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Reaction-Diffusion Equations and the Laplacian in Hilbert Space

By
Shuyu Wang

Thesis Submitted
to the School of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in Mathematics

at the
University of Ottawa



Shuyu Wang, Ottawa, Canada, 1990



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UNIVERSITÉ D'OTTAWA
UNIVERSITY OF OTTAWA

To the memory of my father

Reaction Diffusion Equations and the
Laplacian in Hilbert Space

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Abstract

This dissertation consists of two parts. First, we study some problems associated with reaction-diffusion equations with variables in finite-dimensional space. We investigate the positivity of solutions, the existence of positive invariant regions, and we also make some stability analysis. In part II, we study the Lévy-Laplacian in infinite-dimensional space. We explore some properties of this Laplacian and solve some boundary value problems.

Introduction

In this thesis, two topics are discussed. First, we work on the reaction-diffusion equations in n -dimensional space. Reaction-diffusion equations have lately received a great deal of attention. Their interest lies partially in the fact that they occur in the mathematical models for a wide range of natural processes (see, e.g., [3], [18], [34], [35], and the references given in those papers). Another reason lies in the wealth of mathematical problems about these equations. In the article [9] of Fife, one may find a survey of this subject, as well as an extensive bibliography of related research. In the study of reaction-diffusion equations, most researchers consider weakly-coupled systems, and put their attention on the various restrictions for the reaction terms. It is our purpose in the first part to analyse the effects of cross-diffusions in several aspects. Our results show that cross-diffusion indeed plays an important role in the preservation of positivity, the existence of invariant regions and stability, etc..

In the second part, we study the Laplacians in infinite-dimensional space. The Laplace operator in infinite-dimensional space was originally introduced by P. Lévy, (see [19], [20]). Later some soviet mathematicians contributed to this program of extending the theory of this differential

equation to infinite-dimensional space. (see [S], [25], [30—33]). The original definition of the Laplacian given by Lévy depends on the choice of the basis. Šilov [31] defined a Laplacian on a narrow class of functionals in which the Laplacian is invariant under the change of basis. In our paper, we use the original Lévy definition of C^2 -functionals to derive a Laplacian which does not depend on the choice of basis. Then we explore some important properties of this Laplacian. Finally, we discuss the solvability of boundary value problems associated with linear and semilinear equations.

Part I

**Some Problems on
Reaction-Diffusion Equations
with Cross-Diffusion**

Introduction

In recent years, systems of reaction-diffusion equations have received a great deal of attention by both their widespread occurrence in models of chemical biological phenomena, and by the richness of the structure of their solution sets. In the simplest models, the equations take the form

$$\frac{\partial \mathbf{u}}{\partial t} = D\Delta \mathbf{u} + \mathbf{f}(\mathbf{u}) \quad x \in \Omega \subset R^k, t > 0, \quad (0.1)$$

where $\mathbf{u} \in R^n$, D is an $n \times n$ matrix, Δ is the Laplace operator in the spatial coordinates and $\mathbf{f}(\mathbf{u})$ is a vector-valued smooth function. In applications, the matrix D which describes the dispersal degree of species is called the diffusive matrix and $\mathbf{f}(\mathbf{u})$ which represents the reaction and interaction between species is called the reaction term. Hence, the equation (0.1) is usually called a reaction-diffusion equation (ref. [3], [18]).

Before the 1970's, the role of spatial heterogeneity had mostly been ignored in the literature of mathematical ecology, yet the equation had been studied for more than 50 years. Due to the motivation from ecology, the reaction-diffusion equations have received much attention from mathematicians. A review may be found in Levin [18]. During last two decades, the R-D systems in which only self-diffusion is present (i.e., D is diagonal) have

been studied extensively, (ref. [2], [9], [24], [28], [34]). For the case in which cross-diffusion is present (i.e., D is nondiagonal), few results are available. However, the study of general multicomponent diffusions with nondiagonal diffusion matrix is an important aspect of many physical and biological processes, (ref. [14], [16], [35]), e. g., Diffusive Lotka-Volterra mechanisms arising from ecological and chemical phenomena, Keller-Segel model for the chemotactic movements of cellular slime molds, disease epidemic models, etc.. Recently, some work on this kind of R-D systems has been done. For example, some nonlinear analysis was carried out by Nanjundiah [22] and Childress and Percus [4] ; the stationary problem for the Keller-Segel system has been considered by C. S. Lin, W. M. Ni and I. Takagi [21] . Little is known about the evolution of components for the general system (0.1) with nondiagonal diffusion matrix.

In the following chapters, we will investigate some problems for system (0.1) with a nondiagonal diffusion matrix D . The outline is as follows: In chapter 1, we discuss the preservation of nonnegativity, along with a necessary and sufficient condition for such to hold. In chapter 2, the invariant region method has been used to get global existence of positive solutions. Finally, in chapter 3, we study the effect of cross-diffusion on the stability of equilibria and we also discuss some asymptotic behavior for the solutions.

Chapter 1

Preservation of nonnegativity

Introduction

In the applications, the components of $u(t, x)$ in R-D system (0.1) usually represent the chemical concentrations or biological population densities, and so it is natural to seek the positive solutions. So, the first problem that we want to study is the preservation of nonnegativity of solutions of (0.1) under nonnegative initial and boundary conditions. When D is a diagonal matrix, the preservation of nonnegativity is an immediate consequence of the maximum principle if the reaction term $f(u)$ satisfies some suitable structure conditions. In the presence of cross-diffusion it turns out to be more complicated. C. S. Kahane ([15]) has considered this problem, and he proved a negative result in this direction when D is a constant positive definite nondiagonal matrix. In this chapter, we will give some necessary and sufficient conditions for the preservation of nonnegativity for more general

systems. From the point of view of applications, the necessary condition we derived is in agreement with physical motivation.

For convenience, we state a standard maximum principle for weakly coupled parabolic systems first. Let $H = H(t, x)$ be an $n \times n$ matrix with entries $h_{pq}(t, x)$, where $h_{pq}(t, x)$, $p, q = 1, \dots, n$, are continuous functions defined in a subset $(0, T) \times \Omega$ of R^{1+k} . Consider the system of parabolic inequalities

$$\begin{cases} \Delta u_1 + \sum_{q=1}^n h_{1q} u_q - \frac{\partial u_1}{\partial t} \geq 0 \\ \dots \\ \Delta u_n + \sum_{q=1}^n h_{nq} u_q - \frac{\partial u_n}{\partial t} \geq 0. \end{cases} \quad (1.1)$$

Assume that the off-diagonal terms of the matrix H are nonnegative:

$$h_{pq} \geq 0 \text{ for } p \neq q; \quad p, q = 1, 2, \dots, n. \quad (1.2)$$

We shall use the notation $u = \text{col}(u_1, \dots, u_n)$, $u < 0$ to mean that every component $u_p, p = 1, \dots, n$ is negative. Similarly, $u \leq 0$ means that every component is nonpositive.

[Maximum Principle] ([26]) *Suppose that $u \in (C^{1,2}((0, T) \times \Omega))^n$ satisfies the uniformly parabolic system of inequalities (1.1) in a bounded domain $E = (0, T) \times \Omega$. If $u \leq 0$ at $t = 0$ and on $(0, T) \times \partial\Omega$ and if H satisfies the condition (1.2), then $u \leq 0$ in E . Moreover, if $u_p = 0$ at an interior point (t_0, x_0) , then $u_p = 0$ for $t \leq t_0$.*

Note: In (1.1), $\Delta u_p, (p = 1, \dots, n)$ can be replaced by $L_p u_p$, where L_p are operators which are uniformly elliptic for fixed $t \in (0, T)$. For more details about maximum principles, we refer to [26].

1.1 Linear equations

Consider the system

$$\frac{\partial \mathbf{u}}{\partial t} = A\Delta \mathbf{u} + \sum_{i=1}^k B_i \frac{\partial \mathbf{u}}{\partial x_i} + C\mathbf{u} \quad (1.3)$$

where $\mathbf{u} = \mathbf{u}(t, x)$ is an n -vector valued function with components $u_j(t, x)$, $j = 1, \dots, n$, $(t, x) \in R^+ \times \Omega$ (Ω is a bounded subset in R^k , $R^+ = (0, \infty)$); $\Delta \mathbf{u} = \text{col}(\Delta u_1, \dots, \Delta u_n)$ with the usual Laplace operator Δ in R^k ; $\frac{\partial \mathbf{u}}{\partial x_i} = \text{col}(\frac{\partial u_1}{\partial x_i}, \dots, \frac{\partial u_n}{\partial x_i})$; A , B_i ($i = 1, \dots, k$) and C are $n \times n$ matrices.

We say the system (1.3) has the property of preserving nonnegativity if the solutions of (1.3) with the following initial-boundary conditions

$$\mathbf{u}(t, x) = \phi(t, x) \geq 0 \quad (t, x) \in R^+ \times \partial\Omega \quad (1.4)$$

$$\mathbf{u}(0, x) = \mathbf{u}^0(x) \geq 0 \quad x \in \Omega \quad (1.5)$$

are nonnegative for all $t > 0$.

Note: The boundary condition (1.4) can be replaced by Neumann or Robin boundary conditions.

First we consider the linear systems.

Assumption L: $A = (a_{pq}(t, x))$, $B_i = (b_{pq}^i(t, x))$, ($i = 1, \dots, k$), $C = (c_{pq}(t, x))$ are continuous $n \times n$ matrices in $[0, \infty) \times \Omega$ and A is nonsingular, and $a_{pp} > 0$, ($p = 1, \dots, n$), for all $(t, x) \in R^+ \times \Omega$.

Theorem 1.1 *Let assumption L hold. The system (1.3) has the property of preserving nonnegativity if and only if A, B_i ($i = 1, \dots, k$) are diagonal and the off-diagonal elements of C are nonnegative.*

Proof: Necessity. Assume that system (1.3) preserves nonnegativity. Suppose the theorem is not true. Then there is at least a pair $(p, q), p \neq q$, such that $a_{pq} \neq 0$, or $b_{pq}^i \neq 0$ for some i , or $c_{pq} < 0$ at some point $(t_0, x_0) \in R^+ \times \Omega$, for definiteness, let $(p, q) = (1, 2), t_0 = 0$.

Suppose that $u = u(t, x)$ is a solution of (1.3) with homogeneous Dirichlet boundary condition and initial condition (1.5) in which

$$u^0(x) = \text{col}(0, u_2^0(x), 0, \dots, 0).$$

Consider the first equation of the system (1.3). We have

$$\begin{aligned} \frac{\partial u_1}{\partial t} = & a_{11}\Delta u_1 + \dots + a_{1n}\Delta u_n + \sum_{i=1}^k (b_{11}^i \frac{\partial u_1}{\partial x_i} + \dots + b_{1n}^i \frac{\partial u_n}{\partial x_i}) \\ & + c_{11}u_1 + \dots + c_{1n}u_n. \end{aligned} \quad (1.6)$$

Let $t \rightarrow 0, x = x_0$, to find

$$\frac{\partial u_1}{\partial t} \Big|_{t=0, x=x_0} = \{a_{12}\Delta u_2^0 + \sum_{i=1}^k b_{12}^i \frac{\partial u_2^0}{\partial x_i} + c_{12}u_2^0\}_{x=x_0}. \quad (1.7)$$

Now we choose $u_2^0(x) \geq 0$ suitably such that above quantity is negative, for example, if $c_{12}(0, x_0) < 0$, then choose $u_2^0(x)$ such that

$$u_2^0(x_0) > 0, \Delta u_2^0(x_0) = \frac{\partial u_2^0(x_0)}{\partial x_i} = 0, i = 1, \dots, k.$$

Hence, we get $u_1(t, x_0) < 0$ for all sufficiently small positive t since $u_1(0, x) = 0$. This contradicts the preservation of nonnegativity.

Sufficiency. Assume that $A, B_i, (i = 1, \dots, k)$, are diagonal and $C = (c_{pq})$ with $c_{pq} \geq 0$ for $p \neq q$. Then the system (1.3) becomes a weakly coupled parabolic system (i.e., the system is coupled in the terms which are not differentiated). Hence the sufficiency is an immediate consequence of *theorem 1*, the standard maximum principle. Q.E.D.

1.2 Nonlinear equations

Theorem 1.1 tells us that when we consider the system (1.3) in the presence of cross-diffusion, if we expect the preservation of nonnegativity, then the coefficients of the cross-diffusion must be concentration dependent in some sense. The following theorem gives a necessary condition for the preservation of nonnegativity. For simplicity, we consider the system

$$\frac{\partial \mathbf{u}}{\partial t} = A \Delta \mathbf{u} \quad (1.8)$$

where A depends on either \mathbf{u} or $\nabla \mathbf{u}$, or both.

Theorem 1.2 *A necessary condition for system (1.8) to preserve nonnegativity is that the off-diagonal elements a_{pq} of A satisfy*

$$\lim_{\tilde{u}_p \rightarrow 0} a_{pq} = 0 \quad \forall (t, x) \in \mathbb{R}^+ \times \Omega, \mathbf{u} \geq 0; p \neq q \quad (1.9)$$

where $\tilde{u}_p = (u_p, \nabla u_p)$, and ∇ is the gradient symbol.

Proof: The method is similar to the one used in the proof of the preceding theorem. In fact, if (1.9) is not true, for example, $a_{12}(t, x, \mathbf{u}, \nabla \mathbf{u})|_{\tilde{u}_1=0} \neq 0$, at $(t, x) = (0, x_0)$, it is possible to choose $\mathbf{u}^0 = (0, u_2^0, 0, \dots)$ with $u_2^0(x) \geq 0$ suitably such that

$$a_{12}(0, x_0, \mathbf{u}^0, \nabla \mathbf{u}^0) \Delta u_2^0(x_0) < 0.$$

By the first equation in (1.8),

$$\frac{\partial u_1}{\partial t} = a_{11} \Delta u_1 + \dots + a_{1n} \Delta u_n,$$

and so by letting $t \rightarrow 0$, $x = x_0$, we get

$$\frac{\partial u_1}{\partial t} \Big|_{t=0, x=x_0} = a_{12} \Delta u_2^0(x_0) < 0.$$

Hence $u_1(t, x_0) < 0$ for small t since $u_1(0, x_0) = 0$. This gives a proof of the theorem.

Next we will give a sufficient condition. Consider the system

$$\frac{\partial \mathbf{u}}{\partial t} = A \Delta \mathbf{u} + C \mathbf{u}. \quad (1.10)$$

where A depends on \mathbf{u} , $\nabla \mathbf{u}$ or both and C may also depend on \mathbf{u} , etc..

Theorem 1.3 *Assume that*

(1). *The off-diagonal elements a_{pq} of A have the form*

$$a_{pq} = \alpha_{pq} u_p + \beta_{pq} \cdot \nabla u_p, \quad p \neq q \quad (1.11)$$

where α_{pq}, β_{pq} are continuous functions and n -vector valued functions, respectively.

(2). *The diagonal elements of A are positive.*

(3). *The off-diagonal elements of C are nonnegative.*

Then (1.10) preserves nonnegativity.

Proof: Using condition (1) in the assumption, the p^{th} equation of (1.10) can be rewritten as

$$\frac{\partial u_p}{\partial t} = a_{pp} \Delta u_p + r_p \cdot \nabla u_p + \sum_{q=1}^n s_{pq} u_q \quad (1.12)$$

where

$$r_p = \sum_{q=1, q \neq p}^n \Delta u_q \beta_{pq},$$

$$s_{pq} = c_{pq} \geq 0. \quad (p \neq q),$$

$$s_{pp} = \sum_{q=1, q \neq p}^n \alpha_{pq} \Delta t_q + c_{pp}$$

are bounded for classical solutions. Because of (2). (3) in the assumption, the standard maximum principle can be applied to (1.12) (the system whose p^{th} equation has the form (1.12)), hence we get preservation of nonnegativity. Q.E.D.

Finally, we note that if the diffusive coefficients are smooth enough, e.g., C^1 -class, then in *theorem 1.3*. (1.11) can be replaced by (1.9). In fact, in this case the necessary condition (1.9) is equivalent to (1.11) by the following lemma.

Lemma 1.1 *If $f \in C^1(R)$, and $f(0) = 0$, then there is a continuous function $h(t)$ such that $f(t) = h(t)t$.*

Proof: Define $h(t)$ by

$$h(t) = \frac{f(t)}{t} \quad \text{for } t \neq 0, \quad h(0) = f'(0),$$

then

$$\lim_{t \rightarrow 0} h(t) = \lim_{t \rightarrow 0} \frac{f(t)}{t} = f'(0) = h(0),$$

that is, $h(t)$ is continuous. Q.E.D.

If $a_{pq} = a_{pq}(t, x, u)$ is C^1 smooth in the argument u_p , by *lemma 1.1*, (1.9) is equivalent to (1.11). Hence it is easy to see that the assumption in *theorem 1.3* is also a necessary condition for preserving nonnegativity.

The discussion in this chapter shows that if the cross-diffusion appears, the cross-dispersal terms must be present in the form of a nonlinearity. The models from chemical engineering theory have provided the nice motivation for this result. In [29], a condition for diffusion coefficients was specified in the modeling of multicomponent reaction-diffusion equations,

$$\frac{\partial \mathbf{u}}{\partial t} = D\Delta \mathbf{u} + K(\mathbf{u}).$$

This condition is that the off-diagonal elements of the diffusion matrix must satisfy the following limit

$$\lim_{u_i \rightarrow 0} D_{ij} = 0 \text{ for } i \neq j$$

to ensure species conservation. The description of the chemotactic interaction of amoebae, as mediated by acrasin, is modeled by

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla(D_1 \nabla u - D^* \nabla v) \\ \frac{\partial v}{\partial t} = D_2 \Delta v + u f(v) - k(v)v. \end{cases} \quad (1.13)$$

A possible form of $D^*(u, v)$ is

$$D^* = \delta u/v$$

where δ is a constant. This model was first formulated by E. F. Keller and L. A. Segel, so it is usually called the Keller-Segel model, (ref. [16]).

In the study of reaction-diffusion equations with diagonal diffusion, an extensive useful tool is a kind of comparison theorem, thus monotone methods have often been used to get solutions, (ref. [24], [28]). For the R-D systems with nondiagonal diffusion, we don't have such a comparison theorem, thus this increases the difficulty for the study. In the next chapter, we will use an invariant region method to get an existence theorem.

Chapter 2

Invariant regions and the existence of positive solutions

Introduction

In this chapter we use the invariant region method to get the global existence of positive solutions for a class of reaction-diffusion systems with cross-diffusion:

$$\frac{\partial \mathbf{u}}{\partial t} = D\Delta \mathbf{u} + \mathbf{f}(\mathbf{u}) \quad x \in \Omega \subset R^k, t > 0 \quad (2.1)$$

where $\mathbf{u} \in R^n$, D is an $n \times n$ matrix, and $\mathbf{f}(\mathbf{u})$ is a smooth vector-valued function.

The invariant region method is a method which ensures compactness. Assume the system (2.1) admits bounded invariant regions, i.e., bounded regions Σ in phase space (i.e., \mathbf{u} -space), with the property that if the “data” lie in Σ , then the solution $\mathbf{u}(x, t)$ lies in Σ for all $x \in \Omega$ and all $t > 0$. Thus

Σ provides *a-priori* sup-norm bound on u , and it follows that if the datum lies in Σ , then the solution exists for all $t > 0$. In [34], J. Smoller has given a necessary and sufficient condition for the systems to admit invariant regions. For the systems with only diagonal diffusions, these conditions hold, e.g., choosing Σ as suitable rectangles, (ref. [2], [17], [34], [36]). But do invariant regions exist for systems with non-diagonal diffusions? In this chapter we shall investigate this problem for a class of reaction-diffusion systems with cross diffusions as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = D_1 \Delta u + D^* \Delta v + f(u, v) \\ \frac{\partial v}{\partial t} = + D_2 \Delta v + g(u, v). \end{cases} \quad (2.2)$$

This system arises from the model of chemotactic movements of cellular slime molds (see Keller and Segel [16]).

The outline of this chapter is as follows. In §1, we review some basic concepts and results about invariant regions. Next, in §2, we seek some conditions for ensuring the existence of feasible regions for the system (2.2). Also, we give some examples for which the invariant regions can be constructed geometrically. Finally, an existence theorem is established in §3.

2.1 Preliminaries

In this section we describe the notion of an invariant region and a sequence of fundamental theorems about it provided in [34].

Consider the system

$$\frac{\partial \mathbf{u}}{\partial t} = D\Delta \mathbf{u} + \mathbf{f}(\mathbf{u}) \quad (t, x) \in \mathbb{R}^+ \times \Omega \quad (2.3)$$

together with the initial data

$$\mathbf{u}(0, x) = \mathbf{u}^0(x) \quad x \in \Omega. \quad (2.4)$$

Here Ω is an open subset in \mathbb{R}^k , \mathbf{u} is a vector-valued function from $\mathbb{R}^+ \times \Omega$ to \mathbb{R}^n , $D = D(\mathbf{u})$ is an $n \times n$ matrix with all positive eigenvalues. If Ω is not the whole space \mathbb{R}^k , we will assume that \mathbf{u} satisfies specific boundary conditions, e. g., Dirichlet or Neumann boundary conditions. We assume that this problem has a local (in time) solution on some set X of smooth functions from Ω to \mathbb{R}^n , i.e., given a function $\mathbf{u}_0 \in X$, there is a $\delta > 0$ such that for $t \in [0, \delta)$, such that $\mathbf{u}(t, x) \in X$. The topology on X should be at least as strong as the compact-open topology.

Definition 2.1 *A closed subset $\Sigma \subset \mathbb{R}^n$ is called an invariant region for the local solutions defined by (2.3),(2.4), if any solution $\mathbf{u}(t, x)$ having all of its boundary and initial values in Σ , satisfies $\mathbf{u}(t, x) \in \Sigma$ for all $x \in \Omega$ and for all $t \in [0, \delta)$.*

Here are some notations we will use later. Let $G = G(\mathbf{v}) = G(v_1, \dots, v_n)$ be a smooth function defined in \mathbb{R}^n . Denote by

$$dG = \left(\frac{\partial G}{\partial v_1}, \dots, \frac{\partial G}{\partial v_n} \right) = \nabla G,$$

$$d^2G = \left[\frac{\partial^2 G}{\partial v_i \partial v_j} \right]_{n \times n}.$$

If $\eta = (\eta_1, \dots, \eta_n)$, $\zeta = (\zeta_1, \dots, \zeta_n) \in R^n$, then denote by

$$dG \cdot \eta = \sum_{i=1}^n \eta_i \frac{\partial G}{\partial v_i},$$

$$d^2G(\eta, \zeta) = \eta d^2G \zeta^T,$$

where ζ^T is the transpose of the vector ζ .

The invariant regions Σ will be made up of the intersection of "half spaces", i.e., we consider regions Σ of the form

$$\Sigma = \bigcap_{i=1}^m \{u \in R^n; G_i(u) \leq 0\} \quad (2.5)$$

where G_i are smooth real-valued functions defined in R^n , and for each i , the gradient dG_i never vanishes.

Definition 2.2 *The smooth function $G : R^n \rightarrow R$ is called quasi-convex at v if whenever $dG(v) \cdot \eta = 0$, $\eta \in R^n$, then $d^2G(v)(\eta, \eta) \geq 0$.*

In the following we cite some important facts, as propositions, about invariant regions proved by Smoller [34].

Proposition 2.1 (sufficiency) *Let Σ be defined by (2.5), and suppose that for every $v \in \partial \Sigma$ (so $G_i(v) = 0$ for some i), the following conditions hold:*

1. dG_i at v is a left eigenvector of $D(v)$.
2. If $dG_i D(v) = \mu dG_i$, with $\mu \neq 0$, then G_i is quasi-convex at v .

3. $dG_i \cdot f < 0$ at v . (f is given in the system (2.3).)

Then Σ is an invariant region for (2.3).

Proposition 2.2 (Necessity) Let Σ be defined by (2.5), and suppose that Σ is an invariant region for (2.3). Then the following conditions hold at each point v on $\partial\Sigma$ (say $G_i(v) = 0$):

1. dG_i is a left eigenvector of D at v .
2. G_i is quasi-convex at v .
3. $dG_i \cdot f \leq 0$ at v .

Definition 2.3 The system (2.3) is called f-stable if, whenever f is the limit of functions in the C^1 -topology on compacta, then any solution of (2.3),(2.4) is the limit in the compact-open topology, of solutions of (2.3), (2.4), where f is replaced by f_n .

Proposition 2.3 Let Σ be defined by (2.5), and suppose the system (2.3) is f -stable. Then Σ is an invariant region for (2.3) if and only if the followings hold at each boundary point v of $\partial\Sigma$ (so $G_i(v) = 0$):

1. dG_i is a left eigenvector of D .
2. G_i is a quasi-convex at v .
3. $dG_i \cdot f \leq 0$.

Definition 2.4 A closed bounded subset Σ defined by (2.5) of R^n is called a feasible region for system (2.3) if the first two conditions in the above theorem hold.

2.2 Invariant regions for a class of systems

In this section, we are going to find invariant regions for the Keller-Segel system (2.2). Let

$$D = \begin{pmatrix} D_1 & D^* \\ 0 & D_2 \end{pmatrix} \quad (2.6)$$

where D_1, D_2 are unequal positive constants and $D^* = D^*(u, v) \in C^1$. It is natural to assume that

$$\lim_{u \rightarrow 0} D^*(u, v) = 0 \quad \forall v \geq 0 \quad (2.7)$$

because we are interested in positive solutions (see chapter 1).

First, observe the left-eigenvectors of the positive definite upper triangular matrix

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

We have,

1. if $b = 0$, then $(0, 1)$ and $(1, 0)$ are two independent left eigenvectors of A with eigenvalues c and a , respectively;
2. if $b \neq 0, a = c$, then there is only one left eigenvector $(0, 1)$ (in the independent sense) of A with eigenvalue a ;
3. if $b \neq 0, a \neq c$, then $(0, 1)$ and $(1, b/(a - c))$ are independent left eigenvectors of A with eigenvalues c and a , respectively.

Assume Σ , a closed bounded subset of R^2 , has the form

$$\Sigma = \{(u, v) \in R^2; G_i(u, v) \leq 0, i = 1, 2, 3, 4\} \quad (2.8)$$

where

$$\begin{aligned} G_1 &= -v. \\ G_2 &= v - a. \\ G_3 &= -u \end{aligned}$$

and $G_4 = G_4(u, v)$ is to be determined.

By the assumption on D and the discussion above, $dG_1 = (0, -1)$, $dG_2 = (0, 1)$ are left eigenvectors of D , and $dG_3 = (-1, 0)$ is also a left eigenvector of D at $G_3 = -u = 0$ (since $D^*(0, v) = 0$). And $dG_i, i = 1, 2, 3$ are all quasiconvex because of the linearity of the G_i 's.

Now we find $G_4 = G(u, v)$ such that Σ is a feasible region. To this end we only have to consider the behavior of $G(u, v) = 0$. By *condition 1* in *proposition 2.3*, $dG = (G_u, G_v)$ must be a left eigenvector of D at $G(u, v) = 0$. In order that Σ be bounded, closed, we assume that $G_u \neq 0$. By the implicit function theorem, u can be solved from $G(u, v) = 0$. So, for simplicity, let $G(u, v) = u - h(v)$ where $h(v)$ is to be determined.

By *condition 1* in *proposition 2.3* and the previous discussion (item 3),

$$-h'(v) = \frac{D^*(h(v), v)}{D_1 - D_2}. \quad (2.9)$$

This is an ordinary differential equation of the first order. Hence for $h_0 > 0$ there is a solution $h = h(v) > 0$ of (2.9) with $h(0) = h_0$ for $v \in [0, v_0)$ since $D^* = D^*(u, v) \in C^1$, where v_0 is some positive constant.

We still need to consider the quasi-convexity of the function $G = G(u, v) = u - h(v)$ at $G = 0$. Since

$$dG = (1, -h'(v)),$$

$$d^2G = \begin{pmatrix} 0 & 0 \\ 0 & -h''(v) \end{pmatrix},$$

let $\eta = (\eta_1, \eta_2) (\neq 0) \in R^2$ be such that $dG(\eta) = 0$, i.e., $\eta_1 = h'(v)\eta_2$, then $\eta d^2G \eta^T = -h''(v)\eta_2^2$. Hence the quasiconvexity of G is equivalent to

$$h''(v) \leq 0. \quad (2.10)$$

By (2.9), we get

$$h''(v) = \left\{ \frac{1}{(D_1 - D_2)^2} D^*(u, v) \frac{\partial D^*}{\partial u} - \frac{1}{D_1 - D_2} \frac{\partial D^*}{\partial v} \right\}_{u=h(v)}. \quad (2.11)$$

Hence, by (2.10) and (2.11), the condition for quasiconvexity is

$$D^* \frac{\partial D^*}{\partial u} \leq (D_1 - D_2) \frac{\partial D^*}{\partial v} \quad (2.12)$$

at $u = h(v)$. So we have proved

Theorem 2.1 *Assume that $D^* = D^*(u, v) \in C^1$. If (2.12) holds for $0 < \alpha \leq u \leq \beta$, $0 \leq v \leq \gamma$, then for each $h_0 \in (\alpha, \beta)$, there exists a constant a and $h = h(v) > 0$ for $v \in [0, a] \subset [0, \gamma]$ with $h(0) = h_0$, $\alpha \leq h(v) \leq \beta$ such that Σ defined by (2.8) is a feasible region for system (2.2) with $G_4 = G(u, v) = u - h(v)$.*

Next we observe some special cases to simplify the condition (2.12).

Case 1. Assume $D^* = D^*(u, v) = p(u)$. Clearly, in this case, $\frac{\partial D^*}{\partial v} = 0$.

Hence (2.12) becomes

$$p(u)p'(u) \leq 0$$

for $0 < \alpha \leq u \leq \beta$, where α, β are some positive numbers. Solving the o.d.e. (2.9) by integration, with $h(0) = h_0 > 0$, $p(h_0) \neq 0$, we get

$$\int_{h(0)}^{h(v)} \frac{dt}{p(t)} = \frac{-v}{D_1 - D_2}.$$

If we denote by $P = P(t)$ a primitive function of $\frac{1}{p(t)}$, then we have

$$P(h(v)) = \frac{-v}{D_1 - D_2} + c$$

where c is some constant. Hence

$$h(v) = P^{-1}\left(\frac{-v}{D_1 - D_2} + c\right).$$

Example 1. Let $D^* = \sin u$. Clearly the condition (2.12) holds for $u \in [\pi/2, \pi]$. Since $\int \frac{dt}{\sin t} = \ln \tan \frac{t}{2}$, we get

$$\tan \frac{h(v)}{2} = c \exp \frac{v}{D_2 - D_1},$$

hence

$$h(v) = 2 \arctan \left[c \exp \frac{v}{D_2 - D_1} \right].$$

So, let

$$G = G(u, v) = u - 2 \arctan \left[c \exp \frac{v}{D_2 - D_1} \right], \quad (2.13)$$

and choose c such that

$$\pi/2 \leq u = 2 \arctan \left[c \exp \frac{v}{D_2 - D_1} \right] \leq \pi,$$

i.e.

$$1 \leq c \exp \frac{v}{D_2 - D_1} < \infty$$

for all $v \in [0, a]$, i.e.

$$c \geq \max\{1, \exp \frac{a}{D_1 - D_2}\}.$$

Then Σ defined by (2.8) with $G_4 = G(u, v)$ defined by (2.13) is a feasible region for all positive a . Hence, in this case the system admits an arbitrarily large feasible region since inequality (2.12) holds in $(\pi/2 + 2n\pi, (2n + 1)\pi)$ for any positive integer n , (a is arbitrary).

Case 2. Let $D^* = D^*(u, v) = uq(v)$.

In this case the condition (2.12) is

$$q^2(v) \leq (D_1 - D_2)q'(v)$$

(hence a necessary condition is that $\text{sgn}(q'(v)) = \text{sgn}(D_1 - D_2)$), if $q(v) \neq 0$, then

$$1 \leq (D_1 - D_2) \frac{q'(v)}{q^2(v)},$$

i.e.

$$v \leq (D_1 - D_2)(1/q(0) - 1/q(v))$$

for all $v \geq 0$. In order to get $h = h(v)$, we look at the o.d.e. (2.9),

$$-h'(v) = \frac{h(v)q(v)}{D_1 - D_2}.$$

By integration, we get

$$h(v) = ce^{\frac{1}{D_2 - D_1} \int_0^v q(t) dt}$$

where c is a positive number, $c = h(0)$.

Example 2. $D^* = D^*(u, v) = u/(1+v)$. By a calculation, the condition for the existence of feasible regions is that $D_2 - D_1 \geq 1$. In this case, $G_4 = G(u, v) = u - c(1+v)^{\frac{1}{D_2-D_1}}$.

Case 3. Let $D^* = D^*(u, v) = p(u)q(v)$.

If $p > 0$ for $u \geq \alpha > 0$ and $q(v) \neq 0$ in $[0, \gamma)$, Then condition (2.12) is

$$p'(u) \leq (D_1 - D_2)(-1/q(v))'$$

for $v \geq 0$ and u in some positive interval. From the o.d.e. (2.9), we get

$$h = h(v) = P^{-1}(k(v, c))$$

where $P = P(t)$ is a primitive function of $1/p(t)$, c is a constant and

$$k(v, c) = \frac{1}{D_2 - D_1} \int_0^v q(t)dt + c.$$

By now we have shown that it is possible for system (2.2) to have feasible regions if the condition (2.12) holds for D . In order that Σ be precisely an invariant region we need to check condition 3 in *proposition 2.3*, i.e., $dG_i \cdot (f, g) \leq 0$ for $i = 1, 2, 3, 4$. Now we show that Σ of the form (2.8) is indeed an invariant region for a class of reaction terms.

Assume that $D^*(u, v) \geq 0$, $D_2 > D_1$. Σ has the form of (2.8) where $h = h(v)$ is the solution of the o.d.e. (2.9) with $h(0) = h_0 > 0$. Clearly, $h'(v) > 0$.

Let (f, g) in (2.2) have the form

$$\begin{cases} f(u, v) = v - l(u) \\ g(u, v) = \alpha u - \beta v \end{cases}$$

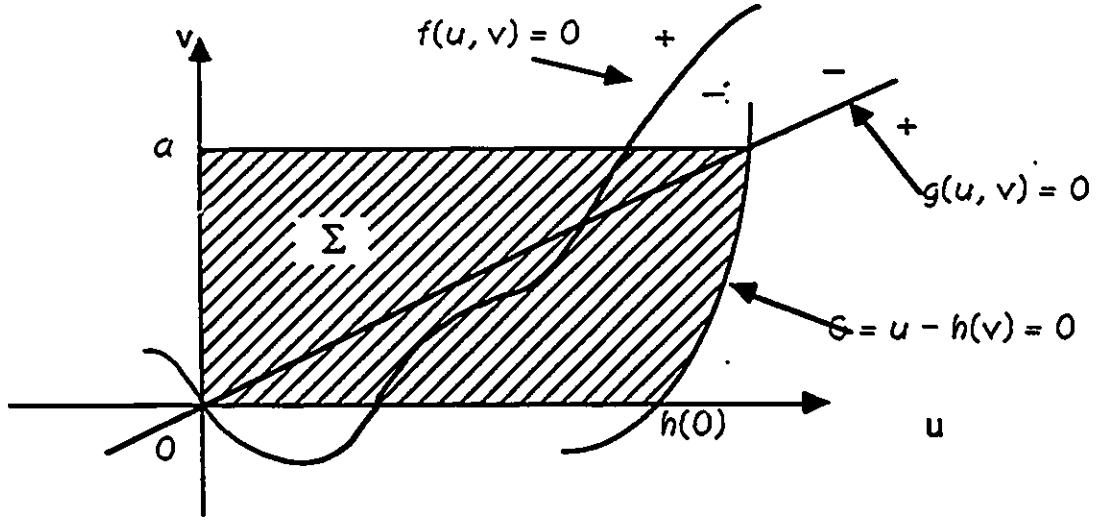


Figure 2.1: Invariant region

with $l(0) = 0, l(u) \geq \frac{\alpha}{\beta}u$ for $u \geq h_0$ and α, β are positive constants. Then by choosing a in (2.8) for Σ such that $h(a) = \frac{\beta}{\alpha}a$, Σ becomes an invariant region for the system (2.2). For example, the zero sets of f, g and the invariant region in $u - v$ phase plane are depicted in figure 2.1.

It suffices to show that $dG_i \cdot (f, g) \leq 0$ at $G_i = 0$. In fact,
 $G_1 = -v, dG_1 = (0, -1),$

$$dG_1 \cdot (f, g)|_{G_1=0} = -g|_{v=0} \leq 0;$$

$$G_2 = v - a, dG_2 = (0, 1),$$

$$dG_2 \cdot (f, g)|_{G_2=0} = g|_{v=a} \leq 0;$$

$$G_3 = -u, dG_3 = (-1, 0),$$

$$dG_3 \cdot (f, g)|_{G_3=0} = -f|_{-u=0} \leq 0$$

$$\text{and } G_4 = u - h(v), dG_4 = (1, -h'(v)),$$

$$dG_4 \cdot (f, g)|_{G_4=0} = f - h'(v)g|_{u=h(v)} \leq 0.$$

Hence, for the system (2.2), if condition (2.12) holds for $u > h_0$ and signs of f, g in $u - v$ phase plane are indicated as in figure 2.1, then for any given "datum" in Σ , it is possible to conclude that the global solution of the system (2.2) exists and is still located in Σ . Details are in the next section.

2.3 Existence of positive solutions

In order to use the invariant region method to get the existence of global solutions, we need to ensure the local existence of solutions. The latter is a classical result. For convenience, we state a result about the local solvability of the Cauchy problem for a quasilinear parabolic system as a lemma here.

Consider the Cauchy problem for the quasilinear parabolic system

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} &= A(t, x, \mathbf{u}, D\mathbf{u})\mathbf{L}\mathbf{u} + \mathbf{f}(t, x, \mathbf{u}, D\mathbf{u}) \\ \mathbf{u}|_{t=t_0} &= \mathbf{u}_0(x) \end{cases} \quad (2.14)$$

in a domain $Q = \{(t, x, z, y); t \in [t_0, T], x \in R^k, z \in R^n, y \in R^\nu, \|z\|, \|y\| \leq M\}$, where $\mathbf{L} = \text{col}(L_1, \dots, L_n)$, $L_i (i = 1, \dots, n)$ are uniformly elliptic operators (quasilinear) for each $t \in [t_0, T]$. $A(t, x, z, y)$, is an $n \times n$ matrix such that (2.14) is a parabolic system, and $\mathbf{f}(t, x, z, y)$ is a vector function, and each is defined in Q .

Proposition 2.4 *Let the entries of A and F be in the space $C^{(0,\alpha,1,1)}(Q)$ (the space of functions which are continuous in t , Hölder continuous with Hölder constant α in x and are Lipschitz continuous in other arguments). The problem (2.14) has a unique solution which belongs to the space $C^{(1,2+\alpha)}(\Pi_\delta)$*

with $\delta > 0$, where $\Pi_\delta = \{(t, x) \in [t_0, \delta] \times R^k\}$. The magnitude of the time interval δ depends on the upper bounds of moduli of coefficients, their derivatives, Hölder constants and initial values ([7]).

This proposition is a special case of theorem 6.3 in the book [7]. For more details about the local existence of solutions of parabolic systems, we also refer to the typical works [11] and [37].

With the aid of the above lemma, we can get the existence of positive solutions for the system (2.2).

Theorem 2.2 *Suppose system (2.2) has a positive invariant region Σ . Then for $T > 0$, the solution of (2.2) with initial data $(u_0(x), v_0(x)) \in \Sigma$ exists for all $t \in [0, T]$.*

Proof: Since (2.2) has a positive invariant region Σ and initial condition $\{u_0(x), x \in R^k\} \subset \Sigma$, we consider the problem for $u \in \Sigma$. Hence all coefficients are bounded uniformly. Using *proposition 2.4*, $\exists \delta > 0$, δ depends on Σ , such that (2.2) along with the initial condition have a solution for $t \in [0, \delta]$, and $u(x, t) \in \Sigma$ since Σ is invariant. Then taking $u(x, \delta)$ as the initial data at $t = \delta$, we use the lemma again, to get a solution as $t \in [\delta, 2\delta]$. We repeat this process to find a solution on $[2\delta, 3\delta]$, and eventually after finite steps, we obtain a solution on $[0, T]$, and it stays in Σ . The theorem is proved.

Combining the results in the previous section with the above theorem, we see that the evolution problem of Keller-Segel is solvable under some suitable conditions.

Example 3: We consider the logarithmic sensitivity ([1]) case, i.e.,

$$\begin{cases} \frac{\partial u}{\partial t} = D_1 \Delta u + \frac{u}{1+v} \Delta v \\ \frac{\partial v}{\partial t} = D_2 \Delta v + \alpha u - \beta v. \end{cases} \quad (2.15)$$

Suppose $D_1 < D_2$, let

$$\Sigma = \{(u, v) \in \mathbb{R}^2; 0 \leq u \leq h(v), 0 \leq v \leq a\}$$

where $a > 0$ satisfies

$$a = (\alpha/\beta)h(a)$$

and

$$h(v) = c(1+v)^{\frac{1}{D_2-D_1}}.$$

It is easy to check that for all $c > 0$, Σ is a positive invariant region for equation (2.15). So, we can find a positive invariant region as large as we want. Hence by *theorem 2.2*, (2.15) has a positive solution for all bounded initial values.

Chapter 3

Asymptotic behaviour and stability analysis

3.1 Asymptotic behaviour

As an application of invariant regions, in this section we analyse the asymptotic behaviour of solutions of reaction diffusion equations which we studied in previous sections. The result in this section is the extension of J. Smoller's result in [34] to nonconstant and nonsymmetric diffusion matrices.

Let Ω be a bounded domain in \mathbb{R}^k . Consider the general Keller-Segel model

$$\frac{\partial \mathbf{u}}{\partial t} = D\Delta \mathbf{u} + F(\mathbf{u}) \quad x \in \Omega, t > 0 \quad (3.1)$$

with homogeneous Neumann boundary condition

$$\frac{\partial \mathbf{u}}{\partial n} = 0 \quad x \in \partial\Omega, t > 0, \quad (3.2)$$

where $\mathbf{u} = \text{col}(u, v)$, $u, v \in C^{1,2}(R^+ \times \Omega)$, and

$$D = \begin{pmatrix} D_1 & D^* \\ 0 & D_2 \end{pmatrix}, F = \begin{pmatrix} f \\ g \end{pmatrix}$$

are given matrix-valued and vector-valued smooth functions, respectively, defined in R^2 . We assume that

$$D_1 > 0, D_2 > 0 \text{ are constants, } \lim_{\mathbf{u} \rightarrow 0} D^*(\mathbf{u}) = 0, \quad (3.3)$$

and (3.1) has a positive invariant region Σ . Let

$$\gamma = 4D_1D_2 - \|D^*\|^2 > 0 \quad (3.4)$$

with $\|D^*\| = \sup_{\mathbf{u} \in \Sigma} |D^*(\mathbf{u})|$. Set

$$M = \sup\left\{\left|\frac{\partial F}{\partial \mathbf{u}}\right|, \mathbf{u} \in \Sigma\right\}. \quad (3.5)$$

Some well-known inequalities follow (see [34]):

Proposition 3.1 *Let $u \in W_2^1(\Omega)$, then if λ is the smallest positive eigenvalue of $-\Delta$ on Ω (with the appropriate boundary conditions), the following Poincaré inequalities hold*

$$\|\nabla u\|^2 \geq \lambda \|u\|^2 \text{ if } u = 0 \text{ on } \partial\Omega; \quad (3.6)$$

$$\|\nabla u\|^2 \geq \lambda \|u - \bar{u}\|^2 \text{ if } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \quad (3.7)$$

where $\bar{u} = (\text{meas.}\Omega)^{-1} \int_{\Omega} u dx$, and

$$\|\Delta u\|^2 \geq \lambda \|\nabla u\|^2 \text{ if } u \in W_2^2(\Omega) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \quad (3.8)$$

In the above theorem the norm $\| \cdot \|$ denotes the L_2 -norm, i.e.

$$\| u \|^2 = \int_{\Omega} |u|^2 dx.$$

In the sequel, if \mathbf{u} is a vector-valued function in Ω , we still denote by $\| \cdot \|$ the usual product L_2 - norm, i.e., if $\mathbf{u} = (u, v)$, then

$$\| \mathbf{u} \|^2 = \int_{\Omega} |u|^2 dx + \int_{\Omega} |v|^2 dx.$$

Define a number

$$\eta = \lambda\alpha - M \quad (3.9)$$

where $\lambda > 0$ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition, and

$$\alpha = \frac{1}{2}(D_1 + D_2 - \sqrt{(D_1 + D_2)^2 - \gamma}). \quad (3.10)$$

Theorem 3.1 *Let $\{u_0(x), x \in \Omega\} \subset \Sigma$ and $u(t, x)$ be the solution of (3.1), (3.2) with $u(0, x) = u_0(x)$. If $\eta > 0$, then $\exists c_i, i = 1, 2, 3$, such that*

$$(i) \quad \| \nabla_x u(t, x) \| \leq c_1 e^{-\eta t}, \quad (3.11)$$

$$(ii) \quad \| u(t, x) - \bar{u}(t) \| \leq c_2 e^{-\eta t} \quad (3.12)$$

with

$$\bar{u}(t) = (\text{meas.}\Omega)^{-1} \int_{\Omega} u(x, t) dx$$

and

$$\frac{d\bar{u}}{dt} = F(\bar{u}) + w(t), \quad (3.13)$$

$$\bar{u}(0) = (\text{meas.}\Omega)^{-1} \int_{\Omega} u_0(x) dx, \quad (3.14)$$

$$|w(t)| \leq c_3 e^{-\eta t}. \quad (3.15)$$

Proof: Define

$$\phi(t) = \frac{1}{2} \|\nabla_x \mathbf{u}(t, \cdot)\|^2 = \frac{1}{2} \int_{\Omega} (\nabla \mathbf{u}, \nabla \mathbf{u}) dx. \quad (3.16)$$

Then

$$\begin{aligned} \dot{\phi}(t) &= \int_{\Omega} (\nabla \mathbf{u}, \nabla \mathbf{u}_t) dx \\ &= \int_{\Omega} (\nabla \mathbf{u}, \nabla (D\Delta \mathbf{u} + F(\mathbf{u}))) dx \\ &= \int_{\Omega} (\nabla \mathbf{u}, \nabla (D\Delta \mathbf{u})) dx + \int_{\Omega} (\nabla \mathbf{u}, \nabla F(\mathbf{u})) dx \\ &= - \int_{\Omega} (\Delta \mathbf{u}, D\Delta \mathbf{u}) dx + \int_{\Omega} (\nabla \mathbf{u}, (dF)\nabla(\mathbf{u})) dx \\ &= - \int_{\Omega} (\Delta \mathbf{u}, D_s \Delta \mathbf{u}) dx + \int_{\Omega} (\nabla \mathbf{u}, (dF)\nabla(\mathbf{u})) dx \end{aligned}$$

where $dF = \frac{\partial F}{\partial \mathbf{u}}$ is the Jacobian matrix, $D_s = (1/2)(D + D^r)$ is the symmetric part of D . By the assumption on D , $D_1 > 0$, and

$$\det(D_s) = D_1 D_2 - (1/4)(D^*)^2 > 0,$$

i.e., D_s is positive definite. Its eigenvalues are

$$\lambda_{s,l} = \frac{1}{2}(D_1 + D_2 \pm \sqrt{(D_1 + D_2)^2 - 4(D_1 D_2 - \frac{1}{4}(D^*)^2)})$$

Hence

$$\alpha \leq \lambda_s < \lambda_l \leq \beta$$

where

$$\begin{aligned} \alpha &= \frac{1}{2}(D_1 + D_2 - \sqrt{(D_1 + D_2)^2 - \gamma}) = \min_{\mathbf{u} \in \Sigma} \lambda_s, \\ \beta &= \frac{1}{2}(D_1 + D_2 + \sqrt{(D_1 + D_2)^2 - \gamma}) = \max_{\mathbf{u} \in \Sigma} \lambda_l \end{aligned}$$

and γ is defined by (3.4). We note that

$$0 < \frac{\gamma}{4(D_1 + D_2)} < \alpha \leq \min\{D_1, D_2\}.$$

From the calculation before, we have

$$\dot{\phi}(t) \leq -\alpha \int_{\Omega} \|\Delta \mathbf{u}\|^2 dx + M \int_{\Omega} (\|\nabla \mathbf{u}\|^2) dx$$

where M is defined by (3.5). By using the Poincaré inequality (3.8) for each component of \mathbf{u} , we get

$$\begin{aligned} \dot{\phi}(t) &\leq -\alpha \lambda \int_{\Omega} \|\nabla \mathbf{u}\|^2 dx + M \int_{\Omega} \|\nabla \mathbf{u}\|^2 dx \\ &= (M - \alpha \lambda) \int_{\Omega} \|\nabla \mathbf{u}\|^2 dx \\ &= -2\eta \phi(t), \end{aligned}$$

i.e., we have

$$\dot{\phi}(t) \leq -2\eta \phi(t).$$

This implies that

$$\phi(t) \leq \phi(0)e^{-2\eta t},$$

i.e.

$$\frac{1}{2} \int_{\Omega} |\nabla \mathbf{u}(t, x)|^2 dx \leq \frac{1}{2} \int_{\Omega} |\nabla \mathbf{u}(0, x)|^2 dx e^{-2\eta t}.$$

Setting $c_1 = \|\nabla \mathbf{u}_0\|$, we get

$$\|\nabla \mathbf{u}\| \leq c_1 e^{-\eta t}.$$

So, (i) is proved.

In order to get the second inequality in the theorem, just use the Poincaré inequality (3.7),

$$\lambda \|\mathbf{u} - \bar{\mathbf{u}}\|^2 \leq \|\nabla \mathbf{u}\|^2$$

where

$$\bar{\mathbf{u}} = (\text{meas.}\Omega)^{-1} \int_{\Omega} \mathbf{u} dx. \quad (3.17)$$

This, along with (3.11), and $c_2 = c_1/\sqrt{\lambda}$ gives

$$\| \mathbf{u} - \bar{\mathbf{u}} \| \leq c_2 e^{-\eta t}.$$

To complete the proof of the theorem, we show (3.13)–(3.15). Differentiating (3.17) with respect to t ,

$$\begin{aligned} \dot{\bar{\mathbf{u}}}(t) &= (\text{meas.}\Omega)^{-1} \int_{\Omega} \mathbf{u}_t(t, x) dx \\ &= (\text{meas.}\Omega)^{-1} \int_{\Omega} (D\Delta \mathbf{u} + F(\mathbf{u})) dx. \end{aligned}$$

We consider the first component $\bar{u}(t)$ of $\bar{\mathbf{u}}$. Then

$$\begin{aligned} \dot{\bar{u}}(t) &= (\text{meas.}\Omega)^{-1} \int_{\Omega} (D_1 \Delta u + D^* \Delta v + f(\mathbf{u})) dx \\ &= (\text{meas.}\Omega)^{-1} \int_{\Omega} (-\nabla_x D^*) \cdot (\nabla v) + f(\mathbf{u}) dx \\ &= (\text{meas.}\Omega)^{-1} \int_{\Omega} \left[-\left(\frac{\partial D^*}{\partial u} \nabla u + \frac{\partial D^*}{\partial v} \nabla v \right) \cdot (\nabla v) + f(\mathbf{u}) \right] dx \\ &= f(\bar{u}, \bar{v}) + w_1(t) \end{aligned}$$

where we set

$$w_1(t) = (\text{meas.}\Omega)^{-1} \int_{\Omega} \left[-\left(\frac{\partial D^*}{\partial u} \nabla u + \frac{\partial D^*}{\partial v} \nabla v \right) \cdot (\nabla v) + f(\mathbf{u}) - f(\bar{\mathbf{u}}) \right] dx.$$

Since F, D^* are smooth, thus, on Σ , $\exists M, M_1$ such that $|dF| \leq M, |\nabla D^*| \leq M_1$, hence

$$\begin{aligned} |w_1(t)| &\leq (\text{meas.}\Omega)^{-1} \int_{\Omega} [M|\mathbf{u} - \bar{\mathbf{u}}| + M_1(|\nabla u \nabla v| + |\nabla v|^2)] dx \\ &\leq (\text{meas.}\Omega)^{-1} [M(\int_{\Omega} 1)^{1/2} \|\mathbf{u} - \bar{\mathbf{u}}\| + M_1(\|\nabla u\| \|\nabla v\| + \|\nabla v\|^2)] \\ &\leq c_3 e^{-\eta t} \end{aligned}$$

for some suitable constant c_3 . (The second inequality above is an application of the Schwarz inequality and the last inequality is obtained by previous results.) Similarly, for the second component $\bar{v}(t)$, we also get

$$\dot{\bar{v}}(t) = g(\bar{\mathbf{u}}) + w_2(t)$$

where $w_2(t)$ satisfies the same inequality as w_1 does. This completes the proof.

This theorem tells us that if $\eta > 0$, then solutions of the reaction-diffusion equations under homogeneous Neumann boundary condition and initial condition get exponentially close to their spatial averages as $t \rightarrow \infty$, i.e., they tend to the spatial homogeneous solutions in $W_2^1(\Omega)$ exponentially fast as $t \rightarrow \infty$. Another result that we can derive from this theorem is that if $\eta > 0$, then there couldn't exist any nonconstant stationary state, i.e., the elliptic system

$$D\Delta \mathbf{u} + f(\mathbf{u}) = 0 \tag{3.18}$$

with homogeneous Neumann boundary condition has no nonconstant solutions! This is because the solutions of (3.18) depend only on x , and *theorem 3.1* implies that they must tend to solutions independent of x as $t \rightarrow \infty$.

In [21], Lin *et al* have explored the existence of stationary solutions of K-S systems with homogeneous Neumann boundary condition. They gave an existence theorem under some restriction on the diffusion coefficients.

3.2 Stability analysis

From both the theoretical and applied standpoints, the stability analysis is one of the important mathematical problems. But it is hard to deal with. In this section, we mainly consider the effects of cross-dispersal on the linear stability of equilibria. Our question is that if u_0 is a stable equilibrium for the associated kinetic system, then how is the stability of u_0 affected by the presence of diffusion?

Before giving the detailed investigation, we state a criterion for linear stability here.

Consider a system

$$u_t = A(u) \tag{3.19}$$

where $u = u(t) \in B$ for each $t \in R^+$, B is a Banach space, A is an operator defined on B .

Suppose $\phi \in B$ is a stationary state of (3.19), i.e., $A(\phi) = 0$. Let S_ϕ be the linearized operator of A about the given stationary state ϕ . Denote by $\sigma(S)$ the spectrum of the operator S .

Criterion for linear stability (ref. [9])

ϕ is stable according to the linearized criterion (l.c.) if $\sigma(S_\phi)$ is in the negative half-plane and is bounded away from the imaginary axis. ϕ is marginally stable (l.c.) if $\sigma(S_\phi)$ is in the negative closed half-plane and is not bounded away from the imaginary axis. ϕ is unstable (l.c.) if $\sigma(S_\phi)$ contains a point in the right open half-plane.

Now we begin our investigation. Consider

$$\mathbf{u}_t = D\Delta\mathbf{u} + F(\mathbf{u}) \quad (3.20)$$

where, for simplicity of calculation, $\mathbf{u} = (u, v)$, D is 2×2 matrix and $F = (f, g)$ is a vector-valued function. Assume \mathbf{u}_0 is an equilibrium, i.e., \mathbf{u}_0 is a constant vector s. t. $F(\mathbf{u}_0) = 0$, and

$$\mathbf{u}_t = F(\mathbf{u}) \quad (3.21)$$

is the corresponding kinetic system. The linearized system of (3.21) at equilibrium \mathbf{u}_0 is

$$\mathbf{w}_t = \frac{\partial F}{\partial \mathbf{u}}(\mathbf{u}_0)\mathbf{w} \quad (3.22)$$

where $\frac{\partial F}{\partial \mathbf{u}}$ is the Jacobian matrix, more precisely,

$$S_d \stackrel{\text{def}}{=} \frac{\partial F}{\partial \mathbf{u}}(\mathbf{u}_0) = \begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix}$$

where $f_1 = \frac{\partial f}{\partial u}(\mathbf{u}_0)$, $f_2 = \frac{\partial f}{\partial v}(\mathbf{u}_0)$, $g_1 = \frac{\partial g}{\partial u}(\mathbf{u}_0)$, $g_2 = \frac{\partial g}{\partial v}(\mathbf{u}_0)$. Clearly,

$$\sigma(S_d) = \{ \text{all eigenvalues of } S_d \}.$$

The characteristic polynomial of S_d is

$$p(\lambda) = \det(S_d - \lambda I) = \lambda^2 - (f_1 + g_2)\lambda + (f_1g_2 - f_2g_1). \quad (3.23)$$

The roots of $p(\lambda)$ are

$$\lambda_{1,2} = (1/2)(f_1 + g_2 \pm \sqrt{(f_1 + g_2)^2 - 4(f_1g_2 - f_2g_1)}). \quad (3.24)$$

Theorem 3.2 *Suppose that*

$$f_1 \leq 0, g_2 \leq 0, f_1 + g_2 < 0. \quad (3.25)$$

Then u_0 is stable (l.c.) for the system (3.21) if

$$f_1 g_2 - f_2 g_1 > 0. \quad (3.26)$$

unstable if

$$f_1 g_2 - f_2 g_1 < 0, \quad (3.27)$$

and marginally stable if

$$f_1 g_2 - f_2 g_1 = 0. \quad (3.28)$$

Proof: In the first case, i.e., (3.25), (3.26) hold, the eigenvalues of S_d , λ_1, λ_2 are both real and negative or a pair of imaginary eigenvalues with negative real part $(f_1 + g_2)/2$ (when the expression in the square root is negative). By the criterion, u_0 is stable (l.c.). For the second case, we have

$$(f_1 + g_2)^2 - 4(f_1 g_2 - f_2 g_1) > (f_1 + g_2)^2.$$

Hence

$$\begin{aligned} \lambda_1 &= (1/2)(f_1 + g_2 + \sqrt{(f_1 + g_2)^2 - 4(f_1 g_2 - f_2 g_1)}) \\ &> (1/2)(f_1 + g_2 + |f_1 + g_2|) \geq 0. \end{aligned}$$

So S_d has a positive eigenvalue, thus u_0 is unstable. In the third case, we have $\lambda_1 = 0, \lambda_2 < 0$, hence u_0 is marginally stable. Q.E.D.

Hence we have proved that u_0 is stable (l.c.) for the kinetic system (3.21) under the assumption that (3.25) and (3.26) hold.

Next, assume (3.25), (3.26) hold: we investigate the stability of u_0 in the presence of dispersals.

Let Ω be a bounded subset of R^k . Consider the reaction-diffusion system

$$u_t = D\Delta u + F(u) \quad x \in \Omega, t > 0 \quad (3.29)$$

with homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad x \in \partial\Omega, t > 0. \quad (3.30)$$

Here $D = (D_{ij}(u))_{2 \times 2}$ is a matrix with positive diagonal elements and positive determinant for each $u \in R^2$. The linearized operator about u_0 of the right-hand side in (3.29) is

$$\begin{aligned} S = S_{u_0} &= D(u_0)\Delta + \frac{\partial F}{\partial u}(u_0) \\ &= \begin{pmatrix} D_{11}\Delta + f_1 & D_{12}\Delta + f_2 \\ D_{21}\Delta + g_1 & D_{22}\Delta + g_2 \end{pmatrix} \end{aligned} \quad (3.31)$$

The arguments in the matrix are evaluated at u_0 . The linearized problem is now

$$w_t = S\Delta w \quad x \in \Omega, t > 0 \quad (3.32)$$

with homogeneous Neumann boundary condition

$$\frac{\partial w}{\partial n} = 0 \quad x \in \partial\Omega, t > 0. \quad (3.33)$$

Let $X = \{u = \text{col}(u, v) \in C^2(\Omega) \times C^2(\Omega), \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}$. Consider S as an operator defined on X .

Let $\{\psi_i, i = 0, 1, 2, \dots, \}$ be a completely orthonormal sequence of eigenvectors of the Laplace operator Δ under the Neumann boundary condition

with corresponding eigenvalues $\{\mu_i\}$, i.e.

$$\Delta\psi_i - \mu_i\psi_i = 0, \quad \frac{\partial\psi_i}{\partial n}|_{\partial\Omega} = 0, \quad i = 0, 1, \dots.$$

Arrange the μ_i 's in the numerical order, so that we have

$$\dots \leq \mu_2 \leq \mu_1 < \mu_0 = 0.$$

For $i = 0, 1, \dots$, we define matrices A_i by

$$A_i = \begin{pmatrix} D_{11}\mu_i + f_1 & D_{12}\mu_i + f_2 \\ D_{21}\mu_i + g_1 & D_{22}\mu_i + g_2 \end{pmatrix}. \quad (3.34)$$

Theorem 3.3 *The operator S has λ as an eigenvalue if and only if λ is an eigenvalue of A_i for some i .*

Proof: Suppose λ is an eigenvalue of S , i.e., there are $col(w_1, w_2) \in X$, $col(w_1, w_2) \neq col(0, 0)$ such that

$$S \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

W.l.o.g, assume $w_1 \neq 0$. Then there is at least one i such that $z_1 = (w_1, \psi_i) \neq 0$, where we use the notation (\cdot, \cdot) as the real inner product in $L_2(\Omega)$, i.e.

$$(\phi, \psi) = \int_{\Omega} \phi(x)\psi(x)dx$$

for $\phi, \psi \in L_2(\Omega)$. Let $z_2 = (w_2, \psi_i)$. It is not difficult to see that

$$A_i \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Hence, λ is an eigenvalue of A_i .

Conversely, suppose λ is an eigenvalue of A_i for some i . Then there is a nonzero vector $col(z_1, z_2) \in R^2$, such that

$$A_i \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Let $w_1 = z_1\psi_i$, $w_2 = z_2\psi_i$, then $w_j \in C^2(\Omega)$, $\frac{\partial w_j}{\partial n}|_{\partial\Omega} = 0$, $j = 1, 2$. A calculation shows that

$$S \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Hence λ is an eigenvalue of S , and the theorem is proved.

According to the *theorem 3.3*, the spectrum of S is

$$\sigma(S) = \{\lambda \in C, \lambda \text{ is an eigenvalue of } A_i \text{ for some } i\}. \quad (3.35)$$

To find the spectrum of S , we only need to observe the eigenvalues of the A_i 's. Let $p_i(\lambda)$ be the characteristic polynomial of A_i . Then

$$p_i(\lambda) = \det(A_i - \lambda I) = \lambda^2 - b_i\lambda + c_i \quad (3.36)$$

where

$$b_i = \text{tr}(A_i) = D_{11}\mu_i + f_1 + D_{22}\mu_i + g_2, \quad (3.37)$$

$$\begin{aligned} c_i &= \det(A_i) \\ &= (D_{11}\mu_i + f_1)(D_{22}\mu_i + g_2) - (D_{12}\mu_i + f_2)(D_{21}\mu_i + g_1). \end{aligned} \quad (3.38)$$

The roots of $p_i(\lambda)$ are

$$\lambda_{1,2}^i = (1/2)(b_i \pm \sqrt{b_i^2 - 4c_i}). \quad (3.39)$$

By the assumption on D ($D_{ii} > 0$) and (3.25), we see that $b_i < 0$. Hence, from the above formula, we see that if for all i , $c_i > 0$, then all $Re(\lambda_{1,2}^i) < 0$, i.e., all eigenvalues of S are in the negative half-plane. In this case, if we can show that they are bounded away from the imaginary axis (we will do this below), then u_0 is stable (*l.c.*). And if there is a $c_i < 0$, then $\lambda_1^i > 0$, so S has a positive eigenvalue. Hence in this case, u_0 is unstable. From this discussion we see that the stability of u_0 is determined by the signs of the c_i 's.

In order to confirm the stability, we need to show that if $\sigma(S)$ is in the negative half-plane, then it is bounded away from the imaginary axis. Recall

$$\sigma(S) = \{\lambda_{1,2}^i, i = 0, 1, \dots, \}$$

where $\lambda_{1,2}^i$ is given by (3.39). We prove the following theorem:

Theorem 3.4 *Suppose $Re(\lambda_{1,2}^i) < 0, i = 0, 1, \dots$. Then there is a positive number α such that $Re(\lambda_{1,2}^i) < -\alpha, i = 0, 1, \dots$.*

Proof: The condition given in the theorem is equivalent to that $c_i > 0$ by the discussion of the previous paragraph.

If $b_i^2 - 4c_i < 0$, then

$$Re(\lambda_{1,2}^i) = (1/2)b_i \leq (1/2)(f_1 + g_2) < 0.$$

If $b_i^2 - 4c_i > 0$, then $\lambda_{1,2}^i$ are real and

$$\frac{2c_i}{|b_i|} < \frac{|b_i|}{2}.$$

Hence

$$\lambda_{1,2}^i \leq (1/2)(b_i + ||b_i| - 2\frac{c_i}{|b_i|}|) = -\frac{c_i}{|b_i|}.$$

Since

$$\frac{c_i}{|b_i|} = \frac{a\mu_i^2 + b\mu_i + c}{|d\mu_i + e|}$$

where $b_i = (D_{11} + D_{22})\mu_i + f_1 + g_2 = d\mu_i + e$ by (3.37) and

$$a = (D_{11}D_{22} - D_{12}D_{21}) = \det(D(u_0)), \quad (3.40)$$

$$b = f_1D_{22} + g_2D_{11} - (f_2D_{21} + g_1D_{12}), \quad (3.41)$$

$$c = f_1g_2 - f_2g_1. \quad (3.42)$$

$$c_i = a\mu_i^2 + b\mu_i + c. \quad (3.43)$$

by (3.38), and $a > 0, c > 0$ by the assumption on D ($\det(D) > 0$) and (3.26), (the condition of stability for the kinetic system), hence

$$\frac{c_i}{|b_i|} \rightarrow +\infty \text{ as } \mu_i \rightarrow -\infty.$$

Thus there is an integer N s. t.

$$\frac{c_i}{|b_i|} > 1. \quad \forall i > N.$$

Let $\beta = \min\{1, c_i/|b_i|, i = 0, 1, \dots, N\}$, then

$$\frac{c_i}{|b_i|} \geq \beta > 0. \quad \forall i.$$

Hence

$$\lambda_{1,2}^i \leq -\beta.$$

So, finally, Let $\alpha = \min\{\beta, (1/2)|f_1 + g_2|\}$, then $\text{Re}\{\lambda_{1,2}^i\} \leq -\alpha < 0$ for all i . This completes the proof of the theorem.

Now we look at how the dispersals affect the stability.

First, we note that if only self-diffusion is present, the stability of u_0 couldn't be affected. For, in this case, $b = f_1 D_{22} + g_2 D_{11} < 0$, thus $c_i > 0$ for all i 's. This means that u_0 is stable.

Consider the polynomial of the second degree (with real coefficients)

$$h(z) = az^2 + bz + c.$$

Its two roots are

$$z_{1,2} = (1/(2a))(-b \pm \sqrt{b^2 - 4ac}).$$

In order to determine the sign of c_i , we analyze the positions of the roots.

(1). If $b < 0$, then either z_1, z_2 are both positive or they are conjugate imaginary. Hence in this case $h(z)$ has no negative roots, thus $c_i = h(\mu_i) > 0$ since $a > 0$ and $\mu_i \leq 0$. —stable.

(2). If $b \geq 0$, $b^2 - 4ac < 0$. Then $h(z)$ has only one pair of conjugate imaginary roots. Also $c_i > 0$. —stable.

(3). If $b > 0$, $b^2 - 4ac = 0$. Then $h(z)$ has exactly one real negative root $z = (-b/(2a))$. If $(-b/(2a)) \neq \mu_i, \forall i$, then $c_i > 0, \forall i$. —stable. If there is an i , such that $(-b/(2a)) = \mu_i$, then $c_i = 0$, $c_j > 0, \forall j \neq i$, —marginally stable.

(4). If $b > 0$, $b^2 - 4ac > 0$. Then $h(z)$ has two negative real roots, i.e.,

$$z_2 = (1/(2a))(-b - \sqrt{b^2 - 4ac}), \quad z_1 = (1/(2a))(-b + \sqrt{b^2 - 4ac})$$

and $z_2 < z_1 < 0$. If there is a $\mu_i \in (z_2, z_1)$, then $c_i < 0$, hence in this case, u_0 is unstable.

We summarize the above results in following theorem.

Theorem 3.5 *Let a, b, c defined by (3.40), (3.41), (3.42). If $b < 2\sqrt{ac}$, then u_0 is stable (l.c.) if it is a stable equilibrium for the kinetic system. If $b > 2\sqrt{ac}$, then the u_0 may be unstable even it is stable in the kinetic system.*

Remark 1: The condition $b < 2\sqrt{ac}$ for preserving the stability can be refined. e.g., in item (4) above, $h(z)$ has two negative roots $z_2 < z_1 < 0$. But if $\mu_1 < z_2$, then we still have that $c_i > 0$ for all i . In this case, for stability we only need

$$b < \frac{c + a\mu_1^2}{|\mu_1|}. \quad (3.44)$$

And this is better because

$$\frac{c + a\mu_1^2}{|\mu_1|} > 2\sqrt{ac}.$$

Anyway, the above analysis shows that the cross-diffusion is indeed the factor of instability.

Remark 2: By using a quadratic form argument, we can also derive a condition which ensures that such an S is negative definite, and thus get the stability of equilibrium. But that is only a sufficient condition for stability.

Example 4. Consider the Keller-Segel model we studied in chapter 2:

$$\begin{cases} u_t = D_1 \Delta u + D^* \Delta v + f(u, v) \\ v_t = D_2 \Delta v + g(u, v) \end{cases} \quad (3.45)$$

where D_1, D_2 are positive constants, $D^* = D^*(u)$ with $D^*(0) = 0$. Suppose

$$f = l(v) - u, \quad g = \alpha u - \beta v$$

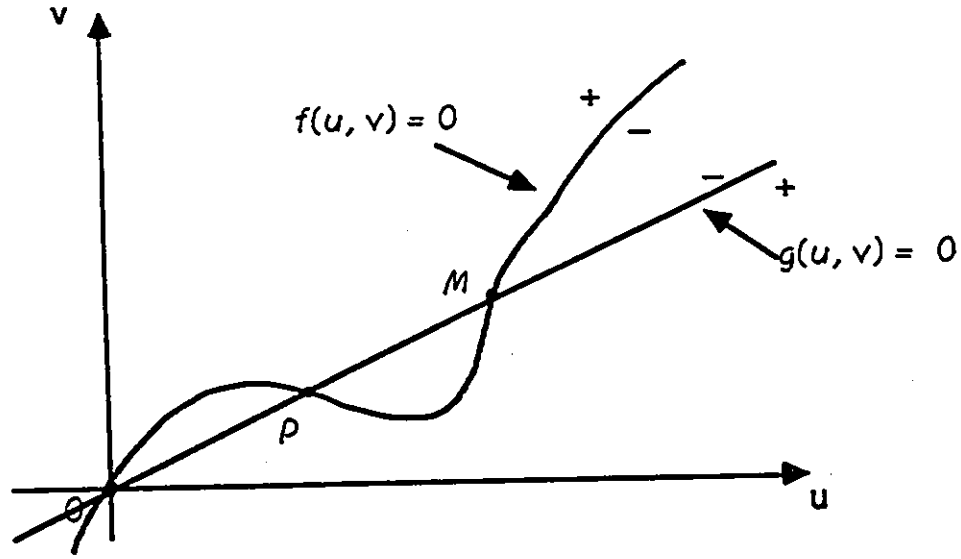


Figure 3.1: Equilibria

where α, β are positive and $l(v)$ has the form shown in figure 3.1.

With the above assumptions, the equilibria are $O = (0, 0)$, P , M . And

$$f_1 = -1, f_2 = l'(v), g_1 = \alpha, g_2 = -\beta$$

Hence, by previous notations,

$$a = \det(D) > 0, c = \beta - \alpha l'(v).$$

Clearly, at O and M , $c > 0$. Thus, O , M are stable equilibria for the corresponding kinetic system.

Now observe the parameter b connected with cross-diffusion coefficients.

$$b = f_1 D_2 + g_2 D_1 - g_1 D^* = -D_2 - \beta D_1 - \alpha D^*(u).$$

If we suppose $D^*(u) \geq 0$. Then $b < 0$, Hence the equilibria O , M are also stable for the R-D system (3.45). i.e., in this case, the cross diffusion does not affect the stability.

However, the following case shows that cross diffusion may affect stability. For simplicity in calculation, suppose, in K-S model, $f = 0$, g as

before, $D^*(u) \leq 0$ in $u \geq 0$. Then

$$a = \det(D) > 0. \quad c = 0.$$

So, all equilibria $(u, (\alpha/\beta)u)$ are marginally stable. And

$$b = g_2 D_1 - g_1 D^* = -\beta D_1 - \alpha D^*.$$

We prove that if

$$|D^*(u_0)| > \frac{-a\mu_1 + \beta D_1}{\alpha}. \quad (3.46)$$

then the equilibrium $(u_0, (\beta/\alpha)u_0)$ is unstable.

In fact, By (3.46), $-D^* = |D^*(u_0)| > \frac{\beta D_1}{\alpha}$, hence $b > 0$. This implies that $h(z)$ has two real roots, $z_1 = 0$, $z_2 = -(b/a) < 0$. By (3.46),

$$-D^*(u_0)\alpha > -a\mu_1 + \beta D_1.$$

i.e.

$$a\mu_1 + b > 0.$$

this implies that $c_1 = a\mu_1^2 + b\mu_1 < 0$ since $\mu_1 < 0$. Hence we get the instability.

Remark: E. F Keller and L. A. Segel analyzed the stability of equilibria for their model [16]. They gave a condition similar to

$$|D^*| > \frac{\beta D_1}{\alpha}$$

from a different way that would ensure instability. Here a more precisely refined criterion is established, i.e.. we conclude that if (3.46) holds, then the instability must occur, otherwise, the equilibrium is still marginally stable.

Part II

On Some Problems Connected with the Laplace Operator in Infinite-Dimensional Space

Introduction

At the beginning of the century Jacques Hadamard proposed a broad program to extend the classical theory to infinite-dimensional spaces. Hadamard called functions with an argument varying in infinite-dimensional space functionals (hence, functional analysis). Hadamard, Gâteaux and Fréchet constructed a theory of functional analysis, the theory of differentiation of functionals, and applied it to the foundation of the calculus of variations. The next natural step would be the systematic discussion of differential equations. This situation turned out to be more complicated even with the second order differential operators, particularly, the Laplace operator.

The first notion of the Laplace operator for functionals in Hilbert space was introduced by Lévy [19], (revised edition [20]). That definition depends on the choice of basis of the space and the domain of this Laplacian was not clear. During the sixties, this problem was actively considered by some Russian mathematicians. G. E. Šilov ([31—33]) discussed a Laplace operator and its inverse on a rather narrow class of functionals. The axiomatic definitions of the Laplace operator were given by Nemirovskii and Šilov[23] and Sikirjavyi [30].

In this article we first develop the definition of the Laplacian (given by

Lévy) from the point of view of invariance under the change of basis. Then we discuss some relative problems. e.g., some properties of the Laplacian, maximum principles, Dirichlet and Poisson problems and boundary value problems for semilinear equations, etc..

We note that the Laplacians with other definitions in Hilbert spaces have been studied by E. M. Poliščuk [25], M. N. Feller [8], and Yu. L. Daletskii [5], etc. .

Chapter 4

Laplacian in Hilbert space

Introduction

In his books [19, 20], Paul Lévy introduced a Laplacian based on the notion of the mean in Hilbert space. That definition depends on the choice of a complete basis. Then G. E. Šilov [31—33] investigated the Lévy-Laplacian by considering it on a very special class of functionals which he called regular functionals.

In this chapter, we begin with the original definition of Lévy and derive a definition of the Laplacian on a class of C_0^2 -functionals such that the Laplacian is invariant under change of basis. Then we explore some properties of the Laplacian. One important property is the maximum principle.

4.1 The mean over a sphere in Hilbert space

Let \mathbf{H} be a real separable Hilbert space with inner product (\cdot, \cdot) . The induced norm of \mathbf{H} is denoted by $|x| = \sqrt{(x, x)}$. Let $E = \{e_k\}_1^\infty$ be an orthonormal basis of \mathbf{H} . We always suppose that G is a subset of \mathbf{H} . For $x_0 \in \mathbf{H}$, $\rho > 0$, we denote by

$$\begin{aligned} P_n^E(x_0) &= \{x \in \mathbf{H}; (x - x_0, e_k) = 0, k = n + 1, n + 2, \dots\}, \\ S(x_0, \rho) &= \{x \in \mathbf{H}; |x - x_0| = \rho\}, \\ S_n^E(x_0, \rho) &= S(x_0, \rho) \cap P_n^E(x_0). \end{aligned}$$

We call $P_n^E(x_0)$ the n -plane through x_0 with respect to the basis E (sometimes the superindex E is omitted), $S(x_0, \rho)$ a sphere at the center x_0 and radius ρ , $S_n = S_n^E(x_0, \rho)$ the n -section of a sphere $S(x_0, \rho)$. Clearly, P_n^E is isomorphic to R^n , S_n is isomorphic to a sphere in R^n . Without confusion, we denote by $|S_n|$ the $(n-1)$ -dimension measure of S_n in R^n , and the mean of a function f_n defined in P_n^E over a sphere S_n is

$$\mathcal{M}_n(f_n, S_n) = \frac{1}{|S_n|} \int_{S_n} f_n(x) d\sigma_n(x) \quad (4.1)$$

where $d\sigma_n(x)$ is surface element in R^n .

Definition 4.1 *Let $f = f(x)$ be a functional defined in G . The mean $\mathcal{M}^E(f, x_0, \rho)$ of f over the sphere $S = S(x_0, \rho) \subset G$ w.r.t. the basis E is defined by the following limit*

$$\mathcal{M}^E(f, x_0, \rho) = \lim_{n \rightarrow \infty} \mathcal{M}_n(f_n, S_n(x_0, \rho)) \quad (4.2)$$

whenever it exists, where $f_n = f|_{F_n^E(x_0)}$.

If the value $\mathcal{M}^E(f, x_0, \rho)$ is independent of the choice of the basis E , then we say f has an invariant mean over the sphere $S = S(x_0, \rho)$ and this invariant mean is denoted by $\mathcal{M}(f, x_0, \rho)$, or $\mathcal{M}(f, S)$ or $\mathcal{M}_{x_0, \rho}(f)$.

If for each point $x \in G$, $\mathcal{M}(f, x, \rho)$ exists for all sufficiently small ρ , then we say that f has an invariant mean in G . The set of all functionals which have an invariant mean in G is denoted by $M(G)$.

Example 5. Let $a \in H$, $f_1(x) = (a, x)$. Then, by definition,

$$\mathcal{M}(f_1, x_0, \rho) = (a, x_0) = f_1(x_0).$$

Example 6. Let S be a Hilbert-Schmidt operator on H . Define a functional by $f_2(x) = (Sx, x)$. Then f_2 has an invariant mean in H and for every $x_0 \in H$ and $\rho > 0$

$$\mathcal{M}(f_2, x_0, \rho) = f_2(x_0).$$

Calculation: Since S is a Hilbert-Schmidt operator, there is an orthonormal basis $E = \{e_k, k = 1, 2, \dots\}$ and a sequence $\{\lambda_k, k = 1, 2, \dots\}$ such that $Se_k = \lambda_k e_k$ and $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$ (ref. [27]). Under this basis E , $f = f_2$ can be written as

$$f(x) = \sum_{k=1}^{\infty} \lambda_k x_k^2$$

where $x_k = (x, e_k)$.

First we calculate the mean $\mathcal{M}^E(f, x_0, \rho)$. Since

$$x_k^2 = (x_k - x_{0k})^2 + 2x_{0k}(x_k - x_{0k}) + x_{0k}^2,$$

then

$$f(x) = f(x_0) + \sum_{k=1}^{\infty} \lambda_k ((x_k - x_{0k})^2 + 2x_{0k}(x_k - x_{0k})).$$

Hence

$$f_n^E(x) = f|_{P_n^E} = f(x_0) + \sum_{k=1}^n \lambda_k ((x_k - x_{0k})^2 + 2x_{0k}(x_k - x_{0k})),$$

and

$$\begin{aligned} \mathcal{M}^E(f, x_0, \rho) &= \lim_{n \rightarrow \infty} \mathcal{M}_n(f_n, S_n(x_0, \rho)) \\ &= f(x_0) + \lim_{n \rightarrow \infty} \mathcal{M}_n\left(\sum_{k=1}^n \lambda_k ((x_k - x_{0k})^2 + 2x_{0k}(x_k - x_{0k})), S_n\right) \\ &= f(x_0) + \lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda_k \mathcal{M}_n((x_k - x_{0k})^2 + 2x_{0k}(x_k - x_{0k}), S_n) \end{aligned}$$

where $S_n(x_0, \rho) = \{x; \sum_{k=1}^n |x_k - x_{0k}|^2 = \rho^2\}$, $x_{0k} = (x_0, e_k)$.

In R^n , it is not difficult to get that

$$\mathcal{M}_n(x_k, S_n(0, \rho)) = 0. \quad (4.3)$$

$$\mathcal{M}_n(x_k^2, S_n(0, \rho)) = \frac{1}{n} \rho^2, \quad (4.4)$$

$$\mathcal{M}_n(x_k x_j, S_n(0, \rho)) = 0, \quad k \neq j. \quad (4.5)$$

Hence, from (4.3), (4.4), we get

$$\mathcal{M}^E(f, x_0, \rho) = f(x_0) + \lim_{n \rightarrow \infty} \frac{1}{n} \rho^2 \sum_{k=1}^n \lambda_k.$$

Since $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$, this implies that $\{\lambda_k\}$ is $(C, 1)$ summable and $\frac{1}{n} \sum_{k=1}^n \lambda_k \rightarrow 0$, hence

$$\mathcal{M}^E(f, x_0, \rho) = f(x_0).$$

Now, for any basis $Q = \{q_j, j = 1, 2, \dots\}$, let $[s_{ij}]$ be the doubly infinite matrix of the operator S under the basis Q , i.e., $s_{ij} = (S q_i, q_j)$. Hence,

$Sq_i = \sum_{j=1}^{\infty} s_{ij}q_j$, $i = 1, 2, \dots$. For any $y \in H$, we write $y = \sum_{i=1}^{\infty} y_i q_i$, $y_i = (y, q_i)$. Then

$$f(y) = (Sy, y) = \sum_{i,j=1}^{\infty} s_{ij}y_i y_j.$$

Let $y_0 \in H$, and rewrite $f(y)$ as follows

$$\begin{aligned} f(y) &= \sum_{i,j=1}^{\infty} s_{ij}((y_i - y_{0i})(y_j - y_{0j}) + y_{0i}(y_j - y_{0j}) \\ &\quad + y_{0j}(y_i - y_{0i}) + y_{0i}y_{0j}) \\ &= \sum_{i,j=1}^{\infty} s_{ij}((y_i - y_{0i})(y_j - y_{0j}) + y_{0i}(y_j - y_{0j}) + y_{0j}(y_i - y_{0i})) \\ &\quad + f(y_0). \end{aligned}$$

Then

$$f_n^Q(y) = f(y_0) + \sum_{i,j=1}^n s_{ij}((y_i - y_{0i})(y_j - y_{0j}) + y_{0i}(y_j - y_{0j}) + y_{0j}(y_i - y_{0i})).$$

Hence

$$\begin{aligned} \mathcal{M}^Q(f(y), y_0, \rho) &= \lim_{n \rightarrow \infty} \mathcal{M}_n(f_n^Q, y_0, \rho) \\ &= f(y_0) + \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\rho^2}{n} s_{ii} \end{aligned}$$

by (4.3), (4.4), (4.5). Since S is a Hilbert-Schmidt operator, i.e.

$$\text{tr}(SS^*) = \sum_{i,j} s_{ij}^2 < \infty,$$

so

$$s_{ii} \rightarrow 0, \tag{4.6}$$

hence

$$\frac{1}{n} \sum_{i=1}^n s_{ii} \rightarrow 0.$$

Hence, we get

$$\mathcal{M}^Q(f, y_0, \rho) = f(y_0).$$

Therefore, f has an invariant mean over any sphere $S(x_0, \rho)$ and

$$\mathcal{M}(f, x_0, \rho) = f(x_0).$$

Remark: Glance at *example 6*, we only used (4.6) in the calculation. If S is a symmetric compact operator on H , the conclusion of *example 6* is still true by the following well-known result.

Lemma 4.1 S is a compact operator on H if and only if for any orthonormal basis $Q = \{q_1, q_2, \dots\}$,

$$(Sq_i, q_i) \rightarrow 0. \tag{4.7}$$

Proof: We only prove the necessity. Since S is symmetric compact operator, there exists an orthonormal basis $E = \{e_k, k = 1, 2, \dots\}$ and a sequence $\{\lambda_k, k = 1, 2, \dots\}$ such that $Se_k = \lambda_k e_k$ with $\lambda_k \rightarrow 0$ (ref. [27]). Let $A = [a_{ij}]$ be the matrix of the transformation from E to Q , i.e.,

$$q_i = \sum_{j=1}^{\infty} a_{ij} e_j, \quad i = 1, 2, \dots$$

We have

$$\sum_{j=1}^{\infty} a_{ij}^2 = (q_i, q_i) = 1, \quad i = 1, 2, \dots,$$

$$\lim_{i \rightarrow \infty} a_{ij} = 0, \quad j = 1, 2, \dots$$

Since $\lambda_j \rightarrow 0$, given $\epsilon > 0$, $\exists N$, s. t. $|\lambda_j| < \epsilon$ when $j \geq N$. Hence

$$\begin{aligned} |(Sq_i, q_i)| &= \left| \sum_{j=1}^{\infty} a_{ij}^2 \lambda_j \right| \\ &\leq \sum_{j=1}^N a_{ij}^2 |\lambda_j| + \sum_{j=N+1}^{\infty} a_{ij}^2 |\lambda_j| \\ &\leq \sum_{j=1}^N a_{ij}^2 |\lambda_j| + \epsilon \\ &\rightarrow \epsilon. \end{aligned}$$

Hence, (4.7) holds since ϵ is arbitrary. Q.E.D.

In fact, (Sq_i, q_i) is the linear mean of $\{\lambda_k\}$ (determined by the matrix $M = [a_{ij}^2]$), and M is a regular matrix. Hence $\lim_{i \rightarrow \infty} (Sq_i, q_i) = \lim_{k \rightarrow \infty} \lambda_k = 0$ (ref. [3S]).

Example 7. Let $f_3(x) = (x, x)$. Then

$$\mathcal{M}(f_3, x_0, \rho) = \rho^2 + |x_0|^2.$$

Theorem 4.1 *Let Γ be a linear symmetric operator on H . Then $f(x) = (\Gamma x, x)$ has an invariant mean iff $\Gamma = \gamma I + T$ where T is a completely continuous operator.*

Proof: Sufficiency is the result of *example 7, 6* and the remark above. Now we give the proof of the necessity. Suppose $\mathcal{M}(f, o, \rho)$ exists for any basis $E = \{e_1, e_2, \dots\}$ and does not depend on the choice of basis. Let $[\gamma_{ij}]$ be the matrix of the operator Γ under the basis E , i.e., $\gamma_{ij} = (\Gamma e_i, e_j)$. From the calculation in *example 6, (p. 55)* we see that

$$\mathcal{M}(f, o, \rho) = \rho^2 \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \gamma_{ii}$$

where $f = (\Gamma x, x)$. Since the limit exists and does not depend on the choice of the basis (especially, not depend on the order of the basis vectors), there must be a number γ s. t. $\gamma_{ii} \rightarrow \gamma$. (Otherwise, if $\lim_{i \rightarrow \infty} \gamma_{ii}$ does not exist, then the (C. 1) sum of the sequence must depend on the order of the sequence. Hence $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \gamma_{ii} = \lim_{i \rightarrow \infty} \gamma_{ii}$ exists ([38]). The left hand side of this equality is independent of the choice of basis, so there is a number γ such that $\gamma_{ii} \rightarrow \gamma$ for any choice of basis.) Let

$$T = \Gamma - \gamma I. \quad T = [t_{ij}].$$

Then $t_{ii} = \gamma_{ii} - \gamma \rightarrow 0$ for any choice of basis. Then by lemma 4.1 T is compact. Q.E.D.

Theorem 4.2 *Let $f_1(x), \dots, f_m(x)$ be finite uniformly continuous functionals which have invariant means in G . and let F be a uniformly continuous function defined in R^m . Then $F(x) = F(f_1(x), \dots, f_m(x))$ also has an invariant mean in G and*

$$\mathcal{M}(F) = F(\mathcal{M}(f_1), \dots, \mathcal{M}(f_m)).$$

In order to prove this theorem, we introduce a notion and a lemma in Lévy[19] here.

Definition 4.2 *Let $S = S(x_0, \rho)$ be a sphere in H and $S_n = S_n(x_0, \rho) = S \cap P_n(x_0)$. Let $f = f(x)$ be a functional defined on S and $f_n = f|_{S_n}$. Let M be some constant. For $\delta > 0$, we use $|S_n(\delta)|$ to denote the measure of*

$S_n(\delta)$, the set of points of S_n for which $|f_n - M| > \delta$. If there is some $\epsilon_0 > 0$ such that when $\delta < \epsilon_0$, we have

$$\lim_{n \rightarrow \infty} \frac{|S_n(\delta)|}{|S_n|} = 0.$$

then we say that the equation $f(x) = M$ holds almost everywhere (a. e.) on the sphere S .

Lemma 4.2 If $f(x)$ is uniformly continuous on the sphere S and $\mathcal{M}(f) = \mathcal{M}(f, S)$ exists, then the equation $f(x) = \mathcal{M}(f)$ holds almost everywhere on the sphere S . (See Lévy [19], part III, chapter 2).

The converse is also true:

Lemma 4.3 If a bounded uniformly continuous functional f satisfies the equation $f = M$ almost everywhere on S , then $\mathcal{M}(f)$ exists and

$$\mathcal{M}(f) = M.$$

Proof: Let $\delta > 0$ be small enough and $K = \|f\|_0 + M$. Then

$$\begin{aligned} |\mathcal{M}_n(f) - M| &= |\mathcal{M}_n(f - M)| \\ &\leq \frac{1}{|S_n|} \int_{S_n} |f - M| \\ &= \frac{1}{|S_n|} \left(\int_{S_n(\delta)} |f - M| + \int_{S_n - S_n(\delta)} |f - M| \right) \\ &\leq K \frac{|S_n(\delta)|}{|S_n|} + \delta \\ &\rightarrow \delta, \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence the lemma 4.3 has been proved since δ is arbitrarily small.

Proof of theorem 4.2: For simplicity, we just let $m = 1$. The other cases are similar. By lemma 4.2, $f = \mathcal{M}(f)$ a.e. on any sphere $S \in G$. Hence $F = F(f) = F(\mathcal{M}(f))$ a.e. on S since F is uniformly continuous. Hence $\mathcal{M}(F) = \mathcal{M}(F(\mathcal{M}(f))) = F(\mathcal{M}(f))$ by lemma 4.3. Q.E.D.

Theorem 4.3 *Let $\{f_n\}$ be a sequence in $M(G)$ and $f_n \rightarrow f$ uniformly. Then $f \in M(G)$, and*

$$\mathcal{M}(f, x_0, \rho) = \lim_{m \rightarrow \infty} \mathcal{M}(f_m, x_0, \rho).$$

Proof: By assumption, given $\epsilon > 0$, $\exists m_0$ such that

$$|f(x) - f_m(x)| < \epsilon, \forall x \in G, m \geq m_0.$$

Let $x_0 \in G$ be fixed but arbitrary. Then for any n , the above inequality is true on the plane $P_n(x_0)$ (with any basis), hence

$$|\mathcal{M}_n(f, x_0, \rho) - \mathcal{M}_n(f_m, x_0, \rho)| < \epsilon \quad \forall n = 1, 2, \dots, m \geq m_0. \quad (4.8)$$

Since f_{m_0} has an invariant mean, hence $\exists n_0$ s. t., when $n, n' \geq n_0$,

$$|\mathcal{M}_n(f_{m_0}, x_0, \rho) - \mathcal{M}_{n'}(f_{m_0}, x_0, \rho)| < \epsilon.$$

Then by a 3ϵ -argument, for $n, n' \geq n_0$,

$$|\mathcal{M}_n(f, x_0, \rho) - \mathcal{M}_{n'}(f, x_0, \rho)| < 3\epsilon.$$

We have proved that the limit $\mathcal{M}_n(f, x_0, \rho)$ exists (under any basis). In (4.8), by letting $n \rightarrow \infty$, and then $m \rightarrow \infty$ we get

$$\mathcal{M}(f, x_0, \rho) = \lim_{m \rightarrow \infty} \mathcal{M}(f_m, x_0, \rho).$$

Hence, $f \in M(G)$. Q.E.D.

Definition 4.3 (Šilov) *Let f be a functional in G . If there are finitely many elements a_1, \dots, a_m in H such that f can be expressed as*

$$f(x) = F((a_1, x), \dots, (a_m, x), |x|^2) \quad x \in G \quad (4.9)$$

where F is a function defined in a suitable subset $Q \subset R^{m+1}$, (e.g., $Q \supset \{((a_1, x), \dots, (a_m, x), |x|^2); x \in G\}$), then we say f is regular in G . Moreover, if in the expression (4.9), F does not depend on $|x|^2$, we say that f is simple.

By theorems 4.2, 4.3, and examples 5, 7 (p. 53-57), the following corollary is clear.

Corollary 4.1 *All regular functionals and uniform limits of regular functionals have invariant means.*

4.2 Definitions of Laplacians and some simple properties

Definition 4.4 (Laplacian) Let $f : G \rightarrow \mathbb{R}^1$, $x_0 \in G$ and let f have an invariant mean at x_0 . Define the Laplacian L of f at x_0 by

$$Lf(x_0) = \lim_{\rho \rightarrow 0} \frac{2}{\rho^2} (\mathcal{M}(f, x_0, \rho) - f(x_0)) \quad (4.10)$$

whenever the limit exists.

Remark: The definition of the Laplacian here is independent of the choice of basis.

Definition 4.5 A functional f is called harmonic if $Lf = 0$. The set of all harmonic functionals in G is denoted by $D(G)$.

Now we look at some examples given in the preceding section (p. 53-57). Clearly, $Lf_1 = 0$, i.e., f_1 is a harmonic functional. Also $Lf_2 = 0$. In fact, $f_2(x)$ is the uniform limit of a sequence of simple functionals.

Generally, all simple functionals and uniform limits of sequences of simple functionals are harmonic. (This is a consequence of *example 5* (p. 53), *theorems 4.2, 4.3* in the preceding section and *definition 4.4*.)

For *example 7* (p. 57), it is easy to get that

$$Lf_3 = 2.$$

Generally, if f is regular and has the form (4.9), then $\forall x \in G$, $\rho > 0$ small enough, by *example 5, 7, theorem 4.2*, we have

$$\mathcal{M}(f, x, \rho) = F((a_1, x), \dots, (a_n, x), |x|^2 + \rho^2).$$

So, if F is differentiable in the last variable ξ_{m+1} , then f can be operated on by the Laplacian and we have

$$Lf(x) = 2 \frac{\partial F}{\partial \xi_{m+1}} \Big|_{\xi_{m+1}=|x|^2} . \quad (4.11)$$

Hence, if we restrict functionals to the class of regular functionals, the *definition 4.4*, (4.10), for the Laplacian coincides with the one that Šilov gave in [31]. Later we will see that our definition is a proper extension of Šilov's.

Now, we consider the Laplacian acting on C^2 -functionals.

Let G be a subset of H . We denote by $C_b(G)$ the set of all uniformly continuous and bounded mappings from G to \mathbb{R} . $C_b(G)$ endowed with the norm

$$\|\phi\|_0 = \sup\{|\phi(x)|, x \in G\}$$

is a Banach space. We shall denote by $C_b^k(G)$, ($k = 1, 2$) the set of all the mappings $\phi : G \rightarrow \mathbb{R}$ which are Fréchet differentiable up to order k and uniformly continuous and bounded with all derivatives of order less than or equal to k . The Fréchet differential operator of order j is denoted by D^j . $C_b^k(G)$ endowed with the norm

$$\|\phi\|_k = \|\phi\|_0 + \sum_{j=1}^k \|D^j \phi\|_0$$

is a Banach space. For any $k \in \mathbb{N}$, let $D_k = \frac{\partial}{\partial x_k}$ be the linear operator in $C_b(G)$ defined by

$$(D_k \phi)(x) = \frac{\partial \phi(x)}{\partial x_k} = \lim_{h \rightarrow 0^+} \frac{1}{h} (\phi(x + h e_k) - \phi(x))$$

for $\phi \in \mathcal{D}(D_k)$, ($\mathcal{D}(\cdot)$ is the domain of the given operator), that is, $D_k\phi$ is the Gâteaux derivative in the direction e_k . If ϕ is Fréchet differentiable, then

$$D_k\phi(x) = (D\phi(x) \cdot e_k).$$

Similarly, for $\phi \in C_b^2$, we denote

$$\frac{\partial^2 \phi}{\partial x_k^2} = D_{kk}\phi = (D^2\phi e_k \cdot e_k).$$

Theorem 4.4 *If $f \in C_b^2(G)$, and Lf exists, then*

$$Lf(x) = \mathcal{M}_{0,1}((D^2f(x)y \cdot y)). \quad (4.12)$$

Moreover, if $E = \{e_k\}_1^\infty$ is an orthonormal basis, then

$$Lf(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\partial^2 f(x)}{\partial x_k^2}. \quad (4.13)$$

Proof: By definition and a calculation.

$$\begin{aligned} Lf(x_0) &= \lim_{\rho \rightarrow 0} \frac{2}{\rho^2} (\mathcal{M}(f, x_0, \rho) - f(x_0)) \\ &= \lim_{\rho \rightarrow 0} \frac{2}{\rho^2} \mathcal{M}_{x_0, \rho}(f(x) - f(x_0)) \\ &= \lim_{\rho \rightarrow 0} \frac{2}{\rho^2} \mathcal{M}_{0,1}(f(x_0 + \rho y) - f(x_0)). \end{aligned}$$

Since $f \in C_b^2(G)$,

$$\begin{aligned} f(x_0 + \rho y) - f(x_0) &= (Df(x_0) \cdot \rho y) + \frac{1}{2}(D^2f(x_0) \cdot \rho y \times \rho y) + o(\rho^2) \\ &= \rho(Df(x_0) \cdot y) + \frac{\rho^2}{2}(D^2f(x_0)y \cdot y) + o(\rho^2) \end{aligned}$$

and $\mathcal{M}(Df(x_0), y) = 0$. Hence we get

$$Lf(x_0) = \mathcal{M}_{0,1}(D^2f(x_0)y, y).$$

For the second conclusion, by the definition of the mean,

$$\begin{aligned} \mathcal{M}_{0,1}(D^2f, y \times y) &= \lim_{n \rightarrow \infty} \mathcal{M}_{0,1,n}(D^2f, \sum_{k=1}^n y_k y_j e_k \times e_j) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathcal{M}_{0,1,n}(y_k y_j)(D^2f, e_k \times e_j) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} (D^2f, e_k \times e_k), \end{aligned}$$

since

$$\mathcal{M}_{0,1,n}(y_k y_j) = \begin{cases} 0 & \text{if } k \neq j \\ \frac{1}{n} & \text{if } k = j. \end{cases}$$

Hence

$$Lf(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\partial^2 f}{\partial x_k^2}.$$

Q.E.D.

From now on, we will use formula (4.12) as the definition of the Laplace operator L on the functionals of C^2 -class and denote by $\mathcal{D}_G(L)$ the set of those functionals whose Laplacian exists and are bounded uniformly continuous in G . About the domain $\mathcal{D}_G(L)$ of the operator L in C^2 , we have the following theorem.

Theorem 4.5 *Let $G \subset \mathbb{H}$, $f \in C_b^2(G)$. Then $f \in \mathcal{D}_G(L)$ if and only if there is a $\gamma(x) \in C_b(G)$ and a compact operator $T(x)$, for each $x \in G$, such that $D^2f(x) = \gamma(x)I + T(x)$ where I is the identity operator on \mathbb{H} . Moreover, $Lf(x) = \gamma(x)$.*

Proof: Just use *theorem 4.1* and *theorem 4.4*. Moreover

$$\begin{aligned} Lf(x) &= \mathcal{M}_{0,1}(D^2f(x)y, y) \\ &= \mathcal{M}_{0,1}((\gamma(x)y, y)) + \mathcal{M}_{0,1}((T(x)y, y)) \\ &= \gamma(x). \end{aligned}$$

The last equality is obtained from the results of *examples 6, 7* (p. 53-57).

Q.E.D.

If we denote by

$$\mathcal{F}(G) = \{f \in C_b^2(G); \exists \gamma \in C_b(G) \ni D^2f - \gamma I \text{ is compact on } \mathbf{H}\},$$

then *theorem 4.5* says $\mathcal{D}_G(L) = \mathcal{F}(G)$.

The following are some simple and useful properties of L .

(1). L is a linear operator. i.e., if $u, v \in \mathcal{D}_G(L)$, α, β are two real numbers, then

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv.$$

(2). L is a derivation. More precisely, if $u, v \in \mathcal{D}_G(L)$, then $uv \in \mathcal{D}_G(L)$, and

$$L(uv) = uLv + (Lu)v.$$

(3). If $u \in \mathcal{D}_G(L)$, $\Phi \in C^2(\mathbf{R})$, then the composition $\Phi(u) \in \mathcal{D}_G(L)$ and

$$L(\Phi(u)) = \Phi_u(u)Lu.$$

The linearity (1) is clear. We give a proof of (3) here. Since

$$\frac{\partial \Phi}{\partial x_k} = \Phi' \frac{\partial u}{\partial x_k},$$

$$\frac{\partial^2 \Phi}{\partial x_k^2} = \frac{\partial}{\partial x_k} \Phi' \frac{\partial u}{\partial x_k} = \Phi'' \left(\frac{\partial u}{\partial x_k} \right)^2 + \Phi' \frac{\partial^2 u}{\partial x_k^2},$$

so

$$\frac{1}{n} \sum_{k=1}^n \frac{\partial^2 \Phi}{\partial x_k^2} = \Phi'' \frac{1}{n} \sum_{k=1}^n \left(\frac{\partial u}{\partial x_k} \right)^2 + \Phi' \frac{1}{n} \sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2},$$

and by letting $n \rightarrow \infty$, we get

$$\Delta \Phi = \Phi' \Delta u.$$

since $D_k u(x) \in \mathbf{H}$, thus $|D_k u| \rightarrow 0$.

To get (2), just do a similar calculation.

The properties (1) and (2) show that $\mathcal{D}_G(L)$ is an algebra over R^1 .

By *theorem 4.5*, we also get

Theorem 4.6 *If $f \in C_b^2(G)$, then f is harmonic in G if and only if $D^2 f(x)$ (as an operator on \mathbf{H}) is compact for each $x \in G$.*

Example 8. Let $\mathcal{E} = \{e_k\}$ be a basis for \mathbf{H} . Let $m \geq 3$, and define

$$f(x) = \sum_{k=1}^{\infty} (x \cdot e_k)^m.$$

By a calculation we find that $D^2 f$ has e_k , $k = 1, 2, \dots$ as its eigenvectors with corresponding eigenvalues $\lambda_k = m(m-1)x_k^{m-2}$ where $x_k = (x \cdot e_k)$, and $\lim_{k \rightarrow \infty} \lambda_k = 0$ ($m > 2$). Hence $D^2 f(x)$ is a compact operator for each x . Hence, $Lf = 0$.

It is not difficult to see that f is neither a simple functional nor a regular one. Assume the contrary that $f(x)$ is regular, say

$$f(x) = F((a_1, x), \dots, (a_n, x), |x|^2), \quad x \in \mathbf{H}$$

for some continuous function F on R^{n+1} . Then

$$1 = f(e_k) = F(a_1, c_k), \dots, (a_n, c_k), 1)$$

which converges to $F(0, \dots, 0, 1)$ as $k \rightarrow \infty$. Therefore $F(0, \dots, 0, 1) = 1$. On the other hand, let $p_k = (c_1 + \dots + c_k)/\sqrt{k}$. Then $|p_k| = 1$ for all k and p_k converges to zero in the weak topology. Thus, again, $f(p_k) = F((a_1, p_k), \dots, (a_n, p_k), 1)$ converges to $F(0, \dots, 0, 1)$ as $k \rightarrow \infty$. But

$$f(p_k) = \sum_{j=1}^{\infty} (p_k \cdot e_j)^m = k^{1-m/2} \rightarrow 0$$

as $k \rightarrow \infty$ and therefore $F(0, \dots, 0, 1) = 0$, a contradiction.

Furthermore, we see that f is the limit but not the uniform limit (in any bounded domain) of a sequence of simple functionals $f_n = \sum_{k=1}^n (x \cdot e_k)^m$, $n = 1, 2, \dots$. This example shows that the Laplacian we just defined is really an extension of one Šilov studied in [31—33].

We will discuss other properties of L in later sections.

Now we introduce a norm on $\mathcal{F}(G)$ as follows

$$\| f \| = \| f \|_0 + \| Lf \|_0 \tag{4.14}$$

where $\| \cdot \|_0$ is the uniform norm.

Denote by $A(G)$ the closure of $\mathcal{F}(G)$ with the norm $\| \cdot \|$.

From the original definition (4.10) of the Lévy-Laplacian, there is no requirement for the second Fréchet derivative of functionals. We shall show that the Laplacian defined by (4.12) for functionals in $\mathcal{F}(G)$ can be extended to $A(G)$. We will do that after maximum principles are established in the next section.

There are several other definitions for the Laplacian in infinite-dimensional space. We mention some here.

Definition 4.6 Let $f \in C_b^2(G)$. The Laplacian L_2 is simply defined as a trace operator of the second Fréchet derivative, that is

$$L_2 f = \text{Trace}(D^2 f). \quad (4.15)$$

This definition coincides with the usual one when the Hilbert space is finite. This has been studied by some mathematicians. We refer to L. Gross [13] and the literature therein. It is noted here that even though the second Fréchet derivative $D^2 f$ exists, we cannot conclude that $D^2 f$ is trace class. For example, if $f(x) = (x, x)$, then $D^2 f$ could not be trace class. In order to avoid this, the other "modified" operator was defined as follows.

Definition 4.7 Let $f \in C_b^2(G)$ and let S be a positive nuclear operator on H . Define a Laplacian L_3 by

$$L_3 f = \text{Trace}(S D^2 f). \quad (4.16)$$

By definition, there is a basis $\{e_k\}$ and a sequence of positive numbers $\{\lambda_k\}$ which is summable such that $S e_k = \lambda_k e_k$, $k = 1, 2, \dots$. Hence L_3 has the form

$$L_3 f = \sum_{k=1}^{\infty} \lambda_k (D^2 f e_k, e_k) \quad (4.17)$$

(ref. Da Prato [6], Yu. L. Daletskii [5]).

If we denote by $\mathcal{D}(L_i)$ ($i = 1, 2, 3$), ($L_1 = L$) the domain of the operator L_i in C_b^2 , then we have

$$\mathcal{D}(L_2) \subset \mathcal{D}(L_1) \subset \mathcal{D}(L_3) = C_b^2. \quad (4.18)$$

4.3 Maximum Principles

The maximum principle is one of the most useful tools in the study of partial differential equations. In finite-dimensional space, there are many known versions of maximum principles for the Laplacian, see [26]. So, it is natural to investigate similar properties for the Laplacians in infinite dimensional space. This is the main subject of this section.

Theorem 4.7 (Maximum Principle) *Let G be a bounded connected subset of H , and let $u \in C_b^2(G) \cap C_b(\bar{G})$ satisfy*

$$-Lu \geq 0, \quad (-Lu \leq 0) \quad \text{in } G, \quad (4.19)$$

then

$$\inf_{\bar{G}} u(x) = \inf_{\partial G} u(x), \quad (\sup_{\bar{G}} u(x) = \sup_{\partial G} u(x)). \quad (4.20)$$

Proof: Let $\{e_k\}$ be any orthonormal basis for H . By *theorem 4.4*,

$$Lu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2}. \quad (4.21)$$

First we consider the case in which $-Lu \geq a > 0$ in G where a is a constant.

Assume the contrary that $u_0 = \inf_{\bar{G}} u(x) < u_1 = \inf_{\partial G} u(x)$.

By the definition of u_0 , we can get a sequence of points $\{x_n\}$ in \bar{G} , such that $u(x_n) \rightarrow u_0$. By the assumption, only a finite number, if any, of points of this sequence are on ∂G . In fact, if there is a subsequence say, $\{x_{n_i}\}$ of $\{x_n\}$ which belongs to ∂G , then $\{u(x_{n_i})\}$ is a bounded sequence of R^1 , thus there is a subsequence which converges to u_0 . Thus, $u_1 \leq u_0$.
—Contradiction.

Also, we can choose a subsequence, still call it $\{x_n\}$, such that $\text{dist}(x_n, \partial G)$ converges to a constant, δ_1 . By the assumption that $u_0 < u_1$, it is not difficult to see that $\delta_1 > 0$ by the fact that u is uniformly continuous and the usual analysis. Now we prove the result by the following 3 steps.

1. So we can assume that there exists a sequence of points $\{x_m\}$ in G with $\text{dist}(\{x_m\}, \partial G) > \delta_1$. And for each sufficiently large m , there is a x_m such that $u(x_m) - u_0 < \frac{1}{m}$. Hence we have

$$u(x_m) < u(x) + \frac{1}{m} \quad \forall x \in G. \quad (4.22)$$

since $u_0 \leq u(x)$ for all $x \in G$.

2. Since $u \in C_b^2(G)$, D^2u is uniformly continuous in G , i.e., $\forall \epsilon > 0$, $\exists \delta_0 > 0$, such that for all $x, x_0 \in G$, $|x - x_0| < \delta_0$

$$\|D^2u(x) - D^2u(x_0)\| < \epsilon.$$

Hence, for all integral k , we have

$$\left| \frac{\partial^2 u(x)}{\partial x_k^2} - \frac{\partial^2 u(x_0)}{\partial x_k^2} \right| < \epsilon. \quad (4.23)$$

Now, for $\epsilon = a/4$, there is a δ_0 , such that when $|x - x_0| < \delta_0$, we have

$$\frac{1}{N} \sum_{k=1}^N \frac{\partial^2 u(x_0)}{\partial x_k^2} - a/4 < \frac{1}{N} \sum_{k=1}^N \frac{\partial^2 u(x)}{\partial x_k^2} < \frac{1}{N} \sum_{k=1}^N \frac{\partial^2 u(x_0)}{\partial x_k^2} + a/4 \quad (4.24)$$

for all N .

3. Let $\delta = \min\{\delta_0, \delta_1\}$ and m be so large that $1/m < (a/8)\delta^2$. We choose a point x_m in G such that (4.22) holds (see item 1.).

Since

$$-\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2}(x) \geq a > 0 \quad x \in G,$$

it is, in particular, true at $x = x_m$. Hence there exists an integral N such that

$$-\frac{1}{N} \sum_{k=1}^N \frac{\partial^2 u(x_m)}{\partial x_k^2} > a/2. \quad (4.25)$$

By the second inequality in (4.24), replacing x_0 with x_m , and using (4.25) we get

$$-\frac{1}{N} \sum_{k=1}^N \frac{\partial^2 u(x)}{\partial x_k^2} > -\frac{1}{N} \sum_{k=1}^N \frac{\partial^2 u(x_m)}{\partial x_k^2} - a/4 \geq a/4 \quad (4.26)$$

for all $x \in B(x_m, \delta) \subset G$.

Now let $B_N(x_m, \delta) = B(x_m, \delta) \cap P_N(x_m)$. Define

$$u_N(x) = u((x, e_1), \dots, (x, e_N), (x_m, e_{N+1}), \dots) \quad x \in G.$$

So we can look at u_N as a function defined in a bounded domain of R^N , and thus we have

$$-\frac{1}{N} \sum_{k=1}^N \frac{\partial^2 u_N(x)}{\partial x_k^2} > a/4. \quad (4.27)$$

Let $w(x) = u_N(x) + (a/S)|x - x_m|^2$, $x \in B_N(x_m, \delta)$. From (4.27) we get

$$-\frac{1}{N} \sum_{k=1}^N \frac{\partial^2 w(x)}{\partial x_k^2} > 0 \quad x \in B_N(x_m, \delta). \quad (4.28)$$

By the maximum principle for elliptic operators in R^N , we get

$$u_N(x) + (a/S)|x - x_m|^2 \geq \min_{\partial B_N} (u_N + (a/S)|x - x_m|^2)$$

for all $x \in B_N$. In particular,

$$u_N(x_m) \geq \min_{\partial B_N} (u_N(x)) + (a/S)\delta^2.$$

Hence there is a $x^* \in \partial B_N$ such that

$$u_N(x_m) \geq u_N(x^*) + (a/S)\delta^2.$$

Hence $x^* \in G$ is such that

$$u(x_m) \geq u(x^*) + (a/S)\delta^2 > u(x^*) + 1/m \quad (4.29)$$

by the choice of m . This contradicts (4.22). Hence we have proved the result when $-Lu \geq a > 0$.

Now we can finish the proof of general case ($-Lu \geq 0$) as usual by introducing the auxiliary function:

$$v(x) = u(x) - \delta|x|^2 \quad \forall x \in G$$

where δ is an arbitrary positive number. Then

$$-Lv = -Lu + 2\delta \geq 2\delta > 0.$$

By the result we derived above, we get

$$\inf_{\bar{G}} v(x) = \inf_{\partial G} v(x).$$

i.e.

$$\begin{aligned} \inf_{\bar{G}} u(x) &= \inf_{\bar{G}} (v(x) + \delta|x|^2) \geq \inf_{\bar{G}} v(x) \\ &= \inf_{\partial G} v(x) = \inf_{\partial G} (u(x) - \delta|x|^2) \\ &\geq \inf_{\partial G} u(x) - \delta \sup_{\bar{G}} |x|^2. \end{aligned}$$

Since G is bounded, $c = \sup_{\bar{G}} |x|^2 < \infty$, and so for each $\delta > 0$

$$\inf_{\bar{G}} u(x) \geq \inf_{\partial G} u(x) - \delta c.$$

Now letting $\delta \rightarrow 0$, we get

$$\inf_{\bar{G}} u(x) \geq \inf_{\partial G} u(x).$$

Q.E.D.

Remark: By the similar argument it is not difficult to see that the maximum principle also holds for other versions of the Laplacian defined by formulas (4.15) and (4.16).

Now consider $u(x) = \sum_{k=1}^{\infty} x_k^4$ where x_k 's are the coordinates of x with reference to an orthonormal complete system in a unit ball $G = B(o, 1)$. Then $u(o) = 0$ and $u(x) > 0$ if $x \neq o$. In *example 8 (p. 67)* we have seen $Lu(x) = 0$. By the maximum principle, we have, for all $x \in G$

$$\inf_{\partial G} u \leq u(x) \leq \sup_{\partial G} u,$$

hence,

$$0 \leq \inf_{|x|=1} u \leq u(o) = 0,$$

i.e.

$$\inf_{|x|=1} u = 0,$$

despite the fact that $u(x) > 0$ on $\{|x| = 1\}$. This result also reflects the fact that the unit ball in infinite-dimensional normed space is not compact.

From the above example, we also see that the strong maximum principle is not true for L , since, in the above example, $Lu = 0$, and $u \neq \text{constant}$, but u has a minimum in the region. What we can conclude is

Theorem 4.8 *If $-Lu > 0$ ($-Lu < 0$) in G , then u could not get a minimum (maximum) value in G .*

Next, we observe the boundary version of the maximum principle.

Let G be a bounded subset of H with smooth boundary ∂G . Let $\bar{n} = (n_1, \dots, n_k, \dots)$ be the unit normal vector in an outward direction at a point P on the boundary of G . We say the vector $\nu = (\nu_1, \nu_2, \dots)$ points outward from G at the point P if $(\nu, \bar{n}) > 0$.

For example, let $G = B_o(R)$, the ball in H with the center at the origin and radius R . The boundary ∂G is given by $g(x) = |x|^2 - R^2 = 0$. Then the unit normal vector in the outward direction at the boundary point $P : x = (x_1, x_2, \dots)$ is

$$\bar{n} = \frac{1}{R^2}(x_1, x_2, \dots).$$

Let u be a function from G to \mathbb{R} . P a point on ∂G . We define

$$\frac{\partial u}{\partial \nu}(P) = \lim_{Q \xrightarrow{\nu} P} \frac{u(P) - u(Q)}{|P - Q|}$$

if the limit exists, where $Q \xrightarrow{\nu} P$ means that Q tends to P along the direction ν . It is easy to see that if $u \in C_b^1(\bar{G})$,

$$\frac{\partial u}{\partial \nu}(P) = (Du, \nu)(P) = \lim_{x \rightarrow P} \sum_{k=1}^{\infty} \nu_k \frac{\partial u}{\partial x_k}(x).$$

We know that the Hopf boundary point version of the maximum principle holds for usual elliptic operators in R^n (see [26]), but in infinite-dimensional space it is not generally true, as we can see, from the following simple example.

Example 9: Consider the problem

$$\begin{cases} Lu = (R - r) & \text{in } B_o(R) \\ u = 0 & \text{on } \partial B \end{cases}$$

By a direct calculation, $u = \frac{1}{2}Rr^2 - \frac{1}{3}r^3 - \frac{1}{6}R^3$ is a solution. Clearly u satisfies $Lu > 0$ in $B_o(R)$ and u attains its maximum at all boundary points, but $u'(r) = r(R - r) \rightarrow 0$ as $r \rightarrow R$.

About the sign of the derivative at the boundary, we have

Theorem 4.9 *Let $Lu \geq \alpha > 0$ in G . Suppose that $u \leq M$ in G and that $u = M$ at a boundary point P . Assume G has a internally tangent ball at P . If u is continuous in $G \cup \{P\}$ and an outward directional derivative $\frac{\partial u}{\partial \nu}$ exists at P , then*

$$\frac{\partial u}{\partial \nu} > 0$$

at the point P (where ν is the outward direction from G at P).

Proof: Let K_1 be a ball internally tangent to ∂G at P .

Define an auxiliary function as follows:

$$v(x) = c^{-r^2} - c^{r_1^2} \quad x \in K_1$$

where r_1 is the radius of the ball K_1 and r is the distance of a point x from the center of K_1 . Then

$$Lv = \frac{1}{r}v'(r) = -2c^{-r^2} < 0 \quad \text{in } K_1$$

and $v|_{\partial K_1} = 0$.

Now let $w = u + \delta v$. Choose δ small enough such that

$$Lw = Lu + \delta Lv \geq 0 \quad \text{in } K_1.$$

This is possible since $Lu \geq \alpha > 0$, and $|Lv| < 2$. On the boundary of K_1 , we have $w(P) = u(P) = M$ and

$$w|_{\partial K_1} = u|_{\partial K_1} \leq M.$$

By the maximum principle (*theorem 4.7*), we have

$$w \leq M \quad \text{in } K_1.$$

Since $w(P) = u(P) = M$, w attains the maximum at the boundary point P , thus

$$\frac{\partial w}{\partial \nu} \Big|_P \geq 0.$$

i.e.

$$\frac{\partial u}{\partial \nu} \Big|_P + \delta \frac{\partial v}{\partial \nu} \Big|_P \geq 0.$$

On the other hand, at the point P , we have

$$\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \bar{n}}(\nu, \bar{n}) = -2r e^{-r^2}(\nu, \bar{n})$$

where \bar{n} is the outward unit normal direction. Since $(\nu, \bar{n}) > 0$, thus

$$\frac{\partial v}{\partial \nu} \Big|_P < 0.$$

Combining the above results, we get $\frac{\partial u}{\partial \nu} \Big|_P > 0$. Q.E.D.

Now we use the maximum principle to derive an "a priori" estimate.

Corollary 4.2 *Let G be a bounded connected subset of H , $u \in C_b^2(G) \cap C_b(\bar{G})$ and $f \in C_b(G)$ such that*

$$Lu = f.$$

Then

$$\|u\|_0 \leq \sup_{\partial G} |u(x)| + c \|f\|_0 \quad (4.30)$$

where c is a constant depending only on G , in fact, we may take

$$c = \sup\{|x|^2/2, x \in G\}.$$

Proof: Let

$$g = g(x) = c - |x|^2/2.$$

$$v = v(x) = \sup_{\partial G} |u(x)| + g(x) \|f\|_0.$$

Then

$$Lv = (Lg(x)) \|f\|_0 = -\|f\|_0.$$

hence

$$-L(v - u) = -Lv + Lu = \|f\|_0 + f \geq 0.$$

$$\begin{aligned} (v - u)|_{\partial G} &= \sup_{\partial G} |u| + \|f\|_0 (g(x)|_{\partial G}) - u(x)|_{\partial G} \\ &\geq \sup_{\partial G} |u| - u(x)|_{\partial G} \geq 0. \end{aligned}$$

By the maximum principle (*theorem 4.7*), we get

$$v(x) - u(x) \geq 0.$$

i.e.,

$$u(x) \leq \sup_{\partial G} |u| + g(x) \|f\|_0.$$

Similarly, replacing u by $-u$, finally, we get inequality (4.30). Q.E.D.

4.4 Extension of the Laplacian

With the help of the maximum principle, we are going to extend the definition of the Laplacian given by (4.12) on C^2 functionals to a broader class of functionals.

Recall the notation of the previous section:

$$\mathcal{F}(G) = \{f \in C_b^2(G), D^2f = \gamma I + T, \gamma \in C_b(G), T \text{ compact}\}$$

with the norm $\|f\| = \|f\|_0 + \|Lf\|_0$ is a normed space. Let $A(G)$ be the closure of $\mathcal{F}(G)$ under the norm $\|\cdot\|$.

In order to extend the definition of L to $A(G)$, we first prove

Theorem 4.10 *Let G be a bounded subset of H . Assume that a sequence of functionals $\{u_n\} \in \mathcal{F}(G)$ satisfies $u_n \rightarrow 0$, and $Lu_n \rightarrow q(x)$ uniformly in G . Then $q(x) \equiv 0$.*

Proof: We prove the conclusion by contradiction. Suppose $q(x) \not\equiv 0$. Without loss of generality, suppose there is a point $x_0 \in G$ such that $q(x_0) > 0$. Let $c = q(x_0)/2$.

1. By continuity of $q(x)$, $\exists \delta_1 > 0$, such that

$$q(x) \geq c > 0 \quad \text{in } B(x_0, \delta_1)$$

2. Since $Lu_n \rightarrow q(x)$ uniformly in G , for any given $\delta_2 > 0$, there is an integer N , such that

$$\|Lu_n - q\| < \delta_2$$

holds for $n \geq N$, i.e.

$$q(x) - \delta_2 < Lu_n(x) < q(x) + \delta_2, \quad \forall x \in G.$$

Take $\delta_2 = c/2$. Then

$$Lu_n(x) > c/2 \quad \text{in } B(x_0, \delta_1), n \geq N,$$

hence

$$L(u_n(x) - c/4|x - x_0|^2) > 0 \quad \text{in } B(x_0, \delta_1), n \geq N.$$

By the maximum principle, we get, for all $x \in B(x_0, \delta_1)$,

$$u_n(x) - (c/4)|x - x_0|^2 \leq \sup_{\partial B(x_0, \delta_1)} (u_n - (c/4)|x - x_0|^2).$$

In particular, if we let $x = x_0$, then for $n \geq N$,

$$u_n(x_0) \leq \sup_{\partial B} u_n - (c/4)\delta_1^2. \quad (4.31)$$

3. Since $u_n \rightarrow 0$ uniformly in G , for every small positive number $\epsilon > 0$, there is an integer N_2 such that

$$\|u_n\| < \epsilon \quad \text{for } n \geq N_2. \quad (4.32)$$

Take $\epsilon < (c/8)\delta_1^2$, $N = \max(N_1, N_2)$. By (4.31), (4.32), we get

$$u_n(x_0) \leq (c/8)\delta_1^2 - (c/4)\delta_1^2 = -(c/8)\delta_1^2 < 0$$

for $n \geq N$. This contradicts the condition that $u_n \rightarrow 0$ uniformly. Q.E.D.

Now, for $f \in A(G)$, G a bounded subset of H , there is a sequence $\{f_n\} \in F(G)$ satisfying $f_n \rightarrow f$ uniformly and $\{Lf_n\}$ is a Cauchy sequence with the uniform norm. Then we define the Laplacian of f by

$$Lf(x) = \lim_{n \rightarrow \infty} Lf_n(x). \quad (4.33)$$

Theorem 4.10 assures us that Lf is well-defined (since the definition is independent of the sequence f_n chosen). Hence the domain of the Laplace operator could be extended to $\mathcal{A}(G)$ and clearly all smooth regular functionals (Šilov's definition) are included in $\mathcal{A}(G)$.

By the standard limit arguments, the properties 1, 2, 3 in §4.2 still hold for the extension L defined in $\mathcal{A}(G)$.

Correspondingly, we can establish the maximum principle for L acting on functionals in $\mathcal{A}(G)$.

Theorem 4.11 *Let $G \subset H$ bounded, $u \in \mathcal{A}(G)$, $Lu \geq 0$. Then*

$$\sup_{\bar{G}} u(x) = \sup_{\partial G} u(x). \quad (4.34)$$

Proof: By a usual limit argument. First assume $Lu \geq a > 0$ where a is a constant. There is sequence $\{u_n\} \in \mathcal{F}(G)$ such that

$$u_n \rightarrow u, \quad Lu_n \rightarrow Lu \quad \text{uniformly.}$$

Letting $\epsilon < a/2$, $\exists N$ such that for $n \geq N$,

$$\|Lu_n - Lu\| < \epsilon.$$

Hence, for $n \geq N$,

$$Lu_n(x) \geq a/2 > 0 \quad \text{in } G.$$

By the maximum principle for functionals in $\mathcal{F}(G)$,

$$\sup_{\bar{G}} u_n(x) = \sup_{\partial G} u_n(x).$$

Letting $n \rightarrow \infty$, we have, since $u_n \rightarrow u$ uniformly,

$$\sup_{\bar{G}} u(x) = \sup_{\partial G} u(x). \quad (4.35)$$

Now assume $Lu \geq 0$. Let $w_\delta(x) = u(x) + \delta|x|^2$ for $\delta < 0$. Then $Lw_\delta \geq \delta > 0$.

By what we have proved,

$$\sup_{\bar{G}} w_\delta = \sup_{\partial G} w_\delta,$$

i.e.

$$\sup_{\bar{G}} (u(x) + \delta|x|^2) = \sup_{\partial G} (u(x) + \delta|x|^2).$$

Since G is bounded and δ is arbitrary small, we finally get

$$\sup_{\bar{G}} u(x) = \sup_{\partial G} u(x).$$

Q.E.D.

Chapter 5

Boundary value problems

Introduction

In this chapter, we investigate the existence of solutions of boundary value problems associated with the Laplacian we discussed in the preceding chapter. G. E. Šilov solved the Dirichlet and Poisson problems for the Laplacian restricted to the regular functionals and a class of simple domains. In §5.1, we establish a necessary and sufficient condition on the domain for the solvability of the Dirichlet problem in fundamental functionals. We also give a more explicit expression for the solutions of Poisson problems (compare to the one given by Šilov in [31]). G. E. Šilov's consideration is a special case of our results here. In §5.2, we look at the radially symmetric problems. An interesting remark is that B. Gidas *et al*'s results for radially symmetric problem in n -dimensional space also hold for this Laplacian in infinite-dimensional space. Finally, more general boundary value problems

of nonlinear equations are studied in §5.3.

5.1 The Dirichlet and Poisson problems

In this section, we consider the Dirichlet problem and the Poisson problem. We will give a necessary and sufficient condition on the domain for the solvability of the Dirichlet problem in the class of fundamental functionals. In our derivation, the maximum principle plays an important role.

The Dirichlet problem is to find a functional $u(x)$ which takes the given values on the boundary ∂G of a given domain $G \subset H$ and is harmonic in this domain, i.e., solve the following boundary value problem

$$Lu = 0 \quad \text{in } G, \quad (5.1)$$

$$u = \phi(x) \quad \text{on } \partial G. \quad (5.2)$$

The Poisson problem is to determine a functional $u(x)$ which satisfies the equation

$$Lu(x) = f(x) \quad \text{in } G \quad (5.3)$$

and the homogeneous boundary condition

$$u = 0 \quad \text{on } \partial G \quad (5.4)$$

where f is a given functional.

First we show the uniqueness of solutions of these problems. It is a natural consequence of the maximum principle.

Theorem 5.1 *The Dirichlet problem (5.1), (5.2) has at most one solution in the functional space $A(G)$.*

Proof: Suppose $u_1, u_2 \in A(G)$ are two solutions of (5.1), (5.2). Then $w = u_1 - u_2$ is also the solution of (5.1) and equals 0 on ∂G . By applying the maximum principle, we get $w \equiv 0$ in G , i.e., $u_1 = u_2$. The uniqueness has been proved.

Similarly, we have

Theorem 5.2 *The Poisson problem (5.3), (5.4) has at most one solution in the functional space $A(G)$.*

Definition 5.1 (Fundamental functional for the Laplacian) *If a functional f defined on a subset G of H can be represented by a composition of a smooth function defined on a subset Q of R^{n+1} and n harmonic functionals defined on G and the basic functional $|x|^2$, i.e., there are $s_i : G \rightarrow R$, $Ls_i = 0, (i = 1, \dots, n)$ and $F = F(\xi_1, \dots, \xi_n, \xi_{n+1}) : Q \rightarrow R$, s. t. $f(x) = F(s_1(x), \dots, s_n(x), |x|^2)$, then f is called a fundamental functional. Denote by $B(G)$ the set of all fundamental functionals.*

It is easy to see that any smooth regular functional is fundamental.

Definition 5.2 (Fundamental domain) *Let G be a bounded subset of H with boundary ∂G . If there is a harmonic functional $s(x)$ such that $s(x) > |x|^2$ in G , $s(x) = |x|^2$ on ∂G , then G is called a fundamental domain, and $s(x)$ is called the representation of the domain G .*

Clearly, the domains considered by G.E. Šilov [31—33],

$$G = \{x \in H; |x|^2 < h((a_1, x), \dots, (a_n, x))\},$$

are fundamental.

Now we prove a necessary condition for the solvability of the Dirichlet problem.

Theorem 5.3 *Let G be a bounded domain in H . If the Dirichlet problem is solvable for all regular ϕ , then G must be fundamental.*

Proof: Let $\phi(x) = |x|^2$. It is regular, so the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } G \\ u = |x|^2 & \text{on } \partial G \end{cases}$$

is solvable. Let $s(x)$ be the unique solution of this problem and define $w(x) = s(x) - |x|^2$, then

$$Lw = Ls - L|x|^2 = -2 < 0 \quad \text{in } G$$

and on ∂G , we have

$$w = s - |x|^2 = 0.$$

Hence, by the maximum principle, $w(x) > 0$ in G , i.e., $s(x) > |x|^2$ in G .

Hence G is fundamental. Q.E.D.

Now we are going to consider the existence of solutions of the Dirichlet problem. Let G be a fundamental domain with representative harmonic

functional $s(x)$. Let ϕ be a fundamental functional where, for simplicity, we assume $\phi(x) = \phi(h(x), |x|^2)$ where $h(x)$ is harmonic. Define

$$u(x) = \phi(h(x), s(x)) \quad x \in G. \quad (5.5)$$

Since $h(x), s(x)$ are harmonic in G , so is $u(x)$. On the boundary ∂G , $s(x) = |x|^2$, thus $u(x) = \phi(h(x), s(x)) = \phi(h(x), |x|^2)$ on ∂G . This shows that u is the solution of (5.1), (5.2). Hence, combining this with the previous theorem, we have proved

Theorem 5.4 *The Dirichlet problem (5.1), (5.2) is solvable in the class of fundamental functionals if and only if the domain G is fundamental.*

Remark: The result here is analogous to the Dirichlet problem for the finite-dimensional Laplacian. In finite-dimensional space, a necessary and sufficient condition for the solvability of the Dirichlet problem is that G must be regular. And in that case, *regularity is equivalent to fundamentality*. In fact, if G is regular, then the Dirichlet problem is solvable; in particular, the problem

$$\begin{cases} \Delta u = 0 & \text{in } G \\ u = |x|^2 & \text{on } \partial G \end{cases}$$

is solvable. Let $s(x)$ be the solution. Then $s(x)$ is exactly the harmonic function required by the fundamentality of G . Conversely, if G is fundamental with the representation $s(x)$, then for each boundary point ξ , define

$$w_\xi(x) = s(x) - |x|^2 + |x - \xi|^2.$$

It is easy to see that $w_\xi(x)$ is the required barrier function for the regularity of G in R^n , i.e.,

$$\Delta w_\xi \leq 0, w_\xi(\xi) = 0, w_\xi > 0 \quad \text{in } \bar{G} - \{\xi\}.$$

Next, we will consider the Poisson problem. We prove the following existence theorem.

Theorem 5.5 *Let G be a fundamental domain with the representation $s(x)$. Let f be a given fundamental functional defined in G , in detail, $f = f(h(x), |x|^2)$. Then the Poisson problem (5.3), (5.4) has a unique solution and it can be written as*

$$u = u(x) = - \int_0^{\sqrt{s(x)-|x|^2}} \rho f(h(x), |x|^2 + \rho^2) d\rho. \quad (5.6)$$

Before giving the detailed proof, we show how it works for a simple example.

Assume f is simple, hence, f is harmonic. By the formula (5.6),

$$u = - \int_0^{\sqrt{s(x)-|x|^2}} \rho f(x) d\rho = (1/2)f(x)(|x|^2 - s(x)).$$

It is not difficult to check that u is the solution of (5.3), (5.4). In fact,

$$Lu = (1/2)[(Lf)(|x|^2 - s(x)) + f(x)L(|x|^2 - s(x))] = f(x)$$

and on the boundary, $s(x) = |x|^2$, thus $u = 0$.

Proof: Make a change of the variable of integration, i.e., let $r = \rho^2$.

Then (5.6) turns to be

$$u = u(x) = -(1/2) \int_0^{s(x)-|x|^2} f(h(x), |x|^2 + r) dr. \quad (5.7)$$

It is easy to see that $u = 0$ on the boundary ∂G since $s(x) = |x|^2$ there.

By using the properties of the operator L (see §4.2), we can do the following calculation.

$$\begin{aligned}
 Lu &= -(1/2)L\left[\int_0^{s(x)-|x|^2} f(h(x), |x|^2 + r) dr\right] \\
 &= -(1/2)[f(h(x), s(x))L(s(x) - |x|^2) + \int_0^{s(x)-|x|^2} Lf(h(x), |x|^2 + r) dr] \\
 &= -(1/2)[f(h(x), s(x))(-2) \\
 &\quad + \int_0^{s(x)-|x|^2} (f_1(h(x), |x|^2 + r)Lh + f_2(h(x), |x|^2 + r)L(|x|^2 + r)) dr] \\
 &= -(1/2)[-2f(h(x), s(x)) + \int_0^{s(x)-|x|^2} 2\frac{\partial f}{\partial r}(h(x), |x|^2 + r) dr] \\
 &= -(1/2)[-2f(h(x), s(x)) + 2(f'(h(x), s(x)) - f(h(x), |x|^2))] \\
 &= f(h(x), |x|^2).
 \end{aligned}$$

In the calculation we have used the notation $f_i = \frac{\partial f(\xi_1, \xi_2)}{\partial \xi_i}$, $i = 1, 2$.

Hence the formula (5.6) gives the unique solution of the Poisson problem. Q.E.D.

By making another change of variable, (5.6) can be rewritten as

$$u(x) = -(1/2) \int_{|x|^2}^{s(x)} f(h(x), r) dr. \quad (5.8)$$

It is easy to see that the solution of the boundary value problem

$$\begin{cases} Lu(x) = f(x) & \text{in } G \\ u(x) = \phi(x) & \text{on } \partial G \end{cases} \quad (5.9)$$

where f , ϕ are given fundamental functionals, is given by

$$u(x) = -(1/2) \int_{|x|^2}^{s(x)} f(h(x), r) dr + \phi(x, s(x)). \quad (5.10)$$

Now we consider the linear equation

$$Lu(x) + cu(x) = f(x) \quad \text{in } G \quad (5.11)$$

where c is a constant, with boundary condition

$$u(x) = o(x) \quad \text{on } \partial G. \quad (5.12)$$

Theorem 5.6 Assume G is a fundamental domain with representation $s(x)$, $f \in B(G)$. Then (5.11), (5.12) have a unique solution which is given by

$$u(x) = -(1/2)e^{-(c/2)|x|^2} \int_{|x|^2}^{s(x)} f(x, r)e^{(c/2)r} dr + e^{(c/2)(s(x)-|x|^2)} \phi(x, s(x)). \quad (5.13)$$

Proof: The proof of the existence (i.e., $u = u(x)$, given by (5.13), is a solution) is similar to the proof of *theorem 5.5*, so we omit it. Here we give a proof of the uniqueness. It is equivalent to showing that the homogeneous problem

$$\begin{cases} Lu(x) + cu(x) = 0 & \text{in } G \\ u(x) = 0 & \text{on } \partial G. \end{cases} \quad (5.14)$$

has only the null solution. Suppose that u is a non-zero solution of problem (5.14). W. l. o. g., suppose $x_0 \in G$ is such that $u(x_0) > 0$. Let Z be a component of $\{x \in G; u(x) < 0\}$, containing x_0 . Then on ∂Z , $u(x) = 0$, in Z , $F(u) = \ln(u)$ satisfies

$$L(F(u(x))) = \frac{1}{u(x)} Lu(x) = -c.$$

The general solution of the above equation is

$$F(u(x)) = c \frac{|x|^2}{2} + H(x),$$

where $H(x)$ represents any harmonic function. So, in Z , $u(x) = H(x)e^{-\frac{c|x|^2}{2}}$. By the boundary condition of u on ∂Z , $H(x)|_{\partial Z} = 0$, hence $u = 0$ in Z . We get a contradiction. Hence $u \equiv 0$ in G . Q.E.D.

5.2 Radially symmetric problems

In the previous chapter and the last section, we have discussed the properties of the Laplace operator L and the corresponding Dirichlet and Poisson problems. In this and the next section we will consider boundary value problems associated with semilinear equations

$$\begin{cases} Lu + f(x, u) = 0 & \text{in } G \\ u = \phi & \text{on } \partial G \end{cases} \quad (5.15)$$

where f and ϕ are known functionals defined on $G \times R$ and G , respectively, and L is the Laplacian defined in the previous chapter. We will see that, for a class of nonlinear terms f , the solvability of problem (5.15) is equivalent to solving a functional equation due to the differential property of L . For the more general case, we shall use the iteration method to get the existence of solutions.

5.2.1 The problem in a ball

In this section we look at a special case of radially symmetric problem:

$$\begin{cases} Lu + f(u) = 0 & \text{in } G = B_o(R) \\ u|_{\|x\|=R} = 0. \end{cases} \quad (5.16)$$

The corresponding problem in finite-dimensional space (L is replaced by the usual Laplace operator Δ in R^n) has attracted a lot of attention. An elegant result which shows that all positive solutions of the symmetric problem are radially symmetric has been obtained by Gidas *et al* [12]. Here, for the

problem in infinite dimensional space, we shall derive an analogous result. By using the special properties of L , the derivation becomes much simpler.

Assume that $f \in C^1((\tau, T))$, $(0 \in (\tau, T) \subset R)$, $f \neq 0$. Let

$$F(t) = \int_0^t \frac{ds}{f(s)}, \quad t \in (\tau, T). \quad (5.17)$$

Now, suppose $u \in A(G)$ is a solution of problem (5.16). Then, by the third property of L in §4.2, $F(u)$ satisfies

$$LF(u) = F'Lu = \frac{1}{f(u)}(-f(u)) = -1,$$

i.e.

$$LF(u) = -1.$$

Letting $v(x) = F(u(x)) + \frac{r^2}{2}$, where $r^2 = |x|^2$, then it is easy to check that v satisfies

$$\begin{cases} Lv = 0 & \text{in } G \\ v = R^2/2 & \text{on } \partial G. \end{cases} \quad (5.18)$$

This implies that

$$F(u(x)) = (1/2)(R^2 - |x|^2) \quad \text{on } \bar{G} \quad (5.19)$$

since $v = (1/2)R^2$ is the unique solution of the Dirichlet problem (5.18).

The above derivation shows that if u is a solution of problem (5.16), then it must satisfy the functional equality (5.19). Hence we can prove the following theorem.

Theorem 5.7 (Uniqueness) *Assume that $f \in C^1((\tau, T))$, $(0 \in (\tau, T) \subset R)$, $f \neq 0$. Then (5.16) has at most one solution. Furthermore, if the solution exists, it must be radially symmetric.*

Proof: Let F be defined by (5.17). Clearly, F is a 1 - 1 mapping. Suppose u_1, u_2 are two solutions of (5.16). Then for each $x \in G$, $F(u_1(x)) = F(u_2(x))$. Hence $u_1(x) = u_2(x)$. Uniqueness is proved. Furthermore, if u is a solution of (5.16), let x, y be any two points in G s.t. $|x| = |y|$, then $F(u(x)) = F(u(y))$, thus $u(x) = u(y)$. hence u is radially symmetric. Q.E.D.

Now we show that the sign of the solution of (5.16) depends only on the sign of $f(0)$.

Theorem 5.8 *If $f(0) > 0$ (< 0), then all solutions (if any) of the problem (5.16) are positive (resp., negative).*

Proof: Assume $f(0) > 0$, we prove that all solutions of problem (5.16) are nonnegative. By the continuity of f , there exists $\epsilon > 0$, s.t., $f(t) > 0, t \in (-\epsilon, 0)$. Let u be a solution of (5.16). So it satisfies the equation (5.19). Suppose $u(x_0) < 0$ for some point $x_0 \in G$. Hence

$$F(u(x_0)) = (1/2)(R^2 - |x_0|^2) > 0,$$

i.e.

$$\int_{u(x_0)}^0 \frac{1}{f(t)} dt < 0$$

by the definition of F and because $u(x_0) < 0$. This implies that

$$u(x_0) < -\epsilon \text{ if } u(x_0) < 0 \tag{5.20}$$

since $f(t) > 0, t \in (-\epsilon, 0)$.

On the other hand, let $l = \{x \in \bar{G}, x = tx_0, t \in R\}$. Along the line l , let $h(t) = u(tx_0)$. Clearly, h is continuous and

$$h(1) = u(x_0) < -\epsilon, h(R/|x_0|) = u((R/|x_0|)x_0) = 0.$$

So, for a $a \in (-\epsilon, 0)$, $\exists c \in (1, R/|x_0|)$ such that $h(c) = a$ by the continuity of h , i.e.,

$$u(cx_0) = a.$$

Hence, we have found a point $x_1 = cx_0 \in G$ such that $-\epsilon < u(x_1) = a < 0$. This contradicts (5.20). Q.E.D.

So we only need to assume f is a function defined on $[0, T)$ if $f(0) > 0$ and then only positive solutions could exist.

Now we are going to show existence.

Theorem 5.9 *Assume that $f \in C^1([0, T))$, $(0 < T \leq \infty)$ and $f(t) > 0$ in $[0, T)$. Then there exists a positive constant $a_0 > 0$ (may be ∞) such that (5.16) has a (unique, positive, radially symmetric) solution $u = u(x) = u(|x|)$ for $R < a_0$ and $u'(r) < 0$ in $0 < r < R$ and no solutions exist for $R > a_0$.*

Example 10: let $f(t) = 1 + t^2$, then for any $R < \sqrt{\pi} = a_0$, $0 \leq r < R$,

$$u = u(r) = \tan \frac{R^2 - r^2}{2}$$

is the solution of (5.16) in which $f(t) = 1 + t^2$.

The proof of the theorem. Let $F(t) = \int_0^t \frac{1}{f(s)} ds$. By the assumption on f , F is monotone increasing, thus $F : [0, T) \rightarrow F([0, T))$ is a bijection. i.e., $F^{-1} : F([0, T)) \rightarrow [0, T)$ exists. Clearly, $F([0, T)) = [0, a)$ where $a = \sup\{F(t), 0 < t < \infty\}$, $0 < a \leq \infty$.

Set $a_0 = \sqrt{2a}$. If $R < a_0$, then

$$\left[0, \frac{R^2}{2}\right] \subset F([0, T]).$$

We can define a functional u in G by

$$u = u(x) = F^{-1}\left(\frac{R^2 - |x|^2}{2}\right). \quad (5.21)$$

We show that u is the solution of (5.16). First it is clear that the boundary condition is satisfied by u . In fact, when $|x| = R$, $u(x) = F^{-1}(0) = 0$ since $F(0) = 0$ and F is 1-1.

Since $F \in C^2$, so, $u \in C^2(G)$. Hence $Lu = LF^{-1}((R^2 - |x|^2)/2)$ exists. Moreover, u satisfies the functional equation

$$F(u(x)) = \frac{R^2 - |x|^2}{2}. \quad (5.22)$$

Hence we have

$$F'(u)Lu = -1, \quad (5.23)$$

i.e.

$$Lu + f(u) = 0$$

Hence u , defined by (5.21), is the solution of problem (5.16).

Clearly, u is radially symmetric. Furthermore, by differentiating in the radial direction $r = |x|$ on both sides of formula (5.22), we get

$$F'(u)u'(r) = -r,$$

i.e.

$$u'(r) = -rf(u),$$

hence, $u'(r) < 0$ in $0 < r < R$ since $f > 0$.

By the previous argument, we have seen that if u is a solution of (5.16), it must satisfy the functional equation (5.22). Hence if $R > a_0$, the problem has no solutions. Q.E.D.

Remark: In the derivation of the functional equation (5.19), we note that we have supposed $f \neq 0$. Otherwise, we may lose some solutions u such that $f(u(x)) = 0$ in some $x \in G$. Generally, we have

Corollary 5.1 *Assume that $f \in C([0, T])$, $f \geq 0$ and $F(t) = \int_0^t \frac{1}{f(s)} ds$ exists for $t \in [0, T]$. Then (5.16) has a unique solution u such that $f(u(x)) \neq 0$ in $B_o(R)$ when $R < a_0$, where $a_0^2 = 2 \sup_{t>0} F(t)$. Such a solution is radially symmetric.*

Proof: Suppose u is a solution of (5.16) such that $f(u(x)) \neq 0$ in $B_o(R)$. Then $F'(u(x)) = \frac{1}{f(u(x))}$ makes sense for $x \in B_o(R)$. The remainder of the proof is the same as the proof of *theorem 5.9*. Q.E.D.

Example 11: let $f(t) = \sqrt{t}$, then for $0 \leq r < R < \infty$,

$$u = u(r) = \frac{1}{16}(R^2 - r^2)^2$$

is one positive solution of problem (5.16). By *corollary 5.1*, the positive solution is unique. Moreover, for any a , $0 \leq a < R$,

$$u = u(r) = \begin{cases} \frac{(a^2 - r^2)^2}{16}, & 0 \leq r \leq a \\ 0, & a \leq r \leq R \end{cases}$$

is also a solution, but $f(u(r)) = 0$ inside the ball $B_o(R)$.

5.2.2 The problem in the entire space

Now we consider the problem in the entire space. i.e.

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{H}, \quad (5.24)$$

$$u(x) \rightarrow 0 \text{ as } \|x\| \rightarrow \infty \quad (5.25)$$

Considering only radial solutions we let $u = u(r)$, $r = \|x\|$, so that the above problem reduces to

$$\begin{cases} u'(r) + rf(u) = 0 & r < \infty \\ u(r) > 0 & r \geq 0 \\ u(r) \rightarrow 0 & \text{as } r \rightarrow \infty. \end{cases} \quad (5.26)$$

It is not difficult to see that if u is a C^2 -solution of (5.26), then it must be a radial solution of (5.24), (5.25).

Now we state a necessary and sufficient condition for the existence of a radial solution of (5.24) and (5.25).

Theorem 5.10 *Suppose $f \in C^1(0, \infty)$, $f(t) > 0$ in $t > 0$. Then the problem (5.24), (5.25) has a radial solution if and only if*

$$F^\delta(t) = \int_t^\delta \frac{1}{f(s)} ds \rightarrow \infty \text{ as } t \rightarrow 0 \quad (5.27)$$

for any $\delta > 0$. Furthermore if (5.27) holds, then for any $\lambda > 0$, there exists a unique positive radial solution of (5.24), (5.25) satisfying $u(0) = \lambda$.

Proof: Let $u = u(r)$ be a solution of (5.26). It satisfies

$$\frac{u'(r)}{f(u)} + r = 0.$$

Integrating from r to $2r$, we get

$$-\int_{u(r)}^{u(2r)} \frac{1}{f(t)} dt = \frac{3}{2}r^2.$$

Given $\delta > 0$, choose r large enough such that $u(r) \leq \delta$, we let t be so small such that $0 < t < u(2r)$, to find that

$$\frac{3}{2}r^2 = \int_{u(2r)}^{u(r)} \frac{1}{f(s)} ds < \int_t^\delta \frac{1}{f(s)} ds.$$

By letting $t \rightarrow 0, r \rightarrow \infty$, we get (5.27).

Conversely, suppose that (5.27) holds. We can rewrite equation (5.26) as

$$\frac{u'(r)}{f(u)} + r = 0.$$

since $f \neq 0$. Integrate it from r to 0 and let $\lambda = u(0)$ so as to obtain

$$\int_{u(r)}^\lambda \frac{1}{f(s)} ds - \frac{r^2}{2} = 0.$$

Hence we deduce from the problem (5.26) the following functional equation

$$F^\lambda(u) = \frac{1}{2}r^2. \quad (5.28)$$

By (5.27), $F^\lambda(u) \rightarrow \infty$ as $u \rightarrow 0$ and

$$\frac{dF^\lambda}{du} = -\frac{1}{f(u)} < 0.$$

We can now use the inverse function in (5.28) to find that

$$u = u(r) = G^\lambda\left(\frac{1}{2}r^2\right)$$

where G^λ is the inverse function of F^λ . Since $F^\lambda(\lambda) = 0$, hence

$$u(0) = G^\lambda(0) = \lambda,$$

and since $F^\lambda(t) \rightarrow \infty$ as $t \rightarrow 0$, hence

$$u(r) \rightarrow 0 \text{ as } r \rightarrow \infty.$$

Differentiation of the inverse function shows that $u = u(r)$ satisfies the equation. Hence u is the solution of (5.26). Q.E.D.

5.3 General boundary value problems

Now we are going to consider problem (5.15) in a general bounded fundamental domain G in H .

5.3.1 A special case

Let G be a bounded fundamental domain in H . Suppose the nonlinear term has the following form

$$f(x, t) = g(x)q(x, t) \quad (5.29)$$

where $g = g(x)$ is a fundamental functional and $q(x, t)$ is harmonic for each $t \in R$ and $q(x, t) \neq 0$, i.e., we consider

$$\begin{cases} Lu + g(x)q(x, u) = 0 & \text{in } G \\ u = 0 & \text{on } \partial G. \end{cases} \quad (5.30)$$

Let

$$Q(x, t) = \int_0^t \frac{1}{q(x, s)} ds, \quad (5.31)$$

Then it is not difficult to see that $Q(x, t)$ is smooth and harmonic in x for each t .

Suppose u is a solution of problem (5.30). Observe $Q(x, u(x))$:

$$\begin{aligned} LQ(x, u(x)) &= Q_u Lu = \frac{1}{q(x, u(x))} (-g(x)q(x, u(x))) = -g(x); \\ Q(x, u(x))|_{\partial G} &= Q(x, 0) = 0 \end{aligned}$$

since $Q(x, t)$ is harmonic in x for each t . Hence $Q(x, u(x))$ is a solution of the problem

$$\begin{cases} Lv = -g(x) & \text{in } G \\ v|_{\partial G} = 0. \end{cases} \quad (5.32)$$

By the uniqueness of solutions of the Poisson problem,

$$Q(x, u(x)) = P(-g(x)) \quad (5.33)$$

where $P(-g(x))$ is the solution of (5.32). P is called the Poisson operator.

Theorem 5.11 *The problem (5.30) is equivalent to the functional equation (5.33), i.e., u is a solution of (5.30) if and only if u is the solution of equation (5.33).*

Proof: The necessity has been obtained from the derivation above. Now we prove the sufficiency. Assume $u = u(x)$ is a solution of (5.33). First we check the boundary condition in the problem. Since $P(-g(x))|_{\partial G} = 0$, so

$$Q(x, u(x))|_{\partial G} = 0.$$

i.e.,

$$\int_0^{u(x)} \frac{1}{q(x, s)} ds = 0, x \in \partial G.$$

Hence $u(x) = 0$ for $x \in \partial G$ since $q(x, t) \neq 0$.

Now let us take the operation of the Laplacian on both sides of equation (5.33) to find that

$$Q_u Lu = -g(x),$$

since $Q_u(x, u(x)) = q(x, u(x))^{-1}$ exists for $x \in G$. Hence Lu exists and

$$Lu + g(x)q(x, u(x)) = 0.$$

Thus, u is a solution of problem (5.30). Q.E.D.

Theorem 5.12 *The problem (5.30) has one and only one solution.*

Proof: If u_1, u_2 are two solutions of (5.30), then

$$Q(x, u_1(x)) = Q(x, u_2(x)).$$

Hence $u_1(x) = u_2(x)$ since $Q(x, t)$ is 1-1.

For the solvability, by *theorem 5.11*, we only need to consider equation (5.33).

Since $q(x, t) \neq 0$, we assume that $q(x, t) > 0$. Fixing an arbitrary $x \in G$, $Q(x, t)$ is monotone in t . From the equation (5.33) we can uniquely determine a u for each x , i.e.,

$$u = u(x) = Q^{-1}(x, P(-g(x))),$$

by the implicit function theorem. So, $u(x)$ is a solution of equation (5.33). Hence it is a solution of (5.30) by *theorem 5.11*.

5.3.2 The general case

In this paragraph we consider problem (5.15) where $f(x, t)$ is fundamental in x for each t , and satisfies a Lipschitz condition. We will prove a uniqueness theorem and use the iteration methods to get a solution.

We say $f(x, t)$ satisfies a Lipschitz condition in G uniformly with respect to t if there exists a constant $c > 0$ such that

$$|f(x, t_1) - f(x, t_2)| \leq c|t_1 - t_2| \tag{5.34}$$

holds for all $x \in G$ and $t_1, t_2 \in R$.

Theorem 5.13 (Uniqueness) *Assume f satisfies the Lipschitz condition (5.34). Then the problem*

$$\begin{cases} Lu = f(x, u) & \text{in } G \\ u = \phi & \text{on } \partial G. \end{cases} \quad (5.35)$$

has at most one solution in the fundamental functional class $B(G)$.

Proof: Suppose $u, v \in B(G)$ are two solutions of (5.35). If $u \not\equiv v$, w.l.o.g., say $u(x_0) > v(x_0)$ for some $x_0 \in G$. Denote by

$$G_+ = \{x \in G: u(x) > v(x)\}.$$

Then G_+ is not empty. Let $Z(x_0)$ be the component containing x_0 of G_+ . Then $u > v$ in Z , $u = v$ on ∂Z . By the equation (5.35), we find

$$L(u - v) + c(u - v) = f(x, u) - f(x, v) + c(u - v), \quad (5.36)$$

where c is the Lipschitz constant in (5.34). In view of equation (5.36) in Z , we have

$$g(x) \stackrel{\text{def}}{=} f(x, u(x)) - f(x, v(x)) + c(u(x) - v(x)) \geq 0$$

in Z , and we note that $u(x) - v(x) = 0$ on ∂Z . From the result in §5.1 (*theorem 5.6*), we find

$$u(x) - v(x) \leq 0, \quad x \in Z.$$

We get a contradiction. Hence $u \equiv v$. Q.E.D.

Finally we use the iteration method to get a solution.

Theorem 5.14 *Let G be a fundamental domain with representative $s(x)$. Suppose $f(\cdot, t) \in B(G)$ for each t and f satisfies a Lipschitz condition in t uniformly on G with Lip constant c . (5.34). Then, if $cM < 2$ where $M = \sup_{x \in G}(s(x) - |x|^2)$, the problem (5.35) has one solution $u \in A(G)$ and u is also fundamental.*

Proof: Let $u_0 = 0$. Define u_n inductively as the solution of the following problem

$$\begin{cases} Lu_n = f(x, u_{n-1}) & \text{in } G \\ u_n = \phi & \text{on } \partial G. \end{cases} \quad (5.37)$$

By the results in §5.1, the sequence $\{u_n\}$ is well defined in $B(G)$, and

$$u_n(x) = -(1/2) \int_{|x|^2}^{s(x)} f(x, r, u_{n-1}(x, r)) dr + \phi(x, s(x))$$

where we write

$$f(x, t) = f(h_1(x), \dots, h_k(x), |x|^2, t) = f(x, |x|^2, t),$$

$$\phi(x) = \phi(h_1(x), \dots, h_k(x), |x|^2) = \phi(x, |x|^2).$$

Then we have

$$|u_{n+1}(x) - u_n(x)| \leq (1/2)c(s(x) - |x|^2) \|u_n - u_{n-1}\|_0.$$

Let $M = \max_{x \in G}(s(x) - |x|^2)$. Then we get

$$\|u_{n+1} - u_n\|_0 \leq (1/2)cM \|u_n - u_{n-1}\|_0. \quad (5.38)$$

This shows that the series $\sum_{k=0}^n (u_{k+1}(x) - u_k(x))$ converges uniformly by the condition in the *theorem*

$$cM < 2. \quad (5.39)$$

Hence, in this case, there exists a $u \in C_b(G)$ s.t.

$$\lim_{n \rightarrow \infty} u_n(x) = u(x) \text{ uniformly in } G.$$

Hence $f(x, u_{n-1}(x))$ converges uniformly to $f(x, u(x))$ since f is continuous.

Thus $Lu_n(x)$ converges uniformly to $f(x, u(x))$. Hence $u \in A(G)$ and

$$Lu(x) = f(x, u(x)) \text{ in } G.$$

Clearly, on ∂G , $u = \phi$. Hence $u \in A(G)$ is the solution of problem (5.35).

Moreover, if $f = f(h(x), |x|^2, t)$, then $u_n(x) = u_n(h(x), s(x), |x|^2)$ where u_n , $n = 1, 2, \dots$, are defined by (5.37). Hence the limit of u_n must be a fundamental functional. Q.E.D.

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