Voronoi Diagrams in Metric Spaces

Jonathan Lemaire-Beaucage

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Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

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Abstract

In this thesis, we will present examples of Voronoi diagrams that are not tessellations. Moreover, we will find sufficient conditions on subspaces of $\mathbb{E}^2$, S^2 and the Poincaré disk and the sets of sites that guarantee that the Voronoi diagrams are pre-triangulations. We will also study g-spaces X, which are metric spaces with 'extendable' geodesics joining any 2 points and give properties for a set of sites in a g-space that again guarantees that the Voronoi diagram is a pre-triangulation.

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Contents

Introduction

How far is the next village ? How far do we have to travel to get to the other side of the ocean ? How far is the moon, the sun ? A lot of humans have asked themselves questions similar to this. To answer those questions, they needed the concept of distance. In mathematics, this concept has been generalized. We define a *metric space* X as a nonempty set X of elements together with a real-valued function d on $X \times X$ such that for all x, y, and z in X :

1. $d(x, y) \geq 0$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) = d(y, x)$;
4. $d(x, y) \leq d(x, z) + d(z, y)$.

We call d the distance function. Once we have a notion of distance, one farmer in a mountainous region can ask himself which village is the closest and thus we can associate each farmer to the closest village. Similarly, we could associate each citizen to the closest metro station to maximize public transport. This concept has also been generalized in Mathematics and is called a Voronoi diagram. Precisely, let X be a metric space and S be a subset of X. For $p \in S$, the *Voronoi cell* $V(p)$ of p is

$$V(p) = \{x \in X; d(x, p) \leq d(x, q),\ \forall\, q \in S\}.$$

The collection $V(S)$ of all Voronoi cells $V(p)$, for $p \in S$ is the *Voronoi diagram of S* and we call S the set of *sites*. Voronoi diagrams are used everywhere. For example, let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. This distance better represents the distance between two points in a north American city than the Euclidean distance. So this distance is used to construct Voronoi diagrams, to see which grocery store is the closest to a consumer. We can also modify this distance function so that it reflects different prices at each grocery store, to obtain different Voronoi diagrams. So Voronoi diagrams are used in market area analysis [6]. Moreover, they are used to approximate the quantity of precipitation in a region. If each site s is a pluviometer, then using the Voronoi diagram we approximate the precipitation in the region by calculating

$$Z = \sum_{s \in S} |V(s)| z(s)$$

where $z(s)$ is the precipitation at s and $|V(s)|$ is the area of the Voronoi cell of s [[3] p. 555]. Also, Voronoi diagrams of spheres are used by the military for the coverage areas of air bases in the world [[3], p. 206]. Voronoi diagrams are also used to choose mooring posts for lifeboats on the river, so that the whole river can be covered by the lifeboats. The flow velocity and the velocity of the lifeboats will determine the Voronoi cell of each mooring post [[3] p. 205]. There are many more examples [3].

With many applications in real life, mathematicians have been motivated to study Voronoi diagram more abstractly. There remains much to be known about Voronoi diagrams in general.

For example, let D be a collection of closed subsets of X that covers X. We say that D is a *tessellation of* X if the pairwise intersection of the interiors of the element of D is empty. The elements of D are called the *tiles* of the tessellation. A *pre-triangulation* $\mathcal{P}$ of X is a tessellation of X into polygons such that :

1) any two polygons in $\mathcal{P}$ intersect in a common face, or not at all, and

2) any bounded set in X intersects only finitely many polygons in $\mathcal{P}$.

It is known that, when $X = \mathbb{E}^2$ and S is finite, then the Voronoi diagram of X is a tessellation [[3], §2, property 6]. Is it a pre-triangulation ? For general metric spaces X and a subset S, do we have a tessellation or a pre-triangulation ? In this thesis, we give examples to show that it is not always the case and find sufficient conditions on the metric space and on the set of sites for the Voronoi diagram to be a tessellation and or a pre-triangulation.

In chapter 1, we recall some properties of metric spaces and compact spaces and introduce a somewhat generalized notion of compact spaces. Moreover, we recall the Ascoli-Arzela Theorem and give a corollary.

In chapter 2, we first introduce Voronoi diagrams and define important properties the set of sites S can have. We then recall the notion of a tessellation of the space. After, we present examples where the Voronoi diagram isn't a tessellation. To remedy this, we then define two properties on X and one on S, which are sufficient for the Voronoi diagram to be a tessellation. We then move on to define the concepts of polygons and faces of polygons in a general setting, so that we may speak of pre-triangulations in this general context. Finally, we find conditions on the metric space and the set of sites that guarantees that the Voronoi diagram is a pre-triangulation, and show this may fail with some examples not satisfying this condition.

In chapter 3, we use our previous results to study Voronoi diagrams in metric subspaces of Euclidean plane, the 2-sphere, and the Poincaré disk. We find sufficient conditions on these subspaces and the sets of sites that again allow the Voronoi diagram to be a pre-triangulation.

In chapter 4, we study g-spaces X, which are metric spaces with 'extendable' geodesics joining any 2 points. It has been shown that for dimensions 1-4 those g-spaces are manifolds [2],[9],[15].

In chapter 5, we give a property for a set of sites in a g-space that again guarantees that the Voronoi diagram is a pre-triangulation. Moreover, we describe the boundary of Voronoi cells in g-spaces.

Chapter 1

Preliminaries

In this chapter, we will review the definition and fundamental results of metric spaces and sequences $\{f_n\}_{n\geq 1}$ of continuous function between metric spaces and recall the concept of path in a metric space. We follow chapter 7, sections 2 and 7 of [13].

1.1 Metric spaces and compact spaces

In this thesis, all the spaces will be metric spaces.

Definition 1.1.1 *A* metric space (X, d) *is a nonempty set* X *together with a real-valued function* d *on* $X \times X$ *such that for all* $x, y,$ *and* z *in* X *:*

1. $d(x, y) \geq 0$*;*
2. $d(x, y) = 0$ *if and only if* $x = y$*;*
3. $d(x, y) = d(y, x)$*;*
4. $d(x, y) \leq d(x, z) + d(z, y)$.

We call d the distance function and note that this function is continuous.

Definition 1.1.2 *A metric space X is called* separable *if it has a countable dense subset D.*

We recall the important notion of compactness.

Definition 1.1.3 *A metric space X is said to be* compact *if every open covering U of X has a finite subcovering, that is, if there is a finite collection $\{O_1, O_2, ..., O_n\} \subset U$ such that $X = \bigcup_{i=1}^{n} O_i$.*

Definition 1.1.4 *A space X is said to be* sequentially compact *if every sequence $\{x_n\}_{n\geq 1}$ of elements of X contains a convergent subsequence $\{x_{n_k}\}_{k\geq 1}$.*

We recall the following important results in the theory of metric spaces.

Theorem 1.1.5 *Let X be a metric space. Then X is compact if and only if X is sequentially compact.*

Proposition 1.1.6 *A compact metric space is separable.*

Proposition 1.1.7 *If f is a continuous function of a compact metric space X into a metric space Y, then f is uniformly continuous.*

We now introduce a property of metric spaces related to compactness that will be important in this thesis.

Definition 1.1.8 *A metric space X is a* proper metric space *if for any closed and bounded subset C of X, every infinite sequence in C has a subsequence that converges in C.*

Remark 1.1.9 1) By Theorem 1.1.5, a metric space X is proper if for any closed and bounded subset C of X, C is compact.
2) Every closed subset of a compact space is compact. Thus, every compact metric space is a proper metric space.

3) A proper metric space is complete. Moreover, as a closed subspace of a complete space is complete, a closed subspace of a proper metric space is proper.

Example 1.1.10 1) A compact metric space is proper.
2) The metric space $\mathbb{E}^n$, (i.e $\mathbb{R}^n$ with the Euclidean metric) is proper.
3) Let $X = l^2_{\mathbb{R}}(\mathbb{N}) = \{\{x_n\}_{n\geq 1}; x_n \in \mathbb{R}, \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ with $d(x,y) = \|x-y\|_2$ where $\|x\|_2 = \left(\sum_{n\geq 1} |x_n|^2\right)^{\frac{1}{2}}$. Define $S(n) \in l^2_{\mathbb{R}}(\mathbb{N})$ by $S(n)_k = \begin{cases} 0, & k \neq n \\ 1, & k = n \end{cases}$, for $k \geq 1$
Let $C = \bigcup_n S(n)$. For $i \neq j$, $d(S(i), S(j)) = \sqrt{2}$, thus C is closed and bounded, but $\{S(n)\}$ doesn't have a subsequence that converges in C. Hence, X is not proper.

We now relax the definition of a proper metric space as follows:

Definition 1.1.11 *A metric space is* semi-proper *if for every bounded subspace C, every infinite sequence in C has a Cauchy subsequence.*

Note that this property is inherited by subspaces : Let us take a sequence $\{y_n\} \subset B \subset Y \subset X$ where B is bounded in Y and X is a semi-proper metric space. Then of course B is also bounded in X. We also have that $\{y_n\} \subset B \subset X$ therefore $\{y_n\}$ has a Cauchy subsequence in B. Thus Y is semi-proper.

Remark 1) Recall that a compact metric space C is sequentially compact and so every sequence in C has a Cauchy subsequence, and thus every compact metric space is semi-proper.
2) A proper metric space is a complete semi-proper metric space.

We now introduce the following definition and proposition, to give an equivalent definition of a semi-proper metric space.

Definition 1.1.12 *A metric space X is* totally bounded *if for every $\varepsilon > 0$ there exists a finite subset A_ε such that $\bigcup_{a\in A_\varepsilon} B(a,\varepsilon) = X$.*

Remark Clearly any compact set in a metric space is totally bounded.

The next result is an exercise in [5].

Proposition 1.1.13 *A metric space X is totally bounded if and only if every sequence in X has a Cauchy subsequence.*

The following proof follows [[4], Theorem 2.6].

Proof: Since X is totally bounded, for every $\varepsilon > 0$, let A_ε be a finite subset of X such that $\bigcup_{a\in A_\varepsilon} B(a,\varepsilon) = X$. If $\{x_n\}$ is any infinite sequence, and $k \in \mathbb{N}$, there will be an infinite subsequence $\{x_{\phi(n)}\}$ of $\{x_n\}$ and an element $a_k \in A_{\frac{1}{k}}$ such that $\{x_{\phi(n)}\} \subset B(a_k, 1/k)$.

Proceeding inductively from $m = 1$, for each $m \in \mathbb{N}$, we can thus find a point $a_m \in A_{1/m}$ and an infinite subsequence $\{x_{\phi_m(n)}\}$ of $\{x_{\phi_{m-1}(n)}\}$ such that $\{x_{\phi_m(n)}\} \subset B(a_m, 1/m)$.

Now consider the sequence $y_n = x_{\phi_n(n)}$ for all $n \in \mathbb{N}$, and let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Note that for $p, q \geq N$, then $\{x_{\phi_p(n)}\}$ and $\{x_{\phi_q(n)}\}$ are both subsequences of $\{x_{\phi_N(n)}\}$ and therefore $\{x_{\phi_p(n)}\}, \{x_{\phi_q(n)}\} \subset B(a_N, 1/N)$ for all $n \in \mathbb{N}$. In particular, $x_{\phi_p(p)}, x_{\phi_q(q)} \in B(a_N, 1/N) \subset B(a_N, \varepsilon)$. This shows that the sequence $\{y_n\}$ is a Cauchy subsequence of $\{x_n\}$.

If X is not totally bounded, there exists $\varepsilon > 0$ such that there is no finite set A_ε with $\bigcup_{a\in A_\varepsilon} B(a,\varepsilon) = X$. Choose $x_1 \in X$, so that there is $x_2 \in X \setminus B(x_1, \varepsilon)$; this implies that $d(x_1, x_2) \geq \varepsilon$. Considering the set $\{x_1, x_2\}$, we can find $x_3 \in X \setminus (B(x_1,\varepsilon) \cup B(x_2,\varepsilon))$ and hence $d(x_2, x_3) \geq \varepsilon$ and $d(x_1, x_3) \geq \varepsilon$. Thus, we can construct a sequence $\{x_n\}$ such that $x_n \notin X \setminus \bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$, and therefore such that $d(x_i, x_n) \geq \varepsilon$ for all $i < n$. This sequence has no Cauchy subsequence. ∎

Remark A metric space is semi-proper, if every bounded subspace is totally bounded.

Example. $X = (0, 1) \subset \mathbb{E}$ is totally bounded but not compact.
Clearly, X is not compact. Moreover, for $\varepsilon > 0$, let $A_\varepsilon = \{n\varepsilon; n \in \mathbb{N}\} \cap X$. Then A_ε is finite, $X \subset \bigcup_{n \geq 1} B(n\varepsilon, \varepsilon)$ and so X is totally bounded.

Example. An infinite space X with the discrete metric $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ is bounded but not totally bounded. Indeed, for any $x \in X$, $X \subset B(x, 2)$ and so X is bounded. But for $\varepsilon = \frac{1}{2}$, the balls $B(x, \varepsilon)$ are disjoint for distinct $x's$, hence X is not totally bounded.

1.2 The Ascoli-Arzela Theorem and an important corollary

First we recall this important definition, that is needed in the Ascoli-Arzela Theorem.

Definition 1.2.1 *Let X and Y be metric spaces and $\{f_n\}$ a sequence of functions, where $f_n : X \to Y$.*
1) The sequence $\{f_n\}$ is said to be equicontinuous *at the point x_0 in X, if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x_0, y) < \delta$ with $x_0, y \in X$ implies that $d(f_n(x_0), f_n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.*
2) If the sequence $\{f_n\}$ is equicontinuous for every $x_0 \in X$, then the sequence is said to be equicontinuous.

We now state the Ascoli-Arzela Theorem that will be used to prove Corollary 1.2.3.

Theorem 1.2.2 [[*13*], p.169] *Let F be an equicontinuous family of functions from a* *separable metric space X to a metric space Y. Suppose $\{f_n\}$ is a sequence in F such that for each x in X, the closure of the set $\{f_n(x) : 0 \leq n < \infty\}$ is compact. Then there is a subsequence $\{f_{n_k}\}$ that converges pointwise to a continuous function f, and the convergence is uniform on each compact subset of X.*

We have the following corollary that will be used to prove Theorem 4.2.3.

Corollary 1.2.3 *Let Y be a proper metric space, X be a compact metric space, and $(f_n)_{n\geq 1}$ be an equicontinuous sequence of functions $f_n : X \to Y$.*

If for each $x \in X$, $\{f_n(x) : 0 \leq n < \infty\}$ is bounded, then there is a subsequence $\{f_m(x)\}$ that converges uniformly to a uniformly continuous function $f : X \to Y$.

Proof: If for each $x \in X$, $\{f_n(x) : 0 \leq n < \infty\}$ is bounded, then $\overline{\{f_n(x) : 0 \leq n < \infty\}}$ is a closed and bounded subset of a proper metric space, and thus is compact. Moreover, if X is compact, this implies that X is separable and f is uniformly continuous. By Ascoli-Arzela Theorem we thus have the Corollary 1.2.3. ∎

1.3 Paths in a metric space

In this section, we recall the concept of path in a metric space.

Definition 1.3.1 *Let X be a metric space.*

1) A path *in X is a continuous function $\gamma : [a, b] \to X$, where a and b are two arbitrary real numbers satisfying $a \leq b$.*

2) If $\gamma(a) = x$ and $\gamma(b) = y$, then we say that x and y are the endpoints *of γ, and that γ* joins *the points x and y.*

3) The image of γ in X is called the trajectory *of γ, and is denoted im (γ).*

4) A path, γ is called an arc *if γ is injective. If $\gamma : [a, b] \to X$ is an arc, then if $a \leq c \leq d \leq b$, $\omega : [c, d] \to X$ defined by $t \to \gamma(t)$ is called a* subarc *of γ.*

We choose to allow arbitrary intervals as domains for our paths γ as this will be more convenient when we investigate segments in chapter 4.

Example 1.3.2 Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be defined by $t \mapsto (\cos t, \sin t)$. Then, γ is a path with both endpoints equal to $(1, 0)$ and its trajectory im (γ) is the unit circle S^1. But γ is not an arc because $\gamma(0) = (1, 0) = \gamma(2\pi)$.

Definition 1.3.3 *Let X be a metric space and $\gamma : [a, b] \to X$ a path. Then the path $\gamma^o : [-b, -a] \to X$ defined by $\gamma^o(t) = \gamma(-t)$ is a path from $\gamma(b)$ to $\gamma(a)$. We will call this path the* opposite of γ.

Chapter 2

Voronoi Diagrams in Metric Spaces

In this chapter, (X, d) will denote a metric space, with d the distance function. We will define Voronoi diagrams in metric spaces and find sufficient conditions to guarantee that those diagrams are tessellations and pre-triangulations of the metric spaces.

2.1 Voronoi Diagrams

In this section, we will introduce the notion of Voronoi diagrams in metric spaces.

Definition 2.1.1 *The* bisector *of two points $p \neq q \in X$ is defined by*

$$b(p,q) = \{x \in X; d(x,p) = d(x,q)\}$$

Then $H(p,q) = \{x \in X; d(x,p) \leq d(x,q)\}$ is the half space *containing p.*

Hence, $H(p,q)^c = \{x \in X; d(x,p) > d(x,q)\} = H(q,p) \setminus b(p,q)$.

Remark 2.1.2 Let p, q be distinct points in X. If $\{x_n\}_{n\geq 1}$ is a sequence in $H(p,q)$ that converges to x, then $x \in H(p,q)$. Hence, $H(p,q)$ is closed. Indeed, $d(x_n, p) \leq d(x_n, q)$ for all $n \geq 1$, implies that $d(x,p) \leq d(x,q)$.

Let S be a subset of X, whose elements shall be called *sites*.

Definition 2.1.3 *For $p \in S$, the* Voronoi cell $V(p)$ *of p is*

$$V(p) = \{x \in X; d(x,p) \leq d(x,q),\ \forall\, q \in S\} = \bigcap_{q \in S} H(p,q)$$

Remark 2.1.4 Since each half space is closed by Remark 2.1.2, so is any Voronoi cell.

Definition 2.1.5 *The collection $V(S)$ of all Voronoi cells $V(p)$, for $p \in S$ is the* Voronoi diagram of S.

Two properties of S will be central in this thesis:

Definition 2.1.6 *Let $K, M > 0$ be real numbers.*
*(i) The set S is M-*separated *if $d(p,q) \geq M$, $\forall p, q \in S, p \neq q$. If S is M-separated, for some $M > 0$, we say that S is* well-separated.
*(ii) The set S is K-*syndetic *if $\bigcup_{p \in S} B(p,K) = X$, where $B(p,K) = \{x \in X; d(p,x) < K\}$. If S is K-syndetic, for some $K > 0$, we say that S is* syndetic.

Remark Let S be a finite set. Then S is well-separated. Moreover, if X is an unbounded set, then S is not syndetic.

Examples The following are examples which illustrate these notions.

1) Let $X = \mathbb{E}$, where $\mathbb{E}$ is $\mathbb{R}$ with the Euclidean metric, and $S = \mathbb{Z}$. Then S is 1-separated and 1-syndetic.

2) Let $X = \mathbb{E}^2$, where $\mathbb{E}^2$ is $\mathbb{R}^2$ with the Euclidean metric, and $S = \{(0,n); n \in \mathbb{N}\}$ for $n \in \mathbb{N}$. Then S is 1-separated, but is not syndetic, since $\forall K > 0$, $(2K, 0) \notin \bigcup_{p \in S} B(p,K)$.

3) Let $X = \mathbb{E}$ and $S = \{\frac{1}{n}; n \geq 1\} \cup \mathbb{Z}$. Then S is not well-separated, but is 1-syndetic.

4) Let $X = \mathbb{E}^2$ and $S = \{(0, \frac{1}{n}); n \geq 1\}$. Then S is not well-separated and not syndetic.

Remark 2.1.7 If $S = X$, then S is clearly K-syndetic for any $K > 0$.

2.2 Voronoi Diagrams as Tessellations in Metric Spaces

In this section, we will define tessellations of metric spaces (Definition 2.2.1) and show in Theorem 2.2.10 that a Voronoi diagram of an infinite set S of sites in a metric space X is a tessellation if S is well-separated and X is semi-proper (Definition 1.1.11) and well-bisected (Definition 2.2.5).

Before defining a tessellation of X, we recall that the interior of a set $A \subset X$ is

$$\mathring{A} = \{x \in X; \exists\, \varepsilon > 0 \text{ such that } B(x, \varepsilon) \subset A\},$$

and that a *cover* of X is a collection of subsets whose union contains X.

Definition 2.2.1 *Let D be a collection of closed subsets of X that covers X. We say that D is a* tessellation of X *if the pairwise intersection of the interiors of the element of D is empty. The elements of D are called the* tiles *of the tessellation.*

Example Let $X = \mathbb{E}$ and $S = \mathbb{Z}$. Then $V(S)$, the Voronoi diagram of S, is a tessellation of X.

The well-bisected property is related to the boundaries of subspaces of X.

Definition 2.2.2 *The* boundary *of $A \subset X$ is*

$$\partial A = \overline{A} \cap \overline{A^c} = \{x \in X; \forall \varepsilon > 0,\ B(x, \varepsilon) \cap A \neq \emptyset \textit{ and } B(x, \varepsilon) \cap A^c \neq \emptyset\}.$$

Lemma 2.2.3 *Let X be a metric space and $A \subset X$. Then $A \setminus \partial A = \mathring{A}$.*

Proof: Let $x \in A \setminus \partial A$. Since $x \in A$, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \cap A^c = \emptyset$, and so $B(x,\varepsilon) \subset A$. Hence $x \in \mathring{A}$.

Conversely, if $x \in \mathring{A}$, then for some $\varepsilon > 0$ we have $B(x,\varepsilon) \subset A$. Clearly $x \in A$ and $B(x,\varepsilon) \cap A^c = \emptyset$, hence $x \in A \setminus \partial A$. ∎

Remark In fact, this lemma holds for any topological space [[16], p.28].

Recall that for $x, y \in X$, the bisector $b(x,y)$ is $\{z \in X; d(z,x) = d(z,y)\}$.

Proposition 2.2.4 *If $p, q \in S$, then $\partial H(p,q) \subset b(p,q)$.*

Proof: If $p = q$, then $b(p,q) = X$ and the proposition is clear. Let $x \in \partial H(p,q)$. Then, there is a sequence $\{x_n\}_{n \geq 1} \subset H(p,q)$ with $x_n \to x$ and a sequence $\{y_n\}_{n \geq 1} \subset H(p,q)^c$ with $y_n \to x$. But for all $n \geq 1$, $d(x_n,p) \leq d(x_n,q)$, and so $d(x,p) \leq d(x,q)$. Similarly, for all $n \geq 1$, $d(y_n,p) > d(y_n,q)$, and so $d(x,p) \geq d(x,q)$. Hence $x \in b(p,q)$. ∎

We now introduce a new (to our knowledge) property of metric spaces.

Definition 2.2.5 *A metric space X is said to be* well-bisected *if*

$$b(x,y) = \partial H(x,y) \cup \partial H(y,x), \text{ for every } x \neq y \in X.$$

Remark 2.2.6 By Proposition 2.2.4, a space is well-bisected if

$$b(x,y) \subset \partial H(x,y) \cup \partial H(y,x), \text{ for all } x \neq y.$$

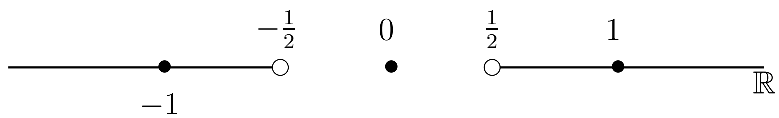
Example 2.2.7 a) Let $X = \mathbb{E}^2$. For every distinct $x, y \in \mathbb{E}^2$, $\partial H(x,y) = b(x,y)$, thus $\mathbb{E}^2$ is well-bisected.

b) Let $X = \mathbb{E} \setminus ([-\frac{1}{2}, 0) \cup (0, \frac{1}{2}])$, with the relative topology. Then,

$$B(0,\frac{1}{4}) \cap H(-1,1)^c = \emptyset \text{ and } B(0,\frac{1}{4}) \cap H(1,-1)^c = \emptyset.$$

So,

$$\partial H(-1,1) \cup \partial H(1,-1) = \emptyset \cup \emptyset \neq \{0\} = b(-1,1)$$



Hence X is not well-bisected.

c) We now give an example of a well-bisected metric space X where $\partial H(p,q) \neq \partial H(q,p)$ for distinct $p, q \in X$. Let $X = \mathbb{E} \setminus (0,1)$ with the relative topology. Then, $b(-1,1) = \{0\}$, $\partial H(-1,1) = \emptyset$ and $\partial H(1,-1) = \{0\}$. Therefore, X is well-bisected.

We now introduce a sufficient condition, using arcs (see Definition 1.3.1, 4)), for a space to be well-bisected.

Lemma 2.2.8 *Let X be a path-connected metric space. Then X is well-bisected if for all distinct points x and y in X, for all $z \in b(x,y)$, there exists an arc $\gamma : [0,1] \to X$ from either x or y to z, such that im $(\gamma) \cap b(x,y) = \{z\}$.*

Proof: Let $x \neq y \in X$. By Remark 2.2.6, we only have to show that $b(x,y) \subset \partial H(x,y) \cup \partial H(y,x)$.

Let $z \in b(x,y)$. By assumption, w.l.o.g. there exists an arc $\gamma : [0,1] \to X$ from x to z such that im $(\gamma) \cap b(x,y) = \{z\}$, where $z = \gamma(1)$. Because γ is continuous, for all $t < 1$, $\gamma(t) \in H(y,x)^c$. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $1 - t < \delta$ implies that $d(\gamma(t), z) < \varepsilon$. Therefore, $\gamma(t) \in B(z,\varepsilon) \cap H(y,x)^c$. Moreover, $z \in B(z,\varepsilon) \cap H(y,x)$ and so $z \in \partial H(y,x)$. Hence X is well-bisected. ∎

Lemma 2.2.9 *Let X be a metric space, S be a subset of sites of X and $p, q \in S$. Then $\partial H(p,q) \cap V(p) \subset \partial V(p)$.*

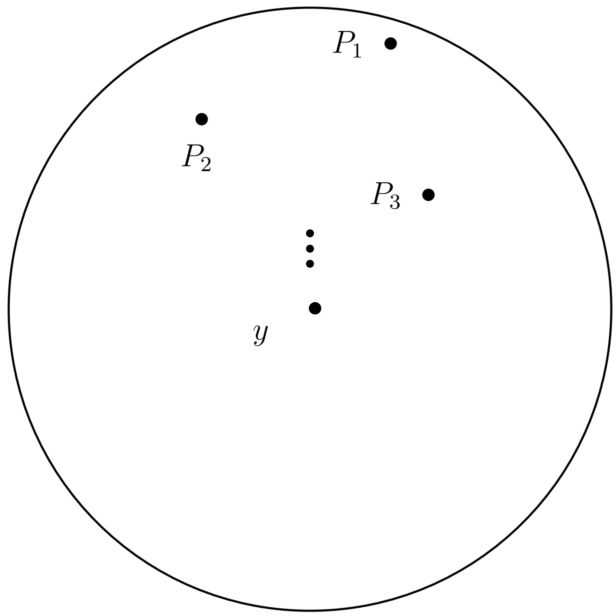
Proof: Let $x \in \partial H(p,q) \cap V(p)$. Then clearly for every $\varepsilon > 0, B(x,\varepsilon) \cap V(p) \neq \emptyset$, and $B(x,\varepsilon) \cap H(p,q)^c \neq \emptyset$, i.e., there exists $y \in X$, such that $y \in B(x,\varepsilon)$, and $d(p,y) > d(q,y)$. Hence, $y \in V(p)^c$ and so $x \in \partial V(p)$, as desired. ∎

We now present the main result of this section.

Theorem 2.2.10 *If X is semi-proper and well-bisected, and S is well-separated, then the Voronoi diagram $V(S)$ is a tessellation of X.*

Proof: As noted earlier (Remark 2.1.4), for $p \in S$, $V(p)$ is closed in X.
Suppose that there exists an $x \in \mathring{V}(p) \cap \mathring{V}(q)$ for distinct $p, q \in S$. Then by Lemma 2.2.3, $x \in (V(p) \setminus \partial V(p)) \cap (V(q) \setminus \partial V(q))$. Because $x \in V(p) \cap V(q)$, this implies that $x \in b(p,q)$. Because X is well-bisected, w.l.o.g., we may suppose that $x \in \partial H(p,q)$, and so by Lemma 2.2.9, $x \in \partial V(p)$. But this is a contradiction, therefore $\mathring{V}(p) \cap \mathring{V}(q) = \emptyset$ for distinct $p, q \in S$.

We now show that $\bigcup_{p \in S} V(p) = X$. Suppose $y \in X$, and that there is no $p \in S$ such that $d(y,p) \leq d(y,q)$, $\forall q \in S$. So for all $p \in S$, there exists a $q(p)$ such that $d(y,p) > d(y,q(p))$. This cannot occur if S is finite, so we may inductively choose an infinite sequence $p_n, n \geq 2$, such that $d(y,p_{n-1}) > d(y,p_n)$.



Thus, $\{p_n\} \subset B(y, p_1)$, a bounded set. As X is semi-proper, the infinite sequence $\{p_n\}$ contains a Cauchy subsequence, which is impossible because S is well-separated. Thus $y \in V(p)$ for some $p \in S$. Therefore $V(S)$ is a tessellation of X. ∎

Remark All three assumptions of Theorem 2.2.10 are necessary.

(I) $\underline{Well - separated}$: Let $S = \{p_n = (0, \frac{1}{n}); n \geq 1\}$ be a sequence of sites in $X = \mathbb{E}^2$. Then X is semi-proper and well-bisected, but S is not well-separated. Moreover, $(0, 0)$ is not in any Voronoi cell $V(0, \frac{1}{n}), \forall n \geq 1$. Hence, $V(S)$ is not a tessellation of $X = \mathbb{E}^2$.

(II) $\underline{Semi - proper}$: For $n \in \mathbb{N}$, define $S(n) \in l^2_{\mathbb{R}}(\mathbb{N})$ by $S(n)_k = \begin{cases} 0 & k \neq n \\ 1 + 1/n & k = n \end{cases}$

Now let $X = (\{S(n); n \in \mathbb{N}\} \cup \{0\}) \subset l^2_{\mathbb{R}}(\mathbb{N})$ and $S = \{S(n); n \in \mathbb{N}\}$. Then for $n > m$, $d(S(n), S(m)) = \sqrt{(1 + \frac{1}{n})^2 + (1 + \frac{1}{m})^2} > \sqrt{2}$, so S is $\sqrt{2} - separated.$

A short computation shows that if $n \neq m, b(S(n), S(m)) = \emptyset$ in X, so by Remark 2.2.6, X is well-bisected.

Clearly $X \subset B(\{0\}, 3)$, thus X is bounded. But for $\varepsilon = \frac{1}{2}$, there is no finite set

$A_{\frac{1}{2}} \subset X$ such that $X \subset \bigcup_{a \in A_{\frac{1}{2}}} B(a, \frac{1}{2})$, because $B(S(n), \frac{1}{2}) = \{S(n)\}$. This implies that X is not totally bounded and so X is not semi-proper.

Finally, $0 \notin V(S)$ because $d(0, S(n)) = 1 + \frac{1}{n}$ and therefore for any $S(n)$ there is an $m > n$ such that $d(0, S(m)) < d(0, S(n))$. Thus $V(S)$ is not a tessellation of X.

(III) $\underline{Well-bisected}$: Let $X = \mathbb{E} \setminus ([-\frac{1}{2}, 0) \cup (0, \frac{1}{2}])$ of Example 2.2.7 b) and $S = \{-1, 1\}$. Then S is clearly 1-separated, X is semi-proper, but is not well-bisected. Moreover, $\overset{\circ}{V}(-1) \cap \overset{\circ}{V}(1) = \{0\}$, and so $V(S)$ is not a tessellation of X.

2.3 Voronoi Diagrams as Pre-triangulations in Metric Spaces

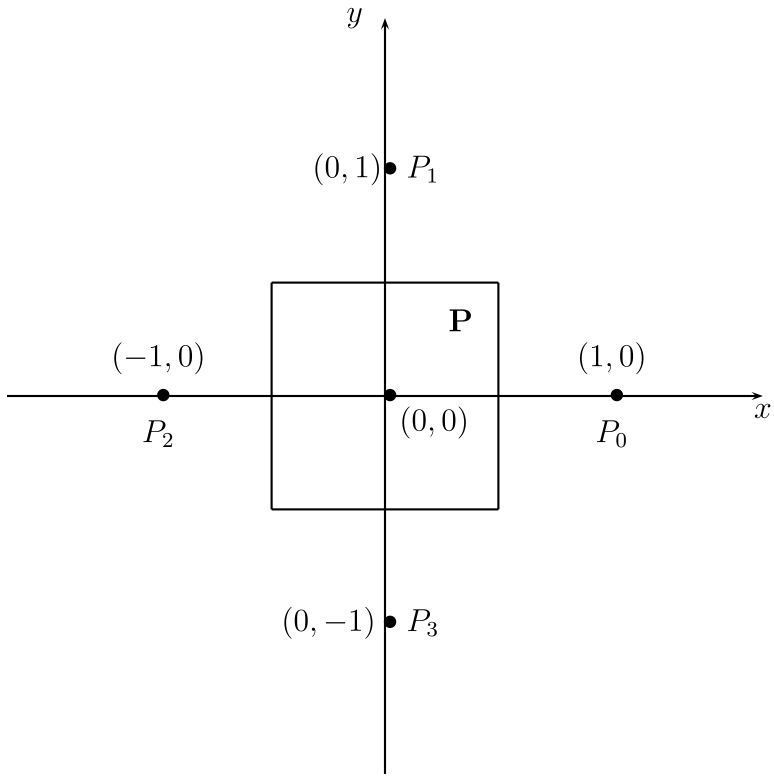
In this section, we will define pre-triangulations (Definition 2.3.6), which are specific tessellations in metric spaces, and show in Theorem 2.3.17 that if X is a proper and well-bisected metric space, S is syndetic and well-separated and the Voronoi cells are Voronoi polygons (Definition 2.3.12), then $V(S)$ is a pre-triangulation of X.

To define pre-triangulations, we will need the notion of polygons in metric spaces.

Definition 2.3.1 *A* polygonal cell *is a set of points that is equal to a finite intersection of half-spaces with non-empty interior. A compact polygonal cell will be called a* polygon.

Examples

1) Let $X = \mathbb{E}^2$, and define $P_i, i = 0, 1, 2, 3$ as in the following diagram. Then $H(0, P_0) \cap H(0, P_2)$ is a polygonal cell, and $P = \bigcap_{i=0}^{3} H(0, P_i)$ is a polygon.



2) The following intersection of infinitely many half-spaces is also an intersection of finitely many half-spaces, and is therefore a polygon.
Let $X = \mathbb{E}$. Then $\bigcap_{k\in\mathbb{Z}} H(0,k) = H(0,-1) \cap H(0,1) = [-\frac{1}{2}, \frac{1}{2}]$ is a polygon.

3) Lastly, we present an infinite intersection of half-spaces that is not a polygon. But before, we have to introduce the followings lemma and remark.

Lemma 2.3.2 *Let X be a topological space. If A, B are closed subsets of X, then*

$$\partial(A \cap B) = (\partial A \cap B) \cup (A \cap \partial B).$$

Proof: Because A, B are closed we have $\overline{A} = A$ and $\overline{A \cap B} = A \cap B$, therefore we get

$$\begin{aligned}\partial(A \cap B) &= \overline{A \cap B} \cap \overline{(A \cap B)^c} \\ &= (A \cap B) \cap \overline{A^c \cup B^c}\end{aligned}$$

$$
\begin{aligned}
&= (A \cap B) \cap (\overline{A^c} \cup \overline{B^c}) \\
&= (A \cap B \cap \overline{A^c}) \cup (A \cap B \cap \overline{B^c}) \\
&= (\overline{A} \cap \overline{A^c} \cap B) \cup (A \cap \overline{B} \cap \overline{B^c}) \\
&= (\partial A \cap B) \cup (A \cap \partial B)
\end{aligned}
$$

∎

Remark 2.3.3 Lemma 2.3.2 generalizes easily to a finite collection of closed sets, i.e. if $A_1, ..., A_n$ are n-closed sets, then $\partial\big(\bigcap_{i=1}^n A_i\big) = \bigcup_{i=i}^n (\partial A_i \bigcap_{1=j,j\neq i}^n A_j)$.

We now define an example which is not a polygon. If $X = \mathbb{E}^2$, then $\bigcap_{z\in S(0,2)} H(0,z) = \overline{B(0,1)}$, where $S(0,n) = \{z \in \mathbb{E}^2; d(z,0) = n\}$. Suppose on the contrary that we have a finite set $N \subset \mathbb{N}$ such that $\bigcap_{i=1}^N H(p_i,q_i) = \overline{B(0,1)}$. In $\mathbb{E}^2$, by Remark 2.3.3

$$\bigcup_{i=1}^{N} \big(\partial H(p_i,q_i) \bigcap_{j=1,j\neq i}^{N} \partial H(p_j,q_j)\big) = \partial \bigcap_{i=1}^{N} H(p_i,q_i) = \partial\overline{B(0,1)} = S(0,1).$$

Moreover, $\partial H(x,y) = b(x,y)$ for distinct $x, y \in \mathbb{E}^2$ and $b(x,y)$ is a straight line. Thus, the circle $S(0,1) = \partial\overline{B(0,1)}$ is a finite intersection of straight lines, which is a contradiction.

We now give an example that shows that Lemma 2.3.2 is false for an infinite intersection.

Example 2.3.4 Let $\{K_n\}_{n\geq 1}$ be defined as $K_m = H(0, -\frac{1}{m}) \cap H(0, \frac{1}{m})$. Then

$$\partial\big(\bigcap_{n\geq 1} K_n\big) = \partial\{0\} = \{0\}.$$

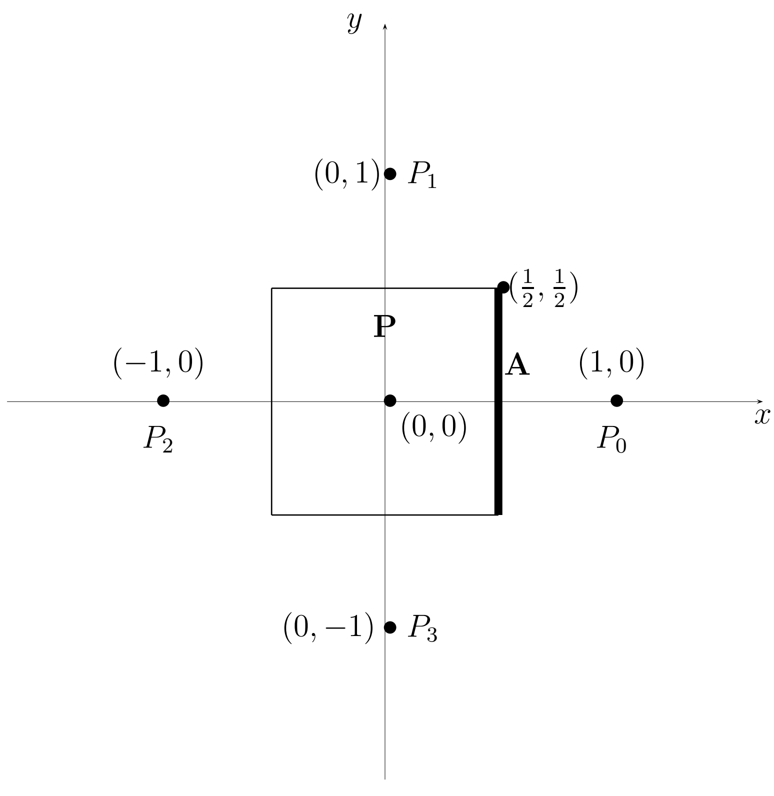
But,

$$\bigcup_{n\geq 1} (\partial K_n \cap \bigcap_{n\neq m} K_m) = \bigcup_{n\geq 1} \Big((\{-\frac{1}{2n}\} \cup \{\frac{1}{2n}\}) \cap \{0\}\Big) = \emptyset$$

So $\partial \bigcap_{n\geq 1} K_n \neq \bigcup_{n\geq 1} (\partial K_n \cap \bigcap_{n\neq m} K_m)$

Definition 2.3.5 *Let X be a metric space, $N \subset \mathbb{N}$ be a finite set and $P = \bigcap_{i=1}^{N} H(p_i, q_i)$ be a polygon in X. A* face *of P is a non-empty intersection $\bigcap_{m \in K} b(p_m, q_m) \cap P$, for some non-empty $K \subset N$.*

Example Let $X = \mathbb{E}^2$, and define P_i for $i = 0, 1, 2, 3$ as in the following diagram. Then $P = \bigcap_{i=0}^{3} H(0, P_i)$ is a polygon. So, $A = b(0, P_0) \cap P = (\frac{1}{2}, t)$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$ is a face of P. Also, $b(0, P_0) \cap b(0, P_1) \cap P = (\frac{1}{2}, \frac{1}{2})$ is a face of P.



We now introduce a more specific tessellation.

Definition 2.3.6 *A* pre-triangulation *$\mathcal{P}$ of X is a tessellation of X into polygons such that :*

1) any two polygons in $\mathcal{P}$ are either disjoint, or intersect in a common face, and

2) any bounded set in X intersects only finitely many polygons in $\mathcal{P}$.

In $\mathbb{E}^2$, the hyperbolic half-plane and the two dimensional sphere, a triangulation is a pre-triangulation of X where each polygon is a triangle.

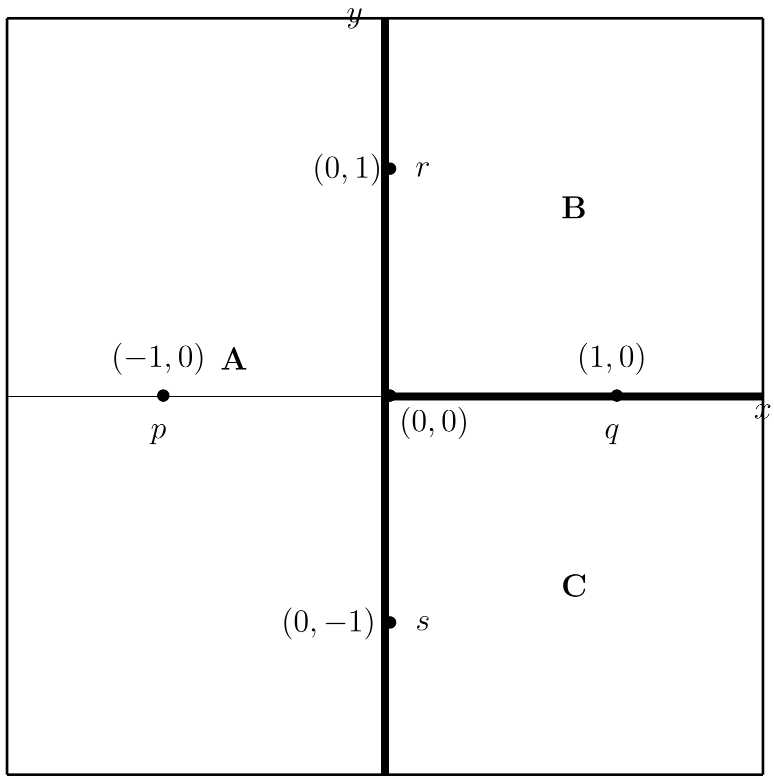
Example

Let $X = \mathbb{E}$. Then $D = \{[k, k+1]\}_{k\in\mathbb{Z}}$ is a tessellation of $\mathbb{E}$, and

$$[k, k+1] = H(k+1, k-1) \cap H(k, k+2)$$

is clearly a polygon. The faces of $[k, k+1]$ are $\{k\}$ and $\{k+1\}$, so the polygons intersect in common faces or not at all. Now, let $p, K \in \mathbb{E}$ and $A \subset B(p, K)$. Then A intersects at most $K+2$ polygons; therefore $D = \{[k, k+1]\}_{k\in\mathbb{Z}}$ is a pre-triangulation of $\mathbb{E}$.

Remark As the following examples show, not every tessellation is a pre-triangulation.
1) If $X = \mathbb{E}^2$, then $D = \{\overline{B(0,1)}, \overline{B(0,1)}^c\}$ is a tessellation of $\mathbb{E}^2$, but $\overline{B(0,1)}$ is not a polygon, and so D is not a pre-triangulation of $\mathbb{E}^2$.
2) In the following example of a tessellation, all tiles are polygons, but there are polygons that do not intersect in common faces.
Let $X = [0,1] \times [0,1] \subset \mathbb{E}^2$, p, q, r, s be defined as in the following diagram and $D = \{A, B, C\}$, where $A = \{(x, y) \in X; x \leq 0\}, B = \{(x, y) \in X; x \geq 0 \text{ and } y \geq 0\}$, and $C = \{(x, y) \in X; x \geq 0 \text{ and } y \leq 0\}$. Clearly D is a tessellation of X and A, B, C are polygons. But $b(p, q)$ is a face of A and $b(p, q) \cap H(s, r)$ is a face of B, but $b(p, q) \cap H(s, r) \neq b(p, q)$ and thus A and B do not intersect in common faces, hence D is not a pre-triangulation of X.



3) This is an example of a polygonal tessellation where a bounded set intersects infinitely many polygons (which however do satisfy condition 1) of the Definition 2.3.6).

Let $X = \{S(n); n \in \mathbb{N}\} \subset l^2_{\mathbb{R}}(\mathbb{N})$ where $S(n)_k = \begin{cases} 0 & k \neq n \\ 1 + 1/n & k = n \end{cases}$.
Let $D = \{H(S(n), S(n+1))\}_{n\in\mathbb{N}}$. We have that $H(S(n), S(n + 1)) = \{S(n)\}$. Therefore $\bigcup_{n\in\mathbb{N}} H(S(n), S(n+1)) = X$ and $H(S(n), S(n+1)) \cap H(S(m), S(m+1)) = \emptyset$ for distinct $n, m \in \mathbb{N}$, thus the intersection of the interior of two tiles is clearly empty, this implies that D is a tessellation of X. As each tile is also a polygon, D is a polygonal tessellation.

Because $b(S(n), S(n+1)) = \emptyset$ we have that the faces never intersect. But $B(0, 3)$ is clearly a bounded set and for every $n \in \mathbb{N}$, $H(S(n), S(n+1)) = \{S(n)\} \subset B(0, 3)$, thus there are infinitely many polygons that intersect the bounded set $B(0, 3)$, so D is not a pre-triangulation of X.

Recall (Theorem 2.2.10) that if X is semi-proper and well-bisected, and S is well-separated, then the Voronoi diagram $V(S)$ is a tessellation of X.

Remark: Not every Voronoi diagram of a well-separated set of sites in a semi-proper and well-bisected space X is a pre-triangulation of X.
Indeed, let $X = (0, 2) \subset \mathbb{E}$ and $S = \{1\}$. Then X is semi-proper and well-bisected and S is well-separated. But $V(1) = (0, 2)$ is not compact and therefore not a polygon. This implies that $V(S)$ is not a pre-triangulation of X.

A necessary (but not sufficient) condition for a Voronoi diagram to be a pre-triangulation is that Voronoi cells are compact. The next lemma shows that if the set of sites is syndetic, then each Voronoi cell is bounded.

Lemma 2.3.7 *If K is syndetic and $p \in K$, then $V(p) \subset B(p, K)$.*

Proof: If not, then there exists $x \in V(p) \backslash B(p, K)$. Thus, $d(x, p) \leq d(x, s)$, $\forall s \in S$ and $d(x, p) > K$, so $d(x, s) > K$ for all $s \in S$, which is a contradiction, because S is K-syndetic. ∎

Remark 2.3.8 Let X be a proper metric space, S be K-syndetic and $p \in S$. By Lemma 2.3.7, $V(p)$ is bounded. As $V(p)$ is closed by Remark 2.1.4 and X is proper, $V(p)$ is compact.

Let X be a proper and well-bisected metric space and S be syndetic and well-separated. Then $V(S)$ is a tessellation and for every $p \in S$, $V(p)$ is compact. But those properties are not sufficient to guarantee that $V(S)$ is a pre-triangulation of X, as the following example illustrates. The next remark and lemma will be used in the example. The example will be a subspace of $\mathbb{H}^2 = \{(x, y); x, y \in \mathbb{R}, y > 0\}$ the Hyperbolic half-plane with metric $ds^2 = \frac{dx^2+dy^2}{y^2}$.

Remark 2.3.9 Let X be a metric space and S be a set of sites. If $S = X$, then for every $p \in S$, $V(p) = \{p\}$.

Proof: If $x \in X$, and $d(x,p) \leq d(x,q)$, for all $q \in S$, then $d(x,p) = 0$. Hence $V(p) = \{p\}$. ∎

Lemma 2.3.10 *Let $X \subset \mathbb{H}^2$ and $V \subset X$. The following conditions are sufficient for V not to be a polygon in X.*

1) The subspace X is infinite.

2) If $r \in \mathbb{H}^2$, and $x(r)$ denotes its x-coordinate, then

$$Inf\{|x(t) - x(s)|; s, t \in X, s \neq t\} > 0.$$

3) For every distinct x and y in X, the bisector $b(x,y)$ is a circle centred on the x-axis and V lies outside the disk bounded by this circle.

4) The subspace V is bounded.

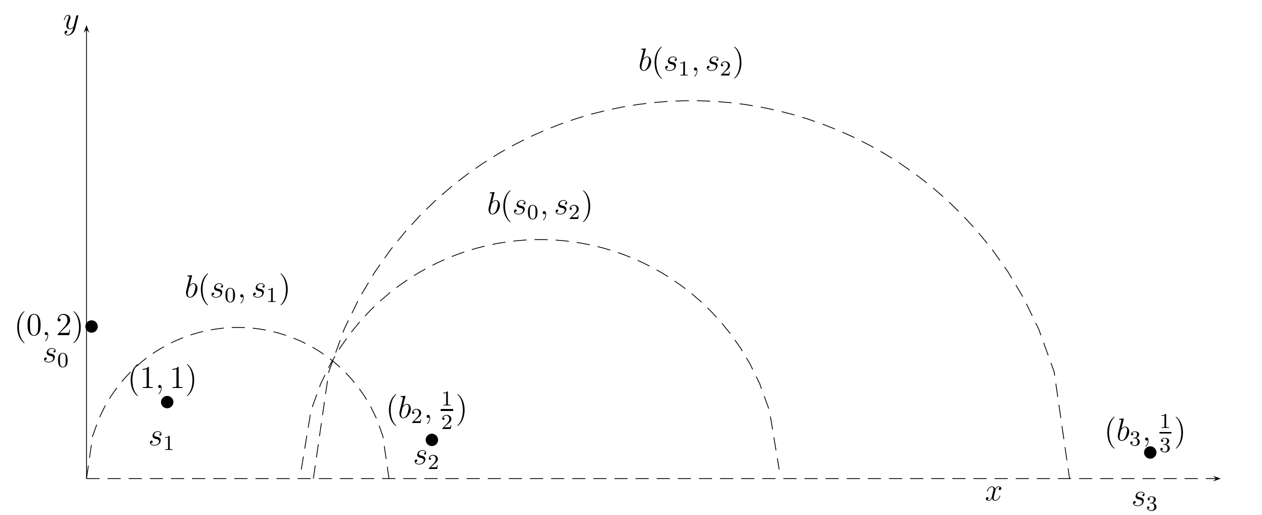
Proof: Suppose on the contrary that V is a polygon (i.e a finite intersection of half-spaces $H(x_i, y_i)$ where $x_i, y_i \in X$ for $i = 1, ..., n$). By condition 3) all the $H(y_i, x_i)$ are the interior in $\mathbb{E}^2$ of the circle $b(x_i, y_i)$ centred on the x-axis. Thus, by condition 2) there are only finitely many points in each $H(y_i, x_i)$ and so finitely many points in each $H(y_i, x_i) \setminus b(y_i, x_i)$. This implies that there are finitely many points in $\bigcup_{i=1}^{n} \left(H(y_i, x_i) \setminus b(y_i, x_i)\right)$. Hence there must be infinitely many points in $\bigcap_{i=1}^{n} H(x_i, y_i) = V$ by condition 1). This is a contradiction by condition 4), because V is bounded, hence V is not a polygon. ∎

Example 2.3.11 We will define a metric space $X \subset \mathbb{H}^2$ and a set of sites S, such that X and $V(s_0)$ for $s_0 \in S$ satisfy the conditions of Lemma 2.3.10. Moreover, S will be syndetic and well-separated, and X will be well-bisected and proper.

Let $s_0 = (0, 2)$ and $s_1 = (1, 1) \in \mathbb{H}^2$. Recall that the bisector $b(s_n, s_m)$ for two points $s_n = (x_1, y_1) \neq s_m = (x_2, y_2) \in \mathbb{H}^2$ with $x_1 \neq x_2$ is a semi-circle centred on the x-axis. We denote by $(c_{n,m}, 0)$ the centre of $b(s_n, s_m)$ and let $r_{n,m}$ be its radius. If $x_2 > x_1$ and $y_1 > y_2$, then $c_{n,m} = \dfrac{x_1 y_2 - y_1 x_2}{y_2 - y_1} > x_2$. Now define $s_2 = (b_2, \frac{1}{2})$ where $b_2 > c_{0,1} + r_{0,1}$. Note that $c_{0,1} > 0$. Inductively we choose b_n so that

$$b_n > \max\left(\{c_{i,j} + r_{i,j}; 0 \leq i < j \leq n-1\} \cup \{b_i + 1; 0 \leq i \leq n-1\}\right)$$

and $\bigcup_{i=0}^{n-1}\{s_i\} \cap \bigcup_{i=0}^{n-1} b(s_i, s_n) = \emptyset$. Now set $s_n = (b_n, \frac{1}{n})$. Then, for $i < j$, $b_j > b_i$, and $c_{i,j} > b_j$.



Let $X = S = \{s_n\}_{n \geq 0}$. We will first prove that X is well-bisected and proper, then prove that S is well-separated and syndetic and finally prove that X and $V(s_0)$ satisfy the conditions of Lemma 2.3.10.

For every $x \neq y \in X$, $b(x,y) = \emptyset$, so by Remark 2.2.6, X is well-bisected. Moreover, X is a closed subset of $\mathbb{H}^2$, so X is proper.

Also, $S = X$ thus S is syndetic by Remark 2.1.7.

To see that S is well-separated, we use estimates from the Appendix. By equation (0.2.3)

$$d_{\mathbb{H}^2}(s_0, s_1) > \frac{d_{\mathbb{E}^2}(s_0, s_1)}{2} = \frac{\sqrt{2}}{2} > \frac{2}{3}.$$

Now, $b_2 > c_{0,1} + r_{0,1} = 2 + 2$, so $b_n > 4$ for $n \geq 2$. Hence, by equation (0.2.4) and (0.2.7),

$$d^2_{\mathbb{H}^2}(s_0, s_n) > \frac{d_{\mathbb{E}^2}(s_0, s_n)}{r^2_{0,n}} > \frac{b_n^2 + (2 - y_n)^2}{\frac{b_n^2}{2} + 2^2} > \frac{b_n^2 + 1}{\frac{b_n^2}{2} + 4}$$

But

$$\frac{b_n^2 + 1}{\frac{b_n^2}{2} + 4} = \frac{2b_n^2 + 2}{b_n^2 + 8} = 2 - \frac{14}{b_n^2 + 8} > 2 - \frac{14}{4^2 + 8} = \frac{17}{12},$$

if $n \geq 2$. Hence $d_{\mathbb{H}^2}(s_0, s_n) > 1$ for $n \geq 2$. Also, for $n > m > 0$,

$$(b_n - b_m)^2 \geq 1 > (\frac{1}{n})^2 - (\frac{1}{m})^2 = y_n^2 - y_m^2.$$

Thus, by equation (0.2.4)

$$d^2_{\mathbb{H}^2}(s_n, s_m) > \frac{d^2_{\mathbb{E}^2}(s_n, s_m)}{r^2_{s_n, s_m}} > \frac{(b_n - b_m)^2 + (y_n - y_n)^2}{\frac{(b_n - b_m)^2}{2} + y_m^2} > \frac{(b_n - b_m)^2}{\frac{(b_n - b_m)^2}{2} + 1} > \frac{2}{3}$$

Thus S is $\frac{2}{3}$- separated.

We now show that $V(s_0)$ and X satisfy the condition of Lemma 2.3.10 and therefore $V(s_0)$ is *not* a polygon. Clearly, X is infinite. Because,

$$b_n > \max\left(\{c_{i,j} + r_{i,j}; 0 \leq i < j \leq n-1\} \cup \{b_i + 1; 0 \leq i \leq n-1\}\right),$$

condition 2) is satisfied. Furthermore, $X = S$, so by Remark 2.3.9, $V(s_0) = \{s_0\}$ and is therefore bounded. We now have to prove condition 3). Consider s_n and s_k with $n < k, n, k \in \mathbb{N}$. Recall that

$$b(s_n, s_k) = \{(x,y) \in \mathbb{H}^2; (x - c_{n,k})^2 + y^2 = r^2_{n,k}\},$$

where

$$c_{n,k}^2 = \Big(\frac{\frac{b_n}{k} - \frac{b_k}{n}}{\frac{1}{k} - \frac{1}{n}}\Big)^2 = \frac{b_n^2 n^2 - 2b_n b_k nk + b_k^2 k^2}{(n-k)^2},$$

and

$$r_{n,k}^2 = \frac{1}{nk}\Big(\frac{b_n - b_k}{\frac{1}{n} - \frac{1}{k}}\Big)^2 + \frac{1}{nk} = \frac{b_n^2 nk - 2b_n b_k nk + b_k^2 nk}{(n-k)^2} + \frac{1}{nk}.$$

If $z = (x_1, y_1)$ satisfies $(x_1 - c_{n,k})^2 + y_1^2 > r_{n,k}^2$, then of course z lies outside of the circle

$$(x - c_{n,k})^2 + y^2 = r_{n,k}^2$$

in $\mathbb{E}^2$. Moreover,

$$(b_n - c_{n,k})^2 + \frac{1}{n^2} = \Big(\frac{\frac{1}{n}(b_k - b_n)}{\frac{1}{n} - \frac{1}{k}}\Big)^2 + \frac{1}{n^2} = \frac{1}{n^2}\Big(\Big(\frac{b_n - b_k}{\frac{1}{n} - \frac{1}{k}}\Big)^2 + 1\Big) > \frac{1}{nk}\Big(\Big(\frac{b_n - b_k}{\frac{1}{n} - \frac{1}{k}}\Big)^2 + 1\Big) = r_{n,k}^2,$$

and

$$(b_k - c_{n,k})^2 + \frac{1}{k^2} = \Big(\frac{\frac{1}{k}(b_k - b_n)}{\frac{1}{n} - \frac{1}{k}}\Big)^2 + \frac{1}{k^2} = \frac{1}{k^2}\Big(\Big(\frac{b_n - b_k}{\frac{1}{n} - \frac{1}{k}}\Big)^2 + 1\Big) < \frac{1}{nk}\Big(\Big(\frac{b_n - b_k}{\frac{1}{n} - \frac{1}{k}}\Big)^2 + 1\Big) = r_{n,k}^2,$$

so $s_n \in \{(x,y) \in \mathbb{H}^2; (x - c_{n,k})^2 + y^2 \geq r_{n,k}^2\}$ and $s_k \in \{(x,y) \in \mathbb{H}^2; (x - c_{n,k})^2 + y^2 \leq r_{n,k}^2\}$, for $n < k$. This implies

$$H(s_n, s_k) = \{(x,y) \in \mathbb{H}^2; (x - c_{n,k})^2 + y^2 \geq r_{n,k}^2\}$$

and

$$H(s_k, s_n) = \{(x,y) \in \mathbb{H}^2; (x - c_{n,k})^2 + y^2 \leq r_{n,k}^2\}$$

for $n < k$. We will now show that $s_0 \in H(s_n, s_k)$ for every $n < k \in F$. By the above, it is equivalent to showing that $2^2 + c_{n,k}^2 > r_{n,k}^2$ for $0 < n < k, n, k \in F$. But,

$$\begin{aligned} c_{n,k}^2 - r_{n,k}^2 &= \frac{b_k^2 k - b_n^2 n}{k - n} - \frac{1}{nk} \\ &\geq \frac{b_k^2 k - b_k^2 n}{k - n} - \frac{1}{nk} \\ &= b_k^2 - \frac{1}{nk} \\ &\geq 16 - \frac{1}{2} > -4. \end{aligned}$$

Hence $s_0 \in H(s_n, s_k)$ for $n < k$. Thus, condition 3) is satisfied. Hence, $V(s_0)$ is not a polygon in X, so $V(S)$ is not a pre-triangulation of X.

Therefore, we will need another condition to guarantee that a Voronoi diagram is a pre-triangulation of X. Let $S \subset X$ be a set of sites. For $p \in S$, recall that $V(p)$ is the intersection of the half-spaces $H(p,q)$, for $q \in S$. As a polygon is a compact finite intersection of half-spaces, this motivates the following definition.

Definition 2.3.12 *Let X be a metric space and S be a set of sites in X. A compact Voronoi cell is a* Voronoi polygon *if it is the intersection of finitely many $H(p,q)$, $q \in S$.*

Remark: A Voronoi cell can be a polygon, without being a Voronoi polygon. It is illustrated by the next example. To define this example, we will use the following lemma which is similar to Lemma 2.3.10.

Lemma 2.3.13 *Let $X \subset \mathbb{H}^2$, S be a set of sites and $p \in S$. The following conditions are sufficient for $V(p)$ not to be a Voronoi polygon in X.*
1) S is infinite.
2) If $r \in \mathbb{H}^2$, and $x(r)$ denotes its x-coordinate, then

$$Inf\{|x(q) - x(r)|; q, r \in S \setminus \{p\}, q \neq r\} > 0.$$

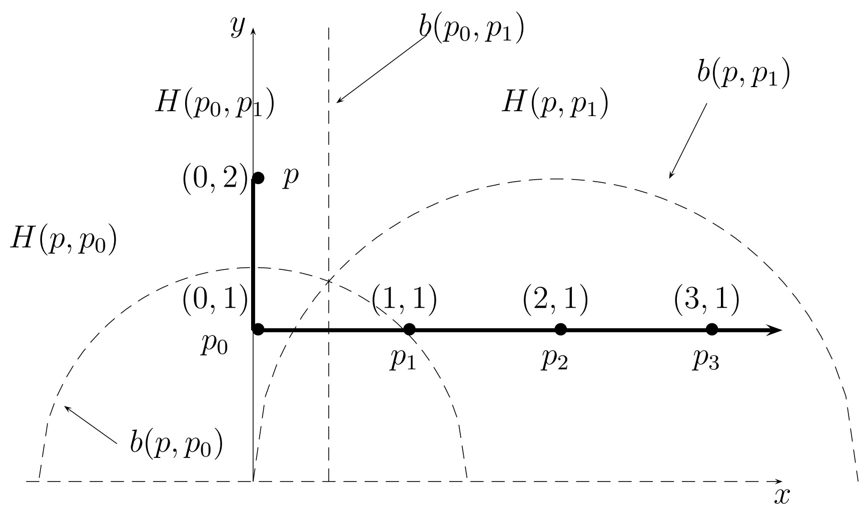
3) For every sites *s in $S \setminus \{p\}$, the bisector $b(p,s)$ is a circle centred on the x-axis and p lies outside the disk bounded by this circle.*

Proof: Suppose on the contrary that $V(p)$ is a Voronoi polygon (i.e a finite intersection of half-spaces $H(p,q)$ where $q \in F \subset S$ and F is finite). By condition 3) all the $H(q,p)$ with $q \in F$ are the interior in $\mathbb{E}^2$ of the circle $b(p,q)$ centred on the x-axis. Thus, by condition 2) there are only finitely many sites in each $H(q,p)$

and so finitely many sites in each $H(q,p) \setminus b(q,p)$ for $q \in S$. This implies that there are finitely many sites in $\bigcup_{q\in F} \big(H(q,p) \setminus b(q,p)\big)$. Hence there must be some sites in $\bigcap_{q\in F} H(p,q) = V(p)$ by condition 1). This is a contradiction, hence V is not a Voronoi polygon. ∎

Example We will define a proper and well-bisected metric space $X \subset \mathbb{H}^2$ and a syndetic and well-separated set of sites S that satisfy conditions of Lemma 2.3.13 for $p \in S$ and $V(p)$ *is* a polygon.

Let $X = \{(0,t)) \in \mathbb{H}^2; t \in [1,2]\} \cup \{(x,1)) \in \mathbb{H}^2; x \in \mathbb{R}^+\}$ and $S = \{(n,1) \in \mathbb{H}^2; n \in \mathbb{N}\} \cup \{(0,2)\}$. Let $p = (0,2)$ and $p_n = (n,1), n \geq 0$. Note that X is path-connected.



First, we will show that S is well-separated and syndetic and then show that X is well-bisected and proper and finally that X and $V(p)$ satisfy the condition of Lemma 2.3.13 and that $V(p)$ is a polygon.

For $n \neq m \in \mathbb{N}, (n-m)^2 \geq 1$, thus by Equations (0.2.4) and (0.2.8) of the Appendix, " Useful estimates for the Hyperbolic half-plane", we have

$$d_{\mathbb{H}^2}(p_n, p_m) > \frac{d_{\mathbb{E}^2}(p_n, p_m)}{r_{p_n,p_m}} > \frac{|m-n|}{\sqrt{1+\frac{(m-n)^2}{2}}} > \frac{|m-n|}{\sqrt{2}|m-n|} = \frac{1}{\sqrt{2}} > \frac{1}{2}. \tag{2.3.1}$$

Also, by Equation (0.2.1) in the Appendix,

$$d_{\mathbb{H}^2}(p, p_0) > \frac{|2-1|}{\max\{2,1\}} = \frac{1}{2} \tag{2.3.2}$$

For $m = 1$, by Equation (0.2.3)

$$d_{\mathbb{H}^2}(p, p_1) > \frac{d_{\mathbb{E}^2}(p, p_1)}{2} = \frac{\sqrt{2}}{2} > \frac{1}{2}. \tag{2.3.3}$$

For, $m > 1$, by Equations (0.2.4) and (0.2.8)

$$d_{\mathbb{H}^2}(p, p_m) > \frac{d_{\mathbb{E}^2}(p, p_m)}{r_{p,p_m}} > \frac{m}{\sqrt{\frac{m^2}{2}+1}} > \frac{\sqrt{2}m}{\sqrt{3}m} > \frac{2}{3} > \frac{1}{2}. \tag{2.3.4}$$

Thus, S is $\frac{1}{2}$-separated.

Moreover, if $(0, y) \in X$, then $d(p, (0, y)) < 1$ by Equation (0.2.2). If $x = (x_1, 1) \in X$ and $x_1 \in [n, n+1)$, then by Equation (0.2.4)

$$d_{\mathbb{H}^2}(p_n, x) < r\theta < (1 + \frac{1}{2})\pi < 2\pi \tag{2.3.5}$$

where θ is the euclidean angle pCx where C is the centre of the geodesic joining p and x. Hence S is syndetic.

We now show that X is well-bisected, using Lemma 2.2.8. If $x = (0, s) \neq (0, t) = y \in X$ with $s < t$, then $b(x, y)$ is a circle centred on the x-axis and $b(x, y) \cap X = \{z, w\}$ where z is on the y-axis and $w = (r, 1)$ for $r \in \mathbb{R}$. The segment in $\mathbb{E}^2$ on the y-axis provides an arc from x to y that meets $b(x, y)$ once and the arcs from x to w meet $b(x, y)$ only once. Let $x = (0, s) \neq y = (t, 1) \in X$. Then $b(x, y)$ is a circle centered on the x-axis and $b(x, y) \cap X = \{z, w\}$. The arcs from y to z and the arcs from y to w intersect $b(x, y)$ only once. Suppose that $x = (s, 1) \neq y = (t, 1) \in X$. Then $b(x, y)$ is a vertical line and $b(x, y) \cap = \{z\}$. The segment in $\mathbb{E}^2$ from x to y provides an arc that meets $b(x, y)$ once. Thus by Lemma 2.2.8, X is well-bisected.

Also, X is proper as $\mathbb{H}^2$ is proper and X is a closed subset of $\mathbb{H}^2$.

Moreover, $V(p) = H(p,p_0) \cap H(p_0,p_1)$, so $V(p)$ is a polygon. Clearly, S is infinite, so condition 1) is satisfy. By definition of the p_m, condition 2) is satisfied. Furthermore, for $m \in \mathbb{N}$,

$$b(p,p_m) = \{(x,y) \in X; (x-2m)^2 + y^2 = 2(m^2+1)\}$$

and $4m^2 + 4 > 2m^2 + 2$, so $p = (0,2)$ lies outside of the circle $b(p,p_m)$ in $\mathbb{E}^2$. Thus, condition 3) is satisfied. Hence, $V(p)$ is not a Voronoi polygon.

In the following, we will prove that if X is a proper and a well-bisected metric space, S is syndetic and well-separated and the Voronoi cells are Voronoi polygons, then $V(S)$ is a pre-triangulation of X. By Theorem 2.2.10, $V(S)$ is a tessellation of X and by Remark 2.3.8 each $V(p)$ is compact. We will now show that if the Voronoi cells are Voronoi polygons then they intersect in common faces.

Proposition 2.3.14 *If X is a metric space and S is a set of sites in X such that $V(S)$ is a tessellation of Voronoi polygons of X, then the Voronoi polygons intersect in common faces.*

Proof: Let $p \in S$ and $L(p)$ be a finite subset of sites such that $V(p) = \bigcap_{q \in L(p)} H(p,q)$. Let $q \in L(p)$, so that $b(p,q) \cap V(p)$ is a face of $V(p)$ and suppose $x \in b(p,q) \cap V(p)$. Then $d(x,q) = d(x,p) \leq d(x,r)$ $\forall r \in S$, so $x \in b(p,q) \cap V(q)$. Hence $b(p,q) \cap V(p) \subset b(p,q) \cap V(q)$. By the same argument $b(p,q) \cap V(q) \subset b(p,q) \cap V(p)$, and so $b(p,q) \cap V(p) = b(p,q) \cap V(q)$. This implies that $b(p,q) \cap V(p)$ is a common face of $V(p)$ and $V(q)$.

∎

Now, we will show that any bounded set intersects only finitely many Voronoi cells.

Lemma 2.3.15 *Let X be a semi-proper metric space and S be a well-separated set of sites in X. If $x \in X$ and $r \in \mathbb{R}$, then there are only finitely many $p \in S$ such that $V(p) \cap B(x,r) \neq \emptyset$.*

Proof: If there were infinitely many sites p in $B(x,r)$, then there would be a Cauchy subsequence of these sites, which is impossible, as S is well-separated.

So now suppose there are infinitely many $p \in S$ such that $V(p) \cap B(x,r) \neq \emptyset$, but only finitely many of which lie in $B(x,r)$. We will show that all the sites p are included in a bounded set, which is a contradiction.
Fix a site $p \notin B(x,r)$. Now, for any site q, with $V(q) \cap B(x,r) \neq \emptyset$, choose $y \in V(q) \cap B(x,r)$. We now show that $d(x,q) < 3d(x,p)$. Suppose on the contrary that $d(x,q) \geq 3d(x,p)$. Then we have that

$$d(y,q) + r > d(y,q) + d(y,x) \geq d(q,x) \geq 3d(x,p)$$

Hence, $d(y,q) > 3d(x,p) - r$. Because $p \notin B(x,r)$, we have that $d(x,p) > r$, so

$$3d(x,p) - r > 2d(x,p) > d(x,p) + r > d(x,p) + d(y,x) \geq d(p,y)$$

Thus $d(y,q) > d(y,p)$, which is a contradiction, so $d(x,q) < 3d(x,p)$. Hence, for all $q \in S$ such that $V(q) \cap B(x,r) \neq \emptyset$ we have that $q \in B(x, 3d(x,p))$. But then there are infinitely many $q \in S$ with $q \in B(x, 3d(x,p))$, which is a contradiction. Hence, there are only finitely many $p \in S$ such that $V(p) \cap B(x,r) \neq \emptyset$. ∎

Corollary 2.3.16 *Let X be a semi-proper metric space and S be a well-separated set of sites in X. Then any bounded set in X intersects only finitely many $V(p)$ with $p \in S$.*

We now present the main result of this section.

Theorem 2.3.17 *Let X be a proper and well-bisected metric space and S be a well-separated and syndetic set of sites in X such that the Voronoi cells are Voronoi polygons. Then $V(S)$ is a pre-triangulation of X.*

Proof: By Theorem 2.2.10, $V(S)$ is a tessellation of X and by Proposition 2.3.14, the Voronoi polygons intersect in common faces. Finally, by Corollary 2.3.16, any bounded set in X intersects only finitely many $V(p)$ with $p \in S$. Hence, $V(S)$ is a pre-triangulation of X. ∎

The difficulty is now to find when the Voronoi cells are Voronoi polygon. We conclude this section with a lemma and a corollary that will be used later to prove that Voronoi cells are Voronoi polygons in some subspaces of $\mathbb{E}^2$ and $\mathbb{D}$.

Lemma 2.3.18 *Let X be a semi-proper metric space, S be a well-separated and syndetic set of sites in X, and let $p \in S$. Suppose there exist finitely many sites $r_1, ..., r_k$ such that $\bigcap_{i=1}^{k} H(p, r_i)$ is bounded. Then $V(p) = \bigcap_{s \in L} H(p, s)$ for a finite $L \subset S$.*

Proof: We assume that the set of sites is infinite, as otherwise the lemma is clear. Suppose $V(p) \neq \bigcap_{r \in L} H(p, r)$ for any finite set L. Since $V(p) = \bigcap_{r \in S} H(p, r)$, we clearly have $V(p) \subset \bigcap_{i=1}^{k} H(p, r_i)$, but our assumption means that there exists $x_{k+1} \in X$ and a site r_{k+1} such that $d(x_{k+1}, p) \leq d(x_{k+1}, r_i)$ for $i = 1, ..., k$ but that $d(x_{k+1}, p) > d(x_{k+1}, r_{k+1})$. Without lost of generality, we may choose r_{k+1} so that $x_{k+1} \in V(r_{k+1})$. We may continue this procedure to obtain infinite sequences $\{x_n\}_{n \geq k+1}$ and $\{r_n\}_{n \geq 1} =: R \subset S$ such that for $n \geq k+1$

1. $d(x_n, p) \leq d(x_n, r_m), \forall m < n$
2. $x_n \in V(r_n)$
3. $d(x_n, p) > d(x_n, r_n)$.

By construction, R is infinite. If R were bounded, R would contain a Cauchy subsequence because X is semi-proper, which contradicts the fact that S is well-separated. Hence R is unbounded, and thus so is $\{d(p, r_n); n \geq k+1\}$.

Condition (1) above shows that $\{x_n; n \geq k+1\} \subset \bigcap_{i=1}^{k} H(p, r_i)$, which is bounded. Hence $\{d(x_n, p); n \geq k+1\}$ is bounded.

The second condition, in light of the fact that S is syndetic and Lemma 2.3.7, shows that $\{d(x_n, r_n); n \geq k+1\}$ is also bounded. However, $\forall n \geq k+1$

$$d(p, r_n) \leq d(p, x_n) + d(x_n, r_n),$$

which implies $\{d(p, r_n); n \geq k+1\}$ is bounded, a contradiction. Hence there is a finite set of sites L such that $\bigcap_{r \in L} H(p, r) = V(p)$.

∎

Immediately from this lemma and Remark 2.3.8. The following result follows.

Corollary 2.3.19 *Let X be a proper metric space, S be a well-separated, syndetic set of sites in X, and $p \in S$. Suppose there exists finitely many sites $r_1, ..., r_k$ such that $\bigcap_{i=1}^{k} H(p, r_i)$ is bounded. Then $V(p)$ is a Voronoi polygon.*

Chapter 3

Voronoi Diagrams in $\mathbb{E}^2$, S^2 and $\mathbb{D}$

In this chapter, we will study Voronoi diagrams in metric subspaces of $\mathbb{E}^2$, S^2 and $\mathbb{D}$. We will find sufficient conditions for the Voronoi cells to be Voronoi polygons. Thus, by Theorem 2.3.17 we will get pre-triangulations.

3.1 Voronoi Diagrams in $\mathbb{E}^2$

In this section, we will show that a well-separated and syndetic set of sites in a closed and well-bisected subspace of $\mathbb{E}^2$ always has Voronoi cells that are Voronoi polygons, allowing us to effectively eliminate one of the hypotheses of Theorem 2.3.17. Thus the Voronoi diagram will be not only a tessellation but a pre-triangulation. Clearly $\mathbb{E}^2$ is a proper metric space, thus any closed subspace X of $\mathbb{E}^2$ is a proper metric space. Let X be well-bisected and S be well-separated and syndetic in X. By Theorem 2.2.10, $V(S)$ is a tessellation of X.

To prove that it is a pre-triangulation, we will show that the Voronoi cells are Voronoi polygons. First we will define the quadrants in $\mathbb{E}^2$.

Definition 3.1.1 *As usual, we define the* four quadrants in $\mathbb{E}^2$ *as*

$$Q_1 = \{(x, y); x > 0, y \geq 0\}$$

$$Q_2 = \{(x,y); x \leq 0, y > 0\}$$
$$Q_3 = \{(x,y); x < 0, y \leq 0\}$$
$$Q_4 = \{(x,y); x \geq 0, y < 0\}.$$

Note that these quadrants are not closed in $\mathbb{E}^2$.

For, completeness, we include a proof of the following (geometrically obvious) result.

Lemma 3.1.2 *Let $a \in Q_1, b \in Q_2, c \in Q_3$ and $d \in Q_4$. Then $\bigcap_{z\in\{a,b,c,d\}} H(0,z)$ is bounded.*

Proof: If $a = (a_1, a_2)$, we have that $H(0,a) = \{x \in \mathbb{E}^2; d(0,x) \leq d(x,a)\}$ and $(x_1, x_2) \in H(0,a)$ iff

$$\begin{aligned} x_1^2 + x_2^2 &\leq (x_1 - a_1)^2 + (x_2 - a_2)^2 \\ &= x_1^2 - 2a_1x_1 + a_1^2 + x_2^2 - 2x_2a_2 + a_2^2 \end{aligned}$$

If $a_2 \neq 0$, because $a_1, a_2 > 0$, this implies that

$$x_2 + \frac{a_1x_1}{a_2} \leq \frac{a_1^2 + a_2^2}{2a_2}$$

If $a_2 = 0$ then

$$x_1^2 + x_2^2 \leq (x_1 - a_1)^2 + x_2^2$$

and so

$$x_1 \leq a_1/2$$

Therefore $H(0,a) = \begin{cases} \{x \in \mathbb{E}^2; x_2 + \frac{a_1x_1}{a_2} \leq \frac{a_1^2+a_2^2}{2a_2}\} \text{ if } a_2 \neq 0 \\ \{x \in \mathbb{E}^2; x_1 \leq \frac{a_1}{2}\} \text{ if } a_2 = 0 \end{cases}$

In the same way, we obtain (with an obvious notation)

$$H(0,b) = \begin{cases} \{x \in \mathbb{E}^2; x_2 + \frac{b_1 x_1}{b_2} \leq \frac{b_1^2+b_2^2}{2b_2}\} \text{ if } b_1 \neq 0 \\ \{x \in \mathbb{E}^2; x_2 \leq \frac{b_2}{2}\} \text{ if } b_1 = 0 \end{cases}$$

$$H(0,c) = \begin{cases} \{x \in \mathbb{E}^2; x_2 + \frac{c_1 x_1}{c_2} \geq \frac{c_1^2+c_2^2}{2c_2}\} \text{ if } c_2 \neq 0 \\ \{x \in \mathbb{E}^2; x_1 \geq \frac{c_1}{2}\} \text{ if } c_2 = 0 \end{cases}$$

$$H(0,d) = \begin{cases} \{x \in \mathbb{E}^2; x_2 + \frac{d_1 x_1}{d_2} \geq \frac{d_1^2+d_2^2}{2d_2}\} \text{ if } d_1 \neq 0 \\ \{x \in \mathbb{E}^2; x_2 \geq \frac{d_2}{2}\} \text{ if } d_1 = 0 \end{cases}$$

If $a \in Q_1$ with $a_2 \neq 0$, then $H(0,a) \cap Q_1$ is bounded, because both $x_1 < \infty$ and $x_2 < \infty$. Now if $a_2 = 0$, then $x_1 < \infty$. If $b_1 = 0$, then $x_2 < \infty$. In any case, this implies that $H(0,a) \cap Q_1 \cap H(0,b)$ is bounded. If $b_1 > 0$, then x_1 is bounded above and so too is x_2, so $H(0,a) \cap Q_1 \cap H(0,b)$ is bounded.

The same argument shows that the intersection of $\bigcap_{z=a,b,c,d} H(0,z)$ with each quadrant is bounded.

∎

Proposition 3.1.3 *Let $X \subset \mathbb{E}^2$, S be a well-separated, K-syndetic set of sites in X and $p \in S$. Then there exists a finite $F \subset S$ such that $\bigcap_{r \in F} H(p,r)$ is bounded.*

Proof: Without lost of generality, we suppose $p = (0,0)$, because we can translate X so $p = (0,0)$. It suffices to show that for each quadrant Q, there is a finite set F_Q of sites such that

$$X \cap Q \cap \bigcap_{r \in F_Q} H(p,r)$$

is bounded. (Since we may take $F = \bigcup_Q F_Q$.)

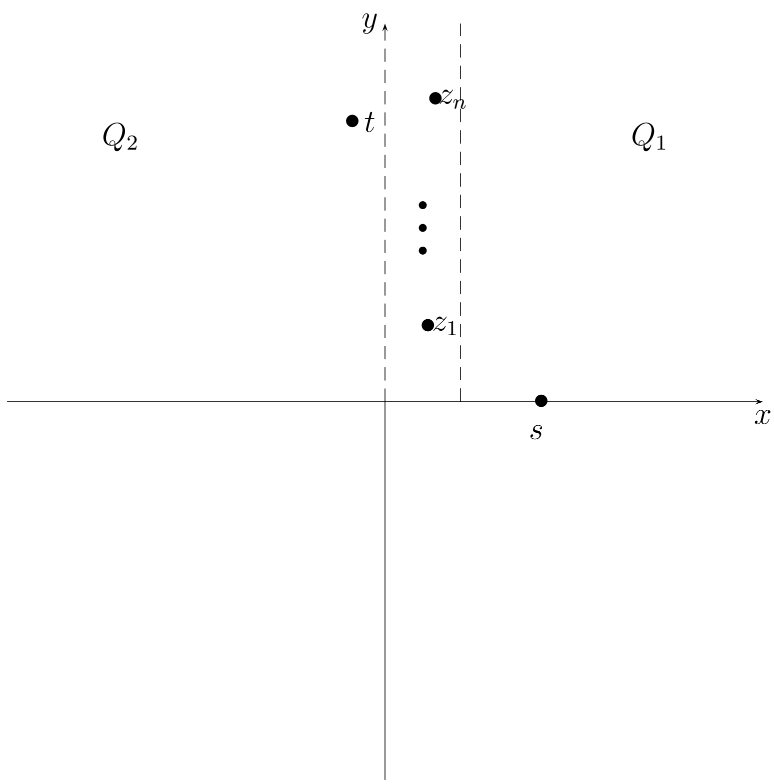
We will show this for $Q = Q_1$. The argument for the other quadrants is similar.

If $X \cap Q_1$ is bounded, take $F_{Q_1} = \emptyset$. Henceforth, we assume $X \cap Q_1$ to be unbounded.

There are 3 cases :

Case (I) There is a site $s = (a_1, a_2) \in Q_1$, with $a_2 > 0$.
Then the argument in Lemma 3.1.2 shows that $Q \cap H(0, s)$ is bounded, and we take $F_Q = \{s\}$.

Case (II) There are only sites of the form $s = (a, 0)$ in Q_1.
Then $(x, y) \in X \cap Q_1 \cap H(0, s)$, implies that $0 < x \leq \frac{a}{2}$. If $X \cap Q_1 \cap H(0, s)$ is bounded, take $F_{Q_1} = \{s\}$. If not, there is a sequence $z_n = (x_n, y_n) \in X \cap Q_1 \cap H(0, s)$ such that $y_n \to \infty$.



Hence, there is $N \in \mathbb{N}$ such that $y_N > 2K$ and since S is K-syndetic, there must be a site $t = (t_1, t_2) \in Q_2$ such that $z_N \in B(t, K)$. Hence, $t_2 > K$ and so by computations in Lemma 3.1.2,

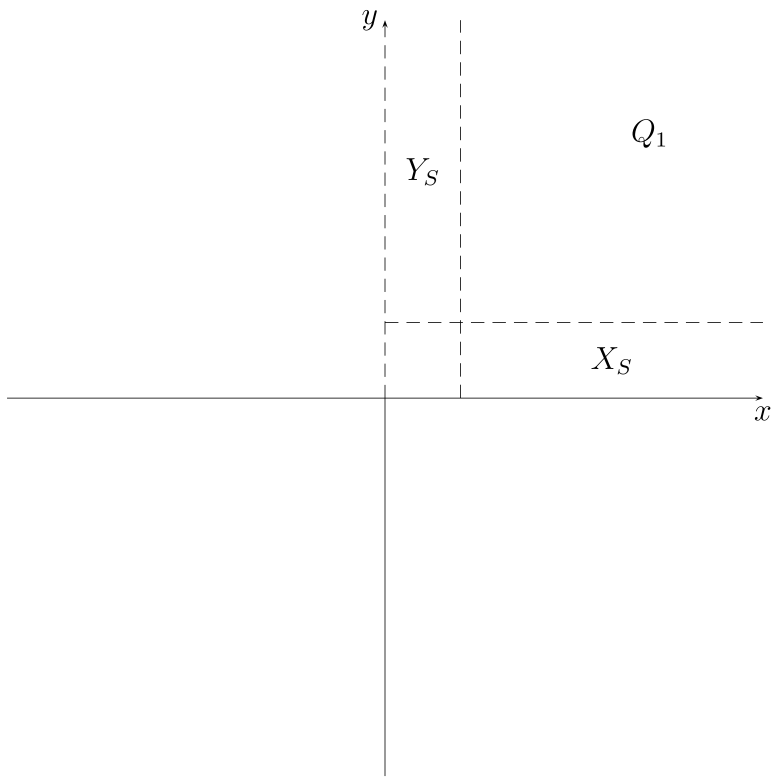
$$\forall (x, y) \in X \cap Q_1 \cap H(0, s) \cap H(0, t)$$

$x < \frac{a}{2}$ and $y \leq \frac{t_1^2+t_2^2}{2t_2} - \frac{xt_1}{t_2} \leq \frac{t_1^2+t_2^2}{2t_2} + \frac{Kt_1}{t_2}$. Thus we may take $F_{Q_1} = \{s, t\}$.

Case (III) There are no sites in $Q_1 \cap X$.

$$\text{Let } Y_S = \{(x, y) \in X \cap Q_1; 0 < x < K\} \text{ and}$$

$$X_S = \{(x, y) \in X \cap Q_1; 0 \leq y < K\}.$$



Because S is K-syndetic, $X \cap Q_1 = X_S \cup Y_S$. Now, an argument similar to that in Case II shows that if Y_S is unbounded, there is a site $r \in Q_2$ such that $Y_S \cap H(0, r)$ is bounded.

Likewise, if X_S is unbounded, there is a site $u \in Q_4$ such that $X_S \cap H(0, s)$ is bounded.

Hence, if both X_S and Y_S are unbounded, let $F_{Q_1} = \{r, u\}$. If X_S is bounded but Y_S is not, $F_{Q_1} = \{r\}$ will suffice. Similarly, if Y_S is bounded but X_S is not, $F_{Q_1} = \{u\}$

does the trick.

This completes the proof of Proposition 3.1.3.

∎

By Corollary 2.3.19 we obtain the following.

Corollary 3.1.4 *Let X be a closed subset of $\mathbb{E}^2$ and S be a well-separated, and syndetic set of sites in X. Then for all $p \in S$, $V(p)$ is a Voronoi polygon.*

If S is not well separated or not syndetic, $V(p)$ may not be a Voronoi polygon as the following examples show:

Example 3.1.5 The following example will show that the condition that S is syndetic is necessary. Let $X = \mathbb{E}^2$ and $S = \{(1,0)\} \cup \{s_n\}$ where $s_n = (0,n)$ for every $n \in \mathbb{N}$. Then S is 1-separated but not syndetic. Let $n \in \mathbb{N}$, then $H((1,0),(0,n)) = \{(x,y); y \leq \frac{x}{n} + \frac{-1+n^2}{2n}\}$. Let

$$(x,y) = (\frac{(n+2)n^2-(n+1)^2(n-1)+1}{2}, \frac{(n+2)n^2-(n+1)^2(n-1)+1}{2n} + \frac{n^2-1}{2n}),$$

then $(x,y) \in H((1,0),(0,n))$. But $y > \frac{x}{n+1} + \frac{-1+(n+1)^2}{2(n+1)}$, so $(x,y) \notin H((1,0),(0,n+1))$ and therefore $H((1,0),(0,n)) \not\subseteq H((1,0),(0,n+1))$. Hence, $V((0,0)) = \bigcap_{n\in\mathbb{N}} H((1,0),(0,n))$, which is not a Voronoi polygon.

Example 3.1.6 The condition that the set of sites is well-separated is necessary. Let $X = \mathbb{E}^2$ and $S((0,0),1) \cup \{(n,m); n,m \in \mathbb{N}\}$ be the set of sites. Recall that $S((0,0),1) = \{(x,y) \in \mathbb{E}^2; d((0,0),(x,y)) = 1\}$. Then the set of sites is 2-syndetic but not well-separated. However, $V((0,0)) = \bigcap_{s\in S((0,0),1)} H(0,s) = \overline{B((0,0),\frac{1}{2})}$ and this is not a Voronoi polygon.

By Theorem 2.3.17 and Corollary 3.1.4 we have the following theorem.

Theorem 3.1.7 *Let X be a closed and well-bisected subset of $\mathbb{E}^2$. If S is well-separated and syndetic in X, then $V(S)$ is a pre-triangulation of X.*

Remark 3.1.8 Clearly $\mathbb{E}^2$ is a well-bisected and proper metric space, thus if S is well-separated and syndetic in $\mathbb{E}^2$, then $V(S)$ is a pre-triangulation of $\mathbb{E}^2$.

3.2 Voronoi Diagrams in S^2

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3; \|(x, y, z)\| = 1\}$ where $\| \cdot \|$ denotes the Euclidean norm. In this section, we will prove that the Voronoi diagram of a set S of well-separated sites in any closed and well-bisected subspace of S^2 is a pre-triangulation. Note that S^2 is bounded, thus every set of sites S is syndetic. Clearly S^2 is proper, thus any closed subspace X of S^2 is proper. Let X be well-bisected and S be well-separated in X. By Theorem 2.2.10, $V(S)$ is a tessellation of X.

To prove that $V(S)$ is a pre-triangulation under these conditions we will show that the Voronoi cells are Voronoi polygons. Note that because S^2 is compact, every closed subset X of S^2 is compact. If S is well-separated in X, then S must be finite. Thus, clearly every Voronoi cell is a Voronoi polygon and we obtain the desired result by Theorem 2.3.17.

Theorem 3.2.1 *Let X be a closed and well-bisected subset of S^2 and S be well-separated in X. Then $V(S)$ is a pre-triangulation of X.*

3.3 Voronoi Diagrams in $\mathbb{D}$

Let $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ with the metric $ds^2 = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$. We call $\mathbb{D}$ the Poincaré disk. It is well-known that $\mathbb{D}$ is isometric to $\mathbb{H}$, the hyperbolic half plane with the isometry $\mathbb{H} \to \mathbb{D}$ defined by $z \mapsto \dfrac{iz+1}{z+i}$ [[14], page 81]. We now present some basic facts about $\mathbb{D}$ that we will use in this section. In $\mathbb{D}$, the isometries are of the form $z \mapsto \frac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}}$ [[1], p. 120]. Note that if f is an isometry of $\mathbb{D}$ and S is syndetic and well-separated, so too is $f(S)$, since these are metric properties. Also, the geodesics are either circular arcs which intersect the boundary of the disk orthogonally, or are line segments passing through the origin [10]. Moreover, there is a unique geodesic joining two points and the intersection of two distinct geodesics is either a point or empty [[7], prop 3.141]. Moreover, if $z_1 = x_1+iy_1, z_2 = x_2+iy_2 \in \mathbb{D}$, then the distance between z_1 and z_2 is

$$d(z_1, z_2) = \tanh^{-1}(|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}|).$$

Every bisector is a geodesic and if $z_1 = 0$ and $z_2 = x_2$ then :

$$\begin{aligned} b(0, z_2') =& \{z = x + iy \in \mathbb{D}; x^2 + y^2 - 2(\frac{1}{x_2})x + 1 = 0\} \\ & \{z = x + iy \in \mathbb{D}; y^2 + (x - \frac{1}{x_2})^2 = \frac{1}{x_2^2} - 1\}. \end{aligned}$$

[[10], Lemma 3] . We thus get $H(0, z_2') = \{z = x + iy \in \mathbb{D}; y^2 + (x - \frac{1}{x_2})^2 \geq \frac{1}{x_2^2} - 1\}$. We define $\underline{\mathbb{D}} = \mathbb{D} \cup S^1$, and, for a set B in $\mathbb{D}$ we define $\underline{B}$ as the Euclidean closure of B in $\underline{\mathbb{D}}$.

In this section, we will first prove that a well-separated and syndetic set of sites in $\mathbb{D}$ always has Voronoi cell that are Voronoi polygons, allowing us to effectively eliminate one of the hypotheses of Theorem 2.3.17. Thus the Voronoi diagram will be not only a tessellation but a pre-triangulation. Secondly, we will introduce a sufficient condition on closed and well-bisected subspaces X of $\mathbb{D}$ that will guarantee that the Voronoi diagram of a set S of well-separated and syndetic sites in X is a pre-triangulation of X.

Since the topology of $\mathbb{D}$ is identical to the Euclidean topology, it is clear that $\mathbb{D}$ is proper, and so any closed subspace X of $\mathbb{D}$ is proper. Moreover, for distinct z_1, z_2

in $\mathbb{D}$ and every $z \in b(z_1, z_2)$, the geodesic between z_1 and z intersects no other point of $b(z_1, z_2)$, thus by Lemma 2.2.8 $\mathbb{D}$ is well-bisected. Hence $V(S)$ is a tessellation of $\mathbb{D}$ if S is well-separated.

To prove that in $\mathbb{D}$, well separated and syndetic sites guarantee polygonal Voronoi cells, we will need some lemmas.

But first, recall that by Lemma 2.3.7, if S is K-syndetic and $p \in S$, then $V(p) \subset B(p, K)$. Hence if $q \in S$ and $V(p) \cap V(q) \neq \emptyset$, then if $x \in V(p) \cap V(q)$

$$2K > d(x,p) + d(x,q) \geq d(p,q).$$

and so $q \in B(p, 2K)$. (Note that this is true in any metric space.)

Lemma 3.3.1 *Let X be a semi-proper metric space and S be well-separated in X, and suppose $p \in S$. Then there exists only finitely many $q \in S$ such that $V(p) \cap V(q) \neq \emptyset$. In particular, this holds in $\mathbb{D}$.*

Proof: Suppose there are infinitely many $q \in S$ such that $V(p) \cap V(q) \neq \emptyset$. Then by our previous comment, $S \cap B(p, 2K)$ is infinite. As X is semi-proper, there is a Cauchy sequence $\{q_n\} \subset S \cap B(p, 2K)$, which is a contradiction because S is well-separated. ∎

Lemma 3.3.2 *Let $X = \mathbb{D}$ and $p, q \in S$. If $V(p) \cap V(q) = \emptyset$, then*

$$V(p) = \bigcap_{r \in S \setminus \{q\}} H(p, r)$$

Proof: Clearly $V(p) \subset \bigcap_{r \in S \setminus \{q\}} H(p, r)$.
Let $x \in \bigcap_{r \in S \setminus \{q\}} H(p, r)$. To see that $x \in V(p)$, suppose on the contrary that $d(x,p) > d(x,q)$. We recall that in $\mathbb{D}$, there is a geodesic $\gamma : [0, d(p,x)] \to \mathbb{D}$ joining

x and p and satisfying

$$d(x, \gamma(t)) + d(\gamma(t), p) = d(x, p), \forall t \in [0, d(x, p)].$$

Consideration of the continuous function $t \mapsto d(p, \gamma(t) - d(q, \gamma(t))$ then shows that there is a point w on this geodesic which belongs to $b(p, q)$. This point w cannot belong to either $V(p)$ or $V(q)$, since then it would belong to $V(p) \cap V(q)$, which is empty. Hence there is another site $r \in S \setminus \{p, q\}$ with $w \in V(r)$ and so $d(w, r) < d(w, q)$. But then,

$$d(r, x) \leq d(r, w) + d(w, x) < d(p, w) + d(w, x) = d(p, x)$$

contradicting the fact that $x \in \bigcap_{t \in S \setminus \{q\}} H(p, t)$. This implies that $d(x, p) \leq d(x, r)\ \forall r \in S$. Hence, $x \in V(p)$ and $\bigcap_{r \in S \setminus \{q\}} H(p, r) = V(p)$. ∎

Proposition 3.3.3 *Let S be a well-separated and syndetic set of sites in $\mathbb{D}$. Then for all $p \in S$, $V(p)$ is a Voronoi polygon.*

Proof: First, we know by Lemma 2.3.7, that $V(p)$ is bounded. Also, $V(p)$ is closed and $\mathbb{D}$ is proper, so $V(p)$ is compact.

By Lemma 3.3.1 there are only finitely many sites q such that $V(p) \cap V(q) \neq \emptyset$. By Lemma 3.3.2, this implies that $V(p) = \bigcap_{r \in L(p)} H(p, r)$ where $L(p) = \{q \in S; V(p) \cap V(q) \neq \emptyset\}$, which is finite. Being the intersection of finitely many half-planes, $V(p)$ is a polygon. ∎

Thus, by Theorem 2.3.17 we have the following theorem.

Theorem 3.3.4 *If S is a well-separated and syndetic set of sites in $\mathbb{D}$, then $V(S)$ is a pre-triangulation of $\mathbb{D}$.*

Remark 3.3.5 The conclusion of Lemma 3.3.2 doesn't hold for all subspaces of $\mathbb{D}$. Let p and q be distinct point in $\mathbb{D}$, $S = \{p, q\}$ and $X = \mathbb{D} \backslash b(p, q)$. Then, $V(p) \cap V(q) = \emptyset$, but

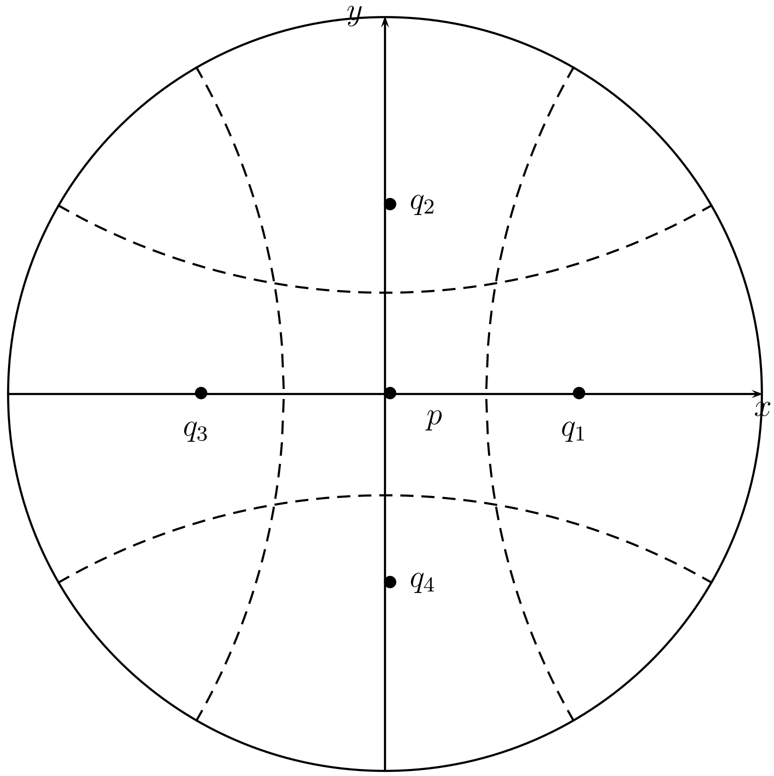
$$V(p) = H(p, q) \cap H(p, p) \neq H(p, p) = X.$$

Let X be a well-bisected and closed subset of $\mathbb{D}$. Because $\mathbb{D}$ is proper, X is proper. Thus, if the S is well-separated in X, then by Theorem 2.2.10, $V(S)$ is a tessellation of X.

In order to guarantee that our Voronoi cells will be polygons, inspired by the Euclidean case, we define sectors around each point in $\mathbb{D}$. In contrast to the Euclidean case, the number of sectors needed will depend on how close the sites are from their "origin".

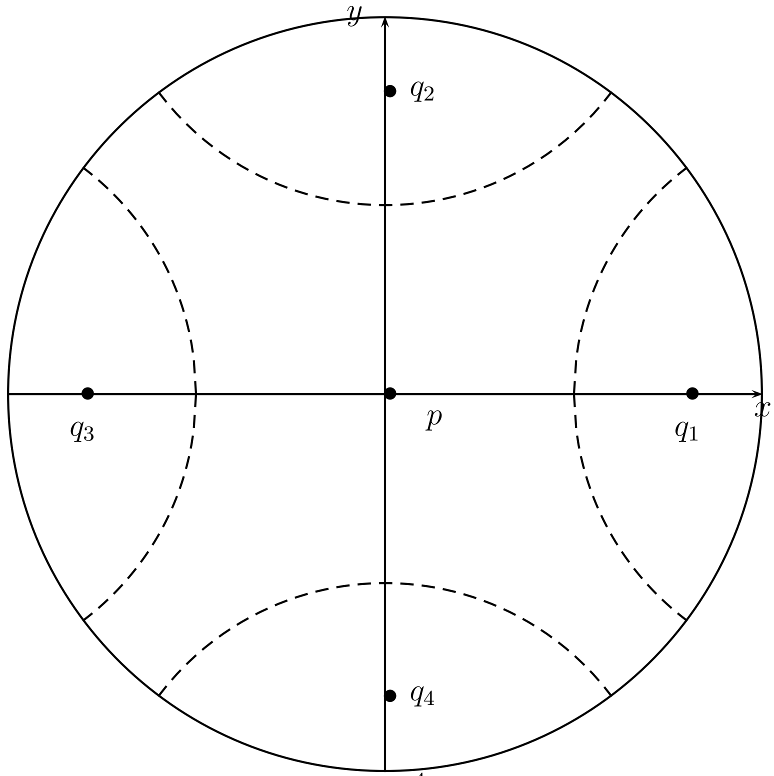
Remark 3.3.6 Clearly, a sequence of points diverges in $\mathbb{D}$ if the sequence tends (in $\mathbb{R}$) to a point in S^1. Thus $B \subset \mathbb{D}$ is unbounded if and only if $\underline{B} \cap S^1 \neq \emptyset$.

Example Let $p = (0, 0), q_1 = (m, 0), q_2 = (0, m), q_3 = (-m, 0), q_4 = (0, -m)$. If $m = \frac{1}{2}$, then $\bigcap_{i=1}^{4} H(p, q_i)$ is bounded, because $\underline{\bigcap_{i=1}^{4} H(p, q_i)} \cap S^1 = \emptyset$.



$m = \frac{1}{2}$

But if $m = \frac{4}{5}$, then $\underline{\bigcap_{i=1}^{4} H(p, q_i)} \cap S^1 \neq \emptyset$, thus $\bigcap_{i=1}^{4} H(p, q_i)$ is unbounded, as the following diagram illustrates.



We now define a property on S which will be sufficient in $\mathbb{D}$ to guarantee we have Voronoi polygons. We will then extend this property to well-bisected subsets of $\mathbb{D}$.

We first define sectors with "origin" $(0,0)$, and will then use properties of the group of isometries of $\mathbb{D}$ to generalize these.

Let $m \in (0,1)$ and define

$$l(m) = \begin{cases} \frac{\pi}{\arccos m} + 1, & \frac{\pi}{\arccos m} \in \mathbb{N} \\ \lceil \frac{\pi}{\arccos m} \rceil, & \text{else} \end{cases} .$$

Note that $3 \leq l(m) < \infty$, and $[0, 2\pi] = \bigsqcup_{n=1,\dots 2l(m)} \left[\frac{\pi(n-1)}{l(m)}, \frac{\pi n}{l(m)}\right)$. Now define sectors $\phi_n(m) \subset \mathbb{D}$ for $n = 1, \dots, 2l(m)$, as follows

$$\phi_n(m) = \{re^{i\delta}; 0 < r \leq m, \delta \in [\frac{\pi(n-1)}{l(m)}, \frac{\pi n}{l(m)})\}.$$

We remark that as $m \to 1$, $2l(m) \to \infty$ and so as the distance $\tanh^{-1}(m)$ in $\mathbb{D}$ between $(0,0)$ and $(m,0)$ increases without bound, so too does the number of sectors $2l(m)$.

Definition 3.3.7 *Let $m \in (0,1)$. A set of sites S containing $(0,0)$ in $\mathbb{D}$ is m-*crowded *at the site $(0,0)$ if there is a site q_n in each sector $\phi_n(m)$, $n = 1, ..., 2l(m)$.*

Recall that the orientation preserving isometries of $\mathbb{D}$ are of the form $z \to \frac{az+b}{\bar{b}z+\bar{a}}$ with $|a|^2 - |b|^2 = 1$ [14]. It is straightforward to show that the subgroup of orientation preserving isometry is transitive on $\mathbb{D}$, and that any such isometries fixing $(0,0)$ is in fact a Euclidean rotation about $(0,0)$.

First note that if ϕ is a Euclidean rotation about $(0,0)$, and S is m-crowded at $(0,0)$, then $\phi(S)$ is also m-crowded at $(0,0)$.

Now, if p is any site in S, and f and g are two orientation preserving isometries with $f(p) = 0 = g(p)$, then both fg^{-1} and gf^{-1} are Euclidean rotations about $(0,0)$. Thus, $g(S) = gf^{-1}(f(S))$ is m-crowded at $(0,0)$ if and only if $f(S) = fg^{-1}(g(S))$ is m-crowded at $(0,0)$. Thus we can make the following definition.

Definition 3.3.8 *Let S be a subset of $\mathbb{D}$ and $m \in (0,1)$.*
*1) We say that S is m-*crowded at *$p \in S$ if, for any orientation preserving isometry f satisfying $f(p) = (0,0)$, we have that $f(S)$ is m-crowded at $(0,0)$.*
2) We say S is m-crowded *if S is m-crowded at p, $\forall p \in S$.*

We now use this notion and invoke Corollary 2.3.19 to show that Voronoi cells in $\mathbb{D}$ are Voronoi polygons.

Proposition 3.3.9 *If S is m-crowded at $p \in S$, then there is a finite set $L = \{q_1, ..., q_{2l(m)}\}$ such that $\bigcap_{q \in L} H(p,q)$ is bounded.*

Proof: Recall that for a set B to be bounded in $\mathbb{D}$, we must have that $\underline{B} \cap S^1 = \emptyset$. Previous remarks show that we may, without lost of generality, choose $p = 0$. If

$m \in (0,1)$, then

$$\underline{H(0,m)} = \{(x,y) \in \underline{\mathbb{D}}; (x - \frac{1}{m})^2 + y^2 \geq \frac{1}{m^2} - 1\}$$

and therefore

$$\underline{H(0,m)} \cap S^1 = \{e^{i\delta}; \delta \in [\arccos m, 2\pi - \arccos m]\}$$

and

$$\underline{b(0,m)} \cap S^1 = \{e^{i(2\pi - \arccos m)}, e^{i(\arccos m)}\}.$$

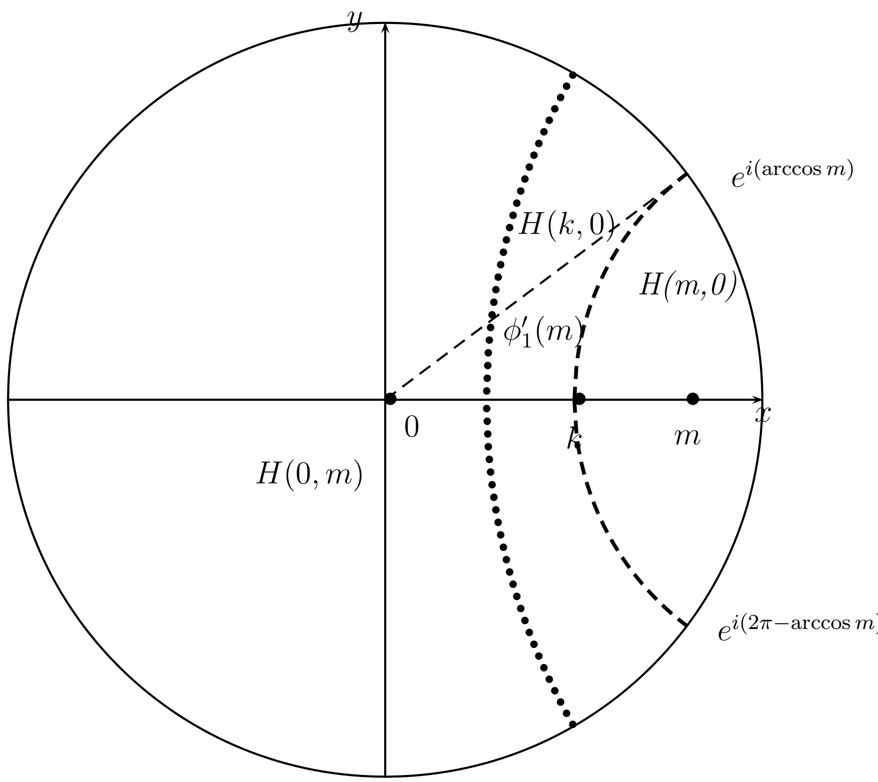
Thus,

$$\underline{H(m,0)} \cap S^1 = \{e^{i\delta}; \delta \in [2\pi - \arccos m, \arccos m]\}.$$

If $0 < k < m$, then

$$\{e^{i\delta}; \delta \in [2\pi - \arccos m, \arccos m]\} \subset \{e^{i\delta}; \delta \in [2\pi - \arccos k, \arccos k]\},$$

so $\underline{H(m,0)} \cap S^1 \subset \underline{H(k,0)} \cap S^1$.



Let us define $\phi'_n(m) = \{re^{i\delta}; r \in (0,1), \delta \in [\frac{\pi(n-1)}{l(m)}, \frac{\pi n}{l(m)})\}$. We will show that if there is a site q in a sector $\phi_n(m)$, then $\phi'_n(m) \cap H(0,q)$ is bounded. This will do, because $\mathbb{D} = \bigcup_{i=1}^{n} \phi'_n(m) \cup \{0\}$.

Without lost of generality we choose $n = 1$. For $\phi'_1(m) \cap H(0,q)$ to be bounded, it suffices to show that $\underline{H(0,q)} \cap \underline{\phi'_1(m)} \cap S^1 = \emptyset$. Therefore it suffices to show that

$$\underline{\phi'_1(m)} \cap S^1 \subset \underline{H(q,0)} \setminus \underline{b(q,0)}$$

We have just seen that for $0 < k < m$, $\underline{H(m,0)} \cap S^1 \subset \underline{H(k,0)} \cap S^1$. Moreover, the rotations $z \mapsto ze^{i\theta}$ on $\mathbb{D}$ are isometries, thus for $0 < k < m$,

$$\underline{H(me^{i\theta},0)} \cap S^1 \subset \underline{H(ke^{i\theta},0)} \cap S^1$$

and

$$\underline{H(me^{i\theta},0)} \cap S^1 = \{e^{i\delta} \in S^1; \delta \in [2\pi - \arccos m + \theta, \arccos m + \theta]\}.$$

with $\underline{b(me^{i\theta},0)} \cap S^1 = \{e^{i(2\pi - \arccos m + \theta)}, e^{i(\arccos m + \theta)}\}$. If $me^{i\theta} \in \phi_1(m)$, then $\theta \in [0, \frac{\pi}{l(m)})$. So, because $l(m) \geq \frac{\pi}{\arccos m}$ we have

$$-\arccos m + \theta < -\arccos m + \frac{\pi}{l(m)} \leq 0$$

and

$$\frac{\pi}{l(m)} \leq (\arccos m) + \theta.$$

Thus, for all $0 < k \leq m$ and $\theta \in [0, \frac{\pi}{l(m)})$,

$$\begin{aligned}
\underline{\phi'_1(m)} \cap S^1 &= \{e^{i\delta}; \delta \in [0, \frac{\pi}{l(m)})\} \\
&\subset \{e^{i\delta}; \delta \in ((2\pi - \arccos m) + \theta, (\arccos m) + \theta)\} \\
&= (\underline{H(me^{i\theta},0)} \setminus \underline{b(me^{i\theta},0)}) \cap S^1 \\
&\subset (\underline{H(ke^{i\theta},0)} \setminus \underline{b(ke^{i\theta},0)}) \cap S^1.
\end{aligned}$$

Hence, $\phi'_1(m) \cap H(0,q)$ is bounded for every $q \in \phi_1(m)$. Similarly for all other sectors. ∎

We now define the crowded property for a subspace of $\mathbb{D}$. Recall that for an $m \in (0,1)$ we have defined the sectors $\phi_n(m) = \{re^{i\delta}; r \in (0,m], \delta \in [\frac{\pi(n-1)}{l(m)}, \frac{\pi n}{l(m)})\}$ and $\phi'_n(m) = \{re^{i\delta}; r \in (0,1), \delta \in [\frac{\pi(n-1)}{l(m)}, \frac{\pi n}{l(m)})\}$ for $n = 1, ..., 2l(m)$.

Definition 3.3.10 *Let $X \subset \mathbb{D}$, S be a set of sites in X and $m \in (0,1)$.*
*1) We say that S is m-*crowded *in X at 0, if, whenever $\phi'_n(m) \cap X \neq \emptyset$, there is a site $q_n \in \phi_n(m)$.*
*2) We say that S is m-*crowded *in X at $p \in S$ if, for any orientation preserving isometry f satisfying $f(p) = (0,0)$, we have that $f(S)$ is m-crowded in $f(X)$ at $(0,0)$.*
3) We say S is m-crowded *in X, if S is m-crowded at p, $\forall p \in S$.*

Thus, by Proposition 3.3.9, we have the following result.

Proposition 3.3.11 *Let $X \subset \mathbb{D}$. If S is m-crowded in X at p, then there is a finite set $L = \{q_1, ..., q_{2l(m)}\}$ such that $\bigcap_{q \in L} H(p,q)$ is bounded.*

Hence, by Corollary 2.3.19 we obtain the following.

Corollary 3.3.12 *Let X be a closed subset of $\mathbb{D}$ and S be a well-separated, syndetic and m-crowded set of sites in X. Then for all $p \in S$, $V(p)$ is a Voronoi polygon.*

By Theorem 2.3.17 we have the following theorem.

Theorem 3.3.13 *Let X be a closed and well-bisected subset of $\mathbb{D}$. If S is well-separated, syndetic and m-crowded in X, then $V(S)$ is a pre-triangulation of X.*

Chapter 4

The world of g-spaces: metric spaces with 'segments'.

A g-space is a metric space in which every pair of points has a 'segment' joining them, and such segments will be unique when the 2 points are sufficiently close. These segments can also be (locally) prolonged uniquely. This begs the question : What is a 'segment' in a metric space ? These will of course be paths whose 'lengths' are the same as the distance between their endpoints. So we will begin by defining the length of a certain "rectifiable" paths in a metric space. In this chapter, the definitions and proofs will come from the book "The geometry of geodesics" by Herbert Busemann [2] and the book "Metric Spaces, Convexity and Nonpositive Curvature". [12], by Athanase Papadopoulos.

4.1 Length of a path

In this section we define the length of a path and collect some basic properties of the length. To do this, first we have to recall the concept of a partition.

Definition 4.1.1 *Let $[a, b]$ be an interval of $\mathbb{R}$.*

1) A partition *of $[a, b]$ is a finite sequence $\{t_0, ..., t_n\} \subset [a, b]$ such that*

$$a = t_0 \leq t_1 \leq ... \leq t_n = b.$$

Note that some partition points t_i may coincide.

2) To each partition σ, we associate its fineness *defined as $|\sigma| = max\{|t_{i+1} - t_i||0 \leq i \leq n-1\}$.*

3) If $\sigma_1 \subset \sigma_2$, when considered as sets, then we write $\sigma_1 \prec \sigma_2$ and say that σ_1 is coarser *then σ_2.*

Remark 4.1.2 If $\sigma_1 \prec \sigma_2$, then $|\sigma_2| \leq |\sigma_1|$.

We can now recall the definition of length of a path.

Definition 4.1.3 *Let $\gamma : [a, b] \to X$ be a path.*

a) If $\sigma = \{t_0, ..., t_n\}$ is a partition of $[a, b]$, define

$$L(\gamma, \sigma) = \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)).$$

b) The length *of γ is then defined as*

$$L(\gamma) = sup_{\sigma} L(\gamma, \sigma)$$

where the supremum is taken over the collection of partitions σ of $[a, b]$.

c) A path is said to be rectifiable *if its length is finite.*

Remark 4.1.4 1) Note that for every partition σ of $[a, b]$, $L(\gamma, \sigma) \leq L(\gamma)$.

2) Note that if σ and τ are partitions of $[a, b]$, and $\tau = \sigma$ when considered as sets, then $L(\gamma, \sigma) = L(\gamma, \tau)$ for any path $\gamma : [a, b] \to X$.

3) The triangle inequality shows that omitting points in a partition $\sigma : a = t_0 \leq t_1 \leq ... \leq t_n = b$ does not increase $L(\gamma, \sigma)$:

$$d(\gamma(t_i), \gamma(t_{i+2})) \leq d(\gamma(t_i), \gamma(t_{i+1})) + d(\gamma(t_{i+1}), \gamma(t_{i+2})).$$

Thus, if $\sigma_1 \prec \sigma_2$, then $L(\gamma, \sigma_1) \leq L(\gamma, \sigma_2)$ and hence the length of a path is bounded below by the distance between its endpoints.
4) Note that if $\gamma : [a, b] \to \mathbb{R}^n$ and γ is differentiable we have that $L(\gamma) = \int_a^b |\gamma'(t)|dt$.

Now we will prove the semicontinuity of length.

Proposition 4.1.5 [[2], (5.8)] *Let X be a metric space and, $\gamma_n : [a, b] \to X$ be rectifiables paths such that $lim_{n\to\infty}\gamma_n(t) = \gamma(t)$ for each $t \in [a, b]$. Then*

$$L(\gamma) \leq \liminf L(\gamma_n).$$

Proof: Let $\varepsilon > 0$ and suppose $L(\gamma) < \infty$. Then there is a partition $\sigma : a = t_0 \leq t_1 \leq ... \leq t_m = b$ such that $L(\gamma, \sigma) > L(\gamma) - \varepsilon$. By the continuity of d there is a N such that $k \geq N$ implies that $d(\gamma(t_i), \gamma_k(t_i)) < \frac{\varepsilon}{2m}$ foreach i, and thus by the triangle inequality,

$$|\sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) - \sum_{i=0}^{m-1} d(\gamma_k(t_i), \gamma_k(t_{i+1}))| < \varepsilon$$

Then with the definition of $L(\gamma)$ we have

$$\begin{aligned} L(\gamma_k) \geq L(\gamma_k, \sigma) &= \sum_{i=0}^{m-1} d(\gamma_k(t_i), \gamma_k(t_{i+1})) \\ &\geq \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) - \varepsilon = L(\gamma, \sigma) - \varepsilon \\ &> L(\gamma) - 2\varepsilon \end{aligned}$$

As ε was arbitrary, we have that $L(\gamma) \leq \liminf L(\gamma_n)$. ∎

Note that if γ and some γ_k are not rectifiable, then the same arguments (beginning with $L(\gamma, \sigma) > M$) yield the same inequality.

Remark 4.1.6 Even for a sequence of differentiable paths γ_n that converges uniformly to γ, $L(\gamma) = \lim_{n\to\infty} L(\gamma_n)$ won't always be true. For example, let $n > 0$ and

$\gamma_n : [0,\pi] \to \mathbb{R}^2$ be defined by $t \mapsto (t, \frac{\cos(n^2 t)}{n})$. Since

$$|\frac{\cos(n^2t)}{n}| \leq \frac{1}{n}, \forall t \in [0,\pi], \forall n > 0,$$

the sequence of paths $\{\gamma_n\}_{n\geq 1}$ converges uniformly to the path $\gamma : [0,\pi] \to \mathbb{R}^2$ defined by $t \mapsto (t,0)$.

But, $L(\gamma_n) \to \infty$ because

$$\begin{aligned} L(\gamma_n) = \int_0^\pi |\gamma_n'(t)|dt &= \int_0^\pi (1 + n^2\sin^2(n^2t))^{1/2}dt \\ &\geq n\int_0^\pi |\sin(n^2t)|dt \\ &= n^3\int_0^{\frac{\pi}{n^2}} \sin(n^2t)dt \\ &= 2n. \end{aligned}$$

4.2 Segments

In this section, we will introduce the concept of a segment, which we will use throughout the remaining of the thesis.

Recall that a map $f : X \to Y$ between metric spaces is an *isometry* if

$$d(f(x_1), f(x_2)) = d(x_1,x_2), \forall x_1, x_2 \in X.$$

Note that every isometry is injective.

Definition 4.2.1 *a) If x and y are points in a metric space X, a* parametrized segment from x to y *is an isometry $\gamma : [a,b] \to X$ from some closed interval $[a,b] \subset \mathbb{R}$ to X with $\gamma(a) = x$ and $\gamma(b) = y$.*
b) A subset $T \subset X$ is a segment from x to y *in a metric space X if T is the image*

of a parametrized segment from x to y. In this case we will sometimes denote T by $T(x,y)$, even though there may be many segments from x to y. The ambiguity of this notation will be eliminated in the contexts where it is used.

Remark 4.2.2 Suppose $\gamma : [a,b] \to X$ is a parametrized segment from x to y.
(i) If σ is any partition of $[a,b]$, then $L(\gamma,\sigma)$ is constant and equal to $b-a$. So every parametrized segment is rectifiable and its length is the length of its domain.
(ii) By (i), $L(\gamma) = b-a = d(f(a), f(b)) = d(x,y)$, and so a parametrized segment is a shortest path from x to y. Thus a parametrized segment from x to y is a path whose length is the distance between its endpoints, and hence a " shortest path" between them.
(iii) It is clear that the opposite γ^o of γ is a parametrized segment from y to x.
(iv) Because a parametrized segment is injective, it is an arc.
(v) Clearly, any subarc of a segment is a segment.

Examples 1) Let $X = S^2$. Then $\gamma : [0,2] \to S^2$ defined by $t \mapsto (\cos t, 0, \sin t)$ is a parametrized segment from $(1,0,0)$ to $(0,0,1)$.
2) Let $X = \mathbb{E}^2$. Then $\gamma : [0,\sqrt{2}] \to X$ defined by $t \mapsto (\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}})$ is a parametrized segment between $(0,0)$ and $(1,1)$. The straight line from $(0,0)$ to $(1,1)$ is the unique segment between $(0,0)$ and $(1,1)$.
3) Let $X = \mathbb{R}^2$ with the metric $d_1((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Then $\gamma_1 : [0,2] \to X$ defined by $t \mapsto (\frac{t}{2}, \frac{t}{2})$, $\gamma_2 : [0,2] \to X$ defined by

$$t \mapsto \begin{cases} (t,0) & \text{for } \ 0 \le t \le 1 \\ (1,t-1) & \text{for } \ 1 \le t \le 2 \end{cases}$$

and $\gamma_3 : [0,2] \to X$ defined by

$$t \mapsto \begin{cases} (0,t) & \text{for } \ 0 \le t \le 1 \\ (t-1,1) & \text{for } \ 1 \le t \le 2 \end{cases}$$

are parametrized segments. Clearly the trajectories of these paths are not the same, and so the segment between $(0,0)$ and $(1,1)$ is not unique. Thus, we see that the segments depend on the distance function.

4) Let $X = \mathbb{R}^2$ with the metric $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Then $\gamma : [0,1] \to \mathbb{R}^2$ defined by $t \mapsto (t,t)$ is a parametrized segment from $(0,0)$ to $(1,1)$. Therefore, the straight line from $(0,0)$ to $(1,1)$ is again a segment between $(0,0)$ and $(1,1)$.

We present an important result on convergence of parametrized segments.

Theorem 4.2.3 *Let X be a proper metric space, $\{\gamma_n\}$ be a sequence of parametrized segments $\gamma_n : [a,b] \to X$. If the initial points x_n of γ_n form a bounded set, then $\{\gamma_n\}$ contains a subsequence $\{\gamma_m\}$ which converges uniformly to a parametrized segment $\gamma : [a,b] \to X$ and*

$$L(\gamma) = \lim_{m\to\infty} L(\gamma_m).$$

Proof: First, we will show that the sequence $\{\gamma_n\}$ is equicontinuous. Let $t_0 \in [a,b]$ and $\varepsilon > 0$. If $|t_0 - t| < \varepsilon$, then $d(\gamma_n(t), \gamma_n(t_0)) = |t_0 - t| < \varepsilon$ for every $n \geq 1$. Because the initial points are bounded and the length of the paths is equal to $b-a$, for every $t \in [a,b]$, $\{\gamma_m(t)\}$ is bounded. Thus, by Corollary 1.2.3 there is a subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ that converges uniformly to a path $\gamma : [a,b] \to X$. Moreover,

$$d(\gamma(t), \gamma(t_1)) = d(\lim_{m\to\infty} \gamma_m(t), \lim_{m\to\infty} \gamma_m(t_1)) = \lim_{m\to\infty} d(\gamma_m(t), \gamma_m(t_1)) = |t - t_1|.$$

Hence, γ is a parametrized segment. Finally, all γ_m have the same length, so $L(\gamma) = \lim_{m\to\infty} L(\gamma_m)$ is clear. ∎

Remark This result can be generalized to sequences of paths in a proper metric space. [(5.16), [2]]. One can also prove the existence of a shortest path, by the preceding remark. $[[2], (5.18)]$

4.3 m-convex sets

Recall that in a Euclidean space, a set is convex if it contains all the line segments between each pair of its points.

In this section, for a metric space X, we recall (Definition 4.3.3) the notion of "an m-convex set", and show in Theorem 4.3.7, that if C is a m-convex set of a proper metric space, then for any pair of points of C, there exists a segment contained in C joining them.

Following [2], we will use the following notation.

Definition 4.3.1 *Let X be a metric space. For any pair of points x and y, a point z is* between *x and y if x, y and z are distinct and*

$$d(x, y) + d(y, z) = d(x, z)$$

We will use (xyz) to mean that y is a point between x and z.

Clearly (xyz) implies (zyx).

We can now introduce the notion of m-convexity.

Definition 4.3.2 *Let X be a metric space.*
a) A subset Y of X is m-convex *if for every pair of points x and y in Y, there exists a point $z \in Y$ between x and y (i.e. with (xyz)).*
b) A subset Y of X is convex *if for any pair of points x and y in Y, there exists a segment from x to y, contained in Y.*

Remark 1) A convex subset of a metric space is always m-convex, but the converse is false. For example $\mathbb{Q}^n \subset \mathbb{R}^n$ is m-convex, but not convex.
2) A stronger notion of convexity could be defined as follows : a subset Y in a metric space X is strongly convex if Y is convex and for each pair x and y in Y, all the segments joining x to y are in Y. Let $X = S^1$. Then $Y = \{e^{i\theta} \in S^1; 0 \leq \theta \leq \pi\}$ is convex, because for every pair of points $e^{i\delta_1}$ and $e^{i\delta_2}$ in Y, there exists a parametrized segment $\gamma : [\delta_1, \delta_2] \to S^1$ defined by $t \mapsto e^{it}$ from $e^{i\delta_1}$ to $e^{i\delta_2}$, but Y is not strongly convex because $\gamma_1 : [0, \pi] \to S^1$ defined by $t \mapsto e^{-it}$ is a parametrized segment from 1 to -1 that is not included in Y.

Using the generalized notion of segment in a metric space X, we present in Theorem 4.3.7 sufficient conditions on X for a m-convex subset to be convex.

We will need some technical lemmas, which we now present.

Lemma 4.3.3 [[2], (6.6)] *Let X be a metric space. If $w, x, y, z \in X$ satisfy (wxy) and (wyz), then they satisfy (xyz) and (wxz).*

Proof: By assumption and the triangle inequality, we have

$$d(w, z) = d(w, y) + d(y, z) = d(w, x) + d(x, y) + d(y, z) \geq d(w, x) + d(x, z) \geq d(w, z)$$

Therefore, $d(w, x) + d(x, z) = d(w, z)$ and so we have (wxz).
We also have

$$d(w, z) = d(w, x) + d(x, y) + d(y, z). \tag{4.3.1}$$

Moreover as (wxz) and by (0.2.1),

$$d(x, z) = d(w, z) - d(w, x) = d(x, y) + d(y, z),$$

which shows (xyz). ∎

We now define the product of two segments, which is the concatenation of two segments having one endpoint in common.

Definition 4.3.4 *Let X be a metric space. If $\gamma_1 : [a,b] \to X$ and $\gamma_2 : [c,d] \to X$ are parametrized segments such that $\gamma_1(b) = \gamma_2(c)$, then we define the* product of the two segments *as $\gamma_1 * \gamma_2 : [a, b+d-c] \to X$ defined by* $t \mapsto \begin{cases} \gamma_1(t) & \textit{for} \quad a \le t \le b \\ \gamma_2(t-b+c) & \textit{for} \quad b \le t \le b+d-c \end{cases}$

Note that this new path is continuous. We now show a sufficient condition for the product of two parametrized segments to be a parametrized segment.

Lemma 4.3.5 *Let X be a metric space. If $x, y, z \in X$, and there exists parametrized segments $\gamma_1 : [a,b] \to X$ from x to y and $\gamma_2 : [c,d] \to X$ from y to z such that (xyz), then $\gamma_1 * \gamma_2$ is a parametrized segment from x to z.*

Proof: As noted previously, $\gamma_1 * \gamma_2$ is a path because it is clearly continuous. We now have to show it is a parametrized segment. If $t, s \in [a,b]$ or $t, s \in [c, b+d-c]$ it is clearly isometric. Let $t \in [a,b]$ and $s \in [b, b+d-c]$. Then,

$$\begin{aligned} d(\gamma_1 * \gamma_2(t), \gamma_1 * \gamma_2(s)) &\le d(\gamma_1 * \gamma_2(t), \gamma_1 * \gamma_2(b)) + d(\gamma_1 * \gamma_2(s), \gamma_1 * \gamma_2(b)) \\ &= b - t + s - b = s - t \end{aligned}$$

If $B = d(\gamma_1 * \gamma_2(s), \gamma_1 * \gamma_2(t)) < s - t$, then

$$d(x,z) \le d(x, \gamma_1 * \gamma_2(t)) + d(\gamma_1 * \gamma_2(t), \gamma_1 * \gamma_2(s)) + d(\gamma_1 * \gamma_2(s), z) = t - a + B + b + d - c - s < b - a + d - c.$$

Which is a contradiction. Hence, $\gamma_1 * \gamma_2$ is a parametrized segment from x to z. ∎

We now prove a lemma that we will use in the proof of Theorem 4.3.7, the main result of this section.

Lemma 4.3.6 [[2], (6.8)] *Let X be a proper, m-convex space. For any pair of distinct points x and y in X, there exists $z \in X$ with (xzy) and $d(x,z) = \frac{1}{2}d(x,y)$.*

Proof: Let x, y be two distinct points of X and B be the set of points z which satisfy $d(x,z) + d(z,y) = d(x,y)$. Define continuous functions $f, q : X \to \mathbb{R}$ by $f(z) = d(x,z)$ and $g(z) = d(x,y) - d(x,z)$. Then

$$B = \{z \in X; d(x,z) + d(z,y) = d(x,y)\} = \{z \in X; f(z) = g(z)\} \subset B(x, d(x,y)).$$

This implies that B is a closed and bounded subset of a proper metric space, therefore B is compact. Moreover, X is m-convex, thus $B \neq \emptyset$. So, the function $\min\{d(z,x), d(z,y)\}$ attains on B its maximum M at some point m, because the minimum and the distance functions are continuous and B is compact.

Suppose that $\min\{d(m,x), d(m,y)\} > \frac{d(x,y)}{2}$. This implies that

$$\min\{d(m,x), d(y,m)\} > \frac{d(x,m) + d(m,y)}{2}$$

Which is a contradiction, therefore $\min\{d(m,x), d(m,y)\} = M \leq \frac{d(x,y)}{2}$. Suppose $M = d(m,x) < d(m,y)$. Then $M < \frac{d(x,y)}{2}$. Let z satisfy (mzy). Then, since (xmy), Lemma 4.3.3 implies (xzy) and (xmz). Suppose $d(z,y) > M$. Then because $\min\{d(z,x), d(z,y)\} \leq M$, we have that $d(x,z) \leq M$. By (xmz) we have that $d(x,m) + d(m,z) = d(x,z)$. We already know that $d(x,m) = M$, hence $M + d(m,z) \leq M$. Thus $d(m,z) = 0$, so $m = z$ and we have a contradiction. Therefore, $d(z,y) \leq M = d(x,m)$. From $M < \frac{d(x,y)}{2}$ we have $d(x,y) - 2M > 0$. Also, because of (xmz), (xzy) and $d(z,y) \leq M = d(x,m)$ we have

$$\begin{aligned} d(m,z) + 2d(m,x) &= d(z,x) + d(m,x) \\ &\geq d(z,x) + d(y,z) \\ &= d(x,y) \end{aligned}$$

This implies that $d(m,z) \geq d(x,y) - 2M > 0$. Let C be the set of points z which satisfy (mzy) together with the point y. By the same argument as for the set B, this set is a closed and bounded subspace of a proper metric space and therefore is compact. Thus, $d : C \to \mathbb{R}$ defined by $z \mapsto d(z, m)$ is a continuous function such that $d(m, z) \geq d(x, y) - 2M > 0$. Hence, this function reaches a positive minimum at some point z_0. Because X is m-convex, there is a point $z' \in X$ such that $(mz'z_0)$; hence $d(m, z') < d(m, z_0)$ and by Lemma 4.3.3, we have $(mz'y)$ so $z' \in C$ and this is a contradiction. Therefore $M = d(m, x) = d(m, y)$ and so $d(x, m) = \frac{1}{2}d(x, y)$. ∎

We now prove the main result of this section, one that gives sufficient conditions to have segments between every points of the space.

Theorem 4.3.7 $[[2], (6.8)]$ *Let X be a proper and m-convex metric space. Then for every $x, y \in X$, there exists a segment in X that joins x to y.*

Proof: Rename x, y as x_0, x_1. We have shown in Lemma 4.3.6 that in X they have a mid-point $x_{\frac{1}{2}}$ such that $d(x_0, x_{\frac{1}{2}}) + d(x_1, x_{\frac{1}{2}}) = d(x_0, x_1)$. By continuing the bisection process, we get a denumerable set of points $C = \{x_{m2^{-n}}; 0 \leq m \leq 2^n, n \in \mathbb{N}\}$. Now if $\alpha = d(x_0, x_1)$, we define the function $f : C \to \mathbb{R}$ as $f(x_{m2^{-n}}) = \alpha m 2^{-n}$. By Lemma 4.3.3, we have that for $x_v, x_u \in C$ with $v < u$,

$$d(x_u, x_v) = d(x_0, x_u) - d(x_0, x_v) = \alpha u - \alpha v = f(x_u) - f(x_v). \tag{4.3.2}$$

Therefore, f is isometric and so C is isometric to the set of points $D = \{m2^{-n}\alpha; 0 \leq m \leq 2^n, n \in \mathbb{N}\} \subset [0, \alpha]$.
Let $u \in [0, \alpha]$. There is a sequence $\{u_k\}_{k \geq 1}$ that converges to u, where the $u_k \in D$. Let the sequence $\{x_{\frac{u_k}{\alpha}}\}_{k \geq 1}$ in C. Then if $k_2 \geq k_1$,

$$d(x_{\frac{u_{k_1}}{\alpha}}, x_{\frac{u_{k_2}}{\alpha}}) \leq d(x_0, x_1) = \alpha$$

thus the sequence $\{x_{\frac{u_k}{\alpha}}\}_{k \geq 1}$ is bounded. Because X is a proper metric space, there is a subsequence of $\{x_{\frac{u_k}{\alpha}}\}_{k \geq 1}$ that converges to a point $x_u \in X$. If $u = m2^{-n}\alpha \in D$,

then $x_u = x_{m2^{-n}}$. Let $g : [0, \alpha] \to X$ be defined by $u \mapsto x_u$.
Let $u, v \in [0, \alpha]$. Then there are two sequences $\{u_k\}_{k\geq 1}$ and $\{v_k\}_{k\geq 1}$ of elements of D, converging respectively to u and v and such that $\lim_{k\to\infty} x_{\frac{u_k}{\alpha}} = x_u$ and $\lim_{k\to\infty} x_{\frac{v_k}{\alpha}} = x_v$. Then by Equation 4.3.2, we have

$$\begin{aligned} d(g(u), g(v)) &= d(x_u, x_v) \\ &= d(\lim_{k\to\infty} x_{\frac{u_k}{\alpha}}, \lim_{k\to\infty} x_{\frac{v_k}{\alpha}}) \\ &= \lim_{k\to\infty} d(x_{\frac{u_k}{\alpha}}, x_{\frac{v_k}{\alpha}}) \\ &= \lim_{k\to\infty} |u_k - v_k| \\ &= |u - v| \end{aligned}$$

Thus, g is an isometry and thus a parametrized segment between x and y. ∎

We now show that if a point is in the image of a segment, then it is between the endpoints of the segment.

Lemma 4.3.8 *Let X be a metric space. For $x, y, z \in X$, if y is in a segment $T(x, z)$, then we have (xyz).*

Proof: Suppose that $y \in T(x, z)$. There exists a parametrized segment $\gamma : [0, a] \to T(x, z)$ such that $L(\gamma) = d(x, y)$ and $\gamma(b) = y$ for $b \in [0, a]$. Then because a subarc of a segment is a segment, we have that $L(\gamma|[0, b]) = d(x, y)$ and $L(\gamma|[b, c]) = d(y, z)$ with $L(\gamma|[a, b]) + L(\gamma|[b, c]) = L(\gamma)$, thus $d(x, y) + d(y, z) = d(x.z)$ and so y satisfies (xyz). ∎

To end this section, we will prove a lemma about the convexity of segments in balls.

Lemma 4.3.9 [[2], (6.9)] *Let p be a point of a proper and m-convex metric space and*

$r > 0$. If x and y are points in the open ball $B(p, r)$, then any segment $T(x, y)$ is contained in $B(p, 2r)$.

Proof: If $z \in T(x, y)$, then by Lemma 4.3.8, z satisfies (xzy). We have

$$\min(d(x, z), d(y, z)) \leq \frac{d(x, y)}{2} \leq \frac{d(x, p) + d(p, y)}{2} < r.$$

Hence, $d(z, p) \leq \min\{d(p, x) + d(x, z), d(p, y) + d(y, z)\} < 2r$. ∎

Keeping the notation of 4.3.9, we could expect that if $x, y \in B(p, r)$, then $T(x, y) \subset B(p, r)$, but this is not always the case. For example, consider $S^1 = \{z \in \mathbb{C}; |z| = 1\}$ with $z_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $z_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. We see that

$$|z_1 - 1| = |z_2 - 1| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}.$$

Let $\sqrt{3} < r < 2$, then $z_1, z_2 \in B(1, r) \cap S^1$. But $\gamma : [\frac{2\pi}{3}, \frac{4\pi}{3}] \to S^1$ defined by $t \to e^{it}$ is a segment and $\gamma(\pi) \notin B(1, r)$.

4.4 Geodesics

In this section, we will define geodesics which arise as (infinite) extension of segments. Then we will state sufficient conditions to guarantee the existence of geodesics and to have unique prolongation.

Definition 4.4.1 *Let X be a metric space. A* geodesic *is a locally isometric map $\gamma : \mathbb{R} \to X$, i.e. for every $t_0 \in \mathbb{R}$, there exists $\varepsilon_{t_0} > 0$ such that if $|t_0 - t_1| \leq \varepsilon_{t_0}$ and $|t_0 - t_2| \leq \varepsilon_{t_0}$, then $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$.*

Examples 1) Let $X = \mathbb{E}^2$. Then $\gamma : \mathbb{R} \to \mathbb{E}^2$ defined by $t \mapsto (\frac{\sqrt{2}}{2}t, \frac{\sqrt{2}}{2}t)$ is a geodesic, because for every $t_1, t_2 \in \mathbb{R}$, $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$.

2) Let $X = \mathbb{E}^2$. Then, $\gamma : \mathbb{R} \to \mathbb{E}^2$ defined by $t \mapsto (t, t^2)$ is not a geodesic, as for $0 \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$d(\gamma(0), \gamma(t)) = |t|\sqrt{1+t^2} \neq |t|.$$

Lemma 4.4.2 [[2], (7.3)] *Let X be a metric space and γ be a geodesic. If $L(\gamma|[t_1, t_2])$ denotes the length of the arc $\gamma(t)$ for $t_1 \leq t \leq t_2$. Then $L(\gamma|[t_1, t_2]) = t_2 - t_1$.*

Proof: As γ is locally isometric, for each $t \in \mathbb{R}$, there exists an open I_t centered on t, such that $\gamma|_I$ is isometric. By the compactness of $[t_1, t_2]$, there are finitely many $t_1 \leq t_j \leq t_2$ with $[t_1, t_2] \subset \bigcup_j I_{t_j}$, and therefore $\gamma|_{[t_1,t_2]}$ is an isometry. ∎

Recall that the isometry group $E(1)$ of $\mathbb{E}$ consists of all the functions of the form $t \mapsto B \pm t$. It is the semi-direct product of $\mathbb{E}$ with the cyclic group with two elements. Then, if γ is a geodesic of a metric space and $\eta \in E(1)$, thus $\gamma \circ \eta = \gamma_1$ is also a geodesic of X, with the same range as γ. We say that γ_1 is a *reparametrization* of γ.

In Theorem 4.4.8, we will show that a segment in a metric space satisfying the following property can be extended in a geodesic.

Definition 4.4.3 *A metric space X has the* local prolongation property *denoted LP-property, if for each point $p \in X$, there exists $\rho > 0$ such that for any pair of distinct points $x, y \in B(p, \rho)$, there exists $z \in X$ with (xyz).*

Let $P(x, y)$ denote the statement: "$\exists z \in X$ such that (xyz)", and for $p \in X$, let

$$\rho(p) = \sup\{\rho; P(x, y) \text{ holds for all } x, y \in B(p, \rho)\}.$$

Clearly $P(x, y)$ holds for all $x, y \in B(p, \rho(p))$.

Lemma 4.4.4 [[2], (7.5)] *Let X be a metric space with the LP-property. Then, the following dichotomy holds: Either,* *i)* $\rho(p)$ *is infinite, for all* $p \in X$.

ii) $\rho(p)$ *is finite, for all* $p \in X$ *and the function* $\rho : X \to (0, \infty)$ *is 1-Lipschitz (i.e.* $|\rho(p) - \rho(q)| \leq d(p,q)$*, for all* $p, q \in X$*).*

Proof: If $x, y \in B(p, \rho(p))$ and $x \neq y$, then for some $\rho < \rho(p)$ we have $x, y \in B(p, \rho)$, and hence $P(x,y)$ holds.

If $\rho(p) = \infty$ for some p, then $P(x,y)$ holds for any distinct $x, y \in X$. Therefore $\rho(q)$ is infinite for all $q \in X$.

So, conversely, we can assume that $\rho(p) < \infty$, for all $p \in X$. Let $p, q \in X$. Without loss of generality, we can assume that $\rho(p) > \rho(q)$. If $d(p,q) \geq \rho(p)$, then $\rho(p) - \rho(q) \leq d(p,q)$, and if $d(p,q) < \rho(p)$, then $B(q, \rho(p) - d(p,q)) \subset B(p, \rho(p))$. Thus, every pair of points $x, y \in B(q, \rho(p) - d(p,q))$ satisfies $P(x,y)$ and therefore $\rho(p) - d(p,q) \leq \rho(q)$. ∎

Lemma 4.4.5 [[2], (7.7)] *Let* X *be a proper metric space with the LP-property. If* M *is a bounded subspace of* X*, then* $\inf_{p \in M} \rho(p) > 0$.

Proof: Since X has the LP-property, $\rho(p) > 0$, for all $p \in X$. As X is proper and $\overline{M}$ is bounded and closed, $\overline{M}$ is compact. As by Lemma 4.4.4, ρ is continuous, the lemma is proved. ∎

Remark 4.4.6 Let X be a proper and m-convex metric space with the LP-property. If x and y are two distinct points in X, with $d(x,y) < \rho(x)$, and if $L \in \mathbb{R}^+$, we will now show that there exists a $z \in X$ such that (xyz) and $d(y,z) \leq L$. Indeed, by the LP-property there is a z such that (xyz). Suppose that $d(y,z) > L$. By Theorem 4.3.7, there exists segments $T(x,y)$ and $T(y,z)$. By Lemma 4.3.5, $T(x,y) \cup T(y,z)$

is a segment. Hence by Lemma 4.3.8, there exists a $z' \in T(x,y) \cup T(y,z)$ such that (xyz') and $d(y,z') \le L$.

We will need the following lemma to prove the existence of geodesics.

Lemma 4.4.7 [[2], (7.8)] *Let X be a proper metric space, which is m-convex, and $p \in X$. If $p, q \in X$ satisfy $0 < d(p,q) < \rho(p)$ and $0 < L < \rho(p)$, then there exists $q' \in X$ with (qpq') and $d(p,q') = L$.*

Proof: Let $K = \{x \in X; (qpx) \text{ and } d(p,x) \le L\}$. Clearly, K is closed and bounded, thus compact. Moreover, $K \neq \emptyset$ by Remark 4.4.6. Hence, $d(p,x)$ attains its maximum at some point z_0 in K. If $d(p,z_0) = L$ take $q' = z_0$. Suppose $d(p,z_0) < L$. Since $q, z_0 \in B(p,\rho(p))$, there is a point z_1, with (qz_0z_1). Thence because we have (qpz_0) and (qz_0z_1), by Lemma 4.3.3 we have (qpz_1) and (pz_0z_1). Since $d(p,z_1) > d(p,z_0)$, $z_1 \notin K$ and we have $d(p,z_1) > L$ because (qpz_1). By Lemma 4.3.7 there is a segment $T(p,z_1)$. This segment contains a point q' with $d(p,q') = L$ because $d(p,z_1) > L$. Since (qpz_1) and $(pq'z_1)$, we have (qpq') by Lemma 4.3.3. Hence, $q' \in K$ and we have a contradiction. ∎

We now prove an important theorem on the existence of geodesics.

Theorem 4.4.8 [[2], (7.9)] *Let X be a proper, m-convex metric space which satisfies the LP-property, and let γ be a segment with $[a,b]$ as domain. Then there exists a geodesic $\alpha : \mathbb{R} \to X$, whose restriction to $[a,b]$ is γ.*

Proof: Let $f : X \to \mathbb{R}$ be defined by $p \mapsto \min(\frac{\rho(p)}{2}, 1)$. Then f is continuous and $f(p) > 0$ for every $p \in X$. Write a_0, b_0 for a, b and $\alpha(t)$ for $\gamma(t)$ when $a \le t \le b$.

Choose t_0 with $a_0 < t_0 < b_0, b_0 - t_0 < f(\alpha(b_0))$. So,

$$d(\alpha(b_0), \alpha(t_0)) = d(\gamma(b_0), \gamma(t_0)) = b_0 - t_0 < f(\alpha(b_0))$$

and $f(\alpha(b_0)) < \rho(b_0)$. Thus Lemma 4.4.7 guarantees the existence of a point q_1, with $(\alpha(t_0)\alpha(b_0)q_1)$ and $d(\alpha(b_0), q_1) = f(\alpha(b_0))$. Let $\alpha : [b_0, b_1] \to X$ represent a parametrized segment with $\alpha(b_1) = q_1$ and $b_1 = b_0 + f(\alpha(b_1))$. Then, by Lemma 4.3.5, the path α with domain $[a_0, b_1]$ represents a segment for $t \in [t_0, b_1]$.

Continuing the process, we find a sequence

$$a_0 < t_0 < b_0 < t_1 < b_1 < t_2 < b_2 < ...$$

with $b_{n+1} = b_0 + \sum_{i=1}^{n} f(\alpha(b_i))$ and $d(\alpha(b_n), \alpha(b_{n+1})) = b_{n+1} - b_n = f(\alpha(b_n))$. Now $b_n \to \infty$. For otherwise $f(\alpha(b_n)) \to 0$; but if $m > n$ then $d(\alpha(b_n), \alpha(b_m)) \leq b_m - b_n$, and so $\alpha(b_n)$ would converge to a point p; but then $f(\alpha(b_n)) \to f(p) > 0$.

Hence α is defined for all $t \in [a, \infty]$, and it represents a segment when $b_n \leq t \leq b_{n+1}$.

Similarly we can construct a sequence $b_0 > t'_0 > a_0 > t'_1 > a_1 > t'_2 > a_2$ with $a_{n+1} = a_0 - \sum_{i=1}^{n} f(\alpha(a_i)) \to -\infty$ and a function $\alpha(t)$ for $-\infty < t \leq b$ which coincide with α with domain $[a_0, b_0]$, and which represents a segment when $t'_n \leq t \leq a_{n-1}$.

It is clear that α is locally isometric by construction and hence it represents a geodesic.

∎

For the uniqueness of prolongation we will add the following property:

Definition 4.4.9 *A metric space X has the* local unique prolongation property *denoted LUP-property, if for each point $p \in X$, there exists a $\delta > 0$ such that any pair of points $y_1, y_2 \in X$ that satisfy $(xpy_1), (xpy_2)$ and $d(p, y_1) = d(p, y_2) < \delta(p)$ for $x \in X$, are equal.*

With this property, we can obtain the following uniqueness result.

Proposition 4.4.10 [[2], (8.3)] *Let X be a m-convex and proper metric space that satisfies the LUP-property. If x, y, z_1, z_2 are distinct points in X that satisfy $(xyz_1), (xyz_2)$ and $d(y, z_1) = d(y, z_2)$, then $z_1 = z_2$.*

Proof: Suppose there exists points x, p, y_1, y_2 with $(xpy_1), (xpy_2), d(p, y_1) = d(p, y_2)$ but $y_1 \neq y_2$. By Theorem 4.3.7 there exists parametrized segments $\gamma_{xp} : [0, a] \to X$ from x to p, $\gamma_{y_1p} : [0, b] \to X$ from p to y_1 and $\gamma_{py_2} : [0, b] \to X$ from p to y_2. By Lemma 4.3.5, there are parametrized segments $\gamma_1 : [0, a+b] \to X$ from x to y_1 with $\gamma_1(a) = y$ and $\gamma_2 : [0, a+b] \to X$ from x to y_2 with $\gamma_2(a) = y$. Let im $(\gamma_1) = T_1$ and im $(\gamma_2) = T_2$. There is a last value $t_0 \in [a + b]$ such that $\gamma_1(t) = \gamma_2(t)$ for $0 \leq t \leq t_0$ and because $y_1 \neq y_2$, we have $a \leq t_0 < a+b$. Put $p' = \gamma_1(t_0) = \gamma_2(t_0)$. By the definition of t_0, there is a t_1 with $t_1 > t_0, t_1 < t_0 + \min(\delta(p')/2, d(p', y_1), d(p', y_2))$ such that $\gamma_1(t_1) \neq \gamma_2(t_1)$. But because $x, p' \in T_1 \cap T_2$, $\gamma_1(t_1) \in T_1$ and $\gamma_2(t_1) \in T_2$ by the LUP-property. By Lemma 4.3.8 we have $(xp'\gamma_1(t_1)), (xp'\gamma_2(t_1))$ and $d(p', \gamma(t_1)) = d(p', \gamma_2(t_1)) < \delta(p')$, which is a contradiction. ∎

Proposition 4.4.10 also gives us the following results on uniqueness of segments.

Lemma 4.4.11 [[2], (8.6)] *Let X be a proper and m-convex metric space that satisfies the LUP-property. If x, y, z are distinct points in X that satisfy (xyz), then the parametrized segments $\gamma_1 : [0, a] \to X$ from x to y and $\gamma_2 : [0, b] \to X$ from y to z are unique and hence $T(x, y) \cup T(y, z)$ is the only segment from x to z that contains y.*

Proof: By Theorem 4.3.7 these parametrized segments exists. Let im $(\gamma_2) = T(y, z)$. Suppose that there is another parametrized segment $\gamma_3 : [0, b] \to X$ from y to z and denote im $(\gamma_3) = T'(y, z)$. By Lemma 4.3.5 there are parametrized segments $\gamma_4 : [0, a + b] \to X$ from x to z with im $(\gamma_4) = T(x, y) \cup T(y, z)$ and $\gamma_5 : [0, a + b] \to X$ from x to z with im $(\gamma_5) = T(x, y) \cup T'(y, z)$. By Lemma 4.3.8, for all $t \in [a, a + b]$ we have $(xy\gamma_4(t)), (xy\gamma_5(t))$ and $d(x, \gamma_4(t)) = d(x, \gamma_5(t))$, thus by

Lemma 4.4.10 $\gamma_4(t) = \gamma_5(t)$. By the same argument we show that $T(x, y)$ is unique. ∎

From this proposition, we have the following result on intersection of segments.

Lemma 4.4.12 [[2], (8.7)] *Let X be a proper and m-convex metric space that satisfies the LUP-property. If two segments T_1, T_2 have more than two common points, then $T_1 \cap T_2$ is a segment; but if they have just two common points these are the end points of both segments.*

Proof: Suppose that x, y, z are common points to T_1 and T_2. Then without loss of generality (xyz), so the segment $T(x, y) \cup T(y, z)$ is unique by Lemma 4.4.11, and hence lies on both T_1 and T_2.

If T_1 and T_2 have just two common points x, y, there can be no point z with (xyz), since $T(x, y)$ would then be unique and part of both T_1 and T_2. Thus y must be an end point of both segments. The same argument shows that x must be a end point of both segments. ∎

Lemma 4.4.13 *Let X be a proper and m-convex metric space that satisfies the LUP-property. If $x, y \in B(p, \rho(p))$, then $T(x, y)$ is unique.*

Proof: Because $x, y \in B(p, \rho(p))$, there exists a $z \in B(p, \rho(p))$ such that (xyz). Thus, by Lemma 4.4.11 $T(x, y)$ is unique. ∎

4.5 g-spaces

In this section, we introduce the class of g-spaces. They will be metric spaces, in which any pair of points is linked by a segment, and this segment will be unique if the points are close enough. We then present examples of such spaces and state in Proposition 4.5.4 a property of their geodesics.

Definition 4.5.1 ([2], §8) *A* g-space *is a proper, m-convex metric space, which has both the local and the unique local prolongation properties. In other words, (X, d) is a proper metric space such that :*

1. *For each pair of distinct points x and y in X, there exists a point $z \in X$ with (xyz).*
2. *For every $p \in X$, there exist $\rho > 0$, such that for each pair of distinct points $x, y \in B(p, \rho)$ there exists $z \in X$ with (xyz).*
3. *For each $p \in X$, there exists $\delta > 0$ such that if $x \in X$, $y_1, y_2 \in B(p, \delta)$ with (xpy_1) and (xpy_2), then $y_1 = y_2$*

Example a) The space $\mathbb{E}^2$ is a g-space. It is clearly a proper and m-convex metric space and it is known that geodesics are line segments. Thus between every pair of points, there is a unique line segment which can be extended uniquely, hence both the local and the unique local prolongation properties are satisfied.

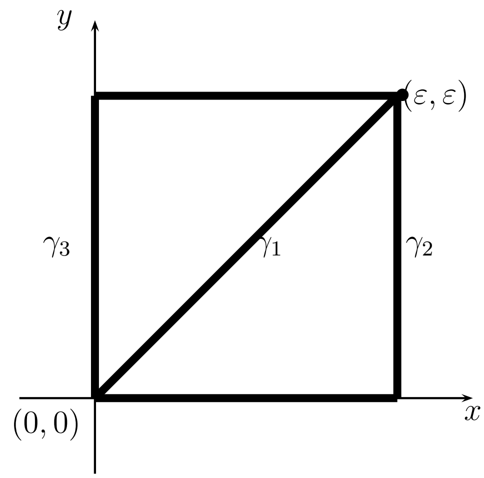
b) However the space $\mathbb{R}^2$ with the metric induced by $\|\cdot\|_1$ is not a g- space. For every $\varepsilon > 0$, $\gamma_1 : [0, 2\varepsilon] \to X$ defined by $t \mapsto (\frac{t}{2}, \frac{t}{2})$,

$$\gamma_2 : [0, 2\varepsilon] \to X \text{ defined by } t \mapsto \begin{cases} (t, 0) & \text{for} \quad 0 \leq t \leq \varepsilon \\ (\varepsilon, t-\varepsilon) & \text{for} \quad \varepsilon \leq t \leq 2\varepsilon \end{cases}$$

and

$$\gamma_3 : [0, 2\varepsilon] \to X \text{defined by } t \mapsto \begin{cases} (0, t) & \text{for} \quad 0 \leq t \leq \varepsilon \\ (t-\varepsilon, \varepsilon) & \text{for} \quad \varepsilon \leq t \leq 2\varepsilon \end{cases}$$

are parametrized segments between $(0,0)$ and $(\varepsilon, \varepsilon)$. Therefore, $(\mathbb{R}^2, \|\cdot\|_1)$ does not satisfy the unique local prolongation property.



Remark 4.5.2 By the previous examples, we see that whether or not X is a g-space depends on the metric and not just the topology induced by the metric.

Example 1) It is easy to see that $\mathbb{E}^n$ is a g-space, but the real Hilbert space $l^2_{\mathbb{R}}(\mathbb{N})$ is not a g-space because that it is not a proper metric space (see Example 1.1.10, (3)).
2) The Poincaré disk $\mathbb{D}$ is a g-space. It is a proper metric space and between every two points there is a unique geodesic [[7], prop 3.141]. The geodesics are either circular arcs which intersect the boundary of the disk orthogonally, or are line segment passing through the origin [10]. Thus every segment can be extended to a unique geodesic.
3) The space S^2 is a g-space. It is a proper metric space since it is compact. Moreover, between every two points with distance less than π there is a unique segment joining them. Also, every segment can be extended in a unique geodesic, so S^2 is a g-space.

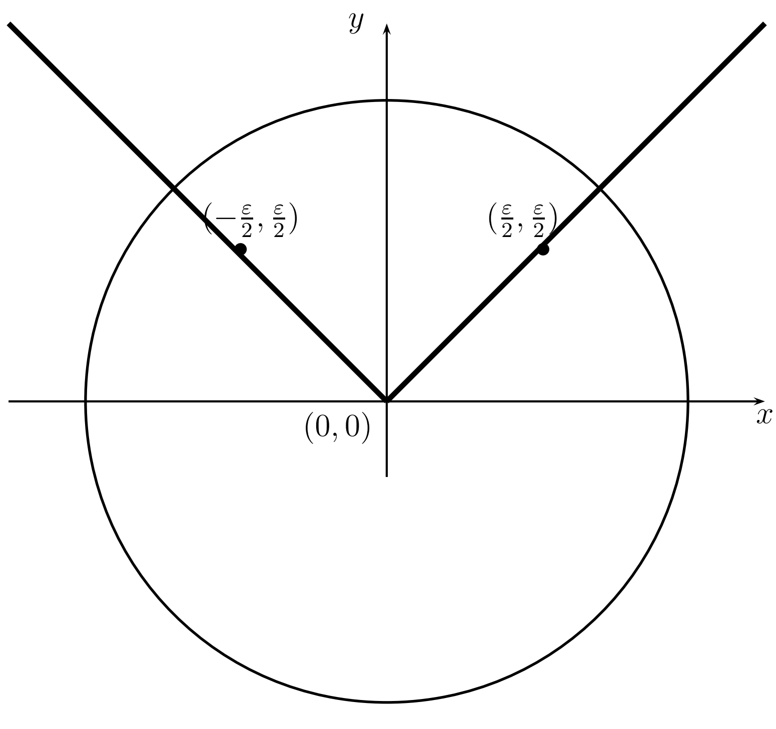
Remark 4.5.3 It has been proved that every two-dimensional g-space is a manifold [[2], (10.4)]. Moreover, it has been shown that all 3-dimensional and 4-dimensional

g-spaces are manifolds ([9] and [15] respectively), and conjectured that all g-spaces are manifolds [[2], (10.4)].

Example We present here examples of spaces that are not g-spaces.

1) Let $X = \mathbb{R} \setminus \{(-1,1)\}$. Then X is a proper metric space and satisfies the LP-property and the LUP-property of a g-space, but there is no y such that $d(-1,y) + d(y,1) = d(-1,1)$ and therefore X is not a g-space. In fact, every disconnected metric space is not a g-space.

2) Let $X = \{(x,y) \in \mathbb{E}^2; y \geq |x|\}$. Then X is a closed subset of $\mathbb{E}^2$, thus is a proper metric space. Moreover, between every pair of points in X there is a unique segment joining them. But, let $\varepsilon > 0$, take $(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \in B((0,0),\varepsilon)$ and $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \in B((0,0),\varepsilon)$, then there is no $z \in B((0,0),\varepsilon) \cap X$ such that $(\ (\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\ (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\ z)$ is satisfied, so X is not a g-space.



From now on, X will denote a g-space.

We now show that segments can be uniquely extended to geodesics.

Proposition 4.5.4 $[[2],(8.4)]$ *Let X be a g-space and $\gamma : [a,b] \to X$ be a segment. Then there is up to a reparametrization exactly one representation α of a geodesic such that $\alpha(t) = \gamma(t)$ for $a \leq t \leq b$.*

Proof: Theorem 4.4.8 guarantees the existence of at least one such geodesic α. If α_1 and α_2 are two distinct parametrization of α, then let $t_0 = \sup\{t \geq b; \alpha_1(t) = \alpha_2(t)\}$ (respectively $t_0' = \inf\{t \leq a; \alpha_1(t) = \alpha_2(t)\}$). If $t_0 < \infty$, choose $\varepsilon > 0$ such that α_1 represents a segment for $|t - t_0| \leq \varepsilon$. Then there exists a $t_1 \in \mathbb{R}$ that satisfies $t_0 < t_1 < t_0 + \varepsilon$ such that $(\alpha_i(t_0 - \varepsilon)\alpha_i(t_0)\alpha_i(t_1))$ for $i = 1, 2$ and $\alpha_1(t_1) \neq \alpha_2(t_1)$. By Proposition 4.4.10 this is a contradiction. If $t_0 = \infty$, then $-\infty < t_1$ and a similar argument finishes the proofs. ∎

Chapter 5

Voronoi Diagrams in a g-space

In this chapter, (X, d) will denote a g-space, where d denotes the distance function and S will be a subset of X. We shall call the points of S *sites*. We will find sufficient conditions on the set of sites for the Voronoi diagram to be a pre-triangulation. Moreover, we will study the faces and the segments in Voronoi polygons.

5.1 Voronoi Diagrams

In this section, we will show that for S well-separated (see Definition [2.1.6,i)]) and K-syndetic (see Definition [2.1.6,ii)]), $V(S)$ is a pre-triangulation (see Definiton 2.3.6) of X.

Firstly, we will prove that $V(S)$ is a tessellation (see Definition 2.2.1) and to do so, we need the following results.

Lemma 5.1.1 *Let X be a g-space and let p, q be a pair of distinct points of X. If $x \in b(p, q)$ and $T(x, q)$ is a segment from x to q, then $T(x, q) \setminus \{x\} \subset H(p, q)^c$ (i.e if $y \in T(x, q) \setminus \{x\}$, then $d(p, y) > d(q, y)$) .*

Proof: First of all, there exists a segment $T(x, q)$ because X is g-space. Let

$y \in T(x,q) \setminus \{x\}$. Suppose that

$$d(x,y) + d(y,p) = d(x,p).$$

Then by Lemma 4.3.8, $d(x,q) = d(x,y) + d(y,q)$ and as $d(x,p) = d(x,q)$, then $d(x,y) + d(y,p) = d(x,y) + d(y,q)$ and thus $d(y,p) = d(y,q)$. Then, by Proposition 4.4.10, we have that $p = q$, which is a contradiction. Therefore

$$d(x,y) + d(y,p) > d(x,p).$$

So, because $d(x,p) = d(x,q)$ and (xyq), we have that $d(x,y) + d(y,p) > d(x,y) + d(y,q)$, which implies that $d(y,p) > d(y,q)$. Therefore, $y \in H(p,q)^c$ and so $T(x,q) \setminus \{x\} \subset H(p,q)^c$. ∎

Proposition 5.1.2 *Let X be a g-space. If p and q are two distinct sites, then the boundary $\partial H(p,q)$ of $H(p,q)$ is equal to $b(p,q)$.*

Proof: By Proposition 2.2.4, we only have to show that $b(p,q) \subset \partial H(p,q)$. Suppose $x \in b(p,q)$. Thus $d(x,p) = d(x,q)$ and clearly $B(x,\varepsilon) \cap H(p,q) \neq \emptyset$. Because X is proper and m-convex, there is a segment $T(x,q)$. By Lemma 5.1.1, $T(x,q) \setminus \{x\} \subset H(p,q)^c$. Let $\gamma : [0, d(x,q)] \to X$ be a parametrized segment with im $(\gamma) = T(x,q)$. For every $\varepsilon > 0$, $y = \gamma(\varepsilon/2) \in T(x,q)$ satisfies $d(x,y) < \varepsilon$ and $y \in H(p,q)^c$. Therefore $B(x,\varepsilon) \cap H(p,q)^c \neq \emptyset$. Thus we get $x \in \partial H(p,q)$, hence $b(p,q) \subset \partial H(p,q)$. ∎

Hence we obtain the following corollary.

Corollary 5.1.3 *If X is a g-space, then X is well-bisected (i.e. for every pair of distinct x and y in X, we have $b(x,y) = \partial H(x,y) \cup \partial H(y,x)$) .*

A g-space is proper and therefore is semi-proper. Because of the preceding corollary a g-space is also well-bisected, therefore the following proposition follows directly from Theorem 2.2.10.

Proposition 5.1.4 *Let X be a g-space and S be well-separated. Then $V(S)$ is a tessellation.*

We will now prove (Proposition 5.1.9) that if S is a well-separated and K-syndetic subset of a g-space X, then the Voronoi cells associated to S are Voronoi polygons (see Definition 2.3.12). Proposition 5.1.9 and its proof generalizes Proposition 3.3.3 and the case of the Poincaré disk.

Recall (Lemma 2.3.7) that in this case, each Voronoi cell is bounded, more precisely $V(p) \subset B(p, K)$, for $p \in S$. As a direct consequence we have that if two Voronoi cells $V(p)$ and $V(q)$ intersect, then $d(p, q) < 2K$.

Lemma 5.1.5 *Let X be a g-space and S be well-separated. If $p \in S$, then there exists only finitely many $q \in S$ such that $V(p) \cap V(q) \neq \emptyset$.*

Proof: Suppose that $\{q \in S; V(p) \cap V(q) \neq \emptyset\}$ is infinite. Then $A = \{q \in S; d(p, q) < 2K\}$ is infinite, and as X is proper, A contains a Cauchy sequence. This contradicts the fact that S is well-separated. ∎

Lemma 5.1.6 *Let X be a proper metric space and S be K-syndetic and well-separated. Then any segment in X intersects only finitely many Voronoi cells.*

Proof: Let $T(x, y)$ be a segment from x to y and suppose that $A = \{p \in S; V(p) \cap T(x, y) \neq \emptyset\}$ is infinite. As $V(p) \subset B(p, K)$ and $\{z \in X; d(z, T(x, y)) \leq K\}$ is bounded, A is a bounded set. As S is well-separated and X is proper, then A is finite. ∎

Lemma 5.1.7 *Let X be a g-space and S be K-syndetic and well-separated. For any pair of distinct sites p and q, and any segment $T(x,y)$ from $x \in V(p)$ and $y \in V(q)$, then $\partial V(q) \cap T(x,y) \neq \emptyset$.*

Proof: Suppose $\partial V(q) \cap T(x,y) = \emptyset$. We know that the functions $f_p : T(x,y) \to \mathbb{R}$ and $f_q : T(x,y) \to \mathbb{R}$ defined respectively by $z \mapsto d(z,p)$ and $z \mapsto d(z,q)$ are continuous, therefore $h = f_p - f_q$ is continuous. Clearly $h(x) < 0$, because $d(x,p) < d(x,q)$ and $h(y) > 0$ because $d(y,p) > d(y,q)$. Therefore, by the continuity of h and the connectedness of $T(x,y)$ there exists a $z \in T(x,y)$ such that $h(z) = 0$. So, $f_p(z) = f_q(z)$ and thus, $d(z,p) = d(z,q)$. But because $\partial V(q) \cap T(x,y) = \emptyset$, there exists a $r \in S \setminus \{p,q\}$ such that $d(z,r) < d(z,q)$ and $z \in V(r)$. So now let us define $h_1 = f_r - f_q$, where $f_r : T(x,y) \to \mathbb{R}$ is defined by $z \mapsto d(z,r)$. We will find another $z_1 \in T(x,y)$ such that $d(z_1,r) = d(z_1,q)$. But again because $\partial V(q) \cap T(x,y) = \emptyset$, there exists a $r_1 \in S \setminus \{p,q,r\}$ such that $d(z_1,r_1) < d(z_1,q)$ and $z_1 \in V(r_1)$. Therefore, there are infinitely many $r \in S$ such that $V(r)$ intersects $T(x,y)$, which is a contradiction by Lemma 5.1.6. Thus there exists a $w \in \partial V(q) \cap T(x,y)$. ∎

Lemma 5.1.8 *Let X be a g-space and S be a set of sites. If $p, q \in S$ and $V(p) \cap V(q) = \emptyset$, then*

$$\bigcap_{r \in S \setminus \{q\}} H(p,r) = \bigcap_{r \in S} H(p,r) = V(p)$$

Proof: Clearly $V(p) \subset \bigcap_{r \in S\setminus\{q\}} H(p,r)$.

Let $x \in \bigcap_{r \in S\setminus\{q\}} H(p,r)$. Then $d(p,x) \leq d(r,x)\ \forall r \in S \setminus \{q\}$. Suppose that $d(x,p) > d(x,q)$. Thus $x \in V(q)$. Because $V(p) \cap V(q) = \emptyset$, for all $y \in \partial V(q)$ there exists a $w \in S \setminus \{q\}$ such that $d(y,p) > d(y,w)$. If $x \in \partial V(q)$, there exists a $w \in S \setminus \{q\}$ such that $d(x,p) > d(x,w) \geq d(x,p)$. This is a contradiction, so we have that $x \in \mathring{V}(q)$.

Because X is a g-space, there exists at least one segment $T(x,p)$ between $x \in V(q)$ and $p \in V(p)$. By Lemma 5.1.7, there exists a $w \in \partial V(q) \cap T(p,x)$. So, because $V(p) \cap V(q) = \emptyset$, there is a $r \in S \setminus \{p,q\}$ such that $d(w,r) = d(w,q) < d(w,p)$. We also have that $d(p,w) + d(w,x) = d(p,x)$. Therefore we have,

$$\begin{aligned} d(r,x) &\leq d(r,w) + d(w,x) \\ &< d(p,w) + d(w,x) \\ &= d(p,x) \end{aligned}$$

We thus have that $d(x,r) < d(x,p)$ which is a contradiction because $x \in \bigcap_{r \in S \setminus \{q\}} H(p,r)$. This implies that $d(x,p) \leq d(x,r)\ \forall r \in S$.

Hence, $x \in V(p)$ and $\bigcap_{r \in S \setminus \{q\}} H(p,r) = V(p)$. ∎

Hence, we have the following important result.

Proposition 5.1.9 *Let X be a g-space and S be well-separated and K-syndetic. Then for all $p \in S$, $V(p)$ is a Voronoi polygon.*

Proof: First, by Lemma 2.3.7 $V(p)$ is bounded. By Remark 2.1.4, $V(p)$ is closed, and X is proper, thus $V(p)$ is compact.

By Lemma 5.1.5 there exists finitely many q such that $V(p) \cap V(q) \neq \emptyset$. By Lemma 5.1.8, if $V(p) \cap V(q) = \emptyset$, then $\bigcap_{r \in S \setminus \{q\}} H(p,r) = V(p)$. This implies that $V(p) = \bigcap_{r \in S} H(p,r) = \bigcap_{r \in L(p)} H(p,r)$ where $L(p) \subset S$ and $L(p) = \{q \in S; V(p) \cap V(q) \neq \emptyset\}$ which is finite. Therefore, $V(p)$ is the intersection of finitely many half-planes and so is a polygon. ∎

Notice that both conditions on S are necessary. Indeed as $\mathbb{E}^2$ is a g-space, the syndeticity is necessary by Remark 3.1.5, and condition to be well-separated, by Remark 3.1.6

By Theorem 2.3.17, we obtain the main result of this section:

Theorem 5.1.10 *Let X be a g-space and S be well-separated and K-syndetic in X. Then $V(S)$ is a pre-triangulation in X.*

5.2 Voronoi cells

In this section, we will make some comments about the faces (see Definition 2.3.5) of a Voronoi polygon in a g-space and about the segments starting from the sites in the Voronoi cells.

Remark 5.2.1 Let X be a g-space and S be K-syndetic and well-separated. If $p \in S$, then $V(p) = \bigcap_{q \in S} H(p,q) = \bigcap_{q \in L(p)} H(p,q)$, where $L(p) = \{q \in S; V(p) \cap V(q) \neq \emptyset\}$ is finite. Thus by Remark 2.3.3

$$\partial V(p) = \bigcup_{q \in L(p)} \Big(\partial H(p,q) \bigcap_{r \in L(p)} H(p,r) \Big)$$

where every $\partial H(p,q) \bigcap_{r \in L(p)} H(p,r)$ is a face of $V(p)$. Thus the boundary of $V(p)$ is the union of its faces.

The following lemma illustrates another way to see the boundary of a Voronoi polygon in a g-space.

Lemma 5.2.2 *Let X be a g-space and S be a syndetic and well-separated set of sites. If $p \in S$, then $\partial V(p) = \bigcup_{q \in L(p)} V(p) \cap V(q)$, for $L(p)$ a finite subset of S.*

Proof: We know that $V(p) = \bigcap_{q \in S} H(p,q) = \bigcap_{q \in L(p)} H(p,q)$, where $L(p) = \{q \in S; V(p) \cap V(q) \neq \emptyset\}$ is finite, because $V(p)$ is a polygon by Proposition 5.1.9. So by Remark 2.3.3 and Lemma 5.1.2 we have

$$\partial V(p) = \bigcup_{q \in L(p)} \Big(\partial H(p,q) \bigcap_{r \in L(p)} H(p,r) \Big)$$

$$= \bigcup_{q \in L(p)} \Big(b(p,q) \bigcap_{r \in L(p)} H(p,r) \Big)$$
$$= \bigcup_{q \in L(p)} \Big(b(p,q) \bigcap_{r \in S} H(p,r) \Big)$$
$$= \bigcup_{q \in L(p)} \{x \in X; d(x,p) = d(x,q) \text{ and } d(x,p) \leq d(x,r) \, \forall r \in S\}$$
$$= \bigcup_{q \in L(p)} V(p) \cap V(q)$$

∎

Definition 5.2.3 *Let X be a g-space, and A a be a subset of X. If $p \in A$, then A is said to be* strongly star-shaped with center p *if for every $x \in A$, all of the segments from p to x are in A.*

Example 5.2.4 1) Any open ball $B(x,r)$ in $\mathbb{E}^2$ is strongly star-shaped with center x.
2) Let $X = S^2$ be the two-sphere in $\mathbb{R}^3$ and $A = \overline{B((0,1,0), \frac{\pi}{2})}$. If $p = (1,0,0)$ and $x = (-1,0,0)$, then $\gamma : [0,\pi] \to S^2$ defined by $t \mapsto (\cos(t), \sin(t), 0)$ is a parametrized segment from p to x in A, but $\gamma : [0,\pi] \to S^2$ defined by $t \mapsto (\cos(t), -\sin(t), 0)$ is a parametrized segment from p to x not included in A. Thus A is not strongly star-shaped with center p.

Lemma 5.2.5 *Let X be a g-space, S be K-syndetic and well-separated and $p \in S$. Then $V(p)$ is strongly star-shaped with center p.*

Proof: If $V(p)$ is not strongly star-shaped with center p, then there exist $y \in V(p)$ and a segment $T(p,y)$ from p to y such that there exists $x \in T(p,y) \cap V(p)^c$. As $x \in V(q)$ for some q, we have

$$d(p,y) = d(x,p) + d(x,y)$$

$$>d(x,q)+d(x,y)$$
$$\geq d(q,y)$$

This is a contradiction, because $y \in V(p)$. ∎

More can be said about segments starting from a site.

Lemma 5.2.6 *Let X be a g-space, S be well-separated and K-syndetic. If $p \in S$, then every segment T such that $L(T) > K$ starting from p intersects $\partial V(p)$ once and only once.*

Proof: Let $T(p,x)$ be a segment such that $L(T(p,x)) > K$. Suppose that $x \in V(p)$, then $K < d(x,p) \leq d(x,q)$ $\forall q \in S$, which is a contradiction because S is K-syndetic. Thus $x \notin V(p)$. Therefore, by Lemma 5.1.7, $T(p,x)$ intersects $\partial V(p)$.

Let $y \in T(p,x) \cap \partial V(p)$ be such that for any $z \in T(p,x) \cap \partial V(p)$, $d(p,y) < d(p,z)$, by Lemma 5.2.2 there $\exists q \in S$ such that $y \in V(q)$. Then for any other $w \in T(p,x)$ such that $d(p,w) > d(p,y)$, we have that

$$d(p,w) = d(p,y) + d(y,w) = d(q,y) + d(y,w) \geq d(q,w)$$

Suppose that $d(q,w) = d(q,y) + d(y,w)$, then there exists a segment $T(q,w)$ such that $y \in T(q,w)$. Thus we have (pyw), (qyw) and $d(p,y) = d(q,y)$ because $y \in V(p) \cap V(q)$, but $p \neq q$ hence we have a contradiction because X is a g-space.
This implies that $d(q,w) < d(q,x) + d(x,w)$ and so $d(p,w) > d(q,w)$ for any $w \in T(y,x) \subset T(p,x)$. Therefore there is no other $w \in \partial V(p)$ on the segment starting at p. ∎

Recall that $\rho(p)$ is the least upper bound of those ρ for which $P(x,y)$ holds in $B(p,\rho)$. Also, by Proposition 4.4.13, if $x,y \in B(p,\rho(p))$, then $T(x,y)$ is unique. Thus we obtain directly the following result.

Lemma 5.2.7 *Let X be a g-space and S be K-syndetic. If $p \in S$ with $\rho(p) > K$, then for every $x \in \partial V(p)$ there is a unique segment $T(p,x)$.*

Chapter 6

Conclusion

In this thesis, we have shown that :

Theorem 6.0.8 *Let X be a proper and well-bisected metric space and S be a well-separated and syndetic set of sites in X such that the Voronoi cells are Voronoi polygons. Then $V(S)$ is a pre-triangulation of X.*

However, we were not able to find sufficient conditions to guarantee that the Voronoi cells are Voronoi polygons, which is not always the case as Example 2.3.11 shows. In Chapter 3, we have shown that the Voronoi diagram of a set S of well-separated and syndetic sites in any closed and well-bisected subspace of $\mathbb{E}^2$ or S^2 is a pre-triangulation. But, this result does not hold in $\mathbb{D}$. It is well known that $\mathbb{E}^2$ has constant curvature, S^2 has positive curvature and $\mathbb{D}$ has negative curvature. One could think that studying the curvature of a space could be useful to find sufficient conditions to guarantee that the Voronoi cells are Voronoi polygons.

In our study of the Voronoi diagram in g-spaces in Chapter 5, we have proved the following results :

Lemma 6.0.9 *Let X be a g-space, S be well-separated and K-syndetic. If $p \in S$,* *then every segment T such that $L(T) \geq 2K$ starting from p intersects $\partial V(p)$ once and only once.*

and

Lemma 6.0.10 *Let X be a g-space and S be K-syndetic. If $p \in S$ with $\rho(p) > K$, then for every $x \in \partial V(p)$ there is a unique segment $T(p, x)$.*

Using those results and the fact that g-spaces of dimensions 1-4 are manifolds, we think that it could be proven that for a g-space of dimension 1-4 and S a well-separated and K-syndetic subset, if $p \in S$ with $\rho(p) > K$, then $V(p)$ is homeomorphic to the closed unit disk.

In the Euclidean case, the Delaunay tessellation [[3], p.55] is the "dual tessellation" of the Voronoi diagram [[3], p. 52]. In further study, it would be interesting to work on the Delaunay tessellations of $\mathbb{D}$, S^2 and g-spaces.

Chapter 7

Appendix

Here we collect some useful properties and estimates about the Hyperbolic upper half-plane.

7.1 Useful properties of the Hyperbolic Half-Plane.

The following properties come from [11]. First, we recall its definition :

Definition 7.1.1 *Let* $\mathbb{H} = \{(x, y); x, y \in \mathbb{R}, y > 0\}$ *be the Riemannian manifold with Riemannaian metric* $ds^2 = \frac{dx^2+dy^2}{y^2}$. *This manifold is called the* Hyperbolic upper half-plane.

Note that the geodesics are of two types :

a) Parts of Euclidean circles $(x - p)^2 + y^2 = r^2 (p, r \in \mathbb{R}, r > 0)$

or

b) Vertical (half-) lines : $x = c$ ($c \in \mathbb{R}$).

Moreover, there is a unique geodesic passing through every two distinct points.

Theorem 7.1.2 *If* $a_1 := (x_1, y_1)$ *and* $a_2 := (x_2, y_2)$ *are 2 points in* $\mathbb{H}$ *the* hyperbolic distance *is defined as*

$$d_{\mathbb{H}}(a_1, a_2) = \int_{\text{the geodesic that connects } a_1 \text{ and } a_2} ds$$
$$= \frac{1}{2} \left| log \frac{A + \sqrt{A^2 - 4y_1^2 y_2^2}}{A - \sqrt{A^2 - 4y_1^2 y_2^2}} \right|$$

where $A = (x_1 - x_2)^2 + (y_1^2 + y_2^2)$.

Fact A bisector $b(x, y) = \{z \in \mathbb{H}; d_{\mathbb{H}}(x, z) = d_{\mathbb{H}}(y, z)\}$ in $\mathbb{H}$ is a geodesic. Indeed, for two points $a_1 = (x_1, y_1), a_2 = (x_1, y_2)$ in H, the bisector $b(a_1, a_2)$ is

$$b(a_1, a_2) = \{(\frac{x_1 + x_2}{2}, y) | y > 0\}$$

when $y_1 = y_2$, and

$$b(a_1, a_2) = \{(x, y); (x - p)^2 + y^2 = y_1 y_2 ((\frac{x_1 - x_2}{y_1 - y_2})^2 + 1)\},$$

when $y_1 \neq y_2$. Here p=$\frac{x_1 y_2 - y_1 x_2}{y_2 - y_1}$.

Definition 7.1.3 *For a geodesic C, each of the two connected components C_1 and C_2 of $\mathbb{H} \setminus C$ is called a* half-space.

Lemma 7.1.4 *"A half-space is a convex set"; if $a_1, a_2 \in C_1$, then $C_{a_1,a_2} \subset C_1$, where C_{a_1,a_2} is the geodesic segment that connects the two points a_1 and a_2.*

Proof: Suppose that C and C_{a_1,a_2} are on circles C' and C'_{a_1,a_2} in the Euclidean plane respectively. The two circles intersect at most 2 points and the centers of the circles C', C'_{a_1,a_2} are on the x-axis. Therefore, one of the intersection is in the upper half-plane and the other is in the lower half-plane. Hence, if $a_1, a_2 \in C_1$, then C_{a_1,a_2} is in C_1.

The case in which C is part of a vertical line $x = c$ can be dealt with in a similar way. ∎

Fact If $a := (x_1, y_1) \in \mathbb{H}$, the hyperbolic circle with centre a and hyperbolic radius r:

$$\{z \in \mathbb{H}; d(z,a) = r\}$$

is a Euclidean circle with equation

$$(x - x_1)^2 + (y - \frac{e^{r/2} + e^{-r/2}}{2} y_1)^2 = (\frac{e^{r/2} - e^{-r/2}}{2} y_1)^2$$

Moreover the centre a is always in the interior of the circle.

7.2 Useful estimates for the hyperbolic half-plane

Recall that the geodesic joining two points in $\mathbb{H}^2$ is either part of a Euclidean circle centred on the x-axis, or is a vertical line. Let $a = (x_1, y_1) \neq b = (x_2, y_2)$ be points in $\mathbb{H}^2$, with $x_1 \neq x_2$. Then the centre $C = (c, 0)$ of this geodesic satisfies

$$c = \frac{x_2^2 + y_2^2 - (x_1^2 + y_1^2)}{2(x_2 - x_1)}.$$

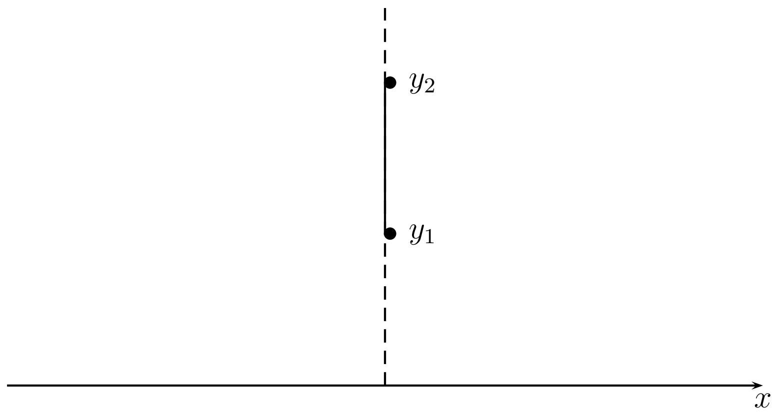
Also the radius r of the geodesic circle satisfies

$$r^2 = \frac{(x_1 - x_2)^2}{4} + \frac{(y_1^2 - y_2^2)^2}{4(x_1 - x_2)^2} + \frac{(y_1^2 + y_2^2)}{2}.$$

Recall that

$$d_{\mathbb{H}^2}(a,b) = \int_{\text{the geodesic that connects } x \text{ and } y} ds = \int_{t_0}^{t_1} \frac{\|\gamma'(t)\|}{y} dt$$

where γ is the geodesic between $a = \gamma(t_0)$ and $b = \gamma(t_1)$ and $dt = \sqrt{dx^2 + dy^2}$. We treat the two cases.



Case I) The geodesic is a vertical line : Suppose $a = (x_1, y_1) \neq b = (x_1, y_2) \in \mathbb{H}^2$. So, $\gamma(t) = (x_1, t)$ with $t_0 = y_1$ and $t_1 = y_2$ where t is not the hyperbolic arc length. Thus,

$$d_{\mathbb{H}}(a, b) = \int_{t_0}^{t_1} \frac{\|\gamma'(t)\|}{y} dt > \frac{\int_{t_0}^{t_1} \|\gamma'(t)\| dt}{\max\{y_1, y_2\}} = \frac{|y_1 - y_2|}{\max\{y_1, y_2\}} \tag{7.2.1}$$

and

$$d_{\mathbb{H}}(a, b) = \int_{t_0}^{t_1} \frac{\|\gamma'(t)\|}{y} dt < \frac{\int_{t_0}^{t_1} \|\gamma'(t)\| dt}{\min\{y_1, y_2\}} = \frac{|y_1 - y_2|}{\min\{y_1, y_2\}} \tag{7.2.2}$$

Case II) The geodesic is part of a circle : Now, suppose that $a = (x_1, y_1) \neq b = (x_2, y_2)$ are 2 points in $\mathbb{H}^2$ with $x_1 < x_2$ and $y_2 \leq y_1$. Then the geodesic joining a and b is the circle with center $C = (\frac{x_2^2 + y_2^2 - (x_1^2 + y_1^2)}{2(x_2 - x_1)}, 0)$ and radius $r^2 = \frac{(x_1 - x_2)^2}{4} + \frac{(y_1^2 - y_2^2)^2}{4(x_1 - x_2)^2} + \frac{(y_1^2 + y_2^2)}{2}$. If $c \geq x_2$, then

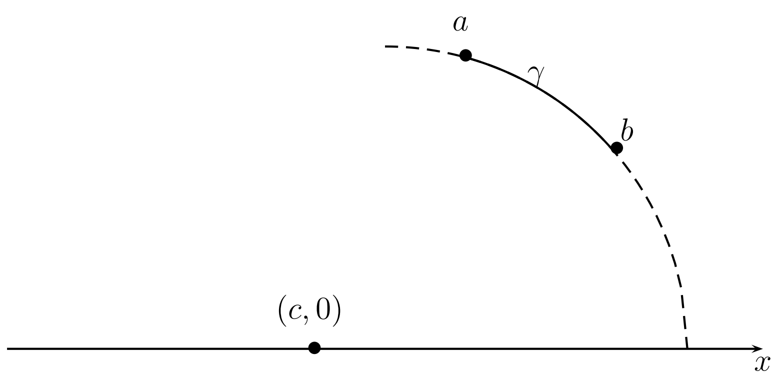
$$\frac{x_2^2 + y_2^2 - (x_1^2 + y_1^2)}{2(x_2 - x_1)} \geq x_2.$$

Therefore,

$$0 \geq y_2^2 - y_1^2 \geq x_2^2 + x_1^2 - 2x_2x_1 = (x_2 - x_1)^2 \geq 0.$$

So, $x_2 = x_1$, but $x_1 \neq x_2$, thus $c < x_2$. There are two possibilities :

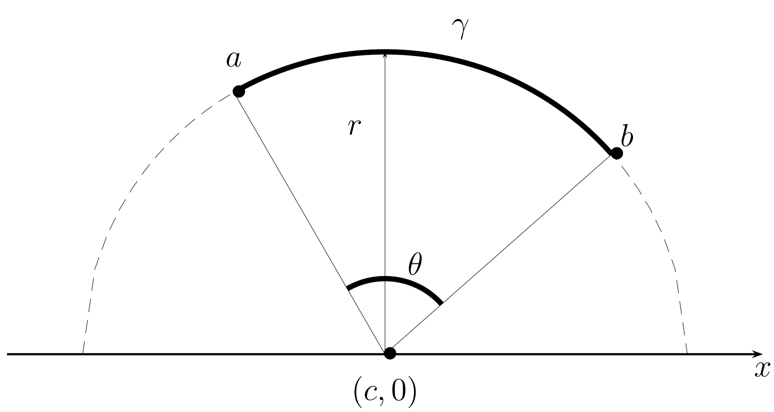
a) If $(x_1 - x_2)^2 < y_1^2 - y_2^2$, then $c < x_1 < x_2$.



So,

$$d_{\mathbb{H}}(a,b) = \int_{t_0}^{t_1} \frac{\|\gamma'(t)\|}{y} dt > \frac{1}{\max\{y_1, y_2\}} \int_{t_0}^{t_1} \|\gamma'(t)\| dt = \frac{d_{\mathbb{E}^2}(a,b)}{y_1} \tag{7.2.3}$$

b) Otherwise, if $(x_1 - x_2)^2 \geq y_1^2 - y_2^2$, then $x_1 < c < x_2$.



Therefore,

$$d_{\mathbb{H}}(a,b) = \int_{t_0}^{t_1} \frac{\|\gamma'(t)\|}{y} dt > \frac{1}{r} \int_{t_0}^{t_1} \|\gamma'(t)\| dt = \frac{d_{\mathbb{E}^2}(a,b)}{r} \tag{7.2.4}$$

Moreover,

$$d_{\mathbb{H}^2}(a,b) = \int_{t_0}^{t_1} \frac{\|\gamma'(t)\|}{y} dt < \frac{1}{\min\{y_1, y_2\}} \int_{t_0}^{t_1} \|\gamma'(t)\| dt = \frac{r(\theta)}{y_2} \tag{7.2.5}$$

We now present some useful lower and upper bounds for the radius r. Let $a = (x_1, y_1) \neq b = (x_2, y_2) \in \mathbb{H}^2$ with $x_1 < x_2$, $y_2 \leq y_1$, and $(x_1 - x_2)^2 \geq y_1^2 - y_2^2$. Recall that

$$r^2 = \frac{(x_1 - x_2)^2}{4} + \frac{(y_1^2 - y_2^2)^2}{4(x_1 - x_2)^2} + \frac{(y_1^2 + y_2^2)}{2}.$$

If $y_1 = y_2$, then

$$r^2 = \frac{(x_1 - x_2)^2}{4} + y_1^2. \tag{7.2.6}$$

If $y_1 \geq y_2$, then

$$r^2 \leq \frac{(x_1 - x_2)^2}{2} + y_1^2, \tag{7.2.7}$$

$$\frac{r^2}{y_1^2} = \frac{(x_1 - x_2)^2}{2y_1^2} + 1 \tag{7.2.8}$$

and

$$r^2 \geq \frac{(y_1^2 - y_2^2)^2}{4(x_1 - x_2)^2} + y_2^2. \tag{7.2.9}$$

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