_s$. The traction $q$ acts in a vertical (downward) direction. From the geometry in Figure 4.3 the tractions in the longitudinal, tangential and radial directions are:

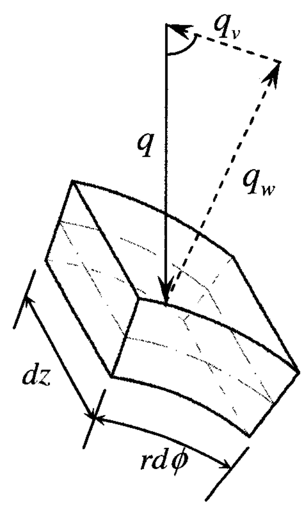


**Figure 4.3** – Tractions $q_v(z,\phi)$ and $q_w(z,\phi)$ as components of pipe self weight

(4.67a) $$q_u(z,\phi) = 0$$

(4.67b) $$q_v(z,\phi) = -t\gamma_s \cos\phi$$

(4.67c) $$q_w(z,\phi) = -t\gamma_s \sin\phi$$

For mode $n=0$, Eq. 2.20 provides the following equilibrium conditions:

(4.68) $$\left[\begin{array}{c:c:c} 2C_1D^2 & 2C_3D & 0 \\ \hdashline 2C_3D & 2C_2+2C_6D^4 & 0 \\ \hdashline 0 & 0 & 2(C_4+C_9)D^2 \end{array}\right]\begin{Bmatrix} a_{0g}(z) \\ f_{0g}(z) \\ c_{0g}(z) \end{Bmatrix} = \{Q_0\} = \begin{Bmatrix} -\int\limits_{\phi=0}^{2\pi} q_u(\phi,z)\,rd\phi \\ \int\limits_{\phi=0}^{2\pi} q_w(\phi,z)\,rd\phi \\ -\int\limits_{\phi=0}^{2\pi} q_v(\phi,z)\,rd\phi \end{Bmatrix}$$

In Eq. 4.68, the general displacements $a_{0g}(z)$, $c_{0g}(z)$ and $f_{0g}(z)$ are the sum of the displacements corresponding to the homogeneous solution and the displacements corresponding to the particular solution:

(4.69a) $$a_{0g}(z) = a_0(z) + a_{0p}(z)$$

(4.69b) $$c_{0g}(z) = c_0(z) + c_{0p}(z)$$

(4.69c) $$f_{0g}(z) = f_0(z) + f_{0p}(z)$$

When the tractions in Eq. 4.67 are substituted into the ride hand side vector $\{Q_0\}$ of Eq. 4.68, the energy equivalent load vector is found to vanish:

(4.70) $$\{Q_0\}_{3\times1} = \begin{Bmatrix} 0 \\ \int\limits_{\phi=0}^{2\pi} -t\gamma_s \sin\phi\, rd\phi \\ -\int\limits_{\phi=0}^{2\pi} -t\gamma_s \cos\phi\, rd\phi \end{Bmatrix}_{3\times1} = \begin{Bmatrix} 0 \\ t\gamma_s r\left[\cos\phi\right]_{\phi=0}^{2\pi} \\ t\gamma_s r\left[\sin\phi\right]_{\phi=0}^{2\pi} \end{Bmatrix}_{3\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}_{3\times1}$$

Eq. 4.68 then simplifies to:

$$(4.71)\quad \left[\begin{array}{c|c|c} 2C_1D^2 & 2C_3D & 0 \\ \hline 2C_3D & 2C_2+2C_6D^4 & 0 \\ \hline 0 & 0 & 2(C_4+C_9)D^2 \end{array}\right]_{3\times3} \begin{Bmatrix} a_{0g}(z) \\ f_{0g}(z) \\ c_{0g}(z) \end{Bmatrix}_{3\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}_{3\times1}$$

The solution of Eq. 4.71 in which the effects of self weight are included is shown to coincide with the homogeneous solution. The particular solutions are $a_{0p}(z)=0$, $c_{0p}(z)=0$ and $f_{0p}(z)=0$ so the resulting general solutions are $a_{0g}(z)=a_0(z)$, $c_{0g}(z)=c_0(z)$ and $f_{0g}(z)=f_0(z)$ in which $a_0(z)$, $c_0(z)$ and $f_0(z)$ are the displacement corresponding to the homogeneous solution provided in Eq. 4.19.

The boundary conditions for the homogeneous case of the problem for mode $n=0$ were given in Eq. 4.27 as $[\Psi_0]_{8\times8}\{\overline{A}_0\}_{8\times1}=\{Q_0\}_{8\times1}$ in which $[\Psi_0]_{8\times8}$ is the matrix of boundary equations, $\{\overline{A}_0\}_{8\times1}$ is the vector of integration constants and $\{Q_0\}_{8\times1}$ is the load vector. For the current problem of self weight, the load vector is $\{Q_0\}_{8\times1}=\{0\}_{8\times1}$. The only possible solution for the boundary equations is that the integration constants must vanish, leading to $a_0(z)=c_0(z)=f_0(z)=0$.

The equilibrium equations for Fourier modes $n\geq1$ were given in Eq. 2.27 as:

$$(4.72)\quad \left[\begin{array}{c|c} \mathbf{X}_n & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{X}_n \end{array}\right]_{6\times6} \begin{Bmatrix} a_{n,g}(z) \\ d_{n,g}(z) \\ f_{n,g}(z) \\ b_{n,g}(z) \\ -c_{n,g}(z) \\ g_{n,g}(z) \end{Bmatrix}_{6\times1} = \{Q_n\}_{6\times1} = \begin{Bmatrix} -\int_{\phi=0}^{2\pi} q_u(\phi,z)\cos(n\phi)\,r\,d\phi \\ \int_{\phi=0}^{2\pi} q_v(\phi,z)\sin(n\phi)\,r\,d\phi \\ \int_{\phi=0}^{2\pi} q_w(\phi,z)\cos(n\phi)\,r\,d\phi \\ -\int_{\phi=0}^{2\pi} q_u(\phi,z)\sin(n\phi)\,r\,d\phi \\ -\int_{\phi=0}^{2\pi} q_v(\phi,z)\cos(n\phi)\,r\,d\phi \\ \int_{\phi=0}^{2\pi} q_w(\phi,z)\sin(n\phi)\,r\,d\phi \end{Bmatrix}_{6\times1}$$

in which

$$(4.73)\quad [\mathbf{X}_n]_{3\times3} = \left[\begin{array}{c|c|c} C_1D^2 - \frac{1}{r^2}C_4n^2 & (C_3+C_5)nD & C_3D \\ \hline (C_3+C_5)nD & \begin{array}{c}(C_2+C_7)n^2 \\ -(C_4+C_9)D^2\end{array} & \begin{array}{c}C_2n+C_7n^3- \\ (C_8+C_9)nD^2\end{array} \\ \hline C_3D & \begin{array}{c}C_2n+C_7n^3- \\ (C_8+C_9)nD^2\end{array} & \begin{array}{c}C_2+C_7n^4+C_6D^4- \\ (2C_8+C_9)n^2D^2\end{array} \end{array}\right]$$

The general displacements $a_{n,g}(z)$, $b_{n,g}(z),\ldots,g_{n,g}(z)$ are the sum of the displacements corresponding to the homogeneous solution, $a_n(z)$, $b_n(z),\ldots,g_n(z)$, and the displacements corresponding to the particular solution, $a_{n,p}(z), b_{n,p}(z),\ldots,g_{n,p}(z)$.

The tractions in Eq. 4.67 are substituted into the vector on the ride hand side Eq. 4.72 to give the load potential vector for mode $n$:

$$(4.74)\quad \{\mathbf{Q}_n\}_{6\times1} = \begin{Bmatrix} -\int_{\phi=0}^{2\pi} q_u(\phi,z)\cos(n\phi)\,rd\phi \\ \int_{\phi=0}^{2\pi} q_v(\phi,z)\sin(n\phi)\,rd\phi \\ \int_{\phi=0}^{2\pi} q_w(\phi,z)\cos(n\phi)\,rd\phi \\ -\int_{\phi=0}^{2\pi} q_u(\phi,z)\sin(n\phi)\,rd\phi \\ -\int_{\phi=0}^{2\pi} q_v(\phi,z)\cos(n\phi)\,rd\phi \\ \int_{\phi=0}^{2\pi} q_w(\phi,z)\sin(n\phi)\,rd\phi \end{Bmatrix}_{6\times1} = \begin{Bmatrix} 0 \\ \int_{\phi=0}^{2\pi} -t\gamma_s\cos\phi\sin(n\phi)\,rd\phi \\ \int_{\phi=0}^{2\pi} -t\gamma_s\sin\phi\cos(n\phi)\,rd\phi \\ 0 \\ -\int_{\phi=0}^{2\pi} -t\gamma_s\cos\phi\cos(n\phi)\,rd\phi \\ \int_{\phi=0}^{2\pi} -t\gamma_s\sin\phi\sin(n\phi)\,rd\phi \end{Bmatrix}_{6\times1}$$

Consider first Fourier modes $n \geq 2$. The load potential vector in Eq. 4.74 becomes:

$$\{\mathbf{Q}_n\}_{6\times1}=\begin{Bmatrix}0\\ \int_{\phi=0}^{2\pi}-t\gamma_s\cos\phi\sin n\phi r d\phi\\ \int_{\phi=0}^{2\pi}-t\gamma_s\sin\phi\cos n\phi r d\phi\\ 0\\ -\int_{\phi=0}^{2\pi}-t\gamma_s\cos\phi\cos n\phi r d\phi\\ \int_{\phi=0}^{2\pi}-t\gamma_s\sin\phi\sin n\phi r d\phi\end{Bmatrix}_{6\times1}=\begin{Bmatrix}0\\ -t\gamma_s r\left[\frac{\cos(1-n)\phi}{2(1-n)}+\frac{\cos(1+n)\phi}{2(1+n)}\right]_{\phi=0}^{2\pi}\\ -t\gamma_s r\left[\frac{\cos(1-n)\phi}{2(1-n)}+\frac{\cos(1+n)\phi}{2(1+n)}\right]_{\phi=0}^{2\pi}\\ 0\\ t\gamma_s r\left[\frac{\sin(1-n)\phi}{2(1-n)}+\frac{\sin(1+n)\phi}{2(1+n)}\right]_{\phi=0}^{2\pi}\\ -t\gamma_s r\left[\frac{\sin(1-n)\phi}{2(1-n)}-\frac{\sin(1+n)\phi}{2(1+n)}\right]_{\phi=0}^{2\pi}\end{Bmatrix}_{6\times1}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\end{Bmatrix}_{6\times1} \tag{4.75}$$

Substitute the load potential vector in Eq. 4.75 into Eq. 4.72 to obtain the equilibrium equations for modes $n \geq 2$:

$$\begin{bmatrix}\mathbf{X}_n & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_n\end{bmatrix}_{6\times6}\begin{Bmatrix}a_{n,g}(z)\\ d_{n,g}(z)\\ f_{n,g}(z)\\ b_{n,g}(z)\\ -c_{n,g}(z)\\ g_{n,g}(z)\end{Bmatrix}_{6\times1}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\end{Bmatrix}_{6\times1} \tag{4.76}$$

Similar to the case of mode $n=0$, the solution for the effects of pipe self weight the coincides with the homogeneous solution and the particular solution vanishes. The boundary conditions for the homogeneous case of the problem for modes $n \geq 2$ were given in Eq. 4.63 as $[\Psi_n]_{8\times8}\{\bar{F}_n\}_{8\times1}=\{Q_n\}_{8\times1}$ in which $[\Psi_n]_{8\times8}$ is the matrix of boundary equations, $\{\bar{F}_n\}_{8\times1}$ is the vector of integration constants and $\{Q_n\}_{8\times1}$ is the load vector. For the current problem, the load vector is $\{Q_n\}_{8\times1}=\{0\}_{8\times1}$. The only possible solution for the boundary equations is that the integration constants must vanish, leading to $a_n(z)=b_n(z)=c_n(z)=d_n(z)=f_n(z)=g_n(z)=0$.

Consider now the first Fourier mode. For mode $n=1$, the load potential vector $\mathbf{Q}_1$ becomes:

$$(4.77)\quad \{\mathbf{Q}_1\}_{6\times1}=\begin{Bmatrix} 0 \\ \int_{\phi=0}^{2\pi} -t\gamma_s \cos\phi \sin\phi r d\phi \\ \int_{\phi=0}^{2\pi} -t\gamma_s \sin\phi \cos\phi r d\phi \\ 0 \\ -\int_{\phi=0}^{2\pi} -t\gamma_s \cos^2\phi r d\phi \\ \int_{\phi=0}^{2\pi} -t\gamma_s \sin^2\phi r d\phi \end{Bmatrix}_{6\times1} = \begin{Bmatrix} 0 \\ -t\gamma_s r\left[\dfrac{\sin^2\phi}{2}\right]_{\phi=0}^{2\pi} \\ -t\gamma_s r\left[\dfrac{\sin^2\phi}{2}\right]_{\phi=0}^{2\pi} \\ 0 \\ t\gamma_s r\left[\dfrac{\phi}{2}+\dfrac{\sin 2\phi}{4}\right]_{\phi=0}^{2\pi} \\ -t\gamma_s r\left[\dfrac{\phi}{2}-\dfrac{\sin 2\phi}{4}\right]_{\phi=0}^{2\pi} \end{Bmatrix}_{6\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \pi t\gamma_s r \\ -\pi t\gamma_s r \end{Bmatrix}_{6\times1}$$

The load potential vector in Eq. 4.77 is substituted into Eq. 4.72 to give the equilibrium equations for mode $n=1$:

$$(4.78)\quad \left[\begin{array}{c|c} \mathbf{X}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{X}_1 \end{array}\right]_{6\times6} \begin{Bmatrix} a_{1,g}(z) \\ d_{1,g}(z) \\ f_{1,g}(z) \\ b_{1,g}(z) \\ -c_{1,g}(z) \\ g_{1,g}(z) \end{Bmatrix}_{6\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \pi t\gamma_s r \\ -\pi t\gamma_s r \end{Bmatrix}_{6\times1}$$

The general displacements $a_{1,g}(z)$, $b_{1,g}(z),\ldots, g_{1,g}(z)$ are the sum of the displacements corresponding to the homogeneous solution, $a_1(z)$, $b_1(z),\ldots, g_1(z)$, and the displacements corresponding to the particular solution, $a_{1,p}(z), b_{1,p}(z),\ldots, g_{1,p}(z)$.

Similar to modes $n=0$ and $n\geq 2$, it is apparent from Eq. 4.78 that $a_{1,g}(z)=d_{1,g}(z)=f_{1,g}(z)=0$. However, the displacements corresponding to the particular solution for $b_{1,p}(z)$, $c_{1,p}(z)$ and $g_{1,p}(z)$ are non-zero. The equilibrium equations in Eq. 4.78 can therefore be simplified as

$$
\left[\begin{array}{c:c:c}
C_1D^2-\dfrac{C_4}{r^2} & (C_3+C_5)D & C_3D \\
\hdashline
(C_3+C_5)D & \begin{array}{c}C_2+C_7-\\(C_4+C_9)D^2\end{array} & \begin{array}{c}C_2+C_7-\\(C_8+C_9)D^2\end{array} \\
\hdashline
C_3D & \begin{array}{c}C_2+C_7-\\(C_8+C_9)D^2\end{array} & \begin{array}{c}C_2+C_7+C_6D^4-\\(2C_8+C_9)D^2\end{array}
\end{array}\right]_{3\times3}
\begin{Bmatrix} b_{1,g}(z) \\ -c_{1,g}(z) \\ g_{1,g}(z) \end{Bmatrix}_{3\times1}
=\begin{Bmatrix} 0 \\ \pi t\gamma_s r \\ -\pi t\gamma_s r \end{Bmatrix}_{3\times1}
\tag{4.79}
$$

The displacement contributions for modes $n=0$ and $n\geq 2$ are zero, so the only mode which contributes to the displacement of the pipe is mode $n=1$, specifically $b_{1,g}(z)$, $c_{1,g}(z)$ and $g_{1,g}(z)$. The displacements corresponding to the homogeneous displacements, $b_1(z)$, $c_1(z)$ and $g_1(z)$, were provided in Eq. 4.34. It is now required to formulate a particular solution for Eq. 4.79. Assume that the particular solution takes the form:

(4.80a) $b_{1p}(z)=T_1z+T_2z^3$

(4.80b) $c_{1p}(z)=T_3z^2+T_4z^4$

(4.80c) $g_{1p}(z)=T_5z^2+T_6z^4$

in which constants $T_1$ through $T_6$ are to be determined by substituting the expressions in Eq. 4.80 into Eq. 4.79 and matching the coefficients of the left hand side of the equations for $z^0$, $z$, $z^2$, $z^3$ and $z^4$ to those of the right hand side. The resulting particular solution is:

(4.81a) $b_{1p}(z)=T_1z-4rT_4z^3$

(4.81b) $c_{1p}(z)=T_3z^2+T_4z^4$

(4.81c) $g_{1p}(z)=T_5z^2+T_4z^4$

Constants $T_1$, $T_3$, $T_4$ and $T_5$ were determined numerically to be:

(4.82a) $T_1=-2.3654\times10^{-7}$

(4.82b) $T_3=1.6053\times10^{-6}\ m^{-1}$

(4.82c) $T_4=-8.1244\times10^{-7}\ m^{-3}$

(4.82d) $T_5 = 1.4883 \times 10^{-7}\ m^{-1}$

The particular solution is added to the homogeneous solution to give the total displacement of the pipe. The homogeneous solution for the current self weight problem is the same as the homogeneous solution for the point load applied at the tip of the cantilever presented in Section 4.3.5. From Eq. 4.34 the homogeneous solution is:

$$(4.83)\qquad \begin{Bmatrix} b_{1h}(z) \\ c_{1h}(z) \\ g_{1h}(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \overline{\overline{A}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \\ -\left\langle \overline{\overline{D}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \\ \left\langle e_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \end{bmatrix} \left\{ \overline{G}_1 \right\}_{8\times 1}$$

in which the vectors $\left\langle \overline{\overline{A}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}}$, $\left\langle \overline{\overline{D}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}}$ and $\left\langle e_1(z) \right\rangle_{1\times 8}^{\mathrm{T}}$ were defined in Eq. 4.35. Hence, the total displacement is given as:

$$(4.84)\qquad \begin{Bmatrix} b_1(z) \\ c_1(z) \\ g_1(z) \end{Bmatrix} = \begin{Bmatrix} b_{1h}(z) + b_{1p}(z) \\ c_{1h}(z) + c_{1p}(z) \\ g_{1h}(z) + g_{1p}(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \overline{\overline{A}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \\ -\left\langle \overline{\overline{D}}_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \\ \left\langle e_1(z) \right\rangle_{1\times 8}^{\mathrm{T}} \end{bmatrix} \left\{ \overline{G}_1 \right\}_{8\times 1} + \begin{Bmatrix} T_1 z - 4rT_4 z^3 \\ T_3 z^2 + T_4 z^4 \\ T_5 z^2 + T_4 z^4 \end{Bmatrix}$$

The unknown integration constants, $\left\{ \overline{G}_1 \right\}_{8\times 1}$, can now be determined from the boundary conditions in Eq. 4.15 and the total displacement expressions in Eq. 4.84. At the fixed end, all displacements vanish. Hence, the boundary conditions at $z = 0$ are:

$$(4.85)\qquad b_1(0) = \left\langle \psi_{1,1} \right\rangle_{1\times 8}^{\mathrm{T}} \left\{ \overline{G}_1 \right\}_{8\times 1} + T_1(0) - 4rT_4(0)^3 = \left\langle \psi_{1,1} \right\rangle_{1\times 8}^{\mathrm{T}} \left\{ \overline{G}_1 \right\}_{8\times 1} = 0$$

$$\text{where } \left\langle \psi_{1,1} \right\rangle_{1\times 8}^{\mathrm{T}} = \left\langle \overline{\overline{A}}_1(0) \right\rangle_{1\times 8}^{\mathrm{T}} = \left\langle 0 \mid \overline{\overline{A}}_{1,1.1} \mid 0 \mid \overline{\overline{A}}_{1,1.2} \mid \overline{\overline{A}}_{1,5} \mid \overline{\overline{A}}_{1,6} \mid \overline{\overline{A}}_{1,7} \mid \overline{\overline{A}}_{1,8} \right\rangle_{1\times 8}$$

$$(4.86)\qquad c_1(0) = \left\langle \psi_{1,2} \right\rangle_{1\times 8}^{\mathrm{T}} \left\{ \overline{G}_1 \right\}_{8\times 1} + T_3(0)^2 + T_4(0)^4 = \left\langle \psi_{1,2} \right\rangle_{1\times 8}^{\mathrm{T}} \left\{ \overline{G}_1 \right\}_{8\times 1} = 0$$

where

$$\langle \psi_{1,2} \rangle_{1\times 8}^{\mathrm{T}} = -\left\langle \bar{\bar{D}}_1(0) \right\rangle_{1\times 8}^{\mathrm{T}} = \left\langle -\bar{\bar{D}}_{1,1.1} \;\vdots\; 0 \;\vdots\; -\bar{\bar{D}}_{1,1.2} \;\vdots\; 0 \;\vdots\; -\bar{\bar{D}}_{1,5} \;\vdots\; -\bar{\bar{D}}_{1,6} \;\vdots\; -\bar{\bar{D}}_{1,7} \;\vdots\; -\bar{\bar{D}}_{1,8} \right\rangle_{1\times 8}$$

(4.87) $$g_1(0) = \langle \psi_{1,3} \rangle_{1\times 8}^{\mathrm{T}} \{\bar{G}_1\}_{8\times 1} + T_5(0)^2 + T_4(0)^4 = \langle \psi_{1,3} \rangle_{1\times 8}^{\mathrm{T}} \{\bar{G}_1\}_{8\times 1} = 0$$

where $\langle \psi_{1,3} \rangle = \langle e_1(0) \rangle_{1\times 8}^{\mathrm{T}} = \langle 1 \;\vdots\; 0 \;\vdots\; 0 \;\vdots\; 0 \;\vdots\; 1 \;\vdots\; 1 \;\vdots\; 1 \;\vdots\; 1 \rangle_{1\times 8}$

(4.88) $$g_1'(0) = \langle \psi_{1,4} \rangle_{1\times 8}^{\mathrm{T}} \{\bar{G}_1\}_{8\times 1} + 2T_5(0) + 4T_4(0)^3 = 0$$

where $\langle \psi_{1,4} \rangle_{1\times 8}^{\mathrm{T}} = \left\langle e_1'(0) \right\rangle_{1\times 8}^{\mathrm{T}} = \langle 0 \;\vdots\; 1 \;\vdots\; 0 \;\vdots\; 0 \;\vdots\; m_{1,5} \;\vdots\; m_{1,6} \;\vdots\; m_{1,7} \;\vdots\; m_{1,8} \rangle_{1\times 8}$

At the free end of the pipe, none of the displacements are specified. Therefore, the natural boundary conditions given in Eq. 4.15 must be applied. From Eq. 4.15j, noting that $\delta b_1(\ell) \neq 0$, the first boundary condition becomes:

(4.89) $$C_1 b_1'(\ell) - C_3 c_1(\ell) + C_3 g_1(\ell) = 0$$

Now substitute the vector expressions for the total displacements $b_1'(\ell), c_1(\ell)$ and $g_1(\ell)$ from Eq. 4.84 into Eq. 4.89.

(4.90) $$C_1\left[b_{1h}'(\ell) + T_1 - 12rT_4\ell^2\right] - C_3\left[g_{1h}(\ell) + T_3\ell^2 + T_4\ell^4\right] + C_3\left[c_{1h}(\ell) + T_5\ell^2 + T_4\ell^4\right] = 0$$

The boundary equation in Eq. 4.90 is written in vector form as:

(4.91) $$\langle \psi_{1,5} \rangle_{1\times 8}^{\mathrm{T}} \{\bar{G}_1\}_{8\times 1} = -C_1\left(T_1 - 12rT_4\ell^2\right) + C_3\left(T_3 - T_5\right)\ell^2$$

where

$$
(4.92)\quad \{\psi_{1,5}\}_{8\times 1} = \left\{ \begin{array}{c} C_3\left(1+\bar{\bar{D}}_{1,1.1}\right) \\ C_3\ell\left(1+\bar{\bar{D}}_{1,2.1}\right) \\ C_1\bar{\bar{A}}_{1,2}+C_3\left(\ell^2+\bar{\bar{D}}_{1,1.2}+\bar{\bar{D}}_{1,3}\ell^2\right) \\ 2C_1\bar{\bar{A}}_{1,3}\ell+C_3\ell\left(\ell^2+\bar{\bar{D}}_{1,2.2}+\bar{\bar{D}}_{1,4}\ell^2\right) \\ \left[C_1 m_{1,5}\bar{\bar{A}}_{1,5}+C_3\left(1+\bar{\bar{D}}_{1,5}\right)\right]e^{m_{1,5}\ell} \\ \left[C_1 m_{1,6}\bar{\bar{A}}_{1,6}+C_3\left(1+\bar{\bar{D}}_{1,6}\right)\right]e^{m_{1,6}\ell} \\ \left[C_1 m_{1,7}\bar{\bar{A}}_{1,7}+C_3\left(1+\bar{\bar{D}}_{1,7}\right)\right]e^{m_{1,7}\ell} \\ \left[C_1 m_{1,8}\bar{\bar{A}}_{1,8}+C_3\left(1+\bar{\bar{D}}_{1,8}\right)\right]e^{m_{1,8}\ell} \end{array} \right\}_{8\times 1}
$$

Similarly, from Eq. 4.15k, with $\delta c_1(\ell)\neq 0$, the boundary equation is:

$$
(4.93a)\quad C_5 b_1(\ell)+(C_4+C_9)c_1'(\ell)-C_9 g_1'(\ell)=0
$$

$$
(4.93b)\quad \begin{array}{l} C_5\left[b_{1h}(\ell)+T_1\ell-4rT_4\ell^3\right]+(rC_5+C_9)\left[c_{1h}'(\ell)+2T_3\ell+4T_4\ell^3\right] \\ \quad -C_9\left[g_{1h}'(\ell)+2T_5\ell+4T_4\ell^3\right]=0 \end{array}
$$

$$
(4.93c)\quad \langle\psi_{1,6}\rangle_{1\times 8}^{\mathrm{T}}\{\bar{G}_1\}_{8\times 1}=-C_5\ell\left(T_1-4rT_4\ell^2\right)-2\ell(rC_5+C_9)\left(T_3+2T_4\ell^2\right)+2C_9\ell\left(T_5+2T_4\ell^2\right)
$$

where

$$
(4.93d)\quad \{\psi_{1,6}\}_{8\times 1} = \left\{ \begin{array}{c} 0 \\ C_5\bar{\bar{A}}_{1,1.1}-(C_4+C_9)\bar{\bar{D}}_{1,2.1}-C_9 \\ C_5\bar{\bar{A}}_{1,2}\ell-2(C_4+C_9)\bar{\bar{D}}_{1,3}\ell-2C_9\ell \\ C_5\left(\bar{\bar{A}}_{1,1.2}+\bar{\bar{A}}_{1,3}\ell^2\right)-(C_4+C_9)\left(\bar{\bar{D}}_{1,2.2}+3\bar{\bar{D}}_{1,4}\ell^2\right)-3C_9\ell^2 \\ \left[C_5\bar{\bar{A}}_{1,5}-(C_4+C_9)m_{1,5}\bar{\bar{D}}_{1,5}-C_9 m_{1,5}\right]e^{m_{1,5}\ell} \\ \left[C_5\bar{\bar{A}}_{1,6}-(C_4+C_9)m_{1,6}\bar{\bar{D}}_{1,6}-C_9 m_{1,6}\right]e^{m_{1,6}\ell} \\ \left[C_5\bar{\bar{A}}_{1,7}-(C_4+C_9)m_{1,7}\bar{\bar{D}}_{1,7}-C_9 m_{1,7}\right]e^{m_{1,7}\ell} \\ \left[C_5\bar{\bar{A}}_{1,8}-(C_4+C_9)m_{1,8}\bar{\bar{D}}_{1,8}-C_9 m_{1,8}\right]e^{m_{1,8}\ell} \end{array} \right\}_{8\times 1}
$$

The boundary condition from Eq. 4.15o.ii, with $\delta g_1(\ell) \neq 0$, gives (note that Eq. 4.15o.ii is used because no point load is present and $P = 0$):

(4.94a) $$-C_6 g_1'''(\ell) - (C_8 + C_9) c_1'(\ell) + (C_8 + C_9) g_1'(\ell) = 0$$

(4.94b) $$\begin{aligned} &-C_6\left[g_{1h}'''(\ell) + 24T_4\ell\right] - (C_8 + C_9)\left[c_{1h}'(\ell) + 2T_3\ell + 4T_4\ell^3\right] \\ &+ (C_8 + C_9)\left[g_{1h}'(\ell) + 2T_5\ell + 4T_4\ell^3\right] = 0 \end{aligned}$$

(4.94c) $$\langle \psi_{1,7} \rangle_{1\times 8}^{\mathrm{T}} \{\bar{G}_1\}_{8\times 1} = 24C_6T_4\ell + 2\ell(C_8 + C_9)(T_3 + 2T_4\ell^2) - 2\ell(C_8 + C_9)(T_5 + 2T_4\ell^2)$$

where

(4.94d) $$\{\psi_{1,7}\}_{8\times 1} = \begin{Bmatrix} 0 \\ (C_8 + C_9)\left(\bar{\bar{D}}_{1,2.1} + 1\right) \\ 2(C_8 + C_9)\left(\bar{\bar{D}}_{1,3} + 1\right)\ell \\ -6C_6 + (C_8 + C_9)\left(3\bar{\bar{D}}_{1,4}\ell^2 + \bar{\bar{D}}_{1,2.2} + 3\ell^2\right) \\ \left[-C_6 m_{1,5}{}^3 + (C_8 + C_9) m_{1,5}\left(\bar{\bar{D}}_{1,5} + 1\right)\right] e^{m_{1,5}\ell} \\ \left[-C_6 m_{1,6}{}^3 + (C_8 + C_9) m_{1,6}\left(\bar{\bar{D}}_{1,6} + 1\right)\right] e^{m_{1,6}\ell} \\ \left[-C_6 m_{1,7}{}^3 + (C_8 + C_9) m_{1,7}\left(\bar{\bar{D}}_{1,7} + 1\right)\right] e^{m_{1,7}\ell} \\ \left[-C_6 m_{1,8}{}^3 + (C_8 + C_9) m_{1,8}\left(\bar{\bar{D}}_{1,8} + 1\right)\right] e^{m_{1,8}\ell} \end{Bmatrix}_{8\times 1}$$

The final boundary condition is obtained from Eq. 4.15p, noting that $\delta g_1'(\ell) \neq 0$:

(4.95a) $$C_6 g_1''(\ell) + C_8 c_1(\ell) - C_8 g_1(\ell) = 0$$

(4.95b) $$C_6\left[g_{1h}''(\ell) + 2T_5 + 12T_4\ell^2\right] + C_8\left[c_{1h}(\ell) + T_3\ell^2 + T_4\ell^4\right] - C_8\left[g_{1h}(\ell) + T_5\ell^2 + T_4\ell^4\right] = 0$$

(4.95c) $$\langle \psi_{1,8} \rangle_{1\times 8}^{\mathrm{T}} \{\bar{G}_1\}_{8\times 1} = -2C_6(T_5 + 6T_4\ell^2) - C_8(T_3 - T_5)\ell^2$$

where

$$(4.95d)\quad \{\psi_{1,8}\}_{8\times1} = \begin{Bmatrix} -C_8\left(\bar{\bar{D}}_{1,1.1}+1\right) \\ -C_8\left(\bar{\bar{D}}_{1,2.1}+1\right)\ell \\ 2C_6 - C_8\left(\bar{\bar{D}}_{1,1.2}+\bar{\bar{D}}_{1,3}\ell^2+\ell^2\right) \\ 6C_6\ell - C_8\left(\bar{\bar{D}}_{1,2.2}+\bar{\bar{D}}_{1,4}\ell^2+\ell^2\right)\ell \\ \left[C_6 m_{1,5}{}^2 - C_8\left(\bar{\bar{D}}_{1,5}+1\right)\right]e^{m_{1,5}\ell} \\ \left[C_6 m_{1,6}{}^2 - C_8\left(\bar{\bar{D}}_{1,6}+1\right)\right]e^{m_{1,6}\ell} \\ \left[C_6 m_{1,7}{}^2 - C_8\left(\bar{\bar{D}}_{1,7}+1\right)\right]e^{m_{1,7}\ell} \\ \left[C_6 m_{1,8}{}^2 - C_8\left(\bar{\bar{D}}_{1,8}+1\right)\right]e^{m_{1,8}\ell} \end{Bmatrix}_{8\times1}$$

The boundary equations developed in Eqs. 4.85 through 4.95 can be written in a consolidated matrix form:

$$(4.96)\quad \left[\Psi_{1,1}\right]_{8\times8}\left\{\bar{G}_1\right\}_{8\times1} = \left\{Q_{1,1}\right\}_{8\times1}$$

where

$$\left[\Psi_{1,1}\right]_{8\times8} = \begin{bmatrix} \langle\psi_{1,1}\rangle^{\mathrm{T}}_{1\times8} \\ \langle\psi_{1,2}\rangle^{\mathrm{T}}_{1\times8} \\ \langle\psi_{1,3}\rangle^{\mathrm{T}}_{1\times8} \\ \langle\psi_{1,4}\rangle^{\mathrm{T}}_{1\times8} \\ \langle\psi_{1,5}\rangle^{\mathrm{T}}_{1\times8} \\ \langle\psi_{1,6}\rangle^{\mathrm{T}}_{1\times8} \\ \langle\psi_{1,7}\rangle^{\mathrm{T}}_{1\times8} \\ \langle\psi_{1,8}\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix}_{8\times8}$$

and

$$\{Q_{1,1}\}_{8\times1} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -C_1\left(T_1 - 12rT_4\ell^2\right) + C_3\left(T_3 - T_5\right)\ell^2 \\ -C_5\ell\left(T_1 - 4rT_4\ell^2\right) - 2\ell\left(rC_5 + C_9\right)\left(T_3 + 2T_4\ell^2\right) + 2C_9\ell\left(T_5 + 2T_4\ell^2\right) \\ 24C_6T_4\ell + 2\ell\left(C_8 + C_9\right)\left(T_3 + 2T_4\ell^2\right) - 2\ell\left(C_8 + C_9\right)\left(T_5 + 2T_4\ell^2\right) \\ -2C_6\left(T_5 + 6T_4\ell^2\right) - C_8\left(T_3 - T_5\right)\ell^2 \end{Bmatrix}$$

The components of the vectors $\langle \psi_{1,i} \rangle^{\mathrm{T}}_{1\times8}$ are defined in Eqs. 4.85 through 4.95 and all of the components of $\{Q_{1,1}\}_{8\times1}$ are known. Hence, the only unknowns in Eq. 4.96 are the elements of the vector $\{\bar{G}_1\}_{8\times1}$. Solving the equation for these unknowns gives:

$$\text{(4.97)} \quad \{\bar{G}_1\}_{8\times1} = \begin{Bmatrix} \bar{G}_{1,1} \\ \bar{G}_{1,2} \\ \bar{G}_{1,3} \\ \bar{G}_{1,4} \\ \bar{G}_{1,5} \\ \bar{G}_{1,6} \\ \bar{G}_{1,7} \\ \bar{G}_{1,8} \end{Bmatrix}_{8\times1} = \begin{Bmatrix} -1.8449\times10^{-6}\, m \\ -6.7798\times10^{-6} \\ -7.8644\times10^{-5}\, m^{-1} \\ 1.2999\times10^{-5}\, m^{-2} \\ 4.1076\times10^{-75} + 7.7211\times10^{-76} i\, m \\ 4.1076\times10^{-75} - 7.7211\times10^{-76} i\, m \\ 9.2245\times10^{-7} - 1.0316\times10^{-6} i\, m \\ 9.2245\times10^{-7} + 1.0316\times10^{-6} i\, m \end{Bmatrix}_{8\times1}$$

With vector $\{\bar{G}_1\}_{8\times1}$ now completely defined, displacements $b_{1h}(\ell)$, $c_{1h}(\ell)$ and $g_{1h}(\ell)$ at the tip of the pipe can be calculated from Eq. 4.84:

$$\text{(4.98a)} \quad b_1(\ell) = b_{1h}(\ell) + b_{1p}(\ell) = \left\langle \bar{\bar{A}}_1(\ell) \right\rangle_{1\times8} \{\bar{G}_1\}_{8\times1} + T_1\ell - 4rT_4\ell^3 = 0.041611\, mm$$

$$\text{(4.98b)} \quad c_1(\ell) = c_{1h}(\ell) + c_{1p}(\ell) = -\left\langle \bar{\bar{D}}_1(\ell) \right\rangle_{1\times8} \{\bar{G}_1\}_{8\times1} + T_3\ell^2 + T_4\ell^4 = -0.63949\, mm$$

$$\text{(4.98c)} \quad g_1(\ell) = g_{1h}(\ell) + g_{1p}(\ell) = \left\langle \bar{e}_1(\ell) \right\rangle_{1\times8} \{\bar{G}_1\}_{8\times1} + T_5\ell^2 + T_4\ell^4 = -0.63950\, mm$$

The displacement of any point on the pipe mid-surface at the free end, $z = \ell$, is found by substituting Eq. 4.98 into Eq. 2.11:

(4.99a) $u(\ell,\phi) = b_1(\ell)\sin\phi = (0.041611\,mm)\sin\phi$

(4.99b) $v(\ell,\phi) = c_1(\ell)\cos\phi = (-0.63949\,mm)\cos\phi$

(4.99c) $w(\ell,\phi) = g_1(\ell)\sin\phi = (-0.63950\,mm)\sin\phi$

The vertical and longitudinal displacement at Point *A*, *B* and *C* on Figure 4.4 are found by evaluating Eq. 4.99 at the values of $\phi = \pi/2$, $\phi = \pi$ and $\phi = 3\pi/2$, respectively. Eq. 4.99a gives the longitudinal displacement at all three points while Eq. 4.99b gives the vertical displacement at Point *B* and Eq. 4.99c gives the vertical displacement at Points *A* and *C*. The displacements are summarized in Table 4.11 below (recall that positive vertical displacement is downward).

**Table 4.11** – Vertical and horizontal displacement for Points *A*, *B* and *C* under pipe self weight

| Location | Vertical Displacement | Longitudinal Displacement |
|---|---|---|
| Point *A* | 0.63950 *mm* | 0.041611 *mm* |
| Point *B* | 0.63949 *mm* | 0 *mm* |
| Point *C* | 0.63950 *mm* | -0.041611 *mm* |

The Euler-Bernoulli beam theory gives the vertical displacement of the tip of the cantilever under its self weight as:

(4.100) $$\Delta_{EB} = \frac{q\ell^4}{8EI}$$

For the case of pipe self weight, the load term $q$ is given as:

(4.101) $$q = 2\pi t r \gamma_s = 2\pi(0.006\,m)(0.200\,m)(78\,kN/m^3) = 0.588\,kN/m = 0.588\,N/mm$$

Hence, the Euler-Bernoulli vertical tip displacement is:

(4.102) $$\Delta_{EB} = \frac{q\ell^4}{8EI} = \frac{(0.588\,N/mm)(4000\,mm)^4}{8(200{,}000MPa)(150.83\times10^6\,mm^4)} = 0.62386\,mm$$

As with the previous case of a cantilever subjected to a point load at the tip, the Euler-Bernoulli beam theory does not account for cross-section deformation. The vertical displacement predicted from this theory is equal for all Points *A*, *B* and *C* in Figure 4.4 and is given in Eq. 4.102.

A shell finite element analysis of the self weight problem was also conducted in ABAQUS. The model parameters, including the element mesh, were identical to those used in Section 4.2 except that the loading was altered. The point load at the tip of the cantilever was removed and replaced with a body force acting on each shell element in the downward direction. The magnitude of the body force was $q = t\gamma_s = 0.468\, kN/m^2$.

The finite element model was able to capture cross-section deformation and provide specific values for each of Points *A*, *B* and *C* in Figure 4.4. The results are summarized in Table 4.12 below.

**Table 4.12** – Displacements of Points *A*, *B* and *C* as determined by ABAQUS shell FEA for pipe self weight

| Location | Vertical Displacement $\Delta_{Vert,FEA}$ | Longitudinal Displacement $\Delta_{Long,FEA}$ |
|---|---|---|
| Point *A* | 0.64554 *mm* | 0.042016 *mm* |
| Point *B* | 0.64553 *mm* | 0 *mm* |
| Point *C* | 0.64554 *mm* | -0.042016 *mm* |

By comparison of Tables 4.11 and 4.12, it is clear that the method developed in the current study provides results very comparable to those obtained with a shell finite element analysis. The difference between the vertical displacements is 0.9% while the difference between the horizontal displacements is 1.0%. Both models provide a more flexible model that the Euler-Bernoulli beam theory, resulting in a larger predicted vertical deflection.

As illustrated in the two examples presented in this Chapter, the closed-form solution using the methodology developed in this study is rather lengthy. From a computational viewpoint, however, it requires significantly less effort than a shell finite element analysis. In the current method, a total of eight degrees of freedom were enough to capture the behaviour of the problem within the assumptions of the current theory (two nodes with four degrees of freedom per node). In contrast, the shell finite element analysis used a total of 16,480 degrees of freedom (3,296 nodes each with five degrees of freedom per node). The following chapter presents a method to automate the solution procedure used in the examples in this chapter.

# CHAPTER 5: FINITE ELEMENT FORMULATION

The solution derived in Chapter 3 is applicable only to the simple problem of a pipe with defined boundary conditions and subjected to general loads. The objective of the current chapter is to extend the solution so that it can be utilized in a finite element application. A solution is developed in a finite element context by expressing the variation of the total potential energy of the system in terms of nodal displacements and evoking the stationarity conditions.

## 5.1 Displacement Fields in Terms of Nodal Displacements

The displacement field contribution for mode $n=0$ is given in Eq. 3.12, while Eqs. 3.31 and 3.37 give the displacement field contributions for modes $n=1$ and $n\geq 2$, respectively, for displacements $a_n(z)$, $d_n(z)$ and $f_n(z)$ and Eqs. 3.36 and 3.41 give the displacement field contributions for modes $n=1$ and $n\geq 2$, respectively, for displacements $b_n(z)$, $c_n(z)$ and $g_n(z)$. The development of the finite element solution to the problem requires the displacement field contributions to be written in terms of the nodal displacements of the element. In the present study, the relationship between the displacement field contributions and the nodal displacements is established by formulating the exact shape functions for the element. This is a difference from previous finite element formulations in which approximate polynomial shape functions are postulated to relate the displacements within the element to nodal displacement.

### *5.1.1 Nodal Displacements*

The nodal displacement vector for the axi-symmetric displacement contribution corresponding to Fourier mode $n=0$ is defined as:

$$\langle\Delta_0\rangle_{1\times 8}^{\mathrm{T}}=\langle a_0(0)\quad c_0(0)\quad f_0(0)\quad f_0'(0)\quad a_0(\ell)\quad c_0(\ell)\quad f_0(\ell)\quad f_0'(\ell)\rangle \tag{5.1}$$

The nodal displacement vector for the general case of mode $n\geq 1$ is defined as:

(5.2) $$\langle \Delta_n \rangle^{\mathrm{T}}_{1\times16} = \left\langle \langle \Delta_{n,1} \rangle^{\mathrm{T}}_{1\times8} \;\middle|\; \langle \Delta_{n,2} \rangle^{\mathrm{T}}_{1\times8} \right\rangle$$

in which

(5.3a) $$\langle \Delta_{n,1} \rangle^{\mathrm{T}}_{1\times8} = \langle a_n(0) \quad d_n(0) \quad f_n(0) \quad f_n'(0) \quad a_n(\ell) \quad d_n(\ell) \quad f_n(\ell) \quad f_n'(\ell) \rangle$$

(5.3b) $$\langle \Delta_{n,2} \rangle^{\mathrm{T}}_{1\times8} = \langle b_n(0) \quad c_n(0) \quad g_n(0) \quad g_n'(0) \quad b_n(\ell) \quad c_n(\ell) \quad g_n(\ell) \quad g_n'(\ell) \rangle$$

### *5.1.2 Relating Integration Constants to Nodal Displacements: Exact Shape Functions*

The vectors of unknown integration constants, $\{\bar{A}_0\}_{8\times1}$, $\{\bar{F}_1\}_{8\times1}$, $\{\bar{F}_n\}_{8\times1}$, $\{\bar{G}_1\}_{8\times1}$ and $\{\bar{G}_n\}_{8\times1}$, are eliminated from the displacement field equations by expressing the constants in terms of the nodal displacements. For a given mode $n$, the relationship between the constants and the nodal displacements is established by introducing matrix $[L_n]_{8\times8}$. For mode $n=0$:

(5.4) $$\{\Delta_0\}_{8\times1} = [L_0]_{8\times8} \{\bar{A}_0\}_{8\times1}$$

The entries of $[L_0]_{8\times8}$ are determined by evaluating the mid-surface displacement functions from Eq. 3.12 at $z=0$ and $z=\ell$ in Eq 5.1.

$$(5.5) \qquad [L_0]_{8\times8} = \begin{bmatrix} \langle e_0(0)\rangle^{\mathrm{T}}_{1\times8} \\ \langle \bar{\bar{C}}_0(0)\rangle^{\mathrm{T}}_{1\times8} \\ \langle \bar{\bar{F}}_0(0)\rangle^{\mathrm{T}}_{1\times8} \\ \langle \bar{\bar{F}}_0'(0)\rangle^{\mathrm{T}}_{1\times8} \\ \langle e_0(\ell)\rangle^{\mathrm{T}}_{1\times8} \\ \langle \bar{\bar{C}}_0(\ell)\rangle^{\mathrm{T}}_{1\times8} \\ \langle \bar{\bar{F}}_0(\ell)\rangle^{\mathrm{T}}_{1\times8} \\ \langle \bar{\bar{F}}_0'(\ell)\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix}_{8\times8}$$

Evaluate the vectors in Eq. 3.12 at the values of $z=0$ and $z=\ell$ and substitute the result into Eq. 5.5 to obtain the expanded version of $[L_0]_{8\times8}$:

$$(5.6) \qquad [L_0]_{8\times8} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \bar{\bar{F}}_{0,1} & \bar{\bar{F}}_{0,3} & \bar{\bar{F}}_{0,4} & \bar{\bar{F}}_{0,5} & \bar{\bar{F}}_{0,6} & 0 & 0 \\ 0 & 0 & m_{0,3}\bar{\bar{F}}_{0,3} & m_{0,4}\bar{\bar{F}}_{0,4} & m_{0,5}\bar{\bar{F}}_{0,5} & m_{0,6}\bar{\bar{F}}_{0,6} & 0 & 0 \\ 1 & \ell & e^{m_{0,3}\ell} & e^{m_{0,4}\ell} & e^{m_{0,5}\ell} & e^{m_{0,6}\ell} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ell \\ 0 & \bar{\bar{F}}_{0,1} & \bar{\bar{F}}_{0,3}e^{m_{0,3}\ell} & \bar{\bar{F}}_{0,4}e^{m_{0,4}\ell} & \bar{\bar{F}}_{0,5}e^{m_{0,5}\ell} & \bar{\bar{F}}_{0,6}e^{m_{0,6}\ell} & 0 & 0 \\ 0 & 0 & m_{0,3}\bar{\bar{F}}_{0,3}e^{m_{0,3}\ell} & m_{0,4}\bar{\bar{F}}_{0,4}e^{m_{0,4}\ell} & m_{0,5}\bar{\bar{F}}_{0,5}e^{m_{0,5}\ell} & m_{0,6}\bar{\bar{F}}_{0,6}e^{m_{0,6}\ell} & 0 & 0 \end{bmatrix}$$

The nodal displacements corresponding to $a_n(z)$, $d_n(z)$ and $f_n(z)$ for modes $n \geq 1$ are established in a similar manner:

$$(5.7a) \qquad \{\Delta_{1,1}\}_{8\times1} = [L_{1,1}]_{8\times8}\{\bar{F}_1\}_{8\times1}$$

(5.7b) $\left\{\Delta_{n,1}\right\}_{8\times1}=\left[L_{n,1}\right]_{8\times8}\left\{\bar{F}_n\right\}_{8\times1} \qquad n=2,3,\ldots,\alpha$

The elements of matrices $\left[L_{1,1}\right]_{8\times8}$ and $\left[L_{n,1}\right]_{8\times8}$ are determined by evaluating the mid-surface displacement functions in Eqs. 3.31 and 3.37, respectively, at the values of $z$ corresponding to the nodal displacements in Eq 5.2.

$$\left[L_{n,1}\right]_{8\times8}=\begin{bmatrix} \left\langle \bar{\bar{A}}_n(0)\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle \bar{\bar{D}}_n(0)\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle e_n(0)\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle e_n'(0)\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle \bar{\bar{A}}_n(\ell)\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle \bar{\bar{D}}_n(\ell)\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle e_n(\ell)\right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle e_n'(\ell)\right\rangle_{1\times8}^{\mathrm{T}} \end{bmatrix}_{8\times8,n\geq1} \tag{5.8}$$

For the case of $n=1$, evaluate the vectors in Eq. 3.31 at the values of $z=0$ and $z=\ell$ and substitute the result into Eq. 5.8 to obtain:

$$\left[L_{1,1}\right]_{8\times8}=\begin{bmatrix} 0 & \bar{\bar{A}}_{1,1.1} & 0 & \bar{\bar{A}}_{1,1.2} & \bar{\bar{A}}_{1,5} & \bar{\bar{A}}_{1,6} & \bar{\bar{A}}_{1,7} & \bar{\bar{A}}_{1,8} \\ \bar{\bar{D}}_{1,1.1} & 0 & \bar{\bar{D}}_{1,1.2} & 0 & \bar{\bar{D}}_{1,5} & \bar{\bar{D}}_{1,6} & \bar{\bar{D}}_{1,7} & \bar{\bar{D}}_{1,8} \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & m_{1,5} & m_{1,6} & m_{1,7} & m_{1,8} \\ 0 & \bar{\bar{A}}_{1,1.1} & \bar{\bar{A}}_{1,2}\ell & \bar{\bar{A}}_{1,1.2}+\bar{\bar{A}}_{1,3}\ell^2 & \bar{\bar{A}}_{1,5}e^{m_{1,5}\ell} & \bar{\bar{A}}_{1,6}e^{m_{1,6}\ell} & \bar{\bar{A}}_{1,7}e^{m_{1,7}\ell} & \bar{\bar{A}}_{1,8}e^{m_{1,8}\ell} \\ \bar{\bar{D}}_{1,1.1} & \bar{\bar{D}}_{1,2.1}\ell & \bar{\bar{D}}_{1,1.2}+\bar{\bar{D}}_{1,3}\ell^2 & \bar{\bar{D}}_{1,2.2}\ell+\bar{\bar{D}}_{1,4}\ell^3 & \bar{\bar{D}}_{1,5}e^{m_{1,5}\ell} & \bar{\bar{D}}_{1,6}e^{m_{1,6}\ell} & \bar{\bar{D}}_{1,7}e^{m_{1,7}\ell} & \bar{\bar{D}}_{1,8}e^{m_{1,8}\ell} \\ 1 & \ell & \ell^2 & \ell^3 & e^{m_{1,5}\ell} & e^{m_{1,6}\ell} & e^{m_{1,7}\ell} & e^{m_{1,8}\ell} \\ 0 & 1 & 2\ell & 3\ell^2 & m_{1,5}e^{m_{1,5}\ell} & m_{1,6}e^{m_{1,6}\ell} & m_{1,7}e^{m_{1,7}\ell} & m_{1,8}e^{m_{1,8}\ell} \end{bmatrix} \tag{5.9}$$

For modes $n \geq 2$ substitute the values of $z=0$ and $z=\ell$ into Eq. 3.37 and insert the result into Eq. 5.8 to obtain:

$$(5.10)\quad \left[L_{n,1}\right]_{8\times8}=\begin{bmatrix} \bar{\bar{A}}_{n,1} & \bar{\bar{A}}_{n,2} & \bar{\bar{A}}_{n,3} & \bar{\bar{A}}_{n,4} & \bar{\bar{A}}_{n,5} & \bar{\bar{A}}_{n,6} & \bar{\bar{A}}_{n,7} & \bar{\bar{A}}_{n,8} \\ \bar{\bar{D}}_{n,1} & \bar{\bar{D}}_{n,2} & \bar{\bar{D}}_{n,3} & \bar{\bar{D}}_{n,4} & \bar{\bar{D}}_{n,5} & \bar{\bar{D}}_{n,6} & \bar{\bar{D}}_{n,7} & \bar{\bar{D}}_{n,8} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & m_{n,5} & m_{n,6} & m_{n,7} & m_{n,8} \\ \bar{\bar{A}}_{n,1}e^{m_{n,1}\ell} & \bar{\bar{A}}_{n,2}e^{m_{n,2}\ell} & \bar{\bar{A}}_{n,3}e^{m_{n,3}\ell} & \bar{\bar{A}}_{n,4}e^{m_{n,4}\ell} & \bar{\bar{A}}_{n,5}e^{m_{n,5}\ell} & \bar{\bar{A}}_{n,6}e^{m_{n,6}\ell} & \bar{\bar{A}}_{n,7}e^{m_{n,7}\ell} & \bar{\bar{A}}_{n,8}e^{m_{n,8}\ell} \\ \bar{\bar{D}}_{n,1}e^{m_{n,1}\ell} & \bar{\bar{D}}_{n,2}e^{m_{n,2}\ell} & \bar{\bar{D}}_{n,3}e^{m_{n,3}\ell} & \bar{\bar{D}}_{n,4}e^{m_{n,4}\ell} & \bar{\bar{D}}_{n,5}e^{m_{n,5}\ell} & \bar{\bar{D}}_{n,6}e^{m_{n,6}\ell} & \bar{\bar{D}}_{n,7}e^{m_{n,7}\ell} & \bar{\bar{D}}_{n,8}e^{m_{n,8}\ell} \\ e^{m_{n,1}\ell} & e^{m_{n,2}\ell} & e^{m_{n,3}\ell} & e^{m_{n,4}\ell} & e^{m_{n,5}\ell} & e^{m_{n,6}\ell} & e^{m_{n,7}\ell} & e^{m_{n,8}\ell} \\ m_{n,1}e^{m_{n,1}\ell} & m_{n,2}e^{m_{n,2}\ell} & m_{n,3}e^{m_{n,3}\ell} & m_{n,4}e^{m_{n,4}\ell} & m_{n,5}e^{m_{n,5}\ell} & m_{n,6}e^{m_{n,6}\ell} & m_{n,7}e^{m_{n,7}\ell} & m_{n,8}e^{m_{n,8}\ell} \end{bmatrix}$$

In a similar way, the nodal displacements corresponding to $b(z)$, $c(z)$ and $g(z)$ for modes $n \geq 1$ are:

$$(5.11a)\quad \left\{\Delta_{1,2}\right\}_{8\times1}=\left[L_{1,2}\right]_{8\times8}\left\{\bar{G}_1\right\}_{8\times1}$$

$$(5.11b)\quad \left\{\Delta_{n,2}\right\}_{8\times1}=\left[L_{n,2}\right]_{8\times8}\left\{\bar{G}_n\right\}_{8\times1} \qquad n=2,3,\ldots,\alpha$$

The entries of matrices $\left[L_{1,2}\right]_{8\times8}$ and $\left[L_{n,2}\right]_{8\times8}$ are determined by evaluating the mid-surface displacement functions in Eqs. 3.36 and 3.41, respectively, at the values of $z$ corresponding to the nodal displacements in Eq 5.2.

$$(5.12)\quad \left[L_{n,2}\right]_{8\times 8} = \begin{bmatrix} \left\langle \bar{\bar{A}}_n(0) \right\rangle^{\mathrm{T}}_{1\times 8} \\ -\left\langle \bar{\bar{D}}_n(0) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle e_n(0) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle e'_n(0) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle \bar{\bar{A}}_n(\ell) \right\rangle^{\mathrm{T}}_{1\times 8} \\ -\left\langle \bar{\bar{D}}_n(\ell) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle e_n(\ell) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle e'_n(\ell) \right\rangle^{\mathrm{T}}_{1\times 8} \end{bmatrix}_{8\times 8, n\geq 1}$$

For the case of $n=1$, evaluate the vectors in Eq. 3.36 at the values of $z=0$ and $z=\ell$ and substitute the result into Eq. 5.12 to obtain:

$$(5.13)\quad \left[L_{1,2}\right]_{8\times 8} = \begin{bmatrix} 0 & \bar{\bar{A}}_{1,1.1} & 0 & \bar{\bar{A}}_{1,1.2} & \bar{\bar{A}}_{1,5} & \bar{\bar{A}}_{1,6} & \bar{\bar{A}}_{1,7} & \bar{\bar{A}}_{1,8} \\ -\bar{\bar{D}}_{1,1.1} & 0 & -\bar{\bar{D}}_{1,1.2} & 0 & -\bar{\bar{D}}_{1,5} & -\bar{\bar{D}}_{1,6} & -\bar{\bar{D}}_{1,7} & -\bar{\bar{D}}_{1,8} \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & m_{1,5} & m_{1,6} & m_{1,7} & m_{1,8} \\ 0 & \bar{\bar{A}}_{1,1.1} & \bar{\bar{A}}_{1,2}\ell & \bar{\bar{A}}_{1,1.2}+\bar{\bar{A}}_{1,3}\ell^2 & \bar{\bar{A}}_{1,5}e^{m_{1,5}\ell} & \bar{\bar{A}}_{1,6}e^{m_{1,6}\ell} & \bar{\bar{A}}_{1,7}e^{m_{1,7}\ell} & \bar{\bar{A}}_{1,8}e^{m_{1,8}\ell} \\ -\bar{\bar{D}}_{1,1.1} & -\bar{\bar{D}}_{1,2.1}\ell & -\bar{\bar{D}}_{1,1.2}-\bar{\bar{D}}_{1,3}\ell^2 & -\bar{\bar{D}}_{1,2.2}\ell-\bar{\bar{D}}_{1,4}\ell^3 & -\bar{\bar{D}}_{1,5}e^{m_{1,5}\ell} & -\bar{\bar{D}}_{1,6}e^{m_{1,6}\ell} & -\bar{\bar{D}}_{1,7}e^{m_{1,7}\ell} & -\bar{\bar{D}}_{1,8}e^{m_{1,8}\ell} \\ 1 & \ell & \ell^2 & \ell^3 & e^{m_{1,5}\ell} & e^{m_{1,6}\ell} & e^{m_{1,7}\ell} & e^{m_{1,8}\ell} \\ 0 & 1 & 2\ell & 3\ell^2 & m_{1,5}e^{m_{1,5}\ell} & m_{1,6}e^{m_{1,6}\ell} & m_{1,7}e^{m_{1,7}\ell} & m_{1,8}e^{m_{1,8}\ell} \end{bmatrix}$$

For modes $n\geq 2$, substitute the values of $z=0$ and $z=\ell$ into Eq. 3.41 and insert the result into Eq. 5.12 to obtain:

$$(5.14)\quad \left[L_{n,2}\right]_{8\times8}=\begin{bmatrix} \bar{\bar{A}}_{n,1} & \bar{\bar{A}}_{n,2} & \bar{\bar{A}}_{n,3} & \bar{\bar{A}}_{n,4} & \bar{\bar{A}}_{n,5} & \bar{\bar{A}}_{n,6} & \bar{\bar{A}}_{n,7} & \bar{\bar{A}}_{n,8} \\ -\bar{\bar{D}}_{n,1} & -\bar{\bar{D}}_{n,2} & -\bar{\bar{D}}_{n,3} & -\bar{\bar{D}}_{n,4} & -\bar{\bar{D}}_{n,5} & -\bar{\bar{D}}_{n,6} & -\bar{\bar{D}}_{n,7} & -\bar{\bar{D}}_{n,8} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & m_{n,5} & m_{n,6} & m_{n,7} & m_{n,8} \\ \bar{\bar{A}}_{n,1}e^{m_{n,1}\ell} & \bar{\bar{A}}_{n,2}e^{m_{n,2}\ell} & \bar{\bar{A}}_{n,3}e^{m_{n,3}\ell} & \bar{\bar{A}}_{n,4}e^{m_{n,4}\ell} & \bar{\bar{A}}_{n,5}e^{m_{n,5}\ell} & \bar{\bar{A}}_{n,6}e^{m_{n,6}\ell} & \bar{\bar{A}}_{n,7}e^{m_{n,7}\ell} & \bar{\bar{A}}_{n,8}e^{m_{n,8}\ell} \\ -\bar{\bar{D}}_{n,1}e^{m_{n,1}\ell} & -\bar{\bar{D}}_{n,2}e^{m_{n,2}\ell} & -\bar{\bar{D}}_{n,3}e^{m_{n,3}\ell} & -\bar{\bar{D}}_{n,4}e^{m_{n,4}\ell} & -\bar{\bar{D}}_{n,5}e^{m_{n,5}\ell} & -\bar{\bar{D}}_{n,6}e^{m_{n,6}\ell} & -\bar{\bar{D}}_{n,7}e^{m_{n,7}\ell} & -\bar{\bar{D}}_{n,8}e^{m_{n,8}\ell} \\ e^{m_{n,1}\ell} & e^{m_{n,2}\ell} & e^{m_{n,3}\ell} & e^{m_{n,4}\ell} & e^{m_{n,5}\ell} & e^{m_{n,6}\ell} & e^{m_{n,7}\ell} & e^{m_{n,8}\ell} \\ m_{n,1}e^{m_{n,1}\ell} & m_{n,2}e^{m_{n,2}\ell} & m_{n,3}e^{m_{n,3}\ell} & m_{n,4}e^{m_{n,4}\ell} & m_{n,5}e^{m_{n,5}\ell} & m_{n,6}e^{m_{n,6}\ell} & m_{n,7}e^{m_{n,7}\ell} & m_{n,8}e^{m_{n,8}\ell} \end{bmatrix}$$

With the non-singular matrices $\left[L_0\right]_{8\times8}$, $\left[L_{n,1}\right]_{8\times8}$ and $\left[L_{n,2}\right]_{8\times8}$ now defined, the unknown integration constants, $\left\{\bar{A}_0\right\}_{8\times1}$, $\left\{\bar{F}_n\right\}_{8\times1}$ and $\left\{\bar{G}_n\right\}_{8\times1}$ can be written in terms of the nodal displacements. For mode $n=0$:

$$(5.15)\quad \left\{\bar{A}_0\right\}_{8\times1}=\left[L_0\right]_{8\times8}^{-1}\left\{\Delta_0\right\}_{8\times1}$$

For mode $n=1$:

$$(5.16a)\quad \left\{\bar{F}_1\right\}_{8\times1}=\left[L_{1,1}\right]_{8\times8}^{-1}\left\{\Delta_{1,1}\right\}_{8\times1} \qquad (5.16b)\quad \left\{\bar{G}_1\right\}_{8\times1}=\left[L_{1,2}\right]_{8\times8}^{-1}\left\{\Delta_{1,2}\right\}_{8\times1}$$

For modes $n\geq2$:

$$(5.17a)\quad \left\{\bar{F}_n\right\}_{8\times1}=\left[L_{n,1}\right]_{8\times8}^{-1}\left\{\Delta_{n,1}\right\}_{8\times1} \qquad (5.17b)\quad \left\{\bar{G}_n\right\}_{8\times1}=\left[L_{n,2}\right]_{8\times8}^{-1}\left\{\Delta_{n,2}\right\}_{8\times1}$$

With the integration constants known in terms of the nodal displacements, the constants $\left\{\bar{A}_0\right\}_{8\times1}$, $\left\{\bar{F}_n\right\}_{8\times1}$ and $\left\{\bar{G}_n\right\}_{8\times1}$ can be eliminated from the displacement field contribution equations in Eqs. 3.12, 3.31, 3.36, 3.37 and 3.41. Hence, the mid-surface displacement contributions of the element for each mode $n$ can be written in terms of the nodal displacement. From Eqs. 3.12a and 5.15, the displacement contributions for $n=0$ are:

$$\text{(5.18)} \qquad \begin{Bmatrix} a_0(z) \\ c_0(z) \\ f_0(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{e}_0(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle \bar{\bar{C}}_0(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle \bar{\bar{F}}_0(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix} [L_0]^{-1}_{8\times8} \{\Delta_0\}_{8\times1} = \begin{bmatrix} \left\langle N_{A,0}(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle N_{C,0}(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle N_{F,0}(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix} \{\Delta_0\}_{8\times1}$$

From Eqs. 3.31, 3.37, 5.16a and 5.17a, the displacement contributions for $a_n(z)$, $d_n(z)$ and $f_n(z)$ for $n \geq 1$, are:

$$\text{(5.19)} \qquad \begin{Bmatrix} a_1(z) \\ d_1(z) \\ f_1(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{\bar{A}}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle \bar{\bar{D}}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle \bar{e}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix} \left[L_{1,1}\right]^{-1}_{8\times8} \left\{\Delta_{1,1}\right\}_{8\times1} = \begin{bmatrix} \left\langle N_{A,1}(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle N_{D,1}(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle N_{F,1}(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix} \left\{\Delta_{1,1}\right\}_{8\times1}$$

$$\text{(5.20)} \qquad \begin{Bmatrix} a_n(z) \\ d_n(z) \\ f_n(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{\bar{A}}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle \bar{\bar{D}}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle \bar{e}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix} \left[L_{n,1}\right]^{-1}_{8\times8} \left\{\Delta_{n,1}\right\}_{8\times1} = \begin{bmatrix} \left\langle N_{A,n}(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle N_{D,n}(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle N_{F,n}(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix} \left\{\Delta_{n,1}\right\}_{8\times1}$$

The displacement contributions for $b_n(z)$, $c_n(z)$ and $g_n(z)$ for modes $n \geq 1$ are established by substituting Eqs. 5.16b and 5.17b into Eqs. 3.36 and 3.41, respectively:

$$\text{(5.21)} \qquad \begin{Bmatrix} b_1(z) \\ c_1(z) \\ g_1(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{\bar{A}}_1(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ -\left\langle \bar{\bar{D}}_1(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle \bar{e}_1(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix} \left[L_{1,2}\right]^{-1}_{8\times8} \left\{\Delta_{1,2}\right\}_{8\times1} = \begin{bmatrix} \left\langle N_{B,1}(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle N_{C,1}(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle N_{G,1}(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix} \left\{\Delta_{1,2}\right\}_{8\times1}$$

$$(5.22)\qquad \begin{Bmatrix} b_n(z) \\ c_n(z) \\ g_n(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{\bar{A}}_n(z) \right\rangle_{1\times8}^{\mathrm{T}} \\ -\left\langle \bar{\bar{D}}_n(z) \right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle \bar{e}_n(z) \right\rangle_{1\times8}^{\mathrm{T}} \end{bmatrix} \left[ L_{n,2} \right]_{8\times8}^{-1} \left\{ \Delta_{n,2} \right\}_{8\times1} = \begin{bmatrix} \left\langle N_{B,n}(z) \right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle N_{C,n}(z) \right\rangle_{1\times8}^{\mathrm{T}} \\ \left\langle N_{G,n}(z) \right\rangle_{1\times8}^{\mathrm{T}} \end{bmatrix} \left\{ \Delta_{n,2} \right\}_{8\times1}$$

In Eqs. 5.18 through 5.22, vectors $\left\langle N_{A,0}(z) \right\rangle_{1\times8}^{\mathrm{T}}$, $\left\langle N_{C,0}(z) \right\rangle_{1\times8}^{\mathrm{T}}$, $\left\langle N_{F,0}(z) \right\rangle_{1\times8}^{\mathrm{T}}$, $\left\langle N_{A,n}(z) \right\rangle_{1\times8}^{\mathrm{T}}$, $\left\langle N_{D,n}(z) \right\rangle_{1\times8}^{\mathrm{T}}$, $\left\langle N_{F,n}(z) \right\rangle_{1\times8}^{\mathrm{T}}$, $\left\langle N_{B,n}(z) \right\rangle_{1\times8}^{\mathrm{T}}$, $\left\langle N_{C,n}(z) \right\rangle_{1\times8}^{\mathrm{T}}$ and $\left\langle N_{G,n}(z) \right\rangle_{1\times8}^{\mathrm{T}}$ consist of exact shape function for the respective displacement contributions corresponding to mode $n$ and are functions only of $z$.

## 5.2 <u>Variation of Potential Energy in Terms of Nodal Displacements</u>

Recall that the total potential energy of an element is the difference between the internal strain energy of the element and work done by the applied loads:

$$(5.23)\qquad \pi = U - W$$

The internal strain energy, $U$, is given in Table 2.1 while the work done by the external loads, $W$, is given in Eq. 2.16. As in Section 2.7, equilibrium of the system is ensured by setting the variation the total potential energy with respect to each function to zero:

$$(5.24)\qquad \delta\pi = \delta U - \delta W = 0$$

However, as opposed to the procedure used in Chapter 2, the variation of the total potential energy will now be written in terms of the nodal displacements.

### *5.2.1 Internal Strain Energy*

For simplicity, consider each internal strain energy term in Table 2.1 individually.

The first term of the internal strain energy is:

$$(5.25)\qquad U_1 = \frac{1}{2} C_1 \int_{z=0}^{l} \left[ 2\left(a_0'(z)\right)^2 + \sum_{n=1}^{\alpha} \left(a_n'(z)\right)^2 + \sum_{n=1}^{\alpha} \left(b_n'(z)\right)^2 \right] dz$$

Take the variation Eq 5.25 with respect to each variable to obtain:

(5.26a) $$\delta U_1 = \frac{C_1}{2}\int_{z=0}^{l}\left[4a_0'(z)\delta a_0'(z)+\sum_{n=1}^{\alpha}2a_n'(z)\delta a_n'(z)+\sum_{n=1}^{\alpha}2b_n'(z)\delta b_n'(z)\right]dz$$

(5.26b) $$\delta U_1 = \frac{1}{2}C_1\int_{z=0}^{l}4a_0'(z)\delta a_0'(z)+\sum_{n=1}^{\alpha}\frac{1}{2}C_1\int_{z=0}^{l}\left[2a_n'(z)\delta a_n'(z)+2b_n'(z)\delta b_n'(z)\right]dz$$

Note that in Eq. 5.26 the order of summation and integration are interchangeable.

Substitute the expressions for $a_n(z)$ and $b_n(z)$ from Eqs. 5.18 through 5.22 into Eq. 5.26b to write the variation of internal strain energy in terms of the nodal displacements.

(5.27) $$\begin{aligned}\delta U_{1,n} &= \frac{1}{2}C_1\int_{z=0}^{l}4\left[\langle\delta\Delta_0\rangle_{1\times8}^{\mathrm{T}}\{N_{A,0}'(z)\}_{8\times1}\langle N_{A,0}'(z)\rangle_{1\times8}^{\mathrm{T}}\{\Delta_0\}_{8\times1}\right]dz\\ &+\sum_{n=1}^{\alpha}\frac{1}{2}C_1\int_{z=0}^{l}2\left[\langle\delta\Delta_{n,1}\rangle_{1\times8}^{\mathrm{T}}\{N_{A,n}'(z)\}_{8\times1}\langle N_{A,n}'(z)\rangle_{1\times8}^{\mathrm{T}}\{\Delta_{n,1}\}_{8\times1}\right.\\ &\left.+\langle\delta\Delta_{n,2}\rangle_{1\times8}^{\mathrm{T}}\{N_{B,n}'(z)\}_{8\times1}\langle N_{B,n}'(z)\rangle_{1\times8}^{\mathrm{T}}\{\Delta_{n,2}\}_{8\times1}\right]dz\end{aligned}$$

Eq. 5.27 can be rewritten as:

(5.28) $$\delta U_{1,n} = \langle\delta\Delta_0\rangle_{1\times8}^{\mathrm{T}}\left[K_{1,0}\right]_{8\times8}\{\Delta_0\}_{8\times1}+\sum_{n=1}^{\alpha}\left[\langle\delta\Delta_{n,1}\rangle_{1\times8}^{\mathrm{T}}\left[K_{1,n,1}\right]_{8\times8}\{\Delta_{n,1}\}_{8\times1}+\langle\delta\Delta_{n,2}\rangle_{1\times8}^{\mathrm{T}}\left[K_{1,n,2}\right]_{8\times8}\{\Delta_{n,2}\}_{8\times1}\right]$$

In Eq. 5.28 $\left[K_{1,0}\right]_{8\times8}$ and $\left[K_{1,n}\right]_{8\times8}$ are the contributions to the total stiffness matrix from the first term of the internal strain energy for modes $n=0$ and $n\geq1$, respectively:

(5.29) $$\left[K_{1,0}\right]_{8\times8} = \left[\int_{z=0}^{l}2C_1\left(\{N_{A,0}'(z)\}_{8\times1}\langle N_{A,0}'(z)\rangle_{1\times8}^{\mathrm{T}}\right)dz\right]_{8\times8}$$

(5.30) $$\left[K_{1,n,1}\right]_{8\times8} = \left[\int_{z=0}^{l}C_1\{N_{A,n}'(z)\}_{8\times1}\langle N_{A,n}'(z)\rangle_{1\times8}^{\mathrm{T}}dz\right]_{8\times8}$$

(5.31) $$\left[K_{1,n,2}\right]_{8\times8} = \left[\int_{z=0}^{l}C_1\{N_{B,n}'(z)\}_{8\times1}\langle N_{B,n}'(z)\rangle_{1\times8}^{\mathrm{T}}dz\right]_{8\times8}$$

A similar manipulation can be done to the other seven internal strain energy contributions given in Table 2.1. The resulting stiffness matrix contributions for Fourier mode $n = 0$ are:

(5.32a) $$\left[K_{2,0}\right]_{8\times8} = \left[\int_{z=0}^{l} 2C_2\left(\left\{N_{F,0}(z)\right\}_{8\times1}\left\langle N_{F,0}(z)\right\rangle_{1\times8}^{\mathrm{T}}\right)dz\right]_{8\times8}$$

(5.32b) $$\left[K_{3,0}\right]_{8\times8} = \left[\int_{z=0}^{l} 2C_3\left(\left\{N'_{A,0}(z)\right\}_{8\times1}\left\langle N_{F,0}(z)\right\rangle_{1\times8}^{\mathrm{T}} + \left\{N_{F,0}(z)\right\}_{8\times1}\left\langle N'_{A,0}(z)\right\rangle_{1\times8}^{\mathrm{T}}\right)dz\right]_{8\times8}$$

(5.32c) $$\left[K_{4,0}\right]_{8\times8} = \left[\int_{z=0}^{l} 2C_4\left(\left\{N'_{C,0}(z)\right\}_{8\times1}\left\langle N'_{C,0}(z)\right\rangle_{1\times8}^{\mathrm{T}}\right)dz\right]_{8\times8}$$

(5.32d) $$\left[K_{5,0}\right]_{8\times8} = \left[\int_{z=0}^{l} 2C_6\left(\left\{N''_{F,0}(z)\right\}_{8\times1}\left\langle N''_{F,0}(z)\right\rangle_{1\times8}^{\mathrm{T}}\right)dz\right]_{8\times8}$$

(5.32e) $$\left[K_{6,0}\right]_{8\times8} = [\mathbf{0}]_{8\times8}$$

(5.32f) $$\left[K_{7,0}\right]_{8\times8} = [\mathbf{0}]_{8\times8}$$

(5.32g) $$\left[K_{8,0}\right]_{8\times8} = \left[\int_{z=0}^{l} 2C_9\left(\left\{N'_{C,0}(z)\right\}_{8\times1}\left\langle N'_{C,0}(z)\right\rangle_{1\times8}^{\mathrm{T}}\right)dz\right]_{8\times8}$$

The remaining stiffness matrix contributions for each Fourier mode $n \geq 1$ are:

(5.33a) $$\left[K_{2,n,1}\right]_{8\times8} = \left[\begin{array}{l}\int_{z=0}^{l} C_2\Big[\left\{N_{F,n}(z)\right\}_{8\times1}\left\langle N_{F,n}(z)\right\rangle_{1\times8}^{\mathrm{T}} + n\Big(\left\{N_{D,n}(z)\right\}_{8\times1}\left\langle N_{F,n}(z)\right\rangle_{1\times8}^{\mathrm{T}} + \\ \qquad \left\{N_{F,n}(z)\right\}_{8\times1}\left\langle N_{D,n}(z)\right\rangle_{1\times8}^{\mathrm{T}}\Big) + n^2\left\{N_{D,n}(z)\right\}_{8\times1}\left\langle N_{D,n}(z)\right\rangle_{1\times8}^{\mathrm{T}}\Big]dz\end{array}\right]_{8\times8}$$

(5.33b) $$\left[K_{2,n,2}\right]_{8\times8} = \left[\begin{array}{l}\int_{z=0}^{l} C_2\Big[\left\{N_{G,n}(z)\right\}_{8\times1}\left\langle N_{G,n}(z)\right\rangle_{1\times8}^{\mathrm{T}} - n\Big(\left\{N_{C,n}(z)\right\}_{8\times1}\left\langle N_{G,n}(z)\right\rangle_{1\times8}^{\mathrm{T}} + \\ \qquad \left\{N_{G,n}(z)\right\}_{8\times1}\left\langle N_{C,n}(z)\right\rangle_{1\times8}^{\mathrm{T}}\Big) + n^2\left\{N_{C,n}(z)\right\}_{8\times1}\left\langle N_{C,n}(z)\right\rangle_{1\times8}^{\mathrm{T}}\Big]dz\end{array}\right]_{8\times8}$$

(5.33c) $$\left[K_{3,n,1}\right]_{8\times8} = \left[\begin{array}{l}\int_{z=0}^{l} C_3\Big[\left\{N'_{A,n}(z)\right\}_{8\times1}\left\langle N_{F,n}(z)\right\rangle_{1\times8}^{\mathrm{T}} + \left\{N_{F,n}(z)\right\}_{8\times1}\left\langle N'_{A,n}(z)\right\rangle_{1\times8}^{\mathrm{T}} + \\ \qquad n\Big(\left\{N'_{A,n}(z)\right\}_{8\times1}\left\langle N_{D,n}(z)\right\rangle_{1\times8}^{\mathrm{T}} + \left\{N_{D,n}(z)\right\}_{8\times1}\left\langle N'_{A,n}(z)\right\rangle_{1\times8}^{\mathrm{T}}\Big)\Big]dz\end{array}\right]_{8\times8}$$

(5.33d) $$\left[K_{3,n,2}\right]_{8\times8}=\left[\begin{array}{l}\int\limits_{z=0}^{l}C_3\Big[\left\{N'_{B,n}(z)\right\}_{8\times1}\left\langle N_{G,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\left\{N_{G,n}(z)\right\}_{8\times1}\left\langle N'_{B,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}-\\ \qquad n\Big(\left\{N'_{B,n}(z)\right\}_{8\times1}\left\langle N_{C,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\left\{N_{C,n}(z)\right\}_{8\times1}\left\langle N'_{B,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big)\Big]dz\end{array}\right]_{8\times8}$$

(5.33e) $$\left[K_{4,n,1}\right]_{8\times8}=\left[\begin{array}{l}\int\limits_{z=0}^{l}\Big[C_4\left\{N'_{D,n}(z)\right\}_{8\times1}\left\langle N'_{D,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}-C_5n\Big(\left\{N_{A,n}(z)\right\}_{8\times1}\left\langle N'_{D,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\\ \qquad\left\{N'_{D,n}(z)\right\}_{8\times1}\left\langle N_{A,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big)+\dfrac{C_4n^2}{r^2}\left\{N_{A,n}(z)\right\}_{8\times1}\left\langle N_{A,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big]dz\end{array}\right]_{8\times8}$$

(5.33f) $$\left[K_{4,n,2}\right]_{8\times8}=\left[\begin{array}{l}\int\limits_{z=0}^{l}\Big[C_4\left\{N'_{C,n}(z)\right\}_{8\times1}\left\langle N'_{C,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+C_5n\Big(\left\{N_{B,n}(z)\right\}_{8\times1}\left\langle N'_{C,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\\ \qquad\left\{N'_{C,n}(z)\right\}_{8\times1}\left\langle N_{B,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big)+\dfrac{C_4n^2}{r^2}\left\{N_{B,n}(z)\right\}_{8\times1}\left\langle N_{B,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big]dz\end{array}\right]_{8\times8}$$

(5.33g) $$\left[K_{5,n,1}\right]_{8\times8}=\left[\int\limits_{z=0}^{l}C_6\left\{N''_{F,n}(z)\right\}_{8\times1}\left\langle N''_{F,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}dz\right]_{8\times8}$$

(5.33h) $$\left[K_{5,n,2}\right]_{8\times8}=\left[\int\limits_{z=0}^{l}C_6\left\{N''_{G,n}(z)\right\}_{8\times1}\left\langle N''_{G,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}dz\right]_{8\times8}$$

(5.33i) $$\left[K_{6,n,1}\right]_{8\times8}=\left[\begin{array}{l}\int\limits_{z=0}^{l}C_7\Big[n^4\left\{N_{F,n}(z)\right\}_{8\times1}\left\langle N_{F,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+n^3\Big(\left\{N_{D,n}(z)\right\}_{8\times1}\left\langle N_{F,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\\ \qquad\left\{N_{F,n}(z)\right\}_{8\times1}\left\langle N_{D,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big)+n^2\left\{N_{D,n}(z)\right\}_{8\times1}\left\langle N_{D,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big]dz\end{array}\right]_{8\times8}$$

(5.33j) $$\left[K_{6,n,2}\right]_{8\times8}=\left[\begin{array}{l}\int\limits_{z=0}^{l}C_7\Big[n^4\left\{N_{G,n}(z)\right\}_{8\times1}\left\langle N_{G,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}-n^3\Big(\left\{N_{C,n}(z)\right\}_{8\times1}\left\langle N_{G,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\\ \qquad\left\{N_{G,n}(z)\right\}_{8\times1}\left\langle N_{C,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big)+n^2\left\{N_{C,n}(z)\right\}_{8\times1}\left\langle N_{C,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big]dz\end{array}\right]_{8\times8}$$

(5.33k) $$\left[K_{7,n,2}\right]_{8\times8}=\left[\begin{array}{l}\int\limits_{z=0}^{l}C_8\Big[-n^2\Big(\left\{N''_{F,n}(z)\right\}_{8\times1}\left\langle N_{F,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\left\{N_{F,n}(z)\right\}_{8\times1}\left\langle N''_{F,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big)-\\ \qquad n\Big(\left\{N_{D,n}(z)\right\}_{8\times1}\left\langle N''_{F,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\left\{N''_{F,n}(z)\right\}_{8\times1}\left\langle N_{D,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big)\Big]dz\end{array}\right]_{8\times8}$$

(5.33l) $$\left[K_{7,n,2}\right]_{8\times8}=\left[\begin{array}{l}\int\limits_{z=0}^{l}C_8\Big[-n^2\Big(\left\{N''_{G,n}(z)\right\}_{8\times1}\left\langle N_{G,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\left\{N_{G,n}(z)\right\}_{8\times1}\left\langle N''_{G,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big)+\\ \qquad n\Big(\left\{N_{C,n}(z)\right\}_{8\times1}\left\langle N''_{G,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}+\left\{N''_{G,n}(z)\right\}_{8\times1}\left\langle N_{C,n}(z)\right\rangle^{\mathrm{T}}_{1\times8}\Big)\Big]dz\end{array}\right]_{8\times8}$$

(5.33m) $$\left[K_{8,n,1}\right]_{8\times8}=\left[\begin{array}{c}\int_{z=0}^{l} C_9\Big[n^2\{N'_{F,n}(z)\}_{8\times1}\langle N'_{F,n}(z)\rangle^{\mathrm{T}}_{1\times8}+n\Big(\{N'_{D,n}(z)\}_{8\times1}\langle N'_{F,n}(z)\rangle^{\mathrm{T}}_{1\times8}+\\ \{N'_{F,n}(z)\}_{8\times1}\langle N'_{D,n}(z)\rangle^{\mathrm{T}}_{1\times8}\Big)+\{N'_{D,n}(z)\}_{8\times1}\langle N'_{D,n}(z)\rangle^{\mathrm{T}}_{1\times8}\Big]dz\end{array}\right]_{8\times8}$$

(5.33n) $$\left[K_{8,n,2}\right]_{8\times8}=\left[\begin{array}{c}\int_{z=0}^{l} C_9\Big[n^2\{N'_{G,n}(z)\}_{8\times1}\langle N'_{G,n}(z)\rangle^{\mathrm{T}}_{1\times8}-n\Big(\{N'_{C,n}(z)\}_{8\times1}\langle N'_{G,n}(z)\rangle^{\mathrm{T}}_{1\times8}+\\ \{N'_{G,n}(z)\}_{8\times1}\langle N'_{C,n}(z)\rangle^{\mathrm{T}}_{1\times8}\Big)+\{N'_{C,n}(z)\}_{8\times1}\langle N'_{C,n}(z)\rangle^{\mathrm{T}}_{1\times8}\Big]dz\end{array}\right]_{8\times8}$$

The total variation of the internal strain energy of the element is now written as:

(5.34) $$\delta U=\langle\delta\Delta_0\rangle^{\mathrm{T}}_{1\times8}[K_0]_{8\times8}\{\Delta_0\}_{8\times1}+\sum_{n=1}^{\alpha}\Big[\langle\delta\Delta_{n,1}\rangle^{\mathrm{T}}_{1\times8}\left[K_{n,1}\right]_{8\times8}\{\Delta_{n,1}\}_{8\times1}+\langle\delta\Delta_{n,2}\rangle^{\mathrm{T}}_{1\times8}\left[K_{n,2}\right]_{8\times8}\{\Delta_{n,2}\}_{8\times1}\Big]$$

in which the stiffness matrices $[K_0]_{8\times8}$, $\left[K_{n,1}\right]_{8\times8}$ and $\left[K_{n,2}\right]_{8\times8}$ are:

(5.35) $$[K_0]_{8\times8}=\sum_{j=1}^{8}\left[K_{j,0}\right]_{8\times8}$$

(5.36) $$\left[K_{n,1}\right]_{8\times8}=\sum_{j=1}^{8}\left[K_{j,n,1}\right]_{8\times8}$$

(5.37) $$\left[K_{n,2}\right]_{8\times8}=\sum_{j=1}^{8}\left[K_{j,n,2}\right]_{8\times8}$$

In Eq. 5.35 matrices $\left[K_{j,0}\right]_{8\times8}$, defined in Eqs. 5.29 and 5.32, are the stiffness contributions of $U_j$ (for $j=1$ to 8) as defined in Table 2.1 to the total stiffness matrix, $[K_0]_{8\times8}$, for mode $n=0$. Similarly, matrices $\left[K_{j,n,1}\right]_{8\times8}$ and $\left[K_{j,n,2}\right]_{8\times8}$ in Eqs. 5.36 and 5.37, defined in Eqs. 5.30, 5.31 and 5.33, are the stiffness contributions of $U_j$ (for $j=1$ to 8) as defined in Table 2.1 to the total stiffness matrices, $\left[K_{n,1}\right]_{8\times8}$ and $\left[K_{n,2}\right]_{8\times8}$ respectively, for each of modes $n\geq1$.

It is apparent from Eq. 5.34 that Each Fourier mode $n$ can be independently considered from other modes $m\neq n$. The result is $\alpha$ independent stiffness matrices (one for each

Fourier mode $n$ with a total of $\alpha$ Fourier modes) and $\alpha$ sets of nodal displacements. The $\alpha$ equations are solved for the $\alpha$ sets of nodal displacements that are then summed to obtain the total nodal displacement.

### *5.2.2 Load Potential Vector*

The total load potential for the externally applied tractions $q_u(z,\phi)$, $q_v(z,\phi)$ and $q_w(z,\phi)$ was derived in Eq. 2.16. Taking the variation of the load potentials with respect to each variable gives:

$$\delta W_u = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_u(z,\phi)\left[\delta a_0(z)+\sum_{n=1}^{\alpha}\cos n\phi\,\delta a_n(z)+\sum_{n=1}^{\alpha}\sin n\phi\,\delta b_n(z)\right]rd\phi dz \tag{5.38a}$$

$$\delta W_v = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_v(z,\phi)\left[\delta c_0(z)+\sum_{n=1}^{\alpha}\cos n\phi\,\delta c_n(z)+\sum_{n=1}^{\alpha}\sin n\phi\,\delta d_n(z)\right]rd\phi dz \tag{5.38b}$$

$$\delta W_w = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_w(z,\phi)\left[\delta f_0(z)+\sum_{n=1}^{\alpha}\cos n\phi\,\delta f_n(z)+\sum_{n=1}^{\alpha}\sin n\phi\,\delta g_n(z)\right]rd\phi dz \tag{5.38c}$$

In Eqs. 5.18 through 5.22 the displacement contributions for each mode $n$ were related to the corresponding nodal displacements. Use these relationships to rewrite the variation of the load potential terms in Eq. 5.38 as:

$$\begin{aligned}\delta W_u = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_u(z,\phi)\Bigg[&\langle\delta\Delta_0\rangle_{1\times 8}^{\mathrm{T}}\{N_{A,0}(z)\}_{8\times 1}+\sum_{n=1}^{\alpha}\cos n\phi\langle\delta\Delta_{n,1}\rangle_{1\times 8}^{\mathrm{T}}\{N_{A,n}(z)\}_{8\times 1}\\ &+\sum_{n=1}^{\alpha}\sin n\phi\langle\delta\Delta_{n,2}\rangle_{1\times 8}^{\mathrm{T}}\{N_{B,n}(z)\}_{8\times 1}\Bigg]rd\phi dz\end{aligned} \tag{5.39a}$$

$$\begin{aligned}\delta W_v = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_v(z,\phi)\Bigg[&\langle\delta\Delta_0\rangle_{1\times 8}^{\mathrm{T}}\{N_{C,0}(z)\}_{8\times 1}+\sum_{n=1}^{\alpha}\cos n\phi\langle\delta\Delta_{n,2}\rangle_{1\times 8}^{\mathrm{T}}\{N_{C,n}(z)\}_{8\times 1}\\ &+\sum_{n=1}^{\alpha}\sin n\phi\langle\delta\Delta_{n,1}\rangle_{1\times 8}^{\mathrm{T}}\{N_{D,n}(z)\}_{8\times 1}\Bigg]rd\phi dz\end{aligned} \tag{5.39b}$$

$$\delta W_w = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_w(z,\phi)\Bigg[\langle\delta\Delta_0\rangle_{1\times8}^{\mathrm{T}}\{N_{F,0}(z)\}_{8\times1} + \sum_{n=1}^{\alpha}\cos n\phi\langle\delta\Delta_{n,1}\rangle_{1\times8}^{\mathrm{T}}\{N_{F,n}(z)\}_{8\times1} \tag{5.39c}$$

$$+\sum_{n=1}^{\alpha}\sin n\phi\langle\delta\Delta_{n,2}\rangle_{1\times8}^{\mathrm{T}}\{N_{G,n}(z)\}_{8\times1}\Bigg]rd\phi dz$$

As with the internal strain energy terms, the order of the summation and the integration in Eq. 5.39 are interchangeable and each Fourier mode $n$ is considered independently. Combining Eqs. 5.39a through 5.39c gives the total variation of the traction load potential of the element. For mode $n=0$:

$$\delta W_0 = \langle\delta\Delta_0\rangle_{1\times8}^{\mathrm{T}}\{Q_0\}_{8\times1} \tag{5.40}$$

in which

$$\{Q_0\}_{8\times1} = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\Big[q_u(z,\phi)\{N_{A,0}(z)\}_{8\times1} + q_v(z,\phi)\{N_{C,0}(z)\}_{8\times1} + q_w(z,\phi)\{N_{F,0}(z)\}_{8\times1}\Big]rd\phi dz \tag{5.41}$$

For each mode $n\geq1$:

$$\delta W_n = \langle\delta\Delta_{n,1}\rangle_{1\times8}^{\mathrm{T}}\{Q_{n,1}\}_{8\times1} + \langle\delta\Delta_{n,2}\rangle_{1\times8}^{\mathrm{T}}\{Q_{n,2}\}_{8\times1} \tag{5.42}$$

in which

$$\{Q_{n,1}\}_{8\times1} = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\Big[q_u(z,\phi)\{N_{A,n}(z)\}_{8\times1}\cos(n\phi) + q_v(z,\phi)\{N_{D,n}(z)\}_{8\times1}\sin(n\phi) \tag{5.43a}$$

$$+ q_w(z,\phi)\{N_{F,n}(z)\}_{8\times1}\cos(n\phi)\Big]rd\phi dz$$

$$\{Q_{n,2}\}_{8\times1} = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\Big[q_u(z,\phi)\{N_{B,n}(z)\}_{8\times1}\sin(n\phi) + q_v(z,\phi)\{N_{C,n}(z)\}_{8\times1}\cos(n\phi) \tag{5.43b}$$

$$+ q_w(z,\phi)\{N_{G,n}(z)\}_{8\times1}\sin(n\phi)\Big]rd\phi dz$$

Thus, the variation of the total load potential for the element is given by:

$$\delta W = \langle\delta\Delta_0\rangle_{1\times8}^{\mathrm{T}}\{Q_0\}_{8\times1} + \sum_{n=1}^{\alpha}\langle\delta\Delta_{n,1}\rangle_{1\times8}^{\mathrm{T}}\{Q_{n,1}\}_{8\times1} + \sum_{n=1}^{\alpha}\langle\delta\Delta_{n,2}\rangle_{1\times8}^{\mathrm{T}}\{Q_{n,2}\}_{8\times1} \tag{5.44}$$

where the vectors $\{Q_0\}_{8\times1}$ , $\{Q_{n,1}\}_{8\times1}$ and $\{Q_{n,2}\}_{8\times1}$ were defined in Eqs. 5.41 and 5.43.

### 5.2.3 *Total Potential Energy*

The variation of the total potential energy of the element, given in Eq. 5.24, is the difference between the variation of internal strain energy and the variation of the load potential. Substitute the expression for the variation of the internal strain energy from Eq. 5.34 and the expression for the variation of the load potential in 5.44 into Eq. 5.24 to obtain the expression for the variation of the total potential energy of the element:

$$\begin{aligned} \delta\pi = {} & \langle \delta\Delta_0 \rangle_{1\times8}^T \left\{ [K_0]_{8\times8} \{\Delta_0\}_{8\times1} - \{Q_0\}_{8\times1} \right\}_{8\times1} \\ & + \sum_{n=1}^{\alpha} \langle \delta\Delta_{n,1} \rangle_{1\times8}^T \left\{ \left[K_{n,1}\right]_{8\times8} \left\{\Delta_{n,1}\right\}_{8\times1} - \left\{Q_{n,1}\right\}_{8\times1} \right\}_{8\times1} \\ & + \sum_{n=1}^{\alpha} \langle \delta\Delta_{n,2} \rangle_{1\times8}^T \left\{ \left[K_{n,2}\right]_{8\times8} \left\{\Delta_{n,2}\right\}_{8\times1} - \left\{Q_{n,2}\right\}_{8\times1} \right\}_{8\times1} = 0 \end{aligned} \tag{5.45}$$

The equilibrium equations for the system can be recovered from Eq. 5.45. It is apparent from Eq. 5.45 that the contribution of each Fourier mode $n$ is independent of the contribution of each other Fourier mode $m \neq n$. Thus, the equilibrium equations for each mode can be written individually. For mode $n = 0$, noting that the vector $\langle \delta\Delta_0 \rangle_{1\times8}^T$ is non-zero, Eq. 5.45 gives the equilibrium equations as:

$$[K_0]_{8\times8} \{\Delta_0\}_{8\times1} - \{Q_0\}_{8\times1} = 0 \tag{5.46}$$

Similarly, for each of the modes $n \geq 1$, noting that the vectors $\langle \delta\Delta_{n,1} \rangle_{1\times8}^T$ and $\langle \delta\Delta_{n,2} \rangle_{1\times8}^T$ are non-zero, the equilibrium equations are:

$$\left[K_{n,1}\right]_{8\times8} \left\{\Delta_{n,1}\right\}_{8\times1} - \left\{Q_{n,1}\right\}_{8\times1} = 0 \tag{5.47a}$$

$$\left[K_{n,2}\right]_{8\times8} \left\{\Delta_{n,2}\right\}_{8\times1} - \left\{Q_{n,2}\right\}_{8\times1} = 0 \qquad \text{for } n = 1, 2, \ldots, \alpha \tag{5.47b}$$

# CHAPTER 6: VERIFICATION PROBLEMS

In this chapter several simple problems are solved with the finite element procedure developed in Chapter 5 to demonstrate the features, validity and convergence characteristics of the new element. The results from the new element are compared to those based on traditional shell finite element models.

The shell finite element models were created using the S4R shell element in ABAQUS. Details of the element and its characteristics were outlined in Chapter 4. The solution developed in Chapter 5 was implemented with Borland C++. Details of the program implementation are included in Appendix 11.

## 6.1 Example 1: Vertical Point Load Applied to Tip of Cantilever

The example solved in Chapter 4 of a cantilever pipe subjected to a vertical point load applied to its tip is now solved using the finite element developed in Chapter 5. The geometry of the problem, shown in Figure 6.1, is the same as that used in Chapter 4. The loading, geometric and material properties of the pipe are: $P = 16\,kN$, $\ell = 4\,m$, $t = 6\,mm$, $r = 200\,mm$, $\mu = 0.3$, $E = 200{,}000\,MPa$ and $G = E/2(1+\mu) = 76{,}925\,MPa$.

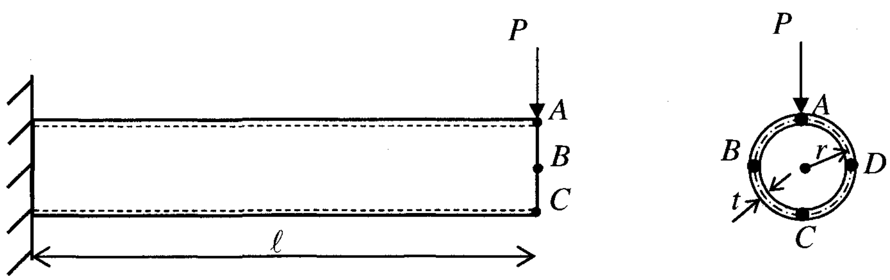


**Figure 6.1** – Example 1: Cantilever pipe with point load

It is required to determine the displacement of Points *A*, *B*, *C* and *D* located on the pipe middle surface at the free end (Figure 6.1).

### 6.1.1 Load Potential Vectors

The loading condition in the current example consists of a point load applied to the element at $z = \ell$ and $\phi = \pi/2$. Hence, the load functions are:

(6.1a) $q_u(z,\phi) = 0$

(6.1b) $q_v(z,\phi) = 0$

(6.1c) $q_w(z,\phi) = -P\delta(z-\ell)\delta\left(\phi - \frac{\pi}{2}\right)$

In Eq. 6.1c, functions $\delta(z-\ell)$ and $\delta\left(\phi-\frac{\pi}{2}\right)$ are the Dirac Delta Functions defined in Section 4.3.1 of Chapter 4.

The load potential vectors were developed in Chapter 5. For mode $n = 0$, Eq. 5.41 gives:

(6.2)
$$\begin{aligned} \{Q_0\}_{8\times1} &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_w(z,\phi)\{N_{F,0}(z)\}_{8\times1} r\,d\phi\,dz \\ &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} -P\delta(z-\ell)\delta\left(\phi-\frac{\pi}{2}\right)\{N_{F,0}(z)\}_{8\times1} d\phi\,dz \\ &= -P\{N_{F,0}(\ell)\}_{8\times1} \end{aligned}$$

For modes $n \geq 1$, the load potential vector $\{Q_{n,1}\}_{8\times1}$ corresponding to displacement contributions $a_n(z)$, $d_n(z)$ and $f_n(z)$ from Eq. 5.43a is:

(6.3)
$$\begin{aligned} \{Q_{n,1}\}_{8\times1} &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_w(z,\phi)\{N_{F,n}(z)\}_{8\times1}\cos(n\phi) r\,d\phi\,dz \\ &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} -P\delta(z-\ell)\delta\left(\phi-\frac{\pi}{2}\right)\{N_{F,n}(z)\}_{8\times1}\cos(n\phi) d\phi\,dz \\ &= -P\{N_{F,n}(\ell)\}_{8\times1}\cos\left(\frac{n\pi}{2}\right) \end{aligned}$$

Similarly, from Eq. 5.43b, the load potential vector $\{Q_{n,2}\}_{8\times1}$ corresponding to displacement contributions $b_n(z)$, $c_n(z)$ and $g_n(z)$ for modes $n \geq 1$ is:

$$\{Q_{n,2}\}_{8\times1} = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_w(z,\phi)\{N_{G,n}(z)\}_{8\times1}\sin(n\phi)rd\phi dz$$

$$(6.4) \qquad = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} -P\delta(z-\ell)\delta\left(\phi-\pi/2\right)\{N_{G,n}(z)\}_{8\times1}\sin(n\phi)d\phi dz$$

$$= -P\{N_{G,n}(\ell)\}_{8\times1}\sin\left(n\pi/2\right)$$

### *6.1.2 Results from Current Finite Element Analysis*

With the load potential vectors completely defined, the nodal displacement contributions for each mode $n$ are determined independently from Eqs. 5.46 and 5.47. The stiffness matrix, $[K_n]_{8\times8}$, for each mode $n$ in Eqs. 5.46 and 5.47 is constructed using the procedure outlined in Chapter 5.

For comparison with the solved example in Chapter 4, the current example is solved with a total of seven Fourier modes (i.e. $\alpha = 6$). The results of the current finite element method, those based on the analytical method in Chapter 4, and those based on the ABAQUS shell finite element analysis, are presented in Table 6.1.

**Table 6.1** – Comparison of results for Current FEA, Analytical Method and ABAQUS shell FEA Example 1

| | | Point $A$ | Point $B$ | Point $C$ | Point $D$ |
|---|---|---|---|---|---|
| Vertical Displacement | Current FEA | 16.317 | 11.265 | 9.380 | 11.265 |
| | Analytical (Ch. 4) | 16.317 | 11.265 | 9.380 | 11.265 |
| | ABAQUS S4R element | 16.755 | 11.392 | 9.599 | 11.392 |
| Longitudinal Displacement | Current FEA | 1.101 | -0.133 | -0.731 | -0.133 |
| | Analytical (Ch. 4) | 1.101 | -0.133 | -0.731 | -0.133 |
| | ABAQUS S4R element | 1.142 | -0.132 | -0.742 | -0.132 |

As expected, the results from the current finite element analysis exactly match those based on the analytical solution in Chapter 4. The equilibrium equations for both methods were established through the same method so it is expected that they would produce the same solution. Both methods provide excellent correlation with the ABAQUS S4R FEA model.

### *6.1.3 Inclined Point Load Applied to Tip of Cantilever*

A variation of the previous example is now presented in which the point load is applied at an angle $\phi = 7\pi/8$ to the horizontal, as shown in Figure 6.2, rather than vertically. The load is again applied in the radial direction with no tangential or axial components present. The magnitude of the load, as well as the geometric and material properties, is the same as those used in Example 1.

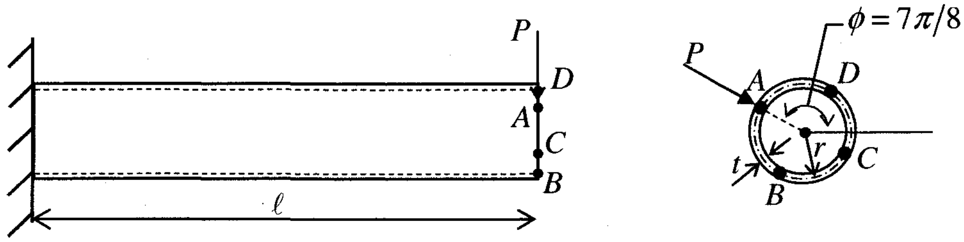


**Figure 6.2** – Inclined point load at tip of cantilever

#### 6.1.3.1 Load Potential Vectors

The procedure for developing the load potential vector is that used in Section 6.1.1 for the vertical load. However, for the case of an inclined load, the load functions are:

(6.5a) $q_u(z,\phi) = 0$

(6.5b) $q_v(z,\phi) = 0$

(6.5c) $q_w(z,\phi) = -P\delta(z-\ell)\delta\left(\phi - \frac{7\pi}{8}\right)$

The resulting load potential vector for mode $n = 0$ from Eq. 5.41 is:

(6.6) $\{Q_0\}_{8\times1} = -P\{N_{F,0}(\ell)\}_{8\times1}$

The load potential vectors for each mode $n \geq 1$ corresponding to displacement contributions $a_n(z)$, $d_n(z)$ and $f_n(z)$ and $b_n(z)$, $c_n(z)$ and $g_n(z)$, respectively, from Eq. 5.43 are:

$$\text{(6.7a)} \quad \{Q_{n,1}\}_{8\times1} = -P\{N_{F,n}(\ell)\}_{8\times1}\cos\left(\frac{7n\pi}{8}\right)$$

$$\text{(6.7b)} \quad \{Q_{n,2}\}_{8\times1} = -P\{N_{G,n}(\ell)\}_{8\times1}\sin\left(\frac{7n\pi}{8}\right)$$

The nodal displacements for modes $n = 0$ and $n \geq 1$ are determined from Eqs. 5.46 and 5.47, respectively, with the load potential vectors given in Eqs. 6.6 and 6.7. The nodal displacement vectors $\{\Delta_{n,1}\}_{8\times1}$ and $\{\Delta_{n,2}\}_{8\times1}$ obtained by solving Eqs. 5.47a and 5.47b are different from the corresponding nodal displacement vectors for the vertically applied point load. However, the total longitudinal, tangential and radial displacements, *u*, *v* and *w*, as determined by Eq. 2.11 are identical. While the displacement contributions $a_n(z)$, $b_n(z),\ldots,\ g_n(z)$ are different for the two loading conditions, the actual total displacements of Points *A*, *B*, *C* and *D* are the same, as expected.

### *6.1.4 Solution with Multiple Finite Elements*

The problem of a vertical point load applied to the top of the tip of a cantilevered pipe was also solved using multiple elements rather than just one. The analysis was conducted with two elements of different length as well as with five elements of equal length. In both cases, the results for the displacement of Points *A*, *B*, *C* and *D* at the free end of the cantilever were identical, to five significant digits, to those obtained using only one element.

These results illustrate that refining the finite element mesh by using a larger number of smaller elements does not increase the accuracy of the analysis and thus is shown to be unnecessary. The additional computational effort of using more elements is not required. A single element produces the same results as multiple elements because the shape functions describing the displacement along the element exactly satisfy the homogeneous form of the equilibrium conditions. In previous finite element work, the shape functions were interpolated with polynomial functions (Ohtsubo and Watanabe, 1978; Bathe and Almeida, 1980; Mitello and Huespe, 1988; Yan et al., 1999). With such interpolation, refining the finite element mesh produces results that are more accurate. However, in the current study the shape functions are exact and refining the mesh is not needed.

The proceeding examples will be modeled with only one element, where appropriate, as no accuracy is gained by using multiple elements.

### *6.1.5 Convergence and Number of Modes*

Convergence toward the "exact" displacement of the pipe is achieved by increasing the number of Fourier modes above the zero mode, $\alpha$, included in the analysis rather than increasing the number of elements. As $\alpha$ increases, the accuracy of the results also increases. The effect of increasing $\alpha$ is shown graphically in Figures 6.3 and 6.4 for both the vertical and longitudinal displacements of Points *A*, *B* and *C* (see Figure 6.1).

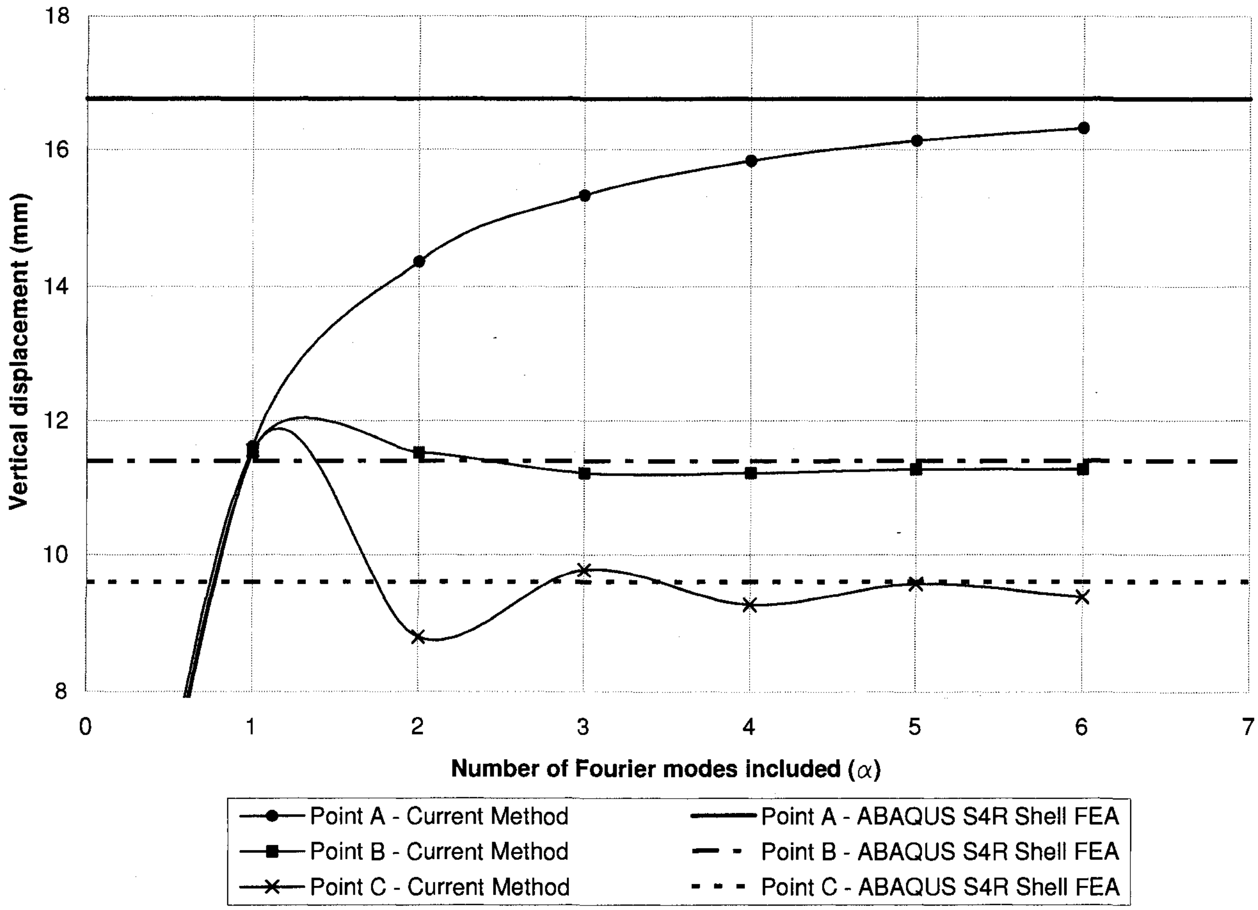


**Figure 6.3** – Cumulative vertical displacements for Points *A*, *B* and *C*

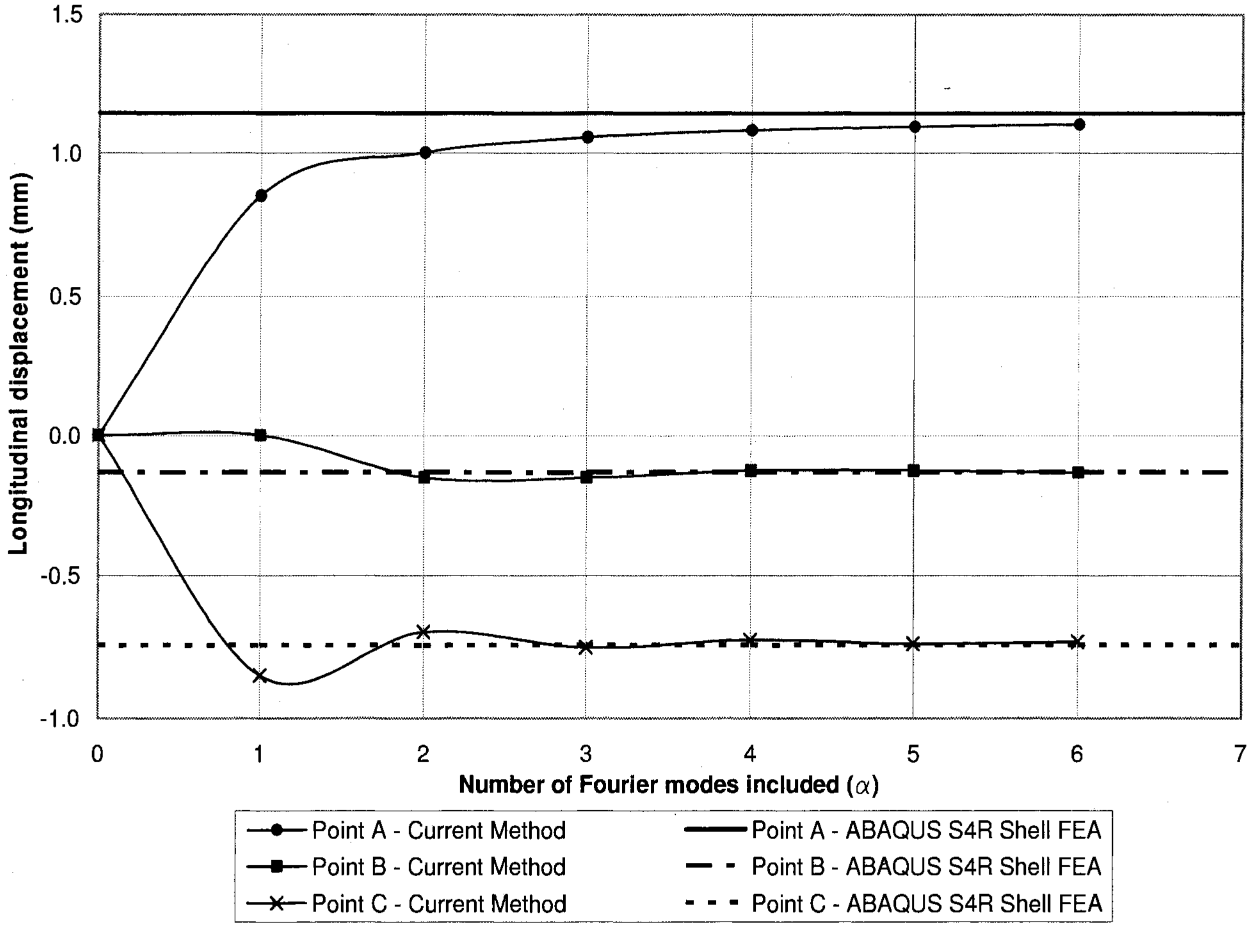


**Figure 6.4** – Cumulative longitudinal displacement for Points *A*, *B* and *C*

It can be seen in the figures that as $\alpha$ increases, the displacement for all three points given by the current analysis approaches the displacement given by the shell finite element analysis.

The benefit of the element developed in the current study is that increased accuracy can be obtained with little additional computational effort. In a traditional shell finite element solution, convergence is achieved by refining the element mesh and increasing the number of degrees of freedom. The mesh is continually refined and the analysis repeated multiple times until convergence is achieved. Successive refinements of the model do not benefit from the results of previous trials.

In contrast, in the current method convergence is achieved by adding the contributions of higher Fourier modes to those of previous Fourier modes to obtain the total displacements, stresses, etc. Thus, the solution based on the sum of the contributions of modes 0, 1,..., $\alpha+1$ is always an improvement over the solution based on the sum of the contributions of modes 0, 1, ..., $\alpha$. The solution based on modes $0,1,\ldots,\alpha+1$ is the solution based on modes $0,1,\ldots,\alpha$ plus the contributions from mode $\alpha+1$.

The element in the current study also benefits in a computational manner from the fact that the displacement contributions from each Fourier mode $n$ are uncoupled from every other Fourier mode $m \neq n$. A total of $\alpha+1$ smaller systems of equations are solved rather than the one larger system of equations that is solved in a traditional shell finite element analysis. The advantages of solving smaller systems of equations are not fully apparent in the current example due to its small size. However, for larger and complex problems, the ability to solve a number of smaller systems of equations efficiently can become a substantial benefit.

### *6.1.6 Stress Analysis*

Once the nodal displacements of an element have been determined, they can be used to calculate the stress at any point within the element. Consider the longitudinal stress in the element, which was given by Hooke's Law in Eq. 2.2a as:

$$\sigma_z(z,\phi,\xi) = \frac{E}{1-\mu^2}\left[\varepsilon_z(z,\phi,\xi) + \mu\varepsilon_\phi(z,\phi,\xi)\right] \tag{6.8}$$

in which Eqs. 2.3a and 2.3b gave $\varepsilon_z(z,\phi,\xi)$ and $\varepsilon_\phi(z,\phi,\xi)$ as:

$$\varepsilon_z(z,\phi,\xi) = \bar{\varepsilon}_z(z,\phi) + \xi\kappa_z(z,\phi) \tag{6.9a}$$

$$\varepsilon_\phi(z,\phi,\xi) = \bar{\varepsilon}_\phi(z,\phi) + \xi\kappa_\phi(z,\phi) \tag{6.9b}$$

As a special case, the longitudinal stress at the pipe middle surface, $\xi = 0$, is considered. Using the relationships for membrane strain from Eq. 2.4, Eq.6.9 reduces to:

$$\varepsilon_z(z,\phi,\xi=0) = \bar{\varepsilon}_z(z,\phi) = \frac{\partial u}{\partial z} \tag{6.10a}$$

$$\varepsilon_\phi(z,\phi,\xi=0) = \bar{\varepsilon}_\phi(z,\phi) \approx \frac{1}{r}\left(w + \frac{\partial v}{\partial \phi}\right) \tag{6.10b}$$

Substitute Eq. 6.10 into Eq. 6.8 to obtain an expression for the longitudinal stress in terms of the displacement at the middle surface of the element:

$$(6.11)\quad \sigma_z(z,\phi)=\frac{E}{1-\mu^2}\left[\frac{\partial u}{\partial z}+\frac{\mu}{r}\left(w+\frac{\partial v}{\partial \phi}\right)\right]$$

Recall from Eq. 2.11 and 2.12:

$$(6.12a)\quad w=f_0(z)+\sum_{n=1}^{\alpha} f_n(z)\cos n\phi+\sum_{n=1}^{\alpha} g_n(z)\sin n\phi$$

$$(6.12b)\quad \frac{\partial u}{\partial z}=a_0'(z)+\sum_{n=1}^{\alpha} a_n'(z)\cos n\phi+\sum_{n=1}^{\alpha} b_n'(z)\sin n\phi$$

$$(6.12c)\quad \frac{\partial v}{\partial \phi}=\sum_{n=1}^{\alpha} -nc_n(z)\sin n\phi+\sum_{n=1}^{\alpha} nd_n(z)\cos n\phi$$

The displacement functions $a(z)$, $b(z)\ldots g(z)$ were defined in terms of the nodal displacements in Chapter 5 through the shape function vectors. Recall from Eqs. 5.18 through 5.22:

$$(6.13a)\quad a_0(z)=\langle N_{A,0}(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_0\}_{8\times 1}\rightarrow a_0'(z)=\langle N_{A,0}'(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_0\}_{8\times 1}$$

$$(6.13b)\quad f_0(z)=\langle N_{F,0}(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_0\}_{8\times 1}$$

$$(6.13c)\quad a_n(z)=\langle N_{A,n}(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_{n,1}\}_{8\times 1}\rightarrow a_n'(z)=\langle N_{A,n}'(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_{n,1}\}_{8\times 1}$$

$$(6.13d)\quad b_n(z)=\langle N_{B,n}(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_{n,2}\}_{8\times 1}\rightarrow b_n'(z)=\langle N_{B,n}'(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_{n,2}\}_{8\times 1}$$

$$(6.13e)\quad c_n(z)=\langle N_{C,n}(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_{n,2}\}_{8\times 1}$$

$$(6.13f)\quad d_n(z)=\langle N_{D,n}(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_{n,1}\}_{8\times 1}$$

$$(6.13g)\quad f_n(z)=\langle N_{F,n}(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_{n,1}\}_{8\times 1}$$

$$(6.13h)\quad g_n(z)=\langle N_{G,n}(z)\rangle_{1\times 8}^{\mathrm{T}}\{\Delta_{n,2}\}_{8\times 1}$$

Substitute Eqs. 6.12 and 6.13 into Eq. 6.11 to express the total longitudinal stress of the pipe middle surface in terms of the nodal displacements:

$$\sigma_z(z,\phi,\xi=0)=\frac{E}{1-\mu^2}\left\langle \langle N'_{A,0}(z)\rangle^{\mathrm{T}}_{1\times8}+\frac{\mu}{r}\langle N_{F,0}(z)\rangle^{\mathrm{T}}_{1\times8}\right\rangle^{\mathrm{T}}_{1\times8}\{\Delta_0\}_{8\times1}$$

$$(6.14)\qquad +\sum_{n=1}^{\alpha}\frac{E\cos n\phi}{1-\mu^2}\left\langle \langle N'_{A,n}(z)\rangle^{\mathrm{T}}_{1\times8}+\frac{n\mu}{r}\langle N_{D,n}(z)\rangle^{\mathrm{T}}_{1\times8}+\frac{\mu}{r}\langle N_{F,n}(z)\rangle^{\mathrm{T}}_{1\times8}\right\rangle^{\mathrm{T}}_{1\times8}\{\Delta_{n,1}\}_{8\times1}$$

$$+\sum_{n=1}^{\alpha}\frac{E\sin n\phi}{1-\mu^2}\left\langle \langle N'_{B,n}(z)\rangle^{\mathrm{T}}_{1\times8}-\frac{n\mu}{r}\langle N_{C,n}(z)\rangle^{\mathrm{T}}_{1\times8}+\frac{\mu}{r}\langle N_{G,n}(z)\rangle^{\mathrm{T}}_{1\times8}\right\rangle^{\mathrm{T}}_{1\times8}\{\Delta_{n,2}\}_{8\times1}$$

The longitudinal stress at the pipe middle surface was determined for Point *A*, at $\phi=\pi/2$, along the length of pipe from $z=0$ to $z=\ell$ using Eq. 6.14. The stress contributions from each mode $n=0$ to $n=6$ were combined to obtain the total stress. The stress contribution from each mode, the total stress and the longitudinal stress at the middle surface as determined by ABAQUS S4R shell finite element analysis are shown in Figure 6.5. Note that the stress contribution for mode $n=0$ is negligible ($<0.7$ MPa) and it is therefore not included in Figure 6.5.

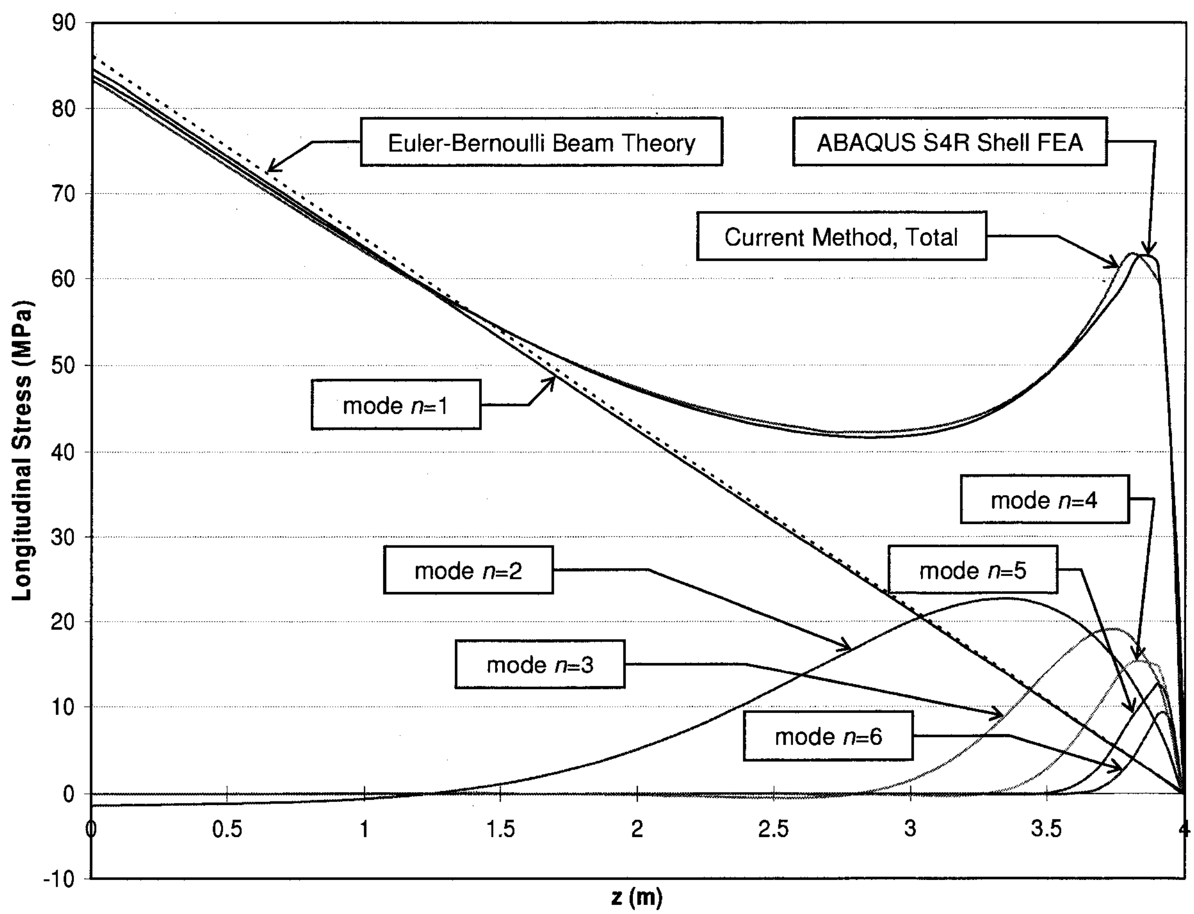


**Figure 6.5** – Longitudinal stress variation of Point *A* along length of pipe

It is apparent from Figure 6.5 that the longitudinal stresses predicted by the current method closely match those determined by the shell finite element. Of particular note is that the current method captures the local stress increase near the applied point load at $z=\ell$. The stress contributions from the higher Fourier modes are responsible for capturing this local stress variation while the lower modes, especially modes $n=1$ and $n=2$, capture the stress variation along the remainder of the length of the pipe. Of interest is to note that the stress converges with as few as seven modes.

The longitudinal stress at Point *A* can be determined from traditional Euler-Bernoulli beam theory as $\sigma = M/S$, where *M* is the bending moment in the pipe and *S* is the elastic modulus of the pipe cross section. The longitudinal stress at Point *A* predicted by the Euler-Bernoulli beam theory along the length of the pipe is also shown in Figure 6.5. As indicated in Chapter 4 (Section 4.3.5.2), the Euler-Bernoulli beam theory is closely approximated by the mode $n=1$ displacement contribution in the current method. From Figure 6.5, it is evident the stress contribution from mode $n=1$ closely estimates the stress predicted by the Euler-Bernoulli beam theory.

Relationships similar to Eq. 6.14 can also be derived for the hoop stress, $\sigma_\phi$, and the shear stress, $\tau_{z\phi}$, using the strain and displacement expressions in Chapters 2 and 5, respectively. Also, the results can be generalized to determine the stresses at any point in the pipe wall thickness (for any $\xi \neq 0$) by retaining the terms $\kappa_z(z,\phi)$, $\kappa_\phi(z,\phi)$ and $\kappa_{z\phi}(z,\phi)$ in the general strain expression in Eq. 2.3.

## 6.2 Example 2: Cantilever Beam Under Self Weight

The example of a cantilever beam subjected to its self-weight that was solved analytically in Section 4.4 is now solved using the finite element formulation developed in Chapter 5. The geometry of the pipe, shown in Figure 6.6, and material properties of the pipe are identical to those used in Section 4.4, namely: $\ell = 4\,m$, $t = 6\,mm$, $r = 200\,mm$, $\mu = 0.3$, $E = 200{,}000\,MPa$, $G = E/2(1+\mu) = 76{,}925\,MPa$ and $\gamma_s = 78\,kN/m^3$.

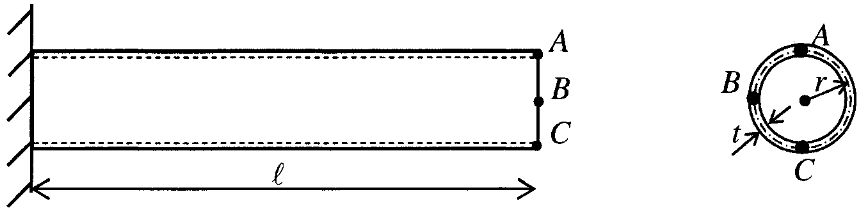


**Figure 6.6** – Example 2: Pipe under self weight

### *6.2.1 Load Potential Vectors*

The load due to the self weight of a pipe, expressed as $q = t\gamma_s$, acts as a vertical traction applied to the mid-surface of the element. By considering an infinitesimal piece of the pipe with volume $t(rd\phi)(dz)$, as shown in Figure 6.7, the load functions in each of the longitudinal, tangential and radial directions are given as:

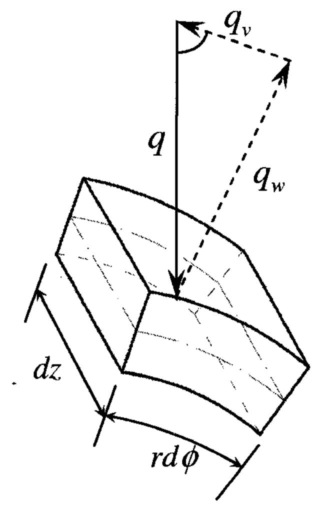


**Figure 6.7** – Tractions $q_V(z,\phi)$ and $q_w(z,\phi)$ as components of pipe self weight

(6.15a) $q_u(z,\phi) = 0$

(6.15b) $q_v(z,\phi) = -t\gamma_s \cos\phi$

(6.15c) $q_w(z,\phi) = -t\gamma_s \sin\phi$

Each Fourier mode $n$ is independent from each other mode $m \neq n$, so each mode will be considered individually. From Eq. 5.41, the load potential vector for Fourier mode $n = 0$ is:

(6.16)
$$\begin{aligned}\{Q_0\}_{8\times1} &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\left[q_u(z,\phi)\{N_{A,0}(z)\}_{8\times1} + q_v z,\phi\{N_{C,0}(z)\}_{8\times1} + q_w z,\phi\{N_{F,0}(z)\}_{8\times1}\right] rd\phi dz \\ &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\left[-t\gamma_s\cos\phi\{N_{C,0}(z)\}_{8\times1} - t\gamma_s\sin\phi\{N_{F,0}(z)\}_{8\times1}\right] rd\phi dz\end{aligned}$$

Performing the integral in Eq. 6.16 with respect to $\phi$ gives:

$$(6.17)\quad \{Q_0\}_{8\times1} = -t\gamma_s \int_{z=0}^{\ell}\left[\{N_{C,0}(z)\}_{8\times1}\int_{\phi=0}^{2\pi}\cos\phi r d\phi + \{N_{F,0}(z)\}_{8\times1}\int_{\phi=0}^{2\pi}\sin\phi r d\phi\right]dz = \{0\}_{8\times1}$$

The load potential vectors for modes $n \geq 1$ are given in Eq. 5.43. From Eq. 5.43a, the load potential vector $\{Q_{n,1}\}_{8\times1}$ is:

$$(6.18)\quad \begin{aligned}\{Q_{n,1}\}_{8\times1} &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\Big[q_u(z,\phi)\{N_{A,n}(z)\}_{8\times1}\cos(n\phi) + q_v(z,\phi)\{N_{D,n}(z)\}_{8\times1}\sin(n\phi) \\ &\qquad + q_w(z,\phi)\{N_{F,n}(z)\}_{8\times1}\cos(n\phi)\Big] r d\phi dz \\ &= -t\gamma_s\int_{z=0}^{\ell}\left[\{N_{D,n}(z)\}_{8\times1}\int_{\phi=0}^{2\pi}\cos\phi\sin(n\phi) r d\phi + \{N_{F,n}(z)\}_{8\times1}\int_{\phi=0}^{2\pi}\sin\phi\cos(n\phi) r d\phi\right]dz\end{aligned}$$

Similarly, Eq. 5.43b gives the load potential vector $\{Q_{n,2}\}_{8\times1}$ as:

$$(6.19)\quad \begin{aligned}\{Q_{n,2}\}_{8\times1} &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\Big[q_u(z,\phi)\{N_{B,n}(z)\}_{8\times1}\sin(n\phi) + q_v(z,\phi)\{N_{C,n}(z)\}_{8\times1}\cos(n\phi) \\ &\qquad + q_w(z,\phi)\{N_{G,n}(z)\}_{8\times1}\sin(n\phi)\Big] r d\phi dz \\ &= -t\gamma_s\int_{z=0}^{\ell}\left[\{N_{C,n}(z)\}_{8\times1}\int_{\phi=0}^{2\pi}\cos\phi\cos(n\phi) r d\phi + \{N_{G,n}(z)\}_{8\times1}\int_{\phi=0}^{2\pi}\sin\phi\sin(n\phi) r d\phi\right]dz\end{aligned}$$

Performing the integrals with respect to $\phi$ in Eqs. 6.18 and 6.19 leads to:

$$(6.20a)\quad \{Q_{n,1}\}_{8\times1} = \{0\}_{8\times1} \qquad \text{for } n \geq 1$$

$$(6.20b)\quad \{Q_{1,2}\}_{8\times1} = -\pi t\gamma_s r\int_{z=0}^{\ell}\{N_{C,1}(z)\}_{8\times1} + \{N_{G,1}(z)\}_{8\times1} dz$$

$$(6.20c)\quad \{Q_{n,2}\}_{8\times1} = \{0\}_{8\times1} \qquad \text{for } n \geq 2$$

### *6.2.2 Results from Current Finite Element*

With the energy equivalent load vectors completely defined, the nodal displacements can now be determined. The equilibrium equations for mode $n=0$ from Eq. 5.46, with the energy equivalent load vector in Eq. 6.17, are:

$$(6.21)\quad \left[K_0\right]_{8\times8}\left\{\Delta_0\right\}_{8\times1}-\left\{0\right\}_{8\times1}=0$$

Given that the stiffness matrix $\left[K_0\right]_{8\times8}$ is non-zero and non-singular, the only possible solution for Eq. 6.21 is the trivial one of $\left\{\Delta_0\right\}_{8\times1}=\left\{0\right\}_{8\times1}$.

Consider now Fourier modes $n\geq2$. Substitute the load potential vectors from Eqs. 6.20a and 6.20c into the equilibrium equations from Eq. 5.47a and 5.47b, respectively, to obtain:

$$(6.22a)\quad \left[K_{n,1}\right]_{8\times8}\left\{\Delta_{n,1}\right\}_{8\times1}-\left\{0\right\}_{8\times1}=0$$

$$(6.22b)\quad \left[K_{n,2}\right]_{8\times8}\left\{\Delta_{n,2}\right\}_{8\times1}-\left\{0\right\}_{8\times1}=0$$

Again, given that the stiffness matrices $\left[K_{n,1}\right]_{8\times8}$ and $\left[K_{n,2}\right]_{8\times8}$ are non-zero and non-singular, the only possible solutions for Eqs. 6.22a and 6.22b are the trivial ones of $\left\{\Delta_{n,1}\right\}_{8\times1}=\left\{0\right\}_{8\times1}$ and $\left\{\Delta_{n,2}\right\}_{8\times1}=\left\{0\right\}_{8\times1}$.

Finally, consider the displacement contribution from Fourier mode $n=1$. The equilibrium equations in Eq. 5.47a with the load potential vector in Eq. 6.20a give:

$$(6.23)\quad \left[K_{1,1}\right]_{8\times8}\left\{\Delta_{1,1}\right\}_{8\times1}-\left\{0\right\}_{8\times1}=0$$

As in the previous cases, the stiffness matrix $\left[K_{1,1}\right]_{8\times8}$ is non-zero and non-singular so the only possible solution to Eq. 6.23 is the trivial one of $\left\{\Delta_{1,1}\right\}_{8\times1}=\left\{0\right\}_{8\times1}$. Substituting the load potential vector in Eq. 6.20b into the equilibrium equations from Eq. 5.47b for mode $n=1$ gives:

$$(6.24)\quad \left[K_{1,2}\right]_{8\times 8}\left\{\Delta_{1,2}\right\}_{8\times 1} = -\pi t \gamma_s r \int_{z=0}^{\ell} \left\{N_{C,1}(z)\right\}_{8\times 1} + \left\{N_{G,1}(z)\right\}_{8\times 1} dz$$

It is clear from Eqs. 6.21 through 6.24 that the only non-zero nodal displacement contribution is from $\left\{\Delta_{1,2}\right\}_{8\times 1}$ corresponding to mode $n=1$. Therefore, the total nodal displacements of the element are contained in the vector $\left\{\Delta_{1,2}\right\}_{8\times 1}$. Solving Eq. 6.24 for the nodal displacements gives the displacements of Points *A*, *B* and *C*. The results based on the current FEA, those based on the analytical solution in Chapter 4 and those of the ABAQUS shell finite element analysis are given in Table 6.2.

**Table 6.2** – Comparison of results for Current FEA, Analytical Method and Shell FEA for Example 2

| | | Point *A* | Point *B* | Point *C* |
|---|---|---|---|---|
| Vertical Displacement | Current FEA | 0.6395 | 0.6395 | 0.6395 |
| | Analytical (Ch. 4) | 0.6395 | 0.6395 | 0.6395 |
| | ABAQUS S4R element | 0.6455 | 0.6455 | 0.6455 |
| Longitudinal Displacement | Current FEA | 0.04161 | 0 | -0.04161 |
| | Analytical (Ch. 4) | 0.04161 | 0 | -0.04161 |
| | ABAQUS S4R element | 0.04201 | 0 | -0.04201 |

As in Example 1, the results from the current finite element method for Example 2 are identical to those from the analytical solution in Chapter 4. Such agreement is expected given that the FEA formulation is based on the exact shape functions.

## 6.3 Example 3: Uniform Torque Applied to Tip of Cantilever

In the current example, it is required to determine the displacements of a cantilever pipe subjected to a uniform torque applied to the free end as shown in Figure 6.8. The loading, geometric and material properties of the pipe are as follows: $T = 10\, kN\cdot m$, $\ell = 0.6\, m$, $t = 5\, mm$, $r = 150\, mm$, $\mu = 0.3$, $E = 200{,}000\, MPa$ and $G = E/2(1+\mu) = 76{,}925\, MPa$.

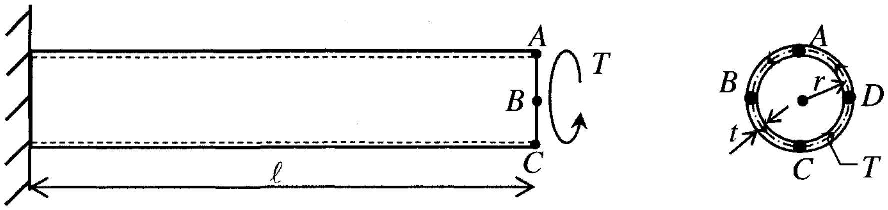


**Figure 6.8** – Example 3: Cantilever with uniform torque

### *6.3.1 Load Potential Vectors*

The loading condition for this example consists of a uniform torsion applied to the tip of a cantilevered pipe. The load functions are given as:

(6.25a) $q_u(z,\phi)=0$

(6.25b) $q_v(z,\phi)=\dfrac{T}{2\pi r^2}\delta(z-\ell)$

(6.25c) $q_w(z,\phi)=0$

In Eq. 6.25b, the expression $\delta(z-\ell)$ is the Dirac Delta function. Consider each Fourier mode $n$ independently to develop the load potential vectors. Eq. 5.41 gives the load potential vector for mode $n=0$ as:

$$
\begin{aligned}
\{Q_0\}_{8\times1} &= \int\limits_{z=0}^{\ell}\int\limits_{\phi=0}^{2\pi} q_v(z,\phi)\{N_{C,0}(z)\}_{8\times1} r\,d\phi\,dz \\
&= \int\limits_{z=0}^{\ell}\int\limits_{\phi=0}^{2\pi} \frac{T}{2\pi r^2}\delta(z-\ell)\{N_{C,0}(z)\}_{8\times1} r\,d\phi\,dz \qquad (6.26) \\
&= \frac{T}{r}\{N_{C,0}(\ell)\}_{8\times1}
\end{aligned}
$$

For modes $n\geq 1$, the load potential vector $\{Q_{n,1}\}_{8\times1}$ corresponding to displacement contributions $a_n(z)$, $d_n(z)$ and $f_n(z)$ from Eq. 5.43a is:

$$
\begin{aligned}
\{Q_{n,1}\}_{8\times1} &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_v(z,\phi)\{N_{D,n}(z)\}_{8\times1}\sin(n\phi)rd\phi dz \\
&= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} \frac{T}{2\pi r^2}\delta(z-\ell)\{N_{D,n}(z)\}_{8\times1}\sin(n\phi)rd\phi dz \qquad (6.27)\\
&= \{0\}_{8\times1}
\end{aligned}
$$

Similarly, from Eq. 5.43b, the load potential vector $\{Q_{n,2}\}_{8\times1}$ corresponding to displacement contributions $b_n(z)$, $c_n(z)$ and $g_n(z)$ for modes $n \geq 1$ is:

$$
\begin{aligned}
\{Q_{n,2}\}_{8\times1} &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_v(z,\phi)\{N_{C,n}(z)\}_{8\times1}\cos(n\phi)rd\phi dz \\
&= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} \frac{T}{2\pi r^2}\delta(z-\ell)\{N_{C,n}(z)\}_{8\times1}\cos(n\phi)d\phi dz \qquad (6.28)\\
&= \{0\}_{8\times1}
\end{aligned}
$$

### *6.3.2 Results from Current Finite Element*

The nodal displacements are now determined from Eqs. 5.46 and 5.47. Consider first modes $n \geq 1$. Substitute the load potential vectors from Eqs. 6.27 and 6.28 into the equilibrium equations from Eq. 5.47a and 5.47b, respectively, to obtain:

(6.29a) $[K_{n,1}]_{8\times8}\{\Delta_{n,1}\}_{8\times1} - \{0\}_{8\times1} = 0$

(6.29b) $[K_{n,2}]_{8\times8}\{\Delta_{n,2}\}_{8\times1} - \{0\}_{8\times1} = 0$

It is apparent, given that the stiffness matrices $[K_{n,1}]_{8\times8}$ and $[K_{n,2}]_{8\times8}$ are non-singular, that the only possible solutions for Eqs. 6.29a and 6.29b are the trivial ones of $\{\Delta_{n,1}\}_{8\times1} = \{0\}_{8\times1}$ and $\{\Delta_{n,2}\}_{8\times1} = \{0\}_{8\times1}$.

For mode $n = 0$, substituting the load potential vector from Eq. 6.26 into the equilibrium equations in Eq. 5.46 gives:

$$(6.30)\quad [K_0]_{8\times8}\{\Delta_0\}_{8\times1} - \frac{T}{r}\{N_{C,0}(\ell)\}_{8\times1} = 0$$

Eq. 6.30 is solved to obtain the nodal displacement contributions for mode $n = 0$, which correspond to the total nodal displacements since the contributions from higher modes vanish. For the loading condition of a uniform torque applied to the free end of a cantilevered pipe, the displacements are entirely axisymmetric.

The resulting tangential displacement is the same at all points on the middle surface at the free end of the cantilever. The value of the tangential displacement is $v(\ell,\phi) = 0.1103\,mm$ giving an angle of twist of $\theta = v(\ell,\phi)/r = 0.1103mm/150mm = 735.3\times10^{-6}\ rad$. From the theory of mechanics of materials, the angle of twist at the free end of the cantilever is:

$$(6.31)\quad \theta_{Theory} = \frac{T\ell}{2\pi G r^3 t} = 735.7\times10^{-6}\ rad$$

The corresponding tangential displacement is $v(\ell,\phi)_{Theory} = r\theta_{Theory} = 0.1103\,mm$. The angle of twist, and corresponding tangential displacement at the free end of the cantilever, determined by the current method is nearly identical to that predicted by the simplified theory of mechanics of materials.

## 6.4 Example 4: Pipe with Two Elements of Different Thickness

Given the convergent characteristic of the element developed, all previous examples were successfully solved using a single element. In contrast, the current example considers a pipe consisting of two segments with different thicknesses. The problem is modeled with two elements. The pipe configuration is shown in Figure 6.9. Both segments of the pipe have $r = 200\,mm$, $\mu = 0.3$, $E = 200{,}000\,MPa$ and $G = E/2(1+\mu) = 76{,}925\,MPa$. While the thickness of the left section is $t_1 = 5\,mm$, that of the right segment is $t_2 = 6\,mm$.

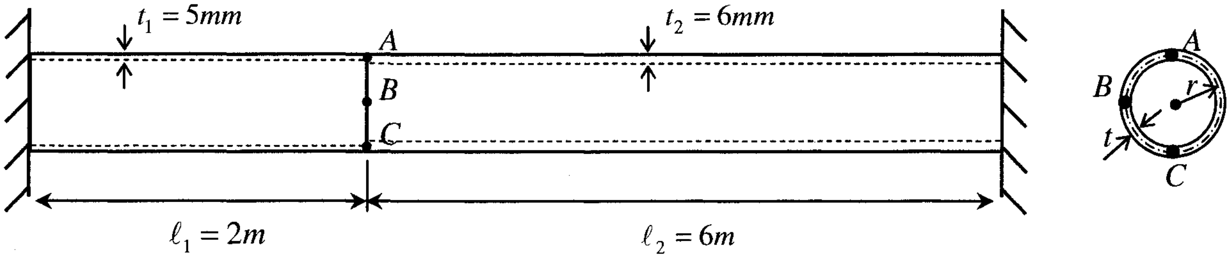


**Figure 6.9** – Example 4: Pipe with two elements of different thickness

### 6.4.1 Load Potential Vectors

The pipe is assumed to be full of highly pressurized fluid of density $\gamma_w$. The pipe is subjected to loading consisting of internal pressure, $p = 1000\,kPa$, self weight of the fluid $\gamma_w = 10\,kN/m^3$ and self weight of the pipe, $\gamma_s = 78\,kN/m^3$. Both the internal pressure and the weight of the fluid have components in the radial direction only. From Figure 6.10, their load functions are:

(6.32a) $q_{u,1}(z,\phi) = 0$

(6.32b) $q_{v,1}(z,\phi) = 0$

(6.32c) $q_{w,1}(z,\phi) = -\gamma_w r(\sin\phi - 1) + p$

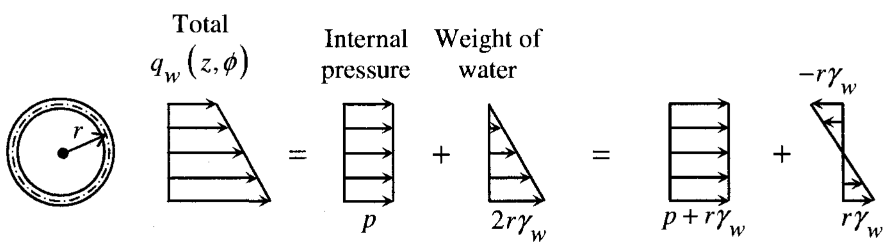


**Figure 6.10**–Example 4: Load function $q_{w,1}(z,\phi)$

The load functions for the self weight of the pipe, $q_{u,2}$, $q_{v,2}$, and $q_{w,2}$ are identical to those of Example 2 (Eq. 6.15):

(6.33a) $q_{u,2}(z,\phi) = 0$

(6.33b) $q_{v,2}(z,\phi) = -t\gamma_s \cos\phi$

(6.33c) $q_{w,2}(z,\phi) = -t\gamma_s \sin\phi$

The total load functions for the problem are obtained by adding Eqs. 6.32 and 6.33:

(6.34a) $q_u(z,\phi) = 0$

(6.34b) $q_v(z,\phi) = -t\gamma_s \cos\phi$

(6.34c) $q_w(z,\phi) = -\gamma_w r(\sin\phi - 1) + p - t\gamma_s \sin\phi$

Consider the load potential vector for each Fourier mode $n$ independently. For mode $n = 0$, Eq. 5.41 gives:

$$\begin{aligned} \{Q_0\}_{8\times1} &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\left[q_v(z,\phi)\{N_{C,0}(z)\}_{8\times1} + q_w(z,\phi)\{N_{F,0}(z)\}_{8\times1}\right] rd\phi dz \\ &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\left[-t\gamma_s\cos\phi\{N_{C,0}(z)\}_{8\times1} + (-\gamma_w r(\sin\phi-1) + p - t\gamma_s\sin\phi)\{N_{F,0}(z)\}_{8\times1}\right] rd\phi dz \\ &= 2\pi r(\gamma_w r + p)\int_{z=0}^{\ell}\{N_{F,0}(z)\}_{8\times1} dz \end{aligned} \tag{6.35}$$

The load potential vectors for modes $n \geq 1$ are given in Eq. 5.43. From Eq. 5.43a, the load potential vector $\{Q_{n,1}\}_{8\times1}$ is:

$$\begin{aligned} \{Q_{n,1}\}_{8\times1} &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\left[q_v(z,\phi)\{N_{D,n}(z)\}_{8\times1}\sin(n\phi) + q_w(z,\phi)\{N_{F,n}(z)\}_{8\times1}\cos(n\phi)\right] rd\phi dz \\ &= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\Big[-t\gamma_s\cos\phi\{N_{D,n}(z)\}_{8\times1}\sin(n\phi) + \\ &\qquad (-\gamma_w r(\sin\phi-1) + p - t\gamma_s\sin\phi)\{N_{F,n}(z)\}_{8\times1}\cos(n\phi)\Big] rd\phi dz \end{aligned} \tag{6.36}$$

Similarly, Eq. 5.43b gives the load potential vector $\{Q_{n,2}\}_{8\times1}$ as:

$$\{Q_{n,2}\}_{8\times1} = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\left[q_v(z,\phi)\{N_{C,n}(z)\}_{8\times1}\cos(n\phi)+q_w(z,\phi)\{N_{G,n}(z)\}_{8\times1}\sin(n\phi)\right]rd\phi dz$$

(6.37)
$$= \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\left[-t\gamma_s\cos\phi\{N_{C,n}(z)\}_{8\times1}\cos(n\phi)+\right.$$
$$\left.(-\gamma_w r(\sin\phi-1)+p-t\gamma_s\sin\phi)\{N_{G,n}(z)\}_{8\times1}\sin(n\phi)\right]rd\phi dz$$

Performing the integrals with respect to $\phi$ in Eqs. 6.36 and 6.37 leads to:

(6.38a) $\{Q_{n,1}\}_{8\times1} = \{0\}_{8\times1}$ for $n \geq 1$

(6.38b) $\{Q_{1,2}\}_{8\times1} = -\pi r t\gamma_s \int_{z=0}^{\ell}\{N_{C,1}(z)\}_{8\times1}dz - \pi r(\gamma_w r + t\gamma_s)\int_{z=0}^{\ell}\{N_{G,1}(z)\}_{8\times1}dz$

(6.38c) $\{Q_{n,2}\}_{8\times1} = \{0\}_{8\times1}$ for $n \geq 2$

### *6.4.2 Results from Current Finite Element Analysis*

The energy equivalent load vectors developed above and the stiffness matrices developed in Chapter 5 are valid for a single element only. The current example uses two elements to model the pipe. The stiffness matrix and energy equivalent load vector for the system will be the addition of the individual matrices and vectors from each individual element. As such, the system stiffness matrix size, for each mode *n*, will be 12x12 and the system energy equivalent load vector size will be 12x1. Correspondingly, the system has three nodes with four displacements per node resulting in the system nodal displacement vector being 12x1 in size.

The nodal displacements can now be determined from the energy equivalent load vectors defined in Eqs. 6.35 and 6.38. The equilibrium equations for mode $n=0$ were given in Eq. 5.46:

(6.39) $[K_0]_{12\times12}\{\Delta_0\}_{12\times1} = \{Q_0\}_{12\times1}$

Solving Eq. 6.39 for the vector $\{\Delta_0\}_{12\times1}$ gives the nodal displacement contributions for mode $n=0$.

Consider now the equilibrium equations corresponding to the nodal displacements $\{\Delta_{n,1}\}_{12\times1}$ for modes $n \geq 1$, which were given in Eq. 5.47a. Introducing the energy equivalent load vector from Eq. 6.38a into the equilibrium equations gives:

$$(6.40) \quad [K_{n,1}]_{12\times12}\{\Delta_{n,1}\}_{12\times1} - \{0\}_{12\times1} = 0$$

Given that the stiffness matrix $[K_{n,1}]_{12\times12}$ is non-zero and non-singular, the only possible solution for Eq. 6.40 is the trivial one of $\{\Delta_{n,1}\}_{12\times1} = \{0\}_{12\times1}$.

Next consider the equilibrium equations corresponding to the nodal displacements $\{\Delta_{n,2}\}_{12\times1}$ for modes $n \geq 2$, which were given in Eq. 5.47b. Insert the energy equivalent load vector from Eq. 6.38c to obtain:

$$(6.41) \quad [K_{n,2}]_{12\times12}\{\Delta_{n,2}\}_{12\times1} - \{0\}_{12\times1} = 0$$

Again, given that the stiffness matrix $[K_{n,2}]_{12\times12}$ is non-zero and non-singular, the only solution for Eq. 6.41 is the trivial one of $\{\Delta_{n,2}\}_{12\times1} = \{0\}_{12\times1}$.

Finally, consider the nodal displacement contribution $\{\Delta_{1,2}\}_{12\times1}$ for Fourier mode $n = 1$. The equilibrium equations in Eq. 5.47a together give:

$$(6.42) \quad [K_{1,2}]_{12\times12}\{\Delta_{1,2}\}_{12\times1} = \{Q_{1,2}\}_{12\times1}$$

The solution to Eq. 6.42 will be $\{\Delta_{1,2}\}_{12\times1} \neq \{0\}_{12\times1}$ because the energy equivalent load vector $\{Q_{1,2}\}_{12\times1}$ is non-zero.

It is evident from Eqs. 6.39 through 6.42 that the only non-zero nodal displacement contributions are from $\{\Delta_0\}_{12\times1}$, corresponding to mode $n = 0$, and $\{\Delta_{1,2}\}_{12\times1}$, corresponding to mode $n = 1$. Solving Eqs. 6.39 and 6.42 for the nodal displacement contribution vectors

$\{\Delta_0\}_{12\times1}$ and $\{\Delta_{1,2}\}_{12\times1}$, respectively, gives the total displacement for the pipe. The total vertical, lateral (horizontal) and longitudinal displacements for Points *A*, *B* and *C* at the intersection of the two elements in the pipe (see Figure 6.10) are summarized and compared to the results of the ABAQUS S4R shell finite element analysis in Table 6.3.

**Table 6.3** – Comparison of results for Current FEA and ABAQUS S4R shell FEA for Example 4

| | | Point *A* | Point *B* | Point *C* |
|---|---|---|---|---|
| Vertical Disp. | Current FEA | 0.4022 | 0.4361 | 0.4688 |
| | ABAQUS S4R element | 0.4143 | 0.4491 | 0.4833 |
| Lateral Disp. | Current FEA | 0 | 0.03331 | 0 |
| | ABAQUS S4R element | 0 | 0.03453 | 0 |
| Long. Disp. | Current FEA | -0.05109 | -0.000672 | 0.04974 |
| | ABAQUS S4R element | -0.05419 | -0.00160 | 0.05099 |

In Table 6.3, a positive value for the vertical displacement indicates downward movement while a positive value for the longitudinal displacement indicates that the pipe fibres are in tension.

The results in Table 6.3 show that the displacements predicted by the current method closely match those of the ABAQUS S4R shell finite element analysis. The current method is able to capture the axisymmetric radial expansion of the pipe due to the internal pressure as well as the vertical displacement due to the weight of the pipe and the fluid with a remarkably small number of degrees of freedom. The ABAQUS solution requires 7,776 degrees of freedom (1296 nodes with six degrees of freedom per node) while the solution from the current method requires only 24 degrees of freedom (three nodes with four degrees of freedom each per mode *n* and a total of two modes, *n*=0 and *n*=1).

# CHAPTER 7: CONCLUSIONS AND RECOMMENDATIONS

## 7.1 Summary

The principle of stationary potential energy was used in conjunction with general Fourier series expansion for the displacement fields to formulate the equilibrium conditions and boundary conditions for elastic prismatic pipes subjected to general loading. The formulation was based on the assumptions of thin-walled shell theory (Ohtsubo and Watanabe, 1978). The general analytical solution of the coupled differential equations of equilibrium derived was developed for all Fourier modes. The analytical solution developed was successfully used to solve two simple problems. Comparisons with established finite element solutions demonstrate the ability of the model to accurately capture the shell behaviour of pipes with a remarkably small number of degrees of freedoms. However, the algebraic manipulations required in the analytical solution were found quite tedious to conduct by hand, even for simple problems.

In order to remedy this limitation and to make the solution scheme amenable to more complex problems, a finite element solution was formulated based on the analytical solution developed. The expressions for the displacement fields obtained in the analytical solution were used to formulate a series of exact shape functions which relate the intermediate displacements within a pipe finite element to the nodal displacements. Using the exact shape functions, the principle of stationary potential energy was adopted to formulate the "exact" stiffness matrices and associated energy equivalent load vectors for common loads. The new finite element was tested for a variety of practical loading conditions including point loads, gravity loads, internal and external pressure, twisting deformation, axial deformation, transverse deformations, and combinations thereof. Results were in excellent agreement with those based on shell finite elements in ABAQUS.

## 7.2 Advantages of the Formulation

The pipe element developed was shown to be free of discretization errors, a direct result of adopting the exact shape functions in the formulation. More refined results within the element are obtained by using higher order harmonics. A main advantage of the new element, in comparison with the shell FEA, is its computational efficiency. This is a direct outcome of the orthogonal properties of the Fourier terms.

The new element can very accurately predict the displacements and stresses of a pipe with just a single element between supports. Excellent displacement results are obtained using a few harmonics and accurate stress values were obtained using only seven harmonics. In comparison, conventional shell FEA would require a number of degrees of freedom a few order of magnitudes higher to obtain results of comparable accuracy. The simplicity of the new element relative to a shell FEA also greatly reduces the modeling effort required by the analyst.

Another feature of the new element is its convergence characteristics. In conventional shell FEA solutions, accuracy of results is improved by refining the element mesh and repeating the analysis until convergence is achieved. In contrast, with the new pipe element, improved accuracy is achieved by including additional Fourier modes in the analysis. The results from the additional modes can simply be added to those of previous modes to obtain the total results. There is no need to repeat the analysis for previous modes.

## 7.3 Recommendation for Future Work

The methodology developed in the present research lends itself to a variety of possibilities for further research. Some possible extensions for the finite element formulations include

1. Pipe bends and elbows rather than straight sections only.
2. Analysis of Pipes under dynamic loading
3. Modelling geometric and material non-linear behaviour. In this type of problems, the use of exact shape functions is not feasible and the analyst would have to resort to more conventional polynomial interpolation functions.
4. Modelling non-prismatic elements in which the cross section radius and/or pipe wall thickness vary along the length of the element. In this type of problems the obtained differential equations of equilibrium will involve non-constant coefficients and formulating the exact shape functions becomes unattainable. Again, the analyst would have to resort to conventional polynomial functions.

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# APPENDIX 1 - PHYSICAL SIGNIFICANCE OF FOURIER MODES

The total displacement of the middle surface in each of the *u*, *v* and *w* directions given in Eq. 2.11 is the sum of the respective displacement contributions of each Fourier mode *n*. As evident from Eq. 2.11, the displacement contribution for any mode $n$ is independent of the displacement contribution for any other mode $m \neq n$. No coupling occurs between the different modes. The displacement contribution for each mode is computed individually and added to the displacements from all other modes to obtain the total displacement. In this appendix, the shape of the displacement contribution for each Fourier mode *n*, as well as the shape of the total displacement, is examined.

In Eq. 2.11 the displacement contribution for each Fourier mode *n* is a function of *z* only multiplied by a function of $\phi$ only (either $\sin(n\phi)$ or $\cos(n\phi)$). The function of *z* describes how the displacement varies along the length of the pipe while the $\sin(n\phi)$ and $\cos(n\phi)$ terms describe how the displacement varies around the circumference of the pipe.

As an example, consider the radial middle surface displacement *w* at a given value of *z*. At a given value of *z* the functions $f_n(z)$ and $g_n(z)$ are constants. The radial displacement, given in Eq. 2.11c, then becomes a function of $\phi$ only:

$$\begin{aligned} w(\phi) &= f_0 + \sum_{n=1}^{\alpha} f_n \cos n\phi + \sum_{n=1}^{\alpha} g_n \sin n\phi \\ &= f_0 + f_1 \cos\phi + g_1 \sin\phi + f_2 \cos 2\phi + g_2 \sin 2\phi + \ldots f_\alpha \cos \alpha\phi + g_\alpha \sin \alpha\phi \end{aligned} \quad \text{(A1.1)}$$

For this example, consider a problem where the total number of Fourier modes included is $\alpha = 3$. The functions $\cos\phi$, $\cos 2\phi$, $\cos 3\phi$, $\sin\phi$, $\sin 2\phi$, and $\sin 3\phi$ are plotted in Figures A1.1 through A1.7 to show the physical shape of the radial displacement contribution, $w_n(\phi)$, for each mode $n = 0$ to 3.

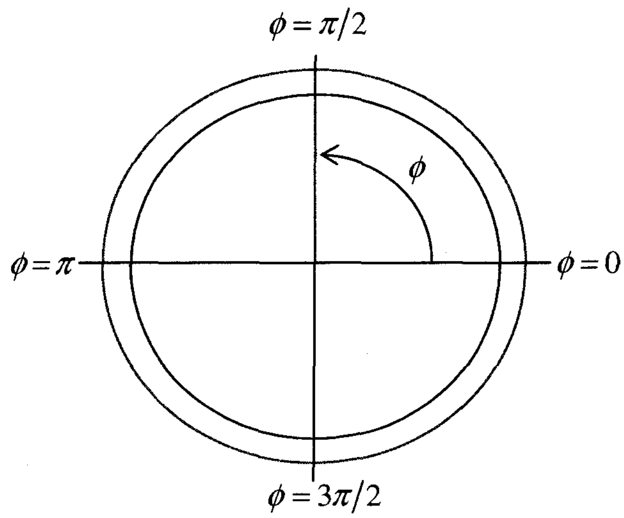


**Figure A1.1 – $n = 0$**

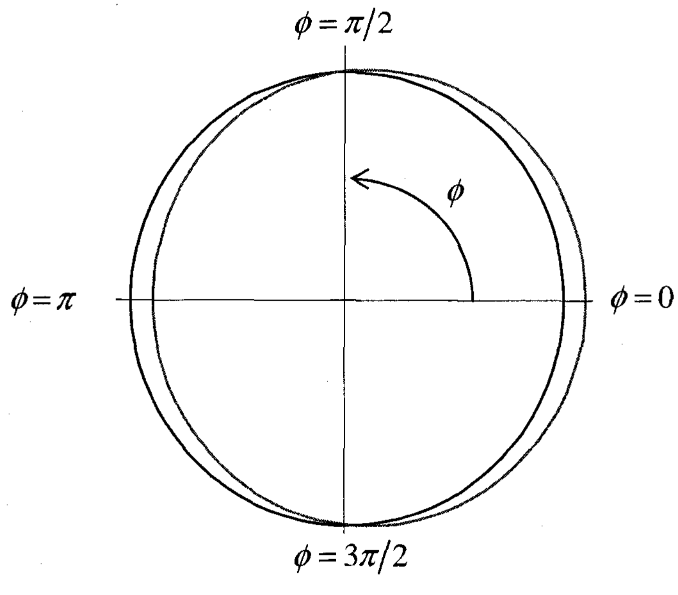


**Figure A1.2 –** $\cos\phi$

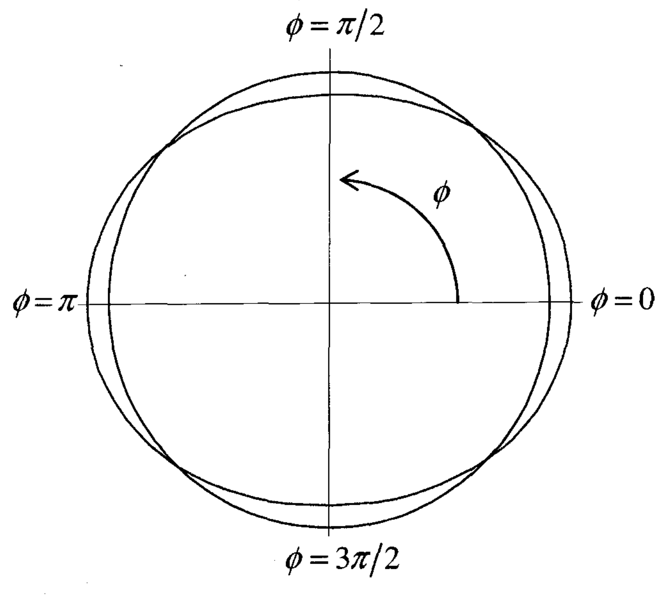


**Figure A1.3 –** $\cos 2\phi$

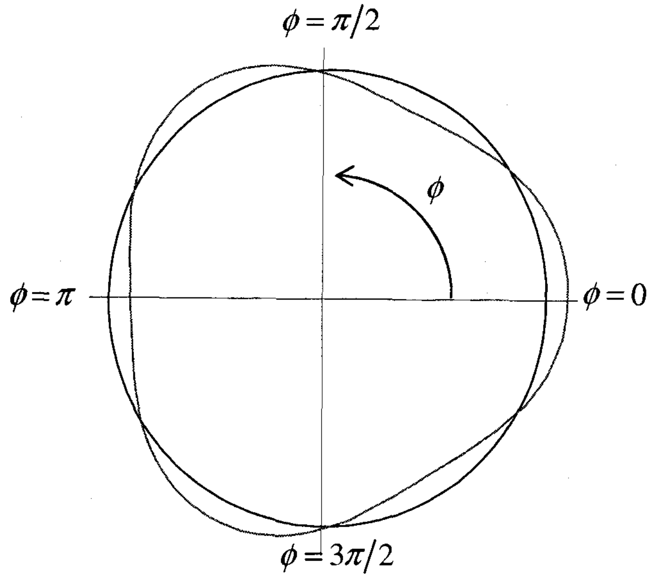


**Figure A1.4**– $\cos 3\phi$

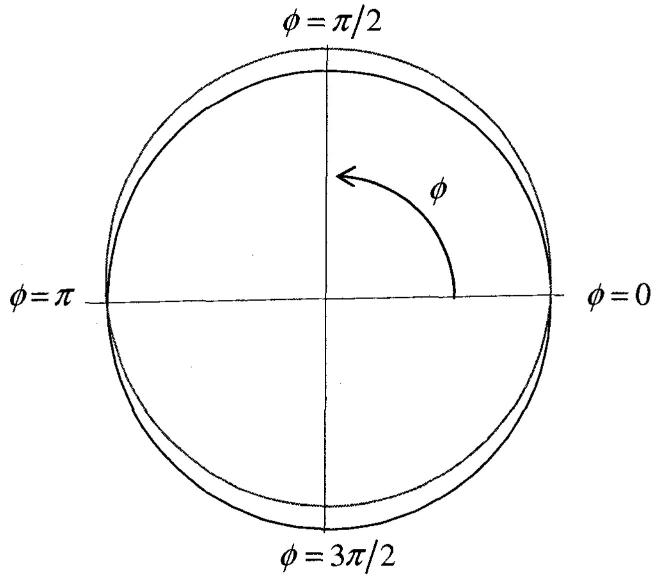


**Figure A1.5** – $\sin \phi$

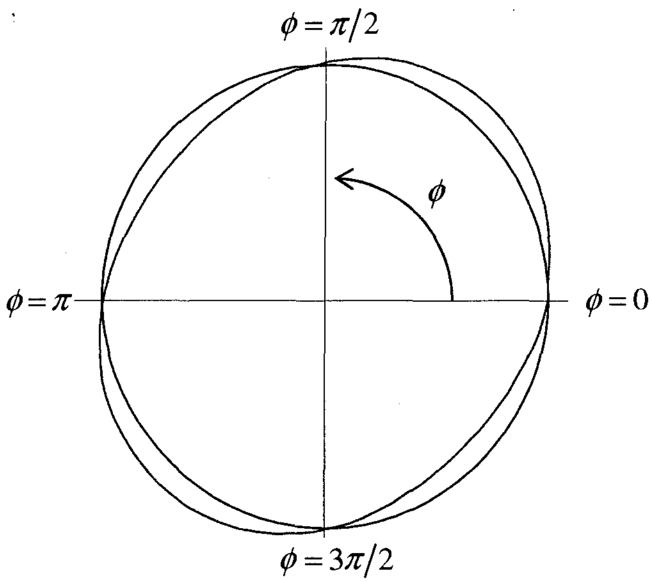


**Figure A1.6** – $\sin 2\phi$

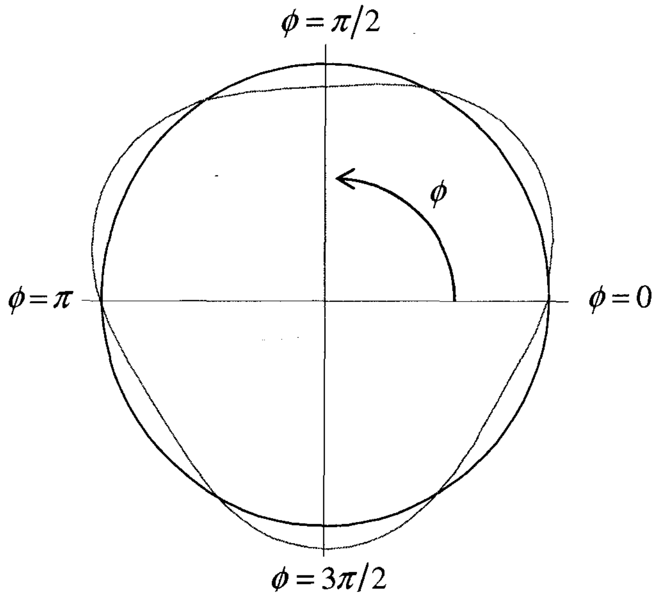


**Figure A1.7** – $\sin 3\phi$

As shown in Figure A1.1 Fourier mode $n=0$ represents uniform axisymmetric displacement. Also, as evident from Figures A1.2 and A1.5, Fourier mode $n=1$ represents displacement that is similar to that described by traditional Euler-Bernoulli beam theory. The higher Fourier modes $(n \geq 2)$ represent deformation of the pipe cross section.

The shape of the total radial displacement of the middle surface is the sum of the above figures multiplied by the appropriate constant values. For this example, assume that $f_0(z) = g_0(z) = c$, where $c$ is an arbitrary constant, and $f_n(z) = g_n(z) = 1/n$ for $n \geq 1$.

The total radial displacement of the middle surface is then:

$$w(\phi) = c + \cos\phi + \sin\phi + \frac{1}{2}\cos 2\phi + \frac{1}{2}\sin 2\phi + \frac{1}{3}\cos 3\phi + \frac{1}{3}\sin 3\phi \tag{A1.2}$$

Plotting Eq. A1.2 gives a graphical representation of the total radial displacement, $w$, for this example, as shown in Figure A1.8.

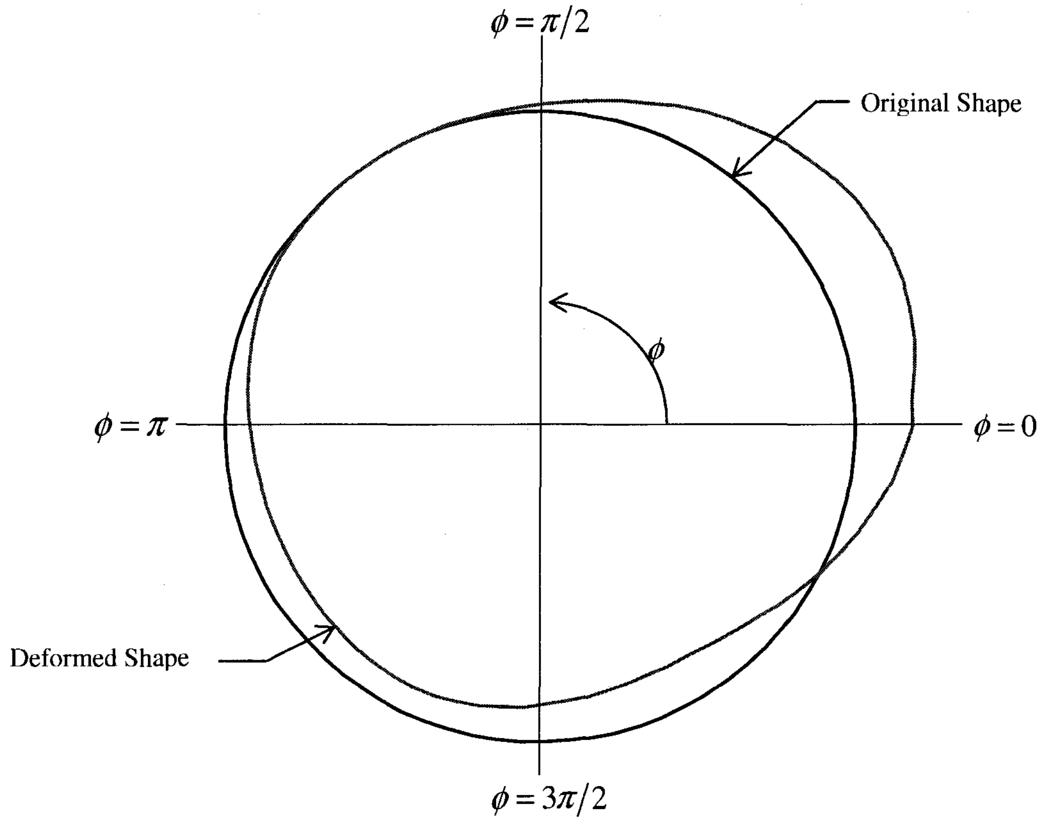


**Figure A1.8** – Total Radial Displacement, *w*

# APPENDIX 2 – INTEGRATING THE PRODUCT OF TWO FOURIER SERIES

This appendix evaluates the integral of the product of two finite Fourier Series in $\phi$ when the limits of $\phi$ are 0 and $2\pi$. The two series do not necessarily need to have an equal number of terms.

Assume the two Fourier Series to be multiplied and integrated are:

$$f(\phi) = a_0 + \sum_{n=1}^{\alpha_a} a_n \cos n\phi + \sum_{n=1}^{\alpha_b} b_n \sin n\phi \tag{A2.1}$$

$$g(\phi) = c_0 + \sum_{m=1}^{\alpha_c} c_m \cos n\phi + \sum_{m=1}^{\alpha_d} d_m \sin n\phi \tag{A2.2}$$

It is required to find the integral for:

$$\int_{\phi=0}^{2\pi} f(\phi) g(\phi) d\phi \tag{A2.3}$$

Start by defining a new function $h(\phi) = f(\phi) g(\phi)$. Substitute the expressions from Eqs. A2.1 and A2.2 to obtain:

$$h(\phi) = \left( a_0 + \sum_{n=1}^{\alpha_a} a_n \cos n\phi + \sum_{n=1}^{\alpha_b} b_n \sin n\phi \right)\left( c_0 + \sum_{m=1}^{\alpha_c} c_m \cos n\phi + \sum_{m=1}^{\alpha_d} d_m \sin n\phi \right) \tag{A2.4}$$

Expand Eq. A2.4 to give:

$$\begin{aligned} h(\phi) = {} & a_0 c_0 + a_0 \sum_{m=1}^{\alpha_c} c_m \cos m\phi + a_0 \sum_{m=1}^{\alpha_d} d_m \sin m\phi + \\ & c_0 \sum_{n=1}^{\alpha_a} a_n \cos n\phi + \left( \sum_{n=1}^{\alpha_a} a_n \cos n\phi \right)\left( \sum_{m=1}^{\alpha_c} c_m \cos m\phi \right) + \left( \sum_{n=1}^{\alpha_a} a_n \cos n\phi \right)\left( \sum_{m=1}^{\alpha_d} d_m \sin m\phi \right) + \\ & c_0 \sum_{n=1}^{\alpha_b} b_n \sin n\phi + \left( \sum_{n=1}^{\alpha_b} b_n \sin n\phi \right)\left( \sum_{m=1}^{\alpha_c} c_m \cos m\phi \right) + \left( \sum_{n=1}^{\alpha_b} b_n \sin n\phi \right)\left( \sum_{m=1}^{\alpha_d} d_m \sin m\phi \right) \end{aligned} \tag{A2.5}$$

Rearrange Eq. A2.5:

$$h(\phi) = a_0c_0 + a_0\sum_{m=1}^{\alpha_c} c_m \cos m\phi + a_0\sum_{m=1}^{\alpha_d} d_m \sin m\phi +$$

(A2.6a)
$$c_0\sum_{n=1}^{\alpha_a} a_n \cos n\phi + \sum_{n=1}^{\alpha_a}\sum_{m=1}^{\alpha_c} a_n c_m \cos n\phi \cos m\phi + \sum_{n=1}^{\alpha_a}\sum_{m=1}^{\alpha_d} a_n d_m \cos n\phi \sin m\phi +$$
$$c_0\sum_{n=1}^{\alpha_b} b_n \sin n\phi + \sum_{n=1}^{\alpha_b}\sum_{m=1}^{\alpha_c} b_n c_m \sin n\phi \cos m\phi + \sum_{n=1}^{\alpha_b}\sum_{m=1}^{\alpha_d} b_n d_m \sin n\phi \sin m\phi$$

$$h(\phi) = a_0c_0 + a_0\sum_{m=1}^{\alpha_c} c_m \cos m\phi + a_0\sum_{m=1}^{\alpha_d} d_m \sin m\phi +$$

(A2.6b)
$$c_0\sum_{n=1}^{\alpha_a} a_n \cos n\phi + \left\langle \sum_{n=1}^{\alpha_a}\sum_{m=1}^{\alpha_c} a_n c_m \cos n\phi \cos m\phi \right\rangle_{n\neq m} + \left\langle \sum_{n=1}^{\alpha_{a,c}} a_n c_n \cos^2 n\phi \right\rangle +$$
$$\sum_{n=1}^{\alpha_a}\sum_{m=1}^{\alpha_d} a_n d_m \cos n\phi \sin m\phi + c_0\sum_{n=1}^{\alpha_b} b_n \sin n\phi + \sum_{n=1}^{\alpha_b}\sum_{m=1}^{\alpha_c} b_n c_m \sin n\phi \cos m\phi +$$
$$\left\langle \sum_{n=1}^{\alpha_b}\sum_{m=1}^{\alpha_d} b_n d_m \sin n\phi \sin m\phi \right\rangle_{n\neq m} + \left\langle \sum_{n=1}^{\alpha_{b,d}} b_n d_n \sin^2 n\phi \right\rangle$$

In Eq. A2.6b $\alpha_{a,c}$ is the smaller of $\alpha_a$ and $\alpha_c$ and similarly for $\alpha_{b,d}$.

Substitute the expression for $h(\phi)$ from Eq. A2.6b into Eq. A2.3 and consider each term of the integral individually:

(A2.7)
$$\int_{\phi=0}^{2\pi} a_0c_0\,d\phi = \left[(a_0c_0)\phi\right]_{\phi=0}^{2\pi} = 2\pi(a_0c_0)$$

(A2.8)
$$\int_{\phi=0}^{2\pi}\left(a_0\sum_{m=1}^{\alpha_c} c_m \cos m\phi\right)d\phi = a_0\sum_{m=1}^{\alpha_c}\left[\frac{c_m}{m}\sin m\phi\right]_{\phi=0}^{2\pi} = a_0\sum_{m=1}^{\alpha_c}\frac{c_m}{m}\left[\sin(2\pi m) - \sin(0)\right] = 0$$

The right hand side of Eq. A2.8 is zero because $\sin(2\pi m) = \sin(0) = 0$ for any integer *m*.

(A2.9) $$\int_{\phi=0}^{2\pi}\left(a_0\sum_{m=1}^{\alpha_d}d_m\sin m\phi\right)d\phi = a_0\sum_{m=1}^{\alpha_d}\left[-\frac{d_m}{m}\cos m\phi\right]_{\phi=0}^{2\pi} = a_0\sum_{m=1}^{\alpha_d}-\frac{d_m}{m}\left[\cos(2\pi m)-\cos(0)\right]=0$$

The right hand side of Eq. A2.9 is zero because $\cos(2\pi m)=\cos(0)=1$ for any integer m.

(A2.10) $$\int_{\phi=0}^{2\pi}\left(c_0\sum_{n=1}^{\alpha_a}a_n\cos n\phi\right)d\phi=0$$

(A2.11) $$\int_{\phi=0}^{2\pi}\left\langle\sum_{n=1}^{\alpha_a}\sum_{m=1}^{\alpha_c}a_nc_m\cos n\phi\cos m\phi\right\rangle_{n\neq m}d\phi=\sum_{n=1}^{\alpha_a}\sum_{m=1}^{\alpha_c}a_nc_m\left[\frac{\sin(n-m)\phi}{2(n-m)}+\frac{\sin(n+m)\phi}{2(n+m)}\right]_{\phi=0,n\neq m}^{2\pi}=0$$

(A2.12a) $$\int_{\phi=0}^{2\pi}\left(\sum_{n=1}^{\alpha_{a,c}}a_nc_n\cos^2 n\phi\right)d\phi=\sum_{n=1}^{\alpha_{a,c}}a_nc_n\left[\frac{\phi}{2}+\frac{\sin(2n\phi)}{4n}\right]_{\phi=0}^{2\pi}=\sum_{n=1}^{\alpha_{a,c}}a_nc_n\left[\frac{2\pi}{2}+\frac{\sin(4n\pi)}{4n}-0\right]$$

(A2.12b) $$\int_{\phi=0}^{2\pi}\left(\sum_{n=1}^{\alpha_{a,c}}a_nc_n\cos^2 n\phi\right)d\phi=\sum_{n=1}^{\alpha_{a,c}}\pi a_nc_n$$

(A2.13) $$\int_{\phi=0}^{2\pi}\left\langle\sum_{n=1}^{\alpha_a}\sum_{m=1}^{\alpha_d}a_nd_m\cos n\phi\sin m\phi\right\rangle_{n\neq m}d\phi=\sum_{n=1}^{\alpha_a}\sum_{m=1}^{\alpha_d}a_nd_m\left[-\frac{\cos(m-n)\phi}{2(m-n)}-\frac{\cos(m+n)\phi}{2(m+n)}\right]_{\phi=0,n\neq m}^{2\pi}=0$$

(A2.14) $$\int_{\phi=0}^{2\pi}\left(\sum_{n=1}^{\alpha_{a,d}}a_nd_n\cos n\phi\sin n\phi\right)d\phi=\sum_{n=1}^{\alpha_{a,d}}a_nd_n\left[\frac{\sin^2 n\phi}{2n}\right]_{\phi=0}^{2\pi}=0$$

(A2.15) $$\int_{\phi=0}^{2\pi}\left(c_0\sum_{n=1}^{\alpha_b}b_n\sin n\phi\right)d\phi=0$$

(A2.16) $$\int_{\phi=0}^{2\pi}\left(\sum_{n=1}^{\alpha_b}\sum_{m=1}^{\alpha_c}b_nc_m\sin n\phi\cos m\phi\right)d\phi=0$$

(A2.17) $$\int_{\phi=0}^{2\pi}\left\langle\sum_{n=1}^{\alpha_b}\sum_{m=1}^{\alpha_d} b_n d_m \sin n\phi \sin m\phi\right\rangle_{n\neq m} d\phi=\sum_{n=1}^{\alpha_b}\sum_{m=1}^{\alpha d} b_n d_m\left[\frac{\sin(n-m)\phi}{2(n-m)}-\frac{\sin(n+m)\phi}{2(n+m)}\right]_{\phi=0,n\neq m}^{2\pi}=0$$

(A2.18) $$\int_{\phi=0}^{2\pi}\left(\sum_{n=1}^{\alpha_{b,d}} b_n d_n \sin^2 n\phi\right)d\phi=\sum_{n=1}^{\alpha_{b,d}} b_n d_n\left[\frac{\phi}{2}-\frac{\sin(2n\phi)}{4n}\right]_{\phi=0}^{2\pi}=\sum_{n=1}^{\alpha_{b,d}} b_n d_n\left[\frac{2\pi}{2}-\frac{\sin(4n\pi)}{4n}-0\right]=\sum_{n=1}^{\alpha_{b,d}}\pi b_n d_n$$

Summing the expressions in Eqs. A2.7 through A2.18 gives the final form of the integral in Eq. A2.3:

(A2.19) $$\int_{\phi=0}^{2\pi} h(\phi)d\phi = 2\pi(a_0c_0)+\sum_{n=1}^{\alpha_{a,c}}\pi a_n c_n+\sum_{n=1}^{\alpha_{b,d}}\pi b_n d_n$$

In Eq. A2.19 $\alpha_{a,c}$ is the smaller of $\alpha_a$ and $\alpha_c$ and $\alpha_{b,d}$ is the smaller of $\alpha_b$ and $\alpha_d$.

For the case of $f(\phi)=g(\phi)$ then $h(\phi)=\left[f(\phi)\right]^2$ and Eq. A2.19 becomes:

(A2.20) $$\int_{\phi=0}^{2\pi}\left[f(\phi)\right]^2 d\phi = 2\pi a_0^{\;2}+\sum_{n=1}^{\alpha_a}\pi a_n^{\;2}+\sum_{n=1}^{\alpha_b}\pi b_n^{\;2}$$

# APPENDIX 3 – VARIATION OF TOTAL POTENTIAL ENERGY

The objective of this appendix is to determine the field equations and boundary terms using the principle of stationary total potential energy. Equilibrium of the system is ensured by applying the stationarity conditions of the total potential energy as defined in Eq. 2.17. The total potential energy is the sum of the internal strain energy, given in Table 2.1, and the load potential terms given in Eq. 2.16.

For simplicity define each internal strain energy term in Table 2.1 independently as:

(A3.1a) $$U_1 = \frac{C_1}{2}\int_{z=0}^{l}\left[2\left(a_0'(z)\right)^2 + \sum_{n=1}^{\alpha}\left(a_n'(z)\right)^2 + \sum_{n=1}^{\alpha}\left(b_n'(z)\right)^2\right]dz$$

(A3.1b) $$\begin{aligned}U_2 = \frac{C_2}{2}\int_{z=0}^{l}\Bigg\{&2(f_0(z))^2 + \sum_{n=1}^{\alpha}(f_n(z))^2 + \sum_{n=1}^{\alpha}(g_n(z))^2\\ &+2\left[\sum_{n=1}^{\alpha} nd_n(z)f_n(z) + \sum_{n=1}^{\alpha} -nc_n(z)g_n(z)\right]\\ &+\sum_{n=1}^{\alpha} n^2(c_n(z))^2 + \sum_{n=1}^{\alpha} n^2(d_n(z))^2\Bigg\}dz\end{aligned}$$

(A3.1c) $$\begin{aligned}U_3 = C_3\int_{z=0}^{l}\Bigg[&2a_0'(z)f_0(z) + \sum_{n=1}^{\alpha} a_n'(z)f_n(z) + \sum_{n=1}^{\alpha} b_n'(z)g_n(z)\\ &+\sum_{n=1}^{\alpha} -nb_n'(z)c_n(z) + \sum_{n=1}^{\alpha} na_n'(z)d_n(z)\Bigg]dz\end{aligned}$$

(A3.1d) $$\begin{aligned}U_4 = \frac{C_4}{2}\int_{z=0}^{l}\Bigg\{&2\left(c_0'(z)\right)^2 + \sum_{n=1}^{\alpha}\left(c_n'(z)\right)^2 + \sum_{n=1}^{\alpha}\left(d_n'(z)\right)^2\\ &+\frac{2}{r}\left[\sum_{n=1}^{\alpha} nb_n(z)c_n'(z) + \sum_{n=1}^{\alpha} -na_n(z)d_n'(z)\right]\\ &+\frac{1}{r^2}\left[\sum_{n=1}^{\alpha} n^2(a_n(z))^2 + \sum_{n=1}^{\alpha} n^2(b_n(z))^2\right]\Bigg\}dz\end{aligned}$$

(A3.1e) $$U_5 = \frac{C_6}{2}\int_{z=0}^{l}\left[2\left(f_0''(z)\right)^2 + \sum_{n=1}^{\alpha}\left(f_n''(z)\right)^2 + \sum_{n=1}^{\alpha}\left(g_n''(z)\right)^2\right]dz$$

(A3.1f)
$$U_6 = \frac{C_7}{2}\int_{z=0}^{l}\Bigg\{\sum_{n=1}^{\alpha} n^4\left(f_n(z)\right)^2 + \sum_{n=1}^{\alpha} n^4\left(g_n(z)\right)^2 - 2\left[\sum_{n=1}^{\alpha} -n^3 d_n(z) f_n(z) + \sum_{n=1}^{\alpha} n^3 c_n(z) g_n(z)\right] + \sum_{n=1}^{\alpha} n^2\left(c_n(z)\right)^2 + \sum_{n=1}^{\alpha} n^2\left(d_n(z)\right)^2\Bigg\}dz$$

(A3.1g)
$$U_7 = C_8\int_{z=0}^{l}\left[\sum_{n=1}^{\alpha} -n^2 f_n''(z) f_n(z) + \sum_{n=1}^{\alpha} -n^2 g_n''(z) g_n(z) - \sum_{n=1}^{\alpha} -n c_n(z) g_n''(z) - \sum_{n=1}^{\alpha} n d_n(z) f_n''(z)\right]dz$$

(A3.1h)
$$U_8 = \frac{C_9}{2}\int_{z=0}^{l}\Bigg\{\sum_{n=1}^{\alpha} n^2\left(f_n'(z)\right)^2 + \sum_{n=1}^{\alpha} n^2\left(g_n'(z)\right)^2 - 2\left[\sum_{n=1}^{\alpha} -n d_n'(z) f_n'(z) + \sum_{n=1}^{\alpha} n c_n'(z) g_n'(z)\right] + 2\left(c_0'(z)\right)^2 + \sum_{n=1}^{\alpha}\left(c_n'(z)\right)^2 + \sum_{n=1}^{\alpha}\left(d_n'(z)\right)^2\Bigg\}dz$$

The total internal strain energy, $U$, is the sum of Eqs. A3.1a through A3.1h.

Take the variation of each term in Eq. A3.1 with respect to each variable:

(A3.2a)
$$\delta U_1 = \frac{C_1}{2}\int_{z=0}^{l}\left[4a_0'\delta a_0' + \sum_{n=1}^{\alpha} 2a_n'\delta a_n' + \sum_{n=1}^{\alpha} 2b_n'\delta b_n'\right]dz$$

(A3.2b)
$$\delta U_2 = \frac{C_2}{2}\int_{z=0}^{l}\Bigg[4f_0\delta f_0 + \sum_{n=1}^{\alpha} 2f_n\delta f_n + \sum_{n=1}^{\alpha} 2g_n\delta g_n + 2\left(\sum_{n=1}^{\alpha} n f_n\delta d_n + \sum_{n=1}^{\alpha} -n g_n\delta c_n + \sum_{n=1}^{\alpha} n d_n\delta f_n + \sum_{n=1}^{\alpha} -n c_n\delta g_n\right) + \sum_{n=1}^{\alpha} 2n^2 c_n\delta c_n + \sum_{n=1}^{\alpha} 2n^2 d_n\delta d_n\Bigg]dz$$

$$\delta U_3 = C_3 \int_{z=0}^{l} \Bigg[ 2f_0\delta a_0' + 2a_0'\delta f_0 + \sum_{n=1}^{\alpha} f_n \delta a_n' + \sum_{n=1}^{\alpha} a_n' \delta f_n$$

$$+\sum_{n=1}^{\alpha} g_n \delta b_n' + \sum_{n=1}^{\alpha} b_n' \delta g_n + \sum_{n=1}^{\alpha} -nc_n \delta b_n' \tag{A3.2c}$$

$$+\sum_{n=1}^{\alpha} -nb_n' \delta c_n + \sum_{n=1}^{\alpha} nd_n \delta a_n' + \sum_{n=1}^{\alpha} na_n' \delta d_n \Bigg] dz$$

$$\delta U_4 = \frac{C_4}{2} \int_{z=0}^{l} \Bigg[ 4c_0'\delta c_0' + \sum_{n=1}^{\alpha} 2c_n'\delta c_n' + \sum_{n=1}^{\alpha} 2d_n'\delta d_n'$$

$$+\frac{2}{r}\left( \sum_{n=1}^{\alpha} nc_n' \delta b_n + \sum_{n=1}^{\alpha} nb_n \delta c_n' + \sum_{n=1}^{\alpha} -nd_n' \delta a_n + \sum_{n=1}^{\alpha} -na_n \delta d_n' \right) + \tag{A3.2d}$$

$$+\frac{1}{r^2}\left( \sum_{n=1}^{\alpha} 2n^2 a_n \delta a_n + \sum_{n=1}^{\alpha} 2n^2 b_n \delta b_n \right) \Bigg] dz$$

$$\delta U_5 = \frac{C_6}{2} \int_{z=0}^{l} \Bigg[ 4f_0''\delta f_0'' + \sum_{n=1}^{\alpha} 2f_n''\delta f_n'' + \sum_{n=1}^{\alpha} 2g_n''\delta g_n'' \Bigg] dz \tag{A3.2e}$$

$$\delta U_6 = \frac{C_7}{2} \int_{z=0}^{l} \Bigg[ \sum_{n=1}^{\alpha} 2n^4 f_n \delta f_n + \sum_{n=1}^{\alpha} 2n^4 g_n \delta g_n -$$

$$-2\left( \sum_{n=1}^{\alpha} -n^3 f_n \delta d_n + \sum_{n=1}^{\alpha} -n^3 d_n \delta f_n + \sum_{n=1}^{\alpha} n^3 g_n \delta c_n + \sum_{n=1}^{\alpha} n^3 c_n \delta g_n \right) \tag{A3.2f}$$

$$+\sum_{n=1}^{\alpha} 2n^2 c_n \delta c_n + \sum_{n=1}^{\alpha} 2n^2 d_n \delta d_n \Bigg] dz$$

$$\delta U_7 = C_8 \int_{z=0}^{l} \Bigg[ \sum_{n=1}^{\alpha} -n^2 f_n \delta f_n'' + \sum_{n=1}^{\alpha} -n^2 f_n'' \delta f_n + \sum_{n=1}^{\alpha} -n^2 g_n \delta g_n'' + \sum_{n=1}^{\alpha} -n^2 g_n'' \delta g_n \tag{A3.2g}$$

$$-\sum_{n=1}^{\alpha} -ng_n'' \delta c_n - \sum_{n=1}^{\alpha} -nc_n \delta g_n'' - \sum_{n=1}^{\alpha} nf_n'' \delta d_n - \sum_{n=1}^{\alpha} nd_n \delta f_n'' \Bigg] dz$$

$$\delta U_8 = \frac{C_9}{2} \int_{z=0}^{l} \Bigg[ \sum_{n=1}^{\alpha} 2n^2 f_n' \delta f_n' + \sum_{n=1}^{\alpha} 2n^2 g_n' \delta g_n'$$

$$-2\left( \sum_{n=1}^{\alpha} -nf_n' \delta d_n' + \sum_{n=1}^{\alpha} -nd_n' \delta f_n' + \sum_{n=1}^{\alpha} ng_n' \delta c_n' + \sum_{n=1}^{\alpha} nc_n' \delta g_n' \right) \tag{A3.2h}$$

$$+4c_0'\delta c_0' + \sum_{n=1}^{\alpha} 2c_n' \delta c_n' + \sum_{n=1}^{\alpha} 2d_n' \delta d_n' \Bigg] dz$$

Derivatives of the functions, having the forms $\delta y_n'$ or $\delta y_n''$, in Eq. A3.2 are eliminated using integration by parts on the relevant terms. The resulting field components of the equations will contain only the functions themselves, $\delta y_n$, with no derivatives. The general rules for integration by parts are:

(A3.3a) $$\int_{z=0}^{l} x_n \delta y_n' dz = [x_n \delta y_n]_{z=0}^{l} - \int_{z=0}^{l} x_n' \delta y_n dz$$

(A3.3b) $$\int_{z=0}^{l} x_n \delta y_n'' dz = [x_n \delta y_n']_{z=0}^{l} - [x_n' \delta y_n]_{z=0}^{l} + \int_{z=0}^{l} x_n'' \delta y_n dz$$

Applying the relations in Eq. A3.3 to Eq. A3.2:

(A3.4a)
$$\delta U_1 = C_1 \Bigg[ [2a_0' \delta a_0]_0^l - \int_{z=0}^{l} 2a_0'' \delta a_0 dz + \sum_{n=1}^{\alpha} \left\{ [a_n' \delta a_n]_0^l - \int_{z=0}^{l} a_n'' \delta a_n dz \right\}$$
$$+ \sum_{n=1}^{\alpha} \left\{ [b_n' \delta b_n]_0^l - \int_{z=0}^{l} b_n'' \delta b_n dz \right\} \Bigg]$$

(A3.4b)
$$\delta U_2 = C_2 \Bigg[ \int_{z=0}^{l} 2f_0 \delta f_0 dz + \sum_{n=1}^{\alpha} \int_{z=0}^{l} f_n \delta f_n dz + \sum_{n=1}^{\alpha} \int_{z=o}^{l} g_n \delta g_n dz$$
$$+ \sum_{n=1}^{\alpha} \int_{z=0}^{l} nf_n \delta d_n dz + \sum_{n=1}^{\alpha} \int_{z=0}^{l} -ng_n \delta c_n dz + \sum_{n=1}^{\alpha} \int_{z=0}^{l} nd_n \delta f_n dz + \sum_{n=1}^{\alpha} \int_{z=0}^{l} -nc_n \delta g_n dz$$
$$+ \sum_{n=1}^{\alpha} \int_{z=0}^{l} n^2 c_n \delta c_n dz + \sum_{n=1}^{\alpha} \int_{z=0}^{l} n^2 d_n \delta d_n dz \Bigg]$$

(A3.4c)
$$\delta U_3 = C_3 \Bigg[ [2f_0 \delta a_0]_0^l - \int_{z=0}^{l} 2f_0' \delta a_0 dz + \int_{z=0}^{l} 2a_0' \delta f_0 dz + \sum_{n=1}^{\alpha} \left\{ [f_n \delta a_n]_0^l - \int_{z=0}^{l} f_n' \delta a_n dz \right\}$$
$$+ \sum_{n=1}^{\alpha} \int_{z=0}^{l} a_n' \delta f_n dz + \sum_{n=1}^{\alpha} \left\{ [g_n \delta b_n]_0^l - \int_{z=0}^{l} g_n' \delta b_n dz \right\} + \sum_{n=1}^{\alpha} \int_{z=0}^{l} b_n' \delta g_n dz$$
$$+ \sum_{n=1}^{\alpha} \left\{ [-nc_n \delta b_n]_0^l - \int_{z=0}^{l} -nc_n' \delta b_n dz \right\} + \sum_{n=1}^{\alpha} \int_{z=0}^{l} -nb_n' \delta c_n dz$$
$$+ \sum_{n=1}^{\alpha} \left\{ [nd_n \delta a_n]_0^l - \int_{z=0}^{l} nd_n' \delta a_n dz \right\} + \sum_{n=1}^{\alpha} \int_{z=0}^{l} na_n' \delta d_n dz \Bigg]$$

$$\delta U_4 = C_4\left[\left[2c_0'\delta c_0\right]_0^l - \int_{z=0}^{l} 2c_0''\delta c_0 dz + \sum_{n=1}^{\alpha}\left\{\left[c_n'\delta c_n\right]_0^l - \int_{z=0}^{l} c_n''\delta c_n dz\right\}\right.$$

$$+\sum_{n=1}^{\alpha}\left\{\left[d_n'\delta d_n\right]_0^l - \int_{z=0}^{l} d_n''\delta d_n dz\right\} + \frac{1}{r}\sum_{n=1}^{\alpha}\int_{z=0}^{l} nc_n'\delta b_n dz + \frac{1}{r}\sum_{n=1}^{\alpha}\left\{\left[nb_n\delta c_n\right]_0^l - \int_{z=0}^{l} nb_n'\delta c_n dz\right\}$$

(A3.4d)

$$+\frac{1}{r}\sum_{n=1}^{\alpha}\int_{z=0}^{l} -nd_n'\delta a_n dz + \frac{1}{r}\sum_{n=1}^{\alpha}\left\{\left[-na_n\delta d_n\right]_0^l - \int_{z=0}^{l} -na_n'\delta d_n dz\right\}$$

$$\left.+\frac{1}{r^2}\sum_{n=1}^{\alpha}\int_{z=0}^{l} n^2 a_n\delta a_n dz + \frac{1}{r^2}\sum_{n=1}^{\alpha}\int_{z=0}^{l} n^2 b_n\delta b_n dz\right\}$$

$$\delta U_5 = C_6\left[\left[2f_0''\delta f_0'\right]_0^l - \left[2f_0'''\delta f_0\right]_0^l + \int_{z=0}^{l} 2f_0''''\delta f_0 dz\right.$$

(A3.4e)

$$+\sum_{n=1}^{\alpha}\left\{\left[f_n''\delta f_n'\right]_0^l - \left[f_n'''\delta f_n\right]_0^l + \int_{z=0}^{l} f_n''''\delta f_n dz\right\}$$

$$+\sum_{n=1}^{\alpha}\left\{\left[g_n''\delta g_n'\right]_0^l - \left[g_n'''\delta g_n\right]_0^l + \int_{z=0}^{l} g_n''''\delta g_n dz\right\}$$

(A3.4f)

$$\delta U_6 = C_7\left[\sum_{n=1}^{\alpha}\int_{z=0}^{l} n^4 f_n\delta f_n dz + \sum_{n=1}^{\alpha}\int_{z=0}^{l} n^4 g_n\delta g_n dz - \sum_{n=1}^{\alpha}\int_{z=0}^{l} -n^3 f_n\delta d_n dz - \sum_{n=1}^{\alpha}\int_{z=0}^{l} -n^3 d_n\delta f_n dz - \right.$$

$$\left.\sum_{n=1}^{\alpha}\int_{z=0}^{l} n^3 g_n\delta c_n dz - \sum_{n=1}^{\alpha}\int_{z=0}^{l} n^3 c_n\delta g_n dz + \sum_{n=1}^{\alpha}\int_{z=0}^{l} n^2 c_n\delta c_n dz + \sum_{n=1}^{\alpha}\int_{z=0}^{l} n^2 d_n\delta d_n dz\right]$$

$$\delta U_7 = C_8\left[\sum_{n=1}^{\alpha}\left\{\left[-n^2 f_n\delta f_n'\right]_0^l - \left[-n^2 f_n'\delta f_n\right]_0^l + \int_{z=0}^{l} -n^2 f_n''\delta f_n dz\right\} + \sum_{n=1}^{\alpha}\int_{z=0}^{l} -n^2 f_n''\delta f_n dz\right.$$

$$+\sum_{n=1}^{\alpha}\left\{\left[-n^2 g_n\delta g_n'\right]_0^l - \left[-n^2 g_n'\delta g_n\right]_0^l + \int_{z=0}^{l} -n^2 g_n''\delta g_n dz\right\} + \sum_{n=1}^{\alpha}\int_{z=0}^{l} -n^2 g_n''\delta g_n dz$$

(A3.4g)

$$-\sum_{n=1}^{\alpha}\left\{\left[-nc_n\delta g_n'\right]_0^l - \left[-nc_n'\delta g_n\right]_0^l + \int_{z=0}^{l} -nc_n''\delta g_n dz\right\} - \sum_{n=1}^{\alpha}\int_{z=0}^{l} -ng_n''\delta c_n dz$$

$$\left.-\sum_{n=1}^{\alpha}\left\{\left[nd_n\delta f_n'\right]_0^l - \left[nd_n'\delta f_n\right]_0^l + \int_{z=0}^{l} nd_n''\delta f_n dz\right\} - \sum_{n=1}^{\alpha}\int_{z=0}^{l} nf_n''\delta d_n dz\right]$$

$$\delta U_8 = C_9\left[\sum_{n=1}^{\alpha}\left\{\left[n^2 f_n'\delta f_n\right]_0^l - \int_{z=0}^{l} n^2 f_n''\delta f_n dz\right\} + \sum_{n=1}^{\alpha}\left\{\left[n^2 g_n'\delta g_n\right]_0^l - \int_{z=0}^{l} n^2 g_n''\delta g_n dz\right\}\right.$$

(A3.4h)
$$-\sum_{n=1}^{\alpha}\left\{\left[-nf_n'\delta d_n\right]_0^l - \int_{z=0}^{l} -nf_n''\delta d_n dz\right\} - \sum_{n=1}^{\alpha}\left\{\left[-nd_n'\delta f_n\right]_0^l - \int_{z=0}^{l} -nd_n''\delta f_n dz\right\}$$

$$-\sum_{n=1}^{\alpha}\left\{\left[ng_n'\delta c_n\right]_0^l - \int_{z=0}^{l} ng_n''\delta c_n dz\right\} - \sum_{n=1}^{\alpha}\left\{\left[nc_n'\delta g_n\right]_0^l - \int_{z=0}^{l} nc_n''\delta g_n dz\right\} + \left[2c_0'\delta c_0\right]_0^l$$

$$\left.-\int_{z=0}^{l} 2c_0''\delta c_0 dz + \sum_{n=1}^{\alpha}\left\{\left[c_n'\delta c_n\right]_0^l - \int_{z=0}^{l} c_n''\delta c_n dz\right\} + \sum_{n=1}^{\alpha}\left\{\left[d_n'\delta d_n\right]_0^l - \int_{z=0}^{l} d_n''\delta d_n dz\right\}\right]$$

The potential energy from the externally applied loads must be added to the internal strain energy to give the total potential energy. The total load potential for an arbitrary loading condition is given as:

(A3.5a) $$W_u = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_u(z,\phi)\left[a_0(z) + \sum_{n=1}^{\alpha} a_n(z)\cos n\phi + \sum_{n=1}^{\alpha} b_n(z)\sin n\phi\right] rd\phi dz$$

(A3.5b) $$W_v = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_v(z,\phi)\left[c_0(z) + \sum_{n=1}^{\alpha} c_n(z)\cos n\phi + \sum_{n=1}^{\alpha} d_n(z)\sin n\phi\right] rd\phi dz$$

(A3.5c) $$W_w = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_w(z,\phi)\left[f_0(z) + \sum_{n=1}^{\alpha} f_n(z)\cos n\phi + \sum_{n=1}^{\alpha} g_n(z)\sin n\phi\right] rd\phi dz$$

In Eq. A3.5, functions $q_u(z,\phi)$, $q_v(z,\phi)$ and $q_w(z,\phi)$ represent the load potential for the traction loads applied to the middle surface of the pipe.

Take the variation of the load potential terms in Eq. A3.5 with respect to each variable:

(A3.6a) $$\delta W_u = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_u(z,\phi)\left[\delta a_0 + \sum_{n=1}^{\alpha}\cos n\phi\delta a_n + \sum_{n=1}^{\alpha}\sin n\phi\delta b_n\right] rd\phi dz$$

(A3.6b) $$\delta W_v = \int_{z=0}^{\ell}\int_{\phi=0}^{2\pi} q_v(z,\phi)\left[\delta c_0 + \sum_{n=1}^{\alpha}\cos n\phi\delta c_n + \sum_{n=1}^{\alpha}\sin n\phi\delta d_n\right] rd\phi dz$$

$$\delta W_w = \int_{z=0}^{\ell} \int_{\phi=0}^{2\pi} q_w(z,\phi) \left[ \delta f_0 + \sum_{n=1}^{\alpha} \cos n\phi \delta f_n + \sum_{n=1}^{\alpha} \sin n\phi \delta g_n \right] r d\phi dz \tag{A3.6c}$$

The variation of the load potential terms, Eq A3.6, is subtracted from the variation of the internal strain energy, Eq. A3.4, to give the variation of the total potential energy of the element. Note that the order of the summation and integral in Eq. A3.6 is interchangeable. Subtracting Eq. A3.6 from A3.4 and grouping like terms together gives:

$$\begin{aligned}
\delta\pi = \int_{z=0}^{l} \Bigg\{ & \left[ 2C_1 a_0'' + 2C_3 f_0' + \int_{\phi=0}^{2\pi} q_u(z,\phi) r d\phi \right] (-\delta a_0) \\
& + \left[ 2C_4 c_0'' + 2C_9 c_0'' + \int_{\phi=0}^{2\pi} q_v(z,\phi) r d\phi \right] (-\delta c_0) \\
& + \left[ 2C_3 a_0' + 2C_2 f_0 + 2C_6 f_0'''' - \int_{\phi=0}^{2\pi} q_w(z,\phi) r d\phi \right] \delta f_0 \Bigg\} dz \\
& + \left[ (2C_1 a_0' + 2C_3 f_0) \delta a_0 \right]_{z=0}^{l} \\
& + \left[ (2(C_4 + C_9) c_0') \delta c_0 \right]_{z=0}^{l} \\
& + \left[ (-2C_6 f_0''') \delta f_0 \right]_{z=0}^{l} \\
& + \left[ (2C_6 f_0'') \delta f_0' \right]_{z=0}^{l}
\end{aligned} \tag{A3.7}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\alpha}\left\langle \int_{z=0}^{l}\left\{\left[C_1a_n''-\frac{C_4n^2}{r^2}a_n+(C_3+C_5)nd_n'+C_3f_n'+\int_{\phi=0}^{2\pi}q_u(z,\phi)\cos(n\phi)rd\phi\right](-\delta a_n)\right.\right.\\
&\qquad+\left[(C_3+C_5)na_n'+(C_2+C_7)n^2d_n-(C_4+C_9)d_n''+\left(C_2n+C_7n^3\right)f_n\right.\\
&\qquad\left.-(C_8+C_9)nf_n''-\int_{\phi=0}^{2\pi}q_v(z,\phi)\sin(n\phi)rd\phi\right]\delta d_ndz\\
&\qquad+\left[C_3a_n'+\left(C_2n+C_7n^3\right)d_n-(C_8+C_9)nd_n''+\left(C_2+C_7n^4\right)f_n\right.\\
&\qquad\left.\left.-(2C_8+C_9)n^2f_n''+C_6f_n''''-\int_{\phi=0}^{2\pi}q_w(z,\phi)\sum_{n=1}^{\alpha}\cos(n\phi)rd\phi\right]\delta f_n\right\}dz\\
&\quad+\left[(C_1a_n'+C_3nd_n+C_3f_n)\delta a_n\right]_{z=0}^{l}\\
&\quad+\left[\left(-C_5na_n+(C_4+C_9)d_n'+C_9nf_n'\right)\delta d_n\right]_{z=0}^{l}\\
&\quad+\left[\left(+(C_8+C_9)nd_n'+(C_8+C_9)n^2f_n'-C_6f_n'''\right)\delta f_n\right]_{z=0}^{l}\\
&\quad\left.+\left[\left(-C_8nd_n-C_8n^2f_n+C_6f_n''\right)\delta f_n'\right]_{z=0}^{l}\right\rangle
\end{aligned}
$$

(A3.7 cont'd)

$$
\begin{aligned}
&+\sum_{n=1}^{\alpha}\left\langle \int_{z=0}^{l}\left\{\left[C_1b_n''-\frac{C_4n^2}{r^2}b_n-(C_3+C_5)nc_n'+C_3g_n'+\int_{\phi=0}^{2\pi}q_u(z,\phi)\sin(n\phi)rd\phi\right](-\delta b_n)\right.\right.\\
&\qquad+\left[(C_3+C_5)nb_n'-(C_2+C_7)n^2c_n+(C_4+C_9)c_n''+\left(C_2n+C_7n^3\right)g_n\right.\\
&\qquad\left.-(C_8+C_9)ng_n''+\int_{\phi=0}^{2\pi}q_v(z,\phi)\cos(n\phi)rd\phi\right](-\delta c_n)\\
&\qquad+\left[C_3b_n'-\left(C_2n+C_7n^3\right)c_n+(C_8+C_9)nc_n''+\left(C_2+C_7n^4\right)g_n\right.\\
&\qquad\left.\left.-(2C_8+C_9)n^2g_n''+C_6g_n''''-\int_{\phi=0}^{2\pi}q_w(z,\phi)\sin(n\phi)rd\phi\right]\delta g_n\right\}dz\\
&\quad+\left[(C_1b_n'-C_3nc_n+C_3g_n)\delta b_n\right]_{z=0}^{l}\\
&\quad+\left[\left(C_5nb_n+(C_4+C_9)c_n'-C_9ng_n'\right)\delta c_n\right]_{z=0}^{l}\\
&\quad+\left[\left(-(C_8+C_9)nc_n'+(C_8+C_9)n^2g_n'-C_6g_n'''\right)\delta g_n\right]_{z=0}^{l}\\
&\quad\left.+\left[\left(C_8nc_n-C_8n^2g_n+C_6g_n''\right)\delta g_n'\right]_{z=0}^{l}\right\rangle=0
\end{aligned}
$$

It is apparent from Eq. A3.7 that each Fourier mode $n$ is independent from every other Fourier mode $m \neq n$. Hence, the summations in the above equations can be removed and the equilibrium equations and boundary conditions written independently for each mode $n$.

Note also that the variation of each displacement function in the field terms of Eq. A3.7 is arbitrary with respect to $z$. Therefore, the bracket multiplying the variation of the function must be zero. The result is a set of equilibrium equations for each mode $n$.

(A3.8a) $$2C_1 a_0'' + 2C_3 f_0' + \int_{\phi=0}^{2\pi} q_u(z,\phi)\, r d\phi = 0$$

(A3.8b) $$2(C_4 + C_9) c_0'' + \int_{\phi=0}^{2\pi} q_v(z,\phi)\, r d\phi = 0$$

(A3.8c) $$2C_3 a_0' + 2C_2 f_0 + 2C_6 f_0'''' - \int_{\phi=0}^{2\pi} q_w(z,\phi)\, r d\phi = 0$$

(A3.8d) $$C_1 a_n'' - \frac{C_4 n^2}{r^2} a_n + (C_3 + C_5) n d_n' + C_3 f_n' + \int_{\phi=0}^{2\pi} q_u(z,\phi) \cos(n\phi)\, r d\phi = 0$$

(A3.8e) $$\begin{aligned}&(C_3 + C_5) n a_n' + (C_2 + C_7) n^2 d_n - (C_4 + C_9) d_n'' + (C_2 n + C_7 n^3) f_n \\ &- (C_8 + C_9) n f_n'' - \int_{\phi=0}^{2\pi} q_v(z,\phi) \sin(n\phi)\, r d\phi = 0\end{aligned}$$

(A3.8f) $$\begin{aligned}&C_3 a_n' + (C_2 n + C_7 n^3) d_n - (C_8 + C_9) n d_n'' + (C_2 + C_7 n^4) f_n - (2C_8 + C_9) n^2 f_n'' \\ &+ C_6 f_n'''' - \int_{\phi=0}^{2\pi} q_w(z,\phi) \sum_{n=1}^{\alpha} \cos(n\phi)\, r d\phi = 0\end{aligned}$$

(A3.8g) $$C_1 b_n'' - \frac{C_4 n^2}{r^2} b_n - (C_3 + C_5) n c_n' + C_3 g_n' + \int_{\phi=0}^{2\pi} q_u(z,\phi) \sin(n\phi)\, r d\phi = 0$$

(A3.8h)
$$\left(C_3+C_5\right)nb_n'-\left(C_2+C_7\right)n^2c_n+\left(C_4+C_9\right)c_n''+\left(C_2n+C_7n^3\right)g_n$$
$$-\left(C_8+C_9\right)ng_n''+\int_{\phi=0}^{2\pi}q_v\left(z,\phi\right)\cos\left(n\phi\right)rd\phi=0$$

(A3.8i)
$$C_3b_n'-\left(C_2n+C_7n^3\right)c_n+\left(C_8+C_9\right)nc_n''+\left(C_2+C_7n^4\right)g_n-\left(2C_8+C_9\right)n^2g_n''$$
$$+C_6g_n''''-\int_{\phi=0}^{2\pi}q_w(z,\phi)\sin\left(n\phi\right)rd\phi=0$$

In the boundary terms in Eq A3.7, the variation of the function can equal zero at either end of the element. Therefore, the term must be retained. Assuming no concentrated loads are applied at the end of the element, such as the loading case presented in Eq. 2.16, the resulting boundary terms for each mode $n$ are:

(A3.9a) $$\left[\left(2C_1a_0'+2C_3f_0\right)\delta a_0\right]_{z=0}^{l}=0$$

(A3.9b) $$\left[\left(2\left(C_4+C_9\right)c_0'\right)\delta c_0\right]_{z=0}^{l}=0$$

(A3.9c) $$\left[\left(-2C_6f_0'''\right)\delta f_0\right]_{z=0}^{l}=0$$

(A3.9d) $$\left[\left(2C_6f_0''\right)\delta f_0'\right]_{z=0}^{l}=0$$

(A3.9e) $$\left[\left(C_1a_n'+C_3nd_n+C_3f_n\right)\delta a_n\right]_{z=0}^{l}=0$$

(A3.9f) $$\left[\left(-C_5na_n+\left(C_4+C_9\right)d_n'+C_9nf_n'\right)\delta d_n\right]_{z=0}^{l}=0$$

(A3.9g) $$\left[\left(\left(C_8+C_9\right)nd_n'+\left(C_8+C_9\right)n^2f_n'-C_6f_n'''\right)\delta f_n\right]_{z=0}^{l}=0$$

(A3.9h) $$\left[\left(-C_8nd_n-C_8n^2f_n+C_6f_n''\right)\delta f_n'\right]_{z=0}^{l}=0$$

(A3.9i) $$\left[\left(C_1b_n'-C_3nc_n+C_3g_n\right)\delta b_n\right]_{z=0}^{l}=0$$

(A3.9j) $$\left[\left(C_5 n b_n + \left(C_4 + C_9\right) c_n' - C_9 n g_n'\right) \delta c_n\right]_{z=0}^{l} = 0$$

(A3.9k) $$\left[\left(-\left(C_8 + C_9\right) n c_n' + \left(C_8 + C_9\right) n^2 g_n' - C_6 g_n'''\right) \delta g_n\right]_{z=0}^{l} = 0$$

(A3.9l) $$\left[\left(C_8 n c_n - C_8 n^2 g_n + C_6 g_n''\right) \delta g_n'\right]_{z=0}^{l} = 0$$

# APPENDIX 4 – REDUCING INTEGRATION CONSTANTS FOR $n=0$

The solution of the equilibrium equations for $n=0$ in Eqs. 3.4 and 3.10 appears to have sixteen integration constants, but only eight boundary conditions are presented in Eq. 2.19. In this appendix the integration constants $\bar{C}_{0,i}$ and $\bar{F}_{0,i}$ are related to $\bar{A}_{0,i}$. As a result, only eight independent integration constants will remain which can be determined from the eight boundary conditions.

The simplified homogeneous equations for displacements $a_0(z)$, $c_0(z)$ and $f_0(z)$ for $n=0$ are given in Eq. 2.20 with the right hand side of the equation being $\{\mathbf{Q_0}\}_{3\times1}=\{\mathbf{0}\}_{3\times1}$:

(A4.1) $$[\mathbf{X_0}]_{3\times3}\begin{Bmatrix} a_0(z) \\ f_0(z) \\ c_0(z) \end{Bmatrix}_{3\times1} = \left[\begin{array}{c|c|c} 2C_1D^2 & 2C_3D & 0 \\ \hline 2C_3D & 2C_2+2C_6D^4 & 0 \\ \hline 0 & 0 & 2(C_4+C_9)D^2 \end{array}\right]\begin{Bmatrix} a_0(z) \\ f_0(z) \\ c_0(z) \end{Bmatrix}_{3\times1} = \{\mathbf{0}\}_{3\times1}$$

Expanding Eq. A4.1 gives:

(A4.2) $$C_1a_0''(z)+C_3f_0'(z)=0$$

(A4.3) $$C_3a_0'(z)+C_2f_0(z)+C_6f_0^{(4)}(z)=0$$

(A4.4) $$2(C_4+C_9)c_0''(z)=0$$

As shown in Chapter 3, Eqs. 3.10 and 3.4, the solution field for Eq. A4.1 is:

(A4.5) $$a_0(z)=\bar{A}_{0,1}+\bar{A}_{0,2}z+\bar{A}_{0,3}e^{m_{0,3}z}+\bar{A}_{0,4}e^{m_{0,4}z}+\bar{A}_{0,5}e^{m_{0,5}z}+\bar{A}_{0,6}e^{m_{0,6}z}$$

(A4.6) $$f_0(z)=\bar{F}_{0,1}+\bar{F}_{0,2}z+\bar{F}_{0,3}e^{m_{0,3}z}+\bar{F}_{0,4}e^{m_{0,4}z}+\bar{F}_{0,5}e^{m_{0,5}z}+\bar{F}_{0,6}e^{m_{0,6}z}$$

(A4.7) $$c_0(z)=\bar{A}_{0,7}+\bar{A}_{0,8}z$$

The integration constants in Eq. A4.7 are independent of all other integration constants.

The derivation of the solution field in Eqs A4.5 and A4.6 is based on the fact that the determinant of matrix $\mathbf{X_0}$ in Eq. A4.1 is zero. Therefore Eqs. A4.2 and A4.3, which are independent of Eq. A4.4, are linearly dependent. To relate constants $\overline{F}_{0,i}$ to constants $\overline{A}_{0,i}$, substitute Eqs. A4.5 and A4.6 into Eq. A4.2 or A4.3 (either one will do). After the substitutions, Eq. A4.2 becomes:

$$C_1\left(\sum_{i=3}^{6} m_{0,i}^2 \overline{A}_{0,i} e^{m_{0,i}z}\right) + C_3\left(\overline{F}_{0,2} + \sum_{i=3}^{6} m_{0,i}\overline{F}_{0,i}e^{m_{0,i}z}\right) = 0 \quad \text{(A4.8)}$$

Expanding Eq. A4.8 gives:

$$\begin{aligned} &C_3\overline{F}_{0,2} + \left(C_1 m_{0,3}^2 \overline{A}_{0,3} + C_3 m_{0,3}\overline{F}_{0,3}\right)e^{m_{0,3}z} + \left(C_1 m_{0,4}^2 \overline{A}_{0,4} + C_3 m_{0,4}\overline{F}_{0,4}\right)e^{m_{0,4}z} \\ &\quad + \left(C_1 m_{0,5}^2 \overline{A}_{0,5} + C_3 m_{0,5}\overline{F}_{0,5}\right)e^{m_{0,5}z} + \left(C_1 m_{0,6}^2 \overline{A}_{0,6} + C_3 m_{0,6}\overline{F}_{0,6}\right)e^{m_{0,6}z} = 0 \end{aligned} \quad \text{(A4.9)}$$

Eq. A4.9 must be valid for any value of $z$, giving the following:

$$C_3\overline{F}_{0,2} = 0 \rightarrow \overline{F}_{0,2} = 0 \quad \text{(A4.10)}$$

$$C_1 m_{0,i}^2 \overline{A}_{0,i} + C_3 m_{0,i}\overline{F}_{0,i} = 0 \qquad \text{for } i = 3 \text{ to } 6 \quad \text{(A4.11)}$$

From Eq. A4.11 $\overline{F}_{0,i}$ can be written in terms of $\overline{A}_{0,i}$ for $i = 3$ to 6:

$$\overline{F}_{0,i} = -\frac{C_1 m_{0,i}}{C_3}\overline{A}_{0,i} \qquad \text{for } i = 3 \text{ to } 6 \quad \text{(A4.12)}$$

Perform a similar substitution into Eq. A4.3 to find an expression for $\overline{F}_{0,1}$ and to check the relations in Eqs. A4.10 and A4.12:

$$C_3\left(\overline{A}_{0,2} + \sum_{i=3}^{6} m_{0,i}\overline{A}_{0,i}e^{m_{0,i}z}\right) + C_2\left(\overline{F}_{0,1} + \overline{F}_{0,2}z + \sum_{i=3}^{6}\overline{F}_{0,i}e^{m_{0,i}z}\right) + C_6\left(\sum_{i=3}^{6} m_{0,i}^4 \overline{F}_{0,i}e^{m_{0,i}z}\right) = 0 \quad \text{(A4.13)}$$

Expand Eq. A4.13:

$$\begin{aligned}(A4.14)\quad & \left(C_3\bar{A}_{0,2}+C_2\bar{F}_{0,1}\right)+C_2\bar{F}_{0,2}z+\left[C_3m_{0,3}\bar{A}_{0,3}+\left(C_2+C_6m_{0,3}^4\right)\bar{F}_{0,3}\right]e^{m_{0,3}z}+\\ & \left[C_3m_{0,4}\bar{A}_{0,4}+\left(C_2+C_6m_{0,4}^4\right)\bar{F}_{0,4}\right]e^{m_{0,4}z}+\left[C_3m_{0,5}\bar{A}_{0,5}+\left(C_2+C_6m_{0,5}^4\right)\bar{F}_{0,5}\right]e^{m_{0,5}z}+\\ & \left[C_3m_{0,6}\bar{A}_{0,6}+\left(C_2+C_6m_{0,6}^4\right)\bar{F}_{0,6}\right]e^{m_{0,6}z}=0\end{aligned}$$

Again, Eq. A4.14 must be valid for any value of $z$, resulting in:

(A4.15) $$\left(C_3\bar{A}_{0,2}+C_2\bar{F}_{0,1}\right)=0\rightarrow\bar{F}_{0,1}=-\frac{C_3}{C_2}\bar{A}_{0,2}$$

(A4.16) $$C_2\bar{F}_{0,2}z=0\rightarrow\bar{F}_{0,2}=0$$ (same expression as Eq. A4.10)

(A4.17) $$C_3m_{0,i}\bar{A}_{0,i}+\left(C_2+C_6m_{0,i}^4\right)\bar{F}_{0,i}=0$$ for $i$ = 3 to 6

From Eq. A4.17 another relationship is established between $\bar{A}_{0,i}$ and $\bar{F}_{0,i}$ for $i$ = 3 to 6:

(A4.18) $$\bar{F}_{0,i}=-\frac{C_3m_{0,i}}{C_2+m_{0,i}{}^4C_6}\bar{A}_{0,i}$$ for $i$ = 3 to 6

Eqs. A4.12 and A4.18 can be used as a check for the four roots, $m_{0,i}$ for $i$ = 3 to 6, given in Eq 3.9. For both Eqs. A4.12 and A4.18 to be valid their right hand sides must be equal:

(A4.19) $$\frac{C_3}{C_1m_{0,i}}=\frac{C_2+m_{0,i}{}^4C_6}{C_3m_{0,i}}$$

Expanding Eq. A4.19 leads to a fourth order equation in $m_{0,i}$:

(A4.20) $$\left(C_1C_2-C_3^2\right)+C_1C_6m_{0,i}{}^4=0$$

The four roots of Eq. A4.20 coincide with the roots determined in Section 3.1 (Eqs. 3.9c through 3.9f). Therefore, both Eqs. A4.12 and A4.18 are valid relationships between $\bar{A}_{0,i}$ and $\bar{F}_{0,i}$ for $i$ = 3 to 6. In fact, the two relationships are identical.

In summary the relationships between $\bar{A}_{0,i}$ and $\bar{F}_{0,i}$ are:

(A4.21) $\bar{F}_{0,1} = -\dfrac{C_3}{C_2}\bar{A}_{0,2} = \bar{\bar{F}}_{0,1}\bar{A}_{0,2}$

(A4.22) $\bar{F}_{0,2} = 0$

(A4.23) $\bar{F}_{0,i} = -\dfrac{C_1 m_{0,i}}{C_3}\bar{A}_{0,i} = -\dfrac{C_3 m_{0,i}}{C_2 + m_{0,i}{}^4 C_6}\bar{A}_{0,i} = \bar{\bar{F}}_{0,i}\bar{A}_{0,i}$ for $i = 3$ to 6

The integration constants $\bar{C}_{0,i}$ and $\bar{F}_{0,i}$ are now known in terms of the constants $\bar{A}_{0,i}$ so only eight independent integration constants are present, $\bar{A}_{0,i}$ for $i = 1$ to 8.

The solution to Eq. A4.1, given in Eqs. A4.5 through A4.7, can now be rewritten in matrix form:

(A4.22a) $$\begin{Bmatrix} a_0(z) \\ c_0(z) \\ f_0(z) \end{Bmatrix} = \begin{bmatrix} \langle e_0(z) \rangle^{\mathrm{T}}_{1\times 8} \\ \langle \bar{\bar{C}}_0(z) \rangle^{\mathrm{T}}_{1\times 8} \\ \langle \bar{\bar{F}}_0(z) \rangle^{\mathrm{T}}_{1\times 8} \end{bmatrix} \{\bar{A}_0\}_{8\times 1}$$

(A4.22b) $$\begin{Bmatrix} a_0(z) \\ c_0(z) \\ f_0(z) \end{Bmatrix} = \begin{bmatrix} 1 & z & e^{m_{0,3}z} & e^{m_{0,4}z} & e^{m_{0,5}z} & e^{m_{0,6}z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \\ 0 & \bar{\bar{F}}_{0,1} & \bar{\bar{F}}_{0,3}e^{m_{0,3}z} & \bar{\bar{F}}_{0,4}e^{m_{0,4}z} & \bar{\bar{F}}_{0,5}e^{m_{0,5}z} & \bar{\bar{F}}_{0,6}e^{m_{0,6}z} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{A}_{0,1} \\ \bar{A}_{0,2} \\ \bar{A}_{0,3} \\ \bar{A}_{0,4} \\ \bar{A}_{0,5} \\ \bar{A}_{0,6} \\ \bar{A}_{0,7} \\ \bar{A}_{0,8} \end{Bmatrix}$$

in which

(A4.23) $\bar{\bar{F}}_{0,1} = -\dfrac{C_3}{C_2}$

(A4.24) $\bar{\bar{F}}_{0,i} = -\dfrac{C_1 m_{0,i}}{C_3}$ for $i = 3$ to 6

# Appendix 5 – Determinant of Field Equation Matrix

The operator matrix for the field equations for Fourier modes $n \geq 1$ is given in Eq. 3.16 as:

(A5.1) $$[\mathbf{Y_n}]_{3\times3} = \left[\begin{array}{c|c|c} C_1 m_n^2 - C_{10,n} & C_{11,n} m_n & C_3 m_n \\ \hline C_{11,n} m_n & C_{12,n} - C_{13} m_n^2 & C_{14,n} - C_{15,n} m_n^2 \\ \hline C_3 m_n & C_{14,n} - C_{15,n} m_n^2 & C_6 m_n^4 - C_{17,n} m_n^2 + C_{16,n} \end{array}\right]$$

In the solution of the homogeneous case it is required to find the values of $m_n$ for which the determinant of $\mathbf{Y_n}$ in Eq. A5.1 is zero. In this appendix the determinant of $\mathbf{Y_n}$ is expanded to give an 8th order polynomial in $m_n$ which is subsequently solved in Chapter 3.

The matrix $\mathbf{Y_n}$ in Eq. A5.1 has the form:

(A5.2) $$\mathbf{C} = \left[\begin{array}{c|c|c} a & b & c \\ \hline b & d & f \\ \hline c & f & g \end{array}\right]$$

Expanding the determinant of $\mathbf{C}$ in Eq. A5.2 gives:

(A5.3) $$|\mathbf{C}| = a\left|\begin{array}{c|c} d & f \\ \hline f & g \end{array}\right| - b\left|\begin{array}{c|c} b & f \\ \hline c & g \end{array}\right| + c\left|\begin{array}{c|c} b & d \\ \hline c & f \end{array}\right|$$

(A5.4) $$|\mathbf{C}| = a\left(dg - f^2\right) - b\left(bg - fc\right) + c\left(bf - dc\right)$$

(A5.5) $$|\mathbf{C}| = adg - af^2 - b^2 g + bfc + cbf - dc^2$$

From Eq. A5.1 the values of *a* through *f* are:

(A5.6) $$\begin{aligned} a &= C_1 m_n^2 - C_{10,n} & d &= C_{12,n} - C_{13} m_n^2 \\ b &= C_{11,n} m_n & f &= C_{14,n} - C_{15,n} m_n^2 \\ c &= C_3 m_n & g &= C_6 m_n^4 - C_{17,n} m_n^2 + C_{16,n} \end{aligned}$$

Substitute Eq. A5.6 into Eq. A5.5 to obtain:

$$
\begin{aligned}
|\mathbf{Y_n}| = {} & \left(C_1 m_n^{\,2} - C_{10,n}\right)\left(C_{12,n} - C_{13} m_n^{\,2}\right)\left(C_6 m_n^{\,4} - C_{17,n} m_n^{\,2} + C_{16,n}\right) - \\
& \left(C_1 m_n^{\,2} - C_{10,n}\right)\left(C_{14,n} - C_{15,n} m_n^{\,2}\right)^2 - \left(C_{11,n} m_n\right)^2 \left(C_6 m_n^{\,4} - C_{17,n} m_n^{\,2} + C_{16,n}\right) + \\
& \left(C_{11,n} m_n\right)\left(C_{14,n} - C_{15,n} m_n^{\,2}\right)\left(C_3 m_n\right) + \left(C_3 m_n\right)\left(C_{11,n} m_n\right)\left(C_{14,n} - C_{15,n} m_n^{\,2}\right) \\
& - \left(C_{12,n} - C_{13} m_n^{\,2}\right)\left(C_3 m_n\right)^2
\end{aligned}
\tag{A5.7}
$$

Expand the brackets of Eq. A5.7 and rearrange:

$$
\begin{aligned}
|\mathbf{Y_n}| = {} & \left(C_1 C_{12,n} m_n^{\,2} - C_1 C_{13} m_n^{\,4} - C_{10,n} C_{12,n} + C_{10,n} C_{13} m_n^{\,2}\right)\left(C_6 m_n^{\,4} - C_{17,n} m_n^{\,2} + C_{16,n}\right) - \\
& \left(C_1 m_n^{\,2} - C_{10,n}\right)\left(C_{14,n}^{\,2} - 2 C_{14,n} C_{15,n} m_n^{\,2} + C_{15,n}^{\,2} m_n^{\,4}\right) - \left(C_{11,n}^{\,2} m_n^{\,2}\right)\left(C_6 m_n^{\,4} - C_{17,n} m_n^{\,2} + C_{16,n}\right) + \\
& \left(C_3 C_{11,n} m_n^{\,2}\right)\left(C_{14,n} - C_{15,n} m_n^{\,2}\right) + \left(C_3 C_{11,n} m_n^{\,2}\right)\left(C_{14,n} - C_{15,n} m_n^{\,2}\right) - \left(C_{12,n} - C_{13} m_n^{\,2}\right)\left(C_3^{\,2} m_n^{\,2}\right)
\end{aligned}
\tag{A5.8a}
$$

$$
\begin{aligned}
|\mathbf{Y_n}| = {} & C_1 C_6 C_{12,n} m_n^{\,6} - C_1 C_{12,n} C_{17,n} m_n^{\,4} + C_1 C_{12,n} C_{16,n} m_n^{\,2} - C_1 C_6 C_{13} m_n^{\,8} + C_1 C_{13} C_{17,n} m_n^{\,6} \\
& - C_1 C_{13} C_{16,n} m_n^{\,4} - C_6 C_{10,n} C_{12,n} m_n^{\,4} + C_{10,n} C_{12,n} C_{17,n} m_n^{\,2} - C_{10,n} C_{12,n} C_{16,n} + C_6 C_{10,n} C_{13} m_n^{\,6} \\
& - C_{10,n} C_{13} C_{17,n} m_n^{\,4} + C_{10,n} C_{13} C_{16,n} m_n^{\,2} - \left(C_1 C_{14,n}^{\,2} m_n^{\,2} - 2 C_1 C_{14,n} C_{15,n} m_n^{\,4} + C_1 C_{15,n}^{\,2} m_n^{\,6}\right. \\
& \left. - C_{10,n} C_{14,n}^{\,2} + 2 C_{10,n} C_{14,n} C_{15,n} m_n^{\,2} - C_{10,n} C_{15,n}^{\,2} m_n^{\,4}\right) - \left(C_6 C_{11,n}^{\,2} m_n^{\,6} - C_{11,n}^{\,2} C_{17,n} m_n^{\,4}\right. \\
& \left. + C_{11,n}^{\,2} C_{16,n} m_n^{\,2}\right) + 2\left(C_3 C_{11,n} C_{14,n} m_n^{\,2} - C_3 C_{11,n} C_{15,n} m_n^{\,4}\right) - \left(C_3^{\,2} C_{12,n} m_n^{\,2} - C_3^{\,2} C_{13} m_n^{\,4}\right)
\end{aligned}
\tag{A5.8b}
$$

Collect like terms of $m_n$ from Eq. A5.8 to obtain the final form of the determinant of $\mathbf{Y_n}$:

$$
\begin{aligned}
|\mathbf{Y_n}| = {} & \left[-C_1 C_6 C_{13}\right] m_n^{\,8} + \\
& \left[C_1 \left(C_6 C_{12,n} + C_{13} C_{17,n} - C_{15,n}^{\,2}\right) + C_6 \left(C_{10,n} C_{13} - C_{11,n}^{\,2}\right)\right] m_n^{\,6} + \\
& \left[C_1 \left(-C_{12,n} C_{17,n} - C_{13} C_{16,n} + 2 C_{14,n} C_{15,n}\right) - C_{10,n} \left(C_6 C_{12,n} + C_{13} C_{17,n} - C_{15,n}^{\,2}\right) + \right. \\
& \left. C_{11,n}^{\,2} C_{17,n} - 2 C_3 C_{11,n} C_{15,n} + C_3^{\,2} C_{13,n}\right] m_n^{\,4} + \\
& \left[C_1 \left(C_{12,n} C_{16,n} - C_{14,n}^{\,2}\right) + C_{10,n} \left(C_{12,n} C_{17,n} + C_{13} C_{16,n} - 2 C_{14,n} C_{15,n}\right) + 2 C_3 C_{11,n} C_{14} - C_{11,n}^{\,2} C_{16,n} \right. \\
& \left. - C_3^{\,2} C_{12,n} - C_{10,n} \left(C_{12,n} C_{16,n} - C_{14,n}^{\,2}\right)\right] m_n^{\,2}
\end{aligned}
\tag{A5.9}
$$

# APPENDIX 6 – QUARTIC EQUATION SOLUTION: FERRARI'S METHOD

The determinant of the operator matrix for the homogeneous case of the field equations reduces to the fourth order polynomial in $M_n$ given in Eq. 3.22. The four roots of the fourth order polynomial (quartic) equation are found by using Ferrari's Method which is described below (Wikipedia, 2006).

In general, assume the polynomial has the following form:

(A6.1) $$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

Divide Eq. A6.1 by $A$ to obtain:

(A6.2) $$x^4 + \frac{B}{A}x^3 + \frac{C}{A}x^2 + \frac{D}{A}x + \frac{E}{A} = 0$$

Define $x$ in terms of a new variable $u$:

(A6.3) $$x = u - \frac{B}{4A}$$

Substitute Eq. A6.3 into Eq. A6.2 to eliminate the $x^3$ term:

(A6.4) $$\left(u - \frac{B}{4A}\right)^4 + \frac{B}{A}\left(u - \frac{B}{4A}\right)^3 + \frac{C}{A}\left(u - \frac{B}{4A}\right)^2 + \frac{D}{A}\left(u - \frac{B}{4A}\right) + \frac{E}{A} = 0$$

Expand the powers of Eq. A6.4 and regroup like powers of $u$ to give:

(A6.5) $$u^4 + \left(\frac{-3B^2}{8A^2} + \frac{C}{A}\right)u^2 + \left(\frac{B^3}{8A^3} - \frac{BC}{2A^2} + \frac{D}{A}\right)u + \left(\frac{-3B^4}{256A^4} + \frac{B^2C}{16A^3} - \frac{BD}{4A^2} + \frac{E}{A}\right) = 0$$

Define new constants for the coefficients of $u$:

(A6.6a) $$\alpha = \frac{-3B^2}{8A^2} + \frac{C}{A}$$

(A6.6b) $\beta = \frac{B^3}{8A^3} - \frac{BC}{2A^2} + \frac{D}{A}$

(A6.6c) $\gamma = \frac{-3B^4}{256A^4} + \frac{B^2C}{16A^3} - \frac{BD}{4A^2} + \frac{E}{A}$

Use the new constants in Eq. A6.6 to rewrite equation A6.5:

(A6.7) $u^4 + \alpha u^2 + \beta u + \gamma = 0$

The objective now is to complete the square in Eq. A6.7 and make the equation a perfect square in terms of $u$. First add the following identity to Eq. A6.7:

(A6.8) $\left(u^2 + \alpha\right)^2 - u^4 - 2\alpha u^2 = \alpha^2$

After the addition Eq. A6.7 becomes:

(A6.9) $\left(u^2 + \alpha\right)^2 + \beta u + \gamma = \alpha u^2 + \alpha^2$

Introduce a new variable $y$ that is defined as follows:

(A6.10) $\left(u^2 + \alpha + y\right)^2 - \left(u^2 + \alpha\right)^2 = 2yu^2 + 2y\alpha + y^2$

(A6.11) $0 = (\alpha + 2y)u^2 - 2yu^2 - \alpha u^2$

Add Eqs. A6.10 and A6.11 to Eq. A6.9:

(A6.12) $\left(u^2 + \alpha + y\right)^2 + \beta u + \gamma = (\alpha + 2y)u^2 + \alpha^2 + 2y\alpha + y^2$

Rearrange Eq. A6.12:

(A6.13) $\left(u^2 + \alpha + y\right)^2 = (\alpha + 2y)u^2 - \beta u + \left(y^2 + 2y\alpha + \alpha^2 - \gamma\right)$

Find the value of $y$ for which the right hand side becomes a perfect square by solving the following equation for $y$:

$$(A6.14)\quad \begin{aligned}&(-\beta)^2-4(2y+\alpha)\left(y^2+2y\alpha+\alpha^2-\gamma\right)\\&=\beta^2-4\left[2y^3+5\alpha y^2+\left(4\alpha^2-2\gamma\right)y+\left(\alpha^3-\alpha\gamma\right)\right]=0\end{aligned}$$

Divide Eq. A6.14 by –8 and rearrange to obtain a cubic equation for $y$:

$$(A6.15)\quad y^3+\frac{5}{3}\alpha y^2+(2\alpha-\lambda)y+\left(\frac{\alpha^3}{2}-\frac{\alpha\gamma}{2}-\frac{\beta^2}{8}\right)=0$$

The cubic equation in Eq. A6.15 for $y$ must be solved. Define $y$ in terms of a new variable $v$:

$$(A6.16)\quad y=v-\frac{5}{6}\alpha$$

Substitute Eq. A6.16 into Eq. A6.15, expand the powers and rearrange to obtain:

$$(A6.17)\quad v^3+\left(-\frac{\alpha^2}{12}-\gamma\right)v+\left(-\frac{\alpha^3}{108}+\frac{\alpha\gamma}{3}-\frac{\beta^2}{8}\right)=0$$

Define new constants for the coefficients of $v$:

$$(A6.18a)\quad P=-\frac{\alpha^2}{12}-\gamma$$

$$(A6.18b)\quad Q=-\frac{\alpha^3}{108}+\frac{\alpha\gamma}{3}-\frac{\beta^2}{8}$$

Use the new constants in Eq. A6.18 to rewrite Eq A6.17:

$$(A6.19)\quad v^3+Pv+Q=0$$

The solution to Eq. A6.19 can be obtained from the solution of the cubic equation. Any root will be sufficient. Define the variable $U$ as:

$$(A6.20)\quad U=\sqrt[3]{\frac{Q}{2}\pm\sqrt{\frac{Q^2}{4}+\frac{P^3}{27}}}$$

The solution to Eq. A6.19 is:

$$v = \frac{P}{3U} - U \quad \text{(A6.21)}$$

The solution for the original nested cubic equation in Eq. A6.15 is:

$$y = -\frac{5}{6}\alpha + \frac{P}{3U} - U \quad \text{(A6.22)}$$

Since the value of $y$ is known the right hand side of Eq. A6.13 can now be written as a perfect square:

$$(\alpha + 2y)u^2 - \beta u + (y^2 + 2y\alpha + \alpha^2 - \gamma) = \left[\left(\sqrt{\alpha + 2y}\right)u + \frac{-\beta}{2\sqrt{\alpha + 2y}}\right]^2 \quad \text{(A6.23)}$$

Eq. A6.13 is rewritten as:

$$\left(u^2 + \alpha + y\right)^2 = \left[\left(\sqrt{\alpha + 2y}\right)u - \frac{\beta}{2\sqrt{\alpha + 2y}}\right]^2 \quad \text{(A6.24)}$$

Take square root of each side of Eq. A6.24:

$$\left(u^2 + \alpha + y\right) = \pm\left[\left(\sqrt{\alpha + 2y}\right)u - \frac{\beta}{2\sqrt{\alpha + 2y}}\right] \quad \text{(A6.25)}$$

Collect like powers of $u$ to express Eq. A6.25 in the form of a quadratic equation in $u$:

$$u^2 + \left(\pm_1 \sqrt{\alpha + 2y}\right)u + \left(\alpha + y \mp_1 \frac{\beta}{2\sqrt{\alpha + 2y}}\right) = 0 \quad \text{(A6.26)}$$

Note that the subscripts on the ± symbols indicate that they are not independent. The values of the two must be opposite; if the value of the first ± is + then the second must be – and vice versa.

Solve Eq. A6.26 by the quadratic equation:

(A6.27a) $$u=\frac{\pm_1\sqrt{\alpha+2y}\pm_2\sqrt{(\alpha+2y)-4\left(\alpha+y\pm_1\frac{\beta}{2\sqrt{\alpha+2y}}\right)}}{2}$$

(A6.27b) $$u=\frac{\pm_1\sqrt{\alpha+2y}\pm_2\sqrt{-\left(3\alpha+2y\pm_1\frac{2\beta}{\sqrt{\alpha+2y}}\right)}}{2}$$

Note that $\pm_1$ must now have the same sign at both places it occurs in Eq. A6.27 (since the first one is multiplied by $-1$ in the quadratic equation) and that $\pm_1$ is independent of $\pm_2$.

Use the relation from Eq. A6.3 to obtain the final form of the roots of the quartic equation:

(A6.28) $$x=-\frac{B}{4A}+\frac{\pm_1\sqrt{\alpha+2y}\pm_2\sqrt{-\left(3\alpha+2y\pm_1\frac{2\beta}{\sqrt{\alpha+2y}}\right)}}{2}$$

The four roots are found from the four possible combinations of $\pm_1$ and $\pm_2$.

# APPENDIX 7 – MULTIPLE ROOTS FOR FOURIER MODE $n=1$

The determinant of the operator matrix for the field equations, Eq. 3.20, must vanish as described in Chapter 3. The resulting equation is an eighth order polynomial in $m_n$, or fourth order in $M_n = m_n^2$. It is shown in this appendix that for mode $n=1$ two of the roots of the polynomial are zero, leaving only a quadratic polynomial in $M_n$. For modes $n \geq 2$, none of the roots of the polynomial are zero and the polynomial is truly fourth order in $M_n$.

The fourth order polynomial in $M_n$ was determined in Eq. 3.22 to be:

(A7.1) $$p_{1,n}M_n^{\ 4} + p_{2,n}M_n^{\ 3} + p_{3,n}M_n^{\ 2} + p_{4,n}M_n + p_{5,n} = 0$$

in which

(A7.2a) $$p_{1,n} = -C_1C_6C_{13}$$

(A7.2b) $$p_{2,n} = C_1\left(C_6C_{12,n} + C_{13}C_{17,n} - C_{15,n}^{\ 2}\right) + C_6\left(C_{10,n}C_{13} - C_{11,n}^{\ 2}\right)$$

(A7.2c) $$\begin{aligned} p_{3,n} &= C_1\left(-C_{12,n}C_{17,n} - C_{13}C_{16,n} + 2C_{14,n}C_{15,n}\right) - C_{10,n}\left(C_6C_{12,n} + C_{13}C_{17,n} - C_{15,n}^{\ 2}\right) \\ &\quad + C_{11,n}^{\ 2}C_{17,n} - 2C_3C_{11,n}C_{15,n} + C_3^{\ 2}C_{13} \end{aligned}$$

(A7.2d) $$\begin{aligned} p_{4,n} &= C_1\left(C_{12,n}C_{16} - C_{14,n}^{\ 2}\right) + C_{10,n}\left(C_{12,n}C_{17,n} + C_{13}C_{16,n} - 2C_{14,n}C_{15,n}\right) + 2C_3C_{11,n}C_{14,n} \\ &\quad - C_{11,n}^{\ 2}C_{16,n} - C_3^{\ 2}C_{12,n} \end{aligned}$$

(A7.2e) $$p_{5,n} = -C_{10,n}\left(C_{12,n}C_{16,n} - C_{14,n}^{\ 2}\right)$$

Multiple roots occur at $M_n = 0$ if certain values of $p_{j,n}$ for $j$ = 1 to 5 are zero. In particular, if $p_{5,n} = 0$ then roots occur at $M_n = 0$ and Eq. A7.1 reduces to a third-order polynomial. Also, if both $p_{4,n} = 0$ and $p_{5,n} = 0$ then double roots occur at $M_n = 0$ and Eq. A7.1 reduces to a quadratic polynomial. It is shown subsequently for mode $n=1$ that $p_{4,1} = p_{5,1} = 0$ and double roots occur at $M_1 = 0$. It is also concurrently shown that, in general, $p_{j,n} \neq 0$ for j=1 to 5 for all $n \geq 2$ and four independent roots $M_n$ exist.

It is required to expand Eqs. A7.2 using the relations in Eqs. 2.15j through 2.25q to determine which of the coefficients $p_{j,n}$ are zero. Consider first the expression for $p_{1,n}$ in Eq. A72.a:

$$\begin{aligned} p_{1,n} &= -C_1 C_6 C_{13} \\ p_{1,n} &= -C_1 C_6 \left(C_4 + C_9\right) \end{aligned} \tag{A7.3}$$

In Eqs. 2.15a through 2.15i, all of the constants $C_1$ through $C_9$ are real and positive. Therefore, it is not possible to have $C_4 = -C_9$, which leads to $p_{1,n} \neq 0$ for all values of $n$.

From Eq. A7.2b, the expression for $p_{2,n}$ is expanded as

$$\begin{aligned} p_{2,n} &= C_1\left(C_6 C_{12,n} + C_{13} C_{17,n} - C_{15,n}{}^2\right) + C_6\left(C_{10,n} C_{13} - C_{11,n}{}^2\right) \\ p_{2,n} &= n^2\left[C_1 C_2\left(C_6 + C_7\right) + C_1 C_4\left(2C_8 + C_9\right) - C_1 C_8{}^2 + \frac{C_4 C_6 C_9}{r^2} - C_3{}^2 C_6 - \frac{2C_3 C_4 C_6}{r}\right] \end{aligned} \tag{A7.4}$$

Again, as the constants are all positive and non-zero it follows that $p_{2,n} \neq 0$ for all values of $n$. Expand $p_{3,n}$ from Eq. A7.2c as

$$\begin{aligned} p_{3,n} &= C_1\left(-C_{12,n} C_{17,n} - C_{13} C_{16,n} + 2C_{14,n} C_{15,n}\right) - C_{10,n}\left(C_6 C_{12,n} + C_{13} C_{17,n} - C_{15,n}{}^2\right) \\ &\quad + C_{11,n}{}^2 C_{17,n} - 2C_3 C_{11,n} C_{15,n} + C_3{}^2 C_{13} \\ p_{3,n} &= C_1\left[2C_2 C_8 n^2\left(1-n^2\right) - C_2 C_9\left(1-n^2\right)^2 - C_4 n^2\left(C_2 + C_7\right)\right] - \frac{C_4 n^2}{r^2}\left[C_6\left(C_2 + C_7\right) - C_8{}^2\right] \\ &\quad + 2C_3{}^2 C_8 n^2\left(n^2-1\right) + \frac{2C_3 C_4 C_8 n^2\left(2n^2-1\right)}{r} + C_3{}^2 C_9\left(n^2-1\right)^2 + \frac{2C_3 C_4 C_9 n^2\left(n^2-1\right)}{r} + C_3{}^2 C_4 \end{aligned} \tag{A7.5}$$

For the general case of $n \geq 2$, it is apparent that $p_{3,n} \neq 0$. However, for the special case of $n = 1$, Eq. A7.5 simplifies to

$$p_{3,1} = C_4\left[-\left(C_1+\frac{C_6}{r^2}\right)(C_2+C_7)+\left(\frac{C_8}{r}+C_3\right)^2\right] \tag{A7.6}$$

Using the relations in Eqs. 2.15a through 2.15h, Eq. A7.6 is further simplified to:

$$p_{3,1} = \frac{Ert\pi}{1+\mu}\left[-\frac{E^2t^2\pi^2\left(12r^2+t^2\right)^2}{144r^2\left(1-\mu^2\right)}\right] \tag{A7.7}$$

For the right hand side of Eq. A7.7 to be zero the relation $12r^2 = -t^2$ must hold, which it does not since both $r$ and $t$ are real numbers. Therefore, it follows that $p_{3,n} \neq 0$ for all values of $n \geq 1$.

The expression in Eq. A7.2d for $p_{4,n}$ is expanded as

$$\begin{aligned} p_{4,n} &= C_1\left(C_{12,n}C_{16}-C_{14,n}{}^2\right)+C_{10,n}\left(C_{12,n}C_{17,n}+C_{13}C_{16,n}-2C_{14,n}C_{15,n}\right)+2C_3C_{11,n}C_{14,n} \\ &\quad -C_{11,n}{}^2C_{16,n}-C_3^2C_{12,n} \\ p_{4,n} &= C_1C_2C_7n^2\left(1-n^2\right)^2+\frac{C_4n^2}{r^2}\left[C_2C_9\left(1-n^2\right)^2-2C_2C_8n^2\left(1-n^2\right)^2\right] \\ &\quad -C_3^2C_7n^2\left(1-n^2\right)^2+\frac{2C_3C_4C_7n^4\left(1-n^2\right)}{r} \end{aligned} \tag{A7.8}$$

For the general case of $n \geq 2$, it is apparent that $p_{4,n} \neq 0$. However, for the special case of $n=1$, the right hand side of Eq. A7.8 vanishes leaving $p_{4,1}=0$.

Finally, from Eq. A7.2e, $p_{5,n}$ is expanded as

$$\begin{aligned} p_{5,n} &= -C_{10,n}\left(C_{12,n}C_{16,n}-C_{14,n}{}^2\right) \\ p_{5,n} &= -\frac{C_4n^2}{r^2}\left[C_2C_7n^2\left(1-n^2\right)^2\right] \end{aligned} \tag{A7.9}$$

It is apparent from Eq. A7.9 that $p_{5,1}=0$ and $p_{5,n} \neq 0$ for all $n \geq 2$.

For Fourier mode $n=1$ two of the polynomial coefficients, namely $p_{4,1}$ and $p_{5,1}$, in Eq. A7.2 are zero. Therefore, Eq. A7.1 reduces to

(A7.10) $$p_{1,1}M_1^{\,4} + p_{2,1}M_1^{\,3} + p_{3,1}M_1^{\,2} = M_1^{\,2}\left(p_{1,1}M_1^{\,2} + p_{2,1}M_1 + p_{3,1}\right) = 0$$

Two of the roots in Eq. A7.10, $M_{1,1}$ and $M_{1,2}$, are zero leading to $m_{1,1} = m_{1,2} = m_{1,3} = m_{1,4} = 0$. The remaining two roots of Eq. A7.10, $M_{1,3}$ and $M_{1,4}$, are determined by solving the quadratic equation $p_{1,1}M_1^{\,2} + p_{2,1}M_1 + p_{3,1} = 0$ as shown in Chapter 3.

For modes $n \geq 2$ all of the polynomial coefficients in Eq. A7.2 are non-zero and no repeated roots occur at $M_n = 0$. However, it is possible for repeated roots to occur at $M_n \neq 0$ for specific geometric and material properties of the pipe. In these circumstances the repeated roots will be apparent after solving the polynomial with Ferrari's Method (see Appendix 6) and handled accordingly on a case by case basis.

# APPENDIX 8 – REDUCING INTEGRATION CONSTANTS FOR $n = 1$

The solution of the equilibrium equations for $n = 1$ in Eq. 3.26 has twenty-four integration constants, but only eight boundary conditions are presented in Eq. 2.19. In this appendix two sets of integration constants, $\overline{A}_{1,i}$ and $\overline{D}_{1,i}$, are related to the third set, $\overline{F}_{1,i}$. As a result, only eight independent integration constants will remain which can be determined from the eight boundary conditions.

The simplified homogeneous equations for displacements $a_1(z)$, $d_1(z)$ and $f_1(z)$ for $n = 1$ are given in Eq. 2.27 with the right hand side of the equation being $\{\mathbf{Q}_1\}_{3\times 1} = \{\mathbf{0}\}_{3\times 1}$:

(A8.1) $$[\mathbf{X}_1]_{3\times 3}\begin{Bmatrix} a_1(z) \\ d_1(z) \\ f_1(z) \end{Bmatrix} = \left[\begin{array}{c|c|c} C_1D^2 - C_{10,1} & C_{11,1}D & C_3D \\ \hline C_{11,1}D & C_{12,1} - C_{13}D^2 & C_{14,1} - C_{15,1}D^2 \\ \hline C_3D & C_{14,1} - C_{15,1}D^2 & C_{16,1} - C_{17,1}D^2 + C_6D^4 \end{array}\right]\begin{Bmatrix} a_1(z) \\ d_1(z) \\ f_1(z) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Expanding Eq. A8.1 gives:

(A8.2) $$C_1a_1''(z) - C_{10,1}a_1(z) + C_{11,1}d_1'(z) + C_3f_1'(z) = 0$$

(A8.3) $$C_{11,1}a_1'(z) + C_{12,1}d_1(z) - C_{13}d_1''(z) + C_{14,1}f_1(z) - C_{15,1}f_1''(z) = 0$$

(A8.4) $$C_3a_1'(z) + C_{14,1}d_1(z) - C_{15,1}d_1''(z) + C_{16,1}f_1(z) - C_{17,1}f_1''(z) + C_6f_1^{(4)}(z) = 0$$

As given in Eq. 3.26, the solution field for Eq. A8.1 is:

(A8.5) $$a_1(z) = \overline{A}_{1,1} + \overline{A}_{1,2}z + \overline{A}_{1,3}z^2 + \overline{A}_{1,4}z^3 + \overline{A}_{1,5}e^{m_{1,5}z} + \overline{A}_{1,6}e^{m_{1,6}z} + \overline{A}_{1,7}e^{m_{1,7}z} + \overline{A}_{1,8}e^{m_{1,8}z}$$

(A8.6) $$d_1(z) = \overline{D}_{1,1} + \overline{D}_{1,2}z + \overline{D}_{1,3}z^2 + \overline{D}_{1,4}z^3 + \overline{D}_{1,5}e^{m_{1,5}z} + \overline{D}_{1,6}e^{m_{1,6}z} + \overline{D}_{1,7}e^{m_{1,7}z} + \overline{D}_{1,8}e^{m_{1,8}z}$$

(A8.7) $$f_1(z) = \overline{F}_{1,1} + \overline{F}_{1,2}z + \overline{F}_{1,3}z^2 + \overline{F}_{1,4}z^3 + \overline{F}_{1,5}e^{m_{1,5}z} + \overline{F}_{1,6}e^{m_{1,6}z} + \overline{F}_{1,7}e^{m_{1,7}z} + \overline{F}_{1,8}e^{m_{1,8}z}$$

The derivation of the solution field in Eqs A8.5 through A8.7 is based on the fact that the determinant of the matrix in the left hand side of Eq. A8.1 is zero. Therefore Eqs. A8.2, A8.3 and A8.4 are linearly dependent. Constants $\bar{A}_{1,i}$ and $\bar{D}_{1,i}$ can be related to constants $\bar{F}_{1,i}$ by substituting Eqs A8.5 through A8.7 into Eqs. A8.2, A8.3 and A8.4. After the substitutions Eq. A8.2 becomes:

$$\begin{aligned}
&C_{1,1}\left(2\bar{A}_{1,3}+6\bar{A}_{1,4}z+m_{1,5}^2\bar{A}_{1,5}e^{m_{1,5}z}+m_{1,6}^2\bar{A}_{1,6}e^{m_{1,6}z}+m_{1,7}^2\bar{A}_{1,7}e^{m_{1,7}z}+m_{1,8}^2\bar{A}_{1,8}e^{m_{1,8}z}\right)-\\
&C_{10,1}\left(\bar{A}_{1,1}+\bar{A}_{1,2}z+\bar{A}_{1,3}z^2+\bar{A}_{1,4}z^3+\bar{A}_{1,5}e^{m_{1,5}z}+\bar{A}_{1,6}e^{m_{1,6}z}+\bar{A}_{1,7}e^{m_{1,7}z}+\bar{A}_{1,8}e^{m_{1,8}z}\right)+\\
&C_{11,1}\left(\bar{D}_{1,2}+2\bar{D}_{1,3}z+3\bar{D}_{1,4}z^2+m_{1,5}{}^2\bar{D}_{1,5}e^{m_{1,5}z}+m_{1,6}{}^2\bar{D}_{1,6}e^{m_{1,6}z}+m_{1,7}{}^2\bar{D}_{1,7}e^{m_{1,7}z}+m_{1,8}{}^2\bar{D}_{1,8}e^{m_{1,8}z}\right)+\\
&C_{3,1}\left(\bar{F}_{1,2}+2\bar{F}_{1,3}z+3\bar{F}_{1,4}z^2+m_{1,5}{}^2\bar{F}_{1,5}e^{m_{1,5}z}+m_{1,6}{}^2\bar{F}_{1,6}e^{m_{1,6}z}+m_{1,7}{}^2\bar{F}_{1,7}e^{m_{1,7}z}+m_{1,8}{}^2\bar{F}_{1,8}e^{m_{1,8}z}\right)=0
\end{aligned} \tag{A8.8}$$

Eq. A8.8 must be valid for any value of $z$. Thus the coefficients of $z^0$, $z$, $z^2$, $z^3$ and $e^{m_{1,i}z}$ must vanish. Rearrange Eq. A8.8 to group like terms of $z$:

(A8.9a) $$2C_1\bar{A}_{1,3}-C_{10,1}\bar{A}_{1,1}+C_{11,1}\bar{D}_{1,2}+C_3\bar{F}_{1,2}=0$$

(A8.9b) $$\left(6C_1\bar{A}_{1,4}-C_{10,1}\bar{A}_{1,2}+2C_{11,1}\bar{D}_{1,3}+2C_3\bar{F}_{1,3}\right)z=0$$

(A8.9c) $$\left(-C_{10,1}\bar{A}_{1,3}+3C_{11,1}\bar{D}_{1,4}+3C_3\bar{F}_{1,4}\right)z^2=0$$

(A8.9d) $$-C_{10,1}\bar{A}_{1,4}z^3=0$$

(A8.9e) $$\left[\left(C_1m_{1,i}^2-C_{10,1}\right)\bar{A}_{1,i}+C_{11,1}m_{1,i}{}^2\bar{D}_{1,i}+C_3m_{1,i}{}^2\bar{F}_{1,i}\right]e^{m_{1,i}z}=0 \quad \text{for } i=5 \text{ to } 8$$

Perform a similar substitution to Eq. A8.3 and rearrange to obtain:

(A8.10a) $$C_{11,1}\bar{A}_{1,2}+C_{12,1}\bar{D}_{1,1}-2C_{13}\bar{D}_{1,3}+C_{14,1}\bar{F}_{1,1}-2C_{15,1}\bar{F}_{1,3}=0$$

(A8.10b) $$\left(2C_{11,1}\bar{A}_{1,3}+C_{12,1}\bar{D}_{1,2}-6C_{13}\bar{D}_{1,4}+C_{14,1}\bar{F}_{1,2}-6C_{15,1}\bar{F}_{1,4}\right)z=0$$

(A8.10c) $\left(3C_{11,1}\bar{A}_{1,4}+C_{12,1}\bar{D}_{1,3}+C_{14,1}\bar{F}_{1,3}\right)z^2=0$

(A8.10d) $\left(C_{12,1}\bar{D}_{1,4}+C_{14,1}\bar{F}_{1,4}\right)z^3=0$

(A8.10e) $\left[C_{11,1}m_{1,i}\bar{A}_{1,i}+\left(C_{12,1}-C_{13}m_{1,i}^2\right)\bar{D}_{1,i}+\left(C_{14,1}-C_{15,1}m_{1,i}^2\bar{F}_{1,i}\right)\right]e^{m_{1,i}z}=0$ for $i$ = 5 to 8

Also perform the same substitution to Eq. A8.4 and rearrange to obtain:

(A8.11a) $C_3\bar{A}_{1,2}+C_{14,1}\bar{D}_{1,1}-2C_{15,1}\bar{D}_{1,3}+C_{16,1}\bar{F}_{1,1}-2C_{17,1}\bar{F}_{1,3}=0$

(A8.11b) $\left(2C_3\bar{A}_{1,3}+C_{14,1}\bar{D}_{1,2}-6C_{15,1}\bar{D}_{1,4}+C_{16,1}\bar{F}_{1,2}-6C_{17,1}\bar{F}_{1,4}\right)z=0$

(A8.11c) $\left(3C_3\bar{A}_{1,4}+C_{14,1}\bar{D}_{1,3}+C_{16,1}\bar{F}_{1,3}\right)z^2=0$

(A8.11d) $\left(C_{14,1}\bar{D}_{1,4}+C_{16,1}\bar{F}_{1,4}\right)z^3=0$

(A8.11e) $\left[C_3m_{1,i}\bar{A}_{1,i}+\left(C_{14,1}-C_{15,1}m_{1,i}^2\right)\bar{D}_{1,i}+\left(C_{16,1}-C_{17,1}m_{1,i}^2+C_6m_{1,i}^4\right)\bar{F}_{1,i}\right]e^{m_{1,i}z}=0$

for $i=5$ to 8

The solution for the case of $i\geq 5$ is identical to the general solution for $n\geq 2$ given in Appendix 9. Only the constants with $i\leq 4$ are of interest here.

Twelve relationships have been developed in Eq. A8.9, A8.10 and A8.11 between the twelve unknown constants. However, because the equations are linearly dependent, the equations are redundant and only four of the equations are independent. The eight redundant equations are used to relate eight of the constants to the other four.

Consider Eqs. A8.9 and A8.10 and note that the equations must be valid for any value of $z$. Since $z$, $z^2$ and $z^3$ are not zero for all values of $z$ the value of the brackets multiplying $z$, $z^2$

and $z^3$ must be zero. The following relationships between $\bar{A}_{1,i}$, $\bar{D}_{1,i}$ and $\bar{F}_{1,i}$ for $i$ = 1 to 4 result:

(A8.12a) $2C_1\bar{A}_{1,3} - C_{10,1}\bar{A}_{1,1} + C_{11,1}\bar{D}_{1,2} + C_3\bar{F}_{1,2} = 0$

(A8.12b) $6C_1\bar{A}_{1,4} - C_{10,1}\bar{A}_{1,2} + 2C_{11,1}\bar{D}_{1,3} + 2C_3\bar{F}_{1,3} = 0$

(A8.12c) $-C_{10,1}\bar{A}_{1,3} + 3C_{11,1}\bar{D}_{1,4} + 3C_3\bar{F}_{1,4} = 0$

(A8.12d) $-C_{10,1}\bar{A}_{1,4} = 0$

(A8.12e) $C_{11,1}\bar{A}_{1,2} + C_{12,1}\bar{D}_{1,1} - 2C_{13}\bar{D}_{1,3} + C_{14,1}\bar{F}_{1,1} - 2C_{15,1}\bar{F}_{1,3} = 0$

(A8.12f) $2C_{11,1}\bar{A}_{1,3} + C_{12,1}\bar{D}_{1,2} - 6C_{13}\bar{D}_{1,4} + C_{14,1}\bar{F}_{1,2} - 6C_{15,1}\bar{F}_{1,4} = 0$

(A8.12g) $3C_{11,1}\bar{A}_{1,4} + C_{12,1}\bar{D}_{1,3} + C_{14,1}\bar{F}_{1,3} = 0$

(A8.12h) $C_{12,1}\bar{D}_{1,4} + C_{14,1}\bar{F}_{1,4} = 0$

From Eq. A8.12 $\bar{A}_{1,i}$ and $\bar{D}_{1,i}$ can be written in terms of $\bar{F}_{1,i}$ for $i$ = 1 to 4.

From A8.12d:

(A8.13) $\bar{A}_{1,4} = 0$

From A8.12h:

(A8.14) $\bar{D}_{1,4} = -\dfrac{C_{14,1}}{C_{12,1}}\bar{F}_{1,4}$

From A8.12g and A8.13:

(A8.15) $\bar{D}_{1,3} = -\dfrac{C_{14,1}}{C_{12,1}}\bar{F}_{1,3}$

From A8.12b, A8.13 and A8.15:

$$\text{(A8.16)} \quad \bar{A}_{1,2} = \frac{2C_{11,1}\bar{D}_{1,3} + 2C_3\bar{F}_{1,3}}{C_{10,1}} = \frac{2}{C_{10,1}}\left( -\frac{C_{11,1}C_{14,1}}{C_{12,1}} + C_3 \right)\bar{F}_{1,3}$$

From A8.12c and A8.14:

$$\text{(A8.17)} \quad \bar{A}_{1,3} = \frac{3C_{11,1}\bar{D}_{1,4} + 3C_3\bar{F}_{1,4}}{C_{10,1}} = \frac{3}{C_{10,1}}\left( -\frac{C_{11,1}C_{14,1}}{C_{12,1}} + C_3 \right)\bar{F}_{1,4}$$

From A8.12e, A8.15 and A8.16:

$$\text{(A8.18a)} \quad \bar{D}_{1,1} = \frac{-C_{11,1}\bar{A}_{1,2} + 2C_{13}\bar{D}_{1,3} - C_{14,1}\bar{F}_{1,1} + 2C_{15,1}\bar{F}_{1,3}}{C_{12,1}}$$

$$\text{(A8.18b)} \quad \bar{D}_{1,1} = \frac{\frac{2C_{11,1}}{C_{10,1}}\left( \frac{C_{11,1}C_{14,1}}{C_{12,1}} - C_3 \right)\bar{F}_{1,3} - 2\frac{C_{13,1}C_{14,1}}{C_{12,1}}\bar{F}_{1,3} - C_{14,1}\bar{F}_{1,1} + 2C_{15,1}\bar{F}_{1,3}}{C_{12,1}}$$

$$\text{(A8.18c)} \quad \bar{D}_{1,1} = -\frac{C_{14,1}}{C_{12,1}}\bar{F}_{1,1} + \frac{2}{C_{12,1}}\left[ \frac{C_{11,1}}{C_{10,1}}\left( \frac{C_{11,1}C_{14,1}}{C_{12,1}} - C_3 \right) - \frac{C_{13}C_{14,1}}{C_{12,1}} + C_{15,1} \right]\bar{F}_{1,3}$$

From A8.12f, A8.14 and A8.17:

$$\text{(A8.19a)} \quad \bar{D}_{1,2} = \frac{-2C_{11,1}\bar{A}_{1,3} + 6C_{13,1}\bar{D}_{1,4} - C_{14,1}\bar{F}_{1,2} + 6C_{15,1}\bar{F}_{1,4}}{C_{12,1}}$$

$$\text{(A8.19b)} \quad \bar{D}_{1,2} = \frac{\frac{6C_{11,1}}{C_{10,1}}\left( \frac{C_{11,1}C_{14,1}}{C_{12,1}} - C_3 \right)\bar{F}_{1,4} - 6C_{13}\frac{C_{14,1}}{C_{12,1}}\bar{F}_{1,4} - C_{14,1}\bar{F}_{1,2} + 6C_{15,1}\bar{F}_{1,4}}{C_{12,1}}$$

$$\text{(A8.19c)} \quad \bar{D}_{1,2} = -\frac{C_{14,1}}{C_{12,1}}\bar{F}_{1,2} + \frac{6}{C_{12,1}}\left[ \frac{C_{11,1}}{C_{10,1}}\left( \frac{C_{11,1}C_{14,1}}{C_{12,1}} - C_3 \right) - \frac{C_{13}C_{14,1}}{C_{12,1}} + C_{15,1} \right]\bar{F}_{1,4}$$

From A8.12a, A8.17 and A8.19:

$$\text{(A8.20a)}\quad \bar{A}_{1,1}=\frac{2C_1\bar{A}_{1,3}+C_{11,1}\bar{D}_{1,2}+C_3\bar{F}_{1,2}}{C_{10,1}}$$

$$\text{(A8.20b)}\quad \begin{aligned}\bar{A}_{1,1}=&\frac{2C_1}{C_{10,1}}\left[\frac{3}{C_{10,1}}\left(-\frac{C_{11,1}C_{14,1}}{C_{12,1}}+C_3\right)\right]\bar{F}_{1,4}+\frac{C_{11,1}}{C_{10,1}}\left\{-\frac{C_{14,1}}{C_{12,1}}\bar{F}_{1,2}\right.\\&\left.+\frac{6}{C_{12,1}}\left[\frac{C_{11,1}}{C_{10,1}}\left(\frac{C_{11,1}C_{14,1}}{C_{12,1}}-C_3\right)-\frac{C_{13}C_{14,1}}{C_{12,1}}+C_{15}\right]\bar{F}_{1,4}\right\}+\frac{C_3}{C_{10,1}}\bar{F}_{1,2}\end{aligned}$$

$$\text{(A8.20c)}\quad \begin{aligned}\bar{A}_{1,1}=&\left\{\frac{C_3}{C_{10,1}}-\frac{C_{11,1}C_{14,1}}{C_{10,1}C_{12,1}}\right\}\bar{F}_{1,2}+\left\{\frac{6C_1}{C_{10,1}^2}\left[-\frac{C_{11,1}C_{14,1}}{C_{12,1}}+C_3\right]\right.\\&\left.+\frac{6C_{11,1}}{C_{10,1}C_{12,1}}\left[\frac{C_{11,1}}{C_{10,1}}\left(\frac{C_{11,1}C_{14,1}}{C_{12,1}}-C_3\right)-\frac{C_{13,1}C_{14,1}}{C_{12,1}}+C_{15,1}\right]\right\}\bar{F}_{1,4}\end{aligned}$$

Now all values of $\bar{A}_{1,i}$ and $\bar{D}_{1,i}$ are known in terms of $\bar{F}_{1,j}$ for $i$ = 1 to 4. In summary:

$$\text{(A8.21a)}\quad \begin{aligned}\bar{A}_{1,1}=&\left\{\frac{C_3}{C_{10,1}}-\frac{C_{11,1}C_{14,1}}{C_{10,1}C_{12,1}}\right\}\bar{F}_{1,2}+\left\{\frac{6C_1}{C_{10,1}^2}\left[-\frac{C_{11,1}C_{14,1}}{C_{12,1}}+C_3\right]\right.\\&\left.+\frac{6C_{11,1}}{C_{10,1}C_{12,1}}\left[\frac{C_{11,1}}{C_{10,1}}\left(\frac{C_{11,1}C_{14,1}}{C_{12,1}}-C_3\right)-\frac{C_{13}C_{14,1}}{C_{12,1}}+C_{15,1}\right]\right\}\bar{F}_{1,4}=\bar{\bar{A}}_{1,1,1}\bar{F}_{1,2}+\bar{\bar{A}}_{1,1,2}\bar{F}_{1,4}\end{aligned}$$

$$\text{(A8.21b)}\quad \bar{A}_{1,2}=\frac{2}{C_{10,1}}\left(-\frac{C_{11,1}C_{14,1}}{C_{12,1}}+C_3\right)\bar{F}_{1,3}=\bar{\bar{A}}_{1,2}\bar{F}_{1,3}$$

$$\text{(A8.21c)}\quad \bar{A}_{1,3}=\frac{3}{C_{10,1}}\left(-\frac{C_{11,1}C_{14,1}}{C_{12,1}}+C_3\right)\bar{F}_{1,4}=\bar{\bar{A}}_{1,3}\bar{F}_{1,4}$$

$$\text{(A8.21d)}\quad \bar{A}_{1,4}=0$$

(A8.21e) $$\bar{D}_{1,1}=-\frac{C_{14,1}}{C_{12,1}}\bar{F}_{1,1}+\frac{2}{C_{12,1}}\left[\frac{C_{11,1}}{C_{10,1}}\left(\frac{C_{11,1}C_{14,1}}{C_{12,1}}-C_3\right)-\frac{C_{13}C_{14,1}}{C_{12,1}}+C_{15,1}\right]\bar{F}_{1,3}=\bar{\bar{D}}_{1,1.1}\bar{F}_{1,1}+\bar{\bar{D}}_{1,1.2}\bar{F}_{1,3}$$

(A8.21f) $$\bar{D}_{1,2}=-\frac{C_{14,1}}{C_{12,1}}\bar{F}_{1,2}+\frac{6}{C_{12,1}}\left[\frac{C_{11,1}}{C_{10,1}}\left(\frac{C_{11,1}C_{14,1}}{C_{12,1}}-C_3\right)-\frac{C_{13}C_{14,1}}{C_{12,1}}+C_{15}\right]\bar{F}_{1,4}=\bar{\bar{D}}_{1,2.1}\bar{F}_{1,2}+\bar{\bar{D}}_{1,2.2}\bar{F}_{1,4}$$

(A8.21g) $$\bar{D}_{1,3}=-\frac{C_{14,1}}{C_{12,1}}\bar{F}_{1,3}=\bar{\bar{D}}_{1,3}\bar{F}_{1,3}$$

(A8.21h) $$\bar{D}_{1,4}=-\frac{C_{14,1}}{C_{12,1}}\bar{F}_{1,4}=\bar{\bar{D}}_{1,4}\bar{F}_{1,4}$$

For i = 5 to 8 the relationships between $\bar{A}_{1,i}$, $\bar{D}_{1,i}$ and $\bar{F}_{1,i}$ are given in Eq. A9.20 and A9.21 in Appendix 9:

(A8.22) $$\bar{A}_{1,i}=\left\{\frac{-\dfrac{(C_{11,1}m_{1,i})}{(C_1m_{1,i}^2-C_{10,1})}\left[\dfrac{(C_{11,1}m_{1,i})(C_3m_{1,i})}{(C_1m_{1,i}^2-C_{10,1})}-(C_{14,1}-C_{15,1}m_{1,i}^2)\right]}{(C_{12,1}-C_{13,1}m_{1,i}^2)-\dfrac{(C_{11,1}m_{1,i})^2}{(C_1m_{1,i}^2-C_{10,1})}}-\frac{(C_3m_{1,i})}{(C_1m_{1,i}^2-C_{10,1})}\right\}\bar{F}_{1,i}=\bar{\bar{A}}_{1,i}\bar{F}_{1,i}$$

(A8.23) $$\bar{D}_{1,i}=\left\{\frac{\dfrac{(C_{11,1}m_{1,i})(C_3m_{1,i})}{(C_1m_{1,i}^2-C_{10,1})}-(C_{14,1}-C_{15,1}m_{1,i}^2)}{(C_{12,1}-C_{13,1}m_{1,i}^2)-\dfrac{(C_{11,1}m_{1,i})^2}{(C_1m_{1,i}^2-C_{10,1})}}\right\}\bar{F}_{1,i}=\bar{\bar{D}}_{1,i}\bar{F}_{1,i}$$

In summary, the equilibrium equations are written in matrix form as:

(A8.24) $$\begin{Bmatrix} a_1(z) \\ d_1(z) \\ f_1(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \overline{\overline{A}}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle \overline{\overline{D}}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} \\ \left\langle \overline{e}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} \end{bmatrix} \left\{ \overline{F}_1 \right\}_{8\times 1}$$

in which:

(A8.25) $$\begin{aligned} \left\langle \overline{\overline{A}}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} = \Big\langle 0 \mid \overline{\overline{A}}_{1,1.1} \mid \overline{\overline{A}}_{1,2} z \mid \left( \overline{\overline{A}}_{1,1.2} + \overline{\overline{A}}_{1,3} z^2 \right) \mid \\ \overline{\overline{A}}_{1,5} e^{m_{1,5} z} \mid \overline{\overline{A}}_{1,6} e^{m_{1,6} z} \mid \overline{\overline{A}}_{1,7} e^{m_{1,7} z} \mid \overline{\overline{A}}_{1,8} e^{m_{1,8} z} \Big\rangle_{1\times 8} \end{aligned}$$

(A8.26) $$\begin{aligned} \left\langle \overline{\overline{D}}_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} = \Big\langle \overline{\overline{D}}_{1,1.1} \mid \overline{\overline{D}}_{1,2.1} z \mid \left( \overline{\overline{D}}_{1,1.2} + \overline{\overline{D}}_{1,3} z^2 \right) \mid \left( \overline{\overline{D}}_{1,2.2} z + \overline{\overline{D}}_{1,4} z^3 \right) \mid \\ \overline{\overline{D}}_{1,5} e^{m_{1,5} z} \mid \overline{\overline{D}}_{1,6} e^{m_{1,6} z} \mid \overline{\overline{D}}_{1,7} e^{m_{1,7} z} \mid \overline{\overline{D}}_{1,8} e^{m_{1,8} z} \Big\rangle_{1\times 8} \end{aligned}$$

(A8.27) $$\left\langle \overline{F}_1 \right\rangle^{\mathrm{T}}_{1\times 8} = \left\langle \overline{F}_{1,1} \quad \overline{F}_{1,2} \quad \overline{F}_{1,3} \quad \overline{F}_{1,4} \quad \overline{F}_{1,5} \quad \overline{F}_{1,6} \quad \overline{F}_{1,7} \quad \overline{F}_{1,8} \right\rangle_{1\times 8}$$

(A8.28) $$\left\langle e_1(z) \right\rangle^{\mathrm{T}}_{1\times 8} = \left\langle 1 \quad z \quad z^2 \quad z^3 \quad e^{m_{1,5} z} \quad e^{m_{1,6} z} \quad e^{m_{1,7} z} \quad e^{m_{1,8} z} \right\rangle_{1\times 8}$$

The values of $\overline{\overline{A}}_{1,i}$ and $\overline{\overline{D}}_{1,i}$ are defined in Eqs. A8.21, A8.22 and A8.23.

Integration constants $\overline{A}_{1,i}$ and $\overline{D}_{1,i}$ are now known in terms of constants $\overline{F}_{1,i}$ so only eight integration constants are present. These unknown constants are the elements of vector $\left\langle \overline{F}_1 \right\rangle^{\mathrm{T}}_{1\times 8}$.

# APPENDIX 9 – REDUCING INTEGRATION CONSTANTS FOR $n \geq 2$

The solution of the equilibrium equations for $n \geq 2$ in Eq. 3.24 appears to have twenty-four integration constants, but only eight boundary conditions are presented in Eq. 2.19. In this appendix two sets of integration constants, $\bar{A}_{n,i}$ and $\bar{D}_{n,i}$, are related to the third set, $\bar{F}_{n,i}$. As a result, only eight integration constants will remain which can be determined from the eight boundary conditions.

The simplified homogeneous equations for displacements $a_n(z)$, $d_n(z)$ and $f_n(z)$ for $n \geq 2$ are given in Eq. 2.27 with the right hand side of the equation being $\{\mathbf{Q}_n\}_{3\times1} = \{\mathbf{0}\}_{3\times1}$:

(A9.1) $$[\mathbf{X}_n]_{3\times3} \begin{Bmatrix} a_n(z) \\ d_n(z) \\ f_n(z) \end{Bmatrix} = \left[\begin{array}{c|c|c} C_1D^2 - C_{10} & C_{11}D & C_3D \\ \hline C_{11}D & C_{12} - C_{13}D^2 & C_{14} - C_{15}D^2 \\ \hline C_3D & C_{14} - C_{15}D^2 & C_{16} - C_{17}D^2 + C_6D^4 \end{array}\right] \begin{Bmatrix} a_n(z) \\ d_n(z) \\ f_n(z) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Expanding Eq. A9.1 gives:

(A9.2) $$C_1a_n''(z) - C_{10,n}a_n(z) + C_{11,n}d_n'(z) + C_3f_n'(z) = 0$$

(A9.3) $$C_{11,n}a_n'(z) + C_{12,n}d_n(z) - C_{13}d_n''(z) + C_{14,n}f_n(z) - C_{15,n}f_n''(z) = 0$$

(A9.4) $$C_3a_n'(z) + C_{14,n}d_n(z) - C_{15,n}d_n''(z) + C_6f_n^{(4)}(z) - C_{17,n}f_n''(z) + C_{16,n}f_n(z) = 0$$

As in Chapter 3 the solution field for Eq. A9.1 is:

(A9.5) $$a_n(z) = \sum_{i=1}^{8} \bar{A}_{n,i}e^{m_{n,i}z}$$

(A9.6) $$d_n(z) = \sum_{i=1}^{8} \bar{D}_{n,i}e^{m_{n,i}z}$$

(A9.7) $$f_n(z)=\sum_{i=1}^{8}\bar{F}_{n,i}e^{m_{n,i}z}$$

The derivation of the solution field in Eqs A9.5 through A9.7 is based on the fact that the determinant of the matrix in the left hand side of Eq. A9.1 is zero. Therefore Eqs. A9.2, A9.3 and A9.4 are linearly dependent. For each root $m_{n,i}$ the constants $\bar{A}_{n,i}$ and $\bar{D}_{n,i}$ can be related to the constants $\bar{F}_{n,i}$ by substituting Eqs A9.5 through A9.7 into Eqs. A9.2, A9.3 and A9.4. After the substitutions and considering each root $i$ independently Eq. A9.2 becomes:

(A9.8) $$\left(C_1 m_{n,i}^{\ 2}-C_{10,n}\right)\bar{A}_{n,i}e^{m_{n,i}z}+\left(C_{11,n}m_{n,i}\right)\bar{D}_{n,i}e^{m_{n,i}z}+\left(C_3 m_{n,i}\right)\bar{F}_{n,i}e^{m_{n,i}z}=0$$

(A9.9) $$\left(C_{11,n}m_{n,i}\right)\bar{A}_{n,i}e^{m_{n,i}z}+\left(C_{12,n}-C_{13,n}m_{n,i}^{\ 2}\right)\bar{D}_{n,i}e^{m_{n,i}z}+\left(C_{14,n}-C_{15,n}m_{n,i}^{\ 2}\right)\bar{F}_{n,i}e^{m_{n,i}z}=0$$

(A9.10) $$\left(C_3 m_{n,i}\right)\bar{A}_{n,i}e^{m_{n,i}z}+\left(C_{14,n}-C_{15,n}m_{n,i}^{\ 2}\right)\bar{D}_{n,i}e^{m_{n,i}z}+\left(C_6 m_{n,i}^{\ 4}-C_{17,n}m_{n,i}^{\ 2}+C_{16,n}\right)\bar{F}_{n,i}e^{m_{n,i}z}=0$$

Eq. A9.8 must be valid for any value of $z$. Since $e^{m_{n,i}z}$ can never be zero Eqs. A9.8 through A9.10 reduce to:

(A9.11) $$\left(C_1 m_{n,i}^{\ 2}-C_{10,n}\right)\bar{A}_{n,i}+\left(C_{11,n}m_{n,i}\right)\bar{D}_{n,i}+\left(C_3 m_{n,i}\right)\bar{F}_{n,i}=0$$

(A9.12) $$\left(C_{11,n}m_{n,i}\right)\bar{A}_{n,i}+\left(C_{12,n}-C_{13,n}m_{n,i}^{\ 2}\right)\bar{D}_{n,i}+\left(C_{14,n}-C_{15,n}m_{n,i}^{\ 2}\right)\bar{F}_{n,i}=0$$

(A9.13) $$\left(C_3 m_{n,i}\right)\bar{A}_{n,i}+\left(C_{14,n}-C_{15,n}m_{n,i}^{\ 2}\right)\bar{D}_{n,i}+\left(C_6 m_{n,i}^{\ 4}-C_{17,n}m_{n,i}^{\ 2}+C_{16,n}\right)\bar{F}_{n,i}=0$$

Rearrange Eq. A9.11 to write $\bar{A}_{n,i}$ in terms of $\bar{D}_{n,i}$ and $\bar{F}_{n,i}$:

(A9.14) $$\bar{A}_{n,i}=\frac{-\left(C_{11,n}m_{n,i}\right)\bar{D}_{n,i}-\left(C_3 m_{n,i}\right)\bar{F}_{n,i}}{\left(C_1 m_{n,i}^{\ 2}-C_{10,n}\right)}$$

Rearrange Eq. A9.12 and substitute Eq. A9.14 to obtain the relationship between $\bar{D}_{n,i}$ and $\bar{F}_{n,i}$:

(A9.15) $$\bar{D}_{n,i} = \frac{-\left(C_{11,n} m_{n,i}\right)\bar{A}_{n,i} - \left(C_{14,n} - C_{15,n} m_{n,i}{}^{2}\right)\bar{F}_{n,i}}{\left(C_{12,n} - C_{13,n} m_{n,i}{}^{2}\right)}$$

(A9.16) $$\bar{D}_{n,i} = \left[\frac{\dfrac{\left(C_{11,n} m_{n,i}\right)\left(C_{3} m_{n,i}\right)}{\left(C_{1} m_{n,i}{}^{2} - C_{10,n}\right)} - \left(C_{14,n} - C_{15,n} m_{n,i}{}^{2}\right)}{\left(C_{12,n} - C_{13,n} m_{n,i}{}^{2}\right) - \dfrac{\left(C_{11,n} m_{n,i}\right)^{2}}{\left(C_{1} m_{n,i}{}^{2} - C_{10,n}\right)}}\right]\bar{F}_{n,i}$$

Substitute Eq. A9.16 into Eq. A9.14 and rearrange to obtain the relationship between $\bar{A}_{n,i}$ and $\bar{F}_{n,i}$:

(A9.17) $$\bar{A}_{n,i} = \left[\frac{-\dfrac{\left(C_{11,n} m_{n,i}\right)}{\left(C_{1} m_{n,i}{}^{2} - C_{10}\right)}\left[\dfrac{\left(C_{11,n} m_{n,i}\right)\left(C_{3} m_{n,i}\right)}{\left(C_{1} m_{n,i}{}^{2} - C_{10,n}\right)} - \left(C_{14,n} - C_{15,n} m_{n,i}{}^{2}\right)\right]}{\left(C_{12,n} - C_{13,n} m_{n,i}{}^{2}\right) - \dfrac{\left(C_{11,n} m_{n,i}\right)^{2}}{\left(C_{1} m_{n,i}{}^{2} - C_{10,n}\right)}} - \frac{\left(C_{3} m_{n,i}\right)}{\left(C_{1} m_{n,i}{}^{2} - C_{10,n}\right)}\right]\bar{F}_{n,i}$$

The integration constants $\bar{A}_{n,i}$ and $\bar{D}_{n,i}$ can now be written in terms of $\bar{F}_{n,i}$ as:

(A9.18) $$\bar{A}_{n,i} = \bar{\bar{A}}_{n,i}\bar{F}_{n,i}$$

(A9.19) $$\bar{D}_{n,i} = \bar{\bar{D}}_{n,i}\bar{F}_{n,i}$$

in which:

(A9.20) $$\bar{\bar{A}}_{n,i} = \frac{-\dfrac{\left(C_{11,n} m_{n,i}\right)}{\left(C_{1} m_{n,i}{}^{2} - C_{10,n}\right)}\left[\dfrac{\left(C_{11,n} m_{n,i}\right)\left(C_{3} m_{n,i}\right)}{\left(C_{1} m_{n,i}{}^{2} - C_{10,n}\right)} - \left(C_{14,n} - C_{15,n} m_{n,i}{}^{2}\right)\right]}{\left(C_{12,n} - C_{13,n} m_{n,i}{}^{2}\right) - \dfrac{\left(C_{11,n} m_{n,i}\right)^{2}}{\left(C_{1} m_{n,i}{}^{2} - C_{10,n}\right)}} - \frac{\left(C_{3} m_{n,i}\right)}{\left(C_{1} m_{n,i}{}^{2} - C_{10,n}\right)}$$

(A9.21) $$\bar{D}_{n,i} = \frac{\dfrac{\left(C_{11,n} m_{n,i}\right)\left(C_3 m_{n,i}\right)}{\left(C_1 m_{n,i}^{\ 2} - C_{10,n}\right)} - \left(C_{14,n} - C_{15,n} m_{n,i}^{\ 2}\right)}{\left(C_{12,n} - C_{13,n} m_{n,i}^{\ 2}\right) - \dfrac{\left(C_{11,n} m_{n,i}\right)^2}{\left(C_1 m_{n,i}^{\ 2} - C_{10,n}\right)}}$$

In summary, the solution field in Eqs. A9.6 through A9.8 is written in matrix form as:

(A9.22) $$\begin{Bmatrix} a_n(z) \\ d_n(z) \\ f_n(z) \end{Bmatrix} = \begin{bmatrix} \left\langle \bar{\bar{A}}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle \bar{\bar{D}}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} \\ \left\langle e_n(z) \right\rangle^{\mathrm{T}}_{1\times8} \end{bmatrix} \left\{ \bar{F}_n \right\}_{8\times1}$$

in which:

(A9.23) $$\left\langle \bar{\bar{A}}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} = \Big\langle \bar{\bar{A}}_{n,1} e^{m_{n,1}z} \quad \bar{\bar{A}}_{n,2} e^{m_{n,2}z} \quad \bar{\bar{A}}_{n,3} e^{m_{n,3}z} \quad \bar{\bar{A}}_{n,4} e^{m_{n,4}z} \quad \bar{\bar{A}}_{n,5} e^{m_{n,5}z} \quad \bar{\bar{A}}_{n,6} e^{m_{n,6}z} \quad \bar{\bar{A}}_{n,7} e^{m_{n,7}z} \quad \bar{\bar{A}}_{n,8} e^{m_{n,8}z} \Big\rangle_{1\times8}$$

(A9.24) $$\left\langle \bar{\bar{D}}_n(z) \right\rangle^{\mathrm{T}}_{1\times8} = \Big\langle \bar{\bar{D}}_{n,1} e^{m_{n,1}z} \quad \bar{\bar{D}}_{n,2} e^{m_{n,2}z} \quad \bar{\bar{D}}_{n,3} e^{m_{n,3}z} \quad \bar{\bar{D}}_{n,4} e^{m_{n,4}z} \quad \bar{\bar{D}}_{n,5} e^{m_{n,5}z} \quad \bar{\bar{D}}_{n,6} e^{m_{n,6}z} \quad \bar{\bar{D}}_{n,7} e^{m_{n,7}z} \quad \bar{\bar{D}}_{n,8} e^{m_{n,8}z} \Big\rangle_{1\times8}$$

(A9.25) $$\left\langle e_n(z) \right\rangle^{\mathrm{T}}_{1\times8} = \Big\langle e^{m_{n,1}z} \quad e^{m_{n,2}z} \quad e^{m_{n,3}z} \quad e^{m_{n,4}z} \quad e^{m_{n,5}z} \quad e^{m_{n,6}z} \quad e^{m_{n,7}z} \quad e^{m_{n,8}z} \Big\rangle_{1\times8}$$

(A9.26) $$\left\langle \bar{F}_n \right\rangle^{\mathrm{T}}_{1\times8} = \Big\langle \bar{F}_{n,1} \quad \bar{F}_{n,2} \quad \bar{F}_{n,3} \quad \bar{F}_{n,4} \quad \bar{F}_{n,5} \quad \bar{F}_{n,6} \quad \bar{F}_{n,7} \quad \bar{F}_{n,8} \Big\rangle_{1\times8}$$

The values of $\bar{\bar{A}}_{n,i}$ and $\bar{\bar{D}}_{n,i}$ in Eqs. A9.23 and A9.24 are given in Eqs. A9.20 and A9.21 respectively.

Integration constants $\bar{A}_{n,i}$ and $\bar{D}_{n,i}$ are now known in terms of constants $\bar{F}_{n,i}$ so only eight integration constants are present. These unknown constants are contained in the vector $\langle \bar{F}_n \rangle^{\mathrm{T}}_{1\times 8}$.

# Appendix 10 – Program Listing for Analytical Solution

The analytical method developed in Chapter 3 was implemented using the C++ programming language and compiled with the software program Borland C++, version 5.02. Included in the subsequent sections are the listing of the program and the program output for the examples given in Chapter 4.

The program utilizes the built-in C++ class "complex" which can carry out all required mathematical manipulations of complex numbers. The geometric and material properties are hard coded into the program. These can be changed and the program re-run to model different pipe geometries. Alternatively, a simple modification could be made to the code to allow the user to input the pipe properties. The latter alternative would be useful for a small problem involving only one or two elements but would quickly become cumbersome for a larger model.

The loading conditions are also hard coded into the program. The main program listing in Section A10.1 includes the code for the loading given in Example 1 in Chapter 4. As with the pipe properties, the loading can be change by modifying the code and re-running the model. The section of code that is required to be changed to alter the loading is relatively small. The code for the loading conditions used for the self weight example in Chapter 4 is also included in this appendix.

Note that the only changes to the program listing required to model the two examples in Chapter 4 are the pipe properties and the loading conditions. All other aspects of the program listing, including the calculation of the all constants and the determination of the stiffness matrix, do not need to be altered.

## A10.1 Program Listing

Below is the listing for the program used to analytically calculate the deflection of the tip of a cantilevered pipe with an applied point load at the tip. The geometry, material properties and loading are those of Example 1 in Chapter 4.

Note that everything in *italics* in the program listing is a comment that is ignored by the C++ compiler. These comments are included for clarification only.

```
#include <complex.h>
#include <math.h>

// Function definitions
void input(double *, double *,double *, double *, double *, double *, int *);
void constants1(double *, double, double, double, double, double);
void constants2(double *, int, double);
void coefficients(double *, double *);
void quartic_roots(complex *, double *);
complex complex_root(double);
void multiply_matrix(complex [][9], complex [][9], complex[][9]);
void multiply_vector(complex [], complex [], complex [][9]);


// Begin main program
main()
{
   double pi=M_PI;
   double r, t, E, G, mu, l;  //physical properties of pipe
   double cn[18];             //array of constants from Eq. 2.15
   double p[6];               //coefficients of quartic equation
   int i,j,k;                 //counters in "for" loops
   int n;                     //current Fourier mode
   int alpha;                 //total number of Fourier modes included in analysis
   complex i_sqrd, im;        //imaginary number defined as sqrt(-1)
   complex M[5];              //roots of quartic equation
   complex m[9];              //sqaure roots of roots of quartic equation
   complex A_bar[9], D_bar[9], F_bar[9], C_bar[9];  //vectors relating integration constants
   complex A1_bar[10], D1_bar[11];    //vectors relating integration constants for mode n=1
   complex H[9][9], H_inv[9][9];      //Matrices of boundary equations
   complex e[9];
   complex Q[9], delta[9];    //load potential and nodal displacement vectors
   complex a, b, c, d, f, g;

//Define imaginary number i and call it "im"
   i_sqrd=-1;
   im=sqrt(i_sqrd);

//Get element data
   input(&r, &t, &l, &E, &G, &mu, &alpha);

//Calculate constants not dependent on "n"
   constants1(cn, r, t, E, G, mu);

//*******************************************************************************************
// *** Axi-symmetric case n=0 ***
//*******************************************************************************************

//Calculate each of the roots for i=3 to 6
    m[1]=-12*(1-pow(mu,2))/pow(r*t,2);
    m[2]=sqrt(m[1]);
    m[3]=sqrt(m[2]);
    m[4]=-1*m[3];
```

```
    m[5]=m[3]*im;
    m[6]=-1*m[3]*im;

//Construct F_bar and C_bar.
    for(i=1;i<=8;i++)
        Fbar[i]=0;
    Fbar[2]=-1*mu*r;
    for(i=3;i<=6;i++)
        Fbar[i]=-1*r*m[i]/mu;

//Construct the transformation matrix H from the boundary conditions

    //Row 1 - Eq. (4.21a)
    H[1][1]=H[1][3]=H[1][4]=H[1][5]=H[1][6]=1;
    H[1][2]=H[1][7]=H[1][8]=0;

    //Row 7 - Eq. (4.21b)
    for(i=1;i<=8;i++)
        H[7][i]=0;
    H[7][7]=1;

    //Row 3 - Eq. (4.21c)
    for(i=1;i<=8;i++)
        H[3][i]=F_bar[i];

    //Row 4 - Eq. (4.21d)
    H[4][1]=H[4][2]=H[4][7]=H[4][8]=0;
    for(i=3;i<=6;i++)
        H[4][i]=m[i]*F_bar[i];

    //Row 2 - Eq. (4.23)
    H[2][1]=H[2][7]=H[2][8]=0;
    H[2][2]=2*cn[1]+2*cn[3]*F_bar[2];
    for(i=3;i<=6;i++)
        H[2][i]=(2*cn[1]*m[i]+2*cn[3]*F_bar[i])*exp(m[i]*l);

    //Row 8 - Eq. (4.24)
    for(i=1;i<=7;i++)
        H[8][i]=0;
    H[8][8]=2*(cn[4]+cn[9]);

    //Row 5 - Eq. (4.25)
    H[5][1]=H[5][2]=H[5][7]=H[5][8]=0;
    for(i=3;i<=6;i++)
        H[5][i]=-2*cn[6]*pow(m[i],3)*F_bar[i]*exp(m[i]*l);

    //Row 6 - Eq. (4.26)
    H[6][1]=H[6][2]=H[6][7]=H[6][8]=0;
    for(i=3;i<=6;i++)
        H[6][i]=2*cn[6]*pow(m[i],2)*F_bar[i]*exp(m[i]*l);

//Invert the martix [H]
    inverse(H, H_inv);

//Create the load vector Q - Eq. (4.27)
    for(i=1; i<=8; i++)
        Q[i]=0;
    Q[5]=-16;

//Calculate the integration constants from {A_bar} = [H_inv]*{Q}
    multiply_vector(A_bar, Q, H_inv);

//Find a(l), c(l) and f(l) from c(l)=<C_bar(l)>{A_bar} in Eq. (4.17)
    a=A_bar[1]+A_bar[2]*2;
    f=A_bar[2]*F_bar[2];
    for(i=3; i<=6; i++)
    {
        a+=A_bar[i]*exp(m[i]*l);
        f+=F_bar[i]*A_bar[i]*exp(m[i]*l);
    }
    c=A_bar[7]+A_bar[8]*l;
```

```
//Output the nodal displacement contributions at z=l for mode n=0 in mm
    cout<<"\n*** Nodal displacement contributions for n=0 ***"<<endl;
    cout<<"a0(l) = "<<a*1000<<" mm"<<endl;
    cout<<"c0(l) = "<<c*1000<<" mm"<<endl;
    cout<<"f0(l) = "<<f*1000<<" mm\n"<<endl;

//*****************************************************************************************
// *** End mode n=0 ***
//*****************************************************************************************

//*****************************************************************************************
// *** Special case of n=1 ***
//*****************************************************************************************

//Calculate constants that depend on 'n' and coefficients of quartic equation
    constants2(cn, 1, r);
    coefficients(cn, p);

//Calculate each of the roots for i=5 to 8
    m[1]=pow(p[2],2)-4*p[1]*p[3];
    m[2]=sqrt(m[1]);
    m[5]=sqrt((-1.0*p[2]+m[2])/(2*p[1]));
    m[6]=sqrt((-1.0*p[2]-m[2])/(2*p[1]));
    m[7]=-1.0*m[5];
    m[8]=-1.0*m[6];

//Construct vectors {A_bar} and {D_bar}
    A1_bar[1]=0;
    A1_bar[2]=cn[3]/cn[10]-(cn[11]*cn[14])/(cn[10]*cn[12]);
    A1_bar[3]=(2/cn[10])*(cn[3]-(cn[11]*cn[14])/cn[12]);
    A1_bar[4]=(3/cn[10])*(cn[3]-(cn[11]*cn[14])/cn[12]);
    D1_bar[1]=-1.0*cn[14]/cn[12];
    D1_bar[2]=-1.0*cn[14]/cn[12];
    D1_bar[3]=-1.0*cn[14]/cn[12];
    D1_bar[4]=-1.0*cn[14]/cn[12];
    for(i=5; i<=8; i++)
    {
        D1_bar[i]=((cn[11]*m[i])*(cn[3]*m[i])/(cn[1]*pow(m[i],2)-cn[10])-(cn[14]-
            cn[15]*pow(m[i],2)))/(cn[12]-cn[13]*pow(m[i],2)-
            pow(cn[11]*m[i],2)/(cn[1]*pow(m[i],2)-cn[10]));
        A1_bar[i]=((-1)*cn[11]*m[i]/(cn[1]*pow(m[i],2)-cn[10]))*D1_bar[i]-
            cn[3]*m[i]/(cn[1]*pow(m[i],2)-cn[10]);
    }
    A1_bar[9]=-6*r*cn[1]/cn[10]+(6*cn[11]/(cn[10]*cn[12]))*(r*cn[11]-cn[4]+cn[8]);
    D1_bar[9]=(2/cn[12])*((cn[11]/cn[10])*(cn[11]*cn[14]/cn[12]-cn[3])-
        (cn[13]*cn[14]/cn[12])+cn[15]);
    D1_bar[10]=3*D1_bar[9];

//*** Calculate first the displacements a1, d1 and f1 ***

//Construct the matrix [H] from the boundary conditions
    //Row 1 - Eq. (4.37c)
    H[1][1]=H[1][5]=H[1][6]=H[1][7]=H[1][8]=1;
    H[1][2]=H[1][3]=H[1][4]=0;

    //Row 2 - Eq. (4.37d)
    H[2][1]=H[2][3]=H[2][4]=0;
    H[2][2]=1;
    for(i=5;i<=8;i++)
        H[2][i]=m[i];

    //Row 3 - Eq. (4.37b)
    H[3][1]=D1_bar[1];
    H[3][2]=H[3][4]=0;
    H[3][3]=D1_bar[9];
    for(i=5;i<=8;i++)
        H[3][i]=D1_bar[i];

    //Row 4 - Eq. (4.37a)
    H[4][2]=A1_bar[2];
```

```
    H[4][1]=H[4][3]=0;
    H[4][4]=A1_bar[9];
    for(i=5;i<=8;i++)
        H[4][i]=A1_bar[i];

    //Row 5 - Eq. (4.39)
    H[5][1]=cn[3]*(1+D1_bar[1]);
    H[5][2]=cn[3]*1*(1+D1_bar[2]);
    H[5][3]=cn[1]*A1_bar[3]+cn[3]*(pow(1,2)*(1+D1_bar[3])+D1_bar[9]);
    H[5][4]=2*cn[1]*A1_bar[4]*1+cn[3]*(pow(1,3)*(1+D1_bar[4])+D1_bar[10]*1);
    for(i=5;i<=8;i++)
        H[5][i]=(cn[1]*m[i]*A1_bar[i]+cn[3]*(1+D1_bar[i]))*exp(m[i]*1);

    //Row 7 - Eq. (4.42)
    H[7][1]=-1*cn[8]*(1+D1_bar[1]);
    H[7][2]=-1*cn[8]*1*(1+D1_bar[2]);
    H[7][3]=2*cn[6]-cn[8]*(pow(1,2)*(1+D1_bar[3])+D1_bar[9]);
    H[7][4]=6*cn[6]*1-cn[8]*(pow(1,3)*(1+D1_bar[4])+D1_bar[10]*1);
    for(i=5;i<=8;i++)
        H[7][i]=(cn[6]*pow(m[i],2)-cn[8]*(1+D1_bar[i]))*exp(m[i]*1);

    //Row 6 - Eq. (4.40)
    H[6][1]=0;
    H[6][2]=-1*cn[5]*A1_bar[2]+(cn[4]+cn[9])*D1_bar[2]+cn[9];
    H[6][3]=1*(-1*cn[5]*A1_bar[3]+2*(cn[4]+cn[9])*D1_bar[3]+2*cn[9]);
    H[6][4]=-1*cn[5]*(A1_bar[9]+A1_bar[4]*pow(1,2))+(cn[4]+cn[9])*(D1_bar[10]+3*D1_bar[4]*
        pow(1,2))+3*cn[9]*pow(1,2);
    for(i=5;i<=8;i++)
        H[6][i]=(-1*cn[5]*A1_bar[i]+(cn[4]+cn[9])*D1_bar[i]*m[i]+cn[9]*m[i])*exp(m[i]*1);

    //Row 8 - Eq. (4.41)
    H[8][1]=0;
    H[8][2]=(cn[8]+cn[9])*(1+D1_bar[2]);
    H[8][3]=(cn[8]+cn[9])*1*2*(1+D1_bar[3]);
    H[8][4]=-6*cn[6]+(cn[8]+cn[9])*(3*pow(1,2)+D1_bar[10]+3*D1_bar[4]*pow(1,2));
    for(i=5;i<=8;i++)
        H[8][i]=(-1*cn[6]*pow(m[i],3)+(cn[8]+cn[9])*(1+D1_bar[i])*m[i])*exp(m[i]*1);

//Invert the martix [H]
    inverse(H, H_inv);

//Create the load vector Q in Eq. (4.43)
    for(i=1; i<=8; i++)
        Q[i]=0;
    Q[8]=-16;

//Calculate the integration constants from {F_bar} = [H_inv]*{Q}
    multiply_vector(F_bar, Q, H_inv);

//Find the vector <e(1)> from Eq. (4.35c)
    e[1]=1;
    e[2]=1;
    e[3]=pow(1,2);
    e[4]=pow(1,3);
    for(i=5;i<=8;i++)
        e[i]=exp(m[i]*1);

//Calculate tip deflection as f=<e(1)>{F} from Eq. (4.34)
    f=0;
    for(i=1;i<=8;i++)
        f+=e[i]*F_bar[i];

//Calculate tip deflection as d=<D1_bar(1)>{f} from Eq. (4.34)
    d=0;
    d+=D1_bar[1]*F_bar[1];
    d+=D1_bar[2]*1*F_bar[2];
    d+=(D1_bar[9]+D1_bar[3]*pow(1,2))*F_bar[3];
    d+=(D1_bar[10]*1+D1_bar[4]*pow(1,3))*F_bar[4];
    for(i=5;i<=8;i++)
        d+=D1_bar[i]*exp(m[i]*1)*F_bar[i];
```

```
//Calculate the axial deflection of the tip as a=<A1_bar(l)>{f} from Eq. (4.34)
    a=0;
    a+=A1_bar[2]*F_bar[2];
    a+=A1_bar[3]*l*F_bar[3];
    a+=(A1_bar[9]+A1_bar[4]*pow(l,2))*F_bar[4];
    for(i=5;i<=8;i++)
        a+=A1_bar[i]*exp(m[i]*l)*F_bar[i];

//Output the nodal displacement contributions at z=l for mode n=1 in mm
    cout<<"*** Nodal displacement components for n=1 ***"<<endl;
    cout<<"a1(l) = "<<a*1000<<" mm"<<endl;
    cout<<"d1(l) = "<<d*1000<<" mm"<<endl;
    cout<<"f1(l) = "<<f*1000<<" mm"<<endl;

//*** Calculate now the displacements b1, c1 and g1 ***

//Construct the matrix [H] from the boundary conditions
    //Row 1 - Eq. (4.37c)
    H[1][1]=H[1][5]=H[1][6]=H[1][7]=H[1][8]=1;
    H[1][2]=H[1][3]=H[1][4]=0;

    //Row 2 - Eq. (4.37d)
    H[2][1]=H[2][3]=H[2][4]=0;
    H[2][2]=1;
    for(i=5;i<=8;i++)
        H[2][i]=m[i];

    //Row 3 - Eq. (4.37b)
    H[3][1]=D1_bar[1];
    H[3][2]=H[3][4]=0;
    H[3][3]=-1*D1_bar[9];
    for(i=5;i<=8;i++)
        H[3][i]= -1*D1_bar[i];

    //Row 4 - Eq. (4.37a)
    H[4][2]=A1_bar[2];
    H[4][1]=H[4][3]=0;
    H[4][4]=A1_bar[9];
    for(i=5;i<=8;i++)
        H[4][i]=A1_bar[i];

    //Row 5 - Eq. (4.39)
    H[5][1]=cn[3]*(1+D1_bar[1]);
    H[5][2]=cn[3]*l*(1+D1_bar[2]);
    H[5][3]=cn[1]*A1_bar[3]+cn[3]*(pow(l,2)*(1+D1_bar[3])+D1_bar[9]);
    H[5][4]=2*cn[1]*A1_bar[4]*l+cn[3]*(pow(l,3)*(1+D1_bar[4])+D1_bar[10]*l);
    for(i=5;i<=8;i++)
        H[5][i]=(cn[1]*m[i]*A1_bar[i]+cn[3]*(1+D1_bar[i]))*exp(m[i]*l);

    //Row 6 - Eq. (4.42)
    H[7][1]=-1*cn[8]*(1+D1_bar[1]);
    H[7][2]=-1*cn[8]*l*(1+D1_bar[2]);
    H[7][3]=2*cn[6]-cn[8]*(pow(l,2)*(1+D1_bar[3])+D1_bar[9]);
    H[7][4]=6*cn[6]*l-cn[8]*(pow(l,3)*(1+D1_bar[4])+D1_bar[10]*l);
    for(i=5;i<=8;i++)
        H[7][i]=(cn[6]*pow(m[i],2)-cn[8]*(1+D1_bar[i]))*exp(m[i]*l);

    //Row 7 - Eq. (4.40)
    H[6][1]=0;
    H[6][2]=-1*cn[5]*A1_bar[2]+(cn[4]+cn[9])*D1_bar[2]+cn[9];
    H[6][3]=l*(-1*cn[5]*A1_bar[3]+2*(cn[4]+cn[9])*D1_bar[3]+2*cn[9]);
    H[6][4]=-1*cn[5]*(A1_bar[9]+A1_bar[4]*pow(l,2))+(cn[4]+cn[9])*(D1_bar[10]+3*D1_bar[4]*
        pow(l,2))+3*cn[9]*pow(l,2);
    for(i=5;i<=8;i++)
        H[6][i]=(-1*cn[5]*A1_bar[i]+(cn[4]+cn[9])*D1_bar[i]*m[i]+cn[9]*m[i])*exp(m[i]*l);

    //Row 8 - Eq. (4.41)
    H[8][1]=0;
    H[8][2]=(cn[8]+cn[9])*(1+D1_bar[2]);
    H[8][3]=(cn[8]+cn[9])*l*2*(1+D1_bar[3]);
    H[8][4]=-6*cn[6]+(cn[8]+cn[9])*(3*pow(l,2)+D1_bar[10]+3*D1_bar[4]*pow(l,2));
```

```
    for(i=5;i<=8;i++)
        H[8][i]=(-1*cn[6]*pow(m[i],3)+(cn[8]+cn[9])*(1+D1_bar[i])*m[i])*exp(m[i]*l);

//Invert the martix [H]
    inverse(H, H_inv);

//Create the load vector Q in Eq. (4.43)
    for(i=1; i<=8; i++)
        Q[i]=0;
    Q[8]=-16;

//Calculate the integration constants from {G_bar} = [H_inv]*{Q}
    multiply_vector(G_bar, Q, H_inv);

//Find the vector <e(l)> from Eq. (4.35c)
    e[1]=1;
    e[2]=l;
    e[3]=pow(l,2);
    e[4]=pow(l,3);
    for(i=5;i<=8;i++)
        e[i]=exp(m[i]*l);

//Calculate tip deflection as g=<e(l)>{G} from Eq. (4.34)
    g=0;
    for(i=1;i<=8;i++)
        g+=e[i]*G_bar[i];

//Calculate tip deflection as c=-1*<D1_bar(l)>{G} from Eq. (4.34)
    c=0;
    c+=D1_bar[1]*G_bar[1];
    c+=D1_bar[2]*l*G_bar[2];
    c+=(D1_bar[9]+D1_bar[3]*pow(l,2))*G_bar[3];
    c+=(D1_bar[10]*l+D1_bar[4]*pow(l,3))*G_bar[4];
    for(i=5;i<=8;i++)
        c+=D1_bar[i]*exp(m[i]*l)*G_bar[i];

//Calculate the axial deflection of the tip as b=<A1_bar(l)>{G} from Eq. (4.34)
    b=0;
    b+=A1_bar[2]*G_bar[2];
    b+=A1_bar[3]*l*G_bar[3];
    b+=(A1_bar[9]+A1_bar[4]*pow(l,2))*G_bar[4];
    for(i=5;i<=8;i++)
        b+=A1_bar[i]*exp(m[i]*l)*G_bar[i];

//Output the nodal displacement contributions at z=l for mode n=1 in mm
    cout<<"*** Nodal displacement components for n=1 ***"<<endl;
    cout<<"b1(l) = "<<b*1000<<" mm"<<endl;
    cout<<"c1(l) = "<<c*1000<<" mm"<<endl;
    cout<<"g1(l) = "<<g*1000<<" mm"<<endl;

//*******************************************************************************************
// *** End mode n=1 ***
//*******************************************************************************************

//*******************************************************************************************
// ***General case of n>=2 ***
//*******************************************************************************************

//Calculate each Fourier mode 'n' independently
    for(n=2; n<=alpha; n++)
    {
    //Calculate constants that depend on 'n' and coefficients of quartic equation
        constants2(cn, n, r);
        coefficients(cn, p);

    //Use Ferrari's method to determine roots M and then take square roots to get m
        quartic_roots(M, p);
        m[5]=sqrt(M[1]);
        m[6]=-1*sqrt(M[1]);
        m[3]=sqrt(M[2]);
        m[4]=-1*sqrt(M[2]);
```

```
        m[1]=sqrt(M[3]);
        m[2]=-1*sqrt(M[3]);
        m[7]=sqrt(M[4]);
        m[8]=-1*sqrt(M[4]);

    //Establish relationships between vectors of integration constants
        for(i=1; i<=8; i++)
        {
            D_bar[i]=((cn[11]*m[i])*(cn[3]*m[i])/(cn[1]*pow(m[i],2)-cn[10])-(cn[14]-
                cn[15]*pow(m[i],2)))/(cn[12]-cn[13]*pow(m[i],2)-
                pow(cn[11]*m[i],2)/(cn[1]*pow(m[i],2)-cn[10]));
            A_bar[i]=((-1)*cn[11]*m[i]/(cn[1]*pow(m[i],2)-cn[10]))*D_bar[i]-
                cn[3]*m[i]/(cn[1]*pow(m[i],2)-cn[10]);
            F_bar[i]=1;
        }

//*** Calculate first the displacements an, dn and fn ***

    //Construct the matrix [H] from the boundary conditions
        for(i=1;i<=8;i++)
        {
            H[1][i]=A_bar[i];
            H[2][i]=D_bar[i];
            H[3][i]=1;
            H[4][i]=m[i];
            H[5][i]=(cn[1]*m[i]*A_bar[i]+cn[3]*(1+n*D_bar[i]))*exp(m[i]*l);
            H[6][i]=(-1*n*cn[5]*A_bar[i]+(cn[4]+cn[9])*m[i]*D_bar[i]+
                n*cn[9]*m[i])*exp(m[i]*l);
            H[7][i]=(-1*cn[6]*pow(m[i],3)+(cn[8]+cn[9])*(n*D_bar[i]+n*n)*m[i])*exp(m[i]*l);
            H[8][i]=(cn[6]*pow(m[i],2)-cn[8]*(n*D_bar[i]+n*n))*exp(m[i]*l);
        }

    //Invert the martix [H]
        inverse(H, H_inv);

    //Create the load vector Q
        for(i=1; i<=8; i++)
            Q[i]=0;
        if(n%2==0)
            Q[7]=-16*cos(n*pi/2);

    //Calculate the integration constants from {F_bar} = [H_inv]*{Q}
        multiply_vector(F_bar, Q, H_inv);

    //Find a(l), d(l) and f(l) from a(l)=<A_bar(l)>{F_bar}
        a=d=f=0;
        for(i=1; i<=8; i++)
        {
            a+=A_bar[i]*exp(m[i]*l)*F_bar[i];
            d+=D_bar[i]*exp(m[i]*l)*F_bar[i];
            f+=F_bar[i]*exp(m[i]*l);
        }

    //Output the nodal displacement contributions at z=l for mode n in mm
        cout<<"*** Nodal displacement components for mode "<<n<<" ***"<<endl;
        cout<<"a"<<n<<"(l) = "<<a*1000<<" mm"<<endl;
        cout<<"d"<<n<<"(l) = "<<d*1000<<" mm"<<endl;
        cout<<"f"<<n<<"(l) = "<<f*1000<<" mm"<<endl;

//*** Calculate now the displacements bn, cn and gn ***

    //Construct the matrix [H] from the boundary conditions
        for(i=1;i<=8;i++)
        {
            H[1][i]=A_bar[i];
            H[2][i]=-1*D_bar[i];
            H[3][i]=1;
            H[4][i]=m[i];
            H[5][i]=(cn[1]*m[i]*A_bar[i]+cn[3]*(1+n*D_bar[i]))*exp(m[i]*l);
            H[6][i]=(n*cn[5]*A_bar[i]-(cn[4]+cn[9])*m[i]*D_bar[i]-n*cn[9]*m[i])*exp(m[i]*l);
            H[7][i]=(-1*cn[6]*pow(m[i],3)+(cn[8]+cn[9])*(n*D_bar[i]+n*n)*m[i])*exp(m[i]*l);
```

```
            H[8][i]=(cn[6]*pow(m[i],2)-cn[8]*(n*D_bar[i]+n*n))*exp(m[i]*l);
        }

    //Invert the martix [H]
        inverse(H, H_inv);

    //Create the load vector Q
        for(i=1; i<=8; i++)
            Q[i]=0;
        if(n%2==0)
            Q[7]=-16*sin(n*pi/2);

    //Calculate the integration constants from {G_bar} = [H_inv]*{Q}
        multiply_vector(G_bar, Q, H_inv);

    //Find b(l), c(l) and g(l) from b(l)=<A_bar(l)>{G_bar}
        b=c=g=0;
        for(i=1; i<=8; i++)
        {
            b+=A_bar[i]*exp(m[i]*l)*G_bar[i];
            c+=D_bar[i]*exp(m[i]*l)*G_bar[i];
            g+=F_bar[i]*exp(m[i]*l);
        }

    //Output the nodal displacement contributions at z=l for mode n in mm
        cout<<"b"<<n<<"(l) = "<<b*1000<<" mm"<<endl;
        cout<<"c"<<n<<"(l) = "<<c*1000<<" mm"<<endl;
        cout<<"g"<<n<<"(l) = "<<g*1000<<" mm"<<endl;
    }

//*************************************************************************************
// *** End mode n ***
//*************************************************************************************

}

//*************************************************************************************
// *** END MAIN PROGRAM ***
//*************************************************************************************

//Function to get the physical properties of the pipe
void input(double *r, double *t, double *l, double *E, double *G, double *mu, int *alpha)
{
   *r=0.2;
   *t=0.006;
   *E=200000000;
   *mu=0.3;
   *G=*E/(2*(1+*mu));
   *l=4;
   *alpha=6;
}

//Function to calculate the constants dependent on physical properties of the pipe
void constants1(double *c, double r, double t, double E, double G, double mu)
{
    double pi=M_PI;
    c[1]=E*r*t*pi/(1-pow(mu,2));
    c[2]=E*t*pi/(r*(1-pow(mu,2)));
    c[3]=E*t*mu*pi/(1-pow(mu,2));
    c[4]=G*r*t*pi;
    c[5]=G*t*pi;
    c[6]=E*r*pow(t,3)*pi/(12*(1-pow(mu,2)));
    c[7]=E*pow(t,3)*pi/(12*pow(r,3)*(1-pow(mu,2)));
    c[8]=E*pow(t,3)*mu*pi/(12*r*(1-pow(mu,2)));
    c[9]=G*pow(t,3)*pi/(3*r);
    c[13]=c[4]+c[9];
}

//Function to calculate constants dependent on Fourier mode n
//Function will be called for each mode n
void constants2(double *c, int n, double r)
```

```
{   c[10]=c[4]*pow((n/r),2);
    c[11]=(c[3]+c[5])*n;
    c[12]=(c[2]+c[7])*pow(n,2);
    c[14]=c[2]*n+c[7]*pow(n,3);
    c[15]=(c[8]+c[9])*n;
    c[16]=c[2]+c[7]*pow(n,4);
    c[17]=(2*c[8]+c[9])*pow(n,2);
}

//Function to calculate the coefficients of the quartic equation
void coefficients(double *c, double *p)
{
    double d1, d2, d3, d4, d5;
    p[1]=(-1)*c[1]*c[6]*c[13];
    p[2]=c[1]*(c[6]*c[12]+c[13]*c[17]-pow(c[15],2))+c[6]*(c[10]*c[13]-pow(c[11],2));
    p[3]=c[1]*((-1)*c[12]*c[17]-c[13]*c[16]+2*c[14]*c[15])-c[10]*(c[6]*c[12]+c[13]*c[17]-
        pow(c[15],2))+pow(c[11],2)*c[17]-2*c[3]*c[11]*c[15]+pow(c[3],2)*c[13];
    d1=c[1]*(c[12]*c[16]-pow(c[14],2));
    d2=c[10]*(c[12]*c[17]+c[13]*c[16]-2*c[14]*c[15]);
    d3=2*c[3]*c[11]*c[14];
    d4=pow(c[11],2)*c[16];
    d5=pow(c[3],2)*c[12];
    p[4]=d1+d2+d3-d4-d5;
    p[5]=(-1)*c[10]*(c[12]*c[16]-pow(c[14],2));
}

//Function to solve the quartic equation with Ferrari's Method for modes n>=2
void quartic_roots(complex *root, double *pp)
{
    int i;
    double alpha, beta, gamma;
    double p, q, radical;
    complex r, u, sqroot, y, w, bracket_pos, bracket_neg, polynomial[6], temp[3];

    alpha = -3*pow(pp[2],2)/(8*pow(pp[1],2))+pp[3]/pp[1];
    beta = pow(pp[2],3)/(8*pow(pp[1],3))-(pp[2]*pp[3])/(2*pow(pp[1],2))+pp[4]/pp[1];
    gamma =-3*pow(pp[2],4)/(256*pow(pp[1],4))+pp[3]*pow(pp[2],2)/(16*pow(pp[1],3))-
        pp[2]*pp[4]/(4*pow(pp[1],2))+pp[5]/pp[1];

    //Check if beta=0
    if (beta<.000001 && beta>-.000001)        //Equation is Biquadratic
    {
        p = pow(alpha,2)-4*gamma;
        if(p>0)
            sqroot = sqrt(p);
        else
            sqroot = complex_root(p);
        bracket_pos = -alpha+sqroot;
        bracket_neg = -alpha-sqroot;
        root[1] = -pp[2]/(4*pp[1])+sqrt(bracket_pos/2.0);
        root[2] = -pp[2]/(4*pp[1])+sqrt(bracket_neg/2.0);
        root[3] = -pp[2]/(4*pp[1])-sqrt(bracket_pos/2.0);
        root[4] = -pp[2]/(4*pp[1])-sqrt(bracket_neg/2.0);
        for(i=1; i<=4; i++)
            if(abs(root[i])<.00001)
                root[i]=0;
    }

    else    //Equation is not Biquadratic - Full Ferrari's method is req'd
    {
        p = -pow(alpha,2)/12-gamma;
        q = -pow(alpha,3)/108+alpha*gamma/3-pow(beta,2)/8;
        radical = pow(q,2)/4+pow(p,3)/27;

        if(radical>0)
            r = q/2.0 + sqrt(radical);
        else
        {
            sqroot = complex_root(radical);
            r = q/2.0 + sqroot;
        }
```

```
        u = pow(r, 1.0/3.0);

        //check if u=0
        if(abs(u)<.000001)
        `   y = -(5.0/6.0)*alpha;
        else
            y = -(5.0/6.0)*alpha - u + p/(3*u);

        w = sqrt(alpha + 2*y);
        bracket_pos = -(3*alpha+2*y+2*beta/w);
        bracket_neg = -(3*alpha+2*y-2*beta/w);

        root[1] = -pp[2]/(4*pp[1])+(w+sqrt(bracket_pos))/2;
        root[2] = -pp[2]/(4*pp[1])-(w+sqrt(bracket_neg))/2;
        root[3] = -pp[2]/(4*pp[1])+(w-sqrt(bracket_pos))/2;
        root[4] = -pp[2]/(4*pp[1])-(w-sqrt(bracket_neg))/2;
    }
}

//Function to compute the complex square root of a number of type "double"
complex complex_root(double x)
{
    complex root, xx=x;
    root = sqrt(xx);
    return root;
}

//Function to compute the inverse of a matrix [a]
void inverse(complex a[][9], complex a_inv[][9])
{
    int i,j,k;
    complex value;

    for(i=1;  i<=8; i++)
    {
        for(j=1; j<=8; j++)
        {
            if(i==j)
                a_inv[i][j]=1.0;
            else
                a_inv[i][j]=0.0;
        }
    }

    for(i=1; i<=8; i++)
    {
        for(j=1; j<=8; j++)
        {
            if(i!=j)
                a[i][j]/=a[i][i];
            a_inv[i][j]/=a[i][i];
        }
        a[i][i]=1.0;
        for(k=1; k<=8; k++)
        {
            if(i!=k)
            {
                value=a[k][i];
                for(j=1; j<=8; j++)
                {
                        a[k][j]-=a[i][j]*value;
                        a_inv[k][j]-=a_inv[i][j]*value;
                }
            }
        }
    }
}

// Calculates the product of 2 8x8 matrices -> [product]=[x][y]
void multiply_matrix(complex product[] [9], complex A[][9], complex B[][9])
```

```
{
    int i, j, k;

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
        {
            product[i][j]=0;
            for(k=1; k<=8; k++)
                product[i][j]+=A[i][k]*B[k][j];
        }
}

// Calculates the product of an 8x8 matrix and an 8x1 vector -> {result}=[matrix]{vector}
void multiply_vector(complex result[], complex vector[], complex matrix[][9])
{
    int i, j;

    for(i=1; i<=8; i++)
    {
        result[i]=0;
        for(j=1; j<=8; j++)
            result[i]+=matrix[i][j]*vector[j];
    }
}

//*********************************************************************************************
//                                    *** END PROGRAM ***
//*********************************************************************************************
```

## A10.2 Output

Below is the output given by the program listed in Secion A10.1:

```
*** Nodal displacement contributions for n=0 ***
a0(1) = 0.000637487 mm
c0(1) = 0 mm
f0(1) = -0.0314979 mm

*** Nodal displacement components for n=1 ***
a1(1) = 0 mm
d1(1) = 0 mm
f1(1) = 0 mm
b1(1) = 0.848928 mm
c1(1) = -11.51643 mm
g1(1) = -11.579102 mm

*** Nodal displacement components for mode 2 ***
a2(1) = -0.152271 mm
d2(1) = -1.3352 mm
f2(1) = 2.74664 mm
b2(1) = 0 mm
c2(1) = 0 mm
g2(1) = 0 mm

*** Nodal displacement components for mode 3 ***
a3(1) = 0 mm
d3(1) = 0 mm
f3(1) = 0 mm
b3(1) = -0.0536703 mm
c3(1) = 0.302682 mm
g3(1) = 0.972001 mm

*** Nodal displacement components for mode 4 ***
a4(1) = 0.0254056 mm
d4(1) = 0.113397 mm
f4(1) = -0.503479 mm
b4(1) = 0 mm
c4(1) = 0 mm
g4(1) = 0 mm

*** Nodal displacement components for mode 5 ***
a5(1) = 0 mm
d5(1) = 0 mm
f5(1) = 0 mm
b5(1) = 0.0132984 mm
c5(1) = -0.0522025 mm
g5(1) = -0.297169 mm

*** Nodal displacement components for mode 6 ***
a6(1) = -0.0073683 mm
d6(1) = -0.0269971 mm
f6(1) = 0.187143 mm
b6(1) = 0 mm
c6(1) = 0 mm
g6(1) = 0 mm
```

## A10.3 Example 2: Cantilever Beam Under Self Weight

Example 2 in Chapter 4 consists of a cantilever beam under self weight. The pipe properties are the same as those used in Example 1 in Chapter 4 and listed in Section A10.1 above. The only non-zero load potential vector is the one corresponding to displacement contributions $b_1$, $c_1$ and $g_1$ for mode $n$=1. The C++ code for the load potential vector must be modified to include the particular solution for the self weight. Also, the particular solution must be included when calculating the tip displacement. The relevant modified sections of the C++ code are:

```
// Create the load vector Q in Eq. (4.95)

//Find coefficients T[i] of particular solution in Eq. 4.80
//LHS matrix and RHS vector
    x[1][1]=-1*cn[5]/r;
    x[1][2]=-2*(cn[3]+cn[5]);
    x[1][3]=-24*r*cn[1];
    x[1][4]=2*cn[3];
    x[2][1]=0;
    x[2][2]=-1*(cn[2]+cn[7]);
    x[2][3]=-12*(r*cn[3]+cn[8]);
    x[2][4]=cn[2]+cn[7];
    x[3][1]=cn[3];
    x[3][2]=2*(cn[8]+cn[9]);
    x[3][3]=24*cn[6];
    x[3][4]=-2*(2*cn[8]+cn[9]);
    x[4][1]=cn[3]+cn[5];
    x[4][2]=2*(r*cn[5]+cn[9]);
    x[4][3]=0;
    x[4][4]=-2*(cn[8]+cn[9]);
    q[1]=q[2]=0;
    q[3]=-1*pi*t*78*r;
    q[4]=pi*t*78*r;

    inverse(x,x_inv);
    multiply_vector(T, q, x_inv);

    for(i=1; i<=4; i++)
        Q[i]=0;
    Q[5]=-1*cn[1]*(T[1]-12*r*T[3]*pow(1,2))+cn[3]*(T[2]-T[4])*pow(1,2);
    Q[6]=-1*cn[5]*1*(T[1]-4*r*T[3]*pow(1,2))-2*1*(r*cn[5]+cn[9])*(T[2]+2*T[3]*pow(1,2))+
        2*1*cn[9]*(T[4]+2*T[3]*pow(1,2));
    Q[7]=24*cn[6]*T[3]*1+2*1*(cn[8]+cn[9])*(T[2]+2*T[3]*pow(1,2))-
        2*1*(cn[8]+cn[9])*(T[4]+2*T[3]*pow(1,2));
    Q[8]=-2*cn[6]*(T[4]+6*T[3]*pow(1,2))-cn[8]*(T[2]-T[4])*pow(1,2);

//Calculate the integration constants from {G_bar} = [H_inv]*{Q}
    multiply_vector(G_bar, Q, H_inv);

//Find the vector <e(1)> from Eq. (4.35c)
    e[1]=1;
    e[2]=1;
    e[3]=pow(1,2);
    e[4]=pow(1,3);
    for(i=5;i<=8;i++)
        e[i]=exp(m[i]*1);

//Calculate tip deflection as g=<e(1)>{G} from Eq. (4.83)
```

```
    g=T[4]*pow(l,2)+T[3]*pow(l,4);
    for(i=1;i<=8;i++)
        g+=e[i]*G_bar[i];

//Calculate tip deflection as c=-<D1_bar(l)>{g} from Eq. (4.83)
    c=T[2]*pow(l,2)+T[3]*pow(l,4);
    c+=-1*D1_bar[1]*G_bar[1];
    c+=-1*D1_bar[2]*l*G_bar[2];
    c+=-1*(D1_bar[9]+D1_bar[3]*pow(l,2))*G_bar[3];
    c+=-1*(D1_bar[10]*l+D1_bar[4]*pow(l,3))*G_bar[4];
    for(i=5;i<=8;i++)
        c+=-1*D1_bar[i]*exp(m[i]*l)*G_bar[i];

//Calculate the axial deflection of the tip as b=<A1_bar(l)>{g} from Eq. (4.34)
    b=T[1]*l-4*r*T[3]*pow(l,3);
    b+=A1_bar[2]*G_bar[2];
    b+=A1_bar[3]*l*G_bar[3];
    b+=(A1_bar[9]+A1_bar[4]*pow(l,2))*G_bar[4];
    for(i=5;i<=8;i++)
        b+=A1_bar[i]*exp(m[i]*l)*G_bar[i];

//Output the nodal displacement contributions at z=l for mode n=1 in mm
    cout<<"*** Nodal displacement components for n=1 ***"<<endl;
    cout<<"b1(l) = "<<b*1000<<" mm"<<endl;
    cout<<"c1(l) = "<<c*1000<<" mm"<<endl;
    cout<<"g1(l) = "<<g*1000<<" mm"<<endl;
```

The output from the program for the self weight of Example 2 is:

```
    *** Nodal displacement components for n=1 ***
    a1(l) = 0 mm
    d1(l) = 0 mm
    f1(l) = 0 mm
    b1(l) = 0.0416114 mm
    c1(l) = -0.639488 mm
    g1(l) = -0.639504 mm
```

# APPENDIX 11 –PROGRAM LISTING FOR FINITE ELEMENT SOLUTION

The finite element method developed in Chapter 5 was implemented using the C++ programming language and compiled with the software program Borland C++, version 5.02. Included in the subsequent sections are the listing of the program and the program output for the examples given in Chapter 6.

The program utilizes the built-in C++ class "complex" which can carry out all required mathematical manipulations of complex numbers. The geometric and material properties are hard coded into the program. These can be changed and the program re-run to model different pipe geometries. Alternatively, a simple modification could be made to the code to allow the user to input the pipe properties. The latter alternative would be useful for a small problem involving only one or two elements but would quickly become cumbersome for a larger model.

The loading conditions are also hard coded into the program. The main program listing in Section A11.1 includes the code for the loading given in Example 1 in Chapter 6. As with the pipe properties, the loading can be change by modifying the code and re-running the model. The section of code that is required to be changed to alter the loading is relatively small. The code for the loading conditions used in each of the examples in Chapter 6 is included in this appendix.

Note that the only changes to the program listing required to model the examples in Chapter 6 are the pipe properties and the loading conditions. All other aspects of the program listing, including the calculation of the all constants and the determination of the stiffness matrix, do not need to be altered.

## A11.1 Program Listing

Below is the listing for the program used to calculate the deflection of the tip of a cantilevered pipe with an applied point load at the tip. The geometry, material properties and loading are those of Example 1 in Chapter 6.

Note that everything in *italics* in the program listing is a comment that is ignored by the C++ compiler. These comments are included for clarification only.

```
#include <complex.h>
#include <math.h>

// Function definitions
void input(double *, double *,double *, double *, double *, double *, int *);
void constants1(double *, double, double, double, double);
void constants2(double *, int, double);
void coefficients(double *, double *);
void quartic_roots(complex *, double *);
complex complex_root(double);
void get_N1_1(complex [][9][9], complex [], complex [], double, double);
void get_N1_2(complex [][9][9], complex [], complex [], double, double);
void get_N1_3(complex [][9][9], complex [], complex [], complex [], double, double);
void get_N1_4(complex [][9][9], complex [], complex [], complex [], double, double, double);
void get_N1_5(complex [][9][9], complex [], double, double);
void get_N1_6(complex [][9][9], complex [], complex [], double, double);
void get_N1_7(complex [][9][9], complex [], complex [], double, double);
void get_N1_8(complex [][9][9], complex [], complex [], double, double);
void get_N1(complex [][9][9], complex [], complex [], double, double);
void get_N2(complex [][9][9], complex [], complex [], complex [], double, double, double);
void get_N3(complex [][9][9], complex [], complex [], complex [], complex [], double,
    double, double);
void get_N4(complex [][9][9], complex [], complex [], complex [], double, double, double,
    double);
void get_N5(complex [][9][9], complex [], complex [], double, double);
void get_N6(complex [][9][9], complex [], complex [], complex [], double, double, double);
void get_N7(complex [][9][9], complex [], complex [], complex [], double, double, double);
void get_N8(complex [][9][9], complex [], complex [], complex [], double, double, double);
void get_sub_sub_N(complex [][9], complex [], complex [], complex [], int, int, double);
void multiply_matrix(complex [][9], complex [][9], complex[][9]);
void multiply_vector(complex [], complex [], complex [][9]);


// Begin main program
main()
{
   double pi=M_PI;
   double r, t, E, G, mu, l;  //physical properties of pipe
   double cn[18];             //array of constants from Eq. 2.15
   double p[6];               //coefficients of quartic equation
   int i,j,k;                 //counters in "for" loops
   int n;                     //current Fourier mode
   int alpha;                 //total number of Fourier modes included in analysis
   complex i_sqrd, im;        //imaginary number defined as sqrt(-1)
   complex M[5];              //roots of quartic equation
   complex m[9];              //sqaure roots of roots of quartic equation
   complex A_bar[9], D_bar[9], F_bar[9], C_bar[9];  //vectors relating integration constants
   complex A1_bar[10], D1_bar[11];    //vectors relating integration constants for mode n=1
   complex L01[9][9], L01_inv[9][9];  //matrices required for modes n=0 and n=1
   complex L02[9][9], L02_inv[9][9];  //matrices required for modes n=0 and n=1
   complex L1[9][9], L1_inv[9][9], L1_invt[9][9];    //matrix [Ln,1] and its inverse
   complex L2[9][9], L2_inv[9][9], L2_invt[9][9];    //matrix [Ln,2] and its inverse
   complex Kn[9][9], Kn_inv[9][9];                   //stiffness matrix and its inverse
```

```
   complex N[9][9], subN[9][9][9], LN[9][9]; //intermediate matrices needed to determine [K]
   complex LL[9][9], prod[9][9];
   complex e[9];
   complex Q[9], delta[9];    //load potential and nodal displacement vectors
   complex a, b, c, d, f, g;

//Define imaginary number i and call it "im"
   i_sqrd=-1;
   im=sqrt(i_sqrd);

//Get element data
   input(&r, &t, &l, &E, &G, &mu, &alpha);

//Calculate constants not dependent on "n"
   constants1(cn, r, t, E, G, mu);

//*******************************************************************************************
// *** Axi-symmetric case n=0 ***
//*******************************************************************************************

//Calculate each of the roots for i=3 to 6
    m[1]=-12*(1-pow(mu,2))/pow(r*t,2);
    m[2]=sqrt(m[1]);
    m[3]=sqrt(m[2]);
    m[4]=-1*m[3];
    m[5]=m[3]*im;
    m[6]=-1*m[3]*im;

//Construct F_bar and C_bar.
    for(i=1;i<=8;i++)
        Fbar[i]=0;
    Fbar[2]=-1*mu*r;
    for(i=3;i<=6;i++)
        Fbar[i]=-1*r*m[i]/mu;

//Construct the matrix [L0]
    for(j=1;j<=8;j++)
    {
        switch(j)
        {
            case 1:    //Column 1
                L[1][j]=L[5][j]=1;
                L[2][j]=L[3][j]=L[4][j]=L[6][j]=L[7][j]=L[8][j]=0;
                break;

            case 2:    //Column 2
                L[1][j]=L[2][j]=L[4][j]=L[6][j]=L[8][j]=0;
                L[3][j]=L[7][j]=Fbar[2];
                L[5][j]=1;
                break;

            case 7:    //Column 7
                L[2][j]=L[6][j]=1;
                L[1][j]=L[3][j]=L[4][j]=L[5][j]=L[7][j]=L[8][j]=0;
                break;

            case 8:    //Column 8
                L[6][j]=1;
                L[1][j]=L[2][j]=L[3][j]=L[4][j]=L[5][j]=L[7][j]=L[8][j]=0;
                break;

            default:   //Columns 3 to 6
                L[1][j]=1;
                L[2][j]=L[6][j]=0;
                L[3][j]=Fbar[j];
                L[4][j]=m[j]*Fbar[j];
                L[5][j]=exp(m[j]*l);
                L[7][j]=Fbar[j]*exp(m[j]*l);
                L[8][j]=m[j]*Fbar[j]*exp(m[j]*l);
                break;
        }
```

```
    }

// *** Need to rearrange [L0] to calculate inverse - cannot have zeroes on diagonal ***
//  Previous order for nodal displacements:  <a(0) c(0) f(0) f'(0) a(1) c(1) f(1) f'(1)>
// New order for nodal displacements:        <a(0) f(1) f(0) f'(0) a(1) f'(1) c(0) c(1)>
    for(i=1; i<=8; i++)
    {
        L01[1][i]=L[1][i];
        L01[2][i]=L[7][i];
        L01[3][i]=L[3][i];
        L01[4][i]=L[4][i];
        L01[5][i]=L[5][i];
        L01[6][i]=L[8][i];
        L01[7][i]=L[2][i];
        L01[8][i]=L[6][i];
    }

//Invert the martix [L0]
    inverse(L01, L01_inv);

// *** Need to rearrange L_inv so it is the inverse of L rather than L0.
// This is necessary because shape function vectors N are written in terms of L, not L0.
// It is accomplished by switching columns in L_inv corresponding to rows switched in L
    for(i=1; i<=8; i++)
    {
        L_inv[i][1]=L01_inv[i][1];
        L_inv[i][2]=L01_inv[i][7];
        L_inv[i][3]=L01_inv[i][3];
        L_inv[i][4]=L01_inv[i][4];
        L_inv[i][5]=L01_inv[i][5];
        L_inv[i][6]=L01_inv[i][8];
        L_inv[i][7]=L01_inv[i][2];
        L_inv[i][8]=L01_inv[i][6];
    }

//Find the transpose of matrix L_inv
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            L_invt[i][j]=L_inv[j][i];

//Calculate the stiffness matrix for mode '0'
//Initialize all sub-matrics subN to zero
    for(k=1;k<=6;k++)
        for(i=1;i<=8;i++)
            for(j=1;j<=8;j++)
                subN[k][i][j]=0;

//Calculate each sub matrix as the integral of the product of the vectors e0, F0 and C0
//  and the constants as per Eqs. 5.31 and 5.32
    subN[1][2][2]=2*c[1]*l;
    subN[2][2][2]=2*c[2]*pow(Fbar[2],2)*l;
    subN[3][2][2]=4*c[3]*Fbar[2]*l;

    for(h=3;h<=6;h++)
    {
        subN[1][2][h]=subN[1][h][2]=2*c[1]*(exp(m[h]*l)-1);
        subN[2][2][h]=subN[2][h][2]=2*c[2]*(Fbar[2]*Fbar[h]/m[h])*(exp(m[h]*l)-1);
        subN[3][2][h]=subN[3][h][2]=2*c[3]*(Fbar[h]/m[h]+Fbar[2])*(exp(m[h]*l)-1);
    }

    for(j=3;j<=6;j++)
        for(h=3;h<=6;h++)
            if(abs(m[j]+m[h])<0.000001)
            {
                subN[1][j][h]=2*c[1]*m[j]*m[h]*l;
                subN[2][j][h]=2*c[2]*Fbar[j]*Fbar[h]*l;
                subN[3][j][h]=2*c[3]*(Fbar[h]*m[j]*Fbar[j]*m[h])*l;
                subN[5][j][h]=2*c[6]*Fbar[j]*Fbar[h]*pow(m[j],2)*pow(m[h],2)*l;
            }
            else
            {
```

```
                subN[1][j][h]=2*c[1]*(m[j]*m[h]/(m[j]+m[h]))*(exp((m[j]+m[h])*l)-1);
                subN[2][j][h]=2*c[2]*(Fbar[j]*Fbar[h]/(m[j]+m[h]))*(exp((m[j]+m[h])*l)-1);
                subN[3][j][h]=2*c[3]*((Fbar[h]*m[j]+Fbar[j]*m[h])/(m[j]+m[h]))*
                    (exp((m[j]+m[h])*l)-1);
                subN[5][j][h]=2*c[6]*(Fbar[j]*Fbar[h]*pow(m[j],2)*pow(m[h],2)/(m[j]+m[h]))*
                    (exp((m[j]+m[h])*l)-1);
            }

    subN[4][8][8]=2*c[4]*l;
    subN[6][8][8]=2*c[9]*l;

//Matrix [N] is sum of all the subN matrices
    for(i=1;i<=8;i++)
        for(j=1;j<=8;j++)
        {
            N[i][j]=0;
            for(k=1;k<=6;k++)
                N[i][j]+=subN[k][i][j];
        }

//Calculate the stiffness matrix as [K]=[Linvt][N][Linv]
    mult(LN, Linvt, N);
    mult(K, LN, Linv);

//Set the nodal load vector Q
//Note that this section is adjusted for different loading conditions
//Below is the load vector corresponding to Example 1 in Chapter 6, Eq. 6.2
//Find {e(l)}
    e[1]=0;
    e[2]=F_bar[2];
    for(i=3; i<=6; i++)
        e[i]=F_bar[i]*exp(m[i]*l);
    e[7]=e[8]=0;

//Set {Q}=-P*[L_invt]*{e(l)}
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-16*L_invt[i][j]*e[j];
    }

//Modify [K] for the boundary conditions of a cantilever with one end completely fixed
    for(i=1; i<=4; i++)
    {
        for(j=1; j<=8; j++)
            if(i!=j)
            {
                Kn[i][j]=0;
                Kn[j][i]=0;
            }
    }

//Invert the modified stiffness matrix to find nodal displacements {delta}
    inverse(Kn, Kn_inv);
    multiply_vector(delta, Q, Kn_inv);

//Find the displacement contributions a, c and f for mode 0
    a=delta[5];
    c=delta[6];
    f=delta[7];

//Output the nodal displacement contributions at z=l for mode n=0 in mm
    cout<<"\n*** Nodal displacement contributions for n=0 ***"<<endl;
    cout<<"a0(l) = "<<a*1000<<" mm"<<endl;
    cout<<"c0(l) = "<<c*1000<<" mm"<<endl;
    cout<<"f0(l) = "<<f*1000<<" mm\n"<<endl;

//*****************************************************************************************
// *** End mode n=0 ***
//*****************************************************************************************
```

```
//**********************************************************************************************
// *** Special case of n=1 ***
//**********************************************************************************************

//Calculate constants that depend on 'n' and coefficients of quartic equation
    constants2(cn, 1, r);
    coefficients(cn, p);

//Calculate each of the roots for i=5 to 8
    m[1]=pow(p[2],2)-4*p[1]*p[3];
    m[2]=sqrt(m[1]);
    m[5]=sqrt((-1.0*p[2]+m[2])/(2*p[1]));
    m[6]=sqrt((-1.0*p[2]-m[2])/(2*p[1]));
    m[7]=-1.0*m[5];
    m[8]=-1.0*m[6];

//Construct vectors {A_bar} and {D_bar}
    A1_bar[1]=0;
    A1_bar[2]=cn[3]/cn[10]-(cn[11]*cn[14])/(cn[10]*cn[12]);
    A1_bar[3]=(2/cn[10])*(cn[3]-(cn[11]*cn[14])/cn[12]);
    A1_bar[4]=(3/cn[10])*(cn[3]-(cn[11]*cn[14])/cn[12]);
    D1_bar[1]=-1.0*cn[14]/cn[12];
    D1_bar[2]=-1.0*cn[14]/cn[12];
    D1_bar[3]=-1.0*cn[14]/cn[12];
    D1_bar[4]=-1.0*cn[14]/cn[12];
    for(i=5; i<=8; i++)
    {
        D1_bar[i]=((cn[11]*m[i])*(cn[3]*m[i])/(cn[1]*pow(m[i],2)-cn[10])-(cn[14]-
            cn[15]*pow(m[i],2)))/(cn[12]-cn[13]*pow(m[i],2)-
            pow(cn[11]*m[i],2)/(cn[1]*pow(m[i],2)-cn[10]));
        A1_bar[i]=((-1)*cn[11]*m[i]/(cn[1]*pow(m[i],2)-cn[10]))*D1_bar[i]-
            cn[3]*m[i]/(cn[1]*pow(m[i],2)-cn[10]);
    }
    A1_bar[9]=-6*r*cn[1]/cn[10]+(6*cn[11]/(cn[10]*cn[12]))*(r*cn[11]-cn[4]+cn[8]);
    D1_bar[9]=(2/cn[12])*((cn[11]/cn[10])*(cn[11]*cn[14]/cn[12]-cn[3])-
        (cn[13]*cn[14]/cn[12])+cn[15]);
    D1_bar[10]=3*D1_bar[9];

//Construct the matrix [L1,1]
    //Column 1
    L1[1][1]=L1[4][1]=L1[5][1]=L1[8][1]=0;
    L1[3][1]=L1[7][1]=1;
    L1[2][1]=L1[6][1]=D1_bar[1];
    //Column 2
    L1[1][2]=L1[5][2]=A1_bar[2];
    L1[2][2]=L1[3][2]=0;
    L1[4][2]=L1[8][2]=1;
    L1[6][2]=D1_bar[2]*1;
    L1[7][2]=1;
    //Column 3
    L1[1][3]=L1[3][3]=L1[4][3]=0;
    L1[2][3]=D1_bar[9];
    L1[5][3]=A1_bar[3]*1;
    L1[6][3]=D1_bar[9]+D1_bar[3]*pow(1,2);
    L1[7][3]=pow(1,2);
    L1[8][3]=2*1;
    //Column 4
    L1[1][4]=A1_bar[9];
    L1[2][4]=L1[3][4]=L1[4][4]=0;
    L1[5][4]=A1_bar[9]+A1_bar[4]*pow(1,2);
    L1[6][4]=D1_bar[10]*1+D1_bar[4]*pow(1,3);
    L1[7][4]=pow(1,3);
    L1[8][4]=3*pow(1,2);
    //Default: Columns 5 through 8
    for(i=5; i<=8; i++)
    {
        L1[1][i]=A1_bar[i];
        L1[2][i]=D1_bar[i];
        L1[3][i]=1;
        L1[4][i]=m[i];
```

```
        L1[5][i]=A1_bar[i]*exp(m[i]*l);
        L1[6][i]=D1_bar[i]*exp(m[i]*l);
        L1[7][i]=exp(m[i]*l);
        L1[8][i]=m[i]*exp(m[i]*l);
    }

//Construct the matrix [L1,2]
    //Column 1
    L2[1][1]=L2[4][1]=L2[5][1]=L2[8][1]=0;
    L2[3][1]=L2[7][1]=1;
    L2[2][1]=L2[6][1]=-1*D1_bar[1];
    //Column 2
    L2[1][2]=L2[5][2]=A1_bar[2];
    L2[2][2]=L2[3][2]=0;
    L2[4][2]=L2[8][2]=1;
    L2[6][2]= -1*D1_bar[2]*l;
    L2[7][2]=l;
    //Column 3
    L2[1][3]=L2[3][3]=L2[4][3]=0;
    L2[2][3]= -1*D1_bar[9];
    L2[5][3]=A1_bar[3]*l;
    L2[6][3]= -1*D1_bar[9]-D1_bar[3]*pow(l,2);
    L2[7][3]=pow(l,2);
    L2[8][3]=2*l;
    //Column 4
    L2[1][4]=A1_bar[9];
    L2[2][4]=L2[3][4]=L2[4][4]=0;
    L2[5][4]=A1_bar[9]+A1_bar[4]*pow(l,2);
    L2[6][4]= -1*D1_bar[10]*l-D1_bar[4]*pow(l,3);
    L2[7][4]=pow(l,3);
    L2[8][4]=3*pow(l,2);
    //Default: Columns 5 through 8
    for(i=5; i<=8; i++)
    {
        L2[1][i]=A1_bar[i];
        L2[2][i]= -1*D1_bar[i];
        L2[3][i]=1;
        L2[4][i]=m[i];
        L2[5][i]=A1_bar[i]*exp(m[i]*l);
        L2[6][i]= -1*D1_bar[i]*exp(m[i]*l);
        L2[7][i]=exp(m[i]*l);
        L2[8][i]=m[i]*exp(m[i]*l);
    }

//Rearrange [L1,1] and [L1,2] to calculate inverse - cannot have zeroes on diagonal
//Previous order for nodal displacements:    <a(0) d(0) e(0) e'(0) a(l) d(l) e(l) e'(l)>
//New order for nodal displacements:         <d(0) a(0) e(l) e'(l) a(l) d(l) e(0) e'(0)>
    for(i=1; i<=8; i++)
    {
        L01[1][i]=L1[2][i];
        L01[2][i]=L1[1][i];
        L01[3][i]=L1[7][i];
        L01[4][i]=L1[8][i];
        L01[5][i]=L1[5][i];
        L01[6][i]=L1[6][i];
        L01[7][i]=L1[3][i];
        L01[8][i]=L1[4][i];
        L02[1][i]=L2[2][i];
        L02[2][i]=L2[1][i];
        L02[3][i]=L2[7][i];
        L02[4][i]=L2[8][i];
        L02[5][i]=L2[5][i];
        L02[6][i]=L2[6][i];
        L02[7][i]=L2[3][i];
        L02[8][i]=L2[4][i];
    }

//Invert the martices [L1,1] and [L1,2]
    inverse(L01, L01_inv);
    inverse(L02, L02_inv);
```

```
//Need to rearrange L_inv so it is the inverse of L rather than L0.
//This is necessary because shape function vectors N are written in terms of L, not L0.
//It is accomplished by switching columns in L_inv corresponding to rows switched in L
    for(i=1; i<=8; i++)
    {
        L1_inv[i][1]=L01_inv[i][2];
        L1_inv[i][2]=L01_inv[i][1];
        L1_inv[i][3]=L01_inv[i][7];
        L1_inv[i][4]=L01_inv[i][8];
        L1_inv[i][5]=L01_inv[i][5];
        L1_inv[i][6]=L01_inv[i][6];
        L1_inv[i][7]=L01_inv[i][3];
        L1_inv[i][8]=L01_inv[i][4];
        L2_inv[i][1]=L02_inv[i][2];
        L2_inv[i][2]=L02_inv[i][1];
        L2_inv[i][3]=L02_inv[i][7];
        L2_inv[i][4]=L02_inv[i][8];
        L2_inv[i][5]=L02_inv[i][5];
        L2_inv[i][6]=L02_inv[i][6];
        L2_inv[i][7]=L02_inv[i][3];
        L2_inv[i][8]=L02_inv[i][4];
    }

//Find the transpose of L1_inv and L2_inv
    for(i=1;i<=8;i++)
        for(j=1;j<=8;j++)
        {
            L1_invt[i][j]=L1_inv[j][i];
            L2_invt[i][j]=L2_inv[j][i];
        }

//*** Calculate first the displacements a1, d1 and f1 ***

//Calculate the stiffness matrix for mode '1'
//Inititalize matrix [N] to zero
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[i][j]=0;

//Calculate Sub Matrices N1_1 through N1_8
    get_N1_1(subN, L1_inv, A1_bar, m, l, cn[1]);
    get_N1_2(subN, L1_inv, D1_bar, m, l, cn[2]);
    get_N1_3(subN, L1_inv, A1_bar, D1_bar, m, l, cn[3]);
    get_N1_4(subN, L1_inv, A1_bar, D1_bar, m, l, cn[4], r);
    get_N1_5(subN, L1_inv, m, l, cn[6]);
    get_N1_6(subN, L1_inv, D1_bar, m, l, cn[7]);
    get_N1_7(subN, L1_inv, D1_bar, m, l, cn[8]);
    get_K1_8(subN, L1_inv, D1_bar, m, l, cn[9]);

//Calculate total stiffness matrix N as sum of N1_1 through N1_8
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            for(k=1; k<=8; k++)
                N[i][j]+=subK[k][i][j];

//Calculate stiffness matrix [K1,1]
    multiply_matrix(LN, N, L1_inv);
    multiply_matrix(Kn, L1_invt, LN);

//Set the nodal load vector Q
//Note that this section is adjusted for different loading conditions
//Below is the load vector corresponding to Example 1 in Chapter 6, Eq. 6.3

//Find {e(l)}
    e[1]=1;
    e[2]=1;
    e[3]=1*l;
    e[4]=1*l*l;
    for(i=5; i<=8; i++)
        e[i]=exp(m[i]*l);
```

```
//Set {Q}=-16*[L_invt]*{e(1)}*cos(pi/2)
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-16*L1_invt[i][j]*e[j]*cos(pi/2);
    }

//Modify [K] for the boundary conditions of cantilever fixed at one end
    for(i=1; i<=4; i++)
    {
        Q[i]=0;
        for(j=1; j<=8; j++)
            if(i!=j)
            {
                Kn[i][j]=0;
                Kn[j][i]=0;
            }
    }

//Invert the modified stiffness matrix to find {delta1}
    inverse(Kn, Kn_inv);
    multiply_vector(delta, Q, Kn_inv);

//Find the displacement contributions a, d and f for mode 1
    a=delta[5];
    d=delta[6];
    f=delta[7];

//Output the nodal displacement contributions at z=1 for mode n=1 in mm
    cout<<"*** Nodal displacement components for n=1 ***"<<endl;
    cout<<"a1(1) = "<<a*1000<<" mm"<<endl;
    cout<<"d1(1) = "<<d*1000<<" mm"<<endl;
    cout<<"f1(1) = "<<f*1000<<" mm"<<endl;

//*** Calculate now the displacements b1, c1 and g1 ***

//Calculate the stiffness matrix for mode '1'
//Inititalize matrix [N] to zero
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[i][j]=0;

//Calculate Sub Matrices N1_1 through N1_8
    get_N1_1(subN, L2_inv, A1_bar, m, l, cn[1]);
    get_N1_2(subN, L2_inv, -1*D1_bar, m, l, cn[2]);
    get_N1_3(subN, L2_inv, A1_bar, -1*D1_bar, m, l, cn[3]);
    get_N1_4(subN, L2_inv, A1_bar, -1*D1_bar, m, l, cn[4], r);
    get_N1_5(subN, L2_inv, m, l, cn[6]);
    get_N1_6(subN, L2_inv, -1*D1_bar, m, l, cn[7]);
    get_N1_7(subN, L2_inv, -1*D1_bar, m, l, cn[8]);
    get_N1_8(subN, L2_inv, -1*D1_bar, m, l, cn[9]);

//Calculate total stiffness matrix N as sum of N1_1 through N1_8
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            for(k=1; k<=8; k++)
                N[i][j]+=subK[k][i][j];

//Calculate stiffness matrix [K1,1]
    multiply_matrix(LN, N, L2_inv);
    multiply_matrix(Kn, L2_invt, LN);

//Set the nodal load vector Q
//Note that this section is adjusted for different loading conditions
//Below is the load vector corresponding to Example 1 in Chapter 6, Eq. 6.4

//Find {e(1)}
    e[1]=1;
    e[2]=1;
    e[3]=1*1;
```

```
    e[4]=l*l*l;
    for(i=5; i<=8; i++)
        e[i]=exp(m[i]*l);

//Set {Q}=-16*[L_invt]*{e(l)}*sin(pi/2)
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-16*L2_invt[i][j]*e[j]*sin(pi/2);
    }

//Modify [K] for the boundary conditions of cantilever fixed at one end
    for(i=1; i<=4; i++)
    {
        Q[i]=0;
        for(j=1; j<=8; j++)
            if(i!=j)
            {
                Kn[i][j]=0;
                Kn[j][i]=0;
            }
    }

//Invert the modified stiffness matrix to find {delta2}
    inverse(Kn, Kn_inv);
    multiply_vector(delta, Q, Kn_inv);

//Find the displacement contributions b, c and g for mode 1
    b=delta[5];
    c=delta[6];
    g=delta[7];

//Output the nodal displacement contributions at z=l for mode n=1 in mm
    cout<<"bl(l) = "<<b*1000<<" mm"<<endl;
    cout<<"cl(l) = "<<c*1000<<" mm"<<endl;
    cout<<"gl(l) = "<<g*1000<<" mm"<<endl;

//*********************************************************************************************
// *** End mode n=1 ***
//*********************************************************************************************

//*********************************************************************************************
// ***General case of n>=2 ***
//*********************************************************************************************

//Calculate each Fourier mode 'n' independently
    for(n=2; n<=alpha; n++)
    {
    //Calculate constants that depend on 'n' and coefficients of quartic equation
        constants2(cn, n, r);
        coefficients(cn, p);

    //Use Ferrari's method to determine roots M and then take square roots to get m
        quartic_roots(M, p);
        m[5]=sqrt(M[1]);
        m[6]=-1*sqrt(M[1]);
        m[3]=sqrt(M[2]);
        m[4]=-1*sqrt(M[2]);
        m[1]=sqrt(M[3]);
        m[2]=-1*sqrt(M[3]);
        m[7]=sqrt(M[4]);
        m[8]=-1*sqrt(M[4]);

    //Establish relationships between vectors of integration constants
        for(i=1; i<=8; i++)
        {
            D_bar[i]=((cn[11]*m[i])*(cn[3]*m[i])/(cn[1]*pow(m[i],2)-cn[10])-(cn[14]-
                cn[15]*pow(m[i],2)))/(cn[12]-cn[13]*pow(m[i],2)-
                pow(cn[11]*m[i],2)/(cn[1]*pow(m[i],2)-cn[10]));
```

```
            A_bar[i]=((-1)*cn[11]*m[i]/(cn[1]*pow(m[i],2)-cn[10]))*D_bar[i]-
                cn[3]*m[i]/(cn[1]*pow(m[i],2)-cn[10]);
            F_bar[i]=1;
        }

    //Construct the matrices [Ln,1] and [Ln,2]
        for(i=1; i<=8; i++)
        {
            L1[1][i]=A_bar[i];
            L1[2][i]=D_bar[i];
            L1[3][i]=1;
            L1[4][i]=m[i];
            L1[5][i]=A_bar[i]*exp(m[i]*l);
            L1[6][i]=D_bar[i]*exp(m[i]*l);
            L1[7][i]=exp(m[i]*l);
            L1[8][i]=m[i]*exp(m[i]*l);
            L2[1][i]=A_bar[i];
            L2[2][i]=D_bar[i];
            L2[3][i]=1;
            L2[4][i]=m[i];
            L2[5][i]=A_bar[i]*exp(m[i]*l);
            L2[6][i]=D_bar[i]*exp(m[i]*l);
            L2[7][i]=exp(m[i]*l);
            L2[8][i]=m[i]*exp(m[i]*l);
        }

    //Invert the martices [Ln,1] and [Ln,2]
            inverse(L1, L1_inv);
            inverse(L2, L2_inv);

    //Find the transpose of L1_inv and L2_inv
        for(i=1;i<=8;i++)
            for(j=1;j<=8;j++)
            {
                L1_invt[i][j]=L1_inv[j][i];
                L2_invt[i][j]=L2_inv[j][i];
            }

//*** Calculate first the displacements an, dn and fn ***

    //Calculate the stiffness matrix for mode 'n'
    //Inititalize matrix [N] to zero
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[i][j]=0;

    //Calculate sub matrices subN[1] through subN[8]
        get_N1(subN, L1_inv, A_bar, m, l, cn[1]);
        get_N2(subN, L1_inv, D_bar, F_bar, m, l, cn[2], n);
        get_N3(subN, L1_inv, A_bar, D_bar, F_bar, m, l, cn[3], n);
        get_N4(subN, L1_inv, A_bar, D_bar, m, l, cn[4], n, r);
        get_N5(subN, L1_inv, F_bar, m, l, cn[6]);
        get_N6(subN, L1_inv, D_bar, F_bar, m, l, cn[7], n);
        get_N7(subN, L1_inv, D_bar, F_bar, m, l, cn[8], n);
        get_N8(subN, L1_inv, D_bar, F_bar, m, l, cn[9], n);

   //Calculate total stiffness matrix N as sum of subN[1] through subN[8]
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            for(k=1; k<=8; k++)
                N[i][j]+=subK[k][i][j];

    //Calculate stiffness matrix [K1,1]
        multiply_matrix(LN, N, L1_inv);
        multiply_matrix(Kn, L1_invt, LN);

    //Set the nodal load vector Q
    //Note that this section is adjusted for different loading conditions
    //Below is the load vector corresponding to Example 1 in Chapter 6, Eq. 6.3

    //Find {e(l)}
```

```
    for(i=1; i<=8; i++)
        e[i]=exp(m[i]*l);

//Set {Q}=-16*[L1_invt]*{e(l)}*cos(n*pi/2)
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-16*L1_invt[i][j]*e[j]*cos(pi/2);
    }

//Modify [K] for the boundary conditions of cantilever fixed at one end
    for(i=1; i<=4; i++)
    {
        Q[i]=0;
        for(j=1; j<=8; j++)
            if(i!=j)
            {
                Kn[i][j]=0;
                Kn[j][i]=0;
            }
    }

//Invert the modified stiffness matrix to find {delta2}
    inverse(Kn, Kn_inv);
    multiply_vector(delta, Q, Kn_inv);

//Find the displacement contributions a, d and f for mode n
    a=delta[5];
    d=delta[6];
    f=delta[7];

//Output the nodal displacement contributions at z=l for mode n in mm
    cout<<"*** Nodal displacement components for mode "<<n<<" ***"<<endl;
    cout<<"a"<<n<<"(l) = "<<a*1000<<" mm"<<endl;
    cout<<"d"<<n<<"(l) = "<<d*1000<<" mm"<<endl;
    cout<<"f"<<n<<"(l) = "<<f*1000<<" mm"<<endl;

//*** Calculate now the displacements bn, cn and gn ***

//Calculate the stiffness matrix for mode 'n'
//Inititalize matrix [N] to zero
for(i=1; i<=8; i++)
    for(j=1; j<=8; j++)
        N[i][j]=0;

//Calculate sub matrices subN[1] through subN[8]
    get_N1(subN, L2_inv, A_bar, m, l, cn[1]);
    get_N2(subN, L2_inv, -1*D_bar, F_bar, m, l, cn[2], n);
    get_N3(subN, L2_inv, A_bar, -1*D_bar, F_bar, m, l, cn[3], n);
    get_N4(subN, L2_inv, A_bar, -1*D_bar, m, l, cn[4], n, r);
    get_N5(subN, L2_inv, F_bar, m, l, cn[6]);
    get_N6(subN, L2_inv, -1*D_bar, F_bar, m, l, cn[7], n);
    get_N7(subN, L2_inv, -1*D_bar, F_bar, m, l, cn[8], n);
    get_N8(subN, L2_inv, -1*D_bar, F_bar, m, l, cn[9], n);

//Calculate total stiffness matrix N as sum of subN[1] through subN[8]
for(i=1; i<=8; i++)
    for(j=1; j<=8; j++)
        for(k=1; k<=8; k++)
            N[i][j]+=subK[k][i][j];

//Calculate stiffness matrix [K2,1]
    multiply_matrix(LN, N, L2_inv);
    multiply_matrix(Kn, L2_invt, LN);

//Set the nodal load vector Q
//Note that this section is adjusted for different loading conditions
//Below is the load vector corresponding to Example 1 in Chapter 6, Eq. 6.4

//Find {e(l)}
```

```
        for(i=1; i<=8; i++)
            e[i]=exp(m[i]*l);

    //Set {Q}=-16*[L2_invt]*{e(l)}*sin(n*pi/2)
        for(i=1; i<=8; i++)
        {
            Q[i]=0;
            for(j=1;j<=8;j++)
                Q[i]+=-16*L2_invt[i][j]*e[j]*sin(pi/2);
        }

    //Modify [K] for the boundary conditions of cantilever fixed at one end
        for(i=1; i<=4; i++)
        {
            Q[i]=0;
            for(j=1; j<=8; j++)
                if(i!=j)
                {
                    Kn[i][j]=0;
                    Kn[j][i]=0;
                }
        }

    //Invert the modified stiffness matrix to find {delta2}
        inverse(Kn, Kn_inv);
        multiply_vector(delta, Q, Kn_inv);

    //Find the displacement contributions b, c and g for mode n
        b=delta[5];
        c=delta[6];
        g=delta[7];

    //Output the nodal displacement contributions at z=l for mode n in mm
        cout<<"b"<<n<<"(l) = "<<b*1000<<" mm"<<endl;
        cout<<"c"<<n<<"(l) = "<<c*1000<<" mm"<<endl;
        cout<<"g"<<n<<"(l) = "<<g*1000<<" mm"<<endl;


    }
//*******************************************************************************************
// *** End mode n ***
//*******************************************************************************************

}

//*******************************************************************************************
// *** END MAIN PROGRAM ***
//*******************************************************************************************

//Function to get the physical properties of the pipe
void input(double *r, double *t, double *l, double *E, double *G, double *mu, int *alpha)
{
   *r=0.2;
   *t=0.006;
   *E=200000000;
   *mu=0.3;
   *G=*E/(2*(1+*mu));
   *l=4;
   *alpha=6;
}

//Function to calculate the constants dependent on physical properties of the pipe
void constants1(double *c, double r, double t, double E, double G, double mu)
{
    double pi=M_PI;
    c[1]=E*r*t*pi/(1-pow(mu,2));
    c[2]=E*t*pi/(r*(1-pow(mu,2)));
    c[3]=E*t*mu*pi/(1-pow(mu,2));
    c[4]=G*r*t*pi;
    c[5]=G*t*pi;
```

```
    c[6]=E*r*pow(t,3)*pi/(12*(1-pow(mu,2)));
    c[7]=E*pow(t,3)*pi/(12*pow(r,3)*(1-pow(mu,2)));
    c[8]=E*pow(t,3)*mu*pi/(12*r*(1-pow(mu,2)));
    c[9]=G*pow(t,3)*pi/(3*r);
    c[13]=c[4]+c[9];
}

//Function to calculate constants dependent on Fourier mode n
//Function will be called for each mode n
void constants2(double *c, int n, double r)
{   c[10]=c[4]*pow((n/r),2);
    c[11]=(c[3]+c[5])*n;
    c[12]=(c[2]+c[7])*pow(n,2);
    c[14]=c[2]*n+c[7]*pow(n,3);
    c[15]=(c[8]+c[9])*n;
    c[16]=c[2]+c[7]*pow(n,4);
    c[17]=(2*c[8]+c[9])*pow(n,2);
}

//Function to calculate the coefficients of the quartic equation
void coefficients(double *c, double *p)
{
    double d1, d2, d3, d4, d5;
    p[1]=(-1)*c[1]*c[6]*c[13];
    p[2]=c[1]*(c[6]*c[12]+c[13]*c[17]-pow(c[15],2))+c[6]*(c[10]*c[13]-pow(c[11],2));
    p[3]=c[1]*((-1)*c[12]*c[17]-c[13]*c[16]+2*c[14]*c[15])-c[10]*(c[6]*c[12]+c[13]*c[17]-
        pow(c[15],2))+pow(c[11],2)*c[17]-2*c[3]*c[11]*c[15]+pow(c[3],2)*c[13];
    d1=c[1]*(c[12]*c[16]-pow(c[14],2));
    d2=c[10]*(c[12]*c[17]+c[13]*c[16]-2*c[14]*c[15]);
    d3=2*c[3]*c[11]*c[14];
    d4=pow(c[11],2)*c[16];
    d5=pow(c[3],2)*c[12];
    p[4]=d1+d2+d3-d4-d5;
    p[5]=(-1)*c[10]*(c[12]*c[16]-pow(c[14],2));
}

//Function to solve the quartic equation with Ferrari's Method for modes n>=2
void quartic_roots(complex *root, double *pp)
{
    int i;
    double alpha, beta, gamma;
    double p, q, radical;
    complex r, u, sqroot, y, w, bracket_pos, bracket_neg, polynomial[6], temp[3];

    alpha = -3*pow(pp[2],2)/(8*pow(pp[1],2))+pp[3]/pp[1];
    beta = pow(pp[2],3)/(8*pow(pp[1],3))-(pp[2]*pp[3])/(2*pow(pp[1],2))+pp[4]/pp[1];
    gamma =-3*pow(pp[2],4)/(256*pow(pp[1],4))+pp[3]*pow(pp[2],2)/(16*pow(pp[1],3))-
        pp[2]*pp[4]/(4*pow(pp[1],2))+pp[5]/pp[1];

    //Check if beta=0
    if (beta<.000001 && beta>-.000001)        //Equation is Biquadratic
    {
        p = pow(alpha,2)-4*gamma;
        if(p>0)
            sqroot = sqrt(p);
        else
            sqroot = complex_root(p);
        bracket_pos = -alpha+sqroot;
        bracket_neg = -alpha-sqroot;
        root[1] = -pp[2]/(4*pp[1])+sqrt(bracket_pos/2.0);
        root[2] = -pp[2]/(4*pp[1])+sqrt(bracket_neg/2.0);
        root[3] = -pp[2]/(4*pp[1])-sqrt(bracket_pos/2.0);
        root[4] = -pp[2]/(4*pp[1])-sqrt(bracket_neg/2.0);
        for(i=1; i<=4; i++)
            if(abs(root[i])<.00001)
                root[i]=0;
    }

    else    //Equation is not Biquadratic - Full Ferrari's method is req'd
    {
        p = -pow(alpha,2)/12-gamma;
```

```
        q = -pow(alpha,3)/108+alpha*gamma/3-pow(beta,2)/8;
        radical = pow(q,2)/4+pow(p,3)/27;

        if(radical>0)
            r = q/2.0 + sqrt(radical);
        else
        {
            sqroot = complex_root(radical);
            r = q/2.0 + sqroot;
        }

        u = pow(r, 1.0/3.0);

        //check if u=0
        if(abs(u)<.000001)
            y = -(5.0/6.0)*alpha;
        else
            y = -(5.0/6.0)*alpha - u + p/(3*u);

        w = sqrt(alpha + 2*y);
        bracket_pos = -(3*alpha+2*y+2*beta/w);
        bracket_neg = -(3*alpha+2*y-2*beta/w);

        root[1] = -pp[2]/(4*pp[1])+(w+sqrt(bracket_pos))/2;
        root[2] = -pp[2]/(4*pp[1])-(w+sqrt(bracket_neg))/2;
        root[3] = -pp[2]/(4*pp[1])+(w-sqrt(bracket_pos))/2;
        root[4] = -pp[2]/(4*pp[1])-(w-sqrt(bracket_neg))/2;
    }
}

//Function to compute the complex square root of a number of type "double"
complex complex_root(double x)
{
    complex root, xx=x;
    root = sqrt(xx);
    return root;
}

//Function to compute the inverse of a matrix [a]
void inverse(complex a[][9], complex a_inv[][9])
{
    int i,j,k;
    complex value;

    for(i=1;  i<=8; i++)
    {
        for(j=1; j<=8; j++)
        {
            if(i==j)
                a_inv[i][j]=1.0;
            else
                a_inv[i][j]=0.0;
        }
    }

    for(i=1; i<=8; i++)
    {
        for(j=1; j<=8; j++)
        {
            if(i!=j)
                a[i][j]/=a[i][i];
            a_inv[i][j]/=a[i][i];
        }
        a[i][i]=1.0;
        for(k=1; k<=8; k++)
        {
            if(i!=k)
            {
                value=a[k][i];
                for(j=1; j<=8; j++)
                {
```

```
                        a[k][j]-=a[i][j]*value;
                        a_inv[k][j]-=a_inv[i][j]*value;
                    }
                }
            }
        }
}

//Functions to compute required integrals
complex int0(complex a, double l)
{
    complex integral, exponent;

   exponent=exp(a*l);
   integral=(exponent-1)/a;
   return integral;
}

complex int1(complex a, double l)
{
    complex integral, exponent;

   exponent=exp(a*l);
   integral=((a*l-1)*exponent+1)/pow(a,2);
   return integral;
}

complex int2(complex a, double l)
{
    complex integral, exponent;

   exponent=exp(a*l);
   integral=pow(l,2)*exponent/a-(2/a)*int1(a,l);
   return integral;
}

complex int3(complex a, double l)
{
    complex integral, exponent;

   exponent=exp(a*l);
   integral=pow(l,3)*exponent/a-(3/a)*int1(a,l);
   return integral;
}

// *** Functions to get matrices subN for case of n=1 ***

//  C1{N_A'}<N_A'> from Eq. 5.30
void get_N1_1(complex N[][9][9], complex A_bar[], complex m[], double length, double
    constant)
{
    int i, j;

    N[1][3][3]=A_bar[3]*A_bar[3]*length;
    N[1][3][4]=N[1][4][3]=A_bar[3]*A_bar[4]*pow(length,2);
    for(i=5;i<=8;i++)
        N[1][3][i]=N[1][i][3]=m[i]*A_bar[3]*A_bar[i]*int0(m[i],length);
    N[1][4][4]=4*A_bar[4]*A_bar[4]*pow(length,3)/3;
    for(i=5;i<=8;i++)
        N[1][4][i]=N[1][i][4]=2*m[i]*A_bar[4]*A_bar[i]*int1(m[i],length);
    for(i=5;i<=8;i++)
        for(j=5;j<=8;j++)
            if(abs(m[j]+m[i])<.0001)
                N[1][i][j]=m[j]*m[i]*A_bar[j]*A_bar[i]*length;
            else
                N[1][i][j]=m[j]*m[i]*A_bar[j]*A_bar[i]*int0((m[i]+m[j]),length);

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[1][i][j]*=constant;
}
```

```
// Eq. 5.33a
void get_N1_2(complex N[][9][9], complex D_bar[], complex m[], double length, double
    constant)
{
    int i, j;
    complex integral;

    N[2][1][1]=(1+2*D_bar[1]+D_bar[1]*D_bar[1])*length;
    N[2][1][2]=N[2][2][1]=(1+D_bar[1]+D_bar[2]+D_bar[1]*D_bar[2])*pow(length,2)/2;
    N[2][1][3]=N[2][3][2]=((1+D_bar[1]+D_bar[3]+D_bar[1]*D_bar[3])*(pow(length,3)/3)+
        (D_bar[9]+D_bar[1]*D_bar[9])*length);
    N[2][1][4]=N[2][4][1]=((1+D_bar[1]+D_bar[4]+D_bar[1]*D_bar[4])*(pow(length,4)/4)+
        (D_bar[10]+D_bar[1]*D_bar[10])*pow(length,2)/2);
    for(i=5;i<=8;i++)
        N[2][1][i]=N[2][i][1]=(1+D_bar[1]+D_bar[i]+D_bar[1]*D_bar[i])*int0(m[i],length);

    N[2][2][2]=(1+2*D_bar[2]+D_bar[2]*D_bar[2])*pow(length,3)/3;
    N[2][2][3]=N[2][3][2]=((1+D_bar[2]+D_bar[3]+D_bar[2]*D_bar[3])*(pow(length,4)/4)+
        (D_bar[9]+D_bar[2]*D_bar[9])*pow(length,2)/2);
    N[2][2][4]=N[2][4][2]=((1+D_bar[2]+D_bar[4]+D_bar[2]*D_bar[4])*(pow(length,5)/5)+
        (D_bar[10]+D_bar[2]*D_bar[10])*pow(length,3)/3);
    for(i=5;i<=8;i++)
        N[2][2][i]=N[2][i][2]=(1+D_bar[2]+D_bar[i]+D_bar[2]*D_bar[i])*int1(m[i],length);

    N[2][3][3]=((1+D_bar[3]+D_bar[3]+D_bar[3]*D_bar[3])*(pow(length,5)/5)+
        (2*D_bar[9]+2*D_bar[3]*D_bar[9])*pow(length,3)/3+D_bar[9]*D_bar[9]*length);
    N[2][3][4]=N[2][4][3]=((1+D_bar[3]+D_bar[4]+D_bar[3]*D_bar[4])*(pow(length,6)/6)+
        (D_bar[9]+D_bar[10]+D_bar[3]*D_bar[10]+D_bar[4]*D_bar[9])*pow(length,4)/4+D_bar[9]*
        D_bar[10]*pow(length,2)/2);
    for(i=5;i<=8;i++)
        N[2][3][i]=N[2][i][3]=(D_bar[9]+D_bar[i]*D_bar[9])*int0(m[i],length)+(1+D_bar[3]+
            D_bar[i]+D_bar[3]*D_bar[i])*int2(m[i],length);

    N[2][4][4]=((1+D_bar[4]+D_bar[4]+D_bar[4]*D_bar[4])*(pow(length,7)/7)+(2*D_bar[10]+
        2*D_bar[4]*D_bar[10])*pow(length,5)/5+D_bar[10]*D_bar[10]*pow(length,3)/3);
    for(i=5;i<=8;i++)
        N[2][4][i]=N[2][i][4]=(D_bar[10]+D_bar[i]*D_bar[10])*int1(m[i],length)+(1+D_bar[4]+
            D_bar[i]+D_bar[4]*D_bar[i])*int3(m[i],length);

    for(i=5;i<=8;i++)
        for(j=5;j<=8;j++)
            if(abs(m[j]+m[i])<.0001)
                N[2][i][j]=(1+D_bar[j]+D_bar[i]+D_bar[j]*D_bar[i])*length;
            else
                N[2][i][j]=(1+D_bar[j]+D_bar[i]+D_bar[j]*D_bar[i])*int0((m[i]+m[j]),length);

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[2][i][j]*=constant;
}

// Eq. 5.33c
void get_N1_3(complex N[][9][9], complex A_bar[], complex D_bar[], complex m[], double
    length, double constant)
{
    int i, j;
    complex integral;

    N[3][1][3]=N[3][3][1]=(A_bar[3]+A_bar[3]*D_bar[1])*length;
    N[3][1][4]=N[3][4][1]=(A_bar[4]+A_bar[4]*D_bar[1])*pow(length,2);
    for(i=5;i<=8;i++)
        N[3][1][i]=N[3][i][1]=(A_bar[i]+A_bar[i]*D_bar[1])*int0(m[i],length);

    N[3][2][3]=N[3][3][2]=(A_bar[3]+A_bar[3]*D_bar[2])*pow(length,2)/2;
    N[3][2][4]=N[3][4][2]=2*(A_bar[4]+A_bar[4]*D_bar[2])*pow(length,3)/3;
    for(i=5;i<=8;i++)
        N[3][2][i]=N[3][i][2]=m[i]*(A_bar[i]+A_bar[i]*D_bar[2])*int1(m[i],length);

    N[3][3][3]=2*(A_bar[3]*D_bar[9]*length+(A_bar[3]+A_bar[3]*D_bar[3])*pow(length,3)/3);
```

```
    N[3][3][4]=N[3][4][3]=((A_bar[3]*D_bar[10]+2*A_bar[4]*D_bar[9])*(pow(length,2)/2)
        +(A_bar[3]+2*A_bar[4]+A_bar[3]*D_bar[4]+2*A_bar[4]*D_bar[3])*pow(length,4)/4);
    for(i=5;i<=8;i++)
        N[3][3][i]=N[3][i][3]=(A_bar[3]+A_bar[3]*D_bar[i]+m[i]*A_bar[i]*D_bar[9])*
            int0(m[i],length)+(m[i]*A_bar[i]*(1+D_bar[3]))*int2(m[i],length);
    N[3][4][4]=((1+D_bar[4])*pow(length,5)/5+D_bar[10]*pow(length,3)/3)*4*A_bar[4];
    for(i=5;i<=8;i++)
        N[3][4][i]=N[3][i][4]=(2*A_bar[4]*(1+D_bar[i])+m[i]*A_bar[i]*D_bar[10])*
            int1(m[i],length)+(m[i]*A_bar[i]*(1+D_bar[4]))*int3(m[i],length);

    for(i=5;i<=8;i++)
        for(j=5;j<=8;j++)
            if(abs(m[j]+m[i])<.0001)
                N[3][i][j]=(m[j]*A_bar[j]*(1+D_bar[i])+m[i]*A_bar[i]*(1+D_bar[j]))*length;
            else
                N[3][i][j]=(m[j]*A_bar[j]*(1+D_bar[i])+m[i]*A_bar[i]*(1+D_bar[j]))*
                    int0((m[i]+m[j]),length);

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[3][i][j]*=constant;
}

// Eq. 5.33e
void get_N1_4(complex N[][9][9], complex A_bar[], complex D_bar[], complex m[], double
    length, double constant, double r)
{

    int i, j;
    complex integral;

    N[4][2][2]=(D_bar[2]*D_bar[2]-(2/r)*(A_bar[2]*D_bar[2])+
        (1/pow(r,2))*A_bar[2]*A_bar[2])*length;
    N[4][2][3]=N[4][3][2]=(length*length/2)*(2*D_bar[2]*D_bar[3]-
        (1/r)*(A_bar[3]*D_bar[2]+2*A_bar[2]*D_bar[3])+(1/pow(r,2))*A_bar[2]*A_bar[3]);
    N[4][2][4]=N[4][4][2]=((D_bar[2]*D_bar[10]-(1/r)*(A_bar[2]*D_bar[10]+A_bar[9]*D_bar[2])+
        (1/pow(r,2))*A_bar[2]*A_bar[9])*length+(3*D_bar[2]*D_bar[4]-
        (1/r)*(3*A_bar[2]*D_bar[4]+A_bar[4]*D_bar[2])+(1/pow(r,2))*A_bar[2]*A_bar[4])*
        pow(length,3)/3);
    for(i=5;i<=8;i++)
        N[4][2][i]=N[4][i][2]=(m[i]*D_bar[2]*D_bar[i]-(1/r)*(m[i]*A_bar[2]*D_bar[i]+
            A_bar[i]*D_bar[2])+(1/pow(r,2))*A_bar[2]*A_bar[i])*int0(m[i],length);

    N[4][3][3]=(4*D_bar[3]*D_bar[3]-(4/r)*(A_bar[3]*D_bar[3])+
        (1/pow(r,2))*A_bar[3]*A_bar[3])*(pow(length,3)/3);
    N[4][3][4]=N[4][4][3]=((2*D_bar[3]*D_bar[10]-(1/r)*(A_bar[3]*D_bar[10]+
        2*A_bar[9]*D_bar[3])+(1/pow(r,2))*A_bar[3]*A_bar[9])*(pow(length,2)/2)+(6*D_bar[3]*
        D_bar[4]-(1/r)*(3*A_bar[3]*D_bar[4]+2*A_bar[4]*D_bar[3])+
        (1/pow(r,2))*A_bar[3]*A_bar[4])*pow(length,4)/4);
    for(i=5;i<=8;i++)
        N[4][3][i]=N[4][i][3]=(2*m[i]*D_bar[3]*D_bar[i]-(1/r)*(m[i]*A_bar[3]*D_bar[i]+
            2*A_bar[i]*D_bar[3])+(1/pow(r,2))*A_bar[3]*A_bar[i])*int1(m[i],length);

    N[4][4][4]=((D_bar[10]*D_bar[10]-(2/r)*(A_bar[9]*D_bar[10])+
        (1/pow(r,2))*A_bar[9]*A_bar[9])*length+(6*D_bar[4]*D_bar[10]-
        (2/r)*(3*A_bar[9]*D_bar[4]+A_bar[4]*D_bar[10])+
        (1/pow(r,2))*2*A_bar[4]*A_bar[9])*pow(length,3)/3+(9*D_bar[4]*D_bar[4]-
        (6/r)*(A_bar[4]*D_bar[4])+(1/pow(r,2))*A_bar[4]*A_bar[4])*pow(length,5)/5);
    for(i=5;i<=8;i++)
        N[4][4][i]=N[4][i][4]=(m[i]*D_bar[10]*D_bar[i]-(1/r)*(m[i]*A_bar[9]*D_bar[i]+
            A_bar[i]*D_bar[10])+(1/pow(r,2))*A_bar[9]*A_bar[i])*int0(m[i],length)+
            (3*m[i]*D_bar[4]*D_bar[i]-(1/r)*(m[i]*A_bar[4]*D_bar[i]+3*A_bar[i]*D_bar[4])+
            (1/pow(r,2))*A_bar[4]*A_bar[i])*int2(m[i],length);

    for(i=5;i<=8;i++)
        for(j=5;j<=8;j++)
            if(abs(m[j]+m[i])<.0001)
                N[4][i][j]=(m[j]*m[i]*D_bar[j]*D_bar[i]-(1/r)*(m[j]*A_bar[i]*D_bar[j]+
                    m[i]*A_bar[j]*D_bar[i])+(1/pow(r,2))*A_bar[j]*A_bar[i])*length;
            else
```

```
            N[4][i][j]=(m[j]*m[i]*D_bar[j]*D_bar[i]-(1/r)*(m[j]*A_bar[i]*D_bar[j]+
                m[i]*A_bar[j]*D_bar[i])+(1/pow(r,2))*A_bar[j]*A_bar[i])*
                int0((m[i]+m[j]),length);

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[4][i][j]*=constant;
}

// Eq. 5.33g
void get_N1_5(complex N[][9][9], complex m[], double length, double constant)
{
    int i, j;
    complex add_value;

    N[5][3][3]=4*length;
    N[5][3][4]=N[5][4][3]=6*pow(length,2);
    for(i=5;i<=8;i++)
        N[5][3][i]=N[5][i][3]=2*pow(m[i],2)*int0(m[i],length);
    N[5][4][4]=12*pow(length,3);
    for(i=5;i<=8;i++)
        N[5][4][i]=N[5][i][4]=6*pow(m[i],2)*int1(m[i],length);

    for(i=5;i<=8;i++)
        for(j=5;j<=8;j++)
            if(abs(m[j]+m[i])<.0001)
                N[5][i][j]=m[j]*m[j]*m[i]*m[i]*length;
            else
                N[5][i][j]=m[j]*m[j]*m[i]*m[i]*int0((m[i]+m[j]),length);

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[5][i][j]*=constant;
}

// Eq. 5.33i
void get_N1_6(complex N[][9][9], complex D_bar[], complex m[], double length, double
    constant)
{
    int i, j;
    complex integral;

    N[6][1][1]=(1+2*D_bar[1]+D_bar[1]*D_bar[1])*length;
    N[6][1][2]=N[6][2][1]=(1+D_bar[1]+D_bar[2]+D_bar[1]*D_bar[2])*pow(length,2)/2;
    N[6][1][3]=N[6][3][2]=((1+D_bar[1]+D_bar[3]+D_bar[1]*D_bar[3])*(pow(length,3)/3)+
        (D_bar[9]+D_bar[1]*D_bar[9])*length);
    N[6][1][4]=N[6][4][1]=((1+D_bar[1]+D_bar[4]+D_bar[1]*D_bar[4])*(pow(length,4)/4)+
        (D_bar[10]+D_bar[1]*D_bar[10])*pow(length,2)/2);
    for(i=5;i<=8;i++)
        N[6][1][i]=N[6][i][1]=(1+D_bar[1]+D_bar[i]+D_bar[1]*D_bar[i])*int0(m[i],length);

    N[6][2][2]=(1+2*D_bar[2]+D_bar[2]*D_bar[2])*pow(length,3)/3;
    N[6][2][3]=N[6][3][2]=((1+D_bar[2]+D_bar[3]+D_bar[2]*D_bar[3])*(pow(length,4)/4)+
        (D_bar[9]+D_bar[2]*D_bar[9])*pow(length,2)/2);
    N[6][2][4]=N[6][4][2]=((1+D_bar[2]+D_bar[4]+D_bar[2]*D_bar[4])*(pow(length,5)/5)+
        (D_bar[10]+D_bar[2]*D_bar[10])*pow(length,3)/3);
    for(i=5;i<=8;i++)
        N[6][2][i]=N[6][i][2]=(1+D_bar[2]+D_bar[i]+D_bar[2]*D_bar[i])*int1(m[i],length);

    N[6][3][3]=((1+D_bar[3]+D_bar[3]+D_bar[3]*D_bar[3])*(pow(length,5)/5)+(2*D_bar[9]+
        2*D_bar[3]*D_bar[9])*pow(length,3)/3+D_bar[9]*D_bar[9]*length);
    N[6][3][4]=N[6][4][3]=((1+D_bar[3]+D_bar[4]+D_bar[3]*D_bar[4])*(pow(length,6)/6)+
        (D_bar[9]+D_bar[10]+D_bar[3]*D_bar[10]+D_bar[4]*D_bar[9])*pow(length,4)/4+
        D_bar[9]*D_bar[10]*pow(length,2)/2);

    for(i=5;i<=8;i++)
        N[6][3][i]=N[6][i][3]=(D_bar[9]+D_bar[i]*D_bar[9])*int0(m[i],length)+(1+D_bar[3]+
            D_bar[i]+D_bar[3]*D_bar[i])*int2(m[i],length);

    N[6][4][4]=((1+D_bar[4]+D_bar[4]+D_bar[4]*D_bar[4])*(pow(length,7)/7)+(2*D_bar[10]+
        2*D_bar[4]*D_bar[10])*pow(length,5)/5+D_bar[10]*D_bar[10]*pow(length,3)/3);
```

```
    for(i=5;i<=8;i++)
        N[6][4][i]=N[6][i][4]=(D_bar[10]+D_bar[i]*D_bar[10])*int1(m[i],length)+(1+D_bar[4]+
            D_bar[i]+D_bar[4]*D_bar[i])*int3(m[i],length);

    for(i=5;i<=8;i++)
        for(j=5;j<=8;j++)
            if(abs(m[j]+m[i])<.0001)
                N[6][i][j]=(1+D_bar[j]+D_bar[i]+D_bar[j]*D_bar[i])*length;
            else
                N[6][i][j]=(1+D_bar[j]+D_bar[i]+D_bar[j]*D_bar[i])*int0((m[i]+m[j]),length);

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[6][i][j]*=constant;
}

// Eq. 5.33k
void get_N1_7(complex N[][9][9], complex D_bar[], complex m[], double length, double
    constant)
{
    int i, j;
    complex integral;

    N[7][1][3]=N[7][3][1]=2*(1+D_bar[1])*length;
    N[7][1][4]=N[7][4][1]=3*(1+D_bar[1])*pow(length,2);
    for(i=5;i<=8;i++)
        N[7][1][i]=N[7][i][1]=pow(m[i],2)*(1+D_bar[1])*int0(m[i],length);

    N[7][2][3]=N[7][3][2]=(1+D_bar[2])*pow(length,2);
    N[7][2][4]=N[7][4][2]=2*(1+D_bar[2])*pow(length,3);
    for(i=5;i<=8;i++)
        N[7][2][i]=N[7][i][2]=pow(m[i],2)*(1+D_bar[2])*int1(m[i],length);

    N[7][3][3]=4*(D_bar[9]*length+(1+D_bar[3])*pow(length,3)/3);
    N[7][3][4]=N[7][4][3]=((D_bar[10]+3*D_bar[9])*pow(length,2)+(8+2*D_bar[4]+6*D_bar[3])*
        pow(length,4)/4);
    for(i=5;i<=8;i++)
        N[7][3][i]=N[7][i][3]=(2+2*D_bar[i]+pow(m[i],2)*D_bar[9])*int0(m[i],length)+
            (pow(m[i],2)*(1+D_bar[3]))*int2(m[i],length);

    N[7][4][4]=2*(6*(1+D_bar[4])*pow(length,5)/5+2*D_bar[10]*pow(length,3));

    for(i=5;i<=8;i++)
        N[7][4][i]=N[7][i][4]=(6+6*D_bar[i]+pow(m[i],2)*D_bar[10])*int1(m[i],length)+
            (pow(m[i],2)*(1+D_bar[4]))*int3(m[i],length);

    for(i=5;i<=8;i++)
        for(j=5;j<=8;j++)
            if(abs(m[j]+m[i])<.0001)
                N[7][i][j]=(m[j]*m[j]*(1+D_bar[i])+m[i]*m[i]*(1+D_bar[j]))*length;
            else
                N[7][i][j]=(m[j]*m[j]*(1+D_bar[i])+m[i]*m[i]*(1+D_bar[j]))*
                    int0((m[i]+m[j]),length);

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[7][i][j]*=constant;
}

// Eq. 5.33m
void get_N1_8(complex N[][9][9], complex D_bar[], complex m[], double length, double
    constant)
{
    int i, j;
    complex integral;

    N[8][2][2]=(1+2*D_bar[2]+D_bar[2]*D_bar[2])*length;
    N[8][2][3]=N[8][3][2]=(1+D_bar[2]+D_bar[3]+D_bar[2]*D_bar[3])*pow(length,2);
    N[8][2][4]=N[8][4][2]=(D_bar[10]*(1+D_bar[2])*length+(1+D_bar[2]+D_bar[4]+D_bar[2]*
        D_bar[4])*pow(length,3));
```

```
    for(i=5;i<=8;i++)
        N[8][2][i]=N[8][i][2]=m[i]*(1+D_bar[2]+D_bar[i]+D_bar[2]*D_bar[i])*int0(m[i],length);

    N[8][3][3]=(4/3)*(1+2*D_bar[3]+D_bar[3]*D_bar[3])*pow(length,3);
    N[8][3][4]=N[8][4][3]=(D_bar[10]*(1+D_bar[3])*pow(length,2)+(3/2)*(1+D_bar[3]+D_bar[4]+
        D_bar[3]*D_bar[4])*pow(length,4));
    for(i=5;i<=8;i++)
        N[8][3][i]=N[8][i][3]=2*m[i]*(1+D_bar[3]+D_bar[i]+D_bar[3]*D_bar[i])*
            int1(m[i],length);

    N[8][4][4]=(D_bar[10]*D_bar[10]*length+2*D_bar[10]*(1+D_bar[4])*pow(length,3)+(9/5)*
        (1+2*D_bar[4]+D_bar[4]*D_bar[4])*pow(length,5));
    for(i=5;i<=8;i++)
        N[8][4][i]=N[8][i][4]=m[i]*D_bar[10]*(1+D_bar[i])*int0(m[i],length)+3*m[i]*
            (1+D_bar[4]+D_bar[i]+D_bar[4]*D_bar[i])*int2(m[i],length);

    for(i=5;i<=8;i++)
        for(j=5;j<=8;j++)
            if(abs(m[j]+m[i])<.0001)
                N[8][i][j]=m[j]*m[i]*(1+D_bar[j]+D_bar[i]+D_bar[j]*D_bar[i])*length;
            else
                N[8][i][j]=m[j]*m[i]*(1+D_bar[j]+D_bar[i]+D_bar[j]*D_bar[i])*
                    int0((m[i]+m[j]),length);

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N[8][i][j]*=constant;
}

// *** Functions to get matrices subN for general case of n>=2 ***

// Eq. 5.30
void get_N1(complex N1[][9][9], complex A_bar[], complex m[], double length, double
    constant)
{
    int i,j;

    get_sub_sub_N(N1[1], A_bar, A_bar, m, 1, 1, length);
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N1[1][i][j]*=constant;
}

// Eq. 5.33a
void get_N2(complex N2[][9][9], complex D_bar[], complex F_bar[], complex m[], double
    length, double constant, double n)
{
    int i,j;
    complex temp_N[5][9][9];

    get_sub_sub_N(temp_N[1], F_bar, F_bar, m, 0, 0, length);
    get_sub_sub_N(temp_N[2], D_bar, F_bar, m, 0, 0, length);
    get_sub_sub_N(temp_N[3], F_bar, D_bar, m, 0, 0, length);
    get_sub_sub_N(temp_N[4], D_bar, D_bar, m, 0, 0, length);
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N2[2][i][j]=constant*(temp_N[1][i][j]+n*(temp_N[2][i][j]+temp_N[3][i][j])+
                n*n*temp_N[4][i][j]);
}

// Eq. 5.33c
void get_N3(complex N3[][9][9], complex A_bar[], complex D_bar[], complex F_bar[], complex
    m[], double length, double constant, double n)
{
    int i,j;
    complex temp_N[5][9][9];

    get_sub_sub_N(temp_N[1], A_bar, F_bar, m, 1, 0, length);
    get_sub_sub_N(temp_N[2], F_bar, A_bar, m, 0, 1, length);
    get_sub_sub_N(temp_N[3], A_bar, D_bar, m, 1, 0, length);
    get_sub_sub_N(temp_N[4], D_bar, A_bar, m, 0, 1, length);
```

```
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N3[3][i][j]=constant*(temp_N[1][i][j]+temp_N[2][i][j]+
                n*(temp_N[3][i][j]+temp_N[4][i][j]));
}

// Eq. 5.33e
void get_N4(complex N4[][9][9], complex A_bar[], complex D_bar[], complex m[], double
    length, double constant, double n, double rad)
{
    int i,j;
    complex temp_N[5][9][9];

    get_sub_sub_N(temp_N[1], D_bar, D_bar, m, 1, 1, length);
    get_sub_sub_N(temp_N[2], A_bar, D_bar, m, 0, 1, length);
    get_sub_sub_N(temp_N[3], D_bar, A_bar, m, 1, 0, length);
    get_sub_sub_N(temp_N[4], A_bar, A_bar, m, 0, 0, length);
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N4[4][i][j]=constant*(temp_N[1][i][j]-(n/rad)*(temp_N[2][i][j]+
                temp_N[3][i][j])+(pow(n,2)/pow(rad,2))*temp_N[4][i][j]);
}

// Eq. 5.33g
void get_N5(complex N5[][9][9], complex F_bar[], complex m[], double length, double
    constant)
{
    int i,j;
    get_sub_sub_N(N5[5], F_bar, F_bar, m, 2, 2, length);
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N5[5][i][j]*=constant;
}

// Eq. 5.33i
void get_N6(complex N6[][9][9], complex D_bar[], complex F_bar[], complex m[], double
    length, double constant, double n)
{
    int i,j;
    complex temp_N[5][9][9];

    get_sub_sub_N(temp_N[1], F_bar, F_bar, m, 0, 0, length);
    get_sub_sub_N(temp_N[2], D_bar, F_bar, m, 0, 0, length);
    get_sub_sub_N(temp_N[3], F_bar, D_bar, m, 0, 0, length);
    get_sub_sub_N(temp_N[4], D_bar, D_bar, m, 0, 0, length);
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N6[6][i][j]=constant*(pow(n,4)*temp_N[1][i][j]+pow(n,3)*(temp_N[2][i][j]+
                temp_N[3][i][j])+pow(n,2)*temp_N[4][i][j]);
}

// Eq. 5.33k
void get_N7(complex N7[][9][9], complex D_bar[], complex F_bar[], complex m[], double
    length, double constant, double n)
{
    int i,j;
    complex temp_N[5][9][9];

    get_sub_sub_N(temp_N[1], F_bar, F_bar, m, 2, 0, length);
    get_sub_sub_N(temp_N[2], F_bar, F_bar, m, 0, 2, length);
    get_sub_sub_N(temp_N[3], D_bar, F_bar, m, 0, 2, length);
    get_sub_sub_N(temp_N[4], F_bar, D_bar, m, 2, 0, length);
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N7[7][i][j]=-1*constant*(pow(n,2)*(temp_N[1][i][j]+temp_N[2][i][j])+
                n*(temp_N[3][i][j]+temp_N[4][i][j]));
}

// Eq. 5.33m
void get_N8(complex N8[][9][9], complex D_bar[], complex F_bar[], complex m[], double
    length, double constant, double n)
```

```
{
    int i,j;
    complex temp_N[5][9][9];

    get_sub_sub_N(temp_N[1], F_bar, F_bar, m, 1, 1, length);
    get_sub_sub_N(temp_N[2], D_bar, F_bar, m, 1, 1, length);
    get_sub_sub_N(temp_N[3], F_bar, D_bar, m, 1, 1, length);
    get_sub_sub_N(temp_N[4], D_bar, D_bar, m, 1, 1, length);
    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
            N8[8][i][j]=constant*(pow(n,2)*temp_N[1][i][j]+n*(temp_N[2][i][j]+
                temp_N[3][i][j])+temp_N[4][i][j]);
}

//Compute the integral of the product of two 8x8 vectors (vectors N in calculating the
//  stiffness matrix).
//'Der' is necessary to do integrals. Value of 'der' is the order of the derivative of
//  exp(mz).
//Der=0 means no derivative; der=1 means first order; der=2 means second order.
void get_sub_sub_N(complex sub_N[][9], complex A_bar[], complex B_bar[], complex m[], int
    derA, int derB, double length)
{
    int i, j, k, h, derivative;
    complex m_prod;

    derivative=derA*10+derB;
    for(i=1; i<=8; i++)
        for(k=1; k<=8; k++)
        {
            sub_N[i][k]=0;
            for(j=1; j<=8; j++)
                for(h=1; h<=8; h++)
                {
                    switch (derivative)
                    {
                        case 0:
                            m_prod=1;
                            break;
                        case 1:
                            m_prod=m[h];
                            break;
                        case 2:
                            m_prod=pow(m[h],2);
                            break;
                        case 10:
                            m_prod=m[j];
                            break;
                        case 11:
                            m_prod=m[j]*m[h];
                            break;
                        case 12:
                            m_prod=m[j]*pow(m[h],2);
                            break;
                        case 20:
                            m_prod=pow(m[j],2);
                            break;
                        case 21:
                            m_prod=pow(m[j],2)*m[h];
                            break;
                        case 22:
                            m_prod=pow(m[j],2)*pow(m[h],2);
                            break;
                    }
                    if(abs(m[j]+m[h])<.000001)
                        sub_N[i][k]+=m_prod*A_bar[j]* B_bar[h]*length;
                    else
                        sub_N[i][k]+=(m_prod*A_bar[j]*B_bar[h]/(m[j]+m[h]))*
                            (exp((m[j]+m[h])*length)-1);
                }
        }
}
```

```
// Calculates the product of 2 8x8 matrices -> [product]=[x][y]
void multiply_matrix(complex product[] [9], complex A[][9], complex B[][9])
{
    int i, j, k;

    for(i=1; i<=8; i++)
        for(j=1; j<=8; j++)
        {
            product[i][j]=0;
            for(k=1; k<=8; k++)
                product[i][j]+=A[i][k]*B[k][j];
        }
}

// Calculates the product of an 8x8 matrix and an 8x1 vector -> {result}=[matrix]{vector}
void multiply_vector(complex result[], complex vector[], complex matrix[][9])
{
    int i, j;

    for(i=1; i<=8; i++)
    {
        result[i]=0;
        for(j=1; j<=8; j++)
            result[i]+=matrix[i][j]*vector[j];
    }
}

//*********************************************************************************************
//                                    *** END PROGRAM ***
//*********************************************************************************************
```

# A11.2 Output

Below is the output given by the program listed in Secion A11.1:

```
*** Nodal displacement contributions for n=0 ***
a0(1) = 0.000637487 mm
c0(1) = 0 mm
f0(1) = -0.0314979 mm

*** Nodal displacement components for n=1 ***
a1(1) = 0 mm
d1(1) = 0 mm
f1(1) = 0 mm
b1(1) = 0.848928 mm
c1(1) = -11.51643 mm
g1(1) = -11.579102 mm

*** Nodal displacement components for mode 2 ***
a2(1) = -0.152271 mm
d2(1) = -1.3352 mm
f2(1) = 2.74664 mm
b2(1) = 0 mm
c2(1) = 0 mm
g2(1) = 0 mm

*** Nodal displacement components for mode 3 ***
a3(1) = 0 mm
d3(1) = 0 mm
f3(1) = 0 mm
b3(1) = -0.0536703 mm
c3(1) = 0.302682 mm
g3(1) = 0.972001 mm

*** Nodal displacement components for mode 4 ***
a4(1) = 0.0254056 mm
d4(1) = 0.113397 mm
f4(1) = -0.503479 mm
b4(1) = 0 mm
c4(1) = 0 mm
g4(1) = 0 mm

*** Nodal displacement components for mode 5 ***
a5(1) = 0 mm
d5(1) = 0 mm
f5(1) = 0 mm
b5(1) = 0.0132984 mm
c5(1) = -0.0522025 mm
g5(1) = -0.297169 mm

*** Nodal displacement components for mode 6 ***
a6(1) = -0.0073683 mm
d6(1) = -0.0269971 mm
f6(1) = 0.187143 mm
b6(1) = 0 mm
c6(1) = 0 mm
g6(1) = 0 mm
```

## A11.3 Example 1: Inclined Point Load Applied to Tip of Cantilever

Example 1 in Chapter 6 also included the results of an inclined point load applied to the tip of the cantilever. The pipe properties are the same as those listed in Section A11.1. The revised code for the load potential term for each mode $n$=0 to $n$=6 are given below.

Mode $n$=0:

```
//Set the nodal load vector Q
//Below is the load vector for the inclined load in Example 1 in Chapter 6, Eq. 6.6

//Find {e(l)}
    e[1]=0;
    e[2]=F_bar[2];
    for(i=3; i<=6; i++)
        e[i]=F_bar[i]*exp(m[i]*l);
    e[7]=e[8]=0;

//Set {Q}=-P*[L_invt]*{e(l)}
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-16*L_invt[i][j]*e[j];
    }
```

Mode $n$=1:

```
//Set the nodal load vector Q
//Below is the load vector for the inclined load Example 1 in Chapter 6, Eq. 6.7a
//      corresponding to displacements a, d and f for mode n=1
//Find {e(l)}
    e[1]=1;
    e[2]=l;
    e[3]=l*l;
    e[4]=l*l*l;
    for(i=5; i<=8; i++)
        e[i]=exp(m[i]*l);

//Set {Q}=-16*[L_invt]*{e(l)}*cos(7*pi/8)
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-16*L1_invt[i][j]*e[j]*cos(7*pi/8);

//Below is the load vector for the inclined load Example 1 in Chapter 6, Eq. 6.7b
//      corresponding to displacements b, c and g for mode n=1
//Find {e(l)}
    e[1]=1;
    e[2]=l;
    e[3]=l*l;
    e[4]=l*l*l;
    for(i=5; i<=8; i++)
        e[i]=exp(m[i]*l);

//Set {Q}=-16*[L_invt]*{e(l)}*sin(7*pi/8)
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-16*L2_invt[i][j]*e[j]*sin(7*pi/8);
```

Modes $n \geq 2$:

```
//Set the nodal load vector Q
//Below is the load vector for the inclined load Example 1 in Chapter 6, Eq. 6.7a
//      corresponding to displacements a, d and f for modes n>=2
//Find {e(l)}
    e[1]=1;
    e[2]=l;
    e[3]=l*l;
    e[4]=l*l*l;
    for(i=5; i<=8; i++)
        e[i]=exp(m[i]*l);

//Set {Q}=-16*[L_invt]*{e(l)}*cos(7*n*pi/8)
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-16*L1_invt[i][j]*e[j]*cos(7*n*pi/8);

//Below is the load vector for the inclined load Example 1 in Chapter 6, Eq. 6.7b
//      corresponding to displacements b, c and g for modes n>=2
//Find {e(l)}
    e[1]=1;
    e[2]=l;
    e[3]=l*l;
    e[4]=l*l*l;
    for(i=5; i<=8; i++)
        e[i]=exp(m[i]*l);

//Set {Q}=-16*[L_invt]*{e(l)}*sin(7*n*pi/8)
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-16*L2_invt[i][j]*e[j]*sin(7*n*pi/8);
```

The output from the program for the inclined load portion of Example 1 is:

```
*** Nodal displacement contributions for n=0 ***
a0(l) = 0.000637487 mm
c0(l) = 0 mm
f0(l) = -0.0314979 mm

*** Nodal displacement components for n=1 ***
a1(l) = -0.784307 mm
d1(l) = -10.6876 mm
f1(l) = 10.7712 mm
b1(l) = 0.324871 mm
c1(l) = -4.42696 mm
g1(l) = -4.46158 mm

*** Nodal displacement components for mode 2 ***
a2(l) = 0.107672 mm
d2(l) = 0.944131 mm
f2(l) = -1.944131 mm
b2(l) = -0.107672 mm
c2(l) = 0.0944131 mm
g2(l) = 1.94217 mm

*** Nodal displacement components for mode 3 ***
a3(l) = -0.0205409 mm
d3(l) = -0.115846 mm
f3(l) = 0.372017 mm
b3(l) = 0.0495902 mm
c3(l) = -0.279676 mm
g3(l) = -0.898128 mm
```

```
*** Nodal displacement components for mode 4 ***
a4(1) = 0 mm
d4(1) = 0 mm
f4(1) = 0 mm
b4(1) = -0.0254056 mm
c4(1) = 0.113397 mm
g4(1) = 0.503479 mm

*** Nodal displacement components for mode 5 ***
a5(1) = 0.00509033 mm
d5(1) = 0.0199827 mm
f5(1) = -0.113756 mm
b5(1) = 0.0122891 mm
c5(1) = -0.0482424 mm
g5(1) = -0.27463 mm

*** Nodal displacement components for mode 6 ***
a6(1) = -0.00521016 mm
d6(1) = -0.0190898 mm
f6(1) = 0.13233 mm
b6(1) = -0.00521016 mm
c6(1) = 0.0190898 mm
g6(1) = 0.13233 mm
```

## A11.4 Example 2: Cantilever Beam Under Self Weight

Example 2 in Chapter 6 consists of a cantilever beam under self weight. The pipe properties are the same as those used in Example 1 in Chapter 6 and listed in Section A11.1 above. The only non-zero load potential vector is $\{Q_{1,2}\}_{8\times1}$ corresponding to displacement contributions $b_1$, $c_1$ and $g_1$ for mode $n$=1. The relevant C++ code for the load potential vector is:

```
//Set the nodal load vector Q
//Below is the load vector for the self weight, Example 2 in Chapter 6, Eq. 6.20b
//      corresponding to displacements b, c and g for mode n=1
//Find {e(l)}
    e[1]=(1-D1_bar[1])*l;
    e[2]=(1-D1_bar[2])*l*l/2;
    e[3]=(1-D1_bar[3])*l*l*l/3-D1_bar[9]*l;
    e[4]=(1-D1_bar[4])*l*l*l*l/4-D1_bar[10]*l*l/2;
    for(i=5; i<=8; i++)
        e[i]=(1-D1_bar[i])*(exp(m[i]*l)-1);

//Set {Q}=-gamma*t*r*pi*[L_invt]*{e(l)}
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-78*t*r*pi*L_invt[i][j]*e[j];
    }
```

The output from the program for the self weight of Example 2 is:

```
*** Nodal displacement components for n=1 ***
a1(1) = 0 mm
d1(1) = 0 mm
f1(1) = 0 mm
b1(1) = 0.0416114 mm
c1(1) = -0.639488 mm
g1(1) = -0.639504 mm
```

## A11.5 Example 3: Uniform Torque Applied to Tip of Cantilever

Example 3 in Chapter 6 consists of a cantilever beam subjected to a uniform torque at its tip. The pipe properties are the same different from those listed in Section A11.1 above. The revised code for the pipe properties is:

```
//Function to get the physical properties of the pipe
void input(double *r, double *t, double *l, double *E, double *G, double *mu, int *alpha)
{
    *r=0.15;
    *t=0.005;
    *E=200000000;
    *mu=0.3;
    *G=*E/(2*(1+*mu));
    *l=0.6;
    *alpha=0;
}
```

The only non-zero load potential vector is $\{Q_0\}_{8\times1}$ corresponding to displacement contributions $a_0$, $c_0$ and $f_0$ for mode $n$=0. The relevant C++ code for the load potential vector is:

```
//Set the nodal load vector Q
//Below is the load vector corresponding to Example 3 in Chapter 6, Eq. 6.26
//Find {e(1)}
    for(i=1; i<=6; i++)
        e[i]=0;
    e[7]=e[8]=1;

//Set {Q}=(10/r)*[L_invt]*{e(1)}
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=(10/r)*L_invt[i][j]*e[j];
    }
```

The output from the program for the torque of Example 3 is:

```
*** Nodal displacement components for n=0 ***
a0(1) = 0 mm
c0(1) = 0.110307 mm
f0(1) = 0 mm
```

## A11.6 Example 4: Multi-Element Pipe

Example 4 in Chapter 6 consists of a multi-element pipe subjected to its self weight, the weight of fluid in the pipe and an internal pressure. The properties of the pipe, namely length and wall thickness, are different for the two elements. The values used are given in Chapter 6.

The only non-zero load potential vectors are $\{Q_0\}_{8\times1}$ and $\{Q_{1,2}\}_{8\times1}$ corresponding to displacement contributions $a_0$, $c_0$ and $f_0$ for mode $n$=0 and $b_1$, $c_1$ and $g_1$ for mode $n$=1, respectively. The relevant C++ code for the load potential vectors is:.

Mode $n$=0:

```
//Set the nodal load vector Q
//Below is the load vector for Example 4 in Chapter 6, Eq. 6.35

//Find {e(l)}
    e[1]=0;
    e[2]=F_bar[2]*l;
    for(i=3; i<=6; i++)
        e[i]=(F_bar[i]/m[i])*(exp(m[i]*l)-1);
    e[7]=e[8]=0;

//Set {Q}}
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=-2*pi*r*(gw*r+p)*L_invt[i][j]*e[j];
    }
```

Mode $n$=1:

```
//Set the nodal load vector Q
//Below is the load vector for the inclined load Example 1 in Chapter 6, Eq. 6.7a
//      corresponding to displacements a, d and f for mode n=1
//Find {db(l)} and {e(l)}
    db[1]=D1_bar[1]*l;
    db[2]=D1_bar[2]*pow(l,2)/2;
    db[3]=D1_bar[9]*l+D1_bar[3]*pow(l,3)/3;
    db[4]=D1_bar[10]*pow(l,2)/2+D1_bar[4]*pow(l,4)/4;
    for(i=5; i<=8; i++)
        db[i]=D1_bar[i]*(exp(m[i]*l)-1)/m[i];
    e[1]=l;
    e[2]=pow(l,2)/2;
    e[3]=pow(l,3)/3;
    e[4]=pow(l,4)/4;
    for(i=5; i<=8; i++)
        e[i]=(exp(m[i]*l)-1)/m[i];

//Set {Q}
    for(i=1; i<=8; i++)
    {
        Q[i]=0;
        for(j=1;j<=8;j++)
            Q[i]+=pi*r*t*gs*L1_invt[i][j]*db[j]*pi*r*(gw*r+t*gs)*L1_invt[i][j]*e[j];
```

Note that the code given above is for the load potential vector for one element only. This code is repeated twice to create a load potential vector for each of the elements. The two vectors are then combined to create the load potential vector for the system.

The output from the program for the multi-element pipe of Example 4 is:

```
*** Nodal displacement components for n=0 ***
a0(1) = 0.000672406 mm
c0(1) = 0 mm
f0(1) = 0.0333118 mm

*** Nodal displacement components for n=1 ***
a1(1) = 0 mm
d1(1) = 0 mm
f1(1) = 0 mm
b1(1) = 0.0504136 mm
c1(1) = -0.436128 mm
g1(1) = -0.435521 mm
```