LEIBNIZ'S PRINCIPLE OF THE IDENTITY OF INDISCERNIBLES:

SYMMETRYAND THE RELATIVITY OF IDENTITY

SHELBY BERTRAND

UNIVERSITY OF OTTAWA

FACULTY OF ARTS

DEPARTMENT OF PHILOSOPHY

THESIS SUBMITTED TO THE UNIVERSITY OF OTTAWA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE MASTER OF ARTS IN PHILOSOPHY WITH THESIS

SUPERVISOR:

DAVID HYDER

TABLE OF CONTENTS

Abstract.....iii
Chapter 1: Leibniz and the PII.....1
§1: The Principle and Numerical Identity.....1
§2: The Principle Applied to Space and Time; Leibniz's *Specimen Dynamicum* and His Debate with Clarke.....11
§3: Objections to Leibniz's Application of the Principle: Counterexamples with Space and Time.....20
Chapter 2: Symmetry.....34
§1: Multiple Indiscernibles as the Symmetrical: Kant, Black, and Wittgenstein.....34
§2: Symmetry in Multiple Perspectives: Frege's "Sense and Reference".....47
Chapter 3: Symmetry and Reality.....54
§1: Klein and Chester: Revisiting Dimensional Motions, Observation and Symmetry.....54
Chapter 4: Conclusion.....63
BIBLIOGRAPHY.....67

Abstract

This thesis examines the relationship between Leibniz's Principle of Identity of Indiscernibles and symmetry. In his 1717 correspondence with Samuel Clarke, Leibniz argued that "There is no such thing as a pair of individuals that are indiscernible from each other" (Leibniz 16). In other words, any objects sharing all their properties are in fact one and the same object. This is Leibniz's Principle of Identity of Indiscernibles (the "PII"). The principle and its converse Leibniz's Law express a conditional relationship between the identity of an object and its properties. Our investigation will use applications of Leibniz's principles from the history of philosophy to examine this relationship and we'll find that imperfect applications result in either perfect qualitative identity between multiple objects ("multiple indiscernibles"), imperfect qualitative identity between multiple objects ("incongruent counterparts"), or, finally, a relative identity between two facets of one object (a "singular discernible"). My project will also trace the historical thread leading from Leibniz to the development of symmetry groups in mathematics. Leibniz's principles are embedded in science's ability to distinguish the objective from the subjective, owing to their usefulness discerning an object's intrinsic properties (properties belonging to the object itself) from extrinsic properties (properties based in relations the object is in with other objects). Symmetry is the relativity of identity, and the PII is an exploratory instrument illuminating this relationship: it injects structure into investigations of identity, but also affords the opportunity to capture pre-existing convictions about identity a thinker brings to the application.

Chapter 1: Leibniz and the PII

§1: The Principle and Numerical Identity

This chapter introduces the Principle of Identity of Indiscernibles, analyses its appearance in Leibniz's correspondence with Clarke, and considers its role in philosophy more broadly. It will largely be examined here, however, as a logical principle, independent of any one historical application. This affords a chance to discuss the relationship to numerical identity as defined by the PII, and the species of identity that arise from modifying the principle. Numerical, qualitative, and relative identity will be central to our investigation of Leibniz's PII and its relationship to symmetry.

Central to our investigation is the relationship of the PII to the concept of identity. Formal expressions of the principle lend themselves better to an examination of this, as they can be broken down into component parts. Thus the formal version of the principle will figure prominently in this section, and an in-depth analysis of an informal application will be reserved for our next section. However, beginning with a very brief look at an application of the principle in an informal context will acquaint readers with its basic structure, better preparing them for the analysis of the formal principle to follow.

In 1716, Leibniz wrote to Samuel Clarke "There is no such thing as a pair of individuals that are indiscernible from each other" (Leibniz 16). This is the Principle of Identity of Indiscernibles (from here on, the "PII"), which appeared repeatedly in Leibniz's works, in various forms. One of Leibniz's most important applications of the PII occurs in his famous debate with Samuel Clarke. Clarke was a Newtonian who exchanged letters with Leibniz from 1715-1716 (Bennett 1). Clarke advocated for Newton's position that space and time are absolute,

or exist in and of themselves apart from objects and events that they contain. Leibniz responded that space and time exist only as relations amongst bodies. Leibniz's argument for this relativist approach hinges on the PII. He argues that the Newtonian conviction that space and time are homogeneous leads to the possibility of multiple indiscernibles, and thus to a conflict with the PII itself.

Leibniz's response begins by asking us to consider a universe that had been created slightly to the left than, or years earlier than, our actual universe. This universe, Leibniz says, would be indiscernible from our actual universe. Observers embedded in these universes, who can observe only relative positions and times, would not find any difference between them. They share all properties. There would be no way for an observer to tell which universe is the one they live in. Thus, the PII tells us that the two hypothetical universes are in fact one and the same. The homogeneity of the two contending backdrops of real space and time means there are no discernibly different properties between them. Leibniz's God is a rational one, and these two backdrops present no genuine choice for Him to make. Hence the "two" contending backdrops are in fact one, and absolute space and time are not real (Leibniz 17-18).

This informal appearance of the PII deserves to be properly examined in a section of its own (see Chapter 1.2). In our present section, an overview of the episode suffices to introduce us to the PII as it appears informally. That said, we also need to become acquainted with a formal appearance of the PII. Long after Leibniz's lifetime, the PII was given a logical formulation by Bertrand Russell and A.N. Whitehead in *Principia Mathematica*. In that work, the PII serves to *define* identity:

> The propositional function '*x* is identical with *y*' will be written '*x=y'*. We shall find that this use of the sign of equality covers all of the common uses of equality that occur in mathematics. The definition is as follows:
>
> *13-01. x=y. =: (Φ):Φ!x. ⊃ .Φ!y Df.
>
> This definition states that *x* and *y* are to be called identical when every predicative function satisfied by *x* is also satisfied by *y* (Russell and Whitehead 176).

As a definition, Russell's formulation involves a biconditional. However, in our project, for the sake of precision, we will often treat the PII and its converse, Leibniz's Law, separately. What's more, seeing as most logicians today do not use the symbolism of *Principia*, Russell's and Whitehead's exact version is substituted for something more contemporary and familiar. The PII may be rendered thus:

$$\forall\Phi(\Phi x \leftrightarrow \Phi y) \rightarrow x = y$$

That is, if for all predicates Φ , x is Φ if and only if y is so as well, then x is y.

And Leibniz's Law, in turn, appears as

$$x = y \rightarrow \forall\Phi(\Phi x \leftrightarrow \Phi y)$$

In syntactic contexts, the principle authorises the inter-substitution of symbols. In other words, as a rule of inference, should we prove biconditionality between the properties of x and y, then we can infer that one can be substituted for the other. Leibniz's application taught us that applying the principle straightforwardly results in a strong conclusion–that "two" things were in fact one and the same. Verbalising our formal expression, $\forall\Phi(\Phi x \leftrightarrow \Phi y) \rightarrow x = y$, results in the following: "If, for any property Φ, x is Φ if and only if y is also Φ , then x and y must be the same". In other words, two objects sharing each and every one of their properties are not in fact

two objects at all; they are one. Applications of the PII with this strong conclusion are applications which capture *numerical identity*.

Numerical identity has sometimes been defined as that relation which everything has with itself and no other thing. It is also notoriously difficult to verbalise in a non-circular way (Noonan §2). One indication of the difficulty of articulating numerical identity is how often the PII serves to define numerical identity, instead of the latter informing us on the former. Black points out that for the consequent to follow, the antecedent must already quantify over the properties of the self-identity of the objects to themselves and the identity or difference of themselves and to each other (Black 153-5). Indeed, many philosophers define numerical identity as that relation which satisfies the PII, or its converse, Leibniz's Law (Noonan §2). This is likely because of the behaviour of the principles when either weakened or invalidated. When weakened, the properties captured by the principle are precisely those that are non-numeric. Thus the concept of numerical identity can still come into relief by virtue of the principle.

Weakening the PII means interpreting the antecedent to not quantify over the objects' self-identity or difference (Noonan §2). The remaining properties for the antecedent to quantify over are what philosophers call *qualitative* properties of the object. Thus it's numerical identity, precisely, that is compromised should the PII be weakened, and the remaining identity captured by the principle is called *qualitative identity.* There also is a name for objects that satisfy this weakened version of the PII. When we find *two* objects that have all non-numerical properties in common, they are called *multiple indiscernibles*–multiple objects which if considered apart from each other could not be told apart (Jorati 926).

Alternatively, when the principle fails on its converse, objects can retain numerical identity despite not sharing every single property. Consider a statue and lump of bronze. Even though the lump of bronze eventually becomes a statue, and even if one supposes the statue is still technically a lump, their properties differ historically. These (numerically) identical things do not share all properties. Our intuition tells us that the statue and lump are somehow *one and the same,* and yet the consequent of Leibniz's Law fails. What type of identity, if any, holds between our statue and lump? Some philosophers say it is *relative identity.*

On this view, identity claims are relative to *sortal terms*, that is to say common nouns such as "tree", "planet", or "tiger". So adjusted, Leibniz's Law prescribes that if two objects *a* and *b* are the same *L*, and are thus identical, and *a* is an *S*, then they must also both be *S.* One and the same lump must also be one and the same statue. But, arguably, they are not. While some may argue the statue remains a lump of sorts, the lump was not a statue. Thus a numerically identical pair of objects need not share all their properties.

My investigation will show that the relationships between Leibniz's Principle and the varieties of identity discussed above amount to the following thesis: symmetry is the relativity of identity. Already clear from the above is that when we call objects multiple indiscernibles, incongruent counterparts, or singular discernibles, we are simultaneously making a claim about their identity. Less clear is that behind these terms is an application of the PII or Leibniz's Law, and that the speaker has therefore granted priority to preestablished convictions about either the identity of the object(s) or their properties.

Newton, Euler, and Clarke all argue for multiple indiscernible parts of space. Newton and Euler (discussed in Chapter 1.2) and arrive at their conclusions by applying Leibniz's principles

to the behaviour of objects, or objects in motion. Newton's argument in his *Principia* argues for absolute space by reductio ad absurdum, or deriving a contradiction from the assumption of relative space. Identity does not in fact hold between a bucket of water that is spinning and one that is not, amounting to a failure of Leibniz's Law. Euler develops an argument in his 1748/50 *Reflections on Space and Time*, from which two applications of Leibniz's Principles may be extracted: one using the PII, indirectly testifying to multiple indiscernibles by drawing numeric identity between our world and one that is Newtonian, and one using Leibniz's Law that testifies to a difference between our world and a Leibnizian one.

Clarke's argument does not directly involve the PII, as qualitatively identical pieces of space and time are presupposed to exist. In other words, multiple indiscernibles figure in the argument as an assumption. Clarke's argument nonetheless contrasts not only with Leibniz's treatment of space and time, but also with his refusal to admit multiple indiscernibles into his universe. Leibniz does not believe God would create two truly identical things, and that we can safely infer from the homogeneity of space *as it appears to us* that "two" pieces of space and time actually have no properties that are different between them. This, according to the PII, makes space and time relative, which point is addressed later by both Euler and Kant.

Kant's argument in *On the First Ground of Distinction of Regions of Space* involves a broader argument for absolute space, employing the PII in a sub-argument. The application supposedly proves the existence of a property differing between two hands, using modus tollens on the PII to straightforwardly deny identity to the hands and then conclude that a property must differ between them. Hence a pair of hands are incongruent counterparts. Incongruent counterparts have *imperfect* qualitative identity. According to Kant, they differ on intrinsic properties, not just extrinsic properties. A later thought-experiment due to Max Black will be

discussed in this chapter to highlight exactly this, since the difference between the spheres is purely numeric. Two perfect spheres, like two parts of space or time, differ in a way two hands do not. Two perfect spheres are like Clarke, Newton and Euler's parts of space and time in that they have perfect qualitative identity. If they could differ, it would only be through extrinsic, or relational, properties. I also in this chapter described the "shock" of Ernst Mach observing the regular and yet seemingly arbitrary swing of a needle above a current. Leibniz's PII and PSR, which eliminate substantial *positions* in space and time for him, predict the needle should be torn between two identical options. But the needle swings. It swings because it seems the universe, as Hermann Weyl says, "contains a screw" (Weyl 19).

The behaviour of objects in the world, as Newton and Euler sought to demonstrate with their thought experiments, betrays differences that evade perception and the description of a static object. For this reason, incongruent counterparts naturally invite questions about the role of observation in questions of identity. Gottlob Frege investigated exactly this question at the turn of the twentieth century. His "Sense and Reference" argued that identity statements are not merely about referents, but involve "senses". Abstract noun clauses fail Leibniz's Law, or the *Principle of Inter-Substitutability Salva Veritate.* Senses can share a referent without sharing all their properties. They are also subjective enough to differ amongst speakers but objective enough to be shared and thus facilitate communication.

In Chapter 2.2, I discuss Black's treatment of the connection between singular discernibles and symmetry. Although dealing directly with perfect multiple indiscernibles, he shows that the identity between two spheres hinges on the symmetrical arrangement. Where we see the connection to Frege's argument is Speaker A's insistence that we cannot rely on names to make our spheres two instead of one until the act of naming actually happens, until an observer

actually arrives at an asymmetric point in this impoverished universe and grants the spheres differing properties, we cannot even refer to a specific sphere (Black 157). The potential extrinsic properties that our sphere would differ by are not, he insists, allowed to bear on questions of identity until they're real. Black was himself responding to Wittgenstein's discussion of Kant's hands, which is considered in 2.1. According to Wittgenstein, the differing properties of leftness or rightness dissolves as soon as you invert a hand through four dimensions like a glove. Counterintuitive as it is, Wittgenstein is right: our visualizations of what Wittgenstein describes are 3D representations of something happening in 4D. The closest we ever get to actually performing such a transformation is to perform it computationally, with a matrix of Cartesian points, and then create from that a projection that allows us to get an *idea* of what would happen to the hand. However, Speaker A would say that until the fourth dimension is somehow accessible to us, this is a moot point.

In the work of Felix Klein (Chapter 3), we find an explicit articulation of what our investigations have indicated: weakening the PII or Leibniz's Law can serve as a method for isolating structure. Klein's *Erlangen Program* sought a unifying model of geometry that could understand Euclidean and Projective geometries as variations on a theme. Transformations performed on shapes either preserve or change the shape, based on the principle group of that geometry.

This alone could sound like the transformations, the groups, swing a gavel on questions of a shape's identity: a geometry's principal group will be exactly those transformations that preserve the identity of a shape (formally: the PII). However, Klein admits that projective geometry was born in part due to some people deciding *after* a transformation that a shape's identity, in fact, has not been changed (formally: Leibniz's Law). This biconditional relationship

of a shape's identity and the changes it survives, along with Klein's employment of algebra in this investigation, mean that identity is relative. It is relative to whatever principal group you anchor your convictions about a shape to. Symmetry is the relativity of identity and is encapsulated by Leibniz's principles.

The findings of my project about identity regarding Klein hold no less for the authors and examples discussed previously. Klein's predecessors do not always discuss or apply Leibniz's principle in the computational language of Klein's work, nor are their applications in formal language of any kind. Additionally, in my project I extract presentations of their arguments that have more formality than the presentation of the authors but are not purely computational. This is not only for the purposes of concision, but also in order to highlight the most noteworthy mechanical features of the authors' reasoning. Nonetheless, my presentations allow for Klein's insight, or that our conclusions about an object's identity reveal prior convictions about exactly that, to be seen operating as a general principle in their explorations. This is due to Leibniz's principle taking the form of a conditional statement, and conditional statements affording two forms of treatment. Allow me to demonstrate, not with the argument of any particular author but with several example proofs illustrating this behaviour of Leibniz's principles as conditionals.

The conditional statement can be featured as an assumption in a reductio ad absurdum proof. In other words, the author who argues in this form is challenging the principle from the outset, showing that a contradiction follows from it.

Example 1)

$\forall\Phi(\varphi x \leftrightarrow \varphi y) \rightarrow x = y$ (assumption)

$x=y$

But $x \neq y$

(!)

$\sim(\forall\Phi(\varphi x \leftrightarrow \varphi y) \rightarrow x = y)$

$\forall\Phi(\varphi x \leftrightarrow \varphi y) \wedge x \neq y$

Example 2)

$x = y \rightarrow \forall\Phi(\varphi x \leftrightarrow \varphi y)$ (assumption)

$\forall\Phi(\varphi x \leftrightarrow \varphi y)$

But $\sim\forall\Phi(\varphi x \leftrightarrow \varphi y)$

(!)

$\sim (x = y \rightarrow \forall\Phi(\varphi x \leftrightarrow \varphi y))$

So $x = y \wedge \exists x((\varphi x \wedge \sim\varphi y) \vee (\sim\varphi x \wedge \varphi y))$

The form of argument in examples one and two results in the whole conditional statement being negated. For the PII, the resultant conjunction denies identity between two objects, but asserts that all properties are shared. For Leibniz's Law, the resultant conjunction results asserts identity between the two objects, but denies shared properties. In this form of argument, the author treats the principle as an assumption in part because they grant priority to whatever features of their investigation amounted to a contradiction. In this example, I used assertions or denials of the consequent in order to contrast with examples three and four.

Examples three and four involve applications of modus tollens. Unlike a reductio, performing modus tollens on a conditional statement is an application that honours the statement as true, rather than setting out to challenge it. In such a case, denying the consequent allows us to infer a negated antecedent.

Example 3)

$\forall\Phi(\varphi x \leftrightarrow \varphi y) \rightarrow x = y$

$x \neq y$

So $\exists x((\varphi x \wedge \sim\varphi y) \vee (\sim\varphi x \wedge \varphi y))$

Example 4)

$x = y \rightarrow \forall\Phi(\varphi x \leftrightarrow \varphi y)$

$\sim(\forall\Phi(\varphi x \leftrightarrow \varphi y))$ (or : $\exists x((\varphi x \wedge \sim\varphi y) \vee (\sim\varphi x \wedge \varphi y))$)

So $x \neq y$

Performing modus tollens on the PII denies identity between two objects and results in denying shared properties between them. Performing modus tollens on Leibniz's Law denies re shared properties and results in denying identity between them. The author who does this is treating Leibniz's principle(s) as true. Noteworthy is that the author of, say, example one and example three both assert the non-identity of the objects, but reach opposite conclusions. Similarly, the authors of examples two and four both deny shared properties between the two objects, but reach opposite conclusions. This is because their arguments are shaped around prior-held convictions about identity and the properties of objects. The result is author one allowing for a world with multiple indiscernibles, and the other not. The author who holds Leibniz's Law to be true reaches the same conclusion as the person who interrogates the PII, and vice versa. Klein would tell us that both authors can be correct.

§2: The Principle Applied to Space and Time; Leibniz's *Specimen Dynamicum* and His Debate with Clarke

Leibniz applied his principle to space and time during the *Leibniz and Clarke Correspondence* and in his *Specimen Dynamicum*. As intangible things, space and time are elusive and philosophically controversial. Principles like the PII, and other concepts from

philosophy's interrogation of identity, are especially incisive for such an area. Thus, this chapter contains an in-depth analysis of these episodes in Leibniz's philosophy.

Leibniz argued that the energy in the universe is the same amount it was at creation by virtue of being the very same energy injected at that time. Leibniz needs to argue this because for him, reality ultimately resides in monads, and monads harbour vis viva, the force that marks the presence of real motion and the reality of the individual thing. In turn, Leibniz needs to argue that said "energy" in the universe cannot be mere locomotion, or the transit of objects from one location to another in time. There cannot be real motion in locomotion for Leibniz, because space and time themselves are not real. This is one reason why, in his debate with Clarke, he applies the PII to space and time in an argument for their unreality.

Leibniz's physics differed from that of Descartes on the question of the "quantity of motion", which Leibniz identified with his "*vis viva*" or living force. We must appreciate the way in which motion involving *force* is distinct from plain *locomotion*, or, as Leibniz refers to it, *quantity of motion*. Leibniz famously criticised what was, in his view, a conflation of these two things on the part of the Cartesians, who considered a measure of one as a measure of the other (Leibniz, *A Brief Demonstration*... 297-301). Such a conflation was no small matter for Leibniz. According to Carolyn Iltis, "what is real in motion [for Leibniz] is force, a momentary state which carries with it a striving toward a future state" (Iltis 35). Crucially, "These units of primitive force can neither be created nor destroyed naturally, and all must begin simultaneously and be annihilated at once" (Iltis 34). Leibniz's *vis viva* directly opposes the Cartesian conviction that the essence of the world's things resides in inert extension. For Leibniz, "What's real in the universe is activity; the essence of substance is action, not extension, as Descartes had insisted" (Iltis 34). Leibniz located the primary residence of this force in *monads* (Iltis 34). Hence

Leibniz's passionate defense of creation having the very same energy present since God made everything. In the dialogue, this translates to Leibniz defending a God that doesn't intervene in creation, save for miracles.

Descartes mathematically represented his "quantity of motion" as m|v|, in which "m" represents mass and "v" velocity, and velocity is always treated as a positive integer. Leibniz represented his vis viva as mv^2, in which velocity is a vector quantity whose direction is variable (Iltis 22). Mathematically, this force would be captured as absolute or positive quantities in the impact of elastic bodies, which will never be taken as null or negative (Iltis 33). In his philosophical system, it was crucial that there be a force present in moving objects that speaks to "[a] primitive force or a striving toward change; [the] innermost nature of a body" (Iltis 34). To demand that this quantity be conserved was to demand that, in a system where objects are exchanging energy and the net total of energy is unchanging, a collision or exchange of energy will never leave the system with a loss or growth in net energy, nor any one monad with a quantity of force below or equal to zero (*Philosophical Essays 125, New Essays Concerning…* 670).

In other words, force anchors reality in motion for Leibniz's philosophy because of his foundational convictions about the locus of reality. By contrast, mere locomotion doesn't satisfy those demands. Leibniz in the *Specimen Dynamicum* refers to locomotion as "the phenomena", arguing that "…from the phenomena one cannot determine with mathematical rigour what is at rest, or the amount of motion with which some body is moved" (*Philosophical Essays* 125). "Phenomena" here clearly refers to those parts of the interaction which are *observable*. He adds that this holds for circular motion as well, "though it appeared otherwise to Isaac Newton…" (*Philosophical Essays* 125).

This remark about Newton and his use of "phenomena" is noteworthy, for it is in this context that he uses the term "discern", observing that "[al]though [Newton] said many superb things about motion, he thought that, with the help of circular motion, he could discern which subject contains motion from centrifugal force, something with which I could not agree…" (*Philosophical Essays* 125). Thus there are no exceptions. Mathematics only describes the transfer of an object from one place to another and from this phenomenon we cannot discern which object has real motion:

> …nevertheless, we can, with good reason, attribute true motion to that subject, which would result in the simplest hypothesis most suitable for explaining the phenomena. For the rest, it is enough for practical purposes for us to investigate not the subject of motion as much as the relative changes of things with respect to one another, since there is no fixed point in the universe (*Philosophical Essays* 125).

Here Leibniz suggests that the relativity of motion is justified by the unreality of absolute place, which he demonstrates in his debate with Clarke. As it stands in the *Specimen Dynamicum*, the consequence of that demonstration alone supports the inference, at least intuitively, to relative space only. That is, if space is unreal and there is no fixed point in relation to which we might describe an object's location, statements regarding an object's location must involve other objects. In consequence, statements regarding moving objects are always relative to other objects. Elsewhere in the same text, however, Leibniz justifies relative motion with the lack of discernment itself. In other words, he applies the PII. Leibniz says:

> However, it is certain that if some people were carried on a large boat (enclosed, if you please, or, at least, set up in such a way that things outside of the boat could not be seen by the passengers), then, however great the speed of the boat might

> be, as long as it moved peacefully and evenly, they would have no criterion for discerning (on the basis of that which happens on the boat, of course) whether the boat was at rest or in motion, even if they played vigorously with a ball or produce other motions (Leibniz *Philosophical Essays* 137).

It may be fair to say that Leibniz is not so much applying the PII as he is referring to a previous application by Galileo in his *Dialogue Concerning the Two Chief World Systems.* The original passage containing this thought experiment will be examined in 2.1, but for now we may summarize it as follows:

1) If two objects share all properties with each other, or are indiscernible, they are identical.

2) In the scenario in which the boat is moving, objects x,y, and z move with relative speeds a, b, and c, and have behaviours p, q, and r.

3) In the scenario in which the boat is at rest, objects x,y, and z move with relative speeds a, b, and c, and have behaviours p, q, and r.

4) Passengers cannot discern from the locomotion of the objects whether the boat moves or rests (2,3)

Conclusion) The two scenarios are actually one (1,4).

Thus, whether the boat moves or whether it doesn't move cannot be determined by the passengers, for the one supposition offers as much (or as little) as the other. Thus locomotion, the mere transit of objects over locations in space and time, cannot reveal to us where real force resides.

Clarke acknowledges exactly this, arguing in his fourth letter that "If a ship is sailing smoothly enough, a man shut up in the cabin can't tell whether it is moving or not; but that doesn't alter the fact that its moving and its not moving are not the same state!" (Clarke 24). But Clarke is unconvinced that the empirical underdetermination of locomotion poses any real threat to locomotion harbouring real motion. For he objects that "if the ship suddenly stopped, that would yield other real effects; as would a sudden stopping of an indiscernible motion of the universe. No answer to this argument has ever been given from Leibniz's side. Newton emphasises this at length in his Mathematical Principles, (Definition 8)..." (Clarke 24). He then points to non-inertial motion as evidence of this and refers to an argument made by Newton that will be analysed in the next section. For now it suffices to note the language Clarke uses in describing Newton's point: "...he shows the difference between real motion (a body's being moved from one part of space to another) and relative motion (bodies merely undergoing a change of the order or situation they have with respect to one other) (Clarke 24). Locomotion for Clarke is indeed the transit of objects over space and time, and is indeed "real motion" (Newton 78, 80). Leibniz will have to go beyond the empirical underdetermination of locomotion to convince Clarke that the constituent parts of "real motion" as he's defined it here, that is the parts of space (and, we presume, time), are not real.

Leibniz's application in his debate with Clarke relies no less on God in its premises. All of creation, for Leibniz, was stamped by God during creation with the *Principle of Sufficient Reason* (the "PSR"). It holds that "for anything that is the case there's a reason why it should be so rather than otherwise" (Leibniz 6). God does not make arbitrary decisions: for Leibniz, God's choosing one outcome over another indicates he perceived an inherent difference in the good of both outcomes. Leibniz anchors the glory of God in His wisdom and design. (Technically

speaking, matters of "design" concern God's decision-making during creation, which accounts for the moment of creation being the setting in which Leibniz's argument will take place).

Clarke disagrees: to trace every decision God makes to some inherent difference in one candidate outcome versus another is to disempower God. God's sheer will means that he can make decisions even in the absence of such circumstances. Clarke diagnoses the recent popularity of attributing materiality to the soul and God to materialist philosophers neglecting the teachings of mathematical principles. Leibniz counters that Clarke's point is a moot one: to bridge from mathematics to "natural philosophy" (studying objects of creation) requires a metaphysical principle, his PSR. He illustrates with the discovery of Archimedes, who inferred that there must exist some difference between two otherwise equal weights to tip a scale (Leibniz 4). To this Clarke responds that while he generally accepts Leibniz's PSR, he believes that the sufficient reason involved in a decision made by God need not be an inherent trait of some candidate object of creation, but rather the will of God. He goes on to illustrate with a highly granular picture of God's decision-making process at creation. It is here that space and time officially enter the debate:

> For an example, consider two material things (particles or complexes) that are exactly alike and are of course in different places. Why are they situated as they are rather than the other way around? Why is x here and y there, rather than y here and x there? So far as bits of matter are concerned, one place is the same as another, so that if the locations of x and y had been switched it would have been exactly the same thing. So the only reason there can be for the two things to be where they are rather than vice versa is the mere will of God. If God couldn't choose without a predetermining cause, any more than a balance can move without an imbalance of weights, this would tend to take away all power of choosing, and to introduce fatality (Clarke 6).

Clarke's argument can be broken down as follows:

1) God does nothing without a sufficient reason

2) Two qualitatively identical things in different places must have a reason for being in different places (1)

3) The reason must either consist in a difference present in the objects themselves or the work of God's will.

4) But there's no difference found in the objects.

Conclusion) God's will must be the reason. (2,3,4)

Thus Leibniz's conviction driving his side of the argument is that thought or decision-making, through which God expresses his wisdom in creation, cannot be undetermined. He thus agrees with Clarke's view given above as the dichotomy in premise two: For God to choose between two truly identical scenarios is for God to not *choose* at all. Choice becomes meaningless, there is no difference involved, and a rational God cannot act in such a way (Leibniz 16).

Leibniz identifies a hidden premise in the argument: Clarke assumes that space is *absolute* (Leibniz 9). That is, space exists as an entity with a being that is independent from the objects that it contains (DiSalle 7, 37). This is what makes the choice God faces, where to put two qualitatively identical objects, a dilemma with any reality for Clarke: there really are two distinct locations with their own existence awaiting an object. In response, Leibniz invokes his

PSR, the PII and the homogeneity of space and time, arguing that there is no discernible difference between our world and a version of it created earlier than ours, or one created in a different part of space. So, were one world created millions of years sooner than another, or at a different location, there is nothing about space itself, time itself, or created objects that could reveal to someone which hypothetical universe they live in.

Thus, Leibniz concludes that space and time are not absolute. As he outlines in his third letter, it is actually the relations between things that gave rise to what we call "space" and "time", since "Space is nothing but an order or set of relations among bodies*"* (Leibniz 9), and "...times, considered without the things or events, are nothing at all, and that they consist only in the successive order of things and events" (Leibniz 10). Space is nothing over and above the relationships between bodies, and time is nothing over and above the relationships between events. They have no properties independently of things and events. Newton's multiple and indiscernible parts of space and time awaiting contents are, in consequence, "an impossible fiction" (Leibniz 16). More formally, Leibniz argues as follows:

1. There is no such thing as two objects that are indiscernible.
2. Space and time are homogeneous, meaning one part has no discernibly different properties from another.
3. The two candidate places or times for creation are in fact one and the same thing (1,2)
4. God does nothing without a sufficient reason.
5. God would have had no sufficient reason to put the world in a different place or time (3,4)
6. Different candidate times or places did not pre-exist the things and events created by God (5)

7) Space and time just are the relationships between events or things (6)

Leibniz also takes issue with Clarke's suggestion that God would ever have even made "two material things (particles or complexes) that are exactly alike". Leibniz does not defend this claim in his debate with Clarke, simply insisting instead that if one looks for two such objects, one won't find any. "Two drops of water or milk will turn out to be distinguishable from each other when viewed with a microscope" (Leibniz 16). Thus, for Leibniz, that God would create and lay out many qualitatively identical yet distinct parts of space would be irrational. The homogeneity of space, underpinning qualitative identity amongst whatever different "parts" we might consider, amounts to its unreality. For, two hypothetical parts of space and time, by virtue of being indiscernible from each other, are "exactly the same" in a far stronger sense. They are numerically identical.

§3: Objections to Leibniz's Application of the Principle: Counterexamples with Space and Time

In the Leibniz and Clarke debate we see a straightforward application of Leibniz's principle of identity of indiscernibles. As mentioned by way of introduction, such straightforward applications of the PII count as such precisely because we have not modified the antecedent of the principle. That is, the properties over which our universal quantifier ranges include numerical ones. Thus these applications entail numerical identity. Two hypothetical worlds with different situations in space and time are in fact one and the same world since, in turn, the several candidate sections of space and time in which creation could occur are homogenous. Since God has no sufficient reason to decide on one over the other, space and time cannot exist. Space and time are nothing over and above the order of bodies and events, what the

realist conceives as their “contents” . The same arrangement cannot be placed in a different region of space or time: the arrangement makes the region, there cannot *be* different regions unless there are different things.

As we saw, weakenings of the PII in a sense deny it, for in such cases the antecedent is satisfied while the consequent fails (that is, $x \neq y$). We allow two things to share some relevant subset of their properties while being numerically distinct, thereby introducing multiple indiscernibles. If they exist, both Clarke's qualitatively identical yet separate parts of space and time and his two material things exactly alike are such multiple indiscernibles. They violate the strong version of the principle, while satisfying the weakened version (in which our antecedent only quantifies over non-numeric properties to start with). This version of the PII captures pairs of numerically distinct objects which are relatively identitical.

Clarke was not the only figure in the history of physics to have challenged Leibniz’s application of the PII in denying the existence of absolute space and time. Two other figures to have done this are Newton (albeit indirectly) and Euler. In 1687, Newton published his *Principia*, and offered a now-famous thought experiment featuring centrifugal force. Although written before Leibniz debated Clarke, his argument is structurally parallel to Euler’s stone-and-water argument. It too challenges the relativity Leibniz would put forward to Clarke. Leonhard Euler challenged Leibniz’s application of the PII, and its resulting consequences for our understanding of space and time, in his *Reflections on Space and Time* (1750), with an argument employing the law of inertia.

In *Principia,* Newton asks us to imagine a vessel (a bucket) with water in it, hanging by a rope that has been tightly wound. At first, the bucket and water inside are both relatively at rest

with regards to one another. However, if the tension of the rope is released and whirls the bucket around, the water will remain still within the bucket and there will therefore be relative motion between the bucket and water. Eventually, the water will also begin to move along with the bucket, while it develops a concavity in its surface. If the water travels at the same speed as the bucket though, sections of the bucket-walls and body of water will travel in alignment with each other. So: there is no relative motion between them.

> ...but the vessel; by gradually communicating its motion to the water, will make it begin sensibly to revolve, and recede by little and little from the middle, and ascend to the sides of the vessel, forming itself into a concave figure (as I have experienced), and the swifter the motion becomes, the higher will the water rise, till at last, performing its revolutions in the same times with the vessel, it becomes relatively at rest in it (Newton 81).

This is equivalent to the first stage examined, the stage of relative rest. But the concave-water-dip prevents these two stages from being called identical. Hence, Newton says "And therefore this endeavour [of the water to recede from its axis of circular motion] does not depend upon any translation of the water in respect of the ambient bodies, nor can true circular motion be defined by such translation" (Newton §79f).

Let's examine a more formal version of Newton's argument:

1) $x = y \rightarrow \forall \Phi(\Phi x \leftrightarrow \Phi y)$

2) An object's relative place is its location in relation to other objects in the world.

3) An object is at relative rest when its position in relation to other objects does not change

(2)

4) The bucket x in stage 1 is at rest relative to the water (3)

5) The bucket x in stage 3 is at rest relative to the water (3)

6) If motion is relative, there is no difference between buckets 1 and 3 (1, 4, 5)

7) Thus, $\exists\Phi(\Phi x \wedge \sim\Phi y)$

8) So $x \neq y$ (1,7)

---

13) So real motion is not relative (6,8)

Thus Newton illustrates that with a successful application of Leibniz's Law, involving properties of relative place between the bucket and water, that buckets x and y would be called identical under Leibniz's view. But then he points out a property that differs between them and, in his view, is apart from the relative positions of the bucket wall and water, namely the concavity. Similarly to what we saw with Euler, establishing the absence of identity happens through denying the consequent with Leibniz's Law.

Leibniz was not persuaded by the bucket argument. As we saw previously, Leibniz responded that he disagrees with Newton's conviction that circular motion can help discern the presence of real motion (in the form of centrifugal force) (*Philosophical Essays* 125). Since we cannot discern true motion merely by observing change of place, Leibniz concluded that it was acceptable to ascribe motion to whatever object offered the simplest and most suitable explanation in the particular situation (*Philosophical Essays* 125). This thesis of Leibniz' is often referred to as the *equivalence of hypotheses.* To explain with a familiar example, it means that Leibniz would consider both the *hypotheses* about whether the ship moves or doesn't move as

equally valid, and decide between them based off of which one, all things considered, offers more explanatory power in the situation (*Philosophical Essays* 131). (Recall that creation having no fixed location in space means that motion as defined by simple change in location is not real period, and so this is not solely a statement about human inability but also the nature of reality). Leibniz believed that the equivalence of hypotheses holds for rectilinear motion (*Philosophical Essays* 136,7). And he also believed that all non-rectilinear motion, like circular motion, is ultimately composed of rectilinear motion (*Philosophical Essays* 135):

> But since rotation also arises only from a combination of rectilinear motions, it follows that if the equivalence of hypotheses is preserved in rectilinear motions, however they might be placed in things, then it will also be preserved in curvilinear motions (*Philosophical Essays* 136,7).

Leibniz's response to the bucket experiment would not consist in denying that force, generally, coincides with presence of real motion. Instead, he denies that pure observation of the changing of locations will reveal where real motion resides, and so and this holds too for the concave dip forming in bucket y's surface. While Leibniz likely would not go so far as to assert the identity of the two buckets, he simply denies centrifugal motion as the presence of absolute motion (and in turn absolute space), because all motion is ultimately rectilinear and therefore relative insofar as mere changes of location (what we can observe!) are concerned. Whether or not this hypothetical rebuttal sufficiently deals with Newton's argument is beyond the scope of our discussion.

As Newton's theory gradually gained the upper hand, mathematical physicists in France and Germany, such as Emilie du Châtelet and Leonhard Euler, took up Newton's arguments in *Principia* anew. Euler argues in this work that the law of inertia proves the existence of absolute

space and time, as we cannot account for the regulation of bodies implied by the law without real space and time (Hyder 255). Despite Euler never naming Leibniz in his *Reflections on Space and Time,* it's quite clear that that is in fact who he's addressing: "I highly doubt that the metaphysicians will risk saying that the equality of spaces must be judged by the equality of the number of monads which fill them…" (Euler §19). Euler speaks of "principles of truth", and this refers to the laws of mechanics, the law of inertia specifically, which Euler takes to be empirically confirmed:

> Far from denying these principles of truth established by mechanics, the metaphysicians try instead to deduce and prove them through their ideas. But they reproach the mathematicians for inappropriately applying these principles to ideas of space and time, which for them are merely imaginary and devoid of all reality (Euler §3).

Euler's presentation of the law of inertia closely follows Newton's *Principia* and runs as follows:

> It is therefore an indisputable truth that a body once at rest will remain perpetually at rest, unless its state is disturbed by some outside force. It is equally certain that a body put into motion will continue to be in motion with the same speed following the same direction, provided that it does not meet any obstacles contrary to the conservation of this state (Euler §3).

Crucial, however, is that when Euler employs the law of inertia in his argument, it is not in a straight-forward fashion. For the purposes of his argument, he deliberately substitutes the Leibnizian notion of relative place into the principle, and it is that version employed as a premise in his argument.

> One will first say that position is nothing more than the relation of a body to other bodies around it. Therefore let us substitute this notion for the idea of position, and we shall be obliged to say that according to this principle a body that is in a certain relation with other bodies surrounding it shall persist in maintaining this relation. That is to say, we must claim that body A surrounded by other bodies B, C, D, E, etc., will try to preserve itself in this same configuration perpetually. And thus when the mathematician says that a body at rest stays in the same position with respect to absolute space, the metaphysician would say that this body retains the same relationship with respect to other bodies in its surroundings (Euler §7).

Euler's full argument utilising this modified law of inertia employs the PII itself. He asks us to imagine a stone immersed in water. We know from experience that should the water begin to move, the stone will *not* move. Yet the law of inertia formulated in terms of Leibnizian relative place predicts that the stone will do just this: it will preserve its place amongst the other bodies (surrounding particles of water) and move with its containing surface.

By contrast, the law of inertia formulated in terms of Newtonian place predicts what we observe: the preserved place of the stone is in absolute space, and it retains that place as the water moves past it (Euler §VIII, IX). Thus, relativists such as Descartes and Leibniz are refuted by experience. The structure present in this argument may be captured as follows:

1. A body at rest in a place shall persist in maintaining its place unless moved by force.

2. The place of the stone, according to a definition of absolute place, is its location in space as a medium sperate from its objects.

3. The place of the stone according to a definition of relative place is its location in relation to other objects in the world.

4. A stone surrounded by particles of water that accelerate will either rest or move.

5. If it rests, it is retaining its position in absolute (Newtonian) space (1,2)

6. If it moves, it is retaining its position in relative (Leibnizian) space (1,3)

7. The stone doesn't move.

8. So the stone rests.

---

9. The stone retains position in absolute, or Newtonian, space (8, 5).

Thus Euler's thought experiment, his argument for the reality of absolute space, takes the form of a constructive dilemma, wherein the final conclusion comes about through a simple modus ponens.

At this level of granularity, the operation of the PII is not visible. Again, absolute space is a structure containing multiple indiscernibles, which are in themselves counterexamples to strong applications of the PII. Recall that for Leibniz, the homogeneity of space means we collapse two hypothetical worlds into one because the "two" expanses of space confer no different properties on the matter that occupies them. A condensed presentation of how the PII technically functioned for Leibniz in his debate with Clarke might be the following:

1. $\forall\Phi(\Phi x \leftrightarrow \Phi y) \rightarrow x = y$
2. If our world x has property φ iff a different hypothetical world y has property φ, then world x and y are one world

3. World x and world y share all their properties.

---

4. So, world x and y are the same world.

The role the PII plays in Euler's argument is similar, in that both compare the actual world we live in with another world, which could either match (be identical to) ours or be different. In this case, the properties being examined to determine identity or difference are not simple arrangements of objects and events placed in space and time at creation–they are the behaviour of those objects in motion. A condensed presentation of one role the PII played in Euler's argument takes this form:

1. $\forall\Phi(\Phi x \leftrightarrow \Phi y) \rightarrow x = y$
2. If our world x has property φ iff Newtonian world y has property φ, then world x and y are one world.
3. World x does in fact have property φ iff world y has property φ.

---

4. So, world x and y are in fact one world.

This is a successful and straightforward application of the PII, one which fulfills the antecedent and consequent of the conditional and collapses "two" things into one, as in Leibniz's uses. It does not, though, *directly prove* the presence of multiple indiscernibles. Multiple indiscernibles are present only insofar as the reader already understands them to be part of the Newtonian system. They do not get explicit mention in the argument above. The argument also doesn't

directly address the world with relative space and time as a contender. There is, however, another argument one could extract from Euler's thought experiment that does this and involves the PII (or at least its converse, Leibniz's Law):

1. $x = y \rightarrow \forall\Phi(\Phi x \leftrightarrow \Phi y)$
2. If our world x and Leibnizian world y are one world, then world x has property φ iff world y has property φ.
3. Our world has property A that the Leibnizian one doesn't.
4. World x and world y do not share all their properties.

---

5. So, world x and y are different worlds.

Thus, the Leibnizian world is eliminated as a possibility by denial of the consequent. We said earlier that the PII might permit multiple indiscernibles should it be weakened, in which case *two* things would share all non-numerical properties. This is not happening above. While we do see a kind of invalidation of a conditional statement, it is not the same conditional statement, nor is it being applied to the subject matter Leibniz directly applied his to. (Note, this point is one that applies to both applications of the PII by Euler shown above). Leibniz did indeed apply the PII to multiple worlds, but those worlds *were just* imagined swaths of space and time, awaiting creation. Thus, had that application failed, the two subjects sharing all properties but remaining two would be regions of space and time. The subjects of the principle for Euler here are non-empty arenas of objects and their behaviours, and thus are not space and time themselves. Whereas the properties in question when Leibniz applied the PII were properties of space in itself, for Euler the properties in question were properties of the world *and the behaviour of the*

objects in it. The application simply amounts to a refutation of the thesis that our world is a Leibnizian one, as we do not see a sharing of properties between them.

Hence multiple indiscernibles emerge for Euler neither through a failure of the PII (as the failure that happens is of Leibniz's Law), nor through any successful "weakened" application that ignores non-numerical properties (as the subject matter of unproblematic application argument here does not refer properly to space and time). They appear for Euler not in the applications of the PII per se, but in the delivery of his thought experiment, in which absolute space is presupposed as a feature of a Newtonian world. On this very point, Euler says

> In mechanics space and position are considered *real*, and on the basis of this principle we maintain that a body at rest in a particular position will stay there perpetually unless it is chased by some outside force. Thus in this case the body will stay in the same position relative to absolute space (§6).

Here, we see that Euler is positing absolute space as the factor that explains the stone's staying put: *the stone retains its position,* as the law of inertia requires, its position being that in absolute space. It's worth briefly noting a possible objection, namely that the force of the water particles hitting it does not hit a threshold that would uproot the rock from its absolute location. Euler addresses this himself:

> But mechanics shows very clearly that the body does not follow the water current so much as it is struck by the water particles, and consequently an outside force puts the body in motion. Therefore without this force the body would remain at rest in flowing water as it does in still water, and thus the body in the conservation of its state of rest does not follow the bodies that immediately surround it. From this it follows that what is called position in mechanics does not allow the

> explanation offered by metaphysics which claims that position is nothing but the relationship of the body with respect to other bodies that surround it (§10).

It should be especially clear following the discussion of §2 that Leibniz's understanding of space and motion not very receptive to this objection. Leibniz would have acknowledged and accepted the role of force in relation to the stone's behaviour without question.

That aside, Euler seems to say something that amounts to an outright misconstrual of the PII: by distinguishing between things that can be known only through reflection and things that can't, he undercuts the very idea of *discernibility* operating in the principle.

> It is true that the senses are not capable of furnishing us with ideas of space and position, and it is only through reflection that we form them. From this metaphysicians conclude that these are merely abstract ideas, similar to the ideas of genus and species, which only exist in our understanding and to which no real object corresponds. But [...] it would be a mistake to support the idea that nothing exists that can only be known through reflection (§14).

This is an aspect of the PII that is easy to forget when considering the logical presentation of it. The antecedent in this version quantifies over *all* properties, whether discerned or not. This distinction between properties that are present versus properties that are merely perceived introduces an epistemological question into the application of Leibniz's principle once it has been weakened to allow multiple indiscernibles.

By contrast, Leibniz did not employ the formal version of the PII in his debate with Clark. There, the application of the PII resulting in relational space and time indeed had discernibility at the forefront. Euler seems to have informally employed the version of the PII that leaves out the matter of human apprehension, the discernment, of those properties. For

Leibniz, human discernibility is at the forefront in that his application asks not simply "what properties are there?", but "what properties can we see?". Whatever properties we see are the properties that are there. Otherwise we couldn't draw an objective conclusion about the object itself. Human perception, discernment, is held in high esteem epistemologically. For Euler, this isn't the case. This may partially explain why his application does not examine two ("two") different swaths of space per se–that they're perceptibly indiscernible wouldn't matter. In this respect he is more similar to Clarke than Leibniz. While Leibniz assumed a human imagining of God's dilemma at creation in his thought experiment really did reflect God's perspective, Clarke felt this was no guarantee. Even if humans may not be able to perceive differences between two candidate locations, God certainly can, and this is what matters. Thus, Euler's application of the PII diverges significantly from Leibniz's.

To recall, Leibniz's application of the PII in his debate with Clarke applies the PII directly to space and time themselves, and produced multiple indiscernibles that way. We saw Euler apply a thought experiment and we uprooted the PII as it functioned in it, but learned that Euler's argument didn't produce multiple indiscernible parts of space and time by a failure of the PII in a direct way, as it was not applied to space and time. Euler's argument applies the law of inertia in such a way so as to neglect nuance in Leibniz's own understanding of motion. The situation mirrors another famous criticism of Leibniz's model of space from Newton, whose bucket experiment also applies the PII to two situations instead of space and time directly, and neglects Leibniz's belief in the equivalence of hypotheses. These criticisms of Leibniz's application not only help us appreciate the role played by multiple indiscernibles in the historical debates concerning space and time, but also the role played by the PII : the behaviour of objects already present in full-fledged creation has invited applications of the PII to clarify the nature of

space and time, even if they don't directly deal with imagined multiple indiscernible parts of space and time themselves.

Chapter 2: Symmetry

§1: Multiple Indiscernibles as the Symmetrical: Kant, Black, and Wittgenstein

Leibniz argued for the non-existence of space and time by applying the PII directly to the multiple indiscernibles that constitute the manifolds of space and time. The homogeneity of space and time licensed this application of the so-called "strong PII", which asserts numerical identity. Euler argued that there *were* such multiple indiscernibles, and that if everything that existed could be known only through reflection, this would undercut the PII. Shortly afterwards, in 1768, Kant wrote his article "On the First Ground of the Distinction of Regions of Space". In it, he defends the existence of absolute space by arguing that only absolute space could account for the existence of right and left. That is to say, he considers cases of partial indiscernibility and relative identity (e.g. between right and left hands) to argue for the existence of a background space.

We saw with Leibniz and Clarke how the homogeneity and intangibility of space and time made discussions of multiple indiscernibles slippery and abstract, with neither side's argument producing an indubitable conclusion. As for Euler, even when features of experience appear to testify to the existence of absolute position, the empirical findings are not enough to identify the multiple and discrete pieces themselves supposedly making up absolute space and time. Kant carefully studied the Leibniz and Clarke debate, as well as Euler's defense of Newtonianism (Janiak, Kant 28). It is no surprise therefore that Kant's application is in fact closest yet to our question, for he shows that for Leibniz to accept a world with multiple indiscernibles is also to accept a world in which there is symmetry. I will speak briefly about the project of Kant's article in general, and then present Kant's argument. We will consider each premise of his argument to carefully trace the inferences made and explore the concepts

involved. I'll then examine how Kant's argument takes the form of an application of the PII, and, finally, what Kant's understanding of space may teach us about Leibniz's original PII application.

Let's first clarify what exactly it means for Kant to argue in this article for absolute space insofar as that concerns "the existence of right and left". Kant distinguishes in this piece between two ways in which objects in the world are arranged. First there are the relative positions of the parts of some extended object, or of space. In keeping with his example of a map of the heavens, such a map could be entirely described in terms of the relations among its objects (Kant 27). In other words, it is possible to completely list which stars and planets are at what angle to each other, and how far away from each other. Second, however, are the "regions" or directions [*Gegende*] towards which this ordering of the parts is oriented, and which we describe intuitively in three-dimensional space as above and below, in front and behind, and right and left. Kant says that our bodies naturally ground our awareness and definition of these fundamental orientations, suggesting that the basic arrangement of the human body is in harmony with the intersecting planes of three-dimensional space (Kant 28-9).

That these two aspects, the raw ordering of parts and the region towards which they are ordered, are separable from each other is supposedly quite clear:

> ...the most complete chart of the heavens, however perfectly I might carry the plan in my mind, would not teach me, from a known region, North say, on which side to look for sunrise, unless, in addition to the positions of the stars in relation to one another, this region were also determined through the position of the plan relative to my hands (Kant 29).

Knowledge of an arrangement of objects, or parts of an object (celestial bodies here), is not decisive unless we also know how that arrangement relates to ultimate regions and in turn how the ultimate regions relate to our bodily ones. To orient oneself, it is not enough to possess a map of celestial bodies (that is, their arrangement). To point to the east, you must also know which of your hands, respectively, is closest to each star, respectively, of an opposing pair on the map. So again, Kant emphasizes in the above passage the special role the body plays in the process. Right and left are necessary to relate to the universe as a whole.

They're necessary on a smaller scale as well. Elsewhere in this article, Kant mentions that activities we do in the world (like writing–changing our orientation to a page of writing makes reading hard or impossible) and even the construction of bodies (lacking perfect symmetry) betray the existence of ultimate regions (Kant 30). Hence, the person wondering where to look for the sunrise uses a map of the stars (mental or physical) to identify a star they can see, and naturally asks themselves, "If my right hand points to *this* star, which way am I facing?". At its broadest level, Kant's argument can be formulated as follows:

1) Space is either absolute, or relative.

2) If it's relative, then either an object's intrinsic or extrinsic properties account for the existence of right and left.

3) Neither intrinsic nor extrinsic properties account for the existence of right and left.

---

4) Space is absolute.

We will examine this argument premise by premise beginning with the relevant terminology, such as “intrinsic” and “extrinsic” as it comes up.

The First Premise

The first premise, as I have phrased it, reads “Space is either absolute, or relative”. How much confidence do we have that Kant intended this exhaustive disjunction? We’ll address and clarify relative and absolute space being mutually exclusive for Kant, despite his acknowledgement of an order or relation present amongst objects that exist. We will also clarify the relationship Kant saw between this order and the regions of absolute space which he indeed took to exist. In the opening passages of his piece, Kant speaks as though he intends to carry out a promised but unexecuted project of Leibniz’, *The Analysis Situs* (Kant 27):

> I am here seeking the first philosophical ground of the possibility of that whose magnitudes Leibniz proposed to determine in a mathematical manner. For the positions of the parts of space in relation to one another presuppose the region towards which they are ordered in such a relation; and this region, in ultimate analysis, consists not in the relation of one thing in space to another (which is properly the concept of position) but in the relation of the system of these positions to the absolute world-space. (Kant 27).

The first thing to notice is that Kant specifically challenges Leibniz’s framework of relative space, opposing it to absolute space. His language here is reminiscent of Leibniz’s definition of space in his debate with Clarke: “Space is nothing but an order or set of relations among bodies*”* (Leibniz 9). Kant is asserting here that the order or relation of things within space, which Leibniz takes for space itself, is in fact dependant on a background of “absolute world-space”. The order of objects supervenes on absolute space and is not space proper. Thus

although there is indeed an order in which objects are embedded in space, it does *not* constitute space.

Kant's words preceding those above, just preceding the example of looking for a sunrise, are even more revealing:

> Even our judgments about the cosmic regions are subordinated to the concept we have of regions in general, in so far as they are determined in relation to the sides of the body. All other relations that we may recognise, in heaven and on Earth, independently of this fundamental conception, are only positions of objects relatively to one another (Kant 29).

It is clear that Kant thinks that subtracting the matter of absolute regions of space from a framework indeed leaves you, unavoidably, with *relative* space. Thus in Kant's argument, Leibniz' framework, relative space, is equivalent to the negation of "absolute space", however delicate Kant tried to be saying as much in his introduction.

Seeing as Kant's argument is driving at the existence of absolute space, he, like Clarke and unlike Leibniz, believes there is in fact a correct answer to questions like "which way is up?". The regions which constitute the answers to such questions also constitute the orientation of the world's objects. Kant says,

> In anything extended the position of parts relatively to one another can be adequately determined from consideration of the thing itself; but the region towards which this ordering of the parts is directed involves reference to the space outside the thing; not, indeed, to points in this wider space –for this would be nothing else but the position of the parts of the thing in outer relation –but to universal space as a unity of which every extension must be regarded as a part (Kant 27).

With the asymmetry of the body specifically in mind, especially universal ones like the asymmetry of the heart, Kant says that the differences in right and left here testify to "a relation to the universal space which geometers postulate", but admits this relation "is such that it cannot itself be immediately perceived" (Kant 31). Like Clarke, Kant is admitting here that the ultimate right and left of absolute space cannot be perceived in relation to these specific instances within reach of human beings. At this point, one may reasonably ask: If the relationship of right and left sides of objects to some ultimate right or left region of absolute space cannot be perceived, how is Kant confident that right and left, in and of themselves, have their own reality of any kind?

The Second Premise

The question is no small matter. The very existence of right and left is implicitly present, or assumed in, our second premise: "If it is relative, then either an object's intrinsic or extrinsic properties account for the existence of right and left". Kant finds the answer in objects whose only difference is, precisely, that one has the property of being right and the other left. Understanding these objects requires distinguishing two kinds of properties they can have, which I call "intrinsic" or "extrinsic".

Kant says "What we do perceive are those differences between bodies which depend exclusively upon the ground which this relation affords" (Kant 31). Kant calls these bodies "incongruent counterparts" (Kant 31). His examples include a right and left–screw, and a pair of hands. Objects like these possess complete similarity. In other words, they are exactly the same regarding their magnitude, the arrangement of parts relative to one another, their proportions, and magnitude of the whole. A list of all such properties, which we may call "intrinsic properties",

amounts to what Kant calls a “complete description”. And it applies to either object in his examples (Kant 31).

Key to these types of objects, Kant says, is that in spite of their similarity they cannot be superimposed, or (if three dimensional) be bound by the same surface. Such incongruent counterparts are different despite their complete similarity (Kant 31). It is this incongruence, the marker of some “inner difference” (Kant 32) beyond their shared description, that marks one object as a “right” one and the other as a “left” one. Thus, in his second premise, Kant is challenging Leibniz to explain the existence of incongruent counterparts. Leibniz could not appeal to absolute space, which would tether the right or left object in a relation of orientation to some ultimate right or left region of the universe. Two options remain for relativists, the first being relations of the object’s parts to other objects in space, which might be called “extrinsic properties” (Kant 32). Although Kant did not use the terminology of “intrinsic and extrinsic properties”, my reason for doing so will become clear when I specifically address Kant’s argument as an application of the PII. The second option is the intrinsic properties, the relations between the parts .

The Third Premise

Kant's third premise, as I've presented it here, is "Neither intrinsic nor extrinsic properties account for the existence of right and left". Here he argues that Leibniz, without absolute space, cannot explain the difference between right and left. Of course, it is technically improper to say that a premise “shows” something–a premise doesn’t demonstrate; an argument does. What I mean is that Kant’s third premise is the result of a demonstration, the component claims of which, at the level of granularity I’ve presented the argument here, are implicit. Let’s examine a

presentation of the argument within premise three, or, what I think Kant's premise three argument would look like presented more formally:

1. Suppose a lone hand is all that exists and we do not know if it is left or right.

2. $\forall\Phi(\Phi x \leftrightarrow \Phi y) \rightarrow x = y$

3. $\forall\Phi(\Phi x \leftrightarrow \Phi y)$ (or: there is no property from the complete description of a right hand that a left does not have, and vice versa).

4. So $x = y$ (or: the imagined hand must be both left and right)

5. But $x \neq y$ (or: we know that claim 4 must be false).

---

6. So $\sim\forall\Phi(\Phi x \leftrightarrow \Phi y)$ (or: there must be a property that is different between two hands of a pair)

More precisely, Kant's argument shows Leibniz cannot explain the "inner difference" of a pair of incongruent counterparts. The intrinsic properties, as just mentioned, are the same between the two objects, meaning that this list of properties on its own cannot tell us whether some hand is a right or left hand. Kant's demonstration proceeds by asking us to imagine that a single hand was all that existed. If that is the case, Leibniz's relativist definitions of space and position mean that such a lone hand would be completely "indeterminate", that is, neither right nor left. This is because the hand itself, its once "intrinsic" properties, would comprise all actual space on Leibniz's model (Kant 32). But we know, rightness and leftness being an "inner difference", that the lone hand would indeed be left or right nonetheless (as stated in line 5). So

properties that might be called “extrinsic” offer Leibniz no suitable recourse (Kant 32). Thus, space is absolute for Kant.

The third premise of Kant’s argument, unpacked above, is expressed by Kant in the following passage:

> What, therefore, we desire to show is that the complete ground of determination of the shape of a body rests not merely upon the position of its parts relatively to one another, but further on a relation to the universal space which geometers postulate (Kant 30-1).

Kant is saying here that his demonstration essentially consists in showing that the aforementioned “complete description” of an incongruent counterpart is not sufficient to determine its shape, whether they’re considered intrinsic or extrinsic (which as we have seen, lose any meaningful difference in this thought experiment). Perhaps ironically, the “complete description” is incomplete. From it we cannot determine which hand of a pair has been created. It cannot be the “determination ground” of the object’s orientation. A truly complete description must incorporate properties lent to it by the regions of the universe in absolute space.

Thus the presence of the PII in premise three takes a form that is more elaborate than others we have seen thus far. Kant’s thought experiment relies on the reader recognizing that the intrinsic properties of, say, a right hand, are the same as the intrinsic properties of, say, a left hand. The arrangement and sizes of parts of the hand relative to one another are shared. Thus, no matter whether you initially picture the lone hand as right or left, that same list of properties holds.

Of course, the PII that is invalidated is the *strong* PII. We explained earlier that weakened applications of the PII permit multiple indiscernibles, granting them a non-numerical identity. However, now we face a certain question: is the factor of their multiplicity truly owed to a simply "numerical" multiplicity? To put it differently, is the difference between right and left qualitative? Kant is clear in his thought experiment that their non-identity is entailed by rightness and leftness. However, a difference of numerical property, plain and simple, could be achieved with two right hands or left hands considered at the same time. After all, Kant mentions in his piece that the counterpart of a counterpart is a copy of the original (Kant 31). Two right hands seem a degree closer to one another than a right and a left. The difference between the two right hands seems indubitably numerical. Thus one may ask: Do symmetrical objects represent a *second* kind of multiple indiscernible? We said earlier multiple indiscernibles violate the PII by differing only numerically. Bilaterally symmetrical objects present a challenge to that claim.

One line of thought suggests that, yes, the properties of right and left have no reality we can give a meaningful description of and are basically numerical. First are some points in favour of this argument contained even in Kant's article itself, despite him ultimately arguing for absolute space. There is the problem pointed out by both Kant and Clark, that if the right or left parts of individual objects, or pairs of objects, here on Earth really do have a relationship to some ultimate right and left region of absolute space, we cannot perceive this relation (Kant 31). Secondly, contemporary physics offers us no more confidence in the existence of right and left than the physics of Leibniz and Euler. A long-standing view in support of Leibniz's strong application of the PII is that the laws of nature are invariant under symmetries such as reflection, translation, rotation etc. (Weyl 19).

Hence the "shock" of Ernst Mach, centuries later, upon learning from Oersted's famous needle-and-current experiment that this is not always so. Oersted discovered that if a magnetized needle is aligned with a wire that has a current running through it, the needle will swing perpendicular to the wire. This is counterintuitive as it betrays the bilateral symmetry of the alignment. An observer would expect the needle to "refuse to decide between left and right, just as scales of equal arms with equal weights" (Weyl 19f). Invisible is the magnetic field of the electrical current. The field travels left to right, that is, north pole to south pole, around the current. The north pole of magnetized needle will swing to the direction of the current (following south). This means that it will always swing right if our current travels away from us and left if it travels towards us. This orientation of a magnetic field to an electric current is now captured as the "right-hand rule", as closing the fingers and opening the thumb of a right hand matches the relationship of (accordingly) the field and current (Sorensen 253). Revealing is the need for a heuristic: The needle swinging right or left relative to the current is as reliable as it is arbitrary. No scientific explanation has been found yet for this relationship between the direction of a current and its magnetic field. Scientists understand that the electrons of magnetic objects spin in the same direction and that this determines the direction of the magnetic field. They do not know why. It seems to be arbitrary. Hermann Weyl says,

> The net result is that in all physics nothing has shown up indicating an intrinsic difference of left and right. Just as all points and all directions in space are equivalent, so are left and right. Position, direction, left and right are *relative* concepts (Weyl 20).

Weyl cites the needle-and-current phenomenon as an example of one that would appear to violate the PII. Were we to reflect the aligned needle and wire on a plane (in other words, examine two with mirror symmetry), they appear to share all properties until the current is

activated and our needles swing different directions because the magnetic field of the current has its direction reversed in the reflection. The perfect qualitative identity, or purely numerical difference, of the physical needle and current is made imperfect by the needle's response to unseen forces.

The difference between a right hand and a left hand seems to be of the same kind. The elusive "inner difference" pointed to by Kant takes the multiple indiscernibles beyond a bare numerical difference and appears qualitative. Max Black in his 1952 article presents a thought experiment, the structure of which is very similar to Kant's argument. Black (rather one of two speakers in his dialogue, "Speaker B") asks readers to imagine that two spheres are all that exist in the universe (Black 156). The two spheres have all properties in common, their intrinsic properties being the same. Even attempts to distinguish the spheres based on location or distance from each other are futile, as are attempts to name them (Black 157-8). Black's spheres strike us as differing in a plainly numerical way that a right hand and left hand, sharing a "complete description" or not, do not because of their incongruence. The inability to put one surface within the bounds of the other seems to be a difference of kind that is different than the spheres, or two right hands and two left hands. One is "left-kind" and the other "right-kind", by virtue of a different quality. Someone may question this line of thinking: how real is this quality? Are the hands and spheres *really* different?

Wittgenstein in his *Tractatus* (1922) rejects Kant's claims about one surface being unable to bound its counterpart's:

> The Kantian problem of the right and left hand which cannot be made to cover one another already exists in the plane, and even in one-dimensional space; where the two congruent figures a and b cannot be made to cover one another

> without moving them out of this space. The right and left hand are in fact completely congruent. And the fact that they cannot be made to cover one another has nothing to do with it. A right-hand glove could be put on a left hand if it could be turned round in four-dimensional space (Wittgenstein 6.36111).

Wittgenstein's point is that limitations on how an object can be manipulated within a certain space should not be deemed relevant to its identity with its counterpart. For Wittgenstein, manipulations ("motions") unavailable in one kind of space but perhaps available in a higher-dimensional one are relevant. The unavailable but theoretically possible motions speak to the object's identity no less than the motions it can actually perform. A right hand is only a right hand until someone inverts it, and then it is a left hand. The hand seems to have lost its essential character as a *right hand* or a *left hand*. Rightness and leftness no longer speak to the kind of thing it fundamentally is. It is now simply *a hand*. For Kant, only *non-dimensional* motions are relevant to the object's identity–indeed, his argument depends on there being no way, at least in our reality, in which a right and left hand can be turned into each other.

Kant's and Wittgenstein's exchange raises a question: do manipulations in higher space deserve to be considered relevant to the identity of a symmetrical object? What do we mean when we talk about "dimensional" or "higher space" motions? Robert E. Frederick explores this exact topic in relation to Kant's argument. He distinguishes between rigid and non-rigid motions. A rigid motion is a manipulation of an object in Euclidean space that preserves relations of distance and angle between parts of an object. Objects that cannot be made congruent by rigid motions in Euclidean space are incongruent counterparts, and objects with mirror symmetry whose manipulations are limited to a certain dimension will fulfill this criterion (Frederick 4).

A *dimensional motion* occurs when we move an N-dimensional object through N+1 dimensional space. For a 2D object, an image on a page, these include rotating it or shifting it. They do not include lifting it off the page and flipping it over. Doing this would be to perform a dimensional motion (Frederick 3,5). Were this object asymmetrical, such a motion would transform it into its previously incongruent counterpart. Alongside dimensional motions in this respect motions are *Mobius motions*. Mobius motions are motions that happen in non-orientable spaces, meaning that no definition of clockwise or counter-clockwise can be established (Munroe 263). Mobius strips are 2D objects that can exist embedded in 3D space, and they demonstrate this impossibility. They too allow incongruent counterparts to be made congruent (Frederick 5).

It appears dimensional or Mobius motions would make incongruent counterparts indiscernibly similar. However, it is not possible within the scope of this project to consider whether Leibniz would have granted full identity to objects when dimensional motions are considered. What should be apparent is that Kant, by defending the reality of absolute space, also illustrates the way in which a world with symmetry is also a world with allows multiple indiscernibles.

§2: Symmetry in Multiple Perspectives: Frege's "Sense and Reference"

For Kant, Leibniz's principle counted two objects as one. The right and left hands present counterexamples and invalidate the PII. Objects with bilateral symmetry, should Leibniz not admit dimensional motions, are multiple indiscernibles and invalidate the strong PII. Frege's *Sense and Reference* contains a thought experiment that, we'll see, illuminates the converse: What does it mean for Leibniz's principle to count one object as two?

Frege interrogates identity per se, and like Kant, has to confront the relevance of extrinsic properties to identity. Leibniz's Law, like the PII for Kant, proves an incisive tool: Frege's application shows that extrinsic properties of an object involving relationships of the object to observers threaten to make one object two under Leibniz's Law. Similarly to dimensional motions, granting these properties validity preserves the oneness of a single object.

First we will examine and unpack Frege's argument about identity-statements, which concludes that identity statements must involve "senses", in his technical sense of the term. Senses mediate between signs and their referents, meaning that it is possible for one referent to relate to observers in different ways, or different signs referring to the same referent by means of different senses. They thus account for differences in truth-value between different nouns that share a referent. Frege's example of "The Morning Star" and "The Evening Star" involves an application of Leibniz's Law, in which the relationship of a referent to an observer can be considered an extrinsic property of the referent.

Frege's "Sense and Reference" begins with the following dilemma: are statements of identity, like "a=a" or "a=b", statements about a relation between the signs, or between their referents? He argues that both options lead to absurdities as follows:

1. The meanings of identity statements assert either relationships (a) between signs as objects, (b) between the referents of signs, or (c) signs as symbols.

2. There is a cognitive difference between statements of identity like "a=a" and "a=b": a=a is a priori, whereas a=b is in general not.

3. If identity statements express a relationship between signs as objects, then "a=a" and "a=b" have the same cognitive value, if they are true.

4. So, identity cannot be a relationship between signs as objects (4,5)

5. If identity statements assert a relation between signs, the signs are considered in their referential relation (as opposed to as mere tokens).

6. Then as the referents they represent, then "a=a" and "a=b" once again have the same cognitive value, if true.

7. So, identity cannot be a relationship either just between signs, nor one just as between their referents (2, 6)

---

8. So, there can be a cognitive difference between statements of identity like "a=a" and "a=b" only if identity statements express a relationship between signs as symbols (senses).

Frege concludes that the difference must lie in the way that the objects are given, the "sense", and that this is associated with the sign. What does Frege mean about a "cognitive difference" between "a=a" and "a=b" and why is it a problem that certain models don't capture this difference? As Frege points out, there is something distinctly informative about "a=b". Whereas "a=a" is a priori knowledge (or analytically true), "a=b" may have to be discovered (Frege 8). The reasons someone chooses to say "a=b" over "a=a" tend to be pragmatic.

When Frege considers the possibility that identity-statements are about "signs as objects", he means the physical, visual shapes on paper making up a word we read (were the example concerning speech, it would be the sounds produced by mouth and voice). Hence the inference between lines three and four involves supposing the sign "a" has the same use as the sign "b", which is to designate their one object. If this were the case, and we knew only this, the statement tells us that "a" can be used anywhere we'd use "b". We are here hearing something about the objects, and it can be expressed completely devoid of any information for the reader on when to

actually use one sign or the other, due to the arbitrary nature of language. Thus “a=a” ought to express nothing different than “a=b” (Frege 8). However, this is not our experience of the expressions.

When Frege considers the possibility that identity-statements are about relationships between the referents of signs, he imagines that the meaning of “a” consists entirely in whatever it is in the world “a” refers to, and the meaning of “b” consists entirely in whatever it is in the world “b” refers to. In this case, “a=a” and “a=b” face the same problem: Nothing different is expressed between “a=a” and “a=b” (Frege 8). However, this too does not match our experience of the expressions. Thus the only suitable model of identity-statements is one in which it is understood as a relationship between signs as symbols, which Frege calls their “sense”. The rest of “Sense and Reference” involves Frege elucidating the exact relevance of signs as symbols to knowledge and identity statements.

Frege quotes Leibniz in “Sense and Reference”: “Eadem sunt, quae sibi mutuo substitui possunt, salva veritate”, or, “Two terms are the same, which can be substituted for one another, saving the truth” (Frege 13). This principle is more commonly known in logic as “intersubstitutivity *salva veritate*”, and is a direct consequence of the Leibniz’s Law. We have seen that substituting one sense for another is often benign. However, Frege discusses how this isn’t so when it comes to abstract noun clauses. Abstract noun clauses embed claims like “The Evening Star is a body illuminated by the sun” within a bigger claim about someone’s relationship of knowledge or belief to the original smaller claim. When senses are substituted in an abstract noun clause, the truth value of the whole claim changes (Frege 15). If The Morning Star is The Evening Star, and it is true that “Frank knows that Venus is The Morning Star”, this does not guarantee “Frank knows that Venus is The Evening Star” is true.

Indeed, we can see that what drives Frege's argument is a failure of Leibniz's Law. Let v=Venus, m=The Morning Star and e= The Evening Star, and Fx= "Frank knows x a priori…".

1) e=m

2) $x=y \rightarrow (\Phi x \leftrightarrow \Phi y)$

3) So $\Phi e \leftrightarrow \Phi m$

4) F(e=e)

5) F(e=m) By, 3

6) ~F(e=m) By hypothesis

---

6) So $\sim(x=y \rightarrow (\Phi x \leftrightarrow \Phi y))$

In short, despite the fact that the two names have the same referent, they differ with respect to Frank's knowledge-state. This application of Leibniz's Law takes into account properties of an object anchored not in the object itself but in relation to other objects, in this case, the mental states of observers to speakers. That is, it takes into account extrinsic properties. Leibniz's Law only fails, or the morning and evening star appear to be numerically two things, when we incorporate properties from the vantage point of specific observers. Put simply, if we associate different perspectives on one object with different names, we can learn that they have the same referent.

Frege's demonstration shows that matters of knowledge are matters in which the sense of a statement is consequential: when speaking about someone's knowledge, the truth-value of that statement hinges on the sense. And as it happens, someone's experience of an object determines their knowledge, or lack thereof, of the object. Hence Frege draws a connection between senses and perspective. He makes this connection rather literal by making a referent, sense, and conception analogous to, accordingly, a planet, its image on a telescope lens, and the planet's image in the observer's mind, Frege says,

> The optical image in the telescope is indeed dependent upon the standpoint of observation; but it is still objective, inasmuch as it can be used by several observers. At any rate it could be arranged for several to use it simultaneously. But each one would have his own retinal image (Frege 11).

One's use of "The Morning Star" in connection to or divorced from their use of "Venus" may speak to their knowledge or ignorance of the shared referent. One's use of "The Morning Star", as opposed to "The Evening Star", while recounting events may be purposeful, as the person may wish to connote the time events occurred at. Whatever the reason, use of one over the other shows that senses document speaker observations and perspectives.

The relevance of symmetry to Kant's application was transparent, given that hands possess bilateral symmetry. In turn, the relevance of symmetry to identity depends upon whether dimensional motions are permitted to make the hands congruent. The relevance of symmetry is less apparent in the case of the morning star and the evening star. The relevance lies in the relationship of the PII to names, and in turn, perspective. Describing his PII to Clarke, Leibniz said "'Suppose x and y are two indiscernible things' comes down to 'Suppose x is y, and that this thing has two names'" (Leibniz 16). Leibniz's view, which we discussed earlier, is that it is

meaningless to *talk* about the two contending spaces and times for creation as different—the supposedly different universes are actually *one* universe, and the Newtonian has mistaken a difference in perspective (due to a family of alternative coordinate systems) for a genuine numerical difference. Arbitrarily choosing *one* violates the PSR. This is the same reason Max Black's Speaker-B challenges A's attempt to distinguish the two spheres by means of different labels: what reason would an observer have to give either name to either sphere? Speaker B says "The important thing, for my purpose, was that the configuration of two spheres was symmetrical" (Black 160) and that "[i]n the absence of any *asymmetric* observer…the spheres would have all their properties in common" (Black 163, emphasis mine). For the spheres to be given names at all, an observer must stand in an asymmetrical relationship to them. In the absence of differing intrinsic properties (or differing relational properties even to each other (Black 159)), there remains only differing extrinsic, relational properties in the form of differing relationships to the third thing that exists. Single discernible Venus and two lone qualitatively identical spheres owe their difference to an asymmetrical universe. Were it not for the possibility of observers experiencing different perceptions of one planet, were it not for one observer to enter Black's two-sphere universe at a point that is not the axis of the plane along which the spheres are reflected, then the PII would count the two senses, and two spheres, as one.

In both cases, Frege's morning and evening star and Black's two spheres, we see extrinsic (perspective, or observer-based) properties accounting for the difference between two objects or two perceptions of one object. Whether or not we grant that these properties the right to speak to the identity of the object or objects is a question is something that applying the PII or Leibniz's Law has brought into higher resolution, but not straightforwardly solved.

Chapter 3: Symmetry and Reality

§1: Klein and Chester: Revisiting Dimensional Motions, Observation and Symmetry

When we discussed Kant's problem of right and left hands as multiple indiscernibles, we briefly considered non-orientable spaces such as the Mobius-strip. Like dimensional motions, it allows for incongruent counterparts to be made congruent. Motions which allow such a thing surfaced in our discussion of Wittgenstein's retort that the problem of right and left hands dissolves if we permit inverting the glove in a space of higher dimension. Mobius's work was part of a larger trend in mathematics in the 19th century, which emerged from the work of authors we have discussed so far such as Leibniz and Kant.

Among these later figures was Felix Klein, who in 1870 launched the "Erlangen Program", a research program dedicated to studying the relationships between emerging fields in geometry. In his 1872 *A Comparative Review of Recent Researches in Geometry*, he explains that the development of projective geometry in his lifetime challenged our understanding of what properties genuinely belonged to figures. Metrical properties are not preserved under projections, raising the question of whether metrical properties are essential to geometry (Klein §1). In two papers called "On the So-called Non-Euclidean Geometry" Klein articulated the basic ideas of his program. He observes that recent challenges to traditional (Euclidean) geometry's means of defining a figure have invited investigation of methods for the definition of figures across geometries in general (Klein §2). Klein's strategy involved "dispos[ing] with the concrete notion of space", and proposing instead a more general one:

> Let us now dispose with the concrete conception of space, which for the mathematician is not essential, and regard it only as a manifoldness of *n* dimensions, that is to say, of three dimensions, if we hold to the usual idea of the

> point as space element. By analogy with the transformations of space we speak of transformations of the manifold; they also form groups (Klein §1).

These groups are "the most essential idea required" (Klein §1). A group is a set of the elements and an operation which produces a third element, also in the group, when performed on two elements. The operation can represent *transformations of coordinates*, and indeed it is these same transformation-groups which are the source of Klein's more general concept. Groups contain an identity element, an element which when combined with another element leaves the element unchanged (combining any rotation with one of 360°, for example). Groups also contain, for each element, an inverse operation: an operation that reverses the effect of whatever operation was first applied to it (Klein §1).

A *principle group* is the group of those motions available in a geometry that can be performed while leaving properties of figures in that geometry unchanged. In other words, the principal group includes all those transformations that preserve whatever properties make the figure the kind of figure it is. Klein gives as examples "transformation into similar configurations, by transformation into symmetrical configurations with regard to a plane (reflection), as well as by any combination of these transformations" (Klein §1). The concept of symmetry and its association with concepts like congruence and similarity is such that mathematicians today refer to symmetries more broadly as products of transformations (functions) that preserve structure. Functions and transformations that preserve structure are often described as carrying the figure "into itself" (Weyl 41, 45). For these reasons many mathematicians, for instance Edward Frenkel, called symmetry "the point of departure" for the Erlangen Program (Frenkel 248). The connection between symmetry and identity is therefore of long standing.

The existence of a principal group comes out of a distinction drawn and defended between space and the figures inside it:

> … there are space-transformations by which the geometric properties of configurations in space remain entirely unchanged. For geometric properties are, from their very idea, independent of the position occupied in space by the configuration in question, of its absolute magnitude, and finally of the sense in which its parts are arranged (Klein §1)

In mathematics, changes performed on an object can be described algebraically thanks to a coordinate system (algebraic variables), in which the individual coordinates themselves bring nothing to bear on intrinsic nature of an object situated there.

Now we can better understand what Klein meant by "By analogy with the transformations of space we speak of transformations of the manifold". Manifolds are generalisations of the concept of space. A manifold of dimension *n* is a second-countable (as in, the sets of individual points are countable) Hausdorff-space (meaning one can define for any two points in the space two disjoint sets of points surrounding them), each point of which has a neighbourhood homeomorphic to (a continuous bijection exists mapping it to) a plane (or, for figures with a boundary, one half of a plane in which all y coordinates equal or are below 0). Although this terminology was not employed by Klein himself, manifolds are now understood as *topological spaces* (Armstrong 11, 15, 169). A topological space is a set whose elements are called points, plus a structure called a topology which can be defined as "a nonempty collection of subsets of X, called open sets, such that any union of open sets is open, any finite intersection of open sets is open, and both X and the empty set are open" (Armstrong 28).

Returning to the connection drawn by way of introduction between incongruent counterparts and the work of Mobius, we can now say that a Mobius-strip is a non-orientable topological space. This means that if an asymmetrical figure, that is, an incongruent counterpart, were present on the original strip, it could now be brought along the strip once 360° to become its counterpart, or mirror-image. A return to the figure's original appearance means travelling 720°. Conversely, this means that if we begin with one version of the figure (say, a 2D right hand) and travelling 360° creates a left hand. It can become its counterpart (Frederick 5, Armstrong 18). To apply Klein's ideas directly here, the identity element of the group for our Mobius-strip hand includes the 720° translation around the band, along with 1440°, 2160°, and other multiples of 720°. Any hand paused at one of these rotations meets the criterion for being "the same hand". This example shows how Klein's models capture and disregard the same properties of an object as the weak PII.

What about Leibniz's Law? What is now important to emphasise is that for Klein, the relation between the identity of the shape and its properties is a biconditional. They determine each other:

> …geometric properties are not changed by the transformations of the principal group. And, conversely, geometric properties are characterized by their remaining invariant under the transformations of the principal group (Klein §1).

In other words, the PII and Leibniz's Law operate here in conjunction. Should all the properties of a shape remain unchanged by transformations, those properties define identity (thus $\forall\Phi(\varphi x \leftrightarrow \varphi y) \rightarrow x = y$). Should we decide beforehand what constitutes the identity of a shape, those properties should remain unchanged by transformations (thus $x = y \rightarrow \forall\Phi(\varphi x \leftrightarrow \varphi y)$). In words closer to the terminology of Klein, a group of transformations that acts on a space preserves

properties that define a geometry. Transformations that preserve the properties with which you seek to define a space form a group (Klein §2).

Euclidean geometry's symmetry group consists of all transformations of Euclidean space that preserve distances and angles. These transformations are compositions of rotations and translations. Non-Euclidean geometries correspond to other symmetry groups, ones that don't preserve distance, but other features instead. As classified by Klein, geometries have a relationship of succession with one another, owing to a larger group of transformations being able to contain a smaller one (Klein §2):

> Every space-transformation not belonging to the principal group can be used to transfer the properties of known configurations to new ones. Thus we apply the results of plane geometry to the geometry of surfaces that can be represented (*abgebildet*) upon a plane; in this way long before the origin of a true projective geometry the properties of figures derived by projection from a given figure were inferred from those of the given figure. But projective geometry only arose as it became customary to regard the original figure as essentially identical with all those deducible from it by projection, and to enunciate the properties transferred in the process of projection in such a way as to put in evidence their independence of the change due to the projection (Klein §3).

Key here is Klein's remark that "projective geometry only arose as it became *customary* to regard the original figure as essentially identical with all those deducible from it by projection" (my emphasis). We are seeing here the experiential implications of the biconditional relationship described above: symmetry groups do not merely determine or construct what properties are associated with an object's identity, they also confirm our pre-existing ideas of what that should be. In other words, it appears that within Klein's framework, we can identify a "weak" version of Leibniz's Law, in which presumed identity between the object before and after a transformation

implies that the list of properties whose survival defines the shape (or, is the criterion for said identity) is decided not by the transformations, but those specifying them.

An important assumption has been made in this last idea. If we are asking questions about the identity of an object to begin with, we must be operating from a framework that allows us to do so. The broader framework is a geometry of a broader group than the object's geometry of origin. This allows us to see an analogy between Kant's hand problem and our Mobius strip. For Wittgenstein, right and left dissolve in the fourth dimension because it allows us to perform a rotation unavailable in three dimensions. Were it not for Wittgenstein's consideration of the hand from a broader viewpoint in which more transformations are available, the dissolving of orientation's relationship to the identity of hands would not have occurred. Right and left would still be prerequisite properties to the identity of two different types of hands. For the Mobius-strip, the features of three-dimensional space we bring with us as observers mean that the 2D hand appears of opposite orientation when on "the other side" of the strip. For the hand, there is no "other side". The Mobius-strip is a one-sided creature. The transformation into its counterpart is only such a thing owing to the relative spatial positioning of an observer in orientable space outside the strip. Mobius motions allow us to dissolve right and left with a change we can't perform in orientable spaces. Conveniently, our example with the Mobius-strip doesn't depend on a fourth dimension with features beyond our capacities of visualisation. What we experience with the Mobius-strip, Wittgenstein conjectures with theory. This is what the mathematics of symmetry groups allow us to do–the extrinsic properties of an object that are dependant on something having a certain relationship to something else, normally accessed with perspective (observation), can be discerned even if experience (the chance to perform the act of observation) of them is off-limits.

The most famous application of Klein's subtractive method, and therefore the concept of symmetry, is Einstein's Theory of Special Relativity, which explored the consequences for the laws of nature if the speed of light were invariant in all inertial frames of reference. In order for these laws to remain universal, observers must sometimes disagree on their measurements of time (Mook 70, 78). The speed of light in this case is an invariant feature across multiple observer frameworks, and this invariance dictates the Lorentz transformations (Weyl 132). Hermann Weyl calls the theory of relativity " but another aspect of symmetry" (Weyl 17).

Our investigation thus far featured an application of the PII that straightforwardly involved discovering an identity: Frege's case of The Morning Star and The Evening Star. A failure of Leibniz's Law arises when one object is known from two vantage points, and abstract noun-clauses, combined with the principle of intersubstitutivity, amount to contradictory observer-based properties. This case of a single object, Venus, having two descriptions presented several intriguing questions: Is this a symmetry? Are there two Venuses that share some traits but differ with some that concern observer experience? Are these observer-based or relational properties relevant to the identity? Our discussion of Frege's thought experiment in the previous section appeared to settle the latter two questions. The observer-based properties ("Frank knows The Morning Star is Venus" and "Frank does not know The Evening Star is Venus") initially contradict and are liable to change in time. These properties of the object seem to become observer-based properties, properties of *observers,* once it is learned that Venus is both The Morning Star and The Evening Star.

Our first question remains, however: is this a case of symmetry? Symmetry by no means only applies to multiple objects with a shared structure. Physicist Marvin Chester explains symmetry as "the end result of identification" (Chester 3). He outlines the relevance of

group theory, the theory initiated by Klein, to the whole enterprise of science and thus charts its relevance to *discovery.* Physicists often study objects they cannot directly observe. Group theory reveals what properties of an object are invariant (Chester 4). Group theory codifies our evaluation of perceptions (or measurements) with the underlying concept of symmetry, because "the essential quality that characterises symmetry is this: the appearance of sameness under altered scrutiny… sameness under altered scrutiny is just what captures the notion of identity" (Chester 5). Group theory mathematically formulates the "internal consistency in the description of things" (Chester 2). In this sense, elementary particles are much like the planet Venus. We must rely on measurements of angular momentum and mass, or patterns of motion in the sky, to establish the object's membership within a kind and/or uniqueness. We rely on measurements of the object whose existence is, at the stage of investigation, only theoretical. Observed effects may prove to be not the properties of one object, but effects of pre-existing ones mistaken for a new object. An observer cataloguing the properties of The Morning Star, if they compare with those of The Evening Star, deems the overlap too big to be coincidence: the "two" objects are one seen from different perspectives, and there is a spatial transformation relating them. Chester reminds us "We know objects by their properties" and come to eventually assign sets of properties "the dignity of an object" (Chester 4). That is, we consider properties which are corroborated in the perception of multiple observers to belong to the object proper, and or we consider properties possessed in common amongst multiple objects to indicate they are members of a kind.

Klein's insight was that that which survives certain transformations and altered scrutiny is that which is intrinsic to the object. These properties are nonetheless tethered to the groups of their local space, meaning that while we can identify intrinsic properties with symmetry groups, questions about their identity, and the properties that constitute it, cannot be neatly answered

without consideration of the geometry in which the object is situated and from which our geometer asks their questions about it. The identity of multiple indiscernibles, in this sense, is relative. The identity of single discernibles is less so, but the role of symmetry is no less prominent. Those perceived properties of the object from one observer's vantage point that are not contained in another observer's experience of it are "relative" in the sense of being extrinsic, not intrinsic properties. They are analogous to the real but arbitrary properties of multiple indiscernibles in geometry–the exact coordinates of the shape, its magnitude, etc., as well as the names given by observers of Black's two spheres. The common properties mark the intrinsic properties of some one object multiple observers are perceiving. Extrinsic, or relational, properties break what would otherwise be a perfect symmetry. Symmetry groups, Chester explains, capture this sameness under altered scrutiny. Symmetry is present in multiple indiscernibles and single discernibles alike.

Chapter 4 –Conclusion

We began our investigation with an examination of the Principle of the Identity of Indiscernibles, considering its logical structure, and several candidate interpretations of its conditional. We saw that the strong PII is associated with numerical identity, and the weakened version with qualitative identity. Failures of the PII produce multiple indiscernibles, objects with qualitative but not numerical identity. Failures of Leibniz's Law capture single discernibles, which have relative identity (in other words: they have numerical identity but unshared properties). In that same discussion, we considered Russell's and Whitehead's formulation of the principle and their application of it as a definition of identity in *Principia Mathematica*. This determined our later discussion: we assessed the PII in terms of its success or failure of the conditional statement.

In chapters 2 and 3, we surveyed applications of Leibniz's principles in the history of philosophy, and their role in the debates on space and time. Leibniz's application of the PII in the *Leibniz/Clarke Correspondence* illustrated the consequences when the strong PII is satisfied: objects we thought were two are in fact numerically identical, leading to the argument that space and time are in fact unreal. This application is strongly related to another, in which the PII entails the equivalence of hypotheses for differing frames of reference and, at least practically for Leibniz, the relativity of motion.

Euler objected to Leibniz's application, but we saw that Euler's application was lower-resolution, in that Euler compared differing worlds rather than parts of space and time themselves. We saw in this as well that multiple indiscernibles emerged neither through the PII per se failing nor through a successful application of the weakened principle. They appear only

insofar as they are presupposed features of the Newtonian world in which, on his argument, we live. A higher-resolution examination features the PII affirming the identity of our world with Newton's, and a failure of Leibniz's Law asserting the difference.

In Chapter 2 and Chapter 3 our discussion turned to applications of the PII in which questions concerning possible transformations or observer-based properties and their connection to indiscernibility and identity were raised. Kant's application revealed that the PII fails to capture multiple indiscernibles that possess bilateral symmetry. Contradiction emerges owing to our raw perception of the hand and our convictions about numerical identity–the lone hand is one of a pair missing its counterpart. In Frege's work, the failure of Leibniz's Law led to the converse situation: observational or extrinsic properties cause us to count one object as two. In the first case, the failure of the PII, a contradiction emerges owing to unshared properties between The Morning Star and The Evening Star, known and unknown by Frank to be Venus. That the star seen in the morning and the star seen in the evening are one object is a discovery, but this is incompatible with the identity becoming analytic under substitution. With Kant's example, we examined the relevance of dimensional motions to the identity of a thing–should changes we can only theoretically apply to the object speak to its identity, like Wittgenstein believed? With Frege's example, the planet Venus, we examined whether observer-based properties should fall under the scope of Leibniz's Law. Max Black's thought experiment with two identical spheres illustrated how an asymmetry introduced by an observer may account for the difference between multiple indiscernibles or different experiences of a single thing.

The rigorous language of symmetry groups offers some clarity on these questions. Klein used the concept of a group in his Erlangen Program, and symmetry is at the root of group theory. In Klein's model, we found that the relationship between the identity of a geometry (or,

its figures' definitional properties) and the symmetry group corresponding to it to be biconditional. Understood in terms of Leibniz's principle, the PII and Leibniz's Law are joined here in a biconditional statement. Questions, or assumptions, about the identity of a figure emerge as a possibility when performing transformations from beyond the symmetry group we have initially considered the figure in. We can ask whether a property, say, rightness or leftness, is relevant to the identity of a figure once it is added or removed. Conversely, we can also declare a figure that has undergone a transformation beyond its original geometry to be essentially the same as its new version.

The mathematics of symmetry groups allows us to distinguish an object's extrinsic and intrinsic properties in the process of identification, where the biconditional PII expresses the identity-conditions. This process of identification could involve deciding whether or not multiple objects meet a criterion of shared properties for qualitative identity (making them multiple indiscernibles). The process could involve deciding whether two perspectives of one object (someone who sees Venus in the morning and another who sees it at night) are in fact two facets of one numerically identical thing (making them singular discernibles). The room for applications involving pre-conceived assumptions about the identity of a figure or about the properties prerequisite for its identity reveal that for objects with imperfect qualitative identity, such as incongruent counterparts, there may be no correct answer: right and left need not be tethered to an object's identity if the observer grants validity to transformations applicable from a broader group (which in experiential instead of computational terms, corresponds to observing the object/s from a vantage point involving asymmetry). When supplemented with formal mathematics, symmetry groups afford explorations of identity by allowing us to cast beyond the limitations of observation or intuition. In addition to applications of symmetry groups (and

therefore Leibniz's principle) in a familiar process of scientific discovery, say, of a new celestial body, there is technically potential to philosophically reify observational extrinsic properties. That is, due to symmetry groups allowing for a decision to be made on the validity of said properties, one may perform modus tollens on Leibniz's Law for the two facets of an object and declare them unidentical, granting a more substantial reality to "The Morning Star" and "The Evening Star" as such.

BIBLIOGRAPHY

Primary Sources

Chester, M. “Is Symmetry Identity?”, *International Studies in the Philosophy of Science*, 2010, 16, 111-124. 10.1080/02698590220145061

Leibniz, Gottfried Wilhelm, and Samuel Clarke. “Exchange of papers between Leibniz and Clarke.” [1717]. Translated by Jonathan Bennett, *Early Modern Texts*, 2017. https://www.earlymoderntexts.com/assets/pdfs/leibniz1715_1.pdf.

Leibniz, Gottfried Wilhelm, "On Copernicanism and The Relativity of Motion ," [1689]. *G.W. Leibniz–Philosophical Texts*, Translated by Franks, Richard, and R.S. Woolhouse. Oxford University Press, New York, 1998. 153-183.

Leibniz, Gottfried Wilhelm, "Specimen Dynamicum," [1695]. *G.W. Leibniz–Philosophical Texts*, Translated by Franks, Richard, and R.S. Woolhouse. Oxford University Press, New York, 1998. 153-183.

Leibniz, Gottfried W.V., and Roger Ariew. *Philosophical Essays*. Edited by Daniel Garber, Hackett Publishing Company, 1989.

Einstein, Albert, et al. “On the Electrodynamics of Moving Bodies.” *The Principle of Relativity*, Methuen and Company, Ltd., London, 1923. *https://www.fourmilab.ch*, Fourmilab, www.fourmilab.ch/etexts/einstein/specrel/specrel.pdf.

Einstein, Albert. *Relativity: The Special and General Theory*. [1916]. CreateSpace Independent Publishing Platform, 2017.

Euler, Leonhard. “Reflections on Space and Time.” *Eulerarchive.maa.org*, 2011. Mathematical Association of America, [1750]. https://scholarlycommons.pacific.edu/cgi/viewcontent.cgi?filename=0&article=1148&context=euler-works&type=additional

Frege, “On Sense and Reference”. [1892]. *Contemporary Analytic Philosophy*, Second Edition, James Baillie (ed). Saddle River, New Jersey: Prentice Hall, 2003, ISBN: 0-13–099068-X , 463. pages 8-24.

Galilei, Galileo. *Dialogue Concerning the Two Chief World Systems— Ptolemaic & Copernican*. Translated by Stillman Drake, University of California Press, 1967.

Kant I. (1991) On the First Ground of the Distinction of Regions in Space. In: Van Cleve J., Frederick R.E. (eds) *The Philosophy of Right and Left*. The University of

Western Ontario Series in Philosophy of Science (A Series of Books in Philosophy of Science, Methodology, Epistemology, Logic, History of Science, and Related Fields), vol 46. Springer, Dordrecht. https://doi.org/10.1007/978-94-011-3736-2_3

Klein, Felix. "A Comparative Review of Recent Researches in Geometry." *Bulletin of The York Mathematical Society*, Translated by M W Haskell, no. 2, 1892, pp. 215–249., doi:arXiv:0807.3161v1.

Klein, Felix. Abstract of "A Comparative Review of Recent Researches in Geometry." *Bulletin of The New York Mathematical Society*, Translated by M W Haskell, no. 2, 1892, pp. 215–249. doi:arXiv:0807.3161v1.

Leibniz G.W. (1989) A Brief Demonstration of a Notable Error of Descartes and Others Concerning a Natural Law, According to Which God Is Said Always To Conserve the Same Quantity of Motion; a Law Which They Also Misuse in Mechanics. In: Loemker L.E. (eds) *Philosophical Papers and Letters*. The New Synthese Historical Library (Texts and Studies in the History of Philosophy), vol 2. Springer, Dordrecht. https://doi.org/10.1007/978-94-010-1426-7_35

Leibniz, Gottfried Wilhelm, and John Locke. *New Essays Concerning Human Understanding, Together With an Appendix Consisting of Some of His Shorter Pieces. Translated from the Original Latin, French and German with Notes by Alfred Gideon Langley*. Translated by Alfred Gideon Langley, 2nd ed., Open Court Pub. Co., 1949.

Russell, Bertrand, and Alfred North Whitehead. *Principia Mathematica*. Vol. 1, Cambridge University Press, 1910.

Newton, Isaac. "Newton's Principia : The Mathematical Principles of Natural Philosophy." *Internet Archive*, Translated by Andrew Motte, New-York : Published by Daniel Adee, 23 Mar. 2006, https://archive.org/details/newtonspmathema00newtrich/page/n189/mode/2up?view=theater&q=vessel.

Wittgenstein, Ludwig. Tractatus Logico-Philosophicus. Translated by C. K. Ogden, m.a. 1922.

Secondary Sources

Armstrong, M. A. *Basic Topology*. Springer, 1983.

Arthur, Richard T.W, 2014, *Leibniz*, Cambridge, Polity Press

Beaney, Michael. *The Frege Reader*, by Blackwell Publishers Ltd, Malden MA, 1997.

Bennett, Jonathan. Preface for "Exchange of papers between Leibniz and Clarke." *Early Modern Texts*, 2017. https://www.earlymoderntexts.com/assets/pdfs/leibniz1715_1.pdf.

Black, Max. “The Identity of Indiscernibles.” *Mind*, vol. 61, no. 242, 1952, pp. 153–164. *JSTOR* www.jstor.org/stable/2252291. Accessed 18 Aug. 2021.

Breger, Herbert. Kontinuum, Analysis, Informales– Beiträge zur Mathematik und Philosophie von Leibniz: Herausgegeben von W. Li. Germany, Springer Berlin Heidelberg, 2016.

Cain, Fraser. “Venus, The Morning Star and Evening Star.” Universe Today, 25 Dec. 2008, https://www.universetoday.com/22570/venus-the-morning-star/.

Clatterbaugh, Kennth C. “Leibniz's Principle of the Identity of Indiscernibles.” Studia Leibnitiana, vol. 3, no. 4, 1971, pp. 241–252. JSTOR, www.jstor.org/stable/40693645. Accessed 13 July 2021.

Coolidge, J. L. “The Rise and Fall of Projective Geometry.” *The American Mathematical Monthly*, vol. 41, no. 4, 1934, pp. 217–28. *JSTOR*, https://doi.org/10.2307/2302023. Accessed 12 Jul. 2022.

DiSalle, Robert. *Understanding Space-Time: The Philosophical Development of Physics from Newton to Einstein.* United Kingdom, Cambridge University Press, 2008.

Frenkel, Edward. *Love and Math: The Heart of Hidden Reality*. Basic Books, 2014.

Huggett, Nick, et al. “Absolute and Relational Space and Motion: Post-Newtonian Theories.” *Stanford Encyclopedia of Philosophy*, Stanford University, 19 July 2021, https://plato.stanford.edu/entries/spacetime-theories/.

Hyder, David. “Time, Norms, and Structure in Nineteenth-Century Philosophy of Science.” *The Oxford Handbook of The History of Analytic Philosophy*, edited by Michael Beaney, Oxford University Press, 2013, pp. 250–279.

Iltis, Carolyn. “Leibniz and the Vis Viva Controversy.” *Isis*, vol. 62, no. 1, [The University of Chicago Press, The History of Science Society], 1971, pp. 21–35, http://www.jstor.org/stable/228997.

Jacob, Pierre, "Intentionality", The Stanford Encyclopedia of Philosophy (Winter 2019 Edition), Edward N. Zalta (ed.), URL <https://plato.stanford.edu/archives/win2019/entries/intentionality/>.

Janiak, Andrew, "Kant’s Views on Space and Time", *The Stanford Encyclopedia of Philosophy* (Summer 2022 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/sum2022/entries/kant-spacetime/>.

Jorati, Julia. “The Contingency of Leibniz’s Principle of the Identity of Indiscernibles.” *Ergo: An Open Access Journal of Philosophy*, vol. 4, no. 31, 2017, pp. 899–929., https://doi.org/https://doi.org/10.3998/ergo.12405314.0004.031.

Linsky, Bernard and Andrew David Irvine, "Principia Mathematica", The Stanford Encyclopedia of Philosophy (Winter 2020 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/win2020/entries/principia-mathematica/>.

Lowe, E. J. (2002). *A Survey of Metaphysics*. Oxford University Press.

Mook, Delo E., and Thomas Vargish. *Inside Relativity*. Princeton Univ. Pr., 1987.

Muehlmann, Robert. “Russell and Wittgenstein on Identity.” The Philosophical Quarterly (1950-), vol. 19, no. 76, 1969, pp. 221–230. JSTOR, www.jstor.org/stable/2218108. Accessed 9 Aug. 2021.

Munroe, M. Evans. *Modern Multidimensional Calculus*. Dover Publications, Inc, 2019.

Noonan, Harold and Ben Curtis, "Identity", *The Stanford Encyclopedia of Philosophy* (Summer 2018 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/sum2018/entries/identity/>.

Norton, John D (1999) Geometries in Collision: Einstein, Klein and Riemann. In: *The Symbolic Universe : Geometry and Physics* 1890-1930. Oxford University Press, Oxford ; New York, 128 –144. ISBN 0198500882

Rayo, Augustin, 2005, “Logicism Reconsidered”, in *The Oxford Handbook of Philosophy of Mathematics and Logic*, Stewart Shapiro, ed., New York: Oxford University Press, 203–235.

Ian Rumfitt, “Frege's Logicism”, *Aristotelian Society Supplementary Volume*, Volume 73, Issue 1, 1 July 1999, Pages 151–180, https://doi.org/10.1111/1467–8349.00053

Rusnock, Paul. *More Logic: Course Notes for Logic 2 (PHI 3171)*. 1.0 ed., University of Ottawa, 2014.

Sorensen, Roy. “Thought experiments.” *American Scientist*, vol. 79, no. 3, Sigma Xi, The Scientific Research Society, 1991, pp. 250–63, http://www.jstor.org/stable/29774372.

Stewart, Matthew. *The Courtier and the Heretic: Leibniz, Spinoza, and the Fate of God in the Modern World*. Yale Univ. Press, 2007.

Torretti, Roberto. *Philosophy of Geometry from Riemann to Poincaré*. N.p, 1978.

Torretti, Roberto, and Bunge, Mario. *Relativity and Geometry: Foundations and Philosophy of Science and Technology Series.* United Kingdom, Elsevier Science, 2014.

van Cleve, J., Frederick, R.E. *The Philosophy of Right and Left: Incongruent Counterparts and the Nature of Space*. University of Western Ontario, 1991.

Yaglom, I. M. “Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century”. Translated by Sergei Sossinsky. *Science*, vol. 246, no. 4932. Birkhäuser, 1988. https://www.science.org/doi/10.1126/science.246.4932.940.b

Weyl, Hermann. *Symmetry*. Princeton University Press, 1952.