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A Survey of Some Problems in Affine Algebraic Geometry

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A SURVEY OF SOME PROBLEMS IN AFFINE ALGEBRAIC GEOMETRY

By

Raman K. Pall, B.Sc.

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A Thesis
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If I cannot brag of knowing something, then I brag of not knowing it.

R. W. Emerson, 1866

Education is an admirable thing, but it is well to remember from time to time that nothing worth knowing can be taught.

Oscar Wilde, 1891

Abstract

We describe the status of the following two questions: What are the automorphisms of the affine n-space $\mathbf{A}^n$? Given $m < n$, is there a unique way (up to automorphisms of $\mathbf{A}^n$) to embed $\mathbf{A}^m$ in $\mathbf{A}^n$?

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Introduction

Given a field k and indeterminates $X_1, X_2, \ldots$, we ask:

1. What are the automorphisms of the k-algebra $k[X_1, \ldots, X_n]$? (In geometric terms: what are the automorphisms of the affine n-space $\mathbf{A}_k^n$?)

2. If $\varepsilon_1, \varepsilon_2 : k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_m]$ are any two surjective homomorphisms of k-algebras, does there exist an automorphism θ of $k[X_1, \ldots, X_n]$ such that $\varepsilon_2 = \varepsilon_1 \circ \theta$? (Geometrically: up to automorphisms of $\mathbf{A}^n$, is there a unique way to embed $\mathbf{A}_k^m$ in $\mathbf{A}_k^n$?)

In the case $n = 2$, both problems have been solved in the twentieth century: (1) is the Jung-van der Kulk Theorem (1942, 1953) (see theorem 26) and (2) is the Abhyankar-Moh-Suzuki Theorem (1974, 1975) (see theorem 39). Both of these results are highly nontrivial and have played a major role in the evolution of affine algebraic geometry. Questions (1) and (2) are still open when $n \geq 3$.

These two questions are in fact the main themes of the present thesis. Our aim is to describe the current status of these questions, to examine the relations between different versions of the problems, to discuss conjectures and to give examples. However we shall not attempt to contribute new results. Note also that we restrict ourselves to the above two questions and that other important problems of affine algebraic geometry are not considered here, notably the Jacobian Problem, the Cancellation Problem, the Characterization of the affine n-space, and so on.

Although we have consulted a variety of sources for preparing this thesis, our main source was van den Essen's book [34] and in some cases (for instance in writing chapter 2) we followed his exposition quite closely.

The thesis is divided into four chapters. In Chapter 1, several basic concepts are introduced: the notion of a coordinate system of a polynomial ring $R[X]$, the concept of a derivation on a ring A, locally nilpotent derivations, and the exponential map associated to a derivation D. In particular, it is shown that if A is a $\mathbb{Q}$-algebra and $D \in \text{Der}A$ is a locally nilpotent derivation on A, then $\exp D : A \to A$ is a ring automorphism on A. Several propositions are proved which prove useful in the later chapters. We end the chapter with a brief discussion of the Classification Problem for locally nilpotent derivations.

Chapter 2 is concerned with the group of automorphisms $\text{Aut}_k k[X_1, \ldots, X_n]$ of a polynomial ring $k[X_1, \ldots, X_n]$ over a field k. We begin with the definition of some key subgroups of $\text{Aut}_k k[X_1, \ldots, X_n]$, namely, the affine subgroup, the de Jonquières subgroup, the elementary subgroup, and the tame subgroup. It is shown that if R is a domain and if $n = 2$, the tame subgroup $T(R, n)$ is equal to the amalgamated free product of the subgroup of affine automorphisms $\text{Aff}(R, n)$ and the subgroup of triangular automorphisms $J(R, n)$ over their intersection.

We describe an algorithm to decide whether a polynomial endomorphism of $R[X, Y]$ is tame. Furthermore, we provide a partial proof for the Jung-van der Kulk theorem stating that $\text{Aut}_k k[X, Y] = T(k, 2)$, where k is a field. The proof depends on Rentschler's theorem, whose proof we outline in the same manner as van den Essen [34]. Some remarks on other proofs of the equality $\text{Aut}_k k[X, Y] = T(k, 2)$ are provided in section 2.2.2.

We next study $T(R, n)$ in the case where R is a commutative $\mathbb{Q}$-algebra and $R[X]$ is a polynomial ring in $n \geq 3$ variables over R. We prove Derksen's result stating that if R is a field, then $T(R, n) = \langle \text{Aff}(R, n), \varepsilon \rangle$ where ε is the non-linear automorphism $(X_1 + X_2^2, X_2, \ldots, X_n)$. We also discuss the Nagata automorphism and the exponential conjecture in sections 2.3.1 and 2.3.2 respectively.

Chapter 3 deals mainly with the Abhyankar-Moh-Suzuki Theorem, stating that if k is a field of characteristic zero and given nonconstant polynomials $f(t), g(t) \in k[t]$ such that $k[f(t), g(t)] = k[t]$, then either $\deg f(t) | \deg g(t)$ or $\deg g(t) | \deg f(t)$.

Some history on the theorem is presented, as well as some of its reformulations (and consequences). We then present some results for the case where k is a field of characteristic $p > 0$. Nagata's example is presented as a counterexample to the Abhyankar-Moh-Suzuki Theorem in characteristic $p > 0$, and Moh's Conjecture is stated. We end the chapter with some generalizations of the theory when the field k is replaced by a ring R.

In Chapter 4, we examine the question whether there is a unique way to embed $\mathbf{A}^m$ in $\mathbf{A}^n$, up to automorphisms of $\mathbf{A}^n$. We define the concepts of embeddings, rectifiablity of embeddings, and equivalence of embeddings.

We discuss the geometric formulation of the Abhyankar-Moh-Suzuki Theorem, stating that if k is an algebraically closed field of characteristic zero, then every embedding of $\mathbf{A}^1$ in $\mathbf{A}^2$ is rectifiable. Moreover, we consider Nagata's example as an embedding of $\mathbf{A}^1_k$ in $\mathbf{A}^2_k$, where k is a field of characteristic $p > 0$.

We consider the results of Craighero, Jelonek, and Srinivas showing that every embedding of $\mathbf{A}^1$ in $\mathbf{A}^n$ is rectifiable when $n \geq 4$. The remaining case, embeddings of $\mathbf{A}^1$ in $\mathbf{A}^3$, is then examined, and Shastri's conjecture is presented.

We end the chapter with an overview of the Abhyankar-Sathaye Conjecture and the Sathaye Conjecture.

Contents

Chapter 1

Preliminaries

1.1 Coordinate systems of $R[X]$

We start with some notation. Let R be an arbitrary commutative ring, and R^* its group of units. We shall denote by $R[X]$ the polynomial ring in n variables over R, i.e., $R[X] = R[X_1, \ldots, X_n]$ unless otherwise stated. Also note that $R[[X]] = R[[X_1, \ldots, X_n]]$ is the ring of formal power series in n variables over R.

We denote the set of all automorphisms of $R[X]$ that fix R (i.e. are identity on R) by $\mathrm{Aut}_R(R[X])$.

We shall often use the same notation for an n-tuple $F = (F_1, \ldots, F_n) \in R[X]^n$ and for the corresponding R-endomorphism $F : R[X] \to R[X]$ mapping each X_i to F_i. Moreover, we use the following non-standard notation, which is customary in the field of polynomial automorphisms:

Warning. If F, G are R-endomorphisms of $R[X]$, then by FG we mean $G \circ F$.

In other words, when the symbol '$\circ$' is present, we mean the usual composition $((F \circ G)(a) = F(G(a)))$, but when it is absent we mean backward composition.

Let $F = (F_1, \ldots, F_n)$ and $G = (G_1, \ldots, G_n)$ be two endomorphisms of $R[X]$. Then we have the following rule for composition:

$$FG = (F_1, \ldots, F_n)(G_1, \ldots, G_n) = (F_1(G_1, \ldots, G_n), \ldots, F_n(G_1, \ldots, G_n)).$$

Evaluating at X_i yields

$$(FG)(X_i) = G(F(X_i)) = G(F_i) = F_i(G(X_1), \ldots, G(X_n)) = F_i(G_1, \ldots, G_n),$$

so the above rule is valid.

We say that F is *invertible* over R or that $(F_1, \ldots, F_n)$ is a *coordinate system* of $R[X]$ if $R[F_1, \ldots, F_n] = R[X_1, \ldots, X_n]$, i.e., if there exist $G_1, \ldots, G_n \in R[X]$ such that $X_i = G_i(F_1, \ldots, F_n)$ for each i.

Proposition 1. *If $F \in R[X]^n$ is invertible, then the $G = (G_1, \ldots, G_n)$ defined as above is uniquely determined and satisfies $F \circ G = X = G \circ F$.*

In clear terms, this proposition asserts that any surjective endomorphism of the R-algebra $R[X]$ is in fact bijective. For a proof, see van den Essen's book [34].

Let $G \in R[X]$. We say that G is a *variable* of $R[X]$ if there exist $F_1, \ldots, F_{n-1} \in R[X]$ such that $(G, F_1, \ldots, F_{n-1})$ is a coordinate system of $R[X]$.

Let $G \in R[X] = R[X_1, \ldots, X_n]$. It is easy to see that if G is a variable of $R[X]$ then $R[X]/(G)$ is isomorphic to a polynomial ring in $n-1$ variables over R. The question whether the converse is true turns out to be surprisingly interesting, as we will see in chapters 3 and 4.

1.2 Derivations

Let A and R be commutative rings. A *derivation on A* is an additive map $D : A \to A$ satisfying the Leibniz rule, i.e.

$$D(a+b) = D(a) + D(b),\ D(ab) = aD(b) + bD(a), \text{ for all } a, b \in A.$$

As an example, consider $A = R[X] = R[X_1, \ldots, X_n]$. Then the usual partial derivative $\frac{\partial}{\partial X_i}$, which we sometimes denote by ∂_i, is a derivation on A.

We find that

$$D^n(ab) = \sum_{i=0}^{n} \binom{n}{i} D^i(a) D^{n-i}(b)$$

for all $a, b \in A$, for all $n \geq 1$.

We denote the set of all derivations on A by Der$A$. Moreover, if A is an R-algebra via a ring homomorphism $f : R \rightarrow A$, we say that D is an *R-derivation* if $D \circ f = 0$. We denote the set of all R-derivations on A by $\mathrm{Der}_R A$ (i.e. the set of all derivations on an R-algebra A that are trivial on R).

We have $D(1) = D(1 \cdot 1) = D(1) + D(1) \implies D(1) = 0$ for all $D \in \mathrm{Der}A$. Hence viewing A as a $\mathbb{Z}$-algebra, we get $\mathrm{Der}A = \mathrm{Der}_{\mathbb{Z}} A$. If D and D' belong to $\mathrm{Der}_R A$, then one can check that the *Lie bracket*

$$[D, D'] = DD' - D'D$$

also belongs to $\mathrm{Der}_R A$. In fact equipped with this bracket $\mathrm{Der}_R A$ forms a *Lie algebra.*

Furthermore, if A is commutative, then $\mathrm{Der}_R A$ is a left A-module with the obvious addition and multiplication by elements of A.

The simple proposition below shall prove useful in allowing us to find a basis for $\mathrm{Der}_R R[X]$.

Lemma 2. *Let G be a generating set for the R-algebra A and let $D \in \mathrm{Der}_R A$. Then D is completely determined by the elements $D(g)$, $g \in G$.*

Proposition 3. *Let $R[X] = R[X_1, \ldots, X_n]$ and let ∂_i denote the partial derivative with respect to X_i. Then $\mathrm{Der}_R R[X]$ is a free $R[X]$-module with basis $\partial_1, \ldots, \partial_n$ and $[\partial_i, \partial_j] = 0$ (i.e. $\partial_i \partial_j = \partial_j \partial_i$) for all i, j.*

The proof of the proposition above results by noting that, given any $D \in \mathrm{Der}_R R[X]$, the derivation $D_0 = D - \sum D(X_i)\partial_i$, is zero on each R-generator X_i, and thus $D_0 = 0$ by the lemma.

Applying the lemma to the (unrelated)derivation $D_0 = [\partial_i, \partial_j]$ completes the proof, by showing that D_0 is 0 on each X_i, and so $D_0 = 0$, as desired.

Proposition 4. (Extension of derivations)
Let A be a commutative ring and $D \in \mathrm{Der}A$.

1. *Let $S \subset A$ be a multiplicatively closed subset. Then the formula*

$$\tilde{D}\left(\frac{a}{s}\right) = \frac{sD(a) - aD(s)}{s^2}$$

defines a derivation $\tilde{D} : S^{-1}A \longrightarrow S^{-1}A$. Moreover, $\tilde{D}$ is the only derivation of $S^{-1}A$ which satisfies $\tilde{D}\left(\frac{a}{1}\right) = \frac{D(a)}{1}$ for all $a \in A$.

If A has no S-torsion then we may view A as a subring of $S^{-1}A$. In this case, we call $\tilde{D}$ the unique extension of D to $S^{-1}A$. In particular, every derivation on a domain has a unique extension to its quotient field.

2. *Let $K \subset L$ be a separable algebraic extension of fields, and let $D \in \mathrm{Der}K$. Then D can be extended uniquely to a derivation $\tilde{D}$ on L.*

For a proof, we refer the reader to Lang [15].

Let $D : A \longrightarrow A$ be a derivation on A. The kernel of D will be denoted by A^D or $\ker(D, A)$. A^D is a subring of A, called the *ring of constants of D*. Note that if A is a field, then so is A^D, and we call it the *field of constants of D*.

Lemma 5. *If A is a domain of characteristic zero, then A^D is algebraically closed in A.*

Proof. Let $0 \neq a \in A$ be algebraic over A^D. Then there exists $n \in \mathbb{N}$ and $c_0, \ldots, c_{n-1}, c_n \in A^D$ such that $c_n \neq 0$ and

$$c_n a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0.$$

Take n to be minimal with this property. Then applying D to this equation gives

$$(nc_n a^{n-1} + (n-1)c_{n-1}a^{n-2} + \cdots + c_1)D(a) = 0.$$

As n was chosen to be minimal, and since A is a domain, we find that $D(a) = 0$. Hence $a \in A^D$. □

1.3 Locally nilpotent derivations

Let A be a commutative ring. We say that a derivation D on A is *locally nilpotent* if for every $a \in A$ there exists an $n \in \mathbb{N}$ such that $D^n(a) = 0$.

Example

The derivation $\frac{\partial}{\partial X_i}$ on $R[X]$ is locally nilpotent.

Let A be generated as an R-algebra by some set G. If we are to study an R-derivation $D : A \longrightarrow A$, it is useful to consider the action of D on elements of G, as the next proposition shows. Its proof is trivial.

Proposition 6. *Let A be generated as an R-algebra by some set G, and let $D : A \longrightarrow A$ be an R-derivation on A. Then D is locally nilpotent if and only if for every $g \in G$ there exists $n \in \mathbb{N}$ with $D^n(g) = 0$.*

From this result we obtain a large class of locally nilpotent derivations on $R[X]$, as seen below:

Corollary 7. *Let D be a* triangular derivation *on $R[X]$, i.e. a derivation of the form*

$$D = a_1(X_2, \ldots, X_n)\partial_1 + \cdots + a_{n-1}(X_n)\partial_{n-1} + a_n\partial_n$$

where each $a_i \in R[X_{i+1}, \ldots, X_n]$ and $a_n \in R$. Then D is locally nilpotent on $R[X]$.

Proof. We proceed by induction on n. The case $n = 1$ is obvious, as $D = a_1\partial_1$, where $a_1 \in R$, which is locally nilpotent on $R[X]$.

Let $n \geq 2$ and write $D = a_1\partial_1 + D_0$. Then D_0 restricts to a triangular derivation of $R[X_2, \ldots, X_n]$, and so by the induction hypothesis D_0 is locally nilpotent on $R[X_2, \ldots, X_n]$. Note that $D(g) = D_0(g)$ for all $g \in R[X_2, \ldots, X_n]$. Additionally, $DX_i = a_1\partial_1(X_i) + D_0X_i \in R[X_2, \ldots, X_n]$ for all i. Hence

$$D^{m+1}(X_i) = D^m(D(X_i)) = D_0^m(D(X_i))$$

which is zero for large m, since D_0 is locally nilpotent on $R[X_2, \ldots, X_n]$, and $DX_i \in R[X_2, \ldots, X_n]$. □

Let $D : A \longrightarrow A$ be a locally nilpotent derivation. An element $s \in A$ is called a *slice of* D if $D(s) = 1$. If a slice exists, then it is uniquely determined up to an element of A^D, in the sense that if s, s' are two slices of D, then $s - s' \in A^D$, as $D(s - s') = D(s) - D(s') = 1 - 1 = 0$.

Note that locally nilpotent derivations do not always have a slice: consider the derivation $X_1\partial_2$ on $\mathbb{C}[X_1, X_2]$. However, those that do have slices have a lot of interesting properties, as we will see later.

From now on we assume A to be a commutative $\mathbb{Q}$-algebra. One of the most useful tools in the study of derivations is the so-called *exponential map* associated to a derivation: let D be a locally nilpotent derivation on A. Consider the map $\exp(D)$: $A \longrightarrow A$, defined by the formula

$$\exp(D)(g) = \sum_{p=0}^{\infty} \frac{1}{p!} D^p(g), \quad \text{for all } g \in A.$$

If D is a locally nilpotent derivation on A, then so is $-D$. Hence it makes sense to consider the map $\exp(-D)$ and the compositions $\exp(D) \circ \exp(-D)$ and $\exp(D) \circ \exp(-D)$. Note that

$$\begin{aligned}
(\exp(D) \circ \exp(-D))(g) &= \exp(D) \left(\sum_{p=0}^{\infty} \frac{1}{p!} (-D)^p(g) \right) \\
&= \sum_{q=0}^{\infty} \frac{1}{q!} D^q \left(\sum_{p=0}^{\infty} \frac{1}{p!} (-D)^p(g) \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{p=0}^{n} \frac{(-1)^p}{p!(n-p)!} D^n(g) \right) \\
&= g + \sum_{n=1}^{\infty} \left(\sum_{p=0}^{n} \frac{(-1)^p}{p!(n-p)!} D^n(g) \right) \\
&= g.
\end{aligned}$$

Similarly, we can find that $(\exp(-D) \circ \exp(D))(g) = g$.

We study this map because of the following proposition.

Proposition 8. *If A is a $\mathbb{Q}$-algebra and $D \in \mathrm{Der} A$ is a locally nilpotent derivation on A, then $\exp D : A \to A$ is a ring automorphism on A.*

Proof. 1. For each $p \geq 0$, $D^p(f+g) = D^p(f) + D^p(g)$, hence $\exp D$ is additive.

2. Let $f, g \in A$. Then

$$\begin{aligned}
\exp D(f) \exp D(g) &= \left(\sum_{p=0}^{\infty} \frac{1}{p!} D^p(f)\right)\left(\sum_{q=0}^{\infty} \frac{1}{q!} D^q(g)\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{p+q=n} \frac{1}{p!q!} D^p(f) D^q(g)\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} D^n(fg) \quad (*) \\
&= \exp D(fg)
\end{aligned}$$

$(*)$ holds as

$$\begin{aligned}
D^n(fg) &= \sum_{p=0}^{n} \binom{n}{p} D^p(f) D^{n-p}(g) \\
&= \sum_{p=0}^{n} \frac{n!}{p!(n-p)!} D^p(f) D^{n-p}(g) \\
&= \sum_{p+q=n} \frac{n!}{p!q!} D^p(f) D^q(g) \\
&= n! \sum_{p+q=n} \frac{1}{p!q!} D^p(f) D^q(g).
\end{aligned}$$

3. Also note that $(\exp(D) \circ \exp(-D))(g) = g$, as seen previously. This causes $\exp(D) \circ \exp(-D)$ to be the identity on A, concluding the proof. □

Let D be a locally nilpotent derivation on A and consider the polynomial ring $A[T]$ in one variable over A. Extend D to a derivation $\tilde{D}$ on $A[T]$ by the formula

$$\tilde{D}\left(\sum_{i=1}^{n} a_i T^i\right) = \sum_{i=1}^{n} D(a_i) T^i,$$

i.e., $\tilde{D}$ is the unique derivation of $A[T]$ which extends D and maps T to 0. Note that $\tilde{D}$ is locally nilpotent: if $\sum_{i=1}^{n} a_i T^i \in A[T]$, then for each i, $D^{m_i}(a_i) = 0$ for some $m_i \in \mathbb{N}$. Write $m = \max\{m_i \mid 1 \leq i \leq n\}$. Then $\tilde{D}^m\left(\sum_{i=1}^{n} a_i T^i\right) = \sum_{i=1}^{n} D^m(a_i)T^i = 0$.

As $\tilde{D}(T) = 0$, it follows that $T\tilde{D} : A[T] \to A[T]$ is a locally nilpotent derivation. Thus, we may consider the automorphism $\exp(T\tilde{D})$ of $A[T]$. Henceforth, given a locally nilpotent derivation $D : A \to A$, we will abuse notation and write $TD : A[T] \to A[T]$ when we mean $T\tilde{D} : A[T] \to A[T]$ as above. Hence, we will talk of the automorphism $\exp TD$ of $A[T]$.

We can see that

$$\frac{d}{dT}(\exp TD(a)) = \exp TD(Da), \text{ for all } a \in A$$

and

$$\exp TD(a)|_{T=0} = a, \text{ for all } a \in A.$$

We will now show that if $D : A \to A$ is a locally nilpotent derivation having a slice s in A, where A is a $\mathbb{Q}$-algebra, then A can be viewed as a a polynomial ring in s over A^D. For each $a \in A$ we define $\pi_a : A[T] \to A$ to be the substitution homomorphism defined by

$$\pi_a(g(T)) = g(a) \text{ for all } g(T) \in A[T].$$

Recall that if D is locally nilpotent on A, then $\phi = \exp TD$ is a ring automorphism of $A[T]$, where we extend D to $A[T]$ by defining $D(T) = 0$.

We denote the composition $\pi_a \circ \phi : A[T] \to A$ by ϕ_a for each $a \in A$.

Lemma 9. *Let A be a $\mathbb{Q}$-algebra, let $D : A \to A$ be a locally nilpotent derivation on A, and let $s \in A$. Then each element $a \in A$ can be expressed as a polynomial in s of the form*

$$a = \sum_{i=0}^{\infty} \frac{1}{i!}\phi_{-s}(D^i(a))s^i.$$

Proof. Let $a \in A$. Then $\phi(a) \in A[T]$, with say $\deg_T \phi(a) = N$. Then

$$\begin{aligned} a &= \exp(-TD)\exp(TD)(a) = \exp(-TD)\phi(a) \\ &= \sum_{i=0}^{N} \frac{1}{i!} \exp(-TD)(D^i(a))T^i \end{aligned}$$

as $\phi = \exp TD$ is a ring automorphism, and $\exp(-TD)(T^i) = T^i$ for all i. Finally, substitute $T = s$. ☐

Theorem 10. *Let $D : A \to A$ be a locally nilpotent derivation having a slice s in A, where A is a $\mathbb{Q}$-algebra. Then A is a polynomial ring in s over A^D, i.e. $A = A^D[s]$, and $D = \frac{d}{ds}$.*

Proof. Let $a \in A$. Then

$$D\phi_{-s}(a) = D\left(\sum_{i=0}^{\infty} \frac{1}{i!} D^i(a)(-s)^i\right) = D\left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} D^i(a)s^i\right)$$

Note that this sum is finite (as $D^i(a) = 0$ for large enough i, as D is locally nilpotent). Hence

$$\begin{aligned} D\left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} D^i(a)s^i\right) &= \sum_{i=0}^{\infty} D\left(\frac{(-1)^i}{i!} D^i(a)s^i\right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} D\left(D^i(a)s^i\right) \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(D^{i+1}(a)s^i + iD^i(a)s^{i-1}D(s)\right) \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} D^{i+1}(a)s^i + \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} iD^i(a)s^{i-1} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} D^{i+1}(a)s^i + \sum_{i=1}^{\infty} \frac{(-1)^i}{(i-1)!} D^i(a)s^{i-1} = 0 \end{aligned}$$

Hence $\phi_{-s}(a) \in A^D$, for all $a \in A$.

By the previous lemma, we know that every element of A is a polynomial in s with coefficients in $\phi_{-s}(A)$, and so in A^D. Consequently, $A = A^D[s]$. By lemma 5, s is transcendental over A^D. It then follows that $D = \frac{d}{ds}$. ☐

Corollary 11. *Let A be an R-algebra (where R is a $\mathbb{Q}$-algebra). Let $D \in \mathrm{Der}_R A$ be a locally nilpotent derivation on A having a slice s in A. Then $A^D = \phi_{-s}(A)$. In particular, if G is a generating set for the R-algebra A, we find that $\phi_{-s}(G)$ is a generating set for the R-algebra A^D.*

1.4 Locally nilpotent derivations on a domain

Henceforth, we assume that A is a domain containing $\mathbb{Q}$. We will be studying locally nilpotent derivations on a domain. The main technique used in the proofs to study these derivations is to localize A at a suitable multiplicatively closed subset such that the extended derivation has a slice in the localized ring.

We first state a result detailing how one can arrive at a locally nilpotent derivation on $S^{-1}A$ from another locally nilpotent derivation on A.

Proposition 12. *Let A be a commutative ring and $D \in \text{Der}A$ a locally nilpotent derivation on A. Let $S \subseteq A$ be a multiplicatively closed subset with $S \subseteq A^D$.*

Then the derivation $\tilde{D} : S^{-1}A \to S^{-1}A$ (cf. proposition 4) is locally nilpotent. Moreover, if A has no S-torsion then $(S^{-1}A)^{\tilde{D}} = S^{-1}(A^D)$.

Proof. By proposition 4, we know that D can be extended to a derivation $\tilde{D}$ on $S^{-1}A$. All that remains is to show that if $S \subseteq A^D$ then $\tilde{D}$ is locally nilpotent.

Consider an element $\frac{a}{s} \in S^{-1}A$. Note that $s \in S \subseteq A^D$, so that $D(s) = 0$. Then

$$\tilde{D}\left(\frac{a}{s}\right) = \frac{D(a)s - aD(s)}{s^2} = \frac{D(a)s}{s^2} = \frac{D(a)}{s}.$$

From this, we find that $\tilde{D}^n\left(\frac{a}{s}\right) = \frac{D^n(a)}{s}$. As D is locally nilpotent, $D^n(a) = 0$ for some $n \in \mathbb{N}$. Thus $\tilde{D}^n\left(\frac{a}{s}\right) = 0$. Hence $\tilde{D}$ is locally nilpotent.

We now show that $(S^{-1}A)^{\tilde{D}} = S^{-1}(A^D)$ if A has no S-torsion. Note that $\frac{a}{s} \in (S^{-1}A)^{\tilde{D}} \iff \tilde{D}\left(\frac{a}{s}\right) = 0 \iff \frac{D(a)}{s} = 0 \iff s'D(a) = 0$ for some $s' \in S$, which is the case if and only if $s' = 0$ or $D(a) = 0$ as A has no S-torsion. As the case $s' = 0$ is impossible, we must have $D(a) = 0$, which happens if and only if $a \in A^D \iff \frac{a}{s} \in S^{-1}(A^D)$. Hence $\frac{a}{s} \in (S^{-1}A)^{\tilde{D}} \iff \frac{a}{s} \in S^{-1}(A^D)$, so $(S^{-1}A)^{\tilde{D}} = S^{-1}(A^D)$. □

We now explain the localization procedure. Let A be a domain containing $\mathbb{Q}$.

Let $0 \neq D$ be a locally nilpotent derivation on A. Then there exists an element $a \in A$ such that $D(a) \neq 0$. Let $n \geq 1$ be the smallest positive integer such that $D^{n+1}(a) = D^n(D(a)) = 0$.

Put $p = D^{n-1}(a)$. Then $d = D(p) \neq 0$ and $D(d) = D^2(p) = 0$. Hence

$$d \in A^D.$$

We call such an element $p \in A$ a *preslice of* D. Put

$$\tilde{A} = A[d^{-1}] \text{ and } s = \frac{p}{d}.$$

Then by proposition 12, we find that we can extend D uniquely to a derivation on $\tilde{A}$. This derivation (which we shall also call D) is locally nilpotent on $\tilde{A}$ as it is locally nilpotent on A, and $S = \{1, d, d^2, d^3, \dots\} \subseteq A^D$. Note now the following interesting fact:

$$D(s) = D\left(\frac{p}{d}\right) = \frac{D(p)d - pD(d)}{d^2} = \frac{d^2 - 0}{d^2} = 1,$$

and so D has a slice in $\tilde{A}$. Furthermore, we find by proposition 12 that

$$\tilde{A}^D = A^D[d^{-1}].$$

Then applying theorem 10 on $\tilde{A}$, we find that

$$\tilde{A} = \tilde{A}^D[s] = (A^D[d^{-1}])[s] \text{ and } D = \frac{d}{ds} \text{ on } \tilde{A}.$$

Then from the fact that $d \in A^D$ and from $A[d^{-1}] = A[d^{-1}][s]$ above, we find that

$$\mathrm{Frac}(A) = \mathrm{Frac}(A^D)(s)$$

where s is transcendental over $\mathrm{Frac}(A^D)$ by lemma 5.

In order to transfer some properties, we use the fact that

$$A \cap \mathrm{Frac}(A^D) = A^D.$$

Proposition 13. *Let k be a field of characteristic 0 contained in a domain A. Let $D \neq 0$ be a locally nilpotent derivation on A. Then:*

1. *A^D is* factorially closed *in A, i.e. if $a, b \in A$ (both non-zero) with $ab \in A^D$, then $a, b \in A^D$.*

2. *The group of units of* A^D *is the same as the group of units of* A*, i.e.* $(A^D)^* = A^*$*. Consequently,* $k \subseteq A^D$*.*

3. *If* $\mathrm{trdeg}_k\mathrm{Frac}(A)$ *is finite, i.e. the cardinality of the transcendental basis of* $\mathrm{Frac}(A)$ *over* k *is finite, then* $\mathrm{trdeg}_k\mathrm{Frac}(A^D) = \mathrm{trdeg}_k\mathrm{Frac}(A) - 1$*.*

4. *(Eigenvalue property). If* $Da = \lambda a$ *for some* $0 \neq a \in A$ *and* $\lambda \in A$*, then* $\lambda = 0$*.*

5. *Let* $a \in A$ *be such that* $Da \neq 0$*. Then* $\rho(D, a)$*, the smallest power of* D *which annihilates* a*, is equal to* $[\mathrm{Frac}(A) : \mathrm{Frac}(A^D)(a)] + 1$*.*

Proof. We may consider $s = \frac{p}{d} \in A[d^{-1}]$ as in the above discussion.

1. Let $a, b \in A$ (both non-zero) with $ab \in A^D$. Again, we can write $a = f(s)$, $b = g(s)$, where $f(T), g(T) \in A^D[d^{-1}][T]$.

   Then $D(ab) = 0$ implies that $(f(s)g(s))' = 0$, which causes $f(s), g(s) \in A^D[d^{-1}]$. As $A^D[d^{-1}] \subseteq \mathrm{Frac}(A^D)$ and as $f(s), g(s) \in A$, we find that $f(s), g(s) \in A \cap \mathrm{Frac}(A^D) = A^D$. So $a, b \in A^D$.

2. We show that $(A^D)^* = A^*$. Let $x \in A^*$. Then $xy = 1 \in A^D$ for some $y \in A$. We then find that $x, y \in A^D$ as A^D is factorially closed in A. Hence $x \in (A^D)^*$. Consequently, any field contained in A is also contained in A^D.

3. As $\mathrm{Frac}(A) = \mathrm{Frac}(A^D)(s)$, where s is transcendental over $\mathrm{Frac}(A^D)$, we find that $\mathrm{trdeg}_k\mathrm{Frac}(A) = \mathrm{trdeg}_k\mathrm{Frac}(A^D) + 1$, from which follows the result.

4. As $\tilde{A} = A^D[d^{-1}][s]$, we can write $a = f(s), \lambda = g(s)$, where $f(T), g(T) \in A^D[d^{-1}][T]$. Since $D = \frac{d}{ds}$ on $\tilde{A}$, we find that $f'(s) = g(s)f(s)$. Examining the degrees of these polynomials in s, we deduce that $f'(s) = 0$. As $a = f(s) \neq 0$ and A is a domain, we find that $\lambda = g(s) = 0$.

5. Write $a = f(s) \in \mathrm{Frac}(A^D)[s]$, say with $\deg_T f(T) = d \geq 1$. Since $D = \frac{d}{ds}$ on $\mathrm{Frac}(A^D)[s]$, we find that $\rho(D, a) = d + 1$. Then notice that

$$[\mathrm{Frac}(A) : \mathrm{Frac}(A^D)(a)] = [\mathrm{Frac}(A^D)(s) : \mathrm{Frac}(A^D)(f(s))] = d$$

   which completes the proof. □

Using part 2 of the previous proposition, we can find that given a ring A such that trdeg_k Frac$(A) = 1$, we have trdeg_k Frac$(A^D) = 0$, and so that A^D is algebraic over k. Hence $k \subseteq A^D$ is an integral extension, and so A^D is a field. Hence d is a unit in $A^D \subseteq A$, so $s = \frac{p}{d} \in A$, and so D has a slice in A.

Using theorem 10, we can find that $A = A^D[s]$. This leads us to the following corollary:

Corollary 14. *Let A be a domain containing a field k of characteristic zero and such that* $\text{trdeg}_k\text{Frac}(A) = 1$. *If $D \neq 0$ is a locally nilpotent derivation of A, then $A = A^D[s]$ where $D(s) = 1$ and $k \subseteq A^D$ is an algebraic field extension. In particular, if k is algebraically closed, then $A = k[s]$, a polynomial ring in one variable over k.*

Using the Eigenvalue property of proposition 13, we can decide when aD is a locally nilpotent derivation, given the knowledge of whether D is locally nilpotent or not:

Corollary 15. *Let $D \neq 0$ be a derivation on A and $0 \neq a \in A$. Then aD is locally nilpotent if and only if D is locally nilpotent and $a \in A^D$.*

Proof. If $a \in A^D$, then we have

$$(aD)^n(b) = a^n D^n(b), \text{ for all } b \in A, \, n \geq 1.$$

Hence if D is locally nilpotent, then so is aD.

We now prove the converse. Let aD be locally nilpotent on A. As $(aD)(a) = D(a)a$, then by the Eigenvalue property from proposition 13, $D(a) = 0$, i.e. $a \in A^D$ and consequently

$$(aD)^n(b) = a^n D^n(b), \text{ for all } b \in A, \, n \geq 1.$$

Since A is a domain, it follows that D is locally nilpotent. □

Part 1 of proposition 13 gives:

Corollary 16. *Let $D \neq 0$ be a locally nilpotent derivation on A. Then $(A^D)^* = A^*$ and every irreducible element of A^D is irreducible in A. In particular, if A is a UFD then so is A^D.*

The following proposition follows from corollary 14:

Proposition 17. *Let* $0 \neq D = a\partial_1 + b\partial_2$ *be a locally nilpotent derivation on* $k[X_1, X_2]$ *with* $\gcd(a, b) = 1$*. Then* D *has a slice in* $k[X_1, X_2]$*.*

Proof. We first prove the case where k is algebraically closed. Choose a preslice $p \in A = k[X_1, X_2]$ with minimal degree. We will show that $D(p) \in k^*$, and thus $s = p/D(p) \in k[X_1, X_2]$ is a slice of D, as

$$D(s) = D\left(\frac{p}{D(p)}\right) = \frac{D(p)D(p) - pD^2(p)}{(D(p))^2} = 1.$$

Assume for contradiction that $D(p) \notin k^*$. Then there exists an irreducible factor $q \in k[X_1, X_2]$ with $q | D(p)$, i.e. $D(p) \in (q)$.

Since $D(p) \in A^D$ (as p is a preslice of D), it then follows from part 1 of proposition 13 that $q \in A^D$. Now consider the induced derivation $\tilde{D} : A/(q) \to A/(q)$. Note that as D is locally nilpotent, then so is $\tilde{D}$.

Suppose $\tilde{D} = 0$. Then $D(X_i) \in (q)$ for $i = 1, 2$. However, as $D(X_1) = a$, $D(X_2) = b$, and $\gcd(a, b) = 1$, we find that $1 \in (q)$, contradicting the fact that q is irreducible. Hence $\tilde{D} \neq 0$. Note that $\operatorname{trdeg}_k \operatorname{Frac}(A/(q)) = 1$. By corollary 14, and since we assumed that k is algebraically closed, we find that $\ker \tilde{D} = k$.

As $D(p) \in (q)$, we find that $\tilde{D}(p + (q)) = 0$, and so $p + (q) = \lambda + (q)$ for some $\lambda \in k$. Hence $p - \lambda = hq$ for some $h \in k[X_1, X_2]$. Applying D and D^2 respectively to both sides of this equality yields the equalities

$$D(p) - D(\lambda) = D(h)q + hD(q),$$
$$D^2(p) - D^2(\lambda) = D^2(h)q + 2D(h)D(q) + hD^2(q).$$

As $D^2(p) = 0$, $D(\lambda) = 0$, and $D(q) = 0$, we find using the second equation that $D^2(h)q = 0$, causing $D^2(h) = 0$ as $q \neq 0$. Using the first equation, we can find that $D(h) \neq 0$ as $D(p) \neq 0$ and $D(q) = D(\lambda) = 0$. Hence we have found that $D(h) \neq 0$, but $D^2(h) = 0$.

As $p - \lambda = hq$, we find that $\deg(h) < \deg(p)$. Hence we have found a preslice h of D with degree less than $\deg(p)$, contradicting the minimality of p. Thus $D(p) \in k^*$, and $s = p/D(p) \in k[X_1, X_2]$ is a slice of D.

Now drop the assumption that k is algebraically closed, consider the algebraic closure $\bar{k}$ of k and extend D to a derivation D' on $\bar{k} \otimes_k k[X_1, X_2] = \bar{k}[X_1, X_2]$. Note that $D' = a\partial_1 + b\partial_2$. As $\gcd(a, b) = 1$ in $k[X_1, X_2]$, it follows that $\gcd(a, b) = 1$ in $\bar{k}[X_1, X_2]$. Since the result is true when k is algebraically closed, we know that D' has a slice $s' \in \bar{k}[X_1, X_2]$. Consequently, $D' : \bar{k}[X_1, X_2] \to \bar{k}[X_1, X_2]$ is surjective. Since D' is obtained by applying the functor $\bar{k} \otimes_k (\_)$ to $D : k[X_1, X_2] \to k[X_1, X_2]$, and since $\bar{k}$ is a free k-module (hence a faithfully flat k-module), it follows that D is surjective and hence has a slice $s \in k[X_1, X_2]$. □

1.5 Locally nilpotent derivations on $k[X]$

Let k be a field of characteristic zero. We are interested in the following problem in the theory of locally nilpotent derivations:

Classification Problem. Describe all locally nilpotent derivations on $k[X]$.

The case $n = 1$ is trivial, as one can easily see that every locally nilpotent derivation of $k[X_1]$ is of the form

$$a\frac{d}{dX_1}, \quad \text{where } a \in k.$$

The case $n = 2$ of the Classification Problem was solved by Rentschler in 1968:

Theorem 18. (Rentschler) *Let $0 \neq D$ be a locally nilpotent derivation on $k[X_1, X_2]$. Then there exists $\phi \in \mathrm{Aut}_k k[X_1, X_2]$ and $f(X_2) \in k[X_2]$ such that $\phi^{-1}D\phi = f(X_2)\frac{\partial}{\partial X_1}$.*

We shall outline the proof of Rentschler's theorem in Chapter 2. This result will be of vital importance when studying the automorphisms of $k[X_1, X_2]$. In fact, it may be of note to remark that Rentschler obtained a stronger result stating that the automorphism ϕ of the theorem may be chosen to be tame. Using this result one can deduce that every element of $\mathrm{Aut}_k k[X_1, X_2]$ is tame, as we shall see next chapter.

The Classification Problem is open for all $n > 2$, and this is in fact a very active research area. One should also note that the Classification Problem is closely related to that of understanding the structure of the group of automorphisms of $k[X]$.

Chapter 2

A Survey of $\mathrm{Aut}_k(k[X_1, \ldots, X_n])$

2.1 Some subgroups of $\mathrm{Aut}_R R[X]$

Let R be a commutative ring, and a domain unless otherwise noted. Let $R[X] = R[X_1, \ldots, X_n]$ be the polynomial ring in n variables over R. Recall that $\mathrm{Aut}_R R[X]$ is defined as the set of all R-automorphisms of $R[X]$.

We will consider the following subgroups of $\mathrm{Aut}_R R[X]$:

- $\mathrm{Aff}(R, n)$ is the *affine subgroup* of $\mathrm{Aut}_R R[X]$, consisting of all R-automorphisms $F = (F_1, \ldots, F_n)$ such that $\deg F_i = 1$ for all $1 \leq i \leq n$.

- $J(R, n)$ is the *de Jonquières* subgroup of $\mathrm{Aut}_R R[X]$, also known as the *triangular subgroup* of $\mathrm{Aut}_R R[X]$, consisting of all R-automorphisms of the form

  $$F = (a_1 X_1 + f_1(X_2, \ldots, X_n), a_2 X_2 + f_2(X_3, \ldots, X_n), \ldots, a_n X_n + f_n)$$

  where each $a_i \in R^*$ and each $f_i \in R[X_{i+1}, \ldots, X_n]$ and $f_n \in R$.

- $E(R, n)$ is the subgroup generated by the *elementary* automorphisms, i.e. the ones of the form

  $$F = (X_1, \ldots, X_{i-1}, X_i + a, X_{i+1}, \ldots, X_n)$$

  where $a \in R[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$. We denote the elementary automorphism F above by $E_i(a)$.

- $T(R,n)$ is the *tame subgroup*, defined as the subgroup of $\mathrm{Aut}_R R[X]$ generated by $\mathrm{Aff}(R,n)$ and $E(R,n)$, i.e. $T(R,n) = \langle \mathrm{Aff}(R,n), E(R,n)\rangle$.

Proposition 19. $T(R,n) = \langle \mathrm{Aff}(R,n), J(R,n)\rangle$.

Proof. Let $F \in J(R,n)$. Say

$$F = (a_1X_1 + f_1(X_2,\ldots,X_n), a_2X_2 + f_2(X_3,\ldots,X_n),\ldots, a_nX_n + f_n),$$

where each $a_i \in R^*$ and each $f_i \in R[X_{i+1},\ldots,X_n]$ and $f_n \in R$. Then

$$F = (E_1(a_1^{-1}f_1)\circ\cdots\circ E_n(a_n^{-1}f_n))\circ(a_1X_1,\ldots,a_nX_n) \in \langle \mathrm{Aff}(R,n), E(R,n)\rangle = T(R,n).$$

Hence $J(R,n) \subseteq T(R,n)$, and so $\langle \mathrm{Aff}(R,n), J(R,n)\rangle \subseteq T(R,n)$.

We now show that each $G \in E(R,n)$ is in $\langle \mathrm{Aff}(R,n), J(R,n)\rangle$. Let

$$G = (X_1,\ldots,X_{i-1}, X_i + a, X_{i+1},\ldots,X_n),$$

where $a \in R[X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n]$.

Define

$$F_1 = (X_i, X_2,\ldots,X_{i-1}, X_1, X_{i+1},\ldots,X_n) \in \mathrm{Aff}(R,n),$$
$$F_2 = (X_1 + a(X_i, X_2,\ldots,X_{i-1},X_{i+1},\ldots,X_n), X_2, X_3,\ldots,X_n) \in J(R,n).$$

Thence

$$F_1F_2F_1 = (X_1,\ldots,X_{i-1}, X_i + a, X_{i+1},\ldots,X_n) = G.$$

As $G = F_1F_2F_1 \in \langle \mathrm{Aff}(R,n), J(R,n)\rangle$, we find that $E(R,n) \subseteq \langle \mathrm{Aff}(R,n), J(R,n)\rangle$. Thus $T(R,n) = \langle \mathrm{Aff}(R,n), E(R,n)\rangle \subseteq \langle \mathrm{Aff}(R,n), J(R,n)\rangle$, and so we find that $T(R,n) = \langle \mathrm{Aff}(R,n), J(R,n)\rangle$. □

2.2 The tame automorphism group of $R[X,Y]$

In this section, we consider the group $\mathrm{Aut}_R R[X,Y]$ where R is an integral domain. We consider two questions: What is the structure of the subgroup $T(R,2)$? When do we have $T(R,2) = \mathrm{Aut}_R R[X,Y]$?

We first define the notion of an amalgamated free product. Let H, K, and U be groups and let $\alpha_1 : U \to H$ and $\alpha_2 : U \to K$ be given homomorphisms. The *amalgamated free product* of H and K over U, denoted $H *_U K$, is defined as a group as having the following presentation:

$$H *_U K = \langle H, K \,|\, rel(H), rel(K), \alpha_1(u)\alpha_2(u)^{-1} \rangle,$$

where $rel(H)$ are the relations of H, $rel(K)$ are the relations of K, and $u \in U$.

We offer a equivalent characterization of the concept of an amalgamated free product which shall prove to be of great use later in the chapter. A proof can be found in [16].

Proposition 20. *Let H and K be subgroups of a group G. Suppose that G is generated by H and K and let U be the intersection of H and K.*

Then $G = H *_U K$ if and only if $h_1k_1 \ldots h_nk_nh_{n+1} \notin K$ whenever $h_i \in H - K$ and $k_i \in K - H$.

We will show that $T(R, 2)$ is the amalgamated free product of $\mathrm{Aff}(R, 2)$ and $J(R, 2)$ over their intersection. We will describe an algorithm which decides if a given endomorphism $F : R[X, Y] \to R[X, Y]$ is a tame automorphism.

We shall see that if R is not a field, then $T(R, 2)$ is a proper subgroup of $\mathrm{Aut}_R R[X, Y]$. If R is a field, then $\mathrm{Aut}_R R[X, Y] = T(R, 2)$, and so $\mathrm{Aut}_R R[X, Y]$ will be the amalgamated free product of $\mathrm{Aff}(R, 2)$ and $J(R, 2)$ over their intersection. We will use the following results.

Lemma 21. *Let G be a group generated by two subgroups H and K. Then every element $g \in G$ can be expressed as*

$$g = h_0k_1h_1 \ldots k_lh_l$$

for some $l \geq 0$, where $h_0, h_l \in H$, $h_i \in H - K$ for $1 \leq i \leq l - 1$, and $k_i \in K - H$ for $1 \leq i \leq l$.

Proof. Let $g \in G$. By a *decomposition* of g we mean a finite sequence

$$(h_0, k_1, h_1, \ldots, k_l, h_l) \tag{1}$$

of elements of G satisfying:

$$l \geq 0,\ h_i \in H,\ k_i \in K \quad \text{and} \quad g = h_0 k_1 h_1 \ldots k_l h_l.$$

It is clear that there exists at least one decomposition of g, so we may consider a decomposition (1) of g of minimal length (the length is the number of terms in the sequence). To prove the lemma, it suffices to verify that this decomposition satisfies

$$h_i \in H - K \text{ for } 1 \leq i \leq l-1 \quad \text{and} \quad k_i \in K - H \text{ for } 1 \leq i \leq l. \tag{2}$$

Note that if (2) is not satisfied then one of the following must hold:

$$\text{there exists } i \text{ such that } 1 \leq i \leq l-1 \text{ and } h_i \in H \cap K; \tag{3}$$

$$\text{there exists } i \text{ such that } 1 \leq i \leq l \text{ and } k_i \in H \cap K. \tag{4}$$

If (3) holds then k_i and k_{i+1} are defined and the product $k_i h_i k_{i+1}$ is an element of K; so $(h_0, k_1, h_1, \ldots, h_{i-1}, k_i h_i k_{i+1}, h_{i+1}, \ldots, k_l, h_l)$ is a decomposition of g which is shorter than (1), which is impossible. Similarly, if (4) holds then $h_{i-1} k_i h_i \in H$, and this gives rise to a contradiction.

So both (3) and (4) are false and consequently (2) is satisfied. □

For the aforementioned algorithm deciding whether a given endomorphism $F : R[X,Y] \to R[X,Y]$ is a tame automorphism, we shall make use of the following notations: let $F = (F_1, F_2) \in R[X,Y]^2$. We then define the following:

$$\begin{aligned} \mathrm{bideg} F &= (\deg F_1, \deg F_2), \\ \deg F &= \max(\deg F_1, \deg F_2), \\ \mathrm{tdeg} F &= \deg F_1 + \deg F_2. \end{aligned}$$

Before we proceed further, let us recall that $T(R,2) = \langle \mathrm{Aff}(R,2), J(R,2) \rangle$, and that if $\lambda \in \mathrm{Aff}(R,2)$ and $\tau \in J(R,2)$, we can write

$$\begin{aligned} \lambda &= (aX + bY + c, dX + eY + f), \text{ for some } a,b,c,d,e,f \in R, \\ \tau &= (aX + f(Y), bY + c), \text{ for some } a,b,c \in R,\ f(Y) \in R[Y]. \end{aligned}$$

We can thus apply lemma 21 to $G = T(R,2)$, $H = \mathrm{Aff}(R,2)$, $K = J(R,2)$ and $g = F$ to write

$$F = \lambda_0 \tau_1 \lambda_1 \ldots \tau_l \lambda_l$$

where $\lambda_0, \lambda_l \in \mathrm{Aff}(R,2)$, $\lambda_i \in \mathrm{Aff}(R,2) - J(R,2)$ for $1 \leq i \leq l-1$ and $\tau_i \in J(R,2) - \mathrm{Aff}(R,2)$ for $1 \leq i \leq l$. We will write

$$\lambda_i = (a_i X + b_i Y + c_i, d_i X + e_i Y + f_i).$$

Lemma 22. *For all $1 \leq i \leq l$ we have*

$$\mathrm{bideg}(\tau_i \lambda_i \ldots \tau_l \lambda_l) = \left(\prod_{j=i}^{l} \deg \tau_j, \prod_{j=i+1}^{l} \deg \tau_j \right)$$

Proof. We proceed by decreasing induction on i, starting with case $i = l$. In the case $i = l$, $\mathrm{bideg}(\tau_l \lambda_l) = (\deg \tau_l, 1)$, which is exactly what we wanted to show.

Now let us assume that the statement is true for some $2 \leq k \leq l$, that is to say that $\mathrm{bideg}(\tau_k \lambda_k \ldots \tau_l \lambda_l) = \left(\prod_{j=k}^{l} \deg \tau_j, \prod_{j=k+1}^{l} \deg \tau_j \right)$. Consider $\mathrm{bideg}(\tau_{k-1} \lambda_{k-1} \ldots \tau_l \lambda_l)$. As $\lambda_{k-1} \notin J(R,2)$, $d_{k-1} \neq 0$. Hence

$$\mathrm{bideg}(\lambda_{k-1} \tau_k \lambda_k \ldots \tau_l \lambda_l) = \left(p_k, \prod_{j=k}^{l} \deg \tau_j \right),$$

where $p_k \leq \prod_{j=k}^{l} \deg \tau_j$. As $\tau_{k-1} \notin \mathrm{Aff}(R,2)$ we have that $\deg \tau_{k-1} \geq 2$. Hence

$$\mathrm{bideg}(\tau_{k-1} \lambda_{k-1} \tau_k \lambda_k \ldots \tau_l \lambda_l) = \left(\prod_{j=k-1}^{l} \deg \tau_j, \prod_{j=k}^{l} \deg \tau_j \right),$$

and so the statement holds for the case $i = k-1$. Thus, by induction, the statement holds for all $1 \leq i \leq l$. □

Corollary 23. *$T(R,2)$ is the amalgamated free product of* $\mathrm{Aff}(R,2)$ *and* $J(R,2)$ *over their intersection.*

That is to say that $T(R,2)$ is generated by these two subgroups and that if $\tau_i \in J(R,2) - \mathrm{Aff}(R,2)$, $\lambda_i \in \mathrm{Aff}(R,2) - J(R,2)$, then $\tau_1 \lambda_1 \ldots \tau_n \lambda_n \tau_{n+1} \notin \mathrm{Aff}(R,2)$.

Proof. We already know that $T(R,2) = \langle \mathrm{Aff}(R,2), J(R,2)\rangle$. By proposition 20, proving the second part of the above corollary will yield that $T(R,2)$ is the amalgamated free product of $\mathrm{Aff}(R,2)$ and $J(R,2)$ over their intersection.

Suppose that

$$\tau_1\lambda_1\ldots\tau_n\lambda_n\tau_{n+1} = \lambda \in \mathrm{Aff}(R,2)$$

where $\tau_i \in J(R,2) - \mathrm{Aff}(R,2)$ and $\lambda_i \in \mathrm{Aff}(R,2) - J(R,2)$ for all i. Then

$$\tau_1\lambda_1\ldots\tau_n\lambda_n\tau_{n+1}\lambda^{-1} = id_{R[X,Y]} = (X,Y).$$

So $\mathrm{bideg}(\tau_1\lambda_1\ldots\tau_n\lambda_n\tau_{n+1}\lambda^{-1}) = \mathrm{bideg}(X,Y) = (1,1)$.

Additionally, $\mathrm{bideg}(\tau_1\lambda_1\ldots\tau_{n+1}\lambda^{-1}) = (\prod_{j=1}^{n+1}\deg\tau_j, \prod_{j=2}^{n+1}\deg\tau_j)$ by the lemma. However, $\deg\tau_1 \geq 2$ as otherwise $\tau_1 \in \mathrm{Aff}(R,2)$. So $\prod_{j=1}^{n+1}\deg\tau_j \geq 2$, a contradiction as $(\prod_{j=i}^{l}\deg\tau_j, \prod_{j=i+1}^{l}\deg\tau_j) = (1,1)$. Thus,

$$\tau_1\lambda_1\ldots\tau_n\lambda_n\tau_{n+1} \notin \mathrm{Aff}(R,2).$$

□

Remark that just because two subgroups generate $T(R,2)$, that does not imply that $T(R,2)$ is their amalgamated free product.

Example

If J_0 is the subgroup of $J(R,2)$ consisting of all elements of the form $(X+g(Y),Y)$ with $g(Y) \in R[Y]$, then it is easy to see that $T(R,2) = \langle \mathrm{Aff}(R,2), J(R,2)\rangle = \langle \mathrm{Aff}(R,2), J_0\rangle$. (It is obvious that $J_0 \subseteq J(R,2)$. All that must be shown is that $J(R,2) \subseteq \langle \mathrm{Aff}(R,2), J_0\rangle$. Let $F = (aX+f(Y), bY+c) \in J(R,2)$, where $a,b \in R^*$, $c \in R$, and $f(Y) \in R[Y]$. Then $F = F_1F_2$, where $F_1 = (aX, bY+c) \in \mathrm{Aff}(R,2)$ and $F_2 = (X + a^{-1}f(Y), Y) \in J_0$. Hence $F \in \langle \mathrm{Aff}(R,2), J_0\rangle$.)

However, $T(R,2)$ is not the amalgamated free product of $\mathrm{Aff}(R,2)$ and J_0 over their intersection. Consider $\tau_1 = (X-Y^2,Y)$, $\lambda_1 = (X,Y+1)$, $\tau_2 = (X+(Y+1)^2,Y)$. Then $\tau_1\lambda_1\tau_2 = \tau_1(X+(Y+1)^2, Y+1) = (X+(Y+1)^2-(Y+1)^2, Y+1) = \lambda_1 \in \mathrm{Aff}(R,2)$. However, $\tau_1,\lambda_1,\tau_2 \notin \mathrm{Aff}(R,2)\cap J_0$.

Recall that since the beginning of this section, we have assumed R to be a domain. If R is not a domain, then lemma 22 fails and $T(R,2)$ is not necessarily the amalgamated free product of $\mathrm{Aff}(R,2)$ and $J(R,2)$ over their intersection.

Example

Let $\tau = (X + aY^2, Y)$, $\lambda = (X, Y + bX)$, where $a, b \neq 0$ are such that $ab = 0$. Note that neither τ nor λ belongs to $\mathrm{Aff}(R,2) \cap J(R,2)$. However,

$$\tau\lambda = (X + a(Y + bX)^2, Y + bX) = (X + aY^2 + 2abXY + ab^2X^2, Y + bX)$$
$$\lambda\tau = (X + aY^2, Y + b(X + aY^2)) = (X + aY^2, Y + bX + abY^2)$$

As $ab = 0$, we find that $\tau\lambda = \lambda\tau \implies \tau\lambda\tau^{-1} = \lambda \in \mathrm{Aff}(R,2)$.

So $T(R,2)$ is not the amalgamated free product of $\mathrm{Aff}(R,2)$ and $J(R,2)$ over their intersection.

Moreover,

$$\begin{aligned}\tau\lambda\tau\lambda &= (X + aY^2, Y + bX)(X + aY^2, Y + bX) \\ &= ((X + aY^2) + a(Y + bX)^2, (Y + bX) + b(X + aY^2)) \\ &= (X + aY^2 + aY^2 + 2abXY + ab^2X^2, Y + bX + bX + abY^2) \\ &= (X + 2aY^2, Y + 2bX)\end{aligned}$$

and so $\mathrm{bideg}(\tau\lambda\tau\lambda) = (2,1)$, whereas lemma 22 would demand that

$$\mathrm{bideg}(\tau\lambda\tau\lambda) = ((\deg\tau)^2, \deg\tau) = (2^2, 2) = (4,2).$$

Hence lemma 22 does not hold in the case where R is not a domain.

Let $F \in T(R,2)$, where R is a domain. By lemma 22, we can write F in the form $F = \lambda_0\tau_1\lambda_1 \ldots \tau_l\lambda_l$, where

$$\mathrm{bideg}(\tau_1\lambda_1 \ldots \tau_l\lambda_l) = \left(\prod_{j=1}^{l} \deg\tau_j, \prod_{j=2}^{l} \deg\tau_j\right).$$

Now write $\lambda_0 = (a_0X + b_0Y + c_0, d_0X + e_0Y + f_0)$. We have three cases to consider:

- If $a_0 = 0$, then $\mathrm{bideg}F = \left(\prod_{j=2}^{l} \deg \tau_j, \prod_{j=1}^{l} \deg \tau_j\right)$.
- If $d_0 = 0$, then $\mathrm{bideg}F = \left(\prod_{j=1}^{l} \deg \tau_j, \prod_{j=2}^{l} \deg \tau_j\right)$.
- If $a_0 \neq 0$ and $d_0 \neq 0$, then $\mathrm{bideg}F = \left(\prod_{j=1}^{l} \deg \tau_j, \prod_{j=1}^{l} \deg \tau_j\right)$.

This proves the first part of the following result; the proof of the second assertion is left to the reader.

Corollary 24. *Let $F = (F_1, F_2) \in T(R, 2)$ with $\mathrm{bideg}F = (d_1, d_2)$. Let h_i denote the homogeneous component of F_i of degree d_i. Then*

1. *$d_1 | d_2$ or $d_2 | d_1$.*
2. *If $\deg F > 1$, then we have*
   (a) *if $d_1 < d_2$, then $h_2 = ch_1^{d_2/d_1}$ for some $c \in R$.*
   (b) *if $d_1 > d_2$, then $h_1 = ch_2^{d_1/d_2}$ for some $c \in R$.*
   (c) *if $d_1 = d_2$, then there exists $\lambda \in \mathrm{Aff}(R, 2)$ such that $\deg F_1' > \deg F_2'$ where $(F_1', F_2') = \lambda F$.*

If $d_1 = d_2$, then in general there does not exist $c \in R$ such that $h_1 = ch_2$ or $h_2 = ch_1$.

Note that if R is not a domain, then the above corollary does not hold in general.

Example
Suppose $ab = 0$ and let $F = (X, Y + aX^3)(X + bY^2, Y) \in T(R, 2)$.

Then $F = (X+bY^2, Y+a(X+bY^2)^3) = (X+bY^2, Y+a(X^3+3bX^2Y^2+3b^2XY^4+b^3Y^6)) = (X + bY^2, Y + aX^3)$ as $ab = 0$. Thence $\mathrm{bideg}F = (2, 3)$, and so $d_1 \nmid d_2$ and $d_2 \nmid d_1$.

Let F be as in corollary 24 part 2. In case (a), let $\tau = (X, Y - cX^{d_2/d_1})$. In case (b), let $\tau = (X - cY^{d_1/d_2}, Y)$. We thus find that $\mathrm{tdeg}(\tau F) < \mathrm{tdeg}F$. In part (c), there exists $\lambda \in \mathrm{Aff}(R, 2)$ such that $\mathrm{tdeg}(\lambda F) < \mathrm{tdeg}F$.

Hence we have the following algorithm, in which we use the notation $\det JF$ to mean the determinant of the Jacobian of F:

Algorithm to decide if $F = (F_1, F_2) \in R[X,Y]^2$ belongs to $T(R,2)$.
Let $F = (F_1, F_2) \in R[X,Y]^2$ with $\mathrm{bideg}(F_1, F_2) = (d_1, d_2)$.

1. If $d_1 = d_2 = 1$ with $\det JF \in R^*$, then $F \in T(R,2)$.

2. If $d_1 \neq d_2$, skip the next step.

3. If there exists $\lambda \in \mathrm{Aff}(R,2)$ with $\mathrm{tdeg}\lambda F < \mathrm{tdeg}F$, replace F by λF and restart. Else $F \notin T(R,2)$.

4. If $d_1 > d_2$, then replace F by (F_2, F_1).

5. If $d_1 | d_2$ and there exists $c \in R$ with $h_2 = c h_1^{d_2/d_1}$, then replace F by $(X, Y - cX^{d_2/d_1})F$ and restart. Else $F \notin T(R,2)$.

The above algorithm to decide whether an element of $R[X,Y]^2$ is in $T(R,2)$ shall prove very useful in what follows.

We can now ask ourselves under which conditions on R is every R-automorphism of $R[X,Y]$ tame, i.e. when do we have $T(R,2) = \mathrm{Aut}_R R[X,Y]$? We shall see that the answer is that the equality holds if and only if R is a field.

In the case where R is not a field, one can consider the Nagata automorphism, defined below, which was constructed by M. Nagata in 1972 [20]. This automorphism has a three-dimensional analogue which we shall study in section 2.3.1.

Proposition 25. *If R is not a field, then $T(R,2) \neq \mathrm{Aut}_R R[X,Y]$. More precisely, if $0 \neq z \in R$ is a non-unit, then consider the* Nagata automorphism

$$F = (X - 2Y(zX + Y^2) - z(zX + Y^2)^2, Y + z(zX + Y^2)).$$

$F \in \mathrm{Aut}_R R[X,Y] - T(R,2)$, so $T(R,2) \neq \mathrm{Aut}_R R[X,Y]$.

Proof. Consider the locally nilpotent derivation $D = (zX + Y^2)(-2Y\partial_X + z\partial_Y) \in \mathrm{Der}_R(R[X,Y])$. We have $\exp D = \sum_n \frac{1}{n!}D^n$, so

$$\begin{aligned}
\exp D(X) &= X + D(X) + \frac{1}{2}D^2(X) + \frac{1}{6}D^3(X) + \ldots \\
&= X - 2Y(zX+Y^2) - 2\frac{1}{2}D(Y(zX+Y^2)) + \frac{1}{6}D^3(X) + \ldots \\
&= X - 2Y(zX+Y^2) - z(zX+Y^2)^2 + 0 + \ldots \\
&= X - 2Y(zX+Y^2) - z(zX+Y^2)^2
\end{aligned}$$

and

$$\begin{aligned}
\exp D(Y) &= Y + D(Y) + \frac{1}{2}D^2(Y) + \frac{1}{6}D^3(Y) + \ldots \\
&= Y + z(zX+Y^2) + \frac{1}{2}D^2(Y) + \frac{1}{6}D^3(Y) + \ldots \\
&= Y + z(zX+Y^2) + 0 + 0 + \ldots \\
&= Y + z(zX+Y^2).
\end{aligned}$$

Hence $\exp D = F$. As the exponential of a locally nilpotent derivation on $R[X,Y]$ is an R-automorphism, $F \in \mathrm{Aut}_R R[X,Y]$.

If $F \in T(R,2)$, then by the corollary we find that $-zY^4 = c(zY^2)^2$ for some $c \in R$. However, this implies that $-z = cz^2$ which implies that $-zc = 1$, a contradiction as z was chosen to be a non-unit. Hence $F \notin T(R,2)$, and so $\mathrm{Aut}_R R[X,Y] \neq T(R,2)$. □

Theorem 26. (Jung, van der Kulk) *If k is a field, then* $\mathrm{Aut}_k k[X,Y] = T(k,2)$. *More precisely,* $\mathrm{Aut}_k k[X,Y]$ *is the amalgamated free product of* $\mathrm{Aff}(k,2)$ *and* $J(k,2)$ *over their intersection.*

We shall prove theorem 26 in the special case where k is a field of characteristic zero; for the other cases, refer to the papers of van der Kulk [35], Makar-Limanov [17], and Dicks [9], whose proof shall be presented later in the chapter. In fact we shall derive theorem 26 from the following strong version of Rentschler's theorem (this is the version that Rentschler proved):

Theorem 27. (Rentschler) *Let k be a field of characteristic zero and let $D \neq 0$ be a locally nilpotent derivation on $k[X,Y]$. Then there exist $h \in T(k,2)$ and $f(Y) \in k[Y]$ such that $h^{-1} \circ D \circ h = f(Y)\partial_X$.*

The proof of theorem 27 shall be presented later in the chapter. Now we give:

Proof of theorem 26 (char$k = 0$). Write $A = k[X,Y]$. We must show that $T(k,2) = \mathrm{Aut}_k A$. Let $F = (F_1, F_2) \in \mathrm{Aut}_k A$. Then $\frac{\partial}{\partial F_1}$ is locally nilpotent, so by theorem 27 there exist $h \in T(k,2)$ and $f(Y) \in k[Y]$ such that $h^{-1} \circ \frac{\partial}{\partial F_1} \circ h = f(Y)\partial_X$.

Let $g \in A$. Then $h^{-1} \circ \frac{\partial}{\partial F_1} \circ h(g) = 0$ if and only if $h(g) \in \ker \frac{\partial}{\partial F_1} = k[F_2]$, i.e. $\ker h^{-1} \circ \frac{\partial}{\partial F_1} \circ h = k[h^{-1}(F_2)]$. Additionally, $\ker f(Y)\partial_X = k[Y]$, so $k[Y] = k[h^{-1}(F_2)]$. This implies that $h^{-1}(F_2) = cY + d$ for some $c \in k^*$ and some $d \in k$. So $F_2 = ch(Y) + d$.

Furthermore, $(h^{-1} \circ \frac{\partial}{\partial F_1} \circ h)(h^{-1}(F_1)) = 1$, so $f(Y)\partial_X(h^{-1}(F_1)) = 1$ (by our assumption). This causes $\partial_X(h^{-1}(F_1)) \in k^*$, so $h^{-1}(F_1) = c'X + d'(Y)$ for some $c' \in k^*$ and some $d'(Y) \in k[Y]$. Hence $F_1 = c'h(X) + d'(h(Y))$. Thence

$$F = (F_1, F_2) = (c'h(X) + d'(h(Y)), ch(Y) + d) = (c'X + d'Y, cY + d)h.$$

Since $h \in T(k,2)$ one then deduces that $F \in T(k,2)$, and so $\mathrm{Aut}_k A \subseteq T(k,2)$. □

2.2.1 Outline of the proof of Rentschler's theorem

We follow the exposition given in van den Essen's book [34]. First we need some background regarding derivations and gradings. We start with the following.

Proposition 28. *Let $R = \bigoplus_{m \in \mathbb{Z}} R_m$ be a graded ring and let $D \neq 0$ be a derivation on R. Suppose that D can be written as a finite sum of derivations of the form*

$$D = D_p + D_{p+1} + \cdots + D_d$$

where $D_n R_m \subset R_{n+m}$ for each $m, n \in \mathbb{Z}$. If D is locally nilpotent, then so is D_d.

Proof. It suffices to show that for each $m \in \mathbb{Z}$, each element of R_m is annihilated by some power of D_d. Let $g \in R_m$. Since D is locally nilpotent, we have $D^N(g) = 0$ for some N. $D_d^N(g)$ is the component of $D^N(g)$ belonging to R_{m+Nd}. As $D^N(g) = 0$ and the sum is direct, we must have $D_d^N(g) = 0$. □

We shall now restrict our vision to the case $R = k[X] = k[X_1,\ldots,X_n]$. Consider a vector $0 \neq w \in \mathbb{Z}^n$. Associate to w the *w-grading on R* by defining for each $d \in \mathbb{Z}$ the k-vector space $R_d(w)$ generated by all monomials X^a with $\langle a, w\rangle = d$ (where $\langle\, ,\rangle$ is the standard inner product of $\mathbb{R}^n$).

We shall sometimes write R_d instead of $R_d(w)$ where no confusion shall arise. We apply the previous proposition using the following:

Proposition 29. *Let $0 \neq w \in \mathbb{Z}^n$ and consider the w-grading on R. Let $D \neq 0$ be a derivation on R. Then D can be written uniquely as a finite sum of derivations $\sum D_p$ such that $D_p R_d \subset R_{p+d}$ for all $p, d \in \mathbb{Z}$.*

Proof. Uniqueness follows from the fact that the sum is direct (as we are examining a grading on R). We must prove existence. Let $T = cX^a\partial_i$ be a term appearing in D. Put $s = a - e_i$, where e_i is a standard basis vector of $\mathbb{R}^n$. Then $T(X^m) \in kX^{m+s}$ for all m.

We call s the *strength of* T and write

$$\operatorname{supp}(D) = \{\, s \in \mathbb{Z}^n \mid D \text{ contains a term of strength } s \,\}.$$

Denote by $D(s)$ the sum of terms in D of strength s. Put

$$D_p = \sum_{\langle s,w\rangle = p} D(s)$$

We obviously have $D = \sum D_p$ as every term in D has some strength. It remains to show that $D_p R_d \subset R_{p+d}$. Equivalently, we must show that $D(s)R_d \subset R_{p+d}$. As $R_d = \langle \{\, X^a \mid a \in \mathbb{Z}^n \text{ with } \langle a, w\rangle = d\}\rangle$, we must show that

$$D(s)X^m \in R_{p+d}$$

for all $X^m \in R_d$ and all $s \in \mathbb{Z}^n$ with $\langle s, w\rangle = p$. Note that $m \in \mathbb{Z}^n$ satisfies $\langle m, w\rangle = d$.

Observe that $D(s)X^m \in kX^{m+s}$ so we need to show that $\langle m+s, w\rangle = p+d$. This is trivial as $\langle m+s, w\rangle = \langle m, w\rangle + \langle s, w\rangle = d + p$. □

The decomposition of D into a finite sum $\sum D_p$ with $D_pR_d \subset R_{p+d}$ as described above is called the *w-homogeneous decomposition of D*. If p is maximal with $D_p \neq 0$, p is called the w-degree of D, denoted $w\deg(D)$. If $w = (1,1,\ldots,1)$, then p is called the *degree of D* and is denoted by $\deg D$.

We shall now consider the case $n = 2$. We write $R = k[X,Y]$ for $k[X_1,X_2]$. Let $D \neq 0$ be a derivation on R. An element of supp$D$ shall be denoted by (s,t). So $s,t \geq -1$. For example, $(s,-1) \in \mathrm{supp}D$ means that D contains a term of the form $cX^s\partial_Y$ with $c \in k^*$.

Proposition 30. *If D is a locally nilpotent derivation on $R = k[X,Y]$, then one of the following hold:*

1. $D = f(Y)\partial_X$ *for some* $f(Y) \in k[Y]$,
2. $D = f(X)\partial_Y$ *for some* $f(X) \in k[X]$,
3. *there exist* $s_0, t_0 \geq 0$ *such that* $(s_0,-1),(-1,t_0) \in \mathrm{supp}D$*; furthermore,* supp$D$ *is contained in the triangle with vertices* $(s_0,-1),(-1,-1),(-1,t_0)$.

Proof. We refer the reader to a proof due to van den Essen [34], Corollary 5.1.16, p91. □

This proposition can then be used to prove Rentschler's theorem.

We offer a sketch of the proof: if we are in cases 1 or 2 of proposition 30, we are done (taking $h = (X,Y)$ or $h = (Y,X)$ for cases 1 and 2 respectively).

Hence we may assume that s_0, t_0 are as in proposition 30. (We will sometimes write $s_0(D), t_0(D)$ if necessary). Let l' be the line passing through both $(s_0,-1)$ and $(-1,t_0)$. Then l' is given by the equation

$$(t_0+1)x + (s_0+1)y = p,$$

where $p = s_0t_0 - 1$.

Let $w = (t_0+1, s_0+1)$, consider the w-grading of $k[X,Y]$ and the corresponding w-homogeneous decomposition of D. By proposition 30, $w\deg D = p$. By proposition

28, D_p is locally nilpotent. Write D_p as $D_p = g\delta$, where $\delta = a\partial_X + b\partial_Y$ satisfies $a, b \in k[X,Y]$ and $\gcd(a,b) = 1$. As D_p is locally nilpotent, so is δ, and $\delta(g) = 0$ (by corollary 15). Since D_p is w-homogeneous, it follows that g and δ are also w-homogeneous.

By proposition 17, δ has a slice, so in particular $a(0) \neq 0$ or $b(0) \neq 0$. We assume the case $a(0) \neq 0$ (the other case is similar). Then δ contains a term of the form $c\partial_X$, where $c \in k^*$. As D_p contains a term of the form $c'X^{s_0}\partial_Y$ and $D_p = g\delta$, δ must contain a term of the form $dX^r\partial_Y$ with $r \geq 0$, $d \in k^*$. As δ is w-homogeneous, suppδ is on a line (which must be the line passing through $(-1,0)$ and $(r,-1)$). As there are no other integral points on this line segment, we must have

$$\delta = c\partial_X + dX^r\partial_Y.$$

Since $\delta(g) = 0$, we find that $g \in \ker\delta = k[Y - \frac{d}{(r+1)c}X^{r+1}]$. The homogeneity of g implies that $g = a(Y - \frac{d}{(r+1)c}X^{r+1})^N$ for some $a \in k^*$ and some $N \in \mathbb{N}$. Thence

$$D_p = a\left(Y - \frac{d}{(r+1)c}X^{r+1}\right)^N (c\partial_X + dX^r\partial_Y).$$

Now take h to be the automorphism given by $(X, Y - \frac{d}{(r+1)c}X^{r+1})$.

We then find that $h^{-1} \circ D_p \circ h = aY^N c\partial_X$, and so $s_0(h^{-1} \circ D \circ h) < s_0(D)$. The theorem then follows by induction on $s_0(D) + t_0(D)$. We refer the reader to [34] for details.

Another proof of the equality $\mathrm{Aut}_k k[X,Y] = T(k,2)$ is a consequence of the Abhyankar-Moh-Suzuki theorem, as we shall see in chapter 3.

We continue with some remarks on other proofs of the equality, due to W. Dicks among others.

2.2.2 Remarks on the proofs of $\mathrm{Aut}_k k[X,Y] = T(k,2)$

Throughout this section, k is an arbitrary field.

Several proofs of the equality $\mathrm{Aut}_k k[X,Y] = T(k,2)$ exist in the literature. In almost all proofs, the crucial step consists in establishing Theorem 31, below. First, some notation.

Given $0 \neq f \in k[X,Y]$, say $f = \sum_{i,j} \lambda_{i,j} X^i Y^j$ with $\lambda_{i,j} \in k$, define the following:

$$\begin{aligned} \mathrm{supp}(f) &= \left\{ (i,j) \in \mathbb{N}^2 \mid \lambda_{i,j} \neq 0 \right\} \\ \Delta(f) &= \left\{ (i,j) \in \mathbb{N}^2 \mid ni + mj \leq mn \right\}, \end{aligned}$$

where $m = \deg_X(f)$, $n = \deg_Y(f)$. Note that $\Delta(f)$ is the triangle with vertices $(0,0)$, $(m,0)$, $(0,n)$.

Theorem 31. *If f is a variable of $k[X,Y]$ such that $\deg(f) > 1$, then*

$$(m,0),(0,n) \in \mathrm{supp}(f) \subseteq \Delta(f) \text{ and } m \mid n \text{ or } n \mid m.$$

A proof of theorem 31 due to Makar-Limanov can be found in [17].

The aim of this section is to explain how the equality $\mathrm{Aut}_k k[X,Y] = T(k,2)$ can be derived from Theorem 31. Assuming that Theorem 31 is true, let us first prove:

Theorem 32. *Let (p,q) be a k-algebra automorphism of $k[X,Y]$ with $\deg p \leq \deg q$. Then either (p,q) is affine or there is a unique $\mu \in k^*$ and $r \geq 0$ such that*

$$\deg(q - \mu p^r) < \deg(q).$$

Proof. As (p,q) is an automorphism, it has an inverse automorphism (f,g). As before, write $f = \sum \lambda_{i,j} X^i Y^j$ for some $\lambda_{i,j} \in k$, with $m = \deg_X f$, $n = \deg_Y f$.

If $\deg(p^m) \neq \deg(q^n)$, then $\deg(f(p,q)) = \max\{\deg(p^m), \deg(q^n)\}$. However, $f(p,q) = X$, and so $\max\{\deg(p^m), \deg(q^n)\} = 1$, and so we have a contradiction.

Now suppose $\deg(p^m) = \deg(q^n)$. Note that as $\deg p \leq \deg q$, we must have $m \geq n$. Assume (p,q) is not affine (as otherwise the conclusion holds). This causes $n|m$ by the previous theorem; define $r = \frac{m}{n}$. Then $\deg(p^r) = \deg(q)$, and so $\deg q > 1$. Since $f(p,q) = X$ we must have p_0 and q_0 algebraically dependent over k, where p_0, q_0 are the highest homogeneous components of p, q respectively, with respect to the standard grading of $k[X,Y]$, i.e. the w-grading of $k[X,Y]$, where $w = (1,1)$.

Thus q_0/p_0^r is algebraic over k, so we may write $q_0/p_0^r = \mu$ for some $\mu \in k$. Thence $\deg(q - \mu p^r) < \deg(q)$, as desired. □

By induction on $\deg(q)$ in the above theorem, it follows that all k-algebra automorphisms of $k[X, Y]$ are compositions of triangular and affine automorpisms, and are thus tame.

Dicks' paper also gives the following decomposition for $\mathrm{Aut}_k k[X, Y]$ [9]:

Theorem 33. $\mathrm{Aut}_k k[X, Y]$ *is the amalgamated free product of* $\mathrm{Aff}(R, 2)$ *and* $J(R, 2)$ *over their intersection.*

Proof. A proof using some facts from combinatorial group theory can be found in [9]. The proof involves finding an explicit example of a tree acted on by $\mathrm{Aut}_k k[X, Y]$ from which the amagamated free product decomposition can then be read off. □

It is worth noting that Makar-Limanov proceeded in a somewhat similar fashion to provide a proof for theorem 33 for arbitary characteristic [17].

2.3 The tame automorphism group in dimension ≥ 3

Let R be a commutative $\mathbb{Q}$-algebra and let $R[X]$ be the polynomial ring in $n \geq 3$ variables over R. Our goal is to study $T(R, n)$. We shall see that this case differs greatly with the $n = 2$ case, as although $T(R, n)$ is generated by $\mathrm{Aff}(R, n)$ and $J(R, n)$, it is not their amalgamated free product over their intersection.

In order to show this, the following example will illustrate that there exist $\tau_i \in J(R, n) - \mathrm{Aff}(R, n)$ and $\lambda_i \in \mathrm{Aff}(R, n) - J(R, n)$ such that

$$\tau_1 \lambda_1 \ldots \tau_n \lambda_n \tau_{n+1} \in \mathrm{Aff}(R, n).$$

Example

(Jacques Alev) Let $n \geq 3$. Consider $\lambda = (X_1, X_3, X_2, X_4, \ldots, X_n)$ and $\tau = \lambda \circ \varepsilon \circ \lambda$, where $\varepsilon = (X_1 + X_2^2, X_2, \ldots, X_n)$.

Then $\tau = (X_1 + X_3^2, X_2, \ldots, X_n)$. Notice that $\tau, \varepsilon \in J(R,n) - \mathrm{Aff}(R,n)$ and that $\lambda \in \mathrm{Aff}(R,n) - J(R,n)$, but

$$\tau^{-1}\lambda\varepsilon = \lambda^{-1} \in \mathrm{Aff}(R,n).$$

Hence $T(R,n)$ is not the amalgamated free product of $\mathrm{Aff}(R,n)$ and $J(R,n)$ over their intersection.

If R is a field, we shall see that $T(R,n) = \langle \mathrm{Aff}(R,n), \varepsilon\rangle$ where ε is the non-linear automorphism $(X_1 + X_2^2, X_2, \ldots, X_n)$. This equality will be seen to generalize to the case where R is any local ring.

In what follows we use the following notations: the elementary automorphism

$$(X_1, \ldots, X_{i-1}, X_i + a, X_{i+1}, \ldots, X_n)$$

where $a \in R[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$ shall be denoted by $E_i(a)$.

If $c \in R^*$, we denote $(X_1, \ldots, X_{i-1}, cX_i, X_{i+1}, \ldots, X_n)$ by $D_i(c)$. Additionally, we write $P_{i,j}$ for the element of $\mathrm{Aff}(R,n)$ that interchanges X_i and X_j.

If G is any group with $g_1, g_2 \in G$, then we denote the *commutator* $g_1g_2g_1^{-1}g_2^{-1}$ by $[g_1, g_2]$. The *commutator subgroup* of G is the subgroup generated by all such commutators, and is denoted by $[G,G]$.

We say that R *contains sufficiently many units* if R^* generates R as a $\mathbb{Q}$-vector space. For example, if R is a local ring (or specifically a field), then R has sufficiently many units.

The following theorem was obtained in 1994 by Harm Derksen, providing some structure for $T(R,n)$ in the case when $n \geq 3$.

Theorem 34. (Derksen) *Let $n \geq 3$. Then $T(R,n)$ is generated by $\mathrm{Aff}(R,n)$ and the elementary automorphisms $E_1(cX_2^2)$ with $c \in R$.*

Furthermore, if R has sufficiently many units, then

$$T(R,n) = \langle \mathrm{Aff}(R,n), \varepsilon\rangle,$$

where $\varepsilon = (X_1 + X_2^2, X_2, \ldots, X_n)$.

We now state a proposition whose proof is similar to that of the above theorem.

Proposition 35. *Let* $n \geq 3$. *Then* $[E(R,n), E(R,n)] = E(R,n)$.

In order to prove theorem 34 and proposition 35, we shall make use of the following lemma (each statement can be verified directly quite easily).

Lemma 36. 1. $D_1(c^{-1})D_2(c^{-1})\varepsilon D_2(c)D_1(c) = E_1(cX_2^2)$ *for all* $c \in R^*$.

2. $P_{i,2}E_1(aX_2^d)P_{i,2} = E_1(aX_i^d)$ *for all* $d \in \mathbb{N}$, $a \in R$, $i \geq 2$.

3. *Let* $2 \leq i \leq n$ *and*

$$L_i = (X_1, \ldots, X_{i-1}, c_2X_2 + \cdots + c_nX_n, X_{i+1}, \ldots, X_n)$$

with $c_j \in R$ *for all* j *and* $c_i \in R^*$. *Then for all* $a \in R$, $d \in \mathbb{N}$ *we have*

$$E_1(a(c_2X_2 + \cdots + c_nX_n)^d) = L_i^{-1}E_1(aX_2^d)L_i.$$

4. $E_1(X_2X_3) = E_1(-(\frac{1}{2}X_2 - \frac{1}{2}X_3)^2)E_1((\frac{1}{2}X_2 + \frac{1}{2}X_3)^2)$.

5. $E_1(aX_2^d) = [E_1(X_2X_3), E_3(aX_2^{d-1})]$ *for all* $a \in R$, $d \geq 1$.

6. $E_i(aX_j) = [E_i(aX_k), E_k(X_j)]$ *for all* $a \in R$ *and all* i, j, k *mutually distinct.*

Proof of theorem 34. Let H be the subgroup of $T(R,n)$ generated by $\mathrm{Aff}(R,n)$ and the elements of the form $E_1(cX_2^2)$, where $c \in R$. We will show that $H = T(R,n)$.

Since $T(R,n) = \langle \mathrm{Aff}(R,n), E(R,n) \rangle$, it suffices to prove that $E_i(g) \in H$ for all $g \in R[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$ and for all $1 \leq i \leq n$. After permutation, it suffices to show that $E_1(g) \in H$ for all $g \in R[X_2, \ldots, X_n]$. Note that we can write g as a sum of monomials, none of which contain X_1.

Using the fact that $E_1(g+h) = E_1(g)E_1(h)$, we must only show that $E_1(m) \in H$ for each monomial m not containing X_1.

If $m = X_2X_3$, then by the lemma parts 3 and 4, we find $m \in H$. By induction on d in part 5 of the lemma, we can deduce that $E_1(aX_2^d) \in H$ for all $d \geq 2$, $a \in R$. Thence by part 3 of the lemma we find that

$$E_1(a(X_2 + c_3X_3 + \cdots + c_nX_n)^d) \in H$$

for all $a, c_i \in R$, $d \geq 2$.

Note that every monomial of degree d in $\mathbb{Q}[X_2,\ldots,X_n]$ is a finite $\mathbb{Q}$-linear combination of elements of the form $(X_2 + q_3X_3 + \cdots + q_nX_n)^d$, $q_i \in \mathbb{Q}$. Hence, $E_1(m) \in H$ for all monomials $m \in R[X_2,\ldots,X_n]$.

All that remains is to show the second part of the theorem, i.e. $E_1(cX_2^2) \in \langle \mathrm{Aff}(R,n), \varepsilon \rangle$ if R has sufficiently many units. If $c \in R^*$, then from part 1 of the lemma, we have

$$E_1(cX_2^2) = D_1(c^{-1})D_2(c^{-1})\varepsilon D_2(c)D_1(c) \in \langle \mathrm{Aff}(R,n), \varepsilon \rangle.$$

So let $c \in R$ be arbitrary. As R has sufficiently many units, $c = c_1 + \cdots + c_t$, with each $c_i \in R^*$. Since $E_1(cX_2^2) = E_1(c_1X_2^2)\ldots E_1(c_tX_2^2)$, we find that

$$E_1(cX_2^2) \in \langle \mathrm{Aff}(R,n), \varepsilon \rangle. \qquad \square$$

Proof of proposition 35. If we argue using the same method as in the proof of theorem 34, then all we must show is that

$$E_1(a(\langle q, X' \rangle)^d) \in [E(R,n), E(R,n)]$$

for all $a \in R$, all $d \geq 2$, and all $0 \neq q \in \mathbb{Q}^{n-1}$.

So let $q = (q_2,\ldots,q_n) \in \mathbb{Q}^{n-1}$, say with $q_i \neq 0$. Note that $E_1(a(\langle q, X' \rangle)^d) = E_1(a(q_2X_2 + \cdots + q_nX_n)^d)$.

Then by the lemma part 3, we find that

$$E_1(a(\langle q, X' \rangle)^d) = L_i^{-1}E_1(aq_i^dX_i^d)L_i$$

where $L_i = (X_1,\ldots,X_{i-1}, q_i^{-1}q_2X_2 + \cdots + q_i^{-1}q_nX_n, X_{i+1},\ldots,X_n)$. Since

$$L_i = E_i(q_i^{-1}q_2X_2)\ldots E_i(q_i^{-1}q_{i-1}X_{i-1})E_i(q_i^{-1}q_{i+1}X_{i+1})\ldots E_i(q_i^{-1}q_nX_n)$$

it follows from the lemma part 6 that

$$L_i \in [E(R,n), E(R,n)].$$

Additionally, $E_1(aq_i^d X_i^d) \in [E(R,n), E(R,n)]$ by the lemma part 5.

As $[E(R,n), E(R,n)]$ is closed and since we have found that $E_1(a(\langle q, X'\rangle)^d) = L_i^{-1}E_1(aq_i^d X_i^d)L_i$, then we have $E_1(a(\langle q, X'\rangle)^d) \in [E(R,n), E(R,n)]$ and the proof is complete. □

2.3.1 The Nagata automorphism

Recall that every automorphism of $k[X,Y]$ is tame.

It was suggested in 1972 by Nagata that the following element of $\mathrm{Aut}_k k[X,Y,Z]$, denoted σ, is not tame [19].

$$\sigma = (X - 2Y(ZX+Y^2)^2 - Z(ZX+Y^2)^2, Y + Z(ZX+Y^2), Z)$$

(Recall that we showed in section 2.2 that the two-dimensional analogue of the Nagata automorphism was not tame over $R[X,Y]$, where R was not a field.) For many years, it was not known whether $\sigma \in \mathrm{Aut}_k k[X,Y,Z]$ was tame, i.e. whether $\sigma \in T(k,3)$.

It was shown by Smith that the Nagata automorphism is stably tame [31], i.e. that it becomes tame after adding another variable. In other words, if we extend σ to $\tilde{\sigma} : k[X,Y,Z,W] \to k[X,Y,Z,W]$ by setting $\tilde{\sigma}(W) = W$, then $\tilde{\sigma} \in T(k,4)$.

Recently, I.P. Shestakov and U.U. Umirbaev found that the tame automorphisms of the polynomial ring $A = k[X,Y,Z]$ over a field k of characteristic zero are algorithmically recognizable [29, 30]. In particular, it was shown that the Nagata automorphism is wild, i.e.

Theorem 37. (Shestakov, Umirbaev) $\sigma \in \mathrm{Aut}_k k[X,Y,Z] - T(k,3)$.

2.3.2 The exponential conjecture

In the case when $R = k$ is a field, there are many open problems concerning the group of automorphisms of $k[X]$. One such open problem is that of a structure theorem for $\mathrm{Aut}_k k[X]$. The exponential conjecture, studied in this section, deals with this question. Note that this section follows the exposition given in van den Essen [34].

Since there exist non-tame automorphisms of $R[X]$, it is natural to replace the tame generators conjecture by a weaker one, the so-called exponential conjecture, which asserts that for any $\mathbb{Q}$-algebra R, every R-automorphism of $R[X]$ is a finite product of exponential automorphisms and affine automorphisms.

One defines the group of exponential automorphisms, a large class of polynomial automorphisms, using locally nilpotent derivations. This group will be shown to include Nagata's example and all elementary automorphisms.

Let A be any $\mathbb{Q}$-algebra, and let D be a locally nilpotent derivation on A. Then by proposition 8 we know that $\exp(D) : A \longrightarrow A$ is an automorphism. The automorphisms of the form $\exp D$, where D is a locally nilpotent derivation on A, are called *exponential automorphisms* of A.

We now describe a criterion for deciding whether a given endomorphism of A is an exponential automorphism. Let $f : A \longrightarrow A$ be any map. We associate a map $E : A \longrightarrow A$ to f given by $E = f - 1_A$. Note that if f is additive, then so is E. One can also note that f is a homomorphism if and only if

$$E(ab) = aE(b) + E(a)b + E(a)E(b), \text{ for all } a, b \in A,$$

or equivalently if E is an *f-derivation* on A, i.e.

$$E(ab) = E(a)f(b) + aE(b), \text{ for all } a, b \in A.$$

Proposition 38. *Let $f : A \longrightarrow A$ be a ring homomorphism. Then f is an exponential automorphism of A if and only if E is locally nilpotent. Moreover, in this case, then the map $D : A \longrightarrow A$ defined by*

$$D(a) = \sum_{i \geq 1} (-1)^{i+1} \frac{E^i(a)}{i}, \text{ for all } a \in A$$

is a locally nilpotent derivation on A and $f = \exp D$.

We do not offer a proof of this assertion; however, the following commentary is noteworthy: the map D above is defined as the power series expansion of the natural logarithm of $f = 1_A + E$.

We now show that all elementary automorphisms are exponential automorphisms.

Example
Let $1 \le i \le n$, $a \in R[X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n]$, and let $D = a\frac{\partial}{\partial X_i}$. Then it is easy to see that D is locally nilpotent. One can see that

$$\exp D(X_j) = X_j \text{ if } j \neq i,$$
$$\exp D(X_i) = X_i + a.$$

Hence $\exp D = (X_1,\ldots,X_{i-1},X_i+a,\ldots,X_n)$. Thus any elementary automorphism is an exponential automorphism.

Not only do the exponential automorphisms contain the elementary automorphisms, but they also contain the Nagata example!

Example
Let $D_0 = -2X_2\partial_1 + X_3\partial_2$ on $R[X_1,X_2,X_3]$. Then D_0 is locally nilpotent as it is a triangular derivation. Moreover, $a = X_1X_3 + X_2^2$ satisfies $D_0(a) = 0$, so by corollary 15, $D = aD_0$ is locally nilpotent.

One can find that $DX_1 = -2aX_2$, $D^2X_1 = -2a^2X_3$, $D^3X_1 = 0$, $DX_2 = aX_3$, $D^2X_2 = 0$, $DX_3 = 0$. Hence

$$\exp DX_1 = X_1 - 2aX_2 - a^2X_3,$$
$$\exp DX_2 = X_2 + aX_3,$$
$$\exp DX_3 = X_3.$$

Thus $\exp D = \sigma$, Nagata's example described earlier.

These examples all suggest replacing the Tame Generators Conjecture (which is now known to be false) with the conjecture below.

Exponential Generators Conjecture. *Let R be any commutative $\mathbb{Q}$-algebra. Then* $\mathrm{Aut}_R R[X]$ *is generated by* $\mathrm{Aff}(R,n)$ *and the exponential automorphisms of* $R[X]$.

In [34], van den Essen provides some more evidence for the above conjecture by showing that the *nilpotency subgroup* of $\mathrm{Aut}_R R[X]$, denoted $N(R,n)$, consisting of all F of the form

$$(X_1+g_1,\ldots,X_n+g_n)$$

where each g_i is a nilpotent element of $R[X]$, is actually a subset of the set of exponential automorphisms of $R[X]$.

We shall provide more evidence for the exponential conjecture by showing that *Anick's example* [34, 31], a candidate non-tame automorphism of $k[X_1,X_2,X_3,X_4]$, is an exponential automorphism.

Example

Consider the polynomial map of $k[X_1,X_2,X_3,X_4]$ given by $F=(X_1-X_4d, X_2+X_3d, X_3, X_4)$, where $d=X_3X_1+X_4X_2$.

This map is an automorphism of $k[X_1,X_2,X_3,X_4]$, and is known as Anick's example. We show that this example is an exponential automorphism.

Construct $E=F-1_{k[X_1,X_2,X_3,X_4]}$. We find that E maps each X_i as follows:

$$\begin{aligned} X_1 &\mapsto -X_4d \\ X_2 &\mapsto X_3d \\ X_3 &\mapsto 0 \\ X_4 &\mapsto 0 \end{aligned}$$

To show that F is an exponential automorphism, we must show that E is locally nilpotent. Note first that

$$\begin{aligned} E(d) &= E(X_3)X_1+E(X_3)E(X_1)+X_3E(X_1)+E(X_4)X_2+E(X_4)E(X_2)+X_4E(X_2) \\ &= X_3E(X_1)+X_4E(X_2) \\ &= -X_3X_4d+X_3X_4d \\ &= 0 \end{aligned}$$

We then have $E^2(X_1) = -X_4E(d) - E(X_4)E(d) - E(X_4)d = 0$ and $E^2(X_2) = X_3E(d) + E(X_3)E(d) + E(X_3)d = 0$. It follows that E is locally nilpotent and hence F is an exponential automorphism. Moreover, $F = \exp(D)$, where $D : k[X_1, X_2, X_3, X_4] \to k[X_1, X_2, X_3, X_4]$ is the locally nilpotent derivation given by

$$D(a) = \sum_{i\geq 1}(-1)^{i+1}\frac{E^i(a)}{i}, \quad \text{for all } a \in A.$$

Explicitly,

$$\begin{aligned}
D(X_1) &= E(X_1) - \frac{E^2(X_1)}{2} + \frac{E^3(X_1)}{3} - \cdots = -X_4d \\
D(X_2) &= E(X_2) - \frac{E^2(X_2)}{2} + \frac{E^3(X_2)}{3} - \cdots = X_3d \\
D(X_3) &= E(X_3) - \frac{E^2(X_3)}{2} + \cdots = 0 \\
D(X_4) &= E(X_4) - \frac{E^2(X_4)}{2} + \cdots = 0
\end{aligned}$$

We then see that

$$\begin{aligned}
\exp D(X_1) &= \sum_{p\geq 0}\frac{1}{p!}D^p(X_1) = X_1 - X_4d \\
\exp D(X_2) &= \sum_{p\geq 0}\frac{1}{p!}D^p(X_2) = X_2 + X_3d \\
\exp D(X_3) &= \sum_{p\geq 0}\frac{1}{p!}D^p(X_3) = X_3 \\
\exp D(X_4) &= \sum_{p\geq 0}\frac{1}{p!}D^p(X_4) = X_4
\end{aligned}$$

and so $F = \exp D$.

Chapter 3

The Abhyankar-Moh-Suzuki Theorem

3.1 History

In this section, k shall denote a field of characteristic zero. We shall study the Abhankar-Moh-Suzuki theorem, which asserts the following:

Theorem 39. (Abhyankar-Moh-Suzuki)
Let k be a field of characteristic zero. Given nonconstant polynomials $f(t), g(t) \in k[t]$ such that $k[f(t), g(t)] = k[t]$, then either $\deg f(t) | \deg g(t)$ or $\deg g(t) | \deg f(t)$.

It is interesting to note that this result was used in 1956 by Segre to "prove" the Jacobian Conjecture in the case $k = \mathbb{C}$ [27]. (Recall that the Jacobian conjecture asserts that every polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ whose Jacobi determinant is a non-zero constant is invertible.) Therefore, this result is sometimes labelled as *Segre's lemma*. In 1970, Canals and Lluis found an error in Segre's proof and published a correction [5].

In the monumental paper by Abhyankar and Moh [2] in 1975, the authors remark that there is an error in the corrected paper of Canals and Lluis. The proof given by the authors is rather complicated and based on their earlier work on the theory of approximate roots and the Tschirnhausen transformation.

Meanwhile, several new proofs have appeared, among them a proof using knot theory by Rudolph [24] in 2002, a topological proof by Żołądek [36] in 1982, and a proof using algebraic surfaces by Gurjar [10] in 2003.

It is noteworthy that whilst theorem 39 was proved in 1975 by Abhyankar and Moh [2], it was obtained independently in 1974 by Suzuki [33], whose proof was based on subharmonic partitions.

3.2 Reformulations of the Abhyankar-Moh-Suzuki Theorem

We prove three reformulations (theorems 40-42) of the Abhyankar-Moh-Suzuki Theorem. We also show that (in characteristic zero) the equality $\mathrm{Aut}_k k[X,Y] = T(k,2)$ is a corollary to the Abhyankar-Moh-Suzuki Theorem.

Theorem 40. *Let k be a field of characteristic zero and let $f(t), g(t) \in k[t]$ be such that $k[f(t), g(t)] = k[t]$. Then there exists an automorphism $\sigma = (U(X,Y), V(X,Y))$ of $k[X,Y]$ satisfying $U(f(t), g(t)) = 0$ and $V(f(t), g(t)) = t$.*

Before the proof, we introduce some notation. Given a pair $(f,g) \in k[t] \times k[t]$ and an automorphism $\sigma = (U(X,Y), V(X,Y))$ of $k[X,Y]$, define

$$(f,g)^\sigma = \big(U(f,g), V(f,g)\big) \in k[t] \times k[t].$$

This notation has the property that, given $\sigma_1, \sigma_2 \in \mathrm{Aut}_k k[X,Y]$,

$$\big((f,g)^{\sigma_1}\big)^{\sigma_2} = (f,g)^{(\sigma_1 \circ \sigma_2)}.$$

We also note that if $(f_1, g_1) = (f,g)^\sigma$ then $k[f_1, g_1] = k[f,g]$.

Proof of Theorem 40. Let $(f,g) \in k[t] \times k[t]$ be such that $k[f,g] = k[t]$. We have to prove the existence of $\sigma \in \mathrm{Aut}_k k[X,Y]$ satisfying $(f,g)^\sigma = (0,t)$. Let $m =$

$\deg f(t)$, $n = \deg g(t)$, and say

$$f(t) = f_0 + f_1 t + \cdots + f_m t^m,$$
$$g(t) = g_0 + g_1 t + \cdots + g_n t^n,$$

where each $f_i, g_j \in k$. We can assume that $m|n$ without loss of generality by the Abhyankar-Moh-Suzuki theorem. Say $\frac{n}{m} = d \in \mathbb{N}$.

Consider the automorphism $\tau = (X, Y - uX^d)$ of $k[X,Y]$, where $u = g_n/f_m^d$, and replace the pair $(f(t), g(t))$ by

$$(f(t), g(t))^\tau = (f(t), g(t) - uf(t)^d) = (f(t), h(t)),$$

where $h(t) = g(t) - uf(t)^d$. Note that $\deg h(t) < \deg g(t)$ and that $k[f(t), h(t)] = k[t]$ so that we may assume by induction that there exists $\sigma_1 \in \mathrm{Aut}_k k[X,Y]$ satisfying $(f,h)^{\sigma_1} = (0,t)$. Then $(f,g)^{(\tau\circ\sigma_1)} = \left((f,g)^\tau\right)^{\sigma_1} = (f,h)^{\sigma_1} = (0,t)$. $\square$

Recall that a *variable* of $k[X,Y]$ is an element $F \in k[X,Y]$ for which there exists a $G \in k[X,Y]$ satisfying $k[F,G] = k[X,Y]$. It is clear that if F is a variable of $k[X,Y]$ then $k[X,Y]/(F)$ is a polynomial ring in one variable over k; the Abhyankar-Moh-Suzuki Theorem gives the converse:

Theorem 41. *Let k be a field of characteristic zero. If $F \in k[X,Y]$ is such that $k[X,Y]/(F)$ is a polynomial ring in one variable over k, then F is a variable of $k[X,Y]$.*

Proof. Composing the canonical epimorphism $k[X,Y] \to k[X,Y]/(F)$ with an isomorphism $k[X,Y]/(F) \cong k[t]$ shows that there exists a surjective k-homomorphism $\varepsilon : k[X,Y] \to k[t]$ with kernel (F). Define $f(t) = \varepsilon(X)$ and $g(t) = \varepsilon(Y)$, and note that

$$F(f(t), g(t)) = F(\varepsilon(X), \varepsilon(Y)) = \varepsilon(F) = 0.$$

We have $k[f(t), g(t)] = k[t]$ (by surjectivity of ε) so Theorem 40 implies that there exists $\sigma = (U,V) \in \mathrm{Aut}_k k[X,Y]$ satisfying $U(f(t), g(t)) = 0$. Then

$$\varepsilon(U) = U(\varepsilon(X), \varepsilon(Y)) = U(f(t), g(t)) = 0,$$

so $U \in (F)$. As U is irreducible, we have $F = \lambda U$ for some $\lambda \in k^*$. Consequently $k[F, V] = k[U, V] = k[X, Y]$ and F is a variable of $k[X, Y]$. □

In the next statement, by an "epimorphism" we mean a surjective homomorphism of k-algebras $k[X, Y] \to k[t]$. Two epimorphisms $\varepsilon_1, \varepsilon_2 : k[X, Y] \to k[t]$ are *equivalent* if there exists $\sigma \in \mathrm{Aut}_k k[X, Y]$ satisfying $\varepsilon_2 = \varepsilon_1 \circ \sigma$. Then we have:

Theorem 42. (Epimorphism Theorem) *Let k be a field of characteristic zero. Then any two epimorphisms $k[X, Y] \to k[t]$ are equivalent.*

Proof. Define the epimorphism $\varepsilon_0 : k[X, Y] \to k[t]$ by $\varepsilon_0(X) = 0$ and $\varepsilon_0(Y) = t$, and let us show that an arbitrary epimorphism $\varepsilon : k[X, Y] \to k[t]$ is equivalent to ε_0.

Define $f(t) = \varepsilon(X)$ and $g(t) = \varepsilon(Y)$, then $k[f, g] = k[t]$ so by Theorem 40 there exists $\sigma = (U, V) \in \mathrm{Aut}_k k[X, Y]$ satisfying $U(f, g) = 0$, $V(f, g) = t$. Then

$$\varepsilon(\sigma(X)) = \varepsilon(U) = U(\varepsilon(X), \varepsilon(Y)) = U(f, g) = 0 = \varepsilon_0(X)$$
$$\varepsilon(\sigma(Y)) = \varepsilon(V) = V(\varepsilon(X), \varepsilon(Y)) = V(f, g) = t = \varepsilon_0(Y),$$

so $\varepsilon \circ \sigma = \varepsilon_0$. □

Let us also mention that the above results have the following interpretation in the language of algebraic geometry. Let $\mathbf{A}^1$ be the affine line and $\mathbf{A}^2$ the affine plane (over a field of characteristic zero); *then, up to an automorphism of $\mathbf{A}^2$, there is only one way to embed $\mathbf{A}^1$ in $\mathbf{A}^2$*. This will be discussed in Chapter 4.

To conclude this section, we show that the equality $\mathrm{Aut}_k k[X, Y] = T(k, 2)$ (in case $\mathrm{char} k = 0$) can be derived from Theorem 39. So this is not a reformulation of the Abhyankar-Moh-Suzuki Theorem, but a consequence of it.

Corollary 43. *If k is a field of characteristic zero then* $\mathrm{Aut}_k k[X, Y] = T(k, 2)$.

Proof. Given $\sigma = (F, G) \in \mathrm{Aut}_k k[X, Y]$, define $\|\sigma\| = (m, n)$ where

$$m = \deg F(X, 0) \quad \text{and} \quad n = \deg G(X, 0).$$

Consider the case where $n < 1$, i.e., $G \in k[Y]$. Since G is a variable of $k[X, Y]$, $k[X, Y]/(G)$ is a polynomial ring in one variable over k; together with $k[X, Y]/(G) \cong \big(k[Y]/(G)\big)[X]$, this implies that $k[Y]/(G) \cong k$, so $\deg G = 1$. Then $k[G] = k[Y]$ and consequently $k[G][F] = k[G][X]$, which implies that $F = aX + b(G)$ for some $a \in k^*$ and $b(G) \in k[G]$. Let $\tau = \big(a^{-1}(X - b(Y)), Y\big) \in J(k, 2)$, then $\tau\sigma = (X, G) \in \mathrm{Aff}(k, 2)$, so $\sigma \in T(k, 2)$.

Similarly, if $m < 1$ then $\sigma \in T(k, 2)$.

Assume that $m \geq 1$ and $n \geq 1$ and note that the assumption $k[F, G] = k[X, Y]$ implies that $k[F(X, 0), G(X, 0)] = k[X]$. Then Theorem 39 implies that $m \mid n$ or $n \mid m$. Assume that $n \mid m$ (the other case is similar) and let $d = m/n$. Then for a suitable $c \in k^*$ we have $\deg \big(F(X, 0) - cG(X, 0)^d\big) < m$. This means that if we define $\tau = (X - cY^d, Y) \in J(k, 2)$, then $\tau\sigma = (F', G)$ where $\deg F'(X, 0) < m$, i.e.,

$$\|\tau\sigma\| = (m', n) \quad \text{where } m' < m \text{ (and } n \text{ is the same as before).}$$

By induction we may assume that $\tau\sigma$ is tame; so $\sigma \in T(k, 2)$ as well. □

3.3 Characteristic $p > 0$

Throughout this section, let k be a field of characteristic $p > 0$. We shall see that Theorems 39–42 are false when k is such a field.

Lemma 44. (Moh, [18]) *Let k be a field of characteristic $p > 0$. Let $f(t)$, $g(t) \in k[t]$. Then $k[f^p, g] = k[t] \iff k[f, g] = k[t]$ and $\frac{dg}{dt} \in k^* = k - \{0\}$.*

Proof. Assume that $k[f^p, g] = k[t]$. As $k[f^p, g] \subseteq k[f, g]$, and $f(t)$, $g(t) \in k[t]$, we find that

$$k[t] = k[f^p, g] \subseteq k[f, g] \subseteq k[t].$$

Thus, we can write t as $t = F(f^p, g)$ for some $F(X,Y) \in k[X,Y]$. Differentiating the above equation with respect to t yields

$$1 = F_g \frac{dg}{dt}$$

as we are working in characteristic p. This in turn causes $\frac{dg}{dt} \in k^*$.

Conversely, assume that $k[f,g] = k[t]$ and $\frac{dg}{dt} \in k^*$. We find that

$$k[f^p, g] \supseteq k[f^p, g^p] = k[t^p].$$

As $\frac{dg}{dt} \in k^*$, we find that g is of the form $g = ct + h(t^p)$, where $c \in k^*$ and $h(X) \in k[X]$. We know that $h(t^p) \in k[t^p] \subseteq k[f^p, g]$ and $g \in k[f^p, g]$, so

$$t = c^{-1}(g - h(t^p)) \in k[f^p, g].$$

Thus $k[t] \subseteq k[f^p, g]$, and so $k[f^p, g] = k[t]$. □

The following example is due to Nagata [20].

Nagata's Example

Let k be a field of characteristic $p > 0$ and let $m > 1$ be an integer such that $p \nmid m$. Let $f = t^{p^2}$, $g = t^{mp} + t$. We claim that $k[f,g] = k[t]$. Indeed, lemma 44 gives:

$$k[t^{p^2}, t^{mp} + t] = k[t] \iff k[t^p, t^{mp} + t] = k[t] \iff k[t, t^{mp} + t] = k[t],$$

and since the last condition is true, we have $k[f,g] = k[t]$. We shall now see that the pair (f,g) gives a counterexample to theorems 39-42 (when $\text{char}k = p > 0$).

Nagata's example provides us with a counterexample to the Abyhankar-Moh-Suzuki Theorem in characteristic $p > 0$. As $\deg f = p^2$, $\deg g = mp$, where $m > 1$ is an integer with $p \nmid m$, we find that $\deg f \nmid \deg g$ and $\deg g \nmid \deg f$. Hence the result of theorem 39 is false in positive characteristic.

Note that Nagata's example gives us a counterexample to theorem 41: Let $F = Y^{p^2} - X^{mp} - X$, then $F \in k[X,Y]$ is irreducible and $F(f,g) = 0$. So (F) is the kernel of the surjective k-homomorphism $\varepsilon : k[X,Y] \to k[t]$ defined by $\varepsilon(X) = f$ and $\varepsilon(Y) = g$. Consequently, $k[X,Y]/(F) \cong k[t]$.

However we have $\deg_X F \nmid \deg_Y F$ and $\deg_Y F \nmid \deg_X F$, so it follows from theorem 31 that F is not a variable of $k[X,Y]$. Hence theorem 41 is false.

Additionally, Nagata's example gives us a counterexample to theorem 40: If there exists an automorphism $\sigma = (U,V)$ of $k[X,Y]$ satisfying $U(f(t),g(t)) = 0$, then $\varepsilon(U) = U(f,g) = 0$, so $F = \lambda U$ for some $\lambda \in k^*$. However, this is impossible as U is a variable but F is not. Hence theorem 40 is false.

Finally, Nagata's example gives us a counterexample to theorem 42: Consider the epimorphisms $\varepsilon_0, \varepsilon : k[X,Y] \to k[t]$ where ε_0 is defined by $\varepsilon_0(X) = 0$ and $\varepsilon_0(Y) = t$, and ε is defined in the above paragraphs.

Then it can be shown that there does not exist a $\sigma \in \mathrm{Aut}_k k[X,Y]$ that satisfies $\varepsilon \circ \sigma = \varepsilon_0$. (Otherwise, σ would map $\ker \varepsilon_0$ onto $\ker \varepsilon$, i.e., the ideal (X) onto the ideal (F). Since F is not a variable, this cannot happen, so σ does not exist.) Hence theorem 42 is false.

One can ask: *What are all pairs $(f,g) \in k[t] \times k[t]$ satisfying $k[f,g] = k[t]$?* (As we will see in Chapter 4, this is equivalent to asking what are all embeddings of $\mathbf{A}^1$ in $\mathbf{A}^2$, i.e., can we classify them?) If $\mathrm{char} k = 0$, Theorem 40 says that the answer is simply $\{(0,t)^\sigma \mid \sigma \in \mathrm{Aut}_k k[X,Y]\}$. However, if $\mathrm{char} k = p > 0$ then we may do the following:

- Pick $\sigma_1 \in \mathrm{Aut}_k k[X,Y]$ such that if we set $(f_1,g_1) = (0,t)^{\sigma_1}$, then $g_1'(t) \in k^*$;
- let $(f_2,g_2) = (f_1^p, g_1)$;
- pick $\sigma_2 \in \mathrm{Aut}_k k[X,Y]$ such that if we set $(f_3,g_3) = (f_2,g_2)^{\sigma_2}$, then $g_3'(t) \in k^*$;
- let $(f_4,g_4) = (f_3^p, g_3)$;
- ...
- let $(f_{2n}, g_{2n}) = (f_{2n-1}^p, g_{2n-1})$;

- choose any $\sigma \in \mathrm{Aut}_k k[X,Y]$ and define $(f_{2n+1}, g_{2n+1}) = (f_{2n}, g_{2n})^\sigma$.

Then Lemma 44 implies that $k[f_{2n+1}, g_{2n+1}] = k[t]$. In 1987, Moh [18] conjectured that all pairs (f,g) satisfying $k[f,g] = k[t]$ can be constructed by the above process. More precisely:

Moh's Conjecture 45. *Let k be an algebraically closed field of characteristic $p > 0$ and let $f, g \in k[t]$ be such that $k[f,g] = k[t]$. Then there exists $\sigma \in \mathrm{Aut}_k k[X,Y]$ such that the pair $(f_1, g_1) = (f,g)^\sigma$ satisfies:*

1. $f_1 \in k[t^p]$,
2. $\left(\frac{1}{p}\deg f_1,\ \deg g_1\right) < \left(\deg f,\ \deg g\right)$,

where the ordering $<$ is lexicographical ordering.

It is interesting to note that Moh's conjecture remains open.

3.4 Generalizations for $R[X,Y]$

We now ask whether Theorems 39 and 41 remain valid when the field k is replaced by a ring R.

We first consider Theorem 39. We contend that the following statement is true if and only if R is an integral domain of characteristic zero:

Given any nonconstant $f(t), g(t) \in R[t]$ satisfying $R[f,g] = R[t]$, we have:

$$\deg f(t) \mid \deg g(t) \ \textit{or}\ \deg g(t) \mid \deg f(t).$$

Indeed, if R is a domain of characteristic zero then $K[f,g] = K[t]$ (where $K = \mathrm{Frac} R$) so applying Theorem 39 to $K[f,g]$ gives the desired conclusion. It is also easy to see that Nagata's example gives a counterexample in the case where R is a domain of characteristic $p > 0$. If R is not an integral domain then the above statement is false, by the following example:

Example

Let R be a commutative ring containing elements $a, b \neq 0$ such that $ab = 0$. Taking

$$F = (X,\, Y + aX^3)(X + bY^2,\, Y),$$

we find that $F \in \text{Aut}_R R[X,Y]$ (as it is the composition of elementary automorphisms). Thence as $ab = 0$,

$$F = (X + bY^2,\, Y + aX^3).$$

Take $f(t) = t + bt^2$, $g(t) = t + at^3$, so that $(f, g) = (t, t)^\sigma$. Thence $R[f, g] = R[t, t] = R[t]$, but $\deg f(t) \not| \deg g(t)$ and $\deg g(t) \not| \deg f(t)$.

So we see that generalizing Theorem 39 is a rather trivial matter. Several authors have sought generalizations of Theorem 41, and this turns out to be more interesting. In this case we ask what is the class of rings R for which the following statement is true:

Each $F \in R[X,Y]$ satisfying $R[X,Y]/(F) \cong R[t]$ is a variable of $R[X,Y]$. $(*)$

Russell and Sathaye [25] showed that $(*)$ is valid whenever R is a locally factorial Krull domain of characteristic zero. Then Bhatwadekar showed in [3] that $(*)$ is valid for any commutative ring R containing $\mathbb{Q}$. In the same paper, Bhatwadekar proved the following result.

Recall that a reduced ring R is said to be *seminormal* if for all $b,\, c \in R$ with $b^3 = c^2$, there is an $a \in R$ such that $a^2 = b$, $a^3 = c$.

In an effort to further generalize the Abyhankar-Moh-Suzuki theorem, Bhatwadekar found that the result was valid for any domain R of characteristic zero under the assumption that R is seminormal [3], i.e. the following theorem was proven:

Theorem 46. *Let R be a seminormal commutative domain of characteristic zero. Let I be an ideal of $R[X,Y]$ such that $R[X,Y]/I = R[Z]$. Then I is a principal ideal, say $I = (F)$, and $R[X,Y] = R[F,G]$ for some $G \in R[X,Y]$.*

Moreover, it was shown that by Bhatwadekar [3] that

Theorem 47. *Let R be a commutative ring containing a field of characteristic zero. Let $F \in R[X,Y]$ be such that $R[X,Y]/(F) = R[Z]$. Then $R[X,Y] = R[F,G]$ for some $G \in R[X,Y]$.*

Chapter 4

Embeddings of $\mathbf{A}^m$ in $\mathbf{A}^n$

Throughout this chapter, k is an algebraically closed field (unless otherwise specified).

We shall examine the relationship between automorphisms and embeddings. More precisely, given integers $0 < m < n$ we ask whether there is a unique way to embed $\mathbf{A}^m$ in $\mathbf{A}^n$, up to automorphisms of $\mathbf{A}^n$. Let us first recall some background.

4.1 Definitions

The set k^n is called the *affine n-space* and is denoted $\mathbf{A}_k^n$ or simply $\mathbf{A}^n$. In particular, we call $\mathbf{A}^1$ the affine line, and $\mathbf{A}^2$ the affine plane.

An *(affine) algebraic set* over an algebraically closed field k is a subset of some affine space $\mathbf{A}^n$ over k, which can be described as the zero set of finitely many polynomials in n variables with coefficients in k.

One can define a subset of affine space $\mathbf{A}^n$ to be *closed* if it is an algebraic set. The closed subsets then actually satisfy the requirements for closed sets in a topology, so this defines a topology on the affine variety known as the *Zariski topology.*

A nonempty subset $Y \subseteq \mathbf{A}^n$ is *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.

An *affine variety* over an algebraically closed field k is an irreducible algebraic set. For example, the locus described by $Y - X^2 = 0$ as a subset of $\mathbb{C}^2$ is an affine variety over the complex numbers. But the locus described by $YX = 0$ is not (as it is the union of the loci $X = 0$ and $Y = 0$).

A *regular map* $\phi : \mathbf{A}^m \to \mathbf{A}^n$ *between affine spaces* over an algebraically closed field is merely a map given by polynomials. That is to say that there are n polynomials $F_1, \ldots, F_n$ in m variables such that the map is given by $\phi(x_1, \ldots, x_m) = (F_1(x), \ldots, F_n(x))$ where x stands for the many components x_i.

A *regular map* $\phi : V \to W$ *between affine varieties* is one which is the restriction of a regular map between affine spaces. That is, if $V \subset \mathbf{A}^m$ and $W \subset \mathbf{A}^n$, then there is a regular map $\psi : \mathbf{A}^m \to \mathbf{A}^n$ with $\psi(V) \subset W$ and $\phi = \psi|_V$. So, this is a map given by polynomials, whose image lies in the intended target.

The above notions of affine varieties and regular maps define the category of affine varieties over an algebraically closed field k. In addition, we shall also need the notion of embedding:

Let $\phi : V \to W$ be a regular map between affine varieties. Then ϕ is called an *embedding* if

1. $\phi(V)$ is closed in the Zariski topology of W,

2. $\phi : V \to \phi(V)$ is an isomorphism, i.e. $\phi^{-1} : \phi(V) \to V$ exists and is a regular map.

Example 1. Let $V \subset \mathbf{A}^n$ be a closed algebraic set. Then the inclusion map $i : V \hookrightarrow \mathbf{A}^n$ is an embedding. Furthermore, if $F \in \mathrm{Aut}_k \mathbf{A}^n$ then $F \circ i : V \to \mathbf{A}^n$ is also an embedding.

2. Let $1 \leq m < n$. Then the map $j : \mathbf{A}^m \to \mathbf{A}^n$ defined by

$$j(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0 \ldots, 0)$$

is an embedding, called the *standard embedding*. Furthermore, if $F \in \mathrm{Aut}_k \mathbf{A}^n$ then $F \circ j : \mathbf{A}^m \to \mathbf{A}^n$ is also an embedding.

The following two definitions will be key in the following disussion identifying the link between polynomial automorphisms and embeddings of affine spaces.

Let $V \subset \mathbf{A}^n$ be a closed algebraic subset and $\phi : V \longrightarrow \mathbf{A}^n$ an embedding. Then ϕ is called an *exotic embedding* if ϕ is not of the form $F \circ i$ as above, i.e. it is not the composition of a polynomial automorphism and the inclusion map.

Let $1 \leq m < n$. An embedding $\phi : \mathbf{A}^m \longrightarrow \mathbf{A}^n$ is called *rectifiable* if it is of the form $F \circ j$, where $F \in \mathrm{Aut}_k \mathbf{A}^n$ and j is as above.

Note that if an embedding $\phi : \mathbf{A}^m \longrightarrow \mathbf{A}^n$ is rectifiable, say $\phi = F \circ j$, then composing ϕ with the automorphism F^{-1} we get $F^{-1} \circ \phi = j$, the standard embedding of $\mathbf{A}^m$ in $\mathbf{A}^n$, explaining the significance of the term "rectifiable."

We say that two embeddings $\phi : \mathbf{A}^m \longrightarrow \mathbf{A}^n$ and $\psi : \mathbf{A}^m \longrightarrow \mathbf{A}^n$ are *equivalent* if there exists a k-automorphism σ such that $\sigma \circ \phi = \psi$.

Thus, an embedding is rectifiable if it is equivalent to the standard embedding.

4.1.1 Remarks on regular maps $\mathbf{A}^m \longrightarrow \mathbf{A}^n$.

We shall now translate some of the above notions into algebraic terms.

By definition of a regular map $\phi : \mathbf{A}^m \longrightarrow \mathbf{A}^n$, there exist $f_1, \ldots, f_n \in k[X_1, \ldots, X_m]$ such that $\phi(x) = (f_1(x), \ldots, f_n(x))$ for all $x \in \mathbf{A}^m$. So we obtain a homomorphism of k-algebras $\phi^* : k[X_1, \ldots, X_n] \rightarrow k[X_1, \ldots, X_m]$ defined by $\phi^*(X_i) = f_i$ for all i. Since k is algebraically closed, it is in particular infinite, so the polynomials $f_1, \ldots, f_n$ are uniquely determined by ϕ and consequently ϕ^* is uniquely determined by ϕ. Moreover, $\phi \mapsto \phi^*$ is a bijection going from the set of regular maps $\mathbf{A}^m \longrightarrow \mathbf{A}^n$ to the set of k-homomorphisms $k[X_1, \ldots, X_n] \rightarrow k[X_1, \ldots, X_m]$.

$$\begin{array}{ccc} \mathbf{A}^m & \xrightarrow{\phi} & \mathbf{A}^n \\ k[X_1, \ldots, X_m] & \xleftarrow{\phi^*} & k[X_1, \ldots, X_n] \end{array}$$

One has the following facts:

1. *$\phi : \mathbf{A}^n \longrightarrow \mathbf{A}^n$ is an automorphism of $\mathbf{A}^n$ if and only if ϕ^* is an automorphism of $k[X_1, \ldots, X_n]$.*

2. *$\phi : \mathbf{A}^m \to \mathbf{A}^n$ is an embedding if and only if $\phi^* : k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_m]$ is an epimorphism (i.e., a surjective k-homomorphism).*

Two epimorphisms $\varepsilon_1, \varepsilon_2 : k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_m]$ are said to be *equivalent* if there exists $\sigma \in \mathrm{Aut}_k k[X_1, \ldots, X_n]$ satisfying $\varepsilon_2 = \varepsilon_1 \circ \sigma$. This corresponds exactly to the already defined notion of equivalence of embeddings, i.e.,

3. *Two embeddings $\phi_1, \phi_2 : \mathbf{A}^m \to \mathbf{A}^n$ are equivalent if and only if the corresponding epimorphisms $\phi_1^*, \phi_2^* : k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_m]$ are equivalent.*

Note the following equivalences:

Lemma 48. *Fix an algebraically closed field k and two integers $1 \leq m \leq n$. Then the following conditions are equivalent:*

(a) Every embedding $\mathbf{A}_k^m \to \mathbf{A}_k^n$ is rectifiable;

(b) for any subvariety W of $\mathbf{A}_k^n$ satisfying $W \cong \mathbf{A}_k^m$, there exists an automorphism of $\mathbf{A}_k^n$ which maps W onto the linear subvariety $V(X_{m+1}, \ldots, X_n)$;

(c) any two epimorphisms $k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_m]$ are equivalent;

(d) for any ideal $\mathfrak{p}$ of $k[X_1, \ldots, X_n]$ satisfying $k[X_1, \ldots, X_n]/\mathfrak{p} \cong k[X_1, \ldots, X_m]$, there exists an automorphism of $k[X_1, \ldots, X_n]$ which maps $\mathfrak{p}$ onto the ideal $(X_{m+1}, \ldots, X_n)$.

Proof. Equivalence (a $\iff$ c) is clear from the discussion preceding the lemma, and (b $\iff$ d) is also just a matter of going back and forth between the geometric and the algebraic viewpoints. So it's enough to prove that (c $\iff$ d). We use the abbreviation $R_i = k[X_1, \ldots, X_i]$.

Suppose that (c) holds and consider $\mathfrak{p} \subset R_n$ satisfying $R_n/\mathfrak{p} \cong R_m$. Let $\varepsilon : R_n \to R_m$ be the composition of the canonical epimorphism $R_n \to R_n/\mathfrak{p}$ with an isomorphism $R_n/\mathfrak{p} \to R_m$. By (c), there exists $\sigma \in \mathrm{Aut}_k R_n$ such that $\varepsilon_0 \circ \sigma = \varepsilon$, where we define:

$$\varepsilon_0 : R_n \to R_m, \qquad \varepsilon_0(X_i) = \begin{cases} X_i, & \text{if } 1 \le i \le m \\ 0, & \text{if } m < i \le n. \end{cases} \tag{5}$$

In particular, σ maps the kernel of ε onto that of ε_0, i.e., it maps $\mathfrak{p}$ onto the ideal $(X_{m+1}, \dots, X_n)$.

Conversely, suppose that (d) holds and consider an arbitrary epimorphism $\varepsilon : R_n \to R_m$. To prove (c), it's enough to show that ε is equivalent to ε_0 defined in (5), above. Let $\mathfrak{p} = \ker \varepsilon$, then obviously $R_n/\mathfrak{p} \cong R_m$, so (d) implies that there exists $\sigma_1 \in \mathrm{Aut}_k R_n$ which maps $(X_{m+1}, \dots, X_n)$ onto $\mathfrak{p}$. Let $\varepsilon_1 = \varepsilon \circ \sigma_1$, then ε_1 is an epimorphism equivalent to ε. Moreover we have $\ker \varepsilon_1 = (X_{m+1}, \dots, X_n)$, which implies that the restriction $\theta : R_m \to R_m$ of ε_1 is surjective, hence $\theta \in \mathrm{Aut}_k R_m$ by Proposition 1. Define $h_1, \dots, h_m \in R_m$ by $h_i = \theta^{-1}(X_i)$, then $k[h_1, \dots, h_m] = R_m$, so

$$k[h_1, \dots, h_m, X_{m+1}, \dots, X_n] = R_m[X_{m+1}, \dots, X_n] = R_n.$$

Consequently, $\sigma := (h_1, \dots, h_m, X_{m+1}, \dots, X_n)$ belongs to $\mathrm{Aut}_k R_n$. Since $\varepsilon_1(h_i) = X_i$ for $i \le m$ and $\varepsilon_1(X_i) = 0$ for $i > m$, we obtain $\varepsilon_1 \circ \sigma = \varepsilon_0$. This shows that ε_1 (and hence ε) is equivalent to ε_0, so we proved that (d) implies (c). □

4.2 Embeddings of the line in the plane

We shall investigate under which conditions an embedding is not exotic, as well as for which m and n every embedding of $\mathbf{A}^m$ in $\mathbf{A}^n$ is rectifiable. Our first case is $m = 1, n = 2$, i.e. the embedding of the line in the plane.

By lemma 48 and theorem 42, it immediately follows:

Theorem 49. *If k is an algebraically closed field of characteristic zero, then every embedding of $\mathbf{A}^1$ in $\mathbf{A}^2$ is rectifiable.*

The above statement is the geometric formulation of the Abhyankar-Moh-Suzuki Theorem. Recall that theorems 39–42 are all false if $\mathrm{char} k = p > 0$; in fact the example of Nagata can be phrased as follows:

Nagata's Example

Let k be an algebraically closed field of characteristic $p > 0$ and let $m > 1$ be such that $p \not| m$. Then the map

$$\begin{aligned} \phi : \mathbf{A}^1 &\longrightarrow \mathbf{A}^2 \\ s &\longmapsto (s^{p^2}, s + s^{mp}) \end{aligned}$$

is an embedding which is not rectifiable. Note that the image of ϕ is a curve $C \subset \mathbf{A}^2$ satisfying $C \cong \mathbf{A}^1$, but which cannot be mapped to the X-axis by an automorphism of $\mathbf{A}^2$. Such a curve is sometimes called a *wild line.*

As mentioned in the discussion of Moh's Conjecture, the problem of classifying embeddings $\mathbf{A}^1 \to \mathbf{A}^2$ (in positive characteristic) is open.

4.3 Embeddings where $n \gg m$

As we now know that any embedding of $\mathbf{A}^1$ in $\mathbf{A}^2$ is rectifiable, the natural question would be to ask whether every embedding $\mathbf{A}^1$ in $\mathbf{A}^n$ is rectifiable in the case where $n \geq 3$.

It was conjectured by Abhyankar that for $n \geq 3$ there exist embeddings of $\mathbf{A}^1$ in $\mathbf{A}^n$ which are not rectifiable [1], where k is a field of characteristic zero. However, Craighero showed that for $n \geq 4$ every embedding of $\mathbf{A}^1$ in $\mathbf{A}^4$ is rectifiable [7], in the case $k = \mathbb{C}$. This result was improved upon by Jelonek, who showed that if $n \geq 2m + 2$, then every embedding of $\mathbf{A}^m$ in $\mathbf{A}^n$ is rectifiable [11], whereas Craighero showed this result only for $n \geq 3m + 1$. Note that this result was proved by Jelonek for an algebraically closed field of any characteristic. Srinivas generalized this result, as we shall soon see.

Let A be a finitely generated k-algebra. Consider the homomorphism $\mu : A \otimes_k A \longrightarrow A$ defined by $\mu(a \otimes b) = ab$. Define $I = \ker \mu$ and $\Omega_{A/k} = I/I^2$. The A-module $\Omega_{A/k}$ is called the *A-module of Kähler differentials.* Put

$$\begin{aligned} m(A) &= \text{the Krull dimension of the symmetric algebra of } \Omega_{A/k} \text{ over } A, \\ n(A) &= \sup\{m(A), 2\dim A + 1\}. \end{aligned}$$

Recall that the the Krull dimension (or simply dimension) of a ring R, denoted $\dim R$, is the supremum of all integers n such that there is an increasing sequence of prime ideals $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of length n in R.

In 1991, Srinivas proved the following theorem [32]:

Theorem 50. *Let k be any infinite field and A a finitely generated k-algebra. Suppose that $f : k[X_1, \ldots, X_n] \to A$ and $g : k[X_1, \ldots, X_n] \to A$ are two surjections of k-algebras with $n > n(A)$. Then there exists $F \in T(k, n)$ such that $f = g \circ F$.*

From this, one can deduce the following result obtained by Jelonek in 1987 [11], which is valid for any algebraically closed field k. In particular it holds in positive characteristic as well, which is noteworthy, given the existence of wild lines in $\mathbf{A}^2$.

Corollary 51. *If $n \geq 2m + 2$, then every embedding of $\mathbf{A}^m$ in $\mathbf{A}^n$ is rectifiable. In particular, if $n \geq 4$ every embedding of a line in $\mathbf{A}^n$ is rectifiable.*

Proof. Let $A = k[X_1, \ldots, X_m]$, then one sees that $n(A) = 2m + 1$, so the assumption on n is that $n > n(A)$. So, by Srinivas' result, any two epimorphisms $k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_m]$ are equivalent. The desired conclusion follows from Lemma 48. □

Recall that if V is an algebraic subset of $\mathbf{A}^n$, it can be described as the zero set of finitely many polynomials in n variables with coefficients in k. We denote this set by $I(V)$. We denote the quotient ring $k[X]/I(V)$ by $A(V)$, and call it the *coordinate ring* of V.

Another consequence of theorem 50 is the following [12]:

Proposition 52. (Kaliman) *Let V, W be closed algebraic subsets of $\mathbf{A}^n$, and let $n > n(A(V))$. Then any isomorphism $\phi : V \to W$ can be extended to an automorphism of $\mathbf{A}^n$.*

Proof. Since $V, W \subseteq \mathbf{A}^n$ we may consider the coordinate rings $A(V) = k[X]/I(V)$ and $A(W) = k[X]/I(W)$ of V and W, where $k[X] = k[X_1, \ldots, X_n]$, and the corresponding canonical epimorphisms $p_V : k[X] \to A(V)$ and $p_W : k[X] \to A(W)$.

Given an isomorphism $\phi : V \longrightarrow W$, we have an induced isomorphism $\phi^* : A(W) \longrightarrow A(V)$, and so we have another surjection $\phi^* \circ p_W : k[X] \longrightarrow A(V)$. Applying Srinivas' result to p_V and $\phi^* \circ p_W$ yields the desired result. □

4.4 Embeddings of the line in $\mathbf{A}^3$

We now have results that show that every embedding of $\mathbf{A}^1$ in $\mathbf{A}^n$ is rectifiable if $n = 2$ (in characteristic zero) or if $n \geq 4$ (in any characteristic). Hence a natural question would be to ask whether every embedding of $\mathbf{A}^1$ in $\mathbf{A}^3$ is rectifiable.

In 1977, Abhyankar conjectured that there exist polynomial embeddings of the affine line $\mathbf{A}^1$ in $\mathbf{A}^3$ which are inequivalent under the polynomial automorphisms of $\mathbf{A}^3$ [1].

More precisely, Abhyankar conjectured the following in the case where $\operatorname{char} k = 0$:

Conjecture 53. *Let $\theta(t) = (f(t), g(t), h(t)) : \mathbf{A}^1 \hookrightarrow \mathbf{A}^3$ be an embedding such that none of the integers $\deg f$, $\deg g$, and $\deg h$ belong to the additive semigroup generated by the other two. Then θ is not rectifiable.*

Conjecture 54. *The embeddings $\theta_n(t) = (t + t^{n+2}, t^{n+1}, t^n)$ of $\mathbf{A}_1$ in $\mathbf{A}_3$ are not rectifiable.*

Craighero has given examples of θ disproving the first conjecture, and has shown that θ_3 is rectifiable, disproving the second [8].

Moreover, Bhatwadekar and Roy showed in 1989 [4] that if $\operatorname{char} k > 0$, then θ_n is rectifiable for all n.

In 1992, Shastri constructed an embedding of the real line in the real 3-space which cannot be rectified (by polynomial automorphisms of $\mathbb{R}^3$) [28] as follows: set

$$f(t) = t^3 - 3,\ g(t) = t^4 - 4t^2,\ h(t) = t^5 - 10t.$$

We can check that the algebra homomorphism $\phi : \mathbb{C}[X, Y, Z] \rightarrow \mathbb{C}[t]$ defined by

$$X \mapsto f(t),\ Y \mapsto g(t),\ Z \mapsto h(t)$$

is surjective, as

$$\begin{aligned}\phi(YZ - X^3 - 5XY + 2Z - 7X) &= g(t)h(t) - f(t)^3 - 5f(t)g(t) + 2h(t) - 7f(t)\\ &= (t^4 - 4t^2)(t^5 - 10t) - (t^3 - 3)^3\\ &\quad - 5(t^3 - 3)(t^4 - 4t^2) + 2(t^5 - 10t) - 7(t^3 - 3)\\ &= t\end{aligned}$$

Hence, the mapping $\alpha : \mathbb{C} \to \mathbb{C}^3$, $\alpha(z) = (f(z), g(z), h(z))$, is an embedding of $\mathbf{A}^1_{\mathbb{C}} = \mathbb{C}$ in $\mathbf{A}^3_{\mathbb{C}} = \mathbb{C}^3$. It is not known whether or not α is rectifiable, but one can remark the following.

Because α is defined by real polynomials, one expects that if there exists $\sigma \in \mathrm{Aut}_{\mathbb{C}}\mathbf{A}^3_{\mathbb{C}}$ such that $\sigma \circ \alpha$ is the standard embedding, then it "should" be possible to find such a σ which is defined by polynomials with *real* coefficients. However, Shastri proved:

There does not exist $\sigma \in \mathrm{Aut}_{\mathbb{C}}\mathbf{A}^3_{\mathbb{C}}$ defined by polynomials with real coefficients and such that $\sigma \circ \alpha$ is the standard embedding. $(*)$

This led him to conjecture the following:

Conjecture 55. *(Shastri) The embedding $\alpha : \mathbf{A}^1_{\mathbb{C}} \to \mathbf{A}^3_{\mathbb{C}}$ is not rectifiable.*

Let us explain the proof of $(*)$. Since α is defined by real polynomials, it is clear that it maps the subset $\mathbb{R}$ of $\mathbb{C}$ into the subset $\mathbb{R}^3$ of $\mathbb{C}^3$. In fact what Shastri showed is that $\alpha(\mathbb{R})$ is a trefoil knot in $\mathbb{R}^3$; this is in fact the only nontrivial part of the argument. It follows that $\mathbb{R}^3 - \alpha(\mathbb{R})$ is not homeomorphic to $\mathbb{R}^3$ minus a straight line.

If $\sigma \in \mathrm{Aut}_{\mathbb{C}}\mathbf{A}^3_{\mathbb{C}}$ is defined by real polynomials then it restricts to a polynomial automorphism of $\mathbb{R}^3$, which is in particular a homeomorphism $H : \mathbb{R}^3 \to \mathbb{R}^3$. If in addition $\sigma \circ \alpha : \mathbb{C} \to \mathbb{C}^3$ is the standard embedding, then H must map $\alpha(\mathbb{R})$ to the x-axis, and hence $\mathbb{R}^3 - \alpha(\mathbb{R})$ to $\mathbb{R}^3$ minus the x-axis. So σ cannot exist and $(*)$ is proved.

Although we have restricted ourselves to the case where k is algebraically closed, one can define $\mathbf{A}^n_k$ and the notion of embedding for any field k. Then Shastri's example shows that there exists a non-rectifiable embedding of $\mathbf{A}^1_{\mathbb{R}}$ in $\mathbf{A}^3_{\mathbb{R}}$. This is interesting, but the real question is: *Does there exist a non-rectifiable embedding of* $\mathbf{A}^1_{\mathbb{C}}$ *in* $\mathbf{A}^3_{\mathbb{C}}$ *?* This is still an open problem.

4.5 Embeddings of $\mathbf{A}^2$ in $\mathbf{A}^3$

We now continue our discussion of embeddings by considering embeddings of $\mathbf{A}^{n-1}$ in $\mathbf{A}^n$. Throughout this section k will denote an algebraically closed field.

Note that if $\phi : \mathbf{A}^{n-1} \to \mathbf{A}^n$ is an embedding, then $\phi(\mathbf{A}^{n-1}) \cong \mathbf{A}^{n-1}$. Moreover, $\phi(\mathbf{A}^{n-1}) = V(f)$ for some irreducible polynomial $f \in k[X]$.

Lemma 56. *The following are equivalent:*

1. *Every embedding of* $\mathbf{A}^{n-1}$ *in* $\mathbf{A}^n$ *is rectifiable.*
2. *If* $f \in k[X_1, \ldots, X_n]$ *is irreducible and such that* $V(f) \cong \mathbf{A}^{n-1}$*, then* f *is a variable of* $k[X_1, \ldots, X_n]$.
3. *Every* $f \in k[X_1, \ldots, X_n]$ *satisfying* $k[X_1, \ldots, X_n]/(f) \cong k[X_1, \ldots, X_{n-1}]$ *is a variable of* $k[X_1, \ldots, X_n]$.

Proof. Two affine varieties are isomorphic if and only if their coordinate rings are isomorphic k-algebras; so $k[X_1, \ldots, X_n]/(f) \cong k[X_1, \ldots, X_{n-1}]$ is equivalent to $V(f) \cong \mathbf{A}^{n-1}$, and consequently ($2 \iff 3$).

Set $m = n - 1$ in lemma 48 and note that the prime ideal $\mathfrak{p}$, in condition (d) of the cited result, must have height one and hence be a principal ideal (f), with $f \in k[X_1, \ldots, X_n]$ irreducible. Then equivalence (a $\iff$ d) of the cited result gives ($1 \iff 3$) here. □

Let $n \geq 3$. Note that if f is a variable in $k[X]$, then $k[X]/(f) \cong k[T_1, \ldots, T_{n-1}]$.

The converse of this statement (i.e. whether $f \in k[X]$ with $k[X]/(f) \cong k[T_1, \ldots, T_{n-1}]$ is a variable) is not known to be true, but was conjectured by Abhyankar and Sathaye, as stated below.

Conjecture 57. (Abhyankar-Sathaye) *Let $n \geq 3$. If $f \in k[X]$ is such that $k[X]/(f) \cong k[T_1, \dots, T_{n-1}]$, then f is a variable.*

In view of lemma 56, this conjecture is equivalent to the assertion that every embedding of $\mathbf{A}^{n-1}$ in $\mathbf{A}^n$ is rectifiable. The conjecture has not been settled to this day, even for $n = 3$, but some partial results have been obtained, which we now discuss.

The following special case of this conjecture was proved by Sathaye and by Russell [26, 25]:

Theorem 58. *Let k be a field and $f \in k[X, Y, Z]$ such that $k[X]/(f) \cong k[T_1, T_2]$. If additionally f is linear in Z, then f is a variable.*

We can verify that the Abhyankar-Sathaye Conjecture implies the following:

Conjecture 59. (Sathaye) *Let $n \geq 3$. If $f \in k[X]$ is irreducible and $f^{-1}(0) \cong \mathbf{A}^{n-1}$, then $f^{-1}(c) \cong \mathbf{A}^{n-1}$ for all $c \in k$.*

Proof. We assume the Abhyankar-Sathaye Conjecture. Let $f \in k[X]$ be irreducible with $f^{-1}(0) \cong \mathbf{A}^{n-1}$. We show that $f^{-1}(c) \cong \mathbf{A}^{n-1}$ for all $c \in k$.

As f is irreducible, the coordinate ring of $f^{-1}(0)$ is $k[X]/(f)$. Note also that the coordinate ring of $\mathbf{A}^{n-1}$ is $k[T_1, \dots, T_{n-1}]$.

Now recall that two varieties are isomorphic if and only if their coordinate rings are isomorphic (as k-algebras). As $f^{-1}(0) \cong \mathbf{A}^{n-1}$, we have that

$$k[X]/(f) \cong k[T_1, \dots, T_{n-1}].$$

By the Abhyankar-Sathaye Conjecture, f is a variable.

Thus $f - c$ is a variable for any $c \in k$, and so

$$k[X]/(f - c) \cong k[T_1, \dots, T_{n-1}]$$

(as the converse of the Abhyankar-Sathaye Conjecture always holds). Note that $k[X]/(f - c)$ is the coordinate ring of $f^{-1}(c)$, and so

$$f^{-1}(c) \cong \mathbf{A}^{n-1}. \qquad \square$$

Some believe that this conjecture, and hence the Abhyankar-Sathaye Conjecture, is not likely to be true. In fact there are several candidate counterexamples (see the paper [6] of Choudary and Dimca for details).

However, there has been some support for the Abhyankar-Sathaye Conjecture, as evidenced by the following theorem, due to Kaliman and Zaidenberg [13, 14].

Theorem 60. (Kaliman, Zaidenberg) *If* $f \in \mathbb{C}[X, Y, Z]$ *is such that* $\mathbb{C}[X, Y, Z]/(f - c) \cong \mathbb{C}[t_1, t_2]$ *for almost all* $c \in \mathbb{C}$*, then* f *is a variable of* $\mathbb{C}[X, Y, Z]$.

Bibliography

[1] S. Abhyankar, On the semigroup of a meromorphic curve I, *Proc. Int. Symp. Alg. Geom.*, Kinokuniya Bookstore, Tokyo, (1978), 249-414.

[2] S. Abhyankar and T. Moh, Embeddings of the line in the plane, *J. Reine Angew. Math.*, 276(1975), 148-166.

[3] S.M. Bhatwadekar, Generalized epimorphism theorem, *Proc. Indian Acad. Sci. Math.*, 98(1988), 109-116.

[4] S.M. Bhatwadekar and A. Roy, Some results on embedding a line in 3-space, *J. Alg.*, 142(1991), 101-109.

[5] I. Canals and E. Lluis, Acerca de un resultado de Segre, *Anal. Inst. Math. Univ. Nacional Autonoma de Mexico*, 10(1970), 1-15.

[6] A. Choudary and A. Dimca, Complex hypersurfaces diffeomorphic to affine spaces, *kodai. Math. J.*, 12(1994), 171-178.

[7] P.C. Craighero, A result on m-flats in $\mathbf{A}_k^n$, *Rend. Sem. Mat. Univ. Padova*, 75(1986), 39-46.

[8] P.C. Craighero, About Abhyankar's Conjectures on space lines, *Rend. Sem. Mat. Univ. Padova*, 74(1985), 115-121.

[9] W. Dicks, Automorphisms of the polynomial ring in two variables, *Publ. Sec. Mat. Univ. Autònoma Barcelona*, 27(1983), 155-162.

[10] R. V. Gurjar, A new proof of the Abhyankar-Moh-Suzuki theorem, *Transform. Groups*, 7(2002), 61-66.

[11] Z. Jelonek, The extension of regular and rational embeddings, *Math. Ann.*, 277(1987), 113-120.

[12] S. Kaliman, Extensions of isomorphisms between affine algebraic subvarieties of k^n to automorphisms of k^n, *Proc. Amer. Math. Soc.*, 113(1991), 325-334.

[13] S. Kaliman, Polynomials with general $\mathbb{C}^2$-fibers are variables, *Pacific J. Math.*, 203(2002), 161-190.

[14] S. Kaliman and M. Zaidenberg, Families of affine planes: The existence of a cylinder, *Michigan Math. J.*, 49(2001), 353-367.

[15] S. Lang, *Algebra*, Addison-Wesley Publishing Company, 1984.

[16] Lyndon, R.C. and Schupp, P.E., Combinatorial Group Theory, *A Series of Modern Surveys in Mathematics*, Springer-Verlag, Berlin, 1977.

[17] L. Makar-Limanov, *On automorphisms of certain algebras*, Ph.D. thesis, Moscow Sate University, 1970.

[18] T.T. Moh, On the classification problem of embedded lines in characteristic p, *Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata*, Kyoto University, Kinokuniya Bookstore (1987), 267-279.

[19] M. Nagata, Polynomial rings and affine spaces, *Expository lectures from CBMS Regional Conference held at Northern Illinois University, July 25-29, 1977*, American Mathematical Society, 1978.

[20] M. Nagata, On the automorphism group of $k[x,y]$, *Lectures in Mathematics*, Kyoto University, Kinokuniya Bookstore, Tokyo, 1972.

[21] M. Nagata and A. Nowicki, Rings of constants for k-derivations in $k[x_1,\ldots,x_n]$, *J. Math. Kyoto Univ.*, 28(1988), 111-118.

[22] J.J. Rotman, *Advanced Modern Algebra*, Prentice Hall, Upper Saddle River, 2002.

[23] R. Rentschler, Opérations du groupe additif sur le plan affine, *C.R. Acad. Sci. Paris*, 267(1968), 384-387.

[24] L. Rudolph, Embeddings of the line in the plane, *J. Reine. Angew. Math.*, 337(1982) 113-118.

[25] P. Russell, Simple birational extensions of two dimensional affine rational domains, *Compositio Mathematica*, 33(1976), 197-208.

[26] A. Sathaye, On linear planes, *Proc. Amer. Math. Soc.*, 56(1976), 1-7.

[27] B. Segre, Corrispondenze di Möbius e transformazioni cremoniane intere, *Atti Accademia delle Scienze di Torino*, 91(1956), 1-17.

[28] A.R. Shastri, Polynomial representations of knots, *Tôhotu Mat. J.*, 44(1992), 11-17.

[29] I.P. Shestakov and U.U. Umirbaev, Poisson brackets and two generated subalgebras of rings of polynomials, *J. Amer. Math. Soc.*, 17(2004), 181-196.

[30] I.P. Shestakov and U.U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables, *J. Amer. Math. Soc.*, 17(2004), 197-227.

[31] M.K. Smith, Stably tame automorphisms, *J. of Pure and Applied Algebra*, 58(1989), 209-212.

[32] V. Srinivas, On the embedding dimension of an affine variety, *Math. Ann.*, 289(1991), 125-132.

[33] M. Suzuki, Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace $\mathbb{C}^2$, *J. Math. Soc. Japan*, 26(1974), No. 3, 241-257.

[34] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, Birkhäuser, Basel, 2000.

[35] W. van der Kulk, On polynomial rings in two variables, *Nieuw Archief voor Wiskunde*, 3(1953), No. 1, 33-41.

[36] Henryk Żołądek, A new topological proof of the Abhyankar-Moh theorem, *Math. Z.* 244(2003), 689-695.