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THE IDENTITIES OF
SYMMETRIC MATRICES

By
WENXIN MA

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Abstract

Let $H_n$ denote the subspace of symmetric matrices of $M_n$, the full $n \times n$ matrix algebra with coefficients in a field $F$. Let

$$T_{2n}(x_1, \ldots, x_{2n-1}; x_0) = \sum_{\sigma \in S_{2n-1}} (-1)^{\sigma+i-1} x_{\sigma(1)} \cdots x_{\sigma(i)} \cdots x_{\sigma(2n-1)},$$

and $e(n) = n$ if $n$ is even, $n + 1$ if $n$ is odd. For all $n \geq 1$, $T_{2n}(x_1, \ldots, x_{2n-1}; [x_{2n}, x_{2n+1}])$ is an identity of $H_n$. If $\text{char} F / e(n)!$, $|F| > 2n$ and $n \neq 3$, then any homogeneous polynomial identity of $H_n$ of degree $2n + 1$ is a consequence of

$$T_{2n}(x_1, \ldots, x_{2n-1}; x_{2n}), T_{2n}(x_1, \ldots, x_{2n-1}; [x_{2n}, x_{2n+1}])$$

by substituting $x_i \circ x_j := x_i x_j + x_j x_i$ or $x_k$ for some of their variables or multiplying them by a variable. For $n = 3$, any identity of $H_3$ of degree 7 is a consequence of

$$T_6(x_1, \ldots, x_6; x_0), T_6(x_1, \ldots, x_6; [x_6, x_7]),$$

$$Q(x_1, \ldots, x_6), [S_3([x_1, x_2], [x_3, x_4], [x_5, x_6]), x_7],$$

where

$$Q(x_1, \ldots, x_6) := \sum_{(123),(456)} \{[x_1, x_2][x_3, x_4][x_5, x_6]\},$$

the commutators are the arguments of the triple product $\{abc\} := abc + cba$, the sum is taken over cyclic permutations of 1 2 3 and 4 5 6, and $S_n$ is the standard polynomial.

To prove the results, a partial ordering on the homogeneous elements of the free associative algebra $F[X]$ over field $F$ with noncommuting generators $X = \{x_1, x_2, \ldots\}$ is defined. Let $f$ be an element of $F[X]$ and $n$ the maximum of the degrees of the variables and the multiplicities of the degrees in $f$. If $f$ is homogeneous and $\text{char} F / n!$ then $f$ can be decomposed into a sum of two polynomials $f_0$ and $f_1$ such that for $0 < m \leq n$, $f_0$ is either symmetric or skew-symmetric in all its arguments of degree $m$ according as $m$ is even or
odd, and $f_1$ is a consequence of polynomials of lower type than $f$. Osborn’s Theorem about the symmetry of the absolutely irreducible polynomial identities is obtained as a corollary.

The method we set up here is applicable not only to searching for identities of matrices but also to find the identities of arbitrary algebras.
Dedication

To the memory of my mother
and
To my
    father,
    wife,
    and my son
for all your love and support.
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Chapter 0

Introduction

0.1 The Identities of $M_n$

Let $F$ be a field, $F[X]$ (respectively $FJ[X]$) denote the free associative (respectively Jordan) algebra over $F$ with generators $X = \{x_1, x_2, \ldots \}$, a countable set of elements $x_1, x_2, \ldots$. Let $f \in F[X]$. Then $f = f(x_1, \ldots, x_n)$ is called a polynomial. Let $A$ be an associative algebra over $F$ and $a_1, a_2, \ldots, a_n \in A$. By the universal property of the free algebra $F[X]$, there exists a homomorphism $\theta$ from $F[X]$ to $A$ such that $\theta(x_i) = a_i$ for $i = 1, 2, \ldots, n$ and $\theta(x_j) = 0$ for $j > n$. Denote $\theta(f)$ by $f(a_1, a_2, \ldots, a_n)$. The polynomial $f$ is called an identity of $A$ if $f(a_1, a_2, \ldots, a_n) = 0$ for any choice of $a_1, a_2, \ldots, a_n \in A$.

Let $M_n$ denote the $n \times n$ matrix algebra with coefficients in the field $F$, $H_n$ (respectively $K_n$) denote the subspace of symmetric (respectively skew symmetric) matrices of $M_n$.

Let

$$S_k(x_1, x_2, \ldots, x_k) := \sum_{\sigma \in S_k} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where $S_k$ is the symmetric group on $k$ objects and $(-1)^\sigma$ the sign of the permutation $\sigma$. $S_k(x_1, x_2, \ldots, x_k)$ is a multilinear alternating polynomial of degree $k$. It is called the standard polynomial.

In 1950, Amitsur and Levitzki determined all polynomial identities of $M_n$ of degree $\leq 2n$ in [AL, Theorem 1 and 3].
Theorem 0.1 [Amitsur, Levitzki]

1. $M_n$ has no identity of degree less than $2n$.

2. $S_{2n}(x_1, x_2, \ldots, x_{2n})$ is an identity of $M_n$.

3. If $|F| > 2$ or $n > 2$, any identity of $M_n$ of degree $2n$ is a scalar multiple of the standard identity $S_{2n}(x_1, x_2, \ldots, x_{2n})$.

Since $S_{2n}(x_1, \ldots, x_{2n})$ is an identity of $M_n$, obviously the polynomials $x_{2n+1}S_{2n}(x_1, \ldots, x_{2n}), S_{2n}(x_1, \ldots, x_{2n}), x_{2n+1}$ and $S_{2n}(x_1, \ldots, x_{2n+1}x_i, \ldots, x_{2n})$ are identities of $M_n$ of degree $2n + 1$. Naturally one may ask the question: Do all identities of $M_n$ of degree $2n + 1$ come from $S_{2n}$ in this way? It was answered positively by Leron in 1973. He proved the following theorem in [LE]:

Theorem 0.2 [Leron] If the characteristic of the field is 0 then every identity of $M_n$ of degree $2n + 1$ is a consequence of the standard identity $S_{2n}$ for $n > 2$, and every identity of $M_2$ of degree 5 is a consequence of the standard identity $S_4$ and $[x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2], [x_3, x_4]x_5]$.

It was also proved by Drensky and Kasparian in [DK] that all identities of $M_3$ of degree 8 are obtained from $S_6$ by substituting some variables of $S_6$ or multiplying $S_6$ by some variables.

0.2 The identities of $H_n$

In 1973, Razmyslov [RZ1] introduced the concept of weak identities, namely, if $V$ is a subspace of an associative algebra $A$ over a field $F$. A polynomial $f \in F[X]$ is a weak identity on $V$ if $f(a_1, a_2, \ldots, a_n) = 0$ for any $a_1, a_2, \ldots, a_n \in V$. Since $H_n$ plays an important role in Jordan theory we are interested in the weak identities of $H_n$ as well as Jordan identities of $H_n$, a weak identity of $H_n$ which is also in $FJ[X]$.

Let $[a, b]$ denote the commutator $ab - ba$ and $\{abc\}$ the triple product $abc + cba$. Let

$$Q(x_1, \ldots, x_6) := \sum_{(123), (456)} \{[x_1, x_2][x_3, x_4][x_5, x_6]\},$$

2
where the commutators are the arguments of the triple product and the sum is taken over cyclic permutations of 1 2 3 and 4 5 6.

Let

\[ T_k^i := \sum_{\sigma \in S_k, \sigma^{-1}(i) \equiv 1, 2 \mod 4} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}, \quad 1 \leq i \leq k, \]

\( S_k \) the symmetric group on \( k \) objects and \((-1)^\sigma\) the sign of the permutation \( \sigma \).

We also define

\[ T_{2n}(x_1, \ldots, x_{2n-1}; x_{2n}) \quad := \quad T_{2n}^{2n}(x_1, \ldots, x_{2n-1}, x_{2n}), \]

\[ T_{2n}(x_1, \ldots, x_{2n-1}; x_{2n}) \quad := \quad S_{2n}(x_{2n}, x_{1}, \ldots, x_{2n-1}) - T_{2n}(x_1, \ldots, x_{2n-1}; x_{2n}). \]

In the paper [MR], M. L. Racine and I have determined all the weak identities of minimal degree for the subspace \( H_n \) of symmetric matrices of the full matrix algebra \( M_n \). The following theorem was proved:

**Theorem 0.3 [MR]** Let \( F \) be a field of arbitrary characteristic. For all \( n \geq 1 \)

1. [Slinsky] \( H_n \) has no weak identity of degree less than \( 2n \).

2. The polynomials \( T_{2n}^i(x_1, x_2, \ldots, x_{2n}) \) are weak identities of \( H_n \) and \( Q(x_1, \ldots, x_6) \) is a weak identity of \( H_3 \).

3. Let \( e(n) \) := \( n \) if \( n \) is even, \( n + 1 \) if \( n \) is odd. If \( \text{char} F \nmid e(n)! \) and \( |F| > 2n \), then for \( n \neq 3 \) all weak identities of \( H_n \) of degree \( 2n \) are consequences of \( T_{2n}^1 \). If \( n = 3 \), then all identities of degree \( 6 \) of \( H_3 \) are consequences of \( T_6^1 \) and \( Q \).

4. There is no Jordan identity of degree \( 2n \) of \( H_n \).

The results in [MR] are analogous to Amitsur and Levitzki's results [AL, Theorem 1 and 3]. Since \( H_n \) is not closed under the usual matrix product but the circle product \( x \circ y := xy + yx \) is closed in \( H_n \), in analogy to Leron's results we may ask the question: Do all weak identities of \( H_n \) of degree \( 2n + 1 \) come from \( T_{2n} \) by substituting \( x \circ y := xy + yx \) for some variables or multiplying them by a variable for \( n \neq 3 \)? For \( n = 3 \) do all identities of \( H_3 \) of degree 7
come from $T_6$ and $Q$ in this way? This is the topic of this thesis. The answer is no! Because in Chapter 2 we shall get a new identity:

$$T_{2n}(x_1, x_2, \ldots, x_{2n-1}; [x_{2n}, x_{2n+1}]),$$

and for $H_3$ we also have a new identity

$$K(x_1, x_2, \ldots, x_7) := \{S_3([x_1, x_2], [x_3, x_4], [x_5, x_6]), x_7\}.$$

Nevertheless, the weak identities of $H_n$ of degree $2n + 1$ will be determined. The key step is an induction argument on $n$ to be found in Chapter 6. Since the identities of $H_3$ do not follow the general scheme, the induction must begin with $n = 4$. This forces us to study the identities of $H_2$, $H_3$ and $H_4$ very carefully (see Chapter 3, 4 and 5).

Let $f(x_1, \ldots, x_m) \in F[X]$ and

$$f = \sum_{\sigma} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}.$$

If $f$ is an identity of $H_n$ then we substitute an element of the form $e_{ji} + e_{jj}$ for each $x_k$ in $f$, $f(x_1, \ldots, x_m) = 0$, where $e_{ij}$ is the usual matrix unit. This yields some homogeneous linear equations which have the $\alpha_\sigma$ as unknowns. So, in principle we can do as many substitutions as we need to determine the coefficients $\alpha_\sigma$. But in practice, it is impossible to get results even using a computer because the number of coefficients $\alpha_\sigma$ is very big. Therefore we have to find ways to cut down the numbers of the unknown. One common way for doing this is following decomposition:

$$f = \frac{1}{2}(f + f^\ast) + \frac{1}{2}(f - f^\ast)$$

is a sum of a symmetric identity and a skew symmetric identity, where $\ast$ is the involution of $F[X]$ defined by $x_i^\ast = x_i$, $\forall x_i \in X$. So we may assume that $f^\ast = \pm f$. Thus, every monomial $x_{\sigma(1)} \cdots x_{\sigma(m)}$ and its reverse $x_{\sigma(m)} \cdots x_{\sigma(1)}$ have the same coefficient in $f$ up to sign. Hence half of the coefficients are cut off. Usually this is still not enough. In Chapter 1, we follow Marshall Osborn’s definition of a partial ordering on the set of homogeneous polynomials of $F[X]$, extend Marshall Osborn’s result to set up a new method of searching for identities of matrices. Then we use it to determine all the weak identities of $H_n$ of degree $2n + 1$ (Chapter 3 to Chapter 6).
Throughout the paper $F$ always denotes a field. The polynomials are in the noncommuting free associative algebra $F[X]$, where $X$ is a set of countable noncommuting indeterminates $x_1, x_2, \ldots$, and identities mentioned are homogeneous weak identities except when otherwise noted.
Chapter 1

A Decomposition of Elements of the Free Algebra

1.1 Introduction

In [OS], Osborn introduced a partial ordering on the free nonassociative algebra $F<X>$ over a field $F$ and proved the following interesting result: if $n$ is a positive integer with $\text{char} F \nmid n!$, and $f$ is an identity of a not necessarily associative algebra $A$ over $F$ such that $A$ has no identity of type lower than $f$ in the partial ordering, then $f$ is either symmetric or skew symmetric in its arguments of degree $n$, depending on whether $n$ is even or odd. This theorem was used in [MR] to determine the identities of degree $2n$ of the space of $n \times n$ symmetric matrices. However, the assumption that $A$ has no identity of lower type limits the use of the result in many cases. In the present chapter, we remove the restriction that $A$ has no identities of lower type and consider polynomials at large, not only polynomial identities. Osborn's Theorem follows as a consequence.

1.2 A Partial Order, $\Delta$-operators and Weak T-ideals

Following Osborn [OS, p.78] we introduce a partial ordering on the set of homogeneous polynomials in $F[X]$, the free associative algebra in infinitely many noncommuting variables $X = \{x_1, x_2, \ldots\}$. The elements of $F[X]$ are called polynomials. If $p(x_1, \ldots, x_m)$ is a homogeneous polynomial of degree $n$, we say that it is of type $[n_1, \ldots, n_m]$ if $n_j$ is the
degree of $x_j$ in $p$ and $n_m \neq 0$ but $n_j = 0$ for $j > m$. Define $[n_1, \ldots, n_m]$ to be $[n_{i_1}, \ldots, n_{i_m}]$, where $n_{i_1} \geq n_{i_2} \geq \cdots \geq n_{i_m}$ and
\[
\{n_{i_1}, \ldots, n_{i_m}\} = \{n_1, \ldots, n_m\}
\]
as multisets. Let $p'$ be another homogeneous polynomial of degree $n'$ and of type $[n'_1, \ldots, n'_m]$. If
\[
[[n'_1, n'_2, \ldots, n'_m]] = [n'_{j_1}, n'_{j_2}, \ldots, n'_{j_m}],
\]
then $p$ is lower than $p'$ in the partial ordering if and only if either (i) $n < n'$ or (ii) $n = n'$ and $n_{i_k} > n'_{j_k}$ for the first integer $k$ such that $n_{i_k} \neq n'_{j_k}$. Otherwise the two polynomials are not comparable. If an integer $n_j$ is repeated $k$ times, we shall denote this by an exponent; for example $[3, 2^2, 1^3]$ means $[3, 2, 2, 1, 1, 1]$.

Let $f(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_m)$ be a homogeneous polynomial of type $[n_1, n_2, \ldots, n_m]$ with coefficients in $F$. Then we define the polynomial
\[
f'(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m) := f(x_1, \ldots, x_i + x_j, \ldots, x_j, \ldots, x_m)
\]
\[
= f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m) + f_1 + \cdots + f_n, \quad (1.1)
\]
where $f_k$ is the homogeneous component of $f'$ of type $[n_1, \ldots, n_i - k, \ldots, n_j + k, \ldots, n_m]$. Following [ZSSS, p.9] we define
\[
\Delta^k(x_i, x_j)f := f_k, \quad k = 1, 2, \ldots, n_i, \quad \Delta^0(x_i, x_j)f := f.
\]
The mapping $\Delta^k$ is called $\Delta$-operator (or a derivation) and $\Delta^k(x_i, x_j)f$ is called a partial linearization of $f$. If $n_i > 0$ and $n_j = 0$ (which is allowed in the above definition), $\Delta^1(x_i, x_j)f$ is of degree $n_i - 1$ in $x_i$ and 1 in $x_j$. Therefore if enough variables of degree 0 were present one can linearize $f$ using successive $\Delta$-operators. We denote by $\Delta f$ the set of all polynomials in $F[X]$ which can be obtained from $f$ by means of repeated $\Delta$-operations:
\[
\Delta f = \{g \in F[X] \mid g = \Delta^{j_1}(x_{i_1}, x_{k_1}) \cdots \Delta^{j_s}(x_{i_s}, x_{k_s})f\}.
\]
Each $f_k$ is obtained by setting $k$ $x_i$'s to be $x_j$ in $f$, and $x_i, x_j$ have degree $n_i - k, n_j + k$ in $f_k$ respectively.

It is well-known that the set $T(A)$ of all polynomial identities of an $F$-algebra is a $T$-ideal, i.e. an ideal of $F[X]$ which is invariant under all endomorphisms of $F[X]$. However, this is no longer true for weak identities. So we introduce the concept of weak $T$-ideals.
Definition 1.2.1 An ideal $W$ of $F[X]$ is called a weak $T$-ideal if $W$ is invariant under every linear mapping $x_i \mapsto \sum \alpha_{ij} x_j$, for $\alpha_{ij} \in F$.

If $S$ is a subset of $F[X]$ then the smallest weak $T$-ideal containing $S$ in $F[X]$ is called the weak $T$-ideal generated by $S$ and denoted by $\langle S \rangle$.

One basic example of weak $T$-ideal is the ideal $T(A,V)$ consisting of all weak polynomial identities on a subspace $V$ of an algebra $A$.

Using a standard Vandermonde argument (e.g., [ZSSS, p.12, Theorem 5]) one can prove

Proposition 1.2.1 Let $f(x_1, \ldots, x_i, \ldots, x_j, \ldots) \in F[X]$ be a homogeneous polynomial of type $[\ldots, n_i, \ldots, n_j, \ldots]$. If $|F| > n_i$ then $\Delta^k(x_i, x_j)f \in \langle f \rangle$, $k = 0, 1, \ldots, n_i$. In particular, if $f \in T(A,V)$ then $\Delta^k(x_i, x_j)f \in T(A,V)$ for $k = 0, 1, \ldots, n_i$.

Let $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ be homogeneous polynomials of $F[X]$. We shall say that $g$ is a consequence of (or comes from) the polynomial $f$ if $g$ belongs to the vector space $\text{Span}\{\Delta f\}$.

Since a partial linearization of an identity $f$ on some algebra is not necessarily an identity, the polynomials in $\Delta f$ need not be identities either. But from Proposition 1.2.1, if $F$ contains enough elements then $\Delta f$ lies in the weak $T$-ideal $\langle f \rangle$, and hence if $g$ comes from $f$ then $g$ is a linear combination of identities in $\langle f \rangle$.

1.3 A Decomposition of Elements of $F[X]$

As in [ZSSS], $F[X]$ has a decomposition into a direct sum of subspaces $V^{[n_1,\ldots,n_k]}[X]$ consisting of the homogeneous polynomials of type $[n_1,\ldots,n_k]$. In this section we shall give a decomposition of the elements of $V^{[n_1,\ldots,n_k]}[X]$ which is useful in the study of identities.

Theorem 1.1 Let $n$ be a positive integer and let $p$ be a homogeneous polynomial with coefficients in a field $F$ of characteristic not dividing $n!$. Let $x$ and $y$ be arguments of degree $n$. Define $p_1$ by

$$p_1 := p(\ldots, y, \ldots, x, \ldots) - (-1)^n p(\ldots, x, \ldots, y, \ldots).$$

Then the following statements hold:
1. The polynomial $p_1$ comes from a polynomial of lower type than $p$.

2. If there exist two variables in $p$ either of even degree in which $p$ is skew symmetric, or of odd degree in which $p$ is symmetric, then $p = \frac{1}{2}p_1$ comes from a polynomial of lower type.

3. If $|F| > 2n - 1$ and $p$ is an identity, then $p_1$ is a linear combination of identities of type lower than $p$.

4. If $\Delta(y,x)p = 0$ then $p$ is either symmetric or skew symmetric in $x$ and $y$ depending on whether $n$ is even or odd.

Proof. Define

$$\tilde{p}_1 := \alpha_0 f_n + \alpha_1 f_{n-1} + \ldots + \alpha_{n-1} f_1,$$

where $\alpha_k = (-1)^k \frac{k!}{n(n-1)\ldots(n-k)!}, 0 \leq k \leq n - 1$,

$$f_1 := \Delta^0(y,x)\Delta^1(x,y)\Delta^1(y,x)p,$$

$$f_2 := \Delta^1(y,x)\Delta^2(x,y)\Delta^1(y,x)p - \binom{n}{1} f_1,$$

$$f_{m+1} := \Delta^m(y,x)\Delta^{m+1}(x,y)\Delta^1(y,x)p - \binom{n}{m} f_1 - \binom{n-1}{m-1} f_2 - \ldots - \binom{n-m+1}{1} f_m. \tag{1.2}$$

The polynomial $\tilde{p}_1$ comes from $\Delta^1(y,x)p$ which is of lower type than $p$.

If $|F| > 2n - 1$ and $p$ is an identity, then for each $i$, $i = 1, \ldots, n$, $f_i \in < \Delta^1(y,x)p >$ by Proposition 1.2.1 and hence $\tilde{p}_1$ is a linear combination of consequences of lower type identities $f_i$. Thus, to prove 1) and 3) of the theorem it suffices to show that

$$p(\ldots, y, \ldots, x \ldots) = (-1)^n p(\ldots, z, \ldots, y, \ldots) + \tilde{p}_1 \tag{1.3}$$

Since $\Delta^k(x,y)$ and $\Delta^m(y,x)$ change only the variables $x$ and $y$, the polynomials $p(\ldots, y, \ldots, x \ldots)$, $(-1)^n p(\ldots, x, \ldots, y, \ldots)$ and $\tilde{p}_1$ are of the same type. So, they are linear combinations of the same monomials. Thus, to show (1.3) it suffices to show that for
each monomial $M$, the coefficient of $M$ in $p(..., y, ..., x, ...) $ and the coefficient of $M$ in 
$(-1)^np(..., x, ..., y, ...) + \tilde{p}_1$ are the same.

First, we define an equivalence relation on the set of monomials occurring in $p$: $M_1 \sim M_2$ if $M_2$ is obtainable from $M_1$ by permuting some $x$'s and $y$'s and leaving the other variables fixed. We denote by $[M]$ the equivalence class of $M$. Let $I$ be the set of all associative words in $x$ and $y$ involving exactly $n$ $x$'s and $n$ $y$'s. Setting the variables, other than $x$ and $y$, equal to 1 in an element of $[M]$ yields a bijection from $[M]$ to $I$; so we may write $[M] = \{M_i|i \in I\}$ and in particular $M = M_w$ for some $w \in I$. Let $I^{(k)}, k = -1, 0, 1, \ldots, n$, be the set of all associative words which have $n + k$ $y$'s and $n - k$ $x$'s. Then $I = I^{(0)}$ and $I' := I^{(-1)}$ is the set of all associative words in $x$ and $y$ involving exactly $n + 1$ $x$'s and $n - 1$ $y$'s. For $i, j \in I' \cup I' \cup I^{(1)} \cup \cdots \cup I^{(n)},$ we define $i \cdot j$ to be the number of positions in which both $i$ and $j$ have a $y$.

Let $c_i$ be the coefficient of $M_i$ in $p$ and $r_j$ be the coefficient of $M_j$ in $p' := \Delta^1(y, x)p$. Then for each $j \in I'$ the total coefficient of $M_j$ in $p'$ will be the sum of the $n + 1$ coefficients $c_i$ where $i$ runs over all those elements of $I$ such that $i \cdot j = n - 1$, i.e.

$$r_j = \sum_{i : i \cdot j = n - 1} c_i, \quad \forall j \in I'. \tag{1.4}$$

If $k = j \cdot w$ then every $i$ occurring in (1.4) satisfies either $i \cdot w = k$ or $i \cdot w = k + 1$. Indeed since $j$ has $n - 1$ elements $y$'s and $i \cdot j = n - 1$, $i$ has a $y$ in a given position whenever $j$ does. So, in the positions where both $j$ and $w$ have a $y$, $i$ has a $y$ also. Therefore $j \cdot w = k$ implies $i \cdot w$ is at least $k$. But $i$ has only one more $y$ than $j$, so $i \cdot w \leq k + 1$. Each $i \in I$ satisfying $i \cdot w = k$ occurs in exactly $n - k$ equations of type (1.4) for which $j \cdot w = k$. Indeed $i$ occurs in (1.4) iff $M_i$ becomes $M_j$ when replacing one $y$ by an $x$ in $M_i$; since $i \cdot w = k$, if we put one $y$ in $M_i$ which occurs in the same position in $w$ to be $x$ to get $M_j$ then $j \cdot w = k - 1$ not $k$. This can't happen so we have only $n - k$ choices and each one produces distinct $M_j$'s, so $c_i$ occurs $n - k$ times in the set of equations of type (1.4) satisfying $j \cdot w = k$. Each $i \in I$ satisfying $i \cdot w = k + 1$ occurs in exactly $k + 1$ equations of type (1.4) satisfying $j \cdot w = k$, for, this time, we can only choose those $y$'s which occur in the same position in $i$ and $w$ in order to get $j \cdot w = k$. 

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Now, we add up all equations of type (1.1) satisfying \( j \cdot w = k \) to get

\[
\sum_{j \cdot w = k} r_j = (n-k)C_k + (k+1)C_{k+1}, \quad k = 0, 1, 2, \ldots, n-1.
\]  

(1.5)

where, for \( 0 \leq m \leq n \), \( C_m := \sum_{i \cdot w = m} c_i \), with \( c_i \) occurring in an equation of type (1.1) satisfying \( j \cdot w = k \). Substituting the \( n \) equations of (1.5) into one another yields

\[
C_0 = (-1)^n C_n + (-1)^{n-1} \frac{(n-1)!}{n(n-1) \cdots 1} \sum_{j \cdot w = n-1} r_j + (-1)^{n-2} \frac{(n-2)!}{n(n-1) \cdots 2} \sum_{j \cdot w = n-2} r_j + \cdots
\]

\[
- \frac{1}{n(n-1)} \sum_{j \cdot w = 1} r_j + \frac{1}{n} \sum_{j \cdot w = 0} r_j.
\]

We know that \( C_n = c_w \) and \( C_0 = c_{w'} \), where \( w' \) is the word obtained from \( w \) by interchanging all the \( x \)'s and \( y \)'s. Thus we have that

\[
c_{w'} = (-1)^n c_w + \alpha_{n-1} \beta_{n-1} + \ldots + \alpha_0 \beta_0,
\]

where \( \alpha_k = (-1)^k \frac{k!}{n-\ldots-(n-k)} \) and \( \beta_k = \sum_{j \cdot w = k} r_j \), \( k = 0, 1, \ldots, n-1 \). Next we show that \( \sum_{i=0}^{n-1} \alpha_i \beta_i \) is the coefficient of the monomial \( M_w \) in \( \hat{p}_1 \), and therefore \( \hat{p}_1 \) comes from \( p' \). By the definition of \( \hat{p}_1 \), it suffices to show that each \( \beta_{n-k} \) is the coefficient of \( M_w \) in \( f_k \).

For \( u, u', v \in I \cup I' \cup I^{(1)} \cup \ldots \cup I^{(n)} \) we define \( \#(v) := \) the degree of \( y \) in \( v \), and we say that \( u \leq u' \) if whenever \( u \) has a \( y \) in a given position, so does \( u' \). Define \( u \cup u' \) to be the word which has a \( y \) in a given position iff \( u \) or \( u' \) does (e.g. if \( u = xxyzyy \) and \( u' = yxyxxy \) then \( u \cup u' = yxyzxyy \) ). From the definition we have that if \( u \leq t \) and \( u' \leq t \) then \( u \cup u' \leq t \) and \( \#(u \cup u') = \#(t) \).

Next let us consider \( \beta_{n-1} \). Since \( f_1 \) is obtained by setting one \( x \) equal to \( y \) in \( p' \), the coefficient of \( M_w \) in \( f_1 \) is

\[
\gamma_w = \sum_{j \cdot w = n-1} r_j = \beta_{n-1}.
\]

Suppose \( \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{n-m} \) are the coefficients of \( M_w \) in \( f_1, f_2, \ldots, f_m \) respectively. Now for \( \beta_{n-m-1} \) putting \( m + 1 \) elements \( x \)'s to be \( y \) in \( \Delta^1(y,x)p \) yields the polynomial \( \Delta^{m+1}(x,y)p \Delta^1(y,x)p \), then putting \( m \) elements \( y \)'s to be \( x \) in \( \Delta^{m+1}(x,y)p \Delta^1(y,x)p \) we get the polynomial \( \Delta^m(y,x) \Delta^{m+1}(x,y)p \Delta^1(y,x)p \). In \( \Delta^{m+1}(x,y) \Delta^1(y,x)p \), the monomial \( M_t \) has coefficient \( \lambda_t = \sum_{j \cdot t = n-1} r_j \), since \( \#(j) = n-1 \) and \( \#(t) = n + m = (n-1) + (m+1) \).
(If $j \cdot t < n - 1$ then putting $m + 1$ $x$'s to be $y$ in $\mathcal{M}_j$ cannot yield $\mathcal{M}_t$ which has $n + m$ $y$'s). Similarly, in $\Delta^m(y,x)\Delta^{m+1}(x,y)\Delta^1(y,x)p$, $M_w$ has coefficient $\mu_w = \sum_{t \in I^{(m)}, t \cdot w = n} \lambda_t$, since 
$\#(t) = n + m$ and $\#(w) = n$. Thus 

$$\mu_w = \sum_{t \in I^{(m)}, t \cdot w = n} \left( \sum_{j \in I', j \cdot t = n - 1} r_j \right) \quad (1.6)$$

We claim that

(*) \{ $j \in I' \mid j \cdot t = n - 1$ for some $t \in I^{(m)}$ and $t \cdot w = n$
= \{ $j \in I' \mid n - m - 1 \leq j \cdot w \leq n - 1$ \}.

For if $j \in I'$ with $j \cdot t = n - 1$, $\#(t) = n + m$ and $t \cdot w = n$ then $j \leq t$, $w \leq t$ and $j \cup w \leq t$. Since $\#(j) = n - 1$, $j \cdot w \leq n - 1$. If $j \cdot w \leq n - m - 2$ then $j \cup w \leq t$ implies that $\#(j \cup w) \leq n + m$. But 

$\#(j \cup w) = n - 1 + (n - j \cdot w) \geq n - 1 + n - n + m + 2 = n + m + 1,$

a contradiction. On the other hand, if $j \cdot w = n - k$, $k = 1, 2, \ldots, m + 1$ then 

$\#(j \cup w) = n - 1 + n - (n - k) = n - 1 + k \leq n + m,$

therefore there exists a word $t \in I^{(m)}$ such that $j \cup w \leq t$. Since $j \leq t$ and $w \leq t$, $j \cdot t = \#(j) = n - 1$ and $w \cdot t = \#(w) = n$. This establishes (*). It also means that $r_j$ occurs in (1.6) iff $j \cdot w = n - k$ for some $k$ with $1 \leq k \leq n - m - 1$.

Each $r_j$ may occur several times in (1.6). We group the $r_j$'s in (1.6) according as $j \cdot w = n - k$ for each $k$. We claim that

(**) each $r_j$ with $j \in I'$ satisfying $j \cdot w = n - k$ occurs \(\binom{n - k + 1}{m - k + 1}\) times in (1.6).

Indeed, the number of the occurrences of $r_j$ in (1.6) is exactly the number of words $t \in I^{(m)}$ such that $t \cdot w = n$ and $j \cdot t = n - 1$. That is $t \in I^{(m)}$ such that $w \cup j \leq t$ since $\#(j) = n - 1$. Now, if $w \cdot j = n - k$ then

$\#(w \cup j) = n + n - 1 - n + k = n + k - 1.$
So, the word obtained by changing \( m - k + 1 \) \( x \)'s in \( w \cup j \) to be \( y \)'s belongs to \( f^{(n)} \) and \( w \cup j \leq t \). However, the number of choices for such a word \( t \) is \( 2n - (n + k - 1) \) choose \( m - k + 1 \), that is \( \binom{n - k + 1}{m - k + 1} \). Hence the (**) holds and

\[
\mu_w = \sum_{t\in n} \sum_{j\in n-1} r_j = \binom{n}{m} \sum_{j\in n-1} r_j + \binom{n-1}{m-1} \sum_{j\in n-2} r_j + \ldots + \binom{n - m + 1}{1} \sum_{j\in n-m} r_j + \sum_{j\in n-m-1} r_j.
\]

Since \( \beta_{n-k} = \sum_{j\in n-k} r_j \),

\[
\beta_{n-m-1} = \mu_w - \binom{n}{m} \beta_{n-1} - \binom{n-1}{m-1} \beta_{n-2} - \ldots - \binom{n - m + 1}{1} \beta_{n-m}.
\]

Thus, by the induction hypothesis and the definition of \( f_{m+1} \), \( \beta_{n-m-1} \) is the coefficient of \( M_w \) in \( f_{m+1} \).

From above we know that \( \sum a_k \beta_k \) is the coefficient of \( M_w \) in \( \tilde{p}_1 \). We also know that \( c_w \) and \( (-1)^n c_w \) are the coefficients of \( M_w \) in \( p(\ldots, y, \ldots, x, \ldots) \) and \( (-1)^n p(\ldots, x, \ldots, y, \ldots) \) respectively. Thus (1.3) holds. So 1) and 3) are proved. The second statement of the theorem follows easily from (1.3).

If \( \Delta(y, x)p \equiv 0 \) then \( \tilde{p}_1 = 0 \) by the definition of \( \tilde{p}_1 \). Hence \( p_1 = 0 \), which implies that \( p \) is either symmetric or skew symmetric in \( x \) and \( y \) depending on whether \( n \) is even or odd.

Theorem 1.1 is also true for the free nonassociative algebra. The proof is exactly the same except the definition of the mapping from \( [M] \) to \( I \). In this case, erasing the brackets and the variables, other than \( x \) and \( y \), in an element of \( [M] \) yields a bijection from \( [M] \) to \( I \) since elements in \( [M] \) have the same distribution of the brackets. Thus from Theorem 1.1 we have

**Corollary 1.3.1** [Osborn's Theorem]. Let \( p \) be a homogeneous identity in a nonassociative algebra \( A \) over a field \( F \) of characteristic not dividing \( n! \). If \( A \) has no identity of type lower than \( p \) then \( p \) is either symmetric or skew symmetric in its arguments of degree \( n \), depending on whether \( n \) is even or odd.
Theorem 1.1 says that for each pair of arguments $x, y$ of degree $n$ there exists a polynomial $p' = \frac{1}{2}(p(\ldots, x, \ldots, y, \ldots) - (-1)^n p(\ldots, y, \ldots, x, \ldots))$ which comes from polynomials of lower type such that $p - p'$ is symmetric or skew symmetric in $x, y$ according as $n$ is even or odd. In fact, we can find a polynomial $p_1$ which comes from some polynomials of type lower than that of $p$ such that $p - p_1$ is symmetric or skew symmetric in all its arguments of degree $n$, according to $n$ is even or odd.

**Theorem 1.2** Let $r$ be an integer and $F$ be a field with char $F \neq r!$. Let $V$ be an $S_r$-module then

1. $V = \{v \in V | \pi v = v, \forall \pi \in S_r\} + \left(\sum_{1 \leq i < j \leq r} \{v \in V | (i, j)v = -v\}\right)$;

and

2. $V = \{v \in V | \pi v = -v, \forall \pi \in S_r\} + \left(\sum_{1 \leq i < j \leq r} \{v \in V | (i, j)v = v\}\right)$.

**Proof.** To prove the theorem we use induction on $r$. In what follows, we agree that the product of two permutations is performed from right to left. For $r = 1$, the theorem holds obviously. For $r = 2$, every $v \in V$ has a unique decomposition

$$v = \frac{1}{2}(v + (12)v) + \frac{1}{2}(v - (12)v).$$

Suppose the result is true for $r - 1$. We imbed $S_{r-1}$ into $S_r$ as $\{\pi \in S_r | \pi(r) = r\}$. Thus a given $S_r$-module is also an $S_{r-1}$-module. Hence by induction,

$$V = \{v \in V | \pi v = v, \forall \pi \in S_{r-1}\} + \left(\sum_{1 \leq i < j \leq r-1} \{v \in V | (i, j)v = -v\}\right).$$

Let $v \in V^{S_{r-1}} := \{v \in V | \pi v = v, \forall \pi \in S_{r-1}\}$, and define

$$w := v - \frac{1}{r} \sum_{1 \leq i < r-1} (v - (ir)v).$$

It suffices to show $w \in V^{S_r}$. We have

$$w = \frac{1}{r}(v + \sum_{1 \leq i \leq r-1} (ir)v),$$
hence, for $1 \leq j \leq r - 1$,

$$(jr)v = \frac{1}{r}[(jr)v + v + \sum_{1 \leq i \leq r-1, i \neq j}(jr)(ir)v]$$

$$= w$$

since $(jr)(ir) = (ir)(ij)$ and thus

$$(jr)(ir)v = (ir)(ij)v = (ir)v$$

using $v \in V^{S_r}$. Because $\{(jr)|1 \leq j \leq r - 1\}$ generates $S_r$, we shown that $w \in V^{S_r}$, completing the proof of part 1 of Theorem 1.2.

Symmetrically, to prove part 2 of the theorem we define

$$w' := v' - \frac{1}{r} \sum_{1 \leq i \leq r-1} (v' + (ir)v'),$$

where $v' \in \{v \in V|\pi v = -v, \forall \pi \in S_r\}$. Now we can repeat the proof of part 1 of the theorem step by step to show that $w' \in \{v \in V|\pi v = -v, \forall \pi \in S_r\}$. This completes the proof of Theorem 1.2.

Let $V = V[m, n, \ldots, u][X]$, and

$$W_1 = \text{Span}\{f(x_1, \ldots, x_r, y_1, \ldots, z_l) \in V|f \text{ is skewsymmetric in } x_i \text{ and } z_j \text{ for some } i \neq j\}.$$ Let $U$ be the set of the elements $f(x_1, \ldots, x_r, y_1, \ldots, z_l) \in V$ which are either symmetric or skew symmetric in some $w_i, w_j$ where $i \neq j$ and $w \in \{x, y, \ldots, z\}$, according as the degree of $w_i$ in $f$ is odd or even. Let $V_1 = \text{Span}U$ then we have

**Theorem 1.3** Let $m_0 = \max\{m, n, \ldots, u, r, s, \ldots, t\}$. If $\text{char} F \nmid m_0$ then every element $f$ in $V$ has a decomposition into a sum of two polynomials of the same type as $f$, $f = f_0 + f_1$, where, for each $k$ with $0 < k \leq m_0$, $f_0$ is either symmetric or skew symmetric in all variables of degree $k$ depending on whether $k$ is even or odd and $f_1 \in V_1$. Moreover $V_1$ is spanned by elements which come from lower type polynomials: if $g \in V_1$ then $g = \sum g_k$, where $g_k \in U$ and $g_k$ is given by formula (1.2).
Proof. Let \( f(x_1, \ldots, x_r, y_1, \ldots, y_s, \ldots, z_1, \ldots, z_t) \) in \( V \). First we show that \( f = f_0 + f_1 \), where \( f_1 \in V_1 \) and \( f_0 \) is either symmetric or skew symmetric in all \( x_i \)'s according as \( m \) is even or odd. For an arbitrary \( \sigma \in S_r \) we define
\[
\sigma(f(x_1, \ldots, x_r, y_1, \ldots, y_s, \ldots, z_1, \ldots, z_t)) := f(x_{\sigma(1)}, \ldots, x_{\sigma(r)}, y_1, \ldots, y_s, \ldots, z_1, \ldots, z_t).
\]
Then \( V \) is a \( S_r \)-module. If \( m \) is even then by 1) of Theorem 1.2, \( f = f_0 + f_1 \), where \( f_1 \in W_1 \) and \( f_0 \) is symmetric in all \( x_i \)'s. Since \( W_1 \subseteq V_1 \) by the definitions of \( W_1 \) and \( V_1 \), we get \( f_1 \in V_1 \). If \( m \) is odd then we use 2) of Theorem 1.2.

In any case, we have that \( f = h_0 + h_1 \), where \( h_1 \in V_1 \) and \( h_0 \) is either symmetric or skew symmetric in all \( x_i \)'s according as \( m \) is even or odd. Next, we consider \( h_0 \) and variables \( y_1, \ldots, y_s \) as above. The above process will not destroy the symmetry of \( h_0 \) in the \( x_i \)'s. Thus after a finite number of steps, we have that \( f = f_0 + f_1 \), where \( f_1 \in V_1 \) and \( f_0 \) is as in the theorem.

The last statement of the Theorem follows from Theorem 1.1 and the the definition of \( U \).

To summarize, if \( f \in T(A, \mathcal{W}) \cap V \) then \( f = f_0 + f_1 \) with \( f_i \) an identity, for \( i = 0, 1 \) by Theorem 1.3 and the fact that \( T(A, \mathcal{W}) \cap V \) is an \( S_r \)-module. We know in all interesting cases \( f_1 \) comes from some lower type identities. Using Proposition 1.2.1 We have

**Corollary 1.3.2** Let \( m_0 = \max\{m, u, \ldots, u, r, s, \ldots, t\} \). If \( \operatorname{char} F \nmid m_0! \) and \( |F| > 2m_0 - 1 \) then every identity \( f \) in \( V \) has a decomposition into a sum of two identities of the same type as \( f \), \( f = f_0 + f_1 \) where, for each \( k \) with \( 0 < k \leq m_0 \), \( f_0 \) is either symmetric or skew symmetric in all variables of degree \( k \) depending on whether \( k \) is even or odd and \( f_1 \) comes from lower type identities.
Chapter 2

The General Results About The Identities of $H_n$

2.1 Some Formulas About The Identities of $H_n$ of Degree $2n + 1$

To find polynomial identities we must determine the coefficients of monomials in a polynomial identity. One basic method to do this is substitution. In this section we obtain some important lemmas using specific substitutions. From now on $e_{ij}$ denotes the matrix $e_{ij} + e_{ji}$ for $i \neq j$ and $e_{ii}$ denotes $e_{ii}$, where $e_{rs}$ is the usual matrix unit.

In what follows, we agree that the product of two permutations is performed from right to left. Let $\sigma \in S_r$. Then, if

$$
\sigma = \begin{pmatrix}
1, \ldots, r \\
\sigma(1), \ldots, \sigma(r)
\end{pmatrix}
$$

we denote this $\sigma$ in the following simply by $\sigma(1) \cdots \sigma(r)$. The permutation $\sigma(r)\sigma(r-1) \cdots \sigma(2)\sigma(1)$ is called the reverse permutation of $\sigma$ and denoted by $\rho \sigma$.

Lemma 2.1.1 Let $p(x_1, \ldots, x_{2n+1})$ be a multilinear identity of $H_n$. If the monomial $x_{\sigma(1)} \cdots x_{\sigma(2n+1)}$ has coefficient $\alpha_\sigma$ in $p$ for some $\sigma \in S_{2n+1}$, the symmetric group on $2n+1$ objects, then the following statements hold:

1. For $i, j$ with $1 \leq i, j \leq n - 1$ and $i \neq j$,

$$
\alpha_\sigma + \alpha_{(2i-1,2i)\sigma} + \alpha_{(2i+2j,2i+2j+1)\sigma} + \alpha_{(2i-1,2i)(2i+2j,2i+2j+1)\sigma} = 0.
$$
2. \( \sum_{\sigma \in S_3} \alpha_{\sigma} = 0 \), where \( S_3 \) is the symmetric group on \( 2i - 1, 2i \) and \( 2i + 1 \).

3. For \( i, j \) with \( 1 \leq i, j \leq n \) and \( i \neq j \),

\[
\alpha_{\sigma} + \alpha(2i-2,2i-1)\sigma + \alpha(2i+2j+1,2i+2j+2)\sigma + \alpha(2i+2j+1,2i+2j+2)(2i-2,2i-1)\sigma + \\
\alpha_{\rho} + \alpha(2i-2,2i-1)\rho + \alpha(2i+2j+1,2i+2j+2)\rho + \alpha(2i+2j+1,2i+2j+2)(2i-2,2i-1)\rho = 0,
\]

where \( \rho \sigma \) denotes the reverse permutation of the permutation \( \tau \).

4. Let \( \sigma(1)\tau(2n+1) := \sigma(1) \cdots \sigma(2n + 1) \in S_{2n+1} \) then

\[
\alpha_{\sigma(1)\rho(2n+1)} + \alpha_{\sigma(1)\rho(2n+1)} + \alpha_{\sigma(1)\rho(2n+1)} + \alpha_{\sigma(1)\rho(2n+1)} + \\
\alpha_{\rho(\sigma(1)\rho(2n+1))} + \alpha_{\rho(\sigma(1)\rho(2n+1))} + \alpha_{\rho(\sigma(1)\rho(2n+1))} + \\
\alpha_{\rho(\sigma(1)\rho(2n+1))} + \alpha_{\rho(\sigma(1)\rho(2n+1))} + \alpha_{\rho(\sigma(1)\rho(2n+1))} + \\
\alpha_{\rho(\sigma(1)\rho(2n+1))} + \alpha_{\rho(\sigma(1)\rho(2n+1))} + \alpha_{\rho(\sigma(1)\rho(2n+1))} = 0.
\]

Let \( f \) be a homogeneous identity of \( H_n \). If \( \text{char} \mathbb{F} \neq 2, 3 \) then the words having the same indeterminate in positions \( 2i - 1 \) and \( 2i \) and the same indeterminate in positions \( 2i + 2j \) and \( 2i + 2j + 1 \), or having the same indeterminate in positions \( 2i - 1, 2i \) and \( 2i + 1 \), have coefficients 0 in \( f \).

**Proof.** Let \( w := x_{\sigma(1)} \cdots x_{\sigma(2n+1)} \). The operator \((i, j)\) acts on \( w \) by permuting \( x_{\sigma(i)} \) and \( x_{\sigma(j)} \). Substituting

\[
e_{[11]}, e_{[12]}, e_{[22]}, e_{[23]}, \ldots, e_{[i - 1, i - 1]}, e_{[i - 1, i]}, e_{[i i]}, e_{[i, i + 1]}, e_{[i + 1, i + 1]}, e_{[i + 1, i + 2]}, \ldots, e_{[i + j, i + j]}, e_{[i + j, i + j]}, e_{[i + j, i + j]}, \ldots, e_{[nn]}
\]

for

\[
x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(2n+1)}
\]

in \( p \) there are only 4 monomials

\[
w, (2i - 1, 2i)w, (2i + 2j, 2i + 2j + 1)w, (2i - 1, 2i)(2i + 2j, 2i + 2j + 1)w
\]

which produce \( e_{1n} \). This implies 1.
Similarly, substituting
\[ e[11], e[12], e[22], \ldots, e[i - 1, i], e[i], e[i], e[i, i + 1], \ldots, e[nn] \]
for \( x_{\sigma(1)}, \ldots, x_{\sigma(2n+1)} \) in \( p \) and considering the coefficient of \( e_{1n} \) yields 2.

Substituting
\[
e[12], e[22], e[23], \ldots, e[i - 1, i], e[i], e[i], e[i, i + 1], \ldots, e[i + j, i + j], e[i + j, i + j], e[i + j, i + j + 1], \ldots, e[nn], e[n1]
\]
for \( x_{\sigma(1)}, \ldots, x_{\sigma(2n+1)} \) in \( p \) and considering the coefficient of \( e_{11} \) yields 3.

Substituting
\[ e[11], e[12], e[22], e[23], \ldots, e[nn], e[n1] \]
for \( x_{\sigma(1)}, \ldots, x_{\sigma(2n+1)} \) in \( p \) and considering the coefficient of \( e_{11} \) yields 4.

The last statement is obtained by linearizing \( f \) and then applying the results 1 and 2 respectively.

As in Lemma 2.2 of [MR], we have

**Lemma 2.1.2** Let \( f(x, y, z, \ldots) \) be an identity of \( H_n \), \( n_1 \) and \( n_2 \) be the degree of \( x \) and \( y \)
in \( f \) respectively. If the characteristic of \( F \) does not divide \((n_1 - 1)!\) and \((n_2 - 1)!\) then the following statements hold:

1. [MR, Lemma 2.2] If \( f = f_1yx + \sim \), where \( \sim \) is the sum of the terms in \( f \) which do not end in \( yz \). Then \( f_1 \) is an identity of \( H_{n-1} \).

2. If \( f = f_1y^2x + \sim \), where \( \sim \) is the sum of the terms in \( f \) which do not end in \( yz^2 \). Then \( f_1 \) is an identity of \( H_{n-1} \).

**Proof.** We prove 2. Linearizing \( x \) and \( y \) in \( f \) completely we get an identity denoted by \( \text{lin}_{x,y}f \). Then from \( f = f_1y^2x + g \),

\[
\text{lin}_{x,y}f(x_1, \ldots, x_n, y_1, \ldots, y_{n_2}, z, \ldots) = \\
\text{lin}_{x,y}f_1(x_3, \ldots, x_n, y_2, \ldots, y_{n_2}, z, \ldots) y_1x_1x_2 + \ldots + \text{lin}_{x,y}g.
\]

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We claim that $\text{lin}_{x,y} f_1$ is an identity of $H_{n-1}$. For, if it is not then there exist $a_3, \ldots, a_{n_1}, b_2, \ldots, b_{n_2}, c, \ldots \in H_{n-1}$ such that

$$\text{lin}_{x,y} f_1(a_3, \ldots, a_{n_1}, b_2, \ldots, b_{n_2}, c, \ldots) = \alpha e_{ij} + A,$$

with $\alpha \neq 0$, $e_{ij}$ does not occur in the matrix $A$ and $A \in H_{n-1}$. Letting $y_1 = e[jn]$ and $x_1 = x_2 = e_{nn}$ then $\text{lin}_{x,y} f(x_1, x_2, a_3, \ldots, a_{n_1}, y_1, b_2, \ldots, b_{n_2}, c, \ldots) = 0$ implies that

$$\text{lin}_{x,y} f_1(a_3, \ldots, a_{n_1}, b_2, \ldots, b_{n_2}, c, \ldots)y_1(x_1 \circ x_2) + 0 = 0.$$

But

$$\text{lin}_{x,y} f_1(a_3, \ldots, a_{n_1}, b_2, \ldots, b_{n_2}, c, \ldots)y_1(x_1 \circ x_2) = 2\alpha e_{in} + B,$$

with $2\alpha \neq 0$ and $e_{in}$ does not occur in the matrix $B$, a contradiction. Thus $\text{lin}_{x,y} f_1$ is an identity of $H_{n-1}$.

Since

$$\text{lin}_{x,y} f_1(x, \ldots, x, y, \ldots, y, \ldots) = (n_1 - 2)! (n_2 - 1)! f_1,$$

and the characteristic of the field $F$ does not divide $(n_1 - 2)!$ and $(n_2 - 1)!$, $f_1$ is an identity of $H_{n-1}$.

Let $V$ be the set of the homogeneous identities of $H_n$ of degree $2n + 1$ and

$$V[n_1, \ldots, n_r] = \{ f \in V \mid f \text{ is of type } [n_1, \ldots, n_r] \}.$$

Then $V[n_1, \ldots, n_r]$ is a subspace of the vector space $V$. Our aim is to find a basis of $V$. Since $V = \oplus V[n_1, \ldots, n_r]$, it suffices to find a basis for each $V[n_1, \ldots, n_r]$. Although the proof is simple, the lemma below gives a way to reduce the calculations when one searches for a basis of $V[n_1, \ldots, n_r]$.

**Lemma 2.1.3** Let $W$ be a subspace of $F[X]$. If $Q_1, \ldots, Q_m \in W$ are linearly independent and there exist polynomials $p_1, \ldots, p_m \in F[X]$ such that $W \subseteq \text{Span}\{p_1, \ldots, p_m\}$, then $\dim W = m$ and $\{Q_1, \ldots, Q_m\}$ is a basis of $W$. 

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2.2 The Razmyslov Transformation

In this section we introduce the Razmyslov transformation of polynomials. In the following all identities mentioned are weak identities.

Definition 2.2.1 Let $f(x, y, \ldots; z)$ be a polynomial of degree $t$ in $z$. If

$$f = \sum_i g_i z h_i,$$

then following [FM] we define

$$f^\#(z) := \sum_i h_i z g_i,$$

which is called the Razmyslov transform of $f$ with respect to $z$.

Obviously, the Razmyslov transform of an identity need not be an identity. However it is still a kind of zero operator. So, we introduce the concept of mixed identity.

Definition 2.2.2 Let $f(x, y, \ldots, z; w)$ be a polynomial of degree one in $w$. If $f(x, y, \ldots, z; w) = 0$ for $x, y, \ldots, z \in H_n$ and $w \in K_n$ then $f$ is called a mixed identity of $H_n$.

Similar to Lemma 2.1.2 and the Lemma 2.2 of [MR] we have

Proposition 2.2.1 Let $f(x_1, x_2, \ldots; x_n)$ be a multilinear mixed identity of $H_n$ and $f = f_1 x_n x_{n-1} + f_2 x_i x_j + \sim$, where $\sim$ denotes the sum of the terms of $f$ which do not end in $x_n x_{n-1}$ or $x_i x_j$ for $i, j \neq n$. Then $f_2$ is a mixed identity and $f_1$ is an identity.

If $f(x_1, x_2, \ldots; y)$ is a mixed identity over a field $F$ of characteristic not dividing the maximum of the degrees of the variables in $f$, and $f = f_1 y x_1^2 + f_2 x_i x_j^2 + \sim$, then $f_1$ is an identity and $f_2$ is a mixed identity.

In following, $\ast$ is the involution on $F[X]$ defined by $x_i^\ast := x_i$, for $i = 1, 2, \ldots$. From the definition of the Razmyslov transformation one can show

Proposition 2.2.2 In a polynomial $f$, if $f^\ast = \varepsilon f$ then $(f^\#(z))^\ast = \varepsilon f^\#(z)$, where $\varepsilon = \pm 1$.

If $f^\ast = f$ then $f$ is an identity (mixed identity) on $H_n$ iff $f^\#(z)$ is an identity (mixed identity) on $H_n$. If $f^\ast = -f$ then $f$ is an identity on $H_n$ iff $f^\#(z)$ is a mixed identity on $H_n$. 

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Proof. Let \( f = \sum_i g_i z h_i \) then \( f^* = \sum_i h_i^* z g_i^* \) and \( f^\#(z) = \sum_i h_i z g_i \). Thus \( f^* = \pm f \) implies that \( (f^*)^\#(z) = \pm f^\#(z) \), that is, \( \sum_i g_i^* z h_i^* = \pm \sum_i h_i z g_i \). However \( (f^\#(z))^* = \sum_i g_i z h_i^* = (f^*)^\#(z) \). So \( (f^\#(z))^* = \pm f^\#(z) \).

To prove the other part of the proposition we use the fact that if \( A \) is an \( n \times n \) symmetric (skew symmetric) matrix then \( A = 0 \) iff the trace of \( A B \) is zero for every \( B \in H_n \) \((B \in K_n)\). Let \( f(x_1, \ldots, x_m, z) \) be *-symmetric and

\[
f = \sum_i g_i z h_i
\]

then

\[
f^\#(z) = \sum_i h_i z g_i
\]

and

\[
f^\#(x_1, \ldots, x_m, z) \in K_n, \forall x_1, \ldots, x_m \in H_n, \forall z \in K_n.
\]

Thus, denoting by \( T \) the trace of a matrix, for \( \forall x_1, \ldots, x_m \in H_n, \) and \( \forall z \in K_n \)

\[
f^\#(x_1, \ldots, x_m, z) = 0 \iff T(f^\#(x_1, \ldots, x_m, z)B) = 0, \forall B, z \in K_n
\]

\[
\iff \sum_i T(h_i z g_i B) = 0, \forall B, z \in K_n
\]

\[
\iff \sum_i T(g_i B h_i z) = 0, \forall B, z \in K_n
\]

\[
\iff T((\sum_i g_i B h_i) z) = 0, \forall B, z \in K_n
\]

\[
\iff f(x_1, \ldots, x_m, B) = 0, \forall x_1, \ldots, x_m \in H_n, \forall B \in K_n.
\]

The rest of the proposition can be proved by using the same argument.

2.3 A New Identity of \( H_n \)

In this section we use the Razmyslov transformation to show that \( T_{2n} \) is a zero operator on \( M_n \). Namely \( T_{2n}(x_1, \ldots, x_{2n-1}; y) = 0 \) for all \( x_i \in H_n \) and \( y \in M_n \). This sharpens the results in [MR], especially the graph-theoretic interpretation of the results. From this result we get a new identity of \( H_n \): \( T_{2n}(x_1, \ldots, x_{2n-1}; [x_{2n}, x_{2n+1}]) = 0 \).
Let us recall the definition of $T_{2n}$. We define

$$T_{2n}^{2n}(x_1, \ldots, x_{2n}) := \sum_{\sigma \in S_{2n}} (-1)^{\sigma} x_{\sigma(1)} \cdots x_{\sigma(2n)}.$$ 

$S_{2n}$ the symmetric group on $2n$ objects and $(-1)^{\sigma}$ the sign of the permutation $\sigma$.

$$T_{2n}(x_1, \ldots, x_{2n-1}; x_{2n}) := T_{2n}^{2n}(x_1, \ldots, x_{2n-1}, x_{2n}).$$

So,

$$T_{2n}(x_1, \ldots, x_{2n-1}; x_{2n}) = \sum_{\sigma \in S_{2n-1}} \sum_{i=1,2 (mod 4)} (-1)^{\sigma} (-1)^{i+1} x_{\sigma(1)} \cdots x_{\sigma(i-1)} x_{2n} \cdot x_{\sigma(i)} \cdots x_{\sigma(2n-1)}.$$

**Theorem 2.1** $T_{2n}(x_1, \ldots, x_{2n-1}; y) = 0$ for all $x_i \in H_n$ and $y \in M_n$.

**Proof.** Since $T_{2n}$ is linear in $y$ and an identity on $H_n$ by [MR], it suffices to show that $T_{2n}$ is a mixed identity. If $n$ is odd then $T_{2n}$ is $+$-skew symmetric. Indeed,

$$T_{2n}(x_1, \ldots, x_{2n-1}; y)^* = \sum_{i=1,2 (mod 4)} (-1)^{\sigma} (-1)^{i+1} x_{\sigma(1)} \cdots x_{\sigma(i-1)} y \cdot x_{\sigma(i)} \cdots x_{\sigma(2n-1)}^*$$

$$= \sum_{i=1,2 (mod 4)} (-1)^{\sigma} (-1)^{i+1} x_{\sigma(2n-1)} \cdots x_{\sigma(i)} y \cdot x_{\sigma(i-1)} \cdots x_{\sigma(1)}$$

$$= \sum_{i=1,2 (mod 4)} (-1)^{\sigma} (-1)^{i+1} (-1)^{n-1} x_{\sigma(1)} \cdots x_{\sigma(2n-i)} y \cdot x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)}$$

$$= \sum_{i=1,2 (mod 4)} (-1)^{\sigma} (-1)^{i+1} x_{\sigma(1)} \cdots x_{\sigma(2n-i)} y \cdot x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)},$$

since $n$ is odd, $n-1$ is even. On the other hand, $n = 2m+1$ implies that $2n+1-i = 4m-i+3$ and

$$2n+1-i \equiv \begin{cases} 2(mod 4) & \text{if } i \equiv 1(mod 4) \\ 1(mod 4) & \text{if } i \equiv 2(mod 4). \end{cases} \quad (2.1)$$

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Hence
\[ x_{\sigma(1)} \cdots x_{\sigma(2n-i)} y x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)} \]
is a term of \( T_{2n} \) and has different sign by the definition of \( T_{2n} \). So \( T_{2n} \) is \( * \)-skew symmetric.

Since \( T_{2n} \) is an identity, \( T_{2n}^\# \) is a mixed identity by Proposition 2.2.2. However,
\[
T_{2n}(x_1, \ldots, x_{2n-1}; y)^\#(y)
= \sum_{\sigma \in S_{2n-1}} (-1)^\sigma (-1)^{i+1} x_{\sigma(i)} \cdots x_{\sigma(2n-i)} y x_{\sigma(1)} \cdots x_{\sigma(i-1)}
\]
\[
= \sum_{\sigma \in S_{2n-1}} (-1)^\sigma (-1)^{i+1+(2n-i)(i-1)} x_{\sigma(1)} \cdots x_{\sigma(2n-i)} y x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)}
\]
\[
= \sum_{\sigma \in S_{2n-1}} (-1)^\sigma (-1)^{i+1} x_{\sigma(1)} \cdots x_{\sigma(2n-i)} y x_{\sigma(2n-i+1)} \cdots x_{\sigma(2n-1)}
\]
\[
= -T_{2n}(x_1, \ldots, x_{2n-1}; y)
\]
from the last equation. Therefore \( T_{2n} \) is also a mixed identity if \( n \) is odd.

If \( n \) is even then \( n + 1 \) is odd, so \( T_{2(n+1)} \) is a mixed identity. However
\[
T_{2(n+1)}(x_1, \ldots, x_{2n+1}; y) = T_{2n}(x_1, \ldots, x_{2n-1}; y)x_{2n}x_{2n+1} + \sim
\]
from [MR]. Hence \( T_{2n} \) is a mixed identity by Proposition 2.2.1 proving the Theorem. 

Let \((V, E)\) be a finite graph with vertices \( V = \{v_1, \ldots, v_n\} \) and (undirected) edges \( E = \{w_1 \ldots, w_r\}, r \geq 2n \), in an arbitrary but fixed ordering. Then unicursal paths correspond to permutations \( \sigma \in S_r \), where the path is \( w_{\sigma(1)} \cdots w_{\sigma(r)} = p_{\sigma} \). Define \( \varepsilon(p_{\sigma}) \) to be \((-1)^\sigma\).

Choose a distinguished edge \( w_i \). Then for any two fixed vertices \( v_j, v_k \) let \( q_{\sigma} \) be the paths from \( v_j \) to \( v_k \) with \( w_i \) in the positions which are congruent to 1 or 2 modulo 4. Theorem 5 in [MR] said that the number of the paths \( q_{\sigma} \) with \( \varepsilon(q_{\sigma}) = 1 \) is equal to the number of paths \( q_{\sigma} \) with \( \varepsilon(q_{\sigma}) = -1 \). In fact, if we direct \( w_i \) then the paths \( q_{\sigma} \) with \( \varepsilon(q_{\sigma}) = 1 \) are divided into two classes depending on the direction of \( w_i \)(that is the edge \( w_i \) is one way) as are the paths \( q_{\sigma} \) with \( \varepsilon(q_{\sigma}) = -1 \) also. We have

**Theorem 2.2** The number of the paths \( q_{\sigma} \) with \( \varepsilon(q_{\sigma}) = 1 \) and the same direction of \( w_i \) is equal to the number of the paths \( q_{\sigma} \) with \( \varepsilon(q_{\sigma}) = -1 \) and the same direction of the \( w_i \).
2.4 Another Proof of The Amitsur Levitzki Theorem

In this section we give another proof of the Amitsur and Levitzki Theorem by using Proposition 1.1 in [MR] and the Razmyslov transformation.

First we note that it can be proved that \( \hat{T}_{2n} \) is also an identity of \( H_n \) by using the proof that \( T_{2n} \) is an identity (see [MR]), where \( \hat{T}_{2n} \) is the polynomial such that \( T_{2n} + \hat{T}_{2n} = S_{2n} \), the standard identity.

**Corollary 2.4.1** \( S_{2n}(x_1, \ldots, x_{2n-1}, x_{2n}) \) is an identity on \( M_n \).

**Proof.** Since \( S_{2n} \) is multilinear alternating, it suffices to show that

\[
S_{2n}(x_1, \ldots, x_r, y_1, \ldots, y_s) = 0, \quad \forall x_i \in H_n, \quad \forall y_j \in K_n, \quad \text{where } r + s = 2n. \quad (2.2)
\]

We show (2.2) by induction on \( s \). If \( s = 0 \) then (2.2) follows from the fact that \( T_{2n} \) and \( \hat{T}_{2n} \) are identities on \( H_n \). Assume that (2.2) holds for all \( n = 1, 2, 3, \ldots \), and \( s - 1 \). Then for \( s \), we show (2.2) holds by distinguishing two cases.

If \( s \) is odd then for all odd \( n \), \( S_{2n} \) is \( * \)-skew symmetric. Hence \( S_{2n}(x_1, \ldots, x_r, y_1, \ldots, y_s) \in H_n \) for \( \forall x_i \in H_n, \forall y_j \in K_n \). Therefore it suffices to show that

\[ T(S_{2n}(x_1, \ldots, x_r, y_1, \ldots, y_s)x) = 0, \forall x \in H_n. \]

We claim that

\[ S_{2n}^* = -S_{2n}, \quad n = 1, 2, 3, \ldots \]

Let \( (x_1, \ldots, x_r, y_1, \ldots, y_s) = (z_1, \ldots, z_{2n-1}, y_s) \). Then

\[
S_{2n}(z_1, \ldots, z_{2n-1}, y_s) =
\]

\[
= \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n}} (-1)^{\sigma}(-1)^{2n-i}z_{\sigma(1)} \cdots z_{\sigma(i-1)} y_s z_{\sigma(i)} \cdots z_{\sigma(2n-1)}
\]

\[ = \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n}} (-1)^{\sigma}(-1)^i z_{\sigma(1)} \cdots z_{\sigma(i-1)} y_s z_{\sigma(i)} \cdots z_{\sigma(2n-1)}. \]
Thus

\[ S_{2n}^\#(y) = \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n-1}} (-1)^{\sigma}(-1)^{i} z_{\sigma(1)} \cdots z_{\sigma(2n-1)} y \] 

\[ \cdot y_{\sigma(2n-i+1)} \cdots y_{\sigma(2n-i)} = \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n-1}} (-1)^{\sigma}(-1)^{i}(2n-i+1) \] 

\[ y_{\sigma(2n-i+1)} \cdots y_{\sigma(2n-i)} \cdot z_{\sigma(2n-i)} = \sum_{i=1}^{2n} \sum_{\sigma \in S_{2n-1}} (-1)^{\sigma}(-1)^{i} z_{\sigma(1)} \cdots z_{\sigma(2n-i)} \] 

\[ y_{\sigma(2n-i+1)} \cdots y_{\sigma(2n-i)} = -S_{2n}(z_1, \ldots, z_{2n-1}, y). \]

Let

\[ S_{2n}(x_1, \ldots, x_r, y_1, \ldots, y_s) = \sum_i g_i y_i h_i. \]

Then from the claim we have for all \( z \in H_n \)

\[ T(S_{2n}(x_1, \ldots, x_r, y_1, \ldots, y_s)z) = \sum_i T(g_i y_i h_i z) \]

\[ = \sum_i T(h_i z g_i y_i) \]

\[ = T(S_{2n}^\#(x_1, \ldots, x_r, y_1, \ldots, y_{s-1}, z) y_s) \]

\[ = T(-S_{2n} y_s) = 0 \]

by the induction hypothesis. For \( n \) even, \( n + 1 \) is odd. Thus

\[ S_{2n+2}(x_1, \ldots, x_{r+1}, y_1, \ldots, y_s) = 0. \]

However

\[ S_{2n+2}(x_1, \ldots, x_{r+2}, y_1, \ldots, y_s) = \pm S_{2n}(x_1, \ldots, x_r, y_1, \ldots, y_s) x_{r+1} x_{r+2} + \sim. \]

So

\[ S_{2n}(x_1, \ldots, x_r, y_1, \ldots, y_s) = 0 \]

by Lemma 2.1.2.

If \( s \) is even then for \( n = 2, 4, 6, \ldots \), \( S_{2n} \) is \( * \) symmetric and \( S_{2n}(x_1, \ldots, x_r, y_1, \ldots, y_s) \in H_n \) for \( \forall x_i \in H_n, \forall y_j \in K_n \). Therefore it suffices to show that

\[ T(S_{2n}(x_1, \ldots, x_r, y_1, \ldots, y_s)z) = 0, \forall z \in H_n, n = 2, 4, 6, \ldots. \]

This can be shown by using the same argument and the claim above. Then using the same argument as above to show the case when \( n \) is odd completes the proof. \( \Box \)
Chapter 3

The Identities of $H_2$ of Degree 5

In this chapter we are going to show that every identity of $H_2$ is a consequence of $T_4$. From Lemma 2.1.1 we know immediately that $H_2$ has no identity of type $[5]$ or $[4,1]$. Thus identities which we need to determine are of type $[3,2]$ or higher. In following, $\text{VIN}_{n}$ denotes the set of the identities of $H_n$ of type $[n_1, \ldots, n_r]$ and $\text{VIN}_{n}[n_1, \ldots, n_r]$ denotes the set of the identities of $H_n$ of type $[n_1, \ldots, n_r]$ which are also symmetric or skew symmetric in all their variables of degree $m$ according as $m$ is even or odd, where $m \in \{n_1, \ldots, n_r\}$.

**Proposition 3.1.1** If the char $\mathbb{F} 
eq 3!$ then every identity of type $[3,2]$ is a scalar multiple of $T_4(x, y, x^2; y)$.

**Proof.** Let $f(x, y) \in \text{VIN}_2[3,2]$ then from Lemma 2.1.1

\[
f = \alpha_1 xyyx + \alpha_2 yzxy + \alpha_3 zxyy + \alpha_4 zyyx + \\
\alpha_5 xyyx + \alpha_6 zxyy + \alpha_7 zxyx + \alpha_8 zyxy,
\]

and

\[
\alpha_1 + \alpha_4 = 0, \quad \alpha_7 + \alpha_8 = 0, \quad \alpha_2 + \alpha_5 + \alpha_7 = 0, \\
\alpha_3 + \alpha_6 + \alpha_8 = 0, \quad \alpha_4 + \alpha_5 + \alpha_6 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0.
\]

Substituting $e[12], e[11]$ for $x, y$ in $f$ and considering the coefficient of $e_{12}$ yield $\alpha_2 + \alpha_7 = 0$. Substituting $e[12], e[22]$ for $y, x$ and considering the coefficient $e_{11}$ yield $\alpha_3 = 0$.
The solution of the equations above is \( \alpha_1 = -\alpha_2 = -\alpha_4 = \alpha_6 = \alpha_7 = \alpha_8 \). So

\[
f = \alpha_1[y,y], [y,x^2] = \alpha_1 T_4(x,y,x^2;y).
\]

completing the proof of the proposition.  

**Proposition 3.1.2** If the char \( F \neq 3 \) then identities

\[
T_4(x,x^2,y;z), T_4(x,x^2,z;y), T_4(x^2,y;z;x), T_4(x,y,z;x)x
\]

form a basis of the vector space \( VIN_2[3,1,1] \).

**Proof.** From Theorem 1.3 it suffices to find a basis of \( VIN_2[3,1,1] \). Let \( f(x,y,z) \in VIN_2[3,1,1] \) then from Lemma 2.1.1

\[
f = (Id - (y,z))\{\alpha_1yzyzx + \alpha_2yzxzx + \alpha_3zyzzz + \alpha_4xyxxz + \alpha_5xyzx + \alpha_6xyxxz + \alpha_7zxyzz + \alpha_8zxzyz\},
\]

where Id is the identity mapping and the operator \((y,z)\) acts on the monomials by permuting \( y \) and \( z \). Using the same arguments as in the proof of Proposition 3.1.1 we have

\[
\begin{align*}
\alpha_1 + \alpha_4 &= 0, \quad \alpha_7 + \alpha_8 = 0, \quad \alpha_2 + \alpha_5 + \alpha_7 = 0, \\
\alpha_3 + \alpha_6 + \alpha_8 &= 0, \quad \alpha_4 + \alpha_5 + \alpha_6 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0.
\end{align*}
\]

The solution of this system is

\[
\begin{align*}
\alpha_1 &= -\alpha_3 + \alpha_5 - \alpha_8, \quad \alpha_2 = -\alpha_5 + \alpha_8, \quad \alpha_4 = \alpha_3 - \alpha_5 + \alpha_8, \quad \alpha_6 = -\alpha_3 - \alpha_8, \quad \alpha_7 = -\alpha_8.
\end{align*}
\]

Thus \( f = \alpha_3 f_1 + \alpha_5 f_2 + \alpha_8 f_3 \), where \( f_i \), for \( i = 1,2,3 \) are independent from \( f \). So \( VIN_2[3,1,1] \) is contained in \( \text{Span}\{f_1,f_2,f_3\} \).

On the other hand, the identities

\[
v_1 := T_4(y,z,x^2;z), \quad v_2 := T_4(x,y,z;x)x, \quad v_3 := T_4(x,y,x^2;z) - T_4(x,z,x^2;y)
\]

belong to \( VIN_2[3,1,1] \) and are linearly independent. For, if there exist scalars \( \beta_1, \beta_2, \beta_3 \in F \) such that \( \sum_{i=1}^3 \beta_i v_i = 0 \) then the coefficients of the monomials \( yzxxz, \, xzxyz \) and \( xxzyz \) are \( 0 \), which yields \( \beta_1 = \beta_2 = \beta_3 = 0 \). Thus \( v_1, v_2, v_3 \) form a basis of \( VIN_2[3,1,1] \).
Since $VI_2[3, 2]$ is spanned by $T_4(x, y, x^2; z)$ by Proposition 3.1.1, it follows from theorem 1.3, $T_4(x, y, z^2; x) + T_4(x, z, x^2; y)$. $v_1$, $v_2$, $v_3$ form a basis of $VI_2[3, 1, 1]$. Obviously the identities

$$h_1 := T_4(x, x^2; y), \ h_2 := T_4(x, x^2, z; y),$$

$$h_3 := T_4(y, z, x^2; z), \ h_4 := T_4(x, y, z; x).$$

form a basis of $VI_2[3, 1, 1]$. 

Note. $xT_4(x, y, z; x) = h_1 - h_2 + h_4$ and $T_4(x, y, z; x^2) = -h_1 + h_2 + h_3 - h_4$.

Let

$$k_1 := T_4(y, x \circ y, x; z), \ k_2 := T_4(y^2, x, z; x), \ k_3 := T_4(x^2, y, z; y),$$

$$k_4 := T_4(x, y, z \circ y; x), \ k_5 := T_4(y, x, z \circ y; x), \ k_6 := T_4(x, y, z; y),$$

$$k_7 := xT_4(x, y, z; y) + yT_4(y, x, z; x), \ k_8 := T_4(x, y, z; x).$$

Then we have

Proposition 3.1.3 If $char F \neq 3!$ then $\{k_i | i = 1, \ldots, 8\}$ is a basis of the vector space $VI_2[2, 2, 1]$.

Proof. Let $A := yxyz, \ B := yxyx, \ C := yxxy$. Let $X_i$ be the monomial which is obtained by putting $z$ in the $i$-th position of $X$, where $X \in \{A, B, C\}$. For instance, $A_2 = yzyzx$. Thus if $f(x, y, z) \in VI_2[2, 2, 1]$ then the monomials of $f$ are $X_i$ for $i = 1, \ldots, 5$ and $X \in \{A, B, C\}$, as well as the monomials which come from the $X_i$'s by exchanging all $x$'s and $y$'s. Let $\alpha_i$ (resp. $\beta_i, \gamma_i$) denote the coefficient of the monomial $A_i$ (resp. $B_i, C_i$) in $f$. Since $f(x, y, 1) \in VI_2[2, 2] = \{0\}$ by Theorem 1.2 in [MR], $f(x, y, 1) = 0$. This yields

$$\sum_{i=1}^5 \alpha_i = 0, \quad \sum_{i=1}^5 \beta_i = 0, \quad \sum_{i=1}^5 \gamma_i = 0. \quad (3.1)$$

From Lemma 2.1.1 we have

$$\alpha_1 + \beta_1 + \gamma_1 = 0, \ \beta_3 + \beta_4 + \beta_5 + \gamma_3 + \gamma_4 + \gamma_5 = 0,$$

$$\alpha_2 + \beta_2 + \gamma_2 = 0, \ \alpha_4 + \beta_4 + \gamma_4 = 0, \ \alpha_5 + \beta_5 + \gamma_5 = 0. \quad (3.2)$$

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where \( \gamma'_4 \) and \( \gamma'_5 \) are the coefficients of the monomials \( xyzxz \), \( xyyzz \) respectively.

Since \( f(x, y, z) \in VI_2[3, 2] \), from Proposition 3.1.1 there exists a scalar \( \alpha \in F \) such that

\[
f(x, y, z) = \alpha T_4(x, x^2, y, y).
\]

(3.3)

In the following we prove that \( \alpha \), \( \alpha_5 \), \( \gamma_4 \), \( \gamma_5 \) are the parameters of the coefficients of \( f \) if \( f \)

is symmetric or skew symmetric in the variables of degree 1. Since \( T_4(x, x^2, y, y) \) is a fixed polynomial, the linear equations obtained from (3.3) have the coefficients of the monomials in \( f \) as unknown and \( \lambda_i \alpha \) for some \( \lambda_i \in F \) as constants. These \( \lambda_i \)'s are independent from \( f \). So we may assume that \( \alpha = 0 \). Hence \( f(x, y, x) = 0 \) yields

\[
\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 &= 0, \quad \alpha_3 + \alpha_4 + \alpha_5 = 0, \quad \alpha_4 + \beta_1 + \beta_2 = 0, \\
\gamma_1 + \gamma_2 &= 0, \quad \beta_3 + \beta_4 + \gamma_1 = 0, \quad \beta_5 + \beta'_1 + \gamma_3 = 0,
\end{align*}
\]

(3.4)

where \( \gamma'_1 \), \( \beta'_1 \) are the coefficients of the monomials \( zxyyx \) and \( zxyzy \).

Now let us argue in two cases. If \( f \) is symmetric in the variables \( y, z \) then \( \gamma'_i = \gamma_i \) and \( \beta'_i = \beta_i \). Solving systems (3.1), (3.2) and (3.4) yields

\[
\begin{align*}
\alpha_1 &= -\gamma_5 - \gamma_4 = -\alpha_2, \quad \alpha_3 = 0, \quad \alpha_4 = \gamma_1 = -\alpha_5, \quad \beta_1 = 2\gamma_5 + \gamma_4, \\
\beta_2 &= -2\gamma_5 + \alpha_2 - \gamma_4, \quad \beta_3 = -\alpha_1 - \gamma_4 + \gamma_5, \quad \beta_4 = \gamma_5 - \gamma_4,
\end{align*}
\]

If \( f \) is skew symmetric in the variables \( y, z \) then \( \gamma'_i = -\gamma_i \) and \( \beta'_i = -\beta_i \). Solving systems (3.1), (3.2) and (3.4) yields

\[
\begin{align*}
\alpha_1 &= \gamma_5 + \gamma_4 = -\alpha_2, \quad \alpha_3 = 0, \quad \alpha_4 = -\alpha_5, \quad \gamma_1 = -\gamma_5, \quad \beta_1 = -\gamma_4, \\
\beta_2 &= \alpha_5 + \gamma_4 = \beta_4, \quad \beta_3 = -\alpha_5 - \gamma_4 + \gamma_5, \quad \beta_5 = -\alpha_5 + \gamma_5 = \gamma_2, \quad \gamma_3 = \alpha_5 - \gamma_4 - \gamma_5.
\end{align*}
\]

Therefore in every case the coefficients are the linear combination of \( \alpha_5 \), \( \gamma_4 \), \( \gamma_5 \) and \( \alpha \). So every \( f \in VI_2[2, 2, 1] \) is a linear combination of 8 polynomials which are independent from \( f \). Thus, from Lemma 2.1.3 to show the proposition it suffices to show that \( k_1, \ldots, k_8 \) are linearly independent.
Suppose there exist scalars \( \delta_i \in F \) such that \( \sum_{i=1}^{8} \delta_i k_i = 0 \). Then considering the coefficients of the monomials
\[
xyz, \ yxz, \ yzx, \ xzy, \ zyx, \ zxy, \ xyz, \ zyx
\]
yields
\[
\delta_1 - \delta_2 + \delta_3 - \delta_4 = 0, \ 2\delta_2 = 0, \ \delta_1 + \delta_3 + \delta_4 + \delta_5 - \delta_6 - \delta_7 = 0, \\
-\delta_2 + \delta_4 - \delta_5 - \delta_6 = 0, \ -\delta_2 + 2\delta_3 - \delta_4 = 0, \ \delta_2 - \delta_3 - \delta_4 + \delta_6 = 0, \\
\delta_3 + \delta_4 + \delta_5 + \delta_6 = 0, \ -\delta_2 + \delta_3 - \delta_6 - \delta_7 = 0.
\]
The only solution is trivial. Thus \( k_1, \ldots, k_8 \) are linearly independent. \( \blacksquare \)

Note:

1. We give the basis of \( V1_2[2, 2, 1] \) here for later use.

2. \( T_4(x, y, z; x \circ y) = -k_1 - k_2 + k_3, \ (T_4(y, z, x; x \circ y))^* = -k_1 - k_4 + k_5 \)
\[
xT_4(x, y, z; y) - yT_4(y, x, z; x) = -2k_1 - k_2 + k_3 - k_4 + k_5 + k_6 - k_8, \\
T_4(x, z, x \circ y; y) = -k_1 - k_2 - k_4 + k_5.
\]

Let
\[
L_1 := zT_4(y_1, y_2, y_3; x), \ L_2 := z(T_4(y_1, y_2, y_3; x))^*, \ L_3 := T_4(y_1, y_2, y_3; x)x, \\
L_4 := (T_4(y_1, y_2, y_3; x))^*x, \ L_5 := T_4(y_1, y_2, y_3; x^2), \ L_6 := (T_4(y_1, y_2, y_3; x^2))^*, \\
L_7 := \sum_{(123)} T_4(x \circ y_1, y_2, y_3; x).
\]

where the sum is taken over all cyclic permutations of 1 2 3. Then we have

**Proposition 3.1.4** If \( \text{char} F \neq 3! \) then \( \{L_i | i = 1, \ldots, 7\} \) is a basis of the vector space \( V1_2[2, 1^3] \).

**Proof.** Let \( f(x, y_1, y_2, y_3) \in V1_2[2, 1^3] \) then from the symmetry
\[
f = \sum_{1 \leq i < j \leq 5} a_{ij} f_{ij}(x, y_1, y_2, y_3),
\]

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where

\[ f_{ij} = \sum_{\sigma \in S_3} (-1)^\sigma y_{(1)} \cdots x_{(i)} \cdots x_{(j)} \cdots y_{(3)}. \]

Let

\[ \alpha_1 = \alpha_{12}, \alpha_2 = \alpha_{13}, \alpha_3 = \alpha_{14}, \alpha_4 = \alpha_{15}, \alpha_5 = \alpha_{23}, \ldots, \alpha_{10} = \alpha_{45}; \]

and

\[ f_1 = f_{12}, f_2 = f_{13}, f_3 = f_{14}, f_4 = f_{15}, f_5 = f_{23}, \ldots, f_{10} = f_{45}. \]

From Lemma 2.1.1 we know that the sum of the coefficients of the monomials which have form \( y_1y_2 \) is 0. So \( \alpha_8 + \alpha_9 + \alpha_{10} = 0 \). Similarly we have

\[ \alpha_1 + \alpha_2 + \alpha_5 = 0, \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 = 0. \]

Solving these equations yields

\[ \alpha_1 = -\alpha_2 - \alpha_5, \alpha_3 = -\alpha_9 - \alpha_{10}, \alpha_3 = -\alpha_4 - \alpha_6 - \alpha_7. \]

Thus

\[ f = \alpha_2(f_2 - f_1) + \alpha_4(f_4 - f_3) + \alpha_5(f_5 - f_1) + \alpha_6(f_6 - f_3) + \alpha_7(f_7 - f_3) + \alpha_9(f_9 - f_8) + \alpha_{10}(f_{10} - f_8). \]

It suffices now to show that \( L_1, \ldots, L_7 \) are linearly independent. If there exist scalars \( \delta_i \in F \) such that \( \sum_{i=1}^7 \delta_i L_i(x, y, z, w) = 0 \) then considering the coefficients of the monomials

\[ xzwxy, yzxwx, zwxyz, xyxwz, yxxzw, zwzyx, yzzwx \]

yields

\[ \delta_2 + \delta_7 = 0, \delta_3 + 2\delta_7 = 0, \delta_4 = 0, \delta_1 - \delta_7 = 0, \delta_5 + 2\delta_7 = 0, \delta_6 = \delta_7 = 0. \]

Obviously the solution is trivial. Thus \( L_1, \ldots, L_7 \) form a basis of \( VIN_2[2, 1^3] \).

Finally, \( VIN_2[1^5] \) has \( S_5(x_1, \ldots, x_5) \) as a basis. From the definition of standard polynomial and Lemma 1.1 of [MR],

\[ S_5 = \sum_{i=1}^5 (-1)^i x_i S_4(x_1, \ldots, x_i, \ldots, x_5) \]

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and

\[ S_4(x_1, \ldots, \varepsilon_i, \ldots, x_5) = \frac{1}{2} \sum_{j \neq i} (-1)^{j-1} T_4(x_1, \ldots, \varepsilon_j, \ldots, \varepsilon_i, \ldots, x_5; x_j). \]

Thus $VIN_2[1^5]$ has a basis which is a consequence of $T_4$. 
Chapter 4

The Identities of $H_3$ of Degree 7

In this chapter we shall determine all identities of $H_3$ of degree 7. In the following, $VI_3[n_1, \ldots, n_r]$ denotes the set of the identities of $H_3$ of type $[n_1, \ldots, n_r]$ and $VIN_3[n_1, \ldots, n_r]$ denotes the set of the identities of $H_3$ of type $[n_1, \ldots, n_r]$ which are also symmetric or skew symmetric in all their variables of degree $m \in \{n_1, \ldots, n_r\}$ according as $m$ is even or odd.

4.1 The Identities of Type $[4, 1^3]$ or Lower

Proposition 4.1.1 If $\text{char} F / 3!$ then there is no identity of $H_3$ of type $[4, 3]$ or lower.

Proof. Let $f(x, y, z)$ be an identity of type $[5, 1, 1]$ then

$$f = f_1xy + f_2yx + f_3xz + f_4zx + f_5yz + f_6zy + f_7yz^2 + f_8zx^2.$$

So $f_i$ is an identity of $H_2$ from Lemma 2.1.2, for $i = 1, \ldots, 8$. However $H_2$ has no identity of degree 5 which has type lower than $[3, 2]$ and no identity of degree 4 which has type $[3, 1]$. Each $f_i = 0$. Thus $f = 0$.

Now let $f(x, y)$ be an identity of type $[4, 3]$. Since $H_2$ has no identity type $[3, 1]$ or $[2, 2]$, $f = f_1xy + f_2yx$. Then $\Delta(y, x)f = 0$, because $\Delta(y, x)f \in VI_3[5, 2]$ and, as we have shown above, $VI_3[5, 1, 1] = \{0\}$. But $\Delta(y, x)f = (f_1 + f_2)x^2 + \sim$, where $\sim$ denotes the sum of the terms which do not end in $x^2$. So $f_1 + f_2 = 0$. 

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Since $f_1$ is an identity by Lemma 2.1.2, $f_1 = \alpha([y, x], [y, x^2])$ for some $\alpha \in F$, by Proposition 3.1.1. So

$$f = \alpha([y, x], [y, x^2])[y, x].$$

Substituting $e[12], e[23]$ for $x, y$ yields $\alpha = 0$, so $f = 0$. □

**Proposition 4.1.2** If $\text{char} F \nmid 3！$ then $Q(x^2, y, x, z, y, x)$ forms a basis of the vector space $V(2, 1)$. 

**Proof.** Let $f(x, y, z) \in V(2, 1)$. Then

$$f = f_1xx + f_2zx + f_3xy + f_4yx + f_5yx^2,$$

since $H_2$ has no identity of degree 4 of type $[2, 2], [4]$ or $[3, 1]$. Since $f_i$ is an identity for $i = 1, \ldots, 5$ by Lemma 2.1.2, $f_5 = \alpha_5([y, x], [x, z])$ follows from [MR], $f_4 = \alpha_4([y, x], [y, x^2])$, for $i = 1, 2, 3 = \sum_{i=1}^{4} \beta_i h_i$ and $f_4 = \sum_{i=1}^{4} \gamma_i h_i$, where $h_1, \ldots, h_4$ are the basis of the space $V(2, 1)$ given in Chapter 3, namely

$$h_1 = T_4(x, x^2, y, z) = [x, y]z + [z, x^2][y, z],
\quad h_2 = T_4(x, x^2, z; y) = [y, x]z + [y, x^2][z, x],
\quad h_3 = T_4(y, z, x; x^2) = [x, y]z + [z, x]z + [x, z]x^2,
\quad h_4 = T_4(x, y, z; x) = [x, y]z + [z, x]x.$$

Since $f(x, y, 1) \in V(2, 2) = \{0\}$, $f(x, y, 1) = 0$ implies that $\alpha_1 + \alpha_2 = 0$, because $h_i(x, y, 1) = 0$ for $i = 1, \ldots, 4$ and $f_5(x, y, 1) = 0$. So

$$f = f_1([y, x], [y, x^2])[x, z] + \alpha_5([y, x], [x, z])yx^2 + f_3xy + f_4yx.$$

Since $\Delta(y, x)f$ is an identity of type $[5, 1, 1]$, it is 0. Therefore, considering the sum of the terms of $\Delta(y, x)f$ which end in $x^2$, yields $f_3 + f_4 + f_5x = 0$. Since $f_5x = -\alpha_5T_4(y, z, x; x)x = -\alpha_5h_4$ and $h_1, h_2, h_3, h_4$ are linear independent, $\beta_i + \gamma_i = 0$ for $i = 1, 2, 3$ and $\beta_4 + \gamma_4 - \alpha_5 = 0$. Thus

$$f = f_1([y, x], [y, x^2])[x, z] + \alpha_5(h_4yx - T_4(y, z, x; x)yxx) + \sum_{i=1}^{4} \beta_i h_i[x, y]. \quad (4.1)$$
Similarly, $\Delta(z, y)f = 0$ implies that $-\alpha_1 + \beta_1 + \beta_2 = 0$. If we write $f = xzxyyz$, then $g$ is an identity of type $[2,2]$ of $H_2$ by Lemma 2.1.2. Since $H_2$ has no identity of type $[2,2]$, $g = 0$. Thus $f$ has no terms which start with $xz$. So considering the coefficient of the monomial $xzxyyz$ yields $\beta_1 = 0$. So $\alpha_1 = \beta_2$ from the equation above.

It can be proved symmetrically that

$$f = \alpha'_1[z, z][y, y], [y, x^2]] + \alpha'_5(zyh_4 - zyzT_4(y, z, x; x)) + \sum_{i=1}^{4} \beta_i[z, y]h_i. \tag{4.2}$$

From (4.1) and (4.2), considering the coefficients of the monomials $xzxyyz$ and $xzxyyz$ yields $\alpha'_1 = \alpha_5$ and $-\alpha'_1 = \beta_3$ since $\beta_1 = 0$. So $\alpha_5 = -\beta_3$. From (4.2) we know the coefficient of the monomial $xzxyyz$ is 0. But this monomial has coefficient $-\beta_4 + \alpha_5$. So $\beta_4 = \alpha_5$.

Let $g(x_1, x_2, x_3, x_4, y_1, y_2, z)$ be the complete linearization of $f$, then $g$ is also an identity. Substituting

$$e[11], e[12], e[22], e[23], e[33]$$

for $z$, $x_1$, $x_2$, $x_3$, $y$, $x_4$, considering the coefficient of $e_{11}$ in $g(x_1, x_2, x_3, x_4, y, y, z)$ yields $\alpha_1 + \alpha_5 = 0$. For, the monomials which evaluate to $e_{11}$ are

$$xz^{e(1)}yz^{e(2)}yz^{e(3)}yz^{e(4)}, xz^{e(1)}yz^{e(2)}yz^{e(3)}yz^{e(4)}, xz^{e(1)}yz^{e(2)}yz^{e(3)}yz^{e(4)}, xz^{e(1)}yz^{e(2)}yz^{e(3)}yz^{e(4)}z,$$

$\forall \sigma \in S_4$. In fact, the coefficient of $e_{11}$ is 4! times the sum of the coefficients of the following monomials in $f$ since $g$ is the linearization of $f$:

$$xzxyyz, xzxyyz, xzxyyz, xzxyyz.$$

This yields $\alpha_1 + \alpha_5 = 0$ because $\beta_1 = 0$ and $\text{char } F \nmid 3$. So $f = \alpha_5 g$, where $g$ is independent from $f$. Thus from Lemma 2.1.3, $Q(x^2, y, x, y, z, z)$ is a basis of $VJ_0[4, 2, 1]$ since it is an nonzero polynomial. \]

**Proposition 4.1.3** If $\text{char } F \nmid 3!$ then

$$T_0(y_1, y_2, y_3, x, x^2; x), K := [S_3([y_1, x], [y_2, x], [y_3, x]), x]$$

form a basis of the vector space $VIN_3[4, 1^3]$.  

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Proof. First $K$ is an identity since $S_3(z_1, z_2, z_3)$ is in the center of $M_3$ $\forall z_i \in K_3$. Indeed, $K_3$ has a basis $e_{12} - e_{21}$, $e_{13} - e_{31}$, $e_{23} - e_{32}$. Substituting these elements for $z_1, z_2, z_3$ yields $S_3(z_1, z_2, z_3) = -2I$, where $I$ is the unit matrix of $M_3$. So $K$ is an identity of $H_3$, which was already observed in Remark 2.5.15 of [RV1].

Next, we show that $T_0(y_1, y_2, y_3, x, x^2; x)$ and $K$ are linearly independent. If there exist scalars $\alpha \in F$ such that $\alpha_1 T_0(y_1, y_2, y_3, x, x^2; x) + \alpha_2 K = 0$ then considering the coefficient of the monomials $y_1 x y_2 x y_3 x x$, $x y_1 x y_2 x y_3 x$ respectively yields $\alpha_1 + \alpha_2 = 0$, $-2\alpha_2 = 0$. So they are linearly independent.

Let $f(x, y_1, y_2, y_3) \in VIN_3[4, 1^3]$ then

$$f = \sum_{(123)} \{f_1(x, y_1, y_2)y_3 x^2 + f_2(x, y_1, y_2)y_3 x + f_3(x, y_1, y_2)y_3\}$$

since $H_2$ has no identity of type $[4,1]$. Moreover

$$f_1(x, y_1, y_2) = \alpha([x, y_1], [x, y_2]), \quad f_2 = \beta_1 (h_1 - h_2) + \beta_3 h_3 + \beta_4 h_4, \quad f_3 = \gamma_1 (h_1 - h_2) + \gamma_3 h_3 + \gamma_4 h_4,$$

where $h_1 - h_2, h_3, h_4$ are the basis of the space $VIN_2[3,1,1]$ given in Proposition 3.1.2.

Since $f(x, y_1, y_2, 1)$ is an identity of type $[4,1,1]$ it is 0 identically. But

$$f(x, y_1, y_2, 1) = ((\beta_1 + \gamma_1)(h_1 - h_2) + (\beta_3 + \gamma_3)h_3 + (\beta_4 + \gamma_4 - \alpha)h_4) x$$

and $h_1 - h_2, h_3, h_4$ are linearly independent, thus $\beta_i + \gamma_i = 0$ for $i = 1, 3$ and $\beta_4 + \gamma_4 = \alpha$.

By Lemma 2.1.1 the sum of the coefficients of the following monomials is 0:

$$xyz y_3 y_1 x x, \quad y_1 y_3 x y_2 x x x, \quad x x x y_2 y_3 y_1,$$

$$x x y_2 y_3 y_1 x, \quad x x y_1 y_3 x y_2 x, \quad x y_1 y_3 x y_2 x.$$

This yields $\alpha - \beta_1 = 0$.

Let $f'(x_1, x_2, x_3, x_4, y_1, y_2, y_3)$ be the complete linearization of $f$. $f'$ is also an identity of $H_3$. Substituting

$$e[12], \quad e[22], \quad e[23], \quad e[33], \quad e[31]$$

for $y_1, x, y_2, y_3, x_4$ and considering the coefficient of $e_{11}$ in $f'(x, x, x, x_4, y_1, y_2, y_3)$ yields

$$6(\beta_3 - \beta_1 + \beta_3 + \beta_4) = 0 \quad \text{since only } y_1 x x x y_2 y_3 x_4, \quad x_4 y_2 x x x y_1 \text{ evaluate to } e_{11}.$$

Since
\[ \beta_1 = \alpha = \beta_4 + \gamma_4, \gamma_4 = 2\beta_3 \text{ from } -\beta_1 + 2\beta_3 + \beta_4 = 0. \] Thus every coefficient of the monomials in \( f \) is a linear combination of \( \beta_3, \beta_4 \). So from Lemma 2.1.3 the proposition holds.

4.2 The Identities of Type [3, ...]

Proposition 4.2.1 If \( \text{char} F \nmid 3! \) then the vector space \( VIN_3[3, 3, 1] = \{0\} \).

Proof. Let \( f(x, y, z) \in VIN_3[3, 3, 1] \). Then

\[
f = f_1(x, y, z)yz - f_1(y, x, z)yx + f_2(x, y)yz + f_2'(x, y)yz - \\
f_2(y, z)yz - f_2'(y, z)zx + f_3(x, y, z)yz^2 - f_3(y, x, z)yz^2.
\] (4.3)

Since \( \Delta(z, y)f = 0 \), considering the sum of the terms of \( \Delta(z, y)f \) which end in \( yy \) yields \( f_2 + f_2' = 0 \) because \( f_3 \) is of type [2,1,1], \( f_3 = \gamma[[y, z], [y, z]] \) for some \( \gamma \in F \) and hence \( \Delta(z, y)f_3(y, x, z) = 0 \). We have \( f_1 = \sum_{i=1}^{k_1} \alpha_i k_i \) and \( f_1' = \sum_{i=1}^{k_1} \alpha'_i k_i \), where \( k_1, \ldots, k_8 \) are the basis of \( VIN_2[2,2,1] \) given in Proposition 3.1.3. Also, \( f_2 = \alpha[[y, z], [y, z^2]] \) since \( f_2 \) is an identity of type [3,2] of \( H_2 \). Symmetrically, \( f = [y, z]f_2 + \sim \) and \( f_2 = \beta[[y, z], [y, z^2]] \).

Since \( f \) has degree 1 in \( z \),

\[
f = \alpha\{[[y, z], [y, z^2]][y, z] - [[x, y], [x, y^2]][x, z]\} + \\
\beta\{[[y, z][[y, z], [y, z^2]] - [x, z][[y, y], [x, y^2]] + \sim\} \quad (4.4)
\]

From (4.3) the monomial \( yzyzyz^2 \) has coefficient \(-\gamma\) but by (4.4) it has coefficient \( \beta \). So \( \beta = -\gamma \). If

\[
f = xzyg(y, x, z) - yzyg(x, y, z) + \sim
\] (4.5)

then \( g(y, x, z) = \delta[[y, z], [y, z]] \) for some \( \delta \in F \). Similarly we have \( \alpha = \delta \) by considering the coefficient of the monomial \( xzyzyzy \) in \( f \) in the two different expressions.

The monomials \( y^2z^2y \) and \( y^2x^2yz \) have coefficients \( \alpha_1 + \alpha_3 + \alpha_6 \) and \( \alpha_1 - \alpha_2 - \alpha_3 \) respectively from (4.3). But, from (4.4) we know that they have coefficients 0 and \( \beta \) respectively. So

\[
\alpha_1 + \alpha_3 + \alpha_6 = 0, \quad \alpha_1 - \alpha_2 - \alpha_3 = \beta.
\] (4.6)
Using the same argument and considering the monomials

\[ zzyzxy, zzyxyx, zxyzzy, zxyyzy, yyyzyzy, yyzxzyx \]

yields

\[
\begin{align*}
\alpha_1 - \alpha_4 - \alpha_8 &= 0, \quad \alpha_1 + \alpha_5 + \alpha_6 = \beta, \quad -\alpha_2 + \alpha_4 = \beta, \\
\alpha_3 - \alpha_5 &= -\beta, \quad \alpha_3 + \alpha_7 = -\alpha, \quad -\alpha_2 - \alpha_7 = \alpha. \tag{4.7}
\end{align*}
\]

Since \( \Delta(y,x)f \in \mathcal{V} \mathcal{I}_{3}[4,2,1] \), there exists a scalar in \( F \) such that \( \Delta(y,x)f \) is a scalar multiple of \( Q(x^2, y, z, x, x, y) \). So the monomial \( xyzzzxy \) has coefficient 0 in \( \Delta(y,x)f \). Since this monomial comes from the monomials

\[ yyyzyzy, zxyzzyx, zxyyzyx, xyzzzxy, \]

which have coefficients in \( f \)

\[-\alpha_3 - \alpha_7, \quad \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8, \quad -\alpha_4 - \alpha_5 + \alpha_6 + \alpha_7, -\gamma.\]

respectively and since \(-\gamma = \beta\) from above, we have

\[ \beta - \alpha_6 + \alpha_8 = 0. \tag{4.8} \]

Since \( \Delta(z,y)f = 0 \), considering the sum of its terms which end in \( xy \) and \( yx \) respectively yield \( f_1(x, y, y) - f_2(y, x) = 0 \) and \( f_1(y, x, y) + f_2(y, x) = 0 \). From these two equations we have

\[
\begin{align*}
\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5 - \alpha &= 0, \quad \alpha_1 + \alpha_3 - \alpha_5 - 2\alpha_4 + \alpha &= 0. \quad \tag{4.9}
\end{align*}
\]

Now we may assume that \( f^* = \pm f \). Then considering the coefficients of the monomials \( yzyx^2yz, zxyzyzyx \) and their reverse yields

\[
\alpha = s\beta, \quad \alpha_4 + \alpha_5 - \alpha_7 + \alpha_8 = -s(\alpha_4 + \alpha_5 - \alpha_7 + \alpha_6), \tag{4.10}
\]

where \( s = \pm 1 \) when \( f^* = \pm f \). The solution of system (4.6), ..., (4.10) is trivial if \( s = \pm 1 \).

So \( f = 0 \).  ■

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Let
\[ Q_1(x, y, z) := Q(x \circ z, y, x, z, y, x), \quad Q_2(x, y, z) := Q(x \circ y, z, x, y, z, x), \]
\[ Q_3(x, y, z) := Q(x^2, y, z, x, y, z), \quad K(x, y, z) := [S_3([x, y], [x, z], [y, z]), z]. \]
They are all identities in \( V I_5[3, 2, 2] \). Moreover, \( Q_1, Q_2, Q_3, K \) are all symmetric in \( y, z \).
In following we are going to show

**Proposition 4.2.2** If \( \text{char} F \not= 3 \) then \( Q_1 + Q_2, Q_3, K \) form a basis of the vector space \( V I_5[3, 2, 2] \).

**Proof.** First we show that \( Q_1 + Q_2, Q_3, K \) are linearly independent. It suffices to show that \( Q_1, Q_2, Q_3, K \) are linearly independent. From the definition of \( Q \),
\[ Q_1 = [[[x \circ y, y], [x, y]], [x, z]] + [[[y, x], [x, z]], [y, z]] + \{(x \circ y, [x, z])[x, y]) + \}
\[ \{y, z][x, z]][y, z]] + \{(x \circ y, [x, z])[x, y]), \]
\[ Q_2 = Q_1(x, z, y), \]
\[ Q_3 = [[[y, z], [z, x^2]], [y, x]] + [[[x, z], [z, x]], [x, z]] + \{(y, z^2, [z, z])[x, y]) + \}
\[ \{x, z^2]][y, z][y, z]]. \]

If \( \sum_{i=1}^3 \alpha_i Q_i + \alpha_4 K = 0 \) for some \( \alpha_i \in F \) then considering the coefficient of the monomial \( xyzzyzz \) yields \( \alpha_1 = 0 \). Considering the coefficient of the monomial \( yzzxzzx \) in \( \alpha_2 Q_2 + \alpha_3 Q_3 + \alpha_4 K \) yields \( \alpha_3 = 0 \). Considering the coefficient of the monomial \( zzzyyyy \) in \( \alpha_2 Q_2 + \alpha_4 K \) yields \( \alpha_4 = 0 \). Since \( Q_2 \not= 0 \), \( \alpha_2 = 0 \). So \( Q_1, Q_2, Q_3, K \) are linearly independent.

Now let \( f(x, y, z) \in V I_5[3, 2, 2] \). Since \( H_2 \) has no identity of type \([2, 2]\) or \([3, 1]\) by \([MR]\), \( f \) has no terms which end in \( y^2 \) or \( z^2 \) by Lemma 2.1.2. So
\[ f = f_1(x, y, z)zx + f_1'(x, y, z)xyz + f_1(x, z, y)xyz + f_2(x, y, z)yz + f_2'(x, z, y)yz + f_3(x, y, z)xz^2 + f_3(x, z, y)yz^2. \]  \(4.11\)

Moreover,
\[ f_1 = \sum_{i=1}^8 \alpha_i k_i, \quad f_1' = \sum_{i=1}^8 \alpha'_i k_i, \quad f_2 = \sum_{i=1}^4 \gamma_i h_i, \quad f_3 = \lambda [[y, z], [y, z]], \]
\[ 40 \]
where $k_1, \ldots, k_3$ and $h_1, \ldots, h_4$ are the basis of the vector spaces $V_{I_2}[2,2,1]$ and $V_{I_2}[3,1,1]$ given in Chapter 3.

From Lemma 2.1.1 the sum of the coefficients of the monomials $yx yz z z, y z x y z z, z y x y z z$ is 0. So from (4.11) we have

$$
\alpha_3 + \alpha'_3 - \alpha_5 - \alpha'_5 = 0. \quad (4.12)
$$

Similarly for each $w = x y z, y x z, z x y$ considering the sum of the coefficients of the monomials of form $w w w w$ with $z, z, x$ in the last three positions yield

$$
\alpha_2 + \alpha'_2 - \alpha_4 - \alpha'_4 = 0, \alpha_4 + \alpha'_4 + \alpha_7 + \alpha'_7 = 0, \alpha_5 + \alpha'_5 + \alpha_7 + \alpha'_7 = 0. \quad (4.13)
$$

Because for each $w = x y z, y z x, z y x$ the sum of the coefficients of the monomials of form $w w w w$ with $x, x, z$ in the last three positions is 0 we have

$$
\begin{align*}
\alpha_1 + \alpha'_1 + \alpha_3 + \alpha'_3 + \alpha_6 + \alpha'_6 + \lambda &= 0, \\
\alpha_4 + \alpha'_4 + \alpha_5 + \alpha'_5 - \alpha_6 - \alpha'_6 + \alpha_7 + \alpha'_7 - \lambda &= 0, \\
\alpha_1 + \alpha'_1 + \alpha_5 + \alpha'_5 - \alpha_6 - \alpha'_6 + \alpha_7 + \alpha'_7 - \lambda &= 0, \\
\alpha_2 + \alpha'_2 + \alpha_4 + \alpha'_4 + \alpha_5 + \alpha'_5 + \alpha_6 + \alpha'_6 + \lambda &= 0. \quad (4.14)
\end{align*}
$$

Solving (4.12), (4.13) and (4.14) yields

$$
\alpha_i + \alpha'_i = 0, i \neq 6, 8, \alpha_6 + \alpha'_6 = -\lambda. \quad (4.15)
$$

Since $f(x, y, y)$ is of type [4,3], it is 0 identically. So from (4.11) and

$$
\begin{align*}
k_i(x, y, y) &= 0, \ i = 3, 6, 7, 8, \ h_j(x, y, y) = 0, \ j = 1, 2, \\
k_1(x, y, y) &= -k_2(x, y, y) = k_4(x, y, y) = \frac{1}{2}k_5(x, y, y) = [[y, x], [y^2, x]], \\
h_3(x, y, y) &= h_4(x, y, y) = [[y, x], [x^2, y]],
\end{align*}
$$

we have

$$
\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5 = 0, \beta_3 + \beta_4 = 0. \quad (4.16)
$$
Using Lemma 2.1.1 and considering the monomials

\[-y_{z}x_{z}y_{z}-, y_{z}y_{z}x_{z}-, x_{y}z_{z}y_{z}-, y_{y}x_{z}x_{z}-, y_{y}y_{z}y_{z}-, y_{y}z_{y}z_{y}-, x_{z}x_{z}y_{z}-, x_{y}y_{z}y_{z}-, x_{z}z_{z}x_{z}-, y_{z}y_{z}x_{z}-,\]

where the empty positions of the first three words are \(x, z\) while others are \(x, z\) yields

\[
\begin{align*}
\alpha_{1} + \alpha_{3} + \alpha_{6} - \beta_{1} &= 0, \quad \alpha_{1} + \alpha_{5} + \alpha_{6} + \beta_{2} - \beta_{3} = 0, \quad \alpha_{1} - \alpha_{4} - \alpha_{8} = 0, \\
-\alpha_{2} + \alpha_{4} &= 0, \quad \alpha_{4} + \alpha_{5} + \alpha_{7} + \alpha_{8}' - \beta_{3} - \lambda = 0, \\
\alpha_{1} - \alpha_{4} + \alpha_{5} - \alpha_{7} - \alpha_{8}' - \beta_{4} + \lambda &= 0, \quad \alpha_{3} + \alpha_{7} - \lambda = 0, \\
\alpha_{2} - \alpha_{3} + \alpha_{4} - \alpha_{6}' - 2\alpha_{7} + \lambda &= 0, \\
-\alpha_{1} + \alpha_{3} - \alpha_{4} - \alpha_{5} + \alpha_{6}' + 3\alpha_{7} + \alpha_{8}' + \beta_{4} - 2\lambda &= 0. 
\end{align*}
\]

(4.17)

For instance, from Lemma 2.1.1 the sum of the coefficients of the monomials

\[
zyyxxxx, zyxyxyz, zzyyxx, zyyyyy, yyyzzzz, zzzzyzz
\]

is 0. Since \(f\) has no terms which end in \(zz\) and symmetrically it has no terms which start with \(zz\) from (4.11), the sum of the coefficients of the monomials \(zyyxxxx, zyxyxyz\) is 0. So \(\alpha_{1} + \alpha_{3} + \alpha_{8} - \beta_{1} = 0\). This is the first equation in system (4.17). The others in (4.17) can be gotten in the same way.

Next, substituting \(e[12], e[22], e[23]\) for \(x, y, z\) in \(f(x, y, z)\) yield

\[
\begin{align*}
-\alpha_{1} - \alpha_{6} + \alpha_{7} - 2\epsilon + \beta_{1} - \beta_{3} &= 0, \quad \alpha_{2} + \alpha_{5} + \alpha_{6}' + 4\alpha_{7} + \alpha_{8}' = 0. 
\end{align*}
\]

(4.18)

The solution of the systems of (4.15), (4.16), (4.17) and (4.18) is

\[
\begin{align*}
\alpha_{8} &= -5\alpha_{7} - 2\beta_{2}, \quad \beta_{1} = -4\alpha_{7} - 2\beta_{2}, \quad \beta_{4} = \alpha_{3} + \alpha_{7}, \quad \alpha_{8}' = -3\alpha_{7} - \beta_{2}, \\
\alpha_{6}' &= -3\alpha_{7}, \quad \alpha_{6} = 2\alpha_{7} - \alpha_{3} + \alpha_{4} = -\alpha_{7}, \quad \alpha_{1} = -6\alpha_{7} - 2\beta_{2}, \\
\beta_{3} &= -\alpha_{3} - \alpha_{7}, \quad \alpha_{2} = -\alpha_{7}, \quad \alpha_{5} = 3\alpha_{7} + \beta_{2}, \quad \lambda = \alpha_{3} + \alpha_{7}.
\end{align*}
\]

Thus \(f = \alpha_{3}g_{1} + \alpha_{7}g_{2} + \beta_{3}g_{3}\), where \(g_{1}, g_{2}, g_{3}\) are independent from \(f\). So \(Q_{1} + Q_{2}, Q_{3}, K\) are the basis of \(VIN_{3}[3, 2, 2]\) by Lemma 2.1.3. \(\blacksquare\)

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Let
\[ Q_1 := Q(x \circ y, z_1, z_2), \quad Q_2 := (\text{Id} - (z_1, z_2))Q(x \circ y, z_1, z_2), \]
\[ Q_3 := (\text{Id} - (z_1, z_2))Q(x^2, z_1, z_2), \quad Q_4 := (\text{Id} - (z_1, z_2))Q(x \circ z_1, z_1, z_2), \]
\[ Q_5 := K(x, y, z_1, z_2), \quad Q_6 := (\text{Id} - (z_1, z_2))K(x, y, z_1, z_2), \]
\[ Q_7 := T_0(x \circ y, z_1, z_2), \quad Q_8 := T_0(x^2, z_1, z_2, x, y), \]
\[ Q_9 := zQ(x, y, z_1, z_2), \quad Q_{10} := Q(x, y, z_1, z_2). \]

where \( K(x_1, x_2, x_3, x_4, x_5, x_6, x_7) := \{ S_3([x_1, x_2], [x_3, x_4], [x_5, x_6]), x_7 \}. \) Then we have

**Proposition 4.2.3** The set \( \{Q_1, \ldots, Q_{10}\} \) is a basis of the vector space \( \text{V} \text{I} \text{N}_3[3, 2, 1, 1] \).

**Proof.** First we show that \( Q_1, \ldots, Q_{10} \) are linearly independent. Let \( \alpha_i \in F \) such that \( \sum_{i=1}^{10} \alpha_i Q_i = 0 \). Then considering the coefficients of the following monomials
\[ yzxyz, yxzyz, yzyxz, xzyxz, xzxyz, \]
\[ yxzyxz, zxyzyz, zyxzyz, yzyxzz, yzxzzy, \]

yields
\[ \alpha_3 + 2\alpha_2 - \alpha_6 = 0, \quad -\alpha_1 - \alpha_4 - \alpha_5 + \alpha_6 + \alpha_7 - \alpha_{10} = 0, \]
\[ \alpha_4 - \alpha_5 - \alpha_6 - \alpha_{10} = 0, \quad 2\alpha_6 - \alpha_9 + \alpha_{10} = 0, \quad -\alpha_4 + \alpha_6 - \alpha_{10} = 0, \]
\[ \alpha_4 + \alpha_5 + \alpha_6 + \alpha_{10} = 0, \quad -\alpha_2 - \alpha_6 - \alpha_7 - \alpha_{10} = 0, \]
\[ \alpha_3 + \alpha_4 + \alpha_6 + \alpha_8 + \alpha_9 = 0, \quad \alpha_1 + \alpha_7 + \alpha_8 = 0, \quad \alpha_3 + 2\alpha_4 + \alpha_5 = 0. \] (4.19)

The solution of this system is trivial. So \( Q_1, \ldots, Q_{10} \) are linearly independent.

Let \( f(x, y, z_1, z_2) \in \text{V} \text{I} \text{N}_3[3, 2, 1, 1] \) then
\[ f = (\text{Id} + (z_1, z_2)) \{ f_1 x z_1 + f_1' x z_1 + f_2 y z_2 + f_2' y z_2 z_1 + f_3 z_1 z_2 x + f_4 z_1 z_2 + f_5 z_1 z_2 y + f_6 z_1 z_2 y x \} + f_4 z_1 z_2 \]
\[ + f_6' z_1 z_2 + f_5' z_1 z_2 + f_5' z_1 z_2 + f_5' z_1 z_2 + f_5' z_1 z_2, \] (4.20)

since \( H_2 \) has no identity of type \([3, 1]\). Moreover
\[ f_1 = \sum_{i=1}^{8} \alpha_i k_i, \quad f_1' = \sum_{i=1}^{8} \alpha_i' k_i, \quad f_2 = \sum_{i=1}^{4} \beta_i h_i, \quad f_2' = \sum_{i=1}^{4} \beta_i' h_i, \]
\[ f_3 = \alpha[[y, x], [y, z_1]], \quad f_4 = \beta[[y, x], [y, x^2]], \quad f_6 = \gamma[[x, z_1], [x, z_2]], \]
where \( k_1, \ldots, k_8 \) and \( h_1, \ldots, h_4 \) are the basis of the spaces \( V L_2[2,2,1] \) and \( V L_2[3,1,1] \) given in proposition 3.1.3 and 3.1.2 respectively.

Since \( f(x, y, z_1, 1) \) is an identity of \( H_3 \) of type \([3,2,1]\), it is 0 identically. From (4.20), \( f_5 \) and \( f_6 \) are of type \([1^4]\) and \([2,1,1]\) respectively. Thus they are linear combinations of the products of two commutators by Corollary 3.2 of [MR]. So \( f_5(x, y, z_1, 1) = f_6(x, y, z_1, 1) = 0. \) From the definitions of \( k_i \) and \( h_i, k_i(x, y, z_1, 1) = h_i(x, y, z_1, 1) = 0. \) So

\[
f(x, y, z_1, 1) = f_1x + f_1'x + f_2y + f_2'y + f_3xx + f_3'xy + f_8yx.
\]

Hence the coefficients of the following monomials are 0:

\[
xyzxxz, \ yxyyzxz, \ yxyzxxz, \ yzyzxx, \ yzxyx, \ xyxyx, \ xyzyx, \ xyzyxz.
\]

Since the last two variables of the monomials above are not \( xy \) and \( yx \), these monomials occur in \( f_1x + f_1'x + f_3xx \). Thus we have

\[
\alpha_2 + \alpha_2' - \alpha_4 - \alpha_4' = 0, \quad \alpha_3 + \alpha_5 - \alpha_5' = 0,
\]

\[
\alpha_5 + \alpha_6 + \alpha_7 + \alpha_7' = 0, \quad \alpha_4 + \alpha_4' + \alpha_7 + \alpha_7' = 0,
\]

\[
\alpha_1 + \alpha_1' + \alpha_6 - \alpha_6' - \alpha_7 + \alpha = 0,
\]

\[
\alpha_1 + \alpha_1' + \alpha_3 + \alpha_3' + \alpha_6 + \alpha_6' + \alpha = 0,
\]

\[
\alpha_4 + \alpha_4' + \alpha_5 + \alpha_5' + \alpha_6 + \alpha_6' - \alpha_7 + \alpha_7' + \alpha = 0,
\]

\[
\alpha_1 + \alpha_1' - \alpha_4 - \alpha_4' + \alpha_5 + \alpha_5' - \alpha_6 - \alpha_6' + \alpha_7 + \alpha_7' - \alpha = 0.
\]

Similarly considering the coefficients of the monomials

\[
xyzxxz, \ xyx1z, \ z_1xyyz
\]

in \( f(x, y, z_1, 1) \) yields

\[
-\beta_1 - \beta_1' - \beta_4 - \beta_4' = 0, \quad \beta_1 + \beta_1' - \beta_3 - \beta_3' = 0, \quad \beta_4 + \beta_4' = 0.
\]

Solving the equations above we have

\[
\alpha_i = -\alpha_i', \ i \neq 6, \ 8, \ \beta_j = -\beta_j', \ j \neq 2.
\]  \hspace{1cm} (4.21)
Now we assume that \( f^* = \pm f \). Let \( g \) be the sum of the terms of \( f \) in which the first two positions and the last two positions are \( xy, yx \) or \( xx \). Since \( f \) is skew symmetric in \( z_1, z_2 \), we need only consider the coefficients of the following monomials in \( g \) when \( f^* = \pm f \):

\[
xyz_1z_2xy, \ yz_1z_2yx, \ yxz_2z_1xy, \ yx_1y_2zyz_1x, \ yz_1x_2z_1xy, \ yz_1x_2z_1y,
\]

\[
yz_2z_1yxx, \ xy_1z_2yxx, \ xz_1z_2xyx, \ xz_1z_2yxx, \ yz_2y_1z_2xy, \ xy_1y_2z_2xx, \ yz_1y_2z_2xx.
\]

Let \( \gamma_i \) denote the coefficients of the first 7 monomials above in the polynomial \( g \) and let \( \delta_i \) denote the coefficients of the last 6 monomials above. Then using Lemma 2.1.1 we have

\[
\delta_1 + \delta_2 = 0, \ \delta_3 + \delta_4 = 0, \ \delta_5 + \delta_6 = 0, \ \gamma_1 + \gamma_2 + \delta_2 = 0, \ \gamma_3 + \gamma_4 + \delta_1 = 0. \quad (4.22)
\]

By Lemma 2.1.1, the sum of the coefficients of the following monomials is 0:

\[
xyz_1y, \ xyz_2y, \ yxz_2y, \ yx_2z_1y,
\]

where \( w = z_1z_2 \). That is,

\[
\gamma_5 \pm (\gamma_5) \cdot \gamma_6 + \gamma_7 = 0 \quad (4.23)
\]

since \( f^* = \pm f \).

Next we distinguish two cases. First assume that \( f^* = f \). Since \( f(x, y, z_1, y) \) and \( f(x, y, z_1, x) \) are identities of type \([3,3,1]\) and \([4,2,1]\) respectively, there exist scalars \( \lambda_i \in F \) such that

\[
f(x, y, z_1, y) = \lambda_1 Q(x \circ y, x, y, z_1, x, y), \ f(x, y, z_1, x) = \lambda_2 Q(x^2, x, y, z_1, x, y).
\]

But \( f^* = f \) and \( Q^* = -Q \), so \( \lambda_i = 0 \) for \( i = 1, 2 \). Hence \( f(x, y, z_1, y) = f(x, y, z_1, x) = 0 \).

Thus considering the coefficients of monomials

\[
xyz_1y, \ xyz_2y, \ yxz_2y, \ yx_2z_1y,
\]

\[
yxz_1y, \ z_1yxyy, \ yx_2z_1y, \ xy_2z_1y, \ xyy_2xy, \ yxy_2xy,
\]

in \( f(x, y, z_1, y) \) yields

\[
\gamma_1 + \alpha_1 - \alpha_3 + \alpha_5 + \alpha_6 - \alpha_7 = 0, \ \gamma_2 - \alpha_1 - \alpha_5 + \alpha_6 = 0.
\]
\[ \gamma_3 + \alpha_1 + \alpha_3 - \alpha_4 - \alpha_5 + \alpha_7 = 0, \quad \gamma_4 - \alpha_1 + \alpha_4 - \alpha_6' = 0, \]
\[ \gamma_5 - \alpha_1 - \alpha_3 - \alpha_8 = 0, \quad \alpha_1 + \alpha_2 - \alpha_4 + \alpha_7 + \alpha_6' - \gamma_5 = 0, \]
\[ \alpha_1 + \alpha_5 + \alpha_6 - \beta_1 = 0, \quad -\alpha_1 + \alpha_4 + \alpha_8 = 0, \]
\[ \alpha_3 - \alpha_5 + \beta - \beta_1 - \beta_2 - \beta_4 = 0, \]
\[ -\alpha_3 - \alpha_7 + \delta_1 + \delta_5 = 0, \]
\[ -\alpha_2 + \alpha_3 - \alpha_6 + \alpha_8 - \gamma_1 = 0. \] (4.24)

For instance, the monomial $xyz_1 yxx y$ comes from the monomials $xxz_1 yxx y$, $xyz_1 zxx y$ and $xyz_1 yxz_2$ of $f$ which have coefficients $-\alpha_3 + \alpha_5$, $\gamma_1$, $\alpha_1 + \alpha_6 - \alpha_7$ respectively by (4.20), the expression of $g$ and $f^* = f$. So we get the first equation of (4.24). The others were obtained in the same way.

Since $f(x, y, z_1, y) = -f_2(x, y, y) y z_1 + f_4[z_1, y] + \sim$, considering the sum of the terms of $f(x, y, z_1, y)$ which end in $yz_1$ yields
\[ \beta - \beta_3 - \beta_4 = 0. \] (4.25)

It can be proved also that
\[ \gamma + \beta_2 + \beta_4' = 0, \] (4.26)
by considering the sum of the terms of $f(x, y, z_1, y)$ which end in $xyy$. Indeed, from (4.20) this kind of terms occur in
\[ f_2(x, y, z_1) y y + f'_2(x, y, z_1) y y + f_7(x, z_1, y) x y y \]
\[ = (\beta_2 + \beta_4' + \gamma) [x, z_1], [x, y] y y. \]

Thus (4.26) holds.

By $f(x, y, z_1, 1) = 0$, considering the coefficients of the monomials $xyz_1 xyz$, $yzx_1 zxy$ and $xyz_1 xxy$ yields
\[ \gamma_6 = \gamma_7 = 0, \quad \gamma_5 - \alpha_8 - \alpha_6' + \beta_2 + \beta_3' = 0. \] (4.27)

From $f^* = f$, considering the coefficient of the monomial $y z_2 y z x z_1 x$ in $f$ yields
\[ -\alpha_1 - \alpha_3 + \alpha_6' = -\beta_1 + \beta_3. \] (4.28)
From \( f(x, y, z_1, x) = 0 \), considering the coefficients of the monomials

\[
yxz_1xyxx, xzx_1xyy, xzx_1yx, xz_1yxx, xz_1yyx, \\
yz_1yx, xz_1xxy, xz_1xyx,
\]

yields

\[
\delta_1 - \beta_4 + \alpha_8 + \alpha_6' = 0, \quad \delta_3 + \beta_3 = 0, \quad \beta_3 - \delta_5 = 0, \quad \delta_6 + \alpha_6 + \alpha_6' = 0.
\]

\[
\beta - \alpha_1 - \alpha_3 + 2\alpha_4 + \alpha_5 = 0, \quad \beta_2 + \gamma_2 = 0, \quad \beta_2' - \beta_4 + \gamma_1 + \gamma_5 = 0.
\]

(4.29)

[Note. When both \(-x_2x\) and \(-x_2xy\) occur in calculating the coefficients of the monomials above we need only consider \(\alpha_i\) and \(\alpha_6'\) for \(i = 6, 8\) by (4.21)].

Next we use Lemma 2.1.1 to produce more equations. Let \(w = z_1z_2xy\) then the sum of the coefficients of the following monomials is 0:

\[
xzxw, xzwx, xzw, xxw, wxx, wxx.
\]

This yields

\[
-\beta + \alpha_2 - \alpha_4 = 0.
\]

(4.30)

Similarly considering \(w = z_1x_2z_2yy, xz_1z_2yy, z_1z_2yy\) yields

\[-\alpha_2 + \alpha_3 - \alpha_4 - \alpha_5 + \alpha_6' + \alpha_7 = 0, \quad \delta_3 - \beta_3 - \alpha_7 = 0, \quad \beta + \delta_4 + \alpha_5 + \alpha_7 = 0.
\]

(4.31)

Solving systems (4.22),..., (4.31) we have

\[
\alpha = \delta_6, \quad \beta = \delta_6, \quad \gamma = \gamma_5, \quad \alpha_1 = \gamma_5 - \frac{1}{2}\delta_6, \quad \alpha_2 = \frac{1}{2}\gamma_5 + \frac{3}{4}\delta_6, \quad \alpha_3 = -\frac{1}{2}\gamma_5 + \frac{7}{4}\delta_6,
\]

\[
\alpha_4 = \frac{1}{2}\gamma_5 - \frac{1}{4}\delta_6, \quad \alpha_5 = -\frac{1}{2}\gamma_5 + \frac{3}{4}\delta_6, \quad \alpha_6 = -\frac{1}{2}\gamma_5 - \frac{5}{4}\delta_6, \quad \alpha_7 = \frac{1}{2}\gamma_5 - \frac{3}{4}\delta_6,
\]

\[
\alpha_6' = \frac{1}{2}\gamma_5 + \frac{1}{4}\delta_6, \quad \alpha_8 = \frac{1}{2}\gamma_5 - \frac{1}{4}\delta_6, \quad \alpha_4' = -\frac{1}{2}\gamma_5 + \frac{1}{4}\delta_6, \quad \beta_1 = 0, \quad \beta_2 = 0,
\]

\[
\beta_3 = -\delta_6, \quad \beta_4 = 2\delta_6,
\]

\[
\beta_2' = -\gamma_5, \quad \gamma_1 = 2\delta_6, \quad \gamma_2 = \gamma_6 = \gamma_7 = 0,
\]

\[
\gamma_3 = -\gamma_5 - 2\delta_6, \quad \gamma_4 = \gamma_5, \quad \delta_1 = 2\delta_6, \quad \delta_2 = -\delta_2, \quad \delta_3 = -\delta_4 = -\delta_5 = \delta_6.
\]

(4.32)
Let
\[ x = e[22] + e[12] + e[13], \quad y = e[23], \quad z_1 = e[11], \quad z_2 = e[33] \]
then \( f(x, y, z_1, z_2) = 0 \) implies that \( \gamma_3 + \delta_8 = 0 \) by considering the coefficients of \( e_{-3} \). Indeed, only the following 11 monomials in \( f \) evaluate to \( e_{23} \):
\[
xyz21xy21, \quad xz1xyyz21, \quad xy2z1xy21, \quad xyz21z1x2, \quad xz1xyyz21, \quad yz1xx22, \quad yz2xyzz21, \quad yz2zz1xx2, \quad yz2zz1yz2, \quad xz1xx22yz2.
\]
Their coefficients in \( f \) are
\[
\alpha_2 - \alpha_4, \quad -\beta_2, \quad \beta_3, \quad -\gamma_5, \quad \alpha_1 - \alpha'_6 - \alpha_7, \quad \beta_2, \quad -\beta_1 - \beta_2 - \beta_4, \quad \alpha_1 + \alpha'_6 - \alpha_3, \quad \beta'_2 - \beta_4, \quad -\alpha_2 - \alpha_7, \quad \alpha_4 + \alpha_5 - \alpha'_6 - \alpha_7.
\]
Thus from (4.32) and adding them up we get \(-4\gamma_5 - 4\delta_8 = 0\).

So far we have determined the coefficients of \( f_1, f'_1, f_2, f'_2, f_3, f_4 \) and \( f_6 \). The coefficients of the monomials in \( f_5yzx, f_7xy \) and \( f_8yx \) have also been determined since the monomials in these polynomials either start with \( z_1z, \quad zz_1, \quad yz_1, \quad z_1y \) or the monomials in \( g \). Since \( f^* = f \), the coefficients of the first kind are the linear combination of \( \alpha_i, \quad \alpha'_i, \quad \beta_i, \quad \beta'_i \). While the second kind have coefficients \( \gamma_j \) or \( \delta_j \). So \( f = \delta_8 h \), where \( h \) is independent from \( f \). On the other hand, \( Q_9 - Q_10 \) is \( * \)-symmetric and skew in \( z_1, \quad z_2 \). From Lemma 2.1.3 the vector space \( W := \{ f \in VIM_3[3, 2, 1, 1] | f^* = f \} \) has a basis \( Q_9 - Q_10 \).

Now let \( f^* = -f \). We also have
\[
f(x, y, z_1, y) = \lambda_1 Q(x \circ y, y, z_1, x, y), \quad f(x, y, z_1, x) = \lambda_2 Q(x^2, x, y, z_1, x, y)
\]
for some \( \lambda_i \in F \). We may assume that \( \lambda_i = 0 \), \( i = 1, 2 \). Repeating the proof of the case \( f^* = f \), we also have (4.23), (4.25) and (4.26). Similarly to the first 7 equations in (4.24) we have
\[
\gamma_1 + \alpha_1 - \alpha_3 + \alpha_5 + \alpha_6 - \alpha_7 = 0, \quad \gamma_2 - \alpha_1 + \alpha_5 + \alpha'_6 + 2\alpha_7 = 0, \\
\gamma_3 + \alpha_1 + \alpha_3 - \alpha_4 + 2\alpha_5 - \alpha_6 + \alpha_7 = 0, \\
\gamma_4 - \alpha_1 + \alpha_4 - 2\alpha_5 - \alpha'_6 - 2\alpha_7 = 0, \quad \gamma_5 - \alpha_1 + 2\alpha_4 - \alpha_5 + 2\alpha_7 - \alpha_8 = 0, \\
-\gamma_6 - \alpha_1 + \alpha_2 - \alpha_4 - \alpha_7 - \alpha'_6 = 0, \quad \alpha_1 + \alpha_3 + \alpha_6 - \beta_1 = 0.
\]
(4.33)
Similarly to (4.29) we have

$$\delta_1 + \beta_4 + \alpha_8 + \alpha'_g = 0, \quad \delta_3 + \beta_3 = 0, \quad \beta_2 + \delta_5 = 0.$$  
$$\delta_6 + \alpha_6 + \alpha'_6 = 0, \quad \beta - \alpha_1 - \alpha_3 + 2\alpha_4 + \alpha_5 = 0.$$  
$$2(\beta_1 + \beta_4) + \beta_2 + \gamma_2 + \gamma_6 = 0, \quad -2\beta_1 + \beta'_2 - \beta_4 + \gamma_1 + \gamma_5 = 0.$$  

(4.34)

From Lemma 2.1.1 if \(w = z_2xxy\) then the sum of the coefficients of the monomials

$$xyz_1w, \ yz_1xw, \ z_1xyw, \ z_1yxw, \ yxz_1w, \ xz_1yw$$

is 0. Similarly if \(w = yzx_1x\) then the sum of the coefficients of the monomials of the form \(w - - -\) is 0, where empty positions are \(x, y, z_2\). These yield

$$\beta_2 + \beta'_2 + \gamma_1 + \gamma_3 - \alpha_8 - \alpha'_g = 0, \quad -\beta_2 - \beta'_2 + \gamma_5 + \gamma_7 + \alpha_8 + \alpha'_g = 0.$$  

(4.35)

From \(f(x,y,z_1,1) = 0\), considering the coefficients of the monomials \(xyz_1xxy, \ yz_1xyx, \ yxz_1xxy\) in it respectively yields

$$\gamma_5 + \alpha_8 + \alpha'_g + \beta_2 + \beta'_2 = 0, \quad \gamma_6 - 2(\alpha_8 + \alpha'_g) = 0, \quad \gamma_7 - 2(\beta_2 + \beta'_2) = 0.$$  

(4.36)

Solving the systems (4.22), (4.23), (4.25), (4.26), (4.33), (4.34), (4.35) and (4.36) we know that

$$\alpha'_g, \ \beta'_2, \ \gamma_4, \ \gamma_6, \ \gamma_7, \ \delta_2, \ \delta_6$$

are the parameters of the coefficients, i.e. every coefficient is a linear combination of

$$\alpha'_g, \ \beta'_2, \ \gamma_4, \ \gamma_6, \ \gamma_7, \ \delta_2, \ \delta_6.$$

Since we assume that \(\lambda_i = 0, \ i = 1, \ 2,\)

$$f = \alpha'_g g_1 + \beta'_2 g_2 + \gamma_4 g_3 + \gamma_5 g_4 + \gamma_7 g_5 + \delta_2 g_6 + \delta_6 g_7 + \lambda_1 g_8 + \lambda_2 g_9,$$

where \(g_1, \ldots, g_9\) are independent from \(f\). Since \(Q_1, \ldots, Q_8, Q_9 = Q_{10}\) are \(*\)-skew symmetric and linearly independent, by Lemma 2.1.3, \(\{Q_1, \ldots, Q_8, Q_9 = Q_{10}\}\) is a basis of the vector space \(U := \{f \in VIN_5[3,2,1,1]|f^* = -f\}\). Hence \(\{Q_1, \ldots, Q_{10}\}\) is a basis of the vector space \(VIN_5[3,2,1,1]\).
Let
\begin{align*}
Q_1 &= T_6(x, y_1, y_2, y_3, y_4; x^2), \quad Q_2 := T_6(x^2, y_1, y_2, y_3, y_4; x), \\
Q_3 &= xT_6(x, y_1, y_2, y_3, y_4; x), \quad Q_4 := T_6(x, y_1, y_2, y_3, y_4; x)x, \\
Q_5 &= \sum_{1234} (-1)^\sigma \Lambda(x, y_1, x, y_2, x, y_3, y_4), \\
Q_6 &= \sum_{1234} (-1)^\sigma T_6(x, x \circ y_1, y_2, y_3, y_4; x), \\
Q_7 &= \sum_{1234} (-1)^\sigma T_6(x, x^2, y_1, y_2, y_3; y_4),
\end{align*}
where the sum is taken over all cyclic permutations of 1234 and \((-1)^\sigma\) is the sign of the cyclic permutation \(\sigma\). Then

**Proposition 4.2.4** The set \(\{Q_1, \ldots, Q_7\}\) is a basis of the vector space \(VIN_3[3, 1^4]\).

**Proof.** Let \(f(x, y_1, y_2, y_3, y_4) \in VIN_3[3, 1^4]\). Then
\begin{align*}
\begin{aligned}
f &= \sum_{\sigma(1) < \sigma(2) < \sigma(3)} (-1)^\sigma f_1(x, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) + \\
&\quad \sum_{\sigma(1) < \sigma(2) < \sigma(3)} (-1)^\sigma \{f_2(x, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)})x y_{\sigma(4)} + f_3(x, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}) y_{\sigma(4)} x x\}. \\
&= \sum_{i=1}^7 \alpha_i h_i, \quad f_1 = \sum_{i=1}^4 \alpha_i h_i, \quad f_2 = \sum_{i=1}^7 \beta_i L_i, \quad f_3 = \sum_{i=1}^7 \beta_i L_i,
\end{aligned}
\end{align*}
Moreover,
\begin{align*}
f_1 &= \sum_{i=1}^4 \alpha_i h_i, \quad f_1 = \sum_{i=1}^4 \alpha_i h_i, \quad f_2 = \sum_{i=1}^7 \beta_i L_i, \quad f_2 = \sum_{i=1}^7 \beta_i L_i,
\end{align*}
where \(h_1, \ldots, h_4\) and \(L_1, \ldots, L_7\) are the bases of \(VI_2[3, 1, 1]\) and \(VIN_2[2, 1, 1, 1]\) given in Chapter 3. \(f_3\) is an identity of \(II_2\) of type \([1^4]\). So it is a linear combination of the products of two commutators by Corollary 3.2 of [MR]. Since \(f_3(x, y_1, y_2, y_3)\) is skew in the \(y_i\)'s,
\begin{align*}
f_3 &= \sum_{123} \{\gamma_1[x, y_1][y_2, y_3] + \gamma_2[y_1, y_2][x, y_3]\}.
\end{align*}
Since \(f_1\) is skew in the variables of degree 1, from the definition of \(h_i\), we have \(\alpha_3 = -\alpha_4\). Also from the definitions of \(h_i\) and \(L_i\) if we let one of degree 1 variables in \(L_i\) or in \(h_j\)
be equal to 1 then $L_i = h_j = 0$, for $i = 1, \ldots, 6$ and $j = 1, \ldots, 4$. Thus from (4.37),

$$f(x, y_1, y_2, y_3, 1) = 0$$

implies that

$$\gamma_1 + \beta_7 + \beta_7' = 0, \quad \gamma_2 = 0, \quad 3\beta_7' + \beta_7 = 0,$$

(4.38)

by considering the coefficients of the monomials $xy_1y_2y_3xx$, $y_1y_2y_3xxx$, $xy_2y_1y_3x$ in $f(x, y_1, y_2, y_3, 1)$ respectively.

Since $f(x, y_1, y_2, y_3, x) \in VIN_3[4, 1, 1, 1]$, there exist $\alpha, \beta \in F$ such that $f(x, y_1, y_2, y_3, x) = \alpha T_0 + \beta K$, where $T_0, K$ are the basis of $VIN_3[4, 1^3]$ given in Proposition 4.1.3. As before we assume that $\alpha = \beta = 0$. Then the sum of the terms in $f(x, y_1, y_2, y_3, x)$ which end in $xx$ is 0 implies that

$$(*) \quad f_3(x, y_1, y_2, y_3)x + f_2 + f_2' = 0.$$ 

Since $f_3(x, y_1, y_2, y_3)x = \gamma_1 L_3(x, y_1, y_2, y_3)$ and $L_1, \ldots, L_7$ are linearly independent, the equation $(*)$ implies that

$$\gamma_1 + \beta_3 + \beta_3' = 0, \quad \beta_i + \beta_i' = 0, \quad i \neq 3.$$ 

(4.39)

The sum of the terms of $f(x, y_1, y_2, y_3, x)$ which end in $y_3x$ is 0. This implies that

$$(***) \quad f_1(x, y_1, y_2)y_3x - f_2'(x, y_1, y_2, x)y_3x = 0.$$ 

This implies that

$$\alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 (h_3 - h_4) + \sum_{i=1}^{6} \beta_i L_i(x, y_1, y_2, x) = 0$$

since $\alpha_7 = 0$ from (4.38) and (4.39). Since

$$L_1(x, y_1, y_2, x) = h_2 + h_3 - h_4, \quad L_2(x, y_1, y_2, x) = -h_2 - h_3 + h_4,$$

$$L_3(x, y_1, y_2, x) = h_2, \quad L_4(x, y_1, y_2, x) = -h_2,$$

$$L_5(x, y_1, y_2, x) = h_1 + h_3 + h_4, \quad L_6(x, y_1, y_2, x) = h_1,$$

and $h_1, \ldots, h_4$ are linearly independent,

$$\alpha_1 + \beta_5 - \beta_6 = 0, \quad \alpha_2 + \beta_1 - \beta_2 + \beta_3 - \beta_4 = 0, \quad \alpha_3 + \beta_1 - \beta_2 - \beta_5 = 0.$$ 

(4.40)

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Let \( w = y_1 x y_2 y_3 \), then from Lemma 2.1.1 the sum of the coefficients of the monomials
\[
\begin{align*}
&x w y_1, x w^2 y_1, x y_1 w, x y_1 w^2, w x y_1, w^2 x y_1, \\
y_4 w x, y_4 w^2 x, y_4 x w, y_4 x w^2, w y_4 x, w^2 y_4 x
\end{align*}
\]
is 0. Since \( \beta_i = -\beta'_i \) for \( i = 1, \ldots, 6 \) from (4.38) and (4.39), the sum of the coefficients of the above 12 monomials is
\[
-2\alpha_1 - \alpha_2 - 2\alpha_4 - \beta_2 + \beta_4 + 2\beta_6 = 0. \tag{4.41}
\]
Solving the systems (4.38), ..., (4.41) we know that \( \beta_2, \ldots, \beta_6 \) are parameters of the coefficients. Since we assume that \( \alpha = \beta = 0 \),
\[
f = \alpha g_0 + \beta g_1 + \sum_{i=2}^{6} \beta_i g_i,
\]
where \( g_0, \ldots, g_8 \) are independent from \( f \).

Next we show that \( Q_1, \ldots, Q_7 \) are linearly independent. If there exist scalars \( \alpha_i \in F \) such that \( \sum_{i=1}^{7} \alpha_i Q_i = 0 \) then considering the coefficients of the monomials
\[
\begin{align*}
&x y_1 y_2 y_3 y_4 x, y_1 y_2 x y_3 y_4 x, x y_1 x y_2 y_3 y_4 x, y_1 y_2 x y_3 x y_4 x, \\
x y_1 x y_2 y_3 y_4 x, x y_1 y_2 x y_3 x y_4 x, x y_1 x y_2 y_3 x y_4 x, y_1 x x y_2 y_3 y_4 x
\end{align*}
\]
yields
\[
-\alpha_1 + \alpha_2 + 2\alpha_7 = 0, -\alpha_2 + \alpha_5 - 2\alpha_6 - 3\alpha_7 = 0, -\alpha_3 + \alpha_5 - \alpha_6 = 0, \\
\alpha_4 - \alpha_5 + \alpha_6 = 0, -\alpha_4 - \alpha_6 = 0, \alpha_3 + \alpha_6 = 0, \alpha_3 + \alpha_5 + 3\alpha_6 = 0, \alpha_1 + \alpha_2 + 2\alpha_6 = 0.
\]
The solution is trivial. So \( Q_1, \ldots, Q_7 \) are linearly independent, completing the proof.

\[\]
Note. Let
\[
\begin{align*}
Q_8 &:= \sum_{\sigma(i) < \sigma(i+1), i=1,3} (-1)^{\sigma} K(x, y_{\sigma(1)}, x, y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}, x), \\
Q_9 &:= \sum_{\sigma(i) < \sigma(i+1), i=1,3} (-1)^{\sigma} Q(x^2, x, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}), \\
Q_{10} &:= \sum_{\sigma(3) < \sigma(4)} (-1)^{\sigma} Q(x, x, y_{\sigma(1)}, y_{\sigma(2)}, x, y_{\sigma(3)}, y_{\sigma(4)}).
\end{align*}
\]
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\[-34Q_1 - 10Q_2 + 8Q_3 + 8Q_4 + 6Q_5 + 9Q_6 + 2Q_7 + 2Q_8 + 13Q_{10} = 0.\]
\[7Q_1 + Q_2 - 2Q_6 + Q_7 - 2Q_{10} = 0.\]
\[\frac{21}{4}Q_1 - \frac{5}{4}Q_2 - 1Q_3 + 3Q_5 - 2Q_6 + \frac{11}{4}Q_7 + Q_8 = 0.\]

4.3 The Identities of $H_3$ of Type $[2, \ldots]$ 

Let 
\[
Q_1 := \sum_{\sigma \in S_3} Q(x_{\sigma(1)} \circ x_{\sigma(2)}, x_{\sigma(1)}, x_{\sigma(3)}, x_{\sigma(2)}, x_{\sigma(3)}, y),
\]
\[
Q_2 := \sum_{(123)} Q(x_1 \circ y, x_2, x_3, x_1, x_2, x_3),
\]
\[
Q_3 := \sum_{(123)} Q(x_1, x_2, x_3, y, x_2, x_3) x_1, Q_4 := \sum_{(123)} x_1 Q(x_1, x_2, x_3, y, x_2, x_3),
\]
\[
Q_5 := \sum_{(123)} Q(x_1^2, x_2, x_3, y, x_2, x_3), Q_6 := K(x_1, x_2, x_1, x_3, x_2, x_3, y),
\]
\[
Q_7 := \sum_{\sigma \in S_3} K(x_{\sigma(1)}, y, x_{\sigma(1)}, x_{\sigma(3)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(2)}).
\]

Then

**Proposition 4.3.1** The set $\{Q_i | i = 1, \ldots, 7\}$ is a basis of the vector space $VIN_3[2, 2, 2, 1]$.

**Proof.** Let $f(x, y, z, w) \in VIN_3[2^3, 1]$ then $f(x, x, z, w)$ and $f(x, y, z, x)$ are identities of type $[4, 2, 1]$ and $[3, 2, 2]$ respectively, and $f(x, y, z, x)$ is symmetric in the variables of degree 2. Thus there exist $\alpha, \lambda_1, \lambda_2, \lambda_3 \in F$ such that

\[
f(x, x, z, w) = \alpha Q(x^2, x, z, w, z, z), \quad f(x, y, z, x) = \sum_{i=1}^{3} \lambda_i y_i,
\]

where $g_1, g_2, g_3$ are the basis of $VIN_3[3, 2, 2]$ given in Proposition 4.2.2.

Since $g_1, g_2, g_3, Q(x^2, z, z, w, z, z)$ are independent from $f$ we assume that $\alpha = \lambda_1 = \lambda_2 = \lambda_3 = 0$. Then $f(x, x, z, w) = f(x, y, z, x) = 0$. 

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The words involving exactly 2 $x$'s and 2 $y$'s are
\[ xxyy, xyxy, xyyx, yyxx, yxyx, yxxy. \]

Let $A := xxyy$, $B := xyxy$, $C := yyyx$. Putting $w$ in the $i$-th position of $A$ we get a new word denoted by $A_i$ (e.g. $A_1 = wxxyy$). Similarly we define $B_i$ and $C_i$. Let $A_{ij}$ (resp. $A_{ijk}$) denote the word which is obtained by inserting $z$ in $j$-th (resp. $k$-th) position of $A_i$ (resp. $A_{ij}$). (e.g. $A_{13} = wxxyy$, $A_{135} = wxzzzyy$). Similarly we can define $B_{ijk}$, $C_{ijk}$. Since $f(x, y, z, w)$ is symmetric in $x, y, z$, it suffices to determine the coefficients of the monomials $A_{ijk}$, $B_{ijk}$, $C_{ijk}$. Let $\alpha_{ijk}$ (resp. $\beta_{ijk}$, $\gamma_{ijk}$) denote the coefficient of the monomial $A_{ijk}$ (resp. $B_{ijk}$, $C_{ijk}$).

For an arbitrary but fixed pair of $j, k$, if we denote $\alpha_{ijk}$ by $\alpha_i$ then $f(x, z, z, w) = 0$ implies that
\[ \alpha_i + \beta_i + \gamma_i = 0, \quad i = 1, 2, \ldots, 5. \quad (4.42) \]

Now $f(x, y, z, z) = 0$ implies that
\[ \alpha_1 + \alpha_2 + \alpha_3 = 0, \quad (4.43) \]

since the monomial $xxzzyy$ (because $z$ is fixed at positions $j, k$ we can forget it) has coefficient 0 in $f(x, y, z, z)$ and this monomial comes from the monomials $wxxy$, $wxzy$, $xwyx$ (the variable $z$ occur in positions $j, k$ we forget it). Similarly considering the monomials
\[ xxyx, xyxy, xyyz, zxyx, xxzy \]
yields
\[ \beta_1 + \beta_2 + \alpha_4 = 0, \quad \gamma_1 + \beta_3 + \beta_4 = 0, \quad \alpha_1 + \gamma_4 + \gamma_5 = 0, \]
\[ \beta_1 + \gamma_3 + \beta_5 = 0, \quad \gamma_1 + \gamma_2 + \alpha_5 = 0. \quad (4.44) \]

[ The symmetry of $f$ in the variables $x, y$ was used here].

Since $f$ is symmetric in $x, y, z$, $f(x, y, z, x) = 0$ implies that $f(x, y, z, y) = 0$. Thus, for an arbitrary but fixed pair of $j$ and $k$, the monomials $xxyy$, $xyzy$, $xyyx$ have coefficients 0 in $f(x, y, z, y)$. This implies that
\[ \alpha_3 + \alpha_4 + \alpha_5 = 0, \quad \beta_2 + \beta_3 + \gamma_5 = 0, \quad \alpha_2 + \beta_4 + \beta_5 = 0, \quad \gamma_2 + \gamma_3 + \gamma_4 = 0. \quad (4.45) \]
The solution of the systems (4.12) . . . (4.45) is
\[
\alpha_1 = -\gamma_4 - \gamma_5, \alpha_2 = -\beta_4 - \beta_5, \alpha_3 = \beta_4 + \beta_5 + \gamma_4 + \gamma_5, \alpha_4 = -\beta_4 - \gamma_4,
\]
\[
\alpha_5 = -\beta_5 - \gamma_5, \beta_1 = -\beta_5 - \gamma_3, \beta_2 = \beta_4 + \beta_5 + \gamma_3 + \gamma_4,
\]
\[
\beta_3 = -\beta_4 - \beta_5 - \gamma_3 - \gamma_4 - \gamma_5, \gamma_1 = \gamma_3 + \gamma_4, \gamma_2 = -\gamma_3 - \gamma_4.
\] (4.46)

From (4.46) we know that it suffices to determine \( \beta_4, \beta_5, \gamma_3, \gamma_4, \gamma_5 \) for each pair of \( j \) and \( k \).

But if in \( B_{ijk} \) one \( z \) occurs in the position 1 or 2, that is \( j \) or \( k \) \( \in \{1, 2\} \) then \( B_{ijk} \) must have \( z \) or \( y \) occurring in the positions \( m_1, m_2 \) in \( B_{ijk} \) with \( m_r \geq 3 \) for \( r = 1, 2 \). So interchanging this variable with \( z \) in \( B_{ijk} \) yields a word \( X_{sjk}ko \) with \( s \leq 4 \) and \( 3 \leq j_0 < k_0 \leq 7 \). By the symmetry of \( f \), \( B_{ijk} \) and \( X_{sjk}ko \) have the same coefficients in \( f \). That is, \( \beta_{ijk} = d_{sjk}ko \), where \( d_{sjk}ko \) is the coefficient of \( X_{sjk}ko \). Therefore from (4.46) we need only to determine \( \beta_{sjk}ko \) with \( 3 \leq j_0 < k_0 \leq 7 \). (e.g. if \( B_{426} = xzywzy \) then interchanging \( y, z \) yields the word \( xzywzy \) which is \( B_{437} \). So \( \beta_{426} = \beta_{437} \).) Similarly, we need only to determine \( \beta_{sjk}, \gamma_{rjk} \) for \( r = 3, 4, 5 \) and \( 3 \leq j < k \leq 7 \). So we have shown that

**Lemma 4.3.1** The coefficients of \( f \) which we need to determine are \( \beta_{rjk}, \gamma_{sjk} \) for \( r = 4, 5, \)
\( s = 3, 4, 5 \) and \( 3 \leq j < k \leq 7 \).

Since \( H_2 \) has no identity of type [2, 2], the symmetry of \( f \) implies that
\[
f = \sum_{(x,y,z)} (f_1(y,z,x)zw + f_1'(y,z,x)wx) +
\[
f_2(x,y,w)yz^2 + \cdots +
\]
\[
f_3(x,y,z,w)yz + \cdots.
\] (4.47)

Moreover
\[
f_1 = \sum_{i=1}^8 \alpha_ik_i, f_1' = \sum_{i=1}^8 \alpha_i'k_i, f_2 = \beta[[x,y],[x,w]].
\]

Since \( f_3(x,y,z,w) \) is an identity of \( H_2 \) of type [2, 1^3] from Theorem 1.2, \( f_3 = q_0 + q_1 + q_2 + q_3 \), where \( q_0 \) is skew in \( y, z, w \) and \( q_1 \) (resp. \( q_2, q_3 \)) is symmetric in the variables \( y, z \) (resp. \( y, w \); and \( x, w \)). Thus from Theorem 1.1
\[
q_1(x,y,z,w) = \sum_{i=1}^8 \gamma_i k_i(x,y,z,w), q_2(x,y,z,w) = \sum_{i=1}^8 \delta_i k_i'(x,y,w,z),
\]
\[
q_3(x,y,z,w) = \sum_{i=1}^8 \mu_i k_i(x,z,w,y),
\]

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where $k_i^r(x, y, z, w) := \Delta(y, z) k_i(x, y, w)$. $q_0 \in V N_2 [2, 1^3]$, so $q_0 = \sum_{i=1}^7 \beta_i L_i$.

Now $f(x, y, z, 1) \in V N_2 [2^3] = \{0\}$, so $f(x, y, z, 1) = 0$. Since

$L_i(x, y, z, 1) = 0, \quad i = 1, 2, \ldots, 6, \quad L_7(x, y, z, 1) = 2T_4(x, y, z; z),$

$k_i^r(x, y, z, 1) = 0, \quad i = 1, \ldots, 8, \quad k_i^r(z, x, 1, y) = 0, \quad k_i^r(z, y, 1, x) = 0, \quad i = 1, 3, 4, 6,$

$k_i^r(x, x, 1, y) = 2T_4(x, z, y; z), \quad i = 2, 5, \quad k_i^r(z, x, 1, y) = T_4(x, z, y; z), \quad i = 7, 8,$

$k_i^r(z, y, 1, x) = 2T_4(y, z, x; z), \quad i = 2, 5, \quad k_i^r(z, z, 1, y) = T_4(y, z, x; z), \quad i = 7, 8.$

Thus from (4.47) the sum of the terms of $f(x, y, z, 1)$ which end in $z$ is 0, which implies that

$$f_1(y, z, x) + f'_1(y, z, x) + f_2(z, y, x, 1)y + f_3(y, z, x, 1)x = 0.$$  

Hence

$$\sum_{i=1}^8 (\alpha_i + \alpha_i') k_i(y, z, x) + \lambda(T_4(y, z, x; z)y + T_4(z, y, x; z)x) = 0,$$

where

$$\lambda := 2\beta_7 - 2\beta_2 - 2\delta_2 - \delta_7 - \delta_8 + 2\mu_2 + 2\mu_5 + \mu_7 + \mu_8.$$  

Since $T_4(y, z, x; z)y = -k_0(y, z, x)$, $T_4(z, y, x; z)x = -k_8(y, z, x)$ and $k_1, \ldots, k_8$ are linearly independent,

$$\alpha_i + \alpha_i' = 0, \quad i \neq 6, 8, \quad \alpha_j + \alpha_j' - \lambda = j = 6, 8.$$  

(4.48)

Since $f_1$ is symmetric in the variables of degree 2, from the definition of $k_i$ we have

$$\alpha_1 = 0, \quad \alpha_2 = \alpha_3, \quad \alpha_4 = \alpha_5, \quad \alpha_6 = \alpha_8, \quad \alpha_6' = \alpha_8'.$$  

(4.49)

Next we show that following equations hold:

$$\alpha_2 = 3\alpha_4, \quad \gamma_{456} = 0, \quad \gamma_{457} - \beta = 0.$$  

(4.50)

**Proof.** Since $f(x, y, z, w) = 0$ by assumption, by (4.47) the sum of the terms of $f(x, y, z, w)$ which end in $zw$ is 0, which implies that $2f_1(x, z, x) = 0$. But we have

$$f_1(x, z, x) = \alpha_3[[x, x^2], [z, x]] + 2\alpha_4[[z, x], [z, x^2]] + \alpha_5[[x, z], [x^2, z]] +$$

$$(\alpha_3 - 3\alpha_4)[[z, x^2], [z, x]]$$

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because \( \alpha_1 = 0 \) and \( \alpha_4 = \alpha_5 \). Thus \( \alpha_3 = 3\alpha_4 \).

\( C_{456} = xyywzzz \). Substituting

\[ e[12], e[22], e[23], e[33], e[31] \]

for \( x_1, y, w, z, x_2 \) in the partial linearization \( g(x_1, x_2, y, z, w) := \text{lin}_x f \), the coefficient of \( e_{11} \) is the sum of the coefficients of the monomials \( x_1 yywzzz \) and \( x_2 zywyyx_1 \) in \( y \), that is, the sum of the coefficients of the monomials \( xyywzzz \) and \( xzwyyx \). By the symmetry of \( f \) in \( y, z \) we have \( \gamma_{456} = 0 \).

From Lemma 2.1.1 the sum of the coefficients of the monomials \( xyywzzz, xzwyyx, xzwyyz \) is 0. This implies that \( \gamma_{456} + \gamma_{457} - \beta = 0 \). Since \( \gamma_{456} = 0, \gamma_{457} = \beta \).

Let \( \tilde{f}(x, y, z, x_1, w) := \Delta(z, z_1) f(x, y, z, w) \). Then \( \tilde{f}(x, y, z, x_1, w) \) is an identity of type \([3, 3, 1]\). So there exists a scalar \( \tilde{\beta} \in F \) such that \( \tilde{f}(x, y, z, x_1, w) = \tilde{\beta} Q(x \circ y, x, y, w, x, y) \).

Then \( \tilde{\beta} = 0 \). Indeed, from (4.47) the sum of the terms of \( \tilde{f}(x, y, z, x_1, w) \) which end in \( xw \) is

\[ \{\tilde{f}_1(y, x, y, x) + f_1(x, y, y)\} xw = \{\sum_{i=2}^{8} \alpha_i \{k_i(y, x, y, x) + k_i(x, y, y)\}\} xw, \]

where \( \tilde{f}_1(y, x, y, x) := \Delta(z, z_1) f_1(y, x, y) \) and \( k'_i(x, y, y_1, z) := \Delta(y, y_1) k_i(x, y, z) \). If we denote \( k_i(x, y, y) \) (resp. \( k'_i(y, x, x, y) \)) by \( k_i \) (resp. \( k'_i \)), then

\[
\begin{align*}
k_2 &= -k_4 = -\frac{1}{2} k_8 = [[x, y^2], [x, y]], k_i = 0, i = 3, 6, 7, 8, \\
k'_2 &= -k'_3 = -k'_4 = -k'_8 = [[x, y^2], [x, y]], k'_i = 0, i = 6, 7, 8.
\end{align*}
\]

In \( \tilde{\beta} Q(x \circ y, x, y, w, x, y) \) the sum of the terms which end in \( xw \) is \( \tilde{\beta} [[x, y^2], [x, y]] xw \). So \( \tilde{f}(x, y, z, x_1, w) = \tilde{\beta} Q(x \circ y, x, y, w, x, y) \), which implies that \( 2\alpha_2 - \alpha_3 - 3\alpha_5 = \tilde{\beta} \). Since \( \alpha_2 = \alpha_3, \alpha_4 = \alpha_5 \) and \( \alpha_3 = 3\alpha_4 \) from (4.50), \( \tilde{\beta} = 0 \).

Next we show that

\[ \beta = 0, \alpha_6 + \alpha'_6 = 0, \alpha_8 + \alpha'_8 = 0. \]  

(4.51)

Since \( f(x, y, z, x) = 0 \) the sum of the terms of \( f(x, y, z, x) \) which end in \( xy, yx, xx \), or \( yz \) are 0. This implies that

\[
\begin{align*}
f_1(x, z, y) + f_3(z, x, y, x) &= 0, \quad f_1(x, z, y) + f_3(z, y, x, x) = 0, \\
f_1(y, z, x) + f'_1(y, z, x) + f_2(z, y, x)y + f_2(y, z, x)z &= 0, \quad f_3(x, y, z, x) = 0.
\end{align*}
\]

(4.52)
By (4.47), the equations in (4.52) imply

\[-\gamma_1 - \mu_1 + 2\delta_1 = 0, \quad -\alpha_3 - \gamma_1 - \mu_1 - \gamma_3 - \mu_4 - 2\mu_5 - 2\gamma_5 + 2\delta_2 = 0,\]
\[-\alpha_2 - \gamma_3 - \mu_3 + 2\delta_3 = 0, \quad -\alpha_5 - \gamma_4 - \mu_4 + 2\delta_4 = 0,\]
\[-\alpha_4 + \gamma_1 + \mu_1 - \gamma_2 - \mu_2 + \gamma_4 + \mu_4 + \gamma_5 + \mu_5 + 2\delta_5 = 0,\]
\[\alpha'_6 - \gamma_6 - \mu_6 + 2\delta_6 = 0, \quad -\alpha_7 - \gamma_7 - \mu_7 + 2\delta_7 = 0, \quad \alpha'_8 - \gamma_8 - \mu_8 + 2\delta_8 = 0. \tag{4.53}\]

\[-\gamma_1 - \delta_1 + 2\mu_1 = 0, \quad \alpha_3 - \gamma_1 - \delta_1 - \gamma_4 - \delta_4 - 2\gamma_5 - 2\delta_5 + 2\mu_2 = 0,\]
\[\alpha_2 - \gamma_3 - \delta_3 + 2\mu_3 = 0, \quad \alpha_5 - \gamma_4 - \delta_4 + 2\mu_4 = 0,\]
\[\alpha_4 + \gamma_1 + \delta_1 - \gamma_2 - \delta_2 + \gamma_4 + \delta_4 + \gamma_5 + \delta_5 + 2\mu_5 = 0,\]
\[\alpha_8 - \gamma_6 - \delta_6 + 2\mu_6 = 0, \quad \alpha_7 - \gamma_7 - \delta_7 + 2\mu_7 = 0, \quad \alpha_6 - \gamma_8 - \delta_8 + 2\mu_8 = 0. \tag{4.54}\]

\[\beta + \alpha_6 + \alpha'_6 = 0, \quad \beta + \alpha_8 + \alpha'_8 = 0. \tag{4.55}\]

\[\beta_5 - \beta_6 + 4\beta_7 - \delta_2 + \delta_3 + \delta_4 - \delta_5 + \mu_2 - \mu_3 - \mu_4 + \mu_5 = 0,\]
\[\beta_1 - \beta_2 + \beta_3 - \beta_4 + \delta_6 - \delta_8 - \mu_6 + \mu_8 = 0,\]
\[\beta_1 - \beta_2 + \beta_5 + \beta_7 + \gamma_1 + \gamma_3 - 2\gamma_4 - \gamma_5 - 2\delta_1 + \delta_3 - \delta_4 + \mu_2 - \mu_3 = 0,\]
\[-\beta_1 + \beta_2 - \beta_5 - \beta_7 + \gamma_1 + \gamma_3 - 2\gamma_4 - \gamma_5 + \delta_2 - \delta_3 - 2\mu_1 + \mu_4 - \mu_5 = 0. \tag{4.56}\]

Since \(\tilde{f}(x, y, x, y, w) = 0\), the sum of the terms of \(\tilde{f}(x, y, x, y, w)\) which end in \(xy\) is 0, which implies that

\[f_2(y, x, w)x + f_2(x, y, w)y + \tilde{f}_3(x, y, x, y, w) + f_3(y, x, x, w) + f_3(x, y, y, w) = 0,\]

where \(\tilde{f}_3(z, z_1, x, y, w) := \Delta(z, z_1)f_3(z, x, y, w)\). Computing coefficients, this equation implies that

\[2\beta_2 - 2\beta_1 - \beta_5 - \beta_6 + 2\beta_7 - \delta_1 + \mu_1 = 0,\]
\[\beta_2 - \beta_1 - \beta_5 - \beta_7 + 2\gamma_1 + \gamma_2 - \gamma_3 + 2\gamma_4 - \delta_1 + 2\delta_4 - \delta_5 + 2\mu_1 + \mu_3 - \mu_4 - \mu_5 = 0,\]
\[\beta_1 - \beta_2 + \beta_5 + \beta_7 + 2\gamma_1 + \gamma_2 - \gamma_3 + 2\gamma_4 + 2\delta_1 + \delta_3 - \delta_4 - \delta_5 - \mu_1 + 2\mu_4 - \mu_5 = 0,\]
\[\beta_2 - \beta_1 - \beta_6 + 3\beta_7 - 2\gamma_1 + \gamma_4 + \gamma_5 - 2\delta_1 - \delta_3 + \delta_4 + \delta_5 + \mu_1 - \mu_2 + \mu_3 - \mu_4 = 0,\]
\[\beta_1 - \beta_2 + \beta_6 - 3\beta_7 - 2\gamma_1 + \gamma_4 + \gamma_5 + \delta_1 - \delta_2 + \delta_3 - \delta_4 - 2\mu_1 - \mu_3 + \mu_4 + \mu_5 = 0,\]
\[\beta_1 - \beta_2 + \beta_3 - \beta_4 + \gamma_6 + \gamma_8 - \delta_6 - \mu_8 = 0,\]
\[\beta_2 - \beta_1 - \beta_3 + \beta_4 + \gamma_6 + \gamma_8 - \delta_8 - \mu_8 = 0. \tag{4.57}\]
From (4.50), $\gamma_{456} = 0$, $\gamma_{457} = \beta$. But calculating the coefficients of the monomials $C_{456}, C_{457}$ from (4.47) yields

$$-\beta_6 - 2\beta_7 - \gamma_3 - \gamma_4 - \gamma_5 - \gamma_6 - \delta_1 + \delta_2 + \delta_3 - \mu_1 + \mu_2 + \mu_8 = 0,$$

$$\beta_7 + 2\beta_7 - \gamma_3 - \gamma_4 - \gamma_5 - \gamma_6 + \delta_3 - \delta_3 + \mu_3 - \mu_8 - \beta = 0. \quad (4.58)$$

Solving the systems (4.49), (4.50), (4.53), (4.54), (4.55), (4.56), (4.57) and (4.58) yields $\beta = 0$. Hence $\alpha_6 + \alpha_6' = \alpha_8 + \alpha_8' = 0$ by (4.55).

We also have

$$\alpha_2 = -\alpha_7. \quad (4.59)$$

Since we have shown that $\beta = 0$ by assuming that $\alpha = \lambda_i = 0$, for $i = 1, 2, 3$, the coefficients of the monomials which end in $xx, yy$ or $zz$ are 0. More precisely, the coefficients are linear combinations of $\alpha, \lambda_1, \lambda_2, \lambda_3$. $f^*$ is also in $VIN_3[2, 2, 2, 1]$. We claim that

$$f^*(x, x, z, w) = -\alpha Q(x^2, x, z, w, x, z), f^*(x, y, z, x) = -3 \sum \lambda_i Q_i,$$

where $Q_1, Q_2, Q_3$ is the basis of $VIN_3[3, 2, 2]$. For the operators $*$ and $\Delta(z, y)$ commute on each monomial, and so commute on every polynomial. Thus

$$f^*(x, x, z, w) = \frac{1}{2} \Delta(y, x) \Delta(y, x) f^*(x, y, z, w)$$

$$= \left(\frac{1}{2} \Delta(y, x) \Delta(y, x) f(x, y, z, w)\right)^*$$

$$= \left(f(x, x, z, w)\right)^*$$

$$= \alpha(Q(x^2, x, z, w, x, z))^*.$$

Similarly we can show that the second equation of the claim holds. Thus we also have $f^*(x, x, z, w) = f^*(x, y, z, x) = 0$ and $f^*(x, y, y, y, w) = 0$, where $f^*(x, y, z, z, z, w) := \Delta(z, z) f^*(x, y, z, w)$ if we assume that $\alpha = \lambda_i = 0$, for $i = 1, 2, 3$. Repeating the proof of $\beta = 0$ we get the result that $f^*(x, y, z, w)$ has no terms which end in $xx, yy$ or $zz$. This implies that $f$ has no term $yyzzzww$. However from (4.47), this monomial has coefficient $-\alpha_2 - \alpha_7$. So $\alpha_2 = -\alpha_7$.

Next we show
Lemma 4.3.2 The coefficients $\alpha_{ijk}$, $\beta_{ijk}$, $\gamma_{ijk}$ are linear combinations of $\alpha_2$, $\alpha_6$, $\gamma_{334}$.

Proof. In the following if a coefficient is a linear combination of $\alpha_i$'s then we say it is congruent 0 (modulo the $\alpha_i$'s) for short. By Lemma 4.3.1, we need only consider the coefficients

$$\beta_{ijk}, \beta_{5jk}, \gamma_{3jk}, \gamma_{4jk}, \gamma_{5jk},$$

where $3 \leq j < k \leq 7$. Since $R_4 = xyzwy$, the monomials $B_{434}$, $B_{435}$, $B_{445}$ occur in $f(x,y,z)wy$. So their coefficients are all linear combinations of $\alpha_i$'s. Since we have shown that

$$\alpha_i' = -\alpha_i, \quad i = 1, \ldots, 8, \quad \alpha_1 = 0, \quad \alpha_3 = \alpha_2, \quad \alpha_6 = \alpha_4 = \frac{1}{3}\alpha_3, \quad \alpha_7 = -\alpha_2, \quad \alpha_8 = \alpha_6,$$

the coefficients $\beta_{434}, \beta_{435}, \beta_{445}$ are all linear combinations of $\alpha_2, \alpha_6$. So they are congruent 0 modulo the $\alpha_i$'s. So are

$$\beta_{434}, \beta_{435}, \beta_{445}, \gamma_{424}, \gamma_{425}, \gamma_{434}, \gamma_{435}, \gamma_{445}. \quad (4.60)$$

We also know from (4.47) and (4.50) that

$$\beta_{5jk}, \gamma_{5jk}, \gamma_{450} = 0, \quad \beta_{467} = \gamma_{467} = \gamma_{457} = \beta = 0,$$

where $i \leq j < k \leq 7$. Thus the coefficients which we need to determine are

$$\gamma_{334}, \gamma_{335}, \gamma_{336}, \gamma_{337}, \gamma_{345}, \gamma_{346}, \gamma_{347}, \gamma_{356}, \gamma_{357}, \gamma_{436},$$

$$\gamma_{437}, \gamma_{446}, \gamma_{447}, \beta_{436}, \beta_{437}, \beta_{446}, \beta_{447}, \beta_{456}, \beta_{457}, \beta_{457}.$$

By Lemma 2.1.1, given a word $w$ the sum of the coefficients of monomials of the form $w - \ldots$ is 0, which implies that $\gamma_{356} + \gamma_{357} \equiv 0$ and $\gamma_{436} + \gamma_{437} \equiv 0$.

Substituting

$$e[12], e[22], e[23], e[33], e[33], e[31]$$

for $x, y, z, w, z_1, x_1$ in $\Delta(x,x_1)\Delta(z,z_1)f$ and considering the coefficient of $e_{11}$ yields the sum of the coefficients of the monomials

$$xyzwx, xyyzwx, xzwyyz, zwzxyz$$
in $f$ is 0. So $\gamma_{446} + \gamma_{445} + \gamma_{224} + \gamma_{234} = 0$. Since $f$ is symmetric in $z, y$, $\gamma_{224} = \gamma_{356}$, $\gamma_{234} = \gamma_{456}$.

So $\gamma_{446} + \gamma_{356} + \gamma_{256} \equiv 0$ by $\gamma_{445} \equiv 0$ in (4.60). But $\gamma_{256} = -\gamma_{356} - \gamma_{456}$ by (4.46) and (4.50), so

$$\gamma_{446} \equiv 0.$$

Thus $\gamma_{224} + \gamma_{234} \equiv 0$, $\gamma_{2jk} \equiv -\gamma_{3jk} - \gamma_{4jk}$ in (4.46) and $\gamma_{445} \equiv \gamma_{434} \equiv \gamma_{424} \equiv 0$ in (4.60) imply that $\gamma_{324} + \gamma_{334} \equiv 0$. However $\gamma_{324} = \gamma_{336}$ by the symmetry of $f$ in $y, z$, so

$$\gamma_{336} + \gamma_{334} \equiv 0.$$

By the symmetry of $f$ in $y, z$ and the formulas in (4.46),

$$\gamma_{346} = \gamma_{225} = -\gamma_{325} - \gamma_{425} \equiv \gamma_{336}$$

since $\gamma_{425} \equiv 0$.

From (4.46), $\gamma_{156} = \gamma_{356} + \gamma_{456} \equiv \gamma_{356}$ since $\gamma_{456} \equiv 0$ by (4.50). Similarly we have

$$\gamma_{134} \equiv \gamma_{334} \equiv \gamma_{334}.$$  Thus

$$\gamma_{356} \equiv \gamma_{334}.$$

If we substitute

$$e[12], e[22], e[23], e[33]$$

for $x, y, z, w$ in $f$ then the monomials which produce $e_{11}$ are $xyzwzx, xyzwzyx, xzwzyyx$.

So the sum of their coefficients is 0. Since $\gamma_{446} \equiv 0$ and since $f$ is symmetric, $\gamma_{335} + \gamma_{256} \equiv 0$.

So

$$\gamma_{335} \equiv -\gamma_{256} \equiv -\gamma_{334}.$$

By Lemma 2.1.1, $\gamma_{335} + \gamma_{336} = 0$. So

$$\gamma_{336} \equiv -\gamma_{335} \equiv \gamma_{334}.$$

From (4.46) and $\gamma_{445} \equiv 0$, $\gamma_{145} \equiv \gamma_{345}$. But from the symmetry of $f$

$$\gamma_{145} = \gamma_{136} = \gamma_{336} + \gamma_{436} \equiv \gamma_{334} + \gamma_{436}.$$

However $\gamma_{436} \equiv -\gamma_{334}$ from above, thus

$$\gamma_{345} \equiv \gamma_{145} \equiv 0.$$
\[ \beta_{5jk} \equiv 0 \text{ implies that } \beta_{147} \equiv -\gamma_{347} \text{ by (4.16). From the symmetry, } \beta_{147} \equiv \beta_{136} \equiv -\gamma_{336} \equiv \gamma_{334}. \text{ So } \gamma_{347} \equiv -\gamma_{334}. \]

Similarly we have

\[ \gamma_{337} \equiv \beta_{325} \equiv -\beta_{125} - \gamma_{325} - \gamma_{425} \equiv \beta_{337} - \gamma_{336} - \gamma_{434} \equiv -\gamma_{336} \equiv -\gamma_{334}. \]

From Lemma 2.1.1 the sum of the coefficients of the monomials of form \( v - - - \) is 0. Since \( C_{447} \equiv xyzwzx, C_{446} \equiv xyzwzx \) and since the coefficients of the monomials

\[ xyzwzx, xyzwzx, xyyzzw, xyyzzw \]

are linear combinations of \( \alpha_i \)'s, \( \gamma_{446} + \gamma_{447} \equiv 0. \) So \( \gamma_{447} \equiv -\gamma_{446} \equiv 0. \) Similarly, we have \( \gamma_{436} + \gamma_{437} \equiv 0, \beta_{437} + \beta_{436} \equiv 0 \) and \( \beta_{446} + \beta_{447} \equiv 0. \)

From Lemma 2.1.1 the sum of the coefficients of the monomials

\[ xzwyzy, yzwyzy, xzwyzy, xzwyzy, ywzyzy, xzyzwy \]

is 0. This implies that \( \alpha_{456} + \gamma_{345} + \gamma_{334} - \beta = 0. \) Since \( \beta = 0 \) and \( \alpha_{456} = -\beta_{456} - \gamma_{456} \) by (4.51) and (4.46) respectively, \( \beta_{456} \equiv \gamma_{345} + \gamma_{334} - \gamma_{456} \equiv \gamma_{334} \) because \( \gamma_{345} \equiv \gamma_{456} \equiv 0. \) By Lemma 2.1.1 the sum of the coefficients of the monomials \( xzwyzy, xzwyzy, xzwyzy \) is 0. So \( \beta_{456} + \beta_{437} \equiv 0 \) since \( \beta = 0. \) Thus \( \beta_{457} \equiv -\beta_{456} \equiv -\gamma_{334}. \)

By the symmetry of \( f \) in \( y, z \) the monomials \( xzwyzy \) and \( xzwyzyz \) have the same coefficients in \( f. \) So \( \beta_{456} \equiv \alpha_{327} \equiv \beta_{427} + \gamma_{427} \) by (4.46) and \( \beta_{5jk} \equiv \gamma_{5jk} \equiv 0. \) From the symmetry of \( f, \) we have \( \beta_{427} \equiv \beta_{436}, \gamma_{427} \equiv \beta_{334}. \) Since \( \beta_{434} \equiv \gamma_{434} \equiv \beta_{5jk} \equiv \gamma_{5jk} \equiv 0, \) from (4.16) \( \beta_{334} \equiv -\gamma_{334}. \) So \( \beta_{456} \equiv \beta_{436} - \gamma_{334} \) and \( \beta_{436} \equiv -\beta_{456} + \gamma_{334} \equiv 0. \)

By the symmetry of \( f \) in \( y, z, \beta_{446} \equiv \alpha_{427}. \) \( \alpha_{427} \equiv -\beta_{427} - \gamma_{427} \) by (4.46). On the other hand from the symmetry of \( f, \beta_{427} \equiv \beta_{436}, \gamma_{427} \equiv \beta_{334}. \) So \( \beta_{446} \equiv -\beta_{436} - \beta_{334} \equiv -\beta_{436} + \gamma_{334} \) because \( \beta_{334} \equiv -\beta_{434} - \gamma_{334} - \gamma_{434} \) by (4.46) and \( \beta_{5jk} \equiv \gamma_{5jk} \equiv \beta_{434} \equiv \gamma_{434} \equiv 0. \) Thus \( \beta_{446} \equiv \gamma_{334} \) completing the proof of the lemma. \( \blacksquare \)

To prove Proposition 4.3.1 we need now only to show that \( Q_1, \ldots, Q_7 \) are linearly independent. Suppose there exist scalars \( \alpha_i \in F \) such that \( \sum_{i=1}^{7} \alpha_i Q_i = 0. \) Then the
coefficient of the monomial $xyzwxyz$ is 0, which implies that $\alpha_1 + \alpha_3 - \alpha_4 = 0$. Since 
$(\sum_{i=1}^7 \alpha_i Q_i)^* = 0$, the monomial $xyzwxyz$ has coefficient 0 in it, which implies that $-\alpha_1 + \alpha_3 - \alpha_4 = 0$. So $\alpha_1 = 0$, $\alpha_3 = \alpha_4$. Since $\alpha_1 = 0$, the monomial $xyzwxyz$ has coefficient 0 in $\sum_{i=1}^7 \alpha_i Q_i$, which implies that $-2\alpha_2 + \alpha_7 = 0$. Similarly considering the coefficients of the monomials $yxxyzw$, $xyxyzw$ yields 
$$-\alpha_2 - \alpha_6 = 0, \quad -\alpha_2 - 0 = 0.$$ 
So $\alpha_5 = 0$. Since $\alpha_1 = \alpha_5 = 0$, considering the coefficients of the monomials $xyzwxyz$, $xxyzwyz$ yields 
$$\alpha_4 + \alpha_7 = 0, \quad 2\alpha_2 + \alpha_3 - \alpha_4 + 2\alpha_7 = 0.$$ 
So $\alpha_i = 0$ for $i = 1, \ldots, 7$ and $Q_1, \ldots, Q_7$ are linearly independent. \[1\] 

We will now determine a basis for the vector space $VIN_3[2,2,1^3]$. So let $f(y, z, x_1, x_2, x_3) \in VIN_3[2,2,1^3]$. Then 
$$f = \sum_{(123)} \{ f_1(y, z, x_1)[x_2, x_3] + (I + (y, z))(f_3(z, x_1, x_2, x_3)y^2 + f_5(y, z, x_1, x_2, x_3)yz + 
\sum_{(123)} \{ f_2(y, x_1, x_2)x_3z^2 + f_4(y, z, x_1, x_2)x_3z + f_6(y, z, x_1, x_2)x_3z \}. \quad (4.61)$$ 
Moreover, since $f_2 \in VIN_2[2,1,1^3]$, $f_3 \in VIN_2[1^4]$ which are skew in $x_i$'s, they are linear combinations of the products of two commutators by Corollary 3.2 of [MR]. So there exist scalars $\alpha, \beta, \gamma \in F$ such that 
$$f_2 = \alpha[[y, x_1], [y, x_2]], \quad f_3 = \beta \sum_{(123)} [y, x_1][x_2, x_3].$$ 
$$f_5(y, z, x_1, x_2, x_3) \in VIN_2[1^5]$ which is skew in $x_i$'s. We need to determine the coefficients of following 20 monomials: 
$$yzx_1 x_2 x_3, \quad yz x_1 x_2 x_3, \ldots, x_1 x_2 x_3 y z, \quad z y x_1 x_2 x_3, \quad z x_1 y x_2 x_3, \quad \ldots, x_1 x_2 x_3 x_3 x_3 y.$$ 
Let $\beta_i, \beta'_i, i = 1, \ldots, 10$ denote their coefficients. 

Since $f(y, z, x_1, x_2, y) \in VIN_3[3,2,1,1]$, there exist scalars $\alpha_1, \ldots, \alpha_{10} \in F$ such that 
$$f(y, z, x_1, x_2, y) = \sum_{i=1}^{10} \alpha_i Q_i,$$ 
where $\{Q_1, \ldots, Q_{10}\}$ is the basis of $VIN_3[3,2,1,1]$ given in
Proposition 4.2.3. Since $f$ is skew in the variables of degree 1, $f(y, z, x_1, y, y) = 0$. This implies that $\sum_{i=1}^{10} \alpha_i Q_i(y, z, x_1, y) = 0$. So by calculation we have $\alpha_2 = 2\alpha_4$.

From now on we assume that $f(y, z, x_1, x_2, y) = 0$. Then the sum of the terms of $f(y, z, x_1, x_2, y)$ which end in $x_2y$ (resp. in $yx_2$, $yz$, $zy$, $yzy$) is 0, which implies

$$
\begin{align*}
  f_1(y, z, x_1) + f_4(z, y, x_1) &= 0, \quad f_2(y, z, x_1) + f_4(z, y, x_1) = 0, \\
  f_4(y, z, x_1, x_2) + f_5(y, z, x_1, x_2) &= 0, \quad f_3(y, z, x_1, x_2) + f_5(y, z, x_1, x_2) = 0, \\
  f_2(y, x_1, x_2) + f_3(y, x_1, x_2) &= 0. \\
\end{align*}
$$

(4.62)

Thus it suffices to determine $\beta_1$, $\beta'_1$ and $\alpha, \beta, \gamma$. From the last equation of (4.62), we have

$$
\alpha - \beta + \gamma = 0. \quad (4.63)
$$

By Lemma 2.1.1, the sum of the coefficients of the monomials

$$
yxz_1x_2x_3yz, \quad yx_2x_3yz, \quad zyx_1x_2x_3yz, \quad yzx_1x_2x_3yz, \quad yzzx_1x_2x_3yz,
$$

is 0. This implies that

$$
\beta_1 + \beta'_1 = 0. \quad (4.64)
$$

The sum of the coefficients of the monomials

$$
x_1x_2yzzx_3yz, \quad x_1x_2yzzx_3yz, \quad x_1x_2yzzx_3yz, \quad x_1x_2yzzx_3yz
$$

is 0, which implies that

$$
\beta_8 + \beta'_8 = 0. \quad (4.65)
$$

The sum of the coefficients of the monomials $wzy$, $wyz$, $wyzz$ is 0 for each

$$
w = yx_1x_2x_3, \quad x_1yxy, \quad x_1x_2x_3, \quad x_1x_2x_3y,
$$

which implies that

$$
\beta_4 + \beta'_4 + \beta = 0, \quad \beta_7 + \beta'_7 - \beta = 0, \quad \beta_8 + \beta'_8 = 0, \quad \beta_10 + \beta'_10 - \gamma = 0. \quad (4.66)
$$
Similarly considering the monomials \(-x_2 x_3 y z\) and \(-x_2 z x_3\), respectively, we have
\[
\beta_2 + \beta'_3 + \beta_5 + \beta'_5 = 0, \quad \beta_3 + \beta'_3 + \beta_6 + \beta'_6 = 0.
\] (4.67)

Since \(f\) is symmetric in \(y, z\), \(f_1(y, z, x_1) = f_1(z, y, x_1)\) by (4.61). Therefore from (4.62)
\[
f_5(z, y, y, x_1, z) - f_5(y, z, z, x_1, y) = 0.
\]
So the coefficients of the monomials
\[
y z z y x_1, \quad y z x_1 z y, \quad x_1 z z y y, \quad y x_1 z y z
\]
are 0 in
\[
f_5(z, y, y, x_1, z) - f_5(y, z, z, x_1, y)
\]
which implies that
\[
\beta_1 + \beta_2 - \beta_6 - \beta_8 + \beta'_1 + \beta'_2 - \beta'_6 - \beta'_8 = 0, \\
\beta_1 - \beta_3 + \beta_7 - \beta_{10} + \beta'_1 - \beta'_3 + \beta'_7 - \beta'_{10} = 0, \\
\beta_6 + \beta_7 + \beta_8 + \beta_9 + \beta'_6 + \beta'_7 + \beta'_8 + \beta'_9 = 0, \\
-\beta_2 + \beta_4 + \beta_8 + \beta_{10} - \beta'_2 + \beta'_4 + \beta'_8 + \beta'_{10} = 0.
\] (4.68)

By Lemma 2.1.1 the sum of the coefficients of the monomials \(y w y, \ y w^* y, \ y w y, \ y y w^*, \ w y y\), \(w^* y y\) is 0 for \(w = x_1 x_2 x_3 z, \ x_1 x_2 x_3 z, \ x_1 x_2 x_3 z\). This implies that
\[
-\beta_3 - \beta_6 + \beta'_1 + 2\beta'_2 + \beta'_3 + \beta'_6 - \beta + \alpha = 0, \\
\beta'_1 + 2\beta'_2 + \beta'_3 + \beta'_4 + \beta'_6 + \beta'_9 - \gamma = 0, \\
-\beta_4 - \beta_7 - \beta'_1 - \beta'_2 + 2\beta'_3 + \beta'_4 - \beta'_6 + \beta'_7 + \gamma - \alpha = 0.
\] (4.69)

Next let us do some substitutions. Substituting
\[
e[12], \ e[13], \ e[11], \ e[22], \ e[23]
\]
for \(y, \ z, \ x_1, \ x_2, \ x_3\) and considering the coefficient \(e_{23}\) we have
\[
-\beta'_1 + \beta'_3 + \beta'_5 + \beta'_6 - \beta'_8 + \beta'_1 + \beta_{10} - \beta_2 - \beta_4 - 2\beta_5 - \beta_6 + \beta_8 = 0.
\] (4.70)
Indeed if a substitution of the matrix $e[ij]$ denotes the path from $i$ to $j$ and if a substitution of matrix $eii$ denotes the loop at point $i$, then the monomials which evaluate to $e_{23}$ are the paths from vertex 2 to vertex 3 which pass through all edges once. These paths are

$$yz_1z_2x_3y_2z, \quad yz_2z_1x_2y_2z_2, \quad x_2y_1x_3zy_2z_2,$$

$$yz_2x_1z_3z_3, \quad x_2y_1z_2y_2z_2, \quad x_2y_1z_1y_2z_2,$$

$$x_3z_2x_1y_2z_3, \quad x_3z_3x_1z_2y_2z_2, \quad x_2x_3z_2y_1y_2z_2.$$

The sum of the coefficients of these monomials in $f$ is the coefficient of $e_{23}$ in this substitution. It yields (4.70).

Similarly substituting

$$e[13], \quad e[33], \quad e[11], \quad e[12], \quad e[22]$$

for $y$, $z$, $x_1$, $x_2$, $x_3$ and considering the coefficients of $e_{12}, e_{21}$ respectively yields

$$\beta'_1 + \beta'_2 + \beta'_3 + \beta'_6 - \beta_6 - \beta_8 - \beta_{10} = 0,$$

$$\beta_8 + \beta_9 - \beta'_{10} = 0, \quad (4.71)$$

since the monomials which produce $e_{12}$ (resp. $e_{21}$) in the substitution are $yz_2y_1x_2z_3$ and $x_1y_2z_2y_2x_3$ (resp. $x_3x_2z_1y_2z_3$ and $x_3x_2z_1y_2y_3z$).

Next we assume that $f^* = \pm f$. If $f^* = -f$ then by symmetry and considering the coefficients of monomials

$$yz_1z_2x_3y_2z, \quad yz_2z_1x_2y_2z_2, \quad x_2y_1x_3z_2y_2z,$$

$$x_1z_2y_2z_3y_2z, \quad x_1z_2x_3y_2z, \quad x_1y_2z_2x_3y_2z,$$

$$x_1y_3x_2z_3y_2z, \quad x_1y_2z_3y_2z, \quad x_1y_2z_2x_3y_2z, \quad x_1y_2z_3x_3y_2z,$$

we have

$$\beta_1 + \beta_2 + \beta'_4 = 0, \quad -\beta_1 + \beta_3 + \beta'_6 = 0, \quad \beta_1 + \beta_4 + \beta'_6 = 0, \quad \beta'_1 + \beta_5 + \beta_7 = 0,$$

$$2\beta_6 - \beta'_1 = 0, \quad -\beta_1 - \beta_3 + \beta_8 - \beta_{10} + \beta'_6 = 0, \quad -\beta_1 + \beta_3 - \beta_8 + \beta_{10} - \beta'_6 = 0,$$

$$\beta'_1 + \beta'_2 + \beta'_4 = 0, \quad 2\beta'_5 - \beta'_1 = 0, \quad \beta_1 - \beta_3 + \beta'_1 + \beta'_5 - \beta'_{10} = 0,$$

$$\beta_1 + \beta_2 - \beta_6 - \beta'_5 + \beta'_{10} = 0. \quad (4.72)$$
The solution of the systems (4.64)..... (4.72) is

\[ \begin{align*}
\beta_1 &= 2\beta'_4 + 2\beta'_5 + 4\beta'_{10} + 2\gamma, \\
\beta_2 &= \beta'_4 - \beta'_5 - 3\beta'_{10} + \gamma, \\
\beta_3 &= -\beta'_4 + \beta'_{10} - 2\gamma, \\
\beta_4 &= \beta'_4 + \beta'_5 + \beta'_{10} + 2\gamma, \\
\beta_5 &= \beta'_4 + 2\beta'_5 + 4\beta'_{10} + \gamma, \\
\beta_6 &= -\beta'_4 - \beta'_5 - 2\beta'_{10} - \gamma, \\
\beta_7 &= \beta'_4 + \gamma, \\
\beta_8 &= \beta'_4 + \beta'_{10} + \gamma, \\
\beta_9 &= -\beta'_4 - \beta'_5 - 2\beta'_{10} - \gamma.
\end{align*} \]

Thus \( \beta'_4, \beta'_5, \beta'_{10}, \gamma, \alpha_i, i \neq 4, 10 \) are 12 parameters of the coefficients of \( f \) when \( f \) is \( \ast \)-skew symmetric. Since \( f^* = -f \) implies that \( f(y, z, x_1, x_2, y)^* = -f(y, z, x_1, x_2, y) \). So \( \sum_{i=1}^{10} \alpha_i Q_i \) is \( \ast \)-skew symmetric. This implies that \( \alpha_{10} = \alpha_9 \). We have \( \alpha_3 = 2\alpha_4 \) already.

If \( f^* = f \) then corresponding to (4.72) we have

\[ \begin{align*}
\beta_1 &= 0, \\
\beta_2 - \beta'_4 &= 0, \\
\beta_3 - \beta'_5 &= 0, \\
\beta_4 - \beta_{10} &= 0, \\
\beta_5 &= \beta_6 = \beta_7 = \beta_8 = \beta_9 = \gamma, \\
\beta_2 - \beta'_2 &= 0, \\
\beta_3 + \beta'_4 &= 0, \\
\beta_4 + \beta'_5 &= 0, \\
\beta_5 + \beta'_{10} &= 0, \\
\beta_6 + \beta'_{10} &= 0, \\
\beta_7 + \beta'_{10} &= 0.
\end{align*} \]

The solution of the systems (4.64), (4.71) and (4.74) is

\[ \begin{align*}
\beta_1 &= \beta'_4 = 0, \\
\beta_2 &= \beta'_5 = -\frac{1}{2}\beta_7 = \beta_7, \\
\beta_3 &= \beta_4 = \beta'_6 = \beta'_{10} = \gamma, \\
\beta_4 &= \beta_5 = \beta_6 = \beta_7 = \beta_8 = \beta_9 = \gamma, \\
\beta_5 &= \beta_8 = \beta_9 = \beta_{10} = \gamma, \\
\beta_8 &= \beta_9 = \beta_{10} = \gamma.
\end{align*} \]

Thus there are 4 parameters \( \gamma, \alpha_9, \beta_8, \beta'_{10} \) when \( f^* = f \). Since if \( f^* = f \) then \( \sum_{i=1}^{10} \alpha_i Q_i \) is symmetric. This implies that \( \alpha_i = 0 \) for \( i \neq 9, 10 \) and \( \alpha_9 = -\alpha_{10} \).
Next we shall find 12 linear independent identities of $VIN_3[2, 2, 1, 1, 1]$ which are $*$-skew symmetric, and 4 linear independent identities of $VIN_3[2, 2, 1, 1, 1]$ which are $*$-symmetric.

Let

\[
Q_1 := \sum_{(123)} (Id + (x, y))Q(z_1 \circ y, y, x, z_2, z_3),
\]
\[
Q_2 := \sum_{(123)} (Id + (x, y))Q(x^2, z_1, y, y, z_2, z_3),
\]
\[
Q_3 := \sum_{\sigma \in S_3} (Id + (x, y))(-1)^\sigma Q(y \circ z_\sigma(1), z_\sigma(2), x, z_\sigma(3), z_\sigma(4)),
\]
\[
Q_4 := \sum_{(123)} (Id + (x, y))T_0(z_1 \circ y, y, x, z_2, z_3; x),
\]
\[
Q_5 := (Id + (x, y))T_0(x \circ y, y, z_1, z_2, z_3; x),
\]
\[
Q_6 := (Id + (x, y))T_0(x^2, z_1, z_2, z_3, y; y),
\]
\[
Q_7 := (Id + (x, y))K(x, z_1, x, z_2, y, z_3, y),
\]
\[
Q_8 := \sum_{(123)} (Id + (x, y))K(x, y, z_2, z_1, x, z_3, y),
\]
\[
Q_9 := \sum_{\sigma \in S_3} (-1)^\sigma K(x, y, x, z_\sigma(1), y, z_\sigma(2), z_\sigma(3)),
\]
\[
Q_{10} := \sum_{(123)} (Id + (x, y))Q(z_1, y, z_2, x, y, z_3)x,
\]
\[
Q_{11} := \sum_{(123)} (Id + (x, y))Q(x, y, z_1, x, y, z_2)z_3,
\]
\[
Q_{12} := \sum_{(123)} (Id + (x, y))Q(z_1, y, z_2, x, y, z_3),
\]
\[
Q_{13} := \sum_{(123)} (Id + (x, y))z_3Q(x, y, z_1, x, y, z_2),
\]
\[
Q_{14} := (Id + (x, y))T_0(x, y, z_1, z_2, z_3; x)y,
\]
\[
Q_{15} := (Id + (x, y))T_0(x, y, z_1, z_2, z_3; x),
\]
\[
Q_{16} := T_0(x, y, z_1, z_2, z_3; [x, y]).
\]

Then $Q_{16}$ is an identity of $H_3$ by Theorem 3.1 and $Q_i$ for $i \neq 16$ is an identity by [MR]. Next we show that $Q_1, \ldots, Q_{16}$ are linearly independent. If there exist scalars $\alpha_i \in F$ such that $\sum_{i=1}^{16} \alpha_i Q_i = 0$ then

\[
\sum_{i=1}^{16} \alpha_i Q_i(x, y, z_1, z_2, x) = 0.
\]
This implies that
\[ 3\alpha_1 + \alpha_2 = 0, \ -2\alpha_1 - \alpha_2 + \alpha_3 = 0, \ \alpha_1 + \alpha_3 = 0, \ 2\alpha_{12} + \alpha_{13} = 0, \]
\[ -\alpha_7 - 2\alpha_8 = 0, \ -\alpha_7 + \alpha_8 + \alpha_9 = 0, \ 2\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 = 0, \]
\[ -2\alpha_1 - \alpha_2 + 2\alpha_4 + \alpha_6 = 0, \ 2\alpha_{11} + \alpha_{12} = 0. \]  
(4.75)

Since if we denote \( Q_i(x, y, z_1, z_2, x) \) by \( g_i \) then
\[ g_1 = Q_1' + 2Q_0 + Q_4', \ g_2 = Q_0' - Q_3', \ g_3 = Q_2' + 2Q_3' + Q_4', \ g_4 = Q_7' + 2Q_8', \ g_5 = Q_7', \]
\[ g_6 = Q_8', \ g_7 = -Q_5' - Q_6', \ g_8 = -2Q_5' + Q_6', \ g_9 = Q_10', \ g_{10} = 2Q_9', \]
\[ g_{11} = 2Q_{10}', \ g_{12} = 2Q_9', \ g_{13} = Q_9', \ g_{14} = g_{15} = g_{16} = 0. \]

where \( Q_0' = Q_1' - Q_2' + Q_3' + Q_4' - Q_5' \) and \( \{Q_1', \ldots, Q_{10}'\} \) is the basis of \( \mathcal{V}_4 N_3[3, 2, 1, 1] \) given in Proposition 4.2.3.

Since \( \sum_{i=1}^{18} \alpha_i Q_i(x, y, z_1, z_2, x) = 0, \ (\sum_{i=1}^{16} \alpha_i Q_i(x, y, z_1, z_2, x))^* = 0. \) This implies that
\[ 2\alpha_{13} + \alpha_{10} = 0, \ 2\alpha_{10} + \alpha_{11} = 0. \]

because \( Q_i^* = -Q_i, \) for \( i \neq 10, 11, 12, 13, 14, 15, 16, Q_{10}^* = -Q_{12}, \ Q_{11}^* = -Q_{13}, \)
\[ Q_{14}^* = -Q_{15} \) and \( Q_{16}^* = Q_{16}. \)

From \( \sum_{i=1}^{18} Q_i(x, x, z_1, z_2, z_3) = 0 \) we have
\[ \alpha_7 = 0, \ -4\alpha_2 + 4\alpha_5 - 2\alpha_6 = 0. \]

Considering the coefficients of the monomials
\[ y^2z_1y^2x^3z_3, \ z_1z_2z_3xyy, y^3x_1z_2z_3, \ y^2x_1z_2z_3y, \ z_1z_2z_3yz_3, \ z_1z_2z_3y^2z_3, \]
in \( \sum_{i=1}^{18} \alpha_i Q_i \) yields
\[ -\alpha_{16} - \alpha_{15} - \alpha_{13} - \alpha_{11} + 2\alpha_8 - 2\alpha_5 + 2\alpha_4 = 0, \ -\alpha_{15} + \alpha_{11} - \alpha_9 + \alpha_5 + \alpha_1 = 0, \]
\[ -\alpha_{18} + \alpha_{13} - \alpha_9 + \alpha_5 + \alpha_1 = 0, \ -2\alpha_{13} - 2\alpha_{11} + 2\alpha_9 + 4 + \alpha_4 = 0, \]
\[ \alpha_5 + \alpha_2 + \alpha_3 + \alpha_1 + \alpha_6 + \alpha_7 = 0, \ -\alpha_{14} - 2\alpha_{10} - \alpha_7 - \alpha_6 + \alpha_5 + \alpha_4 - \alpha_3 - \alpha_2 = 0. \]

The solution of these systems is trivial. So \( Q_1, \ldots, Q_{16} \) are linearly independent. Thus we have shown the following.
Proposition 4.3.2 If \( \text{char}F \not| 3! \) then the set \( \{Q_1, \ldots, Q_{10}\} \) is a basis of the vector space \( VIN_3[2, 2, 1, 1, 1] \).

Let
\[
Q_1 := \sum_{\sigma(i) < \sigma(i+1), i \neq 2} (-1)^{\sigma} Q(y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}),
\]
\[
Q_2 := \sum_{\sigma(i) < \sigma(i+1), i = 1, 2} (-1)^{\sigma} Q(y, x_{\sigma(1)}, x_{\sigma(2)}, y \circ x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}),
\]
\[
Q_3 := \sum_{\sigma(i) < \sigma(i+1), i \neq 2} (-1)^{\sigma} y Q(y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}),
\]
\[
Q_4 := \sum_{\sigma(i) < \sigma(i+1), i \neq 2} (-1)^{\sigma} Q(y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}),
\]
\[
Q_5 := T_0(x_1, x_2, x_3, x_4, x_5; y^2),
\]
\[
Q_6 := \sum_{(12345)} T_0(y \circ x_1, x_2, x_3, x_4, x_5; y),
\]
\[
Q_7 := y T_0(x_1, x_2, x_3, x_4, x_5; y), \quad Q_8 := T_0(x_1, x_2, x_3, x_4, x_5; y),
\]
\[
Q_9 := \sum_{(12345)} x_1 T_0(y, x_2, x_3, x_4, x_5; y),
\]
\[
Q_{10} := \sum_{(12345)} T_0(y, x_2, x_3, x_4, x_5; y)x_1.
\]

Then we have

Proposition 4.3.3 The set \( \{Q_1, \ldots, Q_{10}\} \) is a basis of the vector space \( VIN_3[2, 1^5] \).

Proof. Let \( f(y, x_1, x_2, x_3, x_4, x_5) \in VIN_3[2, 1^5] \). Then we need only to determine the coefficients of the following monomials in \( f \):
\[
yy^2z_1z_2 \cdots z_5, \quad yz_1yz_2z_3 \cdots z_5, \ldots, \quad yz_1 \cdots z_5, \quad x_1yy^2z_2 \cdots z_5, \ldots, \quad z_1 \cdots z_5yy.
\]

Let \( a_1, \ldots, a_{21} \) be their coefficients in \( f \) respectively. From Lemma 2.1.1 the sum of the coefficients of the monomials \( yz_1w, \ yz_1yw, \ x_1yw \) is 0 for \( w = x_2x_3x_4x_5 \). This yields
\[
a_1 + a_2 + a_7 = 0. \tag{4.76}
\]

Similarly we have
\[
a_{12} + a_{13} + a_{16} = 0, \quad a_{19} + a_{20} + a_{21} = 0. \tag{4.77}
\]
By Lemma 2.1.1, the sum of the coefficients of the monomials \( yx_1wxy_s, \\
x_1ywxy_s, x_1ywyx_5, yx_1wyx_5 \) for \( w = x_2x_3x_4 \) is also 0. So
\[
\alpha_5 + \alpha_6 + \alpha_{10} + \alpha_{11} = 0. 
\quad (4.78)
\]
Since the sum of the coefficients of the monomials
\[
x_1x_2x_3x_4x_5, x_1x_2y_3x_4x_5, x_1x_2x_3y_3x_4x_5y, x_1x_2x_3y_4x_1x_5, 
\]
and the sum of the coefficients of the monomials
\[
yx_1x_2y_3x_4x_5, yx_1x_2x_3y_4x_5, x_1yx_2x_3y_3x_4x_5, x_1yx_2x_3y_4x_5, 
\]
are 0, we have that
\[
\alpha_{14} + \alpha_{15} + \alpha_{17} + \alpha_{18} = 0, \quad \alpha_3 + \alpha_4 + \alpha_8 + \alpha_9 = 0. 
\quad (4.79)
\]
Let \( w = x_1x_2x_3x_4x_5 \) then the sum of the coefficients of the monomials \( yyw, yyw^*, wyy, w^*yy, ywy, yw^*y \) is 0, which implies that
\[
\alpha_1 + \alpha_6 + \alpha_{21} = 0.
\quad (4.80)
\]
Similarly, the sum of the coefficients of the monomials of the form \(-w-, -w^*-, \\
- - w, - - w^*, w --, w^* --\) is 0 for
\[
w = x_1x_2y_3x_4, x_1x_2x_3y_4, yx_2x_3x_4x_5,
\]
which implies that
\[
\alpha_3 + \alpha_4 + \alpha_9 + \alpha_{14} + \alpha_{15} + \alpha_{18} = 0, \\
\alpha_2 + \alpha_3 + \alpha_5 + \alpha_8 + 2\alpha_{10} + \alpha_{11} + \alpha_{15} + \alpha_{17} + \alpha_{18} + \alpha_{20} = 0, \\
\alpha_1 + \alpha_2 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_{11} + \alpha_{19} + \alpha_{20} + \alpha_{21} = 0. 
\quad (4.81)
\]
Substituting
\[
e[11], e[12], e[22], e[23], e[33], e[13], e[13]
\]
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for \(x_1, x_2, x_3, x_4, x_5, y, y\), and considering the coefficient of \(e_{31}\) yield

\[
\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + \alpha_7 - \alpha_8 - \alpha_{10} - \alpha_{17} + \alpha_{19} + \alpha_{20} + \alpha_{21} = 0. \tag{4.82}
\]

Indeed if a substitution of the matrix \(e[ij]\) denotes the path from \(i\) to \(j\) and if a substitution of matrix \(e_{ii}\) denotes the loop at point \(i\), then the monomials which evaluate to \(e_{31}\) are the paths from vertex 3 to vertex 1 which pass each edge only once. They are

\[
yx_1x_2x_3x_4y, \quad yx_2x_3x_4xy_3y, \quad x_3yx_1x_2x_4x_5y, \quad yx_2x_4x_5y, \quad x_1x_2x_3x_4yx_5y, \\
x_4x_3x_2xy_5, \quad x_4x_3x_2yx_5y, \quad x_5x_4x_3x_2yy, \quad x_5yx_4x_3x_2y, \\
yx_1y_5x_4x_3x_2, \quad yx_5x_4x_3x_2x_5, \quad x_5yx_1y_4x_3x_2, \quad x_5yyx_1x_3x_2y.
\]

Thus the sum of these monomials is 0 yields (4.82).

Similarly substituting \(e[i1], e[12], e[22], e[23], e[31], e[33]\) for \(x_1, \ldots, x_5, y\) and considering the coefficient of \(e_{11}\) we have

\[
\alpha_7 + \alpha_{12} + \alpha_{16} + \alpha_{19} = 0. \tag{4.83}
\]

The solution of the systems (4.76), \ldots, (4.83) is

\[
\begin{align*}
\alpha_7 &= -\alpha_8 - \alpha_9 + \alpha_{15} + \alpha_{18} + \alpha_{20} + \alpha_{21}, \\
\alpha_2 &= \alpha_8 + \alpha_9 - \alpha_{15} - \alpha_{18} - \alpha_{20} + \alpha_6, \\
\alpha_{12} &= \alpha_8 + \alpha_9 - \alpha_{15} - \alpha_{18} - \alpha_{16}, \\
\alpha_5 &= -\alpha_6 - \alpha_{11}, \quad \alpha_1 = -\alpha_6 - \alpha_{21}, \\
\alpha_{14} &= -\alpha_{15} + \alpha_8 - \alpha_{18}, \quad \alpha_{13} = -\alpha_8 - \alpha_9 + \alpha_{15} + \alpha_{18}, \\
\alpha_3 &= -\alpha_4 - \alpha_8 - \alpha_9, \quad \alpha_{17} = -\alpha_8, \quad \alpha_{10} = 0, \quad \alpha_{19} = -\alpha_{20} - \alpha_{21}.
\end{align*}
\]

So there are 10 parameters.

Next, we show that \(Q_1, \ldots, Q_{10}\) are linearly independent. If there exist scalars \(\alpha_i \in F\) such that \(\sum_{i=1}^{10} \alpha_i Q_i = 0\) then \((\sum_{i=1}^{10} \alpha_i Q_i)^* = 0\). Adding up these 2 equations yields

\[
(\alpha_4 - \alpha_3)(Q_4 - Q_3) + (\alpha_8 - \alpha_7)(Q_8 - Q_7) + (\alpha_{10} - \alpha_9)(Q_{10} - Q_9) = 0. \tag{4.84}
\]
Since
\[ Q_i(y, x_1, x_2, x_3, x_4, y) = 0, \; \text{i = 3, 4,} \]
\[ Q_j(y, x_1, x_2, x_3, x_4, y) = Q'_j, \; \text{j = 7, 9,} \]
\[ Q_k(y, x_1, x_2, x_3, x_4, y) = Q'_k, \; \text{k = 8, 10,} \]
and \( Q'_3, Q'_4 \) are linearly independent. (4.84) implies that
\[ \alpha_8 - \alpha_7 + \alpha_{10} - \alpha_9 = 0. \]

Considering the coefficients of the monomials
\[ yx_1x_2y^2x_3y^2x_4x_5, \; yx_1x_2x_3y^2x_4x_5, \; yx_1x_2x_3y^2x_4x_5y, \; yx_1x_2x_3y^2x_4x_5, \; x_1y^2x_2x_3y^2x_4x_5, \; x_1y^2x_2x_3y^2x_4x_5. \]
\[ yx_1x_2y^2x_3y^2x_4x_5, \; yx_1x_2x_3y^2x_4x_5, \; yx_1y^2x_2x_3y^2x_4x_5, \; x_1y^2x_2x_3y^2x_4x_5. \]

yields
\[ 3\alpha_2 + 2\alpha_3 + 2\alpha_6 + \alpha_{10} = 0, \; 2\alpha_2 - 2\alpha_3 + 2\alpha_6 = 0, \; 6\alpha_2 + 2\alpha_3 + 3\alpha_6 + \alpha_7 = 0, \]
\[ -2\alpha_3 + 2\alpha_4 - \alpha_7 + \alpha_8 = 0, \; -2\alpha_1 - \alpha_2 - \alpha_5 - 2\alpha_6 + \alpha_{10} = 0, \]
\[ 2\alpha_1 + 2\alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 = 0, \; \alpha_2 - 2\alpha_3 + \alpha_6 - \alpha_7 - \alpha_9 = 0, \]
\[ 2\alpha_2 - 2\alpha_6 + \alpha_9 = 0, \; 2\alpha_1 + 3\alpha_2 + \alpha_9 = 0. \]

Since \((\sum_{i=1}^{10} \alpha_i Q_i)^* = 0\) we have
\[ -\alpha_1 Q_1 - \alpha_2 Q_2 - \alpha_3 Q_3 - \alpha_4 Q_4 - \alpha_5 Q_5 - \alpha_6 Q_6 = \]
\[ -\alpha_7 Q_7 - \alpha_8 Q_8 - \alpha_9 Q_{10} - \alpha_{10} Q_9 = 0. \quad (4.85) \]

Thus considering the coefficients of the monomials
\[ yx_1x_2y^2x_3y^2x_4x_5, \; yx_1x_2x_3y^2x_4x_5, \; yx_1y^2x_2x_3y^2x_4x_5, \; x_1y^2x_2x_3y^2x_4x_5, \]
\[ yx_1x_2x_3y^2x_4x_5, \; yx_1x_2x_3y^2x_4x_5, \; yx_1y^2x_2x_3y^2x_4x_5, \; x_1y^2x_2x_3y^2x_4x_5, \]
in (4.85) yields
\[ 3\alpha_2 + 2\alpha_4 + 2\alpha_6 + \alpha_{10} = 0, \; 2\alpha_2 - 2\alpha_4 + 2\alpha_6 = 0, \; 2\alpha_1 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_8 = 0, \]
\[ \alpha_2 - 2\alpha_4 + \alpha_6 - \alpha_8 - \alpha_9 = 0, \; -5\alpha_2 - 2\alpha_6 + \alpha_9 = 0. \]

The solution of these systems is trivial. So \(Q_1, \ldots, Q_{10}\) are linearly independent. \(\blacksquare\)
Chapter 5

The Identities of Degree 9 on $H_4$

5.1 Introduction

In Chapter 4, we have determined all identities of degree 7 of $H_3$. A basis of the vector space $VIN_3[t]$ was given for each type $t$. As in Chapter 4 let $VI_4[n_1, \ldots, n_r]$ denote the set of all identities of degree 9 on $H_4$ which have type $[n_1, \ldots, n_r]$, and $VIN_4[n_1, \ldots, n_r]$ denote the set of all identities of degree 9 on $H_4$ which have type $[n_1, \ldots, n_r]$ and are symmetric or skew symmetric in all their variables of degree $m$ according as $m$ is even or odd for any $m \in \{n_1, \ldots, n_r\}$.

We also recall

$$T_k(x_1, \ldots, x_{k-1}; x_k) :=$$

$$S_k(x_k, x_1, \ldots, x_{k-1}) - T_k(x_1, \ldots, x_{k-1}; x_k)$$

In this chapter we shall find a basis of the vector space $VIN_4[t]$ for each type $t$. The main results are Theorem 5.1 and Theorem 5.2.

**Theorem 5.1** If the characteristic of the field $F$ does not divide 3! and $|F| > 8$ then:

1. For any type $[n_1, \ldots, n_r]$ with $\sum_{i=1}^r n_i = 9$ and

   $$[n_1, \ldots, n_r] \neq [4, 1^5], [3, 2, 1^4], [3, 1^6], [2^3, 1^5]$$

   $$[2^2, 1^5], [2, 1^7] \text{ or } [1^9]$$

   we have $VI_4[n_1, \ldots, n_r] = \{0\}$. 

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2. The vector space \( V I N_4[2^3, 1^3] \) = \( \{0\} \).

**Theorem 5.2** If the characteristic of the field \( F \) does not divide \( 3! \) and \( |F| \nless 8 \) then:

1. \( \{T_8(x^2, x, y_1, \ldots, y_5; x)\} \) is a basis of the vector space \( V I N_4[3, 1^5] \).

2. \( \{T_8(x^2, x, y, z_1, \ldots, z_4; y), T_8(x, y_1, x, y, z_1, \ldots, z_4; x)\} \) is a basis of the space \( V I N_4[3, 2, 1^4] \).

3. The following identities form a basis of the space \( V I N_4[3, 1^6] \)

\[
T_8(x^2, y_1, \ldots, y_6; x), \ T_8(x; y_1, \ldots, y_6; x^2), \ \sum_{i=1}^{6} (-1)^{i-1} T_8(x \circ y_1, \ldots, \widehat{y}_i, \ldots, y_6; x; x), \ \sum_{i=1}^{6} (-1)^{i-1} T_8(x, y_1, \ldots, \widehat{y}_i, \ldots, y_6; y_i), \ T_8(x, y_1, \ldots, y_6; x) = xT_8(x; y_1, \ldots, y_6; x).
\]

4. The identities

\[
T_8(x^2, y, z_1, \ldots, z_5; y) + T_8(y^2, x, z_1, \ldots, z_5; x), \ T_8(x \circ y, x, z_1, \ldots, z_5; y) + T_8(x \circ y, y, z_1, \ldots, z_5; x), \ \sum_{i=1}^{5} (-1)^{i-1} (T_8(x \circ z_i, z_1, \ldots, \widehat{z}_i, \ldots, z_5; x, y; y) + \ T_8(y \circ z_i, z_1, \ldots, \widehat{z}_i, \ldots, z_5; y, x; x)), \ xT_8(x, y, z_1, \ldots, z_5; y) + yT_8(y, x, z_1, \ldots, z_5; x), \ T_8(x, y, z_1, \ldots, z_5; y) \circ x + T_8(y, x, z_1, \ldots, z_5; x) \circ y, \ T_8(x, y, z_1, \ldots, z_5; [x, y]), \ T_8(x, y, z_1, \ldots, z_5; [x, y])
\]

form a basis of the space \( V I N_4[2, 2, 1^5] \).

5. The identities

\[
T_8(y_1, \ldots, y_7; x^2), \ T_8(y_1, \ldots, y_7; x^2), \ \sum_{i=1}^{7} (-1)^{i-1} T_8(x \circ y_1, y_1, \ldots, \widehat{y}_i, \ldots, y_7; x).
\]

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\[
\sum_{i=1}^{7} (-1)^{i-1} \tilde{T}_8(y_i, \ldots, y_\tau; x) \\
\sum_{i=1}^{7} (-1)^{i-1} T_8(x, y_i, \ldots, y_\tau; y_i \circ x),
\]

\[
T_8(y_1, \ldots, y_\tau; x), \quad xT_8(y_1, \ldots, y_\tau; x),
\]

\[
\tilde{T}_8(y_1, \ldots, y_\tau; x), \quad x\tilde{T}_8(y_1, \ldots, y_\tau; x),
\]

\[
\sum_{i=1}^{7} (-1)^{i-1} y_i T_8(y_1, \ldots, y_\tau, x; x),
\]

\[
\sum_{i=1}^{7} (-1)^{i-1} \tilde{T}_8(y_1, \ldots, y_\tau, x; x) y_i,
\]

\[
\sum_{i=1}^{7} (-1)^{i-1} T_8(x, y_1, \ldots, y_\tau; [x, y_i])
\]

form a basis of the space \( VIN_4[2, 1^7] \).

6. \( S_9(x_1, \ldots, x_9) \) forms a basis of the space \( VIN_4[1^9] \).

Thus, all identities of \( H_4 \) of degree 9 are determined.

Throughout the chapter, we assume that the characteristic of the field \( F \) does not divide \( 3! \) and \( |F| > 8 \).

5.2 The Proof of the Theorem 5.1

In this section we shall prove the Theorem 5.1. The proof will be completed in the following Propositions.

Proposition 5.2.1 \( H_4 \) has no identity of degree 9 which is of type [5, 3, 1].

Proof. Let \( f(x, y, z) \in \mathcal{V}I_4[5, 3, 1] \). Then \( f = f_1 xy + f'_1 yz \) because \( H_3 \) has no identity of degree 7 which is lower than \([4, 2, 1]\), and no identity of degree 6 which is lower than \([2, 2, 1, 1]\). Since \( f_1, f'_1 \) are identities of \( H_3 \) of type \([4, 2, 1]\) by Lemma 2.1.2, there exist scalars \( \alpha, \beta \in F \) such that

\[
f_1 = \alpha Q(x^2, x, y, z, z, y), \quad f'_1 = \beta Q(x^2, x, y, z, z, y).
\]
However, from \( f = f_1 xy + f'_1 yx \) we know that \( f \) has no terms which end in \( yx^2 \). \( f^* \) is also an identity of type \([5,3,1]\), thus it has no terms which end in \( yx^2 \). Thus \( f \) has no terms which start with \( x^2 y \). Since \( x^2 y x z x y x y \) has coefficient \(-\alpha\) in \( f \), \( \alpha = 0 \). Similarly we can show that \( \beta = 0 \). So \( f = 0 \). \( \blacksquare \)

**Proposition 5.2.2** The vector space \( \mathcal{V}I_4[5,2,2] = \{0\} \).

**Proof.** Let \( f(x,y,z) \in \mathcal{V}I[5,2,2] \) then by Theorem 1.1, \( f \) is symmetric in \( y,z \) since \( \Delta(y,z)f \in \mathcal{V}I_4[5,3,1] = \{0\} \). Thus

\[
f = f_1(x,y,z)zx + f'_1(x,x,y,z)zy + f_2(x,z,y)xy + f'_2(x,z,y)yx.
\]

Since \( f_1, f'_1 \) are identities of \( H_3 \) of type \([4,2,1]\), using the same argument as in the proof of Proposition 5.2.1 we have \( f_1 = f'_1 = 0 \). So \( f = 0 \). \( \blacksquare \)

**Proposition 5.2.3** The vector space \( \mathcal{V}I_4[5,2,1,1] = \{0\} \).

**Proof.** From Proposition 5.2.2, \( \mathcal{V}I_4[5,2,1^2] = \mathcal{V}IN_4[5,2,1^2] \). Let \( f(x,y,z,w) \in \mathcal{V}IN_4[5,2,1^2] \) then

\[
f = f_1 xy + f'_1 yx + f_2(x,y,w)zx + 
f'_2(x,y,w)zx - f_2(x,y,z)zw - f'_2(x,y,z)wx,
\]

(5.1)

since \( H_3 \) has no identity which has type \([5,1,1],[5,1],[3,2,1]\) or \([4,1,1]\).

Since \( f \) can be decomposed into the sum of a \( \ast \)-symmetric identity in \( \mathcal{V}IN_4[5,2,1,1] \) and a \( \ast \)-skew symmetric identity in \( \mathcal{V}IN_4[5,2,1,1] \), we may assume that \( f^* = \pm f \). From (5.1) we know that every identity of type \([5,2,1,1]\) has no terms which end in \( x^2 \). So arguing as in the proof of Proposition 5.2.1 yields \( f_2 = f'_2 = 0 \). Thus \( f = f_1 xy + f'_1 yx \).

Since \( \Delta(z,y)f \in \mathcal{V}I_4[5,3,1] = \{0\} \), \( \Delta(z,y)f_1 = 0 \) as is seen by considering the sum of the terms of \( \Delta(z,y)f \) which end in \( xy \). Thus \( f_1 \) is skew symmetric in the variables of degree 1. Since \( f^* = \pm f \) and \( f \) has only terms which end in \( xy \) or \( yx \), \( f_1 \) must have only terms which start with \( xy \) or \( yx \). However \( f_1 \) is skew symmetric in the variables \( y,z \), so \( f_1 = 0 \) and \( f = 0 \). \( \blacksquare \)
Proposition 5.2.4 The vector space $VI_4[5, 1^4] = \{0\}$. Therefore there is no identity of $H_4$ which is of type $[5, 1^4]$ or of lower type.

Proof. Since $H_4$ has no identity which is lower than $[5, 1^4]$, $VI_4[5, 1^4] = VIN_4[5, 1^4]$. Let $f(x, y_1, y_2, y_3, y_4) \in VIN_4[5, 1^4]$ then

$$f = \sum_{i=1}^{4} (-1)^i (f_1(x, y_1, \ldots, y_i, \ldots, y_4)x y_i + f'_1(x, y_1, \ldots, y_i, \ldots, y_4)y_i x).$$

Since $\Delta(y_4, 1)f \in VI_4[5, 1^3] = \{0\}$, $\Delta(y_4, 1)f_1(\ldots, y_4, \ldots) \in VI_3[4, 1^2] = \{0\}$ and $\Delta(y_4, 1)f'_1(\ldots, y_4, \ldots) \in VI_3[4, 1^2] = \{0\}$, from the equation above we have $f_1 + f'_1 = 0$. Because $f_1 \in VIN_3[4, 1^2]$ there exist scalars $\alpha, \beta \in F$ such that

$$f_1 = \alpha T_0(y_1, y_2, y_3, x^2, x; x) + \beta [S_3([y_1, x], [y_2, x], [y_3, x], x), x].$$

We may assume that $f^* = \pm f$. Thus $f$ has no terms which end in $x^2$. This implies that the monomial $x^2 y_1 y_2 y_3 x y_4$ has coefficient 0 in $f$. This implies that $\alpha - \beta = 0$. If $f^* = -f$ then the fact that the monomial $x y_1 x y_2 y_3 x y_4 x$ has coefficient 0 in $f$ implies that $\beta = 0$. So $f_1 = 0$ and therefore $f = 0$. If $f^* = f$ then the Razmyslov transform $f^#(y_4)$ is also an identity of type $[5, 1^4]$. Since there is no identity of $H_3$ of degree 6 with type $[3, 1^3]$, $f^#(y_4)$ has no terms which end in $x^2$. So the monomial $y_4 x y_1 x y_2 y_3 x x x$ has coefficient 0 in $f^#(y_4)$. This means the monomial $x y_1 x y_2 y_3 x y_4$ has coefficient 0 in $f$. However it has coefficient $-\beta$, thus $\beta = 0$. So $\alpha = 0$. Hence $f = 0$.

Let $f$ be an identity of degree 9. If $f$ has type lower than $[5, 1^4]$ then by Proposition 1.2.1, one of its partial linearizations is an identity of type $[5, 1^4]$ since $|F| > 8$. Thus this partial linearization is 0 by the result we have proved above. Therefore $f$ is 0 since the partial linearization did not set some variables of $f$ equal and there is no identity of $H_4$ which is lower than $[5, 1^4]$. \qed

Proposition 5.2.5 The vector space $VI_4[4, 3, 1, 1] = \{0\}$.

Proof. Let $f(x, y, z_1, z_2) \in VI_4[4, 3, 1, 1]$. Then

$$f = f_1 x y + f'_1 x z_1 + f_4 y x^2 +$$

$$\sum_{i=1}^{2} (f_{2i} x z_i + f'_{2i} z_i x + f_{3i} y z_i + f'_{3i} z_i y), \quad (5.2)$$

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since $H_3$ has no identity of degree 7 which is lower than $[4, 2, 1]$ and no identity of degree 6 which is lower than $[2, 2, 1, 1]$. Moreover

$$f_{21} = \alpha_1 Q(x \circ y, x, y, z_2, x, y).$$

because they are identities of $H_3$. We may assume that $f^* = \pm f$. Then from (5.2) $f$ has no terms which start with $y^2$. So the monomial $y^2 x z_2 x^2 y z_2$ has coefficient 0 in $f$. But it has coefficient $-\alpha_1$, thus $f_{21} = 0$. Since $\Delta(y, x)f \in V1[5, 2, 1, 1] = \{0\}$, $\Delta(y, x)f_{21} + f_{31} = 0$. So $f_{31} = 0$. Using the same argument one can show that $f_{22} = f'_{22} = f_{32} = f'_{32} = 0$. Thus $f = f_1 x y + f'_1 x y + f_4 y^2$. Since $f^* = \pm f$ and $f$ has no terms which end in $z_1$, the monomial $z_1 x y z_2 y z_2$ must have coefficient 0. But it has coefficient $\beta_i$, so $f_i = 0$ and $f = f_1 x y + f'_1 x y$. Thus $\Delta(y, x)f = 0$ implies that $f_i = -f'_i$. Since $f_1$ is an identity of $H_3$ of type $[3, 2, 1, 1]$, by the decomposition Theorem 1.1 $f_1 = g + h$, where $g \in V1N_3[3, 2, 1, 1]$ and $h$ is a partial linearization of an identity of type $[3, 2, 2]$. Thus $g = \sum_{i=1}^{10} \alpha_i Q_i$ and $h = \sum_{i=1}^{4} \beta_i \bar{Q}_i$, where $\{Q_i | i = 1, \ldots, 10\}$ and $\{\bar{Q}_i(x, y, z, z) | i = 1, \ldots, 4\}$ are the bases of $V1N_3[3, 2, 1, 1]$ and $V1[3, 2, 2]$ given in Proposition 4.2.3 and Proposition 4.2.2 respectively. Since $f$ has no terms which end in $z_2 x, z_2 y, z_1 z_2$ or $z_1 x$ and $f^* = \pm f$, the monomial $w^* x y$ has coefficient zero, where $w$ is either as in the proof of Proposition 4.2.3 or $x x z_1 z_2 y y x$. Thus by the $*$-symmetry of $Q_i$, changing $\alpha_i$ to be $-\alpha_i$, $\alpha_9$ to be $\alpha_{10}$ and $\alpha_{10}$ to be $\alpha_9$ in the system (4.19) except equation $\alpha_3 + \alpha_4 + \alpha_6 + \alpha_8 + \alpha_9 = 0$ will give a new system. Solving it yields $\alpha_i = 0$. Since $f(x, y, z_1, z_2)$ has only terms which end in $x y$ or $y z$ and $f^* = \pm f$, $f(x, y, z, z)$ has only terms which start with $x y$ or $y z$. So $h(x, y, z, z)$ has only terms which start with $x y$ or $y z$. Considering the coefficients of the monomials $x y z x y z x x$, $y z x z x z x x$, $x x y y z x x$ [see the proof of Proposition 4.2.2] yields $\beta_i = 0$. Thus $f = 0$. 

**Proposition 5.2.6** The vector space $V1N_3[4, 2, 2, 1] = \{0\}$.

**Proof.** Let $f(x, y_1, y_2, z_2) \in V1N_3[4, 2, 2, 1]$. Then

$$f = (Id + (1, 2))\{f_1(x, y_2, y_1, z) x y_1 + f'_1(x, y_2, y_1, z) y_1 x + f_3(x, y_1, y_2, z) y_1 y_2 + f_4(x, y_1, y_2) y_2 z + f'_4(x, y_1, y_2) z y_2 + f_5(x, y_2, y_1, z) y_1 x^2 \} + f_2(x, y_1, y_2) x z + f'_2(x, y_1, y_2) x x. \tag{5.3}$$

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where \text{Id} denotes the identity mapping and the operator \((1, 2)\) acts on polynomials to permute \(y_1\) and \(y_2\). Since \(f_1\) is an identity of \(H_3\) of type \([3, 2, 1, 1]\),

\[
f_1 = \sum_{i=1}^{10} \alpha_i Q_i + \sum_{i=1}^{4} \beta_i \tilde{Q}_i,
\]

where \(\{Q_i| i = 1, \ldots, 10\}\) and \(\{\frac{i}{2} \tilde{Q}_i(x, y_2, z, z)|i = 1, \ldots, 4\}\) are respectively the basis of the space \(VIN_3[2, 2, 1, 1]\) and \(VIN_3[3, 2, 2]\) given in Chapter 4. Since \(f_4 \in VIN_3[4, 2, 1]\) and \(f_5 \in VIN_3[2, 2, 1, 1]\),

\[
f_4 = \gamma Q(x^2, x, y_2, y_1, z, z), f_5 = \delta Q(x, y_2, y_1, x, y_2, z).
\]

If we let \(z = 1\) then \(f(x, y_1, y_2, 1) = f_i(x, y_1, y_2, 1) = 0\) for \(i = 1, 3, 5\) because

\[
VIN_3[4, 2, 2] = VIN_3[3, 2, 1] = VIN_3[1, 1, 1] = 0.
\]

Thus from (5.3) we have \(f_i + f_i' = 0\) for \(i = 2, 4\). Since \(\Delta(z, y_1)f\) belongs to \(VIN_3[4, 3, 2]\) and \(\Delta(z, y_2)f\) belong to \(VIN_3[3, 2, 2]\) which has dimension 3. From this we get \(\beta_1 = \beta_2\). Since \(\Delta(z, y_2)f = 0\), considering the sum of the terms of \(\Delta(z, y_2)f\) which end in \(xy_1\) yields \(f_1(x, y_2, y_1, y_2) = 0\). While \(f_1(x, y_2, y_1, y_2) = 0\) and \(\beta_1 = \beta_2\) imply \(\alpha_1 + \alpha_2 - \alpha_4 = 0\). From \(\Delta(z, x)f = 0\) we have \(f_1(x, y_2, y_1, x) - f_4(x, y_2, y_1) = 0\). This equation yields \(-\alpha_3 + 2\alpha_4 - 2\beta_1 + \beta_3 + \gamma = 0\).

We may assume that \(f^* = \pm f\). Thus using the \(*\)-symmetry of \(f\) and considering the coefficients of the monomials

\[
xyzxyzby_1y_2y_1, y_1y_1xy_2xxyy_2, xy_2y_1y_2xxyy_1, \\
yzyzyzby_2y_1zxyy_1, y_2xy_1xy_2xxyy_1, y_2xy_2y_1xxyy_1, \\
xyzzyzby_2xyy_1, y_2zy_2y_1xzyy_1, y_2zy_1xyy_2xxyy_1, \\
xyzzyzyzby_1zy_2xyy_1, y_1zy_2y_2xxyy_1, xxyy_1zy_2xxyy_1, \\
xyzzyzyzyzby_1y_2y_2y_1zxyy_1,
\]

yields a system of linear equations. Solving this system yields \(f_1 = f_5 = 0\). Since \(\Delta(z, y_2)f = 0\) we have \(f_1(x, y_1, y_2, y_2) + f_2 = 0\) and \(f_5(x, y_1, y_2, y_2) + f_4(x, y_2, y_1) = 0\). While \(\Delta(y_1, x)f = 0\) implies \(f_1(x, x, y_2, z) + f_3 = 0\). So \(f_i = 0\) for \(i = 1, 2, 3, 4, 5\). Thus \(f = 0\).
Proposition 5.2.7 The vector space \( V/N_4[1,2,1^3] \) = \( \{0\} \).

Proof. Let \( f(x, y, z_1, z_2, z_3) \in V/N_4[1,2,1^3] \) then

\[
    f = f_1xy + f'_1yx + f_2yx^2 + f_0(x, y, z_1, z_2)yz_3x^2 + \sum_{(1,2,3)} \{f_2'(x, y, z_1, z_2)xzz_3 + f_2''(x, y, z_1, z_2)xyx + f_3(x, y, z_1, z_2)yz_3y + f_4(x, y, z_1)yz_3z_3\}. \tag{5.4}
\]

Since \( \Delta(z_3, y)f \in V/L_4[4,3,1^2] = \{0\} \), from (5.1) we have

\[
    f_2 = -f_1(x, y, z_1, z_2, y), \quad f'_2 = -f'_1(x, y, z_1, z_2, y), \quad f_4 = f_3(x, y, y, z_1). \quad f_3 = -f_3', \quad f_0 = -f_5(x, y, z_1, z_2, y).
\]

Similarly from \( \Delta(y, x)f = 0 \) we have

\[
    f_3(x, y, z_1, z_2) = -\Delta(y, x)f_2(x, y, z_1, z_2).
\]

Thus \( f \) is determined by \( f_1, f'_1 \) and \( f_5 \).

Since \( \Delta(y, 1)f \in V/L_4[4,1^4] = \{0\} \) and \( \Delta(y, 1)f = 0 \), from (5.4) we have

\[
    f_1 + f'_1 + f_5x = 0.
\]

\( f_5 \) is an identity of \( H_3 \) of type \( [2, 1^4] \) and it is skew symmetric in the last 3 variables. Because every identity of \( H_3 \) of degree 6 comes from \( T_6 \) and \( Q \), there exist scalars \( \beta, \gamma \in F' \) such that

\[
    f_5 = \beta T_6(y, z_1, z_2, z_3, x, x) + \gamma \{Q(x, y, z_1, x, z_2, z_3) + Q(x, y, z_2, x, z_3, z_1) + Q(x, y, z_3, x, z_1, z_2)\}.
\]

From the results in Chapter 4 every identity of \( H_3 \) of type \( [3, 1^4] \) is a linear combination of the identities which come from \( T_6 \), \( Q \) and \( K \) by substituting variables, especially the identity \( f_1(x, y, z_1, z_2, z_3) \) which is skew symmetric in the \( z_i \)'s. Let \( f_1 = \sum \alpha_i Q_i \), where \( Q_i \) is an identity which comes from one of \( Q \), \( T_6 \) and \( K \) by substituting some variables. Let \( \sigma \in S_3 \) then

\[
    \sigma(h(x, y, z_1, z_2, z_3)) := h(x, y, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}),
\]

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where \( l \in \{ f_1, Q_i \} \). So

\[
\sum_{\sigma \in S_3} (-1)^\sigma \sigma(f_1(x, y, z_1, z_2, z_3)) = \sum_{\sigma \in S_3} (-1)^\sigma f_1(x, y, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) = \sum_i \alpha_i \{ \sum_{\sigma \in S_3} (-1)^\sigma \sigma(Q_i) \}.
\]

Thus

\[
6f_1 = \sum_i \alpha_i \{ \sum_{\sigma \in S_3} (-1)^\sigma \sigma(Q_i) \}.
\]

Since \( \text{Char} F \not\equiv 3 \), we may assume that

\[
f_1 = \sum_i \alpha_i y_i,
\]

where \( y_i = \sum_{\sigma \in S_3} (-1)^\sigma \sigma(Q_i) \). So \( y_i \) comes from one of \( Q, T_6 \) and \( K \) and is skew symmetric in the variables \( z_1, z_2, z_3 \).

By the symmetry of the variables of \( T_6, Q \) and \( K \), there are only 21 identities which come from \( T_6, Q \) and \( K \) by substituting some variables and which are skew in \( z_i \)’s. Thus \( f_1 = \sum_{i=1}^{21} \alpha_i y_i \).

Now we may assume that \( f^* = \pm f \). Then, by the *-symmetry of \( f \) and since \( f \) has no terms which end in \( y^2 \), considering the coefficients of some monomials yields a system (I) of linear equations.

If \( f^* = -f \) then the solution of the system (I) implies that \( f_1 = f_5 = 0 \) which can be checked by computer easily.

If \( f^* = f \) then the Razmyslov transform \( f^\#(z_1) \) of \( f \) is also an identity of \( H_4 \) by Proposition 2.2.2. Thus the monomial \( z_2 z_3 x y z_1 y z x x \) must have coefficient 0 in \( f^\#(z_1) \) by Lemma 2.1.1. So the monomial \( y z x x z_1 z_2 z_3 x y \) has coefficient 0 in \( f \). That is

\[-\alpha_{17} + \alpha_{16} + 2\alpha_{13} + \alpha_{11} + \alpha_6 - \alpha_5 + \alpha_4 - \alpha_3 = 0.\]

The solution of this equation and the system (I) implies that \( f_1 = f_5 = 0 \). Hence \( f = 0 \).

Proposition 5.2.8 The vector space \( VIN_4[3, 3, 3] = \{0\} \).
Proof. Let \( f(x_1, x_2, x_3) \in \mathcal{V}I\mathcal{N}_4[3, 3, 3] \) then

\[
f = \sum_{i \neq j} \pm f_{ij}(x_i, x_j)x_i x_j
\]

where \((i, j, k)\) is a permutation of 1, 2, 3, since there is no identity of \( H_3 \) of type \([3, 2, 1]\).

Since \( f_1 \) is an identity of \( H_3 \) which has type \([3, 2, 2]\),

\[
f_1 = \alpha_1 Q(x_1 \circ x_3, x_1, x_2, x_3, x_1, x_2) + \alpha_2 Q(x_1 \circ x_2, x_1, x_3, x_2, x_1, x_3) + \alpha_3 Q(x_1^2, x_2, x_3, x_1, x_2, x_3) + \alpha_4 [S_3([x_1, x_2], [x_1, x_3], [x_2, x_3]). x_1].
\]

We may assume that \( f^* = \pm f \). Then by the symmetry of \( f \), \( \alpha(w) + s\alpha(w^*) = 0 \), where \( \alpha(w) \) denotes the coefficient of the monomial \( w \) in \( f \) and \( s := \pm 1 \) if \( f^* = \mp f \). Thus for

\[
w = x_1 x_2 x_1 x_3 x_2 x_3 x_1 x_2 x_3, \quad x_2 x_3 x_1 x_3 x_2 x_1 x_2 x_3, \quad x_1 x_3 x_1 x_2 x_2 x_3 x_1 x_2 x_3,
\]

we have

\[
2\alpha_1 + s(\alpha_4 + \alpha_2 + 2\alpha_1) = 0,
-\alpha_3 + s(-\alpha_4 - \alpha_2 - \alpha_3) = 0,
-\alpha_4 - 2\alpha_2 - \alpha_1 + s\alpha_3 = 0.
\]

(5.5)

Since \( f \) has no terms which end in \( x_3^2 \) and \( f^* = \pm f \), the monomials \( x_1 x_1 x_2 x_3 x_1 x_3 x_2 x_2 x_3 \) and \( x_1 x_1 x_2 x_1 x_2 x_2 x_3 x_2 x_3 \) have coefficient 0. That is,

\[
-\alpha_4 - \alpha_2 - \alpha_3 = 0, \quad \alpha_4 + \alpha_1 + \alpha_3 = 0.
\]

(5.6)

The solutions in both cases are trivial, so \( f = 0 \). 

Let \( f(x_1, x_2, y, z) \) be an identity of \( H_4 \) of type \([3, 3, 2, 1]\) then

\[
f = (Id - (x_1, x_2)) \{ f_1(x_1, x_2, y, z)x_1x_2 + f_2(x_1, x_2, y, z)x_2y + f_4(x_1, x_2, y, z)x_2y + f_5(x_1, x_2, y, z)x_1 x^2 + f_6(x_1, x_2, y, z)x_1 x^2 + f_7(x_1, x_2, y, z)x_1 x^2 \} + f_8(x_1, x_2, y, z)yz + f_9(x_1, x_2, y, z)yz,
\]

(5.7)

since \( H_3 \) has no identities which are lower than \([3, 1, 1, 1]\).
The identity \( f(x_1, x_2, y, y) \) is of type \([3,3,3]\), and so is zero identically. But

\[
f(x_1, x_2, y, y) = f_1(x_1, x_2, y, y)x_1x_2 - f_1(x_2, x_1, y, y)x_2x_1 + \]
\[
(f_2(x_1, x_2, y, y) + f_3(x_1, x_2, y))y + \]
\[
(f_2'(x_1, x_2, y, y) + f_3'(x_1, x_2, y))yx_2 + \]
\[
(f_4(x_1, x_2, y) + f_4'(x_1, x_2, y))y^2.
\]

Thus,

\[
f_2(x_1, x_2, y, y) + f_3(x_1, x_2, y) = 0, f_4 + f_4' = 0, \]
\[
f_2'(x_1, x_2, y, y) + f_3'(x_1, x_2, y) = 0, f_1(x_1, x_2, y, y) = 0. \tag{5.8}
\]

Similarly, by considering \( f(x_1, x_2, y, x_1) = 0 \) we have

\[
f_3(x_1, x_2, y, y) + f_3'(x_1, x_2, y) + f_5(x_2, x_1, y, x_1)x_2 = 0, \]
\[
f_4(x_1, x_2, y, y) - f_4'(x_2, x_1, y, x_1) = 0, f_5(x_1, x_2, y, x_1) = 0, \]
\[
f_2(x_1, x_2, y, x_1) = 0, f_2'(x_1, x_2, y, x_1) = 0. \tag{5.9}
\]

Since \( f_5 \) is an identity of \( II_3 \) of type \([2,2,1,1]\), there exists a scalar \( \gamma \in F \) such that

\[
f_5(x_1, x_2, y, z) = \gamma Q(x_1, y, x_2, x_1, y, z). \]

Hence \( \Delta(z, x_1)f_5(x_2, x_1, y, z) = 0 \), and

\[
f_3(x_1, x_2, y, y) + f_3'(x_1, x_2, y) = 0. \tag{5.10}
\]

The equalities above tell us we need only determine \( f_1, f_2 \) and \( f_5 \). The identity \( f_2(x_1, x_2, y, z) \) is of type \([3,2,1,1]\), so \( f_2 = y_0 + y_1 \), where

\[
g_0 = \frac{1}{2}(f_2(x_1, x_2, y, z) - f_2(x_2, x_1, y, z)), \]
\[
g_1 = \frac{1}{2}(f_2(x_1, x_2, y, z) + f_2(x_2, x_1, y, z)).
\]

Since \( g_0 \) (respectively \( g_1 \)) are skew symmetric (respectively symmetric) in \( y \) and \( z \),

\[
g_0 = \sum_{i=1}^{10} \beta_i Q_i, \quad g_1 = \sum_{i=1}^{4} \gamma_i \Lambda^i(y, z) \tilde{Q}_i(x_1, x_2, y),
\]

where \( Q_1, \ldots, Q_{10} \) are the basis of \( V I N_3[3,2,1,1] \), and \( \tilde{Q}_1, \ldots, \tilde{Q}_4 \) are the basis of \( V I N[3,2,2] \), both of which were given in Chapter 4.
Lemma 5.2.1 \( f_5 = 0 \), that is in \( f \) the coefficient \( \gamma \) is zero.

Proof. The proof will use Razmyslov's transformation. If \( f \) is \( \ast \) symmetric then the Razmyslov transform \( f^{\#}(z) \) of \( f \) is also an identity by Proposition 2.2.2. Since the monomial \( x_2x_1xy^2 \) has coefficient \( \gamma \) in \( f \), \( y^2x_1x_2x_1 \) has coefficient \( \gamma \) in \( f^{\#}(z) \) also. But from (5.7), every identity of type \([3,3.2,1]\) on \( H_4 \) has no terms which start with \( y^2x_1 \) or end in \( x_1y^2 \). Thus \( \gamma = 0 \), since \( f^{\#}(z) \) is an identity of \( H_4 \) of type \([3,3,2,1]\).

If \( f^* = -f \) then \( f^{\#}(z)(x_1, x_2, y, z) = 0 \) for \( x_1, x_2, y \in H_4 \), and \( z \in K_4 \) by Proposition 2.2.2. Since

\[
f^{\#}(z) = g(x_1, x_2, z)x_2y^2 + \gamma,
\]

\( g(x_1, x_2, z) = 0 \) for \( x_1, x_2 \in H_3 \) and \( z \in K_3 \) by Proposition 2.2.1. Next, we show that \( g \) is identically zero. If \( g \) is \( \ast \)-skew symmetric, then \( g^{\#}(z) \) is an identity of type \([3,2,1]\) on \( H_3 \), therefore \( g^{\#}(z) = 0 \), hence \( g \) is identically zero. Thus we may assume that \( g \) is \( \ast \)-symmetric. Since \( H_2 \) has no identities of type lower than \([2,2]\) and has no identity of degree less than 4 either,

\[
g(x, y, z) = y_1xy + y_1'yx + y_2yx^2 + y_3xy^2,
\]

where \( g_2(x, y, z) \) is multilinear and \( g_2(x, y, z) = 0 \) for \( x, y \in H_2 \) and \( z \in K_2 \), and \( g_3(x, z) \) is of type \([2,1]\). Hence

\[
g_3(x, z) = \alpha_1zx^2 + \alpha_2xzx + \alpha_3x^2z.
\]

From (5.11) we know that \( g \) has no terms ending in one of the monomials

\( zz, zx, zx^2, zy, yz, zy^2 \).

Since \( g \) is \( \ast \)-symmetric, it has no terms starting with one of the monomials

\( zx, zx, x^2z, y^2z, zy, yz \).

Thus \( g_3 \) is identically zero. Since \( H_2 \) has no identity of degree less than 4, we may assume that \( g_2 \) is \( \ast \)-symmetric by Proposition 2.2.2. Thus

\[
g_2(x, y, z) = \beta_1(xyz + yzx) + \beta_2(yzx + xzy) + \beta_3(zxy + yxz).
\]
The same argument as above yields that \( g_2 \) is also identically zero. Thus the terms in \( g \) must start with either \( xy \) or \( yx \), and end in either \( xy \) or \( yx \) also. Hence

\[
g = \gamma_1(xyzz^2y + yz^2zyx) + \gamma_2(xyzxzy + yzxyxz) + \gamma_3(xyzxzy + yzzxxy) + \gamma_4(yzxz^2y + yz^2zyx).
\]

Let us consider the complete linearization \( g'(x_1, x_2, x_3, y_1, y_2, z) \) of \( g \). Substituting \( e[12], e[22], e_3 - e_3, e[33], e[33], e[33] \) for \( x_1, y_1, z, x_2, x_3, y_2 \) in \( g' \) and considering the coefficient of \( e_{13} \) yield \( \gamma_1 + \gamma_3 = 0 \).

Substituting \( e[12], e[22], e_3 - e_3, e[33], e[33], e[33] \) for \( y_1, x_1, z, x_2, y_2, x_3 \) in \( g' \) and considering the coefficient of \( e_{13} \) yield \( \gamma_2 + \gamma_4 = 0 \).

Substituting \( e[11], e[11], e_1 - e_1, e[22], e[23], e[33] \) for \( x_1, y_1, z, x_2, x_3, y_2 \) in \( g' \) and considering the coefficient of \( e_{13} \) yield \( \gamma_1 + \gamma_4 = 0 \).

Substituting \( e[11], e[12], e[22], e_3 - e_3, e[33], e[33] \) for \( x_1, y_1, x_2, z, y_2, x_3 \) in \( g' \) and considering the coefficient of \( e_{13} \) yield \( \gamma_2 + \gamma_3 = 0 \).

So

\[
g = \gamma_1(xyzz^2y + yz^2zyx + xyzxxy + yxzyzx - yzxxxyy - yzxx^2y - yx^2zy).
\]

Let \( n = e_{11} + e_{12}, y = e_{23} \) and \( z = e_{13} - e_{31} \). Then \( g(x, y, z) = 0 \) implies \( \gamma_1 = 0 \). So \( g = 0 \) identically.

From \( g = 0 \) we know that \( f^#(z) \) has no terms ending either in \( x_2y^2 \) or starting with \( y^2x_2 \). Thus \( y^2x_1^2x_2^2x_2x_1 \) has coefficient 0 in \( f^#(z) \). However, it has coefficient \( \gamma \) since the monomial \( x_2x_1y^2z^2x_2^2 \) has coefficient \( \gamma \) in \( f \). So, \( \gamma = 0 \). \( \square \)

Since \( \Delta(y, x_2)f \in V^4_{I_4}[4, 3, 1, 1] = \{0\} \), from Lemma 5.2.1 and (5.7), considering the sum of the terms of \( \Delta(y, x_2)f \) which end in \( x_2^2 \) yields \( f_2 + f'_2 = 0 \). While \( \Delta(x_2, x_1)f \in V^4_{I_4}[4, 2, 2, 1] = \{0\} \) implies that \( f_1(x_1, x_2, y, z) \) is symmetric in \( x_1, x_2 \) from Lemma 5.2.1 and (5.7).

**Lemma 5.2.2** The identity \( f_2 \) is zero identically.
Proof. It suffices to show that \( f_2 \) is zero if \( f \) is either + symmetric or * skew symmetric.

In both cases, the coefficients of the following monomials are zero by (5.7) and \( f_3 = 0 \),

\[
\begin{align*}
& x_1^2 x_2 y x_1 x_2 x_2 y, \quad x_1^2 y x_2 x_1 x_2 y, \quad x_1^2 x_1 y x_1 x_2 x_2, \quad x_1^2 x_2 y x_2 x_1 x_2 y, \\
& x_1^2 x_2 y x_1 x_2 x_2, \quad x_2^2 y x_1 x_2 x_1 x_2 y, \quad x_1^2 x_2 x_2 y x_1 y z, \quad x_1^2 x_2^2 y x_2 x_1 y z.
\end{align*}
\]

That is,

\[
\begin{align*}
\beta_2 + 2\beta_3 + \beta_5 + \beta_7 + \beta_9 + \gamma_2 + 2\gamma_3 + \gamma_4 &= 0, \\
2\beta_3 + \beta_4 + \beta_6 + 2\beta_8 + \beta_9 + \gamma_1 + 2\gamma_3 + \gamma_4 &= 0, \\
\beta_1 + \beta_5 - \beta_7 &= 0, \\
\beta_2 + 2\beta_3 + \beta_9 &= 0, \\
\beta_1 + \beta_7 + 2\beta_5 &= 0, \\
\gamma_1 + 2\gamma_3 + \gamma_4 &= 0, \\
\beta_1 + \beta_2 - \beta_4 + \gamma_1 - \gamma_2 &= 0.
\end{align*}
\]  

(5.13)

From \( f_2(x_1, x_2, y, x_1) = 0 \), we have

\[
-\beta_3 + 2\beta_4 - 2\gamma_1 + \gamma_3 = 0.
\]

(5.14)

If \( h(x_1, \ldots, x_9) \) is a multilinear identity on \( H_4 \) then substituting

\[
e[11], e[12], e[22], e[23], e[33], e[34], e[44], e[42]
\]

for \( x_1, \ldots, x_9 \) and considering the coefficient of \( e_{12} \) in \( h \) yields that the sum of the coefficients of the following monomials is 0,

\[
\begin{align*}
x_1 x_2 x_3 x_4 x_4 x_6 x_7 x_8 x_9, \quad x_1 x_2 x_3 x_4 x_6 x_5 x_7 x_8 x_9, \\
x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_3, \quad x_1 x_2 x_4 x_5 x_6 x_7 x_8 x_9 x_3, \\
x_1 x_2 x_9 x_8 x_7 x_6 x_5 x_4 x_3, \quad x_1 x_2 x_9 x_8 x_7 x_5 x_6 x_4 x_3, \\
x_1 x_2 x_3 x_9 x_8 x_7 x_6 x_5 x_4, \quad x_1 x_2 x_3 x_9 x_8 x_7 x_5 x_6 x_4.
\end{align*}
\]

Using this result and Lemma 2.1.1. considering the monomials \( x_1 x_2 x_1 y x x_2 x_1 x_2 y \) and \( x_2 x_1 y x_1 x_1 x_2 x_2 y \) in \( f \) (in fact, in the complete linearization of \( f \)) respectively yield

\[
\beta_9 - \beta_{10} = 0, \quad \beta_2 - 2\beta_3 - 3\beta_4 - 2\beta_5 + \beta_7 + 3\gamma_1 + 3\gamma_2 + 2\gamma_3 + 2\gamma_4 = 0.
\]

(5.15)
If $f$ is $*$-symmetric then the coefficients of the monomials
\[ x_1y_1x_2^2z_1x_2y, \ x_2y_1x_2x_1z_2x_2y, \ y_1x_2x_1x_3z_1x_2y, \ x_2y_1^2z_2x_1x_2y \]
are equal to the coefficients of their image under $*$, respectively, in $f$, which yields
\[
\begin{align*}
\beta_{10} + \beta_1 - 2\beta_2 - 2\beta_3 - 3\beta_4 - \beta_5 - \beta_6 - \beta_7 + 2\beta_8 - 2\gamma_1 - 2\gamma_2 + 2\gamma_3 - \gamma_4 &= 0, \\
-\beta_{10} - \beta_2 - \beta_4 - \beta_5 - 2\beta_6 - \beta_7 - \beta_10 - \gamma_1 - \gamma_2 + 2\gamma_3 &= 0, \\
\beta_{10} - \beta_2 - \beta_1 - \beta_6 - \beta_7 - \gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 &= 0, \\
-\beta_{10} - 4\beta_3 - 3\beta_4 - \beta_5 - \beta_6 - 2\beta_8 + \gamma_1 + \gamma_4 &= 0.
\end{align*}
\tag{5.16}
\]
Solving the equations (5.13), ..., (5.16) yields $f_2 = 0$ if $f$ is $*$-symmetric.

If $f$ is $*$-skew symmetric then (5.16) becomes
\[
\begin{align*}
\beta_{10} - \beta_1 + 2\beta_2 + 2\beta_3 + 3\beta_4 - \beta_5 + \beta_6 + \beta_7 + 2\beta_8 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3 - \gamma_4 &= 0, \\
\beta_{10} + \beta_2 - \beta_4 - \beta_5 + \beta_6 + \beta_7 - \beta_10 - \gamma_1 + \gamma_2 &= 0, \gamma_1 - \beta_4 &= 0, \\
\beta_{10} - \beta_4 - \beta_5 + \beta_6 + 2\beta_8 - \gamma_1 - 4\gamma_3 - \gamma_4 &= 0.
\end{align*}
\tag{5.17}
\]
The solutions of (5.13), (5.14), (5.15), (5.17) are also trivial. Thus in any case $f_2 = 0$.

From $f_2 = 0$ and $f'_2 = -f_2$ we know that for every identity of type $[3, 3, 2, 1]$ which is $*$-skew symmetric in the variables of degree 3 the only nonzero terms must end in $x_1x_2$ or $x_2x_1$. Since $f^*$ is such an identity, its terms end in $x_1x_2$ or $x_2x_1$. This implies that the terms of $f$ must start with $x_1x_2$ or $x_2x_1$. Thus
\[
f = [x_1, x_2][g(x_1, x_2)]
\tag{5.18}
\]
Hence, the polynomial $g(y, x_1, x_2, z)$ is an identity of $H_2$ of type $[2, 1^3]$ and has the same $*$-symmetry as $f$. Since $f$ is skew symmetric in $x_1, x_2$, $g$ is also skew symmetric in $x_1, x_2$ from (5.18). By $f_1(x_1, x_2, y, x_1) = 0$ and (5.18), $g$ is skew in $x_1, z$. Thus $g$ is skew symmetric in all variables of degree 1. Therefore, if $f^* = -f$ then
\[
g = \lambda_1(L_1 - L_4) + \lambda_2(L_2 - L_3) + \lambda_3(L_5 - L_6) + \lambda_4(L_7 - L_7),
\]
where $\{L_i\ i = 1, \ldots, 7\}$ is the basis of $VIN_3[2, 1^3]$ given in Proposition 3.1.4. If $f^* = f$ then
\[
g = \lambda_1(L_1 + L_4) + \lambda_2(L_2 + L_3) + \lambda_3(L_5 + L_6) + \lambda_4(L_7 + L_7).
\]

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Since \( f^\#(z) \) has no terms which end in \( x_1 y^2 \) if \( f^* = \pm f \) (for the case where \( f^* = -f \) see the proof of Lemma 5.2), the monomials

\[
x_1 x_2 y^2 z x_1 x_2 x_1, \quad x_1 x_2 x_1 y^2 z x_2 x_1 x_2
\]

have coefficients zero in \( f \). Thus \( y^2 z x_1 x_2, x_1 y^2 z x_2 \) have coefficients zero in \( g \) if \( g^* = \pm g \). This implies that

\[
\lambda_1 + \lambda_3 + \lambda_4 = 0, \quad \lambda_3 + 2\lambda_4 = 0.
\]

(5.19)

On the other hand, substituting

\[
e_{[12]} + e_{[24]}, \quad e_{[23]}, \quad e_{[34]}, \quad e_{[14]}
\]

for \( x_1, x_2, y, z \) and considering the coefficients of \( e_{32} \) and \( e_{13} \) in \( f \) respectively yields, if \( f \) is skew symmetric,

\[
\lambda_1 + \lambda_2 + \lambda_3 = 0, \quad -2\lambda_1 - \lambda_3 + 2\lambda_4 = 0,
\]

(5.20)

and if \( f \) is symmetric,

\[
\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 = 0, \quad 2\lambda_2 - \lambda_3 + 2\lambda_4 = 0.
\]

(5.21)

In any case, the solution is trivial. Hence \( f = 0 \) identically. Thus we have shown

**Proposition 5.2.9** There is no identity of type \([3,3,2,1]\) on \( H_4 \).

Let \( f(x_1, x_2, y_1, y_2, y_3) \in VIN[3, 3, 1, 1, 1] \). Then

\[
f(x_1, x_2, y_1, y_2, y_3) = \sum_{\sigma \in S_2} (-1)^{\sigma} f_1(x_{\sigma(1)}, x_{\sigma(2)}, y_1, y_2, y_3) x_{\sigma(1)} x_{\sigma(2)} +
\]

\[
\sum_{(1,2,3)} (-1)^{\sigma} f_2(x_{\sigma(1)}, x_{\sigma(2)}, y_1, y_2) x_{\sigma(2)} y_3 +
\]

\[
\sum_{(1,2,3)} (-1)^{\sigma} f_3(x_{\sigma(1)}, x_{\sigma(2)}, y_1, y_2) y_3 x_{\sigma(2)} +
\]

\[
\sum_{(1,2,3)} f_3(x_1, x_2, y_1)[y_2, y_3] +
\]

\[
\sum_{(1,2,3)} (-1)^{\sigma} f_4(x_{\sigma(1)}, x_{\sigma(2)}, y_1, y_2, y_3) x_{\sigma(1)} x_{\sigma(2)}^2.
\]

(5.22)
where $(1, 2, 3)$ denotes the cyclic permutations of $1, 2, 3$. Since $V IN[4, 3, 1, 1] = 0$, $\Delta^1(y_3, x_1)f$ is zero identically. It implies that

$$\begin{align*}
f_2 - \Delta(y_3, x_1)f_1 &= 0, \\
f_2' + \Delta(y_3, x_1)f_1 &= 0, \\
f_2 + f_2' - \Delta(y_3, x_1)f_4 &= 0, \\
f_3 - \Delta(y_3, x_1)f_4' &= 0.
\end{align*}$$  \tag{5.23}

From (5.22), $f_4(x_1, x_2, y_1, y_2, y_3) \in V f_4[2, 1, 1, 1, 1]$ and is skew symmetric in the last three variables. Therefore it is a linear combination of the identities $g_1$ which come from $T_6$ and $Q$ by [MR] and skew symmetric in $y_i$'s. So

$$f_4 = \beta_2 T_5(x_2, y_1, y_2, y_3, x_1; x_1) + \sum_{i=1}^{3} (-1)^{i-1} \beta_i Q(x_1, x_2, y_i, x_1, y_1, \ldots, y_i, \ldots, y_3).$$

Hence $\Delta^1(y_3, x_1)f_4 = 0$ and $f_2 + f_2' = 0$. So, it suffices to determine $f_2$ and $f_4$.

Since $f_2(x_1, x_2, y_1, y_2) \in V IN_4[3, 2, 1, 1]$ there exist scalars $\alpha_i \in F$ such that

$$f_2 = \sum_{i=1}^{10} \alpha_i Q_i,$$

where $\{Q_1, \ldots, Q_{10}\}$ is the basis of $V IN_4[3, 2, 1, 1]$ given in Chapter 4.

**Lemma 5.2.3** The identity $f_2$ is zero identically.

**Proof.** It is sufficient to show the lemma by assuming that $f^* = \pm f$. Since $V f_4[3, 1, 1, 1] = 0$, there are no terms in $f$ and its Razmyslov transform $f^\#(y_i)$ which start with $x^2y_i$ or end in $y_ix^2$ by Proposition 2.1.1, Proposition 2.2.2 and the skew symmetry of $f$ in $y_i$'s. Thus the following monomials have coefficients zero in $f$:

$$x_1^2 y_1 x_2 x_1 x_2 x_1^2 y_3, \quad y_1 x_1 x_2 x_1^2 y_2 x_2 x_1^2 y_3, \quad y_3 x_2^2 y_2 x_2 x_1^2 x_2^2 y_1, \quad x_2 x_1^2 y_1 x_1 y_2 x_2 x_3.$$

It follows that

$$\begin{align*}
\alpha_3 + \alpha_4 + \alpha_6 + \alpha_8 + \alpha_9 &= 0, \\
-\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 + \alpha_7 + \alpha_8 &= 0, \\
\alpha_1 + \alpha_3 + 2\alpha_4 - \alpha_7 + \alpha_8 &= 0, \\
\alpha_2 + \alpha_3 - \alpha_4 - \alpha_5 - \alpha_7 &= 0.
\end{align*}$$

By the proof of Lemma 5.2.2 and considering the coefficient of the monomial $x_1 y_1 x_2^2 y_2 x_2 x_1^2 y_3$ give

$$2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 + 2\alpha_8 + \alpha_9 - \alpha_{10} = 0.$$
If $f^* = f$ then every monomial and its $\ast$ have the same coefficients in $f$. By considering monomials

\[
x_1y_1x_2x_1x_2y_2x_1x_2y_3, \ x_2^2y_1y_2x_1x_2y_3, \ x_2^2x_1^2y_1y_2x_1x_2y_3,
\]

\[
x_2y_1x_2^2y_2x_1x_2y_3, \ x_2y_1x_1x_2^2x_1y_2x_1x_2y_3, \ x_1^2x_2y_1x_2y_2x_1x_2y_3,
\]

\[
y_1x_1^2x_2^2y_2x_1x_2y_3, \ x_2y_1x_1x_2^2x_1y_2x_1x_2y_3, \ y_1x_1x_2x_1x_2y_2x_1x_2y_3,
\]

we have

\[
\alpha_2 + 4\alpha_4 - \alpha_6 + \alpha_7 + \alpha_9 = 0, \ \alpha_1 - \alpha_2 - \alpha_3 + \alpha_6 + \alpha_8 - \alpha_9 = 0,
\]

\[
2\beta_1 + \beta_2 - \alpha_1 - \alpha_5 + \alpha_7 = 0, \ \alpha_4 + \alpha_5 - \alpha_6 - \alpha_8 - \alpha_{10} = 0,
\]

\[
\alpha_2 - \alpha_4 - \alpha_5 + \alpha_7 = 0, \ -\alpha_1 + \alpha_2 + \alpha_3 - \alpha_6 - \alpha_8 + \alpha_9 = 0,
\]

\[
\alpha_1 - 2\alpha_2 - \alpha_3 - 3\alpha_4 + \alpha_8 - \alpha_7 + \alpha_8 - \alpha_{10} = 0,
\]

\[
\alpha_2 - \alpha_4 - \alpha_5 + \alpha_7 = 0, \ \alpha_4 = 0,
\]

\[
\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_5 - 2\alpha_7 = 0.
\]

The solution of the equations above is $\alpha_i = 0, i = 1, \ldots, 10$ and $\beta_2 = -2\beta_1$.

If $f^* = -f$ then considering the monomials

\[
x_1y_1x_2x_1x_2y_2x_1x_2y_3, \ x_1^2x_2^2y_1y_2x_1x_2y_3, \ x_2^2x_1^2y_1y_2x_1x_2y_3,
\]

\[
x_2y_1x_2^2x_2x_1x_2y_3, \ x_2y_1x_1x_2^2x_1y_2x_1x_2y_3, \ x_1^2x_2y_1x_2y_2x_1x_2y_3,
\]

\[
y_1x_1x_2x_1x_2y_2x_1x_2y_3, \ x_2^2x_1^2y_1x_1x_2y_2x_3, \ y_1x_1x_2x_1x_2y_2x_3
\]

yields

\[
\alpha_2 + 3\alpha_6 + \alpha_7 - \alpha_9 - 2\alpha_{10} = 0, \ \alpha_1 - \alpha_2 - \alpha_3 + \alpha_6 + \alpha_8 - \alpha_9 = 0,
\]

\[
2\beta_1 + \beta_2 + \alpha_1 + \alpha_5 - \alpha_7 = 0,
\]

\[
2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + 2\alpha_7 - \alpha_8 + \alpha_{10} = 0,
\]

\[
\alpha_2 + \alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7 - 2\alpha_10 = 0,
\]

\[
-\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 - \alpha_8 - \alpha_{10} = 0,
\]

\[
\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7 - \alpha_{10} = 0, \ \alpha_1 + \alpha_5 - \alpha_7 + \beta_2 + 2\beta_1 = 0,
\]

\[
\alpha_1 - \alpha_2 + \alpha_4 - \alpha_5 + 2\alpha_8 = 0, \ \alpha_4 = 0.
\]
The solution is also $\alpha_i = 0, i = 1, \ldots, 10$ and $\beta_2 = -2\beta_1$.

Now we can prove

**Proposition 5.2.10** The vector space $VL_4[3,3,1,1,1] = \{0\}$.

**Proof.** Since $f_2 = 0$, it follows that $f$ has no term which has $y_i$ either in one of the last two positions or in one of the first two positions. But the coefficient of the monomial $y_1 y_2 y_3 x_1 x_2 x_1^2 x_2^2$ in $f_4$ is $\pm 4\beta_1$, so $\beta_1 = 0$. Therefore

$$f = f_1(x_1, x_2, y_1, y_2, y_3)x_1 x_2 - f_1(x_2, x_1, y_1, y_2, y_3)x_2 x_1.$$  

Since $\Delta(x_2, x_1)f \in VL_4[4,2,1^3] = \{0\}$, $(\Delta(x_2, x_1)f_1(x_1, x_2, y_1, y_2, y_3))x_1 x_2 = 0$. So $\Delta(x_2, x_1)f_1(x_1, x_2, y_1, y_2, y_3) = 0$. Therefore $f_1$ is symmetric in $x_1, x_2$ by Theorem 1.1. So $f = f_1[x_1, x_2]$. Since $f^* = \pm f$, $f = [x_1, x_2]g[x_1, x_2]$. $g$ is multilinear, skew symmetric in the $x_i$'s, and in the $y_i$'s by the symmetry of $f$. In fact $g$ is alternating and therefore is a scalar multiple of $S_5$. This follows from

$$[x_1, x_2]g(x_2, x_1, y_1, y_2, x_1) = f_1(x_2, x_1, y_1, y_2, x_1) = f_2 = 0.$$  

Hence

$$f = \alpha[x_1, x_2]S_5(x_1, x_2, y_1, y_2, y_3)[x_1, x_2].$$

Since $f(x_1, x_2, y_1, y_2, 1) \in VL_4[3,3,1,1,1] = \{0\}$ and $S_5(x_1, x_2, y_1, y_2, 1) = S_4(x_1, x_2, y_1, y_2) \neq 0$ by [DR], $\alpha = 0$. So $f = 0$. \( \square \)

Let $f(x, y_1, y_2, y_3) \in VIN_4[3,2,2,2]$ then

$$f = \sum_{(1,2,3)} f_1(x, y_1, y_2, y_3)y_3 x + \sum_{(1,2,3)} f_1'(x, y_1, y_2, y_3)y_3 x + \sum_{(1,2,3)} f_2(x, y_1, y_2, y_3)y_2 y_3 + \sum_{(1,2,3)} f_3(y_1, y_2, y_3, x)y_3 x^2, \tag{5.24}$$

since $VL_5[3,2,1] = VIN_5[2,2,2] = 0$. 

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Since we have shown that $VI_4[4, 2, 2, 1] = 0$, $\Delta(y_3, x)f(x, y_1, y_2, y_3) \in VI_4[4, 2, 2, 1]$ implies that
\[
\Delta(y_3, x)f_2(x, y_1, y_2, y_3) = 0, \quad \Delta(y_3, x)f_2(x, y_3, y_1, y_2) = 0, \\
\Delta(y_3, x)f_1(x, y_3, y_1, y_2) + f_2(x, y_1, y_3, y_2) = 0, \\
f_1(x, y_1, y_2, y_3) + f_1'(x, y_1, y_2, y_3, x) + \Delta(y_3, x)f_3(y_1, y_2, y_3, x) + f_3(y_1, y_2, y_3, x)x + \Delta(y_3, x)f_3(y_2, y_3, y_1, x)y_1 + \Delta(y_3, x)f_3(y_2, y_3, y_1, x)y_2 = 0. \quad (5.25)
\]
by considering the sums of terms of $\Delta(y_3, x)f$ which end in $y_2y_3, y_1y_2, xy_2$ and $x^2$ respectively. Since $\Delta(y_3, y_1)f \in VIN_4[3, 3, 2, 1] = 0$, the sum of the terms which end in $y_1y_3$ in $\Delta(y_3, y_1)f$ is zero, which yields
\[
\Delta(y_3, y_1)f_2(x, y_2, y_1, y_3) = 0.
\]
So, $f_2$ is skew symmetric in the variables of degree one. Hence, there exist scalars $\alpha_i \in F$ such that
\[
f_2 = \sum_{i=1}^{10} \alpha_i Q_i,
\]
where $Q_1, \ldots, Q_{10}$ is the basis of $VIN_3[3, 2, 1, 1]$ given in Chapter 4. Since $f_3 \in VI_3[2, 2, 1, 1]$, there exists a scalar $\gamma \in F$ such that
\[
f_3(y_1, y_2, y_3, x) = \gamma Q(y_1, y_2, y_3, y_1, y_2, x).
\]
Next we show

**Lemma 5.2.4** The polynomials $f_2 = f_3 = 0$.

**Proof.** From $\Delta(y_3, x)f_2(x, y_1, y_2, y_3) = 0$, $\Delta(y_3, x)f_2(x, y_3, y_1, y_2) = 0$ we have
\[
2\alpha_4 - \alpha_3 = 0, \quad 2\alpha_1 + 2\alpha_7 - \alpha_8 = 0, \\
2\alpha_2 - \alpha_3 - 2\alpha_7 + \alpha_8 = 0, \quad \alpha_5 + 2\alpha_6 = 0. \quad (5.26)
\]
In the following we may assume that $f^* = \pm f$. From (5.24) the coefficient of the monomial $y_1^2y_2y_3y_5y_2$ is zero. So we have
\[
-\alpha_7 + \alpha_5 + \alpha_1 = 0. \quad (5.27)
\]

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Let $\alpha(w)$ denote the coefficient of the monomial $w$ in $f$. Then $\alpha(w) + s_1(w^*) = 0$, where $s = 1$ if $f^* = -f$ and $s = -1$ if $f^* = f$. Thus considering the monomials:

\[
\begin{align*}
xyzyxyyzyz, & \quad xyzyyzyzyz, & \quad xyzyyzyyzyz, & \quad xyzyyzyzyzyz, \\
yzyzyzyzyzy, & \quad yzyzyzyzyzyz, & \quad yzyzyzyzyzyzyz,
\end{align*}
\]

yields

\[
\begin{align*}
-\alpha_9 - \alpha_8 - \alpha_6 - \alpha_4 - \alpha_3 + s\gamma &= 0, \\
-2\alpha_9 - 2\alpha_2 - 2\alpha_3 + 2s\gamma &= 0, \\
\alpha_7 + \alpha_8 + \alpha_1 &= 0, \\
\alpha_7 + \alpha_4 + \alpha_3 + \alpha_2 - s(-\alpha_7 + \alpha_2 + \alpha_6 + \alpha_3 + \alpha_{10}) &= 0, \\
-2\alpha_7 + \alpha_3 - s(\alpha_{10} + \alpha_6 - \alpha_4 - \alpha_3) &= 0.
\end{align*}
\]

(5.28)

The solutions of (5.26), (5.27) and (5.28) is $\alpha_i = 0$ for $i \neq 9$ and $\alpha_9 + \gamma = 0$ when $s = \pm 1$ if $f^* = \mp f$.

Next we show that $\alpha_9 = \gamma = 0$. Since $VF_3[2^3, 1]$ is spanned by the identities which come from $Q$, $K$ and $T_0$ by substituting some variables, by the symmetry of $Q$, $K$ and $T_0$ the subspace consisting of the elements $g(x, y, z, w) \in VF_3[2^3, 1]$ and symmetric in $y, z$ is spanned by following 17 elements:

\[
\begin{align*}
Q(x^2, y, z, y, z, w), & \quad \sigma(Q(y^2, x, z, x, z, w)), & \quad \sigma(Q(y o x, z, y, z, x, w)), \\
\sigma(Q(y o x, z, z, z, y, w)), & \quad \sigma(Q(y o x, z, w, z, x, y)), & \quad \sigma(Q(y o z, x, y, x, w, z)), \\
Q(x o w, y, z, y, z, x), & \quad \sigma(Q(y o w, x, z, z, z, y)), & \quad \sigma(Q(y o w, x, z, x, z, w)), \\
Q(x, y, z, y, z, w)x, & \quad \sigma(Q(y, x, z, z, z, w)), & \quad \sigma(Q(y, x, z, x, z, w)y)), \\
\sigma(T_0(y o x, y, z, w, x, z)), & \quad K(x, y, z, y, z, w), & \quad \sigma(K(x, y, w, z, y, z, x)), \\
\sigma(K(x, y, w, z, z, x, y)), & \quad \sigma(K(x, y, w, z, x, z, y)),
\end{align*}
\]

where $\sigma := Id + (z, y)$. [Note $\sigma(K(y, w, z, z, y, z)) - \sigma(K(x, y, w, z, y, z, x)) = 0$. Let $y_i$ denote these identities. Then $f_1 = \sum_{i=1}^{17} \beta_i y_i$ for some $\beta_i \in F$. By the $*$-symmetry of $f$, considering the coefficients of the monomials

\[
\begin{align*}
yzyzyzyzyz, & \quad yzyzyzyzyzyz, & \quad yzyzyzyzyzyzyz, & \quad yzyzyzyzyzyzyz, \\
yzyzyzyzyzyz, & \quad yzyzyzyzyzyzyz, & \quad yzyzyzyzyzyzyzyz,
\end{align*}
\]

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\[ y_1 y_3 y_2 y_1 x y_2 x y_3, \quad x y_1 y_3 y_2 y_1 x y_2 y_3, \quad y_3 y_2 y_1 x y_1 x y_3, \\
y_1 x y_2 y_3 y_1 x y_2 x y_3, \quad x y_2 y_2 y_1 y_3 y_1 x y_3, \quad y_2 y_1 x y_3 y_1 x y_2 x y_3, \\
y_3 y_2 x y_1 y_3 y_1 x y_3, \quad y_1 y_1 y_2 y_3 x y_1 x x y_3, \quad y_1 x y_2 x y_2 y_1 x y_3 x y_3, \]

yields
\[
\begin{align*}
-\beta_{17} - \beta_{15} - \beta_{11} + \beta_{10} - \beta_3 - \beta_7 &= 0, \\
\beta_{7} - \beta_{17} + \beta_{8} + \beta_{12} - \beta_{6} + \beta_{3} - 2\beta_{11} + 2\beta_{5} + \beta_{4} &= 0, \\
\beta_{6} - \beta_{16} - \beta_{5} - \beta_{13} + \beta_{12} + \beta_{1} - \beta_{4} - \beta_{3} &= 0, \\
\beta_{16} - 2\beta_{8} - \beta_{13} + \beta_{6} + \beta_{13} + \beta_{11} - \beta_{10} - \beta_{3} &= 0, \\
o_{9} + s(-\beta_{12} + \beta_{17} + \beta_{13} - \beta_{6} - \beta_{5} + \beta_{4} + \beta_{3} - \beta_{2}) &= 0, \\
\beta_{16} + 2\beta_{13} + \beta_{5} - \beta_{12} + \beta_{3} &= 0, \\
2\beta_{12} - 2\beta_{11} + s(\beta_{4} + \beta_{17} - 2\beta_{6} + \beta_{16} + \beta_{12} - \beta_{3} - \beta_{11} + \beta_{8} + \beta_{7}) &= 0, \\
\beta_{6} - \beta_{5} + \beta_{13} + \beta_{1} &= 0, -\beta_{12} + \beta_{11} &= 0, \\
-\beta_{4} + \beta_{14} + \beta_{13} + \beta_{8} + \beta_{1} - 2\beta_{5} &= 0, \quad \beta_{17} - \beta_{15} + \beta_{11} - \beta_{10} + \beta_{8} - \beta_{7} = 0, \\
-\beta_{4} + 2\beta_{3} - \beta_{14} + \beta_{13} - \beta_{8} + s(-\beta_{12} + \beta_{11}) &= 0.
\end{align*}
\]

Since \( \Delta(y_3, x)f = 0 \) implies that \( \Delta(y_3, x)f_1 = 0 \), the monomials
\[ y_2 y_1 x y_2 x y_3, \quad x y_1 x y_2 y_1 y_2 x, \quad y_1 x y_1 y_2 y_2 x, \]
have coefficients 0 in \( f_1(x, y_1, y_2, x) \). Since \( f \) has no terms ending in \( y_1^2 \) the monomials
\[ y_2 y_2 x y_1 y_1 y_3, \quad y_2 y_2 y_1 x y_3 y_1 x, \]
have coefficients 0 in \( f_1 \). Thus we have
\[
\beta_{14} - \beta_{8} - \beta_{5} + \beta_{3} + 2\beta_{15} = 0, \quad 2\beta_{8} + 2\beta_{3} - 2\beta_{2} = 0, \quad 2\beta_{7} + \beta_{1} = 0, \\
-\beta_{2} + \beta_{16} + \beta_{13} - \beta_{11} + \beta_{4} = 0, \quad \beta_{17} + \beta_{11} + \beta_{2} + \beta_{6} = 0.
\]

Since we have \( o_{9} + s\gamma = 0 \) already, solving these 2 systems yields \( o_{9} = \gamma = 0 \) when \( s = \pm 1 \).
Thus \( f_2 = f_3 = 0 \).  \( \blacksquare \)

Now we can show
Proposition 5.2.11 The vector space $VIN_4[3, 2, 2, 2] = \{0\}$.

Proof. From $f_2 = f_3 = 0$ and (5.2.11) we have $f_1 + f'_1 = 0$ and $\Delta(y_3, x) f_1(x, y_3, y_1, y_2) = 0$. We also have $f_1(x, y_1, y_2, y_3) \in VIN_3[2, 2, 2, 1]$ and is symmetric in the variables $y_1, y_2$. The fact that $\Delta(y_3, x) f_1(x, y_3, y_1, y_2) = 0$ implies that $f_1$ is symmetric in $x, y_3$ also by Theorem 1.1. Thus $f_1 \in VIN_3[2, 2, 2, 1]$ and

$$f = \sum_{(1,2,3)} f_1(x, y_1, y_2, y_3)[x, y_3]$$

if $f^* = \pm f$. So, $f$ has no terms which start with one of $y_i y_j, x^2 y_i, y_i^2 x, y_i^2 y_j$, by the symmetry of $f$. Hence $f_1$ has no such terms either. This means that only terms of the form $x^i z_j$ or $y_i x^j$ occur in $f_1$. But any monomial of the form $y_i y_j$ has coefficient 0 in $f_1$, and $f_1$ is symmetric in $y_i, x$. Hence the monomials of the form $y_i x^j$ and $x y_i$ also have coefficients zero in $f_1$. Thus $f_1 = 0$, completing the proof.

Proposition 5.2.12 The vector space $VIN_4[3, 2, 2, 1, 1] = \{0\}$.

Proof. Let $f(x, y_1, y_2, z_1, z_2) \in VIN_4[3, 2, 2, 1, 1]$ then

$$f = (Id + (y_1, y_2))(f_1(x, y_1, y_2, z_1, z_2)x y_2 + f'_1(x, y_1, y_2, z_1, z_2)y_2 x +$$
$$f_3(x, y_1, y_2, z_1, z_2)y_1 y_2 + f_0(x, y_1, y_2, z_1, z_2)y_2 x^2 + f_5(x, y_1, z_1, z_2)y_0 y_2^2 +$$
$$(Id - (z_1, z_2))(f_2(x, y_1, y_2, z_1)z_2 x + f_4(x, y_1, y_2, z_1)z_2 y + f_7(x, y_1, y_2, z_1)z_2 y^2 +$$
$$(Id + (y_1, y_2))f_4(x, y_1, y_2, z_1)y_1 z_2 + f_4'(x, y_1, y_2, z_1)z_2 y^2 +$$
$$f_5(x, y_1, y_2)[z_1, z_2],$$

where $Id$ denotes the identity map, $(y_1, y_2)$ denotes the operator which permutes $y_1, y_2$ and $(z_1, z_2)$ denotes the operator which permutes $z_1, z_2$.

Since there are no identities of $H_4$ of degree 9 which are lower than $[3, 2, 2, 1, 1]$, $\Delta(z_1, x)f = \Delta(z_1, y)f = \Delta(y_1, x)f = 0$ imply that

$$f_4(x, y_1, y_2, z_1) = \Delta(y_2, x)f_4(x, y_1, y_2, z_1),$$
$$f_5(x, y_1, y_2) = f_4(x, y_1, y_2, y_2),$$
$$f_7(x, y_1, y_2, z_1) = f_5(x, y_1, y_2, z_1),$$
$$f_4(x, y_1, y_2, z_1) = f_5(x, y_2, y_1, z_1).$$

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Thus we need only determine $f_2, f_3, f_6, f_8$. As before we may assume that $f^* = \pm f$. $f_2(x, y_1, y_2, z_1) \in V_{I_3}[z^3, 1]$ and symmetric in $y_1, y_2$. Since $V_{I_3}[z^3, 1]$ is spanned by the identities which come from $Q, K$ and $T_6$ by substituting some variables, the subspace consisting of the elements $g(x, y, z, w) \in V_{I_3}[z^3, 1]$ and symmetric in $y, z$, is spanned by $g_i, i = 1, \ldots, 17$, which were given in the proof of the Lemma 5.2.4:

\[
\begin{align*}
\sigma(Q(y^2, x, z, x, z, w)), & \sigma(Q(x^2, y, z, x, w)), \sigma(Q(y o x, z, w, z, y, z)), \\
\sigma(Q(y o x, z, y, z, x, w)), & \sigma(Q(y o x, z, z, x, y, w)), \sigma(Q(y o x, z, z, x, w)), \\
\sigma(Q(y o w, x, z, z, y, x)), & Q(x o w, y, z, y, z, x), \sigma(y Q(y, x, z, x, z, w)), \\
\sigma(K(y, x, z, z, w, y)), & \sigma(K(y, z, x, z, x, w, y)), \sigma(K(y, x, y, z, w, x)), \\
\sigma(T_6(y o x, y, z, w, x)), & K(x, y, x, z, z, w, x), \sigma(K(y, w, z, z, z, y, x)).
\end{align*}
\]

where $\sigma := Id + (y, z)$. Since $g_{15} + g_{18} = 0$, we can forget $g_{18}$ and let $f_2 = \sum_{i=1}^{17} \alpha_i g_i$. Similarly, since $f_6(x, y_1, y_2, z_1, z_2)$ is of type $[2, 1^4]$ and is skew symmetric in $z_1$ and $z_2$

\[
f_6(x, y_1, y_2, z_1, z_2) = \beta_1 T_6(x, y_1, y_2, z_1, z_2, y_1) + \beta_2 Q(y_1, y_2, x, y_1, z_1, z_2) + \\
\beta_3 (Q(y_1, x, z_1, y_1, y_2, z_2) + Q(y_1, x, z_2, y_1, y_2, z_1)).
\]

We also have

\[
f_6(x, y_1, z_1, z_2) = \gamma Q(x, y_1, z_1, x, y_1, z_2),
\]

\[
f_6'(x, y_1, y_2, z_1) = -f_4(x, y_1, y_2, z_1) - \gamma Q(x, y_1, z_1, x, y_1, y_2).
\]

By $f^* = \pm f$, monomials which start with $y_1^2y_2$ have coefficients 0 and $\alpha(w) + s\alpha(w^*) = 0$ for any monomial $w$, where $s = \pm 1$ if $f^* = \mp f$. Considering the 48 monomials we have a system (I) of linear equations.

If $f^* = f$ then its Razmyslov transform $f^*(z_2)$ is also an identity. So the monomial $z_2z_1y_1^2y_2^2xxx$ must have coefficient 0 in $f^*(z_2)$ by Lemma 2.1.1. Thus the monomial $z_1y_1y_1y_2y_2xxx$ has coefficient 0 in $f$. Since $f^*(z_1)$ is another identity of $H_4$ of type $[3, 2, 2, 1, 1]$, from (5.29) the monomials of the form $- - - - - - y_2y_1y_1$, must have coefficients 0. Since $f^* = f$ implies that $(f^*)^* = f^*$, the monomial $y_1y_1y_2 - - - - - -$
has coefficient 0 in \( f^\#(z_1) \). This fact implies that the monomial \( xz_1y_1y_1y_2y_2x_2x_2 \) has coefficient 0 in \( f \). We get 2 equations. The solution of (I) and these 2 equations yields \( f_0 = f_1 = f_6 = 0 \).

Next, since \( V I_4[3, 3, 1, 1, 1] = \{0\} \), any identity of type \([3, 2, 2, 1, 1]\) must be symmetric in the variables of degree 2. So \( \Delta(y_2, z_1) f(x, y_1, y_2, z_1, z_2, w) \) is symmetric in \( y_1, z_1 \) and \( \Delta(y_2, z_1) f_2(x, y_1, y_2, z_1) \) is symmetric in \( y_1, z_1 \). Considering the monomial \( xz_1y_1y_1y_2y_2x \) yields another equation. By solving these 3 equations and (I), \( f_2 = 0 \). Therefore

\[
f_i = 0, \quad i = 2, 4, 5, 6, 7, 8.
\]

This implies that \( f_1^* = -f_1 \) and \( f_3 \) is skew symmetric in \( y_1, y_2, z_1, z_2 \). Thus (5.29) becomes

\[
f = (Id + (y_1, y_2))(f_1[x, y_2] + f_3(x, y_1, y_2, z_1, z_2)y_1y_2).
\]

From this equation we know that \( f \) has only terms which end in \( y_1y_2 \), \( xy_1y_2 \), or \( y_1x \). Since \( f \) is *-symmetric, \( f_3 \) has only terms which start with \( y_1y_2 \), \( xy_1y_2 \), or \( y_1x \). But \( f_3 \) is skew symmetric in \( y_1, y_2, z_1, z_2 \), so \( f_3 = 0 \).

If \( f^* = -f \) then solving (I) yields \( f_i = 0, \) \( i = 6, 7, 8 \). This implies that \( f_i^* = -f_i \) for \( i = 1, 2, 4 \) and \( f_3(x, y_1, y_2, z_1, z_2) \) is skew symmetric in \( y_1, y_2 \). Since \( f_3(x, y_1, y_2, z_1, z_2) \) is of type \([3, 1^4]\) and skew symmetric in \( y_1, y_2 \) and in \( z_1, z_2 \), it is a linear combination of 23 identities \( h_i \), which come from \( \mathbb{Q}, T_0 \) and \( K \). Let \( f_3 = \sum_{i=1}^{23} \lambda_i h_i \). Then from \( f^* = -f \), \( f_3(x, y_2, y_1, y_1) + f_4(x, y_1, y_2, z_1) = 0 \) and monomials which start with \( x^2z_1 \) have coefficients 0 we have a system of linear equations (II). Unfortunately, the systems (I) and (II) are not enough. We have to find more equations.

From the results in Chapter 4, any identity \( p(x, y_1, y_2, z_1, z_2) \) of type \([2, 2, 1, 1, 1]\) which is skew symmetric in \( z_1, z_2 \) is a linear combination of 57 such identities \( p_i \), which come from \( \mathbb{Q}, T_0, K \) and \( T_0(x_1, \ldots, x_5; [x_6, x_7]) \). So let \( f_1 = \sum_{i=1}^{57} \delta_i p_i \). Then from (5.29)

\[
\begin{align*}
f_2(x, y_1, y_2, z_1) &= f_1(x, y_1, y_2, z_1, y_2), \\
f_3(x, y_1, y_2, z_1, z_2) &= \Delta(y_1, x) f_1(x, y_1, y_2, z_1, z_2), \\
f_4(x, y_1, y_2, z_1) &= -\Delta(y_2, x) f_2(x, y_1, y_2, z_1), \\
f_5(x, y_1, y_2) &= -f_4(x, z_1, y_2, y_2).
\end{align*}
\]

Using these new definitions of \( f_i \) and \( f^* = -f \) we get a system (III). Solving (I), (II) and (III) yields \( f_2 = f_3 = 0 \). Thus when \( f^* = \pm f \) we have

\[
f = (Id + (y_1, y_2))f_1(x, y_1, y_2, z_1, z_2)[x, y_2].
\]

(5.30)
Since $\Delta(y_1, x)f = 0$, $\Delta(y_1, x)f_1 = 0$. So $f_1$ is symmetric in $x, y_1$. Similarly, it can be shown that $f_1$ is skew in $y_2, z_1$. Thus $f_1 \in VIN_5[2, 2, 1^2]$. Since $f^* = \pm f$, from (5.30) and the symmetry of the variables of $f_1$,

$$f = (Id + (y_1, y_2))[x, y_1]g(x, y_1, y_2, z_1, z_2)[x, y_2],$$

where $g$ is multilinear and skew symmetric in the variables $y_2, z_1, z_2$. Again, from $\Delta(y_2, x)f = 0$ and $\Delta(y_2, y_1)f = 0$ we have $g$ is skew symmetric in $x, y_2, y_1, y_2$. Thus $g$ is alternating. So, there exists a scalar $\alpha \in F$ such that $g = \alpha S_5(x, y_1, y_2, z_1, z_2)$. So

$$f(x, y_1, y_2, z_1, z_2) = \alpha(Id + (y_1, y_2))[x, y_1]S_5(x, y_1, y_2, z_1, z_2)[x, y_2].$$

Since $S_5(x, y_1, y_2, z_1, 1) = S_4(x, y_1, y_2, z_1)$ by [DR],

$$f(x, y_1, y_2, z_1, 1) = \alpha(Id + (y_1, y_2))[x, y_1]S_4(x, y_1, y_2, z_1, 1)[x, y_2]
= \alpha(Id + (y_1, y_2))[x, y_1]S_4(x, y_1, y_2, z_1)[x, y_2].$$

$f(x, y_1, y_2, z_1, 1)$ is an identity of $H_4$ of degree 8 which has type $[3, 2, 2, 1]$. So $f(x, y_1, y_2, z_1, 1) = 0$. Hence $\alpha = 0$ and $f = 0$. □

**Proposition 5.2.13** The vector space $VIN_4[2^4, 1] = \{0\}$.

**Proof.** Let $f(x_1, x_2, x_3, x_4, y) \in VIN_4[2^4, 1]$ then by the symmetry of the variables $x_i$'s of $f$,

$$f = f_1(x_1, x_2, x_3, x_4, y)x_3x_4 + f_1(x_1, x_2, x_3, x_4, y)x_4x_3 + \ldots$$

$$+ f_2(x_1, x_2, x_3, x_3)x_3y + f_2'(x_1, x_2, x_3, x_4)yx_4 + \ldots$$

$$f_3(x_2, x_3, x_4, y)x_4x_2^2 + \ldots$$

(5.31)

$f_1(x_1, x_2, x_3, x_4, y)$ is an identity of $H_3$ of type $[2^2, 1^2]$ and is symmetric in $x_1, x_2$. So from the results in Chapter 4, $f_1$ is a linear combination of $g_i(x_1, x_2, x_3, x_4, y)$ which is symmetric in $x_1, x_2$ and comes from one of $Q, T_6, K$ and $T(x_1, x_2, x_3, x_5, [x_6, x_7])$ by substituting some variables or multiplying by a variable. From the symmetry of the variables of $Q, T_6$ and $K$, there are only 47 such $g_i$'s. So $f_1 = \sum_{i=1}^{47} a_i g_i$. $\Delta(y, z_3)f = 0$ implies that

$$f_2(x_1, x_2, x_3, x_4) = f_1(x_1, x_2, x_4, x_3, x_3).$$

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Since $f_3 \in V I d[2,1,1]$, there exists a scalar $\beta \in F$ such that $f_3 = \beta Q(x_2, x_3, x_4, x_2, x_3, y)$.

As before we may assume that $f^* = \pm f$. So from the $\ast \ast$ symmetry we have a system (I). From (5.31) and the symmetry of the variables of $f$, $f_2$ is symmetric in its variables of degree 2. This yields a system (II). Finally the system (III) is obtained from

$$
\Delta(y,x_1) f_1(x_1, x_2, x_3, x_4, y) = 0.
$$

$$
\Delta(x_4, x_1) f_1(x_1, x_2, x_3, x_4, y) = 0.
$$

$$
\Delta(x_3, x_1) f_1(x_1, x_2, x_3, x_4, y) = 0.
$$

When $f^* = -f$, the solution of (I), (II) and (III) is trivial. So $f_1 = 0$, hence $f = 0$. If $f^* = f$, let $g(x_1, x_2, x_3, x_4, z, y) := \Delta(x_3, z)f$. Then $g$ is an identity of type $[2^3, 1^3]$ and is $\ast \ast$-symmetric. So the Razmyslov transform $g^\#(z)$ is an identity of $H_4$. Hence $g^\#(z)(x_1, x_2, x_3, z, z, y)$ is an identity of type $[2^4, 1]$ and is $\ast \ast$-symmetric. Thus from (5.31) and the fact that every identity of $H_4$ of type $[2^4, 1]$ has no terms which end in $yx_1 y$, $g^\#(z)$ has no terms which start with $x_1^2 y$. Particularly, the monomial $w := x_1 x_1 y x_2 x_4 x_2 x_4 x_4$ has coefficient 0 in $g^\#(z)(x_1, x_2, x_3, z, z, y)$. Since $w$ comes from the monomials $x_1 x_1 y x_2 x_4 x_2 x_4 x_3$ and $x_1 x_1 y x_2 x_4 x_2 x_3 x_4$, the sum of the coefficients in $g^\#(z)(x_1, x_2, x_3, z, y)$ of the monomials $x_1 x_1 y x_2 x_4 x_2 x_4 x_3$ and $x_1 x_1 y x_2 x_4 x_2 x_3 x_4$ is 0. So by the definition of the Razmyslov transformation, the sum of the coefficients of the monomials $x_1 x_1 y x_2 x_4 x_2 x_3 x_4$ and $x_1 x_1 y x_2 x_4 x_2 x_3 x_4$ in $g(x_1, x_2, x_3, z, y) = \Delta(x_3, z)f$, this implies that, in $f$, the sum of the coefficients of the monomials

$$
x_4 x_3 z_1 x_1 y x_2 x_4 x_2, \quad z_1 x_1 x_1 y x_2 x_4 x_2 x_4 x_4,
$$

$$
x_4 z_3 z_1 x_1 y x_2 x_4 x_2, \quad x_3 z_1 x_1 y x_2 x_4 x_2 x_4,
$$

is 0. This yields a system of equations (IV). Solving (I), ... (IV) we get $f_1 = 0$. Hence $f = 0$. ·

**Proposition 5.2.14** The vector space $V I N[2^3, 1^3] = \{0\}$.

**Proof.** Let $f(x_1, x_2, x_3, y_1, y_2, y_3) \in VIN_4[2^3, 1^3]$ then

$$
f = f_1(x_1, x_2, x_3, y_1, y_2, y_3)x_2 x_3 + f_1(x_1, x_2, x_3, y_1, y_2, y_3)x_3 x_2 + \cdots
$$

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\begin{align*}
+f_2(x_1, x_2, x_3, y_1, y_2)y_3x_2 + f_3(x_1, x_2, x_3, y_1, y_2)x_2y_2 + \cdots \\
+f_2(x_1, x_2, x_3, y_1, y_2)y_3x_2 + f_3(x_1, x_2, x_3, y_1, y_2)x_2y_2 + \cdots \\
+f_4(x_2, x_3, y_1, y_2, y_3)x_3x_2^2 + \cdots \\
+f_5(x_2, x_3, y_1, y_2, y_3)y_3x_2^2 + \cdots,
\end{align*}

(5.32)

where \(f_i\) is an identity of \(H_3\) for \(i = 1, \ldots, 5\).

Since \(\Delta(y_1, x_2)f \in V I N_4[3, 2^2, 1^2] = \{0\}\), by considering the sum of the terms of \(\Delta(y_3, x_2)f\) which end in \(x_2x_3, x_3x_2\) and \(x_3x_2^2\) respectively we get

\[
\begin{align*}
f_1(x_1, x_2, x_3, y_1, y_2, x_2) + f_2(x_1, x_2, x_3, y_1, y_2) &= 0, \\
f_1(x_1, x_2, x_3, y_1, y_2, x_2) + f_2(x_1, x_2, x_3, y_1, y_2) &= 0, \\
f_4(x_2, x_3, y_1, y_2, x_3) + f_5(x_2, x_3, y_1, y_2) &= 0.
\end{align*}
\]

(5.33)

Similarly from \(\Delta(y_2, x_3)f = 0\) we have

\[
f_2(x_1, x_2, x_3, y_1, x_3) - f_3(x_1, x_2, x_3, y_1) = 0.
\]

(5.34)

Thus \(f\) is determined by \(f_1\) and \(f_4\).

It is known by \([MR]\) that every identity of degree 6 of \(H_3\) comes from \(T_6\) and \(Q\). So

\[
\begin{align*}
f_4 &= \gamma_1 T_6(x_2, x_3, y_1, y_2, y_3; x_2) + \\
&\quad \gamma_2 (Q(x_2, x_3, y_1, x_2, y_2, y_3) + Q(x_2, x_3, y_2, x_2, y_3, y_1) + Q(x_2, x_3, y_3, x_2, y_1, y_2))
\end{align*}
\]

by the symmetry of the variables of \(T_6\) and \(Q\).

Since \(f_1(x_1, x_2, x_3, y_1, y_2, y_3)\) is skew symmetric in the \(y_i\)'s, it suffices to determine the coefficients of the monomials in which the \(y_i\) occur in the numerical order. Let \(w_1, \ldots, w_{20}\) denote the monomials

\[
x_1x_1y_1y_2y_3, x_1y_1x_1y_2y_3, x_1y_1y_2x_1y_3, \ldots, y_1y_2y_3x_1x_1,
\]

respectively. Thus by inserting one \(x_2\) and one \(x_3\) into each \(w_i\) will give all monomials which we need to determine the coefficients. Let \(w_{ij} (w_{ijk})\) denote the monomial obtained from \(w_i\) (\(w_{ij}\)) by inserting \(x_2\) (\(x_3\)) into the position \(j\) (\(k\)). e.g \(w_{11} = x_2x_1x_1y_1y_2y_3\) and \(w_{114} = x_2x_1x_1x_3y_1y_2y_3\). Let \(\alpha_{ijk}\) denote the coefficient of \(w_{ijk}\) in \(f_1\), then we have
Lemma 5.2.5 For an arbitrary but fixed pair of $j, k$ if $\alpha_1 := \alpha_{ijk}$ then

$$
\begin{align*}
\alpha_1 + \alpha_2 + \alpha_5 &= 0, \quad \alpha_1 - \alpha_3 - \alpha_6 = 0, \\
\alpha_1 + \alpha_4 + \alpha_7 &= 0, \quad -\alpha_2 - \alpha_3 + \alpha_8 = 0, \\
\alpha_2 - \alpha_4 + \alpha_9 &= 0, \quad \alpha_3 + \alpha_4 + \alpha_{10} = 0.
\end{align*}
$$

Proof. Since $\Delta(y_r, x_1)f_1 \in VIN_4[3, 2^2, 1^2] = \{0\}$, $\Delta(y_r, x_1)f_1 = 0$ from (5.32). Thus from $\Delta(y_1, x_1)f_1 = 0$, for an arbitrary but fixed pair of $j, k$ the monomial obtained from $x_1x_1x_1y_2y_3$ by inserting $x_2, x_3$ at positions $j, k$ respectively has coefficient zero in $\Delta(y_1, x_1)f_1$. Since $\Delta(y_1, x_1)$ acts on $f_1$ without changing $x_2, x_3$ we can forget $x_2, x_3$ and simply say $x_1x_1x_1y_2y_3$ has coefficient 0 in $\Delta(y_1, x_1)f_1$. Since $x_1x_1x_1y_2y_3$ has preimage $w_1, w_2, w_3$ under $\Delta(y_1, x_1)$, $\alpha_1 + \alpha_2 + \alpha_5 = 0$.

Similarly $\Delta(y_1, x_1)f_1 = 0$ implies

$$
-\alpha_2 - \alpha_3 + \alpha_8 = 0, \quad \alpha_2 - \alpha_4 + \alpha_9 = 0, \quad \alpha_3 + \alpha_4 + \alpha_{10} = 0.
$$

Since $\Delta(y_2, x_1)f_1 = 0$ and $\Delta(y_3, x_1)f_1 = 0$ we have

$$
\alpha_1 - \alpha_3 - \alpha_6 = 0, \quad \alpha_1 + \alpha_4 + \alpha_7 = 0,
$$

completing the proof of the Lemma. \qed

Lemma 5.2.6 For an arbitrary but fixed $k$ if $\alpha_{ij} := \alpha_{ijk}$ then

$$
\begin{align*}
\alpha_{42} &= -\alpha_{45} - \alpha_{46} - \alpha_{10} - \alpha_{32} - \alpha_{15}, \\
\alpha_{21} &= -\alpha_{34} - \alpha_{35} + \alpha_{32} + \alpha_{15}, \\
\alpha_{44} &= -\alpha_{36} - \alpha_{34} - \alpha_{35} - \alpha_{46} - \alpha_{45}, \\
\alpha_{13} &= \alpha_{32} + \alpha_{15} + \alpha_{25} + \alpha_{33} - \alpha_{12}, \\
\alpha_{22} &= \alpha_{34} - \alpha_{32} - \alpha_{15} - \alpha_{14} + \alpha_{35}, \\
\alpha_{41} &= \alpha_{45} + \alpha_{46} + \alpha_{32} + \alpha_{15}, \\
\alpha_{31} &= -\alpha_{32} - \alpha_{15}, \\
\alpha_{24} &= -\alpha_{25} - \alpha_{23} - \alpha_{33} - \alpha_{34} - \alpha_{35}, \\
\alpha_{11} &= -\alpha_{32} - \alpha_{15} - \alpha_{25} - \alpha_{33}, \\
\alpha_{43} &= \alpha_{45} - \alpha_{26} + \alpha_{46} - \alpha_{25} - \alpha_{33}.
\end{align*}
$$

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Proof. Since

\[ f_1(x_1, x_2, x_3, y_1, y_2, x_2) = -f_2(x_1, x_2, x_3, y_1, y_2) \]

from (5.33) and since \( f_2 \) is symmetric in its variables of degree 2 from (5.32) and \( f \) is symmetric in the variables of degree 2, \( f_1(x_1, x_2, y_1, y_2, x_2) \) must be symmetric in \( x_1, x_2 \). So \( w_{ijk}(x_1, x_2, x_3, y_1, y_2, x_2) \) and \( w_{ijk}(x_2, x_1, x_3, y_1, y_2, x_1) \) have the same coefficients in \( f_1(x_1, x_2, x_3, y_1, y_2, x_2) \) for arbitrary \( i, j, k \), especially for a fixed \( k \) and arbitrary pair of \( i, j \). So for an arbitrary but fixed \( k \) if we let

\[ p(x_1, x_2, y_1, y_2, y_3) = \sum_{i,j,\sigma \in S_3} (-1)^{\sigma} a_{ij} w_{ij}(x_1, x_2, y_1(\sigma), y_2(\sigma), y_3(\sigma)), \]

then \( p(x_1, x_2, y_1, y_2, x_2) \) is symmetric in \( x_1, x_2 \). Hence \( w_{ij}(x_1, x_2, y_1, y_2, x_2) \) for \( i = 1, \ldots, 4 \) and \( j = 1, \ldots, 5 \) have coefficients 0 in \( p(x_1, x_2, y_1, y_2, x_2) - p(x_2, x_1, y_1, y_2, x_1) \). So using Lemma 5.2.5 to write out \( p \) and then using the fact that the coefficients of monomials \( w_{ij}(x_1, x_2, y_1, y_2, x_2) \) for \( i = 1, \ldots, 4 \) and \( j = 1, \ldots, 5 \) have coefficients 0 in \( p(x_1, x_2, y_1, y_2, x_2) - p(x_2, x_1, y_1, y_2, x_1) \) yields 20 equations. By solving the system we get the equations of the Lemma.

Next, we may assume that \( f^* = \pm f \). Then from the *-symmetry of \( f \), by considering the coefficients of the monomials

\[ w_{ijk}, \ i = 1, \ldots, 4, \ j = 1, \ldots, 6, \ k = 1, \ldots, 7, \]

we obtain a system \((I)\). From the symmetry of \( f_3 \) in the variables of degree 2 we have a system \((II)\). Since

\[ f_1(x_1, x_2, 1, y_1, y_2, y_3) + 2f_3(x_1, x_2, y_1, y_2, y_3) = 0, \]

and since \( \Delta(x_3, 1)f \in V I_4[2^2, 1^4] = \{0\} \) by \([MR]\), we have system \((III)\).

Solving \((I),(II),(III)\) yields \( \gamma_1 = \gamma_2 = 0 \) when \( f^* = -f \) and \( \gamma_2 = 0 \) when \( f^* = f \). This easily implies from (5.33) that \( f_5 = 0 \) when \( f^* = \pm f \). Thus by the decomposition of a polynomial into a symmetric and a skew symmetric part, we have

Lemma 5.2.7 Every identity \( f(x_1, x_2, x_3, y_1, y_2, y_3) \in V I N_4[2^3, 1^3] \) has no term which ends in \( y_1 x_1^2 \).
In fact we also have $\gamma_1 = 0$ when $f^* = f$. Indeed, since $f$ is $*$-symmetric, its Razmyslov transform $f^#(y_2)$ is also an identity. Obviously $f^#(y_2)$ is of type $[2^3, 1^3]$ and symmetric in the variables of degree 2. For, if the monomial $V_1(x_i, x_j)y_2V_2(x_i, x_j)$ has coefficient $\alpha$ in $f^#(y_2)$ then by the definition of the Razmyslov transformation, $V_2(x_i, x_j)y_2V_1(x_i, x_j)$ has coefficient $\alpha$ in $f$, where $V_1, V_2$ are monomials. Thus $V_1(x_j, x_i)y_2V_2(x_j, x_i)$ has coefficient $\alpha$ in $f$ since $f$ is symmetric in the $x_i$'s. So $V_2(x_j, x_i)y_2V_1(x_j, x_i)$ has coefficient $\alpha$ in $f^#(y_2)$. That is, $f^#(y_2)$ is symmetric in $x_i, x_j$. Since $VI_4[2^4, 1] = \{0\}$, $f^#(y_2)$ must be skew symmetric in the variables of degree 1 by Theorem 1.1. Thus $f^#(y_2) \in VIN_4[2^3, 1^3]$ and by Lemma 5.2.7 $x_3y_3x_3x_1x_1y_2y_1x_2x_2$ has coefficient 0 in $f^#(y_2)$. This means that $y_1x_2x_2y_2x_3y_3x_3x_1x_1$ has coefficient 0 in $f$. This and $\gamma_2 = 0$ imply that $\gamma_1 = 0$.

Finally, using $\gamma_1 = \gamma_2 = 0$, Lemma 2.1.1 and the following substitutions we get a system $(IV)$: substituting

\[ e[12], e[22], e[23], e[33], e[34], e[44], e[11], e[11] \]

for

\[ x_1, x_2, x_3, y_3, z_1, x_2, y_2, y_1 \]

in $\Delta(x_1, z_1)\Delta(x_2, z_2)f(x_1, x_2, x_3, y_1, y_2, y_3)$ and considering the coefficient of $c_{23}$, substituting

\[ e[33], e[31], e[12], e[22], e[11], e[14], e[44] \]

for

\[ x_1, y_1, x_3, y_3, y_2, x_2, z_1 \]

in $\Delta(x_1, z_1)f(x_1, x_2, x_3, y_1, y_2, y_3)$ and considering the coefficient of $c_{31}$, substituting

\[ e[11], e[12], e[22], e[23], e[33], e[34], e[44] \]

for

\[ x_1, y_1, y_2, x_2, y_3, x_3, z_1 \]

in $\Delta(x_1, z_1)f(x_1, x_2, x_3, y_1, y_2, y_3)$ and considering the coefficient of $e_{12}$, substituting

\[ e[12], e[22], e[23], e[33], e[34], e[44] \]
for \( x_1, y_1, x_2, y_2, x_3, y_3 \)

in \( f(x_1, x_2, x_3, y_1, y_2, y_3) \) and considering the coefficient of \( e_{11} \).

Solving (I), (II), (III) and (IV) we get \( f_1 = 0 \). Thus \( f = 0 \).

This completes the proof of Theorem 5.1.

5.3 The Proof of Theorem 5.2

Proposition 5.3.1 The identity \( T_6(x^2, x, y_1, \ldots, y_5; x) \) forms a basis of the vector space \( VIN_4[4, 1^5] \).

Proof. Let \( f(x, y_1, \ldots, y_5) \in VIN_4[4, 1^5] \) then

\[
f = \sum_{(1, \ldots, 5)} \{ f_1(x, y_1, \ldots, y_4)x y_5 + f_1'(x, y_1, \ldots, y_4)y_5 x + f_3(x, y_1, \ldots, y_4)y_5 x^2 \} + \sum_{i<j} (-1)^{i+j-1} f_2(x, y_1, \ldots, \tilde{y}_i, \ldots, \tilde{y}_j, \ldots, y_5)[y_i, y_j].
\]

Since \( f_2(x, y_1, y_2, y_3) \in VIN_3[4, 1^3] \), \( f_2(x, y_1, y_2, 1) \in VIN_3[4, 1^2] = \{0\} \). Similarly \( f_3(x, y_1, y_2, y_3, 1) = 0 \). Thus \( f(x, y_1, \ldots, y_4, 1) \in VIN_4[4, 1^4] = \{0\} \) implies that

\[
f_1(x, y_1, \ldots, y_4) + f_1'(x, y_1, \ldots, y_4) + f_3(x, y_1, \ldots, y_4)x = 0.
\]

Furthermore, \( f_3 = \beta T_6(x, y_1, \ldots, y_4; x) \) and \( f_1 = \sum_{i=1}^7 \alpha_i Q_i \), where \( \{Q_1, \ldots, Q_7\} \) is the basis of \( VIN_3[3, 1^4] \) given in Chapter 4. \( f_2 = f_1(x, y_1, y_2, y_3, x) \) because \( \Delta(y_4, x)f \in VIN_4[5, 1^3] = \{0\} \). Thus \( f \) is determined by \( f_1 \) and \( f_3 \).

We may assume that \( f^* = \pm f \). Let \( \alpha(w) \) denote the coefficient of the monomial \( w \) in \( f \). Then \( \alpha(w) + s \alpha(w^*) = 0 \) for every monomial \( w \), where \( s := \pm 1 \) if \( f^* = \mp f \). Thus considering the coefficients of the monomials

\[
y_1 y_2 x y_3 y_4 x y_5, \quad y_1 y_2 x y_3 y_4 x y_5, \quad x y_1 y_2 x y_3 y_4 x y_5, \quad x y_1 y_2 x y_3 y_4 x y_5,
\]

\[
y_1 y_2 y_3 y_4 y_5, \quad y_1 y_2 y_3 y_4 y_5, \quad x y_1 y_2 y_3 y_4 x y_5, \quad x y_1 y_2 y_3 y_4 x y_5,
\]

\[
x y_1 y_2 y_3 y_4 y_5, \quad y_1 x y_2 y_3 y_4 y_5, \quad y_1 x y_2 y_3 y_4 y_5.
\]

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yields

\[-3\alpha_7 - 2\alpha_6 + \alpha_5 - \alpha_2 + s(-2\alpha_6 - \alpha_7 - \alpha_5 + \alpha_2) = 0,\]
\[\alpha_6 - \alpha_5 + \alpha_4 + s(\alpha_2 - \alpha_7 + \alpha_5 - 2\alpha_6) = 0,\]
\[2\alpha_7 + \alpha_2 - \alpha_1 + s(-2\alpha_6 - \alpha_2 - \alpha_1) = 0,\]
\[-\alpha_6 + \alpha_5 - \alpha_3 + s(-2\alpha_6 - \alpha_5) = 0,\]
\[-\alpha_6 - \alpha_4 + s(-\beta - \alpha_4 - \alpha_6 - 2\alpha_7 - \alpha_2 + \alpha_1) = 0,\]
\[\alpha_6 + \alpha_3 + s(-3\alpha_7 + \alpha_1 + \alpha_2) = 0,\]
\[3\alpha_6 + \alpha_5 + \alpha_3 = 0,\]
\[\alpha_2 + 2\alpha_6 + \alpha_1 + s(-2\alpha_7 + \alpha_1 - \alpha_2) = 0.\] (5.35)

If \(f^* = -f\) then the solution of (5.35) is

\[\alpha_i = 0, \ i \neq 2,1, \ \beta = \alpha_2 = -\alpha_4.\]

So \(f = \beta g(x,y_1,\ldots,y_5)\), where \(g\) is independent from \(f\). On the other hand \(T_b(x^2,x,y_1,\ldots,y_5;x) \in VIN_4[4,1^5]\) and is \(+\)-skew symmetric. Thus \(T_b(x^2,x,y_1,\ldots,y_5;x)\) forms a basis of the subspace consisting of the \(+\)-skew symmetric elements of \(VIN_4[4,1^5]\).

If \(f^* = f\) then \(f^*(y_2)\) is also an identity by Proposition 2.2.2. Thus the monomial \(y_3y_4xy_5y_2y_1xxx\) must have coefficient 0 in \(f^*(y_2)\) by Lemma 2.1.1. This means the monomial \(y_1xxxxy_2yy_4xx_5\) has coefficient 0 in \(f\). So we have

\[2\alpha_6 + \alpha_2 + \alpha_1 = 0.\]

The solution of (5.35) and this equation is trivial. Thus \(f_1 = 0\) and \(f = 0\). So every identity in \(VIN_4[4,1^5]\) is \(+\)-skew symmetric and \(T_b(x^2,x,y_1,\ldots,y_5;x)\) is a basis of \(VIN_4[4,1^5]\). \(\Box\)

**Proposition 5.3.2** The identities \(T_0(x^2,x,y,z_1,\ldots,z_4; y)\) and \(T_b(x \circ y, x, y, z_1,\ldots,z_4; x)\) form a basis of the vector space \(VIN_4[3,2,1^4]\).

**Proof.** Let \(f(x,y,z_1,\ldots,z_4) \in VIN_4[3,2,1^4]\) then \(\Delta(y,x)f \in VL_4[4,1^5]\) implies that there exists a scalar \(\alpha \in F\) such that \(\Delta(y,x)f = \alpha T_b(x^2,x,y,z_1,\ldots,z_4; x)\). Since

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\[ T_4(x^2, x, y, z_1, \ldots, z_4; y) \text{ belongs to } V IN_4[3, 2, 1^4] \text{ and } \Delta(y, x)(f - \alpha T_4(x^2, x, y, z_1, \ldots, z_4; y) = 0, \text{ we may assume that } \Delta(y, x)f = 0. \text{ We show that the identity} \]

\[ -2T_6(x^2, x, y, z_1, \ldots, z_4; y) + T_6(x \circ y, x, y, z_1, \ldots, z_4; x) \]

\[ \text{forms a basis of the subspace} \]

\[ W := \{ f \in V IN_4[3, 2, 1^4] | \Delta(y, x)f = 0 \}. \]

Let \( f \in W, \) then \( \Delta(y, x)f = 0. \) Since the actions of the operator \( \Delta(y, x) \) and \( \ast \) commute, \( \Delta(y, x)f^* = 0. \) So

\[ f = \frac{1}{2}(f + f^*) + \frac{1}{2}(f - f^*) \]

is a sum of a \( \ast \)-symmetric identity of \( W \) and a \( \ast \)-skew symmetric identity of \( W. \) Therefore, we may also assume \( f^* = \pm f. \)

Since

\[ f = f_1 xy + f_2 yx + f_3 y z x^2 + \]

\[ \sum_{i=1}^{4} (-1)^i (f_2(x, y, z_1, \ldots, z_i, \ldots, z_4) x z_i + f'_2(x, y, z_1, \ldots, z_i, \ldots, z_4) x z_i + f_3(x, y, z_1, \ldots, z_i, \ldots, z_4) y z_i + f'_3(x, y, z_1, \ldots, z_i, \ldots, z_4) y z_i + f_6(x, y, z_1, \ldots, z_i, \ldots, z_4) y z_i) \]

\[ \sum_{i<j} \pm f_4(x, y, z_1, \ldots, z_i, \ldots, z_j, z_i)[z_i, z_j] + f_7 x y^2, \]

(5.36)

\( f_1 \) is an identity of \( H_3 \) of type \([2, 1^3] \) and is skew symmetric in the variables \( z_1, \ldots, z_4, \) it is a linear combination of the identities \( g_i \) which come from \( Q, T_6^*, T_6 (z_1, z_2, x, z_4, x; [x_0, x_7]) \) and \( K \) by substituting some variables and are skew symmetric in \( z_1, z_2, z_3, z_4. \) By the symmetry of the variables of \( Q, T_6 \) and \( K, \) there are only 41 such \( g_i \)'s. So, let \( f_1 = \sum_{i=1}^{41} \alpha_i g_i. \)

Since \( \Delta(z_4, x)f = \Delta(z_4, y)f = 0. \)

\[ f_2 = -f_1(x, y, z_1, z_2, z_3, y), \]
\[ f_3 = -\Delta(y, x)f_2(x, y, z_1, z_2, z_3) \]
\[ = f_1(x, x, z_1, z_2, z_3, y) + f_1(x, y, z_1, z_2, z_3, x), \]
\[ f_4 = -\Delta(z_3, y)f_3 = f_1(x, x, z_1, z_2, y, y) + f_1(x, y, z_1, z_2, y, x), \]
\[ f_6 = -f_3(x, y, z_1, z_2, z_3, y). \]

(5.37)
Since $f_5$ is an identity of $H_3$ of type $[1^4]$, $f_5(x, y, z_1, z_2, z_3, z_4, z_5) = \sum_i \beta_i p_i$, where $p_i$ is an identity of $H_3$ coming from $T_0$ and $Q$ and is skew symmetric in $z_i$'s. From the symmetry of the variables of $Q$ and $T_0$ there are only 6 such $g_i$'s:

$$g_1 = T_0(x, z_1, z_2, z_3, z_4, y), \quad g_2 = T_0(y, z_1, z_2, z_3, z_4, x),$$

$$g_3 = \sum_{i=1}^4 (-1)^i T_0(x, y, z_1, \ldots, \tilde{z}_i, \ldots, z_4; z_i),$$

$$g_4 = \sum_{\sigma(1) < \sigma(2), \sigma(3) < \sigma(4)} (-1)^\sigma Q(x, z_{\sigma(1)}, z_{\sigma(2)}, y, z_{\sigma(3)}, z_{\sigma(4)}),$$

$$g_5 = \sum_{i=1}^4 (-1)^i Q(x, y, z_i, z_1, \ldots, \tilde{z}_i, \ldots, z_4), \quad g_6 = S_0(x, y, z_1, z_2, z_3, z_4).$$

It can be shown that $g_0 + 2g_4 + 2g_5 = 0$, so $f_5 = \sum_{i=1}^5 \beta_i g_i$. By the results in [M/R], $f_7 = \alpha_6 T_0(x, z_1, z_2, z_3, z_4; z).$

Next from $f^* = \pm f$ and

$$f_2 + f_2' + \Delta(z_4, x)(f_5 y - f_5 z - f_5 z_1 + f_5 z_2 - f_5 z_3) + f_6 z = 0,$$

$$f_3 + f_3' + \Delta(z_4, y)f_7 x = 0,$$

we have (I). If $f^* = f$ then the Razmyslov transform $f^*(z_1)$ is also an identity of type $[3, 2, 1^4]$. Thus it has no terms which end in $z_{xy}$. This implies that the coefficient of the monomial $x_{z_{xy}} z_1, z_{xx} z_4$ is 0 which yields an equation (II). Solving (I) and (II), and substituting the solution into $f_1$ yields $f_2 = f_5 = 0$. Thus $f = f_1 x y + f_1' y x$. Since $\Delta(y, x)f = 0$ and $f$ is *- symmetric,

$$f = [x, y]g(x, z_1, z_2, z_3, z_4)[x : :],$$

where $g$ is multilinear. Furthermore it can be shown easily that $g$ is alternating because $\Delta(z_1, x)f = 0$. So there exists a scalar $\alpha \in F$ such that $g = \alpha S_0(x, z_1, z_2, z_3, z_4)$ and $f = \alpha [x, y]S_0(x, z_1, z_2, z_3, z_4)[x, y]$. However $f(x, y, z_1, z_2, z_3, 1)$ is an identity of $H_4$ of type $[3, 2, 1, 1, 1]$, so $f(x, y, z_1, z_2, z_3, 1) = 0$. Hence $\alpha = 0$ and $f = 0$.

If $f^* = -f$ then solving (I) and (II), and substituting the solution into $f_1$ yields $f_2 = \alpha_{z_2} h_2, f_5 = \alpha_{z_2} h_5$. Thus $f_1' = -f_1 + \alpha h_1$ from $\Delta(y, x)f = 0$ and (5.36). So by
\( f^* = -f \), (5.36) and (5.37).

\[
f = [x, y] g[x, y] + \alpha h_2,
\]

(5.38)

where \( g \) is multilinear. Since \( \Delta(y, x)f = 0 \), from (5.38) we have \( \Delta(z_1, x)g(x, z_1, z_2, z_3, z_4) + \alpha h_3 = 0 \). This means modulo \( \alpha \), \( g \) is alternating. So there exists a scalar \( \beta \in F \) such that \( g = \beta S_5(x, z_1, z_2, z_3, z_4) + \alpha h_4 \). Thus

\[
f = \beta [x, y] S_5(x, z_1, z_2, z_3, z_4)[x, y] + \alpha h_5.
\]

(5.39)

Since \( f(x, y, z_1, z_2, z_3, 1) = 0 \) and \( S_5(x, z_1, z_2, z_3, 1) = S_4(x, z_1, z_2, z_3) \), from (5.39) we have \( \beta = m \alpha \) for some \( m \in F \) and \( f = \alpha h \). So the dimension of the subspace \( W \) is at most 1. Since

\[
-2T_8(x^2, x, y_1, \ldots, y_6; y) + T_8(x \circ y, x, y_1, \ldots, y_6; x) \in W,
\]

the dimension of \( W \) is 1. \( \square \)

**Proposition 5.3.3** The following identities form a basis of the space \( VIN_4[3, 1^6] \).

\[
\begin{align*}
T_8(x^2, y_1, \ldots, y_6; x),& \quad T_8(x, y_1, \ldots, y_6; x^2), \\
\sum_{i=1}^{6} (-1)^{i-1} T_8(x \circ y_i, y_1, \ldots, y_i, x; x),& \quad \sum_{i=1}^{6} (-1)^{i} T_8(x^2, x, y_1, \ldots, y_i, y_i), \\
T_8(x, y_1, \ldots, y_6; x),& \quad xT_8(x, y_1, \ldots, y_6; x).
\end{align*}
\]

**Proof.** First, we show that the space \( VIN_4[3, 1^6] \) is spanned by 6 elements. Let \( f(x, y_1, \ldots, y_6) \in VIN_4[3, 1^6] \) then \( \Delta(y_5, x)f \in VIN_4[4, 1^5] \). So there exists a scalar \( \alpha \in F \) such that

\[
\Delta(y_5, x)f = \alpha T_8(x^2, x, y_1, \ldots, y_4, y_6; x).
\]

Thus we can correct \( f \) by subtracting \( \alpha T_8(x^2, x, y_1, \ldots, y_6; x) \) so that

\[
\Delta(y_5, x)(f - \alpha T_8(x^2, x, y_1, \ldots, y_6; x)) = 0.
\]

Thus it suffices to show that the subspace

\[
W := \{ f \in VIN_4[3, 1^6] \mid \Delta(y_5, x)f = 0 \}
\]

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is spanned by 5 elements.

Let \( f(x, y_1, \ldots, y_6) \in W' \) then

\[
f = \sum_{(i, \ldots, \delta)} \{ f_1(x, y_1, \ldots, y_5) y_6 + f'_1(x, y_1, \ldots, y_5) y_6 x + f_3(x, y_1, \ldots, y_5) y_6 x^2 \} + \sum_{i < j} f_2(x, y_1, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_5) [y_i, y_j]. \quad (5.40)
\]

Since \( f_1 \in VIN_3[2, 1^5] \), there exist scalars \( \alpha_i \in F \) such that \( f_1 = \sum_{i=1}^{10} \alpha_i y_i \), where \( \{g_1, \ldots, g_{10}\} \) is the basis of the space \( VIN_3[2, 1^5] \) given in Chapter 4. \( \Delta(y_5, x)f = 0 \) implies that

\[
\Delta(y_5, x)f_1 + f_2 = 0.
\]

from (5.10). Since \( f_3(x, y_1, y_2, y_3, y_4, y_5) \in W_3[1^9] \), \( f_3 = \sum_i \beta_i g_i \), where

\[
g_1 = \sum_{(i, \ldots, \delta)} T_0(x, y_1, y_2, y_3, y_4; y_5), \quad g_2 = T_0(y_1, \ldots, y_5; x).
\]

\[
g_3 = \sum_{i < j} Q(x, y_i, y_j, y_1, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_5).
\]

Since \( g_2 - g_1 - 2g_3 = 0 \), we may assume \( f_3 = \sum_{i=1}^{2} \beta_i g_i \). Thus from the \( * \)-symmetry of \( f \) we have a system of equations (I). Solving (I) for \( f^* = f \) and \( f^* = -f \) respectively we see that there are only 5 parameters in total. So \( W \) is spanned by 5 elements.

On the other hand, if we denote by \( t_i^4 \) the identities given in the Proposition, then

\[
t_i^4 = t_i^2[y_5, y_6], \quad i = 1, 3, 4, 6, \quad t_2^4 = (t_2^2 + t_3^2)[y_5, y_6],
\]

where \( t_i^2, i = 1, \ldots, 6 \) are the elements occurring in the basis of \( VIN_3[3, 1^4] \) given in Chapter 4. So they are linear independent. If \( \sum_{i=1}^{6} \alpha_i t_i^4 = 0 \) then

\[
\sum_{i=1}^{6} \alpha_i t_i^4 = (\alpha_2 (t_2^2 + t_3^2) + \sum_{i \neq 2} \alpha_i t_i^2)[y_5, y_6] + \sim
\]

implies that \( \alpha_i = 0 \), for \( i = 1, \ldots, 6 \), and we have proved the Proposition. \( \blacksquare \)

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Let

\[ t_1 := T_8(x^2, y, z_1, \ldots, z_5; y) + T_8(y^2, x, z_1, \ldots, z_5; x), \]
\[ t_2 := T_8(x, y, z_1, \ldots, z_5; y) + T_8(x, y, z_1, \ldots, z_5; x), \]
\[ t_3 := \sum_{i=1}^{5} (-1)^{i-1}(T_8(x; z_1, \ldots, \widehat{z_i}, \ldots, z_5, y; x) + T_8(y; z_1, \ldots, \widehat{z_i}, \ldots, z_5, y; x)), \]
\[ t_4 := zT_8(x, y, z_1, \ldots, z_5; y) + yT_8(y, x, z_1, \ldots, z_5; x), \]
\[ t_5 := T_8(x, y, z_1, \ldots, z_5; y)x + T_8(y, x, z_1, \ldots, z_5; x)y, \]
\[ t_6 := T_8(x, y, z_1, \ldots, z_5; [x, y]), \]
\[ t_7 := T_8(x, y, z_1, \ldots, z_5; [x, y]). \]

Then

**Proposition 5.3.4** The set \( \{t_1, \ldots, t_7\} \) is a basis of the vector space \( VIN_4[2, 2, 1^5] \).

**Proof.** Let \( f \) be an identity of type \([2, 2, 1^5]\). Then

\[
  f = f_1(x_1, x_2, y_1, \ldots, y_5)x_1x_2 + f_1(x_2, x_1, y_1, \ldots, y_5)x_2x_1 + \\
  f_2y_5x_2 + f'_2x_2y_5 + \cdots \\
  + f_3[y_4, y_5] + \cdots \\
  + f_4(x_1, y_1, \ldots, y_5)x_1x_2^2 + f_4(x_2, y_1, \ldots, y_5)x_2x_1^2 + \\
  f_5y_5x_2 + \cdots. 
\]  

(5.41)

Since \( f_4 \) is an identity of degree 6 on \( H_3 \) by Lemma 2.1.2, it comes from \( T_5 \) and \( Q \) by \( [M, R] \). More precisely, by the symmetry of \( T_6 \) and \( Q \), \( f_4 \) is a linear combination of identities of the form \( T_6(y_1, \ldots, y_5; x_1) \), \( T_6(x_1, y_1, \ldots, y_5; y_5) \) and \( Q(x_1, y_1, y_2, y_3, y_4, y_5) \). But, \( f_4(x_1, y_1, \ldots, y_5) \) is skew symmetric in the variables \( y_i \)'s, from the symmetry of \( T_6 \) and \( Q \) it is a linear combination of the following identities:

\[ g_1 := T_6(y_1, \ldots, y_5; x_1), \quad g_2 := \sum_{i < j} T_6(x_1, y_i, y_j, \ldots, y_5; y_5), \]
\[ g_3 := \sum_{i < j} (-1)^{i+j-1} Q(x_1, y_i, y_j, \ldots, \widehat{y_i}, \ldots, \widehat{y_j}, \ldots, y_5). \]
[Note $4S_6(x_1, y_1, \ldots, y_5) = g_1 - g_2$ by [MR]] It can be checked easily by computer that $g_3 = 2S_3(x_1, y_1, \ldots, y_5)$, thus $-g_1 + g_2 + 2g_3 = 0$ and $f_4 = \beta_1 g_1 + \beta_2 g_2$.

Let $w(i, j)$ denote the monomial $y_1 \cdots \frac{(i)}{x_1} \cdots \frac{(j)}{x_2} \cdots y_5$ and $(1, 2)w(i, j)$ denote the monomial obtained by interchanging $x_1$ and $x_2$ in $w(i, j)$. e.g. $w(1, 3) = x_1 y_1 x_2 y_2 y_3 y_4 y_5$ and $(1, 2)w(1, 3) = x_2 y_1 x_1 y_2 y_3 y_4 y_5$. Let $w_i := w_i(x_1, x_2, y_1, \ldots, y_5), i = 1, \ldots, 21$ denote monomials

\[
w(1, j), j = 2, 3, 4, 5; \quad w(2, j), j = 3, 4, 5; \\
w(3, j), j = 4, 5; \quad w(4, 5); \\
w(i, 6), w(i, 7), i = 1, \ldots, 5; \quad w(6, 7),
\]

respectively and let $w_i, i = 22, \ldots, 42$ denote the monomials $(1, 2)w_i, i = 1, \ldots, 21$, respectively. Then

\[
f_1 = \sum_{i=1}^{42} \alpha_i \left( \sum_{\sigma \in S_5} (-1)^{\sigma} \sigma(w_i) \right),
\]

where $\sigma$ acts on $w_i$ to permute $y_i$'s.

Since $\Delta(y_5, x_1)f \in VIN_4[3, 2, 1^6]$, there exist scalars $\gamma_1, \gamma_2 \in F$ such that

\[
\Delta(y_5, x_1)f = \gamma_1 T_5(x_1^2, x_1, x_2, y_1, \ldots, y_4; x_2) + \gamma_2 T_5(x_1 \circ x_2, x_1, x_2, y_1, \ldots, y_4; x_2)
\]

Let $h := \gamma_1 t_1 + \gamma_2 t_2$ then $\Delta(y_5, x_1)h = -\Delta(y_5, x_1)f$. Hence $\Delta(y_5, x_1)(f + h) = 0$. So we may assume that $\Delta(y_5, x_1)f = 0$. Thus from (5.41) we have

\[
\begin{align*}
f_2(x_1, x_2, y_1, \ldots, y_4) &= -f_1(x_1, x_2, y_1, \ldots, y_4, x_1), \\
f_2'(x_1, x_2, y_1, \ldots, y_4) &= -f_1(x_2, x_1, y_1, \ldots, y_4, x_1), \\
f_3(x_1, x_2, y_1, y_2, y_3) &= -f_1(x_1, x_2, y_1, \ldots, y_4, x_1), \\
f_5(x_1, y_1, y_2, y_3, y_4) &= -f_1(x_1, y_1, y_2, \ldots, y_4, x_1). \\
\end{align*}
\]

(5.42)

It is easy to see that the operator $\Delta$ and involution $*$ commute on every monomial. So they commute on any polynomial. Hence $\Delta(y_5, x_1)f = 0$ implies that $\Delta(y_5, x_1)f^* = 0$. Thus the decomposition $f = \frac{1}{2}(f + f^*) + \frac{1}{2}(f - f^*)$ allows us to discuss only these identities in $VIN_4[2, 2, 1^5]$ which are $*$-skew symmetric or $*$-symmetric and on which $\Delta(y_5, x_1)$ acts as zero.

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So from now on we assume that \( f^* = \pm f \) and \( \Delta(y_5, x_1)f = 0 \).

We also can correct \( f \) by subtracting an identity in such a way that

\[
\alpha_{31} = \alpha_{33} = \alpha_{42} = 0, \text{ if } f^* = -f; \quad \alpha_{31} = \alpha_{33} = 0, \text{ if } f^* = f.
\]

Indeed, when \( f^* = -f \) let

\[
g := \lambda_1 (2t_1 + t_3) + \lambda_2 (t_4 + t_6) + \lambda_3 (t_6 - t_7).
\]

Then \( g \) is \(*\)-skew symmetric and \( \Delta(y_5, x)g = 0 \). In \( g \), the monomial \( w_{31} \) has coefficient \(-\lambda_2 - 2\lambda_3\), while \( w_{42} \) has coefficient \(-\lambda_1 \) and \( w_{33} \) has coefficient \( 4\lambda_1 + \lambda_2 + \lambda_3 \). Let

\[
-\lambda_2 - 2\lambda_3 = \alpha_{31}, \quad -\lambda_1 = \alpha_{42}, \quad 4\lambda_1 + \lambda_2 + \lambda_3 = \alpha_{33}.
\]

This system has a solution, therefore there exist scalars \( \lambda_1, \lambda_2, \lambda_3 \in F \) such that \( w_{31}, w_{42}, w_{33} \) have coefficients zero in \( f - g \). When \( f^* = f \) let

\[
g := y(t_4 - t_5) + z(t_6 + t_7).
\]

Then \( g \) is \(*\)-symmetric and \( \Delta(y_5, x)^*g = 0 \). The monomial \( w_{31} \) has coefficient \(-\lambda_2 - 2\lambda_3\), while \( w_{33} \) has coefficient \( \lambda_2 - \lambda_3 \). Let

\[
\lambda_2 + 2\lambda_3 = \alpha_{31}, \quad \lambda_2 - \lambda_3 = \alpha_{33}
\]

Solving this system there exist scalars \( \lambda_2, \lambda_3 \in F \) such that \( w_{31}, w_{33} \) have coefficients 0 in \( f - g \).

From \( f^* = \pm f \), considering the coefficients of monomials \( wx_1x_2 \) and its \(*\) for

\[
w = w(1,3), \ldots, w(1,7), w(2,4), \ldots, w(2,7),
\]

\[
w(3,4), \ldots, w(3,7), w(4,5), \ldots, w(4,7),
\]

\[
w(5,6), w(5,7), w(1,2), w(1,3)
\]

we obtain a system of equations \((I)\). Since \( f_3 \) is symmetric in \( x_1 \) and \( x_2 \), \( f_3(x_1, x_2, y_1, y_2, y_3) - f_3(x_2, x_1, y_1, y_2, y_3) = 0 \). Thus considering the coefficients of monomials \( w_i(x_1, x_2, y_1, y_2, y_3, x_2, x_1) \) for \( i = 1, \ldots, 21 \) respectively, yields a system \((II)\).

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Next we apply Lemma 2.1.1 to produce more equations. Considering the monomials

\[- y_1 y_2 y_3 y_4 y_5 --, \quad y_1 y_2 -- y_3 y_4 y_5 --, \quad y_1 y_2 y_3 y_4 -- y_5 --,\]

where the blank positions are filled by the variables \(x_1, x_2, x_1, x_2\), yields three equations. For instance, by Lemma 2.1.1, the sum of the coefficients of the monomials

\[x_1 x_2 y_1 y_2 y_3 y_4 y_5 x_1 x_2, \quad x_2 x_1 y_1 y_2 y_3 y_4 y_5 x_1 x_2,\]

\[x_1 x_2 y_1 y_2 y_3 y_4 y_5 x_2 x_1, \quad x_2 x_1 y_1 y_2 y_3 y_4 y_5 x_2 x_1,\]

is 0. Applying the Lemma 2.1.1 to monomials \(wx_2 x_1 x_2\) for

\[w = x_1 y_1 y_2 y_3 y_4 y_5, \quad y_1 x_1 y_2 y_3 y_4 y_5, \quad y_1 y_2 x_1 y_3 y_4 y_5,\]

\[y_1 y_2 x_1 x_2 y_3 y_4 y_5, \quad y_1 y_2 y_3 x_1 x_2 y_4 y_5, \quad y_1 y_2 y_3 y_4 x_1 y_5,\]

produces six equations. That is the sum of the coefficients of monomials \(wx_1 x_2 x_1, \quad wx_1 x_1 x_2, \quad wx_2 x_1 x_1\) is zero.

Applying the Lemma 2.1.1 to the positions 1, 2, 3 of the monomial \(x_1 x_2 y_1 y_2 y_3 y_4 y_5 x_1 x_2\), the positions 3, 4, 5 of the monomial \(y_1 y_2 x_1 x_2 y_3 y_4 y_5 x_1 x_2\) and the position 5, 6, 7 of the monomial \(y_1 y_2 y_3 y_4 x_1 x_2 y_5 x_1 x_2\) yield three equations. Using Lemma 2.1.1, consider monomials \(- w - -,\) where the first two blank positions are variables \(x_1, y_1\) and the last two positions are \(x_1, x_2\) for

\[w = x_2 y_2 y_3 y_4 y_5, \quad y_2 x_2 y_3 y_4 y_5, \quad y_2 y_3 x_2 y_4 y_5,\]

\[y_2 y_3 y_4 x_2 y_5, \quad y_2 y_3 y_4 y_5 x_2,\]

Using Lemma 2.1.1, consider the following monomials and position 3, 4 and 8, 9

\[x_2 y_1 x_1 y_2 y_3 y_4 y_5 x_1 x_2, \quad y_1 x_2 x_1 y_2 y_3 y_4 y_5 x_1 x_2, \quad y_1 y_2 x_1 y_3 x_2 y_4 y_5 x_1 x_2,\]

\[y_1 y_2 x_1 y_3 y_4 x_2 y_5 x_1 x_2, \quad y_1 y_2 x_1 y_3 y_4 y_5 x_2 x_1 x_2.\]
Using Lemma 2.1.1, consider the following monomials and position 5, 6 and 8, 9
\[
E_{21} y_1 y_2 y_3 x_1 y_4 y_5 x_1 x_2, \quad E_{23} y_1 x_2 y_2 y_3 x_1 y_4 y_5 x_1 x_2, \quad E_{23} y_1 y_2 y_3 y_4 x_1 y_5 x_1 x_2,
\]
All these applications of Lemma 2.1.1 produce a system (III).

Let us do the following substitution to produce equations (IV).

1. Substituting
\[ e[12], e[14], e[22], e[23], e[34], e[33] \]
for \( x_1, x_2, y_1, \ldots, y_5 \) and considering coefficients of \( e_{24} \).

2. Substituting
\[ e[11], e[12], e[22], e[23], e[31], e[34], e[44] \]
for \( y_1, x_1, y_2, y_3, x_2, y_4, y_5 \) and considering the coefficient of \( e_{24} \) [see type \([2, 2, 1^3]\) id. of \( H_3 \)].

3. Substituting
\[ e[11], e[12], e[22], e[13], e[33], e[34], e[44] \]
for \( y_1, x_1, x_2, y_2, y_3, y_4, y_5 \) and considering the coefficients of \( e_{14} \) and \( e_{41} \) respectively.

4. Substituting
\[ e[11], e[13], e[32], e[22], e[33], e[34], e[44] \]
for \( y_1, y_2, x_1, x_2, y_3, y_4, y_5 \) and considering the coefficients of \( e_{14} \) and \( e_{41} \) respectively.

5. Substituting
\[ e[11], e[12], e[22], e[13], e[33], e[34], e[44] \]
for \( y_1, x_1, y_2, y_3, y_4, x_2, y_5 \) and considering the coefficient of \( e_{14} \).

6. Substituting
\[ e[11], e[12], e[22], e[23], e[33], e[34], e[44] \]
for \( y_1, y_2, y_3, x_1, y_4, x_2, y_5 \) and considering the coefficients of \( e_{14} \).
Finally, since \( f_1(x_2, x_1, y_1, \ldots, y_4; x_1) + f'_2 = 0 \) and
\[
\begin{align*}
&f_2 + f'_2 + f_4(x_1, y_1, \ldots, y_4, x_2)x_1 + f_5(x_1, y_1, \ldots, y_4)x_2 + \\&f_5(x_1, y_2, y_3, y_4, x_2)y_1 + f_5(x_1, y_3, y_4, x_2, y_1)y_2 + \\&f_5(x_1, y_4, x_2, y_1, y_2)y_3 + f_5(x_1, x_2, y_1, y_2, y_3)y_4 = 0,
\end{align*}
\]
\[f_1(x_2, x_1, y_1, \ldots, y_4, x_1) - f_2 - f_4(x_1, y_1, \ldots, y_4, x_2)x_1 - f_5(x_1, y_1, \ldots, y_4)x_2 - \]
\[f_5(x_1, y_2, y_3, y_4, x_2)y_1 - f_5(x_1, y_3, y_4, x_2, y_1)y_2 + \]
\[f_5(x_1, y_4, x_2, y_1, y_2)y_3 - f_5(x_1, x_2, y_1, y_2, y_3)y_4 = 0.
\]

Thus considering the following monomials yields a system \((V)\)
\[
w_i(x_1, x_2, y_1, \ldots, y_4, x_1), \ i = 1, \ldots, 20.
\]

When \( f^* = -f \), solving \((1), \ldots, (V)\) yield \( f_1 = \beta_2y_1 + \alpha_3\alpha_2 \). Since \( g_i \) is of type \([1, 1, 1^5]\) and skew symmetric in the last 5 variables
\[
g_j = \sum_{\sigma \in S_5} (-1)^\sigma q_j^\sigma,
\]
where
\[
q_j^\sigma := V_j[\sigma(w_1), \ldots, \sigma(w_42)]^t,
\]
and \( \sigma \) acts on \( w_i \) to permute \( y_i \)'s.

On the other hand, every identity of type \([1, 1, 1^5]\) on \( H_3 \) which is skew in the last 5 variables is a linear combination of 40 \( h_i \)'s by the results in Chapter 3 that every identity of degree 7 on \( H_3 \) comes from \( T_0, Q, K \) and \( T_0(x_1, \ldots, x_5; [x_6, x_7]) \). We claim that

\[(*) \quad \text{Both } g_1, g_2 \text{ are not identities.}\]

If, say \( g_1 \) is an identity then there exist scalars \( \beta_i \in F \) such that
\[
g_1 = B_1[h_1, \ldots, h_{40}]^t, \tag{5.43}
\]
where \( B_1 = (\beta_1, \ldots, \beta_{40}) \). Since
\[
h_j = \sum_{\sigma \in S_5} (-1)^\sigma k_j^\sigma,
\]

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where
\[ k_j^q := A_j[\sigma(w_1), \ldots, \sigma(w_{42})]^t, \]
for some $1 \times 42$ matrix $A_j = (a_{js})$, and $\sigma$ acts on $w_i$ to permute $y_i$'s. Thus (5.43) implies that
\[ q_1 = B_1[k_1, \ldots, k_{40}]^t, \quad (5.44) \]
where $r_i := r_{id}$ for $r = q, k$ and id denotes the identity permutation. Since
\[ [k_1, \ldots, k_{40}]^t = A[w_1, \ldots, w_{42}], \]
where $A = [A_1, \ldots, A_{40}]^t = (a_{js})$ which is a $40 \times 42$ matrix, by (5.44) and using the definition of $q_j$ we obtain
\[ V_1 = B_1 A. \]
Hence
\[ A^t B_1^t = V_1^t. \]
That is, $B_1^t$ is a solution of the linear equation $A^t X = V_1^t$. But as can easily be checked on a computer this equation has no solution. Thus the claim and the fact that $f_1 = \beta_2 g_1 + \alpha g_2$ is an identity imply that there exists a scalar $\alpha \in \mathbb{F}$ such that $\beta_2 = \alpha \alpha_{36}$ and $f_1 = \alpha_{36}(g_1 + \alpha g_2)$. But arguing as above, $g_1 + \alpha g_2$ is not an identity since for any $\alpha \in \mathbb{F}$ the equation $A^t X = V_1^t + \alpha V_2^t$ has no solution as was easily checked by computer. Thus $\alpha_{36} = 0$ and hence $\beta_2 = 0$. So $f_1 = f_4 = 0$, and therefore $f = 0$.

When $f^* = f$, solving $(I), \ldots, (V)$ with $s = -1$ yield $f_1 = \beta_2 g_1 + \alpha g_2 + \alpha_{42} g_3$, where
\[ g_i = \sum_{\sigma \in \mathcal{S}_3} (-1)^\sigma U_i[\sigma(w_1), \ldots, \sigma(w_{42})]. \]
As in the proof of skew symmetric case, it can easily be checked by computer that for any scalars $\lambda_i, \lambda \in \mathbb{F}$ the following equations have no solution.

\[
\begin{align*}
A^t X &= U_i^t, \quad i = 1, 2, 3, \\
A^t X &= U_1^t + \lambda_1 U_3^t, \\
A^t X &= U_2^t + \lambda_2 U_3^t, \\
A^t X &= U_1^t + \lambda_1 U_3^t + \lambda(U_2^t + \lambda_2 U_3^t).
\end{align*}
\]

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Hence \( \alpha_8 = \alpha_{42} = 0 \) and \( \beta_2 = 0 \). So \( f_1 = f_4 = 0 \), and therefore \( f = 0 \). \( \blacksquare \)

Let

\[
\begin{align*}
t_1 & := T_8(y_1, \ldots, y_7; x^2), \quad t_2 := \tilde{T}_8(y_1, \ldots, y_7; x^2), \\
t_3 & := \sum_{(1, \ldots, 7)} T_8(x \circ y_1, \ldots, y_7; x), \quad t_4 := \sum_{(1, \ldots, 7)} \tilde{T}_8(x \circ y_1, \ldots, y_7; x), \\
t_5 & := \sum_{(1, \ldots, 7)} T_8(x, y_1, \ldots, y_6; x \circ y_7), \quad t_6 := \sum_{(1, \ldots, 7)} \tilde{T}_8(x, y_1, \ldots, y_6; x \circ y_7), \\
t_7 & := xT_8(y_1, \ldots, y_7; x), \quad t_8 := \tilde{T}_8(y_1, \ldots, y_7; x)x, \\
t_9 & := x\tilde{T}_8(y_1, \ldots, y_7; x), \quad t_{10} := T_8(y_1, \ldots, y_7; x)x, \\
t_{11} & := \sum_{(1, \ldots, 7)} y_1 T_8(x, y_2, \ldots, y_7; x), \\
t_{12} & := \sum_{(1, \ldots, 7)} \tilde{T}_8(x, y_1, \ldots, y_6; x)y_7, \\
t_{13} & := \sum_{(1, \ldots, 7)} y_1 \tilde{T}_8(x, y_2, \ldots, y_7; x), \quad t_{14} := \sum_{(1, \ldots, 7)} T_8(x, y_1, \ldots, y_6; x)y_7, \\
t_{15} & := \sum_{(1, \ldots, 7)} T_8(x, y_1, \ldots, y_6; [x, y_7]), \quad t_{16} := \sum_{(1, \ldots, 7)} \tilde{T}_8(x, y_1, \ldots, y_6; [x, y_7]).
\end{align*}
\]

Then

**Proposition 5.3.5** The set \( \{t_i|i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15\} \) forms a basis of the vector space \( VIN_4[2, 1^7] \).

**Proof.** Let \( V := \text{Span}\{t_i|i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15\} \). We show that \( V = VIN_4[2, 1^7] \). First, we show there exist 12 polynomials such that every \( f \in V \) is a linear combination of these polynomials.

Let \( f \in VIN_n[2, 1^7] \). Then \( \Delta(y_7, x)f \in VIN_4[3, 1^6] \). Hence there exist scalars \( \alpha_1, \ldots, \alpha_6 \in F \) such that \( \Delta(y_7, x)f = \sum_{i=1}^{6} \alpha_i q_i \), where \( \{q_i|i = 1, \ldots, 6\} \) is the basis of \( VIN_4[3, 1^6] \) given in Proposition 5.3.3. Next we shall correct \( f \) so that \( \Delta(y_7, x)f = 0 \).

Let \( \sum_{(1, \ldots, 4)} \) denote the sum over all cyclic permutations of \( 1, 2, \ldots, k \) and

\[
g_1 := \sum_{(1, \ldots, 7)} T_8(x^2, y_1, \ldots, y_6; y_7),
\]

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\[ g_2 := \sum_{(1, \ldots, 7)} T_6(x \circ y_1, \ldots, y_7; x), \]
\[ g_3 := \sum_{(1, \ldots, 7)} \sum_{(1, \ldots, 6)} (-1)^{(1 \cdots 6)} T_6(x \circ y_1, x, y_2, \ldots, y_6; y_7), \]
\[ g_4 := T_8(y_1, \ldots, y_7; x^2), \quad g_5 := T_6(y_1, \ldots, y_7; x)x, \]
\[ g_6 := xT_5(y_1, \ldots, y_7; x). \]

Then
\[
\Delta(y_7, x)g_i = q_i \quad \text{for } i = 4, 5, 6,
\]
\[
\Delta(y_7, x)g_1 = q_1 + q_3, \quad \Delta(y_7, x)g_2 = q_2 + 2q_1, \quad \Delta(y_7, x)g_3 = q_2 + 2q_3. \quad (5.45)
\]

Let
\[
\beta_i := \alpha_i, \quad \beta_3 := \frac{1}{4}(-\alpha_1 + 2\alpha_2 + \alpha_3)
\]
\[
\beta_2 := \alpha_2 - \beta_3, \quad \beta_1 := \alpha_3 - 2\beta_3
\]

and
\[
g := \sum_{i=1}^{6} \beta_i g_i.
\]

Then from (5.45), \( \Delta(y_7, x)g = \Delta(y_7, x)f \). Hence \( \Delta(y_7, x)(f - g) = 0 \). So we may assume that \( \Delta(y_7, x)f = 0 \).

Since the operators \( \Delta \) and \( * \) are linear and they commute on any monomial, \( (\Delta(y_7, x)f)^* = \Delta(y_7, x)(f^*) \). Thus \( \Delta(y_7, x)f = 0 \) implies \( \Delta(y_7, x)f^* = 0 \). Hence \( f \) can be decomposed into the sum of a \( * \)-symmetric identity and a \( * \)-skew symmetric identity \( f = \frac{1}{2}(f + f^*) + \frac{1}{2}(f - f^*) \) and \( \Delta(y_7, x)(\frac{1}{2}(f + f^*)) = 0, \Delta(y_7, x)(\frac{1}{2}(f - f^*)) = 0 \). So we may assume that \( f^* = \pm f \) and \( \Delta(y_7, x)f = 0 \).

Hence
\[
f = \sum_{i=1}^{7} (f_1(x, y_1, \ldots, \widehat{y}_i, \ldots, y_7)xy_i + f'_1(x, y_1, \ldots, \widehat{y}_i, \ldots, y_7)y_1x) + \\
\sum_{i < j} f_2(x, y_1, \ldots, \widehat{y}_i, \ldots, y_j, y_7) [y_i, y_j] + f_3(y_1, \ldots, y_7)x^2,
\]

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by the skew symmetry of $f$ in the variables of $y_i$'s. $f_0 = \beta S_7$ for some $\beta \in F$ and

$$f_1 = \sum_{i=1}^{7} \alpha_i y_1 \cdots \hat{x} \cdots y_7,$$

$$f'_1 = \sum_{i=1}^{7} \alpha'_i y_1 \cdots \hat{x} \cdots y_7.$$

From $\Delta(y_7, x)f = 0$, we have

$$f_1 + f'_1 + \beta S_7(y_1, \ldots, y_6, x) = 0,$$

$$\alpha_i + \alpha'_i + (-1)^{i-1} \beta = 0, \ i = 1, \ldots, 7. \quad (5.46)$$

By the formulas in Lemma 2.1.1 we have

$$\alpha_1 + \alpha_2 + \alpha'_1 + \alpha'_2 = 0, \ \alpha_3 + \alpha_4 + \alpha'_3 + \alpha'_4 = 0, \ \alpha_5 + \alpha_6 + \alpha'_5 + \alpha'_6 = 0. \quad (5.47)$$

Since $\Delta(y_7, x)f = 0$ and $\Delta(y_6, x)f = 0$,

$$f_1(x, y_1, \ldots, y_5, x) + f_2 = 0, \ f'_1(x, y_1, \ldots, y_5, x) - f_2 = 0. \quad (5.48)$$

Let $w_i := y_1 \cdots \hat{x} \cdots y_7$ and $s := \mp 1$. Then (5.48) and $f^* = \pm f$ imply that

$$\alpha(w_i) + s\alpha(w_i^*) = 0,$$

where $\alpha(w)$ denotes the coefficient of $w$ in $f$. Thus by considering each $w_i$ and $w_i^*$ we have

$$\alpha_1 + s(-\alpha'_2) = 0, \ \alpha'_1 + s(-\alpha'_2) = 0, \ \alpha_2 + s(-\alpha_2) = 0,$$

$$\alpha_3 - s(-\alpha_2 - \alpha_7) = 0, \ \alpha'_3 - s(-\alpha_1 + \alpha_7) = 0, \ \alpha_4 - s(\alpha_2 - \alpha_6) = 0,$$

$$\alpha'_4 - s(\alpha_1 + \alpha_6) = 0, \ \alpha_5 - s(-\alpha_2 - \alpha_4) = 0, \ \alpha'_5 - s(-\alpha_1 + \alpha_5) = 0,$$

$$\alpha_6 - s(\alpha_2 - \alpha_4) = 0, \ \alpha'_6 - s(\alpha_1 + \alpha_4) = 0, \ \alpha_7 - s(-\alpha_2 - \alpha_3) = 0,$$

$$\alpha'_7 - s(-\alpha_1 + \alpha_3) = 0. \quad (5.49)$$

Solving the equations (5.46), (5.47) and (5.49) in the $s = \mp 1$ cases respectively yields that in each case there are only three parameters. That is in each case $f$ is a linear combination of three polynomials which are independent from $f$. This means $f - g$ is a linear combination

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of six polynomials which are independent from $f - g$. Since $g$ is a linear combination of six fixed polynomials, $f$ is a linear combination of twelve polynomials which are also independent from $f$. Thus $\text{Dim}(V I N_4[2, 1^7]) \leq 12$.

On the other hand we shall show that there exist 12 such linearly independent identities, so $V = V I N_4[2, 1^7]$.

Suppose there exist scalars $\alpha_i \in F$ such that $\sum_{i \neq 6, 13, 14} \alpha_i t_i = 0$. Then considering the coefficients of the following monomials
\begin{align*}
x^2 y_1 \cdots y_7, & \ y_1 y_2 x^2 y_3 \cdots y_7, \ xy_1 \cdots y_4 y_5 y_6 y_7, \ xy_1 y_2 y_3 \cdots y_7, \\
x y_1 x_2 \cdots y_7, & \ y_1 y_2 x_3 y_4 \cdots y_7, \ y_1 y_2 y_3 \cdots y_6 y_7, \end{align*}
\tag{5.50}
yields
\begin{align*}
\alpha_1 + \alpha_3 - \alpha_5 + \alpha_7 - \alpha_{15} & = 0, \\
\alpha_2 + 2\alpha_4 - \alpha_5 + \alpha_{11} + \alpha_{15} - \alpha_{16} & = 0, \\
3\alpha_3 - 3\alpha_5 + \alpha_7 - \alpha_{15} & = 0, \\
2\alpha_3 + \alpha_4 - \alpha_5 + \alpha_9 - \alpha_{12} - \alpha_{15} & = 0, \\
\alpha_3 - \alpha_7 + \alpha_{12} + 2\alpha_{15} - \alpha_{16} & = 0, \\
\alpha_{11} + \alpha_{12} + 2\alpha_{15} - 2\alpha_{16} & = 0, \\
4\alpha_4 - \alpha_5 + \alpha_{11} + \alpha_{15} - \alpha_{16} & = 0. \tag{5.51}
\end{align*}

Since $t_i^* = t_{i+1}$, for $i = 1, 3, 7, 9, 11, 15$ and $t_5^* = -t_3 - t_4 - t_6$ from $t_5 + t_5^* = -t_3 - t_4,$

\[\sum_{i \neq 6, 13, 14} \alpha_i t_i^* = 0\]

implies
\begin{align*}
\alpha_1 t_2 + \alpha_2 t_1 + (\alpha_4 - \alpha_5) t_3 + (\alpha_3 - \alpha_5) t_4 - & \\
\alpha_5 t_5 + \alpha_6 t_7 + \alpha_7 t_8 + \alpha_{11} t_9 + \alpha_{10} t_{10} + & + \\
\alpha_{12} t_{11} + \alpha_{13} t_{12} + \alpha_{16} t_{15} + \alpha_{15} t_{16} & = 0.
\end{align*}

Thus, considering the coefficients of the monomials in (5.50) in the polynomial $\sum_{i \neq 6, 13, 14} \alpha_i t_i^*$ we have

\[\alpha_2 + \alpha_4 + \alpha_8 - \alpha_{16} = 0,\]

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\[\alpha_1 + 2(\alpha_3 - \alpha_1) + \alpha_5 + \alpha_{12} + \alpha_{16} - \alpha_{15} = 0,\]
\[3\alpha_4 - \alpha_3 + 3\alpha_5 + \alpha_8 - \alpha_{16} = 0,\]
\[2(\alpha_4 - \alpha_5) + \alpha_3 + \alpha_{10} - \alpha_11 - \alpha_{16} = 0,\]
\[\alpha_4 - \alpha_5 - \alpha_8 + \alpha_{11} + 2\alpha_{16} - \alpha_{15} = 0,\]
\[\alpha_{12} + \alpha_{11} + 2\alpha_{16} - 2\alpha_{15} = 0,\]
\[-4(\alpha_3 - \alpha_5) + \alpha_5 + \alpha_{12} + \alpha_{16} - \alpha_{15} = 0.\]  
(5.52)

Let \(\{g_i| i = 1, \ldots, 6\}\) be the basis of \(V^*N_4[3, 1^{1^4}]\) given in Proposition 5.3.3. Then
\[
\Delta(y_7, x)t_2 = \Delta(y_7, x)t'_7 = g_4^* = \frac{1}{4}(-g_1 - g_3 + g_4) - g_4,
\]
\[
\Delta(y_7, x)t_4 = (-g_2 + 2g_1)^* = g_2 + \frac{1}{2}(g_1 + g_3 - g_4) - g_1,
\]
\[
\Delta(y_7, x)t_1 = g_4, \Delta(y_7, x)t_3 = -g_2 + 2g_1,
\]
\[
\Delta(y_7, x)t_5 = 2g_4, \Delta(y_7, x)t_7 = g_6, \Delta(y_7, x)t_8 = g_6^* = -g_5,
\]
\[
\Delta(y_7, x)t_{10} = g_5, \Delta(y_7, x)t_9 = g_5^* = -g_6, \Delta(y_7, x)t_{11} = g_6
\]
\[
\Delta(y_7, x)t_{12} = g_6^* = -g_5, \Delta(y_7, x)t_{15} = \Delta(y_7, x)t_{16} = 0.
\]

Thus \((\sum \alpha_i t_i)^* = 0\) implies
\[
\alpha_1 g_4 + \alpha_2 \left(\frac{1}{4}(-g_1 - g_3 + g_4) - g_4\right) +
\]
\[
\alpha_3(2g_1 - g_2) + \alpha_4(g_2 + \frac{1}{2}(g_1 + g_3 - g_4) - g_1) +
\]
\[2\alpha_5 g_4 + \alpha_7 g_6 - \alpha_8 g_3 - \alpha_9 g_6 + \alpha_{10} g_3 + \alpha_{11} g_6 - \alpha_{12} g_3 = 0.
\]

So
\[
-\frac{1}{4}\alpha_2 + 2\alpha_3 - \frac{1}{2}\alpha_4 = 0, \ -\alpha_3 + \alpha_4 = 0,
\]
\[
-\frac{1}{4}\alpha_2 + \frac{1}{2}\alpha_4 = 0, \ \alpha_1 - \frac{3}{4}\alpha_2 - \frac{1}{2}\alpha_4 + 2\alpha_5 = 0,
\]
\[-\alpha_8 + \alpha_{10} - \alpha_{12} = 0, \ \alpha_7 - \alpha_9 + \alpha_{11} = 0.
\]  
(5.53)

Solving (5.51), (5.52) and (5.53) yields
\[\alpha_i = 0, i \neq 7, 8, 9, 10, 15, 16\]

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and \( \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \alpha_{15} = \alpha_{16} \). Thus among these 13 identities, 12 of them are linearly independent. Since we have shown that \( \text{Dim}(\text{VIN}_4[2,1^7]) \leq 12 \) at the beginning of the proof, \( t_7 + t_8 + t_9 + t_{10} + t_{15} + t_{16} = 0 \). \] \[ \square \]

**Remark:** Proposition 5.3.5 holds for \( H_5 \). The proof is exactly the same.
Chapter 6

The Identities of Degree $2n + 1$ on $H_n$

6.1 Introduction

In Chapter 4 and Chapter 5 we have found out all identities of degree 7 and 9 on $H_3$ and $H_4$ respectively. A basis of $VIN_4[t]$, for each type $t$ was given. In the present chapter the above results for the identities of degree 9 on $H_4$ will be extended to $H_n$ for arbitrary $n \geq 4$. This is the content of Theorem 6.1 and Theorem 6.2.

Let $VIN_n[n_1, \ldots, n_r]$ denote the set of all identities of degree $2n + 1$ on $H_n$ which have type $[n_1, \ldots, n_r]$, and $VIN_n[n_1, \ldots, n_r]$ denote the set of all identities of degree $2n + 1$ on $H_n$ which have type $[n_1, \ldots, n_r]$ and are symmetric or skew symmetric in all their variables of degree $m$ according as $m$ is even or odd for any $m \in \{n_1, \ldots, n_r\}$.

We also define

$$e(n) := \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

**Theorem 6.1** If the characteristic of the field $F$ does not divide $e(n)!$, $|F| > 2n$ and $n \geq 4$ then:

1. For any type $[n_1, \ldots, n_r]$ with $\sum_{i=1}^r n_i = 2n + 1$ if

$$[n_1, \ldots, n_r] \neq [4, 1^{2n-3}], [3, 2, 1^{2n-4}], [3, 1^{2n-2}], [2^3, 1^{2n-5}],$$

$$[2, 2, 1^{2n-3}], [2, 1^{2n-1}] \text{ or } [1^{2n+1}]$$
then \(V I_n[n_1, \ldots, n_r] = \{0\}\).

2. The vector space \(V I N_n[2^3, 1^{2n-5}] = \{0\}\).

**Theorem 6.2** If the characteristic of the field \(F\) does not divide \(e(n)l\) and \(|F| > 2n\) then

1. \(\{T_{2n}(x^2, x, y_1, \ldots, y_{2n-3}; x)\}\) is a basis of the vector space \(V I_n[4, 1^{2n-3}]\).

2. \(\{T_{2n}(x^2, x, y, z_1, \ldots, z_{2n-4}; y), T_{2n}(x \circ y, x, y, z_1, \ldots, z_{2n-4}; x)\}\) is a basis of the space \(V I_n[3, 2, 1^{2n-4}]\).

3. The following identities form a basis of the space \(V I N_n[3, 1^{2n-2}]\):

\[
T_{2n}(x^2, y_1, \ldots, y_{2n-2}; x),\quad T_{2n}(x, y_1, \ldots, y_{2n-2}; x^2),
\]
\[
\sum_{i=1}^{2n-2} (-1)^i T_{2n}(x \circ y_i, y_1, \ldots, y_i, \ldots, y_{2n-2}; x; x),
\]
\[
\sum_{i=1}^{2n-2} (-1)^i T_{2n}(x^2, x, y_1, \ldots, y_i, \ldots, y_{2n-2}; y_i),
\]
\[
T_{2n}(x, y_1, \ldots, y_{2n-2}; x) x,\quad x T_{2n}(x, y_1, \ldots, y_{2n-2}; x).
\]

4. The identities

\[
T_{2n}(x^2, y, z_1, \ldots, z_{2n-3}; y) + T_{2n}(y^2, x, z_1, \ldots, z_{2n-3}; x),
\]
\[
T_{2n}(x \circ y, x, z_1, \ldots, z_{2n-3}; y) + T_{2n}(x \circ y, y, z_1, \ldots, z_{2n-3}; x),
\]
\[
\sum_{i=1}^{2n-3} (-1)^i (T_{2n}(x \circ z_i, z_1, \ldots, z_i, \ldots, z_{2n-3}; x, y; y) + T_{2n}(y \circ z_i, z_1, \ldots, z_i, \ldots, z_{2n-3}; y; x)),
\]
\[
x T_{2n}(x, y, z_1, \ldots, z_{2n-3}; y) + y T_{2n}(y, x, z_1, \ldots, z_{2n-3}; x),
\]
\[
T_{2n}(x, y, z_1, \ldots, z_{2n-3}; x) y + T_{2n}(y, x, z_1, \ldots, z_{2n-3}; y),
\]
\[
T_{2n}(x, y, z_1, \ldots, z_{2n-3}; [x, y]),\quad T_{2n}(x, y, z_1, \ldots, z_{2n-3}; [x, y])
\]
form a basis of the space \(V I N_n[2, 2, 1^{2n-3}]\).
5. The identities

\begin{align*}
T_{2n}(y_1, \ldots, y_{2n-1}; x^2), \\
\sum_{i=1}^{2n-1} (-1)^{i-1} T_{2n}(x \circ y_i, y_1, \ldots, \tilde{y}_i, \ldots, y_{2n-1}; x), \\
T_{2n}(y_1, \ldots, y_{2n-1}; x^2), \\
\sum_{i=1}^{2n-1} (-1)^{i-1} T_{2n}(x \circ y_i, y_1, \ldots, \tilde{y}_i, \ldots, y_{2n-1}; x), \\
T_{2n}(y_1, \ldots, y_{2n-1}; x) x, \\
x T_{2n}(y_1, \ldots, y_{2n-1}; x), \\
T_{2n}(y_1, \ldots, y_{2n-1}; x) x, \\
x T_{2n}(y_1, \ldots, y_{2n-1}; x), \\
\sum_{i=1}^{2n-1} (-1)^{i-1} y_i T_{2n}(y_1, \ldots, \tilde{y}_i, \ldots, y_{2n-1}; x; x), \\
\sum_{i=1}^{2n-1} (-1)^{i-1} T_{2n}(y_1, \ldots, \tilde{y}_i, \ldots, y_{2n-1}; x; y_i), \\
\sum_{i=1}^{2n-1} (-1)^{i-1} T_{2n}(x, y_1, \ldots, \tilde{y}_i, \ldots, y_{2n-1}); [x, y_i])
\end{align*}

form a basis of the space \( VIN_n[2, 1^{2n-1}] \).

6. \( S_{2n+1}(x_1, \ldots, x_{2n+1}) \) forms a basis of the space \( VIN_n[1^{2n+1}] \).

To prove these theorems we use induction on \( n \). For \( n = 4 \) it was shown in Chapter 5 that the Theorems hold. Assume that the theorems are also true for integers which are less than \( n \) then for \( n \) the proof will be completed in following sections.

6.2 The Proof of the Theorem 6.1

Lemma 6.2.1 Let \( f \) be an identity of \( II_n \) of degree \( 2n+1 \) with \( n > 4 \). Assume that \( f^* = \pm f \) and that \( f \) has no terms which start with \( y_{2y_1} \). If \( \text{char} F \nmid e(n)! \) and

\[
f = xyf_1 + \sim, \quad \text{or} \quad f = x^2yf_2 + \sim,
\]

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where \( \sim \) denote the sum of the terms of \( f \) which do not start with \( xy \) and \( x^2y \) respectively, then the following statements hold:

1. If \( f_1 = f_1(z, y_1, y_2, \ldots, y_{2n-3}) \) is of type \([4, 1^{2n-5}]\) or \( f_1 = f_1(z, w, y_1, y_2, \ldots, y_{2n-6}) \) is of type \([3, 2, 1^{2n-6}]\) then \( f_1 = 0 \).

2. If \( f_1 = f_1(z, y_1, \ldots, y_{2n-4}) \) is of type \([3, 1^{2n-4}]\) and skew symmetric in all its variables of degree 1 then \( f_1 \) is a scalar multiple of \( T_{2n-2}(z, y_1, \ldots, y_{2n-4}; z)z \).

3. If \( f_1 = f_1(z, w, y_1, \ldots, y_{2n-5}) \) is of type \([2, 2, 1^{2n-5}]\), symmetric in its variables of degree 2 and skew symmetric in its variables of degree 1 then \( f_1 \) is a scalar multiple of
   \[
   T_{2n-2}(z, w, y_1, \ldots, y_{2n-5}; w)z + T_{2n-2}(w, z, y_1, \ldots, y_{2n-5}; z)w.
   \]

4. If \( f_2 = f_2(z, y_1, y_2, \ldots, y_{2n-4}) \) is of type \([2, 1^{2n-4}]\) then \( f_2 = 0 \).

**Proof.** If \( f \) is an identity of \( H_n \) of degree \( 2n + 1 \) and

\[
f = xyf_1 + \sim, \text{ or } f = x^2yf_2 + \sim,
\]

then \( f_1, f_2 \) are identities of degree \( 2n - 1 \) and degree \( 2n - 2 \) on \( H_{n-1} \) respectively by Lemma 2.1.2.

1. If \( f_1 \) is of type \([4, 1^{2n-5}]\) then by the induction hypothesis there exists a scalar \( \alpha \in F \) such that

\[
f_1 = \alpha T_{2n-2}(z^2, z, y_1, \ldots, y_{2n-5}; z).
\]

So

\[
f_1 = \alpha T_{2n-4}(z^2, z, y_3, \ldots, y_{2n-5}; z)y_1y_2 + \sim
\]

and hence

\[
f = \alpha xy T_{2n-4}(z^2, z, y_3, \ldots, y_{2n-5}; z)y_1y_2 + \sim. \quad (6.1)
\]

Since \( f \) has no terms which start with \( y_2y_1 \) and \( f^* = \pm f \) by assumption, \( f \) has no terms which end in \( y_1y_2 \). So (6.1) implies that \( \alpha = 0 \) and therefore \( f_1 = 0 \).
2. If \( f_1 \) is of type \([3,2,1^{2n-4}]\) then by the induction hypothesis there exists scalars \( \alpha, \beta \in F \) such that

\[
f_1 = \alpha T_{2n-4}(z^2, z, w, y_1, \ldots, y_{2n-6}; w) + \\
\beta T_{2n-4}(z \circ w, z, w, y_1, \ldots, y_{2n-6}; z).
\]

So

\[
f_1 = \{ \alpha T_{2n-4}(z^2, z, w, y_3, \ldots, y_{2n-8}; w) + \\
\beta T_{2n-4}(z \circ w, z, w, y_3, \ldots, y_{2n-6}; z) \} y_1 y_2 + \sim.
\]

Therefore we have

\[
f = xy \{ \alpha T_{2n-4}(z^2, z, w, y_3, \ldots, y_{2n-8}; w) + \\
\beta T_{2n-4}(z \circ w, z, w, y_3, \ldots, y_{2n-6}; z) \} y_1 y_2 + \sim. \tag{6.2}
\]

Using the same argument as in the proof of part 1, (6.2) implies that

\[
\alpha T_{2n-4}(z^2, z, w, y_3, \ldots, y_{2n-6}; w) + \beta T_{2n-4}(z \circ w, z, w, y_3, \ldots, y_{2n-6}; z) = 0,
\]

hence \( \alpha = \beta = 0 \), since for \( n \geq 5 \), it can be shown by induction on \( n \) that

\[
T_{2n-4}(z^2, z, w, y_3, \ldots, y_{2n-6}; w), \ T_{2n-4}(z \circ w, z, w, y_3, \ldots, y_{2n-6}; z)
\]

are linear independent.

Let \( e_1^{n-1}, \ldots, e_6^{n-1} \) be the basis of the space \( V \ N_{n-1}[3,1^{2n-4}] \) given in Theorem 6.2. Then

\[
e_i^{n-1} = e_i^{n-2}[y_1, y_2] + \sim, \quad i = 1, 4, 6,
\]

\[
e_2^{n-1} = \{ e_2^{n-2} + e_5^{n-2} \}[y_1, y_2] + \sim,
\]

\[
e_3^{n-1} = \left\{ \begin{array}{ll}
e_3^{n-2}[y_1, y_2] + \sim & \text{if } n-1 \text{ is even} \\
(e_3^{n-2} - 2S_{2n-4}(z^2, z, y_3, \ldots, y_{2n-4}))[y_1, y_2] + \sim & \text{if } n-1 \text{ is odd.}
\end{array} \right.
\]

By [MR, Lemma 1.4 ]

\[
\sum_{i=1}^{2n} T_{2n}^i(x_1, \ldots, x_{2n}) = e(n) S_{2n}(x_1, \ldots, x_{2n}),
\]

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where \( \epsilon(n) = n \) if \( n \) is even, \( \epsilon(n) = n + 1 \) if \( n \) is odd. So
\[
S_{2n-4}(x^2, z, y_3, \ldots, y_{2n-4}) = \frac{1}{n-2}(t_4^{n-2} - t_1^{n-2} - t_3^{n-1}),
\]
when \( n - 1 \) is odd. Thus if \( n - 1 \) is odd then
\[
t_3^{n-1} = \{(t_3^{n-2} - \frac{2}{n-2}(t_4^{n-2} - t_1^{n-2} - t_3^{n-2}))\}[y_1, y_2] + \sim.
\]
Now, let \( f_1 \in VIN_{n-1}[3, 1^{2n-4}] \). Then \( f_1 = \sum_{i=1}^6 \alpha_i t_i^{n-1} \). To show that \( f_1 = 0 \) different arguments are given according as \( n \) is even or odd.

Case 1. If \( n - 1 \) is even then
\[
f_1 = \{ \sum_{i \neq 2, 5} \alpha_i t_i^{n-2} + \alpha_2(t_2^{n-2} + t_5^{n-2})\}[y_1, y_2] + \sim
\]
and
\[
f = xy\{ \sum_{i \neq 2, 5} t_i^{n-2} + \alpha_2(t_2^{n-2} + t_5^{n-2})\}[y_1, y_2] + \sim.
\]
Therefore \( f \) has no terms which start with \( y_1 y_2 \), and \( f^* = \pm f \) implies that
\[
\sum_{i \neq 2, 5} \alpha_i t_i^{n-2} + \alpha_2(t_2^{n-2} + t_5^{n-2}) = 0.
\]
So \( \alpha_i = 0 \) for \( i \neq 5 \), since \( t_1^{n-2}, \ldots, t_6^{n-2} \) are linearly independent by the induction hypothesis [Note that when \( n - 2 = 3 \), \( t_1^{n-2}, \ldots, t_6^{n-2} \) are also linear independent. See Proposition 4.2.4]. Hence \( f_1 = \alpha_5 T_{2n-2}(x, y_1, \ldots, y_{2n-4}; z)x \).

Case 2. If \( n - 1 \) is odd then we have
\[
\sum_{i \neq 2, 3, 5} \alpha_i t_i^{n-2} + \alpha_2(t_2^{n-2} + t_5^{n-2}) + \alpha_3\{t_3^{n-2} - \frac{2}{n-2}(t_4^{n-2} - t_1^{n-2} - t_3^{n-2})\} = 0.
\]
Thus \( \alpha_i = 0 \) for \( i \neq 5 \) by the linear independence of \( t_1^{n-2}, \ldots, t_6^{n-2} \). So part 2 holds in any case.

3. By the induction hypothesis, if \( f_1 \in VIN_{n-1}[2, 2, 1^{2n-3}] \) then
\[
f_1 = \sum_{i=1}^7 \alpha_i t_i^{n-1} = \{ \sum_{i \neq 3, 5} \alpha_i t_i^{n-2} + \alpha_3(t_3^{n-2} + t_5^{n-2})\}[y_1, y_2] + \sim,
\]
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where \( t_i^k, \ldots, t_k^k \) is the given basis of \( V \mathbb{N}_k[2, 2, 1^{2k-3}] \) for \( k = n - 1, n - 2 \) respectively by the induction hypothesis. So arguing as above we have \( \alpha_i = 0 \) for \( i \neq 5 \). Therefore

\[
f_1 = \alpha_5(\text{z}_2, \text{w}, \text{y}_4, \ldots, \text{y}_{2n-4}; \text{w}) \text{z} + \text{z}_2, \text{w}, \text{y}_4, \ldots, \text{y}_{2n-4}; \text{z} \text{w}).
\]

The proof of 4 is easy, completing the proof of the Lemma. 

**Proposition 6.2.1** If \( \text{char} \ F \not| \ e(n)! \) then the vector space \( VI_n[5, 1^{2n-4}] = \{0\} \). Therefore \( VI_n[5, n_1, \ldots, n_r] = \{0\} \) for any positive integers \( n_i \leq 5, \ i = 1, \ldots, r \) such that \( \sum_{i=1}^{r} n_i = 2n - 4 \).

**Proof.** Let \( f(y_0, y_1, \ldots, y_{2n-4}) \) be an identity of type \([5, 1^{2n-4}]\). By Lemma 2.1.1, if

\[
\begin{align*}
f &= y_i y_j f_1(y_0, y_1, \ldots, \text{y}_{i}, \ldots, y_{2n-4}) + y_i y_k f_2(y_0, y_1, \ldots, \text{y}_{k}, \ldots, y_{2n-4}) + \sim,
\end{align*}
\]

where at least one of \( i, j \) is nonzero and \( \sim \) denotes the sum of the terms of \( f \) which do not start with either \( y_i y_j \) or \( y_i y_k \), then both \( f_1 \) and \( f_2 \) are identities of \( II_{n-1} \). Thus by the induction hypothesis and the fact that \( II_{n-1} \) has no identity of degree \( 2n - 2 \) which is lower than \([2, 1^{2n-4}]\),

\[
f = \sum_{i=1}^{r} (y_0 y_i f_1(y_0, \ldots, y_{i}, \ldots, y_r) + y_i y_0 f_1(y_0, \ldots, y_{i}, \ldots, y_r))
\]

where \( f_i, f_i' \in VI_{n-1}[4, 1^{2n-5}] \). Thus if \( f^* = \pm f \) then \( f_i = f_i' = 0 \) by Lemma 6.2.1. So \( f = 0 \) when \( f^* = \pm f \). Since \( f \) can be decomposed into the sum of a \(+\)-symmetric identity and a \(-\)-skew symmetric identity, \( f = 0 \).

The second part of the proposition follows from the first part. 

Using Proposition 6.2.1 and Proposition 1.2.1 one can show easily that

**Corollary 6.2.1** If \( \text{char} \ F \ not divide \ e(n)! \) and \( |F| > 2n \) then for each \([n_1, \ldots, n_r] \leq [5, 1^{2n-4}] \) the vector space \( VI_n[n_1, \ldots, n_r] = \{0\} \).

From now on we assume that \( F \) does not divide \( e(n)! \) and \( |F| > 2n \).
Proposition 6.2.2 The vector space $VI_n[1, 3, 1^{2n-6}] = \{0\}$.

Proof. Let $f(x, y, z_1, \ldots, z_{2n-6}) \in VI_n[4, 3, 1^{2n-6}]$. Then using the same argument as in the proof of Proposition 6.2.1

$$f = xyf_1(x, y, z_1, \ldots, z_{2n-6}) + yzf_2(x, y, z_1, \ldots, z_{2n-6}),$$

where $f_1, f_2 \in VI_{n-1}[3, 2, 1^{2n-6}]$. We may assume that $f^* = \pm f$. So, the fact that $f$ has no terms which start with $z_iz_j$ implies that $f$ has no terms which end in $z_iz_j$ either. Thus by Lemma 6.2.1, $f_1 = f_2 = 0$. Hence $f = 0$. 

Proposition 6.2.3 The vector space $VI_n[4, 2, 1^{2n-5}] = \{0\}$.

Proof. Let $f(x, y, z_1, \ldots, z_{2n-5})$ be an identity of type $[4, 2, 1^{2n-5}]$. To show $f = 0$ we may assume that $f^* = \pm f$. By the induction hypothesis we have

$$f = xyf_1(x, y, z_1, \ldots, z_{2n-5}) + yzf_1'(x, y, z_1, \ldots, z_{2n-5}) + \sum_{i=1}^{2n-5} z_iy_1(x, y, z_1, \ldots, z_{2n-5}) + \sum_{i=1}^{2n-5} z_iz_1(x, y, z_1, \ldots, z_{2n-5}) + \sum_{i=1}^{2n-5} yiz_1(x, y, z_1, \ldots, z_{2n-5}) + \sum_{i=1}^{2n-5} z_iz_1(x, y, z_1, \ldots, z_{2n-5}) + \sum_{i=1}^{2n-5} z_iz_1(x, y, z_1, \ldots, z_{2n-5}),$$

where $f_1, f_1' \in VI_{n-1}[3, 2, 1^{2n-6}], f_3, f_3' \in VI_{n-1}[4, 1, 1^{2n-6}]$ and $f_1 \in VI_{n-1}[2, 1^{2n-4}]$. So, $f$ has no terms which start with $z_jz_k$ for $i \neq j, k$ by Lemma 6.2.1, this implies that $f_4 = f_{r_1} = f_{r_1}' = 0$ for $r = 2, 3$. Hence

$$f = xyf_1(x, y, z_1, \ldots, z_{2n-5}) + yzf_1'(x, y, z_1, \ldots, z_{2n-5}).$$

Since $\Delta(y, z)f \in VI_n[5, 1^{2n-4}] = \{0\}$, $f_1 = -f_1'$. Thus

$$f = [x, y]f_1(x, y, z_1, \ldots, z_{2n-5}).$$

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Because \( \Delta(z_1, y)f \in V I_n[4, 3, 1^{2n-6}] = \{0\} \) we have \( \Delta(z_1, y)f_1 = 0 \). So \( f_1 \) is skew symmetric in variables \( y, z_1, \ldots, z_{2n-5} \). That is, \( f_1 \in V I N_{n-1}[3, 1^{n-4}] \). Therefore \( f_1 = \alpha T_{2n-2}(x, y, z_1, \ldots, z_{2n-5}; x) x \) and

\[
f = \alpha(x, y) T_{2n-2}(x, y, z_1, \ldots, z_{2n-5}; x) x.
\]

So \( f \) has no terms which do not end in \( x \). Hence, \( f \) has no terms which do not start with \( x \) because \( f^* = \pm f \). Therefore \( \alpha = 0 \) and \( f \) is zero identically. \( \square \)

Using Corollary 6.2.1 and Proposition 6.2.3 one can easily show that

**Corollary 6.2.2** For each type \( [n_1, \ldots, n_r] \leq [4, 2, 1^{2n-5}] \) the vector space \( V I_n[n_1, \ldots, n_r] = \{0\} \).

**Proposition 6.2.4** The vector space \( V I_n[3, 3, 2, 1^{2n-7}] = \{0\} \).

**Proof.** Let \( f(x_1, x_2, y, z_1, \ldots, z_{2n-5}) \) be an identity of type \( [3, 3, 2, 1^{2n-7}] \). Then by the induction hypothesis and since \( H_{n-1} \) has no identity of degree \( 2n - 4 \) of type \( [2, 2, 1^{2n-6}] \) or lower,

\[
f = x_1 x_2 f_1 + x_2 x_1 f_2 + x_1 y f_3 + y x_1 f_4 + x_2 y f_5 + y x_2 f_6,
\]

where \( f_3, f_4, f_5, f_6 \in V I_{n-1}[3, 2, 1^{2n-7}] \). So they are zero by Lemma 6.2.1. Since \( \Delta(y, x_i)f \in V I_n[4, 3, 1^{2n-6}] = \{0\}, \) for \( i = 1, 2 \), \( \Delta(y, x_i)f_1 = 0 \). Hence \( f_1 \) is symmetric in the variables of degree 2 by Theorem 1.1. Moreover \( f_1 \) is also skew symmetric in the variables of degree 1 since \( V I_{n-1}[2^3, 1^{2n-9}] = \{0\} \). Thus \( f_1 \in V I N_{n-1}[2^3, 1^{2n-7}] = \{0\} \). Symmetrically we can show that \( f_2 = 0 \). Thus \( f = 0 \). \( \square \)

**Proposition 6.2.5** The vector space \( V I_n[3, 3, 1^{2n-5}] = \{0\} \).

**Proof.** Let \( f(x, y, z_1, \ldots, z_{2n-5}) \) be an identity of type \( [3, 3, 1^{2n-5}] \). We may assume that \( f^* = \pm f \). Since \( \Delta(z_i, z_j)f \in V I_n[3^2, 2, 1^{2n-7}] = \{0\} \) and \( \Delta(z, y)f \in V I_n[4, 2, 1^{2n-5}] = \{0\} \), \( V I_n[3, 3, 1^{2n-5}] = V I N_n[3, 3, 1^{2n-5}] \). So by the induction hypothesis we have

\[
f = x y f_1(x, y, z_1, \ldots, z_{2n-5}) - y x f_1(y, x, z_1, \ldots, z_{2n-5}) + 
\]
\[ \sum_{i=1}^{2n-5} xz_i f_{2i}(x, y, z_1, \ldots, z_{2n-5}) + \]
\[ \sum_{i=1}^{2n-5} z_i x f'_{2i}(x, y, z_1, \ldots, z_{2n-5}) - \]
\[ \sum_{i=1}^{2n-5} yz_i f_{2i}(x, y, z_1, \ldots, z_{2n-5}) - \]
\[ \sum_{i=1}^{2n-5} z_i y f'_{2i}(x, y, z_1, \ldots, z_{2n-5}) + \]
\[ x^2 y f_3(x, y, z_1, \ldots, z_{2n-5}) - y^2 x f_3(y, x, z_1, \ldots, z_{2n-5}), \]

where \( f_{2i}, f'_{2i} \in VI_{n-1}[3, 2, 1^{2n-6}], f_3 \in VI_{n-1}[2, 1^{2n-4}] \). So, by Lemma 6.2.1, \( f_3 = f_{2i} = f'_{2i} = 0 \), for \( i = 1, \ldots, 2n - 5 \). Hence

\[ f = xy f_1(x, y, z_1, \ldots, z_{2n-5}) - yx f_1(y, x, z_1, \ldots, z_{2n-5}). \]

This implies that \( f \) has no terms which do not start with \( xy \) or \( yx \), therefore it has no terms which do not end in \( xy \) or \( yx \) by \( f^* = \pm f \).

On the other hand, since \( \Delta(x, y)f = 0 \) implies that \( \Delta(x, y)f_1 = 0, f_1 \in VIN_{n-1}[2, 2, 1^{2n-5}] \). So by Lemma 6.2.1

\[ f_1 = \alpha(T_{2n-2}(x, y, z_1, \ldots, z_{2n-5}; y) x + T_{2n-2}(y, x, z_1, \ldots, z_{2n-5}; x)y). \]

So there exists a monomial in \( f_1 \) therefore a monomial in \( f \) which does not end in either \( xy \) or \( yx \) with coefficient \( \alpha \). Thus \( \alpha = 0 \). So \( f = 0 \). \(

**Proposition 6.2.3** The vector space \( VI_n[3, 2^3, 1^{2n-8}] = \{0\} \).

**Proof.** Let \( f(x, y_1, y_2, y_3, z_1, \ldots, z_{2n-8}) \) be an identity of type \( [3, 2^3, 1^{2n-8}] \). Then

\[ f = xy_3 f_1(x, y_1, y_2, y_3, z_1, \ldots, z_{2n-8}) + \cdots \]
\[ + y_3 x f'_{2}(x, y_1, y_2, y_3, z_1, \ldots, z_{2n-8}) x + \cdots \]
\[ + y_2 y_3 f_2(x, y_1, y_2, y_3, z_1, \ldots, z_{2n-8}) + \cdots. \]
We may assume that $f^* = \pm f$. Since $f_2 \in VI_{n-1}[3,2,1^{2n-6}]$ and $f$ has no terms which start with $z_i z_j$, $f_2 = 0$ by Lemma 6.2.1. So

$$f = xy_3 f_1(x, y_1, y_2, z_1, \ldots, z_{2n-8}) + \ldots + y_3 x f'_1(x, y_1, y_2, z_1, \ldots, z_{2n-8}) x + \ldots.$$

Since $\Delta(y_1, x)f \in VI_n[4,2,2,1^{2n-7}] = \{0\}$, $\Delta(y_1, x)f_1 \in VI_n[4,2,2,1^{2n-7}] = \{0\}$. Hence $f_1$ is symmetric in the variables $x$ and $y_1$ by Theorem 1.1. While $\Delta(y_2, x)f = 0$ and $\Delta(z_i, y_3)f = 0$ imply that $f_1$ is symmetric in the variables $x, y_2$, and is skew symmetric in the variables $z_i, y_3$. Thus

$$f_1(x, y_1, y_2, y_3, z_1, \ldots, z_{2n-8}) \in VI_{n-1}[2^3,1^{2n-7}] = \{0\}.$$

Therefore $f_1 = 0$. Similarly we can show that $f'_1 = 0$. So $f = 0$. \hfill \blacksquare

**Proposition 6.2.7** The vector space $VI_n[3,2^2,1^{2n-6}] = \{0\}$. Therefore $VI_n[3,2^2,n_1,\ldots,n_r] = \{0\}$, for any type $[3,2^2,n_1,\ldots,n_r]$ with

$$[4,1^{2n-5}] < [3,2^2,n_1,\ldots,n_r] \leq [3,2^2,1^{2n-6}].$$

**Proof.** Let $f(x, y_1, y_2, z_1, \ldots, z_{2n-6})$ be an identity of type $[3,2^2,1^{2n-6}]$ then

$$f = xy_3 f_1(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \ldots + y_2 x f'_1(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \ldots + x z_1 f_2(x, y_1, y_2, z_2, \ldots, z_{2n-6}) + \ldots + z_1 x f'_2(x, y_1, y_2, z_2, \ldots, z_{2n-6}) + \ldots + y_1 y_2 f_3(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \ldots + y_2 z_1 f_4(x, y_1, y_2, z_2, \ldots, z_{2n-6}) + \ldots + z_1 y_2 f'_4(x, y_1, y_2, z_2, \ldots, z_{2n-6}) + \ldots + x^2 y_2 f_5(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \ldots.$$

We may assume that $f^* = \pm f$. Thus by Lemma 6.2.1, $f$ has no terms which start with $z_i z_j$, which implies that $f_4 = f'_4 = f_5 = 0$ since $f_4, f'_4 \in VI_{n-1}[3,2,1^{2n-6}]$ and $f_5 \in$
\[ V \text{I}_{n-1}[2, 1^{2n-4}] \]. Hence

\[
f = x y_2 f_1(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \cdots + y_2 x f'_1(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \cdots + x z_1 f_2(x, y_1, y_2, z_2, \ldots, z_{2n-6}) + \cdots + z_1 x f'_2(x, y_1, y_2, z_2, \ldots, z_{2n-6}) + \cdots + y_1 y_2 f_3(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \cdots. \tag{6.3}
\]

Since \( \Delta(z_1, y_1)f \in V \text{I}_{n}[3, 3, 2, 1^{2n-7}] = \{0\} \), \( \Delta(z_1, y_1)f_3 = 0 \) by (6.3). Similarly, we have \( \Delta(z_1, y_2)f_3 = 0 \) by (6.3). So \( f_3 \) is skew symmetric in \( y_1, y_2, z_1 \). Since \( V \text{I}_{n}[3, 2^3, 1^{2n-8}] = \{0\} \), \( \Delta(z_1, z_i)f = 0 \) implies that \( \Delta(z_j, z_i)f = 0 \) and hence \( f \) is skew symmetric in all \( z_i \)'s. So \( f_3 \in V \text{I} \text{I}_{n-1}[3, 1^{2n-4}] \). Thus by Lemma 6.2.1

\[ f_3 = \alpha T_{2n-2}(x, y_1, y_2, z_1, \ldots, z_{2n-6}; x) x. \]

The monomial \( x y_1 y_2 z_1 \cdots z_{2n-6} x \) has coefficient \( (-1)^n \alpha \) in \( f_3 \). So the monomial \( y_1 y_2 x y_1 y_2 z_1 \cdots z_{2n-6} x x \) has coefficient \( (-1)^n \alpha \) in \( f \). Thus \( \alpha = 0 \) by (6.3) and \( f^* = \pm f \). Thus \( f_3 = 0 \).

Next we show that \( f_2 = f'_2 = 0 \) by showing that \( f_2, f'_2 \in V \text{I} \text{I}_{n-1}[2^3, 1^{2n-7}] \). By the induction hypothesis \( V \text{I} \text{I}_{n-1}[2^3, 1^{2n-7}] = \{0\} \), and it suffices to show that \( f_2, f'_2 \) is symmetric in \( x, y_1, y_2 \). Since \( \Delta(y_1, x)f = 0 \) and \( f_3 = 0 \), \( \Delta(y_1, x)f_2 = \Delta(y_1, x)f'_2 = 0 \). So \( f_2, f'_2 \) are symmetric in \( x, y_1 \) by Theorem 1.1. Similarly we have \( f_2, f'_2 \) are symmetric in \( x, y_2 \). Thus \( f_2, f'_2 \in V \text{I} \text{I}_{n-1}[2^3, 1^{2n-7}] \). So

\[
f = x y_2 f_1(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \cdots + y_2 x f'_1(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \cdots. \]

This and \( \Delta(y_2, x)f = 0 \) imply that \( f_1 = -f'_1 \). Thus

\[
f = [x, y_2] f_1(x, y_1, y_2, z_1, \ldots, z_{2n-6}) + \cdots. \tag{6.4}
\]

We know \( \Delta(y_1, x)f = 0 \) and \( \Delta(z_1, y_2)f = 0 \) by (6.4). This implies that \( \Delta(y_1, x)f_1 = 0 \) and \( \Delta(z_1, y_2)f_1 = 0 \). Hence \( f_1 \in V \text{I} \text{I}_{n-1}[2, 2, 1^{2n-5}] \). So

\[
f_1 = \alpha(T_{2n-2}(x, y_1, y_2, z_1, \ldots, z_{2n-6}; y_1)x + T_{2n-2}(y_1, x, y_2, z_1, \ldots, z_{2n-6}; x) y_1) \]

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and

\[ f = \alpha[x, y_1](T_{2n-2}(x, y_1, y_2, z_1, \ldots, z_{2n-6}; y_1) x + T_{2n-2}(y_1, x, y_2, z_1, \ldots, z_{2n-6}; x) y_1 + \cdots. \]  

(6.5)

Since the monomial \( x y_2 y_1 x y_1 y_2 z_1 \cdots z_{2n-6}x \) has coefficient \( \alpha \) and the monomial \( x y_2 y_1 x y_1 y_2 z_1 \cdots z_{2n-6} \) has coefficient 0 by (6.5), \( \alpha = 0 \) and \( f = 0 \).

**Proposition 6.2.8** The vector space \( VI_n[2^4, 1^{2n-7}] = \{0\} \). Therefore \( VI_n[2^4, n_1, \ldots, n_r] = \{0\} \) for any type \([2^4, n_1, \ldots, n_r] \) with

\[ [3, 1^{2n-5}] < [2^4, n_1, \ldots, n_r] \leq [2^4, 1^{2n-6}] \].

**Proof.** Let \( f(x_1, x_2, x_3, x_4, z_1, \ldots, z_{2n-7}) \) be an identity of type \([2^4, 1^{2n-7}]\). Then \( \Delta(x_4, x_1)f \in VI_n[3, 2, 2, 1^{2n-6}] \). By Proposition 6.2.7, \( VI_n[3, 2, 2, 1^{2n-6}] = \{0\} \). So \( \Delta(x_4, x_1)f = 0 \) and hence \( f \) is symmetric in \( x_1, x_4 \) by Theorem 1.1. Similarly \( f \) is symmetric in other \( x_i \)'s. So \( f \) is symmetric in all its variables of degree two. Next we claim

If \( f = z_1 z_2 f_1 + z_2 x_4 f_2 + \cdots \), then \( f_1, f_2 \in VIN_{n-1}[2^3, 1^{2n-5}] \).  

(6.6)

Obviously \( f_1, f_2 \in VIN_{n-1}[2^3, 1^{2n-5}] \) by Lemma 2.1.1. \( f_1, f_2 \) are symmetric in the variables of degree 2 since \( f \) is. By the induction hypothesis \( VI_{n-1}[2^4, 1^{2n-7}] = \{0\} \). Hence \( f_1, f_2 \) are skew symmetric in their variables of degree 1. So the claim holds. Since \( VIN_{n-1}[2^3, 1^{2n-7}] = \{0\} \), the claim implies that \( f_1 = f_2 = 0 \). Therefore

\[ f = x_3 x_4 f_1(x_1, x_2, x_3, x_4, z_1, \ldots, z_{2n-7}) + x_4 x_3 f_1(x_1, x_2, x_4, x_3, z_1, \ldots, z_{2n-7}) + \cdots \]

This equation and \( \Delta(x_3, x_4)f = 0 \) imply that

\[ f_1(x_1, x_2, x_3, x_4, z_1, \ldots, z_{2n-7}) + f_1(x_1, x_2, x_4, x_3, z_1, \ldots, z_{2n-7}) = 0. \]

So

\[ f = [x_3, x_4]f_1(x_1, x_2, x_3, x_4, z_1, \ldots, z_{2n-7}) + \cdots. \]
From this equation and \(\Delta(x_1, x_2)f \in VI_n[3, 2, 2, 1^{2n-6}] = \{0\}\) we know that \(f_1\) is symmetric in the variables of degree 2 by Theorem 1.1. While \(\Delta(x_i, x_3)f = 0\), for \(i = 1, \ldots, 2n - 7\) implies that \(\Delta(x_i, x_3)f_1 = 0\). So \(f_1\) is skew symmetric in \(x_3, x_i\). Since \(f_1\) is skew symmetric in \(x_3, x_4\) also, \(f_1 \in VI_{n-1}[2, 2, 1^{2n-5}]\). Thus by Lemma 6.2.1,

\[
f_1 = \alpha(T_{2n-2}(x_1, x_2, x_3, x_4, z_1, \ldots, z_{2n-7}; x_2)z_1 + T_{2n-2}(x_2, x_1, x_3, x_4, z_1, \ldots, z_{2n-7}; x_1)z_2).
\]

Hence

\[
f = \alpha(x_3, x_4)(T_{2n-2}(x_1, x_2, x_3, x_4, z_1, \ldots, z_{2n-7}; x_2)z_1 + T_{2n-2}(x_2, x_1, x_3, x_4, z_1, \ldots, z_{2n-7}; x_1)z_2) + \ldots.
\]

From this equation we know that the monomial \(x_3x_4x_2x_1x_3x_2x_4x_1 \cdots z_{2n-7}z_1\) has coefficient \(-\alpha\) while its \(\ast\) has coefficient 0. So if \(f^* = \pm f\) then \(\alpha = 0\) and therefore \(f = 0\) for any \(f\) in \(VI_n[2^4, 1^{2n-7}]\).

**Proposition 6.2.9** The vector space \(VIN_n[2^3, 1^{2n-5}] = \{0\}\).

**Proof.** Let \(f(x_1, x_2, x_3, z_1, \ldots, z_{2n-5}) \in VIN_n[2^3, 1^{2n-5}]\). It suffices to show \(f = 0\) when \(f^* = \pm f\). Since

\[
VIN_{n-1}[2^3, 1^{2n-7}] = VI_{n-1}[2, 2, 1^{2n-6}] = \{0\},
\]

\[
f = x_2x_3f_1(x_1, x_2, x_3, z_1, \ldots, z_{2n-5}) + x_3x_2f_1(x_1, x_2, x_3, z_1, \ldots, z_{2n-5}) + \ldots + x_1x_2f_1(x_1, x_2, x_3, z_2, z_3, \ldots, z_{2n-5}) + \ldots + x_3x_2f_2(x_1, x_2, z_1, z_2, \ldots, z_{2n-5}) + \ldots + x_3x_2f_3(x_1, x_2, z_1, \ldots, z_{2n-5}) + \ldots.
\]

From Lemma 6.2.1, \(f_3 = 0\). So \(f\) has no terms which start with \(x_1^2\). Thus \(\Delta(z_1, x_3)f = 0\) implies that \(f_2 + f_2' = 0\), and \(\Delta(z_2, x_3)f = 0\) implies that \(f_2\) is skew symmetric in the variables of degree 1. Since \(f\) is symmetric in the variables of degree 2 we have \(f_2\) is also

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symmetric in the variables of degree 2. Hence \( f_2 \in \text{VIN}_{n-1}[2,2,1^{2n-5}] \). So by Lemma 6.2.1,

\[
f_2 = \alpha(T_{2n-2}(x_1, x_2, x_3, z_2, \ldots, z_{2n-5}; x_2)x_1 + T_{2n-2}(x_2, x_1, x_3, z_2, \ldots, z_{2n-5}; x_1)x_2).
\]

In \( f_2 \), the monomial \( x_2x_3x_2^2z_2 \cdots z_{2n-5}x_1x_1 \) has coefficient \(-\alpha\). So the monomial \( x_1x_3x_2^2x_2^2z_2 \cdots z_{2n-5}z_1x_1 \) has coefficient \(-\alpha\) in \( f \). But its reverse has coefficient 0 in \( f \). So \( \alpha = 0 \) and hence \( f_2 = 0 \). Thus

\[
f = x_2x_3f_1(x_1, x_2, x_3, z_2, \ldots, z_{2n-5}) + x_3x_2f_1(x_1, x_3, x_2, z_1, \ldots, z_{2n-5}) + \cdots.
\]

By this expression and \( \Delta(z_1, x_2)f = \Delta(z_1, x_3)f = 0 \) we have \( f_1 \in \text{VIN}_{n-1}[2, 1^{2n-3}] \). Especially, \( f_1 \) is skew symmetric in \( x_2, x_3, z_i \). Since \( f^* = \pm f \) and \( f \) has no terms which start with \( z_iz_j, z_ix_j, z_jx_i \) we obtain that \( f_1 \) has no terms which end in \( z_iz_j, z_ix_j, z_jx_i \). So by the symmetry of \( f_1 \) it also has no terms which end in \( z_iz_j \). Thus \( f_1 = 0 \) and therefore \( f = 0 \). \( \blacksquare \)

From Corollary 6.2.2, Proposition 6.2.7, Proposition 6.2.8 and Proposition 6.2.9 we have Theorem 6.1.

### 6.3 The Proof of the Theorem 6.2

**Lemma 6.3.1** Let \( f(x_1, \ldots, x_s, y_1, \ldots, y_t) \) be an identity of degree \( 2n + 1 \) on \( H_n \) which is skew symmetric in all variables \( y_i \)'s and has degree 1 in these \( y_i \)'s. If \( t \geq 4 \) and

\[
f = \left[y_1, y_2 \right] f_1[y_{t-1}, y_t] + \sim,
\]

where \( \sim \) denotes the sum of the terms of \( f \) which do not start with one of \( y_1y_2, y_2y_1 \) or which do not end with one of \( y_{t-1}y_t, y_ty_{t-1} \), then

1. \( f_1 \) is an identity on \( H_{n-2} \).

2. If \( f^* = \pm f \) then \( f^*_1 = \pm f_1 \).
3. If \( \Delta(y_i, x_j)f = 0 \) then \( \Delta(y_i, x_j)f_1 = 0 \).

The proof is easy. We skip it. \( \blacksquare \)

**Lemma 6.3.2** The identities

\[
T_{2n}(y_1, \ldots, y_{2n-2}, x; x), \quad \widetilde{T}_{2n}(y_1, \ldots, y_{2n-2}, x; x)
\]

are \( \ast \)-skew symmetric.

**Proof.** From the definition of \( T_{2n} \) and \( \widetilde{T}_{2n} \), \( S_{2n} = T_{2n} + \widetilde{T}_{2n} \). Thus

\[
S_{2n}(x, y_1, \ldots, y_{2n-2}, x) = T_{2n}(y_1, \ldots, y_{2n-2}, x; x) + \widetilde{T}_{2n}(y_1, \ldots, y_{2n-2}, x; x)
\]

So

\[
T_{2n}(y_1, \ldots, y_{2n-2}, x; x) + \widetilde{T}_{2n}(y_1, \ldots, y_{2n-2}, x; x) = 0. \quad (6.7)
\]

On the other hand, by the definition of \( T_{2n} \) and \( \widetilde{T}_{2n} \), \( T_{2n} = \widetilde{T}_{2n} \) when \( n \) is even. Thus this equation and (6.7) imply that

\[
T_{2n}(y_1, \ldots, y_{2n-2}, x; x), \quad \widetilde{T}_{2n}(y_1, \ldots, y_{2n-2}, x; x)
\]

are \( \ast \)-skew symmetric. When \( n \) is odd, we always have that \( T_{2n} \) and \( \widetilde{T}_{2n} \) are both \( \ast \)-skew symmetric by their definitions. \( \blacksquare \)

**Proposition 6.3.1** The identity \( T_{2n}(x^2, x, y_1, \ldots, y_{2n-3}; x) \) forms a basis of the vector space \( V I_n[1, 1^{2n-3}] \).

**Proof.** Since \( V I_n[4, 2, 1^{2n-5}] = \{0\} \) by Theorem 6.1, \( V I_n[4, 1^{2n-3}] = V I_n[4, 1^{2n-3}] \). Let \( f(x, y_1, \ldots, y_{2n-3}) \in V I_n[4, 1^{2n-3}] \). Then

\[
f = \sum_{i<j} (-1)^{i+j-1}[y_i, y_j]f_2(x, y_1, \ldots, y_i, \ldots, y_j, \ldots, y_{2n-3}) + \sim
\]

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and \( f_2 \in \text{V}I\text{N}_{n-1}[1, 1^{2n-5}] \). Therefore

\[
f_2(x, y_3, \ldots, y_{2n-3}) = \alpha T_{2n-2}(x^2, x, y_3, \ldots, y_{2n-3}; x)
\]

for some \( \alpha \in F \). Subtracting a multiple of \( T_{2n} \) from \( f \) if necessary, we may assume that \( f_2 = 0 \). Since

\[
T_{2n-2}(x^2, x, y_3, \ldots, y_{2n-3}; x) = T_{2n-4}(x^2, x, y_3, \ldots, y_{2n-5}; x)[y_{2n-4}, y_{2n-3}] + \sim,
\]

\[
f = \alpha[y_1, y_2]T_{2n-4}(x^2, x, y_3, \ldots, y_{2n-5}; x)[y_{2n-4}, y_{2n-3}] + \sim.
\]

Hence from Lemma 6.3.1 and Lemma 6.3.2, \( \alpha = 0 \) if \( f^* = f \).

If \( f^* = -f \) then

\[
f - \alpha T_{2n}(x^2, x, y_1, \ldots, y_{2n-3})
\]

has no terms which start with \( y_1 y_2 \) therefore has no terms which start with \( y_i y_j \) and is \( * \)-skew symmetric. Thus we may assume that \( f^* = \pm f \) and has no terms which start with \( y_i y_j \). So by Lemma 6.2.1, \( f \) has no terms which start with \( x^2 y_i \) and hence

\[
f = \sum_{i=1}^{2n-3} (-1)^{i-1} x y_i \{f_1(x, y_1, \ldots, \widehat{y_i}, \ldots, y_{2n-3}) + y_i x \{f'_1(x, y_1, \ldots, \widehat{y_i}, \ldots, y_{2n-3})\}.
\]

Since \( \Delta(y_1, x) f \in \text{V}I_n[5, 1^{2n-4}] = \{0\} \), \( f_1 + f'_1 = 0 \). So

\[
f = \sum_{i=1}^{2n-3} (-1)^{i-1}[x, y_i]f_1(x, y_1, \ldots, \widehat{y_i}, \ldots, y_{2n-3}).
\]

By Lemma 6.2.1,

\[
f_1 = \alpha T_{2n-2}(x, y_{\sigma(2)}, \ldots, y_{\sigma(2n-3)}; x)x
\]

and

\[
f = \alpha \sum_{i=1}^{2n-3} (-1)^{i-1}[x, y_i]T_{2n-2}(x, y_1, \ldots, \widehat{y_i}, \ldots, y_{2n-3}; x)x. \tag{6.8}
\]

Therefore \( f \) has no terms which end in \( xy_i \) by (6.8). But the monomial \( y_1 x y_2 y_3 \cdots y_{2n-3} x \) has coefficient \(-\alpha \). Thus \( \alpha = 0 \) by \( f^* = \pm f \) and hence \( f = 0 \).
Proposition 6.3.2 The identities

\[ T_{2n}(x^2, x, y, z_1, \ldots, z_{2n-4}; y), \]
\[ T_{2n}(x \circ y, x, y, z_1, \ldots, z_{2n-4}; x) \]

form a basis of the vector space \( VIN_n[3, 2, 1^{2n-4}] \).

Proof. Let

\[ t_1^n := T_{2n}(x^2, x, y, z_1, \ldots, z_{2n-4}; y), \]
\[ t_2^n := T_{2n}(x \circ y, x, y, z_1, \ldots, z_{2n-4}; x). \]

Then

\[ t_1^n = T_{2n-2}(x^2, x, y, z_1, \ldots, z_{2n-6}; y)[z_{2n-5}, z_{2n-4}] + \sim \]
\[ = t_1^{n-1}[z_{2n-5}, z_{2n-4}] + \sim, \]
\[ t_2^n = T_{2n}(x \circ y, x, y, z_1, \ldots, z_{2n-6}; y)[z_{2n-5}, z_{2n-4}] + \sim \]
\[ = t_2^{n-1}[z_{2n-5}, z_{2n-4}] + \sim. \] (6.9)

Since \( t_1^4, t_2^4 \) are linearly independent it can be shown that \( t_1^n, t_2^n \) are linearly independent for \( n \geq 4 \) by induction on \( n \) and (6.9). Thus we need only to show that \( t_1^n, t_2^n \) span \( VIN_n[3, 2, 1^{2n-4}] \).

Let \( f \in VIN_n[3, 2, 1^{2n-4}] \). We may assume that \( f^* = \pm f \). Since \( f \) is skew symmetric in its variables of degree 1,

\[ f = \sum_{1 \leq i < j \leq 2n-4} (-1)^{i+j-1} f_4(x, y, z_1, \ldots, \xi_i, \ldots, \xi_j, \ldots, z_{2n-4})[z_i, z_j] + \sim, \] (6.10)

where \( f_4 \in VIN_{n-1}[3, 2, 1^{2n-9}] \). Hence by the induction \( f_4 = \alpha t_1^{n-1} + \beta t_2^{n-1} \) for some \( \alpha, \beta \in F \).

Next, we correct \( f \) by a suitable linear combination \( g \) of \( t_1^n \) and \( t_2^n \) so that \( f - g \) has no terms which end in \( z_i z_j \), and \( f - g, f \) have the same symmetry.
Let
\[
\tilde{t}_1^n := \tilde{T}_{2n}(x^2, x, y, z_1, \ldots, z_{2n-4}; y),
\]
\[
\tilde{t}_2^n := \tilde{T}_{2n}(x \circ y, x, y, z_1, \ldots, z_{2n-4}; x).
\]

Then
\[
f_4 = [z_1, z_2](\alpha \tilde{t}_1^{n-2} + \beta \tilde{t}_2^{n-2}) + \sim,
\]
and from (6.9)
\[
f = [z_1, z_2](\alpha \tilde{t}_1^{n-2} + \beta \tilde{t}_2^{n-2})[z_{2n-5}, z_{2n-4}] + \sim.
\]

Thus from Lemma 6.3.1, \( \alpha \tilde{t}_1^{n-2} + \beta \tilde{t}_2^{n-2} \) and \( f \) have the same symmetry. Since \( \tilde{t}_1^n = -\tilde{t}_1^n \) and \( \tilde{t}_2^n \) is \(*\)-skew symmetric, \( \alpha \tilde{t}_1^{n-2} + \beta \tilde{t}_2^{n-2} \) is \(*\)-skew symmetric when \( \alpha \) or \( \beta \neq 0 \). So if \( f^* = f \) then \( f_4 = 0 \). That is \( f \) has no terms which end in \( z_1 z_j \). If \( f^* = -f \), let \( g := \alpha \tilde{t}_1^n + \beta \tilde{t}_2^n \) then \( g \) is \(*\)-skew symmetric and \( g \in VJN_n[3, 2, 1^{2n-4}] \). From (6.9) and (6.10), \( f - g \) has no terms which end in \( z_{2n-5} z_{2n-4} \) therefore has no terms which end in \( z_1 z_j \) by the skew symmetry of \( g \) and \( g \) in their variables of degree 1.

Now we may assume that \( f \) has no terms which end in \( z_1 z_j \) and \( f^* = \pm f \). Applying Lemma 6.2.1 to \( f^* \) we have

\[
f = f_1(x, y, z_1, \ldots, z_{2n-4})xy + f'_1(x, y, z_1, \ldots, z_{2n-4})yx +
\]
\[
\sum_{i=1}^{2n-4} (-1)^i (f_2(x, y, z_1, \ldots, \tilde{z}_i, \ldots, z_{2n-4})xz_i + f'_2(x, y, z_1, \ldots, \tilde{z}_i, \ldots, z_{2n-4})yz_i) +
\]
\[
\sum_{i=1}^{2n-4} (-1)^i (f_3(x, y, z_1, \ldots, \tilde{z}_i, \ldots, z_{2n-4})yz_i + f'_3(x, y, z_1, \ldots, \tilde{z}_i, \ldots, z_{2n-4})xz_i) +
\]
\[
f_5(x, y, z_1, \ldots, z_{2n-4})yx^2.
\]

Since \( \Delta(z_1, y) f \in VJ_n[3, 3, 1^{2n-5}] = \{0\} \), by considering the sum of the terms which end in \( yx^2 \) we have \( \Delta(z_1, y) f_5 = 0 \) from (6.11). This implies that \( f_5 \) is skew in \( y, z_1 \), therefore skew in \( y, z_i \). So, since \( f^* = \pm f \) and has no terms which end in \( z_1 z_j \),

\[
f_5 = \alpha \sum_{\sigma \in S_{2n-3}} y_{\sigma(1)} \cdots y_{\sigma(2n-3)} + \beta \sum_{\nu \in S_{2n-3}} y_{\nu(1)} x \cdots y_{\nu(2n-3)},
\]

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where \((y_1, \ldots, y_{2n-3}) = (y, z_1, \ldots, z_{2n-4})\).

We know that \(f_5\) is an identity of degree \(2n - 2\) on \(\tilde{H}_{n-1}\). We apply Lemma 3.3 of [MR] and consider for the monomial \(y_1y_2 \cdots y_{2n-3}\) and the position \((i, j) = (2, 3)\). This yields \(\beta = 0\). We apply Lemma 3.1 of [MR] to the monomial \(y_1y_2 \cdots y_{2n-3}\). This gives \(\alpha = 0\). So

\[
\begin{align*}
    f &= f_1(x, y, z_1, \ldots, z_{2n-4})xy + f'_1(x, y, z_1, \ldots, z_{2n-4})yx + \\
    &+ \sum_{i=1}^{2n-4} (-1)^i (f_2(x, y, z_1, \ldots, z_i, \ldots, z_{2n-4})xz_i + f'_2(x, y, z_1, \ldots, z_i, \ldots, z_{2n-4})zx) + \\
    &+ \sum_{i=1}^{2n-4} (-1)^i (f_3(x, y, z_1, \ldots, z_i, \ldots, z_{2n-4})yz_i + f'_3(x, y, z_1, \ldots, z_i, \ldots, z_{2n-4})yz) \\
    &\quad \text{(6.12)}
\end{align*}
\]

Since \(\Delta(z_i, x)f = 0\) and \(\Delta(z_i, y)f = 0\), from (6.12) we have \(f_i = -f'_i\) for \(i = 2, 3\). We also have that \(f_1 = -f'_1\). Indeed, from the proof of Proposition 6.3.1 we know that every identity \(g\) in \(VIN_{n}[4, 1^{2n-3}]\) which has no terms ending in \(z_iz_j\) is zero identically if \(g^* = \pm g\). Since \(f\) has no terms which end in \(z_iz_j\) and \(f^* = \pm f\), \(\Delta(y, x)f\) has no terms which end in \(z_iz_j\) and \((\Delta(y, x)f)^* = \Delta(y, x)f\). Thus \(\Delta(y, x)f \in VIN_{n}[4, 1^{2n-3}]\) implies that \(\Delta(y, x)f = 0\), hence \(f_1 + f'_1 = 0\) from (6.12). Thus

\[
\begin{align*}
    f &= f_1(x, y, z_1, \ldots, z_{2n-4})[x, y] + \\
    &+ \sum_{i=1}^{2n-4} (-1)^i (f_2(x, y, z_1, \ldots, z_i, \ldots, z_{2n-4})[x, z_i] + \\
    &+ \sum_{i=1}^{2n-4} (-1)^i (f_3(x, y, z_1, \ldots, z_i, \ldots, z_{2n-4})[y, z_i]. \\
    &\quad \text{(6.13)}
\end{align*}
\]

We claim that \(f_3 = 0\). Indeed, from (6.13) and \(\Delta(z_1, y)f = 0\), we have \(f_3 \in VIN_{n-1}[3, 1^{2n-2}]\). Hence applying Lemma 6.2.1 to \(f^*\) yields \((f_3)^* = \alpha T_{2n-2}(x, y, z_1, \ldots, z_{2n-4}, x)\). Thus \(f_3 = -\alpha x T_{2n-2}(x, y, z_1, \ldots, z_{2n-4}, x)\) by Lemma 6.3.2. Thus the expression of \(f_3\), \(f^* = \pm f\) and the fact that \(f\) has no terms which end in \(z^2\) imply that \(\alpha = 0\). So \(f_3 = 0\), and

\[
\begin{align*}
    f &= f_1(x, y, z_1, \ldots, z_{2n-4})[x, y] + \\
    &+ \sum_{i=1}^{2n-4} (-1)^i (f_2(x, y, z_1, \ldots, z_i, \ldots, z_{2n-4})[x, z_i]. \\
    &\quad \text{(6.14)}
\end{align*}
\]
Now using the same argument as in the proof of the fact that \( f_3 \) is 0, one can show \( f_2 = 0 \) and get

\[
f = f_1(x, y, z_1, \ldots, z_{2n-4})[x, y].
\] (6.15)

From (6.15) we know that \( f_1 \) is skew symmetric in \( y \), and the \( z_i \)'s. Thus \( f^* = \pm f \) and \( f \) has no terms which end in \( z_iz_j, xz_i \) or \( zi\bar{x} \) imply that every monomial in \( f_1 \) has coefficient zero. So \( f_1 = 0 \) and hence \( f = 0 \). \( \blacksquare \)

Let

\[
\begin{align*}
t_1^n & := T_{2n}(x^2, y_1, \ldots, y_{2n-2}; x), \\
t_2^n & := \sum_{i=1}^{2n-2} (-1)^i T_{2n}(x \circ yi, y_1, \ldots, y_{2n-2}; x), \\
t_3^n & := \sum_{i=1}^{2n-2} (-1)^i T_{2n}(x^2, x, y_1, \ldots, y_{2n-2}; y_i), \\
t_4^n & := T_{2n}(x, y_1, \ldots, y_{2n-2}; x^2), \\
t_5^n & := T_{2n}(x, y_1, \ldots, y_{2n-2}; x), \\
t_6^n & := xT_{2n}(x, y_1, \ldots, y_{2n-2}; x).
\end{align*}
\] (6.16)

Then

**Proposition 6.3.3** The set \( \{t_1^n, \ldots, t_6^n\} \) is a basis of the vector space \( V \cap N_n[3, 1^{2n-2}] \).

**Proof.** From the definition of \( t_i^n \) we have for \( i = 1, 4, 6 \)

\[
\begin{align*}
t_1^n & = t_1^{n-1}[y_{2n-3}, y_{2n-2}] + \sim, \\
t_2^n & = (t_2^{n-1} + t_5^{n-1})[y_{2n-3}, y_{2n-2}] + \sim, \\
t_3^n & = \begin{cases} t_3^{n-1}[y_{2n-3}, y_{2n-2}] + \sim & \text{if } n \text{ is even} \\
(\frac{t_3^{n-1}}{2} - \frac{2}{n-1}(t_4^{n-1} - t_4^{n-1} - t_5^{n-1}))[y_{2n-3}, y_{2n-2}] + \sim & \text{if } n \text{ is odd}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
&= \begin{cases} t_3^{n-1}[y_{2n-3}, y_{2n-2}] + \sim & \text{if } n \text{ is even} \\
(\frac{t_3^{n-1}}{2} - \frac{1}{n-1}(t_4^{n-1} - t_4^{n-1} - t_5^{n-1}))[y_{2n-3}, y_{2n-2}] + \sim & \text{if } n \text{ is odd}
\end{cases}
\end{align*}
\]

[See the proof of part 4 in Lemma 6.2.1]. Using these formulas one can easily show that \( t_i^n \) for \( i = 1, \ldots, 6 \) are linearly independent by induction on \( n \geq 3 \).

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Next we show these $t_i^n$ span $V/I_N[3,1^{2n-2}]$. Let $\tilde{t}_i^n$ be the identity which is obtained by changing each $T$ in $t_i^n$ to be $\tilde{T}$. Then we have the following formulas. For $i = 1, 4, 6$

\[
t_i^n = \begin{cases} 
[y_1, y_2] \tilde{t}_i^{n-1} + \sim & \text{if } n \text{ is even} \\
[y_1, y_2] (\tilde{t}_2^{n-1} + \tilde{t}_6^{n-1}) + \sim & \text{if } n \text{ is odd}
\end{cases}
\]

Let $k_{n-1} := S_{2n-2}(x^2, x, y_3, \ldots, y_{2n-2})$ then

\[
k_{n-1} = \frac{1}{1-\alpha(n-1)} (t_1^{n-1} + t_3^{n-1} - t_4^{n-1})
\]

(6.17)

by the definition of the $t_i^n$ and Lemma 1.1 of [MR] ($\alpha(n-1) \neq 0$ since we assume that $n > 4$).

We only need to check $t_5^n$. The others are obvious. First, we have

\[
\begin{align*}
\tilde{t}_i^{n-1} + t_i^{n-1} &= (-1)^i k_{2n-2} \quad \text{for } i = 1, 4, \\
\tilde{t}_3^{n-1} + t_3^{n-1} &= -(2n-4) k_{2n-2}.
\end{align*}
\]

(6.18)

Indeed, from the definition of $T_{2n}, \tilde{T}_{2n}$ and $t_i^n$,

\[
T_{2n-2}(x^2, x, y_3, \ldots, y_{2n-2}) + \tilde{T}_{2n-2}(x^2, x, y_3, \ldots, y_{2n-2})
\]

\[
= S_{2n-2}(y_{2n-2}, x^2, x, y_3, \ldots, y_{2n-2})
\]

\[
= -S_{2n-2}(x^2, x, y_3, \ldots, y_{2n-2})
\]

\[
= -(-1)^n S_{2n-2}(x^2, x, y_3, \ldots, y_{2n-2})
\]

Thus

\[
\tilde{t}_3^{n-1} + t_3^{n-1} = -(2n-4) k_{n-1}.
\]

The others in (6.18) are obvious. By the definition of $t_5^n$ we have

\[
t_3^n = [y_1, y_2] (\tilde{t}_3^{n-1} - 2k_{n-1}) + \sim.
\]

On the other hand by (6.17), (6.17), we have

\[
\sum_{i=1,3,4} (-1)^{i-1} (t_i^{n-1} + \tilde{t}_i^{n-1}) = (-2(n-1)) k_{n-1}
\]

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This formula and (6.17) imply

\[ k_{n-1} = \frac{1}{e(n-1) - 2(n-1)}(\tilde{t}_1^{n-1} + \tilde{t}_3^{n-1} - \tilde{t}_4^{n-1}) \]

\[ = \begin{cases} \frac{1}{n-1}(\tilde{t}_3^{n-1} + \tilde{t}_3^{n-1} - \tilde{t}_4^{n-1}) & \text{if } n \text{ is even} \\ \frac{1}{n}(\tilde{t}_1^{n-1} + \tilde{t}_3^{n-1} - \tilde{t}_4^{n-1}) & \text{if } n \text{ is odd} \end{cases} \]  \hspace{1cm} (6.19)

Thus (6.19) implies

\[ t_3^n = [y_1, y_2] \begin{cases} (\tilde{t}_3^n - \frac{2}{n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n)) + \sim & \text{if } n \text{ is even} \\ (\tilde{t}_3^n - \frac{2}{n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n)) + \sim & \text{if } n \text{ is odd} \end{cases} \]

We also claim that

\[ (\ast) \quad \tilde{t}_i^n \text{ for } i = 1, \ldots, 6 \text{ is a basis of } VIN_m[3, 1^{2n-2}], \text{ for } 4 \leq m < n. \]

Indeed, this follows from (6.18), (6.19) and the induction hypothesis.

Now let \( f(x, y_1, y_2, \ldots, y_{2n-3}) \in VIN_n[3, 1^{2n-2}] \) and \( f^* = \pm f \). Then

\[ f = \sum f_1(x, y_1, \ldots, y_{2n-3})x y_{2n-2} + f_1^*(x, y_1, \ldots, y_{2n-3})y_{2n-2} \]

\[ = \sum f_2(x, y_1, \ldots, y_{2n-4})[y_{2n-3}, y_{2n-2}] + \]

\[ = \sum f_3(x, y_1, \ldots, y_{2n-3})y_{2n-2} x^2, \]

where \( f_2(x, y_1, \ldots, y_{2n-4}) \in VIN_{n-1}[3, 1^{2n-4}] \) and \( f_3(x, y_1, \ldots, y_{2n-3}) \in VIN_{n-1}[1^{2n-2}] \).

Thus, there exist \( \alpha_i \in F \) such that

\[ f_2 = \sum_{i=1}^{6} \alpha_i \tilde{t}_i^{n-1} \]

\[ = [y_1, y_2][\alpha_1 \tilde{t}_1^{n-2} + \alpha_2(\tilde{t}_2^{n-2} + \tilde{t}_6^{n-2}) + \]

\[ \alpha_3(\tilde{t}_3^{n-2} - \frac{2}{e(n-1) - 2(n-1)}(\tilde{t}_1^{n-2} + \tilde{t}_3^{n-2} - \tilde{t}_4^{n-2})) + \]

\[ \alpha_4 \tilde{t}_4^{n-2} + \alpha_5 \tilde{t}_5^{n-2} + \sim. \]

So by Lemma 6.3.1, \( f^* = \pm f \) implies that \( g_2^* = \pm g_2 \), where

\[ g_2 := \alpha_1 \tilde{t}_1^{n-2} + \alpha_2(\tilde{t}_2^{n-2} + \tilde{t}_6^{n-2}) + \alpha_4 \tilde{t}_4^{n-2} + \alpha_5 \tilde{t}_5^{n-2} + \]

\[ \alpha_3(\tilde{t}_3^{n-2} - \frac{2}{e(n-2) - 2(n-2)}(\tilde{t}_1^{n-2} + \tilde{t}_3^{n-2} - \tilde{t}_4^{n-2})) \]

\[ + \sim. \]

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\begin{align*}
&= (\alpha_1 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_1^{n-2} + \alpha_2 \tilde{t}_2^{n-2} + \\
&\quad (\alpha_3 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_3^{n-2} + \alpha_5 \tilde{t}_5^{n-2} + \\
&\quad (\alpha_4 + \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_4^{n-2} + \alpha_2 \tilde{t}_6^{n-2}.
\end{align*}

(6.20)

When \(n\) is odd,

\((t_i^n)^* = -t_i^n, i = 1, 2, 3, 4\) \((t_6^n)^* = -t_6^n, (t_5^n)^* = -t_5^n\),

and

\((\tilde{t}_i^n)^* = -\tilde{t}_i^n, i = 1, 2, 3, 4\) \((\tilde{t}_6^n)^* = -\tilde{t}_6^n, (\tilde{t}_5^n)^* = -\tilde{t}_5^n\),

by the definition of \(\tilde{T}_{2n}\). Thus

\[g_2^* = -((\alpha_1 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_1^{n-2} + \alpha_2 \tilde{t}_2^{n-2} + \alpha_5 \tilde{t}_5^{n-2} + \\
\quad (\alpha_3 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_3^{n-2} + (\alpha_4 + \frac{2}{e(n-2) - 2(n-2)} \alpha_3) \tilde{t}_4^{n-2}).\]

Hence by claim (*), \(g^* = \pm g\) implies

\[
\alpha_1 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3 = \mp (\alpha_1 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3),
\]

\[
\alpha_3 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3 = \mp (\alpha_3 - \frac{2}{e(n-2) - 2(n-2)} \alpha_3),
\]

\[
\alpha_4 + \frac{2}{e(n-2) - 2(n-2)} \alpha_3 = \mp (\alpha_4 + \frac{2}{e(n-2) - 2(n-2)} \alpha_3),
\]

\[
(\alpha_2 = \mp \alpha_2 = \mp \alpha_5).
\]

That is, if \(n\) is odd then:

1. When \(f^* = f\), \(\alpha_i = 0.0\) for \(i = 1, \ldots, 5\). So \(f_2 = \alpha_6 t_6^{n-1}\).

2. When \(f^* = -f\), \(\alpha_2 = \alpha_5\). So \(f_2 = \sum_{i \neq 2, 5} \alpha_i t_i^{n-1} + \alpha_2 (t_2^{n-1} + t_5^{n-1})\).

Let \(g := \alpha_6 (t_6^n - t_5^n)\) if \(f^* = f\). Then \(g = f_2[y_{2n-3}, y_{2n-2}] + \sim\) by (6.17). \(g\) is \(\ast\)-symmetric and \(f - g\) is \(\ast\)-symmetric and has no terms which end in \(y_i y_j\). If \(f^* = -f\), letting

\[g := (\alpha_1 - \frac{2\alpha_3}{n + 1}) t_1^n + \alpha_2 t_2^n + \frac{(n-1)\alpha_3}{n+1} t_3^n + (\alpha_4 + \frac{2\alpha_3}{n + 1}) t_4^n + \alpha_6 (t_6^n + t_5^n),\]

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then \( g = \mathcal{S}_2[y_{2n-3}, y_{2n-2}] + \sim \) by (6.17); \( g \) is \( * \)-skew symmetric and \( f - g \) is \( * \)-symmetric and has no terms which end in \( y_i y_j \). Thus we may assume that \( f^* = \pm f \) and \( f \) has no terms which end in \( y_i y_j \) when \( n \) is odd.

Next we discuss the case when \( n \) is even. If \( n \) is even then

\[
(t_1^n)^* = -t_1^n, \quad (t_2^n)^* = -t_2^n.
\]

Since \( n \) is even, \( T_{2n}^* = T_{2n} \). Thus by (6.18) we have

\[
(t_1^n)^* = -k_n - t_1^n, \quad (t_3^n)^* = k_n - t_4^n,
\]

\[
(t_2^n)^* = -(2(n - 1))k_n - t_3^n,
\]

where \( k_n = \mathcal{S}_2(x^2, x, y_1, \ldots, y_{2n-2}) \). So, in the vector space \( \text{Span}\{t_i^n|i = 1, \ldots, 6\} \), every \( * \)-symmetric element is a linear combination of the elements \( t_1^n + t_2^n - t_3^n, \quad t_4^n - t_6^n \) and every \( * \)-skew symmetric element is a linear combination of the elements

\[
t_1^n - 2(n - 1)t_3^n, \quad t_1^n + t_2^n, \quad t_4^n + t_6^n, \quad t_2^n.
\]

If \( n \) is even then \( (t_2^n)^* = -t_2^n \), \( (t_6^n)^* = -t_6^n \). Since \( n \) is even, \( T_{2n}^* = T_{2n} \). Thus by (6.18) and (6.19) we have

\[
(t_1^n)^* = -k_n - t_1^n
\]

\[
= \frac{-1}{e(n) - 2n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n) - \tilde{t}_1^n,
\]

\[
(t_3^n)^* = -(2(n - 1))k_n - \tilde{t}_3^n
\]

\[
= \frac{-2(n - 1)}{e(n) - 2n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n) - \tilde{t}_3^n,
\]

\[
(t_4^n)^* = k_n - \tilde{t}_4^n
\]

\[
= \frac{1}{e(n) - 2n}(\tilde{t}_1^n + \tilde{t}_3^n - \tilde{t}_4^n) - \tilde{t}_4^n.
\]

From (6.20),

\[
g_2 = \frac{2}{e(n - 2) - 2(n - 2)}\alpha_3 \left\{ \frac{-1}{e(n - 2) - 2(n - 2)}(t_1^n - t_3^n - t_4^n - t_6^n) + \frac{-2(n - 3)}{e(n - 2) - 2(n - 2)}(t_1^n + t_3^n - t_4^n - t_6^n) + \frac{1}{e(n - 2) - 2(n - 2)}(t_1^n + t_3^n - t_4^n - t_6^n) - \alpha_2 \tilde{t}_2^n - \alpha_5 \tilde{t}_3^n - \alpha_2 \tilde{t}_5^n \right\}.
\]

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Since \( n - 2 \) is even, \( e(n - 2) = n - 2 \). Let

\[
A_1 := \alpha_1 + \frac{2}{n - 2} \alpha_3, \quad A_3 := \alpha_3 + \frac{2}{n - 2} \alpha_3, \quad A_4 := \alpha_4 - \frac{2}{n - 2} \alpha_3,
\]

then by claim (\(*\)), \( g_2^* = \pm g_2 \) implies that

\[
\frac{1}{n - 2} A_1 + \frac{2n - 6}{n - 2} A_3 - \frac{1}{n - 2} A_4 = \pm A_1,
\]

\[
\frac{1}{n - 2} A_1 + (\frac{2n - 6}{n - 2} - 1) A_3 - \frac{1}{n - 2} A_4 = \pm A_3,
\]

\[
-\frac{1}{n - 2} A_1 - \frac{2n - 6}{n - 2} A_3 + (\frac{1}{n - 2} - 1) A_4 = \pm A_4,
\]

\[
-\alpha_2 = \pm \alpha_2 = \pm \alpha_5.
\]

(6.21)

Thus if \( f^* = f \) then \( g_2^* = g_2 \), solving (6.21) yields

\[
\alpha_2 = \alpha_5 = 0 = A_1 = A_3 = A_4.
\]

Therefore \( \alpha_4 = 0 \) for \( i = 1, \ldots, 5 \), i.e. \( f_2 = \alpha_6 t_6^{n-1} \) when \( f^* = f \). Let \( g := \alpha_6(t_6^n - t_5^n) \) then \( g \) is \( * \)-symmetric and \( y = f_2[y_{2n-3}, y_{2n-2}] + \sim \). Thus \( f - g \) has no terms which end in \( y_i y_j \).

If \( f^* = -f \) then \( g_2^* = g_2 \). Solving (6.21) yields

\[
\alpha_2 = \alpha_5, \quad A_1 + (2n - 6) A_3 - A_4 = 0.
\]

That is

\[
\alpha_2 = \alpha_5, \quad \alpha_4 = \alpha_1 + \frac{(2n - 6)n + 4}{n - 2} \alpha_3
\]

and

\[
f_2 = \frac{\alpha_1 t_1^{n-1} + \alpha_2(t_2^{n-1} + t_5^{n-1}) \alpha_3 t_3^{n-1} (\alpha_1 + \frac{(2n - 6)n + 4}{n - 2} \alpha_3) t_4^{n-1} + \alpha_6 t_6^{n-1}}{n - 2}.
\]

Let

\[
g = \alpha_1(t_1^n + t_4^n) + \alpha_2(t_2^n + t_5^n) + \alpha_3((2 - 2n)t_1^n + t_3^n) + \alpha_6(t_5^n + t_6^n),
\]

then \( g \) is \( * \)-skew symmetric and

\[
g = (\alpha_1(t_1^{n-1} + t_4^{n-1}) + \alpha_2(t_2^{n-1} + t_5^{n-1}) + \alpha_3((2 - 2n)t_1^{n-1} + t_3^{n-1}) + \alpha_6 t_6^{n-1} [y_{2n-3}, y_{2n-2}] + \sim.
\]
Thus by (6.17)

\[ f - g = \gamma \ell_4^{n-2}[y_{2n-3}, y_{2n-2}] + \sim \]

\[ = [y_1, y_2] \gamma \ell_4^{n-2}[y_{2n-3}, y_{2n-2}] + \sim. \]

Since \( f - g \) is skew symmetric, \( \gamma \ell_4^{n-2} \) is skew symmetric by Lemma 6.3.1. This implies that \( \gamma = 0 \) because \( \ell_4^{n-2} \) is not skew symmetric when \( n - 2 \) is even.

Since \( f - g \) is skew symmetric in the variables \( y_i \)'s, \( f - g \) has no terms which end in \( y_i y_j \). We may assume that \( f \) has this property. So we have reduced the proof to the case \( f^* = \pm f \) and \( f \) has no terms which end in \( y_i y_j \). Therefore from Lemma 6.2.1,

\[ f = \sum_{i=1}^{2n-2} (-1)^i f_1(x, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{2n-2}) y_i + f_1'(x, y_1, \ldots, y_{2n-2}) y_i x. \]  

(6.22)

Since \( f \) has no terms which end in \( y_i y_j \), \( \Delta(y_1, x) f \) has no such terms either. However \( \Delta(y_1, x) f \in V I N_n[4, 1^{2n-3}] \). Thus \( \Delta(y_1, x) f = 0 \) by the proof of Proposition 6.3.1.

Applying \( \Delta(y_1, x) \) to formula (6.22) yields \( f_1 = -f_1' \). Thus from (6.22) and \( f^* = \pm f \),

\[ f = \sum_{i < j} (\pm 1)(x, y_i) h(x, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{2n-2}) [x, y_j]. \]  

(6.23)

Since \( \Delta(y_1, x) f = 0 \) and \( f \) is skew in the variables \( y_i \)'s, \( h \) is alternating by (6.23), therefore \( h = \beta S_{2n-3}(x, y_2, \ldots, y_{2n-3}) \) for some \( \beta \in F \) and

\[ f = \beta[x, y_1] S_{2n-3}(x, y_2, \ldots, y_{2n-3}) [x, y_{2n-2}] + \sim. \]

But

\[ S_{2n-3}(x, y_2, \ldots, y_{2n-4}, 1) = S_{2n-4}(x, y_2, \ldots, y_{2n-4}) \]

from [DR]. So

\[ f(x, y_1, \ldots, y_{2n-4}, 1, y_{2n-2}) = \]

\[ \beta[x, y_1] S_{2n-3}(x, y_2, \ldots, y_{2n-3}) [x, y_{2n-2}] + \sim. \]

However,

\[ f(x, y_1, \ldots, y_{2n-4}, 1, y_{2n-2}) \in V I N_n[3, 1^{2n-3}] = \{0\}, \]

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so $\beta = 0$ and hence $f = 0$. Therefore $t_i^n$ for $i = 1, \ldots, 6$ is a basis of $VIN_n[3, 1^{2n-2}]$. \(\Box\)

Let

\[
\begin{align*}
t_1^n & := T_{2n}(x^2, y, z_1, \ldots, z_{2n-3}; y) + T_{2n}(y^2, x, z_1, \ldots, z_{2n-3}; x), \\
t_2^n & := T_{2n}(x \circ y, x, z_1, \ldots, z_{2n-3}; y) + T_{2n}(x \circ y, y, z_1, \ldots, z_{2n-3}; x), \\
t_3^n & := \sum_{i=1}^{2n-3} (-1)^{i-1}(T_{2n}(x \circ z_i, z_1, \ldots, \hat{z}_i, \ldots, z_{2n-3}, x, y; y) + T_{2n}(y \circ z_i, z_1, \ldots, \hat{z}_i, \ldots, z_{2n-3}, y, x; x)), \\
t_4^n & := xT_{2n}(x, y, z_1, \ldots, z_{2n-3}; y) + yT_{2n}(y, x, z_1, \ldots, z_{2n-3}; x), \\
t_5^n & := T_{2n}(x, y, z_1, \ldots, z_{2n-3}; y) + T_{2n}(y, x, z_1, \ldots, z_{2n-3}; x), \\
t_6^n & := T_{2n}(x, y, z_1, \ldots, z_{2n-3}; [x, y]),
\end{align*}
\]

Then we have

**Proposition 6.3.4** \(\{t_1^n, \ldots, t_6^n\}\) is a basis of the vector space $VIN_n[2, 2, 1^{2n-3}]$.

**Proof.** From the definition of $t_i^n$ we have the following formulas:

\[
\begin{align*}
t_i^n & = t_i^{n-1}[z_{2n-4}, z_{2n-3}] + \sim \text{ for } i = 1, 2, 4, 6, 7, \\
t_3^n & = (t_3^{n-1} + t_5^{n-1})[z_{2n-4}, z_{2n-3}] + \sim. \quad (6.24)
\end{align*}
\]

Since $t_1^n, \ldots, t_6^n$ are linear independent, using (6.24) one can show that $t_1^n, \ldots, t_7^n$ are linear independent by induction on $n$.

As in the proof of Proposition 6.3.3 we define $\tilde{t}_i^n$ to be the identity obtained by changing each $T$ in $t_i^n$ to be $\bar{T}$. Since $S_k$ is alternating,

\[
\tilde{t}_i^n + t_i^n = 0, \text{ for } i = 1, 2, 3, 4, 5
\]

Thus

\[
\tilde{t}_i^n = -t_i^n, \text{ for } i = 1, 2, 3, 4, 5, \tilde{t}_6^n = t_7^n, \tilde{t}_7^n = t_8^n. \quad (6.25)
\]

From the definition of $t_i^n$ and $\tilde{t}_i^n$ we also have the formulas

\[
\begin{align*}
t_i^n & = \tilde{t}_i^{n-1}[z_{2n-4}, z_{2n-3}] + \sim, \text{ for } i = 1, 2, 5, \\
t_3^n & = (\tilde{t}_3^{n-1} + \tilde{t}_5^{n-1})[z_{2n-4}, z_{2n-3}] + \sim, \\
t_6^n & = \tilde{t}_7^{n-1}[z_{2n-4}, z_{2n-3}] + \sim, t_7^n = \tilde{t}_6^{n-1}[z_{2n-4}, z_{2n-3}] + \sim. \quad (6.26)
\end{align*}
\]

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Let \( f \in V_{1}N_{1}[1, 2, 1^{2n-3}] \) then

\[
f = 
f_{1}(x_{1}, x_{2}, z_{1}, \ldots, z_{2n-3})x_{1}x_{2} + f_{1}(x_{2}, x_{1}, z_{1}, \ldots, z_{2n-3})x_{2}x_{1} + \
+f_{2}(x_{1}, x_{2}, z_{2}, \ldots, z_{2n-3})z_{2}x_{2} + f_{2}(x_{2}, x_{1}, z_{2}, \ldots, z_{2n-3})x_{2}z_{2} + \cdots \
+f_{4}(x_{1}, z_{1}, \ldots, z_{2n-3})x_{1}x_{2} + f_{4}(x_{2}, z_{1}, \ldots, z_{2n-3})x_{2}x_{2} + \cdots.
\]

(6.27)

So by the induction hypothesis \( f_{3} = \sum_{i=1}^{2} \alpha_{i}t_{i}^{n-1} \). Hence, from (6.25) and (6.26)

\[
f_{3} = [z_{1}, x_{2}]\{ \sum_{i=1,2,5} \alpha_{i}t_{i}^{n-2} + \alpha_{3}(t_{3}^{n} + t_{4}^{n-2}) + \alpha_{6}t_{6}^{n-2} + \alpha_{7}t_{7}^{n-2} \} + \sim \
= [z_{1}, x_{2}]\{ \sum_{i=1,2,5} -\alpha_{i}t_{i}^{n-2} - \alpha_{3}(t_{3}^{n} + t_{4}^{n-2}) + \alpha_{6}t_{6}^{n-2} + \alpha_{7}t_{7}^{n-2} \} + \sim.
\]

Let

\[
g_{3} := \sum_{i=1,2,5} -\alpha_{i}t_{i}^{n-2} - \alpha_{3}(t_{3}^{n} + t_{4}^{n-2}) + \alpha_{6}t_{6}^{n-2} + \alpha_{7}t_{7}^{n-2} + \sim.
\]

We may assume that \( f^{*} = \pm f \). To complete the proof we shall argue in two cases.

Case 1. \( n \) is odd. When \( n \) is odd \( T_{2n}^{*} = -T_{2n} \) and \( \tilde{T}_{2n}^{*} = -\tilde{T}_{2n} \) imply

\[
(t_{i}^{n})^{*} = -t_{i}^{n}, \text{for } i = 1, 2, 3, 6, 7,
\]

and

\[
(t_{4}^{n})^{*} = -t_{3}^{n}, \quad (t_{5}^{n})^{*} = -t_{4}^{n}.
\]

Thus from Lemma 6.3.1, \( f^{*} = \pm f \) implies that \( g_{3}^{*} = \pm g_{3} \). Since

\[
g_{3} := \sum_{i=1,2,5} -\alpha_{i}t_{i}^{n-2} - \alpha_{3}(t_{3}^{n} + t_{4}^{n-2}) + \alpha_{6}t_{6}^{n-2} + \alpha_{7}t_{7}^{n-2} + \sim,
\]

\[
g_{3}^{*} = \alpha_{1}t_{1}^{n-2} + \alpha_{2}t_{2}^{n-2} + \alpha_{3}(t_{3}^{n} + t_{4}^{n-2}) + \alpha_{5}t_{5}^{n-2} - \alpha_{6}t_{6}^{n-2} - \alpha_{7}t_{7}^{n-2} + \sim.
\]

Thus \( f^{*} = \pm f \) implies that

\[
\alpha_{i} = \mp \alpha_{i} \text{ for } i = 1, 2, 3, 6, 7, \quad \alpha_{5} = \mp \alpha_{3}.
\]

(6.28)

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So if $f^* = f$ then from (6.28), $\alpha_i = 0$ for $i = 1, 2, 3, 5, 6, 7$ and $f_3 = \alpha_4 t_4^{n-1}$. Let $g := \alpha_4(t_4^n - t_3^n)$ then $g$ is also $*$-symmetric and $g = \alpha_4 t_4^{n-1}[z_{2n-4}, z_{2n-3}] + \sim$. Hence $f - g$ is $*$-symmetric and has no terms which end in $z_i z_j$. If $f^* = -f$ then $\alpha_3 = \alpha_5$ by (6.28). So let $g := \sum_{i \neq 3, 5} \alpha_i t_i^{n-1} + \alpha_4(t_3^n + t_5^n)$. Then $g$ is $*$-skew symmetric and

$$g = \{ \sum_{i \neq 3, 5} \alpha_i t_i^{n-1} + \alpha_3(t_3^n + t_5^n) \}[z_{2n-4}, z_{2n-3}] + \sim.$$  

Hence $f - g$ has no terms which end in $z_i z_j$. So we may assume that $f$ has no terms which end in $z_i z_j$ and $f^* = \pm f$ when $n$ is odd.

Case 2. $n$ is even. Using the same argument as in case 1, we have

$$f_3 = \begin{cases} 
\alpha_4 t_4^{n-1} + \alpha_6(t_6^{n-1} + t_7^{n-1}) & \text{if } f^* = f \\
\alpha_3(t_3^{n-1} + t_5^{n-1}) + \alpha_6(t_6^{n-1} - t_7^{n-1}) & \text{if } f^* = -f.
\end{cases}$$

Thus let

$$g_3 = \begin{cases} 
\alpha_4(t_4^{n-1} - t_6^{n-1}) + \alpha_6(t_6^{n-1} + t_7^{n-1}) & \text{if } f^* = f \\
\alpha_3(t_3^{n-1} + t_5^{n-1}) + \alpha_6(t_6^{n-1} - t_7^{n-1}) & \text{if } f^* = -f.
\end{cases}$$

Then $f - g$ has no terms which end in $z_i z_j$ in both $f^* = \pm f$ and it has the same $*$-symmetry as $f$.

Now we assume that $f$ has no terms which end in $z_i z_j$. By Lemma 6.2.1, $f_4 = f_5 = 0$. Thus (6.27) becomes

$$f = f_1(x_1, x_2, z_1, \ldots, z_{2n-3})x_1 x_2 + f_1(x_2, x_1, z_1, \ldots, z_{2n-3})x_2 x_1 + 
$$

$$f_2(x_1, x_2, z_2, \ldots, z_{2n-3}, z_1) x_1 x_2 + f_2(x_2, x_2, z_2, \ldots, z_{2n-3}) x_2 x_2 + \cdots. \quad (6.29)$$

Since $\Delta(z_1, z_2) f \in \mathcal{V} \mathcal{I} \mathcal{N}_n[3, 2, 1^{2n-4}]$, it has no terms which end in $z_i z_j$ and is $*$-symmetric or $*$-skew symmetric because $f$ has these properties, $\Delta(z_1, z_2) f = 0$ by the proof of Proposition 6.3.2. Thus $f_2 = -f_2^*$ by (6.29). Since $f$ has no terms which end in $z_i z_j$, $\Delta(z_2, z_2) f = 0$ implies that $\Delta(z_2, z_2) f_2 = 0$ by (6.29). So $f_2$ is skew symmetric in all its variables of degree 1, i.e. $f_2(x_1, x_2, z_2, \ldots, z_{2n-3})$ is skew symmetric in the variables $z_2$ and $z_i$'s. Thus $f^* = \pm f$ and the fact that $f$ has no terms which end or start with $z_i z_j$ imply $f_2$ has no terms which start with $z_i z_j$. But $f_2$ is skew symmetric in $z_2$ and $z_i$'s, $f_2$ has no terms which start with
\[ f_2(x_1, x_2, z_1, \ldots, z_{2n-3}) = [x_1, x_2]h_2(x_1, z_2, \ldots, z_{2n-3}) + \sum_{i=2}^{2n-3} (-1)^{i-1} [x_1, z_i]h_2(x_1, x_2, z_2, \ldots, z_i, \ldots, z_{2n-3}). \]

Thus from (6.29)
\[
f = f_1(x_1, x_2, z_1, \ldots, z_{2n-3})x_1x_2 + f_1(x_2, x_1, z_1, \ldots, z_{2n-3})x_2x_1 + [x_1, x_2]h_2(x_1, z_2, \ldots, z_{2n-3})[z_1, x_2] - [x_1, x_2]h_2(x_1, x_2, z_3, \ldots, z_{2n-3})[z_1, x_2] + \cdots.
\]

(6.30)

Since \( f \) has no terms which start with \([z_2, x_2]\) and end in \([z_1, x_2]\) and \( \Delta(z_2, x_1)f = 0 \), by considering the sum of the terms in \( \Delta(z_2, x_1)f \) which start with \([x_1, x_2]\) and end in \([z_1, x_2]\) yields \( \Delta(z_2, x_1)h_2 = 0 \). Thus \( h_2 \) is alternating, hence there exists a scalar \( \gamma \in F \) such that \( h_2 = \gamma S_{2n-3}(x_1, x_2, z_2, \ldots, z_{2n-3}) \). Next, we show that \( \gamma = 0 \). Since \( H_n \) has no identity of degree \( 2n \) with type lower than \([2, 12n-2]\) for \( n \geq 4 \), \( f(x_1, x_2, z_1, z_3, \ldots, z_{2n-3}) \in VIN_n[2, 2, 12n-4] = \{0\} \). So \( h_2(x_1, 1, z_3, \ldots, z_{2n-3}) = 0 \) by (6.30). This implies that \( \gamma = 0 \) because
\[ S_{2n-3}(x_1, 1, z_3, \ldots, z_{2n-3}) = S_{2n-4}(x_1, z_3, \ldots, z_{2n-3}). \]

Thus
\[ f = f_1(x_1, x_2, z_1, \ldots, z_{2n-3})x_1x_2 + f_1(x_2, x_1, z_1, \ldots, z_{2n-3})x_2x_1, \]

where \( f_1 \) is multilinear. However this equation and \( \Delta(z_1, x_i)f = 0 \) imply that \( \Delta(z_1, x_i)f_1 = 0 \). So \( f_1 \) is alternating. But \( f \) has no terms which start with \( z_1x_j \) so does \( f_1 \). This property and the fact that \( f_1 \) is alternating imply \( f_1 = 0 \). So \( f = 0 \). \( \blacksquare \)

Let
\[
\begin{align*}
t_1 &= T_{2n}(y_1, \ldots, y_{2n-1}; x^2), \quad t_2^2 := T_{2n}(y_1, \ldots, y_{2n-1}; x^2), \\
t_3 &= \sum_{(1, \ldots, 2n-1)} T_{2n}(x \circ y_1, \ldots, y_{2n-1}; x), \\
t_4 &= \sum_{(1, \ldots, 2n-1)} T_{2n}(x \circ y_1, \ldots, y_{2n-1}; x), \\
t_5 &= \sum_{(1, \ldots, 2n-1)} T_{2n}(x, y_1, \ldots, y_{2n-2}; x \circ y_{2n-1}),
\end{align*}
\]

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\[ t^n_0 := \sum_{(1, \ldots, 2n-1)} \tilde{T}_{2n}(x, y_1, \ldots, y_{2n-2}; x \circ y_{2n-1}), \]

\[ t^n_7 := xT_{2n}(y_1, \ldots, y_{2n-1}; x), \quad t^n_8 := \tilde{T}_{2n}(y_1, \ldots, y_{2n-1}; x)x, \]

\[ t^n_9 := x\tilde{T}_{2n}(y_1, \ldots, y_{2n-1}; x), \quad t^n_{10} := T_{2n}(y_1, \ldots, y_{2n-1}; x)x. \]

\[ t^n_{11} := \sum_{(1, \ldots, 2n-1)} y_1 T_{2n}(x, y_2, \ldots, y_{2n-1}; x), \]

\[ t^n_{12} := \sum_{(1, \ldots, 2n-1)} \tilde{T}_{2n}(x, y_1, \ldots, y_{2n-2}; x)y_{2n-1}, \]

\[ t^n_{13} := \sum_{(1, \ldots, 2n-1)} y_1 \tilde{T}_{2n}(x, y_2, \ldots, y_{2n-1}; x), \]

\[ t^n_{14} := \sum_{(1, \ldots, 2n-1)} T_{2n}(x, y_1, \ldots, y_{2n-2}; x)y_{2n-1}, \]

\[ t^n_{15} := \sum_{(1, \ldots, 2n-1)} T_{2n}(x, y_1, \ldots, y_{2n-2}; [x, y_{2n-1}]), \]

\[ t^n_{16} := \sum_{(1, \ldots, 2n-1)} \tilde{T}_{2n}(x, y_1, \ldots, y_{2n-2}; [x, y_{2n-1}]). \]

Then

\[ t^n_3 = (t^n_3 + t^n_{10})[y_{2n-2}, y_{2n-1}] + \sim, \]

\[ t^n_4 = (t^n_4 + t^n_{8})[y_{2n-2}, y_{2n-1}] + \sim, \]

\[ t^n_i = t^n_{i-1}[y_{2n-2}, y_{2n-1}] + \sim, \quad (6.31) \]

where

\[ i = \begin{cases} 1, 2, 5, 7, 9, 11, 15 & \text{if } n \text{ is even} \\ 1, 2, 6, 7, 9, 11, 16 & \text{if } n \text{ is odd} \end{cases} \]

\[ t^n_{12} = \begin{cases} (t^n_{12} + t^n_{10})[y_{2n-2}, y_{2n-1}] + \sim & \text{if } n \text{ is even} \\ (t^n_{12} - t^n_{8})[y_{2n-2}, y_{2n-1}] + \sim & \text{if } n \text{ is odd} \end{cases} \]

We also have

\[ t^n_1 = [y_1, y_2]t^n_2 + \sim, \quad t^n_2 = [y_1, y_2]t^n_1 + \sim, \]

\[ t^n_3 = [y_1, y_2](t^n_3 + t^n_{0}) + \sim, \quad t^n_4 = [y_1, y_2](t^n_{4} + t^n_{7}) + \sim, \]

\[ t^n_5 = [y_1, y_2](t^n_{6} - t^n_{7} - t^n_{5}) + \sim, \quad t^n_6 = [y_1, y_2]t^n_{5} + \sim, \]

\[ t^n_{10} = [y_1, y_2]t^n_{8} + \sim, \quad t^n_{11} = [y_1, y_2](t^n_{10} - t^n_{11}) + \sim, \]

\[ t^n_{12} = -[y_1, y_2]t^n_{12} + \sim, \quad t^n_{16} = [y_1, y_2]t^n_{15} + \sim, \]

\[ t^n_{15} = [y_1, y_2](t^n_{10} + t^n_{7} + t^n_{5}) + \sim. \]
Thus we can show that

\[ t_7^n + t_8^n + t_9^n + t_{10}^n + t_{15}^n + t_{16}^n = 0. \]  (6.32)

Indeed, for \( n = 4 \) this holds [see the proof of Proposition 5.3.5]. Suppose it is also true for \( n - 1 \) then for \( n \)

\[
\begin{align*}
t_7^n + t_8^n + t_9^n + t_{10}^n + t_{15}^n + t_{16}^n &= \\
y_{2n-1}f_1 + y_{2n-1\times}f'_1 + \cdots \\
+ [y_1,y_2](t_7^{n-1} + t_8^{n-1} + t_9^{n-1} + t_{10}^{n-1} + t_{15}^{n-1} + t_{16}^{n-1}) + \cdots \\
+ ax^2S_{2n-1}(y_1,\ldots,y_{2n-1}).
\end{align*}
\]

Let

\[ g := t_7^n + t_8^n + t_9^n + t_{10}^n + t_{15}^n + t_{16}^n. \]

Then by the induction hypothesis

\[ g = xy_{2n-1}f_1 + y_{2n-1\times}f'_1 + \cdots + ax^2S_{2n-1}(y_1,\ldots,y_{2n-1}). \]

From this equation we know that there are no terms which start with \( y_iy_j \) while this implies that \( a = 0 \) since \( g \) is \( * \)-symmetric. By the definitions of \( t_i^n \),

\[
t_7^n + t_8^n + t_9^n + t_{10}^n = \\
xS_{2n-1}(x,y_1,\ldots,y_{2n-1}) + S_{2n-1}(x,y_1,\ldots,y_{2n-1})x
\]

Thus \( \Delta(y_{2n-1},x)(t_7^n + t_8^n + t_9^n + t_{10}^n) = 0. \) Obviously \( \Delta(t_{15}^n + t_{16}^n) = 0, \) thus \( \Delta(y_{2n-2},x)g = 0. \)

So, by (6.32), \( f_1 + f'_1 = 0 \) and

\[ g = [x,y_{2n-1}]f_1 + \cdots. \]

Again, \( \Delta(y_{2n-1},x)y = 0 \) implies \( \Delta(y_1,x)y = 0. \) Thus \( \Delta(y_1,x)f_1 = 0. \) So \( f_1 = \beta S_{2n-1}(x,y_1,\ldots,y_{2n-2}) \) for some \( \beta \in F. \) Therefore the fact that \( g \) has no terms which start in \( y_iy_j \) and \( g^* = g \) imply that \( \beta = 0, \) hence \( g = 0. \)

It can be shown by induction on \( n \) and the formulas above that both \( \{t_i^n| i = 1,2,3,4,5,7,8,9,10,11,12,15\} \) and \( \{t_i^n| i = 1,2,3,4,6,7,8,9,10,11,12,16\} \) are linearly independent if \( n \geq 4. \)
Proposition 6.3.5 The set \( \{ t_i^n \} | i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15 \) is a basis of the space \( \mathcal{V} \mathcal{I} \mathcal{N}_n[2, 1^{2n-1}] \).

Proof. Since for \( n = 5 \) the proposition holds [see the remark after Proposition 5.3.5] we may assume that \( n > 5 \). Let \( f \in \mathcal{V} \mathcal{I} \mathcal{N}_n[2, 1^{2n-1}] \). Then

\[
f = \sum_{i=1}^{2n-1} (f_1 x y_{2n-1} + f'_i y_{2n-1} x) + \sum_{i<j} \pm f_{ij} [y_i, y_j] + \beta S_{2n-1}(y_1, \ldots, y_{2n-1}) x^2
\]

for some \( \beta \in F \). We may assume that \( f^* = \pm f \).

Case 1. If \( n \) is odd then

\[
(t_i^n)^* = -t_i^n, \quad \text{for } i \neq 7, 8, 9, 10, 11, 12,
\]

\[
(t_{10}^n)^* = -t_{10}^n, \quad (t_{11}^n)^* = -t_{11}^n.
\]

(6.33)

Since \( f_2 \in \mathcal{V} \mathcal{I} \mathcal{N}_{n-1}[2, 1^{2n-3}] \), by the induction hypothesis there exist scalars \( \alpha_i \in F \) such that

\[
f_2 = \alpha_1 t_4^{n-1} + \cdots + \alpha_4 t_4^{n-1} + \alpha_6 t_6^{n-1} + \cdots + \alpha_{12} t_{12}^{n-1} + \alpha_{16} t_{16}^{n-1}.
\]

Thus by (6.32) \( f_2 = [y_1, y_2]g_2 + \sim \), where

\[
g_2 = \alpha_1 t_2^{n-2} + \alpha_2 t_9^{n-2} + \alpha_3 (t_4^{n-2} + t_9^{n-2}) + \alpha_4 (t_3^{n-2} + t_7^{n-2}) + \alpha_6 t_5^{n-2} + \alpha_8 t_{10}^{n-2} + \alpha_{10} t_{8}^{n-2} + \alpha_{11} (t_7^{n-2} - t_{11}^{n-2}) - \alpha_{12} t_{12}^{n-2} + \alpha_{16} t_{15}^{n-2}.
\]

Thus by (6.33)

\[
(g_2)^* = -\alpha_1 t_2^{n-2} - \alpha_2 t_9^{n-2} + \alpha_3 (-t_4^{n-2} - t_9^{n-2}) + \alpha_4 (-t_3^{n-2} - t_7^{n-2}) - \alpha_6 t_5^{n-2} - \alpha_8 t_7^{n-2} - \alpha_{10} t_8^{n-2} + \alpha_{11} (-t_{10}^{n-2} + t_{12}^{n-2}) + \alpha_{12} t_{11}^{n-2} - \alpha_{16} t_{15}^{n-2}.
\]

Since \( f^* = \pm f \), by Lemma 6.3.1, \( g_2^* = \pm g_2 \). Thus, when \( g_2^* = g_2 \), \( \alpha = 0 \), for \( i \neq 7, 9, 11, 12 \) and \( \alpha_{11} = -\alpha_{12} \) since \( t_i^{n-2} \) for \( i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15 \) are linear independent when \( n - 2 \geq 3 \). So

\[
f_2 = \alpha_7 t_7^{n-1} + \alpha_9 t_9^{n-1} + \alpha_{12} (-t_{11}^{n-1} + t_{12}^{n-1}).
\]
Let

\[ g := \alpha_7(t^2_7 - t^0_{10}) - \alpha_9(t^6_8 - t^0_0) + \alpha_{12}(-t^1_{11} + t^n_{12}). \]

Then \( g \) is \( \ast \)-symmetric and

\[ g = \{\alpha_7t^{n-1}_7 + \alpha_9t^{n-1}_6 + \alpha_{12}(-t^{n-1}_{11} + t^{n-1}_{12}) - \alpha_9(t^{n-1}_8)\}[y_{2n-2}, y_{2n-1}] + \sim. \]

So

\[ f - g = \sum_{i=1}^{2n-1} (f_1x y_{2n-1} + f^i_1 y_{2n-1} x) + \gamma S_{2n-1}(y_1, \ldots, y_{2n-1})x^2 + \sum_{i<j} \pm(\beta_8t^{n-1}_8 + \beta_{10}t^{n-1}_{10})[y_i, y_j]. \tag{6.34} \]

Similarly, \( g^*_2 = -g_2 \) implies that

\[ f_2 = \alpha_1t^{n-1}_1 + \alpha_2t^{n-1}_2 + \alpha_3(t^{n-1}_3 + t^{n-1}_{10}) + \alpha_4(t^{n-1}_4 + t^{n-1}_6) + \alpha_6t^{n-1}_6 + \alpha_7t^{n-1}_7 + \alpha_9t^{n-1}_9 + \alpha_{10}t^{n-1}_{10} + \alpha_{11}(t^n_{11} + t^n_{12}). \]

Let

\[ g = \alpha_1t^{n-1}_1 + \alpha_2t^{n-1}_2 + \alpha_3t^{n-1}_3 + \alpha_4t^{n-1}_4 + \alpha_6t^{n-1}_6 + \alpha_7t^{n-1}_7 + \alpha_9t^{n-1}_9 + \alpha_{10}t^{n-1}_{10} + \alpha_{11}(t^n_{11} + t^n_{12}). \]

Then \( g^* = -g \) and we also have (6.34).

Case 2. If \( n \) is even then

\[ (t^n_0)^* = t^n_{i+1}, \quad i = 1, 3, 5, 7, 9, 11, 15. \]

Since \( f_2 \in \text{VIN}_{n-1}[2, 1^{2n-3}] \), by the induction hypothesis

\[ f_2 = \alpha_1t^{n-1}_1 + \cdots + \alpha_5t^{n-1}_5 + \alpha_7t^{n-1}_7 + \cdots + \alpha_{12}t^{n-1}_{12} + \alpha_{15}t^{n-1}_{15}. \]

Thus by (6.32) \( f_2 = [y_1, y_2]g_2 + \sim \), where

\[ g_2 = \alpha_1t^{n-2}_2 + \alpha_2t^{n-2}_4 + \alpha_3(t^{n-2}_3 + t^{n-2}_5) + \alpha_4(t^{n-2}_6 + t^{n-2}_7) + \alpha_5(t^{n-2}_8 - t^{n-2}_9 - t^{n-2}_7) + \alpha_8t^{n-2}_{10} + \alpha_{10}t^{n-2}_8 + \alpha_{11}(t^{n-2}_9 - t^{n-3}_{11}) + \alpha_{12}(-t^{n-2}_{12}) + \alpha_{15}(t^{n-2}_{10} + t^{n-1}_7 + t^{n-1}_0). \]
Since \( f^* = \pm f \), \( g_2^* = \pm g_2 \). Thus, \( g_2^* = g_2 \) implies that

\[
\alpha_5 = \alpha_{15} = 0, \; \alpha_1 = \alpha_2, \; \alpha_3 = \alpha_4, \; \alpha_6 = \alpha_3, \\
\alpha_{10} = \alpha_{11} = \alpha_3 + \alpha_{11}, \; \alpha_{11} = \alpha_{12},
\]

and

\[
f_2 = \alpha_1(t_1^{n-1} + t_2^{-1}) + \alpha_3(t_3^{n-1} + t_4^{n-1}) + \alpha_7 t_7^{n-1} + \alpha_9 t_9^{n-1} + \\
\alpha_{10}(t_5^{n-1} + (\alpha_3 + \alpha_{11})t_{10}^{n-1} + \alpha_{11}(t_{11}^{n-1} + t_{12}^{n-1}).
\]

Let

\[
g = \alpha_1(t_1^n + t_2^n) + \alpha_3(t_3^n + t_4^n) + \alpha_7 t_7^n + \alpha_9(t_5^n + t_6^n) + \\
\alpha_{10}(t_5^n + t_{10}^n) + \alpha_{11}(t_{11}^n + t_{12}^n).
\]

Then \( g \) is \(*\)-symmetric and we also have (6.34).

When \( f^* = -f \), \( g_2^* = -g_2 \) implies that

\[
f_2 = \alpha_1(t_1^{n-1} - t_2^{n-1}) + \alpha_3(t_3^{n-1} + t_4^{n-1}) + \alpha_7 t_7^{n-1} + \alpha_9 t_9^{n-1} + \\
\alpha_{10}(t_{10}^{n-1} - t_{11}^{n-1} + t_{12}^{n-1}) + \alpha_{15} t_{15}^{n-1}.
\]

Let

\[
g := \alpha_1(t_1^n - t_2^n) + \alpha_3(t_3^n + t_4^n + t_{11}^n - t_{12}^n) + \alpha_4(t_5^n + t_6^n) + \\
(\alpha_7 - \alpha_{15})(t_7^n - t_8^n) + (\alpha_9 - \alpha_{15})(t_9^n - t_{10}^n) + \\
\alpha_{10}(-t_{11}^n + t_{12}^n) + \alpha_{15}(t_7^n + t_9^n + t_{15}^n).
\]

Since

\[
t_3^n + t_4^n = S_{2n}(x, x \circ y_1, \ldots, y_{2n-1}) = -t_5^n - t_6^n
\]

and (6.32), \( g \) is \(*\)-skew symmetric and (6.34) holds.
Thus we may assume that $f^* = \pm f$ and

$$f = \sum_{i=1}^{2n-1} (f_1^i y_{2n-1} + f_1^i y_{2n-1} x) + \gamma S_{2n-1}(y_1, \ldots, y_{2n-1})x^2 + \sum_{i<j} \pm (\beta_{8i} t_8^{n-1} + \beta_{10i} t_{10}^{n-1})[y_i, y_j].$$

(6.35)

From (6.35), the monomial $y_1 \cdots y_{2n-1}x^2$ has coefficient $\gamma$ in $f$. But its reverse has coefficient 0 because

$$t_8^{n-1} = T_{2n-2}(y_1, \ldots, y_{2n-3}; x) x, \quad t_{10}^{n-1} = T_{2n-2}(y_1, \ldots, y_{2n-3}; x)x.$$  

Using the same argument and considering the coefficients of monomials $w$ and $w^*$ for

$$w = y_1 \cdots y_6 x y_7 \cdots y_{2n-3} x y_{2n-2} y_{2n-1}, \quad w^* = y_4 x y_5 \cdots y_{2n-3} x y_{2n-2} y_{2n-1},$$

respectively yield $\beta_8 = \beta_{10} = 0$. Thus

$$f = \sum_{i=1}^{2n-1} (f_1^i y_{2n-1} + f_1^i y_{2n-1} x).$$

From this equation we know that $f$ has no terms which end in $y_i y_j$. So $\Delta(y_{2n-1}, x)f$ has no such terms also. But $\Delta(y_{2n-1}, x)f \in \mathcal{VIN}[n \in [3, 1^{2n-2}]]$, and $\Delta(y_{2n-1}, x)f = 0$ from the proof of Proposition (6.3.3). Thus $f_1 + f_1^i = 0$. Since $f$ is skew symmetric in the variables $y_i$'s, $\Delta(y_{2n-1}, x)f = 0$ implies $\Delta(y_1, x)f = 0$. So $\Delta(y_1, x)f_1 = 0$. Thus $f_1$ is skew symmetric in $x, y_1, \ldots, y_{2n-2}$ by Theorem 1.1. Hence $f_1 = dS_{2n-1}(x, y_1, \ldots, y_{2n-2})$. However, $f$ has no terms which end in $y_i y_j$ and $f^* = \pm f, d = 0$. So $f = 0$.

Since every identity in $\mathcal{VIN}_n[1^{2n+1}]$ is multilinear and alternating, it is a scalar multiple of the standard identity $S_{2n+1}(x_1, \ldots, x_{2n+1})$. Thus, Theorem (6.2) was proved.
Appendix I

The Dimension of Space $VIN_n[n_1, \ldots, n_r]$

**The identities of $H_2$**

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**The identities of $H_n(n \geq 4)$**

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Appendix II

Reliability of Computations

Like many if not most mathematical results, the contents of this thesis are not presented in the order they were obtained. We set out to prove that all identities of degree $2n + 1$ of $H_n$ follow from those of degree $2n$ ($n > 3$). Corollary 1.3.2 allows us to restrict ourselves to homogeneous polynomials which are symmetric (respectively skew symmetric) in variables of equal even (respectively odd) degree. If need be we can further specialize to $\ast$-symmetric and $\ast$-skew symmetric polynomials with this property, where $\ast$ is the reversal involution.

Such a polynomial is a linear combination of simpler polynomials. For example, if $p(x_1, x_2, x_3, y)$ is of degree 1 in each $x_i$ and $k$ in $y$ and is skew symmetric in $x_1, x_2, x_3$ then $p$ is a linear combination of $\binom{k + 3}{3}$ polynomials each of which corresponds to choosing a place for $x_1, x_2$ and $x_3$ and then taking the alternating sum. Subtracting a linear combination of known identities we may sometimes assume conditions on the coefficients. Then we must make substitutions to obtain further conditions on the coefficients and solve the resulting homogeneous system of linear equations. If it has only the trivial solution we are done and may pass to the next multidegree in our ordering. If not, at least one identity has been obtained or more substitutions must be made to rule it out.

The substitutions step and the solving of systems of linear equations step were done in Mathematica on a Next and Maple on the main frame. Some cases were done on both systems. All substitutions were checked by hand using the graph-theoretical version which allows one to visually decide which monomials correspond to unicursal paths. While we did not double check the second step for all cases, we feel confident in our results since errors are more likely to produce spurious candidate identities which are rejected when we try to show that they are indeed identities. Each linear equation was obtained by requesting the coefficients of a monomial. In the most cases, if we mistype a monomial we get the equation $0 = 0$ and nothing is lost. Moreover, the new class of identities $T_{2n}(x_1, \ldots, x_{2n-1}; [x_{2n}, x_{2n+1}])$ was discovered by implementing the procedure described above.
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