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EXTENSION of  
LINEAR QUADRATIC REGULATOR THEORY  
and its APPLICATIONS

by

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A thesis submitted to the  
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in partial fulfillment of the requirements for the degree of  
Master of Applied Science

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Abdu-Nasser R. Mouadeb, Ottawa, Canada, 1992



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## CHAPTER I

### INTRODUCTION

#### I.0. Introduction.

The demand for more precise control in a number of fields e.g. avionics, missiles guidance, robotics,... etc. and with present day technology the question of finding optimal control law arises. These control functions represent the feedback devices which use the instantaneous state of the system to generate control signals. These control signals steer the system to the target state whenever any disturbance occur in the system state.

In early sixties, the problem of optimal feedback regulation for linear systems was studied by Kalman and reexamined by Lukes and others. Lukes [17] has found a proof of the fact that the stabilizability of the system is equivalent to the solvability of the Kalman-Riccati matrix equation.

Feedback regulation for nonlinear analytical systems was first attempted by Al'brekht around 1961-1963, who discovered the optimal control as a power series. However, number of the authors, who worked in this area, performed their work under the assumption that the state(s) and control(s) remain in the neighborhood of a fixed trajectory, where the system dynamics can be expanded in a power series. For sufficiently 'close' neighborhood, terms higher than the first term can be ignored and the system behavior simulate that of a linear one. This last statment is going to be used in chapter IV.

An outline of the thesis is given below;

In the remaining of chapter I a reiew of the methodology of study of systems, chronological order of the development of the theory of control, optimal control in the presence of control constraints is obtained using Pontryagin minimum principle and

statement of the optimal control is presented.

In chapter II to motivate the development in the following chapters the Linear Quadratic Regulator theory is reviewed, also the optimal control on finite as well as on infinite time intervals is presented. One of the essential control problems "Minimum Energy Control" is also presented. Example using the L.Q.R. approach is presented to highlight some of the benefits obtained by using this approach.

In chapter III the L.Q.R. is considered for the problem when the cost associated with the input is identically zero. Formulation of the optimal regulator is presented. Based on the developed regulator an algorithm to compute the optimal control is presented. Examples are given to demonstrate the system behavior with optimal control.

In chapter IV nonlinear system regulator is developed based on L.Q.R. Examples of nonlinear systems are presented and the effectiveness of the nonlinear system regulator is demonstrated.

In chapter V conclusions based on what was presented in previous chapters.

### Preview.

Engineering is concerned with understanding and controlling the materials available on this planet for the benefit of mankind. Control system engineers are concerned with understanding and controlling segments of their environment, often called systems, in order to provide useful economic products for the society. The twin goals of understanding and control are complementary because, in order to be controlled more effectively, the system under control must be understood and modeled. Furthermore, control engineering often must consider the control of poorly understood systems such as chemical process systems. The present challenge to control engineers is the modeling and control of modern, complex, interrelated systems such as traffic-

control systems, chemical processes, and robotic systems. However, simultaneously, the fortunate engineer has the opportunity to control many very useful and interesting industrial automation systems. Perhaps the most characteristic quality of control engineering is the opportunity to control machines, industrial and economic processes for the benefit of the society.

Control engineering is based on the foundation of feedback theory and linear system analysis, and integrates the concepts of network theory and communication theory. Therefore control engineering is not limited to any engineering discipline but is equally applicable for aeronautical, chemical, mechanical, environmental, civil and electrical engineering. For example, quite often a control system includes electrical, mechanical and chemical components. Furthermore, as the understanding of the dynamics of business, social and political system increases, the ability to control those systems will increase also.

To study and analyze any system different techniques are used. Among those techniques are frequency- and time-domain, however, the limitations of the frequency-domain techniques and the recently acquired attractiveness of the time-domain approach require a reconsideration of the time domain formulation of the equations representing control systems.

The frequency domain techniques are limited in applicability to linear, time-invariant systems. Furthermore, they are particularly limited in their usefulness for multivariable control systems due to the emphasis on the input-output relationship of transfer functions. By contrast, the time domain techniques can be readily utilized for nonlinear, time-varying and multivariable systems. A time-varying control system is a system for which one or more of the parameters of the system may vary as a function of time. For example, the mass of a missile varies as a function of time as the fuel is expended during flight. A multivariable system, is a system with several

input and output signals. The solution of time-domain formulation of a control system problem is facilitated by the use of digital and/or analog computers. Therefore, the time-domain representation of control systems is an essential basis for modern control theory and optimization.

### 1.1. Study of Systems.

Study and design of physical systems can be carried out by empirical methods. We apply various signals to the physical system and measure its response. If we are not satisfied with the performance of the physical system, we may adjust some of its parameters or use a compensator to achieve such goal. In such cases past experience, if any, plays an important role. This approach succeeded in designing many physical systems.

The empirical method has the disadvantage of being unsatisfactory if the requirement on the performance is very precise and stringent. It also may fall short if the physical system is very complicated or relatively expensive or safety requirement will be violated if experimented with. In such cases, analytical methods become indispensable. Generally speaking, analytical study of physical systems is divided into the following stages :

(1) Modeling (including mathematical equations closely describing the system). (2) Analysis. (3) Design.

It is very essential to distinguish between physical systems and models. In analysing any system, the corner stone is the model, and hence for the success of the design the proper model of the physical system is very important.

Depending on the question asked and on the different operational environment the physical system may have different models. For example, a bipolar junction transistor

has a model at audio frequencies and the same transistor has a different model at radio frequencies. A spaceship may be modeled as a particle to study its trajectory, and the same spaceship must be modeled differently to study its orientation, or angular velocity. It is essential to understand the physical system and its operation limits to develop a reliable model that reflects all parameters of importance. At this time it is in order to state the following definition;

A physical system is a device or a collection of devices existing in the real world; a mathematical system is a model of a physical system.

Once a mathematical system is found for a physical system, by applying various physical laws, mathematical equations describing the system can be developed. For example applying Kirchhoff's voltage and/or current law we can describe electrical systems, similarly applying Newton's law we obtain description for mechanical systems. Equations describing systems may assume many forms, they may be linear, nonlinear, differential, integral or a combination thereof. Depending on the question asked, one form of equation may be preferable to another in describing the same system. In summary a system may have many different mathematical descriptions, exactly as a physical system may have many different models.

Once a mathematical description of a system is obtained, its analysis can be performed on qualitative and/or quantitative basis. Exact response of a system due to the application of certain input can be analysed on quantitative basis. This analysis can easily be carried out using a computer. To acquire information about the general properties of the system, such as stability, controllability and observability we use qualitative analysis. This part is very important because design techniques may often evolve from this analysis. If unsatisfactory system response is found, the system has to be improved or optimized. In some cases, adjusting certain parameters improves system response. In other cases, improvement can be obtained only by introduction

of a compensator. Note, as mentioned earlier, the model plays an important role since design is carried out on the basis of the physical system's model. Hence if the model is correctly chosen, the performance of the physical system will be very close to that of the model and the effect of the corresponding parameters adjustment and/or the introduction of the appropriate compensator will produce corresponding improvement.

### I.2 Historical Perspective. [4]

Before World War II, the design of control system was primarily an art. During and after the war, considerable effort was expended on the design of closed-loop feedback control systems, and negative feedback was used to improve performance and accuracy. The first theoretical tools used were based upon the work of Bode and Nyquist. In particular, concepts such as frequency response, bandwidth, gain, and phase margin were used to design servomechanisms in the frequency domain in a more or less trial-and-error fashion. This was, in a sense, the beginning of modern automatic control engineering.

The theory of servomechanisms developed rapidly from the end of the war to the beginning of the fifties. Time-domain criteria, such as rise time, settling time, and peak overshoot ratio, were commonly used, and the introduction of the "root locus" method by Evans in 1948 provided both a bridge between the time- and frequency-domain methods and a significant new design tool. During this period, the primary concern of the control engineer was the design of linear servomechanisms. The relative insensitivity due to the use of negative feedback made it possible to tolerate moderate nonlinearity. With the beginning of aerospace exploration and rapid technological change the demand for stringent accuracy and cost efficiency became an important issue in control system design. Nonlinear control systems were examined using two

approches viz, the describing function and phase-space methods, to meet the new design challenge. The describing function method enabled the engineer to examine the stability of a closed loop nonlinear system from a frequency-domain point of view, while the phase-space method enabled the engineer to design nonlinear control system in the time domain.

Minimum-time control laws ( in terms of switching curves and surfaces) were obtained for a variety of second and third order systems in the early fifties. Heuristic and/or geometric approaches were used to prove optimality. Mathematicians found it very appealing to formulate the optimal control problem so as to optimize system performance with respect to a specific measure.

The time-optimal control problem was extensively studied by mathematicians in the United States and the former Soviet Union. In the period from 1953 to 1957, Bellman, Gamkrelidze, Krasovskii, and LaSalle developed the basic theory of minimum-time problems and presented results concerning the existence, uniqueness, and general properties of the time-optimal control. Control problems then were recognized as calculus of variation problems. Classical variational theory could not handle the "hard" control problems with constraints. This difficulty led Pontryagin et al to provide a proof for the famous minimum principle, which was first announced in 1958.

The rapid development of control theory was accompanied by continuous revolution in computer technology, this revolution has provided a vast computational power and simulation aids to the engineer. Also the need for a closed form solution was reduced owing to the availability of general purpose computer.

Modern control theory can be viewed as the confluence of three diverse streams :

- (1) Theory of servomechanisms,
- (2) Calculus of variations,
- (3) Development of computer.

Using control theory as a design aid gives a better understanding for the optimal control problem to the engineer. In the main optimal feedback systems are nonlinear and hence difficulties in analysing effects of disturbances and variations are encountered.

There are several reasons suggesting that the theory will become useful to the engineer

- (1) Computer aided designs of optimum system can be used in evaluating various alternative designs.
- (2) Knowledge of the optimal solution for a given problem facilitates the task of the engineer in selecting a suboptimal design.
- (3) With the rapid improvement in computer technology it is anticipated that some of the on-line computational difficulties will be solved.

Last, it is a fact that the theory has expanded the horizon of the engineer and made it easy to tackle difficult problems were not previously attackable.

### I.3. Pontryagin Minimum Principle.

It is almost a common practice in designing any control system to consider its cost against the expected benefit of its installation. In practice the economic factors lead to a compromise solution for the controller, which must be reasonably cheap to install, yet relatively satisfactory in its performance. In the absence of economic factors basically "the sky is the limit" i.e. the criterion in which the designer defines "bestness" is the objective. In fields such as space travel, guided weaponry, ...etc., wherein the cost was insignificant compared with the benefits obtained, optimal control theory has originated. In contrast, industrial application of optimal control systems are still comparatively rare, but with the development of powerful mini computers, they are becoming important.

The Hamiltonian formulation of the variational calculus has existed since the early nineteenth century. The most significant contribution in recent times was made by L.S.Pontryagin and his students in the late fifties. Their work made it possible to handle control problems wherein the control and state vectors are constrained.

This class of problems is distinguished from other classes due to the presence of some constraints on the control and/or the state vectors. This restriction has important significance since the controls that can be applied in many physical systems must be constrained in amplitude and/or in the number of feasible control settings.

It is instructive to note the similarity between the Lagrange-multiplier and the adjoint state used in the minimum principle. The Lagrange-multiplier approach can be summarized as follows. If we are given a function  $P(x, y)$  to minimize subject to constraints given by  $f(x, y) = 0$ , and since the increments  $dx, dy$  are not independent, in general, due to

$$\begin{aligned} df &= (f_x, dx) + (f_y, dy) \\ dP &= (P_x, dx) + (P_y, dy), \end{aligned} \tag{I.1}$$

then by introducing an undetermined multiplier  $\lambda$  we obtain an extra degree of freedom, and the multiplier  $\lambda$  can be selected to make the increments  $dx, dy$  behave as if they were independent. Defining the Lagrangian  $\mathcal{L}$  by

$$\mathcal{L}(x, y, \lambda) = P(x, y) + (\lambda, f(x, y))$$

and setting to zero the partial derivatives of  $\mathcal{L}$  with respect to all arguments yields a stationary point. Thus, by the introduction of the Lagrange-multiplier we have been able to replace the problem of minimizing the function  $P(x, y)$  subject to the constraint  $f(x, y) = 0$  with the problem of minimizing the Lagrangian  $\mathcal{L}(x, y, \lambda)$  without constraint.

We define the optimal control problem as the problem of finding a control  $u \in \mathcal{U}_{ad}$

where  $\mathcal{U}_{ad}$  is the set of admissible controls, that transfers the system from state  $x(t_0) = x_0$  at  $t_0$  to state  $x(T) = x_1$  at  $T$  while minimizing the cost functional

$$J(u) = \int_{t_0}^T l(x, u, t) dt + \phi(x(T)), \quad (I.2)$$

subject to the system dynamics

$$\begin{aligned} \dot{x} &= f(x, u, t) \\ x(t_0) &= x_0 \end{aligned} \quad (I.3)$$

Then, defining the Hamiltonian function  $H$  by

$$H(x, u, \psi, t) = (\psi, f(x, u, t)) + l(x, u, t) \quad (I.4)$$

where the vector valued function  $\psi$  is absolutely continuous. Now, since the stationarity condition ( $\frac{\partial H}{\partial u} = 0$ ) is not always true for constrained inputs, a more general condition must be used to define the optimal control, viz

$$H(x^*, u^*, \psi^*, t) \leq H(x^*, v, \psi^*, t) \quad \forall v \in \mathcal{U}_{ad} \quad (I.5)$$

Further, we denote by  $M(x, \psi, t)$  the function

$$M(x, \psi, t) = \min_{u \in \mathcal{U}} H(x, u, \psi, t)$$

equivalently

$$M(x, \psi, t) = H(x^*, u^*, \psi^*, t) \quad (I.6)$$

That is to say, this function ( $M(x, \psi, t)$ ) serves to eliminate the input  $u$  from the following canonical system of equations;

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial \psi} \\ -\dot{\psi} &= \frac{\partial H}{\partial x} \end{aligned} \quad (I.7)$$

Any solution  $u(t), x(t), \psi(t)$  of the above system with a nonzero  $\psi(t)$  that satisfies the boundary conditions

$$x(t_0) = x_0 \text{ and } x(T) = x_1 \quad (I.8)$$

will be called an extremal solution or an extremal of the above optimal problem.

Pontryagin minimum principle asserts that any solution of the optimal problem is contained among the extremals of this problem.

Note, it is required that the function  $\psi(t)$  does not vanish identically [7], because with  $\psi(t) \equiv 0$  and the linearity of  $H$  in  $\psi$  implies that  $H \equiv 0 \forall t \in I$  and hence any solution will satisfy the canonical equations which makes the minimum principle an empty assertion.

The fact that there are no boundary conditions imposed on  $\psi(t)$  is entirely natural. By a simple count of the number of "essential parameters" one can easily convince oneself that the number of "essential parameters" of the system (I.7)-(I.8) is equal to the number of its "independent conditions" and therefore, the system is not overdetermined. However, an addition of any new (independent) condition will lead to an overdetermined system.

Indeed, since equation (I.6) is uniquely solvable in  $u$  for almost all  $t$ , assuming the existence of an optimal control, then, given the initial conditions  $t_0, x(t_0), \psi(t_0)$  the solution  $x(t), \psi(t), u(t)$  of the system (I.6)-(I.7) will evolve in time in a unique way. Since  $t_0, x(t_0)$  are fixed in the boundary conditions, we are left with  $T, \psi(t_0)$ . Using these parameters we are to satisfy the terminal condition, namely,  $x(T) = x_1$  at  $t = T$ .

Since  $H(x, u, \psi, t)$  is linear in  $\psi$ , the function  $M(x, \psi, t)$  is positive homogeneous in  $\psi$ , i.e.

$$M(x, \lambda\psi, t) = \lambda M(x, \psi, t) \quad , \forall \lambda \geq 0$$

and if  $u(t)$ ,  $x(t)$ ,  $\psi(t)$ ,  $t \in I$  is a solution of the system (I.6)-(I.7) then,  $u(t)$ ,  $x(t)$ ,  $\lambda\psi(t)$  is also a solution of the above system. Therefore, there are in fact only  $(n-1)$  essential parameters among the coordinates of the vector  $\psi(t_0)$  and counting  $T$  we have at our disposal only  $n$  essential parameters with the aid of which we must satisfy the boundary condition  $x(T) = x_1$ . In fact this is a Two Point Boundary Value Problem (TPBVP), since at  $t_0$  we have the set of initial conditions  $x(t_0)$  and at  $t = T$  we have the set of terminal conditions  $x(T) = x_1$ .

#### I.4. Statment of The Optimal Control Problem.

Consider the following differential equation representing the dynamics of a given system.

$$\dot{x} = f(x, u, t) \quad (I.9)$$

where

$$x \in R^n, u \in R^m, t \in I \equiv [t_0, T]$$

and  $f$  has a continous derivative with respect to  $x$ .

An arbitrary set  $U \subset R^r$  is given in the "space of parameters"  $R^m$ . This set is said to be the set of admissible values of the control parameter. An arbitrary measurable and bounded function  $u(t)$  which is defined on  $I$  and takes on values from the set  $U$  is said to be an admissible control. This set will be denoted by  $\mathcal{U}_{ad}$  and will be known as the class of admissible controls.

If an arbitrary admissible control  $u^1(t)$  is substituted for the parameter  $u$  in equation (I.9) then we obtain the differential equation

$$\dot{x} = f(x, u^1, t), t \in I \quad (I.10)$$

Note any absolutely continous function  $x^1(t)$ ,  $t \in [0, T] \equiv I$  is said to be a solution

of this equation on the interval  $I$  if it satisfies the equality

$$\dot{x}^1(t) = f(x^1(t), u^1(t), t) \quad (I.11)$$

for almost all  $t \in I$ . By the absolute continuity of  $x(t)$ , this is equivalent to the statement that

$$x(t) = x(0) + \int_0^t f(x(s), u(s), s) ds \quad (I.12)$$

for all  $t \in I$

Given the control function  $u(t)$ ,  $t \in I$ , and the initial state  $x(0)$ , the evolution of the system with time is uniquely determined by the solution of the equation

$$\begin{aligned} \dot{x} &= f(x, u, t) \\ x(0) &= x_0 \end{aligned} \quad (I.13)$$

The optimal control problem can be stated as follows;

From the class of admissible controls find a control that minimizes the cost functional

$$J(u) = \int_0^T l(x, u, t) dt + \phi(x(T)) \quad (I.14)$$

subject to the system dynamics

$$\begin{aligned} \dot{x} &= f(x, u, t), \quad t \in [0, T] \\ x(0) &= x_0 \\ x(T) &= x_1 \end{aligned} \quad (I.15)$$

Here the cost functional is a suitably chosen criterion to be minimized for the given problem.

If we can find a control  $u^* \in \mathcal{U}_{ad}$  satisfying

$$J(u^*) \leq J(u)$$

for all  $u \in \mathcal{U}_{ad}$  and the differential equation (I.15) has a solution  $x^*(t)$  satisfying boundary conditions on the interval  $[0, T]$ , then the control  $u^*(t)$  is called the optimal control for the given problem,  $x^*(t)$  is called the corresponding optimal trajectory and  $J(u^*)$  is the minimum cost.

Sometimes it is useful to narrow down the class  $\mathcal{U}_{ad}$  by considering only specific control functions, e.g. only piecewise-continuous or piecewise-constant controls which takes on their values in  $U$ . In this respect, a control  $u \in \mathcal{U}_{ad}$  is said to be piecewise-continuous if it has a finite number of points of discontinuity on any bounded interval of the  $t$ -axis and if both left and right hand limits exist at every point of discontinuity. A control  $u \in \mathcal{U}_{ad}$  is said to be piecewise-constant if the  $t$ -axis can be partitioned into a finite number of intervals on each of which  $u$  has a constant value.

## CHAPTER II

### LINEAR QUADRATIC REGULATOR

#### II.1. Introduction.

Optimal control theory has been of considerable importance in a wide variety of disciplines. Over the years, the theory has been developed, extended and, although it is by no means complete, it has been applied to a diverse collection of problems. In order to give a better understanding, the basic theory of optimal control is outlined below.

Optimal control theory starting point is the set of equations which describes the behavior of a system. The state equations for a continuous time system are a set of first order differential equations.

$$\dot{x}(t) = f(x, u, t), \quad t \in [0, T] \quad (II.1)$$

where  $x$  is the state-vector,  $u$  is the control-vector (input) and  $f$  is a vector valued function describing the system dynamics.

The problem we have is to find a "suitable" control policy so as to achieve certain specified objectives while satisfying the dynamic constraints.

Optimal control problem solution involves the following steps:

- i. Find a control policy  $u^*$  which will steer the system in the best possible way in some accepted sense, such that a performance criterion is optimized.
- ii. With the help of a controller realize the control policy obtained above.

The following factors are considered basic for the optimum controller design:

- i. The characteristics of the system (system physical limitations).
- ii. The requirement imposed upon the plant (performance function).
- iii. The nature of information fed to the controller about the system ( open or closed loop ).

The performance function is carefully selected to translate the physical requirements of the problem into mathematical terms. It is assumed to be of the form

$$J = \phi(x(T)) + \int_0^T G(x(t), u(t), t) dt \quad (II.2)$$

where  $\phi : R^n \rightarrow R$ ,  $G : R^n \times R^p \times R \rightarrow R$

With the above outline, we can pose the following problem. Find an admissible control  $u^*$  which causes the system (II.1) to follow an admissible trajectory  $x^*$  and optimize the performance function (II.2).

The major techniques for the optimal control system design utilize:

- i. Calculus of variations.
- ii. Pontryagin minimum principle.
- iii. Dynamic programming.

### **Definitions.**

#### **D1. Class of Admissible Controls. $\mathcal{U}_{ad}$**

The class of admissible controls is given by all bounded, measurable and piecewise continuous functions defined on  $R^n \times (0, T]$  with values in  $R^m$  and satisfying the property:

$$\|u(x, t)\| \leq \beta$$

Where  $x(t)$  is the solution of the differential equation (II.1) corresponding to control  $u$  and initial condition  $x(t_0) = x_0$  that lies in the neighbourhood  $N(x^0)$  of the nominal trajectory  $x^0(t)$ ,  $t \in (t_0, T]$  and  $\beta > 0$

## D2. Optimal Feedback Control

A feedback control  $u^* \in \mathcal{U}_{ad}$  is called optimal if

$$J(x_0, u^*, t_0) \leq J(x_0, u, t_0)$$

for all  $u \in \mathcal{U}_{ad}$  for which  $x \in N(x^0)$ . In general, it is observed that it may not be always possible to translate all the performance requirements into a single mathematical criterion. Also, quite often the obtained optimal control is not directly applicable due to sensitivity and other considerations, hence a suboptimal design can be used instead .

As can be seen in the literature [1,4], systems are classified as linear and nonlinear based on the equations describing their dynamics. Similarly, controls are classified as linear and non linear.

Focusing on linear control, the most notable is linear optimal control theory, in particular the Linear Quadratic Regulator (L.Q.R) problem. The nomenclature of Linear Quadratic Regulator problem arises because the system dynamics are governed by linear differential equation while the performance index (or function ) is quadratic in both the state and control. L.Q.R. is probably the most important general problem for the purpose of applications as it is the only control problem that give rise to an optimal control law which is linear in state.

One of the most important subjects in control theory that received and still receiving considerable research effort is stability of systems as they stand, if they are initially unstable, or to improve their stability if transient phenomena do not die sufficiently fast. Stability properties of systems can be improved by state feedback using L.Q.R. theory among a number of other methods. In practice there is always some limitations on system input and output which lead quite naturally to the formulation of the optimization problem. For instance the "nominal" operating condition

of a given system is one where all system inputs, states and outputs are constant in time. Typical examples of such applications are base-load operation of power plant and temperature and humidity control in enviromental systems. In such applications, the role of the control system is primarily that of a regulator to return the system variables to their "nominal" values following a disturbance.

In summary, the linear regulator theory solves the following problem. Find a linear feedback control law  $u(x) = \Gamma x$  that minimizes the performance functional

$$J(u) = \frac{1}{2}(Mx(T), x(T)) + \frac{1}{2} \int_0^T [(Qx(t), x(t)) + (Ru(t), u(t))]dt \quad (II.3)$$

subject to the dynamic constraint

$$\dot{x} = Ax(t) + Bu(t) \quad (II.4)$$

with the matrices  $M$  and  $Q$  symmetric positive semidefinite and the matrix  $R$  symmetric positive definite, all matrices of suitable dimensions. This problem has the known solution given by

$$u(x) = \Gamma x(t)$$

$$\Gamma = -R^{-1}B'K(t)$$

where  $\Gamma$  denotes the feedback matrix, with  $K$  being the solution of the matrix Riccati differential equation,

$$\begin{aligned} \dot{K} + KA + A'K - KBR^{-1}B'K + Q &= 0 \\ K(T) &= M \end{aligned} \quad (II.5)$$

If equation (II.5) has a steady state solution i.e.  $\lim_{t \rightarrow \infty} K(t) = K_0$ , then as  $t \rightarrow \infty$  we will have  $\dot{K}(t) = 0$  and the differential equation reduces to the following algebraic equation

$$KA + A'K - KBR^{-1}B'K + Q = 0 \quad (II.6)$$

this equation is known as Algebraic Riccati Equation (ARE), its symmetric positive solution  $K_0$  guarantees the stability of the closed loop system

$$\dot{x} = (A - BR^{-1}B'K)x \quad (II.7)$$

Lemma II.1.1

There is only one solution,  $K_0$  say, of the ARE that yields a stable closed loop system (II.7) [12] .

Proof.

For proof consulte references [1,12].

## II.2. Optimal Control on a Finite Time Interval.

We start this section by introducing the Hamilton-Jacobi-Bellman equation. We pose the following optimal control problem. For the system

$$\begin{aligned}\dot{x} &= f(x, u, t) \\ x(t_0) &= x_0\end{aligned}\tag{II.8}$$

find the optimal control  $u^*(t)$ ,  $t \in [t_0, T]$  which minimizes

$$J(x_0, u(\cdot), t_0) = \int_{t_0}^T l(x(\tau), u(\tau), \tau) d\tau + \phi(x(T))\tag{II.9}$$

without explicitly defining the degree of smoothness - that is, the number of times quantities should be differentiable- we shall restrict  $f, l$  and  $\phi$  to being smooth functions of their arguments. The functions  $l$  and  $\phi$  will often be non-negative, to imply some physical variables the optimization of which is desired. As we can see, the performance function  $J(x(t_0), u(\cdot), t_0)$  depends on the initial state  $x_0$  and the time  $t_0$  and the control  $u(t)$  for all  $t \in [t_0, T]$ . The optimal control  $u^*$  may be required a priori to lie in some special set with given characteristics  $\mathcal{U}_{ad}$ . If the system starts in state  $x(t)$  at time  $t$ , the minimum value of the performance function is  $J^*(x(t), t)$ .

$$J^*(x(t), t) = \min_u \left\{ \int_t^T l(x(\tau), u(\tau), \tau) d\tau + \phi(x(T)) \right\}\tag{II.10}$$

Let  $\Omega(\tau) \equiv l(x(\tau), u(\tau), \tau)$  and  $u_1 \equiv u(t), t \in [t, t_1]$  and  $u_2 \equiv u(t), t \in [t_1, T]$ .

$$\begin{aligned}J^*(x(t), t) &= \min_{u_1} \{ \min_{u_2} \{ \int_t^{t_1} \Omega(\tau) d\tau + \int_{t_1}^T \Omega(\tau) d\tau + \phi(x(T)) \} \} \\ &= \min_{u_1} \{ \int_t^{t_1} \Omega(\tau) d\tau + J^*(x(t_1), t_1) \}\end{aligned}$$

Let  $t_1 = t + \Delta t$  and using Tylor's series expansion we get the following form

$$\begin{aligned}J^*(x(t), t) &= \min_{u_1} \{ \Delta t \Omega(t) + J^*(x(t), t) + \left( \frac{\partial J^*}{\partial x} \right)' \frac{dx}{dt} \Delta t + \frac{\partial J^*}{\partial t} \Delta t + 0(\Delta t)^2 \} \\ 0 &= \min_{u_1} \{ \Omega(t) + \left( \frac{\partial J^*}{\partial x} \right)' \dot{x} + \frac{\partial J^*}{\partial t} \}\end{aligned}$$

Last equation was obtained through taking the limit  $\Delta t \rightarrow 0$ . On arranging the terms we get,

$$\frac{\partial J^*}{\partial t} + \min_u \{l(x, u, t) + (\frac{\partial J^*}{\partial x}, f(x, u, t))\} = 0 \quad (II.11)$$

which is the known Hamilton-Jacobi-Bellman equation.

Now consider the L.Q.R. problem, from the Hamilton-Jacobi-Bellman equation (II.11) and the linear system given by

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ x(0) &= x_0 \end{aligned} \quad (II.12)$$

and the value function given by

$$J(x, u, t_0) = \frac{1}{2}(Mx(T), x(T)) + \frac{1}{2} \int_{t_0}^T [(Qx(t), x(t)) + (Ru(t), u(t))]dt \quad (II.13)$$

we get the following

$$\frac{\partial J^*}{\partial t} + \min_u \{[(Qx, x) + (Ru, u)] + (\frac{\partial J^*}{\partial x}, Ax + Bu)\} = 0. \quad (II.14)$$

Now, we claim that the optimum value function is quadratic in the state. The necessary conditions for a function  $J^*(x(t), t)$  to be a quadratic form are that  $J^*(x(t), t)$  is continuous in  $x(t)$ , and

$$J^*(\lambda x, t) = \lambda^2 J^*(x, t) \quad \text{for all real } \lambda \quad (II.15)$$

$$J^*(x_1, t) + J^*(x_2, t) = \frac{1}{2}[J^*(x_1 + x_2, t) + J^*(x_1 - x_2, t)] \quad (II.16)$$

Let  $u_x^*$  denote the optimal control over  $[t, T]$  when the initial state is  $x(t)$  at time  $t$ . From the linearity of the system (II.12) and the quadratic nature of (II.13) we get the following,

$$J^*(\lambda x, t) \leq J(\lambda x, \lambda u_x^*, t) = \lambda^2 J^*(x, t)$$

$$\lambda^2 J^*(x, t) \leq \lambda^2 J(x, \frac{1}{\lambda} u_x^*, t) = J^*(\lambda x, t)$$

for all real  $\lambda$ . These imply (II.15).

Similarly

$$\begin{aligned} J^*(x_1, t) + J^*(x_2, t) &= \frac{1}{4} [J^*(2x_1, t) + J^*(2x_2, t)] \\ &\leq \frac{1}{4} [J(2x_1, u_1 + u_2, t) + J(2x_2, u_1 - u_2, t)] \\ &= \frac{1}{2} [J(x_1 + x_2, u_1, t) + J(x_1 - x_2, u_2, t)] \\ &= \frac{1}{2} [J^*(x_1 + x_2, t) + J^*(x_1 - x_2, t)] \end{aligned}$$

where  $u_1$  is the optimal control corresponding to the initial state  $x_1 + x_2$  and  $u_2$  is the optimal control when the initial state is  $x_1 - x_2$ . Following the same reasoning we get

$$\begin{aligned} \frac{1}{2} [J^*(x_1 + x_2, t) + J^*(x_1 - x_2, t)] &\leq \frac{1}{2} [J(x_1 + x_2, \tilde{u}_1, t) + J(x_1 - x_2, \tilde{u}_2, t)] \\ &= J^*(x_1, t) + J^*(x_2, t) \end{aligned}$$

where  $\tilde{u}_1 = u_{x_1}^* + u_{x_2}^*$  and  $\tilde{u}_2 = u_{x_1}^* - u_{x_2}^*$

These imply (II.16). It is trivial to show that  $J^*(x(t), t)$  is continuous in  $x(t)$ .

Hence, we conclude that  $J^*(x(t), t)$  has the form,

$$J^*(x(t), t) = (x(t), K(t)x(t)) \quad (II.17)$$

for some symmetric matrix  $K(t)$ .

Now, we return to our L.Q.R. problem, the optimum value function is given by

$$\begin{aligned} J^*(x(t), t) &= \frac{1}{2} (x(t), P(t)x(t)) \\ \frac{\partial J^*}{\partial x} &= P(t)x(t) \end{aligned}$$

$$\frac{\partial J^*}{\partial t} = \frac{1}{2}(x(t), \dot{P}(t)x(t)).$$

To simplify the expression the time dependance will not be shown in the following two equations.

Hence, equation (II.11) becomes

$$(x, \dot{P}x) + \min_{u \in \mathcal{U}_{ad}} \{[(Qx, x) + (Ru, u)] + (2Px, Ax + Bu)\} = 0$$

$$(x, \dot{P}x) + \min_u \{(u + \Gamma x, R(u + \Gamma x)) + ((Q - P\Gamma + PA + A'P)x, x)\} = 0$$

where  $\Gamma = R^{-1}B'P$ .

Because the matrix  $R$  is positive definite, we can minimize the expression between paranthesis by setting  $u = -R^{-1}B'Px$  in which case we obtain

$$(x(t), \dot{P}(t)x(t) + (Q - PBR^{-1}B'P + PA + A'P)x(t)) = 0$$

Now, this equation holds for all  $x(t)$ , therefore,

$$\dot{P} + PA + A'P - PBR^{-1}B'P + Q = 0 \quad (II.18)$$

which is the matrix Ricatti differential equation with boundary conditions following immediately from the H-J-B boundary condition

$$J^*(x(T), T) = \phi(x(T)) = \frac{1}{2}(P(T)x(T), x(T)). \quad (II.19)$$

### II.3. Optimal Control on an Infinite Time Interval.

Consider the system given by (II.12) with continous entries for  $A(t)$  and  $B(t)$ . Let the matrices  $Q(t)$  and  $R(t)$  be symmetric, have continous entries and be positive semidefinite and positive definite respectively. Define the performance index

$$J(x(t_0), u(\cdot), t_0) = \int_{t_0}^{\infty} [(Qx(t), x(t)) + (Ru(t), u(t))]dt.$$

The problem we have is to find an optimal control  $u^*(t)$ ,  $t \geq t_0$  minimizing  $J$  and the associated optimum performance index  $J^*(x(t_0), t_0)$ .

This problem as stated is not always solvable, this is can be due to the following:

1. The state  $x(t_0)$  is uncontrollable.
2. The uncontrollable part of the system trajectory is unstable.
3. The unstable part of the system trajectory is reflected in the system performance index.

To proceed with the discussion, let us assume the given system is stabilizable [3]. Refer to section (II.2) and let  $P(t, T)$  be the solution of the matrix Ricatti equation (II.15) with boundary condition  $P(T, T) \equiv 0$ , then  $\lim_{T \rightarrow \infty} P(t, T) = \bar{P}$  exists for all  $t$  and is a solution of (II.15). Moreover,  $(x(t), \bar{P}x(t))$  is the optimal performance index  $J^*(x(t), t)$  when the initial time is  $t$  and the initial state is  $x(t)$ . The optimal control at time  $t$  ( assuming  $t$  lies in the optimization interval) is uniquely -up to a set of measure zero- given by

$$u^*(t) = -R^{-1}B'\bar{P}x(t)$$

Note that  $\bar{P}$  can be obtained from the limiting operation as mentioned above, alternatively it can be obtained by integrating the Ricatti differential equation backwards in time from an arbitrary point  $T$  and noting that  $\bar{P} = \lim_{t \rightarrow -\infty} P(t, T)$  which is the steady state value. Yet another approach as in section (II.1) is obtained by noting that the steady state of (II.15) can be characterized by setting  $\dot{P}(t) = 0$ , i.e.  $\bar{P}$  is the symmetric positive-definite solution of **A.R.E.** The corresponding optimal value of the performance index is then given by

$$J(u^*) = \frac{1}{2}(x(t_0), \bar{P}x(t_0))$$

## II.4. Minimum Energy Control.

### II.4.1. Constrained Control Case.

Consider the problem of finding a control that steer the system

$$\begin{aligned}\dot{x} &= f(x, u, t) \\ x(0) &= x_0\end{aligned}\tag{II.20}$$

from initial state  $x_0$  to final state  $x_T$  while minimizing the cost functional

$$J(u) = \phi(x(T)) + \int_0^T l(x, u, t) dt\tag{II.21}$$

and the control is constrained to

$$u(t) \in \Omega(t) \quad \forall t \in [0, T]\tag{II.22}$$

Clearly, the last constraint (control constraint) makes this problem differ from others due to the fact that the stationarity condition i.e ( $\frac{\partial H}{\partial u} = 0$ ) is not always valid for this problem.

For this problem we write the Hamilton-Jacobi-Bellman equation

$$\frac{\partial J^*}{\partial t} + \min_{u \in \Omega} \{l(x, u, t) + (\frac{\partial J^*}{\partial x}, f(x, u, t))\} = 0\tag{II.23}$$

With the exception of very few cases, the solution of this type of problems is extremely difficult if not impossible.

### II.4.2. Unconstrained Control Case.

Consider the linear system given by (II.12) and the associated cost function (II.13). The objective is to transfer a system from a given initial state  $x(t_0) = x_0$  to a specified target set  $S$  with a minimum expenditure of energy. In a number of problems

$\|u(t)\|^2$  is a measure of instantaneous rate of expenditure of energy. To minimize energy expenditure, one may minimize the cost function

$$J(u) = \frac{1}{2} \int_{t_0}^T (Ru, u) dt \quad (II.24)$$

where  $R$  is a positive definite matrix reflect the weight of each component of the vector  $u$ .

This sort of problems can be interpreted as follows;

1. The introduction of the cost function is an attempt to reach the target state (set) using a control of "reasonable" magnitude.
2. The integrand in equation (II.24) is interpreted as a measure of power provided by the controller and hence  $J(u)$  gives the total energy injected by the control input  $u(t)$  on the given interval.

Then the objective of the optimal controller is to minimize the energy required to transfer the system from  $x_0$  to  $x(T) = x_f$ . This problem has the same structure as the L.Q.R. with  $Q \equiv 0$  and  $\phi(x(T)) \equiv 0$  but with the condition  $x(T) = x_f$ . Unfortunately this is only superficial similarity, however this can be used to tackle the problem. It is clear that with those values of  $Q$  and  $\phi$  the solution of the Ricatti differential equation is identically zero for all  $t \in [t_0, T]$  i.e.  $P(t) \equiv 0$  and hence  $u^*(t) \equiv 0$ . This cannot be the case, in general, as a zero input will not normally produce a state trajectory satisfying  $x(T) = x_f$ .

For this problem the solution is given in the following theorem as the solution of the two-point boundary value problem **TPBVP**.

**Theorem II.1** [19]

If  $u^*(t)$  is an optimal control for the above problem and if  $x^*(t)$  is the resultant optimal trajectory, then there is a corresponding costate  $\psi^*(t)$  such that :

- a. The state  $x^*(t)$  and the costate  $\psi^*(t)$  satisfy the canonical system of differential

equations

$$\begin{aligned}\dot{x}^*(t) &= Ax^*(t) + Bu^*(t) \\ \dot{\psi}(t) &= -A'\psi(t)\end{aligned}\tag{II.25}$$

and the boundary conditions

$$x^*(t_0) = x_0$$

$$x^*(T) = x_f$$

b. The following relation holds for all  $u(t) \in R^r$  and  $t \in [t_0, T]$

$$\frac{1}{2}(Ru^*, u^*) + (\psi^*, Bu^*) \leq \frac{1}{2}(Ru, u) + (\psi^*, Bu)\tag{II.26}$$

**Proof**

For the proof consulte one of the references [1,3,4,...].

Example II.1 [4]

Consider the system

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = u(t)$$

$$x_1(0) = x_{10} = 1.0$$

$$x_2(0) = x_{20} = -1.0$$

or in matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

We assume that there are no magnitude constraints on the control  $u(t)$ . The problem we have is to find the control  $u(t)$  which steers the system from the initial state  $x(t_0)$  to the final state  $x(T) = 0$  while minimizing the cost functional

$$J(u) = \frac{1}{2} \int_{t_0}^T u^2(t) dt$$

where  $T$  is specified.

Using the minimum principles, the first step is to write the Hamiltonian,

$$H(x, u, \psi, t) = \frac{1}{2} u^2(t) + x_2(t) \psi_1(t) + u(t) \psi_2(t)$$

The Hamiltonian is quadratic in the control  $u$ , the optimal control can be found by setting  $\frac{\partial H}{\partial u} = 0$  and by checking  $\frac{\partial^2 H}{\partial u^2}$  for positivity.

$$\frac{\partial H}{\partial u} = u(t) + \psi_2(t)$$

$$\frac{\partial^2 H}{\partial u^2} = 1 > 0$$

We conclude that the optimal control is given by

$$u^*(t) = -\psi_2(t) \quad , t \in (t_0, T]$$

The adjoint system must satisfy the following equations;

$$-\dot{\psi} = \frac{\partial H}{\partial x}$$

we have

$$\begin{aligned}\dot{\psi}_1 &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\psi}_2 &= -\frac{\partial H}{\partial x_2} = -(\psi_1(t))\end{aligned}$$

Let the adjoint system initial values be

$$\psi_1(t_0) = C_1 \quad \psi_2(t_0) = C_2$$

Hence,

$$\begin{aligned}\psi_1(t) &= C_1 \\ \psi_2(t) &= C_2 - C_1 t\end{aligned}$$

Thus, the optimal control law is given by,

$$u^*(t) = -C_2 + C_1 t \quad , t \in [t_0, T]$$

It is clear that the optimal control is a linear function of time with slope  $C_1$ . To relate the constants  $C_1, C_2$  to the initial system conditions, we solve the system equations,

$$\begin{aligned}x_1(t) &= x_{10} + x_{20}t - \frac{1}{2}C_2t^2 + \frac{1}{6}C_1t^3 \\ x_2(t) &= x_{20} - C_2t + \frac{1}{2}C_1t^2\end{aligned}$$

The requirment was to steer the system to the origin of the state space. i.e.  $x(T) = 0$ , which leads to the following

$$C_1 = \frac{6}{T^3}(2x_{10} + x_{20}T)$$

$$C_2 = \frac{2}{T^2}(3x_{10} + 2x_{20}T)$$

and the optimal control is given by;

$$u(t) = -\frac{2}{T^2}(3x_{10} + 2x_{20}T) + \frac{6}{T^3}(2x_{10} + x_{20}T)t$$

Substituting this last expression of  $u(t)$  in the cost equation for the optimal cost, we get;

$$J^*(u) = \frac{2}{T^3}(3x_{10}^2 + 3x_{10}x_{20}T + x_{20}^2T^2)$$

It is clear that the minimum cost is highly affected by the choice of  $(T)$ . Looking to figure (II.2) we observe that the minimum energy required decreases rapidly in the beginning and then decreases slowly. For  $T$  very large  $J^*(u) \approx \frac{2}{T}x_{20}^2$ .

Looking to figure (II.1) we observe that the trajectories originating at (1,-1) and ending at (0,0) with different  $(T)$  have distinguished pattern. Although the cost differential between  $T = 3$  and  $T = 10$  is very small, but the trajectories are extremely different. Those two observations bring the sensitivity of optimal problems into focus, quite often, small perturbation in some parameter can affect the minimum cost. This sensitivity issue is a large subject in itself but considering it in an engineering design is essential.

# TRAJECTORY WITH OPTIMAL CONTROL

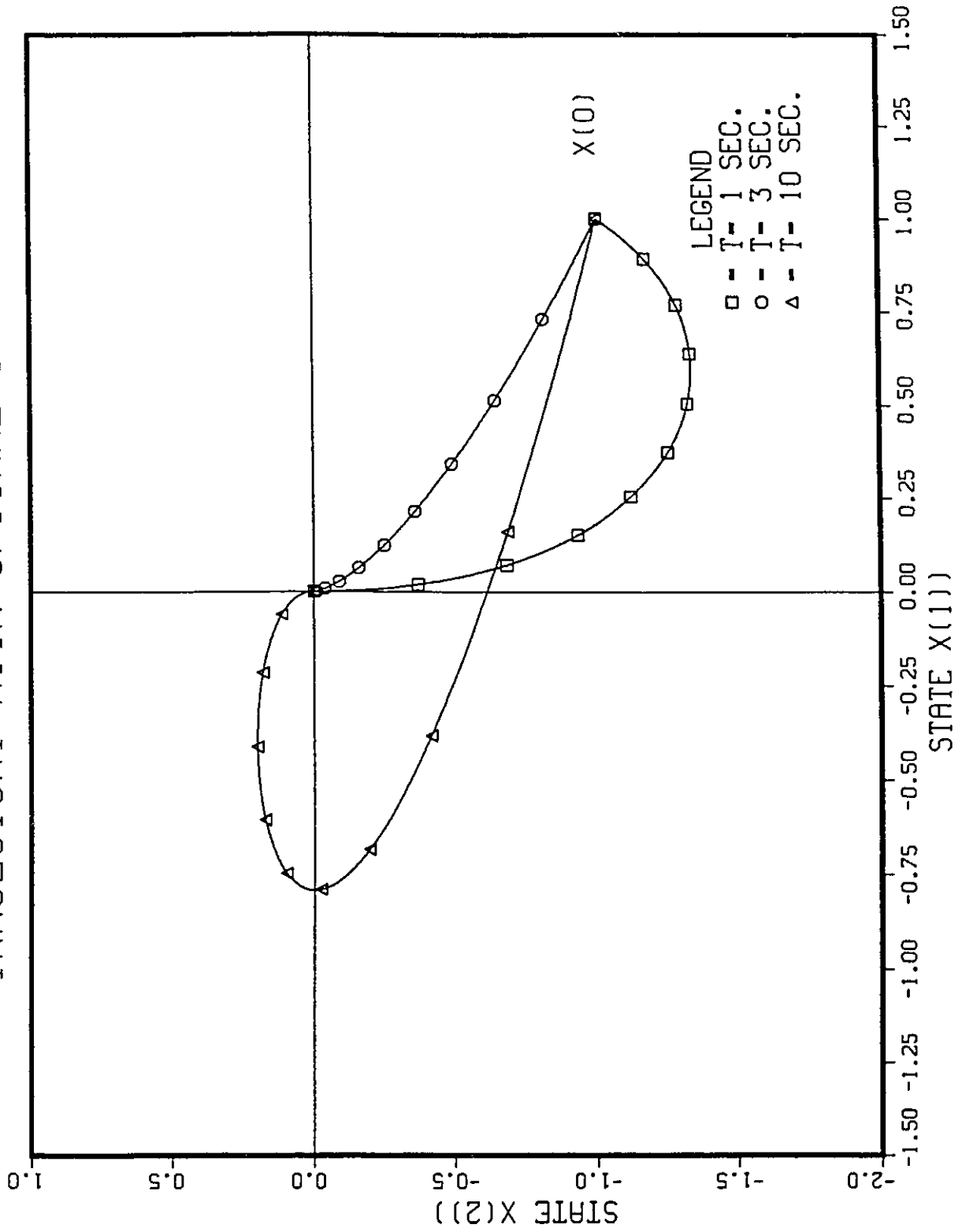


Figure II.1 Optimal Trajectories for Different Final Time ( $T$ ).

# MINIMUM ENERGY OPTIMAL CONTROL

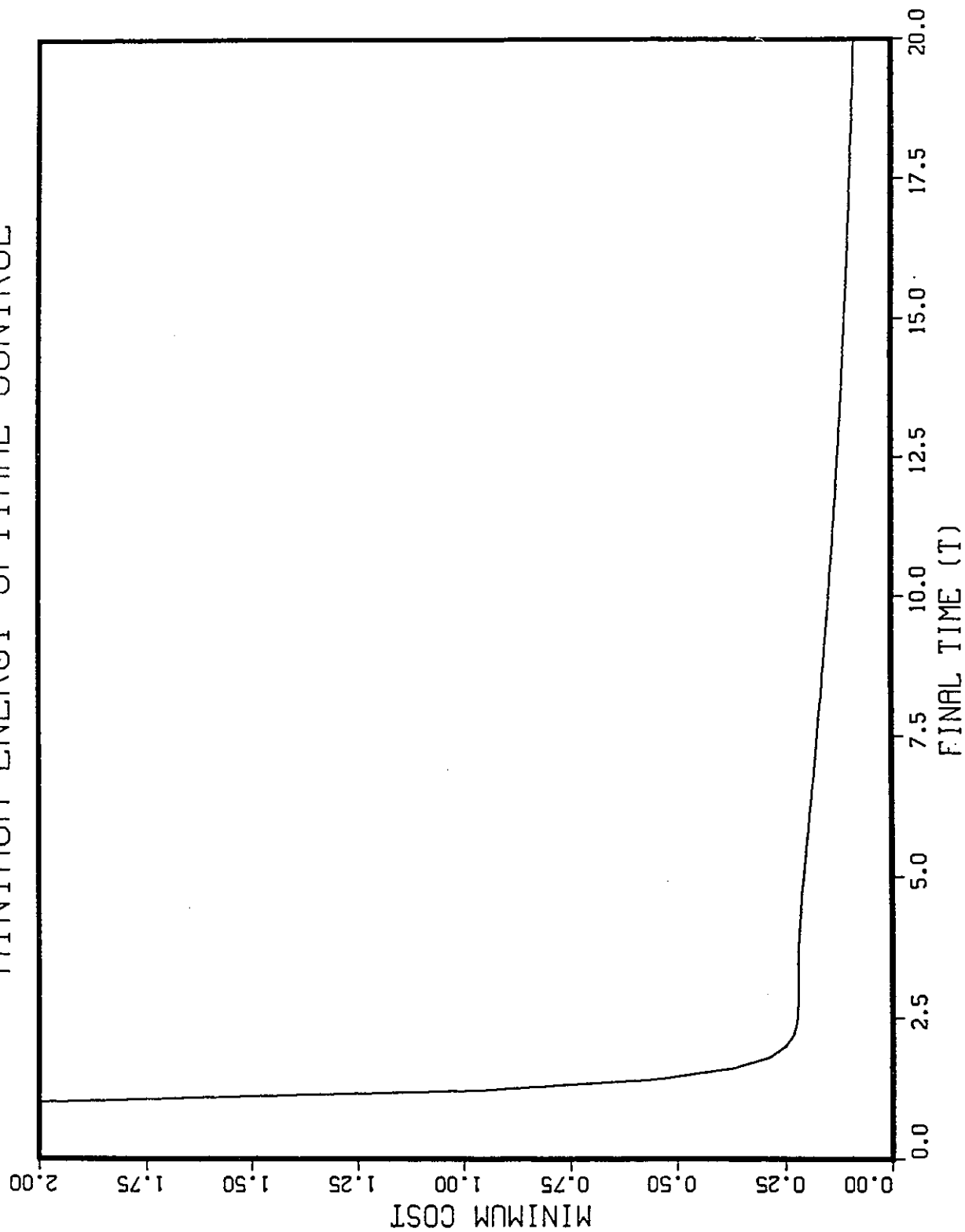


Figure II.2 Minimum Cost with Final Time.

## CHAPTER III

### REGULATORS WITH ZERO CONTROL COST

#### III.1. Introduction.

Study of the response of a system controlled by an optimal linear regulator obtained under the assumption of cheap control parameterized by a scalar weighting factor  $\epsilon$  has attracted the attention of a number of researchers. Motivation for such study stems from the fact that with high-gain feedback fast transient response can be obtained. Such problems are also important in analysing the limiting possibilities of optimal regulators and optimal filters.[14]

High-gain feedback forces the given system to have slow and fast transients coupled with high- and low-amplitude interaction. Due to this, some researchers [18,21] suggested a hierarchy of time-scales and appropriate scaling of variables. By these considerations those interactions are normalized and the problem is decomposed into several subproblems of minimal order each pertaining to only one time scale. However, in this we follow the same line as in [2], wherein we assumed different approach from those mentioned above.

In [2], for the case of constrained control, we obtained a set of coupled Riccati equations whose solution is difficult if not impossible. Using unconstrained control it was possible to recover the classical Riccati differential equation. This is considered in the next section.

#### III.2. Formulation of the Regulator problem.

Before proceeding with the formulation and to motivate the development we will consider the following example.

Find a linear control law that steers the given system from initial state  $x_0$  to final

state  $x_1 = 0$  while minimizing the following cost function

$$J(u) = \frac{1}{2}\beta x^2(T) \quad , \beta > 0 \quad (III.1)$$

subject to the dynamics of the following controllable scalar system;

$$\begin{aligned} \dot{x} &= ax(t) + bu(t) \quad , t \in [0, T] \\ x(0) &= x_0 \\ a &\geq 0 \end{aligned} \quad (III.2)$$

As will be shown later the linear control law is given by

$$u(t) = -\alpha b k x \quad (III.3)$$

where  $\alpha > 0$  and  $k$  is the solution of the scalar Riccati differential equation

$$\begin{aligned} \dot{k} + 2ak - \alpha b^2 k^2 &= 0 \\ k(T) &= \beta \end{aligned} \quad (III.4)$$

Now, for  $T$  large enough, the algebraic Riccati equation will have a solution  $\tilde{k}_\alpha$  which is a function of  $\alpha$ . Then the system is given by

$$\begin{aligned} \dot{x}(t) &= (a - \alpha b^2 \tilde{k}_\alpha) x \\ x(0) &= x_0 \\ \tilde{k} &= \frac{2a}{\alpha b^2} \end{aligned} \quad (III.5)$$

which has a solution;

$$\begin{aligned} x(t) &= e^{(a - \alpha b^2 \tilde{k}_\alpha)t} x_0 \quad , t \in [0, T] \\ &= e^{-at} x_0 \end{aligned} \quad (III.6)$$

Then the cost is given by

$$\begin{aligned} J(u) &= \frac{\beta}{2} e^{2(a-\alpha b^2 \hat{k}_a)T} x_0^2 \\ &= J_0 e^{-2\alpha b^2 \hat{k}_a T} = J_0 e^{-4aT} \end{aligned} \quad (III.7)$$

where

$$J_0 = \frac{\beta}{2} e^{2aT} x_0^2$$

Clearly, since all exponents are at least nonnegative,  $J(u)$  as given above is a non increasing function of  $\alpha$  owing to the dependance of the Riccati equation on the parameter  $\alpha$ . However, numerical examples, given later, show that after some value, increasing  $\alpha$  further has almost no effect on the cost which agrees with the intuition that 'increasing' the control energy indefinitely does not necessarily improve the performance.

With this in mind we return to our aim of formulating the regulator problem.

Find a linear feedback control law  $u(x) = \Gamma x$  that minimizes the cost functional

$$J(u) = \frac{1}{2}(Mx(T), x(T)) + \frac{1}{2} \int_0^T (Qx(t), x(t))dt \quad (III.8)$$

subject to the dynamic constraint

$$\begin{aligned} \dot{x} &= Ax + Bu \\ x(0) &= x_0. \end{aligned} \quad (III.9)$$

Comparing the cost functional (III.8) with (II.3) it is not difficult to see that the matrix  $R$  which reflects the importance of the control vector is identically zero ( $R \equiv 0$ ). Recalling that for the standard LQR problem the optimal linear feedback control law is given by

$$u^o(t) = -R^{-1}B'Kx \quad (III.10)$$

where  $K$  is the solution of the Riccati differential equation (II.5). Clearly, since  $R$  is not invertible, we can not use this form for the feedback control law, and hence a more fundamental approach to solve this problem is sought.

### Lemma III.1

Suppose the pair  $\{A, B\}$  is controllable and the matrix  $M$  is a real positive definite matrix and  $Q$  is positive semidefinite matrix all of appropriate dimension. Then there exists an optimal control  $u \in L_2(I, R^m)$  for the above problem.

### Proof

Refer to reference [2] for the proof.

The controllability is sufficient to have  $J_{min} < \infty$ . The proof of this statement is as follows.[12]

1. Suppose the matrix  $A$  is stable; i.e., all its eigenvalues have negative real parts. Then if we apply no input ( $u \equiv 0$ ), the cost, say  $J_0$ , will obviously be greater than or equal to  $J_{min}$  [due to  $u \equiv 0$  is not the optimal solution]. For stable systems with no input, the state  $x(t)$  tends to zero exponentially as  $t \rightarrow \infty$ . Therefore

$$J_0 = \int_0^\infty (Qx(t), x(t))dt < \infty$$

2. If the matrix  $A$  is unstable, but the pair  $\{A, B\}$  is controllable. Then by a stabilizing feedback we can make new system

$$\dot{\tilde{x}}(t) = (A - B\Gamma)\tilde{x}(t) + Bu(t)$$

such that the matrix  $(A - B\Gamma)$  is stable. Now we use again  $u = 0$  as before. Then the cost will be

$$J_{min} \leq \int_0^\infty [(Qx, x) + (R\tilde{u}, \tilde{u})]dt < \infty,$$

where  $\tilde{u} = \Gamma\tilde{x}$ . The controllability ensures boundedness of  $J_{min}$  whether or not  $A$  is stable. In fact, we need only to assume the stabilizability of  $\{A, B\}$ .

An alternative argument is that a bounded input  $u(t)$  could be used to drive the state of a controllable system to the origin of the state space in a finite time. If we then set  $u \equiv 0$ , the state would continue at the origin, and the cost with this strategy would be finite. Clearly, then, the optimum strategy will have a cost no higher than this, and therefore again we must have  $J_{min} < \infty$ .

### Theorem III.1

For the above problem and under the assumptions of lemma III.1, an optimal control  $u \in L_2(I, R^m)$  exists with  $\|u\| = \gamma$  for some finite  $\gamma > 0$ . Let  $x$  be the solution corresponding to  $u$ . Then the pair  $\{x, u\}$  must satisfy the integral minimum principle [7] in the closed ball

$$B_\gamma \equiv \{w \in L_2(I, R^m) : \|w\| \leq \gamma\}$$

That is, there exists a nontrivial  $\psi \in AC(I, R^m)$  (space of absolutely continuous  $R^n$ -valued functions) such that

$$\int_I H(x(t), u(t), \psi(t)) dt \leq \int_I H(x(t), v(t), \psi(t)) dt \quad (III.11)$$

for all  $v \in B_\gamma$  where the Hamiltonian  $H$  is given by

$$H(\xi, z, \eta) = (A\xi + Bz, \eta) + \frac{1}{2}(Q\xi, \xi) \quad (III.12)$$

and  $\xi, \eta \in R^n$  and  $z \in R^m$ . Furthermore,  $x$  and  $\psi$  must satisfy the canonical equations

$$\begin{aligned}\dot{x} &= H_\psi = H_\psi(x, u, \psi) \\ x(0) &= x_0 \\ \dot{\psi} &= -H_x = -H_x(x, u, \psi) \\ \psi(t) &= Mx(T)\end{aligned}\tag{III.13}$$

Expression (III.11) is equivalent to the inequality

$$l(v) \equiv \int_I (-B'\psi(t), v(t))dt \leq \int_I (-B'\psi(t), u(t))dt \equiv l(u) \tag{III.14}$$

The functional  $l$ , as defined above, is a linear functional on the Hilbert space  $L_2(I, R^m)$  its extremum must be attained on the boundary  $\partial B_\gamma$ , and hence  $u$  must have the form

$$u = -\alpha B'\psi(t) \quad , t \in [0, T], \quad 0 < \alpha < \infty \tag{III.15}$$

satisfying  $\|u\| = \gamma$ .

Let  $\psi(t) = K(t)x(t)$  then

$$\begin{aligned}\dot{\psi} &= \dot{K}(t)x(t) + K(t)\dot{x}(t) \\ &= \dot{K}(t)x(t) + K(t)(A - \alpha BB'K(t))x(t) \\ &= (\dot{K} + KA - \alpha KBB'K)x\end{aligned}$$

$$\text{and also } \dot{\psi} = -(A'Kx + Qx)$$

$$0 = (\dot{K} + KA + A'K - \alpha KBB'K + Q)x.$$

Since  $x$  is arbitrary, we have

$$\begin{aligned}\dot{K} + KA + A'K - \alpha KBB'K + Q &= 0 \\ K(T) &= M\end{aligned}\tag{III.16}$$

Defining

$$\begin{aligned}
 V &= \frac{1}{2}(Kx, x) \\
 \dot{V} &= \frac{1}{2}[(\dot{K}x, x) + (K\dot{x}, x) + (Kx, \dot{x})] \\
 &= \frac{1}{2}[(\dot{K}x, x) + (K(Ax - \alpha BB'Kx), x) + (Kx, Ax - \alpha BB'Kx)] \\
 &= \frac{1}{2}[(\dot{K} + KA + A'K - \alpha KBB'K - \alpha KBB'K)x, x] \\
 &= -\frac{1}{2}[(\alpha KBB'K + Q)x, x] \\
 &= -\frac{1}{2}[\alpha(B'Kx, B'Kx) + (Qx, x)] \\
 &= -\frac{\alpha}{2}\|B'Kx\|^2 - \frac{1}{2}(Qx, x)
 \end{aligned}$$

Hence,  $V$  is a Lyapunov functional for the feedback system

$$\begin{aligned}
 \dot{x} &= Ax - \alpha BB'Kx \\
 &= (A - \alpha BB'K)x \\
 x(0) &= x_0
 \end{aligned} \tag{III.17}$$

Looking at last equation for  $\dot{V}$ , viz

$$\dot{V} = -\frac{\alpha}{2}\|B'Kx\|^2 - \frac{1}{2}(Qx, x)$$

one may think that increasing  $\alpha$  larger and larger would make the decay rate faster and faster. In general, this is not true, since looking at the  $K$  equation (III.16) we see the dependance of  $K$  on  $\alpha$  as well.

### III.3. Computational Algorithm for The Optimal Control.

We know that the optimal control law is given by;

$$u^* = -\alpha B' K_\alpha x$$

so, our goal is to find  $\alpha$  that optimizes the cost functional (III.8).

Step 0. Set  $n=1$  .

Step 1. Guess  $\alpha_n > 0$ .

Step 2. Solve equation (III.16) for  $K_\alpha$  in the proper sense ( $K$  is solved backwards in time) for  $t \in [0, T] \equiv I$ .

Step 3. Solve the state equation (III.17).

Step 4. Compute the cost  $J_\alpha^n(u)$  using equation (III.8).

Step 5. For suitable  $\epsilon, \delta > 0$  and

a. For  $n=1$ , set  $\alpha_{n+1} = \alpha_n + \epsilon$ , and return to step 2.

b. For  $n > 1$ , see  $|J_\alpha^{n+1}(u) - J_\alpha^n(u)| < \delta$  if this is true go to next step, if not, set  $\alpha_{n+1} = \alpha_n + \epsilon$ , and return to step 2.

Step 6. Print  $\alpha, J_\alpha^n(u)$ . Stop.

### III.4. Examples.

For illustration of the results presented in this chapter, we consider the following examples. In all examples the system is given by

$$\dot{x} = Ax + Bu \quad , t \geq 0$$

$$x(0) = x_0$$

where it is required to steer the system from the initial state  $x_0$  to the neighborhood of the origin of the state space while minimizing the quadratic cost functional

$$J = \frac{1}{2}(Mx(T), x(T))$$

for  $M$  positive definite. For simulation purposes we considered the following

$$x(0) = (1.0, 1.5)$$

$$K(T) = M = 5.0 \times I$$

#### **Example III.1.**

In this example, the system parameters are given by;

$$A = \begin{pmatrix} -2.0 & 0.5 \\ 1.0 & -0.5 \end{pmatrix} \quad , B = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}$$

One can easily verify that this system is stable and controllable.(see the eigen values and the controllability matrix ).

As a function of  $\alpha$  we solve the Riccati and the state equations and compute the cost and the control norm which is given by

$$\|u\| = \alpha \sqrt{\frac{1}{T} \int_0^T \|B'Kx\|^2 dt}$$

those are shown in figure (III.1). As expected the cost is a nonincreasing function of  $\alpha$ . As  $\alpha$  assumes large values, the cost tends to saturate and the rate of increase of the norm of control is decreasing.

# REGULATION WITH NO CONTROL COST CONTROLLABLE-STABLE SYSTEM.

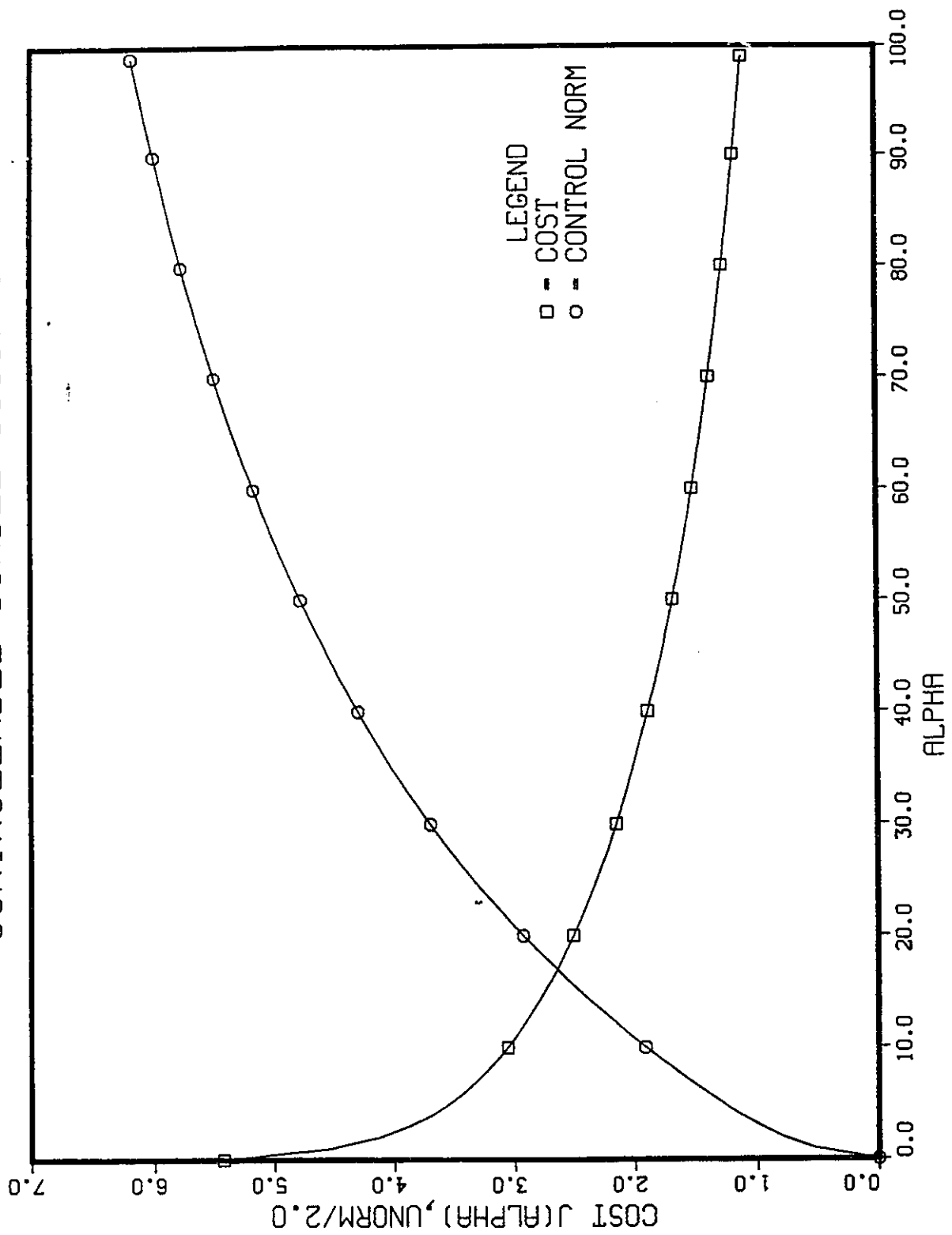


Figure III.1 Cost  $J(\alpha)$  and Control Norm  $\|u\|$  with Control multiplier  $\alpha$  Ex.III.1.

**Example III.2.**

In this example, the system parameters are given by;

$$A = \begin{pmatrix} -1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix} \quad , B = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}$$

This system is unstable, (see the eigen values), but it is controllable, hence it is stabilizable.

By solving the Riccati equation and the state equation we can get the cost and the control norm as a function of  $\alpha$ . ( see figure (III.2)).

Since we are using a linear control law of the form

$$u^* = \Gamma x$$

then, provided  $u^* \neq 0$ , we can say large excursions in the state leads to large excursions in the control, and since for this class of problems (zero control cost problems) the cost is given by

$$J = \frac{1}{2} \int_0^T (Qx, x) dt$$

which implies that large excursions in the state leads to sharp drop in the cost.(see figure (III.2).

For uncontrollable-stable and uncontrollable-unstable refer to reference [2].

# REGULATION WITH NO CONTROL COST CONTROLLABLE-UNSTABLE SYSTEM.

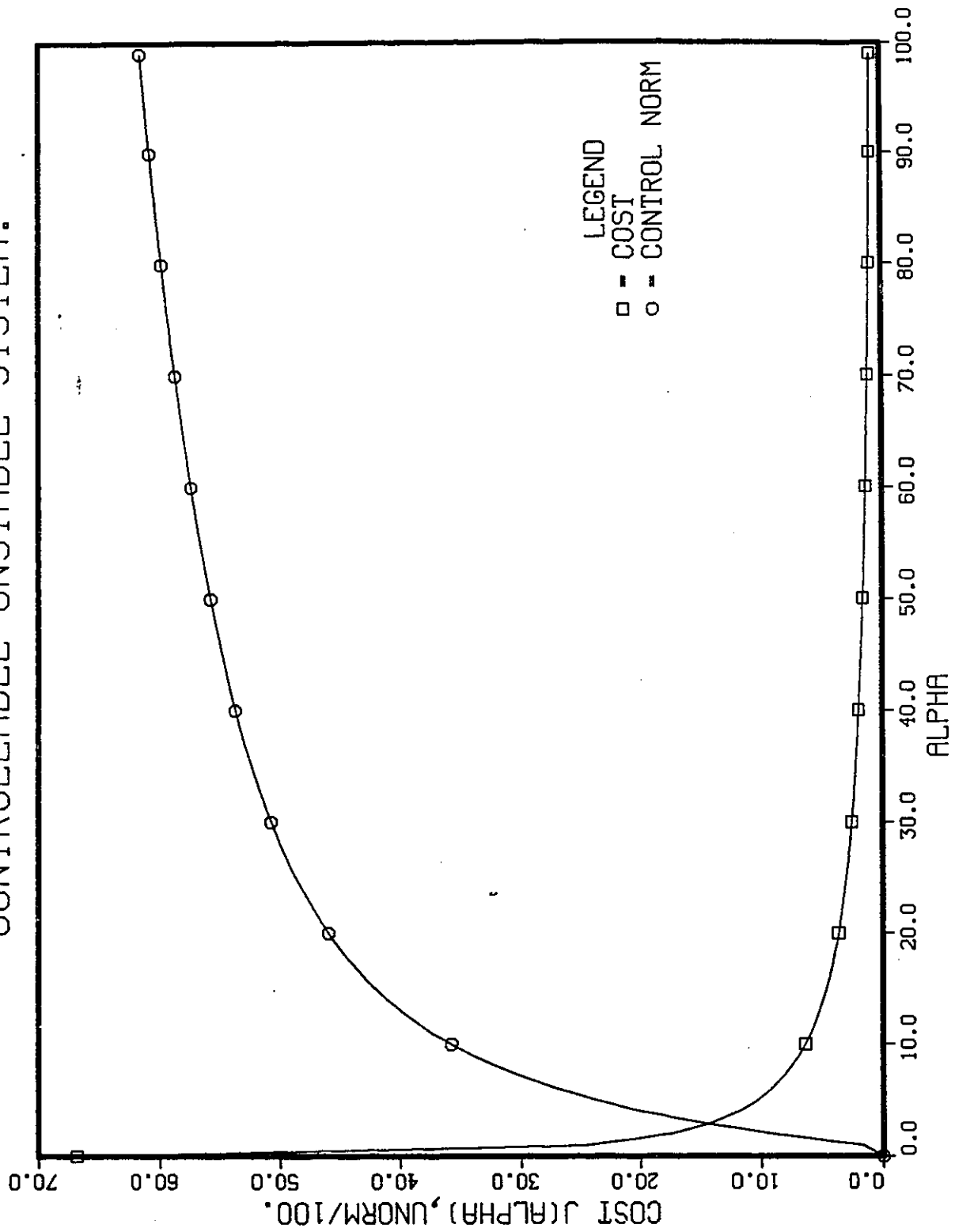


Figure III.2 Cost  $J(\alpha)$  and Control Norm  $\|u\|$  with Control multiplier  $\alpha$ . Ex.III.2.

## CHAPTER IV

### REGULATION OF NONLINEAR SYSTEMS

#### IV.1 Introduction.

In practice, most components and actuators found in physical systems have non-linear characteristics. However, we may find that some devices have moderate nonlinear characteristics or the nonlinear properties would occur if they are driven into certain operating regions. For systems with moderate nonlinearities modeling as linear system may give quite accurate analytical results over a "reasonable" range of operating conditions. For systems with strong nonlinear characteristics linearized model is valid only for a limited range of operating conditions very close to the operating trajectory at which the linearization is carried out. As we present later, the linearized system is a time varying one.

The optimal control law  $u^*(x)$  can be obtained by solving the Hamilton-Jacobi-Bellman equation for the given problem as demonstrated earlier ( for LQR).

We solve the following problem. Find a feedback law  $u^*$  that minimizes the functional

$$J(u, t) = \phi(x(T)) + \int_t^T l(x, u, t) dt \quad (IV.1)$$

subject to the dynamic constraint

$$\begin{aligned} \dot{x} &= f(x, u, t) \\ x(0) &= x_0 \end{aligned} \quad (IV.2)$$

Using the Hamilton-Jacobi equation we get

$$\frac{\partial J^*}{\partial t} + \min_{u(t) \in U_{ad}} \{ l(x, u, t) + \left( \frac{\partial J^*}{\partial x}, f(x, u, t) \right) \} \quad (IV.3)$$

The H.J.B equation is a partial differential equation whose solution is the value function. This is very difficult to solve. Except in a very few special cases, the solution

of the H.J.B equation for systems above the first order is a hopeless task. Even if one could find such an optimal solution, the resulting control law would very likely be difficult if not impossible to realize. Even an approximate solution to the exact optimization problem ( such as the power series ) may have serious convergence difficulties and may be too complex and expensive to realize. Hence one may as well abandon the hope of finding either the exact or the approximate solution to the optimal control problem.

In other words, for the general class of non-linear systems the optimal control formalism has not been very successful in the design of optimal feedback control laws. The construction and realization of optimal control laws for non-linear systems is an active field of control research . There have been a wide variety of techniques and approximations used to circumvent the difficulties associated with the non-linear optimal control problem, the primary emphasis is usually on the synthesis of suboptimal control law.

A large class of systems can be represented by the following state equation,

$$\begin{aligned}\dot{x} &= f(x, u, t) \\ x(t_0) &= x_0\end{aligned}\tag{IV.4}$$

where  $x(t)$  represents the state vector,  $u(t)$  the control vector and  $f(x, u, t)$  denotes a vector valued analytical function describing system dynamics. A significant feature of simple analytical functions is that they are smooth enough to possess a convergent Taylor expansions at all points and consequently can be linearized, so that the full mathematical power of linear system analysis becomes available.

A linearization process that depends on expanding the non-linear state equation into Taylor series about a given trajectory is described below;  
Discard all the terms higher than the first term of the Taylor series, and a linear

approximation of the nonlinear state equation at the given trajectory results.

Let the nominal operating trajectory be denoted by  $x^o(t)$ , which corresponds to the nominal input  $u^o(t)$  and some fixed initial state.

Expanding (IV.4) into a Taylor series about  $x^o(t)$  and neglecting all terms higher than the first term we get;

$$\dot{x}(t) = f(x^o, u^o) + \frac{\partial f}{\partial x}(x - x^o) + \frac{\partial f}{\partial u}(u - u^o) + H.O.T. \quad (IV.5)$$

where

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

and

$$\frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \dots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \dots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} \dots & \frac{\partial f_n}{\partial u_m} \end{pmatrix}$$

where  $HOT$  stands for High Order Terms. Then,

$$\dot{x} - f(x^o, u^o) = A\tilde{x} + B\tilde{u} + H.O.T. \quad (IV.6)$$

where  $A = \frac{\partial f(x^o, u^o)}{\partial x}$        $B = \frac{\partial f(x^o, u^o)}{\partial u}$

$\tilde{x} = x - x^o$        $\tilde{u} = u - u^o$ .

Defining  $\xi = x - x^o$ , equation (IV.6) takes the form

$$\dot{\xi}(t) = A(t)\xi(t) + B(t)v(t) + g(t) \quad (IV.7)$$

where the  $H.O.T.$  are represented by the vector  $g(t)$ . If the Euclidean norm of  $g$  approaches zero " i.e.  $\|g\| \rightarrow 0$  " over a given region close to the nominal trajectory,

then for practical reasons the linearized system (IV.7) will be a good representation of the system (IV.4).

One way of looking at (IV.7) is as follow;

Given the linear time varying system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0\end{aligned}\tag{IV.8}$$

with some uncertainty in the system parameters reflected in the entries of the matrix  $A$ . i.e. we can write the matrix  $A$  as  $A = A^o + \tilde{A}$ , where  $\tilde{A}$  represents the uncertainty in the system. (IV.8) can be written as

$$\dot{x}(t) = A^o x(t) + Bu(t) + \tilde{A}x(t)\tag{IV.9}$$

Comparing equations (IV.7) and (IV.9) we can think of the last term of equation (IV.9) viz " $\tilde{A}(t)x(t)$ " as the vector " $g$ " in equation (IV.9) provided the uncertainty is sufficiently bounded. Similar argument can be used for uncertainty in the matrix  $B$ . Although the last term in the last equation is state dependent, we still can use the function  $g$  as state independant provided the following holds;

For the system

$$\dot{y} = Ay + Bv + c(y)\tag{IV.10}$$

where  $c(y)$  represents a state dependant vector valued function . If the vector valued function  $c(y)$  satisfy the linear growth condition, i.e.

$$\|c(y)\| \leq k(1 + \|y\|)\tag{IV.11}$$

for all  $t \in [t_0, T] \equiv I$  then the attainable set is

$$\mathcal{A}(t) \equiv \{\xi \in R^n : \xi = \phi(t, t_0)y(t_0) + \int_{t_0}^t \phi(t, s)(Bv(s) + c(y))ds\}\tag{IV.12}$$

For  $u \in \mathcal{U}_{ad}$  the attainable set  $\mathcal{A}(t)$  is compact. Then we can write

$$\bigcup_{t \in I} \mathcal{A}(t) \equiv \Gamma \quad (IV.13)$$

Define the set

$$\mathcal{G} \equiv \{\xi \in R^n : \|c(\xi)\| \leq k(1 + \|\xi\|)\} \quad (IV.14)$$

Therefore  $\mathcal{G} \subset \Gamma \subset R^n$  and hence the problem takes the form

$$\dot{x} = A(t)x(t) + B(t)u(t) + g(t)$$

where the function  $g(t)$  is state independent vector valued function taking values from the set  $\mathcal{G}$ .

Another way of looking at (IV.7) is through thinking of the linear system (IV.8) subjected to bounded perturbation represented by  $g(t)$ .

#### Theorem IV.1 [15]

Consider the time-invariant system with state differential equation

$$\dot{x} = f(x(t)) \quad (IV.15)$$

Suppose that the system has an equilibrium state  $x_e$  and that the function  $f$  possesses partial derivatives with respect to the components of  $x$  at  $x_e$ . Suppose the linearized state differential equation about  $x_e$  is

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) \quad (IV.16)$$

where the constant matrix  $A$  is the Jacobian of  $f$  at  $x_e$ . Then if  $A$  is asymptotically stable, the solution  $x(t) = x_e$  is an asymptotically stable solution of the nonlinear equation (IV.4).

For proof consult reference [11, 15]. In fact this theorem is nothing but Lyapunov stability criterion.

Note that we cannot conclude anything about stability in the large simply because linearization is around  $x_e$ .

#### IV.2. Formulation of Regulator Problem.

Consider the linear system

$$\dot{x} = Ax + Bu + g$$

with the associated cost function

$$J = \frac{1}{2} \int_0^T [(Qx, x) + (Ru, u)] dt + \frac{1}{2} (Mx(T), x(T))$$

For this problem the Hamiltonian is given by,

$$H(x, u, \psi) = (Ax + Bu + g, \psi) + \frac{1}{2} [(Qx, x) + (Ru, u)]$$

Applying the minimum principle we get

$$H_u = B'\psi + Ru$$

$$u^o = -R^{-1}B'\psi$$

also

$$\dot{\psi} = -H_x = -(A'\psi + Qx)$$

$$\psi(T) = Mx(T)$$

Now, let

$$\psi = Kx + \beta$$

using the above set of equations we get

$$\dot{\psi} = \dot{K}x + K\dot{x} + \dot{\beta}$$

Substituting and simplifying we get the following set of equations

$$\begin{aligned} \dot{K} + KA + A'K - KBR^{-1}B'K + Q &= 0 \\ K(T) &= M \end{aligned} \tag{IV.17}$$

and

$$\begin{aligned} \dot{\beta} + (A' - KBR^{-1}B')\beta + Kg &= 0 \\ \beta(T) &= 0 \end{aligned} \tag{IV.18}$$

The optimal control law is given by

$$u^o = -R^{-1}B'(Kx + \beta)$$

The resulting state equation is given by

$$\dot{x} = (A - BR^{-1}B'K)x - BR^{-1}B'\beta + g \tag{IV.19}$$

#### IV.3. Regulator for Non-linear system.

In most applications engineers are required to design a controller to steer the system to the required target. The available alternatives for such a problem are [13] ;

1. An on-line digital computer that calculates optimal control signals as the process evolves, and additional hardware to synthesize the control signals.
2. A special purpose digital controller to synthesize an optimal control law that has been precomputed off-line with a general purpose digital computer.
3. A suboptimal, but easily implemented, controller whose configuration and parameters have been precalculated with an off-line computer.

The implications of each of these alternatives are;

\* For the first alternative, it may be difficult to justify economically the presence of an on-line computer. Due to the time required for computation the controllers

implemented in this way are suboptimal specially if the system states change rapidly with time .

\* For the second alternative, although all computations are done off-line which implies that the computer used for computation can be used for other tasks but the controller in this case will require a large amount of storage which most likely be of the rapid-access storage.

\* For the last alternative, owing to its easy implementation, the suboptimal controller is very attractive from a practical point of view. The acceptability of the suboptimal design is based on the comparison of the system performance with optimal and suboptimal controller. Indeed the system performance with the suboptimal should be comparable with that of the optimal controller.

Two problems can be considered here, viz, free end-point problem and fixed end-point problem. The difference between the two is the requirement for the final state to be zero for the fixed end-point problem otherwise they are the same.

The nonlinear system is assumed to be given by simple analytical functions. Hence the system

$$\dot{x} = f(x, u(x, t), t)$$

can be written in the form;

$$\dot{x} = A(t)x(t) + v(x, t)$$

$$x(0) = x_0$$

where the function  $v(x, t)$  contains the higher order terms in  $x$ . It is known that the solution for this problem exists on  $[0, T]$  for  $x_0 \in N_{x^0}$  (neighborhood of the optimal trajectory).[22] Hence,

$$\|v(x, t)\| \leq \Theta(x)\|x(t)\|$$

for  $t \in [0, T]$  where the function  $\Theta$  has the property  $\lim_{x \rightarrow 0} \Theta(x) = 0$ .

From linear system theory we have;

$$x(t) = \Phi(t)\Phi^{-1}(0)x_0 + \int_0^t \Phi(t)\Phi^{-1}(s)v(x(s), s)ds \quad (IV.20)$$

where  $\Phi(t)$  is the fundamental matrix solution of the linear equation

$$\dot{x} = A(t)x(t)$$

and satisfies

$$\dot{\Phi} = A(t)\Phi(t)$$

$$\Phi(0) = I.$$

Note that the continuity of  $\Phi(t)$  on  $[0, T]$  has the result that  $\Phi(t)\Phi^{-1}(s)$  is bounded for all  $(t, s) \in [0, T]$  i.e.

$$\|\Phi(t)\Phi^{-1}(s)\| \leq M$$

where  $M$  is a positive number. Hence,

$$\|x(t)\| \leq M\|x_0\| + \int_0^t M\Theta(x(s))\|x(s)\|ds$$

choose  $\delta > 0$  sufficiently small such that  $|\Theta(x)| \leq 1$  for  $\|x_0\| < \delta$ . Then,

$$\|x(t)\| \leq M\|x_0\| + \int_0^t M\|x(s)\|ds$$

From Gronwall inequality we can write

$$\|x(t)\| \leq M\|x_0\|exp(Mt)$$

for  $\|x_0\| < \delta$ . Furthermore,

$$\sup_{t \in I} \|x(t, x_0) - \Phi(t)x_0\| = o(\|x_0\|).$$

That is to say with the proper selection of  $\delta$  the nonlinear system behavior will be very close to that of the linearized system.

**Lemma 3.1**

For each feedback control  $u \in \mathcal{U}_{ad}$ , with  $u(x, t) = u^* + \delta u$ , for which  $\|\delta u\|$  is small, there exists a neighborhood  $N_{x^0}$  of the optimal trajectory  $x^0$  in which  $J(u) = J(u^*) + G(\delta x, \delta u)$ , where  $u^*$  is the optimal control law of the linearized system, and  $G$  represents the excursion in the cost.

### System Behavior with Suboptimal Control.

Given the dynamical system (IV.4), the linearized system (IV.7) and the assumption that the H.O.T. are bounded in the neighborhood of the reference trajectory we can find the suboptimal control as follows:

1. With proper choice of the matrices  $Q$  and  $R$  solve the LQR problem to get the linear control law  $u^* = -R^{-1}B'K(t)x(t)$  where  $K(t)$  is the solution of the Riccati equation (IV.17),  $t \in [t_0, T]$ . Note that the Riccati equation is solved backward in time.
2. Apply the control obtained in part 1 above subject to the given initial conditions to the non-linear system under consideration.
3. Note the behavior of the system for different initial conditions and perturbations.
4. In part 1, the solution  $K$  of the 'Riccati equation' is time dependent. Since we are seeking a suboptimal solution, we may solve the algebraic Riccati equation instead of the differential one and save computation time specially if the solution period is long enough.

### Issues in selecting $Q$ and $R$ [3]

- a. Designers need to be prepared to combine  $Q, R$  selection with iteration. The selection of  $Q$  and  $R$  as a translation of system specification is imprecise, and so often initially chosen  $Q$  and  $R$  may be inappropriate .
- b. Selection of  $Q, R$  may interact with the state estimator design process.
- c. State vector entries generally have physical significance. Then the choice of  $Q, R$  entries is reflective of physical insight especially if diagonal  $Q, R$  are used.

In reference [24] the degree of positive definiteness of  $Q_1$  in comparison with  $Q's$  is described by  $Q_1 > Q$  if  $Q_1 - Q$  is positive definite. One way of expanding the size of the region of parameter perturbation is by increasing the degree of positive

definiteness of  $Q$ . On the other hand, this is going to increase the cost associated with the problem.

#### IV.4. Examples.

In order to show the applicability of the method we have given above two examples of nonlinear systems are given below.

##### **Example IV.1**

##### **Stabilization of Satellites. [1]**

The dynamics of a satellite in geosynchronous orbit is given by

$$\begin{aligned} I_x \dot{p} + (I_z - I_y)(q - w_0)r &= T_x \\ I_y \dot{q} + (I_x - I_z)qr &= T_y \\ I_z \dot{r} + (I_y - I_x)p(q - w_0) &= T_z \end{aligned} \tag{IV.21}$$

where  $I_x, I_y, I_z$  are the moments of inertia along the  $x, y, z$  directions respectively and  $p, q, r$  are the angular velocities and  $T_x, T_y, T_z$  are the externally applied torques.

In this example it is required to maintain a pointing accuracy and to stall any undesirable motion. The system of equation in compact form is given by;

$$\dot{x} = f(x) + Bu \tag{IV.22}$$

where

$$x = (p, q, r)'$$

$$u = (T_x, T_y, T_z)'$$

where

$$f(x) = \begin{pmatrix} -(I_z - I_y)(q - w_0)r/I_x \\ -(I_x - I_z)pr/I_y \\ -(I_y - I_x)p(q - w_0)/I_z \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{pmatrix}$$

A suitable cost functional, for the satellite in a geosynchronous orbit, which minimizes the kinetic energy and the applied torques is given by;

$$J(u) = \frac{1}{2} \int_0^T [(Qx, x) + (Ru, u)] dt + \frac{1}{2} (Mx(T), x(T)) \quad (IV.23)$$

where  $Q$ ,  $R$  and  $M$  are weighting matrices of the appropriate dimension.

In order to apply the method presented previously it is important to note that, in order to maintain the satellite pointing accuracy, the satellite, first of all, has to have a "nominal" angular velocities, which keep the satellite in a nominal trajectory. However, in case of perturbation a regulator is required to steer the system states (satellite angular velocities) to the nominal trajectory. Since

$$\dot{x} = f(x) + Bu$$

and

$$\frac{d}{dt}(x - x^0) = f(x) - f(x^0) + B(u - u^0) \quad (IV.24)$$

Expanding  $f(x)$  in Taylor's series around  $x^0$  and letting  $v = u - u^0$  we get;

$$\frac{d}{dt}(x - x^0) = f(x^0) + f'(x^0)(x - x^0) + H.O.T. - f(x^0) + Bv \quad (IV.25)$$

In the neighborhood of the nominal trajectory we have for this problem

$$\lim_{\|x\| \rightarrow 0} \frac{\|H.O.T.\|}{\|x\|} = 0.$$

Hence equation (IV.25) can be written in the form

$$\dot{\xi} = A\xi(t) + Bv(t) + g(t) \quad (IV.26)$$

where

$$A = \begin{pmatrix} 0 & 0 & (I_z - I_y)w_0/I_x \\ 0 & 0 & 0 \\ (I_y - I_x)w_0/I_z & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{pmatrix}.$$

For simulation purposes the following values were used;

$$I_x = 645, \quad I_y = 100, \quad I_z = 669, \quad w_0 = 7.29 \times 10^{-5}$$

Now, the control law which will return the perturbed system to the nominal trajectory is given by ;

$$v^* = -R^{-1}B'(Kx + \beta) \quad (IV.27)$$

where  $K$  is the solution of the Ricatti differential equation

$$\begin{aligned} \dot{K} + KA + A'K - KBR^{-1}B'K + Q &= 0 \\ K(T) &= M, \end{aligned} \quad (IV.28)$$

and  $\beta$  is the solution of

$$\begin{aligned} \dot{\beta} + (A' - KBR^{-1}B')\beta + Kg &= 0 \\ \beta(T) &= 0. \end{aligned} \quad (IV.29)$$

With the linear (affine) control law

$$u^*(t) = -R^{-1}B'(Kx + \beta),$$

so obtained, we study the behavior of the full nonlinear system given by;

$$\dot{x} = f(x) + Bu.$$

Without loss of generality the origin of the state space can be considered to be the desired state to steer the system to. Solving these differential equations using the

IMSL routine 'DVERK' ( which uses fifth and sixth order Runge-Kutta method ), with different perturbations, we observe the following;

- i. Under initial conditions with  $\|x_0\| \leq 1.7321$  and different perturbations ranging from 10% to 90% of the initial state  $x_0$  in each component of the state vector, regulation to within 10% of the initial state  $x_0$  is achieved within the solution period. see figures [IV.1,IV.6,IV.11,IV.16]
- ii. The vector  $(\beta)$  which is due to the perturbation, 'remembers' when the perturbation started and ended. The presence of this vector helps in reducing the state excursion and steering the system to its target. see figures [IV.5,IV.10,IV.15,IV.20]. It is worth noting that  $\|\beta\|$  is proportional to the perturbation  $\|g(t)\|$ .
- iii. The state norm  $\|x(t)\|$  is asymptotically decreasing function of time. See figures [IV.2,IV.7,IV.12,IV.17].
- iv. The control vector norm is also asymptotically decreasing function of time. See figures [IV.3,IV.8,IV.13,IV.18] for the control vector and figures [IV.4,IV.9, IV.14,IV.19] for the control vector norm.

Note, for the satellite system given above a Lyapunov function is given by [1, chapter 4]

$$V(x) = \frac{1}{2}[I_x p^2 + I_y q^2 + I_z r^2]$$

$$\dot{V}(x) = T_1 p + T_2 q + T_3 r$$

choosing

$$T_1 = -u_1(p)$$

$$T_2 = -u_2(q)$$

$$T_3 = -u_3(r)$$

where  $u_i(\xi)\xi > 0$  for  $\xi \neq 0$ , we have

$$\dot{V} = -(u_1(p)p + u_2(q)q + u_3(r)r) < 0$$

for  $(p, q, r) \neq 0$ .

Clearly with this control the system is globally asymptotically stable with respect to the origin of the state space. Since the control we used in our previous example is  $u^* = -R^{-1}B'(Kx + \beta)$ , the simulation results agreed with the above statement. Since the  $\beta$  equation is solved backward in time, from the terminal time  $T$  to the initial time  $t_0$ , hence all data used to solve the  $\beta$  equation are somehow carried in this solution, this explains the predictive behavior of this method which is shown in the simulation results. However, as we can see from figures IV.0-1 and IV.0-2 that using the compensator  $\beta \equiv 0$  results in excursion in the state norm about double that when the compensator  $\beta \neq 0$ .

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL.

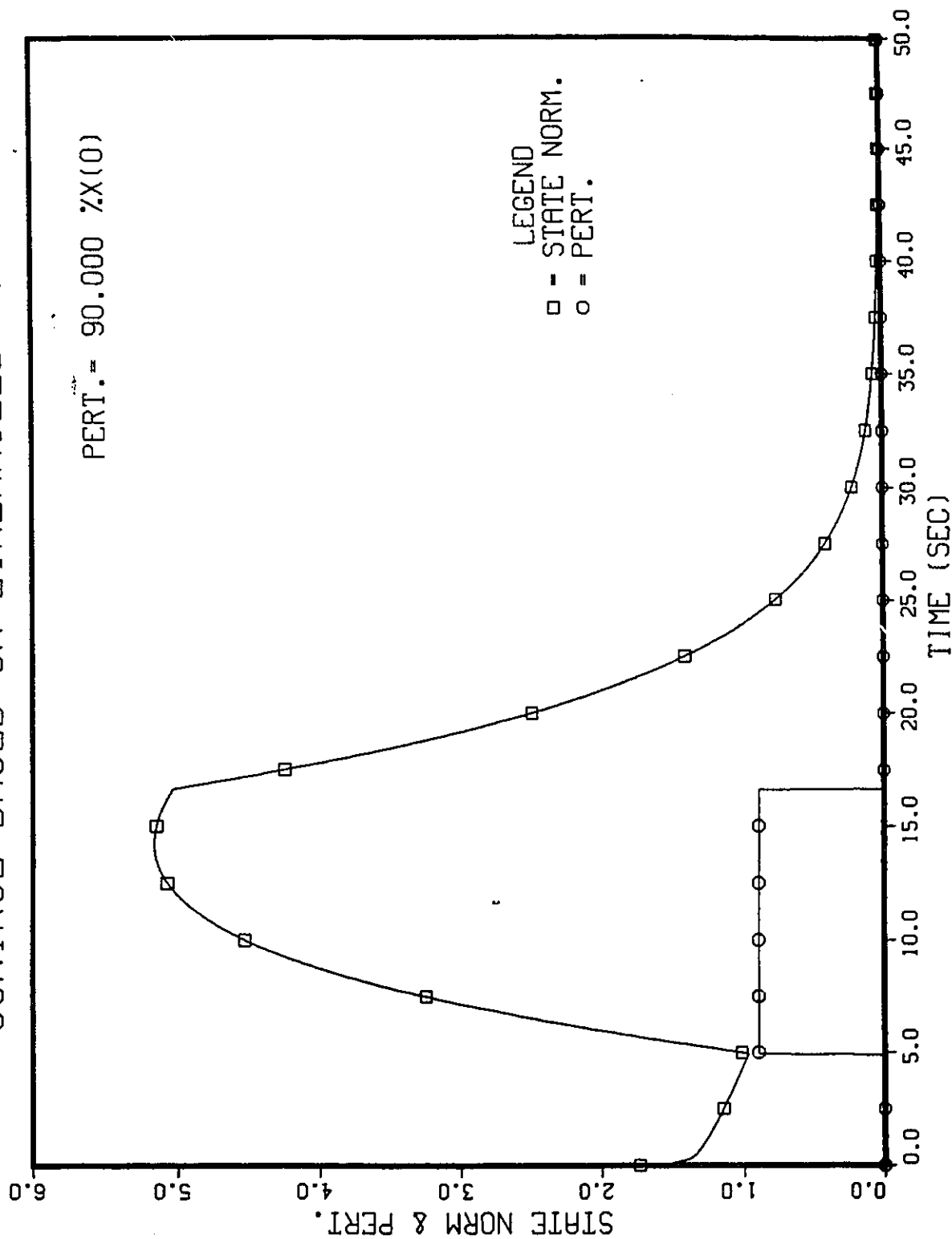


Figure IV.0-1 State Norm  $\|x(t)\|$  for  $\|g(t)\| = 1.5588$  with Time. Ex. IV.1  $\beta \equiv 0$

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL.

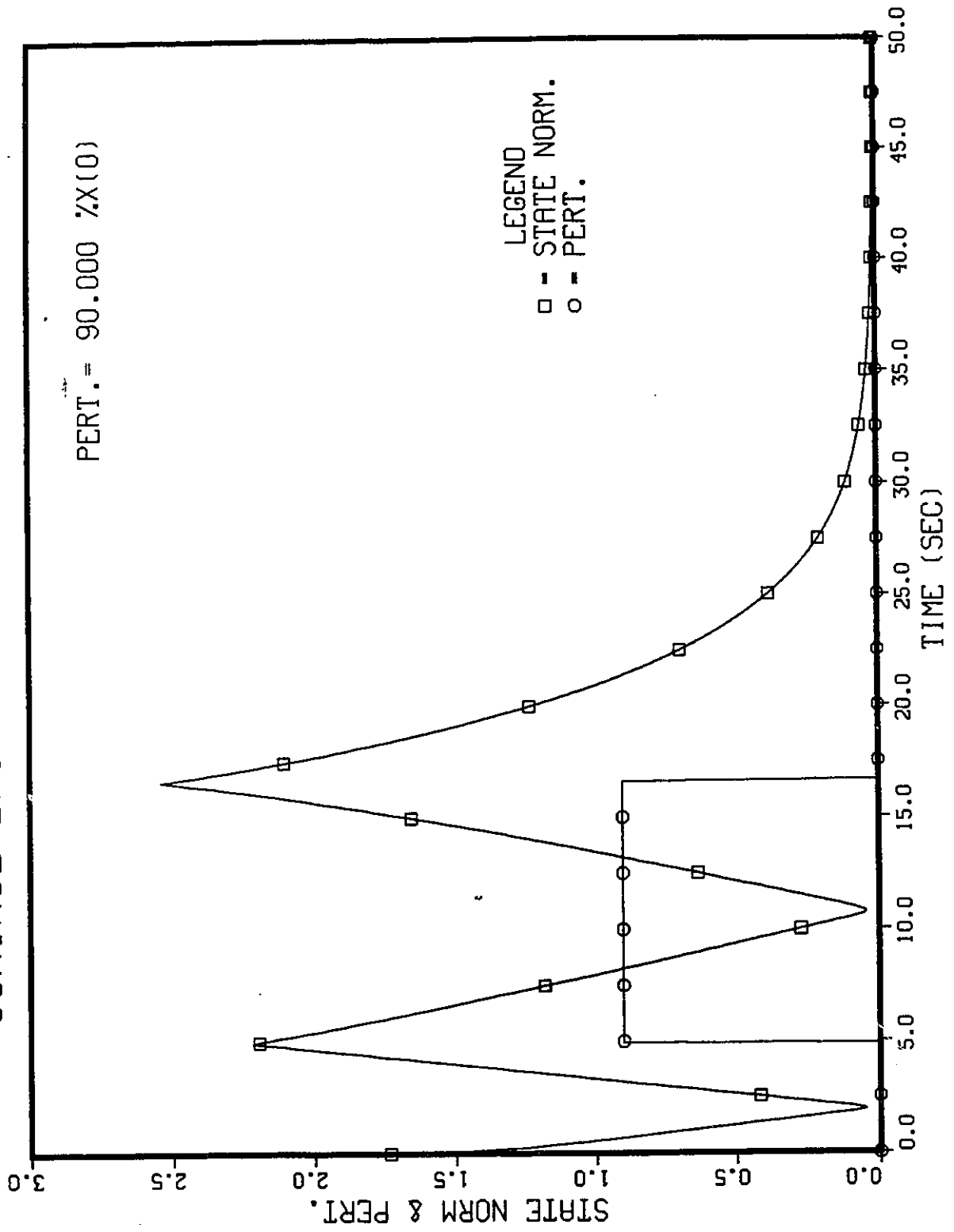


Figure IV.0-2 State Norm  $\|x(t)\|$  for  $\|g(t)\| = 1.5588$  with Time. Ex. IV.1  $\beta \neq 0$

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 10% X0

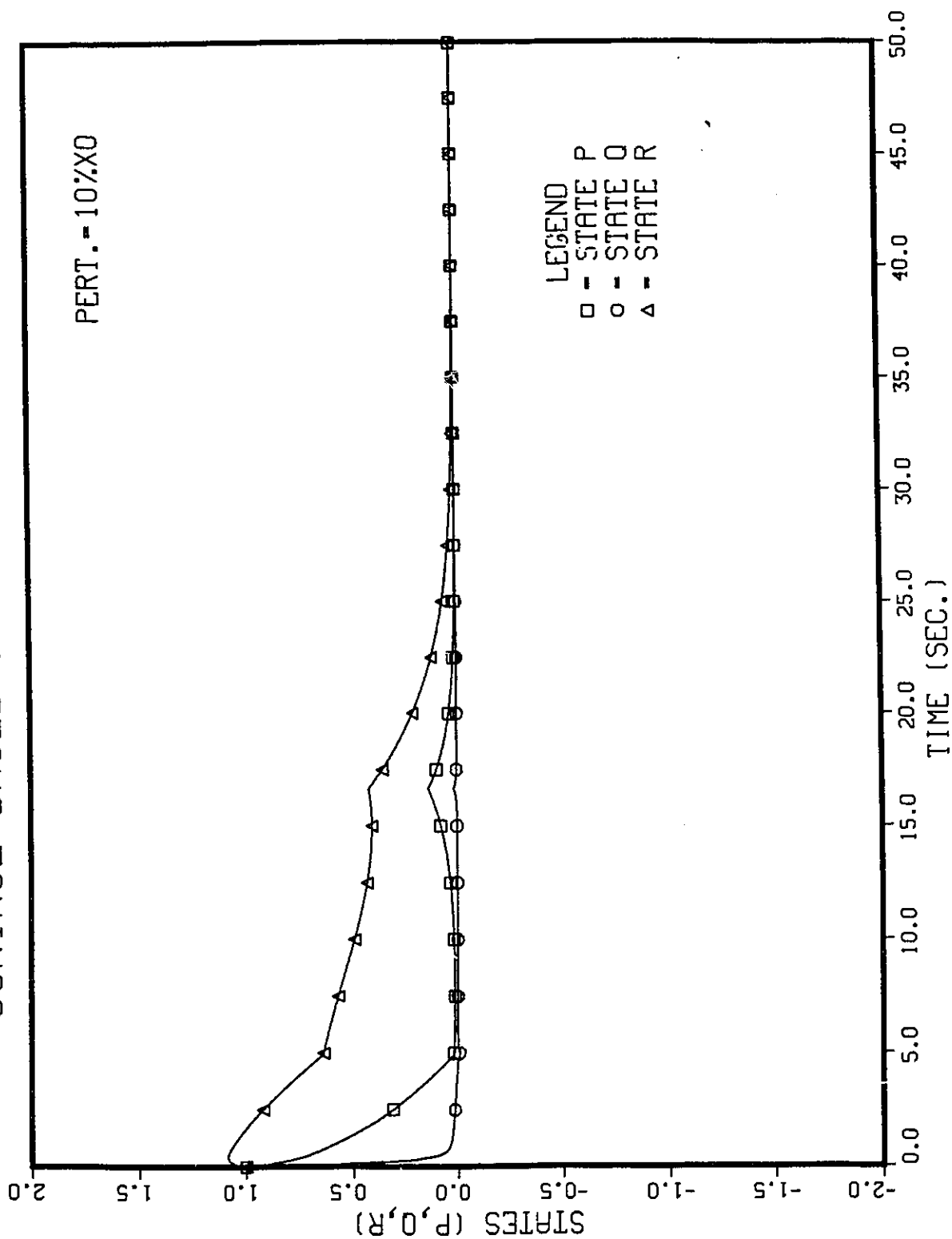


Figure IV.1 State Trajectory for Pert. Norm  $\|g(t)\| = 0.1732$  with Time. Ex. IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. - 10% X0

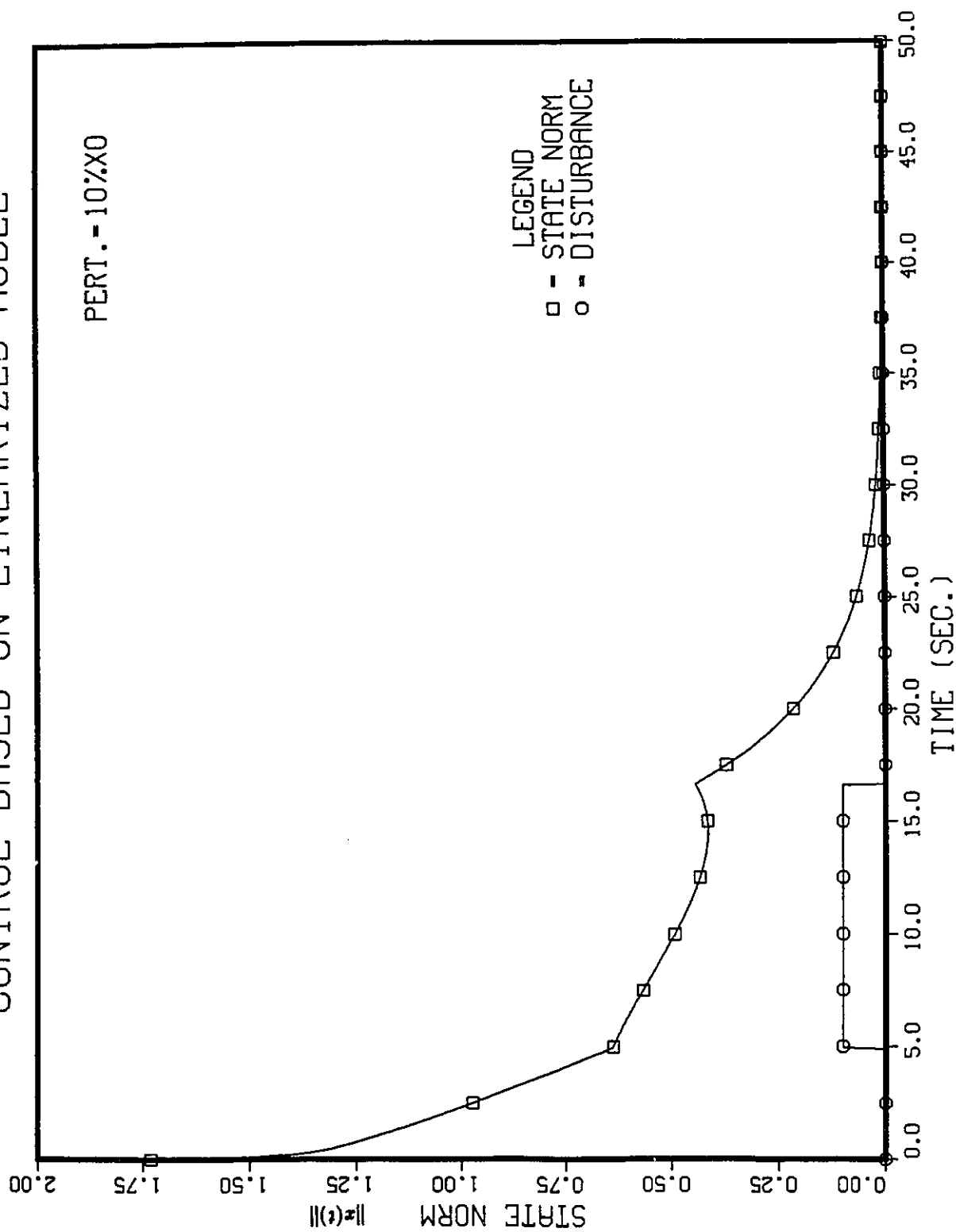


Figure IV.2 State Norm  $\|x(t)\|$  for  $\|g(t)\| = 0.1732$  with Time, Ex. IV.1

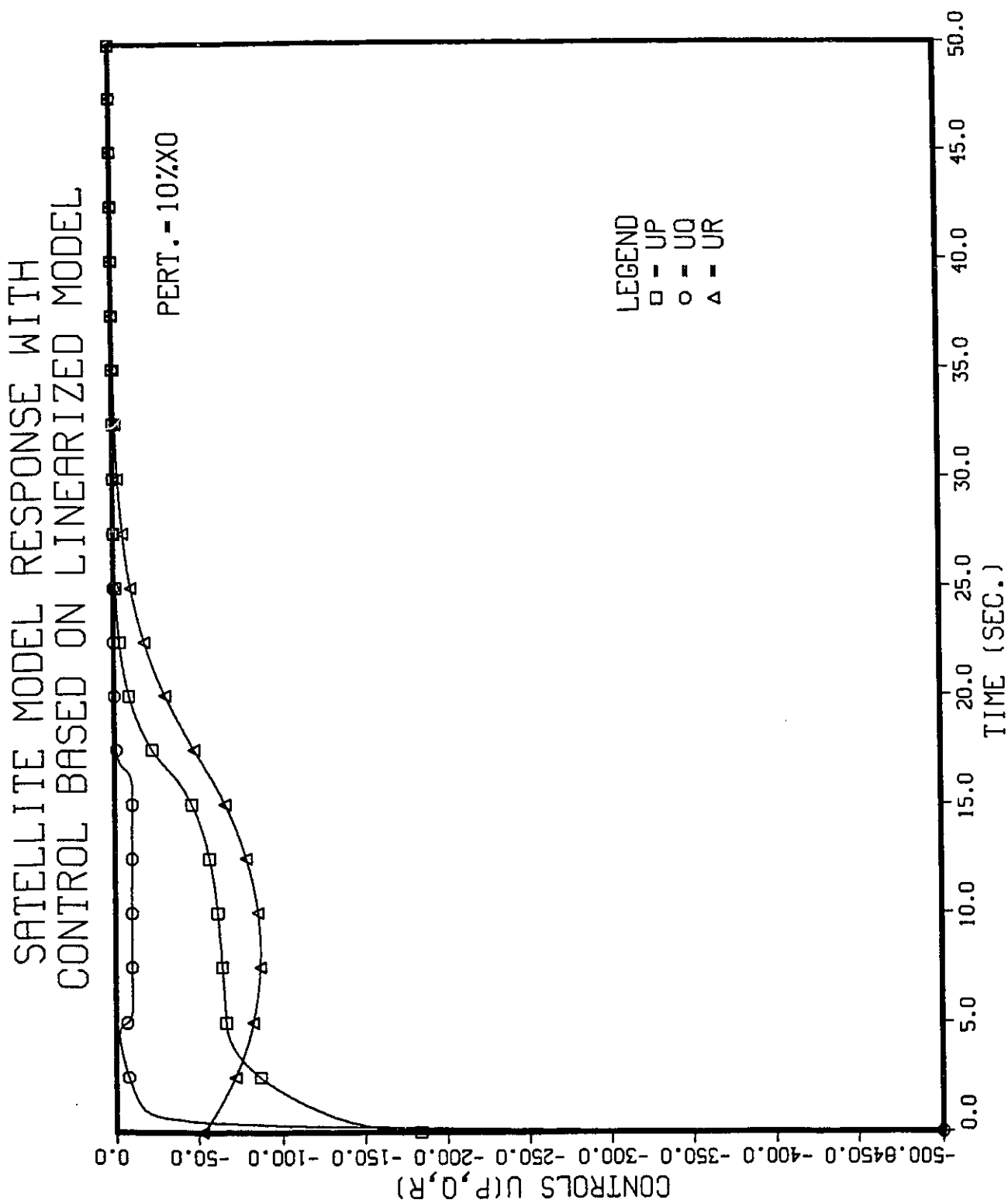


Figure IV.3 Optimal Control for  $\|g(t)\| = 0.1732$  with Time. Ex.IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 10% X 0

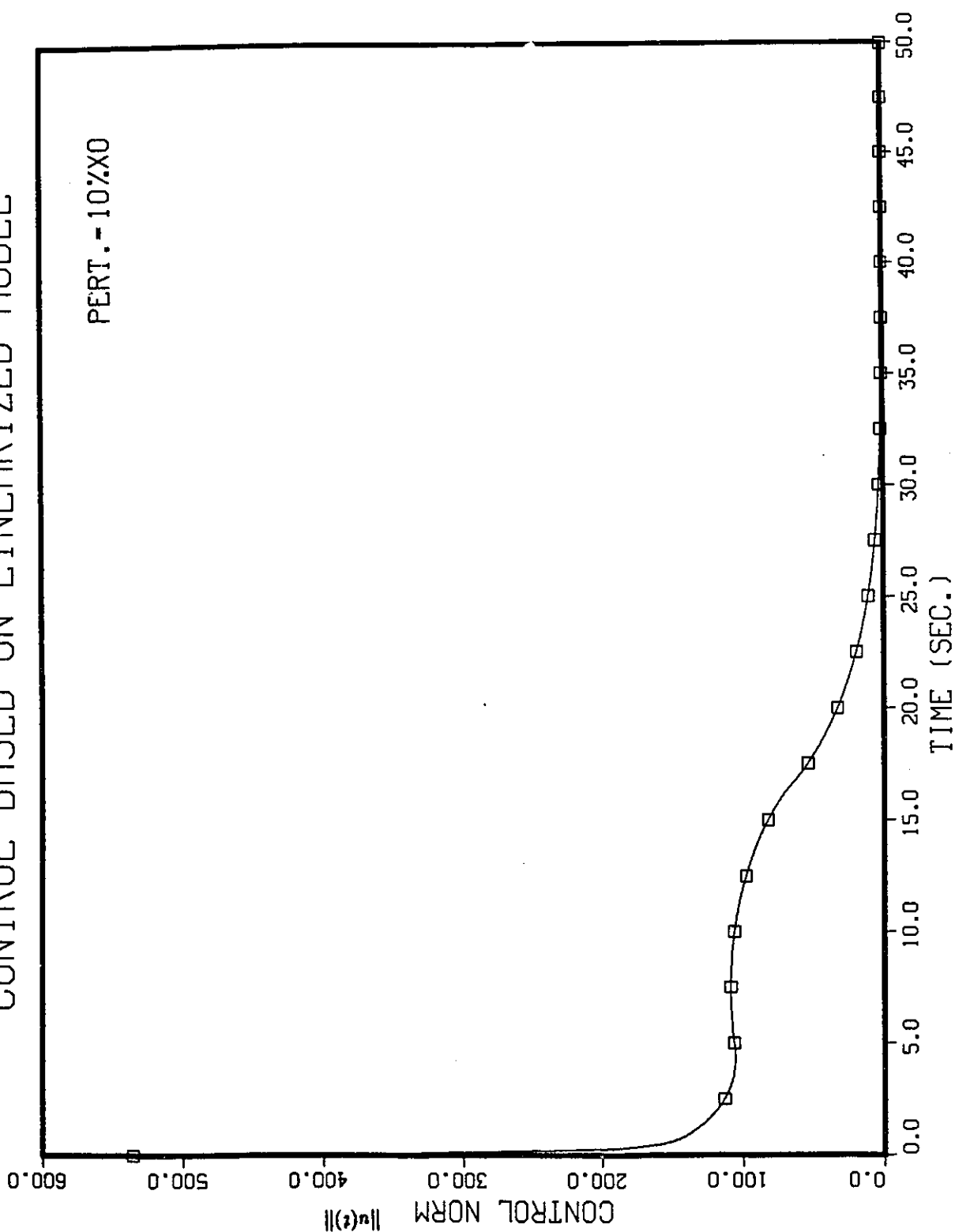


Figure IV.4 Control Norm  $\|u(t)\|$  for  $\|g(t)\| = 0.1732$  with Time. Ex. IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 10% X 0

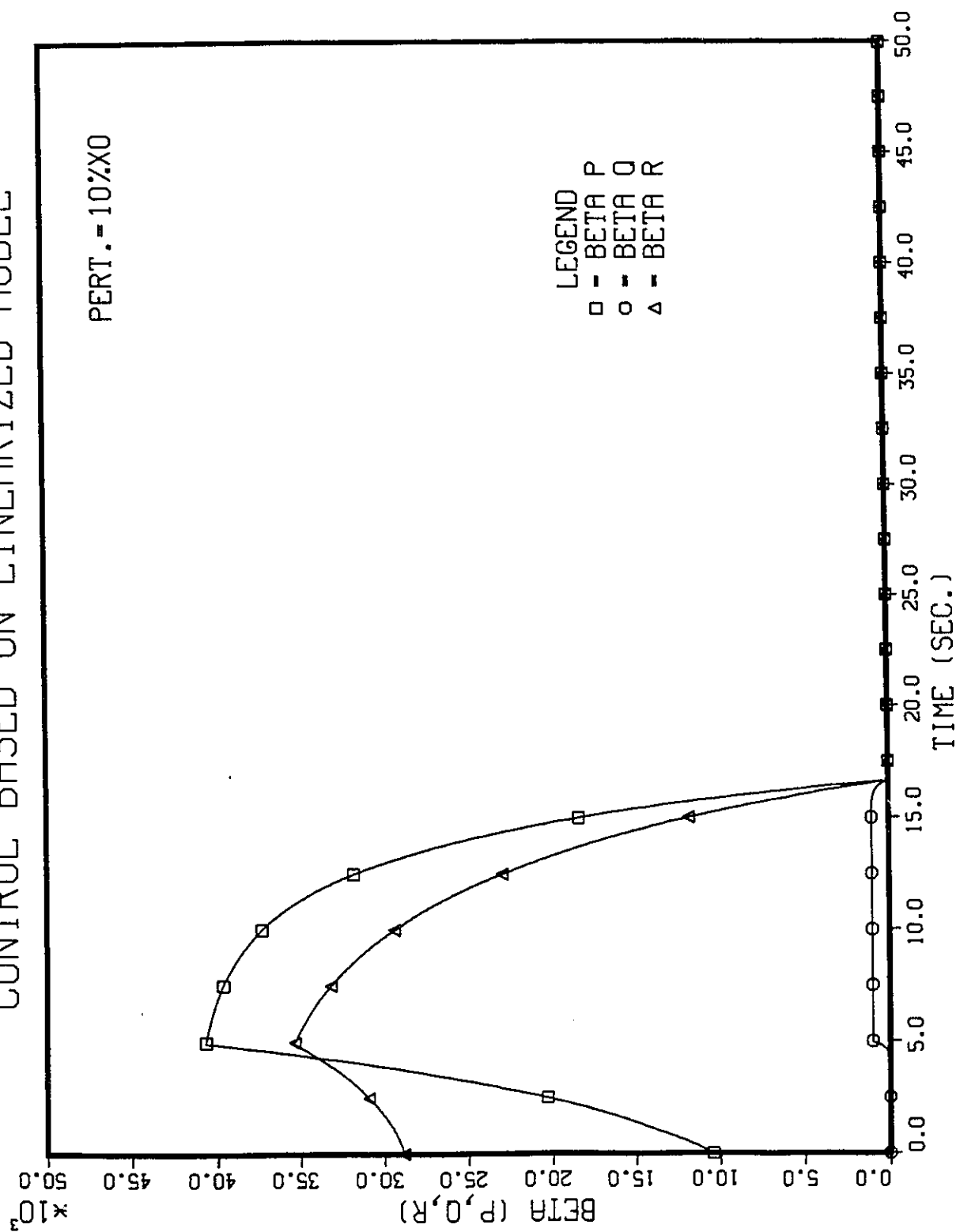


Figure IV.5  $\beta(t)$  for  $\|g(t)\| = 0.1732$  with Time. Ex. IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 30% X0

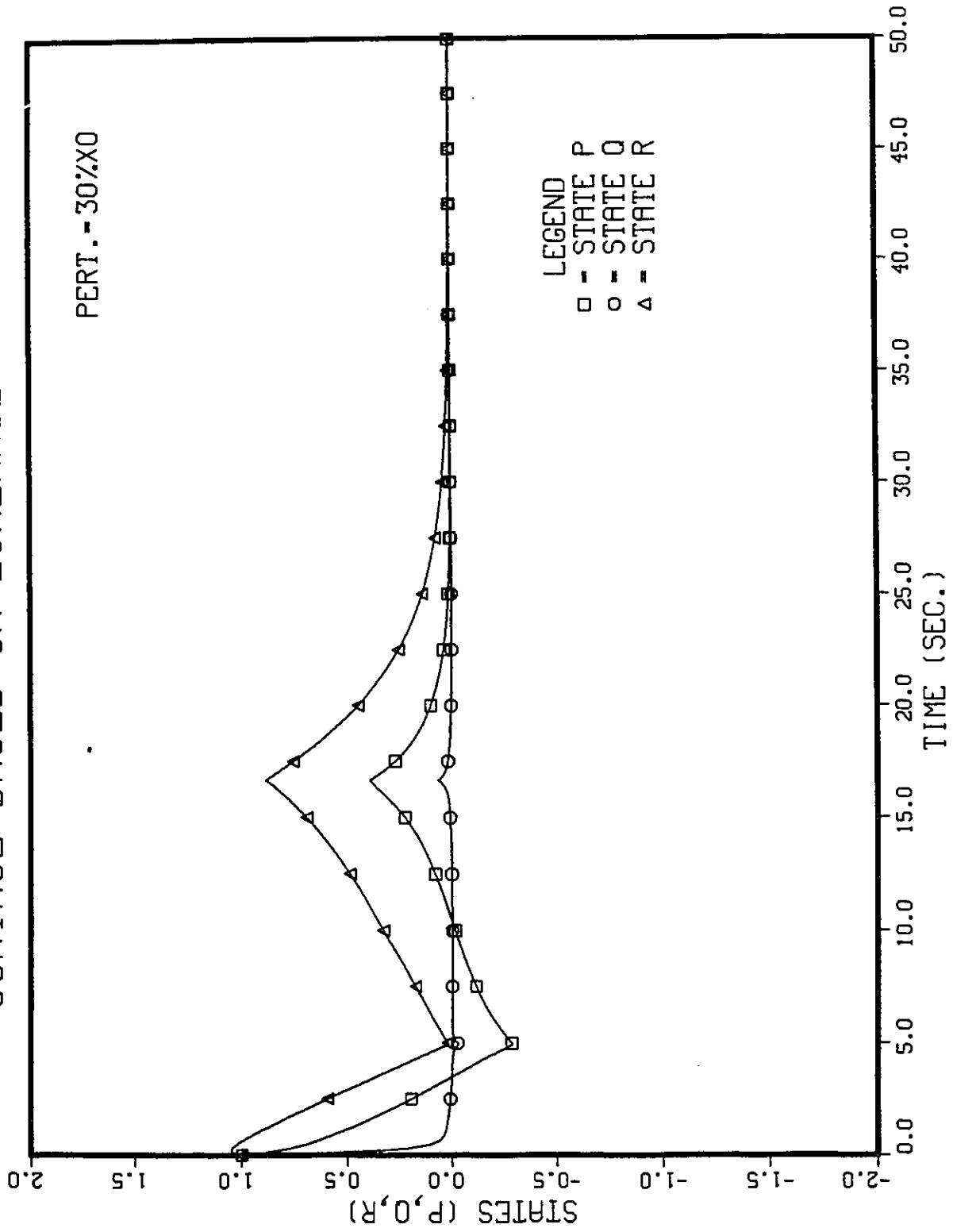


Figure IV.6 State Trajectory for Pert. Norm  $\|g(t)\| = 0.5196$  with Time. Ex.IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 30%X0

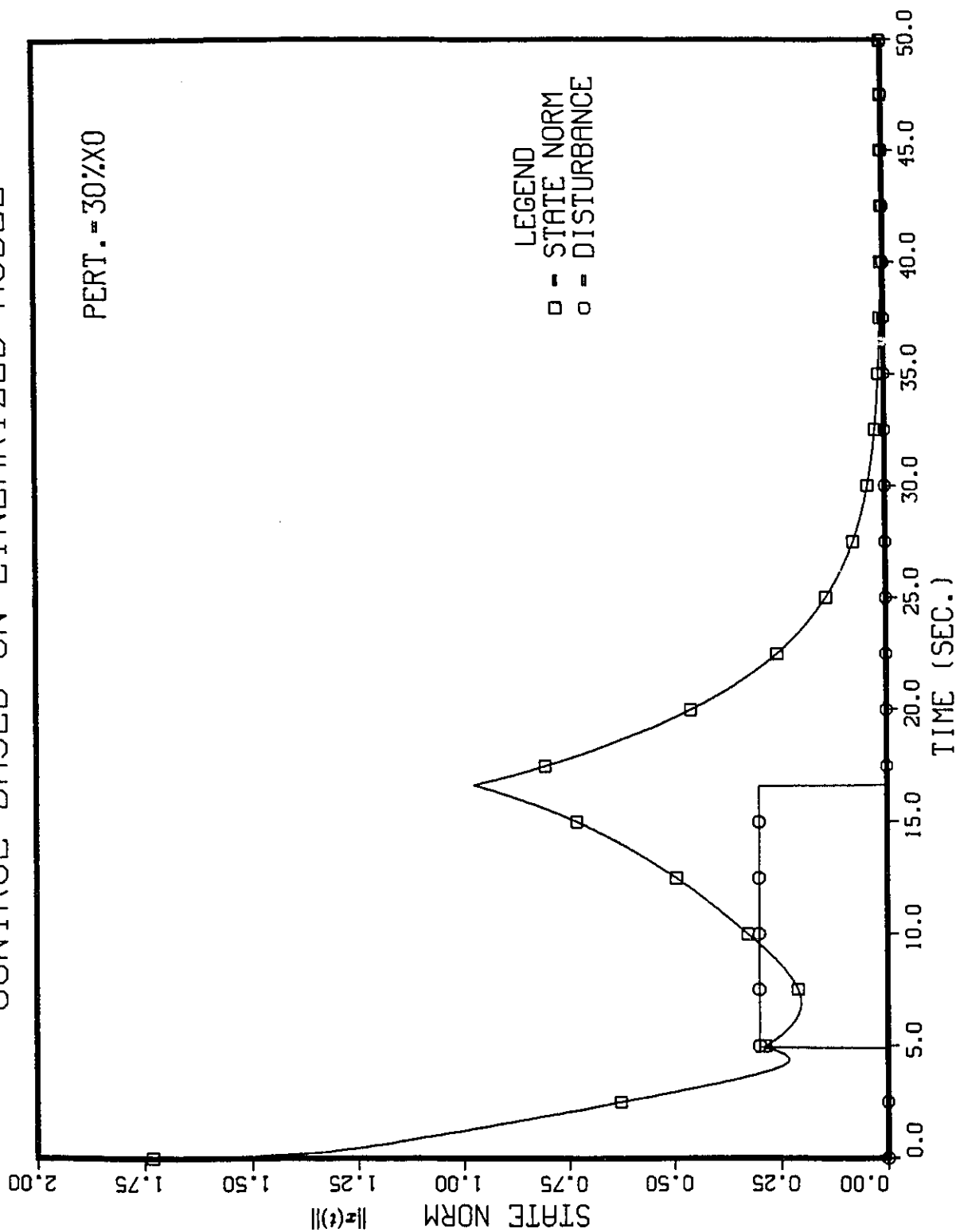


Figure IV.7 State Norm  $\|x(t)\|$  for  $\|g(t)\| = 0.5196$  with Time. Ex.IV.1

SATELLITE MODEL RESPONSE WITH  
CONTROL BASED ON LINEARIZED MODEL

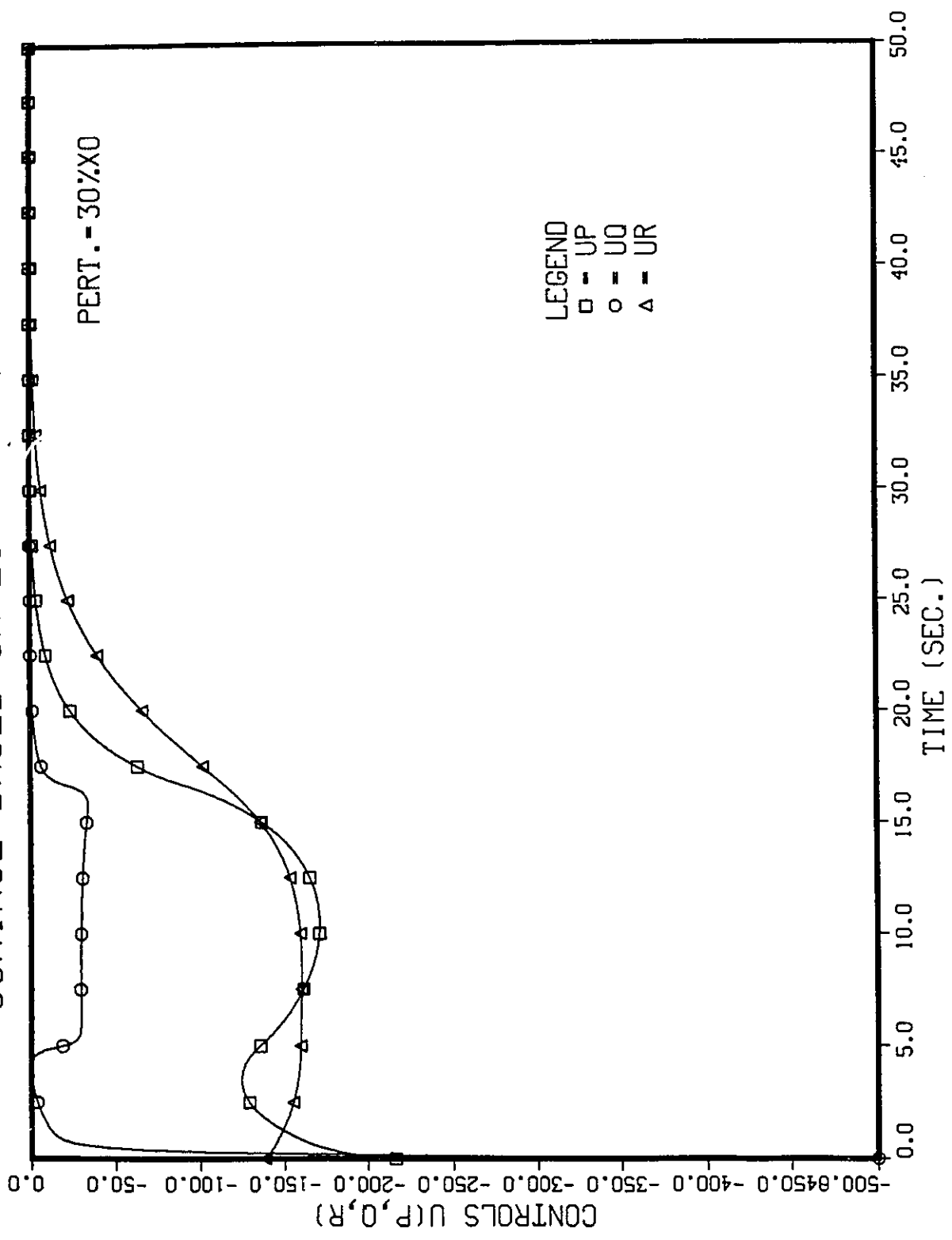


Figure IV.8 Optimal Control for  $\|g(t)\| = 0.5196$ . with Time. Ex. IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 30% X 0

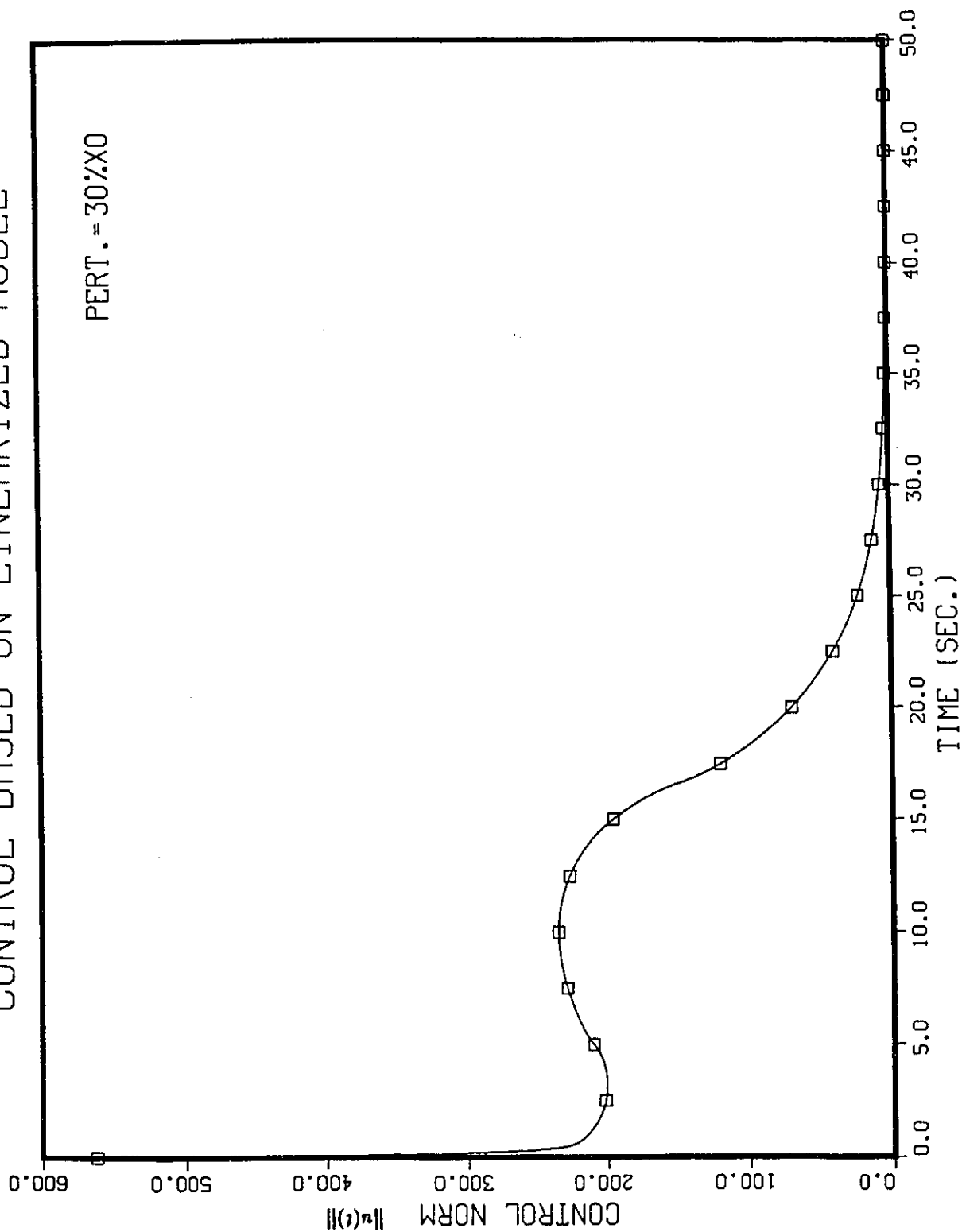


Figure IV.9 Control Norm  $\|u(t)\|$  for  $\|g(t)\| = 0.5196$  with Time. Ex. IV.1

SATELLITE MODEL RESPONSE WITH  
CONTROL BASED ON LINEARIZED MODEL

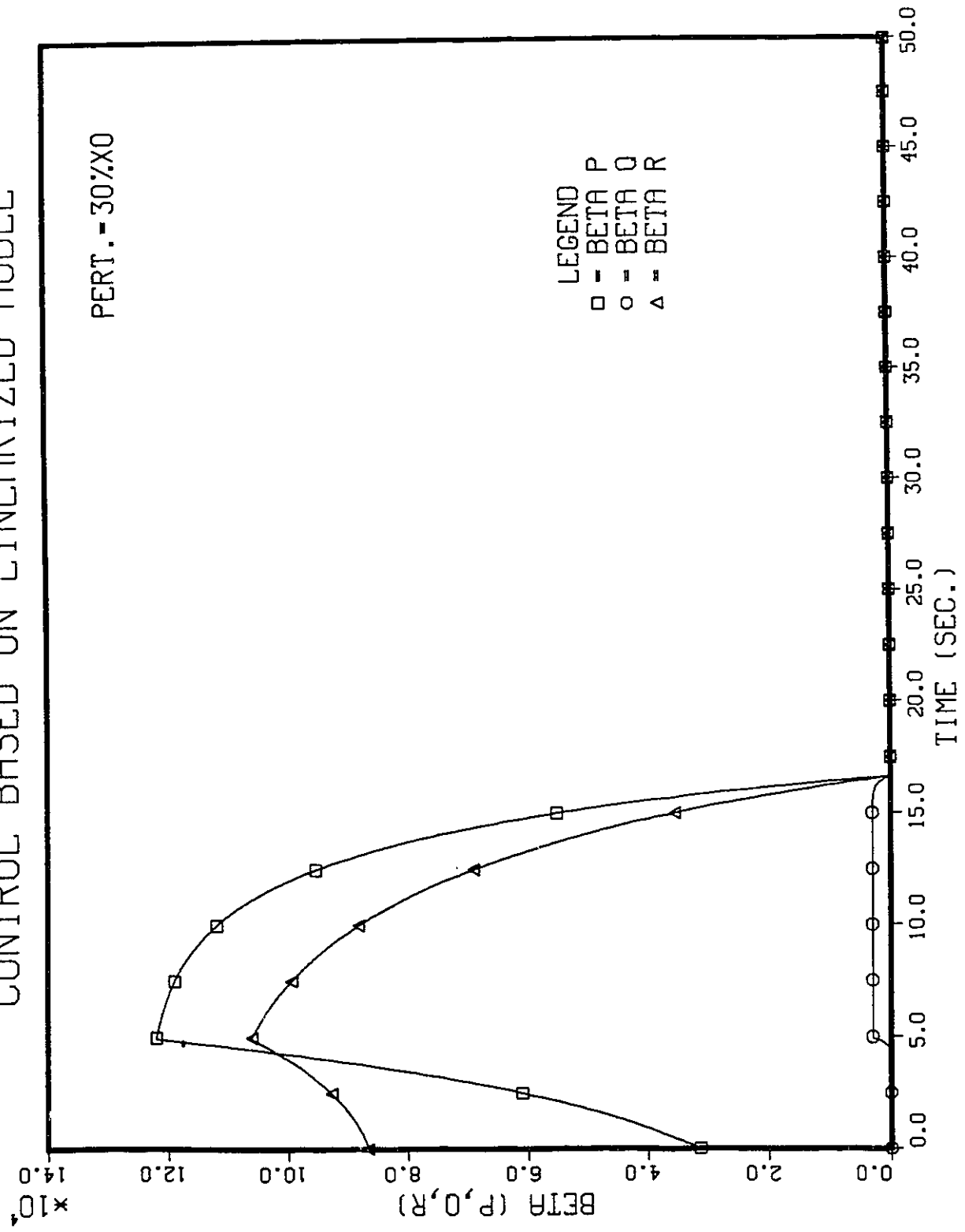


Figure IV.10  $\beta(t)$  for  $\|g(t)\| = 0.5196$  with Time. Ex. IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

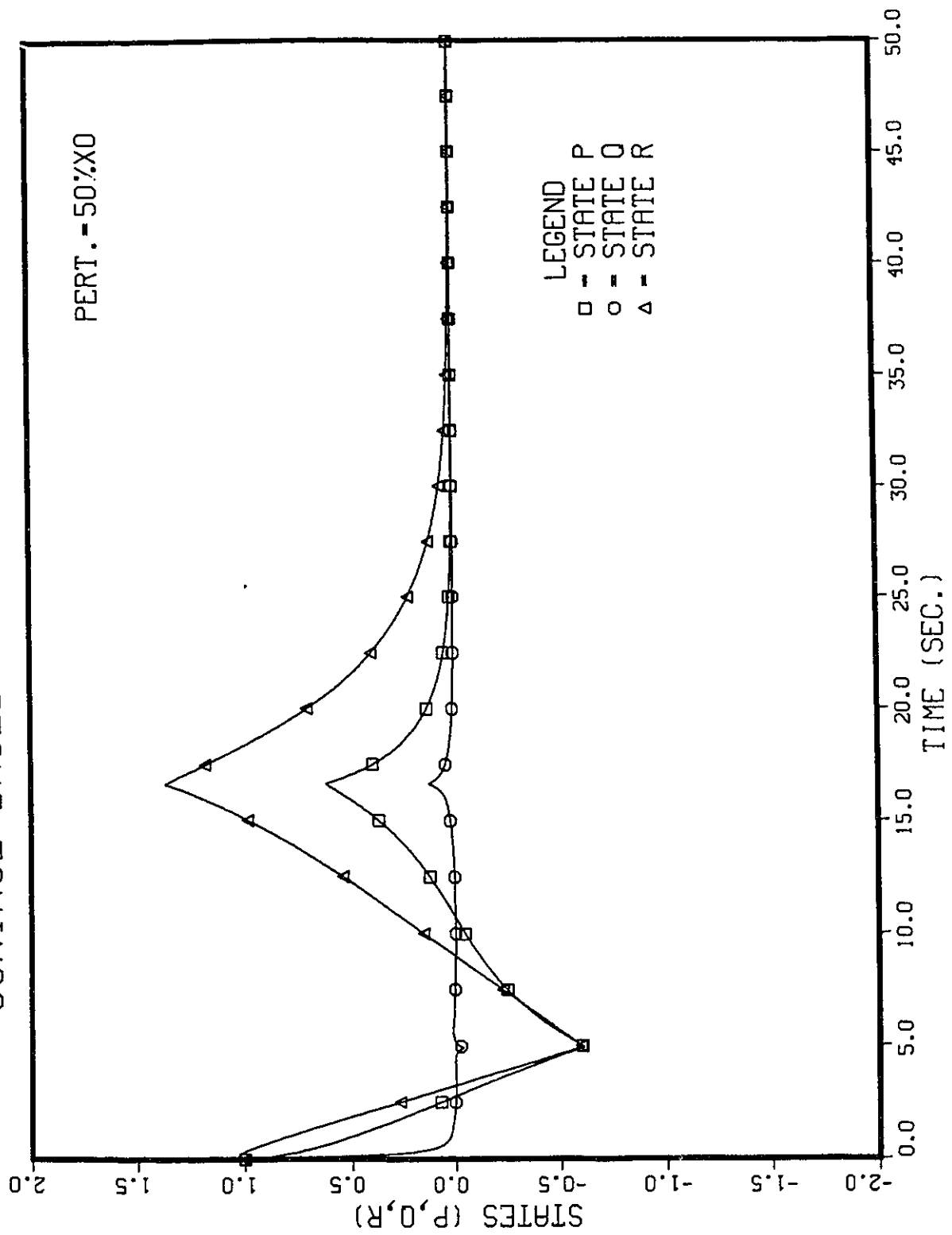


Figure IV.11 State Trajectory for Pert. Norm  $\|g(t)\| = 0.8660$  with Time. Ex. IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. - 50% X0

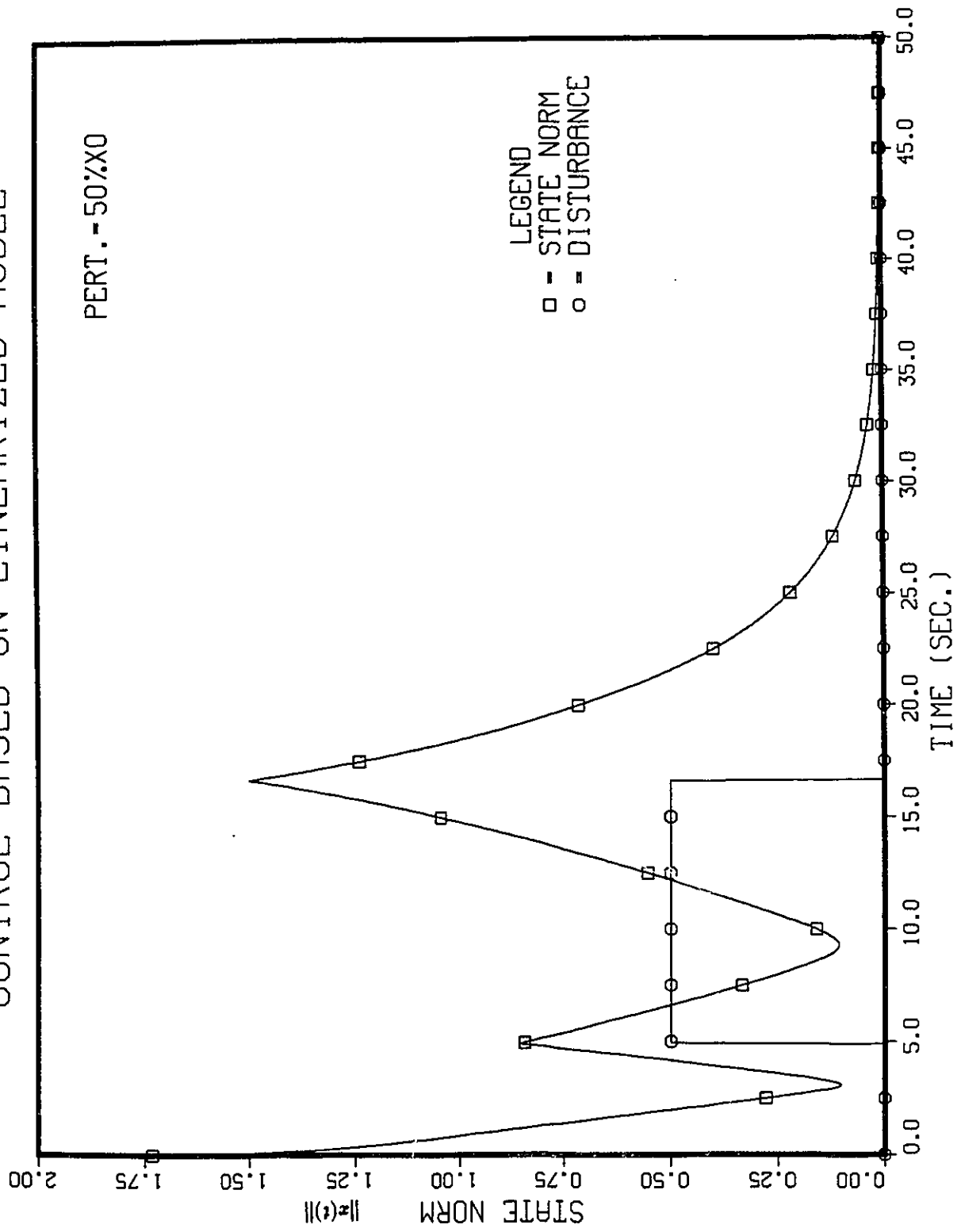


Figure IV.12 State Norm  $\|x(t)\|$  for  $\|g(t)\| = 0.8660$  with Time. Ex. IV.1

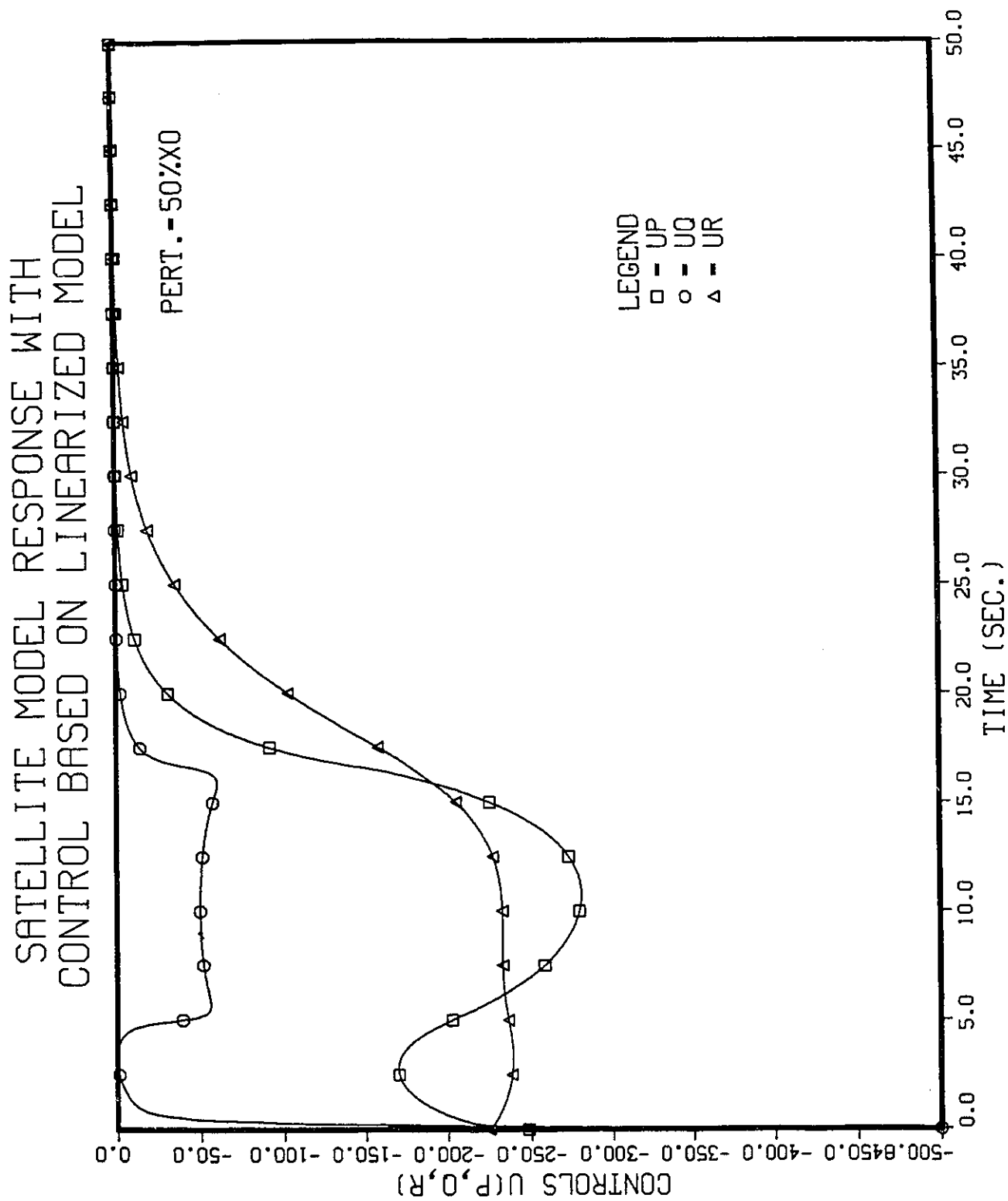


Figure IV.13 Optimal Control for  $\|g(t)\| = 0.8660$ . with Time. Ex. IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 50% X 0

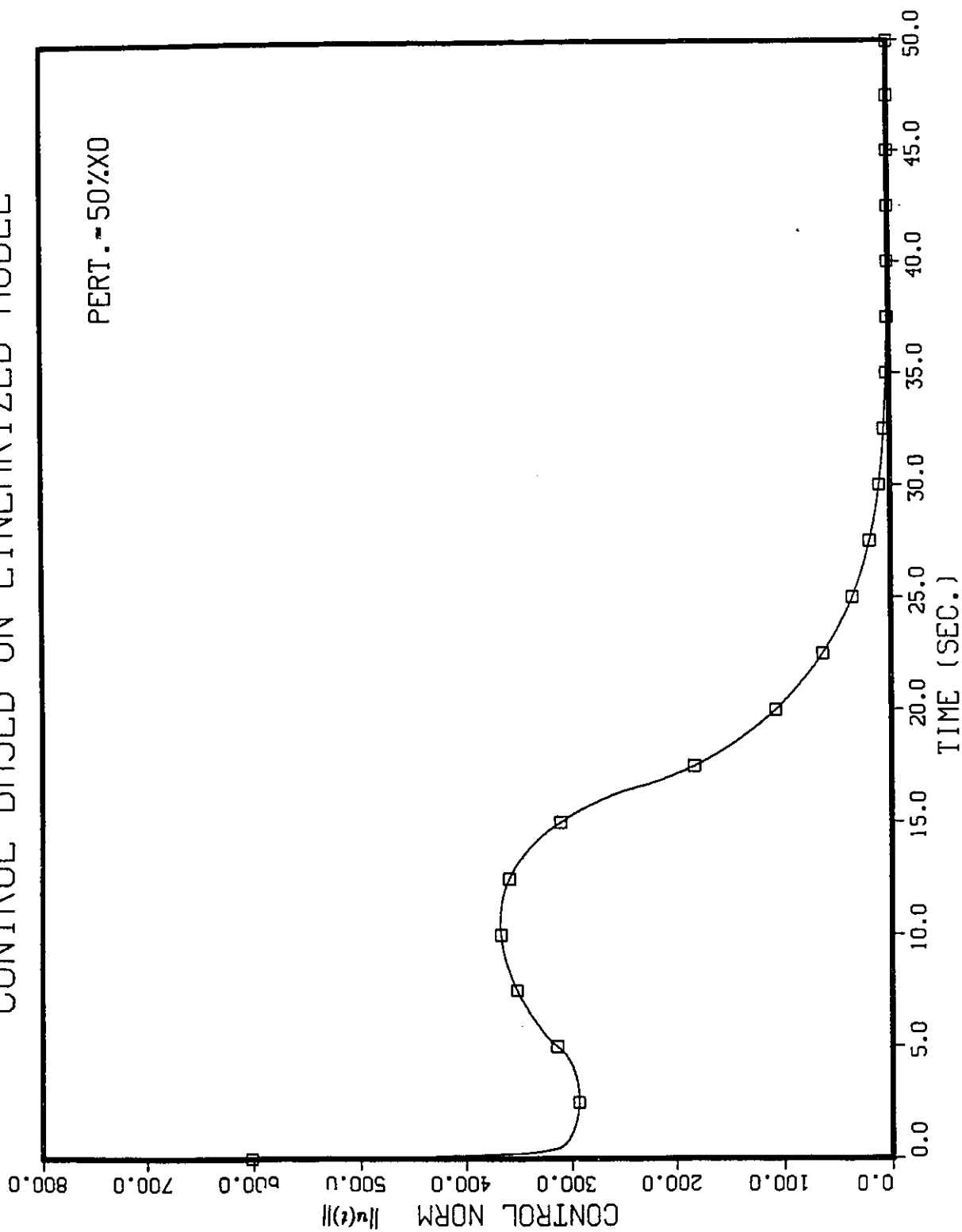


Figure IV.14 Control Norm  $\|u(t)\|$  for  $\|g(t)\| = 0.8660$  with Time. Ex.IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 50% X 0

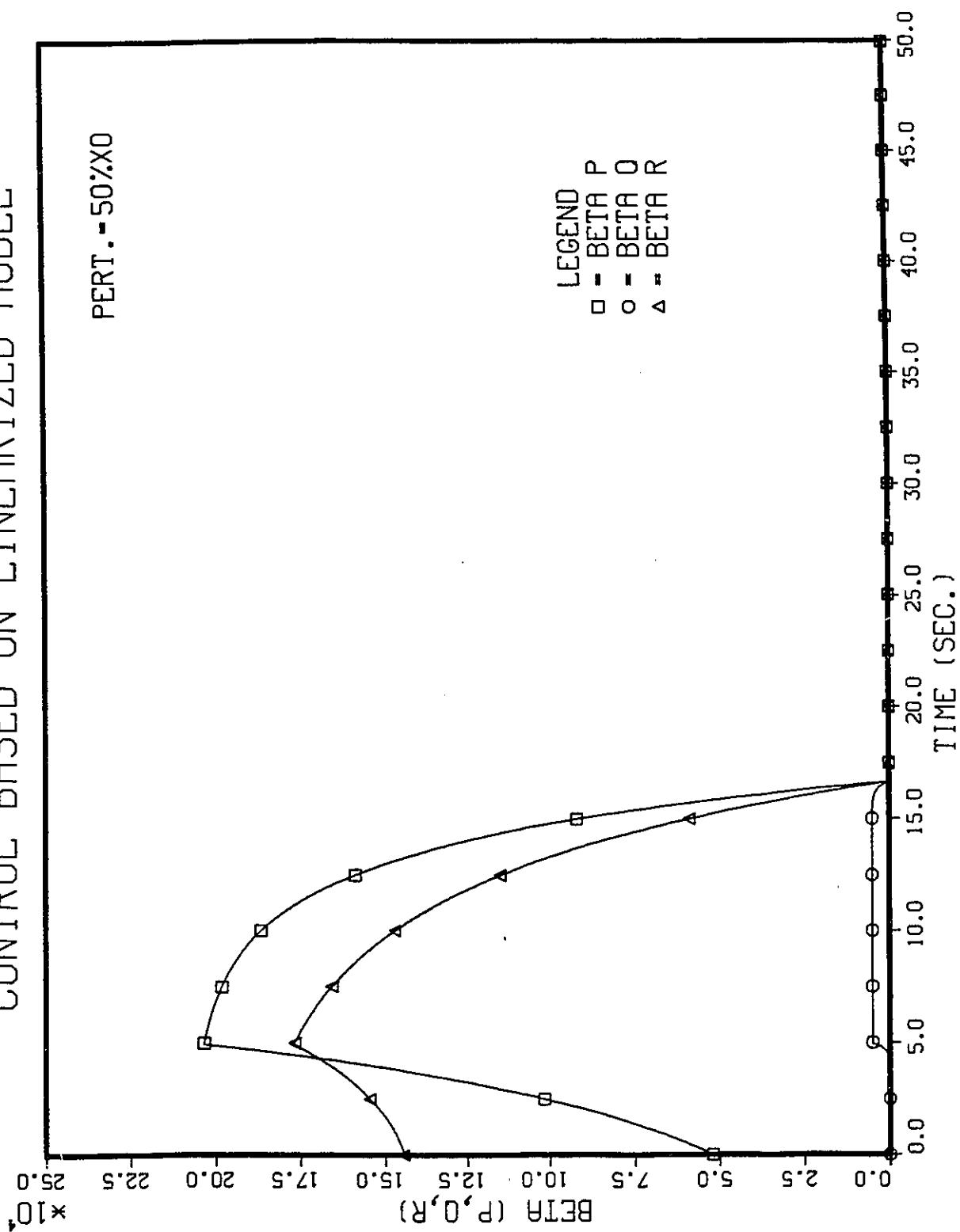


Figure IV.15  $\beta(t)$  for  $\|g(t)\| = 0.8660$  with Time. Ex.IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

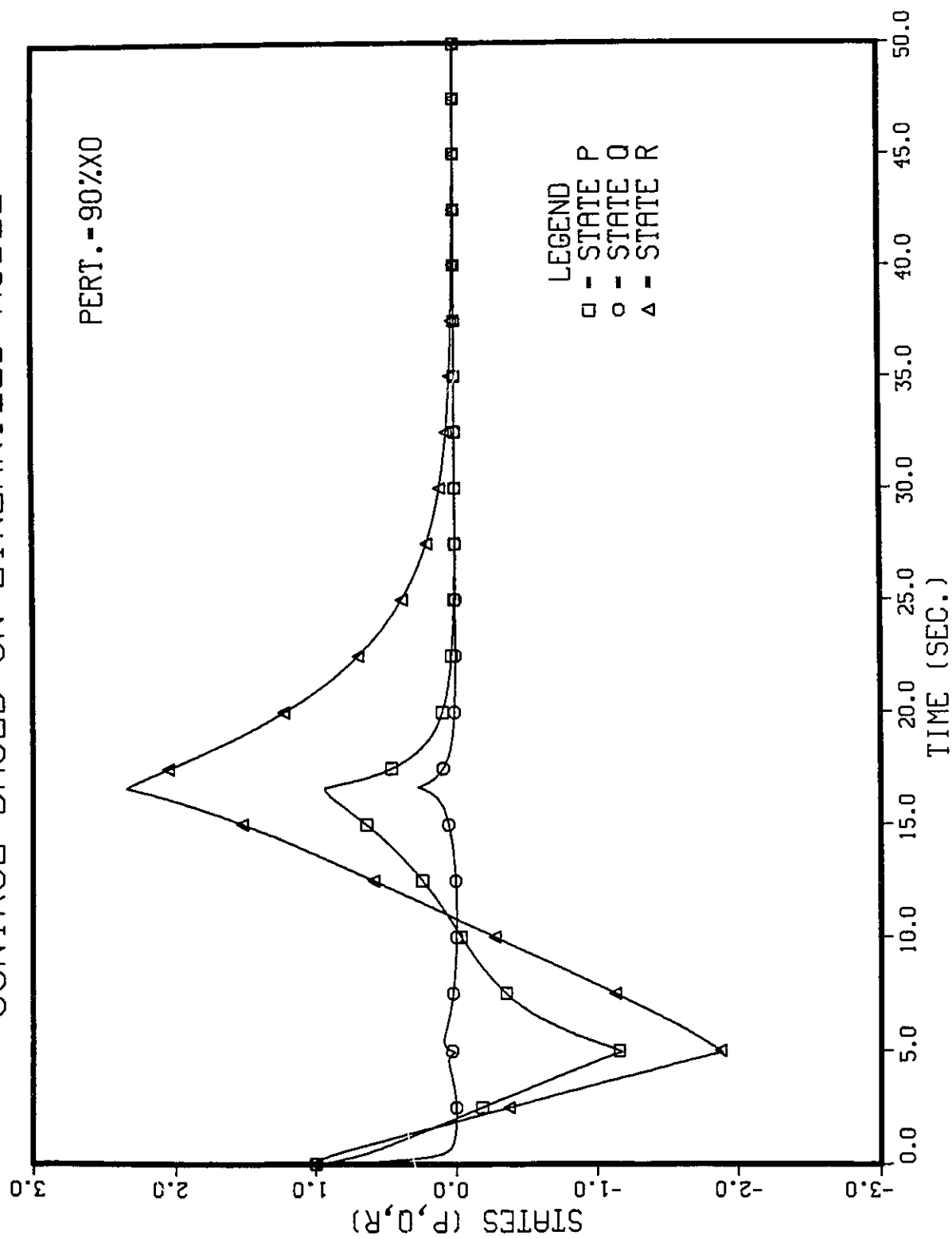


Figure IV.16 State Trajectory for Pert. Norm  $\|g(t)\| = 1.5588$  with Time. Ex.IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 90% X 0

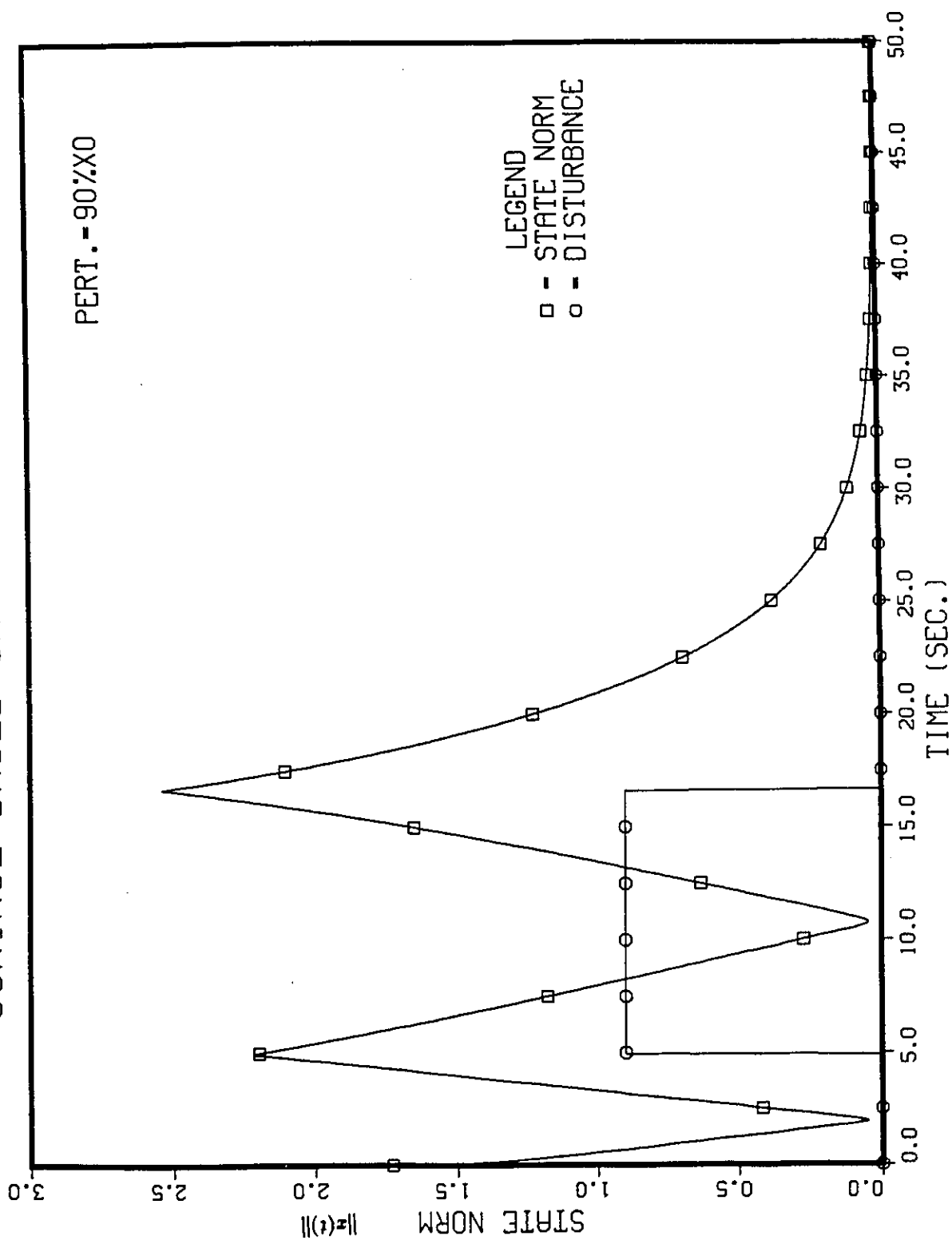


Figure IV.17 State Norm  $\|x(t)\|$  for  $\|g(t)\| = 1.5588$  with Time. Ex. IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

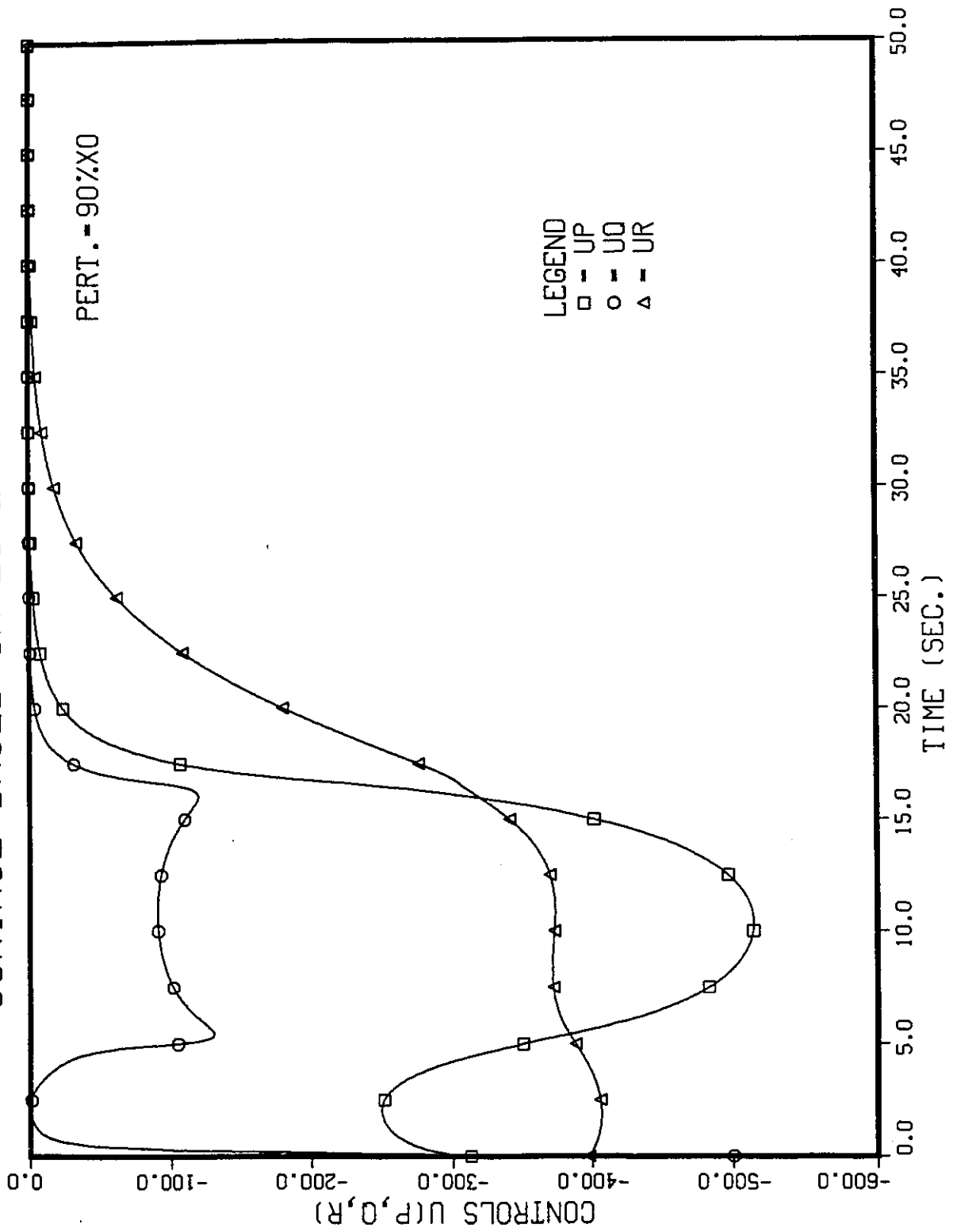


Figure IV.18 Optimal Control for  $\|g(t)\| = 1.5588$ , with Time. Ex.IV.1

# SATELLITE MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 90% X 0

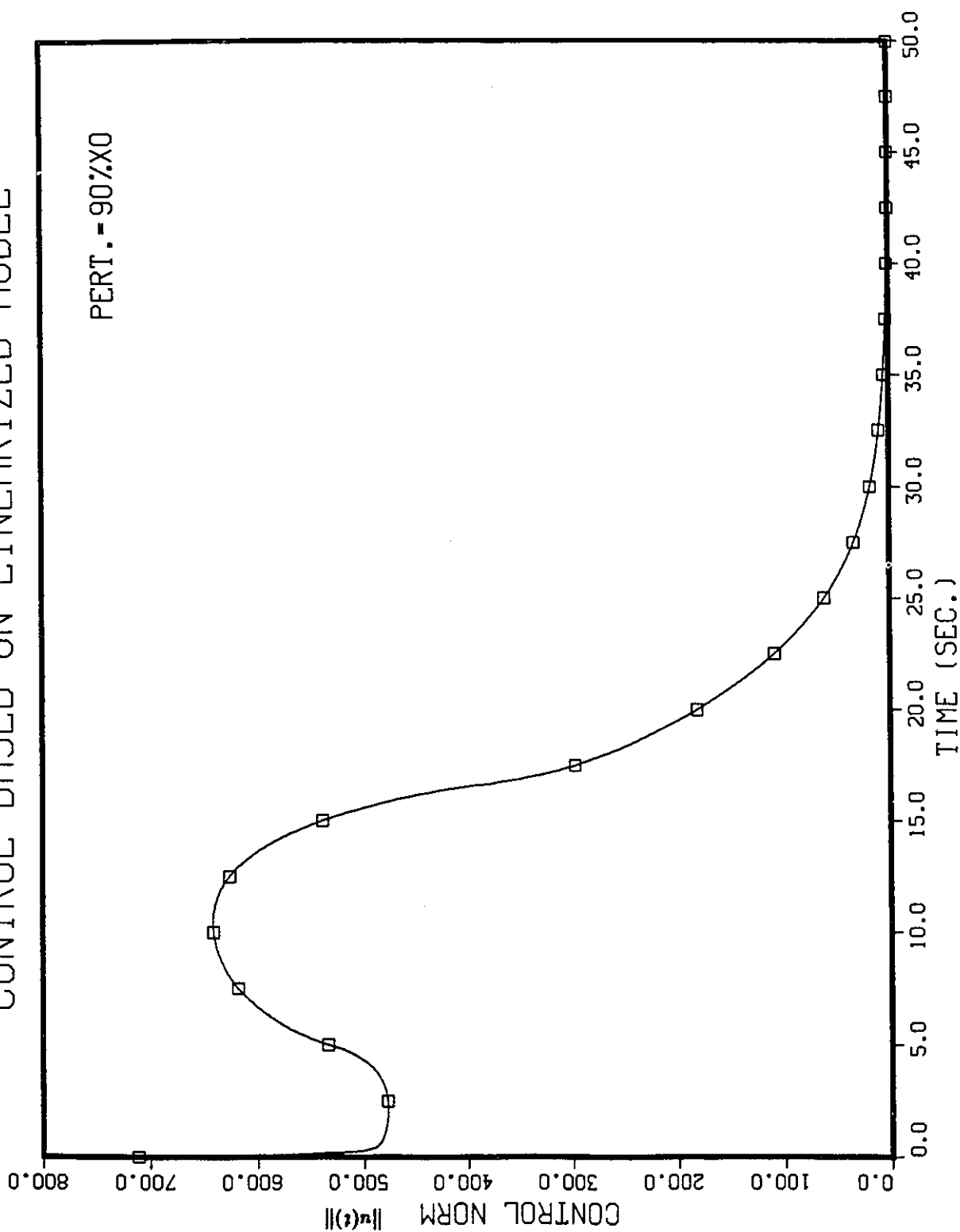


Figure IV.19 Control Norm  $\|u(t)\|$  for  $\|g(t)\| = 1.5588$  with Time. Ex.IV.1

SATELLITE MODEL RESPONSE WITH  
 CONTROL BASED ON LINEARIZED MODEL

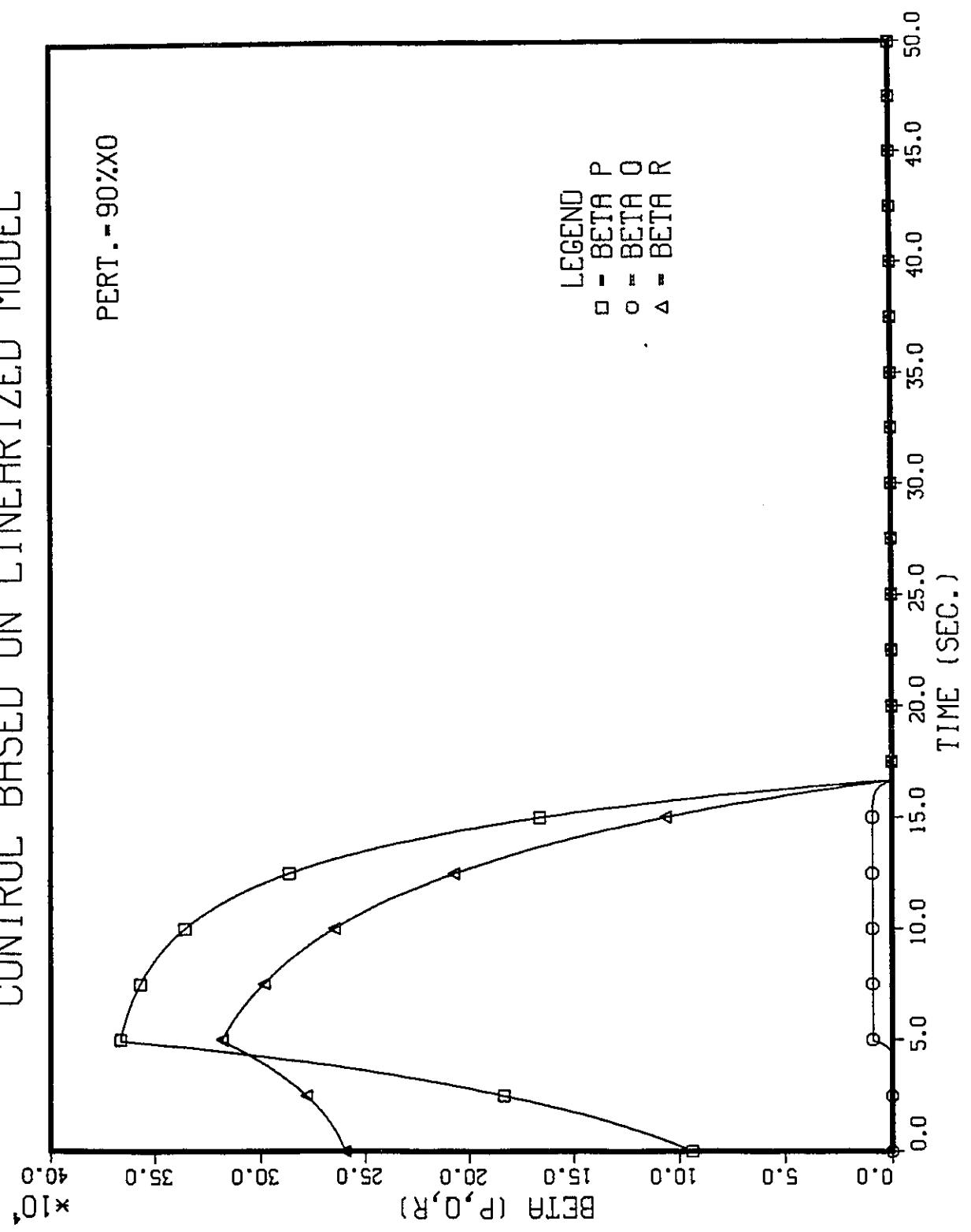


Figure IV.20  $\beta(t)$  for  $\|g(t)\| = 1.5588$  with Time. Ex.IV.1

**Example IV.2 Automelec ACR robot with cylindrical coordinates.[9]**

One of the possible means of increasing the productivity of assembly robot is to minimize the total execution time. This problem has been of great interest to many researchers over the last two decades.[9]

This robot has three degrees of freedom and is governed by the following set of equations

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \{u_1(t) + m_k x_1(t) x_4^2(t) + m_1 [d_0 + x_1(t)] x_4^2(t)\} / (m_k + m_1) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= \{u_2(t) - 2m_k x_1(t) x_2(t) x_4(t) - 2m_1 [d_0 + x_1(t)] x_2(t) x_4(t)\} / M[x_1(t)],\end{aligned}\tag{IV.30}$$

$$M[x_1(t)] = M_z + M_c + m_k x_1^2(t) + m_1 [d_0 + x_1(t)]$$

Here,  $x_1$  is the radial translation measured between the azimuth axis and the centre of gravity of the naked arm;  $x_2$  is the rate of  $x_1$  ;  $x_3$  is the azimuth rotation;  $x_4$  is the rate of  $x_3$  ( angular velocity);  $u_1$  is a limited force driving the radial translation;  $u_2$  is a limited torque driving the azimuth rotation;  $m_k$  is the mass of the naked arm;  $m_1$  is the mass of the hand and the load;  $M_z$  denotes the mass moment of inertia of the robot without the arm, hand, and load, measured with respect to the azimuth axis,  $M_c$  is the mass moment of inertia of the naked arm without the load, measured with respect to the centre of gravity  $G$ , and  $d_0$  is the distance between the point mass  $m_1$  and  $G$ . The control is subject to the constraints

$$|u_1| \leq F_{max} \quad |u_2| \leq T_{max}$$

The initial and terminal positions are given by  $[0.15, 0, 0, 0]$  and  $[0.15, 0, \pi/2, 0]$ , respectively, and the time-optimal control to accomplish this transfer is shown in figure

(IV.22). Now this form of control is open loop which is commonly used for robotic manipulators. The open loop control does not cater for any disturbance that may occur to the control system in contrast to the feedback control which does and gains its advantage from this feature.

In order to apply the LQR methods to this problem we need to linearize around the optimal trajectory obtained by the application of the optimal control given above. The linearized system is given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

where the matrices  $A$  and  $B$  are given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \theta_1 & 0 & 0 & \theta_2 \\ 0 & 0 & 0 & 1 \\ \theta_3 & \theta_4 & 0 & \theta_5 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 \\ \beta & 0 \\ 0 & 0 \\ 0 & \frac{1}{\phi(x_1)} \end{pmatrix}$$

where

$$\theta_1 = x_4^2$$

$$\theta_2 = 2\beta x_4[x_1(\alpha + \gamma) + \gamma a]$$

$$\theta_3 = \frac{1}{(\phi(x_1))^2} \{-2(\alpha + \gamma)\phi(x_1)x_2x_4 + 2[(\alpha + \gamma)x_1 + \gamma a][2((\alpha + \gamma)x_1 + \gamma a)x_2x_4 - u_2]\}$$

$$\theta_4 = \frac{-2x_4}{\phi(x_1)}[(\alpha + \gamma)x_1 + \gamma a]$$

$$\theta_5 = \frac{-2x_2}{\phi(x_1)}[(\alpha + \gamma)x_1 + \gamma a]$$

$$\phi(x_1) = c_1 + \alpha x_1^2 + \gamma(a + x_1)^2$$

For simulation purposes the following values were used;

$$\alpha = 3.7 \quad \gamma = 4.6$$

$$a = 0.37 \quad c_1 = 0.37 \quad \beta = \frac{1}{8.3}$$

With different initial state perturbations we solve the Riccati matrix differential equation. Then based on the LQR presented earlier, we compute the control which will return the system to its nominal trajectory.

Since the time-optimal control shown in figure (IV.22) would steer the system to its target in the least possible time in the case when there is no state perturbation, see figure (IV.21) for state trajectory, this is an open loop control. In the case we are handling there is initial state perturbation which require additional time and/or control to return the system to its nominal trajectory.

To solve this problem, since for time-optimal control problems time is very essential, we will keep the time as in the original problem and define the domain of acceptable perturbation as *the set of perturbations for which the error in the state norm at the given final time is bounded by an acceptable bound. i.e.*

$$P_{ac} \equiv \{x^p(t) : \|x^p(T) - x^0(T)\| \leq \delta\}$$

where  $x^p(t)$  is the perturbed state at time  $t$  and  $x^0(t)$  is the nominal state at time  $t$ . For simulation purposes we considered  $\delta = 0.015$  which led to find out that the domain of acceptable perturbations is given by

$$P_{ac} \equiv \{x^p : \|x^p(0)\| \leq 1.30\}$$

However, one may argue that the control obtained in this way may violate the constraints imposed on the original problem.

One may also, by considering the constraints, set perturbation domain on the basis of

the control required to return the system to its nominal trajectory. However, this last point, although it enjoys some practicality, it violates the basic assumption of the L.Q.R. of the availability of the whole control space to select control function from. Returning to our definition of  $P_{ac}$  and figures (IV.23,IV.24) it is possible to return the system to its nominal trajectory if the perturbation vector  $p(t)$  is bounded by the norm  $\|p(0)\| \leq 1.30$ .

This can be explained simply by the fact that beyond this limit the system behavior is nonlinear and the linearization is not valid. One way to remedy this situation is to consider nonlinear regulator which is too complex and hence practically not so attractive as the linear.

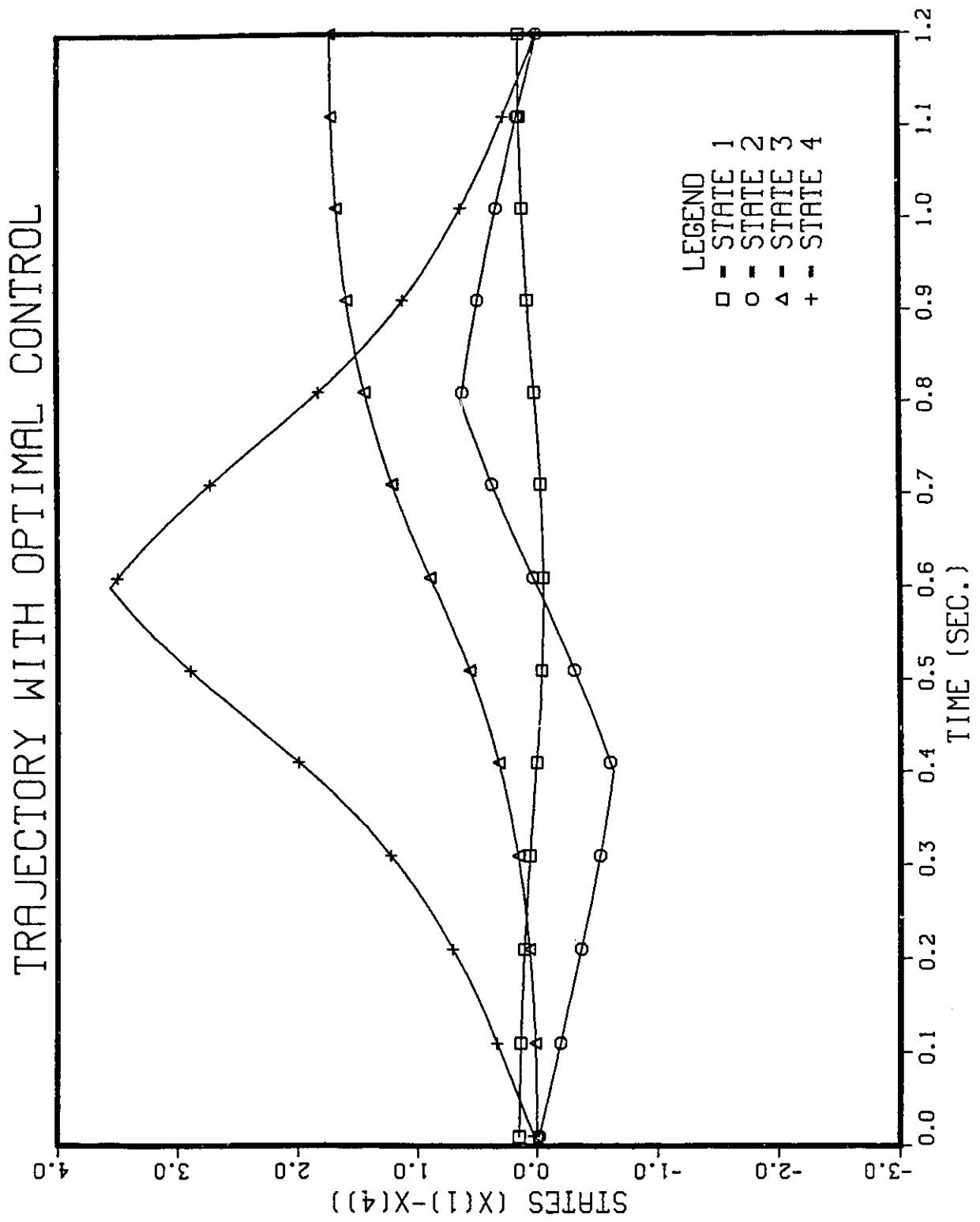


Figure IV.21 Optimal Trajectory with Time. ExIV.2

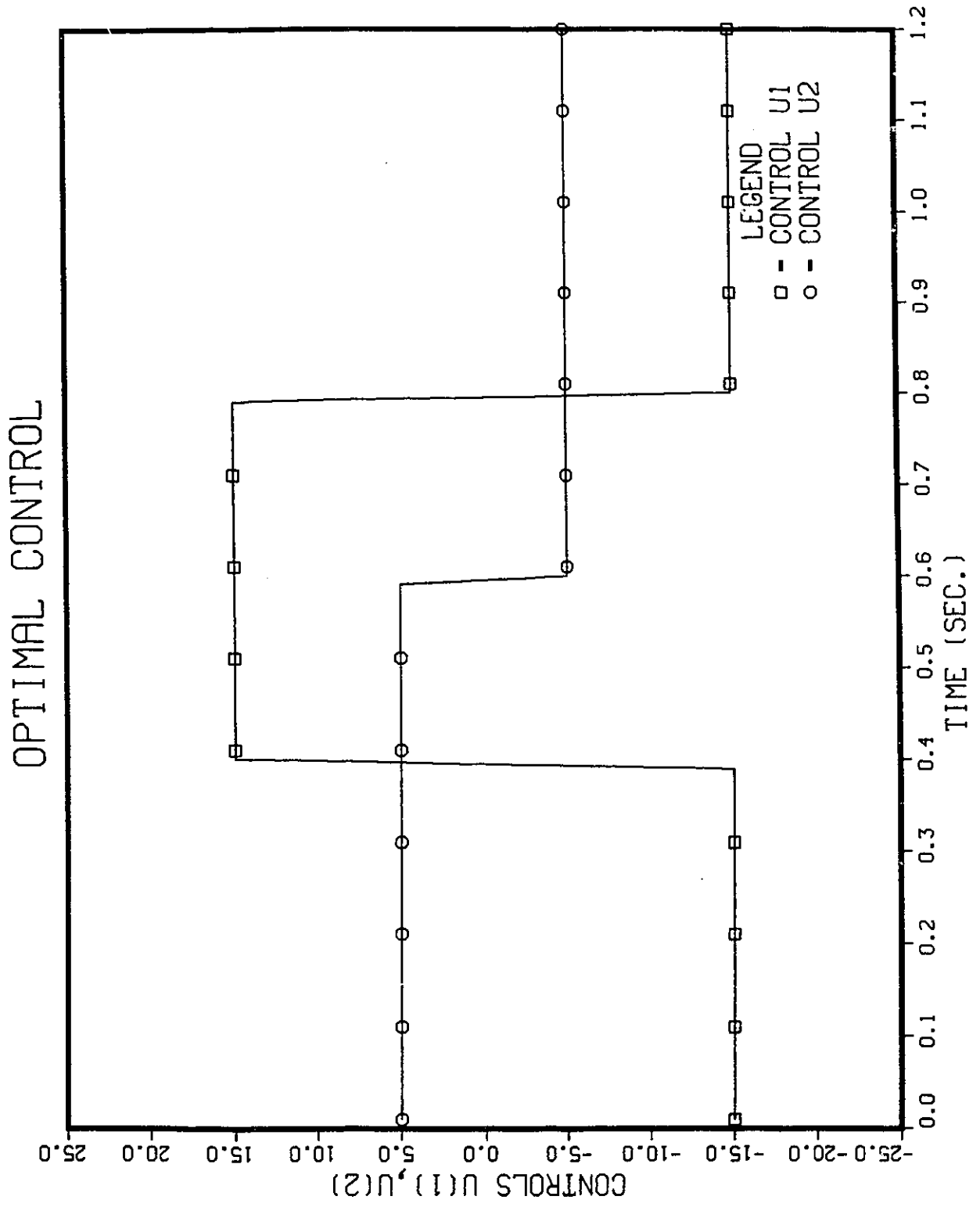


Figure IV.22 Time-Optimal control. Ex.IV.2

# ROBOT MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL

PERT. = 0.6500

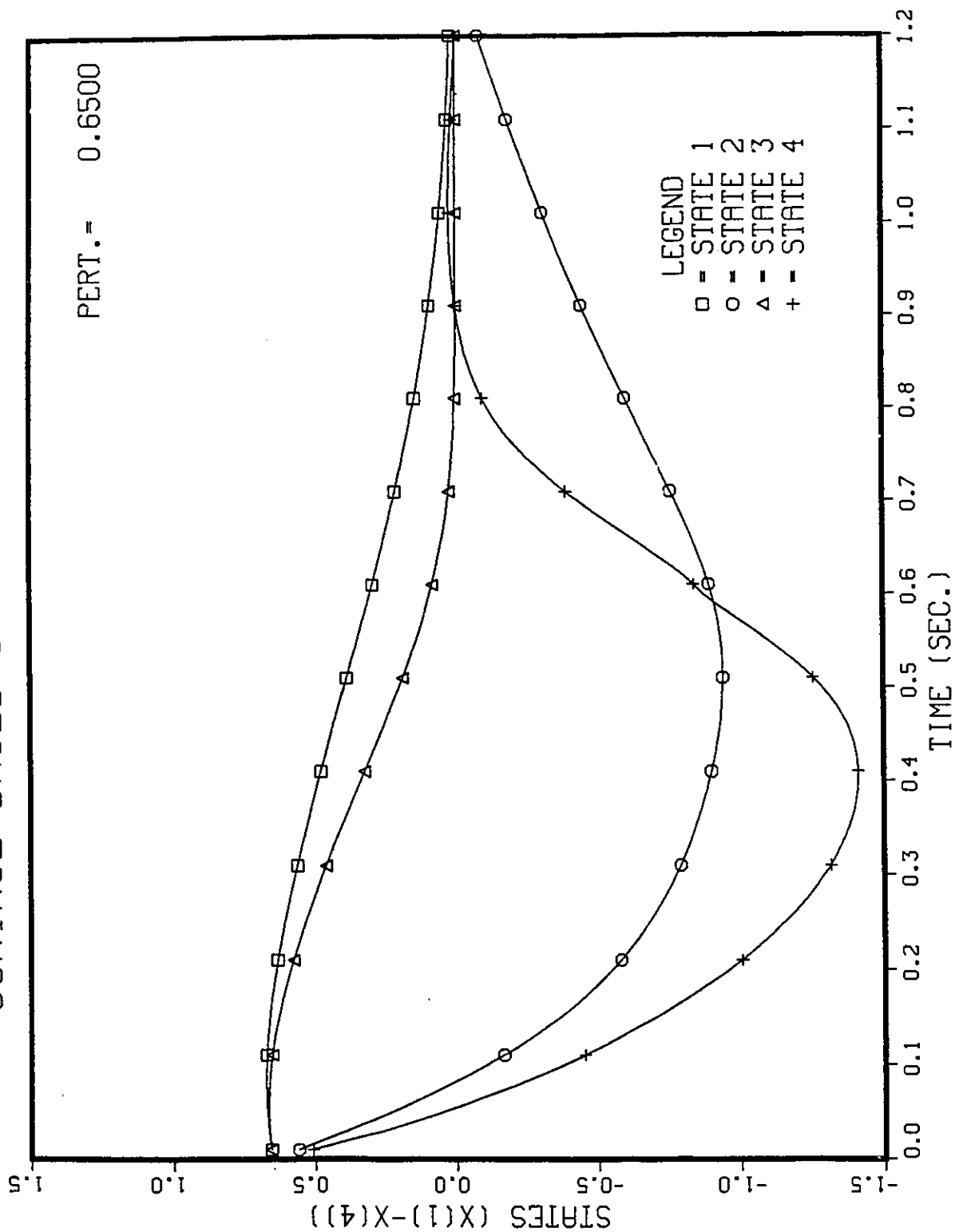


Figure IV.23 State Trajectory for Pert. Norm  $\|p(t)\| = 1.3000$  with Time. Ex. IV.2

# ROBOT MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL.

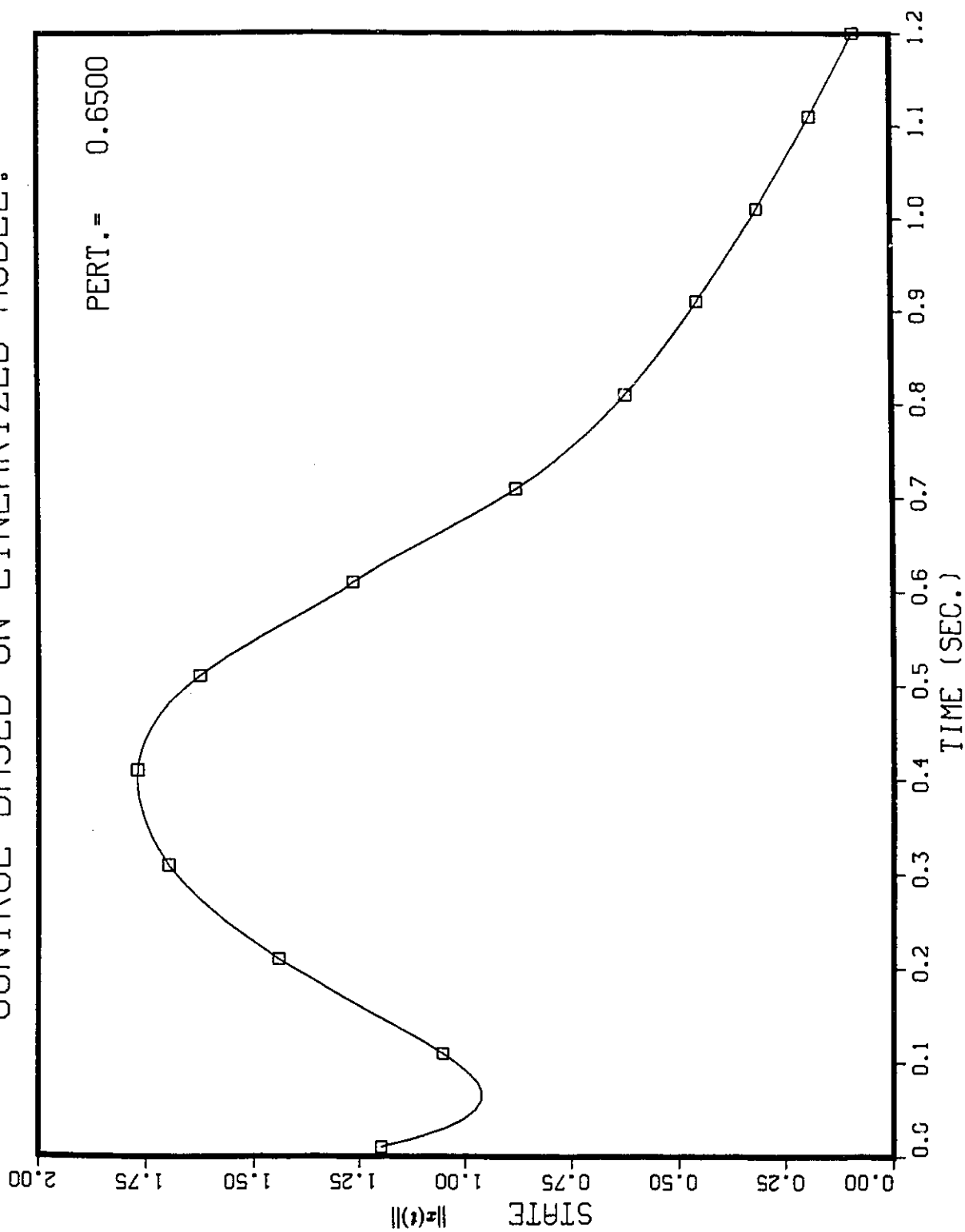


Figure IV.24 State Norm  $\|x(t)\|$  for  $\|p(t)\| = 1.3000$  with Time. Ex. IV.2

# ROBOT MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL.

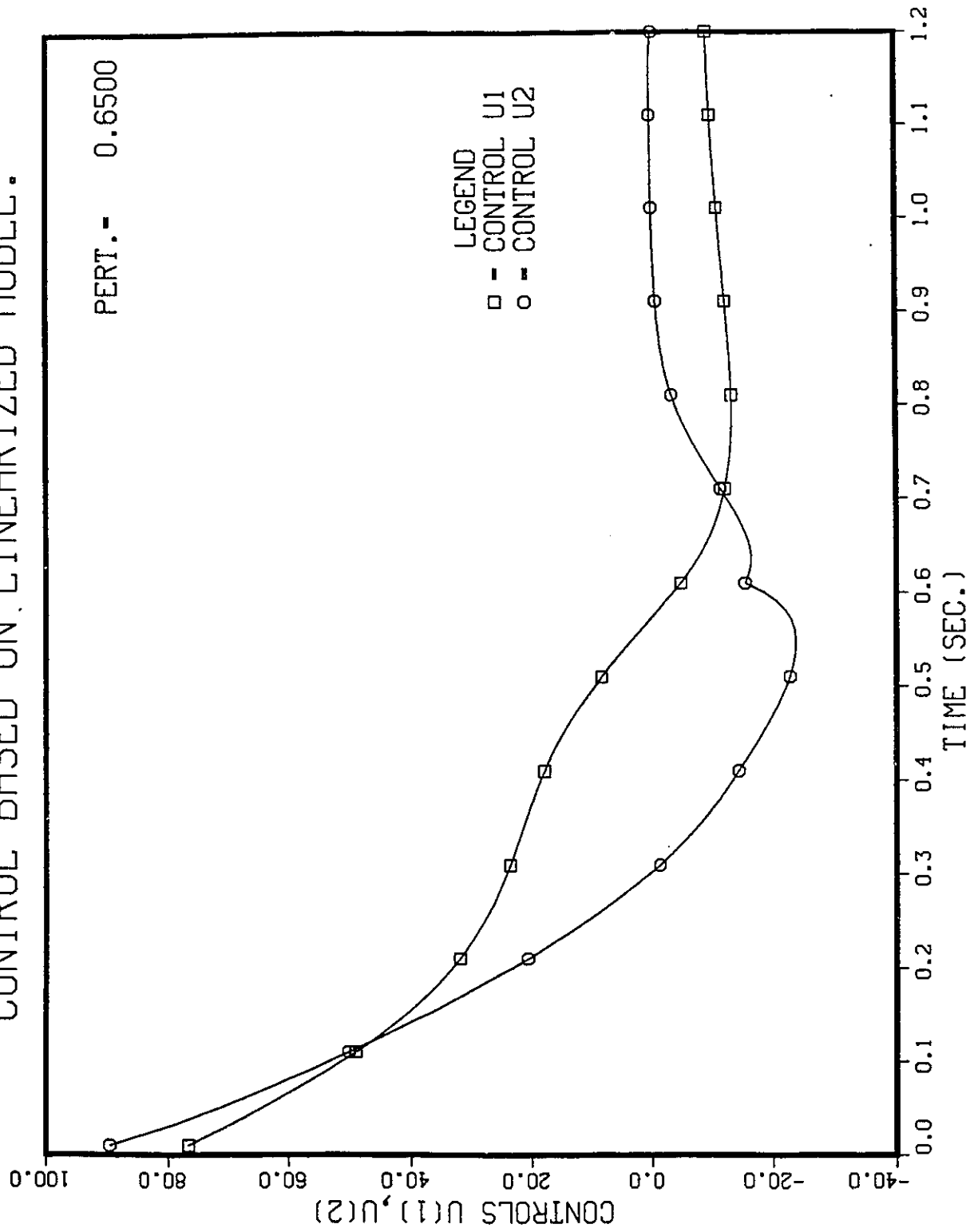


Figure IV.25 Optimal Control for  $\|p(t)\| = 1.3000$ , with Time. Ex. IV.2

# ROBOT MODEL RESPONSE WITH CONTROL BASED ON LINEARIZED MODEL.

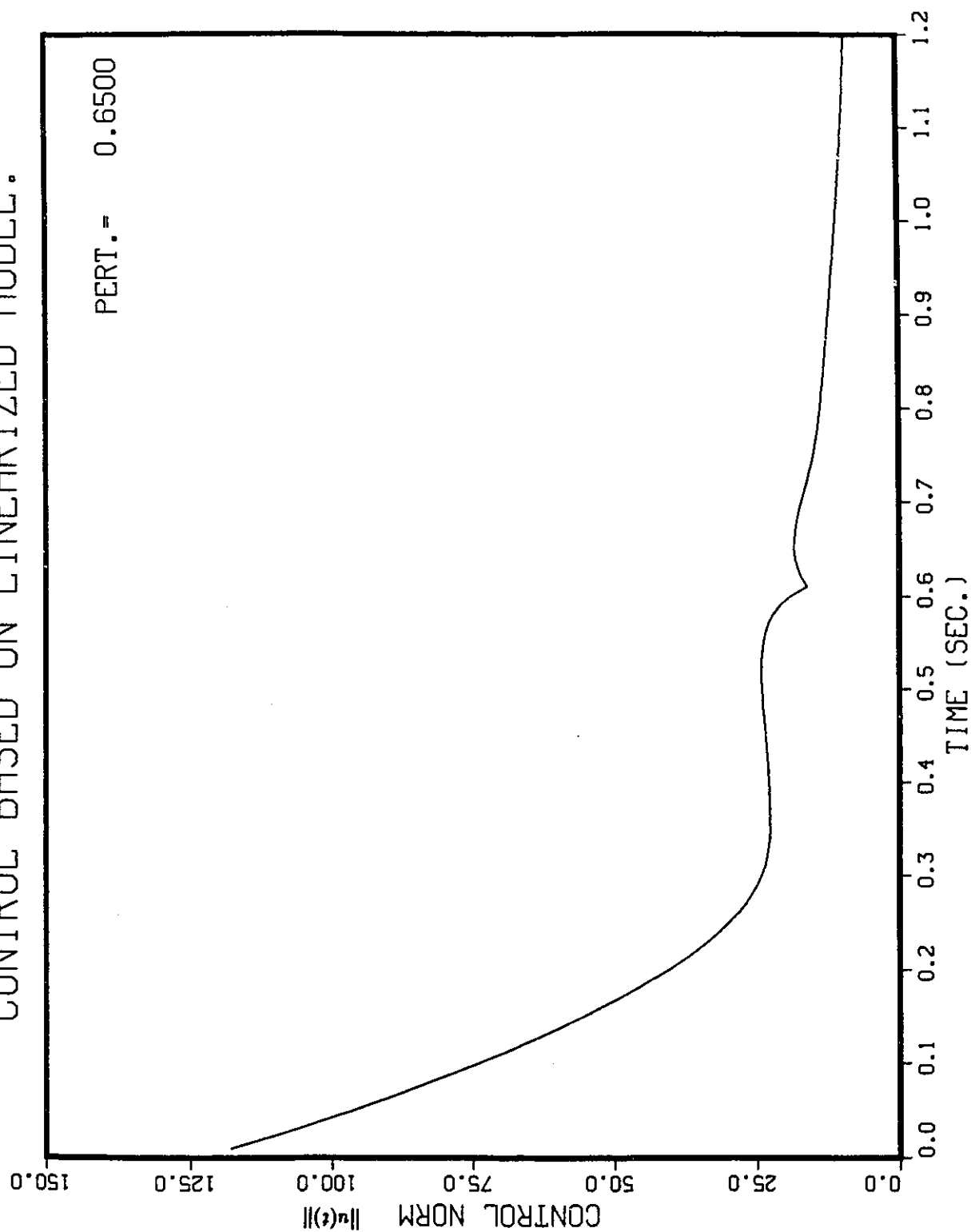


Figure IV.26 Control Norm  $\|u(t)\|$  for  $\|p(t)\| = 1.3000$  with Time. Ex. IV.2.

## CHAPTER V

### CONCLUSIONS.

In this work we have extended the LQR theory to the following;

i. The Zero Control Cost problem, that is the modified LQR problem in which the control has no contribution to the cost functional. This class of problems is the limiting case of the class known as the cheap control problems.[14] Where in the later class, the cheap control class, there is a scalar factor  $\epsilon$  that multiplies the control weighting matrix  $R$ , and the cheap control is obtained by considering the LQR problem when  $\epsilon \rightarrow 0$ . Whereas in our case, we have taken  $R$  identically zero.

ii. Regulation of nonlinear systems. The attractiveness of the LQR stems from the simplicity by which the linear control can be obtained and implemented. On the other hand, the complexity involved in solving the H-J-B equation (IV.3) for nonlinear systems makes it difficult to obtain and implement nonlinear controls. However, since a large number of control systems are basically used to regulate associated systems around a nominal trajectory (see section II.1), therefore for practical considerations linearization in the neighborhood of the nominal trajectory is a "reasonable" approximation for nonlinear systems, and hence, the mathematical power of linear system analysis becomes available.

iii. The regulation of satellite angular velocities is achieved through the use of the linear (affine) control law (IV.27). With this control it was possible to regulate the satellite angular velocities in the presence of perturbations as high as 90% of the initial state perturbation in all three components. (see figures IV.16,IV.17).

Since the satellite system is globally asymptotically stable with the linear control given in [1, chapter 4], it was possible to regulate the satellite angular velocities with this linear control law even with higher perturbations.

iv. The regulation of a robot movement around a time-optimal control trajectory. Although the robot system is nonlinear, but it was possible to use the linearization technique to regulate the movement of this robot around the time-optimal trajectory provided the initial state perturbation is bounded (see figure IV.23). However, increasing the initial state perturbation beyond this bound, it was not possible to achieve an acceptable bound on the state at the final time. The inability to achieve an acceptable bound on the state at the final time is due to driving the system beyond the linear region of the system around the given trajectory .

## REFERENCES

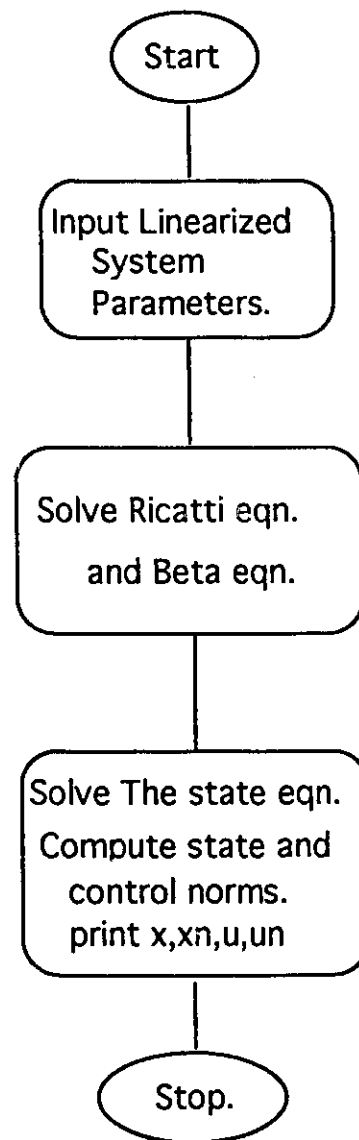
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## APPENDIX

This appendix contains two programs developed for the solution of examples IV.1 and IV.2. The first program for the solution of the satellite problem. This program starts by computing the entries of the linearized system matrices  $A$  and  $B$ , using those values and the initial conditions the program solves both the Ricatti matrix differential equation and  $\beta$  in the proper time sense. After solving those equations and storing them in the proper time order the program solves the state equations based on the linear affine control law (IV.27), compute the control and the state and control norms.

The second program is similar to the first with the exception of we are using only initial state perturbation. However, the flow charts indicate the steps the programs follow to solve the two problems.



Flow chart for the first program (Satellite) .

```

      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y1(6),Y2(3),Y3(3),SK(6,1500),SB(3,1500),F(3),FF(6),
& D(3,1500),A(3,3),AP(3,3),AK(3,3),B(3,3),BETA(3),YY(3),Z(3),
& QQ(3,3),Y4(3),PA(3),AZ(3),AA(3,3),RI(3,3),
& W1(6,9),W2(3,9),W3(3,9),C11(24),C22(24),C33(24)
      EXTERNAL FAK,FAB,FAS
      COMMON /AAA/ A,AP,QQ
      COMMON /BBB/ F,AK,BETA
      COMMON /CCC/ B,RI
      COMMON /DATA1/ PA
      NP = 1000
C INITIALIZATIONS.
      DO 5 J=1,3
      DO 5 I=1,NP
      SK(J,I)= 0.0
      SB(J,I)= 0.0
5      D (J,I)= 0.0
      DO 10 I=1,3
      DO 10 J=1,3
      A(I,J)= 0.0
      AP(I,J)= 0.0
      AK(I,J)= 0.0
      RI(I,J)= 0.0
      QQ(I,J)= 0.0
10     B(I,J)= 0.0
C LINEARIZED SYSTEM EQUATION MATRICES.
      A(1,3)= (669.0-100.0)*(7.29D-5)/645.0
      A(3,1)= (100.0-645.0)*(7.29D-5)/669.0
      B(1,1)= 1.0/645.0
      B(2,2)= 1.0/100.0
      B(3,3)= 1.0/669.0
      CALL MTR(A,AP)
      PP= 0.0
      JP=PP
      WRITE(6,*) JP
      L1 = NP/10
      L2 = NP/3
      DO 15 I=1,3
      DO 15 J= L1,L2
15     D(I,J) = PP/100.
      IND1= 1
      IND2= 1
      IND3= 1
      NW1 = 6
      NW2 = 3
      NW3 = 3
      N1 = 6
      N2 = 3
      N3 = 3
      DO 20 I=1,3
      F(I) = 0.0
      FF(I) = 0.0
      FF(I+3)= 0.0
      RI(I,I)= 1.0
      Y1(I) = 10.0

```

```

TP100010
TP100020
TP100030
TP100040
TP100050
TP100060
TP100070
TP100080
TP100090
TP100100
TP100110
TP100120
TP100130
TP100140
TP100150
TP100160
TP100170
TP100180
TP100190
TP100200
TP100210
TP100220
TP100230
TP100240
TP100250
TP100260
TP100270
TP100280
TP100290
TP100300
TP100310
TP100320
TP100330
TP100340
TP100350
TP100360
TP100370
TP100380
TP100390
TP100400
TP100410
TP100420
TP100430
TP100440
TP100450
TP100460
TP100470
TP100480
TP100490
TP100500
TP100510
TP100520
TP100530
TP100540
TP100550

```

|    |                                                       |          |
|----|-------------------------------------------------------|----------|
| 20 | Y1(I+3)= 0.0                                          | TP100560 |
|    | Y2(I)= 0.0                                            | TP100570 |
|    | Y3(1)= 1.0                                            | TP100580 |
|    | Y3(2)= 1.0                                            | TP100590 |
|    | Y3(3)= 1.0                                            | TP100600 |
|    | C COST FUNCTIONAL WEIGHTING MATRICES.                 | TP100610 |
|    | QQ(1,1)= 9500.0                                       | TP100620 |
|    | QQ(2,2)= 1000.0                                       | TP100630 |
|    | QQ(3,3)= 8000.0                                       | TP100640 |
|    | TOL= 1.0D-9                                           | TP100650 |
|    | C SOLUTION PERIOD TIME.                               | TP100660 |
|    | TAX = 50.0                                            | TP100670 |
|    | DT= TAX/DFLOAT(NP)                                    | TP100680 |
|    | T1= TAX                                               | TP100690 |
|    | T2= TAX                                               | TP100700 |
|    | DO 55 I=1,NP                                          | TP100710 |
|    | J=NP-I+1                                              | TP100720 |
|    | T11=DFLOAT(J-1)*DT                                    | TP100730 |
|    | T12=DFLOAT(J-1)*DT                                    | TP100740 |
|    | XYZ=111.                                              | TP100750 |
| 35 | CALL MFO(Y1,AK)                                       | TP100760 |
|    | C SOLUTION OF THE RICCATI EQUATION.                   | TP100770 |
|    | CALL DVERK(N1,FAK,T1,Y1,T11,TOL,IND1,C11,NW1,I1,IER1) | TP100780 |
|    | IF(IND1.LT.0.OR.IER1.GT.0) GO TO 90                   | TP100790 |
|    | DO 40 K=1,6                                           | TP100800 |
| 40 | SK(K,J)= Y1(K)                                        | TP100810 |
|    | CALL MFO(Y1,AK)                                       | TP100820 |
|    | DO 45 K=1,3                                           | TP100830 |
| 45 | F(K)= D(K,J)                                          | TP100840 |
| 50 | XYZ=222.                                              | TP100850 |
|    | C SOLUTION OF THE BETA EQUATION.                      | TP100860 |
|    | CALL DVERK(N2,FAB,T2,Y2,T12,TOL,IND2,C22,NW2,W2,IER2) | TP100870 |
|    | IF(IND2.LT.0.OR.IER2.GT.0) GO TO 90                   | TP100880 |
|    | DO 55 L=1,3                                           | TP100890 |
|    | SB(L,J) = Y2(L)                                       | TP100900 |
| 55 | CONTINUE                                              | TP100910 |
|    | T3 = 0.0                                              | TP100920 |
|    | COST = 0.0                                            | TP100930 |
|    | CALL MMP(R1,B,AA)                                     | TP100940 |
|    | DO 75 I=1,NP                                          | TP100950 |
|    | T33=DFLOAT(I)*DT                                      | TP100960 |
|    | DO 60 J=1,6                                           | TP100970 |
| 60 | FF(J) = SK(J,I)                                       | TP100980 |
|    | CALL MFO(FF,AK)                                       | TP100990 |
|    | DO 65 J=1,3                                           | TP101000 |
|    | F(J) = D(J,I)                                         | TP101010 |
| 65 | BETA(J)= SB(J,I)                                      | TP101020 |
|    | CALL MYM(AK,Y3,Z)                                     | TP101030 |
|    | DO 66 J=1,3                                           | TP101040 |
| 66 | YY(J)=Z(J)+BETA(J)                                    | TP101050 |
|    | CALL MYM(AA,YY,AZ)                                    | TP101060 |
|    | DO 74 J=1,3                                           | TP101070 |
| 74 | PA(J) = -AZ(J)                                        | TP101080 |
|    | CALL MYM(GG,Y3,Y4)                                    | TP101090 |
|    | C COST FUNCTIONAL COMPUTATION.                        | TP101100 |

```

      DO 76 J=1,3
      COST = COST + (Y3(J)*Y4(J)+PA(J)**2*RI(1,1))*DT/2.0
      SNO = DSQRT(PA(1)**2 +PA(2)**2 +PA(3)**2)
C      WRITE(6,7) T3,(PA(JJ),JJ=1,3)
      WRITE(6,7) T3,SNO
      XYZ=333.
C SOLUTION OF THE STATE EQUATION.
      CALL DVERK(N3,FAS,T3,Y3,T33,TOL,IND3,C33,NW3,W3,IER3)
      IF(IND3.LT.0.OR.IER3.GT.0) GO TO 90
      75 CONTINUE
C      WRITE(6,7) COST,RI(1,1)
      7 FORMAT(2X,4(E14.6,1X))
      GO TO 100
      90 WRITE(6,95) XYZ
      95 FORMAT(2X,' ERROR IN SOLVING THE DIFF. EQUATION',2X,F6.2)
      WRITE(6,*) IND1,IER1,IND2,IER2,IND3,IER3
      100 STOP
      END
      SUBROUTINE FAK(N,X,Y,YDOT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(6),YDOT(6),C1(3,3),C2(3,3),C3(3,3),C4(3,3),QQ(3,3),
& C5(3,3),C6(3,3),C7(3,3),C8(3,3),A(3,3),AP(3,3),AK(3,3),B(3,3)
      COMMON /AAA/ A,AP,QQ
      COMMON /BBB/ F,AK,BETA
      COMMON /CCC/ B,RI
      CALL MMP(AK,A,C1)
      CALL MMP(AP,AK,C2)
      CALL MMP(AK,B,C8)
      CALL MMP(C8,RI,C3)
      CALL MMP(B,AK,C4)
      CALL MMP(C3,C4,C5)
      P= 1.0
      CALL ADD(C5,C1,C6,P)
      CALL ADD(C6,C2,C7,P)
C      DO 10 I=1,3
C 10 YDOT(I)= C7(I,1) -90.0
      YDOT(1)= C7(1,1) -QQ(1,1)
      YDOT(2)= C7(2,2) -QQ(2,2)
      YDOT(3)= C7(3,3) -QQ(3,3)
      YDOT(4)= C7(1,2)
      YDOT(5)= C7(1,3)
      YDOT(6)= C7(2,3)
      RETURN
      END
      SUBROUTINE FAB(N,X,Y,YDOT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(3),YDOT(3),AP(3,3),AK(3,3),B(3,3),F(3),
& C1(3,3),C2(3,3),C3(3,3),C4(3,3),U2(3),U3(3)
      COMMON /AAA/ A,AP,QQ
      COMMON /BBB/ F,AK,BETA
      COMMON /CCC/ B,RI
      CALL MMP(AK,B,C1)
      CALL MMP(C1,RI,C4)
      CALL MMP(C4,B,C2)
      P= -1.0

```

TP101110  
 TP101120  
 TP101130  
 TP101140  
 TP101150  
 TP101160  
 TP101170  
 TP101180  
 TP101190  
 TP101200  
 TP101210  
 TP101220  
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 TP101550  
 TP101560  
 TP101570  
 TP101580  
 TP101590  
 TP101600  
 TP101610  
 TP101620  
 TP101630  
 TP101640  
 TP101650

```

      CALL ADD(AP,C2,C3,P)
      CALL MYM(C3,Y,U2)
      CALL MYM(AK,F,U3)
10    DO 10 I=1,3
      YDOT(1)= -(U2(I)+U3(I))
      RETURN
      END
      SUBROUTINE FAS (N,X,Z,ZDOT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Z(3),ZDOT(3),PA(3),F(3)
      COMMON /BBB/ F,AK,BETA
      COMMON /DATA1/ PA
      AIX= 645.0
      AIY= 100.0
      AIZ= 669.0
      W0 = 7.29E-5
      W0 = 0.0
C      ZDOT(1)= (PA(1)-(AIZ-AIY)*(Z(2)-W0)*Z(3))/AIX +F(1)
      ZDOT(2)= (PA(2)-(AIX-AIZ)*Z(1)*Z(3))/AIY +F(2)
      ZDOT(3)= (PA(3)-(AIY-AIX)*Z(1)*(Z(2)-W0))/AIZ +F(3)
      RETURN
      END
      SUBROUTINE MMP(A,B,C)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(3,3),B(3,3),C(3,3)
      DO 20 I=1,3
      DO 20 J=1,3
      C(I,J)= 0.0
      DO 20 K=1,3
20    C(I,J)= C(I,J) + A(I,K)*B(K,J)
      RETURN
      END
      SUBROUTINE ADD(A,B,C,P)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(3,3),B(3,3),C(3,3)
      REAL P
      DO 20 I=1,3
      DO 20 J=1,3
      C(I,J) = 0.0
20    C(I,J) = A(I,J) + B(I,J)*P
      RETURN
      END
      SUBROUTINE MTR(A,AP)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(3,3),AP(3,3)
      DO 10 I=1,3
      DO 10 J=1,3
10    AP(I,J)= A(J,I)
      RETURN
      END
      SUBROUTINE MYM(A,V,U)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(3,3),V(3),U(3)
      DO 20 I=1,3
      SUM= 0.0

```

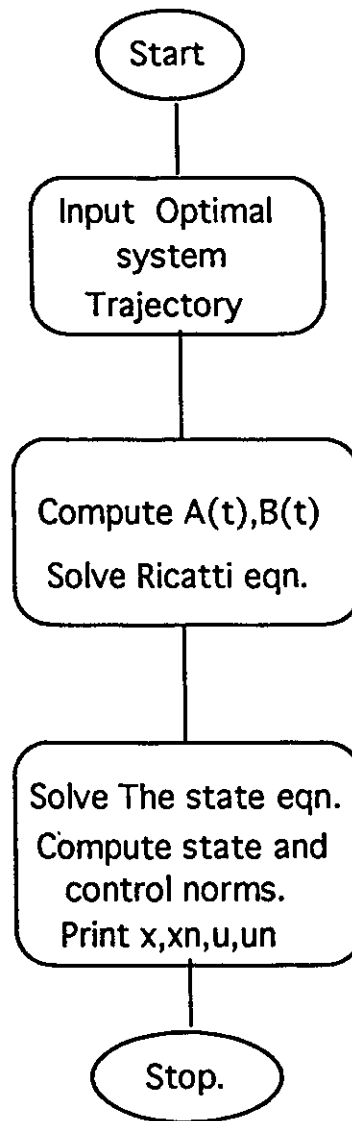
```

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TP101670
TP101680
TP101690
TP101700
TP101710
TP101720
TP101730
TP101740
TP101750
TP101760
TP101770
TP101780
TP101790
TP101800
TP101810
TP101820
TP101830
TP101840
TP101850
TP101860
TP101870
TP101880
TP101890
TP101900
TP101910
TP101920
TP101930
TP101940
TP101950
TP101960
TP101970
TP101980
TP101990
TP102000
TP102010
TP102020
TP102030
TP102040
TP102050
TP102060
TP102070
TP102080
TP102090
TP102100
TP102110
TP102120
TP102130
TP102140
TP102150
TP102160
TP102170
TP102180
TP102190
TP102200

```

```
      DO 10 J=1,3
10     SUM= SUM +A(I,J)*V(J)
20     U(I) = SUM
      RETURN
      END
      SUBROUTINE MFO(Y,A)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(6),A(3,3)
      DO 10 I=1,3
10     A(I,I)= Y(I)
      DO 20 I=2,3
20     A(1,I)=Y(I+2)
      A(3,I-1)=Y(I+3)
      A(2,1)= Y(4)
      A(2,3)= Y(6)
      RETURN
      END
```

```
TP102210
TP102220
TP102230
TP102240
TP102250
TP102260
TP102270
TP102280
TP102290
TP102300
TP102310
TP102320
TP102330
TP102340
TP102350
TP102360
TP102370
```



Flow chart for the second program (Robot) .

```

      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(4),C(24),W(4,9),A(4,4),B(4,2),U(2),R(2,2),Q(4,4),
& P(10),C1(24),W1(10,9),PT(300,10),GA(4,4),BT(2,4),P1(4,4),
& P2(2,4),AT(4,4),G1(4,2),XT(300,4),P3(2,4),TB(200,8)
      COMMON /ABC/ A,AT,Q
      COMMON /AAA/ U
      COMMON /BBB/ GA
      EXTERNAL FUN,F1
      READ(5,*) NP
C FEEDING THE MODEL WITH THE NOMINAL TRAJECTORY AND INITIALIZATION.
      DO 3 I=1,NP
3      READ(5,4) X,(XT(I,J),J=1,4)
4      FORMAT(2X,5(E13.4,1X))
      DO 5 I =1,NP
      DO 5 J =1,10
5      PT(I,J) = 0.0
      DO 6 J=1,10
6      P(J) = 0.0
      NW1=10
      N1 =10
      DO 10 I=1,4
      DO 10 J=1,4
      Q(I,J) = 0.0
      GA(I,J) = 0.0
10     A(I,J) = 0.0
C THE MATRIX P REPRESENTS THE SOLUTION OF THE RICCATI EQUATION
C WITH P(T) = 900.01, WEIGHTING MATRICES ARE Q AND R*-1.
      DO 20 I=1,4
20     P(I) = 900.0
      Q(1,1) = 5000.0
      Q(2,2) = 500.0
      Q(3,3) = 4500.0
      Q(4,4) = 500.0
      A(1,2) = 1.0
      A(3,4) = 1.0
      DO 30 I=1,4
      DO 30 J=1,2
      BT(J,I) = 0.0
30     B(I,J) = 0.0
      B(2,1) = 1.0/8.3
      BT(1,2)=B(2,1)
      DO 35 I=1,2
      DO 35 J=1,2
35     R(I,J) = 0.0
      R(1,1) = 1.0
      R(2,2) = 3.0
      II = 2
      IJ = 4
      TOL = 1.D-7
      X = 1.2
      DO 50 I=1, NP
      IND1 = 1
      K= NP-I+1
      DO 39 J=1,4
39     Y(J) = XT(K,J)

```

```

ROB00010
ROB00020
ROB00030
ROB00040
ROB00050
ROB00060
ROB00070
ROB00080
ROB00090
ROB00100
ROB00110
ROB00120
ROB00130
ROB00140
ROB00150
ROB00160
ROB00170
ROB00180
ROB00190
ROB00200
ROB00210
ROB00220
ROB00230
ROB00240
ROB00250
ROB00260
ROB00270
ROB00280
ROB00290
ROB00300
ROB00310
ROB00320
ROB00330
ROB00340
ROB00350
ROB00360
ROB00370
ROB00380
ROB00390
ROB00400
ROB00410
ROB00420
ROB00430
ROB00440
ROB00450
ROB00460
ROB00470
ROB00480
ROB00490
ROB00500
ROB00510
ROB00520
ROB00530
ROB00540
ROB00550

```

C THE LINEAR MODEL MATRICES A &amp; B AS FUNCTION OF TIME.

```

U2 = -5.0
IF(X.LE.0.60) U2 = 5.0
PHX = 3.7*Y(1)**2 + 4.6*(0.37+Y(1))**2 + 0.37
A(2,1) = Y(4)**2
A(2,4) = 2.0*Y(1)*Y(4) + 0.41012*Y(4)
A1 = 16.6*Y(1) + 3.404
A2 = -16.6*PHX*Y(2)*Y(4) - A1*(U2 - A1*Y(2)*Y(4))
A(4,1) = A2/(PHX**2)
A(4,2) = -Y(4)*A1/PHX
A(4,4) = -Y(2)*A1/PHX
B(4,2) = 1.0/PHX
BT(2,4) = B(4,2)
CALL MMP(B,R,G1,IJ,II,II)
CALL MMP(G1,BT,GA,IJ,II,IJ)
DO 40 L=1,4
DO 40 M=1,4
40 AT(L,M) = A(M,L)
M1=0
DO 48 I1=1,4
DO 45 J1=1,2
L=M1+J1
45 TB(K,L) = B(I1,J1)
48 M1 = M1 + 2

```

C SOLUTION OF THE RICCATI EQUATION BACKWARDS IN TIME.

```

XEND = 1.2-DFLOAT(I)*0.1 D-1
IYZ = 11
CALL DVERK(N1,F1,X,P,XEND,TOL,IND1,C1,NW1,W1,IER1)
IF(IND1.LT.0.OR.IER1.GT.0) GO TO 100
DO 50 J=1,10
50 PT(K,J) = P(J)
NW= 4
N = 4
X = 0.0
DO 51 J=1,4
51 Y(J) = 6.5D-1
WRITE (6,*) Y(1)
DO 70 I=1, NP
IND = 1
DO 60 J=1,10
60 P(J) = PT(I,J)
M1=0
DO 66 I1=1,4
DO 65 J1=1,2
L=M1+J1
65 B(I1,J1) = TB(I1,L)
66 M1 = M1 + 2
DO 68 I1=1,4
DO 68 J1=1,2
68 BT(J1,I1) = B(I1,J1)
CALL MFO(P,P1)
CALL MMP(BT,P1,P2,II,IJ,IJ)
CALL MMP(R,P2,P3,II,II,IJ)
XEND = DFLOAT(I)*0.1D-1
CALL MVM(P3,Y,U,II,IJ)

```

```

ROB00560
ROB00570
ROB00580
ROB00590
ROB00600
ROB00610
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ROB00630
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ROB00650
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ROB00670
ROB00680
ROB00690
ROB00700
ROB00710
ROB00720
ROB00730
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ROB00970
ROB00980
ROB00990
ROB01000
ROB01010
ROB01020
ROB01030
ROB01040
ROB01050
ROB01060
ROB01070
ROB01080
ROB01090
ROB01100

```

```

C SOLUTION OF THE STATE EQUATION FORWARD IN TIME.
      IYZ = 22
      CALL DVERK(N,FUN,X,Y,XEND,TOL,IND,C,NW,W,IER)
      IF(IND.LT.0.OR.IER.GT.0) GO TO 100
C      WRITE (6,900) X,(U(J),J=1,2)
C      WRITE (6,900) X,(Y(J),J=1,4)
      SNO = DSQRT( Y(1)**2 + Y(2)**2 + Y(3)**2 + Y(4)**2)
      CNO = DSQRT( U(1)**2 + U(2)**2)
      WRITE (6,900) X,CNO
      70  CONTINUE
      STOP
      100  WRITE(6,999) IYZ,IND,IER,IND1,IER1,IND2,IER2
      900  FORMAT(2X,5(E13.4,1X))
      999  FORMAT(2X,'ERROR IN D.E. SOL.',7(16,1X))
      END
      SUBROUTINE F1(N,X,Y,YDOT)
C COMPUTING THE DIFFERENTIAL OF K (K DOT) TO BE USED BY THE IMSL DVERK.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(10),YDOT(10),P(4,4),A(4,4),AT(4,4),P1(4,4),P2(4,4),
&      P3(4,4),P4(4,4),P5(4,4),Q(4,4),GA(4,4)
      COMMON /ABC/ A,AT,Q
      COMMON /BBB/ GA
      II = 2
      IJ = 4
      CALL MFO(Y,P)
      CALL MMP(P,A,P1,IJ,IJ,IJ)
      CALL MMP(AT,P,P2,IJ,IJ,IJ)
      CALL MMP(GA,P,P3,IJ,IJ,IJ)
      CALL MMP(P,P3,P4,IJ,IJ,IJ)
      DO 10 I=1,4
      DO 10 J=1,4
      10  P5(I,J) = - (P1(I,J) + P2(I,J) - P4(I,J) + Q(I,J))
      DO 20 I=1,4
      20  YDOT(I) = P5(I,I)
      DO 30 I=1,3
      J=I+4
      30  YDOT(J) = P5(I,I+1)
      YDOT(8) = P5(1,3)
      YDOT(9) = P5(2,4)
      YDOT(10) = P5(1,4)
      RETURN
      END
      SUBROUTINE FUN(N,X,Y,YDOT)
C COMPUTING THE DIFFERENTIAL OF X (X DOT) TO BE USED BY THE IMSL DVERK.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(4),YDOT(4),U(2)
      COMMON /AAA/ U
      U1 = U(1)
      U2 = U(2)
      YDOT(1) = Y(2)
      YDOT(2) = (U1+3.7*Y(1)*Y(4)**2+4.6*(0.37+Y(1))*Y(4)**2)/8.3
      YDOT(3) = Y(4)
      AA = 0.37+3.7*Y(1)**2+4.6*(0.37+Y(1))**2
      YDOT(4) = (U2-7.4*Y(1)*Y(2)*Y(4)-9.2*(0.37+Y(1))*Y(2)*Y(4))/AA
      RETURN

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ROB01110  
 ROB01120  
 ROB01130  
 ROB01140  
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 ROB01180  
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 ROB01210  
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 ROB01230  
 ROB01240  
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 ROB01310  
 ROB01320  
 ROB01330  
 ROB01340  
 ROB01350  
 ROB01360  
 ROB01370  
 ROB01380  
 ROB01390  
 ROB01400  
 ROB01410  
 ROB01420  
 ROB01430  
 ROB01440  
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 ROB01460  
 ROB01470  
 ROB01480  
 ROB01490  
 ROB01500  
 ROB01510  
 ROB01520  
 ROB01530  
 ROB01540  
 ROB01550  
 ROB01560  
 ROB01570  
 ROB01580  
 ROB01590  
 ROB01600  
 ROB01610  
 ROB01620  
 ROB01630  
 ROB01640  
 ROB01650

```
      END
C AUXILARY SUBROUTINES.
      SUBROUTINE MMP(A,B,C,L1,L2,L3)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(L1,L2),B(L2,L3),C(L1,L3)
      INTEGER L1,L2,L3
      DO 20 I=1,L1
      DO 20 J=1,L3
      C(I,J)= 0.1D-9
      DO 20 K=1,L2
10    C(I,J)= C(I,J) + A(I,K)*B(K,J)
      RETURN
      END
      SUBROUTINE MYM(A,V,U,L1,L2)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(L1,L2),V(L2),U(L1)
      DO 10 I=1,L1
      U(I)= 0.0
      DO 10 J=1,L2
10    U(I)=U(I)+A(I,J)*V(J)
      RETURN
      END
      SUBROUTINE MFO(Y,A)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(10),A(4,4)
      DO 10 I=1,4
10    A(I,1)= Y(I)
      DO 20 I=1,3
      J=I+1
20    A(I,J)=Y(I+4)
      A(1,3)= Y(8)
      A(2,4)= Y(9)
      A(1,4)= Y(10)
      DO 30 J=1,4
      DO 30 I=1,4
30    A(I,J)= A(J,I)
      RETURN
      END
```

```
ROB01660
ROB01670
ROB01680
ROB01690
ROB01700
ROB01710
ROB01720
ROB01730
ROB01740
ROB01750
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ROB01770
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ROB01800
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ROB01960
ROB01970
ROB01980
ROB01990
ROB02000
ROB02010
ROB02020
ROB02030
```