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Cycle and Closed Trail Decompositions of Complete Equipartite Graphs and Complete Multigraphs

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Cycle and closed trail decompositions of complete equipartite graphs and complete multigraphs

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Thesis Submitted to the Faculty of Graduate and Postdoctoral Studies
In partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics [1]

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Abstract

An H-decomposition of a graph G is a partition of the edges of G into subsets, each of which induces a graph isomorphic to H. In this thesis, we will study decompositions of certain graphs into cycles and closed trails. In particular, we consider decompositions of complete multigraphs and complete equipartite graphs. The complete multigraph λK_m is the multigraph with m vertices and λ edges between each pair of vertices. The complete equipartite graph $K_m * \overline{K_n}$ with m parts of size n is the simple graph on mn vertices, such that the vertices are partitioned into m parts of size n, and two vertices are adjacent if and only if they are in different parts.

With regard to closed trail decompositions, we find necessary and sufficient conditions for the existence of a decomposition of the complete equipartite graph $K_m * \overline{K_n}$ into closed trails of length k. We then turn our attention to cycle decompositions, and give necessary and sufficient conditions for the existence of a decomposition of the complete multigraph $3K_m$ into cycles of odd length $k \neq 9$. We then show how to use this result to construct k-cycle decompositions of $K_m * \overline{K_3}$ when $k = 9k'$, where $k' | \binom{m}{2}$ and $k' > 3$. In addition, we discuss other cases in which decompositions of $K_m * \overline{K_3}$ into k-cycles can be found.

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Chapter 1

Introduction

1.1 An introduction to cycle and closed trail decompositions

This thesis is concerned with decompositions of graphs into cycles and closed trails. We begin with the relevant notation and definitions. Any concepts not defined in the thesis are standard in graph theory or design theory; the interested reader is invited to consult the textbooks [14] and [53] for definitions of basic concepts.

Given a graph G, we will denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. We use the standard notation K_m to denote the complete graph on m vertices. The notation $K_m - I$ will refer to the complete graph on m vertices with the edges of a 1-factor removed. The complete directed graph on m vertices will be denoted by K_m^*.

Definition 1.1. A sequence $v_0 e_1 v_1 e_2 v_2 \cdots v_{k-1} e_k v_k$, where v_i is a vertex for each $i = 0, 1, \ldots, k$, and for each $j = 1, 2, \ldots, k$, e_j is an edge with end-vertices v_{j-1} and v_j, is called a *closed trail* if $v_k = v_0$ and the edges $e_1, e_2, \ldots, e_k$ are pairwise distinct; a *cycle* if $v_k = v_0$ and the vertices $v_0, v_1, \ldots, v_{k-1}$ are pairwise distinct; and a *path* if the vertices $v_0, v_1, \ldots, v_k$ are pairwise distinct. In simple graphs, the closed trail or cycle

$C = v_0e_1v_1 \cdots v_ke_{k-1}v_0$ will be referred to unambiguously by listing its vertices, and denoted by $C = v_0, v_1, \ldots, v_{k-1}, v_0$. The path $P = v_0e_1v_1 \cdots v_{k-1}e_kv_k$ will be denoted by $P = v_0, v_1, \ldots, v_{k-1}, v_k$. Similar notation will be used for closed trails, cycles and paths in multigraphs when the choice of which edge with end-vertices v_{i-1} and v_i to be taken as the edge e_i is immaterial.

We will also use the terms closed trail, cycle and path to refer to graphs whose vertices and edges form such sequences. In particular, a graph G is called *eulerian* if there is a closed trail T in G which contains all the edges of G. We will use the terms *closed trail* and *eulerian graph* interchangeably.

The *length* of a closed trail, cycle or path is taken as its number of edges. A cycle, closed trail or path of length k is called a k-cycle, closed k-trail, or k-path, respectively. Throughout the thesis, we will let C_k denote a k-cycle, T_k denote a closed k-trail, and P_k denote a k-path.

Definition 1.2. Let G be a graph and let $H_1, H_2, \ldots, H_n$ be subgraphs of G. Then G is said to be *decomposed* into its subgraphs $H_1, H_2, \ldots, H_n$ if the subgraphs H_1, H_2, ..., H_n are pairwise edge disjoint, and $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_n)$. In this case, we write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_n$.

Frequently, the decomposition under question is into subgraphs which are isomorphic. We thus have the following definition.

Definition 1.3. Let G and H be graphs and let $H_1, H_2, \ldots, H_n$ be subgraphs of G such that $H_i \cong H$ for each $i \in \{1, 2, \ldots, n\}$. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_n$, then we say that G is *H-decomposable*, and write $H|G$.

In particular, if G is decomposable into cycles of length k, then we will write $C_k|G$. If G decomposes into closed trails of length k, we will abuse notation slightly and write $T_k|G$, even though the closed trails in the decomposition need not be isomorphic.

A notable type of cycle decomposition is a *Hamilton decomposition*, which is a decomposition of a graph G into cycles of length $|V(G)|$.

Example 1.4. The complete graph K_5 is Hamilton decomposable, as illustrated in Figure 1.1.

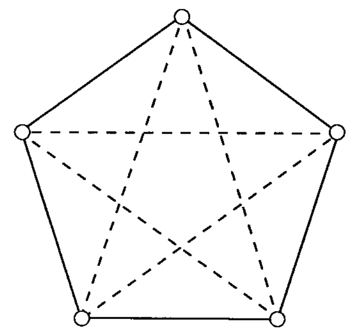

Figure 1.1: A C_5-decomposition of K_5. The solid edges form one 5-cycle, the dashed edges another.

The graphs which we are most interested in decomposing are complete multigraphs and complete equipartite graphs.

Definition 1.5. The *λ-fold complete multigraph*, denoted λK_m, has m vertices, with each pair of vertices connected by λ edges.

Definition 1.6. The *complete multipartite graph*, denoted $K_{n_1,n_2,\ldots,n_m}$, consists of $n_1 + n_2 + \cdots + n_m$ vertices, partitioned into m *parts* $P_1, P_2, \ldots, P_m$, where $|P_i| = n_i$ for each $i = 1, 2, \ldots, m$, and vertices u and v are adjacent if and only if u and v belong to distinct parts. In the case that $n_1 = n_2 = \cdots = n_m = n$, this graph is referred to as the *complete equipartite graph* with m parts of size n, and denoted by $K_m * \overline{K_n}$.

Both of these families of graphs may be viewed as generalizations of complete graphs, as the complete multigraph λK_m is just K_m when $\lambda = 1$, and the complete equipartite graph with m parts of size 1 is isomorphic to K_m. Moreover, the complete equipartite graph with m parts of size 2 is isomorphic to $K_{2m} - I$. Examples of complete multigraphs and complete equipartite graphs may be found in Figure 1.2.

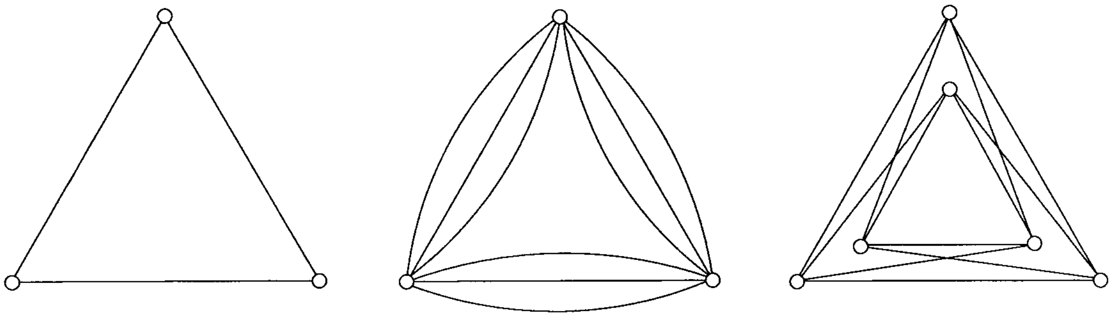

Figure 1.2: From left to right: K_3, $3K_3$, $K_3 * \overline{K_2}$.

Some other basic properties of complete multigraphs and complete equipartite graphs are summarized in the following lemma. Its proof is by straightforward application of the definitions of these graphs.

Lemma 1.7. *Let m, n and λ be positive integers.*

1. *The complete multigraph λK_m is regular of degree $\lambda(m-1)$.*
2. $|V(\lambda K_m)| = m$ *and* $|E(\lambda K_m)| = \lambda\binom{m}{2}$.
3. *The complete equipartite graph $K_m * \overline{K_n}$ is regular of degree $(m-1)n$.*
4. $|V(K_m * \overline{K_n})| = mn$ *and* $|E(K_m * \overline{K_n})| = \binom{m}{2}n^2$

We remark that there are various alternative notations in the literature for the complete equipartite graph $K_m * \overline{K_n}$, such as $K(n, n, \ldots, n)$, $K(n^m)$ and $K_{m(n)}$. The notation $K_m * \overline{K_n}$ which we use reflects the fact that this graph may be viewed as a particular type of graph product.

Definition 1.8. Let G and H be simple graphs. The *wreath product* $G * H$ is a simple graph on vertex set $V(G) \times V(H)$, with (g, h) adjacent to (g', h') if and only if either

$\{g, g'\} \in E(G)$ or else $g = g'$ and $\{h, h'\} \in V(H)$. The wreath product is sometimes also referred to as the *lexicographic product* or *composition* of G and H.

From this definition, it is easy to see that the complete equipartite graph with m parts of size n is isomorphic to the wreath product of K_m with the edgeless graph $\overline{K_n}$. It can be thought of as being obtained from K_m by replacing each vertex with a part of size n, and each edge with the edges of a complete bipartite graph $K_{n,n}$ between two parts. We will occasionally see other wreath products, such as $C * \overline{K_n}$, where C is a cycle.

The question that we are concerned with in this thesis is when a complete multigraph or a complete equipartite graph admit a decomposition into closed trails or cycles of a particular length.

Obvious necessary conditions for a graph G to admit a decomposition into closed trails of length $k \geq 2$ are as follows:

1. Each vertex of G has even degree.
2. k divides $|E(G)|$.

Looking at the particular cases in which G is either $K_m * \overline{K_n}$ or λK_n, we have the following necessary conditions.

Lemma 1.9. *Let $m \geq 2$, $n \geq 1$ and $k \geq 2$ be integers. Suppose $T_k | K_m * \overline{K_n}$. Then:*

1. *$(m-1)n$ is even*
2. *$k | \binom{m}{2} n^2$.*

Moreover, if $m = 2$, then k is even.

The stipulation that k be even in the case $m = 2$ of Lemma 1.9 stems from the fact that a bipartite graph contains no closed trail of odd length.

Lemma 1.10. *Let $m \geq 2$, $\lambda \geq 1$ and $k \geq 2$ be integers. If $T_k|\lambda K_n$, then:*

1. *$\lambda(m-1)$ is even*

2. *$k|\lambda\binom{m}{2}$.*

Moreover, if $k = 2$, then λ is even.

Now consider decomposing a graph G into cycles. As a cycle is also a special type of closed trail, the necessary conditions for closed trail decompositions are also necessary for cycle decompositions. However, further restrictions must be introduced — in particular, obvious necessary conditions for the existence of a C_k-decomposition of a graph G are that

1. Each vertex in G has even degree.

2. k divides $|E(G)|$

3. $k \leq |V(G)|$.

As the second condition implies that $k \leq |E(G)|$, the third condition ensures that $k \leq \min\{|E(G)|, |V(G)|\}$. This restriction is necessary so that it is possible for G to even contain a cycle.

Letting G be a complete multipartite graph or a complete multigraph, we thus have the following two sets of necessary conditions.

Lemma 1.11. *Let $m \geq 2$, $n \geq 1$ and $k \geq 3$ be integers. If $C_k|K_m * \overline{K_n}$, then:*

1. *$(m-1)n$ is even*

2. *$k|\binom{m}{2}n^2$*

3. *$3 \leq k \leq mn$.*

Moreover, if $m = 2$, then k is even.

Lemma 1.12. *Let $m \geq 2$, $\lambda \geq 1$ and $k \geq 2$ be integers. If $C_k|\lambda K_m$, then:*

1. *$\lambda(m-1)$ is even*

2. *$k|\lambda\binom{m}{2}$*

3. *$2 \leq k \leq m$*

Moreover, if $k = 2$, then λ is even.

The question of determining whether these necessary conditions are also sufficient is significantly more difficult and certain cases of this question will be the focus of this thesis. In the next section, we give an overview of the cases in which the question has been answered.

1.2 History

In this section, we describe the cases in which necessary and sufficient conditions have been found for decomposition of complete equipartite graphs and complete multigraphs into cycles and closed trails. Let us first consider the problem of cycle decomposition. We will first discuss the history of decomposing the complete graph into cycles. As this problem remained open for approximately a century-and-a-half, it has on its own a quite remarkable history. For the sake of brevity, we will only highlight a few results towards its solution; the interested reader is invited to read the survey articles [16, 32] for a more detailed view.

The problem of decomposing the complete graph into cycles dates back at least to 1837, when Plücker [39] noticed the existence of a C_3-decomposition of K_9 and stated that a C_3-decomposition of K_m can exist only if $m \equiv 3 \pmod 6$. He later revised this observation to include the correct necessary conditions of $m \equiv 1$ or $3 \pmod 6$ [40]. In 1847, Kirkman [28] proved that $C_3|K_m$ whenever $m \equiv 1$ or $3 \pmod 6$. A decomposition of K_m into cycle of length 3 is now also called a Steiner triple system, named for Jakob Steiner, who posed the question of when Steiner triple systems exist in 1853 [52], apparently unaware of Kirkman's earlier work. In the late 1800's, Walecki, in [35], provided simple constructions for Hamilton decompositions of K_m whenever m is odd and $K_m - I$ (which is isomorphic to $K_{m/2} * \overline{K_2}$) whenever m is even. In 1938, Peltesohn [38] found necessary and sufficient conditions for the existence of a cyclic C_3-decomposition of K_m. The problem of decomposing K_m into cycles of arbitrary lengths was not considered until the 1960s, when the existence of a C_k-decomposition of K_m was established for even k and $m \equiv 1 \pmod{2k}$ by Kotzig and Rosa [29, 41]. In addition, Rosa [42] proved necessary and sufficient conditions for the existence of a 5- or 7-cycle decomposition of K_m in 1966. For odd k, Jackson [27] proved in 1988 that $C_k | K_m$ whenever $m \equiv 1$ or $k \pmod{2k}$. Sufficiency of the obvious necessary conditions for C_k-decomposition of K_m and $K_m - I$ was finally established by Alspach and Gavlas [1] in the case that m and k have the same parity, and by Šajna [45], in the case that m and k are of opposite parity.

One direction for generalization of the Alspach-Gavlas-Šajna result is to consider cycle decompositions of the complete equipartite graph, as the results of Alspach, Gavlas and Šajna show that the obvious necessary conditions are sufficient for C_k-decomposition of $K_m * \overline{K_n}$ when $n \in \{1, 2\}$. If $m = 2$, then the graph $K_2 * \overline{K_n}$ is bipartite, and with the added requirement that k is even (since bipartite graphs contain no odd cycle), the necessary conditions were proven to be sufficient in 1981 by Sotteau [50]. The tripartite case was considered by Cavenagh [18], who proved that $C_k | K_3 * \overline{K_n}$ if and only if $3 \leq k \leq 3n$ and $k | 3n^2$. Later, Billington, Cavenagh and Smith found necessary and sufficient conditions for the existence of a C_k-decomposition of $K_m * \overline{K_n}$ when $m = 4$ [10] and $m = 5$ [11].

Many early results on cycle decompositions of complete equipartite graphs tended to focus on particular small cycle lengths. In 1975, Hanani [21] found necessary and sufficient conditions for the existence of a C_3-decomposition of $K_m * \overline{K_n}$ (which is equivalent to a group divisible design with groups of size n and blocks of size

3). Two years later, Cockayne and Hartnell [20] showed that the obvious necessary conditions are sufficient for cycle length 4. This result was replicated, together with a proof of sufficiency for cycles of lengths 6 and 8, by Cavenagh and Billington [19]. Necessary and sufficient conditions were found when $k = 5$ by Billington, Hoffman and Maenhaut [12]. In 2006, Manikandan and Paulraja [37] proved sufficiency for all prime cycle lengths $p \geq 7$. This result was extended by Smith [47] to cycles of length $2p$, where p is prime.

One variation on the problem of finding a cycle decomposition of $K_m * \overline{K_n}$ is to require that the decomposition be *resolvable*, that is, the cycles may be partitioned into classes such that in each class, the vertex sets of the cycles in the class partition $V(K_m * \overline{K_n})$. Liu [33, 34] showed that a resolvable C_k-decomposition of $K_m * \overline{K_n}$ exists if and only if $(m-1)n$ is even, k divides mn, k is even if $m = 2$, and $(m, n, k) \notin \{(3, 2, 3), (3, 6, 3), (6, 2, 3), (2, 6, 6)\}$. This result, together with the existence of C_3-decompositions of $K_3 * \overline{K_2}$, $K_3 * \overline{K_6}$, $K_6 * \overline{K_2}$ and a C_6-decomposition of $K_2 * \overline{K_6}$ by previously mentioned results, proves that the obvious necessary conditions for the existence of a C_k-decomposition of $K_m * \overline{K_n}$ are sufficient in the case that $k|mn$.

Let us now consider decompositions of the complete multigraph λK_m into cycles. Further to the results on decomposing the complete graph, Alspach, Gavlas, Šajna and Verrall [2] found necessary and sufficient conditions for the existence of a decomposition of the complete directed graph into directed cycles of length k. As a corollary, they determined that the obvious necessary conditions for decomposing $2K_m$ into k-cycles are sufficient. Necessary and sufficient conditions have not, however, been found for greater values of λ.

A fair amount of work has been done on decomposing complete multigraphs into cycles of particular lengths. The obvious necessary conditions were shown to be sufficient for cycles of length 3 by Hanani in 1961 [22]. Huang and Rosa settled the problem when $k = 4$ [25] and $k = 5$ or 6 [44]. Bermond and Sotteau proved sufficiency for odd k with $3 \leq k \leq 7$ [8], and, together with Huang, for even k

with $4 \leq k \leq 16$ [6]. Much more recently, Smith [48] found necessary and sufficient conditions for decomposition of λK_m into cycles of length λ; as a corollary of this result, he proved sufficiency of the obvious necessary conditions for decomposition of λK_m into cycles of prime length.

Decomposition into closed trails is a relaxation of the problem of decomposition into cycles. The problem of finding necessary and sufficient conditions for the existence of a decomposition of λK_m into closed trails of length k was completely solved by Hwang and Lin [26] in 1977. Balister [4] found necessary and sufficient conditions for decomposition of K_m and $K_m * \overline{K_2}$ into closed trails of (possibly different) lengths $k_1, k_2, \ldots, k_r$. Necessary and sufficient conditions for decomposition of the complete bipartite graph $K_2 * \overline{K_n}$ into closed trails of lengths $k_1, k_2, \ldots, k_r$ were found by Horňák and Woźniak [24], while Billington and Cavenagh [9] proved an analogous result for the complete tripartite graph $K_3 * \overline{K_n}$. The problem of finding necessary and sufficient conditions for decomposition of $K_m * \overline{K_n}$ into closed trails of lengths $k_1, k_2, \ldots, k_r$, however, remains unsolved in general. In this thesis, we will restrict our attention to the case in which all closed trails have the same length.

1.3 An outline of the thesis

We begin the remainder of the thesis in Chapter 2, with some preliminary information. Section 2.1 provides additional definitions of concepts which will recur throughout the rest of the thesis, while in Section 2.2, we present some preliminary results on decomposition. In particular, the results contained in Section 2.2 concern forming new decompositions based on known ones.

In Chapter 3, we find necessary and sufficient conditions for the existence of a decomposition of the complete equipartite graph $K_m * \overline{K_n}$ into closed trails of length k. Along the way, we provide an alternative proof of necessary and sufficient conditions for the existence of a decomposition of the complete directed graph K_m^* into closed trails of length k; this result is originally due to Balister [4]. We also detail several methods for lifting a closed trail decomposition of the complete multigraph λK_m into a closed trail decomposition of a complete equipartite graph. The results in this chapter have appeared in the Journal of Combinatorial Designs [17]. Necessary and sufficient conditions for the existence of a T_k-decomposition of $K_m * \overline{K_n}$ have since been independently proven by Smith [49].

Chapter 4 concerns cycle decompositions of the complete multigraph $3K_m$. In this chapter, we prove necessary and sufficient conditions for the existence of a decomposition of $3K_m$ into cycles of odd length $k \neq 9$. This result extends the work of Alspach, Gavlas, Šajna and Verrall [1, 2, 45], who found necessary and sufficient conditions for C_k-decompositions of K_m and $2K_m$.

In Chapter 5, we consider cycle decomposition of the complete equipartite graph $K_m * \overline{K_3}$, with parts of size 3. To decompose this graph into k-cycles, necessary conditions are that $3 \leq k \leq 3m$, m is odd and $k | 9\binom{m}{2}$; we show these conditions are sufficient when $k | 3\binom{m}{2}$ or when $k \neq 27$ is odd. In the case that $9|k$, the proof employs cycle decompositions of $3K_m$ found in Chapter 4.

Chapter 6 presents a further result on multigraph decomposition, namely a construction for k-cycle decompositions of kK_m when k and m are even.

Finally, in Chapter 7, we sum up the results contained in the thesis and state some open problems related to this work.

Chapter 2

Preliminaries

In this chapter, we present in Section 2.1 some additional definitions of concepts which will recur in the remainder of the thesis. Section 2.2 contains some results on creating new decompositions of complete multigraphs and complete multipartite graphs based on known decompositions.

2.1 Definitions

In many of our constructions, we will let $V(K_m) = \mathbb{Z}_m$. In this way, we may view the complete graph as a circulant graph and construct decompositions based on the lengths of edges. Before we present the relevant definitions, let us first state some notation. Let m be a positive integer and $x \in \mathbb{Z}_m$. We use $|x|_m$ to denote the unique element $i \in \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$ such that $x \equiv \pm i \pmod{m}$. If we view the elements of $\mathbb{Z}_m$ as placed on a circle, $|x|_m$ is the least number of steps, clockwise or counterclockwise, beginning at element 0, taken to reach x. When we speak of an *interval* $[a, b]$ in $\mathbb{Z}_m$, where $b - a \neq \frac{m}{2}$, we mean the set chosen from $\{a, a+1, \ldots, b\}$ or $\{b, b+1, \ldots, a\}$ which has smallest size (necessarily $|b - a|_m$). We do not define the interval $[a, b]$ for $b - a = \frac{m}{2}$, where the two sets $\{a, a+1, \ldots, b\}$ and $\{b, b+1, \ldots, a\}$ both have size

$|b-a|_m = \frac{m}{2}$, as we will not have need of such an interval.

Definition 2.1. Let m be a positive integer and $S \subseteq \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$. The *circulant graph* $X(m;S)$ *of order* m *and edge length set* S is the simple graph with vertex set $\mathbb{Z}_m$ such that $\{x,y\} \in E(X(m;S))$ if and only if $|x-y|_m \in S$. (That is, $E(X(m;s)) = \{\{x, x+s\} : x \in \mathbb{Z}_m \text{ and } s \in S\}$.) An edge $\{x,y\}$ in the circulant $X(m;S)$ is said to have *length* $|x-y|_m$.

Example 2.2. The circulant graph $X(7;\{1,3\})$ of order 7 and with edge length set $\{1,3\}$ is shown in Figure 2.1. As examples, the edge $\{0,1\}$ has length 1, and $\{2,5\}$ has length 3.

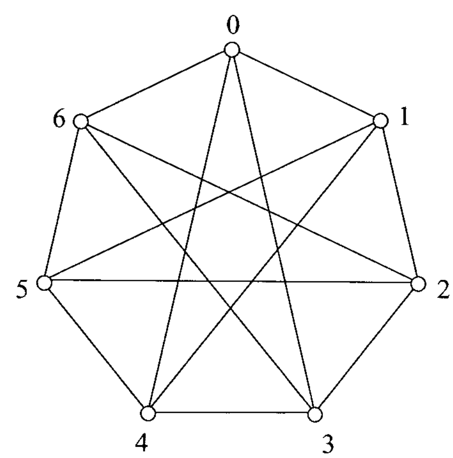

Figure 2.1: The circulant $X(7;\{1,3\})$.

We note that the the definition of a circulant graph $X(m;S)$ may easily be extended to the case in which S is a multiset of edge lengths. In this case, each edge of length $\ell \in S$ occurs in the graph $X(m;S)$ with multiplicity the number of times ℓ is contained in S.

It is also useful to note that the complete graph K_m may be viewed as the circulant $X(m;S)$ where $S = \{1, 2 \ldots, \lfloor \frac{m}{2} \rfloor\}$.

Definition 2.3. Let G be a graph (directed graph) on vertex set $V = V(G)$ and let ρ be a permutation of V. For a sub(di)graph H of G, define $\rho(H)$ to be the graph (directed graph) with vertex set $\rho(V(H))$ and edge set (arc set) $E(\rho(H)) = \{\rho(u)\rho(v) : uv \in E(H)\}$. In general, $\rho(H)$ need not be a sub(di)graph of G. However, clearly if $G = K_n$ or $G = K_n^*$, then for any permutation ρ of V and sub(di)graph H of G, $\rho(H)$ is a sub(di)graph of G.

We note that to decompose a circulant $X(m; S)$ into cycles (closed trails) of length k, it is sufficient to find a set $\{C^{(1)}, C^{(2)}, \ldots, C^{(t)}\}$ of starter k-cycles (closed k-trails) which together contain an edge of each length in S exactly once. Necessarily, $t = |S|/k$. Letting ρ denote the cyclic permutation $(0, 1, \ldots, m-1)$, then the set of cycles (closed trails) $\{\rho^i(C^{(j)}) : 0 \leq i \leq m-1, 1 \leq j \leq t\}$ form a C_k-decomposition of $X(m; S)$. Such a decomposition, in which $\rho(C)$ is a cycle (closed trail) in the decomposition for each cycle (closed trail) C in the decomposition, is called *cyclic.*

Example 2.4. The cycle $C = 0, 1, 3, 0$ is a starter cycle for a cyclic C_3-decomposition of K_7, since it contains an edge of each length $1, 2, 3$. The cycles $\rho^i(C)$, $i = 0, 1, \ldots, 6$, where $\rho = (0, 1, 2, 3, 4, 5, 6)$, form the decomposition. This decomposition is illustrated in Figure 2.2.

A concept which will frequently be used in our constructions is that of the *jump* of an edge in a complete equipartite graph, which is essentially the difference between its positions in the two parts containing its endvertices. As we will see in Section 2.2 and later, we can form cycle or closed trail decompositions of $K_m * \overline{K_n}$ from known cycle or closed trail decompositions of λK_m by assigning to each edge a jump in a particular manner. Let us now specify more precisely the definition of jump.

Definition 2.5. Let the complete directed graph K_m^* have vertex set $\mathbb{Z}_m$. An arc (u, v) in K_m^* will be said to be of *positive direction* if one of the following conditions hold:

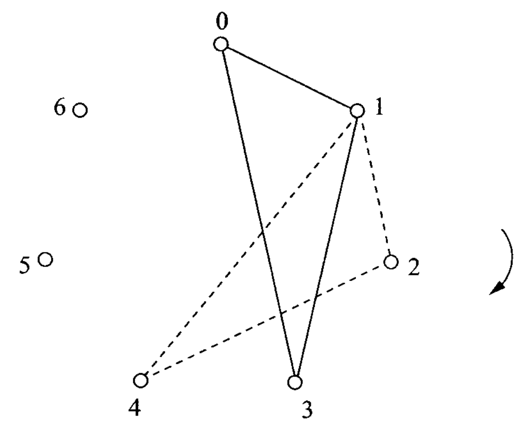

Figure 2.2: Forming a cyclic C_3-decomposition of K_7. The cycles $C = 0, 1, 3, 0$ and $\rho(C) = 1, 2, 4, 1$ are explicitly drawn; the remaining cycles are formed by repeated rotation of C.

1. m is odd and $v - u \in \{1, 2, \ldots, \frac{m-1}{2}\}$
2. m is even and $v - u \in \{1, 2, \ldots, \frac{m-2}{2}\}$
3. m is even, $v - u = \frac{m}{2}$, and $u \in \{0, 1, \ldots, \frac{m-2}{2}\}$.

An arc which is not of positive direction will be said to have *negative direction.*

In essence, that (u, v) has positive direction in the case that $v - u \neq \frac{m}{2}$ means that if the vertices of $\mathbb{Z}_m$ are placed around a circle in order 0, 1, ..., $m - 1$ in the clockwise direction, then the shortest distance around the circle from u to v is in the clockwise direction. If $v - u = \frac{m}{2}$, then the distance from u to v is equal in both the clockwise and counterclockwise directions, so the positive direction is chosen as the one in which the initial vertex u is in $\{0, 1, \ldots, \frac{m-2}{2}\}$.

Definition 2.6. Let the vertex set of K_m be $\mathbb{Z}_m$ and that of K_d be $\mathbb{Z}_d$, so that $K_m * \overline{K_d}$ has vertex set $\mathbb{Z}_m \times \mathbb{Z}_d$. Consider an edge $e = \{(u, i), (u', i')\}$ of $K_m * \overline{K_d}$ where the arc (u, u') of K_m^* is of positive direction. The edge e is said to have *jump* $i' - i$.

Sometimes we will make use of matchings, all of whose edges have a particular jump.

Definition 2.7. A matching M in $K_m * \overline{K_n}$ whose edges have vertices all in two parts will be said to be of *jump j* if each of its edges has jump j.

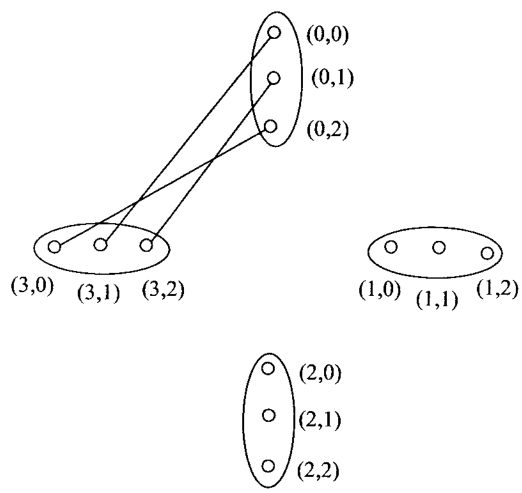

Figure 2.3: A matching of jump 2 in $K_4 * \overline{K_3}$.

We note that the matchings of size n of jumps $0, 1, \ldots, n-1$ partition the edges of the complete bipartite graph $K_{n,n}$.

The jumps of edges in $K_m * \overline{K_n}$ defined in Definition 2.6 will be considered as a *natural jump*; in this context, *jump* may be thought of as the difference in position in their respective parts of the endvertices of an edge. In contrast, we will also consider the concept of an *assigned jump*, which is a label in $\mathbb{Z}_\lambda$ assigned to an edge of λK_m. In practice, we will transform λK_m into $K_m * \overline{K_\lambda}$ by assigning a jump to each edge of λK_m, with λ parallel edges receiving λ distinct labels. We will associate an edge $e = \{u, u'\}$ of λK_m which has assigned jump j with a matching of size n and jump j in $K_m * \overline{K_\lambda}$ whose edges have endvertices in the parts $\{u\} \times \mathbb{Z}_\lambda$ and $\{u'\} \times \mathbb{Z}_\lambda$. In this way, we will speak of *lifting* λK_m to $K_m * \overline{K_\lambda}$.

Definition 2.8. Let $e = \{u, u'\}$ be an edge of λK_m. Suppose that e is traversed in the direction (u, u'), and let e be assigned jump j. If (u, u') is of positive direction, then the *signed jump* of e is j; if (u, u') is of negative direction, then the signed jump of e is $-j$. If $e_1, e_2, \ldots, e_k$ are edges of λK_m, each with a prescribed direction, then the *displacement* over $e_1, e_2, \ldots, e_k$ is the sum of the signed jumps of $e_1, e_2, \ldots, e_k$.

We may similarly define signed jump and displacement for edges of $K_m * \overline{K_d}$: if $\{(u,i),(u',i')\}$ is traversed in the direction $((u,i),(u',i'))$ and has jump $j \in \{i-i', i'-i\}$, then $\{(u,i),(u',i')\}$ has signed jump j (resp. $-j$) if (u, u') has positive (resp. negative) direction. The displacement over a trail $e_1, e_2, \ldots, e_k$ in $K_m * \overline{K_d}$ is the sum of the signed jumps of $e_1, e_2, \ldots, e_k$.

We will also require some further notation related to paths.

Definition 2.9. Let $P = v_0, v_1, \ldots, v_{k-1}, v_k$ be a path of length k. The *reversal of* P is the path $\overleftarrow{P} = v_k, v_{k-1}, \ldots, v_1, v_0$.

Note that P and $\overleftarrow{P}$ represent the same underlying path. However, at times the order of listing of the vertices of a path will be important in incorporating this path in larger paths or cycles.

Definition 2.10. Let $P = u_0, u_1, \ldots, u_k$ and $Q = v_0, v_1, \ldots, v_s$ be paths, such that the terminal vertex of P and the initial vertex of Q coincide, i.e. $u_k = v_0$. The *concatenation of* P *and* Q is $PQ = u_0, u_1, \ldots, u_k, v_1, v_2, \ldots, v_s$. If u_k is the only vertex shared between P and Q, then PQ is also a path. If $v_s = u_0$, and the sets $\{u_1, \ldots, u_{k-1}\}$ and $\{v_1, \ldots, v_{s-1}\}$ are disjoint, then PQ is a cycle.

Example 2.11. Let $V(K_{15}) = \mathbb{Z}_{15}$. Let $P = 0, 1, -1, 5, -2$, $Q = -2, 6$, and $R = 6, -4, 7, 0$. Then PQ is the path 0, 1, -1, 5, -2, 6 and PQR is the cycle 0, 1, -1, 5, -2, 6, -4, 7, 0 in K_{15}.

Definition 2.12. Let $P = v_0, v_1, \ldots, v_k$ be a path. Then vertices $v_0, v_2, v_4, \ldots$ (i.e. the first, third, fifth, ... vertices along P) will be referred to as the *odd-positioned* *vertices of* P, while the vertices $v_1, v_3, v_5, \ldots$ (i.e. the second, fourth, ..., vertices along P) will be referred to as the *even-positioned vertices of* P, with respect to the order of listing of the vertices of P.

Finally, we close this section with one remaining concept.

Definition 2.13. Let G and H be simple graphs. The *join* of G and H, denoted $G \bowtie H$, is the simple graph with vertex set $V(G \bowtie H) = V(G) \cup V(H)$ and edge set $E(G \bowtie H) = E(G) \cup E(H) \cup \{\{g,h\} : g \in V(G) \text{ and } h \in V(H)\}$.

In particular, we will sometimes view K_m as the join $K_{m-1} \bowtie K_1$. In this case, the vertex of K_1 is called *central.* This representation, illustrated in Figure 2.4 for $m = 6$, allows us to consider the edges of K_m as either having length in K_{m-1}, or else being incident with the central vertex.

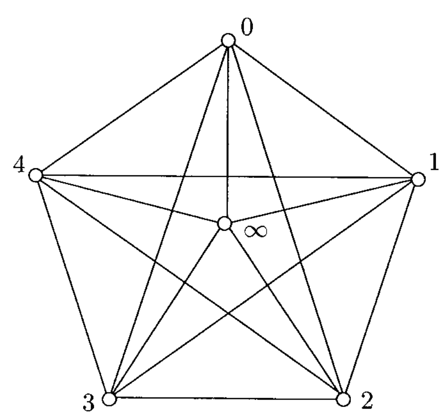

Figure 2.4: $K_6 \cong K_5 \bowtie K_1$. Here, $V(K_6) = \mathbb{Z}_5 \cup \{\infty\}$, where $V(K_5) = \mathbb{Z}_5$ and ∞ is the central vertex.

2.2 New decompositions from old

2.2.1 Blowing up cycles and closed trails in complete equipartite graphs

The next few results show how we may obtain new decompositions based on existing decompositions. We first give some known results on blowing up cycle decompositions.

Lemma 2.14. *[5] Let $k \geq 3$ be a positive integer, and suppose that C is a cycle of length k. Then for any positive integer n, $C_k | C * \overline{K_n}$.*

Lemma 2.15. *[30] Let $k \geq 3$ be a positive integer, and suppose that C is a cycle of length k. Then for any positive integer n, $C_{kn} | C * \overline{K_n}$.*

The following two results are easy consequences of Lemmas 2.14 and 2.15, and may be found in numerous places in the literature (see, for example [18]).

Lemma 2.16. *Let $k \geq 3$, $m \geq 2$ and $n \geq 1$ be positive integers. If $C_k | K_m * \overline{K_n}$, then for any integer $s \geq 1$, $C_k | K_m * \overline{K_{sn}}$.*

Lemma 2.17. *Let $k \geq 3$, $m \geq 2$ and $n \geq 1$ be positive integers. If $C_k | K_m * \overline{K_n}$, then for any integer $s \geq 1$, $C_{sk} | K_m * \overline{K_{sn}}$.*

The decompositions alluded to in Lemmas 2.18 and 2.19 below are formed analogously to the corresponding results on cycle decomposition (Lemmas 2.14 and 2.15).

Lemma 2.18. *Let $k \geq 3$ be a positive integer, and suppose that T is a closed trail of length k in a simple graph. Then for any positive integer n, $T_k | T * \overline{K_n}$.*

Lemma 2.19. *Let $k \geq 3$ be a positive integer, and suppose that T is a closed trail of length k in a simple graph. Then for any positive integer n, $T_{kn} | T * \overline{K_n}$.*

As a direct result of these lemmas, we have the following useful tools for blowing up existing closed trail decompositions.

Corollary 2.20. *Let m, n and $k \geq 3$ be positive integers. If $T_k|K_m * \overline{K_n}$, then for any positive integer s, $T_k|K_m * \overline{K_{sn}}$.*

Corollary 2.21. *Let m, n and $k \geq 3$ be positive integers. If $T_k|K_m * \overline{K_n}$, then for any positive integer s, $T_{sk}|K_m * \overline{K_{sn}}$.*

2.2.2 Converting closed trail decompositions to cycle decompositions

In this section we state several known results which enable, under certain circumstances, the formation of a cycle decomposition of a complete equipartite graph from a known closed trail decomposition.

We begin by stating the following result, due to Cavenagh. Here, $\chi(T)$ represents the chromatic number of a closed trail T and $\Delta(T)$ its maximum degree. This result was employed by Cavenagh in his proof of necessary and sufficient conditions for the existence of a C_k-decomposition of $K_3 * \overline{K_n}$ in [18].

Lemma 2.22. *[18] Suppose T is a closed trail of length k in the complete tripartite graph $K_3 * \overline{K_n}$, i.e. $\chi(T) \leq 3$. If $\Delta(T) \leq 2s$, then $C_k|T * \overline{K_{sn}}$.*

More recently, Smith [47] published the following generalization of Lemma 2.22.

Lemma 2.23. *[47] Let T be a closed trail of length k in a simple graph. Let $\Delta = \Delta(T)$ and $\chi = \chi(T)$. Suppose that $n \geq \frac{\Delta}{2}$. If there exist $(\chi - 2)$ mutually orthogonal Latin squares of order n, then $C_k|T * \overline{K_n}$.*

We note that Lemma 2.23 requires the existence of mutually orthogonal Latin squares. In order to use this lemma, we note the following well-known result.

Theorem 2.24. *[36] Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ be the prime factorization of the integer n, where $p_1 < p_2 < \cdots < p_k$. Then there exist ${p_1}^{\alpha_1} - 1$ mutually orthogonal Latin squares of order n.*

2.2.3 Lifting decompositions of complete multigraphs to decompositions of complete equipartite graphs

Theorem 2.25. *Suppose we have a C_k-decomposition of λK_m. Assign to each cycle an arbitrary direction of traversal. Suppose we can assign a jump in $\mathbb{Z}_\lambda$ to each edge of λK_m such that:*

1. *No two parallel edges of λK_m share a label.*
2. *The total displacement along any cycle in the decomposition is 0.*

Then $C_k | K_m * \overline{K_\lambda}$.

Proof. Let u and v be vertices of λK_m. Notice that the assignment of a jump $j \in \{0, 1, \ldots, m-1\}$ to an edge of λK_m with endvertices u and v gives rise to a matching of size m and jump j in the $K_{\lambda,\lambda}$ induced by vertices in $\{u, v\} \times \mathbb{Z}_\lambda$ in $K_m * \overline{K_\lambda}$. Among the m parallel edges, each jump in $\{0, 1, \ldots, m-1\}$ is assigned to one of them, and so the corresponding matchings partition the $K_{m,m}$ induced by vertices in $\{u, v\} \times \mathbb{Z}_\lambda$ in $K_m * \overline{K_\lambda}$. In this way we see that the jump assignment gives rise to $K_m * \overline{K_\lambda}$, with each edge occurring once as the result of a jump.

Consider a k-cycle $C = v_0 e_1 v_1 e_2 v_2 \cdots v_{k-1} e_k v_0$ in λK_m, with vertices v_0, v_1, …, v_{k-1} and edges $e_1, e_2, \ldots, e_k$. With the assignment of jumps to the edges of C, this cycle gives rise to $k\lambda$ edges in $K_m * \overline{K_\lambda}$; we wish to show that these $k\lambda$ edges may be partitioned into the edges of λ cycles of length k in $K_m * \overline{K_\lambda}$. Let us assign a direction of traversal of C; in particular, we give the edges of C the directions (v_0, v_1), (v_1, v_2), …, (v_{k-2}, v_{k-1}), (v_{k-1}, v_0). Let s_i denote the signed jump on the arc (v_{i-1}, v_i), where $v_k = v_0$, so that $\sum_{i=1}^{k} s_i = 0$. Let $\alpha \in \mathbb{Z}_\lambda$, and consider the vertex (v_0, α). Beginning at this vertex, we traverse a sequence of k edges and vertices as follows. If we are currently at vertex (v_i, β), we follow along the edge $\{(v_i, \beta), (v_{i+1}, \beta + s_{i+1})\}$ to the next vertex $(v_{i+1}, \beta + s_{i+1})$; that is, the part of the next vertex is determined by the next vertex of C, and its position in the part is determined by the jump assigned to the edge

e_{i+1}. Thus, beginning at vertex (v_0, α) and traversing k edges, we obtain a sequence $C_\alpha = (v_0, \alpha), (v_1, \alpha + s_1), (v_2, \alpha + s_1 + s_2), \ldots, (v_{k-1}, \alpha + \sum_{i=1}^{k-1} s_i), (v_0, \alpha + \sum_{i=1}^{k} s_i)$. Since $\sum_{i=1}^{k} s_i = 0$, the terminal vertex is equal to the initial vertex (v_0, α). We note that since C is a cycle in λK_m, the vertices in C_α occur in k different parts; in particular, except that the inital and terminal vertex coincide, the vertices are distinct. It follows that C_α must be a cycle. Finally, we note that if α and β are distinct elements of $\mathbb{Z}_\lambda$, then it is easy to see that the cycles C_α and C_β have no vertex in common. We have thus found k-cycles C_0, C_1, …, C_{m-1} which partition the km edges arising from C.

If we do the same procedure for each cycle in the decomposition of λK_m, then we obtain a k-cycle decomposition of $K_m * \overline{K_\lambda}$. □

Theorem 2.26. *Suppose we have a C_k-decomposition of λK_m. Assign to each cycle an arbitrary direction of traversal. Suppose we can assign a jump in $\mathbb{Z}_\lambda$ to each edge of λK_m such that:*

1. *No two parallel edges of λK_m share a label.*

2. *The total displacement in a cycle is relatively prime to λ.*

Then $C_{k\lambda} | K_m * \overline{K_\lambda}$.

Proof. That each edge of λK_m lifts to λ edges in $K_m * \overline{K_\lambda}$ and that no two edges of λK_m give rise to the same edge in $K_m * \overline{K_\lambda}$ follow as in the proof of Theorem 2.25, as any λ parallel edges receive distinct jumps in $K_m * \overline{K_\lambda}$.

Also, similarly to the proof of Theorem 2.25, a k-cycle $C = v_0 e_1 v_1 e_2 v_2 \cdots v_{k-1} e_k v_0$ gives rise to $k\lambda$ edges in $K_m * \overline{K_\lambda}$; in this case, however, we must show that these $k\lambda$ edges constitute a single $k\lambda$-cycle. Beginning at $(v_0, 0)$, we traverse these $k\lambda$ edges in the following manner. Let s_i denote the signed jump on the arc (v_{i-1}, v_i). If we are at vertex (v_i, β), then we follow along the edge $\{(v_i, \beta), (v_{i+1}, \beta + s_{i+1})\}$. We thus obtain the sequence $C' = (v_0, 0)$, (v_1, s_1), …, $(v_{k-1}, \sum_{i=1}^{k-1} s_i)$, $(v_0, \sum_{i=1}^{k} s_i)$,

$(v_1, s_1+\sum_{i=1}^{k} s_i)$, $(v_2, s_1+s_2+\sum_{i=1}^{k} s_i)$, ..., $(v_{k-1}, \sum_{i=1}^{k-1} s_i+\sum_{i=1}^{k} s_i)$, $(v_0, 2\sum_{i=1}^{k} s_i)$, ..., $(v_{k-1}, \sum_{i=1}^{k-1} s_i+(\lambda-1)\sum_{i=1}^{k} s_i)$, $(v_0, \lambda\sum_{i=1}^{k} s_i)$. Since the second coordinate is in $\mathbb{Z}_\lambda$, the terminal vertex is $(v_0, 0)$. Note that since $\sum_{i=1}^{k} s_i$ is relatively prime to λ, and each part is reached, in order, on vertices (v_i, β), $(v_i, \beta+\sum_{i=1}^{k} s_i)$, $(v_i, \beta+2\sum_{i=1}^{k} s_i)$, $(v_i\beta+3\sum_{i=1}^{k} s_i)$, ..., it follows that each part may be reached $\lambda-1$ times in C' before a vertex in that part is repeated. Beginning at vertex $(v_0, 0)$, the only part which is reached λ times is that containing $(v_0, 0)$, and this vertex is the only one repeated. Hence, other than the fact that the initial and terminal vertices coincide, no other vertex is reached more than once. We now see that C' is a cycle of length $k\lambda$.

Following the procedure for each cycle in the decomposition of λK_m, we obtain a $k\lambda$-cycle decomposition of $K_m * \overline{K_\lambda}$. □

We will see a few further lifting results later in the thesis where they are required, notably in Chapter 3, where we will lift closed trail decompositions of complete multigraphs to closed trail decompositions of complete equipartite graphs.

Chapter 3

Closed trail decompositions of $K_m * \overline{K_n}$

In this chapter, we consider closed trail decompositions of complete equipartite graphs. The main result of this chapter, Theorem 3.15, is the determination of necessary and sufficient conditions for the existence of a decomposition of the complete equipartite graph $K_m * \overline{K_n}$, with m parts of size n, into closed trails of length k. The results of this chapter have appeared in the Journal of Combinatorial Designs [17]. We note also that Theorem 3.15 has since been independently proven by Smith [49].

3.1 Some basic decompositions

We begin by recalling some decomposition results which we will later use in finding decompositions of $K_m * \overline{K_n}$. We note that several of the results on closed trail decompositions are part of more general results regarding decomposition into closed trails whose lengths may vary.

Theorem 3.1. *[24] Suppose $k \geq 3$. The complete bipartite graph $K_{n,n} \cong K_2 * \overline{K_n}$ is decomposable into closed trails of length k if and only if n and k are both* *even and* $k|n^2$.

Theorem 3.2. *[9] Suppose* $k \geq 3$. *The complete tripartite graph* $K_3 * \overline{K_n}$ *is decomposable into closed trails of length* k *if and only if* $k|3n^2$.

Theorem 3.3. *[26] Let* $k \geq 2$, $d \geq 1$, $m \geq 1$ *be integers. Then the complete multigraph* dK_m *is decomposable into closed trails of length* k *if and only if the following conditions all hold:*

- $k|d\binom{m}{2}$,
- d *is even or* m *is odd,*
- *if* $k = 2$, *then* d *is even, and*
- *if* $m = 2$, *then* k *is even.*

Theorem 3.4. *[4] Let* $m, k \geq 3$ *be integers. Then* $T_k|K_m * \overline{K_2}$ *if and only if* $k|2m(m-1)$.

Theorem 3.5. *[3] Let* $m, k \geq 2$ *be integers. Then the complete directed graph* K_m^* *admits a decomposition into directed closed trails of length* k *if and only if* $k|m(m-1)$ *and* $(m, k) \neq (6, 3)$.

Theorem 3.6. *[54] Let* d, k *and* m *be positive integers. The complete multigraph* dK_m *is decomposable into paths of length* k *if and only if* $k|d\binom{m}{2}$ *and* $k \leq m-1$.

For our purposes, we will require closed trail decompositions of dK_m which exhibit certain properties, in particular decompositions each of whose trails contain a pair of parallel edges or whose trails may be partitioned into pairs in a certain manner. In order to obtain decompositions with the desired properties, we offer an alternative proof of Balister's result (Theorem 3.5) on decomposition of K_m^*. The properties which we seek differ depending on the parities of k and m, as do the methods by which we find these decompositions, and so we present this proof via several lemmas.

Before we make this foray into the world of directed graphs, we present some relevant definitions and notation. We will let $\overrightarrow{T_k}$ (resp. $\overrightarrow{C_k}$) denote a directed closed trail (resp. a directed cycle) of length k, and for any undirected graph G, we let G^* denote the directed graph obtained from G by replacing each edge $\{x, y\}$ of G with two arcs (x, y) and (y, x).

Definition 3.7. Let m be a positive integer and $S \subseteq \mathbb{Z}_m - \{0\}$. The *circulant digraph* $\overrightarrow{X}(m; S)$ is the directed graph with vertex set $\mathbb{Z}_m$ such that (i, j) is an arc if and only if $j - i \in S$. An arc (i, j) is said to have *difference* $j - i$.

The first lemma which we present on directed graphs, Lemma 3.8, is a natural application of known decomposition results. Although we will not directly use this lemma towards our decompositions of complete equipartite graphs, we nevertheless include this result for the sake of completeness of our alternative proof of Theorem 3.5.

Lemma 3.8. *Suppose $m \geq 3$ and $k \geq 2$ are integers such that m is odd. If $k|m(m-1)$, then $\overrightarrow{T_k}|K_m^*$.*

Proof. First consider the case that k is odd. We must have that $k|\binom{m}{2}$, so that $T_k|K_m$ by Theorem 3.3; taking each closed trail in this decomposition in both possible orientations, we obtain $\overrightarrow{T_k}|K_m^*$.

Now, we suppose that k is even. Since $k|m(m-1)$, $\frac{k}{2}|\frac{m(m-1)}{2}$, and so, provided $k \geq 6$, $T_{k/2}|K_m$ by Theorem 3.3. Given a closed trail T in this decomposition, if we traverse T twice, once in each direction, we obtain a directed closed trail of length k. It is now easy to see that $\overrightarrow{T_k}|K_m^*$. If $k = 4$, then $P_2|K_m$ by Theorem 3.6; again, traversing each path twice, once in each direction, we obtain $\overrightarrow{T_4}|K_m^*$. If $k = 2$, then the result is obvious. $\square$

Lemma 3.9. *Suppose $m \geq 2$ and $k \geq 2$ are both even. If $k|m(m-1)$, then $\overrightarrow{T_k}|K_m^*$.*

Proof. Let $k' = k/2$. Note that in this case, we need only show how to decompose $K_m = G_1 \oplus G_2 \oplus \cdots \oplus G_r$ where each G_i is a connected subgraph of K_m containing exactly k' edges. If such a decomposition exists, then we may write $K_m^* = G_1^* \oplus G_2^* \oplus \cdots \oplus G_r^*$. Recall that a directed graph is eulerian if and only if for each vertex v, the indegree of v equals the outdegree of v. It is thus easy to see that each G_i^* would be an eulerian subgraph of K_m^* containing exactly k edges, and hence would be a directed closed trail of length k.

If $k \leq m$, then by Theorem 3.6, K_m is decomposable into paths of length k', and we are done. Suppose then that $k > m$. To decompose K_m into connected subgraphs with k' edges, we first write $k' = st$, where $s|(m/2)$ and $t|(m-1)$, and let $V(K_m) = \mathbb{Z}_m$. For $i = 0, 1, \ldots, (m-1)/t - 1$, let

$$P_i = -\frac{it}{2}, \frac{it}{2} + 1, -\left(\frac{it}{2} + 1\right), \frac{it}{2} + 2, \ldots, \frac{it}{2} + \frac{t+1}{2}$$

if i is even and

$$P_i = \frac{it+1}{2}, -\left(\frac{it+1}{2}\right), \frac{it+1}{2} + 1, -\left(\frac{it+1}{2} + 1\right), \ldots, \frac{it+t}{2}, -\left(\frac{it+t}{2}\right)$$

if i is odd. Let ρ denote the cyclic permutation $(0, 1, 2, \ldots, m-1)$, and for $j = 0, 1, \ldots, m/2 - 1$, let $P_{i,j} = \rho^j(P_i)$. Then the t-paths $P_{i,j}$, $i = 0, 1, \ldots, (m-1)/t - 1$, $j = 0, 1, \ldots, m/2 - 1$, decompose K_m. (In essence, we have taken Walecki's decomposition of K_m into Hamilton paths given in [35] and "chopped" each $(m-1)$-path into $(m-1)/t$ paths of length t.) Note that $t \geq 3$, as $t = 1$ implies $k'|(m/2)$ and hence $k \leq m$. It follows that each path $P_{i,j}$ contains two consecutive vertices q and $q+1$, and so $P_{i,j}$ and $P_{i,j+1}$ share a common vertex. For each $i = 0, 1, \ldots, (m-1)/t - 1$, $j = 0, 1, \ldots, m/(2s) - 1$, let $G_{i,j} = \cup_{r=0}^{s-1} P_{i,sj+r}$, so that $G_{i,j}$ is a connected subgraph of K_m containing $st = k'$ edges; the graphs $G_{i,j}$ decompose K_m. □

Corollary 3.10. *Suppose $m \geq 2$ and $k \geq 2$ are both even, and that $k|m(m-1)$. Then $T_k|2K_m$, and moreover, there exists such a decomposition in which each closed trail contains a pair of parallel edges.*

Lemma 3.11. *Suppose $m \geq 2$ is even, $k \geq 3$ is odd, and $(m,k) \neq (6,3)$. If $k|m(m-1)$, then $\overrightarrow{T_k}|K_m^*$.*

Proof. If $k \leq m$, then $\overrightarrow{C_k}|K_m^*$ by the main result of [2], so we need only consider the case where $k > m$. Write $k = st$, where $s|(m/2)$ and $t|(m-1)$. Let $t' = (m-1)/t$. We consider separately the cases $s < m/2$ and $s = m/2$.

Case 1. $s < m/2$.

Note that in this case, since m is even and s is an odd divisor of m, $3s < m$, so we can assume that $t \geq 5$, as $k = st > m$.

Let the vertex set of K_m^* be $\mathbb{Z}_m$, so that each arc has difference occurring in $\{\pm1, \pm2, \ldots, \pm(\frac{m}{2}-1), \frac{m}{2}\}$. Let ρ denote the cyclic permutation $(0, 1, \ldots, m-1)$ of $\mathbb{Z}_m$. The closed trails of length k in our decomposition will be formed in the following manner. We first form $t'+1$ closed trails of length t, each consisting of a directed triangle together with $(t-3)/2$ directed digons. In each of these closed trails of length t, only five arcs, which form a directed triangle and a digon, will be given explicitly. The remaining $t-5$ arcs will be added in pairs forming directed digons and will not need to be fully specified. Two of these basic closed trails of length t will contain arcs of difference $m/2$, and will together use t differences. The remaining $t'-1$ closed trails of length t will each use t differences. We will form closed trails of length $k = st$ by successive rotation of these basic closed trails until each contains st arcs. Finally, further closed trails of length k will be formed by rotation of these closed k-trails to ensure that all arcs of K_m^* are used up. We formalize this procedure as follows.

First, we construct the directed closed 5-trails that will form the anchors for the $t'+1$ closed trails of length t. We start with the two closed 5-trails containing arcs of difference $m/2$. If $m \equiv 0 \pmod 4$, let

$$\begin{aligned} T &= 0, 1, \frac{m}{2}, 0, -1, 0 \\ \text{and} \quad \widetilde{T} &= 0, -\left(\frac{m}{2} - 1\right), 1, 0, \frac{m}{2} - 1, 0. \end{aligned}$$

If $m \equiv 2 \pmod 4$, let

$$\begin{aligned} T &= 0, 1, \frac{m}{2}, 0, -1, 0 \\ \text{and} \quad \widetilde{T} &= 0, \frac{m}{2}, 1, 0, \frac{m}{2} - 1, 0. \end{aligned}$$

In each of these two cases, T and $\widetilde{T}$ together contain two arcs of each of the differences in $\{\pm 1, \pm(\frac{m}{2} - 1), \frac{m}{2}\}$. However, this will not be a problem. Let the difference j of an arc $(i, i+j)$ be called *odd* if i is odd (denoted j_o) and *even* if i is even (denoted j_e). We note that between T and $\widetilde{T}$, each of the differences 1_e, 1_o, $(-1)_e$, $(-1)_o$, $(\frac{m}{2} - 1)_e$, $(\frac{m}{2} - 1)_o$, $-(\frac{m}{2} - 1)_e$, $-(\frac{m}{2} - 1)_o$, $(\frac{m}{2})_e$, $(\frac{m}{2})_o$ occurs exactly once. This fact will ensure that among the trails $\rho^{2i}(T)$ and $\rho^{2i}(\widetilde{T})$, where $0 \leq i \leq \frac{m}{2} - 1$, each arc of difference in $\{\pm 1, \pm(\frac{m}{2} - 1), \frac{m}{2}\}$ occurs exactly once. Also, in the case $m \equiv 2 \pmod 4$, since $\gcd(m, \frac{m}{2} - 1) = \gcd(m, \frac{m-2}{2}) = 2$, it follows that for any closed trail τ in K_m, $\{\tau, \rho^{m/2-1}(\tau), \rho^{2(m/2-1)}(\tau), \ldots, \rho^{(m/2-1)^2}(\tau)\}$ $= \{\tau, \rho^2(\tau), \rho^4(\tau), \ldots, \rho^{2(m/2-1)}(\tau)\}$. Hence among the trails $\rho^{i(m/2-1)}(T)$ and $\rho^{i(m/2-1)}(\widetilde{T})$, where $0 \leq i \leq \frac{m}{2} - 1$, each arc of difference in $\{\pm 1, \pm(\frac{m}{2} - 1), \frac{m}{2}\}$ occurs exactly once if $m \equiv 2 \pmod 4$.

If $t' \geq 2$, we construct a further $t' - 1 = 2(\alpha + \beta)$ directed 5-trails. The numbers α and β are specified as follows:

$$\alpha = \begin{cases} \frac{t'-1}{4}, & \text{if } t' \equiv 1 \pmod 4 \\ \frac{t'+1}{4}, & \text{if } t' \equiv 3 \pmod 4 \end{cases} \quad \text{and} \quad \beta = \begin{cases} \frac{t'-1}{4}, & \text{if } t' \equiv 1 \pmod 4 \\ \frac{t'-3}{4}, & \text{if } t' \equiv 3 \pmod 4 \end{cases}.$$

Let $J = \{5i + 2 : i = 0, 1, \ldots, \alpha - 1\} \cup \{5i + 4 : i = 0, 1, \ldots, \beta - 1\}$. For each $j \in J$, we now construct two directed 5-trails, T_j and $\widetilde{T}_j$.

For $0 \leq i \leq \alpha - 1$, let

$$T_{5i+2} = 5i+2, \frac{m}{2} - \frac{t'+1}{2} + 2i, 0, 5i+2, \frac{m}{2} - \frac{t'-1}{2} + 2i, 5i+2.$$

If $t' \equiv 1 \pmod 4$, let

$$\widetilde{T_2} = 2, 0, -\left(\frac{m}{2} - \frac{t'+5}{2}\right), 2, \frac{5t'+7}{4}, 2,$$

while if $t' \equiv 3 \pmod 4$, let

$$\widetilde{T_2} = 2, 0, -\left(\frac{m}{2} - \frac{t'+5}{2}\right), 2, \frac{5t'+5}{4}, 2.$$

For $1 \leq i \leq \alpha - 1$, let

$$\widetilde{T}_{5i+2} = 5i+2, 0, -\left(\frac{m}{2} - \frac{t'+5}{2} - 3i\right), 5i+2, 1, 5i+2.$$

For $0 \leq i \leq \beta - 1$, let

$$T_{5i+4} = 0, 5i+4, \frac{m}{2} - \frac{t'-1}{2} + 2i, 0, 5i+3, 0$$

and

$$\widetilde{T}_{5i+4} = 5i+4, 0, -\left(\frac{m}{2} - \frac{t'+7}{2} - 3i\right), 5i+4, -1, 5i+4.$$

It is not difficult to check that among the 5-trails T_j and $\widetilde{T_j}$ for $j \in J$, there appears exactly one arc of each difference in

$$\{\pm 2, \pm 3, \ldots, \pm\gamma\} \cup \left\{\pm\delta, \pm(\delta+1), \ldots, \pm\left(\frac{m}{2} - 2\right)\right\},$$

where $\gamma = \frac{5t'-1}{4}$ and $\delta = \frac{m}{2} - \frac{5t'-1}{4}$ if $t' \equiv 1 \pmod 4$, and $\gamma = \frac{5t'-3}{4}$ and $\delta = \frac{m}{2} - \frac{5t'+1}{4}$ if $t' \equiv 3 \pmod 4$.

There remain $m - (5t'+1)$ differences which are not used in any of the 5-trails T, $\widetilde{T}$, T_j, $\widetilde{T_j}$ (for $j \in J$), and these unused differences occur in pairs $\pm d$, where $d \in \{\gamma+1, \gamma+2, \ldots, \delta-1\}$. We extend the 5-trails T, $\widetilde{T}$, T_j, $\widetilde{T_j}$ to closed trails of length t by distributing these differences as follows.

We first extend the 5-trails T and $\widetilde{T}$ by choosing a set S consisting of $(t-5)/2$ unused differences from $\{\gamma+1, \gamma+2, \ldots, \delta-1\}$. For each difference $d \in S$, we add the arcs $(0, d)$ and $(d, 0)$ to T, and arcs $(1, d+1)$ and $(d+1, 1)$ to $\widetilde{T}$. In this way, we obtain closed t-trails T' and $\widetilde{T}'$ from T and $\widetilde{T}$, respectively. Note that the trails $\rho^{2i}(T')$ and $\rho^{2i}(\widetilde{T}')$ for $i \in \{0, 1, \ldots, \frac{m}{2}-1\}$ decompose the circulant digraph $\overrightarrow{X}(m; \pm S \cup \{\pm 1, \pm(\frac{m}{2}-1), \frac{m}{2}\})$.

For each of the remaining closed trails T_j, form T_j' by choosing a set S_j of $(t-5)/2$ unused differences from $\{\gamma+1, \gamma+2, \ldots, \delta-1\}$ and adding the arcs $(0, d)$ and $(d, 0)$ to those of T_j for each $d \in S_j$. Similarly, form $\widetilde{T}_j'$ from $\widetilde{T}_j$ using a set $\widetilde{S}_j$ of $(t-5)/2$ of the remaining unused differences from $\{\gamma+1, \gamma+2, \ldots, \delta-1\}$. We note that the differences in T_j and $\widetilde{T}_j$ are $d_1^j, d_2^j, d_3^j, d_4^j, -d_4^j$ and $-d_1^j, -d_2^j, -d_3^j, d_5^j, -d_5^j$, respectively, for some $\{d_1^j, d_2^j, d_3^j, d_4^j, d_5^j\} \subseteq \{\pm 2, \ldots, \pm\gamma\} \cup \{\pm\delta, \ldots, \pm(\frac{m}{2}-2)\}$. Moreover, the sets $\{d_1^j, \ldots, d_5^j\}$ are pairwise disjoint for distinct j. Hence the circulant digraph $\overrightarrow{X}(m, \pm\{d_1^j, \ldots, d_5^j\} \cup \pm S_j \cup \pm\widetilde{S}_j)$ decomposes as $\bigoplus_{i=0}^{m-1}(\rho^i(T_j') \oplus \rho^i(\widetilde{T}_j'))$.

Notice that the sets S, S_j and $\widetilde{S}_j$, for all $j \in J$, together partition the set $\{\gamma+1, \gamma+2, \ldots, \delta-1\}$. Hence the closed t-trails T', $\widetilde{T}'$, T_j' and $\widetilde{T}_j'$ (for all $j \in J$) use each of the differences in $\{\pm 1, \pm 2, \ldots, \pm(\frac{m}{2}-1), \frac{m}{2}\}$ precisely once.

Finally, we show how to use the t-trails T', $\widetilde{T}'$, T_j' and $\widetilde{T}_j'$ (for all $j \in J$) to construct closed directed k-trails that decompose K_m^*. First, suppose $m \equiv 0 \pmod 4$. Then, since T' contains both vertices -1 and 1, and $\widetilde{T}'$ contains both vertices $\frac{m}{2}-1$ and $-(\frac{m}{2}-1) = \frac{m}{2}+1$, note that for $i \in \{0, 1, \ldots, \frac{m}{2}-2\}$, the underlying graphs of both $\rho^{2i}(T') \oplus \rho^{2i+2}(T')$ and $\rho^{2i}(\widetilde{T}') \oplus \rho^{2i+2}(\widetilde{T}')$ are connected. Let $\mathscr{T} = T' \oplus \rho^2(T') \oplus \cdots \oplus \rho^{2s-2}(T')$ and $\widetilde{\mathscr{T}} = \widetilde{T}' \oplus \rho^2(\widetilde{T}') \oplus \cdots \oplus \rho^{2s-2}(\widetilde{T}')$. Then each of $\mathscr{T}$ and $\widetilde{\mathscr{T}}$ are closed trails of length k, and moreover,

$$\overrightarrow{X}\left(m; \pm S \cup \left\{\pm 1, \pm\left(\frac{m}{2}-1\right), \frac{m}{2}\right\}\right) = \bigoplus_{i=0}^{m/(2s)-1}\left(\rho^{2si}(\mathscr{T}) \oplus \rho^{2si}(\widetilde{\mathscr{T}})\right).$$

In the case that $m \equiv 2 \pmod 4$, note that since T and $\widetilde{T}$ each contain vertices 1 and $m/2$, we have that the underlying graphs of both $\rho^{i(\frac{m}{2}-1)}(T') \oplus \rho^{(i+1)(\frac{m}{2}-1)}(T')$

and $\rho^{i(\frac{m}{2}-1)}(\widetilde{T}') \oplus \rho^{(i+1)(\frac{m}{2}-1)}(\widetilde{T}')$ are connected for each $i \in \{0, 1, \ldots, \frac{m}{2} - 2\}$. Let $\mathscr{T} = T' \oplus \rho^{\frac{m}{2}-1}(T') \oplus \rho^{2(\frac{m}{2}-1)}(T') \oplus \cdots \oplus \rho^{(s-1)(\frac{m}{2}-1)}(T')$ and $\widetilde{\mathscr{T}} = \widetilde{T}' \oplus \rho^{\frac{m}{2}-1}(\widetilde{T}') \oplus \rho^{2(\frac{m}{2}-1)}(\widetilde{T}') \oplus \cdots \oplus \rho^{(s-1)(\frac{m}{2}-1)}(\widetilde{T}')$. Then each of $\mathscr{T}$ and $\widetilde{\mathscr{T}}$ are closed trails of length k, and moreover,

$$\overrightarrow{X}\left(m; \pm S \cup \left\{\pm 1, \pm\left(\frac{m}{2} - 1\right), \frac{m}{2}\right\}\right) = \bigoplus_{i=0}^{m/(2s)-1} \left(\rho^{is(\frac{m}{2}-1)}(\mathscr{T}) \oplus \rho^{is(\frac{m}{2}-1)}(\widetilde{\mathscr{T}})\right).$$

Let $\mathscr{T}_2 = T_2' \oplus \rho^2(T_2') \oplus \cdots \oplus \rho^{2s-2}(T_2')$ and $\widetilde{\mathscr{T}_2} = \widetilde{T}_2' \oplus \rho^2(\widetilde{T}_2') \oplus \cdots \oplus \rho^{2s-2}(\widetilde{T}_2')$. Since T_2 and $\widetilde{T}_2$ contain vertices 0 and 2, $\mathscr{T}_2$ and $\widetilde{\mathscr{T}_2}$ are closed trails of length k. Also,

$$\begin{aligned}&\overrightarrow{X}(m; \pm\{d_1^2, d_2^2, d_3^2, d_4^2, d_5^2\} \cup \pm S_2 \cup \pm \widetilde{S}_2)\\ &= \bigoplus_{i=0}^{m/(2s)-1} \left(\rho^{2si}(\mathscr{T}_2) \oplus \rho^{2si+1}(\mathscr{T}_2) \oplus \rho^{2si}(\widetilde{\mathscr{T}_2}) \oplus \rho^{2si+1}(\widetilde{\mathscr{T}_2})\right).\end{aligned}$$

For $1 \leq i \leq \alpha - 1$, let $\mathscr{T}_{5i+2} = T'_{5i+2} \oplus \rho(T'_{5i+2}) \oplus \cdots \oplus \rho^{s-1}(T'_{5i+2})$ and $\widetilde{\mathscr{T}_{5i+2}} = \widetilde{T}'_{5i+2} \oplus \rho(\widetilde{T}'_{5i+2}) \oplus \cdots \oplus \rho^{s-1}(\widetilde{T}'_{5i+2})$. For $0 \leq i \leq \beta - 1$, let $\mathscr{T}_{5i+4} = T'_{5i+4} \oplus \rho(T'_{5i+4}) \oplus \cdots \oplus \rho^{s-1}(T'_{5i+4})$ and $\widetilde{\mathscr{T}_{5i+4}} = \widetilde{T}'_{5i+4} \oplus \rho(\widetilde{T}'_{5i+4}) \oplus \cdots \oplus \rho^{s-1}(\widetilde{T}'_{5i+4})$. Then for each $j \in J$, since T_j and $\widetilde{T}_j$ contain two vertices that differ by 1, $\mathscr{T}_j$ and $\widetilde{\mathscr{T}_j}$ are both closed trails of length k, and moreover,

$$\overrightarrow{X}(m; \pm\{d_1^j, d_2^j, d_3^j, d_4^j, d_5^j\} \cup \pm S_j \cup \pm \widetilde{S}_j) = \bigoplus_{i=0}^{m/s-1} \left(\rho^{si}(\mathscr{T}_j) \oplus \rho^{si}(\widetilde{\mathscr{T}_j})\right).$$

We can now see that the trails $\rho^{2si}(\mathscr{T})$, $\rho^{2si}(\widetilde{\mathscr{T}})$, $\rho^{2si}(\mathscr{T}_2)$, $\rho^{2si}(\widetilde{\mathscr{T}_2})$, $\rho^{2si+1}(\mathscr{T}_2)$, $\rho^{2si+1}(\widetilde{\mathscr{T}_2})$, where $0 \leq i \leq m/(2s) - 1$, together with $\rho^{si}(\mathscr{T}_j)$, $\rho^{si}(\widetilde{\mathscr{T}_j})$ where $0 \leq i \leq m/s - 1$ and $j \in J$, decompose $K_m^* = \overrightarrow{X}(m; \pm\{1, 2, \ldots, \frac{m}{2}\})$.

Case 2. $s = m/2$.

We first deal with the case $m = 6$. Here $s = 3$ and $t = 5$. The following two directed closed trails of length 15 decompose K_6^*: $0, 1, 2, 3, 4, 5, 0, 3, 5, 2, 4, 1, 0, 4, 3, 0$ and $0, 5, 1, 4, 2, 5, 3, 2, 1, 3, 1, 5, 4, 0, 2, 0$.

Suppose now that $m > 6$ and let $t' = (m-1)/t$, and note that t and t' must have the same congruence modulo 4, since $m - 1 = tt' \equiv 1 \pmod 4$. To manage this case, we begin with the decomposition of K_m^* into directed cycles of length s given in [2, Lemma 5.8], and paste the cycles together, t at a time, to form directed closed trails of length k. The $\overrightarrow{C_s}$-decomposition from [2, Lemma 5.8] is formed in the following manner. First, partition the vertices of K_m^* into two sets, $U = \{u_0, u_1, \ldots, u_{s-1}\}$ and $V = \{v_0, v_1, \ldots, v_{s-1}\}$, and let ω be the permutation $(u_0, u_1, \ldots, u_{s-1})(v_0, v_1, \ldots, v_{s-1})$. Let C_1 be the directed cycle $u_0, v_{s-1}, u_1, v_{s-2}, \ldots, v_{(s+1)/2}, u_{(s-1)/2}, u_0$. The cycle C_2 is C_1 taken in the opposite direction. The cycle C_3 is taken to be a particular cycle $\widetilde{C}$, with the arc (v_0, v_2) replaced by the path v_0, u_0, u_1, v_1, v_2, where $\widetilde{C}$ is a directed cycle of length $s-3$ in $K_m^*[V]$, the subgraph of K_m^* induced by V, containing the arc (v_0, v_2), but not using vertex v_1. The decomposition consists of directed cycles $\omega^i(C_j)$ where $0 \le i \le s-1$ and $j \in \{1, 2, 3\}$, together with two directed Hamilton cycles, C_V and C'_V, in $K_m^*[V]$ and $s-4$ directed Hamilton cycles in $K_m^*[U]$. Of the directed Hamilton cycles in $K_m^*[U]$, one of them, $\hat{C}$, consists of arcs of difference $-1 \in \mathbb{Z}_{m/2}$ in U, while the remaining occur in pairs $\hat{C}_i$, $\hat{C}'_i$, $0 \le i \le (s-7)/2$, where the paired cycles represent the same undirected cycle oriented in each direction. To form closed trails of length k, we seek sets of t of these directed cycles whose union has its underlying graph connected.

Note that

1. For each $i \in \{0, 1, \ldots, s-2\}$, $j \in \{1, 2, 3\}$, $\omega^i(C_j) \cup \omega^{i+1}(C_j)$ has a connected underlying graph, since both cycles contain vertex u_{i+1}.

2. All other cycles are contained either in $K_m^*[U]$ or $K_m^*[V]$.

3. For each $i \in \{0, 1, \ldots, s-2\}$, $\omega^{i+1}(C_3)$ contains the arc (v_{i+1}, u_{i+1}) and $\omega^i(C_3)$ contains the arc (u_{i+1}, v_{i+1}).

We now form our decomposition of K_m^* into directed closed trails of length $k = st$.

If $t' = 1$, then this decomposition consists of two directed closed trails. These are T, which contains directed cycles $\omega^i(C_1)$ for $1 \le i \le s-1$ and $\omega^j(C_2)$ for $0 \le j \le s-1$, and T', which contains the remaining cycles in the decomposition of K_m^* into directed s-cycles, namely C_1, the two directed Hamilton cycles in $K_m^*[V]$, the $s-4$ directed Hamilton cycles in $K_m^*[U]$, and the cycles $\omega^i(C_3)$ for $0 \le i \le s-1$. Notice that the underlying graph of T is connected since $\omega^i(C_j) \oplus \omega^{i+1}(C_j)$ has connected underlying graph for each $i \in \{0, 1, \ldots, s-2\}$ and $j \in \{1, 2\}$, and $\omega^{s-1}(C_1)$ and $\omega^{s-1}(C_2)$ both contain vertex u_0. The underlying graph of T' is connected since C_1 contains vertices in both U and V, and C_1 and C_3 both contain vertex u_0.

If $t' > 1$, we take the following closed trails in our decomposition:

- For each $0 \le i \le (t'-3)/2$, take $T_i = \omega^{it+1}(C_1) \oplus \omega^{it+2}(C_1) \oplus \cdots \oplus \omega^{(i+1)t}(C_1)$.
- For each $0 \le i \le (t'-3)/2$, take $T_i' = \omega^{it+1}(C_2) \oplus \omega^{it+2}(C_2) \oplus \cdots \oplus \omega^{(i+1)t}(C_2)$.
- The remaining unused directed cycles among $\omega(C_1)$, $\omega^2(C_1)$, $\ldots$, $\omega^{s-1}(C_1)$, C_2, $\omega(C_2)$, $\ldots$, $\omega^{s-1}(C_2)$ are $\omega^{s-(t-1)/2}(C_1)$, $\omega^{s-(t-3)/2}(C_1)$, $\ldots$, $\omega^{s-1}(C_1)$, C_2, $\omega^{s-(t-1)/2}(C_2)$, $\omega^{s-(t-3)/2}(C_2)$, $\ldots$, $\omega^{s-1}(C_2)$, which number t. Take $T = (\omega^{s-(t-1)/2}(C_1) \oplus \cdots \oplus \omega^{s-1}(C_1)) \oplus C_2 \oplus (\omega^{s-(t-1)/2}(C_2) \oplus \omega^{s-(t-3)/2}(C_2) \oplus \cdots \oplus \omega^{s-1}(C_2))$. Observe that the underlying graph of T is connected since C_2, $\omega^{s-1}(C_1)$ and $\omega^{s-1}(C_2)$ all contain vertex u_0.
- Take T' to consist of C_1 together with the two directed Hamilton cycles in $K_m^*[V]$ and $t-3$ directed Hamilton cycles in $K_m^*[U]$, that is, $T' = C_1 \oplus C_V \oplus C_V' \oplus \hat{C}_0 \oplus \hat{C}_0' \oplus \hat{C}_1 \oplus \hat{C}_1' \oplus \cdots \oplus \hat{C}_{(t-5)/2} \oplus \hat{C}'_{(t-5)/2}$. Observe that the underlying graph of T' is connected since C_1 contains vertices of both U and V. The number of unused cycles among the Hamilton cycles in $K_m^*[U]$ is now $s-4-(t-3) = s-t-1 = \frac{tt'+1}{2} - t - 1 = \frac{t(t'-2)-1}{2} \ge \frac{t-1}{2}$.
- For each $0 \le i \le (t'-3)/2$, take $\widetilde{T}_i = \omega^{it}(C_3) \oplus \omega^{it+1}(C_3) \oplus \cdots \oplus \omega^{it+(t-1)}(C_3)$. Since C_3 contains vertices u_0 and u_1, it follows that the underlying graph of $\widetilde{T}_i$

is connected. Note that the number of trails formed here, $(t'-1)/2$, is even if $t \equiv 1 \pmod 4$ and odd if $t \equiv 3 \pmod 4$.

- The unused cycles among $C_3, \omega(C_3), \ldots, \omega^{s-1}(C_3)$ are $\omega^i(C_3)$ for $s-1-(t-1)/2 \le i \le s-1$, which number $(t+1)/2$. Take $\widetilde{T}$ to contain these cycles together with $(t-1)/2$ of the Hamilton cycles in $K_m^*[U]$. If $(t-1)/2$ is even, i.e. $t \equiv 1 \pmod 4$, we take these to be $\hat{C}_{(t-3)/2}$, $\hat{C}'_{(t-3)/2}$, $\hat{C}_{(t-1)/2}$, $\hat{C}'_{(t-1)/2}$, $\ldots$, $\hat{C}_{(3t-11)/4}$, $\hat{C}_{(3t-7)/4}$. The inclusion of $\hat{C}_{(3t-7)/4}$ rather than $\hat{C}'_{(3t-11)/4}$ will become important in Corollary 3.12, where it will ensure that there is an arc (x,y) in $\widetilde{T}$ for which the directed closed trail $\hat{T}$, which contains $\hat{C}'_{(3t-11)/4}$, has the arc (y,x). If $(t-1)/2$ is odd, i.e. $t \equiv 3 \pmod 4$, and $t' > 3$, we take the Hamilton cycles in $K_m^*[U]$ which comprise part of $\widetilde{T}$ to be $\hat{C}_{(t-3)/2}$, $\hat{C}'_{(t-3)/2}$, $\hat{C}_{(t-1)/2}$, $\hat{C}'_{(t-1)/2}$, $\ldots$, $\hat{C}_{(3t-13)/4}$, $\hat{C}'_{(3t-13)/4}$, $\hat{C}_{(3t-9)/4}$. If $t' = 3$, then all of the $(t-1)/2$ remaining Hamilton cycles in $K_m^*[U]$, namely $\hat{C}$, $\hat{C}_{(t-3)/2}$, $\hat{C}'_{(t-3)/2}$, $\hat{C}_{(t-1)/2}$, $\hat{C}'_{(t-1)/2}$, $\ldots$, $\hat{C}_{(s-7)/2}$, $\hat{C}'_{(s-7)/2}$, will be used in forming $\widetilde{T}$.

- The remaining Hamilton cycles in $K_m^*[U]$ are grouped into groups of size t. The number of these groups is $(s-4-(t-3)-(t-1)/2)/t = (s-1-(3t-1)/2)/t = ((2s-1)-3t)/(2t) = (tt'-3t)/(2t) = (t'-3)/2$, which is odd if $t \equiv 1 \pmod 4$, and even if $t \equiv 3 \pmod 4$.

  If $t \equiv 1 \pmod 4$, we let the first group, forming $\hat{T}$, contain $\hat{C}$, $\hat{C}'_{(3t-11)/4}$ and $\hat{C}'_{(3t-7)/4}$, together with $\hat{C}_i, \hat{C}'_i$ for $(3t-3)/4 \le i \le (5t-13)/4$. We distribute the cycles $\hat{C}_i$ and $\hat{C}'_i$ for $(5t-9)/4 \le i \le (t'-5)/4 + (5t-13)/4$ into the remaining $(t'-5)/2$ groups which will form trails $\hat{T}_i, \hat{T}'_i$, for $1 \le i \le (t'-5)/4$, so that $\hat{T}_i$ contains cycle $\hat{C}_{(5t-13)/4+i}$ and $\hat{T}'_i$ contains cycle $\hat{C}'_{(5t-13)/4+i}$; the remaining cycles, namely $\hat{C}_{(5t-13)/4+(t'-5)/4+1}$, $\hat{C}_{(5t-13)/4+(t'-5)/4+2}$, $\ldots$, $\hat{C}_{(s-7)/2}$ and $\hat{C}'_{(5t-13)/4+(t'-5)/4+1}$, $\hat{C}'_{(5t-13)/4+(t'-5)/4+2}$, $\ldots$, $\hat{C}'_{(s-7)/2}$, are distributed into the groups forming $\hat{T}_i$ and $\hat{T}'_i$ in any way so that each group contains t cycles in total.

If $t \equiv 3 \pmod 4$ and $t' > 3$, we distribute the cycles $\hat{C}_i$ and $\hat{C}'_i$ for $(3t-5)/4 \leq i \leq (3t-9)/4 + (t'-3)/4$ into groups forming closed trails $\hat{T}_i$ and $\hat{T}'_i$, $1 \leq i \leq (t'-3)/4$, so that $\hat{T}_i$ contains $\hat{C}_{(3t-9)/4+i}$ and $\hat{T}'_i$ contains $\hat{C}'_{(3t-9)/4+i}$. All the remaining cycles are distributed into the groups forming $\hat{T}_i$ and $\hat{T}'_i$ in any way so that each group contains t cycles in total.

□

If we ignore the direction on edges, Lemma 3.11 has the following corollary.

Corollary 3.12. *Let $m \geq 2$ be even and $k \geq 3$ be odd, and suppose that $k|m(m-1)$. Then $T_k|2K_m$. Moreover, there exists such a decomposition whose closed trails may be partitioned into pairs with the following property (called the* pairing property*): if T and T' are paired closed trails, then there is an edge in T and an edge in T' which are parallel.*

Proof. That $T_k|2K_m$ follows directly from Lemma 2.10, except in the case that $(m,k) = (6,3)$. In this case, the following ten 3-cycles decompose $2K_6$ on vertex set $\mathbb{Z}_6$: $0,1,2,0$; $0,1,3,0$; $0,3,4,0$; $2,3,4,2$; $1,2,4,1$; $1,4,5,1$; $0,2,5,0$; $0,4,5,0$; $1,3,5,1$; $2,3,5,2$. It is easy to check that these cycles satisfy the pairing property.

In all the remaining cases except the case $k=3$, we show that there exists a decomposition of K^*_m into directed closed k-trails whose closed trails may be paired in such a way that if T and T' are paired directed closed trails, then T contains the reversal of an arc in T'. The result will then follow immediately. For $k=3$, we will show that there exists a T_3-decomposition of $2K_m$ with the pairing property.

Case 1. $k \leq m$

If $k \leq m$, then the decompositions ensured by Lemma 3.11 were those directed cycle decompositions of K^*_m guaranteed by [2, Theorem 1.1]. In the case $k \geq 5$, these directed cycles are constructed in [2]. If $m = k+1$, then $\overrightarrow{C_k}|K^*_{k+1}$ by [2, Lemma 5.7]. Here, the graph K^*_{k+1} is viewed as $K^*_k \bowtie K^*_1$, with the vertex of K^*_1 called central. The decomposition consists of a single directed cycle $\tilde{C}$ containing k arcs of length $\frac{k-1}{2}$, together with directed cycles C, $\rho(C)$, ..., $\rho^{k-1}(C)$, where C is a cycle of length k containing arcs of differences $\pm 1, \pm 2, \ldots, \pm(\frac{k-3}{2})$ and $\frac{k-1}{2}$, and two arcs incident with the central vertex, and the permutation ρ cyclically permutes the vertices of K_k^* and leaves the central vertex fixed. The cycle C contains the arc $(0,1)$, while $\rho^{(k+1)/2}(C)$ contains the arc $(-1,0)$. Pair $\rho^i(C)$ with $\rho^{i+(k+1)/2}(C)$ for $0 \leq i \leq \frac{k-3}{2}$. This leaves $\rho^{(k-1)/2}(C)$ to be paired with $\tilde{C}$, and $\tilde{C}$ contains an arc opposite the arc of length $\frac{k-1}{2}$ in $\rho^{(k-1)/2}(C)$.

A $\overrightarrow{C_k}$-decomposition of K_{2k}^* in constructed in [2, Lemma 5.8]; this decomposition has already been employed in the proof of Case 2 of Lemma 3.11. Recall that in the construction, the vertices of K_{2k}^* are partitioned into two sets $U = \{u_0, \ldots, u_{k-1}\}$ and $V = \{v_0, \ldots, v_{k-1}\}$, with differences in $\mathbb{Z}_k$ for vertices in the subgraphs induced by U and V. If $k \geq 7$, then the decomposition contains a cycle C_3 which contains arcs of differences $-2, \pm 3, \pm 4, \ldots, \pm k$ in V together with the directed path v_0, u_0, u_1, v_1, v_2 (which includes arcs of difference 1 in U and V), and also cycles $\omega(C_3), \ldots, \omega^{k-1}(C_3)$, where $\omega = (u_0, u_1, \ldots, u_{k-1})(v_0, v_1, \ldots, v_{k-1})$ cyclically permutes U and V. There are also cycles C_V which contains all arcs of difference -1 in V, C_V', which contains all arcs of difference 2 in V, and $\hat{C}$, which contains all arcs of difference -1 in U. Noting that C_3 contains the arc (u_1, v_1) and $\omega(C_3)$ contains the arc (v_1, u_1), we may pair $\omega^{2i}(C_3)$ with $\omega^{2i+1}(C_3)$ for $0 \leq i \leq \frac{m-5}{2}$. Pair $\omega^{m-3}(C_3)$ with C_V (noting that C_V contains the arc opposite the arc of difference 1 in V in $\omega^{m-3}(C_3)$), $\omega^{m-2}(C_3)$ with C_V' (noting that C_V' contains the arc opposite the arc of difference -2 in V in $\omega^{m-2}(C_3)$) and $\omega^{m-1}(C_3)$ with $\hat{C}$ (noting that $\hat{C}$ contains the arc opposite the arc of dfference -1 in U in $\omega^{m-1}(C_3)$). The remaining cycles (or all cycles in the case that $k = 5$) occur in pairs which consist of the same underlying undirected cycle taken in both possible orientations.

In all other cases with $k \leq m < 3k$, a $\overrightarrow{C_k}$-decomposition of K_m^* may be found in [2, Theorem 5.1]. In each case of this theorem, the graph K_m^* is viewed as the join of K^*_{m-1} with K^*_1, where the vertex of K^*_1 is again called central. We let the integer ℓ and permutation ρ be defined as in [2, Theorem 5.1]; that is, ℓ is a particular divisor of $m-1$ and $\rho = (0, 1, \ldots, m-2)$, where $V(K^*_{m-1}) = \mathbb{Z}_{m-1}$. The decomposition consists of directed cycles $\rho^j(C_1)$, $\rho^j(C_1')$, $\rho^j(C_2)$, $\rho^j(C_2')$, …, for $j = 0, 1, \ldots, m-2$ and special directed cycles C, $\rho(C)$, …, $\rho^{\ell-1}(C)$ and C^*, $\rho(C^*)$, …, $\rho^{m-2}(C^*)$, where the cycles generated by C^* contain the central vertex. For all i, the cycles C_i and C_i' are reversals of each other; hence the directed cycles C_1, C_1', C_2, C_2', …, use all differences of the form $\pm d$ for d from a certain subset of $\{1, 2, \ldots, m/2-1\}$. That means that the sum of the differences used in cycles C and C^* is 0. The cycle C uses ℓ distinct differences, none of which is the negative of another, and the sum of the differences used in C is ℓ. This means that the sum of differences used in C^* (not counting the two arcs going to and from the central vertex) is $-\ell$, and the two arcs incident with the central vertex are $(u, 0)$ and $(-\ell, u)$, where u is the central vertex. In addition, C^* must contain a difference that is the negative of a difference in C. Let $C^{**} = \rho^q(C^*)$ be such that, for some arc (v, v') in C, $\rho^q(C^*)$ contains the arc (v', v). Pair C, $\rho(C)$, …, $\rho^{\ell-1}(C)$ with C^{**}, $\rho(C^{**})$, …, $\rho^{\ell-1}(C^{**})$, respectively. We are left with the cycles $\rho^{\ell}(C^{**})$, $\rho^{\ell+1}(C^{**})$, …, $\rho^{m-2}(C^{**})$ to be paired. Noting that $m-1$ is an odd multiple of ℓ, we may pair $\rho^{(2i-1)\ell+j}(C^{**})$, which contains the arc $(u, q+(2i-1)\ell+j)$, with $\rho^{2i\ell+j}(C^{**})$, which contains the arc $(q+(2i-1)\ell+j, u)$, for each $i = 1, \ldots, (m-1-\ell)/(2\ell)$, $j = 0, 1, \ldots, \ell-1$.

If $m \geq 3k$, then the $\overrightarrow{C_k}$-decomposition of K^*_m is given by [2, Lemma 5.6]. This case is handled in this lemma by first decomposing K^*_m into subgraphs of the forms K^*_n where $k|n(n-1)$ and $k < n < 3k$, $(K_q * \overline{K_{2k}})^*$, and the join of K^*_{2k} with either $\overline{K^*_n}$ or $\overline{K^*_{2k+n}}$. Of these subgraphs, K^*_n is decomposed according to [2, Theorem 5.1], and we have already shown that these cycles may be paired in the desired way. The decomposition of $(K_q * \overline{K_{2k}})^*$ is done by decomposing the undirected graph $K_q * \overline{K_{2k}}$ into k-cycles and taking each cycle in both possible orientations; the pairing of these directed cycles may be done in the natural way. It only remains to consider the cycles from the decompositions of $K^*_{2k} \bowtie \overline{K^*_n}$ and $K^*_{2k} \bowtie \overline{K^*_{2k+n}}$, which are taken from decompositions detailed in [2, Propositions 5.4 and 5.5]. The decompositions of [2, Proposition 5.4] consist of cycles C_1, $\omega(C_1)$, $\ldots$, $\omega^{k-1}(C_1)$ and C_2, $\omega(C_2)$, $\ldots$, $\omega^{k-1}(C_2)$, where for each i, $\omega^i(C_1)$ contains an arc which is the reversal of an arc of $\omega^i(C_2)$, and so we may pair $\omega^i(C_1)$ with $\omega^i(C_2)$. The decompositions detailed in [2, Proposition 5.5] consist of a set of undirected cycles oriented in each direction, together with digraphs which are decomposed into directed cycles via [2, Proposition 5.4], and hence the decompositions have the pairing property. Finally, in addition to the cycles from [2, Propositions 5.4 and 5.5], some outlying cases in the lemma have additional directed cycles consisting of the cycles of a $\overrightarrow{C_7}$-decomposition of K^*_8 (this is the case $m = k+1$ mentioned above and may be decomposed with the pairing property), the cycles of a $\overrightarrow{C_5}$-decomposition of K^*_5 (which may be obtained by taking the undirected cycles of a C_5-decomposition of K_5 in both orientations), or two directed 5-cycles which are the reversal of each other.

Suppose now that $k = 3$. In [5, Theorem 2], it is proved by induction that $C_3|2K_m$ whenever $m \equiv 0$ or $1 \pmod 3$. To show that $C_3|2K_m$, let us walk through a similar proof, with the added requirement that the decompositions obtained satisfy the desired pairing property. Changes have been made to the case structure in order to make the exhibition of the pairing property simpler. We note that both odd and even values of m must be included in the induction. We also take note of the result of [21] that $C_3|K_m * \overline{K_n}$ if and only if $(m-1)n$ is even and $3|\binom{m}{2}n^2$.

First, consider the base cases $m = 3, 4, 6$. We note that the two cycles $0, 1, 2, 0$ and $0, 1, 2, 0$ decompose $2K_3$, and clearly the pairing property is satisfied. For $m = 4$, $2K_4$ decomposes into the cycles $0, 1, 2, 0$; $0, 1, 3, 0$; $1, 2, 3, 1$; and $0, 2, 3, 0$, and again the pairing property is clearly satisfied. The case $m = 6$ may be dealt with by the C_3-decomposition of $2K_6$ given at the beginning of this proof.

Now, suppose that $m \equiv 0$ or $1 \pmod 3$, and that for any $n < m$ with $n \equiv 0$ or $1 \pmod 3$, $C_3|2K_n$, and there is such a decomposition in which the cycles may be paired as desired.

If $m \equiv 0 \pmod 9$ or $m \equiv 3 \pmod 9$, then $2K_m$ decomposes as $2K_{m/3} \oplus 2K_{m/3} \oplus 2K_{m/3} \oplus 2(K_3 * \overline{K_{m/3}})$. By the induction hypothesis, $C_3 | 2K_{m/3}$, with the pairing property satisfied. Moreover, $C_3 | K_3 * \overline{K_{m/3}}$ by Corollary 2.20, so we can clearly decompose $2(K_3 * \overline{K_{m/3}})$ into 3-cycles and pair each cycle with a parallel copy.

If $m \equiv 6 \pmod{18}$, then $2K_m$ decomposes into $2(K_{m/6} * \overline{K_6})$ together with $m/6$ subgraphs isomorphic to $2K_6$. We know that $C_3|2K_6$ with the pairing property. Moreover, $C_3|K_{m/6} * \overline{K_6}$ by Corollary 2.20 in conjunction with either Theorem 3.3 or 3.4, so we can clearly decompose $2(K_{m/6} * \overline{K_6})$ into 3-cycles and pair each cycle with a parallel copy.

If $m \equiv 15 \pmod{18}$ or $m \equiv 1 \pmod 6$, then $C_3|K_m$ since m is odd and $3|\binom{m}{2}$. Taking each cycle twice gives the desired decomposition of $2K_m$.

The final case is that $m \equiv 4 \pmod 6$, in which case $2K_m$ decomposes into $2(K_{(m-1)/3} * \overline{K_3})$ together with $(m-1)/3$ subgraphs isomorphic to $2K_4$. We know that $C_3|2K_4$ with the pairing property. Moreover, since $(m-1)/3$ is odd, $K_{(m-1)/3} * \overline{K_3}$ satisfies the necessary and sufficient conditions given in [21] for decomposition into 3-cycles; pair each of these cycles with a parallel copy.

This completes the induction, and the case $k = 3$.

Case 2. $k > m$.

In the case $s < m/2$ of Lemma 3.11, the directed closed trail $\mathscr{T}$ contains the arc $(0,1)$ and $\widetilde{\mathscr{T}}$ contains the arc $(1,0)$. For $j \in \{5i+2 : 0 \le i \le \alpha\} \cup \{5i+4 : 0 \le i \le \beta\}$, $\mathscr{T}_j$ contains the arc $(0,j)$ and $\widetilde{\mathscr{T}_j}$ contains $(j,0)$. That we may pair the closed trails of the $\overrightarrow{T}_k$-decomposition of K_m^* constructed in Lemma 3.11 in the desired manner easily follows from this observation.

Finally, consider the case $s = m/2$. If $t' = 1$, then the decomposition contains only two closed trails, T and T'. This pair is easily seen to have the desired property as T contains the directed cycle C_2 and T' contains the directed cycle C_1; these two directed cycles have the same underlying graph. If $t' > 1$, we may pair the closed trails of the decomposition constructed in Lemma 3.11 as follows. For each $0 \leq i \leq (t'-3)/2$, pair T_i and T_i'; this pairing satisfies the desired property since C_1 and C_2 have the same underlying graph. Pair T (which contains C_2) and T' (which contains C_1). If $t \equiv 1 \pmod 4$, proceed as follows. For $0 \leq i \leq (t'-5)/4$, pair $\widetilde{T}_{2i}$ (which contains the arc (u_{2i+1}, v_{2i+1})) with $\widetilde{T}_{2i+1}$ (which contains the arc (v_{2i+1}, u_{2i+1})). Pair $\widetilde{T}$ with $\hat{T}$; these trails respectively contain the arcs of $\hat{C}_{(3t-7)/4}$ and $\hat{C}'_{(3t-7)/4}$, which represent two orientations of the same undirected cycle. Finally, for $1 \leq i \leq (t'-5)/4$, pair $\hat{T}_i$ with $\hat{T}_i'$, which respectively contain the arcs of $\hat{C}_{(5t-9)/4+i}$ and $\hat{C}'_{(5t-9)/4+i}$; these two directed cycles represent two orientations of the same undirected cycle. If $t' \equiv 3 \pmod 4$, we pair the remaining trails as follows. For $0 \leq i \leq (t'-7)/4$, pair $\widetilde{T}_{2i}$ (which contains the arc (u_{2i+1}, v_{2i+1})) and $\widetilde{T}_{2i+1}$ (which contains the arc (v_{2i+1}, u_{2i+1})). Pair $\widetilde{T}_{(t'-3)/2}$ (which contains the arc $(u_{s-1-(t-1)/2}, v_{s-1-(t-1)/2})$) and $\widetilde{T}$ (which contains the arc $(v_{s-1-(t-1)/2}, u_{s-1-(t-1)/2})$). Finally, if $t' > 3$, then for $1 \leq i \leq (t'-3)/4$, pair $\hat{T}_i$ with $\hat{T}_i'$, which respectively contain the arcs of $\hat{C}_{(3t-9)/4+i}$ and $\hat{C}'_{(3t-9)/4+i}$; these two directed cycles represent two orientations of the same undirected cycle. □

Before we proceed, we note that Lemmas 3.8, 3.9 and 3.11 offer an alternative proof of sufficiency for Theorem 3.5.

3.2 Lifting lemmas

We will sometimes find it useful to first consider a decomposition of dK_m or $2dK_m$ into closed trails, and from it construct a decomposition of $K_m * \overline{K_d}$. The next two lemmas provide a means by which to do this. In each, we associate with each edge of dK_m or $2dK_m$ a matching in $K_m * \overline{K_d}$. The matchings we will use will be of a fixed jump.

Lemma 3.13. *Suppose that there is a decomposition of dK_m into closed trails of length k such that each closed trail in the decomposition contains a pair of parallel edges. Then $T_{dk}|K_m * \overline{K_d}$.*

Proof. Note that the number of trails in either a closed k-trail decomposition of dK_m or a closed dk-trail decomposition of $K_m * \overline{K_d}$ is $\binom{m}{2}d/k$. Hence for each closed trail in the decomposition of dK_m, we seek a single corresponding closed trail of length dk in $K_m * \overline{K_d}$.

For convenience, let the vertex set of $K_m * \overline{K_d}$ be $\mathbb{Z}_m \times \mathbb{Z}_d$, and let the parts of $K_m * \overline{K_d}$ be $\{v\} \times \mathbb{Z}_d$, where $v \in \mathbb{Z}_m$. Let $V(dK_m) = \mathbb{Z}_m$. For two vertices $u, v \in V(dK_m)$, we let the edges between u and v be labelled as $\{u,v\}_0$, $\{u,v\}_1$, ..., $\{u,v\}_{d-1}$. Note that we may label the edges in such a way that each closed trail in the decomposition of dK_m has two parallel edges $\{u,v\}_i$, $\{u,v\}_{i+1}$ with consecutive subscripts. From dK_m, we obtain $K_m * \overline{K_d}$ by associating with each edge of dK_m a matching in $K_m * \overline{K_d}$ in the following manner: the edge $\{u,v\}_i$ in dK_m corresponds with the matching in $K_m * \overline{K_d}$ consisting of all edges of jump i with both end-vertices in $\{u\} \times \mathbb{Z}_d \cup \{v\} \times \mathbb{Z}_d$.

Clearly the subgraphs induced by each set of matchings associated with the edges of a closed trail in the decomposition of dK_m decompose $K_m * \overline{K_d}$.

Consider a particular closed trail T of length k in the decomposition of dK_m, and let T' denote the subgraph of $K_m * \overline{K_d}$ induced by the matchings associated with the edges of T. We wish to show that T' is eulerian, and hence is a closed trail of length dk; once this is shown, we will have a T_{dk}-decomposition of $K_m * \overline{K_d}$. First, since each vertex in T has even degree, it is clear that each vertex in T' has even degree, as $\deg_{T'}(u,j) = \deg_T(u)$. It remains only to show that T' is connected.

Write $T = t_0, t_1, \ldots, t_{k-1}, t_0$ (where the vertices t_0, t_1, ..., t_{k-1} are not necessarily all distinct). By assumption T contains two parallel edges with consecutive labels, say $e = \{t_j, t_{j+1}\} = \{u,v\}_i$, $e' = \{t_{j'}, t_{j'+1}\} = \{u,v\}_{i+1}$. Note that the subgraph S of

$K_m * \overline{K}_d$ induced by the matchings associated with e and e' is connected. Indeed, S is isomorphic to the subgraph of $K_m * \overline{K}_d$ induced by the matchings associated with $\{u, v\}_0$ and $\{u, v\}_1$, which is a $(2d)$-cycle; this subgraph is illustrated in Figure 3.1.

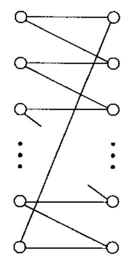

Figure 3.1: The subgraph of $K_m * \overline{K}_d$ induced by the matchings associated with $\{u, v\}_0$ and $\{u, v\}_1$.

Suppose (t, α) and (t', β) are two vertices of T'. Since T is a closed trail, it contains a trail from t to t_j and a trail from t_{j+1} to t'. Hence T' contains a trail P_1 from (t, α) to (t_j, α') and a trail P_2 from (t_{j+1}, β') to (t', β) for some $\alpha', \beta' \in \mathbb{Z}_d$. Since S is connected, it, and hence T', contains a path P from (t_j, α') to (t_{j+1}, β'), and so the concatenation of P_1, P and P_2 forms a trail from (t, α) to (t', β) in T'. Therefore, T' is connected, and so is eulerian. $\square$

Lemma 3.14. *Let $2dK_m$ be 2-edge coloured with colours* s *and* c *such that among every set of $2d$ parallel edges, there are d of each colour. Suppose that there exists a decomposition of $2dK_m$ into closed trails of length k such that the trails in the decomposition may be partitioned into pairs which satisfy the following conditions on some fixed traversal of the trails.*

1. *Suppose T and T' are paired trails. Then there are three consecutively traversed edges of T which are monochromatically coloured, say with colour* s*, for which T' contains three edges parallel to these edges of T, which are also consecutively traversed and which are monochromatically coloured with colour* c*; see*

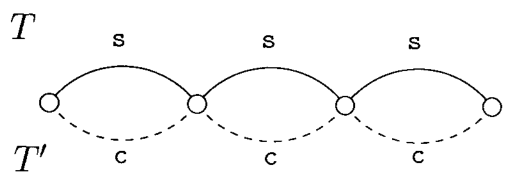

Figure 3.2: Parallel 3-trails

Figure 3.2. The subtrails of T and T' induced by these edges will be referred to as parallel 3-trails. *We note that two or more of the edges of T (or T') in its parallel 3-trail may be parallel edges.*

2. *Each trail T in the decomposition contains a pair of parallel edges, not in its parallel 3-trail, which are traversed consecutively and are monochromatically coloured with either colour* s *or* c. *For each closed trail in the decomposition, we select such a pair and refer to it as the trail's* consecutively-traversed parallel edges, *or* ct-parallel edges.

Then $T_{dk}|K_m * \overline{K_{2d}}$.

Proof. We begin by labelling each edge of $2dK_m$ with an element of the set $\{\mathsf{s}, \mathsf{c}\} \times \{0, 1, \ldots, d-1\}$. The first coordinate is determined by the 2-edge colouring of $2dK_m$ specified in the statement of the lemma. The second coordinates are assigned so that the ct-parallel edges in each trail receive consecutive labels, and so that among any $2d$ parallel edges, each element of $\{\mathsf{s}, \mathsf{c}\} \times \{0, 1, \ldots, d-1\}$ appears as a label. Given a trail T in the decomposition of $2dK_m$, let $c(T)$ denote the number of edges in T, but not in its parallel 3-trail, which are coloured by c.

We can now construct a decomposition of $K_m * \overline{K_{2d}}$. Let the vertex set of $K_m * \overline{K_{2d}}$ be $V(K_m) \times \{0, 1, \ldots, 2d-1\} = \mathbb{Z}_m \times (\{0, 1, \ldots, d-1\} \cup \{d, d+1, \ldots, 2d-1\})$ and $V(2dK_m) = \mathbb{Z}_m$. Each trail in the decomposition of $2dK_m$ will be lifted to two trails in $K_m * \overline{K_{2d}}$. Each edge $e = \{u, v\}$ will be lifted to two matchings of size d, one in each of the two trails.

A matching of size d will be called a *straight-across matching* if, for some $u, v \in V(K_m)$, the end-vertices of each edge in the matching are either all contained in the set $(\{u\} \times \{0, 1, \ldots, d-1\}) \cup (\{v\} \times \{0, 1, \ldots, d-1\})$ or are all contained in $(\{u\} \times \{d, d+1, \ldots, 2d-1\}) \cup (\{v\} \times \{d, d+1, \ldots, 2d-1\})$. A matching of size d will be called a *cross matching* if, for some $u, v \in V(K_m)$, the end-vertices of each edge in the matching are either all contained in the set $(\{u\} \times \{0, 1, \ldots, d-1\}) \cup (\{v\} \times \{d, d+1, \ldots, 2d-1\})$ or are all contained in $(\{u\} \times \{d, d+1, \ldots, 2d-1\}) \cup (\{v\} \times \{0, 1, \ldots, d-1\})$. Each of the matchings that we use will contain edges of a fixed jump in $\mathbb{Z}_d$. Note that, since the values for the jumps of these matchings occur in $\mathbb{Z}_d$ rather than $\mathbb{Z}_{2d}$, the use of the term "jump" here differs slightly from its previous definition. In assigning the jump of a d-matching, we view each of $\{0, 1, \ldots, d-1\}$ and $\{d, d+1, \ldots, 2d-1\}$ as $\mathbb{Z}_d$, so that an edge $\{(u, i), (v, j)\}$ has either jump $i - j \pmod d$ or $j - i \pmod d$, depending on the value of $u - v$. In particular, a straight-across matching of jump i is a d-matching of jump i induced on the $K_{d,d}$ induced by $(\{u\} \times \{0, 1, \ldots, d-1\}) \cup (\{v\} \times \{0, 1, \ldots, d-1\})$ or $(\{u\} \times \{d, d+1, \ldots, 2d-1\}) \cup (\{v\} \times \{d, d+1, \ldots, 2d-1\})$, and a cross-matching of jump j is a d-matching of jump j on the $K_{d,d}$ induced by $(\{u\} \times \{0, 1, \ldots, d-1\}) \cup (\{v\} \times \{d, d+1, \ldots, 2d-1\})$ or $(\{u\} \times \{d, d+1, \ldots, 2d-1\}) \cup (\{v\} \times \{0, 1, \ldots, d-1\})$. Note that the straight-across matchings of jumps $0, 1, \ldots, d-1$ and cross-matching of jumps $0, 1, \ldots, d-1$ partition the edge set of the $K_{2d,2d}$ induced by $(\{u\} \times \{0, 1, \ldots, 2d-1\}) \cup (\{v\} \times \{0, 1, \ldots, 2d-1\})$.

With the exception of the edges in parallel 3-trails, each edge labelled (s, i) will be lifted into two straight-across matchings of jump i, while each edge labelled (c, j) will be lifted into two cross matchings of jump j.

Let us now describe how to lift the parallel 3-trails. Suppose that T and T' are paired k-trails in the decomposition of $2dK_m$. In the fixed traversals of T and T', let the three edges of T contained in its parallel 3-trail be e_1, e_2 and e_3 in the order they are encountered along T, and let those of T' be e'_1, e'_2 and e'_3 in the order they are encountered along T'. We can assume that for $i \in \{1, 2, 3\}$, e_i and e'_i are parallel, as otherwise, we may reverse the direction of the traversal of T'. The three edges e_1, e_2 and e_3 are monochromatically coloured with one of s and c, and the edges e'_1, e'_2 and e'_3 with the other colour. Without loss of generality, we may assume that e_1, e_2 and e_3 are coloured with s, and e'_1, e'_2 and e'_3 with c. For $x = 1, 2, 3$, let the label of e_x be (s, i_x) and the label of e'_x be (c, j_x), as indicated in Figure 3.3.

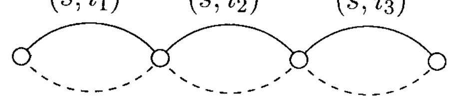

Figure 3.3: The edges of the parallel 3-trails of paired trails T and T'.

If $c(T)$ and $c(T')$ are both even, lift the parallel 3-trails according to the scheme shown in Figure 3.4. Each thick edge represents a straight-across matching or a cross matching with indicated jump. In each of the four diagrams, the upper row of vertices represents the sets of vertices $\{u\} \times \{0, 1, \ldots, d-1\}$ for corresponding vertices u in the parallel 3-trails, and the lower row represents the sets of vertices $\{u\} \times \{d, d+1, \ldots, 2d-1\}$. Each edge of the parallel 3-trails is lifted to two matchings of size d; these two matchings will each form part of a separate closed dk-trail in $K_m * \overline{K_{2d}}$. Hence the three edges of T contained in the parallel 3-trail and the three edges of T' contained in the parallel 3-trail will each give rise to two subgraphs, each with $3d$ edges, of $K_m * \overline{K_{2d}}$. Note that if u, v are the end-vertices in $2dK_m$ of the edges labelled (s, i_t) and (c, j_t) in the parallel 3-trails of T and T', then the scheme of Figure 3.4 uses both straight-across matchings of jump i_t between $\{u\}\times\{0, 1, \ldots, d-1\}$ and $\{v\}\times\{0, 1, \ldots, d-1\}$, and between $\{u\}\times\{d, d+1, \ldots, 2d-1\}$ and $\{v\} \times \{d, d+1, \ldots, 2d-1\}$, and also both cross matchings of jump j_t between vertex sets $\{u\} \times \{0, 1, \ldots, d-1\}$ and $\{v\} \times \{d, d+1, \ldots, 2d-1\}$, and between

$\{u\} \times \{d, d+1, \ldots, 2d-1\}$ and $\{v\} \times \{0, 1, \ldots, d-1\}$

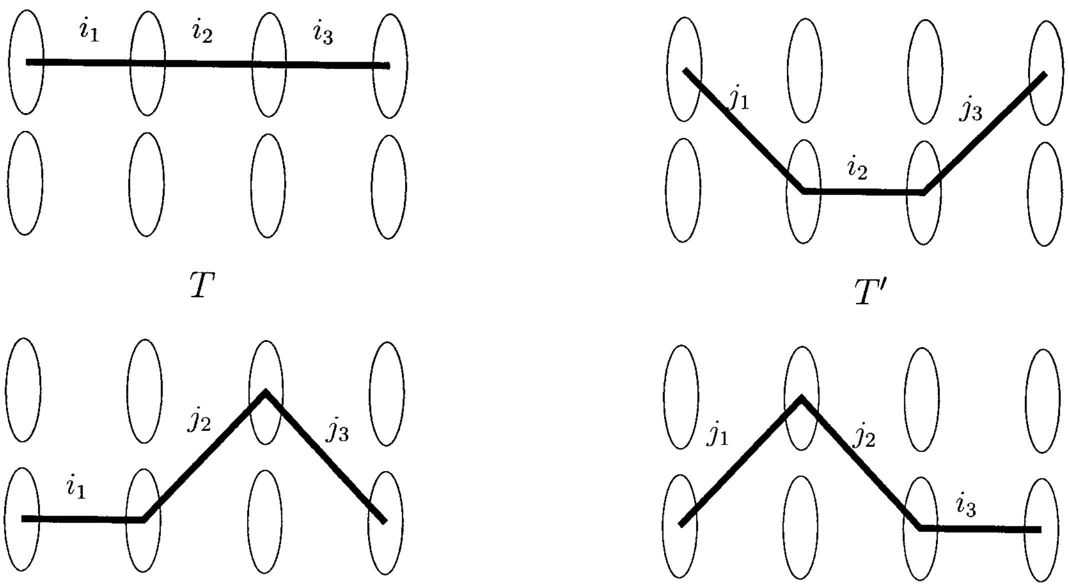

Figure 3.4: Lifting of parallel 3-trails of paired k-trails T and T' in the case that $c(T)$ and $c(T')$ are both even. Each 3-trail is lifted to two different subgraphs with $3d$ edges of $K_m * \overline{K_{2d}}$, each of which will form part of a closed dk-trail.

If $c(T)$ is even and $c(T')$ is odd, we lift the parallel 3-trails according to their labelling, that is, the edge (s, i_t) will be lifted to the straight-across matchings of jump i_t and the edge (c, j_t) to the cross matchings of jump j_t.

If $c(T)$ is odd and $c(T')$ is even, we relabel the parallel 3-trails so that the edges in T receive labels (c, j_1), (c, j_2) and (c, j_3), and the edges in T' receive labels (s, i_1), (s, i_2), (s, i_3). Then lift according to the labelling of each edge.

If $c(T)$ and $c(T')$ are both odd, lift according to the scheme illustrated in Figure 3.5.

Let $T = t_0, t_1, \ldots, t_{k-1}, t_0$ be a closed trail in the decomposition of $2dK_m$. We now describe how to lift T to two closed trails of length dk in $K_m * \overline{K_{2d}}$. Beginning at the set of vertices $\{t_0\} \times \{0, 1, \ldots, d-1\}$, we follow, as we traverse T, along matchings

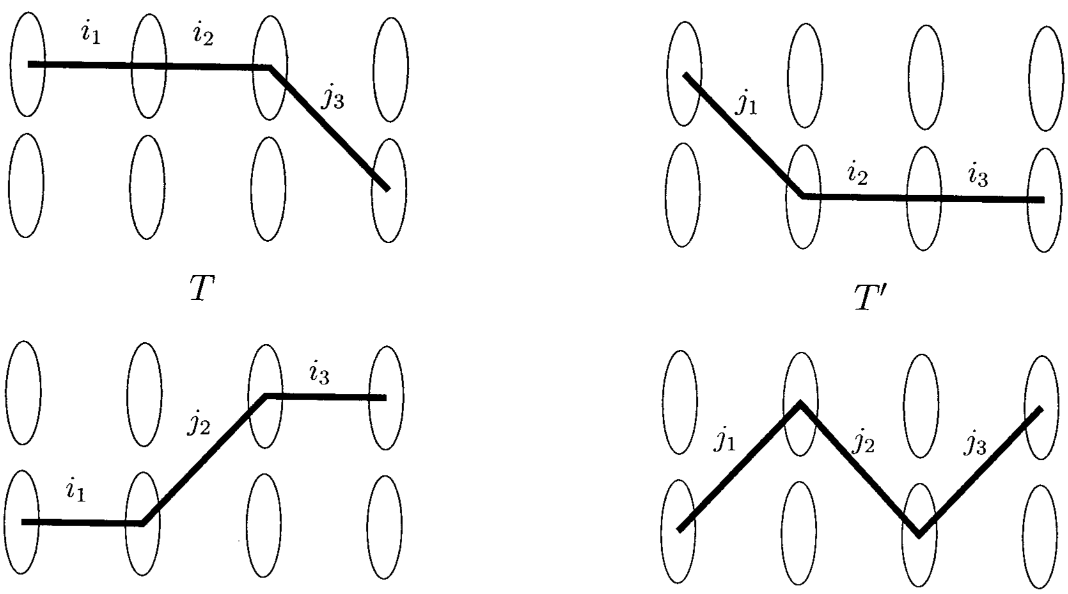

Figure 3.5: Lifting of parallel 3-trails of paired k-trails T and T' in the case that $c(T)$ and $c(T')$ are both odd. Each 3-trail is lifted to two different subgraphs with $3d$ edges of $K_m * \overline{K_{2d}}$, each of which will form part of a closed dk-trail.

in $K_m * \overline{K_{2d}}$ obtained by lifting the edges of T, each time selecting a matching which begins at the set of vertices at which we previously ended. For each edge $\{u, v\}$ of T we choose either a straight-across matching or a cross-matching depending on the colour of the edge (s or c, respectively), except when $\{u, v\}$ lies in the parallel 3-trail of T, in which case it is lifted according to the scheme described above. The matching chosen will have jump i if $\{u, v\}$ has label (x, i) where $x \in \{\mathsf{s}, \mathsf{c}\}$. We thus obtain a subgraph of $K_m * \overline{K_{2d}}$ containing exactly kd edges. The scheme for lifting parallel 3-trails results in each trail giving rise to an even number of cross matchings, which implies that we end at a matching between either $\{t_{k-1}\} \times \{0, 1, \ldots, d-1\}$ or $\{t_{k-1}\} \times \{d, d+1, \ldots, 2d-1\}$ and $\{t_0\} \times \{0, 1, \ldots, d-1\}$, and it follows that each vertex in our subgraph has even degree. Moreover, that the ct-parallel edges of T were labelled with consecutive integers and were either both lifted to straight-across matchings or both to cross matchings ensures that this subgraph will be connected. Hence it is a closed trail of length dk. Similarly, beginning at $\{t_0\}\times\{d, d+1, \dots, 2d-1\}$ and using the remaining matchings, we obtain a second closed trail of length dk, edge-disjoint from the first. Taking all closed trails in $K_m * \overline{K_{2d}}$ obtained in this manner from the closed trails in the decomposition of $2dK_m$, it is clear that any two of these closed trails are edge-disjoint, so that we obtain a decomposition of $K_m * \overline{K_{2d}}$ into closed trails of length dk. □

3.3 Decomposition of $K_m * \overline{K_n}$ into closed trails

Our aim in what follows is to prove the following theorem, which is the main result of this chapter.

Theorem 3.15. *Let $m \geq 2$, $n \geq 1$ and $k \geq 3$ be integers. Then $T_k | K_m * \overline{K_n}$ if and only if $(m-1)n$ is even, $k | \binom{m}{2} n^2$ and if $m = 2$, then k is even.*

Proof. It is obvious that the conditions that $(m-1)n$ is even, $k|\binom{m}{2}n^2$ and that k is even if $m = 2$ are necessary for the existence of a T_k-decomposition of $K_m * \overline{K_n}$.

To avoid the proof of sufficiency of these conditions becoming too cumbersome, we will prove the theorem via a sequence of lemmas. As Theorems 3.1 and 3.2 take care of the cases $m \in \{2, 3\}$, we may assume $m \geq 4$. Also, Theorems 3.3 and 3.4 deal with the case $n \in \{1, 2\}$, so we may also assume $n \geq 3$. Sufficiency for $m \geq 4$ and $n \geq 3$ will follow from Lemmas 3.16, 3.17, 3.18, 3.19, 3.20 and 3.21. □

We now prove Lemmas 3.16-3.21, which will complete the proof of the main result of this chapter. We henceforth assume that $m \geq 4$, $n \geq 3$ and $k \geq 3$ are integers such that $(m-1)n$ is even and $k|\binom{m}{2}n^2$. Before we proceed, let us introduce some notation. Let $d = \gcd(n, k)$, $k' = k/d$ and $n' = n/d$. Let $d' = \gcd(d, k')$ and $k'' = k'/d'$. The cases we consider depend on the values of d, k', d' and k''. We begin with the simplest case.

Lemma 3.16. *If $d = \gcd(n,k) = 1$, then $T_k | K_m * \overline{K_n}$.*

Proof. The condition $d = 1$ implies that $k | \binom{m}{2}$. If m is odd, then $T_k | K_m$ by Theorem 3.3, which immediately implies that $T_k | K_m * \overline{K_n}$ by Corollary 2.20. If m is even, then we note that n is also even. Moreover, $k | \binom{m}{2}$ yields $k | 2m(m-1)$ and so $T_k | K_m * \overline{K_2}$ by Theorem 3.4. Since $2 | n$, we obtain $T_k | K_m * \overline{K_n}$ by Corollary 2.20. □

Proceeding to the case when $\gcd(n,k) > 1$, most of our constructions will involve finding $T_{k'}$-decompositions of either K_m, $K_m * \overline{K_d}$, or $K_m * \overline{K_{2d}}$. Hence we must separately deal with the cases that $k' \in \{1,2\}$.

Lemma 3.17. *If $k' = k/\gcd(n,k) \in \{1,2\}$, then $T_k | K_m * \overline{K_n}$.*

Proof. In the case that $k' = 1$, we have that $k | n$. This condition is sufficient for the existence of a resolvable C_k-decomposition of $K_m * \overline{K_n}$ except if $(m,n,k) \in \{(3,6,3),(2,6,6)\}$ [34]. However, that $T_k | K_m * \overline{K_n}$ in these two cases follows from Theorems 3.1 and 3.2.

Suppose now that $k' = 2$, so that $k = 2d$. If d is even, then $4 | k$. Since $(m-1)(2n')$ is even and $4 | \binom{m}{2}(2n')^2$, there exists a 4-cycle decomposition of $K_m * \overline{K_{2n'}}$ [20]. Since $k/4 = n/(2n')$, it follows from Corollary 2.21 that $T_k | K_m * \overline{K_n}$. If d is odd, then we note that $d \geq 3$ since $k = 2d > 2$. Also, n must be odd since k is even and $d = \gcd(n,k)$ is odd; hence, since $k | \binom{m}{2} n^2$, we have that $2 | \binom{m}{2}$. Let p be a prime dividing d. By the main result of [47], since p is prime and $2p | \binom{m}{2} p^2$, it follows that $C_{2p} | K_m * \overline{K_p}$. By Corollary 2.21, $C_{2d} | K_m * \overline{K_d}$, and so, since $k = 2d$ and $d | n$, Corollary 2.20 gives that $T_k | K_m * \overline{K_n}$. □

We assume from now on that $k' \geq 3$. We now proceed depending on the value of d'.

Lemma 3.18. *If $d = \gcd(n,k) > 1$ and $d' = \gcd(d, k/d) = 1$, then $T_k | K_m * \overline{K_n}$.*

Proof. The condition $d' = 1$ implies that $\binom{m}{2}$ is divisible by $k' = k/d$. If m is odd, then $T_{k'}|K_m$ by Theorem 3.3, and so by Corollary 2.21, $T_{dk'}|K_m * \overline{K_d}$, i.e. $T_k|K_m * \overline{K_d}$. That $T_k|K_m * \overline{K_n}$ follows by Corollary 2.20.

Suppose now that m is even. Then n is also even. If $n' = n/d$ is even, then as $T_{k'}|K_m * \overline{K_2}$ by Theorem 3.4, $T_k|K_m * \overline{K_{2d}}$ by Corollary 2.21, and so $T_k|K_m * \overline{K_n}$ by Corollary 2.20. If n' is odd, then d must be even as n is even. By Theorem 3.4, $T_{2k'}|K_m * \overline{K_2}$, which implies $T_k|K_m * \overline{K_d}$ by Corollary 2.21 and hence $T_k|K_m * \overline{K_n}$ by Corollary 2.20. □

The case that $d' > 1$ requires the greatest effort. To preview this case, we note that since $d = \gcd(n,k)$ and $k|\binom{m}{2}n^2$, $k' = k/d$ must be a divisor of $d\binom{m}{2}$. It follows that $\frac{k'}{d'}|\binom{m}{2}\frac{d}{d'}$, but since $d' = \gcd(d,k')$ we have that $\gcd(\frac{k'}{d'}, \frac{d}{d'}) = 1$, so that $\frac{k'}{d'}|\binom{m}{2}$, i.e. $k''|\binom{m}{2}$. Except in the case that m is even and d is odd, we will construct $T_{k'}$-decompositions of dK_m which we will lift to T_k-decompositions of $K_m * \overline{K_d}$. If m is even and d is odd, then dK_m has odd valency; we will instead lift a $T_{k'}$-decomposition of $2dK_m$ to a T_k-decomposition of $K_m * \overline{K_{2d}}$ and then to a decomposition of $K_m * \overline{K_n}$. Our $T_{k'}$-decompositions will arise from $T_{k''}$-decompositions of K_m or $2K_m$, and so this approach requires that $k'' \geq 3$. We consider the cases $k'' = 1, 2$ separately.

Lemma 3.19. *If $k'' = k'/d' = 1$, then $T_k|K_m * \overline{K_n}$.*

Proof. The condition $k'' = 1$ implies that $k|n^2$. First, suppose that k is even. Necessarily, n must also be even, and we have that $T_k|K_{n,n}$ by Theorem 3.1. But clearly $K_m * \overline{K_n}$ may be decomposed into $\binom{m}{2}$ subgraphs, each isomorphic to $K_{n,n}$, and it follows that $T_k|K_m * \overline{K_n}$.

Consider now the situation that k is odd. If $m \equiv 1$ or $3 \pmod 6$, then it is well-known that $K_3|K_m$, and so $K_3 * \overline{K_n}|K_m * \overline{K_n}$. By Theorem 3.2, $T_k|K_3 * \overline{K_n}$, and it follows that $T_k|K_m * \overline{K_n}$. If $m \equiv 0$ or $4 \pmod 6$, then $3|2m(m-1)$ and so

$T_3|K_m * \overline{K_2}$, i.e. $K_3|K_m * \overline{K_2}$, by Theorem 3.4. As m is even, n must also be even, and hence $K_3 * \overline{K_{n/2}}|K_m * \overline{K_n}$. Since k is odd, it must be that $k|(n/2)^2$. By Theorem 3.2, $T_k|K_3 * \overline{K_{n/2}}$, and hence $T_k|K_m * \overline{K_n}$.

If $m \equiv 5 \pmod 6$, then it is shown in [46] that $K_m = C_3 \oplus C_3 \oplus \cdots \oplus C_3 \oplus C_4$. By borrowing back a 3-cycle in this decomposition which contains one of the diagonal edges of the 4-cycle in the decomposition, we obtain that $K_m = C_3 \oplus C_3 \oplus \cdots \oplus C_3 \oplus G$, where G denotes the graph depicted in Figure 3.6. Hence $K_m * \overline{K_n} = (C_3 * \overline{K_n}) \oplus$

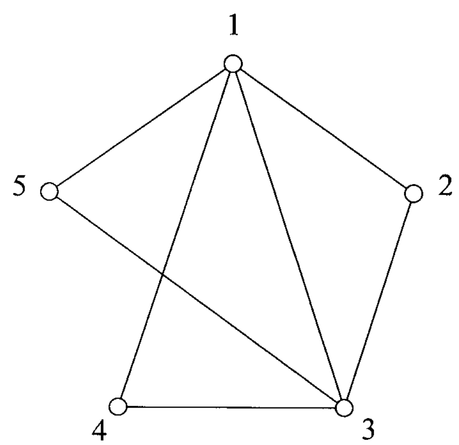

Figure 3.6: The graph G in the case $m \equiv 5 \pmod 6$ of Lemma 3.19.

$\cdots \oplus (C_3 * \overline{K_n}) \oplus (G * \overline{K_n})$. Since $T_k|C_3 * \overline{K_n}$ by Theorem 3.2, it need only be shown that $T_k|G * \overline{K_n}$.

Similarly in the case $m \equiv 2 \pmod 6$, we begin with the decomposition $K_m * \overline{K_2} = C_3 \oplus C_3 \oplus \cdots \oplus C_3 \oplus C_4$ [13]. We note that the diagonals in the 4-cycle-leave of the decomposition exhibited in [13] are indeed edges in the graph $K_m * \overline{K_2}$. (This decomposition is obtained in [13] in the following manner. For $m > 8$, it is noted in [13] that $C_3|(K_m * \overline{K_2}) - E(K_5 * \overline{K_2})$, as there exists a group divisible design with blocks of size 3, one group of size 10, and the remaining groups of size 2. The graph $K_5 * \overline{K_2}$ is decomposed into 12 3-cycles and a 4-cycle. Taking the vertex set of $K_5 * \overline{K_2}$ to be $\mathbb{Z}_5 \times \mathbb{Z}_2$ with parts $\{x\} \times \mathbb{Z}_2$ for $x \in \mathbb{Z}_5$, the vertices of the 4-cycle are contained in $\mathbb{Z}_5 \times \{0\}$ and hence are contained in four distinct parts. If $m = 8$, it is shown in [13] that $K_8 * \overline{K_2} = K_5 \oplus K_3 \oplus \cdots \oplus K_3$. That $K_8 * \overline{K_2} = C_3 \oplus C_3 \oplus \cdots \oplus C_3 \oplus C_4$ follows since $K_5 = C_4 \oplus C_3 \oplus C_3$, and clearly these latter 3-cycles contain the diagonals of the 4-cycle.) Hence, by again borrowing back a 3-cycle containing one of the diagonals, we obtain that $K_m * \overline{K_2} = C_3 \oplus \cdots \oplus C_3 \oplus G$, where the graph G is as in the case $m \equiv 5 \pmod 6$. Since m is even, we have that n is even, and so we may blow up parts by a factor of $n/2$ to obtain $K_m * \overline{K_n} = (C_3 * \overline{K_{n/2}}) \oplus \cdots \oplus (C_3 * \overline{K_{n/2}}) \oplus (G * \overline{K_{n/2}})$. We know that $T_k | C_3 * \overline{K_{n/2}}$ (Theorem 3.2), and we need to show that $T_k | G * \overline{K_{n/2}}$, where k is a divisor of $(n/2)^2$.

We have now reduced this case so that we need only prove that $T_k | G * \overline{K_{\bar{n}}}$ if k is odd and $k | \bar{n}^2$, where we will take $\bar{n} = n$ if $m \equiv 5 \pmod 6$ and $\bar{n} = n/2$ if $m \equiv 2 \pmod 6$. Note that if $k | \bar{n}$, then $k | n$, and so $d = \gcd(n, k) = k$ and $k' = k/d = 1$. As this case has previously been dealt with in Lemma 3.17, we can assume that $k \nmid \bar{n}$. Hence k must be divisible by a square divisor of $\bar{n}^2$ other than 1. Let q^2 be the largest square dividing k, where $q \geq 3$ is odd. Supposing we know that $T_{q^2} | G * \overline{K_q}$, applying Lemma 2.19 gives that $T_{q^2(k/q^2)} | G * \overline{K_{q(k/q^2)}}$, i.e. $T_k | G * \overline{K_{k/q}}$. Since $(k/q) | \bar{n}$, we may apply Lemma 2.18 to obtain $T_k | G * \overline{K_n}$. Hence we need only show that $T_{q^2} | G * \overline{K_q}$ where $q \geq 3$ is odd. We divide the problem into cases depending on the congruence class of q modulo 3. Note that $G * \overline{K_q} = (C_4 * \overline{K_q}) \oplus (C_3 * \overline{K_q})$. Our closed trails are either formed as closed trails of length q^2 in $C_3 * \overline{K_q}$ or are made up of closed trails of smaller length in $C_4 * \overline{K_q}$ together with cycles in $C_3 * \overline{K_q}$. We also remark that $C_3 | C_3 * \overline{K_q}$ by Lemma 2.18. In what follows, we fix a C_3-decomposition of $C_3 * \overline{K_q}$ whose cycles will be used in our construction.

Case 1. $q \equiv 0$ or $1 \pmod 3$, $q > 3$

Note that since q is odd, we have that $q \geq 7$. Let F denote the 3-factor of $K_{q,q}$ formed by matchings of jumps 0, 1, -1. The closed trails of length q^2 are formed as follows.

1. Each of the four edges of the 4-cycle which forms G gives rise to an instance of $K_{q,q}$ in $G*\overline{K_q}$. From each instance of $K_{q,q}$, form a subgraph of $G*\overline{K_q}$ with exactly q^2 edges by taking $K_{q,q} - F$ together with q 3-cycles from the C_3-decomposition of $C_3 * \overline{K_q}$. Note that, since $q > 4$, $K_{q,q} - F$ contains all edges of $K_{q,q}$ of consecutive jumps 2 and 3, and is thus connected. In the C_3-decomposition of $C_3 * \overline{K_q}$, each 3-cycle must contain one edge arising from each of the edges of the 3-cycle used to form G. As this 3-cycle contains the diagonal of the 4-cycle used to form G, each C_3 in $C_3 * \overline{K_q}$ must contain a vertex of $K_{q,q} - F$. Hence the subgraph of $G * \overline{K_q}$ we obtain from $K_{q,q} - F$ and q 3-cycles from $C_3 * \overline{K_q}$ must be connected. It is easy to see that each vertex of this subgraph has even degree. Hence it is eulerian and forms a closed trail of length q^2.

2. There remain $3q^2$ unused edges of $G * \overline{K_q}$, which will form three closed trails of length q^2 in the following way. The remaining edges in $C_4 * \overline{K_q}$ may be decomposed into three $4q$-cycles. (One has matchings of jumps, in order, $0, 0, 0, 1$; one has matchings of jumps, in order, $1, 1, -1, 0$; one has matchings of jumps $-1, -1, 1, -1$. The total jump in each is either 1 or -2, each of which is relatively prime to q.) For each of these cycles, take it together with $(q^2 - 4q)/3$ of the remaining 3-cycles from $C_3 * \overline{K_q}$; the number of 3-cycles needed to complete each of the four closed q^2 trails is an integer since $q \equiv 0$ or $1 \pmod 3$ and is positive since $q > 3$. The resulting three subgraphs of $G * \overline{K_q}$ are each clearly connected since each 3-cycle from $C_3 * \overline{K_q}$ intersects each $4q$-cycle. Since each of the three subgraphs is formed from 3-cycles and a $4q$-cycle, each of its vertices must clearly be of even degree. Hence we have formed three eulerian subgraphs of $G * \overline{K_q}$, each containing q^2 edges, that is three closed q^2-trails.

Case 2. $q \equiv 2 \pmod 3$, $q \geq 23$

Let F denote the 9-factor of $K_{q,q}$ containing matchings of jumps $0, 1, \ldots, 8$. The closed trails of length q^2 are formed as follows.

1. From each $K_{q,q}$ in $C_4 * \overline{K_q}$ which arises from an edge of C_4, take $K_{q,q} - F$ together with $3q$ 3-cycles from $C_3 * \overline{K_q}$. Similarly to Case 1, it is easy to check that each of the four subgraphs obtained is eulerian and hence forms a closed trail of length q^2.

2. The remaining edges of $C_4 * \overline{K_q}$ decompose into $T_{20q} \oplus T_{8q} \oplus T_{8q}$ (T_{20q} containing all matchings of jumps 0,1,2,3,4 and the two T_{8q} containing all matchings of jumps 5,6 and 7,8 respectively), with each of these three closed trails being a connected spanning subgraph of $C_4 * \overline{K_n}$. Take T_{20q} together with $(q^2 - 20q)/3$ of the remaining 3-cycles from $C_3 * \overline{K_q}$, and each T_{8q} together with $(q^2 - 8q)/3$ 3-cycles from $C_3 * \overline{K_q}$ to form the remaining closed trails of length q^2. This step is possible since each of $(q^2 - 20q)/3$ and $(q^2 - 8q)/3$ is an integer because $q \equiv 2 \pmod 3$, and is positive because $q \geq 23$.

To conclude the case $k'' = 1$, we need only check that $T_{q^2} | G * \overline{K_q}$ for $q \in \{3, 5, 11, 17\}$. The following decompositions were obtained by computer. We let the vertex set of $G * \overline{K_q}$ in the given decompositions be $\{1, 2, \ldots, 5q\}$. The parts of $G * \overline{K_q}$ are taken to be the sets $P_1, P_2, \ldots, P_5$, where $P_i = \{5j + i : j = 0, 1, 2, \ldots, q-1\}$.

A T_9-decomposition of $G * \overline{K_3}$:

- 1, 2, 3, 1, 4, 3, 5, 6, 7, 1
- 1, 5, 8, 1, 9, 3, 6, 2, 13, 1
- 1, 10, 3, 7, 8, 2, 11, 3, 12, 1
- 1, 14, 3, 15, 6, 4, 8, 11, 15, 1
- 4, 11, 5, 13, 6, 9, 8, 10, 13, 4
- 6, 8, 12, 6, 10, 11, 7, 13, 14, 6

- 8, 14, 11, 9, 13, 11, 12, 13, 15, 8

A T_{25}-decomposition of $G * \overline{K_5}$:

- 1, 2, 3, 1, 4, 3, 5, 1, 7, 3, 6, 2, 8, 1, 9, 3, 10, 1, 12, 3, 11, 2, 13, 4, 18, 1
- 1, 13, 5, 6, 4, 8, 5, 11, 4, 16, 2, 18, 5, 16, 3, 14, 1, 15, 3, 17, 1, 19, 3, 20, 23, 1
- 1, 20, 6, 7, 8, 6, 9, 8, 10, 6, 12, 8, 11, 7, 13, 6, 14, 8, 15, 6, 17, 8, 16, 13, 22, 1
- 1, 24, 3, 21, 2, 23, 4, 21, 5, 23, 6, 18, 7, 16, 9, 11, 10, 13, 9, 18, 10, 16, 12, 11, 25, 1
- 3, 22, 6, 19, 8, 20, 11, 13, 12, 18, 11, 14, 13, 15, 11, 17, 13, 19, 11, 22, 8, 21, 7, 23, 25, 3
- 6, 24, 8, 25, 13, 20, 16, 14, 18, 15, 16, 17, 18, 16, 19, 18, 20, 21, 9, 23, 10, 21, 22, 16, 25, 6
- 11, 23, 12, 21, 13, 24, 16, 23, 14, 21, 15, 23, 17, 21, 18, 22, 23, 19, 21, 23, 24, 18, 25, 21, 24, 11

A T_{121}-decomposition of $G * \overline{K_{11}}$:

- 1, 2, 3, 1, 4, 3, 5, 1, 7, 3, 6, 2, 8, 1, 9, 3, 10, 1, 12, 3, 11, 2, 13, 1, 14, 3, 15, 1, 17, 3, 16, 2, 18, 1, 19, 3, 20, 1, 22, 3, 21, 2, 23, 1, 24, 3, 25, 1, 27, 3, 26, 2, 28, 1, 29, 3, 30, 1, 32, 3, 31, 2, 33, 1, 34, 3, 35, 1, 37, 3, 36, 2, 38, 1, 39, 3, 40, 1, 42, 3, 41, 2, 43, 1, 44, 3, 45, 1, 47, 3, 46, 2, 48, 1, 49, 3, 50, 1, 52, 3, 51, 2, 53, 1, 54, 3, 55, 6, 4, 8, 5, 6, 7, 8, 6, 9, 8, 10, 6, 13, 55, 1
- 4, 11, 5, 13, 4, 16, 5, 18, 4, 21, 5, 23, 4, 26, 5, 28, 4, 31, 5, 33, 4, 36, 5, 38, 4, 41, 5, 43, 4, 46, 5, 48, 4, 51, 5, 53, 6, 12, 8, 11, 7, 13, 9, 11, 10, 13, 11, 12, 13,

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A T_{289}-decomposition of $G * \overline{K_{17}}$:

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□

Lemma 3.20. *If $d' > 1$ and $k'' = k'/d' = 2$, then $T_k | K_m * \overline{K_n}$.*

Proof. Here we have that $k' = 2d'$. Note that $2d' \nmid d$, since otherwise, $k'|d$, which implies that $d' = \gcd(d, k') = k' = 2d'$, a contradiction. Also, $2|\binom{m}{2}$, so that $P_2|K_m$ by Theorem 3.6. We consider the following two cases.

Case 1. $4|k$.

Note that d is even since $k = 2dd'$, $4|k$ and $d'|d$. Hence d' must be even, as $d'|d$ and $2d' \nmid d$. Since $P_2|K_m$, doubling the edges in each path gives $T_4|2K_m$ and hence, taking each edge of each T_4 of multiplicity $d'/2$, $T_{2d'}|d'K_m$, so that $T_{2d'}|dK_m$, i.e. $T_{k'}|dK_m$, with each closed trail in the latter decomposition containing parallel edges. Again, we obtain $T_k|K_m * \overline{K_d}$ by Lemma 3.13 and hence $T_k|K_m * \overline{K_n}$ by Corollary 2.20.

Case 2. $4 \nmid k$.

Note that $k = 2dd'$, and so the condition $4 \nmid k$ implies that d and d' must both be odd. In particular, since k is even but $d = \gcd(n, k)$ is odd, this means that n must be odd. As $(m-1)n$ is even, it must be that m is also odd, and so the condition $2|\binom{m}{2}$ gives us that $m \equiv 1 \pmod 4$.

We begin by decomposing $d'K_m$ into closed trails of length $T_{2d'}$ and from this decomposition deduce that $T_k|K_m * \overline{K_n}$. First suppose that $m \equiv 1 \pmod 8$. Then $C_4|K_m$ [29], so clearly $d'C_4|d'K_m$. We may decompose $d'C_4$ into two subgraphs as follows. Let $C_4 = u, v, w, x, u$, so that edges of $d'C_4$ take on one of the forms $\{u, v\}$, $\{v, w\}$, $\{w, x\}$ or $\{x, u\}$. For the first subgraph, we take two of the edges joining u and v, $(d'-1)$ of the edges joining u and x, and $(d'-1)$ of the edges joining x and w. The second subgraph contains the remaining edges, that is, $(d'-2)$ edges joining u and v, d' edges joining v and w, one edge joining w and x, and one edge joining x and u. It is easy to see that, since d' is odd, each of these subgraphs has all vertices of even degree; hence each is a closed trail of length $2d'$, and we may conclude $T_{2d'}|d'C_4$, so that $T_{2d'}|d'K_m$. Note moreover that in the decomposition which we have exhibited, each closed trail contains a pair of parallel edges.

Now suppose $m \equiv 5 \pmod 8$, and write $m = 8m' + 5$. We note that $K_{8m'+5} = K_{8m'+1} \oplus K_{4,8m'} \oplus K_5$. We know that $C_4|K_{8m'+1}$ [29] and $C_4|K_{4,8m}$ [50], so that $d'K_{8m'+5} = (d'C_4 \oplus d'C_4 \oplus \cdots \oplus d'C_4) \oplus d'K_5$. We first consider the case $d' > 3$. It is easy to see that $K_5 = C_4 \oplus B$, where B is the bow-tie graph (Figure 3.7). Hence we have that $K_{8m'+5} = (C_4 \oplus C_4 \oplus \cdots \oplus C_4) \oplus B$, so that $d'K_{8m'+5} = (d'C_4 \oplus d'C_4 \oplus \cdots \oplus d'C_4) \oplus d'B$. Let the vertices of B be u, v, w, x, y, as indicated in Figure 3.7. Then we may decompose $d'B$ into three subgraphs G_1, G_2 and G_3 as follows. The

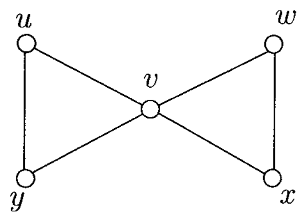

Figure 3.7: The graph B.

subgraph G_1 contains two edges between u and v, $(d'-1)$ edges between u and y, and $(d'-1)$ edges between v and y. The subgraph G_2 contains two edges between v and w, $(d'-1)$ between v and x, and $(d'-1)$ between w and x. The third subgraph, G_3, contains all remaining edges, that is, $(d'-2)$ edges between u and v, $(d'-2)$ edges between v and w, and one edge each between w and x, between v and x, between v and y, and between u and y. Since d' is odd, each of G_1, G_2 and G_3 contains only vertices of even degree, and so each is a closed trail of length $2d'$. Hence $T_{2d'}|d'B$. It is easy to see that $d' > 3$ implies that each of the closed trails in the decomposition exhibited contains a pair of parallel edges. We have already seen the existence of a $T_{2d'}$-decomposition of $d'C_4$, each closed trail of which contains a pair of parallel edges. We conclude that $T_{2d'}|d'K_m$.

If $d' = 3$, we again begin with the decomposition $d'K_{8m'+5} = (d'C_4 \oplus \cdots d'C_4) \oplus d'K_5$, i.e. $3K_{8m'+5} = (3C_4 \oplus 3C_4 \oplus \cdots \oplus 3C_4) \oplus 3K_5$. Letting $d' = 3$ in the $T_{2d'}$-decomposition of $d'C_4$ given earlier yields a decomposition whose closed trails all contain parallel edges. To deal with $3K_5$, we let $V(3K_5) = \mathbb{Z}_5$. Letting $T = 0, 1, 0, 2, 4, 3, 0$, we note that T is a closed trail of length 6 in $3K_5$ which contains exactly three edges of length 1 and three edges of length 2. Hence, letting $\rho = (0, 1, 2, 3, 4)$, we have that $3K_5 = T \oplus \rho(T) \oplus \rho^2(T) \oplus \rho^3(T) \oplus \rho^4(T)$, and each of the latter closed trails contains a pair of parallel edges.

Thus, we have found a decomposition of $d'K_m$ into closed trails of length $2d'$, and moreover we may note that each closed trail in the decomposition contains a pair of parallel edges. By Lemma 3.13, we determine that $T_{2(d')^2}|K_m * \overline{K_{d'}}$. Since $d'|d$, we may apply Corollary 2.21 to obtain that $T_{2d'd}|K_m * \overline{K_d}$, i.e. that $T_k|K_m * \overline{K_d}$, and so by Corollary 2.20, $T_k|K_m * \overline{K_n}$.

□

Lemma 3.21. *If $d' = \gcd(d, k/d) > 1$ and $k'' = k'/d' \geq 3$, then $T_k|K_m * \overline{K_n}$.*

Proof. We consider four cases.

Case 1. m is odd

Since m is odd and $k''|\binom{m}{2}$, we have that $T_{k''}|K_m$ by Theorem 3.3. Taking each $T_{k''}$ of multiplicity d', we obtain $T_{d'k''}|d'K_m$, i.e. $T_{k'}|d'K_m$ with each closed trail in the latter decomposition containing parallel edges. Since $d'|d$, we may take d/d' copies of each closed trail to obtain $T_{k'}|dK_m$. As each closed trail in the latter decomposition contains parallel edges, we have that $T_k|K_m * \overline{K_d}$ by Lemma 3.13, and so $T_k|K_m * \overline{K_n}$ by Corollary 2.20.

Case 2. m is even and k' is even

Note that since m is even, n is also even. Also, k is even, so that d must be even. Since d and k' are even, d' is even.

Since $k''|\binom{m}{2}$, Corollary 3.10 gives us that $T_{2k''}|2K_m$; moreover, each closed trail in this decomposition contains a pair of parallel edges. Taking each closed trail of multiplicity $d'/2$, we obtain that $T_{k'}|d'K_m$, and then taking d/d' copies of each of the closed trails in the latter decomposition, we obtain $T_{k'}|dK_m$. We may now apply Lemma 3.13 and Corollary 2.20 to obtain $T_k|K_m * \overline{K_n}$.

Case 3. m is even, k' is odd and d is even

Since $k''|\binom{m}{2}$, Theorem 3.3 ensures the existence of a $T_{k''}$-decomposition of $2K_m$. Similarly to the previous case, we obtain that $T_{k'}|2d'K_m$ and hence $T_{k'}|dK_m$ since $2d'|d$. Since $d' > 1$, it is easy to ensure that each closed trail in the latter decomposition contains a pair of parallel edges. By Lemma 3.13, $T_k|K_m * \overline{K_d}$ and so $T_k|K_m * \overline{K_n}$ by Corollary 2.20.

Case 4. m is even, k' is odd and d is odd

Since $k''|\binom{m}{2}$, it follows by Corollary 3.12 that $T_{k''}|2K_m$, and we may form such a decomposition whose closed trails may be partitioned into pairs such that if T and T' are paired, then there is an edge of T and an edge of T' which are parallel to each other. Taking each edge of each $T_{k''}$ of multiplicity d', we obtain $T_{d'k''}|2d'K_m$, i.e. $T_{k'}|2d'K_m$. Again, we may take d/d' copies of each $T_{k'}$ to obtain $T_{k'}|2dK_m$. The closed trails in this decomposition may be partitioned into pairs such that for each pair of trails, there are two vertices u and v such that each trail in the pair contains $d' \geq 3$ edges with end-vertices u and v.

We wish to use Lemma 3.14 to obtain a T_k-decomposition of $K_m * \overline{K_{2d}}$. We have partitioned our k'-trails into pairs; we must show that these pairs satisfy the conditions of the lemma. For a closed trail T of length k' in the decomposition of $2dK_m$ with pair $\widetilde{T}$, let $T' = u_0, u_1, \ldots, u_{k''-1}, u_0$ be the closed trail of length k'' in the decomposition of $2K_m$ from which T arises, and $\widetilde{T}' = u_0, u_1, v_2, \ldots, v_{k''-1}, u_0$ be the closed trail of length k'' from which $\widetilde{T}$ arises. We may traverse T as follows. Taking $i = 0, 1, \ldots k'' - 2$ in order, we traverse all d' edges between u_i and u_{i+1}. Finally, traverse all d' edges between $u_{k''-1}$ and u_0. That is,

$$T = u_0, u_1, u_0, u_1, \ldots, u_0, u_1, u_2, u_1, u_2, \ldots, u_1, u_2, \ldots, u_{k''-2}, u_{k''-1}, u_{k''-2}, u_{k''-1},$$
$$u_0, u_{k''-1}, u_0, u_{k''-1}, \ldots, u_0.$$

We similarly traverse $\widetilde{T}$. We take the first three edges in the traversals of T and $\widetilde{T}$ to be the parallel 3-trails of T and $\widetilde{T}$. For the second condition of Lemma 3.14, clearly T and $\widetilde{T}$ each contain a pair of parallel edges not in its parallel 3-trail which are traversed consecutively – in the case of T, we may take two consecutively traversed edges between u_1 and u_2, while in $\widetilde{T}$ we may take two consecutively traversed edges between u_1 and v_2.

It remains to show that $2dK_m$ can be 2-edge coloured as specified in the statement of Lemma 3.14. For each pair of closed trails T and $\widetilde{T}$, colour all edges of T parallel to its parallel 3-trail (including its parallel 3-trail) by colour s and all edges of $\widetilde{T}$ parallel to its parallel 3-trail (including its parallel 3-trail) by colour c. We now must colour the remaining edges. Colour them so that among $2d$ parallel edges, d receive colour s and d receive colour c. Since $d' \geq 3$, there will be among any d' parallel edges, a pair of the same colour. Hence we can clearly ensure that the ct-parallel edges in each closed trail both receive the same colour.

Recalling that $k = dk'$, we may now apply Lemma 3.14 to our $T_{k'}$-decomposition of $2dK_m$ to obtain a T_k-decomposition of $K_m * \overline{K_{2d}}$. We note that n is even, as m is even, and since d is odd and $d|n$, it must be that $2d|n$. Hence by Corollary 2.20, $T_k|K_m * \overline{K_n}$.

□

We have now completed the proof of the main result of this chapter, Theorem 3.15.

Chapter 4

Cycle decompositions of $3K_m$

Let us now turn our attention to cycle decompositions of the complete multigraph λK_m. We begin by recalling known results towards finding necessary and sufficient conditions for decomposing λK_m into cycles of length k.

First, the obvious necessary conditions are known to be sufficient for certain small cycle lengths. The following theorem compiles the work of several authors.

Theorem 4.1. *[22, 25, 44, 8, 6] Suppose $\lambda \geq 1$ and $m \geq 3$ are integers, and $k \in \{3, 4, 5, 6, 7, 8, 10, 12, 14\}$. Then $C_k | \lambda K_m$ if and only if $k \leq m$, $k | \lambda \binom{m}{2}$ and $\lambda(m-1)$ is even.*

Smith [48] was able to find necessary and sufficient conditions for the existence of a λ-cycle decomposition of λK_m; in turn, this result implies that the obvious necessary conditions for decomposition of λK_m into cycles of length p are sufficient, where p is an odd prime.

Theorem 4.2. *[48] Let $\lambda \geq 3$ and $m \geq 3$ be integers. Then $C_\lambda | \lambda K_m$ if and only if $\lambda \leq m$ and $\lambda(m-1)$ is even.*

Corollary 4.3. *[48] Let p be an odd prime, and let $\lambda \geq 1$ and $m \geq 3$ be integers. Then $C_p | \lambda K_m$ if and only if $p \leq m$, $p | \lambda \binom{m}{2}$ and $\lambda(m-1)$ is even.*

Other authors have focussed on determining which cycle decompositions of λK_m exist for certain values of λ. In particular, necessary and sufficient conditions for the existence of a C_k-decomposition of λK_m are known when $\lambda \in \{1, 2\}$, as evidenced by the following results.

Theorem 4.4. *[1, 45] Let m, $k \geq 3$ be integers. Then $C_k | K_m$ if and only if $k \leq m$ and $k | \binom{m}{2}$.*

Theorem 4.5. *[2] Let $m \geq 1$ and $k \geq 2$ be integers. Then $C_k | K_m^*$ if and only if $k \leq m$, $k | m(m-1)$, and $(m, k) \notin \{(4, 4), (6, 3), (6, 6)\}$.*

Corollary 4.6. *[2] Let $m \geq 1$ and $k \geq 2$ be integers. Then $C_k | 2K_m$ if and only if $k \leq m$ and $k | m(m-1)$.*

The main result which will be proved in this section is to extend the results of Theorem 4.4 and Corollary 4.6 to the case that $\lambda = 3$ when k is odd, with one possible exceptional value of k, namely $k = 9$. That is, we will prove the following theorem.

Theorem 4.7. *Let $m \geq 1$ and $k \geq 3$ be integers, where k is odd and $k \neq 9$. Then $C_k | 3K_m$ if and only if $k \leq m$ and $k | 3\binom{m}{2}$.*

Our methods for proving Theorem 4.7 are similar to those used in the proof of Theorems 4.4 and 4.5.

4.1 Induction step

A key step in the proof of sufficiency of the obvious necessary conditions for k-cycle decomposition of K_m and $2K_m$ was the proof that, for a fixed value of k, it is enough to show that the necessary conditions are sufficient for values of m in a particular range (depending on the case, this range either included values of m between k and $2k$, or between k and $3k$). We will require a comparable result for decomposition of $3K_m$.

4.1.1 k odd

The following theorem, due to Hoffman, Lindner and Rodger, represents the induction step for decomposing K_m into cycles of odd length.

Theorem 4.8. *[23] Let $k \geq 3$ be an odd integer. If $C_k|K_v$ whenever $k \leq v < 3k$ and $k|\binom{v}{2}$, then $C_k|K_m$ whenever $k \leq m$ and $k|\binom{m}{2}$*

In this section, we prove an analogous result for decomposition of pK_v, where p is prime. First, let us state some results which will be used in the proof.

Theorem 4.9. *[23] Let $k \geq 3$ and $m \geq 3$ be odd. If $m \leq \frac{k^2+1}{2}$, then $C_k|K_{2k} \bowtie \overline{K_m}$.*

Theorem 4.10. *[23] Let $k \geq 3$ be an odd integer and $m \geq 3$ an integer. Then $C_k|K_m * \overline{K_{2k}}$.*

We can now state the induction step for cycles of odd length.

Theorem 4.11. *Let p be an odd prime and let $k \geq 3$ be an odd integer such that $k \neq p$ if $p \leq 13$, and $k \neq 9$ if $p = 3$. If $C_k|pK_v$ whenever $k \leq v < 3k$ and $k|p\binom{v}{2}$, then $C_k|pK_m$ whenever $k \leq m$ and $k|p\binom{m}{2}$.*

Proof. The argument proceeds as Theorem 4.8 was proved in [23].

First, consider the case in which $k = q^\alpha$ where q is prime, $q \neq p$ and $\alpha \geq 1$. Then $q^\alpha|p\binom{m}{2}$ implies that $q^\alpha|\binom{m}{2}$. Hence $C_{q^\alpha}|K_m$, and it easily follows that $C_{q^\alpha}|pK_m$. This argument shows that the necessary conditions for the existence of a C_k-decomposition of pK_m are sufficient when $k < 15$, with possible exceptions for $k = p$ when $p \leq 13$, and $k = 9$ when $p = 3$.

Now, suppose $k \geq 15$. Suppose that $k|p\binom{m}{2}$ and $k \leq m$. Write $k = 2\ell + 1$ and $m = v + 2kv'$, where $k \leq v < 3k$. Note that

$$p\binom{v}{2} = p\binom{m - 2kv'}{2}$$

$$\begin{aligned}
&= p\left(\frac{(m-2kv')(m-2kv'-1)}{2}\right) \\
&= p\frac{m(m-1)}{2} - mkv' - kv'(m-2kv'-1) \\
&= p\binom{m}{2} - pkv'(2m-2kv'-1),
\end{aligned}$$

and so since $k|p\binom{m}{2}$, we have that $k|p\binom{v}{2}$ as well.

If $v' \geq 3$, then we may decompose

$$pK_m = pK_v \oplus p(K_{v'} * \overline{K_{2k}}) \oplus \left(p(K_{2k} \bowtie \overline{K_v}) \oplus p(K_{2k} \bowtie \overline{K_v}) \oplus \cdots \oplus p(K_{2k} \bowtie \overline{K_v})\right).$$

Now, $v < 3k \leq \frac{k^2+1}{2}$, so that $C_k|K_{2k} \bowtie \overline{K_v}$ by Theorem 4.9, and hence $C_k|p(K_{2k} \bowtie \overline{K_v})$. Since $v' \geq 3$, it follows that $C_k|K_{v'} * \overline{K_{2k}}$ by Theorem 4.10, and hence $C_k|p(K_{v'} * \overline{K_{2k}})$. Finally, $C_k|pK_v$ by hypothesis. We obtain that $C_k|pK_m$.

If $v' = 1$, then $m = v + 2k$. Decompose

$$pK_m = pK_v \oplus p(K_{2k} \bowtie \overline{K_v}).$$

Again, $C_k|pK_v$ by hypothesis, and $C_k|K_{2k} \bowtie \overline{K_v}$ by Theorem 4.9.

If $v' = 2$, decompose

$$pK_m = pK_{v+2k} \oplus p(K_{2k} \bowtie \overline{K_{v+2k}}).$$

Since $v + 2k < 5k \leq k^2/3 \leq (k^2+1)/2$, as $k \geq 15$, we have that $C_k|K_{2k} \bowtie \overline{K_{v+2k}}$ by Theorem 4.9. Also, that $C_k|pK_{v+2k}$ follows from the case $v' = 1$. Hence $C_k|pK_m$. □

We note that for $k \geq 15$, the fact that p is prime in the statement of Theorem 4.11 was irrelevant to the proof. Hence we have the following induction step for general multiplicities.

Theorem 4.12. *Let $k \geq 15$ be an odd integer and $\lambda \geq 1$ an integer. If $C_k|\lambda K_v$ whenever $k \leq v < 3k$ and $k|\lambda\binom{v}{2}$, then $C_k|\lambda K_v$ whenever $k \leq v$ and $k|\lambda\binom{v}{2}$.*

Let us return to the possible exceptions in Theorem 4.11. If $k = 3$, then recall from Theorem 4.1 that necessary and sufficient conditions for the existence of a C_3-decomposition of λK_m are known; this was proven by Hanani [22]. Thus, the case $k = 3$ need not be considered. Considering other prime values of k, we recall that Corollary 4.3, due to Smith, proves sufficiency of the obvious necessary conditions for p-cycle decomposition of λK_m when p is an odd prime, and thus the only meaningful possible exception to Theorem 4.11 occurs when $\lambda = 3$ and $k = 9$. Unfortunately, there appears to be a discrepancy in the literature regarding decomposition of complete multigraphs into cycles of length 9. It has frequently been cited that necessary and sufficient conditions for C_9-decomposition of λK_m appear in [8]; however, closer inspection shows that the authors claim to have solved the problem of C_k-decomposition of λK_m only for cycle lengths 3, 5 and 7. We were not aware of this when we completed the proof Theorem 4.7, and therefore necessary and sufficient conditions for decomposition of $3K_m$ into cycles of length 9 remain to be proved.

As we will be interested in constructing decompositions of $3K_m$ later in this chapter, we now explicitly state the induction step, Theorem 4.11, in the specific case that $\lambda = 3$.

Theorem 4.13. *Let $k \geq 3$ be an odd integer, $k \notin \{3, 9\}$. If $C_k | 3K_v$ whenever $k \leq v < 3k$ and $k | 3\binom{v}{2}$, then $C_k | 3K_m$ whenever $k \leq m$ and $k | 3\binom{m}{2}$.*

4.1.2 k even

Although we will be focussed on odd-cycle decomposition of $3K_m$, we include here for the sake of completeness the induction step for even-cycle decomposition of λK_m. Since the induction step in this case follows easily from known results and we will not use this case further, we omit the proof for the sake of brevity.

The key for odd m is the following theorem, which may be proved in the same manner as the corresponding result for even-cycle decomposition of K_m found in [43].

Theorem 4.14. *Suppose k is even and m is odd. If $C_k|\lambda K_m$, then $C_k|\lambda K_{m+2kq}$ for any positive integer q.*

From this result, the induction step easily follows; it is proved as the corresponding result for C_k-decomposition of K_m when k is even and m odd in [45].

Corollary 4.15. *Suppose $k \geq 2$ is even. If $C_k|\lambda K_v$ whenever $k \leq v < 3k$, $k|\lambda\binom{v}{2}$ and v is odd, then $C_k|\lambda K_m$ whenever $k \leq m$, $k|\lambda\binom{m}{2}$ and m is odd.*

We now consider the case that m is even. When m and k are even, we must have that λ is even for the existence of a C_k-decomposition of λK_m. The induction step in this case is proved as that for C_k-decomposition of K_m^* for even m and k in [2].

Theorem 4.16. *Let m, λ and k be positive even integers. If $C_k|\lambda K_m$ whenever $k \leq m < 2k$ and $k|\lambda\binom{m}{2}$, then $C_k|\lambda K_m$ whenever $k \leq m$ and $k|\lambda\binom{m}{2}$.*

4.2 A word on the techniques to be used for decomposition

The cycles which we will form in the construction of the C_k-decompositions of $3K_m$ when $k < m < 3k$ will be of two types, called peripheral cycles and central cycles. This terminology is due to Šajna [45]. We now describe these two cycle types. The following definition may be found in [45].

Definition 4.17. Let $G = X(n; S)$ be a circulant graph. Let $d = \gcd(n, k)$, $k' = k/d$ and $n' = n/d$. A *peripheral cycle* of length k is a k-cycle C in G that satisfies one of the following two properties:

1. For some path P in G of length k', C is the concatenation of paths P, $\rho^{n'}(P)$, $\ldots$, $\rho^{(d-1)n'}(P)$.

2. For some path P in G of length $k'-1$, C is formed from paths P, $\rho^{n'}(P)$, ..., $\rho^{(d-1)n'}(P)$, together with d edges to link together these paths.

In either case, the path P is said to *generate* the peripheral cycle C.

The peripheral cycles to be used in our constructions will be of the second type of peripheral cycle. In forming such a cycle, we will begin by finding a path P to generate the cycle. This path will be a *zig-zag path*, a concept which we now define. The term "zig-zag path" is due to Šajna [45].

Definition 4.18. Let $G = X(n;S)$ be a circulant. A path P of length k whose edges are of k distinct lengths is called a *zig-zag path*.

In particular, given a set $L_P = \{\ell_1, \dots, \ell_k\}$ of k distinct lengths such that $1 \leq \ell_1 < \ell_2 < \cdots, \ell_k \leq \lfloor \frac{n}{2} \rfloor$ in a circulant $X(n;S)$, the path

$$P = 0, \ell_1, \ell_1 - \ell_2, \ell_1 - \ell_2 + \ell_3, \dots, \ell_1 - \ell_2 + \ell_3 - \cdots + (-1)^{k-1}\ell_k$$

is a zig-zag path. The paths which we construct to generate peripheral cycles will have this form. Note that if C is a peripheral cycle generated by the zig-zag path P and whose linking edges are all of length $\ell \notin L_P$, then the cycles $C, \rho(C), \dots, \rho^{n'-1}(C)$ decompose $X(n; L_P \cup \{\ell\})$.

A $(k'-1)$-path P will generate a peripheral cycle if the paths $\rho^{in'}(P)$ are pairwise vertex-disjoint for each $0 \leq i \leq d-1$, and suitable linking edges can be found. The technique that we will use to find linking edges of appropriate lengths is due to Alspach and Gavlas [1], and will be discussed in Section 4.2.1.

Let us now mention the second type of cycle to occur in the decomposition.

Definition 4.19. A k-cycle in G which is not a peripheral cycle is called a *central cycle* of length k.

The central cycles will vary in form depending on whether $k < m < 2k$ or $2k < m < 3k$. In the former case, central cycles will be formed in K_m, which is viewed as $K_{m-1} \bowtie K_1$, consisting of a path P in K_{m-1} together with its diametrically opposed path $\rho^{(m-1)/2}(P)$, an edge of length $\frac{m-1}{2}$, and two edges incident with the vertex of K_1. In the latter case, central cycles will be formed containing edges of k distinct lengths in K_m. We will discuss the construction of central cycles more thoroughly in each case.

4.2.1 Finding linking edges using auxiliary circulant graphs

This ingenious technique was first used by Alspach and Gavlas [1] in decomposing K_m into odd cycles and $K_m - I$ (the complete graph K_m with the edges of a 1-factor removed) into cycles of even length. It also proved instrumental in the determination of necessary and sufficient conditions for a decomposition of K_m into even cycles or $K_m - I$ into odd cycles [45], and for a decomposition of K_m^* (and hence also $2K_m$) into fixed-length cycles [2].

To form peripheral cycles in $3K_m$, we will begin by forming peripheral cycles in K_m, and then taking each with multiplicity 1, 2 or 3. Thus, we need only consider here how to link the paths which generate peripheral cycles in a circulant $X(t; S)$, which is a simple graph.

Suppose that we are trying to find a k-cycle decomposition of the circulant $X(t; S)$ on vertex set $\mathbb{Z}_t$. Let $d = \gcd(t, k)$, $t' = t/d$ and $k' = k/d$. Suppose we have zig-zag paths $P_{i,0}$, $0 \le i \le b-1$, and we wish to form b peripheral cycles generated by these paths. For each $i \in \{0, 1, \ldots, b-1\}$, we form a family of paths $\mathcal{P}_i = \{P_{i,0}, P_{i,1}, \ldots, P_{i,d-1}\}$ by letting $P_{i,j} = \rho^{jt'}(P_{i,0})$ for each $j \in \{0, 1, \ldots, d-1\}$. We furthermore require these paths to satisfy the following properties:

1. Each of $P_{0,0}$, $P_{1,0}$, $\ldots$, $P_{b-1,0}$ is a zig-zag path of length $k' - 1$.

2. Each of $P_{0,0}$, $P_{1,0}$, $\ldots$, $P_{b-1,0}$ has the same initial vertex 0 and the same terminal vertex q. (Note that this condition implies that the path $P_{i,j}$ has initial vertex jt' and terminal vertex $jt' + q$.)

3. No two paths $P_{i,0}$ and $P_{j,0}$ have an edge of the same length.

4. For each $i \in \{0, 1, \ldots, b-1\}$, the paths in $\mathcal{P}_i$ are pairwise vertex-disjoint.

For each $i \in \{0, 1, \ldots, b-1\}$, we wish to find d edges to link the paths in $\mathcal{P}_i$ together to form a cycle C_i. The linking edges in C_0, C_1, $\ldots$, C_{b-1} should be of b lengths, chosen in such a way that among the peripheral cycles C_0, C_1, $\ldots$, C_{b-1}, we use d linking edges of each length. Once we have formed cycles C_0, C_1, $\ldots$, C_{b-1}, we take, for each $i \in \{0, 1 \ldots, b-1\}$, the cycles C_i, $\rho(C_i)$, $\ldots$, $\rho^{t'-1}(C_i)$. In this way, we will exhaust all edges of lengths which appear in the paths $P_{0,0}$, $P_{1,0}$, $\ldots$, $P_{b-1,0}$ and also all edges whose lengths are used in the linking edges.

We find these edges using Hamilton cycles in an auxiliary circulant graph G of order d on vertex set $\{0, 1, \ldots, d-1\}$. Let C be a Hamilton cycle in G, and assign to C an arbitrary orientation to form a directed cycle $\overrightarrow{C} = j_0, j_1, \ldots, j_{d-1}, j_0$. Fix an $i \in \{0, 1, \ldots, b-1\}$. For each $s \in \{0, 1, \ldots, d-1\}$, we join the terminal vertex of P_{i,j_s} to the initial vertex of $P_{i,j_{s+1}}$ We have now used d distinct edges and formed a cycle C_i. Suppose that $e_s = (j_s, j_{s+1})$ is an arc of C. If $j_{s+1} > j_s$, then the linking edge formed from the arc e_s has length $|(j_{s+1} - j_s)t' - q|_t$, while if $j_{s+1} < j_s$, then the linking edge formed from e_s has length $|(j_{s+1} - j_s)t' + q|_t$. In practice, the values of the parameters in our constructions will ensure $(j_{s+1} - j_s)t' - q$ or $(j_{s+1} - j_s)t' + q$, as appropriate, is in $\{1, 2, \ldots, \lfloor \frac{t}{2} \rfloor\}$, so that the linking edge from e_s has length $(j_{s+1} - j_s)t' - q$ if $j_{s+1} > j_s$, and $(j_{s+1} - j_s)t' + q$ if $j_{s+1} < j_s$. Note that if all edges of C are of the same length, then all of the linking edges formed from it will be of a constant length. Moreover, from one Hamilton cycle C in G, we can link two families of paths, one corresponding to each direction of C.

To apply this method, we need to decompose the circulant G into Hamilton cycles. First, G is decomposed into connected circulants of degree 2 of the form $X(d; \{L\})$ where $\gcd(d, L) = 1$, and degree 4 of the form $X(d; \{L_1, L_2\})$ for some suitable choice of lengths L_1 and L_2 such that this circulant is connected (such as if L_1

and L_2 are consecutive integers or consecutive odd integers). A connected circulant $X(d; \{L\})$ of degree 2 is itself a d-cycle, while any connected circulant of degree 4 can be decomposed into Hamilton cycles by the following theorem of Bermond, Favaron and Maheo.

Theorem 4.20. *[7] Suppose X is a connected 4-regular circulant graph. Then X has a decomposition into two Hamilton cycles.*

In each case we will specify the 2-regular and 4-regular connected circulants which decompose the auxiliary circulant G.

Each of the 2-regular circulants can be used to link one or two families of paths, by using one or both orientations of the corresponding Hamilton cycle. Each of the 4-regular circulants will be used, with Hamilton cycles taken in both directions, to link four families of paths. In all cases the linking edges formed will have length congruent to $\pm q \pmod{t'}$, with lengths $Lt' + q$ and $Lt' - q$ arising from the orientations of an edge of G of length L. In each case, we use d edges of each length that appears in the 2- and 4-regular circulants; when we apply the permutations $\rho, \rho^2, \ldots, \rho^{t'-1}$ to the cycles formed, we will use all $t = dt'$ edges of each of these lengths. As long as we can find enough edge lengths to form these 2- and 4-regular connected circulants to join b families of paths, we can use them to form the edge length set of an appropriate auxiliary circulant G.

4.3 Decomposing $3K_m$ into cycles of odd length k, where $k < m < 3k$

In this section, we will prove Theorem 4.7. By Theorem 4.13, we need only show that $C_k|3K_m$ when m and k are odd, $k|3\binom{m}{2}$ and $k \leq m < 3k$, where $k \notin \{3, 9\}$. The proof is divided into two main cases, namely that $k \leq m < 2k$, and that $2k < m < 3k$.

Let m and $k \geq 3$ be odd integers, with $k \leq m < 3k$. The condition $k|3\binom{m}{2}$ is clearly necessary for the existence of a decomposition of $3K_m$ into cycles of length k. We wish to show that this condition is also sufficient. Note that if $k|\binom{m}{2}$, then $C_k|K_m$ by Theorem 4.4, and so we can assume that $k \nmid \binom{m}{2}$. In particular, this means that $m \neq k$.

4.3.1 The case $k < m < 2k$

Let $m = k + r$, where $0 < r < k$. Note that r is even. Let $k = k'm'$, where $k'|3m$ and $m'|\frac{m-1}{2}$ (i.e. $m' = \gcd(k, m-1)$ and $k' = k/m'$). Since $k \nmid \binom{m}{2}$, we have that $3|k'$. Let $\ell = \frac{m-1}{m'}$, and note that $\ell = k' + \frac{r-1}{m'}$. Since m and m' are odd, ℓ must be even. Note that $r < k$ implies that $\ell = k' + \frac{r-1}{m'} < 2k'$, and so $\ell \leq 2k' - 2$.

We will view K_m as $K_{m-1} \bowtie K_1$, and hence $3K_m$ as $3(K_{m-1} \bowtie K_1)$. Here, we take $V(3K_m) = \mathbb{Z}_{m-1} \cup \{\infty\}$, where ∞ is the central vertex. We let ρ denote the permutation $\rho = (0, 1, \ldots, m-2)(\infty)$ which cyclically permutes the vertices of K_{m-1} and leaves the central vertex fixed.

The following lemma, due to Alspach and Gavlas, shows how central cycles can be formed in this case.

Lemma 4.21. *[1] Suppose k and m are odd, $3 \leq k \leq m < 2k$, and $A \subseteq \{1, 2, \ldots, \frac{m-3}{2}\}$ such that $|A| = \frac{k-3}{2}$. Then $C_k|X(m-1; A \cup \{\frac{m-1}{2}\}) \bowtie K_1$.*

Proof. As the cycles formed from this lemma will appear again in Chapter 5, we provide a sketch of the proof so that the reader can see the form of the cycles. Let $V(X(m-1; A \cup \{\frac{m-1}{2}\}) \bowtie K_1) = \mathbb{Z}_{m-1} \cup \{\infty\}$, where ∞ is the vertex of K_1. Let $A = \{a_1, a_2, \ldots, a_{\frac{k-3}{2}}\}$, where $1 \leq a_1 < a_2 < \cdots < a_{\frac{k-3}{2}} \leq \frac{m-3}{2}$. Let

$$P = 0, a_1, a_1 - a_2, a_1 - a_2 + a_3, \ldots, a_1 - a_2 + a_3 - \cdots + (-1)^{\frac{k-1}{2}} a_{\frac{k-3}{2}},$$

For brevity, let v denote the vertex $a_1 - a_2 + a_3 - \cdots + (-1)^{\frac{k-1}{2}} a_{\frac{k-3}{2}}$. Note that the vertices in the sequence $a_1, a_1 - a_2 + a_3, a_1 - a_2 + a_3 - a_4 + a_5, \ldots$ are strictly increasing, while the vertices in the sequence $0, a_1 - a_2, a_1 - a_2 + a_3 - a_4, \ldots$ are strictly decreasing. Since the longest edge length in P is $a_{\frac{k-3}{2}} < \frac{m-1}{2}$, it follows that P and $\rho^{\frac{m-1}{2}}(P)$ are vertex-disjoint. The endvertices of $\rho^{\frac{m-1}{2}}(P)$ are $\frac{m-1}{2}$ and $v + \frac{m-1}{2}$. Let P' be the path $v, v + \frac{m-1}{2}$ of length 1 and P'' the path $\frac{m-1}{2}, \infty, 0$.

Form a cycle

$$C = PP'\overleftarrow{\rho^{\frac{m-1}{2}}(P)}P''$$

by concatenating P, P', $\overleftarrow{\rho^{\frac{m-1}{2}}(P)}$ (the reversal of $\rho^{(m-1)/2}(P)$) and P''. Then cycles $C, \rho(C), \rho^2(C), \ldots \rho^{\frac{m-3}{2}}(C)$ form the required decomposition. □

Let us use $\langle x_1, x_2, \ldots, x_n \rangle$ to denote the multiset with elements $x_1, x_2, \ldots, x_n$ (not necessarily all distinct).

The number of edge lengths in K_{m-1} is $\frac{m-1}{2}$. We can view $3K_{m-1}$ as having edges of length in the multiset $L = \langle 1, 1, 1, 2, 2, 2, 3, 3, 3, \ldots, \frac{m-1}{2}, \frac{m-1}{2}, \frac{m-1}{2} \rangle$, with $m - 1$ edges of each length in L except the diameter length $\frac{m-1}{2}$. Note that $|L| = 3(\frac{m-1}{2})$. A set $A \cup \{\frac{m-1}{2}\}$ as in Lemma 4.21 contains $\frac{k-1}{2}$ edge lengths, so by applying Lemma 4.21 three times (to account for all diameter edges and all central edges) will use in total $3(\frac{k-1}{2})$ lengths from L (not necessarily all distinct). The remaining edge lengths total

$$3\left(\frac{m-1}{2}\right) - 3\left(\frac{k-1}{2}\right) = 3\left(\frac{m-k}{2}\right) = \frac{3r}{2}.$$

Consider a submultiset B of $L - \langle \frac{m-1}{2}, \frac{m-1}{2}, \frac{m-1}{2} \rangle$ with $|B| = 3r/2$. The elements of $L - B$ can be partitioned into three sets of size $\frac{k-1}{2}$, each containing $\frac{m-1}{2}$, and none containing repeated elements. The edges stemming from these sets of edge lengths, together with the central edges, can be partitioned into cycles of length k by using Lemma 4.21 on each set. Namely, if $L - B = S_1 \cup S_2 \cup S_3$, where $|S_i| = \frac{k-1}{2}$ for each i and $S_i \cap S_j = \{\frac{m-1}{2}\}$ whenever $i \neq j$, then by Lemma 4.21, we have that $C_k | X(m-1; S_i) \bowtie K_1$ for each $i \in \{1, 2, 3\}$. It remains only to find an appropriate submultiset B of $L - \langle \frac{m-1}{2}, \frac{m-1}{2}, \frac{m-1}{2} \rangle$ of size $3r/2$, such that $X(m-1; B)$ can be decomposed into peripheral k-cycles.

Since $k'|3m$ and $k'|k$, we have that $k'|(3m - 3k)$, i.e. $k'|3r$. Moreover, as k' is odd and r is even, it follows that $k'|\frac{3r}{2}$. Note that if $k'|\frac{r}{2}$, then $k'|m$, and so $k|\binom{m}{2}$. This means that $C_k|K_m$ (by Theorem 4.4), and so that $C_k|3K_m$ immediately follows. So we can assume that $\frac{3r}{2k'} \equiv 1$ or $2 \pmod 3$. Let $\frac{3r}{2k'} = 3s + \delta$, where $\delta \in \{1, 2\}$. We will form $\frac{3r}{2k'}$ starter cycles, each of which uses k' lengths from L. We do this by first forming $\lceil \frac{r}{2k'} \rceil = s + 1$ starter cycles in K_m. Let $c = \lceil \frac{r}{2k'} \rceil$ denote the number of starter peripheral cycles. We take $s = c - 1$ of these cycles with multiplicity 3, and one with multiplicity δ, giving in total $\frac{3r}{2k'}$ starter cycles in $3K_m$.

We now obtain properties of the integer c which will be used in many of the cases to follow.

- If $\frac{3r}{2k'} \equiv 2 \pmod 3$, then $c = \lceil \frac{r}{2k'} \rceil = \frac{r}{2k'} + \frac{1}{3} < \frac{m'}{2} + \frac{1}{3} = \frac{m'-1}{2} + \frac{5}{6}$. Since $\lceil \frac{r}{2k'} \rceil, \frac{m'-1}{2} \in \mathbb{Z}$, this means that $c = \lceil \frac{r}{2k'} \rceil \leq \frac{m'-1}{2} < \frac{m'}{2}$, so that $c\ell = \lceil \frac{r}{2k'} \rceil \ell < \frac{m'}{2}\ell = \frac{m-1}{2}$.

- If $\frac{3r}{2k'} \equiv 1 \pmod 3$, then $c = \lceil \frac{r}{2k'} \rceil = \frac{r}{2k'} + \frac{2}{3} < \frac{m'}{2} + \frac{2}{3} = \frac{m'-1}{2} + \frac{7}{6}$. So either $c = \lceil \frac{r}{2k'} \rceil \leq \frac{m'-1}{2}$, in which case again $c\ell = \lceil \frac{r}{2k'} \rceil \ell < \frac{m-1}{2}$, or else $c = \lceil \frac{r}{2k'} \rceil = \frac{m'+1}{2}$. This latter case implies that $r = k - \frac{k'}{3}$.

Thus we have that either $c \leq \frac{m'-1}{2}$, or else $r = k - \frac{k'}{3}$.

We will divide the problem into two main cases depending on the congruence of k' modulo 4. Note that since $3|k'$, we have in particular that $k' \geq 3$.

Case 1. $k' \equiv 3 \pmod 4$

We will further divide this case into a number of subcases. In general, we will divide the $m - 1$ non-central vertices into m' segments of length ℓ. We construct a path $P_{i,0}$ of length $k' - 1$ in the first segment, and form $m' - 1$ other paths by rotation of $P_{i,0}$ by multiples of ℓ (i.e. application of permutations of the form $\rho^{j\ell}$); cycles will be formed from these paths by the addition of linking edges. Given a starter peripheral cycle C formed in this manner, the decomposition will contain cycles C, $\rho(C)$, ..., $\rho^{\ell-1}(C)$.

This approach will be problematic in two instances. First, if $m' = 1$, then $\ell = m - 1$; there is only one segment, and the scheme of finding linking lengths by auxiliary circulants will not work. Secondly, recall that either $c \leq \frac{m'-1}{2}$ or $r = k - \frac{k'}{3}$. The latter case allows $c = \frac{m'+1}{2}$, and using the methods which will be applied in the other cases, we will not be able to form enough starter paths for peripheral cycles. We thus deal with the possibilities that $m' = 1$ and $r = k - \frac{k'}{3}$ as special cases. In all remaining cases, we assume $m' \geq 3$ and $r < k - \frac{k'}{3}$ (noting that in general $r \leq k - \frac{k'}{3}$ as $\frac{k'}{3}|r$ and $r < k$). Note that $\ell = k' + \frac{r-1}{m'} \geq k' + 1$ and that $r < k$ implies $\ell < 2k'$. The three cases are that $\ell = k' + 1$, $k' + 1 < \ell \leq \frac{3k'-1}{2}$ and $\frac{3k'-1}{2} < \ell < 2k'$.

Subcase 1.1. $m' = 1$.

If $m' = 1$, then $k = k'$, so that $k|3m$. Since $k \nmid m$, we may write $k = 3d$, where $d|m$. We may assume that $d \geq 3$, since the required result is known if $k = 3$, by Theorem 4.1. We have that $k < m < 2k$, or $3d < m < 6d$, so that $3 < \frac{m}{d} < 6$. Since m is odd, this means that $\frac{m}{d} = 5$, or $d = \frac{m}{5}$. Hence $k = \frac{3m}{5}$, i.e. $m = \frac{5k}{3}$. So $r = m - k = \frac{5k}{3} - k = \frac{2k}{3} = k - \frac{k}{3} = k - \frac{k'}{3}$.

Note that $\frac{3r}{2k'} = \frac{3(2k/3)}{2k} = 1$. Thus $\frac{r}{2k'} = \frac{1}{3} < 1$, so that $\lceil \frac{r}{2k'} \rceil = 1$. So it suffices to find a single starter peripheral k-cycle in $3K_m$; the remaining lengths can be fit into central cycles by Lemma 4.21.

Let $C = 0, 1, -1, 2, -2, \ldots, (k-3)/4, -(k-3)/4, (k+1)/4, (3k-1)/4, (k+5)/4, (3k-5)/4, (k+9)/4, \ldots, (k-1)/2, (k+1)/2, 0$. Then C is a k-cycle in $3K_m$ which has two edges of each length in $\{1, 2, \ldots, \frac{k-1}{2}\}$, and one edge of length $\frac{k+1}{2}$. We may take C as our starter cycle unless $k = m - 2$ (in which case $\frac{k+1}{2} = \frac{m-1}{2}$; this length must occur in a central cycle). However, since m is odd, in this case $k' = 3d = 3\gcd(m, m-2) = 3$, and so $k = 3$; necessary and sufficient conditions for C_3-decomposition of $3K_m$ are already known (Theorem 4.1).

Subcase 1.2. $\ell > \frac{3k'-1}{2}$, $m' \geq 3$ and $r \neq k - \frac{k'}{3}$.

Let

$$P_{0,0} = 0, 1, -1, 2, -2, \ldots, \frac{k'-3}{4}, -\frac{k'-3}{4}, \frac{k'+5}{4}, -\frac{k'+1}{4}, \frac{k'+9}{4}, -\frac{k'+5}{4}, \ldots, \frac{k'+1}{2}, -\frac{k'-1}{2}.$$

Then $P_{0,0}$ is a path of length $k'-1$ containing edges of lengths $1, 2, \ldots, \frac{k'-3}{2}, \frac{k'+1}{2}, \frac{k'+3}{2}, \ldots, k'$. So $P_{0,0}$ does not contain an edge of length $\frac{k'-1}{2}$, and moreover does not contain an edge of length $\ell - \frac{k'-1}{2}$, since $\ell - \frac{k'-1}{2} > \frac{3k'-1}{2} - \frac{k'-1}{2} = k'$. Note that $P_{0,0}$ and $\rho^{\ell}(P_{0,0})$ are vertex-disjoint, as $\ell - \frac{k'-1}{2} = k' + \frac{r-1}{m'} - \frac{k'-1}{2} = \frac{k'+1}{2} + \frac{r-1}{m'} > \frac{k'+1}{2}$.

For $i = 1, 2, \ldots, c-1$, let $P_{i,0}$ be obtained from $P_{0,0}$ by adding $i\ell$ to the even-positioned vertices of $P_{0,0}$, so that

$$P_{i,0} = 0, 1+i\ell, -1, 2+i\ell, -2, \ldots, \frac{k'-3}{4}+i\ell, -\frac{k'-3}{4}, \frac{k'+5}{4}+i\ell, -\frac{k'+1}{4}, \frac{k'+9}{4}+i\ell, -\frac{k'+5}{4}, \ldots, \frac{k'+1}{2}+i\ell, -\frac{k'-1}{2}.$$

The longest length in $P_{i,0}$ is hence $k' + i\ell$, and then longest length in any of these paths, recalling that $r \neq k - \frac{k'}{3}$ implies that $c \leq \frac{m'-1}{2}$, is

$$k' + (c-1)\ell = \ell - \frac{r-1}{m'} + c\ell - \ell \leq c\ell \leq \frac{m'-1}{2}\ell < \frac{m'}{2}\ell = \frac{m-1}{2}.$$

Note that, since $\ell = k' + \frac{r-1}{m'} > k'$ (as $r \geq 2$), and the longest length in $P_{0,0}$ is k', we have that each vertex of $P_{0,0}$ is in a different congruence class modulo ℓ. So the same must also hold true of $P_{i,0}$ for each i. Hence, the only way $P_{i,0}$ and $P_{i,j} = \rho^{j\ell}(P_{i,0})$ could share a vertex (where $j \in \{1, \ldots, m'-1\}$) is if, for some vertex v of $P_{i,0}$, $v = \rho^{j\ell}(v) = v + j\ell$. This is impossible, since $0 < \ell \leq j\ell \leq (m'-1)\ell = m - 1 - \ell < m - 1$. It follows that $P_{i,0}, P_{i,1}, \ldots, P_{i,m'-1}$ are pairwise vertex-disjoint paths.

For each $i = 0, 1, \ldots, c-1$, let $\mathcal{P}_i = \{P_{i,0}, P_{i,1}, \ldots, P_{i,m'-1}\}$. To link the families $\mathcal{P}_i$ of paths into cycles, the linking edges will be found via auxiliary circulants of order m'. We need to verify that there are enough linking lengths congruent to

$\pm\frac{k'-1}{2} \pmod{\ell}$ to link $c = \lceil\frac{r}{2k'}\rceil$ paths. Note that $\frac{k'-1}{2} \equiv -\frac{k'-1}{2} \pmod{\ell}$ is impossible, since this condition implies $k'-1 \equiv 0 \pmod{\ell}$, but $2 \leq k'-1 < \ell$. The allowable lengths for linking edges are $\ell\pm\frac{k'-1}{2}$, $2\ell\pm\frac{k'-1}{2}$, ..., $\frac{m'-1}{2}\ell\pm\frac{k'-1}{2}$ (noting that $\frac{m'-1}{2}\ell+\frac{k'-1}{2} = \frac{m-1}{2}+\frac{k'-1}{2}-\frac{\ell}{2} < \frac{m-1}{2}$ since $\ell > k'$).

If c is even, then we use linking lengths $\ell\pm\frac{k'-1}{2}, 2\ell\pm\frac{k'-1}{2}, \ldots, \frac{c}{2}\ell\pm\frac{k'-1}{2}$, derived from circulants $X(m';\{1,2\})$, $X(m';\{3,4\})$, ..., $X(m';\{(\frac{c}{2}-1),\frac{c}{2}\})$ if $c \equiv 0 \pmod 4$, and from circulants $X(m';\{1\})$, $X(m';\{2,3\})$, $X(m';\{4,5\})$, ..., $X(m';\{\frac{c}{2}-1,\frac{c}{2}\})$ if $c \equiv 2 \pmod 4$. Note that $c \leq \frac{m'+1}{2}$ implies $\frac{c}{2} \leq \frac{m'+1}{4} \leq \frac{m'-1}{2}$ since $m' \geq 3$, and so all linking lengths are admissible.

If c is odd, then we use linking lengths $\ell-\frac{k'-1}{2}$, $2\ell\pm\frac{k'-1}{2}$, $3\ell\pm\frac{k'-1}{2}$, ..., $\frac{c+1}{2}\ell\pm\frac{k'-1}{2}$, derived from circulants $X(m';\{1\})$, $X(m';\{2,3\})$, $X(m';\{4,5\})$, ..., $X(m';\{\frac{c-1}{2},\frac{c+1}{2}\})$ if $c \equiv 1 \pmod 4$, and from circulants $X(m';\{1\})$, $X(m';\{2\})$, $X(m';\{3,4\})$, $X(m';\{5,6\})$, ..., $X(m';\{\frac{c-1}{2},\frac{c+1}{2}\})$ if $c \equiv 3 \pmod 4$. Note that $c \leq \frac{m'+1}{2}$ implies that $\frac{c+1}{2} \leq \frac{m'+3}{4} \leq \frac{m'-1}{2}$ if and only if $m' \geq 5$, and so the linking edges can be formed if $m' \geq 5$. If $m' = 3$ and $c = \frac{m'+1}{2}$, then $c = 2$ is even. So if $m' = 3$ and c is odd, we must have $c \leq \frac{m'-1}{2} = 1$, so that $c = 1$, and the linking edge used is $\ell-\frac{k'-1}{2}$. Note that $\ell-\frac{k'-1}{2} < \ell < \frac{3\ell}{2} = \frac{m'\ell}{2} = \frac{m-1}{2}$.

Subcase 1.3. $k'+1 < \ell \leq \frac{3k'-1}{2}$, $m' \geq 3$ and $r \neq k-\frac{k'}{3}$.

First, note that $\ell > k'+1$ implies that $\frac{k'+1}{2} < \ell-\frac{k'+1}{2}$, and $\ell \leq \frac{3k'-1}{2}$ implies that $\ell-\frac{k'+1}{2} < k'+1$. Let

$$\begin{gathered}P_{0,0} = 0,1,-1,2,-2,\ldots,\frac{k'-3}{4},-\frac{k'-3}{4},\frac{k'+1}{4},-\frac{k'+5}{4},\frac{k'+5}{4},\\ -\frac{k'+9}{4},\ldots,\frac{\ell}{2}-\frac{k'+5}{4},-\left(\frac{\ell}{2}-\frac{k'+1}{4}\right),\frac{\ell}{2}-\frac{k'-3}{4},-\left(\frac{\ell}{2}-\frac{k'-3}{4}\right),\\ \frac{\ell}{2}-\frac{k'-7}{4},-\left(\frac{\ell}{2}-\frac{k'-7}{4}\right),\ldots,\frac{k'+1}{2},-\frac{k'+1}{2}.\end{gathered}$$

Then the edge lengths in $P_{0,0}$ are all between 1 and $k'+1$, with lengths $\frac{k'+1}{2}$ and $\ell-\frac{k'+1}{2}$ not appearing. The fact that $\ell-\frac{k'+1}{2} > \frac{k'+1}{2}$ means that $P_{0,0}$ and $\rho^{\ell}(P_{0,0})$ are vertex-disjoint.

For $i = 0, 1, \ldots, c-1$, let $P_{i,0}$ be obtained from $P_{0,0}$ by adding $i\ell$ to the even-positioned vertices of $P_{0,0}$. Then $P_{i,0}$ contains no edge of length congruent to $\pm\frac{k'+1}{2}$ (mod ℓ). The longest length in $P_{i,0}$ is $k'+1+i\ell$, and so the longest length in any of these paths is $k'+1+(c-1)\ell = c\ell - \frac{r-1}{m'} + 1$. Since $\ell > k'+1$, we have that $\frac{r-1}{m'} > 1$, so that the longest length is less than $c\ell$, and since $r \neq k - \frac{k'}{3}$, we have that $c\ell < \frac{m-1}{2}$. As before, for each $i \in \{0, 1, \ldots, c-1\}$, we form a family of paths $\mathcal{P}_i = \{P_{i,0}, P_{i,1}, \ldots, P_{i,m'-1}\}$ by letting $P_{i,j} = \rho^{j\ell}(P_{i,0})$ for each $j \in \{0, 1, \ldots, m'-1\}$.

Let us now verify that we have enough linking edge lengths, which must be congruent to $\pm\frac{k'+1}{2}$ (mod ℓ). Note that since $\ell \neq k'+1$, $\frac{k'+1}{2} \not\equiv -\frac{k'+1}{2}$ (mod ℓ). Also, note that $\frac{m'-1}{2}\ell + \frac{k'+1}{2} = \frac{m'\ell}{2} - \frac{\ell}{2} + \frac{k'+1}{2} = \frac{m-1}{2} - \frac{\ell}{2} + \frac{k'+1}{2} < \frac{m-1}{2}$ since $\ell > k'+1$; hence allowable linking edge lengths are $\ell \pm \frac{k'+1}{2}, 2\ell \pm \frac{k'+1}{2}, \ldots, \frac{m'-1}{2}\ell \pm \frac{k'+1}{2}$.

If c is even, then we use linking lengths $\ell \pm \frac{k'+1}{2}, 2\ell \pm \frac{k'+1}{2}, \ldots, \frac{c}{2}\ell \pm \frac{k'+1}{2}$, derived from circulants $X(m'; \{1,2\})$, $X(m'; \{3,4\})$, …, $X(m'; \{\frac{c}{2}-1, \frac{c}{2}\})$ if $c \equiv 0 \pmod 4$, and from circulants $X(m'; \{1\})$, $X(m'; \{2,3\})$, $X(m'; \{4,5\})$, …, $X(m'; \{\frac{c}{2}-1, \frac{c}{2}\})$ if $c \equiv 2 \pmod 4$. Note that $c \leq \frac{m'-1}{2}$ implies $\frac{c}{2} \leq \frac{m'-1}{4} \leq \frac{m'-1}{2}$, and so all linking lengths are admissible.

If c is odd, then we use linking lengths $\ell - \frac{k'+1}{2}$, $2\ell \pm \frac{k'+1}{2}$, $3\ell \pm \frac{k'+1}{2}$, …, $\frac{c+1}{2}\ell \pm \frac{k'+1}{2}$, derived from circulants $X(m'; \{1\})$, $X(m'; \{2,3\})$, $X(m'; \{4,5\})$, …, $X(m'; \{\frac{c-1}{2}, \frac{c+1}{2}\})$ if $c \equiv 1 \pmod 4$, and from circulants $X(m'; \{1\})$, $X(m'; \{2\})$, $X(m'; \{3,4\})$, $X(m'; \{5,6\})$, …, $X(m'; \{\frac{c-1}{2}, \frac{c+1}{2}\})$ if $c \equiv 3 \pmod 4$. Note that $c \leq \frac{m'-1}{2}$ implies that $\frac{c+1}{2} \leq \frac{m'+1}{4} \leq \frac{m'-1}{2}$ since $m' \geq 3$.

Subcase 1.4. $\ell = k'+1$, $m' \geq 3$ and $r \neq k - \frac{k'}{3}$.

The following construction is similar to that in [1, Proof of Theorem 2, Subcase 1.2, case $\ell = a'+1$].

Let
$$P_{0,0} = 0, 1, -1, 2, -2, \ldots, \frac{k'-1}{2}, -\frac{k'-1}{2},$$
and for $i \geq 0$, let $P_{i,0}$ be formed from $P_{0,0}$ by adding $i\ell$ to the even-positioned vertices

of $P_{0,0}$.

Let us determine which lengths are allowable for linking edges. First we derive a bound on the value of c in this case. Note that $\ell = k' + 1$ implies that $r = m' + 1$, so that (since also $k' \geq 3$), $\frac{r}{2k'} = \frac{m'+1}{2k'} \leq \frac{m'+1}{6}$. Hence $c = \lceil \frac{r}{2k'} \rceil \leq \lceil \frac{m'+1}{6} \rceil$. Now, since m' is odd, $\lceil \frac{m'+1}{6} \rceil \leq \frac{m'+5}{6} \leq \frac{m'+1}{4}$ for $m' \geq 7$. It is easy to check directly that $\lceil \frac{m'+1}{6} \rceil \leq \frac{m'+1}{4}$ when $m' = 3$ or 5. We therefore have that $c \leq \frac{m'+1}{4}$.

This means that

$$(2c-1)\ell + \frac{k'-1}{2} = (2c-1)\ell + \frac{\ell-2}{2} = \left(\frac{4c-1}{2}\right)\ell - 1 < \left(\frac{4c-1}{2}\right)\ell \leq \frac{m'}{2}\ell = \frac{m-1}{2}.$$

Thus linking edge lengths may be chosen from $\ell \pm \frac{k'-1}{2}, 2\ell \pm \frac{k'-1}{2}, \ldots, (2c-1)\ell \pm \frac{k'-1}{2}$.

If $c = \lceil \frac{r}{2k'} \rceil \equiv 0 \pmod 4$, then we use families of paths derived from $P_{0,0}$, $P_{1,0}$, …, $P_{c-1,0}$. The largest length in these paths is $(c-1)\ell + k' - 1$. Note that $(c-1)\ell + k' - 1 < c\ell - \frac{k'-1}{2}$ if and only if $\ell > \frac{3(k'-1)}{2}$, so length $c\ell - \frac{k'-1}{2}$ may not be available for linking. Guaranteed linking lengths are $(c+1)\ell \pm \frac{k'-1}{2}, (c+2)\ell \pm \frac{k'-1}{2}, \ldots, (2c-1)\ell \pm \frac{k'-1}{2}$, which number $2(c-1)$. Use linking lengths $(c+1)\ell \pm \frac{k'-1}{2}, \ldots, \frac{3c}{2}\ell \pm \frac{k'-1}{2}$ arising from circulants $X(m'; \{c+1, c+2\}), X(m'; \{c+3, c+4\}), \ldots, X(m'; \{\frac{3c}{2} - 1, \frac{3c}{2}\})$.

If $c \equiv 1 \pmod 4$ and $c > 1$, then we use families of paths derived from $P_{1,0}$, $P_{2,0}$, …, $P_{c,0}$. The shortest length used is $\ell + 1$, and the longest is $c\ell + k' - 1$. Use linking lengths $(c+2)\ell \pm \frac{k'-1}{2}, (c+3)\ell \pm \frac{k'-1}{2}, \ldots, (\frac{3c+1}{2})\ell \pm \frac{k'-1}{2}$, arising from circulants $X(m'; \{c+2, c+3\}), X(m'; \{c+4, c+5\}), \ldots, X(m'; \{\frac{3c-1}{2}, \frac{3c+1}{2}\})$, together with length $\ell - \frac{k'-1}{2} = \frac{k'+3}{2}$ from the circulant $X(m'; \{1\})$. The longest length in peripheral cycles is the linking length $\frac{3c+1}{2}\ell + \frac{k'-1}{2} \leq (2c-1)\ell + \frac{k'-1}{2} < \frac{m-1}{2}$. If $c = 1$, then we use $P_{0,0}$ and linking length $\ell + \frac{k'-1}{2} = (2c-1)\ell + \frac{k'-1}{2}$. The longest length used in peripheral cycles is $(2c-1)\ell + \frac{k'-1}{2} < \frac{m-1}{2}$.

If $c \equiv 2 \pmod 4$ and $c > 2$, then we use families of paths derived from $P_{2,0}$, $P_{3,0}$ …, $P_{c+1,0}$. The shortest length used is $2\ell + 1$, and the longest is $(c+1)\ell + k' - 1$. Use linking lengths $(c+3)\ell \pm \frac{k'-1}{2}, (c+4)\ell \pm \frac{k'-1}{2}, \ldots, (\frac{3c+2}{2})\ell \pm \frac{k'-1}{2}$, arising from circulants $X(m'; \{c+3, c+4\}), X(m'; \{c+5, c+6\}), \ldots, X(m'; \{\frac{3c}{2}, \frac{3c+2}{2}\})$, together with lengths

$\ell \pm \frac{k'-1}{2}$ from the circulant $X(m'; \{1\})$. (Note that $c \geq 6$ implies that $\frac{3c+2}{2} > c+3$.) If $c = 2$, then we use paths $P_{0,0}$ and $P_{2,0}$ to form peripheral cycles, together with linking lengths $\ell + \frac{k'-1}{2}$ and $2\ell - \frac{k'-1}{2}$. Note that $k' - 1 < \ell + \frac{k'-1}{2} < 2\ell - \frac{k'-1}{2} < 2\ell + 1$, and so the linking lengths do not occur in either of $P_{0,0}$ or $P_{2,0}$. Moreover, the longest length used in a peripheral cycle will be $2\ell + k' - 1 = (2c-2)\ell + k' - 1 < (2c-1)\ell < \frac{m-1}{2}$.

If $c \equiv 3 \pmod 4$, then we use families of paths derived from $P_{2,0}, P_{3,0}, \ldots, P_{c+1,0}$. The shortest length used is $2\ell + 1$, and the longest is $(c+1)\ell + k' - 1$. If $c = 3$, then we use the linking lengths $\ell \pm \frac{k'-1}{2}$ which arise from the circulant $X(m'; \{1\})$, and $2\ell - \frac{k'-1}{2}$ from the circulant $X(m'; \{2\})$. Otherwise, use linking lengths $(c+3)\ell \pm \frac{k'-1}{2}$, $(c+4)\ell \pm \frac{k'-1}{2}, \ldots, (\frac{3c+1}{2})\ell \pm \frac{k'-1}{2}$, arising from circulants $X(m'; \{c+3, c+4\})$, $X(m'; \{c+5, c+6\}), \ldots, X(m'; \{\frac{3c-1}{2}, \frac{3c+1}{2}\})$, together with lengths $\ell \pm \frac{k'-1}{2}$from the circulant $X(m'; \{1\})$ and $2\ell - \frac{k'-1}{2}$ from the circulant $X(m'; \{2\})$. Note that $c \geq 7$ implies that $\frac{3c+1}{2} \geq c+3$. Moreover, $2\ell - \frac{k'-1}{2} < 2\ell + 1$, so none of these lengths have previously been used.

Subcase 1.5. $r = k - \frac{k'}{3}$ and $m' \geq 3$.

In this case, most of the paths which form peripheral cycles will be those constructed in Subcase 1.2, which required the condition that $\ell - \frac{k'-1}{2} > k'$, or equivalently $\ell > \frac{3k'-1}{2}$. Note that since $\frac{k'+3}{3m'} \leq \frac{k'+3}{9} \leq \frac{2k'}{9}$, we have that

$$\ell = k' + \frac{r-1}{m'} = k' + \frac{k - k'/3 - 1}{m'} = 2k' - \frac{k'+3}{3m'} \geq \frac{16k'}{9} > \frac{3(k'-1)}{2} > \frac{3k'-1}{2}. \quad (4.3.1)$$

The case that $r = k - \frac{k'}{3}$ allows the possibility that the number of required paths is $c = \lceil \frac{r}{2k'} \rceil = \frac{m'+1}{2}$. This means that, although paths $P_{0,0}, \ldots, P_{c-2,0}$ as in the construction given in Subcase 1.2 can be formed, the longest length in the path $P_{c-1,0}$ exceeds $\frac{m-1}{2}$. To compensate for this, we will replace paths $P_{0,0}$ and $P_{1,0}$ by three paths $P^*_{0,0}$, $P^*_{1,0}$ and $P^{**}_{1,0}$ (i.e. we fit three paths into the first two segments). We must ensure that these three paths each have initial vertex 0 and terminal vertex $-(\frac{k'-1}{2})$; that each of these paths P is pairwise vertex disjoint with $\rho^{j\ell}(P)$; that none of the three paths uses an edge of any linking length; and that the edge lengths in these three paths fall between 1 and 2ℓ. These paths, together with paths $P_{2,0}, \ldots, P_{c-2,0}$, will be used to form peripheral cycles in the usual way.

Let $P^*_{0,0} = 0, 1, -1, 2, -2, \ldots, \frac{k'-1}{2}, -(\frac{k'-1}{2})$. Then $P^*_{0,0}$ contains edges of lengths $1, 2, \ldots, k'-1$. Note that length $\frac{k'-1}{2}$ occurs in $P^*_{0,0}$; however, this length is not used in the linking scheme. We wish to show that $k' - 1 < \ell - \frac{k'-1}{2}$, or that $\ell > \frac{3(k'-1)}{2}$; this ensures that $P^*_{0,0}$ is vertex disjoint with $\rho^{i\ell}(P^*_{0,0})$ for each $1 \leq i \leq m'-1$. It has previously been shown in inequality 4.3.1 that the inequality $\ell > \frac{3(k'-1)}{2}$ is true.

Let

$$\begin{aligned} P^*_{1,0} = {} & 0, k', -1, k'+1, -2, \ldots, \frac{\ell}{2} + \frac{k'-3}{4}, -\left(\frac{\ell}{2} - \frac{3k'-1}{4}\right), \frac{\ell}{2} + \frac{k'+5}{4}, \\ & -\left(\frac{\ell}{2} - \frac{3k'-5}{4}\right), \ldots, \ell - \frac{k'+1}{2}, -\left(\ell - \frac{3k'+1}{2}\right), \ell+2, -\left(\ell - \frac{3k'-1}{2}\right), \\ & \ell+3, -\left(\ell - \frac{3k'-3}{2}\right), \ldots, -\left(\frac{\ell}{2} - \frac{k'+5}{2}\right), \frac{\ell}{2} + k', -\left(\frac{\ell}{2} - \frac{k'+3}{2}\right), \\ & \frac{\ell}{2} + k' + 2, -\left(\frac{\ell}{2} - \frac{k'+1}{2}\right), \frac{\ell}{2} + k' + 3, \ldots, 2k'+2, -\left(\frac{k'-1}{2}\right). \end{aligned}$$

Let us summarize the key properties of the path $P^*_{1,0}$. This path skips vertex $\frac{\ell}{2} - \frac{k'+1}{4}$, so that it does not use an edge of length $\ell - \frac{k'-1}{2}$. Vertices $\ell - \frac{k'-1}{2}, \ell - \frac{k'-3}{2}, \ldots, \ell$ are skipped; this fact will help to ensure that $P^*_{1,0}$ and $\rho^{\ell}(P^*_{1,0})$ are vertex-disjoint. Vertex $\ell + (k' - \frac{\ell}{2}) + 1$ (which is equal to $\frac{\ell}{2} + k' + 1$) is skipped, so that the path does not contain an edge of length $\ell + \frac{k'-1}{2}$. So $P^*_{1,0}$ contains edges of lengths k', $k'+1$, $k'+2$, …, $\ell - \frac{k'+1}{2}$, $\ell - \frac{k'-3}{2}$, $\ell - \frac{k'-5}{2}$, …, $2\ell - 2k' - 1$, $2\ell - \frac{3k'-3}{2}$, $2\ell - \frac{3k'-5}{2}$, …, $\ell + \frac{k'-3}{2}$, $\ell + \frac{k'+1}{2}$, $\ell + \frac{k'+3}{2}$, …, $\frac{5k'+3}{2}$. (Note that the fact that $\ell = k' + \frac{r-1}{m'} \leq 2k' - 2$ implies that $\ell + \frac{k'+1}{2} > 2\ell - \frac{3k'-3}{2}$ and that $\ell + \frac{k'+1}{2} < \frac{5k'+3}{2}$.)

The longest length which occurs in $P^*_{1,0}$ is $2k' + 2 + \frac{k'-1}{2} = \frac{5k'+3}{2}$. Note that

$$\begin{array}{rrcl} & 2k' + 2 + \frac{k'-1}{2} & < & 2\ell - \frac{k'-1}{2} \\ \iff & 2\ell - 2k' & > & k' + 1 \\ \iff & \ell - k' & > & \frac{k'+1}{2} \\ \iff & \frac{r-1}{m'} & > & \frac{k'+1}{2} \\ \iff & r - 1 & > & \frac{k+m'}{2} \\ \iff & k - \frac{k'}{3} - 1 & > & \frac{k+m'}{2} \\ \iff & \frac{k}{2} & > & \frac{k'}{3} + \frac{m'}{2} + 1 \end{array}$$

If $m' \leq k'$, then $\frac{k'}{3} + \frac{m'}{2} + 1 \leq \frac{k'}{3} + \frac{k'}{2} + \frac{k'}{3} = \frac{7k'}{6} < \frac{k}{2}$ since $m' \geq 3 > \frac{14}{6}$. Similarly, if $k' < m'$, then $\frac{k'}{3} + \frac{m'}{2} + 1 < \frac{7m'}{6} < \frac{k}{2}$, so the inequality is true. Hence $P^*_{1,0}$ does not reach length $2\ell - \frac{k'-1}{2}$.

We need to verify that $P^*_{1,0}$ and $\rho^\ell(P^*_{1,0})$ are vertex-disjoint. (Since the longest length in $P^*_{1,0}$ is $\frac{5k'+3}{2} < 2\ell - \frac{k'-1}{2} < 2\ell$, $P^*_{1,0}$ is vertex disjoint from $\rho^{j\ell}(P^*_{1,0})$ whenever $j \geq 2$.) The vertices of $P^*_{1,0}$ are as follows:

- $-(\frac{k'-1}{2}), -(\frac{k'-3}{2}), \ldots, 0$
- $k', k'+1, \ldots, \frac{\ell}{2} + \frac{k'-3}{4}$
- $\frac{\ell}{2} + \frac{k'+5}{4}, \ldots, \ell - \frac{k'+1}{2}$
- $\ell + 2, \ell + 3, \ldots, \frac{\ell}{2} + k'$
- $\frac{\ell}{2} + k' + 2, \ldots, 2k' + 2,$

while the vertices of $\rho^\ell(P^*_{1,0})$ are as follows:

- $\ell - \frac{k'-1}{2}, \ell - \frac{k'-3}{2}, \ldots, \ell$
- $\ell + k', \ell + k' + 1, \ldots, \frac{3\ell}{2} + \frac{k'-3}{4}$
- $\frac{3\ell}{2} + \frac{k'+5}{4}, \ldots, 2\ell - \frac{k'+1}{2}$

- $2\ell + 2, 2\ell + 3, \ldots, \frac{3\ell}{2} + k'$
- $\frac{3\ell}{2} + k' + 2, \ldots, \ell + 2k' + 2$.

Hence to show that $P^*_{1,0}$ and $\rho^\ell(P^*_{1,0})$ are vertex-disjoint, it suffices to show that the following two inequalities hold:

$$\begin{aligned} \ell - \frac{k'-1}{2} &> 0 \\ \ell + k' &> 2k' + 2 \end{aligned}$$

The first is clearly true since $\ell > k'$. To prove the second inequality, note that it is equivalent to the inequality $\ell > k' + 2$. Note, however, that $\ell = k' + \frac{r-1}{m'} = k' + 1$ implies $r = m' + 1 = k - \frac{k'}{3}$. Letting $M = \max\{k', m'\}$, we see, since $k' \geq 3$ and $m' \geq 3$, that $k = m' + \frac{k'}{3} + 1 \leq M + \frac{M}{3} + \frac{M}{3} = \frac{5}{3}M < 3M \leq k$, a contradiction. Hence, $\ell > k' + 1$, and the fact that ℓ is even and k' is odd proves that $\ell \geq k' + 3$.

Next, we form path $P^{**}_{1,0}$. Recall that the longest length in $P^*_{1,0}$ was $\frac{5k'+3}{2}$, which is less than $2\ell - \frac{k'-1}{2}$. Since $\frac{5k'+3}{2}$ and $2\ell - \frac{k'-1}{2}$ are both odd, we begin with length $\frac{5k'+7}{2}$ so that the vertex skipped to avoid using length $2\ell - \frac{k'-1}{2}$ is positive.

Let

$$P^{**}_{1,0} = 0, \frac{5k'+7}{2}, -1, \frac{5k'+9}{2}, -2, \ldots, \ell + k' + 1, -\left(\ell - \frac{3k'+3}{2}\right), \ell + k' + 3,$$
$$-\left(\ell - \frac{3k'+1}{2}\right), \ell + k' + 4, \ldots, 3k' + 3, -\left(\frac{k'-1}{2}\right).$$

Note that $P^{**}_{1,0}$ skips vertex $\ell + k' + 2$ in order to avoid containing an edge of length $2\ell - \frac{k'-1}{2}$.

The path $P^{**}_{1,0}$ contains edges of lengths $\frac{5k'+7}{2}$, $\frac{5k'+9}{2}$, $\ldots$, $2\ell - \frac{k'+1}{2}$, $2\ell - \frac{k'-3}{2}$, $2\ell - \frac{k'-5}{2}$, $\ldots$, $\frac{7k'+5}{2}$. We will now show that the longest length in $P^{**}_{1,0}$, $\frac{7k'+5}{2}$, is less than 2ℓ.

First, note that if $k' = 3$, then $r = k - \frac{k'}{3} = k - 1$, so that $\frac{m-1}{2} = \frac{k+r-1}{2} = k - 1$ is relatively prime to k; hence $m' = 1$, and this case has already been dealt with. So we can assume (as k' is odd and divisible by 3) that $k' \geq 9$. Let $M = \max\{m', k'\} \geq 9$.

Note that

$$\begin{array}{rrcl} & \frac{7k'+5}{2} & < & 2\ell \\ \iff & 7k'+5 & < & 4\ell \\ \iff & 7k'+5 & < & 4k'+4\left(\frac{r-1}{m'}\right) \\ \iff & 3k'+5 & < & 4\left(\frac{r-1}{m'}\right) \\ \iff & 3k+5m' & < & 4(r-1) \\ \iff & 3k+5m' & < & 4\left(k-\frac{k'}{3}-1\right) \\ \iff & 3k+5m' & < & 4k-\frac{4k'}{3}-4 \\ \iff & \frac{4k'}{3}+5m'+4 & < & k \end{array}$$

Now, $\frac{4k'}{3}+5m'+4 \leq \frac{4M}{3}+5M+\frac{4M}{9} = \frac{61M}{9}$. If $M = m'$, then $\frac{k}{M} = k' \geq 9$, so that $\frac{61M}{9} < 9M \leq k$, so the inequality is true. Otherwise, $M = k'$. If $m' \geq 7 > \frac{61}{9}$, then $\frac{61M}{9} < k$, and the inequality is again true. So the only potential problem may occur if $m' \in \{3,5\}$. First, if $m' = 5$, then $\frac{4k'}{3}+5m'+4 = \frac{4k'}{3}+24 < 5k' = k$, since $k' \geq 9$. If $m' = 3$ and $k' \geq 15$, then $\frac{4k'}{3}+5m'+4 = \frac{4k'}{3}+19 \leq \frac{4k'}{3}+\frac{19k'}{15} = \frac{39k'}{15} < 3k' = k$, so the inequality is true. The only remaining case is if $m' = 3$ and $k' = 9$; however, this is impossible, as $\gcd(m', \frac{k'}{3}) = 1$.

So we have that the longest length in $P^{**}_{1,0}$, $\frac{7k'+5}{2}$, is less than 2ℓ. Hence $P^{**}_{1,0}$ is vertex-disjoint with $\rho^{j\ell}(P^{**}_{1,0})$ whenever $j \geq 2$. We need to check that $P^{**}_{1,0}$ and $\rho^{\ell}(P^{**}_{1,0})$ are vertex-disjoint. The vertices which appear in $P^{**}_{1,0}$ are:

- $-\left(\frac{k'-1}{2}\right), \ldots, 0$
- $\frac{5k'+7}{2}, \ldots, \ell+k'+1$
- $\ell+k'+3, \ldots, 3k'+3$,

while those in $\rho^{\ell}(P^{**}_{1,0})$ are:

- $\ell-\left(\frac{k'-1}{2}\right), \ldots, \ell$
- $\ell+\frac{5k'+7}{2}, \ldots, 2\ell+k'+1$

- $2\ell + k' + 3, \ldots, \ell + 3k' + 3$

We have that $\ell < \frac{5k'+7}{2}$ since $\ell < 2k' < \frac{5k'+7}{2}$, and also $3k' + 3 < \ell + \frac{5k'+7}{2} = \frac{7k'+7}{2} + \frac{r-1}{m'} = (3k'+3) + \frac{k'+1}{2} + \frac{r-1}{m'}$. It easily follows that $P^{**}_{1,0}$ and $\rho^{\ell}(P^{**}_{1,0})$ are vertex-disjoint.

The remaining paths used to form peripheral cycles are the paths $P_{2,0}, \ldots, P_{c-2,0}$ defined in Subcase 1.2. If c is even, then we use linking lengths $\ell \pm \frac{k'-1}{2}$, $2\ell \pm \frac{k'-1}{2}$, ..., $\frac{c}{2}\ell \pm \frac{k'-1}{2}$, derived from circulants $X(m'; \{1,2\})$, $X(m'; \{3,4\})$, ..., $X(m'; \{\frac{c}{2} - 1, \frac{c}{2}\})$ if $c \equiv 0 \pmod 4$, and from circulants $X(m'; \{1\})$, $X(m'; \{2,3\})$, $X(m'; \{4,5\})$, ..., $X(m'; \{\frac{c}{2} - 1, \frac{c}{2}\})$ if $c \equiv 2 \pmod 4$. If c is odd, then we use linking lengths $\ell - \frac{k'-1}{2}$, $2\ell \pm \frac{k'-1}{2}$, $3\ell \pm \frac{k'-1}{2}$, ..., $\frac{c+1}{2}\ell \pm \frac{k'-1}{2}$, derived from circulants $X(m'; \{1\})$, $X(m'; \{2,3\})$, $X(m'; \{4,5\})$, ..., $X(m'; \{\frac{c-1}{2}, \frac{c+1}{2}\})$ if $c \equiv 1 \pmod 4$, and from circulants $X(m'; \{1\})$, $X(m'; \{2\})$, $X(m'; \{3,4\})$, $X(m'; \{5,6\})$, ..., $X(m'; \{\frac{c-1}{2}, \frac{c+1}{2}\})$ if $c \equiv 3 \pmod 4$. That the largest linking length is less than $\frac{m-1}{2}$ follows as in Subcase 1.2.

Case 2. $k' \equiv 1 \pmod 4$.

The breakdown into subcases is similar to when $k' \equiv 3 \pmod 4$.

Subcase 2.1. $m' = 1$.

If $m' = 1$, then $k = k'$, and so $k | 3m$. Since $k \nmid m$, we may write $k = 3d$, where $d | m$. We may assume that $d \geq 3$, since the required result is known if $k = 3$ (by Theorem 4.1). We have that $k < m < 2k$, or $3d < m < 6d$, so that $3 < \frac{m}{d} < 6$. Since m is odd, this means that $\frac{m}{d} = 5$, or $d = \frac{m}{5}$. Hence $k = \frac{3m}{5}$, i.e. $m = \frac{5k}{3}$. So $r = m - k = \frac{5k}{3} - k = \frac{2k}{3} = k - \frac{k}{3} = k - \frac{k'}{3}$.

Note that $\frac{r}{2k'} = \frac{r}{2k} < 1$, so that $c = \lceil \frac{r}{2k'} \rceil = 1$. So it suffices to find a single starter peripheral k-cycle in $3K_m$; the remaining lengths can be fit into central cycles by Lemma 4.21.

Let

$$C = 0, 1, -1, 2, -2, \ldots, \frac{k-1}{4}, -\frac{k-1}{4}, -\frac{3(k-1)}{4}, -\frac{k+3}{4},$$

$$-\frac{3k-7}{4}, \dots, -\frac{k+1}{2}, -\frac{k-1}{2}, 0.$$

Then C is a k-cycle in $3K_m$ which has two edges of each length in $\{1, 2, \dots, \frac{k-3}{2}\}$, and three edges of length $\frac{k-1}{2}$. Noting that $\frac{k-1}{2} < \frac{m-1}{2}$, we may take C as our starter peripheral cycle.

Subcase 2.2. $\ell > \frac{3k'+1}{2}$, $m' \geq 3$ and $r \neq k - \frac{k'}{3}$.

Let

$$P_{0,0} = 0, 2, -1, 3, -2, \dots, \frac{k'-1}{4}, -\frac{k'-5}{4}, \frac{k'+7}{4}, -\frac{k'-1}{4}, \frac{k'+11}{4},$$
$$-\frac{k'+3}{4}, \dots, \frac{k'+3}{2}, -\frac{k'-1}{2}.$$

Then $P_{0,0}$ is a path of length $k'-1$ containing edges of lengths 2, 3, ..., $\frac{k'-3}{2}$, $\frac{k'+1}{2}$, $\frac{k'+3}{2}$, ..., $k'+1$ with initial vertex 0 and terminal vertex $-\frac{k'-1}{2}$. The condition $\ell > \frac{3k'+1}{2}$ implies that $\ell - \frac{k'-1}{2} > k'+1$, so that neither length $\frac{k'-1}{2}$ nor $\ell - \frac{k'-1}{2}$ occurs in $P_{0,0}$. Note that $\ell - \frac{k'-1}{2} > k'+1 \geq \frac{k'+5}{2}$, so that $P_{0,0}$ and $\rho^{\ell}(P_{0,0})$ are vertex-disjoint.

For $i = 1, 2, \dots, c-1$, let $P_{i,0}$ be obtained from $P_{0,0}$ by adding $i\ell$ to the even vertices of $P_{0,0}$, so that

$$P_{i,0} = 0, 2+i\ell, -1, 3+i\ell, -2, \dots, \frac{k'-1}{4} + j\ell, -\frac{k'-5}{4}, \frac{k'+7}{4} + j\ell,$$
$$-\frac{k'-1}{4}, \frac{k'+11}{4} + j\ell, -\frac{k'+3}{4}, \dots, \frac{k'+3}{2} + j\ell, -(\frac{k'-1}{2}).$$

The longest length in $P_{i,0}$ is hence $k'+1+i\ell$, and then longest length in any of these paths is

$$\begin{aligned}
k'+1+(c-1)\ell &= \ell - \frac{r-1}{m'} + c\ell - \ell + 1 \\
&= c\ell - \frac{r-1}{m'} + 1 \\
&< c\ell,
\end{aligned}$$

since $\ell = k' + \frac{r-1}{m'} > k'+1$.

We recall that since $r \neq k - \frac{k'}{3}$, we have that $c\ell < \frac{m-1}{2}$, so that the longest length in any of the paths $P_{i,0}$ is less than $\frac{m-1}{2}$. As in previous cases, it may be checked that for each $0 \leq i \leq c-1$, the paths in the family $\mathcal{P}_i = \{P_{i,j} : 0 \leq j \leq m'-1\}$ (where $P_{i,j} = \rho^{j\ell}(P_{i,0})$) are pairwise vertex-disjoint.

To link the families $\mathcal{P}_i$ of paths into cycles, the linking edges will be found via auxiliary circulants of order m'. We need to verify that there are enough linking lengths congruent to $\pm\frac{k'-1}{2} \pmod{\ell}$ to link c families of paths. Note that $\frac{k'-1}{2} \equiv -\frac{k'-1}{2} \pmod{\ell}$ is impossible, since this condition implies $k'-1 \equiv 0 \pmod{\ell}$, but $2 \leq k'-1 < \ell$. The verification that we can find enough linking lengths is as in Subcase 1.2.

Subcase 2.3. $k'+1 < \ell \leq \frac{3k'+1}{2}$, $m' \geq 3$ and $r \neq k - \frac{k'}{3}$.

Let

$$P_{0,0} = 0, 1, -1, 2, -2, \ldots, \frac{k'-1}{4}, -\left(\frac{k'-1}{4}\right), \frac{k'+7}{4}, -\left(\frac{k'+3}{4}\right), \frac{k'+11}{4},$$
$$-\left(\frac{k'+7}{4}\right), \ldots, -\left(\frac{\ell}{2} - \left(\frac{k'+7}{4}\right)\right), \frac{\ell}{2} - \left(\frac{k'-1}{4}\right), -\left(\frac{\ell}{2} - \frac{k'-1}{4}\right),$$
$$\frac{\ell}{2} - \left(\frac{k'-5}{4}\right), -\left(\frac{\ell}{2} - \left(\frac{k'-5}{4}\right)\right), \ldots, \frac{k'+1}{2}, -\left(\frac{k'+1}{2}\right).$$

Here, $P_{0,0}$ is a path of length $k'-1$, with edge lengths between 1 and $k'+1$. The path skips vertices $\frac{k'+3}{4}$ and $-(\frac{\ell}{2} - (\frac{k'+3}{4}))$ so that $P_{0,0}$ contains no edge of length $\frac{k'+1}{2}$ or $\ell - \frac{k'+1}{2}$. Note that the condition $\ell \leq \frac{3k'+1}{2}$ means that $\frac{\ell}{2} - \frac{k'-1}{4} \leq \frac{3k'+1}{4} - \frac{k'-1}{4} = \frac{k'+1}{2}$, so that this second skip is possible.

We remark that $\ell - \frac{k'+1}{2} \geq k'+3 - \frac{k'+1}{2} = \frac{k'+5}{2} > \frac{k'+1}{2}$, so $P_{0,0}$ and $\rho^{\ell}(P_{0,0})$ are pairwise vertex disjoint.

Form paths $P_{i,0}$, $0 \leq i \leq c-1$ as before. The longest length in any of these paths is $k'+1+(c-1)\ell = \ell - \frac{r-1}{m'} + 1 + c\ell - \ell = c\ell - \frac{r-1}{m'} + 1 \leq c\ell$, which is less than $\frac{m-1}{2}$ since $r \neq k - \frac{k'}{3}$.

The verification that there are enough linking lengths is as in Subcase 1.3.

Subcase 2.4. $\ell = k' + 1$, $m' \geq 3$ and $r \neq k - \frac{k'}{3}$.

Exactly as Case 1.4.

Subcase 2.5. $r = k - \frac{k'}{3}$ and $m' \geq 3$.

Let paths $P_{i,0}$ be defined as in Subcase 2.2, noting that here since $\frac{k'+3}{3m'} \leq \frac{k'+3}{9} \leq \frac{2k'}{9}$, we have that

$$\ell = k' + \frac{r-1}{m'} = k' + (k' - \frac{k'}{3m'} - \frac{1}{m'}) = 2k' - \frac{k'+3}{3m'} \geq \frac{16k'}{9} > \frac{3k'+1}{2}. \qquad (4.3.2)$$

Namely, if $k' > 9$, then $m' \geq 3$ implies that $\frac{k'+3}{m'} \leq \frac{k'+3}{3} < \frac{k'-1}{2}$, so that $2k' - \frac{k'+3}{m'} > \frac{3k'+1}{2}$. Otherwise, $k' = 9$ and so $3 \nmid m'$, so that $\frac{k'+3}{m'} \leq \frac{k'+3}{5} < \frac{k'-1}{2}$, and again $2k' - \frac{k'+3}{m'} > \frac{3k'+1}{2}$. It is possible here that $c = \lceil \frac{r}{2k'} \rceil = \frac{m'+1}{2}$, which means that we may not be able to form path $P_{c-1,0}$ as its longest length exceeds $\frac{m-1}{2}$. As in Case 1.5, we replace paths $P_{0,0}$ and $P_{1,0}$ by three paths $P^*_{0,0}, P^*_{1,0}, P^{**}_{1,0}$. These paths, together with paths $P_{2,0}, \ldots, P_{c-2,0}$, will be used to form peripheral cycles in the usual way. We must ensure that these three paths each have initial vertex 0 and terminal vertex $-(\frac{k'-1}{2})$; that each of these paths P is pairwise vertex disjoint with $\rho^{j\ell}(P)$; that none of the three paths uses an edge of any linking length; and that the edge lengths in these three paths fall between 1 and 2ℓ.

Let $P^*_{0,0} = 0, 1, -1, 2, -2, \ldots, \frac{k'-1}{2}, -(\frac{k'-1}{2})$. The largest length in $P^*_{0,0}$ is $k' - 1$, which is even. Then $P^*_{0,0}$ is a zig-zag path containing edges of lengths $1, 2, \ldots, k' - 1$. Note that since $\ell > \frac{3k'+1}{2} > \frac{3(k'-1)}{2}$ by inequality 4.3.2, $P_{0,0}$ contains no edge of length $\ell - \frac{k'-1}{2}$.

In forming $P^*_{1,0}$, we will need to skip the linking lengths $\ell \pm \frac{k'-1}{2}$ and also vertices $\ell - \frac{k'-1}{2}, \ldots, \ell$ (to avoid any overlap between $P^*_{1,0}$ and $\rho^{\ell}(P^*_{1,0})$). Length $\ell - \frac{k'-1}{2}$ is even; to skip a positive vertex to avoid this length, we begin $P^*_{1,0}$ with an edge of even length. Let

$$P^*_{1,0} = 0, k' + 1, -1, k' + 2, -2, \ldots, \frac{\ell}{2} + \frac{k'-1}{4}, -\left(\frac{\ell}{2} - \frac{3k'+1}{4}\right), \frac{\ell}{2} + \frac{k'+7}{4},$$

$$-\left(\frac{\ell}{2}-\frac{3k'-3}{4}\right), \frac{\ell}{2}+\frac{k'+11}{4}, \ldots, -\left(\ell-\frac{3k'+5}{2}\right), \ell-\frac{k'+1}{2}, -\left(\ell-\frac{3k'+3}{2}\right),$$
$$\ell+1, -\left(\ell-\frac{3k'+1}{2}\right), \ell+2, \ldots, \frac{\ell}{2}+k', -\left(\frac{\ell}{2}-\frac{k'+3}{2}\right),$$
$$\frac{\ell}{2}+k'+2, -\left(\frac{\ell}{2}-\frac{k'+1}{2}\right), \frac{\ell}{2}+k'+3, \ldots, 2k'+2, -\left(\frac{k'-1}{2}\right).$$

Let us note the key properties of $P^*_{1,0}$. This path skips vertex $\frac{\ell}{2}+\frac{k'+3}{4}$ in order that it does not contain an edge of length $\ell-\frac{k'-1}{2}$. That vertices $\ell-\frac{k'-1}{2}, \ell-\frac{k'-3}{2}, \ldots, \ell$ do not occur in $P^*_{1,0}$ will help to ensure that the paths $P^*_{1,0}$ and $\rho^\ell(P^*_{1,0})$ are vertex-disjoint. Finally, vertex $\ell+(k'-\frac{\ell}{2})+1 = \frac{\ell}{2}+k'+1$ is skipped so that the path has no edge of length $\ell+\frac{k'-1}{2}$.

So $P^*_{1,0}$ is a zig-zag path with edges of lengths $k'+1$, $k'+2$, …, $\ell-\frac{k'+1}{2}$, $\ell-\frac{k'-3}{2}$, $\ell-\frac{k'-5}{2}$, …, $2\ell-2k'-2$, $2\ell-\frac{3k'+1}{2}$, $2\ell-\frac{3k'-1}{2}$, …, $\ell+\frac{k'-3}{2}$, $\ell+\frac{k'+1}{2}$, $\ell+\frac{k'+3}{2}$, …, $\frac{5k'+3}{2}$. (Recall that $\ell = k'+\frac{r-1}{m'} \le 2k'-2$. This implies that $2\ell-\frac{3k'+1}{2} < \ell+\frac{k'-3}{2}$ and that $\ell+\frac{k'+3}{2} < \frac{5k'+3}{2}$.) The longest length in $P^*_{1,0}$ is $2k'+2+\frac{k'-1}{2} = \frac{5k'+3}{2}$, which is less than $2\ell-\frac{k'-1}{2}$ (see Subcase 1.5).

We need to verify that $P^*_{1,0}$ and $\rho^\ell(P^*_{1,0})$ are vertex-disjoint. The path $P^*_{1,0}$ contains vertices

- $-(\frac{k'-1}{2}), \ldots, 0$
- $k'+1, \ldots, \frac{\ell}{2}+\frac{k'-1}{4}$
- $\frac{\ell}{2}+\frac{k'+7}{4}, \ldots, \ell-\frac{k'+1}{2}$
- $\ell+1, \ldots, \frac{\ell}{2}+k'$
- $\frac{\ell}{2}+k'+2, \ldots, 2k'+2,$

while the vertices of $\rho^\ell(P^*_{1,0})$ are

- $\ell-\frac{k'-1}{2}, \ldots, \ell$

- $\ell + k' + 1, \ldots, \frac{3\ell}{2} + \frac{k'-1}{4}$
- $\frac{3\ell}{2} + \frac{k'+7}{4}, \ldots, 2\ell - \frac{k'+1}{2}$
- $2\ell + 1, \ldots, \frac{3\ell}{2} + k'$
- $\frac{3\ell}{2} + k' + 2, \ldots, \ell + 2k' + 2.$

In order that $P^*_{1,0}$ and $\rho^\ell(P^*_{1,0})$ are vertex-disjoint, we need that $\ell - \frac{k'-1}{2} > 0$, which is true since $\ell \geq k' > \frac{k'-1}{2}$, and $\ell + k' + 1 > 2k' + 2$, which is true since $\ell > \frac{3k'+1}{2} > k' + 1$ by inequality 4.3.2.

We now need to form the path $P^{**}_{1,0}$. The longest length in $P^*_{1,0}$ was $\frac{5k'+3}{2}$, which is less than $2\ell - \frac{k'-1}{2}$. Note that $2\ell - \frac{k'-1}{2}$ and $\frac{5k'+3}{2}$ are both even. We let

$$P^{**}_{1,0} = 0, \frac{5k'+7}{2}, -1, \frac{5k'+9}{2}, -2, \ldots, \ell + k' + 1, -\left(\ell - \frac{3k'+3}{2}\right), \ell + k' + 3,$$
$$-\left(\ell - \frac{3k'+1}{2}\right), \ell + k' + 4, \ldots, 3k' + 3, -\left(\frac{k'-1}{2}\right).$$

as in Subcase 1.5. As before, $P^{**}_{1,0}$ contains edges of lengths $\frac{5k'+7}{2}$, $\frac{5k'+9}{2}$, …, $2\ell - \frac{k'+1}{2}$, $2\ell - \frac{k'-3}{2}$, $2\ell - \frac{k'-5}{2}$, …, $\frac{7k'+5}{2}$. The verification that the largest length in $P^{**}_{1,0}$ is less than 2ℓ and that $P^{**}_{1,0}$ and $\rho^\ell(P^{**}_{1,0})$ are vertex-disjoint is exactly as in Subcase 1.5.

We complete the decomposition as in Subcase 1.5.

Hence we can form starter peripheral cycles in any case. Recall that for each starter peripheral cycle C, the decomposition contains cycles C, $\rho(C)$, …, $\rho^{\ell-2}(C)$; this ensures that for the multiset B of $3r/2$ edge lengths, consisiting of the k' edge lengths used to form each of the $\frac{3r}{2k'}$ starter peripheral cycles, all edges of $3K_m$ arising from B occur in the decomposition. That central cycles can be formed from the remaining edges follows directly from Lemma 4.21.

4.3.2 The case $2k < m < 3k$

Write $m = 2k + r$, where $0 < r < k$. Note that since m is odd, we have that r is odd in this case. Let $d = \gcd(m, k)$, $\tilde{k} = k/d$ (so that $\tilde{k}|3(\frac{m-1}{2})$), $\tilde{m} = m/d$, $\tilde{r} = r/d$ (noting that $d|r$). Notice that $\tilde{k} \geq 3$, since k is odd and $k \neq m$; this fact also implies that $\tilde{m} \geq 7$, since $\tilde{m} = 2\tilde{k} + \tilde{r}$. Also, if $\tilde{k}|\frac{m-1}{2}$, then $k|\binom{m}{2}$, so that $C_k|K_m$ (by Theorem 4.4), and it is trivial that $C_k|3K_m$. Hence, we may assume that $\tilde{k} \nmid \frac{m-1}{2}$; in particular, this means that $3|\tilde{k}$. In this case, we let the vertex set of K_m be $\mathbb{Z}_m$. That is, the construction in this case uses no central vertex. Throughout, we let ρ denote the cyclic permutation $\rho = (0, 1, \ldots, m-1)$.

Note that $3(\frac{m-1}{2}) = 3(\frac{2k+r-1}{2}) = 3k + 3(\frac{r-1}{2})$. In each case, we will form peripheral cycles which use between them $3(\frac{r-1}{2})$ edge lengths from the multiset $\langle 1, 1, 1, 2, 2, 2, \ldots, \frac{m-1}{2}, \frac{m-1}{2}, \frac{m-1}{2} \rangle$ of edge lengths in $3K_m$. In all cases except $d = 1$, there will be $3(\frac{r-1}{2\tilde{k}})$ starter peripheral cycles, which contain $\tilde{k}$ edge lengths per peripheral cycle. In each case, three starter central cycles will be formed, each of which uses k distinct edge lengths. The following lemma, which is inspired by the technique for constructing central cycles in [1], will be integral to our constructions of central cycles. We note, however, that no analogue of Part 3 of the lemma was required in [1].

Lemma 4.22. *Suppose $2k < m < 3k$. Let $L = \{\ell_1, \ell_2, \ldots, \ell_k\}$ be a set of k distinct edge lengths in K_m with $\ell_1 < \ell_2 < \cdots < \ell_k$. Suppose there exists an even integer q such that $\ell_q, \ell_{q+1}, \ldots, \ell_k$ are consecutive integers, and $\ell_k = \frac{m-1}{2}$. Let w be defined by*

$$-w = \ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{q-1} - \ell_{q+1} + \ell_{q+2} - \cdots - \ell_k = \sum_{i=1}^{q-1}(-1)^{i-1}\ell_i + \sum_{i=q+1}^{k}(-1)^i\ell_i.$$

Then:

1. *If $\ell_q \leq w + 1$, then $C_k|X(m; L)$.*

2. *If $w + 1 < \ell_q \leq w + k - q$, then $C_k|X(m; L)$.*

3. If, for some positive j, $\ell_{q-2j}, \ell_{q-2j+1}, \ldots, \ell_{q-1}$ are consecutive and $w+1 < w+k-q < \ell_q \leq w+k-q+2j$, then $C_k | X(m; L)$.

Proof. Throughout the proof, we will use the following notation: for integers i and i' such that $i \leq i'$, let $\sigma_i^{i'}$ denote the alternating sum $\ell_i - \ell_{i+1} + \ell_{i+2} - \ell_{i+3} + \cdots + (-1)^{i'-i}\ell_{i'}$.

1. Note that, since $\ell_q, \ell_{q+1}, \ldots, \ell_k$ are consecutive,

$$\begin{aligned}
& \ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{q-1} - \ell_{q+1} + \ell_{q+2} - \cdots - \ell_k \\
= \; & \ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{q-1} - \ell_q + \ell_{q+1} - \ell_{q+3} + \ell_{q+4} - \cdots - \ell_k \\
= \; & \ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{q-1} - \ell_q + \ell_{q+1} - \ell_{q+2} + \ell_{q+3} - \ell_{q+5} + \ell_{q+6} - \cdots - \ell_k \\
\vdots \; & \\
= \; & \ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{k-2} - \ell_k
\end{aligned}$$

Consider the path

$$0, \sigma_1^1, \sigma_1^2, \ldots, \sigma_1^{q-1}, \sigma_1^{q-1} - \sigma_{q+1}^{q+1}, \sigma_1^{q-1} - \sigma_{q+1}^{q+2}, \ldots, \sigma_1^{q-1} - \sigma_{q+1}^{k}.$$

The vertices of this path occur in the interval $[-w, -w+\ell_k]$, which has length $\ell_k = \frac{m-1}{2}$; since $k \geq 3$, the path contains positive vertices, and thus it follows that $-w + \ell_k = -w + \frac{m-1}{2} \geq 1$, so that $-w \geq -\frac{m-3}{2}$, or $w \leq \frac{m-3}{2} = \ell_{k-1}$. Since also $\ell_q - 1 \leq w$ by assumption, we have that either $w = \ell_q - 1$, or else there exists $j \geq 0$ such that $w = \ell_{q+j}$. In this case, choose $J \in \{j, j+1\}$ such that $q+J$ is even. Note that $q + J \leq k-1$. If $w = \ell_q - 1$, then we set $J = 0$.

Before we proceed, we show that $q + J \leq k - 3$. Suppose to the contrary that $q + J > k-3$, so that $q + J = k-1$. Note that $\ell_{q+J} \in \{w, w+1\}$. If $\ell_{k-1} = \ell_{q+J} = w$, then $\ell_{k-1} = -\ell_1 + \ell_2 - \ell_3 + \cdots - \ell_{k-2} + \ell_k$, so that $\ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{k-2} = \ell_k - \ell_{k-1} = 1$. Since the sequence $\ell_1, \ell_2, \ldots, \ell_k$ of edge lengths is strictly increasing, this can only happen if $k = 3$ and $\ell_1 = 1$. But this is impossible as by assumption $k \neq 3$. Otherwise, $\ell_{k-1} = \ell_{q+J} = w + 1 = -\ell_1 + \ell_2 - \ell_3 + \cdots - \ell_{k-2} + \ell_k + 1$, so that

$\ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{k-2} = \ell_k - \ell_{k-1} + 1 = 2$. Since the sequence $\ell_1, \ell_2, \ldots, \ell_k$ of edge lengths is strictly increasing, this can happen only if either $k = 3$ and $\ell_1 = 2$, or else $k = 5$, $\ell_1 = 1$ and $\ell_3 = \ell_2 + 1$. Again, $k \neq 3$ by assumption. Also, $k \neq 5$ as $3|k$. Hence we have that $q + J \leq k - 3$.

Form a path P as follows:

$$P = 0, \sigma_1^1, \sigma_1^2, \ldots, \sigma_1^{q+J-1}, \sigma_1^{q+J-1} - \sigma_{q+J+1}^{q+J+1}, \sigma_1^{q+J-1} - \sigma_{q+J+1}^{q+J+2}, \ldots, \sigma_1^{q+J-1} - \sigma_{q+J+1}^{k-1}.$$

Note that the even-positioned vertices of P are strictly increasing, while the odd-positioned vertices of P are strictly decreasing. Note that, since k is odd, P ends on positive vertex $\sigma_1^{q+J-1} - \sigma_{q+J+1}^{k-1}$, while the smallest negative vertex of P is $\sigma_1^{q+J-1} - \sigma_{q+J+1}^{k-2}$. Since $\sigma_1^{q+J-1} - \sigma_{q+J+1}^{k-1} - (\sigma_1^{q+J-1} - \sigma_{q+J+1}^{k-2}) = \ell_{k-1} = \frac{m-3}{2} < \frac{m-1}{2}$, there can be no overlap between the positive and negative vertices of P, and so P is indeed a path.

The path P uses edges of each length in L except ℓ_{q+J} and ℓ_k. Adding an edge of length $\frac{m-1}{2} = \ell_k$ gives a path terminating either on vertex

$$\begin{aligned} & \sigma_1^{q+J-1} - \sigma_{q+J+1}^{k} \\ = \; & \ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{q+J-1} - \ell_{q+J+1} + \ell_{q+J+2} - \cdots + \ell_{k-1} - \ell_k \\ = \; & -w, \end{aligned}$$

or on vertex

$$\begin{aligned} & \sigma_1^{q+J-1} - \sigma_{q+J+1}^{k-1} + \frac{m-1}{2} \\ = \; & \ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{q+J-1} - \ell_{q+J+1} + \ell_{q+J+2} - \cdots + \ell_{k-1} - \frac{m+1}{2} \\ = \; & -(w+1). \end{aligned}$$

Thus, we have found paths with initial and terminal vertices 0 and either $-w$ or $-(w+1)$. Choosing whichever path has terminal vertex $-\ell_{q+J}$ and adding the edge $\{-\ell_{q+J}, 0\}$ gives a starter cycle C of length k with edges of lengths $\ell_1, \ell_2, \ldots, \ell_k$. An illustration of central cycles formed in this manner may be found in Figure 5.3.

2. Note that the hypothesis in this case necessitates $w+1 < w+k-q$; hence $q < k-1$. For each $i = 1, 2, \ldots, \frac{1}{2}(k-q-1)$, we will form paths P_i and P_i' with inital vertex 0 and containing an edge of each length in L except ℓ_q. We will show that under the given conditions, one of these paths will have terminal vertex $-\ell_q$, so that a starter cycle can be formed by adding an edge of length ℓ_q. These paths are begun as the path P in Part 1; however, after a certain point, the lengths encountered in the path will be taken in reverse order than those in P. In particular, P_i and P_i' begin with edges in order of lengths $1, 2, \ldots, q-1, q+1, q+2, \ldots, k-1-2i$, and continue with edges (in order) of lengths $\frac{m-1}{2}, \frac{m-3}{2}, \ldots, k-2i$. This alteration will ensure that the terminal vertices of P_i and P_i' are $-(w+2i)$ and $-(w+2i+1)$, respectively (to be shown below). Note that if $k-1-2i = q$, which occurs when $i = \frac{1}{2}(k-q-1)$, then the set $\{q+1, \ldots, k-1-2i\}$ is empty, so that the edge lengths in the paths are $1, 2, \ldots, q-1, \frac{m-1}{2}, \frac{m-3}{2}, \ldots, q+1$.

Form the path P_i in the following manner:

$$\begin{aligned}
P_i \;=\; & 0, \sigma_1^1, \sigma_1^2, \sigma_1^3, \ldots, \sigma_1^{q-1}, \sigma_1^{q-1} - \sigma_{q+1}^{q+1}, \sigma_1^{q-1} - \sigma_{q+1}^{q+2}, \ldots, \sigma_1^{q-1} - \sigma_{q+1}^{k-1-2i}, \\
& \sigma_1^{q-1} - \sigma_{q+1}^{k-1-2i} - \frac{m-1}{2}, \sigma_1^{q-1} - \sigma_{q+1}^{k-1-2i} - \frac{m-1}{2} + \frac{m+3}{2}, \\
& \sigma_1^{q-1} - \sigma_{q+1}^{k-1-2i} - \frac{m-1}{2} + \frac{m+3}{2} - \frac{m+5}{2}, \\
& \sigma_1^{q-1} - \sigma_{q+1}^{k-1-2i} - \frac{m-1}{2} + \frac{m+3}{2} - \frac{m+5}{2} + \frac{m+7}{2}, \\
& \sigma_1^{q-1} - \sigma_{q+1}^{k-1-2i} - \frac{m-1}{2} + \frac{m+3}{2} - \frac{m+5}{2} + \cdots + (m - \ell_{k-2i+1}) - (m - \ell_{k-2i}).
\end{aligned}$$

Let $-w_i$ denote the terminal vertex of P_i.

The path P_i' is formed as P_i, except that $\frac{m-1}{2}$ is replaced by $\frac{m+1}{2}$; this means that the terminal vertex of P_i' is $-w_i - 1 = -(w_i + 1)$. It is easy to see that P_i and P_i' are paths.

Note that

$$\frac{m+3}{2} - \frac{m+5}{2} + \frac{m+7}{2} - \cdots + (m - \ell_{k-2i+1}) - (m - \ell_{k-2i})$$

$$\begin{aligned}
&= \ell_{k-2i} - \ell_{k-2i+1} + \cdots - \frac{m-7}{2} + \frac{m-5}{2} - \frac{m-3}{2} \\
&= \ell_{k-2i} - \ell_{k-2i+1} + \cdots - \ell_{k-3} + \ell_{k-2} - \ell_{k-1} \\
&= -\frac{1}{2}(k-1-(k-2i-1)) \\
&= -i
\end{aligned}$$

Hence

$$\begin{aligned}
-w_i &= \sigma_1^{q-1} - \sigma_{q+1}^{k-1-2i} - \frac{m-1}{2} - i \\
&= \sigma_1^{q-1} - (\sigma_{q+1}^{k-1} + i) - \frac{m-1}{2} - i \\
&= \sigma_1^{q-1} - \sigma_{q+1}^{k-1} - 2i - \frac{m-1}{2} \\
&= -w - 2i
\end{aligned}$$

So from this process, we can obtain paths of length $k-1$, using one edge of each length in $L - \{\ell_q\}$, with terminal vertices $-w-2, -w-3, -w-4, \ldots, -w-(k-q-1)$, $-w-(k-q)$. Since $w+2 \leq \ell_q \leq w+k-q$, we can complete one of these paths to a starter cycle containing exactly one edge of each length in L by adding the edge $\{-\ell_q, 0\}$. An illustration of this type of central cycle may be found in Figure 5.4.

3. Note that again $w+1 < w+k-q$ implies that $q < k-1$. Let j be as defined in the statement of Part 3 of the lemma. For each $i = 1, 2, \ldots, j$, form paths $\tilde{P}_i$ and $\tilde{P}_i'$ as follows:

$$\begin{aligned}
\tilde{P}_i \;=\; & 0, \sigma_1^1, \sigma_1^2, \ldots, \sigma_1^{q-1-2i}, \sigma_1^{q-1-2i} - \frac{m-1}{2}, \sigma_1^{q-1-2i} - \frac{m-1}{2} + \frac{m+3}{2}, \\
& \sigma_1^{q-1-2i} - \frac{m-1}{2} + \frac{m+3}{2} - \frac{m+5}{2}, \ldots, \\
& \sigma_1^{q-1-2i} - \frac{m-1}{2} + \frac{m+3}{2} - \frac{m+5}{2} + \cdots + (m - \ell_{q+2}), \\
& \sigma_1^{q-1-2i} - \frac{m-1}{2} + \frac{m+3}{2} - \frac{m+5}{2} + \cdots + (m - \ell_{q+2}) - (m - \ell_{q+1}), \\
& \sigma_1^{q-1-2i} - \frac{m-1}{2} + \frac{m+3}{2} - \frac{m+5}{2} + \cdots + (m - \ell_{q+2}) - (m - \ell_{q+1})
\end{aligned}$$

$$
\begin{aligned}
&+(m-\ell_{q-1}),\\
\sigma_1^{q-1-2i}\ &-\frac{m-1}{2}+\frac{m+3}{2}-\frac{m+5}{2}+\cdots+(m-\ell_{q+2})-(m-\ell_{q+1})\\
&+(m-\ell_{q-1})-(m-\ell_{q-2}),\ldots,\\
\sigma_1^{q-1-2i}\ &-\frac{m-1}{2}+\frac{m+3}{2}-\frac{m+5}{2}+\cdots+(m-\ell_{q+2})-(m-\ell_{q+1})\\
&+(m-\ell_{q-1})-(m-\ell_{q-2})+\ldots-(m-\ell_{q-2i}).
\end{aligned}
$$

The path $\tilde{P}_i'$ is formed as $\tilde{P}_i$, except that $\frac{m-1}{2}$ is replaced with $\frac{m+1}{2}$.

Then for each i, it is easy to see that $\tilde{P}_i$ and $\tilde{P}_i'$ is a path of length $k-1$ containing an edge of each length in $L-\{\ell_q\}$.

Note that

$$
\begin{aligned}
&\frac{m+3}{2}-\frac{m+5}{2}+\ldots+(m-\ell_{q+2})-(m-\ell_{q+1})\\
= &\ \ell_{q+1}-\ell_{q+2}+\ldots+\frac{m-5}{2}-\frac{m-3}{2}\\
= &\ -\frac{1}{2}(k-1-q)
\end{aligned}
$$

and

$$
\begin{aligned}
&(m-\ell_{q-1})-(m-\ell_{q-2})+\ldots+(m-\ell_{q-2i+1})-(m-\ell_{q-2i})\\
= &\ \ell_{q-2i}-\ell_{q-2i+1}+\ldots+\ell_{q-2}-\ell_{q-1}\\
= &\ -\frac{1}{2}(q-1-(q-2i-1))\\
= &\ -i.
\end{aligned}
$$

So the terminal vertex of $\tilde{P}_i$ is $-\tilde{w}_i=\sigma_1^{q-1-2i}-\frac{1}{2}(k-1-q)-i-\frac{m-1}{2}=\sigma_1^{q-1}+\ell_{q-2i}-\ell_{q-2i+1}+\ldots-\ell_{q-1}-\frac{1}{2}(k-q-1)-i-\frac{m-1}{2}$. Since $\ell_{q-2i},\ldots,\ell_{q-1}$ are consecutive, we have that $-\tilde{w}_i=\sigma_1^{q-1}-\frac{1}{2}(q-1-(q-2i-1))-\frac{1}{2}(k-q-1)-i-\frac{m-1}{2}=\sigma_1^{q-1}-2i-\frac{1}{2}(k-q-1)-\frac{m-1}{2}=-w-k+q+1-2i$. Hence we can reach terminal vertices $-w-k+q-1,-w-k+q-3,\ldots,-w-k+q+1-2j$ using the paths $\tilde{P}_i$ and terminal vertices $-w-k+q-2,-w-k+q-4,\ldots,-w-k+q-2j$ using the paths $\tilde{P}_i'$. Hence we may form such a path with terminal vertex $-\ell_q$ since $\ell_q\leq w+k-q+2j$;

adding the edge $\{-\ell_q, 0\}$ to this path gives a starter cycle of length k with an edge of each length in L. An illustration of a central cycle formed using Part 3 may be found in Figure 5.5. $\square$

Before we proceed, we note that since $\ell_q, \ell_{q+1}, \ldots, \ell_k$ are consecutive and $\ell_k = \frac{m-1}{2}$, we have that $k - q = \frac{m-1}{2} - \ell_q$. Hence the inequality $\ell_q \leq w + k - q$ in Part 2 of Lemma 4.22 is equivalent to $w + \frac{m-1}{2} \geq 2\ell_q$, while the inequality $\ell_q \leq w + k - q + 2j$ in Part 3 is equivalent to $w + \frac{m-1}{2} + 2j \geq 2\ell_q$. These forms of the inequalities will frequently be used in subsequent cases to show that the appropriate inequality holds.

It remains to construct the cycles in each case. We begin by forming starter cycles which use $3(\frac{r-1}{2})$ edge lengths. There remain $3k$ edge lengths from the multiset $\langle 1, 1, 1, 2, 2, 2, \ldots, \frac{m-1}{2}, \frac{m-1}{2}, \frac{m-1}{2} \rangle$, and we finish by showing that we can form three starter central cycles, each containing k of these lengths.

Case $d = 1$.

The following lemma will be used in the construction of cycles in this case.

Lemma 4.23. *Suppose $m > 2k$, $k > 3$, m and k are odd and $3|k$. Then there exists a k-cycle in K_m which contains exactly three edges of each length in the set $\{1, 2, \ldots, \frac{k}{3} - 2, \frac{1}{2}(k-3), k-3\}$.*

Proof. Let $V(K_m) = \mathbb{Z}_m$ and let

$$P = 0, 1, -1, 2, -2, \ldots, \frac{1}{6}(k-9), -\frac{1}{6}(k-9), \frac{1}{6}(k-3).$$

Then P is a zig-zag path of length $\frac{k}{3} - 2$ containing edges of lengths $1, 2, \ldots, \frac{k}{3} - 2$. Let $P' = 0, k-3, \frac{1}{2}(k-3)$ and $P'' = 0, \frac{1}{2}(k-3), -\frac{1}{2}(k-3)$. Note that each of P' and P'' contains an edge of length $\frac{1}{2}(k-3)$ and an edge of length $k-3$.

Let us note that: the initial vertex of $\rho^{\frac{1}{6}(k-3)}(P')$ and terminal vertex of P are both $\frac{1}{6}(k-3)$; the initial vertex of $\rho^{\frac{2}{3}(k-3)}(P)$ and terminal vertex of $\rho^{\frac{1}{6}(k-3)}(P')$ are both $\frac{2}{3}(k-3)$; the initial vertex of $\rho^{\frac{5}{6}(k-3)}(P'')$ and terminal vertex of $\rho^{\frac{2}{3}(k-3)}(P)$ are both $\frac{5}{6}(k-3)$; the initial vertex of $\rho^{\frac{1}{3}(k-3)}(P)$ and the terminal vertex of $\rho^{\frac{5}{6}(k-3)}(P'')$ are both $\frac{1}{3}(k-3)$, and the initial vertex of $\rho^{\frac{1}{2}(k-3)}(P'')$ and terminal vertex of $\rho^{\frac{1}{3}(k-3)}(P)$ are both $\frac{1}{2}(k-3)$. We see from these facts that the initial (resp. terminal) vertex of each path occurs as the terminal (resp. initial) vertex of a single other path, but does not occur as the initial or terminal vertex of any other path. We may concatenate the paths in the following manner to form C.

$$C = P\rho^{\frac{1}{6}(k-3)}(P')\rho^{\frac{2}{3}(k-3)}(P)\rho^{\frac{5}{6}(k-3)}(P'')\rho^{\frac{1}{3}(k-3)}(P)\rho^{\frac{1}{2}(k-3)}(P'').$$

We would like to show that C is the required k-cycle.

First, that C is a closed trail is guaranteed by the fact that the initial vertex of P and terminal vertex of $\rho^{\frac{1}{2}(k-3)}(P'')$ are each 0. To verify that C is indeed a cycle, we need only check that the paths which form C share no vertex other than the shared initial and terminal vertices previously noted. The vertices of P fall in the interval $[-\frac{1}{6}(k-9), \frac{1}{6}(k-3)]$; hence, those of $\rho^{\frac{2}{3}(k-3)}(P)$ lie in $[\frac{1}{2}(k-1), \frac{5}{6}(k-3)]$ and those of $\rho^{\frac{1}{3}(k-3)}(P)$ lie in $[\frac{1}{6}(k+3), \frac{1}{2}(k-3)]$. Since $-\frac{m-1}{2} \leq -k < -\frac{1}{6}(k-9) < \frac{1}{6}(k-3) < \frac{1}{6}(k+3) < \frac{1}{2}(k-3) < \frac{1}{2}(k-1) < \frac{5}{6}(k-3) < k \leq \frac{m-1}{2}$, it is easy to see that the paths P, $\rho^{\frac{1}{3}(k-3)}(P)$ and $\rho^{\frac{2}{3}(k-3)}(P)$ are pairwise vertex-disjoint. The internal vertex of $\rho^{\frac{1}{6}(k-3)}(P')$ is $\frac{7}{6}(k-3)$, while that of $\rho^{\frac{5}{6}(k-3)}(P'')$ is $\frac{4}{3}(k-3)$ and that of $\rho^{\frac{1}{2}(k-3)}(P'')$ is $k-3$. Since $\frac{5}{6}(k-3) < k-3 < \frac{7}{6}(k-3) < \frac{4}{3}(k-3) < m - \frac{1}{6}(k-9)$ (as $m > 2k$), it is easy to see that these vertices are distinct from each other and from the vertices of P, $\rho^{\frac{1}{3}(k-3)}(P)$ and $\rho^{\frac{2}{3}(k-3)}(P)$. It is now easy to see that C is a cycle.

Finally, it is clear from the construction of C that C contains three edges of each length in $\{1, 2, \ldots, \frac{k}{3} - 2, \frac{1}{2}(k-3), k-3\}$. Hence C is the desired cycle. $\square$

In the case $d = 1$, we have that $k | 3(\frac{m-1}{2})$. Since $m = 2k + r$, we also have $3(\frac{m-1}{2}) = 3k + 3(\frac{r-1}{2})$, and so $k | 3(\frac{r-1}{2})$. Since $0 < r < k$, we have that $0 \leq 3(\frac{r-1}{2}) < 3(\frac{k-1}{2}) < 2k$. So $3(\frac{r-1}{2k}) = 0$ or 1. If $3(\frac{r-1}{2k}) = 0$, then $r = 1$, and so $m = 2k+1$; it is known that $C_k | K_{2k+1}$ (by Theorem 4.4), so we need not consider this case. Hence we may assume $3(\frac{r-1}{2k}) = 1$, so that $r = \frac{2}{3}k + 1$. We also assume $k > 3$ since necessary and sufficient conditions for the existence of a C_3-decomposition of $3K_m$ are known (Theorem 4.1).

By Lemma 4.23, there exists a cycle C containing three edges of each length in $L = \{1, 2, \ldots, \frac{k}{3} - 2, \frac{1}{2}(k-3), k-3\}$. Taking C and its translates $\rho(C)$, $\rho^2(C)$, $\ldots$, $\rho^{m-1}(C)$ uses each edge of length in L of multiplicity 3. This leaves lengths in $L' = \{\frac{k}{3}-1, \frac{k}{3}, \ldots, \frac{1}{2}(k-3)-1, \frac{1}{2}(k-3)+1, \frac{1}{2}(k-3)+2, \ldots, k-4, k-2, k-1, \ldots, \frac{m-1}{2}\}$ of multiplicity 3 to be placed in central cycles. Note that $|L'| = \frac{m-1}{2} - \frac{k}{3} = \frac{2k+r-1}{2} - \frac{k}{3} = \frac{2k}{3} + \frac{r-1}{2} = \frac{2k}{3} + \frac{k}{3} = k$. We form a single starter central cycle C' using k edges of each of the k lengths in L'; it and its translates $\rho(C'), \rho^2(C'), \ldots, \rho^{m-1}(C')$ will occur of multiplicity 3 in the decomposition, thus using all remaining edges.

Let the lengths of L', in ascending order, be denoted by $\ell_1, \ell_2, \ldots, \ell_k$, and let $\ell_q = k - 2$ (note: $q = \frac{2k}{3} - 2$ is even). Moreover, lengths $\ell_{q-\frac{1}{2}(k-5)}, \ell_{q-\frac{1}{2}(k-7)}, \ldots \ell_{q-1}$ (i.e. $\frac{1}{2}(k-3)+1, \frac{1}{2}(k-3)+2, \ldots, k-4$) are consecutive.

Let w be as defined in the statement of Lemma 4.22. Note that since $-w$ is the terminal vertex in a zig-zag path of length $k-1$ formed by edges of increasing length in alternating directions, we have that $w \geq \frac{k-1}{2}$; there is no need to explicitly compute the value of w. If the inequality $\ell_q \leq w + k - q$ (which simplifies to $w \geq \frac{2k}{3} - 4$) holds, then we may form a starter central cycle by Part 1 or 2 of Lemma 4.22. In the case that this inequality is not true, we will appeal to Part 3 of Lemma 4.22, as follows.

If $k \equiv 3 \pmod 4$, then let $j = \frac{1}{4}(k-7)$. We have that lengths ℓ_{q-2j}, ℓ_{q-2j+1}, $\ldots$, ℓ_{q-1} are consecutive. By Lemma 4.22 (Part 3), to prove that a starter central cycle can be formed, we need only show that $\ell_q \leq w + k - q + 2j$.

$$\begin{array}{rrcl} & \ell_q & \leq & w + k - q + 2j \\ \iff & k-2 & \leq & w + k - (\frac{2k}{3} - 2) + \frac{1}{2}(k-7) \\ \iff & w & \geq & \frac{k-3}{6} \end{array}$$

Since $w \geq \frac{k-1}{2} \geq \frac{k-3}{6}$, this inequality is true. Hence we may form a starter central cycle using one edge of each length in L'.

If $k \equiv 1 \pmod 4$, then let $j = \frac{1}{4}(k-5)$. So lengths $\ell_{q-2j}, \ell_{q-2j+1}, \ldots, \ell_{q-1}$ are consecutive. By Part 3 of Lemma 4.22, to prove that a starter central cycle can be formed, we need only show that $\ell_q \leq w + k - q + 2j$.

$$\begin{array}{rrcl} & \ell_q & \leq & w + k - q + 2j \\ \iff & k-2 & \leq & w + k - (\frac{2k}{3} - 2) + \frac{1}{2}(k-5) \\ \iff & w & \geq & \frac{k-9}{6} \end{array}$$

Since $w \geq \frac{k-1}{2} \geq \frac{k-9}{6}$, this inequality is true. Hence we may again form a starter central cycle using one edge of each length in L'. $\square$

Case $d \geq 3$.

In the remaining cases, we now assume $d \geq 3$. Before we continue, let us first consider the parameters involved. We have that $V(3K_m) = \mathbb{Z}_m$, and $3K_m$ contains edges of $\frac{m-1}{2}$ distinct lengths, each of multiplicity 3. Note that $3(\frac{m-1}{2}) = 3(\frac{2k+r-1}{2}) = 3k + 3(\frac{r-1}{2})$. We would like to form $3(\frac{r-1}{2\tilde{k}})$ starter peripheral cycles (with $\tilde{k}$ distinct edge lengths per starter peripheral cycle) and three starter central cycles (each containing edges of k distinct lengths). Let $t = 3(\frac{m-1}{2\tilde{k}})$. Note that if $3|t$, then $\tilde{k}|\frac{m-1}{2}$, and since $k = d\tilde{k}$ and $d|m$, we obtain that $k|\binom{m}{2}$; in this case, we know that $C_k|K_m$ (Theorem 4.4), and so trivially $C_k|3K_m$. Hence we may assume that $3 \nmid t$. Let $c = \lceil \frac{r-1}{2\tilde{k}} \rceil$, so that $3(\frac{r-1}{2\tilde{k}}) = 3(c-1) + \delta$, where $\delta \in \{1, 2\}$. We will form c starter peripheral cycles; of these, $c-1$ will occur three times in the decomposition, while the remaining one will occur δ times. The zig-zag paths which generate the peripheral cycles will have length $\tilde{k} - 1$. A peripheral cycle is constructed in the following manner. We construct a path $P_{i,0}$ of length $\tilde{k} - 1$, and form $d-1$ further paths by rotation of $P_{i,0}$ by multiples of $\tilde{m}$. Edges which link these d paths into cycles are formed with the assistance of auxiliary circulants of order d. Once a starter peripheral cycle C has been formed, the decomposition will contain the cycles C, $\rho(C)$, $\ldots$, $\rho^{\tilde{m}-1}(C)$, each with the same multiplicity as C. We will ensure that the linking edge length used in the starter peripheral cycle taken of multiplicity 1 or 2 arises from an auxiliary circulant of degree 2, and thus, for each edge length ℓ, all edges of length ℓ will be used the same number of times in the peripheral cycles – 0, 1, or 3 if there is a peripheral cycle taken only of multiplicity 1, and 0, 2, or 3 if there is a peripheral cycle taken of multiplicity 2.

The following bound on the value of c will be frequently used in the discussions of our constructions, so we derive it here. Since $r < k$, we have that

$$c = \left\lceil \frac{r-1}{2\tilde{k}} \right\rceil \leq \left\lceil \frac{k}{2\tilde{k}} \right\rceil = \left\lceil \frac{d}{2} \right\rceil = \frac{d+1}{2}. \tag{4.3.3}$$

Equivalently, this says that

$$d \geq 2c - 1. \tag{4.3.4}$$

Note that this bound together with the inequality $m < 3k$ means that

$$m + k > \frac{4}{3}m \geq \frac{4}{3}(2c-1)\tilde{m}. \tag{4.3.5}$$

Consider the parameter $t = 3(\frac{m-1}{2\tilde{k}})$. Note that if $t \equiv 1 \pmod 3$, then $3(\frac{r-1}{2\tilde{k}}) = 3(\frac{m-1}{2\tilde{k}} - d) = t - 3d \equiv 1 \pmod 3$ as well. So $c = \lceil \frac{r-1}{2\tilde{k}} \rceil = \frac{r-1}{2\tilde{k}} + \frac{2}{3}$. Also, $\frac{m-1}{2} = k + \frac{r-1}{2} = k + (c-1)\tilde{k} + \frac{\tilde{k}}{3}$. In this case, one peripheral cycle will be taken of multiplicity 1; this cycle will contain edges of $\tilde{k}$ distinct lengths, with d edges of each length. After the formation of the peripheral cycles, let L_1 denote the set of edge lengths which remain of multiplicity 3 (i.e. are unused in peripheral cycles), and L_2, the set of leftover edge lengths of multiplicity 2 (i.e. those lengths which occur in the cycle taken of multiplicity 1 in the decomposition). Note that $|L_1| = \frac{m-1}{2} - (c-1)\tilde{k} - \tilde{k} = k - \frac{2\tilde{k}}{3}$ and $|L_2| = \tilde{k}$. Partition the set L_2 in any way into three subsets S_1, S_2, S_3, each of size $\frac{\tilde{k}}{3}$. We will form three central cycles, each with edge length sets $L_1 \cup S_i \cup S_j$, where $i, j \in \{1, 2, 3\}$ and $i \neq j$. Ensuring that each S_i is used in forming two central cycles, we guarantee that the edge lengths in L_2 will be used the correct number of times. Choose $S, S' \in \{S_1, S_2, S_3\}$, $S \neq S'$, and write $L_1 \cup S \cup S' = \{\ell_1, \ell_2, \ldots, \ell_k\}$, where $\ell_1 < \ell_2 < \cdots < \ell_k$. We will use the procedure described in Lemma 4.22 to form a central cycle with edges of lengths $\ell_1, \ell_2, \ldots, \ell_k$. In the process, the choice of ℓ_q and the consecutive edge lengths relevant to the construction of the cycle depend only on L_1, and not on the choice of S or S'. Therefore, we need only verify that a central cycle C can be formed for a single choice of S and S'; the other two are formed analogously. Once C has been formed, the cycles $C, \rho(C), \ldots, \rho^{m-1}(C)$ will use each edge of length in $L_1 \cup S \cup S'$ once.

Now consider the case in which $t \equiv 2 \pmod 3$. Here, $3(\frac{r-1}{2\tilde{k}}) = 3(\frac{m-1}{2\tilde{k}} - d) = t - 3d \equiv 2 \pmod 3$ as well. So $c = \lceil \frac{r-1}{2\tilde{k}} \rceil = \frac{r-1}{2\tilde{k}} + \frac{1}{3}$. Also, $\frac{m-1}{2} = k + \frac{r-1}{2} = k + (c-1)\tilde{k} + \frac{2\tilde{k}}{3}$. In this case, one peripheral cycle will be taken of multiplicity 2; this cycle will contain edges of $\tilde{k}$ distinct lengths, with d edges of each length. We again let L_1 denote the set of distinct edge lengths which remain of multiplicity 3. In this case, we define the set L_2 to contain the edge lengths which remain of multiplicity 1, i.e. those lengths which occur in the peripheral cycle taken of multiplicity 2 in the decomposition. Note that $|L_1| = \frac{m-1}{2} - c\tilde{k} = k - \frac{\tilde{k}}{3}$ and $|L_2| = \tilde{k}$. Partition the set L_2 in any way into subsets S_1, S_2, S_3, each of size $\frac{\tilde{k}}{3}$. We will form three central cycles, with edge length sets $L_1 \cup S_i$ for $i = 1, 2, 3$. Choose $S \in \{S_1, S_2, S_3\}$, and write $L_1 \cup S = \{\ell_1, \ell_2, \ldots, \ell_k\}$, where $\ell_1 < \ell_2 < \cdots < \ell_k$. We will use the procedure described in Lemma 4.22 to form a central cycle with edges of lengths $\ell_1, \ell_2, \ldots, \ell_k$. In the process, the choice of ℓ_q and the determination of relevant consecutive edge lengths will again depend only on L_1, and not on the choice of $S \in \{S_1, S_2, S_3\}$; therefore, we need only verify the construction for a single central cycle C, and the other two will be formed analogously. Once C has been formed, the cycles $C, \rho(C), \ldots, \rho^{m-1}(C)$ will use each edge of length in $L_1 \cup S$ exactly once.

The construction of peripheral cycles depends on the value of $\tilde{k}$ modulo 4.

Case 1. $\tilde{k} \equiv 3 \pmod 4$ and $d \geq 3$

We begin by defining the paths to be used in the construction of peripheral cycles.

For $\tilde{k} > 3$, let

$$P_{0,0} = 0, 1, -1, 2, -2, \dots, \frac{\tilde{k}-3}{4}, -\frac{\tilde{k}-3}{4}, \frac{\tilde{k}+5}{4}, -\frac{\tilde{k}+1}{4},$$
$$\frac{\tilde{k}+9}{4}, -\frac{\tilde{k}+5}{4}, \dots, \frac{\tilde{k}+1}{2}, -\frac{\tilde{k}-1}{2}.$$

If $\tilde{k} = 3$, then let $P_{0,0} = 0, 2, -1$. We have that $P_{0,0}$ is a path of length $\tilde{k}-1$ containing edges of lengths $1, 2, \dots, \frac{\tilde{k}-3}{2}, \frac{\tilde{k}+1}{2}, \frac{\tilde{k}+3}{2}, \dots, \tilde{k}$.

If $\tilde{m} - \frac{\tilde{k}-1}{2} > 2\tilde{k} - 1$, then we let

$$P^*_{0,0} = 0, \tilde{k}+1, -1, \tilde{k}+2, -2, \dots, \frac{3\tilde{k}-1}{2}, -\frac{\tilde{k}-1}{2}.$$

Otherwise, let

$$P^*_{0,0} = 0, \tilde{k}+1, -1, \tilde{k}+2, -2, \dots, \frac{\tilde{m}}{2} + \frac{\tilde{k}-1}{4}, -\left(\frac{\tilde{m}}{2} - \frac{3\tilde{k}+1}{4}\right), \frac{\tilde{m}}{2} + \frac{\tilde{k}+7}{4},$$
$$-\left(\frac{\tilde{m}}{2} - \frac{3\tilde{k}-3}{4}\right), \frac{\tilde{m}}{2} + \frac{\tilde{k}+11}{4}, \dots, \frac{3\tilde{k}+1}{2}, -\frac{\tilde{k}-1}{2}.$$

Note that the edge lengths in $P^*_{0,0}$ range between $\tilde{k}+1$ and either $2\tilde{k}-1$ or $2\tilde{k}$, but $P^*_{0,0}$ does not contain an edge of length $\tilde{m} - \frac{\tilde{k}-1}{2}$. Thus, the edge length set of $P^*_{0,0}$ does not overlap with that of $P_{0,0}$; nor does it contain any element exceeding $\tilde{m}-1$, as $2\tilde{k} < \tilde{m}$. For $0 \le i \le (c-1)/2$ or $0 \le i \le (c-2)/2$, if c is odd or even, respectively, form $P_{i,0}$ from $P_{0,0}$, resp. $P^*_{i,0}$ from $P^*_{0,0}$, by adding $i\tilde{m}$ to the even-positioned vertices of $P_{0,0}$, resp. $P^*_{0,0}$. Note that for each i, $P_{i,0}$ and $P^*_{i,0}$ have initial vertex 0 and terminal vertex $-(\frac{\tilde{k}-1}{2})$. Thus linking lengths will be congruent to $\pm(\frac{\tilde{k}-1}{2}) \pmod{\tilde{m}}$.

The path $P_{i,0}$ contains edges of lengths $1+i\tilde{m}$, $2+i\tilde{m}$, …, $\frac{\tilde{k}-3}{2}+i\tilde{m}$, $\frac{\tilde{k}+1}{2}+i\tilde{m}$, …, $\tilde{k}+i\tilde{m}$. There are two possibilities for the edge lengths of $P^*_{i,0}$: either $\tilde{k}+1+i\tilde{m}$, $\tilde{k}+2+i\tilde{m}$, …, $2\tilde{k}-1+i\tilde{m}$ or $\tilde{k}+1+i\tilde{m}$, $\tilde{k}+2+i\tilde{m}$, …, $\tilde{m}-\frac{\tilde{k}+1}{2}$, $\tilde{m}-\frac{\tilde{k}-3}{2}$, $\tilde{m}-\frac{\tilde{k}-5}{2}$, …, $2\tilde{k}+i\tilde{m}$. For $0 \le j \le d-1$, let $P_{i,j} = \rho^{j\tilde{m}}(P_{i,0})$ and $P^*_{i,j} = \rho^{j\tilde{m}}(P^*_{i,0})$. Note that the paths $P_{i,0}, \dots, P_{i,d-1}$ are pairwise vertex-disjoint, as are the paths $P^*_{i,0}, \dots, P^*_{i,d-1}$.

The construction of peripheral cycles depends on the parity of c.

Subcase 1.1. c is even.

We use starter paths $P_{i,0}$, $0 \le i \le (c-2)/2$ and $P^*_{i,0}$, $0 \le i \le (c-2)/2$. Note that the longest edge length in any of these paths is at most $2\tilde{k} + (\frac{c-2}{2})\tilde{m}$. One path $P_{i,0}$ will give paths used in a peripheral cycle of multiplicity 1 or 2; the others occur in cycles taken of multiplicity 3. The linking lengths used will be $\tilde{m} \pm \frac{\tilde{k}-1}{2}$, $2\tilde{m} \pm \frac{\tilde{k}-1}{2}$, $\ldots$, $(\frac{c}{2})\tilde{m} \pm \frac{\tilde{k}-1}{2}$. If $c \equiv 0 \pmod 4$, the linking lengths arise from auxiliary circulants

$$X(d;\{1\}), X(d;\{2\}), X(d;\{3,4\}), X(d;\{5,6\}), \ldots, X\left(d;\left\{\frac{c-2}{2}, \frac{c}{2}\right\}\right).$$

If $c \equiv 2 \pmod 4$, the linking lengths arise from auxiliary circulants

$$X(d;\{1\}), X(d;\{2,3\}), \ldots, X\left(d;\left\{\frac{c-2}{2}, \frac{c}{2}\right\}\right).$$

Note that the largest linking length, $\frac{c}{2}\tilde{m} + \frac{\tilde{k}-1}{2}$, is greater than the length of the largest edge in the starter paths, $\frac{c-2}{2}\tilde{m} + 2\tilde{k} - 1$ or $\frac{c-2}{2}\tilde{m} + 2\tilde{k}$, as $\frac{c-2}{2}\tilde{m} + 2\tilde{k} = \frac{c}{2}\tilde{m} + \frac{\tilde{k}-1}{2} - \tilde{m} + \frac{3\tilde{k}+1}{2} < \frac{c}{2}\tilde{m} + \frac{\tilde{k}-1}{2}$.

Consider the starter peripheral cycle C which will be taken of multiplicity 1 or 2. We will link the family of paths used to form C with edges of length $\tilde{m} + \frac{\tilde{k}-1}{2}$ derived from the auxiliary circulant $X(d;\{1\})$ of degree 2. Edges of length $\tilde{m} - \frac{\tilde{k}-1}{2}$ (also from the circulant $X(d;\{1\})$) will occur in a different starter peripheral cycle, which will necessarily be taken of multiplicity 3. Using up edge length $\tilde{m} - \frac{\tilde{k}-1}{2}$ in the peripheral cycles ensures as large a gap as possible between the smallest two edge lengths which remain for use in central cycles; this property will be exploited in the construction of the central cycles.

Note that the lengths used in the paths and linking edges are sufficiently small since $d \ge 3$, as the longest length used is the linking length $\frac{c}{2}\tilde{m} + \frac{\tilde{k}-1}{2} \le \frac{d+1}{4}\tilde{m} + \frac{\tilde{k}-1}{2} \le \frac{d-1}{2}\tilde{m} + \frac{\tilde{k}-1}{2} = \frac{m}{2} - \frac{\tilde{m}}{2} + \frac{\tilde{k}-1}{2} = \frac{m-1}{2} - \frac{\tilde{m}-\tilde{k}}{2} < \frac{m-1}{2}$.

Provided $c > 2$, we choose a path $P_{i,0}$, where $i > 0$, to be taken of multiplicity 1 or 2, so that the two smallest lengths to be used in a central cycle are $\frac{\tilde{k}-1}{2}$ and either $2\tilde{k}$ or $2\tilde{k}+1$. Since $2\tilde{k} - \frac{\tilde{k}-1}{2} = \frac{3\tilde{k}+1}{2}$, there is a gap of $\frac{3\tilde{k}-1}{2}$ vertices between 0 and the first negative vertex in a zig-zag path formed from leftover lengths. If $c = 2$, we take $P_{0,0}$ of multiplicity 3 and $P^*_{0,0}$ of multiplicity 1 or 2.

Before we proceed to the construction of central cycles, which depends on the congruence class of c modulo 4, we notice that if $t \equiv 1 \pmod 3$, then $\frac{m-1}{2} = k + (c-1)\tilde{k} + \frac{\tilde{k}}{3}$ is odd, while if $t \equiv 2 \pmod 3$, then $\frac{m-1}{2} = k + (c-1)\tilde{k} + \frac{2\tilde{k}}{3}$ is even.

Subcase 1.1.1. $c \equiv 0 \pmod 4$

Let $c = 4\alpha$. Here, L_1 contains consecutive lengths $2\alpha\tilde{m} + \frac{\tilde{k}+1}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then let $\ell_q = 2\alpha\tilde{m} + \frac{\tilde{k}+1}{2}$, so $\ell_q, \ell_{q+1}, \ldots, \ell_k$ are consecutive. Since k and $\ell_k = \frac{m-1}{2}$ are odd and ℓ_q is even, it follows that $q = k - \frac{m-1}{2} + \ell_q$ is even. To form central cycles using Part 1 or 2 of Lemma 4.22, we need to show that $w + k - q \geq \ell_q$, or equivalently, $w + \frac{m-1}{2} \geq 2\ell_q = 4\alpha\tilde{m} + \tilde{k} + 1$. We have that $w \geq \frac{k-3}{2} + (\ell_2 - \ell_1) \geq \frac{k-3}{2} + (2\tilde{k} - \frac{\tilde{k}-1}{2}) = \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$, so it will suffice to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq 4\alpha\tilde{m} + \tilde{k} + 1,$$

or equivalently

$$k + m + \tilde{k} \geq 8\alpha\tilde{m} + 5.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = 2\alpha\tilde{m} + \frac{\tilde{k}+3}{2}$. As k is odd, ℓ_k is even and ℓ_q is even, it must be that q is even. By Lemma 4.22 (Parts 1 and 2), we can form the required starter central cycle if we can show that $\ell_q \leq w + k - q$, or $w + \frac{m-1}{2} \geq 2\ell_q$, where w is as defined in Lemma 4.22. Again, $w \geq \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$. Hence it will suffice to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq 4\alpha\tilde{m} + \tilde{k} + 3,$$

or equivalently that

$$k + m + \tilde{k} \geq 8\alpha\tilde{m} + 9.$$

Hence in either case, central cycles can be formed if we can show that $k+m+\tilde{k} \geq 8\alpha\tilde{m} + 9$. From inequality 4.3.5, we have that $m + k > \frac{4}{3}(2c-1)\tilde{m} = \frac{4}{3}(8\alpha - 1)\tilde{m}$, so that

$$m + k + \tilde{k} > \frac{4}{3}(8\alpha - 1)\tilde{m} + \tilde{k} = 8\alpha\tilde{m} + \left(\frac{8\alpha - 4}{3}\right)\tilde{m} + \tilde{k}.$$

Since $\alpha \geq 1$, $\tilde{m} \geq 7$ and $\tilde{k} \geq 3$, we have that $(\frac{8\alpha-4}{3})\tilde{m} + \tilde{k} \geq 9$, and so it follows that $k + m + \tilde{k} > 8\alpha\tilde{m} + 9$.

Subcase 1.1.2. $c \equiv 2 \pmod 4$

Write $c = 4\alpha + 2$. The consecutive lengths in L_1 include $(2\alpha + 1)\tilde{m} + \frac{\tilde{k}+1}{2}$, $(2\alpha + 1)\tilde{m} + \frac{\tilde{k}+3}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then let $\ell_q = (2\alpha + 1)\tilde{m} + \frac{\tilde{k}+3}{2}$, so that $\ell_q, \ell_{q+1}, \ldots, \ell_k$ are consecutive integers. Since k and $\ell_k = \frac{m-1}{2}$ are odd and ℓ_q is even, it follows that $q = k - \frac{m-1}{2} + \ell_q$ is even. We would like to show that $w + \frac{m-1}{2} \geq 2\ell_q = (4\alpha+2)\tilde{m} + \tilde{k} + 3$; if this inequality holds, then central cycles can be formed by Part 1 or 2 of Lemma 4.22. If $\alpha \geq 1$, then we have that $w \geq \frac{k-3}{2} + (\ell_2 - \ell_1) \geq \frac{k-3}{2} + 2\tilde{k} - \frac{\tilde{k}-1}{2} \geq \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$, so it will suffice to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq (4\alpha + 2)\tilde{m} + \tilde{k} + 3,$$

or equivalently, that

$$k + m + \tilde{k} \geq (8\alpha + 4)\tilde{m} + 9.$$

If $\alpha = 0$, then $c = 2$, so the peripheral cycle formed from $P^*_{0,0}$ together with linking length $\tilde{m} + \frac{\tilde{k}-1}{2}$ is taken of multiplicity 1. Here, $w \geq \frac{k-3}{2} + (\ell_2 - \ell_1) \geq \frac{k-3}{2} + (\tilde{k} + 1 - \frac{\tilde{k}-1}{2}) = \frac{k-3}{2} + \frac{\tilde{k}+3}{2}$. It will suffice to show that

$$\frac{k-3}{2} + \frac{\tilde{k}+3}{2} + \frac{m-1}{2} \geq 2\tilde{m} + \tilde{k} + 3,$$

or equivalently, that

$$k + m \geq 4\tilde{m} + \tilde{k} + 7.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = (2\alpha + 1)\tilde{m} + \frac{\tilde{k}+1}{2}$. Since $\ell_k = \frac{m-1}{2}$ is even and k is odd, it follows that $q = k - \frac{m-1}{2} + \ell_q$ must be even. To apply Parts 1 and 2 of

Lemma 4.22, we need to show that $w + \frac{m-1}{2} \geq 2\ell_q$. If $\alpha \geq 1$, then $w \geq \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$, so it will suffice to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq (4\alpha + 2)\tilde{m} + \tilde{k} + 1,$$

or equivalently that

$$k + m + \tilde{k} \geq (8\alpha + 4)\tilde{m} + 5.$$

If $\alpha = 0$, then the peripheral cycle formed from $P^*_{0,0}$ is taken of multiplicity 2, so that again $w \geq \frac{k-3}{2} + (\ell_2 - \ell_1) \geq \frac{k-3}{2} + \frac{\tilde{k}+3}{2}$. Hence it will suffice to show that

$$\frac{k-3}{2} + \frac{\tilde{k}+3}{2} + \frac{m-1}{2} \geq 2\tilde{m} + \tilde{k} + 1,$$

or equivalently that

$$k + m \geq 4\tilde{m} + \tilde{k} + 3.$$

In either case, then, it suffices to show that $k + m + \tilde{k} \geq (8\alpha + 4)\tilde{m} + 9$ if $\alpha = 1$, and $k + m \geq 4\tilde{m} + \tilde{k} + 7$ if $\alpha = 0$.

Let us first suppose $\alpha \geq 1$. From inequality 4.3.5, we have that $k + m > \frac{4}{3}(2c - 1)\tilde{m} = \frac{4}{3}(8\alpha + 3)\tilde{m}$, and so since $\tilde{m} \geq 7$ and $\alpha \geq 1$,

$$k + m + \tilde{k} > \frac{4}{3}(8\alpha + 3)\tilde{m} + \tilde{k} = (8\alpha + 4)\tilde{m} + \frac{8}{3}\alpha\tilde{m} + \tilde{k} > (8\alpha + 4)\tilde{m} + 9.$$

Next, suppose $\alpha = 0$. If $d \geq 5$, then $k+m \geq 5\tilde{k}+5\tilde{m} > 4\tilde{m}+\tilde{k}+4\tilde{k} > 4\tilde{m}+\tilde{k}+7$. However, if $d = 3$, then the inequality $m + k \geq 4\tilde{m} + \tilde{k} + 7$, which simplifies to $2\tilde{k} \geq \tilde{m} + 7$, is false, since $\tilde{m} > 2\tilde{k}$.

Let us consider the values of the parameters involved when $\alpha = 0$ and $d = 3$. If $t \equiv 2 \pmod 3$, then $c = \lceil \frac{r-1}{2\tilde{k}} \rceil = \frac{r-1}{2\tilde{k}} + \frac{1}{3} = 2$ implies that $\frac{r-1}{2\tilde{k}} = \frac{5}{3}$, and so $\frac{m-1}{2\tilde{k}} = \frac{r-1}{2\tilde{k}} + d = \frac{14}{3}$. Solving, we obtain $m = \frac{28}{3}\tilde{k} + 1 > 9\tilde{k} = 3k$, a contradiction.

However, if $t \equiv 1 \pmod 3$, then $c = \frac{r-1}{2\tilde{k}} + \frac{2}{3} = 2$, so that $\frac{r-1}{2\tilde{k}} = \frac{4}{3}$, $\frac{m-1}{2\tilde{k}} = \frac{r-1}{2\tilde{k}} + d = \frac{13}{3}$ and $m = \frac{26}{3}\tilde{k} + 1 < 9\tilde{k}$. In this case, we would like to appeal to Part 3 of Lemma 4.22 in order to form the required starter central cycles. Unfortunately, however, for some central cycles, we have $\ell_{q-2} = \tilde{m} + \frac{\tilde{k}-3}{2}$ and $\ell_{q-1} = \tilde{m} + \frac{\tilde{k}+1}{2}$, which are not consecutive. To get around this, we use a different construction for the peripheral cycles in this case. Let

$$P = 0, 1, -2, 2, -3, 3, -4, \dots, \frac{\tilde{k}-1}{2}, -\left(\frac{\tilde{k}+1}{2}\right).$$

Then P contains an edge of each length $1, 3, 4, \dots, \tilde{k}$. Moreover, since $\tilde{m} > \tilde{k}$, paths P and $\rho^{\tilde{m}}(P)$ are vertex-disjoint.

Recall that $m = \frac{26}{3}\tilde{k} + 1$. Hence $\tilde{m} = \frac{m}{3} = \frac{26}{9}\tilde{k} + \frac{1}{3} = \frac{26\tilde{k}+3}{9} > \frac{5\tilde{k}+1}{2}$, and so $\tilde{m} - \frac{\tilde{k}+1}{2} > \frac{5\tilde{k}+1}{2} - \frac{\tilde{k}+1}{2} = 2\tilde{k}$. Thus, if we form a second starter zig-zag path P' with edges of lengths not exceeding $2\tilde{k}$, then this path will not contain an edge of length $\tilde{m} - \frac{\tilde{k}+1}{2}$ – this fact is crucial, as length $\tilde{m} - \frac{\tilde{k}+1}{2}$ will be used for linking edges. Let

$$P' = 0, \tilde{k}+1, -1, \tilde{k}+2, -2, \dots, \frac{3\tilde{k}-3}{2}, -\left(\frac{\tilde{k}-3}{2}\right), \frac{3\tilde{k}-1}{2}, -\left(\frac{\tilde{k}+1}{2}\right).$$

Then P' contains edges of lengths $\tilde{k}+1, \tilde{k}+2, \dots, 2\tilde{k}-2, 2\tilde{k}$, and contains no edge of length $\tilde{m} - \frac{\tilde{k}+1}{2}$.

The paths P and P' both have initial vertex 0 and terminal vertex $-(\frac{\tilde{k}+1}{2})$; we may use them, together with linking edges of length $\tilde{m} \pm \frac{\tilde{k}+1}{2}$ to form peripheral cycles. (Note that $\frac{\tilde{k}+1}{2} \not\equiv -(\frac{\tilde{k}+1}{2}) \pmod{\tilde{m}}$, since $\tilde{m} | (\tilde{k}+1)$ cannot happen as $\tilde{m} > 2\tilde{k} \geq \tilde{k}+3$.) We take the peripheral cycle formed from P' with linking length $\tilde{m} - \frac{\tilde{k}+1}{2}$ of multiplicity 2, and the cycle formed from P and linking length $\tilde{m} + \frac{\tilde{k}+1}{2}$ of multiplicity 3. The longest length used in a peripheral cycle is the linking length $\tilde{m} + \frac{\tilde{k}+1}{2}$, and so each central cycle will contain edges of consecutive lengths $\tilde{m} + \frac{\tilde{k}+3}{2}, \tilde{m} + \frac{\tilde{k}+5}{2}, \dots, \frac{m-1}{2}$. Let $\ell_q = \tilde{m} + \frac{\tilde{k}+3}{2}$. Since ℓ_q is even, and k and $\ell_k = \frac{m-1}{2}$ are odd, we have that $q = k - \frac{m-1}{2} + \ell_q$ is even. Note that $\ell_2 - \ell_1 \geq (\tilde{k}+1) - 2 = \tilde{k} - 1$, and so we obtain the bound $w + \frac{m-1}{2} \geq \frac{k-3}{2} + \tilde{k} - 1 + \frac{m-1}{2} = \frac{3\tilde{k}-3}{2} + \tilde{k} - 1 + \frac{3\tilde{m}-1}{2} = \frac{5\tilde{k}-5}{2} + \frac{3\tilde{m}-1}{2}$. If $\frac{5\tilde{k}-5}{2} + \frac{3\tilde{m}-1}{2} \geq 2\ell_q = 2\tilde{m} + \tilde{k} + 3$ (which occurs whenever $\tilde{m} \leq 3\tilde{k} - 12$), then we can form central cycles by Part 1 or 2 of Lemma 4.22. Otherwise, we must apply

Part 3 instead. We thus need to find values of j such that $\ell_{q-2j}, \ell_{q-2j+1}, \dots, \ell_{q-1}$ are consecutive.

Let $j = \frac{\tilde{k}-1}{2}$. Then for each central cycle, lengths ℓ_{q-2j}, ℓ_{q-2j+1}, $\dots$, ℓ_{q-1}, i.e. $\tilde{m} - \frac{\tilde{k}-3}{2}$, $\tilde{m} - \frac{\tilde{k}-5}{2}$, $\dots$, $\tilde{m} + \frac{\tilde{k}-1}{2}$, are consecutive. Hence, in order to apply Part 3 to obtain central cycles, we need only show that $w + \frac{m-1}{2} + 2j \geq 2\ell_q$, or $w + \frac{m-1}{2} + \tilde{k} - 1 \geq 2\tilde{m} + \tilde{k} + 3$. Note that $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + (\tilde{k}+1) - 2 = \frac{k-3}{2} + \tilde{k} - 1$. Using this bound on w, we see that it is enough to show that

$$\frac{k-3}{2} + \tilde{k} - 1 + \frac{m-1}{2} \geq 2\tilde{m} + 3,$$

or equivalently, that

$$k + m + 2\tilde{k} \geq 4\tilde{m} + 9.$$

Since $d = 3$, this inequality simplifies to $5\tilde{k} \geq \tilde{m} + 9$. Now, since $\tilde{m} < 3\tilde{k}$, we have that $5\tilde{k} > \tilde{m} + 2\tilde{k} \geq \tilde{m} + 9$ whenever $\tilde{k} > 4$. The only other possibility is that $\tilde{k} = 3$, and so $k = 9$; recall that cycle length 9 has been left as an exception.

Subcase 1.2. c is odd.

Unless otherwise specified as we proceed through the cases, the c peripheral cycles will be formed using paths $P_{i,0}$ for $0 \leq i \leq \frac{c-1}{2}$ and paths $P^*_{i,0}$ for $0 \leq i \leq \frac{c-3}{2}$, and linking lengths will be as follows. If $c \geq 5$, the linking lengths will be $\tilde{m} \pm \frac{\tilde{k}-1}{2}, 2\tilde{m} - \frac{\tilde{k}-1}{2}, 3\tilde{m} \pm \frac{\tilde{k}-1}{2}, \dots, \frac{c+1}{2}\tilde{m} \pm \frac{\tilde{k}-1}{2}$. Since $c \leq \frac{d+1}{2}$, we have that $d \geq 5$, and hence $\frac{c+1}{2}\tilde{m} + \frac{\tilde{k}-1}{2} \leq \frac{\frac{d+1}{2}+1}{2}\tilde{m} + \frac{\frac{\tilde{m}-1}{2}-1}{2} = \frac{d+3}{4}\tilde{m} + \frac{\tilde{m}-3}{4} < \frac{d+4}{4}\tilde{m} - \frac{1}{2} < \frac{d}{2}\tilde{m} - \frac{1}{2} = \frac{m-1}{2}$. The auxiliary circulants used to link the paths are

$$X(d;\{1\}), X(d;\{2\}), X(d;\{3,4\}), X(d;\{5,6\}), \dots, X\left(d;\left\{\frac{c-1}{2}, \frac{c+1}{2}\right\}\right)$$

if $c \equiv 3 \pmod 4$, and

$$X(d;\{2\}), X(d;\{1,3\}), X(d;\{4,5\}), X(d;\{6,7\}), \dots, X\left(d,\left\{\frac{c-1}{2}, \frac{c+1}{2}\right\}\right)$$

if $c \equiv 1 \pmod 4$. If $c = 3$, then we use linking lengths $\tilde{m} \pm \frac{\tilde{k}-1}{2}, 2\tilde{m} - \frac{\tilde{k}-1}{2}$ from auxiliary circulants $X(d;\{1\}), X(d;\{2\})$. Note that since $d \geq 3$, $\tilde{k} \geq 3$ and $c \leq \frac{d+1}{2}$, we have that $2\tilde{m} - \frac{\tilde{k}-1}{2} = \frac{c+1}{2}\tilde{m} - \frac{\tilde{k}-1}{2} \leq \frac{d+3}{4}\tilde{m} - \frac{\tilde{k}-1}{2} < \frac{d}{2}\tilde{m} - \frac{1}{2} = \frac{m-1}{2}$. If $c = 1$, then we link using length $\tilde{m} - \frac{\tilde{k}-1}{2}$ from auxiliary circulant $X(d; \{1\})$. Similarly to before, $\tilde{m} - \frac{\tilde{k}-1}{2} = \frac{c+1}{2}\tilde{m} - \frac{\tilde{k}-1}{2} < \frac{m-1}{2}$.

In any case, we must use one of the edge lengths from an auxiliary circulant of degree 2 in the cycle which will be taken of multiplicity 1 or 2; if $c > 1$, we can easily ensure that this length is $2\tilde{m} - \frac{\tilde{k}-1}{2}$, while $2\tilde{m} + \frac{\tilde{k}-1}{2}$ remains unused; in this way, length $\tilde{m} - \frac{\tilde{k}-1}{2}$ will be used of multiplicity 3 in the peripheral cycles. Moreover, if $c > 1$, the path from which the peripheral cycle taken of multiplicity 1 or 2 arises may be taken as $P_{i,0}$ or $P^*_{i,0}$ for some $i > 0$.

Before we construct the starter central cycles according to the congruence of c modulo 4, we remark that if $t \equiv 1 \pmod 3$, then $\frac{m-1}{2} = k + (c-1)\tilde{k} + \frac{\tilde{k}}{3}$ is even, while if $t \equiv 2 \pmod 3$, then $\frac{m-1}{2} = k + (c-1)\tilde{k} + \frac{2\tilde{k}}{3}$ is odd.

Subcase 1.2.1. $c \equiv 1 \pmod 4$

Let $c = 4\alpha + 1$. We first consider the case $\alpha \geq 1$. Here we have that the consecutive leftover lengths in L_1 include $(2\alpha+1)\tilde{m} + \frac{\tilde{k}+1}{2}, (2\alpha+1)\tilde{m} + \frac{\tilde{k}+3}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then we let $\ell_q = (2\alpha+1)\tilde{m} + \frac{\tilde{k}+1}{2}$. Since ℓ_q is odd, $\ell_k = \frac{m-1}{2}$ is even and k is odd, $q = k - \frac{m-1}{2} + \ell_q$ must be even. To apply Lemma 4.22, Parts 1 and 2, to form central cycles, we must show that $w + \frac{m-1}{2} \geq 2\ell_q = (4\alpha+2)\tilde{m} + \tilde{k} + 1$. Since $\alpha \geq 1$, we have that $w \geq \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$, so it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq (4\alpha+2)\tilde{m} + \tilde{k} + 1,$$

or equivalently

$$k + m + \tilde{k} \geq (8\alpha+4)\tilde{m} + 5.$$

If $t \equiv 2 \pmod 3$, then we let $\ell_q = (2\alpha+1)\tilde{m} + \frac{\tilde{k}+3}{2}$. Since k is odd, ℓ_k is odd and ℓ_q is odd, it follows that q is even. By Lemma 4.22 (Parts 1 and 2), we can form the required starter central cycles if we can show that $w + \frac{m-1}{2} \geq 2\ell_q$, where w is as defined as in Lemma 4.22. Notice that provided $\alpha \neq 0$, we have that

$w \geq \frac{k-3}{2} + (\ell_2 - \ell_1) \geq \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$. Hence we need only show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq (4\alpha+2)\tilde{m} + \tilde{k} + 3,$$

or equivalently that

$$k + m + \tilde{k} \geq (8\alpha+4)\tilde{m} + 9.$$

Hence, if the inequality $k + m + \tilde{k} \geq (8\alpha+4)\tilde{m} + 9$ holds, then central cycles can be formed for either value of t. By inequality 4.3.5, we have that $k + m > \frac{4}{3}(2c-1)\tilde{m} = \frac{4}{3}(8\alpha+1)\tilde{m}$, so that

$$k + m + \tilde{k} \geq \frac{4}{3}(8\alpha+1)\tilde{m} + \tilde{k} = (8\alpha+4)\tilde{m} + \frac{8}{3}(\alpha-1)\tilde{m} + \tilde{k}$$

Since $\tilde{m} \geq 7$, we have that $\frac{8}{3}(\alpha-1)\tilde{m} \geq 9$ if $\alpha \geq 2$. If $\alpha = 1$ and $\tilde{k} \geq 9$, then the desired inequality also holds. Let us now suppose $\alpha = 1$ and $\tilde{k} < 9$. Since $\tilde{k} \equiv 3 \pmod 4$ and $3|\tilde{k}$, we must have $\tilde{k} = 3$. In this case, $2\tilde{k} < \tilde{m} < 3\tilde{k}$ implies $\tilde{m} = 7$. Note also that by inequality 4.3.4, $d \geq 2c - 1 = 9$. The inequality to be proved is $k + m + \tilde{k} \geq 12\tilde{m} + 9$, or equivalently $k + m \geq 90$ or $10d \geq 90$. This is clearly true since $d \geq 9$.

Next, suppose that $\alpha = 0$. Then the only peripheral cycle, taken with multiplicity 1 or 2, is formed from $P_{0,0}$ together with linking edges of length $\tilde{m} - \frac{\tilde{k}-1}{2}$. Consecutive leftover lengths in L_1 include $\tilde{m} - \frac{\tilde{k}-3}{2}, \tilde{m} - \frac{\tilde{k}-5}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then let $\ell_q = \tilde{m} - \frac{\tilde{k}-3}{2}$. Since ℓ_q is odd, $\ell_k = \frac{m-1}{2}$ is even and k is odd, $q = k - \frac{m-1}{2} + \ell_q$ is even. To apply Lemma 4.22, Parts 1 and 2, to form central cycles, we must show that $w + \frac{m-1}{2} \geq 2\ell_q = 2\tilde{m} - \tilde{k} + 3$. Note that $w \geq \frac{k-1}{2}$, so it suffices to show that

$$\frac{k-1}{2} + \frac{m-1}{2} \geq 2\tilde{m} - \tilde{k} + 3,$$

or equivalently that

$$k + m + 2\tilde{k} \geq 4\tilde{m} + 8.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = \tilde{m} - \frac{\tilde{k}-5}{2}$. Since ℓ_q is even, $\ell_k = \frac{m-1}{2}$ is odd and k is odd, $q = k - \frac{m-1}{2} + \ell_q$ must be even. To apply Lemma 4.22, Parts 1 and 2, to form central cycles, we must show that $w + \frac{m-1}{2} \geq 2\ell_q = 2\tilde{m} - \tilde{k} + 5$. Note that $w \geq \frac{k-1}{2}$, so it suffices to show that

$$\frac{k-1}{2} + \frac{m-1}{2} \geq 2\tilde{m} - \tilde{k} + 5,$$

or equivalently that

$$k + m + 2\tilde{k} \geq 4\tilde{m} + 12.$$

Hence in either case, it is enough to show that $k + m + 2\tilde{k} \geq 4\tilde{m} + 12$. Note that this inequality is true if $d \geq 5$, since in this case, $k + m + 2\tilde{k} \geq 5\tilde{m} + 7\tilde{k} \geq 4\tilde{m} + 12$. If $d = 3$, the inequality becomes $5\tilde{k} \geq \tilde{m} + 12$, which is true if $\tilde{k} > 3$, since in this case that $\tilde{k} \equiv 3 \pmod 4$ and $3|\tilde{k}$ implies that $\tilde{k} \geq 15$, so that $5\tilde{k} > \tilde{m} + 2\tilde{k} > \tilde{m} + 12$. Otherwise, if $d = 3$ and $\tilde{k} = 3$, then $k = 9$; recall that cycle length 9 is an exeptional value.

Subcase 1.2.2. $c \equiv 3 \pmod 4$

Write $c = 4\alpha + 3$. First suppose that $\alpha \geq 1$. In this case, we have that consecutive lengths in L_1 include $(2\alpha+2)\tilde{m} + \frac{\tilde{k}+1}{2}, (2\alpha+2)\tilde{m} + \frac{\tilde{k}+3}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then let $\ell_q = (2\alpha+2)\tilde{m} + \frac{\tilde{k}+3}{2}$. Since ℓ_q is odd, $\ell_k = \frac{m-1}{2}$ is even and k is odd, it follows that $q = k - \frac{m-1}{2} + \ell_q$ is even. If we can show that $w + \frac{m-1}{2} \geq 2\ell_q = (4\alpha+4)\tilde{m} + \tilde{k} + 3$, then we can form central cycles by Lemma 4.22, Parts 1 and 2. We have that $w \geq \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$, so it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq (4\alpha+4)\tilde{m} + \tilde{k} + 3,$$

or equivalently

$$k + m + \tilde{k} \geq (8\alpha+8)\tilde{m} + 9.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = (2\alpha+2)\tilde{m} + \frac{\tilde{k}+1}{2}$. Note that ℓ_q is even. Since lengths $\ell_q, \ell_{q+1}, \ldots, \ell_k = \frac{m-1}{2}$ are consecutive, and k and ℓ_k are odd, it follows that

$q = k - \frac{m-1}{2} + \ell_q$ is even. By Lemma 4.22, Parts 1 and 2, we can form the required starter central cycles if we can show that $w + \frac{m-1}{2} \geq 2\ell_q$, where w is as defined in Lemma 4.22. Note that since $\alpha \geq 0$, we have that $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$, and so it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq (4\alpha+4)\tilde{m} + \tilde{k} + 1,$$

or equivalently that

$$k + m + \tilde{k} \geq (8\alpha+8)\tilde{m} + 5.$$

Thus, if we show that $k + m + \tilde{k} \geq (8\alpha+8)\tilde{m} + 9$, then we can form central cycles in either case.

By Inequality 4.3.5, we have that $k + m > \frac{4}{3}(2c-1)\tilde{m} = \frac{4}{3}(8\alpha+5)\tilde{m}$, so that

$$k + m + \tilde{k} \geq \frac{4}{3}(8\alpha+5)\tilde{m} + \tilde{k} = (8\alpha+8)\tilde{m} + \frac{4}{3}(2\alpha-1)\tilde{m} + \tilde{k}.$$

Since $\alpha \geq 1$, we have that $\frac{4}{3}(2\alpha-1)\tilde{m} + \tilde{k} \geq \frac{4}{3}\tilde{m} + \tilde{k} \geq \frac{4}{3}(7) + 3 > 9$, and so the desired inequality holds.

Next, suppose that $\alpha = 0$. Then $c = 3$, and the peripheral cycles use linking edge lengths $\tilde{m} \pm \frac{\tilde{k}-1}{2}$ and $2\tilde{m} - \frac{\tilde{k}-1}{2}$. The paths used in forming peripheral cycles are $P_{0,0}$, $P^*_{0,0}$ and $P_{1,0}$, with $P_{1,0}$ forming the peripheral cycle taken of multiplicity 1 or 2. The longest length in this path is $\tilde{m} + \tilde{k} < 2\tilde{m} - \frac{\tilde{k}-1}{2}$. Hence L_1 contains consecutive lengths $2\tilde{m} - \frac{\tilde{k}-3}{2}, 2\tilde{m} - \frac{\tilde{k}-5}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then let $\ell_q = 2\tilde{m} - \frac{\tilde{k}-5}{2}$. Then ℓ_q is odd, and since $\ell_k = \frac{m-1}{2}$ is even and k is odd, it follows that $q = k - \ell_k + \ell_q$ is even. To show that we can form central cycles using Parts 1 and 2 of Lemma 4.22, we need to show that $w + \frac{m-1}{2} \geq 2\ell_q$. Since $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$, it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq 4\tilde{m} - \tilde{k} + 5,$$

or equivalently that

$$k + m + 5\tilde{k} \geq 8\tilde{m} + 13.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = 2\tilde{m} - \frac{\tilde{k}-3}{2}$, remarking that ℓ_q even implies q is even. We need to show that $w + \frac{m-1}{2} \geq 2\ell_q$, and since $w \geq \frac{k-3}{2} + \frac{3\tilde{k}+1}{2}$, it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq 4\tilde{m} - \tilde{k} + 3,$$

or equivalently that

$$k + m + 5\tilde{k} \geq 8\tilde{m} + 9.$$

Hence if we can show that $k+m+5\tilde{k} \geq 8\tilde{m}+13$, then we can form central cycles in either case.

Note that if $d \geq 9$, then this inequality is true since $k + m + 5\tilde{k} \geq 9\tilde{m} + 14\tilde{k} \geq 8\tilde{m} + 13$. If $d = 7$, the inequality becomes $12\tilde{k} \geq \tilde{m} + 13$, which is true since $3\tilde{k} \geq \tilde{m}$ and $9\tilde{k} \geq 13$. Finally, if $d = 5$, then the inequality becomes $10\tilde{k} \geq 3\tilde{m} + 13$, which holds if $\tilde{k} \geq 13$ since $10\tilde{k} = 9\tilde{k} + \tilde{k} > 3\tilde{m} + \tilde{k}$. Since $3|\tilde{k}$ and $\tilde{k} \equiv 3 \pmod 4$, the only other possibility is that $\tilde{k} = 3$. In this case, $2\tilde{k} < \tilde{m} < 3\tilde{k}$ implies that $\tilde{m} = 7$. Hence, since $d = 5$, we have that $k = 15$ and $m = 35$. Note that $t = 3(\frac{m-1}{2k}) = 17 \equiv 2 \pmod 3$, and so it suffices to show that $k + m + 5\tilde{k} \geq 8\tilde{m} + 9$, which is true since $k + m + 5\tilde{k} = 65 = 8\tilde{m} + 9$.

Finally, we note that by Inequality 4.3.4, $d \geq 2c - 1 = 5$, and so we are done with this case.

Case 2. $\tilde{k} \equiv 1 \pmod 4$ and $d \geq 3$

We begin by defining the paths to be used in the formation of peripheral cycles. Let

$$P_{0,0} = 0, 1, -1, 2, -2, \ldots, \frac{\tilde{k}-1}{4}, -\left(\frac{\tilde{k}-1}{4}\right), \frac{\tilde{k}+7}{4}, -\left(\frac{\tilde{k}+3}{4}\right), \ldots,$$
$$-\left(\frac{\tilde{k}-3}{2}\right), \frac{\tilde{k}+1}{2}, -\left(\frac{\tilde{k}+1}{2}\right).$$

Note that $P_{0,0}$ contains edges of lengths $1, 2, \ldots, \frac{\tilde{k}-1}{2}, \frac{\tilde{k}+3}{2}, \frac{\tilde{k}+5}{2}, \ldots, \tilde{k}-1, \tilde{k}+1$.

If $\tilde{m} - \frac{\tilde{k}+1}{2} > 2\tilde{k} - 1$, let

$$P^*_{0,0} = 0, \tilde{k}, -2, \tilde{k}+1, -3, \ldots, \frac{3\tilde{k}-3}{2}, -\left(\frac{\tilde{k}+1}{2}\right).$$

In this case, the edge lengths which occur in $P^*_{0,0}$ are $\tilde{k}$, $\tilde{k}+2$, $\tilde{k}+3$, $\tilde{k}+4$, …, $2\tilde{k}-1$. If $\tilde{m} - \frac{\tilde{k}+1}{2} \leq 2\tilde{k} - 1$, then let

$$P^*_{0,0} = 0, \tilde{k}, -2, \tilde{k}+1, -3, \ldots, \frac{\tilde{m}}{2} + \frac{\tilde{k}-7}{4}, -\left(\frac{\tilde{m}}{2} - \frac{3\tilde{k}-1}{4}\right), \frac{\tilde{m}}{2} + \frac{\tilde{k}+1}{4},$$
$$-\left(\frac{\tilde{m}}{2} - \frac{3\tilde{k}-5}{4}\right), \frac{\tilde{m}}{2} + \frac{\tilde{k}+5}{4}, \ldots, \frac{3\tilde{k}-1}{2}, -\left(\frac{\tilde{k}+1}{2}\right).$$

In this case, the edge lengths which occur in $P^*_{0,0}$ are $\tilde{k}$, $\tilde{k}+2$, $\tilde{k}+3$, …, $\tilde{m} - \frac{\tilde{k}+3}{2}$, $\tilde{m} - \frac{\tilde{k}-1}{2}$, $\tilde{m} - \frac{\tilde{k}-3}{2}$, …, $2\tilde{k}$. In either case, the largest edge length in $P^*_{0,0}$ is strictly less than $\tilde{m}$. (In particular, this means that $P_{0,0}$ and $P^*_{0,0}$ are vertex-disjoint, respectively, with $\rho^{i\tilde{m}}(P_{0,0})$ and $\rho^{i\tilde{m}}(P^*_{0,0})$.)

For $i \geq 0$, let paths $P_{i,0}$ and $P^*_{i,0}$ be formed from $P_{0,0}$ and $P^*_{0,0}$, respectively, by adding $i\tilde{m}$ to the even-positioned vertices. For $0 \leq j \leq d-1$, let $P_{i,j} = \rho^{j\tilde{m}}(P_{i,0})$ and $P^*_{i,j} = \rho^{j\tilde{m}}(P^*_{i,0})$.

We now specify the construction of peripheral cycles according to the parity of c.

Subcase 2.1. c is even.

Unless otherwise specified as we progress through the possible cases, the c peripheral cycles will be formed from paths $P_{i,0}$ and $P^*_{i,0}$, $0 \leq i \leq \frac{c-2}{2}$. Note that the longest edge length in any of these paths is at most $2\tilde{k} + (\frac{c-2}{2}\tilde{m})$. Linking lengths will be $\tilde{m} \pm \frac{\tilde{k}+1}{2}, 2\tilde{m} \pm \frac{\tilde{k}+1}{2}, \ldots, \frac{c}{2}\tilde{m} \pm \frac{\tilde{k}+1}{2}$. Note that the largest length in a peripheral cycle is the linking length $\frac{c}{2}\tilde{m} + \frac{\tilde{k}+1}{2} \leq \frac{d+1}{4}\tilde{m} + \frac{\tilde{k}+1}{2} \leq \frac{d-1}{2}\tilde{m} + \frac{\tilde{k}+1}{2} = \frac{m}{2} - \frac{\tilde{m}}{2} + \frac{\tilde{k}+1}{2} = \frac{m-1}{2} - \frac{\tilde{m}}{2} + \frac{\tilde{k}+2}{2} < \frac{m-1}{2}$ since $\tilde{m} > 2\tilde{k} > \tilde{k}+2$.

If $c \equiv 0 \pmod 4$, the linking lengths arise from auxiliary circulants

$$X(d;\{1\}), X(d;\{2\}), X(d;\{3,4\}), X(d;\{5,6\}), \ldots, X\left(d;\left\{\frac{c-2}{2}, \frac{c}{2}\right\}\right).$$

If $c \equiv 2 \pmod 4$, the linking lengths arise from auxiliary circulants

$$X(d;\{1\}), X(d;\{2,3\}), \ldots, X\left(d;\left\{\frac{c-2}{2}, \frac{c}{2}\right\}\right).$$

In either case, we use linking edges of length $\tilde{m} + \frac{\tilde{k}+1}{2}$ from the auxiliary circulant $X(d;\{1\})$ in the cycle taken of multiplicity 1 or 2.

Moreover, if $c > 2$, the path from which the peripheral cycle of multiplicity 2 arises may be taken as $P_{i,0}$ or $P^*_{i,0}$ for some $i > 0$. If $c = 2$, we let the cycle arising from $P^*_{0,0}$ be the one taken of multiplicity 1 or 2. Hence if $c > 2$, then $\ell_2 - \ell_1 \geq 2\tilde{k} - \frac{\tilde{k}+1}{2} = \frac{3\tilde{k}-1}{2}$, while if $c = 2$, then $\ell_2 - \ell_1 \geq \tilde{k} - \frac{\tilde{k}+1}{2} = \frac{\tilde{k}-1}{2}$.

We note that if $t \equiv 1 \pmod 3$, then $\frac{m-1}{2} = k + \frac{r-1}{2} = k + (c-1)\tilde{k} + \frac{1}{3}\tilde{k}$ is odd, while if $t \equiv 2 \pmod 3$, then $\frac{m-1}{2} = k + \frac{r-1}{2} = k + (c-1)\tilde{k} + \frac{2}{3}\tilde{k}$ is even.

Let us now proceed to the construction of central cycles. For this purpose, we need to consider the possible values of c modulo 4.

Subcase 2.1.1. $c \equiv 0 \pmod 4$

Let $c = 4\alpha$. Note that L_1 contains consecutive lengths $2\alpha\tilde{m} + \frac{\tilde{k}+3}{2}$, $2\alpha\tilde{m} + \frac{\tilde{k}+5}{2}$, $\ldots$, $\frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then we let $\ell_q = 2\alpha\tilde{m} + \frac{\tilde{k}+3}{2}$. Since ℓ_q is even, $\ell_k = \frac{m-1}{2}$ is odd, and k is odd, we have that $q = k - \frac{m-1}{2} + \ell_q$ must be even. To apply Parts 1 and 2 of Lemma 4.22, we must show that $w + k - q \geq \ell_q$, or equivalently $w + \frac{m-1}{2} \geq 2\ell_q = 4\alpha\tilde{m} + \tilde{k} + 3$. Moreover, $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + \frac{3\tilde{k}-1}{2}$, so we need only show that

$$\frac{k-3}{2} + \frac{3\tilde{k}-1}{2} + \frac{m-1}{2} \geq 4\alpha\tilde{m} + \tilde{k} + 3,$$

or equivalently that

$$k + m + \tilde{k} \geq 8\alpha\tilde{m} + 11.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = 2\alpha\tilde{m} + \frac{\tilde{k}+5}{2}$. Since ℓ_q is odd, $\ell_k = \frac{m-1}{2}$ is even, and k is odd, it follows that $q = k - \frac{m-1}{2} + \ell_q$ is even. Let w be as defined in the statement of Lemma 4.22. To apply Part 1 or 2 of Lemma 4.22, we would like to show that $\ell_q \le w + k - q$, or equivalently $w + \frac{m-1}{2} \ge 2\ell_q = 4\alpha\tilde{m} + \tilde{k} + 5$. Note that $w \ge \frac{k-3}{2} + \ell_2 - \ell_1 \ge \frac{k-1}{2} + \frac{3\tilde{k}-1}{2}$, and so it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}-1}{2} + \frac{m-1}{2} \ge 4\alpha\tilde{m} + \tilde{k} + 5,$$

or equivalently

$$k + m + \tilde{k} \ge 8\alpha\tilde{m} + 15.$$

Hence, if we can show that $k + m + \tilde{k} \ge 8\alpha\tilde{m} + 15$, then we are done in either case.

From Inequality 4.3.5, we have that $k + m > \frac{4}{3}(2c-1)\tilde{m} = \frac{4}{3}(8\alpha - 1)\tilde{m}$. Hence, since $\alpha \ge 1$, $\tilde{m} \ge 7$ and $\tilde{k} \ge 9$, we have that

$$k+m+\tilde{k} > \frac{4}{3}(8\alpha-1)\tilde{m}+\tilde{k} = 8\alpha\tilde{m}+\frac{4}{3}(2\alpha-1)\tilde{m}+\tilde{k} \ge 8\alpha\tilde{m}+\frac{4}{3}(7)+9 > 8\alpha\tilde{m}+15.$$

Subcase 2.1.2. $c \equiv 2 \pmod 4$

Let $c = 4\alpha+2$. Consecutive lengths in L_1 include $(2\alpha+1)\tilde{m}+\frac{\tilde{k}+3}{2}$, $(2\alpha+1)\tilde{m}+\frac{\tilde{k}+5}{2}$, $\ldots$, $\frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then let $\ell_q = (2\alpha+1)\tilde{m}+\frac{\tilde{k}+5}{2}$. Since ℓ_q is even, $\ell_k = \frac{m-1}{2}$ is odd and k is odd, it follows that q must be even. We need to show that $w + \frac{m-1}{2} \ge 2\ell_q$. First consider the case that $\alpha \ge 1$, so that $w \ge \frac{k-3}{2} + \ell_2 - \ell_1 \ge \frac{k-3}{2} + \frac{3\tilde{k}-1}{2}$. Hence we need only show that

$$\frac{k-3}{2} + \frac{3\tilde{k}-1}{2} + \frac{m-1}{2} \ge (4\alpha+2)\tilde{m} + \tilde{k} + 5,$$

or equivalently that

$$k + m + \tilde{k} \ge (8\alpha+4)\tilde{m} + 15.$$

If $\alpha = 0$, then we have $w \ge \frac{k-3}{2} + \ell_2 - \ell_1 \ge \frac{k-3}{2} + \frac{\tilde{k}-1}{2}$, so we wish to show that

$$\frac{k-3}{2} + \frac{\tilde{k}-1}{2} + \frac{m-1}{2} \ge 2\tilde{m} + \tilde{k} + 5,$$

or equivalently that

$$k + m \ge 4\tilde{m} + \tilde{k} + 15.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = (2\alpha+1)\tilde{m} + \frac{\tilde{k}+3}{2}$. Then, since ℓ_q is odd, $\ell_k = \frac{m-1}{2}$ is even and k is odd, it follows that $q = k - \frac{m-1}{2} + \ell_q$ is even. To apply Parts 1 and 2 of Lemma 4.22, we need to show that $w + \frac{m-1}{2} \geq 2\ell_q$. Note that if $\alpha \geq 1$, then $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + \frac{3\tilde{k}-1}{2}$. Thus, it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}-1}{2} + \frac{m-1}{2} \geq (4\alpha+2)\tilde{m} + \tilde{k} + 3,$$

or equivalently

$$k + m + \tilde{k} \geq (8\alpha+4)\tilde{m} + 11.$$

If $\alpha = 0$, then $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + \frac{\tilde{k}-1}{2}$, and so the inequality that we need to prove is

$$\frac{k-3}{2} + \frac{\tilde{k}-1}{2} + \frac{m-1}{2} \geq 2\tilde{m} + \tilde{k} + 3,$$

or equivalently

$$k + m \geq 4\tilde{m} + \tilde{k} + 11.$$

Hence, in either case, if we can show that $k + m + \tilde{k} \geq (8\alpha+4)\tilde{m} + 15$ when $\alpha \geq 1$, and $k + m \geq 4\tilde{m} + \tilde{k} + 15$ when $\alpha = 0$, then we will be done.

First suppose $\alpha \geq 1$. Then by Inequality 4.3.5, we have that $k+m > \frac{4}{3}(2c-1) = \frac{4}{3}(8\alpha+3)$, and so, since $\alpha \geq 1$ and $\tilde{m} \geq 7$, we have that

$$k + m + \tilde{k} > \frac{4}{3}(8\alpha+3)\tilde{m} + \tilde{k} = (8\alpha+4)\tilde{m} + \frac{8\alpha}{3}\tilde{m} + \tilde{k} \geq (8\alpha+4)\tilde{m} + 15.$$

Next consider the case in which $\alpha = 0$. Then if $d \geq 5$, then $k + m \geq 5\tilde{m} + 5\tilde{k} = (4\tilde{m} + \tilde{k}) + \tilde{m} + 4\tilde{k} \geq 4\tilde{m} + \tilde{k} + 15$, since $\tilde{m} \geq 7$ and $\tilde{k} \geq 9$. If $d = 3$, however, the inequality $k + m \geq 4\tilde{m} + \tilde{k} + 15$ is equivalent to $2\tilde{k} \geq \tilde{m} + 15$, which is false since $\tilde{m} > 2\tilde{k}$. Note that if $t \equiv 2 \pmod 3$, then $c = \lceil \frac{r-1}{2\tilde{k}} \rceil = \frac{r-1}{2\tilde{k}} + \frac{1}{3} = 2$ implies $\frac{r-1}{2\tilde{k}} = \frac{5}{3}$, so that $\frac{m-1}{2\tilde{k}} = \frac{r-1}{2\tilde{k}} + d = \frac{14}{3}$ and $m = \frac{28}{3}\tilde{k} + 1 > 9\tilde{k} = 3k$, contradicting that $2k < m < 3k$. So we may assume $t \equiv 1 \pmod 3$ in this case.

We would like to apply Part 3 of Lemma 4.22; however, lengths $\ell_{q-1} = \tilde{m} + \frac{\tilde{k}+3}{2}$ and $\ell_{q-2} = \tilde{m} + \frac{\tilde{k}-1}{2}$ are not consecutive. We thus define an alternative construction for peripheral cycles in this case. First, let us examine the values of the parameters. Since $d = 3$ and $c = \lceil \frac{r-1}{2\tilde{k}} \rceil = \frac{r-1}{2\tilde{k}} + \frac{2}{3} = 2$, we have that $\frac{r-1}{2\tilde{k}} = \frac{4}{3}$, so that $\frac{m-1}{2\tilde{k}} = \frac{r-1}{2\tilde{k}} + d = \frac{13}{3}$. Solving gives $m = \frac{26}{3}\tilde{k} + 1$, and so $\tilde{m} = \frac{m}{3} = \frac{26\tilde{k}+3}{9}$.

Let

$$P = 0, 2, -1, 3, -2, \ldots, \frac{\tilde{k}+1}{2}, -\left(\frac{\tilde{k}-1}{2}\right)$$

and

$$P' = 0, \tilde{k}+1, -1, \tilde{k}+2, -2, \ldots, \frac{3\tilde{k}-1}{2}, -\left(\frac{\tilde{k}-1}{2}\right).$$

Then each of P and P' is a path of length $\tilde{k} - 1$ with inital and terminal vertices 0 and $-(\frac{\tilde{k}-1}{2})$, respectively. The path P contains edges of lengths $2, 3, \ldots, \tilde{k}$, while P' contains edges of lengths $\tilde{k}+1, \tilde{k}+2, \ldots, 2\tilde{k}-1$. Note that as $\tilde{m} > 2\tilde{k}$, we have that $\tilde{m} - \frac{\tilde{k}-1}{2} > \frac{3\tilde{k}-1}{2}$, so P' and $\rho^{\tilde{m}}(P')$ are vertex disjoint. Also, it is easy to see that P and $\rho^{\tilde{m}}(P)$ are vertex disjoint. We would like to use linking edges of lengths $\tilde{m} \pm \frac{\tilde{k}-1}{2}$ derived from auxiliary circulant $X(d; \{1\})$. Note that $\tilde{m} - \frac{\tilde{k}-1}{2} = \frac{26\tilde{k}+3}{9} - \frac{\tilde{k}-1}{2} = \frac{43\tilde{k}+15}{18} > 2\tilde{k} - 1$, and so neither linking edge length occurs in paths P or P'. We form peripheral cycle C from paths $\rho^{i\tilde{m}}(P)$, $0 \le i \le d-1$, together with linking edges of length $\tilde{m} + \frac{\tilde{k}-1}{2}$, and peripheral cycle C' from paths $\rho^{i\tilde{m}}(P')$, $0 \le i \le d-1$, together with linking edges of length $\tilde{m} - \frac{\tilde{k}-1}{2}$. We use C of multiplicity 3 and C' of multiplicity 2. Since $\tilde{m} - \frac{\tilde{k}-1}{2} \ge \frac{3\tilde{k}-1}{2} > \tilde{k}+1$, it follows that the two smallest lengths in $L_1 \cup L_2$ are 1 and $\tilde{k}+1$, and so we have that $w \ge \frac{k-3}{2} + \ell_2 - \ell_1 \ge \frac{k-3}{2} + \tilde{k} + 1 - 1 = \frac{k-3}{2} + \tilde{k}$. We note that L_1 contains consecutive lengths $\tilde{m} + \frac{\tilde{k}+1}{2}, \ldots \frac{m-1}{2}$, and take $\ell_q = \tilde{m} + \frac{\tilde{k}+1}{2}$, remarking that ℓ_q is even, so that q must also be even, since $\ell_k = \frac{m-1}{2}$ is odd and k is odd. To show that $w + \frac{m-1}{2} \ge 2\ell_q$, it thus suffices to show that

$$\frac{k-3}{2} + \tilde{k} + \frac{m-1}{2} \ge 2\tilde{m} + \tilde{k} + 1.$$

or equivalently that $k + m \ge 4\tilde{m} + 6$.

If this inequality holds, then we can use Part 1 or 2 of Lemma 4.22 to form central cycles. However, if not, then we appeal to Part 3 of Lemma 4.22. Hence we must find an appropriate value of j such that for each central cycle, lengths $\ell_{q-2j}, \ell_{q-2j+1}, \ldots, \ell_{q-1}$ are consecutive. Note that each central cycle contains edges of consecutive lengths $\tilde{m} - \frac{\tilde{k}-3}{2}, \tilde{m} - \frac{\tilde{k}-5}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-3}{2}$. Hence, if we let $j = \frac{\tilde{k}-3}{2}$, then $\ell_{q-2j}, \ell_{q-2j+1}, \ldots, \ell_{q-1}$, i.e. $\tilde{m} - \frac{\tilde{k}-5}{2}, \tilde{m} - \frac{\tilde{k}-7}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-3}{2}$ are consecutive. To show that we can form central cycles using Part 3 of Lemma 4.22, we must prove that $w + \frac{m-1}{2} + 2j \geq 2\ell_q$, or $w + \frac{m-1}{2} + \tilde{k} - 3 \geq 2\tilde{m} + \tilde{k} + 1$. Note that $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 = \frac{k-3}{2} + \tilde{k} + 1 - 1 = \frac{k-3}{2} + \tilde{k}$. Hence it suffices to show that

$$\frac{k-3}{2} + \tilde{k} + \frac{m-1}{2} + \tilde{k} - 3 \geq 2\tilde{m} + \tilde{k} + 1,$$

or equivalently that

$$k + m + 2\tilde{k} \geq 4\tilde{m} + 12.$$

Since $d = 3$, this inequality simplifies to $5\tilde{k} \geq \tilde{m} + 12$, which is true since $\tilde{m} < 3\tilde{k}$ and $\tilde{k} \geq 9$ imply that $5\tilde{k} \geq \tilde{m} + 2\tilde{k} \geq \tilde{m} + 12$.

Subcase 2.2. c is odd.

In this case, the c peripheral cycles will be formed using paths $P_{i,0}$ for $0 \leq i \leq \frac{c-1}{2}$ and paths $P^*_{i,0}$ for $0 \leq i \leq \frac{c-3}{2}$, and linking lengths will be as follows. If $c \geq 5$, then note that $d \geq 2c - 1 = 9$. The linking lengths are $\tilde{m} \pm \frac{\tilde{k}+1}{2}$, $2\tilde{m} - \frac{\tilde{k}+1}{2}$, $3\tilde{m} \pm \frac{\tilde{k}+1}{2}$, $4\tilde{m} \pm \frac{\tilde{k}+1}{2}, \ldots, \frac{c+1}{2}\tilde{m} \pm \frac{\tilde{k}+1}{2}$. Note that, using the facts that $c \leq \frac{d+1}{2}$, $\tilde{m} > 2\tilde{k}$, $\tilde{m} \geq 7$ and $d \geq 7$, we have $\frac{c+1}{2}\tilde{m} + \frac{\tilde{k}+1}{2} < \frac{d+3}{4}\tilde{m} + \frac{\tilde{m}+3}{4} = \frac{d+4}{4}\tilde{m} + \frac{3}{4} \leq \frac{d-1}{2}\tilde{m} + \frac{3}{4} = \frac{m}{2} - \frac{\tilde{m}}{2} + \frac{3}{4} < \frac{m-1}{2}$. The auxiliary circulants used to link the paths are

$$X(d; \{1\}), X(d; \{2\}), X(d; \{3, 4\}), X(d; \{5, 6\}), \ldots, X\left(d; \left\{\frac{c-1}{2}, \frac{c+1}{2}\right\}\right)$$

if $c \equiv 3 \pmod 4$, and

$$X(d; \{1, 3\}), X(d; \{2\}), X(d; \{4, 5\}), X(d; \{6, 7\}), \ldots, X\left(d; \left\{\frac{c-1}{2}, \frac{c+1}{2}\right\}\right)$$

if $c \equiv 1 \pmod 4$. (Note that if $c = 5$, then the circulants are $X(d; \{1, 3\})$ and $X(d; \{2\})$.

If $c = 3$, then $d \geq 2c - 1 = 5$. We use linking lengths $\tilde{m} \pm \frac{\tilde{k}+1}{2}$ and $2\tilde{m} - \frac{\tilde{k}+1}{2}$ (from circulants $X(d; \{1\})$ and $X(d; \{2\})$. Note that $2\tilde{m} - \frac{\tilde{k}+1}{2} < 2\tilde{m} \leq \frac{d-1}{2}\tilde{m} < \frac{m-1}{2}$.

If $c = 1$, then the single peripheral cycle is formed from $P_{0,0}$ with linking length $\tilde{m} - \frac{\tilde{k}+1}{2}$ from auxiliary circulant $X(d; \{1\})$. Note that $\tilde{m} - \frac{\tilde{k}+1}{2} < \tilde{m} \leq \frac{m}{3} < \frac{m-1}{2}$. Also, the longest length in $P_{0,0}$ is $\tilde{k} + 1$, which is less than $\tilde{m} - \frac{\tilde{k}+1}{2}$.

In any case, we must use one of the edge lengths from an auxiliary circulant of degree 2 in the cycle which will be taken of multiplicity 1 or 2; if $c > 1$, we can easily ensure that this circulant is not $X(d; \{1\})$, so that length $\tilde{m} - \frac{\tilde{k}+1}{2}$ is used of multiplicity 3. Moreover, if $c > 1$, the path from which the peripheral cycle taken of multiplicity 1 or 2 arises may be taken as $P_{i,0}$ or $P_{i,0}^*$ for some $i > 0$.

Note that if $t \equiv 1 \pmod 3$, then $\frac{m-1}{2} = k + (c-1)\tilde{k} + \frac{\tilde{k}}{3}$ is even, while if $t \equiv 2 \pmod 3$, then $\frac{m-1}{2} = k + (c-1)\tilde{k} + \frac{2\tilde{k}}{3}$ is odd.

We now show how the central cycles may be formed depending on the congruence class of c modulo 4.

Subcase 2.2.1. $c \equiv 1 \pmod 4$

Let $c = 4\alpha + 1$. Let us first suppose that $\alpha \geq 1$. Then L_1 contains consecutive lengths $(2\alpha+1)\tilde{m} + \frac{\tilde{k}+3}{2}, (2\alpha+1)\tilde{m} + \frac{\tilde{k}+5}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then let $\ell_q = (2\alpha+1)\tilde{m} + \frac{\tilde{k}+3}{2}$. Note that ℓ_q is odd, so that $q = k - \frac{m-1}{2} + \ell_q$ is even. Since $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + \frac{3\tilde{k}-1}{2}$, to show that $w + \frac{m-1}{2} \geq 2\ell_q$, it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}-1}{2} + \frac{m-1}{2} \geq (4\alpha+2)\tilde{m} + \tilde{k} + 3,$$

or equivalently that

$$k + m + \tilde{k} \geq (8\alpha+4)\tilde{m} + 11.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = (2\alpha+1)\tilde{m} + \frac{\tilde{k}+5}{2}$. Then, since ℓ_q is even, $\ell_k = \frac{m-1}{2}$ is odd and k is odd, it follows that $q = k - \frac{m-1}{2} + \ell_q$ is even. To apply Parts 1 and 2 of Lemma 4.22, we need to show that $w + \frac{m-1}{2} \geq 2\ell_q$. Since $\alpha \geq 1$, we have $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + \frac{3\tilde{k}-1}{2}$. Hence it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}-1}{2} + \frac{m-1}{2} \geq (4\alpha+2)\tilde{m} + \tilde{k} + 5,$$

or equivalently that

$$k + m + \tilde{k} \geq (8\alpha+4)\tilde{m} + 15.$$

If we can show that $k + m + \tilde{k} \geq (8\alpha+4)\tilde{m} + 15$, then by Parts 1 and 2 of Lemma 4.22, we will be able to form central cycles in either case. By Inequality 4.3.5, $k + m > \frac{4}{3}(2c-1)\tilde{m} = \frac{4}{3}(8\alpha+1)\tilde{m}$, and so

$$k + m + \tilde{k} > \frac{4}{3}(8\alpha+1)\tilde{m} + \tilde{k} = (8\alpha+4)\tilde{m} + \frac{8}{3}(\alpha-1)\tilde{m} + \tilde{k}.$$

Since $\tilde{m} \geq 7$, it is easy to see that $k + m + \tilde{k} \geq (8\alpha+4)\tilde{m} + 15$ if $\alpha \geq 2$ or $\tilde{k} \geq 15$. If, however, $\alpha = 1$ and $\tilde{k} < 15$, that $3|\tilde{k}$ and $\tilde{k} \equiv 1 \pmod 4$ implies that $\tilde{k} = 9$. In this case, the inequality to show is

$$k + m + \tilde{k} \geq 12\tilde{m} + 15.$$

Note that by Inequality 4.3.4, $d \geq 2c-1 = 9$. If $d \geq 13$, then $k+m+\tilde{k} \geq 13\tilde{m}+14\tilde{k} \geq 12\tilde{m} + 15$. If $d = 11$, then the inequality $k + m + \tilde{k} \geq 12\tilde{m} + 15$ is equivalent to $12\tilde{k} \geq \tilde{m} + 15$, which is true since $12\tilde{k} = 3\tilde{k} + 9\tilde{k} > \tilde{m} + 9\tilde{k} > \tilde{m} + 15$. Finally, suppose that $d = 9$, so that $k = 81$. Note that $\tilde{k} = 9$ implies that $18 < \tilde{m} < 27$, or $19 \leq \tilde{m} \leq 25$. The inequality $k + m + \tilde{k} \geq 12\tilde{m} + 15$ is equivalent in this case to $81 + 9\tilde{m} + 9 \geq 12\tilde{m} + 15$, or $3\tilde{m} \leq 75$, which is true.

Now suppose that $\alpha = 0$. Then $c = 1$, and we use $P_{0,0}$ and linking length $\tilde{m} - \frac{\tilde{k}+1}{2}$ to form the single peripheral cycle, which is taken of multiplicity 1 or 2. The longest length used in a peripheral cycle is $\tilde{m} - \frac{\tilde{k}+1}{2}$, and so L_1 contains consecutive lengths $\tilde{m} - \frac{\tilde{k}-1}{2}, \tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then let $\ell_q = \tilde{m} - \frac{\tilde{k}-1}{2}$. Then ℓ_q is odd, so that $q = k - \frac{m-1}{2} + \ell_q$ is even. Considering the value w, we have no lengths which are used of multiplicity 3 in peripheral cycles, and so $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-1}{2}$. Thus, to show that $w + \frac{m-1}{2} \geq 2\ell_q$, it is enough to show that

$$\frac{k-1}{2} + \frac{m-1}{2} \geq 2\tilde{m} - \tilde{k} + 1,$$

or equivalently that

$$k + m + 2\tilde{k} \geq 4\tilde{m} + 4.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = \tilde{m} - \frac{\tilde{k}-3}{2}$. Since ℓ_q is even, $\ell_k = \frac{m-1}{2}$ is odd and k is odd, it follows that q must be even. To apply Parts 1 and 2 of Lemma 4.22, we wish to show that $w + \frac{m-1}{2} \geq 2\ell_q = 2\tilde{m} - \tilde{k} + 3$. Note that here, no edge has been used of multiplicity 3 in the peripheral cycles, and so $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-1}{2}$. Hence, we would like to show that

$$\frac{k-1}{2} + \frac{m-1}{2} \geq 2\tilde{m} - \tilde{k} + 3,$$

or equivalently that

$$k + m + 2\tilde{k} \geq 4\tilde{m} + 8.$$

Thus, if we can show that $k + m + 2\tilde{k} \geq 4\tilde{m} + 8$, then by Parts 1 and 2 of Lemma 4.22, we can form central cycles in either case. Note that if $d \geq 5$, then $k + m + 2\tilde{k} \geq 5\tilde{m} + 7\tilde{k} \geq 4\tilde{m} + 8$. If $d = 3$, then the inequality $k + m + 2\tilde{k} \geq 4\tilde{m} + 8$ is equivalent to $5\tilde{k} \geq \tilde{m} + 8$, which is true since $5\tilde{k} = 3\tilde{k} + 2\tilde{k} > \tilde{m} + 2\tilde{k} \geq \tilde{m} + 8$.

Subcase 2.2.2. $c \equiv 3 \pmod 4$

Let $c = 4\alpha + 3$. First, suppose that $\alpha \geq 1$. In this case, L_1 contains consecutive lengths $(2\alpha + 2)\tilde{m} + \frac{\tilde{k}+3}{2}, (2\alpha + 2)\tilde{m} + \frac{\tilde{k}+5}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then $\ell_q = (2\alpha+2)\tilde{m} + \frac{\tilde{k}+5}{2}$. Then, since ℓ_q is odd, $q = k - \frac{m-1}{2} + \ell_q$ is even. Since $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-1}{2} + \frac{3\tilde{k}-1}{2}$, in order to show that $w + \frac{m-1}{2} \geq 2\ell_q$, it is enough to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}-1}{2} + \frac{m-1}{2} \geq (4\alpha + 4)\tilde{m} + \tilde{k} + 5,$$

or equivalently that

$$k + m + \tilde{k} \geq (8\alpha + 8)\tilde{m} + 15.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = (2\alpha + 2)\tilde{m} + \frac{\tilde{k}+3}{2}$. Note that, since ℓ_q is even, $\ell_k = \frac{m-1}{2}$ is odd, and k is odd, it follows that $q = k - \frac{m-1}{2} + \ell_q$ is even. To apply Lemma 4.22, we must show that $w + \frac{m-1}{2} \geq 2\ell_q = (4\alpha + 4)\tilde{m} + \tilde{k} + 3$. As we have that $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + \frac{3\tilde{k}-1}{2}$, it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}+1}{2} + \frac{m-1}{2} \geq (4\alpha + 4)\tilde{m} + \tilde{k} + 3,$$

or equivalently that

$$k + m + \tilde{k} \geq (8\alpha + 8)\tilde{m} + 11.$$

Hence, if we show that $k + m + \tilde{k} \geq (8\alpha + 8)\tilde{m} + 15$, then by Parts 1 and 2 of Lemma 4.22, we can form central cycles in either case. By Inequality 4.3.5, $k + m > \frac{4}{3}(2c - 1)\tilde{m} = \frac{4}{3}(8\alpha + 5)\tilde{m}$, and so, since $\tilde{m} \geq 7$ and $\tilde{k} \geq 9$, we have that

$$k + m + \tilde{k} > \frac{4}{3}(8\alpha + 5)\tilde{m} + \tilde{k} = (8\alpha + 8)\tilde{m} + \frac{4}{3}(2\alpha - 1)\tilde{m} + \tilde{k} \geq (8\alpha + 8)\tilde{m} + 15.$$

Now suppose that $\alpha = 0$. In this case, $c = 3$, and the linking lengths used are $\tilde{m} \pm \frac{\tilde{k}+1}{2}, 2\tilde{m} - \frac{\tilde{k}+1}{2}$. The paths used in forming peripheral cycles are $P_{0,0}$, $P^*_{0,0}$ and $P_{1,0}$, with the longest edge length occurring in any of these paths $\tilde{m} + \tilde{k} + 1 < 2\tilde{m} - \frac{\tilde{k}+1}{2}$. Hence L_1 contains consecutive lengths $2\tilde{m} - \frac{\tilde{k}-1}{2}, 2\tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \frac{m-1}{2}$.

If $t \equiv 1 \pmod 3$, then let $\ell_q = 2\tilde{m} - \frac{\tilde{k}-3}{2}$. Note that since ℓ_q is odd, we have that $q = k - \frac{m-1}{2} + \ell_q$ is even. Also, $w \geq \frac{k-3}{2} + \ell_2 - \ell_1 \geq \frac{k-3}{2} + \frac{3\tilde{k}-1}{2}$, so to show that $w + \frac{m-1}{2} \geq 2\ell_q$, it is enough to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}-1}{2} + \frac{m-1}{2} \geq 4\tilde{m} - \tilde{k} + 3,$$

or equivalently that

$$k + m + 5\tilde{k} \geq 8\tilde{m} + 11.$$

If $t \equiv 2 \pmod 3$, then let $\ell_q = 2\tilde{m} - \frac{\tilde{k}-1}{2}$. Then ℓ_q is even, and since k and $\ell_k = \frac{m-1}{2}$ are odd, we conclude that $q = k - \frac{m-1}{2} + \ell_q$ is even. We need to show that $w + \frac{m-1}{2} \geq 2\ell_q = 4\tilde{m} - \tilde{k} + 1$. Since $w \geq \frac{k-3}{2} + \frac{3\tilde{k}-1}{2}$, it suffices to show that

$$\frac{k-3}{2} + \frac{3\tilde{k}-1}{2} + \frac{m-1}{2} \geq 4\tilde{m} - \tilde{k} + 1,$$

or equivalently, that

$$k + m + 5\tilde{k} \geq 8\tilde{m} + 7.$$

Hence by Part 1 or 2 of Lemma 4.22, we can form central cycles in either case if we can show that $k + m + 5\tilde{k} \geq 8\tilde{m} + 11$. Before we show this inequality to be true, we note that $d \geq 2c - 1 = 5$. Now, if $d \geq 9$, then $k + m + 5\tilde{k} \geq 9\tilde{m} + 14\tilde{k} \geq 8\tilde{m} + 11$. If $d = 7$, then the inequality $k + m + 5\tilde{k} \geq 8\tilde{m} + 11$ is equivalent to $12\tilde{k} \geq \tilde{m} + 11$; this is true since $12\tilde{k} = 3\tilde{k} + 9\tilde{k} > \tilde{m} + 9\tilde{k} \geq \tilde{m} + 11$. If $d = 5$, then the inequality $k + m + 5\tilde{k} \geq 8\tilde{m} + 11$ is equivalent to $10\tilde{k} \geq 3\tilde{m} + 11$; this is true if $\tilde{k} > 9$, since in this case, $10\tilde{k} = 9\tilde{k} + \tilde{k} > 3\tilde{m} + \tilde{k} \geq 3\tilde{m} + 11$. We note, however, that if $\tilde{k} = 9$ and $t \equiv 2 \pmod 3$, then the inequality $k + m + 5\tilde{k} \geq 8\tilde{m} + 7$, which is equivalent to $10\tilde{k} \geq 3\tilde{m} + 7$ is true, and so peripheral cycles can be formed in this case. Finally, if $t \equiv 1 \pmod 3$, $d = 5$ and $\tilde{k} = 9$, then since $c = \lceil \frac{r-1}{2k} \rceil = \frac{r-1}{2k} + \frac{2}{3} = 3$, we have $\frac{r-1}{2k} = \frac{7}{3}$, so that $\frac{m-1}{2k} = \frac{r-1}{2k} + d = \frac{22}{3}$. Hence $m = \frac{44}{3}\tilde{k} + 1 = 133$, which is impossible since $d = 5 = \gcd(m, k)$.

We have now shown that $C_k | 3K_m$ whenever $k | 3\binom{m}{2}$, $3 \leq k \leq m < 3k$ and $k \neq 9$. By Theorem 4.13, this result implies that $C_k | 3K_m$ whenever $k | \binom{m}{2}$, $3 \leq k \leq m$ and $k \neq 9$. This completes the proof of Theorem 4.7.

Chapter 5

Cycle decompositions of $K_m * \overline{K_3}$

Recall that the obvious necessary conditions for the existence of a C_k-decomposition of the complete equipartite graph $K_m * \overline{K_3}$ with m parts of size 3 are:

1. $3 \leq k \leq 3m$
2. m is odd
3. $k|9\binom{m}{2}$.

In this chapter, we will show that these conditions are sufficient in certain cases, namely: $k|\binom{m}{2}$ and $k \leq m$; $k|3\binom{m}{2}$ but $k \nmid \binom{m}{2}$; $k \in \{2m, 3m\}$; and $k|9\binom{m}{2}$, $9|k$ and $k \neq 27$ is odd. Thus, only three cases remain to be proven: $k = 27$; $k \mid 9\binom{m}{2}$, $k \nmid 3\binom{m}{2}$ and k is even; and $k|\binom{m}{2}$ and $k > m$.

5.1 k-cycle decompositions of $K_m * \overline{K_3}$ when $k|\binom{m}{2}$ or $k|3\binom{m}{2}$

Lemma 5.1. *Let m be odd, $3 \leq k \leq m$ and $k|\binom{m}{2}$. Then $C_k|K_m * \overline{K_3}$.*

Proof. In this case, it is known [1, 45] that $C_k|K_m$. By Lemma 2.16, it follows that $C_k|K_m * \overline{K_3}$. $\square$

The following lemma proves that the necessary conditions are sufficient when $k|3\binom{m}{2}$ but $k \nmid \binom{m}{2}$.

Lemma 5.2. *Let m be odd, $3 \leq k \leq 3m$, $3|k$ and $k|3\binom{m}{2}$. Then $C_k|K_m * \overline{K_3}$.*

Proof. Necessary and sufficient conditions for decomposition of $K_m * \overline{K_n}$ were proven for $k = 3$ and $k = 6$ in [21] and [19], respectively.

Suppose, then, that $k \geq 9$. Since $k|3\binom{m}{2}$ and $3|k$, it follows that $\frac{k}{3}|\binom{m}{2}$. Hence by [1, 45], we have that $C_{k/3}|K_m$. By Lemma 2.17, it follows that $C_k|K_m * \overline{K_3}$. □

We note that the case in which $k|\binom{m}{2}$ and $k > m$ remains unresolved in general. However, we can show that the necessary conditions are sufficient in some special circumstances. We note that if $k = 3m$, then it is known that $C_{3m}|K_m * \overline{K_3}$; indeed, it is shown in [31] that in general $K_m * \overline{K_n}$ is Hamilton decomposable. The following lemma, which deals with the case $k = 2m$, was proved by Smith [47] in the case that m is prime.

Lemma 5.3. *Suppose $m \geq 3$ is odd and $2m|9\binom{m}{2}$. Then $C_{2m}|K_m * \overline{K_3}$.*

Proof. Note that $2|\binom{m}{2}$ and m is odd, so that $m \equiv 1 \pmod 4$. We view K_m as $K_{m-1} \bowtie K_1$, and let the vertex set of K_m be $\mathbb{Z}_{m-1} \cup \{\infty\}$, where ∞ is the vertex in K_1. Let ρ denote the cyclic permutation $(0, 1, \ldots, m-2)$, and let $C = 0, 1, -1, 2, -2, \ldots, \frac{m-3}{2}, -(\frac{m-3}{2}), \frac{m-1}{2}, \infty, 0$. Then $C, \rho(C), \ldots, \rho^{(m-3)/2}(C)$ are the cycles in Walecki's decomposition of K_m into m-cycles given in [35]. For each $i \in \{0, 1, \ldots, \frac{m-5}{4}\}$, let $T^{(i)}$ denote the closed trail $T^{(i)} = \rho^{2i}(C) \oplus \rho^{2i+1}(C)$. So $K_m = T^{(0)} \oplus T^{(1)} \oplus \cdots \oplus T^{(\frac{m-5}{4})}$. Note that each closed trail $T^{(i)}$ has $\Delta(T^{(i)}) = 4$ and by Brooks' Theorem [15], $\chi(T^{(i)}) \leq 4$ whenever $m \neq 5$. Hence by Lemma 2.23, since there exist two mutually orthogonal Latin squares of order 3, we have that $C_{2m}|T^{(i)} * \overline{K_3}$ if $m \neq 5$, and it follows that $C_{2m}|K_m * \overline{K_3}$. If $m = 5$, we note that a C_{10}-decomposition of $K_5 * \overline{K_3}$ was exhibited in [47]. □

From the results in this section, we see that the obvious necessary conditions for the existence of a C_k-decomposition of $K_m * \overline{K_n}$ are sufficient when $k|\binom{m}{2}$ and $k \leq m$, or when $k|3\binom{m}{2}$ and $3|k$. These results are fairly easy consequences of certain known decompositions and lifting results. This leaves under consideration the cases that $k|\binom{m}{2}$ and $k > m$, and $k|9\binom{m}{2}$ but $k \nmid 3\binom{m}{2}$.

The case in which $k|\binom{m}{2}$ and $k > m$ seems to be quite difficult, and remains open except in a few special cases, namely $k \in \{2m, 3m\}$. When $k|\binom{m}{2}$ and $k \leq m$, the fact that $C_k|K_m * \overline{K_n}$ followed easily from the existence of a C_k-decomposition of K_m. However, when $k > m$, no such decomposition exists. It is known that $T_k|K_m$ when m is odd, $k|\binom{m}{2}$ and $k > m$ (see Theorem 3.3); however, it is not immediately clear in many cases how to transform the closed trails of such a decomposition into cycles of length k in $K_m * \overline{K_3}$, as known methods for lifting closed trail decompositions to cycle decompositions of complete equipartite graphs (see Theorem 2.23 require the closed trails to have particular restrictive properties.

In the next section, we take a closer look at the case when $k|9\binom{m}{2}$ but $k \nmid 3\binom{m}{2}$.

5.2 k-cycle decompositions of $K_m * \overline{K_3}$ when k is odd, $k|9\binom{m}{2}$ and $9|k$

The aim in this section is to prove that the obvious necessary conditions for decomposing $K_m * \overline{K_3}$ into k-cycles are sufficient in the case that $k|9\binom{m}{2}$, $9|k$ and k is odd, with the possible exception of $k = 27$. We begin with the case $k = 9$.

Lemma 5.4. *Suppose $m \geq 3$ is odd. Then $C_9|K_m * \overline{K_3}$.*

Proof. If $m \equiv 1$ or $3 \pmod 6$, then it is well-known that $C_3|K_m$ [28], which implies $C_9|K_m * \overline{K_3}$ by Lemma 2.17. Otherwise, it is known that K_m decomposes as $C_3 \oplus C_3 \oplus \cdots \oplus C_3 \ominus C_4$ [46]. Form the graph G, illustrated in Figure 5.1, by taking the 4-cycle together with a 3-cycle from the decomposition containing one of its diagonals. So

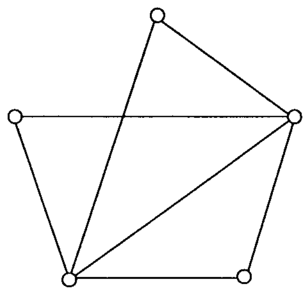

Figure 5.1: The graph G in the proof of Lemma 5.4.

$K_m = C_3 \oplus \cdots \oplus C_3 \oplus G$. The graph $G * \overline{K_3}$ decomposes into 9-cycles as illustrated in Figure 5.2. Note also that by the previous discussion with regard to the case $k = 3m$, $C_9 | K_3 * \overline{K_3}$, and so it follows that $C_9 | K_m * \overline{K_3}$. □

In the remainder of this section, we will prove the following lemma.

Lemma 5.5. *Let $k > 9$ be odd, $3|k$, and $k|3\binom{m}{2}$. Then $C_{3k}|K_m * \overline{K_3}$.*

The idea is as follows. We begin with the C_k-decomposition of $3K_m$ specified in Chapter 4. To each edge of $3K_m$ we assign a jump in $\{0, 1, -1\}$ so that three parallel edges receive three distinct jumps. Associate with each edge $\{x, y\}$ of $3K_m$ a matching in $K_m * \overline{K_3}$ with endvertices in $\{x, y\} \times \mathbb{Z}_3$. If we can label the edges of $3K_m$ in such a way that the total displacement along any cycle in the decomposition is nonzero (i.e. relatively prime to 3), then each k-cycle in $3K_m$ will correspond with a $3k$-cycle in $K_m * \overline{K_3}$; by Lemma 2.26, these cycles decompose $K_m * \overline{K_3}$.

Let us now proceed by walking through the various cases of decomposition of $3K_m$, showing that each k-cycle can be lifted into a $3k$-cycle in $K_m * \overline{K_3}$.

Suppose some cycle C occurs of multiplicity three in the decomposition of $3K_m$. Any such cycle can be easily labelled in the required way as follows. Consider the displacements listed in Table 5.1 for the three copies of C. Each edge may clearly be given appropriate jump (depending on its direction) to obtain this displacement.

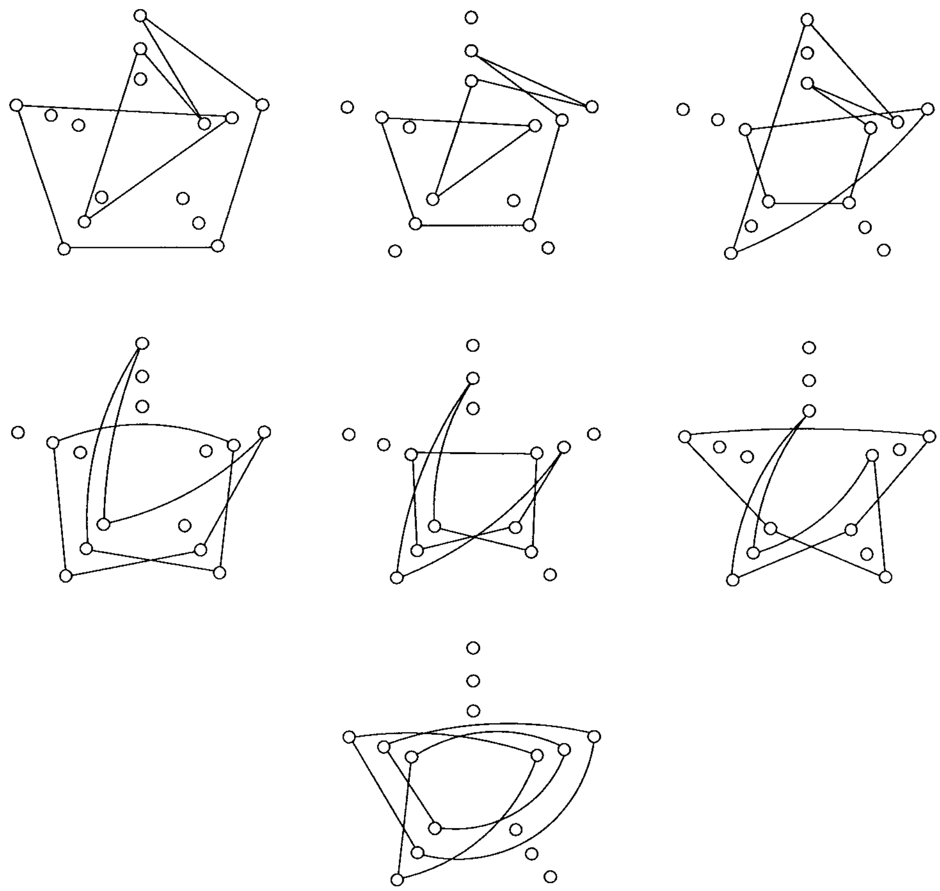

Figure 5.2: A C_9-decomposition of $G * \overline{K_3}$.

So we need only consider cycles which do not occur of multiplicity three in the decomposition of $3K_m$.

5.2.1 The induction step

Note that in the induction step, $3K_m$ is decomposed by taking a decomposition of $3K_v$ (where $k \leq v < 3k$) together with a number of k-cycles each taken of multiplicity three. Since any k-cycle of multiplicity three in $3K_m$ can be lifted into three $3k$-cycles in $K_m * \overline{K_3}$, we need only show that the cycles in a decomposition of $3K_v$ can be lifted,

Cycle	Edge						
	e_1	e_2	$\cdots$	e_{k-2}	e_{k-1}	e_k	Total displacement (mod 3)
Copy 1	0	0	$\cdots$	0	0	1	1
Copy 2	1	1	$\cdots$	1	1	-1	1
Copy 3	-1	-1	$\cdots$	-1	-1	0	1

Table 5.1: Displacements for a k-cycle occuring of multiplicity 3 in the decomposition, where $k \equiv 0 \pmod 3$.

where $k \leq v < 3k$. Note that we may also assume that $k \nmid \binom{v}{2}$, as in this case, each cycle in the decomposition of $3K_v$ is taken of multiplicity three, and so each cycle in the C_k-decomposition of $3K_v$ can be lifted to $3k$-cycles in $K_m * \overline{K_3}$. In particular, this means that we can assume $k < v < 3k$.

5.2.2 The case $k < m < 2k$

Recall that the C_k-decomposition of $3K_m$ from Chapter 4 was formed by viewing $3K_m$ as $3(K_{m-1} \bowtie K_1)$ and specifying the construction of starter central and peripheral cycles; the remaining cycles were formed by rotation of these starter cycles. Each starter peripheral cycle consists of $m' = \gcd(m-1, k)$ rotations of a zig-zag path P of length $k' - 1$, together with m' linking edges to join these paths into cycles. In total $\ell = (m-1)/m'$ cycles are formed by succesive rotation of each starter peripheral cycle under the permutation ρ. The three starter central cycles consist of a zig-zag path P', together with its diagonal opposite $\rho^{(m-1)/2}(P')$, an edge of length $\frac{m-1}{2}$ joining one end-vertex of P' with an end-vertex of $\rho^{(m-1)/2}(P')$, and two edges joining the remaining end-vertices of P' and $\rho^{(m-1)/2}(P')$ to the central vertex.

Note that it suffices to assign jump labels to the starter central and peripheral cycles; unless otherwise specified, each translate is given the same labelling as the starter which generates it. When $k < m < 2k$, there is exactly one peripheral cycle C taken of multiplicity 1 or 2; this cycle consists of m' rotations of a zig-zag path P of length $k' - 1$, each followed by an edge which links it to another rotation of

P. These m' linking edges are all of the same length and in any traversal of C are all traversed in the same direction (positive or negative). When $m' \not\equiv 0 \pmod 3$, we will give P and each of its rotations a labelling with displacement 0, and assign the m' linking edges the same jump chosen from ± 1; in this way, we obtain a total displacement $\pm m' \not\equiv 0 \pmod 3$. If $m' \equiv 0 \pmod 3$, this labelling will be adjusted to ensure the displacement is not 0. We then show how to assign the remaining jumps to the central cycles.

5.2.2.1. C occurs of multiplicity 2

Case 1. $m' \not\equiv 0 \pmod 3$

In the first copy of C, we give all edges jump 1. Since P is formed by zig-zagging so that consecutive edges of P are of opposite direction, and P contains an even number of edges, the displacement along P is 0. Adding the linking edge which follows P in the traversal of C, we obtain displacement ± 1. Giving each rotation of this configuration the same jumps, we obtain a displacement $\pm m' \not\equiv 0 \pmod 3$ in the cycle. In the second copy of C, we give all edges jump -1. Similarly, this yields a nonzero displacement in the cycle.

Now, each starter central cycle has the following form: two diagonally opposed paths together with an edge of diagonal length and two edges joining vertices 0 and $(m-1)/2$ to the central vertex. Note that any two edges of the same length in the diagonally opposed paths are traversed in opposite directions to each other. Hence, giving them the same label contributes 0 to the displacement along the cycle. We label the diagonally opposed paths as follows. Each edge which shares its length with an edge of C is given jump 0. The other edges of lengths in $\{1, 2, \ldots, (m-3)/2\}$ are all given label 0 in one starter central cycle, 1 in another starter central cycle, and -1 in the third. In this way, we obtain displacement 0 from the diagonally opposed paths.

Note that the two edges incident with the central vertex are traversed in opposite directions to each other (one towards the centre vertex, and one away from it). We will view the centre vertex as if it is labelled $(m-1)$, so that the direction of an edge traversed towards the centre vertex is positive and an edge traversed away from the centre vertex has negative direction. We give the remaining three edges labels indicated as follows.

If the diameter edge is traversed in the positive direction, the jumps are assigned as indicated in Table 5.2.

Cycle	Edge	Jump	Displacement
Cycle 1	Diameter	0	0
	Positive central	0	0
	Negative central	-1	1
Total			1
Cycle 2	Diameter	1	1
	Positive central	1	1
	Negative central	1	-1
Total			1
Cycle 3	Diameter	-1	-1
	Positive central	-1	-1
	Negative central	0	0
Total			1

Table 5.2: Jump assignments for diameter and central edges in central cycles when $k < m < 2k$ and the diameter is traversed in the positive direction.

If the diameter edge is traversed in the negative direction, give the central edges the same labelling as before, but replace jump j on the diameter edge by $-j$. This again gives each central cycle displacement 1.

Case 2. $m' \equiv 0 \pmod 3$

Give the two copies of C the jump labelling as in Case 1, except that we switch the labels on one edge between the two copies. (So one copy has all but one edge labelled 1, with one labelled -1, and the other copy has all but one edge labelled -1, with one edge, parallel to that in the first copy with label -1, labelled 1. Note that only one edge in each cycle is affected, not one edge in each rotation of P in the cycle.) This offsets the displacement of $\pm m'$ in each cycle by ± 1, thus giving a nonzero displacement. The central cycles are given jumps as in Case 1.

5.2.2.2. C occurs of multiplicity 1

Case 1. $m' \not\equiv 0 \pmod 3$

Give all edges of C jump 1. As in Case 1 of Section 5.2.2.1, this yields displacement ± 1. The central cycles are labelled similarly to those in Case 1 of Section 5.2.2.1, with diagonally opposed edges receiving the same label. Thus, if t is the length of an edge in C, then among the two central cycles (say, C' and C'') which contain edges of length C, the two edges of length t in one of C' and C'' will both be assigned jump 0, and the two edges of length t in the other will both be assigned jump -1. If $t < \frac{m-1}{2}$ is an edge length which does not occur in any peripheral cycle, then edges of length t will occur in all three central cycles; in this case, one cycle has both its edges of length t assigned jump 0, a second central cycle has both its edges of length t assigned jump 1, and the third central cycle has both its edges of length t assigned jump -1. The diameter edges (those of length $\frac{m-1}{2}$) and edges incident with the central vertex are labelled as in Case 1 of Section 5.2.2.1.

Case 2. $m' \equiv 0 \pmod 3$

Begin by labelling the cycles as in Case 1. As of this point, each central cycle has displacement 1, while C and its translates have displacement 0. Let e' be an edge in a central cycle C' such that e' is parallel to an edge e of C, e' has jump label 0, and e' is traversed in a positive direction. Such a central cycle C' may be guaranteed to exist as follows. Choose an edge e of C. There are two starter central cycles which contain edges of the same length as e. One translate of each of these two starter central cycles contains an edge parallel to e; let C' be the translate in which this edge, e', has jump 0. We can assume that e' is traversed in the positive direction, as otherwise we can reverse the direction of traversal of the central cycles and change the sign of the jumps on the edges of diameter length and those incident with the central vertex, so that e' is traversed in the positive direction and the central cycles currently have total displacement 1.

Switch the jumps on e and e'. Then C has displacement ± 1, depending on the direction of e, and C' has displacement $2 \equiv -1 \pmod 3$. Similarly switch labels on $\rho^i(e)$ and $\rho^i(e')$ for $1 \leq i \leq \ell - 1$. This gives each translate of C displacement ± 1 and each translate of C' either displacement 1 or -1, depending on whether it has been affected by this switching.

5.2.3 The case $2k < m < 3k$

When $2k < m < 3k$, no central vertex is used, so that edge lengths occur in $\mathbb{Z}_m$. We let $d = \gcd(m, k)$, $\tilde{m} = m/d$ and $\tilde{k} = k/d$. Recall that c denotes the number of distinct starter peripheral cycles in the decomposition. The decomposition is done differently in the cases that $d = 1$ and $d > 1$, and so we assign jumps to these cases separately.

Case d = 1.

Let us first suppose that $d = 1$. In this case, the starter cycles for the decomposition consist of a cycle C which contains three edges of each length in the set $\{1, 2, \ldots, \frac{k}{3} - 2, \frac{1}{2}(k-3), k-3\}$, together with a central cycle containing edges of the remaining lengths, which is taken of multiplicity 3. Let these cycles be denoted by C_1, C_2 and C_3.

In C, we have that for each $\ell \in \{1, 2, \ldots, \frac{k}{3} - 2\}$, the three edges of length ℓ are traversed in the same direction. Hence, no matter which edge is assigned which jump in $\mathbb{Z}_3$, the contribution of these edges to the total displacement along C is 0. It remains to label the edges of length $\frac{1}{2}(k-3)$ and $k-3$. There are two edges of length $\frac{1}{2}(k-3)$ traversed in the positive direction and one traversed in the negative direction. Give the two edges of length $\frac{1}{2}(k-3)$ traversed in the positive direction jump labels 0 and 1 in some order, and give the edge of length $\frac{1}{2}(k-3)$ traversed in the negative direction jump label -1. Hence the edges of length $\frac{1}{2}(k-3)$ contribute -1 to the total displacement along C. Finally, there are two edges of length $k-3$ traversed in the negative direction and one traversed in the positive direction. Assign the two edges of length $k-3$ traversed in the negative direction jumps 0 and -1, and the edge of length $k-3$ traversed in the positive direction jump 1. Hence the edges of length $k-3$ contribute -1 to the displacement along C. The total displacement along C is 1.

Let us now consider the cycles C_1, C_2 and C_3. As these cycles are parallel, we have already shown that they can be assigned jumps so that the displacement along each of these cycles is nonzero; see Table 5.1.

Case d $\geq$ 3.

We henceforth assume that $d \geq 3$. Let us now recall the method of construction of peripheral and central cycles in this case. A peripheral cycle is formed by specifying a zig-zag path of length $\tilde{k}-1$, and rotating it by multiples of $\tilde{m}$ vertices, to form d paths. These paths are joined by d linking edges of particular lengths. All except one peripheral cycle is taken with multiplicity 3 in the decomposition; it has already been demonstrated that these cycles taken of multiplicity 3 can be assigned jumps in the required way. The remaining cycle, C, occurs with multiplicity 1 or 2. In C, all linking edges are traversed in the same direction, and all have the same length.

We now recall the statement of Lemma 4.22, which dictates how the central cycles are formed.

Lemma 4.22. *Suppose $2k < m < 3k$. Let $L = \{\ell_1, \ell_2, \ldots, \ell_k\}$ be a set of k distinct edge lengths in K_m with $\ell_1 < \ell_2 < \cdots < \ell_k$. Suppose there exists an even integer q such that $\ell_q, \ell_{q+1}, \ldots, \ell_k$ are consecutive integers, and $\ell_k = \frac{m-1}{2}$. Let w be defined by*

$$-w = \ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{q-1} - \ell_{q+1} + \ell_{q+2} - \cdots - \ell_k = \sum_{i=1}^{q-1}(-1)^{i-1}\ell_i + \sum_{i=q+1}^{k}(-1)^i \ell_i.$$

Then:

1. *If* $\ell_q \leq w+1$, *then* $C_k|X(m;L)$.

2. *If* $w+1 < \ell_q \leq w+k-q$, *then* $C_k|X(m;L)$.

3. *If, for some* j, $\ell_{q-2j}, \ell_{q-2j+1}, \ldots, \ell_{q-1}$ *are consecutive and* $w+1 < w+k-q < \ell_q \leq w+k-q+2j$, *then* $C_k|X(m;L)$.

We define a *central cycle of type* i to be a central cycle formed by using Part i of Lemma 4.22. Recall that according to Lemma 4.22, a central cycle is formed by constructing a particular zig-zag path of length $k-1$ using edges of length in $\{\ell_1, \ell_2, \ldots, \ell_k\}$, but skipping an edge of a certain length ℓ_{q+J}, where J is even. The initial vertex of this path is 0, and its terminal vertex is $-\ell_{q+J}$. Adding the edge $\{-\ell_{q+J}, 0\}$ creates the central cycle. This edge will be called the cycle's *back-connecting edge.* If the cycle is of type 2 or 3, then the back-connecting edge has length ℓ_q, while a central cycle of type 1 has back-connecting edge of length $q+J$, where $J \geq 0$ is even.

Let us note some important properties of Lemma 4.22. First, this lemma is used, with the same value of ℓ_q, to form all three starter central cycles. However, different central cycles may contain edges of different length less than ℓ_q. This means that the value of w may vary with the starter central cycle. In particular, for fixed values of m and k, different central cycles may possibly be of different types. If two central cycles are of type 1, then since w varies, the value of J may also vary; in particular, two central cycles of type 1 may have their back-connecting edge of differing lengths.

If a central cycle is of type 1, then the zig-zag path of length $k-1$ which determines it is formed by traversing edges of increasing length. Edges of lengths $\ell_1, \ell_2, \ldots, \ell_{q+J-1}$ (where $J \geq 0$ is an even integer) are traversed with edges of odd-subscripted length in the positive direction and edges of even-subscripted length in the negative direction, while edges of lengths $\ell_{q+J+1}, \ell_{q+J+2}, \ldots, \ell_{k-1}$ are traversed with edges of even-subscripted length in the positive direction and edges of odd-subscripted length in the negative direction. The general form of a type 1 cycle is illustrated in Figure 5.3.

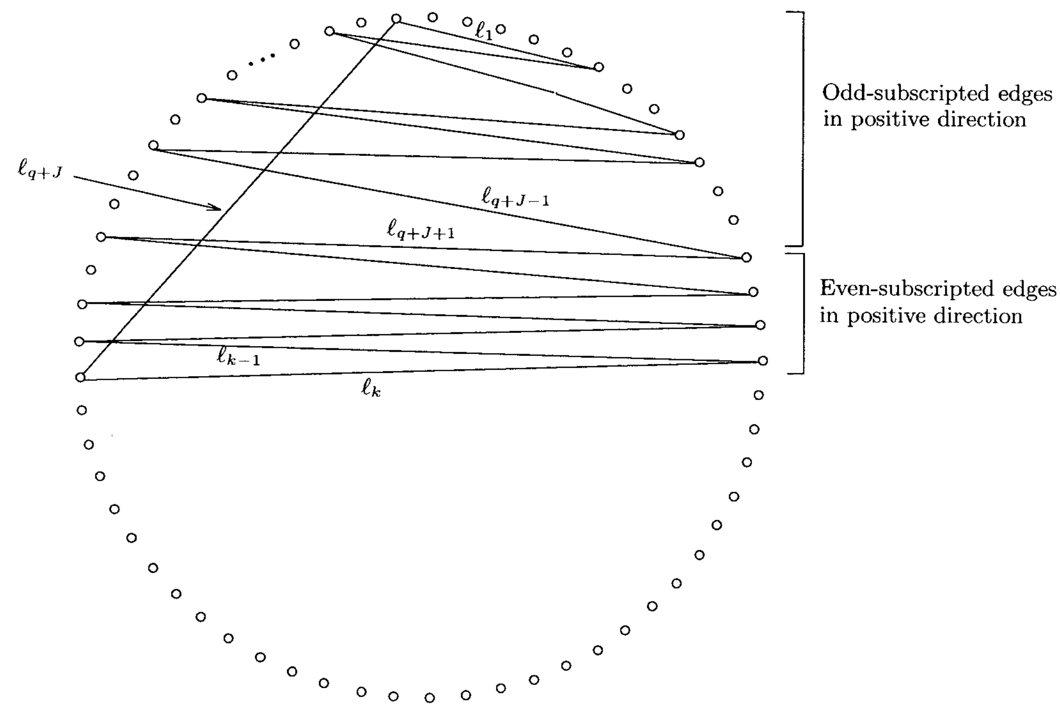

Figure 5.3: A type 1 central cycle

In a type 2 central cycle, the edges are traversed in the order ℓ_1, ℓ_2, $\ldots$, ℓ_{q-1}, ℓ_{q+1}, ℓ_{q+2}, $\ldots$, ℓ_{k-2i-1} for some i, followed by $\ell_k, \ell_{k-1}, \ldots, \ell_{k-2i}$; the point at which the reversal of order of edge length occurs varies according to w, and thus may be different among two different type 2 central cycles formed for the same values of m and k. However, the reversal always takes place after length ℓ_q has been skipped. Edges of lengths ℓ_1, ℓ_2, $\ldots$, ℓ_{q-1} are traversed with edges of odd-subscripted lengths in the positive direction and edges of even-subscripted lengths in the negative direction, while edges of lengths ℓ_{q+1}, ℓ_{q+2}, $\ldots$, ℓ_{k-2i-1} are traversed with edges of even-subscripted lengths in the positive direction and edges of odd-subscripted lengths in the negative direction. After the reversal of order, edges $\ell_k, \ell_{k-1}, \ell_{k-2}, \ldots, \ell_{k-2i}$ occur, with edges of odd-subscripted length (except possibly ℓ_k) in the positive direction and edges of even-subscripted lengths in the negative direction. The form of a type 2 central cycle is illustrated in Figure 5.4.

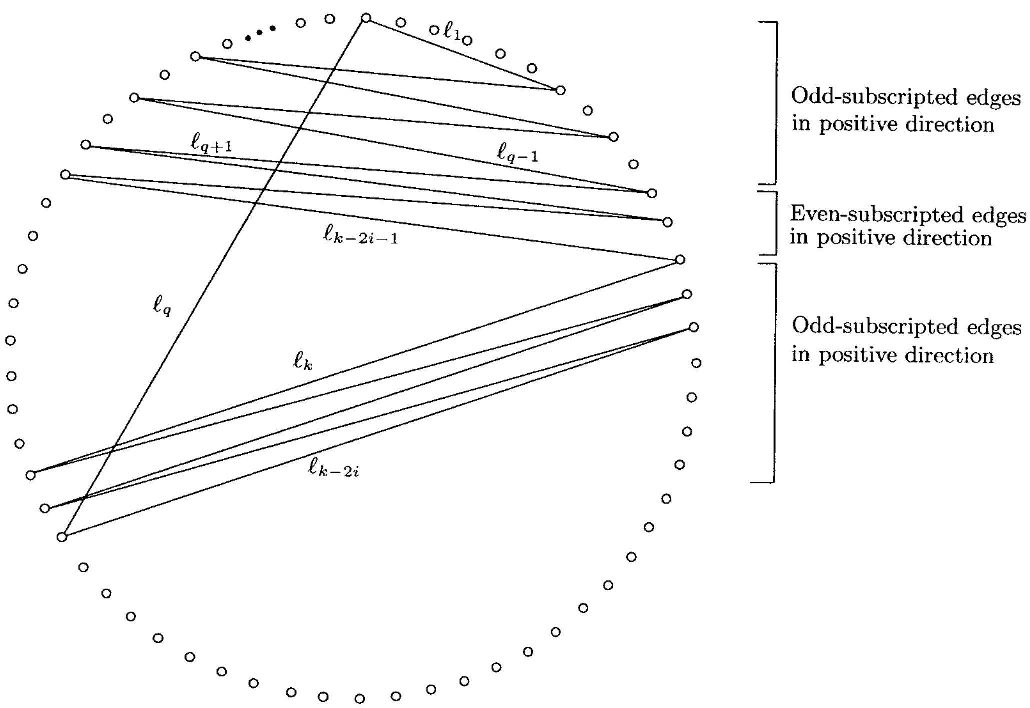

Figure 5.4: A type 2 central cycle

In a type 3 central cycle, the edges are traversed in order $\ell_1, \ell_2, \ldots, \ell_{q-2i-1}$ for some i, followed by $\ell_k, \ell_{k-1}, \ldots, \ell_{q+1}, \ell_{q-1}, \ell_{q-2}, \ldots, \ell_{q-2i}$. In this case, the reversal occurs before the skipping of length ℓ_q. The point of reversal still depends on the value of w, however, and may vary even among two type 3 central cycles formed for the same m and k. In the statement of Lemma 4.22, recall that to form a type 3 cycle, we needed to be guaranteed the existence of a positive integer j such that $\ell_{q-2j}, \ell_{q-2j-1}, \ldots, \ell_{q-1}$ are consecutive and $w+k-q < \ell_q \leq w+k-q+2j$; this integer j is constant among the three starter central cycles. The earliest point at which the reversal can occur is at the edge of length ℓ_{q-2j}. Edges of lengths $\ell_1, \ell_2, \ldots, \ell_{q-2i-1}$

are traversed with edges of odd-subscripted length in the positive direction and edges of even-subscripted length in the negative direction. After the reversal, edges of lengths $\ell_k, \ell_{k-1}, \ell_{k-2}, \dots, \ell_{q+1}$ are also traversed with edges of odd-subscripted length (except possibly ℓ_k) in the positive direction and edges of even-subscripted length in the negative direction, while edges of lengths $\ell_{q-1}, \ell_{q-2}, \dots, \ell_{q-2j}$ are traversed with edges of even-subscripted length in the positive direction and edges of odd-subscripted length in the negative direction. The form of a type 3 cycle is illustrated in Figure 5.5.

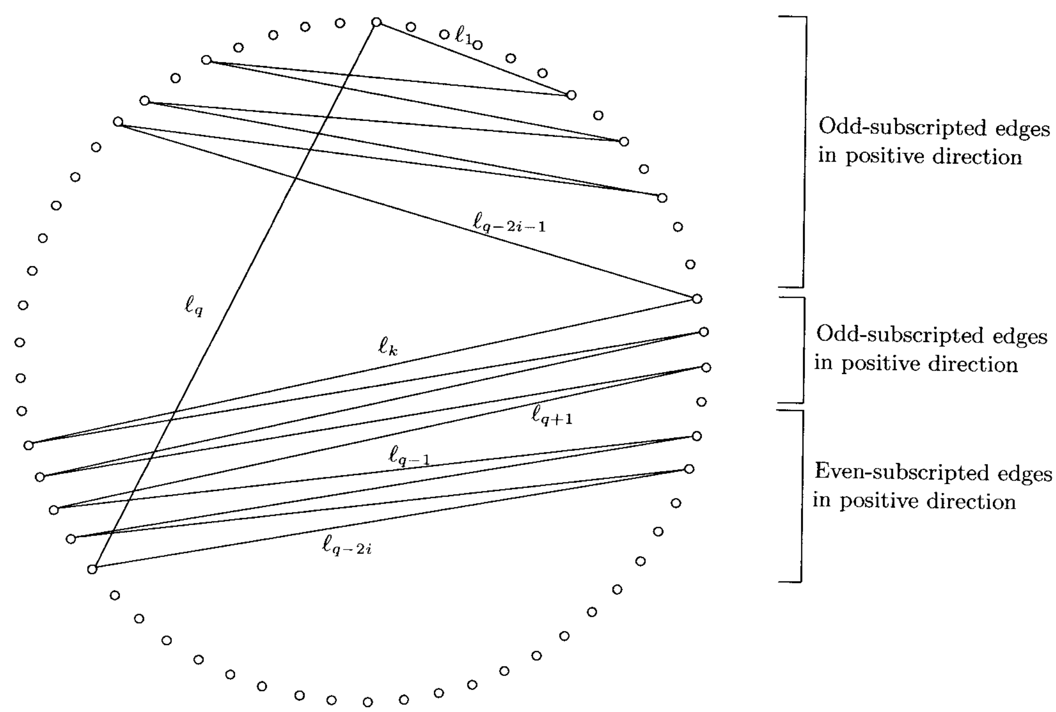

Figure 5.5: A type 3 central cycle

Finally, we note that for any type of central cycle, traversing it beginning at vertex 0, first along the zig-zag path and finally the back-connecting edge, the back-connecting edge is traversed in a positive direction. The direction of the edge of length $\ell_k = \frac{m-1}{2}$, however, may be traversed in a positive or negative direction regardless of the cycle type, and the direction may vary even among cycles of the same type.

5.2.3.1. C occurs of multiplicity 2

Case 1. $d \not\equiv 0 \pmod 3$

In one copy of C, give all edges jump 1, and in the other, all edges jump -1. Then by a similar argument to that in Case 1 of Section 5.2.2.1 (with m' replaced by d), each of these copies has total displacement ± 1.

Now we must consider the three starter central cycles. Each central cycle uses $\frac{\tilde{k}}{3}$ edges whose length is shared with an edge of C; these edges must receive jump 0. The remaining edges, which are even in number (there being $k - \frac{\tilde{k}}{3}$ of them) have lengths in L_1, with no length repeated in this central cycle. Pair the edges of length in $L_1 - \{\frac{m-3}{2}, \frac{m-1}{2}\}$ in any way. Consider a given pair ℓ and ℓ' of edge lengths in $L_1 - \{\frac{m-3}{2}, \frac{m-1}{2}\}$. Each of the central cycles contains an edge of length ℓ and an edge of length ℓ'. There must be two central cycles, say C_1 and C_2, such that C_1 contains an edge e^1_ℓ of length ℓ and an edge $e^1_{\ell'}$ of length ℓ', and C_2 contains an edge e^2_ℓ of length ℓ and an edge $e^2_{\ell'}$ of length ℓ' with one of the following properties:

1. e^1_ℓ is traversed in the same direction (positive or negative) as e^2_ℓ, and $e^1_{\ell'}$ is traversed in the same direction as $e^2_{\ell'}$.

2. e^1_ℓ is traversed in the opposite direction to e^2_ℓ, and $e^1_{\ell'}$ is traversed in the opposite direction to $e^2_{\ell'}$.

This observation may be verified in the following manner. Choose two of the central cycles, say $C^{(1)}$ and $C^{(2)}$. If these cycles have one of the desired properties, then we may take $C_1 = C^{(1)}$ and $C_2 = C^{(2)}$. Otherwise, it must be true that in one of the cycles (say, $C^{(1)}$), the edges of length ℓ and ℓ' are traversed in the same direction (both positive or both negative), and in the other (say, $C^{(2)}$), the edges of length ℓ and ℓ' are traversed with one in the positive direction and one in the negative direction. If the edges of length ℓ and ℓ' in the third cycle, $C^{(3)}$, are traversed in the same direction (either both positive or both negative), then we may take $C_1 = C^{(1)}$ and $C_2 = C^{(3)}$; otherwise, we take $C_1 = C^{(2)}$ and $C_2 = C^{(3)}$.

Now, assign the edges e_ℓ^1 and $e_{\ell'}^1$ jumps chosen from ± 1 so that together they contribute 0 to the displacement along C_1. Then edges e_ℓ^2 and $e_{\ell'}^2$ are given the negative of the jumps on e_ℓ^1 and $e_{\ell'}^1$, respectively; these two edges contribute 0 to the displacement along C_2. Finally, the edges of length ℓ and ℓ' in the third starter central cycle are given jump 0. Repeat this procedure for each pair of edge lengths, noting that for different pairs, the choice of which two cycles have the desired property for labelling them with jumps from ± 1 and which cycle is the "third" cycle may vary. In this way, the edges of length in $L_1 - \{\frac{m-3}{2}, \frac{m-1}{2}\}$ may be labelled so that they contribute 0 to the total jump along each starter central cycle.

It remains only to label the edges of lengths $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in each central cycle so that these give nonzero displacement to each cycle. We will represent the directions of traversal of the edges of length $\frac{m-3}{2}$ and $\frac{m-1}{2}$ as a ± 1-matrix, where the rows represent the edges in a particular cycle (C_1, C_2, or C_3, in order from top to bottom) and the columns represent the edges of a particular length ($\frac{m-3}{2}$ or $\frac{m-1}{2}$, in order from left to right). An entry 1 will signify that an edge is traversed in the positive direction, while -1 means that the edge is traversed in the negative direction. So, for example, the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

represents the situation in which the following occur: in C_1, the edge of lengths $\frac{m-3}{2}$ and $\frac{m-1}{2}$ are both traversed in the positive direction; in C_2, the edge of length $\frac{m-3}{2}$ is traversed in the positive direction and the edge of length $\frac{m-1}{2}$ is traversed in the negative direction; and in C_3, the edge of length $\frac{m-3}{2}$ is traversed in the negative direction and the edge of length $\frac{m-1}{2}$ is traversed in the positive direction.

Suppose we have an orientation of the edges of lengths $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in the central cycles, and let A be the matrix representing this orientation. Suppose also that for this orientation, we can assign jumps to the edges of lengths $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in the central cycles so that the total displacement along the edges of these lengths is nonzero in each central cycle. We note that the following operations on A produce a new possible orientation for which we can also assign jumps to the edges of lengths $\frac{m-3}{2}$ and $\frac{m-1}{2}$ with nonzero total displacement in any central cycle:

1. Permutation of the rows of A.

   Clearly, by permuting the jump assignments on edges of the same length between the cycles corresponding to the permuted rows, we obtain a jump assignment in the new orientation for which each cycle has nonzero displacement.

2. Multiplying a row by -1.

   In this case, we assign the same jump to each edge as in the jump assignment for the orientation corresponding to A. Then, if a row has been unaffected by this matrix operation, the displacement in the corresponding cycle is unchanged. In any row multiplied by -1, the displacement in the corresponding cycle will be multiplied by -1, and will remain nonzero.

Let us now consider four possible orientations, listed by their associated matrices A_1, A_2, A_3 and A_4 in Table 5.3. In this table, we also show how to assign jumps to the edges of lengths $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in the central cycles in the case that these edges are traversed in the directions indicated by the matrices A_1, A_2, A_3 and A_4. Each jump assignment results in a nonzero displacement.

Finally, note that any possible orientation of the edges of lengths $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in the three starter central cycles can be obtained from one of A_1, A_2, A_3 and A_4 by permutation of rows and multiplication of rows by -1. Thus, for any possible

Cycle	Edge direction matrix	Jumps	Displacement in cycle
C_1 C_2 C_3	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix}$	1 1 1
C_1 C_2 C_3	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}$	1 1 1
C_1 C_2 C_3	$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}$	1 −1 −1
C_1 C_2 C_3	$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 0 \end{bmatrix}$	−1 −1 −1

Table 5.3: Jump assignments for the edges of length $(m-3)/2$ and $(m-1)/2$, in the case that $2k < m < 3k$ and C occurs with multiplicity 2

orientation of the edges of these lengths in the central cycles, a jump assignment can be made so that the total displacement in each central cycle is nonzero.

Case 2. $d \equiv 0 \pmod 3$

We begin with the assignment of jumps given in the case that $d \not\equiv 0 \pmod 3$. In this case, each central cycle has nonzero displacement, but the two copies of the peripheral cycle C (which occurs with multiplicity 2) and their translates have displacement 0. We show how to modify the jump assignment so that each cycle has nonzero displacement.

Switch the labels on one edge between the two copies of C. So one copy has all but one edge of jump 1, and the remaining edge labelled -1, and the other copy has all but one edge labelled -1 with the remaining one labelled 1. Note that only one edge in each of these cycles are affected. This offsets the displacement of $\pm d$ in each cycle by ± 1, thus giving a nonzero displacement. The central cycles are given jumps as in Case 1.

5.2.3.2. C occurs of multiplicity 1

Case 1. $d \not\equiv 0 \pmod 3$

All peripheral cycles except C occur with multiplicity 3, and so may be assigned jumps in the appropriate manner. We may assign to each edge of C jump 1. Then an argument similar to that in Case 1 of Section 5.2.2.1 (with m' replaced with d) shows that the total displacement along C is ± 1. Note that the edges other than the linking edges contribute 0 to the total displacement along C, while the linking edges contribute ± 1. This jump assignment will be used in nearly all cases in which c is even, and in one case where c is odd. In the remaining cases, we will need to give C a slightly different jump assignment by replacing the jump on the linking edges with -1. As the non-linking edges contribute 0 to the displacement along C, changing the sign of the displacement along the linking edges does not change the fact that C has nonzero total displacement. This alternative jump assignment occurs in nearly all cases in which c is odd. Thus, in the cases that follow, we may assign to the edges of the central cycles which have the same length as the linking edges of C either jumps 0 and -1, or jumps 0 and 1.

The key to the simple labelling of central cycles in the case that $k < m < 2k$ was that each central cycle contained two edges of each non-diameter length. In this case, the central cycles do not have as nice a form, so we must label them according to case. Before we proceed to the case analysis, we first present several lemmas which will be used to assign jumps to edges of large length in the central cycles.

In general, we will first assign jumps to edges of length at most $\frac{m-7}{2}$ in each starter central cycle so that these edges contribute 0 to the total displacement along each cycle. Then we assign jumps to the edges of lengths $\frac{m-5}{2}$, $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in such a way that these edges make the displacement nonzero. The following lemma will be used to assign jumps to the edges of length at most $\frac{m-7}{2}$.

Lemma 5.6. *Let C_1 and C_2 be k-cycles in λK_m and assign to each a direction of traversal. Suppose there are an even number of edge lengths $\Lambda_1, \Lambda_2, \ldots, \Lambda_{2r}$ such that for each $i \in \{1, 2, \ldots, 2r\}$, there is exactly one edge of length Λ_i in each of C_1 and C_2. Moreover, suppose that one of the following conditions hold.*

1. *For each $i \in \{1, 2, \ldots, 2r\}$, the edge of length Λ_i in C_1 is traversed in the same direction (positive or negative) as the edge of length Λ_i in C_2.*

2. *For each $i \in \{1, 2, \ldots, 2r\}$, the edge of length Λ_i in C_1 is traversed in the opposite direction to the edge of length Λ_i in C_2.*

Then the edges of length in $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_{2r}\}$ in C_1 and C_2 may be assigned jumps chosen from ± 1 such that the following two conditions both hold.

1. *For each $i \in \{1, 2, \ldots, 2r\}$, the edges of length Λ_i in C_1 and C_2 are assigned different jumps.*

2. *The displacement along the edges of length in $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_{2r}\}$ is 0 in each of C_1 and C_2.*

Proof. Partition the lengths in $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_{2r}\}$ into pairs. Consider a fixed pair $\{\Lambda_i, \Lambda_{i'}\}$. In C_1, assign jumps chosen from ± 1 to the two edges of length Λ_i and $\Lambda_{i'}$ such that these edges contribute 0 to the total displacement along C_1. Suppose that the edge of length Λ_i in C_1 has been assigned jump j_i and the edge of length $\Lambda_{i'}$ in C_1 has been assigned jump $j_{i'}$. Assign to the edge of length Λ_i in C_2 jump $-j_i$ and to the edge of length $\Lambda_{i'}$ in C_2 jump $-j_{i'}$. Then the displacement along the edges of length Λ_i and $\Lambda_{i'}$ in C_2 is also 0. Repeating this procedure for each pair, we obtain a jump assignment for all edges of length in $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_{2r}\}$ such that the displacement along the edges of length in $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_{2r}\}$ is 0 in each of C_1 and C_2. Clearly by this procedure, these edges are labelled with jumps chosen from ± 1 such that two edges of the same length in different cycles receive different jumps. $\square$

Lemma 5.7. *Let $k \geq 15$ and m be odd integers such that $2k < m < 3k$, and $k \mid 3\binom{m}{2}$ but $k \nmid \binom{m}{2}$. Let $d = \gcd(m, k)$ and suppose $d > 1$. In the C_k-decomposition of $3K_m$ constructed in Chapter 4, let C_1, C_2 and C_3 be the three starter central cycles constructed by Lemma 4.22.*

Let $\ell_1, \ell_2, \ldots, \ell_k = \frac{m-1}{2}$ be the edge lengths in C_1, where $\ell_1 < \ell_2 < \cdots < \ell_k$; $\ell'_1, \ell'_2, \ldots, \ell'_k = \frac{m-1}{2}$ be the edge lengths in C_2, where $\ell'_1 < \ell'_2 < \cdots < \ell'_k$; and $\ell''_1, \ell''_2, \ldots, \ell''_k = \frac{m-1}{2}$ be the edge lengths in C_3, where $\ell''_1 < \ell''_2 < \cdots < \ell''_k$. Suppose there is an odd element $\mu \in \{1, 2, \ldots, k\}$ such that $\ell_i = \ell'_i = \ell''_i$ for all $i \geq \mu$, and such that $\mu < q$, where q is the even integer defined in the construction so that in each cycle, $\ell_q, \ell_{q+1}, \ldots, \ell_k = \frac{m-1}{2}$ are consecutive. Moreover, if one or more of the cycles C_1, C_2 and C_3 is type 3, suppose that $\mu < q - 2i$, where $q - 2i$ is the minimum value such that ℓ_{q-2i} is the point of reversal of order of traversal of edges in a type 3 cycle.

Then the edges in C_1, C_2 and C_3 of lengths $\ell_\mu, \ell_{\mu+1}, \ldots, \ell_k$ can be assigned jumps in $\mathbb{Z}_3$ such that the following properties hold:

1. *For each $\alpha \in \{\mu, \mu + 1, \ldots, k\}$, the edges of length ℓ_α are assigned different jumps in each of C_1, C_2 and C_3.*
2. *In each of C_1, C_2 and C_3, the displacement along the trail induced by the edges of lengths $\ell_\mu, \ell_{\mu+1}, \ldots, \ell_k$ is nonzero.*

Proof. We first show that in each central cycle, the edge which links back to vertex 0 has length at most $\ell_{k-3} = \frac{m-7}{2}$. First, suppose that a central cycle is of type 1. Then the length of the back-connecting edge is ℓ_{q+J}, where $q + J$ is even and $\ell_{q+J} \in \{-w, -(w+1)\}$, where $-w = \ell_1 - \ell_2 + \ell_3 - \cdots + \ell_{k-2} - \ell_k$. It was shown in the proof of Lemma 4.22, Part 1, that $q + J \leq k - 3$. Since the sequence $\ell_1, \ell_2, \ldots, \ell_k$ is strictly increasing, it follows that $\ell_{q+J} \leq \ell_{k-3}$. If a central cycle is of type 2 or 3, then the fact that $w + 1 < w + k - q$ implies that $q < k - 1$, so that $q \leq k - 3$. Hence $\ell_q \leq \ell_{k-3} = \frac{m-7}{2}$.

Let us now demonstrate the assignment of jumps to the edges of lengths $\ell_\mu, \ell_{\mu+1}, \ldots, \ell_k$. We first show that we can assign a jump to each edge of length $\ell_\mu, \ell_{\mu+1}, \ldots, \ell_{k-3}$ such that the displacement over these edges is 0. First, suppose that the edges of each length in $\{\ell_\mu, \ell_{\mu+1}, \ldots, \ell_{k-3}\}$ are traversed in the same direction in two of the cycles C_1, C_2 and C_3, or that the edges of each length in $\{\ell_\mu, \ell_{\mu+1}, \ldots, \ell_{k-3}\}$ are traversed in opposite directions in two of the cycles. Without loss of generality, suppose these cycles are C_1 and C_2. Note that the number of lengths in $\{\ell_\mu, \ell_{\mu+1}, \ldots, \ell_{k-3}\}$ is even. The existence of the required jump assignment from ± 1 to the edges of these lengths in C_1 and C_2 is guaranteed by Lemma 5.6, while the edges of lengths $\ell_\mu, \ldots, \ell_{k-3}$ in C_3 are all assigned jump 0.

Now let us suppose that no two cycles amongst C_1, C_2 and C_3 have all edges of length in $\{\ell_\mu, \ell_{\mu+1}, \ldots, \ell_{k-3}\}$ traversed in the same directions between the cycles. We note that either all three cycles have the same type, two cycles have the same type, or all three cycles are of different type. Hence we can assume that either two cycles are of type i for some $i \in \{1, 2, 3\}$, or else there is one cycle of type 2 and another of type 3. We will consider these cases separately.

Case A. Two of C_1, C_2 and C_3 are of type 1.

We may assume that C_1 and C_2 are of type 1. Let ℓ_i (in C_1) and $\ell_{i'} = \ell'_{i'}$ (in C_2) be the lengths of the edges which link back to vertex 0. Note that i and i' are both even. Also, if $i = i'$, then all edges in C_1 and C_2 of lengths at least $\ell_\mu = \ell'_\mu$ are traversed in the same direction in C_1 and C_2, so we may assume $i \neq i'$. Without loss of generality we may assume that $i < i'$, so that $\ell_i < \ell_{i'}$. Then the following hold:

1. For each $\alpha \in \{\mu, \mu + 1, \ldots, i - 1\}$, the edge of length ℓ_α in C_1 is traversed in the

same direction as the edge of length ℓ_α in C_2. The number of such elements α is odd.

2. The edge of length ℓ_i in C_1 (the back-connecting edge) is traversed in the positive direction, and the edge of length ℓ_i in C_2 is traversed in the negative direction.

3. For each $\alpha \in \{i+1, i+2, \ldots, i'-1\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such elements α is odd.

4. The edge of length $\ell_{i'}$ is traversed in the positive direction in both C_1 and C_2.

5. For each $\alpha \in \{i'+1, i'+2, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2. The number of such α is even.

Hence there are an even number of edge lengths whose corresponding edges are traversed in opposite directions in C_1 and C_2. The edges of these lengths may be assigned jumps from ± 1 by Lemma 5.6. Similarly, the number of even lengths whose corresponding edges are traversed in the same direction in C_1 and C_2 is even; the edges of these lengths may also be assigned jumps from ± 1 by Lemma 5.6. Assign a jump of 0 to each edge of length in $\{\ell_\mu, \ldots, \ell_{k-3}\}$ in C_3.

Case B. Two of C_1, C_2 and C_3 are of type 2.

We may assume that C_1 and C_2 are of type 2. The back-connecting edge in C_1 and C_2 is of length ℓ_q. Let us recall how the cycles C_1 and C_2 are formed. They each begin by zigzagging edges of length $\ell_1, \ell_2, \ldots, \ell_{q-1}$ such that edges of length with odd subscript are traversed in the positive direction and edges of length with even subscript are traversed in the negative direction. The cycle C_1 is then continued with edges of length $\ell_{q+1}, \ell_{q+2}, \ldots, \ell_{k-2i-1}$ for some i, with edges of length with odd subscript traversed in the negative direction and edges of length with even subscript traversed in the positive direction. These are followed by edges of length $\ell_k, \ell_{k-1}, \ldots, \ell_{k-2i}$, with edges with length of odd subscript (except possibly ℓ_k) traversed in the positive direction and edges of length with even subscript traversed in the negative direction. Finally, an edge of length ℓ_q is traversed in the positive direction. The cycle C_2 is completed in a similar manner, except that the reversed edges begin at $\ell_{k-2i'}$ for some value of i' (not necessarily equal to i).

Assume without loss of generality that $i \geq i'$, so that $k - 2i \leq k - 2i'$. We note the following:

1. For each $\alpha \in \{\mu, \mu+1, \ldots, k-2i-1\} \cup \{k-2i', k-2i'+1, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2. The number of such α is even. In particular, the back-connecting edges of length ℓ_q are traversed in the positive direction in both C_1 and C_2.

2. For each $\alpha \in \{k-2i, k-2i+1, \ldots, k-2i'-1\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such α is even.

Hence there are an even number of edge lengths in $\{\ell_\mu, \ell_{\mu+1}, \ldots, \ell_{k-3}\}$ for which the edges of these lengths are traversed in opposite directions in C_1 and C_2, and an even number of lengths for which the edges are traversed in the same direction in the two cycles. We may assign the jumps from ± 1 to these edges by Lemma 5.6 such that the total displacement along the edges of lengths $\ell_\mu, \ldots, \ell_{k-3}$ is 0 in each of C_1 and C_2. We assign to each edge of C_3 of length in $\{\ell_\mu, \ldots, \ell_{k-3}\}$ jump 0.

Case C. Two of C_1, C_2 and C_3 are of type 3.

Without loss of generality, assume cycles C_1 and C_2 are of type 3. Both of these cycles use an edge of length ℓ_q as a back-connecting edge. In type 3 cycles, the reversal of edges begins on an edge of length ℓ_α for some even α, but occurs before the skipping of the edge of length ℓ_q. Let ℓ_{q-2i} be the smallest length of a reversed edge in C_1 and $\ell_{q-2i'}$ the smallest length of a reversed edge in C_2. We may assume that $i \geq i'$, so that $q-2i \leq q-2i'$. We note the following:

1. For each $\alpha \in \{\mu, \mu+1, \ldots, q-2i-1\} \cup \{q-2i', q-2i'+1, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2. The number of such j is even. Note that ℓ_q, as the length of the back-connecting edges, has its lengths traversed in the positive direction in both cycles.

2. For each $\alpha \in \{q-2i, q-2i+1, \ldots, q-2i'-1\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such α is even.

Hence there are an even number of edge lengths in $\{\ell_\mu, \ell_{\mu+1}, \ldots, \ell_{k-3}\}$ for which edges of these lengths are traversed in opposite directions in C_1 and C_2, and an even number of edge lengths for which edges of these lengths are traversed in the same direction in each of C_1 and C_2. The edges of these lengths may be assigned jumps from ± 1 using Lemma 5.6 such that they contribute 0 to the displacement along each of C_1 and C_2. The edges of C_3 of length in $\{\ell_\mu, \ldots, \ell_{k-3}\}$ are all assigned jump 0.

Case D. There is a cycle of type 2 and a cycle of type 3.

Without loss of generality, we may assume that C_1 is of type 2 and C_2 is of type 3. In both C_1 and C_2, ℓ_q is skipped as a length in the zig-zag forming the cycle and used as a back-connecting edge. In C_1, lengths $\ell_{k-2i}, \ell_{k-2i+1}, \ldots, \ell_k$ are traversed in reverse order, for some i such that $k-2i > q$, and in C_2, lengths $\ell_{q-2i'}, \ell_{q-2i'+1}, \ldots, \ell_{q-1}, \ell_{q+1}, \ell_{q+2}, \ldots, \ell_k$ are reversed for some i'. We note the following.

1. For each $\alpha \in \{\mu, \mu+1, \ldots, q-2i'-1\} \cup \{q\} \cup \{k-2i, k-2i+1, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length

ℓ_α in C_2. In particular, the back-linking edges of length ℓ_q are both traversed in the positive direction. The number of such α is even.

2. For each $\alpha \in \{q-2i', q-2i'+1, \ldots, q-1\} \cup \{q+1, q+2, \ldots, k-2i-1\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction of the edge of length ℓ_α in C_2. The number of such α is even.

Hence the number of lengths in $\{\ell_\mu, \ell_{\mu+1}, \ldots, \ell_{k-3}\}$ whose edges are traversed in opposite directions in C_1 and C_2 is even, and the number of lengths whose edges are traversed in the same direction in each of the two cycles is also even. The edges of length in $\{\ell_\mu, \ell_{\mu+1}, \ldots, \ell_{k-3}\}$ may be assigned jumps from ± 1 using Lemma 5.6 so that they contribute 0 to the displacement along each of C_1 and C_2. The edges of C_3 of length in $\{\ell_\mu, \ldots, \ell_{k-3}\}$ are all assigned jump 0.

We have now completed the jump assignments to the edges of length at most $\frac{m-7}{2}$ so that these edges contribute 0 to the total displacement along each cycle. It only remains to assign a jump to the edges of lengths $\frac{m-5}{2}$, $\frac{m-3}{2}$ and $\frac{m-1}{2}$; in each central cycle, the displacement along these three edges will be the total displacement along the cycle. Note that as a consequence of the construction of central cycles, edges of lengths $\frac{m-5}{2}$ and $\frac{m-3}{2}$ are traversed in opposite directions.

Similarly to when we labelled the edges of length $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in Case 1 of Section 5.2.3.1, we will represent the directions of traversal of the edges of length $\frac{m-5}{2}$, $\frac{m-3}{2}$ and $\frac{m-1}{2}$ as a ± 1-matrix. Here, the rows represent the edges in a particular cycle (C_1, C_2, or C_3, in order from top to bottom) and the columns represent the edges of a particular length ($\frac{m-5}{2}$, $\frac{m-3}{2}$, or $\frac{m-1}{2}$, in order from left to right). As before, an entry 1 will signify that an edge is traversed in the positive direction, while -1 means that the edge is traversed in the negative direction.

Suppose we have an orientation of the edges of lengths $\frac{m-5}{2}$, $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in the central cycles, and let A be the matrix representing this orientation. Suppose also that for this orientation, we can assign jumps to the edges of lengths $\frac{m-5}{2}$, $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in the central cycles so that the total displacement along the edges of these lengths is nonzero in each central cycle. As in Case 1 of Section 5.2.3.1, it is easy to see that permuting the rows of A or multiplying a row of A by -1 creates a new possible orientation for which we can also assign jumps to the edges of lengths $\frac{m-5}{2}$, $\frac{m-3}{2}$ and $\frac{m-1}{2}$ with nonzero total displacement in any central cycle.

Let us now consider four possible orientations, listed by their associated matrices (A_1, A_2, A_3, and A_4) in Table 5.4. In this table, we also show how to assign jumps in the case that the edges of lengths $\frac{m-5}{2}$, $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in the central cycles are traversed in the directions given.

Cycle	Edge direction matrix	Jumps	Displacement in cycle
C_1 C_2 C_3	$A_1 = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$	1 1 1
C_1 C_2 C_3	$A_2 = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$	1 1 1
C_1 C_2 C_3	$A_3 = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$	1 1 1
C_1 C_2 C_3	$A_4 = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$	-1 1 1

Table 5.4: Jump assignments for the edges of lengths $(m-5)/2$, $(m-3)/2$ and $(m-1)/2$ in four basic cases.

Finally, note that, since the edges of lengths $\frac{m-5}{2}$ and $\frac{m-3}{2}$ are traversed in opposite directions in any central cycle, any possible orientation of the edges of lengths $\frac{m-5}{2}$, $\frac{m-3}{2}$ and $\frac{m-1}{2}$ in the three starter central cycles can be obtained from one of A_1, A_2, A_3 and A_4 by permutation of rows and multiplication of rows by -1. Thus, for any possible orientation of the edges of these lengths in the central cycles, a jump assignment can be made so that the total displacement in each central cycle is nonzero. $\square$

Let us now describe our strategy for assigning jumps to the edges of the starter central cycles. In each case, we will begin by assigning jumps to the edges in each central cycle of length at most ℓ, where ℓ is the largest length in L_2. (Recall that L_2 is the set of edge lengths which are not contained in all central cycles, while L_1 is the set of edge lengths which are.) In most cases, after we assign these jumps, the edge of largest length in a central cycle which has been assigned a jump is in an even position along the traversal of the cycle. In such cases, we will apply Lemma 5.7 to assign jumps to the remaining edges. We have that $\ell_\mu = \ell + 1$. Since ℓ is the largest length in L_2, it follows that ℓ_q, which has the property that the lengths $\ell_q, \ell_{q+1}, \ldots, \ell_k$ are consecutive in each of the three central cycles, must be at least $\ell + 1 = \ell_\mu$. Moreover, since μ is odd and q is even, we have that $\ell_q > \ell_\mu$. Thus, as we proceed through the various cases, we do not need to verify each time that the inequality $\ell_q > \ell_\mu$ holds. Note that also, in each case where type 3 cycles are used, the value ℓ_{q-2i}, where an edge of length ℓ_{q-2i} is the point of reversal of traversal of edges, also exceeds ℓ. Indeed, ℓ_α must be in L_1 for each $\alpha \geq q - 2j$, where j is as defined in the statement of Part 3 of Lemma 4.22, since the same value of j is chosen for each starter central cycle. As previously noted, we have that $q - 2i \geq q - 2j$, and so in particular, $\ell_\alpha \in L_1$ for each $\alpha \geq q - 2i$. However, the edge of length ℓ does not occur in all peripheral cycles, and so $\ell < \ell_{q-2i}$. Since $q - 2i$ is also even, we again have $\ell_{q-2i} \geq \ell_\mu + 1$, so that $\mu < q - 2i$. Again, we will not need to verify this fact directly in each case.

In cases where the edge of length ℓ occurs in an odd position of the central cycles, we will use the following lemma to assign jumps to the remaining edges.

Lemma 5.8. *Let $k \geq 15$ and m be odd integers such that $2k < m < 3k$, and $k \mid 3\binom{m}{2}$ but $k \nmid \binom{m}{2}$. Let $d = \gcd(m,k)$ and suppose $d > 1$. In the C_k-decomposition of $3K_m$ constructed in Chapter 4, let C_1, C_2 and C_3 be the three starter central cycles of length k constructed by Lemma 4.22.*

Let $\ell_1, \ell_2, \ldots, \ell_k = \frac{m-1}{2}$ be the edge lengths in C_1, where $\ell_1 < \ell_2 < \cdots < \ell_k$; $\ell'_1, \ell'_2, \ldots, \ell'_k = \frac{m-1}{2}$ be the edge lengths in C_2, where $\ell'_1 < \ell'_2 < \cdots < \ell'_k$; and $\ell''_1, \ell''_2, \ldots, \ell''_k = \frac{m-1}{2}$ be the edge lengths in C_3, where $\ell''_1 < \ell''_2 < \cdots < \ell''_k$. Let q be the even integer defined in the construction so that the lengths in each cycle with subscript at least q are consecutive. Note that for all $i \geq q$, $\ell_i = \ell'_i = \ell''_i$.

Suppose that the edges of lengths $\ell_1, \ldots, \ell_{q-1}$ in C_1, the edges of lengths $\ell'_1, \ldots, \ell'_{q-1}$ in C_2, and the edges of lengths $\ell''_1, \ldots, \ell''_{q-1}$ in C_3 have all been assigned jumps such that the total displacement along these edges is 1 in C_1, -1 in C_2, and 0 in C_3.

Then the edges of lengths $\ell_q, \ell_{q+1}, \ldots, \ell_k$ can be assigned jumps in each central cycle such that the total displacement in each cycle is nonzero.

Proof. We note, as in the proof of Lemma 5.7, that the edge which connects the zig-zag path back to vertex 0 has length at most $\ell_{k-3} = \frac{m-7}{2}$. We will begin by demonstrating an appropriate labelling of the edges of length $\ell_q, \ell_{q+1}, \ldots, \ell_{k-3}$ so that the total displacement in each cycle along the edges of length at most ℓ_{k-3} is 0. In each case, we will assign the edges of lengths in $\{\ell_q, \ell_{q+1}, \ldots, \ell_{k-3}\} = \{\ell''_q, \ell''_{q+1}, \ldots, \ell''_{k-3}\}$ in C_3 jump 0, so it remains to assign the edges of length at least $\ell_q = \ell'_q$ in C_1 and C_2 jumps chosen from ± 1.

We first suppose that for each $\alpha \in \{q, q+1, \ldots, k-3\}$, the edge of length $\ell_\alpha = \ell'_\alpha$ is traversed in the same direction in C_1 and C_2. In this case, we begin by assigning a jump chosen from ± 1 to the edge of length ℓ_q in C_1 such that the signed jump on this edge is -1. Then assign the other jump from ± 1 to the edge of length ℓ_q in C_2; its signed jump will be 1. The displacement along the edges labelled so far in C_1 and C_2 is 0. Now, the number of lengths in $\{q+1, q+2, \ldots, k-3\}$ is even. The edges of these lengths may be labelled in C_1 and C_2 with jumps ± 1 so that the total displacement is 0 by Lemma 5.6. Let us now proceed by considering the possible types of the cycles.

Case A. C_1 and C_2 are both of type 1

Let ℓ_i in C_1 and $\ell_{i'} = \ell'_{i'}$ in C_2 be the lengths of the edges that link back to vertex 0. Note that i and i' are both even. Note that if $i = i'$, then all edges of length at least $\ell_q = \ell'_q$ are traversed in the same direction in C_1 and C_2, so we may assume that $i \neq i'$. Let us first suppose that $i < i'$. Then

1. For each $\alpha \in \{q, q+1, \ldots, i-1\} \cup \{i', i'+1, \ldots, k-3\}$ the edge of length ℓ_α is traversed in the same direction in both C_1 and C_2. In particular, the edge of length $\ell_{i'}$ is traversed in the positive direction in both cycles. The number of such α is odd.

2. For each $\alpha \in \{i, i+1, \ldots, i'-1\}$, the edge of length ℓ_α is traversed in the opposite direction in C_1 and C_2. In particular, the edge of length i is traversed in the positive direction in C_1 and the negative direction in C_2. The number of such α is even.

We first label the edge of length $\ell_{i'}$ in C_1 with jump -1 and the edge of length $\ell_{i'}$ in C_2 with jump 1. Since these edges are both traversed in the positive direction, this adds -1 to the displacement along C_1 and 1 to the displacement along C_2; hence, among the edges which have been assigned jumps so far in C_1 and C_2, the displacement is 0.

There are an even number of lengths whose edges are traversed in opposite direction in C_1 and C_2. These may be assigned jumps in an appropriate manner by Lemma 5.6. Similarly, the remaining lengths whose edges are traversed in the same direction in C_1 and C_2 is even and may be assigned jumps from ± 1 using Lemma 5.6.

If $i > i'$, then the jump assignment is similar, with the roles of i and i' interchanged.

Case B. C_1 and C_2 are both of type 2

In this case, the back-connecting edges are of length ℓ_q in C_1 and C_2; these edges are traversed in the positive direction in both cycles. Assign jump -1 to the edge of length ℓ_q in C_1 and jump 1 to the edge of length ℓ_q in C_2. In each cycle, the total displacement along the edges which have been assigned jumps so far is 0. In C_1, we have that, for some positive integer $i < \frac{k-q}{2}$, edges of length in $\{\ell_{q+1}, \ell_{q+2}, \ldots, \ell_{k-2i-1}\}$ are traversed with edges of even-subscripted length in the positive direction and edges of odd-subscripted length in the negative direction. Edges of length in $\{k-2i, k-2i+1, \ldots, k\}$ are traversed in decreasing order of length and occur (except possibly the edge of length ℓ_k) with edges of odd-subscripted length in the positive direction and edges of even-subscripted length in the negative direction. The traversal of C' is similar, with i replaced by a positive integer $i' < \frac{k-q}{2}$, not necessarily equal to i.

Let us first suppose that $i \geq i'$, so that $k - 2i \leq k - 2i'$. As in the proof of Lemma 5.7, we have that:

1. For each $\alpha \in \{q+1, q+2, \ldots, k-2i-1\} \cup \{k-2i', k-2i'+1, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2. The number of such α is even, so the edges of these lengths can be assigned jumps from ± 1 so that they contribute 0 to the total displacement along each cycle by Lemma 5.6.

2. For each $\alpha \in \{k-2i, k-2i+1, \ldots, k-2i'-1\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. Again, the number of such α is even, so that the edges of these lengths can be assigned jumps from ± 1 by Lemma 5.6 so that they contribute 0 to the total displacement along each cycle.

Hence we have assigned jumps to all edges of C_1 and C_2 of length at most ℓ_{k-3} so that the total displacement along the edges of each cycle labelled so far is 0. The case that $i \leq i'$ is similar.

Case C. C_1 and C_2 are both of type 3

Again, both C_1 and C_2 use an edge of length ℓ_q as their back-connecting edge. A similar analysis to the case in Lemma 5.7 in which two cycles are of type 3 shows that the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2 for each $\alpha \geq q$. We have already shown that in such a case, the edges of length at most ℓ_{k-3} can be assigned jumps so that in each cycle, the total displacement along these edges is 0.

Case D. One of C_1 and C_2 is of type 1 and the other is of type 2

Let us first suppose that C_1 is of type 1 and C_2 is of type 2. Let ℓ_{q+2i} be the length of the back-connecting edge of C_1, where $i \geq 0$. In C_2, the back-connecting edge is of length ℓ_q. In C_1, the edges of length in $\{\ell_1, \ell_2, \dots, \ell_k\} - \{\ell_{q+2i}\}$ are traversed in increasing order of edge length, alternating direction between positive and negative (with the possible exception of the edge of length ℓ_k, which may be traversed in either direction). Hence, for $\alpha < q+2i$, the edge of length ℓ_α is traversed in positive direction if α is odd and in negative direction if α is even. If $\alpha > q + 2i$, the edge of length ℓ_α is traversed in negative direction if α is odd and positive direction if α is even. The edges of length greater than ℓ_q in C_2 are traversed beginning with edges of lengths $\ell_{q+1}, \ell_{q+2}, \dots, \ell_{k-2i'-1}$ for some positive i', with edges of length with odd subscript traversed in the negative direction and edges of length with even subscript traversed in the positive direction, followed in order by edges of lengths $\ell_k, \ell_{k-1}, \dots, \ell_{k-2i'}$, with edges of length with odd subscript (except possibly ℓ_k) traversed in the positive direction and edges of length with even subscript traversed in the negative direction.

Note that $q + 2i$ is even, while $k - 2i'$ is odd. Let us proceed depending on which of $q + 2i$ and $k - 2i'$ is greater.

Subcase D.i. $q + 2i < k - 2i'$

First, suppose $i > 0$. Then:

1. The edge of length ℓ_q in C_1 is traversed in the negative direction, while the edge of length ℓ_q in C_2 is traversed in the positive direction.

2. For $\alpha \in \{q+1, q+2, \ldots, q+2i-1\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. There are an odd number of such α.

3. The edge of length $q + 2i$ in each of C_1 and C_2 is traversed in the positive direction.

4. For $\alpha \in \{q + 2i + 1, q + 2i + 2, \ldots, k - 2i' - 1\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2. The number of such α is even.

5. The edge of length $k - 2i'$ in C_1 is traversed in the negative direction, while the edge of length $k - 2i'$ in C_2 is traversed in the positive direction.

6. For $\alpha \in \{k-2i'+1, k-2i'+2, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such α is odd.

Next suppose that $i = 0$, so that the back-connecting edge in C_1 is ℓ_q. Then:

1. The edge of length ℓ_q in each of C_1 and C_2 is traversed in the positive direction.

2. For $\alpha \in \{q + 1, q + 2, \ldots, k - 2i' - 1\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2. The number of such α is even.

3. The edge of length $k - 2i'$ in C_1 is traversed in the negative direction, while the edge of length $k - 2i'$ in C_2 is traversed in the positive direction.

4. For $\alpha \in \{k-2i'+1, k-2i'+2, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such α is odd.

In either case, we have that the edge of length ℓ_{q+2i} in each of C_1 and C_2 is traversed in the positive direction, that there are in total an odd number of edge lengths whose corresponding edges are traversed in the same direction in both of C_1 and C_2, and that there are an even number of edge lengths whose corresponding edges are traversed in opposite directions in C_1 and C_2.

First, assign jump -1 to the edge of length ℓ_{q+2i} in C_1 and jump 1 to the edge of length ℓ_{q+2i} in C_2. Since these edges are both traversed in the positive direction, this means that the displacement along the edges assigned jumps so far in each cycle is 0. There remain an even number of lengths at most ℓ_{k-3} for which the edges of these lengths are traversed in the same direction in C_1 and C_2. These edges may be labelled with jumps ± 1 so that they contribute 0 to the displacement along each cycle by Lemma 5.6. Also, there are an even number of lengths at most ℓ_{k-3} for which the edges of these lengths are traversed in opposite directions in C_1 and C_2. These edges may also be labelled with jumps ± 1 by Lemma 5.6 so that they contribute 0 to the displacement along each cycle.

Subcase D.ii. $q + 2i > k - 2i'$

Note that this condition implies that $i > 0$, as $k - 2i' > q$. We have that:

1. The edge of length ℓ_q in C_1 is traversed in the negative direction, while the back-connecting edge of length ℓ_q in C_2 is traversed in the positive direction.

2. For $\alpha \in \{q+1, q+2, \ldots, k-2i'-1\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such α is even.

3. The edge of length $\ell_{k-2i'}$ in each of C_1 and C_2 is traversed in the positive direction.

4. For $\alpha \in \{k-2i'+1, k-2i'+2, \ldots, q+2i-1\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2. the number of such α is even.

5. The back-connecting edge of length ℓ_{q+2i} in C_1 is traversed in the positive direction, while the edge of length ℓ_{q+2i} in C_2 is traversed in the negative direction.

6. For $\alpha \in \{q+2i+1, q+2i+2, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such α is even.

Let us now assign jumps to the edges of length in $\{\ell_q, \ell_{q+1}, \ldots, \ell_{k-3}\}$. First, assign a jump of -1 to the edge of length $\ell_{k-2i'}$ in C_1 and 1 to the edge of length $\ell_{k-2i'}$ in C_2. Since each of these edges is traversed in the positive direction, the edges labelled so far (including those of length less than ℓ_q) contribute 0 to the total displacement along each cycle. The remaining lengths for which the edges of these lengths are traversed in the same direction in each of C_1 and C_2 are even in number, and may be assigned jumps from ± 1 so that they contribute 0 to the displacement along each cycle by Lemma 5.6. The lengths for which the edges of these lengths are traversed in opposite directions in C_1 and C_2 are also even in number and may also be assigned jumps chosen from ± 1 by Lemma 5.6 in such a way that they contribute 0 to the displacement along each cycle.

The case that C_1 is of type 2 and C_2 is of type 1 is similar.

Case E. One of C_1 and C_2 is of type 1 and the other is of type 3

Let us first suppose that C_1 is of type 1 and C_2 is of type 3. Let ℓ_{q+2i} be the length of the back-connecting edge in C_1, where $i \geq 0$. The back-connecting edge in

C_2 has length ℓ_q. Note that in C_1, the edges of length in $\{\ell_1, \ldots, \ell_{k-3}\} - \{\ell_{q+2i}\}$ are traversed in increasing order of edge length, with alternating direction. In particular, if $\alpha < q + 2i$, then the edge of length ℓ_α is traversed in the positive direction if α is odd and in the negative direction if α is even. If $\alpha > q + 2i$, then the edge of length ℓ_α is traversed in the negative direction is α is odd and in the positive direction if α is even. In C_2, the edges of length $\ell_1, \ldots, \ell_{q-2i'-1}$, for some positive integer i', are traversed with edges of length with odd subscript in the positive direction and edges of length with even subscript in the negative direction. The cycle next follows edges of length $\ell_k, \ell_{k-1}, \ldots, \ell_{q+1}$, with edges of length with odd subscript (except possibly ℓ_k) traversed in the positive direction and edges of length with even subscript traversed in the negative direction. Finally, lengths $\ell_{q-1}, \ldots, \ell_{q-2i'}$ are traversed with edges of length with odd subscript traversed in the negative direction and edges of length with even subscript traversed in the positive direction.

Let us consider the possible cases depending on the value of $q + 2i$.

Subcase E.i. $i > 0$

If $i > 0$, we have that among the edges of length at least ℓ_q:

1. The edge of length ℓ_q in C_1 is traversed in the negative direction and the back-connecting edge of length ℓ_q in C_2 is traversed in the positive direction.

2. For $\alpha \in \{q + 1, q + 2, \ldots, q + 2i - 1\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2. The number of such α is odd.

3. The edges of length ℓ_{q+2i} in C_1 and C_2 are traversed in the opposite direction.

4. For $\alpha \in \{q+2i+1, q+2i+2, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such α is even.

In this case, the edge of length ℓ_{q+1} is traversed in the positive direction in both cycles. Give the edge of this length in C_1 jump -1 and the edge of this length in C_2 jump 1. As before, this means that the total displacement is 0 among the edges assigned a jump so far in each cycle. There remain an even number of edge lengths whose corresponding edges are traversed in the same direction in each of C_1 and C_2; these edges may be labelled with jumps chosen from ± 1 so that they contribute 0 to the displacement along each cycle by Lemma 5.6. There are also an even number of edge lengths whose corresponding edges are traversed in opposite directions in C_1 and C_2; these edges may also be labelled with jumps chosen from ± 1 so that they contribute 0 to the displacement along each cycle by Lemma 5.6.

Subcase E.ii. $i = 0$

Here, the back-connecting edge of C_1 has length ℓ_q. Note that:

1. The edge of length ℓ_q in each of C_1 and C_2 is traversed in the positive direction.

2. For $\alpha \in \{q+1, q+2, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such α is even.

Hence the edge of length ℓ_q is traversed in the positive direction in both cycles. Give the edge of this length in C_1 jump -1 and the edge of this length in C_2 jump 1. As before, this means that the total displacement is 0 among the edges assigned jump so far in each cycle. Now, there remain an even number of edge lengths whose corresponding edges are traversed in the same direction in each of C_1 and C_2; these edges may be labelled with jumps chosen from ± 1 so that they contribute 0 to the displacement along each cycle by Lemma 5.6. There are also an even number (possibly 0) of edge lengths whose corresponding edges are traversed in the opposite directions in C_1 and C_2; these edges may also be labelled with jumps chosen from ± 1 so that they contribute 0 to the displacement along each cycle by Lemma 5.6.

The case in which C_1 is of type 3 and C_2 is of type 1 is similar.

Case F. One of C_1 and C_2 is of type 2 and the other is of type 3

Let us first suppose that C_1 is of type 2 and C_2 is of type 3. A similar analysis to the case in which there is one cycle of type 2 and one of type 3 in the proof of Lemma 5.7 show that:

1. The back-connecting edge of length ℓ_q in each of C_1 and C_2 is traversed in the positive direction.

2. For each $\alpha \in \{q+1, q+2, \ldots, k-2i-1\}$, the edge of length ℓ_α in C_1 is traversed in the opposite direction to the edge of length ℓ_α in C_2. The number of such α is even.

3. For each $\alpha \in \{k-2i, k-2i+1, \ldots, k-3\}$, the edge of length ℓ_α in C_1 is traversed in the same direction as the edge of length ℓ_α in C_2. The number of such α is even.

Assign to the edge of length ℓ_q in C_1 jump -1 and to the edge of length ℓ_q in C_2 jump 1. Then, since these edges are both traversed in the positive direction, the total displacement along the edges in each cycle so far assigned jumps is 0. There remain an even number of lengths ℓ_α with the property that the edge of length ℓ_α is traversed in the same direction in both C_1 and C_2. The edges of such lengths may be labelled by Lemma 5.6 with jumps ± 1 so that they contribute 0 to the displacement along each cycle. Finally, there are an even number of edges ℓ_α with the property that the edges of length ℓ_α in C_1 and C_2 are traversed in opposite directions. These lengths may also be labelled by Lemma 5.6 with jumps ± 1 so that they contribute 0 to the displacement along each cycle.

The case in which C_1 is of type 3 and C_2 is of type 2 is similar.

We have now exhibited an assignment of jumps to the edges in C_1 and C_2 of length at most ℓ_{k-3} in each case so that the total displacement along the edges labelled so far is 0. The edges of lengths $\ell_{k-2} = \frac{m-5}{2}$, $\ell_{k-1} = \frac{m-3}{2}$ and $\ell_k = \frac{m-1}{2}$ may be labelled as in the proof of Lemma 5.7. □

Jump assignments for central cycles

We now proceed to the assignment of jumps by case.

Subcase 1.1. $\tilde{k} \equiv 3 \pmod 4$

Recall that when $\tilde{k} \equiv 3 \pmod 4$, the zig-zag path $P_{i,0}$ was defined to have edges of lengths $i\tilde{m}+1$, $i\tilde{m}+2$, ..., $i\tilde{m}+\frac{\tilde{k}-3}{2}$, $i\tilde{m}+\frac{\tilde{k}+1}{2}$, $i\tilde{m}+\frac{\tilde{k}+3}{2}$, ..., $i\tilde{m}+\tilde{k}$. If $\tilde{m}-\frac{\tilde{k}-1}{2} > 2\tilde{k}-1$, then the path $P^*_{i,0}$ has edges of lengths $i\tilde{m}+\tilde{k}+1$, $i\tilde{m}+\tilde{k}+2$, ..., $i\tilde{m}+2\tilde{k}-1$, while if $\tilde{m}-\frac{\tilde{k}-1}{2} \leq 2\tilde{k}-1$, then $P^*_{i,0}$ has edges of lengths $i\tilde{m}+\tilde{k}+1$, $i\tilde{m}+\tilde{k}+2$, ..., $(i+1)\tilde{m}-\frac{\tilde{k}+1}{2}$, $(i+1)\tilde{m}-\frac{\tilde{k}-3}{2}$, $(i+1)\tilde{m}-\frac{\tilde{k}-5}{2}$, ..., $i\tilde{m}+2\tilde{k}$. Except in rare instances in which alternative choices of paths were constructed, the paths $P_{i,0}$ and $P^*_{i,0}$ were used to construct the peripheral cycles. If c, the number of starter peripheral cycles, is even, then paths $P_{i,0}$ and $P^*_{i,0}$, for $0 \leq i \leq \frac{c}{2}-1$, are used in the formation of peripheral cycles, together with linking edges of lengths $i\tilde{m} \pm \frac{\tilde{k}-1}{2}$, where $1 \leq i \leq \frac{c}{2}$. If c is odd, then paths $P_{i,0}$ for $0 \leq i \leq \frac{c-1}{2}$ and $P^*_{i,0}$ for $0 \leq i \leq \frac{c-3}{2}$ are used in the formation of peripheral cycles, together with linking edges of lengths $\tilde{m} \pm \frac{\tilde{k}-1}{2}, 2\tilde{m}-\frac{\tilde{k}-1}{2}$ and $i\tilde{m} \pm \frac{\tilde{k}-1}{2}$ for $3 \leq i \leq \frac{c+1}{2}$.

One peripheral cycle, C, occurs with multiplicity 1, while all the other peripheral cycles occur with multiplicity 3 in the decomposition. The set of edge lengths which occur in C is denoted by L_2 (where $|L_2| = \tilde{k}$). The set of edge lengths which do not occur in peripheral cycles is denoted by L_1. Each central cycle contains an edge of each length in L_1, and $\frac{2\tilde{k}}{3}$ edges of length in L_2.

Subcase 1.1.1. c is even

In the construction of peripheral cycles, we can guarantee that $\tilde{m}+\frac{\tilde{k}-1}{2}$ is the linking length in C, the peripheral cycle which occurs of multiplicity 1, while linking length $\tilde{m}-\frac{\tilde{k}-1}{2}$ is used of multiplicity 3 in the peripheral cycles.

Subcase 1.1.1.1. $c \geq 4$

We use a path $P_{i,0}$, where $i \geq 1$, in the formation of C. Since $c > 2$, we may let this path be $P_{1,0}$.

We must now assign jumps to the starter central cycles so that each has nonzero displacement. Note that L_2, the set of edge lengths which occur of multiplicity 1 in peripheral cycles, is given by

$$L_2 = \{\tilde{m}+1, \tilde{m}+2, \ldots, \tilde{m}+\tilde{k}\}.$$

Recall that in the construction of central cycles, we partitioned L_2 into three sets, S_1, S_2 and S_3, each of size $\frac{\tilde{k}}{3}$. We may let $S_1 = \{\tilde{m}+1, \tilde{m}+2, \ldots, \tilde{m}+\frac{\tilde{k}}{3}\}$, $S_2 = \{\tilde{m}+\frac{\tilde{k}}{3}+1, \tilde{m}+\frac{\tilde{k}}{3}+2, \ldots, \tilde{m}+\frac{2\tilde{k}}{3}\}$ and $S_3 = \{\tilde{m}+\frac{2\tilde{k}}{3}+1, \tilde{m}+\frac{2\tilde{k}}{3}+2, \ldots, \tilde{m}+\tilde{k}\}$. Let C_1, C_2 and C_3 denote the three starter central cycles, where C_1 contains edges of lengths in $S_1 \cup S_2$, C_2 contains edges of lengths in $S_1 \cup S_3$, and C_3 contains edges of lengths in $S_2 \cup S_3$.

Recall that L_1 denotes the set of edge lengths which do not occur in peripheral cycles (and thus occur in each central cycle). Which edge lengths occur in L_1 depend on whether $\tilde{m} - \frac{\tilde{k}-1}{2} > 2\tilde{k}-1$ or $\tilde{m} - \frac{\tilde{k}-1}{2} \leq 2\tilde{k}-1$, as different lengths occur in $P^*_{0,0}$ in each case. Note that if $\tilde{k} = 3$, then $2\tilde{k} < \tilde{m} < 3\tilde{k}$ implies that $\tilde{m} = 7$, and in this case $\tilde{m} - \frac{\tilde{k}-1}{2} > 2\tilde{k}-1$.

In the case that $\tilde{m} - \frac{\tilde{k}-1}{2} > 2\tilde{k}-1$, the following are the lengths in L_1 that are less than $\tilde{m}+\tilde{k}$ (these are the ones which may affect the direction of an edge in the peripheral cycles): $\frac{\tilde{k}-1}{2}; 2\tilde{k}, 2\tilde{k}+1, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}; \tilde{m}-\frac{\tilde{k}-3}{2}, \tilde{m}-\frac{\tilde{k}-5}{2}, \ldots, \tilde{m}$. (Note that we have used semicolons for clarity in separating groupings of consecutive edge lengths.) In total the number of these lengths is even. As the largest of these lengths is less than the smallest length in L_2 in this case, the cycle begins with a zig-zag path consisting of edges of these lengths traversed in alternating order. Thus, assigning these edges all jump 0 in one central cycle, jump 1 in a second, and jump -1 in the third means that the edges of length in L_1 contribute 0 to the displacement along each central cycle.

Table 5.5 depicts the direction in which we encounter each edge of length at most $\tilde{m}+\tilde{k}$ in each of the three central cycles in the case that $\tilde{k}>3$ and $\tilde{m}-\frac{\tilde{k}-1}{2} > 2\tilde{k}-1$, together with the assigned jumps and displacements. The values given for displacements are the total displacements for the edges in each row and the total displacement in each cycle. These values are given modulo 3. We note that in Table 5.5, and in the other tables throughout this section, it should be assumed that the edge directions in each row follow an alternating sign pattern; this is due to the zig-zag nature of the formation of each cycle.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	L_1	$2\tilde{k},\dots,\tilde{m}-\frac{\tilde{k}+1}{2}$	$-,+,\dots,-,+$	$0,\dots,0$	0
	L_1	$\tilde{m}-\frac{\tilde{k}-3}{2},\dots,\tilde{m}$	$-,+,\dots,+,-$	$0,\dots,0$	0
	S_1	$\tilde{m}+1,\dots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\dots,-,+$	$0,-1,-1,\dots,-1,0$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\dots,\tilde{m}+\frac{2\tilde{k}}{3}$	$-,+,\dots,+,-$	$0,-1,-1,\dots,-1,0$	-1
Total					0
C_2	L_1	$\frac{\tilde{k}-1}{2}$	$+$	1	1
	L_1	$2\tilde{k},\dots,\tilde{m}-\frac{\tilde{k}+1}{2}$	$-,+,\dots,-,+$	$1,\dots,1$	0
	L_1	$\tilde{m}-\frac{\tilde{k}-3}{2},\dots,\tilde{m}$	$-,+,\dots,+,-$	$1,\dots,1$	-1
	S_1	$\tilde{m}+1,\dots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\dots,-,+$	$-1,0,0,\dots,0,-1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+1,\dots,\tilde{m}+\tilde{k}$	$-,+,\dots,+,-$	$0,-1,-1,\dots,-1,0$	-1
Total					0
C_3	L_1	$\frac{\tilde{k}-1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k},\dots,\tilde{m}-\frac{\tilde{k}+1}{2}$	$-,+,\dots,-,+$	$-1,\dots,-1$	0
	L_1	$\tilde{m}-\frac{\tilde{k}-3}{2},\dots,\tilde{m}$	$-,+,\dots,+,-$	$-1,\dots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\dots,\tilde{m}+\frac{2\tilde{k}}{3}$	$+,-,\dots,-,+$	$-1,0,0,\dots,0,-1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+1,\dots,\tilde{m}+\tilde{k}$	$-,+,\dots,+,-$	$-1,0,0,\dots,0,-1$	-1
Total					0

Table 5.5: Jump assignments for the edges of length at most $\tilde{m}+\tilde{k}$ when $\tilde{k}\equiv 3 \pmod 4$, $c\geq 4$ is even, $\tilde{k}>3$, and $\tilde{m}-\frac{\tilde{k}-1}{2}>2\tilde{k}-1$.

Table 5.6 shows the jump assignments in the case that $\tilde{k}=3$. Note that the facts $2\tilde{k} < \tilde{m} < 3\tilde{k}$ and $\tilde{m}$ is odd imply that $\tilde{m} = 7$. Here, $S_1 = \{8\}$, $S_2 = \{9\}$ and $S_3 = \{10\}$, while the lengths in L_1 less than 10 are 1 and 7. We note that in this case, the linking edge in C, which has length $\tilde{m} - \frac{\tilde{k}-1}{2} = 8$, has been assigned jump -1, so that we may use jump 1 on edges of length 8 in central cycles.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	1	$+$	0	0
	L_1	7	$-$	1	-1
	S_1	8	$+$	0	0
	S_2	9	$-$	-1	1
Total					0
C_2	L_1	1	$+$	-1	-1
	L_1	7	$-$	0	0
	S_1	8	$+$	1	1
	S_3	10	$-$	0	0
Total					0
C_3	L_1	1	$+$	1	1
	L_1	7	$-$	-1	1
	S_2	9	$+$	0	0
	S_3	10	$-$	-1	1
Total					0

Table 5.6: Jump assignments for the edges of length at most $\tilde{m} + \tilde{k}$ when $\tilde{k} = 3$ and $c \geq 4$ is even.

Table 5.7 shows the jump assignments of the edges of length at most $\tilde{m} + \tilde{k}$ in the starter central cycles in the case that $\tilde{m} - \frac{\tilde{k}-1}{2} \leq 2\tilde{k} - 1$. In this case the lengths in L_1 which are less than $\tilde{m} + \tilde{k}$ are

$$\frac{\tilde{k}-1}{2}; 2\tilde{k}+1, 2\tilde{k}+2, \ldots, \tilde{m}.$$

The total number of these lengths is even.

In either case, the number of edges in each starter central cycle which have been assigned jumps so far is even, and the displacement along the edges labelled so far is 0 in each cycle. The remaining edges, which are odd in number, all have length greater than $\tilde{m} + \tilde{k}$, the largest length of an edge assigned jump so far. Moreover,

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$0,\ldots,0$	0
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$0,-1,-1,\ldots,-1,0$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{2\tilde{k}}{3}$	$-,+,\ldots,+,-$	$0,-1,-1,\ldots,-1,0$	-1
Total					0
C_2	L_1	$\frac{\tilde{k}-1}{2}$	$+$	1	1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$1,\ldots,1$	-1
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$-1,0,0,\ldots,0,-1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+1,\ldots,\tilde{m}+\tilde{k}$	$-,+,\ldots,+,-$	$0,-1,-1,\ldots,-1,0$	-1
Total					0
C_3	L_1	$\frac{\tilde{k}-1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$-1,\ldots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{2\tilde{k}}{3}$	$+,-,\ldots,-,+$	$-1,0,0,\ldots,0,-1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+1,\ldots,\tilde{m}+\tilde{k}$	$-,+,\ldots,+,-$	$-1,0,0,\ldots,0,-1$	-1
Total					0

Table 5.7: Jump assignments for the edges of length at most $\tilde{m}+\tilde{k}$ when $\tilde{k}\equiv 3 \pmod 4$, $c\geq 4$ is even, and $\tilde{m}-\frac{\tilde{k}-1}{2}\leq 2\tilde{k}-1$.

the set of lengths of unlabelled edges is the same for each starter central cycle. We may apply Lemma 5.7 to assign jumps to the remaining edges in the starter central cycles so that any two edges of the same length receive different jumps and the total displacement along each cycle is nonzero.

Subcase 1.1.1.2. $c=2$, $d>3$, $\tilde{k}>3$, and $\tilde{m}-\frac{\tilde{k}-1}{2}>2\tilde{k}-1$.

In this case, $P^*_{0,0}$ is the path used to form the peripheral cycle which occurs of multiplicity 1. Since $\tilde{m}-\frac{\tilde{k}-1}{2}>2\tilde{k}-1$ and $\tilde{k}>3$, we have $L_2=S_1\cup S_2\cup S_3$, where

$$\begin{aligned} S_1 &= \left\{\tilde{k}+1,\tilde{k}+2,\ldots,\frac{4\tilde{k}}{3}\right\} \\ S_2 &= \left\{\frac{4\tilde{k}}{3}+1,\frac{4\tilde{k}}{3}+2,\ldots,\frac{5\tilde{k}}{3}\right\} \\ S_3 &= \left\{\frac{5\tilde{k}}{3}+1,\frac{5\tilde{k}}{3}+2,\ldots,2\tilde{k}-1;\tilde{m}+\frac{\tilde{k}-1}{2}\right\} \end{aligned}$$

The lengths of L_1 which are less than $\tilde{m} + \frac{\tilde{k}-1}{2}$ are:

$$\frac{\tilde{k}-1}{2}; 2\tilde{k}, 2\tilde{k}+1, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}; \tilde{m} - \frac{\tilde{k}-3}{2}, \tilde{m} - \frac{\tilde{k}-5}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-3}{2}.$$

Note that the number of lengths between $2\tilde{k}$ and $\tilde{m} - \frac{\tilde{k}+1}{2}$ is even, while the number of lengths between $\tilde{m} - \frac{\tilde{k}-3}{2}$ and $\tilde{m} + \frac{\tilde{k}-3}{2}$ is odd.

Table 5.8 shows the assignment of jumps to the edges of length at most $\tilde{m} + \frac{\tilde{k}-1}{2}$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	S_1	$\tilde{k}+1, \ldots, \frac{4\tilde{k}}{3}$	$-,+,\ldots,+,-$	$0,\ldots,0$	0
	S_2	$\frac{4\tilde{k}}{3}+1, \ldots, \frac{5\tilde{k}}{3}$	$+,-,\ldots,-,+$	$0,\ldots,0$	0
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}$	$-,+,\ldots,-,+$	$0,0,\ldots,0$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-3}{2}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}-1}{2}$	$+$	1	1
	S_1	$\tilde{k}+1, \ldots, \frac{4\tilde{k}}{3}$	$-,+,\ldots,+,-$	$-1,\ldots,-1$	1
	S_3	$\frac{5\tilde{k}}{3}+1, \ldots, 2\tilde{k}-1$	$+,-,\ldots,+,-$	$-1,0,0,\ldots,0$	-1
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}$	$+,-,\ldots,+,-$	$1,1,\ldots,1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-3}{2}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m} + \frac{\tilde{k}-1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}-1}{2}$	$+$	-1	-1
	S_2	$\frac{4\tilde{k}}{3}+1, \ldots, \frac{5\tilde{k}}{3}$	$-,+,\ldots,+,-$	$-1,-1,\ldots,-1$	1
	S_3	$\frac{5\tilde{k}}{3}+1, \ldots, 2\tilde{k}-1$	$+,-,\ldots,+,-$	$0,-1,-1,\ldots,-1$	1
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}$	$+,-,\ldots,+,-$	$-1,-1,\ldots,-1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-3}{2}$	$+,-,\ldots,-,+$	$1,1,\ldots,1$	1
	S_3	$\tilde{m} + \frac{\tilde{k}-1}{2}$	$-$	-1	1
Total					0

Table 5.8: Jump assignments for the edges of length at most $\tilde{m} + \frac{\tilde{k}-1}{2}$ when $\tilde{k} \equiv 3 \pmod 4$, $c = 2$, $d > 3$, $\tilde{k} > 3$, and $\tilde{m} - \frac{\tilde{k}-1}{2} > 2\tilde{k} - 1$

Note that an even number of edges in each starter central cycle has been assigned jumps so far, and the displacement along the labelled edges of each cycle is 0. The remaining edges all have length greater than $\tilde{m} + \frac{\tilde{k}-1}{2}$, the largest length of an edge assigned jump so far. Also, the set of lengths of the unlabelled edges in each starter central cycle is the same. By Lemma 5.7, we may assign jumps to the remaining edges so that the total displacement is nonzero.

Subcase 1.1.1.3. $c = 2$, $d > 3$, $\tilde{k} > 3$, and $\frac{5\tilde{k}}{3} < \tilde{m} - \frac{\tilde{k}-1}{2} \leq 2\tilde{k} - 1$

Here we have that $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned}
S_1 &= \left\{\tilde{k}+1, \tilde{k}+2, \ldots, \frac{4\tilde{k}}{3}\right\} \\
S_2 &= \left\{\frac{4\tilde{k}}{3}+1, \frac{4\tilde{k}}{3}+2, \ldots, \frac{5\tilde{k}}{3}\right\} \\
S_3 &= \left\{\frac{5\tilde{k}}{3}+1, \frac{5\tilde{k}}{3}+2, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}; \tilde{m}-\frac{\tilde{k}-3}{2}, \tilde{m}-\frac{\tilde{k}-5}{2}, \ldots, 2\tilde{k}; \tilde{m}+\frac{\tilde{k}-1}{2}\right\}
\end{aligned}$$

The lengths in L_1 which are less than $\tilde{m} + \frac{\tilde{k}-1}{2}$ are:

$$\frac{\tilde{k}-1}{2}; 2\tilde{k}+1, 2\tilde{k}+2, \ldots, \tilde{m}+\frac{\tilde{k}-3}{2}.$$

Note that the number of lengths between $2\tilde{k}+1$ and $\tilde{m} + \frac{\tilde{k}-3}{2}$ is odd.

Table 5.9 shows the assignment of jumps to the edges of length at most $\tilde{m} + \frac{\tilde{k}-1}{2}$.

Note that an even number of edges in each starter central cycle have so far been assigned jump, with displacement so far being 0 in each cycle. The remaining edges are those of length in L_1 greater than $\tilde{m}+\frac{\tilde{k}-1}{2}$. We may assign jumps to the remaining edges by Lemma 5.7 so that the total displacement in each cycle is nonzero.

Subcase 1.1.1.4. $c = 2$, $d > 3$, $\tilde{k} > 3$, and $\tilde{m} - \frac{\tilde{k}-1}{2} \leq \frac{5\tilde{k}}{3}$

Note that since $\tilde{m} > 2\tilde{k}$, we have that $\tilde{m} - \frac{\tilde{k}-1}{2} > \frac{3\tilde{k}+1}{2} > \frac{4\tilde{k}}{3}$. Hence we may write $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned}
S_1 &= \left\{\tilde{k}+1, \tilde{k}+2, \ldots, \frac{4\tilde{k}}{3}\right\} \\
S_2 &= \left\{\frac{4\tilde{k}}{3}+1, \frac{4\tilde{k}}{3}+2, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}, \tilde{m}-\frac{\tilde{k}-3}{2}, \tilde{m}-\frac{\tilde{k}-5}{2}, \ldots, \frac{5\tilde{k}}{3}+1\right\}
\end{aligned}$$

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	S_1	$\tilde{k}+1,\ldots,\frac{4\tilde{k}}{3}$	$-,+,\ldots,+,-$	$0,\ldots,0$	0
	S_2	$\frac{4\tilde{k}}{3}+1,\ldots,\frac{5\tilde{k}}{3}$	$+,-,\ldots,-,+$	$0,\ldots,0$	0
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}-1}{2}$	$+$	1	1
	S_1	$\tilde{k}+1,\ldots,\frac{4\tilde{k}}{3}$	$-,+,\ldots,+,-$	$-1,\ldots,-1$	1
	S_3	$\frac{5\tilde{k}}{3}+1,\ldots,\tilde{m}-\frac{\tilde{k}+1}{2}$	$+,-,\ldots,+,-$	$-1,0,0,\ldots,0$	-1
	S_3	$\tilde{m}-\frac{\tilde{k}-3}{2},\ldots,2\tilde{k}$	$+,-,\ldots,+,-$	$0,0,\ldots,0$	0
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m}+\frac{\tilde{k}-1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}-1}{2}$	$+$	-1	-1
	S_2	$\frac{4\tilde{k}}{3}+1,\ldots,\frac{5\tilde{k}}{3}$	$-,+,\ldots,+,-$	$-1,\ldots,-1$	1
	S_3	$\frac{5\tilde{k}}{3}+1,\ldots,\tilde{m}-\frac{\tilde{k}+1}{2}$	$+,-,\ldots,+,-$	$0,-1,-1,\ldots,-1$	1
	S_3	$\tilde{m}-\frac{\tilde{k}-3}{2},\ldots,2\tilde{k}$	$+,-,\ldots,+,-$	$-1,-1\ldots,-1$	0
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$+,-,\ldots,-,+$	$1,1,\ldots,1$	1
	S_3	$\tilde{m}+\frac{\tilde{k}-1}{2}$	$-$	-1	1
Total					0

Table 5.9: Jump assignments for the edges of length at most $\tilde{m}+\frac{\tilde{k}-1}{2}$ in the case that $\tilde{k}\equiv 3 \pmod 4$, $c=2$, $d>3$, $\tilde{k}>3$, and $\frac{5\tilde{k}}{3}<\tilde{m}-\frac{\tilde{k}-1}{2}\leq 2\tilde{k}-1$

$$S_3 = \left\{\frac{5\tilde{k}}{3}+2, \frac{5\tilde{k}}{3}+3, \ldots, 2\tilde{k}, \tilde{m}+\frac{\tilde{k}-1}{2}\right\}$$

The lengths in L_1 which are less than $\tilde{m}+\frac{\tilde{k}-1}{2}$ are:

$$\frac{\tilde{k}-1}{2}; 2\tilde{k}+1, 2\tilde{k}+2, \ldots, \tilde{m}+\frac{\tilde{k}-3}{2}.$$

Note that the number of lengths between $2\tilde{k}+1$ and $\tilde{m}+\frac{\tilde{k}-3}{2}$ is odd.

Table 5.10 shows the assignment of jumps to the edges of length at most $\tilde{m}+\frac{\tilde{k}-1}{2}$.

Note that an even number of edges in each starter central cycle have so far been assigned jump, with displacement so far being 0 in each cycle. The remaining edges

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	S_1	$\tilde{k}+1,\ldots,\frac{4\tilde{k}}{3}$	$-,+,\ldots,+,-$	$0,\ldots,0$	0
	S_2	$\frac{4\tilde{k}}{3}+1,\ldots,\tilde{m}-\frac{\tilde{k}+1}{2}$	$+,-,\ldots,-,+$	$0,\ldots,0$	0
	S_2	$\tilde{m}-\frac{\tilde{k}-3}{2},\ldots,\frac{5\tilde{k}}{3}+1$	$-,+,\ldots,-,+$	$0,\ldots,0$	0
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}-1}{2}$	$+$	1	1
	S_1	$\tilde{k}+1,\ldots,\frac{4\tilde{k}}{3}$	$-,+,\ldots,+,-$	$-1,\ldots,-1$	1
	S_3	$\frac{5\tilde{k}}{3}+2,\ldots,2\tilde{k}$	$+,-,\ldots,+,-$	$-1,0,0,\ldots,0$	-1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m}+\frac{\tilde{k}-1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}-1}{2}$	$+$	-1	-1
	S_2	$\frac{4\tilde{k}}{3}+1,\ldots,\tilde{m}-\frac{\tilde{k}+1}{2}$	$-,+,\ldots,+,-$	$-1,\ldots,-1$	1
	S_2	$\tilde{m}-\frac{\tilde{k}-3}{2},\ldots,\frac{5\tilde{k}}{3}+1$	$+,-,\ldots,+,-$	$-1,\ldots,-1$	0
	S_3	$\frac{5\tilde{k}}{3}+2,\ldots,2\tilde{k}$	$+,-,\ldots,+,-$	$0,-1,-1,\ldots,-1$	1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$+,-,\ldots,-,+$	$1,1,\ldots,1$	1
	S_3	$\tilde{m}+\frac{\tilde{k}-1}{2}$	$-$	-1	1
Total					0

Table 5.10: Jump assignments for the edges of length at most $\tilde{m}+\frac{\tilde{k}-1}{2}$ in the case that $\tilde{k}\equiv 3 \pmod 4$, $c=2$, $d>3$, $\tilde{k}>3$, and $\tilde{m}-\frac{\tilde{k}-1}{2}\leq\frac{5\tilde{k}}{3}$

are those of length in L_1 greater than $\tilde{m}+\frac{\tilde{k}-1}{2}$. We may assign jumps to the remaining edges by Lemma 5.7 so that the total displacement in each cycle is nonzero.

Subcase 1.1.1.5. $c=2$, $d>3$ and $\tilde{k}=3$

Note that in this case, the facts that $\tilde{m}$ is odd and $2\tilde{k}<\tilde{m}<3\tilde{k}$ imply that $\tilde{m}=7$. Hence we have that $\tilde{m}-\frac{\tilde{k}-1}{2}=6>2\tilde{k}-1=5$. The path $P^*_{0,0}$, which is used to form the peripheral cycle C, contains edges of lengths 4 and 5, while the linking length used in C is $\tilde{m}+\frac{\tilde{k}-1}{2}=8$. Hence $L_2=S_1\cup S_2\cup S_3$, where $S_1=\{4\}$, $S_2=\{5\}$ and $S_3=\{8\}$. The lengths in L_1 less than 8 are 1 and 7. So C_1 contains edges of lengths $1,4,5,7$, C_2 of lengths $1,4,7,8$ and C_3 of lengths $1,5,7,8$.

Table 5.11 shows how to assign jumps to the edges of length at most 8 in each central cycle.

Cycle		Lengths in cycle	Edge direction	Jump	Displacement
C_1	L_1	1	$+$	0	0
	S_1	4	$-$	0	0
	S_2	5	$+$	0	0
	L_1	7	$-$	0	0
Total					0
C_2	L_1	1	$+$	1	1
	S_1	4	$-$	-1	1
	L_1	7	$+$	1	1
	S_3	8	$-$	0	0
Total					0
C_3	L_1	1	$+$	-1	-1
	S_2	5	$-$	-1	1
	L_1	7	$+$	-1	-1
	S_3	8	$-$	-1	1
Total					0

Table 5.11: Jump assignments for the edges of length at most 8, in the case that $c = 2$, $d > 3$ and $\tilde{k} = 3$.

Note that the number of edges in each starter central cycle that have been assigned jumps so far is even, and these edges contribute 0 to the displacement along each cycle. The edges that remain to be labelled are edges of length in L_1 greater than 8. By Lemma 5.7, we may assign jumps to the remaining edges so that the total displacement along each cycle is nonzero.

Subcase 1.1.1.6. $c = 2$ and $d = 3$

Note that we may assume that $\tilde{k} \neq 3$; otherwise, since $d = 3$, we have that $k = 9$, an exceptional case. In the case that $c = 2$ and $d = 3$, we have used an alternative choice of paths in the construction of peripheral cycles. The first path P contains edges of lengths $1, 3, 4, 5, \dots, \tilde{k}$, while the second path P' contains edges of lengths $\tilde{k}+1, \tilde{k}+2, \dots, 2\tilde{k}-2; 2\tilde{k}$. The linking edge lengths are $\tilde{m} \pm \frac{\tilde{k}+1}{2}$. The peripheral cycle C of multiplicity 1 is formed from P' together with linking length $\tilde{m} - \frac{\tilde{k}+1}{2}$. Note that

$c = \frac{r-1}{2\tilde{k}} + \frac{2}{3} = 2$, $\tilde{k} = \frac{k}{d} = \frac{k}{3}$ and $m = 2k + r$ implies that $m = \frac{26}{3}\tilde{k} + 1$, so that, since $d = 3$, $\tilde{m} = \frac{26}{9}\tilde{k} + \frac{1}{3}$. Hence $\tilde{m} - \frac{\tilde{k}+1}{2} = \frac{43}{18}\tilde{k} - \frac{1}{6} > 2\tilde{k}$. Hence we have that

$$L_2 = \left\{\tilde{k}+1, \tilde{k}+2, \ldots, 2\tilde{k}-2; 2\tilde{k}; \tilde{m} - \frac{\tilde{k}+1}{2}\right\},$$

and so we may write $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \left\{\tilde{k}+1, \tilde{k}+2, \ldots, \frac{4\tilde{k}}{3}\right\} \\ S_2 &= \left\{\frac{4\tilde{k}}{3}+1, \frac{4\tilde{k}}{3}+2, \ldots, \frac{5\tilde{k}}{3}\right\} \\ S_3 &= \left\{\frac{5\tilde{k}}{3}+1, \frac{5\tilde{k}}{3}+2, \ldots, 2\tilde{k}-2; 2\tilde{k}; \tilde{m} - \frac{\tilde{k}+1}{2}\right\} \end{aligned}$$

The edges of length in L_1 less than $\tilde{m} - \frac{\tilde{k}+1}{2}$ are

$$2; 2\tilde{k}-1; 2\tilde{k}+1, 2\tilde{k}+2, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}.$$

Note that since $\tilde{k} \equiv 3 \pmod 4$, the number of lengths between $2\tilde{k}+1$ and $\tilde{m} - \frac{\tilde{k}+3}{2}$ is even.

Table 5.12 shows the assignment of jumps to the edges of length at most $\tilde{m} - \frac{\tilde{k}+1}{2}$.

Note that an even number of edges in each starter central cycle have been assigned jumps so far, and the displacement along these edges is 0 in each cycle. The edges yet to be assigned jumps consist of those of length in L_1 greater than $\tilde{m} - \frac{\tilde{k}+1}{2}$. We may assign jumps to the remaining edges by Lemma 5.7.

Subcase 1.1.2. c is odd

Note that in the cases in which $\tilde{k} \equiv 3 \pmod 4$ and $c \geq 3$ is odd, we assign jump -1 rather than 1 to the linking edges of the peripheral cycle C. As previously stated, this does not affect the fact that the total displacement along C is nonzero. It does, however, mean that we can assign jumps 0 and 1 to the edges in the central cycles that share their length with the linking edges of C. If $c = 1$, then we assign jump 1 to the linking edges of C as before.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	2	$+$	0	0
	S_1	$\tilde{k}+1, \ldots, \frac{4\tilde{k}}{3}$	$-, +, \ldots, +, -$	$-1, 0, 0, \ldots, 0, -1$	-1
	S_2	$\frac{4\tilde{k}}{3}+1, \ldots, \frac{5\tilde{k}}{3}$	$+, -, \ldots, -, +$	$-1, 0, 0, \ldots, 0, -1$	1
	L_1	$2\tilde{k}-1$	$-$	0	0
	L_1	$2\tilde{k}+1, \ldots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$+, -, \ldots, +, -$	$0, 0, \ldots, 0$	0
Total					0
C_2	L_1	2	$+$	1	1
	S_1	$\tilde{k}+1, \ldots, \frac{4\tilde{k}}{3}$	$-, +, \ldots, +, -$	$0, -1, -1, \ldots, -1, 0$	-1
	S_3	$\frac{5\tilde{k}}{3}+1, \ldots, 2\tilde{k}-2$	$+, -, \ldots, -, +$	$-1, -1, \ldots, -1$	-1
	L_1	$2\tilde{k}-1$	$-$	1	-1
	S_3	$2\tilde{k}$	$+$	-1	-1
	L_1	$2\tilde{k}+1, \ldots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$-, +, \ldots, -, +$	$1, 1, \ldots, 1$	0
	S_3	$\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	0	0
Total					0
C_3	L_1	2	$+$	-1	-1
	S_2	$\frac{4\tilde{k}}{3}+1, \ldots, \frac{5\tilde{k}}{3}$	$-, +, \ldots, +, -$	$0, -1, -1, \ldots, -1, 0$	-1
	S_3	$\frac{5\tilde{k}}{3}+1, \ldots, 2\tilde{k}-2$	$+, -, \ldots, -, +$	$0, 0, \ldots, 0$	0
	L_1	$2\tilde{k}-1$	$-$	-1	1
	S_3	$2\tilde{k}$	$+$	0	0
	L_1	$2\tilde{k}+1, \ldots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$-, +, \ldots, -, +$	$-1, -1, \ldots, -1$	0
	S_3	$\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	-1	1
Total					0

Table 5.12: Jump assignments for the edges of length at most $\tilde{m}-\frac{\tilde{k}+1}{2}$, in the case that $\tilde{k} \equiv 3 \pmod 4$, $c = 2$ and $d = 3$.

Subcase 1.1.2.1. $c \geq 5$ and $\tilde{k} > 3$

Since $c \geq 5$, the paths $P_{0,0}$, $P^*_{0,0}$, $P_{1,0}$, $P^*_{1,0}$ and $P_{2,0}$ are all used in the formation of peripheral cycles. Since $c \geq 3$, we may use $P_{1,0}$ in the formation of the peripheral cycle of multiplicity 1. This path contains edges of lengths $\tilde{m}+1$, $\ldots$, $\tilde{m}+\frac{\tilde{k}-3}{2}$; $\tilde{m}+\frac{\tilde{k}+1}{2}$, $\ldots$, $\tilde{m}+\tilde{k}$. The linking length in this cycle may be taken as $2\tilde{m}-\frac{\tilde{k}-1}{2}$. We note that linking lengths $\tilde{m} \pm \frac{\tilde{k}-1}{2}$ are used with multiplicity 3 in the peripheral cycles, while $2\tilde{m}+\frac{\tilde{k}-1}{2}$ does not occur in any peripheral cycle.

Let $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \left\{\tilde{m}+1, \tilde{m}+2, \ldots, \tilde{m}+\frac{\tilde{k}}{3}\right\} \\ S_2 &= \left\{\tilde{m}+\frac{\tilde{k}}{3}+1, \tilde{m}+\frac{\tilde{k}}{3}+2, \ldots, \tilde{m}+\frac{\tilde{k}-3}{2}; \tilde{m}+\frac{\tilde{k}+1}{2}, \tilde{m}+\frac{\tilde{k}+3}{2}, \ldots, \tilde{m}+\frac{2\tilde{k}}{3}+1\right\} \\ S_3 &= \left\{\tilde{m}+\frac{2\tilde{k}}{3}+2, \tilde{m}+\frac{2\tilde{k}}{3}+3, \ldots, \tilde{m}+\tilde{k}; 2\tilde{m}-\frac{\tilde{k}-1}{2}\right\} \end{aligned}$$

First, let us suppose that $\tilde{m}-\frac{\tilde{k}-1}{2} > 2\tilde{k}-1$. In this case, the lengths in L_1 which are less than $2\tilde{m}-\frac{\tilde{k}-1}{2}$ are

$$\frac{\tilde{k}-1}{2}; 2\tilde{k}, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}; \tilde{m}-\frac{\tilde{k}-3}{2}, \ldots, \tilde{m}; 2\tilde{k}+\tilde{m}, \ldots, 2\tilde{m}-\frac{\tilde{k}+1}{2}.$$

Note that there is an even number of lengths between $2\tilde{k}$ and $\tilde{m}-\frac{\tilde{k}+1}{2}$, an odd number of lengths between $\tilde{m}-\frac{\tilde{k}-3}{2}$ and $\tilde{m}$, and an even number of lengths between $2\tilde{k}+\tilde{m}$ and $2\tilde{m}-\frac{\tilde{k}+1}{2}$. Hence the number of lengths in L_1 that are less than $2\tilde{m}-\frac{\tilde{k}-1}{2}$ is even.

Table 5.13 shows the assignment of jumps to the edges of length at most $2\tilde{m}-\frac{\tilde{k}-1}{2}$.

Next, consider the case in which $\tilde{m}-\frac{\tilde{k}-1}{2} \leq 2\tilde{k}-1$. It follows that $\tilde{m}+2\tilde{k}+1 \geq 2\tilde{m}-\frac{\tilde{k}-1}{2}$, and so, since the next-largest length in L_1 after $\tilde{m}$ is $\tilde{m}+2\tilde{k}+1$, we have that the largest length in L_1 which is less than $2\tilde{m}-\frac{\tilde{k}-1}{2}$ is $\tilde{m}$. Hence, the lengths in L_1 which are less than $2\tilde{m}-\frac{\tilde{k}-1}{2}$ are

$$\frac{\tilde{k}-1}{2}; 2\tilde{k}+1, \ldots, \tilde{m}.$$

Note that the number of lengths between $2\tilde{k}+1$ and $\tilde{m}$ is odd. Hence the number of lengths in L_1 which are less than $2\tilde{m}-\frac{\tilde{k}-1}{2}$ is even.

Table 5.14 shows the assignment of jumps to the edges of length at most $2\tilde{m}-\frac{\tilde{k}-1}{2}$ in the starter central cycles.

Note that in either case the number of edges in each starter central cycle which have been assigned jumps so far is even, and these edges contribute 0 to the displacement along each cycle. Those left to be labelled consist of the edges whose length is in L_1 and is greater than $2\tilde{m} - \frac{\tilde{k}-1}{2}$. Using Lemma 5.7, we may assign jumps to the remaining edges so that the total displacement along each cycle is nonzero.

Subcase 1.1.2.2. $c = 3$ and $\tilde{k} > 3$

In this case, only paths $P_{0,0}$, $P^*_{0,0}$ and $P_{1,0}$ are used to form peripheral cycles. However, we may still ensure that $P_{1,0}$ is used to form the peripheral cycle C with multiplicity 1, and that the linking edge used in this cycle is $2\tilde{m} - \frac{\tilde{k}-1}{2}$. The linking edges which are used of multiplicity 3 in peripheral cycles are $\tilde{m} \pm \frac{\tilde{k}-1}{2}$.

First, let us suppose that $\tilde{m} - \frac{\tilde{k}-1}{2} > 2\tilde{k} - 1$. In this case, the lengths in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}-1}{2}$ are

$$\frac{\tilde{k}-1}{2}; 2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}; \tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \tilde{m}; \tilde{m} + \tilde{k} + 1, \ldots, 2\tilde{m} - \frac{\tilde{k}+1}{2}.$$

Note that there is an even number of lengths between $2\tilde{k}$ and $\tilde{m} - \frac{\tilde{k}+1}{2}$, an odd number of lengths between $\tilde{m} - \frac{\tilde{k}-3}{2}$ and $\tilde{m}$, and an even number between $\tilde{m} + \tilde{k} + 1$ and $2\tilde{m} - \frac{\tilde{k}+1}{2}$. Hence the total number of lengths in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}-1}{2}$ is even.

Table 5.15 shows the assignment of jumps to the edges in each central cycle whose length is at most $2\tilde{m} - \frac{\tilde{k}-1}{2}$.

Next, suppose that $\tilde{m} - \frac{\tilde{k}-1}{2} \leq 2\tilde{k} - 1$. Then the lengths in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}-1}{2}$ are

$$\frac{\tilde{k}-1}{2}; 2\tilde{k} + 1, \ldots, \tilde{m}; \tilde{m} + \tilde{k} + 1, \ldots, 2\tilde{m} - \frac{\tilde{k}+1}{2}.$$

Note that the number of lengths between $2\tilde{k} + 1$ and $\tilde{m}$ is odd and the number of lengths between $\tilde{m} + \tilde{k} + 1$ and $2\tilde{m} - \frac{\tilde{k}+1}{2}$ is even, so that the number of edges in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}-1}{2}$ is even.

Table 5.16 shows the assignment of jumps to the edges of length at most $2\tilde{m} - \frac{\tilde{k}-1}{2}$ in each starter central cycle.

Note that in either case, the total displacement of the edges labelled so far in each starter central cycle is 0, and also that an even number of edges have been assigned jumps in each central cycle. The edges remaining to be assigned jumps are those in L_1 of length greater than $2\tilde{m} - \frac{\tilde{k}-1}{2}$. By Lemma 5.7, we can assign jumps to the remaining edges such that the total displacement is nonzero.

Subcase 1.1.2.3. $c \geq 3$ and $\tilde{k} = 3$

Note that since $2\tilde{k} < \tilde{m} < 3\tilde{k}$, we have that $\tilde{m} = 7$, so that $\tilde{m} - \frac{\tilde{k}-1}{2} > 2\tilde{k} - 1$. The path $P_{1,0}$, which is used to form the peripheral cycle C of multiplicity 1, contains edges of lengths 9 and 10, while the linking edge length in C is $2\tilde{m} - \frac{\tilde{k}-1}{2} = 13$. Hence we have $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \{9\} \\ S_2 &= \{10\} \\ S_3 &= \{13\} \end{aligned}$$

Paths $P_{0,0}$ and $P^*_{0,0}$, which contain edges of lengths 2 and 3, and 4 and 5, respectively, occur in peripheral cycles taken with multiplicity 3, as do the linking edge lengths $\tilde{m} - \frac{\tilde{k}-1}{2} = 6$ and $\tilde{m} + \frac{\tilde{k}-1}{2} = 8$.

Let us first suppose that $c = 3$, so that $P_{0,0}$, $P^*_{0,0}$ and $P_{1,0}$ are the only zig-zag paths which are used in the formation of peripheral cycles. The lengths in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}-1}{2} = 13$ are 1, 7, 11, 12.

Table 5.17 shows the jump labels of the edges of each central cycle with length at most 13.

Now suppose that $c \geq 5$. In this case, the path $P^*_{1,0}$, containing edges of length 11 and 12, is also used of multiplicity 3 in the peripheral cycles, so that the lengths in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}-1}{2} = 13$ are 1 and 7. Table 5.18 shows the jump assignments in this case.

In either case, an even number of edges in each starter central cycle have been assigned jump so far, and the displacement along the labelled edges in each central cycle is currently 0. The edges which remain to be assigned jumps are those in

L_1 which have length greater than 13. By Lemma 5.7, the remaining edges can be assigned jumps so that the total displacement along each cycle is nonzero.

Subcase 1.1.2.4. $c = 1$

In this case, there is only one peripheral cycle, formed from $P_{0,0}$ together with linking edges of length $\tilde{m} - \frac{\tilde{k}-1}{2}$. This peripheral cycle is taken of multiplicity 1.

First, suppose that $\tilde{k} = 3$. Here, $c = \lceil \frac{r-1}{2\tilde{k}} \rceil = \frac{r-1}{2\tilde{k}} + \frac{2}{3} = 1$ implies that $r - 1 = \frac{2\tilde{k}}{3} = 2$, so that $m - 1 = 2k + r - 1 = 2k + 2$. Therefore, since k is odd, we have that $d = \gcd(k, m) = \gcd(k, 2k+3) = 3$. Hence $k = 9$, which is a value not covered by the results of Chapter 4; consequently, the existence of a C_{27}-decomposition of $K_m * \overline{K_3}$ has been left as an exceptional case in the statement of Lemma 5.5.

So we may assume that $\tilde{k} > 3$. Here, we have that $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \left\{1, 2, \ldots, \frac{\tilde{k}}{3}\right\} \\ S_2 &= \left\{\frac{\tilde{k}}{3} + 1, \frac{\tilde{k}}{3} + 2, \ldots, \frac{\tilde{k}-3}{2}; \frac{\tilde{k}+1}{2}, \frac{\tilde{k}+3}{2}, \ldots, \frac{2\tilde{k}}{3} + 1\right\} \\ S_3 &= \left\{\frac{2\tilde{k}}{3} + 2, \frac{2\tilde{k}}{3} + 3, \ldots, \tilde{k}; \tilde{m} - \frac{\tilde{k}-1}{2}\right\} \end{aligned}$$

The lengths in L_1 which are less than $\tilde{m} - \frac{\tilde{k}-1}{2}$ are as follows:

$$\frac{\tilde{k}-1}{2}; \tilde{k}+1, \tilde{k}+2, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}.$$

Note that the number of lengths between $\tilde{k}+1$ and $\tilde{m} - \frac{\tilde{k}+1}{2}$ is even, so that an odd number of lengths in L_1 are less than $\tilde{m} - \frac{\tilde{k}-1}{2}$.

Table 5.19 shows the assignment of jumps to the edges from L_2 in each central cycle and those lengths in L_1 which are less than $\tilde{m} - \frac{\tilde{k}-1}{2}$.

An odd number of edges in each starter central cycle has so far been assigned jumps. The total displacement along edges so far labelled is 0 in C_1, -1 in C_2 and 1 in C_3. The remaining edges to be assigned jumps are those in L_1 with length greater than or equal to $\tilde{m} - \frac{\tilde{k}-3}{2} = \ell_q$. By Lemma 5.8, the remaining edges may be labelled so that each cycle has nonzero displacement.

Subcase 1.2. $\tilde{k} \equiv 1 \pmod 4$

Note that in this case, $\tilde{k} \geq 9$. Recall that when $\tilde{k} \equiv 1 \pmod 4$, the zig-zag path $P_{i,0}$ is defined to have edges of lengths $i\tilde{m}+1$, $i\tilde{m}+2$, ..., $i\tilde{m}+\frac{\tilde{k}-1}{2}$, $i\tilde{m}+\frac{\tilde{k}+3}{2}$, $i\tilde{m}+\frac{\tilde{k}+5}{2}$, ..., $i\tilde{m}+\tilde{k}-1$, $i\tilde{m}+\tilde{k}+1$. If $\tilde{m}-\frac{\tilde{k}-1}{2} > 2\tilde{k}-1$, then the path $P^*_{i,0}$ has edges of lengths $i\tilde{m}+\tilde{k}, i\tilde{m}+\tilde{k}+2, i\tilde{m}+\tilde{k}+3, \ldots, i\tilde{m}+2\tilde{k}-1$, while if $\tilde{m}-\frac{\tilde{k}-1}{2} \leq 2\tilde{k}-1$, then $P^*_{i,0}$ has edges of lengths $i\tilde{m}+\tilde{k}$, $i\tilde{m}+\tilde{k}+2$, $i\tilde{m}+\tilde{k}+3$, ..., $(i+1)\tilde{m}-\frac{\tilde{k}+3}{2}$, $(i+1)\tilde{m}-\frac{\tilde{k}-1}{2}$, $(i+1)\tilde{m}-\frac{\tilde{k}-3}{2}$, ..., $i\tilde{m}+2\tilde{k}$. Except in rare instances in which alternative choices of paths were constructed, the paths $P_{i,0}$ and $P^*_{i,0}$ are used to construct the peripheral cycles. If c, the number of starter peripheral cycles, is even, then paths $P_{i,0}$ and $P^*_{i,0}$, for $0 \leq i \leq \frac{c}{2}-1$, are used in the formation of peripheral cycles, together with linking edges of lengths $i\tilde{m} \pm \frac{\tilde{k}+1}{2}$, where $1 \leq i \leq \frac{c}{2}$. If c is odd, then paths $P_{i,0}$ for $0 \leq i \leq \frac{c-1}{2}$ and $P^*_{i,0}$ for $0 \leq i \leq \frac{c-3}{2}$ are used in the formation of peripheral cycles, together with linking edges of lengths $\tilde{m} \pm \frac{\tilde{k}+1}{2}, 2\tilde{m} - \frac{\tilde{k}+1}{2}$ and $i\tilde{m} \pm \frac{\tilde{k}+1}{2}$ for $3 \leq i \leq \frac{c+1}{2}$.

One peripheral cycle, C, occurs with multiplicity 1, while all the other peripheral cycles occur with multiplicity 3 in the decomposition. The set of edge lengths which occur in C is denoted by L_2 (where $|L_2| = \tilde{k}$). The set of edge lengths which do not occur in peripheral cycles is denoted by L_1. Each central cycle contains an edge of each length in L_1, and $\frac{2\tilde{k}}{3}$ edges of length in L_2.

Subcase 1.2.1. c is even

In the construction of peripheral cycles, we can guarantee that $\tilde{m} + \frac{\tilde{k}+1}{2}$ is the linking length in C, the peripheral cycle which occurs of multiplicity 1, while linking lenth $\tilde{m} - \frac{\tilde{k}+1}{2}$ is used of multiplicity 3 in the peripheral cycles.

Subcase 1.2.1.1. $c \geq 4$

Here we may take $P_{1,0}$ to be the path forming the peripheral cycle C which occurs of multiplicity 1. Note that L_2, the set of edge lengths which occur of multiplicity 1 in the peripheral cycles, is given by

$$L_2 = \{\tilde{m}+1, \tilde{m}+2, \ldots, \tilde{m}+\tilde{k}-1; \tilde{m}+\tilde{k}+1\}.$$

Recall that in the construction of central cycles, we partitioned L_2 into three sets S_1, S_2 and S_3, each of size $\frac{\tilde{k}}{3}$. We may let $S_1 = \{\tilde{m}+1, \tilde{m}+2, \ldots, \tilde{m}+\frac{\tilde{k}}{3}\}$, $S_2 = \{\tilde{m}+\frac{\tilde{k}}{3}+1, \tilde{m}+\frac{\tilde{k}}{3}+2, \ldots, \tilde{m}+\frac{2\tilde{k}}{3}\}$ and $S_3 = \{\tilde{m}+\frac{2\tilde{k}}{3}+1, \tilde{m}+\frac{2\tilde{k}}{3}+2, \ldots, \tilde{m}+\tilde{k}-1; \tilde{m}+\tilde{k}+1\}$. Let C_1, C_2 and C_3 denote the three starter central cycles, where C_1 contains edges of lengths in $S_1 \cup S_2$, C_2 contains edges of lengths in $S_1 \cup S_3$ and C_3 contains edges of lengths in $S_2 \cup S_3$.

Recall that L_1 denotes the set of edge lengths which do not occur in peripheral cycles (and thus occur in each central cycle). In the case $\tilde{m} - \frac{\tilde{k}+1}{2} > 2\tilde{k}-1$, the following are the lengths in L_1 which are less than $\tilde{m}+\tilde{k}+1$ (i.e. lengths that may affect the direction an edge of length in L_2 is traversed in a cycle):

$$\frac{\tilde{k}+1}{2}; 2\tilde{k}, 2\tilde{k}+1, \ldots, \tilde{m}-\frac{\tilde{k}+3}{2}; \tilde{m}-\frac{\tilde{k}-1}{2}, \tilde{m}-\frac{\tilde{k}-3}{2}, \ldots, \tilde{m}.$$

Since $\tilde{m}$ is odd, the number of lengths in L_1 between $2\tilde{k}$ and $\tilde{m}$ is odd. Table 5.20 shows the jumps assignments for the edges in L_2 and the edges in L_1 which are less than $\tilde{m}+\tilde{k}+1$ in this case.

In the case $\tilde{m} - \frac{\tilde{k}+1}{2} \leq 2\tilde{k}-1$, the following are the lengths in L_1 less than $\tilde{m}+\tilde{k}+1$:

$$\frac{\tilde{k}+1}{2}; 2\tilde{k}+1, 2\tilde{k}+2, \ldots, \tilde{m}.$$

The number of lengths between $2\tilde{k}+1$ and $\tilde{m}$ is odd. Table 5.21 illustrates the jump assignments for edges in each central cycle of length at most $\tilde{m}+\tilde{k}+1$ in this case.

In either case, the number of edges which have been assigned jumps in each starter central cycle is even, and the displacement along the labelled edges is 0 in each cycle. The edges remaining to be assigned jumps are those in L_1 which have length greater than $\tilde{m}+\tilde{k}+1$. The remaining edges can be assigned jumps by Lemma 5.7 so that the total displacement along each cycle is nonzero.

Subcase 1.2.1.2. $c = 2$, $d > 3$ and $\tilde{m} - \frac{\tilde{k}+1}{2} > 2\tilde{k} - 1$

In this case, the lengths of $P_{0,0}$ are used of multiplicity 3 in the peripheral cycles, and the lengths in $P^*_{0,0}$ are used of multiplicity 1.

Since $\tilde{m} - \frac{\tilde{k}+1}{2} > 2\tilde{k} - 1$, then the following lengths are in L_2:

$$\tilde{k}; \tilde{k}+2, \tilde{k}+3, \ldots, 2\tilde{k}-1; \tilde{m} + \frac{\tilde{k}+1}{2}.$$

We have $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \left\{\tilde{k}; \tilde{k}+2, \tilde{k}+3, \ldots, \frac{4\tilde{k}}{3}\right\} \\ S_2 &= \left\{\frac{4\tilde{k}}{3}+1, \frac{4\tilde{k}}{3}+2, \ldots, \frac{5\tilde{k}}{3}\right\} \\ S_3 &= \left\{\frac{5\tilde{k}}{3}+1, \frac{5\tilde{k}}{3}+2, \ldots, 2\tilde{k}-1; \tilde{m} + \frac{\tilde{k}+1}{2}\right\} \end{aligned}$$

The lengths in L_1 less than $\tilde{m} + \frac{\tilde{k}+1}{2}$ are

$$\frac{\tilde{k}+1}{2}; 2\tilde{k}, 2\tilde{k}+1, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}; \tilde{m} - \frac{\tilde{k}-1}{2}, \tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-1}{2}.$$

Note that since $\tilde{k} \equiv 1 \pmod 4$ and $\tilde{m}$ is odd, the number of lengths between $2\tilde{k}$ and $\tilde{m} - \frac{\tilde{k}+3}{2}$ is even, and the number of lengths between $\tilde{m} - \frac{\tilde{k}-1}{2}$ and $\tilde{m} + \frac{\tilde{k}-1}{2}$ is odd.

Table 5.22 illustrates the assignments of jumps to the edges of length at most $\tilde{m} + \frac{\tilde{k}+1}{2}$.

Note that in each starter central cycle, the number of edges which have so far been assigned jumps is even, and the displacement along these edges is 0. The edges which have yet to be assigned jumps are those in L_1 that have length greater than $\tilde{m} + \frac{\tilde{k}+1}{2}$. We may apply Lemma 5.7 to assign jumps to the remaining edges so that each cycle has nonzero total displacement.

Subcase 1.2.1.3. $c = 2$, $d > 3$ and $\frac{5\tilde{k}}{3} < \tilde{m} - \frac{\tilde{k}+1}{2} \leq 2\tilde{k} - 1$

In the case that $\tilde{m} - \frac{\tilde{k}+1}{2} \leq 2\tilde{k} - 1$, the lengths which occur of multiplicity 1 in peripheral cycles are

$$\tilde{k}; \tilde{k}+2, \tilde{k}+3, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}; \tilde{m} - \frac{\tilde{k}-1}{2}, \tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, 2\tilde{k}; \tilde{m} + \frac{\tilde{k}+1}{2}.$$

We have that $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \left\{\tilde{k}, \tilde{k}+2, \tilde{k}+3, \ldots, \frac{4\tilde{k}}{3}\right\} \\ S_2 &= \left\{\frac{4\tilde{k}}{3}+1, \frac{4\tilde{k}}{3}+2, \ldots, \frac{5\tilde{k}}{3}\right\} \\ S_3 &= \left\{\frac{5\tilde{k}}{3}+1, \frac{5\tilde{k}}{3}+2, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}; \tilde{m} - \frac{\tilde{k}-1}{2}, \tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, 2\tilde{k}; \tilde{m} + \frac{\tilde{k}+1}{2}\right\} \end{aligned}$$

The lengths in L_1 which are less than $\tilde{m} + \frac{\tilde{k}+1}{2}$ are:

$$\frac{\tilde{k}+1}{2}; 2\tilde{k}+1, 2\tilde{k}+2, \ldots, \tilde{m} + \frac{\tilde{k}-1}{2}.$$

Note that the number of lengths between $2\tilde{k}+1$ and $\tilde{m} + \frac{\tilde{k}-1}{2}$ is odd.

Table 5.23 shows the assignment of jumps to the edges of length at most $\tilde{m} + \frac{\tilde{k}+1}{2}$.

Note that in each starter central cycle, the number of edges which have so far been assigned jumps is even, and the displacement along these edges is 0. The edges which have yet to be assigned jumps are those in L_1 that have length greater than $\tilde{m} + \frac{\tilde{k}+1}{2}$. We may apply Lemma 5.7 to assign jumps to the remaining edges so that each cycle has nonzero total displacement.

Subcase 1.2.1.4. $c = 2$, $d > 3$ and $\tilde{m} - \frac{\tilde{k}+1}{2} \leq \frac{5\tilde{k}}{3}$

Note that since $\tilde{m} > 2\tilde{k}$ and $\tilde{k} > 3$, we have that $\tilde{m} - \frac{\tilde{k}+1}{2} > \frac{3\tilde{k}-1}{2} > \frac{4\tilde{k}}{3}$. Hence we may write $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \left\{\tilde{k}; \tilde{k}+2, \tilde{k}+3, \ldots, \frac{4\tilde{k}}{3}\right\} \\ S_2 &= \left\{\frac{4\tilde{k}}{3}+1, \frac{4\tilde{k}}{3}+2, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}; \tilde{m} - \frac{\tilde{k}-1}{2}, \tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \frac{5\tilde{k}}{3}+1\right\} \end{aligned}$$

$$S_3 = \left\{\frac{5\tilde{k}}{3}+2, \frac{5\tilde{k}}{3}+3, \ldots, 2\tilde{k}; \tilde{m}+\frac{\tilde{k}+1}{2}\right\}$$

The lengths in L_1 which are less than $\tilde{m}+\frac{\tilde{k}+1}{2}$ are:

$$\frac{\tilde{k}+1}{2}; 2\tilde{k}+1, 2\tilde{k}+1, \ldots, \tilde{m}+\frac{\tilde{k}-1}{2}.$$

Note that the number of lengths between $2\tilde{k}+1$ and $\tilde{m}+\frac{\tilde{k}-1}{2}$ is odd.

Table 5.24 shows the assignment of jumps to the edges of length at most $\tilde{m}+\frac{\tilde{k}+1}{2}$.

Note that in each starter central cycle, the number of edges which have so far been assigned jumps is even, and the displacement along these edges is 0. The edges which have yet to be assigned jumps are those in L_1 that have length greater than $\tilde{m}+\frac{\tilde{k}+1}{2}$. We may apply Lemma 5.7 to assign jumps to the remaining edges so that each cycle has nonzero total displacement.

Subcase 1.2.1.5. $c = 2$ and $d = 3$

In this case, an alternative choice of path is used in the formation of peripheral cycles. One path, P, contains edges of lengths $2, 3, \ldots, \tilde{k}$ and the other, P', contains edges of lengths $\tilde{k}+1, \ldots, 2\tilde{k}-1$. The linking lengths are $\tilde{m} \pm \frac{\tilde{k}-1}{2}$. The peripheral cycle C which occurs of multiplicity 1 is formed from path P' and linking edges of length $\tilde{m}-\frac{\tilde{k}-1}{2}$. Note that as in the corresponding case in the proof of Theorem 4.7, it can be determined that $\tilde{m} = \frac{26\tilde{k}+3}{9}$. Thus, $\tilde{m}-\frac{\tilde{k}-1}{2} = \frac{43\tilde{k}+15}{18} > 2\tilde{k}-1$. Hence we have that

$$L_2 = \left\{\tilde{k}+1, \tilde{k}+2, \ldots, 2\tilde{k}-1; \tilde{m}-\frac{\tilde{k}-1}{2}\right\},$$

and so we may write $L_2 = S_1 \cup S_2 \cup S_3$, where

$$S_1 = \left\{\tilde{k}+1, \tilde{k}+2, \ldots, \frac{4\tilde{k}}{3}\right\}$$
$$S_2 = \left\{\frac{4\tilde{k}}{3}+1, \frac{4\tilde{k}}{3}+2, \ldots, \frac{5\tilde{k}}{3}\right\}$$

$$S_3 = \left\{\frac{5\tilde{k}}{3}+1, \frac{5\tilde{k}}{3}+2, \ldots, 2\tilde{k}-1; \tilde{m}-\frac{\tilde{k}-1}{2}\right\}$$

The edges of length in L_1 which are less than $\tilde{m} - \frac{\tilde{k}-1}{2}$ are

$$1; 2\tilde{k}, 2\tilde{k}+1, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}.$$

Note that since $\tilde{k} \equiv 1 \pmod 4$, there is an odd number of lengths between $2\tilde{k}$ and $\tilde{m} - \frac{\tilde{k}+1}{2}$.

Table 5.25 shows the assignment of jumps to the edges of length at most $\tilde{m} - \frac{\tilde{k}-1}{2}$.

Note that the number of edges in each central cycle which have already been assigned jumps is even. In each starter central cycle, the displacement along the edges already labelled is 0. The edges to which jumps still need to be assigned are those of length in L_1 greater than $\tilde{m} - \frac{\tilde{k}-1}{2}$. We may assign jumps to the remaining edges by Lemma 5.7 so that the displacement along each cycle is nonzero.

Subcase 1.2.2. c is odd

When $\tilde{k} \equiv 1 \pmod 4$ and c is odd, we assign jump -1 rather than 1 to the linking edges of the peripheral cycle C; as previously mentioned, this does not affect the fact that the total displacement along C is nonzero. However, it does mean that the edges in the central cycles that have the same length as the linking edges of C may be assigned jumps 0 and 1.

We will first consider several cases in which $c \geq 3$; we thus discuss the construction in that case here. If $c \geq 3$, then we may use $P_{1,0}$ in the formation of the peripheral cycle C which occurs of multiplicity 1. This path contains edges of lengths $\tilde{m}+1, \tilde{m}+2, \ldots, \tilde{m}+\frac{\tilde{k}-1}{2}, \tilde{m}+\frac{\tilde{k}+3}{2}, \tilde{m}+\frac{\tilde{k}+5}{2}, \ldots, \tilde{m}+\tilde{k}-1, \tilde{m}+\tilde{k}+1$. Note that if $c = 3$, then $\tilde{m}+\tilde{k} \in L_1$; otherwise, the edges of length $\tilde{m}+\tilde{k}$ occur of multiplicity 3 in the peripheral cycles. The linking length in C may be taken to be $2\tilde{m} - \frac{\tilde{k}+1}{2}$, while edges of length $\tilde{m} \pm \frac{\tilde{k}+1}{2}$ are used of multiplicity 3 in the peripheral cycles, and length $2\tilde{m} + \frac{\tilde{k}+1}{2}$ is in L_1.

We may write $L_2 = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned}
S_1 &= \left\{\tilde{m}+1, \tilde{m}+2, \ldots, \tilde{m}+\frac{\tilde{k}}{3}\right\} \\
S_2 &= \left\{\tilde{m}+\frac{\tilde{k}}{3}+1, \tilde{m}+\frac{\tilde{k}}{3}+2, \ldots, \tilde{m}+\frac{\tilde{k}-1}{2}; \tilde{m}+\frac{\tilde{k}+3}{2}, \tilde{m}+\frac{\tilde{k}+5}{2}, \ldots, \tilde{m}+\frac{2\tilde{k}}{3}+1\right\} \\
S_3 &= \left\{\tilde{m}+\frac{2\tilde{k}}{3}+2, \tilde{m}+\frac{2\tilde{k}}{3}+3, \ldots, \tilde{m}+\tilde{k}-1; \tilde{m}+\tilde{k}+1; 2\tilde{m}-\frac{\tilde{k}+1}{2}\right\}
\end{aligned}$$

Subcase 1.2.2.1. $c \geq 5$ and $\tilde{m} - \frac{\tilde{k}+1}{2} > 2\tilde{k} - 1$

In this case, the lengths in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}+1}{2}$ are

$$\frac{\tilde{k}+1}{2}; 2\tilde{k}, 2\tilde{k}+1, \ldots, \tilde{m}-\frac{\tilde{k}+3}{2}; \tilde{m}-\frac{\tilde{k}-1}{2}, \ldots, \tilde{m}; \tilde{m}+2\tilde{k}, \ldots, 2\tilde{m}-\frac{\tilde{k}+3}{2}.$$

Note that there is an even number of lengths between $2\tilde{k}$ and $\tilde{m} - \frac{\tilde{k}+3}{2}$, an odd number of lengths between $\tilde{m} - \frac{\tilde{k}-1}{2}$ and $\tilde{m}$, and an even number of lengths between $\tilde{m} + 2\tilde{k}$ and $2\tilde{m} - \frac{\tilde{k}+3}{2}$. Hence the total number of lengths in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}+1}{2}$ is even.

Table 5.26 illustrates the assignment of jumps to the edges in the starter central cycles of length at most $2\tilde{m} - \frac{\tilde{k}+1}{2}$.

Note that in each starter central cycle, the number of edges assigned jump so far is even, and the displacement along these edges is 0. The remaining edges are those in L_1 of length greater than $2\tilde{m} - \frac{\tilde{k}+1}{2}$. We may apply Lemma 5.7 to assign jumps to the remaining edges so that each cycle has nonzero displacement.

Subcase 1.2.2.2. $c \geq 5$ and $\tilde{m} - \frac{\tilde{k}+1}{2} \leq 2\tilde{k} - 1$

Let us begin this case by noting that the conditions $\tilde{m} > 2\tilde{k}$ and $\tilde{m} - \frac{\tilde{k}+1}{2} \leq 2\tilde{k} - 1$ imply that $\tilde{m} + \tilde{k} + 1 < 2\tilde{m} - \frac{\tilde{k}+1}{2} \leq \tilde{m} + 2\tilde{k} - 1$. Since the lengths in $P_{1,0}$ and $P_{1,0}^*$ are $\tilde{m}+1, \ldots, 2\tilde{m} - \frac{\tilde{k}+3}{2}; 2\tilde{m} - \frac{\tilde{k}-1}{2}, \ldots, \tilde{m} + 2\tilde{k}$, the lengths in L_1 less than $2\tilde{m} - \frac{\tilde{k}+1}{2}$ are

$$\frac{\tilde{k}+1}{2}; 2\tilde{k}+1, 2\tilde{k}+2, \ldots, \tilde{m}.$$

Note the number of lengths between $2\tilde{k}+1$ and $\tilde{m}$ is odd.

Table 5.27 shows the assignment of jumps to the edges in each central cycle of length at most $2\tilde{m} - \frac{\tilde{k}+1}{2}$.

Note that in each starter central cycle, the number of edges assigned jump so far is even, and the displacement along these edges is 0. The remaining edges are those in L_1 of length greater than $2\tilde{m} - \frac{\tilde{k}+1}{2}$. We may apply Lemma 5.7 to assign jumps to the remaining edges so that each cycle has nonzero displacement.

Subcase 1.2.2.3. $c = 3$ and $\tilde{m} - \frac{\tilde{k}+1}{2} > 2\tilde{k} - 1$

Since $c = 3$, we now have that $\tilde{m} + \tilde{k} \in L_1$, as the path $P^*_{1,0}$ is not used. Hence the lengths in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}+1}{2}$ are

$$\frac{\tilde{k}+1}{2}; 2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}; \tilde{m} - \frac{\tilde{k}-1}{2}, \ldots, \tilde{m}; \tilde{m}+\tilde{k}; \tilde{m}+\tilde{k}+2, \ldots, 2\tilde{m} - \frac{\tilde{k}+3}{2}.$$

Note that there is an even number of lengths between $2\tilde{k}$ and $\tilde{m} - \frac{\tilde{k}+3}{2}$, an odd number of lengths between $\tilde{m} - \frac{\tilde{k}-1}{2}$ and $\tilde{m}$, and an odd number of lengths between $\tilde{m}+\tilde{k}+2$ and $2\tilde{m} - \frac{\tilde{k}+3}{2}$.

Table 5.28 shows the assignment of jumps to the edges in each starter central cycle of length at most $2\tilde{m} - \frac{\tilde{k}+1}{2}$.

Note that the number of edges in each central cycle which have so far been assigned jumps is even. Also, the displacement along the edges of each central cycle which have already been labelled is 0. The edges which still need to be assigned jumps are those in L_1 which have length greater than $2\tilde{m} - \frac{\tilde{k}+1}{2}$. We may apply Lemma 5.7 to assign jumps to the remaining edges such that the total displacement along any cycle is nonzero.

Subcase 1.2.2.4. $c = 3$ and $\tilde{m} - \frac{\tilde{k}+1}{2} \leq 2\tilde{k} - 1$

The lengths in L_1 which are less than $2\tilde{m} - \frac{\tilde{k}+1}{2}$ are

$$\frac{\tilde{k}+1}{2}; 2\tilde{k}+1, \ldots, \tilde{m}; \tilde{m}+\tilde{k}; \tilde{m}+\tilde{k}+2, \ldots, 2\tilde{m} - \frac{\tilde{k}+3}{2}.$$

Note that there is an odd number of lengths between $2\tilde{k}+1$ and $\tilde{m}$, and an odd number of lengths between $\tilde{m}+\tilde{k}+2$ and $2\tilde{m}-\frac{\tilde{k}+3}{2}$.

Table 5.29 shows the assignment of jumps to the edges of length at most $2\tilde{m}-\frac{\tilde{k}+1}{2}$ in each starter central cycle.

Note that the number of edges in each central cycle which have so far been assigned jumps is even. Also, the displacement along the edges of each central cycle which have already been labelled is 0. The edges which still need to be assigned jumps are those in L_1 which have length greater than $2\tilde{m}-\frac{\tilde{k}+1}{2}$. We may apply Lemma 5.7 to assign jumps to the remaining edges such that the total displacement along any cycle is nonzero.

Subcase 1.2.2.5. $c=1$

In this case, there is only one peripheral cycle C. This cycle is formed using the path $P_{0,0}$, with edges of lengths $1,2,\ldots,\frac{\tilde{k}-1}{2},\frac{\tilde{k}+3}{2},\frac{\tilde{k}+5}{2},\ldots,\tilde{k}-1,\tilde{k}+1$. Linking edges in C have length $\tilde{m}-\frac{\tilde{k}+1}{2}$.

We have that $L_2=S_1\cup S_2\cup S_3$, where

$$\begin{aligned}
S_1 &= \left\{1,2,\ldots,\frac{\tilde{k}}{3}\right\}\\
S_2 &= \left\{\frac{\tilde{k}}{3}+1,\frac{\tilde{k}}{3}+2,\ldots,\frac{\tilde{k}-1}{2};\frac{\tilde{k}+3}{2},\frac{\tilde{k}+5}{2},\ldots,\frac{2\tilde{k}}{3}+1\right\}\\
S_3 &= \left\{\frac{2\tilde{k}}{3}+2,\frac{2\tilde{k}}{3}+3,\ldots,\tilde{k}-1;\tilde{k}+1;\tilde{m}-\frac{\tilde{k}+1}{2}\right\}
\end{aligned}$$

The edges of L_1 which are less than $\tilde{m}-\frac{\tilde{k}+1}{2}$ are as follows:

$$\frac{\tilde{k}+1}{2};\tilde{k};\tilde{k}+2,\tilde{k}+3,\ldots,\tilde{m}-\frac{\tilde{k}+3}{2}.$$

Note that the number of lengths between $\tilde{k}+2$ and $\tilde{m}-\frac{\tilde{k}+3}{2}$ is odd.

Table 5.30 shows the assignment of jumps to the edges of each starter central cycle with length at most $\tilde{m}-\frac{\tilde{k}+1}{2}$.

An odd number of edges in each starter central cycle has so far been assigned jumps. The total displacement along edges so far labelled is 0 in C_1, 1 in C_2 and -1 in C_3. The remaining edges to be assigned jumps are those in L_1 with length greater than or equal to $\tilde{m} - \frac{\tilde{k}-1}{2} = \ell_q$. By Lemma 5.8, the remaining edges may be labelled so that each cycle has nonzero displacement.

Case 2. $d \equiv 0 \pmod 3$

Begin by assigning jumps to the cycles in the decomposition as in the $d \not\equiv 0 \pmod 3$ case. The central cycles still have nonzero displacement; however, now the peripheral cycles have total displacement 0. We will show how to modify this labelling so that each cycle has nonzero displacement.

We begin by noting a few cases in which a modification can be done. First, suppose that some central cycle C' with displacement 1 has a positively-traversed edge with length $\ell \in L_2$ assigned jump 0, where the edges of length ℓ in C have been assigned jump 1. Let e be an edge in C of length ℓ. There exists a translate $C'' = \rho^i(C')$ of C' which contains an edge e'' parallel to e, which is traversed in the positive direction and assigned jump 0. Switch the jump assignments on e and e'', and between the edges $\rho^j(e)$ and $\rho^j(e'')$ in translates $\rho^j(C)$ and $\rho^j(C'')$ for $j \in \{0, 1, \ldots, \tilde{m}-1\}$ (i.e. for all the translates of C). Then replacing jump 0 with jump 1 on $\rho^j(e'')$ increases the displacement along $\rho^j(C'')$ by 1, giving $\rho^j(C'')$ displacement 2, which is congruent to $-1 \pmod 3$. Replacing jump 1 with jump 0 on $\rho^j(e)$ subtracts 1 from the displacement along C if $\rho^j(e)$ is traversed in the positive direction, or adds 1 to the displacement if $\rho^j(e)$ is traversed in the negative direction – hence, the new displacement along $\rho^j(C)$ is nonzero in either case. We have thus given all translates of C nonzero displacement, and the translates of C' have displacement 1 or -1, depending on whether they have been affected by the switching.

Next, suppose that there is a central cycle C' with displacement -1 which has a negatively-traversed edge with length $\ell \in L_2$ that has been assigned jump 0, where the edges of length ℓ in C have been assigned jump 1. In this case, we may reverse the direction of traversal of C', so that C' now has displacement 1, and the edge of length ℓ is traversed in the positive direction. The modification of the jump assignment now may be done as in the previous paragraph so that each cycle has nonzero displacement.

Similar switchings may be done if there is a central cycle with displacement 1 with an edge of length in L_2 traversed in the negative direction and assigned jump -1, where the edges of the same length in C have been assigned jump 1 (replacing this jump with 1 gives such a cycle displacement -1, while switching from jump 1 to jump -1 in C makes its displacement nonzero), or if there is a central cycle with displacement -1 which has an edge of length in L_2 traversed in the positive direction and assigned jump -1 (reversing the direction of traversal of such a cycle means that it has displacement 1, and an edge traversed in the negative direction with assigned jump -1).

We now show that in each case, the jump assignment may be modified so that no cycle has displacement 0. First suppose that it is not the case that $\tilde{k} = 3$ and $c \geq 4$ is even. In all other cases, one of starter the central cycles has an edge of length in L_2 traversed in positive direction assigned jump 0 and an edge of length in L_2 traversed in negative direction assigned jump 0. Such a pair of edges may be found in the cycle C_1. Moreover, the edges of these lengths are assigned jumps 0 and -1 in the central cycles, so that the edges of this length in C have jump 1. Thus, no matter the current displacement along C_1 (1 or -1), the assignment of jumps may be modified by an appropriate switching of jumps so that each cycle has nonzero displacement.

The only case left to consider is when $\tilde{k} = 3$ and $c \geq 4$ is even. First, if C_3 has displacement 1, then since the edge of length $9 \in L_2$ is traversed in the positive direction and has jump 0 (noting also that the edge in C of length 9 has jump 1), then the switching may be done as detailed above. Suppose now that C_3 has displacement -1. If C_1 has displacement 1, then noting that C_1 has an edge of length $9 \in L_2$ traversed in the negative direction and assigned jump -1, we see that a switching can also be done. So we now suppose that C_1 and C_3 have displacement -1. If C_2 has displacement -1, then since C_2 contains an edge of length $10 \in L_2$ which is traversed in the negative direction and has jump 0, and also noting that the edge of length 10 in C has jump 1, a switching can again be done. Finally, we suppose that C_1 and C_3 have displacement -1 and C_2 has displacement 1. In this case, we do a switching on edges of length 8. The edge of length 8 in C_2 has jump 1, while the edges of length 8 in C have jump -1. Let e be an edge of length 8 in C, and let $C' = \rho^i(C_2)$ be a translate of C_2 which contains an edge e' parallel to e. Switch jumps between e and e'. Then since e' was traversed in the positive direction with jump 1, and C' originally had displacement 1, replacing the assigned jump of e' with -1 gives that C' now has displacement -1. Also, since C had displacement 0, replacing the jump -1 on e with jump 1 yields a nonzero displacement. Similarly switch jumps between the edge $\rho^j(e)$ in the translate $\rho^j(C)$ of C, for $0 \leq j \leq d-1$, and the edge $\rho^j(e')$. We now obtain a jump assignment such that each cycle in the decomposition has nonzero displacement. □

We have now completed the proof of Lemma 5.5. Combining the results of Lemmas 5.4 and 5.5, we obtain the main result of this section.

Theorem 5.9. *Let $k \neq 27$ be an odd integer which is divisible by 9. Then $C_k|K_m * \overline{K_3}$ if and only if m is odd, $3 \leq k \leq 3m$ and $k|9\binom{m}{2}$.*

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \ldots, -, +$	$0, 0, \ldots, 0$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \tilde{m}$	$-, +, \ldots, +, -$	$0, 0, \ldots, 0$	0
	S_1	$\tilde{m} + 1, \ldots, \tilde{m} + \frac{\tilde{k}}{3}$	$+, -, \ldots, -, +$	$0, 0, \ldots, 0$	0
	S_2	$\tilde{m} + \frac{\tilde{k}}{3} + 1, \ldots, \tilde{m} + \frac{\tilde{k}-3}{2}$	$-, +, \ldots, +, -$	$-1, -1, \ldots, -1$	1
	S_2	$\tilde{m} + \frac{\tilde{k}+1}{2}, \ldots, \tilde{m} + \frac{2\tilde{k}}{3} + 1$	$+, -, \ldots, +, -$	$-1, 0, \ldots, 0$	-1
	L_1	$\tilde{m} + 2\tilde{k}, \ldots, 2\tilde{m} - \frac{\tilde{k}+1}{2}$	$+, - \ldots, +, -$	$0, 0, \ldots, 0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}-1}{2}$	$+$	1	1
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \ldots, -, +$	$1, 1, \ldots, 1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \tilde{m}$	$-, +, \ldots, +, -$	$1, 1, \ldots, 1$	-1
	S_1	$\tilde{m} + 1, \ldots, \tilde{m} + \frac{\tilde{k}}{3}$	$+, -, \ldots, -, +$	$-1, -1, \ldots, -1$	-1
	S_3	$\tilde{m} + \frac{2\tilde{k}}{3} + 2, \ldots, \tilde{m} + \tilde{k}$	$-, +, \ldots, -, +$	$-1, 0, \ldots, 0$	1
	L_1	$\tilde{m} + 2\tilde{k}, \ldots, 2\tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \ldots, -, +$	$1, 1, \ldots, 1$	0
	S_3	$2\tilde{m} - \frac{\tilde{k}-1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}-1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \ldots, -, +$	$-1, -1, \ldots, -1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-3}{2}, \ldots, \tilde{m}$	$-, +, \ldots, +, -$	$-1, -1, \ldots, -1$	1
	S_2	$\tilde{m} + \frac{\tilde{k}}{3} + 1, \ldots, \tilde{m} + \frac{\tilde{k}-3}{2}$	$+, -, \ldots, -, +$	$0, 0, \ldots, 0$	0
	S_2	$\tilde{m} + \frac{\tilde{k}+1}{2}, \ldots, \tilde{m} + \frac{2\tilde{k}}{3} + 1$	$-, +, \ldots, -, +$	$0, -1, -1, \ldots, -1$	-1
	S_3	$\tilde{m} + \frac{2\tilde{k}}{3} + 2, \ldots, \tilde{m} + \tilde{k}$	$-, +, \ldots, -, +$	$0, -1, -1, \ldots, -1$	-1
	L_1	$\tilde{m} + 2\tilde{k}, \ldots, 2\tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \ldots, -, +$	$-1, -1, \ldots, -1$	0
	S_3	$2\tilde{m} - \frac{\tilde{k}-1}{2}$	$-$	1	-1
Total					0

Table 5.13: Jump assignments for the edges of length at most $2\tilde{m} - \frac{\tilde{k}-1}{2}$ in the case that $\tilde{k} \equiv 3 \pmod 4$, $c \geq 5$ is odd, $\tilde{k} > 3$, and $\tilde{m} - \frac{\tilde{k}-1}{2} > 2\tilde{k} - 1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$0,0,\ldots,0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$-,+,\ldots,+,-$	$-1,-1,\ldots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}+1}{2},\ldots,\tilde{m}+\frac{2\tilde{k}}{3}+1$	$+,-,\ldots,+,-$	$-1,0,\ldots,0$	-1
Total					0
C_2	L_1	$\frac{\tilde{k}-1}{2}$	$+$	1	1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$1,1,\ldots,1$	-1
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2,\ldots,\tilde{m}+\tilde{k}$	$-,+,\ldots,-,+$	$-1,0,0,\ldots,0$	1
	S_3	$2\tilde{m}-\frac{\tilde{k}-1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}-1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$-1,-1,\ldots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$+,-,\ldots,-,+$	$0,0,\ldots,0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}+1}{2},\ldots,\tilde{m}+\frac{2\tilde{k}}{3}+1$	$-,+,\ldots,-,+$	$0,-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2,\ldots,\tilde{m}+\tilde{k}$	$-,+,\ldots,-,+$	$0,-1,-1,\ldots,-1$	-1
	S_3	$2\tilde{m}-\frac{\tilde{k}-1}{2}$	$-$	1	-1
Total					0

Table 5.14: Jump assignments for the edges of length at most $2\tilde{m}-\frac{\tilde{k}-1}{2}$ in the case that $\tilde{k}\equiv 3 \pmod 4$, $c\geq 5$ is odd, $\tilde{k}>3$ and $\tilde{m}-\frac{\tilde{k}-1}{2}\leq 2\tilde{k}-1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	L_1	$2\tilde{k}, \dots, \tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \dots, -, +$	$0, 0, \dots, 0$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-3}{2}, \dots, \tilde{m}$	$-, +, \dots, +, -$	$0, 0, \dots, 0$	0
	S_1	$\tilde{m} + 1, \dots, \tilde{m} + \frac{\tilde{k}}{3}$	$+, -, \dots, -, +$	$0, 0, \dots, 0$	0
	S_2	$\tilde{m} + \frac{\tilde{k}}{3} + 1, \dots, \tilde{m} + \frac{\tilde{k}-3}{2}$	$-, +, \dots, +, -$	$-1, -1, \dots, -1$	1
	S_2	$\tilde{m} + \frac{\tilde{k}+1}{2}, \dots, \tilde{m} + \frac{2\tilde{k}}{3} + 1$	$+, -, \dots, +, -$	$-1, 0, 0, \dots, 0$	-1
	L_1	$\tilde{m} + \tilde{k} + 1, \dots, 2\tilde{m} - \frac{\tilde{k}+1}{2}$	$+, -, \dots, +, -$	$0, 0, \dots, 0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}-1}{2}$	$+$	1	1
	L_1	$2\tilde{k}, \dots, \tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \dots, -, +$	$1, 1, \dots, 1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-3}{2}, \dots, \tilde{m}$	$-, +, \dots, +, -$	$1, 1, \dots, 1$	-1
	S_1	$\tilde{m} + 1, \dots, \tilde{m} + \frac{\tilde{k}}{3}$	$+, -, \dots, -, +$	$-1, -1, \dots, -1$	-1
	S_3	$\tilde{m} + \frac{2\tilde{k}}{3} + 2, \dots, \tilde{m} + \tilde{k}$	$-, +, \dots, -, +$	$-1, 0, 0, \dots, 0$	1
	L_1	$\tilde{m} + \tilde{k} + 1, \dots, 2\tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \dots, -, +$	$1, 1, \dots, 1$	0
	S_3	$2\tilde{m} - \frac{\tilde{k}-1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}-1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}, \dots, \tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \dots, -, +$	$-1, -1, \dots, -1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-3}{2}, \dots, \tilde{m}$	$-, +, \dots, +, -$	$-1, -1, \dots, -1$	1
	S_2	$\tilde{m} + \frac{\tilde{k}}{3} + 1, \dots, \tilde{m} + \frac{\tilde{k}-3}{2}$	$+, -, \dots, -, +$	$0, 0, \dots, 0$	0
	S_2	$\tilde{m} + \frac{\tilde{k}+1}{2}, \dots, \tilde{m} + \frac{2\tilde{k}}{3} + 1$	$-, +, \dots, -, +$	$0, -1, -1, \dots, -1$	-1
	S_3	$\tilde{m} + \frac{2\tilde{k}}{3} + 2, \dots, \tilde{m} + \tilde{k}$	$-, +, \dots, -, +$	$0, -1, -1, \dots, -1$	-1
	L_1	$\tilde{m} + \tilde{k} + 1, \dots, 2\tilde{m} - \frac{\tilde{k}+1}{2}$	$-, +, \dots, -, +$	$-1, -1, \dots, -1$	0
	S_3	$2\tilde{m} - \frac{\tilde{k}-1}{2}$	$-$	1	-1
Total					0

Table 5.15: Jump assignments for the edges of length at most $2\tilde{m} - \frac{\tilde{k}-1}{2}$ in the case that $\tilde{k} \equiv 3 \pmod 4$, $c = 3$, $\tilde{k} > 3$ and $\tilde{m} - \frac{\tilde{k}-1}{2} > 2\tilde{k} - 1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$0,0,\ldots,0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$-,+,\ldots,+,-$	$-1,-1,\ldots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}+1}{2},\ldots,\tilde{m}+\frac{2\tilde{k}}{3}+1$	$+,-,\ldots,+,-$	$-1,0,0,\ldots,0$	-1
	L_1	$\tilde{m}+\tilde{k}+1,\ldots,2\tilde{m}-\frac{\tilde{k}+1}{2}$	$+,-,\ldots,+,-$	$0,0,\ldots,0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}-1}{2}$	$+$	1	1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$1,1,\ldots,1$	-1
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2,\ldots,\tilde{m}+\tilde{k}$	$-,+,\ldots,-,+$	$-1,0,0,\ldots,0$	1
	L_1	$\tilde{m}+\tilde{k}+1,\ldots,2\tilde{m}-\frac{\tilde{k}+1}{2}$	$-,+,\ldots,-,+$	$1,1,\ldots,1$	0
	S_3	$2\tilde{m}-\frac{\tilde{k}-1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}-1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$-1,-1,\ldots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{\tilde{k}-3}{2}$	$+,-,\ldots,-,+$	$0,0,\ldots,0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}+1}{2},\ldots,\tilde{m}+\frac{2\tilde{k}}{3}+1$	$-,+,\ldots,-,+$	$0,-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2,\ldots,\tilde{m}+\tilde{k}$	$-,+,\ldots,-,+$	$0,-1,-1,\ldots,-1$	-1
	L_1	$\tilde{m}+\tilde{k}+1,\ldots,2\tilde{k}-\frac{\tilde{k}+1}{2}$	$-,+,\ldots,-,+$	$-1,-1,\ldots,-1$	0
	S_3	$2\tilde{m}-\frac{\tilde{k}-1}{2}$	$-$	1	-1
Total					0

Table 5.16: Jump assignments for the edges of length at most $2\tilde{m}-\frac{\tilde{k}-1}{2}$ in the case that $\tilde{k}\equiv 3 \pmod 4$, $c=3$, $\tilde{k}>3$ and $\tilde{m}-\frac{\tilde{k}-1}{2}\leq 2\tilde{k}-1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displacement
C_1	L_1	1, 7	+, −	0, 0	0
	S_1	9	+	0	0
	S_2	10	−	0	0
	L_1	11, 12	+, −	0, 0	0
Total					0
C_2	L_1	1, 7	+, −	−1, 1	1
	S_1	9	+	−1	−1
	L_1	11, 12	−, +	1, 1	0
	S_3	13	−	0	0
Total					0
C_3	L_1	1, 7	+, −	1, −1	−1
	S_2	10	+	−1	−1
	L_1	11, 12	−, +	−1, −1	0
	S_3	13	−	1	−1
Total					0

Table 5.17: Jump assignments for the edges of length at most 13 in the case that $c = 3$ and $\tilde{k} = 3$.

Cycle		Lengths in cycle	Edge direction	Jump	Displacement
C_1	L_1	1, 7	+, −	0, 0	0
	S_1	9	+	0	0
	S_2	10	−	0	0
Total					0
C_2	L_1	1, 7	+, −	−1, 1	1
	S_1	9	+	−1	−1
	S_3	13	−	0	0
Total					0
C_3	L_1	1, 7	+, −	1, −1	−1
	S_2	10	+	−1	−1
	S_3	13	−	1	−1
Total					0

Table 5.18: Jump assignments for the edges of length at most 13 in the case that $c \geq 5$ is odd and $\tilde{k} = 3$.

Cycle		Lengths in Cycle	Edge direction	Jump	Displ.
C_1	S_1	$1, \ldots, \frac{\tilde{k}}{3}$	$+, -, \ldots, -, +$	$0, 0, \ldots, 0$	0
	S_2	$\frac{\tilde{k}}{3}+1, \ldots, \frac{\tilde{k}-3}{2}$	$-, +, \ldots, +, -$	$-1, -1, \ldots, -1$	1
	L_1	$\frac{\tilde{k}-1}{2}$	$+$	0	0
	S_2	$\frac{\tilde{k}+1}{2}, \ldots, \frac{2\tilde{k}}{3}+1$	$-, +, \ldots, -, +$	$0, 0, \ldots, 0, -1$	-1
	L_1	$\tilde{k}+1, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}$	$-, +, \ldots, -, +$	$0, 0, \ldots, 0$	0
Total:					0
C_2	S_1	$1, \ldots, \frac{\tilde{k}}{3}$	$+, -, \ldots, -, +$	$-1, -1, \ldots, -1$	-1
	L_1	$\frac{\tilde{k}-1}{2}$	$-$	1	-1
	S_3	$\frac{2\tilde{k}}{3}+2, \ldots, \tilde{k}$	$+, -, \ldots, +, -$	$0, 0, \ldots, 0, -1$	1
	L_1	$\tilde{k}+1, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}$	$+, -, \ldots, +, -$	$1, 1, \ldots, 1$	0
	S_3	$\tilde{m}-\frac{\tilde{k}-1}{2}$	$+$	0	0
Total:					-1
C_3	S_2	$\frac{\tilde{k}}{3}+1, \ldots, \frac{\tilde{k}-3}{2}$	$+, -, \ldots, -, +$	$0, 0, \ldots, 0$	0
	L_1	$\frac{\tilde{k}-1}{2}$	$-$	-1	1
	S_2	$\frac{\tilde{k}+1}{2}, \ldots, \frac{2\tilde{k}}{3}+1$	$+, -, \ldots, +, -$	$-1, -1, \ldots, -1, 0$	-1
	S_3	$\frac{2\tilde{k}}{3}+2, \ldots, \tilde{k}$	$+, -, \ldots, +, -$	$-1, -1, \ldots, -1, 0$	-1
	L_1	$\tilde{k}+1, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}$	$+, -, \ldots, +, -$	$-1, -1, \ldots, -1$	0
	S_3	$\tilde{m}-\frac{\tilde{k}-1}{2}$	$+$	-1	-1
Total:					1

Table 5.19: Jump assignments for the edges of length at most $\tilde{m}-\frac{\tilde{k}-1}{2}$ in the case that $\tilde{k} \equiv 3 \pmod 4$, $c = 1$ and $\tilde{k} > 3$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	L_1	$2\tilde{k},\dots,\tilde{m}-\frac{\tilde{k}+3}{2}$	$-,+,\dots,-,+$	$0,0,\dots,0$	0
	L_1	$\tilde{m}-\frac{\tilde{k}-1}{2},\dots,\tilde{m}$	$-,+,\dots,+,-$	$0,0,\dots,0$	0
	S_1	$\tilde{m}+1,\dots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\dots,-,+$	$0,-1,-1,\dots,-1,0$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\dots,\tilde{m}+\frac{2\tilde{k}}{3}$	$-,+,\dots,+,-$	$0,-1,-1,\dots,-1,0$	-1
Total					0
C_2	L_1	$\frac{\tilde{k}+1}{2}$	$+$	1	1
	L_1	$2\tilde{k},\dots,\tilde{m}-\frac{\tilde{k}+3}{2}$	$-,+,\dots,-,+$	$1,1,\dots,1$	0
	L_1	$\tilde{m}-\frac{\tilde{k}-1}{2},\dots,\tilde{m}$	$-,+,\dots,+,-$	$1,1,\dots,1$	-1
	S_1	$\tilde{m}+1,\dots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\dots,-,+$	$-1,0,0,\dots,0,-1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+1,\dots,\tilde{m}+\tilde{k}-1$	$-,+,\dots,-,+$	$0,-1,-1,\dots,-1$	-1
	S_3	$\tilde{m}+\tilde{k}+1$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}+1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k},\dots,\tilde{m}-\frac{\tilde{k}+3}{2}$	$-,+,\dots,-,+$	$-1,-1,\dots,-1$	0
	L_1	$\tilde{m}-\frac{\tilde{k}-1}{2},\dots,\tilde{m}$	$-,+,\dots,+,-$	$-1,-1,\dots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\dots,\tilde{m}+\frac{2\tilde{k}}{3}$	$+,-,\dots,-,+$	$-1,0,0,\dots,0,-1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+1,\dots,\tilde{m}+\tilde{k}-1$	$-,+,\dots,-,+$	$-1,0,0,\dots,0$	1
	S_3	$\tilde{m}+\tilde{k}+1$	$-$	-1	1
Total					0

Table 5.20: Jump assignments for the edges of length at most $\tilde{m}+\tilde{k}+1$ in the case that $\tilde{k}\equiv 1 \pmod 4$, $c\geq 4$ is even and $\tilde{m}-\frac{\tilde{k}+1}{2} > 2\tilde{k}-1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	L_1	$2\tilde{k}+1, \ldots, \tilde{m}$	$-, +, \ldots, +, -$	$0, \ldots, 0$	0
	S_1	$\tilde{m}+1, \ldots, \tilde{m}+\frac{\tilde{k}}{3}$	$+, -, \ldots, -, +$	$0, -1, -1, \ldots, -1, 0$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1, \ldots, \tilde{m}+\frac{2\tilde{k}}{3}$	$-, +, \ldots, +, -$	$0, -1, -1, \ldots, -1, 0$	-1
Total					0
C_2	L_1	$\frac{\tilde{k}+1}{2}$	$+$	1	1
	L_1	$2\tilde{k}+1, \ldots, \tilde{m}$	$-, +, \ldots, +, -$	$1, 1, \ldots, 1$	-1
	S_1	$\tilde{m}+1, \ldots, \tilde{m}+\frac{\tilde{k}}{3}$	$+, -, \ldots, -, +$	$-1, 0, 0, \ldots, 0, -1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+1, \ldots, \tilde{m}+\tilde{k}-1$	$-, +, \ldots, -, +$	$0, -1, -1, \ldots, -1$	-1
	S_3	$\tilde{m}+\tilde{k}+1$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}+1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}+1, \ldots, \tilde{m}$	$-, +, \ldots, +, -$	$-1, -1, \ldots, -1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1, \ldots, \tilde{m}+\frac{2\tilde{k}}{3}$	$+, -, \ldots, -, +$	$-1, 0, 0, \ldots, 0, -1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+1, \ldots, \tilde{m}+\tilde{k}-1$	$-, +, \ldots, -, +$	$-1, 0, 0, \ldots, 0$	1
	S_3	$\tilde{m}+\tilde{k}+1$	$-$	-1	1
Total					0

Table 5.21: Jump assignments for the edges of length at most $\tilde{m}+\tilde{k}+1$ in the case that $\tilde{k} \equiv 1 \pmod 4$, $c \geq 4$ is even and $\tilde{m} - \frac{\tilde{k}+1}{2} \leq 2\tilde{k} - 1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	S_1	$\tilde{k}; \tilde{k}+2, \tilde{k}+3, \ldots, \frac{4\tilde{k}}{3}$	$-, +, \ldots, +, -$	$0, -1, -1, \ldots, -1$	0
	S_2	$\frac{4\tilde{k}}{3}+1, \ldots, \frac{5\tilde{k}}{3}$	$+, -, \ldots, -, +$	$0, -1, -1, \ldots, -1$	0
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}$	$-, +, \ldots, -, +$	$0, 0, \ldots, 0$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-1}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-1}{2}$	$-, +, \ldots, +, -$	$0, 0, \ldots, 0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}+1}{2}$	$+$	1	1
	S_1	$\tilde{k}; \tilde{k}+2, \tilde{k}+3, \ldots, \frac{4\tilde{k}}{3}$	$-, +, \ldots, +, -$	$-1, 0, 0, \ldots, 0$	1
	S_3	$\frac{5\tilde{k}}{3}+1, \ldots, 2\tilde{k}-1$	$+, -, \ldots, +, -$	$-1, -1, \ldots, -1, 0$	-1
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}$	$+, -, \ldots, +, -$	$1, 1, \ldots, 1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-1}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-1}{2}$	$+, -, \ldots, -, +$	$-1, -1, \ldots, -1$	-1
	S_3	$\tilde{m} + \frac{\tilde{k}+1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}+1}{2}$	$+$	-1	-1
	S_2	$\frac{4\tilde{k}}{3}+1, \ldots, \frac{5\tilde{k}}{3}$	$-, +, \ldots, +, -$	$-1, 0, 0, \ldots, 0$	1
	S_3	$\frac{5\tilde{k}}{3}+1, \ldots, 2\tilde{k}-1$	$+, -, \ldots, +, -$	$0, 0, \ldots, 0, -1$	1
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}$	$+, -, \ldots, +, -$	$-1, -1, \ldots, -1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-1}{2}, \ldots, \tilde{m} + \frac{\tilde{k}-1}{2}$	$+, -, \ldots, -, +$	$1, 1, \ldots, 1$	1
	S_3	$\tilde{m} + \frac{\tilde{k}+1}{2}$	$-$	-1	1
Total					0

Table 5.22: Jump assignments for the edges of length at most $\tilde{m} + \frac{\tilde{k}+1}{2}$, in the case that $\tilde{k} \equiv 1 \pmod 4$, $c = 2$, $d > 3$ and $\tilde{m} - \frac{\tilde{k}+1}{2} > 2\tilde{k} - 1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	S_1	$\tilde{k}; \tilde{k}+2, \tilde{k}+3, \dots, \frac{4\tilde{k}}{3}$	$-,+,\dots,+,-$	$0,0,\dots,0$	0
	S_2	$\frac{4\tilde{k}}{3}+1, \dots, \frac{5\tilde{k}}{3}$	$+,-,\dots,-,+$	$0,0,\dots,0$	0
	L_1	$2\tilde{k}+1, \dots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$-,+,\dots,+,-$	$0,0,\dots,0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}+1}{2}$	$+$	1	1
	S_1	$\tilde{k}; \tilde{k}+2, \tilde{k}+3, \dots, \frac{4\tilde{k}}{3}$	$-,+,\dots,+,-$	$-1,-1,\dots,-1$	1
	S_3	$\frac{5\tilde{k}}{3}+1, \dots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$+,-,\dots,+,-$	$-1,0,0,\dots,0$	-1
	S_3	$\tilde{m}-\frac{\tilde{k}-1}{2}, \dots, 2\tilde{k}$	$+,-,\dots,+,-$	$0,0,\dots,0$	0
	L_1	$2\tilde{k}+1, \dots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$+,-,\dots,-,+$	$-1,-1,\dots,-1$	-1
	S_3	$\tilde{m}+\frac{\tilde{k}+1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}+1}{2}$	$+$	-1	-1
	S_2	$\frac{4\tilde{k}}{3}+1 \dots, \frac{5\tilde{k}}{3}$	$-,+,\dots,+,-$	$-1,-1,\dots,-1$	1
	S_3	$\frac{5\tilde{k}}{3}+1, \dots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$+,-,\dots,+,-$	$0,-1,-1,\dots,-1$	1
	S_3	$\tilde{m}-\frac{\tilde{k}-1}{2}, \dots, 2\tilde{k}$	$+,-,\dots,+,-$	$-1,-1,\dots,-1$	0
	L_1	$2\tilde{k}+1, \dots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$+,-,\dots,-,+$	$1,1,\dots,1$	1
	S_3	$\tilde{m}+\frac{\tilde{k}+1}{2}$	$-$	-1	1
Total					0

Table 5.23: Jump assignments for the edges of length at most $\tilde{m}+\frac{\tilde{k}+1}{2}$ in the case that $\tilde{k} \equiv 1 \pmod 4$, $c=2$, $d>3$ and $\frac{5\tilde{k}}{3} < \tilde{m}-\frac{\tilde{k}+1}{2} \le 2\tilde{k}-1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	S_1	$\tilde{k}; \tilde{k}+2, \tilde{k}+3, \ldots, \frac{4\tilde{k}}{3}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
	S_2	$\frac{4\tilde{k}}{3}+1, \ldots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$+,-,\ldots,-,+$	$0,0,\ldots,0$	0
	S_2	$\tilde{m}-\frac{\tilde{k}-1}{2}, \ldots, \frac{5\tilde{k}}{3}+1$	$-,+,\ldots,-,+$	$0,0,\ldots,0$	0
	L_1	$2\tilde{k}+1, \ldots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}+1}{2}$	$+$	1	1
	S_1	$\tilde{k}; \tilde{k}+2, \tilde{k}+3, \ldots, \frac{4\tilde{k}}{3}$	$-,+,\ldots,+,-$	$-1,-1,\ldots,-1$	1
	S_3	$\frac{5\tilde{k}}{3}+2, \ldots, 2\tilde{k}$	$+,-,\ldots,+,-$	$-1,0,0,\ldots,0$	-1
	L_1	$2\tilde{k}+1, \ldots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m}+\frac{\tilde{k}+1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}+1}{2}$	$+$	-1	-1
	S_2	$\frac{4\tilde{k}}{3}+1, \ldots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$-,+,\ldots,+,-$	$-1,-1,\ldots,-1$	1
	S_2	$\tilde{m}-\frac{\tilde{k}-1}{2}, \ldots, \frac{5\tilde{k}}{3}+1$	$+,-,\ldots,+,-$	$-1,-1,\ldots,-1$	0
	S_3	$\frac{5\tilde{k}}{3}+2, \ldots, 2\tilde{k}$	$+,-,\ldots,+,-$	$0,-1,-1,\ldots,-1$	1
	L_1	$2\tilde{k}+1, \ldots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$+,-,\ldots,-,+$	$1,1,\ldots,1$	1
	S_3	$\tilde{m}+\frac{\tilde{k}+1}{2}$	$-$	-1	1
Total					0

Table 5.24: Jump assignment for the edges of length at most $\tilde{m}+\frac{\tilde{k}+1}{2}$, in the case that $\tilde{k} \equiv 1 \pmod 4$, $c=2$, $d>3$ and $\tilde{m}-\frac{\tilde{k}+1}{2} \leq \frac{5\tilde{k}}{3}$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	1	$+$	0	0
	S_1	$\tilde{k}+1, \tilde{k}+2, \ldots, \frac{4\tilde{k}}{3}$	$-, +, \ldots, +, -$	$0, 0, \ldots, 0$	0
	S_2	$\frac{4\tilde{k}}{3}+1, \frac{4\tilde{k}}{3}+2, \ldots, \frac{5\tilde{k}}{3}$	$+, -, \ldots, -, +$	$0, 0, \ldots, 0$	0
	L_1	$2\tilde{k}, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}$	$-, +, \ldots, +, -$	$0, 0, \ldots, 0$	0
Total					0
C_2	L_1	1	$+$	1	1
	S_1	$\tilde{k}+1, \tilde{k}+2, \ldots, \frac{4\tilde{k}}{3}$	$-, +, \ldots, +, -$	$-1, -1, \ldots, -1$	1
	S_3	$\frac{5\tilde{k}}{3}+1, \frac{5\tilde{k}}{3}+2, \ldots, 2\tilde{k}-1$	$+, -, \ldots, +, -$	$-1, 0, 0, \ldots, 0$	-1
	L_1	$2\tilde{k}, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}$	$+, -, \ldots, -, +$	$-1, -1, \ldots, -1$	-1
	S_3	$\tilde{m}-\frac{\tilde{k}-1}{2}$	$-$	0	0
Total					0
C_3	L_1	1	$+$	-1	-1
	S_2	$\frac{4\tilde{k}}{3}+1, \frac{4\tilde{k}}{3}+2, \ldots, \frac{5\tilde{k}}{3}$	$-, +, \ldots, +, -$	$-1, -1, \ldots, -1$	1
	S_3	$\frac{5\tilde{k}}{3}+1, \frac{5\tilde{k}}{3}+2, \ldots, 2\tilde{k}-1$	$+, -, \ldots, +, -$	$0, -1, -1, \ldots, -1$	1
	L_1	$2\tilde{k}, \ldots, \tilde{m}-\frac{\tilde{k}+1}{2}$	$+, -, \ldots, -, +$	$1, 1, \ldots, 1$	1
	S_3	$\tilde{m}-\frac{\tilde{k}-1}{2}$	$-$	-1	1
Total					0

Table 5.25: Jump assignments for the edges of length at most $\tilde{m}-\frac{\tilde{k}-1}{2}$, in the case that $\tilde{k} \equiv 1 \pmod 4$, $c = 2$ and $d = 3$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}$	$-, +, \ldots, -, +$	$0, 0, \ldots, 0$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-1}{2}, \ldots, \tilde{m}$	$-, +, \ldots, +, -$	$0, 0, \ldots, 0$	0
	S_1	$\tilde{m}+1, \ldots, \tilde{m}+\frac{\tilde{k}}{3}$	$+, -, \ldots, -, +$	$0, 0, \ldots, 0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1, \ldots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$-, +, \ldots, +, -$	$0, 0, \ldots, 0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}+3}{2}, \ldots, \tilde{m}+\frac{2\tilde{k}}{3}+1$	$+, -, \ldots, +, -$	$0, 0, \ldots, 0$	0
	L_1	$\tilde{m}+2\tilde{k}, \ldots, 2\tilde{m}-\frac{\tilde{k}+3}{2}$	$+, -, \ldots, +, -$	$0, 0, \ldots, 0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}+1}{2}$	$+$	1	1
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}$	$-, +, \ldots, -, +$	$1, 1, \ldots, 1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-1}{2}, \ldots, \tilde{m}$	$-, +, \ldots, +, -$	$1, 1, \ldots, 1$	-1
	S_1	$\tilde{m}+1, \ldots, \tilde{m}+\frac{\tilde{k}}{3}$	$+, -, \ldots, -, +$	$-1, -1, \ldots, -1$	-1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2, \ldots, \tilde{m}+\tilde{k}-1$	$-, +, \ldots, +, -$	$-1, 0, 0, \ldots, 0$	1
	S_3	$\tilde{m}+\tilde{k}+1$	$+$	0	0
	L_1	$\tilde{m}+2\tilde{k}, \ldots, 2\tilde{m}-\frac{\tilde{k}+3}{2}$	$-, +, \ldots, -, +$	$1, 1, \ldots, 1$	0
	S_3	$2\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}+1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}, \ldots, \tilde{m} - \frac{\tilde{k}+3}{2}$	$-, +, \ldots, -, +$	$-1, -1, \ldots, -1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-1}{2}, \ldots, \tilde{m}$	$-, +, \ldots, +, -$	$-1, -1, \ldots, -1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1, \ldots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$+, -, \ldots, -, +$	$-1, -1, \ldots, -1$	-1
	S_2	$\tilde{m}+\frac{\tilde{k}+3}{2}, \ldots, \tilde{m}+\frac{2\tilde{k}}{3}+1$	$-, +, \ldots, -, +$	$-1, -1, \ldots, -1$	0
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2, \ldots, \tilde{m}+\tilde{k}-1$	$-, +, \ldots, +, -$	$0, -1, -1, \ldots, -1$	0
	S_3	$\tilde{m}+\tilde{k}+1$	$+$	-1	-1
	L_1	$\tilde{m}+2\tilde{k}, \ldots, 2\tilde{m}-\frac{\tilde{k}+3}{2}$	$-, +, \ldots, -, +$	$-1, -1, \ldots, -1$	0
	S_3	$2\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	1	-1
Total					0

Table 5.26: Jump assignments for the edges of length at most $2\tilde{m} - \frac{\tilde{k}+1}{2}$, in the case that $\tilde{k} \equiv 1 \pmod 4$, $c \geq 5$ and $\tilde{m} - \frac{\tilde{k}+1}{2} > 2\tilde{k} - 1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$0,0,\ldots,0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{\tilde{k}-1}{2}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}+3}{2},\ldots,\tilde{m}+\frac{2\tilde{k}}{3}+1$	$+,-,\ldots,+,-$	$0,0,\ldots,0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}+1}{2}$	$+$	1	1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$1,1,\ldots,1$	-1
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2,\ldots,\tilde{m}+\tilde{k}-1$	$-,+,\ldots,+,-$	$-1,0,0,\ldots,0$	1
	S_3	$\tilde{m}+\tilde{k}+1$	$+$	0	0
	S_3	$2\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}+1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$-1,-1,\ldots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{\tilde{k}-1}{2}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_2	$\tilde{m}+\frac{\tilde{k}+3}{2},\ldots,\tilde{m}+\frac{2\tilde{k}}{3}+1$	$-,+,\ldots,-,+$	$-1,-1,\ldots,-1$	0
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2,\ldots,\tilde{m}+\tilde{k}-1$	$-,+,\ldots,+,-$	$0,-1,-1,\ldots,-1$	0
	S_3	$\tilde{m}+\tilde{k}+1$	$+$	-1	-1
	S_3	$2\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	1	-1
Total					0

Table 5.27: Jump assignments for the edges of length at most $2\tilde{m}-\frac{\tilde{k}+1}{2}$, in the case that $\tilde{k}\equiv 1 \pmod 4$, $c\geq 5$ and $\tilde{m}-\frac{\tilde{k}+1}{2}\leq 2\tilde{k}-1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	L_1	$2\tilde{k}, \dots, \tilde{m} - \frac{\tilde{k}+3}{2}$	$-,+,\dots,-,+$	$0,0,\dots,0$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-1}{2}, \dots, \tilde{m}$	$-,+,\dots,+,-$	$0,0,\dots,0$	0
	S_1	$\tilde{m}+1, \dots, \tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\dots,-,+$	$0,-1,-1,\dots,-1,0$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1, \dots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$-,+,\dots,+,-$	$0,0,\dots,0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}+3}{2}, \dots, \tilde{m}+\frac{2\tilde{k}}{3}+1$	$+,-,\dots,+,-$	$-1,0,0,\dots,0$	-1
	L_1	$\tilde{m}+\tilde{k}$	$+$	0	0
	L_1	$\tilde{m}+\tilde{k}+2, \dots, 2\tilde{m}-\frac{\tilde{k}+3}{2}$	$-,+,\dots,+,-$	$0,0,\dots,0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}+1}{2}$	$+$	1	1
	L_1	$2\tilde{k}, \dots, \tilde{m} - \frac{\tilde{k}+3}{2}$	$-,+,\dots,-,+$	$1,1,\dots,1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-1}{2}, \dots, \tilde{m}$	$-,+,\dots,+,-$	$1,1,\dots,1$	-1
	S_1	$\tilde{m}+1, \dots, \tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\dots,-,+$	$-1,0,0,\dots,0,-1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2, \dots, \tilde{m}+\tilde{k}-1$	$-,+,\dots,+,-$	$0,-1,-1,\dots,-1,0$	-1
	L_1	$\tilde{m}+\tilde{k}$	$+$	1	1
	S_3	$\tilde{m}+\tilde{k}+1$	$-$	0	0
	L_1	$\tilde{m}+\tilde{k}+2, \dots, 2\tilde{m}-\frac{\tilde{k}+3}{2}$	$+,-,\dots,-,+$	$-1,-1,\dots,-1$	-1
	S_3	$2\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}+1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}, \dots, \tilde{m} - \frac{\tilde{k}+3}{2}$	$-,+,\dots,-,+$	$-1,-1,\dots,-1$	0
	L_1	$\tilde{m} - \frac{\tilde{k}-1}{2}, \dots, \tilde{m}$	$-,+,\dots,+,-$	$-1,-1,\dots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1, \dots, \tilde{m}+\frac{\tilde{k}-1}{2}$	$+,-,\dots,-,+$	$-1,-1,\dots,-1$	-1
	S_2	$\tilde{m}+\frac{\tilde{k}+3}{2}, \dots, \tilde{m}+\frac{2\tilde{k}}{3}+1$	$-,+,\dots,-,+$	$0,-1,-1,\dots,-1$	-1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2, \dots, \tilde{m}+\tilde{k}-1$	$-,+,\dots,+,-$	$-1,0,0,\dots,0,-1$	-1
	L_1	$\tilde{m}+\tilde{k}$	$+$	-1	-1
	S_3	$\tilde{m}+\tilde{k}+1$	$-$	-1	1
	L_1	$\tilde{m}+\tilde{k}+2, \dots, 2\tilde{m}-\frac{\tilde{k}+3}{2}$	$+,-,\dots,-,+$	$1,1,\dots,1$	1
	S_3	$2\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	1	-1
Total					0

Table 5.28: Jump assignments for the edges of length at most $2\tilde{m} - \frac{\tilde{k}+1}{2}$, in the case that $\tilde{k} \equiv 1 \pmod 4$, $c = 3$ and $\tilde{m} - \frac{\tilde{k}+1}{2} > 2\tilde{k} - 1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$0,-1,-1,\ldots,-1,0$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{\tilde{k}-1}{2}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
	S_2	$\tilde{m}+\frac{\tilde{k}+3}{2},\ldots,\tilde{m}+\frac{2\tilde{k}}{3}+1$	$+,-,\ldots,+,-$	$-1,0,0,\ldots,0$	-1
	L_1	$\tilde{m}+\tilde{k}$	$+$	0	0
	L_1	$\tilde{m}+\tilde{k}+2,\ldots,2\tilde{m}-\frac{\tilde{k}+3}{2}$	$-,+,\ldots,+,-$	$0,0,\ldots,0$	0
Total					0
C_2	L_1	$\frac{\tilde{k}+1}{2}$	$+$	1	1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$1,1,\ldots,1$	-1
	S_1	$\tilde{m}+1,\ldots,\tilde{m}+\frac{\tilde{k}}{3}$	$+,-,\ldots,-,+$	$-1,0,0,\ldots,0,-1$	1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2,\ldots,\tilde{m}+\tilde{k}-1$	$-,+,\ldots,+,-$	$0,-1,-1,\ldots,-1,0$	-1
	L_1	$\tilde{m}+\tilde{k}$	$+$	1	1
	S_3	$\tilde{m}+\tilde{k}+1$	$-$	0	0
	L_1	$\tilde{m}+\tilde{k}+2,\ldots,2\tilde{m}-\frac{\tilde{k}+3}{2}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_3	$2\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	0	0
Total					0
C_3	L_1	$\frac{\tilde{k}+1}{2}$	$+$	-1	-1
	L_1	$2\tilde{k}+1,\ldots,\tilde{m}$	$-,+,\ldots,+,-$	$-1,-1,\ldots,-1$	1
	S_2	$\tilde{m}+\frac{\tilde{k}}{3}+1,\ldots,\tilde{m}+\frac{\tilde{k}-1}{2}$	$+,-,\ldots,-,+$	$-1,-1,\ldots,-1$	-1
	S_2	$\tilde{m}+\frac{\tilde{k}+3}{2},\ldots,\tilde{m}+\frac{2\tilde{k}}{3}+1$	$-,+,\ldots,-,+$	$0,-1,-1,\ldots,-1$	-1
	S_3	$\tilde{m}+\frac{2\tilde{k}}{3}+2,\ldots,\tilde{m}+\tilde{k}-1$	$-,+,\ldots,+,-$	$-1,0,0,\ldots,0,-1$	-1
	L_1	$\tilde{m}+\tilde{k}$	$+$	-1	-1
	S_3	$\tilde{m}+\tilde{k}+1$	$-$	-1	1
	L_1	$\tilde{m}+\tilde{k}+2,\ldots,2\tilde{m}-\frac{\tilde{k}+3}{2}$	$+,-,\ldots,-,+$	$1,1,\ldots,1$	1
	S_3	$2\tilde{m}-\frac{\tilde{k}+1}{2}$	$-$	1	-1
Total					0

Table 5.29: Jump assignments for the edges of length at most $2\tilde{m}-\frac{\tilde{k}+1}{2}$, in the case that $\tilde{k}\equiv 1 \pmod 4$, $c=3$ and $\tilde{m}-\frac{\tilde{k}+1}{2}\leq 2\tilde{k}-1$.

Cycle		Lengths in cycle	Edge direction	Jump	Displ.
C_1	S_1	$1, \dots, \frac{\tilde{k}}{3}$	$+, -, \dots, -, +$	$0, 0, \dots, 0$	0
	S_2	$\frac{\tilde{k}}{3}+1, \dots, \frac{\tilde{k}-1}{2}$	$-, +, \dots, +, -$	$0, 0, \dots, 0$	0
	L_1	$\frac{\tilde{k}+1}{2}$	$+$	0	0
	S_2	$\frac{\tilde{k}+3}{2}, \dots, \frac{2\tilde{k}}{3}+1$	$-, +, \dots, -, +$	$0, 0, \dots, 0$	0
	L_1	$\tilde{k}$	$-$	0	0
	L_1	$\tilde{k}+2, \dots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$+, -, \dots, -, +$	$0, 0, \dots, 0$	0
Total					0
C_2	S_1	$1, \dots, \frac{\tilde{k}}{3}$	$+, -, \dots, -, +$	$-1, -1, \dots, -1$	-1
	L_1	$\frac{\tilde{k}+1}{2}$	$-$	-1	1
	S_3	$\frac{2\tilde{k}}{3}+2, \dots, \tilde{k}-1$	$+, -, \dots, -, +$	$-1, -1, \dots, -1$	-1
	L_1	$\tilde{k}$	$-$	1	-1
	S_3	$\tilde{k}+1$	$+$	-1	-1
	L_1	$\tilde{k}+2, \dots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$-, +, \dots, +, -$	$1, 1, \dots, 1$	-1
	S_3	$\tilde{m}-\frac{\tilde{k}+1}{2}$	$+$	0	0
Total					-1
C_3	S_2	$\frac{\tilde{k}}{3}+1, \dots, \frac{\tilde{k}-1}{2}$	$+, -, \dots, -, +$	$-1, -1, \dots, -1$	-1
	L_1	$\frac{\tilde{k}+1}{2}$	$-$	1	-1
	S_2	$\frac{\tilde{k}+3}{2}, \dots, \frac{2\tilde{k}}{3}+1$	$+, -, \dots, +, -$	$-1, -1, \dots, -1$	0
	S_3	$\frac{2\tilde{k}}{3}+2, \dots, \tilde{k}-1$	$+, -, \dots, -, +$	$0, 0, \dots, 0$	0
	L_1	$\tilde{k}$	$-$	-1	1
	S_3	$\tilde{k}+1$	$+$	0	0
	L_1	$\tilde{k}+2, \dots, \tilde{m}-\frac{\tilde{k}+3}{2}$	$-, +, \dots, +, -$	$-1, -1, \dots, -1$	1
	S_3	$\tilde{m}-\frac{\tilde{k}+1}{2}$	$+$	1	1
Total					1

Table 5.30: Jump assignments for the edges of length at most $\tilde{m}-\frac{\tilde{k}+1}{2}$, in the case that $\tilde{k} \equiv 1 \pmod 4$ and $c = 1$.

Chapter 6

Some further decompositions of λK_m

In this chapter, we prove the following theorem.

Theorem 6.1. *Let k and n be even with $2 \leq k \leq n$. Then $C_k | kK_n$.*

Theorem 6.1 has been independently demonstrated by Smith [48], along with a proof that $C_k | kK_n$ if n is odd; thus Smith's paper completes the proof of necessary and sufficient conditions for the existence of a C_k-decomposition of kK_n. However, our proof of Theorem 6.1 is particularly dear to the author, and so we include it here. We note, moreover, that our method of proof is quite different from Smith's.

Before we prove Theorem 6.1, let us first state some lemmas regarding the existence of particular cycles; these cycles will be used to form our decompositions. Throughout the statements of these lemmas, we use the following notation. We will let n and k be even integers with $4 \leq k < n$. (The cases $k = 2$ and $k = n$ will be dealt with separately from the others in the proof of Theorem 6.1.) The lemmas concern cycles in the graphs K_{n-1} and $K_{n-1} \bowtie K_1$. Throughout, we will let $V(K_{n-1}) = \mathbb{Z}_{n-1}$ and $V(K_{n-1} \bowtie K_1) = \mathbb{Z}_{n-1} \cup \{\infty\}$. We thus consider edge lengths in K_{n-1}, namely $\{1, 2, \ldots, \frac{n}{2} - 1\}$, and edges incident with vertex ∞ will be called *central*. Throughout the following sequence of lemmas, we define integers q and r as follows: write $\frac{n}{2} - 1 = q(\frac{k}{2}) + r$, where $0 \le r < \frac{k}{2}$. Note that $q \ge 1$ since $k \le n-2$.

Lemma 6.2. *Let $i \in \{0, 1, \dots, q-1\}$. There exists a cycle C_i of length k in kK_{n-1} which contains exactly two edges of length ℓ for each $\ell \in \{\frac{ik}{2}+1, \frac{ik}{2}+2, \dots, \frac{(i+1)k}{2}\}$.*

Proof. Let ρ denote the permutation $(0, 1, \dots, n-2)$, which cyclically permutes the vertices of kK_{n-1}.

The construction of cycles depends on whether $\frac{k}{2}$ is even or odd.

Case 1. $k \equiv 0 \pmod 4$

Let

$$P_i = 0, \frac{ki}{2}+1, -1, \frac{ki}{2}+2, -2, \dots, -\left(\frac{k}{4}-1\right), \frac{ki}{2}+\frac{k}{4}.$$

Then P_i is a zig-zag path containing edges of lengths $\frac{ki}{2}+1, \frac{ki}{2}+2, \dots, \frac{k(i+1)}{2}-1$. Note that P_i has initial and terminal vertices 0 and $\frac{ki}{2}+\frac{k}{4}$, and that $\overleftarrow{\rho^{-k(i+1)/2}(P_i)}$ (the reversal of $\rho^{-k(i+1)/2}(P_i)$) has initial and terminal vertices $-\frac{k}{4}$ and $-\frac{k(i+1)}{2}$, respectively. Let $e_i = \frac{ki}{2}+\frac{k}{4}, -\frac{k}{4}$ and $e_i' = -\frac{k(i+1)}{2}, 0$. (Note that e_i and e_i' are paths containing a single edge.)

We define C_i by forming the concatenation of P_i with e_i, $\overleftarrow{\rho^{-k(i+1)/2}(P_i)}$, and e_i', that is,

$$C_i = P_i e_i \overleftarrow{\rho^{-k(i+1)/2}(P_i)} e_i'.$$

The form of a cycle constructed in this manner may be seen in Figure 6.1. A further example of such a cycle may be seen in Figure 6.3. Note that C_i contains exactly two edges of each length in $\{\frac{ki}{2}+1, \frac{ki}{2}+2, \dots, \frac{k(i+1)}{2}\}$. It remains only to show that C_i is indeed a cycle. Since P_i has initial vertex 0, it is clear that C_i is a closed trail. Note that the vertices of P_i are contained in the interval $[-(\frac{k}{4}-1), \frac{ki}{2}+\frac{k}{4}]$ and the vertices of $\overleftarrow{\rho^{-k(i+1)/2}(P_i)}$ are contained in the interval $[-(\frac{ki}{2}+\frac{3k}{4}-1), -\frac{k}{4}]$. Moreover, note that $0 < \frac{ki}{2}+\frac{3k}{4}-1 = \frac{k(i+1)}{2}+\frac{k}{4}-1 \le \frac{kq}{2}+\frac{k}{4}-1 \le \frac{n}{2}-1+\frac{k}{4}-1 < \frac{n}{2}+\frac{n}{4}-2 < n-1$. It is now

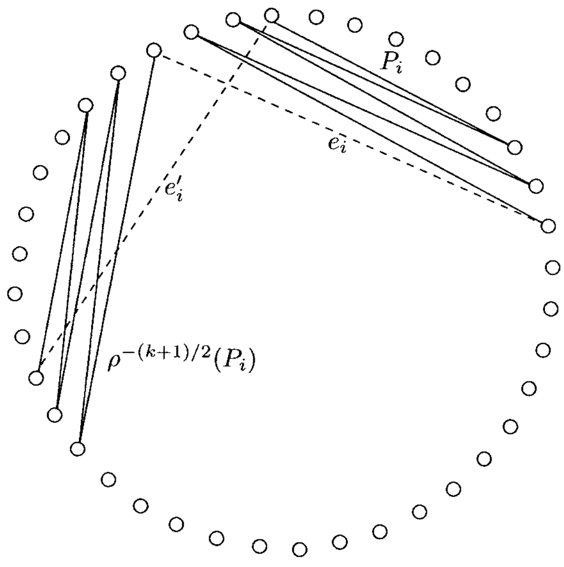

Figure 6.1: The cycle C_i in Case 1 of Lemma 6.2.

easy to see that C_i is a cycle provided the inequality $(n-1)-(\frac{ki}{2}+\frac{3k}{4}-1) > \frac{ki}{2}+\frac{k}{4}$ holds. This inequality is equivalent to $n > ki+k$, which is true since $k(i+1) \leq qk \leq n-2 < n$.

Hence C_i is the desired cycle.

Case 2. $k \equiv 2 \pmod 4$)

Let

$$P_i = 0, \frac{ki}{2}+1, -1, \frac{ki}{2}+2, -2, \ldots, \frac{ki}{2}+\frac{k-2}{4}, -\left(\frac{k-2}{4}\right).$$

Then P_i is a zig-zag path containing edges of lengths $\frac{ki}{2}+1, \frac{ki}{2}+2, \ldots, \frac{k(i+1)}{2}-1$. Note that P_i has initial and terminal vertices 0 and $-(\frac{k-2}{4})$, respectively, and that $\overleftarrow{\rho^{k(i+1)/2}(P_i)}$ (the reversal of $\rho^{k(i+1)/2}(P_i)$) has initial and terminal vertices $\frac{ki}{2}+\frac{k+2}{4}$ and $\frac{k(i+1)}{2}$, respectively. Let $e_i = -(\frac{k-2}{4}), \frac{ki}{2}+\frac{k+2}{4}$ and $e_i' = \frac{k(i+1)}{2}, 0$. (Note that e_i and e_i' are paths of length 1.)

Define C_i as the concatenation of P_i, e_i, $\overleftarrow{\rho^{k(i+1)/2}(P_i)}$ and e_i', that is,

$$C_i = P_i e_i \overleftarrow{\rho^{k(i+1)/2}(P_i)} e_i'.$$

The form of a cycle constructed in this manner may be seen in Figure 6.2. A further example of such a cycle may be seen in Figure 6.4. Note that C_i contains two edges

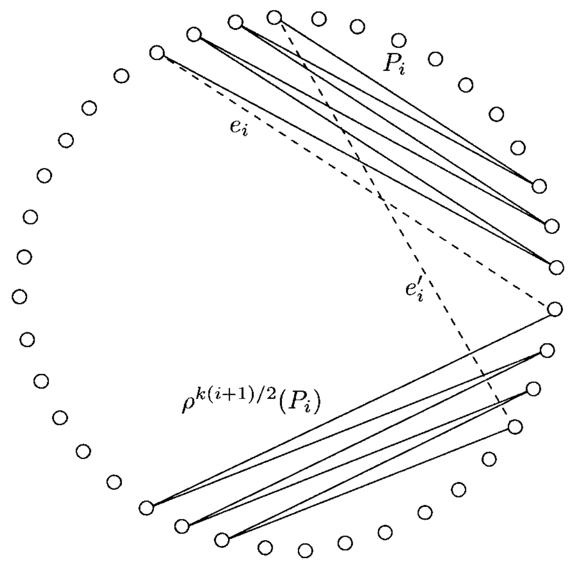

Figure 6.2: The cycle C_i in Case 2 of Lemma 6.2.

of each length in $\{\frac{ki}{2}+1, \frac{ki}{2}+2, \ldots, \frac{k(i+1)}{2}\}$. It remains only to show that C_i is a cycle. Since the initial vertex of P_i is 0, it is clear that C_i is a closed trail. Note that the vertices of P_i are contained in the interval $[-(\frac{k-2}{4}), \frac{ki}{2}+\frac{k-2}{4}]$, and the vertices of $\overleftarrow{\rho^{k(i+1)/2}(P_i)}$ are contained in the interval $[\frac{ki}{2}+\frac{k+2}{4}, ki+\frac{3k-2}{4}]$. Moreover, note that $0 < ki + \frac{3k-2}{4} = k(i+1) - \frac{k+2}{4} \leq kq - 1 - \frac{k-2}{4} < n - 1 - \frac{k-2}{4}$. It is now easy to see that C_i is the desired cycle. $\square$

Lemma 6.3. *Suppose k and n are positive even integers and $k < n$. There is a k-cycle in $k(K_{n-1} \bowtie K_1)$ which contains $(k-2)$ edges of length $\frac{n}{2}-1$ and two central edges.*

Proof. Let $C = 0, \frac{n}{2}-1, n-2, \frac{n}{2}-2, n-3, \ldots, \frac{n}{2}-(\frac{k}{2}-1), n-\frac{k}{2}, \infty, 0$. It is easy to see that C contains $(k-2)$ edges of length $\frac{n}{2}-1$ and two central edges. An example of such a cycle may be seen in Figure 6.3.

The vertices in odd position of C consist of all those between $n-\frac{k}{2}$ and $n-1$ (i.e. 0), inclusive, while the non-central vertices in an even position in C consist of all those in the interval $[\frac{n}{2} - \frac{k}{2} + 1, \frac{n}{2} - 1]$. Note that $\frac{n}{2} - \frac{k}{2} + 1 \geq 1$ as $k \leq n-2$. To see that C is a cycle, it suffices to show that $\frac{n}{2} - 1 < n - \frac{k}{2}$; this inequality is true since $n > k > k-2$. □

Lemma 6.4. *Let $i \in \{1, 2, \ldots, \lfloor \frac{r}{2} \rfloor\}$ be fixed, and let $a_i = q(\frac{k}{2}) + 2i - 1$. There exists a k-cycle in $k(K_{n-1} \bowtie K_1)$ which contains exactly $\frac{k}{2} - 1$ edges of each length a_i and $a_i + 1$, and two central edges.*

Proof. Let $C = 0, a_i, -1, a_i - 1, -2, \ldots, a_i - (\frac{k}{2} - 2), -(\frac{k}{2} - 1), \infty, 0$. It is easy to see that C contains $\frac{k}{2} - 1$ edges of length a_i, $\frac{k}{2} - 1$ edges of length $a_i + 1$, and two central edges. Examples of cycles formed in this manner may be found in Figure 6.4.

To see that C is a cycle, note that the vertices of K_{n-1} encountered in an odd position of C are $0, -1, -2, \ldots, -(\frac{k}{2} - 1)$, and the noncentral vertices in an even position of C are $a_i - (\frac{k}{2} - 2)$, $a_i - (\frac{k}{2} - 3)$, $\ldots$, a_i. Also, note that $n > k$ implies $q \geq 1$, so that $a_i - (\frac{k}{2} - 2) = q(\frac{k}{2}) + 2i - 1 - (\frac{k}{2} - 2) \geq 2i + 1 > 1$, and moreover $a_i = q(\frac{k}{2}) + 2i - 1 \leq q(\frac{k}{2}) + 2(\frac{r}{2}) - 1 = \frac{n}{2} - 2$. So the noncentral vertices in even position of C occur in the interval $[1, \frac{n}{2} - 2]$. Also, since $-(\frac{k}{2} - 1) = n - 1 - (\frac{k}{2} - 1)$ and $n - 1 - (\frac{k}{2} - 1) > n - 1 - (\frac{n}{2} - 1) = \frac{n}{2}$, the vertices in odd position of C occur in the interval $[\frac{n}{2}, n-1] = [\frac{n}{2}, 0]$. There is hence no overlap between the vertices of C, so C is indeed a cycle. □

Lemma 6.5. *If $r > 0$, then there is a k-cycle in $k(K_{n-1} \bowtie K_1)$ containing exactly two edges of each length $q(\frac{k}{2}) + 1, q(\frac{k}{2}) + 2, \ldots, \frac{n}{2} - 1$, exactly $k - 2r - 2$ edges of length 1, and two central edges.*

Proof. Let us first suppose that r is even. In this case, we let $C = 0$, $\frac{n}{2} - 1$, 1, $\frac{n}{2} - 2$, 2, $\ldots$, $\frac{n}{2} - (\frac{r}{2} - 1)$, $\frac{r}{2} - 1$, $\frac{n}{2} - \frac{r}{2}$, $\frac{r}{2}$, $\frac{r}{2} + 1$, $\frac{r}{2} + 2$, $\ldots$, $\frac{r}{2} + \frac{k-2r-4}{2}$, $\frac{r}{2} + \frac{k-2r-2}{2}$, ∞, $-\left(\frac{r}{2} + \frac{k-2r-2}{2}\right)$, $-\left(\frac{r}{2} + \frac{k-2r-4}{2}\right)$, $\ldots$, $-(\frac{r}{2} + 2)$, $-\left(\frac{r}{2} + 1\right)$, $-\frac{r}{2}$, $-\left(\frac{n}{2} - \frac{r}{2}\right)$, $-(\frac{r}{2} - 1)$, $-\left(\frac{n-r+2}{2}\right)$, $\ldots$, -2, $-\left(\frac{n}{2} - 2\right)$, -1, $-\left(\frac{n}{2} - 1\right)$, 0. That is, C contains edges of length, in order along C, $\frac{n}{2} - 1$, $\frac{n}{2} - 2$, $\frac{n}{2} - 3$, $\ldots$, $\frac{n}{2} - r + 1$, $\frac{n}{2} - r$, followed by $(k - 2r - 2)/2$

edges of length 1, two central edges, $(k-2r-2)/2$ edges of length 1, and finally edges of lengths $\frac{n}{2}-r, \frac{n}{2}-r+1, \ldots, \frac{n}{2}-2, \frac{n}{2}-1$. (The second half of C – that occurring after the central vertex – is the mirror image of the first half.) Note that $\{\frac{n}{2}-r, \frac{n}{2}-r+1, \ldots, \frac{n}{2}-1\} = \{q(\frac{k}{2})+1, q(\frac{k}{2})+2, \ldots, \frac{n}{2}-1\}$. We remark that if $k=2r+2$, then the vertices adjacent to the central vertex ∞ are $\frac{r}{2}$ and $-\frac{r}{2}$. In this case, $|\{q(\frac{k}{2})+1, q(\frac{k}{2})+2, \ldots, \frac{n}{2}-1\}| = r = \frac{k}{2}-1$; C contains two edges of each length in $\{q(\frac{k}{2})+1, q(\frac{k}{2})+2, \ldots, \frac{n}{2}-1\}$ and no edges of length 1.

It remains to be shown that C is a cycle. Note that before the central vertex, C contains all vertices in the interval $[0, \frac{r}{2}+\frac{k-2r-2}{2}]$, and all vertices in the interval $[\frac{n}{2}-\frac{r}{2}, \frac{n}{2}-1]$. After the central vertex, C contains all vertices in the interval $[-(\frac{r}{2}+\frac{k-2r-2}{2}), -1]$, as well as all vertices in the interval $[-(\frac{n}{2}-1), -(\frac{n}{2}-\frac{r}{2})]$. Moreover, recalling that $r<\frac{k}{2}$, we have that $0 < \frac{r}{2}+\frac{k-2r-2}{2} = \frac{k}{2}-\frac{r}{2}-1 < \frac{n}{2}-\frac{r}{2} \le \frac{n}{2}-1 < \frac{n}{2} = n-1-(\frac{n}{2}-1) \le \frac{n+r-2}{2} = n-1-(\frac{n}{2}-\frac{r}{2}) < n-1-\frac{k-r-2}{2} \le n-2 < n-1$. It is now easy to see that C is indeed a cycle.

An example of a cycle constructed in this manner may be found in Figure 6.4.

Now suppose that r is odd. In this case, let $C = 0, \frac{n}{2}-1, 1, \frac{n}{2}-2, 2, \ldots, \frac{r-3}{2}, \frac{n}{2}-\frac{r-1}{2}, \frac{r-1}{2}, \frac{n}{2}-\frac{r+1}{2}, \frac{n}{2}-\frac{r+3}{2}, \frac{n}{2}-\frac{r+5}{2}, \ldots, \frac{n}{2}-\frac{k-r-3}{2}, \frac{n}{2}-\frac{k-r-1}{2}, \infty, -(\frac{n}{2}-\frac{k-r-1}{2}), -(\frac{n}{2}-\frac{k-r-3}{2}), \ldots, -(\frac{n}{2}-\frac{r+3}{2}), -(\frac{n}{2}-\frac{r+1}{2}), -(\frac{r-1}{2}), -(\frac{n}{2}-\frac{r-1}{2}), -(\frac{r-3}{2}), \ldots, -2, -(\frac{n}{2}-2), -1, -(\frac{n-2}{2}), 0$. Hence, C contains edges of length, in order along C, $\frac{n}{2}-1, \frac{n}{2}-2, \frac{n}{2}-3, \ldots, \frac{n}{2}-r+1, \frac{n}{2}-r$, followed by $(k-2r-2)/2$ edges of length 1, two central edges, $(k-2r-2)/2$ edges of length 1, and finally edges of lengths $\frac{n}{2}-r, \frac{n}{2}-r+1, \ldots, \frac{n}{2}-2, \frac{n}{2}-1$. Again, note that $\{\frac{n}{2}-r, \frac{n}{2}-r+1, \ldots, \frac{n}{2}-1\} = \{q(\frac{k}{2})+1, q(\frac{k}{2})+2, \ldots, \frac{n}{2}-1\}$. If $k=2r+2$, then the vertices adjacent to the central vertex ∞ are $\frac{n}{2}-\frac{r+1}{2}$ and $-(\frac{n}{2}-\frac{r+1}{2})$. In this case, $|\{q(\frac{k}{2})+1, q(\frac{k}{2})+2, \ldots, \frac{n}{2}-1\}| = r = \frac{k}{2}-1$; C contains two edges of each length in $\{q(\frac{k}{2})+1, q(\frac{k}{2})+2, \ldots, \frac{n}{2}-1\}$ and no edges of length 1.

It need only be shown that C is a cycle. Note that before the central vertex, C contains all vertices in the interval $[0, \frac{r-1}{2}]$ and all vertices in the interval $[\frac{n}{2}-\frac{k-r-1}{2}, \frac{n}{2}-1]$. After the central vertex, C contains all vertices in the interval

$[-(\frac{n}{2}-1), -(\frac{n}{2}-\frac{k-r-1}{2})]$ and all vertices in $[-\frac{r-1}{2}, -1]$. Since $k < n$, $r < \frac{k}{2}$ and $k \geq 4$, it follows that $0 \leq \frac{r-1}{2} < \frac{n}{2} - \frac{k-r-1}{2} \leq \frac{n}{2} - 1 < \frac{n}{2} = n-1-(\frac{n}{2}-1) \leq \frac{n}{2} + \frac{k-r-3}{2} = n-1-(\frac{n}{2}-\frac{k-r-1}{2}) < n-1-\frac{r-1}{2} \leq n-2 < n-1$. It is now easy to see that C is indeed a cycle.

An example of a cycle constructed in this manner may be found in Figure 6.3. □

Lemma 6.6. *There is a k-cycle in $k(K_{n-1} \bowtie K_1)$ containing $(k-2)$ edges of length 1 and two central edges.*

Proof. Let $C = 0, 1, 2, \ldots, k-2, \infty, 0$. Since $k < n$, it is easy to see that C is the desired cycle. □

Lemma 6.7. *There is a k-cycle in $k(K_{n-1} \bowtie K_1)$ containing exactly two edges of each length $2, 3, \ldots, \frac{k}{2}$ and exactly two central edges.*

Proof. If $k \equiv 0 \pmod 4$, let $C = 0,\ 2,\ -1,\ 3,\ -4,\ \ldots,\ -(\frac{k}{4}-1),\ \frac{k}{4}+1,\ \frac{3k}{4}+1,\ \frac{k}{4}+2,\ \frac{3k}{4},\ \frac{k}{4}+3,\ \frac{3k}{4}-1,\ \frac{k}{4}+4,\ \ldots,\ \frac{k}{2},\ \frac{k}{2}+2,\ \infty,\ 0$. An example of such a cycle may be found in Figure 6.3. The vertices in an odd position along C are contained in the intervals $[-(\frac{k}{4}-1), 0]$ and $[\frac{k}{2}+2, \frac{3k}{4}+1]$, while the noncentral vertices in an even position along C are contained in the interval $[2, \frac{k}{2}]$. Note that since $k \leq n-2$, we have that $\frac{3k}{4}+1 = k - \frac{k}{4} + 1 < n - \frac{k}{4} = (n-1) - (\frac{k}{4}-1) = -(\frac{k}{4}-1)$; it is now easy to see that C is a cycle. The lengths used in C, in order along C, are $2,\ 3,\ 4,\ \ldots,\ \frac{k}{2}$, followed by $\frac{k}{2},\ \frac{k}{2}-1,\ \ldots,\ 3,\ 2$, and finally two central edges.

If $k \equiv 2 \pmod 4$, let $C = 0,\ 2,\ -1,\ 3,\ -4,\ \ldots,\ \frac{k+2}{4},\ -(\frac{k-2}{4}),\ -(\frac{3k-2}{4}),\ -(\frac{k+2}{4}),\ -(\frac{3k-6}{4}),\ -(\frac{k+6}{4}),\ \ldots,\ -(\frac{k+2}{2}),\ -(\frac{k-2}{2}),\ \infty,\ 0$. The vertices in an odd position along C are contained in the interval $[-(\frac{k-2}{2}), 0]$, while the noncentral vertices in an even position along C are contained in the intervals $[-(\frac{3k-2}{4}), -(\frac{k+2}{2})]$ and $[2, \frac{k+2}{4}]$. Note that since $k \leq n-2$, $\frac{k+2}{4} = k - \frac{3k-2}{4} < n - 1 - \frac{3k-2}{4} = -(\frac{3k-2}{4})$; it follows that C is indeed a cycle. The lengths used in C, in order along C, are $2,\ 3,\ 4,\ \ldots,\ \frac{k}{2}$, followed by $\frac{k}{2},\ \frac{k}{2}-1,\ \ldots,\ 3,\ 2$, and finally two central edges. □

We now have all the tools necessary to prove Theorem 6.1.

Proof of Theorem 6.1: First if $k = 2$, then clearly $C_2|2K_n$. Next, if $k = n$, then since $C_n|2K_n$ by Corollary 4.6, and since $2|k$, it is easy to see that $C_k|kK_n$. We henceforth assume that $4 \leq k \leq n-2$.

View $K_n = K_{n-1} \bowtie K_1$, so that $kK_n = k(K_{n-1} \bowtie K_1)$. Let $V(K_{n-1}) = \mathbb{Z}_{n-1}$, $V(K_1) = \{\infty\}$, so that $V(kK_n) = \mathbb{Z}_{n-1} \cup \{\infty\}$. The edges in K_{n-1} have lengths 1, 2, ..., $\frac{n}{2} - 1$, while edges incident with ∞ are called central. Let $\frac{n}{2} - 1 = q(\frac{k}{2}) + r$, where $0 \leq r < k/2$. We divide the edge lengths into q sets of size $k/2$, namely $Q_i = \{ik/2+1, ik/2+2, \ldots, (i+1)k/2\}$ for $0 \leq i \leq q-1$, and one of size r, namely $R = \{qk/2+1, qk/2+2, \ldots, (n-2)/2\}$.

We will demonstrate how to form starter cycles for the decomposition. Let ρ denote the permutation $\rho = (0, 1, \ldots, n-2)(\infty)$, which cyclically permutes the elements of $\mathbb{Z}_{n-1}$ and leaves vertex ∞ fixed. For each starter cycle C described below, the decomposition will contain the cycles $\rho^i(C)$ for each $i \in \{0, 1, \ldots, n-2\}$. We wish to form a set of starter cycles containing k edges of each length in $\{1, 2, \ldots, \frac{n}{2} - 1\}$, and k central edges.

We now proceed based on the value of r.

Case 1. $r = 0$

We first form a starter cycle C containing $(k-2)$ edges of length 1 and two central edges, by Lemma 6.6. This cycle occurs with multiplicity 1 in the decomposition. Next, by Lemma 6.7, we can form a starter cycle C' containing two edges of each length in $2, 3, \ldots, \frac{k}{2}$ and two central edges. The cycle C' will occur with multiplicity $\frac{k}{2} - 1$. Note that in the starter cycles formed so far, there appear k central edges and $(k-2)$ edges of each length in Q_0.

For each $i \in \{0, 1, \ldots, q-1\}$ form, by Lemma 6.2, a starter cycle C_i containing two edges of each length in Q_i. The cycle C_0 is taken with multiplicity 1; this means that in the starter cycles, there are k edges of each length in Q_0. The remaining starter cycles C_1, C_2, …, C_{q-1} are taken with multiplicity $\frac{k}{2}$, so that each remaining length occurs k times in the starter cycles.

Case 2. $r = \frac{k}{2} - 1$

Note that since $k > 2$, we have that $r > 0$. If r is even, then partition the lengths in R into pairs. For each pair $\{a, b\}$, form a starter k-cycle by Lemma 6.4 containing $(\frac{k}{2} - 1)$ edges of length a, $(\frac{k}{2} - 1)$ of length b, and two central edges. Each such starter cycle will be taken of multiplicity 2, so that among these cycles, there are $(k - 2)$ edges of each length in R and $2r$ central edges.

If r is odd, then partition the lengths in $R - \{\frac{n}{2} - 1\}$ into pairs. As before, for each pair $\{a, b\}$, form a starter k-cycle by Lemma 6.4 containing $(\frac{k}{2} - 1)$ edges of length a, $(\frac{k}{2} - 1)$ of length b, and two central edges. Again, each such starter cycle will be taken of multiplicity 2, so that among these cycles, there are $(k - 2)$ edges of each length in $R - \{\frac{n}{2} - 1\}$ and $(2r - 2)$ central edges. In addition, we form using Lemma 6.3 a single starter cycle, to be taken of multiplicity 1, containing $(k - 2)$ edges of length $\frac{n}{2} - 1$ and two central edges.

In either case, we have used a total of $(k - 2)$ edges of each length in R and $2r$ central edges in the starter cycles formed so far.

We now construct, by Lemma 6.5, a starter cycle containing two edges of each length in R and two central edges. (Since $k = 2r + 2$, the cycle constructed in Lemma 6.5 contains no edges of length 1.) Taking this cycle with multiplicity 1, we see that the cycles so far contain k edges of each length in R and $2r + 2$, i.e. k, central edges.

Finally, we use Lemma 6.2 to construct, for each $i \in \{0, 1, \ldots, q - 1\}$, a starter cycle containing two edges of each length in Q_i. Each such starter cycle is taken of multiplicity $\frac{k}{2}$, so that each remaining edge length will appear k times in the starter cycles.

Case 3. $0 < r < \frac{k}{2} - 1$

We begin this case similarly to Case 2. That is, if r is even, then we partition the lengths in R into pairs. For each pair $\{a, b\}$, form a starter k-cycle by Lemma 6.4 containing $(\frac{k}{2} - 1)$ edges of length a, $(\frac{k}{2} - 1)$ of length b, and two central edges. Each such starter cycle will be taken of multiplicity 2, so that among these cycles, there are $(k - 2)$ edges of each length in R and $2r$ central edges.

If r is odd, then partition the lengths in $R - \{\frac{n}{2} - 1\}$ into pairs. As before, for each pair $\{a, b\}$, form a starter k-cycle by Lemma 6.4 containing $(\frac{k}{2} - 1)$ edges of length a, $(\frac{k}{2} - 1)$ of length b, and two central edges. Again, each such starter cycle will be taken of multiplicity 2, so that among these cycles, there are $(k - 2)$ edges of each length in $R - \{\frac{n}{2} - 1\}$ and $(2r - 2)$ central edges. In addition, we form using Lemma 6.3 a single starter cycle, to be taken of multiplicity 1, containing $(k - 2)$ edges of length $\frac{n}{2} - 1$ and two central edges.

In either case, we have used a total of $(k - 2)$ edges of each length in R and $2r$ central edges in the starter cycles formed so far.

We next construct, using Lemma 6.5, a starter k-cycle containing two edges of each length in R, $k - 2r - 2$ edges of length 1 and two central edges. This starter cycle occurs with multiplicity 1. We also construct, using Lemma 6.7, a starter k-cycle containing two edges of each length in $\{2, 3, \ldots, \frac{k}{2}\}$ and two central edges. This starter cycle is taken with multiplicity $\frac{k-2r-2}{2}$. Note that so far, we have used $k-2r-2$ edges of each length in Q_0, k edges of each length in R and k central edges.

By Lemma 6.2, we can construct a starter k-cycle containing two edges of each length in Q_0. Taking this starter cycle with multiplicity $(r + 1)$ means that we will now have used each length in Q_0 exactly k times. Finally, we again use Lemma 6.2 to form, for each $i \in \{1, 2, \ldots, q - 1\}$, a starter k-cycle containing two edges of each length in Q_i. These starter cycles occur with multiplicity $\frac{k}{2}$ in the decomposition, so that there are k edges of each remaining length among the starter cycles. $\square$

Example 6.8. The starter cycles of a C_8-decomposition of $8K_{12}$ are shown in Figure 6.3. Here $\frac{n-2}{2} = q\frac{k}{2} + r$, where $q = 1$ and $r = 1$. Note that $0 < r < \frac{k}{2} - 1$.

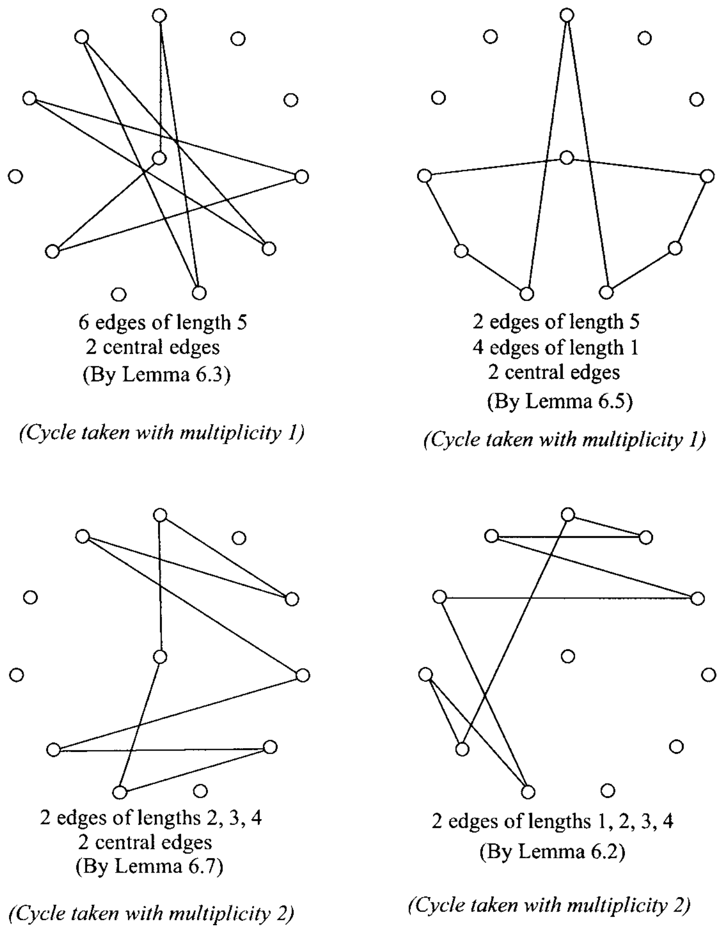

Figure 6.3: Starter cycles for a C_8-decomposition of $8K_{12}$.

Example 6.9. The starter cycles of a C_{10}-decomposition of $10K_{20}$ are shown in Figure 6.4. Here $\frac{n-2}{2} = q\frac{k}{2} + r$, where $q = 1$ and $r = 4$. Note that $r = \frac{k}{2} - 1$.

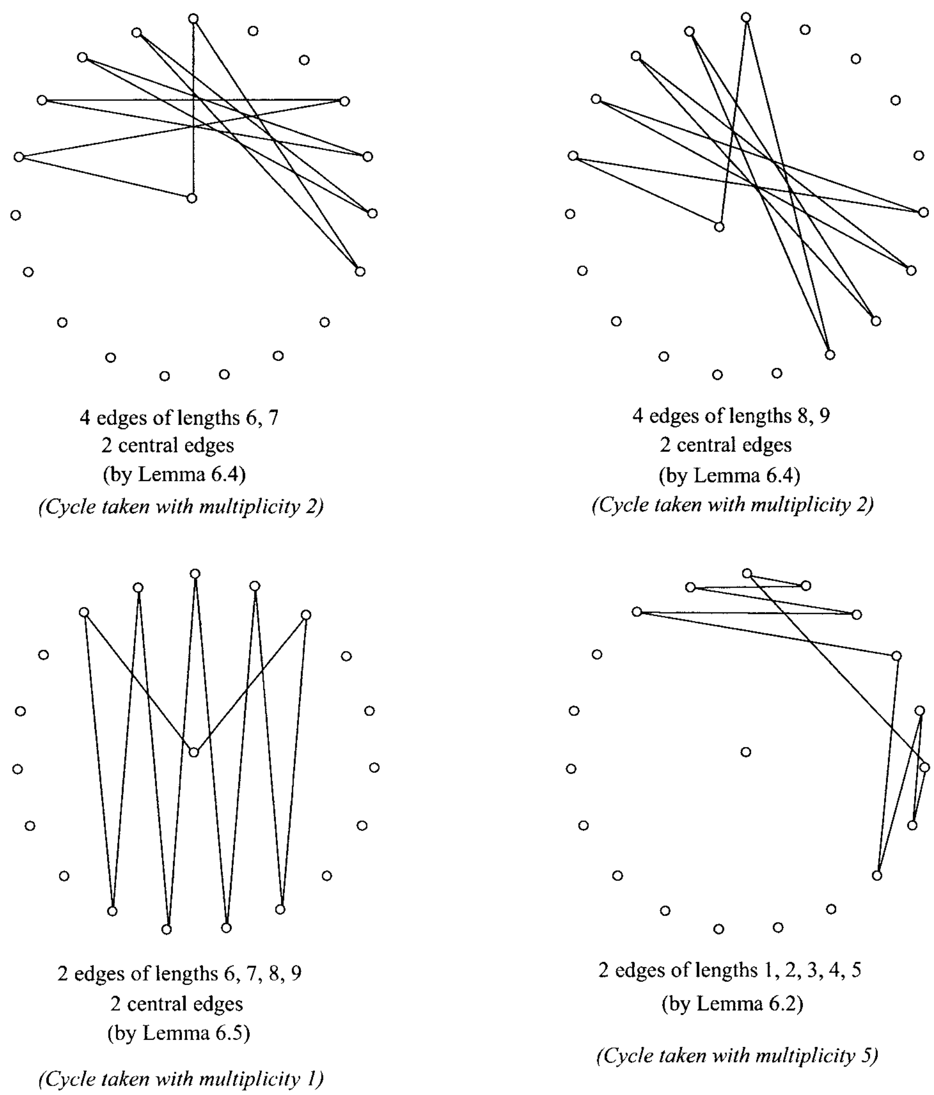

Figure 6.4: Starter cycles for a C_{10}-decomposition of $10K_{20}$.

Chapter 7

Conclusion and Open Problems

Finally, we sum up the results contained in this thesis, and present some open problems which remain to be solved related to the work of the thesis.

In Chapter 2, we presented some preliminary results on obtaining decompositions based on known ones. In particular, we stated known lemmas regarding blowing up cycle and closed trail decompositions of complete equipartite graphs, which were later used to construct such decompositions. We then presented several methods for converting a cycle decomposition of a complete multigraph to a cycle decomposition of a complete equipartite graph.

In Chapter 3, we found necessary and sufficient conditions for the existence of a decomposition of the complete equipartite graph $K_m * \overline{K_n}$ into closed trails of length k. Recall from Chapter 1 that the problem of determining when $K_m * \overline{K_n}$ decomposes into closed trails of lengths $k_1, k_2, \ldots, k_r$ has been solved in the cases that $n \in \{1, 2\}$ [4] and $m \in \{2, 3\}$ [9, 24]. The results contained in Chapter 3 prove necessary and sufficient conditions for closed trail decomposition of $K_m * \overline{K_n}$ in the case that $k_1 = k_2 = \cdots = k_r = k$. Although we did not deal with the case that the lengths of the trails in the decomposition may vary, we state the following open problem.

Question 7.1. Let $k_1, k_2, \ldots, k_r$ be integers (not necessarily all distinct). Obvious necessary conditions for the existence of a decomposition $K_m * \overline{K_n} = T_{k_1} \oplus T_{k_2} \oplus \cdots \oplus T_{k_r}$ are that $(m-1)n$ is even; $k_i \geq 3$ for each $i \in \{1, 2, \ldots, r\}$; $\binom{m}{2}n^2 = k_1 + k_2 + \cdots + k_r$; and if $m = 2$, then $k_1, k_2, \ldots, k_r$ are all even. Are these conditions sufficient?

Chapter 4 considered the problem of cycle decomposition of the complete multigraph $3K_m$. Our main result in this chapter was the establishment of necessary and sufficient conditions for the existence of a decomposition of $3K_m$ into cycles of odd length $k \neq 9$. (Recall that the case $k = 9$ was an exceptional value in which the proof of the induction step failed.) This leaves two cases to be resolved.

Question 7.2. Are the obvious necessary conditions for the existence of a decomposition of $3K_m$ into cycles of length 9, namely that $m \geq 9$ is odd and $9|3\binom{m}{2}$, sufficient?

Question 7.3. Are the obvious necessary conditions for the existence of a C_{2k}-decomposition of $3K_m$, namely that m is odd, $3 \leq 2k \leq m$ and $2k|3\binom{m}{2}$, sufficient?

Our decompositions of $3K_m$ in Chapter 4 were done by methods similar to those in [1, 2, 45] for decomposition of K_m and $2K_m$. We pose the following question.

Question 7.4. Can the methods begun in [1, 2, 45] and continued in Chapter 4 be extended to find cycle decompositions of λK_m for $\lambda > 3$?

The author suspects that similar methods may apply in the case that λ is prime, but this has not yet been verified.

In Chapter 5, we considered k-cycle decompositions of $K_m * \overline{K_3}$, the complete equipartite graph with parts of size 3. We proved that the obvious necessary conditions for the existence of such a decomposition (that $3 \leq k \leq 3m$, m is odd and $k|9\binom{m}{2}$) are sufficient in the following cases:

1. $k|\binom{m}{2}$ and either $k \leq m$ or $m|k$

2. $k|3\binom{m}{2}$ but $k \nmid \binom{m}{2}$

3. $k|9\binom{m}{2}$ but $k \nmid 3\binom{m}{2}$ and $k \neq 27$ is odd.

Sufficiency in the first two cases followed fairly easily from known results. The third case, however, required more work and was proven by lifting $C_{k/3}$-decompositions of $3K_m$ found in Chapter 4 to C_k-decompositions of $K_m * \overline{K_3}$. Several cases which would complete the proof of sufficiency of the obvious necessary conditions for C_k-decomposition of $K_m * \overline{K_3}$ remain open, as follows.

Question 7.5. Suppose m is odd, $k|\binom{m}{2}$ and $m < k \leq 3m$, $k \notin \{2m, 3m\}$. Does there exist a C_k-decomposition of $K_m * \overline{K_3}$?

Question 7.6. Suppose m is odd, $m \geq 9$ and $27|9\binom{m}{2}$. Does there exist a C_{27}-decomposition of $K_m * \overline{K_3}$?

Question 7.7. Suppose m is odd, $4 \leq k < 3m$, k is even, and $k|9\binom{m}{2}$. Does there exist a C_k-decomposition of $K_m * \overline{K_3}$?

Finally, we ended in Chapter 6 with a result on decomposing kK_m into cycles of length k for even m and k.

We conclude the thesis by stating two large open problems which many of the results in this thesis work towards solving.

Question 7.8. Are the obvious necessary conditions for the existence of a C_k-decomposition of λK_m sufficient?

Question 7.9. Are the obvious necessary conditions for the existence of a C_k-decomposition of $K_m * \overline{K_n}$ sufficient?

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