



uOttawa

L'Université canadienne
Canada's university

**FACULTÉ DES ÉTUDES SUPÉRIEURES
ET POSTDOCTORALES**



uOttawa

L'Université canadienne
Canada's university

**FACULTY OF GRADUATE AND
POSTDOCTORAL STUDIES**

Brennan Cornell

AUTEUR DE LA THÈSE / AUTHOR OF THESIS

M.Sc. (Mathematics)

GRADE / DEGREE

Department of Mathematics and Statistics

FACULTÉ, ÉCOLE, DÉPARTEMENT / FACULTY, SCHOOL, DEPARTMENT

An Introduction to Classical Gauge Theory in Mathematics and Physics

TITRE DE LA THÈSE / TITLE OF THESIS

Dr. B. Jessup

DIRECTEUR (DIRECTRICE) DE LA THÈSE / THESIS SUPERVISOR

CO-DIRECTEUR (CO-DIRECTRICE) DE LA THÈSE / THESIS CO-SUPERVISOR

EXAMINATEURS (EXAMINATRICES) DE LA THÈSE / THESIS EXAMINERS

Dr. W. Rossman

Dr. C.K. Fong

Gary W. Slater

Le Doyen de la Faculté des études supérieures et postdoctorales / Dean of the Faculty of Graduate and Postdoctoral Studies

AN INTRODUCTION TO CLASSICAL GAUGE THEORY IN MATHEMATICS AND PHYSICS

By
Brennan Cornell, B.Sc. (RMC)
April 2008

A Thesis
submitted to the School of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Master of Science in Mathematics¹

© Copyright 2008
by Brennan Cornell, B.Sc. (RMC), Ottawa, Canada

¹The M.Sc. Program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics



Library and
Archives Canada

Bibliothèque et
Archives Canada

Published Heritage
Branch

Direction du
Patrimoine de l'édition

395 Wellington Street
Ottawa ON K1A 0N4
Canada

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file Votre référence
ISBN: 978-0-494-41658-7
Our file Notre référence
ISBN: 978-0-494-41658-7

NOTICE:

The author has granted a non-exclusive license allowing Library and Archives Canada to reproduce, publish, archive, preserve, conserve, communicate to the public by telecommunication or on the Internet, loan, distribute and sell theses worldwide, for commercial or non-commercial purposes, in microform, paper, electronic and/or any other formats.

The author retains copyright ownership and moral rights in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

AVIS:

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque et Archives Canada de reproduire, publier, archiver, sauvegarder, conserver, transmettre au public par télécommunication ou par l'Internet, prêter, distribuer et vendre des thèses partout dans le monde, à des fins commerciales ou autres, sur support microforme, papier, électronique et/ou autres formats.

L'auteur conserve la propriété du droit d'auteur et des droits moraux qui protègent cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

In compliance with the Canadian Privacy Act some supporting forms may have been removed from this thesis.

Conformément à la loi canadienne sur la protection de la vie privée, quelques formulaires secondaires ont été enlevés de cette thèse.

While these forms may be included in the document page count, their removal does not represent any loss of content from the thesis.

Bien que ces formulaires aient inclus dans la pagination, il n'y aura aucun contenu manquant.

■ ■ ■
Canada

Abstract

We describe some aspects of classical gauge theory from the perspective of connections on vector bundles. We begin by examining classical electromagnetism, and use it to motivate the development of gauge theory on vector bundles. If G is a Lie group, we review some of the theory of vector G -bundles, their associated principal G -bundles, and the related theory of connections. We then discuss the idea of gauge transformations on principal and vector G -bundles, and view electromagnetism as an example of an abelian gauge theory. We briefly review the action principle in order to describe non-abelian gauge theories such as the Yang-Mills equation. Finally, we present the main results from an article by John Baez entitled “Higher Yang-Mills Theory” [6] where he attempts to abstract Yang-Mills theory using some concepts from category theory.

Acknowledgements

I would like to thank (in no particular order):

1. My supervisor, Dr. Barry Jessup, for his patience and understanding over the past several years, especially concerning my “special circumstances” as a member of the Canadian Forces.
2. The Canadian Forces. In particular, the Directorate of Land Command Systems Program Management (DLCSPM) for sponsoring my post-graduate interests.
3. My very understanding wife, Stephanie, who has patiently put up with my antics over the past eighteen months. It’s not easy to live with an aspiring mathematician.

Dedication

To my son. Your first bedtime story.

Contents

Abstract	ii
Acknowledgements	iii
Dedication	iv
1 Introduction	1
1.1 Background	1
1.2 Scope	5
2 Classical Electromagnetism	6
2.1 Classical Formulation of Maxwell's Equations	6
2.2 Maxwell's Equations via Differential Forms	9
3 Fibre Bundles	15
3.1 Fibre Bundles	15
3.2 Vector Bundles and G -Bundles	21
3.3 Principal and Associated Bundles	24
4 Covariant Derivatives: Connections on Bundles	32
4.1 Connections on Vector Bundles	32
4.2 Connections on Principal Bundles	36
4.3 Connections on Associated Bundles	42
4.4 Connections and Covariant Derivatives	44
4.5 Holonomy	45

4.6	Curvature	46
4.7	The Exterior Covariant Derivative	50
5	Gauge Transformations	55
5.1	Gauge Transformations on Principal Bundles	55
5.2	Gauge Transformations on Vector G -Bundles	58
6	Gauge Theory	62
6.1	Bundle Formulation of Maxwell's Equations	62
6.2	Bohm-Aharonov Effect	65
6.3	The Yang-Mills Equation	72
7	The Action Principle	74
7.1	The Action Principle	74
7.2	The Yang-Mills Equation	76
8	Connections on Higher Bundles	81
8.1	Motivation	81
8.2	Lie 2-Groups and Crossed Modules	86
8.3	Lie 2-Algebras and Differential Crossed Modules	92
8.4	Principal 2-Bundles	99
A	Notation	113
	Bibliography	117

Chapter 1

Introduction

1.1 Background

In this chapter we recount some of the history of gauge theory. Our principal sources are [14], [21] and [23].

In the last century, gauge theory arose as one of the principal tools for modeling the physics of the universe. Success at describing the fundamental forces of nature was achieved through the recognition that the physical laws contained subtle symmetries that could be encoded and exploited using this theory.

In 1918, Hermann Weyl wrote a paper [28] examining the underlying assumptions of Einstein's general relativity. Weyl's motivation was to describe electromagnetism as a consequence of geometry in the same way that Einstein's equations on space-time gave rise to gravity. According to [14], he "imagined a theory in which the scale of length... would vary from point to point in space and in time." This was his notion of "changing the gauge" (or scale) in a non-global manner. While Weyl did succeed in obtaining interesting results, his preliminary formulation was fatally flawed, a problem that was detected by Einstein in early correspondence in 1918. ([21],p.12)

While reactions to Weyl's 1918 paper were mixed ([21],p.12), it received sufficient attention that Weyl's principle of "gauge invariance" was introduced into quantum

mechanics. In a paper in 1927, Fritz London recommended reinterpreting Weyl's results. The metric was replaced by the wave function, and the re-scaling was replaced by a phase change. In 1929, Weyl responded by publishing a seminal paper entitled "Gravitation and the Electron" ([29]) wherein he clearly elucidated his "gauge principle":

The Dirac field-equations for ψ together with the Maxwell equations for the four potentials f_p of the electromagnetic field have an invariance property which is formally similar to the one which I called "gauge invariance" in my 1918 theory of gravitation and electromagnetism; the equations remain invariant when one makes the simultaneous substitutions

$$\psi \text{ by } \exp i\lambda\psi \text{ and } f_p \text{ by } f_p - \frac{\partial\lambda}{\partial x_p},$$

where λ is understood to be an arbitrary function of position in four-space... The connection of this "gauge invariance" to the conservation of electric charge remains untouched... It seems to me that this new principle of "gauge invariance", which follows not from speculation but from experiment, tells us that the electromagnetic field is a necessary accompanying phenomenon, not of gravitation, but of the material wave-field represented by ψ . Since "gauge invariance" involves an arbitrary function λ it has the character of "general" relativity and can naturally only be understood in that context. - [21], p.17-18.

Physicists were surprisingly unaware that they already possessed a physical theory that included gauge invariance. Maxwell's theory of electromagnetism had been around since 1864, however, the underlying symmetries such as Lorentz invariance and $U(1)$ symmetry remained hidden for subsequent decades. It was not until the Einsteinian revolution of special and general relativity that Maxwell's equations were finally viewed in a "geometric" fashion that enabled these properties to be observed ([14],p.956). Nevertheless, it was still not understood whether these properties were a consequence of the manner in which the theory was framed, or reflective of an actual

physical reality. Would re-framing the equations result in a loss of these symmetries? As it turned out, no.

There is a surprising relationship between symmetry in nature and the conservation of particular quantities. This startling result was discovered by Emmy Noëther, a German mathematician at the University of Göttingen in 1918. In general terms, her theorem states that *“to every symmetry..., there corresponds a conserved current.”* Through the application of variational methods, Noëther determined that if a given action is invariant under a change variables, then the resulting equations of motion stemming from that action lead to a conserved quantity ([14],p.957). For example, in classical mechanics, the Lagrangian of a system yields the equations of motion. For an appropriately chosen Lagrangian, the conservation of energy, and linear or angular momentum can deduced from its symmetry in time, translation or rotation.

While Weyl’s 1918 paper failed to describe a geometric convergence of gravity and electromagnetism, subsequent investigations by Theodor Kaluza in 1919 yielded greater success. Kaluza’s insight was to extend space-time from four to five dimensions. By judiciously imposing certain conditions on the space-time manifold, Kaluza was able to split his solutions into gravitic and electromagnetic components. ([21],p.27)

Kaluza’s work was extended by Oscar Klein in 1938 ([14],p.964). Unlike Weyl and Kaluza, Klein was not interested in the intersection of gravity and electromagnetism, but wanted to understand the nuclear forces. He borrowed the concept of gauge invariance from Weyl, and extra dimensions from Kaluza in order to attack the problem. Following a similar developmental process as Kaluza, Klein was able to describe the behaviour of the nucleons (protons and neutrons in the atomic nucleus) by considering them as states of the same field. Unfortunately, to get rid of certain unwanted particles, he had to add a mass term by hand, destroying the gauge symmetry ([14],p.965).

In 1954, Chen Ning Yang and Robert Mills took up Weyl’s mantle of gauge invariance in a paper entitled “Conservation of Isotopic Spin and Isotopic Gauge Invariance.” [30] Yang and Mills were motivated to discover a law of conservation for isotopic spin similar to the conservation of electric charge ([21],p.38). Isotopic spin (or

isospin) was originally introduced by Werner Heisenberg in 1932 to explain symmetries between the proton and the neutron ([5],p.216-218). In particular, the similarity in their masses, strength of reaction to the strong nuclear force, and the mass of the particles mediating their interaction (the pions). Heisenberg recognized that the mathematical formulation for this symmetry was similar to that of normal "spin," and hence the name. Whereas Weyl was able to use gauge invariance to formulate conservation of electric charge, Yang and Mills hoped to achieve something similar with isotopic spin in nucleon-nucleon reactions.

In the development of their ideas, Yang and Mills borrowed heavily from the theory of electromagnetism. Indeed, they systematically replaced elements of electromagnetism with a suitable analogue from the theory of isotopic spin. As a consequence, Yang and Mills obtained equations for the transformation of their fields which are structurally similar to those of electromagnetism. They had succeeded in creating a non-abelian gauge theory.

It was not immediately apparent that Yang and Mills' theory (now known as Yang-Mills theory) was of any real use. The preceding decade had benefited from an explosion of experimental results in particle physics, and Yang-Mills theory appeared to possess practical difficulties that did not correspond with observation. In particular, the requirements of gauge symmetry meant that the particles had to have zero mass ([23],p.198). Fortunately, their results also possessed several significant benefits which eventually allowed theorists to sidestep these difficulties. In 1971, Gerard 't Hooft showed that all Yang-Mills theories were renormalizable ([23],p.205), and that particles acquired mass through the use of the "Higgs mechanism" ([23],p.202). The success of Yang-Mills gauge theory had dawned.

Currently, Yang-Mills theory underlies the Standard Model of physics which describes the electromagnetic, weak, and strong nuclear forces. Although the model has been incredibly successful in describing our universe, it is considered unsatisfactory because it lacks gravity (among other concerns). Recently, interest has been focused on "string theory," in which the concept of the "point particle" is replaced with a vibrating "string." String theory was initially popularized because it is believed to contain a natural way of deriving gravity. Like the Standard Model, string theoretic

formulations are also gauge theories. Unfortunately, the underlying mathematics, and indeed, most of the framework of the theory, remain undeveloped and quantitative predictions are elusive. As a consequence, while it provides a conceptually interesting explanation of nature, it has been incapable of improving upon the Standard Model.

1.2 Scope

This thesis will explore the mathematical foundations underlying our current construction of gauge theory. Using classical electromagnetism as an example of an abelian gauge theory in Chapter 2, we proceed to describe vector bundles in Chapter 3 and the theory of connections in Chapter 4. We define and review the consequence of gauge transformations on connections in Chapter 5, and apply them to electromagnetism, and the development of the Yang-Mills equation in Chapter 6. Using the action principle in Chapter 7, we also show how the Yang-Mills equation can be re-derived for a particular choice of Lagrangian. Finally, we examine John Baez's paper "Higher Yang-Mills Theory" [6] which attempts to abstract Yang-Mills theory using ideas from category theory. We include several proofs not found in his paper.

Chapter 2

Classical Electromagnetism

2.1 Classical Formulation of Maxwell's Equations

Electromagnetism, in its best known form, consists of a collection of four differential equations known as “Maxwell’s equations” which describe the relationship between the electric and magnetic fields.

Let \vec{E} and \vec{B} be the electric and magnetic vector fields respectively defined on a space-time manifold M . For the moment, we will assume that $M = \mathbb{R} \times \mathbb{R}^3$, with one dimension of time and three dimensions of space. Let $\rho : M \rightarrow \mathbb{R}$ and $\vec{j} : M \rightarrow \mathbb{R}^3$ be the electric charge density and the current density respectively. In units where the speed of light is 1, Maxwell’s equations in classical notation are ([5],p.8):

$$\text{ME1. } \vec{\nabla} \cdot \vec{B} = 0$$

$$\text{ME2. } \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\text{ME3. } \vec{\nabla} \cdot \vec{E} = \rho$$

$$\text{ME4. } \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

We remark that in order for the curl and divergence to make sense, we think of \vec{E} and \vec{B} as maps from M into the spatial component of M . In our case, this is just \mathbb{R}^3 , and the curl and divergence operators act normally. Similarly, given a function $f \in C^\infty(M)$, the gradient of f is also only taken over the spatial components.

Maxwell's equations display an inherent symmetry between \vec{E} and \vec{B} , as they both each appear in roughly analogous equations with the exception of a change of sign, and the introduction of the charge and current densities. In a vacuum ($\rho = 0, \vec{j} = 0$), the transformations

$$\vec{B} \rightarrow \vec{E}, \text{ and } \vec{E} \rightarrow -\vec{B}$$

change equations (ME1) and (ME2) into equations (ME3) and (ME4), suggesting some sort of duality between the electric and magnetic fields.

On \mathbb{R}^3 , any curl-free field \vec{F} can be written as the gradient of some function f , and, any divergence free field \vec{G} is the curl of another field. If \vec{B} is time independent, then (ME2) says that \vec{E} can be written as the gradient of some function ϕ . The function ϕ is known as a *scalar potential* or *electric potential* for the electric field. Similarly, since $\vec{\nabla} \cdot \vec{B} = 0$ by (ME1), we can write $\vec{B} = \vec{\nabla} \times \vec{A}$ for some \vec{A} , called the *vector potential*. With this substitution, Maxwell's fourth equation becomes

$$\begin{aligned} \vec{j} &= \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \\ &= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}, \end{aligned}$$

whenever \vec{E} is time independent.

Remark 2.1.1. Notice that any vector potential \vec{A} can be modified by the gradient of an arbitrary scalar potential f , without affecting the magnetic field \vec{B} . That is, $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\vec{A} - \vec{\nabla} f)$. This is a simple consequence of the well-known identity $\vec{\nabla} \times \vec{\nabla} f = 0$ for all $f \in C^\infty(M)$.

Remark 2.1.2. The ability to write \vec{E} and \vec{B} as the gradient and divergence of scalar and vector fields respectively actually depends on the 1st and 2nd de Rham cohomologies of the space M . We discuss this in more detail in section 2.2.

When \vec{B} is time dependent, we can no longer assume that \vec{E} is the divergence of some scalar potential. Fortunately, the vector potential remains a valid means of characterizing the magnetic field, and we can derive a related formula for the electric field using (ME2). Namely,

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0.$$

Hence, for some scalar potential ψ , we obtain

$$\vec{E} = \vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t}.$$

Any modification to the vector potential now has an impact on the electric potential. Let $\vec{A}' = \vec{A} + \vec{\nabla}f$ for some scalar potential $f \in C^\infty(M)$. Then,

$$\begin{aligned} \vec{E} &= \vec{\nabla}\psi - \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}f) \\ &= \vec{\nabla}\left(\psi - \frac{\partial f}{\partial t}\right) - \frac{\partial \vec{A}}{\partial t}. \end{aligned}$$

This gives a pair of coupled transformations for ψ and \vec{A} :

$$\text{CT1. } \vec{A}' = \vec{A} + \vec{\nabla}f$$

$$\text{CT2. } \psi' = \psi - \frac{\partial f}{\partial t}$$

These transformations are known as the *gauge transformations* for the electric and magnetic fields.

If we substitute our equations for \vec{E} and \vec{B} back into Maxwell's equation, we obtain new equations for (ME3) and (ME4) in terms of ψ and \vec{A} :

$$\text{ME3. } \vec{\nabla}^2\psi - \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = \rho, \text{ and}$$

$$\text{ME4. } (\vec{\nabla}^2 A - \frac{\partial^2}{\partial t^2} A) - \vec{\nabla}(\vec{\nabla} \cdot A - \frac{\partial}{\partial t}\psi) = -\vec{j}.$$

By selecting an appropriate scalar potential $f \in C^\infty(M)$, we can transform our scalar and vector potentials via (CT1) and (CT2) such that the new values simplify equations (ME3) and (ME4). This is known as “choosing a gauge.” Two well-known gauges are:

1. The Coloumb gauge ([17], pgs 137-138), where $\vec{\nabla} \cdot \vec{A} = 0$, and
2. The Lorentz gauge ([17], pg 138), where $\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial t}\psi$.

In the case of the Coulomb gauge, (ME3) becomes “Poisson's Equation” while (ME4) becomes a inhomogeneous wave equation. In the case of the Lorentz gauge, (ME3) and (ME4) become inhomogeneous wave equations in ψ and A respectively.

2.2 Maxwell's Equations via Differential Forms

Having reviewed Maxwell's equations in vector notation, we will reformulate them using differential forms. This will enable us to see further the underlying symmetry of the equations. Here, we follow the development by [5], Chapters 4 and 5.

Recall that Maxwell's first two equations were:

$$\begin{aligned}\vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0\end{aligned}$$

There is a relationship between the gradient, curl and divergence operators acting on smooth functions and vectors in \mathbb{R}^3 , and the exterior derivative acting on zero, one and 2-forms in $\Omega(\mathbb{R}^3)$, the “de Rham complex” on \mathbb{R}^3 .

Figure 1 - Relation between forms and vectors on \mathbb{R}^3

$$\begin{array}{ccc} C^\infty(\mathbb{R}^3) & \xrightarrow{\phi_3} & \Omega^3(\mathbb{R}^3) \\ \vec{\nabla} \cdot \uparrow & & \uparrow d \\ \text{Vect}(\mathbb{R}^3) & \xrightarrow{\phi_2} & \Omega^2(\mathbb{R}^3) \\ \vec{\nabla} \times \uparrow & & \uparrow d \\ \text{Vect}(\mathbb{R}^3) & \xrightarrow{\phi_1} & \Omega^1(\mathbb{R}^3) \\ \vec{\nabla} \uparrow & & \uparrow d \\ C^\infty(\mathbb{R}^3) & \xrightarrow{\text{id}} & \Omega^0(\mathbb{R}^3) \end{array}$$

We define the isomorphisms in Figure 1 by:

1. $\phi_1 : \text{Vect}(\mathbb{R}^3) \rightarrow \Omega^1(\mathbb{R}^3)$ is defined by $(v_x, v_y, v_z) \mapsto v_x dx + v_y dy + v_z dz$.
2. $\phi_2 : \text{Vect}(\mathbb{R}^3) \rightarrow \Omega^2(\mathbb{R}^3)$ is defined by $(v_x, v_y, v_z) \mapsto v_x(dy \wedge dz) + v_y(dz \wedge dx) + v_z(dx \wedge dy)$.
3. $\phi_3 : C^\infty(\mathbb{R}^3) \rightarrow \Omega^3(\mathbb{R}^3)$ is defined by $f \mapsto f(dx \wedge dy \wedge dz)$.

With these definitions, all of the squares in Figure 1 commute.

For example, let $\vec{E} = (E_x, E_y, E_z)$ and $\vec{B} = (B_x, B_y, B_z)$ be two vectors on \mathbb{R}^3 where $E_\alpha, B_\beta \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$ for all $\alpha, \beta = \{x, y, z\}$. Then, as time-dependent differential forms in $\Omega(\mathbb{R}^3)$, they become

$$\begin{aligned} E &= E_x dx + E_y dy + E_z dz, \text{ and} \\ B &= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy. \end{aligned}$$

We combine the electric and magnetic fields into the *unified electromagnetic field* F . It is a 2-form on $\mathbb{R} \times \mathbb{R}^3$ defined by

$$F = B + E \wedge dt.$$

In differential form, we will show that Maxwell's first two equations can be written as

$$dF = 0. \tag{1}$$

Using the identity $d^2 = 0$, the Leibnitz formula, and denoting $\partial_\alpha = \frac{\partial}{\partial \alpha}$ for all $\alpha \in \{t, x, y, z\}$, we have

$$\begin{aligned} dF &= (\partial_x B_x + \partial_y B_y + \partial_z B_z) dx \wedge dy \wedge dz + dt \wedge \partial_t B \\ &\quad + (\partial_x E_y - \partial_y E_x) dx \wedge dy \wedge dt + (\partial_z E_x - \partial_x E_z) dz \wedge dx \wedge dt \\ &\quad + (\partial_y E_z - \partial_z E_y) dy \wedge dz \wedge dt \\ &= d_s B + dt \wedge \partial_t B + d_s E \wedge dt \wedge dt \\ &= d_s B + (d_s E + \partial_t B) \wedge dt \end{aligned}$$

where d_s is the exterior derivative over \mathbb{R}^3 instead of $\mathbb{R} \times \mathbb{R}^3$. Thus, $dF = 0$ if and only if the following equations are satisfied:

$$d_s B = 0, \text{ and} \tag{2}$$

$$d_s E + \partial_t B = 0 \tag{3}$$

Using Figure 1, we can see that these are simply equations (ME1) and (ME2) recast in the language of differential forms.

Before continuing, we need to define the "Hodge star operator" on differential forms. Let M be an oriented, n -manifold with semi-Riemannian metric g . Let

$\{e_i\}_{i=\{1,\dots,n\}}$ be a basis of the tangent space in a coordinate chart. Over each $x \in M$, we define a matrix G with entries $G_{ij} = g(e_i, e_j)$. By non-degeneracy of the metric, the matrix G is invertible, and we denote entries of G^{-1} by g^{ij} . Using G^{-1} , we define an inner product on $\Omega(U)$ as follows: Let $\{e^i\}_{i=\{1,\dots,n\}}$ be the dual basis for $\Omega(U)$. Define the inner product of e^i and e^j as

$$\langle e^i, e^j \rangle = g^{ij},$$

and extend it linearly in each term to arbitrary 1-forms. Let $f^1 \wedge \dots \wedge f^p$ and $g^1 \wedge \dots \wedge g^p$ be two p -forms where $f^i, g^j \in \{e^i\}_{i=\{1,\dots,n\}}$. We define their inner product as

$$\langle f^1 \wedge \dots \wedge f^p, g^1 \wedge \dots \wedge g^p \rangle = \det [\langle f^i, g^j \rangle]_{ij},$$

and extend this linearly in each term to arbitrary p -forms. The inner product of a p and q -form is defined to be zero if $p \neq q$.

Example 2.2.1. Let $M = \mathbb{R} \times \mathbb{R}^3$ with the Minkowski metric g defined by

$$g(X, Y) = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3,$$

where $X = (x_0, x_1, x_2, x_3), Y = (y_0, y_1, y_2, y_3)$ in local coordinates. Then,

$$G = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = G^{-1}$$

Let $\mu = dt \wedge dx$ and $\nu = dx \wedge dt$. Then,

$$\begin{aligned} \langle dt \wedge dx, dx \wedge dt \rangle &= \det \begin{bmatrix} \langle dt, dx \rangle & \langle dt, dt \rangle \\ \langle dx, dx \rangle & \langle dx, dt \rangle \end{bmatrix} \\ &= \det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= 1 \end{aligned}$$

Since M is oriented, there exists a volume form on M . A volume form is simply a nowhere vanishing n -form. In local coordinates, it always has the form $f e^1 \wedge \dots \wedge e^n$

for some nowhere zero $f \in C^\infty(U)$. When M has a metric g , the canonical volume form on a coordinate chart of M is defined by

$$\text{vol} = \sqrt{|\det [g(e_i, e_j)]|} e^1 \wedge \dots \wedge e^n.$$

It is a straightforward exercise to show that the volume form is actually independent of the choice of the chart.

Definition 2.2.2. ([5], p.88) Let M be an oriented, semi-Riemannian n -manifold M . We define the *Hodge star operator*

$$\star : \Omega^p(M) \rightarrow \Omega^{(n-p)}(M)$$

to be the unique linear map satisfying $\mu \wedge \star \nu = \langle \mu, \nu \rangle \text{vol}$, for all $\mu, \nu \in \Omega^p(M)$.

It's not immediately obvious that this definition is well-defined. Perhaps a different choice of μ leads to a different value for $\star \nu$? Fortunately, there is a simple algorithm for calculating it in a local coordinate chart. The following development is due to [5], p.89. Let $\{e_i\}_{i \in I}$ be a positively oriented, orthonormal basis of 1-forms on some chart of M . That is, $\langle e^i, e^j \rangle = 0$ if $i \neq j$, and $\langle e^i, e^i \rangle = \epsilon(i)$ where $\epsilon(i) = \pm 1$. Then, for any distinct $1 \leq i_1, \dots, i_p \leq n$,

$$\star(e^{i_1} \wedge \dots \wedge e^{i_p}) = \pm e^{i_{p+1}} \wedge \dots \wedge e^{i_n}$$

where $\{i_{p+1}, \dots, i_n\}$ are the integers from 1 to n not including $\{i_1, \dots, i_p\}$. The sign \pm is given by

$$\text{sign}(i_1, \dots, i_n) \epsilon(i_1) \dots \epsilon(i_p),$$

where $\text{sign}(i_1, \dots, i_n)$ is the sign of the permutation taking $(1, \dots, n)$ to (i_1, \dots, i_n) .

Example 2.2.3. Let M and g be defined as in example 2.2.1. Over a local coordinate chart, it is clear that $\det G = -1$. Then, the canonical volume form is

$$\begin{aligned} \text{vol} &= \sqrt{|\det G|} dt \wedge dx \wedge dy \wedge dz \\ &= dt \wedge dx \wedge dy \wedge dz. \end{aligned}$$

Using the algorithm, we can calculate $\star dx$ as follows.

$$\begin{aligned}\star dx &= \text{sign}(x, t, y, z)\epsilon(x)(dt \wedge dy \wedge dz) \\ &= -dt \wedge dy \wedge dz.\end{aligned}$$

Similarly, if we use the definition, since $\star dx$ is a 3-form, we must consider its wedge product with all possible 1-forms. Fortunately, since the basis is orthonormal, it suffices to consider

$$\begin{aligned}dx \wedge \star dx &= \langle dx, dx \rangle (dt \wedge dx \wedge dy \wedge dz) \\ &= dx \wedge (-dt \wedge dy \wedge dz),\end{aligned}$$

as expected.

As expected, we will use the Hodge star operator to generalize Maxwell's third and fourth equations. To begin, we combine the electric charge density ρ , and the current density (j_x, j_y, j_z) into a 1-form J on $\mathbb{R} \times \mathbb{R}^3$ defined by

$$J = -\rho dt + j_x dx + j_y dy + j_z dz.$$

Then, (ME3) and (ME4) are equivalent to

$$\star d \star F = J, \tag{4}$$

and the verification is straightforward.

Hence, Maxwell's equations can be written as

$$\text{MX1. } dF = 0.$$

$$\text{MX2. } \star d \star F = J.$$

Notice that the equations (MX1) and (MX2) can be written on any oriented, semi-Riemannian manifold M for a 2-form F and a 1-form J .

As we remarked in the previous section, the existence of the scalar potential ψ , and the vector potential A actually depends on the "de Rham cohomology" of the space M . Recall the de Rham complex:

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \dots$$

where d_n denotes the exterior derivative $d_n : \Omega^n(M) \rightarrow \Omega^{(n+1)}(M)$, which satisfies $d_{n+1}d_n = 0$.

Definition 2.2.4. The n^{th} de Rham cohomology group $H^n(M)$ is defined

$$H^n(M) = \ker d_n / \text{im } d_{(n-1)}.$$

That is, for $\mu, \nu \in \ker d_n$, $[\mu] = [\nu]$ if and only if there exists $\eta \in \Omega^{n-1}(M)$ such that $\mu - \nu = d\eta$. We call a differential form μ *closed* if $d\mu = 0$, and *exact* if there exists a form η such that $\mu = d\eta$.

When $H^n(M) = 0$, then every closed n -form is exact. We remark that the de Rham cohomology is a homotopy invariant ([16], p.393). Since \mathbb{R}^n is contractible, then $H^p(\mathbb{R}^n) = 0$ for all $p \geq 1$. In particular, $H^2(M) = 0$. By (MX1), F is a closed 2-form, and there exists $A \in \Omega^1(M)$ such that $F = dA$, as desired.

Using the vector potential A , Maxwell's equations become:

$$\text{MX1. } F = dA \text{ for some } A \in \Omega^1(M).$$

$$\text{MX2. } \star d \star dA = J \text{ for some } J \in \Omega^1(M).$$

We caution the reader that this reformulation is only true when $H^2(M) = 0$.

Interesting situations arise when $H^1(M) \neq 0$ and $H^2(M) = 0$. In this situation, it is possible for the vector potential to be closed, but not exact. This can lead to interesting physical consequences, and we will explore this further in section 6.2 on the Bohm-Aharonov effect.

We remark that the vector potential A can be modified by any closed 1-form μ without affecting (MX1). This corresponds with the gauge transformations of the electric and magnetic field from the last section. When μ is exact, the coupled transformations (CT1) and (CT2) are both contained in the equation $A' = A + d\eta$, where $\mu = d\eta$.

Chapter 3

Fibre Bundles

We will assume that the reader is already familiar with the concepts of smooth manifolds, differential forms, Lie groups and Lie algebras. Please see Kobayashi and Nomizu (KN) [15] or Greub, Halperin and Vanstone (GHV) [12], [13] for a thorough treatment of these topics. In general, we will use the notation of GHV unless otherwise stated.

3.1 Fibre Bundles

A “Gauge Theory” requires a “double” generalization of the directional derivative $X(f)$, where X is a vector field on a manifold M and f a smooth function on M with values in a vector space V . That is, we need to expand both the notion of a “ V -valued functions on M ”, and that of the “directional derivative”. In this chapter, we tackle the first of these issues. The second will be discussed in the next chapter.

To proceed, note that a vector-valued function on a manifold M may be considered to be a map $\sigma : M \rightarrow M \times V$ such that $\pi(\sigma(x)) = x, \forall x \in M$, where $\pi : M \times V \rightarrow M$ is the projection onto M . Then, in the usual notation, for $x \in M$, $\sigma(x) = (x, f(x))$, where f is the corresponding V -valued function over M .

We first generalize the product $M \times V$ to “twisted products” over M by introducing the notion of a “fibre bundle” $\pi : E \rightarrow M$ over M , which is *locally* a product, with “standard fibre” $\pi^{-1}(x) \cong V$ for all $x \in M$. Then, a V -valued function is replaced

with by a “section” $\sigma : M \rightarrow E$ with $\pi\sigma = \text{id}_M$.

We will need fibre bundles where the standard fibre is not a vector space, but a Lie group, so we include a more general definition:

Definition 3.1.1. ([12],p.38) Let M, E and F be smooth manifolds, and let $\pi : E \rightarrow M$ be a smooth map. The quadruple (E, π, M, F) is a smooth *fibre bundle* over M if there is an open covering $\{U_\alpha\}_{\alpha \in I}$ of M and a family $\{\psi_\alpha\}_{\alpha \in I}$ of diffeomorphisms

$$\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F$$

such that $\pi \circ \psi_\alpha^{-1}(x, \xi) = x$ for all $x \in U_\alpha$, and $\xi \in F$.

Remark 3.1.2. We call E the *total space*, M the *base space*, F the *fibre*, and π the *projection*. The collection $\{(U_\alpha, \psi_\alpha) \mid \alpha \in I\}$ is called a *local trivialization* of the fibre bundle, and, for $\alpha \in I$, (U_α, ψ_α) a *local trivialization of E over U_α* . We will generally denote the smooth fibre bundle (E, π, M, F) by E , and will only indicate the base space and fibre as necessary.

Example 3.1.3. Let $E = M \times F$, and $\pi(x, \xi) = x$ for all $(x, \xi) \in M \times F$. Then, E is a fibre bundle called the *trivial fibre bundle* (with trivialization $\{(M, \text{id}_E)\}$).

Example 3.1.4. Let M be the (open) Möbius strip $((0, 1) \times [0, 1]) / \sim$ where $(x, 0) \sim (1 - x, 1)$, and define $\pi : M \rightarrow \mathbf{S}^1 = (\{\frac{1}{2}\} \times [0, 1]) / \sim$ by $[(x, y)] \mapsto [(\frac{1}{2}, y)]$. Then it is straightforward to see that $(M, \pi, \mathbf{S}^1, (0, 1))$ is a smooth fibre bundle over the circle \mathbf{S}^1 . This is the standard example to illustrate the idea that a fibre bundle may be thought of as a “twisted” product.

We are now in a position to define the generalization of an F -valued function on M :

Definition 3.1.5. Let U be an open subset of M . A *local (cross)-section* of a fibre bundle (E, π, M, F) is a smooth map $\sigma : U \rightarrow E$ such that $\pi \circ \sigma = \text{id}_U$. If $U = M$, we call σ a *(global) section* of E .

Remark 3.1.6. We denote the collection of all local sections of E over U by $\Gamma_E(U)$. In the case where $U = M$, this is reduced to $\Gamma(E)$.

Remark 3.1.7. Note that a trivial bundle has many global sections. Indeed, for any $\xi \in F$, $\sigma : M \rightarrow M \times F$ defined by $\sigma(x) = (x, \xi)$ is a section. However, the existence of a global section does not always imply that the bundle is trivial. For example, the Möbius strip is not trivial, but $\sigma : \mathbf{S}^1 \rightarrow M$ defined by $\sigma(y) = [(\frac{1}{2}, y)]$ is a global section.

In order to compare bundles, we use the following definition:

Definition 3.1.8. ([12], p.39) Let (E, π, M, F) and (E', π', M', F') be two fibre bundles. A smooth map $\phi : E \rightarrow E'$ is called a *bundle map* or *fibre preserving* if, whenever $\pi(z_1) = \pi(z_2)$ for $z_1, z_2 \in E$, then $\pi' \circ \phi(z_1) = \pi' \circ \phi(z_2)$.

Any bundle map ϕ induces a smooth map $\varphi : M \rightarrow M'$ making the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \downarrow \pi & & \downarrow \pi' \\ M & \xrightarrow{\varphi} & M' \end{array}$$

Note that for a local trivialization (U, ψ) of E over U , for any $x \in U$ and $\xi \in F$,

$$\varphi(x) = (\pi' \circ \phi \circ \psi^{-1})(x, \xi).$$

In this case, we say that ϕ *covers* φ . If ϕ covers id_M , we call ϕ a *strong* bundle map. A bundle map is an *isomorphism* if ϕ (and hence φ) is a diffeomorphism. We also say that a bundle is *trivial* if it is isomorphic to a trivial bundle.

Let (E, π, M, F) be a fibre bundle over M and $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ a local trivialization of E . For each $\alpha, \beta \in I$, we can define maps

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Diff}(F) \tag{5}$$

by $x \mapsto (\xi \mapsto \psi_\beta \circ \psi_\alpha^{-1}(x, \xi))$, for $x \in M$ and $\xi \in F$. That is, $\psi_{\alpha\beta}$ satisfies

$$\psi_\alpha^{-1}(x, \xi) = \psi_\beta^{-1}(x, \psi_{\alpha\beta}(x)(\xi)), \quad \forall x \in U_\alpha \cap U_\beta, \forall \xi \in F.$$

The maps $\{\psi_{\alpha\beta}\}_{\alpha, \beta \in I}$ (or the maps $\{\psi_\beta \circ \psi_\alpha^{-1}\}_{\alpha, \beta \in I}$) are called the *transition functions* for the trivialization.

We shall be particularly interested in the special cases where each diffeomorphism $\psi_{\alpha\beta}(x)$ preserves some given structure on F . An example we shall consider will be when $F = V$ is a vector space, and $\psi_{\alpha\beta}$ maps $U_\alpha \cap U_\beta$ into some subgroup of $GL(V) \subset \text{Diff}(V)$, which will usually be given as the image of a representation $\rho : G \rightarrow GL(V)$ of a Lie group G .

Conversely, given enough local data like that above, we may re-assemble it to yield a fibre bundle:

Proposition 3.1.9. ([12], p.39) *Let M and F be smooth manifolds, and let E be a set. If there is a surjective set map $\pi : E \rightarrow M$ with the following properties:*

1. *There is an open covering $\{U_\alpha\}_{\alpha \in I}$ of M and a family $\{\phi_\alpha\}_{\alpha \in I}$ of bijections*

$$\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F.$$

2. *For every $x \in U_\alpha$, $y \in F$, then $\pi \circ \psi_\alpha(x, y) = x$.*

3. *The maps $\psi_\beta \circ \psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times F \rightarrow (U_\alpha \cap U_\beta) \times F$ are diffeomorphisms.*

Then, there is exactly one manifold structure on E for which (E, π, M, F) is a fibre bundle with local trivialization $\{(U_\alpha, \psi_\alpha) \mid \alpha \in I\}$.

Proof. An atlas on E is defined using the ψ_α , atlases on both U_α and F , and condition (3). Then, ψ_α is a diffeomorphism, E is a smooth manifold and π is a smooth map. The collection $\{U_\alpha, \psi_\alpha\}_{\alpha \in I}$ is then a local trivialization of E . \square

Example 3.1.10. The Tangent Bundle. A useful example of a fibre bundle constructed in this way is the *tangent bundle* over an n -manifold M . For $x \in M$, let $T_x M$ denote the space of tangent vectors at $x \in M$. Define

$$T(M) = \bigsqcup_{x \in M} T_x M,$$

where \bigsqcup is the disjoint union. We define the projection $\pi : T(M) \rightarrow M$ by $X \mapsto x$ whenever $X \in T_x M$. We can think of every point in $T(M)$ as being a pair (x, X) where $X \in T_x M$.

Suppose that $\{U_\alpha\}_{\alpha \in I}$ is an atlas for M , with local coordinates $x_\alpha^i : U_\alpha \rightarrow \mathbb{R}$, $i = 1, \dots, n$. Then, for $x \in U_\alpha$, a basis of the tangent space at x is given by $\{(\partial_i^\alpha)_x\}_{i=1}^n$, where $\partial_i^\alpha := \frac{\partial}{\partial x_\alpha^i}$. We can define bijections

$$\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^n$$

by $(x, \sum_{i=1}^n v^i (\partial_i^\alpha)_x) \mapsto (x, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix})$. Then, for all α , ψ_α satisfies $\pi \circ \psi_\alpha = \text{id}_M$,

and is indeed linear in its second argument. The transition functions $\psi_\beta \circ \psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ are given by $(x, v) \mapsto (x, w)$ where $w = Av$ for $A = [A_{ij}] \in \text{GL}(n)$ with $A_{ij} = \frac{\partial x_\beta^i}{\partial x_\alpha^j}$, and are clearly diffeomorphisms for all $\alpha, \beta \in I$. By Proposition 3.1.9, $(T(M), \pi, M, \mathbb{R}^n)$ is a fibre bundle known as the *tangent bundle of M* .

Note that, in this case, the transition maps defined at (5) also satisfy $\psi_{\alpha\beta}(x)(v) = A(x)v$, and so

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(n).$$

We shall see that this property makes $T(M)$ a *vector bundle* in a precise sense to be defined soon.

Note that a section $\sigma : M \rightarrow T(M)$ is a vector field on M . By the “Hairy Ball Theorem” (due to Brouwer, [7], p.131-132), \mathbf{S}^2 has no nowhere-vanishing vector fields. This shows that $T(\mathbf{S}^2)$ is not a trivial fibre bundle, since any isomorphism $\phi : \mathbf{S}^2 \times \mathbb{R}^2 \rightarrow T(\mathbf{S}^2)$ would yield the nowhere-vanishing vector field defined by $\sigma(x) = \phi(\varphi^{-1}(x), e_1)$, where ϕ covers φ .

Example 3.1.11. One may analogously define the *cotangent bundle* $(T^*(M), \pi, M, \mathbb{R}^n)$ and its p^{th} exterior power, $(\Lambda^p T^*(M), \pi, M, \Lambda^p \mathbb{R}^n)$ over any n -manifold M . These will also be seen to be examples of vector bundles over M . The sections of $\Lambda^p T^*(M)$ are the differential p -forms on M .

Indeed, anything we can “do naturally” with a vector space V can be applied to $V = T_x M$, and analogous bundles over M constructed, as above. For example a bundle over M usually denoted $T(M) \otimes T^*(M)$, with fibre $T_x(M) \otimes T_x^*(M)$ over

$x \in M$, may be defined in a similar manner. Some examples that will be important later are given below in 3.1.13.

Example 3.1.12. (The Frame Bundle. [15],p.55-56) Let M be an n -manifold. A linear frame u at a point $x \in M$ is an ordered basis $\{X_1, \dots, X_n\}$ of $T_x M$. Define $L(M)$ to be the set of linear frames over all points in M , and define the projection, $\pi : L(M) \rightarrow M$ by $\pi(u) = x$, where u is a linear frame at $x \in M$. Let (x^1, \dots, x^n) be a coordinate system in a chart (V, ϕ) containing x . In the chart, every frame u can be expressed uniquely in the form $u = (X_1, \dots, X_n)$ where $X_j = \sum_i X_{ij} \partial_i$ for some $X_{ij} \in C^\infty(V)$. The map $\psi : \pi^{-1}(V) \rightarrow V \times \text{GL}(n)$ sending $u \mapsto (\pi(u), B)$ where $B = [X_{ij}]$, is a bijection and the transition maps $\psi_{VV'} : V \cap V' \rightarrow \text{Diff}(\text{GL}(n))$ are given by $\psi_{VV'}(B) = AB$ (matrix multiplication), where $A_{ij} = \frac{\partial x_{V'}^i}{\partial x_V^j}$, and are clearly smooth. By 3.1.9, $(L(M), \pi, M, \text{GL}(n))$ is a fibre bundle over M called the (linear) frame bundle of M , and its standard fibre is the Lie group $\text{GL}(n)$. We shall return to this important example later in the section on principal fibre bundles.

Note that Brouwer's theorem also shows that $L(\mathbf{S}^2)$ is not trivial. In fact, it has no global sections, as follows: If $\sigma : \mathbf{S}^2 \rightarrow L(\mathbf{S}^2)$ were a global section, define a smooth vector field $X : M \rightarrow T(M)$ by $X(x) = \sigma(x).e_1$, where, if $u = (X_1, X_2) \in L(\mathbf{S}^2)$ is a frame at $x = \pi(u)$, $u.e_1 = (X_1, X_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = X_1$. Then, X would be a nowhere-vanishing vector field on \mathbf{S}^2 .

Example 3.1.13. (Endomorphism and Automorphism bundles.) As above, for $x \in M$, let $T_x M$ denote the space of tangent vectors at $x \in M$, and let $\text{End}(T_x M)$ and $\text{GL}(T_x M)$ denote the linear endomorphisms and automorphisms of $T_x M$ respectively. Define

$$\text{End}(TM) = \bigsqcup_{x \in M} \text{End}(T_x M),$$

where \bigsqcup is the disjoint union. We define the projection $\pi : \text{End}(TM) \rightarrow M$ by $X \mapsto x$ whenever $X \in \text{End}(T_x M)$. As before, we can think of every point in $\text{End}(TM)$ as being a pair (x, T) where $T \in \text{End}(T_x M)$.

Suppose that $\{U_\alpha\}_{\alpha \in I}$ is an atlas for M , so that a basis of the tangent space at x

is given by $B_x^\alpha := \{(\partial_i^\alpha)_x\}_{i=1}^n$. Now define bijections

$$\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbf{M}(n)$$

by $(x, T) \mapsto (x, [T]_{B_x^\alpha})$, where $[T]_{B_x^\alpha}$ denotes the matrix of T with respect to the basis B_x^α , and $\mathbf{M}(n)$ denotes the $n \times n$ matrices with real entries (that is, $\text{End}(\mathbb{R}^n)$).

Then, for all α , ψ_α is linear in its second argument, and $\pi \circ \psi_\alpha = \text{id}_M$. The transition functions $\psi_\beta \circ \psi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbf{M}(n) \rightarrow (U_\alpha \cap U_\beta) \times \mathbf{M}(n)$ are given by $(x, C) \mapsto (x, ACA^{-1})$ where $A_{ij} = \frac{\partial x_\beta^i}{\partial x_\alpha^j}$ is as above, and are clearly diffeomorphisms for all $\alpha, \beta \in I$. By Proposition 3.1.9, $(\text{End}(TM), \pi, M, \mathbf{M}(n))$ is a fibre bundle known (rather awkwardly) as the *endomorphism bundle of the tangent bundle of M* . It is naturally isomorphic to the bundle $T(M) \otimes T^*(M)$ mentioned in 3.1.11.

It is clear that we can also define a bundle $(\text{GL}(TM), \pi, M, \text{GL}(n))$, called the *automorphism bundle of the tangent bundle of M* , in a similar manner. Note that, unlike the frame bundle over M , (which is also a bundle over M with fibre $\text{GL}(n)$), the automorphism bundle $\text{GL}(TM)$ *always* has a global section defined by $\sigma(x) = \text{id}_{T_x M}$. In particular, $\text{GL}(TS^2) \not\cong L(S^2)$.

Remark 3.1.14. For a fibre bundle E , we can define the endomorphism and automorphism bundles of E in a similar manner. The transition functions also have the same form.

3.2 Vector Bundles and G -Bundles

Many of the examples above have the property that each fibre is naturally a vector space, and the operations vary smoothly from one fibre to another. These are examples of “vector bundles” defined as follows:

Definition 3.2.1. A *vector bundle* is a smooth fibre bundle (E, π, M, V) where:

1. V is a vector space.
2. There is a local trivialization $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ such that the transition functions satisfy

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(V) \subset \text{Diff}(V) \quad (6)$$

That is, for each $x \in U_\alpha \cap U_\beta$, $\psi_{\alpha\beta}(x)$ is a *linear diffeomorphism* (isomorphism) of V .

Alternatively,

Definition 3.2.2. ([12],p.44) A *vector bundle* is a smooth fibre bundle (E, π, M, V) where:

1. V , and the fibres $V_x = \pi^{-1}(x)$, $x \in M$ are real vector spaces.
2. There is a local trivialization $\{(U_\alpha, \psi_\alpha)\}$ such that the maps

$$\psi_\alpha^{-1}|_x : V \rightarrow V_x.$$

are linear isomorphisms for each $x \in U_\alpha$.

Remark 3.2.3. Note that a *morphism of vector bundles* will be a morphism of bundles which is linear on the fibres. That is, the bundle map restricted to any fibre is linear.

A bundle satisfying definition 3.2.2 clearly satisfies the conditions of definition 3.2.1. Now suppose 3.2.1(1&2) holds for all $x \in U_\alpha \subset M$, $k \in \mathbb{R}$ and $e, e' \in V_x$. Let $\pi_2 : U_\alpha \times V \rightarrow V$ denote the projection onto V , and define

$$\begin{aligned} e + e' &:= \psi_\alpha^{-1}(x, \pi_2\psi_\alpha(e) + \pi_2\psi_\alpha(e')), \quad \text{and} \\ k.e &:= \psi_\alpha^{-1}(x, k.\pi_2\psi_\alpha(e)). \end{aligned}$$

To see that addition is well-defined, note that if $x \in U_\alpha \cap U_\beta$, then

$$\begin{aligned} \psi_\alpha^{-1}(x, \pi_2\psi_\alpha(e) + \pi_2\psi_\alpha(e)) &= \psi_\beta^{-1} \circ \psi_\beta \circ \psi_\alpha^{-1}(x, \pi_2\psi_\alpha(e) + \pi_2\psi_\alpha(e)) \\ &= \psi_\beta^{-1}(x, \psi_{\alpha\beta}(x)(\pi_2\psi_\alpha(e) + \pi_2\psi_\alpha(e))) \\ &= \psi_\beta^{-1}(x, \psi_{\alpha\beta}(x)(\pi_2\psi_\alpha(e)) + \psi_{\alpha\beta}(x)(\pi_2\psi_\alpha(e))) \\ &= \psi_\beta^{-1}(x, \pi_2\psi_\beta(e) + \pi_2\psi_\beta(e)) \end{aligned}$$

A similar computation shows that scalar multiplication is well-defined. The maps $\psi_\alpha|_{V_x}$ are then clearly linear isomorphisms.

We note that, in addition to 3.2.1(2), the transition maps for a vector bundle also satisfy the following properties:

$$1. \text{ For every } x \in U_\alpha, \psi_{\alpha\alpha}(x) = \text{id}_V. \quad (7)$$

$$2. \text{ For every } x \in U_\alpha \cap U_\beta \cap U_\gamma, \psi_{\gamma\alpha}(x) \cdot \psi_{\beta\gamma}(x) \cdot \psi_{\alpha\beta}(x) = \text{id}_V. \quad (8)$$

Using Proposition 3.1.9, and either of the definitions above, we see that the tangent bundle $T(M)$, the cotangent bundle $T^*(M)$, $\Lambda^p T^*(M)$ and the endomorphism bundle $\text{End}(T(M))$ are all examples of vector bundles over M .

As mentioned earlier in our discussion of gauge theories, we are particularly interested in vector bundles whose transition functions “factor through a representation of a Lie group” G . These will be called vector G -bundles (or simply G -bundles).

Definition 3.2.4. Let (E, π, M, V) be a vector bundle with transition functions $\{\psi_{\alpha\beta}\}_{\alpha, \beta \in I}$. Suppose G is a Lie group, and $\rho : G \rightarrow \text{GL}(V)$ is a representation. Then, E is a *vector G -bundle* if, for each $\alpha, \beta \in I$, there is a smooth map each $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$, such that

$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & G \\ & \searrow \psi_{\alpha\beta} & \swarrow \rho \\ & \text{GL}(V) & \end{array}$$

where $g_{\gamma\alpha}(x)g_{\beta\gamma}(x)g_{\alpha\beta}(x) = e$. We will also call the maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$ the *transition functions of the G -bundle*.

Remark 3.2.5. We remark that every vector bundle E with fibre V is trivially a $\text{GL}(V)$ -bundle.

Example 3.2.6. Let $\text{GL}^+(n) \hookrightarrow \text{GL}(n)$ denote the subgroup of matrices with positive determinant. A n -manifold M is orientable iff its tangent bundle is a $\text{GL}^+(n)$ -bundle. (This is sometimes taken to be the definition of orientability.)

Example 3.2.7. Let $\mathbf{O}(n) \hookrightarrow \text{GL}(n)$ denote the subgroup of orthogonal matrices. Then, an n -manifold M has a Riemannian metric ([13], p.16) iff its tangent bundle is an $\mathbf{O}(n)$ -bundle: If $T(M)$ is an $\mathbf{O}(n)$ -bundle, one may unambiguously define a

Riemannian metric on M via the formula $\langle \psi_\alpha^{-1}(x, v), \psi_\alpha^{-1}(x, w) \rangle = v \cdot w$, where $x \in M$, $v, w \in \mathbb{R}^n$, and the right-hand side is the usual dot product in \mathbb{R}^n . If M has a Riemannian metric, one can modify the maps in the local trivialization using the metric and the Gram-Schmidt (orthonormalization) algorithm so that the transition functions become orthogonal, as follows.

Let $\psi_\alpha : U_\alpha \rightarrow U_\alpha \times \mathbb{R}^n$ be the local trivialization given in 3.1.10 and e_1, \dots, e_n be the standard ordered basis of \mathbb{R}^n . Apply Gram-Schmidt (in $T_x M$, using the metric) to the (ordered) frame $(\psi_\alpha^{-1}(x, e_1), \dots, \psi_\alpha^{-1}(x, e_n))$. This process depends smoothly on x , and the result, (v_1, \dots, v_n) is the image under $\psi_\alpha^{-1}(x, -)$ of the frame (say) $(f_1(x), \dots, f_n(x)) = A(x) \in \text{GL}(n)$. We then define $\tilde{\psi}_\alpha(u) = A(\pi(u))^{-1} \circ \psi_\alpha$ in an obvious notation.

3.3 Principal and Associated Bundles

Principal bundles are mathematical objects at the heart of gauge theories, but before we present their definition, we give an important and illustrative example.

Example 3.3.1. Let M an n -manifold and let $(L(M), \pi, M, \text{GL}(n))$ be its frame bundle (see 3.1.12). An interesting point we did not make in our previous discussion of this example is that the manifold $L(M)$ actually has a smooth free right action of $\text{GL}(n)$: The matrix $[a_{ij}] = a \in \text{GL}(n)$ acts on a frame $u = (X_1, \dots, X_n)$ at $x \in M$ on the right in the following fashion:

$$\left(X_1 \quad \dots \quad X_n \right) a = \left(\sum_i a_{i1} X_i \quad \dots \quad \sum_i a_{in} X_i \right)$$

Moreover, M can be identified with the space of orbits of this action, and $\pi : L(M) \rightarrow M$ is the canonical map taking a frame to its orbit, since any frame is related to another by a unique element of $\text{GL}(n)$. In addition, the local trivializations are equivariant for this action and the right action in the local trivialization $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \text{GL}(n)$. That is, if $\psi_\alpha(u) = (x, b)$, then $\psi_\alpha(ua) = (x, ba)$ is easily verified. This extra structure on $L(M)$ makes it what we will eventually call a *principal* $\text{GL}(n)$ -bundle over M .

Using this additional structure, and the defining representation of $\mathrm{GL}(n)$, we can see that the frame bundle is intimately related to the tangent bundle, as follows: Let $\mathrm{GL}(n) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the standard (left) action, and consider the smooth action of $\mathrm{GL}(n)$ on $L(M) \times \mathbb{R}^n$ is defined by

$$(u, \xi) \mapsto (ua, a^{-1}\xi), \quad \text{for } a \in \mathrm{GL}(n).$$

The manifold $E = L(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n$, the space of orbits of this action, is the total space of a vector bundle over M as follows: the projection $\pi : E \rightarrow M$ is defined by $\pi([(u, \xi)]) = \pi_{L(M)}(u)$. If $u = (X_1, \dots, X_n)$ and $X_j = \sum_i X_{ij} \partial_i$, in some coordinate neighbourhood, then the vector $\sum_j X_{ij} \xi_j$ clearly depends only on the orbit of (u, ξ) , and local trivializations are given by sending $[(u, \xi)]$ to $(\pi(u), \sum_j X_{ij} \xi_j)$. (The transition functions come from those in $L(M)$ and are easily checked to be of the form $\psi_{\alpha\beta}(x)v = Av$, where A is defined in example 3.1.12.)

The vector bundle $L(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n$ is not mysterious. Indeed, it is not difficult to see that $L(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n \cong T(M)$ as bundles over M : Let $u = (X_1, \dots, X_n)$ be a frame at $x \in M$, and $X \in T_x M$. Then, for some unique $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathbb{R}^n$, $X = \sum_i \xi_i X_i$.

This can be written as

$$\begin{aligned} X &= (X_1, \dots, X_n)\xi \\ &= ((X_1, \dots, X_n)a)(a^{-1}\xi) \end{aligned}$$

for any $a \in \mathrm{GL}(n)$, and a strong bundle isomorphism $L(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n \rightarrow T(M)$ is defined by $[(u, \xi)] \mapsto u.\xi$.

We summarize the discussion above by saying that the $\mathrm{GL}(n)$ -vector bundle $T(M)$ is associated to the principal $\mathrm{GL}(n)$ -bundle $L(M)$ via the representation $\rho = \mathrm{id}_{\mathrm{GL}(n)} : \mathrm{GL}(n) \rightarrow \mathrm{GL}(\mathbb{R}^n)$. We will see how this phenomenon is generalized in the definitions of a principal G -bundle and an associated vector bundle to follow.

Let P be a smooth manifold, and G a Lie group acting smoothly on P on the right. We will denote the action by $u \mapsto ua$ and the right multiplication by $a \in G$ will be written by $R_a : P \rightarrow P$. A left action on a manifold will be denoted in a similar fashion

by L_a . We say that G acts *freely* on P when, for all $u \in P$, $R_a u = u \iff a = e$, where e denotes the identity element in G .

Definition 3.3.2. ([15],p.50) Let M be a manifold, and G be a Lie group, and (P, π, M, G) a fibre bundle over M . It is a *principal (fibre) bundle over M with group G* (or *principal G -bundle*), if there is an action of G on P satisfying the following conditions:

PB1. G acts freely on P on the right.

PB2. M is the space of orbits of the action, and $\pi : P \rightarrow M$ is the canonical projection to the space of orbits.

PB3. The local trivialization $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ of P is equivariant. That is, for every $\alpha \in I$, $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ satisfies $\psi_\alpha(ua) = \psi_\alpha(u)a$, for the trivial right action of G on itself.

We call P the *total space*, M the *base space*, G the *structure group*, and π the *projection*. For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P called the *fibre over x* , and is diffeomorphic to G . Indeed, for any chosen $u \in \pi^{-1}(x)$, we can write the fibre $\pi^{-1}(x) = \{ua | a \in G\}$.

Note that *morphisms of principal bundles* $(P, \pi, M, G) \xrightarrow{\phi} (P', \pi', M', G')$ are morphisms of bundles, together with with a homomorphism $\phi_G : G \rightarrow G'$ such that $\phi(ua) = \phi(u)\phi_G(a)$ for all $u \in P$ and $a \in G$. We say that ϕ is an *isomorphism* if both ϕ and ϕ_G are diffeomorphisms.

Example 3.3.3. Let (E, π_E, M, V) be a vector bundle with local trivializations given by $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ and transition functions $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(V)$. We can define a frame bundle P_E for E in exactly the same way as we did for $L(M)$: Define a frame of E_x to be an ordered basis $b = (v_1, \dots, v_n)$ of E_x , let P_E be the set of all frames over all points in M , and define a projection, $\pi : P_E \rightarrow M$ by $\pi(b) = x$, where b is a frame at $x \in M$. A local trivialization of E over U_α yields one for P_E over U_α of the form $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \text{GL}(V)$, where the transition maps $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Diff}(\text{GL}(V))$ are given by $\phi_{\alpha\beta}(V) = AV$, where $A = \psi_{\alpha\beta}(x)$. By 3.1.9, $(P_E, \pi, M, \text{GL}(V))$ is a fibre

bundle over M . As with $L(M)$, there is an analogous free right $\mathrm{GL}(V)$ action on P_E for which the ϕ_α are equivariant in the sense of (PB3), and it is straightforward to see that $\pi : P_E \rightarrow M$ is the projection to the space of orbits. By 3.3.2, $(P_E, \pi, M, \mathrm{GL}(V))$ is a principal bundle.

Example 3.3.4. Let (E, π_E, M, V) be a vector G -bundle with a faithful representation $\rho : G \rightarrow \mathrm{GL}(V)$, and let (e_1, \dots, e_n) be a fixed ordered basis of V . Then, a frame (v_1, \dots, v_n) is a G -frame of E_x iff $v_i = \sum_j V_{ij} \psi^{-1}(x, e_j)$ with $[V_{ij}] \in \rho(G)$. This definition is independent of the chosen trivialization because the transition functions for E factor through ρ . By considering only the G -frames of E_x , one can proceed exactly as in the previous example to define a principal bundle (P_E^G, π, M, G) with structure group G . This is similar in spirit to the example of the orthonormal frame bundle, which will follow in example 3.3.15.

Given a principal bundle P , and a local trivialization $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$, recall that the transition functions $\psi_{\alpha\beta}$ map into $\mathrm{Diff}(G)$. However, the equivariance implies that, for all $x \in U_\alpha \cap U_\beta$ and $a \in G$,

$$\begin{aligned} \psi_\beta^{-1}(x, \psi_{\alpha\beta}(x)(a)) &= \psi_\alpha^{-1}(x, a) \\ &= \psi_\alpha^{-1}(x, e)a \\ &= \psi_\beta^{-1}(x, \psi_{\alpha\beta}(x)(e))a \\ &= \psi_\beta^{-1}(x, \psi_{\alpha\beta}(x)(e)a). \end{aligned}$$

That is, $\psi_{\alpha\beta}(x)(a) = \psi_{\alpha\beta}(x)(e)a$. In other words, the diffeomorphism $\psi_{\alpha\beta}(x)$ of G is simply left multiplication in G by the element $\psi_{\alpha\beta}(x)(e)$ of G – as it was in the examples of the frame bundles. As the thesis contains some discussion of physics, we cheerfully follow the lead of the physicists and will liberally abuse notation to write $\psi_{\alpha\beta}(x)$ for $\psi_{\alpha\beta}(x)(e)$, and think of the transition functions simply as maps

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$$

for any $u \in \pi^{-1}(x)$. It is obvious that they satisfy the following properties:

TF1. For every $x \in U_\alpha$, $\psi_{\alpha\alpha}(x) = e$, where e is the identity element of G .

TF2. For every $x \in U_\alpha \cap U_\beta \cap U_\gamma$, $\psi_{\gamma\alpha}(x) \cdot \psi_{\beta\gamma}(x) \cdot \psi_{\alpha\beta}(x) = e \in G$.

Moreover, the data of an open cover $\{U_\alpha\}_{\alpha \in I}$ with transition functions $\{\psi_{\alpha\beta}\}_{\alpha, \beta \in I}$ satisfying the conditions directly above are *sufficient* to define a principal bundle:

Theorem 3.3.5. ([15], p.52) *Let M be a manifold, $\{U_\alpha\}_{\alpha \in I}$ an open covering of M and G a Lie group. Suppose that for each pair $(\alpha, \beta) \in I \times I$ there is a smooth mapping $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$ such that the $\psi_{\alpha\beta}$ satisfy properties (TF1) and (TF2). Then, there is a principal bundle $P(M, G)$ for which the $\psi_{\alpha\beta}$ are the transition functions.*

Proof. The essential idea is to define P as the quotient of $\{(\alpha, x, g) \mid \alpha \in I, x \in U_\alpha, g \in G\}$ under the equivalence relation $(\alpha, x, g) \sim (\beta, x, \psi_{\alpha\beta}(x)g)$. The reader may consult [15], p. 52-53 for details. \square

Using 3.3.5, we can construct a principal bundle “associated” with a vector G -bundle, as follows:

Corollary 3.3.6. *Given a vector G -bundle over M , with a faithful representation $\rho : G \rightarrow GL(V)$, and transition data*

$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & G \\ & \searrow \psi_{\alpha\beta} & \swarrow \rho \\ & & GL(V) \end{array}$$

there is a principal fibre bundle with structure group G and transition functions $\{g_{\alpha\beta}\}$.

Proof. Since the $\psi_{\alpha\beta}$ satisfy PB2 and PB2, and ρ is a faithful homomorphism, the maps $\{g_{\alpha\beta}\}$ satisfy (TF1) and (TF2) above. \square

Remark 3.3.7. The principal bundle given by the construction above is strongly isomorphic to that of example 3.3.4. Since every vector bundle (E, π, M, V) is a $GL(V)$ -bundle with $\rho = \text{id}_{GL(V)}$, 3.3.6 yields a principal fibre bundle $(P, \pi, M, GL(V))$ with the “same” transition functions. (We use inverted commas here because, for the principal bundle, the action of $\psi_{\alpha\beta}$ on $GL(V)$ is by left multiplication in $GL(V)$, while for the vector bundle, the action of $\psi_{\alpha\beta}$ on V is by left multiplication in V .) This principal bundle is isomorphic to the frame bundle P_E constructed in the example.

The vector bundle E is *associated* to this P in exactly the same way that $T(M)$ was *associated* to $L(M)$ as described in example 3.3.1. To avoid further use of the italics, we now give the definition of a *vector bundle associated to a principal bundle*.

Let (P, π, M, G) be a principal bundle, V a vector space and $\rho : G \rightarrow \text{GL}(V)$ a faithful representation of G . A smooth action of G on $P \times V$ is defined by $(u, \xi) \mapsto (ua, \rho(a)^{-1}\xi)$ for $u \in P$ and $\xi \in V$. Let $E = P \times_G V$ be the quotient space of orbits. A projection, $\pi_E : E \rightarrow M$ is well-defined by $\pi_E(u\xi) = \pi(u)$, where $u\xi$ denotes the orbit of (u, ξ) .

If $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$, denoted by $u \mapsto (\pi(u), g_\alpha(u))$, is a local trivialization of P over U_α , define $\phi_\alpha : \pi_E^{-1}(U_\alpha) \rightarrow U_\alpha \times V$ by

$$u\xi \mapsto (\pi(u), \rho(g_\alpha(u))\xi).$$

By (PB3), the maps ϕ_α are well-defined, and are clearly surjective. They are bijections, since $\psi_\alpha(u\xi) = \psi_\alpha(u'\xi')$ iff $u' = ua$ for some $a \in G$, and $\rho((g_\alpha(u))\xi) = \rho((g_\alpha(u'))\xi')$. But, as $g_\alpha(u') = g_\alpha(ua) = g_\alpha(u)a$, $\rho(g_\alpha(u))\xi = \rho(g_\alpha(u))\rho(a)\xi'$. So $\xi = \rho(a)\xi'$, and consequently, $u\xi = u'\xi'$.

Moreover, given $x \in U_\alpha$, since ρ is faithful, then $\psi_\alpha^{-1}(x, e)$ is the unique element $u \in \pi^{-1}(x)$ such that $\rho(g_\alpha(u)) = \text{id}_V$. Hence $\phi_\alpha^{-1} : U_\alpha \times V \rightarrow \pi_E^{-1}(U_\alpha)$ is $(x, \xi) \mapsto \psi_\alpha^{-1}(x, e)\xi$, and so $\phi_{\alpha\beta}(x)\xi = \rho(g_\beta(\psi_\alpha^{-1}(x, e)))\xi = \rho(\psi_{\alpha\beta}(x))\xi$, which is clearly linear in ξ . Hence, by Proposition 3.1.9, and definitions 3.2.1 and 3.2.4, (E, π_E, M, V) is a vector G -bundle.

Definition 3.3.8. We call (E, π_E, M, V) the *vector G -bundle associated to the principal fibre bundle P* (via a faithful representation ρ), or simply as an *associated bundle* of P . ([15], p.54-55.)

With this definition we are able to close the circle: a vector G -bundle E with standard fibre V over M gives, via 3.3.6, a principal G -bundle over M (together with a faithful representation $\rho : G \rightarrow \text{GL}(V)$), which in turn, using 3.3.8 yields an associated vector G -bundle with standard fibre V , that is easily proven to be strongly isomorphic as a vector bundle to E :

Proposition 3.3.9. *If (E, π, M, V) is a G -bundle with a faithful representation $\rho : G \rightarrow GL(V)$, and (P, π_P, M, G) is the principal fibre bundle given by 3.3.6, then its associated vector bundle $E'(M, V, G, P)$ is isomorphic to (E, π, M, V) .*

Moreover, we may “begin” this circle with a principal G -bundle P over M and a faithful representation. Then, the principal G -bundle we return to via 3.3.8 and 3.3.6 is strongly isomorphic as a principal bundle to P with $\phi_G = \text{id}_G$.

Remark 3.3.10. In our definition of the vector G -bundle, we demanded that the representation ρ be faithful. As seen, this was necessary to ensure that a principal G -bundle could be constructed. However, in the event that $\rho : G \rightarrow GL(V)$ is not faithful, we remark that since $\ker \rho$ is a closed, normal subgroup of G then $G/\ker \rho$ is also a Lie group, and the map $\tilde{\rho} : G/\ker \rho \rightarrow GL(V)$ given by $[g] \mapsto \rho(g)$ will be faithful ([16],p.232). For the remainder of this document, we will always assume that our vector G -bundle has a faithful representation.

We now briefly conclude this chapter with some examples.

Example 3.3.11. Given a Lie group G and a manifold M , the manifold $P = M \times G$, with the obvious right G -action and projection to M is called the *trivial principal bundle over M with fibre G* .

If $\pi : P \rightarrow M$ is a principal bundle over M , and $U \subset M$ is open in M , $\pi^{-1}(U)$ is a principal bundle over U . Indeed, if (U, ϕ) is a local trivialization of P over U , $\phi : \pi^{-1}(U) \rightarrow U \times G$ is an isomorphism of principal bundles.

Note that $x \mapsto \phi^{-1}(x, e)$, where e is the identity element in G , defines a local section $\sigma : U \rightarrow \pi^{-1}(U)$. It is easy to show that the existence of a global section characterizes trivial principal bundles (amongst principal bundles):

Lemma 3.3.12. *A principal fibre bundle (P, π, M, G) is trivial if and only if it admits a global cross section.*

Remark 3.3.13. If $P = M \times G$ is any trivial principal bundle, and E is any associated bundle with standard fibre V , then $E \cong M \times V$, strongly, as vector bundles.

Recall that the frame bundle $L(M)$ over an n -manifold M is a principal $\mathrm{GL}(n)$ -bundle, and that the tangent bundle $T(M)$ is always an associated vector $\mathrm{GL}(n)$ -bundle of $L(M)$.

Example 3.3.14. When $M = \mathbb{R}^n$, the frame bundle $L(\mathbb{R}^n)$ is a trivial bundle, since $x \mapsto (\partial_1, \dots, \partial_n)$ is a global cross section. Hence, $L(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathrm{GL}(n)$. However, as we saw at the end of example 3.1.12, $L(\mathbb{S}^2)$ is *not* trivial.

Example 3.3.15. The tangent bundle $T(M)$ may also be associated with principal bundles other than $L(M)$. Suppose that M is equipped with a Riemannian metric, and let $O(M) \subset L(M)$ be the collection of orthonormal linear frames on M . These may be assembled into an principal $\mathbf{O}(n)$ -bundle, denoted $\mathbf{O}(M)$, over M , just as in the case of $L(M)$. The inclusion $\mathbf{O}(n) \hookrightarrow \mathrm{GL}(n)$ is, of course, a faithful representation, and the associated vector $\mathbf{O}(n)$ -bundle $E(M, \mathbb{R}^n, \mathbf{O}(n), O(M))$ is strongly isomorphic to $T(M)$, associated to $L(M)$, as example 3.2.7 shows. Indeed, for $u \in \mathbf{O}(M)$ and $\xi \in \mathbb{R}^n$, the map $\phi : E \rightarrow T(M)$ sending $u\xi$ in E to $u\xi$ in $T(M)$ (an orthonormal frame is just a special linear frame) covers the identity, and is a linear isomorphism on the fibres.

Chapter 4

Covariant Derivatives: Connections on Bundles

4.1 Connections on Vector Bundles

In the last chapter, we saw how V -valued functions on a manifold M could be generalized to sections of a vector bundle E over M with standard fibre V . We now turn to the crucial matter of how to “differentiate” them. That is, how to find their “directional derivatives” with respect to vector fields X on M .

A vector field X on M acts on a V -valued function v to return $X(v)$, the derivative of v in the “direction” X , and this operation satisfies certain familiar properties that may be summarized as follows: First, note that the set $V(M)$ of V -valued functions on M is naturally a $C^\infty(M)$ -module (as is $\text{Vect}(M)$). The map

$$\begin{aligned} \text{Vect}(M) \otimes_{\mathbb{R}} V(M) &\rightarrow V(M) \\ \text{defined by } X \otimes v &\mapsto X(v), \end{aligned}$$

is a \mathbb{R} -linear map which is also $C^\infty(M)$ -linear in X , and satisfies the *Leibnitz rule*: If $f \in C^\infty(M)$ and $v \in V(M)$,

$$X(fv) = X(f)v + fX(v).$$

(These follow from analogous properties of $(X, f) \mapsto X(f)$, for $f \in C^\infty(M)$, since $V(M)$ has a global basis over $C^\infty(M)$.)

To generalize this to $\Gamma(E)$, the collection of all sections of a vector bundle (E, π, M, V) , note that $\Gamma(E)$ is also (naturally) a $C^\infty(M)$ -module, so one may simply copy the properties above as a definition of a *covariant derivative*:

Definition 4.1.1. ([5],p.223) A *covariant derivative (or connection)* on a vector bundle E is an \mathbb{R} -linear map

$$D : \text{Vect}(M) \otimes_{\mathbb{R}} \Gamma(E) \rightarrow \Gamma(E)$$

denoted by $X \otimes \sigma \mapsto D_X \sigma,$

which is $C^\infty(M)$ -linear in X , and satisfies the Leibnitz rule:

$$D_X(f\sigma) = X(f)\sigma + fD_X\sigma.$$

In local coordinates over a trivializing neighbourhood of E , these properties yield an expression for $D_X\sigma$ as follows: Let $U \subset M$ be a coordinate neighbourhood of M over which (E, π, M, V) is trivial, and let $\{e_i\}$ be a basis of V . If $\psi : \pi^{-1}(U) \rightarrow U \times V$ is a local trivialization, define local sections $\sigma_i : U \rightarrow E$ by $\sigma_i(x) = \psi^{-1}(x, e_i)$. Then, if σ is any section of E defined over U , $\sigma = \sum_i f_i \sigma_i$ for some $f_i \in C^\infty(M)$ which are uniquely determined by σ . If $X = \sum_j X^j \partial_j \in \text{Vect}(U)$, and D is a connection on E , then

$$\begin{aligned} D_X \sigma &= D_{\sum_j X^j \partial_j} \left(\sum_i f_i \sigma_i \right) \\ &= \sum_j X^j D_{\partial_j} \left(\sum_i f_i \sigma_i \right) \\ &= \sum_{j,i} X^j \partial_j (f_i) \sigma_i + \sum_{j,i} f_i X^j D_{\partial_j} \sigma_i \\ &= \sum_i X(f_i) \sigma_i + \sum_{j,i} f_i X^j D_{\partial_j} \sigma_i \end{aligned}$$

Note that the $C^\infty(M)$ -linearity of $D_X\sigma$ in X allows, for $X_x \in T_x M$, a definition of $D_{X_x}\sigma \in E_x$, if σ is defined in a neighbourhood of x :

$$(D_{X_x}\sigma)(x) := \sum_i X_x(f_i)\sigma_i(x) + \sum_{j,i} f_i(x)X_x^j(D_{\partial_j}\sigma_i)(x)$$

The first term is coordinate free, and, because of $C^\infty(M)$ -linearity of D_X in X , the second term clearly does not depend on the coordinates chosen. So we need only check that the expression is independent of the chosen trivialization ψ of E . For simplicity, let $X_x = (\partial_k)_x$, and denote $D_{(\partial_k)_x}$ by D_k . Let $\tau_i : U' \rightarrow E$ be another basis of local sections and write $\sigma_j = \sum_p A_{pj} \tau_p$, where $A_{pj} \in C^\infty(U \cap U')$ are the transition functions in these bases. Thus, $\sigma = \sum_i f_i \sigma_i = \sum_j g_j \tau_j$ where $g_j \in C^\infty(U')$, and $g_j = \sum_l f_l A_{jl}$. Denote the inverse of $A = [A_{ij}]$ by B , so that $f_i = \sum_j B_{ij} g_j$. Then, suppressing evaluation at $x \in U \cap U'$, we have

$$\begin{aligned}
 \sum_i \partial_k(f_i) \sigma_i + \sum_i f_i (D_{\partial_k} \sigma_i) &= \sum_{i,j,p} \partial_k(B_{ij} g_j) A_{pi} \tau_p + \sum_{i,j,p} B_{ij} g_j D_{\partial_k} (A_{pi} \tau_p) \\
 &= \sum_{i,j,p} \partial_k(B_{ij}) g_j A_{pi} \tau_p + \sum_{i,j,p} B_{ij} \partial_k(g_j) A_{pi} \tau_p \\
 &\quad + \sum_{i,j,p} B_{ij} g_j \partial_k(A_{pi}) \tau_p + \sum_{i,j,p} B_{ij} g_j A_{pi} D_{\partial_k} \tau_p \\
 &= \sum_{i,j,p} \partial_k(B_{ij}) g_j A_{pi} \tau_p + \sum_p \partial_k(g_p) \tau_p \\
 &\quad + \sum_{i,j,p} B_{ij} g_j \partial_k(A_{pi}) \tau_p + \sum_p g_p D_{\partial_k} \tau_p \\
 &= \sum_p \partial_k(g_p) \tau_p + \sum_p g_p (D_{\partial_k} \tau_p),
 \end{aligned}$$

since the first and third terms cancel due to the fact that $\partial_k(A_{pi} B_{ij}) = 0$ for all p, j .

In a similar manner, if σ is defined along a curve $t \mapsto x(t)$ in M , and $X(t) = \dot{x}(t)$ is its tangent, we may also (and will have cause to) define $D_X \sigma$ along this curve.

One can always describe a connection in local coordinates, by writing $D_{\partial_i} \sigma_j = \sum A_{ij}^k \sigma_k$ for some $A_{ij}^k \in C^\infty(U)$ (and keeping track of the way they change from trivializing chart to another). Indeed, in practice, this is often the way it is done. However, some important properties and identities they satisfy can also be viewed if we see what a connection on a vector G -bundle “means” on the principal bundle it yields via 3.3.6 or 3.3.4.

To see how this might be done, it is useful to think first of how to describe $D_X \sigma$ as a limit: i.e. if $t \mapsto x(t)$ is an integral curve of X starting at $x = x(0) \in M$, can we

make sense of

$$(D_{X_x}\sigma)(x) = \lim_{h \rightarrow 0} \frac{\sigma(x(h)) - \sigma(x(0))}{h} \quad (9)$$

The problem, of course, lies in the fact that $\sigma(x(h))$ and $\sigma(x(0))$ lie in *different* fibres of E , so the numerator makes no sense as written. To rectify this, we need some way of (linearly) comparing fibres over different points along a curve.

In a trivial vector bundle there is an unambiguous way to do this. One “transports” vectors in the fibre above one point to a fibre above another point in a “parallel” fashion by declaring that $(x, v) \in E_x$ is parallel to $(y, v) \in E_y$ since they possess the same component in V . In a trivial vector bundle, parallel transport amounts to leaving the V component fixed while traversing a curve in M .

In a non-trivial vector bundle, there is no ready-made mechanism for comparing vectors in two different fibres. However, say, $\sigma(0)$ is “the same” or “is parallel to” $\sigma(1)$ as far as a connection is concerned. This should be the same thing as saying that $D_{\dot{x}(t)}\sigma(t) = 0$ along the curve. In this case, we say that the tangent vector in $T_{\sigma(x(t))}E$ to the curve $t \mapsto \sigma(t) = \sigma(x(t))$ is ‘horizontal’ along the curve. Locally, if $\psi : \pi^{-1}(U) \rightarrow U \times V$ is of the form $e \mapsto (\pi(e), v(e))$, then $T_e E = T_{\pi(e)}U \oplus T_{v(e)}V = T_{\pi(e)}U \oplus V$, and $\sigma(t) = \psi^{-1}(x(t), v(t))$ for some curve $t \mapsto v(t)$ in V .

Let’s apply ψ to the expression we had before:

$$\begin{aligned} \psi(D_{\dot{x}(t)}\sigma(t)) &= (x(t), \pi_2\psi(\sum_i X(f_i)\sigma_i + \sum_{j,i} f_i X^j D_{\partial_j}\sigma_i)) \\ &= (x(t), \sum_i X(f_i)e_i + \sum_{j,i} f_i X^j A_{ji}^k e_k) \end{aligned}$$

where we have written $\pi_2\psi(D_{\partial_j}\sigma_i) = \sum_k A_{ji}^k e_k$ for some $A_{ji}^k \in C^\infty(U)$. If we now define an $\text{End}(V)$ -valued 1-form A on U by $A(X)(\sum_i v_i e_i) := \sum_{i,j,k} v^j X^j A_{ij}^k e_k$, we can write

$$\psi(D_{\dot{x}(t)}\sigma(t)) = (x(t), \sum_i \dot{x}(t)(f_i)e_i + A(\dot{x}(t))(\sum_i f_i e_i)),$$

or indeed

$$\psi(D_X\sigma) = (x, \sum_i X(f_i)e_i + A(X)(\sum_i f_i e_i)), \quad (10)$$

so that locally, the 1-form A is determined by, and determines, the connection. Hence, a section is parallel over the curve $x(t)$ iff

$$\sum_i \dot{x}(t)(f_i)e_i + A(\dot{x}(t))\left(\sum_i f_i e_i\right) = 0.$$

The first term involves derivatives of components of $\sigma(t)$ in the V -direction, while the second involves only derivatives of $\pi(\sigma(t)) = x(t)$. We can summarize this by saying that, for $X_U \in T_x U$ and $X_V \in T_{v(e)} V$, the vector $\psi_*^{-1}(X_U + X_V) \in T_e E$, is horizontal iff

$$X^V + A(X^U)(v) = 0.$$

Of course, this is all local, and one needs to keep track of changes to A when moving from chart to chart. However, connections can be reformulated on the appropriate principal G -bundle [15] *globally* in terms of a \mathfrak{g} -valued 1-form on the frame bundle P_E of E , where \mathfrak{g} denotes the Lie algebra of the Lie group G . The relation with connections on vector bundles is basically that a section is constant with respect to a connection on a vector bundle if its coordinates are kept constant in a *frame* that is “transported in a parallel fashion” on the corresponding principal bundle. We will present this in the following section.

4.2 Connections on Principal Bundles

Let (P, π, M, G) be a principal fibre bundle. Define V_u to be the subspace of $T_u P$ consisting of vectors tangent to the fibre through u . (i.e. $V_u = T_u(\pi^{-1}(x))$ where $\pi(u) = x$.) For every $u \in P$, we call V_u the *vertical subspace* of $T_u P$. Indeed, the right action of G on P induces a homomorphism from the Lie algebra \mathfrak{g} of G into the Lie algebra of vector fields on P that gives an isomorphism $V_u \cong \mathfrak{g}$ as follows.

For fixed $u \in P$, the action induces $L_u : G \rightarrow P$ defined by $a \mapsto ua$, for all $a \in G$. Then, for each $A \in \mathfrak{g} = T_e G$, and $u \in P$, we define

$$A_u^* = (L_u)_* A,$$

where $(L_u)_*$ denotes the tangent map of L_u at $e \in G$.

Definition 4.2.1. The map $u \mapsto A_u^*$ defines a smooth vector field on P which we denote by A^* , and call the *fundamental vector field corresponding to* $A \in \mathfrak{g}$.

That $A \mapsto A_u^*$ induces an isomorphism between \mathfrak{g} and $V_u P$ follows from the freeness of the action on P , and the equivariance of the local trivialization for a principal bundle. Moreover, $A \mapsto A^*$ is a homomorphism of Lie algebras ([15],p.51). We will make use of the following lemma:

Lemma 4.2.2. ([15],p.51) Let $A \in \mathfrak{g}$. Then, for all $a \in G$, $(R_a)_* A^*$ is the fundamental vector field corresponding with $Ad(a^{-1})A$, where Ad is the adjoint representation of G in \mathfrak{g} , and $R_a : P \rightarrow P$ denotes $u \mapsto ua$. That is, $(R_a)_* A^* = (Ad(a^{-1})A)^*$.

Definition 4.2.3. ([15],p.63) A *connection* Γ on P is an assignment of a subspace H_u of $T_u P$ to each $u \in P$ such that

- C1. $T_u P = V_u \oplus H_u$,
- C2. $H_{ua} = (R_a)_* H_u$ for every $u \in P$ and $a \in G$, and
- C3. H_u depends differentiably on $u \in P$.

We call H_u the *horizontal subspace* of $T_u P$ associated to the connection Γ .

A vector $X \in T_u P$ is called vertical or horizontal if it lies in V_u or H_u respectively. (C1) enables us to define projections $v : T_u P \rightarrow V_u$ and $h : T_u P \rightarrow H_u$ which satisfy $v(X) + h(X) = X$ for all $X \in T_u P$. Then, condition (C3) can then be rephrased by requiring that, for every smooth vector field X on P , $h(X)$ is also smooth.

We now arrive at the promised 1-form on P that characterizes a connection on a principal bundle.

Definition 4.2.4. ([15],p.63) Let Γ be a connection on a principal G -bundle P . If $X \in T_u P$, there is a unique $A \in \mathfrak{g}$ depending linearly on X , such that $A_u^* = v(X)$. Then, a \mathfrak{g} -valued *connection (1-)form* ω is defined by

$$\omega(X) = A.$$

Note that X is horizontal iff $\omega(X) = 0$.

Theorem 4.2.5. ([15], p.64) *The connection form ω of (P, π, M, G) satisfies the following conditions:*

CF1. $\omega(A^*) = A$ for every $A \in \mathfrak{g}$

CF2. $(R_a)^*\omega = Ad(a^{-1})\omega$ for every $a \in G$. (i.e. $\omega((R_a)_*X) = Ad(a^{-1})\omega(X)$ where $R_a : P \rightarrow P$ maps $u \mapsto ua$ and $X \in T_uP$.)

Conversely, if there exists a 1-form ω on P satisfying these conditions, then there exists a connection Γ on P whose connection form is ω .

Proof. (CF1) is obvious from the definition of the fundamental vector field corresponding with $A \in \mathfrak{g}$. (CF2) follows easily from properties (C1) and (C2) of connections, and the result of Lemma 4.2.2.

Finally, suppose that ω satisfies properties (CF1) and (CF2). Define the horizontal subspace at $u \in P$ to be the collection of all tangent vectors $X \in T_uP$ such that $\omega(X) = 0$. Consider the onto linear map

$$\omega_u : T_uP \rightarrow \mathfrak{g},$$

defined by $\omega_u(X) = \omega(X)$. Suppose that $X \in T_uP$ and $\omega(X) = A$. Then $v(X) = A_u^*$, $h(X) = X - A_u^*$ and property (C1) is satisfied. (C2) follows easily from (CF2). For (C3), recall that the map $(L_u)_* : \mathfrak{g} \rightarrow T_uP$ given by $A \mapsto A_u^*$ is smooth, so $v(X)$ is smooth as the composition $(L_u)_* \circ \omega$. Then, h is smooth as the difference between the identity map and the vertical map on \mathfrak{g} . \square

Since this will be important when we discuss higher gauge theory in Chapter 8, we consider in detail the case of a connection on a trivial principal bundle.

Lemma 4.2.6. *When $P = M \times G$ is a trivial principal G -bundle, there is a bijection between connection forms on P and \mathfrak{g} -valued 1-forms on M .*

Proof. Let μ be a \mathfrak{g} -valued 1-form on M . Since $P = M \times G$, then, for $(x, a) \in M \times G$, $T_{(x,a)}P \cong T_xM \oplus T_aG$ and $V_{(x,a)} = T_aG$. Let L_a denote the left action of G on itself. Define a smooth \mathfrak{g} -valued 1-form ω on P by

$$\omega(x, a; X + Y) = Ad(a^{-1})\mu(X) + (L_{a^{-1}})_*Y, \quad \text{for } X \in T_xM \text{ and } Y \in T_aG. \quad (11)$$

To see that (CF1) holds for ω , let $A \in \mathfrak{g}$ and denote $(x, a) = u$. By the definition of the fundamental vector field, $A_u^* = (L_u)_*A = (L_a)_*A$. Hence,

$$\begin{aligned}\omega(x, a; A_u^*) &= Ad(a^{-1})\mu(0) + (L_{a^{-1}})_*(L_a)_*A \\ &= A.\end{aligned}$$

For (CF2), let $X + Y$ denote an arbitrary tangent vector in $T_xM \oplus T_aG = T_uP$. We compute the left-hand side of (CF2) as follows:

$$\begin{aligned}(R_g)^*\omega(x, a; X + Y) &= \omega(x, ag; (R_g)_*(X + Y)) \\ &= \omega(x, ag; X + (R_g)_*Y) \\ &= Ad((ag)^{-1})\mu(X) + (L_{(ag)^{-1}})_*(R_g)_*Y \\ &= Ad((ag)^{-1})\mu(X) + Ad(g^{-1})(L_{a^{-1}})_*Y.\end{aligned}$$

On the other hand,

$$\begin{aligned}Ad(g^{-1})\omega(x, a; X + Y) &= Ad(g^{-1})\left(Ad(a^{-1})\mu(X) + (L_{a^{-1}})_*Y\right) \\ &= Ad((ag)^{-1})\mu(X) + Ad(g^{-1})(L_{a^{-1}})_*Y.\end{aligned}$$

Hence, ω satisfies (CF1) and (CF2).

Now, suppose $\tilde{\omega}$ is a connection form on $P = M \times G$. For $X \in T_xM$, define μ to be the \mathfrak{g} -valued 1-form on M given by

$$\mu(x; X) = \tilde{\omega}(x, e; X).$$

It suffices to show that the connection form ω defined by μ using equation (11) is equal to the original connection form $\tilde{\omega}$. Since ω and $\tilde{\omega}$ are both linear, it suffices to show equality for evaluation on vectors $X \in T_xM$ and $Y \in T_aG$. If $Y = (L_a)_*A$ for some $A \in \mathfrak{g}$, then $\omega(x, a; Y) = \omega(x, a; (L_a)_*A) = A = \tilde{\omega}(x, a; Y)$. Moreover, if $X \in T_xM$ then

$$\begin{aligned}\omega(x, a; X) &= Ad(a^{-1})\mu(x; X) = Ad(a^{-1})\tilde{\omega}(x, e; X) = (R_a)^*\tilde{\omega}(x, e; X) \\ &= \tilde{\omega}(x, a; (R_a)_*X) = \tilde{\omega}(x, a; X),\end{aligned}$$

as desired. □

For any principal G -bundle P , the tangent map $\pi_* : T_u P \rightarrow T_{\pi(u)} M$ is surjective, and has kernel V_u . So, for any connection Γ , π_* maps the horizontal subspace H_u isomorphically onto $T_x M$. We can use this isomorphism to uniquely “lift” vectors in $T_x M$ to vectors in H_u . This also permits us to lift curves on M uniquely to “horizontal” curves in P in the following manner:

Definition 4.2.7. ([15],p.68) Let $\lambda : [0, 1] \rightarrow M$ be a smooth curve on M . A *horizontal lift* of λ is a smooth curve α in P such that $\pi(\alpha(t)) = \lambda(t)$, and $\dot{\alpha}(t) \in H_{\alpha(t)}$ for all $t \in [0, 1]$. That is, α is a smooth section over λ satisfying $\omega(\dot{\alpha}(t)) = 0$ for all $t \in [0, 1]$.

Proposition 4.2.8. ([15],p.69) Let $\lambda : [0, 1] \rightarrow M$ be a smooth curve on M . Given $u \in \pi^{-1}(\lambda(0))$, there exists a unique horizontal lift α of λ satisfying $\alpha(0) = u$.

Proof. One proceeds by covering $\lambda([0, 1])$ by a finite number of trivializing neighbourhoods, solving the problem in each, and then gluing the results back together. The local problem is equivalent to one in a trivial bundle, where it reduces to solving a standard differential equation on a Lie group (see example 4.2.13). We refer the reader to [15], p.69, for details. \square

We can also lift vector fields on M to horizontal vector fields on P ([15], p.65). If X^* is the horizontal lift of a vector field X in M , then $(R_a)_* X^* = X^*$ for all $a \in G$, and the integral curve of X^* through $u \in P$ is precisely the horizontal lift (starting at u) of the integral curve of X through the point $\pi(u)$.

We are now in a position to define parallel transport along a curve:

Definition 4.2.9. ([15],p.70) Let $\lambda : [0, 1] \rightarrow M$ be a smooth curve in M and let u_0 be an arbitrary point in $\pi^{-1}(\lambda(0))$. If $t \mapsto \alpha(u_0; t)$ is the unique horizontal lift of λ beginning at $u_0 \in P$, we define $\tau_\lambda(u_0) = \alpha(u_0; 1)$. The resulting map,

$$\tau_\lambda : \pi^{-1}(\lambda(0)) \rightarrow \pi^{-1}(\lambda(1))$$

is called the *parallel transport along the curve* λ .

This map is actually a diffeomorphism of the fibres, and commutes with the action of G on P since the horizontal subspaces are G -invariant ([15],p.70). This parallel transport map, once redefined for the appropriate bundle, is precisely how we will make sense of the numerator in equation (9) of the previous chapter.

The dependence of τ_λ on the path λ will be of interest, so we recall the *Moore composition of paths*. First, note that it is possible to reparameterize any curve, and, as such, we will use intervals $[0, s]$ of varying lengths to do so. Two curves $\lambda_1 : [0, s_1] \rightarrow M$ and $\lambda_2 : [0, s_2] \rightarrow M$, are *composable* if $\lambda_1(s_1) = \lambda_2(0)$. We define their “composition”, $\lambda_1 * \lambda_2 : [0, s_1 + s_2] \rightarrow M$ by

$$(\lambda_1 * \lambda_2)(t) = \begin{cases} \lambda_1(t), & t \in [0, s_1] \\ \lambda_2(t - s_1), & t \in [s_1, s_1 + s_2] \end{cases}$$

The raison d’être for the choice of Moore composition is that it is associative “on the nose”, and not just up to homotopy of paths. Furthermore, the constant path $\hat{x} : [0, 0] \rightarrow M$ at x in M is a right and left identity for this composition. Moreover, for each smooth curve $\lambda : [0, s] \rightarrow M$, we can define an “inverse” $\bar{\lambda} : [0, s] \rightarrow M$ defined by $\bar{\lambda}(t) = \lambda(s - t)$. Of course, the curves $\lambda * \bar{\lambda}$ and $\bar{\lambda} * \lambda$ are *not* constant paths at $\lambda(0)$ or $\lambda(s)$, but they *are* for the purposes of parallel transport.

Proposition 4.2.10. ([15],p.71) *Let λ_1 and λ_2 be composable paths in M , and let $x \in M$. Then*

1. $\tau_{\lambda_1 * \lambda_2} = \tau_{\lambda_2} \circ \tau_{\lambda_1}$,
2. $\tau_{\hat{x}} = \text{id}_{\pi^{-1}(x)}$, and
3. $\tau_{\bar{\lambda}_1} = (\tau_{\lambda_1})^{-1}$

Remark 4.2.11. The collection of all τ_λ for all paths on M with the composition operation above forms a groupoid¹.

¹A *groupoid* H is a collection of objects, and a map $\circ : H \times H \rightarrow H$ satisfying the following properties: For $f, g, h \in H$,

1. Associativity. If $f \circ (g \circ h)$ or $(f \circ g) \circ h$ is defined, then they are both defined and are equal.
2. Right and Left Identities. For every $f \circ g$ defined, there exists $f^{-1}, g^{-1} \in H$ such that

We now briefly consider some examples.

Example 4.2.12. Consider the trivial bundle $P = M \times G$. Let $\lambda : [0, 1] \rightarrow M$ be a smooth curve on M . Let ω be the connection form on P obtained from the \mathfrak{g} -valued 1-form μ on M . The horizontal lift of λ in P starting at $(\lambda(0), e)$ has the form $\alpha(t) = (\lambda(t), g(t))$ where $g(t)$ is a smooth curve in G with $g(0) = e$. Since α is everywhere horizontal, then $\omega(\dot{\alpha}(t)) = 0$ for all $t \in [0, 1]$. That is:

$$\begin{aligned} 0 &= Ad(g(t)^{-1})\mu(\dot{\lambda}(t)) + (L_{g(t)^{-1}})_*\dot{g}(t) \quad \text{or,} \\ \dot{g}(t) &= -(R_{g(t)})_*\mu(\dot{\lambda}(t)) \end{aligned} \tag{12}$$

For instance, suppose that $\mu(\dot{\lambda}(t)) = -K \in \mathfrak{g}$, a constant for all $t \in [0, 1]$. Then, $\dot{g}(t) = (R_{g(t)})_*K$, and so $g(t) = \exp tK$.

Example 4.2.13. Let $P = M \times G$ be a trivial principal bundle with the *standard flat connection*. That is, $H_{(x,a)} = T_xM$ for all $(x, a) \in P$. If μ is the associated \mathfrak{g} -valued 1-form on M , then $\mu = 0$, and so by equation (12) above, the horizontal lift of any curve will have a constant G -component. Therefore, on a trivial principal G -bundle with the standard flat connection, parallel transport amounts to keeping the G -component constant, as expected.

4.3 Connections on Associated Bundles

We now discuss how a connection on a principal bundle leads to one on any associated vector bundle, and how we recover the covariant derivative defined in section 4.1.

Let (E, π_E, V, M) be the vector G -bundle associated with the principal bundle (P, π, M, G) via the representation $\rho : G \rightarrow GL(V)$. Recall the map $\psi : P \times V \rightarrow P \times_G V = E$, $(u, \xi) \mapsto u\xi$ which takes an element to its orbit. For fixed $\xi \in V$, the map $\psi|_{P \times \{\xi\}} = \psi_\xi : P \rightarrow E$ has tangent map $(\psi_\xi)_* : T_uP \rightarrow T_{u\xi}E$. Given a

$$f \circ g \circ g^{-1} = f \text{ and } f^{-1} \circ f \circ g = g.$$

3. Inverse. For every $f \in G$, there exists $f^{-1} \in H$ such that $f \circ f^{-1}$ and $f^{-1} \circ f$ are always defined, and satisfy (2).

The collection of all paths modulo path-homotopy is a standard example. ([19],p.326-327)

connection Γ on P , we define the horizontal subspace $H_{u\xi} \subset T_{u\xi}E$ by

$$H_{u\xi} = \{(\psi_\xi)_*(H_u) \mid \xi \in V\}.$$

This defines the *connection* Γ_E on E associated with the connection Γ on P . As with the principal bundle, the vertical subspace $V_{u\xi}$ is the kernel of the map $(\pi_E)_* : T_{u\xi}E \rightarrow T_xM$ where $\pi_E(u\xi) = \pi(u) = x$.

Proposition 4.3.1. *The connection Γ_E on E satisfies the following properties:*

CE1. $H_{u\xi}$ and $V_{u\xi}$ are independent of the choice of $(u, \xi) \in u\xi$.

CE2. $T_{u\xi}E = H_{u\xi} \oplus V_{u\xi}$ for all $u\xi \in E$.

Proof. These properties follow from the G -invariance of H_u and V_u , and the property $(\psi_{\rho(a)\xi})_* = (\psi_\xi)_*(R_a)_*$. \square

As in P , we can lift a curve λ in M to a horizontal curve in E . Let e be an arbitrary point in $\pi_E^{-1}(\lambda(0))$, and let $(u, \xi) \in P \times V$ be a representative of e . If α is the unique horizontal curve in P over λ satisfying $\alpha(0) = u$, then,

$$\chi(e; t) = \alpha(t)\xi \tag{13}$$

in E is the horizontal lift of λ to E with $\chi(0) = e$. It is straightforward to show that this is independent of the choice of representative of e . Note that χ is linear in ξ .

Parallel transport can be defined as in the case of a principal bundle:

Definition 4.3.2. ([15],p.87-88) Let $\lambda : [0, 1] \rightarrow M$ be a smooth curve in M and let e be an arbitrary point in $\pi_E^{-1}(\lambda(0))$. If $t \mapsto \chi(e; t)$ is the unique horizontal lift of λ in E beginning at $e \in E$, we define $\tau_\lambda(e) = \chi(e; 1)$. The resulting map

$$\tau_\lambda : \pi_E^{-1}(\lambda(0)) \rightarrow \pi_E^{-1}(\lambda(1))$$

is called the *parallel transport along the curve* λ . It is a linear isomorphism of the fibres ([15],p.88).

As one might expect, parallel transport on associated vector bundles satisfies properties identical to those in Proposition 4.2.10.

We now show how connections on associated bundles give a covariant derivative.

4.4 Connections and Covariant Derivatives

Let (P, π, M, G) be a principal G -bundle with connection Γ , and E an associated vector bundle with induced connection Γ_E . Suppose $x \in M$, and σ is a section of E defined in a neighbourhood of $x \in M$. If $X \in \text{Vect}(M)$, denote by λ_s the integral curve of X beginning at x with domain $[0, s]$.

Definition 4.4.1. ([15],p.114) The *covariant derivative* $(\nabla_X \sigma)(x)$ of σ in the direction X at x is defined to be

$$\nabla_X \sigma(x) = \lim_{h \rightarrow 0} \frac{(\tau_{\lambda_h})^{-1} \sigma(\lambda_h(h)) - \sigma(x)}{h}$$

The map $(X, \sigma) \mapsto \nabla_X \sigma$ is clearly \mathbb{R} -linear in σ , satisfies the Leibnitz rule for the usual reasons, and it is straightforward (albeit tedious) to show that it is $C^\infty(M)$ -linear in X [15],p.116). Hence, this agrees with definition 4.1.1, and we henceforth denote it by $(X, \sigma) \mapsto D_X \sigma$, as we did in section 4.1. We note that if we have a section defined over a curve λ , we could use the definition above for the covariant derivative $\mathbf{D}_{\dot{\lambda}(t)} \sigma(\lambda(t))$ since the limit only involves values of σ on λ .

Example 4.4.2. Consider the trivial bundle $P = M \times G$ with connection form ω associated with the \mathfrak{g} -valued 1-form μ on M . Let σ be a smooth section defined over λ . Then, $\sigma(\lambda(t)) = (\lambda(t), f(t))$ where $f(t)$ is a V -valued map defined over λ evaluated at the point $\lambda(t)$. As above, let $\lambda_h = \lambda|_{[0, h]}$. Recall from 4.2.12 and equation (4.5) that the parallel transport

$$\begin{aligned} \tau_{\lambda_t} : \pi_E^{-1}(\lambda(0)) &\rightarrow \pi_E^{-1}(\lambda(t)) \\ \text{is } \xi &\mapsto \rho(g(t))\xi, \end{aligned}$$

where

$$\dot{g}(t) = -(R_{g(t)})_* \mu(\dot{\lambda}(t)) \quad \text{and} \quad g(0) = 1_G. \quad (14)$$

Thus, if $X = \dot{\lambda}(0)$, and ignoring the component in M ,

$$\begin{aligned} \nabla_X \sigma(0) &= \lim_{h \rightarrow 0} \frac{(\tau_{\lambda_h})^{-1} \sigma(\lambda_h(h)) - \sigma(\lambda(0))}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\tau_{\lambda_h})^{-1} f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\rho(g(h)^{-1}) f(h) - f(0)}{h} \\ &= F'(0), \end{aligned}$$

where $F(h) = \rho(g(h)^{-1}) f(h)$. An application of the chain rule, product rules and (14) yields

$$\nabla_X \sigma(0) = X(f)(0) + d\rho(\mu(X))f(0),$$

which agrees with equation 10, as expected.

4.5 Holonomy

Parallel transport around a loop may not be the identity map. We discuss this phenomenon here, briefly.

Suppose Q is a bundle over M , with connection. If λ is a path on M , the parallel transport τ_λ along the curve λ is also known as the *holonomy along the path* λ . The holonomy along a path is an isomorphism of fibres. When λ is a loop at x , the holonomy is an automorphism of the fibre over x . The collection of all holonomies around all loops at x defines a group called the “holonomy group.” The holonomy group is isomorphic to a subgroup of the “structure group”, as we shall see.

For every $x \in M$, we define the *loop space at x* , denoted $C(x)$, to be the collection of all closed, smooth curves in M starting and ending at x . If we restrict holonomy loops in $C(x)$, then the resulting collection is a group, as follows:

Definition 4.5.1. ([15],p.71) Let Q be a bundle with connection Γ_Q . For every $x \in M$, we define the *holonomy group of Γ_Q with reference point x* , denoted $\Phi_Q(x)$ to be the collection of all automorphisms of $\pi_Q^{-1}(x)$ generated by parallel transport over loops in $C(x)$. That is, $\Phi_Q(x) = \{\tau_\lambda | \lambda \in C(x)\} \subset \text{Aut}(\pi_Q^{-1}(x))$.

The collection $\Phi_Q(x)$ really is a group because, since all paths in $C(x)$ start and end at x , any two elements of $\Phi_Q(x)$ will be composable, and Proposition 4.2.10 for principal and associated vector bundles.

On a principal G -bundle P , $\Phi_P(x)$ is isomorphic to a subgroup of G in the following way: let u be a fixed point in $\pi_P^{-1}(x)$. Each $\lambda \in C(x)$ determines a unique element $a \in G$ satisfying $\tau_\lambda(u) = ua$. If $\gamma \in C(x)$ determines an element $b \in G$, then $\tau_\gamma \circ \tau_\lambda(u) = \gamma(ua) = \gamma(u)a = u(ba)$. That is, the loop $\lambda * \gamma$ determines an element ba . This gives an injective homomorphism $\Phi_P(x) \rightarrow G$. Moreover, if E is a vector G -bundle associated to P via a faithful representation, equation shows that $\Phi_E(x)$ is also isomorphic to a subgroup of G . (If ρ is not injective, $\Phi_E(x)$ will be isomorphic to a subgroup of a quotient of G .) Indeed,

Theorem 4.5.2. ([15], p. 73) *The holonomy subgroup is a closed subgroup of G , and hence is a Lie group.*

Example 4.5.3. If $P = M \times G$ is a trivial bundle with the standard flat connection, example 4.2.13 shows that the holonomy group is trivial.

Example 4.5.4. Let $M = \mathbb{R}^2 \setminus \{\text{origin}\}$, $P = M \times S^1$, and $\rho : S^1 \hookrightarrow GL(\mathbb{R}^2)$, the inclusion map. We know that the Lie algebra $\mathfrak{s}^1 \cong \mathfrak{u}(1) \cong \mathbb{R}$. Let $(E, \pi, M, \mathbb{R}^2)$ be the associated vector bundle with connection Γ_E associated with the \mathbb{R} -valued 1-form $\frac{d\theta}{2}$, where $\hat{d}\theta$ is defined as usual by

$$\hat{d}\theta = \frac{xdy - ydx}{x^2 + y^2}.$$

Let $\lambda(t) = (\cos(t), \sin(t))$, and consider the holonomy group based at $(1, 0)$. In P , the horizontal lift of λ to P starting at the identity has the form $\alpha(t) = (\lambda(t), g(t))$ where $\dot{g}(t) = -(R_{g(t)})_* d\theta(\dot{\lambda}(t)) = -\frac{1}{2}g(t)\hat{d}\theta(\frac{\partial}{\partial\theta}) = -\frac{1}{2}g(t)$, and $g(0) = 1$. That is, $g(t) = e^{-\frac{it}{2}}$. Hence, we can see that the holonomy for the loop $\lambda : [0, 2\pi] \rightarrow M$ contains $e^{i\pi}$. Hence, the holonomy group is not trivial, even though the bundle is.

4.6 Curvature

Essentially, the *curvature* (2-form) of a connection at a point measures the “infinitesimal” parallel transport around small loops based at the point. We will make this

precise later. The curvature is an important object that, in essence, tells us how far from being “flat” is a given connection. Recall that we defined the standard flat connection in a trivial principle G -bundle in example 4.2.13, but there are others.

When we discuss electromagnetism in the context of connections on vector G -bundles in the next chapter, we will see that the electromagnetic 2-form F described in Chapter 2 can be thought of as the curvature of a connection related to a vector potential.

Let (P, π, M, G) be a principal G -bundle, and let V be a real vector space. Recall that $h : T_u P \rightarrow H_u$ is the bundle projection taking tangent vectors to their horizontal components.

Definition 4.6.1. ([15],p.77) Let ψ be a V -valued r -form on P . The *exterior covariant derivative* of ψ , denoted $D\psi$, is the $(r + 1)$ -form defined by

$$D\psi(X_1, \dots, X_{r+1}) = d\psi(hX_1, \dots, hX_{r+1}),$$

where X_1, \dots, X_{r+1} are tangent vectors on P .

Definition 4.6.2. ([15],p.77) Let ω be a connection form on (P, π, M, G) . The *curvature form* Ω of ω is defined by

$$\Omega(X, Y) = d\omega(hX, hY).$$

There is an important relationship between the connection form and the curvature form on the P as follows:

Theorem 4.6.3. ([15],p.77) *The connection form ω and the curvature form Ω satisfy the structure equation, namely,*

$$d\omega(X, Y) = -\frac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y) \quad (15)$$

for all $X, Y \in T_u P$, and $u \in P$.

Proof. When X and Y are both horizontal, the first term on the right hand side of the equation is zero, and we recover the definition of curvature. The other cases follow from the following identity ([15],p.36):

$$2d\omega(V, W) = V(\omega(W)) - W(\omega(V)) - \omega([V, W]) \quad (16)$$

where V and W are appropriate vector field extensions of X and Y respectively about the point u , together with the G -invariance of the horizontal subspace. See [15], p.77-78 for further details. \square

We now make precise our earlier description of the curvature as generators of infinitesimal holonomies, and state the well-known characterization of flat connection.

Theorem 4.6.4. ([15], p.89,92) *The Lie algebra of the holonomy group Φ_x is spanned by $\{\Omega(u; X, Y) \mid \pi(u) = x \text{ and } X, Y \in H_u P\}$. Moreover, a connection is flat iff the curvature is zero.*

We will now proceed to the proof of the Bianchi identity ($D\Omega = 0$). When electromagnetism is viewed as a connection on a vector G -bundle, the Bianchi identity is equivalent to Maxwell's first equation in differential form. We first recall the definition of the wedge product of Lie algebra-valued 1-forms: let $\mu, \nu \in \mathfrak{g} \otimes \Omega^1(M)$, and let $X, Y \in \text{Vect}(M)$. Then,

$$2(\mu \wedge \nu)(X, Y) = [\mu(X), \nu(Y)] - [\mu(Y), \nu(X)].$$

Since a connection form ω is a \mathfrak{g} -valued 1-form, then

$$\begin{aligned} (\omega \wedge \omega)(X, Y) &= \frac{1}{2}([\omega(X), \omega(Y)] - [\omega(Y), \omega(X)]) \\ &= [\omega(X), \omega(Y)]. \end{aligned}$$

Therefore, we can re-write the structure equation as

$$d\omega = -\frac{1}{2}\omega \wedge \omega + \Omega.$$

Let $\{e_1, \dots, e_r\}$ be a basis for the Lie algebra \mathfrak{g} , and let c_{ij}^k be the *structure constants* defined by the equations

$$[e_i, e_j] = \sum_k c_{ij}^k e_k$$

where $i, j \in \{1, \dots, r\}$. Then, we can rewrite the connection and the curvature as

1. $\omega = \sum_k \omega^k e_k$, and

$$2. \Omega = \sum_k \Omega^k e_k,$$

where ω^k and Ω^k are appropriate 1- and 2-forms respectively over M .

In this notation, the structure equation becomes

$$d\omega^k = -\frac{1}{2} \sum_{i,j} c_{ij}^k \omega^i \wedge \omega^j + \Omega^k. \quad (17)$$

This will be useful in the following:

Theorem 4.6.5. ([15], p.78) *The curvature Ω on P satisfies the Bianchi Identity. That is,*

$$D\Omega = 0.$$

Proof. This follows from $d^2 = 0$, equation (17), and the fact that $\omega(X) = 0$ whenever X is horizontal. \square

Example 4.6.6. Let $P = M \times G$ be the trivial bundle. Let ω be the connection form associated with the \mathfrak{g} -valued 1-form μ on M . Let X, Y respectively be vector fields on M and G defined near $(x, a) \in P$. Then, from equation (11), we know that $h(X+Y) = h(X) = X - (R_a)_*\mu(X)$. It suffices to compute $\Omega(X, X') = d\omega(h(X), h(X'))$ for $X, X' \in \text{Vect}(M)$. A straightforward calculation involving equations (11), (15), and (16) yields

$$\begin{aligned} 2d\omega(h(X), h(X')) &= -\omega([X - (R_a)_*\mu(X), X' - (R_a)_*\mu(X')]) \\ &= -\omega([X, X'] + [X, (R_a)_*\mu(X')] + [(R_a)_*\mu(X), X'] \\ &\quad + [(R_a)_*\mu(X), (R_a)_*\mu(X')]) \\ &= Ad(a^{-1}) \left(\frac{1}{2} [\mu(X), \mu(X')] + d\mu([X, X']) \right), \end{aligned} \quad (18)$$

Hence,

$$\Omega(X, X') = Ad(a^{-1}) \left(d\mu([X, X']) + \frac{1}{2} [\mu(X), \mu(X')] \right),$$

as one would expect from equation (15). If we have the “standard flat connection” ($\mu = 0$), then $\Omega = 0$.

As with the principal bundle, we can also define the curvature Ω_E of a connection on a vector G -bundle E . Like the covariant derivative, it acts on a section σ of E , and is defined as follows:

Definition 4.6.7. Let X and Y be two vector fields on M , let σ be a smooth section of E defined over M , and let D be a connection on E . The curvature Ω_E is defined as

$$\Omega_E(X, Y)\sigma = D_X(D_Y\sigma) - D_Y(D_X\sigma) - D_{[X, Y]}\sigma.$$

Example 4.6.8. Let $E = M \times V$ be a trivial bundle with connection. Then, we can write $D_X(v) = X(v) + A(X)v$ for some $\text{GL}(V)$ -valued 1-form A on M . A straightforward calculation using the expression for the exterior derivative of an $\text{End}(E)$ -valued 1-form² shows that

$$\Omega_E(X, Y)v = 2dA(X, Y)v + [A(X), A(Y)]v.$$

Clearly, when $A = 0$, the connection is flat, as expected.

4.7 The Exterior Covariant Derivative

The curvature is clearly $C^\infty(M)$ -linear and anti-symmetric in X and Y , and, as such, may be thought of as an $\text{End}(E)$ -valued 2-form on M . The Bianchi identity is a special case of a more general result in this setting that we will now describe (and to which we shall later refer).

The (ordinary) exterior derivative of a function $f \in C^\infty(M)$ is defined, as usual, by $df(X) = X(f)$. If D is a connection on E and $\sigma \in \Gamma(E)$ is a section, we can analogously define the *exterior covariant derivative* of σ to be the $\Gamma(E)$ -valued 1-form on M^3 defined by ([5], p.250)

$$d_D\sigma(X) = D_X\sigma. \tag{19}$$

²The term “ dA ” is the exterior derivative of an $\text{End}(E)$ -valued 1-form, not a real-valued 1-form. In order to be $C^\infty(M)$ -linear in every term, it is defined by

$$2dA(X, Y)v = X(A(Y)v) - A(Y)(Xv) - Y(A(X)v) + A(X)(Yv) - A([X, Y])v.$$

³The exterior covariant derivative of σ is a section of the bundle $T^*M \otimes E$.

Then, mimicking the expression⁴ for the exterior derivative of a 1-form, for a given $\Gamma(E)$ -valued 1-form on M τ , its exterior covariant derivative would be the $\Gamma(E)$ -valued 2-form on M ⁵ defined by

$$2d_D\tau(X, Y) = D_X\tau(Y) - D_Y\tau(X) - \tau([X, Y]). \quad (20)$$

It is easy to check that $d_D\tau$ is $C^\infty(M)$ -linear in X and Y . Hence, if $\sigma \in \Gamma(E)$, then

$$\begin{aligned} 2d_D^2\sigma(X, Y) &= D_X(D_Y\sigma) - D_Y(D_X\sigma) - D_{[X, Y]}\sigma \\ &= \Omega_E(X, Y)\sigma \end{aligned}$$

is precisely the curvature applied to σ .

Note that we may also compute $d_D\tau$ for a $\Gamma(E)$ -valued 1-form τ locally ([5], p.251) as follows. Locally, we may write $\tau = \sigma \otimes \mu$ for a section $\sigma \in \Gamma(E)$ and a 1-form μ . Then, by equation (20)

$$\begin{aligned} 2d_D(\sigma \otimes \mu)(X, Y) &= D_X(\mu(Y)\sigma) - D_Y(\mu(X)\sigma) - \mu([X, Y])\sigma \\ &= \mu(Y)D_X\sigma + X(\mu(Y))\sigma - \mu(X)D_Y\sigma - Y(\mu(X))\sigma - \mu([X, Y])\sigma \\ &= \mu(Y)d_D\sigma(X) - \mu(X)d_D\sigma(Y) + 2d\mu(X, Y)\sigma \\ &= 2(d_D\sigma \wedge \mu)(X, Y) + 2d\mu(X, Y)\sigma, \end{aligned}$$

since the product of two 1-forms α and β is the 2-form defined as usual by

$$(\alpha \wedge \beta)(X, Y) = \frac{1}{2}(\alpha(X)\beta(Y) - \alpha(Y)\beta(X)).$$

That is, we have

$$d_D(\sigma \otimes \mu) = (d_D\sigma \wedge \mu) + \sigma \otimes d\mu, \quad (21)$$

and this could have been used as the definition of d_D .

Moreover, if τ is a $\Gamma(E)$ -valued 1-form on M , recalling the expression for the (ordinary) exterior derivative⁶ of a 2-form η , and mimicking this for the exterior covariant derivative, a straightforward calculation shows that

$$6d_D^2\tau(X, Y, Z) = \Omega_E(X, Y)\tau(Z) + \Omega_E(Z, X)\tau(Y) + \Omega_E(Y, Z)\tau(X). \quad (22)$$

⁴ $2d\omega(V, W) = V(\omega(W)) - W(\omega(V)) - \omega([V, W])$: this is equation (16)

⁵The exterior covariant derivative of τ is a section of the bundle $\wedge^2 T^*M \otimes E$.

⁶ $3d\eta(X, Y, Z) = X(\eta(Y, Z)) + Y(\eta(Z, X)) + Z(\eta(X, Y)) - \eta([X, Y], Z) - \eta([Y, Z], X) - \eta([Z, X], Y)$

Note that Ω_E is an $\text{End}(E)$ -valued 2-form on M . If we define the (exterior) product of an $\text{End}(E)$ -valued 2-form on S with a $\Gamma(E)$ -valued 1-form τ on M to be the $\Gamma(E)$ -valued 3-form given by

$$(S \wedge \tau)(X, Y, Z) = \frac{1}{3} \left(S(X, Y)\tau(Z) + S(Z, X)\tau(Y) + S(Y, Z)\tau(X) \right), \quad (23)$$

then equation (22) becomes

$$2d_D^2\tau = \Omega_E \wedge \tau.$$

We shall also define

$$\begin{aligned} 2(d_D T)(X, Y)\sigma &= D_X(T(Y)\sigma) - D_Y(T(X)\sigma) \\ &\quad - T(Y)(D_X\sigma) + T(X)(D_Y\sigma) - T([X, Y])\sigma. \end{aligned}$$

for an $\text{End}(E)$ -valued 1-form T , and if S is an $\text{End}(E)$ -valued 2-form, then define

$$\begin{aligned} 6(d_D S)(X, Y, Z)\sigma &= D_X(S(Y, Z)\sigma) + D_Y(S(Z, X)\sigma) + D_Z(S(X, Y)\sigma) \\ &\quad - S(Y, Z)(D_X\sigma) - S(Z, X)(D_Y\sigma) - S(X, Y)(D_Z\sigma) \\ &\quad - S([X, Y], Z)\sigma - S([Z, X], Y)\sigma - S([Y, Z], X)\sigma. \end{aligned} \quad (24)$$

In this context, we can also write Bianchi's identity 4.6.5 as

Theorem 4.7.1. ([5], p.255) *The curvature Ω_E on E satisfies*

$$d_D\Omega_E = 0.$$

Proof. Using equation (24) and 4.6.7, we compute

$$\begin{aligned} 6(d_D S)(X, Y, Z)\sigma &= D_X([D_Y, D_Z]\sigma - D_{[Y, Z]}\sigma) + D_Y([D_Z, D_X]\sigma - D_{[X, Z]}\sigma) \\ &\quad + D_Z([D_X, D_Y]\sigma - D_{[X, Y]}\sigma) \\ &\quad - ([D_Y, D_Z] - D_{[Y, Z]})(D_X\sigma) - ([D_Z, D_X] - D_{[X, Z]})(D_Y\sigma) \\ &\quad - ([D_X, D_Y] - D_{[X, Y]})(D_Z\sigma) \\ &\quad - ([D_{[X, Y]}, D_Z] - D_{[[X, Y], Z]})\sigma - ([D_{[Z, X]}, D_Y] - D_{[[Z, X], Y]})\sigma \\ &\quad - ([D_{[Y, Z]}, D_X] - D_{[[Y, Z], X]})\sigma \\ &= 0. \end{aligned}$$

□

As we shall later see, when we rewrite electromagnetism as a connection on a vector G -bundle, the Bianchi identity replaces Maxwell's first equation in differential form.

Example 4.7.2. Let E be a trivial vector G -bundle with the standard flat connection D^0 . That is, $A = 0$. Then,

$$d_{D^0}\sigma(X) = X(\sigma).$$

Therefore, $d_{D^0} = d$, the ordinary exterior derivative.

Some additional properties associated with the exterior covariant derivative will be required in subsequent chapters. Let S be a section of $\text{End}(E)$, σ a section of E , and $\mu, \nu \in \Omega(M)$. We define the wedge product of $S \otimes \mu$ with $\sigma \otimes \nu$ as

$$(S \otimes \mu) \wedge (\sigma \otimes \nu) = S(\sigma) \otimes (\mu \wedge \nu), \quad (25)$$

and extend this linearly in each variable to the arbitrary wedge product of $\text{End}(E)$ and E -valued forms. Note that this is a $\Gamma(E)$ -valued form.

Similarly, let S, T be sections of $\text{End}(E)$, $\mu, \nu \in \Omega(M)$. If ST denotes the composition of S and T , then we define the wedge product of $S \otimes \mu$ with $T \otimes \nu$ as

$$(S \otimes \mu) \wedge (T \otimes \nu) = ST \otimes (\mu \wedge \nu), \quad (26)$$

and extend this linearly in each variable to arbitrary $\text{End}(E)$ -valued forms.

Lemma 4.7.3. *Let ω be an $\text{End}(E)$ -valued p -form, and η an arbitrary $\text{End}(E)$ -valued form. Then,*

$$d_D(\omega \wedge \eta) = d_D\omega \wedge \eta - (-1)^p\omega \wedge d_D\eta.$$

Proof. This is the Leibnitz rule for $\text{End}(E)$ -valued forms. The proof is similar to our construction of equation (21). \square

Lemma 4.7.4. *Let E be a vector G -bundle with connections D and D' , and let η be an $\Gamma(E)$ -valued differential form on M . If $A = D - D'$, then*

$$d_D\eta = d_{D'}\eta + A \wedge \eta.$$

Proof. This follows immediately from (19), and the fact that A is a $\text{End}(E)$ -valued 1-form (as we shall see later in Lemma 5.2.2). \square

In order to obtain a similar result for $\text{End}(E)$ -valued forms, we define the following:

Definition 4.7.5. Let ω and η be $\text{End}(E)$ -valued p and q -forms respectively. We define the *graded commutator* of ω and η as

$$[\omega, \eta] = \omega \wedge \eta - (-1)^{pq} \eta \wedge \omega.$$

This enables us to compare the exterior covariant derivatives of $\text{End}(E)$ -valued forms as follows:

Lemma 4.7.6. *Let E be a vector G -bundle with connections D and D' , and let ω be an $\text{End}(E)$ -valued form on M . If $A = D - D'$, then*

$$d_D \omega = d_{D'} \omega + [A, \omega].$$

Proof. When ω is a 1-form, this follows in a straightforward manner from equations (24) and (26), and A being a $\text{End}(E)$ -valued 1-form (Lemma 5.2.2). The result is valid for all $p \in \mathbb{N}$, and we refer the reader to a similar proof for the usual exterior derivative given in [15], p.36. \square

Remark 4.7.7. On any trivial bundle, we can always define the standard flat connection D^0 . Since, in this case, the exterior covariant derivative is just the ordinary exterior derivative, for $A = D - D^0$, we may write

$$\begin{aligned} d_D \eta &= d\eta + A \wedge \eta & \text{or} \\ d_D \omega &= d\omega + [A, \omega], \end{aligned}$$

as appropriate.

Chapter 5

Gauge Transformations

In Chapter 2, we saw that Maxwell's equations were invariant under certain transformations called "gauge transformations." On a fibre bundle, gauge transformations are just special fibre-preserving maps.

5.1 Gauge Transformations on Principal Bundles

Let P be a principal G -bundle over a manifold M . Then, a gauge transformation is defined as follows:

Definition 5.1.1. ([27],p.4) A *gauge transformation* on a principal G -bundle P over a manifold M is a G -equivariant bundle automorphism. That is, an automorphism $\gamma : P \rightarrow P$ such that $\gamma(ug) = \gamma(u)g$ for all $u \in P, g \in G$, and $\pi(\gamma(u)) = \pi(u)$.

Clearly, if $R_g : P \rightarrow P$ denotes the right action of G on P , then, for all $g \in G$:

$$\gamma R_g = R_g \gamma. \tag{27}$$

Lemma 5.1.2. ([15],p.81) If γ is a gauge transformation of P , and ω is a connection form on P , then $\gamma^*\omega$ is also a connection form.

Proof. It suffices to show that $\gamma^*\omega$ satisfies properties (CF1) and (CF2). For $A \in \mathfrak{g}$, let A_u^* denote the fundamental vector field generated by A at $u \in P$. By (27),

$\gamma_*(A_u^*) = A_{\gamma(u)}^*$. Therefore,

$$\begin{aligned} (\gamma^*\omega)(u; A_u^*) &= \omega(\gamma(u); \gamma_*(A_u^*)) \\ &= \omega(\gamma(u), A_{\gamma(u)}^*) \\ &= A, \end{aligned}$$

and (CF1) is satisfied. Moreover, (27) implies that $\gamma^*\omega$ is Ad-invariant, and property (CF2) is satisfied. Therefore, $\gamma^*\omega$ is also a connection form on P . \square

Lemma 5.1.3. *If ω and ω' are any two connection forms on P , then $\alpha = \omega - \omega'$ is a \mathfrak{g} -valued 1-form on P which is zero on the vertical subspace, and which satisfies $(R_a)^*\alpha = \text{Ad}(a^{-1})\alpha$ for all $a \in G$.*

Proof. For $u \in P$, let $X, X' \in T_uP$. If $\pi_*X = \pi_*X'$, then $(\omega - \omega')(u; X - X') = 0$ since $X - X' = A_u^*$ for some $A \in \mathfrak{g}$. The Ad-invariance follows from the invariance of both ω and ω' . \square

The following describes gauge transformations for trivial principal G -bundles:

Lemma 5.1.4. *Suppose that P is a trivial principal G -bundle, and $\gamma : P \rightarrow P$ is a gauge transformation. Then, there exists a map $\tau : M \rightarrow G$ such that*

$$\gamma(x, a) = (x, \tau(x)a), \quad \forall (x, a) \in P.$$

Conversely, given a map $\tau : M \rightarrow G$, the above equation defines a gauge transformation on a trivial principal G -bundle.

Proof. Suppose that $\gamma : P \rightarrow P$ is a gauge transformation. Then, we know that $\gamma(x, a) = (x, \sigma(x, a))$ for some map $\sigma : M \times G \rightarrow G$. Now, if $a, g \in G$, we must have

$$\begin{aligned} \gamma(x, a)g &= (x, \sigma(x, a))g \\ &= (x, \sigma(x, a)g) \end{aligned}$$

while

$$\gamma(x, ag) = (x, \sigma(x, ag)).$$

Therefore,

$$\sigma(x, ag) = \sigma(x, a)g.$$

In particular,

$$\sigma(x, a) = \sigma(x, e)a.$$

Therefore, we define $\tau : M \rightarrow G$ by

$$x \mapsto \sigma(x, e),$$

which satisfies the map.

Now, given $\tau : M \rightarrow G$, we define $\gamma : P \rightarrow P$ by $\gamma(x, a) = (x, \tau(x)a)$. This is a gauge transformation because left and right multiplication on G commute. \square

Remark 5.1.5. Since all principal G -bundles are locally trivial, then all gauge transformations can be thought of, locally, as left-multiplication on G by some group element determined by $x \in M$.

Example 5.1.6. Let $P = M \times G$ be a trivial principal G -bundle with connection form ω , and let $f : P \rightarrow P$ be a gauge transformation of P . Since P is trivial, the gauge transformation can be written $f(x, a) = (x, \tau(x)a)$ for some map $\tau : M \rightarrow G$. Let $X + Y \in T_u P = T_x M \oplus T_a G$ for $u = (x, a)$. Then, short computations show that

$$\begin{aligned} f_* X &= X + (R_a)_* d\tau(X) \quad \text{and} \\ f_* Y &= (L_{\tau(x)})_* Y. \end{aligned}$$

Thus, using the linearity of $f^*\omega$,

$$(f^*\omega)(x, a; X + Y) = \omega(x, \tau(x)a; X + (R_a)_* d\tau(X) + (L_{\tau(x)})_* Y).$$

Hence, if μ is the \mathfrak{g} -valued 1-form on M associated with ω , then

$$\begin{aligned} (f^*\omega)(x, a; X + Y) &= Ad(\tau(x)a)^{-1}\mu(X) + (L_{\tau(x)a}^{-1})_*(R_a)_* d\tau(X) + (L_{(\tau(x)a)^{-1}})_*(L_{\tau(x)})_* Y \\ &= Ad(a^{-1})Ad(\tau(x)^{-1})\mu(X) + Ad(a^{-1})(L_{\tau(x)^{-1}})_* d\tau(X) + (L_{a^{-1}})_* Y \\ &= Ad(a^{-1})[Ad(\tau(x)^{-1})\mu(X) + (L_{\tau(x)^{-1}})_* d\tau(X)] + (L_{a^{-1}})_* Y \end{aligned}$$

This implies that the \mathfrak{g} -valued 1-form on M associated with $f^*\omega$ is

$$\bar{\mu} = Ad(\tau(x)^{-1})\mu + (L_{\tau(x)^{-1}})_*d\tau. \quad (28)$$

Example 5.1.7. Let $P = \mathbb{R}^4 \times S^1$ be a trivial principal S^1 -bundle with connection form $\omega = d\hat{\theta}$, and let $f : P \rightarrow P$ be a gauge transformation of P . Since \mathbb{R}^4 is simply connected, then $f(x, z) = (x, ze^{i\phi(x)})$ for some $\phi : \mathbb{R}^4 \rightarrow \mathbb{R}$. Similar calculations as above yield

$$\begin{aligned} f_*\left(\frac{\partial}{\partial x^j}\right)_{(x,z)} &= \left(\frac{\partial}{\partial x^j}\right)_{f(x,z)} + \frac{\partial\phi}{\partial x^j}\left(\frac{\partial}{\partial\theta}\right)_{f(x,z)}, \quad \text{and} \\ f_*\left(\frac{\partial}{\partial\theta}\right)_{(x,z)} &= \left(\frac{\partial}{\partial\theta}\right)_{f(x,z)} \end{aligned}$$

Since $\omega = d\hat{\theta}$, then

$$\begin{aligned} (f^*\omega)\left(\frac{\partial}{\partial x^j}\right)_{(x,z)} &= d\phi\left(\frac{\partial}{\partial x^j}\right), \quad \text{and} \\ (f^*\omega)\left(\frac{\partial}{\partial\theta}\right)_{(x,z)} &= 1. \end{aligned}$$

In this case, it is clear that we can write $f^*\omega = \omega + d\phi$. This is similar to our gauge transformation of the vector potential for electromagnetism when $H^2(M) = 0$.

5.2 Gauge Transformations on Vector G -Bundles

Suppose that (E, π_E, M, V) is a vector G -bundle associated with a principal G -bundle P via the faithful representation $\rho : G \rightarrow GL(V)$, and let $\gamma : P \rightarrow P$ be a gauge transformation on P . Note that, for $u \in P, \xi \in V$ and $g \in G$, where $u\xi = (ug)(\rho(g^{-1})\xi) \in E$, we must have

$$\begin{aligned} \gamma(ug)\rho(g^{-1})\xi &= (\gamma(u)g)\rho(g^{-1})\xi \\ &= \gamma(u)(\rho(g)\rho(g^{-1})\xi) \\ &= \gamma(u)\xi. \end{aligned} \quad (29)$$

Definition 5.2.1. Let $\gamma : P \rightarrow P$ be a gauge transformation on a principal G -bundle. The map $\tilde{\gamma} : E \rightarrow E$ defined by

$$\tilde{\gamma}(u\xi) = \gamma(u)\xi.$$

is a *gauge transformation* of E . By (29), the map is well-defined. Note also that $\pi_E \gamma = \pi_E$.

Just as a gauge transformation on a principal G -bundle changes the connection forms, on a vector G -bundle, a gauge transformation similarly changes the covariant derivatives.

Recall from Section 4.1 and definition 4.4.1 that, in a trivial bundle, we can write the covariant derivative of a section σ of E in the direction $X \in \text{Vect}(M)$ as

$$(D_X \sigma) = X(\sigma) + A(X)(\sigma).$$

where the vector potential A as an $\text{End}(E)$ -valued 1-form..

In a non-trivial vector G -bundle E , we can only define the vector potential over a local trivialization U . There is no way to extend it to a globally defined $\text{End}(E)$ -valued 1-form. Instead, we start with a fixed covariant derivative D on a E , and add to it an $\text{End}(E)$ -valued 1-form A . Since their sum, $D + A$, is \mathbb{R} -linear, $C^\infty(M)$ -linear in X , and satisfies the Leibnitz rule, it is a covariant derivative on E . Indeed, all covariant derivatives differ by an $\text{End}(E)$ -valued 1-form A :

Lemma 5.2.2. *Let E be a vector bundle with covariant derivatives D and \bar{D} . Then, there exists a $\text{End}(E)$ -valued 1-form A such that $D - \bar{D} = A$.*

Proof. It suffices to show that A is $C^\infty(M)$ -linear in X . Using the definition,

$$\begin{aligned} A(fX)\sigma &= D_{fX}\sigma - D'_{fX}\sigma \\ &= X(f)\sigma + fD_X\sigma - X(f)\sigma - fD'_X\sigma \\ &= f(D_X\sigma - D'_X\sigma) \\ &= fA(X)(\sigma), \end{aligned}$$

as desired. Since D and \bar{D} are defined over all of M , then so is A . Therefore, we can think of A as being $\text{End}(E)$ -valued 1-form on M . \square

A gauge transformation $\tilde{\gamma}$ of E also gives rise to a new covariant derivative.

Theorem 5.2.3. *Let (E, π, M, V) be a vector G -bundle with covariant derivative D associated with the connection form ω on the principal G -bundle P . Then, for a gauge transformation $\gamma : P \rightarrow P$, the covariant derivative \bar{D} associated with the connection form $\gamma^*\omega$ satisfies*

$$\bar{D}_X = \tilde{\gamma}^{-1} D_X \tilde{\gamma}.$$

Moreover, the local vector potentials transform according to $\bar{A} = \tilde{\gamma}^{-1} A \tilde{\gamma} + \tilde{\gamma}^{-1} d\tilde{\gamma}$.

Proof. We proceed in two steps. It is clear that \bar{D} defined by the right-hand side above is a connection, so we first show that its vector potential \bar{A} has the desired form. We then show that \bar{A} is the local vector potential associated to $f^*\omega$.

In a local trivialization,

$$\begin{aligned} \bar{D}_X(v) &= \tilde{\gamma}^{-1}(X(\tilde{\gamma}v) + A(X)\tilde{\gamma}v) \\ &= X(v) + \tilde{\gamma}^{-1}(X(\tilde{\gamma})v + \tilde{\gamma}(X)v) + \tilde{\gamma}^{-1}A(X)\tilde{\gamma}v \\ &= X(v) + \tilde{\gamma}^{-1}d\tilde{\gamma}(X)v + \tilde{\gamma}^{-1}A(X)\tilde{\gamma}v \\ &= X(v) + \bar{A}(X)v, \end{aligned}$$

as required.

Now, it remains to show that the covariant derivative \bar{D} arises from the connection form $f^*\omega$. Since we're working in a trivialization, there exists a \mathfrak{g} -valued 1-form μ on U associated with the connection form ω , and $\tau : U \rightarrow G$ such that $\gamma(x, a) = (x, \tau(x)a)$ on $U \times G$. Since $\pi_E^{-1}(U)$ is isomorphic to the trivial bundle $U \times V$, the gauge transformation $\tilde{\gamma} : U \times V \rightarrow U \times V$ is $\tilde{\gamma}(x, \xi) = (x, \rho(\gamma(x))\xi)$. If we suppress the first component, then we may write $\tilde{\gamma} = \rho \circ \gamma$.

Recall from 4.4.2 that locally, we have $A = d\rho \circ \mu$. From equation (28), the \mathfrak{g} -valued 1-form $\bar{\mu}$ on U associated with $f^*\omega$ has the form

$$\bar{\mu} = Ad(\tau(x)^{-1})\mu + (L_{\tau(x)^{-1}})_*d\tau.$$

Hence, it suffices to show that $\bar{A} = d\rho \circ \bar{\mu}$. But, $\tilde{\gamma}(x)v = \rho(\tau(x))v$ so

$$d\tilde{\gamma}(X) = d\rho_{\tau(x)} \circ d\tau(X)$$

Since ρ is a homomorphism, $\rho_a = (L_{\rho(a)})_* d\rho_e (L_{a^{-1}})_*$, so

$$\begin{aligned} \tilde{\gamma}(x)^{-1} d\tilde{\gamma}(X) &= \tilde{\gamma}(x)^{-1} d\rho_{\tau(x)} \circ d\tau(X) \\ &= \tilde{\gamma}(x)^{-1} (L_{\rho(\tau(x))})_* \circ d\rho_e \circ (L_{\tau(x)^{-1}})_* \circ d\tau(X) \\ &= \tilde{\gamma}(x)^{-1} (L_{\tilde{\gamma}(x)})_* \circ d\rho \circ (L_{\tau(x)^{-1}})_* \circ d\tau(X) \\ &= d\rho \circ (L_{\tau(x)^{-1}})_* \circ d\tau(X). \end{aligned}$$

On the other hand, again because ρ is a homomorphism, if $Z \in \mathfrak{g}$,

$$d\rho(\text{Ad}(\tau(x)^{-1})Z) = \tilde{\gamma}(x)^{-1} d\rho(Z) \tilde{\gamma}(x)$$

Thus,

$$\begin{aligned} d\rho \circ \bar{\mu}(X) &= d\rho \circ \left(\text{Ad}(\tau(x)^{-1})\mu(X) + (L_{\tau(x)^{-1}})_* d\tau(X) \right) \\ &= \tilde{\gamma}(x)^{-1} d\rho(\mu(X)) \tilde{\gamma}(x) + d\rho \circ (L_{\tau(x)^{-1}})_* d\tau(X) \\ &= A(X) + \tilde{\gamma}(x)^{-1} d\tilde{\gamma}(X) \\ &= \bar{A}(X), \end{aligned}$$

as required. □

Definition 5.2.4. We call two covariant derivatives D and \bar{D} on a vector G -bundle *gauge equivalent* if there exists a gauge transformation $\tilde{\gamma}$ on E for which $\bar{D} = \tilde{\gamma}^{-1} D \tilde{\gamma}$. This allows one to consider the space of all connections modulo gauge transformations.

Suppose that Ω and $\bar{\Omega}$ are the curvature forms for gauge equivalent connections D and \bar{D} . Then, if $\tilde{\gamma}$ is the associated gauge transformation, straightforward calculation shows that

$$\bar{\Omega}(X, Y) = \tilde{\gamma}^{-1} \Omega(X, Y) \tilde{\gamma}.$$

Chapter 6

Gauge Theory

6.1 Bundle Formulation of Maxwell's Equations

As alluded to in the previous chapter, we can formulate electromagnetism as a connection on a vector $U(1)$ -bundle with fibre \mathbb{C} . In the case of a trivial bundle, the electromagnetic field F arises naturally as the curvature form on the fibre.

Let $M = \mathbb{R} \times S$ be a smooth, orientable, semi-Riemannian 4-manifold where we think of \mathbb{R} as time, and S as space. Let P be a trivial principal $U(1)$ -bundle over M , and let $\rho : U(1) \rightarrow \text{GL}(\mathbb{C})$ be the inclusion map. By Proposition 3.3.6, the associated complex vector bundle (E, π, \mathbb{C}, P) is a vector $U(1)$ -bundle. We begin by showing that the curvature form Ω “looks” like the unified electromagnetic field F from Chapter 2.

Recall from example 4.6.8 that the curvature of a trivial vector G -bundle with vector potential A can be written

$$\Omega_E(X, Y) = 2dA(X, Y) + [A(X), A(Y)].$$

By example 4.4.2, $A = d\rho \circ \mu$. Since ρ is a homomorphism of Lie group, then $d\rho$ is a homomorphism of Lie algebras, and $[A(X), A(Y)] = d\rho([\mu(X), \mu(Y)])$. However, since $\mathfrak{u}(1)$ is abelian, then $[\mu(X), \mu(Y)] = 0$. Therefore,

$$\Omega_E = 2dA.$$

Now, let \tilde{A} be the vector potential associated with the unified electromagnetic field F in Chapter 2. Then,

$$F = d\tilde{A}.$$

Comparing this with the result above, we see that F and Ω have exactly the same form (the scalar term can be absorbed in A .) In fact, this relationship is the reason that the form A associated with the connection inherited the name “vector potential” in the first place. Given this similarity, it is possible to *define* the electromagnetic field F on M to be the curvature form of a vector $U(1)$ -bundle E .

Since the electromagnetic field F is now the curvature form of a connection form on E , it satisfies the Bianchi identity, $d_D F = 0$. By analogy, this takes the place of Maxwell’s first equation ($dF = 0$).

Indeed, suppose that $E = \mathbb{R}^4 \times \mathbb{C}$ is the trivial vector $U(1)$ -bundle with the standard flat connection D^0 . Recall from example 4.7.2 that $d_{D^0} = d$, the ordinary exterior derivative on forms. Then, the Bianchi identity reduces to Maxwell’s first equation, $dF = 0$. Furthermore, we can rewrite Maxwell’s second equation as

$$\star d_{D^0} \star F = J.$$

Using the same analogy, this suggests that we define a new equation by

$$\star d_D \star F = J.$$

Indeed, this is known as the *Yang-Mills equation*, and will be discussed in more detail in Section 6.3.

In addition to satisfying Maxwell’s equations, we can also recover the gauge transformations of F . Let $\tilde{\gamma}$ be a gauge transformation applied to the vector potential A as defined in section 5.1. From previous results, we know that

$$\tilde{A} = \tilde{\gamma}^{-1} A \tilde{\gamma} + \tilde{\gamma}^{-1} d\tilde{\gamma}.$$

Since $U(1)$ is abelian, then

$$\tilde{A} = A + \tilde{\gamma}^{-1} d\tilde{\gamma}.$$

Since E is a trivial bundle, then $\tilde{\gamma}(x, z) = (x, \tau(x)z)$ where $\tau : \mathbb{R}^4 \rightarrow S^1$. Since \mathbb{R}^4 is simply connected, τ can be lifted to a function $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ such that $\tilde{\gamma}(x, z) = (x, ze^{f(x)})$ where f is an imaginary-valued function. Consequently,

$$\tilde{\gamma}^{-1}d\tilde{\gamma} = df,$$

and,

$$\bar{A} = A + df.$$

Therefore, a gauge transformation $\tilde{\gamma}$ on a trivial vector $U(1)$ -bundle with connection D amounts to changing the associated vector potential by an exact form df related to $\tilde{\gamma}$. This is similar to our gauge transformations from Section 2.1 except that f is now an imaginary-valued function. Why is this?

Recall from Example 5.1.7 that for a trivial principal S^1 -bundle $P = \mathbb{R}^4 \times S^1$, and gauge transformation $\gamma : P \rightarrow P$ where $\gamma(x, a) = (x, ae^{i\phi(x)})$ for some $\phi : \mathbb{R}^4 \rightarrow \mathbb{R}$, the new connection form is $f^*\omega = \omega + d\phi$. This is much like above, except there is no value of “ i ” introduced.

Now, let $E = M \times \mathbb{C}$ be a complex-valued trivial vector S^1 -bundle over M with representation $\rho : S^1 \hookrightarrow \text{GL}(\mathbb{C})$. Let $g : M \rightarrow \mathbb{C}$ be a complex-valued function corresponding with the section $x \mapsto (x, g(x))$. Let $\tilde{\gamma} : E \rightarrow E$ be a gauge transformation given by $\tilde{\gamma}(x, z) = (x, \tau(x)z)$ for some $\tau : M \rightarrow S^1$. (Really, the map should be $\rho \circ \tau$ but ρ is simply the inclusion map.) If M is simply connected, we can lift τ such that $\tau(x) = e^{it(x)}$ for some map $t : M \rightarrow \mathbb{R}$, and we may write $\tilde{\gamma}(x, z) = (x, ze^{it(x)})$.

Let D be a connection on E associated with the connection form ω on P . Then, for any $X \in \text{Vect}(M)$,

$$D_X g(x) = X(g)(x) + A(x; X)g(x),$$

for some vector potential A . However, from Example 4.4.2, we know that $A = d\rho \circ \mu$ where μ is the $\mathfrak{s}^1 \cong \mathbb{R}$ -valued 1-form associated with ω . But, the map $d\rho = \rho_*(1) : \mathbb{R} \rightarrow \mathfrak{gl}(\mathbb{C})$ is the map sending $s \in \mathbb{R}$ to the map in $\mathfrak{gl}(\mathbb{C})$ given by $w \mapsto isw$. That is, s maps to multiplication by is . With this in mind, if $\tilde{D} = \tilde{\gamma}^{-1}D\tilde{\gamma}$, then

$$\begin{aligned} \tilde{A}(x; X)g(x) &= A(x; X)g(x) + \tilde{\gamma}^{-1}d\tilde{\gamma}g(x) \\ &= A(x; X) + idt(x; X)g(x), \end{aligned}$$

as expected. The factor of “ i ” arises naturally from the derivative of the representation map in a straightforward way.

6.2 Bohm-Aharonov Effect

For a long time, the idea of the “vector potential” in electromagnetism was considered to be a mathematical tool without any physical significance. In the classical equations of motion, only the fields themselves appeared, so the underlying vector potentials were not considered to represent anything physical. However, when quantum theory was developed, the vector potential persisted in the final equations. In 1959, Y. Aharonov and David Bohm published “The Significance of Electromagnetic Potentials in Quantum Theory” [2] in which the vector potential was found to have unexpected physical consequences. In their words:

The essential result ... is that in quantum theory, an electron (for example) can be influenced by the potentials even if all the field regions are excluded from it. In other words, in a field-free multiply-connected region of space, the physical properties of the system still depend on the potentials. - [2], 490.

Bohm and Aharonov cleverly constructed a set of thought experiments that exploited the local nature of field theory. They imagined a situation in which the magnetic field was locally zero, while the vector potential was not. A particle traveling through this space would not be affected by the field, but could be affected by the underlying potential.

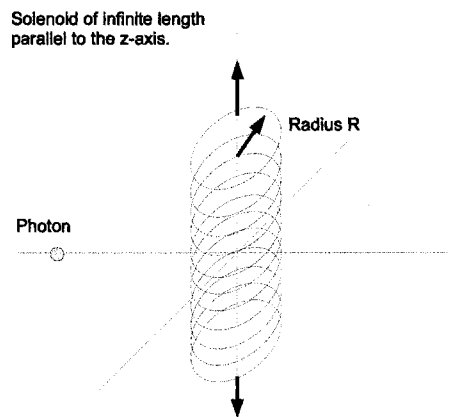
Remark 6.2.1. A proper understanding of the Bohm-Aharonov effect depends on some knowledge of quantum mechanics. We will avoid a detailed discussion of this area, and will refer the interested reader to more comprehensive texts, as necessary.

Consider a particle traveling in space-time between two points. Unlike classical physics, in quantum theory, a particle does not follow a *single* path between two points. Instead, each path has an associated complex phase related to the *probability*

of the particle following that path. Bohm and Aharonov considered the situation of a particle moving in a space-time which was not simply connected.

One of the examples developed by Bohm and Aharonov was a tightly wound solenoid of infinite length stretching along the z -axis that was screened from incoming electrons. That is, electrons could go around the solenoid, but not through it. From Ampere's law, we know that the magnetic field within the solenoid is constant, while outside of the solenoid, the field would be zero.

Figure 2: Bohm-Aharonov Effect - Setup



Mathematically, this situation is analogous to $M = \mathbb{R}^3 \setminus \{z\text{-axis}\}$. This space has $H^2(M) = 0$ and $H^1(M) \cong \mathbb{R}$. By construction, $F = 0$ on M , so the vector potential A such that $F = dA$ is closed, but it need not be zero.

Using cylindrical coordinates (r, θ, z) on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$, we define a 1-form

$$\widehat{d\theta} = \frac{xdy - ydx}{x^2 + y^2},$$

where dx, dy are the standard 1-forms on \mathbb{R}^3 . As in example 4.5.4, a simple calculation shows that $\widehat{d\theta}$ is closed, but not exact.

We will now introduce some concepts from quantum mechanics. The following results are taken from [5], p.137-139. Let $\lambda : [0, T] \rightarrow M$ be a smooth path in M , and let $\psi(a; 0)$ be the wavefunction of an electron at the point $a \in M$ at time $t = 0$. The solution of Schrödinger's wave equation describes the time and space evolution of the wavefunction. At a point $b \in M$ at time T , the wavefunction $\psi(b; T)$ is thought of as the sum of all contributions stemming from the evolution of $\psi(a)$ along all paths λ satisfying $\lambda(0) = a$ and $\lambda(T) = b$, each with a phase factor depending on the action along the path. That is, $\psi(b)$ is an *integral* over the space of all paths λ . However, defining a measure on this space is tremendously difficult. As is often done in the physics literature ([5], p.137) we will sidestep the issue by looking at the contributions on a path-by-path basis, and then use symmetry to argue for the stated solution.

The following results are taken from [25], p.436-438. The Schrödinger wave equation for an electron in the presence of a vector potential A and scalar potential is ([25], p.436):

$$\frac{1}{2m}(-i\hbar\nabla - eA)^2\psi + V\psi = E\psi,$$

where m is the mass of the electron, e is the electric charge, \hbar is Planck's constant divided by 2π , and $V = e\phi$. In a region where the magnetic field is zero ($F = 0$), along a curve $\lambda : [0, T] \rightarrow M$, a solution is

$$\psi_\lambda(b; T) = \psi^0(b; T) \exp\left[\frac{ie}{\hbar} \int_\lambda A\right],$$

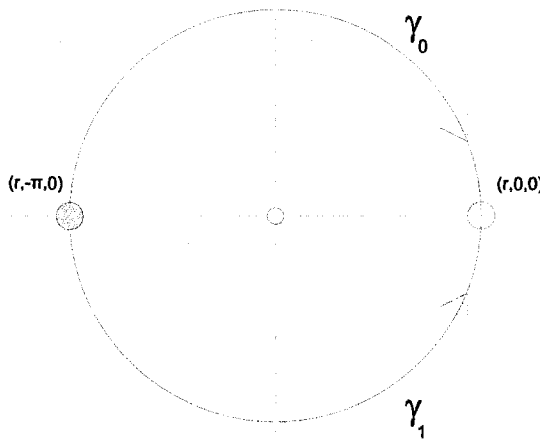
where $\psi^0(b; T)$ satisfies Schrödinger's equation with the same value of ϕ but with $A = 0$, and λ satisfies $\lambda(0) = a$ and $\lambda(T) = b$. Then, the "total" wavefunction at the point b at time T will be the sum of contributions along all such paths λ . We now examine the symmetry arguments alluded to above.

Consider an electron at the point $(r, -\pi, 0)$ in M , and examine all possible paths to the point $(r, 0, 0)$. We can begin by considering two semi-circular paths, γ_0 , and γ_1 , of fixed radius r parameterized solely as functions of θ and defined as follows:

$\gamma_0 : [0, \pi] \rightarrow \mathbb{R}^3$ by $\gamma_0(t) = (r, \pi - t, 0)$ and $\gamma_1 : [0, \pi] \rightarrow \mathbb{R}^3$ by $\gamma_1(t) = (r, \pi + t, 0)$. This situation is summarized in Figure 3.

Figure 3: Bohm-Aharonov Effect

X-Y Plane \ {origin}



Suppose that λ is a second path in M such that the loop $\lambda * \overline{\gamma_0}$ (consisting of doing λ followed by the “inverse” path $\overline{\gamma_0}$) does not encircle the z-axis. Let Σ be the surface bounded by $\lambda * \overline{\gamma_0}$. By Stoke’s theorem,

$$\oint_{\lambda * \overline{\gamma_0}} A = \int_{\Sigma} dA = 0.$$

Thus,

$$\int_{\lambda} A = \int_{\gamma_0} A.$$

Now, we will restrict ourselves to a particular choice of A for which interesting results arise. Let $A = \frac{\hbar}{2e} \widehat{d\theta}$. The path integrals become

$$\begin{aligned} \int_{\gamma_0} A &= \int_{\pi}^0 \frac{\hbar}{2e} \widehat{d\theta} \\ &= -\frac{\hbar\pi}{2e}, \end{aligned}$$

and,

$$\begin{aligned} \int_{\gamma_1} A &= \int_{\pi}^{2\pi} \frac{\hbar}{2e} \widehat{d\theta} \\ &= \frac{\hbar\pi}{2e}. \end{aligned}$$

By symmetry, for any path λ such that $\lambda * \overline{\gamma_0}$ does not encircle the z -axis, the “mirror” path λ_{mirror} taken to be the mirror image through the XZ -plane will satisfy the same condition with $\overline{\gamma_1}$. Consequently, the integrals of λ and λ_{mirror} will also be $-\frac{\hbar\pi}{2e}$ and $+\frac{\hbar\pi}{2e}$ respectively.

If $\lambda * \overline{\gamma_0}$ does loop around the z -axis (say, in the clockwise direction n -times for some $n \in \mathbb{N}$), similar calculations show that

$$\int_{\lambda} A = -\frac{(2n+1)\hbar\pi}{2e},$$

and, for the mirror path,

$$\int_{\lambda_{\text{mirror}}} A = +\frac{(2n+1)\hbar\pi}{2e}.$$

Thus, $e^{\frac{ie}{\hbar} \int_{\lambda} A} = i$ for λ , and $-i$ for λ_{mirror} .

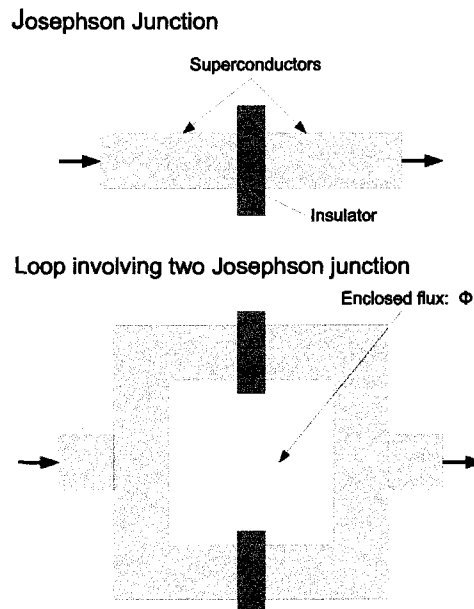
By symmetry, every path has a mirror path for which the integral of A has equal and opposite value. Consequently, we (naively) assume that we can sum each term with its “mirror” term¹, and, by destructive interference, the wavefunction of the electron at $(r, 0, 0)$ must be zero. That is, the probability that the electron can be found at the point $(r, 0, 0)$ is precisely zero.

¹Summing each term with the “mirror” term and hoping that it cancels out is analogous to summing $1 - 1 + 1 - 1 + 1 - 1 \dots$ infinitely. You can argue convincingly that this sum is any finite number. Fortunately, the reasoning is borne out by physical experiment, so one ignores this complication, and plunges ahead.

Bohm and Aharonov came to similar conclusions concerning the solenoid. Namely, that a beam of coherent electrons recombined after travelling along multiple paths suffered from constructive or destructive interference depending on the values of the vector potentials along the paths.

This effect has been used to create “Superconducting Quantum Interference Devices” (SQUIDs). The Bohm-Aharonov effect says that the flux enclosed by a superconducting ring is quantized ([25],p.438). By varying the flux, we vary the current able to flow through a superconducting ring. The SQUID exploits this effect to make extremely sensitive measurements of the magnetic flux using quantum interference in loops built from “Josephson junctions.”

Figure 4: Josephson junction and loop made up of two Josephson junction



The following is taken from [25], p.143-145. In a superconductor, at low enough temperatures, the ground state is one in which the electrons “pair up” to form what are known as *Cooper pairs*. In this state, there is no scattering of electrons, and long-range coherence of the wavefunction is maintained. The current flow in the

superconductor is described by solutions of Schrödinger's equation for Cooper pairs, without worrying about other, complicating factors.

A Josephson junction consists of a thin insulator sandwiched between two superconductors. Let ψ_1 and ψ_2 represent the probability amplitudes for the electron pairs on either side of the junction. If n_1 and n_2 are the pair densities on the two sides, then we may write: ([25],p.145)

$$\psi_1 = \sqrt{n_1}e^{i\theta_1}; \quad \psi_2 = \sqrt{n_2}e^{i\theta_2},$$

for some phases θ_1 and θ_2 . We define the *phase difference* δ for the junction by

$$\delta = \theta_2 - \theta_1.$$

Through quantum tunneling, current stills flow between the two superconductors, and has value

$$J = J_0 \sin \delta,$$

where J_0 is dependent on the tunneling probability of the electron pairs through the junction, and δ is the phase difference for the junction. ([25], p.439)

If we place two Josephson junctions in parallel, their loop can contain a magnetic flux Φ . In the presence of such a flux, the phase difference between the two Josephson junctions becomes quantized ([25],p.438). That is, if δ_a and δ_b are the phase differences at the two junctions respectively, and Φ is the total magnetic flux contained in the loop, then

$$\delta_b - \delta_a = \frac{2e\Phi}{\hbar}.$$

Suppose that the two junctions are identical. In the absence of a flux, let δ_0 denote the phase difference for each junction. In the presence of a flux, we may rewrite the phase differences δ_a and δ_b with respect to δ_0 as

$$\begin{aligned} \delta_b &= \delta_0 + \frac{e\Phi}{\hbar}, \quad \text{and} \\ \delta_a &= \delta_0 - \frac{e\Phi}{\hbar}. \end{aligned} \tag{30}$$

Since the total current through the loop is the sum of the current through the individual junctions, then

$$\begin{aligned} J_{\text{total}} &= J_a + J_b \\ &= J_0 \left[\sin\left(\delta_0 + \frac{e\Phi}{\hbar}\right) + \sin\left(\delta_0 - \frac{e\Phi}{\hbar}\right) \right] \\ &= 2J_0 \sin \delta_0 \cos \frac{e\Phi}{\hbar}. \end{aligned}$$

Hence, the current varies with respect to the magnetic field Φ , reaching a maximum for $\frac{e\Phi}{\hbar} = n\pi$ whenever $n \in \mathbb{N}$. ([25],p.439-440)

A SQUID magnetometer encloses the magnetic flux to be measured in the loop made up of two Josephson junctions. A constant current is applied to the loop, and the voltage difference between the two sides of the loop is measured. The voltage will oscillate very precisely with small changes in the magnetic flux allowing for extremely sensitive measurement. ([25],p.441)

6.3 The Yang-Mills Equation

Having already motivated the form of the Yang-Mills equation in Section 6.1, we will now take some time to define it explicitly.

Let E be a vector G -bundle over an oriented, semi-Riemannian n -manifold M . Let d_D be the exterior covariant derivative on E associated with some covariant derivative D , and let Ω be the associated curvature form.

Define the ‘‘Hodge Star Operator’’ \star acting on a $\text{End}(E)$ -valued differential form to be the unique $C^\infty(M)$ -linear operator such that, for any section ψ of $\text{End}(E)$, and differential form $\eta \in \Omega(M)$,

$$\star(\psi \otimes \eta) = \psi \otimes \star\eta,$$

where $\star\eta$ represents the Hodge Star Operator on ordinary differential forms.

Definition 6.3.1. For any $\text{End}(E)$ -valued 1-form J over M called the (*Yang-Mills current*), the *Yang-Mills equation* is

$$\star d_D \star \Omega = J.$$

As previously discussed, Maxwell's second equation in differential form is the inspiration for the Yang-Mills equation. When coupled with the Bianchi identity, the pair are simply Maxwell's equations with the exterior covariant derivative in place of the exterior derivative, and the electromagnetic field F replaced by the curvature. As discussed previously, for a trivial vector $U(1)$ -bundle with the standard flat connection, the Bianchi identity and the Yang-Mills equation reduce to Maxwell's equations.

The Yang-Mills equation has the important advantage of being *gauge invariant*. That is, the action of any gauge transformation on Ω and J leaves the Yang-Mills equation in the same *form*. We already know that for a gauge transformation $\tilde{\gamma}$, the curvature form transforms as

$$\bar{\Omega} = \tilde{\gamma}^{-1}\Omega\tilde{\gamma}.$$

Let $\bar{J} = \tilde{\gamma}^{-1}J\tilde{\gamma}$, and let \bar{d}_D denote the gauge equivalent exterior covariant derivative of d_D via $\tilde{\gamma}$. Then, as we show below, the Yang-Mills equation becomes

$$\star\bar{d}_D\star\bar{\Omega} = \bar{J}.$$

Therefore, given any pair (Ω, J) which satisfy the Yang-Mills equation, and a gauge transformation $\tilde{\gamma}$, then $(\bar{\Omega}, \bar{J})$ will also be a solution.

In order to see the gauge invariance, it suffices to note that the Hodge star operator commutes with the gauge transformation. Then, the left-hand side of the Yang-Mills equation becomes

$$\begin{aligned} \star(\tilde{\gamma}^{-1}d_D\tilde{\gamma})\star\tilde{\gamma}^{-1}\Omega\tilde{\gamma} &= \tilde{\gamma}^{-1}(\star d_D(\tilde{\gamma}\tilde{\gamma}^{-1}))\star\Omega\tilde{\gamma} \\ &= \tilde{\gamma}^{-1}(\star d_D\star\Omega)\tilde{\gamma} \\ &= \bar{J}. \end{aligned}$$

This development of the Yang-Mills equation was suggested by previous knowledge of Maxwell's equations. However, there is an alternate means of developing the Yang-Mills equation as the consequence of the "action principle." This will be the subject of the next chapter.

Chapter 7

The Action Principle

7.1 The Action Principle

In discussing electromagnetism, we started with Maxwell's equations and deduced the underlying gauge symmetry of the Bianchi identity and the Yang-Mills equation. However, much of modern physics starts with a Lagrangian for a system, and derives the equations of motion as a consequence of the "action principle." Our aim here is to see how the Yang-Mills equations can be obtained from a suitable gauge invariant action.

The following development is taken from [10]. We begin by considering functionals \mathfrak{F} of the form

$$\mathfrak{F}(u) = \int_{\Delta} F(x, u(x), Du(x)) dx,$$

where Δ is some domain of integration, and $u : \bar{\Delta} \rightarrow \mathbb{R}^n$ is smooth. For a suitable choice of F (see [10], p.11-13), the integral for \mathfrak{F} is defined for some neighbourhood of Δ , and we can define a function

$$\Phi(\epsilon) = \mathfrak{F}(u + \epsilon\phi)$$

which is smooth over $(-\epsilon_0, \epsilon)$ for any smooth $\phi : \bar{\Delta} \rightarrow \mathbb{R}^n$ and some $\epsilon < \epsilon_0$ where ϵ_0 arises from our conditions on F .

Definition 7.1.1. ([10],p.12) The *first variation* $\delta F(u, \phi)$ of \mathfrak{F} at u in the direction ϕ is well-defined by

$$\delta \mathfrak{F}(u, \phi) = \Phi'(0).$$

A straightforward computation yields

$$\delta \mathfrak{F}(u, \phi) = \int_{\Delta} \sum_{i, \alpha=1}^n \left(\frac{\partial F}{\partial u^i}(x, u, Du) \phi^i + \frac{\partial F}{\partial p_{\alpha}^i}(x, u, Du) \phi_{x^{\alpha}}^i \right) dx.$$

This suggests that we introduce an expression δF defined by

$$\delta F(u, \phi)(x) = F_u(x, u(x), Du(x))\phi(x) + F_p(x, u(x), Du(x))D\phi(x),$$

where F_u and F_p are the appropriate derivatives of F with respect to components of u and p , and we use the Einstein summation convention to simplify the notation ([10],p.13). We call δF the *first variation of the Lagrangian F at u in the direction ϕ* , and write

$$\delta \mathfrak{F}(u, \phi) = \int_{\Delta} \delta F(u, \phi) dx.$$

We call \mathfrak{F} the *action* of the system described by the Lagrangian F . The *action principle* says that the path u followed by the system is a critical point of the action \mathfrak{F} . ([5],p.272). That is,

$$\delta \mathfrak{F}(u, \phi) = 0,$$

for all smooth $\phi : \bar{\Delta} \rightarrow \mathbb{R}^n$.

In the situation which interests us, let M be a manifold, and let $L : TM \rightarrow \mathbb{R}$ be a Lagrangian for the system. The, the *action* S associated with L is given by

$$S(u) = \int_{\Delta} L(u, Du) dx.$$

(For classical examples of the action, see [26], Chap.16.)

Let V be a finite dimensional real vector space. Let μ and ν be two V -valued differential forms. Since the space of V -valued differential forms is a vector space in an obvious fashion, we may define

$$\mu_s = \mu + s\nu$$

for any $s \in \mathbb{R}$. Then, the *variation of μ in the direction ν* is simply

$$\begin{aligned}\delta\mu &= \left. \frac{d}{ds} \mu_s \right|_{s=0}, \quad \text{or,} \\ &= \nu.\end{aligned}$$

Similarly, let G be a function of the form μ . Then, the *variation of G* is simply

$$\delta G = \left. \frac{d}{ds} G(\mu_s) \right|_{s=0}.$$

This is a simplistic development of the calculus of variations. A more sophisticated approach to the calculus of variations on differential forms requires the development of a theory of infinite dimensional manifolds. Fortunately, our rudimentary definitions will suffice in developing the desired results.

7.2 The Yang-Mills Equation

Let E be a vector G -bundle over a manifold M . To recover the Yang-Mills equation, the key is to define an appropriate Yang-Mills action. In order to do so, we first define the trace of an $\text{End}(E)$ -valued differential form.

Let V be a finite dimensional vector space over a field \mathbb{K} . Recall that the *trace* $\text{tr} : \text{End}(V) \rightarrow \mathbb{K}$ satisfies $\text{tr}(TS) = \text{tr}(ST)$ for all $T, S \in \text{End}(V)$.

For trivializations (U_α, ϕ_α) and (U_β, ϕ_β) of the bundle $\text{End}(E)$, the transition functions map $(x, C) \in (U_\alpha \cap U_\beta) \times \text{End}(E) \mapsto (x, ACA^{-1}) \in (U_\alpha \cap U_\beta) \times \text{End}(E)$. Since, $\text{tr}(ACA^{-1}) = \text{tr}(AA^{-1}C) = \text{tr}(C)$, it is well-defined on sections of $\text{End}(E)$.

Let ψ be a section of $\text{End}(E)$, and let μ be a differential form on M . We define the *trace of $\psi \otimes \mu$* as

$$\text{tr}(\psi \otimes \mu) = (\text{tr } \psi)\mu,$$

and extend it linearly over arbitrary $\text{End}(E)$ -valued differential forms on M .

Definition 7.2.1. Let E be a vector G -bundle with covariant derivative D , and let F be the associated curvature form. We define the *Yang-Mills Lagrangian*, denoted L_{YM} , by

$$L_{YM} = \frac{1}{2} \text{tr}(F \wedge \star F).$$

The *Yang-Mills action* S_{YM} is defined as

$$S_{YM} = \frac{1}{2} \int_M \text{tr}(F \wedge \star F).$$

Remark 7.2.2. This is actually the Yang-Mills action when there are no matter fields. That is, when $J = 0$. We will briefly discuss the general Yang-Mills action at the end of this section.

We will now show that the Yang-Mill equation is equivalent to the action principle. That is, $\delta S_{YM} = 0$ iff the Yang-Mills equation is satisfied. Recall that the covariant derivative can be written $D_X \sigma = X(\sigma) + A(X)\sigma$ for any trivialization of E . Since A determines the covariant derivative on E , we can think of D , and consequently F , as functions of A respectively. Then, a simple calculation reveals

$$\begin{aligned} \delta F &= \frac{d}{ds} F(A_s)|_{s=0} \\ &= \frac{d}{ds} (2dA_s + [A_s, A_s])|_{s=0} \\ &= 2d\delta A + [A, \delta A] \\ &= d_D \delta A, \end{aligned} \tag{31}$$

where $[\cdot, \cdot]$ is the graded commutator.

The series of lemmas that follow will be useful in the application of the action principle. Our proofs will be done locally, but are valid globally.

Lemma 7.2.3. ([5], p.276) *Let ω and η be $\text{End}(E)$ -valued p , and q -forms respectively. Then, their wedge product satisfies*

$$\text{tr}(\omega \wedge \eta) = (-1)^{pq} \text{tr}(\eta \wedge \omega).$$

Proof. Since the equation is linear in each variable, it suffices to examine the case when $\omega = f \otimes \mu$ and $\eta = g \otimes \nu$ where f, g are sections of $\text{End}(E)$, $\mu \in \Omega^p(M)$ and $\nu \in \Omega^q(M)$. Then,

$$\begin{aligned} \text{tr}(\omega \wedge \eta) &= \text{tr}(fg \otimes (\mu \wedge \nu)) \\ &= \text{tr}(fg) \otimes (\mu \wedge \nu) \\ &= \text{tr}(gf) \otimes (-1)^{pq} (\nu \wedge \mu) \\ &= (-1)^{pq} \text{tr}(\eta \wedge \omega), \end{aligned}$$

as desired. \square

Remark 7.2.4. An obvious consequence of the Lemma is that $\text{tr}([\omega, \eta]) = 0$ where μ and ν are any $\text{End}(E)$ -valued p and q -forms respectively, and $[\cdot, \cdot]$ is the graded commutator defined earlier.

Lemma 7.2.5. *Let ω be a $\text{End}(E)$ -valued form on M . Then,*

$$\text{tr}(d_D \omega) = d \text{tr}(\omega).$$

Proof. Since the equation is linear, it suffices to examine the case $\omega = T \otimes \mu$ where T is a section of $\text{End}(E)$ and $\mu \in \Omega(M)$. Then,

$$\begin{aligned} d \text{tr}(T \otimes \mu) &= d(\text{tr}(T) \otimes \mu) \\ &= d(\text{tr}(T)) \wedge \mu + \text{tr}(T) \otimes d\mu \end{aligned}$$

Since trace is linear, it commutes with the covariant derivative, and $d(\text{tr}(T)) = \text{tr}(dT)$. Therefore,

$$d \text{tr}(T \otimes \mu) = \text{tr}(dT \wedge \mu + T \otimes d\mu).$$

Locally, from Lemma 4.7.4, $d_D T = dT + [A, T]$, and

$$\begin{aligned} \text{tr}(d_D(T \otimes \mu)) &= \text{tr}(d_D T \wedge \mu + T \otimes d\mu) \\ &= \text{tr}((dT + [A, T]) \wedge \mu + T \otimes d\mu) \\ &= \text{tr}(dT \wedge \mu + T \otimes d\mu), \end{aligned}$$

as desired. \square

Lemma 7.2.6. *Let M be an oriented, n -manifold. Let ω and η be $\text{End}(E)$ -valued p and q -forms on M respectively. If either ω or η have compact support on M , and $p + q = n - 1$, then*

$$\int_M \text{tr}(d_D \omega \wedge \eta) = (-1)^{(p+1)} \int_M \text{tr}(\omega \wedge d_D \eta).$$

Furthermore, if M has a semi-Riemannian metric, and $p = q$, then

$$\int_M \text{tr}(\omega \wedge \star \eta) = \int_M \text{tr}(\eta \wedge \star \omega).$$

Proof. From Lemma 4.7.3, $d_D(\omega \wedge \eta) = d_D\omega \wedge \eta + (-1)^p\omega \wedge d_D\eta$. In order to show the first equality, it suffices to show that $\int_M \text{tr}(d_D(\omega \wedge \eta)) = 0$. Using Lemma 7.2.5,

$$\int_M \text{tr}(d_D(\omega \wedge \eta)) = \int_M d \text{tr}(\omega \wedge \eta).$$

Using Stoke's theorem,

$$\int_M d \text{tr}(\omega \wedge \eta) = \int_{\partial M} \text{tr}(\omega \wedge \eta).$$

Since $\omega \wedge \eta$ has compact support, the integral over ∂M vanishes, and the first equality is satisfied.

For the second identity, since the equation is linear in each variable, so it suffices to examine the case where $\omega = f \otimes \mu$ and $\eta = g \otimes \nu$ where f, g are sections of $\text{End}(E)$, $\mu, \nu \in \Omega^p(M)$. Since $p = q$, and M is oriented, then $\mu \wedge \star \nu = \langle \mu, \nu \rangle \text{vol} = \nu \wedge \star \mu$. Then,

$$\begin{aligned} \text{tr}(\omega \wedge \star \eta) &= \text{tr}(fg) \otimes (\mu \wedge \star \nu) \\ &= \text{tr}(gf) \otimes (\nu \wedge \star \mu) \\ &= \text{tr}(\eta \wedge \star \omega). \end{aligned}$$

Clearly, the integrals will also be equal, and the second identity is satisfied. \square

Using these lemmas, it is an easy matter to compute the variation of the Yang-Mills action on an n -manifold M :

$$\begin{aligned} \delta S &= \frac{d}{ds} \frac{1}{2} \int_M \text{tr}((F + s\delta F) \wedge (\star F + s\delta \star F))|_{s=0} \\ &= \frac{1}{2} \int_M \text{tr}(\delta F \wedge \star F + F \wedge \delta \star F) \end{aligned} \quad (32)$$

By the second part of Lemma 7.2.6, $\int_M \text{tr}(\delta F \wedge \star F) = \int_M \text{tr}(F \wedge \star \delta F)$. Then, the variation of S becomes

$$\delta S = \int_M \text{tr}(\delta F \wedge \star F).$$

Substituting $\delta F = d_D \delta A$, this becomes

$$\delta S = \int_M \text{tr}(d_D \delta A \wedge \star F).$$

Assuming that δA is compactly supported, then, by the first part of Lemma 7.2.6, we have the final form:

$$\delta S = \int_M \text{tr}(\delta A \wedge d_D \star F).$$

Since this must be zero for all values of the variation δA , then

$$d_D \star F = 0. \quad (33)$$

This is precisely the Yang-Mills equation when there are no matter fields. In order to recover the general Yang-Mills equation, we need to use the general Yang-Mills action: ([11],p.56)

$$S_{GYM}(A) = \frac{1}{g^2} \int_M \text{tr}(F \wedge \star F) + \int_M (d_D \Phi)^+ \wedge \star d_D \Phi - m^2 \int_M \Phi^+ \wedge \Phi,$$

where g is a dimensionless “coupling constant”, Φ is a scalar field, m is the scalar mass, and $+$ denotes the Hermitian conjugate. We will not go into any detail, except to remark that, if γ is a gauge transformation, then the vector potential and the scalar fields transform as:

$$\Phi \mapsto \gamma \Phi \quad \text{and} \quad A \mapsto \gamma A \gamma^{-1} + \gamma d \gamma^{-1}.$$

We say gauge theories whose Lagrangians involve terms of the form “ $\text{tr}(F \wedge \star F)$ ” are Yang-Mills theories. Since our choices of vector G -bundles and connections is so large, there are a great many situations to which the Yang-Mills theories can be applied. In the case of the fundamental forces, Yang-Mills theory has been used successfully to describe the electroweak and strong nuclear forces. Scientists continue to try and use it to tackle other problems in high-energy particle and theoretical physics. We describe one such case in the final chapter.

Chapter 8

Connections on Higher Bundles

8.1 Motivation

This chapter will closely follow the paper “Higher Yang-Mills Theory” by John Baez [6]. We include supplementary details and some detailed proofs of results stated and only briefly explained there. The overall aim of this chapter is to investigate higher order analogues of connections and curvature in a new setting.

In Chapters 2 and 6, we saw that electromagnetism can be described very naturally using a vector potential A . The action on a charged particle is defined using the action on a neutral particle and the path integral of A . When this is abstracted to “strings,” a 2-form B called the *Kalb-Ramond* field plays a similar role, where the integral of B over its “worldsheet” (the area swept out by the string over time) is also a term in the action. This “2-form electromagnetism” (where the electromagnetic force acts on strings), satisfies analogous equations: ([6],p.1-2)

1. $G = dB$
2. $\star d \star G = K,$

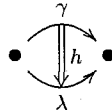
where K is some ‘source’ 2-form for the field. In [6], Baez describes what might be called a first step in a simultaneous generalization of both Yang-Mills and Kalb-Ramond theories.

Recall that the set of holonomies over all paths in a manifold M 4.2.10 forms a

groupoid. If the bundle is trivial, they even form a group. (Note that in this case, we can ‘compose’ holonomies even when we shouldn’t be able too. For instance, it is possible even when the associated *paths* are not composable.) We will see this phenomenon again soon.) We want to do something similar with surfaces. However, whereas paths could only be composed at their end points, two surfaces can be “composed” at a single point, multiple points, or even at paths along their boundary. This makes for some interesting difficulties that are resolved by Baez as follows.

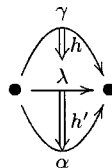
Suppose that $\gamma : [0, 1] \rightarrow M$ and $\lambda : [0, 1] \rightarrow M$ are two paths in M with the same beginning and endpoints (which, of course, may be different.) Let Σ be a smooth surface in M bounded by the loop $\lambda * \gamma^{-1}$. We will *naively* assume that we can assign to Σ some value h in a (‘holonomy’) group \mathfrak{H} , as in figure 5.

Figure 5 - A surface Σ bounded by a path $(\gamma\lambda)$



There are two obvious ways to compose surfaces. First, let α be another path in M sharing beginning and endpoints with γ and λ above, and let Π be a surface bounded by $\lambda * \alpha^{-1}$. We define the “vertical composition” of Σ and Π to be the new surface $\Sigma \cup \Pi$ bounded by $\gamma * \alpha^{-1}$. If Π is assigned $h' \in \mathfrak{H}$, then the new surface has value $h'h \in \mathfrak{H}$. This idea of the *vertical composition* of (the ‘holonomy’ associated to) two surfaces is illustrated in figure 6.

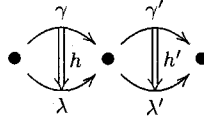
Figure 6 - Vertical composition of two surfaces.



Now, consider a second pair of paths γ' and λ' sharing the same beginning and endpoints, and both starting at the common endpoint of γ and λ (see figure 7). Let Θ be a surface bounded by $(\gamma')^{-1} * \lambda'$. We define the *horizontal composition* of Σ and

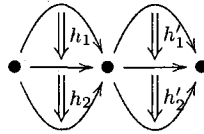
Θ to be the new surface $\Sigma \cup \Theta$ bounded by $(\lambda * \lambda') * (\gamma * \gamma')^{-1}$. If Π is assigned $h' \in \mathfrak{H}$, then the new surface has value $h' \circ h \in \mathfrak{H}$. This idea is illustrated in figure 7.

Figure 7 - Horizontal composition of two surfaces.



When we combine the vertical and horizontal cases, and suppress the notation of the paths, we obtain the following diagram:

Figure 8 - The exchange law.



If we demand that horizontal and vertical multiplication commute, we obtain the *exchange law*:

$$(h'_2 h'_1) \circ (h_2 h_1) = (h'_2 \circ h_2)(h'_1 \circ h_1). \quad (34)$$

A well-known result of Eckmann and Hilton [8] is that, when a group is equipped with two products, so long as they share the same unit and satisfy the exchange law, then the group is abelian, and the products coincide. Indeed, suppose that $h, h' \in \mathfrak{H}$. Then,

$$h \circ h' = (h1) \circ (1h') = (h \circ 1)(1 \circ h') = hh'.$$

Similarly,

$$h \circ h' = (1h) \circ (h'1) = (1 \circ h')(h \circ 1) = h'h.$$

Then, $hh' = h'h$, and $h \circ h' = hh' = h'h = h' \circ h$. So, \mathfrak{H} is abelian with either product.

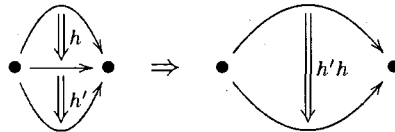
Baez's approach is to think of holonomies of paths and surfaces as taking values in different groups. Further, he thinks of holonomies on a surface as *morphisms* between

holonomies of paths. This kind of idea has been used in many other situations to some effect (e.g. Topological QFT [4]), and is similar in spirit to the idea of a (path) homotopy as being a “path between paths”, or the idea of a path as a map between its endpoints.

In this spirit, we proceed as follows. Let $g \in G$ be assigned to a path from a to b , and let $h \in H$ be assigned to a surface bounded by paths with values g and $g' \in G$. We will denote these by $g : a \rightarrow b$ and $h : g \rightarrow g'$.

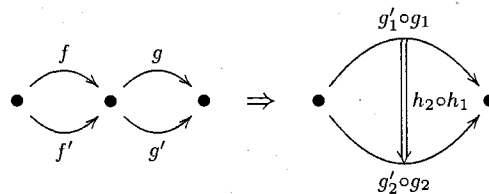
In the case of vertical multiplication, we assume that the three paths from a to b have values g, g' and $g'' \in G$. Then, the surfaces have values $h : g \rightarrow g'$ and $h' : g' \rightarrow g''$. The composite surface $h'' : g \rightarrow g''$ has value $h'h \in H$. This is illustrated in figure 9.

Figure 9 - Vertical composition with two groups.



Similarly, in the case of horizontal multiplication, there are two pairs of paths, denoted by $g_1, g_2 : a \rightarrow b$ and $g'_1, g'_2 : b \rightarrow c$ with associated surfaces $h_1 : g_1 \rightarrow g'_1$ and $h_2 : g_2 \rightarrow g'_2$. The composite surface has value $h_2 \circ h_1 : g_2 \circ g_1 \rightarrow g'_2 \circ g'_1$. This is illustrated in figure 10.

Figure 10 - Horizontal composition with two groups.



Some readers may recognize that this type of structure is known as a *category*. Since we will need the nitty-gritty later, we give the definition in detail:

Definition 8.1.1 ([18]). A category C is a set of objects C_0 , a set of morphisms C_1 and a collection of maps:

1. The source map, $s : C_1 \rightarrow C_0$,
2. The target map, $t : C_1 \rightarrow C_0$,
3. The identity map, $i : C_0 \rightarrow C_1$, and
4. The composition map, $\circ : C_1 \times_{C_0} C_1 \rightarrow C_1$ where $C_1 \times_{C_0} C_1 = \{(f, g) \in C_1 \times C_1 \mid s(f) = t(g)\}$ is called the collection of *composable morphisms*,

satisfying the following identities:

1. $si = ti = \text{id}_{C_0}$
2. If $t(f) = s(g) = b \in C_0$, then $i(b) \circ f = f$ and $g \circ i(b) = g$ for all $f, g \in C_1$.
3. If $t(h) = s(g)$ and $t(g) = s(f)$ then $(f \circ g) \circ h = f \circ (g \circ h)$.

Remark 8.1.2. The identities of a category are similar to those of a groupoid, so it is not unexpected that a category might appear as the ‘holonomy groupoid’ for strings.

In our setting, C_0 and C_1 will be groups, all structure maps will be group homomorphisms, and the exchange law arises from the fact that composition of morphisms is a homomorphism. This is captured by the idea of a “categorical group”, which is an internal category in the category of groups. Explicitly,

Definition 8.1.3. A *categorical group* or *2-group* is an category C where the objects C_0 and morphisms C_1 are themselves groups, and all the structure maps are group homomorphisms.

Baez uses the 2-group as the gauge group in a “higher order” fibre bundle. Specifically, in [6], he treats the case of a trivial principal 2-bundle over a manifold M , which is simply the product of M with the Lie 2-(gauge)group

8.2 Lie 2-Groups and Crossed Modules

We begin by exploring the relationship between a (Lie) 2-group, and another structure known as a “crossed module.” A Lie 2-group is an internal category¹ in the category of Lie groups. For clarity, we define it explicitly as follows:

Definition 8.2.1 ([6], pg. 8). A *Lie 2-group* is a category C where the set of objects C_0 and the set of morphisms C_1 are Lie groups, the source map, target map, and identity map are group homomorphisms, and the composition map is a homomorphism of composable morphisms.

Since composition is a homomorphism, we must have

$$\circ[(f_1 f'_1), (f_2 f'_2)] = (f_1 \circ f_2)(f'_1 \circ f'_2),$$

where $f_1 f'_1$ and $f_2 f'_2$ are group multiplication in C_1 . By writing the left-hand side of the equation as $(f_1 f'_1) \circ (f_2 f'_2)$, the exchange law (equation (34)) is a consequence of the categorical structure.

We define a homomorphism between Lie 2-groups as follows:

Definition 8.2.2. Given two Lie 2-groups C and C' , a *homomorphism* $F : C \rightarrow C'$ is a functor such that $F_0 : C_0 \rightarrow C'_0$ and $F_1 : C_1 \rightarrow C'_1$ are homomorphisms. Two Lie 2-groups are *isomorphic* if F_0 and F_1 are isomorphisms.

The following characterization of the morphisms in a 2-group is central in what follows.

Lemma 8.2.3. [9], p. 8-9] $C_1 \cong (\ker s) \rtimes C_0$

Proof. Define the action of C_0 on $\ker s$ by the map $\alpha : C_0 \rightarrow \text{End}(\ker s)$ where $\alpha(x)[k] = i(x)ki(x)^{-1}$. Since $s(i(x)ki(x)^{-1}) = si(x)s(k)si(x)^{-1} = e \in C_0$, this map is

¹Let E be a category that is finitely complete. An internal category has objects C_0 and morphisms C_1 taken from E , along with the usual categorical operations. When E is the category of groups (**Grp**), the collection of objects and morphisms in an internal category become groups themselves. ([18], p.267-269)

well-defined. We use α to define the semi-direct product in $(\ker s) \rtimes C_0$ as follows: Let $(k, x), (k', x') \in (\ker s) \rtimes C_0$ then

$$(k, x)(k', x') = (k\alpha(x)[k'], xx').$$

For each $f \in C_1$, define $k = f[i(s(f))]^{-1} \in C_1$, and $x = s(f) \in C_0$. Since $s(k) = s(f)[s(i(s(f)))]^{-1} = s(f)s(f)^{-1} = e \in C_1$, then $k \in \ker s$, and $f = ki(x)$.

Define a map, $\phi : C_1 \rightarrow (\ker s) \rtimes C_0$ by $f \mapsto (k, x)$ where $f = ki(x)$, as above. We show that ϕ is a homomorphism as follows: Suppose $f' = k'i(x')$. Then $\phi(ff') = \phi(ki(x)k'i(x')) = \phi(ki(x)k'i(x)^{-1}i(x)i(x')) = \phi(k\alpha(x)[k']i(xx')) = (k\alpha(x)[k'], xx') = (k, x)(k', x') = \phi(f)\phi(f')$, as required. Similarly, one can show that the map $\psi : (\ker s) \rtimes C_0 \rightarrow C_1$ defined by $(k, x) \mapsto ki(x)$ is also a homomorphism. Since the compositions of ϕ and ψ give the appropriate identity maps on C_1 and $(\ker s) \rtimes C_0$, then ϕ is an isomorphism of groups and $C_1 \cong (\ker s) \rtimes C_0$. \square

There is an equivalent characterization of 2-groups in terms of a well-known structure, namely that of a crossed module (see [18]).

Definition 8.2.4 ([6],pg 9.). A *Lie crossed module* is a quadruple (G, H, τ, α) consisting of Lie groups G and H , a group homomorphism $\tau : H \rightarrow G$, and an action of G on H (i.e. a homomorphism, $\alpha : G \rightarrow \text{Aut}(H)$) satisfying the following two conditions: For all $g \in G$, and $h, h' \in H$:

- LCM1. $\tau(\alpha(g)h) = g\tau(h)g^{-1}$ (known as the *equivariance of h with respect to g*), and
- LCM2. $\alpha(\tau(h))h' = hh'h^{-1}$ (known as the *Peiffer Identity*).

Definition 8.2.5. Given two Lie crossed modules (G, H, τ, α) and (G', H', τ', α') , a homomorphism F between them is a commutative square,

$$\begin{array}{ccc} H & \xrightarrow{f} & H' \\ \downarrow \tau & & \downarrow \tau' \\ G & \xrightarrow{\bar{f}} & G' \end{array}$$

satisfying, in addition, that for all $g \in G$ and $h \in H$, $f(\alpha(g)h) = \alpha'(\bar{f}(g))f(h)$. We call F an *isomorphism* if the maps f and \bar{f} are isomorphisms.

Proposition 8.2.6. *[9], pg. 9-11] The category of Lie 2-groups is equivalent to the category of Lie crossed modules.*

Proof. Our proof differs from that in [9]. We make use of a relationship between composition and the group operation on the composable morphisms shown in [18].

Let $C = \{C_0, C_1, s, t, i, \circ\}$ be a Lie 2-group. We will construct an associated Lie crossed module, as follows: Let $G = C_0$ and $H = \ker s \subseteq C_1$. Define $\tau : H \rightarrow G$ to be the restriction of t to $\ker s$. For $g \in G$, define $\alpha(g)[h] = i(g)hi(g)^{-1}$. It is easily checked that $\alpha(g) \in \text{Aut}(H)$, and that τ and α are homomorphisms.

It remains to show the equivariance and Peiffer identity. The equivariance is a trivial consequence of $ti = \text{id}_{C_0}$. The Peiffer identity is more interesting.

Using Lemma 8.2.3, we define new homomorphisms s', t' and i' making the following diagrams commute:

$$\begin{array}{ccccc}
 C_1 & \xrightarrow{s} & C_0 & & C_1 & \xrightarrow{t} & C_0 & & C_0 & \xrightarrow{i} & C_1 \\
 \psi \uparrow & & \downarrow \text{id} & & \psi \uparrow & & \downarrow \text{id} & & \text{id} \uparrow & & \downarrow \phi \\
 \ker s \rtimes C_0 & \xrightarrow{s'} & C_0 & & \ker s \rtimes C_0 & \xrightarrow{t'} & C_0 & & C_0 & \xrightarrow{i'} & \ker s \rtimes C_0
 \end{array}$$

A short calculation shows that

1. $s' : \ker s \rtimes C_0 \rightarrow C_0$ satisfies $(k, x) \mapsto x$,
2. $t' : \ker s \rtimes C_0 \rightarrow C_0$ satisfies $(k, x) \mapsto \tau(k)x$, and
3. $i' : C_0 \rightarrow \ker s \rtimes C_0$ satisfies $(k, x) \mapsto (1, x)$.

The following result is from [18], p.286 - 287. Let (f_1, g_1) and (f_2, g_2) be pairs of composable morphisms. Since C is a category, composable morphisms satisfy the exchange law:

$$(f_1 f_2) \circ (g_1 g_2) = (f_1 \circ g_1)(f_2 \circ g_2).$$

Let (f, g) be a composable pair of morphisms. Let $b = t(g) = s(f) \in C_0$, and

denote $1_b = i(b) \in C_1$. Then,

$$\begin{aligned} f \circ g &= [(1_b 1_b^{-1})f] \circ [g(1_b^{-1} 1_b)] \\ &= [1_b \circ g][(1_b^{-1} f) \circ (1_b^{-1} 1_b)] \quad (\text{exchange law where } f_1 = 1_b \text{ and } g_1 = g) \\ &= [1_b \circ g][1_b^{-1} \circ 1_b^{-1}][f \circ 1_b] \quad (\text{exchange law again}) \\ &= g[1_b^{-1} \circ 1_b^{-1}]f. \end{aligned}$$

Suppose that $f = g = 1_b$, then

$$\begin{aligned} 1_b \circ 1_b &= 1_b[1_b^{-1} \circ 1_b^{-1}]1_b \\ 1_b &= 1_b[1_b^{-1} \circ 1_b^{-1}]1_b \quad (\text{category property}) \\ 1_b^{-1} &= 1_b^{-1} \circ 1_b^{-1}. \quad (\text{multiplication in the group}) \end{aligned}$$

So, $f \circ g = g1_b^{-1}f$ for all composable pairs $(f, g) \in C_1 \times_{C_0} C_1$. Similarly, one can also obtain the reverse, $f \circ g = f1_b^{-1}g$ for all $f, g \in C_1$. In summary, when (f, g) is a composable pair, it satisfies

$$f \circ g = f1_b^{-1}g, \text{ and} \tag{35}$$

$$= g1_b^{-1}f. \tag{36}$$

Using the isomorphism ϕ from Lemma 8.2.3 to identify C_1 with $(\ker s) \rtimes C_0$, we write $\phi(f) = (k, x)$ and $\phi(g) = (l, y) \in \ker s \rtimes C_0$ respectively. Since $s'(k, x) = t'(l, y)$, then $x = \tau(l)y$, $f = (k, \tau(l)y)$, and $1_b^{-1} = (1, (\tau(l)y)^{-1})$. Substituting these values into equation (35) and using the semi-direct product, we obtain

$$\begin{aligned} \phi(f) \circ \phi(g) &= (k, \tau(l)y)(1, (\tau(l)y)^{-1})(l, y) && (\text{equation 35}) \\ &= (k\alpha(\tau(l)y)[1], \tau(l)y(\tau(l)y)^{-1})(l, y) && (\text{semi-direct product}) \\ &= (k, 1)(l, y) && (\alpha(x) \text{ is an isomorphism}) \\ &= (k\alpha(1)[l], y) && (\text{semi-direct product}) \\ &= (kl, y). && (\alpha \text{ is a homomorphism}) \end{aligned}$$

Using equation (36) and similar steps, we also obtain

$$\phi(f) \circ \phi(g) = (l\alpha(\tau(l)^{-1})[k], y).$$

Since these are equal, we must have

$$\alpha(\tau(l)^{-1})k = l^{-1}kl.$$

A substitution of $l \mapsto l^{-1}$ yields the original Peiffer identity. Consequently, every Lie 2-group gives rise to a Lie crossed module.

Now suppose that (G, H, τ, α) is a Lie crossed module. We will construct an associated Lie 2-group as follows: Let $C_0 = G$ and $C_1 = H \rtimes G$ with product

$$(h, g)(h', g') = (h\alpha(g)[h'], gg'). \quad (37)$$

Define $s : H \rtimes G \rightarrow G$ by $(h, g) \mapsto g$, $t : H \rtimes G \rightarrow G$ by $(h, g) \mapsto \tau(h)g$, and $i : G \rightarrow H \rtimes G$ by $g \mapsto (1, g)$. Clearly, s and i are homomorphisms. Equivariance shows that t is also a homomorphism.

Now, define composition $\bullet : C_1 \times_{C_0} C_1 \rightarrow C_1$ by

$$(h, g) \bullet (h', g') = (hh', g').$$

This is a homomorphism if and only if the exchange law is satisfied. Namely,

$$(f_1 f_2) \bullet (g_1 g_2) = (f_1 \bullet g_1)(f_2 \bullet g_2). \quad (38)$$

So, let (f_1, g_1) and (f_2, g_2) be two composable pairs. Write $f_1 = (k, x)$, $f_2 = (l, y)$, $g_1 = (k', x')$, $g_2 = (l', y')$. Since $s(f_1) = t(g_1)$, then $x = \tau(k')x'$, and similarly for the other pair. Substituting these values into the exchange law, we obtain

$$\begin{aligned} (f_1 f_2) \bullet (g_1 g_2) &= (k\alpha(\tau(k')x')[l], \tau(k')x'\tau(l')y') \bullet (k'\alpha(x')[l'], x'y') \quad (\text{product}) \\ &= (k\alpha(\tau(k')x')[l] k'\alpha(x')[l'], x'y') \quad (\text{composition}) \end{aligned}$$

On the other hand,

$$\begin{aligned} (f_1 \bullet g_1)(f_2 \bullet g_2) &= (kk', x')(ll', y') \quad (\text{composition}) \\ &= (kk'\alpha(x')[ll'], x'y') \quad (\text{product}) \\ &= (kk'\alpha(x')[l] ((k')^{-1}k')\alpha(x')[l'], x'y') \quad (\alpha(x) \text{ is a homomorphism}) \\ &= (k\alpha(\tau(k')[\alpha(x')[l]] k'\alpha(x')[l'], x'y') \quad (\text{Peiffer identity}) \\ &= (k\alpha(\tau(k')x')[l] k'\alpha(x')[l'], x'y'). \quad (\alpha \text{ is a homomorphism}) \end{aligned}$$

Therefore, the exchange law is satisfied, and composition is a homomorphism. Consequently, the category $(G, H \rtimes G, s', t', i', \bullet)$ is a Lie 2-group.

It remains to show that there is an equivalence of categories between the Lie 2-groups and the Lie crossed modules. Given a Lie 2-group $(C_0, C_1, s, t, i, \circ)$, we

construct a Lie crossed module (G, H, τ, α) . From the Lie crossed module (G, H, τ, α) , we construct a new Lie 2-group $(C'_0, C'_1, s', t', i', \bullet)$, and show that it is isomorphic to the original Lie 2-group. By construction, the Lie crossed module has the form $(C_0, \ker s, t_{\ker s}, \alpha)$ where $\alpha(x)[f] = i(x)ki(x)^{-1}$ for all $x \in C_0$ and $k \in \ker s$. Then, the constructed Lie 2-group has the form $(C_0, (\ker s) \rtimes C_0, s', t', i', \bullet)$ where $s'(k, x) = x$, $t'(k, x) = t_{\ker s}(k)x = t(k)x$, $i'(x) = (1, x)$ and $(k, x) \bullet (k', x') = (kk', x')$ whenever $((k, x), (k', x'))$ is a composable pair. From Lemma 8.2.3, we have an isomorphism ϕ between C_1 and $(\ker s) \rtimes C_0$. We define a functor F by: $F_0 : C_0 \rightarrow C_0$ is the identity map, and $F_1 : C_1 \rightarrow (\ker s) \rtimes C_0$ is the isomorphism ϕ . With this definition, F is clearly an isomorphism of Lie 2-groups.

Finally, given a Lie crossed module (G, H, τ, α) , we construct a Lie 2-group $(C_0, C_1, s, t, i, \circ)$. From the Lie 2-group, we construct a new Lie crossed module (G', H', τ', α') and show that it is isomorphic to the original Lie crossed module. Using the same ideas as above, it is a simple matter to construct the isomorphism between Lie crossed modules. Therefore, we have equivalence between Lie 2-groups and Lie crossed modules. \square

Example 8.2.7. Let G be any Lie group, H any abelian Lie group and a map $\alpha : G \rightarrow \text{Aut}(H)$. Define $\tau : H \rightarrow G$ by $h \mapsto e$, the trivial homomorphism. Then, $\tau(\alpha(g)[h]) = e = geg^{-1} = g\tau(h)g^{-1}$, and $\alpha(\tau(h))[h'] = \alpha(e)[h'] = h' = (hh^{-1})h' = hh'h^{-1}$, as desired. Therefore, the quadruple (G, H, τ, α) is a Lie crossed module.

Example 8.2.8. Let G be any Lie group, let $H = \mathfrak{g}$ thought of as an additive abelian group, and let α be the adjoint representation of G on \mathfrak{g} . Then, in the example above, $(G, \mathfrak{g}, t, \alpha)$ is a Lie crossed module.

Example 8.2.9. Let H be any Lie group, and let $G = \text{Aut}(H)$, the collection of all automorphisms of H . Let $\tau : H \rightarrow G$ be the homomorphism assigning each element of H to the corresponding inner automorphism (denoted cong_h), and let

$\alpha : G \rightarrow \text{Aut}(H)$ be the identity map. Then, for all $g \in \text{Aut}(H)$, $h, h' \in H$:

$$\begin{aligned} \tau(\alpha(g)[h])(h') &= \tau(g(h))(h') \\ &= \text{cong}_{g(h)}(h') \\ &= g(h)h'(g(h))^{-1} \\ &= g(h)h'g(h^{-1}). \end{aligned}$$

On the other hand,

$$\begin{aligned} (g\tau(h)g^{-1})(h') &= (g \text{cong}_h g^{-1})(h') \\ &= g(hg^{-1}(h')h^{-1}) \\ &= g(h)g(g^{-1}(h'))g(h^{-1}) \\ &= g(h)h'g(h^{-1}), \end{aligned}$$

and the equivariance of h with respect to g is satisfied.

Similarly, for all $h, h' \in H$:

$$\begin{aligned} \alpha(\tau(h))[h'] &= \tau(h)h' \\ &= \text{cong}_h(h') \\ &= hh'h^{-1}, \end{aligned}$$

and the Peiffer identity is satisfied. Therefore, the quadruple $(\text{Aut}(H), H, \tau, \alpha)$ is a Lie crossed module, and the corresponding Lie 2-group is called the “automorphism 2-group”.

8.3 Lie 2-Algebras and Differential Crossed Modules

When given a Lie group, an obvious question to ask is - “What is its Lie algebra?” We can ask a similar question of the Lie 2-group, but to do so, we need to define the analogue of the Lie algebra in this case.

Definition 8.3.1. A *Lie 2-algebra* is an internal category in the category of Lie algebras. That is, a category \mathfrak{c} where the set of objects \mathfrak{c}_0 and the set of morphisms \mathfrak{c}_1 are Lie algebras, the source map, target map, and identity map are Lie algebra homomorphisms, and the composition map is a Lie algebra homomorphism of composable morphisms.

Definition 8.3.2. A homomorphism of Lie 2-algebras \mathfrak{c} and \mathfrak{c}' is a functor $F : \mathfrak{c} \rightarrow \mathfrak{c}'$ which acts on the objects and morphism to give Lie algebra homomorphisms, $F_0 : \mathfrak{c}_0 \rightarrow \mathfrak{c}'_0$ and $F_1 : \mathfrak{c}_1 \rightarrow \mathfrak{c}'_1$ respectively. Two Lie 2-algebras are *isomorphic* if F_0 and F_1 are isomorphisms of Lie algebras.

Given any Lie 2-group, we can describe its Lie 2-algebra in a straightforward way, as follows:

Proposition 8.3.3. *Any Lie 2-group C has a Lie 2-algebra \mathfrak{c} in which \mathfrak{c}_0 is the Lie algebra of the objects C_0 of C , \mathfrak{c}_1 is the Lie algebra of the morphisms C_1 of C , and the source, target, identity and composition maps are the derivatives of the same map from C at the appropriate unit.*

Remark 8.3.4. For the Lie 2-algebra, we will denote the source map by \mathfrak{s} , the target map by \mathfrak{t} , the identity map by \mathfrak{i} , and the composition by \circ .

Proof. The proof is a straightforward consequence of the fact that the tangent map at the identity of a homomorphism of Lie groups is a homomorphism of Lie algebras. \square

There is an analogue of Lemma 8.2.3 with Lie algebras, as follows:

Lemma 8.3.5. $\mathfrak{c}_1 \cong (\ker \mathfrak{s}) \rtimes \mathfrak{c}_0$

Proof. Let $\mathfrak{c} = (\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{s}, \mathfrak{t}, \mathfrak{i}, \circ)$ be a Lie 2-algebra. Define the action of \mathfrak{c}_0 on $\ker \mathfrak{s}$ by the map $\alpha : \mathfrak{c}_0 \rightarrow \text{Aut}(\ker \mathfrak{s})$ where $\alpha(x)[k] = [\mathfrak{i}(x), k]$. Since $\mathfrak{s}([\mathfrak{i}(x), k]) = [\mathfrak{s}(\mathfrak{i}(x)), \mathfrak{s}(k)] = [x, 0] = 0$, the map is well-defined. We use α to define the semi-direct product on $(\ker \mathfrak{s}) \rtimes \mathfrak{c}_0$ as follows: Let $(k, x), (k', x') \in (\ker \mathfrak{s}) \rtimes \mathfrak{c}_0$, then

$$[(k, x), (k', x')] = ([k, k'] + \alpha(x)[k'] - \alpha(x')[k], [x, x']). \quad (39)$$

For each $f \in \mathfrak{c}_1$, define $k = f - \mathfrak{i}(\mathfrak{s}(f)) \in \mathfrak{c}_1$, and $x = \mathfrak{s}(f) \in \mathfrak{c}_0$. Since $\mathfrak{s}(k) = \mathfrak{s}(f) - \mathfrak{s}(\mathfrak{i}(\mathfrak{s}(f))) = 0 \in \mathfrak{c}_1$, then $k \in \ker \mathfrak{s}$, and $f = k + \mathfrak{i}(x)$.

Define a map $\phi : \mathfrak{c}_1 \rightarrow \ker \mathfrak{s} \rtimes \mathfrak{c}_0$ by $f \mapsto (k, x)$ where $f = k + \mathfrak{i}(x)$ as above. We show that ϕ is a homomorphism as follows: Suppose $f' = k' + \mathfrak{i}(x')$. Then $\phi([f, f']) = \phi([k + \mathfrak{i}(x), k' + \mathfrak{i}(x')]) = \phi([k, k'] + [k, \mathfrak{i}(x')] + [\mathfrak{i}(x), k'] + [\mathfrak{i}(x), \mathfrak{i}(x')]) = \phi([k, k'] + \alpha(x)[k'] - \alpha(x')[k] + \mathfrak{i}([x, x'])).$ Since $\mathfrak{s}([\mathfrak{i}(x), k]) = 0$ for all $x \in \mathfrak{c}_0$, and $k \in \ker \mathfrak{s}$, then $\phi([f, f']) = ([k, k'] + \alpha(x)[k'] - \alpha(x')[k], [x, x']) = [(k, x), (k', x')] = [\phi(f), \phi(f')]$, as required. Similarly, one can show that the map $\psi : (\ker \mathfrak{s}) \rtimes \mathfrak{c}_0 \rightarrow \mathfrak{c}_1$ given by $(k, x) \mapsto k + \mathfrak{i}(x)$ is also a homomorphism.

Since the compositions of ϕ and ψ give the appropriate identity maps on \mathfrak{c}_1 and $(\ker \mathfrak{s}) \rtimes \mathfrak{c}_0$, it remains to show that ϕ is an isomorphism of \mathfrak{c}_1 and $(\ker \mathfrak{s}) \rtimes \mathfrak{c}_0$ as vector spaces. Since $\dim(\mathfrak{c}_1) = \dim((\ker \mathfrak{s}) \rtimes \mathfrak{c}_0)$, it suffices to show that $\phi(0_{\mathfrak{c}_1}) = 0_{(\ker \mathfrak{s}) \rtimes \mathfrak{c}_0}$, which is clear. Therefore, ϕ is an isomorphism of \mathfrak{c}_1 and $(\ker \mathfrak{s}) \rtimes \mathfrak{c}_0$ as Lie algebras. \square

As for Lie 2-groups and Lie crossed modules, there is also an equivalence of categories between Lie 2-algebras and a “differential Lie crossed module,” defined as follows:

Definition 8.3.6 ([6], p.13). A *differential Lie crossed module* is a quadruple $(E, F, \delta, \mathfrak{a})$ consisting of Lie algebras E and F , and a pair of Lie algebra homomorphisms, $\delta : F \rightarrow E$, and $\mathfrak{a} : E \rightarrow \text{Der}(F)$ satisfying two conditions: For all $x \in E$, and $y, y' \in F$:

$$\text{DLCM1.} \quad \delta(\mathfrak{a}(x)[y]) = [x, \delta(y)], \text{ and}$$

$$\text{DLCM2.} \quad \mathfrak{a}(\delta(y))[y'] = [y, y'],$$

where $\text{Der}(F)$ denotes the set of all derivations on F . That is, linear maps $f : F \rightarrow F$ such that $f([x, x']) = [f(x), x'] + [x, f(x')]$.

Remark 8.3.7. It is clear that given a Lie crossed module (G, H, τ, α) , we can construct a differential Lie crossed module using the appropriate associated Lie algebras, and derivatives of maps at the identity.

Definition 8.3.8. Given two differential Lie crossed modules (E, F, δ, α) and $(E', F', \delta', \alpha')$, a homomorphism F between them is a commutative square,

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ \downarrow \delta & & \downarrow \delta' \\ E & \xrightarrow{\bar{f}} & E' \end{array}$$

satisfying, in addition, that for all $g \in E$ and $h \in F$, $f(\alpha(g)[h]) = \alpha'(\bar{f}(g))[f(h)]$. We call F an *isomorphism* if the maps f and \bar{f} are isomorphisms.

Proposition 8.3.9. *The category of Lie 2-algebras is equivalent to the category of differential Lie crossed modules.*

Remark 8.3.10. To the best of our knowledge, the following proof is not found in the literature. This proposition is always taken to be a consequence of Proposition 8.2.6. Our proof will avoid framing the Lie 2-algebra and differential Lie crossed modules as the “derivative” of an associated Lie 2-group or Lie crossed module.

Proof. Given a Lie 2-algebra, we will construct a differential Lie crossed module as follows: Let $\mathfrak{c} = \{\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{s}, \mathfrak{t}, \mathfrak{i}, \circ\}$ be a Lie 2-algebra. Denote $E = \mathfrak{c}_0$ and $F = \ker \mathfrak{s}$. Let δ be the restriction of \mathfrak{t} to $\ker \mathfrak{s}$. For $g \in E$, define $\alpha(g)[h] = [\mathfrak{i}(g), h]$. It is easily checked that $\alpha(g) \in \text{Der}(F)$, and that δ and α are homomorphisms. It is straightforward to check that (DLCM1) is satisfied by these definitions. (DLCM2) is more interesting.

Using Lemma 8.3.5, we define new homomorphisms \mathfrak{s}' , \mathfrak{t}' and \mathfrak{i}' making the following diagrams commute:

$$\begin{array}{ccccc} \mathfrak{c}_1 & \xrightarrow{\mathfrak{s}} & \mathfrak{c}_0 & & \mathfrak{c}_1 & \xrightarrow{\mathfrak{t}} & \mathfrak{c}_0 & & \mathfrak{c}_0 & \xrightarrow{\mathfrak{i}} & \mathfrak{c}_1 \\ \psi \uparrow & & \downarrow \text{id} & & \psi \uparrow & & \downarrow \text{id} & & \text{id} \uparrow & & \downarrow \phi \\ F \rtimes E & \xrightarrow{\mathfrak{s}'} & E & & F \rtimes E & \xrightarrow{\mathfrak{t}'} & E & & E & \xrightarrow{\mathfrak{i}'} & F \rtimes E \end{array}$$

A short calculation shows that:

1. $\mathfrak{s}' : F \rtimes E \rightarrow E$ satisfies $(k, x) \mapsto x$,
2. $\mathfrak{t}' : F \rtimes E \rightarrow E$ satisfies $(k, x) \mapsto \delta(k) + x$, and

3. $i' : E \rightarrow F \rtimes E$ satisfies $(k, x) \mapsto (0, x)$.

It is clear that s' and i' are homomorphisms. The map t' is a homomorphism as a consequence of (DLCM1).

Since \mathfrak{c} is a category, from [18], a composable pair (f, g) satisfies:

$$f \circ g = f - i(s(f)) + g \quad (40)$$

Using Lemma 8.3.5, we define composition on the differential Lie crossed module, denoted \bullet , so that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{c}_1 \times_{\mathfrak{c}_0} \mathfrak{c}_1 & \xrightarrow{\phi \times \phi} & (F \rtimes E) \times_E (F \rtimes E) \\ \downarrow \circ & & \downarrow \bullet \\ \mathfrak{c}_1 & \xrightarrow{\phi} & F \rtimes E \end{array}$$

Another short calculation shows that:

$$(k, x) \bullet (l, y) = (k + l, y),$$

whenever $((k, x), (l, y)) \in (F \rtimes E) \times_E (F \rtimes E)$.

We now use the fact that composition in the Lie 2-algebra is a homomorphism to arrive at (DLCM2). Recall that since composition is a homomorphism, it satisfies the exchange law. Namely, for composable pairs $(f_1, g_1), (f_2, g_2) \in \mathfrak{c}_1 \rtimes_{\mathfrak{c}_0} \mathfrak{c}_1$, then

$$[f_1, f_2] \circ [g_1, g_2] = [f_1 \circ g_1, f_2 \circ g_2].$$

Let (f_1, g_1) and (f_2, g_2) be two composable pairs. Using the isomorphism ϕ from Lemma 8.3.5, we write $\phi(f_1) = (k, x), \phi(f_2) = (l, y), \phi(g_1) = (k', x')$, and $\phi(g_2) = (l', y')$. Since $s'(f_1) = t'(g_1)$, then $x = \delta(k') + x'$, and similarly for the other pair. Substituting these values into the exchange law, we obtain for $[\phi(f_1), \phi(f_2)] \bullet [\phi(g_1), \phi(g_2)]$:

$$\begin{aligned} &= ([k, l] + \mathfrak{a}(x)[l] - \mathfrak{a}(y)[k], [x, y]) \bullet \\ &\quad ([k', l'] + \mathfrak{a}(x')[l'] - \mathfrak{a}(y')[k'], [x', y']) \quad (\text{product}) \\ &= ([k, l] + \mathfrak{a}(x)[l] - \mathfrak{a}(y)[k] + [k', l'] + \\ &\quad \mathfrak{a}(x')[l'] - \mathfrak{a}(y')[k'], [x', y']) \quad (\text{composition}) \\ &= ([k, l] + \mathfrak{a}(\delta(k') + x')[l] - \mathfrak{a}(\delta(l') + y')[k] + \\ &\quad [k', l'] + \mathfrak{a}(x')[l'] - \mathfrak{a}(y')[k'], [x', y']) \quad (\text{composable pairs}) \\ &= ([k, l] + \mathfrak{a}(\delta(k'))[l] - \mathfrak{a}(\delta(l'))[k] + \mathfrak{a}(x')[l + l'] \\ &\quad - \mathfrak{a}(y')[k + k'] + [l, l'], [x', y']). \quad (\mathfrak{a}(x') \text{ is a linear map}) \end{aligned}$$

On the other hand, we must also have

$$\begin{aligned} [(k, x) \bullet (k', x'), (l, y) \bullet (l', y')] &= [(k + k', x'), (l + l', y')] && \text{(composition)} \\ &= ([k + k', l + l'] + \mathfrak{a}(x')[l + l'] \\ &\quad - \mathfrak{a}(y')[k + k'], [x', y']). && \text{(product)} \end{aligned}$$

Expanding and canceling terms from both sides, we obtain the following identity:

$$[k', l] + [k, l'] = \mathfrak{a}(t(k))[l'] - \mathfrak{a}(t(l))[k'].$$

Since $k, k' \in F$, and $l, l' \in E$ were arbitrary, and $\tau = t|_{\ker \mathfrak{s}}$, then we must have

$$\mathfrak{a}(\tau(k))[l] = [k, l],$$

which is (DLCM2), as desired. Consequently, the quadruple $(E, F, \delta, \mathfrak{a})$ is a differential Lie crossed module.

Now, given a differential Lie crossed module, we will construct a Lie 2-algebra as follows: Let $(E, F, \delta, \mathfrak{a})$ be a differential Lie crossed module. Let $\mathfrak{c}_0 = E$ and $\mathfrak{c}_1 = F \rtimes E$ where we define the bracket using \mathfrak{a} in the normal fashion:

$$[(h, g), (h', g')] = ([h, h'] + \mathfrak{a}(g)[h'] - \mathfrak{a}(g')[h], [g, g']).$$

This bracket is clearly bilinear and anti-symmetric. It satisfies the Jacobi identity because \mathfrak{a} is a homomorphism and $\mathfrak{a}(x)$ is a derivation.

Define $\mathfrak{s} : F \rtimes E \rightarrow E$ by $(h, g) \mapsto g$, $t : F \rtimes E \rightarrow E$ by $(h, g) \mapsto \delta(h) + g$, and $i : E \rightarrow F \rtimes E$ by $g \mapsto (0, g)$. Clearly, \mathfrak{s} and i are homomorphisms. From (DLCM1), the map t is also a homomorphism.

Let $((h, g), (h', g'))$ be a composable pair in $F \rtimes E$. Define composition by:

$$(h, g) \circ (h', g') = (h + h', g')$$

To show that composition is a homomorphism, it suffices to show that the exchange law is satisfied. Namely, for composable pairs $(f_1, g_1), (f_2, g_2) \in \mathfrak{c}_1 \times_{\mathfrak{c}_0} \mathfrak{c}_1$:

$$[f_1, f_2] \circ [g_1, g_2] = [f_1 \circ g_1, f_2 \circ g_2]'. \quad (41)$$

Write $f_1 = (h, g)$, $f_2 = (l, m)$, $g_1 = (h', g')$, and $g_2 = (l', m')$. Since $\mathfrak{s}(f_1) = \mathfrak{t}(g_1)$, then $g = \delta(h') + g'$, and similarly for the other pair. Substituting these values into the exchange law, we obtain

$$\begin{aligned}
[f_1, f_2] \circ [g_1, g_2] &= ([h, l] + \mathfrak{a}(g)[l] - \mathfrak{a}(m)[h], [g, m]) \circ \\
&\quad ([h', l'] + \mathfrak{a}(g')[l'] - \mathfrak{a}(m')[h'], [g', m']) \quad (\text{product}) \\
&= ([h, l] + \mathfrak{a}(g)[l] - \mathfrak{a}(m)[h] + \\
&\quad [h', l'] + \mathfrak{a}(g')[l'] - \mathfrak{a}(m')[h'], [g', m']) \quad (\text{composition}) \\
&= ([h, l] + \mathfrak{a}(\delta(h') + g')[l] - \mathfrak{a}(\delta(l') + m')[h] \\
&\quad [h', l'] + \mathfrak{a}(g')[l'] - \mathfrak{a}(m')[h'], [g', m']) \quad (\text{composable pairs}) \\
&= ([h, l] + \mathfrak{a}(\delta(h'))[l] - \mathfrak{a}(\delta(l'))[h]) \\
&\quad [h', l'] + \mathfrak{a}(g')[l + l'] - \mathfrak{a}(m')[h + h'], [g', m']) \quad (\mathfrak{a} \text{ is a homomorphism}) \\
&= ([h, l] + [h', l] - [l', h] + [h', l'] + \\
&\quad \mathfrak{a}(g')[l + l'] - \mathfrak{a}(m')[h + h'], [g', m']) \quad (\text{DLCM2}) \\
&= ([h + h', l + l'] + \mathfrak{a}(g')[l + l'] - \\
&\quad \mathfrak{a}(m')[h + h'], [g', m']) \quad (\mathfrak{a}(x) \text{ is a linear map}) \\
&= [(h + h', g'), (l + l', m')] \quad (\text{product}) \\
&= [(h, g) \circ (h', g'), (l, m) \circ (l', m')] \quad (\text{composition}) \\
&= [f_1 \circ g_1, f_2 \circ g_2].
\end{aligned}$$

Therefore, composition is a homomorphism, and the collection $(\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{s}, \mathfrak{t}, \mathfrak{i}, \circ)$ is a Lie 2-algebra.

It remains to show that there is an equivalence of categories between the Lie 2-algebras and the differential Lie crossed modules. Given a Lie 2-algebra $(\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{s}, \mathfrak{t}, \mathfrak{i}, \circ)$, we construct a differential Lie crossed module $(E, F, \delta, \mathfrak{a})$. From the differential Lie crossed module $(E, F, \delta, \mathfrak{a})$, we construct a new Lie 2-algebra $(\mathfrak{c}'_0, \mathfrak{c}'_1, \mathfrak{s}', \mathfrak{t}', \mathfrak{i}', \bullet)$, and show that it is isomorphic to the original Lie 2-algebra. By construction, $E = \mathfrak{c}_0$ and $F = \ker \mathfrak{s}$. The constructed Lie 2-algebra has form $(\mathfrak{c}_0, (\ker \mathfrak{s}) \rtimes \mathfrak{c}_0, \mathfrak{s}', \mathfrak{t}', \mathfrak{i}', \bullet)$. From Lemma 8.3.5, we have an isomorphism ϕ between \mathfrak{c}_1 and $(\ker \mathfrak{s}) \rtimes \mathfrak{c}_0$. As in Proposition 8.2.6, we define a functor F by: $F_0 : \mathfrak{c}_0 \rightarrow \mathfrak{c}_0$ is the identity map, and $F_1 : \mathfrak{c}_1 \rightarrow (\ker \mathfrak{s}) \rtimes \mathfrak{c}_0$ is the isomorphism ϕ . With this definition, F is clearly an isomorphism of Lie algebras.

To finish, given a differential Lie crossed module (E, F, δ, α) , we construct a Lie 2-algebra $(\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{s}, \mathfrak{t}, \mathfrak{i}, \circ)$. From the Lie 2-algebra, we construct a new differential Lie crossed module $(E', F', \delta', \alpha')$, and show that it is isomorphic to the original differential Lie crossed module. Using the same ideas as above, it is a simple matter to construct the isomorphism between the differential Lie crossed modules. Therefore, we have equivalence between the Lie 2-algebras and the differential Lie crossed modules. \square

8.4 Principal 2-Bundles

As in [6], we generalize the concept of a connection and its curvature by replacing the Lie group and its Lie algebras with their 2-analogues. The task of constructing a generalized bundle on which these connections exist is very challenging because of the constraints associated with the “gluing” of the various trivializations together. Fortunately, by restricting ourselves to a trivial “principal 2-bundle,” we can still observe some interesting basic properties.

We remark that, by analogy with a trivial principal bundle, we can think of a connection on a trivial principal 2-bundle as being a Lie 2-algebra valued form on the base space M , as follows:

Definition 8.4.1 ([6], p.14). Let C be a Lie 2-group and (G, H, τ, α) , the corresponding Lie crossed module. A *2-connection* on the manifold M is a pair (A, B) consisting of a \mathfrak{g} -valued 1-form A , and an \mathfrak{h} -valued 2-form B , where \mathfrak{g} and \mathfrak{h} are the Lie algebras of G and H respectively.

Remark 8.4.2. In what follows, we use \mathfrak{t} and \mathfrak{a} to denote the derivatives of τ and α respectively at the appropriate units.

In order to define the curvature of (A, B) , we will need to define the wedge product of \mathfrak{g} and \mathfrak{h} -valued forms, as follows: Let $x, x' \in \mathfrak{g}$ (or \mathfrak{h}), and $\mu, \mu' \in \Omega(M)$. Define the product “ \wedge ” as

$$(x \otimes \mu) \wedge (x' \otimes \mu') = [x, x'] \otimes \mu \wedge \mu',$$

and extend linearly in each argument to all pairs of \mathfrak{g} (or \mathfrak{h})-valued differential forms.

We also define the product “ $\underline{\Delta}$ ” of a \mathfrak{g} - and a \mathfrak{h} -valued differential form as

$$(x \otimes \mu) \underline{\Delta} (y \otimes \nu) = \mathfrak{a}(x)[y] \otimes (\mu \wedge \nu)$$

where $x \in \mathfrak{g}, y \in \mathfrak{h}$ and $\mu, \nu \in \Omega(M)$, and extend linearly in each argument to all pairs of \mathfrak{g} and \mathfrak{h} -valued differential forms. Note that $X \underline{\Delta} Y$ is a \mathfrak{h} -valued form.

Finally, given a \mathfrak{h} -valued differential form $Y = y \otimes \nu$ where $y \in \mathfrak{h}$ and $\nu \in \Omega(M)$, we define a \mathfrak{g} -valued differential form

$$\underline{Y} = \mathfrak{t}(y) \otimes \nu,$$

and extend linearly to all \mathfrak{h} -valued differential forms.

With the above definitions, we define the curvature as follows:

Definition 8.4.3 ([6],pg.14). Let (A, B) be a 2-connection on M . The *curvature* of (A, B) is the pair (F, G) where F is a \mathfrak{g} -valued 2-form defined by

$$F = dA + \frac{1}{2}A \wedge A - \underline{B}, \quad (42)$$

and G is a \mathfrak{h} -valued 3-form defined by

$$G = dB + A \underline{\Delta} B. \quad (43)$$

The definition for F is very similar to the equation for the curvature on a trivial principal G -bundle (example 4.6.6).

Define the exterior covariant derivative of a \mathfrak{g} -valued differential form X as

$$d_A X = dX + A \wedge X.$$

Like our definition for F , this is analogous to a previous result involving bundle-valued differential forms (Lemma 4.7.4).

Similarly, for a \mathfrak{h} -valued differential form Y , the exterior covariant derivative is defined

$$d_A Y = dY + A \underline{\Delta} Y.$$

In this notation, it is clear that $G = d_A B$.

We remark that the products \wedge and $\underline{\Delta}$ are not associative. In fact, there is a graded Jacobi identity which is captured in the following results: Let \mathfrak{e} be a Lie algebra.

Lemma 8.4.4. *Let A and B be \mathfrak{e} -valued j and k -forms respectively. Then,*

$$A \wedge B = (-1)^{jk+1} B \wedge A.$$

Proof. The proof is a straightforward consequence of the anti-symmetry properties of the Lie bracket and the wedge product on differential forms. \square

Lemma 8.4.5. *Let $A, B,$ and C be \mathfrak{e} -valued j, k and l -forms respectively. Then,*

1. $(A \wedge B) \wedge C + (-1)^{l(j+k)}(C \wedge A) \wedge B + (-1)^{j(k+l)}(B \wedge C) \wedge A = 0,$ and
2. $A \wedge (B \wedge C) + (-1)^{l(j+k)}C \wedge (A \wedge B) + (-1)^{j(k+l)}B \wedge (C \wedge A) = 0.$

Proof. The first equation is trilinear in A, B and C , so it suffices to examine $A = x \otimes \mu, B = y \otimes \nu, C = z \otimes \omega$ where $x, y, z \in \mathfrak{e}$, and μ, ν, ω are j, k and l -forms respectively. Then,

$$\begin{aligned} (A \wedge B) \wedge C &= [[x, y], z] \otimes \mu \wedge \nu \wedge \omega \\ (C \wedge A) \wedge B &= [[z, x], y] \otimes \omega \wedge \mu \wedge \nu \\ (B \wedge C) \wedge A &= [[y, z], x] \otimes \nu \wedge \omega \wedge \mu \end{aligned}$$

After reordering the forms on the right,

$$\begin{aligned} (A \wedge B) \wedge C &+ (-1)^{l(j+k)}(C \wedge A) \wedge B + (-1)^{j(k+l)}(B \wedge C) \wedge A \\ &= ([[x, y], z] + [[z, x], y] + [[y, z], x]) \otimes \mu \wedge \nu \wedge \omega \\ &= 0, \end{aligned}$$

by the Jacobi identity in \mathfrak{e} . Thus, we have the first identity. The second identity follows trivially from the application of Lemma 8.4.4 to the first identity. \square

Remark 8.4.6. Let A be any Lie algebra valued differential k -form. By Lemma 8.4.5, we must have $(A \wedge A) \wedge A = 0$. Moreover, if B is a Lie algebra valued differential $(2n)$ -form, then, by Lemma 8.4.4, $B \wedge B = 0$.

Some further identities involving the various products will also be useful in what follows.

Lemma 8.4.7. *Let A be a \mathfrak{g} -valued differential form, and let B and B' be two \mathfrak{h} -valued differential forms. Then,*

$$\begin{aligned} A \wedge \underline{B} &= \underline{A \wedge B}, \text{ and} \\ B \wedge B' &= \underline{B \wedge B'}. \end{aligned}$$

Proof. Since both equations are linear in each variable, it suffices to examine the case when $A = x \otimes \mu$, $B = y \otimes \nu$ and $B' = y' \otimes \nu'$ where $x \in \mathfrak{g}$, $y, y' \in \mathfrak{h}$ and $\mu, \nu, \nu' \in \Omega(M)$. Then, in the first case:

$$\begin{aligned} A \wedge \underline{B} &= (x \otimes \mu) \wedge (\mathfrak{t}(y) \otimes \nu) \\ &= [x, \mathfrak{t}(y)] \otimes (\mu \wedge \nu) \\ &= \mathfrak{t}(\mathfrak{a}(x)[y]) \otimes (\mu \wedge \nu) \quad (\text{DLCM1.}) \\ &= \underline{\mathfrak{a}(x)[y] \otimes (\mu \wedge \nu)} \\ &= \underline{(x \otimes \mu) \wedge (y \otimes \nu)} \\ &= \underline{A \wedge B}. \end{aligned}$$

In the second case:

$$\begin{aligned} \underline{B \wedge B'} &= \underline{(y \otimes \nu) \wedge (y' \otimes \nu')} \\ &= \underline{(\mathfrak{t}(y) \otimes \nu) \wedge (y' \otimes \nu')} \\ &= \underline{\mathfrak{a}(\mathfrak{t}(y))[y'] \otimes (\nu \wedge \nu')} \\ &= \underline{[y, y'] \otimes (\nu \wedge \nu')} \quad (\text{DLCM2.}) \\ &= \underline{(y \otimes \nu) \wedge (y' \otimes \nu')} \\ &= \underline{B \wedge B'}, \end{aligned}$$

as desired. □

Lemma 8.4.8. *Let B be a \mathfrak{h} -valued differential form. Then,*

$$\underline{d_A B} = d_A \underline{B}.$$

Proof. We have

$$\begin{aligned} \underline{d_A B} &= \underline{dB + A \wedge B} \\ &= \underline{dB} + \underline{A \wedge B} \\ &= \underline{dB} + A \wedge \underline{B}. \end{aligned}$$

It suffices to show that $\underline{d}B = d\underline{B}$. Since the equation is linear in B , it suffices to examine the case when $B = y \otimes \mu$ where $y \in \mathfrak{h}$ and $\mu \in \Omega(M)$. Then,

$$\begin{aligned} \underline{d}(y \otimes \mu) &= \underline{(y \otimes d\mu)} \\ &= \mathfrak{t}(y) \otimes d\mu \\ &= d\underline{B}. \end{aligned}$$

Hence, $d_A B = d\underline{B} + A \wedge \underline{B} = d_A \underline{B}$, as desired. \square

Lemma 8.4.9. *Let A be a \mathfrak{g} -valued differential $(2n+1)$ -form, and let B be a \mathfrak{h} -valued differential form. Then,*

$$A \underline{\wedge} (A \underline{\wedge} B) = \frac{1}{2} (A \wedge A) \underline{\wedge} B.$$

Proof. Since the equation is linear in B , it suffices to examine the case when $B = y \otimes \nu$ where $y \in \mathfrak{h}$ and $\nu \in \Omega(M)$, and $A = \sum_{i \in I} x_i \otimes \mu^i$ where $x_i \in \mathfrak{g}$ and $\mu^i \in \Omega(M)$. Then,

$$A \underline{\wedge} (A \underline{\wedge} B) = \sum_{i, j \in I} \mathfrak{a}(x_i) [\mathfrak{a}(x_j) [y]] \otimes (\mu^i \wedge \mu^j \wedge \nu)$$

Since A is an odd form, then $\mu^i \wedge \mu^j = -\mu^j \wedge \mu^i$ and $\mu^i \wedge \mu^i = 0$. Using these properties, we can rewrite the sum as

$$A \underline{\wedge} (A \underline{\wedge} B) = \sum_{i < j \in I} (\mathfrak{a}(x_i) [\mathfrak{a}(x_j) [y]] - \mathfrak{a}(x_j) [\mathfrak{a}(x_i) [y]]) \otimes (\mu^i \wedge \mu^j \wedge \nu).$$

Since \mathfrak{a} is a Lie algebra homomorphism, then $\mathfrak{a}([x_i, x_j]) [y] = [\mathfrak{a}(x_i), \mathfrak{a}(x_j)] [y] = \mathfrak{a}(x_i) [\mathfrak{a}(x_j) [y]] - \mathfrak{a}(x_j) [\mathfrak{a}(x_i) [y]]$. Then, we can rewrite the sum as

$$A \underline{\wedge} (A \underline{\wedge} B) = \sum_{i < j \in I} \mathfrak{a}([x_i, x_j]) [y] \otimes (\mu^i \wedge \mu^j \wedge \nu). \quad (44)$$

Since $\mathfrak{a}([x_i, x_j]) [y] \otimes (\mu^i \wedge \mu^j \wedge \nu) + \mathfrak{a}([x_j, x_i]) [y] \otimes (\mu^j \wedge \mu^i \wedge \nu) = 2\mathfrak{a}([x_i, x_j]) [y] \otimes (\mu^i \wedge \mu^j \wedge \nu)$, we can rewrite the equation as

$$\begin{aligned} A \underline{\wedge} (A \underline{\wedge} B) &= \frac{1}{2} \sum_{i, j \in I} \mathfrak{a}([x_i, x_j]) [y] \otimes (\mu^i \wedge \mu^j \wedge \nu) \\ &= \frac{1}{2} (A \wedge A) \underline{\wedge} B, \end{aligned}$$

as desired. \square

With our definitions and identities in place, we can now develop the higher order Yang-Mills equations over the trivial principal 2-bundle. We begin by examining the Bianchi identity.

Proposition 8.4.10. *If (A, B) is a connection on M , then its curvature (F, G) satisfies the higher Bianchi identities*

$$\text{HB1. } d_A F = -\underline{G}, \text{ and}$$

$$\text{HB2. } d_A G = F \wedge B.$$

Proof. By straightforward calculation and the use of Lemma 8.4.4, Lemma 8.4.5 (see remark), and Lemma 8.4.8, we obtain

$$\begin{aligned} d_A F &= dF + A \wedge F \\ &= -\underline{d_A B}. \end{aligned}$$

Since $G = d_A B$, then $d_A F = -\underline{G}$, as desired.

For (HB2), we examine $d_A G = d_A^2 B$. By straightforward calculation,

$$\begin{aligned} d_A^2 B &= d_A(dB + A \wedge B) \\ &= d(dB + A \wedge B) + A \wedge (dB + A \wedge B) \\ &= d(A \wedge B) + A \wedge dB + A \wedge (A \wedge B). \end{aligned}$$

Since $d(A \wedge B) = dA \wedge B - A \wedge dB$, this becomes

$$d_A^2 B = dA \wedge B + A \wedge (A \wedge B)$$

By Lemma 8.4.9, we can rewrite this as

$$d_A^2 B = dA \wedge B + \frac{1}{2}(A \wedge A) \wedge B.$$

On the other hand,

$$\begin{aligned} F \wedge B &= (dA + \frac{1}{2}A \wedge A - \underline{B}) \wedge B \\ &= dA \wedge B + \frac{1}{2}(A \wedge A) \wedge B - \underline{B} \wedge B. \end{aligned} \tag{45}$$

By Lemma 8.4.7, $\underline{B} \triangle B = B \wedge B$. Since B is an even form, by Lemma 8.4.4, $B \wedge B = 0$. Rewriting the equation, we have

$$\begin{aligned} F \triangle B &= dA \triangle B + \frac{1}{2}(A \wedge A) \triangle B \\ &= d_A^2 B, \end{aligned}$$

as desired. \square

In order to set-up higher order Yang-Mills theory following the steps in [6], we use the action principle to impose restrictions on the connection and the curvature. In order to proceed, we assume that our space-time manifold M is oriented and equipped with a metric. As well, we assume that the gauge group is a Lie 2-group equipped with a pair of non-degenerate symmetric bilinear forms on its Lie 2-algebra. That is, if (G, H, τ, α) is the Lie crossed module associated with the Lie 2-group, then there are non-degenerate, bilinear forms on the Lie algebras \mathfrak{g} and \mathfrak{h} associated with the Lie groups G and H respectively. We require that the bilinear forms be invariant under their respective adjoint actions. In addition, the form on \mathfrak{h} must also be invariant under the action of G on H . We will write the bilinear forms as $\langle \cdot, \cdot \rangle$, and include the subscript \mathfrak{g} or \mathfrak{h} as necessary to indicate the associated Lie algebra.

As a consequence of their invariance under the adjoint action, both bilinear forms satisfy

$$\langle [x, x'], x'' \rangle = -\langle x', [x, x''] \rangle, \quad (46)$$

where $x, x', x'' \in \mathfrak{g}$ (or \mathfrak{h}). The bilinear form on \mathfrak{h} also satisfies

$$\langle \mathfrak{a}(x)y, y' \rangle = -\langle y, \mathfrak{a}(x)y' \rangle, \quad (47)$$

where $x \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$.

In [6], Baez generalizes the classical *trace* on $\text{End}(E)$ as follows: Let \mathfrak{e} be a Lie algebra with bilinear form $\langle \cdot, \cdot \rangle$.

Definition 8.4.11. Let $x \wedge \mu$ and $y \wedge \nu$ be two \mathfrak{e} -valued differential forms, where $x, y \in \mathfrak{e}$, and $\mu, \nu \in \Omega(M)$. We define their *trace*, denoted tr , as

$$\text{tr}((x \otimes \mu), (y \otimes \nu)) = \langle x, y \rangle \otimes \mu \wedge \nu,$$

and extend this linearly in each terms to arbitrary pairs of \mathfrak{e} -valued forms.

Remark 8.4.12. In [6], Baez writes his trace as

$$\mathrm{tr}((x \otimes \mu) \wedge (y \otimes \nu)) = \langle x, y \rangle \otimes (\mu \wedge \nu)$$

in order to recapture the “look-and-feel” of the classical trace ([5] p.273). However, there is some confusion about the order of operations in this notation. Whereas the classical trace is defined for a single $\mathrm{End}(E)$ -valued form, our new definition requires *two* \mathfrak{e} -valued forms to be defined. In Baez’s notation, the trace is no longer defined if we take the wedge product inside brackets first.

The Hodge dual of a Lie algebra-valued differential form is defined as

$$\star(x \otimes \mu) = x \otimes \star\mu,$$

where x is in the Lie algebra, $\mu \in \Omega(M)$, and the definition is extended linearly in each variable over arbitrary Lie algebra-valued differential forms.

Finally, Baez generalizes the classical Yang-Mills action to include the second component of the curvature (A, B) as

$$S(A, B) = \int_M \mathrm{tr}(F, \star F) + \mathrm{tr}(G, \star G),$$

where (F, G) is the curvature of (A, B) . To obtain the Yang-Mills equations, we compute the critical points of the variation δS . As before, we will obtain a series of restrictions on A, B, F and G which, together with the higher Bianchi identities, will form our Yang-Mills equations.

The following lemmas will be useful in what follows. Let \mathfrak{e} be a finite-dimensional Lie algebra.

Lemma 8.4.13. *Let X and Y be two \mathfrak{e} -valued n -forms. Then,*

$$\mathrm{tr}(X, \star Y) = \mathrm{tr}(Y, \star X).$$

Proof. Since the equation is linear in each variable, it suffices to examine the case where $X = x \otimes \mu$ and $Y = y \otimes \nu$ where $x, y \in \mathfrak{e}$ and $\mu, \nu \in \Omega(M)$. Since M

is oriented and semi-Riemannian, and since X and Y are both n -forms, then, by definition of the Hodge star operator, $\mu \wedge \star\nu = \langle\langle\mu, \nu\rangle\rangle \text{vol}$ where $\langle\langle\cdot, \cdot\rangle\rangle$ is the inner product defined from the metric on M . Since the inner product is symmetric, then $\langle\langle\mu, \nu\rangle\rangle \text{vol} = \langle\langle\nu, \mu\rangle\rangle \text{vol} = \nu \wedge \star\mu$. Then,

$$\begin{aligned} \text{tr}(X, \star Y) &= \text{tr}(x \otimes \mu, y \otimes \star\nu) \\ &= \langle x, y \rangle \otimes (\mu \wedge \star\nu) \\ &= \langle y, x \rangle \otimes (\nu \wedge \star\mu) \\ &= \text{tr}(Y, \star X), \end{aligned}$$

as desired. □

Lemma 8.4.14. *Let X, Y and Z be \mathfrak{e} -valued forms. Then,*

$$\text{tr}(X, Y \wedge Z) = \text{tr}(X \wedge Y, Z).$$

Proof. Since the equation is linear in each variable, it suffices to examine the case where $X = x \otimes \mu$, $Y = y \otimes \nu$ and $Z = z \otimes \eta$, where $x, y, z \in \mathfrak{e}$, $\mu, \nu, \eta \in \Omega(M)$. Then,

$$\begin{aligned} \text{tr}(X, Y \wedge Z) &= \text{tr}(x \otimes \mu, (y \otimes \nu) \wedge (z \otimes \eta)) \\ &= \text{tr}(x \otimes \mu, [y, z] \otimes (\nu \wedge \eta)) \\ &= \langle x, [y, z] \rangle \otimes (\mu \wedge \nu \wedge \eta) \\ &= \langle [x, y], z \rangle \otimes (\mu \wedge \nu \wedge \eta) \quad (\text{via equation (46)}) \\ &= \text{tr}(X \wedge Y, Z), \end{aligned}$$

as desired. □

Lemma 8.4.15. *Let X be a \mathfrak{e} -valued p -form, and let Y be an arbitrary \mathfrak{e} -valued form. Then,*

$$d \text{tr}(X, Y) = \text{tr}(dX, Y) + (-1)^p \text{tr}(X, dY).$$

Proof. Since the equation is linear in each variable, it suffices to examine the case

when $X = x \otimes \mu$ and $Y = y \otimes \nu$ where $x, y \in \mathfrak{e}$, $\mu \in \Omega^p(M)$, and $\nu \in \Omega(M)$. Then,

$$\begin{aligned}
 d \operatorname{tr}(X, Y) &= d \operatorname{tr}(x \otimes \mu, y \otimes \nu) \\
 &= d(\langle x, y \rangle \otimes (\mu \wedge \nu)) \\
 &= \langle x, y \rangle \otimes (d\mu \wedge \nu + (-1)^p \mu \wedge d\nu) \\
 &= \langle x, y \rangle \otimes (d\mu \wedge \nu) + (-1)^p \langle x, y \rangle \otimes (\mu \wedge d\nu) \\
 &= \operatorname{tr}(x \otimes d\mu, y \otimes \nu) + (-1)^p \operatorname{tr}(x \otimes \mu, y \otimes d\nu) \\
 &= \operatorname{tr}(d(x \otimes \mu), y \otimes \nu) + (-1)^p \operatorname{tr}(x \otimes \mu, d(y \otimes \nu)) \\
 &= \operatorname{tr}(dX, Y) + (-1)^p \operatorname{tr}(X, dY),
 \end{aligned}$$

as desired. □

Denote the collection of all Lie algebra \mathfrak{e} -valued differential forms on M by $\Omega(M; \mathfrak{e})$. We shall now define a bilinear map

$$\bar{\wedge} : \Omega(M; \mathfrak{h}) \times \Omega(M; \mathfrak{h}) \rightarrow \Omega(M; \mathfrak{g})$$

that is “adjoint” to $\underline{\wedge}$ with respect to the trace, in the sense that it will satisfy

$$\operatorname{tr}(X \underline{\wedge} Y, Z) = \operatorname{tr}(X, Y \bar{\wedge} Z)$$

for $X \in \Omega(M; \mathfrak{g})$ and $Y, Z \in \Omega(M; \mathfrak{h})$.

First, consider the trilinear map

$$\mathfrak{g} \times \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{R},$$

defined by $(x, y, y') \mapsto \langle \mathfrak{a}(x)[y], y' \rangle_{\mathfrak{h}}$. By the universal property of the tensor product, and the fact that $\operatorname{Hom}(V \otimes W, \mathbb{R}) \cong \operatorname{Hom}(W, V^*)$ for finite dimensional vector spaces V and W , this induces a linear map,

$$\tilde{\sigma} : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{g}^*,$$

satisfying $\tilde{\sigma}(y, y')[x] = \langle \mathfrak{a}(x)[y], y' \rangle_{\mathfrak{h}}$ for all $x \in \mathfrak{g}$, and $y, y' \in \mathfrak{h}$. Finally, let $\phi : \mathfrak{g}^* \rightarrow \mathfrak{g}$ be the linear isomorphism satisfying $f(x) = \langle \phi(f), x \rangle_{\mathfrak{g}}$ for all $f \in \mathfrak{g}^*$, and $x \in \mathfrak{g}$. Then,

$$\sigma = \phi \circ \tilde{\sigma},$$

satisfies $\langle x, \sigma(y, y') \rangle_{\mathfrak{g}} = \langle \mathfrak{a}(x)[y], y' \rangle_{\mathfrak{h}}$.

Now, let $Y = y \otimes \mu$ and $Y' = y' \otimes \mu'$ where $y, y' \in \mathfrak{h}$ and $\mu, \mu' \in \Omega(M)$. We define “ $\bar{\wedge}$ ” by

$$Y \bar{\wedge} Y' = \sigma(y, y') \otimes (\mu \wedge \mu'),$$

and extend it bilinearly over arbitrary pairs of \mathfrak{h} -valued forms. We remark that since the bilinear form on \mathfrak{h} is invariant under the action of G (equation (47)), if Y and Y' are \mathfrak{h} -valued p and q -forms respectively, then $Y \bar{\wedge} Y' = (-1)^{pq} Y' \bar{\wedge} Y$.

We also define a linear map

$$\bar{\cdot} : \Omega(M; \mathfrak{g}) \rightarrow \Omega(M; \mathfrak{h})$$

which is adjoint to $\underline{\cdot} : \Omega(M; \mathfrak{h}) \rightarrow \Omega(M; \mathfrak{g})$ with respect to the trace, in the sense that it will satisfy

$$\text{tr}(X, \underline{Y}) = \text{tr}(\bar{X}, Y).$$

Again, consider the unique linear map

$$\tilde{\mathfrak{t}} : \mathfrak{g} \rightarrow \mathfrak{h}^*$$

defined by $\tilde{\mathfrak{t}}(x)[y] \mapsto \langle x, \mathfrak{t}(y) \rangle_{\mathfrak{g}}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$. Let $\psi : \mathfrak{h}^* \rightarrow \mathfrak{h}$ be the linear isomorphism satisfying $f(y) = \langle \psi(f), y \rangle_{\mathfrak{h}}$ for all $f \in \mathfrak{h}^*$, and $y \in \mathfrak{h}$. Then,

$$\mathfrak{t}^* = \psi \circ \tilde{\mathfrak{t}}$$

satisfies $\langle x, \mathfrak{t}(y) \rangle_{\mathfrak{g}} = \langle \mathfrak{t}^*(x), y \rangle_{\mathfrak{h}}$.

Now, let $X = x \otimes \mu$ where $x \in \mathfrak{g}$, and $\mu \in \Omega(M)$. We define “ $\bar{\cdot}$ ” by

$$\bar{X} = \mathfrak{t}^*(x) \otimes \mu,$$

and extend it linearly over arbitrary \mathfrak{g} -valued forms.

With these results in hand, we expand δS as follows:

$$\begin{aligned} \delta S &= \frac{d}{ds} \int_M \text{tr}((F + s\delta F), \star(F + s\delta F)) + \text{tr}((G + s\delta G), \star(G + s\delta G))|_{s=0} \\ &= \int_M \text{tr}(F, \star\delta F) + \text{tr}(\delta F, \star F) + \text{tr}(G, \star\delta G) + \text{tr}(\delta G, \star G). \end{aligned}$$

Using Lemma 8.4.13, $\text{tr}(F, \star \delta F) = \text{tr}(\delta F, \star F)$ and $\text{tr}(G, \star \delta G) = \text{tr}(\delta G, \star G)$. Then,

$$\delta S = 2 \int_M \text{tr}(\delta F, \star F) + \text{tr}(\delta G, \star G). \quad (48)$$

We now calculate δF and δG as functions of A, B and their variations. Straightforward calculation on F gives

$$\begin{aligned} \delta F &= \frac{d}{ds} \left[d(A + s\delta A) + \frac{1}{2}(A + s\delta A) \wedge (A + s\delta A) - \underline{B + s\delta B} \right] \Big|_{s=0} \\ &= d\delta A + A \wedge \delta A - \underline{\delta B} \\ &= d_A \delta A - \underline{\delta B}. \end{aligned}$$

A similar calculation for G gives

$$\delta G = d_A \delta B + \delta A \underline{\wedge} B.$$

Substituting these results into equation (48), we obtain

$$\begin{aligned} \delta S &= 2 \int_M \text{tr}((d_A \delta A - \underline{\delta B}), \star F) + \text{tr}((d_A \delta B + \delta A \underline{\wedge} B), \star G) \\ &= 2 \int_M \text{tr}(d_A \delta A, \star F) - \text{tr}(\underline{\delta B}, \star F) \\ &\quad + \text{tr}(d_A \delta B, \star G) + \text{tr}((\delta A \underline{\wedge} B) \wedge G). \end{aligned} \quad (49)$$

Using the bilinearity of the trace, we can rewrite $\text{tr}(d_A \delta A, \star F)$ as follows:

$$\begin{aligned} \text{tr}(d_A \delta A, \star F) &= \text{tr}(d\delta A + A \wedge \delta A, \star F) \\ &= \text{tr}(d\delta A, \star F) + \text{tr}(A \wedge \delta A, \star F) \end{aligned}$$

By Lemmas 8.4.4 and 8.4.14, $\text{tr}(A \wedge \delta A, \star F) = \text{tr}(\delta A, A \wedge \star F)$. Then,

$$\text{tr}(d_A \delta A, \star F) = d \text{tr}(\delta A, \star F) + \text{tr}(\delta A, A \wedge \star F) \quad (50)$$

By the bilinearity of the trace, and the definition of the exterior covariant derivative, then

$$\text{tr}(\delta A, d_A \star F) = \text{tr}(\delta A, d \star F) + \text{tr}(\delta A, A \wedge \star F). \quad (51)$$

Substituting equation (51) into equation (50), then

$$\mathrm{tr}(d_A \delta A, \star F) = \mathrm{tr}(d \delta A, \star F) + \mathrm{tr}(\delta A, d_A \star F) - \mathrm{tr}(\delta A, d \star F).$$

Since δA is a 1-form, we can use Lemma 8.4.15 to rewrite the equation as

$$\mathrm{tr}(d_A \delta A, \star F) = d \mathrm{tr}(\delta A, \star F) + \mathrm{tr}(\delta A, d_A \star F)$$

Using Stoke's theorem ([6], pg. 119), we integrate over M as follows:

$$\begin{aligned} \int_M \mathrm{tr}(d_A \delta A, \star F) &= \int_M \mathrm{tr}(\delta A, d_A \star F) + \int_M d(\mathrm{tr}(\delta A, \star F)) \\ &= \int_M \mathrm{tr}(\delta A, d_A \star F) + \int_{\partial M} \mathrm{tr}(\delta A, \star F) \end{aligned}$$

If we assume that δA is compactly supported, then $\int_{\partial M} \mathrm{tr}(\delta A, \star F) = 0$. Therefore,

$$\int_M \mathrm{tr}(d_A \delta A, \star F) = \int_M \mathrm{tr}(\delta A, d_A \star F).$$

Repeating these steps using B and G , and assuming that δB is also compactly supported, we obtain

$$\int_M \mathrm{tr}(d_A \delta B, \star G) = \int_M \mathrm{tr}(-\delta B, d_A \star G).$$

Substituting these values into equation (49), we obtain

$$\begin{aligned} \delta S &= 2 \int_M \mathrm{tr}(\delta A, d_A \star F) - \mathrm{tr}(\delta B, \star F) \\ &\quad + \mathrm{tr}(-\delta B, d_A \star G) + \mathrm{tr}((\delta A \wedge B), \star G) \end{aligned}$$

Using the adjoint maps " $\bar{}$ " and " $-$ ", and the bilinearity of the trace, the equation becomes

$$\begin{aligned} \delta S &= 2 \int_M \mathrm{tr}(\delta A, d_A \star F) - \mathrm{tr}(\delta B, \overline{\star F}) \\ &\quad + \mathrm{tr}(-\delta B, d_A \star G) + \mathrm{tr}(\delta A, B \bar{} \star G) \\ &= 2 \int_M \mathrm{tr}(\delta A, d_A \star F + B \bar{} \star G) - \mathrm{tr}(\delta B, d_A \star G + \overline{\star F}) \\ &= 2 \int_M \mathrm{tr}(\delta A, d_A \star F - \star G \bar{} B) - \mathrm{tr}(\delta B, d_A \star G + \overline{\star F}) \end{aligned}$$

Hence, we obtain the following:

Theorem 8.4.16 ([6], p.18). *Suppose that M is an oriented, semi-Riemannian manifold, and suppose that the Lie 2-group C has an associated Lie crossed module (G, H, t, α) for which \mathfrak{g} and \mathfrak{h} are equipped with non-degenerate, invariant, symmetric bilinear forms. Then, the connection (A, B) on M satisfies $\delta S = 0$ (for arbitrary choices of δA and δB) if and only if the **higher Yang-Mills equations** hold:*

$$\text{HYM1.} \quad d_A \star F = \star G \wedge B, \text{ and}$$

$$\text{HYM2.} \quad d_A \star G = -\star \overline{F}.$$

We conclude by remarking that both equations are similar to the Yang-Mills equation derived in Chapter 7. It is interesting to note that in the earlier development, the right-hand side of the equation was zero, reflecting the absence of the matter fields in the Lagrangian. In this development, our Yang-Mills action was also without matter fields, but we obtained terms mixing F and G . Not surprisingly, this suggests that the relationship between the curvature and a higher “Yang-Mills current” is more complicated than previously.

Appendix A

Notation

The following is a list of common notation used within this document. This list is not exhaustive, and notation in the text may differ slightly from what is presented here.

<u>Notation</u>	<u>Definition</u>
M	Oriented, semi-Riemannian, Hausdorff manifold.
U	Subset of M .
G	Lie group.
\mathfrak{g}	Lie algebra of G .
(P, π, M, G)	Principal bundle.
(E, π, M, F)	Fibre bundle.
(E, π, M, V)	Vector bundle.
V	Vector space over \mathbb{R} .
$C^\infty(M)$	Smooth functions from M to \mathbb{R} .
$C^\infty(M; V)$	Smooth functions from M to V .
Γ	Connection on P .
Γ_E	Connection on E .
Ω	Curvature on P .
Ω_E	Curvature on E .
ω	Connection form on P .
Ω, F	Curvature form, electromagnetic field.

<u>Notation</u> (con't)	<u>Definition</u> (cont'd)
$\Gamma(U)$	Sections of P over U .
$\Gamma_E(U)$	Sections of E over U .
$\Omega(M)$	Differential forms on M .
$\Omega^p(M)$	Differential p -forms on M .
$\Omega(M; V)$	V -valued differential forms on M .
$\Omega^p(M; V)$	V -valued differential p -forms on M .
$\text{Vect}(M)$	Vector fields on M .
$T_x M, T_u P, T_{u\xi} E$	Tangent space at $x \in M, u \in P, u\xi \in E$.
$\text{GL}(n)$	n -dimensional general linear group.
$U(n), SU(n)$	n -dimensional unitary, special unitary group.
$O(n), SO(n)$	n -dimensional orthogonal, special orthogonal group.
$[\cdot, \cdot]$	Lie bracket or commutator.
$\langle \cdot, \cdot \rangle$	Bilinear form or inner product.
$\text{End}(M), \text{Aut}(M), \text{Der}(M)$	Endomorphisms, automorphisms, derivations on M .
V_u, H_u	Vertical, horizontal subspaces of $T_u P$.
$V_{u\xi}, H_{u\xi}$	Vertical, horizontal subspaces of $T_{u\xi} E$.

Bibliography

- [1] Abraham, R. and Marsden, J.E. *Foundations of Mechanics*. Reading: Benjamin/Cummings, 1978.
- [2] Aharonov, Y. and Bohm, D., "Significance of Electromagnetic Potentials in the Quantum Theory." *Physical Review*. Vol 115, No. 3, 01 August 1959, 495 - 491.
- [3] Arnol'd, V.I. *Mathematical Methods of Classical Mechanics*. New York: Springer-Verlag, 1989.
- [4] Atiyah, M.F. *The Geometry of Physics and Knots*. Cambridge: Cambridge University Press, 1990.
- [5] Baez, J. and Munian, J.P. *Gauge Fields, Knots and Gravity*. Singapore: World Scientific Press, 1994.
- [6] Baez, J.C. "Higher Yang-Mills Theory." 17 Jun 2002 (v2), <http://arxiv.org/abs/hep-th/0206130v1>.
- [7] Bourgin, D.G. *Modern Algebraic Topology*. New York and London: The MacMillan Company, 1963.
- [8] Eckmann, B. and Hilton, P.J. "Group-like Structures in General Categories." *Mathematische Annalen*. Vol.145, No.3, 1962, 227-255.
- [9] Forrester-Barker, M. "Group Objects and Internal Categories." 4 Dec 2002, <http://arxiv.org/abs/math/0212065>.

- [10] Giaquinta, M. and Hildebrandt, S. *Calculus of Variations I - The Lagrangian Formalism* Germany: Springer-Verlag Berlin Heidelberg, 1996.
- [11] Gökeler, M. and Schücker, T. *Differential Geometry, Gauge Theories, and Gravity*. USA: Cambridge University Press, 1987.
- [12] Greub, W., Halperin, S. and Vanstone, R. *Connections, Curvature and Cohomology - Volume 1*. New York and London: Academic Press, 1973.
- [13] Greub, W., Halperin, S. and Vanstone, R. *Connections, Curvature and Cohomology - Volume 2*. New York and London: Academic Press, 1973.
- [14] Gross, D.J. "Gauge Theory - Past, Present and Future?" *Chinese Journal of Physics* Vol. 30, No. 7, December 1992, 955 - 972.
- [15] Kobayashi, S. and Nomizu, K. *Foundations Differential Geometry - Volume 1*. New York and London: Interscience Publishers, 1963.
- [16] Lee, John M. *Introduction to Smooth Manifolds*. New York: Springer-Verlag, 2003.
- [17] Lopez Davalos, A. and Zanette, D. *Fundamentals of Electromagnetism - Vacuum Electrodynamics, Media, and Relativity*. Germany: Springer-Verlag Berlin Heidelberg, 1999.
- [18] MacLane, S. *Categories for the Working Mathematician* Second Edition. New York: Springer Science and Business Media Inc, 1978.
- [19] Munkres, J.R. *Topology*. Second Edition. Upper Saddle River: Prentice Hall, 2000.
- [20] Nering, E.D. *Linear Algebra and Matrix Theory*. Second Edition. New York: John Wiley and Sons, Inc, 1970.
- [21] O'Raiheartaigh, L. and Straumann, N. "Early History of Gauge Theory and Kaluza-Klein Theories, with a Glance at Recent Developments."

- Accepted for publication in *Review of Modern Physics*. 05 Apr 1999,
<http://arxiv.org/abs/hep-ph/9810524>.
- [22] Peskin, M.E. and Schoeder, D.V *An Introduction to Quantum Field Theory*.
USA: Westview Press, 1995.
- [23] Riordan, M. *The Hunting of the Quark - A True Story of Modern Physics*.
Toronto: Simon and Shuster, Inc, 1987.
- [24] Sagle, A.A. and Walde, R.E. *Introduction to Lie Groups and Lie Algebras*.
New York and London: Academic Press, 1973.
- [25] Singh, J. *Quantum Mechanics - Fundamentals and Applications to Technology*.
New York: John Wiley and Sons, Inc, 1997.
- [26] Synge, J.L and Griffith, B.A. *Principles of Mechanics*. Third Edition. Toron-
ton: McGraw-Hill Book Company Inc, 1959.
- [27] Laquer, H.T. "Invariant Gauge Automorphisms of Homogeneous Principal
Bundles." *Geometriae Dedicata*. Vol 33, No.1, January 1990, 27-35.
- [28] Weyl, H. (1918). Gravitation und Elektrizitat. (translation: Gravitation and
Electricity) Sitzunsber, Akademie der Wissenschaften, Berlin, 1918, 465-480.
- [29] Weyl, H. (1929). "Gravitation and the Electron." *Proceedings of the National
Academy of Sciences of the USA*. Vol. 15, 1929, 323 - 334.
- [30] Yang, C.N. and Mills, R.L. "Conservation of Isotopic Spin and Isotopic Gauge
Invariance." *Physical Review* Vol 96, No. 1, October 01, 1954, 191 - 195.