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SOME CHARACTERIZATIONS OF DUALITY IN B*-ALGEBRAS

A thesis submitted

by

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Abstract

The purpose of the thesis is to assemble together the various known characterizations of duality in B*-algebras. Dual B*-algebras have first been studied by I. Kaplansky. He obtained several characterizations of duality in B*-algebras. He showed for example that a B*-algebra is dual if and only if it is a closed *-subalgebra of the algebra LC(H) of all compact linear operators on a complex Hilbert space H . Also that a B*-algebra A is dual if and only if the socle of A is dense in A .

The rest of the thesis is concerned mainly with results obtained by B. J. Tomiuk, T. Ogasawara and K. Yoshinaga on dual B*-algebras. We show that a B*-algebra is dual if and only if it is complemented or w.c.c.. We also show that a B*-algebra A is dual if and only if every maximal commutative *-subalgebra of A is dual. We end the thesis with a discussion about the successive conjugate spaces of a dual B*-algebra.

Introduction

The aim of this thesis is to give an account of the various characterizations of duality in B*-algebras. Dual B*-algebras have first been introduced and studied by I. Kaplansky.

In Chapter I, we gather together some definitions and basic results in B*-algebras which are used throughout the thesis. In Chapter II , we present I. Kaplansky's work on dual B*-algebras. We show that a simple B*-algebra A is dual if and only if A is *-isomorphic to the algebra $LC(H)$ of all completely continous linear operators on a Hilbert space H . Using this fact, we prove the so-called structure theorem for dual B*-algebras; namely, for every dual B*-algebra A there exists a family $\{ H_\lambda : \lambda \in \Lambda \}$ of Hilbert spaces H_λ such that A is *-isomorphic to the B* (∞)-sum of the family $\{ LC(H_\lambda) : \lambda \in \Lambda \}$. It follows from this theorem that a B*-algebra A is dual if and only if A is *-isomorphic to a closed *-subalgebra of $LC(H)$ for some Hilbert space H . We also include a result due to F. Bonsall and A. W. Goldie which states that an annihilator B*-algebra is dual.

Chapter III is concerned mainly with results obtained by B. J. Tomiuk, T. Ogasawara and K. Yoshinaga on dual B*-algebras. Here we discuss two other types of Banach algebras;

namely, the complemented and w.c.c. Banach algebras. We show that a B*-algebra is dual if and only if it is complemented or w.c.c.. We also show that a B*-algebra A is dual if and only if every maximal commutative *-subalgebra of A is dual.

Chapter IV is devoted to the study of successive conjugate spaces of a dual B*-algebra. We show that the conjugate space of a dual B*-algebra A is isometrically isomorphic to a dual and w.c.c. A*-algebra which is a dense two-sided ideal of A .

Chapter I

B*-Algebra

§ I. Definitions and terminology

Let A be an algebra over the complex field C, a map $x \longrightarrow x^*$ of A onto itself is called an involution provided the following conditions are satisfied :

(i) $(x^*)^* = x$;

(ii) $(x + y)^* = x^* + y^*$;

(iii) $(\lambda x)^* = \bar{\lambda} x^*$;

(iv) $(xy)^* = y^* x^*$,

for all x, y in A and for all λ in C. An algebra with an involution is called a *-algebra. A subset $S \subset A$ is called self-adjoint if $S^* = S$, where $S^* = \{x^* : x \in S\}$. A self-adjoint subalgebra is called a *-subalgebra. Similarly, we define a *-ideal. A homomorphism φ from a *-algebra A into another *-algebra B is called a *-homomorphism if $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. An element x of a *-algebra such that $x^* = x$ is called self-adjoint (s. a.). In a *-algebra, every element x has a unique representation of the form $x = h + ik$ where h, k are s. a. elements ; in fact,

$$h = \tfrac{1}{2}(x + x^*), \quad k = -\tfrac{1}{2}i(x - x^*) .$$

A Banach algebra is an associative algebra A over the complex field C on which there is defined a norm $\|x\|$ such that :

(i) A is a Banach space under the norm $\|x\|$,

(ii) $\|xy\| \leqq \|x\|\|y\|$ $(x , y \in A)$.

Clearly (ii) gives the continuity of multiplication. From now on, all algebras under consideration will be complex algebras, unless mentioned otherwise.

A Banach algebra with an involution is called a <u>Banach *-algebra</u>. A Banach *-algebra A such that $\|x^*x\| = \|x\|^2$ $(x \in A)$, is called a <u>B*-algebra</u>. Clearly, every closed *-subalgebra of a B*-algebra is again a B*-algebra. An element x of a B*-algebra A is called positive if x is of the form $x = yy^*$ $(y \in A)$. It can be shown that every positive element in a B*-algebra has a unique positive square root ([6, Theorem (1.6.1)]). We can verify easily that $B(H)$, the algebra of all bounded linear operators on a Hilbert space H is a B*-algebra, under the operator bound norm and the operation of taking the adjoint as its involution. It can be shown that every B*-algebra is isometrically *-isomorphic to a closed *-subalgebra of $B(H)$, for some Hilbert space H ([12, Theorem (4.8.11)]) .

A Banach *-algebra in which there is defined a second norm $|x|$ which satisfies, in addition to the multiplication $|xy| \leqq |x|\,|y|$, the B*-condition $|x|^2 = |x^*x|$, is called an <u>A*-algebra</u>. This second norm will be called an <u>auxiliary</u>

norm. Note that completeness of the algebra in the auxiliary norm is not required.

Let A be an algebra. We define an operation on A, called the circle operation, by the following relation

$$x \circ y = x + y - xy \quad (x , y \in A) .$$

This operation is associative and has 0 as an identity element. An element of A which has a left (resp. right) inverse relative to the circle operation is said to be left (resp. right) quasi-regular. If it is both left and right quasi-regular, then it is called quasi-regular. An element which is not (left, right) quasi-regular is called (left, right) quasi-singular. If A has an identity element e, then an element x is (left, right) quasi-regular if and only if (e - x) is (left, right) regular. Similarly, x is (left, right) quasi-singular if and only if (e - x) is (left, right) quasi-singular.

Let A be an algebra, and let x be any element of A. The spectrum of x in A is the set $Sp_A(x)$ of all complex numbers λ such that $\lambda^{-1}x$ is quasi-singular, plus zero if x is singular. We shall frequently write Sp(x) in place of $Sp_A(x)$ for the spectrum of an element x in A whenever it is understood clearly what algebra is involved. If A is a Banach algebra, then Sp(x) is a non-empty, bounded closed subset of the complex field C ([12, Theorem (1.6.4)]).

Let A be a Banach algebra. The spectral radius $\nu(x)$ of x in A is defined by $\nu(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$. It follows that $\nu(x) = \max \{ |\lambda| : \lambda \in Sp(x) \}$, and if A is a Banach $*$-algebra then $Sp(x^*) = \overline{Sp(x)}$ (the complex conjugate of Sp(x)) ([12, Lemma (4.1.1)]). If x is a s.a. element of a B^*-algebra, then clearly $\nu(x) = \|x\|$. An element x is said to be topologically-nilpotent if $\nu(x) = 0$. An ideal in a Banach algebra is called a topologically -nil ideal provided it is contained in the set N of all topologically-nilpotent elements.

By a representation π of a Banach algebra A on a normed linear space X , we shall mean a continuous homomorphism $a \longrightarrow \pi(a)$ of A into B(X) , the algebra of all bounded linear operators on X . A representation π of A on X is strictly irreducible if (0) and X are the only linear subspaces of X invariant under $\pi(a)$, for all $a \in A$. A primitive ideal is the kernel of a strictly irreducible representation. Clearly, every primitive ideal is a closed two-sided ideal of A . A Banach algebra is called primitive if A has a faithful irreducible representation. Also, if P is a primitive ideal, the quotient algebra A/P is primitive. (See [12, p.60].)

A left ideal I is said to be modular if there exists $e \in A$ such that $A(1 - e) \subset I$. In other words, the element e is a right identity for A modulo I . Similarly, a right ideal J is modular if there exists a left identity for A modulo J.

Let A be an algebra. An ideal I of A is called <u>minimal</u> if it is different from (0) and does not contain properly any ideal of the same type other than (0). It is called <u>maximal</u> if it is different from the algebra A and is not properly contained in any ideal of the same type other than A. The <u>left</u> (<u>right</u>) <u>socle</u> of the algebra A is the smallest left (right) ideal of A which contains all minimal left (right) ideals of A. If the left and right socles exist and are equal, then the resulting two-sided ideal is called the <u>socle</u> of A. It can be shown that in a B*-algebra A, if the left (right) socle of A exists, then it is equal to the right (left) socle.

The <u>radical</u> R of a Banach algebra A is equal to the intersection of all primitive ideals. It can be shown that R is a topologically-nil left (right) ideal equal to the sum of all topologically-nil left (right) ideals. And if $R \neq A$, it is equal to the intersection of all maximal modular left (right) ideals of A ([12, § 2.3]). A Banach algebra A is called <u>semi-simple</u> if the radical $R = 0$. A is called <u>simple</u> if it is semi-simple and does not contain any closed two-sided ideals other than (0) and itself.

Every B*-algebra A is semi-simple. In fact, let $x \in R$.. Since $x^*x \in R$ and x^*x is s.a., therefore $0 =$

$\nu(x^*x) = \|x^*x\| = \|x\|^2$, which gives $x = 0$. Hence $R = (0)$.

Let $\{ A_\lambda : \lambda \in \Lambda \}$ be a family of B*-algebras and let $(\sum_\lambda A_\lambda)_0$ be the class of all functions (x_λ) defined on Λ such that $x_\lambda \in A$ for each λ and, for every $\varepsilon > 0$, the set $\{ \lambda : \|x_\lambda\| \geq \varepsilon \}$ is finite. $(\sum A_\lambda)_0$ is closed under the following algebraic operations :

$$(x_\lambda) + (y_\lambda) = (x_\lambda + y_\lambda) ;$$
$$\alpha(x_\lambda) = (\alpha x_\lambda) ;$$
$$(x_\lambda)(y_\lambda) = (x_\lambda y_\lambda) ;$$
$$(x_\lambda)^* = (x_\lambda^*) ,$$

for all (x_λ) , $(y_\lambda) \in (\sum A_\lambda)_0$ and all $\alpha \in C$. It is easy to see that with these operations, $(\sum A_\lambda)_0$ is a B*-algebra under the norm given by

$$\|(x_\lambda)\| = \sup_\lambda \|x_\lambda\| .$$

It is called the B* (∞)-sum of A_λ . It is easy to see that the set of all functions (x_λ) which are zero except for a finite number of the indices λ is dense in $(\sum A_\lambda)_0$.

Let $\{ H_\lambda : \lambda \in \Lambda \}$ be a family of Hilbert spaces and let H be the family of all functions (ξ_λ) defined on Λ such that

(i) for each $\lambda \in \Lambda$, $\xi_\lambda \in H_\lambda$,

(ii) $\sum_\lambda \|\xi_\lambda\|^2 < \infty$.

Under the usual operations of addition and multiplication by scalars for functions, H is a vector space. It is a Hilbert space under the inner product given by the relation

$$((\xi_\lambda),(\eta_\lambda)) = \sum_\lambda (\xi_\lambda, \eta_\lambda) .$$

H is called the Hilbert sum of the Hilbert spaces H_λ and denote by

$$H = \bigoplus H_\lambda .$$

§ 2. Some useful results

Let A be a commutative Banach algebra. Let Ω be the set of all non-zero homomorphisms of A into the complex field. Each $x \in A$ defines the function $\hat{x}$ on Ω given by $\hat{x}(\varphi) = \varphi(x)$ $(\varphi \in \Omega)$. We have the following spectral mapping relation :

$$Sp(x) - \{0\} \subset \{ \hat{x}(\varphi) : \varphi \in \Omega \} \subset Sp(x) .$$

Each $\varphi \in \Omega$ is a bounded linear functional on A of norm $\leq$ 1 . Thus Ω is a subset of the closed unit ball S of the conjugate space A' of the Banach algebra A . It is easy to see that the weakest topology on Ω for which all the functions $\hat{x}$ are continuous coincides with the relative topology which Ω has as a subset of A' if A' is given the weak*-topology $\sigma(A',A)$. Since S is $\sigma(A',A)$ compact (Alaoglu theorem), and since $\Omega \cup \{0\}$ is a $\sigma(A',A)$ closed subset of S , it follows that Ω is a locally compact

Hausdorff space. Ω with this topology is called the carrier space of A. (See [12, Chapter III, § 1].)

Let Ω be a locally compact Hausdorff space. A continuous function f defined on Ω is said to vanish at infinity if the set $\{ p : |f(p)| \geqq \varepsilon \}$ is compact for every $\varepsilon > 0$. Let $C_o(\Omega)$ be the set of all complex-valued continuous functions which vanish at infinity. With the usual operations for functions, $C_o(\Omega)$ is a B*-algebra under the sup-norm. If Ω is compact, then $C_o(\Omega)$ agrees with $C(\Omega)$, the algebra of all continuous complex-valued functions on Ω. $C_o(\Omega)$ has an identity element if and only if Ω is compact.

Theorem (1.2.1). (Gelfand Theorem) Every commutative B*-algebra A is isometrically *-isomorphic to $C_o(\Omega)$, where Ω is the carrier space of A.
Proof : cf. [6, Theorem (1.4.1)].

Theorem (1.2.2). Let Φ be an algebraic *-isomorphism of a B*-algebra A into a dense subalgebra of a B*-algebra B. Then Φ maps A onto B and is an isometry.
Proof : cf. [7, Theorem 6.4].

Let A be a Banach algebra. An idempotent e in A such that eAe is a division algebra is said to be minimal. Two idempotents e_1 and e_2 in A such that $e_1e_2 = e_2e_1 = 0$ are said to be orthogonal. Since A is a complex

Banach algebra, $eAe = Ce$ for every minimal idempotent e. If A is semi-simple, then an idempotent e is minimal if and only if Ae (eA) is a minimal left (right) ideal.

Theorem (1.2.3). Let A be a B*-algebra and let I be a minimal left ideal. Then there exists a s.a. idempotent e such that $I = Ae$.

Proof : [12, p.261] Let x be any non-zero element of I and set $h = x^*x$. Then h is a non-zero s.a. element of I. Also, since I is minimal and $h^2 \neq 0$, we have $Ih = I$. Now let z be any element of I such that $zh = 0$. If $z \neq 0$, then $I = Az$. But $I = Ih = Azh = (0)$, a contradiction. In other words, the left annihilator of h in I is zero. Now, since $I = Ih$, there exists $u \in I$ such that $uh = h$. Moreover $(u^2 - u)h = 0$ and hence $u^2 = u$. Since $h \in I$ and $I = Iu$, it follows that $h = hu$. Applying the involution to the equation, we obtain $h = u^*h$. Define $e = u^*u$. Then $e \in I$, $eh = h$, and thus $(e^2 - e)h = 0$, which implies that $e^2 = e$. Furthermore, e is a non-zero element of I so that $I = Ae$. This completes the proof.

The rest of the section contains some spectral theory of bounded linear operators on a Hilbert space H. Most of the results are taken from [14]. An element T in $B(H)$ is called compact (completely continuous) if it takes bounded sets in H into relatively compact sets. Equivalently, T is compact if and only if it maps every weakly convergent sequence of vectors into a strongly convergent sequence.

Notation :

(i) For $g, h \in H$, $g \otimes \bar{h} \in B(H)$ is the linear operator defined by $g \otimes \bar{h}(x) = (x,h)g$ $(x \in H)$; where (,) denotes the given inner product in H. It is clear that $\|g \otimes \bar{h}\| = \|g\| \cdot \|h\|$.

(ii) $F(H)$ = the set of all linear operators on H of finite-ranks.

(iii) $LC(H)$ = the set of all compact linear operators on H .

It can be shown that $LC(H)$ is a closed $*$-subalgebra of $B(H)$ which contains $F(H)$ as a dense two-sided ideal [14, Theorem 5, p.13] . Every compact linear operator T has a unique representation of the form $T = \sum \lambda_i \varphi_i \otimes \overline{\Psi}_i$ where $\{\varphi_i\}$ and $\{\Psi_i\}$ are both orthonormal sequences in H and λ_i's are positive real numbers with $\lambda_i \longrightarrow 0$; moreover, $\|T\| = \sup_i \lambda_i$. This representation is called the polar decomposition of T . (See [14, Theorem 7, p.18].)

Lemma (1.2.4). For every T in $LC(H)$, there exist T_1 and T_2 in $LC(H)$ such that $T = T_1 T_2$.

Proof : Let $T = \sum_1^{\infty} \lambda_i \varphi_i \otimes \overline{\Psi}_i$ be given in its polar decomposition. Let $T_n = \sum_1^{n} \lambda_i \varphi_i \otimes \overline{\Psi}_i$, $A_n = \sum_1^{n} \sqrt{\lambda_i}\, \Psi_i \otimes \overline{\Psi}_i$ and

$B_n = \sum_{1}^{n} \sqrt{\lambda_i}\,\varphi_i \otimes \overline{\varphi}_i$. Clearly, A_n and B_n belong to $F(H)$. Since $\|T_n - T\| = \sup_{i>n} \lambda_i$, $T_n \longrightarrow T$. Since $\|A_n - A_m\| = \sup_{n<i\leq m} \sqrt{\lambda_i}$, therefore $\{A_n\}$ is a Cauchy sequence and so it converges to an operator T_1 in $LC(H)$. Similarly, B_n converges to an operator T_2 in $LC(H)$. Since $(\varphi_i \otimes \overline{\varphi}_i)(\varphi_j \otimes \overline{\varphi}_j) = \delta_{ij}\varphi_i \otimes \overline{\varphi}_j$, we have $A_n B_n = T_n$ and so $A_n B_n \longrightarrow T$. But we also have $A_n B_n \longrightarrow T_1 T_2$. Hence $T = T_1 T_2$.

Theorem (1.2.5). $LC(H)$ is a simple B*-algebra.

Proof : Let I be any non-zero closed two-sided ideal of $LC(H)$. Therefore there exists $T \in I$ such that $Tx = y \neq 0$ for at least one $x \in H$. It is easy to verify that $T(x \otimes \overline{y}) = y \otimes \overline{y}$. Hence $y \otimes \overline{y} \in I$. For g , $h \in H$,

$$\left(g \otimes \frac{\overline{y}}{\|y\|^2}\right)(y \otimes \overline{y})\left(\frac{y}{\|y\|^2} \otimes \overline{h}\right) = g \otimes \overline{h} .$$

Thus $g \otimes \overline{h} \in I$. Since $F(H)$ is generated by operators of the form $g \otimes \overline{h}$ where g , $h \in H$, it follows that $F(H) \subset I$ and hence $I = LC(H)$.

Chapter II

Dual B*-algebras

§ I. Introduction

Let A be a Banach algebra and E an arbitrary subset of A. Denote :

$$L(E) = \{ x \in A : xE = (0) \} .$$
$$R(E) = \{ x \in A : Ex = (0) \} .$$

The sets $L(E)$ and $R(E)$ are called the left and right annihilators of E, respectively. It is clear that $L(E)$ is a closed left ideal and $R(E)$ is a closed right ideal of A ; $E \subset L(R(E))$ and $E \subset R(L(E))$. Clearly if $E_1 \subset E_2$, then $L(E_1) \supset L(E_2)$ and $R(E_1) \supset R(E_2)$. If I is a left ideal of A and such that $I = L(R(I))$, I is called an annihilator left ideal. Similarly, we define an annihilator right ideal of A.

Definition (2.1.1). A Banach algebra A is called an annihilator algebra if, for arbitrary closed left ideal I and closed right ideal J in A, both of the following conditions are satisfied :

(i) $R(I) = (0)$ if and only if $I = A$,

(ii) $L(J) = (0)$ if and only if $J = A$.

Definition (2.1.2). A Banach algebra A is called a dual

<u>algebra</u> if every closed left (right) ideal in A is an annihilator ideal.

Dual algebras were first introduced by I. Kaplansky in [7]. Most of the results presented in this chapter are due to him.

§ 2. The algebra $LC(H)$

Throughout this section, H will denote a complex Hilbert space with inner product $(\, , \,)$ and, for every closed subspace S of H, $S^{\perp}$ will denote the orthogonal complement of S in H. If D is a subset of a topological space X, $cl(D)$ will denote the closure of D in X.

<u>Notation</u> : For every closed subspace S of H, let $\mathcal{J}(S) = \{ T \in LC(H) : T(h) \in S \text{ for } h \in H \}$. For every closed right ideal J of $LC(H)$, let $\mathcal{S}(J)$ be the smallest closed subspace of H that contains the range $T(H)$ of each operator T in J.

<u>Lemma (2.2.1)</u>. For every closed right ideal J of $LC(H)$, $J = \mathcal{J}(\mathcal{S}(J))$; and for every closed subspace S of H, $\mathcal{J}(S)$ is a closed right ideal and $S = \mathcal{S}(\mathcal{J}(S))$.

<u>Proof</u> : [1] It is clear that $\mathcal{J}(\mathcal{S}(J))$ contains J. Let T be any element in $\mathcal{J}(\mathcal{S}(J))$, and let $\{T_n\}$ be a sequence of operators on H of finite rank, such that $T_n \longrightarrow T$. Let P be the projection on $\mathcal{S}(J)$. Since $PT = T$, we have

$PT_n \longrightarrow T$. Since PT_n is of finite rank and whose range is contained in $\mathcal{S}(J)$, it follows from [12, Theorem (2. 4.18)] that $PT_n \in J$ for all n . Since J is a closed right ideal, thus $T \in J$. This shows that $\mathcal{J}(\mathcal{S}(J)) = J$. Let S be a closed subspace of H , it is easy to prove that $\mathcal{J}(S)$ is alclosed right ideal of $LC(H)$ and so $\mathcal{J}(S) = \mathcal{J}(\mathcal{S}(\mathcal{J}(S)))$. It is clear that $\mathcal{S}(\mathcal{J}(S)) \subset S$. Now, if $m \in S$, then $m \otimes \bar{m} \in \mathcal{J}(S)$ and therefore $m \in \mathcal{S}(\mathcal{J}(S))$. Hence $S \subset \mathcal{S}(\mathcal{J}(S))$ and so $\mathcal{S}(\mathcal{J}(S)) = S$.

<u>Corollary (2.2.2)</u>. For every closed left ideal I of $LC(H)$ $I = \{ a \in LC(H) : a(\mathcal{S}(I^*)^{\perp}) = (0) \}$.

<u>Proof</u> : Since I^* is a closed right ideal, the result follows from Lemma (2.2.1) and the fact that, for all a $B(H)$, $Ker(a) = (cl(Range\ a^*))^{\perp}$.

<u>Remark</u> : Lemma (2.2.1) shows that $J \longrightarrow \mathcal{S}(J)$ defines a one-to-one correspondence between the closed right ideals of $LC(H)$ and the closed subspaces of H ; moreover if J_1 , J_2 are closed right ideals such that $J_1 \subset J_2$, then the corresponding closed subspaces S_1 and S_2 in H satisfy the inclusion $S_1 \subset S_2$.

<u>Theorem (2.2.3)</u>. $LC(H)$ is a simple dual B*-algebra.

<u>Proof</u> : [7] Theorem (1.2.5) shows that $LC(H)$ is a simple B*-algebra. Let J be a closed right ideal of $LC(H)$. By Lemma (2.2.1), $J = \mathcal{J}(\mathcal{S}(J))$. Clearly, $L(J) \supset \{ a \in LC(H) : a(\mathcal{S}(J)) = (0) \}$. Now if $b \notin \{ a \in LC(H) : a(\mathcal{S}(J)) = (0) \}$,

then there exists $x \in \mathcal{S}(J)$ such that $b(x) = y \neq 0$. But by Lemma (2.2.1), $x \otimes \bar{y} \in J$ and $b(x \otimes \bar{y}) = y \otimes \bar{y} \neq 0$; so that $b \notin L(J)$. Hence $L(J) \subset \{ a \in LC(H) : a(\mathcal{S}(J)) = (0) \}$ and so $L(J) = \{ a \in LC(H) : a(\mathcal{S}(J)) = (0) \}$. To show that $R(L(J)) = J$, it clearly suffices to prove that $R(L(J)) \subset J$. Let $a \in R(L(J))$. If $a \notin J$, then there exists $x \in H$ such that $a(x) = y \notin \mathcal{S}(J)$ by Lemma (2.2.1). Let $y = y_1 + y_2$, where $y_1 \in \mathcal{S}(J)$ and $y_2 \in \mathcal{S}(J)^{\perp}$; clearly, $y_2 \neq 0$. Since $y_2 \otimes \bar{y}_2(z) = 0$ for all $z \in \mathcal{S}(J)$; and since $L(J) = \{ a \in LC(H) : a(\mathcal{S}(J)) = (0) \}$, we have $y_2 \otimes \bar{y}_2 \in L(J)$. But $(y_2 \otimes \bar{y}_2)a \neq 0$; in fact, $(y_2 \otimes \bar{y}_2)a(x) = (y_2 \otimes \bar{y}_2)y = (y_2 , y_2)y_2 \neq 0$. This is a contradiction since $a \in R(L(J))$. Hence $R(L(J)) = J$, which completes the proof.

Lemma (2.2.4). Every dual B*-algebra A contains minimal left and right ideals.

Proof : Since A is semi-simple, it contains a maximal modular right ideal M . (A contains a right quasi-singular element u and $\{ x - ux : x \in A \}$ is a modular right ideal not containing u .) We claim that $L(M)$ is a minimal left ideal. In fact, let I be a closed left ideal properly contained in $L(M)$. By the duality of A , we have $R(I) \supsetneq R(L(M)) = M$. Hence, by the maximality of M , we have $R(I) = A$, and so $I = (0)$. This completes the proof.

Theorem (2.2.5). Let A be a B*-algebra. Then A is a simple dual B*-algebra if and only if it is isometrically *-

isomorphic to LC(H) , for some Hilbert space H .

Proof : Let A be a simple dual B*-algebra. Lemma (2.2.4) shows that A contains a minimal left ideal H . Theorem (1.2.3) says that $H = Ae$, where e is a s.a. idempotent; moreover, $eAe = Ce$. Therefore for any $x , y \in H$, there exists a complex number (x,y) such that $y^*x = (x,y)e$. It is easy to verify that (x,y) is an inner product on H and moreover, the norm $\|x\|' = (x,x)^{\frac{1}{2}}$, $x \in H$ define by the inner product (,) in H is equal to the given norm in A ; i. e., $\|x\|' = \|x\|$ $(x \in H)$. Hence H is a Hilbert space. Since A is simple and H is a minimal left ideal, the left regular representation T of A on H , $T_a : a \longrightarrow ax$ $(x \in H)$ is faithful and irreducible. Also, $(T_a x,y)e = y^*ax = (a^*y)^*x = (x,a^*y)e = (x,T_{a^*}y)e$. Thus, $T_{a^*} = (T_a)^*$ which shows that T is a *-representation. We claim that the image of A by this representation contains the set F(H) of all operators of finite rank on H ; in fact, since $(g \otimes \overline{h})(x) = (x,h)g = (x,h)ge = g(x,h)e = gh^*x = T_{gh^*}(x)$ $(x \in H)$, we have $T_{gh^*} = g \otimes \overline{h}$, for all $g , h \in H$. Hence $F(H) \subset T(A)$. Since the operator bound norm $\|T_a\| \leq \|a\|$ and since A has the minimal norm property ([2,Theorem 10]) we have $\|T_a\| = \|a\|$ $(a \in A)$. Hence T is an isometry so that T(A) contains LC(H) . But A is a simple algebra, so that T(A) = LC(H) . Thus A is isometrically *-isomorphic to LC(H) . The converse follows from Theorem (2.2.3) and this completes the proof.

Remark : From the proof of the above theorem, we see that every simple B*-algebra with a minimal left (right) ideal is of the form LC(H) .

Theorem (2.2.6). Let A be a primitive B*-algebra with a dense socle, then A is *-isomorphic to LC(H) for some Hilbert space H .

Proof : Let H be a minimal closed left ideal of A . By the proof of Theorem (2.2.5), H can be given an inner product which induces the given norm on H , and the left regular representation $T : A \longrightarrow B(H)$ of A on the Hilbert space H is a *-representation. Moreover, by [12, Corollary (2.4.16)] , T is a faithful irreducible *-representation. By [12, Theorem (2.4.12)] , T maps the socle of A onto the operators of finite rank. Since the socle is dense in A and T is continuous, T maps A into LC(H) . Hence, by Theorem (1.2.2), A is a *-isomorphic to LC(H) .

§ 3. Dual B*-algebras

Theorem (2.3.1). If $\{ A_\lambda : \lambda \in \Lambda \}$ is a family of dual B*-algebras, then $A = (\sum A_\lambda)_0$ is dual.

Proof : [12, p.271] For any element $x \in A_\lambda$, let $\overline{x} = (x_\lambda)$ be the element of A such that $x_\lambda = x$ and $x_\mu = 0$ for $\mu \neq \lambda$. Then $x \longrightarrow \overline{x}$ is obviously a *-isomorphism of A_λ into A . Since these are B*-algebras, this isomorphism is an isometry (Theorem (1.2.2)). In this way, we can (and do) identify A_λ with a subalgebra (actually an ideal) of A. Now let I be a closed left ideal in A and denote by I_λ the image of I in A under the mapping $(x_\lambda) \longrightarrow x_\lambda$ of A onto A_λ. Then it is easily verify that I_λ is a left ideal in A and that $AI_\lambda \subset I$. Since I is closed, we have $I_\lambda \subset I$; in fact, if $x \in I_\lambda$, then there exists $y = (y_\lambda) \in I$ such that $y_\lambda = x$. Since $\bar{x}^*\bar{x} = \bar{x}^*y \in I$, by [12, Corollary (4.9.3)], $\bar{x} \in I$. Identifying $\bar{x}$ with x, we have $x \in I$. It follows now that $I_\lambda = I \cap A_\lambda$ and, in particular, that I_λ is a closed left ideal of A_λ. Next let E be an arbitrary subset of A and denote by E_λ the image of E in A_λ under the mapping $(x_\lambda) \longrightarrow x_\lambda$. Denote the annihilator operations in A_λ by L_λ and R_λ. Then $(L(E))_\lambda = L_\lambda(E_\lambda)$ and $(R(E))_\lambda = R_\lambda(E_\lambda)$. Let us show for example that $(L(E))_\lambda = L_\lambda(E_\lambda)$. Let $x \in L_\lambda(E_\lambda) \Longrightarrow \bar{x}(y_\lambda) = xy_\lambda = 0$ for all $(y_\lambda) \in E \Longrightarrow \bar{x} \in L(E)$. Since $x \in A_\lambda$; $x \in (L(E))_\lambda$. Thus $L_\lambda(E_\lambda) \subset (L(E))_\lambda$. Conversely, suppose $x \in (L(E))_\lambda$. Then there exists $(x_\lambda) \in L(E)$ such that $x_\lambda = x$. This implies that $xE_\lambda = (0)$ which gives $x \in L_\lambda(E_\lambda)$. Thus $(L(E))_\lambda = L_\lambda(E_\lambda)$. Similarly, we can show that $(R(E))_\lambda = R_\lambda(E_\lambda)$. Since A_λ is dual, we obtain $(L(R(I)))_\lambda = L_\lambda(R_\lambda(I_\lambda)) = I_\lambda$; in particular $(L(R(I)))_\lambda \subset I$. Thus I contains the linear subspace B of A generated by the sets $(L(R(I)))_\lambda$ $(\lambda \in \Lambda)$. But the subspace B is dense in $L(R(I))$; in fact, suppose $(x_\lambda) \in L(R(I))$ and denote the non-zero x_λ by $x_{\lambda_1}, x_{\lambda_2}, \cdots$. Identifying x_{λ_n} with the element $\bar{x}_{\lambda_n}$ in A, we see that $y_n = \sum_{1}^{n} x_{\lambda_i} \in B$. Since $\|x_{\lambda_n}\| \longrightarrow 0$ as $n \longrightarrow \infty$ and since $\|(x_\lambda) - y_n\| = \sup_{i>n} \|x_{\lambda_i}\|$, $y_n \longrightarrow (x_\lambda)$ in A. Hence B is dense in $L(R(I))$. Since I is closed and containing B and, since $cl(B) = L(R(I)$, we have $L(R(I)) \subset I$. Now it follows that $I = L(R(I))$. Similarly, if J is a closed right ideal of A, then $J = R(L(J))$.

Corollary (2.3.2). Let $\{ H_\lambda : \lambda \in \Lambda \}$ be a family of Hilbert spaces. Then the B* (∞)-sum $(\sum LC(H_\lambda))_0$ is dual.

Proof : The result follows from Theorems (2.2.3) and (2.3.1).

Lemma (2.3.3). Let A be a B*-algebra and let I and J be closed two-sided ideals of A such that $I \cap J = (0)$ and $I + J$ is dense in A. Then $I + J = A$ and $\|m + n\| = \max(\|m\|, \|n\|)$ for all $m \in I$ and $n \in J$.

Proof : [7] Let A_0 be the direct sum $I \oplus J$ of I and J with the norm defined by $\|m \oplus n\| = \max(\|m\|, \|n\|)$. Clearly A_0 is a B*-algebra under the usual operations for direct sum and the involution given by $m \oplus n \longrightarrow m^* \oplus n^*$. The mapping $\varphi : m \oplus n \longrightarrow m + n$ of A_0 into A defines an algebraic *-isomorphism of A_0 into a dense subalgebra of A. That φ is a *-preserving linear map is clear and, since $I \cap J = (0)$, it is also one-to-one. That it preserve multiplication follows from the fact that $IJ \subset I \cap J = (0)$; in fact

$$\varphi((m_1 \oplus n_1)(m_2 \oplus n_2)) = \varphi(m_1 m_2 \oplus n_1 n_2) = m_1 m_2 + n_1 n_2$$
$$= (m_1 + n_1)(m_2 + n_2) = \varphi(m_1 \oplus n_1) \varphi(m_2 \oplus n_2).$$

Hence, by Theorem (1.2.2), A_0 is isometrically $*$-isomorphic to A .

Lemma (2.3.4). Let A be an annihilator B*-algebra and let I be a closed two-sided ideal of A . Then $I \cap L(I) = (0)$, $R(I) = L(I)$ and $I + L(I) = A$.

Proof : [12, p.99] Let $x \in I \cap L(I)$. Since every closed two-sided ideal of A is a $*$-ideal ([12, Theorem (4.9.2)]). $x^* \in I$ and so $xx^* = 0$, which implies that $x = 0$. Thus $I \cap L(I) = (0)$. Since $R(I)$ and $L(I)$ are both closed two-sided ideals and since $x \in R(I)$ if and only if $x^* \in L(I^*) = L(I)$, it follows that $R(I) = L(I)$. Now $cl(I + L(I))$ is a closed two-sided ideal and $L(I + L(I)) = L(cl(I + L(I))$. If $x \in L(I + L(I))$, then $x \in L(I)$ and $x \in L(L(I))$. Then $x^*x = 0$ and hence $x = 0$. Consequently $L(cl(I + L(I)) = (0)$ and so, since A is an annihilator algebra, we have that $I + L(I)$ is dense in A . By Lemma (2.3.3), it follows that $I + L(I) = A$. This completes the proof.

Lemma (2.3.5). Let I be a closed two-sided ideal of an annihilator B*-algebra A . Then every closed left (right) ideal of the B*-algebra I is a closed left (right) ideal of A .

Proof : Let K be a left ideal of I . Since $I \cap L(I) = (0)$, $L(I)K \subset L(I) \cap I = (0)$. Thus $AK \subset (I + L(I))K = IK + L(I)K \subset IK \subset K$, and hence K is a left ideal of A.

Theorem (2.3.6). The following statements are equivalent :

(i) A is an annihilator B*-algebra.

(ii) A is dual.

(iii) The socle of A is dense in A .

(iv) A is the B* (∞)-sum of $\{ LC(H_\alpha) : \alpha \in \Omega \}$, where H_α are Hilbert spaces.

Proof : (i) $\Longrightarrow$ (ii). Let A be an annihilator B*-algebra and let M be any maximal closed left ideal of A . Let u be any non-zero element of R(M) . Then $M \subset L(R(M)) \subset L(uA)$. Since $L(uA) = \{ x \in A : xu = 0 \}$ is a proper closed left ideal of A , we have $M = L(R(M))$ by the maximality of M . Hence every maximal closed left ideal of A is an annihilator ideal. Let I be a closed left ideal of A . Let $\{ M_\lambda : \lambda \in \Lambda \}$ be the family of all maximal closed left ideals such that $I \subset M_\lambda$ for each $\lambda \in \Lambda$. By [6, Theorem (2.9.5)] , $I = \bigcap_\lambda M_\lambda$. $L(R(I)) \subset L(R(M_\lambda)) = M_\lambda$ $(\lambda \in \Lambda)$. Therefore, $L(R(I)) \subset \bigcap_\lambda M_\lambda = I$ and hence $I = L(R(I))$. Similarly every closed right ideal J of A is an annihilator ideal. This shows that A is dual.

(ii) $\Longrightarrow$ (iii). Let A be a dual B*-algebra and let $\mathfrak{S}$ be the socle of A . Lemma (1.2.3) shows that $\mathfrak{S}$ is the sum of the family $\{ Ae_\gamma : \gamma \in \Gamma \}$, where $\{ e_\gamma \}$ is the family of s.a. minimal idempotents in A . Now, if $x \in R(\mathfrak{S})$, then $x \in R(Ae_\gamma)$, for every $\gamma \in \Gamma$, and so $x \in \bigcap_\gamma R(Ae_\gamma)$. By the duality of A , $\{ R(Ae_\gamma) : \gamma \in \Gamma \}$ is the family of all maximal closed right ideals of A . But every closed maximal right ideal of A is a maximal modular right ideal. Hence

$\{ R(Ae_\gamma) : \gamma \in \Gamma \}$ is the family of all maximal modular right ideals of A. Since A is semi-simple, $\bigcap_\gamma R(Ae_\gamma) = (0)$ and hence $x = 0$. We thus have $R(\mathfrak{S}) = (0)$ which implies that $\mathfrak{S}$ is dense in A.

(iii) $\Longrightarrow$ (iv). Suppose $\mathfrak{S}$ is dense in A. We use the notation in the proof of (ii) $\Longrightarrow$ (iii). Let I be the closed two-sided ideal generated by some Ae_γ. Then I is a minimal closed two-sided ideal of A. In fact, let J be a closed two-sided ideal contained in I. Either $Ae_\gamma \subset J$ or $Ae_\gamma \bigcap J = (0)$. If $Ae_\gamma \subset J$, then clearly $J = I$. If $Ae_\gamma \bigcap J = (0)$, then $Ae_\gamma J \subset Ae \bigcap J = (0)$, so that $Ae_\gamma \subset L(J)$. Since $L(J)$ is a closed two-sided, $I \subset L(J)$. Hence $J \subset L(J)$ and so $J^2 = (0)$, which shows that $J = (0)$. Therefore I is a minimal closed two-sided ideal. By Lemma (2.3.5), Ae_γ is a minimal closed left ideal of the B*-algebra I, and so, by the remark of Theorem (2.2.5), I is isometrically *-isomorphic to $LC(H)$ for some Hilbert space H. Let $\{ I_\lambda : \lambda \in \Omega \}$ be the family of all minimal closed two-sided ideals of A. Since each Ae_γ is contained in some I, $M = \sum_\lambda I_\lambda$ is dense in A; moreover $\sum_\lambda I_\lambda$ is a direct sum since $I_{\lambda 1} \bigcap I_{\lambda 2} = (0)$ for $\lambda_1 \neq \lambda_2$. Let $x \in M$. Since $x = x_{\lambda 1} + \cdots + x_{\lambda n}$, where $x_{\lambda i} \in I_{\lambda i}$ ($i = 1, 2, \cdots, n$), it follows from Lemma (2.3.3) that $\|x\| = \max \{ \|x_{\lambda i}\| : i = 1, 2, \cdots, n \}$. (In fact, by Lemma (2.3.3), $B = \sum_1^n I_{\lambda i}$ is a B*-algebra under the norm given by $\|y\| = \max \{ \|y_{\lambda i}\| : i = 1, 2, \cdots, n \}$, for all $y \in B$.

Hence, in particular, $\|x\| = \max \{ \|x_{\lambda i}\| : i = 1, 2, \cdots, n \}$.) Let M' be the subalgebra of the B*-algebra of the B* (∞)-sum $(\sum_{\lambda} I_\lambda)_0$ consisting of functions which are zero except for a finite number of indices λ ; M' is dense in $(\sum I_\lambda)_0$. Let φ be the mapping of M' into M such that $\varphi((x_\lambda)) = x_{\lambda 1} + \cdots + x_{\lambda n}$ for $(x_\lambda) \in M'$. It is clear that φ is an isometric *-isomorphism of M' onto M . Since M' is dense in $(\sum I_\lambda)_0$, φ has a unique *-isometric extension Φ to all of $(\sum I_\lambda)_0$. Since the range of Φ is in A and contains M and since M is dense in A , Theorem (1.2.2) shows that A is isometrically *-isomorphic to $(\sum I_\lambda)_0$. Since each I_λ is isometrically *-isomorphic to $LC(H_\lambda)$ for some Hilbert space H_λ , it follows that A is isometrically *-isomorphic to $(\sum LC(H_\lambda))_0$.

(iv) $\Longrightarrow$ (i). Suppose $A = (\sum LC(H_\lambda))_0$. Then, by Corollary (2.3.2), A is dual and hence an annihilator algebra.

Corollary (2.3.7). A B*-algebra is dual if and only if its maximal closed left (right) ideals are annihilator ideals.

Proof : Follows from the proof of (i) $\Longrightarrow$ (ii) in the proof of Theorem (2.3.6).

Remark : Statement (iv) of Theorem (2.3.6) is usually referred to as the structure theorem for dual B*-algebras.

§ 4. Duality and closed *-subalgebras of the algebra LC(H)

Lemma (2.4.1). Let $\{ H_\lambda : \lambda \in \Lambda \}$ be a family of Hilbert spaces, let $H = \bigoplus H_\lambda$ the Hilbert sum of H_λ and let $A = (\sum LC(H_\lambda))_0$. Then every element (T_λ) in A can be identified uniquely with an operator $T \in B(H)$ such that $T\xi = (T_\lambda \xi_\lambda)$ (i.e., $T|H_\lambda = T_\lambda$) where $\xi = (\xi_\lambda) \in H$ and $T_\lambda \in LC(H_\lambda)$, and the collection A' of all such operators T in $B(H)$ forms a closed *-subalgebra of the algebra $LC(H)$. The mapping $(T_\lambda) \longrightarrow T$ is an isometric *-isomorphism of A into $LC(H)$.

Proof : Since each T_λ in $(T_\lambda) \in A$ is bounded, the operator T defined on H by $T\xi = (T_\lambda \xi_\lambda)$ $(\xi = (\xi_\lambda) \in H)$, is a bounded linear operator and $\|T\| = \sup \|T_\lambda\|$. It is clear that $(T_\lambda) \longrightarrow T$ is an isometric isomorphism and to show that it is *-isomorphism, we need to show $(T_\lambda^*) \longrightarrow T^*$, since (T_λ^*) is the adjoint of (T_λ) in A . That is, if $T_\lambda = T|H_\lambda$, we need to show that $T_\lambda^* = T^*|H_\lambda$. Suppose that $T^*|H_\lambda$ T_λ' . Then for any $\xi = (\xi_\lambda)$ and $\eta = (\eta_\lambda) \in H$, we have

$$(T\xi, \eta) = \sum (T_\lambda \xi_\lambda, \eta_\lambda) = \sum (\xi_\lambda, T_\lambda^* \eta_\lambda) ,$$

and

$$(\xi, T^*\eta) = \sum (\xi_\lambda, T_\lambda' \eta_\lambda) .$$

Hence

$$\sum (\xi_\lambda, T_\lambda^* \eta_\lambda) = \sum (\xi_\lambda, T_\lambda' \eta_\lambda) ,$$

or

$$\sum (\xi_\lambda, (T_\lambda^* - T_\lambda')\eta_\lambda) = 0 ,$$

for all (ξ_λ) and (η_λ) in H . Hence $T_\lambda^* = T_\lambda'$ for all λ , and so $(T_\lambda) \longrightarrow T$ is a *-isomorphism. In particular, A' is a closed *-subalgebra of $B(H)$ (Theorem (1.2.2)).

We observe that the set B of all elements (T_λ) which are zero except at a finite number of the indices λ, is dense in A. Let B' be the corresponding collection of B in A'. Then B' is dense in A'. Hence to show that $A' \subset LC(H)$, we need to prove every element T in B' is compact. Let

$$\{\xi^{(n)}\} = \{(\xi_\lambda^{(n)})\}$$

be a bounded sequence of elements in H and let $T \in B'$. We shall prove that $\{T\xi^{(n)}\}$ contains a convergent subsequence. We have $T\xi^{(n)} = (T_\lambda \xi_\lambda^{(n)})$ with $T_\lambda \in LC(H_\lambda)$ and $T_\lambda \xi_\lambda^{(n)} = 0$ except at a finite number of the indices λ. Let Γ be the subset of Λ for which $T_\lambda \neq 0$ and let the elements of Γ be $\lambda_1, \lambda_2, \cdots, \lambda_m$. We arrange the elements of $\{T\xi^{(n)}\}$ as follows :

$$T\xi^{(1)} = (T_{\lambda_1}\xi_{\lambda_1}^{(1)}, T_{\lambda_2}\xi_{\lambda_2}^{(1)}, \cdots, T_{\lambda_m}\xi_{\lambda_m}^{(1)}, 0, 0, \cdots)$$

$$T\xi^{(2)} = (T_{\lambda_1}\xi_{\lambda_1}^{(2)}, T_{\lambda_2}\xi_{\lambda_2}^{(2)}, \cdots, T_{\lambda_m}\xi_{\lambda_m}^{(2)}, 0, 0, \cdots)$$

$$\vdots$$

$$T\xi^{(n)} = (T_{\lambda_1}\xi_{\lambda_1}^{(n)}, T_{\lambda_2}\xi_{\lambda_2}^{(n)}, \cdots, T_{\lambda_m}\xi_{\lambda_m}^{(n)}, 0, 0, \cdots)$$

$$\vdots$$

where some of the $T_{\lambda_i}\xi_{\lambda_i}^{(n)}$ may be zero ($i = 1, 2, \cdots, m$; $n = 1, 2, \cdots$). We then have :

(1) $T_{\lambda_1}\xi_{\lambda_1}^{(1)}, T_{\lambda_1}\xi_{\lambda_1}^{(2)}, \cdots, T_{\lambda_1}\xi_{\lambda_1}^{(n)}, \cdots \in H_{\lambda_1}$;

(2) $T_{\lambda_2}\xi_{\lambda_2}^{(1)}, T_{\lambda_2}\xi_{\lambda_2}^{(2)}, \cdots, T_{\lambda_2}\xi_{\lambda_2}^{(n)}, \cdots \in H_{\lambda_2}$;

.

.

(m) $\quad T_{\lambda_m}\xi_{\lambda_m}^{(1)}\ ,\ T_{\lambda_m}\xi_{\lambda_m}^{(2)}\ ,\ \cdots\ ,\ T_{\lambda_m}\xi_{\lambda_m}^{(n)}\ ,\ \cdots \qquad H_{\lambda_m}$:

Since $T_{\lambda_1} \in LC(H_{\lambda_1})$ and $\{\xi_{\lambda_1}^{(n)}\}$ is a bounded sequence of elements in H_{λ_1}, $\{T_{\lambda_1}\xi_{\lambda_1}^{(n)}\}$ contains a convergent subsequence, say

$$\left\{ T_{\lambda_1}\xi_{\lambda_1}^{(n_k^{(1)})} \right\},$$

where $\{n_k^{(1)}\}$ is a subset of $\{1, 2, \cdots\}$. Let the limit of this sequence be η_{λ_1}. Now choose the corresponding elements from (2), i.e., consider the sequence

$$\left\{ T_{\lambda_2}\xi_{\lambda_2}^{(n_k^{(1)})} \right\}.$$

Since $T_{\lambda_2} \in LC(H_{\lambda_2})$ and $\{\xi_{\lambda_2}^{(n_k^{(1)})}\}$ is a bounded sequence of elements in H_{λ_2}, $\{T_{\lambda_2}\xi_{\lambda_2}^{(n_k^{(1)})}\}$ contain a convergent subsequence, say

$$\left\{ T_{\lambda_2}\xi_{\lambda_2}^{(n_k^{(2)})} \right\},$$

where $\{n_k^{(2)}\}$ is a subset of $\{n_k^{(1)}\}$. Let the limit of this sequence by η_{λ_2}. By repeating the same process, we arrive at the m^{th} convergent subsequence of the sequence given by (m), say

$$\left\{ T_{\lambda_m}\xi_{\lambda_m}^{(n_k^{(m)})} \right\},$$

where $\{n_k^{(m)}\}$ is a subset of $\{n_k^{(m-1)}\}$. Let the limit of this sequence by η_{λ_m}. Then the sequence

$$\left\{ T\xi^{(n_k{}^{(m)})} \right\} = \left\{ (T_{\lambda_1}\xi_{\lambda_1}^{(n_k{}^{(m)})}, \cdots, T_{\lambda_m}\xi_{\lambda_m}^{(n_k{}^{(m)})}, 0, 0, \cdots) \right\}$$

is a convergent subsequence of $\left\{ T\xi^{(n)} \right\}$ with the limit given by

$$\eta = (\eta_{\lambda_1}, \eta_{\lambda_2}, \cdots, \eta_{\lambda_m}, 0, 0, \cdots) ,$$

since

$$\| T\xi^{(n_k{}^{(m)})} - \eta \| = \sum_{\lambda \in \Gamma} \| T_\lambda \xi^{(n_k{}^{(m)})} - \eta_\lambda \| .$$

Hence T is a compact operator and this completes the proof.

Theorem (2.4.2). Every dual B*-algebra is (isometrically *-isomorphic to) a closed *-subalgebra of LC(H) for some Hilbert space H .

Proof : Suppose A is dual. Then by Theorem (2.3.6)(iv), A is of the form $(\sum LC(H_\lambda))_0$ where H_λ are Hilbert spaces. Hence from Theorem (2.4.1), it follows that A is isometrically *-isomorphic to a closed *-subalgebra of LC(H) where H is the Hilbert sum of H_λ. This completes the proof.

Theorem (2.4.3). Every closed *-subalgebra B of LC(H) is a dual B*-algebra.

Proof : Let ρ be a non-zero irreducible *-representation of B in H_ρ . Then, by [6, Proposition (2.10.2)], ρ can be extended to an irreducible *-representation π of LC(H) in H_π such that H_ρ is a closed subspace of H_π and $\rho(x) = \pi(x)|H_\rho$ for all x in B . In fact, H_ρ is given by $H_\rho = \mathrm{cl}(\pi(B)\eta)$ for some $\eta \in H_\pi$. (See proof of [6, Prop (2.10.2)]). Since π is equivalent to the identity representation, $\pi(B) \subset LC(H_\pi)$ and so $\rho(B) \subset LC(H_\rho)$. Since ρ is non-zero, by [6, Corollary (4.1.11)], $\rho(B) = LC(H_\rho)$. We show next that ρ is faithful. Let $x \in B$; $x \neq 0$. Since B is semi-simple, there exists $y \in B$ such that $xy \neq 0$. Therefore $\pi(xy) \neq 0$. But $\xi = \pi(y)\eta$ is a non-zero element of H_ρ . Therefore

$$\rho(x)\xi = \pi(x)\xi = \pi(x)\pi(y) = \pi(xy)\eta \neq 0 .$$

Hence ρ is faithful. Thus B is *-isomorphic to a dual B*-algebra. This completes the proof.

It follows immediately from Theorems (2.4.2) and (2.4.3) that :

<u>Theorem (2.4.4)</u>. A B*-algebra is dual if and only if it has a faithful *-representation by completely continuous operators.

Chapter III

Some characterizations of duality in B*-algebras

§ 1. Complemented B*-algebras

Let A be a Banach algebra. Denote the lattice of all closed right (left) ideals in A by L_r (L_l) . Following [16] , we shall call A a right complemented Banach algebra if there exists a mapping $J \longrightarrow J^p$ of L_r into L_r such that :

(C.1) $\quad J \cap J^p = (0) \quad (J \in L_r)$;

(C.2) $\quad (J^p)^p = J \quad (J \in L_r)$;

(C.3) $\quad J + J^p = A \quad (J \in L_r)$;

(C.4) $\quad$ If $J_1 \subset J_2$, then $J_1^p \supset J_2^p \quad (J_1, J_2 \in L_r)$.

Analogously, we define a left complemented Banach algebra. If A is both a left and a right complemented Banach algebra, we shall call A a bicomplemented Banach algebra. The mapping $p : L_r \longrightarrow L_r$ is called a right complementor on A .

Lemma (3.1.1). Let A be a right complemented Banach algebra such that $R(A) = L(A) = (0)$. Then every closed modular right ideal is an annihilator right ideal.

Proof : Let M be a closed modular right ideal and let e' be a left identity modulo M . Then $e' = u + e$ with $u \in M$ and $e \in M^p$. Since $e' \notin M$, $e \neq 0$, It is easy to see that e is a left identity modulo M , i.e., $(1 - e)A \subset M$.

Since $(1-e)x \in M \cap M^p$ for all $x \in M^p$, we have $ex = x$ for all $x \in M^p$ which shows that $e^2 = e$ and $M^p = eA$. Now every $x \in A$ can be written in the form $x = ex + (1-e)x$; hence $A = eA + (1-e)A$. Since $(1-e)A \subset M$, it follows that $M = (1-e)A$. Clearly $L(M) \supset Ae$. Now let $x \in L(M)$. Then $x(1-e)y = 0$ for all $y \in A$ and hence, since $L(A) = (0)$, we obtain that $x = xe$. Thus $L(M) \subset Ae$ and consequently $L(M) = Ae$. It is clear that $R(L(M)) \supset M$. Let $x \in R(L(M))$. Then $yex = 0$ for all $y \in A$ and, since $R(A) = (0)$, $ex = 0$. Thus $x = x - ex \in M$ and therefore $R(L(M)) \subset M$; which completes the proof.

Theorem (3.1.2). Let A be a B*-algebra. Then the following statements are equivalent :

(i) A is dual.

(ii) A is right complemented.

(iii) A is bicomplemented.

Proof : (i) $\Longrightarrow$ (iii). Let A be a dual B*-algebra. We show that the mapping $p : J \longrightarrow L(J)^*$ on L_r is a right complementor on A. Conditions (C.1), (C.4) are easily verified. It is easy to see that condition (C.2) holds. For $L(J)^* = R(J^*)$ and therefore $(J^p)^p = L(J^p)^* = R((J^p)^*) = R((L(J)^*)^*) = R(L(J)) = J$, by the duality of A. We follow the proof of [3] to show that $J + L(J)^* = A$ $(J \in L_r)$. Let $K = J + L(J)^*$. Since $J \cap L(J)^* = (0)$, each $k \in K$ can be written uniquely in the form $k = j + i$ with $j \in J$ and $i \in L(J)^*$. Then $i^*k = i^*i$ and hence $\|i^*\|\|k\| \geq \|i^*k\|$

$= \|i\|^2$, so that $\|k\| \geqq \|i\|$. Similarly $\|k\| \geqq \|j\|$. These inequalities imply that K is closed. Hence K is either A or has a non-zero left annihilator a . In the later case, $aJ = (0)$ and $aL(J)^* = (0)$. Hence $a \in L(J)$, $a^* \in L(J)^*$, which implies the impossible conclusion that $aa^* = 0$. Hence p is a right complementor on A . By the continuity of the involution, A is also a left complemented B*-algebra. And hence A is bicomplemented.

(iii) $\Longrightarrow$ (ii) Trivial.

(ii) $\Longrightarrow$ (i) Follows immediately from Lemma (3.1.1) and Corollary (2.3.7).

Corollary (3.1.3). Every closed *-subalgebra of a right (left) complemented B*-algebra is right (left) complemented.
Proof : Follows from Theorems (2.4.2), (2.4.3) and (3.1.2).

§ 2. Weakly completely continuous B*-algebras

The weak topology on a normed linear space X is the weakest topology on X such that all bounded linear functionals on X are continuous. A net $\{x_\alpha\}$ converges to x weakly if and only if $f(x_\alpha) \longrightarrow f(x)$ for each bounded linear functional f on X . A subspace M of X is closed (in the norm topology) if and only if it is weakly closed. If M is a closed subspace of X , then the relative topology on M induced by the weak topology on X coincides with the weak topology on the normed space M. (See [15, § 3.81].)

An operator T on a normed linear space X to another normed linear space Y is said to be weakly completely continuous (w.c.c.) if it maps every bounded set into relatively weakly compact set.

Let A be a Banach algebra. For each $a \in A$, the mapping $a_\ell : x \longrightarrow ax$ ($x \in A$) is called the left multiplication by a ; similarly we define a_r, the right multiplication by a. An element a of A is called l.w.c.c. (r.w.c.c.) if a_ℓ (a_r) is w.c.c.. Since the involution in a B*-algebra A is continuous, an element a of A is l.w.c.c. if and only if it is r.w.c.c..

Following [10] , we call a Banach algebra A weakly completely continuous (w.c.c.) if the left and right multiplications of every element of A are w.c.c..

<u>Lemma (3.2.1)</u>. A B*-algebra A is finite dimensional if and only if its unit ball S is weakly compact.

<u>Proof</u> : If A is finite dimensional, then S is compact in the norm topology and hence compact in the weak topology. Conversely, if S is weakly compact, then, by [15, Theorem (4.61-C)] , A is reflexive and so weakly sequentially complete ([15, p.210]). Hence, by [13, Theorem 2], A is finite dimensional.

<u>Lemma (3.2.2)</u>. Let Ω be a locally compact Hausdorff space. Then $C_0(\Omega)$ is w.c.c. if and only if Ω is discrete.

Proof : Suppose that $C_o(\Omega)$ is w.c.c.. To prove that is discrete, it suffices to show that every relatively compact open subset G in Ω is finite. (In fact, let $x \in \Omega$ and let U be a relatively compact open neighbourhood of x . Since U is finite and Ω is Hausdorff, each points in U is open. In particular, $\{x\}$ is open, and hence Ω is discrete.) Let $C(G)$ denote the closed subalgebra of $C_o(\Omega)$ consisting of all functions vanishing outside G . G is finite if and only if $C(G)$ is finite dimensional. By Lemma (3.2.1), this is equivalent to saying that the unit ball S in $C(G)$ is weakly compact. Since Ω is locally compact Hausdorff, there exists a function $k \in C_o(\Omega)$ such that $k|G = 1$ ([9, Theorem 18, p.146]). Since $C_o(\Omega)$ is w.c.c., the weak closure of kS , $wcl(kS)$, in $C_o(\Omega)$ is weakly compact in $C_o(\Omega)$. Since the weak topology on the B*-algebra $C(G)$ coincides with the relative weak topology on $C(G)$, $wcl(kS) \cap C(G) = wcl(S) \cap C(G)$ is weakly compact in $C(G)$. Let $f \in wcl(S) \cap C(G)$ and let $\{f_n\}$ be a sequence in S converging weakly to f . For each $x \in G$, let F_x be the linear functional given by $F_x(g) = g(x)$ $(g \in C(G))$. Clearly F_x is bounded on $C(G)$. Moreover, $F_x(f_n) = f_n(x)$ converges to $F_x(f) = f(x)$ in the complex plane. Since $|f_n(x)| \leq 1$ for all $x \in G$, $\|f\| \leq 1$. Thus $wcl(S) \cap C(G) = S$, which shows that the unit ball in $C(G)$ is weakly compact.

Conversely, suppose Ω is discrete. We show that the multiplication operator of every element is completely continuous and hence w.c.c.. In fact, for $f \in C_o(\Omega)$, the set $C_n = \{\alpha \in \Omega : f(\alpha) \geqq 1/n\}$ is finite ($n = 1, 2, \cdots$). Let $C = \bigcup C_n$. Then C is at most countable; denote the elements of C by α_i ($i = 1, 2, \cdots$). Clearly, $f(\alpha) = 0$ if and only if $\alpha \notin C$. To show that f_ℓ is w.c.c., take a bounded sequence $\{g_n\}$ in $C_o(\Omega)$ and consider the following double sequence determined by $\{fg_n\}$ and $\{\alpha_n\}$:

$$
\begin{array}{l}
fg_1(\alpha_1),\ fg_1(\alpha_2),\ \cdots,\ fg_1(\alpha_n),\ \cdots. \\
fg_2(\alpha_1),\ fg_2(\alpha_2),\ \cdots,\ fg_2(\alpha_n),\ \cdots. \\
\cdot \\
\cdot \\
fg_n(\alpha_1),\ fg_n(\alpha_2),\ \cdots,\ fg_n(\alpha_n),\ \cdots. \\
\cdot \\
\cdot
\end{array}
$$

Since each column $\{fg_n(\alpha_i)\}_{n = 1, 2, \cdots}$ is a bounded sequence of complex numbers, it has a convergent subsequence, say

$$\{fg_{n(k,i)}(\alpha_i)\}_{k = 1, 2, \cdots}.$$

Moreover, we can choose the sequence

$$\{fg_{n(k,i+1)}\}_{k = 1, 2, \cdots}$$

to be a subsequence of

$$\{fg_{n(k,i)}\}_{k = 1, 2, \cdots}$$

in $C_0(\Omega)$. Let

$$g(\alpha) = \begin{cases} \lim fg_{n(k,i)}(\alpha_i) & \text{if } \alpha = \alpha_i \quad (i=1,2,\cdots,) \\ 0 & \text{otherwise} \end{cases} .$$

We show now that the subsequence $\{fg_{n(k,k)}\}$ $k = 1, 2, \ldots$ of $\{fg_n\}$ converges uniformly to g . For all $\alpha \in \Omega$, clearly

$$\lim_{k\to\infty} fg_{n(k,k)}(\alpha) = g(\alpha) .$$

Let ε be any arbitrary positive number. Since $f \in C_0(\Omega)$, there exists an m such that $|f(\alpha)| < \varepsilon$, for $\alpha \neq \alpha_1, \alpha_2, \cdots \alpha_m$. Also, since $\{g_n\}$ is bounded by a constant K , $|fg_n(\alpha)| = |f(\alpha)g_n(\alpha)| = |f(\alpha)|\,|g_n(\alpha)| < K\varepsilon$ for $\alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_m$. Clearly $|g(\alpha)| < K\varepsilon$ for $\alpha \neq \alpha_1, \cdots, \alpha_m$, and we have

$$|fg_{n(k,k)}(\alpha) - g(\alpha)| < 2k\varepsilon .$$

For each α_i , let N_i be the positive integer such that for $k > N_i$,

$$|fg_{n(k,k)}(\alpha_i) - g(\alpha_i)| < 2K\varepsilon \quad (i = 1, \cdots, m).$$

Let $N = \max\{N_1, N_2, \cdots, N_m\}$. Then for $k > N$, we have

$$|fg_{n(k,k)}(\alpha) - g(\alpha)| < 2K\varepsilon ,$$

for all $\alpha \in \Omega$. Hence $fg_{n(k,k)}$ converges to g uniformly; i.e., $\lim fg_{n(k,k)} = g$. This prove that f_ℓ is completely continuous and thus completing the proof.

Theorem (3.2.3). Let A be a B*-algebra. Then A is w.c.c. if and only if it is dual.

Proof : Suppose A is w.c.c. and let B be a maximal commutative *-subalgebra of A. Since B is closed, B is w.c.c.. By Theorem (1.2.1) and Lemma (3.2.2), B is *-isomorphic to $C_0(\Omega)$, for some discrete space Ω. Let e_α be the element of B corresponding to the characteristic function of the point α in Ω. Then clearly $\{e_\alpha : \alpha \in \Omega\}$ is a maximal orthogonal family of s.a. minimal idempotents of B. It follows that every $x \in B$ is of the form $x = \sum \lambda_\alpha e_\alpha$, i.e., the series $\sum \lambda_\alpha e_\alpha$ is summable to x in the norm. In fact, let $\varepsilon > 0$ and let $\lambda_\alpha = \hat{x}(\alpha)$ for all $\alpha \in \Omega$. Then there exists a finite subset Ω_1 of Ω such that $|\lambda_\alpha| = |\hat{x}(\alpha)| < \varepsilon$ for $\alpha \notin \Omega_1$. Thus, if Ω_2 is any finite subset of Ω such that $\Omega_1 \cap \Omega_2 = \emptyset$, then $\|\sum_{\alpha\in\Omega_2} \lambda_\alpha e_\alpha\| = \sup_{\alpha\in\Omega_2} |\lambda_\alpha| < \varepsilon$ which shows that $\sum \lambda_\alpha e_\alpha$ is summable. Conversely, if $\sum \lambda_\alpha e_\alpha$ is summable, it represents an element of B. We show that for each $\beta \in \Omega$, e_β is also minimal in A. Let a be a s.a. element of A. Then $e_\beta a e_\beta$ commutes with e_α, for all $\alpha \in \Omega$, and since B coincides with its commutant by maximality, we have $e_\beta a e_\beta \in B$. Thus $e_\beta a e_\beta$ is of the form $e_\beta a e_\beta = \sum \lambda_\alpha e_\alpha$ and so we have $e_\beta a e_\beta = \lambda e_\beta$, where λ is real. Thus $e_\beta A e_\beta = Ce_\beta$ and therefore the family $\{e_\alpha : \alpha \in \Omega\}$ is also a maximal orthogonal family of s.a. minimal idempotents in A.

If $z \in A$, $e_\alpha z = 0$ for all $\alpha \in \Omega$, then $z = 0$. In fact, $e_\alpha z = 0 \Longrightarrow e_\alpha z z^* = 0 \Longrightarrow z z^* e_\alpha = 0 \Longrightarrow z z^*$

commutes with $B \Longrightarrow z^*z \in B$. Hence $z^*z = \sum \lambda_\alpha e_\alpha$ and, since $e_\alpha z^*z = 0$, we have $z^*z = 0$, which gives $z = 0$. Let $\mathfrak{S}$ be the socle of A. We claim that $\mathfrak{S}$ is weakly dense in A. Let $z \in A$ and consider the directed set of finite subsets $\{e_{\alpha_1}, \cdots, e_{\alpha_n}\}$ of $\{e_\alpha : \alpha \in \Omega\}$, directed by inclusion. Since $\|e_{\alpha_1} + \cdots + e_{\alpha_n}\| = 1$, the right multiplication operator z_r transforms the set into a relatively weakly compact set $\{(e_{\alpha_1} + \cdots + e_{\alpha_n})z\}$. Let z' be any limiting point (in the weak topology) of the net $\{(e_{\alpha_1} + \cdots + e_{\alpha_n})z\}$. Since any continuous operator on A is also weakly continuous, it is easy to see that $e_\alpha z = e_\alpha z'$, for all $\alpha \in \Omega$. Hence $z = z'$. Thus, the net $\{(e_{\alpha_1} + \cdots + e_{\alpha_n})z\}$ converges weakly to z, which implies that $\mathfrak{S}$ is weakly dense in A. Since $\mathfrak{S}$ is a subspace of A, $\mathrm{cl}(\mathfrak{S})$ coincides with the weak closure of $\mathfrak{S}$ and therefore $\mathfrak{S}$ is dense in A. Hence, by Theorem (2.3.6), A is dual.

Conversely, suppose A is dual. Then, by Theorem (2.4.2), A is $*$-isomorphic to a closed $*$-subalgebra of $LC(H)$, for some Hilbert space H. Thus it suffices to show that $LC(H)$ is w.c.c.. Let $T \in LC(H)$, by Lemma (1.2.5), $T = T_1T_2$ $(T_1, T_2 \in LC(H))$. Let $\{B_k\}$ be a bounded sequence in $LC(H)$. Then $\{T_2B_n\} \subset S_n = \{T \in B(H) : \|T\| \leq n\}$ for some n. Since S_n is weakly compact in $B(H)$ with respect to the (operator) weak topology ([5, p.33]), there exists a convergent subsequence, say $\{T_2B_{n_k}\}$. Let B

be the limit of $\{T_2 B_{n_k}\}$. Since the left multiplication by T_1 is weakly continuous, $T_1 T_2 B_{n_k}$ converges weakly to $T_1 B$ ([5, p.34]). Let f be any bounded functional on LC(H) , then, by [4, Proposition 8] , $f(TB_{n_k} - T_1 B) \longrightarrow 0$. Thus T is l.w.c.c. ([Hille and Phillips, Functional analysis and semi-groups, Theorem (2.9.6)]). Hence LC(H) is w.c.c..

Corollary (3.2.4). Every closed commutative *-subalgebra of a dual B*-algebra A is *-isomorphic to $C_0(\Omega)$, where Ω is discrete.

Proof : Follows from Theorems (3.2.3), (1.2.1) and Lemma (3.2.2).

§ 3. Maximal commutative *-subalgebras and their carrier spaces

The following theorem is due to T. Ogasawara and K. Yoshinaga [11] .

Theorem (3.3.1). The following statements are equivalent for a B*-algebra A :

(i) A is dual.

(ii) Every s.a. element of A has a spectrum without cluster points other than zero.

(iii) The carrier space of every maximal commutative *-subalgebra is discrete.

(iv) Every maximal commutative *-subalgebra is dual.

Proof : [11] (i) $\Longrightarrow$ (ii). Suppose A is dual and let x be a s.a. element of A . Let B be the closed commtative $*$-subalgebra of A generated by x . Then, by Corollary (3.2.4), B is $*$-isomorphic to $C_0(\Omega)$, where Ω is a discrete space. By [12, Theorem (3.1.6)] , $Sp_B(x) - (0) \subset \{\hat{x}(\varphi) : \varphi \in \Omega\} \subset Sp_B(x)$, where $\hat{x}$ is the function in $C_0(\Omega)$ corresponding to x . Since Ω is discrete and $\hat{x}$ is continuous and vanishing at infinity, the range of $\hat{x}$ has no cluster points other than zero. In fact, if λ_0 is a non-zero cluster point of the range of $\hat{x}$, then every neighbourhood of λ_0 contains infinitely many points of the range of $\hat{x}$. Thus, for every $\varepsilon > 0$, the set $\{\varphi : |\hat{x}(\varphi) - \lambda_0| < \varepsilon\}$ is infinite. As $\lambda_0 \neq 0$, we can choose $\varepsilon > 0$ such that $|\lambda_0| > 2\varepsilon$. Then

$$\{\varphi : |\hat{x}(\varphi) - \lambda_0| < \varepsilon\} \subset \{\varphi : |\lambda_0| - |\hat{x}(\varphi)| < \varepsilon\}$$

$$= \{\varphi : |\lambda_0| - \varepsilon < |\hat{x}(\varphi)|\} \subset \{\varphi : \varepsilon < |\hat{x}(\varphi)|\} .$$

This shows that the set $\{\varphi : |\hat{x}(\varphi)| > \varepsilon\}$ is infinite and hence not compact (since Ω is discrete), contradicting the fact that $\hat{x}$ vanishes at infinity. Hence $Sp_B(x)$ has no cluster points other than zero. Since $Sp_B(x) \cup (0) = Sp_A(x) \cup (0)$ ([12, Corollary(4.8.2)]), this completes the proof of (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii). Suppose (ii) holds and let Ω be the carrier space of a maximal commutative $*$-subalgebra B of A . Let p_0 be any point of Ω and U a compact neighbourhood of p_0. Then there exists a real-valued continuous function $\hat{x}(p)$ defined on Ω such that $0 \leq \hat{x} \leq 1$, $\hat{x}(p_0) = 1$ and $\hat{x}(p) = 0$ $(p \notin U)$ ([9, Theorem 18, p.146]). Since $\hat{x}$ is the Gelfand representation of an element x in B (cf. Theorem (1.2.1)), the range of $\hat{x}$ has no cluster points other than zero. And therefore there exists a neighbourhood V of $\hat{x}(p_0) = 1$ such that $V - \{1\}$ has empty intersection with the range of $\hat{x}$. Thus

$$\{ p : \hat{x}(p) = 1 \} = \{ p : p \in \hat{x}^{-1}(V) \} ,$$

which is obviously a compact open set contained in U. We have thus shown that every compact neighbourhood of a point p in Ω contains a compact open neighbourhood of p. Now if q_0 is a cluster point of Ω, then we can take a sequence of compact open neighbourhood U_n of q_0 in such a way that U_{n+1} is a proper subset of U_n. In fact, let U_1 be a compact open neighbourhood of q_0 in Ω. Since q_0 is a cluster point of Ω, $(U_1 - \{q_0\}) \cap \Omega \neq \emptyset$. Let $p \in (U_1 - \{q_0\}) \cap \Omega$. Then $U' = U_1 - \{p\}$ is an open neighbourhood of q_0. Since Ω is locally compact Hausdorff, there exists a compact neighbourhood U_2' of q_0 such that $q_0 \in U_2' \subset U'$ ([9, Theorem 18, p.146]). Hence there exists a compact open neighbourhood U_2 of q_0 such that $U_2 \subset U_2'$. Clearly, $U_2 \subset U_1$ and proceeding in this way, we obtain our sequence $\{ U_n \}$. Since each U_n is compact open, the characteristic function of U_n is an element of $C_o(\Omega)$. Let e_n be the s.a. element whose Gelfand representation is the characteristic function of U_n. Put

$$y = \sum_1^{\infty} (1/n^2)e_n .$$

Then $y \in B$, since $y_m = \sum_1^m (1/n^2)e_n$ belongs to B and $y = \lim y_m$; and clearly y is s.a.. We claim now that $\sum(1/n^2)$ is a non-zero cluster point of $Sp_B(y)$. In fact, let $\{p_m\}$ be a sequence in Ω such that $p_m \in U_m$ and $p_m \notin U_{m+1}$. Then $\hat{y}(p_m) = \sum_1^m 1/n^2$ is in the range of $\hat{y}$ for each m and $\lim \hat{y}(p_m) = \sum(1/n^2)$. Hence $\sum(1/n^2)$ is a non-zero cluster point of $Sp_B(y)$; a contradiction. Thus Ω has no cluster points and so is discrete.

(iii) $\Longleftrightarrow$ (iv). Follows immediately from Theorem (1.2.1), Lemma (3.2.2) and Theorem (3.2.3).

(iii) $\Longrightarrow$ (i). Suppose (iii) holds. Let x be any s.a. element of A and let B be a maximal commutative *-subalgebra of A containing x. Since the carrier space Ω of B is discrete, it follows from the proof of Theorem (3.2.3) that there exists an orthogonal family of s.a. minimal idempotents $\{e_\alpha\} \subset B$ such that every $z \in B$ can be expressed in the form $\sum \lambda_\alpha e_\alpha$; i.e., the series $\sum \lambda_\alpha e_\alpha$ is summable to z in the norm. It also follows from the proof of Theorem (3.2.3) that the family $\{e_\alpha\}$ is a maximal orthogonal family of s.a. minimal idempotents of A. Since x belongs to B, $x = \sum \lambda_\alpha e_\alpha$ and hence x is in the closure of the socle $\mathfrak{S}$. Since any element of A is a linear combination of s.a. elements, it follows that $\mathfrak{S}$ is dense in A. By Theorem (2.3.6) (iii), A is dual.

Corollary (3.3.2). Let A be a B^*-algebra. If every s.a. element has a finite spectrum, then A is finite dimensional.

Proof : [11] By Theorem (3.3.1), A is dual and therefore by Theorem (2.4.2), A is $*$-isomorphic to a closed $*$-subalgebra of $LC(H)$, for some Hilbert space H. It follows now from the proof of Theorem (2.4.3) that A is $*$-isomorphic to $LC(H_1)$, for some Hilbert space H_1. Hence, A will be finite dimensional if and only if A contains an identity. Let Ω be the carrier space of a maximal commutative $*$-subalgebra B of A. Since Ω is discrete and every s.a. elements of B has a finite spectrum, Ω is finite. In fact if Ω is an infinite set, then it contains a countable subset $\{\alpha_n : n = 1, 2, \cdots\}$ of Ω. Let $\hat{z}$ be the function in $C_o(\Omega)$ such that

$$\hat{z}(\alpha) = \begin{cases} 1/n & \text{if} \quad \alpha = \alpha_n \\ 0 & \text{otherwise} \end{cases} .$$

Then the element $z \in B$ corresponding to $\hat{z}$ is s.a. and, by [12, Theorem (3.1.6)], $Sp_B(z)$ contains the points

$1/n$ ($n = 1, 2, \cdots$). Thus $Sp_B(z)$ is not finite, which is a contradiction. Hence Ω is finite, and therefore B has an identity e . We show that e is also an identity of A . Let x be any element of A and let $y = ex - x$. Then $yy^* = (e - 1)xx^*(e - 1)$ and we have $byy^* = yy^*b = 0$ for all $b \in B$, i.e., yy^* commutes with every element of B . Hence by the maximality of B , $yy^* \in B$. Now $ey = 0$ and therefore $eyy^* = 0$, which shows that $yy^* = 0$. Thus $y = 0$ which implies $ex = x$. Similarly, we can show that e is also a right identity of A . Hence A is finite dimensional.

Chapter IV

The successive conjugate spaces of dual B*-algebras

§ 1. The first conjugate space of a dual B*-algebra

Let H be a Hilbert space and let $\{\xi_\lambda\}$ and $\{\eta_\mu\}$ be any pair of complete orthonormal systems of vectors in H. For any $T \in B(H)$, consider the three sums :

$$\sum_\lambda \|T\xi_\lambda\|^2 , \quad \sum_{\lambda,\mu} |(T\xi_\lambda, \eta_\mu)|^2 , \quad \sum_\mu \|T^*\eta_\mu\|^2 .$$

Using the Parseval equality, we obtain

$$\|T\xi_\lambda\|^2 = \sum_\mu |(T\xi_\lambda, \eta_\mu)|^2 .$$

Therefore,

$$\sum_\lambda \|T\xi_\lambda\|^2 = \sum_{\lambda,\mu} |(T\xi_\lambda, \eta_\mu)|^2 = \sum_\mu |(T^*\eta_\mu, \xi_\lambda)|^2$$
$$= \sum_\mu \|T^*\eta_\mu\|^2 .$$

It follows that the three sums written above are equal to one another (possibly to $+\infty$). Since the two systems $\{\xi_\lambda\}$ and $\{\eta_\mu\}$ are independent of one another, the common value $\sigma(T)^2$ of the sums is independent of the choice of $\{\xi_\lambda\}$ and $\{\eta_\mu\}$. The Schmidt-class $\sigma c(H)$ consists of all those operators $T \in B(H)$ such that $\sigma(T) < \infty$. It follows that $\sigma c(H)$ is a Banach algebra under the norm $\sigma(T)$ (cf. [14, Theorem 3, p.34]). The trace-class $\tau c(H)$ of operators on H consists of all operators of the form $T = T_1T_2$ (T_1, T_2

$\in \sigma c(H)$). The name 'trace-class' derives from the fact that $\tau c(H)$ admits a complex-valued function $t(T)$ which has the characteristic properties of the trace for matrices; $t(T)$ is defined as $t(T) = \sum (T\xi_\lambda, \xi_\lambda)$, where $\{ \xi_\lambda \}$ is a complete orthonormal system of vectors on H . $\tau c(H)$ is a Banach algebra under the norm

$$|T| = t((T^*T)^{\frac{1}{2}}) \quad (T \in \tau c(H)) ,$$

where $(T^*T)^{\frac{1}{2}}$ denotes the positive square root of the element T^*T . Since $\sigma c(H)$ and $\tau c(H)$ are closed under the involution $T \longrightarrow T^*$, they are A^*-algebras, the auxiliary norm being given by the operator bound. Moreover, $\sigma c(H)$ and $\tau c(H)$ are two-sided ideals of $LC(H)$, and contain the set $F(H)$ of all operators of finite rank as a dense subset. (See [14, Theorems 3 and 5, p.34 and 42].)

Now let $\{ H_\lambda : \lambda \in \Lambda \}$ be a family of Hilbert spaces H_λ and let $(\sum \tau c(H_\lambda))_1$ denote the family of all functions f defined on Λ such that $f(\lambda) \in \tau c(H_\lambda)$ for all λ and such that $\sum |f(\lambda)| < \infty$. It follows that $(\sum \tau c(H_\lambda))_1$ is a Banach algebra under the norm $|f| = \sum |f(\lambda)|$, and the usual operations for functions (see p.6) It is clearly a $*$-subalgebra of $(\sum LC(H_\lambda))_0$ and therefore an A^*-algebra.

Theorem (4.1.1). As a Banach space, $(\sum \tau c(H_\lambda))_1$ is isometrically isomorphic to the conjugate space of $(\sum LC(H_\lambda))_0$.

Proof : Let $A_1 = (\sum \tau c(H_\lambda))_1$, $A = (\sum LC(H_\lambda))_0$ and let A' be the conjugate space of A. Let $f \in A_1$ and write f_λ for $f(\lambda)$ ($\lambda \in \Lambda$). By [14, Theorem 3, p.48], each f_λ can be identified (isometrically) with an element of $LC(H_\lambda)'$, the conjugate space of $LC(H_\lambda)$. Considering now f_λ as an element of $LC(H_\lambda)'$, define the function f' on A by the relation

$$f'(x) = \sum f_\lambda(x_\lambda) \qquad (x = (x_\lambda) \in A).$$

Since

$$|f'(x)| \leqq \sum |f_\lambda(x_\lambda)| \leqq \sum |f_\lambda| \, \|x_\lambda\|$$

$$\leqq (\sup \|x_\lambda\|)(\sum |f_\lambda|) = |f| \, \|x\| ,$$

f' is a continuous linear functional on A and $\|f'\| \leqq |f|$, where $\|f'\|$ denotes the operator bound of f'. Since $\sum |f_\lambda| < \infty$, only a countable numbers of $f_\lambda \neq 0$; denote the non-zero f_λ by $f_{\lambda_1}, f_{\lambda_2}, \cdots$. For given $\varepsilon > 0$, choose a positive integer N such that $\sum_N^\infty |f_{\lambda_n}| < \frac{1}{2}\varepsilon$. Since $f_{\lambda_n} \in LC(H_{\lambda_n})'$, there exists an $x_{\lambda_n} \in LC(H_{\lambda_n})$ such that $\|x_{\lambda_n}\| = 1$ and

$$f_{\lambda_n}(x_{\lambda_n}) \geqq |f_{\lambda_n}| - \varepsilon/n^2 \quad (n = 1, 2, \cdots, N)$$

Let $x = (x_\lambda)$, where $x_\lambda = x_{\lambda_n}$, when $\lambda = \lambda_n$ ($n = 1, \cdots, N$) and $x_\lambda = 0$ otherwise. Then $\|x\| = 1$ and

$$|f'(x)| = \sum_1^N f_{\lambda_n}(x_{\lambda_n}) \geqq \sum_1^N |f_{\lambda_n}| - \varepsilon(\sum_1^N 1/n^2)$$

$$\geqq (\sum_1^\infty |f_{\lambda_n}| - \tfrac{1}{2}\varepsilon) - \varepsilon(\sum_1^N 1/n^2)$$

$$\geqq \sum_1^\infty |f_{\lambda_n}| - 2\varepsilon .$$

Since ε is arbitrary, we have $|f| \leqq \|f'\|$ and so $\|f'\| = |f|$.

Let $f' \in A'$ and let f_λ be the restriction of f' to $LC(H_\lambda)$ ($\lambda \in \Lambda$). (We identify $LC(H_\lambda)$ as a subalgebra of A). Then each $f_\lambda \in LC(H_\lambda)'$. Considering now f_λ as an element of $\tau c(H_\lambda)$, let f be the mapping on Λ such that $f(\lambda) = f_\lambda \in \tau c(H_\lambda)$. We show that $f \in A_1$. Let $\{ \lambda_n : n = 1, 2, \cdots, k \}$ be a finite subset of Λ. For given $\varepsilon > 0$, choose $x_{\lambda_n} \in LC(H_{\lambda_n})$ such that $\|x_{\lambda_n}\| = 1$ and

$$f_{\lambda_n}(x_{\lambda_n}) \geqq |f_{\lambda_n}| - \varepsilon/n^2 \qquad (n = 1, 2, \cdots, k)$$

Let $x = (x_\lambda)$, where $x_\lambda = x_{\lambda_n}$, when $\lambda = \lambda_n$ ($n = 1, 2, \cdots, k$), and $x_\lambda = 0$ otherwise. Then

$$f'(x) = f'(x_{\lambda_1}) + \cdots + f'(x_{\lambda_k}) = \sum_{n=1}^{K} f_{\lambda_n}(x_{\lambda_n})$$

$$\geqq \sum_{n=1}^{K} |f_{\lambda_n}| - \varepsilon(\sum_{n=1}^{K} 1/n^2) \geqq \sum_{n=1}^{K} |f_{\lambda_n}| - 2\varepsilon .$$

Thus $\sum_{n=1}^{K} |f_{\lambda_n}| \leqq \|f'\|$ and so $\sum_{\lambda} |f_\lambda| \leqq \|f'\|$. Therefore $f \in A$ and $|f| = \sum_{\lambda} |f_\lambda| = \|f'\|$ (by above argument). Since clearly $f \longrightarrow f'$ is linear, it follows that $f \longrightarrow f'$ is an isometric isomorphism of A_1 onto A .

Theorem (4.1.2). $(\sum \tau c(H_\lambda))_1$ is a dense two-sided ideal of $(\sum LC(H_\lambda))_0$.

Proof : Let $A_1 = (\sum \tau c(H_\lambda))_1$ and $A = (\sum LC(H_\lambda))_0$. Consider any element $x = (x_\lambda) \in A$. For any $\varepsilon > 0$, the set $D = \{ \lambda : \|x_\lambda\| \geqq \varepsilon \}$ is finite; denote the elements of D by $\lambda_1, \lambda_2, \cdots, \lambda_m$. Since $\tau c(H_\lambda)$ is dense in

$LC(H_\lambda)$, there exists $f_{\lambda_i} \in \tau c(H_{\lambda_i})$ such that $\|f_{\lambda_i} - x_{\lambda_i}\| < \varepsilon$ ($i = 1, 2, \cdots, m$). Define f in A by

$$f_\lambda = \begin{cases} f_{\lambda_i} & \text{when } \lambda = \lambda_i \quad (i = 1, 2, \cdots, m) \\ 0 & \text{otherwise} \end{cases} .$$

Then

$$\|f - x\| = \sup_\lambda \|f_\lambda - x_\lambda\| < \varepsilon .$$

Thus A_1 is dense in A . It is also a two-sided ideal of A . In fact, if $x \in A$ and $f \in A_1$, then $(xf)_\lambda = x_\lambda f_\lambda \in \tau c(H_\lambda)$ (since $\tau c(H_\lambda)$ is a two-sided ideal of $LC(H_\lambda)$) and, since

$$\sum |(xf)_\lambda| \leqq \sum \|x_\lambda\| |f_\lambda| \leqq \|x\| \sum |f_\lambda| ,$$

it follows that $xf \in A_1$; similarly $fx \in A_1$. This completes the proof.

<u>Teorem (4.1.3)</u>. $(\sum \tau c(H_\lambda))_1$ is w.c.c..

<u>Proof</u> : Since $A_1 = (\sum \tau c(H_\lambda))_1$ is a dense two-sided ideal of $A = (\sum LC(H_\lambda))_0$ (Theorem (4.1.2)), by [10, Lemma 9] , it suffices to show that A_1^2 is dense in A_1 . Let $f = (f_\lambda) \in A_1$ and denote the non-zero f_λ by $f_{\lambda_1}, f_{\lambda_2} \cdots$. Let $\varepsilon > 0$ be given and choose a positive integer N such that $\sum_{n=N}^{\infty} |f_{\lambda_n}| < \frac{1}{2}\varepsilon$. Since $F(H_{\lambda_n})$ is dense in $\tau c(H_{\lambda_n})$, there exists an element $g_{\lambda_n} \in F(H_{\lambda_n})$ such that $|f_{\lambda_n} - g_{\lambda_n}| < \varepsilon/2^n$ ($n = 1, 2, \cdots, N$). If we express g_{λ_n} in its polar decomposition, $g_{\lambda_n} = \sum_{i=1}^{K} \lambda_i \varphi_i \otimes \overline{\psi}_i$, then we can write $g_{\lambda_n} = g_{\lambda_n} \sum_{i=1}^{K} \psi_i \otimes \overline{\psi}_i$, which shows that $g_{\lambda_n} = g_{\lambda_n} h_{\lambda_n}$, where $h_{\lambda_n} = \sum_{i=1}^{K} \psi_i \otimes \psi_i \in F(H_{\lambda_n})$. Define g and

h in A_1 by

$$g_\lambda = \begin{cases} g_{\lambda n} & \text{if } \lambda = \lambda_n \quad (\, n = 1, 2, \cdots, N \,) \\ 0 & \text{otherwise} \end{cases},$$

and

$$h_\lambda = \begin{cases} h_{\lambda n} & \text{if } \lambda = \lambda_n \quad (\, n = 1, 2, \cdots, N \,) \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that $g = gh$ and therefore $g \in A_1^2$. Also, we have

$$|f - g| = \sum_1^\infty |f_\lambda - g_\lambda| = \sum_{i=1}^{N} |f_{\lambda_i} - g_{\lambda_i}| + \sum_{i=N+1}^{\infty} |f_{\lambda_i}|$$
$$\leq \sum_{n=1}^{N} (1/2^n)\varepsilon + \tfrac{1}{2}\varepsilon < 2\varepsilon ,$$

which shows that A_1^2 is dense in A_1.

<u>Theorem (4.1.4)</u>. $(\sum \tau c(H_\lambda))_1$ is dual.

<u>Proof</u> : Since $A_1 = (\sum \tau c(H_\lambda))_1$ is a dense two-sided ideal of a dual B*-algebra $A = (\sum LC(H_\lambda))_0$, by [10, Lemma 8-(3)] , A_1 is dual if and only if, for every $f \in A_1$, $f \in cl(fA_1)$, the closure of fA_1 in A_1. We show first that $\tau c(H)$ has this property. Let $g \in \tau c(H)$, and write g in its polar decomposition, $g = \sum_{i=1}^{\infty} \lambda_i \varphi_i \otimes \overline{\psi}_i$. By [14, Theorem 2, p.41] , we have $|g| = \sum_{i=1}^{\infty} \lambda_i$, so that $\sum_{i=n}^{\infty} \lambda_i \longrightarrow 0$ as $n \longrightarrow \infty$. Now

$$g_n = \sum_{i=1}^{n} \lambda_i \varphi_i \otimes \overline{\psi}_i = g \sum_{k=1}^{n} \psi_k \otimes \overline{\psi}_k$$

belongs to $g \cdot \tau c(H)$ and, since $|g - g_n| = \sum_{i=n}^{\infty} \lambda_i$, the sequence $\{g_n\}$ converges to g in $\tau c(H)$. Hence $g \in cl(g \cdot \tau c(H))$.

Since $f \in A_1$, we have only a countable numbers of $f_\lambda \neq 0$; denote these by $f_{\lambda_1}, f_{\lambda_2}, \cdots$. For a given $\varepsilon > 0$, choose a positive integer N such that $\sum_{n=N}^{\infty} |f_{\lambda_n}| < \frac{1}{2}\varepsilon$. Since $f_{\lambda_n} \in \mathrm{cl}(f_{\lambda_n}\tau c(H_{\lambda_n}))$ ($n = 1, \cdots, N$), there exists a $g_{\lambda_n} \in \tau c(H_{\lambda_n})$ such that

$$|f_{\lambda_n} - f_{\lambda_n} g_{\lambda_n}| < (1/2^{(n+1)})\varepsilon \quad (n = 1, \cdots, N) .$$

Define g and f in A by

$$g = \begin{cases} g_{\lambda n} & \text{when } \lambda = \lambda_n \quad (n = 1, \cdots, N) \\ 0 & \text{otherwise} \end{cases}$$

$$h = \begin{cases} f_{\lambda n} g_{\lambda n} & \text{when } \lambda = \lambda_n \quad (n = 1, \cdots, N) \\ 0 & \text{otherwise} . \end{cases}$$

Clearly $h = fg$ and therefore $h \in fA_1$. Since

$$|f - h| = \sum_{\lambda} |f_\lambda - h_\lambda| = \sum_{n=1}^{N} |f_{\lambda n} - f_{\lambda n} g_{\lambda n}| + \sum_{n=N+1}^{\infty} |f_{\lambda n}|$$

$$\leq \sum_{n=1}^{N} (1/2^{n+1})\varepsilon + \tfrac{1}{2}\varepsilon < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon ,$$

we have that $f \in \mathrm{cl}(fA_1)$, which completes the proof.

The conjugate space of a dual B*-algebra $A = (\sum LC(H_\lambda))_0$ can be given an algebraic structure induced by A_1. With this identification, A' is an A*-algebra. The above theorems in combination with Theorem (2.3.6) give us the following results:

<u>Theorem (4.1.5)</u>. Let A be a dual B*-algebra. Then the conjugate space A is a w.c.c. dual A*-algebra which is a dense two-sided ideal of A.

§ 2. The second conjugate space of a dual B*-algebra

By [14, Theorem 3, p.48] , the second conjugate space of LC(H) is isometrically isomorphic to B(H) . It follows that the second conjugate space of LC(H) also has an algebraic structure induced by B(H) .

Let $\{ A_\lambda : \lambda \in \Lambda \}$ be a family of B*-algebras. Denote by $\sum A_\lambda$ the class of all functions F defined on Λ with $F_\lambda = F(\lambda) \in A_\lambda$ ($\lambda \in \Lambda$) , and such that the quantity $\| F \|$ defined by

$$\| F \| = \sup \| F_\lambda \|$$

is finite. Under the usual operations for functions and involution as defined on p.6, $\sum A_\lambda$ is a *-algebra. It is easy to see that $\| F \|$ is a norm of $\sum A_\lambda$ under which it is a B*-algebra. This B*-algebra we call the <u>B*-sum</u> of the family $\{ A_\lambda \}$.

<u>Theorem (4.2.1)</u>. Let $A = (\sum LC(H_\lambda))_0$. As Banach space, $\sum B(H_\lambda)$ is isometrically isomorphic to the second conjugate space A'' of A .

<u>Proof</u> : In what follows, we identify A' with $A_1 = (\sum \tau c(H_\lambda))_1$. Then, to each $F = (F_\lambda) \in \sum B(H_\lambda)$, we can assigne (uniquely) a linear functional $\tilde{F}$ on A' given by

$$\tilde{F}(f) = \sum F_\lambda (f_\lambda) \quad (f = (f_\lambda) \in A').$$

Since $F_\lambda (f_\lambda)$ represents a bounded linear functional on

$\tau c(H_\lambda)$ and since

$$|\tilde{F}(f)| = |\sum F_\lambda(f_\lambda)| \leqq \sum |F_\lambda(f_\lambda)| \leqq \sum \|F_\lambda\| \, |f_\lambda|$$

$$\leqq \sup \|F_\lambda\| \sum |f_\lambda| = \|F\| \, |f| < \infty ,$$

we see that $\tilde{F}$ is a bounded linear functional on A'. Clearly, $\|\tilde{F}\|_0 \leqq \|F\|$, where $\|\tilde{F}\|_0$ denotes the operator bound norm of $\tilde{F}$. Since, for each $\lambda \in \Lambda$, $\|F_\lambda\| = \sup_{|f_\lambda|=1} |F_\lambda(f_\lambda)|$, it follows that

$$\|F\| = \sup \|F_\lambda\| = \sup_\lambda \left\{ \sup_{|f_\lambda|=1} |F_\lambda(f_\lambda)| \right\}$$

Identifying f_λ as an element of A', we can write $F_\lambda(f_\lambda) = \tilde{F}(f_\lambda)$ and, obtain

$$\|F\| = \sup_\lambda \left\{ \sup_{|f_\lambda|=1} |\tilde{F}(f_\lambda)| \right\} \leqq \sup_\lambda \|\tilde{F}\|_0 = \|\tilde{F}\|_0$$

i.e., $\|F\| \leqq \|\tilde{F}\|_0$. Hence $\|\tilde{F}\|_0 = \|F\|$. Conversely, let $\tilde{F} \in A''$. Then $F_\lambda = \tilde{F}|\tau c(H_\lambda)$ is clearly a bounded linear functional on $\tau c(H_\lambda)$. Since $LC(H_\lambda)''$ is isometrically isomorphic to $B(H_\lambda)$, we may identify F_λ as an element of $B(H_\lambda)$ (and we do so). It is easy to see that $F = (F_\lambda) \in \sum B(H_\lambda)$; in fact, $\|F_\lambda\| \leqq \|\tilde{F}\|_0$ for each λ, and hence $\|F\| = \sup \|F_\lambda\| \leqq \|\tilde{F}\|_0$, which shows that $F \in \sum B(H_\lambda)$. From the above argument, we also have $\|\tilde{F}\|_0 = \|F\|$. Since clearly, $F \longrightarrow \tilde{F}$ is linear, it follows that $F \longrightarrow \tilde{F}$ is an isometric isomorphism of $\sum B(H_\lambda)$ onto A''.

<u>Corollary (4.2.2)</u>. Let A be a dual B*-algebra. The second

conjugate space of A is a B*-algebra which contains A as a closed two-sided ideal.

Proof : Let A'' be the second conjugate space of A. By Theorem (2.3.6) and Theorem (4.2.1), A'' is isometrically isomorphic to $\sum B(H_\lambda)$ where $\{H_\lambda\}$ is a family of Hilbert spaces. With the algebraic structure induced by $\sum B(H_\lambda)$, A'' is clearly a B*-algebra and contains A as a closed two-sided ideal, since A is *-isomorphic to $(\sum LC(H_\lambda))_0$. This completes the proof.

Remark : There is little that we can say about the characterization of the spaces $LC(H)^{(n)}$ (the nth-conjugate space of $LC(H)$). For $n = 3$, we see that $LC(H)^{(3)}$ is isometrically isomorphic to the conjugate space of $B(H)$. Dixmier [4] has shown that every continuous linear functional F on $B(H)$ may be represented in one and only one way in the form $F = F_1 + F_2$, where F_1 belongs to $\tau c(H)$ and F_2 belongs to the subspace $LC(H)^{\perp}$ which consists of all bounded linear functionals on $B(H)$ that vanish identically on $LC(H)$. Moreover $\|F\| = \|F_1\| + \|F_2\|$.

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