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Dynamical Systems and Wavelets

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Dedication

In memory of my parents and mother-in-law. To my wife Xiaoyan and my daughter Zheng, for their constant encouragements and patience.

Abstract

The first part of this thesis is concerned with Baker's Conjecture (1984) which says that two permutable transcendental entire functions have the same Julia set. To this end, we shall exhibit that two permutable transcendental entire functions of a certain type have the same Julia set. So far, this is the best result to the conjecture.

The second part relates to Newton's method to find zeros of functions. We shall look for the locations of the limits of the iterating sequence of the relaxed Newton function on its wandering domains. A relaxed Newton function with corresponding properties is constructed.

The third part relates to the dynamics of ordinary differential equations and inverse problems. Given a target solution, we shall construct second-order differential equations with Legendre polynomial basis to approximate the target solution. An algorithm and numerical solutions are provided. Examples show that the approximations we have found are much better than the known results obtained by means of first-order differential equations. We shall also discuss approximation using a wavelet basis. MATLAB is used to compute the numerical results.

In the fourth part, we deal with variational problems in signal and image processing. For a given signal or image represented by a function, we shall provide a good approximation to the function, which minimizes a given functional.

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Chapter 1

Introduction

1.1 Background

Dynamical systems are currently the focus of great theoretical and applicable considerations. We will consider two kinds of dynamics: Complex dynamics and inverse problems to differential equations.

The contemporary study of complex dynamics (or iteration theory) originated with the work of Julia ([29]) and Fatou ([15]) between 1917 and 1920. In late 1917. P. Fatou and G. Julia each announced several results regarding the iteration of rational functions of a single complex variable in the *Comptes rendus* of the French Academy of Sciences. In 1918. Julia published a long and fascinating treatise on the subject. which was followed in 1919 by an equally remarkable study. the first instalment of a three-part memoir by Fatou. One of the reason why Fatou and Julia decided to study iteration is the 1915 announcement by the French Academy of Sciences announced that it would award its 1918 *Grand Prix des Sciences Mathématiques* for the study of iteration. One may wonder: why did the Academy offer such a prize? The answer is not obvious. Part of the Academy's decision might have been prompted by Henri Poincaré's use of iteration in his studies of celestial mechanics. Another reason might have been that in France there was a longstanding interest in the iteration of complex maps. beginning with the studies of Gabriel Koenigs in the mid-1880's. The Academy announced at its December 2. 1918 meeting that Julia's paper [1918] was the winning paper. Meanwhile. Fatou and Lattès were awarded 2000 francs each. and Montel got 1000 francs.

The development of the theory of complex dynamics is divided into two main aspects: the iteration of rational functions and. thereafter. of transcendental functions. The dynamics of transcendental functions is quite different from the dynamics of rational functions. mainly because of the existence of essential singularities. Sullivan's so-called No Wandering Theorem. for instance. conjectured by Fatou in 1926 for rational functions does not hold for transcendental functions.

One important thing in complex dynamics is to study the uniqueness problem: when two functions have the same Julia set. Another important thing is to find the location of the limits of the function f on Fatou components (cf. Degui. Hua and Vaillancourt [13]. Howland and Vaillancourt [21]. Howland–Thompson–Vaillancourt [20]. Hua [22]. Hua and Yang [24. 26. 25. 27]).

In recent years. contraction maps have been used in fractal image and signal compression. Some scientists use this idea to study inverse problems for ordinary differential equations (cf. Kunze and Vrscay [33]). Contraction maps have been used to speed up the iterative solution and reduce the difference between data and models. This is very useful in practice. For example. it is useful in the modelling of mechanical systems. biological systems. computer tomography. control system.

Wavelets were introduced relatively recently. at the beginning of the 1980s. The name "wavelet" is due to the fact that every wavelet has to have at least some oscillations. Its first mention appeared in an appendix to the thesis of A. Haar (1909). A standard argument for using wavelets rather than the Fourier basis for signal processing is that wavelets are localized. For many decades. scientists wanted more appropriate functions to approximate choppy signals. By wavelet analysis. we can approximate data with sharp discontinuities. This makes wavelets an excellent tool in the field of data processing and data compression. Nowadays, wavelets are widely used in applied fields such as astronomy, nuclear engineering, sub-band coding, signal and image processing. In pure mathematics, wavelets can be used to solve differential equations (cf. P.P. Vaidyanathan [44] and M. Vetterli [31] in signal processing and I. Daubechies [12] and Y. Meyer [35] in mathematics).

There are at least two approaches to wavelet analysis. The first is the interpretation of the wavelet transform as a time-frequency analysis tool. The second approach uses wavelet analysis as a mathematical microscope, which is closely linked to harmonic analysis and approximation theory (see [2, 12, 42, 44]).

1.2 The Organization of the Thesis

The organization of the thesis is as follows. In Chapter 2, we study Baker's conjecture and give an affirmative answer for a family of functions. An example shows that our results are better than others'. In Chapter 3, we study Newton's method. For a relaxed Newton function of a given entire function, we shall obtain the locations of the limits of the iterating sequence of the function on its wandering domains. We also construct a relaxed Newton function with corresponding properties. In Chapter 4, we discuss inverse problems. Given target solutions, we shall construct second-order differential equations with Legendre polynomial basis to approximate the target solutions. An algorithm and numerical solutions are provided. We shall also discuss approximation by using a wavelet basis instead of the Legendre polynomial basis. In Chapter 5, we deal with variational problems in signal and image processing. For a given signal or image represented by a function, we shall provide a good approximation to the function, which minimizes the functional. Finally we provide observations on future work.

Chapter 2

Baker Conjecture

2.1 Preliminaries

Let f be a non-constant meromorphic function and the sequence of iterates of f be denoted by

$$f^0 = \text{id}, \quad f^1 = f, \quad \ldots, \quad f^{n+1} = f^n(f), \quad \ldots$$

The Fatou set of a non-constant meromorphic function f is the set

$$F = F(f) = \{z \in \mathbb{C} : \text{ the sequence } \{f^n\} \text{ is defined and normal at } z\}$$

and the *Julia set* of f is the set

$$J = J(f) = \mathbb{C} - F(f).$$

Here "normal" means for each sequence in $\{f^n\}$, there exists subsequence such that it converges uniformly at the point. According to the definition, F is open (possibly empty) and J is closed. The components of $F(f)$ are called Fatou components (or stable domains): they are maximal domains of normality. Each component D of $F(f)$ is mapped by f into some component U of $F(f)$. If each $f^n(D)$ belongs to a different component of $F(f)$, the component D is called wandering; otherwise, D is called pre-periodic.

A point z_0 is called a fixed point of f if $f(z_0) = z_0$. The point z_0 is said to be attractive if $|f'| < 1$ and repulsive if $|f'| > 1$.

A complex number $\alpha \neq \infty$ is called an *asymptotic value* of $f^n(z)$ at z_0 if $f^n(z) \to \alpha$ as $z \to z_0$ along a path Γ terminating at z_0. If $(f^n)'(z_0) = 0$ at the point z_0, then $(f^n)(z_0)$ is called a critical value of f^n. Singular values of a function f consist of critical values and asymptotic values of f. We denote by $\text{sing}(f^{-1})$ the set of all finite singular values of f.

2.2 Main results

Let f and g be two nonconstant meromorphic functions. If

$$f \circ g = g \circ f. \tag{2.1}$$

then we call f and g *permutable*.

Fatou [15] proved the following result:

Theorem A. *If two rational functions, R_1 and R_2, are permutable, then their Fatou sets $F(R_1)$ and $F(R_2)$ are equal.*

The following question is natural (see Baker [3]):

Open Question 2.1. *Are the Fatou sets of two permutable transcendental entire functions, f and g, equal, that is, $F(f) = F(g)$?*

This question was resolved in the affirmative for some special cases and a complete answer to the question is yet to be found.

Theorem B (Baker [3]) *Suppose that f and g are permutable transcendental entire functions, and $f = g + c$ for some constant c, then $F(f) = F(g)$.*

A point a is called a singular value of a function f if it is either a critical value or an asymptotic value of f. We denote by $\text{sing}(f^{-1})$ the set of all finite singular values of f.

If the set $\operatorname{sing}(f^{-1})$ is bounded. then we say f is of bounded type. in particular. if the set $\operatorname{sing}(f^{-1})$ is finite, then f is said to be of finite type. We denote them by $f \in B$ and $f \in S$ respectively (cf. [25]).

Theorem C. (Poon and Yang [25]) *Suppose that f and g are permutable transcendental entire functions. Assume that one of the following holds:*

(i) For some constants c and d, $g(z) = cf(z) + d$;

(ii) Both $\operatorname{sing}(f^{-1})$ and $\operatorname{sing}(g^{-1})$ are isolated in the finite complex plane.

Then $F(f) = F(g)$.

Langley [34] studied functions of finite order and proved the following theorem.

Theorem D. *If f and g are permutable transcendental entire functions of finite order. without wandering domains. then their Julia sets. $J(f)$ and $J(g)$. are equal.*

In this chapter of the thesis. we shall prove the following two theorems.

Theorem 2.1. *Let f and g be two permutable transcendental entire functions of finite order. Suppose that*

$$f(z) = p(z) + p_1(z)\, e^{q(z)}, \tag{2.2}$$

where $p(z)$. $p_1(z)$ and $q(z)$ are polynomials. Then $J(f) = J(g)$.

We denote by $d(p)$ the *degree* of the polynomial p.

Theorem 2.2. *Let f and g be two permutable transcendental entire functions of finite order. Suppose that*

$$f(z) = p(z) + p_1(z)\, e^{q(z)} + p_2(z)\, e^{-q(z)}. \tag{2.3}$$

where $p(z)$, $p_1(z)$, $p_2(z)$, and $q(z)$ are polynomials. $(d(p_1) + d(p_2))/2 + d(q) - 1 \neq d(p)$. Then $J(f) = J(g)$.

Remark 2.1. *The function f in our theorem may have wandering domains.*

Example 2.1. *Let $p(z) = z + 2\pi i - 1$. $p_1(z) = 1$. $q(z) = -z$. $p_2(z) = 0$. Then $f(z) = z + 2\pi i - 1 + e^{-z}$. This function f has wandering domain (see [18] or [25, p. 90]).*

Remark 2.2. *From $f \circ g = g \circ f$ one cannot conclude that either $g(z) = af(z) + b$ for constants a and b or g is an iterate of f.*

Example 2.2. *Let*

$$\begin{aligned} f(z) &= ic\left[\exp\left(\frac{(4k+3)\pi}{8c^2} iz^2\right) + \exp\left(-\frac{(4k+3)\pi}{8c^2} iz^2\right)\right], \\ g(z) &= c\left[\exp\left(\frac{(4k+3)\pi}{8c^2} iz^2\right) - \exp\left(-\frac{(4k+3)\pi}{8c^2} iz^2\right)\right]. \end{aligned}$$

Then $f \circ g = g \circ f$.

2.3 Nevanlinna's Characteristic Function

As a quantitative generalization of Picard's theorem, the theory of the distribution of values of meromorphic functions, developed by R. Nevanlinna and his student, L. Ahlfors, was one of the most outstanding achievements of mathematics in the 20th century (see Nevanlinna[37]). The most important function in Nevanlinna's theory is Nevanlinna's characteristic function, which we now introduce.

Definition 2.1. *Let $f(z)$ be meromorphic in $|z| \le R < \infty$. For $0 < r \le R$, we denote by $n(r, f)$ the number of poles of $f(z)$ in $|z| < r$, counted according to multiplicities. Setting $\log^+ x = \max(\log x, 0)$, we define*

$$\begin{aligned} N(r, f) &= \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r, \\ m(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \\ T(r, f) &= m(r, f) + N(r, f), \end{aligned}$$

where $N(r, f)$, $m(r, f)$ and $T(r, f)$ are called counting function, proximity function and Nevanlinna characteristic function, respectively. The order $\lambda(f)$ and the lower order $\rho(f)$ of f are defined, respectively, as follows:

$$\lambda(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

$$\rho(f) = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

2.4 Some Lemmas

Let $T(r, \cdot)$ be the Nevanlinna characteristic function in Definition 2.1. Then we have the following theorem.

Lemma 2.1 (Gross-Osgood [17]). *Let $G_0(z), \ldots, G_m(z)$ be non-zero entire functions and $h_0(z), \ldots, h_m(z)$ $(m \geq 1)$ be non-zero meromorphic functions. Suppose that $f(z)$ is a non-constant entire function. K is a positive real number and $\{r_j\}$ is an unbounded monotone increasing sequence of positive real numbers such that, for each j,*

$$T(r_j, h_i) \leq KT(r_j, f) \quad (i = 0, \ldots, m),$$

$$T(r_j, f') \leq (1 + o(1))T(r_j, f).$$

If

$$h_0 G_0(f) + \cdots + h_m G_m(f) \equiv 0,$$

then there exist polynomials $P_0(z), \ldots, P_m(z)$, not all identically zero, such that

$$P_0(z)G_0(z) + \cdots + P_m(z)G_m(z) \equiv 0.$$

Lemma 2.2 (Borel's Lemma, see [10]). *Let $q_1(z), \ldots, q_m(z)$ be entire functions, and let $h_1(z), \ldots, h_m(z)$ be meromorphic functions such that*

$$T(r, h_i) = o(\min\{T(r, e^{q_j - q_k}) : i, j, k = 1, \ldots, m; j \neq k\}). \tag{2.4}$$

If

$$h_1 e^{q_1} + \cdots + h_m e^{q_m} \equiv 0,$$

then $h_1(z) \equiv \cdots \equiv h_m(z) \equiv 0$.

Remark 2.3. *From the proof it is easily seen that condition (2.4) needs only to be satisfied on a sequence of r tending to ∞.*

Lemma 2.3. *([45, p.325]) Let f and g be two permutable entire functions such that both f and g have finite order and the lower order of f is positive. Then there exists a sequence $\{r_j\}$ tending to $+\infty$ such that*

$$T(r_j, g^{(n)}) \leq KT(r_j, f),$$

where $n \geq 0$ and K is a positive constant.

Lemma 2.4. *Let f and g be two permutable transcendental entire functions of finite order. Suppose that*

$$f(z) = p(z) + p_1(z)\, e^{q_1(z)} + p_2(z)\, e^{q_2(z)}, \tag{2.5}$$

where $p(z)$ is a polynomial, $p_1(z)$, $p_2(z)$, $q_1(z)$, $q_2(z)$ are nonzero polynomials and $q_1(z) - q_2(z) \neq$ constant. Then there exist two nonzero polynomials $Q_0(z)$ and $Q(z)$ such that

$$\frac{Q(f)}{Q_0(f)} = \frac{P(g)}{P_0(g)}, \tag{2.6}$$

where

$$\begin{aligned} P_0 &= -(p_1'' + 2p_1'q_1' + p_1q_1'' + p_1q_1'^2)(p_2' + p_2q_2') \\ &\quad + (p_2'' + 2p_2'q_2' + p_2q_2'' + p_2q_2'^2)(p_1' + p_1q_1'), \end{aligned} \tag{2.7}$$

$$\begin{aligned} P &= p''P_2 + (p_1'' + 2p_1'q_1' + p_1q_1'' + p_1q_1'^2)[(p_2' + p_2q_2')p - p_2p'] \\ &\quad + (p_2'' + 2p_2'q_2' + p_2q_2'' + p_2q_2'^2)[-(p_1' + p_1q_1')p + p_1p'], \end{aligned} \tag{2.8}$$

$$P_2 = p_1p_2' - p_1'p_2 - p_1p_2(q_1' - q_2'). \tag{2.9}$$

Proof. From (2.1) we have

$$f'(g) = \frac{f'}{g'}\, g'(f) \tag{2.10}$$

and, hence,

$$f''(g) = \frac{f''g' - f'g''}{g'^3} g'(f) + \left(\frac{f'}{g'}\right)^2 g''(f). \tag{2.11}$$

From

$$f(z) = p(z) + p_1(z)\, e^{q_1(z)} + p_2(z)\, e^{q_2(z)}$$

we get

$$f'(z) = p'(z) + \left[p_1'(z) + p_1(z)q_1'(z)\right] e^{q_1(z)} + \left[p_2'(z) + p_2(z)q_2'(z)\right] e^{q_2(z)} \tag{2.12}$$

and

$$\begin{aligned} f''(z) &= p''(z) + \left[p_1''(z) + 2p_1'(z)q_1'(z) + p_1 q_1''(z) + p_1(z)q_1'(z)^2\right] e^{q_1(z)} \\ &\quad + \left[p_2''(z) + 2p_2'(z)q_2'(z) + p_2 q_2''(z) + p_2(z)q_2'(z)^2\right] e^{q_2(z)}. \end{aligned} \tag{2.13}$$

By eliminating the factors $e^{q_1(z)}$ and $e^{q_2(z)}$ from the three equations (2.5), (2.12) and (2.13), we get

$$P_2(z)f''(z) + P_1(z)f'(z) + P_0(z)f(z) + P(z) = 0, \tag{2.14}$$

where

$$P_2 = p_1p_2' - p_1'p_2 - p_1p_2(q_1' - q_2'), \tag{2.15}$$

$$\begin{aligned} P_1 &= -p_1p_2'' + p_1''p_2 - 2p_1p_2'q_2' + 2p_1'p_2q_1' + p_1p_2(q_1'' - q_2'') + p_1p_2(q_1'^2 - q_2'^2) \\ &= -P_2' - P_2(q_1' + q_2'), \end{aligned} \tag{2.16}$$

$$\begin{aligned} P_0 &= -(p_1'' + 2p_1'q_1' + p_1q_1'' + p_1q_1'^2)(p_2' + p_2q_2') \\ &\quad + (p_2'' + 2p_2'q_2' + p_2q_2'' + p_2q_2'^2)(p_1' + p_1q_1'), \end{aligned} \tag{2.17}$$

$$\begin{aligned} P &= p''P_2 + (p_1'' + 2p_1'q_1' + p_1q_1'' + p_1q_1'^2)[(p_2' + p_2q_2')p - p_2p'] \\ &\quad + (p_2'' + 2p_2'q_2' + p_2q_2'' + p_2q_2'^2)[-(p_1' + p_1q_1')p + p_1p']. \end{aligned} \tag{2.18}$$

Obviously, $P_2 \not\equiv 0$. In fact, if, on the contrary, $P_2 = 0$, then

$$p_1p_2' - p_1'p_2 = p_1p_2(q_1' - q_2').$$

Since p_i and q_i are polynomials for $i = 1, 2$, it follows that $q_1' = q_2'$, which contradicts the assumption. Replacing z by $g(z)$ in equation (2.14) yields

$$P_2(g)f''(g) + P_1(g)f'(g) + P_0(g)f(g) + P(g) = 0. \tag{2.19}$$

Combining this with (2.1). (2.10) and (2.11). we get

$$P_2(g)\left(\frac{f'}{g'}\right)^2 g''(f) + \left[P_2(g)\frac{f''g'-f'g''}{g'^3} + P_1(g)\frac{f'}{g'}\right] g'(f) + P_0(g)g(f) + P(g) = 0. \quad (2.20)$$

From Nevanlinna's theory (see [37]) and Lemma 2.3 we see that all the coefficients in (2.20) satisfy the assumptions of Lemma 2.1. Then by Lemma 2.1. there exist four polynomials $Q(z)$. $Q_0(z)$. $Q_1(z)$ and $Q_2(z)$. at least one of them not identically zero. such that

$$Q_2(z)g''(z) + Q_1(z)g'(z) + Q_0(z)g(z) + Q(z) = 0. \quad (2.21)$$

Substituting z by $f(z)$ in this equation. we get

$$Q_2(f)g''(f) + Q_1(f)g'(f) + Q_0(f)g(f) + Q(f) = 0.$$

Eliminating the term containing $g''(f)$ from this equation and (2.20). we get

$$H_1 g'(f) + H_0 g(f) + H = 0. \quad (2.22)$$

where

$$H_1 = Q_1(f)P_2(g)\left(\frac{f'}{g'}\right)^2 - Q_2(f)\left[P_2(g)\frac{f''g'-f'g''}{g'^3} + P_1(g)\frac{f'}{g'}\right]. \quad (2.23)$$

$$H_0 = Q_0(f)P_2(g)\left(\frac{f'}{g'}\right)^2 - Q_2(f)P_0(g). \quad (2.24)$$

$$H = Q(f)P_2(g)\left(\frac{f'}{g'}\right)^2 - Q_2(f)P(g). \quad (2.25)$$

From (2.1). (2.10) and (2.22) we deduce that

$$H_1\frac{g'}{f'}f'(g) + H_0 f(g) + H = 0. \quad (2.26)$$

Replacing z by $g(z)$ in equations (2.5) and (2.12) first and then substituting them into (2.26). we obtain

$$H_1\frac{g'}{f'}p'(g) + H_0 p(g) + H + \left[H_1\frac{g'}{f'}(p_1'(g) + p_1(g)q_1'(g)) + H_0 p_1(g)\right] e^{q_1(g)}$$

$$+[H_1 \frac{g'}{f'}(p_2'(g) + p_2(g)q_2'(g)) + H_0 p_2(g)]\, e^{q_2(g)} = 0.$$

Since g and f are permutable. and f has finite order. we deduce that g cannot be a polynomial. Thus g should be transcendental. and so both $e^{q_1(g)}$ and $e^{q_1(g)}$ will have infinite order. Hence the coefficients in the above equation satisfy the assumptions of Lemma 2.2. It follows from Lemma 2.2 that

$$H_1 \frac{g'}{f'}(p_1'(g) + p_1(g)q_1'(g)) + H_0 p_1(g) = 0 \tag{2.27}$$

and

$$H_1 \frac{g'}{f'}(p_2'(g) + p_2(g)q_2'(g)) + H_0 p_2(g) = 0. \tag{2.28}$$

We claim that $H_1 = 0$. In fact. if $H_1 \not\equiv 0$. then from (2.27) and (2.28) we get

$$\frac{p_1'(g) + p_1(g)q_1'(g)}{p_1(g)} = \frac{p_2'(g) + p_2(g)q_2'(g)}{p_2(g)}.$$

Thus,

$$\frac{p_1'(z)}{p_1(z)} + q_1'(z) = \frac{p_2'(z)}{p_2(z)} + q_2'(z)$$

and we have

$$p_1(z)\, e^{q_1(z)} = cp_2(z)\, e^{q_2(z)}$$

for some constant c. It follows from this. Lemma 2.2 and the part that p_i and q_i are polynomials for $i = 1.2$ that $q_1 - q_2$ must be a constant. which contradicts the assumption. Therefore $H_1 = 0$. By (2.26). we have

$$H_0 f(g) + H = 0.$$

It follows from Lemma 2.2 that

$$H_0 \equiv H \equiv 0 \tag{2.29}$$

Hence. by (2.24) and (2.25). we obtain (2.6).

We now show that $P_0 \not\equiv 0$. In fact. if on the contrary. $P_0 \equiv 0$. then from (2.17) we deduce that

$$\frac{(p_1' + p_1q_1')'}{p_1' + p_1q_1'} - \frac{(p_2' + p_2q_2')'}{p_2' + p_2q_2'} = q_1' - q_2'.$$

which yields

$$\frac{p_1' + p_1 q_1'}{p_2' + p_2 q_2'} = c e^{q_1 - q_2}.$$

This implies that $q_1 - q_2$ is a constant, which is impossible.

Next we show that $Q \not\equiv 0$ and $Q_0 \not\equiv 0$. If $Q \equiv 0$, then by (2.24), (2.25) and (2.29) we have $Q_0 \equiv Q_2 \equiv 0$. It follows from (2.21) that $Q_1 g' = 0$. Note that g' is transcendental, thus $Q_1 = 0$, and so, $Q_2 = Q_1 = Q_0 = Q = 0$, which is impossible. Therefore, $Q_0 \not\equiv 0$ and $Q \not\equiv 0$.

Remark 2.4. *Some special cases of Lemma 2.4 were considered by Kobayashi [30] and Zheng and Zhou [45].*

The following result can be found in [45, Theorem 3].

Lemma 2.5. *Let f and g be two permutable transcendental entire functions of finite order. Suppose that $f(z)$ is of the form (2.2), then $g(z) = cf(z) + d$ for two constants $c \neq 0$ and d.*

Lemma 2.6. *Let f and g be two permutable transcendental entire functions satisfying (2.5). Then $q_i(f) = cq_j(g) + d$ for some constants c and d, where $q_i, q_j \in \{q_1, q_2\}$.*

Proof. If $q_1' - q_2' \equiv 0$, then the function f of the form (2.5) reduces to a function of the form (2.2). The conclusion follows from Lemma 2.5. Next we suppose that $q_1' - q_2' \not\equiv 0$. By (2.17) and (2.18) we see that the highest-order terms in P_0 and P are $p_1 p_2 q_1' q_2'(q_1' - q_2')$ and $p p_1 p_2 q_1' q_2'(q_1' - q_2')$, respectively. This implies that P/P_0 is a non-constant rational function, and so is Q/Q_0. We denote $R_1 = P/P_0$ and $R_2 = Q/Q_0$. From (2.6) we have

$$R_2(f) = R_1(g).$$

Thus

$$R_2(f(f)) = R_1(g(f)) = R_1(f(g)).$$

By this and (2.18) we get

$$R_2\left(p(f)+p_1(f)e^{q_1(f)}+p_2(f)e^{q_2(f)}\right)=R_1\left(p(g)+p_1(g)e^{q_1(g)}+p_2(g)e^{q_2(g)}\right). \qquad (2.30)$$

Without loss of generality, we may assume that $\deg q_1 \geq \deg q_2$. According to the numerators and the denominators of R_1 and R_2, we can rewrite (2.30) in the form:

$$\tilde{p}_1(f,g)\,e^{a_1q_1(f)+b_1q_1(g)}+\ldots=\tilde{p}_2(f,g)\,e^{a_2q_1(f)+b_2q_1(g)}+\ldots,$$

where $\tilde{p}_1$ and $\tilde{p}_2$ are polynomials of f and g, and a_1, a_2, b_1 and b_2 are constants, not all of them are zero. It follows from Lemma 2.2 that $q_1(f)=cq_1(g)+d$ for some constants c and d. The lemma is proved.

Remark 2.5. *The following example of a permutable pair f, g (due to T. W. Ng) shows that polynomials q_i and q_j in Lemma 2.6 can be different.*

Example 2.3. *For two permutable entire functions: $f(z)=\sin(\pi/2z^2)$ and $g(z)=\cos(\pi/2z^2)$, we see that $f(f)=g(g)$. Let $q_1(z)=\pi i/2z^2$ and $q_2(z)=-\pi i/2z^2$. Then $q_1(f)=q_2(g)+1$.*

Lemma 2.7. *Let f and g be two permutable transcendental entire functions satisfying the hypotheses of Theorem 2.2. Then C_f and C_g have no limit points in the finite plane, where C_f and C_g denote the set of critical values of f and g, respectively.*

Proof. From

$$f(z)=p(z)+p_1(z)e^{q(z)}+p_2(z)\,e^{-q(z)} \qquad (2.31)$$

we get

$$f'(z)=p'(z)+\left[p_1'(z)+p_1(z)q'(z)\right]e^{q(z)}+\left[p_2'(z)-p_2(z)q'(z)\right]e^{-q(z)}. \qquad (2.32)$$

If the set C_f is finite, then the conclusion of the lemma is obvious. Hence we can suppose that the set C_f is infinite. Then the set of critical points is infinite and tending to ∞. We denote this set by $\{z_j\}$. Then

$$f'(z_j)=0. \qquad (2.33)$$

Next we shall prove that

$$f(z_j) \to \infty. \tag{2.34}$$

By (2.31)-(2.33) and eliminating e^q, we deduce that

$$f(z_j) = p(z_j) - \tilde{p}_1(z_j) \pm \tilde{p}_2(z_j), \tag{2.35}$$

where

$$\tilde{p}_1(z_j) = \frac{p'(z_j)}{2} \frac{p_1(z_j)p_2'(z_j) + p_1'(z_j)p_2(z_j)}{[p_1'(z_j) + p_1(z_j)q'(z_j)][p_2'(z_j) - p_2(z_j)q'(z_j)]} \tag{2.36}$$

and

$$\tilde{p}_2(z_j) = \frac{s(z_j)}{2} \frac{-2p_1(z_j)p_2(z_j)(q'(z_j)) + p_1(z_j)p_2'(z_j) - p_1'(z_j)p_2(z_j)}{[p_1'(z_j) + p_1(z_j)q'(z_j)]\,[p_2'(z_j) - p_2(z_j)q'(z_j)]} \tag{2.37}$$

with

$$s(z_j) = \left(p'(z_j)^2 - 4[p_1'(z_j) + p_1(z_j)q'(z_j)]\,[p_2'(z_j) - p_2(z_j)q'(z_j)]\right)^{1/2}.$$

Note that the degrees of the given polynomials satisfy the following equalities:

$$d(p_1p_2' + p_1'p_2) = d(p_1) + d(p_2) - 1,$$

and

$$d\{[p_1' + p_1q']\,[p_2' - p_2q']\} = d(p_1) + d(p_2) + 2d(q) - 2. \tag{2.38}$$

It follows from these two equations and (2.36) that, as $z_j \to \infty$,

$$\tilde{p}_1 \sim c_1 z_j^{d(p)-1+d(p_1)+d(p_2)-1-(d(p_1)+d(p_2)+2d(q)-2)} = c_1 z_j^{d(p)-2d(q)}, \tag{2.39}$$

where c_1 is a nonzero constant.

To deal with $\tilde{p}_2$, we consider the following two cases:

(i) $d(p_1) + d(p_2) + 2d(q) \leq 2d(p)$. Then

$$d(p) \geq 1 \tag{2.40}$$

and

$$d\left([p_1' + p_1q']\,[p_2' - p_2q']\right) \leq d(p'^2).$$

From this and (2.37), as $z_j \to \infty$, we have

$$\tilde{p}_2(z_j) = o\left(z_j^{d(p)}\right).$$

Combining this, (2.35) and (2.39), we see that

$$f(z_j) \sim p(z_j) \to \infty, \qquad \text{as} \quad z_j \to \infty.$$

This gives (2.34).

(ii) $d(p_1) + d(p_2) + 2d(q) > 2d(p)$. This and the assumption yield

$$\frac{d(p_1) + d(p_2)}{2} + d(q) - 1 > d(p) \tag{2.41}$$

and

$$d([p_1' + p_1 q'] [p_2' - p_2 q']) > d(p'^2).$$

This, (2.37) and (2.38) imply that

$$\tilde{p}_2 \sim c_2 z_j^{(d(p_1)+d(p_2))/2+d(q)-1} \tag{2.42}$$

as $z_j \to \infty$, where c_2 is a non-zero constant. Combining (2.39), (2.41) and (2.42) we deduce that

$$f(z_j) \sim c_2 z_j^{(d(p_1)+d(p_2))/2+d(q)-1}, \qquad \text{as} \quad z_j \to \infty.$$

This also gives (2.34).

By (2.34), C_f has no limit points in the finite plane.

Next we deal with the set C_g. Let $\{w_j\}$ be the set of the critical points of g, i.e., $g'(w_j) = 0$. If the set $\{w_j\}$ is finite, then C_g has no finite limit points. If the set $\{w_j\}$ is infinite, then $w_j \to \infty$. By Lemma 2.6, there exist constants c and d such that $q(f) = cq(g) + d$, which gives

$$q'(f)f' = cq'(g)g'.$$

This implies that $q'(f(w_j))f'(w_j) = 0$. We consider two cases.

1) All but finitely many w_j satisfy $f'(w_j) = 0$. Then all these w_j are critical points of f, and so, by the previous proof, $f(w_j) \to \infty$. It follows that $g(w_j) \to \infty$.

2) There exists a subsequence in $\{w_j\}$, which, for convenience, we still denote by $\{w_j\}$, such that $q'(f(w_j)) = 0$. Since $q(z)$ is a polynomial, $\{f(w_j)\}$ is a finite set $\{V_1, \ldots, V_t\}$, which is a subset of the set of zeros of $q'(z) = 0$. It follows from $f(g(w_j)) = g(f(w_j)) = g(V_i)$ $(i = 1, \ldots, t)$ that $g(w_j)$ is a solution of the equation $f(z) = g(V_i)$ for some $1 \leq i \leq t$. Thus the set $\{g(w_j)\}$ is either finite or tends to ∞.

Combining the two cases above we conclude that the set C_g has no limit points in the finite plane.

Lemma 2.8. *Let f and g be two entire functions of finite order. Then both f and g have finitely many asymptotic values.*

Proof. Since f and g have finite order, it follows from the Denjoy–Carleman–Ahlfors theorem (see [37] or [25, p. 67]) that both f and g have finitely many asymptotic values.

2.5 Proof of the Theorems

Theorem 2.1 follows from Theorem C (i) and Lemma 2.5.

To prove Theorem 2.2, from Lemmas 2.7 and 2.8 we see that both sing(f^{-1}) and sing(g^{-1}) are isolated in the finite complex plane, thus the conclusion of Theorem 2.2 follows from Theorem C (ii).

Chapter 3

Newton's Method

3.1 Preliminaries

As we know, Newton's method to find zeros of functions is very important.

Let g be a meromorphic function. The (unrelaxed) Newton's method for finding the zeros of g consists of iterating the meromorphic function f defined by

$$f(z) = z - \frac{g(z)}{g'(z)}.$$

If w is a zero of g, then w is an attracting fixed point of f, and vice versa. Thus $w \in F(f)$. This implies that if $z \in J(f)$, then $f^n(z)$ cannot converge to a zero of g. In addition, if z tends to w, then $f^n(z) \to w$ as $n \to \infty$ (see [6], [20] or [25]).

The relaxed Newton method is defined by

$$f_\lambda(z) = z - \lambda\frac{g(z)}{g'(z)}, \tag{3.1}$$

where $\lambda \in \mathbb{C}$ and $|\lambda - 1| < 1$.

If w is a zero of g, then w is a fixed point of f_λ, and vice versa.

For function f, we denote

$$E(f) = \bigcup_{n\geq 0} f^n \operatorname{sing}(f^{-1})).$$

which is the forward orbit of singular points. The post-singular set of f is defined to be the closure:

$$\overline{E} := \overline{E}(f) = E(f) \cup E'(f).$$

where $E'(f)$ is the set of the limit points of $E(f)$.

Remark 3.1. *If a domain $D \subset \widehat{\mathbb{C}}$ contains no critical values and no asymptotic values of $f^n(z)$ then $f : f^{-n}(D) \to D$ is a covering. This justifies the name "singularities of f^{-n}".*

Baker [3] has proved the following result on the location of constant limits related to singularities of a transcendental entire function.

Theorem 3.1. *For a transcendental entire function f, any constant limit of a sequence $f^{n_k}(z)$ in a component of $F(f)$ belongs to $\overline{E}(f) \cup \{\infty\}$.*

Moreover, one has the following special result (see Bergweiler–Haruta–Kriete–Meier–Terglane [8]).

Theorem 3.2. *For a transcendental entire function f, let U be a wandering domain of f. Then all the limits of $\{f^n\}$ in U are constants and are contained in $(E'(f) \cap J(f)) \cup \{\infty\}$.*

We shall prove the following two results.

Theorem 3.3. *Let g be an entire function, f_λ be as in (3.1) and let U be a wandering domain of f_λ. Then all the limits of $\{f_\lambda^n|_U\}$ are contained in $E'(f_\lambda) \cup \{\infty\}$.*

Theorem 3.4. *There exists an entire function g such that its relaxed Newton function f_λ has wandering domains and the limit of $\{f_\lambda^n\}$ on the wandering domains is ∞.*

3.2 Proof of Theorem 3.3

A fixed point z_0 of f is called weakly repelling if $|f'(z_0)| \geq 1$.

In order to prove the theorem, we need the following result.

Lemma 3.1. *([7. Theorem 1]) Let f be a transcendental meromorphic function and suppose that f has a multiply-connected wandering domain. Then f has at least one weakly repelling fixed point.*

Lemma 3.2 (Koebe 1/4 Theorem). *Let $D(a,r) = \{z : |z - a| < r\}$. If $f(z)$ is an univalent function (i.e., analytic and one-to-one) in $D(a,r)$, then the image $f(D(a,r))$ covers the open disk $D(f(a), |f'(a)|r/4)$.*

Remark 3.2. *The Koebe function $f(z) = z/(1-z)^2$ shows that 1/4 is optimal.*

Lemma 3.3. *Let f_λ be the relaxed Newton function of g with $|\lambda - 1| < 1$. Then any zero z_0 of g is an attractive fixed point of f_λ and $z_0 \in F(f_\lambda)$.*

Proof. If z_0 is a zero of g with order k, then

$$|f_\lambda'(z_0)| = |1 - \lambda/k|.$$

Since $|\lambda - 1| < 1$, by letting $\lambda = a + ib$ we imply that $(a-1)^2 + b^2 < 1$, i.e., $a > 0$ and $a^2 + b^2 < 2a$. Hence

$$\begin{aligned} |f_\lambda'(z_0)| &= |1 - (a+bi)/k| = \sqrt{1 - \frac{2a}{k} + \frac{a^2+b^2}{k^2}} \\ &< \sqrt{1 - \frac{2a}{k} + \frac{2a}{k^2}} = \sqrt{1 - \frac{2a}{k}\left(1 - \frac{1}{k}\right)} \\ &< 1. \end{aligned}$$

This implies that z_0 is an attractive fixed point of f_λ.

Proof of Theorem 3.3. A fixed point of f_λ is a zero of g. From Lemma 3.3 we see that f_λ has no weakly repelling fixed point.

Since rational functions have no wandering domains, we can assume that f_λ is a transcendental meromorphic function. By Lemma 3.1, we see that f_λ has no multiply-connected wandering domain.

Suppose that U is a wandering domain of f_λ and $a \in \mathbb{C} \setminus E'(f_\lambda)$ is a limit of $\{f_\lambda^n|_U\}$, say $f^{n_k} \to a$ in U.

By hypothesis, $U \cap E(f_\lambda) = \emptyset$ and $U_n \cap E(f_\lambda) = \emptyset$ for all $n \in \mathbb{N}$, where $U_n = f_\lambda^n(U)$. In fact, if there exist b and $m \in \mathbb{N}$ such that $b \in U_m \cap E(f_\lambda)$, then $f_\lambda^n(b) \in U_{m+n} \cup E(f_\lambda)$. Note that $f_\lambda^n(b) \neq f_\lambda^l(b)$ if $n \neq l$, we imply $a \in E'(f_\lambda)$, which is a contradiction. Therefore, f_λ^{-n} exists locally on all U_n and can be continued analytically in U_n to an univalent function, that is, $f_\lambda^n|_U$ is univalent.

We choose $z_0 \in \mathbb{C}$ and $R > 0$, such that $\overline{D}(z_0, R) \subset U$, (here $D(z, r)$ denotes the disc with center z and radius r). Without loss of generality, we assume that $a = 0$. Next we choose $r > 0$ such that

$$D(0, r) \cap E(f_\lambda) \setminus \{0\} = \emptyset.$$

Since U is a wandering of $F(f_\lambda)$, we can assume that

$$f_\lambda^{n_k}(D(z_0, R)) \subset D(0, r) \setminus \{0\}.$$

By Lemma 3.2 (Koebe's 1/4-Theorem),

$$\frac{1}{4}|(f_\lambda^{n_k})'(z_0)|R \leq \mathrm{dist}(f_\lambda^{n_k}(z_0), \partial(f_\lambda^{n_k}(D(z_0, R)))) \leq |f_\lambda^{n_k}(z_0)|.$$

Therefore

$$|(f_\lambda^{n_k})'(z_0)| \leq 4|f_\lambda^{n_k}(z_0)|/R.$$

Define $H = \{z : \Re z < \log r\}$ and $g_k : D(z_0, R) \to H$ by $g_k(z) = \log f_\lambda^{n_k}(z)$, for some branch of the logarithm. Then

$$|g_k'(z_0)| = \frac{|(f_\lambda^{n_k})'(z_0)|}{|f_\lambda^{n_k}(z_0)|} \leq \frac{4}{R}.$$

Since H is simply connected, the inverse function of g_k can be continued analytically to a single-valued function h_k in H, that is, $h_k : H \to \mathbb{C}$ and $h_k(g_k)(z) = z$ for all $z \in D(z_0, R)$. Next we consider two cases.

(i) The function h_k is univalent in H. Assume that h_k takes the value $w_k = g_k(z_0)$. Then $w_k \in H$, that is, $\Re w_k < \log r$. By Koebe's 1/4-Theorem,

$$h_k(H) \supset h_k(D(w_k, \log r - \Re w)) \supset D\left(z_0, \frac{1}{4}|h_k'(w_k)|(\log r - \Re w_k)\right).$$

Note that for any transcendental meromorphic function f_λ and $n \geq 2$, f_λ has infinitely many periodic points of minimal period n (see [6], Theorem 2). Thus we may let $\{p, q\}$ be a periodic cycle of order 2 with $D(0, r) \cap \{p, q\} = \emptyset$, that is, $f_\lambda(p) = q$ and $f_\lambda(q) = p$. Then

$$h_k(H) \cap \{p, q\} = \emptyset,$$

it follows that

$$\frac{1}{4}|h_k'(w_k)|(\log r - \Re w_k) \leq M,$$

where $M = \min\{|z_0 - p|, |z_0 - q|\}$. Since $\Re w_k \to -\infty$ as $k \to \infty$, we conclude that $h_k'(w_k) \to 0$. But $h_k(g_k(z)) = z$ for $z \in D(z_0, R)$, so $h_k'(w_k)g_k'(z_0) = 1$ and this gives a contradiction.

(ii) The function h_k is not univalent in H. Then there exists $l_k \in \mathbb{N}$ such that h_k is periodic with period $2\pi l_k i$ and h_k is univalent in the half-strip

$$\{w : \Re w < \log r; c < \Im w < c + 2\pi l_k\}$$

for a real number c. If $l_k \to \infty$, by Koebe's 1/4-Theorem we have

$$h_k(H) \supset h_k(D(w_k, R_k)) \supset D\left(z_0, \frac{1}{4}|h_k(w_k)|R_k\right)$$

and

$$\frac{1}{4}|h_k'(w_k)|R_k \leq M,$$

where $R_k = \min\{\log r - \Re w_k, \pi l_k\}$. Since $\log r - \Re w_k \to \infty$ and $\pi l_k \to \infty$ as $k \to \infty$, then $R_k \to \infty$. Thus, $h_k'(w_k) \to 0$, which results in a contradiction as in the case (i). Hence $l_k \not\to \infty$. By restricting to a subsequence of $\{l_k\}$ if necessary, we may assume that $l_k = l$ for all k.

We now consider $G_k = \exp(g_k/l)$, $r' = r^{1/l}$ and the function $H_k : D(0, r') \setminus \{0\} \to \mathbb{C}$ defined by $H_k(z) = h_k(l \log z)$. Clearly, $H_k(G_k(z)) = z$ for $z \in D(z_0, R)$. By Koebe's 1/4-Theorem,

$$|G_k(z_0)| = |G_k(z_0)| \frac{1}{l} |g_k'(z_0)| \leq \frac{4|G_k(z_0)|}{lR}$$

so that $|G_k'(z_0)| \to 0$.

Since H_k is univalent, 0 is not an essential singularity of H_k. Suppose that 0 is a (simple) pole of H_k. Then $H_k(D(0, r')\setminus\{0\})$ contains a neighborhood of infinity. However, every neighborhood of infinity contains periodic cycles of f_λ, which, as noted above, cannot be contained in $H_k(D(0, r')\setminus\{0\})$ since they have an empty intersection with $D(0, r)$. This gives a contradiction. Hence H_k has an analytic (and univalent) continuation to $D(0, r')$. Define $z_k = G_k(z_0)$. We deduce from Koebe's 1/4-Theorem that

$$|H_k'(z_k)| \leq \frac{4M}{r' - |z_k|}.$$

Since $|z_k| \to 0$, we have $|H_k'(z_k)| \leq 8M/r'$ for sufficiently large k. This contradicts to the fact that $H_k'(z_k)G_k'(z_0) = 1$. The proof is complete.

3.3 Proof of Theorem 3.4

For any given λ satisfying $|\lambda - 1| < 1$, we fix two constants a and b such that

$$\left|1 + 2\pi i + \frac{4ab\pi^2}{\lambda}\right| < 1. \tag{3.2}$$

Let

$$g(z) = \exp\{a(e^z - bz)\}.$$

Then by (3.1),

$$f(z) := f_\lambda(z) = z - \frac{\lambda}{a(e^z - b)}. \tag{3.3}$$

For $h(z) := f(z) + 2\pi i$, it is easy to check that h and f are permutable, that is,

$$h(f) = f(h).$$

It follows from Baker [5] that f and h have the same Fatou set. i.e..

$$F(f) = F(h). \tag{3.4}$$

Let z_0 be a solution of the equation

$$e^z = b + \frac{\lambda}{a2\pi i} \tag{3.5}$$

and let

$$z_k = z_0 - 2k\pi i, \qquad k = 1, 2, \ldots.$$

We deduce that

$$h(z_k) = z_k, \quad h'(z_k) = 1 + 2\pi i + \frac{4ab\pi^2}{\lambda}.$$

Combining these and (3.2) we see that z_k ($k = 1, 2, \ldots$) are attractive fixed points of $h(z)$. Let U_k be the Fatou components of $F(h)$ such that $z_k \in U_k$ ($k = 1, 2, \ldots$). Obviously

$$h(U_k) \subset U_k, \quad h^n(U_k) \to z_k$$

as $n \to \infty$. Note that

$$f(z_k) = h(z_k) - 2\pi i = z_k - 2\pi i = z_{k+1}.$$

This yields

$$f(U_k) \subset U_{k+1}.$$

By (3.4), we see that $U_k (k = 1, 2, \ldots)$ are also Fatou components of $F(f)$. Therefore, $U_k (k = 1, 2, \ldots)$ are wandering domains of f. Note that

$$f^n(z) = h^n(z) - 2n\pi i.$$

This implies that

$$f^n(U_k) \to \infty, \qquad \text{as} \quad n \to \infty.$$

The proof is complete.

Chapter 4

Inverse Problems

4.1 Preliminaries

As we know, contraction maps can be used in fractal image compression. Some scientists use this idea to study inverse problems for ordinary differential equations (cf. Kunze and Vrscay [33]).

Consider the initial value problem:

$$\frac{d^2x}{dt^2} = f(x,t), \qquad x(t_0) = x_0, \quad x'(t_0) = x_1. \tag{4.1}$$

The solution $x(t)$ can be expressed in the form

$$x(t) = x_0 + x_1(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{u} f(x(v), v)\, dv\, du. \tag{4.2}$$

Based on this observation, we denote the corresponding Picard operator T as follows:

$$(Tx)(t) = x_0 + x_0'(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{u} f(x(v), v)\, dv\, du. \tag{4.3}$$

It is easy to see that a solution of the equation (4.1) is a fixed point of $T(x)$, i.e., $T(x) = x$.

Let $A \subset \mathbb{R}$ be an interval on the real line and let $C(A)$ be the space of continuous functions $x(t)$ on A with norm

$$\|x\|_p = \left\{ \int_A |x(s)|^p ds \right\}^{1/p}, \qquad 1 \le p < \infty. \tag{4.4}$$

and

$$\|x\|_\infty = \sup_{t\in A} |x(t)|. \tag{4.5}$$

The Picard operator T satisfies

$$T : C(A) \to C(A).$$

To prove the following proposition, we need the Cauchy–Schwarz inequality:

$$\int fg\,dt \le \left(\int f^2 dt\right)^{1/2} \left(\int g^2\,dt\right)^{1/2}.$$

Proposition 4.1. *Let f be a contracting mapping in an interval $A \in \mathbb{R}$ with contraction coefficient c such that*

$$k = \frac{c}{2\sqrt{2}}\, m(A)^2 < 1,$$

where $m(A)$ is the measure of A. Then the Picard operator T is also contracting with coefficient k.

Proof. Without loss of generality, we may suppose that $t_0 = 0$ and $A = [0, 1]$. Then

$$\begin{aligned}
\|Tx - Ty\|_2^2 &= \int_A (Tx - Ty)^2 dt \\
&= \int_A \left\{ \int_0^t \int_0^u [f(x(v)) - f(y(v))]\, dv\, du \right\}^2 dt \\
&\le c^2 \int_A \left\{ \int_0^t \int_0^u |x(v) - y(v)|\, dv\, du \right\}^2 dt \\
&\le c^2 \int_A \left\{ \int_0^t \left[\int_0^u 1^2\, dv\right]^{1/2} \left[\int_0^u |x(v) - y(v)|^2\, dv\right]^{1/2} du \right\}^2 dt \\
&= c^2 \int_A \left\{ \int_0^t u^{1/2} \left[\int_0^u |x(v) - y(v)|^2\, dv\right]^{1/2} du \right\}^2 dt \\
&\le c^2 \int_A \left\{ \left[\int_0^t u\, du\right]^{1/2} \left[\int_0^t \left(\int_0^u |x(v) - y(v)|^2\, dv\right) du\right]^{1/2} \right\}^2 dt \\
&= \frac{c^2}{2} \int_A \left\{ t \left[\int_0^t \int_0^u |x(v) - y(v)|^2\, dv\, du\right]^{1/2} \right\}^2 dt \\
&\le \frac{c^2}{6} \|x - y\|_2^2.
\end{aligned}$$

From this we get the desired result.

4.2 Legendre Equation and Legendre Polynomials

Consider the Legendre equation:

$$(1-x^2)y'' - 2xy' + n(n-1)y = 0, \qquad -1 < x < 1, \quad n \in \mathbb{R}. \tag{4.6}$$

The series solution $y_n(x)$ of (4.6) is of the form:

$$y_n(x) = ay_{n,1}(x) + by_{n,2}(x),$$

where

$$y_{n,1}(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \dots,$$

and

$$y_{n,2}(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \dots$$

Note that $y_{n,1}(x)$ and $y_{n,2}(x)$ are even and odd functions, respectively. Obviously, $y_{n,1}(x)$ and $y_{n,2}(x)$ are linearly independent.

If we normalize y_n such that $y_n(1) = 1$, then for $n = 0, 1, 2, \dots$, we get the solutions

$$\begin{aligned}
P_0(x) &= y_0(x) = 1, \\
P_1(x) &= y_1(x) = x, \\
P_2(x) &= y_2(x) = \frac{1}{2}(3x^2 - 1), \\
P_3(x) &= y_3(x) = \frac{1}{2}(5x^3 - 3x), \\
&\vdots
\end{aligned}$$

where the $P_n(x)$ are Legendre polynomials of degree n.

As opposed to the basis $\{1, x, x^2, \ldots\}$, the Legendre polynomials satisfy the following orthogonality relations (see [32, Section 4.5]):

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{2}{2n+1}, & \text{if } m = n, \end{cases}$$

which are easily derived from the Legendre equation written in divergence form,

$$((1-x^2)y')' + n(n-1)y = 0,$$

and Rodrigues' formula,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right],$$

or the generating function,

$$\frac{1}{\sqrt{1-2tx+x^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

4.3 Algorithm for the Inverse Problem

Now we consider a target solution $y(t)$. Our aim is to construct a differential equation of the form (4.1):

$$\frac{d^2x}{dt^2} = f(x,t), \qquad x(t_0) = x_0, \quad x'(t_0) = x_1, \tag{4.7}$$

such that the solution $x(t)$ of this equation is closed to the target solution in the sense of the norm.

Kunze–Vrscay [33] use first-order differential equations with the basis $\{1, x, x^2, \ldots\}$:

$$\frac{dx}{dt} = \sum_{n=0}^{N} c_n x^n$$

for some $N > 0$.

For simplicity, we define

$$I(h) = \int_A h(t)\,dt.$$

Theorem 4.1. *For a target solution $y(t)$ with $t \in A = [a, b]$, there exists a differential equation of the form*

$$\frac{d^2x}{dt^2} = \sum_{n=0}^{N} c_n P_n(x(t)), \qquad x(t_0) = x_0, \quad x'(t_0) = x_1,$$

where the P_n are Legendre polynomials of degree n, such that the solution of this equation can approximate the target solution $y(t)$, where x_0, x_1 and the coefficients c_n satisfy the following linear system:

$$\begin{array}{lllllll}
I(1)x_0 & + I(t-t_0)x_1 & + I(h_0)c_0 & + \cdots + & I(h_N)c_N & = & I(y), \\
I(t-t_0)x_0 & + I((t-t_0)^2)x_1 & + I((t-t_0)h_0)c_0 & + \cdots + & I((t-t_0)h_N)c_N & = & I((t-t_0)y), \\
I(h_0)x_0 & + I((t-t_0)h_0)x_1 & + I(h_0h_0)c_0 & + \cdots + & I(h_0h_N)c_N & = & I(yh_0), \\
I(h_1)x_0 & + I((t-t_0)h_1)x_1 & + I(h_1h_0)c_0 & + \cdots + & I(h_1h_N)c_N & = & I(yh_1), \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
I(h_N)x_0 & + I((t-t_0)h_N)x_1 & + I(h_Nh_0)c_0 & + \cdots + & I(h_Nh_N)c_N & = & I(yh_N),
\end{array}$$

and

$$h_n(t) = \int_{t_0}^{t}\int_{t_0}^{u} P_n(y(v))\,dv\,du.$$

Proof. Consider the following Picard operator:

$$T(y(t)) = x_0 + x_1(t-t_0) + \sum_{n=0}^{N} c_n h_n(t).$$

As in [33], let the squared collage distance be defined by

$$\Delta = \|y - T(y)\|_2 = \left\{\int_A [y(t) - T(y)(t)]^2\,dt\right\}^{1/2}. \qquad (4.8)$$

Then

$$
\begin{aligned}
\Delta^2 &= \int_A \left[y(t) - x_0 - x_1(t-t_0) - \sum_{n=0}^{N} c_n h_n(t) \right]^2 dt \\
&= I(y^2) + I(1)x_0^2 + I((t-t_0)^2)x_1^2 + \sum_{j=0}^{N}\sum_{k=0}^{N} c_j c_k I(h_j h_k) \\
&\quad - 2I(y)x_0 - 2I((t-t_0)y)x_1 + 2I(t-t_0)x_0 x_1 - 2\sum_{j=0}^{N} c_j I(y h_j) \\
&\quad + 2\sum_{j=0}^{N} x_0 c_j I(h_j) + 2\sum_{j=0}^{N} x_1 c_j I((t-t_0)h_j).
\end{aligned} \tag{4.9}
$$

Consider the stationarity conditions:

$$\frac{\partial \Delta^2}{\partial x_0} = 0. \tag{4.10}$$

$$\frac{\partial \Delta^2}{\partial x_1} = 0. \tag{4.11}$$

$$\frac{\partial \Delta^2}{\partial c_j} = 0. \qquad j = 0, 1, \ldots, N. \tag{4.12}$$

From (4.9) and (4.10) we imply that

$$I(1)x_0 - I(y) + I(t-t_0)x_1 + \sum_{j=0}^{N} c_k I(h_j) = 0. \tag{4.13}$$

By (4.9) and (4.11),

$$I((t-t_0)^2)x_1 - I((t-t_0)y) + I(t-t_0)x_0 + \sum_{j=0}^{N} c_j I((t-t_0)h_j) = 0. \tag{4.14}$$

Furthermore, by (4.9) and (4.12),

$$\sum_{k=0}^{N} c_k I(h_j h_k) - I(y h_j) + x_0 I(h_j) + x_1 I((t-t_0)h_j) = 0. \quad j = 0, 1, \ldots, N. \tag{4.15}$$

Combining (4.13)–(4.15), we get the conclusion.

Example 4.1. *Let $y(t) = t^2$ be the target solution on the interval $A = [0, 1]$, $N = 2$ and $t_0 = 0$.*

Then

$$h_0 = \frac{1}{2}t^2, \qquad h_1 = \frac{1}{12}t^4, \qquad h_2 = \frac{1}{20}t^6 - \frac{1}{4}t^2.$$

From this we get the following linear system:

$$\begin{pmatrix} 1 & 1/2 & 1/6 & 1/60 & -8/105 \\ 1/2 & 1/3 & 1/8 & 1/72 & -9/160 \\ 1/6 & 1/8 & 1/20 & 1/168 & -1/45 \\ 1/60 & 1/72 & 1/168 & 1/1296 & -1/385 \\ -8/105 & -9/160 & -1/45 & -1/385 & 29/2925 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/4 \\ 1/10 \\ 1/84 \\ -2/45 \end{pmatrix}.$$

whose solution is

$$(x_0, x_1, c_0, c_1, c_2) = (0, 0, 2, 0, 0).$$

Then we get the differential equation

$$\frac{d^2x}{dt^2} = 2, \quad x(0) = 0, x'(0) = 0.$$

The solution of the equation is $x(t) = t^2$, which is the same as the target solution $y(t) = t^2$. Thus, for this kind of target solution, there is no difference between it and the solution of the equation, that is, the approximation is exact.

Example 4.2. *Let $y(t) = t^3$ be the target solution on the interval $A = [0, 1]$, $N = 1$ and $t_0 = 0$.*

Then

$$h_0 = \frac{1}{2}t^2, \qquad h_1 = \frac{1}{20}t^5,$$

and we have the following linear system:

$$\begin{pmatrix} 1 & 1/2 & 1/6 & 1/120 \\ 1/2 & 1/3 & 1/8 & 1/140 \\ 1/6 & 1/8 & 1/20 & 1/320 \\ 1/120 & 1/140 & 1/320 & 1/4400 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/5 \\ 1/12 \\ 1/180 \end{pmatrix}.$$

whose solution is

$$(x_0, x_1, c_0, c_1) = (1/75, -7/30, 16/9, 308/45).$$

The corresponding differential equation will be

$$\frac{d^2x}{dt^2} = \frac{16}{9} + \frac{308}{45}x, \qquad x_0 = \frac{1}{75}, \quad x_1 = -\frac{7}{30}.$$

The solution of this equation is

$$x(t) = -\frac{20}{77} + \frac{1577}{5775}\cosh\left(\frac{2}{15}385^{1/2}t\right) - \frac{385^{1/2}}{220}\sinh\left(\frac{2}{15}385^{1/2}t\right).$$

The corresponding Picard operator will be

$$T(y(t)) = \frac{1}{75} - \frac{7}{30}t + \frac{8}{9}t^2 + \frac{77}{225}t^5.$$

If we let $x_0 = 0$, then we have the following linear system:

$$\begin{pmatrix} 1/3 & 1/8 & 1/140 \\ 1/8 & 1/20 & 1/320 \\ 1/140 & 1/320 & 1/4400 \end{pmatrix} \begin{pmatrix} x_1 \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1/12 \\ 1/180 \end{pmatrix}.$$

The solution of this linear system is:

$$(x_1, c_0, c_1) = (-0.1556, 1.5802, 7.6049).$$

The corresponding differential equation will be

$$\frac{d^2x}{dt^2} = 1.5802 + 7.6049x, \qquad x_0 = 0, \quad x_1 = -0.1556.$$

with solution

$$x(t) = -\frac{15802}{76049} - \frac{389}{1901225}\sinh\left(\frac{1}{100}76049^{1/2}t\right)76049^{1/2} + \frac{15802}{76049}\cosh\left(\frac{1}{100}76049^{1/2}t\right).$$

The corresponding Picard operator is

$$T(y(t)) = -0.1556t + \frac{1.5802}{2}t^2 + \frac{7.6049}{20}t^5.$$

Furthermore. if we let $x_0 = x_1 = 0$. then we have the following linear systems:

$$\begin{pmatrix} 1/20 & 1/320 \\ 1/320 & 1/4400 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1/12 \\ 1/180 \end{pmatrix}.$$

with solution

$$(c_0. c_1) = (80/81. 880/81).$$

The corresponding differential equation will be

$$\frac{d^2x}{dt^2} = -\frac{16}{9} + \frac{308}{45}x, \qquad x_0 = x_1 = 0.$$

The solution of this equation is

$$x(t) = -\frac{1}{11} + \frac{1}{11}\cosh\left(\frac{4}{9}55^{1/2}t\right).$$

In addition. we have the following corresponding Picard operator:

$$T(y(t)) = \frac{40}{81}t^2 + \frac{44}{81}t^5.$$

The data are listed in Table 4.1.

Remark 4.1. *If we use a first-order differential equation for the same target function $y(t) = t^3$ with N=1. then $h_0 = t. h_1 = t^4/4$.*

The corresponding linear system is:

$$\begin{pmatrix} 1 & 1/2 & 1/20 \\ 1/2 & 1/3 & 1/24 \\ 1/20 & 1/24 & 1/144 \end{pmatrix} \begin{pmatrix} x_0 \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/5 \\ 1/32 \end{pmatrix}.$$

with solution

$$(x_0. c_0. c_1) = (-1/32. 9/40. 27/8).$$

Then we get the following differential equation

$$\frac{dx}{dt} = \frac{9}{40}P_0(x) + \frac{27}{8}P_1(x) = \frac{9}{40} + \frac{27}{8}x, \quad x(0) = -\frac{1}{32}.$$

with solution

$$x(t) = -\frac{1}{15} + \frac{17}{480}e^{27t/8}.$$

The Picard operator T is of the form

$$T(y(t)) = -\frac{1}{32} + \frac{9}{40}t + \frac{27}{32}t^4.$$

The corresponding results are listed Table 4.2.

Table 4.1: Results by using a second-order differential equation

Order of equation	$x(0)$	$x'(0)$	$\|x(t) - y(t)\|$	$\|T(y(t)) - y(t)\|$
2	0	0	0.0406	0.0140
2	0	free	0.0103	0.0056
2	free	0	0.0164	0.0105
2	free	free	0.0040	0.0042

Table 4.2: Results by using a first-order differential equation

Order of equation	$x(0)$	$\|x(t) - y(t)\|$	$\|T(y(t)) - y(t)\|$
1	0	0.1624	0.9002
1	free	0.0298	0.0142

Remark 4.2. *Comparing Table 4.1 with Table 4.2 we see that, when a second-order equation is used to approximate the target solution, we obtain a better estimation.*

Example 4.3. *Let $y(t) = t^3$ be the target solution on the interval $A = [0, 1]$, $N = 2$, $t_0 = 0$.*

Then

$$h_0 = \frac{1}{2}t^2, \quad h_1 = \frac{1}{20}t^5, \quad h_2 = \frac{3}{112}t^8 - \frac{1}{4}t^2,$$

and we have the following linear system:

$$\begin{pmatrix} 1 & 1/2 & 1/6 & 1/120 & -9/112 \\ 1/2 & 1/3 & 1/8 & 1/140 & -67/1120 \\ 1/6 & 1/8 & 1/20 & 1/320 & -293/12320 \\ 1/120 & 1/140 & 1/320 & 1/4400 & -23/15680 \\ -9/112 & -67/1120 & -293/12320 & -23/15680 & 132823/11728640 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/5 \\ 1/12 \\ 1/180 \\ -53/1344 \end{pmatrix}.$$

whose solution is

$$(x_0, x_1, c_0, c_1, c_2) = (0.0062, -0.1389, -2.6821, 13.3086, -8.0802).$$

The corresponding differential equation is

$$\frac{d^2x}{dt^2} = 1.358 + 13.3086 * x - 12.1203 * x^2$$

with initial conditions

$$x_0 = 0.0062, \quad x_1 = -0.1389.$$

The Picard operator is:

$$T(y(t)) = 0.0062 - 0.1389t + 0.6790t^2 + 0.6654t^5 - 0.2164t^8.$$

We list numerical results in Table 4.3. From Figure 4.1 we can see that the numerical solution and the target solution are very close.

Table 4.3: Numerical results for Example 4.3

Order of equation	$x(0)$	$x'(0)$	$\|x(t) - y(t)\|$	$\|T(y(t)) - y(t)\|$
2	free	free	0.00021930	0.00020791

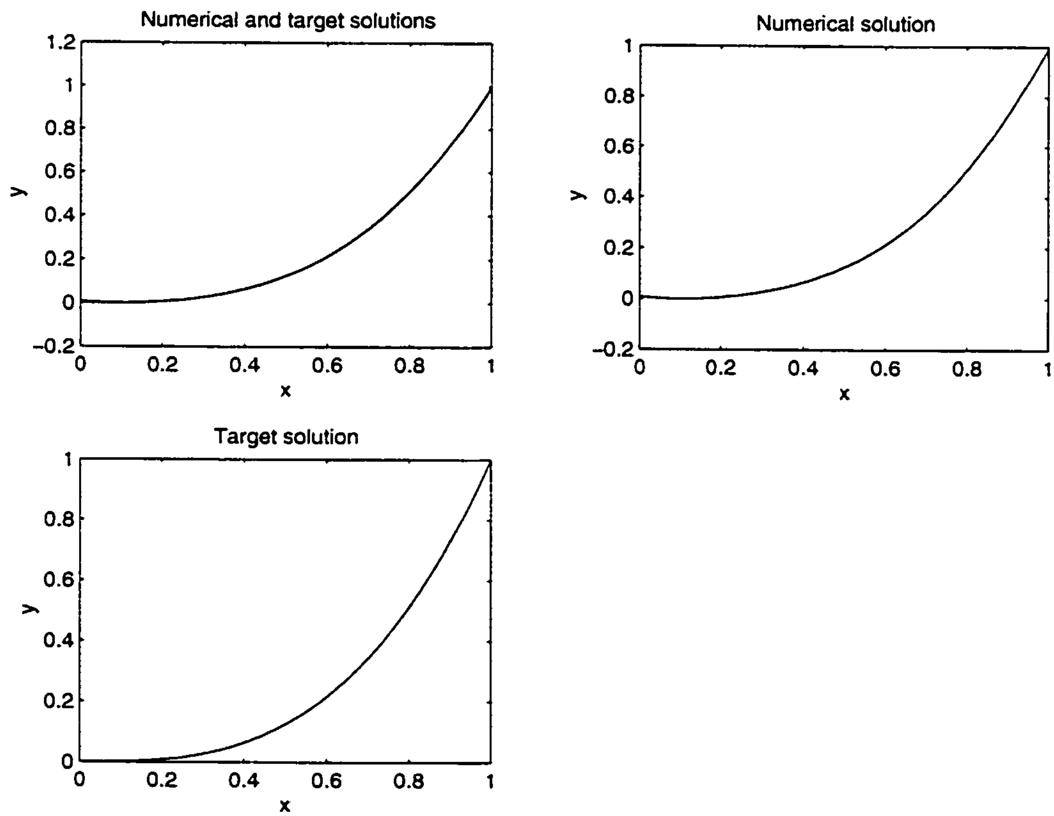

Figure 4.1: Target and numerical solutions.

4.4 Constructing ODE by Wavelet Basis

In this section, we will construct the differential equation by using wavelet basis.

As we know, wavelets are functions satisfying certain conditions. For simplicity, we begin with a simple function, that is, the Haar scaling function:

$$\phi(t) = \begin{cases} 1 & t \in [0,1), \\ 0 & \text{otherwise.} \end{cases} \tag{4.16}$$

From this scaling function, we have the scaled and shifted functions

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k), \qquad j = 0, 1, \ldots \text{ and } k = 0, 1, \ldots, 2^j - 1. \tag{4.17}$$

For fixed k, the set $\{\phi_{j,k}(t) : j = 0, 1, \ldots\}$ is orthonormal. We define the vector space V_j as

$$V_j = \text{span}\{\phi_{j,k} : \ k = 0, \ldots, 2^j - 1\}, \tag{4.18}$$

where "span" means linear combinations. The following property holds for V_j:

$$V_j \subseteq V_{j+1} \ . \tag{4.19}$$

Associated with ϕ is the Haar wavelet function:

$$\psi(t) = \begin{cases} 1 & t \in [0, 1/2), \\ -1 & t \in [1/2, 1), \\ 0 & \text{otherwise.} \end{cases} \tag{4.20}$$

Set

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), \qquad j = 0, 1, \ldots \text{ and } k = 0, 1, \ldots, 2^j - 1 \ . \tag{4.21}$$

By the choice of the constant $2^{j/2}$, the system $\{\psi_{j,k}\}$ is orthonormal. If one considers a wavelet function on other intervals than [0,1], the normalization constant will change. Now let W_j denote the subspace of $L_2(\mathbb{R})$ generated by

$$W_j = \text{span}\{\psi_{j,k} : \ k = 0, \ldots, 2^j - 1\}. \tag{4.22}$$

Then the space $\{W_j : j = 0, 1, \ldots\}$ forms an orthonormal basis for $L_2(\mathbb{R})$, and the following property holds:

$$W_j \subseteq W_{j+1} \,. \tag{4.23}$$

The Haar basis has the following important property:

$$V_{j+1} = V_j \oplus W_j. \tag{4.24}$$

To construct two-dimensional wavelets, we just need to use the one-dimensional ϕ and ψ. For example, for $x = (x_1, x_2) \in \mathbb{R}^2$, we take (see [9, p. 3]):

$$\begin{aligned} \psi_1(x_1, x_2) &:= \psi(x_1)\phi(x_2), \\ \psi_2(x_1, x_2) &:= \psi(x_2)\phi(x_1), \\ \psi_3(x_1, x_2) &:= \psi(x_1)\psi(x_2). \end{aligned}$$

Now we put

Table 4.4: Two-dimensional wavelets

x_1	x_2	$\psi_1(x_1, x_2)$	$\psi_2(x_1, x_2)$	$\psi_3(x_1, x_2)$
$(0, 1/2]$	$(0, 1/2]$	1	1	1
$(1/2, 1)$	$(0, 1/2]$	-1	1	-1
$(0, 1/2]$	$[1/2, 1)$	1	-1	-1
$[1/2, 1)$	$[1/2, 1)$	-1	-1	1

$$\Psi := \{\psi_1, \psi_2, \psi_3\}.$$

Then the set

$$\{\psi_{j,k} = 2^{j/2}\psi(2^j x - k) : \psi \in \Psi, j \in \mathbb{Z}, k \in \mathbb{Z}^2\}$$

forms an orthonormal basis in $L_2(\mathbb{R}^2)$.

Consider the Haar wavelets defined in 4.21, we have the following result.

Theorem 4.2. *For a target solution $y(t) \in C([0,1])$, there exists a differential equation*

$$\frac{d^2x}{dt^2} = \sum_{j=0}^{J}\sum_{k=0}^{2^J-1} c_{jk}\psi_{jk}(t)$$

with initial conditions

$$x(t_0) = x_0, x'(t_0) = x_1$$

such that the solution $x(t)$ is an approximation to the target solution $y(t)$, where x_0, x_1 and the coefficients c_{jk} satisfy the following linear equations:

$$\begin{array}{lllll} I(1)x_0 & + \quad I(t-t_0)x_1 & + & \sum_{j=0}^{J}\sum_{k=0}^{2^J-1} I(h_{jk})c_{jk} & = \quad I(y), \\ I(t-t_0)x_0 + & I((t-t_0)^2)x_1 & + & \sum_{j=0}^{J}\sum_{k=0}^{2^J-1} I((t-t_0)h_{jk})c_{jk} & = I((t-t_0)y), \\ I(h_{00})x_0 & + \quad I((t-t_0)h_{00})x_1 & + & \sum_{j=0}^{J}\sum_{k=0}^{2^J-1} I(h_{00}h_{jk})c_{jk} & = \quad I(h_{00}y), \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I(h_{JK})x_0 & + I((t-t_0)h_{JK})x_1 & + & \sum_{j=0}^{J}\sum_{k=0}^{2^J-1} I(h_{JK}h_{jk})c_{jk} & = \quad I(h_{JK}y), \end{array}$$

here, $K = 2^J - 1$ and

$$h_{jk}(t) = \int_{t_0}^{t}\int_{t_0}^{u} \psi_{jk}(v)dvdu. \tag{4.25}$$

Proof. Consider the following Picard operator:

$$T(y(t)) = x_0 + x_1(t-t_0) + \sum_{j=0}^{J}\sum_{k=0}^{2^J-1} c_{jk}h_{jk}(t).$$

By using the following squared collage distance

$$\Delta = \|y - T(y)\|_2 = \left\{\int_0^1 [y(t) - T(y)(t)]^2 dt\right\}^{1/2},$$

and the procedure in the proof of Theorem 4.1, we can get the result from the following stationarity conditions:

$$\frac{\partial \Delta^2}{\partial x_0} = 0, \tag{4.26}$$

$$\frac{\partial \Delta^2}{\partial x_1} = 0, \tag{4.27}$$

$$\frac{\partial \Delta^2}{\partial c_{jk}} = 0, \qquad 0 \le j \le J, \quad 0 \le k \le K = 2^J - 1. \tag{4.28}$$

Remark 4.3. *The basis in the theorem can be replaced by any other wavelets basis.*

Example 4.4. *Let* $y(t) = t + 1$ *be the target solution on the interval* $[0, 1]$. *Now we take* $J = 1$.

Then

$$\psi_{00}(t) := \begin{cases} 1, & 0 \le t < 1/2. \\ -1, & 1/2 \le t < 1. \end{cases} \tag{4.29}$$

$$\psi_{10}(t) := \begin{cases} \sqrt{2}, & 0 \le t < 1/4. \\ -\sqrt{2}, & 1/4 \le t < 1/2. \\ 0, & \text{other cases.} \end{cases} \tag{4.30}$$

$$\psi_{11}(t) := \begin{cases} \sqrt{2}, & 1/2 \le t < 3/4. \\ -\sqrt{2}, & 3/4 \le t < 1. \\ 0, & \text{other cases.} \end{cases} \tag{4.31}$$

From (4.25) we get the following:

$$h_{00}(t) = \begin{cases} t^2/2, & 0 \le t < 1/2. \\ -1/4 + t - t^2/2, & 1/2 \le t < 1. \end{cases} \tag{4.32}$$

$$h_{10}(t) = \begin{cases} \sqrt{2}/2t^2, & 0 \le t < 1/4. \\ \sqrt{2}/2(-1/8 + t - t^2), & 1/4 \le t < 1/2. \\ \sqrt{2}/16, & 1/2 \le t < 1. \end{cases} \tag{4.33}$$

$$h_{11}(t) = \begin{cases} 0, & 0 \le t < 1/2. \\ \sqrt{2}/2(1/4 - t + t^2), & 1/2 \le t < 3/4. \\ \sqrt{2}/2(-7/8 + 2t - t^2), & 3/4 \le t < 1. \end{cases} \tag{4.34}$$

We derive the following linear system:

$$\begin{pmatrix} 1 & 1/2 & 1/8 & 3\sqrt{2}/64 & \sqrt{2}/64 \\ 1/2 & 1/3 & 17/192 & 89\sqrt{2}/3072 & 41\sqrt{2}/3072 \\ 1/8 & 17/192 & 23/960 & 31\sqrt{2}/4096 & 15\sqrt{2}/4096 \\ 3\sqrt{2}/64 & 89\sqrt{2}/3072 & 31\sqrt{2}/4096 & 83/15360 & 1/512 \\ \sqrt{2}/64 & 41\sqrt{2}/3072 & 15\sqrt{2}/4096 & 1/512 & 23/15360 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ c_{00} \\ c_{10} \\ c_{11} \end{pmatrix} = \begin{pmatrix} 3/2 \\ 5/6 \\ 41/192 \\ 233\sqrt{2}/3072 \\ 89\sqrt{2}/3072 \end{pmatrix}.$$

whose solution is:

$$(x_0, x_1, c_{00}, c_{10}, c_{11}) = (1, 1, 0, 0, 0).$$

Then we have the following differential equation

$$\frac{d^2x}{dt^2} = 0\psi_{00}(t) + 0\psi_{10}(t) + 0\psi_{11}(t) = 0$$

with initial condition

$$(x_0, x_1) = (1, 1).$$

The solution is

$$x(t) = t + 1$$

which is the same as the target solution. Thus

$$\|x(t) - y(t)\|_2 = 0.$$

Chapter 5

Wavelet and Variational Problems

5.1 Preliminaries

In signal and image processing, the following type of variational problems have become fairly common, see, e.g., [39]. Given an image, a signal, or noisy data $f(x)$ defined for x in the interval $[0, 1]$ or the square $I = [0, 1] \times [0, 1]$ or other domains, we want to find a function $\tilde{f}$ such that it minimizes the following functional over a kind of space Y:

$$m(f) := \min_{g} \left\{ \|f - g\|^2_{L_p(I)} + c\|g\|^s_Y \right\}, \tag{5.1}$$

where c is a non-negative parameters, s is an exponent, Y is a space of functions, $L_p(I)$ is the usual L_p space. The solution of this minimization problem interprets wavelet shrinkage.

In Rudin–Osher–Fatemi [40], they set Y to be the space of functions of bounded variation $BV(I)$ for images. In R. A. DeVore and B. J. Lucier [14], they use spaces Y for which the norm of g in Y is equivalent to a sequence norm of the wavelet coefficients of g.

5.2 An Approximation to a Variational Problem

Consider the orthonormal wavelet basis $\{\psi_{j,k}\}$. the functions f and g can be expanded as follows:

$$f = \sum_{j,k,\psi} a_{j,k,\psi} \psi_{j,k}, \tag{5.2}$$

and

$$g = \sum_{j,k,\psi} b_{j,k,\psi} \psi_{j,k}. \tag{5.3}$$

From Chambolle–DeVore–Lee–Lucier [9]. we can deduce the following result.

Theorem E. *An approximation solution to the variational problem (5.1) is: either*

$$\hat{b}_{j,k,\psi} = a_{j,k,\psi} \quad or \quad \hat{b}_{j,k,\psi} = 0$$

for any j, k, and ψ.

For simplicity, in the sequel, we let Y to be the space where the norm is the sequence norm of the wavelet coefficients. The result is also suitable for some other spaces.

Next we will provide a better estimation on the variational problem.

Theorem 5.1. *For $s = 2$, we get an approximation to the variational problem:*

$$\hat{b}_{j,k,\psi} := \frac{1}{1+c} a_{j,k,\psi}$$

for any j, k, and ψ. The corresponding function is:

$$\hat{f} := \sum_{j,k,\psi} \frac{1}{1+c} a_{j,k,\psi} \psi_{j,k}. \tag{5.4}$$

Furthermore,

$$m(f) = \left(\frac{c}{1+c}\right)^2 \|f\|^2_{L_2(I)}.$$

Proof. Set

$$m(f,g) := \|f-g\|^2_{L_p(I)} + c\|g\|^s_Y. \tag{5.5}$$

Substituting (5.2) and (5.3) into (5.5) we get

$$\begin{aligned} m(f,g) &= \sum_{j,k,\psi} |a_{j,k,\psi} - b_{j,k,\psi}|^2 + c \sum_{j,k,\psi} |b_{j,k,\psi}|^s \\ &= \sum_{j,k,\psi} \left(|a_{j,k,\psi} - b_{j,k,\psi}|^2 + c|b_{j,k,\psi}|^s \right). \end{aligned} \tag{5.6}$$

Let

$$E_{j,k,\psi}(b_{j,k,\psi}) = |a_{j,k,\psi} - b_{j,k,\psi}|^2 + c|b_{j,k,\psi}|^s. \tag{5.7}$$

To minimize $m(f,g)$, we need only minimize each $E_{j,k,\psi}(b_{j,k,\psi})$. Consider

$$\begin{aligned} E_{j,k,\psi}(ta_{j,k,\psi}) &= |a_{j,k,\psi} - ta_{j,k,\psi}|^2 + c|ta_{j,k,\psi}|^s \\ &= (1-t)^2 |a_{j,k,\psi}|^2 + ct^s |a_{j,k,\psi}|^s. \end{aligned} \tag{5.8}$$

where t is a parameter to be determined. From

$$\frac{d\left(E_{j,k,\psi}(ta_{j,k,\psi})\right)}{dt} = 0$$

we get

$$-2(1-t)|a_{j,k,\psi}|^2 + cst^{s-1}|a_{j,k,\psi}|^s = 0,$$

which is equivalent to

$$1 = t + \frac{cs}{2} t^{s-1} |a_{j,k,\psi}|^{s-2}. \tag{5.9}$$

When $s = 2$, by (5.9) we have

$$t = \frac{1}{1+c}.$$

This implies that

$$\begin{aligned} E_{j,k,\psi}\left(\frac{1}{1+c} a_{j,k,\psi}\right) &= \left(1 - \frac{1}{1+c}\right)^2 |a_{j,k,\psi}|^2 + c\left(\frac{1}{1+c}\right)^2 |a_{j,k,\psi}|^2 \\ &= \left(\frac{c}{1+c}\right)^2 |a_{j,k,\psi}|^2. \end{aligned} \tag{5.10}$$

Note that,

$$E_{j,k,\psi}\left(0 * a_{j,k,\psi}\right) = |a_{j,k,\psi}|^2$$

and

$$E_{j,k,\psi}\left(1 * a_{j,k,\psi}\right) = c|a_{j,k,\psi}|^2.$$

Thus

$$E_{j,k,\psi}\left(\frac{1}{1+c}a_{j,k,\psi}\right) < E_{j,k,\psi}\left(0 * a_{j,k,\psi}\right), E_{j,k,\psi}\left(1 * a_{j,k,\psi}\right). \tag{5.11}$$

Therefore we take

$$\hat{b}_{j,k,\psi} := \frac{1}{1+c}a_{j,k,\psi}$$

and

$$\hat{f} := \sum_{j,k,\psi} \frac{1}{1+c}a_{j,k,\psi}\psi_{j,k}.$$

Substitute this into (5.1) we obtain

$$m(f) = \left(\frac{c}{1+c}\right)^2 \|f\|^2_{L_2(I)}.$$

From this we can see that $m(f)$ can be arbitrarily small when we make c to be sufficiently small. The proof is complete.

Remark 5.1. *From (5.11) we see that our approximation is better.*

Chapter 6

Conclusions and Future Work

6.1 Conclusions

We have studied dynamical systems and wavelets.

We solved Baker's conjecture for a family of entire functions. that is. some permutable entire functions will have the same Julia set.

For the relaxed Newton function of a given entire function we have found the location of the limits of the iterating sequence on its wandering domains. We also constructed an example with the corresponding properties.

Concerning inverse problems. we have constructed second-order differential equations with a Legendre polynomial basis and a wavelet basis to approximate the target solution. Numerical solutions show that our approximations are much better and faster than the known results.

In regard to variational problems in signal and image processing. we have obtained a good approximation to the function formed by the signal or the image.

Lastly. we introduced a new and interesting research area: wavelets on local fields.

6.2 Future Work

6.2.1 Complex dynamics

In the future, we hope to completely solve Baker's conjecture, which is an important open problem in complex dynamics.

6.2.2 Inverse problems

In the area of inverse problems, we will try to replace our previous basis by Chebyshev polynomials and other wavelets. In addition, we also want to study similar questions for complex differential equations (cf. [23]).

6.2.3 Variational problem

Concerning the variational problem, we will continue to study its applications to the representation of signal and images, the removal of noise, and so on.

As an application, we will try to use wavelets to solve some differential equations.

6.2.4 Wavelets on Local Fields

The theory of wavelet analysis in $\mathbb{R}^n$ has been rapidly and widely developed. There are also some results in other spaces such as Abelian groups, abstract Hilbert spaces, compact Gelfand pairs (cf. [11, 19, 36, 38]). However, one finds almost no research results on wavelet analysis in local fields. A local field has the same algebraic structure as the real number field and the complex number field. So it is natural to consider wavelet analysis on local fields (Fourier analysis on local fields can be found in [43]).

Next we introduce local fields (see [43]).

Let $\mathbb{Z}$ be the set of integers. For a prime number b, let X_b be the collection of formal power series, i.e.,

$$X_b = \left\{ x : \quad x = \sum_{k=s}^{+\infty} a_k b^k,\ a_k \in \{0, \cdots, b-1\} \right\},$$

where $x = 0$ if and only if $a_k = 0$ for all k. The b-adic norm $\|\cdot\|_{b-\text{adic}}$ is defined as follows:

$$\|x\|_{b-\text{adic}} := \begin{cases} 0, & \text{if} \quad x = 0, \\ b^{-s}, & \text{if} \quad x = \sum_{k=s}^{+\infty} a_k b^k, a_s \neq 0. \end{cases} \tag{6.1}$$

where $s \in \mathbb{Z}$.

Remark 6.1. *The number b is called the characteristic of X_b.*

For X_b, we use the usual addition and multiplication. For any

$$x = \sum_{k=s}^{+\infty} a_k b^k \in X_b,$$

we set $o(x) = s$. Then for $x, y, x_0, y_0 \in X_b$, we have

$$\begin{aligned} o((x-y)-(x_0-y_0)) &\geq \min\{o(x-x_0), o(y-y_0)\}, \\ o(xy - x_0y_0) &\geq \min\{o(x-x_0)o(y), o(y-y_0)o(x_0)\}. \end{aligned}$$

These two inequalities imply the continuity of addition and multiplication.

For $x, y \in X_b$, we define a metric as follows.

$$d(x, y) = \|x - y\|_{b-\text{adic}}.$$

Then X_b becomes a topological space.

Let K be such a field with the above topology, let K^+ and K^* be the additive group and multiplicative group of K.

For a locally compact, non-discrete and topologically complete field K, we have two cases as follows:

- If K is connected, then $K = \mathbb{R}$ or $K = \mathbb{C}$.

- If K is not connected, then K is totally disconnected. In this case, if K is of finite characteristic, then K is a field of formal power series over the finite field $F(b^c)$ for some positive number c.

A local field K is an algebraic field and a topological space, which is locally compact, non-discrete, complete and disconnected. An element in K with maximum norm is called a prime element of K.

A local field K has the following properties (see Taibleson [43]):

- K is totally disconnected.

- $\|x\|_{b-\text{adic}} \geq 0$ and $\|x\|_{b-\text{adic}} = 0$ if and only if $x = 0$;

- $\|xy\|_{b-\text{adic}} = \|x\|_{b-\text{adic}}\|y\|_{b-\text{adic}}$;

- $\|x + y\|_{b-\text{adic}} \leq \max(\|x\|_{b-\text{adic}}, \|y\|_{b-\text{adic}})$ (ultrametric inequality);

- K^+ is a locally compact Abelian group. Thus we may choose a Haar measure dx on K^+. A Haar measure on K^* is $dx/\|x\|_{b-\text{adic}}$.

To construct wavelets on local fields, as usual, we first establish multiresolution analysis on local fields.

Set

$$\overline{D} = \{x \in K : \|x\|_{b-\text{adic}} \leq 1\}.$$

$$\partial D = \{x \in K : \|x\|_{b-\text{adic}} = 1\}$$

and

$$D = \{x \in K : \|x\|_{b-\text{adic}} < 1\}.$$

The set $\overline{D}$ is called the ring of integers in K and D is a prime ideal in K.

Let c be a positive integer and b a prime number as before. Then any integer n satisfying $0 \le n < b^c$ can be expanded as follows:

$$n = a_0 + a_1 b + \cdots + a_{c-1} b^{c-1}, \tag{6.2}$$

where $0 \le a_k < b$ for $k = 0, 1, \ldots, c-1$. Note that $\overline{D}/D \cong \mathrm{F}(b^c)$ and $\mathrm{F}(b^c)$ is a c-dimensional vector space. Thus for a canonical homomorphism f from $\overline{D}$ on to $\mathrm{F}(b^c)$, we can choose a set $\{d_0 = 1, \cdots, d_{c-1}\} \subset \partial D$ such that $\{f(d_j) : j = 0, 1, \ldots, c-1\}$ is a basis of $\mathrm{F}(b^c)$. Now by (6.2), for a prime element δ of K, we define

$$k(n) = (a_0 d_0 + a_1 d_1 + \cdots + a_{c-1} d_{c-1})\delta^{-1} \qquad (0 \le n < b^c).$$

Then $k(0) = 0$, $\|k(j)\|_{b\text{-adic}} = b^c$ $(1 \le j < b^c)$, and $\{k(j) : j = 0, 1, \ldots, b^c - 1\}$ is a complete set of coset representatives of $\overline{D}$ in D^{-1}. This way, we can construct $\{k(j) : j = 0, 1, \ldots, \infty\}$ which is a complete set of coset representatives of $\overline{D}$ in K^+.

Definition 6.1. *Let K be a local field and δ a prime element of K. A multiresolution analysis of $L_p(K)$ consists of a sequence of closed subspaces V_j $(j \in \mathbb{Z})$ of $L_p(K)$ satisfying*

(i) $V_j \subset V_{j+1}$;

(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_p(K)$;

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iv) $f(x) \in V_j$ *if and only if* $f(\delta^{-1}x) \in V_{j+1}$;

(v) *There is a scaling function $\phi(x) \in V_0$ such that $\{\phi(x - k(j)) : j = 0, \ldots\}$ is an orthonormal basis of V_0.*

For every $j \in \mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1}, i.e.,

$$V_{j+1} = V_j \bigoplus W_j.$$

Then

$$W_k \perp W_j \qquad \text{if} \quad k \neq j.$$

Thus we can decompose $L_p(K)$ into mutually orthogonal subspaces:

$$L_p(K) = \bigoplus_{j \in \mathbb{Z}} W_j.$$

A simple example is the following wavelet function which is similar to the Haar wavelet. Let the scaling function be defined as follows:

$$\phi(t) = \begin{cases} 1 & t \in D, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Note that, if $g \in L_p(K)$ with $\|g\|_{L_p} = 1$, then $\|g_{j,n}\|_{L_p} = 1$, where

$$g_{j,n}(x) = \delta^{-j/p} g(\delta^{-j} x - k(n)).$$

Based on this fact, we can get a sequence of subspaces V_j of $L_p(K)$ as usual. The corresponding wavelet function is

$$\psi(x) = \phi(\delta^{-1} x) - \phi(\delta^{-1} x - 1).$$

From this ψ we can construct a basis of W_0. However, this basis is not orthogonal. So we propose the following open questions.

Open Question 6.1. *Can we construct an orthogonal wavelet basis for W_0?*

Open Question 6.2. *Can we get some similar properties for wavelet analysis on $L_p(K)$?*

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