

CAHIER DE RECHERCHE #2113E
DÉPARTEMENT DE SCIENCE ÉCONOMIQUE
FACULTÉ DES SCIENCES SOCIALES
UNIVERSITÉ D'OTTAWA

WORKING PAPER #2113E
DEPARTMENT OF ECONOMICS
FACULTY OF SOCIAL SCIENCES
UNIVERSITY OF OTTAWA

Charitable Giving and NPOs Investment Decision in a Stochastic Dynamic Economy

Han Jiang and Aggey Simons¹

November 2021

¹ Department of Economics, University of Ottawa hjian035@uottawa.ca and Aggey.Simons@uottawa.ca. We are grateful to Victoria Barham, Anthony Heyes, Jean-Francois Tremblay, Louis-Philippe Morin and Frances Woolley for comments and suggestions. All errors are ours.

Abstract

This paper presents a dynamic model of charitable giving. At each period, donors contribute to a non-profit organization's (NPO) endowment; the NPO provides a charitable good and invests in the financial market. Investments are made in a risky asset and a risk-free asset. To account for uncertainty, we introduce two types of shocks: donors' income shock and financial market fluctuations.

We show that the optimal share of disposable endowment, invested in risky asset, is constant. Donors' strategy, whether to contribute or free-ride on the NPO's investments, depends on donors' shadow prices. Donors contribute when NPO's endowment is relatively low. Large contribution levels encourage the NPO to participate in the capital market at the expense of providing charitable good. We show that the NPO prefers an environment with a lower rate of return on risk-free asset. NPO's risk exposure to the financial market affects both NPO's and donors' decisions. However, risk exposures on donors' side do not have any impact on parties' decisions. Regulation analysis suggests that portfolio ceiling and provision floor are achievable.

JEL Classification: H41. H49

Keywords: Voluntary Contribution, Public Goods, Charitable Investment, Risk Preference, Stochastic Differential Game, Gauss-Markov Process, Hamilton-Jacobi-Bellman Equations.

1 Introduction

The philanthropic giving market plays an increasingly more significant role in the modern economy than ever before. Particularly, there is a positive dynamics in charitable giving in the USA, which has doubled since 1995 ([List 2011](#)). Charitable individual givings by Americans to non-profit organizations (NPOs) now exceed 2 percent of GDP. In 2018 individual givings amounted to 292 billion dollars. From the supply side, research on NPOs' strategies mainly focus on fundraising activities that include direct mailing, telemarketing, face-to-face solicitations, staffing ([Kamdar et al. 2015](#), [Andreoni and Payne 2011, 2003](#)). Meanwhile, investment income contributes to a sizable proportion of the revenue of NPOs. [Figure 1](#) shows that in 2005 in the USA around 7 percent of the total revenue of NPOs came from investment income. Investment activity allows for a more stable provision of charitable goods ([Sherlock and Gravelle 2009](#), [Weisbrod 2009](#), [Song et al. 2008](#), [Scott 2006](#)). However, an important role that NPOs play in capital markets and the impact on the charitable giving market are mostly neglected ([Jegers and Verschueren 2006](#), [Bowman 2002](#), [Wedig 1994](#)).

To address the role of NPOs investments in the charitable market we need to emphasize the dynamic and uncertain environment in which NPOs and financial markets operate. Charitable and financial markets evolve over time. Donors and NPOs observe the history of giving, charitable good provision and investment strategies over time. Also, charitable givings are sensitive to economic fluctuations. Individual givings are affected by these permutations ([List 2011](#)). Other conventional funding sources, such as private payments for services, government grants, and other revenues, are also affected by these fluctuations ([Sherlock and Gravelle 2009](#)). These revenue fluctuations motivate NPOs to invest in financial markets to reduce the risk associated with reliance on conventional funding sources. Such investments create a certain degree of volatility, which will, in turn, affects the market for charitable giving ([Cettolin et al. 2017](#), [Lohse et al. 2012](#)).

This paper contributes to the literature by considering NPOs' investment activity to finance their philanthropic requirements. We model a charitable market consisting of an NPO and donors as a dynamic non-cooperative stochastic differential game. It is natural to examine macroeconomic

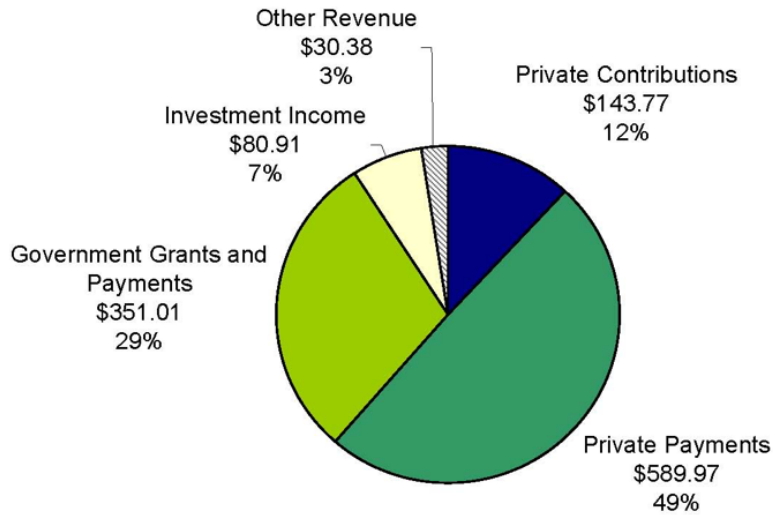


Figure 1: NPO Revenue by Source, 2005 (billions of dollars).

Source: Wing, Pollak, and Blackwood, *The Nonprofit Almanac*, 2008, and CRS calculations.

shocks (Imbs 2007, Ramey and Ramey 1994). However, to account for income disturbances and financial market volatility, we uniquely consider fluctuations of the rate of return of the risky asset and donors' incomes generated by the Itô diffusions. The NPO allocates its resources between a charitable good and investment in risk-free (for example, T-bills) and risky assets. Donors dispatch their incomes between consumption and charitable giving.

The results show that the NPO invests a constant share of disposable endowment in the risky asset. This share depends on the return on the risk free asset. If the return on risk-free asset is high, the NPO does not need to bear the risk in the financial market and it invests all disposable endowment in the risk-free asset. For the medium range of the return on the risk-free asset, the NPO invests a proper proportion to the risky asset. The impact of the rate of return of the risk-free asset on the NPO's public good provision is non-monotonic. A low rate of return on T-bills encourages the NPO to invest all of its disposable endowment in the risky asset thereby increasing its expected endowment. Donors contribute to the NPO if it does not have a sufficiently high level of endowment. An accumulation of endowment may discourage donors to contribute further. The givings by donors are discontinuous, leading to a jump of the NPO's provision of charitable goods when its endowment level exceeds the critical point. A higher contribution level by donors encourages the NPO to participate in the financial market at the expense of providing the public

good. It is interesting to note that while the investment shock affects both NPOs' and donors' decisions, income shock does not affect their decisions. We also show that the NPO exhibits risk-averse behavior, i.e., a more extensive portfolio selection requires a higher risk premium to compensate for the possible investment loss. The empirical test using a panel data of health and hospital NPOs from the National Center for Charitable Statistics (NCCS) over the period 1987-2014 supports the finding of NPOs risk-aversion.

The remainder of the paper is organized as follows: Section 2 presents a literature review. Section 3 describes the model of two types of interacting agents. We solve the model and derive a feedback Nash-equilibrium using the Hamilton-Jacobi-Bellman approach in Section 4. An empirical test is in Section 5. Section 6 summarizes the primary conclusions. All the proofs are relegated to Appendix 7.

2 Literature Review

There are two main strands in the literature on charitable giving. In the first strand, giving is considered as an individual economic decision, where donations are determined by maximizing donors' utility subject to a budget constraint. NPOs are treated as a conduit to channel funds to the desired public projects. Once NPOs receive donations, the provision of public goods is instantaneous (Varian 1994, Andreoni 1988, Bergstrom et al. 1986). The second strand of literature defines giving as the result of strategic interactions. NPOs choose fundraising efforts and mechanisms (Andreoni 2006). Both strands of literature do not consider the role of NPOs in the financial market.

Recent papers focus on the free-riding problem (Yildirim 2006, Marx and Matthews 2000) and the design of the efficient fundraising campaign to influence donors' contribution behaviors (Kamdar et al. 2015, Kumru and Vesterlund 2010, Waters 2008, Rondeau and List 2008). Efficiency is attainable in the limit if contributors punish past free riders by not contributing to the public project (Marx and Matthews 2000).

In a dynamic setting, contributions accumulating over time accentuate the free-riding problem (Fershtman and Nitzan 1991).¹ Uncertainty makes individual donors less motivated to contribute,

¹“...in dynamic context, an individual has the opportunity to learn the response of other players to his and

and the free-rider problem is exacerbated (Wang and Ewald 2010a). Marsiliani and Renstroem (2010) establish that donors' capital stocks will converge due to saving, which eventually leads donors to contribute the same amount of public good. This novel finding contrasts with the static model, where only rich people contribute (Bergstrom et al. 1986).

The existing literature on NPOs' investment decisions is sparse. To our knowledge, a few papers consider NPOs as active investors in financial markets. Jiang (2018) examines the interaction between donors' giving decisions and the NPO's investment strategy in a static and deterministic model. Donors contribute to the endowment but may not contribute to the annual campaign. Agents' behavior differs when annual campaigns are permitted or not. With the permitted annual campaign, the NPO's portfolio selection is a discontinuous, decreasing function of the endowment; donors contribute less to an aggressive NPO and more to a cautious one. Surprisingly, donors and the NPO generally disagree about whether the charity should run an annual campaign. This result complements Kamdar et al. (2015). However, this paper does not incorporate random shocks in the financial market and donors saving behavior to hedge against the income shocks.

There is a line of theoretical and empirical literature related to risky investment by firms. The mean-variance approach is a fundamental framework for optimal portfolio selection (Markowitz 1952). Investors can reduce their exposure to individual asset risk by constructing a combination of instruments that are not perfectly positively correlated (Xing et al. 2010, Tobin 1958, Markowitz 1952). These papers do not consider strategic relations between NPOs' investment behavior and donors' willingness to contribute. Recent empirical papers establish the impact of the risky environment on either donors' contribution (Cettolin et al. 2017) or NPOs borrowing and investing behavior (Rosen and Sappington 2016, Song et al. 2008).

3 Model Setting

There are two players in the model: a NPO and a cohort of homogeneous donors. The NPO's strategy consists of a pair (G_t, β_t) , where G_t is the provision of the public (charitable) good and β_t refers to the proportion of funds invested in a risky asset at time t . Donors select consumption,

others, à la Becker (1974) that this can increase the severity of the free-riding problem, might eventually choke off voluntary participation in the collective action," Fershtman and Nitzan (1991).

C_t , and contribution level, D_t .

3.1 NPO's Payoff and Program

To reduce the risk associated with the fluctuation of returns on risky asset, the NPO determines the optimal allocation of the disposable endowment in the portfolio frontier. The disposable endowment refers to the NPO's endowment fund after the provision of the charitable good. Suppose that the efficient portfolio consists of a risky asset and a risk-free asset (T-bills).

Definition 1. (NPO's investment strategy) *An investment strategy at time t consists of allocating β_t proportion of its disposable endowment fund to the risky asset and $1 - \beta_t$ proportion to the risk-free asset. Assume $\beta_t \in \mathcal{S}$, which is a compact and convex set defined in $[0, 1]$.*

The NPO does not render all of its endowment as the public good but instead invests a proportion of the disposable endowment to participate in the financial market. The remaining proportion is held as an additional fund balance to hedge against the risk of possible financial downturns and ensure there is adequate liquidity available for future operations.

Formally, the NPO chooses a pair (G_t, β_t) at each time t to maximize its present value of expected lifetime utility, $U^n = \int_0^\infty e^{-\rho_c t} u^n(G_t) dt$, where ρ_c is the NPO's discount rate and U_t^n is instantaneous utility given by,

$$U_t^n = u^n(G_t) \quad u_{G_t}^n(G_t) > 0, \quad u_{G_t, G_t}^n(G_t) < 0, \quad (1)$$

where $u_{G_t}^n(G_t)$ and $u_{G_t, G_t}^n(G_t)$ denote NPO's first and second-order derivatives, respectively.

We introduce the following notations:

- s_t = a state variable, representing NPO's total financial endowment at time t ,
- G_t = NPO's provision of the public good at time t ,
- β_t = proportion invested in the risky asset at time t ,
- σ_r = volatility of the risky asset,
- r = rate of return on the risk-free asset (T-bills),
- \bar{r} = long-term mean of the rate of return on the risky asset,
- r_t = rate of return on the risky asset at time t ,
- D_t = donors' contribution to the NPO at time t .

We assume that the NPO takes the exogenous rate of return of risky asset as given. The NPO's investment portfolio at time t consists of two assets, risk and risk-free assets, with proportions β_t and $1 - \beta_t$ respectively. Assume further that the rate of return on T-bill is a constant r and the rate of return on the risky asset, r_t , is generated by the Itô process in the form of Ornstein–Uhlenbeck.² At time interval $[t, t + \Delta t]$, its discrete form is,

$$r_t \Delta t = \theta (\bar{r} - r_t) \Delta t + \sigma_r \varepsilon_t \sqrt{\Delta t}, \quad (2)$$

where the parameter $\theta > 0$ represents how "strongly" the random rate of return generates system's reaction to perturbations, $\varepsilon_t \sim \mathcal{N}(0, 1)$.

Denote by W_{tr} the standard Wiener process and by w_t donors' wealth. Donors' contribution function, $D_t(w_t, s_t) : (w_t, s_t) \rightarrow D_t$, and $D_t(w_t, s_t)$ maps onto $[0, +\infty)$. We have the following:

Proposition 1. *Given donors' consumption and contribution strategy (C_t, D_t) , NPOs' best-response problem consists in choosing the path of the optimal bundle, (G_t^*, β_t^*) , such that the present value of expected lifetime utility is maximized,*

$$\max_{G_t, \beta_t} \mathbb{E}(U^n) = \mathbb{E} \left\{ \int_0^\infty e^{-\rho c t} u^n(G_t) dt \right\}, \quad (3)$$

s. t.

$$ds_t = \left(\left(\beta_t \left(\frac{\theta}{1 + \theta} \bar{r} - r \right) + r \right) s_t - (G_t - D_t(w_t, s_t)) \right) dt + \beta_t s_t \sigma_r dW_{tr}. \quad (4)$$

Proof: See Appendix 7

Note that given donors' choice (C_t, D_t) , the NPO's strategy only affects its endowment. Hence, the NPO's problem is constrained only by its endowment equation (4). Also, note that the variance of the NPO's endowment, $\beta_t^2 s_t^2 \sigma_r^2$, depends on not only the variance of the risky asset, but also on the portfolio selection and the endowment level. For instance, a higher level of endowment and a larger proportion invested in the risky asset generate more fluctuations than smaller ones..

²Ornstein-Uhlenbeck process is the continuous-time analog of the discrete-time $AR(1)$ process. It has stationary, Gaussian, and Markovian properties and approaches its long-term mean \bar{r} with the degree of volatility σ_r . The stationary (long-term) variance is given by $var(r_t) = \sigma_r^2 / 2\theta$.

3.2 Donors' Payoff and Program

Consider a cohort of homogeneous donors with mass 1. An instantaneous utility of a representative donor is quasi-linear in the private consumption C_t and is given by

$$U_t^d = C_t + u^d(G_t), \quad (5)$$

where $u^d(G_t)$ is an increasing and concave function of argument G_t .

Parameters and variables of donor's program are defined as:

C_t = donors' consumption at time t ,

D_t = donors' contribution to the endowment at time t ,

\bar{I} = donors' average income,

I_t = donors' income at time t ,

σ_I = the volatility of income I_t around average income \bar{I} caused by shocks,

w_t = a state variable, representing donors' total wealth level at time t .

Assume that $C_t \in (0, \eta_c w_t)$, where parameter $\eta_c \in (0, 1)$. Donors consume consumptions at any time t and the upper bound $\eta_c w_t$ is a fixed proportion of wealth level. Donors' contribution, D_t , is set within $[0, \eta_d s_t]$, where parameter $\eta_d \in [0, 1)$. The upper bound $\eta_d s_t$ represents donors' required level of the charitable good.

Donors' per unit time income evolves at time interval $[t, t + \Delta t]$ following the process,

$$I_t \Delta t = \bar{I} \Delta t + \sigma_I \varepsilon_t \sqrt{\Delta t}, \quad (6)$$

where $\varepsilon_t \sim \mathcal{N}(0, 1)$. When donors' decision (C_t, D_t) is realized, the corresponding endowment residue receives a risk-free rate of return, r .

For simplicity, assume that both the NPO and donors have the same discount rate, which is equal to the rate of return on T-bill: $\rho_c = \rho_d = r$. Suppose further that Wiener processes W_{tI} and W_{tr} are not correlated. We have the following:

Proposition 2. *Given the NPO's strategy (G_t, β_t) , the donor's best-response problem consists in*

maximizing the present value of expected lifetime utility,

$$\max_{\substack{C_t \in (0, \eta_c w_t] \\ D_t \in [0, \eta_d s_t]}} \mathbb{E}(U^d) = \mathbb{E} \left\{ \int_0^\infty e^{-\rho t} (u^d(G_t) + C_t) dt \right\}, \quad (7)$$

s.t.

$$ds_t = \left(\left(\beta_t \left(\frac{\theta}{1+\theta} \bar{r} - r \right) + r \right) s_t - (G_t - D_t(s_t)) \right) dt + \beta_t s_t \sigma_r dW_{tr}, \quad (8)$$

and

$$dw_t = (rw_t + \bar{I} - D_t - C_t) dt + \sigma_I dW_{tI}. \quad (9)$$

Proof: See Appendix 7

It is important to note that the donors' problem is constrained by two differential equations (8) and (9) because given the NPO's strategy (G_t, β_t) , donors' decision (C_t, D_t) affects both NPO's endowment and donors' wealth evolvments. Note also that donor's wealth fluctuations depend only on income volatility.

4 Equilibrium Analysis

4.1 Resolving Model

To solve the model we use the Feedback Nash equilibrium concept.³ In this equilibrium concept, players' strategies are contingent to the state of the game, S_t . The solution satisfies the Hamilton-Jacobi-Bellman (HJB) equations, which incorporate players' conditional behavior in the Nash equilibrium.

The NPO's best response problem is to choose the (G_t^*, β_t^*) , given the donors' choice, (C_t^*, D_t^*) , at each time t . The same procedure applies to donors' best-response problem. We have the following:

³The choice of equilibrium concept, for example, open-loop Nash or Feedback Nash, is dictated by the assumptions regarding pre-commitment of actions, and information structure. Without a central authority, commitments are not enforceable, making Feedback Nash more realistic to solve the dynamic differential game. In this concept agents adjust their strategies contingent on the system's current state, in the sense of a stationary Markovian Nash equilibrium (Wang and Ewald 2010b,a, Kossioris et al. 2008, Wedig 1994, Fershtman and Nitzan 1991, Yeung and Petrosjan 2006).

Definition 2. Denote by $\Gamma(S_0, \infty - 0)$ the non-cooperative stochastic differential game with equations (3), (4), (7), (8), (9), and initial state $S_0 = \{s_0, w_0\}$. Denote by $\Gamma(S_\tau, \infty - \tau)$ the game with the initial state $S_\tau = \{s_\tau, w_\tau\} \in S$ and $\tau \in [0, \infty)$. The NPO's and donors' value functions, which are the current value of the expected payoffs from $t \rightarrow \infty$, can be written as,

$$\Phi^{(\tau)}(t, s_t) = \mathbb{E} \left\{ \int_t^\infty e^{-r(y-\tau)} u^n(G_y) dy \right\}, \quad (10)$$

and

$$\Psi^{(\tau)}(t, s_t, w_t) = \mathbb{E} \left\{ \int_t^\infty e^{-r(y-\tau)} (u^d(G_y) + C_y) dy \right\}, \quad (11)$$

for $\tau \in [0, \infty)$, and $t \in [\tau, \infty)$ respectively.

It can be concluded that, using equations $\Phi^{(\tau)}(t, s_t) = e^{-(t-\tau)} \Phi^{(t)}(t, s_t)$, and $\Psi^{(\tau)}(t, s_t, w_t) = e^{-(t-\tau)} \Psi^{(t)}(t, s_t, w_t)$, the game, $\Gamma(S_\tau, \infty - \tau)$, has the property that the discounted value of any subgame value function from $t \in [t, \infty)$ is equal to the current value of the value function at time $t \in [t, \infty)$.

Definition 3. (Shadow prices of the NPO and donors) *Donors' shadow prices refer to the partial derivatives of the value function, $\Psi(s_t, w_t)$, with respect to their wealth level and the NPO's endowment, denoted by Ψ_{s_t} and Ψ_{w_t} correspondingly. The NPO's shadow price is the derivative of its value function, $\Phi(s_t)$, with respect to endowment s_t , denoted by Φ_{s_t} .*

Note that value functions $\Psi(s_t, w_t)$ and $\Phi(s_t)$ are increasing and strictly concave in arguments s_t and w_t . Therefore, shadow prices are positive and decreasing.

To solve for the equilibrium of the game, we need the following:

Proposition 3. *A strategy profile $\{(G_t^*, \beta_t^*), (C_t^*, D_t^*)\}$ is the equilibrium of the game $\Gamma(S_0, \infty - 0)$ if there exist two times differentiable functions $\Phi(s_t)$ and $\Psi(s_t, w_t)$ satisfying the following Hamilton-Jacobi-Bellman equations,*

$$\begin{aligned} r\Phi(s_t) = \max_{G_t, \beta_t} & \left\{ u^n(G_t) + \Phi_{s_t} \left(\left(\beta_t \left(\frac{\theta}{1+\theta} \bar{r} - r \right) + r \right) s_t - (G_t - D_t^*) \right) \right. \\ & \left. + \frac{1}{2} \Phi_{s_t, s_t} \beta_t^2 s_t^2 \sigma_r^2 \right\}, \end{aligned} \quad (12)$$

and

$$r\Psi(s_t, w_t) = \max_{\substack{C_t \in (0, \eta_c w_t] \\ D_t \in [0, \eta_d s_t]}} \left\{ u^d(G_t^*) + C_t + \Psi_{s_t} \left((\beta_t^* (\frac{\theta}{1+\theta} \bar{r} - r) + r) s_t - (G_t^* - D_t) \right) \right. \\ \left. + \Psi_{w_t} (r w_t + \bar{I} - D_t - C_t) + \frac{1}{2} \Psi_{s_t, s_t} \beta_t^{*2} s_t^2 \sigma_r^2 + \frac{1}{2} \Psi_{w_t, w_t} \sigma_I^2 \right\}. \quad (13)$$

Proof: See Appendix 7.

Given donors' strategy, (C_t^*, D_t^*) , the NPO chooses (G_t^*, β_t^*) such that the right side of the HJB equation is maximized. Hence, we obtain the first-order conditions for the NPO's problem,

$$u_{G_t^*}^n(G_t^*) - \Phi_{s_t} = 0, \quad (14)$$

$$\Phi_{s_t} \left((\frac{\theta}{1+\theta} \bar{r} - r) s_t \right) + \Phi_{s_t, s_t} \beta_t^* s_t^2 \sigma_r^2 = 0. \quad (15)$$

These equations characterize the NPO's best-response. Similarly, donors' first-order conditions with respect to C_t and D_t are,

$$1 - \Psi_{w_t} > 0 \text{ or } < 0, \quad (16)$$

$$\Psi_{s_t} - \Psi_{w_t} > 0 \text{ or } < 0. \quad (17)$$

These inequalities describe donors' best-response (C_t^*, D_t^*) . Thus, donors' private consumption, C_t^* , and contribution, D_t^* , must take corner values. By inequality (16), we obtain $\Psi_{w_t} < 1$ to ensure that donors have private consumption at time t . Since the value function $\Psi(s_t, w_t)$ is increasing and strictly concave in its arguments, the following properties are satisfied: $\lim_{w_t \rightarrow 0} \Psi_{w_t} = 1$ and $\lim_{w_t \rightarrow +\infty} \Psi_{w_t} = 0$. For the shadow price Ψ_{s_t} , it must be true that $\lim_{s_t \rightarrow 0} \Psi_{s_t} = \infty$ and $\lim_{s_t \rightarrow +\infty} \Psi_{s_t} = 0$. There exists a threshold, \bar{s} , which is uniquely determined by the condition: $\Psi_{s_t} = 1$, such that $\Psi_{s_t} > \Psi_{w_t}$ when $s_t < \bar{s}$, as shown in Figure 2.

To proceed further, define the set of rate of returns on T-bills.

Definition 4. (Effective space of the return rate on T-bills) *The effective space, \mathcal{R}_e , is a compact and convex set in \mathbb{R}^+ . \mathcal{R}_e is a set of rates of return on T-bill, r . $\forall r \in \mathcal{R}_e$, the NPO chooses an interior optimal portfolio; otherwise, the corner solution, $\beta_t^* = 0$ or 1 applies. Figure 4 illustrates.*

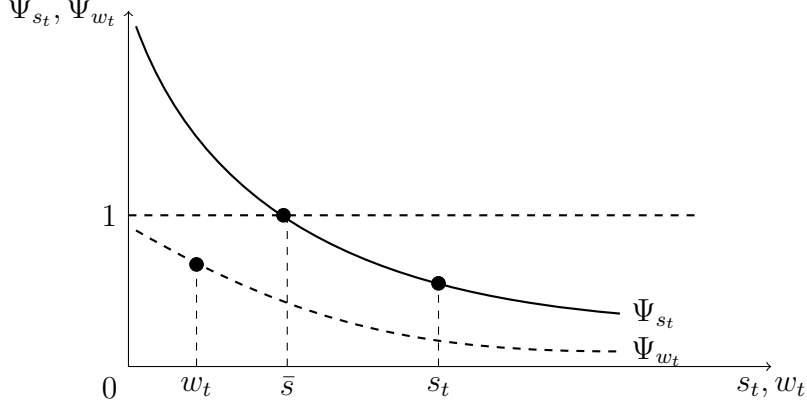


Figure 2: Donors Shadow Price Curves Ψ_{s_t} and Ψ_{w_t}

Note: the solid curve represents donors' shadow price of NPO's endowment, whereas the dashed one is donors' shadow price of their wealth level. $\Psi_{s_t} > \Psi_{w_t}$ if $s_t < \bar{s}$; when $s_t > \bar{s}$, for a arbitrary pair of (s_t, w_t) , $\Psi_{s_t} > < \Psi_{w_t}$.

The next proposition describes the equilibrium of the game.

Proposition 4. *There exists a unique threshold, \bar{s} , such that*

a) *the NPO provides a constant proportion of its current endowment as public good, G_t^* , given by*

$$G_t^* = \begin{cases} \left(r - \frac{\alpha}{1-\alpha} \eta_d - \frac{\alpha(\frac{\theta}{1+\theta} \bar{r} - r)^2}{2(1-\alpha)^2 \sigma_r^2} \right) s_t & \text{if } s_t < \bar{s}, \\ \left(r - \frac{\alpha(\frac{\theta}{1+\theta} \bar{r} - r)^2}{2(1-\alpha)^2 \sigma_r^2} \right) s_t & \text{if } s_t > \bar{s}, \text{ and } \Psi_{s_t} < \Psi_{w_t}, \\ \left(r - \frac{\alpha}{1-\alpha} \eta_d - \frac{\alpha(\frac{\theta}{1+\theta} \bar{r} - r)^2}{2(1-\alpha)^2 \sigma_r^2} \right) s_t & \text{if } s_t > \bar{s} \text{ and } \Psi_{s_t} > \Psi_{w_t}. \end{cases} \quad (18)$$

b) *the NPO exhibits risk-aversion: a higher proportion of the disposable endowment invested in the risky asset requires a larger risk premium. The proportion, β_t^* , is fixed and given by*

$$\beta_t^* = \begin{cases} \frac{\frac{\theta}{1+\theta} \bar{r} - r}{(1-\alpha) \sigma_r^2} & \text{if } r \in \mathcal{R}_e, \\ 0, 1 & \text{if } r \notin \mathcal{R}_e, \end{cases} \quad (19)$$

c) *donors' contributions, D_t^* , are given by*

$$D_t^* = \begin{cases} \eta_d s_t & \text{if } s_t < \bar{s}, \\ 0 & \text{if } s_t > \bar{s} \text{ and } \Psi_{s_t} < \Psi_{w_t} \\ \eta_d s_t & \text{if } s_t > \bar{s} \text{ and } \Psi_{s_t} > \Psi_{w_t}. \end{cases} \quad (20)$$

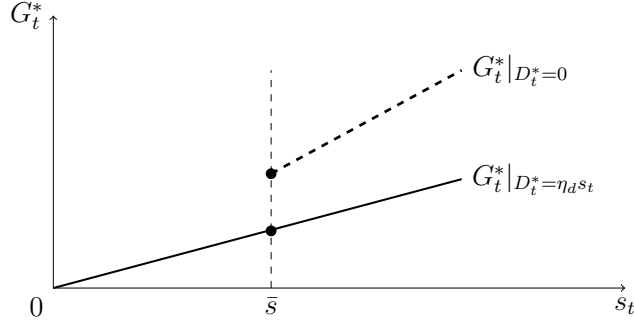


Figure 3: NPO Optimal Provision of Public Good, G_t^* , Conditional on Donors' Optimal Contribution, D_t^* .

The solid bold line represents the NPO's optimal provision of public good when donors contribute; the dashed line represents the provision of public good when donors do not contribute. s_t is the NPO's endowment level at time t , D_t^* is donors' optimal contribution, which equals $\eta_d s_t$ or 0.

d) finally, donors consume a fixed proportion of their wealth, $C_t^* : C_t^* = \eta_c w_t$.

Proof: See Appendix 7

Proposition 4 establishes that the NPO provides a higher proportion of its present endowment as public good if donors do not contribute. The intuition is that more public good provided by NPOs will shrink its endowment, which therefore increases donors' shadow price, Ψ_{s_t} ,⁴ and incentivizes donors to give in the future. Figure 3 illustrates this property. Donors may free ride on the NPO's investment when $s_t > \bar{s}$. In this case, donors' shadow price from w_t exceeds the shadow price from s_t .

The NPO's optimal portfolio selection does not depend on donors' contribution. This result is consistent with Samuelson (1975) and Merton (1971), who show that for CRRA utility, the portfolio-selection decision is independent of the consumption in discrete and continuous-time models. Note also that a sufficiently low rate of return on T-bill will encourage the NPO to invest all its disposable endowment in the risky portfolio. Alternatively, a higher enough rate of return on T-bill will discourage the NPO not to participate in any risky financial activities. For any rate of return on T-bills, $r \in \mathcal{R}$, the NPO chooses either the interior solution, $\beta_t^* = (\frac{\theta}{1+\theta}\bar{r} - r)/((1-\alpha)\sigma_r^2)$, or the corner solution, $\beta_t^* = 0$, or 1, as shown in Figure 4. It is worth noting that the expected rate of return on the risky asset is an increasing function of β_t^* , which shows that the NPO is risk-averse.

⁴Note that donors' first-order condition with respect to contribution equals the difference of shadow price, $\Psi_{s_t} - \Psi_{w_t}$. The properties of decreasing Ψ_{s_t} and Ψ_{w_t} directly lead to the result that lower NPO's endowment encourages donors to contribute.

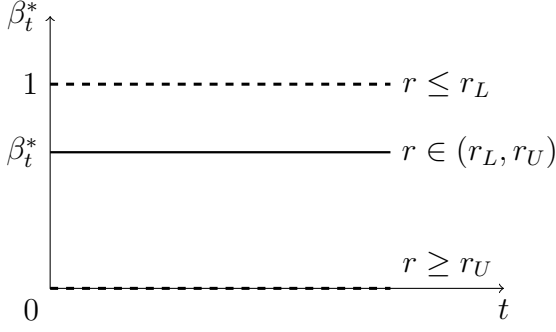


Figure 4 - I

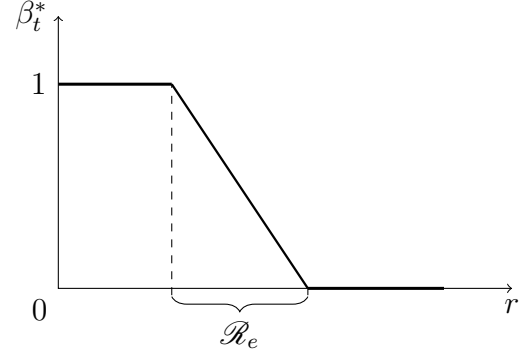


Figure 4 - II

Figure 4: NPO Optimal Portfolio Selection β_t^*

Note that $\beta_t^* = \min \left\{ \max \left\{ \frac{\frac{\theta}{1+\theta}\bar{r}-r}{(1-\alpha)\sigma_r^2}, 0 \right\}, 1 \right\}$. The effective space $\mathcal{R}_e = \{r \in \mathbb{R}^+ \mid r_L < r < r_U\}$, where $r_L = \frac{\theta}{1+\theta}\bar{r} - (1-\alpha)\sigma_r^2$, $r_U = \frac{\theta}{1+\theta}\bar{r}$.

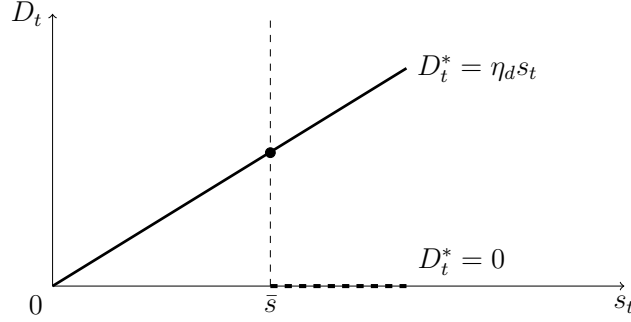


Figure 5: Donors' Optimal Contribution D_t^*

Note that if donors' shadow price from the NPO's endowment, s_t , is greater than the one from their income, ω_t , donors contribute $D_t^* = \eta_d s_t$; otherwise, $D_t^* = 0$. The solid line represents that donors contribute $D_t^* = \eta_d s_t$; the dashed one indicates that donors free ride the NPO.

Figure 5 shows that donors' contribution decision depends on whether the shadow price from NPO's endowment, Ψ_{s_t} , exceeds the one from their income, Ψ_{ω_t} . Otherwise, donors prefer to free ride on the NPO instead of giving. Therefore, donors contribute $D_t^* = \eta_d s_t$ if $s < \bar{s}$. When $s > \bar{s}$, the optimal contribution would be either $D_t^* = \eta_d s_t$ or $D_t^* = 0$.

Remark 1. *The optimal strategy $\{(G_t^*, \beta_t^*), (C_t^*, D_t^*)\}$ depends only on the present state; therefore, it satisfies the Markov property.*

Proof: See Appendix 7

Denote by $f(r) = r - \frac{\alpha}{1-\alpha}\eta_d - \alpha \left(\frac{\theta}{1+\theta}\bar{r} - r \right)^2 / (2(1-\alpha)^2\sigma_r^2)$ the gap between the rate of return on T-bills, r , and the adjusted risk premium, $\alpha \left(\frac{\theta}{1+\theta}\bar{r} - r \right)^2 / (2(1-\alpha)^2\sigma_r^2)$, plus donors' contribution factor,

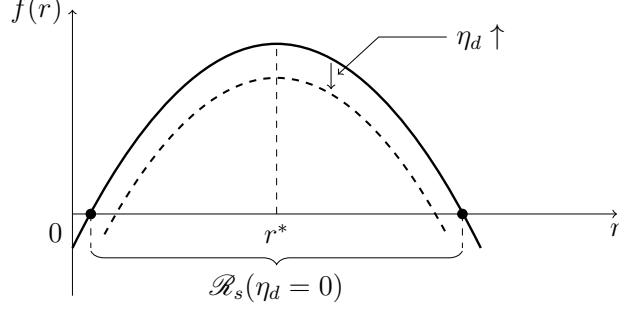


Figure 6: The Subspace of Rate of Return on T-bill, $\mathcal{R}_s(\eta_d)$

Note: The subspace, $\mathcal{R}_s(\eta_d = 0)$, equals $\{r \in \mathbb{R}^+ \mid f(r) > 0 \cap \eta_d = 0\} \neq \emptyset$, where $f(r) = r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r}-r\right)^2}{2(1-\alpha)^2\sigma_r^2}$. $r^* = \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}$. The dashed parabola is the case when η_d is increasing.

$$\frac{\alpha}{1-\alpha}\eta_d.$$

Corollary 1. a) *The interior solution for the provision of the public good, G_t^* , exists if and only if the subspace of the rate of return on T-bill, $\mathcal{R}_s = \{r \in \mathbb{R}^+ \mid f(r) > 0\}$ is non-empty.* b) *The NPO is always willing to provide the public good when donors do not contribute.* c) *The higher contribution proportion, η_d , leads the NPO to provide less public good.*

Proof: See Appendix 7

Providing the public good adds a positive value to NPO's value function, $\Phi(s_t)$, which requires that $\mathcal{R}_s = \{r \in \mathbb{R}^+ \mid f(r) > 0\}$ is non-empty. Corollary 1 establishes that the NPO always provides the public good when donors do not contribute; in this case, the subspace $\mathcal{R}_s(\eta_d = 0) \neq \emptyset$. The higher contribution proportion, η_d , decreases $f(r)$ and monotonically shrinks the subspace, \mathcal{R}_s . Therefore, the NPO is more willing to get involved in the financial market instead of providing the public good. There exists a threshold of the contribution proportion, $\hat{\eta}_d$, such that the NPO will allocate all its endowment to the financial market and will not provide the public good: $G_t^* = 0$ when $\eta_d \geq \hat{\eta}_d$. Figure 6 illustrates.

Corollary 2. *The subspace of the rate of return on T-bill, $\mathcal{R}_s = \{r \in \mathbb{R}^+ \mid f(r) > 0\}$, increases with parameters σ_r , \bar{r} , θ and decreases with NPO's risk parameter α .*

Proof: See Appendix 7

This property is attributed to the fact that the change of parameters $\{\alpha, \theta, \bar{r}, \sigma_r\}$ leads the $f(r)$ curve to shift not only vertically but also horizontally. For example, as the NPO's risk parameter

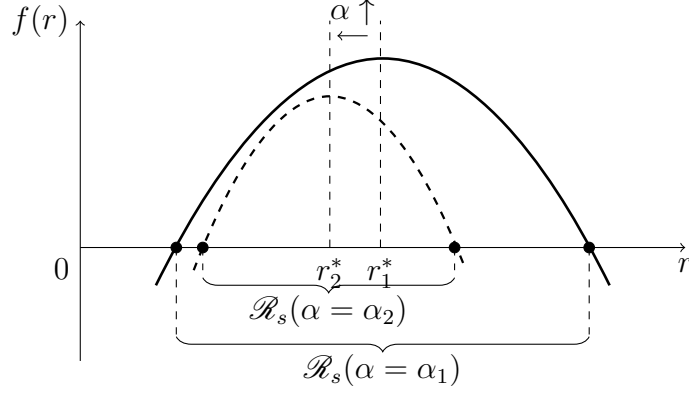


Figure 7: The Impact of Risk Parameter, α , on Dominance of Function $f(r)$

Note: The subspace, \mathcal{R}_s , equals $\{r \in \mathbb{R}^+ \mid f(r) > 0 \cap \eta_d \geq 0\} \neq \emptyset$, where $f(r) = r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha(\frac{\theta}{1+\theta}\bar{r}-r)^2}{2(1-\alpha)^2\sigma_r^2}$, $r^* = \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}$, and $f(r^*) = \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{2\alpha} - \frac{\alpha}{1-\alpha}\eta_d$. The dashed parabola is the case when NPO's risk parameter, α , increases from α_1 to α_2 .

α increases from α_1 to α_2 , the critical point $(r^*, f(r^*))$ shifts lower left, and the subspace, \mathcal{R}_s , shrinks, which is different from the impact of the contribution ratio η_d (See Figures 6 and 7).

4.2 Risk Exposure

It is natural to ask whether NPO's risk exposure impact differs from the donors'. It requires examining how the variance of risky investment and income influences the strategy profile.

4.2.1 Benchmark

In the benchmark, defined in Sections (3.1), (3.2), NPOs can actively choose the risky portfolio to diversify a certain degree of the investment risk; donors expose to the income risk and accumulate the residual wealth by investing in the risk-free asset. It follows,

Proposition 5. *Investments risk affects both the NPO and donors' decisions. The increased risk exposure of NPOs encourages public goods provision, reduces the proportion of disposable endowment invested in the risky asset, and incentivizes donors to contribute less. Donors' income risk exposure does not have any effect on parties' decisions.*

Proof: See Appendix 7

Proposition 5 establishes that the NPO's or donors' risk exposure plays a different role in determining decision profiles. A more volatile risky investment forces the NPO to lower its portfolio

selection and provide more charitable goods, which decreases its endowment level. By Proposition 4, donors, therefore, cut down contributions. On the other hand, donors' wealth variation equals income variation σ_I^2 and is not correlated to donors' decisions $\{C_t^*, D_t^*\}$ (See Proposition 2). The negative variance impact $\frac{1}{2}\Psi_{w_t, w_t}\sigma_I^2$ on donors' value function $\Psi(s_t, w_t)$ is irrelevant to donors' first-order conditions (16, 17). Therefore, the donors' income risk exposure does not affect the equilibrium of the game.

The rationale behind this contrasting difference is attributed to the fact that donors put wealth residual in the risk-free asset passively instead of actively diversifying the residual such that the variance of wealth level is relevant to donors' decisions.

4.2.2 Extended Risk Exposure

It is also interesting to examine donors' risk exposure impact when donors can apply the wealth residual to the risky investment instead of the risk-free asset. Suppose that the rate of return on the risky asset, r_t , is generated by the Itô process in the form of Ornstein–Uhlenbeck,

$$dr_t = \vartheta(\tilde{r} - r_t)dt + \tilde{\sigma}dW_t. \quad (21)$$

where \tilde{r} represents the mean value of the rate of return on the risky asset. $\tilde{\sigma}$ is the standard deviation, and W_t is the standard Wiener process.

Corollary 3. *Suppose that donors expose to the income risk and the residual wealth is invested in the risky asset. Donors risk exposures do not influence agents' optimal decisions.*

Proof: See Appendix 7

Donors invest the entire residual wealth in the risky asset. Without actively diversifying, the variance of wealth level is not relevant to the donors' decisions such that the optimal choices by donors do not affect the negative variance impact on the value function $\Psi(s_t, w_t)$. The findings by Proposition 5 and Corollary 3 are consistent with the experimental evidence by Cettolin et al. (2017), who show that the increased risk exposure of givers keeps silent, but a rise in beneficiaries' risk exposure discourages giving.

4.3 The Expected Stock of Endowment and Its Limitation

Proposition 6. *The NPO's expected stock of endowment is contingent on the rate of return on T-bill. It converges either to infinity or to a level $\bar{s}_\tau \in (\bar{s}_L, \bar{s}_U]$.*⁵

$$\mathbb{E}\{s_t\} = \begin{cases} s_0 e^{\left(\frac{\theta}{1+\theta}\bar{r}-r+\frac{\eta_d}{1-\alpha}+\frac{\alpha}{2}\sigma_r^2\right)t} & \text{if } r \leq r_L, \\ s_0 e^{\left(\frac{\eta_d}{1-\alpha}+\frac{(2-\alpha)(\frac{\theta}{1+\theta}\bar{r}-r)^2}{2(1-\alpha)^2\sigma_r^2}\right)t} & \text{if } r \in \mathcal{R}_e, \\ s_0 e^{\frac{\eta_d}{1-\alpha}t} & \text{if } r \geq r_U, \end{cases}$$

$$\lim_{t \rightarrow +\infty} \mathbb{E}\{s_t\} = \begin{cases} \infty & \text{if } r < r_U, \\ \bar{s}_\tau & \text{if } r \geq r_U. \end{cases}$$

Proof: See Appendix 7

Proposition 6 shows that given the risky investment project, the NPO's risk parameter α , and donors' contributions proportion η_d , the rate of return on T-bill r plays an important role in determining $\mathbb{E}\{s_t\}$ and $\lim_{t \rightarrow +\infty} \mathbb{E}\{s_t\}$. A not sufficiently large value of r ensures that the NPO will participate in the risky investment activities and provide partial investment proceeds as the public good, which eventually leads the expected endowment, $\mathbb{E}\{s_t\}$, to accumulate continuously, as shown in Figures 8 - I, II. Alternatively, for a sufficiently large value, the NPO prefers to invest all its disposable endowment to the T-bill. As the endowment accumulates, donors stop contributing, and the NPO provides all of the net proceeds generated from T-bill as the public good until donors contribute again. This cycle continues until the NPO's endowment s_t reaches the level, $\bar{s}_\tau \in (\bar{s}_L, \bar{s}_U]$. This property helps to explain that $\mathbb{E}\{s_t\}$ increases from the initial level, s_0 , and converges to $\bar{s}_\tau \in (\bar{s}_L, \bar{s}_U]$ (See Figure 8 - III).

Note that for a relatively lower rate of return on the T-bill, the NPO's expected endowment, $\mathbb{E}\{s_t\}$, reaches the threshold level, \bar{s}_L , faster due to allocating a higher proportion of disposable endowment to the risky asset. The $\mathbb{E}\{s_t\}$ curve becomes flat after that, where donors' contributions

⁵Thresholds \bar{s}_L and \bar{s}_U are uniquely determined by $\Psi_{s_t} = 1$ and $\Psi_{s_t} = \Psi_{w_t}$ respectively. Donors contribute when $s_t \leq \bar{s}_L$ and free ride the NPO when $s_t \geq \bar{s}_U$. Donors either contribute or free ride when $\bar{s}_L < s_t < \bar{s}_U$ depending on the sign of $\Psi_{s_t} - \Psi_{w_t}$. Note: $\bar{s}_\tau \in (\bar{s}_1, \bar{s}_2]$ is the minimum level of the NPO's endowment such that donors do not contribute $\forall s_t > \bar{s}_\tau$.

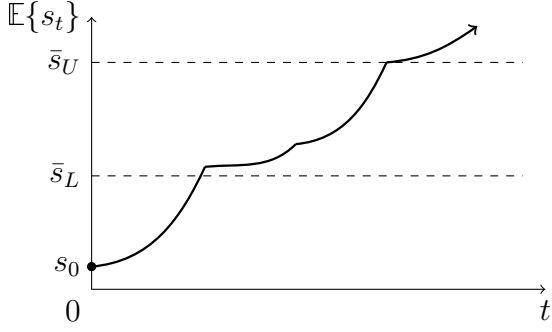


Figure 8 - I

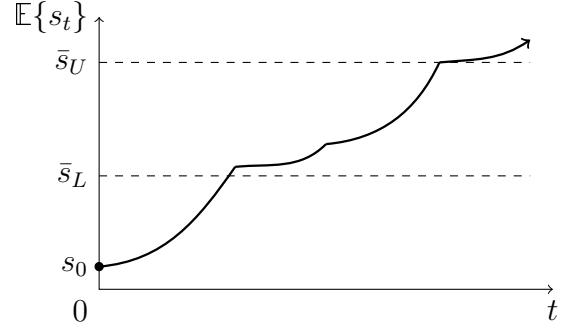


Figure 8 - II

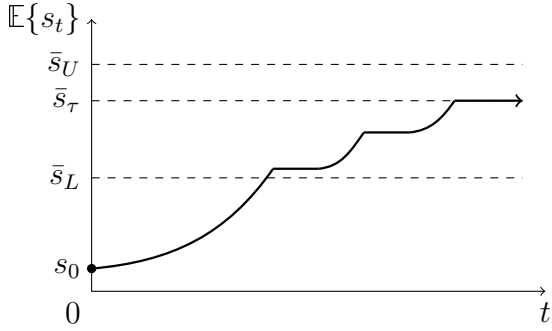


Figure 8 - III

Figure 8: NPO's Expected Endowment $\mathbb{E}\{s_t\}$ and its Limitation

Note: s_0 is the NPO's initial endowment. Figures 8-I, II, III are cases when $r \leq r_L$, $r \in \mathcal{R}_e$ and $r \geq r_U$, respectively.

cease. Recall by Corollary 1 that the lower the contribution ratio, η_d , the higher the NPO's willingness to provide the public good is going to be. Therefore, a sufficiently small η_d leads the NPO to invest less. Thus, the surplus of investment proceeds after the provision of public good accumulates less, which eventually flattens the NPO's expected endowment curve. Figures 8-I, II, and III illustrate.

4.4 Numerical Simulations

Let $\bar{r} = 0.15$, $r = 0.05$, $\theta = 0.85$, $\alpha = 0.35$, $\sigma_r = \sigma_I = 0.5$, $s_0 = 0.2$, and $\bar{s} = 0.6$. Note that qualitatively similar figures are obtained when using other sets of parameters. By Proposition 4, the NPO chooses a constant fraction,

$$\beta_t^* = \min \left\{ \max \left\{ \frac{\frac{\theta}{1+\theta}\bar{r} - r}{(1-\alpha)\sigma_r^2}, 0 \right\}, 1 \right\}. \quad (23)$$

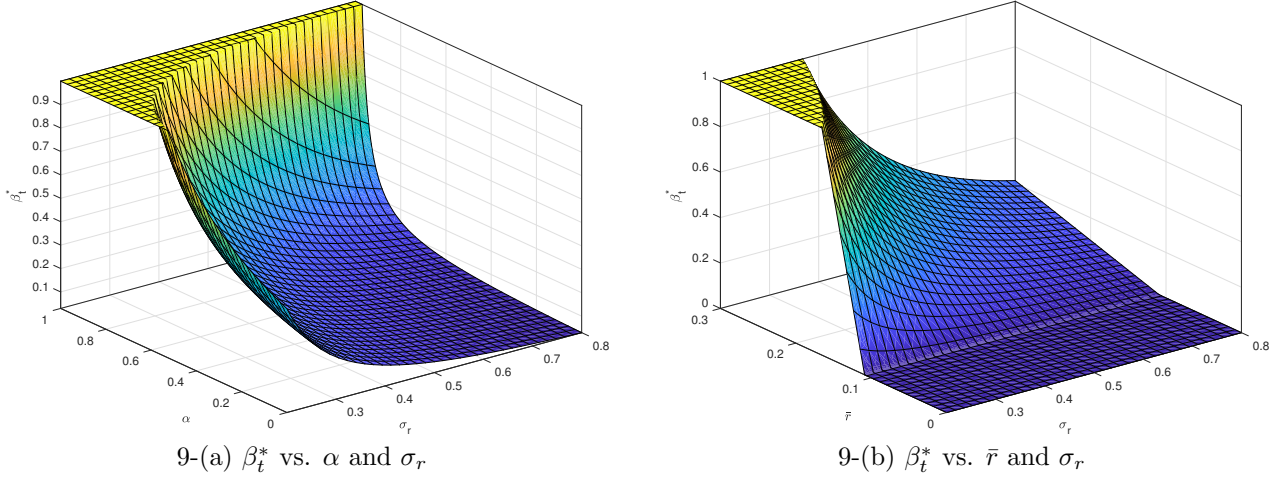


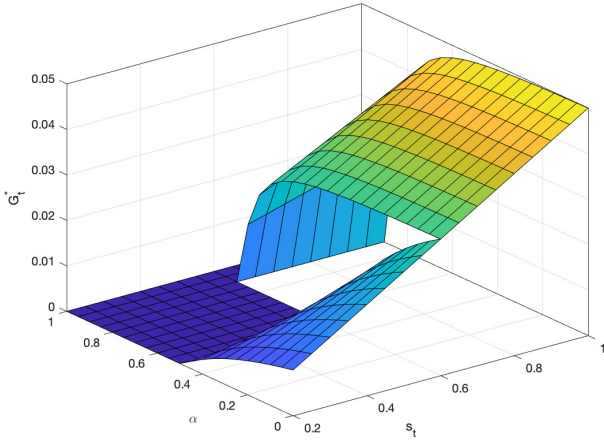
Figure 9: Simulation of NPOs' Optimal Portfolio Selection, β_t^*

Noting that the interior solution of β_t^* requires the rate of return on T-bill, $r \in \mathcal{R}_e = \{r \in \mathbb{R}^+ \mid r_L < r < r_U\}$, where $r_L = \frac{\theta}{1+\theta}\bar{r} - (1-\alpha)\sigma_r^2$, and $r_U = \frac{\theta}{1+\theta}\bar{r}$; otherwise, the NPO chooses corner solution, $\beta_t^* = 0$ or 1.

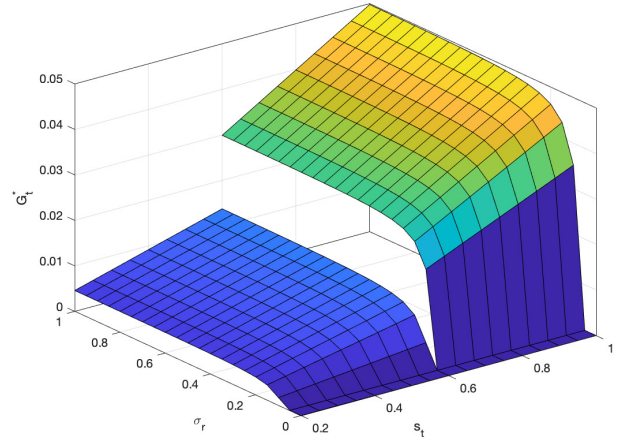
Figure 9-(a) shows that given the parameters \bar{r} , r , and θ , the NPO's optimal portfolio selection, β_t^* , increases with its risk parameter, α , but decreases with the standard deviation of the investment project, σ_r . Note that a sufficiently large risk parameter, α , and a small standard deviation, σ_r , lead the NPO to invest all its disposable endowment in the risky asset.

Figure 9-(b) illustrates that given the rate of return on T-bill, r , and the NPO's risk parameter, α , the optimal β_t^* increases in the long-term mean of the rate of return on risky tangency portfolio, \bar{r} , and decreases with σ_r . Nevertheless, a high enough long-term mean, \bar{r} , motivates the NPO to choose $\beta_t^* = 1$, and, therefore, to allocate all its disposable endowment in the risky asset. A low enough \bar{r} drives the NPO to choose $\beta_t^* = 0$.

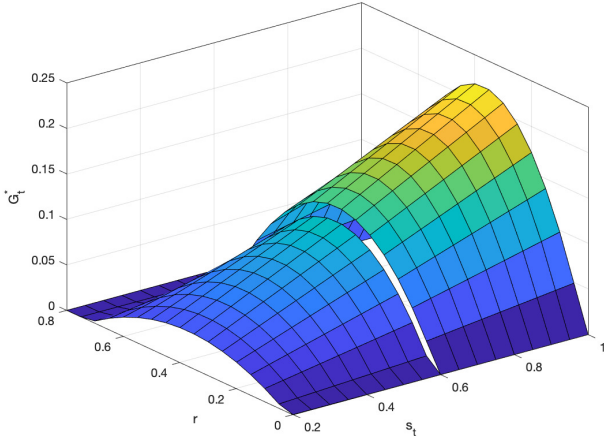
Figures 10-(a), (b) show that the NPO provides less public good when it becomes less risk-averse, but is willing to provide more when the financial investment is more volatile. The intuition is that the NPO with a higher risk parameter, α , prefers to participate in the financial market and to allocate more of its endowment to the risky portfolio. Hence, it chooses a lower level of the public good, G_t^* . Investments with higher volatility drive the NPO to provide more public good and to allocate less endowment to risky assets. Note that a large enough risk parameter, α , or low enough volatility, σ_r , force the NPO not to provide the public good at all. Figure 10-(c) shows that the impact of the rate of return on T-bill, r , is not monotonic. A sufficiently low or a sufficiently high rate of return on T-bills, r , increases the investment risk premium. This encourages the



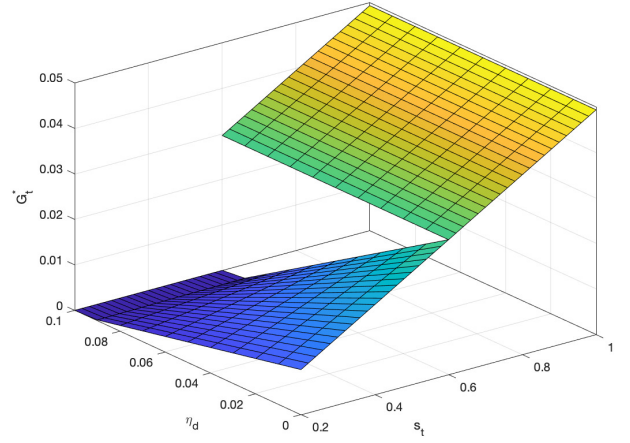
10-(a) G_t^* vs. α and s_t



10-(b) G_t^* vs. σ_r and s_t



10-(c) G_t^* vs. r and s_t



10-(d) G_t^* vs. η_d and s_t

Figure 10: Simulation of NPOs' Optimal Provision of the Public Good, G_t^* .

Noting that $G_t^* = f(r)s_t$. There is an interior solution for the provision of the public good, G_t^* , iff the subspace of the rate of return on T-bill, $\mathcal{R}_s = \{r \in \mathbb{R}^+ \mid f(r) > 0\} \neq \emptyset$, where function, $f(r)$, represents the gap between the rate of return on T-bill, r , and the adjusted risk premium, $\frac{\alpha(\frac{\theta}{1+\theta}\bar{r}-r)^2}{2(1-\alpha)^2\sigma_r^2}$ plus donors' contribution factor, $\frac{\alpha}{1-\alpha}\eta_d$.

NPO to participate in the financial market, and to provide less public good. Figure 10-(d) shows that the NPO will decrease the public good provision when donors increase the contributions proportion, η_d . Note that the results depicted in Figure 10-(c), (d) are consistent with Corollary 1 (See Figure 6). We have the following comparative statics,

Proposition 7. *Parameters, $\{\alpha, \eta_d, \theta, \bar{r}, \sigma_r\}$, affect monotonically public good provision. The impact of the T-bill return rate on the provision of the public good is non-monotonic.*

Proof: See Appendix 7

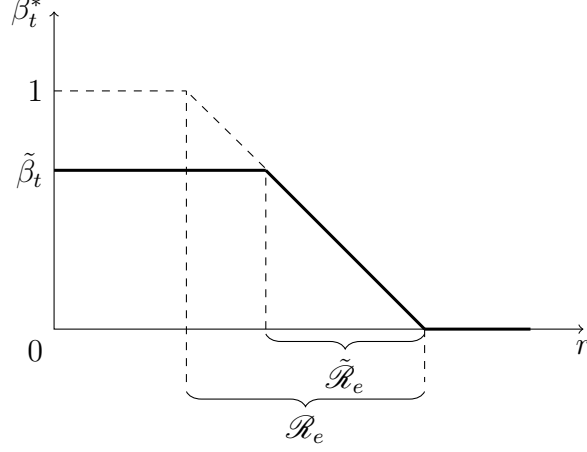


Figure 11: NPO Optimal Portfolio Selection β_t^* with Portfolio Ceiling $\tilde{\beta}_t$

Note that $\beta_t^* = \min \left\{ \max \left\{ \frac{\theta}{1+\theta} \bar{r} - r, 0 \right\}, 1 \right\}$ without portfolio ceiling. The effective space $\mathcal{R}_e = \{r \in \mathbb{R}^+ \mid r_L < r < r_U\}$, where $r_L = \frac{\theta}{1+\theta} \bar{r} - (1-\alpha)\sigma_r^2$, $r_U = \frac{\theta}{1+\theta} \bar{r}$. The solid kinked line represents the optimal portfolio selection with portfolio ceiling $\tilde{\beta}_t$, where $\tilde{\mathcal{R}}_e = \{r \in \mathbb{R}^+ \mid \tilde{r}_L < r < r_U\}$ and $\tilde{r}_L = \frac{\theta}{1+\theta} \bar{r} - (1-\alpha)\sigma_r^2 \tilde{\beta}_t^2$, $r_U = \frac{\theta}{1+\theta} \bar{r}$.

Equation (A17) indicates that the function $f(r)$ has the quadratic form in the rate of return on T-bill, r (See Figure 6). Hence, the impact of the rate of return on T-bill is non-monotonic. The change of other parameters shifts the function $f(r)$ curve (See Figure 7). However, the dominance of function $f(r)$ in its domain, \mathcal{R}_s , is consistent. Thus, the impact of parameters, except r , is monotonic, as illustrated in Figure 9.

5 Regulations

To maximize the present value of expected payoff, it is likely that NPOs invest a large proportion of endowment in the risky asset and therefore incur an unexpected investment loss *ex post*. Meanwhile, NPOs may provide charitable goods insufficiently, and fewer philanthropic activities are financed. In order for NPOs to face meaningful regulations, e.g., portfolio ceiling $\tilde{\beta}_t$ and provision floor \tilde{G}_t must be binding.

5.1 Portfolio Ceiling

Proposition 8. *With portfolio ceiling $\tilde{\beta}_t$, NPO chooses $\tilde{\beta}_t$ as its suboptimal portfolio selection. The effective space \mathcal{R}_e shrinks to $\tilde{\mathcal{R}}_e$.*

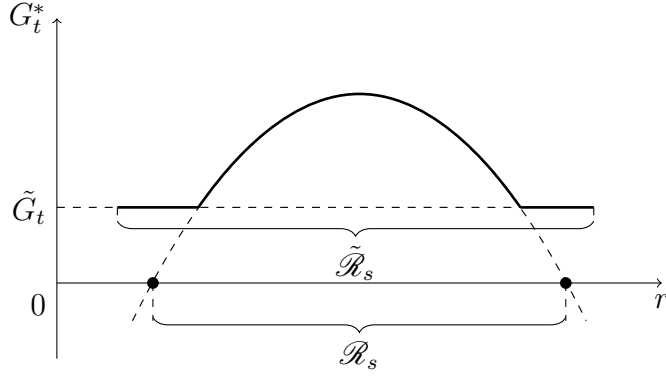


Figure 12: NPOs Provision of Charitable Good with Provision Floor \tilde{G}_t

Note that without the regulation of provision floor, the subspace \mathcal{R}_s equals $\{r \in \mathbb{R}^+ \mid f(r) > 0 \cap \eta_d \geq 0\} \neq \emptyset$, where $G_t^* = (r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha(\frac{\theta}{1+\theta}\bar{r}-r)^2}{2(1-\alpha)^2\sigma_r^2})s_t$. The solid piecewise curve is the case where provision floor \tilde{G}_t is applied.

Proof: See Appendix 7

Proposition 8 shows, as required by the regulation, NPOs apply the portfolio ceiling proportion of disposable endowment to the risky asset. It is helpful to stress that NPOs' portfolio selection is a kinked piecewise function of the rate of return on the risk-free asset (See Figure 11), which suggests that to manage the potential investment loss effectively, the setting of portfolio ceiling needs to take account of the impact of the rate of return on portfolio selection. e.g., NPO lowers its optimal risky portfolio when the rate of return on the risk-free asset goes up, which requires the portfolio ceiling to be set lower for binding.

5.2 Provision Floor

Recall from Proposition 7 and discussions in section 4.1, without provision floor, NPOs may undersupply charitable goods at equilibrium when parameters, $\{\alpha, \eta_d, \theta, \bar{r}, \sigma_r\}$, satisfy certain conditions. It is essential to regulate NPOs to provide charitable goods to a higher level than the equilibrium level. It follows,

Proposition 9. *With provision floor \tilde{G}_t , NPOs deliver the charitable good at the level not less than \tilde{G}_t instead of the optimal level G_t^* . Subspace \mathcal{R}_s expands.*

Proof: See Appendix 7

6 Conclusion

We consider a dynamic charitable market with two types of random shocks: fluctuations of the investment rate of return and income shock. Donors contribute to an NPO's endowment, and the NPO provides the charitable good, which is financed by donors' contributions and investment proceeds. The results show that the NPO chooses a constant share of the risky asset. Donors' contributions are discontinuous; they either hit the upper bound of giving or free-ride the NPO, which leads to the jump in the provision of charitable good by NPOs. The NPO always provides public good when donors do not contribute, but this does not necessarily apply if donors contribute with a higher contribution ratio.

We also show that the environment with a lower rate of return on T-bill is favorable to the NPO's expected endowment. NPO's investment risk affects both NPO's and donors' decision profiles; however, donors' income and passive investment risk exposures keep silent, which complements the experimental evidence by [Cettolin et al. \(2017\)](#).

In future extensions, it would be interesting to examine the causal impact of political 'type' of incumbents on NPOs' behaviors, e.g., the level of risky investment, contribution received and paid, the required risk premium. One can also consider how a NPO designs its investment portfolio policy in the context of competition for donor base, which not only allows the NPO to put its risk attitude on the risk portfolio policy, but also requires the investment to be committed with social responsibilities ([Sparkes 2008](#), [Rodgers 1995](#)).

7 Appendix

7.1 Proof of Proposition 1

Proof. Rewrite equation (2) as:

$$r_t \Delta t = \frac{\theta}{1+\theta} \bar{r} \Delta t + \frac{\sigma_r}{1+\theta} \varepsilon_t \sqrt{\Delta t}. \quad (\text{A1})$$

The total change of NPO's endowment at the time interval $[t, t + \Delta t]$ can be rewritten as:

$$\begin{aligned} s_{t+\Delta t} - s_t &= \sum_{j=1}^m \beta_{tj} (e^{r_j \Delta t} - 1) (s_t + D_t \Delta t - G_t \Delta t) - (G_t - D_t) \Delta t \\ &= (\beta_t (e^{\frac{\theta}{1+\theta} \bar{r} \Delta t + \frac{\sigma_r}{1+\theta} \varepsilon_t \sqrt{\Delta t}} - 1) + (1 - \beta_t) (e^{r \Delta t} - 1)) (s_t - (G_t - D_t) \Delta t) \\ &\quad - (G_t - D_t) \Delta t. \end{aligned} \quad (\text{A2})$$

Using the Taylor expansion, $e^x = 1 + x$, and the fact that the mean of $\varepsilon_t \sqrt{\Delta t}$ is zero, we have two implications of equation (A2):

$$\begin{aligned} \mathbb{E}_t \{ s_{t+\Delta t} - s_t \} &= \left((\beta_t (\frac{\theta}{1+\theta} \bar{r} - r) + r) s_t - (G_t - D_t) \right) \Delta t + O((\Delta t)^2); \\ \mathbb{E}_t \{ (s_{t+\Delta t} - s_t)^2 \} &= \beta_t^2 s_t^2 \sigma_r^2 \Delta t + O((\Delta t)^2), \end{aligned}$$

where $O(\cdot)$ is the asymptotic order symbol representing “the same order as”, and \mathbb{E}_t is the expectation operator conditional on the information at time t . The stochastic differential equation of NPO's total financial endowment at time t can be written as:

$$ds_t = \left((\beta_t (\frac{\theta}{1+\theta} \bar{r} - r) + r) s_t - (G_t - D_t) \right) dt + \beta_t s_t \sigma_r dW_{tr}, \quad (\text{A3})$$

where W_t is the standard Wiener process.⁶ This completes the proof of Proposition 1. ■

⁶The Wiener process follows a normal distribution $\mathcal{N}(0, t)$, and is a continuous-time Markovian stochastic process; its increment satisfies $dW_t \equiv \lim_{\Delta t \rightarrow 0} \varepsilon_t \sqrt{\Delta t}$ with the analog of a discrete-time random walk $W_{t+1} - W_t = \varepsilon_t$, where $\varepsilon_t \sim \mathcal{N}(0, 1)$, and $W(0) = 0$.

7.2 Proof of Proposition 2

Proof. From equation (6), it follows that the change of the wealth level, Δw_t , can be written as:

$$w_{t+\Delta t} - w_t = (e^{r\Delta t} - 1)(w_t + (I_t - D_t - C_t)\Delta t) - (D_t + C_t - I_t)\Delta t. \quad (\text{A4})$$

Inserting equation (6) into equation (A4), the conditional expectations of $w_{t+\Delta t} - w_t$ and $(w_{t+\Delta t} - w_t)^2$ at time t can be written as:

$$\begin{aligned} \mathbb{E}_t\{w_{t+\Delta t} - w_t\} &= (rw_t + \bar{I} - D_t - C_t) \Delta t + O((\Delta t)^2), \\ \mathbb{E}_t\{(w_{t+\Delta t} - w_t)^2\} &= \sigma_I^2 \Delta t + O((\Delta t)^2). \end{aligned}$$

The corresponding stochastic differential equation of the state variable w_t is, therefore,

$$dw_t = (rw_t + \bar{I} - D_t - C_t) dt + \sigma_I dW_{tI}. \quad (\text{A5})$$

This proves Proposition 2. ■

7.3 Proof of Proposition 3

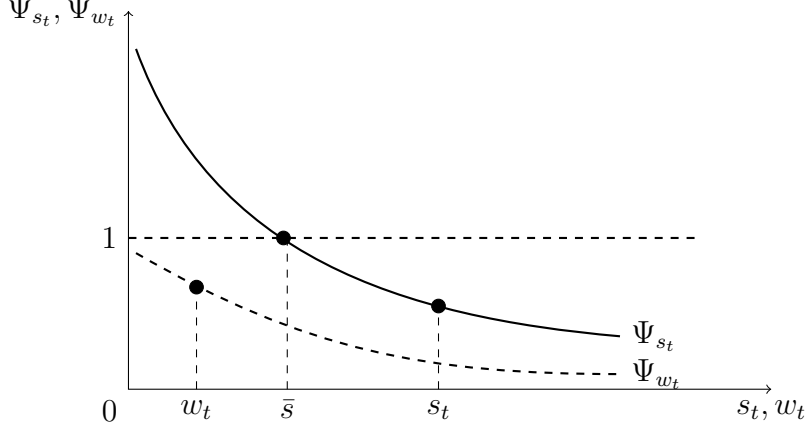
Proof. The proof follows from equations (3, 6) and the theorem from [Yeung and Petrosjan \(2006\)](#). ■

7.4 Proof of Proposition 4

Proof. Consider the donors' first-order conditions with respect to C_t and D_t . By donors' *Hamilton-Jacobi-Bellman* equation (13), the optimal choice pair (C_t^*, D_t^*) satisfies the following inequalities

$$1 - \Psi_{w_t} > 0 \text{ or } < 0, \quad (\text{A6})$$

$$\Psi_{s_t} - \Psi_{w_t} > 0 \text{ or } < 0. \quad (\text{A7})$$



Donors Shadow Price Curves Ψ_{s_t} and Ψ_{w_t}

Note: the solid curve represents donors' shadow price of NPO's endowment, whereas the dashed one is donors' shadow price of their wealth level. $\Psi_{s_t} > \Psi_{w_t}$ if $s_t < \bar{s}$; when $s_t > \bar{s}$, for a arbitrary pair of (s_t, w_t) , $\Psi_{s_t} < \Psi_{w_t}$.

which indicates that donors' private consumption, C_t^* , and contributions, D_t^* , must take the corner solutions instead of interior ones.

Assume that donors' value function, $\Psi(s_t, w_t)$, is increasing and strictly concave in both arguments s_t and w_t . Note that the donors' first-order conditions with respect to consumption, C_t , must be positive to ensure private consumptions at time t with any level of wealth, w_t . That is Ψ_{w_t} is monotonically decreasing and set within $(0, 1)$. Besides, Ψ_{w_t} follows the properties: $\lim_{w_t \rightarrow 0} \Psi_{w_t} = 1$ and $\lim_{w_t \rightarrow +\infty} \Psi_{w_t} = 0$. By equation (A6), the optimal private consumption, C_t^* , would hit the upper bound level, $\eta_c w_t$, where the parameter $\eta_c \in (0, 1)$ represents donors' propensity for consumption.

Donors' value function $\Psi(w_t, s_t)$ is concave in s_t and, therefore, Ψ_{s_t} is a monotonically decreasing function in s_t . It follows that $\lim_{s_t \rightarrow 0} \Psi_{s_t} = +\infty$ and $\lim_{s_t \rightarrow +\infty} \Psi_{s_t} = 0$, as shown in the Figure above. Consequently, there must exist a unique level of NPO's endowment, \bar{s} , such that $\Psi_{s_t} = 1$, which is greater than Ψ_{w_t} . Thus, $\Psi_{s_t} - \Psi_{w_t} > 0$ when $s_t < \bar{s}$. That is the donors' first-order condition with respect to contribution, D_t , is positive, which motivates donors to contribute to the upper bound, $D_t^* = \eta_d s_t$, where parameter $\eta_d \in (0, 1)$ represents the donors' desired level of public good. On the other hand, $\Psi_{s_t} - \Psi_{w_t} < 0$ when $s_t > \bar{s}$. Thus, donors do not contribute if $\Psi_{s_t} < \Psi_{w_t}$, whereas donors contribute $\eta_d s_t$ if $\Psi_{s_t} > \Psi_{w_t}$.

$$C_t^* = \eta_c w_t, \quad (\text{A8})$$

$$D_t^* = \begin{cases} \eta_d s_t & \text{if } s_t < \bar{s} \\ 0 & \text{if } s_t > \bar{s} \text{ and } \Psi_{s_t} < \Psi_{w_t} \\ \eta_d s_t & \text{if } s_t > \bar{s} \text{ and } \Psi_{s_t} > \Psi_{w_t}. \end{cases} \quad (\text{A9})$$

Next, consider the NPO's optimal problem. Following [Merton \(1971\)](#) we assume that the NPO's instantaneous utility function takes the form of CRRA: $u^n(G_t) = G_t^\alpha/\alpha$, where $\alpha \in (0, 1)$. By the NPO's first-order conditions equations ([14](#), [15](#)), G_t^* and β_t^* can be expressed as,

$$G_t^* = \Phi_{s_t}^{1/(\alpha-1)}, \quad (\text{A10})$$

$$\beta_t^* = -\frac{\Phi_{s_t}(\frac{\theta}{1+\theta}\bar{r} - r)}{\Phi_{s_t, s_t} s_t \sigma_r^2}. \quad (\text{A11})$$

Inserting equations ([A10](#), [A11](#)) into the NPO's *HJB* equation ([12](#)), yields,

$$r\Phi(s_t) = \begin{cases} \frac{1-\alpha}{\alpha}\Phi_{s_t}^{\frac{\alpha}{\alpha-1}} + \Phi_{s_t} r s_t - \frac{1}{2} \left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2 \frac{\Phi_{s_t}^2}{\Phi_{s_t, s_t} \sigma_r^2} & \text{if } D_t^* = 0, \\ \frac{1-\alpha}{\alpha}\Phi_{s_t}^{\frac{\alpha}{\alpha-1}} + \Phi_{s_t} (r + \eta_d) s_t - \frac{1}{2} \left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2 \frac{\Phi_{s_t}^2}{\Phi_{s_t, s_t} \sigma_r^2} & \text{if } D_t^* = \eta_d s_t. \end{cases} \quad (\text{A12})$$

Assume that the NPO's *BJH* equation, equation ([A12](#)), has the explicit solution, which takes the form of $\Phi(s_t) = \Lambda s_t^\alpha$ (See [Merton \(1971\)](#)). The function, $\Phi(s_t) = \Lambda s_t^\alpha$, will be the solution of equation ([A12](#)) if the parameter Λ satisfies the following conditions

$$\Lambda = \begin{cases} \frac{1}{\alpha} \left(r - \frac{\alpha(\frac{\theta}{1+\theta}\bar{r} - r)^2}{2(1-\alpha)^2 \sigma_r^2} \right)^{\alpha-1} & \text{if } D_t^* = 0, \\ \frac{1}{\alpha} \left(r - \frac{\alpha}{1-\alpha} \eta_d - \frac{\alpha(\frac{\theta}{1+\theta}\bar{r} - r)^2}{2(1-\alpha)^2 \sigma_r^2} \right)^{\alpha-1} & \text{if } D_t^* = \eta_d s_t. \end{cases} \quad (\text{A13})$$

Hence, by equations (A10, A11, A13), we obtain

$$G_t^* = \begin{cases} \left(r - \frac{\alpha(\frac{\theta}{1+\theta}\bar{r}-r)^2}{2(1-\alpha)^2\sigma_r^2}\right)s_t & \text{if } D_t^* = 0 \\ \left(r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha(\frac{\theta}{1+\theta}\bar{r}-r)^2}{2(1-\alpha)^2\sigma_r^2}\right)s_t & \text{if } D_t^* = \eta_d s_t, \end{cases} \quad (\text{A14})$$

$$\beta_t^* = \min \left\{ \max \left\{ \frac{\frac{\theta}{1+\theta}\bar{r} - r}{(1-\alpha)\sigma_r^2}, 0 \right\}, 1 \right\} \quad \text{if } D_t^* \geq 0. \quad (\text{A15})$$

This proves Proposition 4. ■

7.5 Proof of Remark 1

Proof. According to equations (A8, A9, A14, A15), both the donors' and the NPO's strategies only depend on the state variables s_t , and w_t , representing the Markov property of memoryless, which completes the proof of Remark 1. ■

7.6 Proof of Corollary 1

Proof. Define the function of the rate of return on T-bill, r ,

$$f(r) = r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha\left(\frac{\theta}{1+\theta}\bar{r} - r\right)^2}{2(1-\alpha)^2\sigma_r^2}.$$

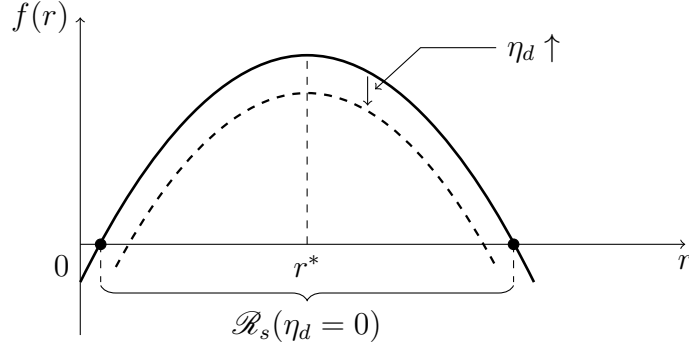
Rewrite $f(r)$ as:

$$f(r) = \frac{\alpha}{2(1-\alpha)^2\sigma_r^2} \left(-r^2 + 2\left(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}\right)r - \left(\frac{\theta}{1+\theta}\bar{r}\right)^2 - 2(1-\alpha)\sigma_r^2\eta_d \right). \quad (\text{A16})$$

By equation (A14), for any given parameter vector of agents and risky asset, $\{\alpha, \eta_d, \theta, \bar{r}, \sigma_r\}$, function f must map onto \mathbb{R}^+ . It follows from equation (A16) that we have $B^2 - 4AC > 0$, that is, parameters, $\{\alpha, \eta_d, \theta, \bar{r}, \sigma_r\}$, must satisfy the following inequality

$$\frac{1-\alpha}{\alpha} \left(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{2\alpha} \right) - \eta_d > 0. \quad (\text{A17})$$

This proves that G_t^* exists if and only if the subspace of the rate of return on T-bill, $\mathcal{R}_s =$



The Subspace of Rate of Return on T-bill, $\mathcal{R}_s(\eta_d)$

Note: The subspace, $\mathcal{R}_s(\eta_d = 0)$, equals $\{r \in \mathbb{R}^+ \mid f(r) > 0 \cap \eta_d = 0\} \neq \emptyset$, where $f(r) = r - \frac{\alpha}{1-\alpha}\eta_d - \frac{\alpha(\frac{\theta}{1+\theta}\bar{r}-r)^2}{2(1-\alpha)^2\sigma_r^2}$. $r^* = \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}$. The dashed parabola represents the case when η_d is increasing.

$$\{r \in \mathbb{R}^+ \mid f(r) > 0\} \neq \emptyset.$$

Donors do not contribute when parameter η_d equals zero. Straightforward calculations show that the subspace of rate of return on T-bill $\mathcal{R}_s(\eta_d = 0) = \frac{2(1-\alpha)\sigma_r}{\alpha} \sqrt{\frac{2\theta}{1+\theta}\bar{r}\alpha + (1-\alpha)^2\sigma_r^2}$ is not empty. This establishes that there exists a subspace \mathcal{R}_s such that the NPO is willing to provide the public good when donors do not contribute.

Note that the function $f(r)$ has a critical point, $r^* = \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}$, which does not depend on η_d ; and $\mathcal{R}_s = \frac{2(1-\alpha)\sigma_r}{\alpha} \sqrt{\frac{2\theta}{1+\theta}\bar{r}\alpha + (1-\alpha)^2\sigma_r^2 - \frac{2\alpha^2}{(1-\alpha)}\eta_d}$. One can verify that \mathcal{R}_s has the following property: $\frac{\partial \mathcal{R}_s}{\partial \eta_d} < 0$. As a result, increasing η_d continuously pushes the solid parabola curve downwards, which monotonically shrinks the subspace, \mathcal{R}_s (See Figure below). Therefore, there exists $\hat{\eta}_d$, such that $\mathcal{R}_s = \{r \in \mathbb{R}^+ \mid f(r) > 0\} = \emptyset$. By equation (A17), the threshold $\hat{\eta}_d = \frac{1-\alpha}{\alpha} \left(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{2\alpha} \right)$. This completes the proof of Corollary 1. \blacksquare

7.7 Proof of Corollary 2

Proof. Note that for any given parameters, $\{\alpha, \eta_d, \theta, \bar{r}, \sigma_r\}$, the function $f(r)$ has the critical point $(r^*, f(r^*)) = (\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}, \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{2\alpha} - \frac{\alpha}{1-\alpha}\eta_d)$. The subspace of the rate of return on T-bill is,

$$\mathcal{R}_s = \frac{2(1-\alpha)\sigma_r}{\alpha} \sqrt{\frac{2\theta}{1+\theta}\bar{r}\alpha + (1-\alpha)^2\sigma_r^2 - \frac{2\alpha^2}{(1-\alpha)}\eta_d}.$$

This subspace \mathcal{R}_s has the property,

$$\begin{cases} \frac{\partial \mathcal{R}_s}{\partial \alpha} < 0, \\ \frac{\partial \mathcal{R}_s}{\partial \sigma_r} > 0, \frac{\partial \mathcal{R}_s}{\partial \bar{r}} > 0, \frac{\partial \mathcal{R}_s}{\partial \theta} > 0. \end{cases}$$

This establishes that the subspace of rate of return on T-bill, \mathcal{R}_s , increases with σ_r , \bar{r} , and θ ; and decreases with the NPO's risk parameter, α .

Direct calculations show that the critical point $(r^*, f(r^*))$ shifts when parameters $\{\alpha, \theta, \bar{r}, \sigma_r\}$ change:

$$\begin{cases} \frac{\partial r^*}{\partial \alpha} < 0, \frac{\partial r^*}{\partial \sigma_r} > 0, \frac{\partial r^*}{\partial \bar{r}} > 0, \frac{\partial r^*}{\partial \theta} > 0, \\ \frac{\partial f(r^*)}{\partial \alpha} < 0, \frac{\partial f(r^*)}{\partial \sigma_r} > 0, \frac{\partial f(r^*)}{\partial \bar{r}} > 0, \frac{\partial f(r^*)}{\partial \theta} > 0. \end{cases}$$

Hence, the $f(r)$ curve shifts to lower left or upper right. This finishes the proof of Corollary 2. ■

7.8 Proof of Proposition 5

Proof. By Proposition 3, given the NPO's optimal strategy $\{G_t^*, \beta_t^*\}$, donors' optimal decision $\{C_t^*, D_t^*\}$ is such that value function $\Psi(s_t, w_t)$ must satisfy

$$\begin{aligned} r\Psi(s_t, w_t) = & \max_{\substack{C_t \in (0, \eta_c w_t] \\ D_t \in [0, \eta_d s_t]}} \left\{ u^d(G_t^*) + C_t + \Psi_{s_t} \left(\left(\beta_t^* \left(\frac{\theta}{1+\theta} \bar{r} - r \right) + r \right) s_t - (G_t^* - D_t) \right) \right. \\ & \left. + \Psi_{w_t} (r w_t + \bar{I} - D_t - C_t) + \frac{1}{2} \Psi_{s_t, s_t} \beta_t^{*2} s_t^2 \sigma_r^2 + \frac{1}{2} \Psi_{w_t, w_t} \sigma_I^2 \right\}. \end{aligned} \quad (\text{A18})$$

By equation (9), the donors' wealth variation is exactly the same as their income volatility, σ_I^2 , which is not related to the donors' decision $\{C_t^*, D_t^*\}$. Thus, the change of the term $\frac{1}{2} \Psi_{w_t, w_t} \sigma_I^2$ in equation (A18) does not affect the donors' first-order conditions (See equations (A6, A7)). That is the change of donors' income risk exposure does not have any effect on the equilibrium of the game.

From Proposition 4, a more volatile investment project leads the NPO to lower its portfolio selection and provide more charitable goods, which decreases its endowment level. Meanwhile,

donors' contribution shrinks. Proposition 5 complements the experimental evidence by Cettolin et al. (2017). This completes the proof of Proposition 5. ■

7.9 Proof of Corollary 3

Proof. At time interval $[t, t + \Delta t]$, the discrete forms of donors' income and rate of return differential equations are,

$$\begin{cases} I_t \Delta t = \bar{I} \Delta t + \sigma_I \varepsilon_t \sqrt{\Delta t}, \\ r_t \Delta t = \vartheta (\tilde{r} - r_t) \Delta t + \tilde{\sigma} \varepsilon_t \sqrt{\Delta t}. \end{cases} \quad (\text{A19})$$

It follows that the change of the wealth level, Δw_t , can be written as:

$$w_{t+\Delta t} - w_t = (e^{r_t \Delta t} - 1)(w_t + (I_t - D_t - C_t) \Delta t) - (D_t + C_t - I_t) \Delta t. \quad (\text{A20})$$

Inserting equation (A19) into equation (A20), the conditional expectations of $w_{t+\Delta t} - w_t$ and $(w_{t+\Delta t} - w_t)^2$ at time t can be written as:

$$\begin{aligned} \mathbb{E}_t \{ w_{t+\Delta t} - w_t \} &= \left(\frac{\vartheta}{1 + \vartheta} \tilde{r} w_t + \bar{I} - D_t - C_t \right) \Delta t + O((\Delta t)^2), \\ \mathbb{E}_t \{ (w_{t+\Delta t} - w_t)^2 \} &= \left(\frac{1}{(1 + \vartheta)^2} w_t^2 \tilde{\sigma}^2 + \sigma_I^2 \right) \Delta t + O((\Delta t)^2). \end{aligned}$$

The corresponding stochastic differential equation of the state variable w_t is, therefore,

$$dw_t = \left(\frac{\vartheta}{1 + \vartheta} \tilde{r} w_t + \bar{I} - D_t - C_t \right) dt + \sqrt{\frac{1}{(1 + \vartheta)^2} w_t^2 \tilde{\sigma}^2 + \sigma_I^2} dW_t. \quad (\text{A21})$$

By Proposition 3, one can derive that NPOs and donors have the same first-order conditions as in the benchmark case. This proves Corollary 3. ■

7.10 Proof of Proposition 6

Proof. Substituting β^* , G_t^* , and D_t^* into equation (4) leads to

$$ds_t = \left(\beta_t^* \left(\frac{\theta}{1+\theta} \bar{r} - r \right) + r - \left((\Lambda\alpha)^{1/(\alpha-1)} - \eta_d \right) \right) s_t dt + \beta_t^* s_t \sigma_r dW_{tr},$$

where $\eta_d \geq 0$. Following [Kuo \(2006\)](#), this stochastic differential equation has the following solution:

$$\begin{aligned} s_t &= s_0 e^{\left(\beta_t^* \left(\frac{\theta}{1+\theta} \bar{r} - r \right) + r - \left((\Lambda\alpha)^{1/(\alpha-1)} - \eta_d \right) - 0.5 \beta_t^{*2} \sigma_r^2 \right) t + \beta_t^* \sigma_r W_{tr}} \\ &\quad + \int_0^t 0 \times e^{\left(\beta_\tau^* \left(\frac{\theta}{1+\theta} \bar{r} - r \right) + r - \left((\Lambda\alpha)^{1/(\alpha-1)} - \eta_d \right) - 0.5 \beta_\tau^{*2} \sigma_r^2 \right) (t-\tau) + \beta_\tau^* \sigma_r (W_{tr} - W_{\tau r})} d\tau. \end{aligned}$$

We take the expectation of both sides conditional on the information at time t . By the property of geometric Brownian motion, we have

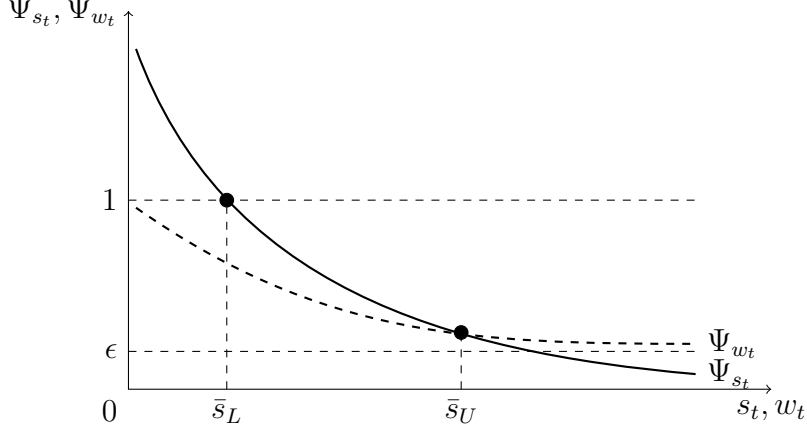
$$\begin{aligned} \mathbb{E}_t \{ s_t \} &= \mathbb{E}_t \left\{ s_0 e^{\left(\beta_t^* \left(\frac{\theta}{1+\theta} \bar{r} - r \right) + r - \left((\Lambda\alpha)^{1/(\alpha-1)} - \eta_d \right) \right) t} \right\} \\ &= s_0 e^{\left(\beta_t^* \left(\frac{\theta}{1+\theta} \bar{r} - r \right) + r - \left((\Lambda\alpha)^{1/(\alpha-1)} - \eta_d \right) \right) t}, \end{aligned} \tag{A22}$$

where $(\alpha\Lambda)^{1/(\alpha-1)} = r - \frac{\alpha}{1-\alpha} \eta_d - \frac{\alpha}{2} \sigma_r^2 \beta_t^2$, and β_t^* equals 0, $\frac{\frac{\theta}{1+\theta} \bar{r} - r}{(1-\alpha)\sigma_r^2}$, 1 for $r > r_U$, $r \in \mathcal{R}_e$, and $r < r_L$, respectively.

By equation (A22), the NPO's expected endowment can be simplified as,

$$\mathbb{E} \{ s_t \} = s_0 e^{\left(\beta_t^* \left(\frac{\theta}{1+\theta} \bar{r} - r \right) + \frac{1}{1-\alpha} \eta_d + \frac{\alpha}{2} \sigma_r^2 \beta_t^2 \right) t}. \tag{A23}$$

By Proposition 4, the NPO chooses the optimal share of the risky asset equal to 1, β_t^* , and 0, respectively, contingent on the rate of return on T-bill. It follows,



Donors Shadow Price Curves Ψ_{s_t} and Ψ_{w_t}

Note: the solid curve represents the donors' shadow price of NPO's endowment, whereas the dashed one represents the donors' shadow price of their wealth level. $\Psi_{s_t} > \Psi_{w_t}$ if $s_t \leq \bar{s}_L$; when $\bar{s}_L < s_t < \bar{s}_U$, for an arbitrary w_t , $\Psi_{s_t} > \Psi_{w_t}$; $\Psi_{s_t} < \Psi_{w_t}$ if $s_t > \bar{s}_U$.

$$\mathbb{E}\{s_t\} = \begin{cases} s_0 e^{\left(\frac{\theta}{1+\theta}\bar{r}-r+\frac{\eta_d}{1-\alpha}+\frac{\alpha}{2}\sigma_r^2\right)t} & \text{if } r \leq r_L, \beta_t^* = 1, \\ s_0 e^{\left(\frac{\eta_d}{1-\alpha}+\frac{(2-\alpha)\left(\frac{\theta}{1+\theta}\bar{r}-r\right)^2}{2(1-\alpha)^2\sigma_r^2}\right)t} & \text{if } r \in \mathcal{R}_e, \beta_t^* = \frac{\theta}{1+\theta}\frac{\bar{r}-r}{(1-\alpha)\sigma_r^2}, \\ s_0 e^{\frac{\eta_d}{1-\alpha}t} & \text{if } r \geq r_U, \beta_t^* = 0. \end{cases}$$

It is innocuous to assume that the donors' shadow price of wealth level, Ψ_{w_t} , follows $\lim_{t \rightarrow +\infty} \Psi_{w_t} = \epsilon$, where $0 < \epsilon < 1$ (See Figure above). Thus, we derive the following properties,

$$D_t^* = \begin{cases} \eta_d s_t & \text{if } s_t \leq \bar{s}_L, \\ \eta_d s_t \text{ or } 0 & \text{if } \bar{s}_L < s_t < \bar{s}_U, \\ 0 & \text{if } s_t \geq \bar{s}_U. \end{cases}$$

Therefore, the limit of $\mathbb{E}\{s_t\}$ is given by

$$\lim_{t \rightarrow +\infty} \mathbb{E}\{s_t\} = \begin{cases} \infty & \text{if } r < r_U, \\ \bar{s}_\tau & \text{if } r \geq r_U. \end{cases}$$

where $\bar{s}_\tau \in (\bar{s}_1, \bar{s}_2]$ is the minimum level of the NPO's endowment such that donors do not contribute $\forall s_t > \bar{s}_\tau$. This proves Proposition 6. ■

7.11 Proof of Proposition 7

Proof. By equation (A17), function $f(r)$ takes the quadratic form,

$$f(r) = \frac{\alpha}{2(1-\alpha)^2\sigma_r^2} \left(-r^2 + 2\left(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}\right)r - \left(\frac{\theta}{1+\theta}\bar{r}\right)^2 - 2(1-\alpha)\sigma_r^2\eta_d \right),$$

with the critical point $(r^*, f(r^*)) = \left(\frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{\alpha}, \frac{\theta}{1+\theta}\bar{r} + \frac{(1-\alpha)^2\sigma_r^2}{2\alpha} - \frac{\alpha}{1-\alpha}\eta_d\right)$. It follows that the impact of r on the optimal provision of charitable good, G_t^* , is non-monotonic.

However, the change of other parameters shifts the function $f(r)$ curve to the lower left, vertically, or to the upper right (See Figure 6), but the dominance of function $f(r)$ in its domain, \mathcal{R}_s , is consistent, which shows that the impact of parameters except r is monotonic, as shown in Figure 9. This completes the proof of Proposition 7. ■

7.12 Proof of Proposition 8

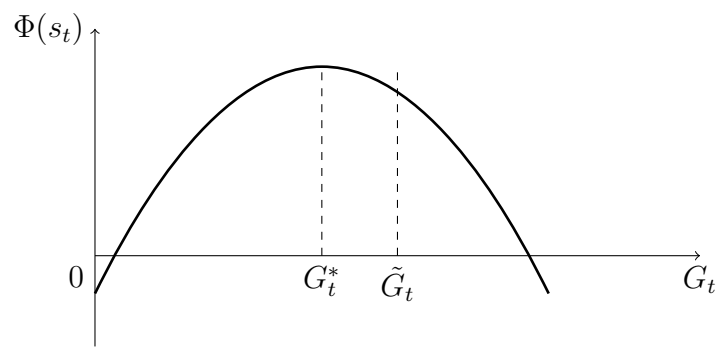
Proof. By the NPO's first-order condition with respect to β_t (See equation (15)), it is direct that if the portfolio ceiling is binding ($\tilde{\beta}_t < \beta_t^*$), the NPO chooses $\tilde{\beta}_t$ instead of β_t^* . This completes the proof of Proposition 8. ■

7.13 Proof of Proposition 9

Proof. Insert NPO's instantaneous utility function and value function into its first-order condition with respect to D_t , equation (14) turns to,

$$G_t^{\alpha-1} - \Lambda\alpha s_t^{\alpha-1} = 0.$$

The NPO is indeed worse off when the when the provision floor is greater than the optimal level. Thus, the best choice for NPO is the provision floor. This completes the proof of Proposition 9. ■



The Impact of Provision Floor, \tilde{G}_t

References

- Andreoni, J.: 1988, Privately provided public goods in a large economy: the limits of altruism, *Journal of public Economics* **35**(1), 57–73.
- Andreoni, J.: 2006, Philanthropy, *Handbook of the economics of giving, altruism and reciprocity* **2**, 1201–1269.
- Andreoni, J. and Payne, A. A.: 2003, Do government grants to private charities crowd out giving or fund-raising?, *American Economic Review* **93**(3), 792–812.
- Andreoni, J. and Payne, A. A.: 2011, Is crowding out due entirely to fundraising? evidence from a panel of charities, *Journal of public Economics* **95**(5-6), 334–343.
- Becker, G.: 1974, A theory of social interactions, *Journal of Political Economy* **82**(6), 1063–1093.
- Bergstrom, T., Blume, L. and Varian, H.: 1986, On the private provision of public goods, *Journal of public economics* **29**(1), 25–49.
- Bowman, W.: 2002, The uniqueness of nonprofit finance and the decision to borrow, *Nonprofit Management and Leadership* **12**(3), 293–311.
- Cettolin, E., Riedl, A. and Tran, G.: 2017, Giving in the face of risk, *Journal of risk and uncertainty* **55**(2-3), 95–118.
- Fershtman, C. and Nitzan, S.: 1991, Dynamic voluntary provision of public goods, *European Economic Review* **35**(5), 1057–1067.
- Imbs, J.: 2007, Growth and volatility, *Journal of Monetary Economics* **54**(7), 1848–1862.
- Jegers, M. and Verschueren, I.: 2006, On the capital structure of non-profit organisations: An empirical study for californian organisations, *Financial Accountability & Management* **22**(4), 309–329.
- Kamdar, A., Levitt, S. D., List, J. A., Mullaney, B. and Syverson, C.: 2015, Once and done: Leveraging behavioral economics to increase charitable contributions.

- Kossioris, G., Plexousakis, M., Xepapadeas, A., de Zeeuw, A. and Mäler, K.-G.: 2008, Feedback nash equilibria for non-linear differential games in pollution control, *Journal of Economic Dynamics and Control* **32**(4), 1312–1331.
- Kumru, C. S. and Vesterlund, L.: 2010, The effect of status on charitable giving, *Journal of Public Economic Theory* **12**(4), 709–735.
- Kuo, H.-H.: 2006, Stochastic differential equations, *Introduction to Stochastic Integration* pp. 185–230.
- List, J. A.: 2011, The market for charitable giving, *Journal of Economic Perspectives* **25**(2), 157–80.
- Lohse, T., Robledo, J. R. and Schmidt, U.: 2012, Self-insurance and self-protection as public goods, *Journal of Risk and Insurance* **79**(1), 57–76.
- Markowitz, H.: 1952, Portfolio selection, *The journal of finance* **7**(1), 77–91.
- Marsiliani, L. and Renstroem, T.: 2010, Privately provided public goods in a dynamic economy.
- Marx, L. M. and Matthews, S. A.: 2000, Dynamic voluntary contribution to a public project, *The Review of Economic Studies* **67**(2), 327–358.
- Merton, R.: 1971, Optimum consumption and portfolio-rules in a continuous-time framework, *Journal of Economic Theory* (December 1971) .
- Ramey, G. and Ramey, V. A.: 1994, Cross-country evidence on the link between volatility and growth.
- Rodgers, S. K.: 1995, Developing the investment policy statement, *The Philanthropist* **12**(3), 9–12.
- Rondeau, D. and List, J. A.: 2008, Matching and challenge gifts to charity: evidence from laboratory and natural field experiments, *Experimental economics* **11**(3), 253–267.
- Rosen, H. S. and Sappington, A. J.: 2016, To borrow or not to borrow? an analysis of university leverage decisions, *Research in Economics* **70**(1), 170–185.

- Samuelson, P. A.: 1975, Lifetime portfolio selection by dynamic stochastic programming, *Stochastic Optimization Models in Finance*, Elsevier, pp. 517–524.
- Scott, K.: 2006, *The nonprofit and voluntary sector in Ontario: Regional highlights of the National Survey of Nonprofit and Voluntary Organizations*, Imagine Canada.
- Sherlock, M. F. and Gravelle, J. G.: 2009, An overview of the nonprofit and charitable sector, *Washington DC: Congressional Research Office* .
- Song, P. H., Smith, D. G. and Wheeler, J. R.: 2008, It was the best of times, it was the worst of times: A tale of two years in not-for-profit hospital financial investments, *Health Care Management Review* **33**(3), 234–242.
- Sparkes, R.: 2008, Socially responsible investment, *Handbook of finance* **2**.
- Tobin, J.: 1958, Liquidity preference as behavior towards risk, *The review of economic studies* **25**(2), 65–86.
- Varian, H. R.: 1994, Sequential contributions to public goods, *Journal of Public Economics* **53**(2), 165–186.
- Wang, W.-K. and Ewald, C.-O.: 2010a, Dynamic voluntary provision of public goods with uncertainty: a stochastic differential game model, *Decisions in Economics and Finance* **33**(2), 97–116.
- Wang, W.-K. and Ewald, C.-O.: 2010b, A stochastic differential fishery game for a two species fish population with ecological interaction, *Journal of Economic Dynamics and Control* **34**(5), 844–857.
- Waters, R. D.: 2008, Applying relationship management theory to the fundraising process for individual donors, *Journal of Communication Management* **12**(1), 73–87.
- Wedig, G. J.: 1994, Risk, leverage, donations and dividends-in-kind: A theory of nonprofit financial behavior, *International Review of Economics & Finance* **3**(3), 257–278.
- Weisbrod, B. A.: 2009, *The nonprofit economy*, Harvard University Press.

Xing, H., Lai, T. L. and Chen, Z.: 2010, Mean-variance portfolio optimization when means and covariances are unknown, *Available at SSRN 1694057* .

Yeung, D. W. and Petrosjan, L. A.: 2006, *Cooperative stochastic differential games*, Springer Science & Business Media.

Yildirim, H.: 2006, Getting the ball rolling: Voluntary contributions to a large-scale public project, *Journal of Public Economic Theory* **8**(4), 503–528.

URL: <http://dx.doi.org/10.1111/j.1467-9779.2006.00275.x>