Distortional Lateral Torsional Buckling Analysis of Beams with Overhangs

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Abstract

The present study investigates the effect of web distortion on the lateral torsional buckling strength of Gerber systems. Towards this goal, a number of modifications are introduced into two finite element formulations for the distortional and non-distortional lateral torsional buckling analysis. The distortional formulation treats the web as a thin plate and the flanges as Gjelsvik members, and captures load height effects. The non-distortional formulation is based on the Vlasov beam kinematics and enables the enforcement of lateral restraints offset from the shear center while preserving the positive definiteness of the stiffness matrices. Both models are validated against shell finite element solutions and then utilized to develop moment gradient coefficients for Gerber beams, assess the web distortional effects, and quantify the influence of various lateral bracing scenarios, on the elastic lateral torsional buckling strength. Unlike rolled simply supported beams where web distortion is considered to be insignificant, the present study indicates that web distortion heavily influences the lateral torsional buckling strength of Gerber beams.

Keywords: Lateral torsional buckling, distortional buckling, beams with overhangs, load height, finite element, warping

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Introduction

The Gerber system (Fig. 1) consists of beams with overhangs connected to two adjacent suspended spans at the overhang tips. Typically, such beams have wide flange cross-sections and are thus prone to lateral torsional buckling (LTB). The Gerber system (Fig. 1) consists of a central span (or back-span) and two overhangs (cantilever spans). Traditionally, Gerber system spans are engineered so that the negative bending moments at the supports are nearly equal to the positive moments at the middle of the central span, with the intent to fully utilize the cross-sectional capacity in the regions of maximum positive and negative moments. Loads acting on Gerber systems arise from: (1) open web steel joists (OWSJ in Fig. 1) that frame perpendicularly to the beam axis, thus applying a series of point loads on the back-span and on the overhangs, and (2) the action of the suspended spans which give rise to point loads at the overhang tips. At the column location, the top and bottom chords of the OWSJ framing into the beam are typically connected to the Gerber beam flanges to provide lateral restraints (Fig. 1). At the connection between the end of the suspended span to the cantilever tip, the OWSJ framing into the cantilever tip provides lateral restraint either at the top flange alone (when connected to the beam top flange) or at both flanges (when both chords of the OWSJ are connected to the beam flanges). It is of interest to assess the effect of both bracing scenarios on the lateral torsional buckling capacity of Gerber systems, in a classical (non-distortional) sense and under a distortional lateral torsional buckling framework.

In classical lateral torsional buckling, the beam cross-section is assumed to displace laterally and twist as a rigid disk, thus omitting possible web distortion. Classical LTB solutions have been established in Barsoum and Gallagher (1970) and Trahair (1993). They form the basis of present
design provisions in structural steel standards for rolled members (e.g., CAN-CSA S16-16 (2016),
that web distortional effects are negligible. Recent studies suggest that web distortion can
influence the LTB strength of rolled steel members, including cantilevers (Pezeshky et al. (2017))
and simply supported beams with cleat angles (Hassan and Mohareb (2015)).

Fig. 1. Gerber beam system

Approximate distortional buckling solutions include the work of (Hancock (1978)) who developed
a finite strip solution and (Hancock et al. (1980)) who developed an energy-based approximate
solution. Wang et al. (1991) investigated the influence of web distortion on the LTB strength for
simply supported beams with mono-symmetric sections under uniform moments and extended
their work for beams under point loads (Hughes and Ma (1996a)) and distributed loads (Ma and
Hughes (1996b)). Samanta and Kumar (2006a) investigated the effect of web distortion on the
LTB of simply supported beams and cantilevers (Kumar and Samanta (2006b). Finite element
formulations for the problem include the work of Bradford and Trahair (1981) who developed a
12 degree of freedom (DOF) element, and (Bradford and Trahair (1982) and Chin et al. (1992))
who developed a 16 DOF element. A number of distortional buckling solutions were developed
for beams supported on seats (Bradford Mark and Trahair Nicholas (1983)), with mono-symmetric
sections (Bradford 1985,1988a), tee sections (Bradford 1988b, 1990a), beam-columns (Bradford
1990b and Vrcelj and Bradford 2006a), cantilevers (Bradford 1992a, 1999), and I-beams with
rotational and translational restraints (Bradford 1992b, Ronagh and Bradford 1994, and Bradford
polynomials to characterize the variation of the lateral displacement along the section depth and
sinusoidal functions in the longitudinal direction. A distortional model by Dekker and Kemp
(1998) adopted continuous springs between the top and bottom flanges to characterize web
distortion. Pi and Trahair (2000) introduced the effective torsional constant concept to characterize
distortional critical moments and investigated the effects of warping rigidity provided by end
supports. Experimental studies in (Zirakian and Showkati (2006)) investigated the distortional
LTB of castellated I-beams. Also, Zirakian and Showkati (2007) investigated the distortional
effects in simply supported I-beams with mid-span concentrated loads. Using 3D shell models in
Abaqus, Samanta and Kumar (2008) investigated the effect of lateral bracing and load height
effects on the distortional LTB of cantilevers. The Generalized Beam Theory (GBT) methodology
has also been adopted to develop distortional buckling models (e.g., Basaglia and Camotim (2013),
de Miranda et al. (2013) and Vieira et al. (2017)). Pezeshky and Mohareb (2014) developed a
distortional theory for the static analysis of mono-symmetric beams and formulated finite element
solutions (Pezeshky and Mohareb (2015)).
Within the above context, the present study introduces modifications and adds new features to past
distortional and non-distortional LTB finite element formulations and then adopts the modified
solutions to investigate the LTB behavior of Gerber beams.

Statement of the problem

A doubly symmetric wide flange steel section (Fig. 2a) is subjected to a general transverse line
load \( q_z(x) \) (Fig. 2b) acting at a load height \( z_q(x) \). The beam is assumed to be laterally restrained
only at selected discrete points at the top and/or bottom flanges. It is required to determine the
critical load \( \lambda q_z(x) \) at which the beam has a tendency to buckle either in a distortional lateral
torsional buckling mode, or in a non-distortional mode (Fig. 3). Towards this goal, modifications
and new features are introduced into the distortional the elements of Bradford and Trahair (1982)
and the non-distortional element by Barsoum and Gallagher (1970). The modified models are subsequently used to assess the LTB strength of Gerber systems.

**Assumed Kinematics**

A right-handed coordinate system (Fig. 2a) is adopted in which origin ‘O’ lies on web mid-height. A local right-handed coordinate system is also assigned to each of the flanges and to the web. Points A, B and C within the top flange, bottom flange, and the web are respectively offset from the middle surface and have local coordinates \((y, n_F), (y, n_F)\) and \((z, n)\). The un-deformed beam is shown in Configuration 1 in Fig. 3. Under the reference load \(q_z(x)\), the beam moves to Configuration 2 by undergoing transverse displacement \(w(x)\). During the pre-buckling deformation stage (Configuration 1 to 2), transverse response \(w(x)\) of the beam is assumed to follow that of the conventional Euler beam theory. The pre-buckling analysis yields the internal forces (moments, shears, and associated stresses) needed to conduct the buckling analysis. The load is then assumed to increase to \(\lambda q_z(x)\) and the beam moves to Configuration 3 at the onset of buckling. Assuming a linear force-displacement relationship, the transverse displacement of the beam increased to \(\lambda w(x)\). The beam then buckles under no increase in loading by displacing laterally from Configuration 3 to Configuration 4.

For the case of distortional buckling, Configuration 4 is characterized by the lateral displacement \(v(x, z)\) where the web is assumed to act as a thin plate, while the flanges are assumed to act as a Gjelsvik thin-walled members (Gjelsvik (1981)) under compression (Fig. 3). The top flange displaces laterally through \(v_T(x) = v(x, h/2)\) and twists through \(\theta_T(x) = -v_{z}(x, z)_{z=h/2}\) while the bottom flange displaces laterally through \(v_B(x) = v(x, -h/2)\) and twists through
\[ \theta_B(x) = -v_z(x,z)_{z=h/2} \]. As a matter of convention ‘,’ denotes partial differentiation with respect to the argument coordinate. Also, all fields with subscript ‘’ denote variables pertaining to the top flange and those with subscript ‘ B ’ pertain to the bottom flange.

For the case of classical non-distortional buckling, the section is assumed to displace as rigid disk and Configuration 4 is characterized by a lateral displacement \( v_0(x) \) at the section shear center and an angle of twist \( \theta(x) \). Under both distortional and non-distortional buckling analyses, it is required to determine the load multiplier \( \lambda \) at which the system will buckle.

**Pre-buckling Analysis**

The conventional two-node Euler Bernoulli beam element is used to extract the nodal shear forces and bending moments \( V_1, M_1, V_2, \) and \( M_2 \) at member ends. Linear interpolation is used to approximate the shear forces and bending moments at Configuration 3, i.e.,

\[
\lambda M_y(x) = \lambda M_1 \left( 1 - \frac{x}{L} \right) - \lambda M_2 \left( \frac{x}{L} \right), \quad \lambda V(x) = \lambda V_1 \left( 1 - \frac{x}{L} \right) - \lambda V_2 \left( \frac{x}{L} \right)
\]  

(1)

where the positive sign conventions for the shear force \( V(x) \) and bending moment \( M(x) \) are shown in Fig. 4. The corresponding normal and shearing stresses as given by the beam theory are

\[
\lambda \sigma_x(x,z) = -\lambda M_y(x) \frac{z}{I_y} \quad \lambda \tau_{xz}(x,z) = \frac{\lambda V(x)Q_y(z)}{I_y t_w}
\]  

(2)a-b

and the negative sign in Eq. (2)a ensures that positive moments (Fig. 4) induce compressive stresses at the top flange \( z = +h/2 \), \( I_y \) is the major moment of inertia, \( h \) is the section height, and \( Q_y(z) \) is the corresponding first moment of inertia about the y axis given by

\[
Q_y(z) = \left( bt - \frac{ht}{2} \right) t_w \left( \frac{h-t}{2} - z \right) \left( z+\frac{h-t}{4} \right)
\]  

(3)
For the flanges, the contribution of the shear stresses is assumed negligible i.e., \( \tau_{xz}(x,z) \approx 0 \) compared to those in the web.

**Fig. 2.** a) Cross section geometry b) general type of loads

**Fig. 3.** Deformation stages of buckling

The normal stresses at top flange \( \sigma_{xt}(x,z = h/2) = -(h/2)M(x)/I_y \) and bottom flange \( \sigma_{xb}(x,z = -h/2) = (h/2)M(x)/I_y \) are obtained by setting the \( z \) coordinate as \( z = +h/2, -h/2 \), respectively. From Eqs. (1) and (3) by substituting into Eqs. (2)a-b, one obtains the normal and shear stresses within the web, i.e.

\[
\lambda \sigma_x(x,z) = -\frac{\lambda M(x)}{I_y} z = \lambda \frac{1}{I_y} \bigg[ M_1 \left( 1 - \frac{x}{L} \right) - M_2 \left( \frac{x}{L} \right) \bigg] z - \frac{h}{2} \leq z \leq \frac{h}{2}
\]

\[
\lambda \tau_{xz}(x,z) = \frac{\lambda V(x) Q_z(z)}{I_y t_w} = \frac{\lambda}{I_y t_w} \bigg[ bt + t_w \left( \frac{h-t}{2} - z \right) \left( z + \frac{h-t}{4} \right) \bigg] \left[ V_1 \left( 1 - \frac{x}{L} \right) - V_2 \left( \frac{x}{L} \right) \right]
\]

**Distortional Buckling Analysis**

**Overview of relevant past work**

Bradford and Trahair (1982) developed a distortional lateral torsional buckling element for monosymmetric thin-walled beams with a flexible web and two symmetric flange assemblies, each consisting of folded plates. When developing the underlying variational expression, the Bradford and Trahair element (subsequently referred to as BT) treats the web as a thin plate. To account for the flexibility of the web, the study assumed the lateral displacements of the web to follow a cubic distribution along the height, while treating each of the flange assemblies as Vlasov beams, hence capturing their global warping behavior.
The present Distortional formulation (subsequently referred as D solution) tackles doubly symmetric I-sections. In a manner identical to BT solution, the D solution treats the web as a thin plate with a cubic distribution for the lateral displacement. In contrast to the BT solution, it treats the flanges as Gjelsvik members (thus accounts for their local warping effects) subjected to longitudinal stresses. Additional features of the present D solution are that it captures (1) the quadratic distribution of the shear stresses along the web height (as opposed to a constant shear stress in the BT solution) and (2) the destabilizing effect due to load height, a feature not supported in past distortional lateral torsional buckling beam formulation.

**Modified Variational Formulation**

The total potential energy $\Pi$ in going from Configuration 3 to 4 is summation of

$$\Pi = \Pi_w + \Pi_r + \Pi_\delta + V$$

where $\Pi_w$, $\Pi_r$, and $\Pi_\delta$ are the total potential energies within the web, top and bottom flanges and $V$ is the load potential increase of loads undergoing vertical displacements due to the twist of the section. The web is treated as a thin plate as discussed under the section on kinematic assumptions. The total potential energy $\Pi_w$ for the web, as given in (e.g., Timoshenko and Woinowsky-Krieger (1956), Bradford and Trahair 1982), is

$$\Pi_w = \int_0^L \int_{-h/2}^{h/2} \left\{ \frac{D}{2} \left[ (v_{xx})^2 + (v_{zz})^2 + 2\mu v_{xx}v_{zz} + 2(1-\mu)(v_{zz})^2 \right] + \frac{A}{2} \left[ t_w \sigma_{xz} v_{xx} + 2t_w r_{xz} v_{xx} v_{zz} \right] \right\} dz dx$$

(5)

where $D = E\lambda^2 / 12(1-\nu^2)$ is the flexural rigidity of the plate, $E$ is Young modulus, $\mu$ is Poisson’s ratio, and $\lambda \sigma_x (x, z), \lambda \tau_{xz} (x, z)$ are the normal and shear stresses in the plate. The first integral in Eq. (5) is the internal strain energy due to bending of the web and the second term is the destabilizing term due to internal normal stress $\sigma_x$ and shear stress $\tau_{xz}$. 
As mentioned under the kinematic assumptions, the flanges are considered to behave as Gjelsvik beams under axial stresses. Throughout buckling, the flanges undergo lateral bending and twist. Hence the total potential energy terms $\Pi_r$, $\Pi_b$ of the top and bottom flanges are expressed as

$$\Pi_r = \frac{1}{2} \int_0^L \left( E I_{zz,F} v_{zz,F}^2 + E I_{ww,F} v_{zzz,F}^2 + G J_F v_{zzz,F}^2 \right) dx + \frac{\lambda}{2} \int_0^L \left[ \sigma_{xt} (x, z = h/2) \left( I_{F_{zzz,F}} v_{zzz,F} + A_F v_{zz,F}^2 \right) \right] dx$$

(6)

$$\Pi_b = \frac{1}{2} \int_0^L \left( E I_{zz,F} v_{zz,F}^2 + E I_{ww,F} v_{zzz,F}^2 + G J_F v_{zzz,F}^2 \right) dx + \frac{\lambda}{2} \int_0^L \left[ \sigma_{sb} (x, z = -h/2) \left( I_{F_{zzz,F}} v_{zzz,F} + A_F v_{zz,F}^2 \right) \right] dx$$

(7)

where $G$ is the shear modulus, and relevant sectional properties for the flanges are

$$(A_F, I_{zz,F}, I_{pF,F}, J_F, I_{ww,F}) = \int_{A_F} (1, y^2, y^2 + n_F^2, 4n_F^2, n_F^2 y^2) dA_F = \left( bt, b^3 t / 12, (b^3 t + b t^3) / 12, b t^3 / 3, b^3 t^3 / 144 \right)$$

where $n_F$ is the normal local coordinate for the flanges (Fig. 2a), $A_F$ is the cross section area, $I_{zz,F}$ is the strong axis moment of inertia, $I_{pF,F}$ is the polar moment of inertia, $J_F$ is the Saint-Venant torsional constant and $I_{ww,F}$ is the local warping constant, all defined for a single flange.

The reference line load $q_z(x)$ is applied at a height $z_q(x)$ above the shear center (Fig. 2).

Throughout buckling (i.e., in deforming from Configuration 3 to 4), load $q_z(x)$ undergoes a vertical displacement $w_b(x)$ (Fig. 5b) and the corresponding total potential gain is obtained by the product of load increment $q_z(x) \, dx$ and the displacement $w_b(x) = -\frac{1}{2} \int_{z=0}^{z(x)} \left[ v_{zz} (x, z) \right]^2 dz$. Integrating with respect to $x$ one recovers the total load potential energy gain as

$$V = \lambda \int_0^L q_z(x) w_b(x) dx = -\frac{\lambda}{2} \int_0^L q_z(x) \int_{z=0}^{z(x)} \left[ v_{zz} (x, z) \right]^2 dz \, dx$$

(8)

**Fig. 5.** Vertical distance travelled by load throughout buckling

In summary, by adding the total potential energy of the web $\Pi_w$ Eq. (5), flanges $\Pi_r$ and $\Pi_b$ Eqs. (6)-(7) and load potential due to external loads $V$ Eq. (8) one recovers the total potential energy as
Finite Element Formulation

A finite element with four nodes \( j = 1, 2, 3, 4 \) with 4 degrees of freedom (DOFs) per node is developed (Fig. 6) for the buckling analysis. The four degrees of freedom of node \( j \) are the lateral displacement \( v_j \), the rotation \( \theta_j \), the angle of twist \( \theta_j \) and the rate of change of the angle of twist \( \frac{d\theta_j}{dx} \) with respect to the longitudinal coordinate.

**Fig. 6.** Definition of the 16 DOFs in the distortional buckling element

**Contribution of the web**

The displacement field \( v(x, z) \) for the web can be expressed in terms of the nodal displacements of the element through cubic Hermitian polynomials in both directions, i.e.,

\[
v(x, z) = M_{16 \times 16}^T (x, z) u_{b_{16 	imes 1}} \tag{10}
\]

where \( M(x, y) \) is the vector of shape functions resulting from the product of Hermitian functions in both directions, i.e.,

\[
M_{16 \times 16} (x, z) = \begin{pmatrix} H_1(z) \bar{N}^T (x) & H_2(z) N^T (x) & H_3(z) \bar{N}^T (x) & H_4(z) N^T (x) \end{pmatrix}_{16 \times 16} \tag{11}
\]

and the shape functions \( H_j(z) \) and \( N_{1 \times 4} (x) \) are given by
\[ H_1(z) = \frac{1}{2} + \frac{3z}{2h} - \frac{2z^3}{h^3}, \quad H_2(z) = -\frac{h}{8} - \frac{z}{4} + \frac{z^2}{2h} + \frac{z^3}{h^2} \]
\[ H_3(z) = \frac{1}{2} - \frac{3z}{2h} + \frac{2z^3}{h^2}, \quad H_4(z) = \frac{h}{8} - \frac{z}{4} - \frac{z^2}{2h} + \frac{z^3}{h^2} \]

and \( \mathbf{N}(x)^T_{1\times 4} = \begin{pmatrix} 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 & 1 - x - 2\left(\frac{x}{L}\right) + \left(\frac{x^3}{L^2}\right) & 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right) - \left(\frac{x^3}{L^2}\right) & 3\left(\frac{x}{L}\right)^2 - \left(\frac{x^3}{L^2}\right) - x + \left(\frac{x^3}{L^2}\right) \end{pmatrix} \)

\[ \mathbf{N}(x)^T_{1\times 4} = \begin{pmatrix} 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 & 1 - x - 2\left(\frac{x}{L}\right) + \left(\frac{x^3}{L^2}\right) & 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right) - \left(\frac{x^3}{L^2}\right) & 3\left(\frac{x}{L}\right)^2 - \left(\frac{x^3}{L^2}\right) - x + \left(\frac{x^3}{L^2}\right) \end{pmatrix} \]

In Eq. (10), vector \( \mathbf{u}_{b(1\times 16)}^T = (\mathbf{u}_{b1}^T \mathbf{u}_{b2}^T \mathbf{u}_{b3}^T \mathbf{u}_{b4}^T) \) contains the nodal displacements vectors defined as

\[ (\mathbf{u}_{b1}^T, \mathbf{u}_{b2}^T, \mathbf{u}_{b3}^T, \mathbf{u}_{b4}^T) = \begin{pmatrix} (v_1, v_{x2}, v_{y1}, v_{y2}, \theta_{x1}, \theta_{y1}, \theta_{x2}, \theta_{y2}) \end{pmatrix} \]

and the interpolation functions \( \mathbf{M}(x, z) \) appearing in Eq. (10) relate the lateral displacement of the web to the nodal displacements \( \mathbf{u}_{bk} \) through \( \mathbf{v}(x, z) = \sum_{k=1}^{16} \mathbf{M}_k(x, z) \mathbf{u}_{bk} \).

From Eq. (10), by substituting into Eq. (5), one can express the total potential energy for the web as

\[ \Pi = \frac{1}{2} \mathbf{u}_b^T \left[ (k_1 + k_2 + k_3 + k_4) + \lambda (k_{1g} + k_{2g}) \right] \mathbf{u}_b \]

where the elastic matrices \( k_1 - k_4 \) and geometric matrices \( k_{1g}, k_{2g} \) are defined as

\[ [k_1, k_2, k_3, k_4] = D \int_0^L \int_{-h/2}^{h/2} \left[ M_{xx} M_{yy}, M_{yx} M_{xy}, 2\mu M_{xx} M_{xy}, 2(1-\mu) M_{xx} M_{xy} \right] dxdy \]

\[ (k_{1g}, k_{2g}) = \int_0^L \int_{-h/2}^{h/2} \left[ N_x M_{xx}, N_{xx} M_{xx}, N_{xx} M_{xy} \right] dxdy \]

where, \( N_x = t_w \sigma_x \) and \( N_{xx} = 2t_w \tau_{xz} \).
Contributions of the flange:

In order to express the total potential energy of the flange, the field displacement $v$ is obtained at the top flange by setting $z = h/2$ in the shape functions (i.e., $M_h(x, h/2)$). From Eq. (10) by substituting into Eq. (6), one rewrites the total potential energy of the top flange as

$$\Pi_T = \frac{1}{2} u_b^T \left[ (k_s + k_6 + k_7) + \lambda \left( k_{3g} + k_{4g} \right) \right]_{(z=h/2)} u_b$$

(15)

where the elastic matrices $k_s - k_7$ and geometric matrices $k_{3g} - k_{4g}$ are defined as

$$\left( k_s, k_6, k_7 \right) = E \int_0^L \left( I_z M_{xx} M_{xx}^T, I_{oo} M_{xx} M_{xx}^T, (GJ/E) M_{xx} M_{xx}^T \right)_{(z=h/2)} dx$$

$$\left( k_{3g}, k_{4g} \right) = \int_{-h/2}^{h/2} \left( I_{pG} \sigma_{xx} (x, z = h/2) M_{xx} M_{xx}^T, A_p \sigma_{xx} (x, z = h/2) M_{xx} M_{xx}^T \right) dx$$

Analogous expressions can be obtained for the bottom flange by replacing subscript ‘$T$’ with ‘$B$’ and setting $z = -h/2$ instead of $z = h/2$ in Eqs. (15), leading to

$$\Pi_B = \frac{1}{2} u_b^T \left[ (k_s + k_9 + k_10) + \lambda \left( k_{5g} + k_{6g} \right) \right]_{(z=-h/2)} u_b$$

where the elastic matrices $k_s - k_{10}$ and geometric matrices $k_{5g} - k_{6g}$ relate to the bottom flange.

Geometric stiffness matrix due to load height effect

From Eq. (10) by substituting into Eq.(8), the load potential energy at Stage 4 (i.e. buckling configuration) is expressed in terms of nodal displacement as

$$V = \frac{1}{2} \lambda u_b^T K_{g7} u_b$$

(16)

where the geometric matrix $K_{g7}$ is defined by

$$K_{g7} = -\int_0^L q_z (x) \int_{z=0}^{z_i(x)} M_{xx} M_{xx}^T dz dx$$

Stationary condition

The stationary condition is evoked by setting the variation of total potential energy to zero, i.e.,

$$\delta \Pi = \delta (\Pi_w + \Pi_T + \Pi_B + V) = 0$$

(17)
From Eqs. (14)-(16), by substituting into Eq. (17) and taking the variation with respect to the nodal 
displacements, one recovers the eigenvalue problem such as

$$\delta \Pi = \delta \mathbf{u}_b^T \left[ \sum_{i=1}^{10} k_i + \lambda \sum_{j=1}^{7} k_{j,g} \right] \mathbf{u}_b = 0$$

**Non-distortional Solution**

*Overview of relevant past work*

The literature on non-distortional beam buckling solutions is vast. Most notable is the classical 
finite element by Barsoum and Gallagher (1970). More recent developments include the work of 
Erkmen and Mohareb (2006a), Erkmen and Mohareb (2006b), Wu and Mohareb (2011a) and Wu 
and Mohareb (2011b). For comprehensive reviews on the subject, the interested reader is referred 
to the recent reviews in Sahraei and Mohareb (2016) and Sahraei et al. (2018). The proposed non-
distortional solution herein is based on a modification of the element of (Barsoum and Gallagher 
1970). The element as originally formulated is based on the lateral displacement of the shear center 
and the angle of twist and is thus able to naturally model lateral braces at the shear center and 
twisting braces. In its original form, it cannot directly model lateral restraints at the top or bottom 
flanges. This feature is added in the present work. The total potential energy in going from the 
point of onset of buckling to the buckled configuration is

$$\Pi_{\text{eb}} = (U_v + U_{sv} + U_w) + V_b$$

(18)

where $U_v$, $U_{sv}$, $U_w$ are the internal strain energy stored due to weak axis bending, conventional 
Saint-Venant torsion, and warping torsion, respectively. Also, the destabilizing load potential 
energy terms $V_b = V_m + V_{ql}$ consists of a component $V_m$ due to the bending moments and another 
one $V_{ql}$ due to load position effect. By expressing the above energy terms in terms of the lateral 
displacement $v$ and the angle of twist $\theta$, one obtains
\[
\left[ U_v, U_{v_x}, U_{w_v}, V_m, V_{q_e} \right] = \frac{1}{2} \int_0^L \left[ EI_z v'^2_x, GJ \theta_x^2, EC_w \theta_{xx}^2, \lambda M_y(x) \theta v_{,xx}, \lambda q(x) z \theta'^2 \right] dx
\]  
(19)

In which, \( J \) is the Saint-Venant torsional constant, \( I_z \) is the moment of inertia of the section about z-axis, and \( C_w \) is the warping constant of the section. The finite element has two nodes, each having four buckling degrees of freedom; the lateral displacement \( v \), the weak-axis rotation \( v_{,x} \), the angle of twist \( \theta \) and the warping deformation \( \theta_x \). The lateral displacement \( v \) is related to the nodal displacements and rotations \( v^T = \langle v_1, v_{,x1}, v_2, v_{,x2} \rangle \) and the angle of twist \( \theta \) are related in terms of nodal rotations and warping deformation \( \theta^T = \langle \theta_1, \theta_{,x1}, \theta_2, \theta_{,x2} \rangle \) using Hermitian polynomials as

\[
v(x) = \tilde{N}(x) v^T_{4x1}, \quad \theta(x) = N(x) \theta^T_{4x1} \quad \text{(20)a-b}
\]

where \( N(x) \) and \( \tilde{N}(x) \) have been defined in Eq. (13)a-b. From Eqs. (19), (20)a-b, by substituting into Eq. (18) one recovers

\[
\Pi_{eb} = \left( \frac{1}{2} \right) u^T_{b18} \left[ k_{E8\times8} - \lambda k_{G8\times8} \right] u_{b8\times1}
\]  
(21)

where \( k_E \) is the elastic stiffness matrix, and \( k_G \) is the geometric stiffness matrix and the corresponding nodal displacement vector is \( u^T_b = \langle (v \theta v_{,x} \theta_{,x}), (v \theta v_{,x} \theta_{,x}) \rangle \). The above element based on the displacement vector \( u^T_b \) allows the modelling of braces at the section shear center and will be referred to as the “BG element” subsequently.

\textbf{Modifications to previous work}

In the BG element, the nodal lateral displacements \( v_i (i=1,2) \) in vector \( u^T_b \) are defined at the shear center and \( \theta_i \) are the angles of twist of the cross-section. In situations where one of the flanges is
laterally restrained while the other is non-restrained (Fig. 7a), rather than characterizing the motion of the cross-section using nodal displacements $v_i$ and $\theta_i$, it becomes more convenient to adopt the displacements of the top and bottom flanges $v_{Ti}$ and $v_{Bi}$ instead. The lateral displacements of the top and bottom flanges are related to the shear center lateral displacement and angle of twist of the section (Fig. 7b) through the relations $v_{Ti,Bi} = v_i \mp (h/2) \theta_i$. When a lateral brace is provided, for example at the top flange, one can express the kinematic constraint $v_{Ti} = 0$. Such a constraint can be enforced, for example, using Lagrange multipliers (Sahraei and Mohareb (2016)). However, the process increases the size of the structure stiffness matrix and the resulting stiffness matrix loses its positive definite character, both aspects being undesirable in commercial software. Thus, a technique is developed herein to preserve the positive definite nature of the structure stiffness matrix. Three cases of practical interest (Fig. 7c) may arise. These are:

**Case 1** – Both ends of the element have braces at one of (or both) flanges. In this case, the element needs to be developed in terms of the top and bottom flanges displacements at both nodes and the nodal displacement vector would be

$$\mathbf{u}_{b1}^T = \left\{ \mathbf{u}_{m1,1}^T \mid \mathbf{u}_{m1,2}^T \right\} = \left( \begin{array}{c} v_T \\ v_{xT} \\ v_B \\ v_{xB} \end{array} \right)_1 \left( \begin{array}{c} v_T \\ v_{xT} \\ v_B \\ v_{xB} \end{array} \right)_2 \right\}. \quad \text{The resulting element will be termed as the BG1 element and the corresponding stiffness and geometric matrices will be developed subsequently.}

**Case 2** – The first end of the element is braced at the shear center and the other end of the element is braced at one of the flanges (Fig. 7c-BG2). In such a case, the element matrices need to be formulated in terms of the nodal displacements $\mathbf{u}_{b2}^T = \left\{ \begin{array}{c} v \\ \theta \\ v_x \\ \theta_x \end{array} \right\}_1 \left( \begin{array}{c} v_T \\ v_{xT} \\ v_B \\ v_{xB} \end{array} \right)_2$. Such an element can serve as a transition element between a BG zone of elements and another BG1 zone of elements (Fig. 7a) and will be referred to as the BG2 element.
Case 3- The first end of the element is braced at one of the flanges and the other end of the element
is braced at the shear center (Fig. 7c-BG3). In such a case, the element matrices need to be
formulated in terms of nodal displacements \( \mathbf{u}_{b_3}^T = \begin{pmatrix} v_T & v_{x,T} & v_B & v_{x,B} \end{pmatrix}^T \). The
element can serve as a transition element between a BG1 zone of element and another BG zone of
elements (Fig. 7a). It is noted that in a zone of no lateral braces, no displacement constraints need
to be imposed, and any of the elements BG, BG1, BG2, and BG3 can be used to model the
segment. For all three elements BG1, BG2, or BG3, the nodal displacement vector
\( \mathbf{u}_{b_i}^T (i = 1 \equiv BG1, 2 \equiv BG2, 3 \equiv BG3) \) can be related to the displacements \( \mathbf{u}_b \) through
\[ \mathbf{u}_{b_{i=1}} = \mathbf{A}_{b_{i=1}} \mathbf{u}_{b_{i=1}} \quad (22) \]
where \( \mathbf{A}_i \) is a transformation matrix depending on the section height \( h \) given in the following
section. From Eq. (22), by substituting into Eq. (21), the total potential energy \( \Pi_{eb} \) takes the form
\[ \Pi_{eb} = \frac{1}{2} \mathbf{u}_b^T \left[ \mathbf{k}_E - \lambda \mathbf{k}_G \right] \mathbf{u}_b = \frac{1}{2} \mathbf{u}_{b_i}^T \mathbf{A}_i^T \left[ \mathbf{k}_E - \lambda \mathbf{k}_G \right] \mathbf{A}_i \mathbf{u}_{b_i} = \frac{1}{2} \mathbf{u}_{b_i}^T \left[ \mathbf{k}_{E_i} - \lambda \mathbf{k}_{G_i} \right] \mathbf{u}_{b_i} \quad (23) \]
where the modified elastic and geometric matrices \( \mathbf{k}_{E_i} \) and \( \mathbf{k}_{G_i} \) are given by
\[ \mathbf{k}_{E_i} = \mathbf{A}_i^T \mathbf{k}_E \mathbf{A}_i, \quad \mathbf{k}_{G_i} = \mathbf{A}_i^T \mathbf{k}_G \mathbf{A}_i \]

Transformation matrices

Case 1:

For the first node, one can write \( v_{T1} = v_1 - (h/2) \theta_1 \), \( v_{B1} = v_1 + (h/2) \theta_1 \), \( v_{xT1} = v_{x1} - (h/2) \theta_{x1} \) and
\( v_{xB1} = v_{x1} + (h/2) \theta_{x1} \). In a matrix form, one has \( \mathbf{u}_{m_{i=1}} = \mathbf{a}_{i}^{-1} \mathbf{u}_{b_{1}} \) where \( \mathbf{u}_{m_{i=1}}^T = \begin{pmatrix} v_T & v_{x,T} & v_B & v_{x,B} \end{pmatrix}^T \),
and \( \mathbf{u}_{b_{1}}^T = \begin{pmatrix} v & \theta & v_x & \theta_x \end{pmatrix}^T \). By solving for \( \mathbf{u}_{b_{1}} \), one has
A similar equation can be written for the second node \( u_{b2} = a_2 u_{m1,2} \) where

\[
\mathbf{u}_{b1} = a_1 \mathbf{u}_{m1,1}, \quad a_1 = \frac{1}{2h} \begin{bmatrix} h & 0 & h & 0 \\ -2 & 0 & 2 & 0 \\ 0 & h & 0 & h \\ 0 & -2 & 0 & 2 \end{bmatrix}
\]

(24a-b)

A similar treatment leads to the following expressions for BG1 and BG2 elements

\[
\mathbf{u}_{m1,2}^T = \begin{bmatrix} v_x \\ v_{xT} \\ v_B \\ v_{x,y} \end{bmatrix}, \quad \mathbf{u}_{b2}^T = \begin{bmatrix} v \\ \theta \\ v_x \\ \theta_x \end{bmatrix}_2. \quad \text{Hence, one rewrites in a matrix form}
\]

\[
\mathbf{u}_b = \begin{bmatrix} \mathbf{u}_{b1} \\ \mathbf{u}_{b2} \end{bmatrix} = A_1 \begin{bmatrix} \mathbf{u}_{m1,1} \\ \mathbf{u}_{m1,2} \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{m1,1} \\ \mathbf{u}_{m1,2} \end{bmatrix}.
\]

(25a-b)

**Cases 2 and 3:**

A similar treatment leads to the following expressions for BG1 and BG2 elements

**Distortional effects on LTB of simply supported beam**

A simply supported beam with W410x39 cross section (\( h = 399, b = 140, t = 8.8, t_w = 6.4 \) mm) and span \( L = 8.00 \) m is subjected to a point load located at 0.6L acting at the shear center. The modulus of elasticity is \( E = 200 \text{GPa} \) and shear modulus is \( G = 76.9 \text{GPa} \). Four bracing scenarios are investigated at the section of the point load; 1) both flanges, 2) only the top flange, 3) only the bottom flange, and 4) no bracing. The critical loads are determined based on 3D shell model under Abaqus, the present distortional (D) and non-distortional modified BG formulations. In the non-distortional solution, the lateral-torsional simply supported end conditions were enforced by restraining the lateral displacement and the angle of twist at both ends. For the distortional and shell solutions, analogous simply supported conditions were achieved by restraining the lateral displacements at the flange-to-web junctions. In the the non-distortional solution, the warping
degree of freedom was released at both ends. Correspondingly, in the distortional model, end
section global warping was allowed by setting free the weak axis rotations at both flanges. Also,
flange local warping was set free by releasing the derivative of the angle of twist at both flanges.
Analogous end conditions were adopted in the shell finite model by releasing the longitudinal
displacement and the three rotations at all flange nodes offset from the flange-to-web junctions.
In the Abaqus model, the S4R shell element was used to mesh the beam. The S4R is a 4-node,
quadrilateral shell element with 6 degrees of freedom per node and reduced integration.
The beam is meshed using 160 elements along the span, 8 elements across the height, and 4
elements to model the flanges. In order to suppress localized buckling of the web under the applied
point load, half of the transverse load $P$ is assigned to each of the web-to-flange junctions. Ten
elements were found enough to attain convergence in the distortional model. In a similar manner,
8 elements were found enough to attain convergence in the modified BG element. In comparison
to the 3D FEA predictions, the present distortional (D) model over-predicts the critical moments
by 3-4% (Table 1) while the modified BG model over-predicts the critical moments by 6-7%. For
the case of no bracing, the ratio of the critical moment as predicted by the modified BG solution
to that of the AISC (ANSI/AISC 360-16 (2016)) equation (also based on a non-distortional model)
is 1.01%. The results indicate that for a simply supported beam, a typical rolled section (e. g., the
W410x39 investigated in this example), distortional effects play a minor role on the magnitude of
the critical moments predicted. In such situations, the modified BG formulation, as well as the
present non-distortional lateral torsional buckling provisions in design standards tend to slightly
overestimate the predicted critical moments. As will be shown in the following section, this
observation is non-universal and the influence of web distortion on critical moments varies
depending on the boundary conditions, bracing scenarios, cross-sectional geometry, and span.
Possible bracing scenarios at the cantilever tip include (1) no bracing (denoted as NO in Fig. 8d), (2) top and bottom bracing (TB), (3) only top bracing (T), and (4) only bottom bracing (B). The series of point loads arising from the actions of the OWSJ can be idealized by a uniformly distributed load $q$ acting on the back-span and overhangs. Intermediate OWSJ between columns and between cantilever tip and columns are assumed to provide no lateral restraint to the Gerber beam. This will be the case as long as no additional inclined horizontal braces frame at the OWSJ-beam junction. Assuming a modular system with constant spans $L_2$ between columns and cantilever spans $L_1$, the suspended span is $L_2 - 2L_1$ (Fig. 8b) and the corresponding reactions $P$ acting on the tip of the Gerber system (Fig. 8c) are also

$$P = q(L_2 - 2L_1)/2$$

(26)

A system with a back-span $L_2 = 8.00 m$ and overhangs $L_1 = 1.50 m$ with a W410x39 cross-section is considered as a reference configuration. It is required to obtain the critical moments and associated mode shapes. Three types of solutions are sought: (1) the modified BG non-distortional solution, (2) the distortional solution (D) developed in the present study, and (3) a shell 3D FEA in based on the commercial software Abaqus (SIMULIA (2011)).

**Description of the models**

In the distortional model, 15 elements were found enough to attain convergence. However, the beam was subdivided into 220 elements to match the discretization of shell model, and thus facilitate the mode shapes comparisons between both models. Also, 10 elements were found enough to attain convergence for the BG solution though the beam was also modelled using 220 elements. At the location of lateral restrains at the flanges, BG1 type elements were used to provide
natural means to model the restraints. The required number of DOFs for convergence using the shell model is 22,542 DOFs. This contrasts with 128 DOFs required in the distortional model and 44 DOFs in the non-distortional BG model.

**Results**

The critical moments based on the BG element, the distortional element (D) and the 3D shell FEA show that the BG provides the highest capacity, followed by the present distortional element prediction, while the shell solution based on Abaqus provides the lowest capacity (Table 2).

**Fig. 8.** a) Gerber system, b) suspended span, c) back-span and overhang beams, d) boundary conditions at tips of the overhang e) geometric dimensions of the cross section

Compared to the 3D FEA results, the BG predictions overestimate the critical moments by 11%, 9%, 11%, and 11%, respectively for the cases of no lateral bracing, top flange bracing, bottom flange bracing, and both flanges bracing. In contrast, the present distortional model predictions are only 5% higher than shell FEA predictions for all bracing scenarios. In contrast to the case of simply supported beam, the results show the importance of incorporating distortional effects in the present problem. Such effects are captured in the 3D FEA shell analysis and the present distortional solution, and hence the proximity of their predictions, but not in the conventional non-distortional solution forming the basis of the BG-type element(s). Also, of interest is to note the drastic effect of tip bracing, where restraining both flanges laterally increased the critical moment predictions by over 150% according to all three models. A comparison of mode shapes as characterized by the lateral displacements at the top and bottom flanges $u_t$ and $u_b$, and the angles of twist $\theta_t$ and $\theta_b$ (Fig. 9a-c) are in nearly perfect agreement in all three solutions.

**Table 2.** Comparison of the critical moments
Fig. 9. a) Lateral displacements of the case (NO), b) lateral displacements of the case (TB), c) angle of twist of the case (NO), d) angle of twist for the case (TB)

**Parametric study:**

In order to study the effects of geometric parameters; flange width $b$ and thickness $t$, web height $h$ and thickness $t_w$ (all shown in Fig. 8e), and spans $L_1, L_2$ on the critical moment, five dimensionless geometric ratios $b/t$, $h/t_w$, $b/h$, $L_2/h$ and $L_2/L_1$ are investigated. The beam geometry is taken identical to that defined as a reference case in previous section. The beam has the following dimensionless ratios $b/t = 15.91$, $h/t_w = 60.97$, $b/h = 0.36$, $L_2/h = 20.50$, and $L_2/L_1 = 5.33$. A parametric study is conducted by keeping four of the above five ratios constant, and varying the remaining parameter within the ranges of common sections in the Handbook of Steel Construction (2016). In the first set of runs, the back-span to overhang ratio $L_2/L_1$ is varied by changing the overhang span $L_1$ in the ranges of Column 1 in Table 3 while keeping the ratios ($b/t$, $h/t_w$, $b/h$, $L_2/h$) identical to those of the reference case. Secondly, the flange width-thickness ratio $b/t$ is varied by changing the flange thickness $t$ according to the values in Column 2 of Table 3 while keeping other ratios ($h/t_w$, $b/h$, $L_2/h$ and $L_2/L_1$) identical to those of the reference case. Next, the web height-thickness ratio $h/t_w$ is varied by changing the web thickness $t_w$ according to Column 3 in Table 3 while keeping other ratios constant. In the fourth set of runs, the flange width-web height $b/h$ is varied by keeping the web height constant while varying the flange width $b$ (Column 4), flange thickness $t$ (Column 2) to preserve the width-thickness ratio $b/t$ identical to that of the reference case. Lastly, the back-span to web height ratio $L_2/h$ was varied.
by varying the height $h$ and varying $b, t$ and $t_w$ in the ranges of Table 3 to keep the ratios $b/t$, $h/t_w$, and $b/h$ equal to those of the reference case.

**Table 3. Geometric parameters ranges**

The critical moments $M_{BG-x}$ based on the BG model and $M_{D-x}$ based on the distortional model are sought for various bracing scenarios $x$ at the cantilever tips, where $x$ takes the values NO, TB, T, B to respectively denote the cases of no bracing, top and bottom bracings, only top bracing, and only bottom bracing (Fig. 8d). Most structural steel design standards (CAN-CSA S16-16 (2016), ANSI/AISC 360-16 (2016), AS-4100 (1998), and EN-1993-1-1-0 2005) define the critical moment as $M_{cr} = C_b M_u$, where $C_b$ is the moment gradient factor and $M_u$ is the elastic critical moment calculated for the case of uniform moments as given by

$$M_u = \pi / L \sqrt{EI_G J J + \left(\pi E / L\right)^2 I_c J}$$

in which span $L$ is the distance between the points of lateral and torsional restraints. For a given moment distribution, standards provide expressions for $C_b$ for simple boundary conditions which enable the designer to estimate the critical moments. The present models enable the direct computation of the critical moments $M_{cr-BG}$ (based on the non-distortional model) and $M_{cr-D}$ (based on the distortional moment) for the Gerber system. The critical moments computed are then divided by the uniform moment as computed by (27) (by taking the span $L = L_2$ as a reference span) to back-calculate moment gradients $C_{b-BG}$ or $C_{b-D}$, from the equations

$$M_{cr-BG} = C_{b-BG} M_u, \quad M_{cr-D} = C_{b-D} M_u$$

where $C_{b-BG}$ is a moment gradient factor that omits distortional effects and $C_{b-D}$ is a moment gradient factor that incorporates distortion, and the ratio $D = C_{b-D} / C_{b-BG}$ can be thought of as a
coefficient that quantifies the effect of distortion on the critical moment. Fig. 10 and Fig. 11a-d depict both moment gradient factors for all four boundary conditions considered.

The cantilever span is varied from $L_1 = 900\,mm$ to $L_1 = 2000\,mm$ (corresponding to $L_2 / L_1 = 8.89$ to $L_2 / L_1 = 4.00$) while the tip loads $P$ were determined according to Eq. (26). The moment gradient factors as determined from Eqs. (28)a-b versus the span ratio $L_2 / L_1$ are depicted in Fig. 10. As expected, the case of no bracing (NO) corresponds to the lowest moment gradient factor and the addition of a top bracing (T) or bottom bracing (B) is found to increase the moment gradient compared to the case of no bracing. In all cases, for a given bracing scenario and span ratio $L_2 / L_1$, the distortional moment gradient is consistently lower than the non-distortional moment gradient. This is a natural outcome of the fact that the distortional model provides a more flexible representation of the section deformation than the BG type solution where the section is assumed to displace and twist as a rigid disk.

The distortional model shows that top bracing increases the moment gradient $C_{b-D}$ by factor of 116% to 168% which is comparable to the increase of 116% to 167% a. predicted by the non-distortional moment gradient $C_{b-BG}$. As a general observation, top bracing becomes less effective as the cantilever span decreases (i.e., $L_2 / L_1$ increases). Bottom bracing is found ineffective as it increases the moment gradient moment distortional by a factor not exceeding 103% based distortional and non-distortional models. In contrast, top and bottom flanges bracing (TB) significantly increases the moment gradient factor compared to the case of no bracing. The increase predicted by the distortional model ranges from 142% to 180% while that predicted by the non-distortional model ranges from 149% to 185%.
The effects of $b/t, h/t_w, b/h, L_2/h$ on the moment gradients $C_{b-BG}$ and $C_{b-D}$ are depicted on Fig. 11a-d. In general, when both flanges are braced (TB), the moment gradient factor $C_{b-BG}$ based on the non-distortional model are found insensitive to $b/t, h/t_w, b/h, L_2/h$ as observed by the flat curves on Fig. 11a-d. In contrast, the moment gradient factor $C_{b-D}$ based on the distortional model shows a positive trend with $b/t$ and $L_2/h$ and a negative trend with $h/t_w$ and $b/h$. For the other three bracing scenarios (B, T, NO), the moment gradient factors $C_{b-BG}$ and $C_{b-D}$ reduce with $b/t$, $h/t_w$ and $b/h$, and increase with $L_2/h$.

The effect of distortion as characterized by the ratio of the moment gradients $C_{b-D}/C_{b-BG}$ is observed to be most pronounced for top and bottom flanges bracing (TB). This is evidenced by the consistently lower $C_{b-D}/C_{b-BG}$ ratio relative to those based on other bracing scenarios (T, B, and NO). Figure 11a indicated that TB bracing distortional effects are insensitive to the flange slenderness since the $C_{b-D}/C_{b-BG}$ ratio varies within the narrow range of 0.92-0.94 when $b/t = 9.3 - 23.3$. In contrast, Figures 11b-c suggest that distortional effects are sensitive for the web slenderness $h/t_w$ and the cross-section aspect ratio $b/h$. In Fig. 11b, ratio $C_{b-D}/C_{b-BG}$ is found to vary from 0.86 (at $h/t_w = 97.6$) to 0.98 (at $h/t_w = 39.0$) and in Fig. 11c, it varies from 0.79 (at $b/h = 0.64$) to 0.96 (at $b/h = 0.31$). Figure 11d shows that distortion is sensitive to the span to depth ratio $L_2/h$ as $C_{b-D}/C_{b-BG}$ varies from 0.83 (at $L_2/h = 13.3$) to 0.97 at ($L_2/h = 26.7$).

For the case of no bracing (NO), distortion is found to be only mildly sensitive to changes in $b/t$, $h/t_w$, $b/h$, and $L_2/h$, as the $C_{b-D}/C_{b-BG}$ ratio is found to vary within a relatively narrow ranges: from 0.94 (at $b/t = 9.3$) to 0.98 at ($b/t = 23.3$) in Fig. 11a, from 0.92 (at $h/t_w = 97.6$) to 0.99 (at
$h/t_w = 39.0$ in Fig. 11b, from 0.87 (at $b/h = 0.64$) to 0.98 (at $b/h = 0.31$) in Fig. 11c, and from 0.93 (at $L_z/h = 13.3$) to 0.99 (at $L_z/h = 26.7$) in Fig. 11d. The distortional effects for top flange bracing (T) and bottom flange bracing (B) are observed to lie between the cases of both flange bracing (TB) and no bracing (NO).

For top flange bracing (T) and for bottom flange bracing (B), ratios $C_{h-D}/C_{h-BG}$ were essentially identical to the those of the case of no bracing (NO) in all cases, suggesting that the degree of distortion in the T and B bracing is comparable to that of the no bracing.

Fig. 10. Moment gradient factor versus back span to overhang ratio $L_z/L_1$

Fig. 11. Moment gradient factors versus (a) flange width to thickness ratio $b/t$, (b) web height to thickness ratio $h/t_w$, (b) flange width to web height ratio $b/h$, and (d) back span to web height ratio $L_z/h$, (all vertical lines correspond to the reference case)

Summary and Conclusions

1) The present study modified the distortional finite element formulation by Bradford and Trahair (1982) to model doubly symmetric wide flange sections, account for flange local warping, account for load height destabilizing effect, and account for the parabolic distribution of the shear stresses along the web height. The formulation treats the web as a thin plate and thus captures web distortion within the plane of the cross-section and treats both flanges as Gjelsvik members with unequal angles of twist and lateral displacement fields. Modifications are also made to the conventional non-distortional lateral torsional buckling finite element of Barsoum and Gallagher (1970) to enable the analyst to model the lateral braces at either of the two flanges, while preserving the positive definite nature of the resulting stiffness matrix.
2) In all examples investigated, the lowest critical moments predicted were obtained by the shell finite element solution, closely followed by the present distortional solution, and the highest critical moments were obtained by the non-distortional solution.

3) For the simply supported beam investigated, the distortional solution is found to predict critical moments in close agreement with those predicted by the shell finite element solution while the non-distortional finite element solution slightly overestimates the critical moments as predicted by Abaqus, but in very close agreement to the present ANSI/AISC 360-16 (2016) predictions. The results suggest that the conventional non-distortional model adequately predicts the critical moment for the problem, albeit it errs on the un-conservative side.

4) The distortional and non-distortional models were used to develop moment gradient factors for the Gerber system in a parametric study. The ratio of the non-distortional to distortional moment gradients is indicative of the influence of distortion on the predicted critical moments.

5) The following observations based on the parametric study can be of practical interest when engineering Gerber beam systems:

i. In general, the conventional non-distortional model grossly overestimates the critical moment compared to the distortional model. As such, it is recommended to adopt a distortional solution when determining the capacity of Gerber systems.

ii. Bracing both flanges laterally at the tips of the cantilever span is found to be particularly effective in increasing the critical moments compared to the case of no lateral bracing at the tip. In comparison, top flange bracing is found moderately effective.

iii. For a given Gerber beam, distortional effects tend to be most significant when the tips of both flanges are laterally braced.
For a given span ratio \( L_2/L_1 \), distortional effects tend to increase (i.e., \( C_{b-D}/C_{b-BG} \) tend to decrease) with the web slenderness ratio \( h/t_w \) and the section aspect ratio \( b/h \). Also, distortional effects significantly decrease with the span to depth ratio \( h/L_2 \) and mildly decrease with the flange to thickness ratio \( b/t \).

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**References**


AS/NZS 4600-05 (2005). "AS/NZS 4600 Cold Formed Steel Structures." Standards Australia (SA), Sydney, Australia.


Table 1. Comparison of critical loads for various bracing scenarios (simply supported beam)

<table>
<thead>
<tr>
<th>Bracing Location</th>
<th>Critical load (kN)</th>
<th>Ratio</th>
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<tbody>
<tr>
<td></td>
<td>Shell 3D FEA</td>
<td>Distortional D</td>
</tr>
<tr>
<td>Top Flange</td>
<td>99.7</td>
<td>104.1</td>
</tr>
<tr>
<td>Bottom Flange</td>
<td>28.6</td>
<td>29.4</td>
</tr>
<tr>
<td>Both Flanges</td>
<td>100.4</td>
<td>104.9</td>
</tr>
<tr>
<td>No Bracing</td>
<td>26.2</td>
<td>27.1</td>
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</table>

Table 2. Comparison of the critical moments

<table>
<thead>
<tr>
<th>Lateral bracing at overhang tips</th>
<th>Critical moments (kNm)</th>
<th>Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BG</td>
<td>D</td>
</tr>
<tr>
<td>No bracing</td>
<td>44.7</td>
<td>42.3</td>
</tr>
<tr>
<td>Top Flange</td>
<td>59.9</td>
<td>57.9</td>
</tr>
<tr>
<td>Bottom Flange</td>
<td>44.7</td>
<td>42.4</td>
</tr>
<tr>
<td>Top and Bottom</td>
<td>70.43</td>
<td>65.9</td>
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*FEA based on the 3D FEA Abaqus model

Table 3. Geometric parameters ranges

<table>
<thead>
<tr>
<th>(1) span $L_i$ (mm)</th>
<th>(2) Flange thickness $t_f$ (mm)</th>
<th>(3) Web thickness $t_w$ (mm)</th>
<th>(4) Flange Width $b$ (mm)</th>
<th>(5) Section height $h$ (mm)</th>
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<tbody>
<tr>
<td>900</td>
<td>6</td>
<td>4.0</td>
<td>120</td>
<td>200</td>
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<tr>
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<td>4.5</td>
<td>130</td>
<td>250</td>
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<tr>
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<td>300</td>
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<tr>
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<td>150</td>
<td>350</td>
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<td>500</td>
</tr>
<tr>
<td>1700</td>
<td>12</td>
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<td>200</td>
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<tr>
<td>2000</td>
<td>15</td>
<td>10.0</td>
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<td>700</td>
</tr>
</tbody>
</table>

Fig. 12. Gerber beam system
Fig. 13. a) Cross section geometry b) general type of loads

Fig. 14. Deformation stages of buckling

Fig. 15. Sign convention for stress resultants
Fig. 16. Vertical distance travelled by load throughout buckling

Fig. 17. Definition of the 16 DOFs in the distortional buckling element

Fig. 18. a) Various type of BG elements, b) buckling kinematics, c) Buckling degrees of freedom
Fig. 19. a) Gerber system, b) suspended span, c) back-span and overhang beams, d) boundary conditions at tips of the overhang e) geometric dimensions of the cross section

Fig. 20. a) Lateral displacements of the case (NO), b) lateral displacements of the case (TB), c) angle of twist of the case (NO), d) angle of twist for the case (TB)
Fig. 21. Moment gradient factor versus back span to overhang ratio $L_2/L_1$.

Fig. 22. Moment gradient factors versus (a) flange width to thickness ratio $b/t$, (b) web height to thickness ratio $h/t_w$, (c) flange width to web height ratio $b/h$, and (d) back span to web height ratio $L_2/h$, (all vertical lines correspond to the reference case).