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LA THÈSE A ÉTÉ  
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ON CONSTRUCTIONS OF TITS AND FAULKNER

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by

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## Introduction

J. Tits gave a construction of Lie algebras (for example [1], Page 122) from composition algebras and Jordan algebras of degree 3 over a field of characteristic  $\neq 2, 3$ . By this construction some exceptional algebras of type  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$  and  $E_8$  are obtained.

John R. Faulkner in [7] gave a construction of Lie algebras from a ternary algebra with a skew bilinear form satisfying certain axioms. Some ternary algebras were obtained from Jordan algebras of degree three over a field of arbitrary characteristic. In particular in characteristic 0 exceptional Jordan algebras give rise to Lie algebras of type  $E_8$ .

Classification theory guarantees the existence of an isomorphism between any two  $E_8$ 's, at least over an algebraically closed field of characteristic 0. Our purpose in this thesis is to construct for any Jordan algebra  $J$  of degree 3 over a field  $\phi$  of characteristic  $\neq 2, 3$  an explicit isomorphism between the algebra obtained from  $J$  by Faulkner's construction and the algebra obtained by

from the split octonions and  $J$  by Tits' Construction.

In Chapter I results from the theory of composition algebras and Jordan algebras are recalled.

In Chapter II the constructions of Tits and Faulkner are described.

In Chapter III the isomorphism mentioned above is given.

## Chapter I

### Preliminaries

#### 1. Composition Algebras

1.1 Definition: Let  $\phi$  be a field and  $V$  be a  $\phi$ -vector space. A map  $q: V \rightarrow \phi$  is called a quadratic form if the following are satisfied:

- (1)  $q(\alpha v) = \alpha^2 q(v)$ , for all  $\alpha$  in  $\phi$ ,  $v$  in  $V$
- (2)  $q(v,w) = q(v+w) - q(v) - q(w)$  is a symmetric bilinear form on  $V$ .

A quadratic form  $q$  is called non-degenerate if  $q(v) = 0$  and  $q(v,w) = 0$  for all  $w$  in  $V$  imply  $v = 0$ .

$$\begin{aligned} \text{We note that } q(v,v) &= q(v+v) - q(v) - q(v) \\ &= 4q(v) - 2q(v) = 2q(v) \end{aligned}$$

1.2 Definition: An algebra  $A$  over a field  $\phi$  is a  $\phi$ -vector space satisfying the following conditions:

- (1) a multiplication in  $A$  is defined such that  $(a+b)c = ac + bc$ ,  $a(b+c) = ab + ac$  for all  $a, b, c$  in  $A$ ;

$$(2) \quad \alpha(ab) = (\alpha a)b = a(\alpha b) \text{ for all } \alpha \text{ in } \phi, a, b \text{ in } A.$$

An algebra  $A$  is said to be associative if it satisfies the associative law

$$(3) \quad (ab)c = a(bc) \text{ for all } a, b, c \text{ in } A.$$

An algebra  $A$  is said to be alternative if it satisfies the left and right alternative laws

$$(4) \quad a^2b = a(ab) \text{ for all } a, b \text{ in } A,$$

$$(5) \quad ba^2 = (ba)a \text{ for all } a, b \text{ in } A.$$

An involution of an algebra  $A$  is a linear operator  $a \rightarrow \bar{a}$  on  $A$  satisfying

$$(6) \quad \overline{ab} = \bar{b}\bar{a}, \quad \bar{\bar{a}} = a \text{ for all } a, b \text{ in } A.$$

We are concerned with an involution satisfying

$$(7) \quad a + \bar{a} \in \phi 1, \quad a\bar{a} = (\bar{a}a) \in \phi 1 \text{ for all } a \text{ in } A.$$

Clearly this implies

$$(8) \quad a^2 - t(a)a + n(a)1 = 0, \quad t(a), n(a) \text{ in } \phi,$$

with (9)  $a + \bar{a} = t(a)1, a\bar{a} (= \bar{a}a) = n(a)1$  for all  $a$  in  $A$ .

$t(a)$  and  $n(a)$  are respectively called the trace and norm

of  $a$ . Since  $\bar{1} = 1$ , we have  $t(\alpha 1) = 2\alpha, n(\alpha 1) = \alpha^2$  for

all  $\alpha$  in  $\phi$ .

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Let  $a$  be any element of an algebra  $A$  over  $\phi$ . The right multiplication  $R_a$  of  $A$  determined by  $a$  is defined by

$$(10) R_a: x \rightarrow xa \text{ for all } x \text{ in } A$$

Clearly  $R_a$  is a linear operator on  $A$ . Similarly, the left multiplication  $L_a$ , defined by

$$(11) L_a: x \rightarrow ax \text{ for all } x \text{ in } A,$$

is a linear operator on  $A$ .

Let  $A$  be any algebra over  $\phi$ . By a derivation of  $A$  is meant a linear operator  $D$  on  $A$  satisfying

$$(12) (ab)D = (aD)b + a(bD) \text{ for all } a, b \text{ in } A.$$

This may be expressed in terms of right or left multiplication of  $A$ : A linear operator  $D$  on  $A$  is a derivation of  $A$  if and only if

$$(13) [R_y, D] = R_{yD} \text{ for all } y \text{ in } A;$$

equivalently, a linear operator  $D$  on  $A$  is a derivation of  $A$  if and only if

$$(14) [L_x, D] = L_{xD} \text{ for all } x \text{ in } A.$$

Let  $A$  be an alternative algebra over an arbitrary field

$\phi$ . For any  $x, z$  in  $A$ , define

$$(15) D_{x,z} = [R_x, R_z] + [L_x, R_z] + [L_x, L_z]$$

It can be shown that  $[R_y, D_{x,z}] = R_y D_{x,z}$  for all  $y$  in  $A$  (see [1], Page-77), so that  $D_{x,z}$  is a derivation of  $A$ .

1.3 Definition: A composition algebra  $C$  over a field  $\phi$  is an algebra with unit 1 and a non-degenerate quadratic form  $n$  satisfying

- (1)  $n(ab) = n(a)n(b)$  for all  $a, b$  in  $C$ ,
- (2)  $n(1) = 1$ .

It has been shown in [2], Theorem 7.5 that "Any composition algebra  $C$  is alternative and has an involution  $j: a \rightarrow \bar{a}$  such that  $\bar{a}a = n(a)1$ ".

By the Cayley-Dickson process ([1], Page-45) one can find a complete list of the composition algebras over a field  $\phi$  of characteristic  $\neq 2$ , namely: (A)  $\phi 1$  of dimension one; (B) quadratic algebras  $L$ , of dimension two; (C) quaternion algebras  $D$ , of dimension four; (D) octonion or Cayley algebras  $C$ , of dimension eight. It can also be shown that  $D$  is associative, but not commutative, and  $C$  is not associative ([1], Page-47).

Let  $C$  be a composition algebra and  $n$  be the non-degenerate quadratic form on  $C$ . Then in [3], page-169, theorem 7 states: "The following conditions on a composition algebra  $C$  are equivalent: (a)  $n(x) = 0$ , for some  $x \neq 0$  in  $C$ ; (b)  $C$  has zero divisors  $\neq 0$  ( $xy = 0$ ,  $x \neq 0$ ,  $y \neq 0$ ); (c)  $C$  is not a division algebra; (d)  $n$  has maximum Witt index which is positive. Moreover, any two composition algebras of the same dimensionality (over  $\phi$ ) satisfying the conditions are isomorphic".

The composition algebras satisfying the conditions of the above theorem are called split composition algebras. For dimension 2 such an algebra is a direct sum of two copies of  $\phi$  with the involution which exchanges the two components. For dimension 4 such an algebra is isomorphic to  $\phi_2$ , the algebra of  $2 \times 2$  matrices with entries in  $\phi$ , with the symplectic involution

$$x \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ in } \phi_2.$$

For dimension 8 such an algebra is isomorphic to the algebra of zero vector-matrices  $\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$ ,  $\alpha, \beta \in \phi$ ,  $a, b \in \phi^3$ , with

involution

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \rightarrow \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix}$$

The product in this algebra is given by

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + (a,d) & \alpha\delta + \delta a - (b\wedge d) \\ \gamma b + \beta d + (a\wedge c) & \beta\delta + (b,c) \end{pmatrix}$$

where  $(a\wedge c)$  and  $(a,c)$  denote, respectively, the customary vector and scalar products in  $\phi^3$ .

So, we have discussed all possible split composition algebras, namely, split quadratic algebra, split quaternion algebra, split Cayley algebra with their respective involutions.

1.4 Derivation algebra of C, the algebra of split octonions. In this section we shall review some properties of DerC and express the results in a table.

There exists a basis  $\{e_1, e_2, f_1, f_2, f_3, g_1, g_2, g_3\}$  for C whose multiplication table is given in table 6, ([6], Page-105).

Table 1

	$e_1$	$e_2$	$f_1$	$f_2$	$f_3$	$s_1$	$s_2$	$s_3$
$e_1$	$e_1$	0	$f_1$	$f_2$	$f_3$	0	0	0
$e_2$	0	$e_2$	0	0	0	$s_1$	$s_2$	$s_3$
$f_1$	0	$f_1$	0	$s_3$	$-s_2$	$-e_1$	0	0
$f_2$	0	$f_2$	$-s_3$	0	$s_1$	0	$-e_1$	0
$f_3$	0	$f_3$	$s_2$	$-s_1$	0	0	0	$-e_1$
$s_1$	$s_1$	0	$-e_2$	0	0	0	$f_3$	$-f_2$
$s_2$	$s_2$	0	0	$-e_2$	0	$-f_3$	0	$f_1$
$s_3$	$s_3$	0	0	0	$-e_2$	$f_2$	$-f_1$	0

Using this table we shall find the action of  $\text{Der}C$  on  $C$  and give the results in table 2. We have from 1.2(15) that for  $c_1, c_2 \in C$

$$D_{c_1, c_2} = [R_{c_1}, R_{c_2}] + [L_{c_1}, R_{c_2}] + [L_{c_1}, L_{c_2}]$$

is a derivation of  $C$ . Using this we find:

Table 2

	$D_{e_1, f_1}$	$D_{e_2, g_1}$	$D_{f_1, g_1}$	$D_{f_1, g_{i+1}}$	$D_{f_1, g_{i+2}}$
$e_1$	$f_1$	$-g_1$	0	0	0
$e_2$	$-f_1$	$g_1$	0	0	0
$f_1$	0	$e_2 - e_1$	$2f_1$	0	0
$f_{i+1}$	$g_{i+2}$	0	$-f_{i+1}$	$3f_1$	0
$f_{i+2}$	$-g_{i+1}$	0	$-f_{i+2}$	0	$3f_1$
$g_1$	$e_1 - e_2$	0	$-2g_1$	$-3g_{i+1}$	$-3g_{i+2}$
$g_{i+1}$	0	$f_{i+2}$	$g_{i+1}$	0	0
$g_{i+2}$	0	$-f_{i+1}$	$g_{i+2}$	0	0

The indices are computed modulo 3.

$$\text{Moreover, } D_{f_1, f_2} = -D_{e_2, g_3}, \quad D_{f_1, f_3} = D_{e_2, g_2},$$

$$D_{f_2, f_3} = -D_{e_2, g_1}, \quad D_{g_1, g_2} = -D_{e_1, f_3}, \quad D_{g_1, g_3} = D_{e_1, f_2},$$

$$D_{g_2, g_3} = D_{e_1, f_1}, \quad D_{e_1, g_1} = -D_{e_2, g_1}, \quad D_{e_2, f_1} = -D_{e_1, f_1}$$

Also,  $D_{f_1, g_1} + D_{f_2, g_2} + D_{f_3, g_3} = 0$ . Thus,  $\text{Der } C$  is

14-dimensional. Let  $C_0$  be the space of all elements in  $C$  having trace 0.  $C_0$  is spanned by  $\{e_2 - e_1, f_1, f_2, f_3, g_1, g_2, g_3\}$ .

Proposition 3 of [4] tells that the derivations of  $C$  which annihilate  $(e_2 - e_1)$  form a 8-dimensional Lie subalgebra of  $\text{Der } C$  of type  $A_2$ . Thus, from table 2 this subalgebra is spanned by

$$\{D_{f_1, g_1}, D_{f_2, g_2} - D_{f_3, g_3}, D_{f_1, g_2}, D_{f_2, g_3}, D_{f_3, g_1}, \\ D_{f_1, g_3}, D_{f_2, g_1}, D_{f_3, g_2}\}.$$

## 2. Jordan Matrix Algebras

2.1 Definition: A Jordan algebra  $J$  is an algebra over a field  $\phi$  of characteristic  $\neq 2$  with a product composition, denoted by  $a \cdot b$ , satisfying

$$(1) \quad a \cdot b = b \cdot a,$$

$$(2) \quad (a \cdot^2 \cdot b) \cdot a = a \cdot^2 \cdot (b \cdot a)$$

where  $a \cdot^2 = a \cdot a$  for  $a, b$  in  $J$ .

Let  $U$  be an associative algebra over a field  $\phi$  of characteristic  $\neq 2$  with product composition denoted by  $ab$ .

Set  $a \cdot b = \frac{1}{2}(ab+ba)$ . Then it is easy to verify that if we replace the given product in  $U$  by the product  $a \cdot b$  then we obtain a Jordan algebra  $U^+$ .  $U^+$  and its subalgebras are called special Jordan algebras. Jordan algebras which are not special will be called exceptional.

2.2 Let  $U$  be an arbitrary algebra with identity element  $1$  and let  $U_n$  denote the algebra of  $n \times n$  matrices with entries in  $U$ . If  $U$  has an involution  $a \rightarrow \bar{a}$  it is easy to check that  $A = (a_{ij}) \rightarrow \bar{A}^t = (\bar{a}_{ij})^t$ , where  $t$  denotes the transpose

matrix, is an involution in  $U_n$ . We call this the standard involution in  $U_n$  associated with the given involution in  $U$ .

Let  $H(U_n)$  denote the set of symmetric elements  $(\bar{A}^t = A)$  under the standard involution in  $U_n$ . Then it is clear that  $H(U_n)$  is a subalgebra of  $U_n^+$ .

In [3], Page-127, Theorem 1 states "Let  $U$  be an algebra over a field of characteristic not two with an identity element and an involution  $j$ , and let  $J_a$  be a canonical involution\* in  $U_n$ . Then  $H(U_n, J_a)$  for  $n \geq 3$  is Jordan if and only if either  $U$  is associative or  $n = 3$  and  $U$  is alternative with symmetric elements in the nucleus  $N(U)$ ." Here  $N(U) = \{n \in U \mid [n,x,y] = [x,n,y] = [x,y,n] = 0, \forall x, y \in U\}$ , where  $[n,x,y] = (nx)y - n(xy)$ .

We shall call the algebras  $H(U_n, J_a)$  which are Jordan, Jordan matrix algebras.

For example, let us take  $K$ , the algebraic closure of  $\phi$ ;  $\mathcal{L}$ , the split quadratic algebra;  $\mathcal{D}$ , the split

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\* Let  $x \rightarrow \bar{x}^t$  be the standard involution in  $U_n$ . We consider involutions  $J_a: x \rightarrow a^{-1}x^{-t}a$  where  $a = \text{diag}\{a_1, a_2, \dots, a_n\}$  and the  $a_i$  satisfy:  $\bar{a}_i = a_i, a_i \in N(U)$ , the nucleus of  $U$  where  $a_j^{-1}$  exists in the associative algebra  $N(U)$ . We call such involutions canonical and we let  $H(U_n, J_a)$  denote the subalgebra of  $U_n^+$  of elements symmetric relative to  $J_a$ .



algebra and  $C$ , the split Cayley algebra.  $K$ ,  $Z$  and  $D$  are associative. Thus, by the above theorem  $H(K_3)$ ,  $H(Z_3)$ ,  $H(D_3)$ , with respect to their standard involutions, are Jordan algebras, and these Jordan algebras are special. The Cayley algebra  $C$  is not associative, but it is alternative and its symmetric elements with respect to its involution are in its nucleus.

For, if  $\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in C$ , and if

$$\overline{\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}} = \begin{pmatrix} \beta & -a \\ -b & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}; \text{ then}$$

$$\alpha = \beta, \quad a = b = 0$$

Hence the symmetric elements in  $C$  are the scalar matrices  $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ , which associate with every pair of elements in  $C$  and hence lie in the nucleus of  $C$ .

Thus,  $H(C_3)$ , with respect to its standard involution, is a Jordan algebra. This Jordan algebra is exceptional.

The dimensions of  $H(K_3)$ ,  $H(Z_3)$ ,  $H(D_3)$  and  $H(C_3)$  are

respectively 6, 9, 15 and 27.

2.3 Let  $U$  be a finite dimensional Jordan algebra over a field  $\phi$  and let  $\{u_1, u_2, \dots, u_n\}$

be a basis for  $U/\phi$ . Let  $P = \phi(\xi_1, \xi_2, \dots, \xi_n)$  where  $\xi_i$  are algebraically independent over  $\phi$  and consider the algebra  $U \otimes_{\phi} P = U_P$ . We shall call the element

$x = \sum_{i=1}^n \xi_i u_i$  of  $U_P$  a generic element of  $U$ . Let  $m_x(\lambda)$  be the

minimum polynomial of  $x$  (in  $U_P$ ). So,  $m_x(\lambda) \in P[\lambda]$  and has leading co-efficient 1.

$m_x(\lambda)$  has the form

$$m_x(\lambda) = \lambda^m - \sigma_1(\xi)\lambda^{m-1} + \sigma_2(\xi)\lambda^{m-2} - \dots + (-1)^m \sigma_m(\xi)$$

where  $\sigma_i(\xi) = \sigma_i(\xi_1, \dots, \xi_n) \in \phi[\xi_1, \dots, \xi_n]$ . It has been

shown in [3], Page 222, that for  $a = \sum_{i=1}^n \alpha_i u_i \in U$

$$m_a(\lambda) = \lambda^m - \sigma_1(a)\lambda^{m-1} + \dots + (-1)^m \sigma_m(a), \text{ and } m_a(a) = 0$$

in  $U$ , where  $\sigma_i(a) = \sigma_i(\xi)$  for,  $\xi_i = \alpha_i$ .

We shall call the polynomial  $m_a(\lambda) \in \phi[\lambda]$  the generic

polynomial of  $a$  and the element  $\sigma_1(a)$  the generic trace of  $a$ . Also, we shall call the degree of  $m$  of the  $m_2(\lambda)$  the generic degree of the algebra  $U$ .

In [3], §6.4, the generic minimum polynomials of all finite dimensional simple Jordan algebras over algebraically closed fields have been derived. Also, it has been shown that each of the algebras  $H(K_3)$ ,  $H(Z_3)$ ,  $H(D_3)$  and  $H(C_3)$  has generic degree 3.

We note here that if the base field is algebraically closed then any composition algebra  $(C, j)$  with  $\dim C > 1$  is split, since any quadratic form on a finite-dimensional vector space of dimensionality  $> 1$  over an algebraically closed field has maximal Witt index. Also, according to Corollary 2, Page-204 in [3], the above Jordan algebras are simple.

### 3. Quadratic Jordan algebras

3.1 Let  $\phi$  be a commutative associative ring with unit 1.

A left  $\phi$ -module is an abelian group  $M$  together with a map  $(a, x) \rightarrow ax$  of  $\phi \times M$  into  $M$  satisfying the following conditions:

- (1)  $a(x+y) = ax + ay$ ,
- (2)  $(a+b)x = ax + bx$ ,
- (3)  $(ab)x = a(bx)$ ,

for  $x, y$  in  $M$   $a, b$  in  $\phi$ .

A left  $\phi$ -module  $M$  is said to be unital if

- (4)  $1_x = x$  for all  $x$  in  $M$ .

Let  $M$  be a unital  $\phi$ -module. A quadratic representation on  $M$  is a quadratic mapping  $U: x \rightarrow U_x$  of  $M$  into  $\text{Hom}_{\phi}(M, M)$ ; that is,

(5)  $U_{\alpha x} = \alpha^2 U_x$  and

(6)  $U_{x,y} = U_{x+y} - U_x - U_y$  is bilinear,  $\alpha$  in  $\phi$ ,  $x, y$  in  $M$ .

If  $U$  is a quadratic representation on  $M$ , then we have a cubic representation  $yU_x$  and a trilinear form

$$(7) \quad \{xyz\} = yU_{x,z}, \text{ which is symmetric in } x \text{ and } z. \text{ We}$$

also introduce the operator  $V_{y,z}$  defined by

$$(8) \quad xV_{y,z} = \{xyz\} = yU_{x,z}$$

A quadratic Jordan algebra with identity over  $\phi$  is a triple  $(J, U, 1)$  where  $J$  is a unital left- $\phi$ -module,  $1$  a distinguished element of  $J$ , and  $U$  a quadratic representation on  $J$  which satisfy:

$$(9) \quad U_1 = 1$$

$$(10) \quad U_a U_b U_a = U_b U_a$$

$$(11) \quad U_b V_{a,b} = V_{b,a} U_b = U_a U_{b,b}$$

(12) Under any extension of  $\phi$ , (9)-(11) remain valid.

Examples: (a) The Jordan algebra  $U^+$  defined in §2.1 of this chapter is a quadratic Jordan algebra.

(b)  $H(U_n, *)$  as defined in §2.2 of this chapter is a quadratic Jordan algebra.

### 3.2 Quadratic Jordan algebras obtained from cubic forms:

Let  $J$  be a finite dimensional left  $\phi$ -module,  $\phi$  as in §3.1.

A map  $N: J \rightarrow \phi$  is a cubic form on  $J$  if  $N$  is homogeneous of degree 3 and  $N(x+\lambda y) = N(x) + \lambda \Delta_x^y N + \lambda^2 \Delta_x^x + \lambda^3 N(y)$ ,

where  $\Delta_x^y N$ , the differential of  $N$  in the direction of  $y$  evaluated at  $x$  (see [3], Chapter VI), is linear in  $y$  and quadratic in  $x$ . Let  $N$  be a cubic form on  $J$ ,  $T$  a symmetric bilinear form on  $J$ ,  $x \rightarrow x^{\#}$  a quadratic mapping on  $J$ , and a distinguished element  $1 \in J$  satisfying

- (1)  $x^{\#\#} = N(x)x$ ,
- (2)  $N(1) = 1$ ,
- (3)  $T(x^{\#}/y) = \Delta_x^y N$ , where  $T(x,y) = -\Delta_1^x \Delta_1^y \log N$ ,\*
- (4)  $1^{\#} = 1$ ,
- (5)  $1 \times y = T(y)1 - y$ , where  $T(y) = T(y,1)$  and  
 $x \times y = (x+y)^{\#} - x^{\#} - y^{\#}$

Assume that these hold under all scalar extension of  $\phi$ .

Introduce a  $U$ -operator by

$$(6) \quad yU_x = T(x,y)x - x^{\#}xy.$$

McCrimmon has shown in [5] that  $(J, U, 1)$  is a quadratic Jordan algebra, which we denote by  $J(N, \#, 1)$ .

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\*  $\Delta_1^x \Delta_1^y \log N = \Delta_1^x \Delta_1^y N - (\Delta_1^x N)(\Delta_1^y N)$ .

A number of important identities valid in  $J(N, \#, 1)$  for  $\phi$  a field can be found in [5] or [8].

For  $x, y, z \in J_n$ ;

$$(7) \quad T(1) = 3,$$

$$(8) \quad T(x \times y) = T(x)T(y) - T(x, y),$$

$$(9) \quad T(x \times y, z) = T(x, y \times z),$$

$$(10) \quad x^3 - T(x)x^2 + S(x)x - N(x)1 = 0, \text{ where}$$

$$(11) \quad S(x) = T(x^\#),$$

$$(12) \quad x^\# = x^2 - T(x)x + S(x)1.$$

$J(N, \#, 1)$  is called the quadratic Jordan algebra with unity obtained from the cubic form  $N$ .

#### 4. Lie Algebras

4.1 Definition: A Lie algebra  $L$  over a field  $\phi$  is an algebra over  $\phi$  in which the multiplication is anticommutative, that is,

$$(1) \quad x^2 = 0 \quad (\text{implying } xy = -yx),$$

and the Jacobi identity

$$(2) \quad (xy)z + (yz)x + (zx)y = 0 \quad \text{for all}$$

$x, y, z$  in  $L$  is satisfied.

If  $U$  is any associative algebra over  $\phi$  then the commutator

$$(3) \quad [x, y] = xy - yx$$

satisfies

$$(4) \quad [x, x] = 0$$

and (5)  $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$

Thus the algebra  $U^-$  obtained by defining a new multiplication

(3) in the same vector space as  $U$  is a Lie algebra over  $\phi$ .

Also any subspace of  $U$  which is closed under commutation (3) gives a subalgebra of  $U^-$ , hence a Lie algebra over  $\phi$ .

The set  $\mathcal{D}(U)$  of all derivations of  $U$  into  $U$  is a subspace of the associative algebra  $C = C(U)$  of all linear operators on  $U$ . Since the commutator  $[D, D']$  of two derivations  $D, D'$  is a derivation of  $U$ ,  $\mathcal{D}(U)$  is a Lie subalgebra of  $C$ ; that is,  $\mathcal{D}(U)$  is a Lie algebra, called the derivation algebra of  $U$ .

In the literature the notation  $[x, y]$ , without regard to (3), is frequently used instead of  $xy$  to denote the product in an arbitrary Lie algebra.

#### 4.2 Classical Lie Algebras

Let  $V$  be a finite dimensional vector space over the field  $\phi$  and  $\text{End } V$  be the set of all linear transformations  $V \rightarrow V$ .  $(\text{End } V)^-$  is a Lie algebra and denoted by  $\mathfrak{gl}(V)$ .  $\mathfrak{gl}(V)$  is known as the general linear algebra.

We sometimes find it convenient to use matrices in place of linear transformations. In the present case we can identify  $\mathfrak{gl}(V)$  with the set of  $n \times n$  matrices, fixing a basis for  $V$  ( $\dim V = n$ ). We now denote  $\mathfrak{gl}(V)$  by  $\mathfrak{gl}(n, \phi)$ .

There are four families  $A_\ell, B_\ell, C_\ell, D_\ell$  of Lie algebras called the classical algebras. Each of these algebras is a subalgebra of  $gl(n, \phi)$ .

$A_\ell$ : Let  $\dim V = \ell + 1$ . We denote by  $sl(\ell+1, \phi)$ , the set of endomorphisms of  $V$  having trace zero. Then  $sl(\ell+1, \phi)$  is a subalgebra of  $gl(n, \phi)$  and it is called the special linear algebra.

$C_\ell$ : Let  $\dim V = 2\ell$ . Define a non-degenerate skew symmetric form  $f$  on  $V$  by the matrix  $S = \begin{pmatrix} 0 & 1_\ell \\ -1_\ell & 0 \end{pmatrix}$ . Let us denote

by  $sp(2\ell, \phi)$ , the symplectic algebra, which by definition consists of all endomorphisms  $x$  of  $V$  satisfying  $f(x(v), w) = -f(v, x(w))$ . It can be easily verified that  $sp(2\ell, \phi)$  is closed under the commutator operation.

$B_\ell$ : Let  $\dim V = 2\ell + 1$  be odd and take  $f$  to be the non-degenerate symmetric bilinear form on  $V$  whose matrix is

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_\ell \\ 0 & 1_\ell & 0 \end{pmatrix}$$

Let us denote by  $o(2\ell+1, \phi)$  the collection of all endomorphisms

of  $V$  satisfying  $f(x(v), w) = -f(v, x(w))$ . These form a Lie algebra called the orthogonal algebra.

$D_\ell$ : Here we obtain another orthogonal algebra. The construction is identical to that of  $B_\ell$ , except that  $\dim V = 2\ell$  is even

and  $S$  has the simpler form  $\begin{pmatrix} 0 & 1_\ell \\ 1_\ell & 0 \end{pmatrix}$ . This Lie algebra is

denoted by  $O(2\ell, \phi)$ .

In [6], Page 2 and 3, it has been shown that  $\dim \mathfrak{sl}(\ell+1, \phi) = \ell(\ell+2)$ ,  $\dim \mathfrak{sp}(2\ell, \phi) = 2\ell^2 + \ell$ ,  $\dim O(2\ell+1, \phi) = 2\ell^2 + \ell$ , and  $\dim O(2\ell, \phi) = 2\ell^2 - \ell$ .

#### 4.3 Exceptional Lie algebras

Over an algebraically closed field  $\phi$  of characteristic 0 there are five exceptional simple Lie algebras: the 14-dimensional algebra  $G_2$ , the 52-dimensional algebra  $F_4$  and three others ( $E_6, E_7, E_8$ ) of dimensions 78, 133 and 248 respectively.

Let  $C$  be a Cayley algebra over a field  $\phi$  of characteristic  $\neq 2, 3$ . Then in [1], Theorem 3.28 states that the derivation algebra  $D(C)$  of  $C$  is a 14-dimensional central simple Lie algebra (of type  $G_2$ ).

Let  $J$  be an exceptional central simple Jordan algebra over  $\phi$  of characteristic  $\neq 2, 3$ . Then in [1], Theorem 4.9 states that the derivation algebra  $D(J)$  of  $J$  is a 52-dimensional central simple Lie algebra (of type  $F_4$ ).

In [1], theorem 4.12 states: "Let  $J$  be any finite-dimensional, exceptional, central, simple Jordan algebra over  $\phi$  of characteristic  $\neq 2, 3$ , and  $L = L(J)$  be the Lie multiplication algebra of  $J$ . Then

$$L = \phi 1_J \oplus L'$$

where the derived algebra

$L' = D(J) + \{R_x \mid \text{trace } R_x = 0\}$  (direct sum) is a 78-dimensional central simple Lie algebra (of type  $E_6$ ) over  $\phi$ ."

Constructions of  $E_7$  and  $E_8$ , due to Tits, will be given in Chapter II.

## Chapter II.

### Constructions yielding exceptional Lie algebras

#### 1. Tits' Construction

1.. Let  $U$  be a composition algebra over a field  $\phi$  of characteristic  $\neq 2, 3$ . Then there exists a linear function  $t$  and quadratic form  $n$  on  $U$  such that

$$(1) \quad a^2 - t(a)a + n(a)1 = 0, \quad a \in U.$$

$t$  and  $n$  are respectively the generic trace and norm on  $U$ .

Let  $U_0 = \{a \mid t(a) = 0\}$ . So  $U_0$  is a subspace and

$U = U_0 \oplus 1$ . In fact, if  $a \in U$ , then  $a = \frac{1}{2}t(a)1 + a_0$ , where

$a_0 = a - \frac{1}{2}t(a)1 \in U_0$ , since  $t(1) = 2$  by 1.2(9) of chapter I.

Hence we can define a bilinear product  $*$  in  $U_0$  by

$$(2) \quad a*b = ab - \frac{1}{2}t(a,b)1, \quad a, b \in U_0,$$

where (3)  $t(a,b) = t(ab)$ ,  $a, b \in U$ , is a symmetric bilinear form on  $U$ , which is associative in the sense that

$$(4) \quad t(ab,c) = t(a,bc), \quad a, b, c \in U.$$

It has been shown in ([4]; page 90-91) that

$$a*b = -b * a, \quad D_{a,b} = -D_{b,a}, \quad \text{for } a, b \in U_0,$$

where  $D_{a,b}$  is defined by 1.2(15) of chapter I.

1.2 Let  $J$  be a central simple Jordan algebra of degree 3 over  $\phi$ . Then we have a linear function  $T(x)$ , a quadratic form  $S(x)$  and a cubic form  $N(x)$  such that

$$(1) \quad x^3 - T(x)x + S(x)x - N(x)1 = 0, \quad x \in J.$$

Let  $J_0 = \{x | T(x) = 0\}$ . If  $x \in J$  we have  $x = \frac{1}{3}T(x)1 + x_0$ ,

where  $x_0 = x - \frac{1}{3}T(x)1 \in J_0$ . Since  $T(1) = 3$ , by 3:2(7) of chapter I,  $J = J_0 + \phi 1$  and we can

$$(2) \quad x*y = x \cdot y - \frac{1}{3}T(x \cdot y)1, \quad x, y \in J_0,$$

where  $T(x,y) = T(x \cdot y)$  is a symmetric bilinear form on  $J$ .

Then  $x*y \in J_0$ .

1.3 Let (1)  $L = \mathcal{D}(U) \oplus (U_0 \oplus J_0) \oplus \mathcal{D}(J)$ ,  $U, J, U_0$  and  $J_0$  as defined in §1.1 and §1.2 of this chapter.  $\mathcal{D}(U)$  and  $\mathcal{D}(J)$

denote the derivation algebras as defined in §4, Chapter 1, and  $U_0 \otimes J_0$  the tensor product between  $U_0$  and  $J_0$ . Then  $L$  is made into a Lie algebra over  $\phi$  by defining a multiplication  $[,]$  in  $L$  which is bilinear and anti-commutative, which agrees with the ordinary commutator in  $\mathcal{D}(U)$  and  $\mathcal{D}(J)$ , and which satisfies (2)  $[\mathcal{D}(U), \mathcal{D}(J)] = 0$ ,

$$(3) \quad [a \otimes x, D] = aD \otimes x$$

for all  $D$  in  $\mathcal{D}(U)$ ,  $a$  in  $U_0$ ,  $x$  in  $J_0$ ,

$$(4) \quad [a \otimes x, E] = a \otimes xE$$

for all  $E$  in  $\mathcal{D}(J)$ ,  $a$  in  $U_0$ ,  $x$  in  $J_0$ ,

$$\text{and (5) } [a \otimes x, b \otimes y] = \frac{1}{12}t(x,y)D_{a,b} + (a*b) \otimes (x*y)$$

$$- \frac{1}{2}t(a,b)[R_x, R_y]$$

for all  $a, b \in U_0$ ,  $x, y$  in  $J_0$ .

Theorem (Tits): The algebra  $L$  in (1) is a Lie algebra. If  $K$  is the algebraic closure of  $\phi$ , then corresponding to the four (alternative) composition algebras  $U$  over  $K$  and the five indicated Jordan algebras  $J$  over  $K$  (that is,  $K_1$  and the four central simple algebras  $H(K_3)$ ,  $H(I_3)$ ,  $H(D_3)$  and  $H(C_3)$ ), the Lie algebras  $L_K$  are given in the table:

Table 3

U/J,	K1	H(K <sub>3</sub> )	H(Z <sub>3</sub> )	H(D <sub>3</sub> )	H(C <sub>3</sub> )
K1	0	SL(2)	SL(3)	SL(6)	F <sub>4</sub>
Z	0	SL(3)	SL(3) ⊕ SL(3)	SL(6)	E <sub>6</sub>
D	SL(2)	Sp(6)	SL(6)	O(12)	E <sub>7</sub>
C	G <sub>2</sub>	F <sub>4</sub>	E <sub>6</sub>	E <sub>7</sub>	E <sub>8</sub>

SL(n) denotes the special linear Lie algebra SL(V),

dim V = n: Hence L is central simple except for three cases:

dim U = dim J = 1 ; dim U = 2, dim J = 1 ; dim U = 2, dim J = 9.

In [1], page-122 the above theorem has been proved.

## 2. Faulkner's Construction

### 2.1 A class of ternary algebras.

We shall be interested in a module M over an arbitrary commutative ring φ with 1 which possesses an alternating bilinear form <, > and a ternary product <,, > which satisfy:

$$(1) \quad \langle x, y, z \rangle = \langle y, x, z \rangle + \langle x, y, z \rangle \text{ for } x, y, z \in M ;$$

$$(2) \quad \langle x, y, z \rangle = \langle x, z, y \rangle + \langle y, z \rangle x \quad \text{for } x, y, z \in M ;$$

$$(3) \quad \langle \langle x, y, z \rangle, w \rangle = \langle \langle x, y, w \rangle, z \rangle + \langle x, y \rangle \langle z, w \rangle \quad \text{for } x, y, z, w \in M ;$$

$$(4) \quad \langle \langle x, y, z \rangle, v, w \rangle = \langle \langle x, v, w \rangle, y, z \rangle + \langle x, \langle y, v, w \rangle, z \rangle + \langle x, y, \langle z, w, v \rangle \rangle \quad \text{for } x, y, z, v, w \in M .$$

The  $\phi$ -module  $M$  is known as a ternary algebra.

Example: Let  $J = J(N, 1)$  be a quadratic Jordan algebra with 1 over a field  $\phi$  constructed as in [5] from an admissible non-degenerate cubic form  $N$  with base point 1. Recall  $yU_x = T(x, y)x - x^\# \times y$  where  $T(\cdot)$  and  $x \rightarrow x^\#$  are respectively the associated non-degenerate bilinear form and quadratic mapping and  $x \times y = (x+y)^\# - x^\# - y^\#$  (§3.2 of Chapter I).

Let

$$M = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mid \alpha, \beta \in \phi ; a, b \in J \right\}.$$

$$\text{For } x_1 = \begin{pmatrix} \alpha_1 & a_1 \\ b_1 & \beta_1 \end{pmatrix} \in M$$

$$\text{we define: (5) } \langle x_1, x_2 \rangle = \alpha_1 \beta_2 - \alpha_2 \beta_1 - T(a_1, b_2) + T(a_2, b_1) ,$$

$$(6) \quad \langle x_1, x_2, x_3 \rangle = \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix},$$

$$\text{where } \gamma = \alpha_1 \beta_2 \alpha_3 + 2\alpha_1 \alpha_2 \beta_3 - \alpha_3 T(a_1, b_2) - \alpha_2 T(a_1, b_3) - \\ - \alpha_1 T(a_2, b_3) + T(a_1, a_2 \times a_3),$$

$$c = (\alpha_2 \beta_3 + T(b_2, a_3))a_1 + (\alpha_1 \beta_3 + T(b_1, a_3))a_2 \\ + (\alpha_1 \beta_2 + T(b_1, a_2))a_3 - \alpha_1 b_2 \times b_3 - \alpha_2 b_1 \times b_3 \\ - \alpha_3 b_1 \times b_2 - \{a_1 b_2 a_3\} - \{a_1 b_3 a_2\} - \{a_2 b_1 a_3\},$$

$$\delta = -\gamma^\sigma, \quad d = -c^\sigma, \quad \text{where } \sigma = (\alpha\beta)(ab).$$

(Note  $\gamma^\sigma$  is the term obtained from  $\gamma$  by interchanging  $\alpha$  and  $\beta$  as well as  $a$  and  $b$ .)

It has been shown in [7], §1 that  $M$  is a ternary algebra.

## 2.2 Construction of Lie algebras

Starting with a module  $M$  over a commutative associative ring  $\Phi$  with 1 which possesses an alternating bilinear form  $\langle, \rangle$  and a ternary product  $\langle, \rangle$  which is a ternary algebra, we shall construct some Lie algebras.

First, we construct  $R = M \oplus \Phi u$  and the associative

subalgebra  $U(M)$  of  $\text{Hom}_\phi(R, R)$  consisting of  $A \in \text{Hom}_\phi(R, R)$  such that  $uA \in \phi u$  and  $MA \subseteq M$ : We let  $U(M)^-$  denote the Lie algebra structure on  $U(M)$  where  $[AB] = AB - BA$ .

If we define  $U \in U(M)^-$  by  $uU = 2u$  and  $xU = x$  for  $x \in M$ , then it is clear that  $U$  is in the centre of  $U(M)^-$ . We may also define  $\rho(A) \in \phi$  for  $A \in U(M)^-$  by

$$(1) \quad uA = \rho(A)u$$

If  $A \in U(M)^-$ , we set

$$(2) \quad A' = A - \rho(A)U.$$

We next define  $R(x, y) \in U(M)^-$  for  $x, y \in M$  by

$$(3) \quad \begin{aligned} uR(x, y) &= \langle x, y \rangle u, \\ zR(x, y) &= \langle z, x, y \rangle \text{ for } z \in M. \end{aligned}$$

Let  $R^*(M)$  consists of those  $R \in U(M)^-$  such that

$$(4) \quad [R(x, y)R] = R(xR, y) + R(x, yR')$$
 for  $x, y \in M$ .

One checks immediately that  $R^*(M)$  is a Lie subalgebra of  $U(M)^-$  containing  $U$ .

Using equations (1.1), (1.3) in [7], §1 and 2.1(4) of

This chapter we can write

$$(5) \quad [R(x_1, x_2)R(x_3, x_4)] = R(x_1 R(x_3, x_4), x_2) + \\ + R(x_1, x_2 R(x_4, x_3)) \quad \text{for } x_i \in M, i = 1, 2, 3, 4.$$

Hence,  $R(x, y) \in R^*(M)$  for  $x, y \in M$ .

It is clear from 2.1(1) of this chapter that

$$(6) \quad R(x, y) - R(y, x) = \langle x, y \rangle U.$$

Hence  $R'(x, y) = R(y, x)$ .  $\{R(x, y) | x, y \in M\} \cup \{U\}$  spans an ideal  $R(M)$  of  $R^*(M)$ . We note that if  $\langle, \rangle$  represents 1, then  $R(M)$  is spanned by

$$\{R(x, y) | x, y \in M\}$$

Now let  $R'$  be any Lie subalgebra of  $R^*(M)$  containing  $R(M)$  and let  $\tilde{R}$  denote a second copy of  $R$ . Form

$$S(M, R') = R \oplus \tilde{R} \oplus R' \\ = M \oplus \tilde{M} \oplus \phi u \oplus \phi \tilde{u} \oplus R'$$

We define a Lie product on  $S = S(M, R')$  by

$$\begin{aligned}
 (7) \quad & [x_1 + \bar{y}_1 + \alpha_1 u + \beta_1 \bar{u} + R_1, \quad x_2 + \bar{y}_2 + \alpha_2 u + \beta_2 \bar{u} + R_2] \\
 & = (x_1 R_2 - x_2 R_1 + \alpha_1 y_2 - \alpha_2 y_1) + (y_1 R_2 - y_2 R_1 + \beta_2 x_1 - \beta_1 x_2) \\
 & + (\langle x_1, x_2 \rangle + \alpha_1 \rho(R_2) - \alpha_2 \rho(R_1)) u \\
 & + (\langle y_1, y_2 \rangle - \beta_1 \rho(R_2) + \beta_2 \rho(R_1)) \bar{u} \\
 & + (R(x_1, y_2) - R(x_2, y_1) + (\alpha_1 \beta_2 - \alpha_2 \beta_1) U + [R_1, R_2])
 \end{aligned}$$

for  $x_i, y_i \in M, \alpha_i, \beta_i \in \Phi, R_i \in R'$

Then it has been shown in [7], §2 that

$S = S(M, R')$  is a Lie algebra. It has also been shown in [7], §2, Theorem 1 that  $S$ , constructed as above, is a simple Lie algebra if and only if  $\langle, \rangle$  on  $M$  is non-degenerate.

### 2.3 Identification of Lie algebras

We wish to identify the simple Lie algebra  $S(M, R(M))$  constructed as in §2.2 from the ternary algebra of the example illustrated in §2.1 of this chapter with  $J$  an exceptional simple Jordan algebra of dimension 27. We shall do this for

$\phi$  a field of characteristic 0. Since  $\langle, \rangle$  remains non-degenerate upon extension of the base field, we may assume that  $\phi$  is algebraically closed.

We first consider the derivation algebra of  $M$ , which we identified with  $\mathcal{D} = \{D \in R^*(M) \mid \rho(D) = 0\}$ . (see [7], §3).

It has been shown in [9] that  $\mathcal{D}$  is a simple Lie algebra of type  $F_7$ , and  $\dim \mathcal{D} = 133$ . It has been shown in [7], §3 that  $R^*(M) = \phi U + \mathcal{D}$ . Since  $\mathcal{D}$  is simple,  $R^*(M) = R(M) = R'$ , ([7], §4). Therefore,  $S(M, R(M)) = M \oplus M \oplus \phi u \oplus \phi \tilde{u} \oplus \phi U \oplus \mathcal{D}$ .

Hence  $\dim S(M, R(M)) = 2(56) + 3 + 133 = 248$ . Then by the classification of simple Lie algebras, we see that  $S(M, R(M))$  is of type  $E_8$ . Thus this algebra must be isomorphic to the algebra  $E_8$  in Tits' construction. In the next chapter we construct such an isomorphism.

## Chapter III

### An Isomorphism Theorem

#### 1. Construction of $G_2$ using Faulkner's method

1.1 Faulkner in [7] gave a construction of Lie algebras from a ternary algebra. We have discussed this construction in section 2 of chapter II. The example of a ternary algebra given in §2.1, chapter II, plays an important part in the whole process. Using  $J = \phi 1$  in this example and following Faulkner's process we shall construct a 14-dimensional algebra and afterwards, in a subsequent section, we shall show this is isomorphic with  $\text{Der } C$ , where  $C$  is the algebra of split octonions.  $\text{Der } C$  is a Lie algebra of type  $G_2$ .

Let  $M_1$  be the ternary algebra obtained by using  $J = \phi 1$ . Then

$$M_1 = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mid \alpha, \beta, a, b \in \phi \right\}$$

For  $x_i = \begin{pmatrix} \alpha_i & a_i \\ b_i & \beta_i \end{pmatrix} \in M_1$

We have from 2.1(5) and 2.1(6) of chapter II that

$$(1) \langle x_1, x_2 \rangle = \alpha_1 \beta_2 - \alpha_2 \beta_1 - 3a_1 b_2 + 3a_2 b_1 \text{ and}$$

$$(2) \langle x_1, x_2, x_3 \rangle = \begin{pmatrix} \gamma \\ d \\ c \end{pmatrix} \text{ where}$$

$$\begin{aligned} \gamma &= \alpha_1 \beta_2 \alpha_3 + 2\alpha_1 \alpha_2 \beta_3 - 3\alpha_3 a_1 b_2 - 3\alpha_2 a_1 b_3 - \\ &\quad - 3\alpha_1 a_2 b_3 + 6a_1 a_2 a_3 \end{aligned}$$

$$c = (\alpha_2 \beta_3 + 3b_2 a_3) a_1 + (\alpha_1 \beta_3 + 3b_1 a_3) a_2 + (\alpha_1 \beta_2 + 3b_1 a_2) a_3$$

$$- 2\alpha_1 b_2 b_3 - 2\alpha_2 b_1 b_3 - 2\alpha_3 b_1 b_2 - 2a_1 b_2 a_3 - 2a_1 b_3 a_2 -$$

$$- 2a_2 b_1 a_3, \quad \delta = -\gamma^\sigma, \quad d = -c^\sigma, \text{ where } \sigma \text{ denotes } (\alpha\beta)(ab)$$

the map interchanging  $\alpha$  and  $\beta$ ,  $a$  and  $b$ .

From §2.3 of Chapter II,

$$S_1 = S_1(M_1, R(M_1)) = M_1 \oplus \tilde{M}_1 \oplus \phi U \oplus \phi \bar{U} \oplus \phi U \oplus \mathcal{D}_1,$$

where  $\mathcal{D}_1$  is the algebra of derivations of  $M_1$ . We have that

$$R^*(M_1) = R(M_1) = \phi U \oplus \mathcal{D}_1 \text{ and } R(M_1), \text{ the span of}$$

$\{R(x_1, x_2) \mid x_1, x_2 \in M_1\} \cup \{U\}$ . In order to study  $R(M_1)$  we

use the table 4 below. This table gives the image of an

element  $z = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$  of  $M_1$  mapped by  $R(l_{ij}, l_{kl})$  in the

position corresponding to  $l_{ij}$  and  $l_{kl}$ .  $l_{ij}$  is the  $2 \times 2$

matrix with 1 in the  $(i,j)$ th position and 0 elsewhere. We

note that  $zR(l_{ij}, l_{kl}) = \langle z, l_{ij}, l_{kl} \rangle$ , by 2.2(3), Chapter II.

The table 5 gives values of  $\rho(l_{ij}, l_{kl}) = \langle l_{ij}, l_{kl} \rangle$ . We

know that  $R(l_{ij}, l_{kl})$  is a derivation of  $M_1$  iff  $\rho(l_{ij}, l_{kl}) = 0$ .

Using these tables we ultimately obtain  $zR(x_1, x_2)$  and

$\rho(x_1, x_2)$ ,  $x_1, x_2 \in M_1$ .

It can be checked by the equation (1) above that for

$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $R(1,1)$  is a derivation of  $M_1$ . It can also be

computed that  $zR(1,1) = \begin{pmatrix} 3\alpha & a \\ -b & -3\beta \end{pmatrix}$ . From table 4, we see

that  $R(l_{11}, l_{21})$ ,  $R(l_{22}, l_{12})$  and

Table 4

$\longleftrightarrow l_{k\ell}$

$zR(l_{ij}, l_{k\ell})$	$l_{11}$	$l_{12}$	$l_{21}$	$l_{22}$
$l_{11}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -3a & -2b \\ -\beta & 0 \end{pmatrix}$	$\begin{pmatrix} 2\alpha & a \\ 0 & -\beta \end{pmatrix}$
$l_{12}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 6a & 4b \\ 2\beta & 0 \end{pmatrix}$	$\begin{pmatrix} -3\alpha & -2a \\ -b & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \alpha \\ 2a & 3b \end{pmatrix}$
$l_{21}$	$\begin{pmatrix} -3a & -2b \\ -\beta & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \alpha \\ 2b & 3\beta \end{pmatrix}$	$\begin{pmatrix} 0 & -2\alpha \\ -4a & -6b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$l_{22}$	$\begin{pmatrix} \alpha & 0 \\ -b & -2\beta \end{pmatrix}$	$\begin{pmatrix} 0 & \alpha \\ 2a & 3b \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\updownarrow l_{ij}$

Table 5

$\longleftrightarrow l_{k\ell}$

$\rho(l_{ij}, l_{k\ell})$	$l_{11}$	$l_{12}$	$l_{21}$	$l_{22}$
$l_{11}$	0	0	0	1
$l_{12}$	0	0	-3	0
$l_{21}$	0	3	0	0
$l_{22}$	-1	0	0	0

$\updownarrow l_{ij}$

$R(1,1)$  are linearly independent and each of those  $R(1_{ij}, 1_{kl})$  which are derivations is a linear combination of these three. Hence the algebra  $\mathcal{D}_1$  of derivations of  $M_1$  is spanned by  $R(1_{11}, 1_{21})$ ,  $R(1_{22}, 1_{12})$  and  $R(1,1)$  and the  $\dim \mathcal{D}_1 = 3$ . Thus, we find  $R(M_1) = \phi U + \mathcal{D}_1$ . Now clearly,  $\dim S_1 = \dim S_1(M_1 + R(M_1)) = 4 + 4 + 1 + 1 + 1 + 3 = 14$ .

1.2 Let  $J$  be an arbitrary quadratic Jordan algebra with 1 of degree 3 over a field  $\phi$ , constructed as in [5].  $\phi 1$  is contained in  $J$ . For

$$M = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mid \alpha, \beta \in \phi; a, b \in J \right\}$$

let  $S(M, R(M))$  denotes the Lie algebra constructed by Faulkner's process.. Then

$$S = S(M, R(M)) = M \oplus \tilde{M} + \phi u + \phi \tilde{u} + R(M)$$

$$\text{where } R(M) = \phi U \oplus \mathcal{D} = \left\{ \sum_{i,j} R(x_i, x_j) \mid x_i, x_j \in M \right\} \cup \{U\}$$

To find out  $\phi U + \mathcal{D}$  we see that  $R(x_1, x_2)$ ,  $x_1, x_2 \in M$ , act on an arbitrary element  $z = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$  of  $M$  and compute

$\rho(x_1, x_2)$  giving the results in tables 6 and 7. In these tables  $q_{12}$  stands for  $\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}$ , where  $q \in J$  and so on. As in the preceding section,  $U = R(l_{11}, l_{22}) - R(l_{22}, l_{11})$  and the members in  $\mathcal{D}$  can be obtained from table 7.

Table 6

$zR(x_1, x_2)$	$l_{11}$	$s_{12}$	$t_{21}$	$l_{22}$
$l_{11}$	0 0 0 0	0 0 0 0	-T(a,t) -bxt -st 0	2a a 0 -B
$q_{12}$	0 0 0 0	T(a, qxs) T(b,s)q+ T(b,q)s- {qbs} Bqxs 0	-aT(q,t) -{atq} {bqt}-bT(q,t) 0	0 aq axq T(b,q)
$r_{21}$	-T(a,r) -bxr -Br 0	0 aT(r,s)- {ars} {bsr} BT(r,s)	0 -arxt {rat}- T(a,t)r- -T(b,rxt) T(a,r)t	0 0 0 0
$l_{22}$	a 0 -b -2B	0 as axs T(b,s)	0 0 0 0	0 0 0 0

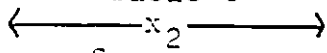
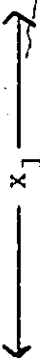


Table 7

$\longleftrightarrow x_2 \longleftrightarrow$

$\rho(x_1, x_2)$	$l_{11}$	$s_{12}$	$t_{21}$	$l_{22}$
$l_{11}$	0	0	0	1
$q_{12}$	0	0	$-T(q, t)$	0
$r_{21}$	0	$T(s, r)$	0	0
$l_{12}$	-1	0	0	0

$\updownarrow x_1$

Let us now consider  $J_0$ , the subspace of trace zero elements of  $J$ . In  $S = S(M, R(M))$  we single out seven copies of  $J_0$ . Clearly,  $M$  contains two copies of  $J_0$  namely

$$(J_0)_{12} = \begin{pmatrix} 0 & J_0 \\ 0 & 0 \end{pmatrix} \text{ and } (J_0)_{21} = \begin{pmatrix} 0 & 0 \\ J_0 & 0 \end{pmatrix} . \text{ Similarly, } \tilde{M}$$

contains two copies of  $J_0$  namely  $(\tilde{J}_0)_{12}$  and  $(\tilde{J}_0)_{21}$ .

When we work with  $J_0$  we see from table 7 that

$R((J_0)_{ij}, l_{kl}), i, j, k, l = 1, 2$ , are derivations of  $M$  and

clearly they yield copies of  $J_0$ . For  $x \in J_0$ , we get eight

non-zero derivations of  $M$ , namely

$$R(x_{12}, l_{22}) = R(x_{21}, l_{21}) = R(l_{22}, x_{12}), R(x_{21}, l_{11}) = R(x_{12}, l_{12})$$

$= R(l_{11}, x_{21})$  and  $R(x_{12}, l_{21}) = R(x_{21}, l_{12})$  and each of these

is an element in a copy of  $J_0$  of above type. We use table.6 to obtain the above equalities. These eight derivations of  $M$  and in fact three classify the copies of  $J_0$  in  $\mathcal{D}$ . These are clearly  $R((J_0)_{12}, l_{22})$ ,  $R((J_0)_{21}, l_{11})$  and  $R((J_0)_{12}, l_{21})$ .

Thus, we have only three copies of  $J_0$  in  $\mathcal{D}$ . For  $x \in J_0$ , we can give names to these copies as  $x_{12}$ ,  $x_{21}$ ,  $\tilde{x}_{12}$ ,  $\tilde{x}_{21}$ ,  $R(x_{12}, l_{22})$ ,  $R(x_{21}, l_{11})$ ,  $R(x_{12}, l_{21})$ .

### 1.3 Lie product of the copies of $J_0$ with elements in $S_1$ .

We shall find out the Lie product of the copies of  $J_0$  with the each of the generators of  $S_1$ . We use 2.2(6) of Chapter 2.  $l_{11}$ ,  $l_{12}$ ,  $l_{21}$ ,  $l_{22}$ ,  $\tilde{l}_{11}$ ,  $\tilde{l}_{12}$ ,  $\tilde{l}_{21}$ ,  $\tilde{l}_{22}$ ,  $u$ ,  $\tilde{u}$ ,  $U$ ,  $R(l_{11}, l_{21})$ ,  $R(l_{22}, l_{12})$ ,  $R(l, l)$  form a basis of  $S_1$ .

$$(a) [x_{12}, l_{11}] = [x_{12}, l_{12}] = [x_{12}, l_{21}] = [x_{12}, l_{22}] = 0 ;$$

$$[x_{12}, \tilde{l}_{11}] = 0, [x_{12}, \tilde{l}_{12}] = R(x_{21}, l_{11}), [x_{12}, \tilde{l}_{21}] =$$

$$= R(x_{12}, l_{21})$$

$$[x_{12}, \tilde{l}_{22}] = R(x_{12}, l_{22}) ;$$

$$[x_{12}, u] = 0, [x_{12}, \tilde{u}] = \tilde{x}_{12}, [x_{12}, U] = x_{12} ;$$

$$[x_{12}, R(1_{11}, 1_{21})] = 0, [x_{12}, R(1_{22}, 1_{12})] = -x_{21}$$

$$[x_{12}, R(1, 1)] = x_{12} .$$

$$(b) [x_{21}, 1_{11}] = [x_{21}, 1_{12}] = [x_{21}, 1_{21}] = [x_{21}, 1_{22}] = 0 ;$$

$$[x_{21}, \tilde{1}_{11}] = R(x_{21}, 1_{11}), [x_{21}, \tilde{1}_{12}] = R(x_{12}, 1_{21}),$$

$$[x_{21}, \tilde{1}_{21}] = R(x_{12}, 1_{22}), [x_{21}, \tilde{1}_{22}] = 0 ;$$

$$[x_{21}, u] = 0, [x_{21}, \tilde{u}] = \tilde{x}_{21}, [x_{21}, U] = x_{21} ;$$

$$[x_{21}, R(1_{11}, 1_{21})] = x_{12}, [x_{21}, R(1_{22}, 1_{12})] = 0 ,$$

$$[x_{21}, R(1, 1)] = -x_{21} .$$

$$(c) [\tilde{x}_{12}, 1_{11}] = 0, [\tilde{x}_{12}, 1_{12}] = -R(x_{21}, 1_{11}), [\tilde{x}_{12}, 1_{21}] = \\ = -R(x_{12}, 1_{21})$$

$$[\tilde{x}_{12}, 1_{22}] = -R(x_{12}, 1_{22});$$

$$[\tilde{x}_{12}, \tilde{1}_{11}] = [\tilde{x}_{12}, \tilde{1}_{12}] = [\tilde{x}_{12}, \tilde{1}_{21}] = [\tilde{x}_{12}, \tilde{1}_{22}] = 0 ;$$

$$[\tilde{x}_{12}, u] = -x_{12}, [\tilde{x}_{12}, \tilde{u}] = 0, [\tilde{x}_{12}, U] = -\tilde{x}_{12} ;$$

$$[\tilde{x}_{12}, R(1_{11}, 1_{21})] = 0, [\tilde{x}_{12}, R(1_{22}, 1_{12})] = -\tilde{x}_{21} ,$$

$$[\tilde{x}_{12}, R(1, 1)] = \tilde{x}_{12} .$$

$$(d) [\tilde{x}_{21}, l_{11}] = -R(x_{21}, l_{11}), [\tilde{x}_{21}, l_{12}] = -R(x_{12}, l_{21}),$$

$$[\tilde{x}_{21}, l_{21}] = -R(x_{12}, l_{22}), [\tilde{x}_{21}, l_{22}] = 0 ;$$

$$[\tilde{x}_{21}, \tilde{l}_{11}] = [\tilde{x}_{21}, \tilde{l}_{12}] = [\tilde{x}_{21}, \tilde{l}_{21}] = [\tilde{x}_{21}, \tilde{l}_{22}] = 0 ;$$

$$[\tilde{x}_{21}, u] = -x_{21}, [\tilde{x}_{21}, \tilde{u}] = 0, [\tilde{x}_{21}, U] = -\tilde{x}_{21} ;$$

$$[\tilde{x}_{21}, R(l_{11}, l_{21})] = \tilde{x}_{12}, [\tilde{x}_{21}, R(l_{22}, l_{12})] = 0,$$

$$[\tilde{x}_{21}, R(1, 1)] = -\tilde{x}_{21} .$$

$$(e) [R(x_{12}, l_{22}), l_{11}] = -x_{12}, [R(x_{12}, l_{22}), l_{12}] = x_{21} ,$$

$$[R(x_{12}, l_{22}), l_{21}] = [R(x_{12}, l_{22}), l_{22}] = 0 ;$$

$$[R(x_{12}, l_{22}), \tilde{l}_{11}] = -\tilde{x}_{12}, [R(x_{12}, l_{22}), \tilde{l}_{12}] = \tilde{x}_{21} .$$

$$[R(x_{12}, l_{22}), \tilde{l}_{21}] = [R(x_{12}, l_{22}), \tilde{l}_{22}] = 0 ;$$

$$[R(x_{12}, l_{22}), u] = [R(x_{12}, l_{22}), \tilde{u}] = [R(x_{12}, l_{22}), U] = 0 ;$$

$$[R(x_{12}, l_{22}), R(l_{11}, l_{21})] = -R(x_{12}, l_{21}),$$

$$[R(x_{12}, l_{22}), R(l_{22}, l_{12})] = 0$$

$$[R(x_{12}, l_{22}), R(1, 1)] = -2R(x_{12}, l_{22}).$$

$$(f) [R(x_{21}, l_{11}), l_{11}] = [R(x_{21}, l_{11}), l_{12}] = 0 ,$$

$$[R(x_{21}, l_{11}), l_{21}] = -x_{12}, [R(x_{21}, l_{11}), l_{22}] = x_{21} ;$$

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$$[R(x_{21}, l_{11}), \tilde{l}_{11}] = [R(x_{21}, l_{11}), \tilde{l}_{12}] = 0,$$

$$[R(x_{21}, l_{11}), \tilde{l}_{21}] = -\tilde{x}_{12}, [R(x_{21}, l_{11}), \tilde{l}_{22}] = \tilde{x}_{21};$$

$$[R(x_{21}, l_{11}), u] = [R(x_{21}, l_{11}), \tilde{u}] = [R(x_{21}, l_{11}), U] = 0;$$

$$[R(x_{21}, l_{11}), R(l_{11}, l_{21})] = 0$$

$$[R(x_{21}, l_{11}), R(l_{22}, l_{12})] = R(x_{12}, l_{21})$$

$$[R(x_{21}, l_{11}), R(1, 1)] = 2R(x_{21}, l_{11})$$

$$(g) [R(x_{12}, l_{21}), l_{11}] = [R(x_{12}, l_{21}), l_{22}] = 0,$$

$$[R(x_{12}, l_{21}), l_{12}] = 2x_{12}, [R(x_{12}, l_{21}), l_{21}] = 2x_{21};$$

$$[R(x_{12}, l_{21}), \tilde{l}_{11}] = [R(x_{12}, l_{21}), \tilde{l}_{22}] = 0,$$

$$[R(x_{12}, l_{21}), \tilde{l}_{12}] = 2\tilde{x}_{12}, [R(x_{12}, l_{21}), \tilde{l}_{21}] = 2\tilde{x}_{21};$$

$$[R(x_{12}, l_{21}), u] = [R(x_{12}, l_{21}), \tilde{u}] = [R(x_{12}, l_{21}), U] = 0;$$

$$[R(x_{12}, l_{21}), R(l_{11}, l_{21})] = -2R(x_{21}, l_{11}),$$

$$[R(x_{12}, l_{21}), R(l_{22}, l_{12})] = 2R(x_{12}, l_{22}),$$

$$[R(x_{12}, l_{21}), R(1, 1)] = 0.$$

We note that if  $x \in J_0$  then the  $\phi$ -vector space spanned by the seven copies of  $x$  is a  $S_1$ -module.

## 2. Tits and Faulkner's constructions

### 2.1 Isomorphism between $\text{Der } C$ and $S_1$

Let  $C$  be the algebra of split octonions and  $J$  be an arbitrary Jordan algebra of degree 3. Then according to Tits

$L = \text{Der}(C) \otimes C_0 \otimes J_0 \otimes \text{Der}(J)$  is a Lie algebra, where  $[a \otimes x, D] = aD \otimes x$ , for all  $D$  in  $\text{Der} C$ ,  $a$  in  $C_0$ ,  $x$  in  $J_0$ .

We have that a derivation of  $C$  leaves  $C_0$  invariant. Thus,  $aD \otimes x \in C_0 \otimes J_0$ , that is, the action of  $\text{Der } C$  on  $J_0$  is an element on  $C_0 \otimes J_0$ .

$C_0$  is seven dimensional and this tells us that  $L$  contains seven copies of  $J_0$ , namely  $e_2 - e_1 \otimes x$ ,  $f_1 \otimes x$ ,  $f_2 \otimes x$ ,  $f_3 \otimes x$ ,  $g_1 \otimes x$ ,  $g_2 \otimes x$ ,  $g_3 \otimes x$ ,  $x \in J_0$ . From 1.3(5) of Chapter 2 we have

$$[a \otimes x, b \otimes y] = \frac{1}{12} T(xy) D_{a,b} + a*b \otimes x*y - \frac{1}{2} T(ab) [R_x, R_y],$$

for all  $a, b \in C_0$ ;  $x, y \in J_0$ . Using this we obtain

$$[(e_2 - e_1) \circ x, (e_2 - e_1) \circ y] = [R_x, R_y] \in \text{Der } J,$$

$$[f_1 \circ x, f_1 \circ y] = [f_2 \circ x, f_2 \circ y] = [f_3 \circ x, f_3 \circ y] = \\ = [g_1 \circ x, g_1 \circ y] = [g_2 \circ x, g_2 \circ y] = [g_3 \circ x, g_3 \circ y] = \\ = 0, \quad x, y \in J_0.$$

According to Faulkner we have for an arbitrary Jordan algebra  $J$  of degree 3 the Lie algebra

$$S = S(M, R(M)) = M + \bar{M} + \phi u + \phi \bar{u} + R(M).$$

Since  $\phi 1 \in J$ , we see that the Lie algebra  $S_1$  constructed in §1.1 of this chapter is contained in  $S$ .

$S$  contains seven copies of  $J_0$  namely  $x_{12}, x_{21}, \bar{x}_{12}, \bar{x}_{21}, R(x_{12}, l_{22}), R(x_{21}, l_{11}), R(x_{12}, l_{21})$  as obtained in §1.2 of this chapter. Using the Lie product rule in Faulkner's construction we have

$$[x_{12}, y_{12}] = \langle x_{12}, y_{12} \rangle u = 0 \text{ using 2.1(5), chapter II.}$$

$$\text{Similarly, } [x_{21}, y_{21}] = \langle x_{21}, y_{21} \rangle u = 0, [\bar{x}_{12}, \bar{y}_{12}] = \\ = \langle x_{12}, y_{12} \rangle \bar{u} = 0, [\bar{x}_{21}, \bar{y}_{21}] = \langle x_{21}, y_{21} \rangle \bar{u} = 0; \text{ for}$$

$$\begin{aligned}
Z &= \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in M, \quad Z[R(x_{12}, l_{22}), R(y_{12}, l_{22})] = \\
&= ZR(x_{12}, l_{22})R(y_{12}, l_{22}) - ZR(y_{12}, l_{22})R(x_{12}, l_{22}) = \\
&= \begin{pmatrix} 0 & 0 \\ axy & T(axx, y) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ ayx & T(axy, x) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ for}
\end{aligned}$$

$xy = yx$  and  $T(axx, y) = T(a, xxy)$ . Thus,

$[R(x_{12}, l_{22}), R(y_{12}, l_{22})]$  is a zero map. Similarly,

$[R(x_{21}, l_{11}), R(y_{21}, l_{11})]$  is a zero map, for

$$Z[R(x_{21}, l_{11}), R(y_{21}, l_{11})] = \begin{pmatrix} T(bxx, y) & Sxy \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} T(bxy, x) & Syxx \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \text{ Finally, } Z[R(x_{12}, l_{21}), R(y_{12}, l_{21})] =$$

$$= \begin{pmatrix} 0 & \{ax\}y - \{ay\}x \\ \{bx\}y - \{by\}x & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 4a[R_x, R_y] \\ 4b[R_x, R_y] & 0 \end{pmatrix} = 4 \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} [R_x, R_y] \text{ which is not}$$

equal to zero, we use above  $\{ax\} = \{ax\} = 2aR_x$ .

Thus, we obtain a similar behaviour between  $(e_2 - e_1) \times x$  and  $R(x_{12}, l_{21})$ .

The annihilator of  $(e_2 - e_1)$  is also the annihilator of  $(e_2 - e_1) \otimes x$ . From 1.3(g) of this chapter the annihilator of  $R(x_{12}, l_{21})$  is  $\{l_{11}, l_{22}, \tilde{l}_{11}, \tilde{l}_{22}, u, \tilde{u}, U, R(1,1)\}$ . Again from §1.4 of chapter 1 the annihilator of  $(e_2 - e_1) \otimes x$  is  $\{D_{f_1, g_1}, D_{f_2, g_2}, D_{f_3, g_3}, D_{f_1, g_2}, D_{f_2, g_3}, D_{f_3, g_1}, D_{f_1, g_3}, D_{f_2, g_1}, D_{f_3, g_2}\}$ .

The annihilator of  $R(x_{12}, l_{21})$  act on the rest six copies of  $J_0$  contained in  $S$  as in §1.3(a) to §1.3(f) of this chapter.

The annihilator of  $(e_2 - e_1) \otimes x$  act on the rest six copies of  $J_0$  contained in  $L$  as in table 2 where we replace  $f_1, g_1$  by  $f_1 \otimes x, g_1 \otimes x$  and apply the product  $[a \otimes x, D] = aD \otimes x$ ,  $a \in C_0, x \in J_0, D \in \text{Der } C$ .

The similarity between these actions appears and we see that ( $\approx$  means corresponds to)

$$(1) \quad f_1 \circ x \approx R(x_{21}, l_{11}) \quad g_1 \circ x \approx R(x_{12}, l_{22})$$

$$f_2 \circ x \approx \tilde{x}_{21} \quad g_2 \circ x \approx -x_{12}$$

$$f_3 \circ x \approx x_{21} \quad g_3 \circ x \approx \tilde{x}_{12}$$

Also,  $D_{f_1, g_2} \approx -3l_{11}$ ,  $D_{f_3, g_1} \approx 3l_{22}$ ,  $D_{f_2, g_3} \approx 3\tilde{u}$ .

$$(2) \quad D_{f_1, g_3} \approx 3\tilde{l}_{11}, \quad D_{f_2, g_1} \approx 3\tilde{l}_{22}, \quad D_{f_3, g_2} \approx -3u,$$

$$D_{f_1, g_1} \approx R(1,1) \text{ and } D_{f_2, g_2} \approx -D_{f_3, g_3} \approx -3U.$$

Next, we also see from §1.3(a) to 1.3(f) of this chapter and table 2 that

$$D(e_2, g_1) \approx R(l_{22}, l_{12}) \quad D(e_1, f_1) \approx R(l_{11}, l_{21})$$

$$D(e_2, g_2) \approx -l_{12} \quad D(e_1, f_2) \approx \tilde{l}_{21}$$

$$D(e_2, g_3) \approx \tilde{l}_{12} \quad D(e_1, f_3) \approx l_{21}.$$

Thus, we have found a correspondance between the generators of Der C and those of  $S_1$  and a correspondance between

the copies of  $J_0$  in  $L$  and those of  $J_0$  in  $S$ .

In  $L$  we know that  $\text{Der } C$  act on the copies of  $J_0$  giving an element in the copies of  $J_0$ . §1.3(a) to §1.3(g) of this chapter show that  $S_1$  act on the copies of  $J_0$  in  $S$  giving an element in the copies of  $J_0$ .

Hence we conclude that the behaviour of  $S_1$  in  $S$  is exactly same as that of  $\text{Der } C$  in  $L$ .

We shall end this section by defining a map

$\phi_1: \text{Der } C \rightarrow S_1$  and  $\phi_2: \text{Der } C \text{ module} \rightarrow S_1 \text{ module}$  we discussed before.

To define  $\phi_1$  and  $\phi_2$  we use the correspondences we found above between the generators of  $\text{Der } C$  and those of  $S_1$  and the generators of  $\text{Der } C \text{ module}$  in  $L$  and those of  $S_1$ -module in  $S$  for a fixed  $x \in J_0$  respectively. Therefore, we define

$$D_{f_1, g_2} \phi_1 = -3l_{11}, D_{f_3, g_1} \phi_1 = 3l_{22} \text{ and so on and}$$

$$(f_1 \otimes x) \phi_2 = R\left(\frac{1}{2} x_{21}, l_{11}\right), (f_2 \otimes x) \phi_2 = \frac{1}{2} \bar{x}_{21} \text{ and so on. } \phi_1$$

is injective: Let  $D_1 \phi_1 = D_2 \phi_2$  for  $D_1, D_2 \in \text{Der } C$  and

$$D_1 = D_2. \text{ We can write } D_1 \phi_1 = (\alpha_1 D_{f_1, g_2} + \alpha_2 D_{f_3, g_1} +$$

$$+ \dots + \alpha_{14} D_{e_1, f_3}) \phi_1 = \alpha_1 D_{f_1, g_2} \phi_1 + \alpha_2 D_{f_3, g_1} \phi_1 +$$

$$+ \dots + \alpha_{14} D_{e_1, f_3} \phi_1) = \alpha_1 (-3l_{11}) + \alpha_2 (3l_{22}) + \dots$$

$$\dots + \alpha_{14} l_{21} = S_1 \text{ (say), where } \alpha_i \in \phi \text{ and } \phi_1 \text{ is clearly}$$

linear. Similarly, we shall obtain  $D_2 \phi_2 = \beta_1 (-3l_{11}) +$

$$+ \beta_2 (3l_{22}) + \dots + \beta_{14} l_{21} = S_2 \text{ (say), } \beta_i \in \phi. \text{ But}$$

$D_1 \phi_1 = D_2 \phi_2$  implies  $S_1 = S_2$ . Thus,  $\alpha_i = \beta_i$  which shows

that  $D_1 = D_2$ .

$\phi_1$  is surjective; Let  $S \in S_1$  and

$$S = \alpha_1(-31_{11}) + \alpha_2(31_{22}) + \dots + \alpha_{14}1_{21}$$

$$= \alpha_1 D_{f_1, g_2} \phi_1 + \alpha_2 D_{f_3, g_1} \phi_1 + \dots + \alpha_{14} D_{e_1, f_3} \phi_1$$

$$= (\alpha_1 D_{f_1, g_2} + \alpha_2 D_{f_3, g_1} + \dots + \alpha_{14} D_{e_1, f_3}) \phi_1$$

=  $D\phi_1$  (say). But  $D \in \text{Der } C$ .  $\phi_1$  preserves Lie product:

We are to show that for  $D_1, D_2 \in \text{Der } C$ ,  $[D_1\phi_1, D_2\phi_1] =$

$= [D_1, D_2]\phi_1$ . First of all we note that for a  $\otimes x \in C_0 \otimes J_0$

and  $D \in \text{Der } C$ ,

(3)  $[(a \otimes x)\phi_2, D\phi_1] = [a \otimes x, D]\phi_2 = (aD \otimes x)\phi_2$ . This is the

similarity we talk about between the table 2 and §1.3(a)

to 1.3(g) of this chapter, where we find correspondances

between the generators of  $\text{Der } C$  and those of  $S_1$ , and

at the same time between the generators of  $\text{Der } C$ -module

in  $L$  and those of  $S_1$ -module in  $S$  for a fixed  $x \in J_0$

defining  $\phi_1$  and  $\phi_2$ .

$a \otimes x, D_1, D_2$  as elements in  $L$  and  $(a \otimes x)\phi_2, D_1\phi_1,$

$D_2\phi_1$  as elements in  $S$  satisfy the Jacobi identity. Thus,

$$[[a \otimes x, D_1], D_2] + [[D_1, D_2], a \otimes x] + [D_2, a \otimes x], D_1] = 0$$

$$\text{i.e. } [a \otimes x, [D_1, D_2]] = [[a \otimes x, D_1], D_2] - [[a \otimes x, D_2], D_1]$$

$$\text{i.e. } a[D_1, D_2] \otimes x = [aD_1 \otimes x, D_2] - [aD_2 \otimes x, D_1]$$

$$\text{i.e. } a[D_1, D_2] \otimes x = aD_1D_2 \otimes x - aD_2D_1 \otimes x .$$

$$\text{Also, } [(a \otimes x)\phi_2, [D_1\phi_1, D_2\phi_1]] = [[(a \otimes x)\phi_2, D_1\phi_1], D_2\phi_1] \\ - [[(a \otimes x)\phi_2, D_2\phi_1], D_1\phi_1].$$

To show  $[D_1\phi_1, D_2\phi_1] = [D_1, D_2]\phi_1$ , it is necessary that

$$[(a \otimes x)\phi_2, [D_1\phi_1, D_2\phi_1]] = [(a \otimes x)\phi_2, [D_1, D_2]\phi_1] \text{ for arbitra-}$$

ry  $a \otimes x \in C_0 \otimes J_0$ .

The right hand side is  $[a \otimes x, [D_1, D_2]]\phi_2$  by (3) above, and again by (3) the left side is

$$[(a \otimes x)\phi_2, [D_1\phi_1, D_2\phi_1]] = [[(a \otimes x)\phi_2, D_1\phi_1], D_2\phi_1] -$$

$$[[a \otimes x)\phi_2, D_2\phi_1], D_1\phi_1] = [[a \otimes x, D_1]\phi_2, D_2\phi_1] -$$

$$[[a \otimes x, D_2]\phi_2, D_1\phi_1] = [aD_1 \otimes x, D_2]\phi_2 - [aD_2 \otimes x, D_1]\phi_2$$

$$= (aD_1D_2 \otimes x)\phi_2 - (aD_2D_1 \otimes x)\phi_2 = (aD_1D_2 \otimes x - aD_2D_1 \otimes x)\phi_2$$

$$= (a[D_1, D_2] \otimes x)\phi_2 = [a \otimes x, [D_1, D_2]]\phi_2 .$$

$\phi_2$  is bijective:  $C_0 \otimes J_0$  is a Der C module of  $\phi$ -dimension  $\dim_{\phi} C_0 \times \dim_{\phi} J_0$  with basis  $\{a \otimes x \mid a \text{ and } x \text{ are fixed bases of } C_0 \text{ and } J_0 \text{ respectively}\}$ . Similarly,  $\text{Im} \phi_2$  is a  $S_1$ -module in  $S$  which has the same dimension because it is the direct sum of  $7(=\dim C_0) \dim J_0$  dimensional vector spaces. Therefore, in the same way  $\phi_1, \phi_2$  is bijective.

Therefore, we have proved that  $\phi_1$  is an isomorphism and  $\phi_2$  is a bijective mapping.

2.2 The centralizer of  $S_1$ .

The centralizer of  $S_1$  in  $S$  is

cent  $S_1 = \{S \in S \mid [SS_1] = 0\}$ . Let  $S = y + \bar{z} + \alpha u + \beta \bar{u} + R \in S$

and  $S_1 = y_1 + \bar{z}_1 + \alpha_1 u + \beta_1 \bar{u} + R_1 \in S_1$ . We shall find some conditions when  $[S, S_1] = 0$ .  $[S, S_1] = 0$  implies

$$(1) [y + \bar{z} + \alpha u + \beta \bar{u} + R, y_1] = 0$$

$$\text{i.e. } \langle y, y_1 \rangle u - R(y_1, z) \sim \beta \bar{y}_1 - y_1 R = 0$$

$$\text{i.e. } \langle y, y_1 \rangle = 0, y_1 R = -R(y_1, z), \beta = 0.$$

$$(2) [y + \bar{z} + \alpha u + \beta \bar{u} + R, \bar{z}_1] = 0$$

$$\text{i.e. } R(y, z_1) + \langle z, z_1 \rangle \bar{u} + \alpha z_1 - (z_1 R') \sim = 0$$

$$\text{i.e. } \langle z, z_1 \rangle = 0, \alpha = 0, (z_1 R') \sim = R(y, z_1)$$

$$(3) [y + \bar{z} + \alpha u + \beta \bar{u} + R, u] = 0$$

$$\text{i.e. } -z - \beta u - \rho(R)u = 0$$

$$\text{i.e. } z = 0, \beta = 0, \rho(R) = 0$$

$$(4) [y+\bar{z}+\alpha u+\beta \bar{u}+R, \bar{u}] = 0$$

$$\text{i.e. } \bar{y} + \alpha U + \rho(R)\bar{u} = 0$$

$$\text{i.e. } \bar{y} = 0, \alpha = 0, \rho(R) = 0$$

$$(5) [y+\bar{z}+\alpha u+\beta \bar{u}+R, R_1] = 0$$

$$\text{i.e. } yR_1 + (zR_1')^{\sim} + \alpha\rho(R_1) - \beta\rho(R_1) + [R, R_1] = 0$$

$$\text{i.e. } yR_1 = 0, zR_1' = 0, \alpha - \beta = 0, [R, R_1] = 0$$

We have that  $\bar{y} = 0 \Rightarrow y = 0$  and  $z = 0 \Rightarrow \bar{z} = 0$ . From (1) to (5) above we get  $y_1R = 0$ ,  $(z_1R_1')^{\sim} = 0 \Rightarrow z_1R_1' = 0$ ,  $\rho(R) = 0 \Rightarrow R$  is a derivation and hence  $R = R'$  and  $[R, R_1] = 0$ . Thus, we have  $S = \sqrt{R}$  where  $R$  is a derivation of  $M$ ,  $M_1R = 0$  and  $[R, R_1(M)] = 0$  in order to satisfy  $[S, S_1] = 0$  because  $y = z = \alpha = \beta = 0$ . In other words,

$$\text{cent } S_1 = \{R \in \text{Der } M \mid M_1R = 0 \text{ and } [R, R_1] = 0\}.$$

Again if  $R_1 = R_1(x_1, y_1)$  where  $x_1, y_1 \in M_1$  then

$$[R, R_1(x_1, y_1)] = -[R_1(x_1, y_1), R] = R_1(x_1R, y_1) +$$

$$+ R_1(x_1, y_1R) = R_1(0, y_1) + R_1(x_1, 0) = 0 \text{ for } M_1R = 0.$$

Thus,  $M_1 R = 0$  implies  $[R, R_1] = 0$ . Hence, the centralizer of  $S_1$  becomes

$$\text{cent } S_1 = \{R \in \text{Der } M \mid M_1 R = 0\}$$

For example we shall show that  $[R(x_{12}, l_{21}), R(y_{12}, l_{21})]$

lies in  $\text{cent } S_1$ ,  $x, y \in J_0$ . Since  $R(x_{12}, l_{21}),$

$R(y_{12}, l_{21}) \in \text{Der } M$ ,  $[R(x_{12}, l_{21}), R(y_{12}, l_{21})] \in \text{Der } M$ .

Again, for  $z = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in M$ ,

$$(6) \quad z[R(x_{12}, l_{21}), R(y_{12}, l_{21})] = \begin{pmatrix} 0 & \{\{ax\}ly\} - \{\{aly\}lx\} \\ \{\{bx\}ly\} - \{\{by\}xl\} & 0 \end{pmatrix}$$

$= 4 \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} [R_x, R_y]$  as in section 2.1. Since multiplication in Jordan algebras is commutative, it is easily verified that  $M_1 [R(x_{12}, l_{21}), R(y_{12}, l_{21})] = 0$ . Thus,

$$(7) \quad [R(x_{12}, l_{21}), R(y_{12}, l_{21})] = z(R(x_{12}, y_{21}) - R((xy)_{12}, l_{21})),$$

using 2.2(5) of chapter 2, is in  $\text{cent } S_1$ .

$[R_x, R_y] = D_{x,y}$  is a derivation of  $J$ , for

$x, y \in J_0$ . We have that for  $x \in J$ ,

(8)  $x = \frac{1}{3} T(x)1 + x_0$ ,  $x_0 \in J_0$ . Therefore, for  $x, y \in J$

$$\begin{aligned} [R_x, R_y] &= [R_{\frac{1}{3} T(x)1 + x_0}, R_y] = [R_{\frac{1}{3} T(x)1}, R_y] + \\ &+ [R_{x_0}, R_y] = \\ &= [R_{x_0}, R_y] = [R_{x_0}, R_{\frac{1}{3} T(y)1 + y_0}] = [R_{x_0}, R_{y_0}], \end{aligned}$$

$y_0 \in J_0$ . Hence by the theorem 13, page-258 in [3]

$[R_x, R_y]$  and thus  $[R_{x_0}, R_{y_0}]$  span  $\text{Der } J$ . We have, for

$$z = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in M, \quad z[R(\frac{1}{2} x_{12}, l_{21}), R(\frac{1}{2} y_{12}, l_{21})] =$$

$$= \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} D_{x,y}. \quad \text{From this we can define a mapping}$$

$$\phi_3: \text{Der } J \rightarrow \text{cent } S_1 \text{ such that } (D_{x,y})\phi_3 =$$

$$= [R(\frac{1}{2} x_{12}, l_{21}), R(\frac{1}{2} y_{12}, l_{21})].$$

$\phi_3$  is injective: Let  $(D_{x,y})\phi_3 = 0$ . Then

$[R(\frac{1}{2}x_{12}, l_{21}), R(\frac{1}{2}y_{12}, l_{21})]$  is a zero map. That is

$$z[R(\frac{1}{2}x_{12}, l_{21}), R(\frac{1}{2}y_{12}, l_{21})] = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} D_{x,y} = 0. \text{ This}$$

implies  $D_{x,y} = 0$  for  $a$  and  $b$  are not equal to zero.

We have that  $S = M \oplus \tilde{M} \oplus \phi u \oplus \phi \tilde{u} \oplus \phi U \oplus \mathcal{D}$  where

$$\mathcal{D} = \{ \sum_i R(x_i, y_i) \mid \sum_i \langle x_i, y_i \rangle = 0 \}, \text{ (\S 4 in [7]).}$$

$$S_1 = M_1 \oplus \tilde{M}_1 \oplus \phi u \oplus \phi \tilde{u} \oplus \phi U \oplus \mathcal{D}_1. \text{ This shows that we can}$$

split  $S$  into three parts and we can write  $S$  as the direct sum of these. These three parts are  $S_1$ , the vector space generated by the copies of  $J_0$  in  $S$  and the rest derivations in  $\mathcal{D}$ . We shall show that an element in  $\mathcal{D}$  can be written as the direct sum of derivations of  $M$  in these three parts.

For some  $i$ , let

$$R(x_i, y_i) = R\left(\begin{pmatrix} 1 & x \\ x' & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ y' & 1 \end{pmatrix}\right), \quad x, x', y, y' \in J.$$

$$= R\left(1_{11}, \begin{pmatrix} 1 & y \\ y' & 1 \end{pmatrix}\right) + R\left(x_{12}, \begin{pmatrix} 1 & y \\ y' & 1 \end{pmatrix}\right) + R\left(x'_{21}, \begin{pmatrix} 1 & y \\ y' & 1 \end{pmatrix}\right) +$$

$$+ R\left(1_{22}, \begin{pmatrix} 1 & y \\ y' & 1 \end{pmatrix}\right)$$

$$\begin{aligned}
 &= R(1_{11}, 1_{11}) + R(1_{11}, y_{12}) + R(1_{11}, y'_{21}) + R(1_{11}, 1_{22}) \\
 &+ R(x_{12}, 1_{11}) + R(x_{12}, y_{12}) + R(x_{12}, y'_{21}) + R(x_{12}, 1_{22}) \\
 &+ R(x'_{21}, 1_{11}) + R(x'_{21}, y_{12}) + R(x'_{21}, y'_{21}) + R(x'_{21}, 1_{22}) \\
 &+ R(1_{22}, 1_{11}) + R(1_{22}, y_{12}) + R(1_{22}, y'_{21}) + R(1_{22}, 1_{22})
 \end{aligned}$$

we have from (8) above that

$$y = \frac{1}{3} T(y)1 + y_0; \quad y_0 \in J_0.$$

$$\text{Therefore, } R(1_{11}, y_{12}) = R(1_{11}, (\frac{1}{3} T(y)1 + y_0)_{12}) =$$

$$= \frac{1}{3} T(y) R(1_{11}, 1_{12}) + R(1_{11}, (y_0)_{12}). \quad \text{Similarly,}$$

$$R(1_{11}, y'_{21}) = \frac{1}{3} T(y') R(1_{11}, 1_{21}) + R(1_{11}, (y'_0)_{21}), \text{ a sum of}$$

two elements one in  $S_1$  and the other in the copies of  $J_0$

in  $S$ . From the table 6,

$$zR(x_{12}, y_{12}) = \begin{pmatrix} T(a, xxy) & (xxy) \times b \\ \beta xxy & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} T(a, (x+y)^\#) & (x+y)^\# \times b \\ \beta(x+y)^\# & 0 \end{pmatrix} - \begin{pmatrix} T(a, x^\#) & x^\# \times b \\ \beta x^\# & 0 \end{pmatrix}$$

$$- \begin{pmatrix} T(a, y^\#) & y^\# \times b \\ \beta y^\# & 0 \end{pmatrix}$$

$$= zR(-(x+y)^\#_{21}, l_{11}) + zR(x^\#_{21}, l_{11}) + zR(y^\#_{21}, l_{11})$$

Thus,  $R(x_{12}, y_{12})$  can be written as a sum of elements in  $S_1$  and copies of  $J_0$  in  $S$ . Similarly,  $R(x'_{21}, y'_{21})$ . We

have the equation (7) above that

$$[R(\frac{1}{2}(x_0)_{12}, l_{21}), R(\frac{1}{2}(y'_0)_{12}, l_{21})] = \frac{1}{2}(R((x_0)_{12}, (y'_0)_{21}) -$$

$$- R((x_0 y'_0)_{12}, l_{21})). \text{ That is, } R((x_0)_{12}, (y'_0)_{21}) =$$

$$= 2[R(\frac{1}{2}(x_0)_{12}, l_{21}), R(\frac{1}{2}(y'_0)_{12}, l_{21})] + R((x_0 y'_0)_{12}, l_{21}).$$

$$\text{By (8) above, } R(x_{12}, y'_{21}) = R((\frac{1}{3} T(x)1+x_0)_{12}, (\frac{1}{3} T(y')1+y'_0)_{21})$$

$$= \frac{1}{3} T(x) R(1_{12}, (\frac{1}{3} T(y')1+y'_0)_{21}) + R((x_0)_{12}, (\frac{1}{3} T(y')1+y'_0)_{21}).$$

$$= \frac{1}{9} T(x)T(y')R(1_{12}, 1_{21}) + \frac{1}{3} T(x)R(1_{12}, (y'_0)_{21}) + \\ + \frac{1}{3} T(y')R((x_0)_{12}, 1_{21}) + R((x_0)_{12}, (y'_0)_{21}).$$
 This shows that

$R(x_{12}, y'_{21})$  can be written as a sum of elements in  $S_1$ ,

copies of  $J_0$  in  $S$  and  $\text{Im } \phi_3$ . Since  $R(x'_{21}, y_{12}) =$

$$= R(y_{12}, x'_{21}) + \langle x'_{21}, y_{12} \rangle U,$$
 by 2.2(6) of Chapter II, we have

the same conclusion for  $R(x'_{21}, y_{12})$  as for  $R(x_{12}, y'_{21})$ . Thus

we see that for each  $i$ ,  $R(x_i, y_i)$  can be written as the direct

sum of  $S_1$ , the vector space generated by the copies of  $J_0$

in  $S$ , i.e.  $\text{Im } \phi_2$  and  $\text{Im } \phi_3$  and hence  $\sum_i R(x_i, y_i) \in \mathcal{D}$ . It

is a direct sum because  $S_1 \cap \text{Im } \phi_2 \cap \text{Im } \phi_3 = \{0\}$ .

$$S_1 \oplus \text{Im } \phi_2 \oplus \text{Im } \phi_3 = S.$$

A similar argument shows that

$S_1 \oplus \text{Im } \phi_2 \oplus \text{cent } S_1 = S$ , because

$[R(\frac{1}{2}(x_0)_{12,1_{21}}), R(\frac{1}{2}(y_0)_{12,1_{21}})] \in \text{cent } S_1$ . Thus

$\text{Im } \phi_3 = \text{cent } S_1$ . That is,  $\phi_3$  is surjective.

$\phi_3$  preserves Lie products:

Let  $A = [R(\frac{1}{2} x_{12,1_{21}}), R(\frac{1}{2} y_{12,1_{21}})]$ ,  $x, y \in J_0$ .

and  $B = [R(\frac{1}{2} x'_{12,1_{21}}), R(\frac{1}{2} y'_{12,1_{21}})]$ ,  $x', y' \in J_0$ .

For  $z = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in M$ ,

$$z[A, B] = zAB - zBA$$

$$= \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} D_{x,y} B - \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} D_{x',y'} A$$

$$= \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} D_{x,y} D_{x',y'} - \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} D_{x',y'} D_{x,y}$$

$$= \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} [D_{x,y}, D_{x',y'}]$$

Then by definition,  $[D_{x,y}, D_{x',y'}] \phi_3 = [A, B] =$

$$= [[R(\frac{1}{2} x_{12}, l_{21}), R(\frac{1}{2} y_{12}, l_{21})], [R(\frac{1}{2} x'_{12}, l_{21}), R(\frac{1}{2} y'_{12}, l_{21})]]$$

$$= [(D_{x,y})\phi_3, (D_{x',y'})\phi_3]$$

This proves that  $\phi_3$  is an isomorphism.

Before ending this section we shall see how the cent  $S_1$  acts on the copies of  $J_0$  in  $S$ . The copies of  $J_0$  are given in §1.2 of this chapter. Using equation (6) of this section

$$z[R(x_{12}, l_{21}), R(y_{12}, l_{21})] = \begin{pmatrix} 0 & T(a,x)y - T(a,y)x + (a \times x) \times y \\ & -(a \times y) \times x \\ T(b,x)y - T(b,y)x + (b \times x) \times y & 0 \\ & -(b \times y) \times x \end{pmatrix}$$

$$x, y \in J_0, z = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in M.$$

$$z_{12}[R(x_{12}, l_{21}), R(y_{12}, l_{21})] = \begin{pmatrix} 0 & T(z,x)y - T(z,y)x + (z \times x) \times y \\ & -(z \times y) \times x \\ 0 & 0 \end{pmatrix}$$

lies in  $(J_0)_{12}$ . For,

$$T(T(z,x)y - T(z,y)x + (z \times x) \times y - (z \times y) \times x) = T(z,x)T(y)$$

$$- T(z,y)T(x) - T(z \times x, y) + T(z \times y, x) = 0$$

$$[\therefore T(z \times x, y) = T(z, x \times y) = T(z, y \times x)].$$

$$\begin{matrix} & & & 0 & & 0 \\ z_{21}[R(x_{12}, l_{21}), R(y_{12}, l_{21})] = & ( & & & & ) \\ & & T(z,x)y - T(z,y)x + & & 0 \\ & & (z \times x) \times y - (z \times y) \times x & & \end{matrix}$$

similarly lies in  $(J_0)_{21}$ .

In the same way,  $\tilde{z}_{12}[R(x_{12}, l_{21}), R(y_{12}, l_{21})]$  and

$\tilde{z}_{12}[R(x_{12}, l_{21}), R(y_{12}, l_{21})]$  lie in  $(\tilde{J}_0)_{12}$  and  $(\tilde{J}_0)_{21}$

respectively.

Using the equation (7) of this section,

$$[R(z_{12}, l_{22}), [R(x_{12}, l_{21}), R(y_{12}, l_{21})]]$$

$$= 2([R(z_{12}, l_{22}), R(x_{12}, y_{21})] - [R(z_{12}, l_{22}), R((xy)_{12}, l_{21})])$$

$$= 2(R(z_{12}, R(x_{12}, y_{21}), l_{22}) + R(z_{12}, l_{22}, R(x_{12}, y_{21}))) -$$

$$\begin{aligned}
 & - R(z_{12} R((xy)_{12,1_{21}}, 1_{22}) - R(z_{12,1_{22}} R((xy)_{12,1_{21}}))) \\
 & = 2(R(-\{zyx\}_{12,1_{22}}) - R(-\{z1(xy)\}_{12,1_{22}})) \\
 & = 2R(\{z1(xy)\} - \{zyx\}_{12,1_{22}}) \text{ lies in } R((J_0)_{12,1_{22}}).
 \end{aligned}$$

For,  $T(\{z1(xy)\} - \{zyx\}) = T(2z(xy) - T(z,x)y - T(z,y)x +$   
 $+ (z \times x) \times y) = 2T(z(xy)) - T(z \times x, y) = 2T(z, xy) - T(z, x \times y)$   
 $= 2T(z, xy) - T(z, 2xy - T(xy)) = 0$ .

We use,  $T(x, y) = T(xy)$ ,  $x \times y = 2xy - T(xy)$ .

$$\begin{aligned}
 & [R(z_{21,1_{11}}), [R(x_{12,1_{21}}), R(y_{12,1_{21}})]] \\
 & = 2([R(z_{21,1_{11}}), R(x_{12,1_{21}})] - [R(z_{21,1_{11}}), R((xy)_{12,1_{21}})]) \\
 & = 2(R(z_{21} R((x_{12}, y_{21}), 1_{11}) + R(z_{21,1_{11}} R((x_{12}, y_{21}))) \\
 & - R(z_{21} R((xy)_{12,1_{21}}, 1_{11}) - R(z_{21,1_{11}} R((xy)_{12,1_{21}})))
 \end{aligned}$$

$$= 2(R(\{zxy\} - zT(x,y))_{21,1_{11}}) + R(z_{21}, -T(x,y)1_{11}) \\ - R(\{z(xy)1\} - zT(xy))_{21,1_{11}} - R(z_{21}, -T(xy)1_{11}))$$

$$= 2(R(\{zxy\} - \{z(xy)1\})_{21,1_{11}}) \text{ lies in } R((J_0)_{21,1_{11}}),$$

because,  $T(\{zxy\} - \{z(xy)1\}) = 0$  as in the preceding case.

$$[R(z_{12}, l_{21}), [R(x_{12}, l_{21}), R(y_{12}, l_{21})]]$$

$$= 2([R(z_{12}, l_{21}), R(x_{12}, y_{21})] - [R(z_{12}, l_{21}), R((xy)_{12}, l_{21})])$$

$$= 2(R(z_{12}R(x_{12}, y_{21}), l_{21}) + R(z_{12}, l_{21}(R(x_{12}, y_{21})))$$

$$- R(z_{12}R((xy)_{12}, l_{21}), l_{21}) - R(z_{12}, l_{21}R((xy)_{12}, l_{21})))$$

$$= 2(R(-\{zyx\}_{12}, l_{21}) + R(z_{12}, (\{lxy\} - T(x,y))_{21}))$$

$$- R(-\{zl(xy)\}_{12}, l_{21}) - R(z_{12}, (\{l(xy)1\} - lT(xy))_{21}))$$

$$= 2(R(\{zl\{xy\} - \{zyx\}\}_{12}, l_{21}) \quad (\because \{lxy\} = 2xy)$$

lies in  $R((J_0)_{12}, l_{21})$ , for  $T(\{zl(xy)\} - \{zyx\}) = 0$ .

Thus, the Lie product of cent  $S_1$  with the copies of  $J_0$  in

$S$  is an element in the copies of  $J_0$ . (We use  $z_{12}, z_{21}, \tilde{z}_{12}, \tilde{z}_{21}, R(z_{12}, l_{22}), R(z_{21}, l_{11})$  and  $R(z_{12}, l_{21})$  as copies of  $J_0$ ). That is, the behavior of cent  $S_1$  in  $S$  is same as that of  $\text{Der } J$  in  $L$ .

### 2.3 Isomorphism between $L$ and $S$ .

Let us define a map  $\phi: L \rightarrow S$  such that

$$\phi = \begin{cases} \phi_1, & \text{when } \phi \text{ acts on } \text{Der } C, \\ \phi_2, & \text{when } \phi \text{ acts on } C_0 \otimes J_0, \\ \phi_3, & \text{when } \phi \text{ acts on } \text{Der } J. \end{cases}$$

$\text{Der } C$  and  $\text{Der } J$  are Lie algebras

Therefore,  $\phi$  is an isomorphism if the restrictions of  $\phi$  on  $\text{Der } C$  and  $\text{Der } J$  are isomorphisms and the restriction of  $\phi$  on  $C_0 \otimes J_0$  is bijective and preserves Lie products.

$\phi_1$  and  $\phi_3$ , the restrictions of  $\phi$  on  $\text{Der } C$  and  $\text{Der } J$ , respectively, are isomorphisms as we have already proved.  $\phi_2$  is injective and surjective implies that the restriction of

$\phi$  on  $C_0 \otimes J_0$  is injective and surjective. Therefore, the only thing left to prove  $\phi$  is an isomorphism is that the restriction of  $\phi$  on  $C_0 \otimes J_0$  preserves Lie products. To do this we shall write  $\phi$  for the restrictions of  $\phi$  to  $C_0 \otimes J_0$ ,  $\text{Der } C$ , or  $\text{Der } J$ .

To show that  $\phi$  preserves Lie products we shall compute every possible Lie product in  $\text{Der } C$ -module in  $L$  and  $S_1$ -module in  $S$  and check that  $\phi$  preserves the products.

From [1], page 22,

$$\begin{aligned} [f_1 \otimes x, g_2 \otimes y] &= \frac{1}{12} \langle x, y \rangle D_{f_1, g_2} + f_1^* g_2 \otimes x^* y \\ &\quad - (f_1, g_2) [R_x, R_y], \quad x, y \in J_0 \\ &= \frac{1}{12} T(xy) D_{f_1, g_2} + (f_1 g_2 - \frac{1}{2} T(f_1 g_2)) \otimes x^* y \\ &\quad + \frac{1}{2} T(f_1 g_2) [R_x, R_y] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{12} T(xy) D_{f_1 g_2} \quad (\because f_1 g_2 = 0, (f_1, g_2) = \frac{1}{2} T(f_1, \bar{g}_2)) \\
 &= \frac{1}{2} T(f_1, -g_2) \\
 &= -\frac{1}{2} T(f_1 g_2)
 \end{aligned}$$

$$[f_1 \otimes x, g_2 \otimes y] \phi = \left( \frac{1}{12} T(xy) D_{f_1, g_2} \right) \phi = \frac{1}{12} T(xy) (-31_{11})$$

$$[(f_1 \otimes x) \phi, (g_2 \otimes y) \phi] = [R(\frac{1}{2} x_{12}, 1_{11}), -\frac{1}{2} y_{12}]$$

$$= \frac{1}{2} y_{12} R(\frac{1}{2} x_{12}, 1_{11}) \quad (\text{from 2.2(7) of chapter 2})$$

$$= \begin{pmatrix} -\frac{1}{4} T(y, x) & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \frac{1}{12} T(xy) (-31_{11}).$$

We use  $T(x, y) = T(xy)$ . For,

$$x^\# = x^2 - T(x)x + s(x)1 \quad (\text{Eq. 3.2(12) of chapter 1})$$

Linearizing this we get

$$\begin{aligned} xxy &= 2xy - T(x)y - T(y)x + T(xxy)1 \\ &= 2xy + T(xxy)1 \end{aligned}$$

$$\therefore T(xxy) = 2T(xy) + 3T(xxy)$$

$$\text{i.e. } 2T(x,y) = 2T(xy)$$

$$\text{i.e. } T(x,y) = T(xy)$$

$$\left( \begin{array}{l} \therefore T(xxy) = -T(x,y) \\ \text{for } x,y \in J_0 \end{array} \right)$$

$$\text{Thus, } [f_1 \otimes x, g_2 \otimes y] \phi = [(f_1 \otimes x) \phi, (g_2 \otimes y) \phi]$$

Similarly, we compute

$$[f_1 \otimes x, g_3 \otimes y] = \frac{1}{12} T(xy) D_{f_1, g_3}$$

$$\therefore [f_1 \otimes x, g_3 \otimes y] \phi = \left( \frac{1}{12} T(xy) D_{f_1, g_3} \right) \phi = \frac{1}{12} \tilde{T}(xy) (3\tilde{1}_{11})$$

$$[(f_1 \otimes x) \phi, (g_3 \otimes y) \phi] = [R(\frac{1}{2}x_{21}, 1_{11}), \frac{1}{2}\tilde{y}_{12}]$$

$$= \left( -\frac{1}{2}y_{12} R(\frac{1}{2}x_{21}, 1_{11}) \right) \sim$$

$$= \begin{pmatrix} \frac{1}{4}T(y,x) & 0 \\ 0 & 0 \end{pmatrix} \sim = \frac{1}{4}T(x,y)\tilde{1}_{11}$$

$$= \frac{1}{12} T(xy) 3\tilde{1}_{11} .$$

$$[f_2 \otimes x, g_1 \otimes y] \phi = \left( \frac{1}{12} T(xy) D_{f_2, g_1} \right) \phi = \frac{1}{12} T(xy) 3\tilde{1}_{22} .$$

$$[f_2 \otimes x] \phi, (g_1 \otimes y) \phi = \left[ \frac{1}{2} \tilde{x}_{21}, R\left(\frac{1}{2} y_{12}, 1_{22}\right) \right]$$

$$= \left( \frac{1}{2} x_{21} R\left(\frac{1}{2} y_{12}, 1_{22}\right) \right) \sim$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4} T(x, y) \end{pmatrix} \sim = \frac{1}{4} T(xy) \tilde{1}_{22}$$

$$= \frac{1}{12} T(xy) (3\tilde{1}_{22}) .$$

$$[f_2 \otimes x, g_3 \otimes y] \phi = \left( \frac{1}{12} T(xy) D_{f_2, g_3} \right) \phi = \frac{1}{12} T(xy) (3\tilde{u})$$

$$[f_2 \otimes x] \phi, (g_3 \otimes y) \phi = \left[ \frac{1}{2} \tilde{x}_{21}, \frac{1}{2} \tilde{y}_{12} \right] = \left\langle \frac{1}{2} x_{21}, \frac{1}{2} y_{12} \right\rangle \tilde{u}$$

$$= \frac{1}{4} T(y, x) \tilde{u} = \frac{1}{12} T(xy) (3\tilde{u}) .$$

$$[f_3 \otimes x, g_1 \otimes y] \phi = \left( \frac{1}{12} T(xy) D_{f_3, g_1} \right) \phi = \frac{1}{12} T(xy) (31_{22})$$

$$[(f_3 \otimes x) \phi, (g_1 \otimes y) \phi] = \left[ \frac{1}{2} x_{21}, R\left(\frac{1}{2} y_{12}, 1_{22}\right) \right]$$

$$= \frac{1}{2} x_{21} R\left(\frac{1}{2} y_{12}, 1_{22}\right)$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4} T(x, y) \end{pmatrix} = \frac{1}{4} T(x, y) 1_{22}$$

$$= \frac{1}{12} T(xy) (31_{22})$$

$$[f_3 \otimes x, g_2 \otimes y] \phi = \left( \frac{1}{2} T(xy) D_{f_3, g_2} \right) \phi = \frac{1}{12} T(xy) (-3u)$$

$$[(f_3 \otimes x) \phi, (g_2 \otimes y) \phi] = \left[ \frac{1}{12} x_{21}, -\frac{1}{2} y_{12} \right]$$

$$= \left\langle \frac{1}{2} x_{21}, -\frac{1}{2} y_{12} \right\rangle u = -\frac{1}{4} T(y, x) u$$

$$= \frac{1}{12} T(xy) (-3u)$$

We also have

$$\begin{aligned}
 [f_1 \otimes x, f_2 \otimes y] &= \frac{1}{12} \langle x, \tilde{y} \rangle D_{f_1, f_2} + f_1^* f_2 \otimes x^* y \\
 &\quad - (f_1, f_2) [R_x, R_y] \\
 &= \frac{1}{12} T(xy) D_{f_1, f_2} + (g_3 - \frac{1}{2} T(g_3)) \otimes x^* y + \frac{1}{2} T(g_3) [R_x, R_y] \\
 &= -\frac{1}{12} T(xy) D_{e_2, g_3} + g_3 \otimes x^* y
 \end{aligned}$$

$$\begin{aligned}
 [f_1 \otimes x, f_2 \otimes y] \phi &= (-\frac{1}{12} T(xy) D_{e_2, g_3}) \phi + (g_3 \otimes x^* y) \phi \\
 &= -\frac{1}{12} T(xy) \tilde{1}_{12} + \frac{1}{2} (x^* y) \tilde{1}_{12} \\
 &= (-\frac{1}{12} T(xy) + \frac{1}{2} (x^* y)) \tilde{1}_{12} \\
 &= \frac{1}{2} (xy - \frac{1}{2} T(xy)) \tilde{1}_{12} \\
 &= \frac{1}{4} xy \tilde{1}_{12} \quad (\because xy = 2xy - T(xy))
 \end{aligned}$$

$$\begin{aligned}
 [(f_1 \otimes x) \phi, (f_2 \otimes y) \phi] &= [R(\frac{1}{2} x_{21}, 1_{11}), \frac{1}{2} \tilde{y}_{21}] \\
 &= (-\frac{1}{2} y_{21} R'(\frac{1}{2} x_{21}, 1_{11})) \tilde{\phantom{y}}
 \end{aligned}$$

$$= (-\frac{1}{2}y_{21}R(\frac{1}{2}x_{21}, l_{11})^{\sim})$$

$$= \begin{pmatrix} 0 & \frac{1}{4}y_{xx} \\ 0 & 0 \end{pmatrix} = \frac{1}{4}x_{xy} \cdot \tilde{l}_{12}$$

We use  $D_{f_1, f_2} = -D_{e_2, g_3}$  (from table 2)

$$T(f_1 f_2) = T(g_3) = 0$$

Similarly,

$$[f_1 \otimes x, f_2 \otimes y] \phi = (\frac{1}{12}T(xy)D_{f_1, f_3})\phi - (g_2 \otimes x^*y)\phi$$

$$= (\frac{1}{12}T(xy)D_{e_2, g_2})\phi - (g_2 \otimes x^*y)\phi$$

$$= \frac{1}{12}T(xy)(-1_{12}) + \frac{1}{2}(x^*y)_{12}$$

$$= (-\frac{1}{12}T(xy) + \frac{1}{2}(x^*y))1_{12}$$

$$= \frac{1}{2}(xy - \frac{1}{2}T(xy))1_{12} = \frac{1}{4}x_{xy}1_{12}$$

$$\begin{aligned}
 [(f_1 \otimes x)\phi, (f_3 \otimes y)\phi] &= [R(\frac{1}{2}x_{21}, l_{11}), \frac{1}{2}y_{21}] \\
 &= -\frac{1}{2}y_{21}R(\frac{1}{2}x_{21}, l_{11}) \\
 &= \begin{pmatrix} 0 & yxx \\ 0 & 0 \end{pmatrix} = \frac{1}{4}xy l_{12}
 \end{aligned}$$

$$\begin{aligned}
 [f_2 \otimes x, f_3 \otimes y]\phi &= (\frac{1}{12}T(xy)D_{f_2, f_3})\phi + (g_1 \otimes x^*y)\phi \\
 &= (-\frac{1}{12}T(xy)D_{e_2, g_1})\phi + (g_1 \otimes x^*y)\phi \\
 &= -\frac{1}{12}T(xy)R(l_{22}, l_{12}) + R(\frac{1}{2}(x^*y)_{12}, l_{22})
 \end{aligned}$$

Now for  $z = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in M$ ,

$$\begin{aligned}
 z([f_2 \otimes x, f_3 \otimes y]\phi) &= -\frac{1}{12}T(xy) \begin{pmatrix} 0 & \alpha \\ T(a)-a & T(b) \end{pmatrix} \\
 &\quad + \frac{1}{2} \begin{pmatrix} 0 & \alpha x^*y \\ axx^*y & T(b, x^*y) \end{pmatrix}
 \end{aligned}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & \alpha(xy - \frac{1}{2}T(xy)) \\ axxy - \frac{1}{2}T(a)T(xy) - \frac{1}{2}aT(xy) & T(b, xy) - \frac{1}{2}T(b)T(xy) \end{pmatrix}$$

But  $z[(f_2 \otimes x)\phi, (f_3 \otimes y)\phi] = z[\frac{1}{2}\tilde{x}_{21}, \frac{1}{2}y_{21}]$

$= -zR(\frac{1}{2}y_{21}, \frac{1}{2}x_{21})$

$$= - \begin{pmatrix} 0 & -\frac{1}{4}axxy \\ \frac{1}{2}ya\frac{1}{2}x & -\frac{1}{4}T(b, xxy) \\ -\frac{1}{4}T(a, x)y & \\ -\frac{1}{4}T(a, y)x & \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 0 & axxy \\ ax(xy) & T(b, xxy) \end{pmatrix}$$

$a \times (xxy) = 2a \times xy - a \times T(xy) = 2a \times xy - T(a)T(xy)$

$+ aT(xy), T(b, xxy) = T(b, 2xy - T(xy))$

$= 2T(b, xy) - T(b)T(xy)$

Thus,  $[f_2 \otimes x, f_3 \otimes y]\phi = [(f_2 \otimes x)\phi, (f_3 \otimes y)\phi]$

$$\begin{aligned}
 [g_1 \otimes x, g_2 \otimes y] \phi &= \left( \frac{1}{12} T(xy) D_{g_1, g_2} \right) \phi + (f_3 \otimes x^* y) \phi \\
 &= \left( - \frac{1}{12} T(xy) D_{e_1, f_3} \right) \phi + (f_3 \otimes x^* y) \phi \\
 &= - \frac{1}{12} T(xy) l_{21} + \frac{1}{2} (x^* y)_{21} \\
 &= \left( \frac{1}{2} (x^* y) - \frac{1}{12} T(xy) \right) l_{21} \\
 &= \frac{1}{2} (xy - \frac{1}{2} T(xy)) l_{21} = \frac{1}{4} x^* y l_{21}
 \end{aligned}$$

$$\begin{aligned}
 [(g_1 \otimes x) \phi, (g_2 \otimes y) \phi] &= [R(\frac{1}{2} x_{12}, l_{22}), -\frac{1}{2} y_{12}] \\
 &= \frac{1}{2} y_{12} R(\frac{1}{2} x_{12}, l_{22}) \\
 &= \frac{1}{4} \begin{pmatrix} 0 & 0 \\ y \times x & 0 \end{pmatrix} = \frac{1}{4} x^* y l_{21}
 \end{aligned}$$

$$\begin{aligned}
 [g_1 \otimes x, g_3 \otimes y] \phi &= \left( \frac{1}{12} T(xy) D_{g_1, g_3} \right) \phi - (f_2 \otimes x^* y) \phi \\
 &= \left( \frac{1}{12} T(xy) D_{e_1, f_2} \right) \phi - (f_2 \otimes x^* y) \phi \\
 &= \frac{1}{12} T(xy) l_{21} - \frac{1}{2} (x^* y)_{21}
 \end{aligned}$$

$$= \left( \frac{1}{12} T(xy) - \frac{1}{2} x^* y \right) \tilde{l}_{21}$$

$$= \frac{1}{2} \left( \frac{1}{2} T(xy) - xy \right) \tilde{l}_{21}$$

$$= -\frac{1}{4} x^* y \tilde{l}_{21}$$

$$[(g_1 \otimes x) \phi, (g_3 \otimes x) \phi] = [R(\frac{1}{2} x_{12}, l_{22}), \frac{1}{2} \tilde{y}_{12}]$$

$$= \left( -\frac{1}{2} y_{12} R(\frac{1}{2} x_{12}, l_{22}) \right) \sim$$

$$= -\frac{1}{4} \begin{pmatrix} 0 & 0 \\ y \times n & 0 \end{pmatrix} \sim = -\frac{1}{4} x^* y \tilde{l}_{21}$$

$$[g_2 \otimes x, g_3 \otimes y] \phi = \left( \frac{1}{12} T(xy) D_{g_2, g_3} \right) \phi + (f_1 \otimes x^* y) \phi$$

$$= \left( -\frac{1}{12} T(xy) D_{e_1, f_1} \right) \phi + (f_1 \otimes x^* y) \phi$$

$$= -\frac{1}{12} T(xy) R(l_{11}, l_{21}) + R(\frac{1}{2} (x^* y)_{21}, l_{11})$$

$$z([g_2 \otimes x, g_3 \otimes y] \phi) = -\frac{1}{12} T(xy) (zR(l_{11}, l_{21}))$$

$$+ zR(\frac{1}{2} (x^* y)_{21}, l_{11})$$

$$= -\frac{1}{12}T(xy) \begin{pmatrix} -T(a) & -T(b)+b \\ -\beta & 0 \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} -T(a, x^*y) & -b \times (x^*y) \\ -\beta x^*y & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \frac{1}{2}T(a)T(xy) - T(a, xy) & -b \times xy + \frac{1}{2}T(b)T(xy) \\ -\frac{1}{2}\beta T(xy) & 0 \\ -\frac{1}{2}\beta xxy & 0 \end{pmatrix}$$

$$z[(g_2 \otimes x)\phi, (g_3 \otimes y)\phi] = z[-\frac{1}{2}x_{12}, \frac{1}{2}y_{12}]$$

$$= zR(-\frac{1}{2}x_{12}, \frac{1}{2}y_{12})$$

$$= \begin{pmatrix} -\frac{1}{4}T(a, xxy) & -\frac{1}{4}T(b, y)x - \frac{1}{4}T(b, x)y \\ & + \{\frac{1}{2}xb\frac{1}{2}y\} \\ -\frac{1}{4}\beta xxy & 0 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} -T(a, xxy) & -b \times (xxy) \\ -\beta xxy & 0 \end{pmatrix}$$

This is similar to  $[f_2 \otimes x, f_3 \otimes y] \phi = [(f_2 \otimes x) \phi, (f_3 \otimes y) \phi]$

In order to compute  $[f_1 \otimes x, g_1 \otimes y]$ ,  $[f_2 \otimes x, g_2 \otimes y]$  and  $[f_3 \otimes x, g_3 \otimes y]$  we need to know  $(D_{f_1, g_1}) \phi$ ,  $(D_{f_2, g_2}) \phi$  and  $(D_{f_3, g_3}) \phi$ . But we already know that  $(D_{f_1, g_1}) \phi = R(1, 1)$ .

We shall show below that  $(D_{f_2, g_2}) \phi = +R(1_{12}, 1_{21})$  and

$$(D_{f_3, g_3}) \phi = R(1_{21}, 1_{12}).$$

It can be easily verified from table 2 that

$$[D_{e_1, f_1}, D_{e_2, g_2}] = D_{f_2, g_2} \text{ and } [D_{e_1, f_3}, D_{e_2, g_3}] = D_{f_3, g_3}$$

$$\text{Therefore, } (D_{f_2, g_2}) \phi = [D_{e_1, f_2}, D_{e_2, g_2}] \phi =$$

$$= [(D_{e_1, f_2}) \phi, (D_{e_2, g_2}) \phi]$$

$$= [1_{21}, -1_{12}] = -R(-1_{12}, 1_{21}) = R(1_{12}, 1_{21}) \text{ (by 2.2(7) of}$$

$$\text{chapter 2)} \quad (D_{f_3, g_3}) \phi = [D_{e_1, f_3}, D_{e_2, g_3}] \phi = [(D_{e_1, f_3}) \phi, (D_{e_2, g_3}) \phi]$$

$$= [1_{21}, 1_{12}] = R(1_{21}, 1_{12}).$$

The restriction of  $\phi$  to Der C is  $\phi_1$ , which has already been shown to preserve Lie products.

$$[f_1 \otimes x, g_1 \otimes y] = \frac{1}{12} T(xy) D_{f_1, g_1} + f_1 * g_1 \otimes x * y + \frac{1}{2} T(f_1, g_1) [R_x, R_y]$$

$$= \frac{1}{12} T(xy) D_{f_1, g_1} + (-e_1 + \frac{1}{2}) \otimes x * y - \frac{1}{2} [R_x, R_y]$$

$$= \frac{1}{12} T(xy) D_{f_1, g_1} + \frac{1}{2} (e_2 - e_1) \otimes x * y + \frac{1}{2} [R_y, R_x]$$

$$[f_1 \otimes x, g_1 \otimes y] \phi = \frac{1}{12} T(xy) (D_{f_1, g_1}) \phi + (\frac{1}{2} (e_2 - e_1) \otimes x * y) \phi + \frac{1}{2} (D_{y, x}) \phi$$

$$= \frac{1}{12} T(xy) R(1, 1) + \frac{1}{2} R(\frac{1}{2} (x * y)_{12, 1_{21}}) +$$

$$\frac{1}{2} [R(\frac{1}{2} y_{12, 1_{21}}), R(\frac{1}{2} x_{12, 1_{21}})]$$

$$= \frac{1}{12} T(xy) R(1, 1) + \frac{1}{4} R((x * y)_{12, 1_{21}}) + \frac{1}{8} [R(y_{12, 1_{21}}), R(x_{12, 1_{21}})]$$

$$= \frac{1}{12} T(xy) R(1, 1) + \frac{1}{4} R((xy - \frac{1}{3} T(xy) 1)_{12, 1_{21}}) + \frac{1}{4} R(y_{12, x_{21}})$$

$$- R((xy)_{12, 1_{21}})$$

(by 2.2(7) of this chapter)

$$= \frac{1}{12}T(xy)R(1,1) + \frac{1}{4}R((xy)_{12}, 1_{21}) - \frac{1}{12}T(xy)R(1_{12}, 1_{21})$$

$$+ \frac{1}{4}(R(y_{12}, x_{21}) - R((xy)_{12}, 1_{21}))$$

$$= -\frac{1}{12}T(xy)R(1,1) - \frac{1}{12}T(xy)R(1_{12}, 1_{21}) + \frac{1}{4}R(y_{12}, x_{21})$$

For  $z = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \in \bar{M}$ ,

$$z([f_1 \otimes x, g_1 \otimes y] \phi) = \frac{1}{12}T(xy) \begin{pmatrix} 3\alpha & a \\ -b & -3\beta \end{pmatrix} - \frac{1}{12}T(xy) \begin{pmatrix} -3\alpha & -2a \\ -b & 0 \end{pmatrix}$$

$$+ \frac{1}{4} \begin{pmatrix} -\alpha T(x,y) & -\{axy\} \\ \{byx\} - bT(x,y) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}\alpha T(xy) & \frac{1}{4}aT(xy) - \frac{1}{4}\{axy\} \\ -\frac{1}{4}bT(xy) + \frac{1}{4}\{byx\} & -\frac{1}{4}\beta T(xy) \end{pmatrix}$$

$$[(f_1 \otimes x) \phi, (g_1 \otimes y) \phi] = [R(\frac{1}{2}x_{21}, 1_{11}), R(\frac{1}{2}y_{12}, 1_{22})]$$

$$= \frac{1}{4}(R(x_{21}R(y_{12}, l_{22}), l_{11}) + R(x_{21}, l_{11}R(y_{12}, l_{22})))$$

$$= \frac{1}{4}(R((T(x,y)l)_{22}, l_{11}) + R(x_{21}, y_{12}))$$

$$= \frac{1}{4}T(xy)R(l_{22}, l_{11}) + \frac{1}{4}R(x_{21}, y_{12})$$

$$z[(f_1 \otimes x)\phi, (g_1 \otimes y)\phi] = \frac{1}{4}T(xy) \begin{pmatrix} \alpha & 0 \\ -b & -2\beta \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & aT(x,y) - \{axy\} \\ \{byx\} & \beta T(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}\alpha T(xy) & \frac{1}{4}aT(xy) - \frac{1}{4}\{axy\} \\ -\frac{1}{4}bT(xy) + \frac{1}{4}\{byx\} & -\frac{1}{4}\beta T(xy) \end{pmatrix}$$

Thus,  $[f_1 \otimes x, g_1 \otimes y]\phi = [(f_1 \otimes x)\phi, (g_1 \otimes y)\phi]$

$$[f_2 \otimes x, g_2 \otimes y] = \frac{1}{12}T(xy)D_{f_2, g_2} + f_2 * g_2 \otimes x * y + \frac{1}{2}T(f_2 g_2)[R_x, R_y]$$

$$= \frac{1}{12}T(xy)D_{f_2, g_2} + \frac{1}{2}(e_2 - e_1) \otimes x * y + \frac{1}{2}[R_y, R_x]$$

$$[f_2 \otimes x, g_2 \otimes y] \phi = \frac{1}{12} T(xy) R(1_{12}, 1_{21}) + \frac{1}{2} R\left(\frac{1}{2}(x*y)_{12}, 1_{21}\right)$$

$$+ \frac{1}{8} [R(y_{12}, 1_{21}), R(x_{12}, 1_{21})]$$

$$= \frac{1}{12} T(xy) R(1_{12}, 1_{21}) + \frac{1}{4} R((xy)_{12}, 1_{21}) - \frac{1}{2} T(xy) R(1_{12}, 1_{21})$$

$$+ \frac{1}{4} (R(y_{12}, x_{21}) - R((xy)_{12}, 1_{21}))$$

$$= \frac{1}{4} R(y_{12}, x_{21})$$

$$[(f_2 \otimes x) \phi, (g_2 \otimes y) \phi] = \left[ \frac{1}{2} x_{21}, -\frac{1}{2} y_{12} \right] = -R\left(-\frac{1}{2} y_{12}, \frac{1}{2} x_{21}\right)$$

$$= \frac{1}{4} R(y_{12}, x_{21})$$

$$[f_3 \otimes x, g_3 \otimes y] = \frac{1}{12} T(xy) D_{f_3, g_3} + \frac{1}{2} (e_2 - e_1) \otimes x*y$$

$$+ \frac{1}{2} [R_y, R_x]$$

$$[f_3 \otimes x, g_3 \otimes y] \phi = \frac{1}{12} T(xy) R(1_{21}, 1_{12}) + \frac{1}{2} R\left(\frac{1}{2}(x*y)_{12}, 1_{21}\right)$$

$$+ \frac{1}{8} [R(y_{12}, 1_{21}), R(x_{12}, 1_{21})]$$

$$= \frac{1}{12} T(xy) R(1_{21}, 1_{12}) + \frac{1}{4} R((xy)_{12}, 1_{21}) - \frac{1}{12} T(xy) R(1_{12}, 1_{21})$$

$$+ \frac{1}{4} (R(y_{12}, x_{21}) - R((xy)_{12}, 1_{21}))$$

$$= \frac{1}{12} T(xy) (R(1_{21}, 1_{12}) - R(1_{12}, 1_{21})) + \frac{1}{4} R(y_{12}, x_{21})$$

$$z[(f_3 \otimes x, g_3 \otimes y)\phi] = \frac{1}{12} T(xy) \left( \begin{pmatrix} 0 & a \\ 2b & 3\beta \end{pmatrix} - \begin{pmatrix} -3a & -2a \\ -b & 0 \end{pmatrix} \right)$$

$$+ \frac{1}{4} \begin{pmatrix} -aT(x,y) & -\{axy\} \\ \{byx\} - bT(x,y) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{4} aT(xy) - \frac{1}{4} \{axy\} \\ \frac{1}{4} \{byx\} & \frac{1}{4} \beta T(xy) \end{pmatrix}$$

$$z[(f_3 \otimes x)\phi, (g_3 \otimes y)\phi] = z\left[\frac{1}{2}x_{21}, \frac{1}{2}y_{12}\right] = \frac{1}{4} zR(x_{21}, y_{12})$$

$$= \frac{1}{4} \begin{pmatrix} 0 & aT(xy) - \{axy\} \\ \{byx\} & \beta T(xy) \end{pmatrix}$$

$$[(e_2 - e_1) \otimes x, f_1 \otimes y] = \frac{1}{12} T(xy) D_{e_2 - e_1, f_1} + (e_2 - e_1) * f_1 \otimes x * y$$

$$+ \frac{1}{2} T((e_2 - e_1) f_1) [R_x, R_y]$$

$$= -\frac{1}{6} T(xy) D_{e_1, f_1} - f_1 \otimes x * y \quad (\because D_{e_2 - e_1, f_1} = -2D_{e_1, f_1})$$

$$[(e_2 - e_1) \otimes x, f_1 \otimes y] \phi = -\frac{1}{6} T(xy) R(1_{11}, 1_{21}) - R\left(\frac{1}{2}(x * y)_{21}, 1_{11}\right)$$

$$= -\frac{1}{6} T(xy) R(1_{11}, 1_{21}) - \frac{1}{2} R((xy)_{21}, 1_{11}) + \frac{1}{6} T(xy) R(1_{21}, 1_{11})$$

$$= -\frac{1}{2} R((xy)_{21}, 1_{11}) \quad (\because R(1_{11}, 1_{21}) = R(1_{21}, 1_{11}))$$

$$z([(e_2 - e_1) \otimes x, f_1 \otimes y] \phi) = \frac{1}{2} \begin{pmatrix} T(a, xy) & b * xy \\ 8xy & 0 \end{pmatrix}$$

$$[(e_2 - e_1) \otimes x] \phi, [(f_1 \otimes y) \phi] = [R\left(\frac{1}{2} x_{12}, 1_{21}\right), R\left(\frac{1}{2} y_{21}, 1_{11}\right)]$$

$$= \frac{1}{4} (R(x_{12} R(y_{21}, 1_{11}), 1_{21}) + R(x_{12}, 1_{21} R(y_{21}, 1_{11})))$$

$$= \frac{1}{4}(R((-T(xy)1)_{11}, l_{21}) + R(x_{12}, y_{12}))$$

$$= \frac{1}{4}(-T(xy)R(l_{11}, l_{21}) + R(x_{12}, y_{12}))$$

$$z[((e_2 - e_1) \otimes x) \phi, (f_1 \otimes y) \phi] = \frac{1}{4}(-T(xy) \begin{pmatrix} -T(a) & b - T(b) \\ -\beta & 0 \end{pmatrix})$$

$$+ \begin{pmatrix} T(a, xxy) \\ \beta xxy \end{pmatrix} \begin{matrix} (xxy) \times b \\ 0 \end{matrix}$$

$$= \frac{1}{4}T(xy) \begin{pmatrix} T(a) & T(b) - b \\ \beta & 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} T(a, 2xy - T(xy)) & b \chi(2xy) \\ \beta(2xy - T(xy)) & -T(xy) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} T(a, xy) & bxy \\ \beta xy & 0 \end{pmatrix}$$

$$[(e_2 - e_1) \otimes x, g_1 \otimes y] = \frac{1}{12}T(xy)D_{e_2 - e_1, g_1} + (e_2 - e_1) * g_1 \otimes x * y$$

$$+ \frac{1}{2}T((e_2 - e_1)g_1)[R_x, R_y]$$

$$= \frac{1}{6}T(xy)D_{e_2, g_1} + g_1 \otimes x * y \quad (\because D_{e_2 - e_1, g_1} = 2D_{e_2, g_1})$$

$$\begin{aligned}
 [(e_2 - e_1) \otimes x, g_1 \otimes y] \phi &= \frac{1}{6} T(xy) R(1_{22}, 1_{12}) + R(\frac{1}{2}(x \star y)_{12}, 1_{22}) \\
 &= \frac{1}{6} T(xy) R(1_{22}, 1_{12}) + \frac{1}{2} R((xy)_{12}, 1_{22}) - \frac{1}{6} T(xy) R(1_{12}, 1_{22}) \\
 &= \frac{1}{2} R((xy)_{12}, 1_{22}) \quad (\because R(1_{12}, 1_{22}) = R(1_{22}, 1_{12}))
 \end{aligned}$$

$$z([(e_2 - e_1) \otimes x, g_1 \otimes y] \phi) = \frac{1}{2} \begin{pmatrix} 0 & \alpha xy \\ \alpha xy & T(b, xy) \end{pmatrix}$$

$$[((e_2 - e_1) \otimes x) \phi, (g_1 \otimes y) \phi] = [R(\frac{1}{2}x_{12}, 1_{21}), R(\frac{1}{2}y_{12}, 1_{22})]$$

$$= \frac{1}{4} (R(x_{12} R(y_{12}, 1_{22}), 1_{21}) + R(x_{12}, 1_{21} R(y_{12}, 1_{22})))$$

$$= \frac{1}{4} (R((x \star y)_{21}, 1_{21}) + R(x_{12}, (T(y) 1)_{22}))$$

$$= \frac{1}{4} (R(2xy - T(xy) 1)_{21}, 1_{21})$$

$$= \frac{1}{4} (2R((xy)_{21}, 1_{21}) - T(xy) R(1_{21}, 1_{21}))$$

$$z([(e_2 - e_1) \otimes x] \phi, (g_1 \otimes y) \phi] = \frac{1}{2} \begin{pmatrix} 0 & -\alpha(T(xy) - xy) \\ -\alpha(T(xy) - xy) & -T(b, T(xy) - xy) \end{pmatrix}$$

$$-\frac{1}{4}T(xy) \begin{pmatrix} 0 & -2a \\ -2(T(a)-a) & -2T(b) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \alpha xy \\ \alpha xy & T(b, xy) \end{pmatrix}$$

$$[(e_2 - e_1) \otimes x, f_2 \otimes y] = \frac{1}{12} T(xy) D_{e_2 - e_1, f_2} + (e_2 - e_1) * f_2 \otimes x * y$$

$$+ \frac{1}{2} T((e_2 - e_1) f_2) [R_x R_y]$$

$$= -\frac{1}{6} T(xy) D_{e_1, f_2} - f_2 \otimes x * y$$

$$[(e_2 - e_1) \otimes x, f_2 \otimes y] \phi = -\frac{1}{6} T(xy) \tilde{l}_{21} - \frac{1}{2} (x * y)_{21}$$

$$= (-\frac{1}{6} T(xy) - \frac{1}{2} (x * y)) \tilde{l}_{21}$$

$$= (-\frac{1}{6} T(xy) - \frac{1}{2} xy + \frac{1}{6} T(xy)) \tilde{l}_{21}$$

$$= -\frac{1}{2} xy \tilde{l}_{21}$$

$$[(e_2 - e_1) \otimes x] \phi, (f_2 \otimes y) \phi = [R(\frac{1}{2} x_{12}, l_{21}), \frac{1}{2} \tilde{y}_{21}]$$

$$= \frac{1}{4} (-y_{21} R(x_{12}, l_{21}))$$

$$= -\frac{1}{4} \begin{pmatrix} 0 & 0 \\ \{yx1\} & 0 \end{pmatrix}$$

$$= -\frac{1}{4}\{yx1\}\bar{l}_{21}$$

$$= -\frac{1}{2}xy\bar{l}_{21}$$

$$[(e_2-e_1) \otimes x, f_3 \otimes y] = \frac{1}{12}T(xy)D_{e_2-e_1, f_3} + (e_2-e_1 * f_3 \otimes x * y)$$

$$+ \frac{1}{2}T((e_2-e_1)f_3)[R_x, R_y]$$

$$= -\frac{1}{6}T(xy)D_{e_1, f_3} - f_3 \otimes x * y$$

$$[(e_2-e_1) \otimes x, f_3 \otimes y] \phi = -\frac{1}{6}T(xy)l_{21} - \frac{1}{2}(x * y)_{21}$$

$$= (-\frac{1}{6}T(xy) - \frac{1}{2}(x * y))l_{21}$$

$$= -\frac{1}{2}xy l_{21}$$

$$[((e_2-e_1) \otimes x) \phi, (f_3 \otimes y) \phi] = [R(\frac{1}{2}x_{12}, l_{21}), \frac{1}{2}y_{21}]$$

$$= \frac{1}{4}(-y_{21}R(x_{12}, l_{21}))$$

$$= -\frac{1}{4}(\{yx1\})_{21} = -\frac{1}{4}\{yx1\}1_{21}$$

$$= -\frac{1}{2}xy1_{21}$$

$$[(e_2 - e_1) \otimes x, g_2 \otimes y] = \frac{1}{12}T(xy)D_{e_2 - e_1, g_2} + (e_2 - e_1)^* g_2 \otimes x^* y$$

$$+ \frac{1}{2}T((e_2 - e_1)g_2) [R_x, R_y]$$

$$= \frac{1}{6}T(xy)D_{e_2, g_2} + g_2 \otimes x^* y$$

$$[(e_2 - e_1) \otimes x, g_2 \otimes y] \phi = \frac{1}{6}T(xy)(-1_{12}) - \frac{1}{2}(x^* y)_{12}$$

$$= (-\frac{1}{6}T(xy) - \frac{1}{2}(x^* y))1_{12}$$

$$= -\frac{1}{2}xy1_{12}$$

$$[((e_2 - e_1) \otimes x) \phi, (g_2 \otimes y) \phi] = [R(\frac{1}{2}(x_{12}, 1_{21}), -\frac{1}{2}y_{12})]$$

$$= \frac{1}{4}y_{12}R(x_{12}, 1_{21}) = \frac{1}{4}(-\{y1x\})_{12}$$

$$= -\frac{1}{2}xy1_{12}$$

$$[(e_2 - e_1) \otimes x, g_3 \otimes y] = \frac{1}{12} T(xy) D_{e_2 - e_1, g_3} + (e_2 - e_1) * g_3 \otimes x * y$$

$$+ \frac{1}{2} T((e_2 - e_1) g_3) [R_x, R_y]$$

$$= \frac{1}{6} T(xy) D_{e_2, g_3} + g_3 \otimes x * y$$

$$[(e_2 - e_1) \otimes x, g_3 \otimes y] \phi = \frac{1}{6} T(xy) \tilde{l}_{12} + \frac{1}{2} (x * y)_{12}$$

$$= (\frac{1}{6} T(xy) + \frac{1}{2} (x * y)) \tilde{l}_{12} = \frac{1}{2} xy \tilde{l}_{12}$$

$$[((e_2 - e_1) \otimes x) \phi, (g_3 \otimes y) \phi] = [R(\frac{1}{2} x_{12}, l_{21}), \frac{1}{2} \tilde{y}_{12}]$$

$$= \frac{1}{4} (-y_{12} R(x_{12}, l_{21})) \tilde{\phantom{y}}$$

$$= \frac{1}{4} (y_{12} x) \tilde{l}_{12} = \frac{1}{2} xy \tilde{l}_{12}$$

Thus,  $\phi$  preserves Lie products when it is restricted in  $C_0 \otimes J_0$ . So, we have proved that the restrictions of  $\phi$  to  $\text{Der } C$  and  $\text{Der } J$  are isomorphisms and that the restriction of  $\phi$  to  $C_0 \otimes J_0$  is bijective and preserves Lie products.

Thus,  $\phi$  is an isomorphism.

#### 2.4 Conclusion:

Both Tits and Faulkner have given constructions of exceptional Lie algebras from an arbitrary Jordan algebra of degree 3 (in the case of Tits' construction an arbitrary octonion algebra enters into the construction). We have provided an explicit isomorphism between  $L(C, J)$  of Tits ( $C$  the split octonions) and  $S(J, R(J))$  of Faulkner, which depends only on  $J$ .

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