

TIME-OPTIMAL SYNTHESIS OF A WIND-TUNNEL DRIVE

by

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ABSTRACT

This thesis presents the time-optimal feed back controller synthesis of the wind-tunnel drive for different operating speeds (i. e. 30, 40, 50 and 60 r/s) based on Pontryagin's Maximum Principle. Normally, the deviations in the operating speed (due to disturbances) lie in a bounded region and the optimal feed back controller is required to correct these deviations in the shortest possible time. The torque-speed characteristic of the air compressor (load) is approximated by an appropriate straight line as well as a quadratic function around each of the operating speeds ω_o^i , $i = 1, \dots, 4$. The optimal controllers for the resulting linear and nonlinear systems are derived by translating them to the corresponding operating speeds. A slight improvement in optimal times is observed in the nonlinear systems over the corresponding linear system, to force the system from a given initial state in R_o^i , to the respective operating speed ω_o^i .

An approximate method is suggested for realization of the switching curves.

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CHAPTER 1

PRELIMINARY DESIGN CONSIDERATION OF TIME-
OPTIMAL SPEED CONTROL OF WIND-TUNNEL
DRIVE AND FUNDAMENTAL RESULTS OF PONTRYAGIN
et al IN TIME-OPTIMAL CONTROL THEORY

1.1 Practical Considerations Arising in the Time-optimal Control of Wind-tunnel Drive System

Introduction

The wind-tunnel is a device for testing aircraft models and their components in a controlled air stream under laboratory conditions so that their behaviour can be analyzed. Wind-tunnels play an important role in aerodynamic research, design and development as they offer a rapid, economical and accurate means of acquiring data on aircraft models.

One of the fundamental requirements of the wind-tunnel work is that the velocity of air in the working section should be kept constant during the period when balance readings are being recorded. Apart from drag and lift, the various important forces that are measured [13] are viscous force (proportional to $\sigma v \ell$), inertia force (proportional to $\rho \ell^2 v^2$) and elastic force (proportional to $\rho a^2 \ell^2$), where σ = coefficient of viscosity, ρ = air density, v = velocity of air, ℓ = characteristic length of the aircraft model and a = velocity of sound in the working section. The other factors requiring consideration in the wind-tunnel work are 'Mach number' and 'Reynold's number'. Mach number is defined as the ratio of inertia force to elastic force (v/a). The Reynold's number is related with the scale effect and is expressed as the ratio of inertia force to viscous force for any given flight condition ($\rho v \ell / \sigma$). It is clear from the above expressions that any variation in the velocity of air in the working section will affect the balance readings and hence the data does not represent the true conditions. Since the design and development of aircraft is mainly based on the wind-

tunnel data, any error in the latter will introduce inaccuracies in the design. Hence it is required to keep the velocity of air in the working section constant. Further, the wind-tunnel work can be broadly divided into two parts (i) Supersonic work and (ii) Subsonic work. In the supersonic work the velocity of air is greater than the velocity of sound, while in subsonic work it is less than the velocity of sound. In subsonic work the velocity of air in the working section is directly related to the compressor speed, and thus an accurate control of compressor speed is required.

A series of tests are to be conducted on the model at various Mach numbers (for example at 30, 40, 50 and 60 rad/sec) to study the performance of the model. In order to record data accurately during each test, any deviation in compressor speed due to sudden changes of aerodynamic loading must be brought back to the set value in the shortest possible time. This leads to the time-optimal speed control of the wind-tunnel drive. Drake et al [3] discussed the design and operation of the speed control scheme for a 8 ft x 8 ft wind-tunnel. The prime drawback of the above scheme is that its response to the input quantities (excitation voltage) is not time-optimal. This is due to the fact that the classical control theory is inadequate to treat minimal time problems.

The purpose of the present work is to obtain a time-optimal feed back controller synthesis [6], [7] for the wind-tunnel drive using Pontryagin's Maximum Principle [12]. It is interesting to note that there are many efficient computational methods [1], [2], [4], [5], [10] for calculating the time-optimal control of linear systems. But these methods result in open loop (time-optimal) control i. e. the control does not depend upon the state of the systems.

Their use is only limited to systems without disturbances and errors and with fixed initial and final states. But, mostly in engineering practice, as encountered above, the systems are subjected to disturbances and hence are required to use optimal feed back controllers. In the next section, we describe briefly the main design consideration of the feed back controller.

Basic Design Considerations

In actual practice, the system is required to operate at three or four different speeds (30, 40, 50 and 60 rad/sec) and the deviation of speed and acceleration due to disturbances usually lie in a certain bounded region of the state space, the bound determined by the intensity of disturbances. Based on practical considerations a deviation of ± 4 r/s and ± 4 r/s² for speed and acceleration respectively, allowed for all the operating speeds under consideration, is known to be fairly realistic.

Though the deviation in the states of the system lie in the bounded region (called the field of disturbances) $R_o^i = \{ (x^i, y^i) : |x^i| \leq 4 \text{ r/s and } |y^i| \leq 4 \text{ r/s}^2 \} \subset R^2$ for all 'i', the optimal trajectories that take the system from those states in R_o^i , in particular $(x^i, y^i) \in \partial R_o^i$ (∂R_o^i is the boundary of R_o^i), to the operating value, may leave R_o^i . This is due to the fact that the optimal control tends to accelerate the process and throws the system out of the region R_o^i .

Though the 'T- ω ' characteristic (see fig. 2. 2. 1) is nonlinear, for each operating point with certain region allowed for deviation, the 'T- ω ' characteristic can be approximated fairly well by an appropriate straight line and the system, can be treated

as linear around the given operating point.

If the $G(\omega)$ ('T- ω ' characteristic) is approximated for the specified deviation of ± 4 r/s around each operating speed, then the resulting differential equations S_i (2.3.3, ... 2.3.6) and \tilde{S}_i (3.1.3, ..., 3.1.6) may not represent the system, for that time, when the time-optimal trajectories leave the region R_o^i . So for each operating speed ω_o^i , the $G(\omega)$ characteristic has to be approximated over a larger (compared to the specified value) interval of deviation and a value of ± 6 r/s satisfies the requirement.

Based on ± 6 r/s deviation, for each of the operating speeds ω_o^i , $i = 1, \dots, 4$ (i. e. 30, 40, 50 and 60 r/s), the $G(\omega)$ characteristic is approximated by four best straight lines and the optimal controller for the resulting linear systems derived. This procedure leads to one feed back controller for each operating speed and consequently requires as many controllers as there are operating speeds. The advantage, is that the above approximation, though costly, is superior to one single linear approximation of the entire 'T- ω ' characteristic.

In this thesis, a ^{Ward} Ward-Leonard speed control scheme is considered for the wind-tunnel drive. The mathematical model of the system is derived. The differential equation describing the system is given by

$$S : \quad \ddot{\omega} + f(\omega, \dot{\omega}) = a_1 v_1 ,$$

where ' v_1 ' is the control voltage, ' ω ' is the speed of the drive and

$$f(\omega, \dot{\omega}) = a_5 \omega + a_4 G(\omega) + a_3 \dot{\omega} + a_2 \frac{d}{dt} G(\omega) .$$

The coefficients a_1, a_2, \dots, a_5 are parameters of the system to be defined in the next chapter and $G(\omega)$ is the air compressor (load) torque given as a function of speed i. e. Torque-speed ('T- ω ') characteristic (see fig. 2.2.1).

The synthesis of the time-optimal feed back controller of systems based on the linear piecewise approximation of the 'T- ω ' characteristic is presented in chapter II.

In chapter III, the 'T- ω ' characteristic is approximated by a quadratic function around each of the operating speeds as before and the corresponding feed back controllers for the nonlinear systems are derived.

We present a brief review of Pontryagin's Maximum Principle in the next section.

1.2 Brief Review of Pontryagin's Maximum Principle

In this section we will state without proof the Pontryagin's Maximum Principle and the basic theorems concerning existence and uniqueness of time-optimal control etc.. These results are used in chapter II and III.

Basic Assumptions : Assume that the controlled object's law of motion can be described by a system of differential equations:

$$\begin{aligned} \frac{d}{dt} x^i &= f^i(x_1, x_2, \dots, x_n, u_1, \dots, u_r) \\ &= f^i(x, u), \quad i = 1, 2, \dots, n \end{aligned} \quad (1.2.1)$$

or in the vector form

$$\frac{d}{dt} \mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \quad (1.2.2)$$

where $\mathbf{f}(\mathbf{x}, \mathbf{u}) = \{f^i(\mathbf{x}, \mathbf{u})\}$, $i = 1, 2, \dots, n$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is the state vector and $\mathbf{u} = (u_1, u_2, \dots, u_r)$ is the control vector.

The functions f^i are defined for $\mathbf{x} \in X$ and for $\mathbf{u} \in U$ where U is the control region an arbitrary subset of E^r . It is also assumed that $\frac{\partial}{\partial x_j} f^i(x_1, x_2, \dots, x_n, \mathbf{u})$; $i, j = 1, 2, \dots, n$ is continuous on the direct product $X \times U$.

In technical problems the control region U is not arbitrary but a closed and bounded set in E^r .

Formulation of the Optimal Problem : Let M^0 denote the set of all bounded measurable functions defined on $I = [t_0, t_1]$ to E^r such that $\mathbf{u}(t) \in U$ a. e. This set will be called the class of admissible controls.

The functional to be minimized is $J = \int_{t_0}^{t_1} f^0(\mathbf{x}, \mathbf{u}) dt$, where it is assumed that $f^0(\mathbf{x}, \mathbf{u})$ satisfies the same conditions as the functions $f^i(\mathbf{x}, \mathbf{u})$, $i = 1, 2, \dots, n$.

Define a new phase co-ordinate $x^0 = f^0(\mathbf{x}, \mathbf{u})$.

By adjoining this new co-ordinate, we obtain the $(n+1)$ dimensional phase space \bar{X} . Then equation (1.2.2) takes the form

$$\frac{d}{dt} \bar{\mathbf{x}} = \bar{\mathbf{f}}(\mathbf{x}, \mathbf{u}), \quad \text{where } \bar{\mathbf{x}} = (x^0, x_1, x_2, \dots, x_n) \quad (1.2.3)$$

and $\bar{\mathbf{f}}(\mathbf{x}, \mathbf{u}) = (f^0(\mathbf{x}, \mathbf{u}), f^1(\mathbf{x}, \mathbf{u}), \dots, f^n(\mathbf{x}, \mathbf{u}))$.

The problem may then be posed as follows :

Statement of the Fundamental Problem : In the $(n+1)$ dimensional phase space \bar{X} the point $\bar{x}_0 = (0, x_0)$ and the line $\bar{\Pi}$ are given. The line $\bar{\Pi}$ is assumed to be parallel to the x^0 axis, and to pass through the point $(0, x_1)$. Among the admissible controls $u = u(t)$, having the property that the corresponding solution $\bar{x}(t)$ of equation (1.2.3) with initial conditions $\bar{x}(t_0) = \bar{x}_0$ intersects $\bar{\Pi}$, find one whose point of intersection with $\bar{\Pi}$ has the smallest co-ordinate x^0 .

Fundamental Theorems of Pontryagin et al

Before presenting the theorem that solves the fundamental problem, we consider the following : The Hamiltonian function \bar{H} is written as

$$\bar{H} = \langle \bar{\psi}, \bar{f}(x, u) \rangle = \sum_{v=0}^n \psi_v f^v(x, u)$$

where ψ is auxiliary variables $= (\psi_0, \psi_1, \dots, \psi_n)$, satisfying the differential equation

$$\frac{d}{dt} \psi_i = - \sum_{v=0}^n \frac{\partial}{\partial x_i} f^v(x, u) \psi_v \quad (1.2.4)$$

It can be seen that the equation (1.2.3) and (1.2.4) may be rewritten with the help of the functions \bar{H} in the form of Hamiltonian system :

$$\frac{d}{dt} x_0^o = \frac{\partial}{\partial \psi_0} \bar{H}, \quad \frac{d}{dt} x_i = \frac{\partial}{\partial \psi_i} \bar{H}, \quad i = 1, 2, \dots, n \quad (1.2.5)$$

$$\frac{d}{dt} \psi_0 = - \frac{\partial}{\partial x_0} \bar{H}, \quad \frac{d}{dt} \psi_i = - \frac{\partial}{\partial x_i} \bar{H}, \quad i = 1, 2, \dots, n \quad (1.2.6)$$

Theorem 1.2.1 (Maximum Principle, Pontryagin et al [12] ,
Th 8, pp 81)

Let $u(t)$, $t_0 \leq t \leq t_1$, be an admissible control such that the corresponding trajectory $\bar{x}(t)$ (see 1.2.5) which begins at the point \bar{x}_0 at the time t_0 is defined on the interval $t_0 \leq t \leq t_1$, and passes, at the time t_1 through a point on the line Π . In order that $u(t)$ and $\bar{x}(t)$ be optimal it is necessary that there exist a nonzero absolutely continuous vector functions $\bar{\psi}(t) = (\psi_0(t), \psi_1(t), \dots, \psi_n(t))$ corresponding to the functions $u(t)$ and $\bar{x}(t)$ (see 1.2.6) such that :

(1) the function $\bar{H}(\bar{\psi}(t), x(t), u)$ of the variable $u \in U$ attains its maximum at the point $u = u(t)$ almost every where in the interval $t_0 \leq t \leq t_1$,

$$\bar{H}(\bar{\psi}(t), x(t), u(t)) (=) \Xi(\bar{\psi}(t), x(t)); \quad (1.2.7)$$

(2) at the terminal time t_1 the relations

$$\psi_0(t_1) \leq 0, \quad \Xi(\bar{\psi}(t_1), x(t_1)) = 0 \quad (1.2.8)$$

are satisfied. Furthermore, it turns out that if $\bar{\psi}(t)$, $\bar{x}(t)$, and $u(t)$ satisfy systems (1.2.5) and (1.2.6) and condition (1), the time functions $\psi_0(t)$ and $\Xi(\bar{\psi}(t), x(t))$ are constant. Thus (1.2.8) may be verified at any time t , $t_0 \leq t \leq t_1$ and not just at t_1 .

For the time optimal case, $f^0(x, u) \equiv 1$, and we consider the Hamiltonian function

$$H = \psi_0 + \sum_{v=1}^n \psi_v f^v(x, u), \text{ where } \psi \text{ is a 'n' dimensional}$$

vector i.e. $\psi = (\psi_1, \psi_2, \dots, \psi_n)$. Here we consider 'n'

dimensional phase space only. Since ψ_0 is constant, the Hamiltonian system becomes

$$\frac{d}{dt} x_i = \frac{\partial}{\partial \psi_i} H \quad i = 1, 2, \dots, n \quad (1.2.9)$$

$$\frac{d}{dt} \psi_i = -\frac{\partial}{\partial x_i} H \quad i = 1, 2, \dots, n \quad (1.2.10)$$

The theorem for time-optimality is given below :

Theorem 1.2.2 (Pontryagin et al [12], Th 2, pp 20-21)

Let $u(t)$, $t_0 \leq t \leq t_1$, be an admissible control which transfers the phase point from x_0 to x_1 , and let $x(t)$ be the corresponding trajectory (see 1.2.9) so that $x(t_0) = x_0$, $x(t_1) = x_1$. In order that $u(t)$ and $x(t)$ be time-optimal it is necessary that there exist a nonzero, continuous vector function $\psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_n(t))$ corresponding to $u(t)$ and $x(t)$ (see 1.2.10) such that :

(1) for all t , $t_0 \leq t \leq t_1$, the function $H(\psi(t), x(t), u(t))$ of the variable $u \in U$ attains its maximum at the point $u = u(t)$:

$$H(\psi(t), x(t), u(t)) = M(\psi(t), x(t)) ; \quad (1.2.11)$$

(2) at the terminal time t_1 the relation

$$M(\psi(t_1), x(t_1)) \geq 0 \quad (1.2.12)$$

is satisfied. Furthermore it turns out that if $\psi(t)$, $x(t)$, and $u(t)$ satisfy system (1.2.9), (1.2.10) and condition (1), the time function $M(\psi(t), x(t))$ is constant. Thus, (1.2.12) may be verified at any time t , $t_0 \leq t \leq t_1$, and not just at t_1 .

The above theorems are, in general, applicable to both linear and nonlinear processes. But the existence, uniqueness of time-optimal control and the theorem relating the number of switchings of time-optimal control have been established for linear process [12], and we present those results here. In order to state the above results, we need the following preparation.

Consider the objects whose law of motion may be described by a linear vector differential equation as shown below :

$$\dot{x} = Ax + Bu \quad (1.2.13)$$

where A is a $(n \times n)$ constant matrix, B is a $(n \times r)$ constant matrix, x is a $(n \times 1)$ state vector and u is a $(r \times 1)$ control vector.

It is assumed that the control region U is a convex, closed, and bounded polyhedron in the r -dimensional vector space E^r with co-ordinates u_1, u_2, \dots, u_r .

A general position condition will be assumed in proving all the theorems (1.2.3), (1.2.4), (1.2.5), which is imposed on the coefficients of equation (1.2.13) and on the form of the polyhedron U .

For every vector z , which has the direction of one of the edges of U , the vector Bz has the property that it does not belong to any proper subspace of X which is invariant under the operator A ; i. e. the vectors $Bz, ABz, \dots, A^{n-1}Bz$ (1.2.14) are linearly independent in X .

Theorem 1.2.3 (Pontryagin et al [12], Th 10, pp 120)

Suppose that the control region U is the parallelepiped ($\alpha_k \leq u_k \leq \beta_k$, $k = 1, 2, \dots, r$), and that all the eigen values of the matrix 'A' of equation (1.2.13) are real. Then, for each nontrivial solution $\psi(t)$ of equation (1.2.15), relation $\langle \psi(t), Bu(t) \rangle$ uniquely determines the control function $u(t) = (u_1(t), u_2(t), u_3(t), \dots, u_r(t))$. In addition, it turns out that each function $u_k(t)$, $k = 1, \dots, r$, is piecewise constant, takes on only the values α_k and β_k , and does not have more than 'n-1' switchings (i. e. not more than 'n' intervals on which it is constant), where 'n' is the order of system (1.2.13).

$$\dot{\psi}(t) = -A^* \psi(t), \quad (1.2.15)$$

where A^* is the conjugate transpose of the matrix A .

Theorem 1.2.4 (Existence Theorem, Pontryagin et al [12], Th 13, pp 127)

If there exists, for a process described by equation (1.2.13), at least one control which transfers the phase point from x_0 to x_1 , there also exists an optimal control which transfers the phase point from x_0 to x_1 .

Finally we present the uniqueness theorem.

Theorem 1.2.5 (Pontryagin et al [12], Th 11, pp 124)

Let $u^1(t)$ and $u^2(t)$ be two optimal controls (given on the intervals $t_0 \leq t \leq t_1$ and $t_0 \leq t \leq t_2$, respectively) which

transfer the point x_0 to the same point x_1 . Then, these controls coincide i. e. $t_1 = t_2$ and $u^1(t) \equiv u^2(t)$ for $t_0 \leq t \leq t_1$.

CHAPTER II

DESIGN OF TIME-OPTIMAL FEED BACK CONTROLLER BASED ON LINEAR PIECEWISE APPROXIMATION

Introduction

In this chapter the time-optimal feed back controller synthesis of the wind-tunnel drive for different operating speeds (i. e. 30, 40, 50 and 60 r/s) is obtained, by approximating the 'T- ω ' characteristic of air compressor by an appropriate straight line around each of the above operating speeds. The Ward-Leonard speed control scheme is briefly described in section 2.1 and the mathematical model of the same is derived in section 2.2. The $G(\omega)$ ('T- ω ' characteristic) is approximated by four best straight lines using least square approximation and the problem is formulated and then precisely stated in section 2.3 . A detailed description of the time-optimal feed back controller synthesis for an operating speed of 30 r/s, using Pontryagin's Maximum Principle is presented in section 2.4 . The results for other operating speeds are given in the same section. The operation of the time-optimal feed back controller is described in section 2.5 .

2.1 Description of the Wind-tunnel Drive

The Ward-Leonard speed control system is shown in fig. 2.1.1 . The d. c. generators G_1 , G_2 are connected in series with the d. c. motors M_1 , M_2 , and are driven at constant speed by a synchronous motor (not shown in fig. 2.1.1). The d. c. motors are coupled to the air compressor (load) via a gear box. The excitation voltage $v_1(t)$ is the input of the system and the speed ' ω ' of the load is the output of the system.

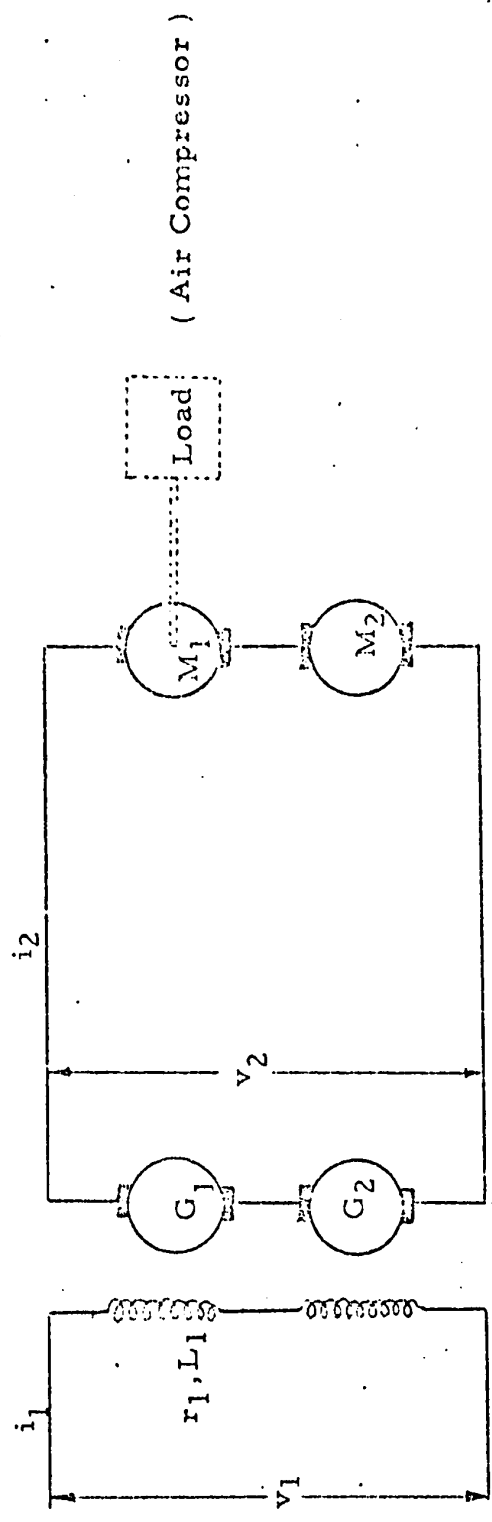


Fig. 2.1.1.1 Ward - Leonard Speed Control System .

2.2 Derivation of the Mathematical Model of the System

The following assumptions are made in deriving the mathematical model of the system :

(a) the d. c. generators G_1, G_2 are driven at constant speed.

(b) the effect of armature reaction, armature inductance and brush contact resistances of both motors and generators are neglected.

(c) though the field inductance of the generator is a function of field current, we consider it to be constant and

(d) the motors M_1, M_2 are supplied with constant excitation.

Then the dynamical equations that describe the wind-tunnel drive system are as follows :

$$(i) \quad v_1 = 2 i_1 r_1 + 2 L_1 \frac{d}{dt} i_1$$

$$(ii) \quad v_2 = k_1 i_1$$

$$(iii) \quad v_2 = i_2 r_2 + k_2 \omega$$

$$(iv) \quad k_3 i_2 = G(\omega) + J \frac{d\omega}{dt}$$

The variables appearing in the above equations have the following physical meaning : v_1 is the control voltage (appearing across the two fields of the generators, (volts)), i_1 is the field current of the generators (amps), v_2 is the generator output voltage (volts), ω is the speed of the load (radians per second);

r_1 , L_1 are the field circuit resistance and inductance of the generator respectively, k_1 is the generator gain (volts per rad/sec), k_3 is the Torque constant of the motor (newton-meter/ampere), $G(\omega)$ is the air compressor (load) torque as given by 'Torque-speed' characteristic ('T- ω ') (see fig. 2.2.1), J is the moment of inertia of load and motor as referred to the d. c. motor end.

Since we are interested in the time-optimal control of the speed ' ω ' by appropriate choice of excitation voltage v_1 , it is convenient to choose $(\omega, \dot{\omega})$ as the state variables of the system. The dynamical equations (i) - (iv) may then be reduced to the differential equation given below :

$$S : \quad \frac{d^2 \omega}{dt^2} + f(\omega, \dot{\omega}) = a_1 v_1 \quad (2.2.1)$$

$$\text{where } f(\omega, \dot{\omega}) = a_5 \omega + a_4 G(\omega) + a_3 \frac{d\omega}{dt} + a_2 \frac{dG(\omega)}{dt} \quad (2.2.2)$$

$$\text{where } a_1 = \frac{k_3 k_1}{2L_1 r_2 J}, \quad a_2 = \frac{1}{J}$$

$$a_3 = \left(\frac{k_2 k_3}{r_2 J} + \frac{r_1}{L_1} \right), \quad a_4 = \frac{r_1}{L_1 J} \quad \text{and} \quad a_5 = \frac{r_1 k_2 k_3}{L_1 r_2 J}$$

It can be seen that $f(\omega, \dot{\omega})$ depends upon $G(\omega)$ and its derivative $\frac{dG(\omega)}{dt}$. Though $G(\omega)$ is continuous, its derivative at $\omega = 15.70$ r/s (150 rpm) is discontinuous (see fig. 2.2.1). Hence $f(\omega, \dot{\omega})$ is discontinuous in ω .

Since $f(\omega, \dot{\omega})$ is discontinuous, the Pontryagin's Maximum Principle cannot be readily applied to the above system. In order to apply the Pontryagin's Maximum Principle the R. H. S. of the differential equation should be continuous and continuously

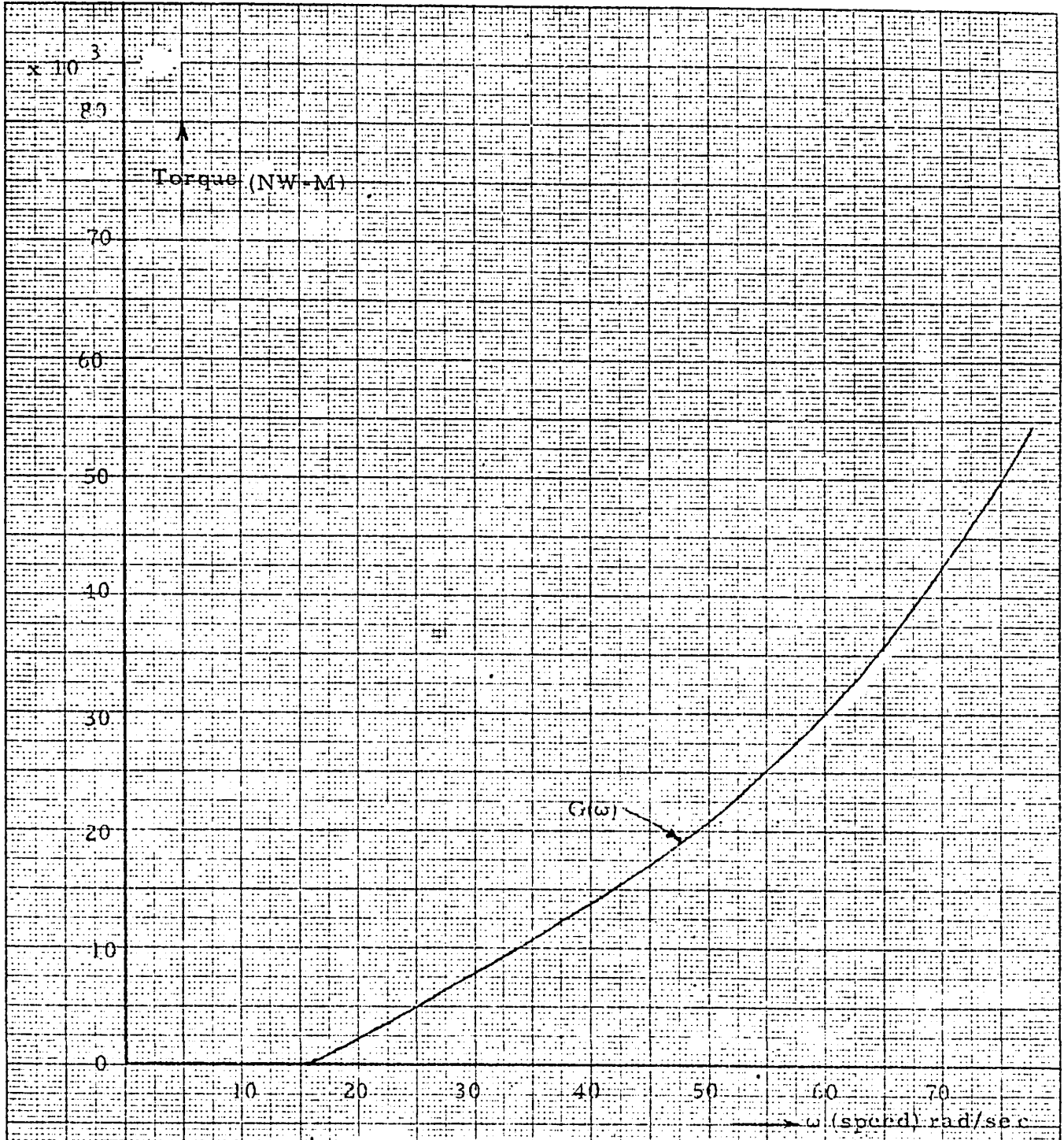


Fig. 2.2.1 Torque-speed characteristic of air compressor (load)

differentiable (see equation 1.2.1 , chapter I) with respect to the states of the system. It can be seen from fig. 2.2.1 that the D.C. machines run under no load up to $\omega = 15.70$ r/s and the air compressor (load) is coupled to the machine at that speed. Due to this action the 'T- ω ' characteristic $G(\omega)$ does not have continuous derivative with respect to ' ω ' at $\omega = 15.70$ r/s .

2.3 Formulation of the Synthesis Problem

The accurate speed control requirement of the wind-tunnel drive at the set speed poses the following problem :

The system is required to operate at 30, 40, 50 and 60 r/s only (as metioned earlier) and the feed back controller is required to maintain the system at the above operating speeds. Further, the feed back controller must bring back any deviation (caused due to disturbances) in the state (speed, acceleration) of the system to the operating point, in minimal time.

If the system were linear, then one feed back controller with arrangements for setting the operating speeds would suffice . Mathematically, this amounts to translating the switching curve (defined in section 2.4 of this chapter) along the speed axis.

Since $G(\omega)$ ('T- ω ' characteristic) is nonlinear, it is not possible to apply the linear theory. Furthermore, it is not possible to design a single time-optimal feed back controller for all the operating speeds based on one linear approximation (as 'T- ω ' characteristic is nonlinear) for the entire 'T- ω ' characteristic. Hence a linear piecewise approximation will be made around each

operating speed ω_o^i ($i = 1, 2, \dots, 4$) i.e. 30, 40, 50 and 60 r/s respectively as described below.

For reasons of safety, reliability and above all the accuracy required of the measurements being carried out on the air craft models, it is undesirable to allow the system to undergo large deviations of speed and acceleration from the operating value. Therefore any deviation in the operating speed must be corrected in the shortest possible time.

From engineering point of view, for a given system, it is possible to specify the maximum possible deviation of its states from the operating value.

In the wind-tunnel work, the large deviations in the working speeds are mainly caused by the abrupt change of aerodynamic loading on the compressor. The same phenomena does not occur in the subsonic work. So, in the case of subsonic studies, the deviations are small and a practical value of ± 4 r/s and ± 4 r/s² for speed and acceleration respectively as mentioned earlier is realistic.

In this thesis, we consider the time-optimal speed control of the wind-tunnel drive operating in subsonic range. In supersonic studies, the velocity of air in the working section is dependent, for a given compressor speed, on the throat characteristics of the convergent/divergent nozzle mounted on the up stream of the model. However, the air velocity in the supersonic range, can be time-optimally controlled in a similar way, given the dynamics of the nozzle and its driving source.

As mentioned earlier, based on ± 6 r/s deviation around each operating speed (30, 40, 50 and 60 r/s), the $G(\omega)$ characteristic (given in fig. 2.2.1) is approximated by four best straight lines using least square approximation [14], as shown in fig. 2.2.2 .

The straight lines thus obtained are given below.

- (1) $G(\omega) \approx - 8892.9 + 553.7 \omega$ for 30 rad/sec
- (2) $G(\omega) \approx - 12929 + 669 \omega$ for 40 rad/sec
- (3) $G(\omega) \approx - 17768 + 776 \omega$ for 50 rad/sec
- (4) $G(\omega) \approx - 37148 + 1125 \omega$ for 60 rad/sec

Remark : The 'T- ω ' characteristic of air compressor (load) given in fig. 2(b) (Drake et al [3] , pp 205) is designed for the total combination of synchronous motor and Ward-Leonard system. Since we consider only the Ward-Leonard system, the above characteristic has been scaled down in proportion to their respective ratings ($\frac{6,000}{60,000}$), and is shown in fig. 2.2.1 .

As shown above, $G(\omega)$ can be described, in general by a straight line $G^i(\omega) = d_o^i + d_1^i \omega$ around each operating speed ω_o^i ($i = 1, \dots, 4$), where d_o^i and d_1^i are the corresponding coefficients. The $G^i(\omega)$ is continuous and continuously differentiable. Substituting the approximated value of $G^i(\omega)$ and its derivative in equations (2.2.1) and (2.2.2) , we obtain the following differential equation.

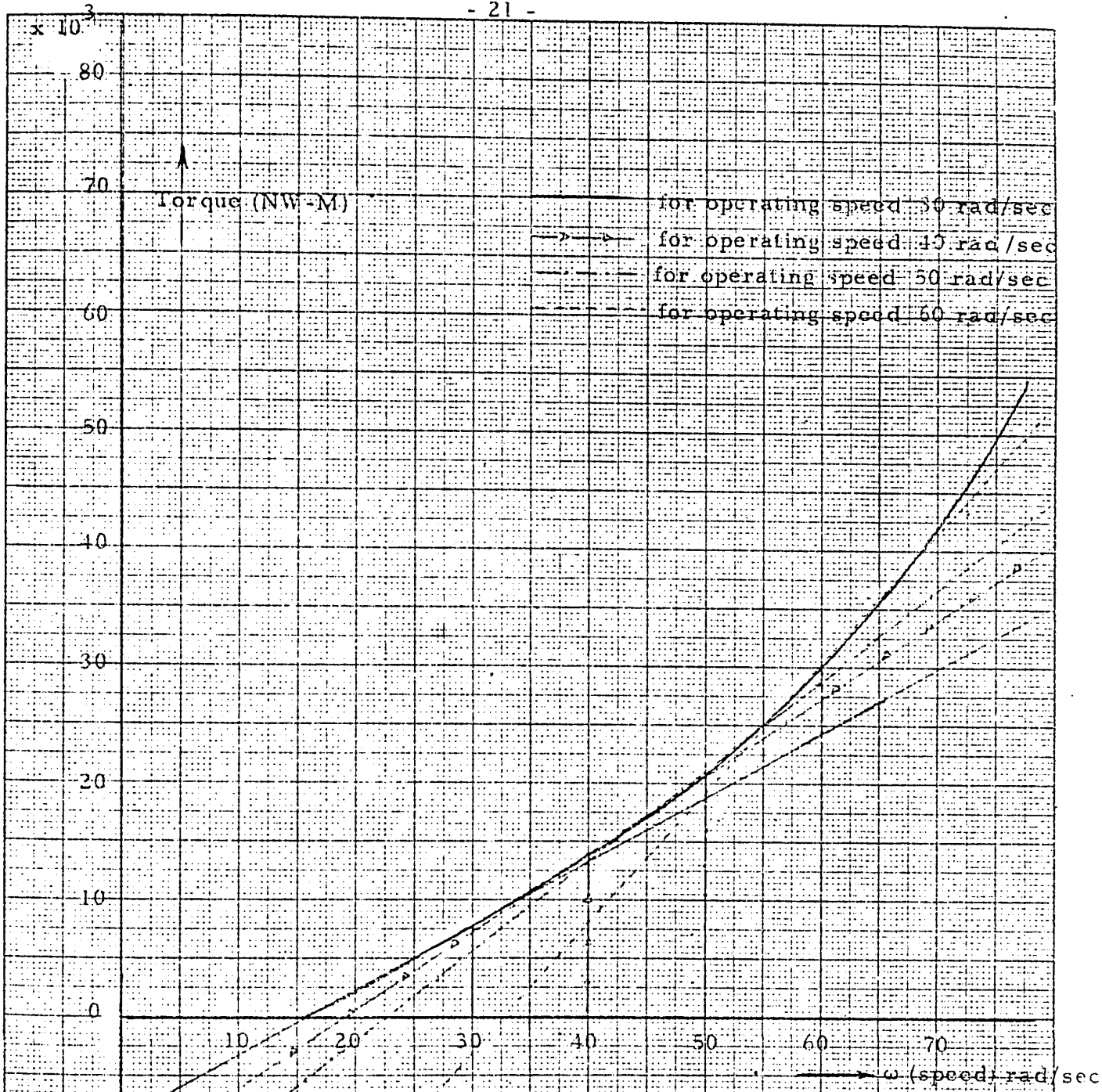


Fig. 2.2.2 Linear Piecewise Approximation of 'T- ω ' characteristic.

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$$\frac{d^2 \omega}{dt^2} + a_3^i \frac{d \omega}{dt} + a_2^i \omega = a_1^i v_1^i - a_0^i \quad (2.3.1)$$

where $a_0^i = \frac{d_0^i}{\tau_1 J}$, $a_1^i = \frac{k_3 k_1}{2L_1 r_2 J}$

$$\tau_1 \left(\begin{array}{l} \text{electrical field time} \\ \text{constant of generator} \end{array} \right) = \frac{L_1}{r_1} \quad (2.3.2)$$

$$\tau_2 \left(\text{mechanical time constant} \right) = \frac{r_2^J}{k_2 k_3}$$

$$a_2^i = \frac{1}{\tau_1 \tau_2} + \frac{d_1^i}{\tau_1 J} \quad \text{and} \quad a_3^i = \frac{1}{\tau_2} + \frac{1}{\tau_1} + \frac{d_1^i}{J}$$

The parameters thus defined in equation (2.3.2) are calculated for all the operating speeds as described in appendix A.1.

The problem of time-optimal synthesis of the wind-tunnel drive system reduces to the problem of the same for four different systems S_i ($i = 1, 2, \dots, 4$) corresponding to the operating speeds ω_i^0 $i = 1, \dots, 4$ respectively as given below :

$$S_1 : \ddot{\omega} + 0.8779 \dot{\omega} + 0.1692 \omega = 0.007978 v_1 + 1.1884$$

for the operating speed 30 rad/sec. (2.3.3)

$$S_2 : \ddot{\omega} + 0.9319 \dot{\omega} + 0.1846 \omega = 0.007978 v_1 + 1.7271$$

for the operating speed 40 rad/sec. (2.3.4)

$$S_3 : \ddot{\omega} + 0.9819 \dot{\omega} + 0.1989 \omega = 0.007978 v_1 + 2.3741$$

for the operating speed 50 rad/sec. (2.3.5)

$$S_4 : \ddot{\omega} + 1.1451 \dot{\omega} + 0.2455 \omega = 0.00678 v_1 + 4.9646$$

for the operating speed 60 rad/sec. (2.3.6)

The control voltage \bar{v}_1^i required to run the air compressor at the steady state operating speed ω_o^i can be calculated from the differential equation S_i , $i = 1, \dots, 4$ as shown below.

$$\bar{v}_1^i = \frac{a_2^i \omega_o^i + a_o^i}{a_1^i} \quad i = 1, \dots, 4, \quad (2.3.7)$$

where (a_o^i, a_1^i, a_2^i) are defined in equation (2.3.2). The calculated values of \bar{v}_1^i are tabulated below.

Table 2.3.1

i	set speed ω_o^i	steady state control voltage \bar{v}_1^i (volts)
1	30.0	487.0
2	40.0	709.0
3	50.0	948.85
4	60.0	1531.0

These are the voltages that are to be applied to the fields of the generators G_1, G_2 , in order to run the air compressor at the respective speeds.

Let us consider the differential equation that describes

the wind-tunnel drive around an operating speed ω_o^i

$$S_i : \frac{d^2 \omega^i}{dt^2} + a_3^i \frac{d\omega^i}{dt} + a_2^i \omega^i = a_1^i v_1^i - a_o^i \quad (2.3.8)$$

The corresponding steady state operating speed ω_o^i also satisfies the above differential equation, i. e.

$$\frac{d^2 \omega_o^i}{dt^2} + a_3^i \frac{d\omega_o^i}{dt} + a_2^i \omega_o^i = a_1^i \bar{v}_1^i - a_o^i \quad (2.3.9)$$

Since ω_o^i is constant, the first two terms in the L. H. S. of the above differential equation are zero.

By translating the system (2.3.8) to the operating speed ω_o^i , we obtain the following differential equation :

$$\bar{S}_i : \frac{d^2 (\omega - \omega_o^i)}{dt^2} + a_3^i \frac{d(\omega - \omega_o^i)}{dt} + a_2^i (\omega - \omega_o^i) = a_1^i (v_1 - \bar{v}_1^i) \quad (2.3.10)$$

which may be written as

$$\bar{S}_i : \frac{d^2 x^i}{dt^2} + a_3^i \frac{dx^i}{dt} + a_2^i x^i = a_1^i u^i \quad (2.3.11),$$

where $u^i \triangleq (v_1^i - \bar{v}_1^i)$ is the variation of the applied voltage around that required for steady state operation and $x^i = (\omega^i - \omega_o^i)$ is the corresponding variation of the speed around the operating value ω_o^i . It is natural to expect that u^i will be constrained for practical reasons. Let it be $|u^i| \leq C^i$, where C^i is the additional voltage to be added

or subtracted from the steady state operating voltage \bar{v}_1^i for controlling the deviation of ± 4 r/s. Usually C^i will be less than \bar{v}_1^i (see appendix A: 2 for the determination of the value C^i).

As described in appendix A. 2 , $C^i = 150$ volts, $i = 1, \dots, 4$ and hence $|u^i| \leq 150$ for all i .

The translation of the systems S_i , ($i = 1, \dots, 4$) to the respective operating speeds ω_o^i , result in the following systems \bar{S}_i ($i = 1, \dots, 4$).

$$\bar{S}_1 : \ddot{x} + 0.8779 \dot{x} + 0.1692 x = 0.007978 u \quad (2.3.12)$$

$$\bar{S}_2 : \ddot{x} + 0.9319 \dot{x} + 0.1846 x = 0.007978 u \quad (2.3.13)$$

$$\bar{S}_3 : \ddot{x} + 0.9819 \dot{x} + 1989 x = 0.007978 u \quad (2.3.14)$$

$$\bar{S}_4 : \ddot{x} + 1.1451 \dot{x} + 0.2455 x = 0.00638 u \quad (2.3.15)$$

with the above preparation, the problem may then be precisely stated as follows :

Statement of the Problem

As defined earlier, $R_o^i = \{ (x_1^i, x_2^i) \in E^2 : |x_1^i| \leq 4 \text{ r/s and } |x_2^i| \leq 4 \text{ r/s}^2 \}$ is the bounded region called the field of disturbances of the systems \bar{S}_i , $i = 1, \dots, 4$ around each of the operating speeds ω_o^i ($i = 1, \dots, 4$).

For each system \bar{S}_i , let us define the set

$$R_i = \{ (x_1^i, x_2^i) \in E^2 : |x_1^i| < 6 \text{ r/s} \} \subset E^2.$$

This is the set in which all the solutions of the system \bar{S}_i , ($i = 1, \dots, 4$), corresponding to the controls $\{u : |u| \leq 150\}$ are confined.

Given the system \bar{S}_i ($i = 1, \dots, 4$) with $|u| \leq 150$, find an admissible control u , which forces the system from any position in R_0^i to the origin (i.e. operating speed ω_0^i ($i = 1, \dots, 4$)) in the shortest possible time, after a Perturbation has taken place.

(This statement holds good for the nonlinear systems to be considered in chapter III as well. To do that replace \bar{S}_i by S_n^i in the above statement).

2.4 Synthesis of Optimal Feed Back Controller

In this section we will present a detailed description of the time-optimal feed back controller synthesis for the system \bar{S}_1 (2.2.13) and later present the results for the other systems \bar{S}_2 to \bar{S}_4 . (2.2.14 - 2.2.16).

The following definition will be useful. in the sequel.

Let U be the subset of E defined by

$U = \{u \in E : |u| \leq 150\}$ and M the space of measurable functions defined for $t \geq 0$ with values in E .

Definition 2.4 : For the class of admissible controls we define the set

$$M_0 = \{u \in M : u(t) \in U \text{ a.e., } t \geq 0\}.$$

Thus a control u defined for $t \geq 0$ and taking values in E is said to be admissible if $u \in M_0$.

The procedure for obtaining the synthesis by Pontryagin's Maximum Principle [12] (Th 1.2.2 of chapter I) consists of the following steps :

- (1) Formulate the Hamiltonian, obtain the adjoint differential equation of the given system and its solution. Then determine the nature of optimal control.
- (2) Solve the system differential equation with $u = \gamma(\pm 150)$ control. Then construct the switching curve.
- (3) Derive the synthesizing function.

The translated system for the operating speed of 30 rad/sec is given by

$$\bar{S}_1 : \ddot{x} + 0.8779 \dot{x} + 0.1692 x = 0.007978 u \quad (2.4.1)$$

$$|u| \leq 150,$$

Define $\tilde{u} = \frac{u}{150}$, then $|\tilde{u}| \leq 1$.

Rewriting equation (2.4.1) in state space form, we obtain

$$\dot{x} = Ax + B\tilde{u}, \quad |\tilde{u}| \leq 1 \quad (2.4.2)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -0.1692 & -0.8779 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1.1967 \end{bmatrix}$$

It may be noted that the system (2.4.2) is completely controllable

(Ogata [11], Th 7-9 , pp 389-390). The time-optimal control exists and is unique (Ths 1.2.4 and 1.2.5 of chapter I).

Step 1 : The Hamiltonian function H for the system (2.4.2) is given by

$$\begin{aligned} H &= \langle \psi, f \rangle = \sum_{v=1}^n \psi_v f^v \\ &= \psi_1 x_2 + \psi_2 (-0.1692 x_1 - 0.8779 x_2 + 1.1967 \tilde{u}). \end{aligned}$$

The control that maximizes H is given by

$$\tilde{u} = \text{Sgn} (1.1967 \psi_2).$$

The adjoint differential equation (see equations 1.2.10 and 1.2.15 of previous chapter) of system (2.4.2) is

$$\begin{aligned} \dot{\psi}_1 &= - \frac{\partial H}{\partial x_1} = 0.1692 \psi_2 \\ \dot{\psi}_2 &= - \frac{\partial H}{\partial x_2} = - \psi_1 + 0.8779 \psi_2 \end{aligned} \tag{2.4.3}$$

The solution of the adjoint differential system (2.4.3) is given by

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 1.9324e^{0.2857t} & -0.9324e^{0.5921t} & 0.5522(e^{0.5921t} - e^{0.2857t}) \\ 3.2637(e^{0.2857t} - e^{0.5921t}) & 1.9324e^{0.5921t} & -0.9324e^{0.2857t} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

where $\psi_1(0) = \mu_1$, $\psi_2(0) = \mu_2$.

The function $\psi_2(t)$ is given by the relation

$$\psi_2(t) = 3.2637\mu_1(e^{0.2857t} - e^{0.5921t}) + \mu_2(1.9324e^{0.5921t} - 0.9324e^{0.2857t})$$

The control $\tilde{u}(t)$ is then given by

$$\tilde{u}(t) = \text{Sgn} \{ 1.1967 [3.2637\mu_1 (e^{0.2857t} - e^{0.5921t}) + \mu_2 (1.9324e^{0.5921t} - 0.9324e^{0.2857t})] \} \quad (2.4.5)$$

Since the eigen values of the system (2.4.2) are real and distinct, the control $\tilde{u}(t)$ can have at most one switching (Th 1.2.3 , chapter I).

Thus $\tilde{u}(t)$ is piecewise constant and takes on values $\tilde{u}(t) = \pm 1$ on each interval.

Step 2 : The solution of the system equation (2.4.2) with $\tilde{u} = \gamma = \pm 1$ can be written as

$$x(t) = \Phi(t) x(0) + \int_0^t \Phi(t - \tau) B \gamma d\tau \quad (2.4.6)$$

where $\Phi(t)$ is the fundamental matrix of (2.4.2) given by

$$\Phi(t) = \begin{bmatrix} 1.9324e^{-0.2857t} - 0.9324e^{-0.5921t} & 3.2637(e^{-0.2857t} - e^{-0.5921t}) \\ 0.5522(e^{-0.5921t} - e^{-0.2857t}) & 1.9324e^{-0.5921t} - 0.9324e^{-0.2857t} \end{bmatrix} \quad (2.4.7)$$

Then equation (2.4.6) becomes

$$\begin{aligned}
 x_1(t) = & \xi_1 (1.9324 e^{-0.2857t} - 0.9324 e^{-0.5921t}) \\
 & + 3.2637 \xi_2 (e^{-0.2857t} - e^{-0.5921t}) \\
 & + 3.9058 \gamma (1.6889 e^{-0.5921t} - 3.5 e^{-0.2857t} + 1.8111)
 \end{aligned}
 \tag{2.4.8}$$

$$\begin{aligned}
 x_2(t) = & 0.5522 \xi_1 (e^{-0.5921t} - e^{-0.2857t}) \\
 & + \xi_2 (1.9324 e^{-0.5921t} - 0.9324 e^{-0.2857t}) \\
 & + 3.9058 \gamma (e^{-0.2857t} - e^{-0.5921t})
 \end{aligned}
 \tag{2.4.9}$$

where $x_1(0) = \xi_1$ and $x_2(0) = \xi_2$

Note that $\lim_{t \rightarrow \infty} x_1(t) = 7.1548 \gamma$

and $\lim_{t \rightarrow \infty} x_2(t) = 0$

For the description of step 3, that yields the switching function we need the following definitions. Using these definitions, we will be able to construct the switching function required for the design of the feed back controller.

Definition 2.4.1. : Let L_+ denote the set of points in E^2 from which the system \bar{S}_1 , given by equation (2.4.2) can be forced to the origin (i. e. operating speed $\omega_o^1 = 30 \text{ r/s}$) by the control $\tilde{u} \equiv \gamma = +1$ in finite time. Similarly L_- denote the set of points in E^2 from which the system \tilde{S}_1 given by (2.4.2) can be forced

to the origin by the control $\tilde{u} \equiv \gamma = -1$ in finite time.

We will present a brief description for the construction of the sets L_{\pm} . Since the control $\tilde{u}(t)$ can switch at most once and is piecewise constant on each interval, the length of interval over which it is constant depends upon the initial state of the system and hence is arbitrary but finite. Choose an arbitrary interval $[0, t_1]$, $0 \leq t_1 < \infty$. Take $t^1 \in [0, t_1]$. Then set $x_1(t^1) = x_2(t^1) = 0$ in equation (2.4.8) and (2.4.9), giving

$$\begin{aligned} 0 = & \epsilon_1 (1.9324 e^{-0.2857t} - 0.9324 e^{-0.5921t}) \\ & + 3.2637 \epsilon_2 (e^{-0.2857t} - e^{-0.5921t}) \\ & + 3.9058 \gamma (1.6889 e^{-0.5921t} - 3.5 e^{-0.2857t} + 1.8111) \end{aligned} \quad (2.4.10)$$

$$\begin{aligned} 0 = & 0.5522 \epsilon_1 (e^{-0.5921t} - e^{-0.2857t}) \\ & + \epsilon_2 (1.9324 e^{-0.5921t} - 0.9324 e^{-0.2857t}) \\ & + 3.9058 \gamma (e^{-0.2857t} - e^{-0.5921t}) \end{aligned} \quad (2.4.11)$$

Put $\tilde{u} = \gamma = +1$ and solve equations (2.4.10) and (2.4.11) for (ϵ_1, ϵ_2) which gives an initial state from which the system (2.4.2) can be driven to the origin under the influence of control $\tilde{u} = +1$ in time interval t^1 .

This procedure is repeated for sufficient number of points on the interval $[0, t_1]$ including the last point t_1 and L_{+}

is the locus of all the points thus obtained. Similarly, L_- can be constructed with $\tilde{u} \equiv \gamma = -1$.

Definition 2.4.2 : The switching curve L is given by i. e. $L = L_+ \cup L_-$. The switching curve L separates the plane into two regions, R^+ and R^- so that $E^2 = R^+ \cup R^-$, where R^+ is the region below the switching curve L and R^- is the region above it. The switching curve is shown in fig. 2.4.1.

With this preparation we are now able to present the final step.

Step 3 : The synthesizing function $F(x_1, x_2)$ is defined as

$$F(x_1, x_2) = \begin{cases} +1 & \text{for } (x_1, x_2) \in (R^+ \setminus L) \cup L_+ \\ -1 & \text{for } (x_1, x_2) \in (R^- \setminus L) \cup L_- \end{cases}$$

The time-optimal control \tilde{u}^* as a function of states (x_1, x_2) is given by $\tilde{u}^*(x_1, x_2) = F(x_1, x_2)$.

Similarly the switching curves for the systems \tilde{S}_i (2.3.13) - (2.3.15) are obtained as above and are shown in fig. 2.4.2. The optimal trajectories leading to the origin (i. e. the operating speed ω_o^i) from the perturbed states (for example) $(4, 4)$ and $(4, -4)$ for the above systems \tilde{S}_i ($i = 2, 3, 4$) are also shown in fig. 2.4.2. In addition, the region R_o^i (field of disturbances) and the solution constraint set R_i for $i = 2, 3, 4$ are also indicated by means of dotted lines in the same figure. It can be seen that the switching curves of the above systems differ from each other.

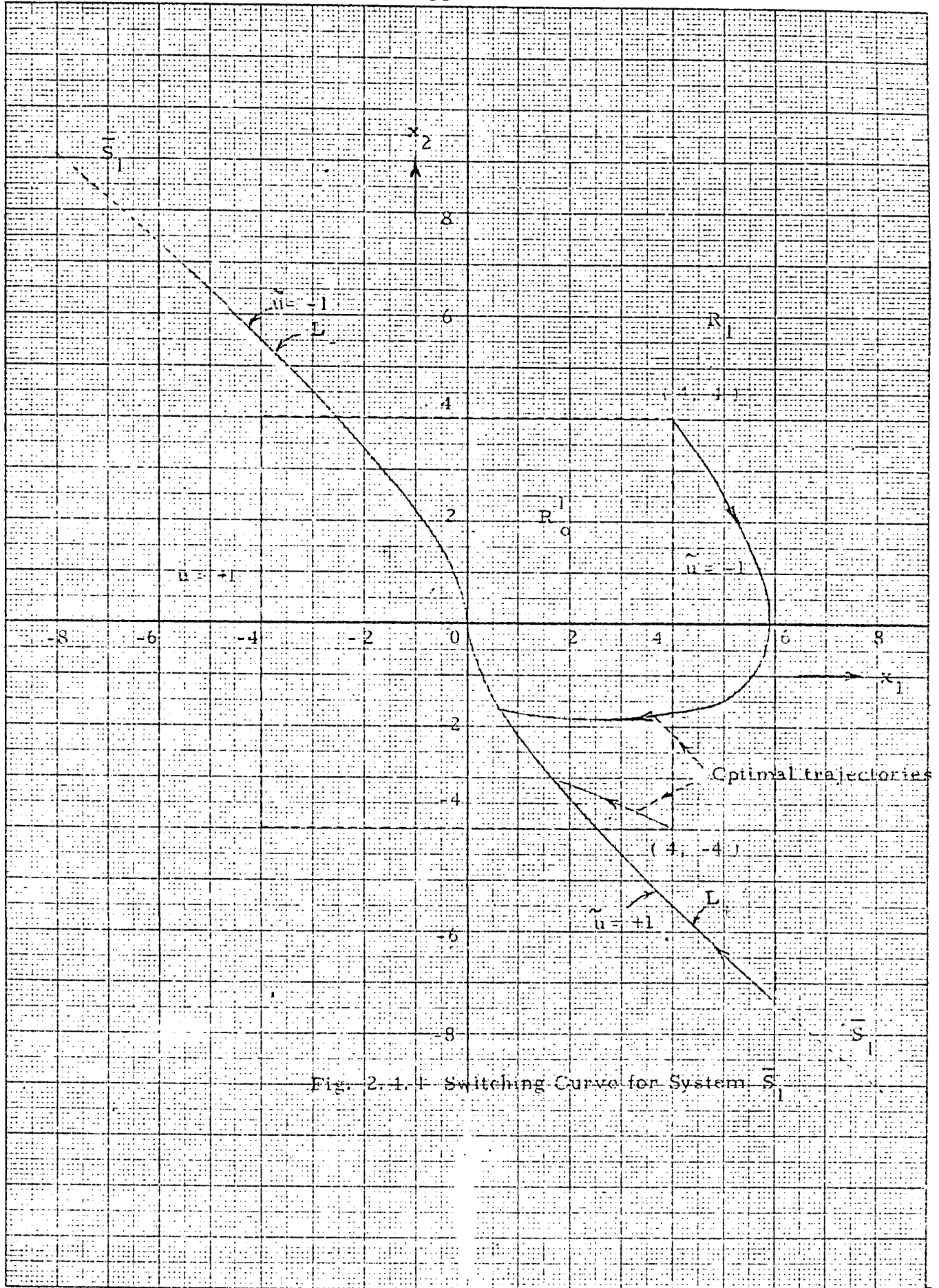


Fig. 2.1.1 Switching Curve for System S_1

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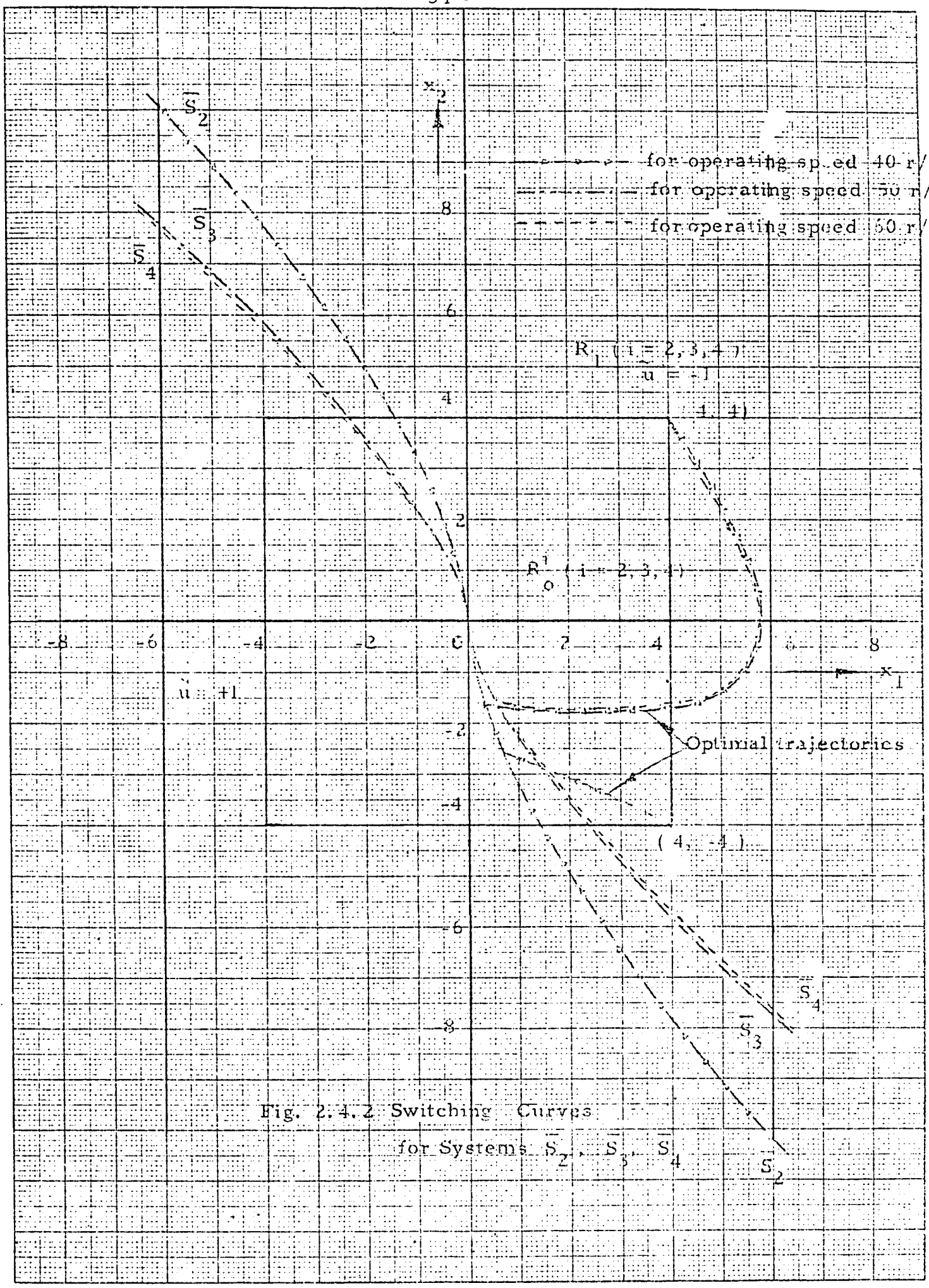


Fig. 2.4.2 Switching Curves

for Systems S_2, S_3, S_4

The schematic diagram of the time-optimal feed back controller is shown in fig. 2.5.1.

2.5 Operation of the Time-optimal Feed Back Controller

In this section we briefly describe the operation of the time-optimal feed back controller.

A schematic diagram of the time-optimal wind-tunnel drive system is shown in fig. 2.5.1. The time-optimal feed back controller unit is represented by T and is shown in the dotted line region. It consists of four nonlinear function generators or devices (as shown in boxes 1, 2, 3, 4) each corresponding to a single operating speed ω_o^i , an ideal relay and a master speed reference unit. The master speed reference 'MSR' is an independent unit which produces (precomputed) signals proportional to the operating speeds ω_o^i , $i = 1, \dots, 4$ and is used for translating the system to the corresponding operating speed ω_o^i . The variable stabilized power supply unit feeds the fields of the generators at different set voltages \bar{v}_1^i , $i = 1, \dots, 4$ (as computed earlier, section 2.3) in order to run the system at the steady state operating speeds ω_o^i . The MSR unit and the power supply unit are mutually coupled and MSR unit controls the operation of the latter.

The action of control with respect to one of the stable operating speeds, say 30 r/s is briefly described below. The speed and acceleration of the compressor may be measured by an a. c. tachogenerator and a differentiating circuit. The output of the above devices are rectified and smoothed out to yield the

state variables ω_1, ω_2 . Assume that the compressor is running at 30 r/s by the application of the steady state control voltage \bar{v}_1^1 (i. e. 487.0 volts) to the generator fields. Mathematically, the steady state operating speed ($\omega_o^1 = 30$ r/s , $\omega_o^2 = 0.0$) represents the stable equilibrium state of the system S_1 (2.3.3) which corresponds to the origin ($x_1 = 0, x_2 = 0$) of the translated system \bar{S}_1 (2.3.12) in the phase space E^2 . At this moment, the feed back controller is plugged to the system, under the control of MSR unit, by setting its dial at 30 r/s (it means simultaneous closure of switches SW1 and AT1 - AT1). Suppose while measuring the data on the model (due to disturbances in the air circuit of the tunnel), the speed of the compressor changes, for example, increases to a new value say, $\omega_1 = 34.0$ r/s , $\omega_2 = 4.0$ r/s² i. e. $x_1 = 4.0$ r/s (under the translation).

The nonlinear device (shown in box 1) produces a signal x_2 depending upon the above input quantity x_1 , which is added to ω_2 resulting in $(x_2 - \omega_2)$ as shown in fig. 2.5.1. If this signal is positive, then $(x_1, x_2) \in R^+$, if it is negative, then $(x_1, x_2) \in R^-$. In this case it is negative and hence relay supplies - 150 v.. The net voltage v_1^1 (i. e. 488 - 150 = 338 volts) is thus reduced, there by decelerating the system, the trajectory of which is shown in fig. 2.4.1. When this optimal trajectory reaches the state $x_1 = 0.65$, the driving signal of the relay becomes zero (i. e. $x_2 - \omega_2 = 0$). The ideal relay switches to + 150 v thereby increasing $v_1^1 = 488 + 150 = 638$ volts and forces the system to the origin (operating speed 30 rps) in minimum time along the curve L_+ .

It may be noted that if all the components of the

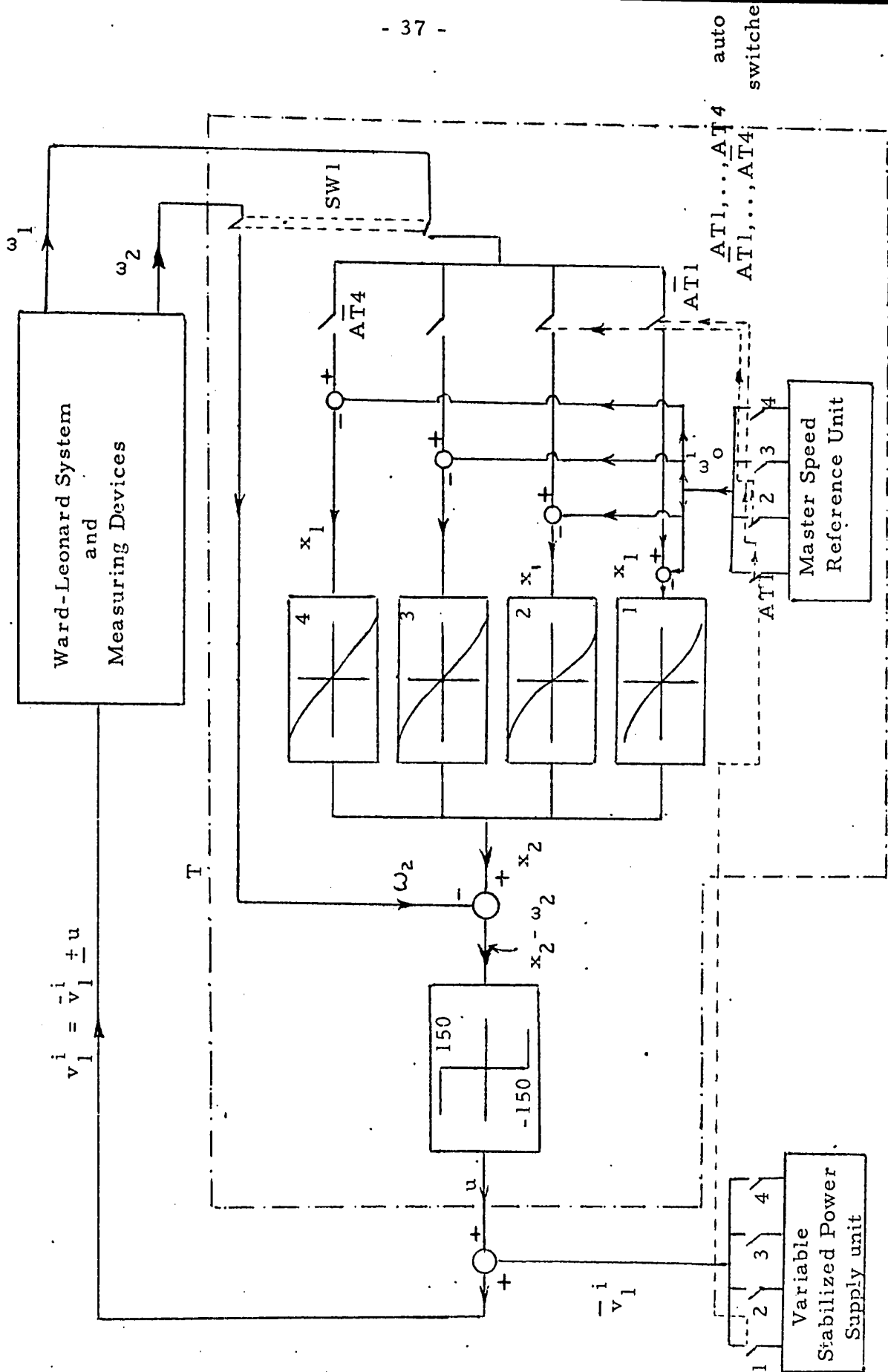


Fig. 2.5.1 Complete Arrangement of Time-Optimal Feed Back Controller

system were ideal, then the signal $(x_2 - \omega_2)$ will be zero on the switching curve 'L'. In such a case two different subsystems have to be designed to generate the time-optimal controls when ever $(x_1, x_2) \in L_+$ or L_- . However, in practice, the noise and non-ideal equipment will never make the state to remain on either L_+ or L_- . So the relay, usually, switches when the state is just above or below the switching curve, so that the actual response of the system will be close to but not exactly the same as the theoretical one. Therefore, an actual system can be expected to exhibit limit cycles about the origin (equilibrium point), the size of the limit cycle will depend upon the amount of deviation of the equipment used in the actual system from the idealization of the theoretical system.

For the case, when the speed drops below the operating value, the feed back controller will act in a similar fashion, to correct it in minimal time.

The above procedure, holds true, for all other operating speeds as well.

CHAPTER III

DESIGN OF TIME-OPTIMAL FEED BACK CONTROLLER
BASED ON QUADRATIC APPROXIMATION

Introduction

In the previous chapter, we have obtained the time-optimal synthesis of the wind-tunnel drive by approximating the 'T- ω ' characteristic of air compressor by an appropriate straight line around each of the operating speeds ω_o^i , $i = 1, \dots, 4$.

In this chapter, we consider the quadratic approximation of the 'T- ω ' characteristic around each operating speed by least square approximation and obtain the time-optimal synthesis of the resulting nonlinear differential equation of the wind-tunnel drive system.

The mathematical model of the system based on quadratic approximation is obtained in section 3.1 and the representation of the model around an operating speed is shown in section 3.2. The time-optimal synthesis of the system for an operating speed of 30 r/s and the results for other operating speeds (i.e. 40, 50 and 60 r/s) are presented in section 3.3. The realization of switching curve is described in section 3.4.

3.1 Derivation of the Mathematical Model Based on Quadratic Approximation

In this section, we obtain the mathematical model of the wind-tunnel drive, when 'T- ω ' characteristic is approximated by a quadratic function, around each operating speed ω_o^i , ($i = 1, \dots, 4$).

The four appropriate quadratic functions that approximate the 'T- ω ' characteristic around 30, 40, 50 and 60 r/s are given below.

$$G(\omega) \approx -6254.5 + 374.5 \omega + 2.9844 \omega^2 \quad \text{for 30 r/s}$$

$$G(\omega) \approx -5764.2 + 307.8 \omega + 4.5225 \omega^2 \quad \text{for 40 r/s}$$

$$G(\omega) \approx 206.4 + 53.18 \omega + 7.2366 \omega^2 \quad \text{for 50 r/s}$$

$$G(\omega) \approx 74599.8 - 2616.1 \omega + 31.1 \omega^2 \quad \text{for 60 r/s}$$

For a given operating speed ω_0^i , the $G(\omega)$ characteristic has the form $G^i(\omega) = d_2^i + d_3^i \omega + d_4^i \omega^2$, where $(d_2, d_3, d_4)^i$ are the corresponding coefficients.

Then the $G^i(\omega)$ is continuous and continuously differentiable around its operating point ω_0^i . Substituting the value of $G^i(\omega)$ and its derivative $\frac{d}{dt} G^i(\omega)$ in equations (2.2.1), (2.2.2) of chapter II, result in the following nonlinear differential equation.

$$\ddot{\omega} + a_4^i \omega \dot{\omega} + a_5^i \dot{\omega} + a_6^i \omega + a_7^i \omega^2 = a_8^i v_1^i - a_9^i \quad (3.1.1)$$

where
$$a_4^i = \frac{2 d_4^i}{J}, \quad a_5^i = \frac{d_3^i}{J} + \frac{1}{\tau_2} + \frac{1}{\tau_1}$$

$$a_6^i = \frac{1}{\tau_1 \tau_2} + \frac{d_3^i}{\tau_1 J}, \quad a_7^i = \frac{d_4^i}{\tau_1 J} \quad (3.1.2)$$

$$a_8^i = \frac{k_3 k_1}{2L_1 r_2^2 J} \quad \text{and} \quad a_9^i = \frac{d_2^i}{\tau_2 J}$$

(τ_1, τ_2 are defined in chapter II).

The coefficients $a_4^i, a_5^i, \dots, a_9^i$ are calculated for all the operating speeds (using the procedure given in appendix A. 1) and the corresponding nonlinear differential equations representing the systems at the above speeds are given below :

$$\begin{aligned} \tilde{S}_1 &: \ddot{\omega} + 0.002791 \omega \dot{\omega} + 0.7941 \dot{\omega} + 0.1452 \omega + 0.0003988 \omega^2 \\ &= 0.007978 v_1 + 0.8358 \end{aligned} \quad (3.1.3)$$

for operating speed 30 r/s

$$\begin{aligned} \tilde{S}_2 &: \ddot{\omega} + 0.00423 \omega \dot{\omega} + 0.7629 \dot{\omega} + 0.1363 \omega + 0.000604 \omega^2 \\ &= 0.007978 v_1 + 0.7705 \end{aligned} \quad (3.1.4)$$

for operating speed 40 r/s

$$\begin{aligned} \tilde{S}_3 &: \ddot{\omega} + 0.00677 \omega \dot{\omega} + 0.6438 \dot{\omega} + 0.1023 \omega + 0.000966 \omega^2 \\ &= 0.007978 v_1 - 0.0275 \end{aligned} \quad (3.1.5)$$

for operating speed 50 r/s

$$\begin{aligned} \tilde{S}_4 &: \ddot{\omega} + 0.029 \omega \dot{\omega} - 0.5096 \dot{\omega} - 0.254 \omega + 0.004156 \omega^2 \\ &= 0.00638 v_1 - 9.9692 \end{aligned} \quad (3.1.6)$$

for operating speed 60 r/s

The steady state control voltage \bar{v}_1^i , $i = 1, \dots, 4$ for the four operating speeds are calculated from the differential equations \tilde{S}_i as shown below :

$$\bar{v}_1^i = \frac{a_6^i \omega_0^i + a_7^i (\omega_0^i)^2 + a_9^i}{a_8^i} \quad (3.1.7)$$

The voltages thus calculated are tabulated below.

Table 3.1.1

i	steady state speed ω_o^i rad/sec	steady state voltage \bar{v}_1^i volts
1	30.0	486.1
2	40.0	707.95
3	50.0	947.2
4	60.0	1590.0

It may be noted that the above figures for \bar{v}_1^i are approximately the same as that obtained in linear case (table 2.3.1) except for the top speed 60 r/s .

3.2, Representation of Nonlinear Wind-tunnel Drive Systems Around the Operating Speeds (ω_o^i)

The nonlinear differential equation of the wind-tunnel drive around a given operating speed ω_o^i has the following form :

$$S_n^i : \frac{d^2 \omega}{dt^2} + a_4^i \omega \dot{\omega} + a_5^i \dot{\omega} + a_6^i \omega + a_7^i \omega^2 = a_8^i v_1^i - a_9^i \quad (3.2.1)$$

The corresponding steady state operating speed ω_o^i also satisfies the above differential equation.

$$\frac{d^2 \omega_o^i}{dt^2} + a_4^i \omega_o^i \dot{\omega}_o^i + a_5^i \dot{\omega}_o^i + a_6^i \omega_o^i + a_7^i (\omega_o^i)^2 = a_8^i v_1^{-i} - a_9^i \quad (3.2.2)$$

Since ω_o^i is constant, $\frac{d^2 \omega_o^i}{dt^2}$ and $\frac{d \omega_o^i}{dt}$ are zero in the above equation. Subtracting equation (3.2.2) from (3.2.1) and after algebraic manipulation of the resulting equation, we obtain an equation that describes the motion of system around the operating speed ω_o^i as given below:

$$S_n^i : \frac{d^2 x^i}{dt^2} + a_4^i x^i \dot{x}^i + (a_5 + a_{11})^i \dot{x}^i + (a_6 + a_{10})^i x^i + a_7^i (x^i)^2 = a_8^i u^i \quad (3.2.3)$$

where $a_{10}^i = 2 a_7^i \omega_o^i$, $a_{11}^i = a_4^i \omega_o^i$,

$$x^i = (\omega - \omega_o)^i \quad \text{and} \quad u^i = v_1^i - v_1^{-i}.$$

Like in the linear system, we consider $|u^i| \leq 150$ for all i .

The nonlinear differential equations that represent the system around each of the operating speed ω_o^i ($i = 1, \dots, 4$) are given below :

$$S_n^1 : \ddot{x} + 0.002791 x \dot{x} + 0.8778 \dot{x} + 0.1691 x + 0.0003988 x^2 = 0.007978 u$$

for 30 r/s (3.2.4)

$$S_n^2 : \ddot{x} + 0.00423 x \dot{x} + 0.9321 \dot{x} + 0.1846 x + 0.000604 x^2 = 0.007978 u$$

for 40 r/s (3.2.5)

$$S_n^3 : \ddot{x} + 0.0067 x \dot{x} + 0.978 \dot{x} + 0.1983 x + 0.000966 x^2 = 0.007978 u$$

för 50 r/s (3.2.6)

$$S_n^4 : \ddot{x} + 0.029 x \dot{x} + 0.2304 \dot{x} + 0.2447 x + 0.004156 x^2 = 0.00638 u$$

for 60 r/s (3.2.7)

3.3 The Synthesis

In this section, we briefly describe the time-optimal feed back controller synthesis for the system S_n^1 (corresponding to operating speed 30 r/s) and later present the results for other systems. (i.e. equations 3.2.5 - 3.2.7).

The nonlinear differential equation of the system S_n^1 is given by

$$S_n^1 : \ddot{x} + 0.002791 x \dot{x} + 0.8778 \dot{x} + 0.1691 x + 0.003988 x^2 = 0.007978 u$$

$|u| \leq 150$ (3.3.1)

We proceed to obtain the time-optimal synthesis of the above system by following the three steps given in section 2.4 of chapter II.

Define $\tilde{u} = \frac{u}{150}$, then $|\tilde{u}| \leq 1$.

The equation (3.3.1) then becomes

$$S_n^1 : \ddot{x} + 0.002791 x \dot{x} + 0.8778 \dot{x} + 0.1691 x + 0.0003988 x^2 = 1.1967 \tilde{u}$$

(3.3.2)

We assume that the optimal control for the system S_n^1 (3.3.2) exists (Lee and Markus [8], Th 3, pp 254) and the system is controllable (Lee and Markus [8], Th. 2, pp 429).

Step 1 : The Hamiltonian function H for the system S_n^1 is given by

$$H = \psi_1 x_2 + \psi_2 (1.1967 \tilde{u} - 0.002791 x_1 x_2 - 0.8778 x_2 - 0.1691 x_1 - 0.0003988 x_1^2)$$

The function H will be maximum when

$$\tilde{u}(t) = \text{Sgn} (1.1967 \psi_2(t))$$

The adjoint differential system of S_n^1 is given

$$\dot{\psi}_1 = - \frac{\partial H}{\partial x_1} = \psi_2 (0.1692 + 0.0007976 x_1 + 0.002791 x_2)$$

$$\dot{\psi}_2 = - \frac{\partial H}{\partial x_2} = - \psi_1 + \psi_2 (0.8778 + 0.002791 x_1)$$

(3.3.3)

Since $\dot{\psi}_1$ and $\dot{\psi}_2$ are functions of x_1 and x_2 , it is difficult to obtain the solution (ψ_1, ψ_2) analytically in terms of their unknown initial conditions μ_1, μ_2 where $\psi_1(0) = \mu_1$ and $\psi_2(0) = \mu_2$. The number of switchings of the control depends upon the zeros of the function $\psi_2(t)$.

Step 2 : We solve the following state equations to obtain the trajectories in E^2 for $\tilde{u} = \gamma = \pm 1$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 1.1967 \gamma - 0.002791 x_1 x_2 - 0.8778 x_2 \\ &\quad - 0.1691 x_1 - 0.0003988 x_1^2 \end{aligned} \quad (3.3.4)$$

Equations(3.3.4) are nonlinear simultaneous differential equations which are difficult to solve analytically. Hence numerical integration techniques are used to solve it.

The construction of the switching curve 'L' is briefly described below. Unlike linear systems, it is not feasible here to obtain the solution of equations (3.3.4) in terms of arbitrary initial conditions, from which the curves L_{\pm} can be determined.

To obtain L_{+} , put $\tilde{u} = \gamma = +1$ in equations (3.3.4) Since, for each time 't', the terminal condition being $(0,0) \in E^2$, the equations (3.3.4) can be integrated backwards in time to obtain the initial state $(x_1, x_2) \in E^2$ that forces the system to the origin under the influence of control $\tilde{u} = +1$ in time t. This may be done by changing the variable of integration from t to τ i.e. $\tau = t^* - t$, where t^* is the length of the interval over which the control \tilde{u} remains constant. Then the equations (3.3.4) reduce to

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= - \{ 1.1967 \gamma - 0.002791 x_1 x_2 - 0.8778 x_2 \\ &\quad - 0.1691 x_1 - 0.0003988 x_1^2 \} \end{aligned} \quad (3.3.5)$$

The above equations are then integrated over some arbitrary interval $[0, t^*]$ using standard Runge-Kutta fixed step size method. The solution thus obtained is the required locus L_+ . Note that the zeros of $\psi_2(t)$ (i. e. time backward solution of equations 3.3.3) coincide with the zeros of $x_2(t)$ (i. e. solution of equations 3.3.5) if the respective solutions start at the initial condition $(0, 0, \pm 1, 0) = (x_1(0), x_2(0), \psi_1(0), \psi_2(0))$ (Lee and Markus [8], Th 1, pp 427). To test it a large value of t^* is chosen and the nature of solution of equations 3,3.5) is observed. It is found that curve L_+ does not reintersect the x_1 axis. It can be concluded that the control $\tilde{u}(t)$ may not have multiple switchings.

To determine whether there are pieces of the switching curve sticking back from infinity the following procedure is followed. Select few points on L_+ . From each point on L_+ , solve equations 3.3.3 and 3.3.4 backwards in time with $\psi_1(0) = -1$, $\psi_2(0) = 0$ until $\psi_2(t)$ becomes zero for certain time \bar{t} , $\bar{t} < 0$ and mark this point in the (x_1, x_2) plane as belonging to switching curve. About four points i. e. $(0.4999, -1.3870)$, $(2.0280, -3.3881)$, $(4.6131, -6.0503)$ and $(6.0187, -7.3687)$ were chosen and by following the above said procedure, it is found that $\psi_2(t)$ never become zero. Hence there were no other pieces of switching curve sticking from infinity. It may be noted that it is not necessary to check whether there are any other switching curves emanating from infinity, as the operating region of the present system is limited. However, this procedure assures that there are no multiple switchings for the control $\tilde{u}(t)$.

Similarly L_- is constructed with $\tilde{u} = -1$. The integration is carried out on the digital computer (IBM System/360) using CSMP.

Step 3 : Same as in linear case i. e. section 2.4 of chapter II.
The switching curve 'L' is shown in fig. 3.3.1 .

The switching curves for other systems (3.2.5) - (3.2.7) are calculated as described above and are shown in fig. 3.3.2 .

In the case of the system S_n^4 i. e. for operating speed 60 r/s, it is observed that the lobes L_+ and L_- reintersect the x_1 axis. It indicates that the control \tilde{u} may have multiple switchings.

Since the range of operation $\{(x_1, x_2) \in R_4\}$ of the system is limited, the control can switch at most once for the initial states lying in R_0 .

Like in linear case, the optimal trajectories leading to the operating speed ω_0^i from the initial states, for example, $(4, 4)$ and $(4, -4)$, in R_0^i are shown in fig. 3.3.2 along with the regions R_0^i and R_i for all i .

It can be seen from fig. 3.3.2 that the switching curves for the operating speeds 30, 40 and 50 r/s are approximately the same unlike in linear case (see fig. 2.4.2). The optimal times are calculated for a typical initial state $(4, -4)$ in R_0^i for both linear systems and nonlinear case and are tabulated below.

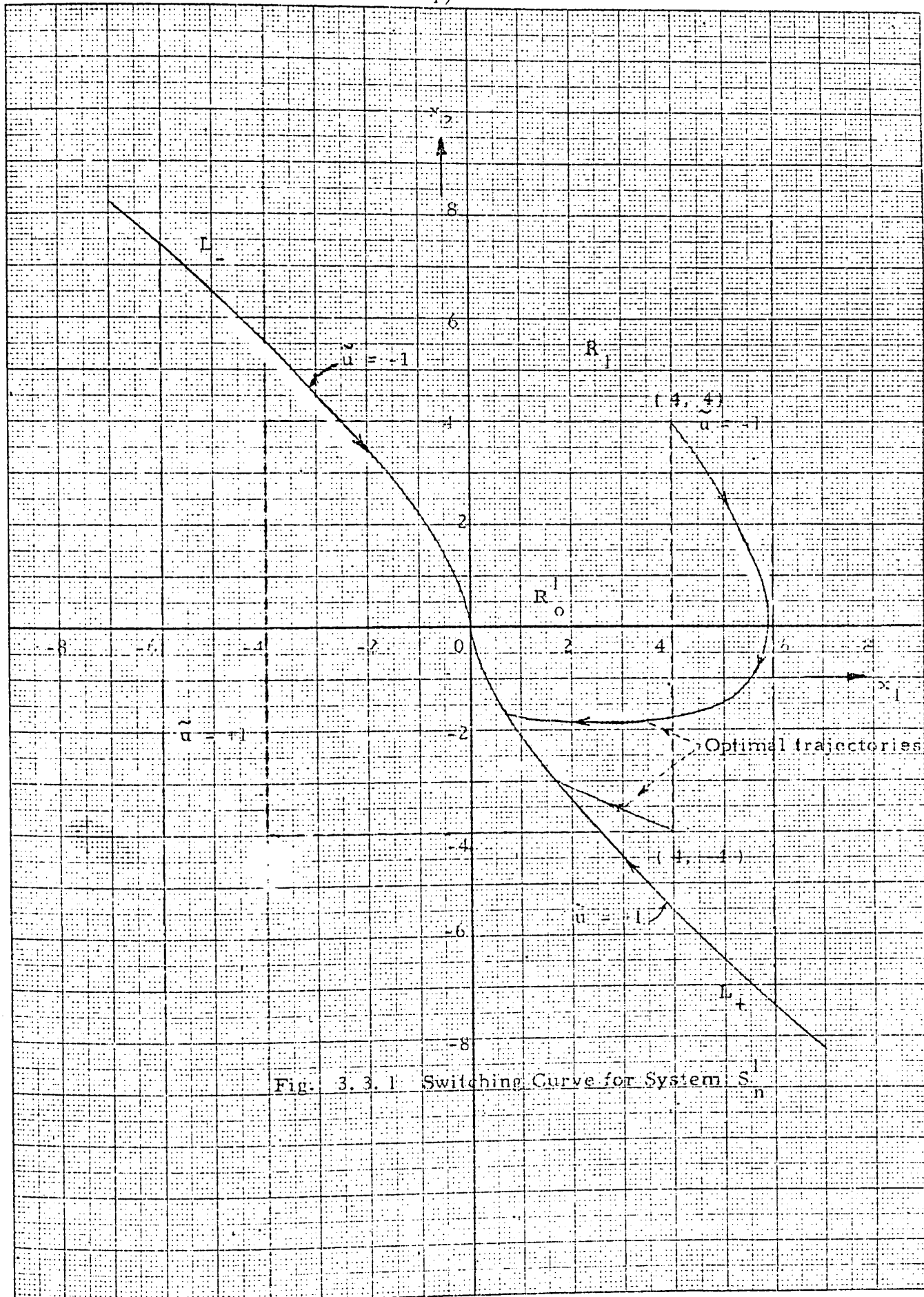


Fig. 3.3.1 Switching Curve for System S_n^1

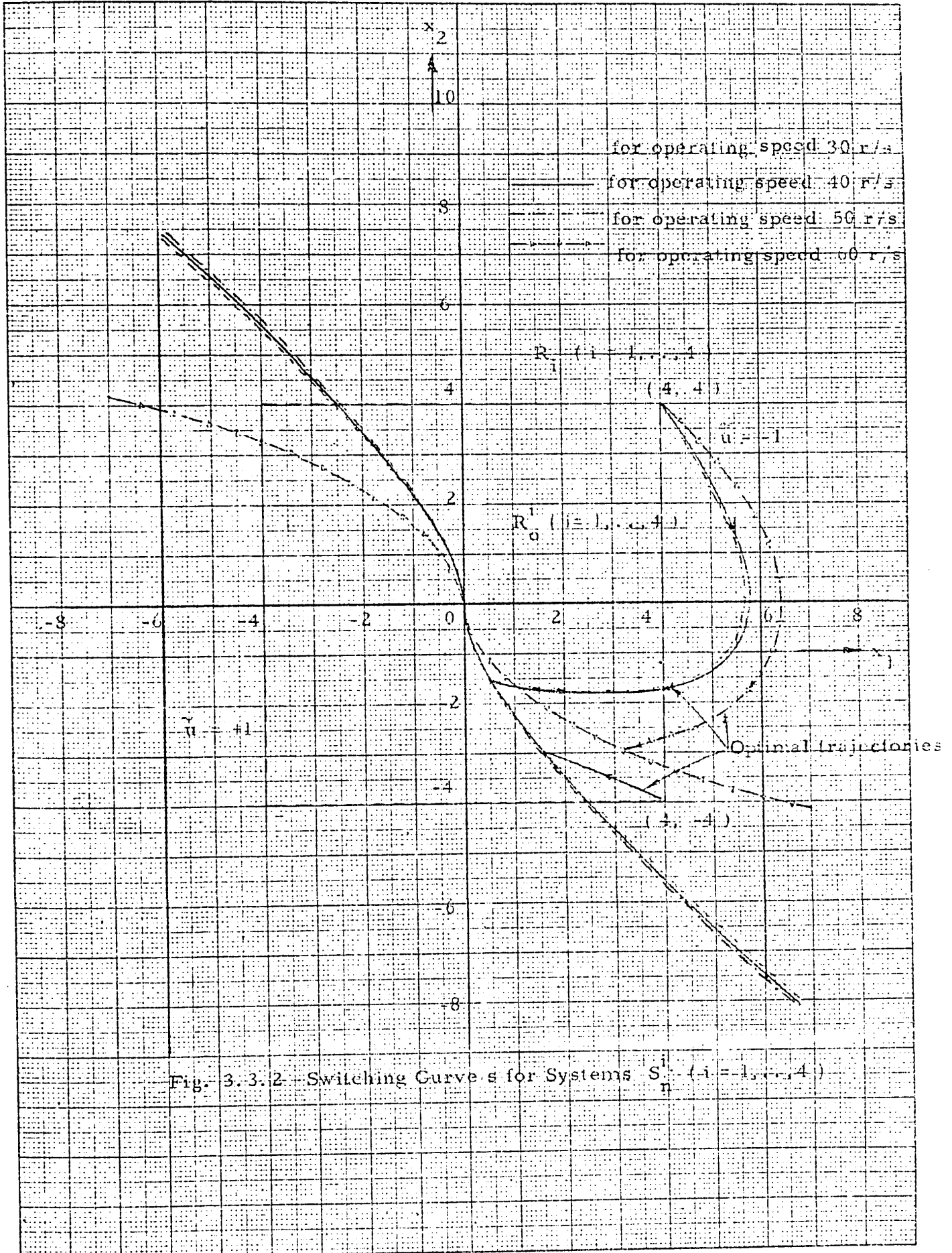


Fig. 3.3.2 - Switching Curves for Systems $S_n^i (i=1, \dots, 4)$

Table 3.3.1

Operating speed (rad/sec)	Optimal time linear system (sec)	Optimal time nonlinear system (sec)
30.0	5.50	5.50
40.0	5.61	5.43
50.0	5.395	5.39
60.0	5.835	5.34

It can be readily seen from the above table that the nonlinear system takes slightly less or the same time for the speeds 30, 40 and 50 r/s and in the case of 60 r/s , nonlinear system is faster than the corresponding linear case. It may be noted that the structure of optimal feed back controller will be exactly similar to that given in fig. 2.5.1 except for the switching curves.

3.4 Approximate Realization of Switching Curves.

In the present section we attempt to present an approximate method for the construction of switching curves. In general, the switching curves are complex nonlinearities (for example, the systems having multiple switching) and it is difficult to construct them by electronic devices. An approximate method is suggested here.

As mentioned earlier, the operating region of the system is limited to R_i ($i = 1, \dots, 4$) for each i in linear as well as nonlinear systems and the switching curve lying in this region only has to be realized. It can be seen from figs. (2.4.2) and (3.3.2) that the switching curves lying in R_i ($i = 1, \dots, 4$) are continuous and single valued. They may be uniformly approximated by a polynomial (Weirstrass approximation theorem)

$$\text{i. e. } h(x) = \sum_{j=0}^n b_j x^j ,$$

where $h(x)$ is the switching curve lying in R_i and b_j , $j = 0, 1, \dots, N$ are constant coefficients to be determined for given ξ (> 0), the criteria for accuracy.

The switching curves of the systems \bar{S}_1 (2.3.12) and S_n^1 (3.2.4) (for the operating speed 30 r/s) are approximated by an eighthorder polynomial and the corresponding coefficients are presented below :

$$h_1 \approx 0.001648 - 2.1048 x - 0.002704 x^2 + 0.09212 x^3 + 0.0003749 x^4 \\ - 0.003325 x^5 - 0.00001535 x^6 + 0.00003946 x^7 + 0.00000018992 x^8 \\ \text{for } \bar{S}_1 \text{ (2.3.12)}$$

$$h_1 \approx 0.002122 - 2.1713 x - 0.004437 x^2 + 0.1192 x^3 + 0.00072416 x^4 \\ - 0.004615 x^5 - 0.00003384 x^6 + 0.00006237 x^7 + 0.0000004769 x^8 \\ \text{for } S_n^1 \text{ (2.3.4)}$$

Similarly the coefficients b_j , $j = 0, 1, \dots, 8$ calculated for the operating speed 40 r/s .

Table 3.4.1

Coefficients	Linear system (2.3.13)	Nonlinear system (3.2.5)
b_0	3.0412×10^{-3}	$- 1.055 \times 10^{-3}$
b_1	$- 3.0950$	$- 2.2032$
b_2	$- 5.665 \times 10^{-3}$	2.483×10^{-3}
b_3	0.14461	0.1102
b_4	8.128×10^{-4}	$- 4.191 \times 10^{-4}$
b_5	$- 5.213 \times 10^{-3}$	$- 4.603 \times 10^{-3}$
b_6	$- 3.392 \times 10^{-5}$	2.037×10^{-5}
b_7	6.192×10^{-5}	6.295×10^{-5}
b_8	4.247×10^{-8}	$- 2.956 \times 10^{-8}$

These polynomials may then be easily realized by multipliers. This is shown in fig. 3.4.1 .

The main limitation of this method is that the switching curve lying in the operating region should be continuous.

A linear piecewise approximation method may be useful, when the switching curves are single valued and monotonic functions. In such case, the diode function generators [15] can be used for realization. But this method is not applicable for complicated switching curves.

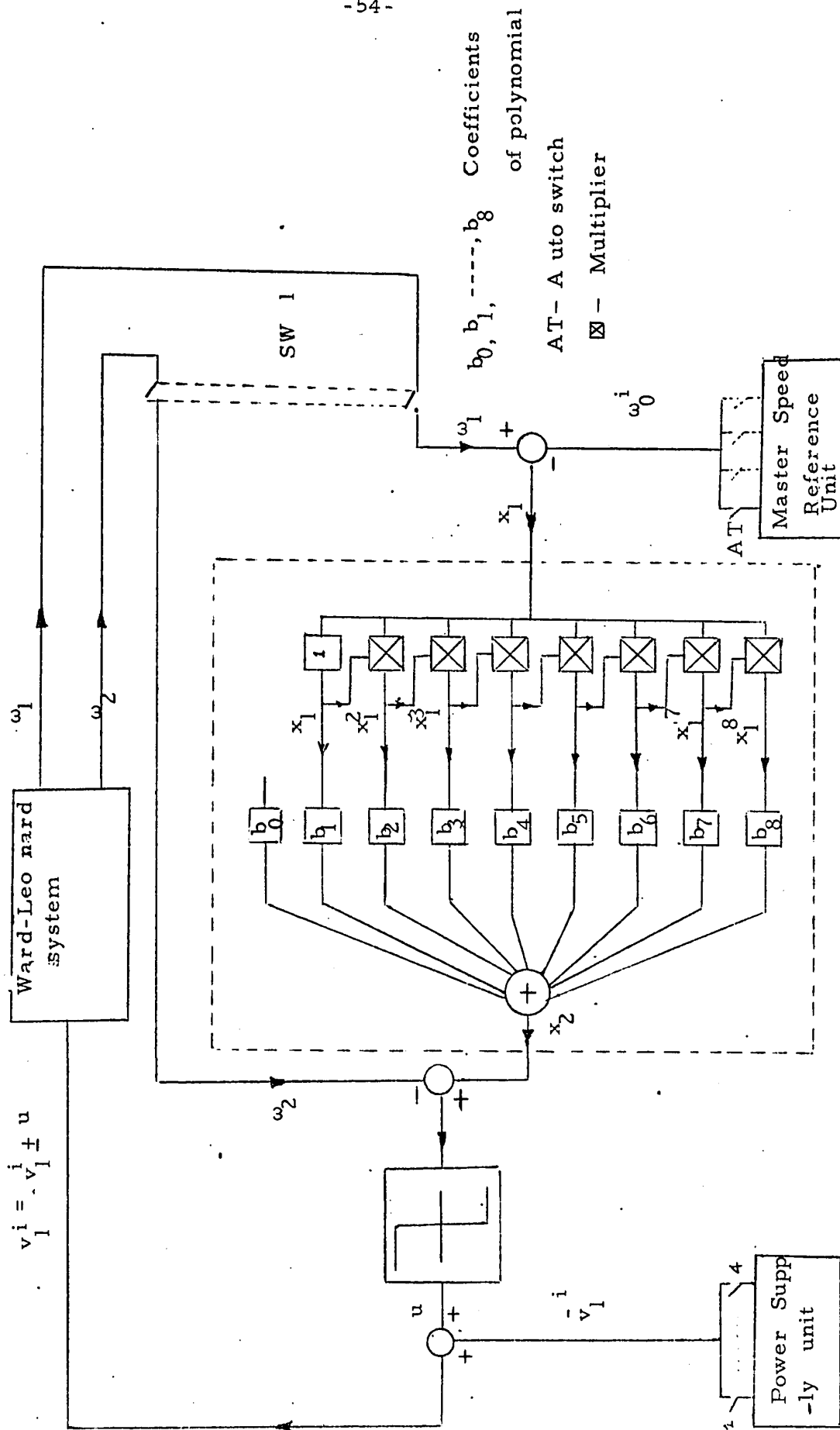


Fig. 3.4.1. Realisation of Switching curve by polynomial

Of eighth order

CONCLUSIONS

The time-optimal feed back controllers, based on linear piecewise as well as quadratic approximation of the 'T- ω ' characteristic of air compressor (for a deviation of ± 6 r/s), are obtained using Pontryagin's Maximum Principle. It is found that the switching curves for the operating speeds 30, 40 and 50 r/s. are approximately the same (see fig. 3.3.2) for the nonlinear systems S_n^i ($i = 1, 2, 3$). This shows that, in building the optimal feedback controller, the different switching functions used for operating speeds 30, 40 and 50 r/s (obtained based on linear piecewise approximation of 'T- ω ' characteristic, see figs. 2.4.2 and 2.5.1 , chapter II) can be replaced by a single switching function derived based on quadratic approximation to cover all the above three operating speeds. Clearly this is an advantage. This is due to the fact that the 'T- ω ' characteristic (fig. 2.2.1) is very close to a quadratic function over that range.

It is observed that there is a slight improvement in optimal times in the case of nonlinear systems compared to the corresponding linear systems (see table 3.3.1).

If the optimal trajectories were also to be confined to $R_o^i = \{ (x_1^i, x_2^i) : |x_1^i| \leq 4 \text{ r/s}, |x_2^i| \leq 4 \text{ r/s}^2 \}$ for all i , then we have state constrained time-optimal control problem. It can be seen from fig. 2.4.1 and fig. 3.3.1 that the optimal control will not be bang-bang for the states (x_1^i, x_2^i) lying on the boundary (∂R_o^i) and it would be interesting to investigate the synthesis for the above problem .

The dynamical equations (i) - (iv) given in section 2.2 of chapter II, actually reduce to the following first order differential equation :

$$S_a : J \frac{d\omega}{dt} + G(\omega) + p\omega = q \int_0^t K(t, \tau) v_1(\tau) d\tau ,$$

where $K(t, \tau) = e^{-\lambda(t-\tau)}$, $\lambda = \frac{1}{\tau_1}$, $p = \frac{k_2 k_3}{r_2}$ and

$$q = \frac{k_1 k_3}{2 r_1 r_2 \tau_1} , \quad (\text{assumed } i_1(0) = 0) .$$

Since $G(\omega)$ is not continuously differentiable and the control voltage v_1 appears under the integral sign, the Pontryagin's Maximum Principle cannot be applied readily to the above system. Even if we assume $G(\omega)$ continuous and continuously differentiable around the operating speeds, it is not strictly correct to apply Pontryagin's Maximum Principle for the system S_a . However, this problem may be solved as indicated below :

Solve the system \bar{S}_a

$\bar{S}_a : J \frac{d\omega}{dt} + G(\omega) + p\omega = u$ for the optimal control u^* , using Pontryagin's Maximum Principle. Then solve the integral equation

$$u^* = q \int_0^t K(t, \tau) v_1^*(\tau) d\tau \quad \text{for } v_1^* .$$

It is very difficult to solve the above volterra-integral equation of the first kind. Because the optimal control obtained by Pontryagin's Maximum Principle is a measurable

function and usually discontinuous, where as the R.H.S. of the above integral equation is a continuous function. So u^* may be approximated by a continuous function and then v_1^* can be solved.

It would be interesting to compare the speed of response of the system obtained from the approximate switching curve with that of the original switching curve.

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APPENDIX

A.1 Calculation of Parameters $a_j, j = 0, 1, \dots, 11$.

Consider the differential equation of the wind-tunnel drive given by (2.3.1)

$$\frac{d^2 \omega}{dt^2} + a_3^i \frac{d\omega}{dt} + a_2^i \omega = a_1^i v_1 - a_0^i,$$

where

$$a_3^i = \frac{1}{\tau_2} + \frac{1}{\tau_1} + \frac{d_1^i}{J}, \quad \tau_2 = \frac{r_2 J}{k_2 k_3},$$

$$a_2^i = \frac{1}{\tau_1 \tau_2} + \frac{d_1^i}{\tau_1 J}, \quad \tau_1 = \frac{L_1}{r_1}$$

$$a_1^i = \frac{k_3 k_1}{2L_1 r_2 J} \tag{A.1.1}$$

First consider L_1, r_1 : Actually the inductance L_1 of the field circuit is a function of the field current and is not constant. It may be noted that it is difficult to estimate the dynamic inductance $L_1(i_1)$ exactly (which is expected to be highly nonlinear). In this thesis, for the purposes of simple analysis, we assume L_1 to be constant. Though r_1 changes with temperature, for all practical purposes, it can be taken as constant. Hence τ_1 is constant.

$$\tau_1 = 3.5 \text{ sec} \quad (\text{Drake et al [3], pp 214}).$$

Generator gain constant k_1 : A typical saturation characteristic of D.C. generator [9] ('E-I' characteristic, E = induced emf

and I is the field current) is shown in fig. A. 1. 1 . It can be seen from fig. A. 1. 1 that k_1 is not really constant for the entire range of field current. In this thesis a piecewise approximation is made and two typical values are taken. i. e. $k_1/r_1 = 2.0$ for the speeds 30 - 50 rad/sec and $k_1/r_1 = 1.6$ for 60 r/s.

The more accurate method could be to make piecewise approximation of 'E-I' characteristic for each operating field current (which can be calculated given the operating speed and torque) and obtain k_1 thereafter.

Flux Constant k_2 : The 'E- ω ' (Back emf of the D.C. motor - speed) characteristic will not be linear due to armature reaction and hence a similar piecewise approximation may be applied to calculate the constant k_2 . Since the effect of armature reaction is neglected, we take k_2 to be constant in this work.

$$k_2 = \frac{\text{Rated terminal voltage (volts)}}{\text{Rated speed (rad/sec)}} = \frac{750}{62.8}$$

$$= 11.94$$

$$r_2 = 0.2 \text{ ohms}$$

The mechanical time constant $\tau_2 = 3.0 \text{ sec}$ (Drake et al [3], pp 214)

where

$$\tau_2 = \frac{r_2 J}{k_2 k_3} \quad \text{but } k_2 = k_3 = 11.94$$

$$\therefore J = 2138 \text{ Kg-m}^2$$

using the above values, the parameters a^i , $i = 0, 1, \dots, 3$ are calculated for the system \bar{S}_1 (2. 3. 12) as shown below :

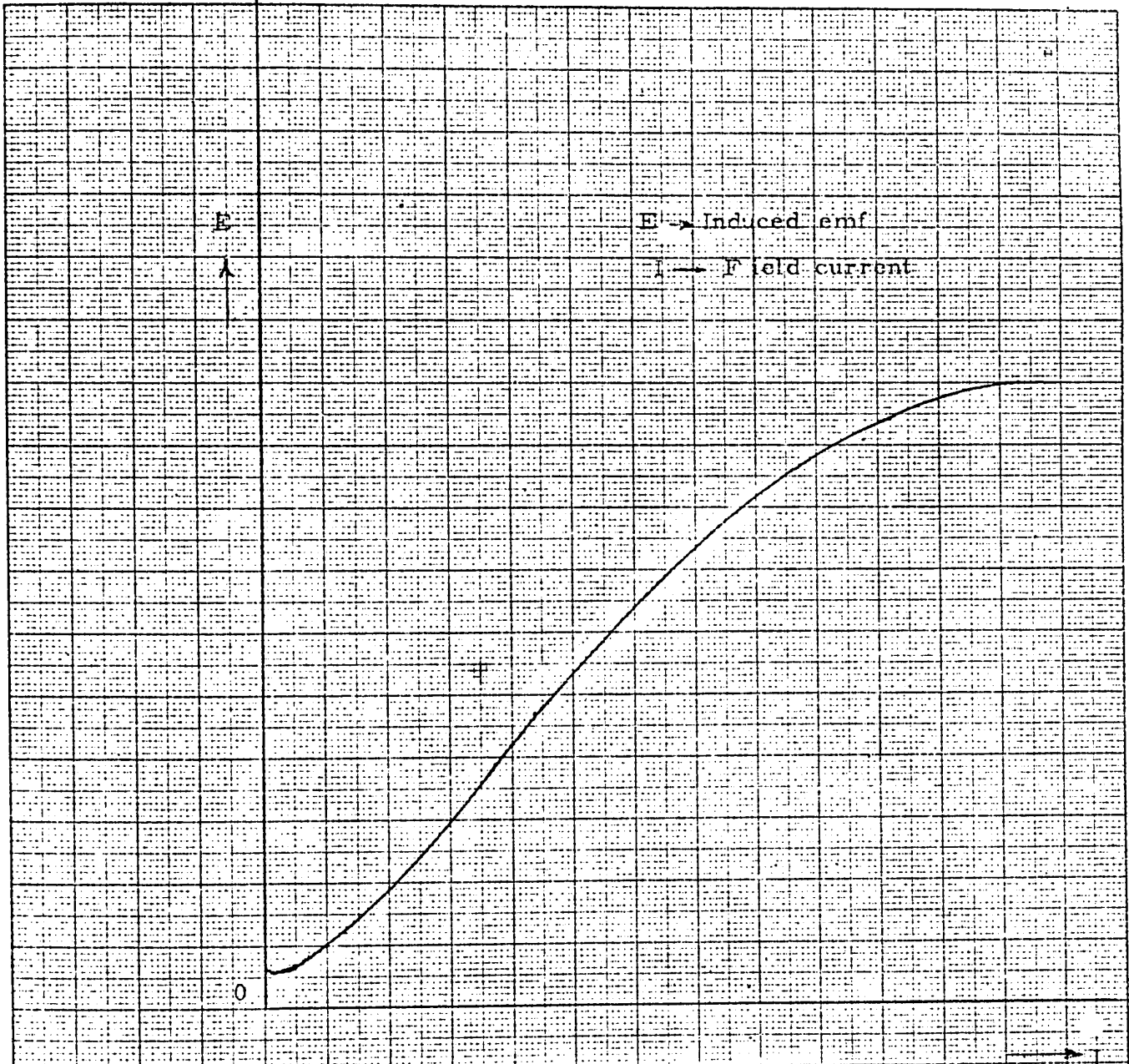


Fig. A. 1. 1. Saturation characteristic of generator

$$G^1(\omega) = -8892.9 + 553.7 \omega$$

$$a_3^1 = \frac{1}{\tau_2} + \frac{1}{\tau_1} + \frac{d_1^1}{J} = \frac{1}{3} + \frac{1}{3.5} + \frac{553.7}{2138} = 0.8978$$

$$a_2^1 = \frac{1}{\tau_1 \tau_2} + \frac{d_1^1}{J} = \frac{1}{3 \times 3.5} + \frac{553.7}{2138 \times 3.5} = 0.1692$$

$$a_1^1 = \frac{k_3 k_1}{2L_1 r_2 J} = \frac{11.94 \times 2.0}{2 \times 3.5 \times 0.2 \times 2138} = 0.007978$$

$$a_0^1 = \frac{d_0^1}{\tau_1 J} = -\frac{8992.9}{2138 \times 3.5} = -1.1884$$

Similarly the coefficients $(a_4, a_5, a_6, \dots, a_9)^i$ appearing in nonlinear differential equation (3.1.1) and (3.1.2) are calculated below for the system S_n^1 (3.1.3).

$$G^1(\omega) = -6254.5 + 374.5 \omega + 2.9844 \omega^2$$

$$a_4^1 = 2 \frac{d_4^1}{J} = \frac{2 \times 2.9844}{2138} = 0.002791$$

$$a_5^1 = \frac{d_3^1}{J} + \frac{1}{\tau_1} + \frac{1}{\tau_2} = \frac{374.5}{2138} + \frac{1}{3.5} + \frac{1}{3.0} = 0.7941$$

$$a_6^1 = \frac{1}{\tau_1 \tau_2} + \frac{d_3^1}{\tau_1 J} = \frac{1}{3 \times 3.5} + \frac{374.5}{2138} = 0.1452$$

$$a_7^1 = \frac{d_4^1}{\tau_1 J} = \frac{2.9844}{3.5 \times 2138} = 0.0003988$$

$$a_8^1 = \frac{k_3 k_1}{2L_1 r_2 J} = \frac{11.94 \times 2.0}{2 \times 3.5 \times 0.2 \times 2138} = 0.007978$$

$$a_9^1 = \frac{d_2^1}{\tau_2 J} = \frac{-6254.5}{3.5 \times 2138} = -0.8358 .$$

A. 2 Determination of C^i ($i = 1, \dots, 4$)

C^i is the additional voltage (over and above the steady state voltage \bar{v}_1^i) to be supplied for the systems \bar{S}_i or S_n^i for controlling the disturbances lying in R_o^i for each i in such a way that the resulting optimal trajectories lie in the set R_i .

It may be noted that it is difficult to determine the exact bounds on u^i that satisfies the above requirements. However it might be possible to determine the appropriate values of C^i from the practical considerations such as reliability and safety of operation of the field system. In the Ward-Leonard system considered, the rated field voltage of each generator is 750 v. and the two field coils are connected in series as shown in fig. 2.1.1. The calculations in chapters II and III show that the maximum steady state voltage (\bar{v}_1^i) required for the top operating speed 60 r/s is 1590 volts, which already exceeded the rated value (1500 v). From the reliability of operation of field system and to reduce the high magnitudes of induced voltages in the fields due to switching a value of $C^i = 150$ v appears to be fairly acceptable figure for the operating speed 60 r/s. It may be noted that C^i need not be the same for all the operating speeds. However, it is convenient to choose the above value i. e. $C^i = 150$ v for all i so as to reduce further complications in the design of feed back controller.

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