CYCLE RANK OF TRANSITION GRAPHS

and the

STAR HEIGHT OF REGULAR EVENTS

by

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Ottawa, Ontario

Cycle Rank of Transition Graphs
and the
Star Height of Regular Events

by

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the requirements for the degree of
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Department of Mathematics
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University of Ottawa
Ottawa, Ontario
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Abstract

This thesis is concerned with the (restricted) star height of regular events, as defined by Eggan. Eggan has established the existence of events of arbitrary star height and has shown that the star height of a given event equals the minimum cycle rank of all transition graphs recognizing the event. The problem of establishing the star height of regular events was further investigated by McNaughton and by Dejean and Schlützenberger.

In this thesis some new approaches to the star height problem are taken, and the relationship between the star height of a given event and the structure of its reduced state graph is further explored. First a summary of previous results is outlined and some general properties of star height are studied. The notions of confinal sets of trails and simple regular expressions are then introduced. Necessary conditions for the star height of an event to be no less than some integer $k > 0$ are developed and a family of events whose star height equals the rank of the corresponding reduced state graph is exhibited.

Rank-non-increasing transformations on transition graphs are investigated, providing a stronger version of Eggan's theorem. By this version, the star height of a regular event equals the smallest rank of all non-deterministic reduced state graphs representing the event. This result makes possible the introduction of some new concepts, such as the $R$-projection functions and covering of state graphs, by means of which a number of auxiliary results on star height is obtained. A technique for establishing the star height of an event is developed, as well as necessary conditions for the star height of an event defined by a reset-
free state graph $G$ to be no less than the rank of some subgraph of $G$. The thesis concludes with a discussion of remaining problems.
Acknowledgements

I wish to express my deep appreciation to my supervisor, Dr. Janusz A. Brzozowski, for his invaluable guidance, encouragement, criticism and many helpful suggestions made during the research work and writing of this thesis.

Gratitude is also expressed to Mrs. Erna Unrau for her outstanding typing of this manuscript.
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List of Symbols and Notation

Latin Alphabet

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<thead>
<tr>
<th>Symbol</th>
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<td>A</td>
<td>finite alphabet</td>
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<td>$\mathcal{A}$</td>
<td>automaton</td>
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<tr>
<td>a</td>
<td>alphabet letter</td>
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<tr>
<td>B</td>
<td>set of branches in a digraph</td>
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<td>$B_L$</td>
<td>set of labelled branches in a transition graph</td>
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<td>D</td>
<td>digraph</td>
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<tr>
<td>$D_q$</td>
<td>the derivative (left-quotient) corresponding to state q in a reduced automaton.</td>
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<td>E</td>
<td>regular expression</td>
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<td>F</td>
<td>set of final states in an automaton</td>
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<td>G</td>
<td>transition graph, state graph</td>
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<td>$G_0(R)$</td>
<td>reduced deterministic state graph recognizing R</td>
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<td>h</td>
<td>star height</td>
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<td>$\ell$</td>
<td>length of a word</td>
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<td>M</td>
<td>transition function of an automaton</td>
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<td>$M_{\mathcal{A}}$</td>
<td>syntactic monoid of an automaton $\mathcal{A}$</td>
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<td>N</td>
<td>set of nodes in a graph</td>
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<td>m,n</td>
<td>nodes</td>
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<td>P(G)</td>
<td>set of paths in a transition graph G</td>
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<td>p</td>
<td>path</td>
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<td>Q</td>
<td>set of states in an automaton</td>
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<td>state</td>
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R — regular event
r — cycle rank
S — strongly connected subgraph
T — set of trails in a state graph
T(\mathcal{A}) — event accepted by automaton \mathcal{A}
\{u,v,w\} — words

Greek Alphabet

γ, η, φ — mappings
\hbar^\alpha (E) — apparent star height of a regular expression E
ε — null path
\lambda — empty word
\Lambda — empty event
\phi — empty set, empty event

 Auxiliary Symbols

\#N — number of elements in set N
\cup — union
\cap — intersection
\complement — complement
. — concatenation
* — star operator
\forall — for every
\land — and
\lor — or
∈ — belongs to
\check{G} — the dual of the graph G
\( \hat{R} \) - the reverse event of \( R \)

\( x \setminus R / y \) - two-sided quotient of \( R \) by \( (x, y) \)

\( x \setminus R, R / x \) - left and right quotients of \( R \) by \( x \)

\( |E| \) - the event denoted by the regular expression \( E \)

\( 2^Q \) - set of all subsets of \( Q \).

\( \hat{\gamma}_0(R) \) - reduced incomplete automaton accepting \( R \)

**Notational Conventions**

1. Composition of mappings is denoted by juxtaposition \( (\gamma \eta(x)) = \gamma(\eta(x)) \)

2. When no confusion arises, a set \( S \) consisting of a single element \( s \) will sometimes be denoted by \( s \) instead of \( \{s\} \).
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Introduction

Since 1956, when regular expressions were first introduced by Kleene [K1], a great deal of research has been centered around them. Their structural properties were investigated and their relation to finite automata has been thoroughly studied. As it turns out, most interesting problems concerning regular expressions arise due to the fact, that there exist infinitely many regular expressions denoting each event, varying in size and structure. Although for some special families of events unique canonical regular expressions have been found ([Br 3], [Yo]), no canonical form for regular expressions is known for the general case. In spite of the fact that methods for checking equivalence of regular expressions are well-known, relatively little is known about the infinite family of all regular expressions denoting a given event. One of the problems requiring the study of this infinite family is the star height problem, namely the problem of determining the star height of a regular event. This problem is of importance because the star height represents a measure of complexity of regular events and the minimum star height would be very desirable in any canonical form for regular expressions.

The notion of star height was first introduced by Eggan [E] who showed that there exist regular events of arbitrary star height. This was done by exhibiting, for each n, events of star height n. A disadvantage in Eggan's examples is the fact that the size of the input alphabet increases with n. Dejean and Schützenberger [DS] have shown that for each
n there exists an event over the two-letter alphabet that has star height \( n \) (it is easy to show that every event over the one-letter alphabet has star height 0 or 1). Eggn has also introduced the notion of (cycle) rank of a digraph and has established a fundamental theorem, by which the star height of a regular event equals the smallest rank of all transition graphs recognizing the event.

Eggn's theorem was used by McNaughton in [Mc 3] for developing graph-theoretical methods for establishing the star height of regular events. It was also proved in that paper, that every regular event of star height \( n \) can be mapped by a homomorphism onto an event of star height \( n \) over the two-letter alphabet. Pathwise homomorphisms between transition graphs were introduced in [Mc 4]. Such homomorphisms were shown to be rank-non-increasing and were utilized for establishing the existence of an algorithm for determining the star height of pure-group events.

There still remains, however, the difficult open problem of how to determine the star height of a regular event in the general case. In this thesis some new concepts are introduced and some new tools developed for attacking this problem. A number of results on star height of certain families of events is obtained, and some techniques for determining star height are presented.

In Chapters 1, 2 the fundamental definitions of automata, regular expressions, transition graphs and related notions are given and some basic results summarized. The star height problem is introduced
in Chapter 3 and a survey of previous results is outlined. A study of general properties of star height is made in Section 3.3, and some results concerning the changes in star height caused by various operations on events are presented.

In Chapter 4 the notions of trails, cofinal sets of trails in a state graph and simple regular expressions are introduced. The main Lemma is then proved in Section 4.3, relating the apparent star height of a regular expression to the rank of its corresponding set of trails in a given state graph. In Section 4.4 sufficient conditions are found for the star height of an event to be no less than some \( k > 0 \) and a family of events whose star height equals the rank of the corresponding reduced state graph is exhibited. Section 4.5 deals with the star height of events defined by digraphs. Each such event consists of the set of all paths in a digraph connecting two sets of nodes. A full characterization of the star height of such events is obtained and two questions raised by Eggan in [E] are answered.

Chapter 5 is concerned with rank-non-increasing transformations on transition graphs. In Section 5.1 some elementary modifications on digraphs are introduced and their effect on the rank of the digraph is studied. In Section 5.2 a reduced transition graph is defined and a rank-non-increasing reduction procedure of transition graphs, based on the results of the previous section, is developed. A process of elimination of all empty-word-transitions in a transition graph \( G \) is described in Section 5.3. The resulting transition graph is equivalent to \( G \), has no
more nodes than $G$ and no greater rank. It follows that every transition graph can be replaced by an equivalent non-deterministic reduced state graph with no more nodes and of no higher rank. As a result, a stronger version of Eggan's theorem is obtained: The star height of a regular event $R$ equals the smallest rank of all non-deterministic reduced state graphs recognizing $R$. It follows as a by-product, that the minimum number of nodes in any transition graph recognizing $R$ equals the minimum number of nodes in any non-deterministic state graph recognizing $R$.

Chapter 6 opens with the definition of the $R$-projection functions and of covering a state graph by another one, two notions which help relate an arbitrary non-deterministic state graph to the given reduced deterministic state graph recognizing $R$. These notions, as well as the notion of pathwise homomorphism, are utilized for establishing two fundamental lemmas, by which most results of this chapter are proved. A technique for establishing star height of regular events is developed and some lower bounds to the star height of events recognized by reset-free state graphs are obtained. The star height of any group-free event recognized by a reset-free state graph $G$ is shown to equal the rank of $G$.

The thesis concludes with a short summary on existing methods for determining star height, and some conjectures concerning the existence of a general algorithm for determining the star height of regular events are discussed. Some other open problems are also mentioned.
Chapter 1

Regular Events and Automata

In this chapter the basic material that is common to all chapters is presented. The reader is referred to [Bo], [Ha] and [Gs] for additional details.

1.1. Regular Expressions and Regular Events

Definition 1.1: Let $A$ be a finite non-empty set of symbols, called letters, and let $A^*$ be the free monoid (i.e. semigroup with identity) generated by $A$, where the operation is called product or concatenation and is denoted by juxtaposition. The elements of $A^*$ are finite sequences of letters from $A$ and are called words on the alphabet $A$. The identity element of $A^*$ is the empty word and is denoted by $\lambda$. Thus for every $w \in A^*$, $\lambda w = w \lambda = w$. The length of a word $w = a_1 a_2 \ldots a_k$, $a_i \in A$ for all $i = 1, \ldots, k$, is the number $k$ of letters in it and is denoted by $\ell(w)$. Clearly $\ell(\lambda) = 0$ and for any two words $w_1, w_2 \in A^*$, $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Any subset of $A^*$ is called an event (or a language). The empty event is denoted by $\emptyset$, and $\Lambda = \{\lambda\}$. Let $U$ and $V$ be events; then $U \cup V$, $U \cap V$ and $UV$ are the union, intersection and concatenation (product) respectively of $U$ and $V$, where

$$UV = \{w \mid w = uv, u \in U, v \in V\}.$$ 

The difference $U - V$ is defined by:

$$U - V = \{w \mid w \in U, w \notin V\}$$
and the complement of \( U \) is defined by \( \bar{U} = A^* - U \).

Define for any event \( U \), \( U^0 = \Lambda \), \( U^1 = U \) and for any \( n > 1 \), \( U^{n+1} = UU^n \). Define the \textit{star operator} (abbreviated star) on any event \( U \) by:

\[
U^* = \Lambda \cup U \cup U^2 \cup \ldots \cup U^n \cup \ldots = \bigcup_{n=0}^{\infty} U^n
\]

Clearly for any event \( U \), \( U^* \) is a monoid contained in \( A^* \); in fact, it is the submonoid of \( A^* \) generated by \( U \).

For any word \( w = a_1a_2\ldots a_k \in A^* \), the \textit{reverse} of \( w \) is the word \( \hat{w} = a_k\ldots a_2a_1 \), and for the empty word \( \lambda \), \( \hat{\lambda} = \lambda \). The reverse of an event \( U \) is defined by

\[
\hat{U} = \{ u \mid u = \hat{w} \text{ for some } w \in U \}
\]

\textbf{Definition 1.2:} Let \( U \subseteq A^* \) and let \( x, y \in A^* \). Define the \textit{two-sided quotient} of \( U \) by the pair \((x, y)\) by:

\[
x \setminus U \div y = \{ w \mid xwy \in U \}
\]

The left quotient of \( U \) by \( x \) is \( x \setminus U = x \setminus U \div \lambda \) and the right quotient of \( U \) by \( y \) is \( U \div y = \lambda \setminus U \div y \).

\textbf{Definition 1.3 [Kl]:} Let \( A \) be a finite non-empty alphabet and let \( \lambda \) and \( \phi \) be two distinct symbols not in \( A \). Let \( \cup, \cap \) and \( \cdot \) be associative binary operators and let \( \ast \) and \( ^{-} \) be unary operators. A \textit{regular expression} (on \( A \)) is defined inductively as follows:

1. \( a \in A \), \( \lambda \) and \( \phi \) are regular expressions.
2. If \( P \) and \( Q \) are regular expressions then so are \( P \cup Q \), \( P \cap Q \), \( PQ \) (the dot is usually omitted) \( P^\ast \) and \( \bar{P} \).
(3) Nothing else is a regular expression unless its being so follows from a finite number of applications of Rules (1) and (2).

Definition 1.4: Define a mapping $||$ from the set of all regular expressions (over the alphabet $A$) into the set of all subsets of $A^*$ inductively as follows:

(i) For all $a \in A$, $|a| = \{a\}$, $|\lambda| = A$, $|\emptyset| = \emptyset$.

(ii) If $P$ and $Q$ are regular expressions then

$|P \cup Q| = |P| \cup |Q|$

$|P \cap Q| = |P| \cap |Q|$

$|PQ| = |P||Q|$

$|P^*| = |P|^*$

$|\overline{P}| = |P|$  

An event $R$ is regular iff there exists a regular expression $E$ such that $|E| = R$. We say that $E$ denotes the regular event $R$. Thus the class of regular events is the range of the function $||$ just defined.

For example, the regular expression $E = (0 \cup 10^*)^*$ over the alphabet $A = \{0,1\}$ denotes the set of all words over $A$ with an even number of appearances of the letter 1.

Theorem 1.1: The class of regular events is a Boolean algebra of sets.

Definition 1.5: Regular expressions with no occurrences of the operators $\cap$ and $-$ are called restricted regular expressions. Thus every restricted regular expression is obtained from the letters of the alphabet $A$, $\lambda$ and $\emptyset$ by a finite number of applications of the operators $\cup$, $\cdot$, $*$ only.
Theorem 1.2 [M Y]: Every regular event can be denoted by a restricted regular expression.

Theorem 1.3: The family of regular events is the smallest family of events containing all finite events and closed under the formation of union, concatenation and star.

In the sequel, by a "regular expression" we shall mean only a restricted regular expression. Regular expressions with the operators $\cap$ and $-$ will be referred to as extended regular expressions.

Notation 1.1: Throughout this paper there will be an ambiguity in notation. When no confusion arises, the brackets $|$ will be omitted and a regular expression $E$ will be understood as the corresponding event $|E|$. Thus the notation $E = F, E \subseteq F, w \in E$ for $E$ and $F$ regular expressions and a word $w$ is to be interpreted $|E| = |F|, |E| \subseteq |F|, w \in |E|$ respectively.

For any function $f$ defined on the domain of regular events, $f(E)$ will mean $f(|E|)$, etc. Another ambiguity is that a word on the alphabet and the singleton set of words containing only that word will be represented by the same symbol. There will be no difference in the symbol for a letter of the alphabet and a word of length one. In each case it should be clear from context what meaning is being used at a given time.

Definition 1.6: The reverse of a regular expression $E$ is denoted by $\hat{E}$ and is defined recursively as follows:

$$\hat{a} = a \text{ for all } a \in A, \hat{\emptyset} = \emptyset \text{ and } \hat{\lambda} = \lambda$$

For all regular expressions $E_1, E_2$, 

}\]
\[
\hat{E}_1 \cup \hat{E}_2 = \hat{E}_1 \cup \hat{E}_2 \\
\hat{E}_1 \hat{E}_2 = \hat{E}_2 \hat{E}_1 \\
\hat{E}_1^* = (\hat{E}_1)^*
\]

**Definition 1.7:** The unary operator \(\delta\) is defined for any regular event \(R\) by:

\[
\delta(R) = \begin{cases} 
\lambda & \text{if } \lambda \in R \\
\phi & \text{otherwise}
\end{cases}
\]

For a regular expression \(E\), \(\delta(E) = \delta(|E|)\).

### 1.2. Finite Automata

**Definition 1.8:** A **finite non-deterministic automaton** (or simply automaton) \(A\) over the (finite) alphabet \(A\) is a quadruple \(A = (Q, M, Q_1, F)\), where \(Q\) is a finite non-empty set of **states**, \(Q_1 \subseteq Q\) is the set of **initial states**, \(F \subseteq Q\) is the set of **final states** (or output states) and \(M\) is the **transition function**: \(M : Q \times A \rightarrow 2^Q\). If \(#Q_1 = 1\) and for every \(q \in Q\), \(a \in A\), \(#(M(q,a)) = 1\), then \(A\) is said to be a **deterministic automaton**. If \(#Q_1 = 1\) and for all \(q \in Q\), \(a \in A\), \(#(M(q,a)) \leq 1\), then \(A\) is called an **incomplete automaton**. Thus an incomplete automaton is a special case of a non-deterministic automaton, and a deterministic automaton is a special case of an incomplete automaton.

Usually \(A\) is referred to as the **input alphabet** of the automaton. All automata as well as all events considered from now on will be assumed to be over a **fixed alphabet** \(A\), unless otherwise specified.
Extend the transition function to subsets of \( Q \). If \( Q' \subseteq Q \) then for each \( a \in A \),
\[
M(Q',a) = \bigcup_{q \in Q'} M(q,a)
\]
Next extend \( M \) to words \( w \in A^* \) as follows: for every \( a \in A \), \( x \in A^* \), \( Q' \subseteq Q \)
\[
M(Q',xa) = M(M(Q',x),a)
\]
and for the empty word \( \lambda \)
\[
M(Q',\lambda) = Q'
\]
**Notation 1.2:** When no confusion arises, a singleton subset of \( Q \), namely \( Q' = \{q\}, q \in Q \), will be denoted by \( q \) (with brackets omitted).

**Definition 1.9:** The automaton \( \mathcal{A} \) accepts a word \( x \in A^* \) iff \( M(Q_1,x) \cap F \neq \emptyset \). The set of all words accepted by \( \mathcal{A} \) is called the event accepted by \( \mathcal{A} \) and is denoted by \( T(\mathcal{A}) \) (sometimes the word 'accepted' is replaced by 'recognized' or 'defined').

**Theorem 1.4 [KL, RS]:** The event accepted by any automaton is regular.

**Definition 1.10:** Two automata are called equivalent iff they accept the same event.

**Definition 1.11:** A state \( q \) of an automaton \( \mathcal{A} = (Q, M, Q_1, F) \) is said to be accessible iff there exist \( q_1 \in Q_1 \) and \( x \in A^* \) such that \( q \in M(q_1,x) \). The automaton \( \mathcal{A} \) is called connected iff all of its states are accessible. For any automaton \( \mathcal{A} = (Q, M, Q_1, F) \), let \( \mathcal{A}_c = (Q', M', Q_1, F') \), where \( Q' \subseteq Q \) is the set of all accessible states in \( \mathcal{A} \), \( M' \) is the restriction of \( M \) to \( Q' \) and \( F' = F \cap Q' \). Clearly \( \mathcal{A}_c \) is a connected automaton.
Theorem 1.5: For any automaton $\mathcal{A}$, $\mathcal{A}_c$ is equivalent to $\mathcal{A}$.

From now on, all automata considered are assumed to be connected and incomplete, unless otherwise specified. The (single) initial state of the automaton will be denoted by $q_1$, unless otherwise specified.

Definition 1.12 [KW]: Let $\mathcal{A} = (Q, M, q_1, F)$ be an automaton and let $q \in Q$. The accepted event of $q$ in $\mathcal{A}$, $A_c(\mathcal{A})$, or $A_c(q)$, when $\mathcal{A}$ is understood, is defined by:

$$A_c(q) = \{ x : x \in A^* \land M(q, x) \in F \}$$

Theorem 1.6 [Br 2]: Let $\mathcal{A} = (Q, M, q_1, F)$ be a deterministic automaton and let $T(\mathcal{A}) = R$. Then:

(a) for every state $q \in Q$ there exists a word $x \in A^*$ such that $A_c(q) = x \setminus R$ (Definition 1.2)

and

(b) for every $x \in A^*$ there exists at least one state $q \in Q$ such that $x \setminus R = A_c(q)$.

If $\mathcal{A}$ is only an incomplete automaton, then part (a) of Theorem 1.6 still holds but part (b) has to be modified as follows:

(b) for every $x \in A^*$ such that $x \setminus R \neq \emptyset$, there exists $q \in Q$ such that $x \setminus R = A_c(q)$.

Corollary 1.1: If $R = T(\mathcal{A})$ for some automaton $\mathcal{A}$ then $R$ has a finite number of distinct left quotients.

Definition 1.13: An automaton $\mathcal{A} = (Q, M, q_1, F)$ is called reduced iff for every pair of states $q, q' \in Q$, $A_c(q) \neq A_c(q')$. 

Theorem 1.7: For every automaton $\mathcal{A}$, there exists a unique reduced deterministic automaton equivalent to $\mathcal{A}$.

Theorem 1.8 [Br 2]: For every event $R$ with a finite number of distinct left quotients, there can be effectively constructed a reduced deterministic automaton accepting $R$, provided that it is possible to ascertain, for each $a \in A$ and for each left quotient $R_i$, which left quotient equals $a \setminus R_i$.

Outline of Proof: Let $R = R_1, R_2, \ldots, R_n$ be the set of all distinct left quotients of $R$. Construct the automaton $\mathcal{A}_0(R) = (\{R_1, \ldots, R_n\}, M, R_1, F)$, where

$$F = \{ R_i \mid 1 \leq i \leq n, \lambda \in R_i \}$$

and the transition function $M$ is defined by $M(R_i, a) = R_j, a \in A, 1 \leq i, j \leq n$ and $R_j = a \setminus R_i$. One can easily verify that in the automaton $\mathcal{A}_0(R)$, for every state $R_i, 1 \leq i \leq n, A_e(R_i) = R_i$. Hence $\mathcal{A}_0(R)$ is reduced and accepts $R$.

Theorem 1.9 [Br 2]: Every regular event has a finite number of distinct left quotients.

Outline of proof: Let $E$ be a regular expression denoting the given event $R$. Then for each word $x \in A^*$, a regular expression $D_x E$ denoting the left quotient $x \setminus R$, can be obtained from $E$ by recursive application of the following rules:

1. For $a \in A, D_a \lambda = \lambda, D_a \phi = \phi$ for $b = \lambda, \phi$ or $b \in A$ and $b \neq a$. 

For any regular expressions $P$ and $Q$,

1. $D_a (P \cup Q) = D_a P \cup D_a Q$

2. $D_a (PQ) = (D_a P)Q \cup \delta(P)D_a Q$

3. $D_a (P^*) = (D_a P)^*$

4. $D_\lambda P = P$

5. $D_{ua} P = D_a (D_u P)$, $u \in A^*$

It can be verified that for any $x \in A^*$ and $R = |E|$, $|D_x E| = x \setminus R$. The regular expression $D_x E$ obtained from $E$ in the above fashion is called a derivative of $E$ with respect to the word $x$.

Thus in order to obtain all left quotients of $R$, one computes the derivatives $D_x E$ for words $x$ taken in a lexicographic order. It is shown in [Br 2] that after a finite number of steps no new derivatives are found, and it follows that $R$ has a finite number of distinct left quotients.

---

**Example 1.1**: Let $A = \{0,1\}$ and let $R = 10^*11$. The derivatives of $R$ are:

- $D_\lambda R = R$
- $D_0 R = \emptyset = D_0 x R$, all $x \in A^*$
- $D_1 R = 0^*11$
- $D_10 R = D_1 R$
- $D_{11} R = 1$
- $D_{110} R = D_0 R$
- $D_{111} R = \lambda$
- $D_{1110} R = D_0 R$
- $D_{1111} R = D_0 R$
Thus no new derivatives of $R$ can be obtained and the set of distinct left quotients of $R$ is: $R_1 = R$, $R_2 = 0 \setminus R = \emptyset$, $R_3 = 1 \setminus R = 0^{*}11$, $R_4 = 11 \setminus R = 1$ and $R_5 = 111 \setminus R = \Lambda$. Using the construction of Theorem 1.8, we obtain the reduced deterministic automaton accepting $R$: $A_0(R) = (\{R_1, \ldots, R_5\}, M, R_1, F)$ where $F = \{R_5\}$ and $M$ is given by the table in Fig. 1.1 (such a table is called the transition table of the automaton $A_0(R)$.)

**Corollary 1.2:** Let $W \subseteq A^\ast$. The following conditions are equivalent:

(a) $W$ is regular.

(b) There exists an automaton $A$ such that $T(A) = W$.

(c) $W$ has a finite number of distinct left quotients.

From Theorem 1.7 and Theorem 1.8 we get:

**Corollary 1.3:** The automaton $A_0(R)$ defined in Theorem 1.8 is the unique reduced deterministic automaton accepting the regular event $R$.

**Definition 1.14:** Let $q$ be a state in an automaton $A = (Q, M, q_1, F)$. $q$ is called a dead state (or trap state) iff $A_c(q) = \emptyset$.

Clearly if $A$ is reduced then there can be at most one dead state $q_\emptyset$ in $A$, and there is one only if the event $R = T(A)$ has a left-quotient equal to $\emptyset$. One can easily verify that by removing from $A$ the dead state $q_\emptyset$ and all transitions leading to it and from it, one obtains an incomplete reduced automaton $\hat{A}_0(R)$ which is equivalent to $A$ and has no dead state. Furthermore, for any reduced incomplete automaton $A' = (Q', M', q'_1, F)$ there can be constructed an equivalent reduced deterministic automaton $A = A_0(R) = (Q, M, q_1, F)$, where
\[ M(q,a) \]

Fig. 1.1
\[ Q = \begin{cases} Q' & \text{if } \mathcal{A}' \text{ has a dead state } q_\phi \\ Q' \cup \{q_\phi\}, \text{ where } q_\phi \notin Q' \end{cases} \]

and for all \( a \in A \), \( q \in Q' \), \( M(q, a) = M'(q, a) \) iff \( \#(M'(q, a)) = 1 \), \( M(q, a) = q_\phi \) otherwise, and \( M(q_\phi, a) = q_\phi \) for all \( a \in A \). By Theorem 1.6 and the above argument, we obtain the following:

**Proposition 1.1:** For every regular event \( R \) there exists a unique reduced incomplete automaton \( \mathcal{A}_0(R) \) having minimal number of states and accepting \( R \). \( \mathcal{A}_0(R) \) can be obtained from \( \mathcal{A}_0(R) \) by removing from the latter the dead state together with all transitions leading to and from it.

**Notation 1.3:** \( \mathcal{A}_0(R) \) will be called the minimal incomplete automaton accepting the event \( R \). Let \( q \) be any state in \( \mathcal{A} = \mathcal{A}_0(R) \) (or \( \mathcal{A}_0(R) \)). Denote by \( D_q \) the corresponding left quotient of \( R \), i.e.

\[ D_q = A_c \mathcal{A}(q) = x \setminus R \quad \text{for some } x \in A^* \]

Thus in \( \mathcal{A}_0(R) \) (\( \mathcal{A}_0(R) \)) the set \( \{D_q \mid q \in Q\} \) coincides with the set of all non-empty (possibly empty) distinct left quotients of \( R \).

**Definition 1.15:** Let \( \mathcal{A} = (Q, M, q_1, F) \), where \( Q = \{q_1, q_2, \ldots, q_n\} \).

Extend further the transition function \( M \) as follows:

\[ M: 2^Q \times 2^{A^*} \to 2^Q, \text{ where for every } Q' \subseteq Q, W \subseteq A^* \]

\[ M(Q', W) = \{q \mid q \in Q \land q = M(q_1, w) \text{ for some } q_1 \in Q', w \in W\} \]

For each \( i, 1 \leq i \leq n \), define the function \( M_i: 2^{A^*} \to 2^Q \) by:

\[ M_i(W) = M(q_i, W), \text{ all } W \subseteq A^* \]
Lemma 1.1: Let $\mathcal{A} = (Q, M, q_1, F)$ and let $R, P, Q$ be events such that $R = PQ$. Then

$$M_i(R) = \bigcup_{\text{j} \in M_i(P)} M_j(Q), \quad i = 1, 2, \ldots, n$$

Proof: Follows immediately from the above definition.

Definition 1.16: A regular expression $E$ of the form $E = P^*$, is called a star expression. An event $W \subseteq A^*$ is a star event iff $W = V^*$ for some event $V$. $V$ is called the root of $W$. In [Br 4] it was shown that for every star event $W$ there exists a unique minimal root $V_m$ such that $W = V_m^*$ and $V_m$ is contained in all other roots of $W$.

Theorem 1.10 [Br 4]: The following equivalent conditions are necessary and sufficient for $W \subseteq A^*$ to be a star event:

(a) $W = W^*$

(b) $W = W^2$

(c) $\lambda \in W$ and for each $u \in A^*$, $\lambda \in u \setminus W$ iff $W \subseteq u \setminus W$

Definition 1.17: Let $R$ be a regular event and let $\mathcal{A} = (Q, M, q_1, F)$ be an automaton accepting $R$. Define for any subset $B \subseteq Q$,

$$\text{str } B = \{x \mid x \in A^* \land M(B, x) \subseteq B\}$$

Proposition 1.2: str $B$ is a regular star event. Moreover, if $q_1 \in B \subseteq F$ then str $B$ $\subseteq$ $R$.

Proof: Clearly $\text{str } B = \cap_{q \in B} T(\mathcal{A}(q, B))$, where $\mathcal{A}(q, B) = (Q, M, q, B)$.

Hence str $B$ is regular. Now obviously $\lambda \in \text{str } B$, and for any $x, y \in \text{str } B$, $q \in Q$ we have: $M(q, xy) = M(M(q, x), y) \in B$, which shows
that also $xy \in \text{str } B$. Hence $\text{str } B = (\text{str } B)^2$ and by Theorem 1.10(b) \str B is a star event. If $q_1 \in B \subseteq F$ then $M(q_1, x) \in B \subseteq F$ for all $x \in \text{str } B$ and therefore $\text{str } B \subseteq R$.

**Definition 1.18 ([RS], [Me]):** Let $\mathcal{A} = (Q, M, q_1, F)$ be an automaton. Define a mapping $\eta$ from $A^*$ into the set of all transformations of $Q$ into itself as follows: for every $w \in A^*$, $\eta(w) = m_w: Q \rightarrow Q$, where $m_w(q) = M(q, w)$ for all $q \in Q$. One can easily verify that $\eta$ is a homomorphism from the free monoid $A^*$ onto a finite monoid $\mathcal{M}_\mathcal{A}$ of transformations of $Q$ into itself, with the operation of composition of transformations. $\mathcal{M}_\mathcal{A}$ is called the **syntactic monoid** of the automaton $\mathcal{A}$. The syntactic monoid of a regular event $R$ is the syntactic monoid of its reduced automaton $\mathcal{A}_0(R)$.

**Example 1.2:** Let $A = \{0,1\}$ and let $R = (0 \cup 1)^*00(0 \cup 1)^*$. The transition table of the automaton $\mathcal{A} = \mathcal{A}_0(R)$ is shown in Fig. 1.2, where $p$ is the initial state and $F = \{r\}$ is the set of final states. The syntactic monoid of $R$ consists of the following transformations:

- $i = \begin{pmatrix} p & q & r \\ p & q & r \end{pmatrix} = m_\lambda$
- $a = \begin{pmatrix} p & q & r \\ q & r & r \end{pmatrix} = m_0 = m_{010}$
- $b = \begin{pmatrix} p & q & r \\ p & p & r \end{pmatrix} = m_1 = m_{11} = m_{101} = m_{111}$
- $c = \begin{pmatrix} p & q & r \\ r & r & r \end{pmatrix} = m_{00} = m_{000} = m_{001} = m_{100}$
- $d = \begin{pmatrix} p & q & r \\ p & r & r \end{pmatrix} = m_{01} = m_{011}$
- $e = \begin{pmatrix} p & q & r \\ q & q & r \end{pmatrix} = m_{10} = m_{110}$
Fig. 1.2

Fig. 1.3
The multiplication table of $M_\mathcal{A}$ is shown in Fig. 1.3.

**Definition 1.19:** A regular event $R$ is group-free [PM] iff its syntactic monoid contains no non-trivial subgroups. The automaton $\mathcal{A}_0(R)$ is group-free iff $R$ is group-free.

**Definition 1.20:** An automaton $\mathcal{A} = (Q, M, q_1, F)$ is called permutation-free iff for any subset $Q' \subseteq Q$ with at least two elements, and for any word $w \in A^*$, the restriction of $M_w$ to $Q'$ is not a cyclic permutation of $Q'$ onto itself. The regular event $R$ is permutation-free iff $\mathcal{A}_0(R)$ is permutation-free.

**Theorem 1.11 [PM]:** A regular event $R$ is group-free iff it is permutation-free.

**Theorem 1.12 (Schützenberger [Sch], Papert and McNaughton [PM]):** The family of group-free events over the alphabet $A$ is a boolean algebra of sets, and a monoid under the operation of concatenation.

Theorem 1.12 is proved by showing that the family of group-free events coincides with the family of all events that can be denoted by an extended regular expression without the star operator, i.e., a regular expression using only the operators $\cup$, $\cap$, $\cdot$, and $\cdot$.

**Definition 1.21:** An automaton $\mathcal{A}$ is a pure-group automaton iff $M_\mathcal{A}$ is a group. A regular event $R$ is a pure-group event iff $\mathcal{A}_0(R)$ is a pure-group automaton.

Apparently $\mathcal{A}$ is a pure-group automaton iff all transformations in $M_\mathcal{A}$ are permutations of $Q$ onto itself.
Definition 1.22 [RS]: Let \( \mathcal{A} = (Q, N, q_1, G) \) and \( \mathcal{B} = (P, M, p_1, F) \) be two deterministic automata. Define their direct product
\[
\mathcal{A} \times \mathcal{B} = (Q \times P, N \times M, (q_1, p_1), G \times F)
\]
where the transition function \( N \times M \) is defined by:
\[
(N \times M)((q, p), a) = (N(q, a), M(p, a))
\]
for all pairs \( (q, p) \in Q \times P \) and \( a \in A \).

Theorem 1.13 [RS]: \( \mathcal{A} \times \mathcal{B} \) accepts the event \( T(\mathcal{A}) \cap T(\mathcal{B}) \).

Lemma 1.2: If \( \mathcal{A} \) and \( \mathcal{B} \) are pure-group automata then so is \( \mathcal{A} \times \mathcal{B} \).

Proof: Apparently the syntactic monoid of the direct product of two automata, \( M_{\mathcal{A} \times \mathcal{B}} \), equals the direct product of the two syntactic monoids, \( M_{\mathcal{A}} \times M_{\mathcal{B}} \) [L]. The result now follows immediately.

Theorem 1.14: The family of pure-group events is closed under intersection.
Chapter 2

Digraphs, Transition Graphs and State Graphs

2.1. Finite Digraphs and Cycle Rank

Definition 2.1: A finite directed graph (or simply digraph) is an ordered pair \( D = (N,B) \), where \( N \) is a finite set of nodes and \( B \), the set of branches of \( D \), is a subset of \( N \times N \). An element \((m,n) \in N \times N \) of \( B \) will be called a branch from \( m \) to \( n \) and will be denoted by \( b_{mn} \). We call \( m \) the initial node and \( n \) the terminal node of the branch \( b_{mn} \).

Definition 2.2: A path \( p \) in a digraph \( D = (N,B) \) is any finite sequence \( b_1b_2\ldots b_k \) of branches \( b_i \in B \), \( i = 1, \ldots, k \), \( k \geq 0 \), such that the terminal node of \( b_i \) is the initial node of \( b_{i+1} \), \( i = 1, 2, \ldots, k - 1 \). \( p \) is a path from \( m \) to \( n \) \((m,n \in N) \) if \( m \) is the initial node of \( b_1 \) and \( n \) is the terminal node of \( b_k \). The nodes \( m \) and \( n \) are the initial and terminal nodes, respectively, of the path \( p \). If \( k = 0 \) then the path is null. If a non-null path \( p \) has the same initial and terminal node, then \( p \) is called a loop.

Let \( p_1 \) and \( p_2 \) be two paths in \( D \) such that the terminal node of \( p_1 \) is the initial node of \( p_2 \), i.e. \( p_1 = b_{n_1n_2n_3\ldots b_{n_kn_{k+1}}} \) and \( p_2 = b_{n_{k+1}n_{k+2}n_{k+3}\ldots n_\lambda n_{\lambda+1}} \), where \( \lambda \geq k \geq 0 \). Then their concatenation is the path \( p_1p_2 = b_{n_1n_2n_3\ldots b_{n_kn_{k+1}}b_{n_{k+1}n_{k+2}n_{k+3}\ldots n_{\lambda}n_{\lambda+1}}} \).

Definition 2.3: Let \( D = (N,B) \) and let \( n \in N \). Denote by \( B(n) \) the set of all branches in \( B \) which contain \( n \) as either a terminal node or as an
initial node. For any \( N' \subseteq N \) define

\[
D - [N'] = (N - N', B - \cup_{n \in N'} B(n))
\]

thus \( D - [N'] \) is simply the graph obtained from \( D \) by eliminating
the nodes of \( N' \) together with all branches incident to them.

**Definition 2.4:** Let \( D = (N, B) \) be a digraph and let \( n \in N \). Define
\[
\pi^D(n) \uparrow, \text{ the set of predecessors of } n \text{ in } D, \text{ by:}
\]

\[
\pi^D(n) = \{n' \in N \mid (n', n) \in B\}
\]

and \( \sigma^D(n) \uparrow, \text{ the set of successors of } n \text{ in } D, \text{ by}
\]

\[
\sigma^D(n) = \{n' \in N \mid (n, n') \in B\}
\]

**Definition 2.5:** Two digraphs \( D = (N, B) \) and \( D' = (N', B') \) are called
isoomorphic iff there exists a \( 1 \)-\( 1 \) mapping \( \phi \) of \( N \) onto \( N' \) such that
\( B' = \{(\phi(n), \phi(n')) \mid (n, n') \in B\} \).

**Definition 2.6:** A digraph \( D' = (N', B') \) will be called a subgraph \( \uparrow\uparrow \)
of \( D = (N, B) \) iff \( \phi \neq N' \subseteq N \) and \( D' = D - [N - N'] \).

**Definition 2.7:** A digraph \( D = (N, B) \) is strongly connected (s.c.) iff
for any pair of nodes, \( m, n \in N \), there exists a path in \( D \) from \( m \) to \( n \).

A section of \( D \) is any s.c. subgraph of \( D \) which is not properly
contained in any other s.c. subgraph of \( D \). It follows that the sections
in any digraph \( D \) are disjoint, and a node \( n \) of \( D \) is contained in a section
iff there exists in \( D \) a non-null loop from \( n \) to \( n \) (i.e. a loop on \( n \)).

\( \uparrow \) When no confusion arises, we shall omit the superscript \( D \).

\( \uparrow\uparrow \) The word "subgraph" has restricted use here.
Definition 2.8: The (cycle) rank of a digraph (after Eggam [E]):

(a) Let $D = (N, B)$ be a strongly connected digraph. The (cycle) rank, $r(D)$, of $D$ is defined inductively as follows:

(i) $r(D) = 1$ iff there exists a node $n \in N$ such that $D - [n]$ contains no s.c. subgraph.

(ii) $r(D) = k$ iff $r(D)$ is not less than $k$ and for some $n \in N$ all sections of $D - [n]$ have rank at most $k - 1$.

(b) Let $D = (N, B)$ be any digraph. Define $r(D) = 0$ iff $D$ has no sections and

$$r(D) = \max \{r(D') \mid D' \text{ a section of } D\}$$

otherwise.

Thus the rank of a digraph is a measure of the complexity of its loops.

Example 2.1: Let $D = (N, B)$ be the digraph shown in Fig. 2.1. The sections of $D$ are $D_1 = D - [n_3]$ and $D_2 = D - \{[n_1, n_2]\}$. It is easily verified by inspection that $r(D_2) = 1$ and $r(D_1) = 2$. Hence $r(D) = \max \{r(D_1), r(D_2)\} = 2$.

The following lemma is an immediate consequence of the above definition.

Lemma 2.1: Let $D = (N, B)$ and $D' = (N', B')$ be two digraphs such that $N' \subseteq N$ and $B' \subseteq B$. Then $r(D') \leq r(D)$.

Definition 2.9: For any digraph $D = (N, B)$, define $\hat{D}$, the dual of $D$, to be the digraph $\hat{D} = (N, \hat{B})$, where $\hat{B} = \{(n', n) \in N \times N \mid (n, n') \in B\}$. 
The following lemma is easily verified:

**Lemma 2.2**: For any digraph \( D \), \( r(D) = r(\hat{D}) \), and for every section \( S \) of \( D \), \( \hat{S} \) is a section of \( \hat{D} \).

**Definition 2.10**: Let \( D = (N,B) \) be a s.c. digraph. A node \( n \in N \) is
called a **cycle center** of \( D \) if \( r(D - [n]) < r(D) \). Now let \( D \) be any
digraph. Then \( n \) is a cycle center of \( D \) if \( n \) is contained in a section
\( S \) of \( D \) such that \( r(S) = r(D) \) and \( n \) is a cycle center of \( S \).

The next two propositions are direct consequences of the
above definitions.

**Proposition 2.1**: Let \( n \) be a cycle center of a s.c. digraph \( D \). Then
\( r(D - [n]) = r(D) - 1 \).

**Proposition 2.2**: Let \( D = (N,B) \) be a s.c. digraph. Then \( r(D) = k \geq 1 \)
iff (a) there exists a node \( n \) in \( D \) such that \( r(D - [n]) = k - 1 \),
and (b) for all \( m \in N \), \( r(D - [m]) \geq k - 1 \).

**Definition 2.11**: Let \( N \) be a set of nodes with \( \#N = k \). The digraph
\( D_k = (N,N \times N) \) is called the **complete digraph on \( k \) nodes**.

**Proposition 2.3**: The complete digraph on \( k(>0) \) nodes has rank \( k \).

**Proof**: Let \( k = 1 \). Then \( D_1 = (\{n\}, \{(n,n)\}) \) clearly has rank 1.

The proof proceeds by induction on \( k \); suppose \( r(D_{k-1}) = k - 1 \)
and let \( D_k = (N,N \times N) \). Then for any \( n \in N \), \( D_k - [n] = (N-\{n\}, (N-\{n\}) \times (N-\{n\})) \),
which is clearly isomorphic with \( D_{k-1} \). Thus \( r(D_k - [n]) = k - 1 \) for all
\( n \in N \) and hence \( r(D_k) = k \).
2.2 Transition Graphs and State Graphs

Definition 2.12: Let $L$ be any finite non-void set of symbols, called labels. A finite labelled digraph over $L$ is an ordered pair $G_L = (N, B_L)$, where $N$ is a finite non-void set of nodes and $B_L$, the set of labelled branches (or simply branches) of $G_L$, is a subset of $N \times L \times N$. An element $(n_1, a, n_2) \in N \times L \times N$ of $B_L$ is called a branch from $n_1$ to $n_2$ labelled by $a$.

A finite transition graph (or simply transition graph) over a finite alphabet $A$ is a quadruple $G = (N, B_L, N_1, N_2)$, where $G_L = (N, B_L)$ is a finite labelled digraph over the set of labels $L = A \cup \{\lambda\}$, and $N_1, N_2$ are subsets of $N$. $N_1$ is called the set of initial nodes and $N_2$ - the set of terminal (or final) nodes, of $G$.

A branch $(n_1, a, n_2) \in N \times A \times N$ in a transition graph $G$ will be called an $a$-transition (or a transition labelled by $a$) from $n_1$ to $n_2$ in $G$. A branch $(n_1, \lambda, n_2) \in N \times \{\lambda\} \times N$ will be called a $\lambda$-transition from $n_1$ to $n_2$. For any branch $b = (n_1, x, n_2) \in B_L$, $n_1(n_2)$ is the initial node (terminal node) of $b$. If $(n_1, x, n_2) \in B_L$ then $n_2$ is an $x$-successor of $n_1$, and $n_1$ is an $x$-predecessor of $n_2$ in $G$. For each node $n \in N$ in $G$ and for each $x \in A \cup \{\lambda\}$, define $\sigma^G_x(n)^+$ to be the set of all $x$-successors of $n$ in $G$, and $\pi^G_x(n)^+$ to be the set of all $x$-predecessors of $n$ in $G$.

Convention: All transition graphs, as well as all automata, regular expressions and regular events considered in this text are assumed to be over some fixed finite alphabet $A$, unless otherwise indicated.

\[\text{\textsuperscript{+} When no confusion arises, the superscript } G \text{ will be omitted.}\]
Notation 2.1: In the figures, a terminal node of a transition graph \( G = (N, B_L, N_1, N_2) \) is denoted by a double circle, a non-terminal node is indicated by a single circle, a branch \((n_1, x, n_2) \in B_L\) is represented by an arrow from \(n_1\) to \(n_2\) labelled by \(x\) and an initial node has a short unlabelled arrow (which is not a branch) pointing to the node.

Example 2.2: The transition graph over the alphabet \(A = \{0, 1\}\) shown in Fig. 2.2 is \(G = (N, B_L, N_1, N_2)\), where \(N = \{n_1, n_2, n_3\}\), \(B_L = \{(n_1, \lambda, n_1), (n_1, 0, n_2), (n_2, 1, n_3), (n_3, \lambda, n_1), (n_3, 0, n_3), (n_3, 0, n_2)\}\), \(N_1 = \{n_1, n_2\}\) and \(N_2 = \{n_2\}\).

Definition 2.13: A path \(p\) in a transition graph \(G = (N, B_L, N_1, N_2)\) is any finite sequence of branches \(b_1 b_2 \ldots b_k, b_i \in B_L, i = 1, \ldots, k, k \geq 0,\) such that the terminal node of \(b_1\) is the initial node of \(b_{i+1}, i = 1, \ldots, k - 1.\) The number of branches in the path, \(k\), is called the length of \(p\) and denoted by \(\lambda(p)\). If \(\lambda(p) = 0\) then \(p\) is the null path, and is denoted by \(\varepsilon\). If \(m \in N\) is the initial node of \(b_1\) and \(n \in N\) is the final node of \(b_k\) then \(p\) is called a path from \(m\) to \(n\), \(m\) is called the initial node, and \(n\) - the final node, of \(p\). A path from a node \(n\) to itself is called a loop on \(n\). A loop of length 1 is called a self-loop. If \(p\) is a loop all of whose branches are labelled by \(\lambda\), then \(p\) is a \(\lambda\)-loop.

Definition 2.14: Let \(p_1\) and \(p_2\) be two non-null paths in \(G\) such that the final node of \(p_1\) coincides with the initial node of \(p_2\), i.e. \(p_1 = (n_1, x_1, n_2) \ldots (n_k, x_k, n_{k+1})\) and \(p_2 = (n_{k+1}, x_{k+1}, n_{k+2}) \ldots (n_\lambda, x_\lambda, n_{\lambda+1})\), where \(\lambda \geq k \geq 0\). Define their concatenation \(p_1 \cdot p_2\) to be the path \((n_1, x_1, n_2) \ldots (n_k, x_k, n_{k+1})(n_{k+1}, x_k, n_{k+2}) \ldots (n_\lambda, x_\lambda, n_{\lambda+1}).\)

† Usually the dot will be omitted.
For any path $p$, define $p \cdot \varepsilon = \varepsilon \cdot p = p$.

Denote the set of all paths of $G$ by $P(G)$.

Let $P_1, P_2 \subseteq P(G)$. Define $P_1 \cdot P_2^+$, the concatenation of $P_1$ by $P_2$, by:

$$P_1 \cdot P_2 = \{ p \mid p \in P(G) \land p = p_1 \cdot p_2 \text{ for some } p_1 \in P_1, p_2 \in P_2 \}$$

It should be noted that $P_1 \cdot P_2$ may be empty even if both $P_1$ and $P_2$ are non-empty sets.

Let $p = b_1 b_2 \ldots b_k \in P(G)$, $k > 0$. Then any path $p' = b_i b_{i+1} \ldots b_j$, $1 \leq i \leq j \leq k$, is called a subpath of $p$. If $j = k$ then $p'$ is a suffix of $p$, and if $i = 1$ then $p'$ is a prefix of $p$.

**Definition 2.15:** Let $w = a_1 a_2 \ldots a_m$ be a word over the alphabet $A$, and let $p = b_1 b_2 \ldots b_k$ be a path in a transition graph $G = (N, B_L, N_1, N_2)$ (over $A$). We say that $w$ is spelled out by $p$ iff $k \geq m$ and there exists a sequence $1 \leq j_1 < j_2 < \ldots < j_m \leq k$ such that for all $i = 1, \ldots, m$, $b_{j_i}$ is a branch labelled by $a_i$, and all other branches $b_t$, $t \neq j_1, \ldots, j_m$, are labelled by $\lambda$. The empty word $w = \lambda$ is spelled out by all paths all of whose branches are labelled by $\lambda$, and by the null path $\varepsilon$.

**Definition 2.16:** A path $p$ in $G = (N, B_L, N_1, N_2)$ is an admissible path iff the initial node of $p$ belongs to $N_1$ and the final node of $p$ is in $N_2$, i.e. $p$ is a path from an initial node of $G$ to a final node of $G$.

**Definition 2.17:** Let $G = (N, B_L, N_1, N_2)$ be a transition graph. Define for every node $n \in N$, the accepted event $A_G(n)^{+\dagger}$, of $n$ in $G$, as the

\[\dagger \text{ Usually the dot will be omitted.}\]

\[\dagger\dagger \text{ When no confusion arises, the subscript } G \text{ will be omitted.}\]
set of all words \( w \) spelled out by at least one path \( p \) from \( n \) to a
terminal node of \( G \). The preceding event \( P_r^G(n) \) of node \( n \) in \( G \), is
the set of all words \( w \) spelled out by at least one path from an initial
node of \( G \) to \( n \).

The event \( R(G) \) accepted (recognized, or defined) by the
transition graph \( G \) is defined by:

\[
R(G) = \cup_{n \in N_1} A_c^G(n) = \cup_{n \in N_2} P_r^G(n)
\]

Thus \( R(G) \) is the set of all words \( w \in A^* \) spelled out by at
least one admissible path of \( G \).

**Example 2.3:** Consider again the transition graph in Fig. 2.2. In this
case \( R(G) = P_r^G(n_2) \), thus \( R(G) \) consists of all words \( w \in (0 \cup 1)^* \) spelled
out by at least one path from either \( n_1 \) or \( n_2 \) to \( n_2 \). For instance,
\( w = 010 \in R(G) \) since \( w \) is spelled out by the admissible path
\( p = (n_1,0,n_2)(n_2,1,n_3)(n_3,\lambda,n_1)(n_1,0,n_2) \).

**Remark 2.1:** Sometimes sets of paths in a transition graph (or a digraph)
will be treated as events over the alphabet of branches \( B_L \) (or \( B \) resp.).
Thus the regular operators \( \cup, \cdot, \ast \) defined for events, will be used for
sets of paths as well. For instance, in the last example, the expression
\((n_1,\lambda,n_1) \ast (n_1,0,n_2)(n_2,1,n_3)[(n_3,\lambda,n_1)(n_1,\lambda,n_1) \ast (n_1,0,n_2) \cup (n_3,0,n_2)]\)
indicates the set of all admissible paths in \( G \) spelling out the word 010.

The following proposition is a direct consequence of the above
definitions.

\[ \dagger \text{ When no confusion arises, the superscript} G \text{ will be omitted.} \]
Proposition 2.4: Let \( G = (N, B_L, N_1, N_2) \) and let \( (m, \lambda, n) \in B_L \). Then:

1. \( A_c^G(n) \subseteq A_c^G(m) \)
2. \( P_r^G(m) \subseteq P_r^G(n) \)

Definition 2.18: A node \( n \) in a transition graph \( G \) is called **admissible** [Mc 3] iff there exists in \( G \) an admissible path touching \( n \). A branch \( b \) in \( G \) is admissible iff there is an admissible path in \( G \) one of whose branches is \( b \). \( G \) is called an **all-admissible transition graph** iff all branches and nodes of \( G \) are admissible.

Clearly, all inadmissible branches and nodes of a transition graph \( G \) can be removed from it without modifying the event accepted by \( G \). In the sequel, all transition graphs dealt with will be assumed to be all-admissible.

Definition 2.19: Two transition graphs \( G \) and \( G' \) are called **equivalent** iff \( R(G) = R(G') \).

Definition 2.20: A transition graph \( G \) without any \( \lambda \)-transition (i.e. no branch is labelled by \( \lambda \)) is called a **non-deterministic state graph**. A non-deterministic state graph \( G = (N, B_L, N_1, N_2) \) with \( \#N_1 = 1 \) and with the property that for every \( n \in N \) and for every \( a \in A \), \( \#(\sigma_a^G(n)) \leq 1 \), is called an **incomplete state graph**, and if for each \( n \in N \), \( a \in A \), \( \#(\sigma_a^G(n)) = 1 \), then \( G \) is a **deterministic state graph**. The nodes of a state graph are usually called **states**.

Definition 2.21: Let \( \mathcal{A} = (Q, M, Q_1, F) \) be a non-deterministic automaton (over alphabet \( A \)). Define a non-deterministic state graph, \( G_{\mathcal{A}} \), called
the state graph of $\mathcal{A}$, as follows:

$$G_\mathcal{A} = (Q, B_L, Q_1, F)$$

where

$$B_L = \{(q, a, q') \in Q \times A \times Q \mid q' \in M(q, a)\}$$

Thus for every transition in $\mathcal{A}$ from a state $q$ to a state $q'$ by input $a$, there corresponds in $G_\mathcal{A}$ a branch $(q, a, q')$.

Obviously the correspondence between an automaton $\mathcal{A}$ and its state graph $G_\mathcal{A}$ is 1:1; in fact a state graph can be considered as merely a convenient way of describing a finite automaton. Clearly the automaton is deterministic (incomplete) iff its corresponding state graph is deterministic (incomplete). Moreover, the event $T(\mathcal{A})$ accepted by $\mathcal{A}$ coincides with the event $R(G_\mathcal{A})$.

Notation 2.2: If $\mathcal{A} = \mathcal{A}_0(R)$ is the reduced deterministic automaton recognizing $R$, then $G_\mathcal{A}$ will be denoted by $G_0(R)$ and will be called the reduced deterministic state graph recognizing $R$. In a similar way, the minimal incomplete state graph $\tilde{G}_0(R)$ recognizing $R$ will be defined by:

$$\tilde{G}_0(R) = G_\mathcal{A}_0(R) \ (\text{Definition 1.14}).$$

Remark: For convenience, $G_0(R)(\tilde{G}_0(R))$ will sometimes be identified with $\mathcal{A}_0(R)(\tilde{A}_0(R))$, and automaton notation will be used for it rather than graph notation.

Theorem 2.1([RS], [OP]): For every transition graph $G$ there can be constructed a deterministic state graph equivalent to $G$. 

Corollary 2.1: (a) The event recognized by any transition graph is regular. (b) The accepted events and the preceding events of the nodes in a transition graph are regular.

Notation 2.3: Let $\mathcal{A} = (Q, M, q_1, F)$ where $Q = \{q_1, q_2, \ldots, q_n\}$ and let $G = G_\mathcal{A}$. Denote by $P_i(G)$, $i = 1, \ldots, n$, the set of all paths in $G$ with initial node $q_i$.

Definition 2.22: Let $G = (N, B_L, N_1, N_2)$ be any transition graph. Associate with $G$ a digraph $D_G = (N, B)$, where $B$ is the set of all pairs $(n_1, n_2) \in N \times N$ such that $(n_1, x, n_2) \in B_L$ for some $x \in A \cup \{\lambda\}$. $D_G$ is called the digraph associated with $G$.

Definition 2.23: The (cycle) rank, $r(G)$, of a transition graph $G = (N, B_L, N_1, N_2)$ is defined to be the rank of the digraph $D_G$ associated with $G$. The rank $r(\mathcal{A})$ of an automaton $\mathcal{A}$ is the rank of its state graph $G_\mathcal{A}$.

Definition 2.24: Let $G = (N, B_L, N_1, N_2)$ and let $N' \subseteq N$. Define

$$G - [N'] = (N-N', B_L-\overline{B}_L, N_1-N', N_2-N')$$

where

$$\overline{B}_L = \{(m, x, n) \mid x \in A \cup \{\lambda\}, m \in N' \text{ or } n \in N'\}$$

Definition 2.25: A transition graph $G' = (N', B'_L, N'_1, N'_2)$ is called a subgraph of $G = (N, B_L, N_1, N_2)$ iff $N' \subseteq N$ and $G' = G - [N-N']$.

Definition 2.26: A transition graph $G = (N, B_L, N_1, N_2)$ is strongly connected (s.c.) iff for every $m, n \in N$ there exists in $G$ a path from $m$ to $n$. A section of $G$ is any maximal s.c. subgraph of $G$. 
Definition 2.27: Let \( G = (N, B_L, N_1, N_2) \) be a transition graph. Define the dual of \( G \), \( \tilde{G} \), to be: \( \tilde{G} = (N, \tilde{B}_L, N_2, N_1) \), where

\[
\tilde{B}_L = \{(m, x, n) \mid (n, x, m) \in B_L\}
\]

Clearly \( \tilde{G} = G \). The next lemma follows directly from the definitions:

**Lemma 2.3:** For every transition graph \( G = (N, B_L, N_1, N_2) \),

(a) \( r(G) = r(\tilde{G}) \)

(b) \( R(G) = R(\tilde{G}) \)

(c) \( A_c^G(n) = \underbrace{P_r^{\tilde{G}}(n)}_{A_c^{\tilde{G}}(n)} \), for all \( n \in N \)

(d) \( P_r^G(n) = \underbrace{A_c^{\tilde{G}}(n)}_{P_r^{\tilde{G}}(n)} \), for all \( n \in N \)

2.3. Homomorphisms of Transition Graphs

The next definitions as well as Theorem 2.2 are due to McNaughton and appear in [Mc 4].

**Definition 2.28:** A **homomorphism** from a transition graph \( G = (N, B_L, N_1, N_2) \) onto a transition graph \( G' = (N', B'_L, N'_1, N'_2) \) is a mapping \( \phi: N \cup B_L \overset{\text{onto}}{\rightarrow} N' \cup B'_L \), which satisfies the following conditions:

(a) for every \( n \in N \), \( \phi(n) \in N' \).

(b) for every \((m, x, n) \in B_L\), either

\[
\phi(m, x, n) = (\phi(m), x, \phi(n)) \in B'_L, \quad \text{or}
\]

\[
\phi(m, x, n) = \phi(m) = \phi(n) \in N'
\]
$G' = \phi(G)$ is called a **homomorphic image** of $G$. The homomorphism $\phi$

is called **full** iff for every $(m, x, n) \in B_L$, $\phi(m, x, n) = (\phi(m), x, \phi(n))$

(i.e. no branch of $G$ is mapped onto a node of $G'$). If $\phi$ is a full 1-1
homomorphism and if $\phi(N_i) = N'_i$ for $i = 1, 2$, then $\phi$ is called an

**isomorphism**.

Let $\phi$ be a homomorphism from $G$ onto $G'$ as above. Extend $\phi$
to a mapping from $P(G)$ into $P(G')$ as follows: for every path

$p = b_1 b_2 \ldots b_k$ in $G$ ($b_i \in B_L$), let $\phi(p) = p' = \phi(b_{i_1}) \phi(b_{i_2}) \ldots \phi(b_{i_r})$, where

$1 \leq i_1 < i_2 < \ldots < i_r \leq k$, for each $t$, $1 \leq t \leq r$, $\phi(b_{i_t})$ is a branch in $G'$

and for all $j$, $1 \leq j \leq k$ and $j \neq i_t$ for $t = 1, \ldots, r$, $\phi(b_j)$ is a node in

$G'$. Clearly $\phi(p)$ is a path in $G'$. Thus for every path in $G$ there corresponds

uniquely a path in $G'$. The converse, however, is not always true.

**Definition 2.29:** A homomorphism $\phi$ from $G$ to $G'$ is a **pathwise homomorphism**
iff for every path $p'$ in $G'$ there exists a path $p$ in $G$ such that $\phi(p) = p'$.

**Theorem 2.2** (McNaughton's Pathwise Homomorphism Theorem): If there exists
a pathwise homomorphism from $G$ onto $G'$, then $r(G') \leq r(G)$.

Next we consider two types of homomorphisms of a transition graph

$G$ which leave the event accepted by $G$ unchanged.

**Theorem 2.3:** Let $\phi$ be a full homomorphism from $G = (N, B_L, N_1, N_2)$ onto

$G' = (N', B'_L, N'_1, N'_2)$, such that $\phi(N_i) = N'_i$, $i = 1, 2$, and such that for

all $m, n \in N$, $\phi(m) = \phi(n)$ implies $A_c^G(m) = A_c^G(n)$ [alternatively, for all

$m, n \in N$, $\phi(m) = \phi(n)$ implies $P_r^G(m) = P_r^G(n)$]. Then $G'$ is equivalent
to $G$. 
Proof: Every homomorphism \( \phi \) of the above type can be decomposed into a product \( \phi_k \phi_{k-1} \cdots \phi_1 \) of full homomorphisms such that each \( \phi_i \), \( 1 \leq i \leq k \), merges a pair of nodes \( n_i, n'_i \) of \( G_{i-1} = \phi_{i-1} \phi_{i-2} \cdots \phi_1 (G) \) (i.e. maps them onto some new node \( n_i \)) for which

\[
A_c G_{i-1} (n_i) = A_c G_{i-1} (n'_i),
\]

maps all other nodes of \( G_{i-1} \) onto themselves and each branch \( (n, x, n') \) of \( G_{i-1} \) onto the corresponding branch \( (\phi_i (n), x, \phi_i (n')) \) in \( G_i \). Now it can be easily verified that for every \( i, 1 \leq i \leq k \), and for every node \( n \) of \( G_{i-1} \), \( A_c G_{i-1} (n) = A_c G_i (\phi_i(n)) \). It follows that \( G_i \) is equivalent to \( G_{i-1} \) for all \( i = 1, \ldots, k \) and hence \( G' = \phi(G) = \phi_k \phi_{k-1} \cdots \phi_1 (G) = G_k \) is equivalent to \( G \).

By replacing in the above proof each accepted event \( A_c G_i (n) \) (\( i = 1, \ldots, k \)) by the corresponding preceding event \( P_r G_i (n) \), one can obtain a proof for the second part of the theorem.

Theorem 2.4: Let \( \phi \) be a pathwise homomorphism from \( G = (N, B_L, N_1, N_2) \) onto \( G' = (N', B_L', N'_1, N'_2) \) such that (1) for each branch \( (m, x, n) \in B_L' \), \( \phi(m, x, n) \in N' \) implies \( x = \lambda \), and (2) for all \( n', n'' \in N \), \( \phi(n') = \phi(n'') \) implies \( n' \in N_i \) iff \( n'' \in N_i \), \( i = 1, 2 \). Then \( G' \) is equivalent to \( G \).

Proof: Let \( w \) be any word accepted by \( G \). Then there exists a path

\[
p = (n_1, x_1, n_2) \ldots (n_k, x_k, n_{k+1}) \text{ admissible in } G \text{ and spelling out } w.
\]

Now if \( x_i \neq \lambda \) then by assumption, \( \phi(n_i, x_i, n_{i+1}) = (\phi(n_i), x_i, \phi(n_{i+1})) \).

Thus the path \( p' = \phi(p) \) also spells out the same word \( w \). Moreover, because of condition (2) of the theorem, \( p' \) is also an admissible path in \( G' \) and it follows that \( G' \) accepts \( w \).
Now let $w'$ be any word spelled out by some admissible path $p'$ in $G'$. Then since $\phi$ is a pathwise homomorphism, there exists a path $p$ in $G$ such that $\phi(p) = p'$. By the same argument as above, and condition (2) of the theorem, $p$ spells out the same word $w$ and is an admissible path in $G$. Hence $w'$ is accepted by $G$ as well. It follows that $G$ and $G'$ are equivalent.
Chapter 3

The Star Height Problem

3.1. The Star Height of a Regular Event

Definition 3.1: The apparent star height $h_\alpha$ of a regular expression $E$ is defined inductively:

- $h_\alpha(a) = 0$ for $a \in A$,
- $h_\alpha(\lambda) = h_\alpha(\emptyset) = 0$
- $h_\alpha(E_1 \cup E_2) = \max \{h_\alpha(E_1), h_\alpha(E_2)\}$
- $h_\alpha(E_1E_2) = \max \{h_\alpha(E_1), h_\alpha(E_2)\}$
- $h_\alpha(E^*) = h_\alpha(E) + 1$

Definition 3.2: The star height $h(R)$ of a regular event $R$ is defined by

$$h(R) = \min \{h_\alpha(E) \mid E \text{ is a regular expression and } |E| = R\}$$

Thus for any regular expression $E$, $h_\alpha(E)$ is the maximum length of a sequence of stars in the expression $E$, such that each star is in the scope of the star that follows it. $h(E)$, however, indicates the star height of the event $|E|$ denoted by $E$ (see Notation 1.1) as defined above.

The next proposition is obvious.

Proposition 3.1: Let $R$ be a regular event. Then $h(R) \geq 1$ iff $R$ is infinite.

Example 3.1: Let $E = (10^*1)^*$. Then $h_\alpha(E) = 2$ is the apparent star height of $E$. However, $h(E) = 1$ because $E_1 = \lambda \cup 1(0 \cup 11)^*1$ is an expression
equivalent to $E$, i.e. $|E_1| = |E|$.

**Example 3.2:** Let $E = (0 \cup 10^*1)^*$. Here again $h_\alpha(E) = 2$. Moreover, this event can be shown to be of star height 2 by using Theorem 4.1 of Chapter 4 (the reduced automaton $\mathcal{F}_0^2(R)$, $R = |E|$, is shown in Fig. 3.1). We present here an outline of a relatively simple algebraic proof to the same effect. $^\dagger$

Suppose, by contradiction, that there exists a regular expression $E'$ of apparent star height 1 such that $|E'| = |E|$. Let $H_i^*, i = 1, \ldots, n$, $n \geq 1$, be the set of all star expressions appearing in $E'$. Then the roots $H_i$ (Definition 1.16) of these star expressions must be finite events. Thus let

$$|H_i| = \sum_{j=1}^{k} w_{ij}, i = 1, \ldots, n$$

where each $w_{ij}$ is a word. It can be easily seen that each $w_{ij}$ must contain an even number of 1's (otherwise there would be $|E'|$ words with an odd number of 1's). Let $r$ be the maximal length of a sequence of consecutive 0's appearing in any $w_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, k_i$, and let $t$ be the total number of letters 0,1 appearing in $E'$ outside the scope of any star operator. Then a word of the form $u_m = 0^m1^m1$, where $m > r$, cannot be a subword of any $w_{ij}$, nor of any product of words $w_{ij}$, i.e. $u_m$, $m > r$, is not a subword of any word in $|H_i^*|$, $i = 1, \ldots, n$.

It follows that a word $(u_m)^k$, where $m > r$ and $k > n + t$ cannot be denoted by the regular expression $E'$. But this contradicts the fact, that $(u_m)^k \in |E'|$ for all integers $m,k$.

$^\dagger$ Unfortunately, such algebraic proofs can be obtained only for some special cases ([E],[DS]) and do not seem to be extensible to general families of events.
3.2. A Survey of Previous Results

The notion of star height of regular events was first introduced by Eggnan in [E]. In that paper it is established that there exist regular events of arbitrarily large star height. This result is proved by exhibiting for each integer \( k > 0 \), an event \( R_k \) of star height \( k \). Eggnan's proof is purely algebraic and quite cumbersome. Moreover, the number of letters in the alphabet \( A_k \) over which \( R_k \) is constructed grows together with \( k \). Thus Eggnan has raised the question of whether or not there exist events of arbitrarily large star height over the two-letter alphabet.† In [DS] Dejean and Schuzenberger have come up with a positive answer to this question, constructing for each \( k > 0 \), an event \( F_k \) over the alphabet \( \{0, 1\} \) of star height \( k \). \( F_k \) \( (k = 1, 2, \ldots) \) is defined to be the event accepted by the automaton \( \mathcal{A}_k = (Q, M, q_0, \{q_0\}) \), where \( Q = \{q_0, q_1, \ldots, q_{2^k-1}\} \) and the transition function \( M \) is defined by:

\[
M(q_i, 0) = q_{i+1} \pmod{2^k}, \quad i = 0, \ldots, 2^k - 1
\]

\[
M(q_i, 1) = q_{i-1} \pmod{2^k}, \quad i = 0, \ldots, 2^k - 1.
\]

Thus \( F_k \) is the set of all sequences of 0's and 1's, in which the difference between the number of occurrences of 0 and the number of occurrences of 1 is a multiple of \( 2^k \).

The second part of Eggnan's paper introduces the important notion of (cycle) rank of digraphs and the following fundamental theorem is established.

† It is not difficult to show, that a regular event over the single-letter alphabet is of star height at most 1.
Theorem 3.1: The star height of a regular event R does not exceed the rank of any transition graph recognizing R.

Theorem 3.1 is proved by constructing, for every transition graph G, a regular expression of apparent star height \( r(G) \) denoting the event \( R(G) \) recognized by G.

Thus for any regular event R, \( r(G_0(R)) \) is an upper bound to the star height of R. Unfortunately, there exist many cases in which \( h(R) < r(G_0(R)) \). For instance, let \( R = (0 \cup 1)*1 \) over the alphabet \( \{0,1\} \). Then the reduced state graph (shown in Fig. 3.2) has rank 2, whereas R is clearly of star height 1. In the next section it is shown that \( r(G_0(R)) - h(R) \) can be arbitrarily large (Corollary 3.4). Thus examining merely the reduced state graph of R does not generally suffice for determining the star height of R. Egan has shown, however, that the star height of R can be determined by studying the family of all transition graphs recognizing R. This result follows from the next lemma.

Lemma 3.1: For any regular event R there exists a transition graph G recognizing R such that \( r(G) = h(R) \).

Lemma 3.1 is proved by constructing, for any given regular expression E, a transition graph \( G_E \) with a single initial node and a single terminal node, accepting the event \( |E| \) and having rank \( r(G_E) = h_\alpha(E) \). The construction is based on the method of Ott and Feinstein [OT], and induction on the shape of the regular expression E is used.

(1) If \( E = \emptyset, \lambda \) or \( E = a, a \in A \), then the problem is trivial.
(2) If there is a transition graph \( G_E \) recognizing \(|E|\), then the transition graph \( G_{E^*} \) recognizing \( E^* \) is shown in Fig. 3.3(a).†

(3) If transition graphs \( G_{E_1} \) and \( G_{E_2} \) recognizing \(|E_1|\) and \(|E_2|\) resp. have been constructed, the transition graph \( G_{E_1 \cup E_2} \) recognizing \(|E_1 \cup E_2|\) is obtained as shown in Fig. 3.3(b).

(4) The transition graph \( G_{E_1E_2} \) recognizing \(|E_1E_2|\) is constructed from \( G_{E_1} \) and \( G_{E_2} \) as shown in Fig. 3.3(c).

One can see, that the only time a loop is created in \( G_E \) is when one encounters a star in \( E \). Hence \( r(G_E) = h_\alpha(E) \) and Lemma 3.1 is proved. Combining this lemma with Theorem 3.1, Eggan's Star Height Theorem is achieved.

Eggan's Star Height Theorem: The star height of a regular event \( R \) equals \( \min \{r(G)\} \), where \( \mathcal{F}_R \) is the family of all transition graphs recognizing \( R \).

\[ G \in \mathcal{F}_R \]

It will be shown in Chapter 5 that a proper subfamily of \( \mathcal{F}_R \) is sufficient for determining \( h(R) \).

A new technique for establishing the star height of regular events, based on Eggan's Theorem, has been introduced by McNaughton in [Mc 3]. The following fundamental theorem constitutes the essence of the graph-theoretic methods developed in that paper.

† In Fig. 3.3, a \( \lambda \)-transition entering (leaving) the square representing \( G_E \) should be interpreted as a \( \lambda \)-transition entering (leaving) the initial (terminal) node of \( G_E \).
Fig. 3.3
Theorem 3.2: If $S_1, \ldots, S_m$ are strongly connected subgraphs of a strongly connected subgraph of a transition graph $G$, and if there is no node common to more than $m - 1$ of the $S_i$'s then the rank of $G$ is at least one more than the minimum rank of all the $S_i$'s.

McNaughton's technique will now be illustrated by the following example, which appears in [Mc 3].

Example 3.4: Let $R = (0^{*}10^{*}1)^{*}$ and let $G = (N, B_L, N_1, N_2)$ be an arbitrary transition graph recognizing $R$. Let $N'(N'')$ be the set of all nodes $n$ in $G$ such that there exists a path to a terminal node from $n$ spelling out a word with an odd number of 1's (an even number of 1's), and any number of 0's interspersed. Clearly $N'$ and $N''$ are disjoint and so are the corresponding subgraphs $G' = G - [N - N']$ and $G'' = G - [N - N'']$. Now suppose $G$ has $k$ nodes. Then any path starting at a node of $N'$ ($N''$) and spelling out the word $0^{k+1}$ must be contained in $G'(G'')$ and must repeat a node. Hence $G'(G'')$ has at least one section. Thus let $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ be all sections of $G'$ and $G''$ respectively. The word $w = (0^{k+1}10^{k+1})^{k+1}$ is contained in $R$. Let $p$ be an admissible path in $G$ spelling out $w$. $p$ must intersect some $A_i$ ($1 \leq i \leq m$) more than once and between times some $B_j$ ($1 \leq j \leq n$). $A_i$ and $B_j$ are disjoint, each is s.c. and there is a path from each to the other, i.e. both are contained in a s.c. subgraph of $G$. It follows by Theorem 3.2 that $G$ is of rank $\geq 2$. Since $G$ was arbitrarily chosen, the star height of $R$ cannot be less than 2 by Eggan's Theorem. Hence $h(R) = 2$.

McNaughton's proof techniques are applicable to a great variety of events. The examples of events $F_k$ of star height $k$ constructed in [DS] are re-established in [Mc 3] by means of these methods, as well as a new
way of creating events of arbitrarily large star height, as indicated by the next theorem.

**Theorem 3.3:** Let $R_1$ and $R_2$ be regular events of star height $h$ whose respective alphabets $A_1$ and $A_2$ are disjoint, and suppose neither $a$ nor $b$ is in $A_1 \cup A_2$. Then the regular event $R = (aR_1 bR_2)^*$ over the alphabet $A_1 \cup A_2 \cup \{a, b\}$ has star height $h + 1$.

Another significant result in [Mc 3] is that for any alphabet $A = \{a_1, \ldots, a_n\}$, there exists a homomorphism $\phi: A^* \rightarrow (0 \cup 1)^*$ which preserves star height, i.e. for any $R \subseteq A^*$, $h(\phi(R)) = h(R)$. $\phi$ is determined by the $n$ equations

$$\phi(a_i) = 10^i10^{n-i+1}1, \quad i = 1, \ldots, n$$

Thus every event $R$ over the $n$-letter alphabet can be coded down to an event $\phi(R)$ over the 2-letter alphabet having the same star height.

In the last section of [Mc 3], techniques for reducing the apparent star height of a given regular expression are described. A few isolated principles are obtained (for instance, $(\alpha \beta^* \gamma)^* = \lambda \cup \alpha(\beta \cup \gamma \alpha)^* \gamma$ for all regular events $\alpha$, $\beta$ and $\gamma$) and a number of illuminating examples are presented, in which such reduction is obtained by looking at the reduced state graph of the event.

Another method for reducing the apparent star height of a given regular expression $E$ is discussed in [Mc 2]. This reduction technique involves a series of transformations on a transition graph $G_E$ representing

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$^*$ An ordinary homomorphism between two free monoids, which is extended to subsets of the monoids in the natural way.
E, leading to an equivalent transition graph $G'$ of smaller rank, from which a regular expression $E'$ of smaller apparent star height can be derived. However, these transformations are carried out on an intuitive basis and are helpful mainly when the regular expression $E'$ is known beforehand. In Chapter 5 of this thesis a similar approach has been adopted, several basic types of rank-non-increasing transformations on transition graphs have been introduced, by means of which any arbitrary transition graph can be converted into an equivalent one with some special properties. The author strongly believes that a more inclusive and powerful set of transformations on transition graphs could be defined, by means of which any arbitrary transition graph recognizing the event $R$ can be modified into an equivalent one of no greater rank, whose number of nodes is bounded by some fixed number $k_R$, which is a function of the number of states in the reduced automaton $A_0(R)$. This would yield a finite algorithm for finding the star height of any arbitrary regular event $R$. Such an algorithm, to the best of the author's knowledge, has not yet been established.

There has been found, however, an algorithm for determining the star height of the members of some subfamily of the family of regular events, namely the family of all pure-group events (Definition 1.21). This algorithm is presented in McNaughton's paper [Mc 4]. The algorithm involves constructing a certain finite family of transition graphs, called $\mu$-graphs, for the given pure-group event $R$. These $\mu$-graphs are derived in a certain manner from the syntactic monoid of $R$. It is shown that if $h(R) = k$ then there exists a subset of this family of $\mu$-graphs whose
union graph $^\dagger$ has rank $k$ and recognizes the event $R$. Thus in order to verify that the star height of $R$ is $\leq i$ ($i = 1, 2, ...$), one constructs the union graph of all $\mu$-graphs of rank $\leq i$ and checks whether this graph accepts the whole event $R$.

Another result in [Mc 4] is the following.

Theorem 3.4: If the reduced state graph $G_0(R)$ of a pure-group event $R$ has exactly one terminal node, then the star height of $R$ equals the rank of $G_0(R)$.

This theorem will be obtained in Chapter 4 as a special case of a more general result (Theorem 4.1) which is proved in a totally different way. Note that the class of events $E_k$ of star height $k$ mentioned before [DS] happens to be a special case of Theorem 3.4.

In [D] Dejean proves, that all pure-group events of star height 1 over an alphabet $A$ are of the form:

$$(A^k)^* (A^1 \cup A^2 \cup ... \cup A^n)$$

where for each $j$, $j = 1, ..., n$, $0 \leq i_j \leq k - 1$.

3.3 Some General Properties of Star Height

In this section the effect of various operations on the star height of regular events is examined.

We first state some results, due to Brzozowski, showing that star height is preserved by quotients.

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$^\dagger$ By a union graph of several graphs is meant the result of considering these graphs together as a single graph. A node is initial, or terminal, iff it was so before.
Theorem 3.5: Let \( R \) be a regular event and let \( w \in A^* \). Then
\[
h(w \setminus R) \leq h(R).
\]
Furthermore, for all integers \( n \geq 0 \), there exists a word \( x \) of length \( n \) such that \( h(x \setminus R) = h(R) \).

Clearly Theorem 3.5 holds for right quotients as well, and since
\[
x \setminus R / y = (x \setminus R) / y = x \setminus (R / y)
\]
for all events \( R \) and \( x, y \in A^* \), we have:

Corollary 3.1: For any regular event \( R \) and for any \( x, y \in A^* \),
\[
h(x \setminus R / y) \leq h(R).
\]

Corollary 3.2: Let \( R \) be a regular event and let \( G = G_0(R) \). Then if \( G \) is strongly connected, \( h(x \setminus R) = h(R) \) for all \( x \in A^* \).

Next we examine the following problem. Let \( T_1 \) and \( T_2 \) be regular events and let \( R \) be an event obtained from \( T_1, T_2 \) by a finite number of applications of the operators \( \cup, \cap, -, \cdot \), and \( \ast \). How is the star height of \( R \) related to those of \( T_1 \) and \( T_2 \)? - The answer is rather disappointing because in most cases very little connection appears to exist between \( h(T_1), h(T_2) \) and \( h(R) \). We now proceed to examine specific cases.

Proposition 3.2: For any regular events \( T_1, T_2 \), \( h(T_1 \cup T_2) \leq \max \{h(T_1), h(T_2)\} \).

Moreover, for every integer \( n \geq 0 \) there exist events \( T_1 \) and \( T_2 \) such that \( h(T_1) = n \) and \( h(T_1 \cup T_2) = 1 \).

Proof: The first assertion is trivial. As for the second, simply take any event \( T_1 \) of star height \( n \) and \( T_2 = \overline{T_1} \). Then \( h(T_1 \cup T_2) = h(A^*) = 1 \).

Proposition 3.3: For any regular events \( T_1, T_2 \), \( h(T_1 \setminus T_2) \leq \max \{h(T_1), h(T_2)\} \), and for any integer \( n \geq 0 \) there exist \( T_1, T_2 \) such that \( h(T_1) = n \) and
\[ h(T_1 T_2) = 1. \]

**Proof:** The first assertion is obvious. As for the second, let \( T_1 \) be any event of star height \( n \) containing the empty word \( \lambda \) and let \( T_2 = A^* \). Then \( h(T_1 T_2) = 1. \)

**Proposition 3.4:** Let \( T \) be a regular event. Then \( h(T^*) \leq h(T) + 1 \) and for any integer \( n \geq 0 \) there exists an event \( T \) of star height \( n \) such that \( h(T^*) = 1. \)

**Proof:** Let \( A = \{0, 1\} \) and let \( T = OT_1 \cup 1 \), where \( h(T_1) = n \) and \( \lambda \in T_1 \). Now \( 0 \setminus T = T_1 \) showing that \( h(T) = h(T_1) = n \) (Theorem 3.5). However, \( T^* = (0 \cup 1)^* \) is of star height 1.

**Proposition 3.5:** For every integer \( m \geq 0 \), there exists a regular event \( R \) such that \( h(R) - h(\overline{R}) \geq m \).

**Proof:** Let \( A = \{a_1, a_2, \ldots, a_n\} \) be the alphabet, where \( n = m + 2 \geq 2 \), and let \( \mathcal{A}_i = (Q_i, M_i, q^i_1, \{q^i_1\}) \), \( i = 1, 2, \ldots, n \), be the reduced deterministic automaton recognizing the event \( R_i \), consisting of all words over \( A \) with an even number of occurrences of the letter \( a_i \) (see Example 3.2). Let \( \mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n = (Q, M, q_1, F) \) (Definition 1.22), where \( Q = Q_1 \times \ldots \times Q_n \), \( q_1 = (q^1_1, \ldots, q^n_1) \) and \( F = \{q_1\} \). Since each \( \mathcal{A}_i \) is a pure-group automaton, so is \( \mathcal{A} \) (Theorem 1.14) and by Theorem 3.4, \( h(R) = r(G, \mathcal{A}) \), where \( R \) is the event recognized by \( \mathcal{A} \). If we show that \( r(G, \mathcal{A}) \geq n \) then we have \( h(R) \geq n \). Furthermore, since \( R = R_1 \cap R_2 \cap \ldots \cap R_n \), by DeMorgan's law, \( \overline{R} = \overline{R_1} \cup \overline{R_2} \cup \ldots \cup \overline{R_n} \), and since each \( \overline{R}_i \) is of star height 2 (again by Theorem 3.4), we get \( h(\overline{R}) \leq 2 \). Hence \( h(R) - h(\overline{R}) \geq n - 2 = m \) as desired.
Thus it remains to show that \( r(G_{\mathcal{A}}) \geq n \). We shall prove this by induction on \( n \). For \( n = 2 \), \( \mathcal{A}_1 \times \mathcal{A}_2 \) is shown in Fig. 3.4 and apparently has rank 2. Now suppose the assertion holds for \( n - 1 \) and let \( \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \). Then for any node \( q = (q^1, \ldots, q^n) \) of \( G_{\mathcal{A}} \), the graph \( G_{\mathcal{A} - [q]} \) certainly contains the subgraph \( G_{\mathcal{A} - [N']} \), where

\[
N' = \{(p_1, \ldots, p_n) \mid p_i \in Q_i, i = 1, \ldots, n, \text{ and } p_n = q^n\}
\]

But since \( Q_n \) (as well as the other \( Q_i \)'s) consists of only two states, this graph \( G_{\mathcal{A} - [N']} \) can be easily seen to be isomorphic with

\[
G_{\mathcal{A}_1 \times \cdots \times \mathcal{A}_{n-1} - [q]}
\]

Hence by the induction hypothesis and Lemma 2.1,

\[
r(G_{\mathcal{A} - [q]}) \geq r(G_{\mathcal{A} - [N']}) \geq n - 1
\]

for any state \( q \) of \( \mathcal{A} \), and therefore \( r(G_{\mathcal{A}}) \geq n \). This completes the proof.

---

**Corollary 3.3:** For every integer \( n \geq 0 \), there exist regular events \( R_1, R_2, \ldots, R_n \) such that

\[
h(R_1 \cap R_2 \cap \cdots \cap R_n) - \max_{1 \leq i \leq n} \{h(R_i)\} \geq n - 2.
\]

**Corollary 3.4:** For any integer \( k > 0 \) there exists a regular event \( R \)
such that

\[
r(G_0(R)) - h(R) > k
\]

**Proof:** The event \( \overline{R} = \overline{R}_1 \cup \overline{R}_2 \cup \cdots \cup \overline{R}_n \) defined in the proof of Proposition 3.5 can be easily shown to be accepted by the automaton \( \overline{\mathcal{A}} \) obtained from \( \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \) by interchanging the set \( F \) of final states with the
Fig. 3.4.
set \( Q - F \). Obviously \( \overline{A} \) is also reduced and the result follows.

The next proposition is obvious.

**Proposition 3.6:** For every regular event \( R \), \( h(\overline{R}) = h(R) \).

The next two propositions are due to Brzozowski.

**Proposition 3.7:** Let \( R \) and \( T \) be two regular events such that \( h(T) = 0 \). Then \( h(R) = h(R - T) = h(R \cup T) \).

**Proposition 3.8:** Let \( A \) be an alphabet with at least two letters and let \( R \) be a regular event over \( A \). If the automaton \( A_0(R) \) is strongly connected then for any \( a \in A \),

\[
h(R - aA^*) = h(R).
\]

In Proposition 3.7 it was shown that removing from \( R \) an event \( T \) of star height 0 does not affect star height. The question naturally arises, whether the star height of \( R \) can be affected by subtracting from it any subevent \( T \subseteq R \) of star height \( h(T) < h(R) \).

**Proposition 3.9:** Let \( R \) and \( T \) be any regular events such that \( T \subseteq R \) and \( h(T) < h(R) \). Then \( h(R - T) \geq h(R) \).

**Proof:** If \( h(R - T) < h(R) \) then \( R = (R - T) \cup T \) would imply \( h(R) \leq \max \{ h(R - T), h(T) \} < h(R) \).

We conjecture:

**Conjecture 3.1:** Let \( R \) and \( T \) be as in Proposition 3.9. Then \( h(R - T) = h(R) \).

We have been able to prove only the following special case of

\[\dagger\] Recently R. McNaughton has disproved this conjecture by a counter example (see Appendix A).
Conjecture 3.1.

Theorem 3.6: Let $R$ be a regular event of star height $k > 1$ such that
for any two words $x, y \in A^*$, either $x \setminus R = y \setminus R$ or $x \setminus R \cap y \setminus R = \emptyset$.
Then for every subset $T \subseteq R$ such that $h(T) < k$ and $r(C_0(T)) \leq k$,
h($R - T$) = h($R$).

Proof: By Proposition 3.9, h($R - T$) $\geq$ h($R$). Consider the reduced
automaton $\mathcal{A} = \mathcal{A}_0(R) = (Q, M, q_1, F)$. As shown in Chapter 4,
Theorem 4.1, $r(\mathcal{A}) = r(\mathcal{A}_0(R)) = k$. Let $\mathcal{A}' = \mathcal{A}_0(T) = (P, N, p_1, G)$
be the reduced deterministic automaton recognizing $T$, and let
$\mathcal{B} = \mathcal{A} \times \mathcal{A}' = (Q \times P, M \times N, (q_1, p_1), F \times G)$ (Definition 1.22). Let
$\mathcal{B}'$ be the automaton obtained from $\mathcal{B}$ by changing the set of final states
from $F \times G$ to $F \times (P - G)$, and then removing all inaccessible states
(Definition 1.11). Let the set of states of $\mathcal{B}'$ be $S$.

It can be easily verified that the automaton $\mathcal{B}'$ thus obtained
accepts the event $R - T$. We claim that $r(\mathcal{B}') \leq k$. To see this, let
$(q, p) \in S$ be any state of $\mathcal{B}'$. Then there exists a word $w$ such that
$(M \times N)(((q_1, p_1), w) = (M(q_1, w), N(p_1, w)) = (q, p)$. Now since $T \subseteq R$,
w$w \in T$ implies $w \in R$ for any word $x$. Hence all words $x$ taking $p$ to a
final state in $\mathcal{A}'$, also take $q$ to a final state in $\mathcal{A}$, i.e. $A_C^{A'}(p) \subseteq A_C^A(q)$.
But since for every two states $q', q''$ in $Q$, $A_C^{A'}(q') \cap A_C^A(q'') = \emptyset$ (see
Theorem 1.6 and Definition 1.13), it follows that for every state $p$ in $\mathcal{A}'$
such that $A_C^{A'}(p) \neq \emptyset$, there exists at most one state $q \in Q$ such that
$(q, p) \in S$. The only state $p$ in $P$ which may appear in more than one pair
$(q, p)$ of $S$ is the dead state $p_\emptyset$, if such a state exists in $\mathcal{A}'$. Thus
decompose $S$ into two disjoint subsets: $S = S' \cup S''$, where $S'$ is the set
of all pairs \((q, p) \in S\) such that \(p \neq p_\phi\), and \(S''\) is the set of all pairs of the form \((q, p_\phi)\) appearing in \(S\). We claim:

1. There does not exist in \(\mathcal{B}'\) any transition from a state of \(S''\) to a state of \(S'\).
2. The subgraph \(G_{\mathcal{B}_1'} - [S'']\) of \(G_{\mathcal{B}_1'}\) is isomorphic with some subgraph of \(G_{\mathcal{A}_1'}\).
3. The subgraph \(G_{\mathcal{B}_1'} - [S']\) of \(G_{\mathcal{B}_1'}\) is isomorphic with some subgraph of \(G_{\mathcal{A}_1'}\).

The first assertion follows from the fact, that for any word \(x, N(p_\phi, x) = p_\phi\). As for (2), let \((q, p) \in S'\). Then for any \(a \in A\), \((M \times N)((q, p), a) = (M(q, a), N(p, a))\). If \(N(p, a) = p_\phi\) then this transition is not contained in \(G_{\mathcal{B}_1'} - [S'']\). Thus suppose \(N(p, a) \neq p_\phi\).

Define a mapping \(\phi: S' \rightarrow P - \{p_\phi\}\) by \(\phi(q, p) = p\) for all \((q, p) \in S'\). Then by the above remark, \(\phi\) is a 1-1 mapping. Moreover, if for some \((q, p) \in S'\) and \(a \in A\), \((M \times N)((q, p), a) = (q', p')\), then by definition, \(p' = N(p, a)\). Hence \(\phi\) is an isomorphism of \(G_{\mathcal{B}_1'} - [S'']\) onto some subgraph of \(G_{\mathcal{A}_1'}\), as required.

To show (3), let \(Q' = \{q \in Q \mid (q, p_\phi) \in S''\}\). Define a mapping \(\psi: S'' \rightarrow Q'\) by \(\psi(q, p_\phi) = q\) for all \((q, p_\phi) \in S''\). Obviously \(\psi\) is an isomorphism of \(G_{\mathcal{B}_1'} - [S']\) onto the subgraph \(G_{\mathcal{A}_1'} - [Q - Q']\), which proves (3).

Now by (1), the set of states of any section in \(G_{\mathcal{B}_1'}\) must be contained in either \(S'\) or \(S''\). Thus:

\[ r(G_{\mathcal{B}_1'}) = \max \{r(G_{\mathcal{B}_1'} - [S']), r(G_{\mathcal{B}_1'} - [S''])\} \leq \max \{r(G_{\mathcal{A}_1'}), r(G_{\mathcal{A}_1'})\} \leq k. \]

Hence by Theorem 3.1, \(h(R - T) \leq k\). This completes the proof.
In connection with Conjecture 3.1 we would like to mention another open problem suggested by Brzozowski.

Let \( R \) be a regular event of star height \( \geq 1 \), and let \( T \) be a finite event. Is it always true that \( h(TR) = h(R) = h(RT) ? \)

Although it seems rather plausible that this equality always holds, so far we have been unable to give it a formal proof.†

**Definition 3.3:** Let \( A \) and \( B \) be two finite alphabets. A mapping \( \phi : 2^{A^*} \to 2^{B^*} \) is called a **generalized homomorphism** iff

1. \( \phi(\Lambda) = \Lambda \)
2. for any letter \( a \in A \), \( \phi(a) \) is a finite event over \( B \).
3. for any word \( w = a_1 a_2 \cdots a_n \in A^* \),
   \[ \phi(w) = (\phi(a_1)) (\phi(a_2)) \cdots (\phi(a_n)) \]
4. for any \( W \subseteq A^* \), \( \phi(W) = \bigcup_{w \in W} \phi(w) \)

An event \( R' \subseteq B^* \) is called a **homomorphic image** of an event \( R \subseteq A^* \) iff there exists a generalized homorphism \( \phi \) from \( 2^{A^*} \) into \( 2^{B^*} \) such that \( R' = \phi(R) \).

The following proposition can be easily verified.

**Proposition 3.10:** If \( R' \) is a homomorphic image of \( R \) then \( h(R') \leq h(R) \).

† R. McNaughton has provided an example of events \( R, T \) with the above properties for which this equality does not hold (see Appendix B).
Chapter 4

Star Height of Certain Families of Events

Introduction

In this chapter the relationships between the apparent star height of a given regular expression and the structure of its reduced state graph are investigated.

Sections 4.1 and 4.2 introduce the notions of trails (i.e. state sequences) in a state graph, and simple regular expressions with respect to a given state graph. In section 4.3 the definition of cofinal sets of trails is given, and the Main Lemma, relating the apparent star height of a given regular expression with some properties of its corresponding set of trails in some state graph, is obtained. By means of the Main Lemma, in section 4.4 sufficient conditions are found for the star height of an event \( R \) to equal the rank of its reduced state graph \( \chi_0(R) \). Thus families of events of arbitrarily large star height can be easily constructed. Moreover, the rank of \( \chi_0(R) \) is shown to be a lower bound to the star height of certain subsets of \( R \).

Section 4.5 deals with the class of events defined by digraphs. Each member of this class consists of all paths connecting two sets of nodes in some digraph \( D \). It is shown that the star height of each such event \( R \) equals the rank of the smallest subgraph of \( D \) containing all paths of \( R \). As a by-product, it is shown that every regular event \( R \) is a homomorphic image of a group-free event whose star height equals the rank
4.1. Trails in a State Graph

Throughout this section and the next two sections let $G$ denote an incomplete state graph without a dead state (Definition 1.14) whose corresponding automaton is $\mathcal{A} = (Q, M, q_1, F)$, where $Q = \{q_1, q_2, \ldots, q_n\}$.

**Definition 4.1:** A trail (state sequence) in $G$ is any sequence of states, $t = (q_{i_1}, q_{i_2}, \ldots, q_{i_m})$, where $q_{i_j} \in Q$, $m \geq 1$ and for each $1 \leq j \leq m - 1$, there exists in $G$ a transition from $q_{i_j}$ to $q_{i_{j+1}}$. $q_{i_1}$ is called the initial node of $t$ and $q_{i_m}$ is the terminal node. A subtrail of $t$ is any trail $t' = (q_{i_1}, q_{i_2}, \ldots, q_{i_l})$ where $1 \leq k \leq l \leq m$.

Given two trails, $t_1 = (q_{i_1}, q_{i_2}, \ldots, q_{i_k})$ and $t_2 = (q_{i_k}, q_{i_{k+1}}, \ldots, q_{i_l})$, where $1 \leq k \leq l$, define their concatenation:

$$t_1 \cdot t_2^+ = (q_{i_1}, q_{i_2}, \ldots, q_{i_k}, q_{i_{k+1}}, \ldots, q_{i_l}).$$

Let $t$ be a trail of the form: $(q_{i_1}, q_{i_2}, \ldots, q_{i_1})$. Then $t$ is called a loop in $G$. Define: $t^0 = (q_{i_1})$, $t^1 = t$, and inductively, $t^k = t \cdot t^{k-1}$ for all $k = 1, 2, \ldots$. A trail of unit length $(q_{i_1})$ is called a trivial trail or a trivial loop.

Denote the set of all trails in $G$ by $T(G)$.

$\dagger$ The dot will sometimes be omitted.
Let $T, T' \subseteq T(G)$ such that all trails of $T$ have a common terminal node, which also coincides with the common initial node of all trails of $T'$. Define the concatenation (or product)

$$T \cdot T' = \{t \cdot t' \mid t \in T, t' \in T'\}.$$ 

Let $L \subseteq T(G)$ be a set of loops in $G$ from state $q_i$ to itself. Define $L^0 = (q_i)$ and by induction: $L^k = L \cdot L^{k-1}$. The star operation is defined by $L^* = \bigcup_{k=0}^{\infty} L^k$.

Denote the set of all trails in $T(G)$ with initial node $q_i$ by $T_i(G)$.

**Definition 4.2:** For each $i$, $1 \leq i \leq n$, define the event

$$W_i(G) = \{w \in A^* \mid \#M(q_i, w) = 1\}$$

Thus $W_i(G)$ is the set of all input words $w$ applicable to the automaton when started at state $q_i$.

The following properties of $W_i(G)$ can easily be verified:

for each $i$, $1 \leq i \leq n$,

1. $W_i(G)$ is regular.
2. $\lambda \in W_i(G)$.
3. If $w \in W_i(G)$ then all prefixes of $w$ also belong to $W_i(G)$.
4. If $G$ is a deterministic state graph then $W_i(G) = A^*$.
5. Let $R$ be the event recognized by $G$. Then $R \subseteq W_i(G)$.

**Definition 4.3:** For every $i$, $1 \leq i \leq n$, define a mapping $\tau_i : W_i(G) \rightarrow T_i(G)$ as follows: $\tau_i(\lambda) = (q_i)$ and for every non-$\lambda$ word $w = a_1a_2...a_k \in W_i(G)$,
\[ \tau_i(w) = (q_{i,1}, q_{j,1}, q_{j,2}, \ldots, q_{j,k}) \text{, where } q_j = M_i(a_1 a_2 \ldots a_r) \text{ for all } 1 \leq r \leq k. \] Thus \( \tau_i \) maps each word \( w \) of \( W_i(G) \) into the trail of states the automaton passes through when started at state \( q_i \) and scanning \( w \).

Extend the mapping \( \tau_i \) to any event \( V \subseteq W_i(G) \):

\[ \tau_i(V) = \{ \tau_i(w) \mid w \in V \} \subseteq T_i(G). \]

The following three lemmas are direct consequences of the above definitions:

**Lemma 4.1:** The mapping \( \tau_i : W_i(G) \rightarrow T_i(G) \) is onto for all \( i = 1, \ldots, n \).

**Lemma 4.2:** Let \( R \subseteq W_i(G) \) and \( R = R_1 \cup R_2 \). Then \( \tau_i(R) = \tau_i(R_1) \cup \tau_i(R_2) \).

**Lemma 4.3:** Let \( R \subseteq W_i(G) \) and \( R = R_1 R_2 \) such that \( M_i(R_1) = \{ q_j \} \) for some \( q_j \in Q \). Then \( \tau_i(R) = \tau_i(R_1) \cdot \tau_i(R_2) \).

### 4.2. Simple Regular Expressions

**Definition 4.4:** The string form \( E_s \) of a regular expression \( E \) is defined inductively as follows:

1. If \( h_\alpha(E) = 0 \) then \( E_s = w_1 \cup w_2 \cup \ldots \cup w_\ell \) where \( w_j \) are words, \( j = 1, \ldots, \ell, \ell \geq 0 \).

2. If \( h_\alpha(E) = k \) then \( E_s = F_1 \cup F_2 \cup \ldots \cup F_p, p > 0 \) where each \( F_j \) is of the form

\[ w_{1,1} w_{1,2}^{r} \ldots w_{1,m}^{r} w_{2,1} w_{2,2}^{r} \ldots w_{2,m}^{r} \ldots w_{m,1}^{r} \ldots w_{m,m}^{r} w_{m+1} \] where \( r > 0 \), \( m \geq 0 \), \( w_j \) are words, \( H_j \) are in string form and \( h_\alpha(H_j) \leq k - 1, j = 1, \ldots, m \).
Thus $E_s$ is formed by a finite number of strings of words and star expressions, that latter having roots which are again made up from a finite number of strings of words and star expressions, and so on. In other words, $E$ and all roots of star expressions appearing in $E$ are represented as a finite union of strings of words and star expressions. The strings $F_i$, $i = 1, \ldots, p$, will be called \textit{first level strings of $E$}.

Obviously, because of the distributivity of concatenation over union, every regular expression $E$ may be transformed into its string form $E_s$.

\textbf{Example 4.1:} Let $E = (1^* (1 \cup 0) \cup 2)^* (2*1 \cup (0 \cup 2)* (22 \cup 2)1) \cup 00$. Then $E_s = (1^* 1 \cup 1*0 \cup 2)^* 2*1 \cup (1^* 1 \cup 1*0 \cup 2)^* (0 \cup 2)* 21 \cup 00$.

Clearly the transformation of $E$ into $E_s$ does not involve any change in the apparent star height. Thus we have:

\textbf{Lemma 4.4:} For any regular expression $E$ \begin{enumerate}
\item[(a)] $|E_s| = |E|$
\item[(b)] $h_\alpha (E_s) = h_\alpha (E)$
\end{enumerate}

Therefore, as far as the apparent star height is concerned, we can assume without loss of generality that all regular expressions dealt with are in string form. \textit{In the sequel, let $E$ always denote a regular expression in string form, unless otherwise specified.}

Let $E$ be a regular expression and $H^*$ a star expression appearing in $E$. $H^*$ may occur in $E$ more than once. In the following, we make a distinction among all appearances of $H^*$ in $E$, and refer to $H^*$ as a \textit{p-star (position-star)} in $E$ when a certain appearance of it is considered.
Thus the number of p-stars in \( E \) is the number of symbols * appearing in it, whereas the number of distinct star expressions in \( E \) may be smaller. For example, if \( E = (1*1 \cup 1*0 \cup 2)* \), the p-stars in \( E \) are \( 1^* \), \( 1^* \), and \( E \) itself.

We shall call the p-stars \( H^* \) appearing as factors in the first-level strings of \( E \) p-stars of level 1 in \( E \). Inductively, a p-star in \( E \) is of level \( k \) iff it occurs as a factor in one of the strings forming the root of a p-star of level \( k - 1 \) in \( E \). Thus in Example 4.1 all appearances of \((1*1 \cup 1*0 \cup 2)^*\), \((0 \cup 2)^*\) and \(2^*\) are of level 1 whereas \(1^*\) appears always in level 2.

**Definition 4.5:** Let \( E \) be a regular expression and \( H^* \) a p-star of level \( k \) in \( E \). Define the **predecessor of \( H^* \) in \( E \)**, \( \pi_E(H^*) \), by induction on \( k \):

1. \( k = 1 \). Then \( H^* = H_1^* \) appears as a factor in a string 
   \[ w_1 H_1^* w_2 \ldots w_n H_1^* \]
   Define:
   \[ \pi_E(H^*) = w_1 H_1^* w_2 \ldots w_n H_1^* \]

2. \( k > 1 \) and assume the predecessor is defined for all p-stars of level less than \( k \). \( H^* \) appears in \( E \) within the scope of a p-star of lower level, say \( H_1^* \). Define:
   \[ \pi_E(H^*) = \pi_E(H_1^*), \pi_H_1^*(H^*) \]

Define the **successor of \( H^* \) in \( E \)**, \( \sigma_E(H^*) \), by:
   \[ \sigma_E(H^*) = \pi_E(H^*) \]

**Lemma 4.5:** Let \( G \) be a state graph recognizing the event \( R \) and let \( E \) be a regular expression denoting a subset of \( R \). Then for every p-star \( H^* \) in \( E \):
(a) Both events \(|\pi_E(H^*)|\) and \(|\sigma_E(H^*)|\) are infinite.

(b) \(|\pi_E(H^*) \cdot \sigma_E(H^*)| \subseteq R

(c) \(|\sigma_E(H^*)| = \bigcap_{q \in M_1(\pi_E(H^*))} D_q \) (See Notation 1.3)

Proof: (a) Obviously \(\pi_E(H^*)\) is not empty and by definition

\[\pi_E(H^*) = \pi_E(H^*) \cdot H^*,\text{ thus } |\pi_E(H^*)| \text{ is infinite. The same proof applies to } |\sigma_E(H^*)|\.

(b) Follows directly from the definitions.

(c) By (b), \(M_1(\pi_E(H^*) \cdot \sigma_E(H^*))\) is contained in the set \(F\) of final states of \(G\). We have:

\[M_1(\pi_E(H^*) \cdot \sigma_E(H^*)) = M(M_1(\pi_E(H^*), \sigma_E(H^*)) \subseteq F.\]

The last equation shows that for every state \(q\) in \(M_1(\pi_E(H^*))\), \(|\sigma_E(H^*)|\) is contained in the corresponding derivative \(D_q\) and hence the result follows.

---

**Definition 4.6:** Let \(G\) be a state graph and let \(E\) be a regular expression.

A p-star \(H^*\) of \(E\) is called **simple with respect to \(G\)** iff \(M_1(\pi_E(H^*))\) is a singleton subset of \(Q\). \(E\) is a **simple regular expression with respect to \(G\)** iff \(|E| \subseteq W_1(G)\) and every p-star of \(E\) is simple w.r.t. \(G\).

Thus if \(E\) is simple w.r.t. \(G\) then every position \([\text{MY}]\) in \(E\) corresponds to a single state of \(G\).

**Example 4.2:** Let \(G\) be the state graph in Fig. 4.1, and let

\[E = (0^*2 \cup 01)^*0 \cup (0 \cup 2)^*02.\]  

We have \(\pi_E((0^*2 \cup 01)^*) = (0^*2 \cup 01)^*;\)

\(\pi_E((0 \cup 2)^*) = (0 \cup 2)^*\) and \(\pi_E(0^*) = (0^*2 \cup 01)^*0^*\). By inspection,
\( M_1((0*2 \cup 01)*) = \{A\}, M_1((0 \cup 2)*) = \{A,B\} \) and \( M_1((0*2 \cup 01)^*0*) = \{A,B\} \).

Thus \( (0*2 \cup 01)^* \) is a simple star w.r.t. \( G \), while \( (0 \cup 2)^* \) and \( 0^* \) are not. Hence \( E \) is not a simple regular expression w.r.t. \( G \). However, one can easily show that \( E' = (00*2 \cup 01)^* \), for instance, is simple w.r.t. \( G \).

The following lemma can be easily verified:

**Lemma 4.6:** Let \( E \) be a regular expression and let \( H^* \) be a p-star in \( E \) simple w.r.t. the state graph \( G \), such that \( M_1(\pi_E(H^*)) = \{q_r\}, q_r \in Q \), and \( \pi_E(H^*) \subseteq W_r(G) \). Then:

(a) \( \tau_r(H^*) \) is a set of loops from \( q_r \) to itself.

(b) \( \tau_r(H^*) = (\tau_r(H))^* \)

**Lemma 4.7:** Let \( E \) be a regular expression simple w.r.t. \( G \) and let \( H^* \) be a p-star of \( E \) such that \( M_1(\pi_E(H^*)) = \{q_r\}, q_r \in Q \). Then \( H \) is a simple regular expression w.r.t. \( G_r \), where \( G_r \) is the state graph derived from \( G \) by changing the initial state from \( q_1 \) to \( q_r \).

**Proof:** Let \( H_1^* \) be any p-star in \( H \). By definition: \( \pi_E(H^*) \cdot \pi_H(H_1^*) = \pi_E(H_1^*) \).

Applying the function \( M_1 \) we get:

\[
M_1(\pi_E(H_1^*)) = M_1(\pi_E(H^*) \cdot \pi_H(H_1^*)) = M(M_1(\pi_E(H^*)), \pi_H(H_1^*)) = M(q_r, \pi_H(H_1^*)) = \pi_H(H_1^*).
\]

Since \( E \) is a simple regular expression w.r.t. \( G \), \( M_1(\pi_E(H_1^*)) \) is a singleton. Hence so is \( M_r(\pi_H(H_1^*)) \), which shows that \( H_1^* \) is a simple p-star in \( H \) w.r.t. \( G_r \). It follows that \( H \) is a simple regular expression w.r.t. \( G_r \).
4.3. Cofinal Sets of Trails and Star Height of Simple Regular Expressions

Definition 4.7: Let $G$ be a state graph and $G'$ a s.c. subgraph of $G$. For every positive integer $m$ define in $G$ an $m$-trail w.r.t. $G'$ and a trail of exponent $m$ w.r.t. $G'$ by induction on the rank of $G'$:

(a) Let $r(G') = 1$. Then a trail in $G$ is an $m$-trail w.r.t. $G'$ iff it contains a subtrail which is a non-trivial loop in $G'$.\[\dagger\] A trail $t \in T(G)$ is of exponent $m$ w.r.t. $G'$ iff $t = t_1 t_2 \ldots t_m$, where each $t_j (j = 1, 2, \ldots, m)$ is an $m$-trail w.r.t. $G'$.

(b) Let $r(G') = k > 1$. Then $t' \in T(G)$ is an $m$-trail w.r.t. $G'$ iff for every state $q$ in $G'$ there exists a subtrail $(q, q_1, \ldots, q_2, q)$ of $t'$ such that $(q_1, q_2, \ldots, q_2)$ is a trail in $G' - [q]$ and is of exponent $m$ w.r.t. some s.c. subgraph of $G' - [q]$ of rank $k - 1$. A trail $t \in T(G)$ is of exponent $m$ w.r.t. $G'$ iff $t = t_1 t_2 \ldots t_m$, where each $t_j (j = 1, 2, \ldots, m)$ is an $m$-trail w.r.t. $G'$.

Example 4.3: Let $G$ be the graph in Fig. 4.1. Obviously $G$ has rank 2. Let $m = 3$ and consider the subgraph $G_1 = G - [B]$. Clearly $r(G_1) = 1$ and two of the loops in it are $(A,A)$ and $(A,A)$. Thus every trail containing one of these as subtrail will be a 3-trail w.r.t. $G_1$. In particular, $(A,A)$ and $(A,A)$ are themselves 3-trails w.r.t. $G_1$. A trail of exponent 3 w.r.t. $G_1$ is any trail which is the product of 3 3-trails; for instance, $(A,A,A,A_1) = (A,A)(A,A)(A,A)$ or $(C,A,C,A,A,B,B,A,A) = (C,A,C,A)(A,A,B,B,A)(A,A)$.

\[\dagger\] The definition of an $m$-trail in the case of $r(G') = 1$ does not depend on $m$. The name $m$-trail is introduced here merely for the sake of uniformity of notation.
Now consider the s.c. subgraph $G_2 = G - [C]$ which is clearly of rank 2.
In order to construct a 3-trail $t$ w.r.t. $G_2$ we first construct for
every state $q$ of $G_2$ a trail in $G_2 - [q]$ which is of exponent 3 w.r.t.
a s.c. subgraph of rank 1 of $G_2 - [q]$. Thus $t_1 = (A,A,A,A)$ is of
exponent 3 w.r.t. $G_2 - [B] = G - \{B,C\}$, whereas $t_2 = (B,B,B,B)$ is of
exponent 3 w.r.t. $G_2 - [A] = G - \{A,C\}$. Now, by definition, $t_1$ must
appear in $t$ between two occurrences of $B$ and $t_2$ must appear in $T$ between
two occurrences of $A$. Thus, for instance, $t = (A,B,B,B,A,A,A,A,B)$ is
a 3-trail w.r.t. $G_2$ (in fact, this is the shortest possible one). Another
such trail is $t' = (A,B,B,B,B,A,C,B,A,A,A,B,A)$. A trail of exponent 3
w.r.t. $G_2$ is a multiple of 3 3-trails w.r.t. $G_2$; for instance, $(t')^3$ is
one such trail, or $[t \cdot (B,A)]^3$ is another one.

The next lemma follows directly from the above definition.
Assume all $m$-trails as well as all trails of exponent $m$ are w.r.t. some
fixed s.c. subgraph of $G$.

Lemma 4.8: (a) If $t$ is an $m$-trail then $t$ is also an $n$-trail for all
$n \leq m$.

(b) If $t$ is of exponent $m$ then $t$ is also of exponent $n$ for
all $n \leq m$.

(c) If $t'$ is a subtrail of $t$ and $t'$ is an $m$-trail then $t$ is
also an $m$-trail.

(d) If $t'$ is a subtrail of $t$ and $t'$ is a trail of exponent $m$
then also $t$ is of exponent $m$.

Lemma 4.9: For every s.c. subgraph $G'$ of a state graph $G$, and for every
positive integer $m$, there exists in $G$ a trail of exponent $m$ w.r.t. $G'$.
Proof: Let \( r(G') = 1 \). Then \( G' \) has at least one non-trivial loop and a trail of exponent \( m \) w.r.t. \( G' \) can be easily constructed. Assume the lemma is true for \( r(G') < k \). Let \( r(G') = k \). Then for every state \( q \) of \( G' \), \( G' - [q] \) has rank at least \( k - 1 \), and by assumption there can be constructed in it a trail \( t \) of exponent \( m \) w.r.t. some s.c. subgraph of rank \( k - 1 \). Since \( G' \) is s.c., one can find a trail \( t_1 \), from \( q \) to the initial node of \( t \), and another one from the terminal node of \( t \) back to \( q \), and form their product: \( t_1 \cdot t \cdot t_2 \). By constructing such trails for all nodes \( q \) in \( G' \) and then by concatenating them all one after the other (by adding subtrails connecting the terminal node of the one to the initial node of the next one), an \( m \)-trail w.r.t. \( G' \) is obtained. Hence a trail of exponent \( m \) w.r.t. \( G' \) can be constructed.

Definition 4.8: A set of trails \( T' \subseteq T(G) \) is called cofinal w.r.t. \( G' \) iff \( T' \) contains trails of exponent \( m \) w.r.t. \( G' \) for all integers \( m \). A cofinal set of trails in \( G \) w.r.t. a s.c. subgraph of rank \( k \) will be called a set of trails of rank \( k \) in \( G \).

Lemma 4.10: If \( \tilde{G} = \tilde{G}_0(R) \) then \( \tau_1(R) \) is cofinal w.r.t. every s.c. subgraph \( G' \) of \( \tilde{G} \).

Proof: Let \( G' \) be any s.c. subgraph of \( \tilde{G} \). By Lemma 4.9 there exists for every integer \( m \) a trail \( t \) of exponent \( m \) w.r.t. \( G' \). Let the initial node of \( t \) be \( q_i \) and the terminal node \( q_j \). Then by Lemma 4.1 there exists a word \( w \in A^k \) such that \( \tau_1(w) = t \). Now let \( w_1 \) be a word such that \( M_1(w_1) = q_i \) and \( w_2 \) a word such that \( M_j(w_2) \) is a final state. Then \( w_1 \cdot w \cdot w_2 \in R \) and since \( \tau_1(w_1 \cdot w \cdot w_2) = \tau_1(w_1) \cdot \tau_1(w) \cdot \tau_j(w_2) \) by
Lemma 4.3, t is a subtrail of $\tau_1(w_1 \cdot w \cdot w_2)$ and therefore the latter is also a trail of exponent $m$ w.r.t. $G'$ (Lemma 4.8). Hence $\tau_1(R)$ contains trails of exponent $m$ w.r.t. $G'$ for every integer $m$ and is cofinal w.r.t. $G'$.

**Lemma 4.11:** Let $G$ be a state graph and let $E$ be a regular expression simple w.r.t. $G$ such that $\tau_1(E)$ is of rank $k > 0$ in $G$. Then there exists a $p$-star of level 1, $H^*$, of $E$ such that $\tau_1(H^*)$ is of rank $k$ in $G$, where $r$ is defined by: $\{q_r\} = M_1(\tau_1^E(H^*))$.

**Proof:** By Lemma 4.4 we may assume $E$ is given in string form. Thus $E = \bigcup_{j=1}^{p} F_j$ and by Lemma 4.2, $\tau_1(E) = \bigcup_{j=1}^{p} \tau_1(F_j)$. By assumption $\tau_1(E)$ is cofinal w.r.t. some s.c. subgraph $G'$ of $G$ of rank $k$ and from the definition of a cofinal set it follows that one of the sets $\tau_1(F_j)$ is also cofinal w.r.t. $G'$.

Now $F_j$ is a string of words and stars, $F_j = w_1H^*_1w_2H^*_2 \cdots w_lH^*_l$. By assumption $E$ is simple w.r.t. $G$. Hence each of the sets: $M_1(w_1H^*_1 \cdots w_i)$, $M_1(w_1H^*_1 \cdots w_iH^*)$, $i = 1, 2, \ldots, l$,

consists of a single state. Let these states be:

$M_1(w_1H^*_1 \cdots w_i) = M_1(w_1H^*_1 \cdots w_iH^*) = \{q_{j_1}^{}\}$. By Lemma 4.3 we get:

$\tau_1(F_j) = \tau_1(w_1H^*_1 \cdots w_2H^*_2 \cdots \tau_1^{-1}(w_{l-1}H^*_1 \cdots w_lH^*) \tau_1^{-1}(w_{l+1})).$

Now $\tau_1(F_j)$ contains trails of exponent $m$ w.r.t. $G'$ for arbitrarily large $m$, i.e. it contains for any $m$ a trail $t$ which is the multiple of $m$ m-trails w.r.t. $G'$. Let $c$ be the total length of all words $w_i$ ($i = 1, 2, \ldots, l + 1$). Then out of the $m$ m-trails in $t$ at least $m - c$ m-trails must be contained in the loops $\tau_{j_i}(H^*_i)$, $i = 1, 2, \ldots, l$. 
Thus for some \( h, 1 \leq h \leq k \), \( \tau^*_{j_h} \) contains at least \( \lceil \frac{m-g}{k} \rceil ^* = m' \) m-trails w.r.t. \( G' \). But since \( m \) grows arbitrarily, so does \( m' \), and recalling than an m-trail is also an \( m' \)-trail (Lemma 4.8) we conclude that \( \tau^*_{j_h} \) contains trails of exponent \( m' \) w.r.t. \( G' \) for arbitrarily large \( m' \). Hence \( \tau^*_{j_h} \) is a set of trails of rank \( k \) in \( G \), where \( \{ q_j \} = \{ q_r \} = M_1(\pi_E(h^*)) \) and \( h^* \) is the required p-star of level \( l \) of \( E \).

**Main Lemma:** Let \( G \) be any state graph and let \( E \) be a regular expression simple w.r.t. \( G \) such that \( \tau^*_{\alpha}(E) \) has rank \( k > 0 \) in \( G \). Then \( h^*_{\alpha}(E) \geq k \).

**Proof:** The proof will proceed by induction on \( k \).

Let \( k = 1 \). Then there exists a s.c. subgraph \( G' \) of \( G \) with \( r(G') = 1 \) such that \( \tau^*_{\alpha}(E) \) is cofinal w.r.t. \( G' \). This implies that \( |E| \) is infinite and hence \( h^*_{\alpha}(E) \geq 1 \).

Now let \( k > 1 \) and assume the lemma holds for \( k - 1 \). Let \( E \) be an expression simple w.r.t. \( G \) such that \( \tau^*_{\alpha}(E) \) is of rank \( k \) in \( G \). By Lemma 4.11 there exists in \( E \) a p-star \( H^* \) of level \( l \) such that \( \tau^*_{\alpha}(H^*) \) is of rank \( k \), where \( r \) is given by: \( \{ q_r \} = M_1(\pi_E(H^*)) \). Clearly \( \tau^*_{\alpha}(H^*) \) is a set of loops from \( q_r \) to itself and by Lemma 4.6, \( \tau^*_{\alpha}(H^*) = (\tau^*_{\alpha}(H))^* \).

Consider the state graph \( G_r \) obtained from \( G \) by changing the initial state from \( q_1 \) to \( q_r \). It follows from Lemma 4.7 that \( H \) is a simple regular expression w.r.t. \( G_r \). It will next be shown that \( \tau^*_{\alpha}(H) \) has rank \( k - 1 \) in \( G_r \).

So far we have seen that \( (\tau^*_{\alpha}(H))^* \) has rank \( k \) in \( G \). This implies the existence of a s.c. subgraph \( G' \) of \( G \) which has rank \( k \) such

\[^* \] denotes the largest integer contained in the number \( r \).
that \((τ_r(H))^*\) is cofinal w.r.t. \(G'\). Thus for every integer \(m\), \((τ_r(H))^*\) contains a trail \(t\) of the form: \(t = t_1 \ast t_2 \ast \ldots \ast t_m\), where each \(t_j\) is an \(m\)-trail w.r.t. \(G'\). It follows from the definition of an \(m\)-trail that for every \(t_j\), \(1 \leq j \leq m\), there exists a subtrail of the form:

\(t' = (q_{r_1}, q_{j_1}, q_{j_2}, \ldots, q_{j_k}, q_r)\) such that \((q_{j_1}, q_{j_2}, \ldots, q_{j_k})\) is a trail in \(G' - [q_r]\) and is of exponent \(m\) w.r.t. some s.c. subgraph of rank \(k - 1\) of \(G' - [q_r]\). Now since \(τ_r(H^*)\) is a set of loops from \(q_r\) to itself, so is \(τ_r(H)\), and since \(q_{j_i} \neq q_r\ (i = 1, \ldots, k)\), the above subtrail \(t'\) must be contained in a trail of \(τ_r(H)\). Thus \(τ_r(H)\) contains, for arbitrarily large \(m\), a trail of exponent \(m\) w.r.t. some s.c. subgraph of rank \(k - 1\) of \(G\). But since the number of such subgraphs is finite, \(τ_r(H)\) also contains, for infinitely many \(m\), trails of exponent \(m\) w.r.t. one of these subgraphs. Hence \(τ_r(H)\) is a set of trails of rank \(k - 1\) in \(G\), and also in \(G_r\). Recalling that \(q_r\) is the initial state of \(G_r\) and that \(H\) is simple w.r.t. \(G_r\), we apply the induction hypothesis to \(H\) and deduce that \(h_α(H) \geq k - 1\). Hence \(h_α(E) \geq k\).

4.4. Star Height of Events with the Finite Intersection Property

Definition 4.9: Let \(R\) be a regular event. \(R\) has the finite intersection property (f.i.p.) iff the intersection of every pair of distinct left quotients of \(R\) is finite, i.e. for any pair of words \(x, y\), either \(x \setminus R = y \setminus R\) or \((x \setminus R) \cap (y \setminus R)\) is finite.
Example 4.4: Let \( R = 012^* \cup 2 \cup \lambda \). The left quotients of \( R \) (other than \( R \)) are:

\[
\begin{align*}
0 \backslash R &= 12^* \\
1 \backslash R &= \emptyset = 00 \backslash R = 02 \backslash R = 20 \backslash R = 21 \backslash R = 22 \backslash R \\
2 \backslash R &= \lambda \\
01 \backslash R &= 2^* = 012 \backslash R
\end{align*}
\]

Thus:

\[
R \cap (2 \backslash R) = \lambda = (01 \backslash R) \cap (2 \backslash R)
\]

\[
R \cap (01 \backslash R) = 2 \cup \lambda
\]

and all other intersections of pairs of left quotients are empty.

Hence \( R \) has the f.i.p.

Lemma 4.12: Let \( R \) be a regular event with the f.i.p. and let \( \widehat{\mathcal{C}} = \widehat{\mathcal{C}_0}(R) \). Then every regular expression \( E \) denoting a subset of \( R \) is simple w.r.t. \( \widehat{\mathcal{C}} \).

Proof: Let \( E \) be a regular expression denoting a subset of \( R \) and let \( H^* \) be a p-star in \( E \). Suppose \( M_1(\pi_E(H^*)) \) contains two distinct states \( q, q' \). Then by Lemma 4.5 \( |\sigma_E(H^*)| \) is contained in \( D_q \cap D_{q'} \), and it follows that \( D_q \cap D_{q'} \) is infinite. But this contradicts the f.i.p. of \( R \). Hence for every p-star \( H^* \) of \( E \), \( M_1(\pi_E(H^*)) \) is a singleton and therefore \( E \) is a simple regular expression w.r.t. \( \widehat{\mathcal{C}} \).

Theorem 4.1: Let \( R \) be a regular event with the f.i.p. and let \( \widehat{\mathcal{C}} = \widehat{\mathcal{C}_0}(R) \). Then every regular event \( R' \subseteq R \) such that \( \tau_1(R') \) has rank \( k \) in \( \mathcal{H} \) is of star height at least \( k \). In particular, the star height of \( R \) equals the rank of \( \widehat{\mathcal{C}} \).
Proof: By Lemma 4.12 every regular expression denoting a subset \( R' \) of \( R \) is simple w.r.t. \( \mathcal{C} \). Now, using the Main Lemma, we deduce that in case \( \tau_1(R') \) has rank \( k \), every regular expression denoting \( R' \) must be of apparent star height at least \( k \). Hence \( h(R') \geq k \).

Concerning \( R \) itself, Lemma 4.10 implies that \( \tau_1(R) \) has rank equal to \( r(\mathcal{C}) \) and therefore \( h(R) \geq r(\mathcal{C}) \). But by Eggan's theorem the star height of \( R \) cannot exceed the rank of any state graph recognizing \( R \). Hence \( h(R) = r(\mathcal{C}) \).

We next consider a special case of Theorem 4.1, which was also proved in [Mc 4].

Definition 4.10: Let \( \mathcal{A} = (Q, M, q_0, F) \) be an automaton. \( \mathcal{A} \) is called reset-free iff for every \( a \in A, q, q' \in Q, q \neq q' \) implies \( M(q, a) \neq M(q', a) \) or \( M(q, a) = M(q', a) = \emptyset \). A state graph is reset-free iff its corresponding automaton is reset-free.

Example 4.5: The automaton shown in Fig. 4.2 is reset-free, whereas the one in Fig. 4.3 is not.

Lemma 4.13: Every event recognized by a reset-free automaton with a single output state has the f.i.p.

Proof: Let \( R \) be recognized by the reset-free automaton \( \mathcal{A} = (Q, M, q_0, F) \), where \( F = \{ q_f \} \). By Theorem 1.6, for every \( q \in Q \), \( A_c(q) = x \setminus R \) for some word \( x \), and for every \( y \in A^* \), \( y \setminus R = A_c(q) \) for some \( q \in Q \), or \( y \setminus R = \emptyset \). We now show that for every pair of states \( q, q' \in Q \), \( A_c(q) \cap A_c(q') = \emptyset \). To see this, suppose for some word \( w = a_1 a_2 \ldots a_m, w \in A_c(q) \cap A_c(q') \). Then
$M(q,w) = M(q',w) = q_f$. But since $\mathcal{A}$ is reset-free, $M(q,a_1) \neq M(q',a_1)$, $M(q,a_1a_2) \neq M(q',a_1a_2)$ and by induction $M(q,w) \neq M(q',w)$, which is a contradiction. Also $\lambda$ cannot be included in the intersection of the two accepted events since there is only a single output state. It follows that $R$ has the f.i.p.

\textbf{Theorem 4.2:} Let $G$ be a reset-free state graph with a single output state, and let $G$ recognize an infinite event $R$. Then $h(R) = r(G)$.

\textbf{Proof:} By the proof of the previous lemma, distinct states of $G$ correspond to distinct left quotients of $R$, and hence $G$ is reduced. Moreover, $G$ cannot have a dead state $q_\phi$. For the existence of $q_\phi$ implies the existence of two transitions in $G$, one from a state $q \neq q_\phi$ to $q_\phi$ and the other from $q_\phi$ to itself, both labelled by the same input letter. But this contradicts the fact that $G$ is reset-free. Hence $G$ coincides with the incomplete reduced state graph $\hat{G}_0(R)$ recognizing $R$. Also, by the last lemma, $R$ has the f.i.p., and it follows from Theorem 4.1 that $h(R) = r(G)$.

Clearly every pure-group automaton is also reset-free. Thus Theorem 3.4 (Section 3.2) is a special case of the above theorem.

\textbf{Example 4.6:} Let $\hat{G} = \hat{G}_0(R)$ be the state graph shown in Fig. 4.3. It can be easily verified that $R$ has the f.i.p. and hence $h(R) = r(\hat{G}) = 3$ by Theorem 4.1.
Example 4.7: The event $R$ recognized by the state graph $\tilde{G}$ in Fig. 4.2 has star height $h(R) = r(\tilde{G}) = 3$ by Theorem 4.2.

Example 4.8: Let $R' = (11\ast02\ast0 \cup 00)^\ast$. In order to determine the star-height of $R'$ we first find the quotients of $R'$:

\[
\begin{align*}
0 \setminus R' &= \emptyset R' = A \\
1 \setminus R' &= 1\ast102\ast0 R' = B \\
2 \setminus R' &= \emptyset \\
00 \setminus R' &= R' = C \\
01 \setminus R' &= 10 \setminus R' = \emptyset \\
11 \setminus R' &= 1\ast02\ast0 R' = D = 111 \setminus R' \\
110 \setminus R' &= 2 \ast R' = E \\
1100 \setminus R' &= R' \\
1101 \setminus R' &= \emptyset \\
1102 \setminus R' &= 110 \setminus R'
\end{align*}
\]

The reduced state graph $\hat{G}' = \hat{G}_0(R')$ is shown in Fig. 4.4. Clearly $r(\hat{G}') = 2$ and hence $h(R') \leq 2$. But $R'$ does not have the f.i.p.; in fact, $(0 \setminus R') \cap (110 \setminus R') = 0 \setminus R' = OR'$, which is infinite. Thus we cannot determine the star height of $R'$ just by observing $G'$ and using Theorem 4.1. However, let us now focus our attention to the state graph $\hat{G} = \hat{G}_0(R)$ shown in Fig. 4.5. Obviously $R = (02\ast0 \cup 1)^\ast$ and Theorem 4.2 $h(R) = 2$ because $\hat{G}$ is reset-free with a single output state and $r(\hat{G}) = 2$. It can be easily verified that $R' \subseteq R$, and furthermore, since $R'$ contains all words $(1^m02^m0)^m$ for $m \geq 2$, $\tau_1^G(R')$ contains the set of trails $
\{(A,A)^m(A,B)(B,B)^m(B,A)^m \mid m \geq 2\}$ which is clearly of rank 2 in $G$.

Hence by Theorem 4.1 and Lemma 4.13 $h(R') \geq r(\hat{G}) = 2$ and therefore $h(R') = 2$. 

Disappointingly, Theorem 4.2 cannot be extended to the case of more than one output state. This is illustrated by the following example:

Example 4.9: Let G be the state graph in Fig. 4.6. Clearly G represents a pure-group automaton and \( r(G) = 3 \). Yet the event R recognized by G can be denoted by:

\[
E = (1 \cup 2 \cup 0(1 \cup 2)*0)* \cup (0 \cup 2 \cup 1(0 \cup 2)*1)*
\]

which is of apparent star height 2. Hence \( h(R) < r(G) \).

However, some partial extensions of Theorem 4.2 to reset-free state graphs with more than one terminal node are possible, as will be shown in Chapter 6.

4.5. Star-Height of Events Defined by Digraphs

In [E] Eggan considers a class of events defined by digraphs in the following way: let \( D = (N, B) \) be a digraph; then any path

\[
p = b_{n_1}^{n_2} n_2^{n_3} \cdots b_{n_{k-1}}^{n_k}
\]

in D can be considered as a word over the alphabet B, and for any \( m, n \in N \), the set of all paths from \( m \) to \( n \) constitutes an event over B. Clearly such an event is regular; in fact, every digraph can be regarded as an incomplete state graph (without initial and final states specified), where for every branch \( (m,n) \in B \), there exists a transition from \( m \) to \( n \) labelled by \( b_{mn} \). Obviously no two transitions are labelled by the same letter; hence such a state graph is always reset-free.

The following two questions were raised by Eggan in [E]:
(1) Is the set of paths from a node to itself in a complete digraph on k nodes of star height k?

(2) For a digraph of rank k, do there necessarily exist nodes m and n such that the set of paths from m to n is an event of star height k?

A positive answer to both questions is provided by the next corollary, which is an immediate consequence of Theorem 4.2.

Corollary 4.1: Let D = (N,B) be a digraph of rank k. Then:

(a) For any \( n \in N \) contained in a section of D of rank k, the set of all paths from n to itself is of star height k.

(b) For any \( m, n \in N \) such that there exists a path from m to n passing through a section of D of rank k, the set of all paths from m to n is of star height k.

Now since the complete digraph on k nodes is strongly connected and has rank k (Proposition 2.3), it follows that for any pair of nodes m, n (not necessarily distinct) in it, the set of all paths from m to n has star height k.

Thus the complete digraph on k nodes can be considered as an incomplete automaton with k states recognizing an event of star height k over an alphabet of \( k^2 \) letters. Furthermore, by appropriate labelling of the branches of the complete digraph, one can reduce the number of alphabet letters to k without decreasing the star height of the corresponding event, and moreover, the resulting automaton will be a
pure-group automaton. This is shown in the next corollary.

**Corollary 4.2:** For every integer \( k \) there exists a pure-group automaton with \( k \) states recognizing an event of star height \( k \) over the \( k \)-letter alphabet.

**Proof:** Let \( A_k = (Q, M, q_0, F) \) be an automaton over the alphabet \( A = \{0, \ldots, k-1\} \), where \( Q = \{q_0, q_1, \ldots, q_{k-1}\} \), \( F = \{q_0\} \) and the transition function \( M \) is defined by: \( M(q_i, j) = q_{i+j \pmod{k}} \), for all \( i, j = 0, 1, \ldots, k-1 \). Clearly \( A_k \) is a pure-group automaton of rank \( k \) with a single output state, and Theorem 4.2 yields the result.

For example, the automaton \( A_3 \) is as shown in Fig. 4.7.

**Definition 4.11:** Let \( D = (N, B) \) be a digraph and let \( N_1, N_2 \) be two non-empty subsets of \( N \). Define \( P(N_1, N_2) \) to be the set of all paths in \( D \) with initial node in \( N_1 \) and terminal node in \( N_2 \). Let \( D_{N_1N_2} \) denote the smallest subgraph of \( D \) containing all paths of \( P(N_1, N_2) \). Thus \( D_{N_1N_2} = D - [N''] \), where \( N'' \) consists of all nodes \( n' \in N \) such that no path of \( P(N_1, N_2) \) passes through \( n' \).

We now generalize the result of Corollary 4.2 to all events \( P(N_1, N_2) \).

**Lemma 4.14:** Let \( D = (N, B) \) and let \( n_1 \in N \) and \( \emptyset \neq N_2 \subseteq N \). Then \( h(P(n_1, N_2)) = r(D_{n_1N_2}) \).
Proof: As has been stated above, any digraph can be regarded as an incomplete state graph, in which the branch from node $m$ to node $n$ is labelled by $b_{mn}$. Now consider the digraph $D' = D_{n_1N_2} = (N', B')$ as an incomplete state graph with initial state $n_1$ and a set of final states $N_2$. Clearly the event accepted by $D'$ is $P = P(n_1, N_2)$. Now by Theorem 1.6, the accepted event $A_c(n)$ of any node $n \in N'$ in $D'$ equals one of the left quotients of $P$, and for every non-empty left quotient $x \setminus P$, $x \in (B')^*$, there exists $n \in N'$ such that $A_c(n') = x \setminus P$. Thus let $m_1, m_2 \in N'$, $m_1 \neq m_2$. Then for $i = 1, 2$, all branches leaving $m_i$ are labelled by letters $b_{m_in}$, $n \in N'$. Thus two branches, one leaving $m_1$ and the other leaving $m_2$ cannot be labelled by the same letter and hence $A_c(m_1) \cap A_c(m_2) \subseteq \Lambda$. This implies that the event $P$ has the f.i.p. Let $\bar{D}' = \bar{D}_{n_1N_2}$ be the digraph obtained from $D'$ by merging all nodes whose accepted events equal $\Lambda$. Apparently $\bar{D}'$ is a reduced state graph without a dead state recognizing $P$ and $r(\bar{D}') = r(D')$. Hence $\bar{D}'$ coincides with $\mathcal{G}_0(P)$ and the result then follows from Theorem 4.1.

---

Lemma 4.15: Let $D = (N, B)$ and let $N_1, N_2$ be two non-empty subsets of $N$. Then

$$h(P(N_1, N_2)) = \max \{ h(P(n, N_2)) \mid \exists n_1 \in N_1, b_{n_1n} \in B \}$$

Proof: We first claim that for any $n_1 \in N_1$,

$$b_{n_1n} \setminus P(N_1, N_2) = \begin{cases} P(n, N_2), & b_{n_1n} \in B \\ \emptyset, & \text{otherwise} \end{cases}$$
This follows easily from the fact that for all \( n \in \mathbb{N} \), there can be at most one branch (namely \( b^1_{1\cdot n} \)) leaving some node of \( N_1 \) and labelled by \( b^1_{1\cdot n} \). Now by Theorem 3.5,

\[
h(P(N_1, N_2)) = \max \{ h(b^1_{1\cdot n} \setminus P(N_1, N_2)) \mid n_1 \in N_1, b^1_{1\cdot n} \in B \} = \\
= \max \{ h(P(n, N_2)) \mid n_1 \in N_1, b^1_{1\cdot n} \in B \}.
\]

---

**Theorem 4.3:** Let \( D = (N, B) \) be a digraph. Then for any two non-empty subsets \( N_1, N_2 \) of \( N \),

\[
h(P(N_1, N_2)) = r(D^1_{1\cdot N_2}).
\]

**Proof:** By Lemmas 4.15 and 4.14,

\[
h(P(N_1, N_2)) = \max \{ r(D^1_{nN_2}) \mid \exists n_1 \in N_1, b^1_{1\cdot n} \in B \}
\]

Thus it remains to show that

\[(*) \quad r(D^1_{1\cdot N_2}) = \max \{ r(D^1_{nN_2}) \mid \exists n_1 \in N_1, b^1_{1\cdot n} \in B \}\]

Obviously every graph \( D^1_{nN_2} \) for which \( b^1_{1\cdot n} \in B \) for some \( n_1 \in N_1 \) must be contained in \( D^1_{1\cdot N_2} \). Thus \( r(D^1_{nN_2}) \leq r(D^1_{1\cdot N_2}) \) for all such graphs \( D^1_{nN_2} \). Moreover, we claim that every section \( S \) of \( D^1_{1\cdot N_2} \) is contained in at least one such graph \( D^1_{nN_2} \). To see this, let \( S \) be any section contained in \( D^1_{1\cdot N_2} \). By Definition 4.12, there exists in \( D \) a path

\[p = b^1_{m_1 m_2 m_3 \cdots m_{k-1} m_k} \text{ with } m_1 \in N_1, m_k \in N_2 \text{ and with at least one}\]
node $m^i_1$, $1 \leq i \leq k$, in $S$. Without loss of generality we may assume $i > 1$; for if $m^1_1$ is the only node of $p$ contained in $S$, we may concatenate $p$ on the left by some non-trivial loop $\lambda$ from $m^1_1$ to itself (which certainly exists since $S$ is strongly connected), thus obtaining another path $\lambda p$ from $m^1_1$ to $m^2_2$ satisfying the above requirement. Now the path $b_{m^2_2}m^3_2 \ldots b_{m^k_1}m^1_1$ must be contained in $D_{m^2_2 N^2_2}$, and since one of its nodes belongs to the strongly connected subgraph $S$, there exist in $D$ paths from $m^2_2$ to $m^k_1$ touching any node of $S$. This implies that the whole section $S$ must be contained in $D_{m^2_2 N^2_2}$. But since $b_{m^1_1 m^2_2} \in B$ and $m^1_1 \in N^1_1$, $D_{m^2_2 N^2_2}$ is one of the graphs appearing in ($*$), and hence

$$r(D_{N^1_1 N^2_2}) \leq \max \{r(D_{n N^2_2}) \mid \exists n^1_1 \in N^1_1, b_{n^1_1 n} \in B\}$$

and the result follows.

**Example 4.10:** Let $D' = (N, B)$ be the digraph shown in Fig. 4.8. Let $N^1_1 = \{A, C, G\}$ and $N^2_2 = \{B, H\}$. Then $D'_{N^1_1 N^2_2} = D'$ and hence $h(P(N^1_1, N^2_2)) = r(D') = 3$.

Now consider $N^1_1 = \{A, B\}$, $N^2_2 = \{A, C\}$; then clearly $D'_{N^1_1 N^2_2} = D' - \{D, E, F, G, H\}$ and $h(P(N^1_1, N^2_2)) = 1$; in fact,

$$P(\{A, B\}, \{A, C\}) = (b_{BA} \cup \lambda)(b_{AB} b_{BA} \cup b_{AC} b_{CB} b_{BA})^\lambda (\lambda \cup b_{AC}).$$

**Lemma 4.16:** For any digraph $D = (N, B)$ and for any $N^1_1, N^2_2 \subseteq N$, $P(N^1_1, N^2_2)$ is a group-free event.

**Proof:** Clearly $P(N^1_1, N^2_2) = \bigcup_{n \in N^1_1} P(n, N^2_2)$. By Theorem 1.12 every event which is a finite union of group-free events is itself group-free. Thus it suffices to show that each event $P(n, N^2_2)$ is group-free. Now by
Theorem 1.11 an event $R$ is group-free iff it is permutation-free, i.e., in the reduced state graph for $R$, no word $w$ induces a non-trivial permutation on any subset of the set of states. Now it has been proved in Lemma 4.14 that for any $n \in N$, $N_2 \subseteq N$, $\overline{D_{nn_2}}$ can be regarded as the reduced (incomplete) state graph recognizing $P(n, N_2)$, with initial state $n$ and the set of final states $\overline{N_2}$. Thus suppose there exists a word $w = b_1b_2\ldots b_k$ over the alphabet $B$, and a subset $N' \subseteq N$ with $\#	ext{N'} \geq 2$ such that $w$ induces a permutation on $N'$. Then there exist two nodes $n_1, n_2 \in N'$ such that there are in $D$ two branches labelled by $b_1$ leaving $n_1$ and $n_2$ respectively. But this is impossible since no two branches are labelled by the same letter.

An important result of Lemmas 4.14 and 4.16 is the following:

**Theorem 4.4:** Every regular event $R$ is a homomorphic image of a group-free event $R'$ such that $h(R') = r(G_0^*(R))$.

**Proof:** Let $G = G_0^*(R) = (Q, M, q_1, F)$ and let $D = D_{q_1}^F = (Q, B)$ be the digraph corresponding to $G$ (Definition 2.22). Since $G_0^*(R)$ is the reduced (incomplete) state graph recognizing $R$, we have $D_{q_1}^F = D$ (Definition 4.12). Hence by Lemma 4.14, $h(P(q_1, F)) = r(D)$ and moreover, by Lemma 4.16, $P(q_1, F)$ is a group-free event. Now define a mapping $\phi$ from the set of branches $B$ of $D$ to the set of all non-empty subsets of the alphabet $A$:

$$\phi : B \rightarrow 2^A - \emptyset$$

$$\forall b\ q_i q_j \in B, \ \phi(b\ q_i q_j) = \{a \in A \mid M(q_i, a) = q_j\}$$
Clearly $\phi(b_{q_i q_j}) \neq \emptyset$ for any $b_{q_i q_j} \in B$. Now extend $\phi$ to a generalized homomorphism from $2^B$ into $2^A$. It is easily seen that $\phi(P(q_1, F)) = R$. Hence $R' = P(q_1, F)$ is the required group-free event.
Chapter 5

Rank Non-Increasing Transformations on Transition Graphs

Introduction:

As was shown in Chapter 4, in some cases the star height of a regular event $R$ can be determined merely by studying the structure of its reduced state graph $\hat{G}_0(R)$. However, as Eggan's Star Height Theorem indicates, in order to determine the star height of $R$ in the general case, one has to examine the vast family of all transition graphs recognizing $R$. Our aim in this chapter is to show that it suffices to consider only a proper subset of this family, namely the family of all reduced non-deterministic state graphs accepting $R$ (roughly speaking, a transition graph $G$ is reduced if every two accepted events, or preceding events, of $G$ are distinct).

In section 5.1 some elementary modifications of a digraph are introduced and their effect on the rank of the digraph is investigated. These transformations form the foundation for the constructions presented in the rest of the chapter.

In section 5.2 a reduced transition graph is defined and a reduction procedure for an arbitrary transition graph is presented, in which the rank of the graph cannot be increased.

A rank-preserving process of elimination of all $\lambda$-transitions in a transition graph is described in section 5.3. This process is based on successive application of basic rank-non-increasing transformations.
to the given transition graph \( G \), in which all \( \lambda \)-transitions are eventually replaced by some other non-\( \lambda \) transitions. Thus every transition graph \( G \) can be replaced by an equivalent non-deterministic reduced state graph \( G' \), whose rank does not exceed the rank of \( G \). As a result, Eggan's Star Height Theorem can be strengthened as mentioned above. Another consequence is, that for any regular event \( R \), the smallest number of nodes of any transition graph accepting \( R \) equals the number of states in a minimal non-deterministic automaton recognizing \( R \) \([\text{KN}]\).

5.1. Elementary Modifications of Digraphs

In this section two elementary modifications of digraphs are defined and some results concerning the changes in rank caused by these modifications are obtained. These results will be used in the next two paragraphs, when similar modifications of transition graphs will serve as the main tools for proving the end results of this chapter.

**Definition 5.1**: Let \( D = (N, B) \) be a digraph and let \( m, n \in N \). Define two digraphs, \( D^1(m, n) \) and \( D^2(m, n) \), as follows:

\[
D^1(m, n) = (N, B \cup \{(m, n') \mid n' \in \sigma(n)\})
\]
\[
D^2(m, n) = (N, B \cup \{(m', n) \mid m' \in \pi(m)\})
\]

Thus \( D^1(m, n) \) is obtained from \( D \) by adding all branches leaving \( m \) and entering successors of \( n \), whereas \( D^2(m, n) \) is obtained from \( D \) by adding all branches leaving predecessors of \( m \) and entering \( n \).
Example 5.1: Let \( D \) be the digraph in Fig. 5.1 a. \( D^1(m, n) \) and \( D^2(m, n) \) are shown in Fig. 5.1 b and Fig. 5.1 c respectively.

Lemma 5.1: \( r(D) \leq r(D^i(m, n)) \leq r(D) + 1, \ i = 1, 2. \)

Proof: When a node and all branches touching it are removed, the rank is decreased by at most one.

Lemma 5.2: \( D^2(m, n) = [(D^1)^1(n, m)] \)

In the following lemmas, let \( D = (N, B) \) be a digraph and let \( n_1, n_2 \in N. \)

Lemma 5.3: (a) If there does not exist any path in \( D \) from \( n_2 \) to \( n_1 \) then all sections of \( D^i(n_1, n_2) \) are identical with those of \( D, \ i = 1, 2. \)

(b) If \( n_1 \) and \( n_2 \) belong to the same section \( S \) of \( D \), then \( T_i = S^i(n_1, n_2) \), where \( T_i \) is the section containing \( n_1 \) and \( n_2 \) in \( D^i(n_1, n_2) \) \( (i = 1, 2) \), and all other sections of \( D^i(n_1, n_2) \) are identical with those of \( D. \)

Proof: By definition of \( D^1(n_1, n_2) \), the set of branches added to \( D \) to obtain \( D^1(n_1, n_2) \) is: \( B_1 = \{(n_1, n') \mid n' \in \sigma(n_2)\} \). Since all branches in \( B_1 \) are incident to \( n_1 \), none of them can be contained in a section of \( D^1(n_1, n_2) \) which does not contain \( n_1 \). Hence all sections not containing \( n_1 \) remain unchanged when \( D^1(n_1, n_2) \) is constructed, and the only section which can possibly be extended is the one containing \( n_1 \). Thus let \( S \) and \( T \) be the sections containing \( n_1 \) in \( D \) and \( D^1(n_1, n_2) \) respectively, or \( S = \phi^+(T = \phi) \) if \( n_1 \) does not belong to any section of \( D \) (\( D^1(n_1, n_2) \)).

\( \dagger \) \( \phi \) denotes here the empty digraph with no nodes and no branches.
Fig. 5.1
Clearly $S \subseteq T$. Now suppose $S \neq T$. Then either (i) there exists in $T$ a node $n_0$ which is not contained in $S$, or (ii) $S$ and $T$ have the same set of nodes but $T$ has a branch which is not contained in $S$.

In case (i), since $n_0$ belongs to $T$, there exists in $T$ a loop $p$ on $n_1$ passing through $n_0$. Since this loop cannot possibly be contained in $S$, at least one of its branches must belong to $B_1$. Thus let $p = p_1 \cdot p_2$, where $p_2 = b_{n_1 n'} \cdot p'$, and $b_{n_1 n'}$ is the last branch in $p$ contained in $B_1$. Since $n' \in \sigma^D(n_2)$, the path $p'' = b_{n_2 n'} \cdot p'$ is a path in $D$ from $n_2$ to $n_1$. But this contradicts our assumption that no such path exists, and hence case (i) is impossible.

As for case (ii), since $S$ and $T$ have the same set of nodes, any branch in $T$ which is not contained in $S$ must clearly belong to $B_1$. Thus suppose $b_{n_1 n'} \in B_1$ is contained in $T$ and not in $S$. Then $n'$, which is a successor of $n_2$ in $D$, is contained in $T$, and hence in $S$. But then there is in $S$ a path $p$ from $n'$ to $n_1$, and thus the path $b_{n_2 n'} \cdot p$ is a path in $D$ from $n_2$ to $n_1$, which is again a contradiction.

We conclude that $S = T$ and therefore $D$ and $D^1(n_1, n_2)$ have identical sections. The proof for $D^2(n_1, n_2)$ is obtained in a dual way (Lemma 5.2).

(b) Since $n_1$ and $n_2$ are in the same section $S$ of $D$, there exists in $D$ a path $p_{n_1 n_2}$ from $n_1$ to $n_2$.

Starting with $i = 1$, we first show that $T_1$ and $S$ have a common set of nodes. Obviously every node of $S$ is also contained in $T_1$. Thus let $n_0$ be any node in $T_1$. Then there exists in $D^1(n_1, n_2)$ a loop on $n_1$ passing through $n_0$. Replacing in this loop each branch
b_{n_1n'},\text{ which also passes through } n_0 \text{ and is contained in } D. \text{ Hence } n_0 \text{ also belongs to } S, \text{ and } T_1 \text{ and } S \text{ have a common set of nodes. Thus } T_1 \text{ is obtained from } S \text{ by adding all branches } b_{n_1n'} \text{ such that } n' \in O^D(n_2) \text{ and } n' \text{ is also contained in } S, \text{ namely } n' \in O^S(n_2). \text{ It follows that } T_1 = S^1(n_1, n_2). \text{ Since all branches added to } D \text{ in constructing } D^1(n_1, n_2) \text{ are incident to } n_1, \text{ all sections of } D \text{ other than } S \text{ remain unchanged in } D^1(n_1, n_2).

The proof for } i = 2 \text{ is obtained in a dual way.}

Lemma 5.4: If there does not exist in } D \text{ any path from } n_2 \text{ to } n_1 \text{ then}

\[ r(D^1(n_1, n_2)) = r(D^2(n_1, n_2)) = r(D). \]

Proof: This is a direct consequence of Lemma 5.3(a) and the definition of the rank.

Lemma 5.5: If } n_1 \text{ and } n_2 \text{ belong to the same section } S \text{ of } D \text{ and } r(S) < r(D) \text{ then:}

\[ r(D^1(n_1, n_2)) = r(D^2(n_1, n_2)) = r(D^1(n_2, n_1)) = r(D^2(n_2, n_1)) = r(D). \]

Proof: By Lemma 5.3(b), } T_i = S^i(n_1, n_2), \text{ where } T_i \text{ is the section of } D^i(n_1, n_2) \text{ containing } n_1 \text{ and } n_2(i = 1, 2). \text{ By Lemma 5.1, } r(T_i) \leq r(S) + 1.
and hence \( r(T_1) \leq r(D) \). Since all other sections of \( D^i(n_1, n_2) \) are identical with those of \( D \), \( r(D^i(n_1, n_2)) = r(D) \). By interchanging \( n_1 \) and \( n_2 \) we get \( r(D^i(n_2, n_1)) = r(D) \), \( i = 1, 2 \).

---

**Lemma 5.6:** If \( n_1 \) and \( n_2 \) belong to the same section of \( D \) and if \( n_1 \) is a cycle center of \( D \) then:

\[
r(D^1(n_1, n_2)) = r(D^2(n_2, n_1)) = r(D).
\]

**Proof:** Let \( S \) and \( T \) be the sections containing \( n_1 \) in \( D \) and \( D^1(n_1, n_2) \) \( (D^2(n_2, n_1)) \) respectively. Then by assumption \( r(S) = r(D) \) and \( r(S - [n_1]) = r(S) - 1 \). But by Lemma 5.3, \( S \) and \( T \) have a common set of nodes, and since \( D^1(n_1, n_2) (D^2(n_2, n_1)) \) is obtained from \( D \) by adding only branches incident to \( n_1 \), \( S - [n_1] = T - [n_1] \). Hence \( r(T - [n_1]) = r(S) - 1 \) and \( r(T) = r(S) \). Since, by Lemma 5.3, all other sections of \( D^1(n_1, n_2)(D^2(n_2, n_1)) \) are identical with those of \( D \), we get \( r(D^1(n_1, n_2)) = r(D)(r(D^2(n_2, n_1)) = r(D)) \).

---

**Lemma 5.7:** Let \( n_0 \) be a node of \( D \), \( n_0 \neq n_1, n_2 \). Then:

\[
(D - [n_0])^i(n_1, n_2) = D^i(n_1, n_2) - [n_0], \ i = 1, 2.
\]

**Proof:** This follows directly from Definition 5.1 and Definition 2.24.
Theorem 5.1: Let $D = (N, B)$ be a digraph and let $n_1, n_2 \in N$. Then at least one of the following conditions must hold:

(i) $r(D^i(n_1, n_2)) = r(D)$, $i = 1$ and 2
(ii) $r(D^i(n_2, n_1)) = r(D)$, $i = 1$ and 2
(iii) $r(D^1(n_1, n_2)) = r(D^2(n_2, n_1)) = r(D)$.
(iv) $r(D^2(n_1, n_2)) = r(D^1(n_2, n_1)) = r(D)$.

Proof: We consider several cases:

Case (a): There does not exist a path from $n_2$ to $n_1$ (from $n_1$ to $n_2$). Then by Lemma 5.4 condition (i) ((ii)) must be satisfied.

Case (b): $n_1$ and $n_2$ are both contained in a section $S$ of $D$ and $r(S) < r(D)$. By Lemma 5.5 all the above conditions (i) - (iv) are satisfied.

Case (c): $n_1(n_2)$ is a cycle center of $D$. Then by Lemma 5.6 condition (iii) ((iv)) holds.

Case (d): $n_1$ and $n_2$ are contained in a section $S$ of $D$, and there exists a cycle center $n_0 \neq n_1, n_2$ of $S$ such that every path in $S$ from $n_2$ to $n_1$ (from $n_1$ to $n_2$) passes through $n_0$.

Let $T_i$ be the section containing $n_1$ in $D_i(n_1, n_2)$, $i = 1, 2$.

By Lemma 5.3(b), $T_i = S_i(n_1, n_2)$. Now consider the graph $S - [n_0]$.

Clearly $r(S - [n_0]) = r(S) - 1$. By Lemma 5.7 we have:

$$S_i(n_1, n_2) - [n_0] = T_i - [n_0] = (S - [n_0])^i (n_1, n_2).$$
Furthermore, since all paths of $S$ from $n_2$ to $n_1$ go through $n_0$, in $S - [n_0]$ there is no path from $n_2$ to $n_1$. Applying Lemma 5.4 to $S - [n_0]$ we get:

$$r(D - [n_0])^i(n_1, n_2) = r(D - [n_0]), \quad i = 1, 2$$

and hence

$$r(D - [n_0]) = r(D - [n_0]) = r(S) - 1.$$ 

But then $r(D - [n_0]) = r(S)$, and since all other sections of $D^i(n_1, n_2)$ are identical with those of $D$, condition (i) holds.

By interchanging $n_1$ and $n_2$ in the above proof, we get condition (ii) for the symmetric case.

**Case (e):** $n_1$ and $n_2$ are contained in the same section $S$ of $D$ such that $r(S) = r(D)$, neither $n_1$ nor $n_2$ is a cycle center of $S$, and for every cycle center $n_0$ of $S$ there exists in $S - [n_0]$ a loop going through $n_1$ and $n_2$.

The proof is by induction on $r(D)$.

**Basis:** $r(D) = 1$. Then $r(S) = 1$ and if $n_0$ is a cycle center of $S$ then $r(S - [n_0]) = 0$. Hence there cannot be any loop in $S - [n_0]$ and the above condition never holds. Thus the theorem is vacuously satisfied.

**Induction step:** Suppose the theorem holds for all graphs of rank $k - 1$ and let $r(D) = k$. Let $n_0$ be any cycle center of $S$; then $r(S - [n_0]) = k - 1$. Now if $S - [n_0]$ belongs also to Case (e), then by induction hypothesis one of the conditions (i) - (iv) holds for it.

If, on the other hand, the graph $S - [n_0]$ belongs to one of the cases (a) - (d), then it has been proved that one of the conditions (i) - (iv) must be satisfied for it. Now by an argument similar to the
one in the proof for Case (d), with the aid of Lemmas 5.3 and 5.7, it follows that the same condition out of (i) - (iv) must also hold for S. For instance, if condition (i) holds for S - [n₀], then it will also hold for S, i.e. \( r(S^i(n_1, n_2)) = r(T_i) = r(S), \) i = 1, 2. Thus the rank of the new section \( T_i \) replacing S in \( D^i(n_1, n_2) \) is also k, and since by Lemma 5.3 the other sections of \( D^i(n_1, n_2) \) remained unchanged, (i) will also hold for D. The proof for the other cases (ii) - (iv) is the same.

---

**Example 5.2:** Let D be the digraph on Fig. 5.2 (a). The graphs \( D^1(n_1, n_2), D^2(n_1, n_2), D^1(n_2, n_1) \) and \( D^2(n_2, n_1) \) are shown on Fig. 5.2 (b), (c) (d) and (e) respectively. It is easily seen that \( r(D^1(n_1, n_2)) = r(D^2(n_1, n_2)) = 2 \) whereas \( r(D^1(n_2, n_1)) = r(D^2(n_2, n_1)) = 1 = r(D) \). Thus condition (ii) is satisfied whereas conditions (i), (iii) and (iv) are not.

**Corollary 5.1:** Let \( D = (N, B) \) and \( n_1, n_2 \in N \) such that \( (n_1, n_2) \in B \). Then either

(A) \( r(D^1(n_1, n_2)) = r(D) \)

or

(B) \( r(D^2(n_1, n_2)) = r(D) \)

**Proof:** By Theorem 5.1, at least one of the equalities (i) - (iv) must hold. Out of these, the only one which does not imply (A) or (B) is (ii). Recalling the proof of Theorem 5.1, one can see that,
Fig. 5.2
with the additional assumption \((n_1, n_2) \in B\), one of conditions (i) (iii) or (iv) must be obtained, and hence either (A) or (B) holds.

The second corollary follows immediately from Theorem 5.1:

**Corollary 5.2:** Let \(D = (N, B)\) and \(n_1, n_2 \in N\). Then for \(i = 1\) and \(i = 2\):

\[
r(D^i(n_1, n_2)) = r(D) \text{ or } r(D^i(n_2, n_1)) = r(D)
\]

5.2. Rank Non-Increasing Reduction of Transition Graphs

**Definition 5.2:** A transition graph \(G = (N, B_L, N_1, N_2)\) is called reduced iff

(i) \(A_c^G(n_1) \neq A_c^G(n_2)\) for all \(n_1, n_2 \in N\)

(ii) \(P_r^G(n_1) \neq P_r^G(n_2)\) for all \(n_1, n_2 \in N\)

(iii) \((n, \lambda, n) \notin B_L\) for all \(n \in N\)
Note that (i) of the above definition corresponds to the notion of reduced deterministic automaton (Definition 1.13), whereas condition (ii) is always satisfied for deterministic automata, but not necessarily for transition graphs. Condition (iii), not allowing the appearance of self-loops labelled by $\lambda$, has been added for convenience. Obviously such self-loops are unnecessary and can be removed from any transition graph without affecting its behaviour.

A reduction procedure for an arbitrary transition graph $G = (N, B_L, N_1, N_2)$ can be easily obtained by applying successively the following basic construction:

Let $K \subseteq N$ be a set of nodes of $G$, all having the same accepted event (preceding event). Define a mapping $\phi$ from $N$ onto a set $N' = (N - K) \cup \{n_K\}$, where $n_K \notin N$, by:

$$\phi(n) = n \quad \text{if } n \in N - K$$

$$\phi(n) = n_K \quad \text{if } n \in K$$

Extend $\phi$ to a full homomorphism (Definition 2.28) from $G$ onto the graph $G' = (N', B_L', \phi(N_1), \phi(N_2))$, where for every $(n, x, n') \in B_L$.

$$\phi(n, x, n') = (\phi(n), x, \phi(n'))$$ and $B_L' = \phi(B_L)$. By Theorem 2.3, $G'$ is equivalent to $G$. Thus we may repeat this procedure until a graph $G''$ is obtained which is equivalent to $G$ and in which every pair of nodes have distinct preceding events and distinct accepted events. Removing all $\lambda$-self-loops from $G''$ yields a reduced transition graph $G_R$ equivalent to $G$ as required.

The only disadvantage in this method is in that the graph $G_R$
might have greater rank than $G$. Such a case is shown in the following example.

**Example 5.3:** Let $G$ be the graph in Fig. 5.3(a). Clearly $r(G) = 1$, and it can be easily verified that $A^G_c(n_1) = A^G_c(n_2)$. Thus merging these two nodes yields the equivalent graph $G'$ (Fig. 5.3(b)) which has rank 2.

Our aim in this section is to develop a reduction procedure of transition graphs, in which the rank of the graph cannot be increased.

**Definition 5.3:** Let $G = (N, B^*_L, N_1, N_2)$ be a transition graph and let $m, n \in N$. Define two transition graphs $G^1(m, n)$ and $G^2(m, n)$ as follows:

$$G^i(m, n) = (N, B_L^i, N_1, N_2), \quad i = 1, 2,$$

where

$$B_L^1 = B_L \cup \{(m, x, n') \mid (n, x, n') \in B_L^*, x \in A \cup \{\lambda}\}$$

$$B_L^2 = B_L \cup \{(m', x, n) \mid (m', x, m) \in B_L^*, x \in A \cup \{\lambda}\}$$

Thus $G^1(m, n)$ is obtained from $G$ by adding, for all $x \in A \cup \{\lambda\}$, $x$-transitions from $m$ to all nodes $n'$ in $G^*_x(n)$. Similarly, $G^2(m, n)$ is obtained from $G$ by adding, for all $x \in A \cup \{\lambda\}$, $x$-transitions from all $x$-predecessors of $m$ to $n$.

**Example 5.4:** Let $G$ be the graph on Fig. 5.4(a). $G^1(m, n)$ and $G^2(m, n)$ are shown in Fig. 5.4(b) and (c) respectively.

The next two lemmas follow directly from Definition 5.3.

† Throughout the rest of this chapter, whenever two nodes $m, n \in N$ of $G$ are given, it is always assumed that $m \neq n$. 
(a)

(b)

Fig. 5.3
Fig. 5.4
Lemma 5.8: \[ G^2(m, n) = (G)^{-1}(n, m) \]

Lemma 5.9: Let \( G = (N, B_L, N_1, N_2) \), \( m, n \in N \) and let:

\[ G^1 = G^1(m, n), \quad G^2 = G^2(m, n) \]
\[ G^3 = (G^1)^{-1}(n, m), \quad G^4 = (G^2)^{-1}(n, m) \]
\[ G^5 = (G^2)^{-1}(n, m), \quad G^6 = (G^1)^{-1}(n, m) \]

Then for all \( x \in A \cup \{\lambda\} \):

1. \( \sigma_x G^1(n) \subseteq \sigma_x G^1(m) \)
2. \( \pi_x G^2(m) \subseteq \pi_x G^2(n) \)
3. \( \sigma_x G^3(n) = \sigma_x G^3(m) \)
4. \( \pi_x G^4(n) = \pi_x G^4(m) \)
5. \( \sigma_x G^5(m) \subseteq \sigma_x G^5(n) \)
6. \( \pi_x G^5(m) \subseteq \pi_x G^5(n) \)

Lemma 5.10: Let \( G = (N, B_L, N_1, N_2) \) and let \( m, n \in N \) such that \( \sigma_x G(m) \subseteq \sigma_x G(n) \) and \( \pi_x G(m) \subseteq \pi_x G(n) \) for all \( x \in A \cup \{\lambda\} \). Then \( G - [m] \) is equivalent to \( G \).
Proof: Let \( p \) be any admissible path in \( G \) spelling out a word \( w \). Replace in \( p \) every branch of the form \((m', x, m)\) by \((m', x, n)\) and any branch of the form \((m, y, m'')\) by \((n, y, m'')\). The new path thus obtained is also admissible and spells out \( w \) in \( G - [m] \).

---

**Lemma 5.11:** Let \( G = (N, B_L, N_1, N_2) \) and let \( m, n \in N \). Then:

(a) either \( r(G^i(m, n)) = r(G) \) or \( r(G^i(n, m)) = r(G) \) for \( i = 1, 2 \).

(b) If for some \( x \in A \cup \{\lambda\} \), \((n, x, m) \in B_L \) then either \( r(G^1(n, m)) = r(G) \) or \( r(G^2(n, m)) = r(G) \).

Proof: It can be easily verified that the directed graph \( D^i_G(m, n) \) associated with \( G^i(m, n) \) [Definition 2.22] is \((D^i_G(m, n)) \) [Definition 5.1]. Hence the result follows directly from Corollary 5.2 and Corollary 5.1.

---

**Lemma 5.12:** Let \( G = (N, B_L, N_1, N_2) \) and let \( m, n \in N \) such that \( A^G_c(m) \supseteq A^G_c(n) \) \([P^G_r(m) \subseteq P^G_r(n)\)]. Then:

(a) The graph \( G^2 = G^2(m, n) \) \([G^1 = G^1(m, n)\)] is equivalent to \( G \).

(b) \( A^G_c(n') = A^G_c(n') \) \([P^G_r(n') = P^G_r(n')\)] for all nodes \( n' \in N \).
Proof: (b) Let \( G^2 = (N, B_L^2, N_1, N_2) \) and assume \( A_c^G(m) \supseteq A_c^G(n) \).

By definition \( B_L^2 \supseteq B_L \), and all branches in \( B_L^2 - B_L \) are of the form (\( m', x, n \)), where \( x \in A \cup \{ \lambda \} \) and \( m' \in \pi_x^G(m) \). Let \( n' \) be any node of \( G \). Then clearly \( A_c^G(n') \subseteq A_c^{G^2}(n') \). Now suppose \( w \in A_c^{G^2}(n') \). Then there exists in \( G^2 \) a path \( p \) from \( n' \) to a final node spelling out \( w \).

If \( p \) is also a path in \( G \) then \( w \in A_c^G(n') \). Otherwise \( p \) must have branches from \( B_L^2 - B_L \). Thus suppose \( p \) has \( k > 0 \) branches from \( B_L^2 - B_L \) and let \( (m', x, n) \) be the rightmost one, i.e. \( p = p' \cdot (m', x, n) \cdot p'' \), where \( (m', x, n) \in B_L^2 - B_L \) and \( p'' \) is a path in \( G \). Let \( w = w' \cdot w'' \) be the corresponding decomposition of \( w \). Clearly \( w'' \in A_c^G(n) \) and hence also \( w'' \in A_c^G(m) \); thus there exists in \( G \) a path \( p_1'' \) from \( m \) to a final node spelling out \( w'' \). Now the path \( p_1 = p' \cdot (m', x, m) \cdot p_1'' \) is also a path in \( G^2 \) from \( n' \) to a final node spelling out \( w \). Moreover, \( p_1 \) has only \( k - 1 \) branches which are not contained in \( G \). Thus after repeating this procedure \( k - 1 \) more times we finally obtain a path from \( n' \) to a final node, which is contained in \( G \) and spells out \( w \). Hence \( w \in A_c^G(n') \) and \( A_c^G(n') = A_c^{G^2}(n') \).

(a) Since \( G^2 \) has the same initial nodes as \( G \), the result follows directly from (b).

The proof for \( G^1 \) and \( P_r^G(n'), P_r^G(n') \), is obtained by duality.

Lemma 5.13: Let \( G = (N, B_L, N_1, N_2) \) and let \( \ell = (n_1, \lambda, n_2)(n_2, \lambda, n_3)(n_k, \lambda, n_1) \) be a non-trivial \( \lambda \)-loop in \( G \). Let \( \phi \) be a homomorphism from \( G \) onto a graph \( \phi(G) = G' = (N', B_L', N_1', N_2') \) defined by:

\[
N' = N - \{ n_1, n_2, \ldots, n_k \} \cup \{ m \}
\]
where $m \notin N$,

$$\phi(n_1) = \phi(n_2) = \ldots = \phi(n_k) = m$$

$$\phi(n) = n, \quad n \neq n_i, \quad i = 1, \ldots, k$$

$$\forall i = 1, \ldots, k-1, \quad \phi(n_i, \lambda, n_{i+1}) = m$$

$$\phi(n_k, \lambda, n_1) = m$$

and for all other branches $(n, x, n') \in B_L$,

$$\phi(n, x, n') = (\phi(n), x, \phi(n'))$$

and

$$N'_i = \phi(N_i), \quad i = 1, 2$$

Then $r(G') \leq r(G)$, and $G'$ is equivalent to $G$.

**Proof:** Let $C = \{n_1, n_2, \ldots, n_k\}$. Define two new sets of initial and terminal nodes in $G$ by:

$$N''_i = \begin{cases} N_i & \text{if } N_i \cap C = \emptyset \\ N_i \cup C & \text{otherwise} \end{cases}$$

$i = 1, 2$, and let $G'' = (N, B_L, N'_1, N'_2)$. One can easily verify that $G''$ is equivalent to $G$. Furthermore, the mapping $\phi$ is a pathwise homomorphism from $G''$ to $G'$ which satisfies the conditions of Theorem 2.4. To see this, let $p'$ be any path in $G'$. Replace in $p'$ any branch $b'_i$ entering $m$ by a corresponding branch $b_i \in \phi^{-1}(b'_i)$ in $G$ entering one of the $n_j$'s, say $n_t (1 \leq t \leq k)$. Then replace the succeeding branch $b'_{i+1}$ leaving $m$ by a corresponding branch $b_{i+1} \in \phi^{-1}(b'_{i+1})$ leaving one of the $n_j$'s, say $n_r$, preceded by the $\lambda$-path from $n_t$ to $n_r$. In this way one
obtains a path $p$ in $G''$ such that $\phi(p) = p'$. Hence $\phi$ is a pathwise homomorphism. By McNaughton's Pathwise Homomorphism Theorem, $r(G') \leq r(G'') = r(G)$, and by Theorem 2.4 $G'$ is equivalent to $G''$, and hence to $G$.

Lemma 5.14: Let $G = (N, B_L, N_1, N_2)$ and let $m, n \in N$ such that:

(a) $A^G_c(m) \leq A^G_c(n)$ \quad [$P^G_x(m) \leq P^G_x(n)$]

(b) $\forall x \in A \cup \{\lambda\}, \pi^G_x(m) \leq \pi^G_x(n)$ \quad [$\sigma^G_x(m) \leq \sigma^G_x(n)$]

(c) There does not exist in $G$ a path from $n$ to $m$ spelling out $\lambda$ \quad [a path from $m$ to $n$ spelling out $\lambda$]

(d) If $m \in N_1$ then also $n \in N_1$ \quad [if $m \in N_2$ then also $n \in N_2$].

Then the graph $G' = G - [m]$ is equivalent to $G$.

Proof: Obviously the event accepted by $G$ contains the event accepted by $G'$. Now let $w$ be a word accepted by $G$. Then there exists in $G$ an admissible path $p$ spelling out $w$. If $p$ does not touch $m$ then it is also a path in $G'$ and then $w$ is accepted also by $G'$. Thus suppose $p$ passes through $m$. Then we have two cases: (i) $p$ starts at $m$.

(ii) $p$ starts at some other node. In Case (i), since $A^G_c(m) \leq A^G_c(n)$, we may replace $p$ by some other admissible path starting at $n$ and spelling out $w$ (recall that by (d) if $m$ is an initial node, $m \in N_1$, then so is $n$). Thus we may assume we have Case (ii). So if $p$ touches $m$, let
\[ p = p' \cdot (m', x, m) \cdot (m, y, m'') \cdot p'', \] where \( p' \) is the maximal prefix of \( p \) that does not touch \( m, x, y \in A \cup \{\lambda\} \) and \( m', m'' \in N \). Let \( w = w'xyw'' \) be the corresponding decomposition of \( w \). Clearly \( yw'' \in A_c^G(m) \) and thus \( yw'' \in A_c^G(n) \). Thus there exists in \( G \) a path \( p''_1 \) from \( n \) to a final node spelling out \( yw'' \). Moreover, since by \( (b) \ \pi_x^G(m) \subseteq \pi_x^G(n) \), we have \( (m', x, n) \in B_L \). Then the path \( p_1'' = p' \cdot (m', x, n) \cdot p''_1 \) is also an admissible path of \( G \) spelling out \( w \). If \( p_1'' \) still touches \( m \), we repeat the same procedure once more. Thus let

\[ p_1'' = p_1' \cdot (m', x', m) \cdot (m, y', m_1'') \cdot p_2'' \],
where \( p_2' \) is the maximal prefix of \( p_1'' \) not touching \( m \). Let the corresponding decomposition of \( w'' \) be: \( w'' = w'\_2'x'y'w_2'' \). Since by assumption \( (c) \) there does not exist in \( G \) any path from \( n \) to \( m \) spelling out \( \lambda \), the word \( w_2'x' \), which is spelled out by \( p_2' \cdot (m'_1, x', m) \), cannot be \( \lambda \). Now by the same argument as above, the suffix \( (m'_1, x', m) \cdot (m, y', m_1'') \cdot p_2'' \) can be replaced by a path \( (m'_1, x', n) \cdot p_3'' \), which also leads to a final node and spells out \( x'y'w_2'' \). Thus we obtain a new path \( p_2 = p' \cdot (m', x, n) \cdot p_2' \cdot (m'_1, x', n) \cdot p_3'' \), in which only the suffix \( p_3'' \) may touch \( m \). But this suffix spells out the word \( y'w_2'' \), which is clearly shorter than the word \( yw'' \) spelled out by the corresponding suffix \( p_1'' \) of \( p_1 \). This shows that repeating this procedure enough times, we eventually end up with a path \( p_k(k \geq 1) \) which is admissible in \( G \) and spells out \( w \), in which the length of the suffix touching \( m \) is 0, i.e. \( p_k \) does not touch \( m \) at all. Hence \( p_k \) is also an admissible path in \( G' = G - [m] \) and \( w \) is accepted also by \( G' \). This completes the proof of the first part of the theorem. The proof of the second part (with \( p_r^G(m) \subseteq p_r^G(n), \) etc.) is obtained in a dual way.
Theorem 5.2: For any transition graph $G = (N, B_L, N_1, N_2)$ there exists an equivalent reduced transition graph $G' = (N', B_L, N_1', N_2')$ such that $\#N' \leq \#N$ and $r(G') \leq r(G)$.

Proof: The reduction of $G$ will be achieved by recursive application of the following two basic steps:

Step (A): Let $G_i$ $(i = 0, 1, \ldots)$ be the graph obtained from $G$ in this process after $i$ basic steps, and let $m, n$ be two nodes of $G_i$ such that $A_c^i(m) = A_c^i(n)$. Then by Lemma 5.11 (a) either $r(G_1^2(m, n)) = r(G_i)$ or $r(G_1^2(n, m)) = r(G_i)$. Without loss of generality we may assume the former. By Lemma 5.12, $G_{i2} = G_1^2(m, n)$ is equivalent to $G_i$ and has the same set of accepted events. Thus we may replace $G_i$ by $G_{i2}$ and $m$ and $n$ have equal accepted events also in $G_{i2}$. Furthermore, by Lemma 5.9 (2), $\pi^G_{i2}(m) \leq \pi^G_{i2}(n)$ for all $x \in A \cup \{\lambda\}$. Now we have two cases:

Case I: there does not exist any path in $G_{i2}$ from $n$ to $m$ spelling out $\lambda$. Then all requirements of Lemma 5.14 are satisfied and hence $G_{i2} - [m]$ is equivalent to $G_{i2}$. In this case define $G_{i+1} = G_{i2} - [m]$.

Case II: there exists in $G_{i2}$ a path $p$ from $n$ to $m$ spelling out $\lambda$. If $\ell(p) \geq 2$, then we have: $p = (n, \lambda, m_1)(m_1, \lambda, m_2) \ldots (m_k, \lambda, m)$ for some $k \geq 1$. By Proposition 2.4:

$A_c^G(m) \geq A_c^G(m_1) \geq \ldots \geq A_c^G(m_k) \geq A_c^G(m)$, and since $m$ and $n$ have equal accepted events, all the above accepted events are equal. In this case start Step A again, replacing $m$ by $m_1$. Thus we may assume that $p$ has
length 1, i.e. there exists in $G_{i2}$ a $\lambda$-transition $(n, \lambda, m)$. Then if there is also a path from $m$ to $n$ spelling out $\lambda$ in $G_{i2}$, then there is a non-trivial $\lambda$-loop in $G_{i2}$. In that case define $G_{i+1}$ to be the graph obtained from $G_{i2}$ by shrinking this $\lambda$-loop as indicated in Lemma 5.13.

Thus assume that $(n, \lambda, m)$ is in $G_{i2}$ but that there does not exist in it any path from $m$ to $n$ spelling out $\lambda$. Now by Lemma 5.11 (b), either (1) $r(G_{i2}^1(n, m)) = r(G_{i2})$ or (2) $r(G_{i2}^2(n, m)) = r(G_{i2})$. If (2) is satisfied, we may replace $G_{i2}$ by $G_{i3} = G_{i2}^2(n, m)$, which, by Lemma 5.12 (a), is equivalent to $G_{i2}$. Now by Lemma 5.12 (b), $m$ and $n$ still have equal accepted events in $G_{i3}$, and by Lemma 5.9 (2),

$$\pi_x^{G_{i3}}(n) \leq \pi_x^{G_{i3}}(m)$$

for all $x \in A \cup \{\lambda\}$. Moreover, if $n$ is an initial node in $G_{i3}$, we can define also $m$ to be an initial node in $G_{i3}$ without affecting the event accepted by $G_{i3}$. Now all requirements of Lemma 5.14 are satisfied and hence the graph $G_{i3} - [n]$ is equivalent to $G_{i3}$. Define in this case $G_{i+1} = G_{i3} - [n]$.

Now suppose (1) is satisfied. Since, by our assumption, the $\lambda$-transition $(n, \lambda, m)$ is in $G_{i2}$, we have, by Proposition 2.4 (2) (page 28), $P_r G_{i2}(m) \supseteq P_r G_{i2}(n)$.

Thus by Lemma 5.12 (a), the graph $G_{i5} = (G_{i2})^1(n, m)$ is equivalent to $G_{i2}$ and we may replace the latter by the former. Now by Lemma 5.9 (5),

$$\sigma_x^{G_{i5}}(m) \subseteq \sigma_x^{G_{i5}}(n)$$

and

$$\pi_x^{G_{i5}}(m) \subseteq \pi_x^{G_{i5}}(n)$$

and by Lemma 5.10, $G_{i5} - [m]$ is equivalent to $G_{i5}$. Define in this case $G_{i+1} = G_{i5} - [m]$.

**Step (B):** Let $m, n$ be two nodes in $G_i$ such that $P_r G_i(m) = P_r G_i(n)$. $G_{i+1}$ is obtained from $G_i$ in a way dual to that defined in Step (A) (Lemma 5.8).
Our reduction procedure is as follows:

(i) Set \( G = G_0 \)

(ii) If there exist in \( G_i \) (\( i = 0, 1, \ldots \)) any non-trivial \( \lambda \)-loops, shrink them as indicated by Lemma 5.13.

(iii) If there exist in \( G_i \) two nodes \( m, n \) such that

\[ A_c^i(m) = A_c^i(n) \]

apply Step (A) to \( G_i \) and return to (ii) with \( G_{i+1} \).

(iv) If there exist nodes \( m, n \) in \( G_i \) such that

\[ P_r^i(m) = P_r^i(n) \]

apply Step (B) to \( G_i \) and return to (ii) with \( G_{i+1} \).

(v) Remove all \( \lambda \)-self-loops from \( G_i \) and define \( G_i = G' \).

Obviously each graph \( G_i \) obtained in the above process is equivalent to \( G \), \( r(G_i) \leq r(G) \) and \( G_i \) has at least \( i \) nodes less than \( G \). Hence the process must terminate after a finite number of steps, yielding the desired reduced graph \( G' \).

5.3. Elimination of \( \lambda \)-transitions

We now use the results of the previous sections to obtain a procedure for replacing all \( \lambda \)-transitions in a given transition graph \( G \) by some other non-\( \lambda \) transitions. The resulting graph will be a non-deterministic state graph, equivalent to \( G \) and of rank no greater than the
rank of $G$.

We first present a construction by which a single $\lambda$-transition in a transition graph can be replaced by a set of some other transitions (some of which might be again labelled by $\lambda$) in such a way that the rank of the graph will not be increased. Such a replacement will be called legitimate.

**Definition 5.4:** Let $G = (N, B_L, N_1, N_2)$ be a transition graph and let $m, n \in N$ such that $(m, \lambda, n) \in B_L$. Let $G_i(m, n) = (N, B^i_L, N^i_1, N^i_2)$, $i = 1, 2$, (Definition 5.3). We now define two more graphs, $G_{mn}(1)$ and $G_{mn}(2)$ as follows:

$$G_{mn}(i) = (N, B^i_L \setminus \{(m, \lambda, n)\}, N^i_1, N^i_2), i = 1, 2$$

where:

$N^1_1 = N_1$

$N^2_1 = N_2$

$N^1_2 = \begin{cases} N_1 \cup \{n\} & \text{if } m \in N_1 \\ N_1 & \text{otherwise} \end{cases}$

$N^2_2 = \begin{cases} N_2 \cup \{m\} & \text{if } n \in N_2 \\ N_2 & \text{otherwise} \end{cases}$

Thus $G_{mn}(i)$ is obtained from $G^i(m, n)$ by removing the $\lambda$-transition $(m, \lambda, n)$ and possibly adding $m$ to $N_2$ or $n$ to $N_1$. 
Example 5.6: Let $G$ be the graph in Fig. 5.4 (a) (Page 100). Since there exists in $G$ a $\lambda$-transition from $m$ to $n$, $G_{mn}(1)$ and $G_{mn}(2)$ can be constructed, and are shown in Fig. 5.4 (d) and (e) respectively.

Lemma 5.15: For any transition graph $G$ and for any $\lambda$-transition $(m, \lambda, n)$ in $G$,

$$
\hat{G}_{mn}(1) = \hat{G}_{nm}(2)
$$

Proof: This follows directly from Lemma 5.8 and the above definition.

Lemma 5.16: If $G = (N, B_L, N_1, N_2)$ is a reduced transition graph, then there does not exist in $G$ any $\lambda$-loop.

Proof: Let $p = (m_1, \lambda, m_2) \ldots (m_k, \lambda, m_1)$ be any $\lambda$-loop of $G$. Then by Proposition 2.4, $A_c^G(m_1) = A_c^G(m_2) = \ldots = A_c^G(m_k)$ and $G$ cannot be reduced. Also a $\lambda$-self-loop cannot exist in $G$ since $G$ is reduced.

Lemma 5.17: Let $G = (N, B_L, N_1, N_2)$ be a reduced transition graph and let $(m, \lambda, n) \in B_L$. Then:

1. $G_{mn}(1)$ and $G_{mn}(2)$ are reduced and equivalent to $G$.
2. Either $r(G_{mn}(1)) \leq r(G)$ or $r(G_{mn}(2)) \leq r(G)$.
Proof: The second part follows directly from the above definitions and the application of Corollary 5.1 and Lemma 2.1.

As for (1), let \( n' \) be a node of \( G \) and let \( w \) be any word accepted by \( n' \). Then \( w \) is spelled out by a path \( p = b_1 b_2 \ldots b_k \) leading from \( n' \) to a terminal node. Modify \( p \) in the following way: for every \( b_i, 1 \leq i \leq k \), such that \( b_i = (m, \lambda, n) \) and \( b_{i+1} = (n, x, n'') \) for some \( n'' \in N \) and \( x \in A \cup \{ \lambda \} \), replace \( b_i b_{i+1} \) by \( b'_i = (m, x, n'') \). Since \( G \) is reduced, \( b_{i+1} \neq (n, \lambda, n) \); hence \( b'_i \neq (m, \lambda, n) \), and it follows from the definition of \( G_{mn}(1) \) that \( b'_i \) is one of its branches. In addition, if \( b_k = (m, \lambda, n) \), then remove \( b_k \) from \( p \). Since \( n \) must be in this case a terminal node of \( G \), \( m \) is a terminal node of \( G_{mn}(1) \). Hence the modified path \( p' \) obtained from \( p \) in this manner is a path in \( G_{mn}(1) \) from \( n' \) to a terminal node, spelling out the same word \( w \). Hence \( A_c G(n') \subseteq A_c G_{mn}(1)(n') \).

Now let \( w \) be a word accepted by \( n' \) in \( G_{mn}(1) \). Then \( w \) is spelled out by a path \( p = b_1 b_2 \ldots b_k \) in \( G_{mn}(1) \) leading from \( n' \) to a terminal node. Replace in \( p \) every branch \( b_i = (m, x, n'') \), which does not belong to \( G \), by \( b'_i = (m, \lambda, n)(n, x, n'') \), and also in case \( b_k = (m', x, m) \) for some \( m' \in N \), \( x \in A \cup \{ \lambda \} \), concatenate \( p \) by \( b_{k+1} = (m, \lambda, n) \). The new path thus obtained is clearly a path in \( G \) from \( n' \) to a terminal node of \( G \), spelling out \( w \). Thus we get: \( A_c G_{mn}(1)(n') = A_c G_{mn}(1)(n') \) for all \( n' \in N \), and hence \( G \) and \( G_{mn}(1) \) have the same set of accepted events. Also, by definition, the set of initial nodes of \( G \) is identical with that of \( G_{mn}(1) \). Furthermore, \( G_{mn}(1) \) cannot have any \( \lambda \)-self-loop \( (n', \lambda, n') \) because \( G \) does not have any, and if \( (n', \lambda, n') \) were one of the new branches added to \( G \) in constructing \( G_{mn}(1) \), we would have \( n' = m, (n, \lambda, m) \in B_L \) and hence \( (m, \lambda, n)(n, \lambda, m) \) would be a \( \lambda \)-loop contained in \( G \), contradicting the fact that \( G \) is reduced.
(Lemma 5.16). All these imply that $G_{mn}(1)$ is equivalent to $G$ and is also reduced.

The proof for $G_{mn}(2)$ is obtained by duality (Lemma 5.15), and in this case we get coincidence of the preceding events of $G_{mn}(2)$ with those of $G$.

Thus every $\lambda$-transition $(m, \lambda, n)$ in a transition graph $G$ can be replaced by some other transitions so that (after appropriate changes in the sets of initial and final nodes) the resulting graph is equivalent to $G$ and has no higher rank.

**Definition 5.5:** Let $G = (N, B_L, N_1, N_2)$ and let $(m, \lambda, n) \in B_L$. The construction of either $G_{mn}(1)$ or $G_{mn}(2)$ is called a *replacement of $(m, \lambda, n)$ in $G$*. A replacement is called *legitimate* iff the corresponding graph $G_{mn}(i)$ has rank no greater than $r(G)$. The sets of branches replacing $(m, \lambda, n)$ in $G_{mn}(1)$ and $G_{mn}(2)$ are, respectively: $B_1 = \{(m, x, n') \mid (n, x, n') \in B_L\}$ and $B_2 = \{(m', x, m) \mid (m', x, n) \in B_L\}$.

Thus every $\lambda$-transition in $G$ has two replacements, at least one of which, by the previous lemma, is legitimate.

**Example 5.7:** For the replacements of $(m, \lambda, n)$ shown in Fig. 5.4 (a), (d) and (e) (page 100), $B_1 = \{(m, 0, p)\}$ and $B_2 = \{(p, 0, n), (m, 1, n)\}$. One can easily verify that $r(G_{mn}(1)) = r(G) = 1$ and $r(G_{mn}(2)) = 2 > r(G)$. Hence the replacement of $(m, \lambda, n)$ by $B_1$ is legitimate whereas the second replacement, by $B_2$, is not.
Throughout the rest of this section let $G = (N, B_L, N_1, N_2)$ be a fixed reduced transition graph.

**Definition 5.6:** Let $<_G$ be the relation on the set of nodes $N$ defined by: $\forall m, n \in N, m <_G n$ iff there exists in $G$ a non-null path from $m$ to $n$ spelling out the word $\lambda$.

**Lemma 5.18:** $<_G$ is a (strong) partial order on $N$.

**Proof:** For any $n \in N$, $n \not<_G n$ by Lemma 5.16. For any pair of nodes $m, n \in N$, $m <_G n$ and $n <_G m$ implies the existence of a $\lambda$-loop through $m$ and $n$, which is again impossible by Lemma 5.16. The transitivity follows directly from the definition.

Define $m \leq_G n$ iff $m <_G n$ or $m = n$.

**Definition 5.7:** The relation $<_G$ is defined on the set $N \times \{\lambda\} \times N$ by: $(m_1, \lambda, n_1) <_G (m_2, \lambda, n_2)$ iff either $m_1 \leq_G m_2$ and $n_2 <_G n_1$ or $m_1 <_G m_2$ and $n_2 \leq_G n_1$.

**Lemma 5.19:** $<_G$ is a strong partial order on $N \times \{\lambda\} \times N$.

**Proof:** Follows directly from the above definition and Lemma 5.18.

**Example 5.8:** Let $G = (N, B_L, N_1, N_2)$ be the transition graph on Fig. 5.6. The partial order $<_G$ is given by:

$$<_G = \{(p, m), (p, q), (q, m), (q, r), (p, r), (n, m), (n, r), (m, r)\}$$
Fig. 5.6.
The restriction of the partial order $<_G$ to the set of all $\lambda$-transitions of $G$ is:

$$<_G/(N \times \{\lambda\} \times N) \cap B_L =$$

$$= \{((p, \lambda, r), (p, \lambda, m)), ((p, \lambda, r), (p, \lambda, q)), ((p, \lambda, r), (m, \lambda, r)), ((p, \lambda, m), (p, \lambda, q)), ((p, \lambda, m), (q, \lambda, m)), ((p, \lambda, r), (q, \lambda, m))\}$$

An example of a related pair of $\lambda$-transitions, one of which is not contained in $G$, is: $(p, \lambda, r) <_G (p, \lambda, n)$.

**Lemma 5.20:** Let $(m, \lambda, n) \in B_L$ and let $B_i$ be the set of branches replacing $(m, \lambda, n)$ in $G_{mn(i)}$, $i = 1, 2$. Then for any $(m', \lambda, n')$ in $B_i$, $(m', \lambda, n') <_G (m, \lambda, n)$.

**Proof:** By definition, $m' = m$ if $(m', \lambda, n')$ is in $B_1$ and $n' = n$ if this branch is in $B_2$. In the former case $(n, \lambda, n') \in B_L$ and hence $n <_G n'$, and in the latter case $(m', \lambda, m) \in B_L$ and thus $m' <_G m$. In both cases we get $(m', \lambda, n') <_G (m, \lambda, n)$. 
Lemma 5.21: If $G'$ is a transition graph obtained from $G$ by a finite series of replacements then:

(a) For any $m', n' \in N$, $m' \prec_G n'$ implies $m' \prec_G n'$.

(b) For any two $\lambda$-transitions $b, b' \in N \times \{\lambda\} \times N$, $b \prec_G b'$ implies $b' \prec_G b'$.

Proof: It suffices to prove (a) and (b) only for the case when $G'$ is obtained from $G$ by a single replacement. The result then follows by induction. Thus assume that $(m, \lambda, n) \in B_L$ and let $G' = G_{mn(i)}$ for $i = 1$ or 2.

(a) For $i = 1$: Let $m' \prec_G n'$ for some $m', n' \in N$. Then there exists in $G'$ a non-null path $p$ from $m'$ to $n'$ spelling out the word $\lambda$. If $p$ is also a path in $G$ then clearly $m' \prec_G n'$.

If, however, $p$ is not a path in $G$, then $p$ contains branches of the replacement of $(m, \lambda, n)$, which are of the form $(m, \lambda, n')$, where $(n, \lambda, n') \in B_L$. Replacing each such branch $(m, \lambda, n')$ in $p$ by $(m, \lambda, n)(n, \lambda, n')$, one obtains a new path $p'$ from $m'$ to $n'$, which is a path in $G$ and spells out $\lambda$. Hence $m' \prec_G n'$ as required. The proof for $i = 2$ follows by duality (Lemma 5.15).
(b) for i = 1: Let \( b = (m_1, \lambda, n_1) \), \( b^i = (m_2, \lambda, n_2) \) and suppose \( b^i \leq_G b \). By definition, either \( m_2 \leq_G m_1 \) and \( n_1 \leq_G n_2 \), or \( m_2 \leq_G m_1 \) and \( n_1 \leq_G n_2 \). By part (a) of this lemma, we get in the former case \( m_2 \leq_G m_1 \) and \( n_1 \leq_G n_2 \) and in the latter case \( m_2 \leq_G m_1 \) and \( n_1 \leq_G n_2 \), both of which yield the result.

---

**Theorem 5.3**: For any reduced transition graph \( G \), there exists an equivalent reduced non-deterministic state graph \( G' \) with the same number of nodes and no greater rank.

**Proof**: We shall first prove the following assertion:

\((**)\) For any reduced transition graph \( G = (N, B_L, N_1, N_2) \) and for any set of \( \lambda \)-transitions \( \{b_1, b_2, \ldots, b_k\} \subseteq N \times \{\lambda\} \times N \) (\( k = 1, 2, \ldots \)), \( G \) can be converted by a series of legitimate replacements into a graph \( G^k = (N, B_L^k, N_1^k, N_2^k) \) such that \( b_1 \not\in B_L^k \) for all \( i = 1, \ldots, k \).

The proof will proceed by induction on \( k \).
Basis: \( k = 1 \). If \( b_1 \notin B_1 \) then \( G^1 = G \) is the required graph. If
\( b_1 \in B_1 \) then by Lemma 5.16 there exists a legitimate replacement of
\( b_1 \). Let the corresponding graph be \( G_{mn(i)} \), where \( b_1 = (m, \lambda, n) \) and
\( i = 1 \) or \( 2 \). By definition, \( G_{mn(i)} \) does not contain \( b_1 \) and hence
\( G^1 = G_{mn(i)} \) is the required graph.

Induction Step: Assume (**) holds for \( k - 1 \). Let
\( \{b_1, b_2, \ldots, b_k\} \subseteq N \times \{\lambda\} \times N \) be any set of \( k \) \( \lambda \)-transitions.
Since this set is finite and partially ordered by the relation \( \prec \),
there exists in it an element which is minimal under this partial
ordering, i.e. for some \( i, 1 \leq i \leq k \), \( b_j \not\prec_G b_1 \) for all \( j = 1, \ldots, k \).
Without loss of generality we may assume that \( b_k \) is such a minimal
element. Now, applying the induction hypothesis to \( \{b_1, \ldots, b_{k-1}\} \),
convert \( G \) by a finite series of legitimate replacements into a graph \( G^{k-1} \),
which contains none of the branches \( b_1, \ldots, b_{k-1} \). If also \( b_k \) is not a
branch of \( G^{k-1} \) then \( G^k = G^{k-1} \) is the required graph. Otherwise, replace
\( b_k \) in \( G^{k-1} \) by a legitimate replacement, and let \( B \) be the set of branches
replacing \( b_k \) in this replacement. By Lemma 5.20, \( b \not\prec_{G^{k-1}} b_k \) for any
\( \lambda \)-transition \( b \) in \( B \). On the other hand it follows from Lemma 5.21 that
\( b_k \) is a minimal element in the set \( \{b_1, \ldots, b_k\} \) also with respect to
the partial order \( \prec_{G^{k-1}} \). We conclude that \( B \cap \{b_1, \ldots, b_k\} = \emptyset \). Hence
the graph \( G^k \) obtained from \( G^{k-1} \) by a legitimate replacement of \( b_k \) does
not have any of \( b_1, \ldots, b_k \) among its branches and is therefore the
required graph.
Now apply (***) to \( \{b_1, \ldots, b_k\} = N \times \{\lambda\} \times N \), where
\[ k = (\#N)^2. \]
Then the graph \( G' = G^k \) does not contain any \( \lambda \)-transitions and is therefore a non-deterministic state graph. Moreover, since \( G' \) has been obtained from \( G \) by a series of legitimate replacements, Lemma 5.17 implies that \( G' \) is reduced, equivalent to \( G \) and \( r(G') \leq r(G) \) as required.

---

**Example 5.9:** Let \( G \) be the reduced graph shown in Fig. 5.7 (a).

First replace the \( \lambda \)-transition \((n, \lambda, p)\) by constructing the graph \( G^1 = G_{np}(2) \) (Fig. 5.7 (b)). Since \( r(G^1) = 2 = r(G) \) this is a legitimate replacement. The set of branches replacing \((n, \lambda, p)\) in \( G^1 \) is \( B^1 = \{(m, \lambda, p), (q, \lambda, p), (p, 3, p)\} \), thus \( G^1 \) has two new \( \lambda \)-transitions. Go on replacing \((q, \lambda, p)\) by the single branch \((p, 2, p)\), thus obtaining the graph \( G^2 = (G^1)_{qp}(2) \) (Fig. 5.7 (c)). Now replace \((m, \lambda, n)\) by constructing \( G^3 = (G^2)_{mn}(1) \) (Fig. 5.7 (d)), in which \( m \) is also a terminal node. Proceed by replacing \((m, \lambda, p)\) and then \((q, \lambda, n)\), obtaining the graphs \( G^4 = (G^3)_{mp}(2) \) and \( G^5 = (G^4)_{qn}(2) \) (Fig. 5.7 (e), (f)). The graph \( G^5 = G' \) is the required non-deterministic state graph equivalent to \( G \).

The following corollary is an immediate result of Theorem 5.2 and Theorem 5.3.

**Corollary 5.3:** For any transition graph \( G \), there exists an equivalent reduced non-deterministic state graph \( G' \) having no more nodes than \( G \) and no higher rank.
Fig. 5.7
(d): $G^3$

(e): $G^4$

(f): $G^5 = G'$

Fig. 5.7
5.4. Applications

An important application of the results obtained in the previous sections is the following stronger version of Eggan's Star Height Theorem.

**Theorem 5.4:** The star height of a regular event $R$ equals $\min \{ r(G) \}, \quad G \in \mathcal{F}_R$

where $\mathcal{F}_R$ is the family of all non-deterministic reduced state graphs recognizing $R$.

We next consider an interesting by-product of Corollary 5.3.

**Definition 5.8:** For any regular event $R$, let $M_{nda}(R)$ be the smallest number of states of any non-deterministic automaton recognizing $R$. Let $M_{tg}(R)$ be the smallest number of nodes of any transition graph accepting $R$.

An algorithm for finding $M_{nda}(R)$ and some of the corresponding minimum-state non-deterministic automata accepting the event $R$ is discussed in [KW]. As a consequence of Corollary 5.3, the same algorithm applies also for the more general case of finding minimum-state transition graphs recognizing $R$.

**Corollary 5.4:** For every regular event $R$, $M_{nda}(R) = M_{tg}(R)$. 
Chapter 6

Further Methods for Establishing Star Height

Introduction: In Chapter 4 the star height of certain families of regular events has been established by relating the rank of the corresponding reduced state graphs to the apparent star height of regular expressions denoting these events. In this chapter a different approach, based on Theorem 5.4, is adopted. Instead of investigating the structure of a regular expression denoting the given event \( R \), we study the rank of a non-deterministic state graph recognizing \( R \). Thus \( h(R) = k \) will be proved by showing that any arbitrary non-deterministic state graph accepting \( R \) has rank no less than \( k \). This approach is similar to the one employed by McNaughton in [Mc 3,4].

In Section 6.1 two functions, called the \( R \)-projection functions, are introduced. These functions, relating the nodes and paths of an arbitrary state graph with sets of nodes and paths in the reduced state graph \( G_0(R) \), constitute the foundation for the methods developed in this chapter. By means of these functions, the concept of one state graph covered by another is defined, and in Section 6.2 two fundamental lemmas concerning this concept are proved.

Section 6.3 presents a new proof technique for establishing the star height of a regular event. In some cases only lower bounds to the star height can be obtained but in many cases the exact star height
is found. This technique is illustrated by examples.

In Section 6.4 some results are obtained for the case when \( \mathcal{G}_0(R) \) is a reset-free state graph. As we have seen in Chapter 4, if \( \mathcal{G}_0(R) \) is reset-free and has only a single final state, then the star height of \( R \) equals the rank of \( \mathcal{G}_0(R) \). However, this is not always the case when \( \mathcal{G}_0(R) \) is reset-free with more than one output state. Nevertheless, it is shown that a partial extension of the above result can be achieved if we consider only state graphs \( \mathcal{G}_0(R) \) which are both reset-free and permutation-free. Some other cases are also considered and sufficient conditions for the star height of \( R \) to be no less than the rank of some section of \( \mathcal{G}_0(R) \) are found.

Throughout this chapter let \( R \) be any fixed regular event, let \( G = G_0(R) = (Q, \bar{M}, q_1, F) \) be the reduced deterministic state graph recognizing \( R \) and let \( \mathcal{G} = \mathcal{G}_0(R) = (Q, M, q_1, F) \) be the reduced incomplete state graph recognizing \( R \). Thus \( Q = Q \cup \{ q_1 \} \) in case \( G \) has a dead state \( q_1 \), or \( Q = Q \) (and hence \( \mathcal{G} = G \)) if there is no dead state in \( G \). In the former case, \( M(q, a) = \emptyset \) iff \( \bar{M}(q, a) = q_1 \) for all \( q \in Q \), \( a \in A \) and \( M(q, a) = \bar{M}(q, a) \) otherwise. Let \( Q = \{ q_1, q_2, \ldots, q_{m_0} \} \) and assume \( m_0 \geq 2 \).

In the sequel, let \( G' = (N, B_L, N_1, N_2) \) be any non-deterministic all-admissible state graph (not necessarily equivalent to \( G \)).
6.1. The R-Projection Functions and Covering of State Graphs

Definition 6.1: Let \( R \) be a regular event and let \( G' = G_0(R) = (Q, M, q_1, F) \). For any non-deterministic state graph \( G' = (N, B_L, N_1, N_2) \) define a function \( f^{G'}_R : N \rightarrow 2^Q \) as follows:† for each node \( n \) of \( G' \),

\[
f^{G'}_R(n) = \{ q \mid q \in Q \land (P^G_r(q) \cap P^G_r(n)) \neq \emptyset \}
\]

Thus \( f^{G'}_R(n) \) is the set of all states \( q \) in \( G' \) such that \( q = M(q_1, w) \) for some word \( w \in P^G_r(n) \). Using the function \( M_1 \) introduced in Definition 1.15 we obtain:

\[
f^{G'}_R(n) = M_1(P^G_r(n))
\]

Clearly for all initial nodes \( n_1 \in N_1 \) of \( G' \), \( q_1 \in f^{G'}_R(n_1) \).

Example 6.1: Let \( G_0(R) \) and \( G' \) be the state graphs shown in Fig. 6.1 and 6.2 respectively. For each node \( n \) of \( G' \), the states of \( f_R(n) \) are indicated inside the circle representing \( n \).

The function \( f_R \) has the following properties.

Lemma 6.1: Let \( G' = (N, B_L, N_1, N_2) \) as above. If there exists in \( G' \) a path from \( n \) to \( n' \) (\( n, n' \in N \)) spelling out the word \( w \), then \( M(f_R(n), w) \subseteq f_R(n') \).

Proof: For every \( q \in f_R(n) \) there exists a word \( u \) such that \( M_1(u) = q \) and \( u \in P^G_r(n) \). Let \( q' = M(q, w) \). If \( q' = \emptyset \), then clearly \( M(q, w) = \emptyset \subseteq f_R(n') \). Otherwise \( q' = M_1(uw) \) and \( uw \in P^G_r(n') \), which implies \( q' \in f_R(n') \) as required.

† When no confusion arises, the superscript \( G' \) will be omitted.
Fig. 6.1

Fig. 6.2
Lemma 6.2: If \( G' = (N, B_L, N_1, N_2) \) recognizes \( R \) then:

(a) For any terminal node \( n \in N_2 \), \( f_R(n) \subseteq F \).

(b) If there exists in \( G' \) a path starting at a node \( m \) and spelling out the word \( w \), then \( M(q, w) \neq \emptyset \) for all states \( q \in f_R(m) \). Consequently \( f_R(m) = M_1(P_r^{G'}(m)) \neq \emptyset \) for all nodes \( m \in N \).

Proof: (a) Let \( n \in N_2 \) and let \( q \in f_R(n) \). Then there exists a word \( u \) such that \( M_1(u) = q \) and \( u \in P_r^{G'}(n) \). The latter implies that \( u \) is accepted by \( G' \) and hence \( M_1(u) = q \in F \).

(b) Suppose \( M(q, w) = \emptyset \) for some \( q \in f_R(m) \). Then \( M(q, w) = q_{\emptyset} \) and also \( \overline{M}(q, wv) = q_{\emptyset} \) for any word \( v \). Let \( n \) be the terminal node of the path starting at \( m \) and spelling out \( w \), and let \( v \in A_C^{G'}(n) \) (such a word \( v \) exists since \( G' \) is all-admissible by assumption). Let \( u \in P_r^{G'}(m) \cap P_r^{\tilde{G}}(q) \); then the word \( u w v \) is accepted by \( G' \) and hence by \( \tilde{G} \). But \( M_1(u w v) = M(M_1(u), w v) = M(q, w v) = \emptyset \), a contradiction.

Definition 6.2: Let \( \tilde{G} = G_0(R) \) and let \( G' = (N, B_L, N_1, N_2) \) be any state graph as above. Define a function \( \tilde{g}_R^{G'}: P(G') \rightarrow 2^P(\tilde{G}) \) as follows. For every path \( p' \) in \( G' \), starting at a node \( n' \) and spelling out a word \( w \), let \( \tilde{g}_R^{G'}(p') \) (or \( g_R(p') \), when the reference to \( G' \) is understood) be the set of all paths \( p \) in \( \tilde{G} \) spelling out \( w \) and starting at nodes contained in \( f_R(n') \). Define \( g_R(\varepsilon) = \varepsilon \).

Thus if \( p' = (n', a_1, n_2) \ldots (n_k, a_k, n_{k+1}) \) and
p = (q_1, c_1, q_2, ..., q_k, c_k, q_{k+1}) \text{ then } p \in g_R(p') \text{ iff } k = \ell, a_i = c_i \text{ for all } i = 1, \ldots, k \text{ and } q^1 \in f_R(n). \text{ Furthermore, by Lemma 6.1 we also have } q^i \in f_R(n_i) \text{ for all } i = 2, \ldots, k + 1. 

Clearly \( g_R(p') \) is always a finite subset of \( P(G) \); moreover, in case no path in \( G \) spells out the word \( w \), which is spelled out by \( p' \), \( g_R(p') = \emptyset \). 

Extend the function \( g_R \) to subsets \( P' \) of \( P(G') \) by:

\[
\bar{g}_R(P') = \bigcup_{p' \in P'} g_R(p')
\]

The following lemma is a direct consequence of the above definitions.

**Lemma 6.3:** Let \( p' \) be a path in \( G' \) and let \( p \in g_R(p') \). If \( p = p_1p_2 \) for some \( p_1, p_2 \in P(G) \) then there exist \( p'_1, p'_2 \) in \( P(G') \) such that \( p' = p'_1p'_2 \) and \( p_i \in g_R(p'_i), i = 1, 2. \)

Thus the functions \( f_R \) and \( g_R \) map nodes and paths in \( G' \) onto the corresponding sets of nodes and paths (resp.) in \( G_0(R) \); they will be called the \( R \)-projection functions.

Before proceeding with the definition of graph covering, we generalize the notion of subgraph as follows.

**Definition 6.3:** Let \( G_1 = (N, B_L, N_1, N_2) \) and \( G_2 = (N', B'_L, N'_1, N'_2) \) be two transition graphs. We say that \( G_2 \) is a partial graph of \( G_1 \) iff \( N' \subseteq N, B'_L \subseteq B_L \) and \( N'_i = N_i \cap N', i = 1, 2. \)

Clearly every subgraph of \( G_1 \) is also a partial graph of \( G_1 \). Moreover, if \( G_2 \) is a partial graph of \( G_1 \) then \( P(G_2) \subseteq P(G_1) \) and \( r(G_2) \leq r(G_1) \).
Definition 6.4: A partial graph $G''$ of $G'$ covers a partial graph $G_1$ of $\hat{G}_0(R)$ iff $P(G_1) \subseteq g_R^{G'}(P(G''))$. In other words, every path $p$ in $G_1$ is contained in $g_R^{G'}(p')$ for at least one path $p'$ in $G''$.

Example 6.2: Let $G'$ and $\hat{G}$ be the graphs in Fig. 6.1 and 6.2 (page 127) respectively, and let $p' = (D, 0, A)(A, 1, A)(A, 0, D)(D, 1, C)$. Then $g_R(p') = \{p_1, p_2\}$, where

$$p_1 = (q, 0, q)(q, 1, p)(p, 0, q)(q, 1, p)$$

$$p_2 = (r, 0, r)(r, 1, q)(q, 0, q)(q, 1, p)$$

The subgraph $G'' = G' - \{B, C\}$ of $G'$ can be seen to cover the subgraph $S = G - \{p, r\}$ of $G$. In fact, every path in $(q, 0, q)^*$ is contained in $g_R(P')$, where

$$P' = [(A, 0, D)(D, 0, A)]^*[(A, 0, D)] \cup \varepsilon \subseteq P(G'').$$

6.2. Two Fundamental Lemmas

Throughout the rest of this chapter it is assumed that the functions $f_R$ and $g_R$ are always those associated with the given non-deterministic state graph $G'$, and not with any of its partial graphs. Thus the superscript $G'$ will be omitted.

We now proceed with the two main lemmas of this chapter.

Lemma 6.4: Let $G'$ be a non-deterministic state graph recognizing $F$ and let $\hat{G} = \hat{G}_0(R)$. Then:

(a) $G'$ covers $\hat{G}$

(b) For every s.c. subgraph $S$ of $\hat{G}$ there exists a section $S'$ of $G'$ such that $S'$ covers $S$. 
Proof: (a) Let p be any path in \( \hat{\gamma} \). Since \( \hat{\gamma} \) is all-admissible (Definition 2.18), there exists an admissible path \( p_0 \) in \( \hat{\gamma} \) containing p as a subpath. Clearly the initial node of \( p_0 \) is the initial state \( q_1 \) of \( \hat{\gamma} \), and the word \( w_0 \) spelled out by \( p_0 \) is accepted by \( \hat{\gamma} \), and hence also by \( G' \). Thus there is in \( G' \) a path \( p' \) from an initial node \( n_1 \) to a terminal node spelling out \( w_0 \). Since \( q_1 \in f_R(n_1) \) by Lemma 6.1, we get \( p_0 \in g_R(p') \). But then there must clearly exist a subpath \( p'' \) of \( p' \) such that \( p \in g_R(p'') \). Hence \( G' \) covers \( \hat{\gamma} \).

(b) Let \( S_1, S_2, \ldots, S_m \) be all sections of \( G' \) and assume that for some s.c. subgraph S of \( \hat{\gamma} \), none of the \( S_i \)'s covers S. Then there exist paths \( p_1, p_2, \ldots, p_m \) in S such that \( g_R(P(S_i)) \) does not contain \( p_i \) for all \( i = 1, \ldots, m \). Since S is strongly connected, one can construct a loop p in S containing all \( p_i \)'s (\( i = 1, \ldots, m \)) as subpaths. Now by part (a) of this lemma, there exists for every integer \( k > 0 \), a path \( t_k \) in \( G' \), such that the path \( (p)^k \) is contained in \( g_R(t_k) \).

Let \( r \) be the total number of nodes in \( G' \) and let \( k > r \). By Lemma 6.3, there exist paths \( t'_1, t'_2, \ldots, t'_k \) in \( G' \) such that \( t_k = t'_1 t'_2 \cdots t'_k \) and \( p \in g_R(t'_i) \) for all \( i = 1, 2, \ldots, k \). Since \( k > r \), at least one of the paths \( t'_i \), \( 1 \leq i \leq k \), must be contained in a section \( S_j \) of \( G' \) (\( 1 \leq j \leq m \)). It follows that \( p \in g_R(P(S_j)) \) and since \( p_j \) is a subpath of \( p \), also \( p_j \in g_R(P(S_j)) \), which contradicts our assumption.

The proof of the next lemma resembles the proof of Theorem 9 in McNaughton's paper [Mc 4].
Lemma 6.5: Let $G'$ be any non-deterministic state graph and let $S'$ be a s.c. subgraph of $G'$ which covers some s.c. subgraph $S$ of $\tilde{G}_0(R)$. If for each node $n$ in $S'$, $f_R(n)$ contains at most one node from $S$, then $r(S') \geq r(S)$.

**Proof:** Let $G' = (N, B_L, N_1, N_2)$ and let $Q_S$ be the set of all nodes of $S$. Let $N'$ be the set of all nodes $n$ in $S'$ for which $f_R(n) \cap Q_S = \emptyset$. Since $S'$ covers $S$, there exist nodes in $S'$ which are not in $N'$. Consider the subgraph $\tilde{G} = S' - [N'] = (\tilde{N}, \tilde{B}_L, \tilde{N}_1, \tilde{N}_2)$ of $S'$. Let $\tilde{E}_L'$ be the set of all branches $(n, a, n')$ in $\tilde{G}$, such that for the state $q$ defined by:

$$\{q\} = f_R(n) \cap Q_S, M(q, a) \neq \emptyset$$

for every branch $(n, a, n')$ in $\tilde{E}_L'$, $q' = M(q, a) \in f_R(n')$.

Let $\tilde{G}'$ be the partial graph obtained from $\tilde{G}$ by removing all branches not in $\tilde{E}_L'$, i.e. $\tilde{G}' = (\tilde{N}, \tilde{E}_L', \tilde{N}_1, \tilde{N}_2)$. We claim that $\tilde{G}'$ also covers $S$. To see this, let $p = (q^1, a_1, q^2) \ldots (q^k, a_k, q^{k+1})$ be any path in $\tilde{G}$ such that $p \in \tilde{G}_R(p')$. Thus $q^i \in f_R(n_i)$ for all $i = 1, \ldots, k+1$ which shows that $\{q^i\} = f_R(n_i) \cap Q_S \neq \emptyset$ and hence $n_i \in \tilde{N}$, $i = 1, \ldots, k+1$. Moreover, for any branch $b_i = (n_i, a_i, n_{i+1})$ in $p'$, $M(q_i, a_i) = q_{i+1} \in Q_S$ and hence $b_i$ is in $\tilde{E}_L'$. Thus the whole path $p'$ is contained in $\tilde{G}'$ and therefore $\tilde{G}'$ also covers $S$.

Define a homomorphism $\phi$ from $\tilde{G}'$ onto $S$ as follows: for every $n$ in $\tilde{N}$, $\phi(n) = q$, where $\{q\} = f_R(n) \cap Q_S$. For any branch $(n, a, n')$ in $\tilde{E}_L'$, let $\phi((n, a, n')) = (\phi(n), a, \phi(n'))$. To see that the latter is indeed a branch in $S$, recall that whenever $(n, a, n')$ is in $\tilde{E}_L'$, $q' = M(q, a)$ is contained in $Q_S$, where $q = \phi(n)$, and $q' \in f_R(n')$. But then $q' = \phi(n')$ and hence the above branch is in $S$. 

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We now show that $\phi$ is a pathwise homomorphism from $\bar{G}$

to $S$. Extend $\phi$ to a mapping from $P(\bar{G}')$ to $P(S)$ (as indicated in
Definition 2.28) and let $p = (q^1, a_1, q^2) \ldots (q^k, a_k, q^{k+1})$ be any
path in $S$. Since $\bar{G}$ covers $S$, there exists a path
$p' = (n_1, a_1, n_2) \ldots (n_k, a_k, n_{k+1})$ in $\bar{G}'$ such that $q^i \in f_R(n_i)$ for all $i = 1, \ldots, k + 1$.
But then $q^i = \phi(n_i)$ for each $i = 1, \ldots, k + 1$ and consequently $p = \phi(p')$,
which shows that $\phi$ is a pathwise homomorphism. By McNaughton's Pathwise
Homomorphism Theorem, $r(S) \leq r(\bar{G}')$ and since $r(\bar{G}') \leq r(S')$ we obtain
$r(S) \leq r(S')$ as required.

---

6.3. A Proof Technique for Establishing Star Height

Definition 6.5: Let $G = G_0(R)$ and $\bar{G} = \bar{G}_0(R)$ as above. For any
state $q_i$ in $\bar{G}$, $1 \leq i \leq m_0$, define a mapping $p_i : A^* \rightarrow P_i(G)$
(notation 2.3) as follows:

(a) $p_i(\lambda) = \varepsilon$

(b) for any non-empty word $w = a_1a_2\ldots a_k$,

$p_i(w) = (q_i, a_1, q^1)(q^1, a_2, q^2) \ldots (q^{k-1}, a_k, q^k),$

where $q^j = M(q_i, a_1 \ldots a_j)$ for all $j$, $1 \leq j \leq k$.

Thus $p_i$ maps every word $w$ to the path in $G$ starting at $q_i$
and spelling out $w$.

For any event $W \subseteq A^*$, let

$p_i(W) = \{p_i(w) \mid w \in W\} \subseteq P_i(G).$
Definition 6.6: For any subset \( B \) of the set of states \( Q \) of \( G \), define two sets of paths of \( G \):

\[
L_B = \bigcup_{q_i \in B} p_i(\text{str } B)
\]

(see Definition 1.17) and

\[
D_B = \bigcup_{q_i \in B} p_i(\bigcap_{q_j \in B} D_j)
\]

Example 6.3: Let \( \hat{G} \) be the state graph shown in Fig. 6.4 (page 139) and let \( B = \{q_1, q_2\} \). Then \( \text{str } B = 1^* \) and \( D_{q_1} \cap D_{q_2} = 1^*2(0 \cup 1 \cup 2)^* \).

Hence:

\[
L_B = [\varepsilon \cup (q_1, 1, q_2)][(q_2, 1, q_1)(q_1, 1, q_2)]*[\varepsilon \cup (q_2, 1, q_1)]
\]

and

\[
D_B = [(\varepsilon \cup (q_1, 1, q_2)][(q_2, 1, q_1)(q_1, 1, q_2)]*[ (q_2, 2, q_3) \cup (q_2, 1, q_1)

(q_1, 2, q_3) \cup (q_1, 2, q_3)]*(q_3, 0, q_3) \cup (q_3, 1, q_3) \cup (q_3, 2, q_3)]*
\]

Definition 6.7: Let \( S \) be any subgraph of a state graph \( G' \) and let \( P' \subseteq P(G') \). We say that \( P' \) covers \( S \) iff every path \( p \) in \( S \) is a subpath of at least one path \( p' \) in \( P' \).

Example 6.4: Let \( G' \) be the graph in Fig. 6.3 and let \( S = G - [t] \). Then the set of paths \( P' = [((r, 1, r)\ast(r, 0, q)(q, 0, r))^3\ast(r, 2, t)(t, 0, t)^2 \) covers \( S \).

The following lemma is an immediate consequence of the above definitions.
Lemma 6.6: Let $G'$ be any state graph and let $S'$ be a section of $G'$.
Then for every section $S$ of $\wedge = \wedge_0(R)$, $S'$ covers $S$ iff for any set
of paths $P' \subseteq P(G')$ covering $S'$, $g_R(P')$ covers $S$.

Lemma 6.7: Let $G'$ be any state graph recognizing $R$ and let $S'$ be a
section of $G'$. For any node $n$ in $S'$, let $L(n)(D(n))$ denote the set
of all paths in $G'$ from $n$ to itself (from $n$ to a terminal node of $G'$).
Then $g_R(L(n)) \subseteq L_B$ and $g_R(D(n)) \subseteq D_B$, where $B = f_R(n)$.

Proof: Let $p = (q^1, a_1, q^2) \ldots (q^k, a_k, q^{k+1})$ be a path in $g_R(L(n))$.
Let $p' = (n, a_1, n_2)(n_2, a_2, n_3) \ldots (n_k, a_k, n)$ be a loop in $L(n)$
such that $p \in g_R(p')$. Then by definition both $q^1$ and $q^{k+1}$ belong to
$B = f_R(n)$. Furthermore, for any state $q_i \in B$, the path in $\wedge$ starting
at $q_i$ and spelling out the word $w = a_1 a_2 \ldots a_k$ is contained in $g_R(p')$ and
hence the terminal node of the path must also be a member of $B$. This
implies that $w \in \text{str } B$, and since $q^1 \in B$ we have $p \in L_B$. Hence
$g_R(L(n)) \subseteq L_B$.

The proof for $g_R(D(n))$ is similar. Let $p$ be any path in
$g_R(D(n))$, with initial node $q^1$, terminal node $q^{k+1}$, spelling out the
word $w = a_1 \ldots a_k$. As before, let $p' = (n, a_1, n_2) \ldots (n_k, a_k, n_{k+1})$
be a path in $G'$ such that $p \in g_R(p')$ and $p' \in D(n)$. Then $q^1 \in f_R(n) = B
and q^{k+1} \in f_R(n_{k+1})$. Since $n_{k+1}$ is a terminal state of $G'$, by Lemma 6.2
$q^{k+1}$ is a final state of $\wedge$. Moreover, for any state $q_i \in B$, the path in
$\wedge$ starting at $q_i$ and spelling out $w$ is in $g_R(p')$ and therefore must
terminate at a final state of $\wedge$. This shows that $w \in q_i \in B \quad D_q_i$
and since $q^1 \in B$, $p \in D_B$ as required.
Theorem 6.1: Let $G = G_0(R) = (Q, M, q_1, F)$, and let $S$ be a s.c. subgraph of $G$. If for every subset $B \subseteq Q$ with at least two elements, the set $L_B$ does not cover $S$, then $h(R) \geq r(S)$. The theorem remains true if the set $L_B$ is replaced by $D_B$.

Proof: We shall prove that, under the above assumptions, every non-deterministic state graph $G'$ recognizing $R$ has rank no less than $r(S)$. The result then follows by Theorem 5.4.

Thus let $G' = (N, B_L, N_1, N_2)$ be any non-deterministic state graph recognizing $R$. By Lemma 6.4, there exists a section $S'$ of $G'$ covering $S$. We claim that for any node $n$ of $S'$, $f_R(n)$ is a singleton.

Suppose, on the contrary, that for some node $n$ in $S'$, $\#B > 1$, where $B = f_R(n)$. Let $L(n)$ and $D(n)$ be defined as in the last lemma. Apparently any path $p'$ of $S'$ is a subpath of some path in $L(n)$, and since $G'$ is assumed to be an all-admissible state graph, $p'$ is also a subpath of some path in $D(n)$. Thus each of the sets $D(n)$ and $L(n)$ covers $S'$, and since $S'$ covers $S$, Lemma 6.6 implies that both sets $g_R(L(n))$ and $g_R(D(n))$ cover $S$. But by Lemma 6.7 the former set is contained in $L_B$ as well as the latter is contained in $D_B$. It follows that both $L_B$ and $D_B$ cover $S$, which contradicts the assumption of the theorem. Hence $f_R(n)$ is a singleton for every node $n$ of $S'$, and by Lemma 6.5, $r(S') \geq r(S)$. Consequently $r(G') \geq r(S)$, which completes the proof.
Corollary 6.1: Let $\hat{G} = G_0(R)$ as above and let $S_1, \ldots, S_k$ be all sections of $\hat{G}$ such that $r(S_i) = r(\hat{G})$, $i = 1, \ldots, k$. If for some $S_i$, $1 \leq i \leq k$, there does not exist a subset $B \subseteq Q$ with at least two elements such that $L_B(D_B)$ covers $S_i$, then the star height of $R$ equals the rank of $\hat{G}$. In particular, if for every $B \subseteq Q$, $\#B > 1$, the set $\text{str} \ B \ (\cap_{q \in B} \ D_q^\dagger)$ is finite, then $h(R) = r(\hat{G})$.

Theorem 6.1 and Corollary 6.1 provide us with a powerful technique for establishing the star height of regular events. This technique will now be illustrated by the following examples.

Example 6.5: Let $\hat{G} = G_0(R)$ be the state graph shown in Fig. 6.4. One can easily verify that:

$$\text{str} \ \{q_1, q_2, q_3\} = 1^* (\lambda \cup 2(0 \cup 1 \cup 2)^*)$$

$$\text{str} \ \{q_1, q_2\} = 1^*$$

$$\text{str} \ \{q_1, q_3\} = (0 \cup 11)^* (\lambda \cup (2 \cup 12)(0 \cup 1 \cup 2)^*)$$

$$\text{str} \ \{q_2, q_3\} = (10^*1)^* (\lambda \cup (2 \cup 10^*2)(0 \cup 1 \cup 2)^*)$$

By inspection, none of the sets $L_B$, where $B = \{q_1, q_2, q_3\}$, $\{q_1, q_2\}$, $\{q_1, q_3\}$ or $\{q_2, q_3\}$, covers the s.c. subgraph $S = \hat{G} - [q_3]$ (no path of $S$ containing the branch $(q_2, 3, q_2)$ is included in these sets).

Thus by Corollary 6.1, $h(R) = r(S) = 2$.

Example 6.6: Consider the state graph $\hat{G}$ shown in Fig. 6.5, and let $S = \hat{G} - \{\{q_3, q_4\}\}$. One can easily see that for $B = \{q_1, q_2\}$, $\text{str} \ B = (0 \cup 1)^*$ and thus $L_B$ covers $S$. However, $D_{q_1} \cap D_{q_2} = \emptyset$ by inspection and therefore

$\dagger$ This is the case when $R$ has the finite intersection property, which has also been covered by Theorem 4.1.
D_B does not cover S. Furthermore, for any other subset B' \subseteq \{q_1, q_2, q_3, q_4\} with \#B' > 1, \bigcap_{q \in B'} D does not contain any word with the subword 010 and thus D_B' cannot cover S. Hence by Corollary 6.1 the event accepted by \( \hat{G} \) has star height 2.

Example 6.7: Let \( \hat{G} = \hat{G}_0(R) \) be the state graph shown in Fig. 6.6. By inspection, the section S = \( \hat{G} - \{q_1, q_5, q_6\} \) is the only section of rank \( r(S) = r(\hat{G}) = 3 \). However, Corollary 6.1 cannot be applied directly to R since both sets L_B and D_B, where B = \{q_1, q_2\}, cover S. Thus consider the event \( R' = 0 \setminus R / 1 \) (Definition 1.2). Since the only branches entering the final states q_5, q_6 and labelled by 1 are \( (q_3, 1, q_5) \) and \( (q_4, 1, q_6) \), \( R' \) is exactly the set of all words spelled out by paths from q_2 to either q_3 or q_4. Thus the state graph S, with initial state q_2 and final states q_3 and q_4, recognizes \( R' \), and since it is reduced we have \( S = \hat{G}_0(R') \). Now one can easily show that for any \( B \subseteq \{q_2, q_3, q_4\} \), with at least two elements, str \( B = \Lambda \). Hence by Corollary 6.1, \( h(R') = 3 \). But \( h(R') = h(0 \setminus R / 1) \leq h(R) \) by Corollary 3.1, and since \( r(\hat{G}) = 3 \) we conclude that \( h(R) = 3 \).

Consider any set of paths \( P' \subseteq P(\hat{G}) \) in \( \hat{G} \) as an event over the alphabet of labelled branches of \( \hat{G} \). Every path \( p \) in \( \hat{G} \) represents a word over this alphabet. Thus let \( P' / p \) denote the right quotient (Definition 1.2) of \( P' \) over the path \( p \), i.e.

\[
P' / p = \{ p' \in P(G) \mid p'p \in P' \}
\]

We now present an extension of Theorem 6.1.
Fig. 6.6
Theorem 6.2: Let \( \hat{G} = \hat{G}_0(R) \) as above, let \( S \) be any section of \( \hat{G} \) and let \( p \) be any path leading from a state of \( S \) to a final state of \( \hat{G} \). If for no subset \( B \leq Q \), \( |B| > 1 \), the set \( D_B / p \) covers \( S \), then \( h(R) \geq r(S) \).

Proof: Let \( Q_S \) be the set of states of \( S \), let \( G' = (N, B_L, N_1, N_2) \) be any non-deterministic state graph recognizing \( R \) and let \( S_1, S_2, \ldots, S_m \) be all sections of \( G' \) covering \( S \) (there exists at least one by Lemma 6.4).

If for one of these sections \( S_i \), \( f_R(n_i) = 1 \) for all nodes \( n \) of \( S_i \), then by Lemma 6.5 \( r(S_i) \geq r(S) \) and the result follows. Thus suppose on the contrary, that every \( S_i, 1 \leq i \leq m \), contains a node \( n_i \) such that \( f_R(n_i) > 1 \), and let \( f_R(n_i) = B_i \). By the assumption of the theorem, \( D_{B_i} / p \) does not cover \( S \) for all \( i = 1, \ldots, m \). Then there exist paths \( p_i \) in \( S \) such that \( p_i \) is not the subpath of any path in \( D_{B_i} / p \), \( i = 1, \ldots, m \). Let \( S_{m+1}, \ldots, S_{m+h} \) be all the remaining sections of \( G' \); since none of these covers \( S \), there exist paths \( r_1, \ldots, r_h \) in \( S \) such that \( r_i \notin g_R(P(S_{m+i})) \) for all \( i = 1, \ldots, h \).

Thus construct a loop \( \lambda \) in \( S \) from the initial node \( q^0 \) of \( p \) to itself, such that each \( p_i, i = 1, \ldots, m \) and each \( r_j, j = 1, \ldots, h \), is a subpath of \( \lambda \). Let \( p_0 \) be any path in \( \hat{G} \) from \( q_1 \) to \( q^0 \). Then for any integer \( k > s \), where \( s \) denotes the total number of nodes in \( G' \), the path \( p' = p_0(\lambda)^k p \) is an admissible path in \( \hat{G} \). Thus there exists an admissible path \( t' \) in \( G' \) spelling out the same word as \( p' \), and since \( q_1 \in f_R(n) \) for any \( n \in N_1 \), we have \( p' \in g_R(t') \). By Lemma 6.3, \( t' = t_0t_1\cdots t_k \) such that \( p_0 \in g_R(t_0) \), \( \lambda \in g_R(t_i) \) for \( i = 1, \ldots, m \) and \( p \in g_R(t) \). Since \( k > s \), at least one \( t_j, 1 \leq j \leq k \), must be contained in some section \( S' \) of \( G' \). Now suppose \( S' = S_i \) for some \( i, 1 \leq i \leq m \). Let \( n \) be the initial node of \( t_j \) and let \( t'' \) be a path from \( n_1 \) to \( n \). Then the path \( t^1 = t'' t_jt_{j+1} \cdots t_kt \) is a path
from \( n_i \) to a terminal node of \( G' \), and hence is contained in \( D(n_i) \).

But then, by Lemma 6.7, \( g_R(t_i) \in D_{B_i} \) and hence \( p''(x)k^{-j+1}p \in D_{B_i} \) for some path \( p'' \in g_R(t'') \) terminating at \( q_0 \). It follows that \( p''(x)k^{-j+1}p \in D_{B_i} / p \) and since \( p_i \) is a subpath of \( x \), \( p_i \) is a subpath of the above path, which is contained in \( D_{B_i} / p \). But this contradicts our assumption.

Thus suppose that \( S' = S_{m+i} \) for some \( i = 1, ..., h \). Then since \( x \in g_R(t_j) \), the subpath \( r_i \) of \( x \) is contained in \( g_R(t'_j) \) for some subpath \( t'_j \) of \( t_j \). But \( t'_j \) is a path in \( S_{m+i} \) and this contradicts our assumption that \( r_i \notin g_R(P(S_{m+i})) \).

Hence \( \#f_{R}(n) = 1 \) for all nodes of at least one section \( S_i \), \( 1 \leq i \leq m \), and then the result follows by Lemma 6.5 as indicated above.

Example 6.8: Let \( \mathcal{G}_0(R) \) be the state graph in Fig. 6.7. One can verify that \( \text{str} \{ m, n \} = \text{str} \{ m, p \} = \text{str} \{ n, p \} = \text{str} \{ m, n, p \} = (0 \cup 1)^* \), and hence for all these sets, \( L_B \) covers the section \( S = \mathcal{G}_0 - [q] \). Also for any of these sets \( B \), \( \cap_{q_i \in B} D_{q_i} = (0 \cup 1)^*2(0 \cup 2 \cup 3)^* \), and hence \( D_B \) also covers \( S \). However, if we consider the path \( p = (n, 3, q) \), then none of the sets \( D_B / p, B \leq Q, \#B > 1 \), covers \( S \). Hence \( h(R) = r(S) = 2 \).
6.4. Results on Reset-Free State Graphs

In this section the state graph $\tilde{G} = \tilde{G}_0(R) = (Q, M, q_1, F)$ is always assumed to be reset-free (Definition 4.10), unless otherwise specified.

Lemma 6.8: If $\tilde{G} = \tilde{G}_0(R)$ as above and if $G'$ is any non-deterministic state graph recognizing $R$, then for any pair of nodes $n, n'$ belonging to one section of $G'$, the following conditions must hold:

(a) $\#f_R(n) = \#f_R(n')$.

For every section $S$ of $\tilde{G}_0(R)$ with set of states $Q_S$,

(b) $\#(f_R(n) \cap Q_S) = \#(f_R(n') \cap Q_S)$

(c) If $u$ is a word spelled out by some path from $n$ to $n'$, then $M((f_R(n) \cap Q_S), u) = f_R(n') \cap Q_S$

Proof: (a) Since $n$ and $n'$ belong to one section of $G'$, there exists a word $u$ spelled out by some path $p$ from $n$ to $n'$, and a word $v$ spelled out by a path $p'$ from $n'$ to $n$. By Lemma 6.1, $M(f_R(n), u) \subseteq f_R(n')$ and $M(f_R(n'), v) \subseteq f_R(n)$. Furthermore, since $\tilde{G}$ is reset-free and since, by Lemma 6.2 (b), $M(q, u) \neq \emptyset$ for any state $q$ in $f_R(n)$ and $M(q', v) \neq \emptyset$ for any $q' \in f_R(n')$, we have $\#f_R(n) = \#M(f_R(n), u) \leq \#f_R(n') = \#M(f_R(n'), v) \leq \#f_R(n)$, and equality follows.

(c) We first claim that for any subset $B \subseteq f_R(n)$ and for any word $w$ spelled out by a path in $G'$ starting at $n$, $\#M(B, w) = \#B$. This follows again from Lemma 6.2 (b) and the fact that $\tilde{G}$ is reset-free.
Now let $B = f_R(n) \cap Q_S$ and $B' = f_R(n') \cap Q_S$, and let $u$ and $v$ be words spelled out by paths $p$ and $p'$ as above. By Lemma 6.1, $M(\text{B}, u) \subseteq f_R(n')$ and $M(\text{B'}, v) \subseteq f_R(n)$. If $B' \neq M(\text{B}, u)$ then there are two cases: (i) $M(\text{B}, u) \subseteq B'$, or (ii) $M(\text{B}, u) \nsubseteq B'$. In case (i) we get: $#B = #M(\text{B}, u) < #B' = #M(\text{B'}, v)$; it follows that $M(\text{B'}, v) \nsubseteq B$, which, by exchanging $B$ and $B'$, $u$ and $v$, corresponds to case (ii). Thus we may assume that (ii) $M(\text{B}, u) \nsubseteq B'$. Then, since $M(\text{B}, u) \nsubseteq f_R(n')$, we get $M(\text{B}, u) \nsubseteq Q_S$. Thus there exists a state $q^1$ in $B$ such that $M(q^1, u) \nsubseteq Q_S$, and then also $q^2 = M(q^1, uv) \nsubseteq Q_S$. Let $q^i = M(q^1, (uv)^{i-1})$, $i = 1, 2, ...$ (by Lemma 6.2(b), none of the sets $M(q^1, (uv)^{i-1})$ is empty). Then $q^1 \neq q^2$ and we claim that $q^i \neq q^j$ for all $0 < i < j$. Clearly for $j > 1$, $q^j \nsubseteq Q_S$ and thus $q^1 \neq q^j$. Now suppose, by contradiction, that $k$ is the smallest integer such that for some $m > k$, $q^k = q^m$. Then $k > 1$ and $q^{k-1} \neq q^{m-1}$; but $M(q^{k-1}, uv) = M(q^{m-1}, uv) = q^k$, which contradicts the fact that $\gamma$ is reset-free. Hence $q^i \neq q^j$ for all $0 < i < j$, which is of course impossible. Consequently $M(\text{B}, u) = B'$, which proves (c).

(b) Follows immediately from (c) and the above proof.

Theorem 6.3: If $\gamma = \gamma_0(R) = (Q, M, q_1, F)$ is reset-free, and if for some state $q$ in a section $S$ of $\gamma$, $[D_q - (\cup_{q' \in Q \setminus q} D_{q'})] \neq \emptyset$, then $h(R) \geq r(S)$. 

Proof: Let $G' = (N, B_L, N_1, N_2)$ be any non-deterministic state graph recognizing $R$ and assume $r(G') = h(R)$. Let $S_1, ..., S_m$ be all sections of $G'$ which do not cover $S$. Then there clearly exist paths $p_1, ..., p_m$ in $S$ such that $p_i \notin g_R(P(S_i))$ for all $i = 1, ..., m$. Now let

$w_0 \in D_0 - (\cup_{q \neq q' \in Q} D_{q'}),$, and let $p_0$ be the path in $G$ starting at $q$, terminating at a final state and spelling out $w_0$. Since all paths $p_i, i = 1, ..., m$, are in the section $S$, one can construct in $S$ a loop $\ell$ from $q$ to itself, containing each $p_i, i = 1, ..., m$, as subpath. Let $s$ be the total number of nodes in $G'$ and let $k$ be any fixed integer greater than $s$. Since $G$ is all-admissible, there exists in it a path $p'$ from the initial state $q'_1$ to $q$. Now define the path $p = p'(\ell)^k p_0$. Clearly $p$ is an admissible path in $G$. Thus the word $u$ spelled out by $p$ is in $R$, and there must exist in $G'$ an admissible path $t$ spelling out the same word $u$. Let $n_0(n_t)$ be the initial (terminal) node of $t$. Then

$n_0 \in N_1, n_t \in N_2$ and by Lemma 6.2, $q_1 \in f_R(n_0)$ and $f_R(n_t) \subseteq F$. Hence $p \in g_R(t)$. By Lemma 6.3, there exists a decomposition of $t$ into subpaths:

$t = t'_1 t'_2 ... t'_k t_0$ such that $\ell \in g_R(t'_i)$ for all $i = 1, ..., k, p' \in g_R(t'_i)$ and $P_0 \in g_R(t'_0)$. But since $k > s$, at least one of the subpaths $t_j (1 \leq j \leq k)$ must be contained in some section of $G'$. Let this section be $S'$. Suppose $S' = S_i$ for some $i, 1 \leq i \leq m$. Then, since $\ell \in g_R(t'_i)$ and since $p_i$ is a subpath of $\ell$, there exists a subpath $t'_j$ of $t_j$ such that $P_i \in g_R(t'_j)$. But this contradicts our assumption that $p_i \notin g_R(P(S_i))$. Hence $S' \neq S_i$ for all $i = 1, ..., m$ and consequently $S'$ covers $S$.

We claim that for all nodes $n$ of $S'$, $f_R(n)$ is a singleton.
Suppose this is not so; then by Lemma 6.8 (a) there exists some fixed integer \( h \geq 2 \) such that for each node \( n \) in \( S' \), \( f_R(n) \) has cardinality \( h \). Let \( n_1 \) be the initial node of the path \( t_j \). Since \( \ell \in s_R(t_j) \) and \( \ell \) starts at \( q \), we have \( q \in f_R(n_1) \), and since \( n_1 \) is in \( S' \), \( \# f_R(n_1) \geq 2 \).

Let \( n_k \) be the terminal node of \( t_k \); then again \( q \in f_R(n_k) \). Furthermore, by Lemma 6.1, \( M(f_R(n_1), w) \subseteq f_R(n_k) \), where \( w \) is the word spelled out by the path \( t_j, t_j+1, \ldots, t_k \). But since \( \mathcal{G} \) is reset-free, \( f_R(n_1) \) cannot be mapped by any such input \( w \) into a set of smaller cardinality (see proof of Lemma 6.8 (c)); thus \( \# f_R(n_k) \geq 2 \). Then there exists in \( f_R(n_k) \) at least one more state \( q' \neq q \). Now since \( p_0 \) spells out the word \( w_0 \), we have, again by Lemma 6.1, \( M(f_R(n_k), w_0) \subseteq f_R(n_k) \). But the latter is contained in \( F \) and hence \( M(q', w_0) \subseteq M(f_R(n_k), w_0) \subseteq F \). This shows that \( w_0 \notin D_q \), which contradicts our assumption that \( w_0 \notin D_q \), for any \( q' \neq q \). We conclude that \( f_R(n) \) is a singleton for all nodes \( n \) in \( S' \). Combining this with the fact that \( S' \) covers \( S \), we get \( r(S') \geq r(S) \) by Lemma 6.5. Hence \( h(R) = r(G') \geq r(S') \geq r(S) \) as required.

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**Example 6.9:** Let \( \mathcal{G} = G_0(R) \) be the state graph in Fig. 6.8. One can verify that \( \mathcal{G} \) is reset-free strongly connected of rank 3. Moreover, the word \( w = 21 \) belongs to \( D_q^3 \), but is not contained in \( D_q^i \) for \( i = 1, 2, 4 \). Hence by the last theorem and Theorem 3.1, \( h(R) = 3 \).
Theorem 6.4: Let $\mathcal{G} = \mathcal{G}_0(R)$ be reset-free and let $S$ be a section of $\mathcal{G}$. If for some state $q$ in $S$ and for some word $w_0$, $\bar{M}(q, w_0)$ is in $S$ but $\bar{M}(q', w_0)$ is not in $S$ for all other states $q'$ of $S$, then the star height of $R$ is no less than $r(S)$.

Proof: Let $G'$ be a non-deterministic state graph recognizing $R$ and assume $h(R) = r(G')$. By Lemma 6.4 there exists a section $S'$ of $G'$ such that $S'$ covers $S$. We claim that for any node $n$ in $S'$, $|f_R(n) \cap Q_S| = 1$, where $Q_S$ denotes the set of all states of $S$. By Lemma 6.8 (b) all such sets $f_R(n) \cap Q_S$ (where $n$ is in $S'$) have the same cardinality, say $k$. Apparently if $k = 0$ then $S'$ cannot cover $S$. Thus suppose $k \geq 2$. Let $p$ be the path in $S$ starting at $q$ and spelling out $w_0$. Then there exists in $S'$ a path $p'$ such that $p \in g_R(p')$. Let $n'(n'')$ be the initial (terminal) node of $p'$. Then clearly $q \in f_R(n')$ and the sets $B' = f_R(n') \cap Q_S$ and $B'' = f_R(n'') \cap Q_S$ have $k \geq 2$ elements each. Furthermore, by Lemma 6.8 (c), $B'' = \bar{M}(B', w_0)$. But this is impossible since for all states $q'$ in $B'$ except $q$, $\bar{M}(q', w_0)$ is not contained in $Q_S$ by the assumption of the theorem. Hence $k = 1$ and the result follows immediately from Lemma 6.5.

Example 6.10: Let $\mathcal{G} = \mathcal{G}_0(R)$ be the state graph shown in Fig. 6.9 and let $S = \mathcal{G} - \{s, t\}$. One can see that $\bar{M}(p, 2) = r$ is in $S$ whereas $\bar{M}(q, 2) = s$ and $\bar{M}(r, 2) = q_\phi$ are not in $S$ ($q_\phi$ is not shown in the graph). Hence $h(R) \geq r(S) = 2$ and since $r(\mathcal{G}) = 2$, we get $h(R) = 2$. 
Fig. 6.9
Definition 6.8: Let $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_0(R)$, let $S$ be any subgraph of $\tilde{\mathcal{G}}$, and let its set of states be $Q_S$. $S$ is called permutation-free iff for any subset $Q' \subseteq Q_S$ with $\#Q' > 1$ and for any word $w$, the mapping $m_w: q \mapsto \tilde{M}(q, w)$, $q \in Q'$, is not a cyclic permutation of $Q'$ onto itself (see Definition 1.20).

Lemma 6.9: Let $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_0(R)$ be as above and let $S$ be a permutation-free section of $\tilde{\mathcal{G}}$. Then there exists a word $w_0$ and a state $q$ in $S$ such that $\tilde{M}(q, w_0)$ is in $S$ whereas $\tilde{M}(q', w_0)$ is not in $S$ for all other states $q'$ in $S$.

Proof: If $S$ consists of a single state then the lemma is trivial. Thus denote the set of states of $S$ by $Q_S$ and suppose $\#Q_S \geq 2$. We first prove the following assertion:

(*) For any two states $q_1$, $q_2$ in $Q_S$, there exists a word $w$ such that either $\tilde{M}(q_1, w) \in Q_S$ and $\tilde{M}(q_2, w) \notin Q_S$ or $\tilde{M}(q_1, w) \notin Q_S$ and $\tilde{M}(q_2, w) \in Q_S$.

To prove (*), assume, on the contrary, that for some pair of states $q_1$, $q_2$ in $S$, $\tilde{M}(q_1, w) \in Q_S$ iff $\tilde{M}(q_2, w) \in Q_S$ for all words $w$. Let $x$ be a word such that $\tilde{M}(q_1, x) = q_2$ (such a word clearly exists since $q_1$ and $q_2$ belong to the same section $S$). By assumption, $q_3 = \tilde{M}(q_2, x) = \tilde{M}(q_1, x^2)$ is also contained in $Q_S$, and inductively, $q_i = \tilde{M}(q_2, x^{i-2}) = \tilde{M}(q_1, x^{i-1}) \in Q_S$ for all $i = 1, 2, \ldots$. 

Now we claim that for all $i, j$, $1 \leq i < j < \infty$, $q_i \neq q_j$. For suppose $q_i = q_j$ for some $i, j$, $i < j$. If $j = i + 1$ then $i > 1$ and $\tilde{M}(q_{i-1}, x) = q_i = q_{i+1} = \tilde{M}(q_i, x)$ which shows that $q_{i-1}$ and $q_i$ are both mapped by $x$ to $q_i$. Since we may assume $q_{i-1} \neq q_i$ (by taking $i$ to be the
smallest integer for which \( q_i = q_{i+1} \), it follows that \( x \) maps two
distinct states onto the same state, contrary to the assumption that
\( Q \) is reset-free.

Thus suppose \( j - i \geq 2 \). Then the word \( x \) induces on
\( \{q_i, q_{i+1}, \ldots, q_{j-1}\} \) the non-trivial cyclic permutation:

\[
q_i \not\sim q_{i+1} \not\sim \ldots \not\sim q_{j-1} \not\sim q_i
\]

contradicting the fact that \( S \) is permutation-free. Thus \( q_i \neq q_j \) for
all positive integers \( i, j, i \neq j \), which is clearly impossible. Hence
(*) must hold.

Now let \( w_1 \) be a word such that for some pair of states
\( q_1, q_1' \in Q_S, \tilde{M}(q_1, w_1) \in Q_S \) whereas \( \tilde{M}(q_1', w_1) \notin Q_S \). Define

\[
Q^1 = \{q \mid q \in Q_S \land \tilde{M}(q, w_1) \in Q_S\}
\]

and \( Q_1 = \tilde{M}(Q^1, w_1) \). Then \( 0 < \#Q^1 < \#Q_S \) and \( \#Q^1 = \#Q_1 \) because \( S \) is
reset-free. If \( \#Q_1 > 1 \) then there exist two states \( q_2, q_2' \) in \( Q_1 \) and
a word \( w_2 \) such that \( \tilde{M}(q_2, w_2) \in Q_S \) and \( \tilde{M}(q_2', w_2) \notin Q_S \). Define

\[
Q^2 = \{q \mid q \in Q_1 \land \tilde{M}(q, w_2) \in Q_S\}
\]

and \( Q_2 = \tilde{M}(Q^2, w_2) \). Then \( 0 < \#Q^2 = \#Q_2 < \#Q_1 \) as before. By induction,
for any \( i = 2, 3, \ldots \), such that \( \#Q_{i-1} > 1 \), let \( w_i \) be a word such that
the set

\[
Q^i = \{q \mid q \in Q_{i-1} \land \tilde{M}(q, w_i) \in Q_S\}
\]
is a non-empty proper subset of \( Q_{i-1} \). Then \( 0 < \#Q^i = \#Q_i < \#Q_{i-1} \), where
\( Q_i = \tilde{M}(Q^i, w_i) \). It follows that for some positive integer \( k \), \( \#Q_k = 1. \)
Thus let $Q_k = \{q^k\}$, where $q^k \in Q_S$. By definition of the sets $Q^i$ and $Q_i$, for every $q' \in Q_i$ (1 < i ≤ k) there exists a state $q'' \in Q_{i-1}$ such that $\tilde{M}(q'', w_i) = q'$. It follows by induction, that for every state $q'$ in $Q_i$ as above, there exists a state $q^1$ in $Q_1$ such that $\tilde{M}(q^1, w_1 w_2 \ldots w_i) = q'$. Thus let $q$ be a state in $Q_1$ such that $\tilde{M}(q, w_1 w_2 \ldots w_k) = q^k$. Since $\tilde{G}$ is reset-free, for all other states $q'$ in $Q_3$, $q'' = \tilde{M}(q', w_1 w_2 \ldots w_k) \neq q^k$. If $q''$ is contained in $Q_S$, the definition of the sets $Q^i$ and $Q_i$ implies that $q''$ should be contained in $Q_k$, that is, should coincide with $q^k$. Hence for all states $q'$ of $Q_S$ other than $q$, $\tilde{M}(q', w_1 w_2 \ldots w_k)$ is not contained in $Q_S$. Thus $w_0 = w_1 w_2 \ldots w_k$ is the required word and the proof is completed.

Theorem 6.5: If $\tilde{G} = \tilde{G}_0(R)$ is reset-free, then

$$h(R) \geq \max \{r(S) \mid S \text{ a permutation-free section of } \tilde{G}\}$$

Proof: Theorem 6.4 and Lemma 6.9

Corollary 6.2: If $\tilde{G}_0(R)$ is both reset-free and permutation-free then

$$h(R) = r(\tilde{G}_0(R)).$$
Chapter 7

Conclusions and Further Problems

In this thesis the (restricted) star height of regular events has been studied. General properties of star height were investigated as well as some rank-non-increasing transformations on transition graphs, which led to a stronger version of Eggan's Star Height Theorem. The problem of determining the star height of a given event was attacked, both by examining the apparent star height of an arbitrary regular expression denoting the event, and by studying the rank of an arbitrary non-deterministic state graph recognizing the event. The former approach, using the concepts of cofinal sets of trails and simple regular expressions, led to some families of events, whose star height equals the rank of the corresponding reduced deterministic state graphs. The latter approach, utilizing the R-projection functions and the notion of covering of state graphs, provided a more extensive technique for determining star height and proved to be quite powerful.

We now summarize the existing methods for determining the star height of a given regular event R.

(a) If $R$ is a finite event, $h(R) = 0$. Otherwise $h(R) \geq 1$.

(b) If $R$ is an event over the single letter alphabet then $h(R) \leq 1$. 
(c) For any event \( R \), \( h(R) \leq r(G_0(R)) \)

(d) If \( R \) is a pure-group event, McNaughton's algorithm
[Mc 4] can be used to determine \( h(R) \). If, in addition,
\( G_0(R) \) has a single output state, then \( h(R) = r(G_0(R)) \).

(e) If \( R \) has the finite intersection property, then \( h(R) = r(G_0(R)) \)
by Theorem 4.1.

(f) If there exists in \( G_0(R) \) a reset-free section \( S \) of rank
greater than 1, then any of the theorems 6.1, 6.2, 6.5
may be applicable to \( R \).

(g) If \( G_0(R) \) is itself reset-free with more than one final
state, Theorems 6.3, 6.4 and Corollary 6.2 should be tried.

(h) If none of (a) - (g) applies to \( R \), one should try to apply
the graph-theoretical technique introduced in [Mc 3].
Alternatively, find an event \( R' \supseteq R \) such that Theorem 4.1
can be applied to \( G_0(R') \) and \( R \) (see Example 4.8). In some
special cases, \( R \) might belong to the families of events of
arbitrarily large star height constructed in either \([E]\) or
[Mc 3].

Unfortunately, the above-mentioned techniques for determining
the star height of a regular event do not apply to all cases. To the
best of our knowledge, the existence of a general algorithm for finding
the star height of any given event has not yet been established. We do
believe that such an algorithm exists, and would like to present some
further conjectures.
Definition 7.1: Let $\hat{G} = \hat{G}_0(R) = (Q, M, q_1, F)$. For any integer $k > 0$, define the subset graph $\hat{G}_k^v(R)$ of order $k$ of $R$ as follows:

$$\hat{G}_k^v(R) = (N_k^v, B_L^k, N_1^k, N_2^k)$$

where

$$N_k^v = \{Q_{(i)}' | \emptyset \neq Q' \subseteq Q, 1 \leq i \leq k\}$$

$$B_L^k = \{(Q_{(i)}', a, Q''_{(j)}) | 1 \leq i, j \leq k, a \in A, M(Q', a) \subseteq Q''\}$$

$$N_1^k = \{Q_{(i)}' | 1 \leq i \leq k, q_1 \in Q'\}$$

$$N_2^k = \{Q_{(i)}' | 1 \leq i \leq k, Q' \subseteq F\}$$

Thus $N_k^v$ consists of $k$ duplicates of each non-empty subset of $Q$, and there is a branch labelled by $a \in A$ from any duplicate of a set $Q'$ to any duplicate of a set $Q''$ iff $Q'$ is mapped by the input $a$ into $Q''$. The initial nodes are all duplicates of subsets $Q'$ containing $q_1$ and the terminal nodes are all duplicates of subsets $Q'$ contained in $F$.

One can easily verify, that any partial graph of $\hat{G}_k^v(R)$ ($k = 1, 2, \ldots$) recognizes a subset of $R$.

Furthermore, if $G' = (N, B_L, N_1, N_2)$ is any non-deterministic state graph recognizing $R$, then the $R$-projection function $f_R$ associates with each node $n$ of $G'$, a subset $f_R(n)$ of $Q$ (which from now on will be called the $R$-projection set of $n$). By Lemma 6.1, $M(f_R(n), a) \subseteq f_R(n')$ whenever $(n, a, n')$ is a branch in $G'$, and by Lemma 6.2, for all initial nodes $n \in N_1$, $f_R(n)$ contains $q_1$ and for all terminal nodes $n' \in N_2$, $f_R(n') \subseteq F$. It follows that every non-deterministic state graph recognizing $R$ can be
regarded as a partial graph of \( \overline{G}_k(R) \), for a large enough \( k \).

Now consider any state graph \( G' = (N, B_L, N_1, N_2) \) recognizing \( R \), and suppose that two nodes \( n \) and \( n' \) of \( G' \) have the same \( R \)-projection set, i.e., \( f^R_R(n) = f^R_R(n') \). One can easily verify that the graph \( G'' = (N', B'_L, N'_1, N'_2) \) obtained from \( G' \) by a full homomorphism \( \phi \), which merges \( n \) and \( n' \) into some new node, say \( m \), and leaves all other nodes fixed, as well as all initial and terminal nodes, is equivalent to \( G' \). Furthermore, for every node \( n \) of \( G' \), \( f^G_R(n) = f^G_R(\phi(n)) \).

By repeating this procedure for all pairs of nodes with equal \( R \)-projection sets, one finally obtains an equivalent state graph \( \overline{G} = (\overline{N}, \overline{B}_L, \overline{N}_1, \overline{N}_2) \) in which \( f^R_R(n) \neq f^R_R(n') \) for all \( n, n' \in \overline{N} \). However, \( \overline{G} \) might have higher rank than \( G' \). Although in most cases the constructions presented in Chapter 5 enable us to obtain from \( G' \) an equivalent graph \( \overline{G} \) of no greater rank, whose \( R \)-projection sets are all distinct, this is not always the case. Two examples will now be presented, one, in which such a construction is possible and another one, in which it is not.

Note that a state graph \( G' \) all of whose \( R \)-projection sets are distinct, can be regarded as a partial graph of the subset graph \( \overline{G}_1(R) \) of order 1 of \( R \).

**Example 7.1:** Let \( \overline{G}_0(R) \) and \( G' \) be the graphs shown in Fig. 7.1 and Fig. 7.2 respectively. The \( R \)-projection sets of the nodes of \( G' \) are indicated inside the circles representing the nodes. One can easily verify that \( G' \) is equivalent to \( \overline{G}_0(R) \) (in fact both graphs accept the event \((0 \cup 1)^*1 \)). Moreover, \( f^R_R(n_2) = f^R_R(n_3) = \{p, q\} \). Thus the full
homomorphism $\phi$ merging $n_2$ with $n_3$ can be carried out, yielding the
equivalent graph $\tilde{G}$ shown in Fig. 7.3. One can see that $r(\tilde{G}) = r(G') = 1$
(alternatively, one can use Lemma 5.14 to remove node $n_2$ from the graph $G'$
without affecting its behaviour. The resulting graph would be $G' - \{n_2\}$).
$\tilde{G}$ constitutes a state graph of minimum rank accepting $R$, in which all
$R$-projection sets are distinct. Thus $\tilde{G}$ is a partial graph of $\hat{G}_1(R)$.

**Example 7.2** (farther-fetched after McNaughton's Far-Fetched Example[Mc 3]):
Consider the graphs in Figs. 7.4 and 7.5 respectively. To verify that $G'$
is equivalent to $\hat{G}(=\hat{G}_0(R))$, let $w \in R$, and let $p = p_1(w)$ be the admissible
path in $\hat{G}$ spelling out $w$. There are two cases:

**Case (i):** $p$ does not contain the subpath $(q, 1, q)^2$. In that case there
exists a path $p'$ in $G_1 = G' - \{n_1, n_2\}$ such that $p = g_R(p')$. To see
this, note that $G_1$ resembles $\hat{G}$, except for the missing 1-self-loop on $q$,
which is replaced by another "duplicate" of $q$ and a 1-transition from one
duplicate to the other. This modification causes the graph $G_1$ to be of
rank 1, although $\hat{G}$ has rank 2.

**Case (ii):** $p$ contains the subpath $(q, 1, q)^2$. In that case one can
express $p$ as $p = p_1 \cdot (q, 1, q)^2 \cdot p_2$, where $p_2$ does not contain $(q, 1, q)^2
as a subpath. Since the graph $G_2 = G' - \{n_2, n_3, n_4, n_5, n_6\}$ obviously
covers $\hat{G}$, one can find a path $p'_1$ in $G_2$ such that $p_1 \in g_R(p'_1)$. As for the
path $(q, 1, q)^2$, it is certainly contained in $g_R((n_1, 1, n_2)(n_2, 1, n_3))$.
Also $p_2$, which starts at $q$ and does not contain $(q, 1, q)^2$, is clearly
equal to $g_R(p'_2)$ for some path $p'_2$ in $G_1$ starting at $n_3$. Hence the path
$p' = p'_1 \cdot (n_1, 1, n_2)(n_2, 1, n_3) \cdot p'_2$ is an admissible path in $G'$ and
Fig. 7.4

Fig. 7.5
spells out \( w \).

It can be easily verified that \( G' \) does not accept any word not in \( R \). Thus \( G' \) is equivalent to \( \tilde{G} \) and since \( r(G') = 1 \), we conclude that \( h(R) = 1 \). Moreover, one can verify (by exhausting all cases) that there does not exist any non-deterministic state graph of rank 1 recognizing \( R \), for which all \( R \)-projection sets \( f_R(n) \) are distinct. In other words, no partial graph of \( \tilde{G}_{1}(R) \) of rank 1 accepts the whole event \( R \).

Furthermore, for any integer \( t > 0 \), an example of an event \( R_t \) of similar nature can be constructed, such that every non-deterministic state graph of minimum rank recognizing \( R_t \) has \( t \) duplicates of at least one \( R \)-projection set \( f_{R_t}(n) \), i.e. no partial graph of \( \tilde{G}_{t-1}(R_{t-1}) \) recognizing \( R_t \) has rank \( h(R_t) \). However in those examples, the number of states in \( \tilde{G}_0(R_t) \) grows together with \( t \). It seems that for a given number of states, say \( m_0 \), in the reduced state graph \( \tilde{G}_0(R) \), there exists an upper bound on the number of duplicates of a set \( f_R(u) \) necessary for constructing a minimum-rank state graph accepting \( R \). This upper bound should be a function of \( m_0 \) alone.

**Conjecture 7.1:** If \( \tilde{G}_0(R) \) has \( m_0 \) states, then there exists a partial graph \( G' \) of the subset graph \( \tilde{G}_k(R) \) of order \( k = 2^{m_0} - 1 \) such that \( r(G') = h(R) \) and \( G' \) recognizes \( R \).

As for reset-free state graphs \( \tilde{G}_0(R) \), we conjecture:

**Conjecture 7.2:** If \( \tilde{G}_0(R) \) is reset-free then there exists a partial graph of \( \tilde{G}_{1}(R) \) recognizing \( R \) of rank \( h(R) \).
We have been able to establish Conjecture 7.2 only for
reset-free state graphs with no more than two final states.

We would like to mention another interesting open problem
concerning regular events, namely the extended star height problem.
The apparent star height of an extended regular expression (i.e.
using operators $\cup$, $\cap$, $\cdot$, $\ast$) is defined in an analogous way to that
of a restricted regular expression. The extended star height $h_{ex}(R)$
of a regular event $R$ is the smallest apparent star height of all extended
regular expressions denoting $R$. Clearly $h_{ex}(R) \leq h(R)$ for all events $R$.
In many cases $h_{ex}(R)$ is indeed less than $h(R)$ (take, for instance,
$R = (0 \cup 1)^* = \emptyset$; then $h_{ex}(R) = 0$ and $h(R) = 1$); in fact, the difference
$h(R) - h_{ex}(R)$ can grow arbitrarily. In [Sch] Schützenberger has proved
that $h_{ex}(R) = 0$ if and only iff $R$ is group-free (see Theorem 1.12). This
also establishes the existence of events of extended star height 1. However,
to the best of our knowledge, nobody as yet has been able to prove that
there exists a regular event of extended star height greater than 1. We
believe that there exist events of arbitrarily large extended star height,
although other people have made contradictory conjectures.

The star height problem could also be extended to certain
families of context-free languages, which can be described by some
generalized regular expressions.

† See (1) Yntema, M. K.: "A generalized regular notation used to
define some families of context-free languages and to obtain their
closure properties and containment relations" Ph.D. dissertation,
University of Illinois, 1965. (2) Brzozowski, J. A.: "Regular-like
expressions for some irregular languages" to appear in the Proceedings
of the 1968 Ninth Annual Symposium on Switching and Automata Theory.
Appendix A

The following is a counter-example to Conjecture 3.1 (page 50), suggested by R. McNaughton.

Let \( A = \{a_1, a_2, a_3, a_4\} \), let \( R_1 \) be the set of all words over \( A \) having an even number of \( a_2 \)'s and an even number of \( a_3 \)'s, let \( T \) be the set of all words over \( A \) having an odd number of \( a_1 \)'s. Let \( R = R_1 \cup T \). Note that \( R - T \) is the set of all words over \( A \) having an even number of \( a_1 \)'s, an even number of \( a_2 \)'s and an even number of \( a_3 \)'s. Now \( h(R) = 3 \), \( h(T) = 2 \) and \( h(R - T) = 4 \).
Appendix B

The following example, due to R. McNaughton, settles the open problem suggested by Brzozowski (Page 53, line 3).

Let $A = \{a_1, a_2, a_3\}$ and let $R$ be the set of all words over $A$ having an even number of at least two of the three letters. Thus, if
$$E_1 = ((A \setminus \{a_1\}) \ast a_1(A \setminus \{a_1\}) \ast a_1 \ast (A \setminus \{a_1\}) \ast,$$
then
$$R = (E_1 \cap E_2) \cup (E_1 \cap E_3) \cup (E_2 \cap E_3).$$
Now it can be verified (using the methods introduced in Mc4) that $h(R) = 3$. Let $T = \lambda \cup a_1 \cup a_2 \cup a_3$. Then $TR = \lambda \cup a_1(E_2 \cup E_3) \cup a_2(E_1 \cup E_3) \cup a_3(E_2 \cup E_3)$. Hence $h(TR) \leq 2$. With a little more effort it can be shown that $h(TR) = 2$. Similarly, $h(RT) = 2$. 
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[Br] Brzozowski, J. A.:


[C] Cohen, R. S.:

2. Transition Graphs and the Star Height Problem. Submitted to the 1968 Ninth Annual Symposium on Switching and Automata Theory.

[C Br] Cohen, R. S., and J. A. Brzozowski:
1. On the Star Height of Regular Events. IEEE Conference Record of 1967 Eighth Annual Symposium on Switching and Automata Theory, Austin, Texas.


[Gz] Ginzburg, A:


[Mc] McNaughton, R.:


[PP] Paz, A., and B. Peleg:


ERRATUM

Page 71, lines 10T† - 16T should be replaced by:

"If \( G \) has a dead state \( q_\phi \), then clearly \( G - [q_\phi] \) is identical with \( \hat{\mathcal{C}}_0(R) \). Moreover, since \( R \) is infinite, \( h(R) \geq 1 \) and hence \( r(G - [q_\phi]) = r(\hat{\mathcal{C}}_0(R)) \geq 1 \). But since \( q_\phi \) cannot be contained in a section of rank \( > 1 \) of \( G \), it follows that \( r(G - [q_\phi]) = r(G) = r(\hat{\mathcal{C}}_0(R)) \). If \( G \) has no dead state then \( G \) itself coincides with \( \hat{\mathcal{C}}_0(R) \). In both cases, since \( R \) has the f.i.p. by the last lemma, we get \( h(R) = r(\hat{\mathcal{C}}_0(R)) = r(G) \) by Theorem 4.1."

Page 92, line 9B† should be replaced by:

"Case (c): \( n_1 \) and \( n_2 \) are both contained in a section \( S \) of \( D \) and \( n_1(n_2) \) is a cycle center of \( D \). Then by Lemma 5.6 condition ..."

Page 107, line 4B should be replaced by:

"... some \( k \geq 1 \). We may clearly assume that \( m_1 \neq n \). By Proposition 2.4:"

† 10T: 10-th line from top

9B: 9-th line from bottom