# The Chromatic Category: A Connection Between Planar Graph Colouring And Representation Theory

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# Abstract

The chromatic category $\mathcal{C}(\delta)$ is a diagrammatic monoidal category that encodes information about the proper colourings of the duals of planar graphs. In this work, we show that the chromatic category is pivotal, provide a basis for its morphisms spaces, and show that it is related to several other categories in the literature. In particular, under some assumptions, we show that $\mathcal{C}(\delta)$ is isomorphic to certain monoidal subcategories of $\mathfrak{sl}_2$-mod and $U_q(\mathfrak{sl}_2)$-mod, monoidally generated by the irreducible 3-dimensional representation in both cases. We show $\mathcal{C}(\delta)$ exhibits a close relationship to the Temperley–Lieb category, which is also isomorphic to a category of representations (instead monoidally generated by the 2-dimensional irreducible representations of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$). We also demonstrate that $\mathcal{C}(\delta)$ has some relation to the Kauffman skein category. We extend many results of Fendley and Krushkal from the chromatic algebra to the chromatic category.

# Résumé

La catégorie chromatique $\mathcal{C}(\delta)$ est une catégorie monoïdale diagrammatique qui code des informations sur les colorations propres des duaux des graphes planaires. Dans ce travail, nous montrons que la catégorie chromatique est une catégorie pivot, fournissons une base pour ses espaces de morphismes, et montrons qu'elle est liée à plusieurs autres catégories dans la littérature. En particulier, sous certaines hypothèses, nous montrons que $\mathcal{C}(\delta)$ est isomorphe à certaines sous-catégories monoïdales de $\mathfrak{sl}_2$-mod et $U_q(\mathfrak{sl}_2)$-mod, généré par la représentation tridimensionnelle irréductible dans les deux cas. Nous montrons que $\mathcal{C}(\delta)$ présente une relation étroite avec la catégorie de Temperley–Lieb, qui est également isomorphe à une catégorie de représentations (généré à la place par les représentations irréductibles bidimensionnelles de $\mathfrak{sl}_2$ et $U_q(\mathfrak{sl}_2)$). Nous démontrons également que $\mathcal{C}(\delta)$ est lié à la catégorie des écheveaux de Kauffman. Nous étendons plusieurs résultats de Fendley et Krushkal de l'algèbre chromatique à la catégorie chromatique.

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# Contents

# Chapter 1

# Introduction

The goal of this thesis is to study in detail the diagrammatic monoidal category called the *chromatic category*, which we denote by $\mathcal{C}(\delta)$. In addition to studying the category itself, we highlight its connection to other well-known categories in the literature, notably to certain categories of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$ representations and the Temperley–Lieb category $\mathcal{TL}(\delta)$. The chromatic category and the chromatic algebra of order $n$ (the endomorphism algebra of a certain object in the chromatic category) already appear in the literature in several places. Most notably:

- The chromatic category, under certain non-degeneracy assumptions, is a trivalent category in the sense defined in [MPS17].
- In [FK10], the chromatic algebra is shown to be related to the Temperley–Lieb and SO(3) BMW algebras.
- In [FK09], the chromatic algebra is utilized to obtain identities satisfied by chromatic polynomials.
- The chromatic algebra appears in some more recent papers, such as [AK19], which studies a more general *flow category*, and [Liu24], which studies certain properties of the chromatic algebra of order $n$, including computing its dimension.

Chapters 2 to 5 introduce the background necessary in order to read this work. The material presented in these chapters is not original, and references are given when possible. The remaining chapters contain the main contents of the thesis.

In Section 6.1, we define and prove several properties of the chromatic category $\mathcal{C}(\delta)$, in particular that it is a pivotal monoidal category. This category depends on the single parameter $\delta$, which should be thought of as the number of colours.

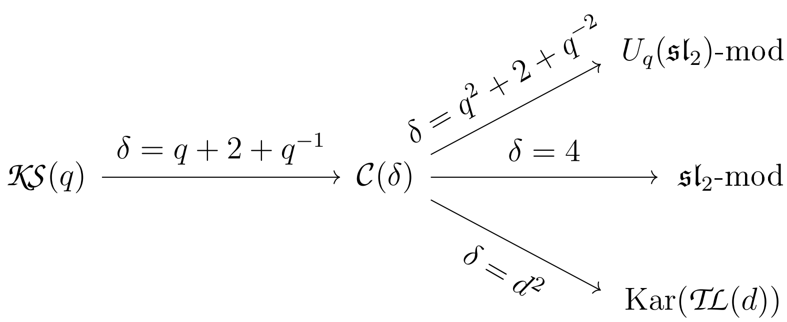
In Section 6.2, we give an alternate definition of the chromatic category, which we call the *planar chromatic category* $\mathcal{P}(\delta)$. This definition matches more closely with the definitions of the chromatic algebra in the literature [FK10; FK09; AK19; Liu24]. We show that $\mathcal{P}(\delta)$ and $\mathcal{C}(\delta)$ are isomorphic and use an argument similar to the one used in [FK10, Section 4] to obtain a basis for the morphism spaces of $\mathcal{P}(\delta)$. We then utilize this result to give many bases for the morphism spaces of $\mathcal{C}(\delta)$.

In Sections 7.1 and 7.2, we explicitly define functors from the chromatic category to the categories $\mathfrak{sl}_2$-mod and $U_q(\mathfrak{sl}_2)$-mod respectively (with the appropriate choice of the parameter $\delta$), where $U_q(\mathfrak{sl}_2)$ is the *quantized enveloping algebra* of $\mathfrak{sl}_2$. Under these functors, the generating object $X$ of $\mathcal{C}(\delta)$ is sent to the 3-dimensional irreducible representations of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$, respectively. In the non-quantum setting, this representation is the adjoint representation of $\mathfrak{sl}_2$, and in the quantum setting, it is the quantum analogue of the adjoint representation. The trivalent vertex of $\mathcal{C}(\delta)$ under these functors is sent to the Lie bracket and "quantum Lie bracket" respectively. Although the existence of these functors is apparently known (see [MPS17, Section 2] for example), an explicit description of them like we give here seems to be missing from the literature. These results are analogous to the well-known equivalence of the Temperley–Lieb category $\mathcal{TL}(\delta)$ with a category of representations of $U_q(\mathfrak{sl}_2)$ wherein the generating object $X$ of $\mathcal{TL}(\delta)$ is sent to the 2-dimensional irreducible representation (the fundamental representation) of $U_q(\mathfrak{sl}_2)$.

In Section 7.3, we exploit the equivalence of $\mathcal{TL}(\delta)$ with a category of $U_q(\mathfrak{sl}_2)$ representations to obtain a functor $\mathcal{C}(\delta^2) \to \mathrm{Kar}(\mathcal{TL}(\delta))$. Versions of this map have appeared in the literature in various contexts. On the level of algebras, this map appears as an algebra homomorphism from the chromatic algebra to the Temperley–Lieb algebra in [FK10, Lemma 6.2].

In Section 7.4, we define a functor from a certain Kauffman skein category $\mathcal{KS}(q)$ to $\mathcal{P}(\delta)$ (which also yields a functor from $\mathcal{KS}(q)$ to $\mathcal{C}(\delta)$). This category $\mathcal{KS}(q)$ was introduced in [Tur89]. The endomorphism algebras of the objects in $\mathcal{KS}(q)$ are the SO(3) BMW algebras. On the level of algebras, this map appears as an algebra homomorphism in [FK10, Theorem 5.1] from the SO(3) BMW algebra to the chromatic algebra.

To summarize the results of Chapter 7, we have the following diagram of functors:




In Section 8.1, we compute the dimension of the morphism spaces of $\mathcal{C}(\delta)$. Following this, in Section 8.2, we prove several facts involving the functors appearing in Sections 7.1 to 7.3. In particular, we show that, for certain choices of the parameter $\delta$, the chromatic category $\mathcal{C}(\delta)$ is isomorphic to the full monoidal subcategory of $U_q(\mathfrak{sl}_2)$-mod generated by the irreducible 3-dimensional representation of $U_q(\mathfrak{sl}_2)$. In the case where $\delta = 4$ (i.e. four colours), the chromatic category $\mathcal{C}(4)$ is isomorphic to the full monoidal subcategory of $\mathfrak{sl}_2$-mod generated by the adjoint representation of $\mathfrak{sl}_2$. Lastly, in Section 8.3, we discuss some possible future work related to the chromatic category.

# Chapter 2

# Graph theory

In this chapter, we give some necessary background on graph theory. Most of the material recalled in this chapter can be found in any standard textbook on graph theory, such as [BM08] or [Wes96].

## 2.1 Definitions and terminology

In this section we introduce graphs and the some of the definitions and terminology that come with them.

**Definition 2.1.1.** A *graph* $G$ consists of a set of *vertices* (also called *nodes*) $V(G)$, a set of *edges* $E(G)$, and an incidence function $\psi_G \colon E(G) \to 2^{V(G)}$ (where $2^S$ is the set of all subsets of a set $S$). The function $\psi_G$ associates to every edge $e \in E(G)$, a set $\psi_G(e) = \{u, v\}$, for some vertices $u, v \in V(G)$. In this work, we will assume all graphs have a finite number of vertices and edges.

We now introduce some standard terminology related to this definition. Let $G$ be a graph, $e, f \in E(G)$, and $u, v \in V(G)$.

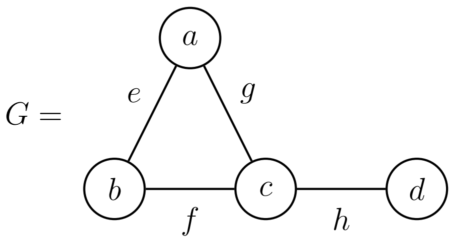
- The edge $e$ is *incident* upon $u$ if $u \in \psi_G(e)$.
- The vertices $u$ and $v$ are *adjacent* if there exists an edge $h \in E(G)$ with $\psi_G(h) = \{u, v\}$.
- The edges $e$ and $f$ are *parallel* if $\psi_G(e) = \psi_G(f)$ and $e \neq f$.
- The edge $e$ is a loop if $\psi_G(e) = \{w\}$ for some vertex $w \in V(G)$.
- The graph $G$ is called *simple* if it contains no loops and no parallel edges.

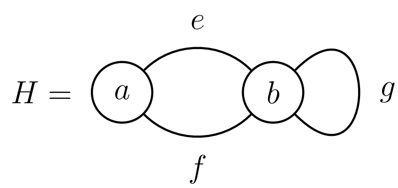
**Definition 2.1.2.** Let $G$ be a graph and $v \in V(G)$. Define the *degree* (or *valency*) of the vertex $v$ in $G$ to be

$$\begin{aligned}\deg_G(v) = \ & |\{e \in E(G) \colon e \text{ is not a loop, and } e \text{ is incident on } v\}| \\ & + \ 2 \cdot |\{e \in E(G) \colon e \text{ is a loop, and } e \text{ is incident on } v\}|.\end{aligned}$$

**Remark 2.1.3.** Given a graph $G$, we can (and usually will) encode the data of the graph $G$ in a drawing, rather than providing the data in Definition 2.1.1. In the drawing, vertices of the graph are drawn as points (or sometimes circles with labels), and edges as curves connecting the points corresponding to the vertices the edge is incident upon. Throughout most of this thesis, we will sometimes give a drawing of the graph $G$ without explicitly defining the sets $V(G)$ and $E(G)$, and the incidence function $\psi_G$.

**Example 2.1.4.** Consider the following drawings of the graphs $G$ and $H$.




In both cases, we can recover the data required to define a graph as in Definition 2.1.1 from the drawing. For the graph $G$, we have $V(G) = \{a, b, c, d\}$, $E(G) = \{e, f, g, h\}$, and

$$\psi_G(e) = \{a, b\},\ \psi_G(f) = \{b, c\},\ \psi_G(g) = \{a, c\},\ \psi_G(h) = \{c, d\}.$$

For $H$, we have $V(G) = \{a, b\}$, $E(G) = \{e, f, g\}$, and

$$\psi_H(e) = \{a, b\},\ \psi_H(f) = \{a, b\},\ \psi_H(g) = \{b\}.$$

Note the following about these two examples:

- In $G$, the edge $g$ is incident upon the vertices $a$ and $c$. In $H$, $g$ is incident upon only $b$.
- In $G$, $a$ is adjacent to both $b$ and $c$, but not to $d$.
- The edges $e$ and $f$ are parallel in $H$. However, $G$ has no parallel edges.
- There are no loops in $G$. The only loop in $H$ is $g$.
- The graph $G$ is simple since it has no loops and no parallel edges.

Note that the degrees of the vertices in the two graphs are the following:

$$\deg_G(a) = 2,\ \deg_G(b) = 2,\ \deg_G(c) = 3,\ \deg_G(d) = 1,\ \deg_H(a) = 2,\ \text{and}\ \deg_H(b) = 4.$$

**Definition 2.1.5.** Let $G$ be a graph and $u, v \in V(G)$. A *uv-path* in $G$ of length $r > 0$ is a sequence that alternates between vertices and edges $v_0 e_0 v_1 e_1 \cdots e_{r-1} v_r$ (i.e. $v_j \in V(G)$ for all $j \in \{0, 1, \cdots, r\}$ and $e_j \in E(G)$ for all $j \in \{0, 1, \cdots, r-1\}$) such that

- $v_0 = u$ and $v_r = v$,
- for all $j \in \{0, 1, \cdots, r-1\}$, we have $\psi_G(e_j) = \{v_j, v_{j+1}\}$, and

- the edges $e_0, e_1, \cdots, e_{r-1}$ are distinct and the vertices $v_0, v_1, \cdots, v_r$ are distinct, except for possibly $v_0 = v_r$.

In the case that $v_0 = v_r$, we call the path a *cycle*.

**Definition 2.1.6.** Let $G$ be a graph. Define a relation $\sim_G$ on $V(G)$ such that

$$v \sim_G u \iff \text{there exists a } uv\text{-path in } G.$$

One can verify that $\sim_G$ is an equivalence relation. We call the equivalence classes of $\sim_G$ the *connected components of* $G$, and denote the number of such classes as $c(G)$. If $c(G) = 1$, then $G$ has one connected component, in which case we say $G$ is *connected*. Otherwise, if $c(G) > 1$, then we say $G$ is *disconnected*.

**Example 2.1.7.** The graphs $G$ and $H$ in Example 2.1.4 are both connected. The graph $G$ contains a cycle of length 3 with vertices $a, b, c$, and $H$ contains cycles of two different lengths; one of length 2 with vertices $a, b$ and one of length 1 using only the vertex $b$. Note that if we have a cycle of length $r > 1$ in a graph, we can obtain many more cycles by choosing a different starting point and orientation, resulting in $2r$ distinct cycles.

Now we define the notion of a subgraph of a graph.

**Definition 2.1.8.** Let $G$ be a graph. A graph $H$ is a *subgraph* of the graph $G$ if the following conditions are satisfied:

- $V(H) \subseteq V(G)$,
- $E(H) \subseteq E(G)$, and
- $\forall e \in E(H), \quad \psi_H(e) = \psi_G(e)$.

If $H$ is a subgraph of $G$ such that $V(G) = V(H)$, then we call $H$ a *spanning subgraph* of $G$. If $H$ is a subgraph of $G$ with vertex set $X = V(H)$ such that $E(H) = \{e \in E(G) \colon \psi_G(e) \subseteq X\}$, then we call $H$ the *induced subgraph* on the vertex set $X$, and denote this by $H = G[X]$.

**Remark 2.1.9.** If $G$ is a graph, any spanning subgraph $H$ by definition satisfies $V(H) = V(G)$. Thus, it is enough to specify the edge set when defining a spanning subgraph $H$ of a graph $G$. It will be convenient sometimes to identify this subgraph with this set of edges.

**Definition 2.1.10.** Let $G$ be a graph and $H$ be a spanning subgraph of $G$. Let $e \in E(G)$. Then, we define the following operations.

- Edge deletion: We define $G \setminus e$ to be the graph obtained from $G$ by deleting the edge $e$. That is, we have $V(G \setminus e) = V(G)$, $E(G \setminus e) = E(G) \setminus \{e\}$, and $\psi_{G\setminus e} = \psi_G\big|_{E(G)\setminus\{e\}}$.

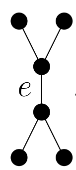
- Edge contraction: We define $G/e$ to be the graph obtained from $G$ by *contracting* the edge $e$. By this we mean the edge $e$ is deleted and the vertices it is incident upon are merged together. Thus, if we suppose $\psi_G(e) = \{u, v\}$ and let $v'$ be a vertex not in $V(G)$, then we can define $V(G/e) = V(G) \setminus \{u, v\} \cup \{v'\}$, $E(G/e) = E(G) \setminus e$. Then, for all edges $f \in E(G/e)$, we have

$$\psi_{G/e}(f) = \begin{cases} \psi_G(f) & \text{if } \{u, v\} \cap \psi_G(f) = \emptyset, \\ \psi_G(f) \setminus \{u, v\} \cup \{v'\} & \text{otherwise.} \end{cases}$$

- Edge addition: Suppose $H$ is a spanning subgraph of $G$. We define $H \cup e$ to be the subgraph $H$ in which we have added in the edge $e$. That is, $E(H \cup e) = E(H) \cup \{e\}$, and for all edges $f \in E(H \cup e)$

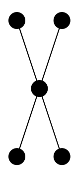
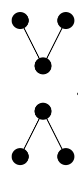
$$\psi_{H \cup e}(f) = \psi_G(f).$$

**Example 2.1.11.** Consider the following graph $G$ with the edge $e$ labeled below

$G =$ 

.

Then the graphs obtained from $G$ by contracting and deleting the edge $e$ respectively are

$G/e =$  and $G \setminus e =$  .

**Definition 2.1.12.** Let $G$ and $H$ be graphs. We say $G$ and $H$ are *isomorphic*, denoted by $G \simeq H$, if there exists bijections $f \colon V(G) \to V(H)$ and $g \colon E(G) \to E(H)$ such that for all edges $e \in E(G)$, $\psi_G(e) = \{u, v\}$ if and only if $\psi_H(g(e)) = \{f(u), f(v)\}$.

## 2.2 Trees, spanning trees, and fundamental cycles

In this section, we recall some definitions and results related to trees, spanning trees, and the cycle space of a graph.

**Definition 2.2.1.** A graph is called *acyclic* if it contains no cycles. A graph is called a *tree* if it is acyclic and connected.

Note that if a graph $G$ is acyclic, then every connected component of $G$ is connected and acyclic. Consequently, we may also refer to an acyclic graph as a *forest*, since each of its connected components are trees. We now recall the following equivalent characterizations of a tree:

**Proposition 2.2.2** ([Wes96, Theorem 2.1.4.]). *Let $T$ be a graph. Then, the following are equivalent:*

*(i) $T$ is a tree.*

*(ii) $T$ is acyclic and connected.*

*(iii) $T$ is acyclic and $|E(T)| = |V(T)| - 1$.*

*(iv) $T$ is connected and $|E(T)| = |V(T)| - 1$.*

**Definition 2.2.3.** Let $G$ be a connected graph. A *spanning tree* of $G$ is a spanning subgraph $T$ of $G$ that is a tree.

We now wish to introduce a notion of the number of "independent" cycles in a graph. To do so, we will first justify how we can view any subset of the edges of a given graph $G$ as a vector in an $\mathbb{F}_2$-vector space.

**Definition 2.2.4.** Let $G$ be a graph and let $W_G = 2^{E(G)}$ be the set of all subsets of the edges of $G$. We equip $W_G$ with an $\mathbb{F}_2$-vector space structure by defining the addition of two sets $S_1$ and $S_2$ to be the symmetric difference $S_1 \Delta S_2$. We call this vector space the *edge space* of the graph $G$. Given a cycle $C$ in $G$, we can identify $C$ with its set of edges and thus we can think of each cycle of the graph $G$ as a vector in $W_G$. We define the *cycle space* of $G$, denoted by $\mathrm{Cyc}_G$, to be span of all of its cycles in $W_G$.

**Remark 2.2.5.** In view of Remark 2.1.9, one can think of the edge space $W_G$ of a graph $G$ as the space of all spanning subgraphs of $G$.

**Definition 2.2.6.** Let $G$ be a connected graph, and $T$ be a spanning tree of $G$. Define the *co-tree* of $T$ to be the set $\overline{T} = E(G) \setminus E(T)$.

**Lemma 2.2.7.** *Let $G$ be a connected graph, and $T$ be a spanning tree of $G$. Let $e \in \overline{T}$ be an element of the co-tree of $T$, then $T \cup \{e\}$ contains a unique cycle, which we denote by $C_{e,T}$. We call $C_{e,T}$ the* fundamental cycle *of $G$ with respect to the edge $e$ and spanning tree $T$.*

**Proof.** Suppose that $\psi_G(e) = \{u, v\}$. Since $T$ is connected, we know there exists a $uv$-path in $T$. Adding in the edge $e$ to this path would yield a cycle in $T \cup e$ as required. If there were two cycles $C_1$ and $C_2$ in $T \cup e$, both cycles must use the edge $e$ since $T$ is acyclic. Let $P_1$ and $P_2$ be the $uv$-paths in $T$ obtained from removing the edge $e$ from $C_1$ and $C_2$ respectively. If these paths were distinct, then concatenating them would yield a walk (a path allowing repeated vertices and edges) which could be refined to a cycle, contradicting the fact that $T$ is acyclic. Thus we must have $C_1 = C_2$. □

**Theorem 2.2.8** ([BM08, Corollary 4.11]). *Let $G$ be a connected graph and let $T$ be a spanning tree of $G$, then the set of fundamental cycles of $G$ with respect to the spanning tree $T$ form a basis for $\mathrm{Cyc}_G$. That is,*

$$\{C_{e,T} \colon e \in \overline{T}\}$$

*is a basis for $\mathrm{Cyc}_G$.*

**Corollary 2.2.9.** *For any graph $G$, we have* $\dim(\mathrm{Cyc}_G) = |E(G)| - |V(G)| + c(G)$.

**Proof.** Let $n = |V(G)|$, $m = |E(G)|$, and $r = c(G)$. We must show that $\dim(\mathrm{Cyc}_G) = m - n + r$. Let $G_1, G_2, \cdots, G_r$ be the connected components of $G$. For $i = 1, 2, \cdots r$, let $n_i = |V(G_i)|$, $m_i = |E(G_i)|$, and pick a spanning tree $T_i$ of $G_i$. For each such $i$, we have by Theorem 2.2.8 that $\dim(\mathrm{Cyc}_{G_i}) = \left|\overline{T_i}\right|$. Consequently, we have

$$\dim(\mathrm{Cyc}_G) = \sum_{i=1}^{r} \dim(\mathrm{Cyc}_{G_i})) = \sum_{i=1}^{r} \left|\overline{T_i}\right| \overset{\text{Proposition 2.2.2}}{=} \sum_{i=1}^{r} (m_i - (n_i - 1)) = m - n + r,$$

as required. □

**Remark 2.2.10.** We call $\dim(\mathrm{Cyc}_G)$ the *cycle number* or *nullity* of the graph $G$ and denote it by $n(G)$.

## 2.3 Planar graphs and proper colourings

In this section, we discuss planar graphs and recall the concept of a proper colouring of a graph.

**Definition 2.3.1.** A graph $G$ is *planar* if it can be drawn in the plane (in the sense of Remark 2.1.3) without any edge crossings. In other words, a planar graph is a graph that has an embedding into the plane, such that no edges cross each other. A *plane graph* is a planar graph with a fixed planar embedding.

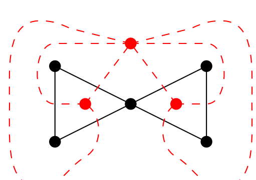
Given a plane graph $G$, if we remove the points and curves associated to its vertices and edges, respectively, from the plane, then the plane is split into many path-connected open sets. We call these connected components the *faces* of the plane graph $G$. Given a plane graph, the curve associated with one of its edges separates two (not necessarily distinct) faces. This motivates the following definition.

**Definition 2.3.2.** Let $G$ be a plane graph and let $\mathcal{F}$ be its set of faces. The *dual graph* of $G$, denoted by $G^*$, with vertex set $V(G^*) = \mathcal{F}$ and $E(G^*) = \{e^* \colon e \in E(G)\}$ where $e^*$ is the curve associated with the edge $e$ from the planar embedding of $G$. Given an edge $e \in E(G)$ such that $\psi_G(e) = \{u, v\}$ and a face $F \in \mathcal{F}$, we define $e^*$ to be incident upon $F$ if

$$(e^* \setminus \{p_u, p_v\}) \cap \partial(F) \neq \emptyset$$

where $\partial(F)$ is the boundary of the set $F$, and $p_u$ and $p_v$ are the points associated to the vertices $u$ and $v$ respectively from the planar embedding of $G$.

**Example 2.3.3.** The following is an example of a plane graph $G$ (in black) and its dual $G^*$ (in red):

 .

The graph $G$ has 3 faces, thus $G^*$ has 3 vertices.

**Proposition 2.3.4** ([BM08, Propositions 10.12, 10.13])**.** *Let $G$ be a connected plane graph.*

- *If $e$ is an edge of $G$ such that $G \setminus e$ is also connected, then $(G \setminus e)^* \simeq G^*/e^*$.*
- *If $e$ is an edge of $G$ which is not a loop, then $(G/e)^* \simeq G^* \setminus e^*$.*

Next, we recall the concept of a proper colouring of a graph.

**Definition 2.3.5.** Let $G$ be a graph. A *$k$-colouring* of the graph $G$ is a function $f\colon V(G) \to \{1, 2, \cdots, k\}$. If $f$ is a $k$-colouring and $v \in V(G)$, we call $f(v)$ the *colour* of $v$. A *proper $k$-colouring* of the graph $G$ is a $k$-colouring $f$ such that for all vertices $v, u \in V(G)$, if $u$ and $v$ are adjacent, then $f(u) \neq f(v)$. We denote the number of proper $k$-colourings of a graph $G$ by $\chi(G; k)$.

**Proposition 2.3.6.** *Let $G$ be a graph and suppose $e \in E(G)$. Then we have the following recursion for $\chi(G, k)$:*

$$\chi(G; k) = \chi(G \setminus e; k) - \chi(G/e; k).$$

**Proof.** Suppose $\psi_G(e) = \{u, v\}$. It is clear that a proper $k$-colouring $f$ of $G \setminus e$ will be a proper $k$-colouring of $G$ if and only if $f(u) \neq f(v)$. Therefore $\chi(G \setminus e; k)$ is equal to the number of proper $k$-colourings of $G \setminus e$ where $u$ and $v$ are assigned the same colour plus $\chi(G; k)$. Let $v'$ be the vertex in $G/e$ that is the result of merging $u$ and $v$ after contracting $e$. Let $X$ be the set of proper $k$-colourings of $G \setminus e$ where $u$ and $v$ are assigned the same colour and $Y$ be the set of proper $k$-colourings of $G/e$. Define a function $\ell\colon X \to Y$ by

$$\ell(f)(x) = \begin{cases} f(v) & \text{if } x = v', \\ f(x) & \text{otherwise,} \end{cases}$$

for all $x \in V(G/e)$. One can easily verify that $\ell$ is a bijection. Consequently, we have

$$\chi(G \setminus e; k) = \chi(G; k) + \chi(G/e; k) \Longrightarrow \chi(G; k) = \chi(G \setminus e; k) - \chi(G/e; k). \qquad \square$$

**Corollary 2.3.7.** *Let $G$ be a connected plane graph and suppose $e \in E(G)$ is not a loop and such that $G \setminus e$ is connected. Then we have the following:*

$$\chi(G^*; k) = \chi((G/e)^*; k) - \chi((G \setminus e)^*; k).$$

**Proof.** By Proposition 2.3.4, we have $\chi((G/e)^*; k) = \chi(G^* \setminus e^*; k)$ and $\chi((G \setminus e)^*; k) = \chi(G^*/e^*; k)$. It follows that

$$\chi(G^*; k) = \chi(G^* \setminus e^*; k) - \chi(G^*/e^*; k) = \chi((G/e)^*; k) - \chi((G \setminus e)^*; k). \qquad \square$$

# Chapter 3

# Combinatorics and the Riordan numbers

The purpose of this chapter is to give an introduction to the sequence of numbers $\{R_n\}_{n\geq 0}$ known as the Riordan numbers. This Catalan-like sequence arises in many different counting problems. This chapter is intended to be mostly self-contained, so we provide proofs for completeness.

## 3.1 The Riordan numbers: A first look

In this section, we give a definition and first formula for the Riordan numbers. We also see how this counting sequence is related to the well-known Catalan numbers.

**Definition 3.1.1.** A *partition* of a set $S$ is a set $\mathcal{S} \subseteq 2^S$ such that

$$\forall A, B \in \mathcal{S}, \quad A \neq B \Longrightarrow A \cap B = \emptyset,$$

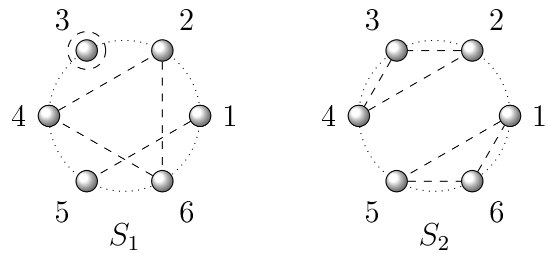
$$\forall A \in \mathcal{S}, \quad A \neq \emptyset,$$

and

$$\bigcup_{A \in \mathcal{S}} A = S.$$

A *partition of size* $n$ is a partition of $\{1, 2, \cdots, n\}$. Let $M \subseteq \mathbb{N}$. A partition $\mathcal{S}$ of $M$ is said to have a *crossing* if for some $A, B \in \mathcal{S}$ with $A \neq B$, there exists $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $a_1 < b_1 < a_2 < b_2$. If a partition has no crossings, we call it *non-crossing.*

**Remark 3.1.2.** Given a partition $\mathcal{S}$ of $\{1, 2, \cdots, n\}$, we can represent $\mathcal{S}$ by a diagram with $n$ points on a circle labeled $1, 2 \cdots, n$ in order going clockwise around the circle. For each part $S \in \mathcal{S}$, we draw the convex hull of the points labeled with the elements of $S$. For example, to the partitions $S_1 = \{\{1, 5\}, \{3\}, \{2, 4, 6\}\}$ and $S_2 = \{\{2, 3, 4\}, \{1, 5, 6\}\}$ we associate the following diagrams:




where the convex hulls for parts have been drawn with a dashed line (and circled with a dash line for singletons). With this representation of a partition, it is easy to tell if the partition has a crossing—a partition will have a crossing if and only if the convex hulls of two distinct parts overlap. For example, $S_1$ has a crossing and $S_2$ does not.

The number of non-crossing partitions of size $n$ is the $n^{\text{th}}$ Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ [Ber99, Section 3.1]. Next, we introduce a related sequence of numbers, called the *Riordan numbers*, which are also counted by a certain subset of the partitions of $\{1, 2, \cdots, n\}$.

**Definition 3.1.3.** We define the $n^{th}$ *Riordan number* $R_n$ to be the number of non-crossing partitions of size $n$ with no parts that are singletons.

Note that if $M \subseteq \mathbb{N}$ with $|M| = n$, then the number of non-crossing partitions of $M$ with no singletons is also $R_n$. By using the principal of inclusion/exclusion, we can obtain the following well-known (for example, see [Ber99, Section 5]) expression for $R_n$ in terms of the Catalan numbers.

**Proposition 3.1.4.** *Let $n$ be a non-negative integer. Then we have the following formula for the $n^{th}$ Riordan number:*

$$R_n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} C_{n-m}.$$

**Proof.** Let $\mathcal{A}$ be the set of non-crossing partitions of size $n$ and $\mathcal{A}_k$ be the set of non-crossing partitions of size $n$ that have the singleton $\{k\}$ as a part. Let $S_n = \{1, 2, \cdots, n\}$. If $\mathcal{S} \in \mathcal{A}_k$, then it is clear that $\mathcal{S} \setminus \{\{k\}\}$ would be an arbitrary non-crossing partition on the numbers $S_n \setminus \{k\}$. Consequently, we have $|\mathcal{A}_k| = C_{n-1}$. Next, notice that if $T \subseteq S_n$, then

$$\bigcap_{j \in T} \mathcal{A}_j = \{\mathcal{S} \in \mathcal{A} \colon \forall j \in T, \{j\} \in \mathcal{S}\}$$

is the set of partitions of size $n$ that have $\{j\}$ as a singleton for each $j \in T$. If we remove the singleton parts $\{j\}$ from such a partition for each such $j$, it would yield an arbitrary partition of the set $S_n \setminus T$. Thus $\left|\bigcap_{j \in T} \mathcal{A}_j\right| = C_{n-|T|}$.

Hence, by the principal of inclusion/exclusion, we have the following

$$\left|\bigcup_{j\in T}\mathcal{A}_j\right| = \sum_{m=1}^{n}(-1)^{m-1}\sum_{\substack{T\subseteq S_n\\|T|=m}}\left|\bigcap_{j\in T}\mathcal{A}_j\right| = \sum_{m=1}^{n}(-1)^{m-1}\sum_{\substack{T\subseteq S_n\\|T|=m}}C_{n-m} = \sum_{m=1}^{n}(-1)^{m-1}\binom{n}{m}C_{n-m}.$$

The set $\bigcup_{j\in T}\mathcal{A}_j$ is the set of partitions of size $n$ with at least one part that is a singleton. Thus we have

$$R_n = |\mathcal{A}| - \left|\bigcup_{j\in T}\mathcal{A}_j\right| = C_n - \sum_{m=1}^{n}(-1)^{m-1}\binom{n}{m}C_{n-m} = \sum_{m=0}^{n}(-1)^{m}\binom{n}{m}C_{n-m}.$$

as required. □

## 3.2 The trinomial difference formula for $R_n$

In this section, we give a different counting problem leading to the Riordan numbers and a trinomial difference formula for the $n^{\text{th}}$ Riordan number. The trinomial difference formula is analogous to the binomial difference formula for the $n^{\text{th}}$ Catalan number. We will see that the trinomial difference formula presented here is more useful than the definition of $R_n$ for showing that the dimensions of certain vector spaces appearing in Sections 4.2 and 4.3 are $R_n$.

**Definition 3.2.1.** Let $U = (1,1)$, $F = (1,0)$, $D = (1,-1)$ and let $n$ and $k$ be non-negative integers.

- A *trinomial path of length* $n$ is a sequence of $n$ steps $M_1, M_2, \cdots M_n$, where $M_j \in \{U, F, D\}$ for all $j \in \{1, 2, \cdots, n\}$. We call $M_j$ the $j^{\text{th}}$ *step* or *move* of the path. We say the path has *height* $k$ if
$$\sum_{j=1}^{n} M_j = (n,k),$$
and we say the $m^{\text{th}}$ step is at *height* $k$ if
$$\sum_{j=1}^{m-1} M_j = (m-1,k).$$
Denote the set of trinomial paths of length $n$ and height $k$ by $\mathcal{T}(n,k)$.

- We call a trinomial path *non-negative* if it has non-negative height and all its steps are at a non-negative height.

- A *binomial path* is a trinomial path using only the moves $U$ and $D$. Denote the set of binomial paths of length $n$ and height $k$ by $\mathcal{B}(n,k)$.

- A *Dyck path* is a non-negative binomial path of height 0. Denote the set of Dyck paths of length $n$ by $\mathcal{D}(n)$.

- A *Riordan path* is a non-negative trinomial path with no $F$-moves at height 0.

Note that if $A(z) = \sum_{k \geq k_0} a_k z^k$ is a formal Laurent series (with $k_0 \in \mathbb{Z}$), we denote its $k^{\text{th}}$ coefficient $a_k$ for $k \geq k_0$ by:

$$[z^k]A(z) := a_k.$$

**Remark 3.2.2.** The number of binomial walks of length $2n$ and height $k$ can be shown to be the binomial coefficient $\binom{2n}{n+k}$. This is because, any binomial path length $2n$ and height $k$ must contain $n+k$ $U$-moves. Consequently, one has $\binom{2n}{n+k}$ choices for the positions of these $U$-moves after which the path is determined. Similarly, the number of trinomial paths of length $n$ and height $k$ can be shown to be equal to the trinomial coefficient given by $[z^k](z+1+z^{-1})^n$. One can prove (see [Ber99, Section 5]) the identity

$$C_n = \binom{2n}{n} - \binom{2n}{n-2}$$

by giving a bijection

$$\mathcal{B}(2n, 0) \setminus \mathcal{D}(2n) \to \mathcal{B}(2n, -2).$$

In this section, we use a similar approach to obtain the analogous identity for the $n^{\text{th}}$ Riordan number $R_n$.

Before moving on, we justify the term *Riordan path* with the following proposition. The following fact is proved in [Men], however we present a proof of this fact for completeness.

**Proposition 3.2.3** ([Men, Section 5.2])**.** *There is a bijection between the set of Riordan paths of length $n$ and the non-crossing partitions of size $n$ with no singletons. Consequently, the number of Riordan paths of length $n$ is $R_n$.*

**Proof.** Fix a non-negative integer $n$ and let $P$ be a non-crossing partition of $\{1, 2, \cdots, n\}$ with no singletons. For each $k \in \{1, 2, \cdots, n\}$, we will say $k$ is *initial* in the partition $P$ if it is the minimum of one of the parts of $P$. Similarly, call we will call $k$ *terminal* if it is the maximum of some part of $P$. Lastly, if $k$ is not initial or terminal we will call it neutral. Since the partition $P$ has no singletons, we know that the maximum and minimum of each part of $P$ are distinct from each other. Consequently, each $k \in \{1, 2, \cdots, n\}$ will be exactly one of initial, terminal, or neutral.

Now given a non-crossing partition $P$ of $\{1, 2, \cdots, n\}$ without singletons, we will define a path $R_P = (M_1, M_2, \cdots, M_n)$ of length $n$ whose $k^{\text{th}}$ move is:
(1) $M_k = U$ if $k$ is initial in $P$,
(2) $M_k = D$ if $k$ is terminal in $P$, and
(3) $M_k = F$ if $k$ is neutral in $P$.

I claim $R_P$ is a Riordan path. To see this, first note that every part contains exactly one initial element and one terminal element. Thus, there are an equal amount of $U$ and $D$ moves in $R_P$, so $R_P$ has height 0. Furthermore, since the minimum of each part occurs before the maximum, we know that the path must be non-negative. Now, suppose $k$ is such that $M_k = F$ and let $T$ be the part of the partition $P$ such that $k \in T$. Let $a = \min(T)$

and $b = \max(T)$ and note that we must have $a < k < b$ since $k \in T$. Now, suppose $T'$ is a part such that there is some $k' \in T'$ with $a < k' < b$. Let $a' = \min(T')$ and $b' = \max(T')$, then since $P$ is a non-crossing partition, we must have $a < a' < b' < b$. It follows that there are an equal number of initial and terminal integers in the positions from $a+1$ to $b-1$. Furthermore, each initial integer can be paired with a terminal integer such that the initial one comes first. Thus, each of the moves $M_{a+1}, M_{a+2}, \cdots, M_{b-1}$ must occur at a height of at least 1 (since $M_a = U$ and $R_P$ is non-negative). In particular, $M_k$ has a height greater than 0 as required. Consequently, $R_P$ is a Riordan path as required.

Now, given a Riordan path $R$, we can construct a partition $P_R$ as follows: if step $k$ is a $U$, we label $k$ as initial. If it is a $D$, we label it as terminal. Otherwise we label it as neutral. One can pair the initial and terminal integers similar to how one would pair parentheses in a balanced parenthesis system. Those integers paired together will be in their own part and an integer $k$ corresponding to a $F$-move will be in a part with initial integer $a$ and terminal integer $b$ if $(a, b)$ is one of the pairs above such that $b - a$ is smallest and $a < k < b$. That is, $k$ is not contained in the scope of any "smaller" pair.

Let $\mathcal{R}$ be the set of Riordan paths of length $n$ and $\mathcal{P}$ be the set of non-crossing partitions of size $n$ with no singletons. Then it should be clear that the assignments

$$\mathcal{P} \to \mathcal{R} \text{ given by } P \mapsto R_P \quad \text{and} \quad \mathcal{R} \to \mathcal{P} \text{ given by } R \mapsto P_R$$

are inverses of each other. Thus there is a bijection between the set of Riordan paths of length $n$ and the non-crossing partitions of $\{1, 2, \cdots, n\}$ without singletons. □

We are now ready to present the trinomial difference formula for the $n^{\text{th}}$ Riordan number. The idea for this proof comes from [Cal06], however we include a proof for completeness.

**Proposition 3.2.4** ([Cal06])**.** *Let $n$ be a non-negative integer. Then the $n^{th}$ Riordan number can be expressed as a difference of trinomial coefficients. That is:*

$$R_n = [z^0](z + 1 + z^{-1})^n - [z^{-1}](z + 1 + z^{-1})^n.$$

**Proof.** Let $\mathcal{RP}(n)$ be the set of Riordan paths of length $n$ and let $T(n, k) = |\mathcal{T}(n, k)| = [z^k](z + 1 + z^{-1})^n$. Then we must show that:

$$R_n = T(n, 0) - T(n, -1) \iff T(n, 0) - R_n = T(n, -1).$$

Notice that Riordan paths of length $n$ are also trinomial paths of length $n$, i.e. $\mathcal{RP}(n) \subseteq \mathcal{T}(n, 0)$. From Proposition 3.2.3 we have that $|\mathcal{RP}(n)| = R_n$. Consequently, to prove the above formula, we need only to give a bijection between $\mathcal{T}(n, 0) \setminus \mathcal{RP}(n)$ and $\mathcal{T}(n, -1)$.

Take a path $P$ in $\mathcal{T}(n, 0) \setminus \mathcal{RP}(n)$ and then consider the last move of this path. If the last move of $P$ is a $U$ or an $F$, one can change this path into an element of $\mathcal{T}(n, -1)$ by making the following change to the last step: $U \to F$ and $F \to D$. It is clear that all paths in $\mathcal{T}(n, -1)$ which end in a $F$ or a $D$ can be obtained in this way from a path in $\mathcal{T}(n, 0) \setminus \mathcal{RP}(n)$ which ends in a $U$ or $F$. If instead the last move of our path is a $D$, then consider the longest consecutive sub-path $R$ of $P$ including this step that is a Riordan path. This path is not the whole path $P$ since $P$ is not a Riordan path, consequently we can express $P$ as $P = P'FR$ or $P = P'UR$ where $R$ is a Riordan path.

In these cases, we can make the following changes to convert $P$ to an element of $\mathcal{T}(n, -1)$:

$$P'FR \to P'D\overline{R} \text{ and } P'UR \to P'F\overline{R}.$$

where $\overline{R}$ is the path obtained from $R$ by swapping all $U$-moves with $D$-moves and vice-versa.

It is clear that this yields an element of $\mathcal{T}(n, -1)$ that ends in an $U$. Furthermore, this process in reversible since if we're given an such an element of $\mathcal{T}(n, -1)$ which ends in a $U$, then the sub-path $\overline{R}$ can by identified uniquely by the fact that the step immediately preceding $\overline{R}$ will be either: (i) the last $F$ step at height $-1$ or (ii) the last $D$ step going from height 0 to height -1.

Thus these assignments together yield a bijection between the two sets, proving that:

$$T(n, 0) - R_n = T(n, -1),$$

and consequently

$$R_n = [z^0](z + 1 + z^{-1})^n - [z^{-1}](z + 1 + z^{-1})^n. \qquad \square$$

# Chapter 4

# Representation theory

In this chapter, we recall some necessary background on the Lie algebra $\mathfrak{sl}_2$ and its "quantum deformation" $U_q(\mathfrak{sl}_2)$ (also called the *quantum of enveloping algebra* of $\mathfrak{sl}_2$). We begin by discussing Lie algebras in general.

## 4.1 Representations of Lie algebras

In this section, we recall some facts about Lie algebras and their representations. Many of these definitions and standard results can be found in [Hum78].

**Definition 4.1.1.** A *Lie algebra* over the field $\Bbbk$ is an $\Bbbk$-vector space $\mathfrak{g}$ equipped with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. This bilinear map is denoted by $(a, b) \mapsto [a, b]$ and it must satisfy

$$[a, a] = 0, \quad \forall a \in \mathfrak{g}, \tag{Alternating}$$

and

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0, \quad \forall a, b, c \in \mathfrak{g}. \tag{Jacobi identity}$$

We call the bilinear form $[\cdot, \cdot]$ the *Lie bracket* of $\mathfrak{g}$.

Note that if $\Bbbk$ does not have characteristic 2, then one can show that the following implies the alternating property:

$$[a, b] = -[b, a], \quad \forall a, b \in \mathfrak{g}. \tag{Anticommutativity}$$

We will sometimes write $[\cdot, \cdot]_{\mathfrak{g}}$ for the Lie bracket on $\mathfrak{g}$ to avoid confusion when working with more than one Lie algebra.

**Example 4.1.2.** The following are examples of Lie algebras:

- Any vector space $V$ can be made into a Lie algebra with the Lie bracket defined by $[a, b] = 0$, for all $a, b \in V$ (such a Lie algebra is called *Abelian*).

- For any $n \geq 1$, the *general linear algebra* over the field $\Bbbk$, denoted by $\mathfrak{gl}_n(\Bbbk)$, is the Lie algebra of $n \times n$ matrices over $\Bbbk$ with Lie bracket given by the *commutator* $(A, B) \mapsto [A, B] = AB - BA$.

- Let $V$ be an $\Bbbk$-vector space. The *general linear algebra over* $V$, denoted by $\mathfrak{gl}(V)$, is the Lie algebra of endomorphisms of $V$ (that is, linear maps $V \to V$) with the Lie bracket given by $(T, S) \mapsto [T, S] = TS - ST$.

- For any $n \geq 1$, the *special linear algebra* over the field $\Bbbk$ defined by
$$\mathfrak{sl}_n(\Bbbk) = \{A \in \mathfrak{gl}_n(\Bbbk)\colon \operatorname{tr}(A) = 0\}$$
is a Lie algebra with the same Lie bracket as $\mathfrak{gl}_n(\Bbbk)$.

**Definition 4.1.3.** A *representation* of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ over $\Bbbk$ is a pair $(V, \varphi)$, where $V$ is a vector space and $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a linear map satisfying:
$$\varphi([x, y]_{\mathfrak{g}}) = [\varphi(x), \varphi(y)]_{\mathfrak{gl}(V)}.$$

**Remark 4.1.4.** Given a representation $(V, \varphi)$ of a Lie algebra $\mathfrak{g}$, we can also think of $V$ as an $\mathfrak{g}$-*module*, where the action of $x \in \mathfrak{g}$ on the vector $v$ is given by $x \cdot v := \varphi(x)v$. Consequently, when it is convenient, we will often use the language of modules when talking about representations of a Lie algebra.

**Definition 4.1.5.** Let $\mathfrak{g}$ be a Lie algebra and $V$ be a representation of $\mathfrak{g}$. A *subrepresentation* of the representation $V$ of $\mathfrak{g}$ is a subspace $U \subseteq V$ such that
$$x \cdot u \in U, \quad \forall x \in \mathfrak{g}, \forall u \in U.$$

**Example 4.1.6.** For any representation $V$ of a Lie algebra $\mathfrak{g}$ over $\Bbbk$, $\{0\}$ and $V$ are always subrepresentations of $V$. The field $\Bbbk$ is also a representation of $\mathfrak{g}$ with action given by $x \cdot \alpha = 0$ for all $x \in \mathfrak{g}$ and $\alpha \in \Bbbk$. The representation $\Bbbk$ is called the *trivial representation* of $\mathfrak{g}$.

**Definition 4.1.7.** Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra over $\Bbbk$.

- A representation $V$ is called *irreducible* if $\{0\}$ and $V$ are its only subrepresentations.

- The *adjoint representation* of $\mathfrak{g}$ is the representation $(V, \mathrm{ad}_{\mathfrak{g}})$ with $V = \mathfrak{g}$ and $\mathrm{ad}_{\mathfrak{g}}$ defined by
$$\mathrm{ad}_{\mathfrak{g}}(x)y = [x, y], \quad \forall x \in \mathfrak{g}, \forall y \in V.$$

- Let $V$ and $U$ be representations of $\mathfrak{g}$. (i) $V \otimes U$ is defined to be the representation of $\mathfrak{g}$ with action given by:
$$x \cdot (v \otimes u) = (x \cdot u) \otimes v + u \otimes (x \cdot v), \quad \forall x \in \mathfrak{g}, \forall u \in U, \forall v \in V.$$
(ii) The dual $V^*$ of $V$ can be made into a representation of $\mathfrak{g}$ with
$$(x \cdot f)(v) = -f(x \cdot v), \quad \forall x \in g, \forall f \in V^*, \forall v \in V.$$
(iii) $\mathrm{Hom}_{\Bbbk}(U, V)$ is a representation of $\mathfrak{g}$ with
$$(x \cdot f)(u) = x \cdot f(u) - f(x \cdot u), \quad \forall x \in \mathfrak{g}, \forall f \in \mathrm{Hom}_{\Bbbk}(U, V), \forall u \in U.$$

- Let $V$ and $U$ be representations of $\mathfrak{g}$. A map $\varphi\colon V \to U$ is called a *homomorphism of representations/$\mathfrak{g}$-modules* (or *$\mathfrak{g}$-linear map*) if it is linear and

$$\varphi(x \cdot v) = x \cdot \varphi(v), \quad \forall x \in \mathfrak{g}, \forall v \in V.$$

We denote the set of $\mathfrak{g}$-linear maps from $V$ to $U$ by $\mathrm{Hom}_{\mathfrak{g}\text{-mod}}(V, U)$.

- A $\mathfrak{g}$-linear map is an *isomorphism* if it is a bijection. We say $V$ and $U$ are *isomorphic* if there exists an isomorphism $V \to U$. In this case, we denote this by $V \simeq U$.

- The *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form $K_{\mathfrak{g}}(\cdot,\cdot)\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{k}$ defined by

$$K_{\mathfrak{g}}(x, y) := \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)\,\mathrm{ad}_{\mathfrak{g}}(y)).$$

**Lemma 4.1.8.** *Let $\mathfrak{g}$ be a Lie algebra and $V$ be a finite-dimensional representation of $\mathfrak{g}$. Then there is an isomorphism*

$$F\colon V^* \otimes V \to \mathrm{End}_{\mathbb{k}}(V) \quad \textit{given by} \quad F(f \otimes v)(w) = f(w)v.$$

**Proof.** First, let $x \in \mathfrak{g}$, $f \in V^*$, and $v, w \in V$. Then we have

$$\begin{aligned} F(x \cdot (f \otimes v))(w) &= F((x \cdot f) \otimes v + f \otimes (x \cdot v))(w) = (x \cdot f)(w)v + f(w)(x \cdot v) \\ &= f(w)(x \cdot v) + (x \cdot f)(w)v = x \cdot (f(w)v) - f(x \cdot w)v = x \cdot (F(f \otimes v)(w)) - F(f \otimes v)(x \cdot w) \\ &= (x \cdot F(f \otimes v))(w). \end{aligned}$$

This shows that $F$ is a map of representations. One can verify that that $F$ is also injective. From this, we know $F$ must be an isomorphism since

$$\dim(V^* \otimes V) = \dim(\mathrm{End}_{\mathbb{k}}(V)) = (\dim V)^2.$$

□

We now recall the following facts about the Killing form.

**Lemma 4.1.9.** *Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra over a field $\mathbb{k}$ of characteristic not equal to 2. Then, (i) the Killing form $K_{\mathfrak{g}}(\cdot,\cdot)$ is a $\mathfrak{g}$-invariant bilinear form. That is,*

$$K_{\mathfrak{g}}([x, y], z) = K_{\mathfrak{g}}(x, [y, z]), \quad \forall x, y, z \in \mathfrak{g}.$$

*(ii) Let $V$ be the adjoint representation of $\mathfrak{g}$. If $K_{\mathfrak{g}}$ is non-degenerate, then the map $G\colon V \to V^*$ given by*

$$G(x)(y) = K_{\mathfrak{g}}(x, y), \quad \forall x, y \in \mathfrak{g}$$

*is an isomorphism of representations.*

**Proof.** For part (i), let $x, y, z \in \mathfrak{g}$. Using the fact that $\mathrm{tr}(ST) = \mathrm{tr}(TS)$, we have

$$\begin{aligned} K_{\mathfrak{g}}([x, y], z) &= \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}([x, y])\,\mathrm{ad}_{\mathfrak{g}}(z)) = \mathrm{tr}([\mathrm{ad}_{\mathfrak{g}}(x), \mathrm{ad}_{\mathfrak{g}}(y)]\,\mathrm{ad}_{\mathfrak{g}}(z)) \\ &= \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)\,\mathrm{ad}_{\mathfrak{g}}(y)\,\mathrm{ad}_{\mathfrak{g}}(z)) - \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(y)\,\mathrm{ad}_{\mathfrak{g}}(x)\,\mathrm{ad}_{\mathfrak{g}}(z)) \\ &= \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)\,\mathrm{ad}_{\mathfrak{g}}(y)\,\mathrm{ad}_{\mathfrak{g}}(z)) - \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)\,\mathrm{ad}_{\mathfrak{g}}(z)\,\mathrm{ad}_{\mathfrak{g}}(y)) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(x)[\mathrm{ad}_{\mathfrak{g}}(y), \mathrm{ad}_{\mathfrak{g}}(z)]) \end{aligned}$$

$$= \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(x)\operatorname{ad}_{\mathfrak{g}}([y,z])) = K_{\mathfrak{g}}(x,[y,z]).$$

Next, for part (ii), let $x, y, z \in \mathfrak{g}$. Then we have

$$G(x \cdot y)(z) = K_{\mathfrak{g}}(x \cdot y, z) = K_{\mathfrak{g}}([x,y],z) = -K_{\mathfrak{g}}([y,x],z) = -K_{\mathfrak{g}}(y,[x,z]) = -G(y)(x \cdot z)$$
$$= (x \cdot G(y))(z).$$

Consequently, $G$ is a map of representations. Since $K_{\mathfrak{g}}$ is non-degenerate, one can show that $G$ is in fact a bijection, and thus an isomorphism. □

The following facts are certainly well known, however, it may be difficult to find a source with them presented as we have below. Consequently, we provide our own proof.

**Proposition 4.1.10.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra such that $K_{\mathfrak{g}}$ is non-degenerate, and $\Bbbk$ has characteristic not equal to 2. Let $V$ be the adjoint representation of $\mathfrak{g}$.*

*(a) The map $V \otimes V \to V$ given by*

$$\varphi_1 \colon x \otimes y \mapsto [x,y]$$

*is a map of representations.*

*(b) The map $V \otimes V \to \Bbbk$ given by*

$$\varphi_2 \colon x \otimes y \mapsto K_{\mathfrak{g}}(x,y)$$

*is a map of representations.*

*(c) Let $(b_1, b_2, \cdots, b_r)$ be an ordered basis of $V$ and $(b_1^\vee, b_2^\vee, \cdots, b_r^\vee)$ be the corresponding dual basis with respect to the killing form $K_{\mathfrak{g}}$. That is, we have*

$$K_{\mathfrak{g}}(b_i, b_j^\vee) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

*Then the map $\Bbbk \to V \otimes V$ given by*

$$\varphi_3 \colon c \mapsto c \sum_{i=1}^{r} b_i \otimes b_i^\vee$$

*is a map of representations.*

**Proof.** First, for part (a) let $x, a, b \in \mathfrak{g}$. Then from the (Jacobi identity) and (Anticommutativity) we have

$$[x,[a,b]] + [a,[b,x]] + [b,[x,a]] = 0 \Longrightarrow [x,[a,b]] = [a,[x,b]] + [[x,a],b]$$
$$\Longrightarrow x \cdot \varphi_1(a \otimes b) = \varphi_1(x \cdot a \otimes b).$$

Next, for part (b), we have by Lemma 4.1.9 that

$$\begin{aligned}\varphi_2(x\cdot(a\otimes b)) &= \varphi_2([x,a]\otimes b + a\otimes[x,b]) = K_{\mathfrak{g}}([x,a],b) + K_{\mathfrak{g}}(a,[x,b])\\ &= K_{\mathfrak{g}}([x,a],b) + K_{\mathfrak{g}}([a,x],b) = K_{\mathfrak{g}}([x,a],b) - K_{\mathfrak{g}}([x,a],b) = 0 = x\cdot\varphi_2(a\otimes b).\end{aligned}$$

For the last part, let $\alpha\colon V^*\otimes V\to \mathrm{End}_{\Bbbk}(V)$ be the isomorphism from Lemma 4.1.8 and $\beta\colon V\to V^*$ be the isomorphism from Lemma 4.1.9(ii). Now, define a map $\gamma\colon \Bbbk\to\mathrm{End}_{\Bbbk}(V)$ given by $\gamma(c) = c\,\mathrm{Id}_V$. Then, claim that the map $\varphi_3\colon \Bbbk\to V\otimes V$ is given by

$$\varphi_3 = (\beta^{-1}\otimes \mathrm{Id}_V)\circ\alpha^{-1}\circ\gamma.$$

This will show that $\varphi_3$ is a map of representations. For any $j\in\{1,2,\cdots r\}$ we have

$$\begin{aligned}(\alpha\circ(\beta\otimes\mathrm{Id}_V))\left(\sum_{i=1}^r b_i\otimes b_i^\vee\right)(b_j^\vee) &= \alpha\left(\sum_{i=1}^r \beta(b_i)\otimes b_i^\vee\right)(b_j^\vee) = \sum_{i=1}^r \beta(b_i)(b_j^\vee)b_i^\vee\\ = \sum_{i=1}^r K_{\mathfrak{g}}(b_i,b_j^\vee)b_i^\vee = b_j^\vee &\implies (\alpha\circ(\beta^{-1}\otimes\mathrm{Id}_V))\left(c\sum_{i=1}^r b_i\otimes b_i^\vee\right) = c\,\mathrm{Id}_V = \gamma(c)\\ &\implies \left((\beta^{-1}\otimes\mathrm{Id}_V)\circ\alpha^{-1}\circ\gamma\right)(c) = c\sum_{i=1}^r b_i\otimes b_i^\vee. \qquad\square\end{aligned}$$

**Example 4.1.11.** Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. This Lie algebra has a basis given by the three matrices

$$e = \begin{pmatrix}0&1\\0&0\end{pmatrix},\ f = \begin{pmatrix}0&0\\1&0\end{pmatrix},\ \text{and } h = \begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

One can verify that we have $[e,f]=h$, $[h,e]=2e$ and $[h,f]=-2f$. Consequently, on the basis $\{e,h,f\}$, the linear transformations $\mathrm{ad}_{\mathfrak{g}}(e), \mathrm{ad}_{\mathfrak{g}}(h)$, and $\mathrm{ad}_{\mathfrak{g}}(f)$ act on $\mathfrak{g}$ by

$$\begin{pmatrix}0&-2&0\\0&0&1\\0&0&0\end{pmatrix},\ \begin{pmatrix}2&0&0\\0&0&0\\0&0&-2\end{pmatrix},\ \text{and}\ \begin{pmatrix}0&0&0\\-1&0&0\\0&2&0\end{pmatrix},$$

respectively. From this we can compute the Killing form on $\mathfrak{sl}_2(\mathbb{C})$:

$$\begin{aligned}K_{\mathfrak{g}}(e,f) &= \mathrm{tr}\left(\begin{pmatrix}0&-2&0\\0&0&1\\0&0&0\end{pmatrix}\cdot\begin{pmatrix}0&0&0\\-1&0&0\\0&2&0\end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix}2&0&0\\0&2&0\\0&0&0\end{pmatrix} = 4,\\ K_{\mathfrak{g}}(h,h) &= \mathrm{tr}\left(\begin{pmatrix}2&0&0\\0&0&0\\0&0&-2\end{pmatrix}\cdot\begin{pmatrix}2&0&0\\0&0&0\\0&0&-2\end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix}4&0&0\\0&0&0\\0&0&4\end{pmatrix} = 8.\end{aligned}$$

Since $K_{\mathfrak{g}}$ is symmetric, we also have $K_{\mathfrak{g}}(f,e) = K_{\mathfrak{g}}(e,f) = 4$. Lastly, since the matrices associated to $\mathrm{ad}_{\mathfrak{g}}(e)$, $\mathrm{ad}_{\mathfrak{g}}(h)$, $\mathrm{ad}_{\mathfrak{g}}(f)$ are upper-triangular, diagonal, and lower-triangular respectively, it is clear that $K_{\mathfrak{g}}(x,y)=0$ for all other choices of $x,y\in\{e,h,f\}$.

Lastly, we recall the following well-known result about irreducible representations.

**Proposition 4.1.12** (Schur's lemma)**.** *Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$ and $V$ and $U$ be irreducible representations of $\mathfrak{g}$.*

*(a) If $f \in \operatorname{Hom}_{\mathfrak{g}\text{-}mod}(V, U)$, then $f$ is either an isomorphism, or the zero map.*

*(b) If $\Bbbk$ is algebraically closed then $\operatorname{Hom}_{\mathfrak{g}\text{-}mod}(V, V) = \{\lambda \operatorname{Id}_V \mid \lambda \in \Bbbk\}$ and*

$$\dim \operatorname{Hom}_{\mathfrak{g}\text{-}mod}(V, U) = \begin{cases} 1 & \text{if } V \simeq U, \\ 0 & \text{if } V \not\simeq U. \end{cases}$$

## 4.2 $\mathfrak{sl}_2$ representation theory

In this section, we provide some background on the representation theory of the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$. We will mainly work over the field $\mathbb{C}$ from here on. Recall from Example 4.1.2 that $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of traceless $2 \times 2$ matrices over $\mathbb{C}$, that is

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{C}, a + d = 0 \right\}.$$

Recall that this Lie algebra has a basis given by the three matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Although we will mainly be interested in the adjoint representation of $\mathfrak{sl}_2$ and its tensor powers later, it will be useful to know how these tensor powers decompose into irreducible representations. The classification of the finite-dimensional irreducible representations of $\mathfrak{sl}_2$ is well known. We summarize some of the basic facts about these irreducible representations.

**Definition 4.2.1.** Let $V$ be a representation of $\mathfrak{sl}_2$ and $v \in V$. A vector $v$ such that $h \cdot v = \lambda v$ for some $\lambda \in \mathbb{C}$ is called a *weight vector* of the representation with *weight* $\lambda$. In other words, $v$ is a weight vector with weight $\lambda$ if it is a $\lambda$-eigenvector for the action of $h$ on $V$. If $v$ is a weight vector such that $e \cdot v = 0$, then we call $v$ a *highest weight vector* of the representation.

**Proposition 4.2.2** ([Kas95, Prop V.4.4.])**.** *For each $n \geq 0$, there is a unique (up to isomorphism) $(n+1)$-dimensional irreducible representation $V_n$ of $\mathfrak{sl}_2$ with basis $v_0, v_1, \cdots, v_n$ such that*

$$\begin{aligned} &e \cdot v_0 = 0, e \cdot v_k = (n - (k-1))v_{k-1}, 0 < k \leq n, \\ &f \cdot v_n = 0, f \cdot v_k = (k+1)v_{k+1}, 0 \leq k < n, \\ &h \cdot v_k = (n - 2k)v_k, 0 \leq k \leq n. \end{aligned}$$

*Furthermore, we have $v_k = \frac{f^k}{k!} v_0$.*

**Remark 4.2.3.** One can determine the irreducible summands of any finite dimension representation of $\mathfrak{sl}_2$ if we have the list of weights of any basis of the representation consisting of weight vectors. For example, if a representation has exactly $k$ basis vectors of maximum weight $n$, then the representation must have exactly $k$ copies of the irreducible representation $V_n$ as irreducible summands. If one removes $k$ copies of the weights $n, n-2, \cdots, -(n-2), -n$ from the list of weights, one can repeat this process to determine the representation's isomorphism class.

Next, the following result tells us explicitly how tensor powers of the irreducible representations in Proposition 4.2.2 decompose into a direct sum of irreducibles.

**Proposition 4.2.4** ([Kas95, Prop V.5.1.])**.** *Let $n \geq m$ be non-negative integers. Then we have the following*

$$V_n \otimes V_m \simeq V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m}.$$

**Theorem 4.2.5.** *Let $V = V_2$ be the adjoint representation of $\mathfrak{sl}_2$ and let $n$ be a non-negative integer. Then we have*

$$\dim \operatorname{Hom}_{\mathfrak{sl}_2}(V^{\otimes n}, \mathbb{C}) = R_n.$$

*That is, the number of copies of the trivial representation $V_0$ in the $n^{th}$ tensor power of $V = V_2$ is the $n^{th}$ Riordan number $R_n$.*

**Proof.** Let $W = V^{\otimes n}$ and let $d_\lambda$ be the dimension of the $\lambda$-weight space in $W$ (i.e. the dimension of the $\lambda$-eigenspace of the action of $h$ on $W$). By Proposition 4.1.12, we know that $\dim \operatorname{Hom}_{\mathfrak{sl}_2}(W, \mathbb{C})$ is the number of copies of the trivial representation in the decomposition of $W$. Recall that $V$ has basis $\{e, h, f\}$ which have weights $2, 0, -2$ respectively. Thus, $W$ has $\{x_1 \otimes x_2 \otimes \cdots \otimes x_n \mid x_i \in \{e, h, f\}\}$ as a basis. If $x_i \in \{e, h, f\}$ for each $i \in \{1, 2, \cdots, n\}$, then one can verify that $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ has weight equal to the sum of the weights of the $x_i$'s. Therefore, the number of such $\lambda$-weight vectors$\{x_1 \otimes x_2 \otimes \cdots \otimes x_n \mid x_i \in \{e, h, f\}\}$ is equal to $d_\lambda = [z^\lambda](z^2 + 1 + z^{-2})^n$. To obtain the number of trivial representations contained in $V^{\otimes n}$ from this, we can note that the dimension of the 0-weight space in $V_n$ is one if $n$ is even, zero otherwise. Thus, all 0-weight vectors in $V^{\otimes n}$ come from a copy of $V_0$ or $V_{2k}$ with $k \geq 1$. For each $k \geq 1$, $V_{2k}$ has $-2$-weight space of dimension one. Consequently, the number of copies of the trivial representation in $W$ will be $d_0 - d_{-2}$. It follows from *Proposition* 3.2.4 that

$$\begin{aligned}\dim \operatorname{Hom}_{\mathfrak{sl}_2}(V^{\otimes n}, \mathbb{C}) = d_0 - d_{-2} &= [z^0](z^2 + 1 + z^{-2})^n - [z^{-2}](z^2 + 1 + z^{-2})^n \\ &= [z^0](z + 1 + z^{-1})^n - [z^{-1}](z + 1 + z^{-1})^n = R_n. \quad \square\end{aligned}$$

## 4.3 $U_q(\mathfrak{sl}_2)$ representation theory

In this section we recall the definition of $U_q(\mathfrak{sl}_2)$ and some results about its representation theory. One can find more details about this algebra and its representations in chapters VI and VII of [Kas95]. In this section suppose $q \in \mathbb{C}$ is non-zero and that $q$ is not a root of unity.

**Definition 4.3.1** (Quantum Integers)**.** For any integer $n \in \mathbb{Z}$, define the *quantum integer* or *quantum analogue* of $n$ to be:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-(n-3)} + q^{-(n-1)}.$$

If $n \geq 0$, we recursively define the *quantum factorial* of $n$ to be:

$$[n]! = \begin{cases} 1 & \text{if } n = 0, \\ [n] \cdot [n-1]! & \text{if } n > 0. \end{cases}$$

**Definition 4.3.2** (Quantum $\mathfrak{sl}_2$)**.** The *quantized enveloping algebra* of $\mathfrak{sl}_2$ (or just *quantum* $\mathfrak{sl}_2$) is the $\mathbb{C}$-algebra $U_q(\mathfrak{sl}_2)$ generated by $E, F, K, K^{-1}$, subject to the relations:

$$KK^{-1} = 1 = K^{-1}K,$$
$$KEK^{-1} = q^2 E,\ KFK^{-1} = q^{-2} F,$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

**Proposition 4.3.3** ([Kas95, Thm VI.3.5])**.** *For each $n \geq 0$, there is an $(n+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module $W_n$ with basis $w_0, w_1, w_2, \cdots, w_n$ such that:*

$$E \cdot w_0 = 0, E \cdot w_k = [n - (k-1)] w_{k-1}, 0 < k \leq n,$$
$$F \cdot w_n = 0, F \cdot w_k = [k+1] w_{k+1}, 0 \leq k < n,$$
$$K \cdot w_k = q^{n-2k} w_k, 0 \leq k \leq n.$$

*Further, one has $w_k = \frac{F^k}{[k]!} \cdot w_0$.*

**Remark 4.3.4.** In particular, let $W = W_2$ and set $v_0 := w_0, v_1 := -w_1, v_2 := -w_2$. Then $E$, $F$, and $K$ act on $W$ with respect to the ordered basis $(v_0, v_1, v_2)$ by

$$\begin{pmatrix} 0 & -[2] & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & [2] & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{pmatrix},$$

respectively. This representation $W$ can be thought of as the quantum analogue of the adjoint representation of $\mathfrak{sl}_2$.

**Proposition 4.3.5** ([Kas95, Proposition VII.1.1])**.** *$U_q(\mathfrak{sl}_2)$ can be equipped with the structure of a Hopf algebra with the co-multiplication and co-unit defined as follows:*

$$\Delta(E) = 1 \otimes E + E \otimes K, \epsilon(E) = 0,$$
$$\Delta(F) = K^{-1} \otimes F + F \otimes 1, \epsilon(F) = 0,$$
$$\Delta(K) = K \otimes K, \epsilon(K) = 1,$$
$$\Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \epsilon(K^{-1}) = 1.$$

**Remark 4.3.6.** We can define a generalized comultiplication recursively as follows:

$$\Delta_1 = \Delta,\ \Delta_{n+1} = (\mathrm{Id} \otimes \Delta) \circ \Delta_n.$$

**Definition 4.3.7.** Given $U_q(\mathfrak{sl}_2)$-modules $A_1, A_2, \cdots, A_n$ where $n \geq 1$, we define the $U_q(\mathfrak{sl}_2)$-module structure on $A_1 \otimes A_2 \otimes \cdots A_n$ by:

$$u \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_n) := \Delta_n(u) \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_n, \forall u \in U_q(\mathfrak{sl}_2), \forall a_i \in A_i.$$

**Theorem 4.3.8** ([Kas95, Proposition VII.7.1])**.** *Let $n \geq m$ be integers. Then we have an isomorphism of $U_q(\mathfrak{sl}_2)$-modules:*

$$W_n \otimes W_m \simeq W_{n+m} \oplus W_{n+m-2} \oplus \cdots \oplus W_{n-m}.$$

**Corollary 4.3.9.** *For any integer $n \geq 0$, if $W_2^{\otimes n}$ has $W_k$ as a direct summand, then $k$ is even.*

**Theorem 4.3.10.** *Let $W = W_2$ and $n$ be a non-negative integer. Then we have*

$$\dim \mathrm{Hom}_{U_q(\mathfrak{sl}_2)\text{-mod}}(W^{\otimes n}, \mathbb{C}) = R_n.$$

*That is, the number of copies of $W_0$ in the $n^{th}$ tensor power of $W = W_2$ is $R_n$.*

**Proof.** The proof of this fact is completely analogous to the proof of Theorem 4.2.5 with the weights $2, 0, -2$ replaced by $q^2, q^0, q^{-2}$. □

# Chapter 5

# Category theory and string diagrams

In this chapter, we recall some necessary background on monoidal categories and the language of string diagrams. We begin this discussion by recalling some basic definitions related to abstract categories.

## 5.1 Categories

In this section, we recall the notion of a category and some related definitions that will be necessary for this work. One can find a more detailed introduction to these topics in [Mac98].

**Definition 5.1.1.** A category $\mathcal{C}$ consists of the following data:

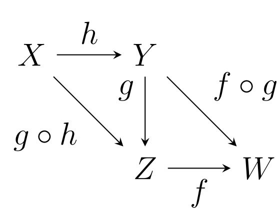
- A class of *objects*,
- a *set of morphisms* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any two objects $X$ and $Y$, and
- a *composition* map

  $$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathbb{C}}(X, Z), \quad (f, g) \mapsto f \circ g,$$

  for all objects $X, Y$, and $Z$ of $\mathcal{C}$. When $f \in \mathrm{Hom}_{\mathbb{C}}(X, Y)$ for some objects $X$ and $Y$, we often will denote this by $f\colon X \to Y$.

These data are subject to the following conditions.

- The composition map must be *associative*, that is: $(f \circ g) \circ h = f(g \circ h)$ for morphisms $f, g, h$ such that the compositions are defined. Equivalently, the following diagram commutes.



- For each object $X$ of $\mathcal{C}$, there is a morphism $\mathrm{Id}_X \colon X \to X$ such that $f \circ \mathrm{Id}_X = f$ and $\mathrm{Id}_X \circ g = g$ for morphisms $f \colon X \to Y$ and $g \colon Y \to X$.

We will sometimes just write $fg$ instead of $f \circ g$ for the composition of morphisms in a category.

**Remark 5.1.2.** The definition above is actually the definition of a *small category*—a category where $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set for all objects $X$ and $Y$. In general, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ need not be a set. However, most of the categories we will discuss will be small.

**Example 5.1.3.** Categories appear in many different areas of mathematics. The following are examples of categories:

| **Category** | **Objects** | **Morphisms** |
|---|---|---|
| **Set** | Sets | Functions |
| $\mathbf{Vect}_{\Bbbk}$ | $\Bbbk$-vector spaces | $\Bbbk$-linear transformations |
| $\mathfrak{g}$-**mod** | Representations of $\mathfrak{g}$ | $\mathfrak{g}$-linear maps |
| **Grp** | Groups | Group homomorphisms |
| **Top** | Topological spaces | Continuous functions |
| **SGraph** | Simple graphs | Graph homomorphisms |

A *graph homomorphism* from $G$ to $H$ is an adjacency preserving function $V(G) \to V(H)$.

**Definition 5.1.4.** Let $\mathcal{C}$ be a category, and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ where $X$ and $Y$ are objects of $\mathcal{C}$. The morphism $f$ is an *isomorphism* if there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $fg = \mathrm{Id}_Y$ and $gf = \mathrm{Id}_X$. In this case, we say $X$ and $Y$ are *isomorphic* and denote this by $X \simeq Y$.

**Definition 5.1.5.** Let $\mathcal{C}$ and $\mathcal{D}$ be a categories.

- The *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ and for every morphism $f \colon X \to Y$ of $\mathcal{C}$, there is a morphism $f^{\mathrm{op}} \colon Y \to X$ of $\mathcal{C}^{\mathrm{op}}$. Composition is defined by:

$$f^{\mathrm{op}} \circ^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}.$$

- The *product category* $\mathcal{C} \times \mathcal{D}$ has objects $X \times Y$ for all objects $X$ in $\mathcal{C}$ and $Y$ in $\mathcal{D}$. For objects $X_1, X_2$ in $\mathcal{C}$ and $Y_1, Y_2$ in $\mathcal{D}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2)$ we have a morphism $f \times g \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}(X_1 \times Y_1, X_2 \times Y_2)$. Composition in $\mathcal{C} \times \mathcal{D}$ is defined by:

$$(f_1 \times g_1) \circ (f_2 \times g_2) = (f_1 f_2) \times (g_1 g_2)$$

  for composable morphisms.

**Definition 5.1.6.** A *functor* $\mathbf{F}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of the following data:

- An object $\mathbf{F}X$ in the category $\mathcal{D}$ for each object $X$ of $\mathcal{C}$ and
- a morphism $\mathbf{F}f\colon \mathbf{F}X \to \mathbf{F}Y$ in the category $\mathcal{D}$ for each morphism $f\colon X \to Y$ in $\mathcal{C}$.

These data must satisfy the following properties:

- The functor must respect the composition of morphisms, that is, $(\mathbf{F}g)(\mathbf{F}f) = \mathbf{F}(gf)$ for all morphisms $f, g$ such that the composition is defined, and
- the functor must send identity morphisms in $\mathcal{C}$ to identity morphisms in $\mathcal{D}$, that is, $\mathbf{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathbf{F}X}$ for each object $X$ of $\mathcal{C}$.

If $\mathbf{F}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$, then we denote this by $\mathbf{F}\colon \mathcal{C} \to \mathcal{D}$.

**Example 5.1.7.** The following are some examples of functors:

- For any category $\mathcal{C}$, the identity endofunctor $\mathbf{Id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is defined by $\mathbf{Id}_{\mathcal{C}}X = X$ and $\mathbf{Id}_{\mathcal{C}}f = f$ for all objects $X$ and morphisms $f$ of the category $\mathcal{C}$.
- We can define a functor $\mathbf{P}\colon \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ where, for a set $X$, we define $\mathbf{P}X$ to be the power set $2^X$ of $X$. For a function $f\colon X \to Y$, we define $\mathbf{P}f^{\mathrm{op}}\colon 2^Y \to 2^X$ to be the function
$$\mathbf{P}f^{\mathrm{op}}A = f^{-1}A, \quad \forall A \in 2^Y.$$

**Definition 5.1.8.** Let $\mathbf{F}$ be a functor from $\mathcal{C}$ to $\mathcal{D}$, then for objects $X$ and $Y$ define

$$\mathbf{F}_{X,Y}\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{F}X, \mathbf{F}Y), \quad \mathbf{F}_{X,Y}(f) = \mathbf{F}f.$$

- The functor $\mathbf{F}$ is *full* if for all objects $X$ and $Y$ of $\mathcal{C}$ the map $\mathbf{F}_{X,Y}$ is surjective.
- The functor $\mathbf{F}$ is *faithful* if for all objects $X$ and $Y$ of $\mathcal{C}$ the map $\mathbf{F}_{X,Y}$ is injective.
- The functor $\mathbf{F}$ is an *isomorphism* if there exists a functor $\mathbf{G}\colon \mathcal{D} \to \mathcal{C}$ such that $\mathbf{FG} = \mathbf{Id}_{\mathcal{D}}$ and $\mathbf{GF} = \mathbf{Id}_{\mathcal{C}}$.
- The functor $\mathbf{F}$ is an *equivalence* of the categories $\mathcal{C}$ and $\mathcal{D}$ if it is full, faithful, and for each object $Y$ of $\mathcal{D}$ there exists an object $X$ of $\mathcal{C}$ such that $\mathbf{F}X \simeq Y$.

**Remark 5.1.9.** The definition of an equivalence of categories above is not the usual one, but it is an equivalent condition for a functor to be an equivalence of categories by [Mac98, Section IV.4., Thm 1].

**Definition 5.1.10.** Let $\mathcal{C}$ be a category. The idempotent completion (also called the *Karoubi envelope*) of the category $\mathcal{C}$ is a category $\mathrm{Kar}(\mathcal{C})$ whose objects are pairs $(A, e)$ where $A$ is an object of $\mathcal{C}$ and $e\colon A \to A$ is an idempotent. The morphisms in $\mathrm{Kar}(\mathcal{C})$ are triples $(e, f, e')\colon (A, e) \to (A', e')$, where $f\colon A \to A'$ and $f = e' \circ f \circ e$.

Given a category $\mathcal{C}$, the category $\mathrm{Kar}(\mathcal{C})$ should be thought of as a category in which we have added in all “images” of the idempotents of $\mathcal{C}$.

## 5.2 Diagrammatic monoidal categories

In this section, we recall the notion of a monoidal category and the string diagram calculus for monoidal categories. One can read [Sav21] for a more detailed introduction of the contents of this section and the following section.

**Definition 5.2.1.** A (strict) *monoidal category* is a category $\mathcal{C}$ equipped with

- a bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product*, and
- a *unit object* $\mathbb{1}$.

For all objects $X, Y, X$ in $\mathcal{C}$, we must have

- $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$,
- $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$, and

for all morphisms $f, g, h$ in $\mathcal{C}$

- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
- $\mathrm{Id}_{\mathbb{1}} \otimes f = f = f \otimes \mathrm{Id}_{\mathbb{1}}$.

**Remark 5.2.2.** For a non-strict monoidal category, the equalities in the definition above must be replaced by isomorphisms and additional coherence conditions must be assumed (see [Mac98, Chapter VII, Section 1] for more details). However, most of the monoidal categories throughout this thesis will be strict by definition, so we do not say much more about this distinction.

**Definition 5.2.3.** Let $\Bbbk$ be a field. A $\Bbbk$-*linear* monoidal category is a monoidal category $\mathcal{C}$ such that

- $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\Bbbk$-vector space for all objects and $X$ and $Y$ in $\mathcal{C}$, and
- The tensor product and composition are bilinear maps on the morphisms of the category.

**Definition 5.2.4.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories and $\mathbf{F} \colon \mathcal{C} \to \mathcal{D}$. Then, $\mathbf{F}$ is a *monoidal functor* if

- it respects the tensor product of morphisms, that is:

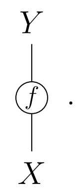
  $$\mathbf{F}(f \otimes g) = \mathbf{F}(f) \otimes \mathbf{F}(g),$$

  and
- $\mathbf{F}(\mathbb{1}_{\mathcal{C}}) = \mathbb{1}_{\mathcal{D}}$.

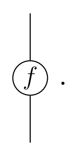
**Example 5.2.5.** The following are examples of (non-strict) $\Bbbk$-linear monoidal categories:

- The category $\mathbf{Vect}_\Bbbk$ with the usual tensor product of vector spaces as the bifunctor.
- The category of representations of $\mathfrak{g}$, $\mathfrak{g}$**-mod**, for a given Lie-algebra $\mathfrak{g}$ over $\Bbbk$.

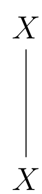
One main advantage of monoidal categories is that they have two notions of multiplication—one given by the tensor product, and another from composition. This allows for one to do two-dimensional algebra on the morphisms which can often yield efficient presentations of certain categories. Strict monoidal categories in particular are well-suited to be described in the language of *string diagrams*. We call such a category a *diagrammatic monoidal category*. String diagrams (sometimes called Penrose diagrams) have their origins from physics in the work of Roger Penrose [Pen71]. In a diagrammatic monoidal category, one can use the calculus of string diagrams to make arguments about the category which are often much more transparent and easy to verify than those written in the conventional one-line algebraic notation. We represent the morphism $f\colon X \to Y$ by the diagram with a coupon as follows:



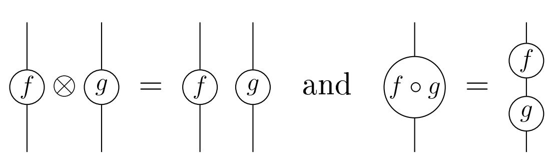

In our diagrams, the domain is at the bottom of the diagram and the codomain is at the top of the diagram. Note that other authors may use different conventions for this. We will often omit the object labels on the top and bottom of the diagram, so that the diagram instead looks like




A strand without a coupon will represent the identity morphism on the given object. That is, for all objects $X$ we have

$\mathrm{Id}_X =$  .


However, in a monoidal category, we will always represent the identity morphism for the unit object $\mathbb{1}$ by the empty diagram. The tensor product and composition of morphisms in the language of string diagrams is given by horizontal and vertical stacking of diagrams respectively. That is, we have

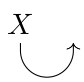


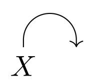
**Remark 5.2.6.** We will often define strict monoidal categories in terms of a set of generating objects and morphisms, and a set of relations on the morphisms. In this setting, the morphisms of the category are actually equivalence classes of diagrams up to the relations imposed on the diagrams, but it will still be useful to discuss properties related to the diagrams themselves. We will sometimes use the terminology "The diagrams in the category satisfying property **P**". By this we will mean all equivalence classes containing a diagram satisfying property **P**.

## 5.3 Pivotal categories

In this section, we recall the notion of a *pivotal category*. We will see that when a diagrammatic monoidal category is pivotal, we will be permitted to apply topological arguments to the morphisms of our category (in fact, this is true for a more general class of categories that we will discuss in this section).

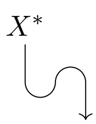
**Definition 5.3.1.** Let $\mathcal{C}$ be a strict monoidal category and let $X$ and $X^*$ be objects of $\mathcal{C}$. We say $X^*$ is *right dual* to $X$ (and $X$ is *left dual* to $X^*$) if we have morphisms



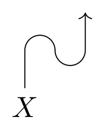
$: \mathbb{1} \to X^* \otimes X$ and 

$: X \otimes X^* \to \mathbb{1}$

such that



$=$ 

$= \mathrm{Id}_{X^*}$ and 

$=$ 

$= \mathrm{Id}_X$ .

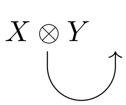
A monoidal category in which all objects have both a left and right dual is called *autonomous* (also called *rigid*).

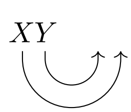
**Theorem 5.3.2** ([Sel11, Thm 4.5])**.** *A well-formed equation between morphisms in the language of autonomous categories follows from the axioms of autonomous categories if and only if it holds in the graphical language up to planar isotopy.*

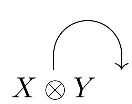
**Definition 5.3.3.** A strict monoidal category $\mathcal{C}$ is a *strict pivotal category* if every object $X$ has a right dual $X^*$ with fixed morphisms $X \otimes X^* \to \mathbb{1}$ and $\mathbb{1} \to X^* \otimes X$ as in Definition 5.3.1 and the following conditions are satisfied:

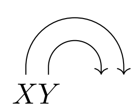
(a) For all objects $X$ and $Y$ in $\mathcal{C}$, $(X^*)^* = X$, $(X \otimes Y)^* = Y^* \otimes X^*$, and $\mathbb{1}^* = \mathbb{1}$.

(b) For all objects $X$ and $Y$ in $\mathcal{C}$, we have



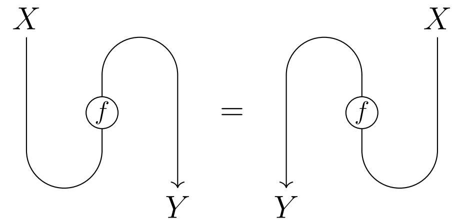
$=$ 

and 

$=$ 

.

(c) For all morphisms $f\colon X \to Y$ we have




**Lemma 5.3.4.** *Let $\mathcal{C}$ be a strict monoidal category. Suppose that every object $X$ of $\mathcal{C}$ has a right dual $X^*$, and for all objects $X$, we have $(X^*)^* = X$. Then, $\mathcal{C}$ is autonomous.*

**Proof.** Since every object $X$ has a right dual, it suffices to show that every object $X$ has a left dual. For every object $X$, we have $(X^*)^* = X$, which means the right dual to $X^*$ is $X$. This is a equivalent $X^*$ being the left dual to $X$. Consequently, $\mathcal{C}$ is autonomous as required. □

**Corollary 5.3.5.** *Every pivotal category is autonomous.*

**Proof.** A pivotal category satisfies the conditions of the lemma, and thus is autonomous. □

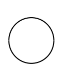
**Remark 5.3.6.** Since every pivotal category is autonomous, by utilizing Theorem 5.3.2, one is also permitted to employ topological arguments on the morphisms of the category. That is, two diagrams in the string diagram language that are the same up to planar isotopy are in fact equal in the category.

## 5.4 The Temperley–Lieb category

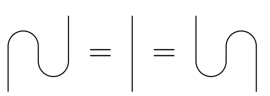
**Definition 5.4.1.** Define the *Temperley–Lieb category* $\mathcal{TL}(\delta)$ to be the strict $\Bbbk$-linear monoidal category generated by one object $X$ and the two generating morphisms

 $: \mathbb{1} \to X \otimes X$ and  $: X \otimes X \to \mathbb{1}$

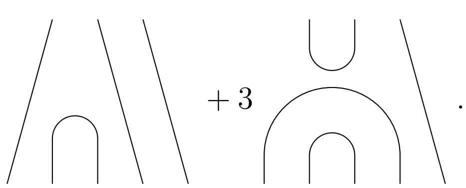
These morphisms are subject to the following relations:

 $= \delta,$

and



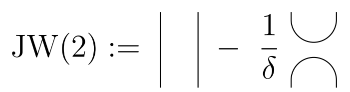
**Example 5.4.2.** The following is an example of a morphism $X^{\otimes 5} \to X^{\otimes 3}$ in $\mathcal{TL}(\delta)$:



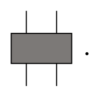
**Remark 5.4.3.** The Temperley–Lieb category is well-studied in the literature. We recall a few important facts about it below.

- $\mathcal{TL}(\delta)$ is a pivotal category [Abr08, Section 3.1].
- Let $n$ and $m$ be non-negative integers and let $D_{n,m}$ be the set of diagrams in $\mathrm{Hom}_{\mathcal{TL}(\delta)}(X^{\otimes n}, X^{\otimes m})$ that contain no loops (see Remark 5.2.6). Then, $D_{n,m}$ is a basis for $\mathrm{Hom}_{\mathcal{TL}(\delta)}(X^{\otimes n}, X^{\otimes m})$ and one can verify that $|D_{n,m}| = C_{n+m}$, the $(n+m)^{\text{th}}$ Catalan number [Che14, Section 2.1].

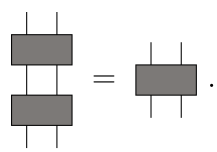
**Definition 5.4.4.** Define the $2^{\text{nd}}$ *Jones–Wenzl idempotent* $\mathrm{JW}(2) \in \mathrm{End}_{\mathcal{TL}(\delta)}(X \otimes X)$ to be




In our diagrams, we will often denote JW(2) by



Note that JW(2) satisfies the following relation (i.e. it is an idempotent):



(Idem)

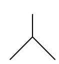
# Chapter 6

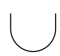
# The chromatic category

## 6.1 The chromatic category $\mathcal{C}(\delta)$

In this section, we define the chromatic category $\mathcal{C}(\delta)$, prove several properties about it, and show that it is a pivotal monoidal category. Our definition differs from those given in the literature [FK10; FK09; AK19] in which morphisms are defined up to planar isotopy. We will see later that the definition below yields the same category up to isomorphism. We define $\mathcal{C}(\delta)$ over an arbitrary field $\Bbbk$ here but later we will assume $\Bbbk = \mathbb{C}$.

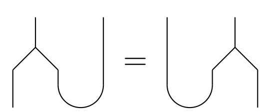
**Definition 6.1.1** (Chromatic Category)**.** Let $\Bbbk$ be a field and let $\delta \in \Bbbk$. Define the *chromatic category* to be the strict $\Bbbk$-linear monoidal category $\mathcal{C}(\delta)$ with one generating object $X$ and generating morphisms

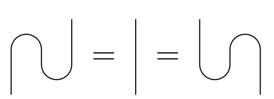
 $: X \otimes X \to X,$ (Merge)

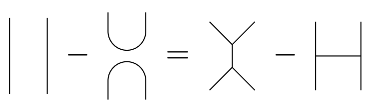
 $: \mathbb{1} \to X \otimes X,$ (Cup)

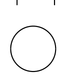
 $: X \otimes X \to \mathbb{1},$ (Cap)

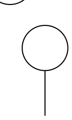
subject to the following relations

 (Cup-Merge)

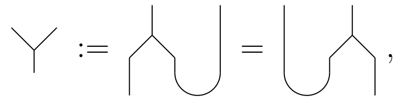
 (ZigZag)

 (Chromatic)

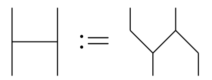
 $= \delta - 1,$ (Bubble)

 $= 0,$ (Lollipop)

where we define the two additional morphisms as follows:

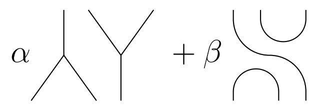


(Split)



(H)

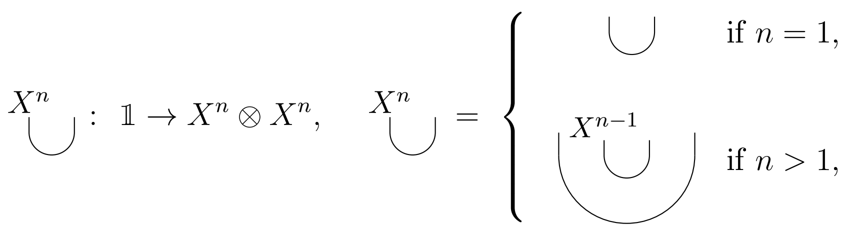
**Example 6.1.2.** The morphisms of $\mathcal{C}(\delta)$ are obtained first by taking all possible compositions and tensor products of the generating morphisms (Merge), (Cup), and (Cap) (we call these morphisms *diagrams*). Then, one can take all $\Bbbk$-linear combinations of the diagrams to obtain all morphisms in the category. For example, the following would be an example of a morphism in $\mathrm{Hom}_{\mathcal{C}(\delta)}(X^{\otimes 3}, X^{\otimes 3})$:



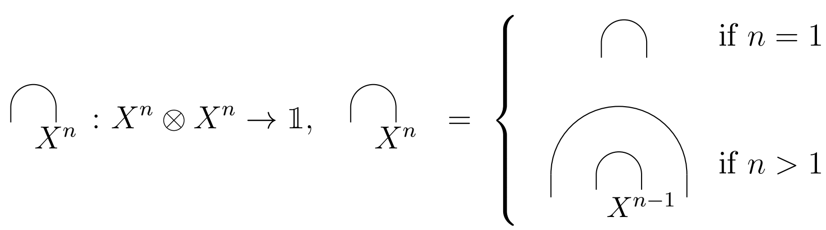

for all $\alpha, \beta \in \Bbbk$.

We will begin our study of this category by showing that $\mathcal{C}(\delta)$ is pivotal. Note that we will sometimes denote the $n^{\text{th}}$ tensor product of $X$ with itself as $X^n := X^{\otimes n}$.

**Definition 6.1.3.** Let $n$ be a positive integer. We define the following morphisms recursively:

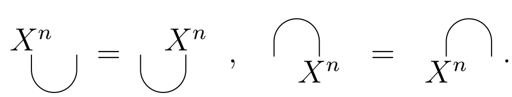


($n$-cup)

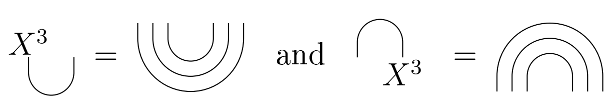


($n$-cap)

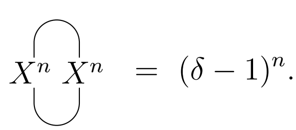
Note that we will use the following notations interchangeably:



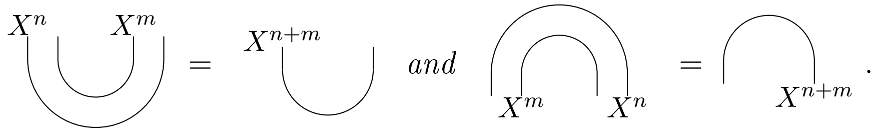

**Example 6.1.4.** When $n = 3$ we have the morphisms



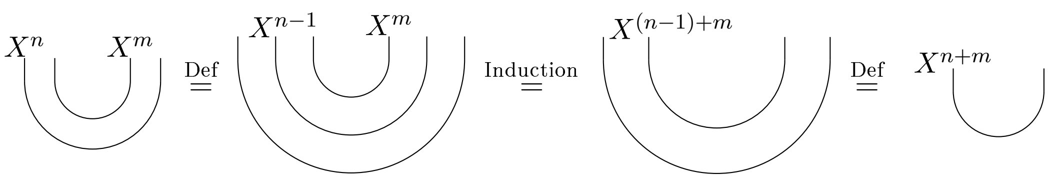

One may also verify that we have the following extended bubble identity



**Proposition 6.1.5.** *Let $n$ and $m$ be positive integers, then*

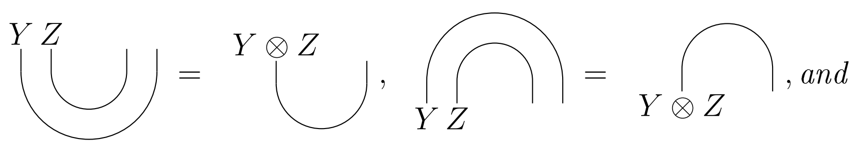



**Proof.** We prove the result by induction on $n$. If $n = 1$, the result follows by the definition of the $(m+1)$-cup and $(m+1)$-cap. Now if $n > 1$ and the result is true for positive integers pairs $(n', m')$ with $n' < n$, then have the following:

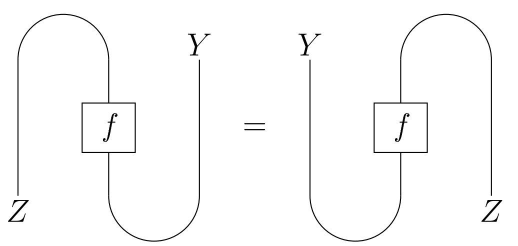



Thus the result follows by induction. The corresponding result for caps follows from an analogous argument. □

**Theorem 6.1.6.** *The chromatic category $\mathcal{C}(\delta)$ is a pivotal category where every object is self-dual. That is, for all objects $Y$, $Z$ in $\mathcal{C}(\delta)$, the following two conditions hold:*

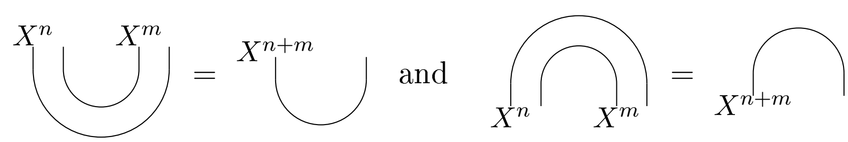


(1)

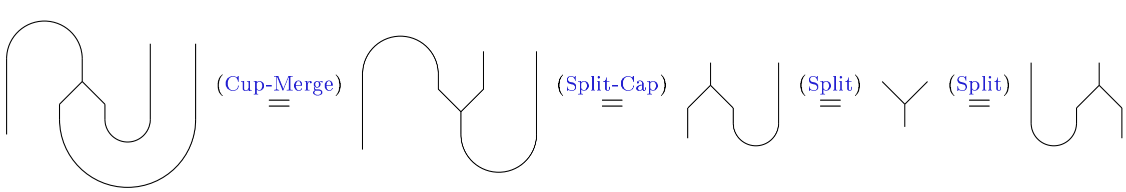


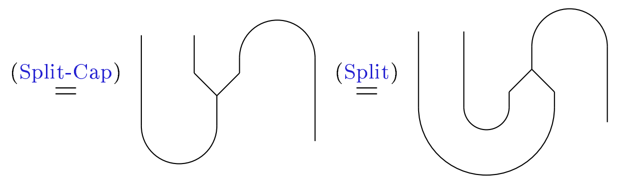

*for all morphisms $f \in \mathcal{C}(\delta)(Y, Z)$.* (2)

**Proof.** It is clear that (1) is true since all objects in $\mathcal{C}(\delta)$ are tensor powers of $X$ and by Proposition 6.1.5 we have

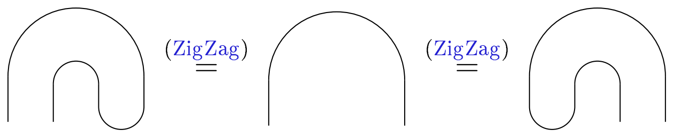



for all non-negative integers $n, m$. It remains to show that (2) is true. To do this, it suffices to show that property (2) is true for the generating morphisms of $\mathcal{C}(\delta)$, that is, the (Merge), (Cup), and (Cap) morphisms. First, we verify property (2) for the (Merge) morphism:

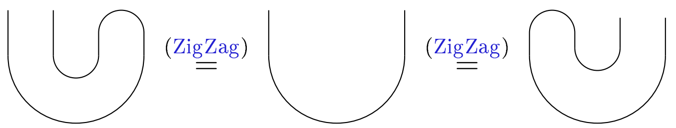





Next, for the (Cup) morphism we have the following computation:

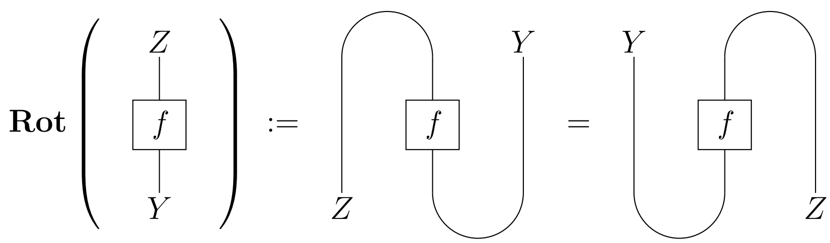



Lastly, by an analogous computation we see property (2) is also satisfied by the (Cap) morphism:




□

**Corollary 6.1.7.** *The map* **Rot**: $\mathcal{C}(\delta) \to \mathcal{C}(\delta)^{\mathrm{op}}$ *that sends every object of* $\mathcal{C}(\delta)$ *to itself and such that*

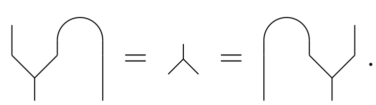



*is a monoidal functor from* $\mathcal{C}(\delta)$ *to* $\mathcal{C}(\delta)^{\mathrm{op}}$.

**Remark 6.1.8.** The result of applying the functor **Rot** to a diagram is a 180 degree rotation of the entire diagram.

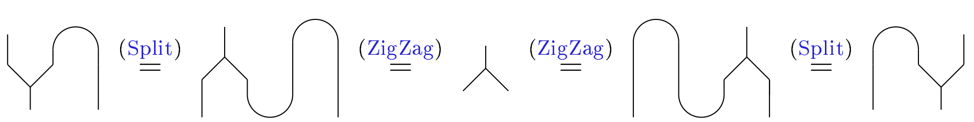
We now derive some additional relations on the morphisms in $\mathcal{C}(\delta)$ which are implied by the defining relations of $\mathcal{C}(\delta)$.

**Lemma 6.1.9.** *The following relations holds in* $\mathcal{C}(\delta)$*:*



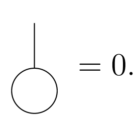

(Split-Cap)

**Proof.** We have

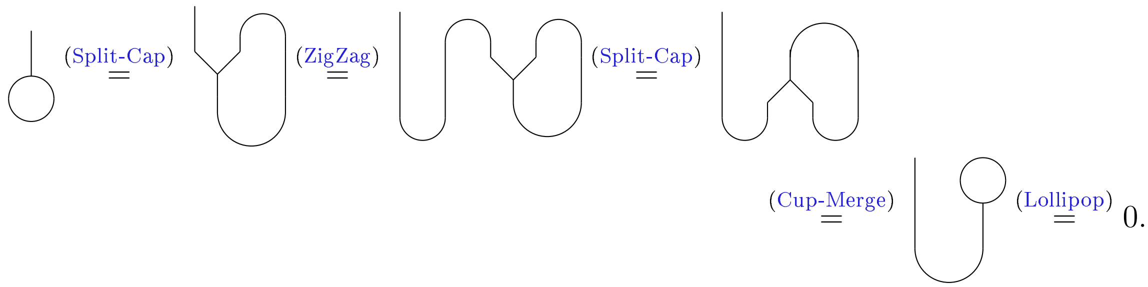



□

**Lemma 6.1.10.** *The following relation holds in* $\mathcal{C}(\delta)$*:*

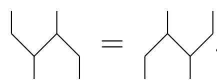


**Proof.** We have



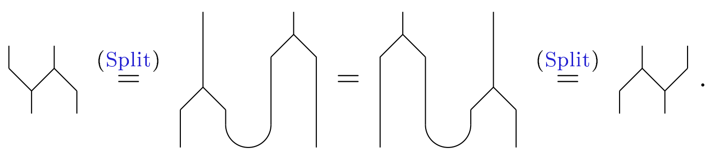

□

**Lemma 6.1.11.** *The following relation holds in $\mathcal{C}(\delta)$.*



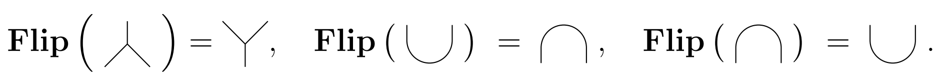
(Split-Merge)

**Proof.** We have




□

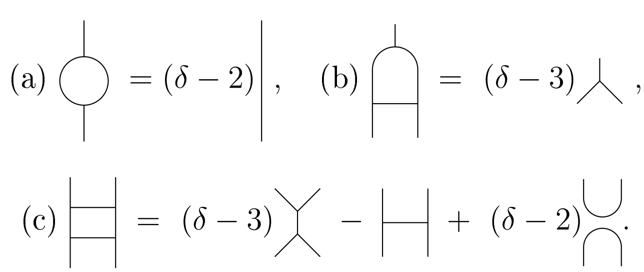
**Definition 6.1.12.** Define a monoidal functor **Flip**: $\mathcal{C}(\delta) \to \mathcal{C}(\delta)^{\mathrm{op}}$ that sends each object of $\mathcal{C}(\delta)$ to itself, and then on the generators of $\mathcal{C}(\delta)$, **Flip** does the following



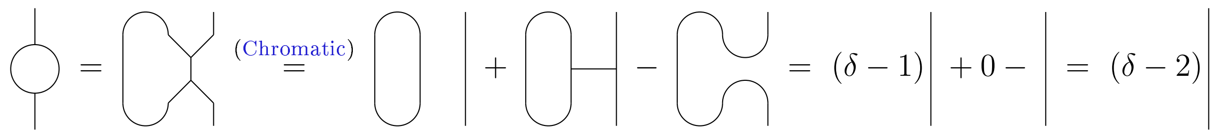

**Definition 6.1.13.** Define a monoidal functor **Rev**: $\mathcal{C}(\delta) \to \mathcal{C}(\delta)^{\mathrm{rev}}$ that sends each object of $\mathcal{C}(\delta)$ to itself, fixes the generating morphisms of $\mathcal{C}(\delta)$, and reverses the order of tensor products of morphisms.

One can easily verify that the maps above respect the defining relations on $\mathcal{C}(\delta)$, and thus they both give well-defined functors. The functor **Flip** corresponds to a reflection of diagrams in a horizontal line, and the functor **Rev** corresponds a reflection of diagrams in a vertical line.

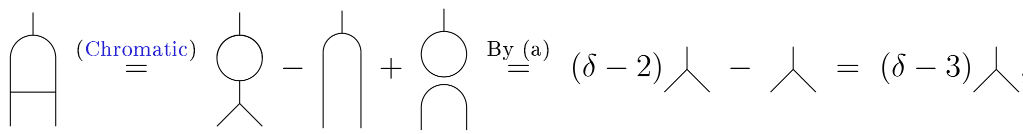
**Proposition 6.1.14.** *In $\mathcal{C}(\delta)$, we have the following relations:*



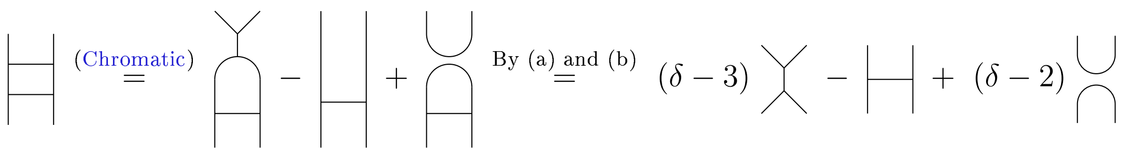
**Proof.** To show relation (a) holds, we have




For the relation in (b), we have




Lastly, for the relation in (c), we have




□

## 6.2 The planar chromatic category $\mathcal{P}(\delta)$

The chromatic category $\mathcal{C}(\delta)$ introduced in Definition 6.1.1 was built from trivalent, planar graphs modulo some relations. In this section we consider an analogous category $\mathcal{P}(\delta)$ that is built instead from all planar graphs modulo planar isotopy and a few other relations. The definition of $\mathcal{P}(\delta)$ matches more closely with the definition of the chromatic algebra in [FK10; FK09; AK19], with a small adjustment similar to the definition of the chromatic algebra in [Liu24]. We will see that $\mathcal{P}(\delta)$ is isomorphic to the chromatic category $\mathcal{C}(\delta)$ defined in Definition 6.1.1 with the matching parameter. In [FK10, Section 4], Fendley and Krushkal give a basis for the chromatic algebra. In our work in this section, we adapt their proof of this fact to obtain a basis for the morphism spaces of $\mathcal{P}(\delta)$. From this basis result, we obtain an isomorphism $C(\delta) \to \mathcal{P}(\delta)$ and use this isomorphism to obtain a basis for the morphism spaces of the chromatic category $\mathcal{C}(\delta)$. Throughout this section fix $\Bbbk$ to be an arbitrary field with $\delta \in \Bbbk^\times$ and assume $\delta \neq 1$.

**Remark 6.2.1.** Recall that a plane graph is a planar graph with a fixed planar embedding. Note that we will allow plane graphs that have loops with no vertices.

**Definition 6.2.2** (Planar Chromatic Category)**.** Fix non-negative integers $n$ and $m$. Define $G_{n,m}$ to be the set of plane graphs in the unit square $[0,1]\times[0,1]$, modulo planar isotopy, and such that the intersection of the graph and the boundary of the square consists of $n$ vertices of degree 1 on the bottom and $m$ vertices of degree 1 at the top. Define the *planar chromatic category* to be the strict $\Bbbk$-linear monoidal category $\mathcal{P}(\delta)$ with one generating object $X$ and, for non-negative integers $n$ and $m$, $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$ is defined to be formal $\Bbbk$-linear combinations of elements in $G_{n,m}$ modulo the following relations for all $G \in G_{n,m}$:

for any inner edge $e$ of $G$ that is not a loop, we have

$$G = G/e - G \setminus e, \tag{P1}$$

for any inner edge $e$ of $G$ that is a loop, we have

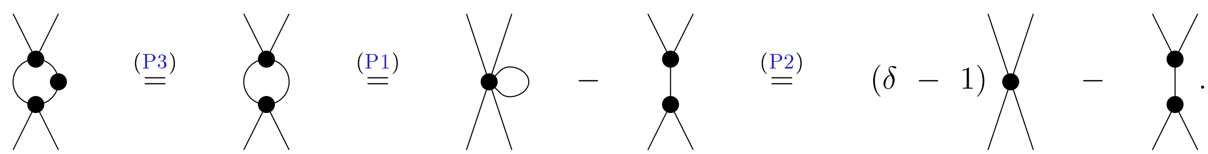
$$G = (\delta - 1)G \setminus e, \tag{P2}$$

and if $G$ contains a vertex of degree 2, and $G'$ is the result of removing this vertex and merging the two edges incident upon it, then we have

$$G = G'. \tag{P3}$$

We define an *inner edge* of a graph $G \in G_{n,m}$ to be an edge that does not touch the border of the unit square containing $G$. Composition of morphisms for $\mathcal{P}(\delta)$ is given by vertical stacking of graphs (and connecting points appropriately), and the tensor product of morphisms is given by horizontal stacking.

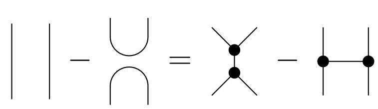
**Example 6.2.3.** As an example, consider the following computation that can be done on a morphism in $\mathcal{P}(\delta)$ using the relations (P1), (P2), and (P3):




Note that we omit the vertices of degree 1 on the top and bottom of the diagram.

We will see at the end of this section that the relations imposed on the morphisms of $\mathcal{P}(\delta)$ encode the chromatic polynomial of the duals of planar graphs as an invariant of the category. This has a precise meaning, explained in [MPS17, Appendix A], where it is shown that there is a bijective correspondence between multiplicative invariants of the planar graphs and trivalent categories.

**Definition 6.2.4** (Trivalent Planar Chromatic Category)**.** Fix non-negative integers $n$ and $m$. Define $T_{n,m}$ to be the set of trivalent plane graphs in the unit square $[0,1] \times [0,1]$, modulo planar isotopy, and such that the intersection of the graph and the boundary of the square consists of $n$ vertices of degree 1 on the bottom and $m$ vertices of degree 1 at the top. Define the *trivalent planar chromatic category* to be the strict $\Bbbk$-linear monoidal category $\mathcal{TP}(\delta)$ with one generating object $X$ and, for non-negative integers $n$ and $m$, $\mathrm{Hom}_{\mathcal{TP}(\delta)}(X^{\otimes n}, X^{\otimes m})$ is defined to be formal $\Bbbk$-linear combinations of elements in $T_{n,m}$ modulo the following relations for all $G \in T_{n,m}$:



(T1)

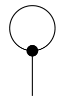
for any inner edge $e$ of $G$ that is a loop, we have

$$G = (\delta - 1)G \setminus e \tag{T2}$$

if $G$ contains a vertex of degree 2, and $G'$ is the result of removing this vertex and merging the two edges incident upon it, then

$$G = G', \tag{T3}$$

and



$$= 0. \tag{T4}$$

Composition of morphisms for $\mathcal{TP}(\delta)$ is given by vertical stacking of graphs (and connecting points appropriately), and the tensor product of morphisms is given by horizontal stacking.

**Remark 6.2.5.** It will occasionally be necessary to make a distinction between a graph and the diagram representing it in each of the categories above. Consequently, we introduce the following maps. Let $\Bbbk G_{n,m}$ be the free $\Bbbk$-vector space on the set $G_{n,m}$ and $\Bbbk T_{n,m}$ be the free $\Bbbk$-vector space on the set $T_{n,m}$. Then, we have surjective linear maps $\pi_{n,m} \colon \Bbbk G_{n,m} \to \mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$ and $\tau_{n,m} \colon \Bbbk T_{n,m} \to \mathrm{Hom}_{\mathcal{TP}(\delta)}(X^{\otimes n}, X^{\otimes m})$ that are given by sending a graph to their associated diagram in the categories $\mathcal{P}(\delta)$ and $\mathcal{TP}(\delta)$.

**Lemma 6.2.6.** *The following two relations hold in $\mathcal{P}(\delta)$:*

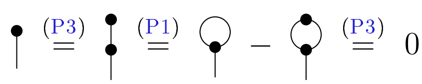
*if $G$ contains a vertex of degree 1 that is not on the border of the unit square, then*

$$G = 0, \tag{R1}$$

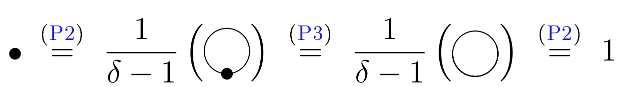
*and if $v$ is a vertex of degree 0 in $G$, then we have*

$$G = G \setminus v. \tag{R2}$$

**Proof.** (R1) holds in $\mathcal{P}(\delta)$:



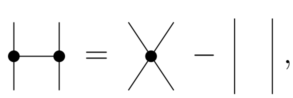

(R2) holds in $\mathcal{P}(\delta)$:



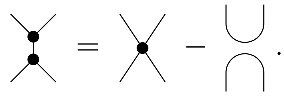

□

**Proposition 6.2.7.** *There is a functor $\mathbf{I} \colon \mathcal{TP}(\delta) \to \mathcal{P}(\delta)$ given by the inclusion of the set of trivalent plane graphs in the set of all plane graphs.*

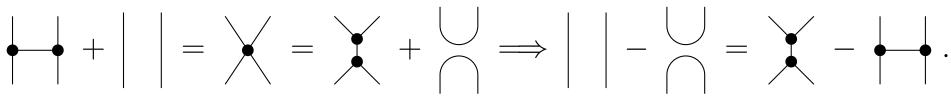
**Proof.** First, the relations (T2) and (T3) are implied by the relations (P2) and (P3) respectively. Next, we know that (T4) is implied by the relation (R1), which $\mathcal{P}(\delta)$ satisfies by Lemma 6.2.6. It then remains to show that (T1) is implied by the relation (P1) of $\mathcal{P}(\delta)$. Notice that from (P1), we have the following two relations (by contracting and deleting the inner edge in each case):



and

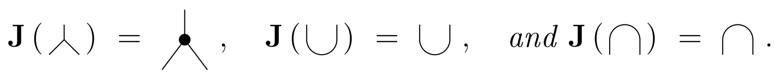


Thus we obtain the relation (T1) as follows:



Consequently $\mathbf{I}$ is a well defined functor. □

**Proposition 6.2.8.** *There is an isomorphism of categories* $\mathbf{J}\colon \mathcal{C}(\delta) \to \mathcal{TP}(\delta)$ *defined by*

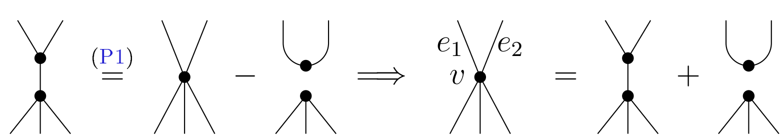



**Proof.** To prove this fact we will also define a functor $\mathbf{L}\colon \mathcal{TP}(\delta) \to \mathcal{C}(\delta)$. First, we show that $\mathbf{J}$ is well-defined. The (Cup-Merge) and (ZigZag) relations of $\mathcal{C}(\delta)$ are implied by planar isotopy. The remaining relations (Chromatic), (Bubble), and (Lollipop) of $\mathcal{C}(\delta)$ are implied by (T1), (T2), and (T4) of $\mathcal{TP}(\delta)$ respectively. Consequently, $\mathbf{J}$ respects the defining relations of $\mathcal{C}(\delta)$ and so $\mathbf{J}$ is a well-defined functor. Given a trivalent plane graph in $T_{n,m}$, we can obtain a unique morphism in $\mathcal{C}(\delta)(X^{\otimes n}, X^{\otimes m})$. This assignment respects planar isotopy by Remark 5.3.6 since $\mathcal{C}(\delta)$ is pivotal by Theorem 6.1.6. It also respects the relations (T1), (T2), and (T4) since these relations are implied by the (Chromatic), (Bubble), and (Lollipop) relations respectively. It is clear that we have $\mathbf{L} \circ \mathbf{J} = \mathbf{Id}_{\mathcal{C}(\delta)}$ and $\mathbf{J} \circ \mathbf{L} = \mathbf{Id}_{\mathcal{TP}(\delta)}$, thus $\mathbf{J}$ is an isomorphism as required. □

**Lemma 6.2.9.** *Any diagram in* $\mathcal{P}(\delta)$ *can be written as a linear combination of trivalent plane graphs. Consequently, the functor* $\mathbf{I}$ *is full.*

**Proof.** For any graph $G$ in $G_{n,m}$, define $V_{>3}(G)$ to be the set of vertices of the associated graph with degree greater than 3. We will show the desired result by induction on $S_{>3}(G) := \sum_{v \in V_{>3}(G)} (\deg(v) - 3)$. Note that by definition, if $v \in V_{>3}(G)$, then $\deg(v) > 3 \implies \deg(v) - 3 > 0$. Thus we would have $S_{>3}(G) > 0$. Consequently, we have $S_{>3}(G) = 0$ iff $D$ is a trivalent plane graph. In the case that $S_{>3}(G) = 0$, the result is then trivial.

Now suppose we have $S_{>3}(G) = k > 0$ and that for any graph $G'$ with $S_{>3}(G') < k$, $\pi_{n,m}(G')$ can be written as a linear combination of trivalent plane graphs. Then, since $k > 0$, the graph $G$ must have at least one vertex $v$ with degree greater than 3. Then using the relation (P1), we can make the following local change at this vertex by arbitrarily choosing two edges $e_1$ and $e_2$ that are incident on $v$:




From this, we can choose graphs $G_1$ and $G_2$ such that $\pi_{n,m}(G) = \pi_{n,m}(G_1) + \pi_{n,m}(G_2)$ where $S_{>3}(G_1)$ and $S_{>3}(G_2)$ are both strictly less than $k$. Thus our induction hypothesis applies and we may conclude that $\pi_{n,m}(G)$ can be written as a linear combination of trivalent graphs. □

**Definition 6.2.10.** Let $n$ and $m$ be non-negative integers. Define $B_{n,m}$ to be the set of graphs $b \in G_{n,m}$ such that

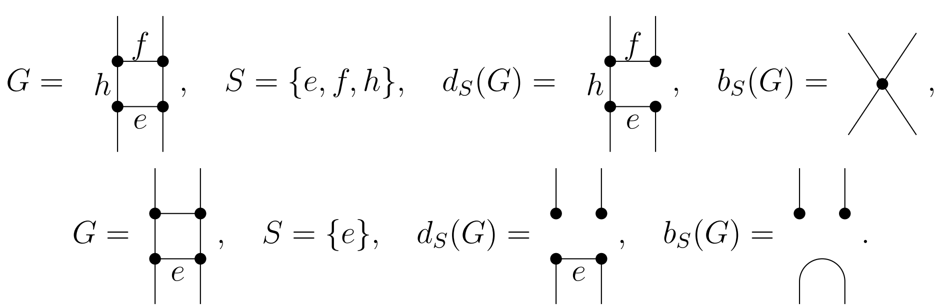
- $b$ has no inner edges,
- $b$ has no vertices of degree 1, other than the ones on the border of the diagram, and
- $b$ has no vertices of degree 2.

Furthermore, let $\overline{B}_{n,m}$ be the diagrams in $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$ corresponding to the graphs in $B_{n,m}$ (that is, $\overline{B}_{n,m} = \pi_{n,m}(B_{n,m})$).

**Lemma 6.2.11.** *The set* $\overline{B}_{n,m}$ *is a basis for* $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$.

**Proof.** We first give a brief overview of the idea of this proof. The main difficulty in this proof is to show that $\dim \mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m}) \geq |B_{n,m}|$. We accomplish this by defining linear maps $\phi\colon \Bbbk B_{n,m} \to \mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$ and $\overline{\Psi}\colon \mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m}) \to \Bbbk B_{n,m}$ such that $\overline{\Psi} \circ \phi$ is an isomorphism. We obtain $\overline{\Psi}$ by first defining a linear map $\Psi\colon \Bbbk G_{n,m} \to \Bbbk B_{n,m}$ and showing that $\Psi$ respects the relations (P1), (P2), and (P3). Recall that $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$ is a quotient of $G_{n,m}$ by the relations (P1), (P2), and (P3). So, if the map $\Psi$ respects these relations, it induces a linear map on $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$. This induced map will be our map $\overline{\Psi}$.

To define the linear map $\Psi$, we first have a few definitions. For a graph $G \in G_{n,m}$ we denote its set of inner edges as $\mathrm{Inner}(G)$. Now fix a graph $G \in G_{n,m}$ and a subset $S \subseteq \mathrm{Inner}(G)$. Define $d_S(G)$ to be the graph obtained from $G$ by deleting all the inner edges in $E(G) \setminus S$. Additionally, define $b_S(G)$ to be the graph obtained from $d_S(G)$ by contracting all the edges in $S$ and removing any vertices of degree 2 (by merging the edges incident upon the vertex) and any vertices of degree 0. Consider the following example in which we compute $d_S(G)$ and $b_S(G)$ for a given $G \in G_{2,2}$ and two different subsets $S$ of the inner edges of $G$.




Let $\Bbbk B_{n,m}$ be the free vector space on the set $B_{n,m}$ with basis $\{v_b : b \in B_{n,m}\}$. For any $G \in G_{n,m} \setminus B_{n,m}$ we will also define $v_G = 0$. Notice that sometimes by deleting edges in $G$, the resulting graph may have 1-valent vertices. In these cases, $b_S(G)$ is not in $B_{n,m}$, and consequently we will have $v_{b_S(G)} = 0$.

Now we will define a linear map $\Psi\colon \Bbbk G_{n,m} \to \Bbbk B_{n,m}$. For a graph $G \in G_{n,m}$ that has no vertices of degree 2 we define:

$$\Psi(G) := \sum_{S \subseteq \text{ inner edges of } G} (-1)^{|E(G)|-|S|} \delta^{n(d_S(G))} v_{b_S(G)}.$$

The definition of $\Psi$ is motivated by the analogous map defined in [FK10, Lemma 4.], where they give a basis for the chromatic algebra. Recall that $n(H)$ is the *nullity* of the graph $H$ introduced in Remark 2.2.10—the number of independent cycles in $H$. For any graph $G \in G_{n,m}$, define $G' \in G_{n,m}$ to be the graph obtained from $G$ by removing all vertices of degree 2 in $G$ and merging the edges incident upon each such vertex. For graphs $G \in G_{n,m}$ with vertices of degree 2, we define $\Psi(G) := \Psi(G')$. The map $\Psi$ is then extended linearly to all elements in $\Bbbk G_{n,m}$.

Next we will show that $\Psi$ respects the relations (P1) and (P2) (note that it respects (P3) by definition), which will show that $\Psi$ induces a map $\overline{\Psi}\colon \mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m}) \to \Bbbk B_{n,m}$ such that:

$$\overline{\Psi} \circ \pi_{n,m} = \Psi.$$

The map $\Psi$ preserves relation (P1): Let $G$ be a graph in $G_{n,m}$ and $f$ an inner edge of $G$ that is not a loop. Then by definition we have:

$$\begin{aligned}
\Psi(G) &= \sum_{S \subseteq \mathrm{Inner}(G)} (-1)^{|E(G)|-|S|} \delta^{n(d_S(G))} v_{b_S(G)} \\
&= \sum_{\substack{S \subseteq \mathrm{Inner}(G) \\ \text{such that } f \in S}} (-1)^{E(G)-|S|} \delta^{n(d_S(G))} v_{b_S(G)} + \sum_{\substack{S \subseteq \mathrm{Inner}(G) \\ \text{such that } f \notin S}} (-1)^{E(G)-|S|} \delta^{n(d_S(G))} v_{b_S(G)} \\
&= \sum_{S \subseteq \mathrm{Inner}(G/f)} (-1)^{(E(G/f)+1)-(|S|+1)} \delta^{n(d_S(G/f))} v_{b_S(G/f)} \\
&\qquad + \sum_{S \subseteq \mathrm{Inner}(G \backslash f)} (-1)^{(E(G\backslash f)+1)-|S|} \delta^{n(d_S(G\backslash f))} v_{b_S(G\backslash f)} \\
&= \sum_{S \subseteq \mathrm{Inner}(G/f)} (-1)^{E(G/f)-|S|} \delta^{n(d_S(G/f))} v_{b_S(G/f)} \\
&\qquad - \sum_{S \subseteq \mathrm{Inner}(G \backslash f)} (-1)^{E(G\backslash f)-|S|} \delta^{n(d_S(G\backslash f))} v_{b_S(G\backslash f)} \\
&= \Psi(G/f) - \Psi(G \setminus f).
\end{aligned}$$

Thus $\Psi$ preserves the relation (P1).

The map $\Psi$ preserves relation (P2): Suppose $G$ is a graph in $G_{n,m}$ with $e$ a loop of $G$. Then we have:

$$\begin{aligned}
\Psi(G) &= \sum_{S \subseteq \mathrm{Inner}(G)} (-1)^{E(G)-|S|} \delta^{n(d_S(G))} v_{b_S(G)} \\
&= \sum_{\substack{S \subseteq \mathrm{Inner}(G) \\ \text{such that } e \in S}} (-1)^{E(G)-|S|} \delta^{n(d_S(G))} v_{b_S(G)} + \sum_{\substack{S \subseteq \mathrm{Inner}(G) \\ \text{such that } e \notin S}} (-1)^{E(G)-|S|} \delta^{n(d_S(G))} v_{b_S(G)} \\
&= \sum_{S \subseteq \mathrm{Inner}(G\backslash e)} (-1)^{(E(G\backslash e)+1)-(|S|+1)} \delta^{n(d_S(G\backslash e))+1} v_{b_S(G\backslash e)} \\
&\qquad + \sum_{S \subseteq \mathrm{Inner}(G\backslash e)} (-1)^{(E(G\backslash e)+1)-|S|} \delta^{n(d_S(G\backslash e))} v_{b_S(G\backslash e)}
\end{aligned}$$

$$
\begin{aligned}
=&\delta \sum_{S\subseteq \mathrm{Inner}(G\setminus e)} (-1)^{E(G\setminus e)-|S|}\delta^{n(d_S(G\setminus e))} v_{b_S(G\setminus e)} \\
&\quad - \sum_{S\subseteq \mathrm{Inner}(G\setminus e)} (-1)^{E(G\setminus e)-|S|}\delta^{n(d_S(G\setminus e))} v_{b_S(G\setminus e)} \\
=& (\delta - 1)\Psi(G \setminus e).
\end{aligned}
$$

Thus $\Psi$ preserves the relation (P2).

Notice that if $b \in B_{n,m}$, then we must have $\Psi(b) = \pm v_b$ since $b$ has no inner edges and no cycles. Define a linear map $\phi\colon \Bbbk B_{n,m} \to \mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$ by $\phi(v_b) = \pi_{n,m}(b), \forall b \in B_{n,m}$. Consequently, we have

$$\overline{\Psi}(\phi(v_b)) = \overline{\Psi}(\pi_{n,m}(b)) = \Psi(b) = \pm v_b, \quad \forall b \in B_{n,m}.$$

Thus $\overline{\Psi} \circ \phi$ is an isomorphism. Now, since $\overline{B}_{n,m}$ spans $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$ (one may repeatedly apply (P1) and (P2) to express any element of $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})$ as a linear combination of graphs with no inner edges and then use (P3) to remove any vertices of degree 2), we have $\dim(\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})) \leq \left|\overline{B}_{n,m}\right| \leq |B_{n,m}|$. Then, since $\overline{\Psi} \circ \phi$ is an isomorphism, we must have $\dim(\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m})) \geq \dim V = |B_{n,m}|$, completing the proof. □

**Remark 6.2.12.** Notice that in the case that $n = 0$ and $m = 0$, we have that $B_{n,m}$ is the set containing the empty graph. Consequently, by Lemma 6.2.11, we know that every element of $\mathrm{End}_{\mathcal{P}(\delta)}(\mathbb{1})$ is a scalar multiple of the empty diagram.

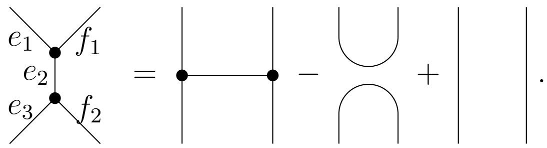
**Lemma 6.2.13.** *Let $n$ and $m$ be non-negative integers. Then, for any trivalent graph $T \in T_{n,m}$, $\tau_{n,m}(T)$ can be written as a linear combination of acyclic trivalent graphs in $\mathrm{Hom}_{\mathcal{TP}(\delta)}(X^{\otimes n}, X^{\otimes m})$ each with at most the same number of trivalent vertices as $T$.*

**Proof.** For any trivalent graph $T$ in $T_{n,m}$, define $r(T)$ to be the number of faces in the graph whose edges are all inner edges (equivalently, the number of chordless cycles—cycles that have no edges of the graph crossing them). Suppose that the Lemma 6.2.13 is not true. Then, choose a counterexample $T \in T_{n,m}$ with minimum possible $r(T)$. We will show that we can reduce the number of such cycles, yielding a contradiction.

First pick a chordless cycle $C$ in $T$ with the fewest number of edges (it must have a cycle, otherwise $T$ would be acyclic), and label its edges $e_1, e_2, \cdots, e_k$ so that they appear in that order going clockwise around the cycle. Also assume that the vertices these edges are adjacent to are trivalent (otherwise, if there is one of degree two, we merge the two edges incident to it). Then, if $k \leq 4$, we can use the relations in Proposition 6.1.14 to write $\tau_{n,m}(T)$ as a linear combination of trivalent graphs with fewer chordless cycles, yielding a contradiction. Note that while Proposition 6.1.14 was proved for $\mathcal{C}(\delta)$, this result remains true for $\mathcal{TP}(\delta)$ in view of the isomorphism in Proposition 6.2.8.

Next, if $k > 4$, let $f_1$ be the third edge incident on the same vertex that $e_1$ and $e_2$ are incident on, and $f_2$ be the edge incident on the same vertex that $e_2$ and $e_3$ are incident on.

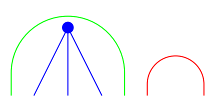
Then we can make the following local change using the relation (T1):




As a result of this change, we can write $\tau_{n,m}(T) = \tau_{n,m}(T_1) - \tau_{n,m}(T_2) + \tau_{n,m}(T_3)$. Note that $T_2$ and $T_3$ each have fewer chordless cycles than $T$. In $T_1$, the edge $e_2$ was removed from $C$, thus reducing the length of this cycle (note that this can increase the length of other cycles that were in $T$, but it will not add new cycles!). We can repeat this process to the trivalent graph $T_1$ with the same cycle, to eventually write $\tau_{n,m}(T_1) = S + \tau_{n,m}(T_4)$ where $S$ is a linear combination of trivalent plane graphs with fewer chordless cycles than $T$, and $T_4$ has the same number of cycles as $T$, but at least one of which that has at most 4 edges. Then, we can again utilize one of the relations in Proposition 6.1.14 to write $\tau_{n,m}(T)$ as a linear combination of trivalent graphs with fewer chordless cycles, yielding a contradiction. □

**Lemma 6.2.14.** *Let $n$ and $m$ be non-negative integers, and $T_1$ and $T_2$ be acyclic trivalent plane graphs in $T_{n,m}$ such that the result of contracting every inner edge in $T_i$ is the element $b \in B_{n,m}$ for each $i \in \{1,2\}$. Then, in $\mathrm{Hom}_{\mathcal{TP}(\delta)}(X^{\otimes n}, X^{\otimes m})$, we can write $\tau_{n,m}(T_1) = \tau_{n,m}(T_2) + S$ where $S \in \mathrm{Hom}_{\mathcal{TP}(\delta)}(X^{\otimes n}, X^{\otimes m})$ is a linear combination of trivalent plane graphs each with fewer trivalent vertices than either of $T_1$ or $T_2$.*

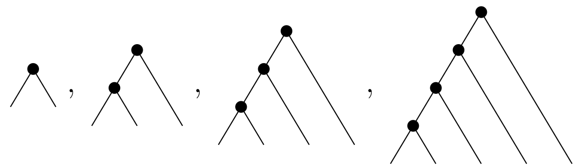
**Proof.** Without loss of generality we may assume that $m = 0$. This is because, one can redirect the $m$ strands on the top of the morphisms in $\mathrm{Hom}_{\mathcal{TP}(\delta)}(X^{\otimes n}, X^{\otimes m})$ to the bottom using $m$ caps, yielding a morphism in $\mathrm{Hom}_{\mathcal{TP}(\delta)}(X^{\otimes(n+m)}, \mathbb{1})$. Then, for any $G \in G_{n,0}$, we can let $v_1, v_2, \cdots, v_n$ be the vertices on the bottom of the diagram ordered from left to right. From this ordering on the vertices we can order the connected components of the graph based on the smallest-index vertex that appears in the connected component. For example,

$b =$  $\in \mathcal{G}_{7,0}$

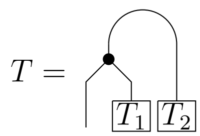
has three connected components that are, in order: green, blue, and red.

Now, suppose we have two acyclic trivalent graphs $T_1$ and $T_2$ in $T_{n,0}$ such that contracting all inner edges in each graph yields the graph $b \in B_{n,0}$. Then, it is clear that $T_1$, $T_2$, and $b$ must have the same number of connected components. Let $k$ be the number of connected components, and then let $T_i^1, T_i^2, \cdots, T_i^k$ be the connected components of $T_i$ $(i = 1, 2)$ in the order discussed above. Additionally, let $b_1, b_2, \cdots, b_k$ be the connected components of $b$ ordered in the same way. Then, since contracting all inner edges of $T_1$ and $T_2$ respectively yields $b$, it must be the case that, for $j \in \{1, 2, \cdots, k\}$, contracting all the inner edges of $T_1^j$ and $T_2^j$ respectively yields $b_j$. Consequently, we can reduce the problem to the case where $T_1$ and $T_2$ both have one connected component, i.e. they are both trees.

Now, we can further reduce the problem to the case where $T_2$ is a specific tree $T_0$ in the set of trees that contract to yield $b$. If this were the case, for $i = 1, 2$ we could write $\tau_{n,m}(T_i) = \tau_{n,m}(T_0) + S_i$, where $S_i$ has fewer trivalent vertices than that of $\tau_{n,m}(T_i)$. Hence we would have $\tau_{n,m}(T_1) - \tau_{n,m}(T_2) = S_1 - S_2$, proving the more general statement. Thus we may assume $T_2$ is a binary rooted tree, wherein the strand going down and to the right at each vertex has no other trivalent vertices on it. For example, these trees with less than 4 trivalent vertices are:



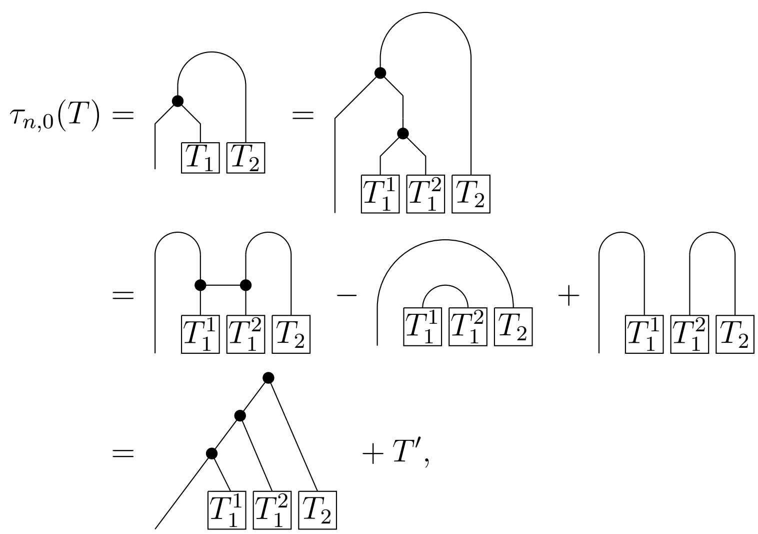
Now, take a tree $T$ in $T_{n,0}$. If $n = 2$, the only such graph is the first one above, so there is nothing to prove in that case. Next, suppose that the result is true for all trees in $T_{k,0}$ for $k < n$. Then we can write $T$ as:




where $T_1 \in T_{k_1,1}$ and $T_2 \in T_{k_2,1}$ are trees with $1 + k_1 + k_2 = n$, where $k_1$ and $k_2$ are positive integers. If $T_1$ has no trivalent vertices, we can apply the induction hypothesis to $\bigcap \circ (\mathrm{Id}_X \otimes T_2)$. Otherwise, we can write:

$$T_1 = \curlywedge \circ (T_1^1 \otimes T_1^2),$$

where $T_1^1 \in T_{s_1,1}$ and $T_1^2 \in T_{s_2,1}$ are trees with $s_1 + s_2 = k_1$, where $s_1$ and $s_2$ are positive integers. In this case, we can use the relation (T1) to make the following change:

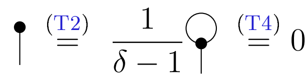



where $T'$ is a linear combination of acyclic trivalent graphs with fewer trivalent vertices than $T$. Furthermore, since $T_1^1$ has fewer trivalent vertices than $T_1$, we may repeat this process until the second-leftmost subtree has no trivalent vertices, in which case we can apply the induction hypothesis as before. □

**Theorem 6.2.15.** *Let $n$ and $m$ be non-negative integers. For every element $b$ of $B_{n,m}$, choose $T_b \in T_{n,m}$ to be an acyclic trivalent plane graph such that the result of contracting all the inner edges of $T_b$ is $b$. Then the set $M = \{\tau_{n,m}(T_b) : b \in B_{n,m}\}$ is a basis for* $\mathrm{Hom}_{\mathcal{TP}(\delta)}(X^{\otimes n}, X^{\otimes m})$.

**Proof.** From Lemma 6.2.13, we know that $\mathrm{Hom}_{\mathcal{TP}(\delta)}(X^{\otimes n}, X^{\otimes m})$ is spanned by the set of acyclic trivalent graphs. Our approach is to show that every acyclic trivalent graph $G$ is in the span of $M$ by induction on the number of trivalent vertices of the graph $G$.

First, assume we have an acyclic trivalent graph $G$ with no trivalent vertices. Assume $G$ has no vertices of degree 2, else we can remove them using the relation (T3). Note that the graph $G$ can have no inner edges since they would form a cycle. If $G$ has a vertex of degree 1 that is not on the border of the diagram, then $\tau_{n,m}(G) = 0$ and we would be done. One can see this is true as follows (note that we have assumed $\delta \neq 1$ at the beginning of this section):




Consequently, we must have $G \in B_{n,m}$. Now I claim $T_G$ has no vertices of degree 3. This is because, if $T_G$ had any vertices of degree 3, then $G$ would have to have a vertex of degree 3 or higher (recall $G$ is obtained from $T_G$ by contracting all of the inner edges of $T_G$). Then, since $T_G$ is acyclic, we must have that $T_G \in B_{n,m}$ as well. So we must have $T_G = G$.

Now suppose that $T$ is an acyclic trivalent graph with $k > 0$ trivalent vertices. Further assume inductively that if $T'$ is any acyclic trivalent graph with fewer than $k$ trivalent vertices, then $\tau_{n,m}(T')$ is in the span of $M$. As in the above case, we may assume $T$ has no vertices of degree 1 that are not on the border of the diagram, and no vertices of degree 2. Let $b \in B_{n,m}$ be the result of contracting all the inner edges of $T$. Then $T_1 := T$ and $T_2 := T_b$ satisfy the conditions of Lemma 6.2.14, so we can write $T = T_b + S$ where $S$ is a linear combination of trivalent graphs with fewer trivalent vertices than $T$ or $T_b$. Then, Lemma 6.2.13 allows us to assume the trivalent graphs above are acyclic. Thus, we can use the induction hypothesis to conclude that $S$ is in the span of $M$. Consequently, $T$ is in the span of $M$ as required, since $T_b \in M$. □

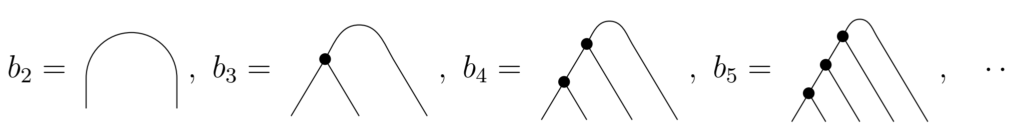
**Corollary 6.2.16.** *The functor* $\mathbf{I}$ *is an isomorphism.*

**Proof.** By Lemma 6.2.9, we know that the functor $\mathbf{I}$ is full. This, combined with the fact that $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^n, X^m)$ and $\mathrm{Hom}_{\mathcal{TP}(\delta)}(X^n, X^m)$ have the same dimension, shows that $\mathbf{I}$ must be an isomorphism. □

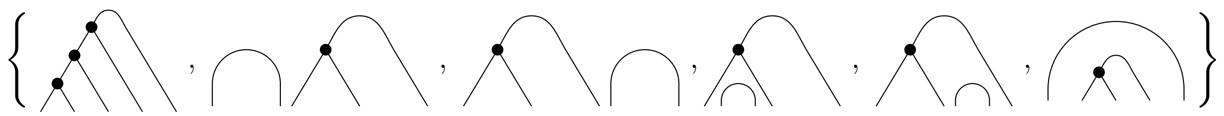
**Corollary 6.2.17.** *There is a monoidal isomorphism from* $\mathcal{C}(\delta)$ *to* $\mathcal{P}(\delta)$ *sending* $X$ *to itself and every diagram in* $\mathcal{C}(\delta)$ *to its isotopy class.*

**Proof.** This functor is $\mathbf{I} \circ \mathbf{J}$. The functor $\mathbf{I}$ is an isomorphism by Corollary 6.2.16 and $\mathbf{J}$ is an isomorphism by Proposition 6.2.8. □

**Remark 6.2.18.** In Theorem 6.2.15, there are many possible choices for the acyclic trivalent graphs $T_b$. For $\mathcal{TP}_{n,0}$, one possible choice of basis is given by choosing a tree that looks like the following for each connected component:



For example, the basis for $\mathcal{TP}_{5,0}$ chosen in this way would be:



**Remark 6.2.19.** The planar chromatic category $\mathcal{P}(\delta)$ (and thus the chromatic category $\mathcal{C}(\delta)$ as well by Corollary 6.2.17) has a close connection to the planar graph colouring. When $\delta > 1$ is an integer, $\mathcal{C}(\delta)$ encodes information about the proper $\delta$-colourings of the dual of planar graphs. Details of this can be found in [FK10, Proposition 3.4.], however we close this section by sketching the idea of this connection.

The connection proceeds in several steps. First, recall that $G_{0,0}$ (introduced in Definition 6.2.2) is the set of plane graphs up to planar isotopy. Fix an integer $\delta \geq 1$. The number of proper colourings using $\delta$ colours of the dual of a plane graph is clearly invariant under planar isotopy. Thus we have a well-defined map:

$$f\colon G_{0,0} \to \Bbbk, \quad f(H) = \frac{1}{\delta}\chi(H^*;\delta), \quad \forall H \in G_{0,0}.$$

We can extend this map linearly to the vector space $\Bbbk G_{0,0}$ of all formal linear combinations of plane graphs (see Remark 6.2.5). That is, we have a linear map:

$$F\colon \Bbbk G_{0,0} \to \Bbbk, \quad F(H) = \frac{1}{\delta}\chi(H^*;\delta), \quad \forall H \in G_{0,0}.$$

The scalar $\frac{1}{\delta}$ is there to guarantee that we if $H \in G_{0,0}$ has connected components $H_1, H_2, \cdots, H_r$, then $F(H) = F(H_1)F(H_2)\cdots F(H_r)$. This should be clear since the quantity $\frac{1}{\delta}\chi(H^*;\delta)$ is the number of proper colourings of the dual of $H$ where we do not assign a colour to the *outer face* (the only unbounded face of the plane graph).

Next, we can show that $F$ actually preserves the defining relations of $\mathcal{P}(\delta)$. The linear map $F$ preserves (P3) since vertices of degree 2 do not affect the number of proper $\delta$-colourings of the dual of a plane graph. Now, suppose that $H \in G_{0,0}$ is a plane graph with an edge $e$ that is a loop. If one colours the faces of $H$ outside of the loop first, one then has $\delta - 1$ choices for the face on the inside of the loop that $e$ is adjacent to. Consequently, we have $\chi(H^*;\delta) = (\delta-1)\chi((H \setminus e)^*;\delta)$. Thus $F$ preserves the relation (P2). Lastly, suppose $H \in G_{0,0}$ is a plane graph with edge $e$ that is not a loop. Let $H'$ be the connected component containing the edge $e$ and $H''$ be the disjoint union of the remaining connected components of $H$. If $e$ is such that $H' \setminus e$ is connected, then one has by Corollary 2.3.7 that

$$\chi((H')^*;\delta) = \chi((H'/e)^*;\delta) - \chi((H' \setminus e)^*;\delta) \Longrightarrow F(H') = F(H'/e) - F(H' \setminus e)$$

and thus

$$F(H) = F(H')F(H'') = F(H'/e)F(H'') - F(H' \setminus e)F(H'') = F(H/e) - F(H \setminus e).$$

In the case where $H' \setminus e$ is disconnected, we have $F(H') = 0 \Longrightarrow F(H) = 0$ since $e^*$ would be a loop in $H^*$ (consequently, $H^*$ has zero proper $\delta$-colourings). Lastly, one can show that $H/e$ and $H \setminus e$ must have the same number of proper $\delta$-colourings in this case, thus $F(H) = 0 = F(H/e) - F(H \setminus e)$.

Therefore $F$ respects the relations (P1), (P2), and (P3). Consequently, $F$ induces a map

$$\overline{F}\colon \operatorname{End}_{\mathcal{P}(\delta)}(\mathbb{1}) \to \Bbbk, \quad \overline{F}(\pi_{0,0}(H)) = \frac{1}{\delta}\chi(H^*;\delta).$$

Now, we know by Remark 6.2.12 that every element of $\operatorname{End}_{\mathcal{P}(\delta)}(\mathbb{1})$ is a scalar multiple of the empty diagram. Using our work above, we can now determine what that scalar is. Let $H \in G_{0,0}$ and denote the empty diagram in $\operatorname{End}_{\mathcal{P}(\delta)}$ by $D$. Then we have $\pi_{0,0}(H) = \lambda D$ for some scalar $\lambda \in \Bbbk$. Consequently, we have

$$\lambda = \lambda\overline{F}(D) = \overline{F}(\lambda D) = \overline{F}(\pi_{0,0}(H)) = F(H) = \frac{1}{\delta}\chi(H^*;\delta).$$

What this means is that one can compute the chromatic polynomial of $H^*$ within the planar chromatic category $\mathcal{P}(\delta)$. One does this by expressing $H$ as a scalar multiple of the empty diagram using the relations (P1), (P2), and (P3). This scalar is $\frac{1}{\delta}\chi(H^*;\delta)$ as explained above, and then you can multiply this by $\delta$ to get the chromatic polynomial of $H^*$.

# Chapter 7

# Four functors

In this chapter, we demonstrate that the chromatic category is closely related to several other categories in the literature. In the following sections we define several functors and we will show later in Section 8.2 that some of these functors satisfy some additional properties.

## 7.1 A functor from $\mathcal{C}(4)$ to $\mathfrak{sl}_2$-mod

In this section, we consider the chromatic category $\mathcal{C}(\delta)$ when $\delta = 4$. Recall from Remark 6.2.19 that the category $\mathcal{C}(4)$ encodes information about the proper 4-colourings of the duals of planar graphs. We will define a functor from $\mathcal{C}(4)$ to $\mathfrak{sl}_2$-mod, in the case where $\Bbbk = \mathbb{C}$. Recall that the Lie algebra $\mathfrak{sl}_2$ has a basis $\{e, f, h\}$ with the Lie bracket defined by:

$$[e, f] = h, \ [h, e] = 2e, \text{ and } [h, f] = -2f.$$

Let $\varphi\colon \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathbb{C}$ be the symmetric bilinear form on $\mathfrak{sl}_2$ defined by

$$\varphi(e, f) = 2, \varphi(h, h) = 4, \text{ and } \varphi(e, e) = \varphi(f, f) = \varphi(e, h) = \varphi(f, h) = 0.$$

Recall that we computed the killing form $K_{\mathfrak{sl}_2}$ of $\mathfrak{sl}_2$ in Example 4.1.11 and notice that we have $\varphi = \frac{1}{2}K_{\mathfrak{sl}_2}$. Also recall that for each $k \geq 0$, there is an irreducible representation $V_k$ of $\mathfrak{sl}_2$ of dimension $k+1$. Let $V = V_2$ be the adjoint representation of $\mathfrak{sl}_2$. Then from Proposition 4.1.10 we have $\mathfrak{sl}_2$-module homomorphisms

$$V \otimes V \to V, \quad V \otimes V \to \mathbb{C}, \quad \text{and} \quad \mathbb{C} \to V \otimes V.$$

We will now show that the subcategory of $\mathfrak{sl}_2$-mod generated by tensors powers of $V$ admits a nice diagrammatic description. This description is given via a functor from $\mathcal{C}(4)$ to $\mathfrak{sl}_2$-mod involving (scalar multiples of) the three $\mathfrak{sl}_2$-module homomorphisms above.

**Theorem 7.1.1.** *Let $V = \mathfrak{sl}_2$ be the adjoint representation of $\mathfrak{sl}_2$. Let $\{b_1, b_2, b_3\}$ be a basis for $V$, and let $\{b_1^\vee, b_2^\vee, b_3^\vee\}$ be its dual basis with respect to the bilinear form $\varphi$ of $\mathfrak{sl}_2$. Then there is a monoidal functor $\mathbf{\Phi}\colon \mathcal{C}(4) \to \mathfrak{sl}_2$-mod such that:*

$$\mathbf{\Phi}(X) = V,$$

$$\mathbf{\Phi}\left(\curlywedge\right)(x\otimes y)=[x,y],\ \forall x,y\in V,$$
$$\mathbf{\Phi}\left(\bigcup\right)(\alpha)=\alpha\sum_{i=1}^{3}b_i\otimes b_i^{\vee},\ \forall\alpha\in\Bbbk,$$
$$\mathbf{\Phi}\left(\bigcap\right)(a\otimes b)=\varphi(a,b),\ \forall x,y\in V.$$

**Proof.** By Proposition 4.1.10, the maps $\mathbf{\Phi}(\curlywedge)$, $\mathbf{\Phi}(\bigcap)$, and $\mathbf{\Phi}(\bigcup)$ are all $\mathfrak{sl}_2$-module homomorphisms. Thus, to prove Theorem 7.1.1, it remains to show that $\mathbf{\Phi}$ respects all the defining relations of $\mathcal{C}(4)$. First, we show that the map $\mathbf{\Phi}$ preserves the (Cup-Merge) relation, that is

$$\mathbf{\Phi}\left(\curlywedge\,\bigcup\right)=\mathbf{\Phi}\left(\bigcup\,\curlywedge\right).$$

Let $\{b_1,b_2,b_3\}$ be a basis for $V$, and let $\{b_1^{\vee},b_2^{\vee},b_3^{\vee}\}$ be its dual basis with respect to $\varphi$. Since $\mathbf{\Phi}(\bigcup)$ is a $\mathfrak{sl}_2$-module homomorphism, we know that for all $x\in\mathfrak{sl}_2$ we have

$$x\cdot\sum_{i=1}^{3}b_i\otimes b_i^{\vee}=0\Longrightarrow\sum_{i=1}^{3}[x,b_i]\otimes b_i^{\vee}+\sum_{i=1}^{3}b_i\otimes[x,b_i^{\vee}]=0\Longrightarrow\sum_{i=1}^{3}[x,b_i]\otimes b_i^{\vee}=\sum_{i=1}^{3}b_i\otimes[b_i^{\vee},x]$$

consequently we have

$$\mathbf{\Phi}\left(\curlywedge\,\bigcup\right)(x)=\mathbf{\Phi}\left(\bigcup\,\curlywedge\right)(x).$$

Next, for the bubble relation we see that for all $\alpha\in\mathbb{C}$:

$$\mathbf{\Phi}\left(\bigcirc\right)\alpha=\mathbf{\Phi}\left(\bigcap\right)\left(\alpha\sum_{i=1}^{3}b_i\otimes b_i^{\vee}\right)=\alpha\sum_{i=1}^{3}\varphi(b_i,b_i^{\vee})=3\alpha=\mathbf{\Phi}(4-1)\alpha.$$

Thus the (Bubble) relation is preserved by $\mathbf{\Phi}$. Next, $\mathbf{\Phi}$ respects the (Lollipop) relation since there are no non-zero $\mathfrak{sl}_2$-module homomorphisms from $V$ to the trivial representation (this is because they are non-isomorphic irreducible representations).

Now, we need to verify that the (ZigZag) relation is respected. As above, choose a basis $\{b_1,b_2,b_3\}$ of $V$, and let $\{b_1^{\vee},b_2^{\vee},b_3^{\vee}\}$ be its dual basis with respect to $\varphi$. Then we have

$$\begin{aligned}\Phi\left(\bigcap\!\bigcup\right)(x)&=\Phi\left(\bigcap\,|\right)\left(x\otimes\left(\sum_{i=1}^{3}b_i\otimes b_i^{\vee}\right)\right)=\Phi\left(\bigcap\,|\right)\left(\sum_{i=1}^{3}x\otimes b_i\otimes b_i^{\vee}\right)\\&=\sum_{i=1}^{3}\varphi(x,b_i)b_i^{\vee}=x=\Phi\left(\,|\,\right)(x)=x=\sum_{i=1}^{3}b_i\varphi(b_i^{\vee},x)=\Phi\left(|\,\bigcap\right)\left(\sum_{i=1}^{3}b_i\otimes b_i^{\vee}\otimes x\right)\\&=\Phi\left(|\,\bigcap\right)\left(\left(\sum_{i=1}^{3}b_i\otimes b_i^{\vee}\right)\otimes x\right)=\Phi\left(\bigcup\!\bigcap\right)(x).\end{aligned}$$

Lastly, we must show the (Chromatic) relation is preserved. Before doing so, we will first show that we have the following relations in $\mathfrak{sl}_2$-mod:

$$\mathbf{\Phi}\left(\bigcap\right)\circ\mathbf{\Phi}\left(\curlyvee\right)=0\quad\text{and}\quad\mathbf{\Phi}\left(\curlywedge\right)\circ\mathbf{\Phi}\left(\bigcup\right)=0.\tag{7.1.1}$$

The first equality is true since $\mathbf{\Phi}$ respects the (Lollipop) relation. On the left side of the second equality in (7.1.1) we have an $\mathfrak{sl}_2$-module homomorphism from $V_0 \to V$, which must be zero since $V_0$ and $V$ are non-isomorphic irreducible representations. Next, we show that

$$\mathbf{\Phi}\left(\boxminus\!\!\!\!\bigcap\right) = 2\mathbf{\Phi}\left(\bigcap\right). \tag{7.1.2}$$

The morphism on the left side of this equality is a $\mathfrak{sl}_2$-module homomorphism from $V \otimes V$ to $V_0$. Since $V \otimes V$ contains only a single copy of $V_0$ as a direct summand we must have

$$\mathbf{\Phi}\left(\bigcap\right) \circ \mathbf{\Phi}\left(\vdash\!\dashv\right) = t_1 \mathbf{\Phi}\left(\bigcap\right)$$

for some scalar $t_1$. Evaluating on a specific vector will allow us to identify this scalar. Note that the ordered basis $(e, f, h)$ has dual basis $e^\vee = \frac{1}{2}f$, $f^\vee = \frac{1}{2}e$, and $h^\vee = \frac{1}{4}h$ with respect to the bilinear form $\varphi$ on $V$. Consequently we have that

$$\begin{aligned}
t_1 = \frac{1}{2}t_1\varphi(e,f) &= \left(\frac{1}{2}t_1\mathbf{\Phi}\left(\bigcap\right)\right)(e\otimes f) = \left(\frac{1}{2}\mathbf{\Phi}\left(\bigcap\right)\circ\mathbf{\Phi}\left(\vdash\!\dashv\right)\right)(e\otimes f)\\
&= \left(\frac{1}{2}\mathbf{\Phi}\left(\bigcap\right)\circ\mathbf{\Phi}\left(\curlywedge\!\smile\!\curlywedge\right)\right)(e\otimes f)\\
&= \frac{1}{2}\mathbf{\Phi}\left(\bigcap\right)\left([e,e]\otimes[e^\vee,f] + [e,f]\otimes[f^\vee,f] + [e,h]\otimes[h^\vee,f]\right)\\
&= \frac{1}{2}\mathbf{\Phi}\left(\bigcap\right)\left([e,e]\otimes\left[\frac{1}{2}f,f\right] + [e,f]\otimes\left[\frac{1}{2}e,f\right] + [e,h]\otimes\left[\frac{1}{4}h,f\right]\right)\\
&= \frac{1}{2}\mathbf{\Phi}\left(\bigcap\right)\left(0 + h\otimes\left(\frac{1}{2}h\right) + (-2e)\otimes\left(-\frac{1}{2}f\right)\right) = \frac{1}{4}\varphi(h,h) + \frac{1}{2}\varphi(e,f) = \frac{1}{4}4 + \frac{1}{2}2 = 2.
\end{aligned}$$

This proves the identity in (7.1.2). We now use a similar approach to show the following identity:

$$\mathbf{\Phi}\left(\boxminus\!\!\!\!\curlywedge\right) = \mathbf{\Phi}\left(\curlywedge\right). \tag{7.1.3}$$

The morphism on the left side of the equality is a $\mathfrak{sl}_2$-module homomorphism from $V \otimes V$ to $V$. Since $V \otimes V$ contains only a single copy of $V$ as a direct summand we must have

$$\mathbf{\Phi}\left(\curlywedge\right) \circ \mathbf{\Phi}\left(\vdash\!\dashv\right) = t_2 \mathbf{\Phi}\left(\curlywedge\right)$$

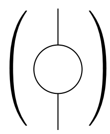
for some scalar $t_2$. As before, evaluating on a specific vector will allow us to identify the unknown scalar. We have that

$$\begin{aligned}
t_2 h = t_2[e,f] &= \left(t_2\mathbf{\Phi}\left(\curlywedge\right)\right)(e\otimes f) = \left(\mathbf{\Phi}\left(\curlywedge\right)\circ\mathbf{\Phi}\left(\vdash\!\dashv\right)\right)(e\otimes f)\\
&= \left(\mathbf{\Phi}\left(\curlywedge\right)\circ\mathbf{\Phi}\left(\curlywedge\!\smile\!\curlywedge\right)\right)(e\otimes f)\\
&= \mathbf{\Phi}\left(\curlywedge\right)\left([e,e]\otimes[e^\vee,f] + [e,f]\otimes[f^\vee,f] + [e,h]\otimes[h^\vee,f]\right)
\end{aligned}$$

$$\begin{aligned}
&= \Phi\left(\curlywedge\right)\left([e,e]\otimes\left[\frac{1}{2}f,f\right]+[e,f]\otimes\left[\frac{1}{2}e,f\right]+[e,h]\otimes\left[\frac{1}{4}h,f\right]\right)\\
&= \Phi\left(\curlywedge\right)\left(0+h\otimes\left(\frac{1}{2}h\right)+(-2e)\otimes\left(-\frac{1}{2}f\right)\right)=\frac{1}{2}[h,h]+[e,f]\\
&\qquad = \frac{1}{2}0+h=1h.
\end{aligned}$$

Thus $t_2 = 1$, proving the identity in (7.1.3).

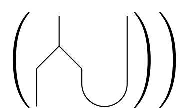
Next, we will show that

$$\Phi\left(\right.$$

$$\left.\right) = 2\Phi\left(\,|\,\right). \qquad (7.1.4)$$

The morphism on the left side of the equation above is a $\mathfrak{sl}_2$-module homomorphism from $V$ to $V$. Since $V$ is irreducible, we have that

$$\Phi\left(\curlywedge\right)\circ\Phi\left(\curlyvee\right)=t_3\,\mathrm{Id}_V=t_3\Phi\left(\,|\,\right)$$

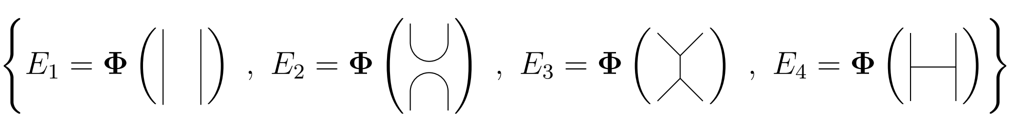
for some scalar $t_3$. We have

$$t_3 h = t_3\Phi\left(\,|\,\right)(h)=\left(\Phi\left(\curlywedge\right)\circ\Phi\left(\curlyvee\right)\right)(h)=\left(\Phi\left(\curlywedge\right)\circ\Phi\left(\right.\right.$$

$$\left.\left.\right)\right)(h)$$

$$\begin{aligned}
&= \Phi\left(\curlywedge\right)\left([h,e]\otimes e^\vee+[h,f]\otimes f^\vee+[h,h]\otimes h^\vee\right)\\
&= \Phi\left(\curlywedge\right)\left([h,e]\otimes\left(\frac{1}{2}f\right)+[h,f]\otimes\left(\frac{1}{2}e\right)+[h,h]\otimes\left(\frac{1}{4}h\right)\right)\\
&= \Phi\left(\curlywedge\right)(e\otimes f-f\otimes e)=[e,f]-[f,e]=h-(-h)=2h.
\end{aligned}$$

Thus $t_3 = 2$, proving the identity in (7.1.4).

Now, recall that $V\otimes V\simeq V_0\oplus V_2\oplus V_4$, thus $\mathrm{End}_{\mathfrak{sl}_2\text{-mod}}(V\otimes V)$ is 3-dimensional. We have that



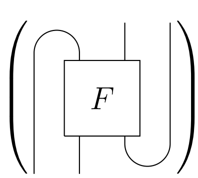

is a spanning set of $\mathrm{End}_{\mathfrak{sl}_2\text{-mod}}(V\otimes V)$. This is because $E_1=\mathrm{Id}_{V\otimes V}$, $E_2$ is a non-zero scalar multiple of the projection onto the $V_0$ component of $V\otimes V$, and $E_3$ is a non-zero scalar multiple of the projection onto the $V_2$ component of $V\otimes V$. Now define an endomorphism $R$ of $\mathrm{End}_{\mathfrak{sl}_2\text{-mod}}(V\otimes V)$ by:

$$R(f)=\Phi\left(\right.$$

$$\left.\right)\circ(\mathrm{Id}_V\otimes f\otimes\mathrm{Id}_V)\circ\Phi\left(\right.$$

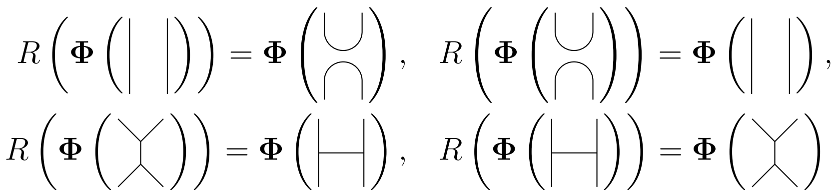
$$\left.\right),\quad \forall f\in\mathrm{End}_{\mathfrak{sl}_2\text{-mod}}(V\otimes V).$$

For a morphism $F\in\mathrm{End}_{\mathcal{C}(\delta)}(X\otimes X)$, note that we have:

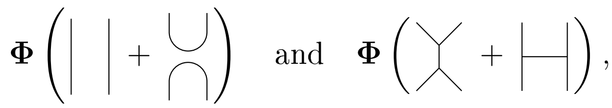
$$R(\Phi(F))=\Phi\left(\right.$$


$$\left.\right).$$

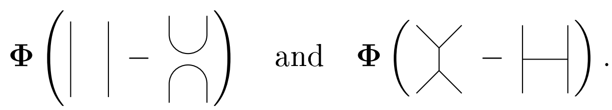
Notice that $R$ acts on the spanning set for $W$ above in the following way:



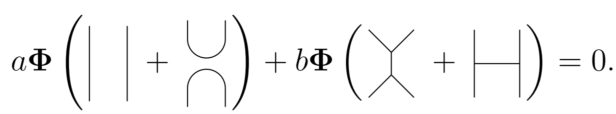
Thus $R$ has 1-eigenvectors:



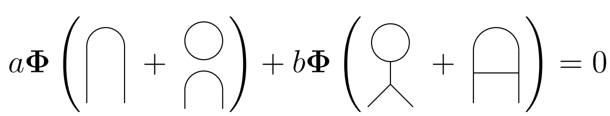
and $(-1)$-eigenvectors:



Since these four eigenvectors also span $W$, we know that one of the eigenspaces must have dimension equal to 1. Suppose that $a, b \in \mathbb{C}$ such that

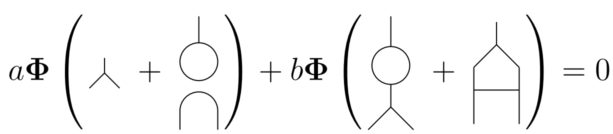


Applying a $\cap$ and a $\curlywedge$ respectively to the top of this equation yields the following equations in $a$ and $b$:



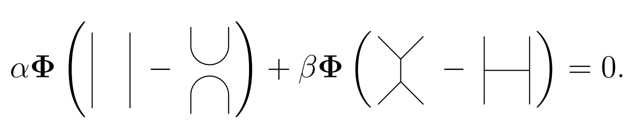
$$\overset{(7.1.1),\ (7.1.2)}{\Longrightarrow} \qquad (a + 3a + 0b + 2b)\mathbf{\Phi}\,(\cap) = 0 \qquad \Longrightarrow \qquad 4a + 2b = 0.$$

and



$$\overset{(7.1.1),\ (7.1.3),\ (7.1.4)}{\Longrightarrow} \qquad (1a + 0a + 2b + 1b)\mathbf{\Phi}\,(\curlywedge) = 0 \qquad \Longrightarrow \qquad a + 3b = 0.$$

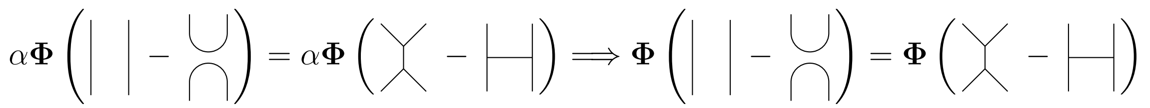
These two equations in $a$ and $b$ together imply that $a = b = 0$. Thus the 1-eigenspace of $R$ has dimension 2. Therefore, there must be scalars $\alpha, \beta \in \mathbb{C}$ that are not both zero such that:



Similar to the previous computation, applying a cap to the top of this equation yields of the following relationship between $\alpha$ and $\beta$:

$$\alpha - 3\alpha + 0 - 2\beta = 0 \Longrightarrow -2\alpha - 2\beta = 0 \Longrightarrow \beta = -\alpha.$$

Consequently we have



□

## 7.2 A functor from $\mathcal{C}(\delta)$ to $U_q(\mathfrak{sl}_2)$-mod

In this section, we will show that there is a functor from $\mathcal{C}(\delta)$ to $U_q(\mathfrak{sl}_2)$-mod for certain choices of $\delta$. In this section, we assume $\Bbbk = \mathbb{C}$ with $q \in \mathbb{C}^\times$ and suppose $q$ is not a root of unity.

Recall that $U_q(\mathfrak{sl}_2)$ is defined to be the $\mathbb{C}$-algebra generated by $E, F, K, K^{-1}$, subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

Recall from Proposition 4.3.3, for each integer $k \geq 0$ there is a finite dimensional irreducible representations $W_k$ of $U_q(\mathfrak{sl}_2)$, where $W_k$ is of dimension $k+1$. Additionally, by Theorem 4.3.8 we have that

$$W_n \otimes W_m \simeq W_{n+m} \oplus W_{n+m-2} \oplus \cdots \oplus W_{n-m}.$$

As in the non-quantum case, the 3-dimensional irreducible representation $W = W_2$ of $U_q(\mathfrak{sl}_2)$ will be of special interest to us.

Recall from Remark 4.3.4 that we can choose an ordered basis $(v_0, v_1, v_2)$ for the representation $W$ such that, with respect to this ordered basis, $E$, $F$, and $K$ act on $W$ by:

$$\begin{pmatrix} 0 & -[2] & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & [2] & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{pmatrix}$$

respectively (recall that $[2] = q + q^{-1}$ as defined in Definition 4.3.1).

**Theorem 7.2.1.** *There is a monoidal functor* $\mathbf{\Phi}_q \colon \mathcal{C}(q^2 + 2 + q^{-2}) \to U_q(\mathfrak{sl}_2)$*-mod such that* $\mathbf{\Phi}_q(X) = W$*, and*

$$\begin{aligned}
&\mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_0) = 0, && \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_1) = -[2]qv_0, && \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_2) = qv_1,\\
&\mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_0) = [2]q^{-1}v_0, && \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_1) = (q^{-2} - q^2)v_1, && \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_2) = -[2]qv_2,\\
&\mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_0) = -qv_1, && \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_1) = [2]q^{-1}v_2, && \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_2) = 0,\\
&\mathbf{\Phi}_q(\frown)(v_0 \otimes v_2) = [2]q^2, && \mathbf{\Phi}_q(\frown)(v_1 \otimes v_1) = [2]^2, && \mathbf{\Phi}_q(\frown)(v_2 \otimes v_0) = [2],
\end{aligned}$$

$$\mathbf{\Phi}_q(\smile)(1) = \frac{1}{[2]^2}v_1 \otimes v_1 + \frac{1}{[2]}v_0 \otimes v_2 + \frac{q^{-2}}{[2]}v_2 \otimes v_0,$$

*and for* $(i,j) \notin \{(0,2),(1,1),(2,0)\}$, $\quad \mathbf{\Phi}_q(\frown)(v_i \otimes v_j) = 0.$

**Proof.** From the *quantum Clebsch-Gordan* formula Theorem 4.3.8, we have that $W_2 \otimes W_2 \simeq W_4 \oplus W_2 \oplus W_0$. Thus, up to scaling, there are unique $U_q(\mathfrak{sl}_2)$-module homomorphisms $W_2 \otimes W_2 \to W_2$, $W_0 \to W_2 \otimes W_2$, and $W_2 \otimes W_2 \to W_0$. These maps are exactly where $\mathbf{\Phi}_q$ sends our split, cup, and cap morphisms.

To show that $\mathbf{\Phi}_q$ is a well-defined functor, we must first show that the definitions of $\mathbf{\Phi}_q(\curlywedge)$, $\mathbf{\Phi}_q(\cup)$, and $\mathbf{\Phi}_q(\cap)$ given above yield well-defined $U_q(\mathfrak{sl}_2)$-module homomorphisms.

To this end, we start with the $\curlywedge$ morphism. First, it is easily checked that each pure tensor $v_i \otimes v_j$ is a weight vector of weight $q^{4-2i-2j}$. Note that a *weight vector* here is just an eigenvector for the action of $K$, and its *weight* is the corresponding eigenvalue. Next, $\mathbf{\Phi}_q(\curlywedge)$ sends each of these pure tensors to a vector in $V$ of the same weight (or to 0). Consequently, $\mathbf{\Phi}_q(\curlywedge)$ commutes with the action of $K$ and $K^{-1}$. Next we check that this is true for the remaining generators of $U_q(\mathfrak{sl}_2)$, $E$ and $F$. Note that we have $\mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_1) = [2](q^{-1} - q)v_1$.

$$\begin{aligned}
\mathbf{\Phi}_q(\curlywedge)(E \cdot v_0 \otimes v_0) &= \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes 0 + 0 \otimes (q^2 v_0)) = \mathbf{\Phi}_q(\curlywedge)(0) = 0 = E \cdot 0 \\
&= E \cdot \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_0), \\
\mathbf{\Phi}_q(\curlywedge)(E \cdot v_0 \otimes v_1) &= \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes (-[2]v_0) + 0 \otimes v_1) = -[2]\mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_0) = 0 \\
&= E \cdot -[2]qv_0 = E \cdot \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_1), \\
\mathbf{\Phi}_q(\curlywedge)(E \cdot v_0 \otimes v_2) &= \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_1 + 0 \otimes (q^{-2}v_2)) = \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_1) = -[2]qv_0 \\
&= q(-[2]v_0) = q(E \cdot v_1) = E \cdot qv_1 = E \cdot \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_2), \\
\mathbf{\Phi}_q(\curlywedge)(E \cdot v_1 \otimes v_0) &= \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes 0 + (-[2]v_0) \otimes (q^2 v_0)) = -[2]q^2\mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_0) = 0 \\
&= E \cdot [2]q^{-1}v_0 = E \cdot \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_0), \\
\mathbf{\Phi}_q(\curlywedge)(E \cdot v_1 \otimes v_1) &= \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes (-[2]v_0) + (-[2]v_0) \otimes v_1) \\
&= -[2]\mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_0 + v_0 \otimes v_1) = -[2]([2]q^{-1}v_0 - [2]qv_0) \\
&= [2](q^{-1} - q)(-[2]v_0) = E \cdot [2](q^{-1} - q)v_1 = E \cdot \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_1), \\
\mathbf{\Phi}_q(\curlywedge)(E \cdot v_1 \otimes v_2) &= \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_1 + (-[2]v_0) \otimes (q^{-2}v_2)) \\
&= \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_1 - [2]q^{-2}v_0 \otimes v_2) = [2](q^{-1} - q)v_1 - [2]q^{-2}(qv_1) \\
&= -[2]qv_1 + [2]q^{-1}v_1 - [2]q^{-1}v_1 = -[2]qv_1 = E \cdot -[2]qv_2 \\
&= E \cdot \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_2), \\
\mathbf{\Phi}_q(\curlywedge)(E \cdot v_2 \otimes v_0) &= \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes 0 + v_1 \otimes (q^2 v_0)) = q^2\mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_0) = q^2[2]q^{-1}v_0 \\
&= -q(-[2]v_0) = E \cdot -qv_1 = E \cdot \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_0), \\
\mathbf{\Phi}_q(\curlywedge)(E \cdot v_2 \otimes v_1) &= \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes (-[2]v_0) + v_1 \otimes v_1) = \mathbf{\Phi}_q(\curlywedge)(-[2]v_2 \otimes v_0 + v_1 \otimes v_1) \\
&= -[2](-qv_1) - [2](q - q^{-1})v_1 = [2]qv_1 - [2]qv_1 + [2]q^{-1}v_1 = [2]q^{-1}v_1 \\
&= E \cdot [2]q^{-1}v_2 = E \cdot \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_1), \\
\mathbf{\Phi}_q(\curlywedge)(E \cdot v_2 \otimes v_2) &= \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_1 + v_1 \otimes (q^{-2}v_2)) = \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_1 + q^{-2}v_1 \otimes v_2) \\
&= [2]q^{-1}v_2 + q^{-2}(-[2]qv_2) = [2]q^{-1}v_2 - [2]q^{-1}v_2 = 0 = E \cdot 0 \\
&= E \cdot \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_2) \\
\mathbf{\Phi}_q(\curlywedge)(F \cdot v_0 \otimes v_0) &= \mathbf{\Phi}_q(\curlywedge)((q^{-2}v_0) \otimes (-v_1) + (-v_1) \otimes v_0)
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{\Phi}_q(\curlywedge)(-q^{-2}v_0 \otimes v_1 - v_1 \otimes v_0) = -q^{-2}(-[2]qv_0) - ([2]q^{-1}v_0) \\
&= [2]q^{-1}v_0 - [2]q^{-1}v_0 = 0 = F \cdot 0 = F \cdot \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_0), \\
\mathbf{\Phi}_q(\curlywedge)(F \cdot v_0 \otimes v_1) &= \mathbf{\Phi}_q(\curlywedge)((q^{-2}v_0) \otimes ([2]v_2) + (-v_1) \otimes v_1) \\
&= \mathbf{\Phi}_q(\curlywedge)([2]q^{-2}v_0 \otimes v_2 - v_1 \otimes v_1) = [2]q^{-2}(qv_1) - ([2](q^{-1} - q)v_1) \\
&= [2]q^{-1}v_1 + [2]qv_1 - [2]q^{-1}v_1 = [2]qv_1 = F \cdot -[2]qv_0 \\
&= F \cdot \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_1), \\
\mathbf{\Phi}_q(\curlywedge)(F \cdot v_0 \otimes v_2) &= \mathbf{\Phi}_q(\curlywedge)((q^{-2}v_0) \otimes 0 + (-v_1) \otimes v_2) = \mathbf{\Phi}_q(\curlywedge)(-v_1 \otimes v_2) \\
&= -(-[2]qv_2) = [2]qv_2 = q([2]v_2) = F \cdot qv_1 = F \cdot \mathbf{\Phi}_q(\curlywedge)(v_0 \otimes v_2), \\
\mathbf{\Phi}_q(\curlywedge)(F \cdot v_1 \otimes v_0) &= \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes (-v_1) + ([2]v_2) \otimes v_0) = \mathbf{\Phi}_q(\curlywedge)(-v_1 \otimes v_1 + [2]v_2 \otimes v_0) \\
&= -([2](q^{-1} - q)v_1) + [2](-qv_1) = [2]qv_1 - [2]q^{-1}v_1 - [2]qv_1 = -[2]q^{-1}v_1 \\
&= F \cdot [2]q^{-1}v_0 = F \cdot \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_0), \\
\mathbf{\Phi}_q(\curlywedge)(F \cdot v_1 \otimes v_1) &= \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes ([2]v_2) + ([2]v_2) \otimes v_1) = [2]\mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_2 + v_2 \otimes v_1) \\
&= [2](-[2]qv_2 + [2]q^{-1}v_2) = [2](q^{-1} - q)([2]v_2) = F \cdot [2](q^{-1} - q)v_1 \\
&= F \cdot \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_1), \\
\mathbf{\Phi}_q(\curlywedge)(F \cdot v_1 \otimes v_2) &= \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes 0 + ([2]v_2) \otimes v_2) = [2]\mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_2) = 0 \\
&= F \cdot -[2]qv_2 = F \cdot \mathbf{\Phi}_q(\curlywedge)(v_1 \otimes v_2), \\
\mathbf{\Phi}_q(\curlywedge)(F \cdot v_2 \otimes v_0) &= \mathbf{\Phi}_q(\curlywedge)((q^2v_2) \otimes (-v_1) + 0 \otimes v_0) = -q^2\mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_1) \\
&= -q^2([2]q^{-1}v_2) = -q([2]v_2) = F \cdot -qv_1 = F \cdot \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_0), \\
\mathbf{\Phi}_q(\curlywedge)(F \cdot v_2 \otimes v_1) &= \mathbf{\Phi}_q(\curlywedge)((q^2v_2) \otimes ([2]v_2) + 0 \otimes v_1) = [2]q^2\mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_2) = 0 \\
&= F \cdot [2]q^{-1}v_2 = F \cdot \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_1), \\
\mathbf{\Phi}_q(\curlywedge)(F \cdot v_2 \otimes v_2) &= \mathbf{\Phi}_q(\curlywedge)((q^2v_2) \otimes 0 + 0 \otimes v_2) = \mathbf{\Phi}_q(\curlywedge)(0) = 0 = F \cdot 0 \\
&= F \cdot \mathbf{\Phi}_q(\curlywedge)(v_2 \otimes v_2).
\end{aligned}$$

Thus $\mathbf{\Phi}_q(\curlywedge)$ is a $U_q(\mathfrak{sl}_2)$-module homomorphism.

Next, we check that $\mathbf{\Phi}_q(\bigcap)$ is a $U_q(\mathfrak{sl}_2)$-module homomorphism. First, the only pure tensors that $\mathbf{\Phi}_q(\bigcap)$ does not send to zero are $v_0 \otimes v_2, v_1 \otimes v_1$, and $v_2 \otimes v_0$, which all have weight $\epsilon(K) = \epsilon(K^{-1}) = 1$. Thus, the action of $K$ commutes with $\mathbf{\Phi}_q(\bigcap)$. Now we check this fact for $E$ and $F$. Note that $\epsilon(E) = \epsilon(F) = 0$ so it suffices to show that the cap sends anything acted upon by $E$ or $F$ to 0. Furthermore, for any $i, j \in \{0, 1, 2\}$, $E \cdot v_i \otimes v_j$ will be sent to zero by the cap unless it has weight 1, and the same is true for $F \cdot v_i \otimes v_j$. Thus for $E$, it suffices to check $\mathbf{\Phi}_q(\bigcap)(E \cdot v_i \otimes v_j) = 0$ when $v_i \otimes v_j$ has weight $q^{-2}$ and for $F$, it suffices to check $\mathbf{\Phi}_q(\bigcap)(F \cdot v_i \otimes v_j) = 0$ when $v_i \otimes v_j$ has weight $q^2$.

$$\begin{aligned}
\mathbf{\Phi}_q(\bigcap)(E \cdot v_1 \otimes v_2) &= \mathbf{\Phi}_q(\bigcap)(v_1 \otimes v_1 + (-[2]v_0) \otimes (q^{-2}v_2)) \\
&= \mathbf{\Phi}_q(\bigcap)(v_1 \otimes v_1 - [2]q^{-2}v_0 \otimes v_2) = [2]^2 - [2]q^{-2}([2]q^2) = [2]^2 - [2]^2 \\
&= 0, \\
\mathbf{\Phi}_q(\bigcap)(E \cdot v_2 \otimes v_1) &= \mathbf{\Phi}_q(\bigcap)(v_2 \otimes (-[2]v_0) + v_1 \otimes v_1) = \mathbf{\Phi}_q(\bigcap)(-[2]v_2 \otimes v_0 + v_1 \otimes v_1) \\
&= -[2][2] + [2]^2 = 0, \\
\mathbf{\Phi}_q(\bigcap)(F \cdot v_0 \otimes v_1) &= \mathbf{\Phi}_q(\bigcap)((q^{-2}v_0) \otimes ([2]v_2) + (-v_1) \otimes v_1) \\
&= \mathbf{\Phi}_q(\bigcap)([2]q^{-2}v_0 \otimes v_2 - v_1 \otimes v_1) = [2]q^{-2}([2]q^2) - [2]^2 = [2]^2 - [2]^2
\end{aligned}$$

$$
\begin{aligned}
&= 0,\\
\mathbf{\Phi}_q\left(\bigcap\right)(F \cdot v_1 \otimes v_0) &= \mathbf{\Phi}_q\left(\bigcap\right)(v_1 \otimes (-v_1) + ([2]v_2) \otimes v_0)\\
&= \mathbf{\Phi}_q\left(\bigcap\right)(-v_1 \otimes v_1 + [2]v_2 \otimes v_0) = -[2]^2 + [2][2] = 0.
\end{aligned}
$$

Thus $\mathbf{\Phi}_q\left(\bigcap\right)$ is a $U_q(\mathfrak{sl}_2)$-module homomorphism as claimed.

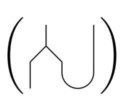
Lastly, $\mathbf{\Phi}_q\left(\bigcup\right)(1)$ is a weight $\epsilon(K) = \epsilon(K^{-1}) = 1$ vector, thus $\mathbf{\Phi}_q\left(\bigcup\right)$ commutes with the action of $K$ and $K^{-1}$. To show it commutes with the action of $E$ and $F$ as well, we must show $E \cdot \mathbf{\Phi}_q\left(\bigcup\right)(1) = 0$ and $F \cdot \mathbf{\Phi}_q\left(\bigcup\right)(1) = 0$.

$$
\begin{aligned}
E \cdot \mathbf{\Phi}_q\left(\bigcup\right)(1) &= E \cdot \left(\frac{1}{[2]^2} v_1 \otimes v_1 + \frac{1}{[2]} v_0 \otimes v_2 + \frac{q^{-2}}{[2]} v_2 \otimes v_0\right)\\
&= \frac{1}{[2]^2} E \cdot v_1 \otimes v_1 + \frac{1}{[2]} E \cdot v_0 \otimes v_2 + \frac{q^{-2}}{[2]} E \cdot v_2 \otimes v_0\\
&= \frac{1}{[2]^2}(v_1 \otimes (-[2]v_0) + (-[2]v_0) \otimes v_1) + \frac{1}{[2]}(v_0 \otimes v_1 + 0 \otimes (q^{-2}v_2))\\
&+ \frac{q^{-2}}{[2]}(v_2 \otimes 0 + v_1 \otimes (q^2 v_0))\\
&= -\frac{1}{[2]} v_1 \otimes v_0 - \frac{1}{[2]} v_0 \otimes v_1 + \frac{1}{[2]} v_0 \otimes v_1 + \frac{1}{[2]} v_1 \otimes v_0 = 0,\\
F \cdot \mathbf{\Phi}_q\left(\bigcup\right)(1) &= F \cdot \left(\frac{1}{[2]^2} v_1 \otimes v_1 + \frac{1}{[2]} v_0 \otimes v_2 + \frac{q^{-2}}{[2]} v_2 \otimes v_0\right)\\
&= \frac{1}{[2]^2} F \cdot v_1 \otimes v_1 + \frac{1}{[2]} F \cdot v_0 \otimes v_2 + \frac{q^{-2}}{[2]} F \cdot v_2 \otimes v_0\\
&= \frac{1}{[2]^2}(v_1 \otimes v_2 + v_2 \otimes v_1) + \frac{1}{[2]}((q^{-2}v_0) \otimes 0 + (-v_1) \otimes v_2)\\
&+ \frac{q^{-2}}{[2]}((q^2 v_2) \otimes (-v_1) + 0 \otimes v_0)\\
&= \frac{1}{[2]^2} v_1 \otimes v_2 + \frac{1}{[2]^2} v_2 \otimes v_1 - \frac{1}{[2]} v_1 \otimes v_2 - \frac{1}{[2]} v_2 \otimes v_1 = 0.
\end{aligned}
$$

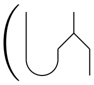
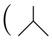
Hence $\mathbf{\Phi}_q\left(\bigcup\right)$ is a $U_q(\mathfrak{sl}_2)$-module homomorphism.

Now we must verify that $\mathbf{\Phi}_q$ preserves all the defining relations of $\mathcal{C}(q^2 + 2 + q^{-2})$. To simplify the remaining calculations, we use the fact that $\{v_0\}$ generates the $U_q(\mathfrak{sl}_2)$-module $W$ and $\{v_1 \otimes v_0, v_1 \otimes v_1, v_1 \otimes v_2\}$ generates the $U_q(\mathfrak{sl}_2)$-module $W \otimes W$.

(1) $\mathbf{\Phi}_q$ preserves the (Cup-Merge) relation.

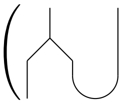
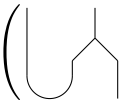
$\mathbf{\Phi}_q$ 

$$
\begin{aligned}
(v_0) &= (\mathbf{\Phi}_q(\curlywedge) \otimes \mathrm{Id}_V)\left(\frac{1}{[2]^2} v_0 \otimes v_1 \otimes v_1 + \frac{1}{[2]} v_0 \otimes v_0 \otimes v_2 + \frac{q^{-2}}{[2]} v_0 \otimes v_2 \otimes v_0\right)\\
&= \frac{1}{[2]^2}(-[2]qv_0) \otimes v_1 + \frac{q^{-2}}{[2]} 0 \otimes v_0 + \frac{q^{-2}}{[2]}(qv_1) \otimes v_0\\
&= \frac{q^{-1}}{[2]} v_1 \otimes v_0 - \frac{q}{[2]} v_0 \otimes v_1,
\end{aligned}
$$

$\mathbf{\Phi}_q$ (  ) $(v_0) = (\mathrm{Id}_V \otimes \mathbf{\Phi}_q$ (  )) $\left(\frac{1}{[2]^2} v_1 \otimes v_1 \otimes v_0 + \frac{1}{[2]} v_0 \otimes v_2 \otimes v_0 + \frac{q^{-2}}{[2]} v_2 \otimes v_0 \otimes v_0\right)$

$$= \frac{1}{[2]^2} v_1 \otimes ([2]q^{-1}v_0) + \frac{1}{[2]} v_0 \otimes (-qv_1) + \frac{q^{-2}}{[2]} v_2 \otimes 0$$
$$= \frac{q^{-1}}{[2]} v_1 \otimes v_0 - \frac{q}{[2]} v_0 \otimes v_1.$$

Consequently, we have

$\mathbf{\Phi}_q$ (  ) $= \mathbf{\Phi}_q$ (  ).

From the computation above, we know that $\mathbf{\Phi}_q$ (  ) acts on $W$ in the following way:

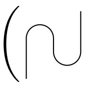
$\mathbf{\Phi}_q$ (  ) $(v_0) = \frac{q^{-1}}{[2]} v_1 \otimes v_0 - \frac{q}{[2]} v_0 \otimes v_1,$

Additionally, one can show that:

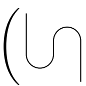
$\mathbf{\Phi}_q$ (  ) $(v_1) = \frac{q^{-1} - q}{[2]} v_1 \otimes v_1 + q^{-1} v_0 \otimes v_2 - q^{-1} v_2 \otimes v_0,$

$\mathbf{\Phi}_q$ (  ) $(v_2) = \frac{q^{-1}}{[2]} v_2 \otimes v_1 - \frac{q}{[2]} v_1 \otimes v_2.$

(2) $\mathbf{\Phi}_q$ preserves the (ZigZag) relation.

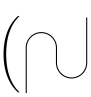
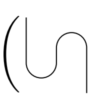
$\mathbf{\Phi}_q$ (  ) $(v_0) = (\mathbf{\Phi}_q$ (  ) $\otimes \mathrm{Id}_V) \left(\frac{1}{[2]^2} v_0 \otimes v_1 \otimes v_1 + \frac{1}{[2]} v_0 \otimes v_0 \otimes v_2 + \frac{q^{-2}}{[2]} v_0 \otimes v_2 \otimes v_0\right)$

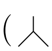
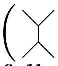
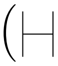
$$= \frac{1}{[2]^2}(0)v_1 + \frac{1}{[2]}(0)v_2 + \frac{q^{-2}}{[2]}([2]q^2)v_0 = v_0,$$

$\mathbf{\Phi}_q$ (  ) $(v_0) = (\mathrm{Id}_V \otimes \mathbf{\Phi}_q$ (  )) $\left(\frac{1}{[2]^2} v_1 \otimes v_1 \otimes v_0 + \frac{1}{[2]} v_0 \otimes v_2 \otimes v_0 + \frac{q^{-2}}{[2]} v_2 \otimes v_0 \otimes v_0\right)$

$$= \frac{1}{[2]^2} v_1(0) + \frac{1}{[2]} v_0([2]) + \frac{q^{-2}}{[2]} v_2(0) = v_0.$$

Consequently, we have

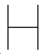
$\mathbf{\Phi}_q$ (  ) $= \mathbf{\Phi}_q$ (  ) $= \mathbf{\Phi}_q$ (  ).

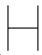
Before moving on, note that from the actions of $\mathbf{\Phi}_q$ (  ) and $\mathbf{\Phi}_q$ (  ) we can compute the actions of $\mathbf{\Phi}_q$ (  ) and $\mathbf{\Phi}_q$ (  ) on the generating set $\{v_1 \otimes v_0, v_1 \otimes v_1, v_1 \otimes v_2\}$ of $W \otimes W$ to be the following

$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_0) = q^{-2} v_1 \otimes v_0 - v_0 \otimes v_1,$

$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_1) = (q^{-1} - q)^2 v_1 \otimes v_1 + [2](1 - q^{-2})(v_2 \otimes v_0 - v_0 \otimes v_2),$

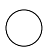
$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_2) = q^2 v_1 \otimes v_2 - v_2 \otimes v_1.$

$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_0) = (q^{-2} - 1) v_1 \otimes v_0 - v_0 \otimes v_1,$

$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_1) = (q^{-1} - q)^2 v_1 \otimes v_1 + [2] v_2 \otimes v_0 + [2] q^{-2} v_0 \otimes v_2,$

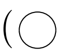
$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_2) = (q^2 - 1) v_1 \otimes v_2 - v_2 \otimes v_1.$

(3) $\mathbf{\Phi}_q$ preserves the (Bubble) relation.

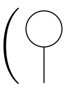
$\mathbf{\Phi}_q$ (  ) $(1) = \mathbf{\Phi}_q$ (  ) $\left( \frac{1}{[2]^2} v_1 \otimes v_1 - \frac{1}{[2]} v_0 \otimes v_2 - \frac{q^{-2}}{[2]} v_2 \otimes v_0 \right)$

$$= \frac{1}{[2]^2}([2]^2) + \frac{1}{[2]}([2]q^2) + \frac{q^{-2}}{[2]}([2])$$
$$= q^2 + 1 + q^{-2} = [3] = \mathbf{\Phi}_q([3])(1).$$

Consequently, we have:

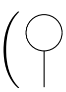
$\mathbf{\Phi}_q$ (  ) $= \mathbf{\Phi}_q([3]).$

(4) $\mathbf{\Phi}_q$ preserves the (Lollipop) relation.

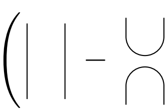
$\mathbf{\Phi}_q$ (  ) $(v_0) = \mathbf{\Phi}_q$ (  ) $\left( \frac{q^{-1}}{[2]} v_1 \otimes v_0 - \frac{q}{[2]} v_0 \otimes v_1 \right)$

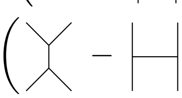
$$= \frac{q^{-1}}{[2]} 0 - \frac{q}{[2]} 0 = 0.$$

Consequently, we have:

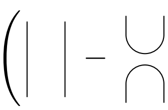
$\mathbf{\Phi}_q$ (  ) $= \mathbf{\Phi}_q(0).$

(5) Next, we show that $\mathbf{\Phi}_q$ preserves the (Chromatic) relation.

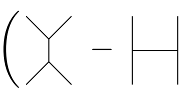
$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_0) = v_1 \otimes v_0 - 0 = v_1 \otimes v_0,$

$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_0) = (q^{-2} v_1 \otimes v_0 - v_0 \otimes v_1) - \left((q^{-2} - 1) v_1 \otimes v_0 - v_0 \otimes v_1\right)$

$$= v_1 \otimes v_0,$$

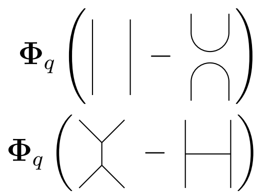
$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_1) = v_1 \otimes v_1 - [2]^2 \left( \frac{1}{[2]^2} v_1 \otimes v_1 + \frac{1}{[2]} v_0 \otimes v_2 + \frac{q^{-2}}{[2]} v_2 \otimes v_0 \right),$

$$= v_1 \otimes v_1 - v_1 \otimes v_1 - [2] v_0 \otimes v_2 - q^{-2}[2] v_2 \otimes v_0$$
$$= -[2] v_0 \otimes v_2 - [2] q^{-2} v_2 \otimes v_0,$$

$\mathbf{\Phi}_q$ (  ) $(v_1 \otimes v_1) = \left((q^{-1} - q)^2 v_1 \otimes v_1 + [2](1 - q^{-2})(v_2 \otimes v_0 - v_0 \otimes v_2)\right)$

$$- \left((q^{-1} - q)^2 v_1 \otimes v_1 + [2] v_2 \otimes v_0 + [2] q^{-2} v_0 \otimes v_2\right)$$

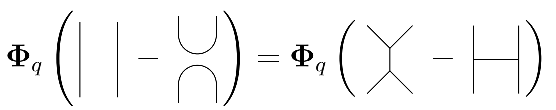
$$\begin{aligned}
&= [2](1-q^{-2})v_2 \otimes v_0 - [2](1-q^{-2})v_0 \otimes v_2 \\
&\quad - [2]v_2 \otimes v_0 - [2]q^{-2}v_0 \otimes v_2 \\
&= \; - [2]q^{-2}v_2 \otimes v_0 - [2]v_0 \otimes v_2 \\
&= \; - [2]v_0 \otimes v_2 - [2]q^{-2}v_2 \otimes v_0,
\end{aligned}$$



$$(v_1 \otimes v_2) = v_1 \otimes v_2 - 0 = v_1 \otimes v_2,$$

$$(v_1 \otimes v_2) = \big(q^2 v_1 \otimes v_2 - v_2 \otimes v_1\big) - \big((q^2-1)v_1 \otimes v_2 - v_2 \otimes v_1\big) = v_1 \otimes v_2.$$

Consequently, we have:



□

## 7.3 A functor from $\mathcal{C}(\delta^2)$ to $\mathrm{Kar}(\mathcal{TL}(\delta))$

In this section we show that there is a monoidal functor from $\mathcal{TL}(\delta)$ to $\mathcal{C}(\delta)$, and from $\mathcal{C}(\delta^2)$ to $\mathrm{Kar}(\mathcal{TL}(\delta))$ (recall for a category $\mathcal{C}$, $\mathrm{Kar}(\mathcal{C})$ is the idempotent completion of $\mathcal{C}$, defined in Definition 5.1.10). Before defining this monoidal functor, we recall a few other definitions and facts that will be helpful.

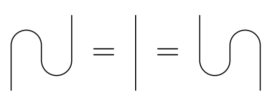
Recall that the Temperley–Lieb category $\mathcal{TL}(\delta)$ is the strict $\Bbbk$-linear monoidal category generated by one object $X$ and the two generating morphisms

$$\cup : \mathbb{1} \to X \otimes X \quad \text{and} \quad \cap : X \otimes X \to \mathbb{1}$$

subject to the relations:

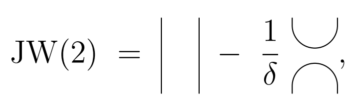
$$\bigcirc \; = \; \delta,$$

and

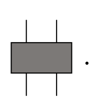


**Remark 7.3.1.** Given a parameter $d$, there is an obvious functor from $\mathcal{TL}(d)$ to $\mathcal{C}(d+1)$ that sends the cup in $\mathcal{TL}(d)$ to the cup in $\mathcal{C}(d+1)$ and the cap in $\mathcal{TL}(d)$ to the cap in $\mathcal{C}(d+1)$.

Recall that the 2[nd] Jones–Wenzl idempotent $\mathrm{JW}(2) \in \mathrm{End}_{\mathcal{TL}(\delta)}(X \otimes X)$ was defined to be



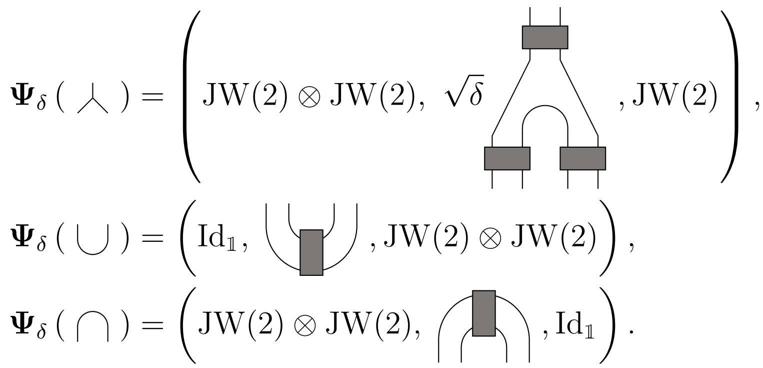
and will denote JW(2) by the following in our diagrams:



Also note that since $\mathcal{TL}(\delta)$ is a pivotal category (see Remark 5.4.3), and its morphisms are invariant under planar isotopy by Remark 5.3.6.

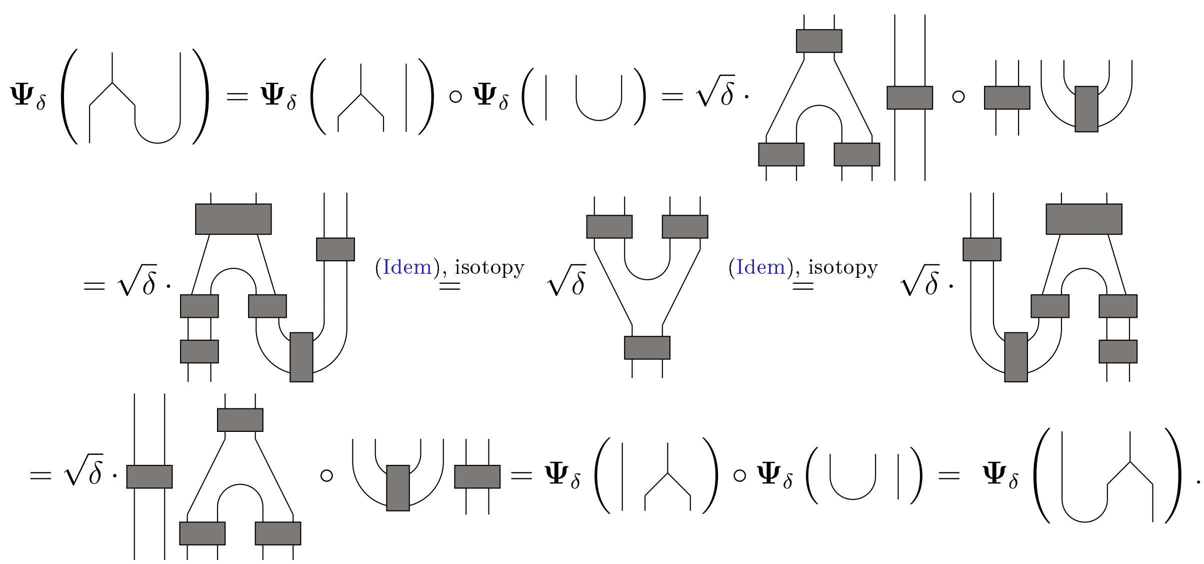
**Theorem 7.3.2.** *There is a monoidal functor from* $\mathbf{\Psi}_\delta \colon C(\delta^2) \to \mathrm{Kar}(\mathcal{TL}(\delta))$ *such that:*

$$\mathbf{\Psi}_\delta(X) = (X \otimes X, \mathrm{JW}(2)),$$

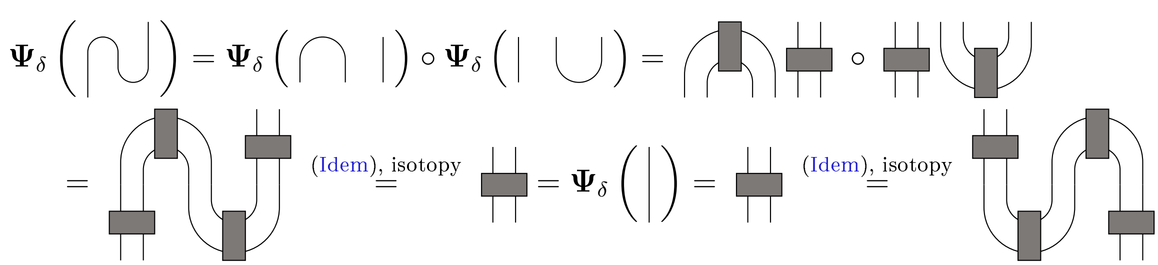


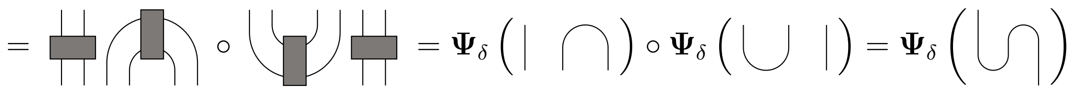

**Proof.** We proceed to show that $\mathbf{\Psi}_\delta$ respects all the relations of $\mathcal{C}(\delta^2)$. To simplify the computations below, we omit the idempotents in the triples above and just write the morphism. The idempotents in question will always be tensor powers of the 2nd Jones–Wenzl idempotents, so there should be no confusion.

(1) $\mathbf{\Psi}_\delta$ respects the (Cup-Merge) relation:

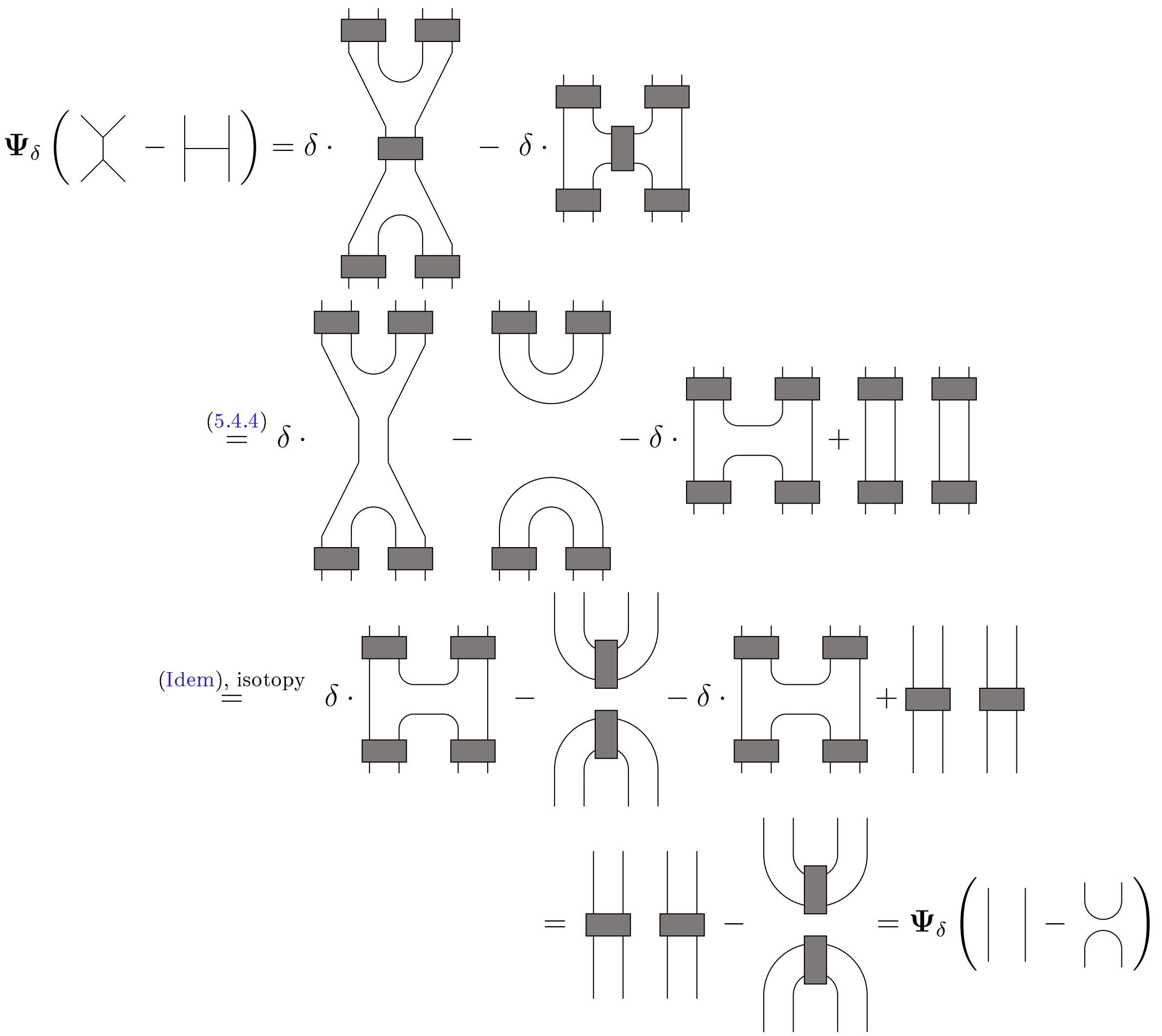



(2) $\mathbf{\Psi}_\delta$ respects the (ZigZag) relation:

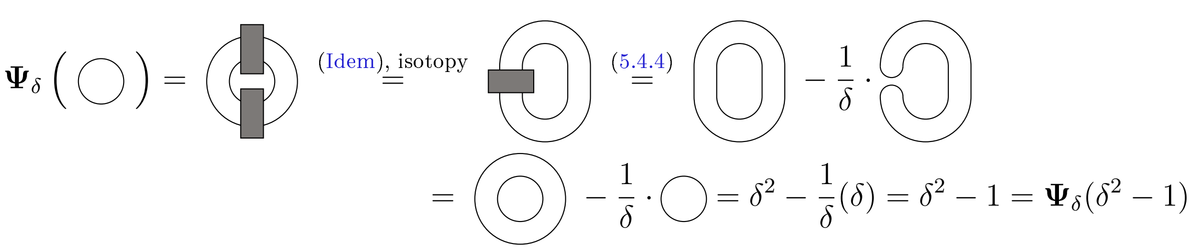




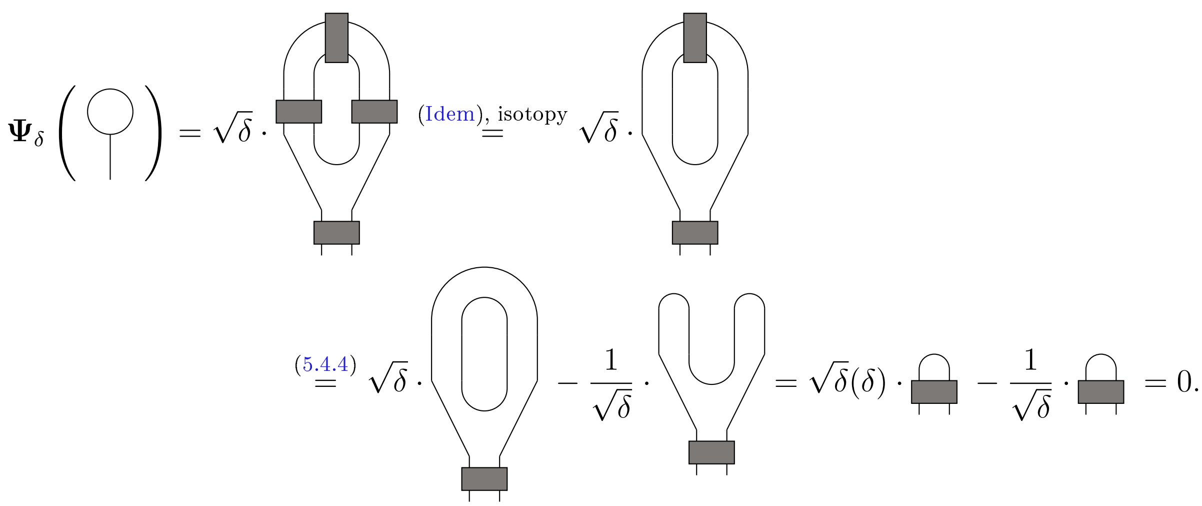

(3) $\mathbf{\Psi}_\delta$ respects the (Chromatic) relation:



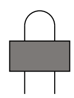

(4) $\mathbf{\Psi}_\delta$ respects the (Bubble) relation:




(5) $\mathbf{\Psi}_\delta$ respects the (Lollipop) relation:




The last equality follows since

 $= \;0.$ ∎

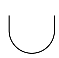
## 7.4 A functor from $\mathcal{KS}(q)$ to $\mathcal{P}(\delta)$

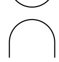
In this section, we define a functor from the Kauffman skein category to the planar chromatic category $\mathcal{P}(\delta)$. This functor, combined with the isomorphism in Corollary 6.2.17, yields a functor from the Kauffman skein category to the chromatic category $\mathcal{C}(\delta)$.

**Definition 7.4.1.** Let $\Bbbk$ be a field and $q \in \Bbbk$. Define the *Kauffman skein category* $\mathcal{KS}(q)$ to be the strict $\Bbbk$-linear monoidal category with one generating object $X$ and generating morphisms

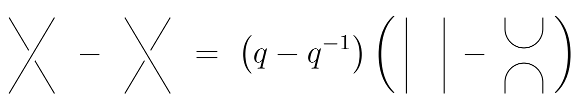
 $: X \otimes X \to X \otimes X,$

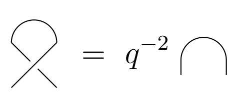
 $: X \otimes X \to X \otimes X,$

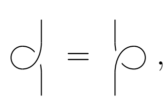
 $: X \otimes X \to \mathbb{1},$

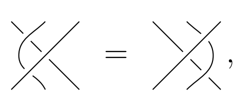
 $: \mathbb{1} \to X \otimes X.$

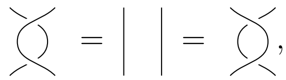
Then we impose the following relations:

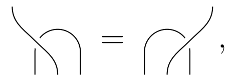
 (KS1)

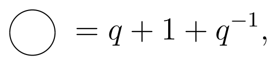
 (KS2)

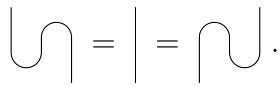
 (KS3)

 (KS4)


(KS5)

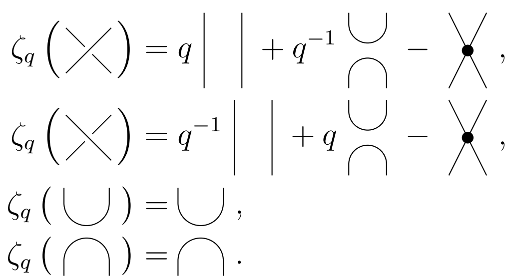

(KS6)


(KS7)

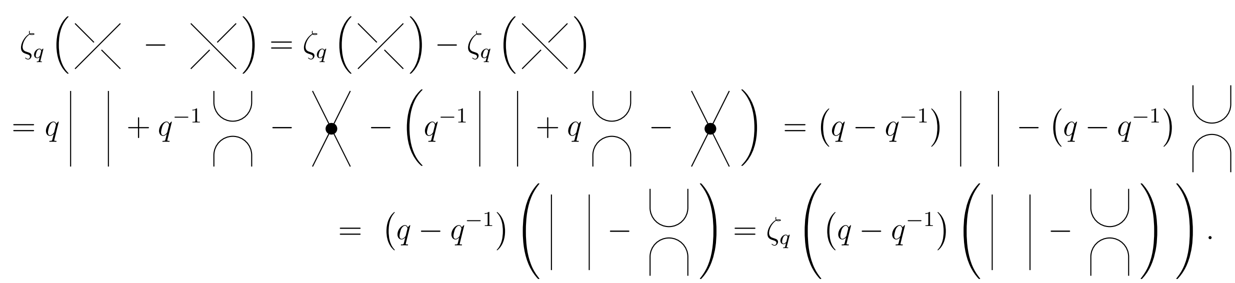

(KS8)

**Remark 7.4.2.** The category $\mathcal{KS}(q)$ belongs to a family of categories $\mathcal{KS}_\epsilon(D; z, t, \delta)$ (see [MS21, Section 8.3]) introduced in [Tur89], where in our case $\epsilon = -1, z = q - q^{-1}, t = q^2$, and $\delta = q + 1 + q^{-1}$.

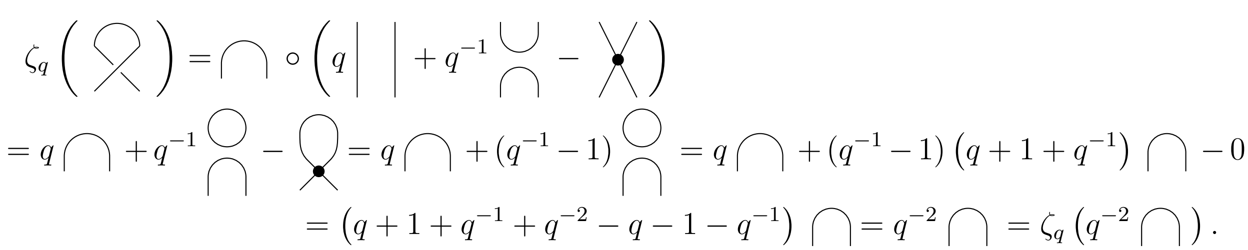
**Theorem 7.4.3.** *There is a monoidal functor* $\zeta_q \colon \mathcal{KS}(q) \to \mathcal{P}(q + 2 + q^{-1})$*, defined by:*



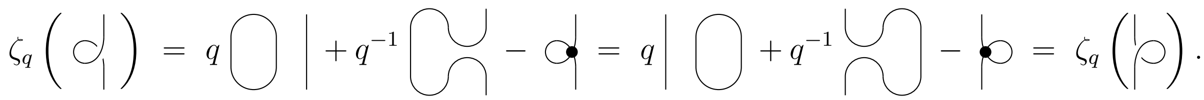
**Proof.** We must check that $\zeta_q$ respects the defining relations of $\mathcal{KS}(q)$. It is clear that it respects all relations involving only cups and caps (that is, the relations (KS7) and (KS8)). Thus we need only show $\zeta_q$ respects the first 6 relations of $\mathcal{KS}(q)$.
(1) $\zeta_q$ respects the relation (KS1) of $\mathcal{KS}(q)$:



(2) $\zeta_q$ respects the relation (KS2) of $\mathcal{KS}(q)$:



(3) $\zeta_q$ respects the relation (KS3) of $\mathcal{KS}(q)$:

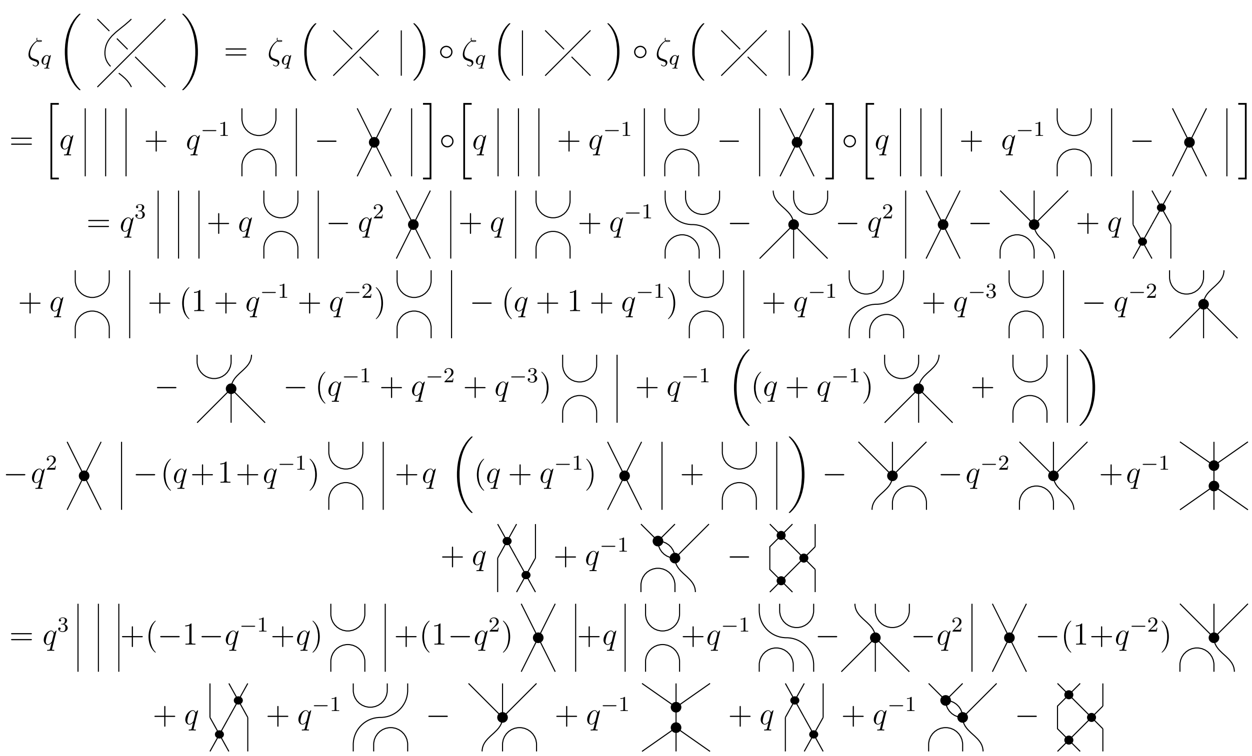


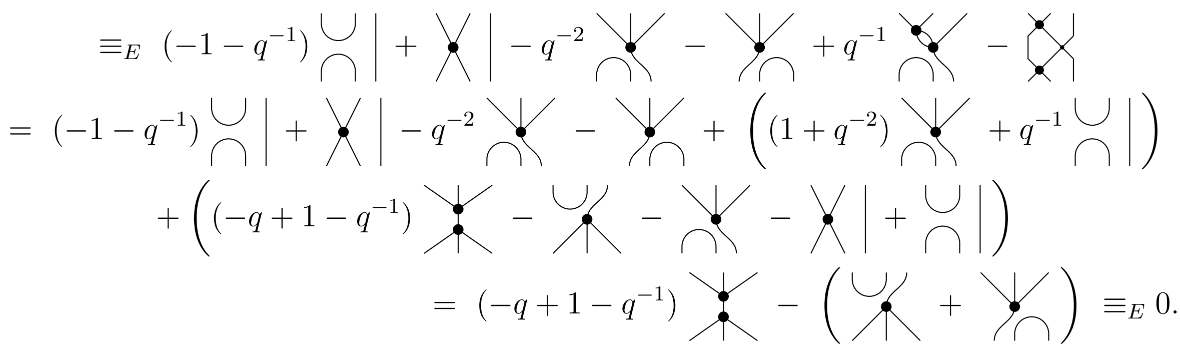
Let $\delta = q + 2 + q^{-1}$. Before moving on, note that $\mathcal{KS}(q)$ and $\mathcal{P}(\delta)$ are both pivotal categories. Thus a 180 degree rotation of diagrams on each category yields functors $\mathcal{KS}(q) \to \mathcal{KS}(q)^{\mathrm{op}}$ and $\mathcal{P}(\delta) \to \mathcal{P}(\delta)^{\mathrm{op}}$. Let $n$ and $m$ be non-negative integers. From the rotation functors on each category we have linear maps $R_1\colon \mathrm{Hom}_{\mathcal{KS}(q)}(X^{\otimes n}, X^{\otimes m}) \to \mathrm{Hom}_{\mathcal{KS}(q)}(X^{\otimes m}, X^{\otimes n})$ and $R_2\colon \mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m}) \to \mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes m}, X^{\otimes n})$. Then, since $\zeta_q$ is monoidal by definition, the following diagram must commute:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{KS}(q)}(X^{\otimes m}, X^{\otimes n}) & \xrightarrow{R_1} & \mathrm{Hom}_{\mathcal{KS}(q)}(X^{\otimes n}, X^{\otimes m}) \\
\zeta_q \big\downarrow & & \big\downarrow \zeta_q \\
\mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes n}, X^{\otimes m}) & \xrightarrow{R_2} & \mathrm{Hom}_{\mathcal{P}(\delta)}(X^{\otimes m}, X^{\otimes n})
\end{array}$$

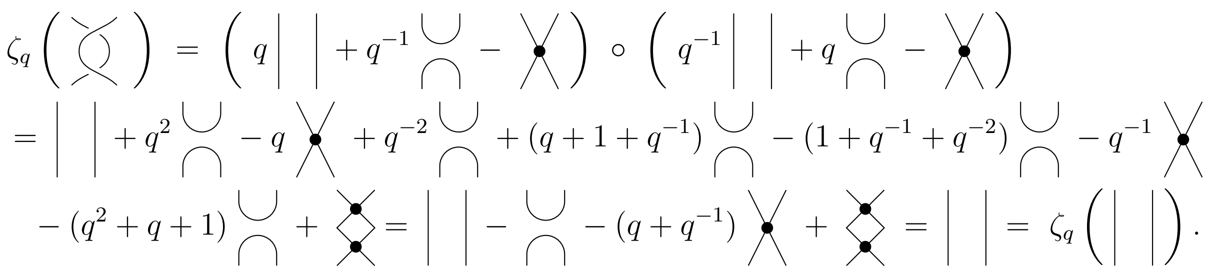
We make use of this fact in showing that $\zeta_q$ respects relations (KS4) and (KS5).
(4) $\zeta_q$ respects the relation (KS4) of $\mathcal{KS}(q)$:

Notice that the right side of (KS4) can be obtained by a 180 degree rotation of the left side (and vice-versa). Consequently, to show they are equal, we need only show that the left side of (KS4) is invariant under 180 degree rotation. Let $E$ be the subspace of $\mathcal{P}(\delta)(X^{\otimes 3}, X^{\otimes 3})$ consisting of morphisms that are invariant under 180 degree rotation, and let $\equiv_E$ denote equivalence modulo $E$. It follows that:



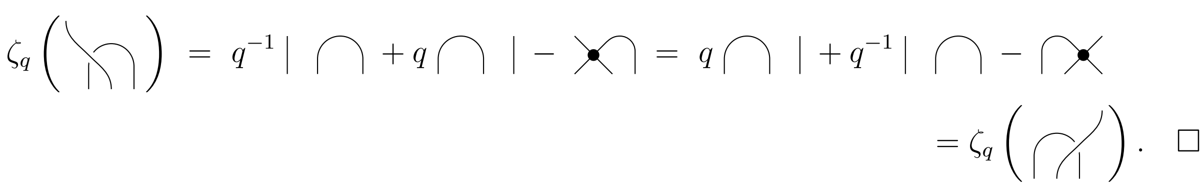


(5) $\zeta_q$ respects the relation (KS5) of $\mathcal{KS}(q)$:

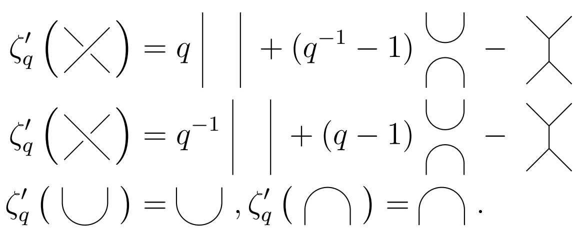


The the 2nd relation in (KS5) is true since it is the same as the first equality but rotated 180 degrees.

(6) $\zeta_q$ respects the relation (KS6) of $\mathcal{KS}(q)$:



**Corollary 7.4.4.** *There is a monoidal functor* $\zeta'_q \colon \mathcal{KS}(q) \to \mathcal{C}(q+2+q^{-1})$ *defined by:*



**Proof.** From Corollary 6.2.17, we have a functor $\mathbf{f}\colon \mathcal{C}(q+2+q^{-1}) \to \mathcal{P}(q+2+q^{-1})$. Then, the monoidal functor $\zeta'_q$ above is $\zeta'_q = \mathbf{f}^{-1} \circ \zeta_q \colon \mathcal{KS}(q) \to \mathcal{C}(q+2+q^{-1})$. □

# Chapter 8

# Consequences of the basis theorem and future work

In this chapter, we compute the dimension of the morphism spaces of $\mathcal{C}(\delta)$ and discuss various consequences of Theorem 6.2.15. This theorem gives us a basis for the morphism spaces of the trivalent chromatic category $\mathcal{TP}(\delta)$, and thus a basis for the morphism spaces of $\mathcal{C}(\delta)$ can be obtained from this via the isomorphism in Proposition 6.2.8. We finish this final chapter by indicating some possible future work.

## 8.1 Dimension of $\mathrm{Hom}_{\mathcal{C}(\delta)}(X^{\otimes n}, X^{\otimes m})$

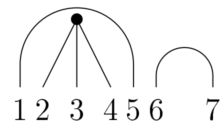
We begin this chapter by computing the dimension $\mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X^{\otimes m})$.

Recall that a partition $\mathcal{S}$ of $M \subseteq \mathbb{N}$ is said to have a *crossing* if for some $A, B \in \mathcal{S}$ with $A \neq B$, there exists $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $a_1 < b_1 < a_2 < b_2$. If a partition has no crossings, we call it *non-crossing*. Also recall that we defined the $n^{\text{th}}$ Riordan number $R_n$ to be the number of non-crossing partitions of the set $\{1, 2, \cdots, n\}$.

**Proposition 8.1.1.** *Let $n$ and $m$ be non-negative integers. Then the dimension of the morphism space $\mathrm{Hom}_{C(\delta)}(X^n, X^m)$ is the Riordan number $R_{n+m}$.*

**Proof.** Since $\mathcal{C}(\delta)$ is pivotal, it suffices to prove this in the case that $m = 0$. We will show that there is a bijection between the basis for $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^n, 1)$ appearing in Lemma 6.2.11 and the set of non-crossing partitions of size $n$ without singletons. Then, by Corollary 6.2.17, we know that $\dim \mathrm{Hom}_{\mathcal{C}(\delta)}(X^n, 1) = \dim \mathrm{Hom}_{\mathcal{P}(\delta)}(X^n, 1)$. Thus, $R_n$ will be the dimension of $\mathrm{Hom}_{\mathcal{C}(\delta)}(X^n, 1)$ as well.

Given an element of the basis $B := \overline{B_{n,0}}$ of $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^n, 1)$ described in Lemma 6.2.11, we associate to it a non-crossing partition as follows: first, label the points at the bottom of the diagram with the numbers $1, 2, \cdots, n$ in order. Then, we form a partition of $[n]$ by putting numbers in the same part if they are in the same connected component of the diagram. For example, to the diagram




we would associate the partition $\{\{1,5\},\{2,3,4\},\{6,7\}\}$. Since our diagrams have no 1-valent vertices and no crossings, it is clear that this construction yields a non-crossing partition with no singletons.

If we start with a non-crossing partition of size $n$ with no singletons, then we construct an element of $B$ as follows. First, draw $n$ points on a horizontal line, and label them with the numbers $1, 2, \cdots, n$ in order. Then, draw a point $P_1$ above the point labeled with the number 1 and connect $P_1$ to every point labeled with a number in the part containing the number 1 with a straight line. Drawing these lines will split the open upper half plane into several connected components. We then repeat this process, by picking the smallest number $k$ with a line not connected to it, and drawing a point $P_k$ directly above it in the connected component directly above it, and then drawing a line from the point $P_k$ to every point labeled with a number in the same part as $k$. Since the given partition is non-crossing, there will be no crossings in the resulting diagram. Similarly, since there are no singletons in the partition, there will be no 1-valent vertices as well. Thus the resulting diagram will be an element of $B$. It is clear that this assignment of partitions to elements of $B$ and assignment of elements of $B$ to partitions are the inverses of each other, thus they yield a bijection between the basis $B$ of $\mathrm{Hom}_{\mathcal{P}(\delta)}(X^n, 1)$ and the non-crossing partitions of size $n$ without singletons. Consequently, by Corollary 6.2.17

$$\dim \mathrm{Hom}_{\mathcal{C}(\delta)}(X^n, 1) = \dim \mathrm{Hom}_{\mathcal{P}(\delta)}(X^n, 1) = R_n. \qquad \square$$

## 8.2 More consequences of the basis theorem

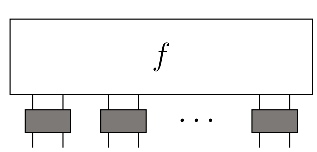
In this section, we show that the functors in Sections 7.1 to 7.3 satisfy some additional properties.

**Proposition 8.2.1.** *Let $\delta = q + q^{-1}$, where $q \in \mathbb{C}$ is not a root of unity. Then the functor $\mathbf{\Psi}_\delta$ in Theorem 7.3.2 is full. That is, for non-negative integers $n$ and $m$, the map given by*

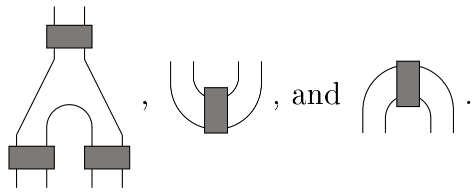
$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(\delta^2)}(X^n, X^m) &\to \mathrm{Hom}_{\mathrm{Kar}(\mathcal{TL}(\delta))}((X \otimes X, \mathrm{JW}(2))^{\otimes n}, (X \otimes X, \mathrm{JW}(2))^{\otimes m}),\\ \lambda &\mapsto \mathbf{\Psi}_\delta(\lambda),\end{aligned}$$

*is surjective.*

**Proof.** It suffices to show that for all integers $n \geq 0$, and all $f \in \mathrm{Hom}_{\mathcal{TL}(\delta)}(X^{\otimes 2n}, \mathbb{1})$, the diagram



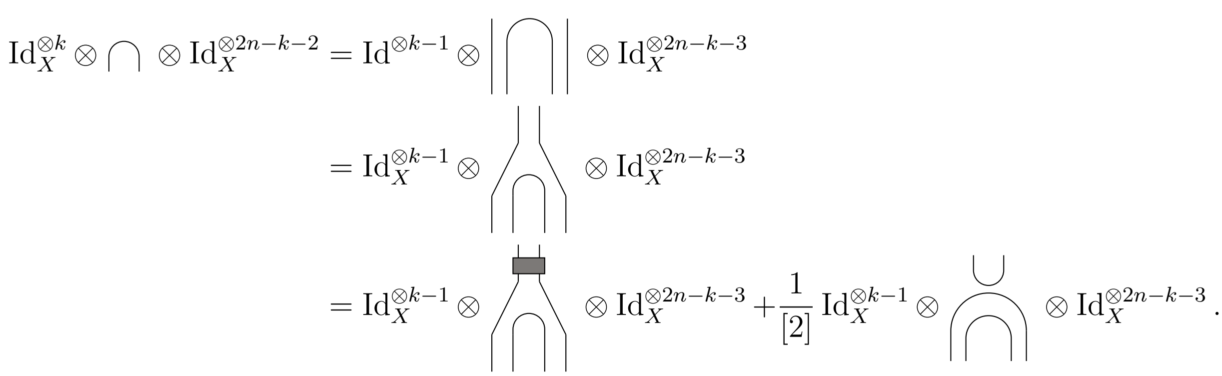
can be monoidally generated from the diagrams



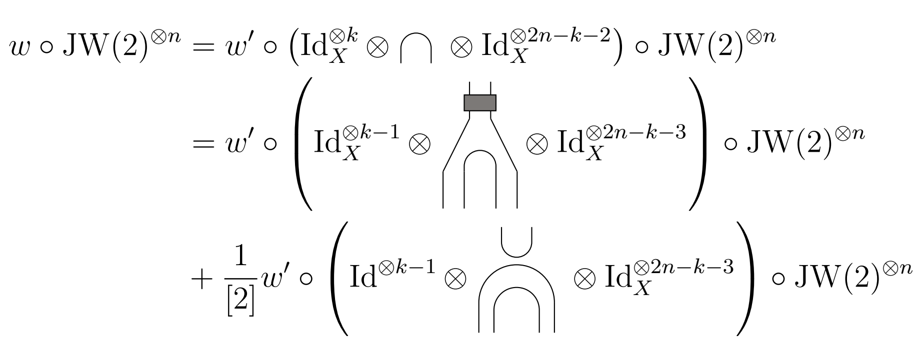

To this end, let $L_n$ be the set of diagrams of $\mathrm{Hom}_{\mathcal{TL}(\delta)}(X^{\otimes 2n}, \mathbb{1})$ with no loops (see Remark 5.2.6), and recall from Remark 5.4.3 that this is a basis for this space. Then, let $W_n$ be the subset of $L_n$ consisting of diagrams such that for each $k \in \{1, 2, \cdots, n\}$, points $2k-1$ and $2k$ at the bottom of the diagram do not belong to the same connected component. Equivalently, in any diagram in $W_n$, there is not a cap on the points $2k-1$ and $2k$ for $k = 1, 2, \cdots, n$. Then, let $\overline{W}_n = \{(\mathrm{JW}(2)^{\otimes n}, w \circ \mathrm{JW}(2)^{\otimes n}, \mathrm{Id}_{\mathbb{1}}) \mid w \in W_n\}$.

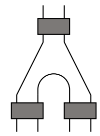
We know that $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{TL}(\delta))}((X^{\otimes 2n}, \mathrm{JW}(2)^{\otimes n}), (\mathbb{1}, \mathrm{Id}_{\mathbb{1}}))$ is spanned by $\{(\mathrm{JW}(2)^{\otimes n}, w \circ \mathrm{JW}(2)^{\otimes n}, \mathrm{Id}_{\mathbb{1}}) \mid w \in L_n\}$ since $L_n$ spans $\mathrm{Hom}_{\mathcal{TL}(\delta)}(X^{\otimes 2n}, \mathbb{1})$. However since $\cap \circ \mathrm{JW}(2) = 0$, any element of $L_n \setminus W_n$ when pre-composed with $\mathrm{JW}(2)^{\otimes n}$ will yield zero. Consequently, $\overline{W}_n$ also spans $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{TL}(\delta))}((X^{\otimes 2n}, \mathrm{JW}(2)^{\otimes n}), (\mathbb{1}, \mathrm{Id}_{\mathbb{1}}))$.

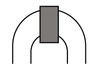
By induction on $n$, we can show that every element of $\overline{W}_n$ is contained in $\mathbf{\Psi}_\delta\left(\mathrm{Hom}_{\mathcal{C}(\delta^2)}(X^{\otimes n}, \mathbb{1})\right)$. The result is clear if $n \le 1$. Thus, take $w \in W_n$ with $n > 1$. Then, we can write $w = w' \circ \left(\mathrm{Id}_X^{\otimes k} \otimes \cap \otimes \mathrm{Id}_X^{\otimes 2n-k-2}\right)$ for some $k$ and some $w' \in W_{n-1}$. Since $w \in W_n$, $k$ must be odd. Therefore we can write:




Now, let $w'' = \frac{1}{[2]} w' \circ \left(\mathrm{Id}_X^{\otimes k-1} \otimes \cup \otimes \mathrm{Id}_X^{\otimes 2n-k-3}\right)$. Then from the above computation, we have that:



$= w' \circ \Big(\mathrm{JW}(2)^{\otimes \frac{k-1}{2}} \otimes$  $\otimes \mathrm{JW}(2)^{\otimes \frac{2n-k-3}{2}}\Big)$

$+ w'' \circ \Big(\mathrm{JW}(2)^{\otimes \frac{k-1}{2}} \otimes$  $\otimes \mathrm{JW}(2)^{\otimes \frac{2n-k-3}{2}}\Big).$

Since $w'$ and $w''$ start on fewer points than $w$, we know inductively that they can be generated using the 3 morphisms above. The result follows. □

**Corollary 8.2.2.** *The monoidal functor $\mathbf{\Psi}_\delta \colon \mathcal{C}(\delta^2) \to \mathrm{Kar}(\mathcal{TL}(\delta))$ is an isomorphism onto the full monoidal subcategory generated by $(X \otimes X, \mathrm{JW}(2))$ where $\delta = q + q^{-1}$ with $q \in \mathbb{C}^\times$ not a root of unity.*

**Proof.** Let $W_k$ be the $(k+1)$-dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$. Then there is an equivalence of categories between $\mathcal{TL}(q + q^{-1})$ and the full monoidal subcategory of $U_q(\mathfrak{sl}_2)$-mod generated by the $W_1$ that sends $X$ to $W_1$ [Kho97, Section 1.2.3]. Note that $W_1 \otimes W_1 \simeq W_0 \oplus W_2$. Through the equivalence above, $\mathrm{JW}(2)$ corresponds to the projection from $W_1 \otimes W_1$ onto $W_2$, followed by its inclusion into $W_1 \otimes W_1$. Consequently, from this we have:

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{Kar}(\mathcal{TL}(q))}&((X \otimes X, \mathrm{JW}(2))^{\otimes n}, (X \otimes X, \mathrm{JW}(2))^{\otimes m}) \\ &= \dim \mathrm{Hom}_{U_q(\mathfrak{sl}_2)\text{-mod}}(W_2^{\otimes n}, W_2^{\otimes m}) = R_{n+m}.\end{aligned}$$

Thus, we know that

$$\dim \mathrm{Hom}_{\mathcal{C}(\delta)}(X^n, X^m) = R_{n+m} = \dim \mathrm{Hom}_{\mathrm{Kar}(\mathcal{TL}(q))}((X \otimes X, \mathrm{JW}(2))^n, (X \otimes X, \mathrm{JW}(2))^m).$$

Thus we know $\mathbf{\Psi}_q$ is full and the dimensions of $\mathrm{Hom}_{\mathcal{C}(\delta)}(X^n, X^m)$ and $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{TL}(q))}((X \otimes X, \mathrm{JW}(2))^n, (X \otimes X, \mathrm{JW}(2))^m)$ match. From this we can conclude that $\mathbf{\Psi}_q$ is an isomorphism on each morphism space, completing the proof. □

**Proposition 8.2.3.** *The monoidal functors $\mathbf{\Phi} \colon \mathcal{C}(4) \to \mathfrak{sl}_2$-mod and $\mathbf{\Phi}_q \colon \mathcal{C}(q^2 + 2 + q^{-2}) \to U_q(\mathfrak{sl}_2)$-mod (where $q \in \mathbb{C}^\times$ is not a root of unity) are both full onto the full monoidal subcategories of $\mathfrak{sl}_2$-mod and $U_q(\mathfrak{sl}_2)$-mod generated by $V_2$ and $W_2$ respectively.*

**Proof.** Fix $q \in \mathbb{C}^\times$ that is not a root of unity. Recall that $W_k$ is the $(k+1)$-dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$ and $V_k$ is the $(k+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$.

Then, via the monoidal equivalence between $\mathcal{TL}(q + q^{-1})$ and the full monoidal subcategory of $U_q(\mathfrak{sl}_2)$-mod generated by $W_1$, we can obtain an equivalence between the full monoidal subcategory $\mathcal{A}_1$ of $\mathrm{Kar}(\mathcal{TL}(q + q^{-1}))$ generated by $(X \otimes X, \mathrm{JW}(2))$ and the full monoidal subcategory $\mathcal{A}_2$ of $U_q(\mathfrak{sl}_2)$-mod generated by $W_2$ that takes $S = (X \otimes X, \mathrm{JW}(2))$ to $W_2$.

By Proposition 8.2.1 we know that morphisms in $\mathcal{A}_1$ are monoidally generated by the unique (up to scaling) maps $S \otimes S \to S$, $\mathbb{1} \to S \otimes S$, and $S \otimes S \to \mathbb{1}$. Consequently,

$\mathcal{A}_2$ is generated by the unique (up to scaling) maps $W_2 \otimes W_2 \to W_2$, $\mathbb{1} \to W_2 \otimes W_2$, and $W_2 \otimes W_2 \to \mathbb{1}$. Since the image of $\mathbf{\Phi}_q$ contains these maps, $\mathbf{\Phi}_q$ is full.

Now, since $\mathfrak{sl}_2$-fdmod is monoidally equivalent to $U_q(\mathfrak{sl}_2)$-fdmod (with $V_k$ corresponding to $W_k$), we know that the full monoidal subcategory of $\mathfrak{sl}_2$-mod generated by $V_2$ is also generated by the unique (up to scaling) maps $V_2 \otimes V_2 \to V_2$, $\mathbb{1} \to V_2 \otimes V_2$, and $V_2 \otimes V_2 \to \mathbb{1}$. This shows that $\mathbf{\Phi}$ is full as well. □

**Corollary 8.2.4.** *The monoidal functors $\mathbf{\Phi}$ and $\mathbf{\Phi}_q$ (where $q \in \mathbb{C}^\times$ is not a root of unity) are both isomorphisms onto the full monoidal subcategories of $\mathfrak{sl}_2$-mod and $U_q(\mathfrak{sl}_2)$-mod generated by $V_2$ and $W_2$ respectively.*

**Proof.** Fix non-negative integers $n$ and $m$. Then by Proposition 8.2.3 we know $\mathbf{\Phi}$ yields a surjective linear map from

$$\mathrm{Hom}_{\mathcal{C}(4)}(X^n, X^m) \to \mathrm{Hom}_{\mathfrak{sl}_2\text{-mod}}(V_2^{\otimes n}, V_2^{\otimes m}).$$

Similarly, $\mathbf{\Phi}_q$ gives a surjective linear map

$$\mathrm{Hom}_{\mathcal{C}(q^2+2+q^{-2})}(X^n, X^m) \to \mathrm{Hom}_{U_q(\mathfrak{sl}_2)\text{-mod}}(W_2^{\otimes n}, W_2^{\otimes m}).$$

Then, we know each of these four morphism spaces have dimension $R_{n+m}$ (this is by Proposition 8.1.1, Theorem 4.2.5, and Theorem 4.3.10), the two maps above must be isomorphisms of vector spaces as required. □

In view of the results above, we have the following analogy between the chromatic category and the Temperley–Lieb category for $q \in \mathbb{C}^\times$ not a root of unity:

| Category | $\mathcal{C}(q^2+2+q^{-2})$ | $\mathcal{TL}(q+q^{-1})$ |
|---|---|---|
| Generated by... | cups, caps, and merges | cups and caps |
| Subcategory of $U_q(\mathfrak{sl}_2)$-mod... | generated by $W_2$ | generated by $W_1$ |
| $\dim(\mathrm{Hom}(X^{\otimes n}, X^{\otimes m}))$ | The Riordan number $R_{n+m}$ | The Catalan number $C_{n+m}$ |

## 8.3 Future work

We close this final chapter by discussing some possible future work related to the chromatic category.

First, one may recall that $W_1 \otimes W_1 \simeq W_0 \oplus W_2$. Under the equivalence of $\mathcal{TL}(q+q^{-1})$ with the category of $U_q(\mathfrak{sl}_2)$ representations monoidally generated by $W_1$, the Jones–Wenzl idempotent $\mathrm{JW}(2) \in \mathrm{End}_{\mathcal{TL}(q+q^{-1})}(X \otimes X)$ corresponds to the projection map $W_1 \otimes W_1 \to W_2$, followed by its inclusion back into $W_1 \otimes W_1$. Similarly, the $n^{th}$ *Jones–Wenzl idempotent* $\mathrm{JW}(n)$ corresponds to the projection map $W_1^{\otimes n} \to W_n$, followed by the inclusion of $W_n \to W_1^{\otimes n}$. One may wish study the analogous idempotents in the chromatic category. Through the isomorphism $\mathbf{\Phi}_q$, we can define a sequence of *chromatic idempotents* $CI(n) \in \mathrm{End}_{\mathcal{C}(q^2+2+q^{-2})}$

such that $\mathbf{\Phi}_q(CI(n))$ is the projection of $W_2^{\otimes n} \to W_{2n}$, followed by the inclusion of $W_{2n}$ into $W_2^{\otimes n}$. Fendley and Krushkal have studied these idempotents and have even given a recursive formula for them ([FK09, Lemma 5.3]). However, it would be very interesting to develop a more explicit formula for these idempotents, similar to what has been done with the Jones–Wenzl idempotents (such as in [Mor17]).

Another natural extension of this work would be to study some possible generalizations of the chromatic category. Although one builds the chromatic category using planar graphs, we could look at a similar category in which we allow crossings of edges so that morphisms linear combinations of arbitrary graphs instead. This category appears as the *flow category* in [AK19] and it has the *flow polynomial* as an invariant of the category. It would be interesting to study this category in more detail. Another interesting direction would be to investigate if there is a good diagrammatic description of the monoidal subcategory of $U_q(\mathfrak{sl}_2)$-mod generated by $W_k$ for other $k$ like there is for $k = 1$ and $k = 2$. For $k = 1$ this category is isomorphic to $\mathcal{TL}(q + q^{-1})$ and for $k = 2$ this category is isomorphic to $\mathcal{C}(q^2 + 2 + q^{-2})$. For $k = 3$, it should be possible but more complicated to give a nice diagrammatic presentation for this category of representations.

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