Strong Positivity Results for Polynomials of Bounded Degree

Alan Walter Kelm

A Ph.D. Thesis

submitted to the School of Graduate Studies and Research

in partial fulfillment of the requirements for

the degree of Doctor of Philosophy in Mathematics*

University of Ottawa

Ottawa, Ontario

Canada

December 1993

*The Ph.D. Program is a joint program with

Carleton University, administered by the Ottawa-Carleton

Institute of Mathematics and Statistics

ISBN 0-315-95928-2

ACKNOWLEDGEMENTS

I would like to thank David Handelman for his patient and helpful supervision throughout the course of my work. He has been a most agreeable person to work under.

I wish to thank my wife, Angie, for her support throughout the latter stages of thesis work.

Thanks also to Valerio De Angelis for several interesting discussions and helpful comments.

Table of Contents

0. Abstract

Given a polynomial f possibly having negative coefficients, and a polynomial P having only positive coefficients, consider the problem of determining whether,

$$\text{for some } n, \text{ the product } P^n f \text{ has no negative coefficients.} \qquad (*)$$

If $P^n f$ has no negative coefficients, then it has no positive real roots, and so f cannot have any positive real roots. Thus a necessary condition on f is that $f(r) > 0$ for all r in $(0, \infty)$. In 1883 Poincaré [Pn] showed that under this condition on f, there exists a polynomial P for which Pf has no negative coefficients.

The case of several variables has been well-studied. Meissner [Mn] in 1911 studied what conditions on f will guarantee that $(*)$ holds, for the special case of $P = 1 + x + y + xy$. Pólya [Pó] in 1928 gave similar results for $P = 1 + x + y$. A complete solution, giving conditions on both P and f which are necessary and sufficient for $(*)$ to hold, was given by David Handelman in [H1] and [H2]. In [H3] connections between these positivity problems and several areas of mathematics are described.

Generalizations of the positivity problem can be made in several ways. De Angelis [DA1] has dropped the condition that P be positive, requiring instead that $|P(re^{i\theta})| < P(r)$ for all $r > 0$ and $0 < \theta < 2\pi$. Another generalization involves permitting P to vary. That is, to replace P^n by a product $P_1 P_2 P_3 \cdots P_n$ of positive polynomials. This was first considered in [H3, Appendix C], which gives some results for the case of several variables. The appropriate generalization of the problem is to take a sequence $\{P_i\}$ of positive polynomials and to ask whether, for all eligible polynomials f,

$$\begin{array}{l} \text{for every } k \text{ in } \mathbf{N} \text{ there is some } n, \text{ for which the product} \\ P_{k+1} P_{k+2} \cdots P_{k+n} \cdot f \ \text{ has no negative coefficients.} \end{array} \qquad (**)$$

In [BH] the problem $(**)$ is studied extensively for the case of polynomials in a single variable x. The sequence $\{P_i\}$ of polynomials is said to be *strongly positive* if $(**)$ holds for all f for which $f|_{(0,\infty)}$ is strictly positive.

It is frequently convenient to permit the polynomials P_i and f to be *Laurent polynomials*; that is, polynomials admitting both positive and negative powers of the indeterminate x. This permits us to regard the coefficients of each P_i as a probability distribution on the integers, provided we normalize the coefficients to sum to 1. With this viewpoint, the P_i's define a time-dependent random walk on the integers, and strong positivity refers to an eventual cancellation of the negative mass in f by the positive mass.

In [BH] three related numerical invariants of a Laurent polynomial P are introduced: $r(P)$, $c(P)$ and $C(P)$, and it is shown that the condition $\sum_i C(P_i) = \infty$ is sufficient to guarantee strong positivity of the sequence $\{P_i\}$ [BH, theorem 1.11]. We introduce a new invariant $\mathcal{R}(P)$ which is related to those given in [BH], and for which $\sum_i \mathcal{R}(P_i) = \infty$ is also sufficient to guarantee strong positivity of $\{P_i\}$ (Theorem 4.12). The principal advantage of $\mathcal{R}$ is the fact that it grows nearly additively as polynomials are multiplied: $\mathcal{R}(P_1 P_2 \cdots P_n) \geq \tanh\left(\sum_i \mathcal{R}(P_i)\right)$, (Theorem 4.7).

[BH] considers certain sequences for which $\sum_i C(P_i)$ converges, in an effort to determine whether they are strongly positive. In the case of symmetric unimodal polynomials, they obtain fairly complete results. The general case is much more difficult. In [BH] it is observed that if the coefficients of the product $P_1 P_2 \cdots P_n$ converge (after appropriate 'regularization': shifting and scaling of coefficients) to a formal Laurent power series F, and if F converges on some disk, then the sequence $\{P_i\}$ is not strongly positive.

If the degrees of the polynomials P_i are not bounded, it is observed in [BH] that even in the symmetric unimodal case, there are very delicate phenomena affecting the strong positivity of a sequence. The non-symmetric non-unimodal case becomes very difficult to approach if one does not assume a bound on the degrees of the P_i. It is for this reason that most of strong positivity results in this thesis require a bound on the degrees of the P_i.

Before studying strong positivity, it is important to be familiar with what is called the *fluctuation* of a Laurent polynomial. If $P = \sum_i p_i x^i$, then the fluctuation $\mathcal{F}(P)$ of P is the number $\frac{\sum_i |p_i - p_{i+1}|}{\sum_i p_i}$. It measures the differences between adjacent coefficients of P as a fraction of the total mass of P. A necessary condition for a sequence $\{P_j\}$ to be strongly positive is that

$$\mathcal{F}(P_{k+1}P_{k+2}\cdots P_{k+n}) \to 0, \quad \text{as} \quad n \to \infty, \qquad \text{for every } k. \tag{+}$$

One way to view this property is to regard it as *mass cancellation*: $\mathcal{F}(P)$ measures the fraction of the mass of P which remains after multiplying P by $(1-x)$. A small value of $\mathcal{F}(P)$ indicates that most of the mass has been cancelled out.

Another viewpoint is given in [BH], which shows that property (+) is equivalent to a condition on the states of a dimension group associated with the sequence $\{P_i\}$. Specifically, the fluctuation tends to zero if and only if point evaluation at 1 is a pure (extremal) state.

The notion of fluctuation has a counterpart which we call the *persistence* of a (Laurent) polynomial. It is defined by $\mathcal{P}(P) = \frac{\sum_i \min(p_i, p_{i+1})}{\sum_i p_i}$, and measures the degree to which mass persists from one coefficient to the next in P. A basic result about fluctuation, due to Mineka [Mi], is the fact that (+) holds if $\sum_i \mathcal{P}(P_i) = \infty$. Using this it is not difficult to show that (+) is equivalent to the existence of a blocking (telescoping) of $\{P_i\}$ into a sequence of products $\{Q_i\}$ for which $\sum_i \mathcal{P}(Q_i) = \infty$ (Theorem 2.10). This fact enables us to show that many sequences satisfy (+). For example, the sequence given by

$$P_n = x^{-2} + \sqrt{n} + x^3$$

has $\mathcal{P}(P_n) = 0$ for all n, but blocking it into pairs P_nP_{n+1} we find that $\mathcal{P}(P_nP_{n+1})$ is of the order of $\frac{1}{n}$ and so $\{P_n\}$ satisfies (+). The technique used in this example breaks down when we change the spacing, or rate of growth of the center term. Consider

$$P_n = x^{-2} + \sqrt{n} + x^5 \qquad \text{or} \qquad P_n = x^{-2} + n + x^3.$$

We can show that no blocking of the latter sequence into products Q_i of a fixed number of terms will make $\sum_i \mathcal{P}(Q_i)$ divergent. It turns out, however, that both of these sequences satisfy (+).

To show this, we need to introduce the idea of fluctuation and persistence over larger step sizes, rather than just between adjacent terms. For $P = \sum_i p_i x^i$ and $w \in \mathbf{Z}$ we define

$$\mathcal{F}_w(P) = \frac{\sum_i |p_i - p_{i+w}|}{\sum_i p_i} \quad \text{and} \quad \mathcal{P}_w(P) = \frac{\sum_i \min(p_i, p_{i+w})}{\sum_i p_i}.$$

For the sequence given by $P_n = x^{-2} + n + x^3$, it is immediate that $\sum_n \mathcal{P}_2(P_n) = \infty$ and $\sum_n \mathcal{P}_3(P_n) = \infty$. An extended version of Mineka's theorem (Corollary 2.8) then tells us that fluctuation over spacings 2 and 3 goes to 0 (the analogous statements to (+)). We show (Corollary 3.5) that if fluctuation goes to zero on two relatively prime spacings, then (+) holds.

If the degrees of the P_i are bounded, another result (Theorem 3.9) tells us that (+) holds if and only if the subgroup of $\mathbf{Z}$ generated by the set

$$\left\{ w \in \mathbf{Z} \,\middle|\, \sum_n \mathcal{P}_w(P_n) = \infty \right\}$$

is all of $\mathbf{Z}$.

Armed with the ability to distinguish which sequences satisfy (+), we are able to discern sequences which are not eligible to be strongly positive, since they fail to satisfy (+).

The property of being a zero-fluctuation sequence is fairly robust, in that ℓ^1 perturbations of a sequence (appropriately normalized) retain this property (see Theorem A.11). In fact for upward perturbations, the property $\sum_i \mathcal{P}(P_i) = \infty$ is retained even under ℓ^∞ perturbations (Lemma A.7)

Strong positivity, on the other hand, is nowhere near as robust. In fact, *arbitrarily small* upward perturbations can cause or destroy strong positivity. The sequence defined by $P_i = \alpha_i + x^2 + x^3$ is strongly positive only if infinitely many of the α_i's are zero (regardless of how small any nonzero α_i's are). On the other hand, the sequence given by

$$P_i = \alpha_i x^{-1} + 1 + x^2 + x^3$$

is strongly positive if and only if infinitely many of the α_i's are *nonzero* (Proposition 5.16). For convenience we adopt a notation which would rewrite this last polynomial as

$$P_i = \begin{array}{cccc} \alpha_i & 1 & 1 & 1 \\ \hline -1 & 0 & 2 & 3 \end{array} \ ;$$

below the line are the positions of the coefficients, and above the line are the coefficients themselves.

Since being a zero-fluctuation sequence is a prerequisite for being strongly positive, and since Mineka's criteria guarantees that a sequence will be zero-fluctuation if $\sum_i \mathcal{P}(P_i) = \infty$, we are led to ask what additional conditions should be imposed in order to gain strong positivity. The sequence given by $P_n = \begin{array}{cccc} 1 & 1 & & \frac{1}{n} \\ \hline 0 & 1 & & 3 \end{array}$ is a zero-fluctuation sequence, but it is not strongly positive because of the isolated coefficient. The polynomial $P_n = \begin{array}{ccccc} 1 & 1 & & \frac{1}{n^2} & 1 \\ \hline 0 & 1 & & 3 & 4 \end{array}$ fails to be strongly positive, again because of the 'essentially isolated' coefficient. It comes as somewhat of a surprise that the sequence $P_n = \begin{array}{ccccc} 1 & 1 & & 1 & e^{-n} \\ \hline 0 & 1 & & 3 & 4 \end{array}$ *is* strongly positive. The first general result along this line is Theorem 5.7, and substantial arguments and techniques are required to prove it. These techniques are built up towards the end of chapter 4 and the start of chapter 5. We generalize the result in steps until we arrive at an analog of Mineka's theorem, for strong positivity (Proposition 5.14 and Theorem 5.15):

For a sequence of positive Laurent polynomials of bounded degree, for which $\sum_i \mathcal{P}(P_i) = \infty$, we additionally require a condition on the ratio of the next-to-end coefficient and the end coefficient at each end. These two ratios, and the persistence $\mathcal{P}(P_i)$ must diverge together.

The boundedness of the degree can be relaxed slightly in this result. An abundance of examples are used throughout chapter 5, and we refer the reader to them.

We end chapter 5 by giving a few circumstances in which we can guarantee strong positivity even though $\sum_i \mathcal{P}(P_i)$ is finite. These are generally cases in which blockings into products will readily yield divergent persistence. We also note that these cases tend to require that the dominant coefficients have relatively prime spacing. Failing this, the sequence is often not even a zero-fluctuation sequence.

An underlying topic which surfaces at various points throughout the thesis is strong unimodality of a Laurent polynomial (also known as log-concavity). In chapter 7 we prove a generalization of a theorem due to Odlyzko and Richmond [OR]. They proved that under suitable conditions on the positioning of the nonzero coefficients of P, all large powers P^n will be strongly unimodal. Our generalization proves the strong unimodality of all products $P_1 P_2 \cdots P_n$ for n sufficiently large, subject to appropriate conditions on the sequence $\{P_i\}$.

In chapter 6 we prove another strong unimodality result, this time for sums of products of linear polynomials. The result is phrased as strong unimodality of various cross-sections of a two-dimensional triangular grid of coefficients. Results of this type come into play when seeking to estimate coefficients of powers of certain univariate polynomials.

The reader will find it helpful to have the List of Theorems (page 113) in hand while reading the thesis, as it gives a synopsis of most of the results. The Index is helpful for locating the definitions of terms and symbols.

1. Introduction

In this chapter we introduce some of the basic objects, terms and concepts which will be referred to throughout this thesis. Each subsequent chapter begins with an overview of the main contents of the chapter.

When dealing with a set S having an additive group structure and a partial order, we will denote by S^+ the positive elements of the set, namely $S^+ = \{ x \in S \mid x \geq 0 \}$. In particular, $\mathbf{R}^+ = \{ x \in \mathbf{R} \mid x \geq 0 \}$, and $\mathbf{Z}^+ = \{0, 1, 2, 3, \ldots\}$. We will take the set of natural numbers to be $\mathbf{N} = \{1, 2, 3, \ldots\}$.

We frequently encounter infinite sums, $\sum_{i \in I} a_i$, having only finitely many nonzero terms, and we define such a sum to be simply the finite sum of the nonzero terms, or 0 in the case that every a_i is zero.

Laurent Polynomial Algebras. While Laurent polynomials may be naively construed as 'polynomials admitting negative integer powers', it is useful to be more precise in our definitions and notation. While most of our results deal with real Laurent polynomials in a single indeterminate, many of the fluctuation results deal with sequences of polynomials involving possibly an infinite number of indeterminates. The following definition of Laurent polynomials is the obvious generalization of the usual definition to the case of arbitrarily many indeterminates. In this situation the additive group formed by the exponents is a direct sum of copies of $\mathbf{Z}$.

Let I be a set. Define the group G to be $G = \oplus_{i \in I} Z_i$ (direct sum of groups) where each $Z_i = \mathbf{Z}$ is a copy of the additive group of integers. If $z \in G$ has non-zero components $z_1, z_2, \ldots, z_k$, with each $z_j \in Z_{i_j}$, we will write $z = x_{i_1}^{z_1} x_{i_2}^{z_2} \cdots x_{i_k}^{z_k}$ or even $z = x^z$ (we also permit some of the z_i's to be zero). In this notation, we will write the group operation as multiplication rather than addition, so if $z = x_{i_1}^{z_1} x_{i_2}^{z_2} \cdots x_{i_k}^{z_k}$ and $w = x_{i_1}^{w_1} x_{i_2}^{w_2} \cdots x_{i_k}^{w_k}$, then $w+z = x^w x^z = x^{w+z} = x_{i_1}^{w_1+z_1} x_{i_2}^{w_2+z_2} \cdots x_{i_k}^{w_k+z_k}$. We thus regard G as a multiplicative group, with identity $1 = x^0$. We define the *algebra of Laurent polynomials over* $\mathbf{R}$ *with indeterminates indexed by* I to be the group algebra $\mathbf{R}[G]$. We denote it by $\mathbf{R}[x_i^{\pm 1}]_{i \in I}$. If $I = \{1, 2, \ldots, n\}$, we may denote it also by $\mathbf{R}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. We may replace the indeterminates x_i by other letters to obtain, for example, $\mathbf{R}[x, y, v]$. $\mathbf{R}[x_i^{\pm 1}]_{i \in I}$ is a unital commutative $\mathbf{R}$-algebra.

For $z \in G$ and $P \in \mathbf{R}[x_i^{\pm 1}]_{i \in I}$, we denote the coefficient of x^z in P by $\langle P, x^z \rangle$.
For $P \in \mathbf{R}[x_i^{\pm 1}]_{i \in I}$ we define

$$\operatorname{Log} P = \{ z \in G = \oplus_{i \in I} Z_i \mid \langle P, x^z \rangle \neq 0 \} .$$

In the univariate case where $P \in \mathbf{R}[x^{\pm 1}]$ this becomes $\operatorname{Log} P = \{ k \in \mathbf{Z} \mid \langle P, x^k \rangle \neq 0 \}$.

For any $P \in \mathbf{R}[x_i^{\pm 1}]_{i \in I}$ we may write $P = \sum_{z \in G} p_z x^z$, with each $p_z \in \mathbf{R}$, and this sum is unique, since necessarily $p_z = \langle P, x^z \rangle$ for all $z \in G$. Of course we also have $P = \sum\limits_{z \in \operatorname{Log} P} p_z x^z$.

We define the degree of a monomial $z = x_{i_1}^{z_1} x_{i_2}^{z_2} \cdots x_{i_k}^{z_k} \in G$, in which $i_1, i_2, \ldots, i_k$ are distinct, by

$$\deg\left(x_{i_1}^{z_1} x_{i_2}^{z_2} \cdots x_{i_k}^{z_k}\right) = |z_1| + |z_2| + \cdots + |z_k|,$$

and then define the *degree* of a non-zero Laurent polynomial $f \in \mathbf{R}[x_i^{\pm 1}]_{i \in I}$ to be

$$\deg f = \max_{z \in \operatorname{Log} f} \deg(x^z).$$

If $f, g \in \mathbf{R}[x_i^{\pm 1}]_{i \in I} \backslash \{0\}$, then

$$\deg(fg) \leq \deg f + \deg g. \tag{1.1}$$

We define the positive cone $\mathbf{R}[x_i^{\pm 1}]_{i \in I}^+$ of $\mathbf{R}[x_i^{\pm 1}]_{i \in I}$ by

$$\mathbf{R}[x_i^{\pm 1}]_{i \in I}^+ = \left\{ P \in \mathbf{R}[x_i^{\pm 1}]_{i \in I} \;\middle|\; \langle P, x^z \rangle \in \mathbf{R}^+, \forall z \in G \right\},$$

and this makes $\mathbf{R}[x_i^{\pm 1}]_{i \in I}$ into a partially ordered $\mathbf{R}$-algebra where we define $P \geq Q$ if and only if $P - Q \in \mathbf{R}[x_i^{\pm 1}]_{i \in I}^+$. The cone $\mathbf{R}[x_i^{\pm 1}]_{i \in I}^+$ is closed under addition, multiplication, and scalar multiplication by elements of $\mathbf{R}^+$.

When dealing with a sequence of polynomials $\{P_n\}_{n \in \mathbf{N}}$ in $\mathbf{R}[x_i^{\pm 1}]_{i \in I}$, we define, for $t \in \mathbf{Z}^+$ and $N \in \mathbf{N}$,

$$P^{N,t} = \prod_{j=t+1}^{t+N} P_j = P_{t+1} P_{t+2} \cdots P_{t+N}.$$

Definition: If $\{P_i\}_{i \in \mathbf{N}}$ is a sequence in $\mathbf{R}[x_i^{\pm 1}]_{i \in I}^+$ and we have a sequence $1 = N_o < N_1 < N_2 < \cdots$ in $\mathbf{N}$, then we may set

$$Q_j = \prod_{N_{j-1} \leq i < N_j} P_i = \prod_{i=N_{j-1}}^{N_j - 1} P_i, \qquad \text{for } j = 1, 2, 3, \ldots$$

The sequence $\{Q_i\}_{i \in \mathbf{N}}$ is called a *blocking* of $\{P_i\}_{i \in \mathbf{N}}$.

Laurent Polynomials in one Variable. In order to highlight the coefficients of a Laurent polynomial in $\mathbf{R}[x^{\pm 1}]$ we will sometimes use the notation $\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_k \\ \hline e_1 & e_2 & e_3 & \cdots & e_k \end{array}$ to denote the polynomial $P = \sum_{i=1}^{k} a_i x^{e_i}$. Here there is no stipulation that the e_i's be distinct, although normally they will be distinct and increasing. As an example, we could write $nx^{-3} + n^2 x^{-2} + 1 + 5nx^2$ as $\begin{array}{cccc} n & n^2 & 1 & 5n \\ \hline -3 & -2 & 0 & 2 \end{array}$. To emphasize the 'gaps' between coefficients we might also write this as $\begin{array}{cccccc} n & n^2 & & 1 & & 5n \\ \hline -3 & -2 & & 0 & & 2 \end{array}$.

Definition: Suppose $P = \sum_{i \in \mathbf{Z}} p_i x^i \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$. If $i, k \in \mathbf{Z}$ with $i \leq k$ and $p_{i-1} < p_i = p_{i+1} = \cdots = p_k > p_{k+1}$, then each of the numbers $i, i+1, \ldots k$ is called a *peak* of P. The consecutive integers $i, i+1, \ldots, k$ together constitute a *mode* of P. Each $P \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ has at least one mode, and each mode contains at least one peak. If P has only one mode, then it is said to be *unimodal*. By the *first peak* of P we will mean the smallest $i \in \mathbf{Z}$ which is a peak of P. Similarly, by the *last peak* of P we will mean the largest $i \in \mathbf{Z}$ which is a peak of P.

Definition: Suppose $P \in \mathbf{R}[x^{\pm 1}]$ and that there are integers $i, k \in \operatorname{Log} P$ such that $k - i > 1$ and $j \notin \operatorname{Log} P$ whenever $i < j < k$. In this situation the consecutive integers $i+1, i+2, \ldots, k-1$ together constitute a *gap* of P. The number $k - i + 1$ is the *size of the gap*. A polynomial that has no gaps is said to be *gapless*.

Strong Unimodality.

Definition: A polynomial $P = \sum_{i \in \mathbf{Z}} p_i x^i \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ is said to be *strongly unimodal* (or *log-concave*) if P is gapless and

$$p_i^2 \geq p_{i-1} p_{i+1}, \qquad \forall i \in \mathbf{Z}. \tag{1.2}$$

An immediate consequence of inequality (1.2) is that $\frac{p_i}{p_{i-1}} \geq \frac{p_{i+1}}{p_i}$ (provided $i-1$ and i are in $\operatorname{Log} P$). In particular, the sequence $\left\{\frac{p_{i+1}}{p_i}\right\}_{i \in \operatorname{Log} P}$ is a non-increasing sequence. Every strongly unimodal polynomial is unimodal.

Strongly unimodality plays a role in numerous problems arising in various areas of mathematics, as is evident in the survey paper [S]. The following result appears in numerous places ([BH, Proposition 1.5], [KG, Theorem 3], [S, Proposition 3]). An early version of this result, in the context of continuous distributions, appeared in [Ib].

Proposition 1.1: Let $P \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$. Then P is strongly unimodal if and only if PQ is unimodal for every unimodal $Q \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$.

An often-used consequence of the foregoing result is the fact that any product of strongly unimodal polynomials is strongly unimodal.

Strong Positivity.

Definition: A sequence $\{P_i\}_{i \in \mathbf{N}}$ in $\mathbf{R}[x^{\pm 1}]^+$ is said to be *strongly positive* if for every polynomial $f \in \mathbf{R}[x^{\pm 1}]$ which satisfies $f(s) > 0$ for all $s \in (0, \infty)$, and for every $t \in \mathbf{Z}^+$, there exists an n in $\mathbf{N}$ for which $f \cdot P^{n,t} \in \mathbf{R}[x^{\pm 1}]^+$. (Recall from page 6 that $P^{n,t} = \prod_{j=t+1}^{t+n} P_j$).

The next three results are routine and will be used extensively in chapter 5. Proposition 1.5 will also be used frequently to build up new strongly positive sequences from simpler ones.

Proposition 1.2: If $\{P_i\}_{i \in \mathbf{N}}$ is a sequence in $\mathbf{R}[x^{\pm 1}]^+$ having a subsequence which is strongly positive, then $\{P_i\}_{i \in \mathbf{N}}$ is itself strongly positive.

Proof: Let f be given as in the definition of strong positivity, and let I be an infinite subset of $\mathbf{N}$ for which $\{P_i\}_{i \in I}$ is a strongly positive sequence. Then for any $t \in \mathbf{Z}^+$, there is an $m > t$ such that

$$f \cdot \prod_{\substack{i \in I \\ t < i \leq m}} P_i \in \mathbf{R}[x^{\pm 1}]^+$$

Since each $P_i \in \mathbf{R}[x^{\pm 1}]^+$, and $\mathbf{R}[x^{\pm 1}]^+$ is closed under multiplication, it follows that

$$f \cdot P^{m-t,t} = f \prod_{t < i \leq m} P_i = \left[f \cdot \prod_{\substack{i \in I \\ t < i \leq m}} P_i \right] \left[\prod_{\substack{i \in \mathbf{N} \backslash I \\ t < i \leq m}} P_i \right] \in \mathbf{R}[x^{\pm 1}]^+$$

Taking $n = m - t$, we see that $f \cdot P^{n,t} \in \mathbf{R}[x^{\pm 1}]^+$. ∎

Proposition 1.3: Any reordering of a strongly positive sequence in $\mathbf{R}[x^{\pm 1}]^+$ is strongly positive.

Proof: Suppose $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive and $\phi : \mathbf{N} \to \mathbf{N}$ is a bijection. We show that $\{P_{\phi(i)}\}_{i\in\mathbf{N}}$ is strongly positive. Let $t \in \mathbf{Z}^+$ be given. It suffices to show that there is some $m > t$ such that $f \cdot \prod_{t<i\leq m} P_{\phi(i)} \in \mathbf{R}[x^{\pm 1}]^+$.

Let $t' = \max\{\, \phi^{-1}(i) \,|\, i \leq t \,\}$, and choose $n \in \mathbf{N}$ such that $f \cdot P^{n,t'} \in \mathbf{R}[x^{\pm 1}]^+$. Letting $I = \{\, \phi(i) \,|\, t' < i \leq t' + n \,\}$ and setting $m = \max I$, we note that

$$f \cdot \prod_{t<i\leq m} P_{\phi(i)} = f \cdot \left[\prod_{\substack{t<i\leq m \\ i\in I}} P_{\phi(i)} \right] \prod_{\substack{t<i\leq m \\ i\notin I}} P_{\phi(i)} = f \cdot P^{n,t'} \cdot \prod_{\substack{t<i\leq m \\ i\notin I}} P_{\phi(i)}$$

and this is clearly in $\mathbf{R}[x^{\pm 1}]^+$. ∎

Proposition 1.4: If $\{P_i\}_{i\in\mathbf{N}}$ is a strongly positive sequence in $\mathbf{R}[x^{\pm 1}]^+$ then the following sequences are also strongly positive:

(1) $\{P_iQ_i\}_{i\in\mathbf{N}}$, where $\{Q_i\}_{i\in\mathbf{N}}$ is any sequence in $\mathbf{R}[x^{\pm 1}]^+$.

(2) $\left\{P_i^{(r)}\right\}_{i\in\mathbf{N}}$, where $\left\langle P_i^{(r)}, x^j \right\rangle = \langle P_i, x^{-j}\rangle$, $\forall i \in \mathbf{N}$, $\forall j \in \mathbf{Z}$.

Proof: Given $f \in \mathbf{R}[x^{\pm 1}]$ with $f|_{(0,\infty)} > 0$, and $t \in \mathbf{Z}^+$, there is some $n \in \mathbf{N}$ for which $f \cdot P^{n,t} \in \mathbf{R}[x^{\pm 1}]^+$. It follows that $f \cdot \prod_{i=t+1}^{t+n} P_iQ_i = f \cdot P^{n,t}Q^{n,t} \in \mathbf{R}[x^{\pm 1}]^+$, and so (1) holds.

Define $f^{(r)} \in \mathbf{R}[x^{\pm 1}]$ by $\langle f^{(r)}, x^j\rangle = \langle f, x^{-j}\rangle$, $\forall j \in \mathbf{Z}$, and note that $f^{(r)}(s) = f(1/s) > 0$ for all $s \in (0,\infty)$. There is an $m \in \mathbf{N}$ for which $f^{(r)} \cdot P^{m,t} \in \mathbf{R}[x^{\pm 1}]^+$. Since the map $x \mapsto x^{-1}$ induces an automorphism on $\mathbf{R}[x^{\pm 1}]$ that carries $\mathbf{R}[x^{\pm 1}]^+$ into $\mathbf{R}[x^{\pm 1}]^+$, it follows that $f \cdot Q^{m,t}$ is in $\mathbf{R}[x^{\pm 1}]^+$, and (2) follows. ∎

The following result will be useful for making new strongly positive sequences from old ones. It (and its proof) is a slight simplification of Lemma 1.3 in [BH].

Proposition 1.5: Suppose that for each $i \in \mathbf{N}$, we have an $M(i) \in \mathbf{N}$ and Laurent polynomials $T_{j,i} \in \mathbf{R}[x^{\pm 1}]^+$ for $j = 0, 1, 2, \ldots M(i)$. For each $i \in \mathbf{N}$ let

$$P_i = \sum_{0\leq j\leq M(i)} T_{j,i}$$

Suppose also that whenever $k = (k(1), k(2), \ldots)$ is a sequence in $\mathbf{Z}^+$ with $0 \leq k(i) \leq M(i)$ for each i, then the sequence $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive. Then the sequence of sums $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive.

Proof: Let $f \in \mathbf{R}[x^{\pm 1}]$ be as in the definition of strong positivity and let $t \in \mathbf{Z}^+$ be given. For each $i \in \mathbf{N}$ define $K_i = \{0, 1, 2, \ldots, M(i)\}$, and let K denote the cartesian product of the K_i's, with the product topology (assuming the discrete topology on each K_i). By Tychonoff's theorem,

K is compact. Given k in K, there exists $n(k) \in \mathbf{N}$ such that $f \cdot \prod_{i=t+1}^{t+n(k)} T_{k(i),i} \in \mathbf{R}[x^{\pm 1}]^+$, and we form the clopen set

$$U_k = \{\, k' \in K \mid k'(j) = k(j) \text{ for all } j \leq n(k) \,\}.$$

The family $\{\, U_k \mid k \in K \,\}$ is an open covering of K. As K is compact, there is a finite set $\{k^{(1)}, k^{(2)}, \ldots, k^{(m)}\}$ of elements of K such that

$$\bigcup_{j=1}^{m} U_{k^{(j)}} = K.$$

Let $n = \max\{n(k^{(1)}), n(k^{(2)}), \ldots, n(k^{(m)})\}$. Then $f \cdot \prod_{i=t+1}^{t+n} T_{k(i),i} \in \mathbf{R}[x^{\pm 1}]^+$ for every k in K. It follows that $f \cdot P^{n,t}$, being a sum of such products, also lies in $\mathbf{R}[x^{\pm 1}]^+$. We have proved that $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive. ∎

Peaks and Ratios. In investigations of strong positivity and strong unimodality, it is frequently necessary to estimate the ratios of adjacent coefficients of a polynomial, especially near the peaks of the polynomial. Frequently such estimates involve products of linear polynomials. In this section we obtain some basic results about these ratios, and how they are affected by certain perturbations. While the proofs are elementary, the examples following Proposition 1.7 and Proposition 1.9 show that we must be careful in making our hypotheses.

Definition: If P and Q are in $\mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$, then Q is said to be a *right-positive perturbation* of P if $Q - P \in \mathbf{R}[x^{\pm 1}]^+$ and $\langle P, x^i \rangle = \langle Q, x^i \rangle$ for all $i < \max \operatorname{Log} P$.

Lemma 1.6: Suppose that P and R belong to $\mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$, R is strongly unimodal, and $d = \max \operatorname{Log} P$. Suppose also that $Q = P + \alpha x^t$ with $\alpha \geq 0$ and $t \geq d$. For any m in $\mathbf{Z}$ and θ in $(0, \infty)$, if $\langle PR, x^m \rangle < \theta \langle PR, x^{m+1} \rangle$, then $\langle QR, x^m \rangle < \theta \langle QR, x^{m+1} \rangle$.

Proof: We can write $P = \sum_{i \in \mathbf{Z}} p_i x^i$ and $R = \sum_{i \in \mathbf{Z}} r_i x^i$. Note that if $r_i > \theta r_{i+1}$ then $\frac{r_{i+1}}{r_i} < 1/\theta$, so by strong unimodality of R, $r_j \geq \theta r_{j+1}$ for all $j \geq i$. Suppose $\langle PR, x^m \rangle < \theta \langle PR, x^{m+1} \rangle$. Then it cannot be that $r_{m-d} > \theta r_{m-d+1}$, since then by strong unimodality of R we would have $r_{m-i} \geq \theta r_{m-i+1}$ whenever $i \leq d$, which would give $\langle PR, x^m \rangle = \sum_{i \leq d} p_i r_{m-i} \geq \theta \sum_{i \leq d} p_i r_{m-i+1} = \theta \langle PR, x^{m+1} \rangle$, which cannot be. Thus $r_{m-d} \leq \theta r_{m-d+1}$. Since $0 < \langle PR, x^{m+1} \rangle = \sum_{i \leq d} p_i r_{m-i+1}$, it follows that $r_j > 0$ for some $j \geq m - d + 1$. Thus, if $r_{m-d} = 0$, then $r_j = 0$ for all $j \leq m - d$. On the other hand, if $r_{m-d} \neq 0$, then strong unimodality of R yields that $r_j \leq \theta r_{j+1}$ for all $j \leq m - d$. Thus, in any case, as $t \geq d$, we have $r_{m-t} \leq \theta r_{m-t+1}$. This gives

$$\begin{aligned} \langle QR, x^m \rangle = \langle PR, x^m \rangle + \alpha r_{m-t} < \theta \langle PR, x^{m+1} \rangle + \alpha r_{m-t} &\leq \theta \langle PR, x^{m+1} \rangle + \theta \alpha r_{m-t+1} \\ &= \theta \langle QR, x^{m+1} \rangle, \end{aligned}$$

as required. ∎

Proposition 1.7: Let $P, Q, R \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ with R strongly unimodal and suppose Q is a right-positive perturbation of P. Then for $m \in \mathbf{Z}$ and $\theta \in (0, \infty)$,

$$\langle QR, x^m \rangle < \theta \langle QR, x^{m+1} \rangle \quad \text{whenever} \quad \langle PR, x^m \rangle < \theta \langle PR, x^{m+1} \rangle.$$

In particular, if $\langle PR, x^m \rangle \neq 0$, then $\frac{\langle PR, x^{m+1} \rangle}{\langle PR, x^m \rangle} \leq \frac{\langle QR, x^{m+1} \rangle}{\langle QR, x^m \rangle}$.

Proof: If $P = Q$ there is nothing to show. Assume $P \neq Q$. Let $\mathrm{Log}(Q-P) = \{i_1, i_2, \ldots, i_k\}$ where $i_1 < i_2 < \cdots < i_k$. For $j = 0, 1, 2, \cdots, k$, define $Q_j = P + \sum_{1 \le m \le j} \langle Q - P,\, x^{i_m} \rangle\, x^{i_m}$. Then $Q_0 = P$ and $Q_k = Q$. For $j = 1, 2, \ldots, k$ we can apply Lemma 1.5 to see that for $m \in \mathbf{Z}$ and $\theta \in (0, \infty)$, if $\langle Q_{j-1}R,\, x^m \rangle < \theta \langle Q_{j-1}R,\, x^{m+1} \rangle$, then $\langle Q_j R,\, x^m \rangle < \theta \langle Q_j R,\, x^{m+1} \rangle$. The desired inequality follows immediately.

If $\langle PR,\, x^m \rangle \neq 0$, then $\langle QR,\, x^m \rangle \neq 0$. Since, for $\rho \in \mathbf{R}$, $\rho < \frac{\langle PR,\, x^{m+1} \rangle}{\langle PR,\, x^m \rangle} \implies \rho < \frac{\langle QR,\, x^{m+1} \rangle}{\langle QR,\, x^m \rangle}$, it follows that $\frac{\langle PR,\, x^{m+1} \rangle}{\langle PR,\, x^m \rangle} \le \frac{\langle QR,\, x^{m+1} \rangle}{\langle QR,\, x^m \rangle}$. ∎

The strong unimodality of R in Lemma 1.6 and Proposition 1.7 cannot be weakened to require only unimodality. This may be seen by considering $R = 1 + x + 3x^2$ with $P = 1 + x$ and $Q = 1 + (1+\epsilon)x$, and noting that $\frac{\langle QR,\, x^2 \rangle}{\langle QR,\, x^1 \rangle} = \frac{4+\epsilon}{2+\epsilon} < \frac{4}{2} = \frac{\langle PR,\, x^2 \rangle}{\langle PR,\, x^1 \rangle}$. Another example, with a different perturbation, is obtained by taking $P = 1 + x$ and $Q = 1 + x + \epsilon x^2$, with $R = 1 + x + 3x^2 + 3x^3$ and noting that $\frac{\langle QR,\, x^3 \rangle}{\langle QR,\, x^2 \rangle} = \frac{6+\epsilon}{4+\epsilon} < \frac{6}{4} = \frac{\langle PR,\, x^3 \rangle}{\langle PR,\, x^2 \rangle}$.

If R is unimodal we can, however, show that the first peak (i.e. the leftmost peak) of PR can only move to the right when P is replaced by a right-positive pertubation of itself. This is the content of Proposition 1.9.

Lemma 1.8: Suppose $P, R \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$, R is unimodal, and $d = \max \mathrm{Log}\, P$. If $Q = P + \alpha x^t$ with $\alpha \ge 0$ and $t \ge d$, then the first peak of QR is greater than or equal to the first peak of PR.

Proof: We can write $P = \sum_{i \in \mathbf{Z}} p_i x^i$ and $R = \sum_{i \in \mathbf{Z}} r_i x^i$. Let p be the first peak of PR.

Suppose that $r_{m-d} > r_{m-d+1}$ for some $m < p$ (we will deduce a contradiction). This gives $p_d r_{m-d} > p_d r_{m-d+1}$, and by unimodality of R, $r_{m-i} \ge r_{m-i+1}$ whenever $i \le d$. This yields

$$\langle PR,\, x^m \rangle = \sum_{i \le d} p_i r_{m-i} > \sum_{i \le d} p_i r_{m-i+1} = \langle PR,\, x^{m+1} \rangle,$$

which cannot be, since p is the first peak of PR, and $m < p$. It follows that $r_{m-d} \le r_{m-d+1}$ for all $m < p$; that is, $r_i \le r_{i+1}$ whenever $i < p - d$.

Let $n < p$. As $t \ge d$, we have $n - t < p - d$, so $r_{n-t} \le r_{n-t+1}$. Thus

$$\begin{aligned} \langle QR,\, x^n \rangle = \langle PR,\, x^n \rangle + \alpha r_{n-t} &\le \langle PR,\, x^{n+1} \rangle + \alpha r_{n-t} \\ &\le \langle PR,\, x^{n+1} \rangle + \alpha r_{n-t+1} = \langle QR,\, x^{n+1} \rangle. \end{aligned}$$

Since $\langle PR,\, x^{p-1} \rangle < \langle PR,\, x^p \rangle$, we have $\langle QR,\, x^{p-1} \rangle < \langle QR,\, x^p \rangle$. It follows that QR has no peak less than p. ∎

Proposition 1.9: If $P, Q, R \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$, R is unimodal, and Q is a right-positive perturbation of P, then the first peak of QR is greater than or equal to the first peak of PR.

Proof: If $P = Q$ there is nothing to show. Assume $P \neq Q$. Let $\mathrm{Log}(Q-P) = \{i_1, i_2, \ldots, i_k\}$ where $i_1 < i_2 < \cdots < i_k$. For $j = 0, 1, 2, \ldots, k$, define $Q_j = P + \sum_{1 \le m \le j} \langle Q - P,\, x^{i_m} \rangle\, x^{i_m}$. Then $Q_0 = P$ and $Q_k = Q$. For $j = 1, 2, \ldots, k$ we can apply Lemma 1.8 to see that the first peak of $Q_j R$ is not less than the first peak of $Q_{j-1}R$. Thus the first peak of QR is not less than the first peak of PR. ∎

The unimodality of R cannot be dropped from the conditions of Lemma 1.8 or Proposition 1.9. For example, if $R = 1 + \frac{1}{2}x + x^2$, $P = 1 + \frac{1}{2}x$ and $Q = 1 + \frac{3}{2}x$, then the first peak of PR is 3, while the first peak of QR is 2.

Returning to the strongly unimodal case, we obtain the following elementary result. Recall from page 6 that $P^{n,0}$ denotes the product $P_1 P_2 \cdots P_n$.

Corollary 1.10: Let P_i and Q_i be strongly unimodal polynomials in $\mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ for $i = 1, 2, 3, \ldots, n$, and suppose that Q_i is a right-positive perturbation of P_i for each i. Then for each $m \in \operatorname{Log} P^{n,0}$,

$$\frac{\langle P^{n,0},\, x^{m+1}\rangle}{\langle P^{n,0},\, x^m\rangle} \le \frac{\langle Q^{n,0},\, x^{m+1}\rangle}{\langle Q^{n,0},\, x^m\rangle}.$$

Proof: For notational convenience, for $k = 1, 2, \ldots, n$, define $R^{(k)} = \left(\prod_{i=1}^{k-1} Q_i\right)\left(\prod_{i=k+1}^{n} P_i\right)$, where the empty product is 1 by convention. Note that each $R^{(k)}$ is strongly unimodal, and that $P^{n,0} = P_1 R^{(1)}$, and $Q^{n,0} = Q_n R^{(n)}$. Suppose $m \in \operatorname{Log} P^{n,0}$. Then $m \in \operatorname{Log}\left(P_k R^{(k)}\right)$ for $k = 1, 2, \ldots, n$, and by Proposition 1.7, we have

$$\frac{\langle P_k R^{(k)},\, x^{m+1}\rangle}{\langle P_k R^{(k)},\, x^m\rangle} \le \frac{\langle Q_k R^{(k)},\, x^{m+1}\rangle}{\langle Q_k R^{(k)},\, x^m\rangle}$$

for $k = 1, 2, \ldots, n$. Since $Q_k R^{(k)} = P_{k+1} R^{(k+1)}$ for $k = 1, 2, \ldots, n-1$, we have

$$\frac{\langle P_k R^{(k)},\, x^{m+1}\rangle}{\langle P_k R^{(k)},\, x^m\rangle} \le \frac{\langle P_{k+1} R^{(k+1)},\, x^{m+1}\rangle}{\langle P_{k+1} R^{(k+1)},\, x^m\rangle}$$

for $k = 1, 2, \ldots, n-1$. Combining these inequalities we obtain

$$\frac{\langle P^{n,0},\, x^{m+1}\rangle}{\langle P^{n,0},\, x^m\rangle} = \frac{\langle P_1 R^{(1)},\, x^{m+1}\rangle}{\langle P_1 R^{(1)},\, x^m\rangle} \le \frac{\langle P_n R^{(n)},\, x^{m+1}\rangle}{\langle P_n R^{(n)},\, x^m\rangle} \le \frac{\langle Q_n R^{(n)},\, x^{m+1}\rangle}{\langle Q_n R^{(n)},\, x^m\rangle} = \frac{\langle Q^{n,0},\, x^{m+1}\rangle}{\langle Q^{n,0},\, x^m\rangle}$$

■

The upper bound on ratios provided by Proposition 1.11 is sharp in the case that all the β_i's are equal, where it gives essentially the ratio of successive binomial coefficients. Among other things, it could be applied to estimate the ratios of Stirling numbers of the second kind. If all of the β_i's lie in a common interval $[b, B]$, with $b > 0$, similar ratio estimates (and bounds) can be developed by careful use of the analytical methods of Chapter 7.

Proposition 1.11: Suppose $\beta_1, \beta_2, \ldots, \beta_n \in (0, \infty)$. Let $P^{(n)} = \prod_{i=1}^{n}(1 + \beta_i x)$ and let $S_n = \sum_{i=1}^{n} \beta_i$. Then

$$\frac{\langle P^{(n)},\, x^{k+1}\rangle}{\langle P^{(n)},\, x^k\rangle} \le \frac{S_n}{k+1}\,\frac{n-k}{n}, \qquad \text{for } k = 0, 1, \ldots, n.$$

This inequality becomes an equality if all the β_i's are equal.

Proof: The method of proof employed here was suggested by Dr. A. Joffe, and is shorter than my original argument. Writing $P^{(n)} = \sum_{k=1}^{n} \binom{n}{k} a_k x^k$, so that $a_k = \frac{\langle P^{(n)},\, x^k\rangle}{\binom{n}{k}}$, for each k, we have the inequality

$$a_{k-1} a_{k+1} \le a_k^2, \qquad k = 1, 2, \ldots, n-1,$$

from page 504 of [S] (or equivalently, equation (6) on page 11 of [Be]). This yields $\frac{a_{k+1}}{a_k} \le \frac{a_k}{a_{k-1}}$. Applying this inequality repeatedly we obtain $\frac{a_{k+1}}{a_k} \le \frac{a_k}{a_{k-1}} \le \cdots \le \frac{a_1}{a_0} = \frac{S_n/n}{1} = \frac{S_n}{n}$. That is,

$$\frac{\langle P^{(n)}, x^{k+1}\rangle}{\langle P^{(n)}, x^k\rangle} \frac{\binom{n}{k}}{\binom{n}{k+1}} \le \frac{S_n}{n}.$$

This yields

$$\frac{\langle P^{(n)}, x^{k+1}\rangle}{\langle P^{(n)}, x^k\rangle} \le \frac{S_n}{n} \frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{S_n}{n} \frac{n-k}{k+1} = \frac{S_n}{k+1} \frac{n-k}{n}.$$

If all the β_i's are equal, say $\beta_i = \beta$ for $i = 1, 2, \ldots, n$, then $a_k = \beta^k$ for each k, and all of our inequalities become equalities. ∎

Proposition 1.11 showed that for a fixed value of $\sum_{i=1}^n \beta_i$, the ratios of successive coefficients of $\prod_{i=1}^n (1+\beta_i x)$ are greatest when all the β_i's are equal. An immediate consequence of this is the fact that choosing all of the β_i's equal will also maximize the coefficients themselves. In Proposition 1.12 we also develop a lower bound on the coefficients $\langle \prod_{i=1}^n (1+\beta_i x), x^k\rangle$, at least for values of k which are small relative to $\sum_{i=1}^n \beta_i$. This lower bound will be needed in chapter 7 (in the proof of Lemma 7.3). Using the lower and upper bounds on the coefficients together, one can also obtain a lower bound on the ratios of coefficients.

Proposition 1.12: Suppose $\beta_1, \beta_2, \ldots, \beta_n$ are numbers in $(0, \infty)$, and let $S_n = \sum_{i=1}^n \beta_i$. Then for $k = 0, 1, 2, \ldots, n$,

$$\left\langle \prod_{i=1}^n (1+\beta_i x), x^k \right\rangle \le \binom{n}{k}\left(\frac{S_n}{n}\right)^k \le \frac{1}{k!} S_n^k. \tag{1.3}$$

For $k \ge 2$ we also have

$$\frac{1}{k!}\left[S_n^k - \binom{k}{2}\left(\sum \beta_i^2\right) S_n^{k-2}\right] \le \left\langle \prod_{i=1}^n (1+\beta_i x), x^k \right\rangle. \tag{1.4}$$

Proof: For $k = 0, 1, 2, \ldots, n$, let $\rho_k = \langle \prod_{i=1}^n (1+\beta_i x), x^k\rangle$. Equation (1.3) clearly holds for $k = 0$ and $k = 1$. Assume $k \ge 2$. Noting that $\rho_0 = 1$, we use Proposition 1.11 to find that

$$\rho_k = \prod_{i=0}^{k-1} \frac{\rho_{i+1}}{\rho_i} \le \prod_{i=0}^{k-1} \frac{S_n}{i+1} \frac{n-i}{n} = S_n^k \cdot \frac{1}{k!} \cdot \frac{1}{n^k} \cdot \frac{n!}{(n-k)!} = \left(\frac{S_n}{n}\right)^k \binom{n}{k}$$

As $\frac{1}{n^k} \cdot \frac{n!}{(n-k)!} \le 1$, we also deduce that $\binom{n}{k}\left(\frac{S_n}{n}\right)^k \le \frac{S_n^k}{k!}$.

If we set $I_n = \{1, 2, \ldots, n\}$, we see that

$$\rho_k = \sum_{\substack{K \subseteq I_n \\ |K| = k}} \prod_{i \in K} \beta_i = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}.$$

We compare this to the expression $E = S_n^k = \left(\sum_{i=1}^n \beta_i\right)^k$. Note that E, when multiplied out, has n^k terms. Not all of these terms have distinct indices, but each set of distinct indices occurs $k!$ times. Thus $\rho_k \le \frac{1}{k!} S_n^k$. We wish to remove from E all terms having a repeated index. Counting such terms, we obtain $\binom{k}{2}\left(\sum \beta_i^2\right)\left(\sum \beta_i\right)^{k-2}$, where some terms have possibly been counted more than once (those with more than a single repeated index). The sum of the terms of E with distinct indices is thus at least $E - \binom{k}{2}\left(\sum \beta_i^2\right)\left(\sum \beta_i\right)^{k-2}$. Hence

$$\rho_k \geq \frac{1}{k!}\left[E - \binom{k}{2}\left(\sum \beta_i^2\right)\left(\sum \beta_i\right)^{k-2}\right] = \frac{1}{k!}\left[S_n^k - \binom{k}{2}\left(\sum \beta_i^2\right) S_n^{k-2}\right]$$

∎

2. Fluctuation and Persistence

In this chapter we introduce the notion of fluctuation of a Laurent polynomial, and the companion notion of persistence. In its simplest form, the fluctuation, $\mathcal{F}(P)$ of a Laurent polynomial P is a number which measures how the differences between adjacent coefficients of P compare to the total mass of P. While fluctuation looks at the differences between adjacent coefficients, persistence $\mathcal{P}(P)$ measures the mass which adjacent coefficients have in common.

As polynomials P_i are multiplied into a product $P_1 P_2 \cdots P_n$, the persistence of the product grows, and the fluctuation decreases. It turns out to be important to know whether, for given k, the fluctuation of the product $P^{n,k} = P_{k+1}P_{k+2}\cdots P_{k+n}$ decreases to zero as n tends to infinity. This is a necessary condition for the sequence $\{P_i\}$ to be strongly positive (see Theorems 2.1 and 2.5 in [BH]; we also prove this in Corollary 4.22).

In [BH] it is shown that the problem of deciding whether the fluctuation $\mathcal{F}(P^{n,k})$ tends to zero (for all k) is equivalent to answering a question about the states on a particular dimension group associated with the sequence $\{P_i\}$. Specifically, the fluctuation tends to zero if and only if point evaluation at 1 is a pure (extremal) state.

The fluctuation of positive Laurent polynomials in a single indeterminate is considered in [BH]. In [MH] this is generalized to the case of polynomials in a finite number of indeterminates. We introduce a definition of fluctuation (and persistence) which is generalized in three ways. Most importantly we define fluctuation over various step sizes (or directions) rather than considering only adjacent coefficients. Fluctuation over such step sizes turns out to be the crucial ingredient needed for the new fluctuation results of chapter 3, such as Corollary 3.6 and Theorem 3.9. Our definition of fluctuation also covers polynomials with negative coefficients and allows an infinite number of indeterminates. While these two aspects do not add anything new to our knowledge of the single-variable positive case, they do provide a few interesting features which shall be commented upon in due course.

As already mentioned, when polynomials P_i are multiplied into a product $P_1P_2\cdots P_n$, the persistence accumulates, reducing the fluctuation of the product. A basic result about this accumulation is the fact that the condition $\sum_i \mathcal{P}(P_i) = \infty$ is sufficient to guarantee that the fluctuation of $P^{n,k}$ tends to 0 as n grows. This is known as Mineka's theorem [BH, theorem 1.6]. A proof of it in the case of finitely many variables is the subject of [MH]. We extend this proof to obtain two corresponding results in our more general setting (Theorem 2.6 and Corollary 2.8). We also derive a bound on how quickly the fluctuation decays, as hinted at in [MH]. By considering blocking of a sequence into products, we obtain a necessary and sufficient condition for fluctuation to go to zero (Theorem 2.10), and this leads to a $0-2$ law for fluctuation (Corollary 2.11).

Throughout this chapter $\mathbf{R}[x_i^{\pm 1}]_{i\in I}$ will be the algebra of Laurent polynomials over $\mathbf{R}$ with indeterminates indexed by some nonvoid set I, and G will denote the direct sum $\oplus_{i\in I}\mathbf{Z}$, which is the group of allowed exponents of x for monomials in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}$. Readers not interested in fluctuation in higher dimensions will lose little by replacing $\mathbf{R}[x_i^{\pm 1}]_{i\in I}$ by $\mathbf{R}[x^{\pm 1}]$ throughout and taking $G = \mathbf{Z}$.

First we make the Laurent polynomial algebra into a normed algebra by introducing a norm which is the restriction of the ℓ^1 norm on $\mathbf{Z}^I$. Define a function $\|\cdot\| : \mathbf{R}[x_i^{\pm 1}]_{i\in I} \to \mathbf{R}$ by

$$\left\|\sum_{z\in G}\alpha_z x^z\right\| = \sum_{z\in G}|\alpha_z|.$$

Proposition 2.1: Let $f, g \in \mathbf{R}[x_i^{\pm 1}]_{i \in I}$ and $\alpha \in \mathbf{R}$. Then

(1) $\|f\| \geq 0$.
(2) $\|f\| = 0 \iff f = 0$.
(3) $\|\alpha f\| = |\alpha| \, \|f\|$.
(4) $\|f + g\| \leq \|f\| + \|g\|$.
(5) $\|fg\| \leq \|f\| \, \|g\|$.
(6) $\|x^z f\| = \|f\|, \ \forall z \in G$.
(7) $\|\mathbf{1}\| = 1$.

In particular, $\mathbf{R}[x_i^{\pm 1}]_{i \in I}$ is a normed $\mathbf{R}$-algebra. For elements of $\mathbf{R}[x_i^{\pm 1}]_{i \in I}^{+}$ we have more:

(8) $\|f + g\| = \|f\| + \|g\|$, if $f, g \geq 0$.
(9) $\|fg\| = \|f\| \cdot \|g\|$, if $f, g \geq 0$.

Proof: Write $f = \sum_{z \in G} f_z x^z$ and $g = \sum_{z \in G} g_z x^z$, where $G = \bigoplus_{i \in I} \mathbf{Z}$. Then

(1): $|f_z| \geq 0, \ \forall z \in G \implies \|f\| = \sum_{z \in G} |f_z| \geq 0$.

(2): $\|f\| = 0 \implies \sum_{z \in G} |f_z| \leq 0 \implies |f_z| = 0, \ \forall z \in G \implies f_z = 0, \ \forall z \in G \implies f = 0$.
$f = 0 \implies f_z = 0, \forall z \in G \implies \|f\| = \sum_{z \in G} 0 = 0$.

(3): $\|\alpha f\| = \|\sum_{z \in G} \alpha f_z x^z\| = \sum_{z \in G} |\alpha f_z| = \sum_{z \in G} |\alpha| \, |f_z| = |\alpha| \sum_{z \in G} |f_z| = |\alpha| \, \|f\|$.

(4): $\|f + g\| = \|\sum_{z \in G} (f_z + g_z) x^z\| = \sum_{z \in G} |f_z + g_z| \leq \sum_{z \in G} |f_z| + |g_z| = \sum_{z \in G} |f_z| + \sum_{z \in G} |g_z| = \|f\| + \|g\|$.

(5): $\|fg\| = \left\|\sum_{w,z \in G} f_w g_z \cdot x^{w+z}\right\| = \|\sum_{v \in G} \left(\sum_{w \in G} f_w g_{v-w}\right) x^v\| = \sum_{v \in G} \left|\sum_{w \in G} f_w g_{v-w}\right|$
$\leq \sum_{v \in G} \sum_{w \in G} |f_w g_{v-w}| = \sum_{v \in G} \sum_{w \in G} |f_w| \, |g_{v-w}| = \sum_{w \in G} |f_w| \sum_{v \in G} |g_{v-w}|$
$= \sum_{w \in G} |f_w| \sum_{v \in G} |g_v| = \|f\| \, \|g\|$.

(6): $\|x^z f\| = \|\sum_{w \in G} f_w x^{w+z}\| = \sum_{w \in G} |f_w| = \|f\|$.

If $f, g \in \mathbf{R}[x_i^{\pm 1}]_{i \in I}^{+}$, then each f_z and g_z is in $\mathbf{R}^{+}$, and so we obtain (8):

$$\|f + g\| = \left\|\sum_{z \in G} (f_z + g_z) x^z\right\| = \sum_{z \in G} |f_z + g_z| = \sum_{z \in G} |f_z| + |g_z| = \sum_{z \in G} |f_z| + \sum_{z \in G} |g_z| = \|f\| + \|g\|,$$

and applying this we get (9):

$$\|fg\| = \left\|\sum_{w,z \in G} f_w g_z \cdot x^{w+z}\right\| = \left\|\sum_{w \in G} f_w \sum_{z \in G} g_z \cdot x^{w+z}\right\| = \sum_{w \in G} \left\|f_w \sum_{z \in G} g_z \cdot x^{w+z}\right\|$$

$$= \sum_{w \in G} |f_w| \left\|\sum_{z \in G} g_z \cdot x^{w+z}\right\| = \sum_{w \in G} |f_w| \sum_{z \in G} |g_z| = \|f\| \, \|g\|$$

∎

Definition: If I is a nonvoid set, $\mathbf{R}[x_i^{\pm 1}]_{i\in I}$ the corresponding Laurent polynomial algebra, and $G = \oplus_{i\in I}\mathbf{Z}$, then for $w \in G$ and $f \in \mathbf{R}[x_i^{\pm 1}]_{i\in I}$ we define the *fluctuation of f in the direction w*, denoted $\mathcal{F}_w(f)$, by

$$\mathcal{F}_w(f) = \begin{cases} \frac{\|(1-x^w)f\|}{\|f\|} & \text{if } f \neq 0 \\ 0 & \text{if } f = 0 \end{cases}$$

and the *persistence of f in the direction w*, denoted $\mathcal{P}_w(f)$, by

$$\mathcal{P}_w(f) = 1 - \frac{\mathcal{F}_w(f)}{2}. \tag{2.1}$$

We will also have occasion to use the *unnormalized fluctuation* $\mathcal{F}_w^*(f)$, defined by

$$\mathcal{F}_w^*(f) = \|(1 - x^w)f\|.$$

In the special case where I is a singleton and $w = 1$, we will write $\mathcal{F}(f)$ for $\mathcal{F}_w(f)$ and $\mathcal{P}(f)$ for $\mathcal{P}_w(f)$, and call them simply the *fluctuation* and *persistence* of f.

Definitions of fluctuation in more restricted settings have been given in [BH] and [MH]. In [BH] (p.12) the fluctuation $\mathcal{F}(f)$ of a positive Laurent polynomial in a single indeterminate is defined as above (w=1). In [MH] fluctuation is defined for positive Laurent polynomials in finitely many indeterminates for the case where $x^w = x_i^1$. In [MH] fluctuation is only used in the situation where $\|f\| = 1$, and there it coincides with both $\mathcal{F}^*$ and $\mathcal{F}$ above.

The generality of the direction of w in our definition turns out to be very useful and yields results for fluctuation in the basic directions, even in the single variable case (compare Example 3.3 and Example 3.8).

In most cases results for $\mathcal{F}^*$ can be reformulated in terms of $\mathcal{F}$, and *vice versa*. A notable exception occurs when dealing with products of polynomials involving both positive and negative coefficients. This difference is highlighted by Example 2.7, and the differences between Theorem 2.6 and Corollary 2.8. In most cases we prefer to deal with the normalized fluctuation $\mathcal{F}_w(f)$.

Finally, Example 3.13 gives a non-trivial example involving fluctuation results for polynomials in countably many indeterminates.

Properties of Fluctuation: Let I be a non-void set. Then for $f, g \in \mathbf{R}[x_i^{\pm 1}]_{i\in I}$, $w \in G = \oplus_{i\in I}\mathbf{Z}$, and $\alpha \in \mathbf{R}\backslash\{0\}$, we have

(0) $0 \leq \mathcal{F}_w(f) \leq 2$.

(1) $\mathcal{F}_w(f+g) \leq \max(\mathcal{F}_w(f), \mathcal{F}_w(g))$, if $f, g \geq 0$.

(2) $\mathcal{F}_w(\alpha f) = \mathcal{F}_w(f)$.

(3) $\mathcal{F}_w(x^u f) = \mathcal{F}_w(f)$, $\forall u \in G$.

(4) $\mathcal{F}_w(fg) \leq \min(\mathcal{F}_w(f), \mathcal{F}_w(g))$, if $f, g \geq 0$.

(5) If $\langle f, x^z\rangle = \langle g, x^{-z}\rangle$, $\forall z \in G$, then $\mathcal{F}_w(f) = \mathcal{F}_w(g)$.

(6) $\mathcal{F}_{-w}(f) = \mathcal{F}_w(f)$.

(7) $\mathcal{F}_w(f) = \frac{\sum_{z\in G} |\langle f, x^z\rangle - \langle f, x^{z+w}\rangle|}{\|f\|}$, if $f \neq 0$.

Proof: Set $f_v = \langle f, x^v\rangle$ and $g_v = \langle g, x^v\rangle$ for all $v \in G$. We use Proposition 2.1 extensively:

(0): $0 \le \|(1 - x^w)f\| \le \|1 - x^w\|\|f\| \le (\|1\| + \|x^w\|)\|f\| = 2\|f\|$.

(2): $\frac{\|(1-x^w)\alpha f\|}{\|\alpha f\|} = \frac{|\alpha|\,\|(1-x^w)f\|}{|\alpha|\,\|f\|} = \frac{\|(1-x^w)f\|}{\|f\|}$.

(3): $\frac{\|(1-x^w)x^u f\|}{\|x^u f\|} = \frac{\|(1-x^w)f\|}{\|f\|}$.

(6): $\|(1 - x^{-w})f\| = \|x^w(1 - x^{-w})f\| = \|(x^w - 1)f\| = \|(1 - x^w)f\|$.

(5): There is an automorphism, h, of the algebra $\mathbf{R}[x_i^{\pm 1}]_{i\in I}$ sending x^u into x^{-u} for every $u \in G$. It is clear, from the definition of the norm in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}$, that h is norm-preserving. Note that $h(f) = g$, and that $h(1 - x^w) = 1 - x^{-w}$. Thus $h((1 - x^w)f) = (1 - x^{-w})g$, so $\|(1 - x^w)f\| = \|(1 - x^{-w})g\|$. Since $\|g\| = \|h(f)\| = \|f\|$, we have, provided $f \neq 0$, that

$$\frac{\|(1 - x^w)f\|}{\|f\|} = \frac{\|(1 - x^{-w})g\|}{\|g\|}.$$

Thus $\mathcal{F}_w(f) = \mathcal{F}_{-w}(g)$. By property (6), $\mathcal{F}_{-w}(g) = \mathcal{F}_w(g)$.

(7): $\|(1 - x^w)f\| = \|\sum_{z\in G} f_z x^z - \sum_{v\in G} f_v x^{v+w}\| = \|\sum_{z\in G} f_z x^z - \sum_{z\in G} f_{z-w} x^z\|$
$= \|\sum_{z\in G}(f_z - f_{z-w})x^z\| = \sum_{z\in G} |f_z - f_{z-w}| = \sum_{z\in G} |f_{z-w} - f_z|$
$= \sum_{z\in G} |f_z - f_{z+w}|$.

We now assume $f, g \in \mathbf{R}[x_i^{\pm 1}]^+_{i\in I}$ and prove (1) and (4). These clearly hold if either f or g is zero, so we may assume $f \neq 0 \neq g$. We use the fact that $\|f + g\| = \|f\| + \|g\|$ to prove (1)

$$\mathcal{F}_w(f + g) = \frac{\|(1 - x^w)(f + g)\|}{\|f + g\|} = \frac{\|(1 - x^w)f + (1 - x^w)g\|}{\|f\| + \|g\|} \le \frac{\|(1 - x^w)f\| + \|(1 - x^w)g\|}{\|f\| + \|g\|}$$
$$\le \max\left(\mathcal{F}_w(f), \mathcal{F}_w(g)\right).$$

Using this and properties (2) and (3) we prove property (4):

$$\mathcal{F}_w(fg) = \mathcal{F}_w\left(f \sum_{u\in G} g_u x^u\right) = \mathcal{F}_w\left(\sum_{u\in G} f g_u x^u\right) \le \max_{u\in G} \mathcal{F}_w(f g_u x^u) = \max_{u\in G} \mathcal{F}_w(f) = \mathcal{F}_w(f)$$

Likewise, $\mathcal{F}_w(fg) \le \mathcal{F}_w(g)$, so $\mathcal{F}_w(fg) \le \min(\mathcal{F}_w(f), \mathcal{F}_w(g))$. ∎

Remark: That property 1 of fluctuation would not be true without the condition $g \geq 0$ is clear from considering $f = 1 + x + x^2$ and $g = -x - x^2 - x^3$, and noting that $\mathcal{F}(f) = \frac{2}{3} = \mathcal{F}(g)$, while $\mathcal{F}(f + g) = 2$. That the requirement $g \geq 0$ is necessary for property 4 of fluctuation is seen by considering $f = 1 + x$ and $g = 1 - x - x^2$, and noting that $fg = 1 - 2x^2 - x^3$, so that $\mathcal{F}(fg) = \frac{3}{2}$, while $\mathcal{F}(f) = 1$ and $\mathcal{F}(g) = \frac{4}{3}$.

We now present a formula for computing persistence. It will be used repeatedly, especially in giving examples.

Proposition 2.2: If $f = \sum_{z\in G} f_z x^z \in \mathbf{R}[x_i^{\pm 1}]_{i\in I}$ and $w \in G$, then

$$\|f\|\,\mathcal{P}_w(f) = \sum_{z\in G} \chi^+(f_z f_{z+w}) \min\left(|f_z|, |f_{z+w}|\right),$$

where we define, for $\alpha \in \mathbf{R}$, $\chi^+(\alpha) = \begin{cases} 1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha \le 0 \end{cases}$.

If $f > 0$, this simplifies to

$$\mathcal{P}_w(f) = \frac{\sum_{z\in G} \min\left(f_z, f_{z+w}\right)}{\|f\|}. \tag{2.2}$$

Proof: If $f = 0$ the result is trivial. Assume $f \neq 0$. Write $f = \sum_{z \in G} f_z x^z$. Noting that for every $r, s \in \mathbf{R}^+$ we have $\max(r, s) = \min(r, s) + |r - s|$, we get

$$r + s = \min(r, s) + \max(r, s) = 2 \min(r, s) + |r - s|. \tag{2.3}$$

If $a, b \in \mathbf{R}$ satisfy $\chi^+(ab) = 1$, then they have the same sign, and so $\big| |a| - |b| \big| = |a - b|$. Taking $r = |a|$ and $s = |b|$, (2.3) yields

$$|a| + |b| = 2 \min(|a|, |b|) + \big| |a| - |b| \big| = 2 \min(|a|, |b|) + |a - b|$$

On the other hand, if $\chi^+(ab) = 0$, then $|a - b| = |a| + |b|$. We conclude that for any $a, b \in \mathbf{R}$,

$$|a| + |b| = 2\chi^+(ab) \min(|a|, |b|) + |a - b|$$

This yields

$$\begin{aligned} 2\,\|f\| = \sum_{z \in G} |f_z| + |f_{z+w}| &= \sum_{z \in G} 2\chi^+(f_z f_{z+w}) \min(|f_z|, |f_{z+w}|) + |f_z - f_{z+w}| \\ &= 2 \sum_{z \in G} \chi^+(f_z f_{z+w}) \min(|f_z|, |f_{z+w}|) + \sum_{z \in G} |f_z - f_{z+w}| \end{aligned}$$

By property 7 of fluctuation, $\sum_{z \in G} |f_z - f_{z+w}| = \|f\|\, \mathcal{F}_w(f)$, and so

$$2 \sum_{z \in G} \chi^+(f_z f_{z+w}) \min(|f_z|, |f_{z+w}|) = 2\|f\| - \|f\|\, \mathcal{F}_w(f) = \|f\| \big(2 - \mathcal{F}_w(f)\big) = 2\|f\|\, \mathcal{P}_w(f)$$

∎

Properties of Persistence: Let I be a non-void set. Then taking $f, g \in \mathbf{R}[x_i^{\pm 1}]_{i \in I}$, $w \in G = \oplus_{i \in I} \mathbf{Z}$, and $\alpha \in \mathbf{R} \backslash \{0\}$, we have

(0) $0 \le \mathcal{P}_w(f) \le 1$.

(1) $\mathcal{P}_w(f + g) \ge \min(\mathcal{P}_w(f), \mathcal{P}_w(g))$, if $f, g \ge 0$.

(2) $\mathcal{P}_w(\alpha f) = \mathcal{P}_w(f)$.

(3) $\mathcal{P}_w(x^u f) = \mathcal{P}_w(f)$, $\forall u \in G$.

(4) $\mathcal{P}_w(fg) \ge \max(\mathcal{P}_w(f), \mathcal{P}_w(g))$, if $f, g \ge 0$.

(5) $\mathcal{P}_w(f + g) \ge \mathcal{P}_w(f) \frac{\|f\|}{\|f\| + \|g\|}$, if $f, g \ge 0$, not both zero.

(6) $\mathcal{P}_w(f) \ge \mathcal{P}_w(f + g) - 2\frac{\|g\|}{\|f\|}$, if $f > 0$ and $g \ge 0$.

(7) $\mathcal{P}_{-w}(f) = \mathcal{P}_w(f)$.

(8) If $\langle f,\, x^z \rangle = \langle g,\, x^{-z} \rangle$, $\forall z \in G$, then $\mathcal{P}_w(f) = \mathcal{P}_w(g)$.

Proof: The results are clear when $f = 0$ or $g = 0$, so we assume both are nonzero. Set $f_v = \langle f,\, x^v \rangle$ and $g_v = \langle g,\, x^v \rangle$ for all $v \in G$.

Properties 0, 2, 3, 7 and 8 follow immediately from equation (2.1) and properties of fluctuation.

To prove property 1, we use property 1 of fluctuation: $\mathcal{P}_w(f + g) = 1 - \frac{1}{2}\mathcal{F}_w(f + g) \ge 1 - \frac{1}{2}\max\big(\mathcal{F}_w(f), \mathcal{F}_w(g)\big) = \min\big(\mathcal{P}_w(f), \mathcal{P}_w(g)\big)$. Similarly, property 4 follows from property 4 of fluctuation. To prove (5), note that

$$\begin{aligned} \|f + g\|\, \mathcal{P}_w(f + g) &= \sum_{z \in G} \min(|f_z + g_z|, |f_{z+w} + g_{z+w}|) \\ &= \sum_{z \in G} \min(f_z + g_z,\, f_{z+w} + g_{z+w}) \ge \sum_{z \in G} \min(f_z,\, f_{z+w}) = \mathcal{P}_w(f)\, \|f\|. \end{aligned}$$

Combining this together with the fact that $\|f+g\| = \|f\| + \|g\|$ completes the proof of (5).
To prove (6), note that for every $z \in G$,

$$\begin{aligned}
&\min(|f_z+g_z|, |f_{z+w}+g_{z+w}|) - \min(|f_z|, |f_{z+w}|) \\
&\quad = \min(|f_z|+|g_z|, |f_{z+w}|+|g_{z+w}|) - \min(|f_z|, |f_{z+w}|) \\
&\quad \le \max(|g_z|, |g_{z+w}|) \le |g_z| + |g_{z+w}|
\end{aligned}$$

Summing this inequality over all $z \in G$ we find that

$$\|f+g\|\,\mathcal{P}_w(f+g) - \|f\|\,\mathcal{P}_w(f) \le 2\|g\|.$$

As $\|f+g\| = \|f\| + \|g\| \ge \|f\|$, we have $\|f\|\,\mathcal{P}_w(f+g) - \|f\|\,\mathcal{P}_w(f) \le 2\|g\|$, so $\mathcal{P}_w(f+g) - 2\frac{\|g\|}{\|f\|} \le \mathcal{P}_w(f)$. ∎

Fluctuation of Sequences. We note that for any sequence $\{P_i\}_{i\in\mathbf{N}}$ in $\mathbf{R}[x_i^{\pm 1}]^+_{i\in I}$, any $w \in G$ and any $t \in \mathbf{Z}^+$, the sequence $\{\mathcal{F}_w(P^{N,t})\}_{N\in\mathbf{N}}$ is (by property 4 of fluctuation) a non-increasing sequence in $[0,2]$. We wish to know under what circumstances the limit of this sequence is 0. An important result in this direction is given in Theorem 2.6, but first we introduce four results that will be needed for its proof.

The upper bound in the following lemma is not particularly sharp, but gets the job done (and is true for $k=3$). Its proof is in the Appendix.

Lemma A.1: For every $k \in \mathbf{N}$,

$$\binom{k}{\lfloor k/2 \rfloor} \le \frac{2^{k+1}}{\sqrt{k\pi}},$$

where $\lfloor k/2 \rfloor$ is the largest integer less than or equal to $k/2$.

The following result and its proof are based on an argument on page 613 of [McD].

Proposition 2.3: Let $\beta_1, \beta_2, \ldots, \beta_n \in [0,1]$, and for $i = 1, 2, \ldots, n$ define $Q_i \in \mathbf{R}[x^{\pm 1}]^+$ by $Q_i = (1-\beta_i) + \beta_i x$. If $B = \sum_{i=1}^n \beta_i > 0$, then

$$\sum_{i \le B-r} \langle Q_1 Q_2 \cdots Q_n, x^i \rangle \le e^{-\frac{r^2}{2B}}, \quad \forall r \in [0,\infty).$$

Proof: Note that if any of the β_i's is zero, excluding Q_i and β_i does not change the inequality to be proved. Thus we may assume that all β_i's are non-zero.

It is easy to show that $1+t \le e^t \le 1+t+t^2/2$ for all $t \in (-\infty, 0]$. Using this fact, we have, for $t \in (-\infty, 0]$,

$$\begin{aligned}
e^{-\beta_i t} Q_i(e^t) &= e^{-\beta_i t}\big[(1-\beta_i) + \beta_i e^t\big] = e^{-\beta_i t}\big[1 + \big(-\beta_i(1-e^t)\big)\big] \\
&\le e^{-\beta_i t} e^{-\beta_i(1-e^t)} = e^{\beta_i(e^t-(1+t))} \\
&\le e^{\beta_i t^2/2}.
\end{aligned}$$

We may apply Theorem A.3 with $S = -\infty$, $T = 0$ and $\alpha_j = \beta_j$ for $j = 1, 2, \ldots, n$, to deduce that

$$\sum_{i \le r+B} \langle Q_1 Q_2 \cdots Q_n, x^i \rangle \le e^{-r^2/2B}, \quad \forall r \in (-\infty, 0],$$

which yields the required result. ∎

The fluctuation of a unimodal polynomial is particularly simple to calculate:

Lemma 2.4: If $f \in \mathbf{R}[x_i^{\pm 1}]_{i\in I}^{+}\backslash\{0\}$ is such that $f = x^v \sum_{k\in\mathbf{Z}} a_k x^{kw}$ for some unimodal sequence $\{a_k\}_{k\in\mathbf{Z}}$ and some $v, w \in G = \oplus_{i\in I}\mathbf{Z}$, then

$$\mathcal{F}_w(f) = \frac{2}{\|f\|}\max_{k\in\mathbf{Z}} a_k.$$

Proof: Let $g = \sum_{k\in\mathbf{Z}} a_k x^{kw}$. Using properties 3 and 7 of fluctuation we note that

$$\mathcal{F}_w(f) = \mathcal{F}_w(x^v g) = \mathcal{F}_w(g) = \frac{1}{\|g\|}\sum_{z\in G}\left|\langle g, x^z\rangle - \langle g, x^{z+w}\rangle\right|$$

$$= \frac{1}{\|g\|}\sum_{k\in\mathbf{Z}}\left|\langle g, x^{kw}\rangle - \langle g, x^{kw+w}\rangle\right| = \frac{1}{\|g\|}\sum_{k\in\mathbf{Z}}|a_k - a_{k+1}|$$

Let $m \in \mathbf{Z}$ be such that $a_m = \max_{k\in\mathbf{Z}} a_k$. Then $a_k - a_{k+1} \le 0$ for $k < m$, and $a_k - a_{k+1} \ge 0$ for $k \ge m$. It follows that

$$\sum_{k\in\mathbf{Z}}|a_k - a_{k+1}| = \sum_{k<m}|a_k - a_{k+1}| + \sum_{k\ge m}|a_k - a_{k+1}| = \sum_{k<m}(a_{k+1} - a_k) + \sum_{k\ge m}(a_k - a_{k+1})$$

$$= a_m + a_m$$

Thus $\mathcal{F}_w(f) = \frac{1}{\|g\|}\,2a_m = \frac{2}{\|f\|}\,a_m$. ∎

A similar decomposition to that given in Lemma 2.5 will appear in chapter 4 in our study of strong positivity (see comments preceding Proposition 4.2).

Lemma 2.5: Suppose $P \in \mathbf{R}[x_i^{\pm 1}]_{i\in I}$ with $\|P\| = 1$. Then for any $w \in G = \oplus_{i\in I}\mathbf{Z}$, we can decompose P as

$$P = (1 + x^w)Q + R,$$

where $\|Q\| = \frac{1}{2}\mathcal{P}_w(P)$, $\|R\| = \frac{1}{2}\mathcal{F}_w(P)$, and $\|P\| = \|(1 + x^w)Q\| + \|R\|$.

Proof: Write $P = \sum_{z\in G} P_z x^z$. As in Proposition 2.2, we define, for $\alpha \in \mathbf{R}$, $\chi^+(\alpha) = \begin{cases} 1 & \text{if } \alpha > 0 \\ 0 & \text{if } a \le 0 \end{cases}$, and we also define, for $\alpha, \beta \in \mathbf{R}$, $\phi(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha, \beta > 0 \\ -1 & \text{if } \alpha, \beta < 0 \\ 0 & \text{otherwise} \end{cases}$. For each $z \in G$ we set

$$Q_z = \frac{1}{2}\phi(P_z, P_{z+w})\,\min(|P_z|, |P_{z+w}|).$$

Let $Q = \sum_{z\in G} Q_z x^z$. We observe that for each z, $|Q_z| \le \frac{1}{2}|P_z|$ and $|Q_z| \le \frac{1}{2}|P_{z+w}|$. Thus

$$\left|\langle (1 + x^w)Q, x^z\rangle\right| = |Q_z + Q_{z-w}| \le |Q_z| + |Q_{z-w}| \le \tfrac{1}{2}|P_z| + \tfrac{1}{2}|P_z| = |P_z|$$

for each $z \in G$. We set $R = P - (1 + x^w)Q$, and write $R = \sum_{z\in G} r_z x^z$. From the definition of Q_z it is clear that if Q_z is nonzero, then it has the same sign as both P_z and P_{z+w}. One consequence of this is that if Q_z and Q_{z-w} are both nonzero, then their signs are both the same as that of P_z. In particular, $|Q_z + Q_{z-w}| = |Q_z| + |Q_{z-w}|$. Also, if $\langle (1 + x^w)Q, x^z\rangle = Q_z + Q_{z-w}$ is nonzero, its sign is the same as that of P_z. Since also $\left|\langle (1 + x^w)Q, x^z\rangle\right| \le |P_z|$ and $R + (1 + x^w)Q = P$, we conclude that if R_z is nonzero, its sign is the same as that of both $\langle (1 + x^w)Q, x^z\rangle$ and P_z. It follows that $|R_z| + \left|\langle (1 + x^w)Q, x^z\rangle\right| = |P_z|$. Summing over all values of $z \in G$, we obtain $\|R\| + \|(1 + x^w)Q\| = \|P\|$.

Noting that $\phi(\alpha,\beta) = \chi^+(\alpha\beta)$ for any $\alpha, \beta \in \mathbf{R}$, and applying Proposition 2.2, we have

$$\|Q\| = \sum_{z\in G} |Q_z| = \sum_{z\in G} \frac{1}{2}\chi^+(P_zP_{z+w}) \min(|P_z|, |P_{z+w}|) = \frac{1}{2}\|P\|\,\mathcal{P}_w(P) = \frac{1}{2}\mathcal{P}_w(P).$$

Since $|Q_z + Q_{z-w}| = |Q_z| + |Q_{z-w}|$ holds for all $z \in G$, by summing over all z, we find that $\|(1+x^w)Q\| = \|Q\| + \|Q\| = 2\|Q\|$. Thus

$$\|R\| = \|P\| - \|(1+x^w)Q\| = 1 - 2\|Q\| = 1 - \mathcal{P}_w(P) = 1 - \left(1 - \frac{\mathcal{F}_w(P)}{2}\right) = \frac{1}{2}\mathcal{F}_w(P).$$

∎

The following theorem and its proof have been generalized from [MH], where the implication (2.4) is proved for the case of non-negative polynomials in finitely many indeterminates, with w being an element of the standard base of $\mathbf{Z}^n$. The implication (2.4), for the case where I is a singleton and $w = 1$, is due to Mineka [Mi]. Note that the right-hand side of the implication (1) deals with the unnormalized fluctuation $\mathcal{F}_w^*$. This cannot be replaced by the normalized fluctuation $\mathcal{F}_w$, as is shown in example Example 2.7. A corresponding result for $\mathcal{F}_w$ in a more restricted setting is given in Corollary 2.8.

Theorem 2.6: Let I be a nonvoid set and let $G = \oplus_{i\in I}\mathbf{Z}$. If $w \in G$, and $\{P_n\}_{n\in\mathbf{N}}$ is any sequence in $\mathbf{R}[x_i^{\pm1}]_{i\in I}$ satisfying $\|P_n\| = 1$, $\forall n$, then

$$\sum_{i\in\mathbf{N}} \mathcal{P}_w(P_i) = \infty \implies \lim_{N\to\infty} \mathcal{F}_w^*(P^{N,t}) = 0,\ \forall t \in \mathbf{Z}^+. \tag{2.4}$$

Moreover, for $t \in \mathbf{Z}^+$, $N \in \mathbf{N}$ and $B_{N,t} = \sum_{i=t+1}^{t+N} \mathcal{P}_w(P_i) > 0$,

$$\mathcal{F}_w^*(P^{N,t}) \le 2e^{-(1/2)s^2 B_{N,t}} + \frac{4}{\sqrt{B_{N,t}(1-s)\pi}}, \qquad \forall s \in (0,1). \tag{2.5}$$

Proof: The result is trivially true if $w = 0$. Assume $w \neq 0$. Since (2.5) is trivially satisfied if $P_i = 0$ for some i with $t < i \le t+N$, and the right-hand side of (2.4) holds if $P_i = 0$ for some $i > t$, we may assume without loss of generality, that each P_i under consideration is nonzero.

Our method of approach is to rewrite P_i as a sum of two parts, one of which has high persistence. We then show that in the product the persistent part propogates and eventually dominates the product.

Using Lemma 2.5, we decompose each P_i as $P_i = (1+x^w)Q_i + R_i$, where $\|Q_i\| = \frac{1}{2}\mathcal{P}_w(P_i)$, $\|R_i\| = \frac{1}{2}\mathcal{F}_w(P_i)$, and $\|(1+x^w)Q_i\| + \|R_i\| = \|P_i\| = 1$.

Let $t \in \mathbf{Z}^+$ be given, and for any $n \in \mathbf{N}$ define $I_n = \{t+1, t+2, \ldots t+n\}$. Since

$$\begin{aligned} P^{n,t} = \prod_{i=t+1}^{t+n} [(1+x^w)Q_i + R_i] &= \sum_{K\subseteq I_n} \left(\prod_{i\in K}(1+x^w)Q_i\right) \prod_{j\in I_n\setminus K} R_j \\ &= \sum_{K\subseteq I_n} (1+x^w)^{|K|} \left(\prod_{i\in K} Q_i\right) \prod_{j\in I_n\setminus K} R_j, \end{aligned}$$

we may apply properties 4 and 5 of Proposition 2.1 to obtain

$$\begin{aligned}
\left\|(1-x^w)P^{n,t}\right\| &= \left\|(1-x^w)\sum_{K\subseteq I_n}(1+x^w)^{|K|}\left(\prod_{i\in K}Q_i\right)\prod_{j\in I_n\setminus K}R_j\right\| \\
&\le \sum_{K\subseteq I_n}\left\|(1-x^w)(1+x^w)^{|K|}\left(\prod_{i\in K}Q_i\right)\prod_{j\in I_n\setminus K}R_j\right\| \\
&\le \sum_{K\subseteq I_n}\left\|(1-x^w)(1+x^w)^{|K|}\right\|\cdot\left\|\left(\prod_{i\in K}Q_i\right)\prod_{j\in I_n\setminus K}R_j\right\| \\
&\le \sum_{k=0}^{n}\sum_{\substack{K\subseteq I_n\\|K|=k}}\left\|(1-x^w)(1+x^w)^{|K|}\right\|\cdot\left(\prod_{i\in K}\|Q_i\|\right)\prod_{j\in I_n\setminus K}\|R_j\| \\
&= \sum_{k=0}^{n}\left\|(1-x^w)(1+x^w)^{k}\right\|\sum_{\substack{K\subseteq I_n\\|K|=k}}\left(\prod_{i\in K}\|Q_i\|\right)\prod_{j\in I_n\setminus K}\|R_j\|
\end{aligned}$$

We now examine $g=(1+x^w)^k$. We may write $g=\sum_{j=0}^{k}\binom{k}{j}x^{jw}$. Applying Lemma 2.4 we find that

$$\|(1-x^w)g\| = \|g\|\,F_w(g) = 2\max_{1\le j\le k}\binom{k}{j} = 2\binom{k}{\lfloor k/2\rfloor},$$

where $\lfloor k/2\rfloor$ denotes the greatest integer less than or equal to $k/2$. For each $i>t$, we let $\beta_i=\mathcal{P}_w(P_i)=2\|Q_i\|$, and note that $\|R_i\|=\frac{1}{2}\mathcal{F}_w(P_i)=1-\mathcal{P}_w(P_i)=1-\beta_i$. Using this, our previous inequality becomes

$$\begin{aligned}
\left\|(1-x^w)P^{n,t}\right\| &\le \sum_{k=0}^{n}2\binom{k}{\lfloor k/2\rfloor}\sum_{\substack{K\subseteq I_n\\|K|=k}}\left(\prod_{i\in K}\frac{\beta_i}{2}\right)\prod_{j\in I_n\setminus K}(1-\beta_j) \\
&\le 2\sum_{k=0}^{n}\binom{k}{\lfloor k/2\rfloor}\frac{1}{2^k}\sum_{\substack{K\subseteq I_n\\|K|=k}}\left(\prod_{i\in K}\beta_i\right)\prod_{j\in I_n\setminus K}(1-\beta_j)
\end{aligned}$$

Consider the polynomial $T\in\mathbf{R}[x^{\pm1}]^+$ defined by

$$T=\prod_{i=t+1}^{t+n}\left((1-\beta_i)+\beta_i x\right),$$

and note that for $k=0,1,\ldots n$,

$$\langle T, x^k\rangle = \sum_{\substack{K\subseteq I_n\\|K|=k}}\left(\prod_{i\in K}\beta_i\right)\prod_{j\in I_n\setminus K}(1-\beta_j).$$

Our inequality now becomes

$$\left\|(1-x^w)P^{n,t}\right\| \le 2\sum_{k=0}^{n}\binom{k}{\lfloor k/2\rfloor}\frac{1}{2^k}\langle T, x^k\rangle$$

As each $\beta_i\le 1$, we may apply Proposition 2.3 and we note that if $B_n=\sum_{i=t+1}^{t+n}\beta_i>0$, then

$$\sum_{i\le B_n-r}\langle T, x^i\rangle \le e^{-r^2/(2B_n)},\quad \forall r\in[0,\infty).$$

Taking $r = sB_n$ for some $s \in (0,1)$ we obtain

$$\sum_{i \le B_n(1-s)} \langle T, x^i \rangle \le e^{-(sB_n)^2/(2B_n)} = e^{-s^2 B_n/2}.$$

By Lemma A.1,

$$\binom{k}{\lfloor k/2 \rfloor} \frac{1}{2^k} \le \frac{2}{\sqrt{k\pi}}, \quad \forall k \in \mathbf{N}.$$

Since it is clear that $\binom{k}{\lfloor k/2 \rfloor} \frac{1}{2^k} \le 1$ for all $k \in \mathbf{Z}^+$, and likewise $\langle T, x^k \rangle \le T(1) = 1$, we find that

$$\begin{aligned}
\mathcal{F}_w^*(P^{n,t}) = \|(1-x^w)P^{n,t}\| &\le 2\sum_{k=0}^{n} \binom{k}{\lfloor k/2 \rfloor} \frac{1}{2^k} \langle T, x^k \rangle \\
&\le 2 \sum_{0 \le k \le B_n(1-s)} \langle T, x^k \rangle + 2 \sum_{B_n(1-s) < k \le n} \frac{2}{\sqrt{k\pi}} \langle T, x^k \rangle \\
&\le 2e^{-s^2 B_n/2} + 2 \sum_{B_n(1-s) < k \le n} \frac{2}{\sqrt{B_n(1-s)\pi}} \langle T, x^k \rangle \\
&\le 2e^{-s^2 B_n/2} + \frac{4}{\sqrt{B_n(1-s)\pi}} \sum_{B_n(1-s) < k \le n} \langle T, x^k \rangle \\
&\le 2e^{-s^2 B_n/2} + \frac{4}{\sqrt{B_n(1-s)\pi}}
\end{aligned}$$

Fixing s, and noting that $B_n = \sum_{i=t+1}^{t+n} \mathcal{P}_w(P_i)$ tends to infinity as $n \to \infty$, we see that $\lim_{n \to \infty} \mathcal{F}_w^*(P^{n,t}) = 0$. ∎

Example 2.7: The unnormalized fluctuation $\mathcal{F}^*$ in the conclusion of implication (2.4) in Theorem 2.6 cannot, in general, be replaced by the normalized fluctuation $\mathcal{F}$. Consider the sequence of polynomials $\{P_i\}_{i \in \mathbf{N}}$ in $\mathbf{R}[x^{\pm 1}]$ defined by

$$P_i = \begin{cases} \frac{1}{2}(1+x) & \text{if } i \text{ is odd} \\ \frac{1}{2}(1-x) & \text{if } i \text{ is even} \end{cases}$$

and note that for any $t \in \mathbf{Z}^+$,

$$P^{2n,t} = \prod_{i=t+1}^{t+2n} P_i = \frac{1}{4^n}\left(1 - x^2\right)^n,$$

and since $\operatorname{Log} P^{2n,t}$ contains only even integers, $\mathcal{F}(P^{2n,t}) = 2$, for all $n \in \mathbf{N}$. On the other hand, by Proposition 2.2, $\mathcal{P}\left(\frac{1}{2}(1+x)\right) = \frac{1}{2}$, and so $\sum_{i \in \mathbf{N}} \mathcal{P}(P_i) = \infty$.

We *can* replace the unnormalized fluctuation $\mathcal{F}^*$ in the conclusion of implication (2.4) with the normalized fluctuation $\mathcal{F}$ if we require all polynomials to have only nonnegative coefficients. In this case we may drop the condition that $|P_i| = 1$. These are the features that differentiate Corollary 2.8 from Theorem 2.6. We could summarize this difference by saying that if negative coefficients are allowed, Mineka's theorem is true for unnormalized fluctuation, and untrue for normalized fluctuation.

Corollary 2.8: Let I be a nonvoid set and let $G = \oplus_{i\in I}\mathbf{Z}$. If $w \in G$, and $\{P_n\}_{n\in\mathbf{N}}$ is any sequence in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}^{+}$ then

$$\sum_{i\in\mathbf{N}} \mathcal{P}_w(P_i) = \infty \implies \lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0, \ \forall t \in \mathbf{Z}^+.$$

Moreover, for $t \in \mathbf{Z}^+$, $N \in \mathbf{N}$ and $B_{N,t} = \sum_{i=t+1}^{t+N} \mathcal{P}_w(P_i) > 0$,

$$\mathcal{F}_w(P^{N,t}) \le 2e^{-(1/2)s^2 B_{N,t}} + \frac{4}{\sqrt{B_{N,t}(1-s)\pi}}, \qquad \forall s \in (0,1).$$

Proof: We may restrict our attention to the case where all P_i's are nonzero, since the occurence of zero polynomials makes the result trivially true. For each $i \in \mathbf{N}$ let $Q_i = \frac{1}{\|P_i\|} P_i$, and apply Theorem 2.6 to the sequence $\{Q_i\}_{i\in\mathbf{N}}$. The fact that each $P_i > 0$ allows us to apply Proposition 2.1, parts (9) and (3) to obtain

$$\mathcal{F}_w(P^{n,t}) = \frac{\left\|(1-x^w)\prod_{i=t+1}^{t+n} P_i\right\|}{\left\|\prod_{i=t+1}^{t+n} P_i\right\|} = \frac{\left\|(1-x^w)\prod_{i=t+1}^{t+n} P_i\right\|}{\prod_{i=t+1}^{t+n} \|P_i\|} = \left\|\frac{(1-x^w)\prod_{i=t+1}^{t+n} P_i}{\prod_{i=t+1}^{t+n} \|P_i\|}\right\|$$
$$= \left\|(1-x^w)\prod_{i=t+1}^{t+n} \frac{P_i}{\|P_i\|}\right\| = \left\|(1-x^w)\prod_{i=t+1}^{t+n} Q_i\right\| = \mathcal{F}_w^{*}(Q^{n,t})$$

for each $t \in \mathbf{Z}^+$ and $n \in \mathbf{N}$. Alse $\mathcal{F}_w(Q_i) = \frac{\|(1-x^w)Q_i\|}{1} = \left\|(1-x^w)\frac{P_i}{\|P_i\|}\right\| = \frac{\|(1-x^w)P_i\|}{\|P_i\|} = \mathcal{F}_w(P_i)$, so $\mathcal{P}_w(Q_i) = \mathcal{P}_w(P_i)$, for each $i \in \mathbf{N}$, and our result follows from Theorem 2.6. ∎

The next two results show how blockings of a sequence (defined on page 6) relate to fluctuation.

Lemma 2.9: If $\{P_i\}_{i\in I}$ is a sequence in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}$, and $\{Q_i\}_{i\in\mathbf{N}}$ is a blocking of $\{P_j\}_{i\in\mathbf{N}}$, then for $w \in G$,

$$\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0 \quad \forall t \in \mathbf{Z}^+ \qquad \iff \qquad \lim_{N\to\infty} \mathcal{F}_w(Q^{N,t}) = 0 \quad \forall t \in \mathbf{Z}^+.$$

Proof: As in the definition of blocking, take $Q_j = \prod_{N_{j-1}\le i<N_j} P_i$, for all $j \in \mathbf{N}$.

Suppose $\lim_{n\to\infty} \mathcal{F}_w(P^{N,t}) = 0$, $\forall t \in \mathbf{Z}^+$. Then $\{\mathcal{F}_w(Q^{N,t})\}_{N\in\mathbf{N}}$ is a subsequence of the sequence $\{\mathcal{F}_w(P^{N,N_t-1})\}_{N\in\mathbf{N}}$, and hence converges to 0.

Conversely, suppose $\lim_{n\to\infty} \mathcal{F}_w(Q^{N,t}) = 0, \forall t \in \mathbf{Z}^+$. Given t, choose $j \in \mathbf{N}$ such that $N_j > t$. Then using property (4) of fluctuation we see that $0 \le \mathcal{F}_w(P^{N+(N_j-t),t}) \le \mathcal{F}_w(P^{N,N_j-1})$, $\forall N \in \mathbf{N}$. But $\{\mathcal{F}_w(P^{N,N_j-1})\}_{N\in\mathbf{N}}$ is a non-increasing sequence whose subsequence $\{\mathcal{F}_w(Q^{N,j})\}_{N\in\mathbf{N}}$ goes to zero. Thus $\mathcal{F}_w(P^{N,t}) \to 0$ as $N \to \infty$. ∎

Theorem 2.10: Let $\{P_i\}_{i\in\mathbf{N}}$ be a sequence in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}^{+}$. Then for $w \in G$,

$$\{P_i\}_{i\in\mathbf{N}} \text{ has a blocking } \{Q_i\}_{i\in\mathbf{N}} \text{ such that } \sum_{i=1}^{\infty} \mathcal{P}_w(Q_i) = \infty$$

$$\iff \lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0, \ \forall t \in \mathbf{Z}^+.$$

Proof: Suppose there is a blocking $\{Q_j\}$ of $\{P_j\}$ such that $\sum_{j=1}^{\infty} \mathcal{P}_w(Q_j) = \infty$. Then, by Corollary 2.8, $\mathcal{F}_w(Q^{N,t}) \to 0$ as $N \to \infty$ for all $t \in \mathbf{Z}^+$. By Lemma 2.9, we conclude that $\mathcal{F}_w(P^{N,t}) \to 0$ as $N \to \infty$ for all $t \in \mathbf{Z}^+$.

Conversely, suppose $\mathcal{F}_w(P^{N,t}) \to 0$ as $N \to \infty$, $\forall t \in \mathbf{N}$. Let $N_0 = 1$. Having defined N_j, we take $N_{j+1} > N_j$ such that $\mathcal{F}_w\left(\prod_{N_j \le i < N_{j+1}} P_i\right) \le 1$. Letting $\{Q_j\}$ be the blocking of $\{P_j\}$ determined by the sequence $N_0, N_1, N_2, \ldots$, that is, letting $Q_j = \prod_{N_{j-1} \le i < N_j} P_i$, for $j = 1, 2, 3, \ldots$, we get $\mathcal{F}_w(Q_j) \le 1$, $\forall j \in \mathbf{N}$, so by equation (2.1), $\mathcal{P}_w(Q_j) \ge 1 - \frac{1}{2} = \frac{1}{2}$. It follows that $\sum_{j=1}^{\infty} \mathcal{P}_w(Q_j) = \infty$. ∎

We now state a 0-2 law for fluctuation.

Corollary 2.11: Let I be a nonvoid set, and let $\{P_i\}_{i\in\mathbf{N}}$ be a sequence in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}^{+}$. Then for any $w \in \oplus_{i\in I} \mathbf{Z}$, either

$$\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0, \quad \forall t \in \mathbf{Z}^+$$

or

$$\lim_{t\to\infty} \lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 2.$$

Proof: For any $t \in \mathbf{Z}^+$, the sequence $\{\mathcal{F}_w(P^{n,t})\}_{n\in\mathbf{N}}$ is (by properties 0 and 4 of fluctuation) a non-increasing sequence in $[0,2]$, and so it has a limit β_t in $[0,2]$. Whenever $s, t \in \mathbf{Z}^+$ with $s > t$, we have, by property 4 of fluctuation,

$$\mathcal{F}_w(P^{n+s-t,t}) = \mathcal{F}_w(P^{n,s}\, P^{s-t,t}) \le \mathcal{F}_w(P^{n,s}),$$

for all $n \in \mathbf{N}$. Taking limits we conclude that $\beta_t \le \beta_s$, and so $\{\beta_t\}_{t\in\mathbf{Z}^+}$ is a non-decreasing sequence in $[0,2]$. Let $\beta = \lim_{t\to\infty} \beta_t$.

If $\beta \ne 2$, choose any γ with $\beta < \gamma < 2$. Then for every $t \in \mathbf{Z}^+$, $\beta_t < \gamma$ and so there exists $M_t \in \mathbf{N}$ such that $\mathcal{F}_w(P^{M_t,t}) < \gamma$. We can use this fact to obtain a blocking $\{Q_i\}_{i\in\mathbf{N}}$ of $\{P_i\}_{i\in\mathbf{N}}$ such that $\mathcal{F}_w(Q_i) < \gamma$ for all $i \in \mathbf{N}$. Then by equation (2.1),

$$\sum_{i=1}^{\infty} \mathcal{P}_w(Q_i) \ge \sum_{i=1}^{\infty} 1 - \frac{\gamma}{2} = \infty.$$

Applying Theorem 2.10 we conclude that

$$\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0, \quad \forall t \in \mathbf{Z}^+.$$

∎

3. Further Results on Fluctuation of Products

In chapter 2 we were looking for techniques for determining for which sequences $\{P_i\}$ the fluctuation $\mathcal{F}_w(P^{n,t})$ tends to 0 as $n \to \infty$. Theorem 2.10 characterized such sequences as those having a blocking $\{Q_i\}$ for which $\sum_i \mathcal{P}_w(Q_i) = \infty$. In this chapter we consider some examples, and observe that beyond some simple cases, it is difficult to show that an appropriate blocking exists.

We find, however, that for many cases which are difficult to decide using the criteria of the previous chapter, we may readily reach a conclusion by considering fluctuation over various step sizes (or directions) w. The key to doing this lies in the observation that if we know some directions in which fluctuation tends to zero, the fluctuation will tend to zero in every direction which is a **Z**-linear combination of these directions (Corollary 3.5). Combining this observation with Corollary 2.8, we find that it is often sufficient to ask for which directions w the sum of persistence, $\sum_i \mathcal{F}_w(P_i)$, diverges (Corollary 3.6). For polynomials of bounded degree this gives us a condition which is both necessary and sufficient (Theorem 3.9 and Corollary 3.10).

At the end of the chapter we comment on how fluctuation is affected by reparametrization. We also note that fluctuation going to zero is a property that is invariant under ℓ^∞ *upward* perturbations, and is also invariant under ℓ^1 perturbations after appropriate normalization. We conclude with two ratio limit theorems which follow from fluctuation considerations.

Example 3.1: In practice it is usually fairly easy to check whether a given sequence $\{P_i\}_{i\in\mathbf{N}}$ satisfies the condition

$$\sum_{i\in\mathbf{N}} \mathcal{P}_w(P_i) = \infty,$$

by applying the fact that $\mathcal{P}_w(P_i) = \frac{\sum_{z\in G} \min\left(\langle P_i, x^z\rangle, \langle P_i, x^{z+w}\rangle\right)}{\|P_i\|}$ (from equation (2.2) of Proposition 2.2). Thus, for example, the sequence in $\mathbf{R}[x^{\pm 1}]^+$ obtained by taking

$$P_n = \begin{array}{ccccc} n & 0 & 0 & n & 1 \\ \hline 0 & 1 & 2 & 3 & 4 \end{array}$$

has $\mathcal{P}(P_n) = \frac{1}{2n+1}$, and so $\sum_{n\in\mathbf{N}} \mathcal{P}(P_n) = \infty$. (This notation for writing a polynomial was defined on page 6). From Corollary 2.8 it follows that $\mathcal{F}(P^{n,t}) \to 0$ as $n \to 0$, for all $t \in \mathbf{Z}^+$.

Example 3.2: The sequence defined by taking $P_n = \begin{array}{ccc} 1 & \sqrt{n} & 1 \\ \hline -2 & 0 & 3 \end{array}$ is one for which $\mathcal{P}(P_n) = 0$ for all $n \in \mathbf{N}$. Nevertheless we note that

$$P_nP_{n+1} = \begin{array}{cccccc} 1 & \sqrt{n}+\sqrt{n+1} & \sqrt{n^2+n} & 2 & \sqrt{n}+\sqrt{n+1} & 1 \\ \hline -4 & -2 & 0 & 1 & 3 & 6 \end{array},$$

and so $\mathcal{P}(P_nP_{n+1}) = \frac{2}{\|P_nP_{n+1}\|}$. As $\|P_nP_{n+1}\| = O(n)$ it follows that $\sum_{i=1}^{\infty} \mathcal{P}(P_{2i-1}P_{2i}) = \infty$, and so applying Theorem 2.10 to this blocking, we conclude that $\lim_{n\to\infty} \mathcal{F}(P^{n,t}) = 0$, for all $t \in \mathbf{Z}^+$.

Example 3.3: The success of the previous example depended heavily upon the spacing of the terms. If we consider the sequence defined by $P_n = \underset{-2}{\overset{1}{\vdash}}\!-\!\underset{0}{\overset{\sqrt{n}}{+}}\!-\!\underset{5}{\overset{1}{\dashv}}$, the technique would fail. If, on the other hand, we increased the disparity between the coefficients, to have, for example, $P_n = \underset{-2}{\overset{1}{\vdash}}\!-\!\underset{0}{\overset{n}{+}}\!-\!\underset{3}{\overset{1}{\dashv}}$, the technique would also fail. In this case we can show that no blocking of bounded size will produce sufficient persistance. Specifically, we show that $\sum_{i \in \mathbf{N}} \mathcal{P}(P^{k,(i-1)k}) < \infty$ for every $k \in \mathbf{N}$. To do this, we take a fixed $k \in \mathbf{N}$, and note that the coefficients $\langle P^{k,n}, x^i \rangle$ of $P^{k,n}$ are polynomials in n. No two such coefficients of degree $k-1$ or k (in n) are adjacent. If follows that, for large n, and all $i \in \mathbf{Z}$, $\min\left(\langle P^{k,n}, x^i \rangle, \langle P^{k,n}, x^{i+1} \rangle\right)$ is a polynomial of degree at most $k-2$ (in n). Since $\|P^{k,n}\|$ is a polynomial of degree k in n, it follows from formula (2.2) that, for large n, $\mathcal{P}(P^{k,n})$ is a rational polynomial in n whose numerator has degree at most $k-2$, and whose denominator has degree k. It follows that $\sum_{i \in \mathbf{N}} \mathcal{P}(P^{k,(i-1)k}) < \infty$ for any $k \in \mathbf{N}$.
In spite of the obstacles pointed out here, both sequences in this example satisfy $\lim_{n \to \infty} \mathcal{F}(P^{n,t}) = 0$ for all $t \in \mathbf{Z}^+$. To show this, we will develop a technique that relies upon persistence over larger steps ($\mathcal{P}_w$ for $w > 1$). Example 3.8 will show how both of these sequences can be easily handled using Corollary 3.6.

We next make the important observation that if fluctuation tends to zero in several directions, it will also tend to zero in any direction which is a $\mathbf{Z}$-linear combination of these directions.

Lemma 3.4: Let I be a nonvoid set and let $G = \oplus_{i \in I} \mathbf{Z}$. Suppose $w \in G$ can be written as $w = \sum_{i=1}^{t} \sigma_i w_i$, where $w_1, w_2, \ldots w_n$ are (not necessarily distinct) elements of G, and each σ_i is either 1 or -1. If $\{P_i\}_{i \in \mathbf{N}}$ is a sequence in $\mathbf{R}[x_i^{\pm 1}]_{i \in I}$ with

$$\mathcal{F}_{w_i}(P_n) \to 0 \text{ as } n \to \infty, \qquad \text{for } i = 1, 2, \ldots, n$$

then

$$\mathcal{F}_w(P_n) \to 0 \text{ as } n \to \infty.$$

Proof: We begin by proving the following formula by induction on t:

$$x^w - 1 = x^{\left(\sum_{i=1}^{t} \sigma_i w_i\right)} - 1 = \sum_{j=1}^{t} \left(\prod_{k=j+1}^{t} x^{\sigma_k w_k} \right) [x^{\sigma_j w_j} - 1].$$

The formula is clearly true when $t = 1$. Assume it holds for $t-1$, and note that

$$\begin{aligned}
\sum_{j=1}^{t}\left(\prod_{k=j+1}^{t} x^{\sigma_k w_k}\right)[x^{\sigma_j w_j}-1] &= [x^{\sigma_t w_t}-1] + \sum_{j=1}^{t-1}\left(\prod_{k=j+1}^{t} x^{\sigma_k w_k}\right)[x^{\sigma_j w_j}-1] \\
&= [x^{\sigma_t w_t}-1] + x^{\sigma_t w_t}\sum_{j=1}^{t-1}\left(\prod_{k=j+1}^{t-1} x^{\sigma_k w_k}\right)[x^{\sigma_j w_j}-1] \\
&= [x^{\sigma_t w_t}-1] + x^{\sigma_t w_t}\left[x^{\left(\sum_{i=1}^{t-1}\sigma_i w_i\right)}-1\right] \\
&= x^{\sigma_t w_t} - 1 + x^{\left(\sum_{i=1}^{t}\sigma_i w_i\right)} - x^{\sigma_t w_t} \\
&= x^{w} - 1
\end{aligned}$$

Let $n \in \mathbf{N}$. If $P_n = 0$, then $\mathcal{F}_w(P_n) = 0$. If $P_n \neq 0$, then we have

$$\begin{aligned}
\mathcal{F}_w(P_n) = \frac{\|(1-x^w)P_n\|}{\|P_n\|} = \frac{\|(x^w-1)P_n\|}{\|P_n\|} &= \frac{1}{\|P_n\|}\left\|\sum_{j=1}^{t}\left(\prod_{k=j+1}^{t} x^{\sigma_k w_k}\right)[x^{\sigma_j w_j}-1]P_n\right\| \\
\leq \frac{1}{\|P_n\|}\sum_{j=1}^{t}\left\|\left(\prod_{k=j+1}^{t} x^{\sigma_k w_k}\right)[x^{\sigma_j w_j}-1]P_n\right\| &= \frac{1}{\|P_n\|}\sum_{j=1}^{t}\|(x^{\sigma_j w_j}-1)P_n\|
\end{aligned}$$

If $\sigma_j = -1$, then by property 6 of fluctuation we have

$$\|(x^{\sigma_j w_j}-1)P_n\| = \|(x^{-w_j}-1)P_n\| = \mathcal{F}_{-w_j}(P_n)\,\|P_n\|, = \mathcal{F}_{w_j}(P_n)\,\|P_n\|,$$

and if $\sigma_j = 1$, then

$$\|(x^{\sigma_j w_j}-1)P_n\| = \|(x^{w_j}-1)P_n\| = \mathcal{F}_{w_j}(P_n)\,\|P_n\|.$$

Thus, we have

$$\mathcal{F}_w(P_n) \leq \frac{1}{\|P_n\|}\sum_{j=1}^{t}\|(x^{\sigma_j w_j}-1)P_n\| = \frac{1}{\|P_n\|}\sum_{j=1}^{t}\mathcal{F}_{w_j}(P_n)\,\|P_n\|. = \sum_{j=1}^{t}\mathcal{F}_{w_j}(P_n)$$

Since the inequality

$$\mathcal{F}_w(P_n) \leq \sum_{j=1}^{t}\mathcal{F}_{w_j}(P_n)$$

is valid also when $P_n = 0$, we may complete our proof by noting that

$$\sum_{j=1}^{t}\mathcal{F}_{w_{i_j}}(P_n) \to 0 \text{ as } n \to \infty,$$

which implies that

$$\mathcal{F}_w(P_n) \to 0 \text{ as } n \to \infty.$$

∎

Corollary 3.5: Let I be a nonvoid set and let $G = \oplus_{i\in I}\mathbf{Z}$. If $\{w_k\}_{k\in K}$ is a family of elements of G which generates a subgroup H of G, and $\{P_n\}_{n\in\mathbf{N}}$ is a sequence in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}$ with

$$\mathcal{F}_{w_k}(P_n) \to 0 \text{ as } n \to \infty, \quad \forall k \in K,$$

then

$$\mathcal{F}_w(P_n) \to 0 \text{ as } n \to \infty, \quad \forall w \in H.$$

Proof: Let $w \in H$ be given. Since $\{w_k\}_{k\in K}$ generates H, we may write

$$w = \sum_{j=1}^{t} \sigma_j w_{i_j}$$

where each $i_j \in K$, and each σ_j is either 1 or -1 (we allow the possibility that some of the i_j are the same). From Lemma 3.4 we know that $\mathcal{F}_w(P_n) \to 0$ as $n \to \infty$. ∎

Corollary 3.6: Suppose $\{P_i\}_{i\in\mathbf{N}}$ be a sequence in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}^{+}$, and let $G = \oplus_{i\in I}\mathbf{Z}$. Set

$$W = \left\{ w \in G \;\middle|\; \sum_{i=1}^{\infty} \mathcal{P}_w(P_i) = \infty \right\},$$

and let $Z(W)$ be the subgroup of G generated by W. Then for all $v \in Z(W)$,

$$\lim_{n\to\infty} \mathcal{F}_v(P^{n,t}) = 0, \; \forall t \in \mathbf{N}.$$

Proof: By Corollary 2.8, for every $w \in W$, $\lim_{n\to\infty}\mathcal{F}_v(P^{n,t}) = 0, \; \forall t \in \mathbf{N}$. Applying Corollary 3.5 completes the proof. ∎

Definition: We say that a sequence $\{P_n\}_{n\in N}$ in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}^{+}$ is a *zero-fluctuation sequence* if, for every $w \in G = \oplus_{i\in I}\mathbf{Z}$,

$$\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0, \quad \forall t \in \mathbf{Z}^+.$$

Proposition 3.7: A sequence $\{P_n\}_{n\in N}$ in $\mathbf{R}[x^{\pm 1}]^+$ is a zero-fluctuation sequence if and only if

$$\lim_{N\to\infty} \mathcal{F}(P^{N,t}) = 0, \quad \forall t \in \mathbf{Z}^+.$$

Proof: Recall that in the case of polynomials in a single indeterminate, $\mathcal{F}(P)$ was defined to be $\mathcal{F}_1(P)$. For each $t \in \mathbf{Z}^+$, applying Corollary 3.5 to the sequence $\{P^{n,t}\}_{n\in\mathbf{N}}$, and noting that 1 generates $\mathbf{Z}$ as an additive group, completes the proof. ∎

Example 3.8: For μ and ν relatively prime integers, the sequence $\{P_n\}_{n\in\mathbf{N}}$ in $\mathbf{R}[x^{\pm 1}]^+$ defined by $P_n = \overset{1}{\underset{\mu}{\vdash}} \!-\!\!-\! \overset{n}{\underset{0}{+}} \!-\!\!-\! \overset{1}{\underset{\nu}{\dashv}}$ is a zero-fluctuation sequence. To see this, note that by equation (2.2) of Proposition 2.2, $\mathcal{P}_\mu(P_n) = \frac{1}{n+2} = \mathcal{P}_\nu(P_n)$, for all $n \in \mathbf{N}$. Thus $\sum_{i=1}^{\infty}\mathcal{P}_\mu(P_i) = \infty$ and $\sum_{i=1}^{\infty}\mathcal{P}_\nu(P_i) = \infty$. As 1 is in the additive subgroup of $\mathbf{Z}$ generated by μ and ν, Corollary 3.6 guarantees that our sequence is a zero-fluctuation sequence. The same argument readily shows that both sequences of Example 3.3 are zero-fluctuation sequences.

We next present a converse to Corollary 3.6 in the case of polynomials of bounded degree, provided the number of indeterminates is finite. This gives an easily checked condition which is both necessary and sufficient for the fluctuation in a given direction to go to zero. The proof of Theorem 3.9 shows that a sequence will fail to have fluctuation tending to zero, if (after a possible translation of the polynomials) almost all of the mass of the polynomials is concentrated at coefficients of x^u for u lying in some proper subgroup of $\mathbf{Z}^n$. That is, $P^{n,t}$ is a perturbation of a polynomial $Q^{n,t}$ with $\operatorname{Log} Q^{n,t}$ lying in some proper subgroup of $\mathbf{Z}^n$.

Theorem 3.9: Let $\{P_i\}_{i\in\mathbf{N}}$ be a sequence in $\mathbf{R}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]^+ \setminus \{0\}$ for which $\sup_{i\in\mathbf{N}} \deg(P_i) < \infty$. Let

$$W = \left\{ w \in \mathbf{Z}^n \,\middle|\, \sum_{i=1}^{\infty} \mathcal{P}_w(P_i) = \infty \right\},$$

and let $Z(W)$ be the subgroup of $\mathbf{Z}^n$ generated by W. Then

$$\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0,\ \forall t \in \mathbf{Z}^+ \iff w \in Z(W).$$

Proof: If $w \in Z(W)$ then Corollary 3.6 guarantees that $\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0$, $\forall t \in \mathbf{Z}^+$. To complete the proof it suffices to show that for every $w \in \mathbf{Z}^n \setminus Z(W)$ there is some $t \in \mathbf{Z}^+$ for which $\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) \neq 0$.

Note that $0 \in W$, so W is nonvoid.

We first 'regularize' each P_i so that $\langle P_i, x^o\rangle \geq \langle P_i, x^u\rangle$, $\forall u \in \mathbf{Z}^n$, and $\sum_{w\in Z(W)} \langle P_i, x^w\rangle = 1$. This may be accomplished by multiplying P_i by αx^v for v an appropriate element of $\mathbf{Z}^n$ and α an appropriate element of $\mathbf{R}^+$. By properties (2) and (3) of fluctuation and persistence, this does not change W or any $\mathcal{F}_w(P^{N,t})$.

Let $d = \sup_{i\in\mathbf{N}} \deg(P_i)$. Then $\|P_i\| \leq (2d+1)^n \langle P_i, x^o\rangle \leq (2d+1)^n$.

Notice that if both w and $u+w$ are elements of $\operatorname{Log} P_i$, for some i, then by equation (1.1), $\deg x^u = \deg(x^{u+w}x^{-w}) \leq \deg(x^{u+w}) + \deg(x^{-w}) \leq d + d = 2d$. It follows from this and equation (2.2) of Proposition 2.2 that $\mathcal{P}_u(P_i) = 0$ whenever $\deg x^u > 2d$.

Let

$$S = \{ u \in \mathbf{Z}^n \mid u \notin Z(W) \text{ and } \deg x^u \leq 2d \}.$$

If $u \in S$, then, by definition of W, $\sum_{i=1}^{\infty} \mathcal{P}_u(P_i) < \infty$. Since S is finite, for t sufficiently large we have

$$\sum_{i=t+1}^{\infty} \mathcal{P}_u(P_i) < \frac{\ln\left(\frac{3}{2}\right)}{(2d+1)^{2n}}, \quad \forall u \in S. \tag{3.1}$$

For every $u \in S$ and $i \in \mathbf{N}$ we have, by equation (2.2) of Proposition 2.2,

$$\mathcal{P}_u(P_i) = \frac{\sum_{v\in\mathbf{Z}^n} \min(\langle P_i, x^v\rangle, \langle P_i, x^{v+u}\rangle)}{\|P_i\|} \geq \frac{\min(\langle P_i, x^o\rangle, \langle P_i, x^u\rangle)}{\|P_i\|} = \frac{\langle P_i, x^u\rangle}{\|P_i\|}.$$

As $\|P_i\| \leq (2d+1)^n$, using (3.1), we obtain, for $u \in S$ and $N \in \mathbf{N}$,

$$\frac{\ln\left(\frac{3}{2}\right)}{(2d+1)^{2n}} > \sum_{i=t+1}^{t+N} \mathcal{P}_u(P_i) \geq \sum_{i=t+1}^{t+N} \frac{\langle P_i, x^u\rangle}{\|P_i\|} \geq \frac{1}{(2d+1)^n} \sum_{i=t+1}^{t+N} \langle P_i, x^u\rangle.$$

Thus

$$\sum_{i=t+1}^{t+N} \langle P_i, x^u \rangle < \frac{\ln\left(\frac{3}{2}\right)}{(2d+1)^n}, \qquad \forall u \in S.$$

Since S has fewer than $(2d+1)^n$ elements, we obtain

$$\sum_{u\in S} \sum_{i=t+1}^{t+N} \langle P_i, x^u \rangle < (2d+1)^n \frac{\ln\left(\frac{3}{2}\right)}{(2d+1)^n} = \ln\left(\frac{3}{2}\right),$$

so

$$\sum_{i=t+1}^{\infty} \sum_{u\in S} \langle P_i, x^u \rangle \le \ln\left(\frac{3}{2}\right).$$

For $i = t+1, t+2, \ldots$, let

$$\beta_i = \sum_{u\in S} \langle P_i, x^u \rangle .$$

Since $\sum_{u\in Z(W)} \langle P_i, x^u \rangle = 1$ and $\langle P_i, x^u \rangle = 0$ if $u \notin S \cup Z(W)$, we have $\|P_i\| = 1 + \beta_i$. As $\sum_{i=t+1}^{\infty} \beta_i \le \ln \frac{3}{2} < 1$, it follows that $\beta_i < 1$ for every $i > t$.

For $i \in \mathbf{N}$ let

$$Q_i = \sum_{w\in Z(W)} \langle P_i, x^w \rangle\, x^w.$$

Note that $Q_i \le P_i$ and that $\mathrm{Log}(Q_i) \subseteq Z(W)$. Since $Z(W)$ is a group, $\mathrm{Log}(Q^{N,t}) \subseteq Z(W)$. Also,

$$\|Q_i\| = \sum_{w\in Z(W)} \langle P_i, x^w \rangle = 1.$$

Since $0 \le Q_i \le P_i$ for each i, it follows that $Q^{N,t} \le P^{N,t}$. Now

$$\sum_{w\in Z(W)} \langle P^{N,t}, x^w \rangle \ge \sum_{w\in Z(W)} \langle Q^{N,t}, x^w \rangle = \|Q^{N,t}\| = \prod_{i=t+1}^{t+N} \|Q_i\| = 1.$$

Thus,

$$\sum_{w\in \mathbf{Z}^n\setminus Z(W)} \langle P^{N,t}, x^w \rangle = \|P^{N,t}\| - \sum_{w\in Z(W)} \langle P^{N,t}, x^w \rangle \le \|P^{N,t}\| - 1$$

Note that for $x \in [0,1)$, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, and so $\ln(1+x) \le x$. It follows that

$$\ln\|P^{N,t}\| = \ln \prod_{i=t+1}^{t+N} \|P_i\| = \ln \prod_{i=t+1}^{t+N} (1+\beta_i) = \sum_{i=t+1}^{t+N} \ln(1+\beta_i) \le \sum_{i=t+1}^{t+N} \beta_i \le \sum_{i=t+1}^{\infty} \beta_i \le \ln\frac{3}{2}.$$

Thus $\|P^{N,t}\| \le \frac{3}{2}$.

Now suppose that $w \in \mathbf{Z}^n \setminus Z(W)$. We will show that $\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) \neq 0$. For every

$z \in \mathbf{Z}^n$, either $z \notin Z(W)$, or $z+w \notin Z(W)$. Using this fact, we see the following (for $N \in \mathbf{N}$):

$$\begin{aligned}
\mathcal{P}_w(P^{N,t}) &= \frac{\sum_{z\in\mathbf{Z}^n} \min\left(\langle P^{N,t}, x^z\rangle, \langle P^{N,t}, x^{z+w}\rangle\right)}{\|P^{N,t}\|} \\
&= \frac{1}{\|P^{N,t}\|}\left[\sum_{z\in Z(W)} \min\left(\langle P^{N,t}, x^z\rangle, \langle P^{N,t}, x^{z+w}\rangle\right)\right. \\
&\qquad \left. + \sum_{z\in\mathbf{Z}^n\setminus Z(W)} \min\left(\langle P^{N,t}, x^z\rangle, \langle P^{N,t}, x^{z+w}\rangle\right)\right] \\
&\le \frac{1}{\|P^{N,t}\|}\left[\sum_{z\in Z(W)} \langle P^{N,t}, x^{z+w}\rangle + \sum_{z\in\mathbf{Z}^n\setminus Z(W)} \langle P^{N,t}, x^z\rangle\right] \\
&= \frac{1}{\|P^{N,t}\|}\left[\sum_{z+w\in\mathbf{Z}^n\setminus Z(W)} \langle P^{N,t}, x^{z+w}\rangle + \sum_{z\in\mathbf{Z}^n\setminus Z(W)} \langle P^{N,t}, x^z\rangle\right] \\
&= \frac{1}{\|P^{N,t}\|}\, 2 \sum_{z\in\mathbf{Z}^n\setminus Z(W)} \langle P^{N,t}, x^z\rangle \le \frac{1}{\|P^{N,t}\|}\, 2\left(\|P^{N,t}\| - 1\right)
\end{aligned}$$

For any real number $\alpha > 0$,

$$\frac{2\alpha - 2}{\alpha} > \frac{2}{3} \implies 2\alpha - 2 > \frac{2}{3}\alpha \implies \frac{4}{3}\alpha > 2 \implies \alpha > \frac{6}{4} = \frac{3}{2}.$$

Since $\|P^{N,t}\| \le \frac{3}{2}$, we must thus have (for $N \in \mathbf{N}$)

$$\frac{2\|P^{N,t}\| - 2}{\|P^{N,t}\|} \le \frac{2}{3}.$$

Thus

$$\mathcal{P}_w(P^{N,t}) \le \frac{2\|P^{N,t}\| - 2}{\|P^{N,t}\|} \le \frac{2}{3}.$$

Hence, by equation (2.1),

$$\mathcal{F}_w(P^{N,t}) = 2 - 2\mathcal{P}_w(P^{N,t}) \ge 2 - 2\cdot\frac{2}{3} = \frac{2}{3}, \quad \forall N \in \mathbf{N}.$$

Thus $\mathcal{F}_w(P^{N,t})$ does not tend to 0 as $N \to \infty$. ∎

Corollary 3.10: A sequence $\{P_i\}_{i\in\mathbf{N}}$ in $\mathbf{R}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]^+ \setminus \{0\}$ with

$$\sup_{i\in\mathbf{N}} \deg(P_i) < \infty$$

is a zero-fluctuation sequence if and only if

$$\left\{ w \in \mathbf{Z}^n \;\middle|\; \sum_{i=1}^{\infty} \mathcal{P}_w(P_i) = \infty \right\}$$

generates $\mathbf{Z}^n$ as an additive group.

Example 3.11: For many sequences of polynomials of bounded degree in $\mathbf{R}[x^{\pm 1}]^+$ the criteria of Corollary 3.10 provides an easy and conclusive way to determine whether or not the sequence is a zero-fluctuation sequence. The sequence $\{P_n\}_{n\in\mathbf{N}}$ of Example 3.8, namely $P_n = \begin{array}{ccc} 1 & n & 1 \\ \hline \mu & 0 & \nu \end{array}$, is a zero-fluctuation sequence if and *only if* μ and ν are relatively prime.

The sequence defined by $P_n = \begin{array}{ccccccc} 1 & n & 1 & n^{5/4} & 1 & n & 1 \\ \hline -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array}$ is not a zero-fluctuation sequence because $\sum_{i=1}^{\infty} \mathcal{P}_w(P_i) = \infty$ only for $w = 2$ and $w = 4$. There is not sufficient mass off the subgroup of even integers.

Why do all of our fluctuation results refer to $\mathcal{F}(P^{n,t})$ tending to 0 *for all t*, rather than simply speaking about $\mathcal{F}(P^{n,t})$ tending to zero for *some t*? A partial answer to this question is found in the fact that changing a single term in a sequence can sometimes dramatically change $\mathcal{F}(P^{n,t})$. It stands to reason that if we want fluctuation to be a characteristic of the sequence as a whole, rather than being possibly the artifact of a very few terms, we must allow finitely many terms to be discarded. The following example illustrates this sensitivity.

Example 3.12: Consider the sequence of Laurent polynomials defined by $P_i = 1 + x^2$, for all $i \in \mathbf{N}$. It is clear that $\mathcal{F}(P^{n,t}) = 2$ for all $t \in \mathbf{Z}^+$ and all $n \in \mathbf{N}$. If we change a single P_j to be $1 + x$, then for all $n \geq j$, the polynomial $P^{n,0}$ is unimodal with largest coefficient $\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$. Lemma 2.4 tells us that $\mathcal{F}(P^{n,0}) = \frac{2}{2^n}\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$, which tends to 0 as $n \to \infty$.

Example 3.13: Here we give an example of a sequence of polynomials $\{P_k\}_{k\in\mathbf{N}}$ in countably many indeterminates whose fluctuation goes to 0 in every direction (not just the cardinal directions, but every $\mathbf{Z}$-linear combination of them). Let $I = \mathbf{N}$, and consider $\mathbf{R}[x_i^{\pm 1}]_{i\in\mathbf{N}}$, with $G = \oplus_{i\in\mathbf{N}} \mathbf{Z}$. Let $\phi : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ be any bijection, and write $\phi(k) = \big(m(k), n(k)\big)$ for every $k \in \mathbf{N}$. We define $P_k = 1 + x_{m(k)}$, and note that if $x^w = x_i^1$ for some $i \in \mathbf{N}$, then $\sum_{k\in\mathbf{N}} \mathcal{P}_w(P_k) = \infty$, and so, by Corollary 2.8, $\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0$ for all $t \in \mathbf{Z}^+$. Since such w generate all of G, we conclude, by Corollary 3.5, that for all $w \in G$, $\lim_{N\to\infty} \mathcal{F}_w(P^{N,t}) = 0$ for all $t \in \mathbf{Z}^+$.

In [BH] a fair amount of attention is devoted to *reparametrizations* of a polynomial. If $\lambda \in (0, \infty)$, and P is in $\mathbf{R}[x^{\pm 1}]$, then $P(\lambda\cdot)$ is the polynomial obtained by replacing x by λx in P. Reparametrization is defined analogously for polynomials in more than one indeterminate.

[BH] shows that to be strongly positive, the sequence $\{P_i\}$ and all its reparametrizations must be zero-fluctuation sequences. Corollary 3.14 tells us that for sequences of bounded degree one reparametrization is zero-fluctuation if and only if all the others are.

Corollary 3.14: Suppose $\{P_i\}_{i\in\mathbf{N}}$ is a sequence in $\mathbf{R}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]^+ \setminus \{0\}$ with $\sup_{i\in\mathbf{N}} \deg(P_i) < \infty$. If $\{P_i\}_{i\in\mathbf{N}}$ is a zero-fluctuation sequence, then the reparametrized sequence $\{P_i(\lambda\cdot)\}_{i\in\mathbf{N}}$ is also zero-fluctuation sequence for every $\lambda \in (0, \infty)^n$.

Proof: Let

$$W = \left\{ w \in \mathbf{Z}^n \,\middle|\, \sum_{i=1}^{\infty} \mathcal{P}_w(P_i(\lambda\cdot)) = \infty \right\}.$$

By Corollary 3.10, W generates $\mathbf{Z}^n$ as an additive group. Let $d = \sup_{i\in\mathbf{N}} \deg(P_i)$ and let $L = \max_{1\le i\le n} \max(\lambda_i, \lambda_i^{-1})$, where $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_n)$. For $w \in \mathbf{Z}^n$, if $i \in \mathbf{N}$ and P_i has non-zero degree, then using equation (2.2) one can show that

$$\frac{1}{L^{2d+\deg x^w}} \mathcal{P}_w(P_i) \le \mathcal{P}_w(P_i(\lambda\cdot)) \le L^{2d+\deg x^w} \mathcal{P}_w(P_i).$$

This inequality is also true (trivially) if P_i has degree 0. Let

$$W' = \left\{ w \in \mathbf{Z}^n \,\middle|\, \sum_{i=1}^{\infty} \mathcal{P}_w(P_i(\lambda\cdot)) = \infty \right\}.$$

Now $w \in W \implies \sum_{i=1}^{\infty} \mathcal{P}_w(P_i) = \infty \implies \sum_{i=1}^{\infty} \mathcal{P}_w(P_i(\lambda\cdot)) = \infty \implies w \in W'$. Therefore $W \subseteq W'$. Since W generates $\mathbf{Z}^n$, so does W'. Applying Corollary 3.10, we find that $\{P_i(\lambda\cdot)\}_{i\in\mathbf{N}}$ is a zero-fluctuation sequence. ∎

The following result says that an ℓ^∞ *upward* perturbation of a sequence $\{P_i\}$ for which $\sum_i \mathcal{P}_w(P_i) = \infty$ yields a sequence which also has divergent persistence. It is proved in the appendix.

Lemma A.7: Suppose that $\{P_j\}_{j\in\mathbf{N}}$ and $\{R_j\}_{j\in\mathbf{N}}$ are sequences in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}^{+}$ and that $\sup_j \frac{\|R_j\|}{\|P_j\|} < \infty$. Then for any $w \in \oplus_{i\in I} \mathbf{Z}$,

$$\sum_{j=1}^{\infty} \mathcal{P}_w(P_j) = \infty \quad \implies \quad \sum_{j=1}^{\infty} \mathcal{P}_w(P_j + R_j) = \infty.$$

Further perturbation results are developed in the appendix. In particular, Theorem A.11 asserts that if $\{P_i\}$ and $\{Q_i\}$ are two sequences in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}^{+}\setminus\{0\}$ and $\sum_i \frac{\|P_i - Q_i\|}{\|P_i\|} < \infty$, then one sequence is a zero-fluctuation sequence if and only if the other one is.

Ratio Limit Theorems. We can use fluctuation and persistence to deduce results about the ratios of adjacent coefficients in a product of polynomials.

Lemma 3.15: Suppose $\{P_n\}$ is a sequence of strongly unimodal polynomials in $\mathbf{R}[x^{\pm 1}]^{+} \setminus \{0\}$ for which $\sum_n \mathcal{P}(P_n) = \infty$, and let U be any open subset of $\mathbf{R}$ containing 1. Then the cardinality of the set

$$U_n = \left\{ i \in \mathbf{Z} \,\middle|\, \frac{\langle P^{n,0}, x^i \rangle}{\langle P^{n,0}, x^{i-1} \rangle}, \frac{\langle P^{n,0}, x^{i+1} \rangle}{\langle P^{n,0}, x^i \rangle} \in U \right\}$$

goes to infinity as $n \to \infty$. Furthermore, the fraction of the mass of $P^{n,0}$ which lies at coefficients in U_n goes to 1 as $n \to \infty$.

Proof: First normalize each P_n by multiplying by $\frac{1}{P_n(1)}$, so that $P_n(1) = 1$ for all $n \in \mathbf{N}$. This normalization does not change $\mathcal{P}(P_n)$, or any of the ratios used to define U_n. Now for each $n \in \mathbf{N}$ let $t_n = \max_{i \in \mathbf{Z}} \langle P^{n,0}, x^i \rangle$. Since $\sum_n \mathcal{P}(P_n) = \infty$, by Corollary 2.8 we have $\lim_{n \to \infty} \mathcal{F}(P^{n,0}) = 0$. Lemma 2.4 tells us that the fluctuation of a normalized unimodal polynomial is twice its highest coefficient, so we have $\mathcal{F}(P^{n,0}) = 2\, t_n$. Thus $t_n \to 0$ as $n \to \infty$.

For $\alpha \in (0,1)$, let

$$V_\alpha = \left\{ i \in \mathbf{Z} \;\middle|\; \frac{\langle P^{n,0}, x^i \rangle}{\langle P^{n,0}, x^{i-1} \rangle}, \frac{\langle P^{n,0}, x^{i+1} \rangle}{\langle P^{n,0}, x^i \rangle} \in (\alpha, 1/\alpha) \right\}$$

Since each $P^{n,0}$ is strongly unimodal, each sequence $\left\{ \frac{\langle P^{n,0}, x^i \rangle}{\langle P^{n,0}, x^{i-1} \rangle} \right\}_{i \in \mathbf{Z}}$ is a decreasing sequence, for those values of i for which the ratios defined. Let

$$k_\alpha = \sup \left\{ i \in \mathbf{Z} \;\middle|\; \frac{\langle P^{n,0}, x^i \rangle}{\langle P^{n,0}, x^{i-1} \rangle} \geq \frac{1}{\alpha} \right\} \quad \text{and} \quad m_\alpha = \inf \left\{ i \in \mathbf{Z} \;\middle|\; \frac{\langle P^{n,0}, x^{i+1} \rangle}{\langle P^{n,0}, x^i \rangle} \leq \alpha \right\},$$

where the supremum and infimum of the empty set are taken to be $-\infty$ and ∞, if applicable. If k_α is finite, then for each $i \leq k_\alpha$ we have $\langle P^{n,0}, x^{i-1} \rangle \leq \alpha \langle P^{n,0}, x^i \rangle$. Thus

$$\sum_{i \leq k_\alpha} \langle P^{n,0}, x^i \rangle \leq \langle P^{n,0}, x^{k_\alpha} \rangle (1 + \alpha + \alpha^2 + \cdots) = \langle P^{n,0}, x^{k_\alpha} \rangle \frac{1}{1-\alpha} \leq \frac{t_n}{1-\alpha}.$$

Likewise, $\sum_{i \geq m_\alpha} \langle P^{n,0}, x^i \rangle \leq \frac{t_n}{1-\alpha}$. Since $V_\alpha = (k_\alpha, m_\alpha) \cap \mathbf{Z}$, we have

$$\sum_{i \in V_\alpha} \langle P^{n,0}, x^i \rangle \geq 1 - \frac{2t_n}{1-\alpha}.$$

Since $t_n \to 0$ as $n \to \infty$, it is clear that the mass at coefficients in V_α approaches 1 as $n \to \infty$. Denoting the cardinality of V_α by $|V_\alpha|$, we note that the total mass at points in V_α is at most $|V_\alpha| \cdot t_n$. Since $t_n \to 0$, it follows that $|V_\alpha|$ tends to infinity as $n \to \infty$. Since U must contain some interval of the form $(\alpha, 1/\alpha)$, for $\alpha \in (0,1)$, our result follows. ∎

While the above result tells us that in the limit all mass lies at coefficients with with ratios near 1, it is worth noting that there will also be an abundance of coefficients having ratios quite different from 1. The following result in this direction will be put to use in the proof of Theorem 5.7.

Corollary 3.16: Suppose $\{\beta_i\}_{i \in \mathbf{N}}$ is a sequence in $[0, \infty)$ with $\sum_{i=1}^{\infty} \beta_i = \infty$. For any open subset U of $\mathbf{R}^+$, the cardinality of the set

$$U_n = \left\{ i \in \mathbf{Z} \;\middle|\; \frac{\left\langle \prod_{j=1}^{n} (1 + \beta_j x), x^{i+1} \right\rangle}{\left\langle \prod_{j=1}^{n} (1 + \beta_j x), x^i \right\rangle} \in U \right\}$$

tends to infinity as $n \to \infty$.

Proof: U contains an interval of the form $(t, t+\epsilon)$, for some $t, \epsilon > 0$. Let $V = \left(\frac{t}{t+\epsilon/2}, \frac{t+\epsilon}{t+\epsilon/2} \right)$, and for $i \in \mathbf{N}$ define the polynomial $P_i = 1 + \frac{\beta_i}{t+\epsilon/2} x$. Since P_i is simply a reparametrization of $1+\beta_i x$ by the factor $\frac{1}{t+\epsilon/2}$, it is clear that

$$\frac{\left\langle \prod_{j=1}^{n} P_j,\, x^{i+1} \right\rangle}{\left\langle \prod_{j=1}^{n} P_j,\, x^{i} \right\rangle} \in V \implies \frac{\left\langle \prod_{j=1}^{n} (1+\beta_j x),\, x^{i+1} \right\rangle}{\left\langle \prod_{j=1}^{n} (1+\beta_j x),\, x^{i} \right\rangle} \in (t, t+\epsilon)$$

Lemma 3.15 tells us that the number of $i \in \mathbf{Z}$ for which $\frac{\left\langle \prod_{j=1}^{n} P_j, x^{i+1} \right\rangle}{\left\langle \prod_{j=1}^{n} P_j, x^{i} \right\rangle} \in V$ tends to infinity as $n \to \infty$. It follows that the cardinality of U_n tends to infinity as $n \to \infty$. ∎

More could be said about the relationship between fluctuation and ratios. It is possible to characterize zero-fluctuation sequences in $\mathbf{R}[x^{\pm 1}]^+$ in terms of ratios. Such results are rather inelegant, however, and have been omitted.

4. Contiguity and Strong Positivity

In this chapter we introduce our most basic tool for approaching strong positivity problems: the contiguity $\mathcal{R}(P)$ of a Laurent polynomial P. While $\mathcal{R}(P)$ is similar to the numerical invariants $c(P)$ and $C(P)$ used in [BH], its principal advantage is the fact that multiplication of polynomials cannot decrease $\mathcal{R}$ (Corollary 4.6) and, moreover, we can guarantee that $\mathcal{R}(P_1P_2\cdots P_n)$ grows in relation to the sum $\sum_{i=1}^{n}\mathcal{R}(P_i)$ (Theorem 4.7). This will be used in an essential way in proving the key results in the next chapter, such as Theorem 5.7.

After noting that the constant sequence given by $P_i = 1+rx$ is strongly positive (Corollary 4.10, a result proved by Meissner [Mn]), we can readily prove strong positivity for any sequence $\{P_i\}$ in $\mathbf{R}[x^{\pm 1}]^+$ for which $\sum_i \mathcal{R}(P_i) = \infty$ (Theorem 4.12), a result similar to 1.8, 1.9 and 1.11 of [BH]. The chapter concludes with Theorem 4.21, a characterization of strong positivity in terms of $\mathcal{R}$.

Definition: For f any unimodal Laurent polynomial in $\mathbf{R}[x^{\pm 1}]^+\backslash\{0\}$, we define $r(f)$ as follows. Writing $f(x) = \sum_i f_i x^i$, we let j be such that $f_j \geq f_i$, $\forall i \in \mathbf{Z}$. Then we define $r(f) = \max\left(\frac{f_{j-1}}{f_j}, \frac{f_{j+1}}{f_j}\right)$. This is well-defined. For convenience we define $r(0) = 1$.

Properties of $r(\,)$:

Let $f, g \in \mathbf{R}[x^{\pm 1}]^+$ be unimodal. Then

(0) $0 \leq r(f) \leq 1$.

(1) $r(\alpha f) = r(f)$, $\quad \forall \alpha \in (0,\infty)$.

(2) $r(x^i f) = r(f)$, $\quad \forall i \in \mathbf{Z}$.

(3) If $\langle g, x^i\rangle = \langle f, x^{-i}\rangle$, $\forall i \in \mathbf{Z}$, then $r(g) = r(f)$.

(4) $r\big((a+bx)+(c+dx)\big) \geq \min\{r(a+bx), r(c+dx)\}$, $\quad \forall a,b,c,d \in [0,\infty)$.

Proof: Properties (0)–(3) are clear. (4) is clear if one of a, b, c or d is zero. Assume they are all non-zero. Consider first the case in which $a \leq b$ and $c \leq d$, so that $r\big((a+bx)+(c+dx)\big) = \frac{a+c}{b+d}$. If $\frac{a}{b} \leq \frac{c}{d}$ then $c \geq \frac{ad}{b}$, so $\frac{a+c}{b+d} \geq \frac{a+\frac{ad}{b}}{b+d} = \frac{\frac{a}{b}(b+d)}{b+d} = \frac{a}{b}$. Similarly, if $\frac{c}{d} \leq \frac{a}{b}$, then $\frac{a+c}{b+d} \geq \frac{c}{d}$, so (4) holds in either situation. Using this and properties (2) and (3) we can deduce (4) in the case where $a \geq b$ and $c \geq d$. In the case where $a \leq b$ and $c \geq d$, we have $\frac{a+c}{b+d} \geq \frac{a+d}{b+d} \geq \frac{a}{b}$ and $\frac{b+d}{a+c} \geq \frac{a+d}{a+c} \geq \frac{d}{c}$, and so (4) holds in this case. Exchanging the roles of $a+bx$ and $c+dx$ proves the case in which $c \leq d$ and $a \geq b$. ∎

Property (4) does not generalize to polynomials with large support. For example, if $f = 1+10x+5x^2$ and $g = 5+10x+x^2$, then $r(f) = \frac{1}{2} = r(g)$ but $r(f+g) = \frac{3}{10}$.

Definition: For any $f \in \mathbf{R}[x^{\pm 1}]^+$ we define the contiguity of f, denoted $\mathcal{R}(f)$, by

$$\mathcal{R}(f) = \sup\left\{ \min_{i\in I} r(a_i x^i + b_i x^{i+1}) \;\middle|\; f = \sum_{i\in I} a_i x^i + b_i x^{i+1},\ I \subset \mathbf{Z},\ a_i, b_i \in [0,\infty) \right\}.$$

Properties of $\mathcal{R}(\,)$:

If $f, g \in \mathbf{R}[x^{\pm 1}]^+$, then

(0) $0 \leq R(f) \leq 1$.
(1) $\mathcal{R}(\alpha f) = \mathcal{R}(f), \qquad \forall \alpha \in (0, \infty)$.
(2) $\mathcal{R}(x^i f) = \mathcal{R}(f), \qquad \forall i \in \mathbf{Z}$.
(3) If $\langle g, x^i \rangle = \langle f, x^{-i} \rangle$, $\forall i \in \mathbf{Z}$, then $\mathcal{R}(g) = \mathcal{R}(f)$.
(4) If $f = a + bx$ then $\mathcal{R}(f) = r(f)$.
(5) $\mathcal{R}(f+g) \geq \min(\mathcal{R}(f), \mathcal{R}(g))$.

Proof: Properties (0)–(3) are a consequence of the corresponding properties for $r(\,)$. (4) is clear. (5) is a consequence of properties (3) and (4) of $r(\,)$. ∎

Proposition 4.1: The 'sup' in the definition of $\mathcal{R}(\,)$ may be replaced by 'max'. Specifically, for every $f \in \mathbf{R}[x^{\pm 1}]^+$ there exists a decomposition $f = \sum_{i \in I} a_i x^i + b_i x^{i+1}$, with $I \subset \mathbf{Z}$, such that

$$\mathcal{R}(f) = \min_{i \in I} r\left(a_i x^i + b_i x^{i+1}\right).$$

Proof:

Claim: If $c_0, c_1, \ldots, c_k \in (0, \infty)$, then $\mathcal{R}\left(\sum_{i=0}^{k} c_i x^i\right)$ is realized by a decomposition, and is a continuous function of c_k.

Proof of claim:

For $k = 0$, $\mathcal{R}(c_0) = 0$, and any decomposition works. For $k = 1$, by property (4) of $\mathcal{R}(\,)$, $\mathcal{R}(c_0 + c_1 x) = r(c_0 + c_1 x) = \min\left(\frac{c_0}{c_1}, \frac{c_1}{c_0}\right)$, and this is a continuous function of c_1. We now assume the result for k, and prove it for $k+1$. If $f = \sum_{i=0}^{k+1} c_i x^i$ is decomposed as $\sum_{i=0}^{k} a_i x^i + b_i x^{i+1}$, then

$$\min_{0 \leq i \leq k} r\left(a_i x^i + b_i x^{i+1}\right) \leq \min\left\{\mathcal{R}\left(\left[\sum_{i=0}^{k-1} c_i x^i\right] + b_{k-1} x^k\right),\ r\left(a_k x^k + b_k x^{k+1}\right)\right\} \qquad (+)$$

We denote the right-hand side of this inequality by $(*)$. By the induction hypothesis, the inequality (+) can be made into an equality by appropriate choice of $a_0, a_1, \ldots, a_{k-1}$ and $b_0, b_1, \ldots b_{k-2}$. Also, the $\mathcal{R}(\cdots)$ term is a continuous function of b_{k-1}. Since $b_{k-1} = c_k - a_k$, and $r\left(a_k x^k + b_k x^{k+1}\right)$ is a continuous function of a_k, it follows that $(*)$ is a continuous function of a_k. It is clear that $a_k \leq c_k$ and $b_k = c_{k+1}$, and we may restrict our attention to a_k for which $r(a_k x^k + c_{k+1} x^{k+1}) \geq \mathcal{R}(f)/2$, so, in particular, $a_k \geq c_{k+1}\mathcal{R}(f)/2$. Now since $(*)$ is a continuous function of a_k on the closed interval $a_k \in [c_{k+1}\mathcal{R}(f)/2,\ c_k]$, its supremeum is attained for some value of a_k. Corresponding to this value of a_k is a decomposition of f for which (+) becomes an equality. We conclude that the maximum value of $(*)$ is in fact $\mathcal{R}(f)$, and that the value of $\mathcal{R}(f)$ is realized by this decomposition.

It remains to show that $\mathcal{R}(f)$ is a continuous function of c_{k+1}. Given $\beta > 0$, note that if $0 < \alpha \leq \beta$, then $\frac{d}{d\alpha}\left(\frac{\alpha}{\beta}\right) = \frac{1}{\beta} \leq \frac{1}{\alpha}$, and that if $\alpha \geq \beta$, then $\left|\frac{d}{d\alpha}\left(\frac{\beta}{\alpha}\right)\right| = \left|\frac{-\beta}{\alpha^2}\right| \leq \frac{1}{\alpha}$.

It follows that if we are given $0 < \epsilon < 1$ and $\alpha_o > 0$, then whenever α is such that $|\alpha_o - \alpha| < \frac{\alpha_o \epsilon}{2}$ (so that, in particular $|\alpha_o - \alpha| < \frac{\alpha_o}{2}$, so $\alpha > \frac{\alpha_o}{2}$ and $\frac{1}{\alpha} < \frac{2}{\alpha_o}$) then

$$\left|\min\left(\frac{\alpha_o}{\beta}, \frac{\beta}{\alpha_o}\right) - \min\left(\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right)\right| < \frac{\alpha_o \epsilon}{2}\frac{2}{\alpha_o} = \epsilon.$$

In particular, if we vary $b_k = c_{k+1}$ by less than $\frac{c_{k+1}\epsilon}{2}$, then $(*)$ will change by less than ϵ for each value of a_k, and thus the maximum value of $(*)$, namely $\mathcal{R}(f)$, will change by less than ϵ. This proves that $\mathcal{R}(f)$ is a continuous function of c_{k+1}, completing the proof of the claim.

∎

Property (2) of $\mathcal{R}(\,)$ allows us to generalize the result of the above claim to any polynomial $f \in \mathbf{R}[x^{\pm 1}]^+$ for which $\operatorname{Log} f$ is contiguous (no gaps). A completely arbitrary $f \in \mathbf{R}[x^{\pm 1}]^+$ can have its Log-set broken up into contiguous blocks, and an optimal decomposition chosen on each block. Putting these decompositions together yields a decomposition of f which realizes $\mathcal{R}(f)$, which is the minimum of the $\mathcal{R}()$ values over the various blocks. ∎

Definition: A polynomial f in $\mathbf{R}[x^{\pm 1}]^+$ is said to be *fully contiguous* if $\mathcal{R}(f) = 1$.

Proposition 4.2 tells us that $(1 + x)$ is a factor of any nonzero fully contiguous polynomial. The decomposition in equation (4.2) is very similar to that in Lemma 2.5. It is subtly different, however. For example, Lemma 2.5 will decompose $\frac{1}{2} + \frac{1}{2}x$ as $(1 + x)\frac{1}{4} + (\frac{1}{4} + \frac{1}{4}x)$, while (4.2) would decompose it as $(1 + x)\frac{1}{2} + 0$. The factor of $\frac{1}{2}$ in inequality (4.1) is necessary, as may be seen by taking $f = 1 + x$ and noting that $\mathcal{P}(f) = \frac{1}{2}$ while $\mathcal{R}(f) = 1$. In fact there is no such thing as a "fully persistent" polynomial: $\mathcal{P}(f)$ can be arbitrarily close to 1 but can never equal 1. $\mathcal{P}(1 + x + x^2 + \cdots + x^n) = \frac{n-1}{n}$. Inequality (4.1) will (eventually) be used to prove that every strongly positive sequence is a zero-fluctuation sequence (Corollary 4.22; also proved in [BH]).

Proposition 4.2: Let $f \in \mathbf{R}[x^{\pm 1}]^+$. Then

$$\mathcal{P}(f) \geq \frac{1}{2}\mathcal{R}(f). \tag{4.1}$$

Furthermore, f can be decomposed as

$$f = (1 + x)g + h \tag{4.2}$$

where g and h are in $\mathbf{R}[x^{\pm 1}]^+$ and $\|(1 + x)g\| = \mathcal{R}(f)\,\|f\|$. In particular, if $\mathcal{R}(f) = 1$, then $f = (1 + x)g$.

Proof: Let $t = \mathcal{R}(f)$. Write $f = \sum_{i \in I} a_i x^i + b_i x^{i+1}$ as in the statement of Proposition 4.1. For each i in I let $c_i = \min(a_i, b_i)$. Note that by choice of the decomposition, $c_i \geq t a_i$ and $c_i \geq t b_i$. Let $g = \sum_{i \in I} c_i$ and $h = \sum_{i \in I}(a_i - c_i)x^i + (b_i - c_i)x^{i+1}$ and note that $h \geq 0$ and

$$(1 + x)g + h = (1 + x)\sum_{i \in I} c_i x^i + \sum_{i \in I}(a_i - c_i)x^i + (b_i - c_i)x^{i+1} = \sum_{i \in I} a_i x^i + b_i x^{i+1} = f.$$

Also, by Proposition 2.1 (6) and (8) we have

$$\|(1 + x)g\| = \|g\| + \|xg\| = 2\|g\| = \sum_{i \in I} 2c_i \geq \sum_{i \in I}(ta_i + tb_i) = t\sum_{i \in I} a_i + b_i = t\|f\|.$$

If $\mathcal{R}(f) = 1$ this forces $\|h\| = 0$ and so $f = (1 + x)g$.

By (2.2) in Proposition 2.2, if $f \neq 0$, we have

$$\|f\| \cdot \mathcal{P}(f) = \sum_{i \in \mathbb{Z}} \min\left(\langle f, x^i \rangle, \langle f, x^{i+1} \rangle\right) \geq \sum_{i \in \mathbb{Z}} \min(a_i, b_i) = \sum_{i \in \mathbb{Z}} c_i = \|g\| = \frac{\mathcal{R}(f)\, \|f\|}{2},$$

so $\mathcal{P}(f) \geq \frac{\mathcal{R}(f)}{2}$. If $f = 0$ then $\mathcal{P}(f) = 1 = \mathcal{R}(f)$. ∎

Proposition 4.3: If $a, b, c \in [0, \infty)$, then

$$\mathcal{R}(a + bx + cx^2) = \min\left(\frac{a+c}{b}, \frac{b}{a+c}\right),$$

where we use the convention that $\frac{x}{0} \geq 0$ when $x > 0$, and $\frac{0}{0} = 1$.

Proof:

Claim: $\mathcal{R}(a + x + cx^2) = \min\left(a + c, \frac{1}{a+c}\right)$, for all $a, c \in [0, \infty)$.

Proof of claim:

If either a or c is zero, the result is clear, so we may assume $a \neq 0$ and $c \neq 0$. Note that

$$\begin{aligned}\mathcal{R}(a + x + cx^2) &= \sup_{0 \leq s \leq 1} \min\left\{r(a + sx),\, r\left((1-s)x + cx^2\right)\right\} \\ &= \sup_{0 < s < 1} \min\left\{\frac{s}{a}, \frac{a}{s}, \frac{1-s}{c}, \frac{c}{1-s}\right\}\end{aligned}$$

We let $w(s) = \min\left\{\frac{s}{a}, \frac{a}{s}, \frac{1-s}{c}, \frac{c}{1-s}\right\}$, and note that if we choose $s = \frac{a}{a+c}$, then $\frac{s}{a} = \frac{1}{a+c}$, and $\frac{a}{s} = a + c$, while $1 - s = 1 - \frac{a}{a+c} = \frac{c}{a+c}$, so $\frac{1-s}{c} = \frac{1}{a+c}$, and $\frac{c}{1-s} = a + c$. It follows that $w\left(\frac{a}{a+c}\right) = \min\left\{a + c, \frac{1}{a+c}\right\}$.

We next show that no other choice of s gives a larger value of $w(s)$. If $s > \frac{a}{a+c}$, then $\frac{a}{s} < a + c$, and $1 - s < 1 - \frac{a}{a+c} = \frac{c}{a+c}$, so $\frac{1-s}{c} < \frac{1}{a+c}$, and thus $w(s) < \min\left\{a + c, \frac{1}{a+c}\right\}$. On the other hand, if $s < \frac{a}{a+c}$, then $\frac{s}{a} < \frac{1}{a+c}$, and $1 - s > 1 - \frac{a}{a+c} = \frac{c}{a+c}$, so $\frac{c}{1-s} < a + c$, and thus $w(s) < \min\left\{a + c, \frac{1}{a+c}\right\}$.

We have proved that $\mathcal{R}(a + x + cx^2) = \min\left\{a + c, \frac{1}{a+c}\right\}$, and that this minimum is attained for the decomposition $a + x + cx^2 = \left(a + \frac{a}{a+c}\, x\right) + \left(\frac{c}{a+c}\, x + cx^2\right)$.

∎

Returning to the proof of the proposition, we note that if $a = b = c = 0$, the result holds, as $\mathcal{R}(0) = 1$. If $b = 0$ but a and c are not both zero, then $\mathcal{R}(a + bx + cx^2) = 0$. On the other hand, if $b \neq 0$ then by property (1) of $\mathcal{R}$ we have

$$\mathcal{R}(a + bx + cx^2) = \mathcal{R}\left(\frac{a}{b} + x + \frac{c}{b}x^2\right) = \min\left\{\frac{a}{b} + \frac{c}{b}, \frac{1}{\frac{a}{b} + \frac{c}{b}}\right\} = \min\left\{\frac{a+c}{b}, \frac{b}{a+c}\right\}.$$

∎

We use Proposition 4.3 to obtain some results about the $\mathcal{R}$-value of products. The right-hand side of equation (4.3) in Proposition 4.4 is reminiscent of the addition formula for the hyperbolic tangent:

$$\tanh(a + b) = \frac{\tanh a + \tanh b}{1 + (\tanh a)(\tanh b)}.$$

This similarity is more than superficial, and leads to the fact that the $\mathcal{R}$-value of a product grows as quickly as the hyperbolic tangent of the sum of the $\mathcal{R}$-values (Theorem 4.7).

Proposition 4.4: If $a + bx, c + dx \in \mathbf{R}[x^{\pm 1}]^+$, then

$$\mathcal{R}\big((a+bx)(c+dx)\big) = \frac{\mathcal{R}(a+bx) + \mathcal{R}(c+dx)}{1 + \mathcal{R}(a+bx)\mathcal{R}(c+dx)}. \tag{4.3}$$

Proof: If either $a + bx = 0$ or $c + dx = 0$ the result is clear from the fact that $\mathcal{R}(0) = 1$. Otherwise, by property (1) of $\mathcal{R}()$, we can replace $a + bx$ by $\frac{1}{\max(a,b)}(a+bx)$ and $c + dx$ by $\frac{1}{\max(c,d)}(c+dx)$ without affecting either side of the equation which we are proving. Thus we may assume without loss of generality that $\max(a, b) = 1$ and $\max(c, d) = 1$.

Let α and β be any numbers in $[0, 1]$. Then $(1-\beta)(1-\alpha) \geq 0 \implies 1 - (\alpha+\beta) + \alpha\beta \geq 0 \implies 1 + \alpha\beta \geq \alpha + \beta$. Thus by Proposition 4.3, we have

$$\mathcal{R}\big((1+\alpha x)(1+\beta x)\big) = \mathcal{R}\big(1 + (\alpha+\beta)x + \alpha\beta x^2\big) = \frac{\alpha+\beta}{1+\alpha\beta} = \frac{\mathcal{R}(1+\alpha x) + \mathcal{R}(1+\beta x)}{1 + \mathcal{R}(1+\alpha x)\mathcal{R}(1+\beta x)}.$$

Using this fact, and applying properties (2) and (3) of $\mathcal{R}()$, and the fact that the map $\sum_i f_i x^i \mapsto \sum_i f_i x^{-i}$ is an automorphism of $\mathbf{R}[x^{\pm 1}]$, we note that

$$\begin{aligned}\mathcal{R}\big((\alpha+x)(\beta+x)\big) &= \mathcal{R}\big((\alpha+x^{-1})(\beta+x^{-1})\big) = \mathcal{R}\big((\alpha x+1)(\beta x+1)\big)\\ &= \frac{\mathcal{R}(1+\alpha x) + \mathcal{R}(1+\beta x)}{1 + \mathcal{R}(1+\alpha x)\mathcal{R}(1+\beta x)} = \frac{\mathcal{R}(\alpha+x) + \mathcal{R}(\beta+x)}{1 + \mathcal{R}(\alpha+x)\mathcal{R}(\beta+x)}.\end{aligned}$$

The fact that $\alpha + \beta \leq 1 + \alpha\beta$, and Proposition 4.3 together imply that

$$\mathcal{R}\big((1+\alpha x)(\beta+x)\big) = \mathcal{R}\big(\beta + (1+\alpha\beta)x + \alpha x^2\big) = \frac{\alpha+\beta}{1+\alpha\beta} = \frac{\mathcal{R}(1+\alpha x) + \mathcal{R}(\beta+x)}{1 + \mathcal{R}(1+\alpha x)\mathcal{R}(\beta+x)}.$$

Applying these facts applied to our given polynomials completes the proof. ∎

Proposition 4.5: Let $f, g \in \mathbf{R}[x^{\pm 1}]^+$. Then

$$\mathcal{R}(fg) \geq \frac{\mathcal{R}(f) + \mathcal{R}(g)}{1 + \mathcal{R}(f)\mathcal{R}(g)}.$$

Proof: If either f or g is zero the result is clear, as $\mathcal{R}(0) = 1$. Assume they are both nonzero. By Proposition 4.1 there exist decompositions $f = \sum_{i \in I} a_i x^i + b_i x^{i+1}$ and $g = \sum_{i \in J} c_i x^i + d_i x^{i+1}$ with $\min_{i \in I} r(a_i x^i + b_i x^{i+1}) = \mathcal{R}(f)$ and $\min_{j \in J} r(c_i x^i + d_i x^{i+1}) = \mathcal{R}(g)$. For each $i \in I$ let $P_i = a_i + b_i x$, and for each $j \in J$ let $Q_j = c_j + d_j x$, so that $f = \sum_{i \in I} P_i x^i$ and $g = \sum_{j \in J} Q_j x^j$. We may assume (by restricting I and J) that $P_i \neq 0$, $\forall i \in I$, and $Q_j \neq 0$, $\forall j \in J$. Applying Proposition 4.4 and properties (5) and (2) of $\mathcal{R}()$, we have

$$\begin{aligned}\mathcal{R}(fg) = \mathcal{R}\left(\sum_{i \in I, j \in J} P_i Q_j x^{i+j}\right) &\geq \min_{i \in I, j \in J} \mathcal{R}\big(P_i Q_j x^{i+j}\big) = \min_{i \in I, j \in J} \mathcal{R}(P_i Q_j)\\ &= \min_{i,j} \frac{\mathcal{R}(P_i) + \mathcal{R}(Q_j)}{1 + \mathcal{R}(P_i)\mathcal{R}(Q_j)}.\end{aligned}$$

As $\mathcal{R}(P_i) \geq \mathcal{R}(f)$ for each $i \in I$, and $\mathcal{R}(Q_j) \geq \mathcal{R}(g)$ for each $j \in J$, we may use the fact that $\frac{x+y}{1+xy}$ is a non-decreasing function of both x and y on $[0,1] \times [0,1]$, to conclude that $\frac{\mathcal{R}(P_i)+\mathcal{R}(Q_j)}{1+\mathcal{R}(P_i)\mathcal{R}(Q_j)} \geq \frac{\mathcal{R}(f)+\mathcal{R}(g)}{1+\mathcal{R}(f)\mathcal{R}(g)}$ for all $i \in I$ and $j \in J$. Thus $\mathcal{R}(fg) \geq \frac{\mathcal{R}(f)+\mathcal{R}(g)}{1+\mathcal{R}(f)\mathcal{R}(g)}$, as required. ∎

Corollary 4.6: If $f, g \in \mathbf{R}[x^{\pm 1}]^+$ then $\mathcal{R}(fg) \geq \min\{\mathcal{R}(f), \mathcal{R}(g)\}$. In particular, if $\mathcal{R}(f) = 1$ then $\mathcal{R}(fg) = 1$.

Proof: By Proposition 4.5 we have

$$\mathcal{R}(fg) - \mathcal{R}(f) \geq \frac{\mathcal{R}(f) + \mathcal{R}(g)}{1 + \mathcal{R}(f)\mathcal{R}(g)} - \mathcal{R}(f) = \frac{\mathcal{R}(f) + \mathcal{R}(g) - \mathcal{R}(f) - \left(\mathcal{R}(f)\right)^2 \mathcal{R}(g)}{1 + \mathcal{R}(f)\mathcal{R}(g)}$$

$$= \frac{\mathcal{R}(g)\left[1 - \left(\mathcal{R}(f)\right)^2\right]}{1 + \mathcal{R}(f)\mathcal{R}(g)} \geq 0,$$

so $\mathcal{R}(fg) \geq \mathcal{R}(f)$. Similarly, $\mathcal{R}(fg) \geq \mathcal{R}(g)$, and so $\mathcal{R}(fg) \geq \min\{\mathcal{R}(f), \mathcal{R}(g)\}$. If $\mathcal{R}(f) = 1$, then by property (0) of $\mathcal{R}()$ we have $1 \geq \mathcal{R}(fg) \geq \mathcal{R}(f) = 1$, so $\mathcal{R}(fg) = 1$. ∎

Theorem 4.7: If $f_1, f_2, \ldots, f_n \in \mathbf{R}[x^{\pm 1}]^+$ then

$$\mathcal{R}(f_1 f_2 \cdots f_n) \geq \tanh\left(\sum_{i=1}^{n} \mathcal{R}(f_i)\right)$$

Proof: Let $R_i = \mathcal{R}(f_i)$ for $i = 1, 2, \ldots, n$. We use induction to prove the inequality

$$\mathcal{R}(f_1 f_2 \cdots f_n) \geq \frac{\prod_{i=1}^{n}(1 + R_i) - \prod_{i=1}^{n}(1 - R_i)}{\prod_{i=1}^{n}(1 + R_i) + \prod_{i=1}^{n}(1 - R_i)} = \frac{1 - \prod_{i=1}^{n} \frac{1-R_i}{1+R_i}}{1 + \prod_{i=1}^{n} \frac{1-R_i}{1+R_i}}. \tag{4.4}$$

If $n = 1$, the inequality (4.4) clearly holds. Assuming that it holds for $n = k$, we show that it holds for $k + 1$.

Applying Proposition 4.5 and the fact that $\frac{a+b}{1+ab}$ is a non-decreasing function of both a and b for $a, b \in [0, 1]$, (as $\frac{\partial}{\partial a}\left(\frac{a+b}{1+ab}\right) = \frac{1-b^2}{(1+ab)^2} \geq 0$), and writing $P = \prod_{i=1}^{k} \frac{1-R_i}{1+R_i}$ we have

$$\mathcal{R}(f_1 f_2 \cdots f_{k+1}) = \mathcal{R}\big((f_1 f_2 \ldots f + k) f_{k+1}\big) \geq \frac{\mathcal{R}(f_1 f_2 \cdots f_k) + \mathcal{R}(f_{k+1})}{1 + \mathcal{R}(f_1 f_2 \cdots f_k)\mathcal{R}(f_k + 1)}$$

$$\geq \frac{\frac{1-P}{1+P} + R_{k+1}}{1 + \frac{1-P}{1+P} R_{k+1}} = \frac{1 - P + R_{k+1} + R_{k+1}P}{1 + P + R_{k+1} - R_{k+1}P}$$

$$= \frac{(1 + R_{k+1}) - P(1 - R_{k+1})}{(1 + R_{k+1}) + P(1 - R_{k+1})} = \frac{1 - P\frac{1-R_{k+1}}{1+R_{k+1}}}{1 + P\frac{1-R_{k+1}}{1+R_{k+1}}} = \frac{1 - \prod_{i=1}^{k+1} \frac{1-R_i}{1+R_i}}{1 + \prod_{i=1}^{k+1} \frac{1-R_i}{1+R_i}}.$$

This establishes the inequality (4.4).

If some $R_i = 1$, then $\mathcal{R}(f_1 f_2 \cdots f_n) = 1$, by Corollary 4.6, and the our result is clear, as $\tanh s < 1$ for all $s \in \mathbf{R}$. Assume that each $R_i \neq 1$. Letting $S = \sum_{i=1}^{n} R_i$, we note that

$$\ln\left[\prod_{i=1}^{n} \frac{1 - R_i}{1 + R_i}\right] = \sum_{i=1}^{n} \ln\left(\frac{1 - R_i}{1 + R_i}\right) = \sum_{i=1}^{n} -2\left[R_i + \frac{R_i^3}{3} + \frac{R_i^5}{5} + \cdots\right] \leq -2\sum_{i=1}^{n} R_i = -2S.$$

Thus $\prod_{i=1}^{n} \frac{1-R_i}{1+R_i} \leq e^{-2S}$. Using the fact that $\frac{d}{dx}\left(\frac{1-x}{1+x}\right) = -\frac{2}{(1+x)^2} < 0$ for $x \geq 0$, and applying inequality (4.4), we have

$$\mathcal{R}(f_1 f_2 \cdots f_n) \geq \frac{1 - \prod_{i=1}^{n} \frac{1-R_i}{1+R_i}}{1 + \prod_{i=1}^{n} \frac{1-R_i}{1+R_i}} \geq \frac{1 - e^{-2S}}{1 + e^{-2S}} = \frac{e^S - e^{-S}}{e^S + e^{-S}} = \tanh S.$$

∎

Our next task is to show how $\mathcal{R}()$ can be used to identify strong positivity of a sequence in $\mathbf{R}[x^{\pm 1}]^+$. The following sequence of results leads to Theorem 4.12. They could each be deduced from Corollary 1.9 in [BH], but we have decided to derive them from elementary principles. Lemma 4.8 is due to Meissner [Mn], while the proof of it given here was inspired by the proof of Lemma 3 in [H4].

Lemma 4.8: Let $f = a-bx+cx^2$ with $a, b, c > 0$ and $b^2-4ac < 0$. Then for sufficiently large n, the polynomial $(1+x)^n f$ is gapless and has only nonnegative coefficients.

Proof: Rescaling by $\frac{1}{a}$, it suffices to prove the result for the case $a = 1$, that is, $f = 1-bx+cx^2$ (and $b^2-4c < 0$). The end coefficients of the product, namely $\langle (1+x)^n f,\, x^k \rangle$ for $k = 0, 1, n+1$ and $n+2$ are, respectively, 1, $n-b$, $nc-b$ and c, and it is clear that for sufficiently large n each of these is strictly positive. It remains to show that $\langle (1+x)^n f,\, x^k \rangle > 0$ for $2 \leq k \leq n$, for sufficiently large n. Now for $2 \leq k \leq n$,

$$\langle (1+x)^n f,\, x^k \rangle = \left\langle \sum_{i=0}^{n} \binom{n}{i} x^i (1-bx+cx^2),\, x^k \right\rangle = \binom{n}{k} - b\binom{n}{k-1} + c\binom{n}{k-2}$$
$$= \binom{n}{k}\left[1 - b\frac{k}{n-k+1} + c\frac{k(k-1)}{(n-k+1)(n-k+2)}\right]$$

Replacing k by γn, we let

$$g_n(\gamma) = 1 - b\frac{\gamma n}{n-\gamma n+1} + c\frac{\gamma n(\gamma n-1)}{(n-\gamma n+1)(n-\gamma n+2)}$$
$$= 1 - b\frac{\gamma}{1-\gamma+\frac{1}{n}} + c\frac{\gamma(\gamma-\frac{1}{n})}{(1-\gamma+\frac{1}{n})(1-\gamma+\frac{2}{n})}.$$

We show that for all sufficiently large n, $g_n(\gamma) > 0$ for all $\gamma \in [\frac{2}{n}, 1]$. Assume that $\gamma \in [\frac{2}{n}, 1]$. Let $\gamma_1 = \frac{1}{1+b}$ and note that for $\gamma \leq \gamma_1$ we have

$$g_n(\gamma) \geq 1 - b\frac{\gamma}{1-\gamma+\frac{1}{n}} > 1 - b\frac{\gamma}{1-\gamma} \geq 1 - b\frac{\frac{1}{1+b}}{1-\frac{1}{1+b}} = 1 - b\frac{1}{(1+b)-1} = 1 - \frac{b}{b} = 0,$$

so $g_n(\gamma) > 0$ if $\gamma \leq \gamma_1$.

Let $\gamma_2 = \frac{b+\frac{c}{2}}{b+c}$, and assume that $n > \frac{2(c+2b)}{c}$, so that $\frac{1}{n}(c+2b) < \frac{c}{2}$. Then for $\gamma \geq \gamma_2$ we have

$$g_n(\gamma) - 1 = -b\frac{\gamma}{1-\gamma+\frac{1}{n}} + c\frac{\gamma(\gamma-\frac{1}{n})}{(1-\gamma+\frac{1}{n})(1-\gamma+\frac{2}{n})}$$
$$= \frac{\gamma}{(1-\gamma+\frac{1}{n})(1-\gamma+\frac{2}{n})}\left[-b\left(1-\gamma+\frac{2}{n}\right) + c\left(\gamma-\frac{1}{n}\right)\right]$$

and

$$-b\left(1-\gamma+\frac{2}{n}\right) + c\left(\gamma-\frac{1}{n}\right) = -b + b\gamma - \frac{2b}{n} + c\gamma - \frac{c}{n} = (b+c)\gamma - b - \frac{1}{n}(c+2b)$$
$$> (b+c)\gamma - b - \frac{c}{2} \geq (b+c)\frac{b+\frac{c}{2}}{b+c} - \left(b+\frac{c}{2}\right) = 0$$

so $g_n(\gamma) > 0$ if $\gamma \geq \gamma_2$ and $n > \frac{2(c+2b)}{c}$.

It remains to show that for sufficiently large n, $g_n(\gamma) > 0$ for all $\gamma \in [\gamma_1, \gamma_2] \cap [\frac{2}{n}, 1]$. Since by assumption $b^2 - 4c < 0$, it follows that $c > \frac{b^2}{4}$, and so we have $c = \frac{b^2}{4}\alpha$ for some $\alpha > 1$. If γ and n are such that

$$\frac{\gamma}{\gamma - \frac{1}{n}} \cdot \frac{1 - \gamma + \frac{2}{n}}{1 - \gamma + \frac{1}{n}} < \alpha \tag{$*$}$$

then

$$\begin{aligned} g_n(\gamma) &= 1 - b\frac{\gamma}{1 - \gamma + \frac{1}{n}} + \frac{b^2}{4}\alpha\,\frac{\gamma(\gamma - \frac{1}{n})}{(1 - \gamma + \frac{1}{n})(1 - \gamma + \frac{2}{n})} \\ &> 1 - b\frac{\gamma}{1 - \gamma + \frac{1}{n}} + \frac{b^2}{4}\,\frac{\gamma}{\gamma - \frac{1}{n}} \cdot \frac{1 - \gamma + \frac{2}{n}}{1 - \gamma + \frac{1}{n}}\,\frac{\gamma(\gamma - \frac{1}{n})}{(1 - \gamma + \frac{1}{n})(1 - \gamma + \frac{2}{n})} \\ &= 1 - b\frac{\gamma}{1 - \gamma + \frac{1}{n}} + \frac{b^2}{4}\,\frac{\gamma^2}{(1 - \gamma + \frac{1}{n})^2} = \left(1 - b\frac{\gamma}{1 - \gamma + \frac{1}{n}}\right)^2 \geq 0 \end{aligned}$$

We note that if $\gamma_1 \leq \gamma \leq \gamma_2$ then

$$\frac{\gamma}{\gamma - \frac{1}{n}} \cdot \frac{1 - \gamma + \frac{2}{n}}{1 - \gamma + \frac{1}{n}} = \frac{1}{1 - \frac{1}{\gamma n}} \cdot \frac{(1 - \gamma) + \frac{2}{n}}{(1 - \gamma) + \frac{1}{n}} \leq \frac{1}{1 - \frac{1}{\gamma_1 n}} \cdot \frac{(1 - \gamma_2) + \frac{2}{n}}{(1 - \gamma_2) + \frac{1}{n}}$$

As the right-hand side of this inequality approaches 1 as $n \to \infty$, we may choose n so large that it is less than α. For such n and for $\gamma \in [\gamma_1, \gamma_2]$, inequality $(*)$ holds, and so $g_n(\gamma) > 0$. We have thus shown that for sufficiently large n, $g_n(\gamma) > 0$ for all $\gamma \in [\frac{2}{n}, 1]$, which completes the proof. ∎

Corollary 4.9: Let $f = a - bx + cx^2$ with $a, b, c > 0$ and $b^2 - 4ac < 0$, and let $r \in (0, \infty)$. Then for sufficiently large n, the polynomial $(1 + rx)^n f$ is gapless and has only nonnegative coefficients.

Proof: We define a new polynomial $g = a - \frac{b}{r}x + \frac{c}{r^2}x^2$, and note that it satisfies the condition of Lemma 4.8, as $\left(\frac{b}{r}\right)^2 - 4a\left(\frac{c}{r^2}\right) = \frac{1}{r^2}(b^2 - 4ac) < 0$. Lemma 4.8 guarantees that for sufficiently large n, $\left\langle (1 + x)^n g, x^k \right\rangle > 0$ for $k = 0, 1, 2, \ldots, n + 2$. The map $x \mapsto rx$ induces an automorphism on $\mathbf{R}[x^{\pm 1}]$ which carries $\mathbf{R}[x^{\pm 1}]^+$ into $\mathbf{R}[x^{\pm 1}]^+$, and this automorphism sends $(1 + x)^n g$ into $(1 + rx)^n f$. It follows that for sufficiently large n, $\left\langle (1 + rx)^n f, x^k \right\rangle = r^k \left\langle (1 + x)^n g, x^k \right\rangle > 0$ for $k = 0, 1, 2, \ldots, n + 2$. ∎

Corollary 4.10: Let $f \in \mathbf{R}[x^{\pm 1}]$ be such that $f(x) > 0$ for all $x \in (0, \infty)$, and let $r \in (0, \infty)$. Then for sufficiently large n, the polynomial $(1 + rx)^n f$ is gapless and has no negative coefficients.

Proof: Multiplying by an appropriate power of x, we may assume, without loss of generality, that the elements of $\operatorname{Log} f$ are non-negative integers, and that $\langle f, x^o \rangle \neq 0$. Since $f(x) > 0$ for all $x \in (0, \infty)$, the highest coefficient of f must be positive. If $\operatorname{Log} f = \{0\}$ we are done. Otherwise, we may factor f (over $\mathbf{R}$) into a product of linear and irreducible quadratic factors with positive leading coeffients, say $f = f_1 f_2 \cdots f_m$. It suffices to show that each one of these factors becomes gapless and nonnegative after multiplcation by $(1 + rx)^n$ for sufficiently large n. As f has no positive real roots, any linear factors must lie in $\mathbf{R}[x^{\pm 1}]^+$, and are necessarily gapless. The irreducible quadratic factors must be of the form $ax^2 - bx + c$ with $b^2 - 4ac < 0$.

As $a > 0$ it follows that also $c > 0$. If $b \leq 0$ then the quadratic is already in $\mathbf{R}[x^{\pm 1}]^+$, and multiplying by $1 + rx$ (in case b is 0) makes it gapless. It suffices to show that for any quadratic $f_i = ax^2 - bx + c$, with $a, b, c > 0$ and $b^2 - 4ac < 0$, the product $(1 + rx)^n f_i$ is gapless and nonnegative for sufficiently large n; but this is a consequence of Corollary 4.9. ∎

Corollary 4.11: Let $f \in \mathbf{R}[x^{\pm 1}]$ be such that $f(x) > 0$ for all $x \in (0, \infty)$, and let $\{a_i + b_i x\}_{i \in \mathbf{N}}$ be a sequence in $\mathbf{R}[x^{\pm 1}]^+$ for which

$$\inf_{i \in \mathbf{N}} r(a_i + b_i x) > 0.$$

Then for sufficiently large m, the product

$$\left(\prod_{i=1}^{m} (a_i + b_i x) \right) f$$

is gapless and has no negative coefficients.

Proof: This proof mimics the argument of Lemma 1.2 in [BH]. If $a_i + b_i x = 0$ for some i, the result is trivially true. Otherwise, multiplying each $a_i + b_i x$ by $\frac{1}{a_i}$, we may assume, without loss of generality, that $a_i = 1$ for all $i \in \mathbf{N}$. Letting $s = \inf_{i \in \mathbf{N}} r(a_i + b_i x)$, we see that $b_i \in [s, \frac{1}{s}]$ for all i. Clearly the sequence $\{b_i\}_{i \in \mathbf{N}}$ has a cluster point r in $[s, \frac{1}{s}]$. By Corollary 4.10, for sufficiently large n, the polynomial $P = (1 + rx)^n f$ is gapless and has no negative coefficients. If we let $k_1 = \min \operatorname{Log} f$ and $k_2 = \max \operatorname{Log} f$, then as P is gapless, we have $\operatorname{Log} P = \{k_1, k_1 + 1, k_1 + 2, \ldots, k_2 + n\}$. Let $\epsilon = \min_{k \in \operatorname{Log} P} \langle P, x^k \rangle$. Since for each $k \in \operatorname{Log} P$ the coefficient of x^k in $(\prod_{i=1}^{n} (1 + r_i x)) f$ is a polynomial in $r_1, r_2, \ldots, r_n$, it follows, by continuity, that there is a $\delta > 0$ such that

$$\left| \left\langle \left(\prod_{i=1}^{n} (1 + r_i x) \right) f, x^k \right\rangle - \langle ((1 + rx)^n f, x^k \rangle \right| < \epsilon$$

for all $k \in \operatorname{Log} P$, whenever $|r_i - r| < \delta$ for $i = 1, 2, \ldots, n$. Let $i_1, i_2, \ldots, i_n$ be an increasing sequence of integers such that $|b_{i_j} - r| < \delta$ for $j = 1, 2, \ldots, n$. Then the coefficient $\left\langle \left(\prod_{j=1}^{n} (1 + b_{i_j} x) \right) f, x^k \right\rangle$ is strictly positive if $k_1 \leq k \leq k_2 + n$, and is zero otherwise. It follows that for $m \geq i_n$, the product $(\prod_{i=1}^{m} (1 + b_i x)) f$ is gapless and has no negative coeffiecients.
∎

We are now ready to state how $\mathcal{R}$ can be used to identify strongly positive sequences. Theorem 4.12 is equivalent to Theorem 1.11 in [BH]. Later (Theorem 4.21) we show that $\mathcal{R}$ can be used to characterize strongly positive sequences.

Theorem 4.12: Every sequence $\{P_i\}_{i \in \mathbf{N}}$ in $\mathbf{R}[x^{\pm 1}]^+$ for which

$$\sum_{i=1}^{\infty} \mathcal{R}(P_i) = \infty$$

is strongly positive.

Proof: Using Theorem 4.7, we note that for any $i, n \in \mathbf{N}$,

$$\mathcal{R}(P_iP_{i+1}P_{i+2}\cdots P_{i+n}) \geq \tanh\left(\sum_{j=i}^{i+n}\mathcal{R}(P_j)\right),$$

so we can form a blocking $\{Q_i\}_{i\in\mathbf{N}}$ of the sequence $\{P_i\}_{i\in\mathbf{N}}$, with $\mathcal{R}(Q_i) > \frac{1}{2}$ for each $i \in \mathbf{N}$. By Proposition 4.1, each Q_i can be decomposed as $Q_i = \sum_{j\in I}(a_i + b_ix)x^i$ with $I \subset \mathbf{Z}$ and $\min_{j\in I} r(a_i + b_ix) = \mathcal{R}(Q_i) > \frac{1}{2}$. Applying Corollary 4.11 and Proposition 1.5, shows that the sequence $\{Q_i\}_{i\in\mathbf{N}}$ is strongly positive. It follows that the original sequence $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive. ∎

Corollary 4.13: If $\{P_i\}_{i\in\mathbf{N}}$ is a sequence in $\mathbf{R}[x^{\pm 1}]^+\backslash\{0\}$ and there exists $\epsilon > 0$ such that for every tail $\{P_i\}_{i>t}$ of the given sequence there is a $n \in \mathbf{N}$ with

$$\mathcal{R}\left(\prod_{i=t+1}^{t+n} P_i\right) > \epsilon,$$

then the sequence is strongly positive.

Proof: Form a blocking $\{Q_i\}_{i\in\mathbf{N}}$ of the sequence $\{P_i\}_{i\in\mathbf{N}}$ with $\mathcal{R}(Q_i) > \epsilon$ for each i. Theorem 4.12 guarantees the strong positivity of $\{Q_i\}_{i\in\mathbf{N}}$, and hence that of the original sequence. ∎

The next two results concern the stability of $\mathcal{R}()$ under perturbations. Proposition 4.14 says that if we perturb the coefficients of a polynomial by multiplicative factors, but no two adjacent coefficients are altered, then the $\mathcal{R}$-value cannot decrease by more than the largest of these multiplicative factors. Corollary 4.15 allows all coefficients of the polynomial to be perturbed. We will make use of these results in chapter 5.

Proposition 4.14: Suppose f and g are in $\mathbf{R}[x^{\pm 1}]^+\backslash\{0\}$ and satisfy

(1) $\operatorname{Log} f = \operatorname{Log} g$

(2) There exists $\alpha \in (0, 1]$ such that $\frac{\langle f, x^i\rangle}{\langle g, x^i\rangle} \in \left[\alpha, \frac{1}{\alpha}\right]$, $\forall i \in \operatorname{Log} f$.

(3) For every $i \in \mathbf{Z}$ either $\langle f, x^i\rangle = \langle g, x^i\rangle$ or $\langle f, x^{i+1}\rangle = \langle g, x^{i+1}\rangle$.

Then

$$\mathcal{R}(g) \geq \alpha\,\mathcal{R}(f).$$

Proof: Write $f = \sum_{i\in\mathbf{Z}} f_ix^i$ and $g = \sum_{i\in\mathbf{Z}} g_ix^i$. If $\mathcal{R}(f) = 0$, the result is clear. Assume $\mathcal{R}(f) \neq 0$. By Proposition 4.1 we can write $f = \sum_{i\in I} a_ix^i + b_ix^{i+1}$ with $\mathcal{R}(f) = \min_{i\in I} r(a_ix^i + b_ix^{i+1})$. For $i \in \mathbf{Z}\setminus I$ we define $a_i = 0$ and $b_i = 0$, so that $f = \sum_{i\in\mathbf{Z}} a_ix^i + b_ix^{i+1}$, and $f_i = a_i + b_{i-1}$ for all $i \in \mathbf{Z}$. Clearly $I \subseteq \operatorname{Log} f$.

For $i \in \operatorname{Log} f$ we define $c_i = a_i\frac{g_i}{f_i}$ and $d_{i-1} = b_{i-1}\frac{g_i}{f_i}$, and we set $c_i = 0 = d_{i-1}$ for $i \in \mathbf{Z}\backslash\operatorname{Log} f$. Note that c_i can be nonzero only when $i \in \operatorname{Log} f$ and a_i is nonzero, so that $i \in I$.

Likewise d_{i-1} can be nonzero only when $i \in \operatorname{Log} f$ and b_{i-1} is non-zero, so that $i - 1 \in I$. Thus $\sum_{i \in \mathbf{Z}} c_i x^i + d_i x^{i+1} = \sum_{i \in I} c_i x^i + d_i x^{i+1}$. For $i \in \operatorname{Log} f$ we have

$$g_i = f_i \frac{g_i}{f_i} = (a_i + b_{i-1}) \frac{g_i}{f_i} = a_i \frac{g_i}{f_i} + b_{i-1} \frac{g_i}{f_i} = c_i + d_{i-1} = \left\langle \sum_{j \in \mathbf{Z}} c_j x^j + d_j x^{j+1},\, x^i \right\rangle$$

$$= \left\langle \sum_{j \in I} c_j x^j + d_j x^{j+1},\, x^i \right\rangle.$$

As $\operatorname{Log} g = \operatorname{Log} f$, we conclude that $g = \sum_{i \in I} c_i x^i + d_i x^{i+1}$.

Note that for $i \in I$, as either $f_i = g_i$ or $f_{i+1} = g_{i+1}$, we have

$$r\big(c_i x^i + d_i x^{i+1}\big) = \min\left(\frac{c_i}{d_i}, \frac{d_i}{c_i}\right) = \min\left(\frac{g_i\, f_{i+1}}{f_i\, g_{i+1}} \frac{a_i}{b_i}, \frac{f_i\, g_{i+1}}{g_i\, f_{i+1}} \frac{b_i}{a_i}\right)$$
$$\geq \alpha \min\left(\frac{a_i}{b_i}, \frac{b_i}{a_i}\right) = \alpha\, r\big(a_i x^i + b_i x^{i+1}\big)$$

Thus

$$\mathcal{R}(g) \geq \min_{i \in I} r\big(c_i x^i + d_i x^{i+1}\big) \geq \alpha \min_{i \in I} r\big(a_i x^i + b_i x^{i+1}\big) = \alpha\, \mathcal{R}(f).$$

∎

Corollary 4.15: Suppose f and g are in $\mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ and satisfy

(1) $\operatorname{Log} f = \operatorname{Log} g$

(2) There exists $\alpha \in (0, 1]$ such that $\frac{\langle f, x^i \rangle}{\langle g, x^i \rangle} \in \left[\alpha, \frac{1}{\alpha}\right]$, $\forall i \in \operatorname{Log} f$.

Then

$$\frac{\mathcal{R}(f)}{\alpha^2} \geq \mathcal{R}(g) \geq \alpha^2\, \mathcal{R}(f).$$

Proof: We define a new polynomial h which has odd coefficients equal to those of f and even coefficients equal to those of g. By Proposition 4.14, $\mathcal{R}(g) \geq \alpha \mathcal{R}(h)$ and $\mathcal{R}(h) \geq \alpha \mathcal{R}(f)$, so that $\mathcal{R}(g) \geq \alpha^2 \mathcal{R}(f)$. Similarly $\mathcal{R}(f) \geq \alpha^2 \mathcal{R}(g)$, and the result follows. ∎

The following result is a characterization of fully contiguous Laurent polynomials. It and the following results through Lemma 4.20 are fairly elementary, but will find uses in chapter 5.

Proposition 4.16: If $f \in \mathbf{R}[x^{\pm 1}]^+$ then $\mathcal{R}(f) = 1$ if and only if $f(-1) = 0$.

Proof: If $\mathcal{R}(f) = 1$, then by Proposition 4.2 f can be written as $f = (1 + x)g$ for some $g \in \mathbf{R}[x^{\pm 1}]^+$, and so $f(x) = 0$.

To prove the reverse implication, assume that $f(-1) = 0$. By appropriate choice of k, we can guarantee that $\operatorname{Log}(x^k f) \subset \mathbf{Z}^+$, and so $x^k f \in \mathbf{R}[x]$. By the factor theorem we can write $x^k f = \big(x - (-1)\big)g = (x + 1)g$, for some $g \in \mathbf{R}[x]$. By property (2) of $\mathcal{R}(\,)$, and Corollary 4.6, we have $\mathcal{R}(f) = \mathcal{R}(x^k f) = \mathcal{R}\big((x + 1)g\big) \geq \mathcal{R}(x + 1) = 1$. It follows that $\mathcal{R}(f) = 1$. ∎

Lemma 4.17: Let $f = \sum_{i=1}^{n} f_i x^i \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$, be unimodal with peak at 1. Then there is a number $\alpha \in [0,\, f_1]$ such that $\mathcal{R}(\alpha + f) = 1$.

Proof: Let $f_i = 0$ for $i = n+1, n+2, \ldots$, and define $s_i = f_i - f_{i+1}$ for $i \geq 1$. By unimodality, each s_i is nonnegative, and $\sum_{i=1}^{\infty} s_i = f_1$. Note that $f(-1) = \sum_{i=1}^{\infty}(-1)^i f_i = \sum_{i=1}^{\infty}(f_{2i} - f_{2i-1}) = \sum_{i=1}^{\infty} -s_{2i-1} \leq 0$. Multiplying through by -1 yields $0 \leq -f(-1) = \sum_{i=1}^{\infty} s_{2i-1} \leq \sum_{i=1}^{\infty} s_i = f_1$. Let $\alpha = -f(-1)$, and define $g = \alpha + f$. We note that $g(-1) = \alpha + f(-1) = -f(-1) + f(1) = 0$, and so by Proposition 4.16, $\mathcal{R}(g) = 1$, as required. ∎

Lemma 4.18: Suppose $f = \sum_{i=1}^{n} f_i x^i \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ is unimodal with peak at 1. If α is any number with $\frac{1}{2} f_1 \leq \alpha \leq f_1$, then $\mathcal{R}(\alpha + f) \geq \frac{1}{2}$.

Proof: We may assume, that $f_n \neq 0$. If $n = 1$, the result is trivially true. Assume $n \geq 2$. We define numbers $a_n, a_{n-1}, a_{n-2}, \ldots, a_1$ with $0 \leq a_i \leq \frac{1}{2} f_i$, by setting $a_n = 0$ and using the recursive formula $a_{i-1} = \frac{1}{2}(f_i - a_i)$, for $i = n, n-1, n-2, \ldots, 2$, noting that the relationship $0 \leq a_i \leq \frac{1}{2} f_i$ is preserved at each stage, since the fact that $0 \leq a_i \leq f_i$ guarantees that $a_{i-1} \geq 0$, and unimodality guarantees that $a_{i-1} \leq \frac{1}{2} f_i \leq \frac{1}{2} f_{i-1}$. Now

$$\sum_{i=1}^{n} a_i x^i = \sum_{i=1}^{n-1} a_i x^i = \sum_{i=1}^{n-1} \frac{1}{2}(f_{i+1} - a_{i+1}) x^i = \sum_{i=2}^{n} \frac{1}{2}(f_i - a_i) x^{i-1},$$

so

$$\begin{aligned} \alpha + f &= \alpha + \sum_{i=1}^{n} f_i x^i = \alpha + \sum_{i=1}^{n} (f_i - a_i) x^i + \sum_{i=1}^{n} a_i x^i = \alpha + \sum_{i=1}^{n} (f_i - a_i) x^i + \sum_{i=2}^{n} \frac{1}{2}(f_i - a_i) x^{i-1} \\ &= \alpha + (f_1 - a_1) x + \sum_{i=2}^{n} (f_i - a_i)\left(x^i + \frac{1}{2} x^{i-1}\right) \end{aligned}$$

The contiguity of each term in the summation is $\frac{1}{2}$. To prove that $\mathcal{R}(\alpha + f) \geq \frac{1}{2}$, it remains to show that the contiguity of $g = \alpha + (f_1 - a_1) x$ is at least $\frac{1}{2}$. Since $0 \leq a_1 \leq \frac{1}{2} f_1$ and $\frac{1}{2} f_1 \leq \alpha \leq f_1$, we have $\frac{f_1 - a_1}{\alpha} \geq \frac{f_1 - \frac{1}{2} f_1}{f_1} = \frac{1}{2}$, and $\frac{\alpha}{f_1 - a_1} \geq \frac{\frac{1}{2} f_1}{f_1} = \frac{1}{2}$. It follows that $\mathcal{R}(g) = r(g) \geq \frac{1}{2}$. ∎

It is worth noting that the conditions on α in the preceding lemma cannot be weakened to allow $\alpha > f_1$, as may be seen by considering the example $f = x + x^2$ and noting that $\mathcal{R}(\alpha + f) = \frac{1}{1+\alpha}$ (by Proposition 4.3). That $\alpha < \frac{1}{2} f_1$ is not allowed is evident from considering $f = x$.

Proposition 4.19: Let $f \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ be unimodal with peak at k. If $\mathcal{R}(f) < \frac{1}{2}$ then $\langle f, x^k \rangle > \langle f, x^{k-1} \rangle + \langle f, x^{k+1} \rangle$.

Proof: By property (2) of $\mathcal{R}$, it suffices to prove the result for the case where the peak is at 0. Write $f = \sum_{i \in \mathbf{Z}} f_i x^i$. It suffices to show that $\mathcal{R}(f) \geq \frac{1}{2}$ whenever $f_o \leq f_{-1} + f_1$. We let $\alpha = \frac{f_o}{f_{-1} + f_1} f_1$ and $\beta = \frac{f_o}{f_{-1} + f_1} f_{-1}$, and note that $\alpha + \beta = f_o$. As $f_o \leq f_{-1} + f_1$, it follows that $\alpha \leq f_1$ and $\beta \leq f_{-1}$. By unimodality, $\frac{\alpha}{f_1} = \frac{f_o}{f_{-1} + f_1} \geq \frac{f_o}{f_o + f_o} = \frac{1}{2}$. Similarly $\frac{\beta}{f_{-1}} \geq \frac{1}{2}$. Take $g = \sum_{i \geq 1} f_i x^i$, and $h = \sum_{i \leq -1} f_i x^i$. If $g \neq 0$, we may use the fact that $\frac{1}{2} f_1 \leq \alpha \leq f_1$ and Lemma 4.18 to conclude that $\mathcal{R}(\alpha + g) \geq \frac{1}{2}$. On the other hand, if $g \neq 0$, we may use Lemma 4.18 and the fact that $\frac{1}{2} f_{-1} \leq \beta \leq f_{-1}$, to conclude that $\mathcal{R}(\beta + h(x^{-1})) \geq \frac{1}{2}$, which implies, by property (3) of $\mathcal{R}()$, that $\mathcal{R}(\beta+h) \geq \frac{1}{2}$. Since $f = (h+\beta)+(\alpha+g)$, we conclude, by property (5) of $\mathcal{R}()$, that $\mathcal{R}(f) \geq \frac{1}{2}$. ∎

Lemma 4.20: If $f \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ and $\langle f, x^i \rangle \leq \langle f, x^{i-1} \rangle + \langle f, x^{i+1} \rangle$ for all $i \in \mathbf{Z}$, then $\mathcal{R}(f) \geq \frac{1}{2}$.

Proof: We prove the result by induction on the number of modes in f. If f is unimodal, then the result follows from Proposition 4.19 and property (2) of $\mathcal{R}()$.

Suppose that f has $n+1$ modes, and that the result has been proved for the case of n modes. Write $f = \sum_{i\in\mathbf{Z}} f_i x^i$. Let j be the last peak of the first mode of f, and let k be the first peak in the second mode of f (counting from the left). From the definition of mode it is clear that $j+1 \leq k-1$.

If $j+1 < k-1$, we let m be an integer between j and k with f_m minimal. If $m = j+1$, we let $P = \sum_{i\leq m} f_i x^i$ and $Q = \sum_{i>m} f_i x^i$. Otherwise we take $P = \sum_{i<m} f_i x^i$ and $Q = \sum_{i\leq m} f_i x^i$. In either case P is unimodal and Q has n modes, and by the induction hypothesis, we obtain that $\mathcal{R}(P) \geq \frac{1}{2}$ and $\mathcal{R}(Q) \geq \frac{1}{2}$. As $f = P+Q$, it follows, from property (5) of $\mathcal{R}()$, that $\mathcal{R}(f) \geq \frac{1}{2}$.

It remains to prove the case in which $j+1 = k-1$. From the definition of mode we know that $f_k > f_{k-1}$ and $f_k \geq f_{k+1}$, but $f_k \leq f_{k-1} + f_{k+1}$. We let $P = \sum_{i<k} f_i x^i + (f_k - f_{k+1})x^k$, and $Q = f_{k+1}x^k + \sum_{i>k} f_i x^i$. From $f_k \leq f_{k-1}+f_{k+1}$ it follows that $f_{j+1} = f_{k-1} \geq f_k - f_{k+1}$, and as $f_j > f_{j+1}$, it is clear that P is unimodal. Q, on the other hand, has n modes. By the induction hypothesis we have $\mathcal{R}(P) \geq \frac{1}{2}$ and $\mathcal{R}(Q) \geq \frac{1}{2}$, and since $f = P+Q$, we conclude that $\mathcal{R}(f) \geq \frac{1}{2}$. ∎

The following result characterizes strong positivity in terms of $\mathcal{R}$. Recall that for any sequence $\{P_i\}_{i\in\mathbf{N}}$ in $\mathbf{R}[x^{\pm 1}]$, and for any $t \in \mathbf{Z}^+$ and $n \in \mathbf{N}$, we have defined $P^{n,t} = \prod_{i=t+1}^{t+n} P_i$.

Theorem 4.21: Let $\{P_i\}_{i\in\mathbf{N}}$ be a sequence in $\mathbf{R}[x^{\pm 1}]^+\backslash\{0\}$. The following are equivalent.

(1) $\{P_i\}_{i\in\mathbf{N}}$ is a strongly positive sequence.

(2) $\forall t \in \mathbf{Z}^+$, $\lim_{n\to\infty} \mathcal{R}(P^{n,t}) = 1$.

(3) $\{P_i\}_{i\in\mathbf{N}}$ has a blocking $\{Q_i\}_{i\in\mathbf{N}}$ with $\sum_{i=1}^{\infty} \mathcal{R}(Q_i) = \infty$.

(4) There is an $\epsilon > 0$ such that for any $t \in \mathbf{Z}^+$, there is an $n \in \mathbf{N}$ with $\mathcal{R}(P^{n,t}) > \epsilon$.

(5) $\exists \epsilon > 0$ such that $\forall t \in \mathbf{Z}^+$, there exist $t' \geq t$ and $n \in \mathbf{N}$ with $\mathcal{R}\left(P^{n,t'}\right) > \epsilon$.

Proof: We first show that (1) $\implies$ (5). Assume $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive. Let $\epsilon = \frac{1}{3}$, and let $t \in \mathbf{Z}^+$ be given. By strong positivity, there is an $n \in \mathbf{N}$ such that $(1-x+x^2)\left(\prod_{i=t+1}^{t+n} P_i\right) \in \mathbf{R}[x^{\pm 1}]^+$. Let $R_n = \prod_{i=t+1}^{t+n} P_i$. For each $j \in \mathbf{Z}$ we have $0 \leq \langle (1-x+x^2)R_n, x^{j+1} \rangle = \langle R_n, x^{j+1} \rangle - \langle R_n, x^j \rangle + \langle R_n, x^{j-1} \rangle$, so $\langle R_n, x^j \rangle \leq \langle R_n, x^{j+1} \rangle + \langle R_n, x^{j-1} \rangle$. By Lemma 4.20 we have $\mathcal{R}(R_n) \geq \frac{1}{2}$, so taking $t' = t$ we see that $\mathcal{R}(\prod_{i=t'+1}^{t'+n} P_i) \geq \frac{1}{2} > \epsilon$.

We next show that (5) $\implies$ (4). Assuming (5), we take ϵ as in (5), and let $t \in \mathbf{Z}^+$ be given. By (5) there is a $t' \geq t$ and an $n \in \mathbf{N}$ with $\mathcal{R}(P^{n,t'}) > \epsilon$. Note that $P^{n+t'-t,t} = \left(\prod_{i=t+1}^{t'+n} P_i\right) = \left(\prod_{i=t'+1}^{t'+n} P_i\right)\left(\prod_{i=t+1}^{t'} P_i\right) = P^{n,t'}P^{t'-t,t}$, and apply Corollary 4.6 to obtain $\mathcal{R}(P^{n+t'-t,t}) \geq \mathcal{R}(P^{n,t'}) > \epsilon$, so (4) holds.

We now show that (4) $\implies$ (3). Let ϵ be as in (3). Taking $N_1 = 0$ and applying (4) we find an $N_2 > N_1$ such that $\mathcal{R}\left(\prod_{i=N_1+1}^{N_2} P_i\right) > \epsilon$. Continuing in this way, having defined N_j we obtain an $N_{j+1} > N_j$ for which $\mathcal{R}\left(\prod_{i=N_j+1}^{N_{j+1}} P_i\right) > \epsilon$. Letting $Q_j = \prod_{i=N_j+1}^{N_{j+1}} P_i$, for $j \in \mathbf{N}$, we note that $\sum_{i=1}^{n} \mathcal{R}(Q_i) \geq n\epsilon$, and so (3) holds.

Assuming (3), we now prove (2). Letting $\{Q_i\}_{i\in\mathbf{N}}$ be as in (3), and let $t \in \mathbf{Z}^+$ be given. Assume that $Q_j = \prod_{i=N_j+1}^{N_{j+1}} P_i$, for $j \in \mathbf{N}$. Take j with $N_j \geq t$, and note that for $n \in \mathbf{N}$, by property (5) of $\mathcal{R}(\,)$ and Theorem 4.7,

$$\mathcal{R}\left(\prod_{i=t+1}^{N_{j+n}} P_j\right) \geq \mathcal{R}\left(\prod_{i=N_j+1}^{N_{j+n}} P_j\right) = \mathcal{R}\left(\prod_{i=j}^{j+n-1} Q_i\right) \geq \tanh\left(\sum_{i=j}^{j+n-1} \mathcal{R}(Q_i)\right).$$

(2) follows from this fact, Corollary 4.6 and property (1) of $\mathcal{R}(\,)$.

Taking $\epsilon = \frac{1}{2}$, it is clear that (2) $\implies$ (4). The implication (4) $\implies$ (1) is the content of Corollary 4.13, and completes the proof. ∎

The following result can also be proved by studying the extremal states of certain dimension groups related to a sequence of polynomials (see Theorems 2.1 and 2.5 in [BH]). The following proof relies only on the elementary methods of chapters 2 through 4 in this thesis.

Corollary 4.22: Every strongly positive sequence in $\mathbf{R}[x^{\pm 1}]^+$ is a zero-fluctuation sequence.

Proof: If the sequence $\{P_j\}$ has $P_i = 0$ for infinitely many i, then it is trivially a zero-fluctuation sequence. Otherwise, we may take a tail of the sequence for which each P_i is nonzero. Applying Theorem 4.21 (3) we obtain a blocking $\{Q_i\}$ of $\{P_i\}$ for which $\sum_i \mathcal{R}(Q_i) = \infty$. By inequality (4.1) of Proposition 4.2, we obtain $\sum_i \mathcal{P}(Q_i) = \infty$. Applying Theorem 2.10 we conclude that $\{P_i\}$ is a zero-fluctuation sequence. ∎

5. Main Strong Positivity Results

We have seen that in order for the sequence $\{P_i\}$ to be strongly positive, it must be a zero-fluctuation sequence (Corollary 4.22). We have also seen that the condition $\sum_i \mathcal{P}(P_i) = \infty$ is sufficient to guarantee that $\{P_i\}$ is a zero-fluctuation sequence, while $\sum_i \mathcal{R}(P_i) = \infty$ is sufficient to guarantee that it is strongly positive (Corollary 2.8 and Theorem 4.12). Since we have already remarked on the similarity of $\mathcal{P}$ and $\mathcal{R}$ — they both measure "how much (1+x)" is in a polynomial (as noted when we compared the decompositions of Lemma 2.5 and Proposition 4.2), we are tempted to ask what the difference between them is. The answer to this question lies in the fact that $\mathcal{P}$ is a global measure that is concerned with the amount of persistence overall, while $\mathcal{R}$ focuses on the contiguity in the worst region (as is clear from the definition of $\mathcal{R}$). While we have seen that fluctuation is fairly robust under perturbations, particularly upward perturbations, we find that strong positivity may be gained or lost by arbitrarily small upward perturbations (see the discussion preceding Proposition 5.16).

Examples such as $P_n = \begin{array}{ccccc} 1 & 1 & & & 1 \\ \hline 0 & 1 & & & 4 \end{array}$ and $P_n = \begin{array}{ccccc} 1 & 1 & & \frac{1}{n^2} & 1 \\ \hline 0 & 1 & & 3 & 4 \end{array}$ have divergent persistence but are not strongly positive (hence they have convergent contiguity). On the other hand, we will discover (after Theorem 5.7) that $P_n = \begin{array}{ccccc} 1 & 1 & & 1 & \frac{1}{n^5} \\ \hline 0 & 1 & & 3 & 4 \end{array}$ is strongly positve. (although its contiguity is still convergent). Our first task in this chapter will be to demonstrate and explain this sort of phenomenon.

Noting that $\sum_i \mathcal{P}(P_i) = \infty$ guarantees a sequence to be a zero-fluctuation sequence, we ask what additional conditions will guarantee that $\{P_i\}$ is strongly positive. Our best answers will be in Proposition 5.14 and Theorem 5.15.

The first six results of this section are technical lemmas needed to prove Theorem 5.7. The most noteworthy among them is Proposition 5.2, which says essentially that if some unimodal polynomials are added to a highly contiguous polynomial, and the peaks of the unimodal polynomials get 'swamped' by the mass of the sum, then the sum is highly contiguous. This is one of the key ideas in the proof of Theorem 5.7, and could potentially be used to produce other similar strong positivity results.

Lemma 5.1: Suppose $P, Q \in \mathbf{R}[x^{\pm 1}]^+$ and $\mathcal{R}(P) \geq \frac{1}{2}$. Suppose also that $\operatorname{Log} Q$ contains no two consecutive integers, and that $\langle P, x^{i-1}\rangle \geq \langle Q, x^i\rangle$ for all $i \in \mathbf{Z}$. Then $\mathcal{R}(P+Q) \geq \frac{1}{3}$.

Proof: If $Q = 0$ the result is trivial. Assume $Q \neq 0$. We can write $P + Q = \left(P - \frac{1}{3}Qx^{-1}\right) + \left(\frac{1}{3}Qx^{-1} + Q\right)$. It is clear that $P - \frac{1}{3}Qx^{-1} \in \mathbf{R}[x^{\pm 1}]^+$, and by Proposition 4.14 we have that $\mathcal{R}\left(P - \frac{1}{3}Qx^{-1}\right) \geq \frac{2}{3}\mathcal{R}(P) \geq \frac{1}{3}$. On the other hand, by Corollary 4.6 we have $\mathcal{R}\left(\frac{1}{3}Qx^{-1} + Q\right) = \mathcal{R}\left(\left(\frac{1}{3}x^{-1} + 1\right)Q\right) \geq \mathcal{R}\left(\frac{1}{3}x^{-1} + 1\right) = \frac{1}{3}$. It follows from property (5) of $\mathcal{R}()$ that $\mathcal{R}(P+Q) \geq \frac{1}{3}$. ∎

The following result deals with the contiguity of the sum of many unimodal polynomials and a polynomial of substantial contiguity.

Proposition 5.2: Suppose I is a finite subset of $\mathbf{Z}$, and for each $m \in I$, $f_m \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ is a unimodal Laurent polynomial with peak at m. Suppose also that $g \in \mathbf{R}[x^{\pm 1}]^+$ satisfies $\mathcal{R}(g) \geq \frac{1}{2}$. Let $h = g + \sum_{m \in I} f_m$. If either

$$\langle h, x^{m-1} \rangle \geq \langle f_m, x^m \rangle, \quad \forall m \in I$$

or

$$\langle h, x^{m+1} \rangle \geq \langle f_m, x^m \rangle, \quad \forall m \in I,$$

then $\mathcal{R}(h) \geq \frac{1}{3}$.

Proof: We assume that $\langle h, x^{m-1} \rangle \geq \langle f_m, x^m \rangle$, $\forall m \in I$, since the other case follows from this one by applying the homomorphism of $\mathbf{R}[x^{\pm 1}]^+$ which sends x to x^{-1}.

Our approach will be to throw parts of the f_m's in with g, until h is a simple perturbation of g. First of all, if $\mathcal{R}(f_m) \geq \frac{1}{2}$ for some $m \in I$, then we may replace g by $g + f_m$, and drop m from I. The relationship $h = g + \sum_{m \in I} f_m$ is still preserved, as is the fact that $\mathcal{R}(g) \geq \frac{1}{2}$, and we now have $\mathcal{R}(f_m) < \frac{1}{2}$, for each $m \in I$. Combining Proposition 4.19 with property (2) of $\mathcal{R}()$, we find that $\langle f_m, x^m \rangle > \langle f_m, x^{m-1} \rangle + \langle f_m, x^{m+1} \rangle$ for each $m \in I$. We will 'chop off' part of the peak of f_m. Let $\alpha_m = \langle f_m, x^m \rangle - \langle f_m, x^{m-1} \rangle - \langle f_m, x^{m+1} \rangle$. By Proposition 4.19, $\mathcal{R}(f_m - \alpha_m x^m) \geq \frac{1}{2}$. We may thus replace g by $g + (f_m - \alpha_m x^m)$, and replace f_m by $\alpha_m x^m$. We now have $h = g + \sum_{m \in I} \alpha_m x^m$, and $\langle h, x^{m-1} \rangle \geq \alpha_m$ for all $m \in I$.

If, for some m, both m and $m+1$ are in I, then if $\alpha_m \leq \alpha_{m+1}$ we replace g by $g + (\alpha_m x^m + \alpha_m x^{m+1})$, drop m from I, and replace α_{m+1} by $\alpha_{m+1} - \alpha_m$. If, on the other hand, $\alpha_m > \alpha_{m+1}$, we replace g by $g + (\alpha_{m+1} x^m + \alpha_{m+1} x^{m+1})$, drop $m+1$ from I, and replace α_m by $\alpha_m - \alpha_{m+1}$. After applying this procedure, we once again have $h = g + \sum_{m \in I} \alpha_m x^m$, and $\langle h, x^{m-1} \rangle \geq \alpha_m$ for all $m \in I$, and, in addition, I contains no two consecutive integers. For each $m \in I$ we have $\langle h, x^{m-1} \rangle = \langle g, x^{m-1} \rangle$, and so $\langle g, x^{m-1} \rangle \geq \alpha_m$. We apply Lemma 5.1 with $P = g$ and $Q = \sum_{m \in I} \alpha_m x^m$, to obtain that $\mathcal{R}(P + Q) \geq \frac{1}{3}$. That is, $\mathcal{R}(h) \geq \frac{1}{3}$. ∎

Lemma 5.3: Suppose that we have a finite family $\{f_i\}_{i \in I}$ of polynomials in $\mathbf{R}[x^{\pm 1}]^+$, where for each $i \in I$, f_i is either $a_i(1 + b_i x)$ or $a_i(1 + b_i x^{-1})$, for numbers $a_i \in [0, \infty)$ and $b_i \in [0, 1]$. Let $T = \prod_{i \in I} f_i$.

(i) If $\|T\| \geq \alpha(\prod_{i \in I} a_i)$ for some $\alpha \geq 1$, then $\mathcal{R}(T) \geq \tanh \ln \alpha$.

(ii) If $\mathcal{R}(T) < \frac{3}{5}$, then T is strongly unimodal with peak at 0.

Proof: Let $A = \prod_{i \in I} a_i$. If $A = 0$, then $T = 0$, and the result is trivially true, as $\mathcal{R}(0) = 1$. Assume $A \neq 0$. Note that $\mathcal{R}(f_i) = b_i$ for each $i \in I$. Using Lemma A.2 we find that

$$\|T\| = T(1) = \prod_{i \in I} \|f_i\| = \prod_{i \in I} f_i(1) = A \prod_{i \in I} (1 + b_i) = A \prod_{i \in I} (1 + \mathcal{R}(f_i)) \leq A \exp\left[\sum_{i \in I} \mathcal{R}(f_i)\right]$$

If $\|T\| \geq \alpha A$ for some $\alpha \geq 1$, then $\exp\left[\sum_{i \in I} \mathcal{R}(f_i)\right] \geq \alpha$, and so $\sum_{i \in I} \mathcal{R}(f_i) \geq \ln \alpha$. Theorem 4.7 tells us that $\mathcal{R}(T) \geq \tanh\left(\sum_{i \in I} \mathcal{R}(f_i)\right)$. It follows that $\mathcal{R}(T) \geq \tanh \ln \alpha$, proving (i).

From (i) we note that if $\frac{\|T\|}{A} \geq 2$, then $\mathcal{R}(T) \geq \tanh \ln 2 = \frac{3}{5}$. It follows that if T is such that $\mathcal{R}(T) < \frac{3}{5}$, then $\|T\| < 2A$. Since $\langle T, x^o \rangle \geq A$, we see that more than half of the mass of T lies on $\langle T, x^o \rangle$. As T is strongly unimodal, it follows that T must have peak at 0. ∎

The following lemma says, essentially, that if a polynomial is the sum of a contiguous part and a unimodal part, and the peak of the unimodal part is swamped by mass from the contiguous part, then the polynomial has high contiguity.

Lemma 5.4: Suppose $g, h \in \mathbf{R}[x^{\pm 1}]^+ \setminus \{0\}$ and g is unimodal with peak at p, while $\mathcal{R}(h) \geq \frac{3}{5}$. Suppose also that $\langle h, x^{p+\sigma} \rangle \geq 3 \langle g, x^p \rangle$ with $\sigma \in \{-1, 1\}$. Then $\mathcal{R}(g+h) \geq \frac{1}{2}$.

Proof: Write $g = \sum_{i \in \mathbf{Z}} g_i x^i$ and $h = \sum_{i \in \mathbf{Z}} h_i x^i$. If $\mathcal{R}(g) \geq \frac{1}{2}$, we are done, by property (5) of $\mathcal{R}$. If $\mathcal{R}(g) < \frac{1}{2}$, then by Proposition 4.19, $g_p > g_{p-1} + g_{p+1}$. Let $c = g_p - (g_{p-1} + g_{p+1})$. By Lemma 4.20 we have $\mathcal{R}(g - cx^p) \geq \frac{1}{2}$. We also have $\mathcal{R}(cx^p + \frac{c}{2} x^{p+\sigma}) = \frac{1}{2}$. As $c \leq g_p \leq \frac{h_{p+\sigma}}{3}$, it follows that $\frac{c}{2} \leq \frac{h_{p+\sigma}}{6}$, and so by Proposition 4.14, $\mathcal{R}(h - \frac{c}{2} x^{p+\sigma}) \geq \frac{5}{6} \mathcal{R}(h) \geq \frac{5}{6} \cdot \frac{3}{5} = \frac{1}{2}$. Since $g + h = (g - cx^p) + (cx^p + \frac{c}{2} x^{p+\sigma}) + (h - \frac{c}{2} x^{p+\sigma})$, we conclude that $\mathcal{R}(g+h) \geq \frac{1}{2}$. ∎

The lower bound in the following lemma could be improved, but is sufficient for our purposes.

Lemma 5.5: Suppose that I is a finite set of n elements, and that for each $i \in I$, $\gamma_i \in [0, 1]$. Let $\rho_j = \langle \prod_{i \in I} (1 + \gamma_i x), x^j \rangle$ for $j \in \mathbf{Z}$. If J is a subset of I of size p, and $s \in \mathbf{Z}$ is such that $\rho_{s-1} < \rho_s$, then

$$\left\langle \prod_{i \in I \setminus J} (1 + \gamma_i x), x^s \right\rangle \geq \rho_s - 2^p \rho_{s-1}.$$

Proof: For convenience, we define, for $j \in \mathbf{Z}$,

$$\sigma_j = \left\langle \prod_{i \in J} (1 + \gamma_i x), x^j \right\rangle \qquad \text{and} \qquad \tau_j = \left\langle \prod_{i \in I \setminus J} (1 + \gamma_i x), x^j \right\rangle.$$

Note that $\rho_s = \sum_{j \in \mathbf{Z}} \sigma_j \tau_{s-j} = \sum_{j=0}^{p} \sigma_j \tau_{s-j}$. As the ρ_i are coefficients of a strongly unimodal polynomial, and $\rho_{s-1} < \rho_s$, it follows that $\rho_i \leq \rho_{s-1}$ for all $i \leq s-1$. We also have $\tau_i \leq \rho_i$ for all $i \in \mathbf{Z}$. Using these facts, we find that

$$\rho_s = \sum_{j=0}^{p} \sigma_j \tau_{s-j} = \sigma_o \tau_s + \sum_{j=1}^{p} \sigma_j \tau_{s-j} \leq \sigma_o \tau_s + \sum_{j=1}^{p} \sigma_j \rho_{s-j} \leq \sigma_o \tau_s + \sum_{j=1}^{p} \sigma_j \rho_{s-1}.$$

Since $\sigma_o = 1$, and

$$\sum_{i=1}^{p} \sigma_j < \sum_{i=0}^{p} \sigma_j = \prod_{i \in J} (1 + \gamma_i) \leq (1+1)^p = 2^p,$$

we conclude that

$$\rho_s \leq \sigma_o \tau_s + \sum_{j=1}^{p} \sigma_j \rho_{s-1} = \tau_s + \rho_{s-1} \sum_{j=1}^{p} \sigma_j \leq \tau_s + \rho_{s-1} 2^p.$$

Rearranging this to get $\tau_s \geq \rho_s - 2^p \rho_{s-1}$ completes the proof. ∎

Lemma 5.6: Suppose P is any strongly unimodal polynomial in $\mathbf{R}[x^{\pm 1}]^{+}\backslash\{0\}$ and $\gamma \in [0,1]$. Let $Q = (1+\gamma x)P$. Then for any $s \in \operatorname{Log} P$,

$$\langle P, x^s\rangle \geq \langle Q, x^{s+1}\rangle \cdot \min\left(\frac{1}{3}, \frac{\langle Q, x^{s-1}\rangle}{2\langle Q, x^s\rangle}\right).$$

Proof: Write $P = \sum_{i\in\mathbf{Z}} p_i x^i$ and $Q = \sum_{i\in\mathbf{Z}} q_i x^i$. If $q_{s-1} = 0$ the result is trivially true. Assume $q_{s-1} \neq 0$. Note that $q_i \geq p_i$ for all $i \in \mathbf{Z}$.

First consider the case in which $q_{s-1} \leq \frac{q_s}{2}$. In this situation, $q_s = p_s + \gamma p_{s-1} \leq p_s + p_{s-1} \leq p_s + q_{s-1} \leq p_s + \frac{q_s}{2}$. It follows that $p_s \geq q_s - \frac{q_s}{2} = \frac{q_s}{2}$. Since Q is strongly unimodal, we know that $\frac{q_s}{q_{s-1}} \geq \frac{q_{s+1}}{q_s}$. This yields $p_s \geq \frac{q_s}{2} = \frac{q_{s-1}}{2}\cdot\frac{q_s}{q_{s-1}} \geq \frac{q_{s-1}}{2}\cdot\frac{q_{s+1}}{q_s} = q_{s+1}\cdot\frac{q_{s-1}}{2q_s}$, and the desired inequality holds.

It remains to consider the case $q_{s-1} > \frac{q_s}{2}$. By the strong unimodality of Q this gives $2 > \frac{q_s}{q_{s-1}} \geq \frac{q_{s+1}}{q_s}$. As $1+\gamma x$ is a right-positive perturbation of 1 (defined on page 9), using Corollary 1.10 we find that $\frac{p_{s+1}}{p_s} \leq \frac{q_{s+1}}{q_s}$. Thus $p_{s+1} < 2p_s$. Since $q_{s+1} = p_{s+1} + \gamma p_s \leq p_{s+1} + p_s \leq 2p_s + p_s = 3p_s$, we conclude that $p_s \geq \frac{1}{3} q_{s+1}$, which completes the proof. ∎

A sequence of polynomials of the form $P_i = \begin{array}{ccc} 1 & \gamma_i & \beta_i \\ \hline 0 & 2 & 3 \end{array}$ cannot be strongly positive. In particular the coefficient of x^1 in $(1-x+x^2)P^{n,t}$ will be negative for every choice of n and t. The obstacle to strong positivity is the fact that $\operatorname{Log} P^{n,t}$ always has a gap at the left end. Filling in this gap, we note that the sequence of polynomials given by $P_i = \begin{array}{cccc} 1 & \frac{1}{i^2} & \gamma_i & \beta_i \\ \hline 0 & 1 & 2 & 3 \end{array}$ still cannot be strongly positive, as the coefficient of x^1 in the product $(1-x+x^2)P^{n,t}$ is $-1 + \sum_{i=t+1}^{t+n} \frac{1}{i^2}$, which will be negative if $t \geq 1$. The obstacle to strong positivity is the fact that we still effectively have a gap, as the mass on x^1 is summable. Putting more mass on x^1 can make the sequence strongly positive: the sequence given by $P_i = \begin{array}{cccc} 1 & \frac{1}{i} & \frac{1}{i} & 1 \\ \hline 0 & 1 & m & m+1 \end{array}$ is strongly positive for any choice of m (by Theorem 4.12). These examples might lead us to conjecture that a necessary condition for strong positivity would be to require the sum of the contiguity at the leftmost two coefficients of P_i to be divergent. This is in fact not the case. Theorem 5.7 shows that adding *any* nonzero mass on the *outside* of the isolated coefficient can give rise to strong positivity. It shows, for example, that the sequences of polynomials given by $P_i = \begin{array}{cccc} \frac{1}{i^7} & 1 & 1 & \frac{1}{i} \\ \hline -1 & 0 & 2 & 3 \end{array}$ and $Q_i = \begin{array}{cccc} 5^{-i} & 1 & i^{-1} & i^{-1/2} \\ \hline -1 & 0 & 4 & 5 \end{array}$ are both strongly positive.

Theorem 5.7 is proved by breaking the product into a sum consisting of a highly contiguous part and a number of unimodal parts. We show that the peaks of the unimodal parts get absorbed by other mass from the product, and so, using Proposition 5.2, we conclude that the product has high contiguity.

Theorem 5.7: Suppose that for each $i \in \mathbf{N}$ we have numbers α_i, β_i and γ_i in $[0,1]$, and that infinitely many of the α_i's are nonzero and $\sum_{i=1}^{\infty}\alpha_i < \infty$, while $\sum_{i=1}^{\infty}\gamma_i\beta_i = \infty$. Suppose also that $\sigma \in \{-1,1\}$. Then for any integer $m \geq 2$, the sequence $\{P_i\}_{i\in\mathbf{N}}$

defined by

$$P_i = \alpha_i x^{-1} + 1 \ + \ \gamma_i x^m + \gamma_i \beta_i x^{m+\sigma}, \qquad i \in \mathbf{N}$$

is strongly positive.

Proof: By Theorem 4.21 (4) it suffices to show that for any $t \in \mathbf{Z}^+$ there is an $n \in \mathbf{N}$ for which $\mathcal{R}(P^{n,t}) \geq \frac{1}{3}$. Let t be given.

Let $A = \sum_{i=t+1}^{\infty} \alpha_i$, and for each $n \in \mathbf{N}$ let $A_n = \prod_{i=t+1}^{t+n}(1+\alpha_i x)$. Now $A < \infty$, and we note that $\langle A_n, x^1 \rangle = \sum_{i=t+1}^{t+n} \alpha_i \leq A$ for all $n \in \mathbf{N}$. Since each A_n is strongly unimodal, we have that $\frac{\langle A_n, x^{i+1}\rangle}{\langle A_n, x^i\rangle} \leq \frac{\langle A_n, x^i\rangle}{\langle A_n, x^{i-1}\rangle} \leq A$ for $i = 1, 2, \ldots, n$. It follows that $\langle A_n, x^i \rangle \leq A^i$ for all $i \in \mathbf{Z}^+$. In particular, $\langle A_n, x^{m-1} \rangle \leq A^{m-1}$. As $\langle A_n, x^{m-1} \rangle$ is a non-decreasing sequence in n that is bounded above, it has a limit λ. Since infinitely many of the α_i's are nonzero, $\lambda > 0$.

Let $L = \frac{2\sqrt{3}}{\lambda} + 2^{m-1}$. By Corollary 3.16, there is an $n_o \in \mathbf{N}$ such that the set

$$U_n = \left\{ i \in \mathbf{Z} \;\middle|\; \frac{\left\langle \prod_{j=t+1}^{t+n}(1+\gamma_j x),\, x^{i+1} \right\rangle}{\left\langle \prod_{j=t+1}^{t+n}(1+\gamma_j x),\, x^{i} \right\rangle} \in (L,\, 2L) \right\}$$

has at least $m+3$ elements whenever $n \geq n_o$. Take $n > n_o$ large enough that $\langle A_n, x^{m-1} \rangle > \lambda/2$, and $\sum_{i=t+1}^{t+n} \gamma_i \beta_i > 32L$. We show that $\mathcal{R}(P^{n,t}) \geq \frac{1}{3}$.

Our first step is to decompose $P^{n,t}$ into a sum of simpler polynomials. We define, for $i \in \mathbf{N}$, $Q_i = \gamma_i + \gamma_i \beta_i x^\sigma$ and $R_i = \alpha_i x^{-1} + 1$, so that $P_i = R_i + Q_i x^m$. We let $I_n = \{t+1, t+2, t+3, \ldots, t+n\}$, and for each $K \subseteq I_n$ and each $i \in I_n$ we define

$$W_{K,i} = \begin{cases} Q_i & \text{if } i \in K \\ R_i & \text{if } i \notin K \end{cases}$$

Now letting $P = P^{n,t} = \prod_{i=t+1}^{t+n} P_i$, we observe that

$$P = \prod_{i \in I_n} P_i = \sum_{K \subseteq I_n} \left(\prod_{i \in I_n \setminus K} R_i \right) \left(\prod_{i \in K} Q_i x^m \right) = \sum_{K \subseteq I_n} x^{m|K|} \prod_{i \in I_n} W_{K,i}.$$

For each $K \subseteq I_n$ we let $T_K = \prod_{i \in I_n} W_{K,i}$, and for $k = 0, 1, 2, \ldots n$, we let $T_k = \sum_{\substack{K \subseteq I_n \\ |K|=k}} x^{mk} T_K$,

so that $P = \sum_{k=0}^{n} T_k$. For each $K \subseteq I_n$ we define $G_K = \prod_{i \in K} \gamma_i$, and for $j \in \mathbf{Z}$ we define

$$\rho_j = \left\langle \prod_{i \in I_n} (1+\gamma_i x),\, x^j \right\rangle = \sum_{\substack{K \subseteq I_n \\ |K|=j}} G_K.$$

Recall that by choice of n, the set U_n has a least $m+3$ elements. Let k_o be the third smallest element of U_n, so that, in particular, $k_o - 2 \in U_n$ and $k_o \leq n - m$. We show that $\mathcal{R}(T_k) \geq \frac{1}{2}$ whenever $k_o \leq k \leq n$.

Take k with $k_o \leq k \leq n$, and set $B_k = \left\{ K \subseteq I_n \,\middle|\, |K| = k,\ \mathcal{R}(T_K) < \frac{3}{5} \right\}$. If B_k is empty, then $\mathcal{R}(T_k) \geq \frac{3}{5} > \frac{1}{2}$. If, on the other hand, B_k is not empty, then Lemma 5.3 tells us that for every $K \in B_k$, T_K is unimodal with peak at 0, and that $\|T_K\| < 2G_K$ (since $\tanh \ln 2 = \frac{3}{5}$).

Letting $g_k = \sum_{K\in B_k} x^{mk}T_K$, we note that g_k is unimodal with peak at mk, and that

$$\langle g_k, x^{mk}\rangle = \left\langle \sum_{K\in B_k} x^{mk}T_K, x^{mk}\right\rangle = \left\langle \sum_{K\in B_k} T_K, x^o\right\rangle = \sum_{K\in B_k} \langle T_K, x^o\rangle$$
$$\leq \sum_{K\in B_k} \|T_K\| \leq \sum_{K\in B_k} 2G_K \leq 2\sum_{\substack{K\subseteq I_n\\ |K|=k}} G_K = 2\rho_k$$

We show that this coefficient of g_k is swamped by the mass around it in T_k. Note that for $K \subseteq I_n$, in the ordering on $\mathbf{R}[x^{\pm 1}]$ we have $T_K \geq \prod_{i\in K} Q_i = G_K \prod_{i\in K}(1+\beta_i x^\sigma)$ and so $\langle T_K, x^\sigma\rangle \geq G_K \sum_{i\in K}\beta_i$. Thus

$$\langle T_k, x^{mk+\sigma}\rangle = \left\langle \sum_{\substack{K\subseteq I_n\\ |K|=k}} x^{mk}T_K, x^{mk+\sigma}\right\rangle = \sum_{\substack{K\subseteq I_n\\ |K|=k}} \langle T_K, x^\sigma\rangle$$
$$\geq \sum_{\substack{K\subseteq I_n\\ |K|=k}} G_K \sum_{i\in K}\beta_i = \sum_{\substack{K\subseteq I_n\\ |K|=k}} \sum_{i\in K}\gamma_i\beta_i \prod_{j\in K\setminus\{i\}} \gamma_j = \sum_{i\in I_n}\gamma_i\beta_i \sum_{\substack{K\subseteq I_n\setminus\{i\}\\ |K|=k-1}} \prod_{j\in K}\gamma_j$$
$$= \sum_{i\in I_n}\gamma_i\beta_i \left\langle \prod_{j\in I_n\setminus\{i\}} (1+\gamma_j x), x^{k-1}\right\rangle$$

We apply Lemma 5.6, with $s = k-1$, to find that for any $i \in I_n$,

$$\left\langle \prod_{j\in I_n\setminus\{i\}} (1+\gamma_j x), x^{k-1}\right\rangle \geq \rho_k \cdot \min\left(\frac{1}{3}, \frac{\rho_{k-2}}{2\rho_{k-1}}\right)$$

Since the sequence $\{\rho_j\}_{j\in\mathbf{Z}}$ is strongly unimodal, and $k_o - 2 \in U_n$, we have $\frac{\rho_{k-1}}{\rho_{k-2}} \leq \frac{\rho_{k_o-1}}{\rho_{k_o-2}} < 2L$, and so $\frac{\rho_{k-2}}{2\rho_{k-1}} > \frac{1}{4L}$, and

$$\langle T_k, x^{mk+\sigma}\rangle \geq \sum_{i\in I_n}\gamma_i\beta_i \cdot \min\left(\frac{1}{3}, \frac{1}{4L}\right)\rho_k = \frac{\rho_k}{4L}\sum_{i\in I_n}\gamma_i\beta_i.$$

Since n was chosen so that $\sum_{i\in I_n}\gamma_i\beta_i > 32L$, we obtain $\langle T_k, x^{mk+\sigma}\rangle \geq 8\rho_k$. As g_k is unimodal with peak at mk, and $\langle g_k, x^{mk}\rangle \leq 2\rho_k$, it follows that $\langle T_k - g_k, x^{mk+\sigma}\rangle \geq 8\rho_k - 2\rho_k = 6\rho_k \geq 3\langle g_k, x^{mk}\rangle$. Applying Lemma 5.4, with $h = T_k - g_k$ and $p = mk$, we conclude that $\mathcal{R}(T_k) \geq \frac{1}{2}$, as desired.

We have shown that $\mathcal{R}(T_k) \geq \frac{1}{2}$ whenever $k_o \leq k \leq n$.

We now consider values of k with $0 \leq k < k_o$. For such k we define

$$D_k = \left\{ K \subseteq I_n \,\middle|\, |K| = k,\ \mathcal{R}(T_K) < \frac{1}{2} \right\}.$$

Lemma 5.3 tells us that for every $K \in D_k$, T_K is unimodal with peak at 0, and that $\|T_K\| < \sqrt{3}G_K$ (since $\tanh\ln\sqrt{3} = \frac{1}{2}$). If D_k is not empty, set $f_{mk} = \sum_{K\in D_k} x^{mk}T_K$, and note that

f_{mk} is unimodal with peak at mk. The coefficient at this peak satisfies

$$\begin{aligned}\langle f_{mk}, x^{mk}\rangle &= \left\langle \sum_{K\in D_k} x^{mk}T_K, x^{mk}\right\rangle = \left\langle \sum_{K\in D_k} T_K, x^o\right\rangle = \sum_{K\in D_k} \langle T_K, x^o\rangle \\ &\le \sum_{K\in D_k} \|T_K\| \le \sum_{K\in D_k} \sqrt{3}G_K \le \sqrt{3}\sum_{\substack{K\subseteq I_n\\|K|=k}} G_K = \sqrt{3}\rho_k\end{aligned}$$

We show that this peak is swamped by the mass $\langle P, x^{mk+1}\rangle$. Note that in the ordering on $\mathbf{R}[x^{\pm 1}]$, $Q_i \ge \gamma_i$ for each $i \in I_n$, and so for any $K \subseteq I_n$,

$$T_K = \left(\prod_{i\in K} Q_i\right)\left(\prod_{i\in I_n\setminus K} R_i\right) \ge \left(\prod_{i\in K}\gamma_i\right)\prod_{i\in I_n\setminus K}(\alpha_i x^{-1}+1).$$

Thus for $0 \le j \le n - |K|$,

$$\langle T_K, x^{-j}\rangle \ge G_K \left\langle \prod_{i\in I_n\setminus K}(\alpha_i x^{-1}+1), x^{-j}\right\rangle = G_K \sum_{\substack{J\subseteq I_n\setminus K\\|J|=j}}\prod_{i\in J}\alpha_i.$$

Since we are considering k with $0 \le k < k_o$, and we know that $k_o \le n - m$, it follows that $m - 1 \le n - (k+1)$, and so

$$\begin{aligned}\langle P, x^{mk+1}\rangle &\ge \langle T_{k+1}, x^{mk+1}\rangle = \left\langle \sum_{\substack{K\subseteq I_n\\|K|=k+1}} x^{mk+m}T_K, x^{mk+1}\right\rangle = \sum_{\substack{K\subseteq I_n\\|K|=k+1}} \langle T_K, x^{1-m}\rangle \\ &\ge \sum_{\substack{K\subseteq I_n\\|K|=k+1}} G_K \sum_{\substack{J\subseteq I_n\setminus K\\|J|=m-1}}\prod_{i\in J}\alpha_i = \sum_{\substack{J\subseteq I_n\\|J|=m-1}}\sum_{\substack{K\subseteq I_n\setminus J\\|K|=k+1}} G_K\prod_{i\in J}\alpha_i \\ &= \sum_{\substack{J\subseteq I_n\\|J|=m-1}}\left(\prod_{i\in J}\alpha_i\right)\sum_{\substack{K\subseteq I_n\setminus J\\|K|=k+1}} G_K = \sum_{\substack{J\subseteq I_n\\|J|=m-1}}\left(\prod_{i\in J}\alpha_i\right)\left\langle\prod_{i\in I_n\setminus J}(1+\gamma_i x), x^{k+1}\right\rangle\end{aligned}$$

By the choice of k_o we have $\frac{\rho_{k_o+1}}{\rho_k} > L$. Since the sequence $\{\rho_j\}_{j\in\mathbf{Z}}$ is strongly unimodal, and $0 \le k < k_o$, we have $\frac{\rho_{k+1}}{\rho_k} \ge \frac{\rho_{k_o+1}}{\rho_{k_o}} > L$. Thus $\rho_k < \frac{\rho_{k+1}}{L} \le \rho_{k+1}$, and applying Lemma 5.5, with $p = m-1$ and $s = k+1$, we find that, for each subset $J \subseteq I_n$ of cardinality $m-1$,

$$\begin{aligned}\left\langle\prod_{i\in I_n\setminus J}(1+\gamma_i x), x^{k+1}\right\rangle &\ge \rho_{k+1} - 2^{m-1}\rho_k = \left[\frac{\rho_{k+1}}{\rho_k} - 2^{m-1}\right]\rho_k \\ &\ge \left[L - 2^{m+1}\right]\rho_k = \frac{2\sqrt{3}}{\lambda}\rho_k\end{aligned}$$

It follows that for $0 \le k < k_o$,

$$\langle P, x^{mk+1}\rangle \ge \sum_{\substack{J\subseteq I_n\\|J|=m-1}}\left(\prod_{i\in J}\alpha_i\right)\left[\frac{2\sqrt{3}}{\lambda}\rho_k\right] = \langle A_n, x^{m-1}\rangle\left[\frac{2\sqrt{3}}{\lambda}\rho_k\right] \ge \frac{\lambda}{2}\frac{2\sqrt{3}}{\lambda}\rho_k = \sqrt{3}\rho_k$$

As $\langle f_{mk}, x^{mk}\rangle \le \sqrt{3}\rho_k$, we conclude that $\langle P, x^{mk+1}\rangle \ge \langle f_{mk}, x^{mk}\rangle$ whenever $0 \le k < k_o$.

Let $I = \{\, mk \mid 0 \le k < k_o,\ D_k \ne \emptyset\,\}$, and let $E = \{\, K \subseteq I_n \mid |K| < k_o,\ \mathcal{R}(T_K) \ge \frac{1}{2}\,\}$. If we set $g = \sum_{K\in E} T_K + \sum_{k=k_o}^{n} T_k$, then $P = g + \sum_{i\in I} f_i$, and $\mathcal{R}(g) \ge \frac{1}{2}$. Since $\langle P, x^{i+1}\rangle \ge$

$\langle f_i, x^i \rangle$ for each $i \in I$, we may apply Proposition 5.2 to find that $\mathcal{R}(P) \geq \frac{1}{3}$, completing the proof. ∎

We now generalize the result of Theorem 5.7 in stages. Our first step will be to allow variability as to which of the two rightmost coefficients is larger.

Corollary 5.8: Suppose $\{P_i\}_{i \in \mathbf{N}}$ is a sequence of polynomials in $\mathbf{R}[x^{\pm 1}]^+$ given by

$$P_i = \begin{array}{cccc} a_i & 1 & b_i & c_i \\ \hline -1 & 0 & p & p+1 \end{array},$$

where $p \geq 1$ is a fixed integer and $b_i,\ c_i \in (0, 1]$ and $a_i \in [0, 1]$, for each i in $\mathbf{N}$. Suppose also that infinitely many of the a_i's are nonzero and that $\sum_{i=1}^{\infty} a_i < \infty$. If $\sum_{i=1}^{\infty} \min(b_i, c_i) = \infty$ then $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive.

Proof: Let $I = \{\, i \in \mathbf{N} \mid b_i \geq c_i \,\}$ and $J = \{\, i \in \mathbf{N} \mid b_i < c_i \,\}$. Since $\sum_{i=1}^{\infty} \min(b_i, c_i) = \infty$, it follows that either $\sum_{i \in I} \min(b_i,\ c_i) = \infty$ or $\sum_{i \in J} \min(b_i,\ c_i) = \infty$, or both.

First consider the case where $\sum_{i \in J} \min(b_i,\ c_i) = \infty$. For each $i \in J$ we let $\alpha_i = a_i$, $\gamma_i = c_i$ and $\beta_i = \frac{b_i}{c_i}$ and take $m = p + 1$. Then $P_i = \alpha_i x^{-1} + 1 + \gamma_i x^m + \gamma_i \beta_i x^{m-1}$, and by Theorem 5.7 we conclude that the sequence $\{P_i\}_{i \in J}$ is strongly positive. It follows from Proposition 1.2 that $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive.

Next consider the case where $\sum_{i \in I} \min(b_i,\ c_i) = \infty$ and $p \geq 2$. For each $i \in I$ we let $\alpha_i = a_i$, $\gamma_i = b_i$ and $\beta_i = \frac{c_i}{b_i}$ and take $m = p$. Then $P_i = \alpha_i x^{-1} + 1 + \gamma_i x^m + \gamma_i \beta_i x^{m+1}$, and by Theorem 5.7 we conclude that the sequence $\{P_i\}_{i \in I}$ is strongly positive. It follows from Proposition 1.2 that $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive.

It remains to consider the case where $\sum_{i \in I} \min(b_i,\ c_i) = \infty$ and $p = 1$. Writing $P_i = \left[a_i x^{-1} + \frac{a_i}{2}\right] + \left[(1 - \frac{a_i}{2}) + \frac{b_i}{2} x\right] + \left[\frac{b_i}{2} x + c_i x^2\right]$, we see that $\mathcal{R}(P_i) \geq \frac{1}{2} \min(b_i, c_i)$. It follows that $\sum_{i \in I} \mathcal{R}(P_i) = \infty$, and by Theorem 4.12 we conclude that $\{P_i\}_{i \in I}$ is strongly positive. It follows from Proposition 1.2 that $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive. ∎

Next we generalize by removing the condition that the mass at x^o be the largest mass in the polynomial. Corollary 5.9 yields the strong positivity of sequences such as that given by

$$P_i = \begin{array}{cccc} i^{-5} & i^{-3} & 1 & i^{-1} \\ \hline -1 & 0 & 7 & 8 \end{array}.$$

Corollary 5.9: Suppose $\{P_i\}_{i \in \mathbf{N}}$ is a sequence of polynomials in $\mathbf{R}[x^{\pm 1}]^+$ given by

$$P_i = \begin{array}{cccc} \alpha_i & \beta_i & \gamma_i & \delta_i \\ \hline -1 & 0 & p & p+1 \end{array},$$

where $p \geq 1$ is a fixed integer, and for all $i \in \mathbf{N}$, $\alpha_i, \beta_i, \gamma_i, \delta_i \in (0, 1]$. Suppose that $\sum_{i=1}^{\infty} \frac{\alpha_i}{\beta_i} < \infty$. If $\sum_{i=1}^{\infty} \min(\gamma_i, \delta_i) = \infty$ then $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive.

Proof: As $\sum_{i=1}^{\infty} \frac{\alpha_i}{\beta_i} < \infty$, we may assume, by taking an appropriate tail of the sequence, that $\alpha_i \leq \beta_i$ for all i in $\mathbf{N}$. For each $i \in \mathbf{N}$ let $w_i = \max(\beta_i,\ \gamma_i,\ \delta_i)$ and note that $w_i < 1$. For $i \in \mathbf{N}$ define

$$Q_i = \frac{\alpha_i}{\beta_i} x^{-1} + 1 + \frac{\gamma_i}{w_i} x^p + \frac{\delta_i}{w_i} x^{p+1} \quad \text{and} \quad R_i = \left(\frac{1}{\beta_i} - \frac{1}{w_i}\right)\gamma_i x^p + \left(\frac{1}{\beta_i} - \frac{1}{w_i}\right)\delta_i x^{p+1},$$

and note that $R_i \geq 0$, and $Q_i + R_i = \frac{1}{\beta_i} P_i$. Note also that each coefficient of Q_i is in $(0, 1]$.

In keeping with the notation of Proposition 1.5, we let $T_{0,i} = Q_i$ and $T_{1,i} = R_i$ for all $i \in \mathbf{N}$. Let $k = (k(1), k(2), \ldots)$ be any sequence in $\{0,1\}^{\mathbf{N}}$, and set $I = \{\, i \in \mathbf{N} \mid k(i) = 0 \,\}$.

If it happens that $\sum_{i \in I} \min(\gamma_i, \delta_i) = \infty$, then $\sum_{i \in I} \min\left(\frac{\gamma_i}{w_i}, \frac{\delta_i}{w_i}\right) = \infty$. and applying Corollary 5.8 we find that the sequence $\{Q_i\}_{i \in I}$ is strongly positive. It follows (from Proposition 1.2) that $\{T_{k(i),i}\}_{i \in \mathbf{N}}$ is strongly positive.

If, on the other hand, $\sum_{i \in I} \min(\gamma_i, \delta_i) < \infty$, then $\sum_{i \in \mathbf{N} \setminus I} \min(\gamma_i, \delta_i) = \infty$. Noting that when $R_i \neq 0$,

$$\mathcal{R}(R_i) = \mathcal{R}(\gamma_i x^p + \delta_i x^{p+1}) = \frac{\min(\gamma_i, \delta_i)}{\max(\gamma_i, \delta_i)} \geq \min(\gamma_i, \delta_i),$$

and that $\mathcal{R}(0) = 1$, we find that $\sum_{i \in \mathbf{N} \setminus I} \mathcal{R}(R_i) \geq \sum_{i \in \mathbf{N} \setminus I} \min(\gamma_i, \delta_i) = \infty$. It follows that $\sum_{i \in \mathbf{N}} \mathcal{R}(T_{k(i),i}) = \infty$, and so $\{T_{k(i),i}\}_{i \in \mathbf{N}}$ is strongly positive, by Theorem 4.12.

Applying Proposition 1.5, we conclude that the sequence $\{T_{0,i} + T_{1,i}\}_{i \in \mathbf{N}}$ is strongly positive. It follows that $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive. ∎

The condition that $\sum_{i=1}^{\infty} \frac{\alpha_i}{\beta_i} < \infty$ in Corollary 5.9 was unnecessarily restrictive. The single condition of Corollary 5.10 admits every sequence that satisfied the conditions of Corollary 5.9 (where $\frac{\beta_i}{\alpha_i} \leq 1$) while also permitting such sequences as $\underset{-1}{1}\;\underset{0}{\tfrac{1}{i}}\;\underset{3}{\tfrac{1}{i}}\;\underset{4}{1}$ and even $\underset{-1}{1}\;\underset{0}{\tfrac{1}{i \ln i}}\;\underset{3}{\tfrac{1}{i}}\;\underset{4}{1}$, while excluding sequences such as $\underset{-1}{1}\;\underset{0}{\tfrac{1}{i^2}}\;\underset{3}{\tfrac{1}{i}}\;\underset{4}{1}$, which is not strongly positive (as remarked earlier). The advantage of Corollary 5.10 over Corollary 5.9 is the fact that it is simpler and more comprehensive, although many of the new sequences it covers, such as the first two sited above, can be proved strongly positive using earlier results such as Theorem 4.12.

Corollary 5.10: Let $\{P_i\}_{i \in \mathbf{N}}$ be a sequence of polynomials in $\mathbf{R}[x^{\pm 1}]^+$ given by

$$P_i = \underset{0}{\alpha_i}\;\underset{1}{\beta_i}\;\underset{p}{\gamma_i}\;\underset{p+1}{\delta_i},$$

where $p \geq 2$ is a fixed integer, and $\alpha_i, \beta_i, \gamma_i, \delta_i \in (0, 1]$ for all i in $\mathbf{N}$.
If $\sum_{i=1}^{\infty} \min\left(\gamma_i,\, \delta_i,\, \frac{\beta_i}{\alpha_i}\right) = \infty$ then $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive.

Proof: For each $i \in \mathbf{N}$ let $\rho_i = \min\left(\gamma_i,\, \delta_i,\, \frac{\beta_i}{\alpha_i}\right)$, $u_i = \frac{1}{2} \min(\alpha_i, \beta_i)$, and $w_i = \frac{\alpha_i \beta_i}{(1+i)^2}$, and set

$$Q_i = w_i + (\beta_i - u_i)x + \gamma_i x^p + \delta_i x^{p+1} \qquad \text{and} \qquad R_i = (\alpha_i - w_i) + u_i x$$

so that $Q_i + R_i = P_i$.

As $\rho_i \leq \frac{\beta_i}{\alpha_i}$, we obtain $\beta_i \geq \rho_i \alpha_i$, and so $u_i \geq \frac{1}{2} \min(\alpha_i,\, \rho_i \alpha_i) = \frac{1}{2} \rho_i \alpha_i$. For each $i \in \mathbf{N}$, we also have $w_i \leq \frac{\alpha_i}{(i+1)^2} \leq \frac{\alpha_i}{4}$, and so $\alpha_i - w_i \geq \alpha_i - \frac{\alpha_i}{4} = \frac{3\alpha_i}{4} > \frac{\alpha_i}{2} \geq u_i$. It follows that $\mathcal{R}(R_i) = \frac{u_i}{\alpha_i - w_i} \geq \frac{\frac{1}{2}\rho_i \alpha_i}{\alpha_i} = \frac{1}{2} \rho_i$.

In keeping with the notation of Proposition 1.5, we let $T_{0,i} = Q_i$ and $T_{1,i} = R_i$ for all $i \in \mathbf{N}$. Let $k = (k(1), k(2), \ldots)$ be any sequence in $\{0,1\}^{\mathbf{N}}$, and set $I = \{\, i \in \mathbf{N} \mid k(i) = 0 \,\}$.

If $\sum_{i \in \mathbf{N} \setminus I} \rho_i = \infty$, then $\sum_{i \in \mathbf{N}} \mathcal{R}(T_{k(i),i}) \geq \sum_{i \in \mathbf{N} \setminus I} \mathcal{R}(R_i) \geq \sum_{i \in \mathbf{N} \setminus I} \frac{1}{2} \rho_i = \infty$, and so $\{T_{k(i),i}\}_{i \in \mathbf{N}}$ is strongly positive, by Theorem 4.12.

If, on the other hand, $\sum_{i\in\mathbf{N}\setminus I} \rho_i < \infty$, then we must have $\sum_{i\in I} \rho_i = \infty$. For each i in I we have $\beta_i - u_i \geq \beta_i - \frac{\beta_i}{2} = \frac{\beta_i}{2} \geq w_i$ and $\frac{w_i}{\beta_i - u_i} \leq \frac{w_i}{\beta_i/2} = \frac{2\alpha_i}{(i+1)^2} \leq \frac{2}{(i+1)^2}$. It follows that $\sum_{i\in I} \frac{w_i}{\beta_i - u_i} < \infty$. As $\sum_{i=1}^{\infty} \min(\gamma_i, \delta_i) \geq \sum_{i=1}^{\infty} \rho_i = \infty$, the sequence $\{Q_i\}_{i\in I}$ satisfies the conditions of Corollary 5.9, and is therefore strongly positive. By Proposition 1.2 we conclude that $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive.

We have shown that for any sequence $k = (k(1), k(2), \ldots)$ in $\{0,1\}^{\mathbf{N}}$, the sequence $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive. It follows from Proposition 1.5 that $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive. ∎

In Corollary 5.11 we extend the preceding results by allowing polynomials with more than 4 nonzero terms and varying (but bounded) degrees. At the same time we add the restriction that the polynomials be gapless. This restriction will be removed in Proposition 5.14. We also reverse the coefficients of the polynomial so that the part with guaranteed contiguity lies at the left. This result will give contiguity for sequences such as that given by $P_i = \begin{array}{ccccc} 1 & i^{-1} & i^{-5} & i^{-9} & 3^{-i} \\ \hline 0 & 1 & 2 & 3 & 4 \end{array}$.

Corollary 5.11: Let $\{P_i\}_{i\in\mathbf{N}}$ be a sequence of polynomials in $\mathbf{R}[x^{\pm 1}]^+\setminus\{0\}$ such that for each i, $\|P_i\| \leq 1$ and $\operatorname{Log} P_i = \{0, 1, \ldots, d(i)\}$ for some $d(i) \geq 3$. Suppose that $\sup_{i\in\mathbf{N}} d(i) < \infty$, and that

$$\sum_{i=1}^{\infty} \min\left(\langle P_i,\, x^o \rangle,\, \langle P_i,\, x^1 \rangle,\, \frac{\langle P_i,\, x^{d(i)-1} \rangle}{\langle P_i,\, x^{d(i)} \rangle} \right) = \infty$$

Then $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive.

Proof: For each $i \in \mathbf{N}$ we write $P_i = \sum_{j\in\mathbf{Z}} P_{i,j} x^j$, and define $A_{i,j} = \frac{1}{2}\min(P_{i,j-1}, P_{i,j})$ for $3 \leq j < d(i)$, and set $A_{i,2} = 0$ and $A_{i,d(i)} = P_{i,d(i)}$. We then define

$$T_{j,i} = \frac{1}{d(i) - 2}\left[P_{i,0} + P_{i,1}x\right] + (P_{i,j-1} - A_{i,j-1})x^{j-1} + A_{i,j}x^j, \qquad j = 3, 4, \ldots, d(i).$$

and note that

$$\begin{aligned}
\sum_{j=3}^{d(i)} T_{j,i} &= \left[P_{i,0} + P_{i,1}x\right] + \sum_{j=3}^{d(i)} \left[(P_{i,j-1} - A_{i,j-1})x^{j-1} + A_{i,j}x^j\right] \\
&= P_{i,0} + P_{i,1}x + \sum_{j=2}^{d(i)-1} (P_{i,j} - A_{i,j})x^j + \sum_{j=3}^{d(i)} A_{i,j}x^j \\
&= P_{i,0} + P_{i,1}x + (P_{i,2} - A_{i,2})x^2 + \sum_{j=3}^{d(i)-1} P_{i,j}x^j + A_{i,d(i)}x^d \\
&= P_{i,0} + P_{i,1}x + P_{i,2}x^2 + \sum_{j=3}^{d(i)-1} P_{i,j}x^j + P_{i,d(i)}x^d \;=\; P_i
\end{aligned}$$

Let $\rho_i = \min\left(P_{i,0},\, P_{i,1},\, \frac{P_{i,d(i)-1}}{P_{i,d(i)}}\right)$ for each i, so that $\sum_{i=1}^{\infty} \rho_i = \infty$. Observe that for each i, and for $3 \le j \le d(i)$,

$$\min\left(\langle T_{j,i},\, x^0\rangle,\, \langle T_{j,i},\, x^1\rangle\right) = \frac{1}{d(i)-2}\min\left(P_{i,0},\, P_{i,1}\right) \ge \frac{\rho_i}{d(i)-2}$$

Also, for each $i \in \mathbf{N}$,

$$\frac{\langle T_{j,i},\, x^{j-1}\rangle}{\langle T_{j,i},\, x^j\rangle} = \frac{P_{i,j-1} - A_{i,j-1}}{A_{i,j}} \ge \frac{P_{i,j-1} - \frac{1}{2}P_{i,j-1}}{\frac{1}{2}P_{i,j-1}} = 1, \qquad \text{for } 3 \le j < d(i),$$

and for $j = d(i)$ we have

$$\frac{\langle T_{j,i},\, x^{j-1}\rangle}{\langle T_{j,i},\, x^j\rangle} = \frac{P_{i,j-1} - A_{i,j-1}}{A_{i,j}} \ge \frac{P_{i,j-1} - \frac{1}{2}P_{i,j-1}}{A_{i,j}} = \frac{\frac{1}{2}P_{i,d(i)-1}}{P_{i,d(i)}} \ge \frac{\rho_i}{2}.$$

Thus

$$\min\left(\langle T_{j,i},\, x^0\rangle,\, \langle T_{j,i},\, x^1\rangle,\, \frac{\langle T_{j,i},\, x^{j-1}\rangle}{\langle T_{j,i},\, x^j\rangle}\right) \ge \frac{\rho_i}{2d(i)}$$

for $3 \le j \le d(i)$ and all $i \in \mathbf{N}$.

Let $k = (k(1), k(2), \ldots)$ be any sequence of integers with $3 \le k(i) \le d(i)$ for each i. For each $m \in \mathbf{Z}^+$ with $m \le \sup_i d(i)$, we let $I(m) = \{\, i \in \mathbf{N} \mid k(i) = m \,\}$. As the union of these sets $I(m)$ is $\mathbf{N}$, we must have $\sum_{i \in I(m)} \rho_i = \infty$ for some m. From the fact that $\sup_{i \in \mathbf{N}} d(i) < \infty$ it follows that $\sum_{i \in I(m)} \frac{\rho_i}{2d(i)} = \infty$. Applying Proposition 1.4 and Corollary 5.10 to the sequence $\{T_{m,i}\}_{i \in I(m)}$, we conclude that it is strongly positive. Since it is a subsequence of $\{T_{k(i),i}\}_{i \in \mathbf{N}}$, we conclude that $\{T_{k(i),i}\}_{i \in \mathbf{N}}$ is also strongly positive. Applying Proposition 1.5 we conclude that $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive. ∎

We are now ready to remove the condition that the contiguous mass be generated from the end of the polynomial. Instead, we require only that the persistence of the sequence diverge, that is, $\sum_i \mathcal{P}(P_i) = \infty$. (Divergent persistence has been implicit in the conditions of every result since Theorem 5.7). We also need to guarantee that the inward ratio at either end does not get too small, and so the condition (5.1) of Corollary 5.12 guarantees that the persistence and the end ratios diverge simultaneously. An example of a new sequence admitted by Corollary 5.12 is the sequence $P_i = \begin{array}{ccccccc} i^{-7} & i^{-1} & 1 & i^{-7} & 1 & i^{-7} & i^{-6} \\ \hline -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{array}$. Note that in this example $\mathcal{R}(P_i) \le \frac{2}{i^7}$ and so $\sum_{i=1}^{\infty} \mathcal{R}(P_i) < \infty$, while $\mathcal{P}(P_i) \ge \frac{i^{-1}}{\|P_i\|} \ge \frac{1}{7i}$ and so $\sum_{i=1}^{\infty} \mathcal{P}(P_i) = \infty$.

Corollary 5.12: Let $\{P_i\}_{i \in \mathbf{N}}$ be a sequence of gapless polynomials in $\mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$. Suppose that $\sup_{i \in \mathbf{N}} \deg P_i < \infty$, and let $m(i) = \min \operatorname{Log} P_i$ and $M(i) = \max \operatorname{Log} P_i$, for each i in $\mathbf{N}$. If

$$\sum_{i=1}^{\infty} \min\left(\mathcal{P}(P_i),\, \frac{\langle P_i,\, x^{m(i)+1}\rangle}{\langle P_i,\, x^{m(i)}\rangle},\, \frac{\langle P_i,\, x^{M(i)-1}\rangle}{\langle P_i,\, x^{M(i)}\rangle}\right) = \infty, \tag{5.1}$$

then $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive.

Proof: For each $i \in \mathbf{N}$ we write $P_i = \sum_{j \in \mathbf{Z}} P_{i,j} x^j$. Without loss of generality we may assume that $\|P_i\| = 1$ and that $\min(P_{i,0}, P_{i,1}) \ge \min(P_{i,j},\, P_{i,j+1})$ for all $j \in \mathbf{Z}$. This may be achieved by muliplying P_i by $\frac{1}{\|P_i\|} x^{u_i}$, for an appropriate choice of u_i, and replacing $m(i)$ and $M(i)$ by $m(i) - u_i$ and $M(i) - u_i$ respectively.

For each $i \in \mathbf{N}$ we set

$$L_i = \sum_{j<0} P_{i,j}x^j + \frac{1}{2}[P_{i,0} + P_{i,1}x] \qquad \text{and} \qquad R_i = \frac{1}{2}[P_{i,0} + P_{i,1}x] + \sum_{j>1} P_{i,j}x^j$$

and note that $P_i = L_i + R_i$.

In keeping with the notation of Proposition 1.5, we let $T_{0,i} = L_i$ and $T_{1,i} = R_i$ for all $i \in \mathbf{N}$. Let $k = (k(1), k(2), \ldots)$ be any sequence in $\{0,1\}^{\mathbf{N}}$. By Proposition 1.5, we will be done if we can show that the sequence $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive.

Let $\rho_i = \min\left(\mathcal{P}(P_i), \frac{P_{i,m(i)+1}}{P_{i,m(i)}}, \frac{P_{i,M(i)-1}}{P_{i,M(i)}}\right)$ for each i in $\mathbf{N}$, so that $\sum_{i=1}^{\infty} \rho_i = \infty$.

Applying equation (2.2) in Proposition 2.2, we observe that

$$\mathcal{P}(P_i) = \sum_{m(i)\leq j<M(i)} \min(P_{i,j}, P_{i,j+1}) \leq [M(i) - m(i)] \cdot \min(P_{i,0}, P_{i,1}),$$

and so $\min(P_{i,0}, P_{i,1}) \geq \frac{\mathcal{P}(P_i)}{M(i)-m(i)}$ for each $i \in \mathbf{N}$.

Let $I = \{\, i \in \mathbf{N} \mid k(i) = 1 \,\}$.

First consider the case where $\sum_{i\in I} \rho_i = \infty$. Let $J = \{\, i \in I \mid \deg R_i \geq 4 \,\}$. If it happens that $\sum_{i\in J} \rho_i = \infty$, then we note that

$$\min\left(\langle R_i, x^0\rangle, \langle R_i, x^1\rangle, \frac{\langle R_i, x^{M(i)-1}\rangle}{\langle R_i, x^{M(i)}\rangle}\right) = \min\left(\frac{P_{i,0}}{2}, \frac{P_{i,1}}{2}, \frac{P_{i,M(i)-1}}{P_{i,M(i)}}\right) \geq \frac{\rho_i}{2[M(i) - m(i)]}$$

As $\sum_{i\in J} \rho_i = \infty$ and $\sup_{i\in\mathbf{N}} \deg P_i < \infty$, it follows that

$$\sum_{i\in J} \min\left(\langle R_i, x^0\rangle, \langle R_i, x^1\rangle, \frac{\langle R_i, x^{M(i)-1}\rangle}{\langle R_i, x^{M(i)}\rangle}\right) = \infty,$$

and so we may apply Corollary 5.11 to conclude that $\{R_i\}_{i\in J}$ is strongly positive, from which the strong positivity of $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ follows, by Proposition 1.2.

If $\sum_{i\in I} \rho_i = \infty$ but $\sum_{i\in J} \rho_i < \infty$, then let $K = \{\, i \in I \mid \deg R_i \leq 3 \,\}$, and note that $\sum_{i\in K} \rho_i = \infty$. By Proposition 4.3 we have, for $i \in K$,

$$\mathcal{R}(R_i) = \min\left(\frac{\langle R_i, x^o\rangle + \langle R_i, x^2\rangle}{\langle R_i, x^1\rangle}, \frac{\langle R_i, x^1\rangle}{\langle R_i, x^o\rangle + \langle R_i, x^2\rangle}\right)$$

$$\geq \min\left(\frac{\langle R_i, x^o\rangle}{1}, \frac{\langle R_i, x^1\rangle}{1}\right) = \frac{1}{2}\min(P_{i,0}, P_{i,1}) \geq \frac{\rho_i}{2}.$$

It follows that $\sum_{i\in K} \mathcal{R}(R_i) = \infty$, and so, by Theorem 4.12, $\{R_i\}_{i\in K}$ is strongly positive. Thus $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive, by Proposition 1.2.

We have shown that $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ will be strongly positive whenever $\sum_{i\in I} \rho_i = \infty$. It remains to consider the case where $\sum_{i\in I} \rho_i < \infty$. In this case, $\sum_{i\in\mathbf{N}\setminus I} \rho_i = \infty$. If for each $i \in \mathbf{N} \setminus I$ we define $S_i \in \mathbf{R}[x^{\pm 1}]^+$ by $\langle S_i, x^j\rangle = \langle L_i, x^{1-j}\rangle$, $\forall j \in Z$, then the argument used above for $\{R_i\}_{i\in I}$ will yield the strong positivity of $\{S_i\}_{i\in\mathbf{N}\setminus I}$. It follows that the sequence $\{L_i\}_{i\in\mathbf{N}\setminus I}$ is strongly positive, and so $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive, by Proposition 1.2. ∎

Example 5.13: In Proposition 5.14 we drop the condition that the polynomials be gapless. This gives us the most general strong positivity result so far. It enables us to deduce the strong positivity of sequences such as that given by $P_n = \begin{array}{ccccc} e^{-n} & 1 & 1 & 1 & \frac{1}{n} \\ \hline -3 & -2 & 0 & 7 & 8 \end{array}$ (extra spaces have been added to emphasize the position of the gap). Here there is an isolated coefficient in the middle, and almost no contiguity at the left end; the divergent contiguity on the right end is what ensures strong positivity.

We may also concentrate the bulk of the mass on an isolated coefficient, as in the sequence given by $P_n = \begin{array}{ccccc} e^{-n} & 1 & \sqrt{n} & 1 & \frac{1}{\sqrt{n}} \\ \hline -3 & -2 & 0 & 7 & 8 \end{array}$.

Alternatively, we could have the persistence arising from the middle rather than an end, as in the sequence given by $P_n = \begin{array}{cccccc} e^{-n} & 1 & 1 & \frac{1}{n} & \frac{1}{n} & e^{-n} \\ \hline -3 & -2 & 0 & 1 & 7 & 8 \end{array}$. Note also that the ends may be much larger than the persistent center, as in $P_n = \begin{array}{cccccc} e^{-n} & n & 1 & \frac{1}{\ln n} & \frac{1}{n^5} & e^{-n} \\ \hline -3 & -2 & 0 & 1 & 7 & 8 \end{array}$.

Proposition 5.14: Let $\{P_i\}_{i \in \mathbf{N}}$ be a sequence of polynomials in $\mathbf{R}[x^{\pm 1}]^+ \setminus \{0\}$. Suppose that $\sup_{i \in \mathbf{N}} \deg P_i < \infty$, and let $m(i) = \min \operatorname{Log} P_i$ and $M(i) = \max \operatorname{Log} P_i$ for each i.

If

$$\sum_{i=1}^{\infty} \min\left(\mathcal{P}(P_i), \frac{\langle P_i, x^{m(i)+1}\rangle}{\langle P_i, x^{m(i)}\rangle}, \frac{\langle P_i, x^{M(i)-1}\rangle}{\langle P_i, x^{M(i)}\rangle}\right) = \infty,$$

then $\{P_i\}_{i \in \mathbf{N}}$ is strongly positive.

Proof: For each i in N define $\sigma_i = \min\left(\mathcal{P}(P_i), \frac{\langle P_i, x^{m(i)+1}\rangle}{\langle P_i, x^{m(i)}\rangle}, \frac{\langle P_i, x^{M(i)-1}\rangle}{\langle P_i, x^{M(i)}\rangle}\right)$, and note that $\sum_{i=1}^{\infty} \sigma_i = \infty$. Our approach will be to find a subsequence of polynomials all having the same Log set, and for which the sum of σ_i's diverges. Let $S = \{\operatorname{Log} P_i \mid i \in \mathbf{N}\}$, and note that S is finite, as $\sup_{i \in \mathbf{N}} \deg P_i < \infty$. For each $L \in S$ let $I(L) = \{i \in \mathbf{N} \mid \operatorname{Log} P_i = L\}$, and note that $\sum_{i \in I(L)} \sigma_i = \infty$ for some $L \in S$. Fixing this L, let $m = \min L$ and $M = \max L$, and note that $m + 1$ and $M - 1$ are in L, since otherwise we would have $\sigma_i = 0$ for each $i \in I(L)$. If $M - m \leq 2$, then P_i is gapless for each $i \in I(L)$, and we may apply Corollary 5.12 to deduce the strong positivity of $\{P_i\}_{i \in I(L)}$ and hence the strong positivity of $\{P_i\}_{i \in \mathbf{N}}$ (by Proposition 1.2)

If $M - m \geq 3$, then let $k = M - m - 2$, and let $\{Q_i\}_{i \in \mathbf{N}}$ be the subsequence of $\{P_i\}_{i \in \mathbf{N}}$ consisting of those P_i for which $i \in I(L)$. Set $\rho_i = \min\left(\mathcal{P}(Q_i), \frac{\langle Q_i, x^{m+1}\rangle}{\langle Q_i, x^{m}\rangle}, \frac{\langle Q_i, x^{M-1}\rangle}{\langle Q_i, x^{M}\rangle}\right)$, for each i in N, and note that $\sum_{i=1}^{\infty} \rho_i = \infty$. Note also that $\operatorname{Log} Q_i = L$ for all $i \in \mathbf{N}$. By Proposition 1.2 the strong positivity of $\{P_i\}_{i \in \mathbf{N}}$ will follow if we can show that $\{Q_i\}_{i \in \mathbf{N}}$ is strongly positive. For each i in N we define $R_i = \prod_{i=ik-k+1}^{ik} Q_i$.

We claim that each R_i is gapless. To prove this, it suffices to show that every integer n with $km \leq n \leq kM$ can be written as $n = \sum_{i=1}^{k} n_i$ where each n_i is in the set $\{m, m+1, M-1, M\}$. We prove this by induction on n. For $n = km$ this is clear. Assuming that we have written $n = \sum_{i=1}^{k} n_i$ and that $n < kM$, we show that $n+1$ can also be written in this form. If one of the n_i's is m, we simply replace it by $m+1$. Otherwise, if one of the n_i's is $M-1$, we replace it by M. If all of the n_i are either $m+1$ or M, we take the first n_i that is equal to $m+1$ and replace it by $M-1$, and replace every other n_i by $n_i - 1$. This operation adds $M - 1 - (m+1) = k$ to one of the n_i's and decreases each of the other $k-1$ n_i's by 1. The new value of $\sum_i n_i$ is thus $n+1$.

We have proven that each R_i is gapless. Using property (4) of persistence (page 18), we note that

$$\mathcal{P}(R_i) \geq \max_{ik-k<j\leq ik} \mathcal{P}(Q_j) \geq \max_{ik-k<j\leq ik} \rho_j \geq \frac{1}{k} \sum_{ik-k<j\leq ik} \rho_j$$

for each $i \in \mathbf{N}$. We also have

$$\frac{\langle R_i, x^{kM-1}\rangle}{\langle R_i, x^{kM}\rangle} = \sum_{j=ik-k+1}^{ik} \frac{\langle Q_j, x^{M-1}\rangle}{\langle Q_j, x^{M}\rangle} \geq \sum_{j=ik-k+1}^{ik} \rho_j$$

and

$$\frac{\langle R_i, x^{km+1}\rangle}{\langle R_i, x^{km}\rangle} = \sum_{j=ik-k+1}^{ik} \frac{\langle Q_j, x^{m+1}\rangle}{\langle Q_j, x^{m}\rangle} \geq \sum_{j=ik-k+1}^{ik} \rho_j$$

It follows that

$$\sum_{i=1}^{\infty} \min\left(\mathcal{P}(R_i), \frac{\langle R_i, x^{kM-1}\rangle}{\langle R_i, x^{kM}\rangle}, \frac{\langle R_i, x^{km+1}\rangle}{\langle R_i, x^{km}\rangle}\right) \geq \sum_{i=1}^{\infty} \frac{1}{k} \sum_{j=ik-k+1}^{ik} \rho_j = \frac{1}{k} \sum_{j=1}^{\infty} \rho_j = \infty.$$

Applying Corollary 5.12, we conclude that $\{R_i\}_{i\in\mathbf{N}}$ is strongly positive. As $\{R_i\}_{i\in\mathbf{N}}$ is merely a blocking of $\{Q_i\}_{i\in\mathbf{N}}$, ilt follows that $\{Q_i\}_{i\in\mathbf{N}}$ is strongly positive, which completes the proof. ∎

Recall from Corollary 2.8 that a sufficient condition for a sequence $\{P_i\}_{i\in\mathbf{N}}$ in $\mathbf{R}[x^{\pm 1}]^+$ to be a zero-fluctuation sequence is that $\sum_{i=1}^{\infty} \mathcal{P}(P_i) = \infty$. Being a zero-fluctuation sequence is a prerequisite for strong positivity, but we have already seen that a zero-fluctuation sequence will fail to be strongly positive if it has gaps at the end, or if the mass on the next-to-end coefficient is convergent relative to the mass on the end coefficient, as is the case for the sequences given by $P_i = \begin{array}{cccc} 1 & 1 & \frac{1}{i^2} & 1 \\ \hline 0 & 1 & 2 & 3 \end{array}$ and $P_i = \begin{array}{cccc} 1 & 1 & \frac{1}{i} & i \\ \hline 0 & 1 & 2 & 3 \end{array}$. Given a sequence with $\sum_{i=1}^{\infty} \mathcal{P}(P_i) = \infty$ we might wonder what additional conditions will guarantee strong positivity. Theorem 5.15 gives an answer to this question for sequences of bounded (or essentially bounded) degree. It requires that all the peaks of each P_i lie in some fixed interval, and permits unimodal tails outside this interval, provided their mass is not too large (conditions (1) and (2)). As for a condition on the ends we require that if one of the end coefficients is a peak of P_i, then its neighbour should not be too much smaller than it (the first and last peak of a polynomial are defined on page 6). As the proof of Theorem 5.15 shows, conditions (3) and (4) may be replaced by the weaker condition

$$\sum_{i=1}^{\infty} \min\left(\mathcal{P}(P_i), \frac{\langle P_i, x^{n(i)+1}\rangle}{\langle P_i, x^{n(i)}\rangle}, \frac{\langle P_i, x^{N(i)-1}\rangle}{\langle P_i, x^{N(i)}\rangle}\right) = \infty,$$

where $n(i)$ and $N(i)$ are the lowest and highest degrees in $\operatorname{Log} P_i$. This weakened condition will admit polynomials such as $P_n = \underset{-5}{1}\;\underset{-4}{\frac{1}{n\ln n}}\;\underset{0}{n}\;\underset{1}{1}\;\underset{3}{\frac{1}{n^2}}\;\underset{4}{e^{-n}}$, which would be excluded otherwise (although covered by Proposition 5.14).

Theorem 5.15: Let $\{P_i\}_{i\in\mathbf{N}}$ be a sequence of polynomials in $\mathbf{R}[x^{\pm1}]^+\backslash\{0\}$, with $\sum_{i=1}^{\infty}\mathcal{P}(P_i)=\infty$. Then $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive, provided that there are integers m and M and numbers $B, C \geq 1$, such that for each $i \in \mathbf{N}$ the following are satisfied:

(1) Every peak of P_i lies in the interval (m, M).

(2) $\displaystyle\sum_{j<m}\langle P_i,\, x^j\rangle \leq B\,\langle P_i,\, x^m\rangle$ and $\displaystyle\sum_{j>M}\langle P_i,\, x^j\rangle \leq B\,\langle P_i,\, x^M\rangle$.

(3) If k is the first peak of P_i and $\langle P_i,\, x^{k-1}\rangle = 0$, then $\mathcal{P}(P_i)\,\langle P_i,\, x^k\rangle \leq C\,\langle P_i,\, x^{k+1}\rangle$.

(4) If k is the last peak of P_i and $\langle P_i,\, x^{k+1}\rangle = 0$, then $\mathcal{P}(P_i)\,\langle P_i,\, x^k\rangle \leq C\,\langle P_i,\, x^{k-1}\rangle$.

Proof: We first replace each P_i by $\frac{1}{\|P_i\|}P_i$. This does not affect any of the hypotheses. We may thus assume that $\|P_i\| = 1$ for each i. We also write $P_i = \sum_{j\in\mathbf{Z}} P_{i,j}$ for all $i \in \mathbf{N}$.

For each i in $\mathbf{N}$ let $N(i) = \max\operatorname{Log} S_i$ and $n(i) = \min\operatorname{Log} S_i$, and set

$$\rho_i = \min\left(\mathcal{P}(P_i),\ \frac{P_{i,n(i)+1}}{P_{i,n(i)}},\ \frac{P_{i,N(i)-1}}{P_{i,N(i)}}\right)$$

Without appealing to conditions (3) and (4) we show that the condition $\sum_{i=1}^{\infty}\rho_i = \infty$ is strong enough to guarantee the strong positivity of $\{P_i\}_{i\in\mathbf{N}}$. Then, at the end of the proof we note that conditions (3) and (4) guarantee that $\sum_{i=1}^{\infty}\rho_i = \infty$.

We next define, for each $i \in \mathbf{N}$,

$$Q_i = \sum_{j<m} P_{i,j}x^j + \frac{P_{i,m-1}}{2}x^m \quad\text{and}\quad R_i = \sum_{j>M} P_{i,j}x^j + \frac{P_{i,M+1}}{2}x^M,$$

and set $S_i = P_i - Q_i - R_i$. As every peak of P_i lies in (m, M), it follows that $P_{i,M-1} \geq P_{i,M} \geq P_{i,M+1}$ and $P_{i,m+1} \geq P_{i,m} \geq P_{i,m-1}$. From this we see that S_i has no negative coefficients. If $R_i \neq 0$, then $\sum_{j>M}\langle P_i,\, x^j\rangle x^j$ is unimodal, and so by Lemma 4.18, we find that $\mathcal{R}(R_i) \geq \frac{1}{2}$. If $R_i = 0$ then $\mathcal{R}(R_i) = 1$, so in any case we have $\mathcal{R}(R_i) \geq \frac{1}{2}$. Likewise $\mathcal{R}(Q_i) \geq \frac{1}{2}$.

Equation (2.2) in Proposition 2.2, tells us that $\mathcal{P}(P_i) = \sum_{j\in\mathbf{Z}}\min(P_{i,j},\, P_{i,j+1})$. It also yields $\|S_i\|\,\mathcal{P}(S_i) \geq \sum_{m<j<M-1}\min(P_{i,j},\, P_{i,j+1})$. Combining these facts with condition (2) we obtain

$$\begin{aligned}
\mathcal{P}(P_i) &= \sum_{j\leq m}\min(P_{i,j},\, P_{i,j+1}) + \sum_{j\geq M-1}\min(P_{i,j},\, P_{i,j+1}) + \sum_{m<j<M-1}\min(P_{i,j},\, P_{i,j+1}) \\
&= \sum_{j\leq m} P_{i,j} + \sum_{j\geq M-1} P_{i,j+1} + \sum_{m<j<M-1}\min(P_{i,j},\, P_{i,j+1}) \\
&\leq \sum_{j\leq m} P_{i,j} + \sum_{j\geq M} P_{i,j} + \|S_i\|\,\mathcal{P}(S_i) \\
&\leq (B+1)P_{i,m} + (B+1)P_{i,M} + \mathcal{P}(S_i)
\end{aligned}$$

Since $P_{i,M} \geq P_{i,M+1}$ we have $\langle S_i, x^M \rangle = P_{i,M} - \frac{1}{2} P_{i,M+1} \geq \frac{1}{2} P_{i,M}$. Similarly, $\langle S_i, x^m \rangle \geq \frac{1}{2} P_{i,m}$. Thus

$$\mathcal{P}(S_i) \geq \|S_i\| \mathcal{P}(S_i) \geq \min\left(\langle S_i, x^m \rangle, \langle S_i, x^{m+1} \rangle\right) + \min\left(\langle S_i, x^{M-1} \rangle, \langle S_i, x^M \rangle\right)$$
$$= S_{i,m} + S_{i,M} \geq \frac{1}{2}\left(P_{i,m} + P_{i,M}\right)$$

We now have

$$\mathcal{P}(P_i) \leq (B+1)(P_{i,m} + P_{i,M}) + \mathcal{P}(S_i) \leq 2(B+1)\mathcal{P}(S_i) + \mathcal{P}(S_i) = (2B+3)\mathcal{P}(S_i).$$

It follows that $\mathcal{P}(S_i) \geq \frac{\mathcal{P}(P_i)}{2B+3} \geq \frac{\rho_i}{2B+3}$.

Note that $\langle S_i, x^{n(i)} \rangle \leq P_{i,n(i)}$ and $P_{i,n(i)+1} \leq 2 \langle S_i, x^{n(i)+1} \rangle$, and so $\frac{\langle S_i, x^{n(i)+1} \rangle}{\langle S_i, x^{n(i)} \rangle} \geq \frac{P_{i,n(i)+1}}{2P_{i,n(i)}} \geq \frac{\rho_i}{2}$. Similarly, $\frac{\langle S_i, x^{N(i)-1} \rangle}{\langle S_i, x^{N(i)} \rangle} \geq \frac{\rho_i}{2}$.

We now assume that $\sum_{i=1}^{\infty} \rho_i = \infty$ and deduce the strong positivity of $\{P_i\}_{i \in \mathbb{N}}$. At the end of the proof we show that conditions (3) and (4) imply $\sum_{i=1}^{\infty} \rho_i = \infty$.

In keeping with the notation of Proposition 1.5, we let $T_{0,i} = Q_i$ and $T_{1,i} = S_i$ and $T_{2,i} = R_i$ for all $i \in \mathbb{N}$. Let $k = (k(1), k(2), \ldots)$ be any sequence in $\{0,1,2\}^{\mathbb{N}}$, and set $I = \{ i \in \mathbb{N} \mid k(i) = 1 \}$.

If $\mathbb{N} \setminus I$ is infinite, then $\sum_{i \in \mathbb{N} \setminus I} \mathcal{R}(T_{k(i),i}) \geq \sum_{i \in \mathbb{N} \setminus I} \frac{1}{2} = \infty$ and so by Theorem 4.12 $\{T_{k(i),i}\}_{i \in \mathbb{N} \setminus I}$ is strongly positive. Thus $\{T_{k(i),i}\}_{i \in \mathbb{N}}$ is strongly positive.

If $\mathbb{N} \setminus I$ is finite, then $\sum_{i \in I} \rho_i = \infty$, and so

$$\sum_{i \in I} \min\left(\mathcal{P}(S_i), \frac{\langle S_i, x^{n(i)+1} \rangle}{\langle S_i, x^{n(i)} \rangle}, \frac{\langle S_i, x^{N(i)-1} \rangle}{\langle S_i, x^{N(i)} \rangle}\right) \geq \sum_{i \in I} \min\left(\frac{\rho_i}{2B+3}, \frac{\rho_i}{2}\right) = \infty,$$

by Proposition 5.14, we conclude that $\{S_i\}_{i \in I}$ is strongly positive. It follows that $\{T_{k(i),i}\}_{i \in \mathbb{N}}$ is strongly positive.

We have shown that, subject to the condition $\sum_{i=1}^{\infty} \rho_i = \infty$, the sequence $\{T_{k(i),i}\}_{i \in \mathbb{N}}$ is strongly positive for any sequence $k = (k(1), k(2), \ldots)$ in $\{0,1,2\}^{\mathbb{N}}$. It follows, by Proposition 1.5, that $\{P_i\}_{i \in \mathbb{N}}$ is strongly positive.

We now show that the condition $\sum_{i=1}^{\infty} \rho_i = \infty$ is a consequence of $\sum_{i=1}^{\infty} \mathcal{P}(P_i) = \infty$ and conditions (3) and (4).

If $n(i)$ is not the first peak of P_i, then $P_{i,n(i)+1} \geq P_{i,n(i)}$, so $\frac{P_{i,n(i)+1}}{P_{i,n(i)}} \geq 1 \geq \mathcal{P}(P_i)$. On the other hand, if $n(i)$ is the first peak of P_i, then by condition (3) we have $\mathcal{P}(P_i)\, P_{i,n(i)} \leq C\, P_{i,n(i)+1}$, which yields $\frac{P_{i,n(i)+1}}{P_{i,n(i)}} \geq \frac{\mathcal{P}(P_i)}{C}$. Similarly, using condition (4) we find that $\frac{P_{i,N(i)-1}}{P_{i,N(i)}} \geq \frac{\mathcal{P}(P_i)}{C}$. It follows that $\rho_i \geq \frac{\mathcal{P}(P_i)}{C}$, and so the divergence of $\sum_{i=1}^{\infty} \rho_i$ follows from that of $\sum_{i=1}^{\infty} \mathcal{P}(P_i)$. ∎

In certain cases, strong positivity may be lost by arbitrarily small positive perturbations. For example, the sequence given by $P_i = \alpha_i + x^2 + x^3$, $i \in \mathbb{N}$, is strongly positive only if infinitely many of the α_i's are zero, and is otherwise not strongly positive (regardless of how small any nonzero α_i's are). On the other hand, strong positivity may sometimes be obtained from arbitrarily small positive perturbations, as the following result shows:

Proposition 5.16: Suppose $\{\alpha_i\}_{i\in\mathbf{N}}$ is a sequence in $[0,1]$. Then the sequence in $\mathbf{R}[x^{\pm 1}]^+$ given by

$$P_i = \alpha_i x^{-1} + 1 + x^2 + x^3, \qquad i \in \mathbf{N}$$

is strongly positive if and only if infinitely many of the α_i's are nonzero.

Proof: First suppose that at most finitely many α_i's are nonzero. Then choosing $t \in \mathbf{Z}^+$ such that $\alpha_i = 0,\ \forall i > t$, we note that the coefficient of x^1 in $(1 - x + x^2)P^{n,t}$ is -1 for all $n \in \mathbf{N}$, so $\{P_i\}_{i\in\mathbf{N}}$ is not strongly positive.

Now assume that infinitely many of the α_i's are nonzero. If $\sum_{i=1}^{\infty} \alpha_i = \infty$, then we have $\sum_{i=1}^{\infty} \mathcal{R}(P_i) = \sum_{i=1}^{\infty} \alpha_i = \infty$, and by Theorem 4.12 we conclude that $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive. If, on the other hand, $\sum_{i=1}^{\infty} \alpha_i < \infty$, then we may apply Theorem 5.7 to conclude that $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive. ∎

In Proposition 5.17 we generalize the situation of Proposition 5.16 by replacing the $\alpha_i x^{-1} + 1$ by a unimodal polynomial, and replacing the $x^2 + x^3$ by polynomials which have sufficient mass at x^2 and have $\mathcal{R}$ value not too small. We require no bound of any kind on the degrees.

Proposition 5.17: Suppose that for each i in $\mathbf{N}$ we have $f_i, g_i \in \mathbf{R}[x^{\pm 1}]^+ \backslash \{0\}$ such that g_i is unimodal with largest coefficient $\langle g_i, x^o \rangle = 1$. Suppose also that $\sum_{i\in\mathbf{N}} \langle f_i, x^2 \rangle = \infty$ and that there is an $\epsilon > 0$ such that $\mathcal{R}(f_i) \geq \epsilon$ for all i. Then the sequence $\{f_i + g_i\}_{i\in\mathbf{N}}$ is strongly positive if $g_i \not\equiv 1$ for infinitely many i.

Proof: Let $P_i = f_i + g_i$ for all $i \in \mathbf{N}$.

If $\sum_{i=1}^{\infty} \mathcal{R}(g_i) = \infty$, then using property (5) of $\mathcal{R}$ (page 38) we have $\sum_{i=1}^{\infty} \mathcal{R}(P_i) \geq \sum_{i=1}^{\infty} \min(\mathcal{R}(g_i), \epsilon) = \infty$, and by Theorem 4.12 we conclude that $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive.

It remains to consider the case where $\sum_{i=1}^{\infty} \mathcal{R}(g_i) < \infty$. Since $\mathcal{R}(g_i) \geq \frac{1}{2}$ for at most finitely many i, we may discard these terms without affecting the strong positivity of the sequence. In keeping with the notation of Proposition 1.5, we define, for each i in $\mathbf{N}$,

$$T_{0,i} = \frac{1}{2} \langle g_i, x^1 \rangle x^0 + \sum_{j\geq 1} \langle g_i, x^j \rangle x^j$$

and note that if $T_{0,i}$ is nonzero, then it is unimodal with peak at 1, and by Lemma 4.18 we find that $\mathcal{R}(T_{0,i}) \geq \frac{1}{2}$. Since $\mathcal{R}(0) = 1$, we conclude that in any case $\mathcal{R}(T_{0,i}) \geq \frac{1}{2}$. Similarly, we set

$$T_{1,i} = \sum_{j\leq -2} \langle g_i, x^j \rangle x^j + \frac{1}{2} \langle g_i, x^{-2} \rangle x^{-1},$$

and note that $\mathcal{R}(T_{1,i}) \geq \frac{1}{2}$. We let $h_i = g_i - T_{0,i} - T_{1,i}$ and note that h is nonnegative and $\operatorname{Log} h_i \in \{-1, 0\}$, and that $\langle h_i, x^o \rangle \geq \frac{1}{2}$. Since $\mathcal{R}(g_i) < \frac{1}{2}$, we must have $\mathcal{R}(h_i) < \frac{1}{2}$. In particular, $\langle h_i, x^{-1} \rangle < \langle h_i, x^o \rangle$. We now set $T_{2,i} = h_i + f_i$, and note that if $k = (k(1), k(2), \ldots)$ is any sequence in $\{0,1,2\}^{\mathbf{N}}$ for which $k(i)$ is different from 2 for infinitely many i, then $\sum_{i=1}^{\infty} \mathcal{R}(T_{k(i),i}) = \infty$, and so by Theorem 4.12 the sequence $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive. In order to be able to apply Proposition 1.5 to show that $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive, it remains to show that if $k = (k(1), k(2), \ldots)$ is any sequence in $\{0,1,2\}^{\mathbf{N}}$ for which $k(i)$ is different from 2 for only finitely many i, then $\{T_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive. This will follow if we show that $\{T_{2,i}\}_{i\in\mathbf{N}}$ is strongly positive.

If it happens that $\sum_{i=1}^{\infty}\langle f_i, x^1\rangle = \infty$, then for each i in $\mathbf{N}$ we set $\beta_i = \frac{1}{2}\min\left(1, \langle f_i, x^1\rangle\right)$ and note that $\mathcal{R}(f_i - \beta_i x^1) \geq \frac{\epsilon}{2}$, by Proposition 4.14. Meanwhile

$$h_i + \beta_i x^1 = \left[\langle h_i, x^{-1}\rangle\left(x^{-1} + \frac{1}{4}\right)\right] + \left[\left(\langle h_i, x^o\rangle - \frac{1}{4}\langle h_i, x^{-1}\rangle\right) + \beta_i x^1\right]$$

and so $\mathcal{R}(h_i + \beta_i x) \geq \min\left(\frac{1}{4}, \beta_i\right)$. It follows that $\sum_{i=1}^{\infty}\mathcal{R}(h_i + \beta_i x) = \infty$. From this we see that $\sum_{i=1}^{\infty}\mathcal{R}(T_{2,i}) = \sum_{i=1}^{\infty}\mathcal{R}\big((h_i + \beta_i x) + (f - \beta_i x)\big) = \infty$, and so by Theorem 4.12, the sequence $\{T_{2,i}\}_{i\in\mathbf{N}}$ is strongly positive.

It remains to prove the strong positivity of $\{T_{2,i}\}_{i\in\mathbf{N}}$ in the case where $\sum_{i=1}^{\infty}\langle f_i, x^1\rangle < \infty$. By Proposition 4.1 we note that each f_i can be decomposed as $f_i = \sum_j a_{i,j}x^j + b_{i,j}x^{j+1}$ where, $r(a_{i,j} + b_{i,j}x) \geq \epsilon$ for each j in $\mathbf{Z}$ (remember that $r(0) = 1$). Since $\sum_i a_{i,1} \leq \sum_i \langle f_i, x^1\rangle < \infty$, and since $b_{i,1} \leq \frac{a_{i,1}}{\epsilon}$, it follows that $\sum_i b_{i,1} < \infty$. As $b_{i,1} + a_{i,2} = \langle f_i, x^2\rangle$ and $\sum_{i=1}^{\infty}\langle f_i, x^2\rangle = \infty$, it follows that $\sum_{i=1}^{\infty} a_{i,2} = \infty$, and so $\sum_{i=1}^{\infty}\min(a_{i,2}, b_{i,2}) = \infty$. If $\delta_i = \max(a_{i,2}, b_{i,2})$ is greater than $\frac{1}{2}$, then we set $R_i = \frac{1}{2\delta_i}\left(a_{i,2}x^2 + b_{i,2}x^3\right)$. Otherwise we set $R_i = \left(a_{i,2}x^2 + b_{i,2}x^3\right)$. Note that $\mathcal{R}(f_i - R_i) \geq \epsilon$. For each i in $\mathbf{N}$ we let $S_{0,i} = h_i + R_i$, and note that, if we first divide each $S_{0,i}$ by $\langle h_i, x^o\rangle$, we may apply Corollary 5.8 to deduce that the sequence $\{S_{0,i}\}_{i\in\mathbf{N}}$ is strongly positive. Setting $S_{1,i} = f_i - R_i$ for each $i \in \mathbf{N}$ we note, as before that if $k = (k(1), k(2), \ldots)$ is any sequence in $\{0,1\}^{\mathbf{N}}$ for which $k(i)$ is different from 0 for infinitely many i, then $\sum_{i=1}^{\infty}\mathcal{R}(S_{k(i),i}) = \infty$, and so by Theorem 4.12 the sequence $\{S_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive. If $k = (k(1), k(2), \ldots)$ is any sequence in $\{0,1\}^{\mathbf{N}}$ for which $k(i)$ is different from 0 for only finitely many i, then $\{S_{k(i),i}\}_{i\in\mathbf{N}}$ is strongly positive, since $\{S_{0,i}\}_{i\in\mathbf{N}}$ is strongly positive. Applying Proposition 1.5 we conclude that $\{S_{0,i} + S_{1,i}\}_{i\in\mathbf{N}}$ is strongly positive, but $S_{0,i} + S_{1,i} = h_i + f_i = T_{2,i}$. Thus the sequence $\{T_{2,i}\}_{i\in\mathbf{N}}$ is strongly positive, and we are done. ∎

When Persistence Converges.

As has been discussed, a necessary condition for a sequence $\{P_i\}_{i\in\mathbf{N}}$ in $\mathbf{R}[x^{\pm 1}]^+$ to be strongly positive is that the sequence be a zero-fluctuation sequence. We have seen that strong positivity additionally requires that there not be isolated masses at the ends of the polynomial. In the preceding section we investigated how non-contiguous the ends of polynomials in a strongly positive sequence could be, under the assumption that $\sum_{i=1}^{\infty}\mathcal{P}(P_i) = \infty$. We now remove the condition that the persistence diverge, and look for conditions which will guarantee strong positivity. We will see that the spacing of the coefficients plays much more of a role here than in the preceding section. The spacing considerations reflect similar conditions which arose in the study of fluctuation, especially relative primeness of the positions of the dominant coefficients.

Example 5.18: The sequence $\{P_n\}_{n\in\mathbf{N}}$ given by

$$P_n = \begin{array}{ccccc} e^{-n} & 1 & \sqrt{n} & 1 & e^{-n} \\ \hline -\mu & -\mu+1 & 0 & \mu & \mu+1 \end{array}$$

is strongly positive for any $\mu \geq 0$, although $\mathcal{P}(P_n) < \frac{2}{e^n}$ if $\mu \geq 2$. This follows from Proposition 5.20 (after factoring out $\sqrt{n}$ from P_n).

We first need a little numerical lemma:

Lemma 5.19: Let $\{\beta_i\}_{i\in\mathbf{N}}$ be a sequence of numbers in $(0,\infty)$ with $\sum_{i=1}^{\infty}\beta_i = \infty$. Then given any k in $\mathbf{N}$ there is an injective function $\phi : \mathbf{N} \to \mathbf{N}$ such that

$$\sum_{i=1}^{\infty} \min\left\{ \beta_{\phi(j)} \,\middle|\, (i-1)k < j \le ik \right\} = \infty.$$

Proof: If $\{\beta_i\}$ has a cluster point $c > 0$, then let $\phi(1)$ be the first i in $\mathbf{N}$ for which $\beta_i \ge \frac{c}{2}$. Having chosen $\phi(j)$, let $\phi(j+1)$ be the smallest $i > \phi(j)$ for which $\beta_i \ge \frac{c}{2}$. Having defined ϕ in this way we note that $\phi(j) \ge \frac{c}{2}$ for all $j \in \mathbf{N}$, and the result follows.

If $\{\beta_i\}$ has no nonzero cluster point, then we choose i so that $\beta_i = \max\{\beta_j \mid j \in \mathbf{N}\}$, and let $\phi(1) = i$. Having chosen $\phi(j)$, take $i \in \mathbf{N}$ different from $\phi(m)$ for $m = 1,2,\ldots,j$ and such that $\beta_i = \max\{\beta_m \mid m \in \mathbf{N} \setminus \{(\phi(1),\phi(2),\ldots,\phi(j)\}\}$. Since $\sum_{i=1}^{n}\beta_{\phi(i)} \ge \sum_{i=1}^{n}\beta_i$, it follows that $\sum_{i=1}^{\infty}\beta_{\phi(i)} = \infty$. As $\{\beta_{\phi(i)}\}_{i\in\mathbf{N}}$ is a nonincreasing sequence it is clear that if we set $S_m = \sum_{i\equiv m (\mathrm{mod}\, k)} \beta_{\phi(i)}$ for $m = 1,2,\ldots k$, then $S_1 \ge S_2 \ge \ldots \ge S_k \ge S_1 - \beta_{\phi(1)}$. Since $\sum_{m=1}^{k} S_m = \infty$, we conclude that $S_k = \infty$, and we are done. ∎

Numerous results may be obtained about sequences which will satisfy the conditions of Proposition 5.14 after blocking into products of a bounded number of terms. The following is an example of such a result. Using Lemma 5.19 many others could be written. As an example, Proposition 5.20 will show the strong positivity of the sequence

$$P_i = \begin{array}{ccccc} \frac{1}{2^i} & \frac{1}{i} & 1 & \frac{1}{\ln i} & \frac{1}{5^i} \\ \hline -50 & -49 & 0 & 50 & 51 \end{array}.$$

Note that $\sum_i \mathcal{P}(P_i) < \infty$.

Proposition 5.20: Suppose that for each i in $\mathbf{N}$,

$$P_i = \begin{array}{ccccc} \rho_i & \sigma_i & 1 & \tau_i & \omega_i \\ \hline -\mu & -\mu+1 & 0 & \mu & \mu+1 \end{array}$$

Where $\mu \ge 0$ is fixed, $0 < \rho_i \le \sigma_i \le 1$ and $0 < \omega_i \le \tau_i \le 1$ for each i. If

$$\sum_{i=1}^{\infty} \sigma_i\tau_i = \infty,$$

then $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive.

Proof: Setting $\beta_j = \sigma_j\tau_j$ for all j in $\mathbf{N}$, we may use Lemma 5.19 to obtain an injective map $\phi : \mathbf{N} \to \mathbf{N}$ for which $\sum_j \min\left(\beta_{\phi(2j-1)}, \beta_{\phi(2j)}\right) = \infty$.

We let $Q_i = P_{\phi(i)}$ for each i. If we can show the strong positivity of $\{Q_i\}$, that of $\{P_j\}$ will follow. Since we don't need $\{P_i\}$ any more, we renumber all the coefficients of Q_i so that

$$Q_i = \begin{array}{ccccc} \rho_i & \sigma_i & 1 & \tau_i & \omega_i \\ \hline -\mu & -\mu+1 & 0 & \mu & \mu+1 \end{array}$$

for all i. We then have

$$\sum_{j \text{ even}} \min\left(\sigma_{j-1}\tau_{j-1}, \sigma_j\tau_j\right) = \infty$$

Note that

$$\begin{aligned}\langle Q_{j-1}Q_j, x^1\rangle = \sigma_{j-1}\tau_j + \sigma_j\tau_{j-1} &\geq \min(\sigma_{j-1},\sigma_j)\tau_j + \min(\sigma_{j-1},\sigma_j)\tau_{j-1} \\ &= \min(\sigma_{j-1},\sigma_j)\cdot(\tau_j+\tau_{j-1}) \geq \min(\sigma_{j-1}\tau_{j-1}, \sigma_j\tau_j).\end{aligned}$$

Thus by equation (2.2), $\mathcal{P}(Q_{j-1}Q_j) \geq \frac{\min(\sigma_{j-1}\tau_{j-1},\sigma_j\tau_j)}{25}$, and so

$$\sum_{j \text{ even}} \mathcal{P}(Q_{j-1}Q_j) = \infty.$$

Using Proposition 5.14 we conclude that $\{Q_{2i-1}Q_{2i}\}_{i\in\mathbf{N}}$ is strongly positive, so $\{Q_i\}$ is strongly positive. ∎

The smallest coefficients need not lie on the outside. For example the sequence given by $P_i = \begin{array}{ccccc} n & 1 & n^{3/2} & 1 & n \\ \hline -5 & -4 & 0 & 5 & 6 \end{array}$ is strongly positive, as may be seen by blocking it in pairs and applying Proposition 5.14.

Notice in the following result that the positions of the dominant masses are required to be relatively prime. Without this condition the sequence would risk not being even a zero-fluctuation sequence.

Proposition 5.21: Suppose that $\mu, \nu > 0$ are relatively prime, and for each $i \in \mathbf{N}$ we have

$$P_i = \begin{array}{ccccc} \rho_i & \sigma_i & 1 & \tau_i & \omega_i \\ \hline -\nu-1 & -\nu & 0 & \mu & \mu+1 \end{array}$$

where $0 < \rho_i \leq \sigma_i \leq 1$ and $0 < \omega_i \leq \tau_i \leq 1$ for each i. Let p and q be positive integers such that $p\mu - q\nu \in \{-1, 1\}$. If

$$\sum_{i=0}^{\infty} \left(\prod_{j=1}^{q} \sigma_{i+j}\right)\left(\prod_{j=1}^{p} \tau_{i+q+j}\right) = \infty,$$

then $\{P_i\}_{i\in\mathbf{N}}$ is strongly positive.

Proof: First note that by equation (2.2), for each i in $\mathbf{N}$,

$$\mathcal{P}\left(\prod_{j=1}^{p+q} P_{i+j}\right) \geq \frac{\left(\prod_{j=1}^{q} \sigma_{i+j}\right)\left(\prod_{j=1}^{p} \tau_{i+q+j}\right)}{5^{p+q}}.$$

It follows that $\sum_{i=1}^{\infty} \mathcal{P}\left(\prod_{j=1}^{p+q} P_{i+j}\right) = \infty$. There is therefore a way of blocking $\{P_i\}_{i\in\mathbf{N}}$, into products Q_i of either 1 or $p+q$ terms, so that $\sum_i \mathcal{P}(Q_i) = \infty$. It follows from Proposition 5.14 that $\{Q_i\}$ is strongly positive. The strong positivity of $\{P_i\}$ follows. ∎

Example 5.22: While in the preceding results we were able to deduce strong positivity by telescoping the sequence into blocks of bounded length, this is not always possible. Substantial effort has been expended to try to determine whether sequences such as

$$P_i = \begin{array}{ccccc} \frac{1}{i^2} & \frac{1}{i} & 1 & \frac{1}{i} & \frac{1}{i^2} \\ \hline -3 & -2 & 0 & 3 & 4 \end{array}$$

are strongly positive. (This is a zero-fluctuation sequence, as -2 and 3 are relatively prime). In fact it has been shown (in 12 pages of detailed estimates and bounds) that if we take this sequence and telescope into blocks of exponentially growing length defined by $n_i = 2^{i-1} - 1$, $(i \in \mathbf{N})$, then $\sum_{k=1}^{\infty} \mathcal{R}\big(\prod_{i=n_k+1}^{n_{k+1}} P_i\big)$ is still convergent.

Example 5.23: The sequence given by

$$P_i = \begin{array}{ccccccc} i^{-\frac{2}{3}} & 1 & & \sqrt{i} & & 1 & i^{-\frac{2}{3}} \\ \hline -3 & -2 & & 0 & & 2 & 3 \end{array}$$

is not strongly positive.

[This may be shown directly, by noting that the proportion of the mass off the sublattice of even integers is convergent, or may be deduced from the fact that this is not a zero-fluctuation sequence.]

6. Strong Unimodality of Certain Sums of Products

Given numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ in $(0, \infty)$, we consider, for $0 \le s \le k \le n$ the coefficient

$$\rho_{k,s} = \left\langle \sum_{|K|=k} \prod_{i \in K} (\alpha_i + \beta_i x),\ x^s \right\rangle \tag{6.1}$$

where the summation ranges over the subsets of $\{1, 2, \ldots n\}$ of size k. We show that for fixed k, these coefficients form a strongly unimodal sequence in s (Corollary 6.4), and that for fixed s the coefficients form a strongly unimodal sequence in k (Corollary 6.3). Placing the coefficients $\rho_{k,s}$ for $0 \le s \le k \le n$ on a grid, they form a triangular array, and the above results correspond to strong positivity along each horizontal and vertical cross-section. We also prove strong unimodality along the diagonals parallel to the third side of the triangle; that is, each sequence $\{\rho_{k+m,s+m}\}_{m \in \mathbf{Z}}$ is strongly unimodal (Corollary 6.5). These considerations arose from investigations of the strong positivity of sequences of the form

$$P_i = 1 + (x^\nu + x^\mu)(\alpha_i + \beta_i x).$$

In Proposition 6.6 we prove an additional related result which plays a role in the product of the P_i's.

Our proofs are combinatorial, and for each of the strong unimodality results we actually prove a somewhat stronger fact, as will be noted in due course. We begin by identifying the $\rho_{k,s}$ with coefficients of a product of simple bivariate polynomials.

Given numbers $\sigma_1, \sigma_2, \ldots \sigma_n$ and $\tau_1, \tau_2, \ldots \tau_n$ in $(0, \infty)$, we define

$$W_n = \prod_{i=1}^{n} (y + \tau_i + \tau_i \sigma_i x), \qquad n \in \mathbf{N}. \tag{6.2}$$

We also define the coefficients of W_n by

$$\rho_{k,s}(n) = \left\langle W_n,\ y^{n-k} x^s \right\rangle, \qquad k, s \in \mathbf{Z}. \tag{6.3}$$

We sometimes shorten $\rho_{k,s}(n)$ to simply $\rho_{k,s}$, when n is clear from context. Note that $\rho_{k,s}(n)$ is nonzero if and only if $0 \le s \le k \le n$. Using (6.3) and (6.2) we find that

$$\begin{aligned}
\rho_{k,s}(n+1) &= \left\langle W_{n+1},\ y^{n+1-k} x^s \right\rangle \\
&= \left\langle (y + \tau_{n+1} + \tau_{n+1}\sigma_{n+1} x) W_n,\ y^{n+1-k} x^s \right\rangle \\
&= \left\langle y W_n,\ y^{n+1-k} x^s \right\rangle + \left\langle \tau_{n+1} W_n,\ y^{n+1-k} x^s \right\rangle + \left\langle \tau_{n+1}\sigma_{n+1} x W_n,\ y^{n+1-k} x^s \right\rangle \\
&= \left\langle W_n,\ y^{n-k} x^s \right\rangle + \tau_{n+1} \left\langle W_n,\ y^{n-(k-1)} x^s \right\rangle + \tau_{n+1}\sigma_{n+1} \left\langle W_n,\ y^{n-(k-1)} x^{s-1} \right\rangle
\end{aligned}$$

which yields the recursion formula

$$\rho_{k,s}(n+1) = \rho_{k,s}(n) + \tau_{n+1}[\rho_{k-1,s}(n) + \sigma_{n+1}\rho_{k-1,s-1}(n)], \qquad \forall k, s \in \mathbf{Z}. \tag{6.4}$$

We let $N = \{1, 2, 3, \ldots, n\}$, and note that

$$\rho_{k,s} = \left\langle W_n,\ y^{n-k} x^s \right\rangle = \sum_{\substack{K \subseteq N \\ |K|=k}} \sum_{\substack{J \subseteq K \\ |J|=s}} \left(\prod_{i \in K \setminus J} \tau_i \right) \left(\prod_{i \in J} \tau_i \sigma_i \right) = \sum_{\substack{K \subseteq N \\ |K|=k}} \sum_{\substack{J \subseteq K \\ |J|=s}} \left(\prod_{i \in K} \tau_i \right) \left(\prod_{i \in J} \sigma_i \right).$$

We see that $\rho_{k,s}$ is identical with the coefficient in (6.1), if we identify α_i with τ_i and β_i with $\tau_i\sigma_i$. Defining, for $0 \le s \le k \le n$ and $0 \le p \le m \le n$,

$$\begin{aligned} f(k,s,m,p) &= \rho_{k,s}\rho_{m,p} \\ &= \sum_{\substack{K_1\subseteq N \\ |K_1|=k}} \sum_{\substack{J_1\subseteq K_1 \\ |J_1|=s}} \sum_{\substack{K_2\subseteq N \\ |K_2|=m}} \sum_{\substack{J_2\subseteq K_2 \\ |J_2|=p}} \left(\prod_{i\in K_1}\tau_i\right)\left(\prod_{i\in J_1}\sigma_i\right)\left(\prod_{i\in K_2}\tau_i\right)\left(\prod_{i\in J_2}\sigma_i\right) \end{aligned} \tag{6.5}$$

we may regard $f(k,s,m,p)$ as a polynomial in indeterminates $\tau_1, \tau_2, \ldots, \tau_n$ and $\sigma_1, \sigma_2, \ldots, \sigma_n$. Each monomial in this polynomial has an integer coefficient. It will be useful for us to be able to show that each such monomial in certain differences of the form $f(k,s,m,p) - f(k',s',m',p')$ has a nonnegative coeficient. To this end, we consider the number of a times a fixed monomial occurs in $f(k,s,m,p)$. For a given monomial, each σ_i and τ_i occurs with an exponent between 0 and 2. For $0 \le \mu \le \nu \le 2$, we let $S_{\mu,\nu}$ be the set of integers $i \in \mathrm{N} = \{1, 2, \ldots, n\}$ for which σ_i^{μ} and τ_i^{ν} appear in the given monomial. We let $s_{\mu,\nu} = |S_{\mu,\nu}|$.

Lemma 6.1: Suppose $0 \le s \le k \le n$ and $0 \le p \le m \le n$, and let $f(k,s,m,p)$ be defined as in (6.5), and for a given monomial in $f(k,s,m,p)$, let $s_{\mu,\nu}$ be defined as above, for $0 \le \mu \le \nu \le 2$. Then the following hold:

$$2(s_{02} + s_{12} + s_{22}) + (s_{01} + s_{11}) = k + m \tag{6.6}$$

$$(s_{11} + s_{12}) + 2s_{22} = s + p \tag{6.7}$$

If we let

$$L = \max\left(-\frac{s_{12}}{2} + \frac{p-s}{2},\ -\frac{s_{11}}{2},\ -\frac{s_{01}}{2} + \frac{m-k}{2}\right)$$

and

$$U = \min\left(\frac{s_{12}}{2} + \frac{p-s}{2},\ \frac{s_{11}}{2},\ \frac{s_{01}}{2} + \frac{m-k}{2}\right),$$

and set $J = \{L, L+1, L+2, \ldots, U\}$, then the coefficient of the given monomial in $f(k,s,m,p)$ is

$$\sum_{\eta\in J} \binom{s_{11}}{\frac{s_{11}}{2}+\eta}\binom{s_{01}}{\frac{s_{01}}{2}+\eta-\frac{m-k}{2}}\binom{s_{12}}{\frac{s_{12}}{2}+\eta-\frac{p-s}{2}}$$

Proof: First suppose we are given a monomial in $f(k,s,m,p)$. This monomial is completely characterized by the sets S_{01}, S_{02}, S_{11}, S_{12} and S_{22}. Let K_1, K_2, J_1 and J_2 be any sets as in (6.5) which give rise to the given monomial. Precisely stated, this means that

$$\left.\begin{aligned} S_{22} &= J_1 \cap J_2 \\ S_{12} &= (K_1 \cap K_2) \cap [(J_1 \cup J_2) \setminus (J_1 \cap J_2)] \\ S_{02} &= (K_1 \cap K_2) \setminus (J_1 \cup J_2) \\ S_{11} &= (J_1 \setminus K_2) \cup (J_2 \setminus K_1) \\ S_{01} &= [(K_1 \setminus (J_1 \cup K_2)] \cup [(K_2 \setminus (J_2 \cup K_1)] \end{aligned}\right\} \tag{6.8}$$

Equation (6.6) follows from the fact that $|K_1| + |K_2| = k + m$. Equation (6.7) follows from the fact that $|J_1| + |J_2| = s + p$.

We next deduce a few facts about the three quantities $a = |S_{11} \cap J_1|$, $b = |S_{01} \cap K_1|$ and $c = |S_{12} \cap J_2|$.

Since $J_1 = S_{22} \cup (S_{11} \cap J_1) \cup (S_{12} \setminus J_2)$ is a disjoint union, taking cardinalities of both sides we obtain the equality $s = s_{22} + a + (s_{12} - c)$, which yields

$$c = s_{12} + s_{22} - s + a \tag{6.9}$$

Since $K_1 \setminus J_1 = S_{02} \cup (S_{01} \cap K_1) \cup (S_{12} \cap J_2)$ is also a disjoint union, we obtain the equality $k - s = s_{02} + b + c$, which can be rewritten as

$$c = k - s - s_{02} - b \tag{6.10}$$

Since $0 \leq a \leq s_{11}$ and $0 \leq b \leq s_{01}$, equations (6.9) and (6.10) yield the inequalities $s_{12} + s_{22} - s \leq c \leq s_{12} + s_{22} - s + s_{11}$ and $k - s - s_{02} - s_{01} \leq c \leq k - s - s_{02}$. Combining this with the inequality $0 \leq c \leq s_{12}$, we can conclude that $c_l \leq c \leq c_h$, where

$$c_l = \max(0,\ s_{12} + s_{22} - s,\ k - s - s_{02} - s_{01})$$

and

$$c_h = \min(s_{12},\ s_{12} + s_{22} - s + s_{11},\ k - s - s_{02}).$$

We wish to calculate the coefficient of our given monomial in $f(k, s, m, p)$. This involves counting all 4-tuples (K_1, J_1, K_2, J_2) of subsets of N which satisfy the inclusions $J_1 \subseteq K_1$ and $J_2 \subseteq K_2$, have the appropriate cardinalities, and give rise to the sets S_{01}, S_{02}, S_{11}, S_{12} and S_{22}, as specified in (6.8). Let γ be any integer with $c_l \leq \gamma \leq c_h$, and let

$$\beta = k - s - s_{02} - \gamma \qquad \text{and} \qquad \alpha = \gamma - (s_{12} + s_{22} - s).$$

Now $\gamma \leq c_h \implies \gamma \leq k - s - s_{02} = (\beta + \gamma) \implies 0 \leq \beta$, and $\gamma \geq c_l \implies \gamma \geq k - s - s_{02} - s_{01} = (\beta + \gamma) - s_{01} \implies \beta \leq s_{01}$, so $0 \leq \beta \leq s_{01}$. Similarly, $\gamma \leq c_h \implies \gamma \leq s_{12} + s_{22} - s + s_{11} = (\gamma - \alpha) + s_{11} \implies \alpha \leq s_{11}$ and $\gamma \geq c_l \implies \gamma \geq s_{12} + s_{22} - s = \gamma - \alpha \implies \alpha \geq 0$, so $0 \leq \alpha \leq s_{11}$.

If A, B and C are, respectively, subsets of S_{11}, S_{01} and S_{12}, of respective cardinalities α, β and γ, then we may define

$$\begin{aligned} J_1 &= S_{22} \cup A \cup (S_{12} \setminus C) \\ J_2 &= S_{22} \cup (S_{11} \setminus A) \cup C \\ K_1 &= S_{02} \cup S_{12} \cup S_{22} \cup A \cup B \\ K_2 &= S_{02} \cup S_{12} \cup S_{22} \cup (S_{11} \setminus A) \cup (S_{01} \setminus B) \end{aligned}$$

It is clear that $J_1 \subseteq K_1$ and $J_2 \subseteq K_2$. Since each of these sets is a disjoint union, we may compute the cardinalities:

$$|J_1| = s_{22} + \alpha + (s_{12} - \gamma) = s_{22} + [\gamma - (s_{12} + s_{22} - s)] + s_{12} - \gamma = s,$$

$$|K_1| = s_{02} + s_{12} + s_{22} + \alpha + \beta = s_{02} + s_{12} + s_{22} + [\gamma - (s_{12} + s_{22} - s)] + (k - s - s_{02} - \gamma) = k.$$

Using (6.7) we get

$$|J_2| = s_{22} + (s_{11} - \alpha) + \gamma = s_{22} + s_{11} - [\gamma - (s_{12} + s_{22} - s)] + \gamma = s_{11} + s_{12} + 2s_{22} - s = p.$$

Using (6.6) we get

$$\begin{aligned} |K_2| &= s_{02} + s_{12} + s_{22} + (s_{11} - \alpha) + (s_{01} - \beta) \\ &= s_{02} + s_{12} + s_{22} + s_{11} - [\gamma - (s_{12} + s_{22} - s)] + s_{01} - (k - s - s_{02} - \gamma) \\ &= [2(s_{02} + s_{12} + s_{22}) + s_{01} + s_{11}] - k = m. \end{aligned}$$

It is a routine verification to show that the equations (6.8) hold for our sets J_1, K_1, J_2 and K_2, which means that this choice of sets does indeed give rise to our given monomial. Observe that $|S_{11} \cap J_1| = |A| = \alpha$, $|S_{01} \cap K_1| = |B| = \beta$ and $|S_{12} \cap J_2| = |C| = \gamma$. Since $c_l \leq \gamma \leq c_h$, and the definitions of α and β give the equations $\gamma = s_{12}+s_{22}-s+\alpha$ and $\gamma = k-s-s_{02}-\beta$, which are analogues to (6.9) and (6.10), it follows that every choice of sets (K_1, J_1, K_2, J_2) is realized by some choice of A, B and C. The number of ways to choose A is $\binom{s_{11}}{\alpha} = \binom{s_{11}}{s-s_{12}-s_{22}+\gamma}$, the number of ways to choose B is $\binom{s_{01}}{\beta} = \binom{s_{01}}{k-s-s_{02}-\gamma}$. Thus the coefficient of the given monomial in $f(k,s,m,p)$ is

$$\sum_{\gamma=c_l}^{c_h} \binom{s_{11}}{s-s_{12}-s_{22}+\gamma}\binom{s_{01}}{k-s-s_{02}-\gamma}\binom{s_{12}}{\gamma}.$$

We now attempt to simplify this summation. For this purpose, we let $\eta = \frac{s_{12}}{2}+\frac{p-s}{2}-\gamma$. Taking (6.7), namely $s_{11}+s_{12}+2s_{22} = s+p$, and adding $(s_{12}-s_{11}-2s)$ to both sides we obtain $2s_{12}+2s_{22}-2s = s_{12}+p-s-s_{11}$. Dividing by 2 we obtain

$$s_{12}+s_{22}-s = \frac{s_{12}}{2}+\frac{p-s}{2}-\frac{s_{11}}{2} = \eta+\gamma-\frac{s_{11}}{2}. \tag{6.11}$$

Subtracting (6.7) from (6.6) yields $2s_{02}+s_{12}+s_{01} = k+m-s-p$. Adding $(k-m+p-s-2s_{02})$ to both sides yields $k-m+p-s+s_{12}+s_{01} = 2k-2s-2s_{02}$. Dividing by 2 we obtain

$$k-s-s_{02} = \frac{k-m}{2}+\frac{p-s}{2}+\frac{s_{12}}{2}+\frac{s_{01}}{2} = \eta+\gamma+\frac{s_{01}}{2}+\frac{k-m}{2} \tag{6.12}$$

Using (6.11) and (6.12), together with the fact that $0 = \eta+\gamma-\frac{s_{12}}{2}-\frac{p-s}{2}$, we find that

$$\begin{aligned}
c_l &= \max(0,\ s_{12}+s_{22}-s,\ k-s-s_{02}-s_{01})\\
&= \max\left(\eta+\gamma-\frac{s_{12}}{2}-\frac{p-s}{2},\ \eta+\gamma-\frac{s_{11}}{2},\ \left[\eta+\gamma+\frac{s_{01}}{2}+\frac{k-m}{2}\right]-s_{01}\right)\\
&= \eta+\gamma+\max\left(-\frac{s_{12}}{2}-\frac{p-s}{2},\ -\frac{s_{11}}{2},\ -\frac{s_{01}}{2}+\frac{k-m}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
c_h &= \min(s_{12},\ s_{12}+s_{22}-s+s_{11},\ k-s-s_{02})\\
&= \min\left(s_{12}+\left[\eta+\gamma-\frac{s_{12}}{2}-\frac{p-s}{2}\right],\ \left[\eta+\gamma-\frac{s_{11}}{2}\right]+s_{11},\ \eta+\gamma+\frac{s_{01}}{2}+\frac{k-m}{2}\right)\\
&= \eta+\gamma+\min\left(\frac{s_{12}}{2}-\frac{p-s}{2},\ \frac{s_{11}}{2},\ \frac{s_{01}}{2}+\frac{k-m}{2}\right)
\end{aligned}$$

Since $c_l \le \gamma$, it follows that

$$\begin{aligned}
\eta \le \eta+\gamma-c_l &= -\max\left(-\frac{s_{12}}{2}-\frac{p-s}{2},\ -\frac{s_{11}}{2},\ -\frac{s_{01}}{2}+\frac{k-m}{2}\right)\\
&= \min\left(\frac{s_{12}}{2}+\frac{p-s}{2},\ \frac{s_{11}}{2},\ \frac{s_{01}}{2}-\frac{k-m}{2}\right).
\end{aligned}$$

Likewise, since $c_h \ge \gamma$, it follows that

$$\begin{aligned}
\eta \ge \eta+\gamma-c_h &= -\min\left(\frac{s_{12}}{2}-\frac{p-s}{2},\ \frac{s_{11}}{2},\ \frac{s_{01}}{2}+\frac{k-m}{2}\right)\\
&= \max\left(-\frac{s_{12}}{2}+\frac{p-s}{2},\ -\frac{s_{11}}{2},\ -\frac{s_{01}}{2}-\frac{k-m}{2}\right).
\end{aligned}$$

Letting $L = \max\left(-\frac{s_{12}}{2}+\frac{p-s}{2},\ -\frac{s_{11}}{2},\ -\frac{s_{01}}{2}+\frac{m-k}{2}\right)$ and $U = \min\left(\frac{s_{12}}{2}+\frac{p-s}{2},\ \frac{s_{11}}{2},\ \frac{s_{01}}{2}+\frac{m-k}{2}\right)$, and setting $J = \{L, L+1, L+2, \ldots, U\}$, the coefficient of the given monomial in $f(k,s,m,p)$

can be rewritten as

$$\sum_{\eta\in J}\binom{s_{11}}{\frac{s_{11}}{2}-\eta}\binom{s_{01}}{\frac{s_{01}}{2}+\eta+\frac{k-m}{2}}\binom{s_{12}}{\frac{s_{12}}{2}-\eta+\frac{p-s}{2}}$$
$$=\sum_{\eta\in J}\binom{s_{11}}{\frac{s_{11}}{2}+\eta}\binom{s_{01}}{\frac{s_{01}}{2}+\eta-\frac{m-k}{2}}\binom{s_{12}}{\frac{s_{12}}{2}+\eta-\frac{p-s}{2}}$$

∎

Lemma 6.2: Let k, m and n be nonnegative integers of the same parity. For $t \in \mathbf{Z}$, set

$$L_t = \max\left(-\frac{k}{2},\ -\frac{m}{2},\ -\frac{n}{2}+t\right) \quad \text{and} \quad U_t = \min\left(\frac{k}{2},\ \frac{m}{2},\ \frac{n}{2}+t\right).$$

Letting $J_t = \{L_t, L_t+1, L_t+2, \ldots U_t\}$, we define

$$w_t = \sum_{j\in J_t}\binom{k}{\frac{k}{2}+j}\binom{m}{\frac{m}{2}+j}\binom{n}{\frac{n}{2}+j-t}.$$

The sequence $\{w_t\}_{t\in\mathbf{Z}}$ is a strongly unimodal sequence symmetric about w_0. In particular $w_0 \geq w_t$ for all $t \in \mathbf{Z}$.

Proof: We first consider the case where all of k, m and n are even. In this case we define, for $j \in \mathbf{Z}$,

$$a_j = \begin{cases} \binom{k}{\frac{k}{2}+j}\binom{m}{\frac{m}{2}+j} & \text{if } |j| \leq \min\left(\frac{k}{2}, \frac{m}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_j = \begin{cases} \binom{n}{\frac{n}{2}-j} & \text{if } |j| \leq \frac{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\{b_j\}_{j\in\mathbf{Z}}$ is a strongly unimodal sequence. The sequence $\{a_j\}_{j\in\mathbf{Z}}$ is also strongly unimodal, since for $|j| \leq \min\left(\frac{k}{2}, \frac{m}{2}\right) - 1$ we have

$$\frac{a_j^2}{a_{j-1}a_{j+1}} = \frac{\binom{k}{\frac{k}{2}+j}^2\binom{m}{\frac{m}{2}+j}^2}{\binom{k}{\frac{k}{2}+j-1}\binom{m}{\frac{m}{2}+j-1}\binom{k}{\frac{k}{2}+j+1}\binom{m}{\frac{m}{2}+j+1}}$$
$$= \frac{\binom{k}{\frac{k}{2}+j}^2}{\binom{k}{\frac{k}{2}+j-1}\binom{k}{\frac{k}{2}+j+1}} \cdot \frac{\binom{m}{\frac{m}{2}+j}^2}{\binom{m}{\frac{m}{2}+j-1}\binom{m}{\frac{m}{2}+j+1}} \geq 1 \cdot 1 = 1$$

We have symmetry, since for $j \in \mathbf{Z}$, $a_j = a_{-j}$ and $b_j = b_{-j}$. Let $\{c_j\}_{j\in\mathbf{Z}}$ be the convolution product of the sequences $\{a_j\}_{j\in\mathbf{Z}}$ and $\{b_j\}_{j\in\mathbf{Z}}$. This sequence is also symmetric, as

$$c_j = \sum_{i\in\mathbf{Z}} a_i b_{j-i} = \sum_{i\in\mathbf{Z}} a_{-i} b_{-j+i} = c_{-j}, \qquad j \in \mathbf{Z}.$$

Since it is also strongly unimodal, its peak must be c_0. Note that for $t \in \mathbf{Z}$, we have

$$\begin{aligned}
c_t &= \sum_{j\in\mathbf{Z}} a_j b_{t-j} = \sum_{\substack{|j|\le \min\left(\frac{k}{2},\frac{m}{2}\right)\\ |j-t|\le\frac{n}{2}}} a_j b_{t-j}\\
&= \sum_{\max\left(-\frac{k}{2},-\frac{m}{2},-\frac{n}{2}+t\right)\le j\le \min\left(\frac{k}{2},\frac{m}{2},\frac{n}{2}+t\right)} \binom{k}{\frac{k}{2}+j}\binom{m}{\frac{m}{2}+j}\binom{n}{\frac{n}{2}-(t-j)}\\
&= \sum_{j\in J_t} \binom{k}{\frac{k}{2}+j}\binom{m}{\frac{m}{2}+j}\binom{n}{\frac{n}{2}+j-t}\\
&= w_t
\end{aligned}$$

Thus $\{w_t\}_{t\in\mathbf{Z}}$ is a strongly unimodal sequence symmetric about w_0. It remains to show that this is also true when k, m and n are odd.

In the case that k, m and n are odd, we define, for $j \in \mathbf{Z}$,

$$a_j = \begin{cases} \binom{k}{\frac{k}{2}+j+\frac{1}{2}}\binom{m}{\frac{m}{2}+j+\frac{1}{2}} & \text{if } \left|j+\frac{1}{2}\right| \le \min\left(\frac{k}{2},\frac{m}{2}\right)\\ 0 & \text{otherwise}\end{cases}$$

and

$$b_j = \begin{cases} \binom{n}{\frac{n}{2}-j-\frac{1}{2}} & \text{if } \left|j+\frac{1}{2}\right| \le \frac{n}{2}\\ 0 & \text{otherwise}\end{cases}$$

It is clear that $\{b_j\}_{j\in\mathbf{Z}}$ is a strongly unimodal sequence. The sequence $\{a_j\}_{j\in\mathbf{Z}}$ is also strongly unimodal, since for $|j+\frac{1}{2}| \le \min\left(\frac{k}{2},\frac{m}{2}\right) - 1$ we have

$$\begin{aligned}
\frac{a_k^2}{a_{k-1}a_{k+1}} &= \frac{\binom{k}{\frac{k}{2}+j+\frac{1}{2}}^2\binom{m}{\frac{m}{2}+j+\frac{1}{2}}^2}{\binom{k}{\frac{k}{2}+j-\frac{1}{2}}\binom{m}{\frac{m}{2}+j-\frac{1}{2}}\binom{k}{\frac{k}{2}+j+\frac{3}{2}}\binom{m}{\frac{m}{2}+j+\frac{3}{2}}}\\
&= \frac{\binom{k}{\frac{k}{2}+j+\frac{1}{2}}^2}{\binom{k}{\frac{k}{2}+j-\frac{1}{2}}\binom{k}{\frac{k}{2}+j+\frac{3}{2}}}\cdot\frac{\binom{m}{\frac{m}{2}+j+\frac{1}{2}}^2}{\binom{m}{\frac{m}{2}+j-\frac{1}{2}}\binom{m}{\frac{m}{2}+j+\frac{3}{2}}} \ge 1\cdot 1 = 1
\end{aligned}$$

The sequence $\{b_j\}_{j\in\mathbf{Z}}$ is symmetric (about $-\frac{1}{2}$), since if $|j+\frac{1}{2}| \le \frac{n}{2}$, then $\left|(-j-1)+\frac{1}{2}\right| = \left|-j-\frac{1}{2}\right| = \left|j+\frac{1}{2}\right| \le \frac{n}{2}$, and

$$b_{-j-1} = \binom{n}{\frac{n}{2}-(-j-1)-\frac{1}{2}} = \binom{n}{\frac{n}{2}+j+\frac{1}{2}} = \binom{n}{\frac{n}{2}-j-\frac{1}{2}} = b_j$$

Similarly, if $\left|j+\frac{1}{2}\right| \le \min\left(\frac{k}{2},\frac{m}{2}\right)$, then $\left|(-j-1)+\frac{1}{2}\right| = \left|-j-\frac{1}{2}\right| = \left|j+\frac{1}{2}\right| \le \min\left(\frac{k}{2},\frac{m}{2}\right)$ and

$$\begin{aligned}
a_{-j-1} &= \binom{k}{\frac{k}{2}+(-j-1)+\frac{1}{2}}\binom{m}{\frac{m}{2}+(-j-1)+\frac{1}{2}}\\
&= \binom{k}{\frac{k}{2}-j-\frac{1}{2}}\binom{m}{\frac{m}{2}-j-\frac{1}{2}} = \binom{k}{\frac{k}{2}+j+\frac{1}{2}}\binom{m}{\frac{m}{2}+j+\frac{1}{2}}\\
&= a_j
\end{aligned}$$

Let $\{c_j\}_{j\in\mathbf{Z}}$ be the convolution product of the sequences $\{a_j\}_{j\in\mathbf{Z}}$ and $\{b_j\}_{j\in\mathbf{Z}}$. This sequence is symmetric about -1, since

$$c_j = \sum_{i\in\mathbf{Z}} a_i b_{j-i} = \sum_{i\in\mathbf{Z}} a_{-i-1}b_{-(j-i)-1} = \sum_{i\in\mathbf{Z}} a_{-(i+1)}b_{(-j-2)+(i+1)} = c_{-j-2}.$$

Since it is also strongly unimodal, its peak must be c_{-1}, so that $c_{-1} \geq c_0 \geq c_1 \geq c_2 \geq \cdots$. Note that for $t \in \mathbf{Z}$, if we let $\beta = j + \frac{1}{2}$, we have

$$\begin{aligned}
c_{t-1} &= \sum_{j \in \mathbf{Z}} a_{-j} b_{(t-1)-j} = \sum_{\substack{|j+\frac{1}{2}| \leq \min\left(\frac{k}{2}, \frac{m}{2}\right) \\ |((t-1)-j)+\frac{1}{2}| \leq \frac{n}{2}}} a_j b_{(t-1)-j} \\
&= \sum_{\substack{|j+\frac{1}{2}| \leq \min\left(\frac{k}{2}, \frac{m}{2}\right) \\ |t-j-\frac{1}{2}| \leq \frac{n}{2}}} \binom{k}{\frac{k}{2}+j+\frac{1}{2}} \binom{m}{\frac{m}{2}+j+\frac{1}{2}} \binom{n}{\frac{n}{2}-(t-1-j)-\frac{1}{2}} \\
&= \sum_{\substack{|\beta| \leq \min\left(\frac{k}{2}, \frac{m}{2}\right) \\ |t-\beta| \leq \frac{n}{2}}} \binom{k}{\frac{k}{2}+\beta} \binom{m}{\frac{m}{2}+\beta} \binom{n}{\frac{n}{2}+\beta-t} \\
&= \sum_{\beta \in J_t} \binom{k}{\frac{k}{2}+\beta} \binom{m}{\frac{m}{2}+\beta} \binom{n}{\frac{n}{2}+\beta-t} \\
&= w_t
\end{aligned}$$

Thus $\{w_t\}_{t \in \mathbf{Z}}$ is a strongly unimodal sequence symmetric about w_0. ∎

Corollary 6.3 and Corollary 6.4 assert the strong unimodality of the sequences $\{\rho_{k,s}\}_{k \in \mathbf{Z}}$ and $\{\rho_{k,s}\}_{s \in \mathbf{Z}}$, as is clear from taking $t = 1$.

Corollary 6.3: If $k, s \in \mathbf{Z}$, and $\rho_{k,s} = \rho_{k,s}(n)$ is as defined as in (6.3), then

$$\rho_{k,s}^2 - \rho_{k-t,s}\rho_{k+t,s} \geq 0 \tag{6.13}$$

for all $t \in \mathbf{N}$. Moreover $\rho_{k,s}^2 - \rho_{k-t,s}\rho_{k+t,s}$, regarded as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $\tau_1, \tau_2, \ldots, \tau_n$, has no negative coefficients.

Proof: If $\rho_{k,s} = 0$, then at least one of the two terms $\rho_{k-t,s}$ and $\rho_{k+t,s}$ is also zero, so (6.13) holds. It remains to consider the case where all three quantities are nonzero. That is, $0 \leq s \leq k-t$, and $k+t \leq n$. Using the notation defined in (6.5), note that $\rho_{k,s}^2 - \rho_{k-t,s}\rho_{k+t,s} = f(k,s,k,s) - f(k-t,s,k+t,s)$. It suffices to show that this difference, regarded as a polynomial in $\tau_1, \tau_2, \ldots, \tau_n$ and $\sigma_1, \sigma_2, \ldots, \sigma_n$, has no negative coefficients. To do this, it is sufficient to show that for any given monomial in $f(k-t,s,k+t,s)$, its coefficient in $f(k,s,k,s)$ is at least as large as its coefficient in $f(k-t,s,k+t,s)$. Let a monomial in $f(k-t,s,k+t,s)$ be chosen, and use the definition of $s_{\mu,\nu}$ for $0 \leq \mu \leq \nu \leq 2$ which was used in Lemma 6.1. For $i = 0, t$, let w_i be the coefficient of the given monomial in $f(k-i,s,k+i,s)$. By Lemma 6.1, we have that

$$w_i = \sum_{\eta \in J} \binom{s_{11}}{\frac{s_{11}}{2}+\eta} \binom{s_{01}}{\frac{s_{01}}{2}+\eta-i} \binom{s_{12}}{\frac{s_{12}}{2}+\eta},$$

where $J = \{L, L+1, L+2, \ldots, U\}$, and $L = \max\left(-\frac{s_{12}}{2}, -\frac{s_{11}}{2}, -\frac{s_{01}}{2}+i\right)$ and $U = \min\left(\frac{s_{12}}{2}, \frac{s_{11}}{2}, \frac{s_{01}}{2}+i\right)$. By Lemma 6.2, we have that $w_0 \geq w_t$, and our result follows. ∎

Corollary 6.4: If $k, s \in \mathbf{Z}$, and $\rho_{k,s} = \rho_{k,s}(n)$ is as defined as in (6.3), then

$$\rho_{k,s}^2 - \rho_{k,s-t}\rho_{k,s+t} \geq 0 \tag{6.14}$$

for all $t \in \mathbf{N}$. Moreover $\rho_{k,s}^2 - \rho_{k,s-t}\rho_{k,s+t}$, regarded as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $\tau_1, \tau_2, \ldots, \tau_n$, has no negative coefficients.

Proof: As in the proof of Corollary 6.3, it suffices to consider the case where the three quantities $\rho_{k,s}$, $\rho_{k,s-t}$ and $\rho_{k,s+t}$ are all nonzero, and to show that for any given monomial in $f(k, s-t, k, s+t)$, its coefficient in $f(k, s, k, s)$ is at least as large as its coefficient in $f(k, s-t, k, s+t)$. For $i = 0, t$, let w_i be the coefficient of the given monomial in $f(k, s-i, k, s+i)$. By Lemma 6.1, we have that

$$w_i = \sum_{\eta \in J} \binom{s_{11}}{\frac{s_{11}}{2} + \eta} \binom{s_{01}}{\frac{s_{01}}{2} + \eta} \binom{s_{12}}{\frac{s_{12}}{2} + \eta - i},$$

where $J = \{L, L+1, L+2, \ldots, U\}$, and $L = \max\left(-\frac{s_{12}}{2} + i,\ -\frac{s_{11}}{2},\ -\frac{s_{01}}{2}\right)$ and $U = \min\left(\frac{s_{12}}{2} + i,\ \frac{s_{11}}{2},\ \frac{s_{01}}{2}\right)$. By Lemma 6.2, we have that $w_0 \geq w_t$, and our result follows. ∎

We now prove strong unimodality in the diagonal direction: each sequence $\{\rho_{k+m,s+m}\}_{m \in \mathbf{Z}}$ is strongly unimodal.

Corollary 6.5: If $k, s \in \mathbf{Z}$, and $\rho_{k,s} = \rho_{k,s}(n)$ is as defined as in (6.3), then

$$\rho_{k,s}^2 - \rho_{k-t,s-t}\rho_{k+t,s+t} \geq 0 \tag{6.15}$$

for all $t \in \mathbf{N}$. Moreover $\rho_{k,s}^2 - \rho_{k-t,s-t}\rho_{k+t,s+t}$, regarded as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $\tau_1, \tau_2, \ldots, \tau_n$, has no negative coefficients.

Proof: As in the proof of Corollary 6.3, it suffices to consider the case where the three quantities $\rho_{k,s}$, $\rho_{k-t,s-t}$ and $\rho_{k+t,s+t}$ are all nonzero, and to show that for any given monomial in $f(k-t, s-t, k+t, s+t)$, its coefficient in $f(k, s, k, s)$ is at least as large as its coefficient in $f(k-t, s-t, k+t, s+t)$. For $i = 0, t$, let w_i be the coefficient of the given monomial in $f(k-i, s-i, k+i, s+i)$. By Lemma 6.1, we have that

$$w_i = \sum_{\eta \in J} \binom{s_{11}}{\frac{s_{11}}{2} + \eta} \binom{s_{01}}{\frac{s_{01}}{2} + \eta - i} \binom{s_{12}}{\frac{s_{12}}{2} + \eta - i},$$

where $J = \{L, L+1, L+2, \ldots, U\}$, and $L = \max\left(-\frac{s_{12}}{2} + i,\ -\frac{s_{11}}{2},\ -\frac{s_{01}}{2} + i\right)$ and $U = \min\left(\frac{s_{12}}{2} + i,\ \frac{s_{11}}{2},\ \frac{s_{01}}{2} + i\right)$. We let $\lambda = i - \eta$, and note that

$$\lambda = i - \eta \leq i - L = i - \max\left(-\frac{s_{12}}{2} + i,\ -\frac{s_{11}}{2},\ -\frac{s_{01}}{2} + i\right) = \min\left(\frac{s_{12}}{2},\ \frac{s_{11}}{2} + i,\ \frac{s_{01}}{2}\right)$$

and

$$\lambda = i - \eta \geq i - U = i - \min\left(\frac{s_{12}}{2} + i,\ \frac{s_{11}}{2},\ \frac{s_{01}}{2} + i\right) = \max\left(-\frac{s_{12}}{2},\ -\frac{s_{11}}{2} + i,\ -\frac{s_{01}}{2}\right)$$

Letting $I = \{(i-U), (i-U)+1, (i-U)+2, \ldots, (i-L)\}$, we may write

$$w_i = \sum_{\lambda \in I} \binom{s_{11}}{\frac{s_{11}}{2} + i - \lambda} \binom{s_{01}}{\frac{s_{01}}{2} + (i-\lambda) - i} \binom{s_{12}}{\frac{s_{12}}{2} + (i-\lambda) - i}$$

$$= \sum_{\lambda \in I} \binom{s_{11}}{\frac{s_{11}}{2} + i - \lambda} \binom{s_{01}}{\frac{s_{01}}{2} - \lambda} \binom{s_{12}}{\frac{s_{12}}{2} - \lambda}$$

$$= \sum_{\lambda \in I} \binom{s_{11}}{\frac{s_{11}}{2} + \lambda - i} \binom{s_{01}}{\frac{s_{01}}{2} + \lambda} \binom{s_{12}}{\frac{s_{12}}{2} + \lambda}$$

By Lemma 6.1, it follows that $w_0 \geq w_t$, which proves our result. ∎

In the case $t = 1$, Corollary 6.3, Corollary 6.4 and Corollary 6.5 assert the strong unimodality of the distribution $\{\rho_{k,s}(n)\}_{k,s \in \mathbf{Z}}$ along the two cross-sections and in the diagonal direction. Inequalities (6.13) and (6.14) can be rewritten as

$$\frac{\rho_{k,s}}{\rho_{k-1,s}} \geq \frac{\rho_{k+1,s}}{\rho_{k,s}} \tag{6.16}$$

and

$$\frac{\rho_{k,s}}{\rho_{k,s-1}} \geq \frac{\rho_{k,s+1}}{\rho_{k,s}}$$

wherever each pair of ratios is defined. If we write $q_{k,s} = \frac{\rho_{k,s}}{\rho_{k-1,s}}$, whenever this quotient is defined, then (6.16) can be rewritten as $q_{k,s} \geq q_{k+1,s}$. The following proposition says, essentially, that $q_{k,s} \geq q_{k+1,s+1}$; that is,

$$\frac{\rho_{k,s}}{\rho_{k-1,s}} \geq \frac{\rho_{k+1,s+1}}{\rho_{k,s+1}}.$$

Proposition 6.6: With $\rho_{k,s} = \rho_{k,s}(n)$ defined as in (6.3),

$$\rho_{k,s}\rho_{k,s+1} - \rho_{k-1,s}\rho_{k+1,s+1} \geq 0, \qquad \forall k, s \in \mathbf{Z}. \tag{6.17}$$

Before proving Proposition 6.6, we need to introduce the following numerical result:

Lemma 6.7: If numbers $a, b, c, d, a', b', c', d' \in [0, \infty)$ satisfy

$$ac \leq a'c' \tag{6.18}$$

$$bc \leq b'c' \tag{6.19}$$

$$bd \leq b'd' \tag{6.20}$$

and

$$ad \leq b'c', \tag{6.21}$$

then

$$(a+b)(c+d) \leq (a'+b')(c'+d'). \tag{6.22}$$

proof of Lemma 6.7: This proof was provided by Valerio De Angelis, and is much simpler than my original proof.

Let $m = \min(ad, bc)$ and $M = \max(ad, bc)$, so that $m + M = ad + bc$ and $mM = adbc$. If $M = 0$ then $ad = 0 = bc$ and the result follows from (6.18) and (6.20). Assume that $M > 0$.

From (6.19) and (6.21) we find that $M \le b'c'$, and so $t = \frac{b'c'}{M} \ge 1$. Let $\gamma = \frac{m}{M} = \frac{mM}{M^2} = \frac{adbc}{M^2}$. As $m \le M$ we have $\gamma \le 1 \le t$. Thus

$$0 \le (t-\gamma)(t-1) = t^2 - (1+\gamma)t + \gamma = \frac{(b'c')^2}{M^2} - \left(1 + \frac{m}{M}\right)\frac{b'c'}{M} + \frac{adbc}{M^2}.$$

Multiplying through by M^2 we obtain

$$0 \le (b'c')^2 - (M+m)\,b'c' + adbc = (b'c')^2 - (ad+bc)\,b'c' + (ac)(bd).$$

Using (6.18) and (6.20) we find that $(ac)(bd) \le a'c'b'd' = b'c'a'd'$. We thus obtain

$$0 \le (b'c')^2 - (ad+bc)\,b'c' + b'c'a'd' = b'c'[b'c' + a'd' - ad - bc].$$

As $0 < M \le b'c'$ we may divide through by $b'c'$ to get the inequality $ad + bc \le a'd' + b'c'$. Combining this with (6.18) and (6.20) we obtain (6.22). ∎

proof of Proposition 6.6:

We use induction on n. The result is clear for $n = 1$, since $\rho_{k-1,s}\rho_{k+1,s+1}$ is always zero in this case. We assume the result is true for n, and show that it is true for $n+1$. Writing $\rho'_{k,s}$ for $\rho_{k,s}(n+1)$, we must show that

$$\rho'_{k,s}\rho'_{k,s+1} - \rho'_{k-1,s}\rho'_{k+1,s+1} \ge 0, \qquad \forall k, s \in \mathbf{Z} \tag{6.23}$$

using (6.17). For convenience we write σ and τ in place of σ_{n+1} and τ_{n+1}. Using (6.4), the left side of (6.23) can be expanded as

$$\begin{aligned}
&\rho'_{k,s}\rho'_{k,s+1} - \rho'_{k-1,s}\rho'_{k+1,s+1} \\
&= (\rho_{k,s} + \tau[\rho_{k-1,s} + \sigma\rho_{k-1,s-1}])(\rho_{k,s+1} + \tau[\rho_{k-1,s+1} + \sigma\rho_{k-1,s}]) \\
&\qquad - (\rho_{k-1,s} + \tau[\rho_{k-2,s} + \sigma\rho_{k-2,s-1}])(\rho_{k+1,s+1} + \tau[\rho_{k,s+1} + \sigma\rho_{k,s}]) \\
&= \rho_{k,s}\rho_{k,s+1} - \rho_{k-1,s}\rho_{k+1,s+1} \\
&\qquad\quad + \rho_{k,s}\tau[\rho_{k-1,s+1} + \sigma\rho_{k-1,s}] + \rho_{k,s+1}\tau[\rho_{k-1,s} + \sigma\rho_{k-1,s-1}] \\
&\qquad\quad - \rho_{k-1,s}\tau[\rho_{k,s+1} + \sigma\rho_{k,s}] - \rho_{k+1,s+1}\tau[\rho_{k-2,s} + \sigma\rho_{k-2,s-1}] \\
&\qquad + \tau^2[\rho_{k-1,s} + \sigma\rho_{k-1,s-1}][\rho_{k-1,s+1} + \sigma\rho_{k-1,s}] \\
&\qquad - \tau^2[\rho_{k-2,s} + \sigma\rho_{k-2,s-1}][\rho_{k,s+1} + \sigma\rho_{k,s}] \\
&= [\rho_{k,s}\rho_{k,s+1} - \rho_{k-1,s}\rho_{k+1,s+1}] \\
&\qquad\quad + \tau[\rho_{k,s}\rho_{k-1,s+1} + \rho_{k,s+1}\rho_{k-1,s} - \rho_{k-1,s}\rho_{k,s+1} - \rho_{k+1,s+1}\rho_{k-2,s}] \\
&\qquad\quad + \tau\sigma[\rho_{k,s}\rho_{k-1,s} + \rho_{k,s+1}\rho_{k-1,s-1} - \rho_{k-1,s}\rho_{k,s} - \rho_{k+1,s+1}\rho_{k-2,s-1}] \\
&\qquad + \tau^2\big[(\rho_{k-1,s+1} + \sigma\rho_{k-1,s})(\rho_{k-1,s} + \sigma\rho_{k-1,s-1}) \\
&\qquad\qquad - (\rho_{k,s+1} + \sigma\rho_{k,s})(\rho_{k-2,s} + \sigma\rho_{k-2,s-1})\big]
\end{aligned}$$

To establish (6.23), it suffices to show that the four last square-bracketed expressions in the preceding display are nonnegative.

Our inductive assumption guarantees that $\rho_{k,s}\rho_{k,s+1} - \rho_{k-1,s}\rho_{k+1,s+1} \ge 0$.

The second of our bracketed expressions, that multiplied by τ, simplifies to $\rho_{k,s}\rho_{k-1,s+1} - \rho_{k+1,s+1}\rho_{k-2,s}$. To show that this is nonnegative, it suffices to consider the case where $\rho_{k+1,s+1}$ and $\rho_{k-2,s}$ are nonzero. If $\rho_{k+1,s+1} \ne 0$, then $k+1 \le n$, while if $\rho_{k-2,s}$ is nonzero, then $0 \le s \le k-2$. These facts guarantee that the three coefficients $\rho_{k-1,s}$, $\rho_{k,s+1}$ and $\rho_{k-1,s+1}$ are all nonzero. Now

$$\begin{aligned}
\rho_{k,s}\rho_{k-1,s+1} - \rho_{k+1,s+1}\rho_{k-2,s} &= \rho_{k-1,s+1}\rho_{k-2,s}\left(\frac{\rho_{k,s}}{\rho_{k-1,s}} \cdot \frac{\rho_{k-1,s}}{\rho_{k-2,s}} - \frac{\rho_{k+1,s+1}}{\rho_{k,s+1}} \cdot \frac{\rho_{k,s+1}}{\rho_{k-1,s+1}}\right) \\
&= \rho_{k-1,s+1}\rho_{k-2,s}(q_{k,s}\, q_{k-1,s} - q_{k+1,s+1}\, q_{k,s+1})
\end{aligned}$$

This is nonnegative, since by our inductive assumption, $q_{k,s} \geq q_{k+1,s+1}$ and $q_{k-1,s} \geq q_{k,s+1}$.

The third of our bracketed expressions, that multiplied by $\tau\sigma$, simplifies to $\rho_{k,s+1}\rho_{k-1,s-1} - \rho_{k+1,s+1}\rho_{k-2,s-1}$. To show that this is nonnegative, it suffices to consider the case where both $\rho_{k+1,s+1}$ and $\rho_{k-2,s-1}$ are nonzero. If $\rho_{k+1,s+1} \neq 0$, then $k+1 \leq n$, while if $\rho_{k-2,s-1} \neq 0$, then $0 \leq s-1 \leq k-2$. These facts guarantee that $\rho_{k,s+1}$ is nonzero. Now

$$\begin{aligned} \rho_{k,s+1}\rho_{k-1,s-1} - \rho_{k+1,s+1}\rho_{k-2,s-1} &= \rho_{k,s+1}\rho_{k-2,s-1}\left(\frac{\rho_{k-1,s-1}}{\rho_{k-2,s-1}} - \frac{\rho_{k+1,s+1}}{\rho_{k,s+1}}\right) \\ &= \rho_{k,s+1}\rho_{k-2,s-1}(q_{k-1,s-1} - q_{k+1,s+1}) \end{aligned}$$

This is nonnegative, since by our inductive assumption, $q_{k-1,s-1} \geq q_{k,s} \geq q_{k+1,s+1}$.

It remains to show that our fourth bracketed expression, that multiplied by τ^2, is nonnegative. That is, we must show that

$$(\rho_{k,s+1} + \sigma\rho_{k,s})(\rho_{k-2,s} + \sigma\rho_{k-2,s-1}) \leq (\rho_{k-1,s+1} + \sigma\rho_{k-1,s})(\rho_{k-1,s} + \sigma\rho_{k-1,s-1}). \quad (6.24)$$

To do this, we first establish four simpler inequalities, and then invoke Lemma 6.7 to prove (6.24). The inequality

$$\rho_{k,s+1}\rho_{k-2,s} \leq \rho_{k-1,s+1}\rho_{k-1,s} \quad (6.25)$$

is equivalent to the inequality $\rho_{k-1,s}\rho_{k-1,s+1} - \rho_{k-2,s}\rho_{k,s+1} \geq 0$, which holds by our inductive assumption. The inequality

$$(\sigma\rho_{k,s})\rho_{k-2,s} \leq (\sigma\rho_{k-1,s})\rho_{k-1,s} \quad (6.26)$$

follows from (6.13) (with $t = 1$). The equation

$$(\sigma\rho_{k,s})(\sigma\rho_{k-2,s-1}) \leq (\sigma\rho_{k-1,s})(\sigma\rho_{k-1,s-1}) \quad (6.27)$$

is equivalent to the inequality $\rho_{k-1,s-1}\rho_{k-1,s} - \rho_{k-2,s-1}\rho_{k,s} \geq 0$, which holds by our inductive assumption. Finally, the inequality

$$\rho_{k,s+1}(\sigma\rho_{k-2,s-1}) \leq (\sigma\rho_{k-1,s})\rho_{k-1,s} \quad (6.28)$$

is equivalent to $\rho_{k-1,s}^2 - \rho_{k-2,s-1}\rho_{k,s+1} \geq 0$, which follows from (6.15) (with $t = 1$). Comparing inequality (6.24) with inequality (6.22), and inequalities (6.25)–(6.28) with inequalities (6.18)–(6.21), we observe that (6.24) is a consequence of Lemma 6.7. We have thus established (6.23), and Proposition 6.6 is proved. ∎

7. Eventual Strong Unimodality of Products

Odlyzko and Richmond [OR] have proved that if P is a polynomial of degree d in $\mathbf{R}[x]^+$, and $0, 1, d-1, d$ are all in $\operatorname{Log} P$, then P^n is strongly unimodal for sufficiently large n. In this chapter we generalize this result, showing that if $\{P_j\}$ is a sequence in $\mathbf{R}[x]^+$ satisfying appropriate conditions, then for all sufficiently large n the product $P^{n,0} = P_1 P_2 \cdots P_n$ is strongly unimodal. The conditions we require are that the degrees d_j of P_j have an upper bound, and that the coefficients at 0, 1, $d_j - 1$ and d_j be (uniformly) bounded below away from zero.

The proof in [OR] consists of two parts, each with conditions somewhat relaxed from those of the overall result. The first, simpler part shows strong unimodality at coefficients k near the ends of $\operatorname{Log} P^n$, while the more difficult part of the proof shows strong unimodality for k in the central region.

The results given in this chapter use the techniques of [OR], adapting and supplementing them as necessary to cover the case of differing polynomials P_j.

The proof of strong unimodality near the ends involves approximating each P_i by its constant and linear terms, and hence approximating $P^{n,0}$ by $Q^{n,0}$, the product of the linear polynomials. A product of linear polynomials is 'better' than strongly unimodal in the sense that if $\sum_{j=0}^{n} b_j x^j$ is such a polynomial, then

$$b_j^2 \geq b_{j-1} b_{j+1} \left(1 + \frac{1}{j}\right)\left(1 + \frac{1}{n-j}\right), \qquad \text{for } 0 < j < n,$$

which is stronger than the strong unimodality condition $b_j^2 \geq b_{j-1} b_{j+1}$ (see p. 504 in [S]). Calculuating the coefficients of $P^{n,0}$ by using a contour integral over a suitable small circle in the complex plane, we can make the coefficients of $P^{n,0}$ close enough to those of $Q^{n,0}$ that the 'extra' strong unimodality of $Q^{n,0}$ produces strong unimodality in $P^{n,0}$.

The proof of strong unimodality in the central region of the product is also approached by calculating the coefficients of $P^{n,0}$ using a contour integral in the complex plane. We use this integral to smoothly interpolate the coefficients of $P^{n,0}$, and eventually prove log concavity of this interpolating function (and hence strong positivity of the $P^{n,0}$). Writing the integrand of the complex integral as the exponential of a Taylor series of three terms, we find that by appropriate choice of the circle of integration, the linear term can be made to disappear. We also find that the contour integral can be well approximated by integrating over a very small portion of the circle, and that on this part of the circle the third order term in negligible. The integral of the remaining quadratic term can be calculated fairly precisely. The same techniques allow us to estimate the derivates of the coefficient-interpolating function accurately enough to deduce log concavity, and hence strong unimodality.

Before embarking upon our calculations, we remark that with extra care the methods given here yield estimates (and bounds) on the coefficients of $P^{n,0}$. The derivative estimates yield estimates of the ratios of successive coefficients of $P^{n,0}$.

These estimates have other uses. For example, Proposition 4 in [H4] shows that subject to appropriate conditions on f, the product $(1+x)^n \cdot f$ will be strongly unimodal for large n. Using the estimates developed here, this could be generalized to certain products of the form $P_1 P_2 \cdots P_n \cdot f$.

Valerio De Angelis has (independently) derived asymptotic estimates for the coefficients of large powers P^n of a polynomial P, using methods similar to those employed here. His results are valid in the central region of P^n (but not at the ends), and permit P to have negative coefficients, provided that $|P(re^{i\theta})| < P(r)$ for all $r > 0$ and $0 < \theta < 2\pi$. Strong unimodality of P^n in the central region is an immediate consequence of his asymptotic estimates. He also obtains Stirling's formula as an easy consequence.

We now procede with our results, deriving strong unimodality first for the 'ends' of the product (Theorem 7.4), and then for the centre (Theorem 7.14). No bound on the degree of the P_j is required for Theorem 7.4, and only the coefficients of x^0 and x^1 need be bounded away from 0. The hypotheses needed for the central region are somewhat stronger, requiring a bound on the degrees d_j, and a lower bound on coefficient of x^{d_j}. We finally combine both components to obtain the overall strong unimodality result, Theorem 7.15.

Strong Unimodality in the Ends of the Product. Suppose that for each j in $\mathbf{N}$, P_j is a polynomial in $\mathbf{R}[x]^+$, and for convenience write

$$P_j = \sum_{i\geq 0} P_{j,i}x^i.$$

Suppose also that

$$\sum_{i\geq 0} P_{j,i} = 1, \qquad \forall j \in \mathbf{N} \tag{7.1}$$

and that there is a constant $\delta > 0$ such that

$$P_{j,0} \geq \delta \qquad \text{and} \qquad P_{j,1} \geq \delta, \qquad \forall j \in \mathbf{N}. \tag{7.2}$$

We are interested in the coefficients of $P^{n,0} = P_1P_2\cdots P_n$ for large values of n. For each j in $\mathbf{N}$ we define $Q_j \in \mathbf{R}[x]^+$ by setting $Q_j = P_{j,0} + P_{j,1}x$. We will use the strong unimodality of $Q^{n,0}$ to deduce strong unimodality near the end of $P^{n,0}$.

Notational note: We use the notation $C(E)$ to denote any quantity x which satisfies $|x| \leq E$. This notation is anologous to $O(E)$, but gives greater precision.

Lemma 7.1: There is a constant $L_1 > 0$ such that if n is in $\mathbf{N}$ and $r \in \left(0, \frac{1}{2\sqrt{n}}\frac{\delta}{1-\delta}\right)$, then

$$P^{n,0}(re^{i\theta}) = \left[\prod_{i=1}^{n}\left(P_{j,0} + P_{j,1}re^{i\theta}\right)\right]\left(1 + nr^2C(L_1)\right), \qquad \forall\theta \in \mathbf{R}.$$

Proof: Let $z = re^{i\theta}$, and for each j in $\mathbf{N}$ let $A_j = P_{j,0} + P_{j,1}z$ and $B_j = P_j(z) - A_j$. As $1 \geq P_{j,0} + P_{j,1} \geq \delta + \delta$, it is clear that $\delta \leq \frac{1}{2}$, and so $\frac{\delta}{1-\delta} \leq 1$. Thus $r < \frac{1}{2}$, and so we have, for each j in $\mathbf{N}$,

$$|B_j| = \left|\sum_{m\geq 2} P_{j,m}z^m\right| \leq \sum_{m\geq 2} P_{j,m}r^m \leq \sum_{m\geq 2} r^m = r^2\sum_{m\geq 0} r^m \leq r^2\sum_{m\geq 0}\frac{1}{2^m} = 2r^2.$$

Note that as $P_{j,0} + P_{j,1} \leq 1$ and $P_{j,0} \geq \delta$, we have $P_{j,1} \leq 1 - P_{j,0} \leq 1 - \delta$. Thus $|P_{j,1}\, z| = P_{j,1}r \leq (1-\delta)\frac{1}{2}\frac{\delta}{1-\delta} = \frac{\delta}{2} \leq \frac{P_{j,0}}{2}$ so that

$$\frac{\delta}{2} \leq P_{j,0}/2 \leq |P_{j,0} + P_{j,1}\, z| \leq 3P_{j,0}/2, \quad \forall j \in \mathbf{N}.$$

In particular, $|A_j| \geq \frac{\delta}{2} > 0$.

For I any subset of $S = \{1, 2, \ldots, n\}$, we define $A_I = \prod_{i \in I} A_i$ and $B_I = \prod_{i \in I} B_i$. By convention we take the empty product to equal 1. We also denote $S \backslash I$ by I'.

$$\left|\prod_{j=1}^{n} P_j(z) - \prod_{j=1}^{n} A_j\right| = \left|\prod_{j=1}^{n}(A_j + B_j) - \prod_{j=1}^{n} A_j\right| = \left|\sum_{I \subseteq S} A_I B_{I'} - A_S\right| = \left|\sum_{\substack{I \subseteq S \\ I \neq S}} A_I B_{I'}\right|$$

$$= \left|A_S \sum_{\substack{I \subseteq S \\ I \neq S}} \frac{B_{I'}}{A_{I'}}\right| \leq |A_S| \sum_{\substack{I' \subseteq S \\ I' \neq \emptyset}} \left|\frac{B_{I'}}{A_{I'}}\right| \leq |A_S| \sum_{\substack{I' \subseteq S \\ I' \neq \emptyset}} \left(\frac{2r^2}{\delta/2}\right)^{|I'|}$$

$$= |A_S| \sum_{i=1}^{n} \binom{n}{i} \left(\frac{2r^2}{\delta/2}\right)^i = |A_S| \sum_{i=1}^{n} \frac{n!}{(n-i)!i!} \left(\frac{4r^2}{\delta}\right)^i$$

$$\leq |A_S| \sum_{i=1}^{n} \frac{n^i}{i!} \left(\frac{4r^2}{\delta}\right)^i = |A_S| \sum_{i=1}^{n} \frac{1}{i!}(r^2 n)^i \left(\frac{4}{\delta}\right)^i$$

As $r\sqrt{n} < \frac{1}{2}\frac{\delta}{1-\delta}$, we have $nr^2 < \delta^2(1-\delta)^{-2}/4$. Letting $\kappa = \max\{1, \delta^2(1-\delta)^{-2}/4\}$, we obtain

$$\left|\prod_{j=1}^{n} P_j(z) - \prod_{j=1}^{n} A_j\right| \leq |A_S| \sum_{i=1}^{n} \frac{1}{i!}(r^2 n)^i \left(\frac{4}{\delta}\right)^i \leq |A_S| \sum_{i=1}^{n} \frac{1}{i!} r^2 n \, \kappa^{i-1} \left(\frac{4}{\delta}\right)^i$$

$$\leq |A_S| \, nr^2 \sum_{i=1}^{n} \frac{1}{i!} \left(\frac{4\kappa}{\delta}\right)^i \leq |A_S| \, nr^2 \, e^{4\kappa/\delta}$$

Letting $L_1 = e^{4\kappa/\delta}$, we have $|P^{n,0}(z) - A_S| \leq |A_S| nr^2 L_1$, so $P^{n,0}(z) = A_S + A_S nr^2 C(L_1)$, as desired. ∎

We need the following numerical lemma:

Lemma 7.2: There is a positive constant L_2 such that if n and k are in $\mathbf{N}$ and $k \leq \sqrt{n} - 1$, then

$$e^k \leq L_2 \binom{n}{k} \frac{k^{k+1/2}}{n^k}$$

Proof: Recall Stirling's Formula ([ST], page 253):

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} e^{-n} n^n} = 1.$$

This implies that the sequence $\frac{n!}{\sqrt{2\pi n} e^{-n} n^n}$ is bounded and there is some K' in $\mathbf{N}$ for which

$$\frac{1}{K'} < \frac{n!}{\sqrt{2\pi n} e^{-n} n^n} < K', \quad \forall n \in \mathbf{N}.$$

From this we deduce that for all $n \in \mathbf{N}$

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{(n-k)!k!} > \frac{1}{K'^{3}} \frac{\sqrt{2\pi n}\, e^{-n} n^{n}}{\sqrt{2\pi k}\, e^{-k} k^{k} \sqrt{2\pi(n-k)}\, e^{-(n-k)} (n-k)^{n-k}} \\
&= \frac{1}{\sqrt{2\pi} K'^{3}} \frac{1}{k^{k+1/2}} \sqrt{\frac{n}{n-k}} \frac{n^{k} n^{n-k}}{(n-k)^{n-k}} \\
&> \frac{1}{\sqrt{2\pi} K'^{3}} \frac{n^{k}}{k^{k+1/2}} \left(\frac{n}{n-k}\right)^{n-k} = \frac{1}{\sqrt{2\pi} K'^{3}} \frac{n^{k}}{k^{k+1/2}} \left(1 + \frac{k}{n-k}\right)^{n-k}
\end{aligned}$$

Note that for $b \in \mathbf{N}$ and $a \in \left[0, \sqrt{b}\right]$,

$$\begin{aligned}
\ln\left[\left(1 + \frac{a}{b}\right)^{b}\right] &= b \ln\left(1 + \frac{a}{b}\right) = b\left\{\frac{a}{b} - \frac{1}{2}\left(\frac{a}{b}\right)^{2} + \frac{1}{3}\left(\frac{a}{b}\right)^{3} - \frac{1}{4}\left(\frac{a}{b}\right)^{4} + \cdots\right\} \\
&\geq b\left\{\frac{a}{b} - \frac{1}{2}\left(\frac{a}{b}\right)^{2}\right\} = a - \frac{1}{2}\frac{a^{2}}{b} \geq a - \frac{1}{2}
\end{aligned}$$

Thus

$$\left(1 + \frac{a}{b}\right)^{b} \geq e^{-1/2} e^{a}.$$

Since $k \leq \sqrt{n} - 1$, we have $k^2 \leq (\sqrt{n} - 1)^2 = n - 2\sqrt{n} + 1 \leq n - k$, and we obtain

$$\binom{n}{k} > \frac{1}{\sqrt{2\pi} K'^{3}} \frac{n^{k}}{k^{k+1/2}} \left(1 + \frac{k}{n-k}\right)^{n-k} \geq \frac{1}{\sqrt{2\pi} K'^{3}} \frac{n^{k}}{k^{k+1/2}} e^{-\frac{1}{2}} e^{k},$$

Thus

$$e^{k} \leq \sqrt{2\pi e}\, K'^{3} \binom{n}{k} \frac{k^{k+1/2}}{n^{k}}$$

∎

Lemma 7.3: There is a constant $L_3 > 0$ such that if n and k are in $\mathbf{N}$ and $0 \leq k < \frac{\delta^2}{2}\sqrt{n}$, then

$$\left\langle P^{n,0}, x^{k} \right\rangle = \left\langle Q^{n,0}, x^{k} \right\rangle \left[1 + C(L_3)\frac{k^{5/2}}{n}\right].$$

Proof: For $k = 0$ the result is clear. Assume $k \geq 1$. For $n \in \mathbf{N}$ define

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \frac{P_{j,1}}{P_{j,0}}.$$

From equation (7.2) it is clear that $\delta \leq P_{j,0}, P_{j,1} \leq 1 - \delta$ for all $j \in \mathbf{N}$, and so $\frac{\delta}{1-\delta} \leq \mu_n \leq \frac{1-\delta}{\delta}$, $\forall n \in \mathbf{N}$.
We set

$$r = \frac{k}{n\mu_n},$$

and note that $r < \frac{\delta^2 \sqrt{n}}{2n\mu_n} \leq \frac{\delta^2}{2\sqrt{n}} \frac{1-\delta}{\delta} < \frac{1}{2\sqrt{n}} \frac{\delta}{1-\delta}$. Recalling that $Q_j = P_{j,0} + P_{j,1}x$ for all j in $\mathbf{N}$, Lemma 7.1 tells us that $P^{n,0}(re^{i\theta}) = Q^{n,0}(re^{i\theta})\left[1 + nr^2 C(L_1)\right]$. for all θ in $\mathbf{R}$. Applying

Laurent's theorem, we get

$$\begin{aligned}\langle P^{n,0}, x^k\rangle &= \frac{1}{2\pi i}\int_{|z|=r}\frac{P^{n,0}(z)}{z^{k+1}}\,dz = \frac{1}{2\pi i}\int_{|z|=r}\frac{Q^{n,0}(z)\left[1+nr^2C(L_1)\right]}{z^{k+1}}\,dz.\\ &= \frac{1}{2\pi i}\int_{|z|=r}\frac{Q^{n,0}(z)}{z^{k+1}}\,dz + \frac{1}{2\pi i}\int_{|z|=r}\frac{Q^{n,0}(z)nr^2C(L_1)}{z^{k+1}}\,dz\\ &= \langle Q^{n,0}, x^k\rangle + \frac{nr^2}{2\pi i}\int_{|z|=r}\frac{Q^{n,0}(z)C(L_1)}{z^{k+1}}\,dz.\end{aligned}$$

Writing $z = re^{2\pi i\theta}$, we find that $dz = 2\pi i r e^{2\pi i\theta}\,d\theta$, and so

$$\begin{aligned}\langle P^{n,0}, x^k\rangle - \langle Q^{n,0}, x^k\rangle &= \frac{nr^2}{2\pi i}\int_0^1 \frac{Q^{n,0}(re^{2\pi i\theta})C(L_1)}{(re^{2\pi i\theta})^{k+1}}\,2\pi i r e^{2\pi i\theta}\,d\theta\\ &= nr^2\int_0^1 \frac{Q^{n,0}(re^{2\pi i\theta})C(L_1)}{(re^{2\pi i\theta})^{k}}\,d\theta\end{aligned}$$

Now

$$\left|\int_0^1 \frac{Q^{n,0}(re^{2\pi i\theta})C(L_1)}{(re^{2\pi i\theta})^{k}}\,d\theta\right| \le \int_0^1 \left|\frac{Q^{n,0}(re^{2\pi i\theta})C(L_1)}{(re^{2\pi i\theta})^{k}}\right|\,d\theta \le \int_0^1 \left|\frac{Q^{n,0}(r)\,L_1}{r^k}\right|\,d\theta$$

and so

$$\langle P^{n,0}, x^k\rangle - \langle Q^{n,0}, x^k\rangle = nr^2\,\frac{Q^{n,0}(r)\,C(L_1)}{r^k} = nr^{2-k}\,Q^{n,0}(r)\,C(L_1).$$

Noting that $Q^{n,0}(0) = \prod_{j=1}^n P_{j,0}$, we apply Lemma A.2 to find that

$$\begin{aligned}Q^{n,0}(r) &= \prod_{j=1}^{n}(P_{j,0}+P_{j,1}r) = Q^{n,0}(0)\prod_{j=1}^{n}\left(1+\frac{P_{j,1}}{P_{j,0}}r\right)\\ &\le Q^{n,0}(0)\,\exp\left[\sum_{j=1}^{n}\frac{P_{j,1}}{P_{j,0}}\,r\right] = Q^{n,0}(0)\,\exp\left[n\mu_n r\right] = Q^{n,0}(0)\,e^k.\end{aligned}$$

Using Lemma 7.2 we now obtain

$$\langle P^{n,0}, x^k\rangle - \langle Q^{n,0}, x^k\rangle = nr^{2-k}\,C(L_1)\,Q^{n,0}(0)\,e^k = nr^{2-k}\,C(L_1L_2)\,Q^{n,0}(0)\binom{n}{k}\frac{k^{k+1/2}}{n^k}.$$

As $r = \frac{k}{n\mu_n}$, this becomes

$$\begin{aligned}\langle P^{n,0}, x^k\rangle - \langle Q^{n,0}, x^k\rangle &= n\left(\frac{k}{n\mu_n}\right)^{2-k} C(L_1L_2)\,Q^{n,0}(0)\binom{n}{k}\frac{k^{k+1/2}}{n^k}\\ &= C(L_1L_2)\,\frac{k^{5/2}}{n}\,Q^{n,0}(0)\binom{n}{k}\mu_n^{k-2}\end{aligned}$$

We now let $\beta_j = \frac{P_{j,1}}{P_{j,0}}$ for all j in N, and set $S_n = \sum_{j=1}^n \beta_j$. Since $\mu_n = \frac{S_n}{n}$, Recall from equation (1.4) in Proposition 1.12 that, for $k \ge 2$,

$$\begin{aligned}\left\langle \prod_{i=1}^{n}(1+\beta_i x),\, x^k\right\rangle &\ge \frac{1}{k!}\left[S_n^k - \binom{k}{2}\left(\sum \beta_i^2\right)S_n^{k-2}\right] \ge \frac{1}{k!}\left[S_n^k - \frac{k^2}{2}nS_n^{k-2}\right]\\ &= \frac{S_n^k}{k!}\left[1-\frac{k^2n}{2S_n^2}\right] = \frac{(n\mu_n)^k}{k!}\left[1-\frac{k^2n}{2(n\mu_n)^2}\right] = \frac{n^k\mu_n^k}{k!}\left[1-\frac{k^2}{2n\mu_n^2}\right]\end{aligned}$$

As $\frac{\delta}{1-\delta} \le \mu_n$, we have $k < \frac{\delta^2}{2}\sqrt{n} \le \delta\sqrt{n} \le \sqrt{n}\,\mu_n$, and so

$$\frac{\langle Q^{n,0},\, x^k\rangle}{Q^{n,0}(0)} = \left\langle \prod_{i=1}^{n}(1+\beta_i x),\, x^k \right\rangle \ge \frac{n^k\mu_n^k}{k!}\left[1-\frac{1}{2}\right] = \frac{\mu_n^k}{} \frac{n^k}{k!} \ge \frac{\mu_n^k}{2}\,\frac{n!}{k!(n-k)!} = \frac{\mu_n^k}{2}\binom{n}{k}.$$

(This last inequality holds even for $k = 1$). Thus

$$\begin{aligned}\langle P^{n,0},\, x^k\rangle - \langle Q^{n,0},\, x^k\rangle &= C(L_1L_2)\,\frac{k^{5/2}}{n}\,Q^{n,0}(0)\binom{n}{k}\mu_n^{k-2}\\ &= C(L_1L_2)\,\frac{k^{5/2}}{n}\,\frac{2}{\mu_n^2}\,\langle Q^{n,0},\, x^k\rangle\end{aligned}$$

As $\mu_n \ge \frac{\delta}{1-\delta}$, we can set $L_3 = \frac{2L_1L_2(1-\delta)^2}{\delta^2}$ to obtain

$$\langle P^{n,0},\, x^k\rangle = \langle Q^{n,0},\, x^k\rangle\left[1 + C(L_3)\,\frac{k^{5/2}}{n}\right].$$

∎

We can now deduce strong unimodality of coefficients near the end of $P^{n,0}$.

Theorem 7.4: Let $\delta > 0$. There is a constant $L(\delta) > 0$ such that if $\{P_j\}_{j\in\mathbb{N}}$ is any sequence of polynomials in $\mathbf{R}[x]^+$ for which

$$P_j(1) = 1, \qquad \forall j,$$

and

$$\langle P_j,\, x^0\rangle \ge \delta \qquad \text{and} \qquad \langle P_j,\, x^1\rangle \ge \delta, \qquad \forall j \in \mathbb{N},$$

then

$$\left(\langle P^{n,0},\, x^k\rangle\right)^2 \ge \langle P^{n,0},\, x^{k-1}\rangle\,\langle P^{n,0},\, x^{k+1}\rangle \tag{7.3}$$

for every integer k for which $k \le L(\delta)\,n^{2/7} - 1$.

Proof: We take $L = L(\delta) = \min\left\{\frac{d^2}{2},\, (4L_3)^{-2/7}\right\}$, where L_3 was given in Lemma 7.3. Set $Q = \prod_{j=1}^{n}(P_{j,0} + P_{j,1}x)$. Equation (7.3) is clearly true for $k \le 0$. Assume $k \ge 1$. As Q has only real roots, Theorem 2 in [S] (p. 504) and the comment following it, show that

$$\left(\langle Q,\, x^k\rangle\right)^2 \ge \langle Q,\, x^{k-1}\rangle\,\langle Q,\, x^{k+1}\rangle\left(1+\frac{1}{k}\right).$$

This can also be proved by regarding $\langle Q,\, x^m\rangle$ as a polynomial the $P_{j,0}$'s and $P_{j,1}$'s, and counting the number of times each monomial appears in $\left(\langle Q,\, x^k\rangle\right)^2$ and $\langle Q,\, x^{k-1}\rangle\,\langle Q,\, x^{k+1}\rangle$.

Writing $\rho_j = \langle P^{n,0},\, x^j\rangle$ for $j = 0, 1, 2, \ldots, n$, we obtain from Lemma 7.3,

$$\begin{aligned}\frac{\rho_k^2}{\rho_{k-1}\rho_{k+1}} &\ge \frac{\left(\langle Q,\, x^k\rangle\left[1 - L_3\,k^{5/2}\,n^{-1}\right]\right)^2}{\left(\langle Q,\, x^{k-1}\rangle\left[1 + L_3\,(k-1)^{5/2}\,n^{-1}\right]\right)\left(\langle Q,\, x^{k+1}\rangle\left[1 + L_3\,(k+1)^{5/2}\,n^{-1}\right]\right)}\\ &\ge \frac{\left[1 - L_3\,(k+1)^{5/2}\,n^{-1}\right]^2}{\left[1 + L_3\,(k+1)^{5/2}\,n^{-1}\right]^2}\cdot\left(1+\frac{1}{k}\right)\end{aligned}$$

Let $w = L_3\,(k+1)^{5/2}\,n^{-1}$, and note that as $|w| < 1$, we have $1 \ge 1 - w^2 = (1-w)(1+w)$, so $\frac{1}{1+w} \ge 1 - w$. As $(1-w)^2 = 1 - 2w + w^2 \ge 1 - 2w$, we have

$$\frac{(1-w)^2}{(1+w)^2} \ge (1-w)^2(1-w)^2 \ge (1-2w)^2 \ge 1 - 4w.$$

Thus

$$\frac{\rho_k^2}{\rho_{k-1}\rho_{k+1}} \ge \frac{(1-w)^2}{(1+w)^2}\cdot\left(1+\frac{1}{k}\right) \ge (1-4w)\,\frac{k+1}{k}$$

As $k \le Ln^{2/7} - 1$, we get $k + 1 \le Ln^{2/7} \le \left(\frac{n}{4L_3}\right)^{2/7}$, and $(k+1)^{7/2} \le \frac{n}{4L_3}$. Thus

$$(k+1)w = (k+1)^{7/2}L_3/n \le \frac{n}{4L_3}\cdot\frac{L_3}{n} = \frac{1}{4}.$$

This gives $4w \le \frac{1}{k+1}$, so

$$(1-4w)\,\frac{k+1}{k} \ge \left(1 - \frac{1}{k+1}\right)\cdot\frac{k+1}{k} = 1.$$

Thus $\rho_k^2 \ge \rho_{k-1}\rho_{k+1}$. ∎

Strong Unimodality in the Middle of the Product. Suppose that for each j in $\mathbf{N}$, P_j is a polynomial in $\mathbf{R}[x]^+$ of degree d_j, and that $d = \sup\{\, d_j \mid j \in \mathbf{N}\,\}$ is finite. For convenience write $P_j = \sum_{i=0}^{d} P_{j,i}x^i$ for each j in $\mathbf{N}$. Suppose also that

$$\sum_{i=0}^{d} P_{j,i} = 1, \qquad \forall j \in \mathbf{N} \tag{7.4}$$

and that there are constants $B_0, B_1, B_d > 0$ such that

$$P_{j,0} \ge B_0, \qquad P_{j,1} \ge B_1, \qquad \text{and} \qquad P_{j,d_j} \ge B_d, \qquad \forall j \in \mathbf{N}. \tag{7.5}$$

We wish to consider the product $P^{n,0} = P_1P_2\cdots P_n$, for large n. We let δ_n denote the degree of $P^{n,0}$, that is, $\delta_n = \sum_{j=1}^{n} d_j$.

We begin by establishing a function to interpolate the coefficients of $P^{n,0}$.

Proposition 7.5: For each $n \in \mathbf{N}$ the function $a_n(\alpha, k)$ defined by

$$a_n(\alpha,k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\pi}^{\pi} P^{n,0}(e^{\alpha+i\theta})\,e^{-ik\theta}\,d\theta, \qquad \alpha, k \in \mathbf{R} \tag{7.6}$$

is a C^∞ function of k for each α. Also, for each $k \in \mathbf{Z}$ we have

$$\left\langle P^{n,0},\, x^k\right\rangle = a_n(\alpha,k), \qquad \forall \alpha \in \mathbf{R}.$$

Furthermore, for $\alpha, k \in \mathbf{R}$,

$$a_n(\alpha,k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\pi}^{\pi} e^{\left[\sum_{j=1}^{n}\log P_j(e^{\alpha+i\theta})\right] - ik\theta}\,d\theta.$$

Proof: Since $\left[P^{n,0}(e^{\alpha+i\theta})\right]e^{-ik\theta}$ is continuous on $\mathbf{R}^3$ (in α, k and θ), Lemma A.4 implies that $a_n(\alpha,k)$ is a well-defined and continuous from $\mathbf{R}^2$ into $\mathbf{C}$. Applying Lemma A.5 shows that for fixed α, $a_n(\alpha,k)$ is a C^∞ function of k on each interval $[k_0 - 1, k_0 + 1]$, and hence on all of $\mathbf{R}$.

By Laurent's theorem, for $k \in \mathbf{Z}$ we have

$$\langle P^{n,0}, x^k \rangle = \frac{1}{2\pi i} \int_{|z|=r} \frac{P^{n,0}(z)}{z^{k+1}}\, dz,$$

for any $r > 0$. For any $\alpha \in \mathbf{R}$, if we take $r = e^{\alpha}$ and let $z = re^{i\theta}$, we have

$$dz = ire^{i\theta} d\theta = ie^{\alpha}e^{i\theta} d\theta = ie^{\alpha+i\theta} d\theta,$$

so

$$\begin{aligned}\langle P^{n,0}, x^k \rangle &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{P^{n,0}(e^{\alpha+i\theta})}{(e^{\alpha+i\theta})^{k+1}}\, ie^{\alpha+i\theta}\, d\theta \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P^{n,0}(e^{\alpha+i\theta})}{(e^{\alpha+i\theta})^{k}}\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P^{n,0}(e^{\alpha+i\theta})}{e^{\alpha k}e^{ik\theta}}\, d\theta \;=\; \frac{e^{-\alpha k}}{2\pi} \int_{-\pi}^{\pi} P^{n,0}(e^{\alpha+i\theta})\, e^{-ik\theta}\, d\theta \;=\; a_n(\alpha, k).\end{aligned}$$

Equation (7.6) defines the function $a_n : \mathbf{R}^2 \to \mathbf{C}$. Recalling that for any $w \in \mathbf{C} \setminus \{0\}$, $e^{\log w} = w$, Equation (7.6) becomes (since $P^{n,0}(z)$ has only finitely many zeroes on $|z| = r$)

$$a_n(\alpha, k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\pi}^{\pi} e^{\log(P^{n,0}(e^{\alpha+i\theta}))-ik\theta}\, d\theta.$$

As $wv = e^{\log w+\log v}$ for $v, w \in \mathbf{C} \setminus \{0\}$, we have

$$a_n(\alpha, k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\pi}^{\pi} e^{\left[\sum_{j=1}^{n} \log P_j(e^{\alpha+i\theta})\right]-ik\theta}\, d\theta.$$

∎

The functions J_0, J_1 and J_2 defined in Lemma 7.6 are related to a_n and its derivatives.

Lemma 7.6: For $m \in \mathbf{Z}^+$, the function J_m defined by

$$J_m(\alpha, n, k) = \int_{-\pi}^{\pi} \theta^m \left[P^{n,0}(e^{\alpha+i\theta})\right] e^{-ik\theta}\, d\theta, \qquad n \in \mathbf{N},\ \alpha, k \in \mathbf{R} \tag{7.7}$$

is a real-valued function if m is even, and is pure imaginary if m is odd.

Proof: Letting $F_m(\theta) = \theta^m\left[P^{n,0}(e^{\alpha+i\theta})\right]e^{-ik\theta}$, we note that for any θ, $F_m(-\theta) = (-1)^m\overline{F_m(\theta)}$. Thus, if m is even,

$$\begin{aligned}\mathrm{Im}(J_m) &= \mathrm{Im} \int_{-\pi}^{\pi} F_m(\theta)\, d\theta = \int_{-\pi}^{\pi} \mathrm{Im}\, F_m(\theta)\, d\theta = \int_{-\pi}^{0} \mathrm{Im}\, F_m(\theta)\, d\theta + \int_{0}^{\pi} \mathrm{Im}\, F_m(\theta)\, d\theta \\ &= \int_{\pi}^{0} \mathrm{Im}\, F_m(-\theta)\, (-d\theta) + \int_{0}^{\pi} \mathrm{Im}\, F_m(\theta)\, d\theta \\ &= \int_{0}^{\pi} \mathrm{Im}\, \overline{F_m(\theta)}\, d\theta + \int_{0}^{\pi} \mathrm{Im}\, F_m(\theta)\, d\theta = -\int_{0}^{\pi} \mathrm{Im}\, F_m(\theta)\, d\theta + \int_{0}^{\pi} \mathrm{Im}\, F_m(\theta)\, d\theta \\ &= 0\end{aligned}$$

and if m is odd,

$$\begin{aligned}
\mathrm{Re}(J_m) &= \mathrm{Re}\int_{-\pi}^{\pi} F_m(\theta)\,d\theta = \int_{-\pi}^{\pi} \mathrm{Re}\,F_m(\theta)\,d\theta = \int_{-\pi}^{0} \mathrm{Re}\,F_m(\theta)\,d\theta + \int_{0}^{\pi} \mathrm{Re}\,F_m(\theta)\,d\theta \\
&= \int_{\pi}^{0} \mathrm{Re}\,F_m(-\theta)\,(-d\theta) + \int_{0}^{\pi} \mathrm{Re}\,F_m(\theta)\,d\theta \\
&= \int_{0}^{\pi} \mathrm{Re}(-1)^m\overline{F_m(\theta)}\,d\theta + \int_{0}^{\pi} \mathrm{Re}\,F_m(\theta)\,d\theta \\
&= -\int_{0}^{\pi} \mathrm{Re}\,F_m(\theta)\,d\theta + \int_{0}^{\pi} \mathrm{Re}\,F_m(\theta)\,d\theta \\
&= 0
\end{aligned}$$

Thus J_m is a real function if m is even, and is pure imaginary if m is odd. ∎

The following result will (eventually) enable us to choose the radius of the circle of integration in such a way that the first order term in the Taylor expansion of the integrand becomes negligible. $r\delta_n$ will be chosen to be close the coefficient of $P^{n,0}$ which we are approximating, and e^{α_r} will be the radius of the circle of integration.

Lemma 7.7: Let $\gamma = \max\left\{0,\ \ln\frac{3d(1-B_d)}{B_d}\right\}$. For each n in N and $r \in \left(0,\ \frac{3}{4}\right]$ there is a smallest number $\alpha_r \in (-\infty, \gamma]$ such that

$$\sum_{j=1}^{n} \frac{P_j'(e^{\alpha_r})e^{\alpha_r}}{P_j(e^{\alpha_r})} = r\,\delta_n. \tag{7.8}$$

Proof: Let $s = \max\left\{\frac{3d(1-B_d)}{B_d},\ 1\right\}$. For an arbitrary fixed j take $b = d_j$. From equations (7.4) and (7.5) it is clear that $P_{j,b} \geq B_d$ and that $P_{j,m} \leq 1 - B_d$ for $m = 0, 1, 2, \ldots, b-1$. Using these facts we find that

$$\begin{aligned}
sP_j'(s) - \frac{3b}{4}P_j(s) &= \sum_{m=0}^{b} mP_{j,m}\,s^m - \frac{3b}{4}\sum_{m=0}^{b} P_{j,m}\,s^m \;\geq\; bP_{j,b}\,s^b - \frac{3b}{4}\sum_{m=0}^{b} P_{j,m}\,s^m \\
&\geq \frac{b}{4}P_{j,b}\,s^b - \frac{3b}{4}\sum_{m=0}^{b-1} P_{j,m}\,s^m \;\geq\; \frac{b}{4}B_d\,s^b - \frac{3b}{4}\sum_{m=0}^{b-1}(1-B_d)\,s^{b-1} \\
&\geq \frac{b}{4}B_d\,s^b - \frac{3b}{4}d\,(1-B_d)\,s^{b-1} \;=\; \frac{b\,B_d\,s^b}{4}\left[1 - \frac{3d\,(1-B_d)}{B_d}\frac{1}{s}\right] \;\geq\; 0
\end{aligned}$$

From this we obtain $\frac{sP_j'(s)}{P_j(s)} \geq \frac{3b}{4} = \frac{3d_j}{4}$. Since j was arbitrary, we may sum this inequality to obtain

$$\sum_{j=1}^{n} \frac{sP_j'(s)}{P_j(s)} \geq \sum_{j=1}^{n} \frac{3d_j}{4} = \frac{3}{4}\,\delta_n.$$

As $s = e^\gamma$, we have shown that $\sum_{j=1}^{n} \frac{e^\gamma P_j'(e^\gamma)}{P_j(e^\gamma)} \geq \frac{3}{4}\,\delta_n$. Since $\lim_{x\to 0}\frac{xP_j'(x)}{P_j(x)} = 0$ for each j, it is clear that $\lim_{\alpha\to-\infty}\sum_{j=1}^{n} \frac{e^\alpha P_j'(e^\alpha)}{P_j(e^\alpha)} = 0$. It follows that for each $r \in \left(0,\ \frac{3}{4}\right]$ there is an $\alpha \in (-\infty, \gamma]$ such that $\sum_{j=1}^{n} \frac{P_j'(e^\alpha)e^\alpha}{P_j(e^\alpha)} = r\,\delta_n$. We take α_r to be the smallest such α. ∎

We will need Lemma 7.8 for obtaining the Taylor expansion of our integrand.

Lemma 7.8: For $\alpha \in \mathbf{R}$ and $j \in \mathbf{N}$ the function $f_{\alpha,j}$ defined by

$$f_{\alpha,j}(\theta) = \log\left(P_j(e^{\alpha+i\theta})\right). \tag{7.9}$$

is infinitely differentiable on the interval $\left[-\frac{\pi}{2d}, \frac{\pi}{2d}\right]$. Furthermore, there are constants K_1, K_2 and K_3 in $(0,\infty)$ such that

$$-\frac{f''_{\alpha,j}(0)}{e^{\alpha}} \in [K_1, K_2], \qquad \forall j \in \mathbf{N},\ \alpha \in (-\infty, \gamma]$$

and

$$\left|f'''_{\alpha,j}(\theta)\right| \le e^{\alpha} K_3, \qquad \forall j \in \mathbf{N},\ \alpha \in (-\infty,\gamma],\ \theta \in \left[-\frac{\pi}{2d}, \frac{\pi}{2d}\right].$$

Proof: Note that for $\alpha \in \mathbf{R}$ and $|\theta| \le \frac{\pi}{2d}$,

$$\begin{aligned}\operatorname{Re} P_j(e^{\alpha+i\theta}) &= \operatorname{Re}\sum_{l=0}^{d} P_{j,l}\, e^{\alpha l}\,(e^{i\theta})^l = \sum_{l=0}^{d} \operatorname{Re} P_{j,l}\, e^{\alpha l}\,(\cos l\theta + i\sin l\theta)\\ &= P_{j,0} + \sum_{l=1}^{d} P_{j,l}\, e^{\alpha l}\cos l\theta \ge P_{j,0}\end{aligned}$$

Thus $\log\left(P_j(e^z)\right)$ is an analytic function of z on a region containing the set

$$\left\{ z = \alpha + i\theta \,\middle|\, \alpha \in \mathbf{R},\ \theta \in \left[-\frac{\pi}{2d}, \frac{\pi}{2d}\right] \right\}.$$

Now

$$f'_{\alpha,j}(\theta) = \frac{P'_j(e^{\alpha+i\theta})e^{\alpha+i\theta}}{P_j(e^{\alpha+i\theta})}\, i,$$

so

$$\begin{aligned}f''_{\alpha,j}(\theta) = &\frac{-e^{\alpha+i\theta}}{[P_j(e^{\alpha+i\theta})]^2}\\ &\cdot \left[P_j(e^{\alpha+i\theta})P''_j(e^{\alpha+i\theta})e^{\alpha+i\theta} + P_j(e^{\alpha+i\theta})P'_j(e^{\alpha+i\theta}) - \left[P'_j(e^{\alpha+i\theta})\right]^2 e^{\alpha+i\theta}\right]\end{aligned}$$

We now consider, for $x > 0$, the expression

$$
\begin{aligned}
&P_j(x)P_j''(x)x + P_j(x)P_j'(x) - \left[P_j'(x)\right]^2 x \\
&= \left[\sum_{l=0}^{d} P_{j,l}x^l\right]\left[\sum_{t=1}^{d-1}(t+1)tP_{j,t+1}x^t + \sum_{t=0}^{d-1}(t+1)P_{j,t+1}x^t\right] \\
&\qquad - \left[\sum_{l=1}^{d} lP_{j,l}x^l\right]\left[\sum_{t=0}^{d-1}(t+1)P_{j,t+1}x^t\right] \\
&= \sum_{l=0}^{d}\sum_{t=0}^{d-1} P_{j,l}P_{j,t+1}\,(t+1)\,[t+1-l]\,x^{l+t} \;=\; \sum_{l=0}^{d}\sum_{m=1}^{d} P_{j,l}P_{j,m}\,m\,[m-l]\,x^{l+m-1} \\
&= \sum_{0\le l<m\le d} P_{j,l}P_{j,m}\left[m(m-l)x^{l+m-1} + l(l-m)x^{m+l-1}\right] \\
&= \sum_{0\le l<m\le d} P_{j,l}P_{j,m}\,(m-l)^2\,x^{l+m-1}
\end{aligned}
$$

It follows that

$$f''_{\alpha,j}(0) = \frac{-e^{\alpha}}{\left[P_j(e^{\alpha})\right]^2}\cdot \sum_{0\le l<m\le d} P_{j,l}P_{j,m}\,(m-l)^2\,(e^{\alpha})^{l+m-1}$$

Thus, since for α in $(-\infty,\gamma]$, $P_j(e^{\alpha}) \le P_j(e^{\gamma}) = \sum_{l=0}^{d} P_{j,l}\,e^{\gamma l} \le e^{\gamma d}\sum_{l=0}^{d} P_{j,l} = e^{\gamma d}$, we have,

$$
\begin{aligned}
-\frac{f''_{\alpha,j}(0)}{e^{\alpha}} &= \frac{1}{\left[P_j(e^{\alpha})\right]^2}\sum_{0\le l<m\le d} P_{j,l}P_{j,m}\,(m-l)^2\,(e^{\alpha})^{l+m-1} \\
&\ge \frac{1}{\left[e^{\gamma d}\right]^2}\left[P_{j,0}P_{j,1}(1-0)^2\,x^{0+1-1}\right] \;\ge\; \frac{B_0B_1}{e^{2\gamma d}}
\end{aligned}
$$

and

$$-\frac{f''_{\alpha,j}(0)}{e^{\alpha}} \le \frac{1}{\left[P_{j,0}\right]^2}\sum_{0\le l<m\le d} 1\cdot 1\,(d)^2\,(e^{\gamma})^{2d} \le \frac{d^4e^{2\gamma d}}{B_0^2}$$

From the form of $f''_{\alpha,j}(\theta)$ it is clear that $f'''_{\alpha,j}(\theta)$ will be equal to $\frac{e^{\alpha+i\theta}}{[P_j(e^{\alpha+i\theta})]^4}$ multiplied by some polynomial in $P_j'''(e^{\alpha+i\theta})$, $P_j''(e^{\alpha+i\theta})$, $P_j'(e^{\alpha+i\theta})$, $P_j(e^{\alpha+i\theta})$ and $e^{\alpha+i\theta}$. If $P_j^{(m)}$ denotes the m'th derivative of P_j, for $m = 0,1,2,3$, then for α in $(-\infty,\gamma]$ and j in $\mathbf{N}$ we have

$$
\begin{aligned}
\left|P_j^{(m)}(e^{\alpha+i\theta})\right| &\le P_j^{(m)}\left(|e^{\alpha+i\theta}|\right) = P_j^{(m)}(e^{\alpha}) \le P_j^{(m)}(e^{\gamma}) \\
&\le \sum_{l=0}^{d} d^m P_{j,l}e^{\gamma l} \le d^m e^{\gamma d}\sum_{l=0}^{d} P_{j,l} = d^m e^{\gamma d}.
\end{aligned}
$$

Since also $\left|P_j(e^{\alpha+i\theta})\right| \ge \operatorname{Re} P_j(e^{\alpha+i\theta}) \ge P_{j,0} \ge B_0$ whenever $|\theta| \le \frac{\pi}{2d}$, so that $\left|\frac{e^{\alpha+i\theta}}{[P_j(e^{\alpha+i\theta})]^4}\right| \le \frac{e^{\alpha}}{B_0^4}$, it follows that there is some constant K_3 in $(0,\infty)$ such that $|f'''_{\alpha,j}(\theta)| \le e^{\alpha}K_3$ for all j in $\mathbf{N}$, α in $(-\infty,\gamma]$, and $\theta \in \left[-\frac{\pi}{2d},\frac{\pi}{2d}\right]$. ∎

Notational note: We use the notation $C(E)$ to denote any quantity x which satisfies $|x| \le E$. This is anologous to $O(E)$, but gives greater precision.

We now reduce our integrals to cover only a portion of the circle of integration. We will show in Lemma 7.10 that the integrand is small outside of this region.

Lemma 7.9: For $m \in \mathbf{Z}^+$, $\alpha \in \mathbf{R}$, $n \in \mathbf{N}$, $\kappa \in \mathbf{R}$ and $0 \le \theta_o \le \min\left(1, \frac{\pi}{2d}\right)$ define

$$J_m^*(\alpha, n, \kappa, \theta_o) = \int_{-\theta_o}^{\theta_o} \theta^m \left[P^{n,0}(e^{\alpha+i\theta})\right] e^{-i\kappa\theta}\, d\theta. \tag{7.10}$$

For any r in $\left(0, \frac{3}{4}\right]$, if we set $H_{n,r} = -\frac{1}{2}\sum_{j=1}^n f''_{\alpha_r,j}(0)$, then

$$J_m^*(\alpha_r, n, \kappa, \theta_o) = P^{n,0}(e^{\alpha_r}) \int_{-\theta_o}^{\theta_o} \theta^m\, e^{[r\delta_n - \kappa]i\theta - H_{n,r}\theta^2 + C(ne^{\alpha_r}K_3/3)\,|\theta|^3}\, d\theta. \tag{7.11}$$

Proof: We first restrict α to lie in $(-\infty, \gamma]$. As Lemma 7.8 guarantees the differentiability of $f_{\alpha,j}(\theta)$, we may apply Proposition A.6 to obtain

$$\begin{aligned} f_{\alpha,j}(\theta) &= f_{\alpha,j}(0) + f'_{\alpha,j}(0)\,\theta + \frac{f''_{\alpha,j}(0)}{2}\,\theta^2 + C\left(2 \sup_{\tau \in [-\frac{\pi}{2d}, \frac{\pi}{2d}]} \left|\frac{f'''_{\alpha,j}(\tau)}{6}\,\theta^3\right|\right) \\ &= f_{\alpha,j}(0) + f'_{\alpha,j}(0)\,\theta + \frac{f''_{\alpha,j}(0)}{2}\,\theta^2 + C\left(\frac{e^\alpha K_3}{3}\right)|\theta|^3 \end{aligned}$$

for all $\theta \in \left[-\frac{\pi}{2d}, \frac{\pi}{2d}\right]$.

As $P^{n,0}(e^{\alpha+i\theta}) = e^{\sum_{j=1}^n f_{\alpha,j}(\theta)}$ and $P^{n,0}(e^\alpha) = e^{\sum_{j=1}^n f_{\alpha,j}(0)}$, equation (7.10) becomes

$$\begin{aligned} J_m^*(\alpha, n, \kappa, \theta_o) &= \int_{-\theta_o}^{\theta_o} \theta^m\, e^{\sum_{j=1}^n \left[f_{\alpha,j}(0) + f'_{\alpha,j}(0)\,\theta + f''_{\alpha,j}(0)\,\theta^2/2 + C(e^\alpha K_3/3)\,|\theta|^3\right]}\, e^{-i\kappa\theta}\, d\theta. \\ &= \int_{-\theta_o}^{\theta_o} \theta^m P^{n,0}(e^\alpha)\, e^{\sum_{j=1}^n \left[f'_{\alpha,j}(0)\,\theta + f''_{\alpha,j}(0)\,\theta^2/2 + C(e^\alpha K_3/3)\,|\theta|^3\right]}\, e^{-i\kappa\theta}\, d\theta. \\ &= \int_{-\theta_o}^{\theta_o} \theta^m P^{n,0}(e^\alpha)\, e^{\left[-i\kappa + \sum_{j=1}^n f'_{\alpha,j}(0)\right]\theta + \sum_{j=1}^n \left[f''_{\alpha,j}(0)\,\theta^2/2 + C(e^\alpha K_3/3)\,|\theta|^3\right]}\, d\theta. \end{aligned}$$

Now $f'_{\alpha,j}(\theta) = \frac{P_j'(e^{\alpha+i\theta})e^{\alpha+i\theta}}{P_j(e^{\alpha+i\theta})}\, i$, so $f'_{\alpha,j}(0) = \frac{P_j'(e^\alpha)e^\alpha}{P_j(e^\alpha)}\, i$. In accordance with Lemma 7.7 we may take $\alpha = \alpha_r$, for any $r \in \left(0, \frac{3}{4}\right]$, and (7.8) becomes

$$\sum_{j=1}^n f'_{\alpha_r,j}(0) = \sum_{j=1}^n \frac{P_j'(e^{\alpha_r})e^{\alpha_r}}{P_j(e^{\alpha_r})}\, i = r\,\delta_n\, i.$$

This yields

$$\begin{aligned} J_m^*(\alpha_r, n, \kappa, \theta_o) &= \int_{-\theta_o}^{\theta_o} \theta^m P^{n,0}(e^{\alpha_r})\, e^{[r\delta_n - \kappa]i\theta + \sum_{j=1}^n \left[f''_{\alpha_r,j}(0)\,\theta^2/2 + C(e^{\alpha_r} K_3/3)\,|\theta|^3\right]}\, d\theta. \\ &= P^{n,0}(e^{\alpha_r}) \int_{-\theta_o}^{\theta_o} \theta^m\, e^{[r\delta_n - \kappa]i\theta - H_{n,r}\theta^2 + C(ne^{\alpha_r}K_3/3)\,|\theta|^3}\, d\theta. \end{aligned}$$

where $H_{n,r} = -\frac{1}{2}\sum_{j=1}^n f''_{\alpha_r,j}(0)$. ∎

Lemma 7.10: There is a constant $K_4 > 0$ such for every θ_o in $(0, 1]$ and n in $\mathbf{N}$,

$$\left|P^{n,0}(e^{\alpha+i\theta})\right| \le P^{n,0}(e^\alpha)\, e^{-nK_4 e^\alpha \theta_o^2}, \qquad \forall \theta \in [\theta_o, 2\pi - \theta_o],\ \alpha \in (-\infty, \gamma].$$

Proof: For $a, b \in (0,\infty)$ and $\theta \in [\theta_o, 2\pi - \theta_o]$, with $\theta_o \leq 1$, we have

$$\begin{aligned}
|a + be^{i\theta}|^2 &= (a+be^{i\theta})(a+be^{-i\theta}) = a^2 + ab(e^{i\theta}+e^{-i\theta}) + b^2 = a^2 + 2ab\cos\theta + b^2 \\
&= (a+b)^2 + 2ab(\cos\theta - 1) \\
&\leq (a+b)^2 + 2ab(\cos\theta_o - 1) = (a+b)^2 + 2ab\left(-\frac{\theta_o^2}{2!} + \frac{\theta_o^4}{4!} - \frac{\theta_o^6}{6!} + \cdots\right) \\
&\leq (a+b)^2 + 2ab\left(-\frac{\theta_o^2}{2} + \frac{\theta_o^2}{24}\right) \leq (a+b)^2 + 2ab\left(-\frac{\theta_o^2}{3}\right) \\
&= (a+b)^2\left[1 + 2\frac{ab}{(a+b)^2}\left(-\frac{\theta_o^2}{3}\right)\right] \\
&\leq (a+b)^2\left[1 + 2\frac{ab}{(a+b)^2}\left(-\frac{\theta_o^2}{3}\right) + \left(\frac{ab}{(a+b)^2}\frac{\theta_o^2}{3}\right)^2\right] \\
&= (a+b)^2\left[1 - \frac{ab}{3(a+b)^2}\theta_o^2\right]^2
\end{aligned}$$

This yields

$$|a + be^{i\theta}| \leq (a+b)\left[1 - \frac{ab}{3(a+b)^2}\theta_o^2\right] = (a+b) - \frac{ab}{3(a+b)}\theta_o^2$$

If for some j in $\mathbf{N}$ we take $a = P_{j,0}$ and $b = P_{j,1}e^{\alpha}$, for α in $(-\infty, \gamma]$, this becomes

$$|P_{j,0} + P_{j,1}e^{\alpha+i\theta}| \leq (P_{j,0} + P_{j,1}e^{\alpha}) - \frac{P_{j,0}P_{j,1}e^{\alpha}}{3(P_{j,0}+P_{j,1}e^{\alpha})}\theta_o^2 \tag{7.12}$$

Since $P_j(e^{\alpha}) \leq P_j(e^{\gamma}) = \sum_{l=0}^{d} P_{j,l}e^{\gamma l} \leq e^{\gamma d}\sum_{l=0}^{d} P_{j,l} = e^{\gamma d}$, it follows that

$$\frac{P_{j,0}P_{j,1}e^{\alpha}}{3(P_{j,0}+P_{j,1}e^{\alpha})} \geq \frac{P_{j,0}P_{j,1}e^{\alpha}}{3(P_{j,0}+P_{j,1}e^{\alpha})}\cdot\frac{P_j(e^{\alpha})}{e^{\gamma d}} \geq \frac{B_0B_1}{3(1+e^{\gamma})e^{\gamma d}}\cdot e^{\alpha}P_j(e^{\alpha})$$

Setting $K_4 = \frac{B_0B_1}{3(1+e^{\gamma})e^{\gamma d}}$ we can use (7.12) to get

$$\begin{aligned}
\left|P_j(e^{\alpha+i\theta})\right| &\leq \left|P_{j,0} + P_{j,1}e^{\alpha+i\theta}\right| + \left|\sum_{l=2}^{d} P_{j,l}\,(e^{\alpha+i\theta})^l\right| \\
&\leq P_{j,0} + P_{j,1}e^{\alpha} - P_j(e^{\alpha})K_4e^{\alpha}\theta_o^2 + \sum_{l=2}^{d} P_{j,l}\,(e^{\alpha})^l = P_j(e^{\alpha})\left[1 - K_4e^{\alpha}\theta_o^2\right]
\end{aligned}$$

Since for $s \in [0,1]$ we have $e^{-s} = 1 - s + \frac{s^2}{2!} - \frac{s^3}{3!} + \cdots \geq 1 - s$, and since $\theta_o \leq 1$ and $K_4e^{\alpha} \leq 1$, we obtain

$$\left|P_j(e^{\alpha+i\theta})\right| \leq P_j(e^{\alpha})\left[1 - K_4e^{\alpha}\theta_o^2\right] \leq P_j(e^{\alpha})e^{-K_4e^{\alpha}\theta_o^2}$$

which yields

$$\left|P^{n,0}(e^{\alpha+i\theta})\right| = \prod_{j=1}^{n}\left|P_j(e^{\alpha+i\theta})\right| \leq \prod_{j=1}^{n}\left[P_j(e^{\alpha})e^{-K_4e^{\alpha}\theta_o^2}\right] = P^{n,0}(e^{\alpha})\,e^{-nK_4e^{\alpha}\theta_o^2}$$

∎

Lemma 7.11: If $z \in \mathbf{C}$ with $|z| \leq 1$, then $e^z = 1 + C(2|z|)$.

Proof: Since
$$\left|\sum_{n=2}^{\infty}\frac{z^n}{n!}\right| \le \sum_{n=2}^{\infty}\frac{|z|^n}{n!} \le \frac{1}{2}\sum_{n=2}^{\infty}\frac{|z|^n}{(n-1)!} = \frac{|z|}{2}\sum_{n=2}^{\infty}\frac{|z|^{n-1}}{(n-1)!} \le \frac{|z|}{2}\sum_{n=1}^{\infty}\frac{1^n}{n!} = \frac{|z|(e-1)}{2} < |z|,$$
we have $e^z = 1 + z + C(|z|) = 1 + C(2|z|)$. ∎

Collecting everything we have done so far, we obtain J_m as a simplified integral with error terms. It is the careful choice of θ_o that will make these error times simultaneously smaller than the main integral.

Lemma 7.12: There are constants K_5, K_6 and K_7 in $(0,\infty)$ such that if n in $\mathbf{N}$ is greater than K_7, and $r \in \left[\frac{n^{1/4}}{\delta_n}, \frac{3}{4}\right]$, then for every κ in $[r\delta_n - 2, r\delta_n + 2]$,
$$\frac{J_m(\alpha_r, n, \kappa)}{P^{n,0}(e^{\alpha_r})} = \int_{-\theta_o}^{\theta_o} \theta^m\, e^{-H_{n,r}\theta^2}\left[1 + C\left(K_5\,|\theta|\,n^{1/15}\right)\right] d\theta + C\left(\frac{2\pi^{m+1}}{m+1}\,e^{-K_6 n^{1/15}}\right),$$
where $\theta_o = n^{1/30}(r\delta_n)^{-1/2}$.

Proof: Since $r \in (0, 3/4]$, Lemma 7.7 furnishes a number α_r in $(-\infty, \gamma]$ for which
$$r\,\delta_n = e^{\alpha_r}\sum_{j=1}^{n}\frac{P_j'(e^{\alpha_r})}{P_j(e^{\alpha_r})}.$$
As $B_0 \le P_j(e^{\alpha_r}) \le P_j(e^{\gamma}) = \sum_{l=0}^{d} P_{j,l}e^{\gamma l} \le e^{\gamma d}\sum_{l=0}^{d} P_{j,l} = e^{\gamma d}$, and $B_1 \le P_j'(e^{\alpha_r}) \le d e^{\gamma d}$, it follows that
$$r\,\delta_n \le e^{\alpha_r}\sum_{j=1}^{n}\frac{de^{\gamma d}}{B_0} = ne^{\alpha_r}\,\frac{de^{\gamma d}}{B_0} \tag{7.13}$$
and
$$r\,\delta_n \ge e^{\alpha_r}\sum_{j=1}^{n}\frac{B_1}{e^{\gamma d}} = ne^{\alpha_r}\,\frac{B_1}{e^{\gamma d}}. \tag{7.14}$$
Let
$$K_7 = \max\left\{\left(\frac{2K_3e^{\gamma d}}{3B_1}\right)^{40}, (4d)^{12}\right\}.$$
Let $\theta_o = n^{1/30}\,(r\delta_n)^{-1/2}$, and note that if $|\theta| \le \theta_o$, then using equation (7.14) we find
$$ne^{\alpha_r}\,\frac{K_3}{3}\,|\theta|^3 \;\le\; ne^{\alpha_r}\,\frac{K_3}{3}\,\theta_o^2|\theta| \;=\; ne^{\alpha_r}\,\frac{K_3}{3}\,\frac{n^{1/15}}{r\delta_n}\,|\theta| \;\le\; \frac{K_3e^{\gamma d}}{3B_1}\,|\theta|\,n^{1/15} \tag{7.15}$$
As $r \ge \frac{n^{1/4}}{\delta_n}$, we have $r\delta_n \ge n^{1/4}$, so
$$\theta_o = \frac{n^{1/30}}{(r\delta_n)^{1/2}} \le \frac{n^{1/30}}{n^{1/8}} = n^{\frac{4}{120}-\frac{15}{120}} = n^{-11/120}.$$
As $n \ge K_7$ we get $n^{1/40} \ge \frac{2K_3e^{\gamma d}}{3B_1}$, so if $|\theta| \le \theta_o$, equation (7.15) becomes
$$ne^{\alpha_r}\,\frac{K_3}{3}\,|\theta|^3 \;\le\; \frac{K_3e^{\gamma d}}{3B_1}\,n^{-11/120}\,n^{1/15} \;=\; \frac{K_3e^{\gamma d}}{3B_1}\,n^{-1/40} \;\le\; \frac{K_3e^{\gamma d}}{3B_1}\,\frac{3B_1}{2K_3e^{\gamma d}} = \frac{1}{2}. \tag{7.16}$$

As $n \geq K_7 \geq (4d)^{12}$, we find that

$$\theta_o \leq n^{-11/120} = \frac{1}{n^{11/120}} \leq \frac{1}{((4d)^{12})^{11/120}} = \frac{1}{(4d)^{11/10}} \leq \frac{1}{4d}.$$

Taking $\kappa \in [r\delta_n - 2, r\delta_n + 2]$, it follows that for $|\theta| \leq \theta_o$,

$$\left|(r\delta_n - \kappa)i\theta\right| \leq 2\theta_o \leq \frac{1}{2d}.$$

Combining this with (7.16) and applying Lemma 7.11 and using (7.15) we obtain, for $|\theta| \leq \theta_o$,

$$\begin{aligned} e^{(r\delta_n-\kappa)i\theta+C(ne^{\alpha_r}K_3/3)\,|\theta|^3} &= 1 + 2C\big(2|\theta| + ne^{\alpha_r}(K_3/3)\,|\theta|^3\big) \\ &= 1 + 2C\left(2|\theta| + \frac{K_3e^{\gamma d}}{3B_1}\,|\theta|\,n^{1/15}\right) = 1 + C\Big(K_5\,|\theta|\,n^{1/15}\Big), \end{aligned}$$

where $K_5 = 4 + \frac{2K_3e^{\gamma d}}{3B_1}$. In this setting equation (7.11) becomes

$$\begin{aligned} J_m^*(\alpha_r, n, \kappa, \theta_o) &= P^{n,0}(e^{\alpha_r}) \int_{-\theta_o}^{\theta_o} \theta^m \; e^{[r\delta_n-\kappa]i\theta - H_{n,r}\theta^2 + C(ne^{\alpha_r}K_3/3)\,|\theta|^3} \; d\theta \\ &= P^{n,0}(e^{\alpha_r}) \int_{-\theta_o}^{\theta_o} \theta^m \; e^{-H_{n,r}\theta^2}\Big[1 + C\Big(K_5\,|\theta|\,n^{1/15}\Big)\Big] \; d\theta \end{aligned} \tag{7.17}$$

Using Lemma 7.10 we calculate

$$\begin{aligned} \left|\int_{-\pi}^{-\theta_o} \theta^m\big[P^{n,0}(e^{\alpha_r+i\theta})\big]\,e^{-i\kappa\theta}\;d\theta\right| &\leq \int_{-\pi}^{-\theta_o} \left|\theta^m\big[P^{n,0}(e^{\alpha_r+i\theta})\big]\,e^{-i\kappa\theta}\right|\,d\theta \\ &= \int_{-\pi}^{-\theta_o} |\theta|^m \left|P^{n,0}(e^{\alpha_r+i\theta})\right|\,d\theta \\ &\leq \int_{-\pi}^{-\theta_o} |\theta|^m\,P^{n,0}(e^{\alpha_r})e^{-nK_4e^{\alpha_r}\theta_o^2}\;d\theta \\ &\leq P^{n,0}(e^{\alpha_r})e^{-nK_4e^{\alpha_r}\theta_o^2}\,\frac{\pi^{m+1}}{m+1}. \end{aligned}$$

Likewise,

$$\left|\int_{\theta_o}^{\pi} \theta^m\big[P^{n,0}(e^{\alpha_r+i\theta})\big]\,e^{-i\kappa\theta}\;d\theta\right| \leq P^{n,0}(e^{\alpha_r})e^{-nK_4e^{\alpha_r}\theta_o^2}\,\frac{\pi^{m+1}}{m+1}.$$

Using this and equation (7.10), we can expand equation (7.7) to yield

$$J_m(\alpha_r, n, \kappa) = J_m^*(\alpha_r, n, \kappa, \theta_o) + 2P^{n,0}(e^{\alpha_r})C\left(e^{-nK_4e^{\alpha_r}\theta_o^2}\,\frac{\pi^{m+1}}{m+1}\right)$$

By (7.13) we have

$$nK_4e^{\alpha_r}\theta_o^2 = nK_4e^{\alpha_r}\frac{n^{1/15}}{r\delta_n} \geq nK_4e^{\alpha_r}\frac{n^{1/15}B_0}{ne^{\alpha_r}de^{\gamma d}} = \frac{K_4B_0}{de^{\gamma d}}\,n^{1/15}$$

Letting $K_6 = \frac{K_4B_0}{de^{\gamma d}}$ and using (7.17), we obtain

$$\frac{J_m(\alpha_r, n, \kappa)}{P^{n,0}(e^{\alpha_r})} = \int_{-\theta_o}^{\theta_o} \theta^m\;e^{-H_{n,r}\theta^2}\Big[1 + C\Big(K_5\,|\theta|\,n^{1/15}\Big)\Big]\;d\theta + C\left(\frac{2\pi^{m+1}}{m+1}\,e^{-K_6n^{1/15}}\right).$$

∎

We now calculate the integrals, and show that J_0 and J_2 are strictly positive in the desired region.

Lemma 7.13: There is a constant M in $(0,\infty)$ such that if n is in $\mathbf{N}$ with $n \geq M$, and $r \in \left[\frac{n^{1/4}}{\delta_n}, \frac{3}{4}\right]$, then for every κ in $[r\delta_n - 2, r\delta_n + 2]$,

$$J_0(\alpha_r, n, \kappa) > 0, \qquad \text{and} \qquad J_2(\alpha_r, n, \kappa) > 0.$$

Proof: For $\rho > 0$ we define

$$D_\rho = \{ (x,y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq \rho^2 \}$$
$$R_\rho = \{ (x,y) \in \mathbf{R}^2 \mid |x|, |y| \leq \rho \}.$$

Then for $K \in (0,\infty)$,

$$\begin{aligned}\int\int_{D_\rho} e^{-K(x^2+y^2)} dx\, dy &= \int\int_{D_\rho} e^{-Kr^2} r\, dr\, d\theta = \int_0^{2\pi} \int_0^\rho e^{-Kr^2} r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^{K\rho^2} e^{-t} \frac{1}{2K}\, dt\, d\theta = \frac{2\pi}{2K} \left[-e^{-t}\Big|_{t=0}^{t=K\rho^2}\right. \\ &= \frac{\pi}{K}\left[1 - e^{-K\rho^2}\right]\end{aligned}$$

If we let $I_\rho = \int_{-\rho}^\rho e^{-Kx^2}\, dx$, then

$$\begin{aligned}I_\rho^2 &= \int_{-\rho}^\rho e^{-Kx^2}\, dx \int_{-\rho}^\rho e^{-Ky^2}\, dy = \int_{-\rho}^\rho \int_{-\rho}^\rho e^{-Kx^2 - Ky^2}\, dx\, dy \\ &= \int\int_{R_\rho} e^{-K(x^2+y^2)}\, dx\, dy.\end{aligned}$$

Since $D_\rho \subset R_\rho \subset D_{\sqrt{2}\rho}$, we have

$$\sqrt{\frac{\pi}{K}\left[1 - e^{-K\rho^2}\right]} \leq \int_{-\rho}^\rho e^{-Kx^2}\, dx \leq \sqrt{\frac{\pi}{K}\left[1 - e^{-2K\rho^2}\right]} \leq \sqrt{\frac{\pi}{K}}$$

Note that

$$\int_{-\infty}^\infty e^{-Kx^2}\, dx = \sqrt{\frac{\pi}{K}}.$$

Since for $a \in [0,1]$, we have $\sqrt{a} \geq a$, it follows that $(1-\sqrt{a})^2 = 1 - 2\sqrt{a} + a \leq 1 - 2a + a = 1 - a$, and so $\sqrt{1-a} \geq 1 - \sqrt{a}$. Thus

$$\int_{-\rho}^\rho e^{-Kx^2}\, dx \geq \sqrt{\frac{\pi}{K}\left[1 - e^{-K\rho^2}\right]} \geq \sqrt{\frac{\pi}{K}}\left(1 - e^{-K\rho^2/2}\right) \tag{7.18}$$

Also, $\int x e^{-Kx^2}\, dx = -\frac{1}{2K} e^{-Kx^2}$, so

$$\int_0^\infty x e^{-Kx^2}\, dx = \frac{1}{2K}. \tag{7.19}$$

For $m \geq 2$ we use integration by parts with $u = x^{m-1}$ and $dv = xe^{-Kx^2}\, dx$ to find that

$$\begin{aligned}\int x^m e^{-Kx^2}\, dx &= x^{m-1}\left(-\frac{1}{2K} e^{-Kx^2}\right) - \int -\frac{1}{2K} e^{-Kx^2} (m-1) x^{m-2}\, dx \\ &= -\frac{x^{m-1} e^{-Kx^2}}{2K} + \frac{m-1}{2K} \int x^{m-2} e^{-Kx^2}\, dx\end{aligned} \tag{7.20}$$

For $m = 3$ this yields

$$\int_0^\infty x^3 e^{-Kx^2} dx = \frac{3-1}{2K} \int_0^\infty x e^{-Kx^2} dx = \frac{2}{2K} \cdot \frac{1}{2K} = \frac{1}{2K^2}. \tag{7.21}$$

Using (7.20) with $m = 2$, and (7.18), we find that

$$\int_{-\rho}^{\rho} x^2 e^{-Kx^2}dx = -\frac{xe^{-Kx^2}}{2K}\Bigg|_{-\rho}^{\rho} + \frac{1}{2K}\int_{-\rho}^{\rho} e^{-Kx^2}dx = \frac{1}{2K}\left[\int_{-\rho}^{\rho} e^{-Kx^2}dx - 2\rho e^{-K\rho^2}\right]$$
$$\geq \frac{1}{2K}\left[\sqrt{\frac{\pi}{K}}\left(1 - e^{-K\rho^2/2}\right) - 2\rho e^{-K\rho^2}\right]$$
$$= \frac{\sqrt{\pi}}{2K^{\frac{3}{2}}}\left[1 - e^{-K\rho^2/2} - \frac{2\sqrt{K\rho^2}}{\sqrt{\pi}}e^{-K\rho^2}\right]$$

Thus

$$\int_{-\rho}^{\rho} x^2 e^{-Kx^2}dx \geq \frac{\sqrt{\pi}}{2K^{\frac{3}{2}}}\cdot\frac{1}{3}, \qquad \text{whenever } K\rho^2 \geq 2. \tag{7.22}$$

Also, for $m \geq 0$,

$$\int_{-\rho}^{\rho} |x|^m e^{-Kx^2}dx = 2\int_0^{\rho} |x|^m e^{-Kx^2}dx = 2\int_0^{\rho} x^m e^{-Kx^2}dx \leq 2\int_0^{\infty} x^m e^{-Kx^2}dx$$

We now assume that $n \geq K_7$. From Lemma 7.6 we recall that J_0 and J_2 are real-valued functions. Setting $\theta_o = n^{1/30}(r\delta_n)^{-1/2}$, we have, from Lemma 7.12,

$$\begin{aligned}\frac{J_2(\alpha_r, n, \kappa)}{p^{n,0}(e^{\alpha_r})} &= \int_{-\theta_o}^{\theta_o} \theta^2\, e^{-H_{n,r}\theta^2}\left[1 + C\left(K_5\,|\theta|\,n^{1/15}\right)\right]\,d\theta + C\left(\frac{2\pi^3}{3}\,e^{-K_6 n^{1/15}}\right)\\ &\geq \int_{-\theta_o}^{\theta_o} \theta^2\, e^{-H_{n,r}\theta^2}\,d\theta - K_5\,n^{1/15}\int_{-\theta_o}^{\theta_o} |\theta|^3\, e^{-H_{n,r}\theta^2}\,d\theta - \pi^3\,e^{-K_6 n^{1/15}}\end{aligned} \tag{7.23}$$

Using (7.21), we find that

$$\int_{-\theta_o}^{\theta_o} |\theta|^3\, e^{-H_{n,r}\theta^2}\,d\theta \leq 2\int_0^{\infty} \theta^3\, e^{-H_{n,r}\theta^2}\,d\theta = 2\cdot\frac{1}{2H_{n,r}^2} = \frac{1}{H_{n,r}^2} \tag{7.24}$$

Recalling from Lemma 7.9 that $H_{n,r} = -\frac{1}{2}\sum_{j=1}^{n} f''_{\alpha_r,j}(0)$, and recalling from Lemma 7.8 that $-f''_{\alpha_r,j}(0) \in [e^{\alpha_r}K_1,\, e^{\alpha_r}K_2]$, we find that

$$ne^{\alpha_r}K_1/2 \leq H_{n,r} \leq ne^{\alpha_r}K_2/2 \tag{7.25}$$

From (7.13) we get $ne^{\alpha_r} \geq \frac{B_0}{de^{\gamma d}}\, r\delta_n$. Combining this with (7.25) and setting $K_9 = \frac{K_1 B_0}{2de^{\gamma d}}$ we obtain

$$H_{n,r} \geq ne^{\alpha_r}K_1/2 \geq \frac{K_1 B_0}{2de^{\gamma d}}\, r\delta_n = K_9\, r\delta_n.$$

Recalling that $\theta_o = n^{1/30}(r\delta_n)^{-1/2}$, we have

$$H_{n,r}{\theta_o}^2 \geq K_9\, r\delta_n\,\frac{n^{1/15}}{r\delta_n} = K_9\, n^{1/15}. \tag{7.26}$$

On the other hand, using the fact that $r \geq \frac{n^{1/4}}{\delta_n}$, we obtain

$$H_{n,r} \geq K_9\, r\delta_n \geq K_9\, n^{1/4}. \tag{7.27}$$

From (7.14) and the fact that $r \leq \frac{3}{4} < 1$ we get $ne^{\alpha_r} \leq \frac{e^{\gamma d}}{B_1}\, r\delta_n \leq \frac{e^{\gamma d}}{B_1}\, dn$. Combining this with (7.25) and setting $K_8 = \frac{K_2 de^{\gamma d}}{2B_1}$ yields

$$H_{n,r} \leq ne^{\alpha_r}K_2/2 \leq \frac{K_2 de^{\gamma d}}{2B_1}\, n = K_8\, n. \tag{7.28}$$

From Lemma 7.12, using (7.18) and (7.19) we have

$$\begin{aligned}
\frac{J_0(\alpha_r,n,\kappa)}{P^{n,0}(e^{\alpha_r})} &= \int_{-\theta_o}^{\theta_o} e^{-H_{n,r}\theta^2}\Big[1+C\Big(K_5\,|\theta|\,n^{1/15}\Big)\Big]\,d\theta + C\Big(2\pi\,e^{-K_6 n^{1/15}}\Big) \\
&\geq \int_{-\theta_o}^{\theta_o} e^{-H_{n,r}\theta^2}\,d\theta - K_5\,n^{1/15}\int_{-\theta_o}^{\theta_o}|\theta|\,e^{-H_{n,r}\theta^2}\,d\theta - 2\pi\,e^{-K_6 n^{1/15}} \\
&\geq \sqrt{\frac{\pi}{H_{n,r}}}\Big(1-e^{-H_{n,r}\theta_o{}^2/2}\Big) - K_5\,n^{1/15}\,\frac{2}{2H_{n,r}} - 2\pi\,e^{-K_6 n^{1/15}} \\
&= \sqrt{\frac{\pi}{H_{n,r}}}\left[1-e^{-H_{n,r}\theta_o{}^2/2} - \frac{K_5\,n^{1/15}}{\sqrt{\pi}\sqrt{H_{n,r}}} - 2\sqrt{\pi}\sqrt{H_{n,r}}\,e^{-K_6 n^{1/15}}\right]
\end{aligned}$$

Using (7.26), (7.27) and (7.28) this yields

$$\sqrt{\frac{H_{n,r}}{\pi}}\,\frac{J_0(\alpha_r,n,\kappa)}{P^{n,0}(e^{\alpha_r})} \geq \left[1-e^{-K_9 n^{1/15}/2} - \frac{K_5\,n^{1/15}}{\sqrt{\pi}\sqrt{K_9 n^{1/4}}} - 2\sqrt{\pi}\sqrt{K_8 n}\,e^{-K_6 n^{1/15}}\right]$$

Since the three last terms in right-hand side of the preceding expression tend to 0 as n gets large, it is clear that there is a number $M_1 > K_7$ such that the right-hand side of this inequality is strictly positive for all $n \geq M_1$. It follows that $J_0(\alpha_r,n,\kappa) > 0$ whenever $n \geq M_1$.

We continue, assuming $n \geq M_1$, and assuming that $n \geq (2/K_9)^{15}$, so that $K_9 n^{1/15} \geq 2$, and (7.26) guarantees that $H_{n,r}\theta_o{}^2 \geq 2$. This allows us to apply (7.22) so that, using (7.24), the ineqality (7.23) gives

$$\begin{aligned}
\frac{J_2(\alpha_r,n,\kappa)}{P^{n,0}(e^{\alpha_r})} &\geq \int_{-\theta_o}^{\theta_o}\theta^2\,e^{-H_{n,r}\theta^2}\,d\theta - K_5\,n^{1/15}\int_{-\theta_o}^{\theta_o}|\theta|^3\,e^{-H_{n,r}\theta^2}\,d\theta - \pi^3\,e^{-K_6 n^{1/15}} \\
&\geq \frac{\sqrt{\pi}}{6H_{n,r}^{3/2}} - K_5\,n^{1/15}\,\frac{1}{H_{n,r}^2} - \pi^3\,e^{-K_6 n^{1/15}}
\end{aligned}$$

Thus using (7.27) and (7.28) we obtain

$$\begin{aligned}
\frac{H_{n,r}^{3/2}\,J_2(\alpha_r,n,\kappa)}{P^{n,0}(e^{\alpha_r})} &\geq \frac{\sqrt{\pi}}{6} - K_5\,n^{1/15}\,\frac{1}{\sqrt{H_{n,r}}} - \pi^3\,H_{n,r}^{3/2}\,e^{-K_6 n^{1/15}} \\
&\geq \frac{\sqrt{\pi}}{6} - K_5\,n^{1/15}\,\frac{1}{\sqrt{K_9 n^{1/4}}} - \pi^3\,(K_8 n)^{3/2}\,e^{-K_6 n^{1/15}}
\end{aligned}$$

As the last two terms on the right-hand side of the preceding expression tend to 0 as n gets large, it is clear that there is a number M greater than both M_1 and $(2/K_9)^{15}$ such that the right-hand side of this inequality is strictly positive for all $n \geq M_2$. It follows that $J_2(\alpha_r,n,\kappa) > 0$ whenever $n \geq M$. ∎

We are ready to conclude strong unimodality at coefficients in the central region of $P^{n,0}$:

Theorem 7.14: Suppose that the sequence $\{P_i\}$ in $\mathbf{R}[x]^+$ satisfies (7.4) and (7.5). Then there is a constant $M > 0$ such that whenever n and m are integers with $n \geq M$ and $n^{1/4} \leq m \leq \frac{3}{4}\deg P^{n,0}$, then

$$\left[\langle P^{n,0},\,x^m\rangle\right]^2 > \langle P^{n,0},\,x^{m-1}\rangle\,\langle P^{n,0},\,x^{m+1}\rangle > 0.$$

Proof: Take M as in Lemma 7.13. Let $r = \frac{m}{\delta_n}$ and note that $r \in \left[\frac{n^{1/4}}{\delta_n}, \frac{3}{4}\right]$. By Lemma 7.13, $J_0(\alpha_r, n, k) > 0$ and $J_2(\alpha_r, n, k) > 0$ for all k in $[m-2, m+2]$. Comparing equations (7.6) and (7.7) we note that

$$a_n(\alpha, k) = \frac{e^{-\alpha k}}{2\pi} J_0(\alpha, n, k), \qquad \forall \alpha \in \mathbf{R}. \tag{7.29}$$

In particular, $a_n(\alpha_r, k) > 0$ for all $k \in [m-2, m+2]$, and so, by Proposition 7.5,

$$\left\langle P^{n,0}, x^k \right\rangle = a_n(\alpha_r, k) > 0, \qquad \text{for } k = m-1, m, m+1. \tag{7.30}$$

We now fix $\alpha = \alpha_r$. Applying Leibnitz's Rule (see Lemma A.5) to (7.29), we obtain

$$\begin{aligned}\frac{\partial}{\partial k} J_0(\alpha, n, k) &= \int_{-\pi}^{\pi} \frac{\partial}{\partial k}\left[P^{n,0}(e^{\alpha+i\theta})\, e^{-ik\theta}\right] d\theta \;=\; \int_{-\pi}^{\pi} P^{n,0}(e^{\alpha+i\theta})\, e^{-ik\theta}(-i\theta)\, d\theta \\ &= -i\, J_1(\alpha, n, k).\end{aligned}$$

Likewise,

$$\frac{\partial}{\partial k} J_1(\alpha, n, k) = -i\, J_2(\alpha, n, k).$$

Since $a_n(\alpha, k) > 0$ for $k \in [m-2, m+2]$, it follows that $\log a_n(\alpha, k)$ is defined, and is differentiable, for k in this interval.

$$\begin{aligned}\frac{\partial}{\partial k} \log a_n(\alpha, k) &= \frac{2\pi}{e^{-\alpha k} J_0(\alpha, n, k)} \, \frac{e^{-\alpha k}(-\alpha) J_0(\alpha, n, k) - i e^{-\alpha k} J_1(\alpha, n, k)}{2\pi} \\ &= -\alpha - \frac{i J_1(\alpha, n, k)}{J_0(\alpha, n, k)}\end{aligned}$$

and

$$\frac{\partial^2}{\partial k^2} \log a_n(\alpha, k) = -\frac{J_2(\alpha, n, k) J_0(\alpha, n, k) - J_1(\alpha, n, k) J_1(\alpha, n, k)}{\left[J_0(\alpha, n, k)\right]^2}.$$

Since, by Lemma 7.6, J_0 and J_2 are real, and J_1 is pure imaginary, it follows that $\frac{\partial}{\partial k}[a_n(\alpha, k)]$ and $\frac{\partial^2}{\partial k^2}[a_n(\alpha, k)]$ are both real, and that

$$\frac{\partial^2}{\partial k^2} \log a_n(\alpha, k) < 0 \iff \left[J_1(\alpha, n, k)\right]^2 < J_0(\alpha, n, k) J_2(\alpha, n, k).$$

Since J_1 is pure imaginary, $J_1^2(\alpha_k, n, k) \leq 0$. As $J_0(\alpha, n, k) > 0$ and $J_2(\alpha, n, k) > 0$ for $k \in [m-2, m+2]$, we conclude that $\frac{\partial^2}{\partial k^2} \log a_n(\alpha, k) < 0$ for k in this interval. We have shown that for sufficiently large n, $J_0(\alpha_k, n, k)$ and $J_2(\alpha_k, n, k)$ are strictly positive, provided $k \in [n^{1/4}, \frac{3dn}{4}]$. Thus, for k in this range, if $a_n(\alpha_k, k)$ is non-zero, then $\frac{\partial^2}{\partial k^2} \log a_n(\alpha, k) < 0$. This 'log-concavity' is all we need, since, by the mean value theorem we obtain

$$\log a_n(\alpha, m+1) - \log a_n(\alpha, m) < \log a_n(\alpha, m) - \log a_n(\alpha, m-1).$$

This yields $\frac{a_n(\alpha, m+1)}{a_n(\alpha, m)} < \frac{a_n(\alpha, m)}{a_n(\alpha, m-1)}$, and by (7.30) we are done. ∎

We now return to the overall problem.

Overall Strong Unimodality Result. Collecting together the strong unimodality results for the ends and central region of $P^{n,0}$, we obtain the following:

Theorem 7.15: Suppose that for each j in $\mathbf{N}$, P_j is a polynomial of degree d_j in $\mathbf{R}[x]^+$ and that $P_j(1) = 1$. Suppose also that

$$\sup \{\, d_j \mid j \in \mathbf{N} \,\} \text{ is finite}$$

and that

$$\inf \{\, \langle P_j, x^k \rangle \mid j \in \mathbf{N},\ k \in \{0, 1, d_j - 1, d_j\} \,\} > 0.$$

Then the product $P_1 P_2 \cdots P_n$ is strongly unimodal for all sufficiently large n.

Proof: Let δ denote the indicated infimum. For each P_j we let Q_j be defined by

$$\langle Q_j, x^k \rangle = \langle P_j, x^{d_j - k} \rangle, \qquad \forall k \in \mathbf{Z}.$$

Applying Theorem 7.4, we choose M_o so large that $M_o^{1/4} < L(\delta)\, M_o^{2/7} - 1$. We apply Theorem 7.14 twice to obtain numbers M_1 and M_2 such that $P^{n,0}$ and $Q^{n,0}$ are strongly positive on the appropriate region whenever $n \geq M_1$ and $n \geq M_2$, respectively. Taking $N = \max \{M_o, M_1, M_2\}$, we see that $P^{n,0}$ is strongly unimodal whenever $n \geq N$. ∎

A. Appendix

This appendix contains a number of technical results used at various points in the thesis. It also contains a section dealing with how perturbations affect the fluctuation of a product.

The constant in the upper bound of the following lemma is not particularly sharp, but is sufficient for our purposes. A sharper bound would immediately improve the bound given in Theorem 2.6.

Lemma A.1: For every $k \in \mathbf{N}$,

$$\binom{k}{\lfloor k/2 \rfloor} \leq \frac{2^{k+1}}{\sqrt{k\pi}},$$

where $\lfloor k/2 \rfloor$ is the largest integer less than or equal to $k/2$.

Proof: First verify directly that the equation holds for $k = 1, 2$ and 3. We now show that $\binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{m\pi}}$ for all $m = 2, 3, \ldots$. Stirling's formula (p. 253 in [St]) says that for n any integer greater than 1,

$$\sqrt{2\pi n}\, e^{-n} n^n \exp\left(\frac{1}{12n + 1/4}\right) < n! < \sqrt{2\pi n}\, e^{-n} n^n \exp\left(\frac{1}{12n}\right).$$

Thus for $m \geq 2$,

$$\binom{2m}{m} = \frac{(2m)!}{m!m!} \leq \frac{\sqrt{2\pi(2m)}\, e^{-2m}(2m)^{2m} \exp\left[\frac{1}{12(2m)}\right]}{\left(\sqrt{2\pi m} e^{-m} m^m \exp\left[\frac{1}{12m+1/4}\right]\right)^2} = \frac{2\sqrt{\pi m}\, e^{-2m}(2m)^{2m} \exp\left[\frac{1}{24m}\right]}{2\pi m e^{-2m} m^{2m} \exp\left[\frac{2}{12m+1/4}\right]}$$

$$= \frac{2^{2m}}{\sqrt{m\pi}} \exp\left[\frac{1}{24m} - \frac{2}{12m + 1/4}\right]$$

But $\frac{1}{24m} - \frac{2}{12m+1/4} \leq 0 \iff \frac{1}{24m} \leq \frac{2}{12m+1/4} \iff 12m + 1/4 \leq 48m \iff 1/4 < 36m$, which certainly holds for $m \geq 2$. Thus $\binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{m\pi}} e^0 = \frac{2^{2m}}{\sqrt{m\pi}}$. Noting that

$$\frac{\binom{2m+1}{m}}{\binom{2m}{m}} = \frac{\frac{(2m+1)!}{m!(m+1)!}}{\frac{(2m)!}{m!m!}} = \frac{2m+1}{m+1} < 2,$$

we obtain, for $m = 2, 3, \ldots$,

$$\binom{2m+1}{m} \leq 2\binom{2m}{m} \leq 2\frac{2^{2m}}{\sqrt{m\pi}} = \frac{2^{2m+1}}{\sqrt{m\pi}}.$$

If k is an integer greater than 3, then either $k = 2m$ or $k = 2m + 1$, for some $m \geq 2$. In the former case we have

$$\binom{k}{\lfloor k/2 \rfloor} = \binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{m\pi}} = \frac{2^k}{\sqrt{\frac{k}{2}\pi}} = \frac{2^{k+1/2}}{\sqrt{k\pi}} \leq \frac{2^{k+1}}{\sqrt{k\pi}}.$$

In the latter case,

$$\binom{k}{\lfloor k/2 \rfloor} = \binom{2m+1}{m} \leq \frac{2^{2m+1}}{\sqrt{m\pi}} = \frac{2^k}{\sqrt{\frac{k-1}{2}\pi}} = \frac{2^k\sqrt{2}}{\sqrt{k\pi}}\sqrt{\frac{k}{k-1}} < \frac{2^{k+1}}{\sqrt{k\pi}}.$$

∎

The following simple lemma is used in the proof of Lemma 5.3.

Lemma A.2: Given numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $[0, \infty)$,

$$1 + \sum_{i=1}^{n} \alpha_i \leq \prod_{i=1}^{n} (1 + \alpha_i) \leq \exp\left(\sum_{i=1}^{n} \alpha_i\right)$$

Proof: The left-hand inequality is clear. As $\ln(1 + \alpha_i) \leq \alpha_i$, it follows that

$$\ln\left[\prod_{i=1}^{n}(1 + \alpha_i)\right] = \sum_{i=1}^{n} \ln(1 + \alpha_i) \leq \sum_{i=1}^{n} \alpha_i,$$

from which the lemma follows. ∎

The following result and its proof are a special case of theorem 3.15 of Petrov [P], and have been adapted to the present context. It is used in proving Proposition 2.3.

Theorem A.3: Suppose $Q_1, Q_2, \ldots, Q_n \in \mathbf{R}[x^{\pm 1}]^{+} \backslash \{0\}$ satisfy $Q(1) = 1$ for $j = 1, 2, \ldots, n$, and that there exist $S, T \in \mathbf{R} \cup \{-\infty, \infty\}$ with $S \leq 0 \leq T$ such that for $j = 1, 2, \ldots, n$ there exist constants $\alpha_j \in (0, \infty)$ and $\beta_j \in \mathbf{R}$ with

$$e^{-\beta_j t} Q_j(e^t) \leq e^{\alpha_j t^2/2}, \quad \forall t \in [S, T] \cap \mathbf{R}.$$

Then if we let $A = \sum_{i=1}^{n} \alpha_i$, $B = \sum_{i=1}^{n} \beta_i$ and $P = Q_1 Q_2 \cdots Q_n$, we have, for $r \in \mathbf{R}$,

$$\sum_{i \geq r+B} \langle P, x^i \rangle \leq e^{-r^2/2A}, \quad \text{if } 0 \leq r \leq AT$$

$$\sum_{i \geq r+B} \langle P, x^i \rangle \leq e^{-Tr/2}, \quad \text{if } r \geq AT$$

$$\sum_{i \leq r+B} \langle P, x^i \rangle \leq e^{-r^2/2A}, \quad \text{if } AS \leq r \leq 0$$

$$\sum_{i \leq r+B} \langle P, x^i \rangle \leq e^{-Sr/2}, \quad \text{if } r \leq AS.$$

Proof: Whenever $t \in [0, T] \cap \mathbf{R}$ and $i \geq r + B$, we have $e^{t(i-(r+B))} \geq 1$, and so

$$\begin{aligned}
\sum_{i \geq r+B} \langle P, x^i \rangle &\leq \sum_{i \geq r+B} \langle P, x^i \rangle e^{t(i-(r+B))} = e^{-t(r+B)} \sum_{i \geq r+B} \langle P, x^i \rangle e^{ti} \\
&\leq e^{-tr} e^{-tB} \sum_{i \in \operatorname{Log} P} \langle P, x^i \rangle (e^t)^i = e^{-tr} \exp\left(-t \textstyle\sum_{i=1}^{n} \beta_i\right) P(e^t) \\
&= e^{-tr} \left(\prod_{i=1}^{n} e^{-t\beta_i}\right) \left(\prod_{i=1}^{n} Q_i(e^t)\right) = e^{-tr} \prod_{i=1}^{n} e^{-t\beta_i} Q_i(e^t) \\
&\leq e^{-tr} \prod_{i=1}^{n} e^{\alpha_i t^2/2} = e^{-tr} e^{At^2/2} = e^{-tr + At^2/2}.
\end{aligned}$$

We wish to choose t in $[0, T] \cap \mathbf{R}$ in such a way as to minimize this upper bound. Since $A > 0$, the function $-tr + At^2/2$ has minimum value when $-r + At = 0$, that is, when $t = r/A$. If $0 \leq r \leq AT$, then $0 \leq \frac{r}{A} \leq T$, so our bound is minimized when $t = r/A$, and we have $\sum_{i\geq r+B}\langle P, x^i\rangle \leq e^{-(r/A)r+A(r/A)^2/2} = e^{-r^2/2A}$. If $r \geq AT$, then $t \leq T \leq \frac{r}{A}$, so $-tr + At^2/2$ is minimized for $t = T$, and we get $\sum_{i\geq r+B}\langle P, x^i\rangle \leq e^{-Tr+AT^2/2} \leq e^{-Tr+rT/2} = e^{-Tr/2}$.

To prove the second pair of inequalities, define, for $j = 1, 2\dots, n$ the polynomial $R_j \in \mathbf{R}[x^{\pm 1}]^+\setminus\{0\}$ by $R_j(x) = Q_j(x^{-1})$, and set $\gamma_j = -\beta_j$. Define also $U = R_1R_2\cdots R_n$. Note that for $t \in [-T, -S]\cap \mathbf{R}$ we have $e^{-\gamma_j t}R_j(e^t) = e^{-\beta_j(-t)}Q_j(e^{-t}) \leq e^{\alpha_j(-t)^2/2} = e^{\alpha_j t^2/2}$. Applying the first part of the proof we obtain the inequalities

$$\sum_{i\geq r-B}\langle U, x^i\rangle \leq e^{-r^2/2A}, \qquad \text{for } 0 \leq r \leq A(-S),$$

and

$$\sum_{i\geq r-B}\langle U, x^i\rangle \leq e^{-(-S)r/2}, \qquad \text{for } r \geq A(-S).$$

Noting that for each $i \in \mathbf{Z}$, $\langle U, x^i\rangle = \langle P, x^{-i}\rangle$, and letting $s = -r$, we obtain

$$\sum_{j\leq s+B}\langle P, x^j\rangle = \sum_{-i\leq s+B}\langle P, x^{-i}\rangle = \sum_{i\geq -s-B}\langle U, x^i\rangle = \sum_{i\geq r-B}\langle U, x^i\rangle.$$

Thus if $AS \leq s \leq 0$, then $0 \leq r \leq A(-S)$, so $\sum_{j\leq s+B}\langle P, x^j\rangle \leq e^{-r^2/2A} = e^{-s^2/2A}$, and if $s \leq AS$, then $r \geq A(-S)$, so $\sum_{j\leq s+B}\langle P, x^j\rangle \leq e^{Sr/2} = e^{-Ss/2}$. ∎

We now furnish some technical results needed in Chapter 7.

Lemma A.4: Suppose X is a topological space, and $f : X\times T \to \mathbf{C}$ is continuous, where T is a compact subset of $\mathbf{R}$. Then the function $g : X \to \mathbf{C}$ defined by $x \mapsto \int_T f(x,t)\,dt$ is well-defined and continuous.

Proof: Assume that the measure of T, $\mu(T)$, is non-zero. Let $\epsilon > 0$ and $x \in X$ be given. For each $t \in T$ there exists a basic neighborhood $U_t \times V_t$ of (x,t) in $X\times T$ such that

$$|f(x,t) - f(x',t')| < \epsilon/\mu(T) \qquad ,\forall (x',t') \in U_t\times V_t.$$

By compactness of T, we have $t_1, t_2, \dots, t_k$ such that $\bigcup_{i=1}^k V_{t_i} = T$. Let $U = \bigcap_{i=1}^k U_{t_i}$. Then if $x' \in U$, for each $t \in T$ we have $t \in V_{t_i}$ for some i, whence $|f(x',t) - f(x,t)| < \epsilon/\mu(T)$. Therefore

$$\left|\int_T f(x',t)dt - \int_T f(x,t)dt\right| = \left|\int_T [f(x',t) - f(x,t)]dt\right| \leq \int_T |f(x',t) - f(x,t)|dt$$
$$< \int_T \epsilon/\mu(T)\,dt = \epsilon.$$

That is, $|g(x') - g(x)| < \epsilon$. ∎

Lemma A.5: Let $\phi : [a,b]\times[c,d] \to \mathbf{C}$ be a function such that $\frac{\partial^m\phi}{\partial t^m}$ exists and is continuous on $[a,b]\times[c,d]$ for all $m \in \mathbf{N}$, then

$$\int_a^b \phi(s,t)\,ds$$

is a C^∞ function on $[c,d]$.

Proof: Recall Leibnitz's Rule (page 68 in [Co]) :
If $\phi : [a,b] \times [c,d] \to \mathbf{C}$ is continuous, and $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$,

$$\int_a^b \phi(s,t)\, ds$$

is a continuously differentiable function of t, and

$$\frac{\partial}{\partial t} \int_a^b \phi(s,t)\, ds = \int_a^b \frac{\partial}{\partial t} \phi(s,t)\, ds.$$

Repeated application of Leibnitz's Rule proves our result. ∎

We now state a version of Taylor's theorem which holds for complex functions of a real variable, and provides a useful bound on the truncation error:

Proposition A.6: Suppose that $f : [s,t] \to \mathbf{C}$ is $n+1$ times differentiable on $[s,t]$, and that $x, a \in [s,t]$, and let $R_{n,a}(x)$ be defined by

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x).$$

Then

$$|R_{n,a}(x)| \leq 2 \sup_{\tau \in [s,t]} \left| \frac{f^{(n+1)}(\tau)}{(n+1)!}(x-a)^{n+1} \right|$$

Proof: Let $g = \operatorname{Re} f$ and $h = \operatorname{Im} f$. Then g and h are each $n+1$ times differentiable on $[s,t]$, and $f^{(k)} = g^{(k)} + ih^{(k)}$ for $k = 0, 1, 2, \ldots, n+1$. Taylor's theorem for real variables (see [Sp], pp. 291-292) gives the equations

$$g(x) = g(a) + g'(a)(x-a) + \cdots + \frac{g^{(n)}(a)}{n!}(x-a)^n + \frac{g^{(n+1)}(t_1)}{(n+1)!}(x-a)^{n+1}$$

$$h(x) = h(a) + h'(a)(x-a) + \cdots + \frac{h^{(n)}(a)}{n!}(x-a)^n + \frac{h^{(n+1)}(t_2)}{(n+1)!}(x-a)^{n+1}$$

where t_1 and t_2 are in $[s,t]$ and depend on x. Thus,

$$R_{n,a}(x) = \frac{g^{(n+1)}(t_1) + ih^{(n+1)}(t_2)}{(n+1)!}(x-a)^{n+1},$$

and the result follows. ∎

Fluctuation and Perturbations.

Lemma A.7: Suppose that $\{P_j\}_{j \in \mathbf{N}}$ and $\{R_j\}_{j \in \mathbf{N}}$ are sequences in $\mathbf{R}[x_i^{\pm 1}]_{i \in I}^{+}$ and that $\sup_j \frac{\|R_j\|}{\|P_j\|} < \infty$. Then for any $w \in \oplus_{i \in I} \mathbf{Z}$,

$$\sum_{j=1}^{\infty} \mathcal{P}_w(P_j) = \infty \implies \sum_{j=1}^{\infty} \mathcal{P}_w(P_j + R_j) = \infty.$$

Proof: Let $\alpha = \sup_j \frac{\|R_j\|}{\|P_j\|}$. Recall from property (5) of persistence (page 18) that $\mathcal{P}_w(P_j + R_j) \geq \mathcal{P}(P_j) \frac{\|P_j\|}{\|P_j\| + \|R_j\|}$. Therefore $\mathcal{P}_w(P_j + R_j) \geq \mathcal{P}_w(P_j) \frac{1}{1 + \frac{\|R_j\|}{\|P_j\|}} \geq \mathcal{P}_w(P_j) \frac{1}{1+\alpha}$. Divergence of $\sum_j \mathcal{P}_w(P_j)$ thus implies divergence of $\sum_j \mathcal{P}_w(P_j + R_j)$, as claimed. ∎

Lemma A.7 shows that if $\sum_i \mathcal{P}(P_i) = \infty$, then the same is true for any ℓ^∞ *positive* perturbation of $\{P_i\}$. If we admit negative as well as positive perturbations, this result no longer holds, as may be seen by perturbing $P_i = 1 + \frac{1}{i}x$ to the constant sequence whose every term is 1, and noting that the resulting sequence is no longer a zero-fluctuation sequence. On the other hand, ℓ^1 perturbations transform zero-fluctuation sequences into zero-fluctuation sequences, as we shall see.

Define a relation $\sim$ on the sequences in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}^+$ having only finitely many zero polynomials by

$$\{P_j\} \sim \{Q_j\} \iff \text{a tail of } \left\{\frac{\|P_j - Q_j\|}{\|P_j\|}\right\}_{j\in\mathbf{N}} \text{ is } \ell^1 \tag{A.1}$$

(A sequence $\{\alpha_j\}_{j\in\mathbf{N}}$ in $\mathbf{R}$ is ℓ^1 if $\sum_{j\in\mathbf{N}} |\alpha_j| < \infty$).

Lemma A.8: The relation $\sim$ defined by (A.1) is an equivalence relation.

Proof: Clearly $\sim$ is reflexive. Now suppose $\{P_j\} \sim \{Q_j\}$. Then there exists m in $\mathbf{N}$ such that $\frac{\|P_j-Q_j\|}{\|P_j\|} < \frac{1}{2}$, and $Q_j \neq 0\ \forall j \geq m$. Then for $j \geq m$, $\|P_j\| = \|(P_j - Q_j) + Q_j\| \leq \|P_j - Q_j\| + \|Q_j\|$, so $1 \leq \frac{\|P_j-Q_j\|}{\|P_j\|} + \frac{\|Q_j\|}{\|P_j\|}$. Thus $\frac{\|Q_j\|}{\|P_j\|} > \frac{1}{2}$, so $\frac{1}{2}\frac{\|P_j\|}{\|Q_j\|} < 1$. Therefore for $j \geq m$,

$$\frac{\|P_j - Q_j\|}{\|P_j\|} \geq \frac{1}{2}\frac{\|P_j\|}{\|Q_j\|}\frac{\|P_j - Q_j\|}{\|P_j\|} = \frac{1}{2}\frac{\|P_j - Q_j\|}{\|Q_j\|} = \frac{1}{2}\frac{\|Q_j - P_j\|}{\|Q_j\|}$$

Thus a tail of $\left\{\frac{\|Q_j-P_j\|}{\|Q_j\|}\right\}_{j\in\mathbf{N}}$ is ℓ^1, so $\{Q_j\} \sim \{P_j\}$, and $\sim$ is symmetric.

Now suppose $\{P_j\} \sim \{Q_j\}$ and $\{Q_j\} \sim \{R_j\}$. By symmetry $\{Q_j\} \sim \{P_j\}$, that is, a tail of $\left\{\frac{\|P_j-Q_j\|}{\|Q_j\|}\right\}$ is ℓ^1. Note that for j is sufficiently large, $P_j \neq 0$ and

$$\frac{\|P_j - R_j\|}{\|P_j\|} = \frac{\|(P_j - Q_j) + (Q_j - R_j)\|}{\|P_j\|} \leq \frac{\|P_j - Q_j\|}{\|P_j\|} + \frac{\|Q_j - R_j\|}{\|P_j\|}$$

Since there exists an m in $\mathbf{N}$ such that $\frac{\|P_j\|}{\|Q_j\|} > \frac{1}{2}, \forall j \geq m$, it follows that for j sufficiently large,

$$\begin{aligned}\frac{\|P_j - R_j\|}{\|P_j\|} \leq \frac{\|P_j - Q_j\|}{\|P_j\|} + \frac{\|Q_j - R_j\|}{\|P_j\|} &\leq 2\frac{\|P_j\|}{\|Q_j\|}\left[\frac{\|P_j - Q_j\|}{\|P_j\|} + \frac{\|Q_j - R_j\|}{\|P_j\|}\right] \\ &= 2\frac{\|P_j - Q_j\|}{\|Q_j\|} + 2\frac{\|Q_j - R_j\|}{\|Q_j\|}\end{aligned}$$

Since tails of both $\left\{\frac{\|P_j-Q_j\|}{\|Q_j\|}\right\}$ and $\left\{\frac{\|Q_j-R_j\|}{\|Q_j\|}\right\}$ are ℓ^1, it follows that a tail of $\left\{\frac{\|P_j-R_j\|}{\|P_j\|}\right\}$ is ℓ^1. Thus $\sim$ is transitive. ∎

Proposition A.9: If $\{P_j\}_{j\in\mathbf{N}}$ and $\{R_j\}_{j\in\mathbf{N}}$ are sequences in $\mathbf{R}[x_i^{\pm 1}]_{i\in I}^+$ containing no zero polynomials, and $\{P_j\} \sim \{R_j\}$, then for any $w \in \oplus_{i\in I}\mathbf{Z}$,

$$\sum_j \mathcal{P}_w(P_j) = \infty \quad \iff \quad \sum_j \mathcal{P}_w(R_j) = \infty.$$

Proof: By the symmetry of the equivalence relation $\sim$, it suffices to show that $\sum_j \mathcal{P}_w(P_j) = \infty$ implies $\sum_j \mathcal{P}_w(R_j) = \infty$.

Now rescale both P_j and R_j by multiplying by $\frac{1}{\|P_j\|}$, so we may assume that $\|P_j\| = 1, \forall j$.

Letting $G = \oplus_{i \in I} \mathbf{Z}$, we define a new sequence $\{S_j\} \subset \mathbf{R}[x_i^{\pm 1}]_{i \in I}^+$ by setting

$$\langle S_j, x^u \rangle = \min\left(\langle P_j, x^u \rangle, \langle R_j, x^u \rangle \right), \qquad \forall u \in G,\ j \in \mathbf{N}.$$

Now as $P_j - S_j \geq 0$ for each j,

$$\|P_j - S_j\| = \sum_{u \in G} \langle P_j - S_j, x^u \rangle \ \leq \sum_{u \in G} \left| \langle P_j - R_j, x^u \rangle \right| = \|P_j - R_j\|,$$

so $\sum_j \|P_j - S_j\|$ is finite (as $\{P_j\} \sim \{R_j\}$ implies $\sum_j \|P_j - R_j\| < \infty$). This implies that at most finitely many S_j are zero, and so we may write $\{P_j\} \sim \{S_j\}$, and by symmetry of $\sim$, $\{S_j\} \sim \{P_j\}$, so a tail of $\left\{\frac{\|P_j - S_j\|}{\|S_j\|}\right\}$ is ℓ^1. Take $j_o \in \mathbf{N}$ such that $S_j \neq 0$ for all $j \geq j_o$, and note that by property (6) of persistence (page 18) we obtain

$$\mathcal{P}_w(S_j) \geq \mathcal{P}_w(P_j) - 2\,\frac{\|P_j - S_j\|}{\|S_j\|}, \qquad \forall j \geq j_o,$$

and so

$$\sum_{j=j_o}^{\infty} \mathcal{P}_w(P_j) \leq \sum_{j=j_o}^{\infty} \left[\mathcal{P}_w(S_j) + 2\,\frac{\|P_j - S_j\|}{\|S_j\|} \right].$$

As $\sum_{j=j_o}^{\infty} \frac{\|P_j - S_j\|}{\|S_j\|} < \infty$ and $\sum_{j=j_o}^{\infty} \mathcal{P}_w(P_j) = \infty$, it follows that $\sum_{j=j_o}^{\infty} \mathcal{P}_w(S_j) = \infty$.

By Proposition 2.1 (8) we have that $\|P_j\| = \|(P_j - S_j) + S_j\| = \|P_j - S_j\| + \|S_j\|$. As $\{P_j\} \sim \{S_j\}$ this gives $\sum_j \|P_j\| - \|S_j\| = \sum_j \|P_j - S_j\| < \infty$. As $\|P_j\| = 1$ for all $j \in \mathbf{N}$, there exists a $j_1 \geq j_o$ such that $\|S_j\| > \frac{1}{2}$ for all $j \geq j_1$. As $\|R_j\| \leq \|R_j - P_j\| + \|P_j\| = \|R_j - P_j\| + 1$ and $\sum_j \|R_j - P_j\| < \infty$, it follows that there is a $j_2 \geq j_1$ such that $\|R_j\| < 2$ for all $j \geq j_2$. For $j \geq j_2$ we have

$$\frac{\|R_j - S_j\|}{\|S_j\|} \leq \frac{\|R_j\| + \|S_j\|}{\|S_j\|} < \frac{2+1}{\frac{1}{2}} = 6.$$

Applying Lemma A.7 to the tails $\{R_j - S_j\}_{j \geq j_2}$ and $\{S_j\}_{j \geq j_2}$, we conclude that

$$\sum_{j=j_2}^{\infty} \mathcal{P}_w(S_j + (R_j - S_j)) = \infty,$$

so $\sum_{j=1}^{\infty} \mathcal{P}_w(R_j) = \infty$. ∎

Lemma A.10: Suppose $\{a_n\}_{n \in \mathbf{N}}$ is a sequence in $[0, \infty)$, and $\sum_{n=1}^{\infty} a_n$ converges. If $m_1 < m_2 < m_3 < \cdots$ is any sequence in $\mathbf{N}$, then

$$\sum_{i=1}^{\infty} \left(\prod_{j=m_i}^{m_{i+1}-1} (1 + a_j) \ - 1 \right) \leq \prod_{j=m_1}^{\infty} (1 + a_n) \ - 1 < \infty.$$

Proof: As $\sum_{i=1}^{\infty} \alpha_i$ converges, it follows that $\prod_{i=1}^{\infty}(1 + a_i) < \infty$ by Theorem 7.28 in [St]. Note that for any $i \in \mathbf{N}$, if we set $A = \prod_{j=m_i}^{m_{i+1}-1}(1 + a_j) \ - 1$ and $B = \prod_{j=m_{i+1}}^{m_{i+2}-1}(1 + a_j) \ - 1$, then the inequality $(1 + A)(1 + B) - 1 = A + B + AB \geq A + B$ may be rewritten as

$$\left(\prod_{j=m_i}^{m_{i+1}-1} (1 + a_j) \prod_{j=m_{i+1}}^{m_{i+2}-1} (1 + a_j) \right) - 1 \geq \left(\prod_{j=m_i}^{m_{i+1}-1} (1 + a_j) \ - 1 \right) + \left(\prod_{j=m_{i+1}}^{m_{i+2}-1} (1 + a_j) \ - 1 \right).$$

Applying this argument repeatedly, we have that for any M,

$$\left(\prod_{i=1}^{M}\prod_{j=m_i}^{m_{i+1}-1}(1+a_j)\right)-1\geq\sum_{i=1}^{M}\left(\prod_{j=m_i}^{m_{i+1}-1}(1+a_j)\ -1\right).$$

Letting $M\rightarrow\infty$ completes the proof of the claim. ∎

Theorem A.11: If $\{P_j\}_{j\in\mathbf{N}}$ and $\{R_j\}_{j\in\mathbf{N}}$ are sequences in $\mathbf{R}[x_i^{\pm1}]_{i\in I}^{+}$ containing no zero polynomials, and $\{P_j\}\sim\{R_j\}$, then for any w in $G=\oplus_{i\in I}\mathbf{Z}$,

$$\lim_{n\rightarrow\infty}\mathcal{F}_w(P^{N,t})=0,\ \forall t\in\mathbf{Z}^+\quad\Longleftrightarrow\quad\lim_{n\rightarrow\infty}\mathcal{F}_w(R^{N,t})=0,\ \forall t\in\mathbf{Z}^+.$$

Proof: By the symmetry of the equivalence relation $\sim$, it suffices to show that if $\lim\limits_{n\rightarrow\infty}\mathcal{F}_w(P^{N,t})=0,\ \forall t\in\mathbf{Z}^+$, then $\lim\limits_{n\rightarrow\infty}\mathcal{F}_w(R^{N,t})=0,\ \forall t\in\mathbf{Z}^+$.

We rescale each P_j and R_j by multiplying by $\frac{1}{\|P_j\|}$, so we may assume $\|P_j\|=1,\ \forall j$.

By Theorem 2.10, $\{P_j\}$ has a blocking $\{Q_j\}$ such that $\sum_{j=1}^{\infty}\mathcal{P}_w(Q_j)=\infty$. Define a sequence $\{S_j\}$ in $\mathbf{R}[x_i^{\pm1}]_{i\in I}^{+}$ by

$$\langle S_j,x^u\rangle=\min\left(\langle P_j,x^u\rangle,\langle R_j,x^u\rangle\right),\qquad\text{for } u\in G,\ j\in\mathbf{N}.$$

Let $\alpha_j=\|P_j\|-\|S_j\|$ for $j\in\mathbf{N}$. Now as $P_j-S_j\geq0$ for all j,

$$\|P_j-S_j\|=\sum_{u\in G}\langle P_j-S_j,x^u\rangle\ \leq\sum_{u\in G}|\langle P_j-R_j,x^u\rangle|=\|P_j-R_j\|,$$

so $\sum_j\|P_j-S_j\|$ is finite (as $\{P_j\}\sim\{R_j\}$ implies $\sum_j\|P_j-R_j\|<\infty$). By Proposition 2.1 (8) we have $\|P_j\|=\|(P_j-S_j)+S_j\|=\|P_j-S_j\|+\|S_j\|$ so $\alpha_j=\|P_j-S_j\|$, and $\sum_j\alpha_j<\infty$. Let $D_j=P_j-S_j$, $j=1,2,3,\ldots$. Then $\sum_j\|D_j\|=\sum_j\alpha_j<\infty$.

Since $\alpha_j+\|S_j\|=\|D_j\|+\|S_j\|=\|P_j\|=1$, it follows that $\|S_j\|\leq1,\ \forall j$. For any t in $\mathbf{Z}^+$ and N in $\mathbf{N}$, if we set $J(N,t)=\{t+1,t+2,\ldots,t+N\}$, then by Proposition 2.1 (8) we have

$$\begin{aligned}\|P^{N,t}\|&=\left\|\prod_{j\in J(N,t)}(S_j+D_j)\right\|=\prod_{j\in J(N,t)}(\|S_j\|+\|D_j\|)=\prod_{j\in J(N,t)}(\|S_j\|+\alpha_j)\\&=\sum_{K\subseteq J(N,t)}\left[\prod_{j\in K}\|S_j\|\right]\prod_{j\in J(N,t)\setminus K}\alpha_j\\&=\left[\prod_{j\in J(N,t)}\|S_j\|\right]+\sum_{\substack{K\subsetneq J(N,t)\\K\neq\emptyset}}\left[\prod_{j\in K}\|S_j\|\right]\prod_{j\in J(N,t)\setminus K}\alpha_j\\&\leq\left[\prod_{j\in J(N,t)}\|S_j\|\right]+\sum_{\substack{K\subsetneq J(N,t)\\K\neq\emptyset}}\prod_{j\in J(N,t)\setminus K}\alpha_j=\left\|\prod_{j\in J(N,t)}S_j\right\|+\prod_{j\in J(N,t)}(1+\alpha_j)\ -1\\&=\|S^{N,t}\|+\left(\prod_{j=t+1}^{t+N}(1+\alpha_j)\ -1\right).\end{aligned}$$

Thus, as $P^{N,t} - S^{N,t} \geq 0$,

$$\|P^{N,t} - S^{N,t}\| = \|P^{N,t}\| - \|S^{N,t}\| \leq \prod_{j=t+1}^{t+N} (1+\alpha_j) - 1.$$

As in the definition of blocking (page 6), let $1 = N_0 < N_1 < N_2 < \cdots$ be the sequence which determines the blocking $\{Q_j\}$, that is, $Q_j = \prod_{i=N_{j-1}}^{N_j-1} P_i$, for j in N. Let $\left\{\hat{Q}_j\right\}$ be the corresponding blocking of the sequence $\{S_j\}$.

We have shown that $\left\|Q_j - \hat{Q}_j\right\| \leq \prod_{k=N_{j-1}}^{N_j-1}(1+\alpha_k) - 1$. Thus, for any $M \in$ N,

$$\sum_{j=1}^{M} \left\|Q_j - \hat{Q}_j\right\| \leq \sum_{j=1}^{M} \prod_{k=N_{j-1}}^{N_j-1} (1+\alpha_k) - 1.$$

By Lemma A.10 this yields

$$\sum_{j=1}^{\infty} \left\|Q_j - \hat{Q}_j\right\| \leq \prod_{j=1}^{\infty}(1+\alpha_j) - 1 < \infty.$$

Now since $\|Q_j\| = 1$ for all j, we have shown that $\{Q_j\} \sim \left\{\hat{Q}_j\right\}$ A finite number of the $\hat{Q}_j$ may be zero, so taking tails of the sequences $\{Q_j\}$ and $\left\{\hat{Q}_j\right\}$, and applying Proposition A.9, we conclude that $\sum_j \mathcal{P}_w(\hat{Q}_j) = \infty$. By Theorem 2.10 we obtain

$$\lim_{N \to \infty} \mathcal{F}_w(S^{N,t}) = 0, \quad \forall t \in \mathrm{N}.$$

Now we rescale S_j, R_j and P_j by multiplying them by $\frac{1}{\|R_j\|}$ so that $R_j(1) = 1$ for $j = 1, 2, 3, \ldots$. Letting $\beta_j = \|R_j - S_j\| = \|R_j\| - \|S_j\|$, for $j \in$ N, we note that $\beta_j = \sum_w |\langle R_j - S_j,\ x^w\rangle| \leq \sum_w |\langle R_j - P_j,\ x^w\rangle| = \|R_j - P_j\|$, and since $\{R_j\} \sim \{P_j\}$, the series $\sum_j \beta_j$ converges. If we let $\left\{\check{Q}_j\right\}$ be the blocking of $\{R_j\}$ corresponding to the blocking $\left\{\hat{Q}_j\right\}$ of $\{S_j\}$, we can repeat an argument given above to obtain

$$\sum_{j=1}^{\infty} \left\|\check{Q}_j - \hat{Q}_j\right\| \leq \prod_{j=1}^{\infty}(1+\beta_j) - 1 < \infty.$$

This together with the fact that $\left\|\check{Q}_j\right\| = 1, \forall j$, tells us that $\left\{\check{Q}_j\right\} \sim \left\{\hat{Q}_j\right\}$. Applying Proposition A.9 to tails of the sequences $\left\{\check{Q}_j\right\}$ (we take tails to guarantee that every $\hat{Q}_j$ is non-zero) and $\left\{\hat{Q}_j\right\}$, we may use the fact that $\sum_j \mathcal{P}_w(\hat{Q}_j) = \infty$ to deduce that $\sum_j \mathcal{P}_w(\check{Q}_j)) = \infty$. This in turn implies, by Theorem 2.10, that

$$\lim_{N \to \infty} \mathcal{F}_w(R^{N,t}) = 0, \quad \forall t \in \mathrm{N}.$$

∎

B. Bibliography

[Be] E. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin, 1965.

[BH] B.M. Baker and D.E. Handelman, *Positive polynomials and time dependent integer-valued random variables*, Can. J. Math. **44**(1992), 3–41.

[Co] J.B. Conway, *Functions of one complex variable*, second edition, Springer-Verlag, New York, 1986.

[DA1] V. De Angelis, *Coefficients of large powers of a polynomial*, preprint.

[DA2] V. De Angelis, *An asymptotic expansion and* log *concavity of polynomials*, in preparation.

[KG] J. Keilson and H. Gerber, *Some Results for Discrete Unimodality*, J. Amer. Stat. Assoc. **66**(1971), 386–389.

[L] S. Lang, *Algebra*, second edition, Addison-Wesley, Menlo Park, 1984.

[H1] D. Handelman, *Positive polynomials and product type actions of compact groups*, Memoirs of the A.M.S. **320**(1985), 79 p. + xi.

[H2] D. Handelman, *Deciding eventual positivity of polynomials*, Ergodic Theory and Dynamical Systems **6**(1986), 57–79.

[H3] D. Handelman, *Positive polynomials, convex integral polytopes, and a random walk problem*, Springer-Verlag Lecture Notes in Mathematics **1282**, 1987, 142 p.

[H4] D. Handelman, *Spectral radii of primitive integral companion matrices and log concave polynomials*, Contemporary Mathematics **135**, 1992.

[Ib] I.A. Ibragimov, *On the composition of unimodal distributions*, Theory of Probability and its Applications **1**(1956), 255–260.

[McD] D.R. McDonald, *On local limit theorem for integer valued random variables*, Theory of Probability and its Applications, **34**(1979), 607–614.

[MH] D.R. McDonald and D.E. Handelman, *Fluctuation in several variables*, unpublished (originally written as an appendix to [BH]), 121–124.

[Mn] E. Meissner, *Über positive Darstellung von Polynomen*, Math. Ann. **70**(1911), 223–235.

[Mi] J. Mineka, *A criterion for tail events of independent random variables*, Z. Wahrscheinlichkeitstheorie verw. Geb. **25**(1973), 167–170.

[OR] A. Odlyzko and L. Richmond, *On the unimodality of high convolutions of discrete distributions*, Annals of Probability **13**(1985), 299–306.

[P] V.V. Petrov, *Sums of independent random variables*, Ergebnisse d. Math., Springer-Verlag, 1975.

[Pn] H. Poincaré, *Sur les équations algébriques*, C. R. Acad. Sci., Paris **97**(1883), 1418.

[Pó] G. Pólya, *Über positive Darstellung von Polynomen*, Vier. d. natur. Gesellsch. Zürich **73**(1928), 141–5.

[Sp] M. Spivak, *Calculus*, second edition, Publish or Perish, Berkeley, 1980.

[S] R.P. Stanley, *Log concave and unimodal sequences in algebra, combinatorics and geometry*, in Graph Theory and its Applications: East and West, Annals of the New York Academy of Sciences **576**(1989), 500–535.

[St] K.R. Stromberg, *Introduction to classical real analysis*, Wadsworth, Belmont, 1981.

List of Theorems (excluding minor results)

Index