Twisted quadratic foldings of root systems and combinatorial Schubert calculus

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Abstract

This thesis builds on the connection of two widely studied objects in the literature, that is, foldings of finite root systems and the structure algebras of moment graphs associated with finite root systems.

Given a finite crystallographic root system Φ whose Dynkin diagram has a nontrivial automorphism, it yields a new root system Φ_τ by a so-called classical folding. On the other hand, Lusztig's folding (1983) folds the root system of type E_8 to type H_4 starting from an automorphism of the root lattice of type E_8. Lanini–Zainoulline (2018) developed the notion of a *twisted quadratic folding* of a root system, which describes both the classical foldings and Lusztig's folding on the same footing.

Our second key object of study is the *structure algebra* $\mathcal{Z}(\mathcal{G})$ of the moment graph $\mathcal{G}$ associated with a finite root system and its reflection group W. The structure algebra $\mathcal{Z}(\mathcal{G})$ is an algebra over a certain polynomial ring $\mathcal{S}$ whose underlying module is free with a distinguished basis $\{\sigma^{(w)} \mid w \in W\}$ called *combinatorial Schubert classes*. Each Schubert class $\sigma^{(w)}$ is an $\mathcal{S}$-valued function on W, whose value is explicitly known for any finite reflection group W.

Lanini–Zainoulline (2018) showed that a twisted quadratic folding $\Phi \rightsquigarrow \Phi_\tau$ induces an embedding of the respective Coxeter groups $\varepsilon : W_\tau \hookrightarrow W$ and a ring homomorphism $\varepsilon^* : \mathcal{Z}(\mathcal{G}) \to \mathcal{Z}(\mathcal{G}_\tau)$ between the corresponding structure algebras.

This thesis investigates how the induced map ε^* relates the Schubert classes of the original structure algebra $\mathcal{Z}(\mathcal{G})$ to those of the folded structure algebra $\mathcal{Z}(\mathcal{G}_\tau)$. In particular, we will provide a combinatorial criterion for a Schubert class $\sigma_\tau^{(u)}$ of $\mathcal{Z}(\mathcal{G}_\tau)$ to admit a Schubert class $\sigma^{(w)}$ of $\mathcal{Z}(\mathcal{G})$ such that the relation $\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)}$ holds for some nonzero scalar c. We will also prove that ε^* is surjective after an appropriate extension of the coefficient ring.

Dedications

To my Dad, whose curiosity in mathematics and unique collection of books in the family bathroom inspired me to study this subject and see the world it offers.

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Contents

General Notation

$\mathbb{C}$	Complex numbers
$\mathbb{R}$	Real numbers
$\mathbb{Q}$	Rational numbers
$\mathbb{Z}$	Integers
$\mathbb{Z}_{\geq m}$	Integers greater than or equal to m
$R^{\times}$	Units of a ring R
$\mathrm{Stab}_G(x)$	Stabilizer subgroup of a group G fixing x
$\sqcup$	Disjoint union of sets
$\lvert X\rvert$	Cardinality of the set X

Chapter 1

Introduction

This thesis builds on the meeting point of two major threads in the literature, namely, *foldings* of finite root systems and the study of the *structure algebras* of moment graphs associated with finite root systems. The former has long been folklore in the literature (cf. Steinberg [35]), and the latter has been an object of a recent active study in the context of theory of sheaves on moment graphs (cf. Fiebig [18], [19], Lanini [28], [29]). The direct foundation of our work is the paper [30] by Lanini and Zainoulline, where the authors showed that a folding of a root system induces a ring homomorphism between the corresponding structure algebras. In this thesis, we will investigate their induced map from the perspective of *combinatorial Schubert classes*.

Foldings of root systems and their twisted formulation

Given a finite crystallographic root system Φ whose Dynkin diagram admits a non-trivial automorphism (subject to certain conditions), it induces a so-called *folding* of the root system Φ. For example, let us consider the following configuration of vectors in the Euclidean space $\mathbb{R}^{2n}$ ($n \geq 2$):

$$\Phi = \{e_i - e_j \mid 1 \leq i, j \leq 2n,\ i \neq j\},$$

where $\{e_i \mid 1 \leq i \leq 2n\}$ denotes the standard basis of $\mathbb{R}^{2n}$. This is the root system of Dynkin type A_{2n-1} realized in the space $\mathbb{R}^{2n}$ in Bourbaki's standard manner (Bourbaki [8, Plate I]), with a set of simple roots $\Delta = \{\alpha_i := e_i - e_{i+1} \mid 1 \leq i \leq 2n-1\}$. The Dynkin diagram of type A_{2n-1} admits an involutive automorphism by interchanging the pair of simple roots α_i and α_{2n-i} for all $1 \leq i \leq n-1$ and fixing the middle simple root α_n. This involution of the Dynkin diagram extends to the following linear automorphism $\mathcal{T}$ on the whole space $\mathbb{R}^{2n}$, which stabilizes the set Φ:

$$\mathcal{T}(e_i) = -e_{2n-i+1} \quad (1 \leq i \leq 2n).$$

By taking the average of each $\mathcal{T}$-orbit in Φ, we obtain a new collection of vectors

$$\Phi_{\mathcal{T}} = \{\tfrac{1}{2}\big(\alpha + \mathcal{T}\alpha\big) \mid \alpha \in \Phi\}.$$

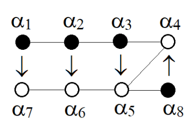
This set forms the root system of Dynkin type C_n realized in $\mathbb{R}^{2n}$ having the following set as simple roots:

$$\Delta_{\mathcal{T}} = \{\tfrac{1}{2}(\alpha_i + \mathcal{T}\alpha_i) \mid 1 \le i \le n\}.$$

This passage from Φ to $\Phi_{\mathcal{T}}$ is called a *folding* of the root system Φ (see Steinberg [35, pp. 175–176], Springer [34, Section 10], Stembridge [37]).

There are precisely four families of such foldings of finite crystallographic root systems: foldings from type A_{2n-1} to C_n ($n \ge 2$), from type D_{n+1} to B_n ($n \ge 3$), from type E_6 to F_4, and from type D_4 to G_2 (see [34, Section 10] for example). We will refer to these foldings as *classical* foldings. In all the classical foldings except the last one, the linear map $\mathcal{T}$ satisfies $\mathcal{T}^2 = \mathrm{id}$.

On the other hand, in his paper [31, Section 3.9(b)], Lusztig discovered a folding construction of the root system of type E_8 to the root system of type H_4. In this folding construction, instead of starting with an automorphism of the Dynkin diagram, one begins with an automorphism of the root lattice of the root system of type E_8. Under this automorphism, half of the simple roots (black dots in the diagram below) are mapped to the opposing simple roots (white dots), and the other half (white dots) are mapped to some linear combinations of simple roots.



The Lusztig folding has since been studied by many authors including Shcherbak [33, Section 2], Moody-Patera [32], Dyer [17, Section 1], and Dechant [12].

More recently, in [30], Lanini and Zainoulline introduced the notion of a *twisted quadratic folding* of a root system, in which both the classical quadratic foldings (all that are listed above except the folding from type D_4 to G_2) and the Lusztig folding are found as particular examples. Central to their formalism is the notion of a *folded representation* of a root system. Let K be a unital subring of $\mathbb{R}$, and let L be a free quadratic K-algebra given by $L = K[x]/(p(x))$ for some monic separable K-polynomial $p(x) = x^2 - c_1 x + c_2$. Let τ be one of the two distinct roots of $p(x)$ in L. We consider a free K-module U obtained from some free L-module $\mathcal{U}$ by restriction of scalars. Then U is equipped with a K-linear map $\mathcal{T}$, called the τ-operator, which is multiplication by τ on $\mathcal{U}$, and a symmetric K-bilinear form $(-,-)_\tau$, called the τ-*form*, which is derived from a dot product on $\mathcal{U}$. In a folded representation, we realize a finite crystallographic root system Φ as a subset of U so that its root lattice is stable under $\mathcal{T}$ and the set of simple roots of Φ admits a certain partition with respect to $\mathcal{T}$ ([30, Definition 3.1]). Then the τ-*twisted folding* of Φ is the procedure to pass from Φ to a new finite root system Φ_τ under the projection onto the τ-eigenspace of $\mathcal{T}$. In this framework, the classical foldings are obtained by taking $p(x) = x^2 - 1$, and the Lusztig's folding is obtained by taking $p(x) = x^2 - x - 1$ as described in [30, Examples 4.3–4.5].

Combinatorial Schubert calculus

Our second main object of study, the notion of a *structure algebra* and its *Schubert classes*, stem from the study of equivariant cohomology rings of flag varieties and its generalization to arbitrary finite Coxeter groups. In order to provide the context, we first briefly review the foundational results for the ordinary cohomology rings, and then discuss Kaji's work [26] on the equivariant setting, which provides us with the notion of combinatorial Schubert classes and their formula.

Let G be a connected semisimple linear algebraic group over $\mathbb{C}$ with a maximal torus T of rank n, and let B be a Borel subgroup containing T. Let W be the *Weyl group* of G with respect to T. The cohomology ring $H^\bullet(G/B, \mathbb{Q})$ of the flag variety G/B has two well-known descriptions. One is via the so-called Schubert classes, that is, the $\mathbb{Q}$-basis of $H^\bullet(G/B)$ obtained from the Schubert cell decomposition of the flag variety G/B parametrized by the elements of W (see Borel [7] for example). The other is by Borel's isomorphism which identifies $H^\bullet(G/B)$ with the *coinvariant ring* $\mathcal{S}_W$ of W, that is, the quotient of the ring $\mathcal{S} = \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$, where $\alpha_1, \cdots, \alpha_n$ are a set of simple roots of W, by the ideal generated by the W-invariant polynomials with trivial constant terms (Borel [6]). An explicit connection between these two descriptions was established by Bernstein-Gel'fand-Gel'fand [2] and Demazure [13], where they defined recursive operators, called *divided difference operators*, on the coinvariant ring $\mathcal{S}_W$ (cf. [2, Definition 3.2]), and used them to inductively define the polynomials corresponding to the Schubert classes in $H^\bullet(G/B)$ under Borel's isomorphism (cf. [2, Theorem 3.14, 3.15]). This bridge made the notion of Schubert classes intrinsic to the coinvariant ring $\mathcal{S}_W$, depending solely on the finite Coxeter group W. In his book [24, Chapter IV], Hiller proved that the polynomials $P_w\,(w \in W)$ introduced in [2] form an $\mathbb{R}$-basis of $\mathcal{S}_W$ for an arbitrary finite Coxeter group W, where $\mathcal{S}$ is now taken to be $\mathcal{S} = \mathbb{R}[\alpha_1, \cdots, \alpha_n]$.

Kaji [26] extended Hiller's result to the equivariant setting. Similarly to the case of ordinary cohomology, the T-equivariant cohomology ring $H_T^\bullet(G/B, \mathbb{R})$ has an $\mathcal{S}$-module basis corresponding to the Schubert cell decomposition of G/B. There is also a description of $H_T^\bullet(G/B)$ analogous to the Borel picture (cf. [26, Proposition 4.1]). That is,

$$H_T^\bullet(G/B) \simeq \mathbb{R}[t_1, \cdots, t_n, x_1, \cdots, x_n]/I_W =: \mathbb{R}_W[t; x] \tag{1.1}$$

as $\mathcal{S}$-algebras, where I_W is the ideal generated by polynomials of the form

$$f(x_1, \cdots, x_n) - f(t_1, \cdots, t_n)$$

for nonconstant W-invariant polynomials f. The ring $\mathbb{R}_W[t; x]$ in (1.1) is called the *double coinvariant ring* of W. Kaji generalized Hiller's polynomials and constructed an explicit $\mathcal{S}$-module basis $\{\mathfrak{S}_w(t; x) \mid w \in W\}$ (*Schubert classes*) for the double coinvariant ring $\mathbb{R}_W[t; x]$ of an arbitrary finite Coxeter group W ([26, Theorem 4.5]).

On the other hand, $H_T^\bullet(G/B)$ has yet another, combinatorially given description called a *structure algebra*, which will be the main object of study of this thesis. The *structure algebra* associated with the Weyl group W is given by

$$\mathcal{Z}(\mathcal{G}) = \{\xi \in \bigoplus_{v \in W} \mathcal{S} \mid \xi(v) - \xi(v') \text{ is divisible by } \alpha \in \Phi^+ \text{ with } v <_\alpha v'\},$$

where $<_\alpha$ denotes the Bruhat cover relation on W and $\mathcal{G}$ denotes a certain labelled graph called a *moment graph*, whose vertices are the elements of W (Fiebig [18, Sections 3.1–3.2] or Definitions 3.1.1, 3.2.1). Each element ξ of $\mathcal{Z}(\mathcal{G})$ is an $\mathcal{S}$-valued function on W, and its value at $v \in W$ is denoted by $\xi(v)$. Originated in the paper [21] by Goresky-Kottwitz-MacPherson, the notion of a moment graph and its structure algebra now forms a part of the actively studied theory of sheaves on moment graphs (see Braden-MacPherson [9], Fiebig [18], [19], Lanini [28], [29]).

The structure algebra $\mathcal{Z}(\mathcal{G})$ is an $\mathcal{S}$-algebra under coordinate-wise addition and multiplication. We have the following isomorphism of $\mathcal{S}$-algebras:

$$H_T^\bullet(G/B) \simeq \mathcal{Z}(\mathcal{G}) \tag{1.2}$$

via the so-called localization map ([21, Theorem 7.2]). The ordinary cohomology ring $H^\bullet(G/B)$ is recovered from $H_T^\bullet(G/B)$ by taking the quotient by the ideal generated by polynomials with trivial constant terms (cf. [21, (1.2.4)]). In terms of the structure algebra,

$$H^\bullet(G/B) \simeq \frac{\mathcal{Z}(\mathcal{G})}{\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G})},$$

where $\mathfrak{a}$ denotes the augmentation ideal of $\mathcal{S}$. The $\mathcal{S}$-algebra on the right-hand side is called the *augmented structure algebra* of $\mathcal{G}$ (Lanini–Zainoulline [30, Section 6]). The notions of a moment graph and its (augmented) structure algebra depend only on the Weyl group W and its realization as a real reflection group, and thus they are well-defined for any finite Coxeter group W.

Kaji's result [26, Corollary 4.10] combines the Borel perspective (1.1) and the moment graph perspective (1.2) of $H_T^\bullet(G/B)$ via the generalized localization map

$$\oplus_{w \in W} i_w^* : \mathbb{R}_W[t;x] \to \mathcal{Z}(\mathcal{G}) : f(t;x) \mapsto \big(f(t; w^{-1}(t))\big)_{w \in W}, \tag{1.3}$$

which is an isomorphism of $\mathcal{S}$-algebras. Further, [26, Corollary 4.11] provides an explicit formula for the image $\sigma^{(w)}$ of the Schubert class $\mathfrak{S}_w(t;x)$ under this isomorphism: for an element $v \in W$ and an arbitrary reduced expression $v = s_{i_1} \cdots s_{i_l}$, the value of $\sigma^{(w)}$ at v is given by

$$\sigma^{(w)}(v) = \sum \beta_{j_1} \cdots \beta_{j_{l(w)}}, \tag{1.4}$$

where $\beta_k = s_{i_1} \cdots s_{i_{k}-1}(\alpha_{i_k})$, α_{i_k} is a simple root corresponding to s_{i_k}, and the sum runs over all $1 \leq j_1 < \cdots < j_{l(w)} \leq l$ such that $s_{i_{j_1}} \cdots s_{i_{j_{l(w)}}} = w$. As Kaji observed, (1.4) generalizes Billey's celebrated formula [4, Theorem 4] for the Kostant-Kumar polynomial $\xi^w(v)$ to an arbitrary finite Coxeter group W.

In this thesis, we will refer to the functions $\{\sigma^{(w)} \mid w \in W\}$ as *(combinatorial) Schubert classes* of the structure algebra $\mathcal{Z}(\mathcal{G})$, which recover the geometric Schubert classes of $H_T^\bullet(G/B)$ under (1.1) and (1.2) when W is crystallographic.

Schubert calculus of the folding map

Let $\Phi \rightsquigarrow \Phi_\tau$ be a twisted quadratic folding of a finite crystallographic root system. Let W and W_τ be the finite reflection groups of Φ and Φ_τ respectively. The folding $\Phi \rightsquigarrow \Phi_\tau$ induces an embedding $\varepsilon : W_\tau \hookrightarrow W$ of Coxeter groups (see Lusztig [31, Corollary 3.3]). If we denote the moment graph associated with the original root system Φ by $\mathcal{G}$ and the one associated with the τ-folded root system Φ_τ by $\mathcal{G}_\tau$, the main result of Lanini–Zainoulline [30, Theorem 6.2] states that there is a canonical ring homomorphism $\varepsilon^* : \mathcal{Z}(\mathcal{G}) \to \mathcal{Z}(\mathcal{G}_\tau)$ induced from the embedding $\varepsilon : W_\tau \hookrightarrow W$, which further restricts to the augmented structure algebras $\varepsilon^* : \overline{\mathcal{Z}}(\mathcal{G}) \to \overline{\mathcal{Z}}(\mathcal{G}_\tau)$.

The theme of the present thesis is the Schubert calculus of the folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}) \to \mathcal{Z}(\mathcal{G}_\tau)$. We will take (1.4) as definition of a Schubert class of $\mathcal{Z}(\mathcal{G})$ (Definition 3.3.1), and investigate how the Schubert classes of the original structure algebra $\mathcal{Z}(\mathcal{G})$ relate to those of the folded structure algebra $\mathcal{Z}(\mathcal{G}_\tau)$ under the folding map ε^*. We will look for the description of the ε^*-preimage of a Schubert class of the folded structure algebra $\mathcal{Z}(\mathcal{G}_\tau)$, and in particular find a criterion (see Theorem 1.1 below) for a Schubert class $\sigma_\tau^{(u)}$ of $\mathcal{Z}(\mathcal{G}_\tau)$ to admit a Schubert class $\sigma^{(w)}$ of $\mathcal{Z}(\mathcal{G})$ such that

$$\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)} \text{ for some nonzero constant } c \in L. \tag{1.5}$$

When the relation (1.5) holds between $\sigma^{(w)}$ and $\sigma_\tau^{(u)}$, we say that $\sigma^{(w)}$ is a *lifting* of $\sigma_\tau^{(u)}$ (Definition 5.1.1).

Our interesting discovery is that for a given $u \in W_\tau$, the condition for the Schubert class $\sigma_\tau^{(u)}$ of $\mathcal{Z}(\mathcal{G}_\tau)$ to admit a lifting in $\mathcal{Z}(\mathcal{G})$ involves only the existence of an element $w \in W$ satisfying certain combinatorial conditions. More precisely, we will introduce a map $\varphi : S \to S_\tau$ on the Coxeter generating sets of W and W_τ which serves as a combinatorial counterpart of the folding map ε^* (Section 5.3.1), and define the *folding subset* $W^{\mathcal{F}}$ of W in terms of how φ behaves on the reduced expressions of each element $w \in W$ (Definition 5.4.2). Our first main result can be stated as follows:

Theorem 1.1 (Corollary 5.4.9, Proposition 5.4.12)**.** *Let $u \in W_\tau$ and $w \in W$. Let $w = s_{i_1} \cdots s_{i_l}$ be any W-reduced expression. Then the Schubert class $\sigma^{(w)} \in \mathcal{Z}(\mathcal{G})$ is a lifting of $\sigma_\tau^{(u)} \in \mathcal{Z}(\mathcal{G}_\tau)$ if and only if $w \in W^{\mathcal{F}}$ and $u = \varphi(s_{i_1}) \cdots \varphi(s_{i_l})$. Moreover, the lifting coefficient c in (1.5) equals τ^m for some unique nonnegative integer m, where τ is the parameter appearing in the definition of the twisted quadratic folding.*

If every Schubert class of $\mathcal{Z}(\mathcal{G}_\tau)$ admits a lifting, then we say that ε^* has the *lifting property* (Definition 5.1.1). In general, ε^* does not have this property. For instance, in the smallest example of a twisted quadratic folding from type A_3 to C_2, all the Schubert classes of the folded structure algebra $\mathcal{Z}(\mathcal{G}_\tau)$ admit liftings except the one corresponding to the longest element u_0 of W_τ. Exactly the same statement is true for the folding from type A_4 to H_2 (Example 5.4.14). In general, for the longest element u_0 of W_τ, the Schubert class $\sigma_\tau^{(u_0)}$ is not liftable. As the size of the folded root system increases, there will be more and more Schubert classes in $\mathcal{Z}(\mathcal{G}_\tau)$ that are not liftable as Corollary 5.4.21 suggests. However, by passing to the parabolic setting, in some cases, it is possible to choose a parabolic subgroup $W_P \subsetneq W$ so that the folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ does have the lifting property (Example 5.4.15). In Section 5.5, we will provide a complete list of parabolic subgroups W_P for each folding listed in Section 4.3 such that the folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ has the lifting property.

We will also address the question of surjectivity of the folding map ε^* using the Schubert classes. It turns out that the coordinate formula (1.4) of $\sigma^{(w)}$, achieved by Billey [4] for W crystallographic, and by Kaji [26] for an arbitrary finite reflection group W, is the right tool to prove the surjectivity of ε^* over L. This leads to our second main result:

Theorem 1.2 (Theorem 5.6.2)**.** *The folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}) \otimes_K L \to \mathcal{Z}(\mathcal{G}_\tau)$ is surjective.*

Structure of the thesis

The structure of the thesis is as follows. Chapter 2 reviews the notion of a finite root system in a way suitable to twisted quadratic foldings (Section 2.1), and summarizes the properties of finite Coxeter groups necessary to the main discussion of this thesis (Section 2.2).

Chapter 3 introduces the main objects of our study; a moment graph $\mathcal{G}^P$ associated with the pair (Φ, Φ_P) of a finite root system Φ and its subsystem Φ_P (Section 3.1), the structure algebra $\mathcal{Z}(\mathcal{G}^P)$ over the moment graph $\mathcal{G}^P$ (Section 3.2), and the notion of Schubert classes of $\mathcal{Z}(\mathcal{G}^P)$ (Section 3.3). In Section 3.3, we provide a self-contained summary of some essential properties of the Schubert classes $\sigma^{(w)}$ ($w \in W$) of $\mathcal{Z}(\mathcal{G}^P)$.

Chapter 4 summarizes the construction of a twisted quadratic folding of a finite crystallographic root system developed in [30] (Sections 4.1–4.2), provides a list of examples (Section 4.3), and discusses the induced folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ (Section 4.4). In Section 4.3, we list all the classical foldings (except for the one from type D_4 to G_2), the Lusztig folding from type E_8 to H_4 and its two subfoldings, and describe them in the framework of twisted quadratic foldings. Among them, the description of the folding from type E_6 to F_4 is a new contribution to the literature.

Our main chapter, Chapter 5, is dedicated to the Schubert calculus of the folding map ε^*. After stating the problems (Section 5.1), we compute the ε^*-images of the Schubert classes of $\mathcal{Z}(\mathcal{G}^P)$ for some examples of foldings (Section 5.2). We then introduce an auxiliary map $\varphi : S \to S_\tau$, which captures foldings on the level of Coxeter generating sets of W and W_τ and serves as a primary technical tool in the proofs of our main results (Section 5.3). In Section 5.4, we provide a concrete combinatorial criterion for the liftability of the Schubert classes in $\mathcal{Z}(\mathcal{G}_\tau^P)$ (Theorem 5.4.8, Corollary 5.4.9). In Section 5.5, we return to the examples of foldings listed in Section 4.3, and in each case, provide a complete description of parabolic subgroups W_P of W such that the folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ has the lifting property. In Section 5.6, using the Schubert classes of $\mathcal{Z}(\mathcal{G}^P)$, we prove that the folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ is always surjective after extending the coefficients to L (Theorem 5.6.2).

Finally, Appendix A depicts the Coxeter diagrams of the foldings discussed in Section 4.3, and Appendix B collects all the moment graphs appearing in Section 5.2.

Chapter 2

Coxeter groups and root systems

Every topic investigated in this thesis builds on two fundamental notions: finite root systems and finite Coxeter groups. This chapter provides a summary of these notions in a way that will be adopted in later chapters (Chapters 3–5) and establishes the notation.

2.1 Root data and root systems

In this thesis, we will work with two approaches to the notion of a finite root system. We will first define a root datum following [14, Exposé XXI] and [34, Section 7.4], and then define a finite root system following [25, Section 1.2]. For a unital subring $K \subset \mathbb{R}$, we will define a *geometric K-representation* of a root datum (Definition 2.1.3) and a *K-realized* root system (Definition 2.1.5), where the notions of a root datum and a root system are brought on the same footing. They become essential in Chapter 4 when we define a twisted quadratic folding of a root system.

2.1.1 Definition

Definition 2.1.1 (Root datum)**.** Let Λ be a free $\mathbb{Z}$-module of finite rank. Let $\Lambda^\vee := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ be the dual $\mathbb{Z}$-module. Let $\Phi \subset \Lambda$ be a nonempty finite subset, and consider an embedding $\Phi \hookrightarrow \Lambda^\vee$. We denote the image of each $\alpha \in \Phi$ by $\alpha^\vee$, and set $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$. For each $\alpha \in \Phi$, define $s_\alpha \in \mathrm{GL}_{\mathbb{Z}}(\Lambda)$ and $s_{\alpha^\vee} \in \mathrm{GL}_{\mathbb{Z}}(\Lambda^\vee)$ by the following formulae:

$$\begin{aligned} s_\alpha(x) &= x - \alpha^\vee(x)\alpha \quad (x \in \Lambda), \\ s_{\alpha^\vee}(y) &= y - y(\alpha)\alpha^\vee \quad (y \in \Lambda^\vee). \end{aligned} \tag{2.1.1}$$

We will call the embedding $\Phi \hookrightarrow \Lambda^\vee : \alpha \mapsto \alpha^\vee$ a *root datum* if it satisfies the following conditions:

(RD1) $\alpha^\vee(\alpha) = 2$ for all $\alpha \in \Phi$.

(RD2) $s_\alpha(\Phi) \subset \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$ for all $\alpha \in \Phi$.

(RD3) If $\Phi \cap c\Phi \neq \emptyset$ for some $c \in \mathbb{Z}$, then $c = \pm 1$.

The elements of Φ are called *roots*, and those of $\Phi^\vee$ are called *coroots*. The $\mathbb{Z}$-submodule $\Lambda_r := \mathrm{span}_{\mathbb{Z}}(\Phi)$ is called the *root lattice*. The *Weyl group of* Φ, denoted by $W(\Phi)$, is the subgroup of the automorphism group $\mathrm{GL}_{\mathbb{Z}}(\Lambda)$ generated by s_α $(\alpha \in \Phi)$.

By (RD2), the set of roots Φ is stable under the action of $W(\Phi)$. The set Φ admits a subset Δ which is a $\mathbb{Z}$-module basis of the root lattice Λ_r such that each $\gamma \in \Phi$ has a (unique) expression $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$ where either all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$. Such a set Δ is called a *simple system* of the root datum and its elements are called *simple roots*. If all $c_\alpha \geq 0$ in the above expression, γ is called a *positive root*. If all $c_\alpha \leq 0$, γ is called a *negative root*. The set of positive (resp. negative) roots is denoted by Φ^+ (resp. Φ^-). The *Cartan matrix* of a root datum is the matrix $\mathfrak{C} := (c_{ij})$ where $c_{ij} := {\alpha_j}^\vee(\alpha_i)$ $(\alpha_i, \alpha_j \in \Delta)$.

Remark 2.1.2. (1) Some authors do not require (RD3) in the definition of a root datum. In that case, a root datum satisfying (RD3) is called *reduced* ([34, p. 125]).

(2) Any two simple systems Δ, Δ' are conjugate under the action of $W(\Phi)$, that is, $w(\Delta) = \Delta'$ for some $w \in W$ ([14, Exposé XXI, Corollary 3.3.7]).

Definition 2.1.3 (Geometric representation of a root datum)**.** Let $K \subset \mathbb{R}$ be a (unital) subring, and let $\Phi \hookrightarrow \Lambda^\vee$ be a root datum. Let U be a free K-module such that U contains $\Lambda \otimes_{\mathbb{Z}} K$ as a K-submodule. Let $(-,-)$ denote a dot product on U. If $\alpha^\vee(x) = \frac{2(\alpha,x)}{(\alpha,\alpha)}$ for all $\alpha \in \Phi$ and $x \in \Lambda$, then we call the inclusion $\Phi \subset \Lambda \hookrightarrow \Lambda \otimes_{\mathbb{Z}} K \subset U$ a *geometric K-representation* of the root datum and denote it by $(\Phi, U)_K$. By extending the formula (2.1.1) over U, we obtain $s_\alpha \in \mathrm{GL}_K(U)$. We define the Weyl group associated with the geometric K-representation $(\Phi, U)_K$ to be the subgroup $\langle s_\alpha \mid \alpha \in \Phi \rangle$ of $\mathrm{GL}_K(U)$.

Definition 2.1.4 (Finite root system)**.** Let $\mathbb{E}$ be an $\mathbb{R}$-vector space equipped with an inner product $(-,-)$. Let $\Phi \subset \mathbb{E} \backslash \{0\}$ be a nonempty finite subset. Let $\mathrm{O}(\mathbb{E})$ denote the orthogonal transformation group of $\mathbb{E}$. For each $\alpha \in \Phi$, define $s_\alpha \in \mathrm{O}(\mathbb{E})$ by the following formula:

$$s_\alpha(x) = x - \tfrac{2(\alpha,x)}{(\alpha,\alpha)}\alpha \quad (x \in \mathbb{E}).$$

We will call the set Φ a *root system* if it satisfies the following conditions:

(R1) $s_\alpha(\Phi) \subset \Phi$ for all $\alpha \in \Phi$.

(R2) If $\Phi \cap c\Phi \neq \emptyset$ for some $c \in \mathbb{R}$, then $c = \pm 1$.

The elements of Φ are called *roots*. The *Weyl group of* Φ, denoted by $W(\Phi)$, is the subgroup of $\mathrm{O}(\mathbb{E})$ generated by s_α ($\alpha \in \Phi$). The group $W(\Phi)$ is also called *finite real reflection group*.

By (R1), the root system Φ is stable under the action of $W(\Phi)$. Similarly to the case of root datum, the set of roots Φ admits a subset Δ which is a basis of the subspace $U_r := \mathrm{span}_{\mathbb{R}}(\Phi)$ of U such that each $\gamma \in \Phi$ has a (unique) expression $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$ where either all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$. Then we define the terms *simple system*, *simple roots*, *positive (resp. negative) roots* in the same way as we did for the root datum. An element $x \in U_r$ is called *dominant with respect to* Δ if $(x, \alpha) \geq 0$ for all $\alpha \in \Delta$. The *Cartan matrix* of the root system Φ is the matrix $\mathfrak{C} := (c_{ij})$ where $c_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ ($\alpha_i, \alpha_j \in \Delta$). The *minimal coefficient ring* of the root system Φ is the minimal subring $K_{min} \subset \mathbb{R}$ containing all the values c_{ij} of the Cartan matrix $\mathfrak{C}$. The root system Φ is called *crystallographic* if $K_{min} = \mathbb{Z}$.

Let us observe that Definition 2.1.4 remains valid if we replace $\mathbb{E}$ by a free module over any subring K of $\mathbb{R}$ containing K_{min}.

Definition 2.1.5 (K-realization of a root system)**.** Let Φ be a root system with the minimal coefficient ring K_{min}. Let $K \subset \mathbb{R}$ be any subring containing K_{min}. Let U be a free K-module equipped with a dot product $(-,-)$. The root system Φ admits a realization inside U, that is, $\Phi \subset U$ satisfying (R1)–(R2) of Definition 2.1.4. The realization $\Phi \subset U$ will be called a *K-realization* of the root system. The *Weyl group* of Φ with respect to the K-realization $\Phi \subset U$ is the subgroup of $\mathrm{GL}_K(U)$ generated by s_α ($\alpha \in \Phi$).

Let Φ be a crystallographic root system, and let $K \subset \mathbb{R}$ be any (unital) subring. Let U be a free K-module equipped with a dot product $(-,-)$. Let $\Phi \subset U$ be a K-realization of the root system with a simple system Δ. Let $\Lambda \subset U$ be a free $\mathbb{Z}$-submodule generated by the basis Δ (i.e., $\Lambda = \mathrm{span}_{\mathbb{Z}} \Delta$) and set $\Lambda^\vee := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Define $\alpha^\vee(x) = \frac{2(\alpha, x)}{(\alpha, \alpha)}$ for $\alpha \in \Phi, x \in \Lambda$. Since Φ is crystallographic, the assignment $\Phi \hookrightarrow \Lambda^\vee : \alpha \mapsto \alpha^\vee$ is well-defined. Then the inclusion $\Phi \subset \Lambda \subset U$ is a geometric K-representation of the root datum $\Phi \hookrightarrow \Lambda^\vee$ as defined in Definition 2.1.3.

Conversely, let $K \subset \mathbb{R}$ be a unital subring and let $(\Phi, U)_K$ be a geometric K-representation of a root datum $\Phi \hookrightarrow \Lambda^\vee$. Let $\mathbb{E} = U \otimes_K \mathbb{R}$. We have $\Phi \subset U \subset \mathbb{E}$ (the first inclusion is by Definition 2.1.3). Then $\Phi \subset \mathbb{E}$ is a root system as defined in Definition 2.1.4, and $\Phi \subset U$ is a K-realization of the root system as defined in Definition 2.1.5.

The two distinct notions as defined in Definitions 2.1.3 and 2.1.5 will be used when we discuss the construction of a twisted quadratic folding in Chapter 4, where the input of a folding is a root datum and the output is a not necessarily crystallographic finite root system.

2.1.2 Examples

Example 2.1.6 (Type $A_n (n \geq 1)$)**.** Let U be a free $\mathbb{Z}$-module of rank $n+1$. Let $\{e_i \mid 1 \leq i \leq n+1\}$ denote a basis of U and let $(-,-)$ be the associated dot product. Define a submodule Λ by

$$\Lambda = \{x \in U \mid (x, e_1 + \cdots + e_{n+1}) = 0\}.$$

This is a free $\mathbb{Z}$-module of rank n. Let us take a finite subset $\Phi := \{e_i - e_j \mid i \neq j\} \subset \Lambda$ and define a $\mathbb{Z}$-linear functional $\alpha^\vee(x) = \frac{2(\alpha,x)}{(\alpha,\alpha)}$ for $\alpha \in \Phi, x \in \Lambda$. Then the assignment $\Phi \hookrightarrow \Lambda^\vee : \alpha \mapsto \alpha^\vee$ satisfies the conditions (RD1)–(RD3) of Definition 2.1.1. By setting $K = \mathbb{Z}$, $(\Phi, U)_K$ is a geometric K-representation of the root datum.

Example 2.1.7 (Type $D_n (n \geq 4)$)**.** Let Λ be a free $\mathbb{Z}$-module of rank n. Let $\{e_i \mid 1 \leq i \leq n\}$ denote a basis of Λ and let $(-,-)$ be the associated dot product. Define a finite subset $\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \subset \Lambda$ and a $\mathbb{Z}$-linear functional $\alpha^\vee(x) = \frac{2(\alpha,x)}{(\alpha,\alpha)}$ for $\alpha \in \Phi, x \in \Lambda$. Then the assignment $\Phi \hookrightarrow \Lambda^\vee : \alpha \mapsto \alpha^\vee$ satisfies the conditions (RD1)–(RD3) of Definition 2.1.1. By setting $K = \mathbb{Z}$, $(\Phi, \Lambda)_K$ is a geometric K-representation of the root datum. The following set of elements forms a simple system of Φ.

$$\begin{aligned}
\alpha_1 &= (1, -1, 0, 0, \cdots, 0) = e_1 - e_2, \\
\alpha_2 &= (0, 1, -1, 0, \cdots, 0) = e_2 - e_3, \\
&\cdots \\
\alpha_{n-2} &= (0, 0, 0, \cdots, 1, -1, 0) = e_{n-2} - e_{n-1}, \\
\alpha_{n-1} &= (0, 0, 0, \cdots, 0, 1, -1) = e_{n-1} - e_n, \\
\alpha_n &= (0, 0, 0, \cdots, 0, 1, 1) = e_{n-1} + e_n.
\end{aligned}$$

Example 2.1.8 (Type E_6)**.** Let $\mathbb{E} = \mathbb{R}^8$. Let $\{e_i \mid 1 \leq i \leq 8\}$ denote a basis of $\mathbb{E}$ and let $(-,-)$ be the associated dot product. Let us define $\Phi \subset \mathbb{E}$ by

$$\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 5\} \cup \{\pm \tfrac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i) \mid \sum_{i=1}^{5} \nu(i) \text{ is even}\}.$$

This set satisfies (R1)–(R2) of Definition 2.1.4, and Φ is a root system with the following set of elements as a simple system ([8, Plate V]):

$$\begin{aligned}
\alpha_1 &= (\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}) \\
&= \tfrac{1}{2}(e_1 + e_8) - \tfrac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6 + e_7), \\
\alpha_2 &= (1, 1, 0, 0, 0, 0, 0, 0) = e_1 + e_2, \\
\alpha_3 &= (-1, 1, 0, 0, 0, 0, 0, 0) = e_2 - e_1, \\
\alpha_4 &= (0, -1, 1, 0, 0, 0, 0, 0) = e_3 - e_2, \\
\alpha_5 &= (0, 0, -1, 1, 0, 0, 0, 0) = e_4 - e_3, \\
\alpha_6 &= (0, 0, 0, -1, 1, 0, 0, 0) = e_5 - e_4.
\end{aligned}$$

Let us define $\alpha^\vee(x) = \frac{2(\alpha,x)}{(\alpha,\alpha)}$ for $\alpha \in \Phi, x \in \mathbb{E}$. Since ${\alpha_i}^\vee(\alpha_j) \in \mathbb{Z}$ for all $1 \leq i, j \leq 6$, Φ is a crystallographic root system. Let Λ be a free $\mathbb{Z}$-submodule of $\mathbb{E}$ generated by the basis $\Delta := \{\alpha_i \mid 1 \leq i \leq 6\}$. Since Φ is crystallographic, the assignment $\Phi \hookrightarrow \Lambda^\vee : \alpha \mapsto \alpha^\vee$ is well-defined and satisfies (RD1)–(RD3) of Definition 2.1.1. Finally, let $K = \mathbb{Z}[\frac{1}{2}]$ and $U = K^8$. Then U is a free K-submodule of $\mathbb{E}$ containing Λ. Thus $(\Phi, U)_K$ is a geometric K-representation of the root datum $\Phi \hookrightarrow \Lambda^\vee$.

Example 2.1.9 (Type H_4)**.** Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Let us denote the division $\mathbb{R}$-algebra of quaternions by $\mathbb{H}$ with a basis $\{1, \mathbb{i}, \mathbb{j}, \mathbb{k}\}$ over $\mathbb{R}$ and the associated dot product $(-,-)$. Consider the three vectors

$$\{1, \tfrac{1}{2}(1 + \mathbb{i} + \mathbb{j} + \mathbb{k}), \tfrac{1}{2}(\tau + \mathbb{i} + (\tau - 1)\mathbb{j})\} \subset \mathbb{H},$$

and define the set Φ of all vectors obtained from these three by applying even permutations of coordinates and arbitrary sign changes. Then Φ satisfies the conditions (R1)–(R2) of Definition 2.1.4 (see [25, Section 2.13]). The following four vectors form a simple system of Φ.

$$\begin{aligned}
\alpha_1 &= \tfrac{1}{2}\big(\tau - \mathbb{i} + (\tau - 1)\mathbb{j}\big) = \big(\tfrac{1}{2}\tau, -\tfrac{1}{2}, \tfrac{1}{2}(\tau - 1), 0\big), \\
\alpha_2 &= \tfrac{1}{2}\big(-\tau + \mathbb{i} + (\tau - 1)\mathbb{j}\big) = \big(-\tfrac{1}{2}\tau, \tfrac{1}{2}, \tfrac{1}{2}(\tau - 1), 0\big), \\
\alpha_3 &= \tfrac{1}{2}\big(1 + (\tau - 1)\mathbb{i} - \tau\mathbb{j}\big) = \big(\tfrac{1}{2}, \tfrac{1}{2}(\tau - 1), -\tfrac{1}{2}\tau, 0\big), \\
\alpha_4 &= \tfrac{1}{2}\big(-1 - \tau\mathbb{i} + (\tau - 1)\mathbb{k}\big) = \big(-\tfrac{1}{2}, -\tfrac{1}{2}\tau, 0, \tfrac{1}{2}(\tau - 1)\big).
\end{aligned}$$

We have $\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}[\tau]$ for all $1 \leq i, j \leq 4$. Hence the minimal coefficient ring of Φ is $K_{min} = \mathbb{Z}[\tau]$. Let $K = \mathbb{Z}[\frac{1}{2}][\tau]$ and $U = \mathrm{span}_K\{1, \mathbb{i}, \mathbb{j}, \mathbb{k}\}$. Then $\Phi \subset U$ is a K-realization of the root system Φ.

2.2 Coxeter groups

We define the notion of a finite Coxeter group and list its basic properties. They will be the key language to be used in Chapters 3 and 5 when we discuss moment graphs, their associated structure algebras and their Schubert classes.

2.2.1 Definition

Definition 2.2.1. A *Coxeter system* is defined to be a pair (W, S) consisting of a group W (whose identity element we denote by e) and a distinguished generating set $S \subset W$ which is subject only to relations of the following form:

$$(st)^{m(s,t)} = e \quad (s, t \in S), \tag{2.2.1}$$

where $m(s,t) = 1$ if $s = t$ and $m(s,t) \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ otherwise. When no relation holds between $s, t \in S$, we write $m(s,t) = \infty$. The group W is called the *Coxeter group* and the set S is called the *Coxeter generating set.* The *rank* of (W, S) is defined to be the cardinality of S. A Coxeter system (W, S) is called *finite* if the Coxeter group W is finite.

Remark 2.2.2. (a) When $S = \emptyset$, we have $W = \{e\}$.

(b) The Coxeter generating set S is not uniquely determined by W. When we refer to a group W as a Coxeter group on its own, we implicitly assume that a Coxeter generating set S is chosen for W and fixed.

Here we provide an explicit description of W as a group presented by the generating set S and relations (2.2.1). Let $F(S)$ be the free monoid generated by S, that is, $F(S)$ is the set of words on the alphabet S with product given by concatenation of words. Define an equivalence relation $\equiv$ on $F(S)$ by declaring for $\lambda, \mu \in F(S)$, $\lambda \equiv \mu$ if one is obtained from the other by removing or inserting words of the form $(st)^{m(s,t)} = stst \cdots st$ (st concatenated $m(s,t)$ times) for all $s, t \in S$ such that $s \neq t$ and $m(s,t) \neq \infty$. Then W is isomorphic to $F(S)/\equiv$. We have the following:

Proposition 2.2.3 ([5, Proposition 1.1.1], [5, Corollary 1.4.8])**.** *Let (W, S) be a Coxeter system, and let $p : F(S) \twoheadrightarrow W$ be the canonical map. Then*

(a) *Let $s, t \in S$. If $s \neq t$, then their images under p are distinct in W.*

(b) *The image $p(S)$ is a minimal generating set for W,* i.e., *no Coxeter generator can be expressed in terms of the others.*

This allows us to identify S with its image in W, as we did in Definition 2.2.1.

A Coxeter system has a combinatorial description by means of a *Coxeter graph.*

Definition 2.2.4. A *Coxeter graph* is an undirected labelled graph Γ with no loops and no multiple edges, where the edges are labelled by the elements of $\mathbb{Z}_{\geq 3} \cup \{\infty\}$.

There is a canonical correspondence between a Coxeter system and a Coxeter graph. Given a Coxeter system (W, S), we may construct an undirected graph Γ with the vertex set S, where we connect two distinct vertices $s, t \in S$ by an edge if and only if $m(s,t) \geq 3$ and label the edge by the number $m(s,t)$. Conversely, given a Coxeter graph Γ with a vertex set S, we construct a group $W := \langle S \mid (st)^{m(s,t)} = e \,(s, t \in S)\rangle$, where $m(s,s) = 1$ $(s \in S)$, and for $s \neq t \in S$, $m(s,t) = 2$ if there is no edge between them, and $m(s,t)$ equals the label of the edge between s and t if there is an edge between them. By convention, the label $m(s,t) = 3$ in a Coxeter graph is omitted.

The above correspondence between Coxeter systems and Coxeter graphs is bijective up to isomophism of Coxeter systems, since distinct Coxeter graphs give arise to non-isomorphic Coxeter systems as can be deduced from the following proposition:

Proposition 2.2.5 ([5, Proposition 1.1.1])**.** *Let (W, S) be a Coxeter system determined by a Coxeter graph as above. Let s, t be distinct elements of S. Then the order of the group element st in W equals the number $m(s,t)$ in (W, S).*

A Coxeter system (as well as the Coxeter group) is called *irreducible* if the associated Coxeter graph is connected. Finite irreducible Coxeter systems are classified in terms of their Coxeter graphs.

Proposition 2.2.6 ([25, Theorem 2.7])**.** *Each finite irreducible Coxeter system has one of the Coxeter graphs in Figure 2.1, and two such Coxeter systems are isomorphic if and only if they have the same Coxeter graph.*

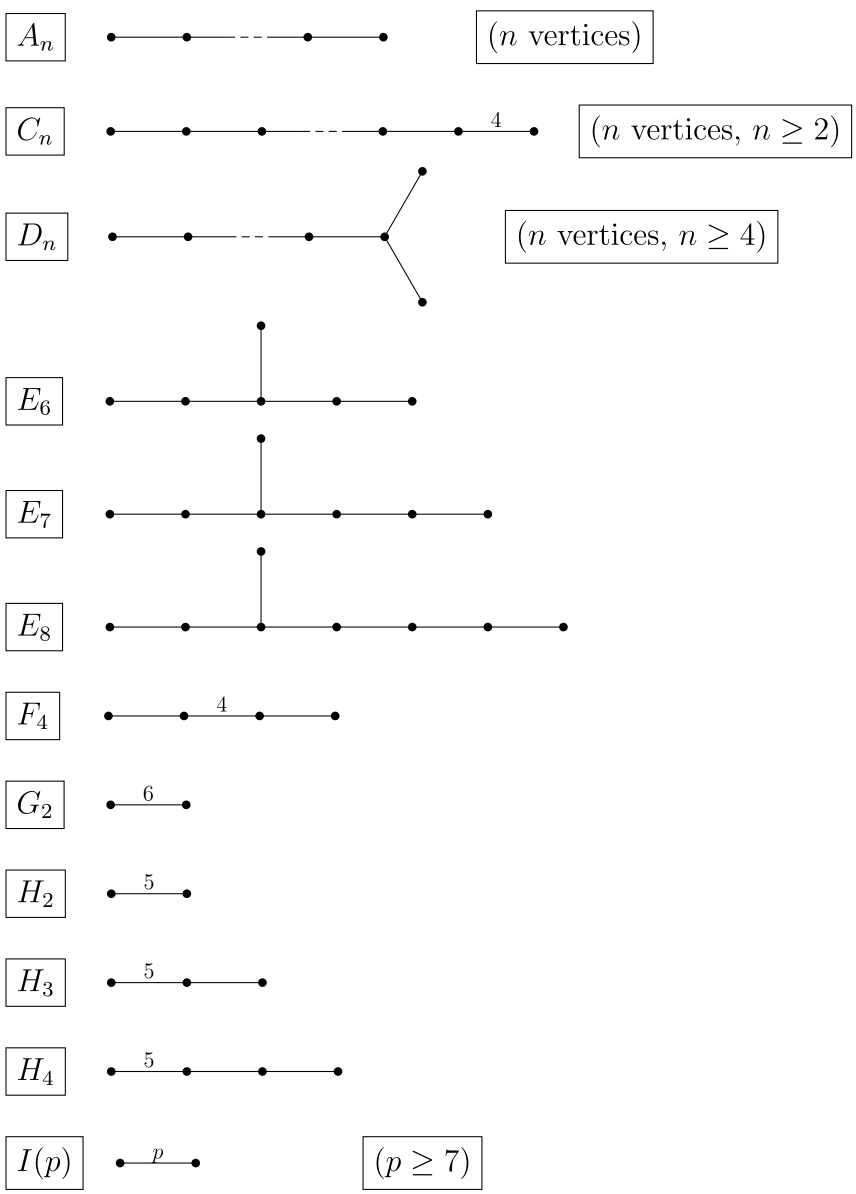


Figure 2.1: Classification of finite Coxeter groups

2.2.2 Examples

Example 2.2.7 (Symmetric groups)**.** Let $\mathfrak{S}_n$ be the group of permutations on n objects. This group is generated by the set S of $n-1$ transpositions

$$S = \{(i, i+1) \mid 1 \leq i \leq n-1\}.$$

Set $s_i := (i, i+1)$ for each i. Then we have $s_i^2 = e$, $(s_i s_{i+1})^3 = e$ for all $1 \leq i \leq n-1$, and $(s_i s_j)^2 = e$ for all $1 \leq i, j \leq n-1$ such that $|i-j| > 1$. Thus $(\mathfrak{S}_n, S)$ forms a Coxeter system.

Notation (One-line notation)**.** Each element $\sigma \in \mathfrak{S}_n$ can be expressed as a $2 \times n$ matrix

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

By the term *one-line notation* of σ, we mean the row matrix consisting solely of the second row of the above matrix, that is, $\sigma(1)\ \sigma(2)\ \cdots\ \sigma(n)$.

Example 2.2.8 (Dihedral groups)**.** Let D_{2m} be the group of symmetries of a regular m-gon on a Euclidean plane $\mathbb{E}$. This group is generated by a reflection $s \in \mathrm{O}(\mathbb{E})$ stabilizing the m-gon and a rotation $r \in \mathrm{O}(\mathbb{E})$ through the angle $\frac{2\pi}{m}$:

$$D_{2m} = \langle s, r \mid s^2 = e, r^m = e \rangle.$$

In order to view this group as a Coxeter group, set $s_1 := s$ and $s_2 := sr$. Then

$$\begin{aligned} s_1^2 &= s^2 = e, \quad (s_1 s_2)^m = (ssr)^m = r^m = e, \\ s_2^2 &= (sr)(sr) = (r^{m-1}s)(sr) = r^m = e. \end{aligned}$$

Thus $(D_{2m}, \{s_1, s_2\})$ forms a Coxeter system.

2.2.3 Length function

Let (W, S) be a Coxeter system. Let $w \in W$ and $w \neq e$. Then the element w can be written as a finite product $w = s_1 s_2 \cdots s_k$ where each $s_i \in S$ (not necessarily distinct) and $k \geq 1$. When k is the smallest possible, the expression is called *reduced* and the number k is called the *length* of w, denoted by $l(w)$. We have $l(e) = 0$. The map $l : W \to \mathbb{Z}_{\geq 0}$ assigning $l(w)$ to each $w \in W$ is called the *length function* of W.

Remark 2.2.9. In the above definition, we refer to the sequence $s_1 \cdots s_k \in F(S)$ itself as a *reduced word*. The term *reduced expression of* $w \in W$ means a reduced word $s_1 \cdots s_k$ such that $s_1 \cdots s_k = w$.

The following are some elementary properties of the length function.

Proposition 2.2.10 ([25, Section 5.2]). *Let $w, w' \in W$ and $s \in S$.*

(a) $l(w^{-1}) = l(w)$.

(b) $l(w) = 1$ *if and only if* $w \in S$.

(c) $l(w) - l(w') \leq l(ww') \leq l(w) + l(w')$.

(d) $l(ws) = l(w) \pm 1$.

When W is finite, W admits a unique element of maximum length called the *longest element*. We will denote it by w_0.

2.2.4 Geometric representation

Let (W, S) be a Coxeter system. The Coxeter group W admits a construction called the *geometric representation*, which is described as follows. Let U be a vector space over $\mathbb{R}$ with a distinguished basis $\{\alpha_s \mid s \in S\}$ parametrised by the generating set S. Then define a symmetric bilinear form $B(-,-) : U \times U \to \mathbb{R}$ by setting

$$B(\alpha_s, \alpha_t) = -\cos \tfrac{\pi}{m(s,t)} \quad (s, t \in S)$$

and extend it bilinearly over U. By convention, we set $B(\alpha_s, \alpha_t) = -1$ if $m(s,t) = \infty$. We have $B(\alpha_s, \alpha_s) = 1, s \in S$ and hence in particular, the vector α_s is non-isotropic. Hence if we define a subspace $H_s := \{u \in U \mid B(u, \alpha_s) = 0\}$ for each $s \in S$, we have $\mathbb{R}\alpha_s \oplus H_s = U$. We may now define a *reflection* $\varrho_s \in \mathrm{GL}(U)$ corresponding to $s \in S$ by

$$\varrho_s(u) = u - 2B(u, \alpha_s)\alpha_s \quad (u \in U), \tag{2.2.2}$$

which indeed satisfies the properties $\varrho_s(\alpha_s) = -\alpha_s$ and $\varrho_s(u) = u$ for all $u \in H_s$.

Proposition 2.2.11 ([25, Proposition 5.3 and Proposition 5.4]). *There exists a unique faithful representation $\varrho : W \to \mathrm{GL}(U)$ that sends $s \in S$ to $\varrho_s \in \mathrm{GL}(U)$.*

The group embedding ϱ is called the *geometric representation* of (W, S). Since

$$B(\varrho_s(u), \varrho_s(v)) = B(u, v)$$

for all $s \in S, u, v \in U$, the bilinear form $B(-,-)$ on U is W-invariant under this representation.

2.2.5 Roots and reflections

Let $\varrho : W \to \mathrm{GL}(U)$ be the geometric representation of (W, S). Each $w \in W$ acts on U via ϱ. Let us write $w(u)$ to denote $\varrho(w)(u)$ for $u \in U$. We define the *root system of* W to be the subset $\Phi := \{w(\alpha_s) \mid s \in S, w \in W\} \subset U$ and we call its elements *roots*. The set Φ has the property that each $\alpha \in \Phi$ is expressed as a linear combination $\alpha = \sum_{s \in S} c_s \alpha_s$ where either all $c_s \geq 0$ or all $c_s \leq 0$. In the former case, α is called a *positive root* and in the latter case, it is called a *negative root*. The set of all positive (resp. negative) roots is denoted by Φ^+ (resp. Φ^-). We have the following:

Proposition 2.2.12 ([25, Theorem 6.4]). *The following conditions on a Coxeter system (W, S) are equivalent:*

(a) *The Coxeter group W is finite.*

(b) *The bilinear form $B(-,-)$ is positive definite.*

(c) *The root system Φ is finite.*

If one of the above equivalent conditions holds, then U is a Euclidean space, and the finite subset $\Phi \subset U$ satisfies (R1)–(R2) of Definition 2.1.4. The group $\varrho(W)$ coincides with the reflection group $W(\Phi)$ in the sense of Definition 2.1.4. Conversely, we have

Proposition 2.2.13 ([25, Theorem 1.9]). *Let Φ be a finite root system as defined in Definition 2.1.4 with a simple system Δ. Let $W := W(\Phi)$ and $S := \{s_\alpha \mid \alpha \in \Delta\}$. Then the pair (W, S) forms a Coxeter system.*

Remark 2.2.14. (a) By Propositions 2.2.12 and 2.2.13, finite Coxeter groups are identified with finite real reflection groups defined in Definition 2.1.4, and the bilinear form $B(-,-)$ is identified with $(-,-)$. From now on, the perspectives of finite Coxeter groups and finite real reflection groups will be used interchangeably unless otherwise stated.

(b) A finite root system Φ with a simple system Δ is called *simply laced* if in the Coxeter system $(W(\Phi), S)$ as given in Proposition 2.2.13, $m_{\alpha\beta} \in \{2, 3\}$ holds for every Coxeter relation $(s_\alpha s_\beta)^{m_{\alpha\beta}} = e$.

Recall that each $s \in S$ acts as a reflection on U under ϱ (see formula (2.2.2)). We associate a *reflection* with an arbitrary root $\alpha \in \Phi$ as follows: we have $\alpha = w(\alpha_s)$ for some $s \in S, w \in W$. Set $s_\alpha := wsw^{-1}$. Then for $u \in U$

$$\begin{aligned} s_\alpha(u) &= wsw^{-1}(u) \\ &= w\big(w^{-1}(u) - 2B(w^{-1}(u), \alpha_s)\alpha_s\big) \\ &= u - 2B(w^{-1}(u), \alpha_s)w(\alpha_s) \\ &= u - 2B(u, \alpha)(\alpha), \end{aligned}$$

and s_α acts as a reflection corresponding to α: s_α sends α to $-\alpha$ and fixes pointwise the hyperplane orthogonal to α. Let us denote the set of all reflections of W by T. Explicitly,

$$T = \bigcup_{w \in W} wSw^{-1} = \{s_\alpha \mid \alpha \in \Phi^+\}.$$

We have the following statements regarding the relationship between the length function $l : W \to \mathbb{Z}_{\geq 0}$ and reflections of W.

Proposition 2.2.15 ([25, Proposition 5.7])**.** *Let $w \in W$, $\alpha \in \Phi^+$. Then $l(ws_\alpha) > l(w)$ if and only if $w(\alpha) \in \Phi^+$. Equivalently, $l(s_\alpha w) > l(w)$ if and only if $w^{-1}(\alpha) \in \Phi^+$.*

Proposition 2.2.16 (Strong Exchange Condition, [25, Theorem 5.8])**.** *Let $w = s_{i_1} \cdots s_{i_r}$ be a (not necessarily reduced) expression for $w \in W$ and let $t \in T$ be a reflection. If $l(wt) < l(w)$, then $wt = s_{i_1} \cdots \widehat{s_{i_p}} \cdots s_{i_r}$ (s_{i_p} omitted) for some $1 \leq p \leq r$. If $w = s_{i_1} \cdots s_{i_r}$ is reduced, then such p is unique.*

2.2.6 Parabolic subgroups

Let P be a (possibly empty) subset of S. The *parabolic subgroup* W_P is the subgroup of W generated by the elements of $P \subset W$. The pair (W_P, P) forms a Coxeter system, and its length function coincides with the restriction of $l : W \to \mathbb{Z}_{\geq 0}$ to W_P (see [25, Theorem 5.5]).

Notation. If $S = \{s_1, \cdots, s_n\}$ and I is a (possibly empty) subset of $\{1, \cdots, n\}$, then we write P_I to denote the subset $\{s_i \mid i \notin I\} \subset S$. For example, $P_{1,n} = \{s_2, \cdots, s_{n-1}\}$. This will be used when we work with examples in Chapters 3–5.

Let us define the subset $W^P := \{w \in W \mid l(ws) > l(w) \text{ for all } s \in P\}$.

Proposition 2.2.17 ([5, Proposition 2.4.4])**.** *Each $w \in W$ can be written as the product $w = vx$ for some unique $v \in W^P$ and $x \in W_P$. Moreover, $l(w) = l(v) + l(x)$.*

The set W^P is thus the unique set of minimal length representatives of the left coset space W/W_P.

Notation. For $w \in W$, we will denote the unique element of $wW_P \cap W^P$ by $\overline{w}$.

We remark that when W is finite, W^P also admits a unique element of maximum length, namely, $\overline{w_0}$. This can be understood as follows. Suppose there exists $w \in W^P$ such that $l(w) > l(\overline{w_0})$. We have $w_0 = \overline{w_0}x$ for some unique $x \in W_P$ and $l(w_0) = l(\overline{w_0}) + l(x)$ by the above proposition. On the other hand, we also have $l(wx) = l(w) + l(x) > l(\overline{w_0}) + l(x) = l(w_0)$, a contradiction. Hence $l(\overline{w_0}) > l(w)$ for all $w \in W^P$.

Let $\varrho : W \to \mathrm{GL}(U)$ be the geometric representation of (W,S). Let $U^\vee = \mathrm{Hom}_{\mathbb{R}}(U,\mathbb{R})$ be the dual vector space of U, and let $\langle -,- \rangle : U \times U^\vee \to \mathbb{R}$ be a bilinear form given by $\langle u, f\rangle = f(u)$. Let us consider the contragredient action of W on $U^\vee$: $(w.f)(u) = f\big(w^{-1}(u)\big)$, $w \in W, f \in U^\vee, u \in U$. For each $s \in S$, define subsets of $U^\vee$ as follows:

$$Z_s := \{f \in U^\vee \mid \langle \alpha_s, f\rangle = 0\},$$
$$O_s := \{f \in U^\vee \mid \langle \alpha_s, f\rangle > 0\}.$$

and define

$$C_P := \big(\bigcap_{s\in P} Z_s\big) \cup \big(\bigcap_{s \in S\setminus P} O_s\big).$$

Then we have the following characterization of the parabolic subgroup W_P.

Proposition 2.2.18 ([25, Theorem 5.13 (a)])**.** *Let $f_P \in C_P$. Then the stabilizer subgroup of f_P equals W_P: $\mathrm{Stab}_W(f_P) = W_P$.*

2.2.7 Bruhat order

For each $w \in W$, consider the following binary relation:

$$w \le s_\alpha w \quad \text{if} \quad \alpha \in \Phi^+ \quad \text{and} \quad l(w) < l(s_\alpha w).$$

By taking the transitive closure, we obtain a partial order $\le$ on W, called the *(strong) Bruhat order* of W.

Notation. Let $w, w' \in W$. We will write $w < w'$ (resp. $w > w'$) to denote $w \le w'$ and $w \neq w'$ (resp. $w \ge w'$ and $w \ne w'$).

Let $w \in W$ and let $w = s_1 \cdots s_k$ be a reduced expression. A *subexpression* of this reduced expression is defined to be a (not necessarily reduced) sequence $s_{a_1} \cdots s_{a_r}$ where $1 \le a_1 < \cdots < a_r \le k$. We will frequently use the following characterization of the Bruhat order in Chapter 5.

Proposition 2.2.19 ([25, Theorem 5.10])**.** *Let $w, w' \in W$. Let $w = s_1 \cdots s_k$ be a reduced expression. Then $w' \le w$ if and only if w' admits a reduced expression which is a subexpression of $s_1 \cdots s_k$.*

2.2.8 Word property

Let (W,S) be a Coxeter system and let $F(S)$ be the free monoid generated by S. For each pair $s,t \in S$ such that $m(s,t) < \infty$, we adopt the notation $[s,t]_{m(s,t)}$ for the word $stst\cdots \in F(S)$ of length $m(s,t)$. We consider the following two kinds of operations on $F(S)$: the *nil-move* where we replace the occurrence of ss by the empty word $\emptyset$ for each $s \in S$ and the *braid-move* where we replace the occurrence of $[s,t]_{m(s,t)}$ by $[t,s]_{m(s,t)}$ for each pair $s,t \in S$ with $m(s,t) < \infty$. Let us define a binary relation (*braid relation*) on $F(S)$ by declaring $\lambda \sim \mu$ for $\lambda, \mu \in F(S)$ if and only if λ can be obtained from μ by some braid move. The transitive closure of this relation is an equivalence relation called the *braid relation.* The equivalence class of a word $\lambda \in F(S)$ is called a *braid equivalence class* of λ.

Proposition 2.2.20 ([5, Theorem 3.3.1])**.** *Let $w \in W$. Then*

(a) *Each expression $w = s_1 \cdots s_k$ can be transformed into a reduced expression by a finite sequence of nil-moves and braid-moves.*

(b) *The set of all reduced words representing w forms a single braid equivalence class in $F(S)$.*

Notation. We denote the set of all reduced words representing $w \in W$ by $\mathcal{R}(w)$. We have $\mathcal{R}(w) \subset F(S)$. We define

$$\mathcal{R}(W) := \bigcup_{w \in W} \mathcal{R}(w).$$

2.2.9 Orbit method

Let (W,S) be a finite Coxeter system, and let P be a (possibly empty) subset of S. Let $\varrho : W \hookrightarrow \mathrm{GL}(U)$ be the geometric representation, and let $\Delta, \Phi \subset U$ denote the simple system and the root system of W respectively. By Proposition 2.2.18, we may take a dominant vector $\theta \in U$ with respect to Δ such that its stabilizer subgroup in W is W_P.

The W-orbit $W\theta$ is equipped with the partial order given by taking the transitive closure of the following binary relation:

$$x < s_\beta(x) \text{ for all } \beta \in \Phi^+ \text{ with } B(x,\beta) > 0. \tag{2.2.3}$$

We have the following result. A proof is included for the convenience of the reader (also taken from [36, Proposition 1.1]).

Proposition 2.2.21 ([36, Proposition 1.1])**.** *The orbit map $w \mapsto w(\theta)$ restricts to a poset isomorphism $(W^P, \leq) \simeq (W\theta, \leq)$.*

Proof: The claim that the restriction of the orbit map $W^P \to W\theta$ is bijective follows from the fact that W_P is the stabilizer subgroup of θ in W. What needs to be shown is that the map $W^P \to W\theta : w \mapsto w\theta$ and its inverse both preserve the partial orders on W^P and on $W\theta$. Note also that it suffices to prove the statement for the cover relations.

First direction: We aim to show that the orbit map $W^P \to W\theta : w \mapsto w\theta$ is order-preserving. Let $w < \overline{s_\beta w}$ where $w \in W^P$ and $\beta \in \Phi^+$, which implies $w < s_\beta w$ and $w \neq \overline{s_\beta w}$. Since $l(w) < l(s_\beta w)$, we have $w^{-1}(\beta) \in \Phi^+$ (see Proposition 2.2.15). Hence $B(w\theta, \beta) = B(\theta, w^{-1}(\beta)) \geq 0$ (in the last inequality, the fact that θ is dominant and $w^{-1}(\beta) \in \Phi^+$ is used). If $B(w\theta, \beta) = 0$, then $s_\beta(w\theta) = w\theta$ hence $w^{-1}s_\beta w \in W_P$. If we write $w^{-1}s_\beta w = w_p$ for some $w_P \in W_P$, we obtain $w = s_\beta w w_P$, hence $w = \overline{s_\beta w}$, a contradiction to our assumption. Thus $B(w\theta, \beta) > 0$. Now $s_\beta(w\theta) > w\theta$ follows (see (2.2.3)).

Second direction: We aim to show that the inverse of the orbit map is also order-preserving. Let $x < s_\beta(x)$ where $x \in W\theta$ and $\beta \in \Phi^+$. By (2.2.3), we have $B(x, \beta) > 0$. Let us write $x = w\theta$ for some (unique) $w \in W^P$. Then $B(w\theta, \beta) = B(\theta, w^{-1}(\beta)) > 0$. Since θ is dominant, we obtain $w^{-1}(\beta) \in \Phi^+$. By Proposition 2.2.15, $l(w) < l(s_\beta w)$ and hence $w < s_\beta w$ follows. Finally, let $v \in W_P$ be the unique element such that $(s_\beta w)v = \overline{s_\beta w}$. Since $w \in W^P$, we have $w \leq wv$. The two elements wv and $s_\beta(wv)$ are related by Bruhat cover relation. If $s_\beta(wv) < wv$ in W, then under the orbit map, it would yield $s_\beta(wv)\theta < (wv)\theta$ in $W\theta$, which is equivalent to $s_\beta w\theta < w\theta$, but this contradicts our assumption. Hence $wv < s_\beta(wv)$. Thus we have obtained $w < s_\beta(wv) = \overline{s_\beta w}$ in W^P. ∎

Chapter 3

Moment graphs and structure algebras

In this chapter, we introduce the notion of a *moment graph* $\mathcal{G}^P$ associated with a pair (Φ, Φ_P) of a finite root system Φ and its subsystem Φ_P (Section 3.1). We define the *structure algebra* $\mathcal{Z}(\mathcal{G}^P)$ over the moment graph $\mathcal{G}^P$ (Section 3.2), then provide a self-contained account of the *Schubert classes* of $\mathcal{Z}(\mathcal{G}^P)$ (Section 3.3).

3.1 Moment graphs

We will first give an abstract definition of a moment graph, then specialize it to a moment graph associated with a pair (Φ, Φ_P) of a finite root system Φ and its subsystem Φ_P.

3.1.1 Definition

Definition 3.1.1 (Moment graph, cf. [18, Section 3.1], [20, Section 2.1])**.** Let R be a commutative ring and M an R-module. A *moment graph over* M, denoted by $\mathcal{G}$, is a directed labelled graph with no loops and no multiple edges which satisfies the conditions listed below. By the term *loop*, we mean an edge which starts and ends at the same vertex. We write $v \to v'$ to denote an edge starting from vertex v and ending at vertex v'.

(M1) The vertex set of $\mathcal{G}$ is a poset $(\mathcal{V}, \leq)$,

(M2) The direction of an edge is compatible with the underlying partial order, *i.e.*, if $v \to v'$ is an edge of $\mathcal{G}$, then $v \leq v'$,

(M3) Each edge is labelled by an element of $M \backslash \{0\}$.

We denote the edge set by $\mathcal{E}$, and the labelling function by $\mathcal{L} : \mathcal{E} \to M\backslash\{0\}$.

Remark 3.1.2. The reader will notice that in the definition above, M can be any set. However, in this thesis, M is always naturally an R-module as is shown below.

We now introduce an important class of moment graphs, namely the one associated with finite root systems. Let $\Phi \subset U$ be a K-realized finite root system (see Definition 2.1.5) with a chosen simple sytem $\Delta \subset \Phi$ and let $W = W(\Phi)$ be its Weyl group. We may associate with it a moment graph $\mathcal{G} = ((\mathcal{V}, \leq), \mathcal{E}, \mathcal{L})$ by setting

- the vertex set to be the Bruhat poset $(W, \leq)$,
- the edge set to be $\mathcal{E} := \{w \to s_\alpha w \mid w \in W, w < s_\alpha w, \alpha \in \Phi^+\}$,
- the labelling function to be $\mathcal{L}(w \to s_\alpha w) = \alpha$ for each $\alpha \in \Phi^+$.

Here we are setting $R = K$ and $M = \operatorname{span}_K \Phi$ in Definition 3.1.1.

Take a (possibly empty) subset $\Delta_P \subset \Delta$, and denote by W_P the subgroup of W generated by $P := \{s_\alpha \mid \alpha \in \Delta_P\}$. Let $W^P \subset W$ be the unique set of minimal length representatives of the left coset space W/W_P (see Section 2.2.6 before Proposition 2.2.17). The Bruhat poset $(W, \leq)$ restricts to the subposet $(W^P, \leq)$. Recall that for each $w \in W$, $\overline{w}$ denotes the unique minimal length representative of the coset wW_P. We may now obtain a moment graph $\mathcal{G}^P$ associated with the pair (Δ, Δ_P) by setting

- the vertex set to be the poset $(W^P, \leq)$,
- the edge set to be $\mathcal{E} := \{w \to \overline{s_\alpha w} \mid w \in W^P, \alpha \in \Phi^+, w < \overline{s_\alpha w}\}$,
- the labelling function to be $\mathcal{L}(w \to \overline{s_\alpha w}) = \alpha$ for each $\alpha \in \Phi^+$.

The following lemma is folklore.

Lemma 3.1.3. *The labelling function $\mathcal{L} : \mathcal{E} \to \Phi^+$ is well-defined.*

Proof: Let $w \to \overline{s_\alpha w}, w \to \overline{s_\beta w} \in \mathcal{E}$ with $\overline{s_\alpha w} = \overline{s_\beta w}$. We aim to show that $\alpha = \beta$. Let $V := U \otimes_K \mathbb{R}$ and take $\theta \in V$ to be a dominant vector with respect to Δ such that $\operatorname{Stab}_W(\theta) = W_P$. By the poset isomorphism $(W^P, \leq) \simeq (W\theta, \leq)$ discussed in Proposition 2.2.21, we have

$$\begin{aligned} \overline{s_\alpha w} = \overline{s_\beta w} &\Leftrightarrow (s_\alpha w)\theta = (s_\beta w)\theta \\ &\Leftrightarrow s_\alpha \mu = s_\beta \mu \quad (\mu := w\theta) \\ &\Leftrightarrow \frac{(\mu, \alpha)}{(\alpha, \alpha)}\alpha = \frac{(\mu, \beta)}{(\beta, \beta)}\beta. \end{aligned}$$

Since $w \to \overline{s_\alpha w} \in \mathcal{E}$, we have the Bruhat relation $w < \overline{s_\alpha w}$ in the poset $(W^P, \leq)$, which under the poset isomorphism of Proposition 2.2.21 transports to $\mu < s_\alpha \mu$ in the poset $(W\theta, \leq)$. In particular, $(\mu, \alpha) > 0$. Since Φ is reduced and α, β are both positive roots, the above equality yields $\alpha = \beta$. ∎

Remark 3.1.4. (a) The moment graph $\mathcal{G}$ associated with a finite root system Φ coincides with the *Bruhat graph* as defined in [5, Definition 2.1.1].

(b) By taking $\Delta_P = \emptyset$ in the definition of $\mathcal{G}^P$, we recover $\mathcal{G}$. That is, $\mathcal{G}^\emptyset = \mathcal{G}$.

3.1.2 Examples

Let Φ_1 be the root system of type A_1 with a simple root denoted by α_1, and let Φ_2 be the root system of type A_2 with simple roots denoted by $\{\alpha_1, \alpha_2\}$. The corresponding Weyl groups are $W(\Phi_1) = \{e, s_1\}$ and $W(\Phi_2) = \{e, s_1, s_2, s_2s_1, s_1s_2, s_1s_2s_1\}$, where each s_i denotes the simple reflection associated with the simple root α_i. The moment graphs of type A_1 and type A_2 are shown in the Figures 3.1 and 3.2 respectively. Each edge is implicitly directed from the vertex with shorter length to the one with longer length.

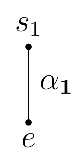


Figure 3.1: Moment graph of type A_1

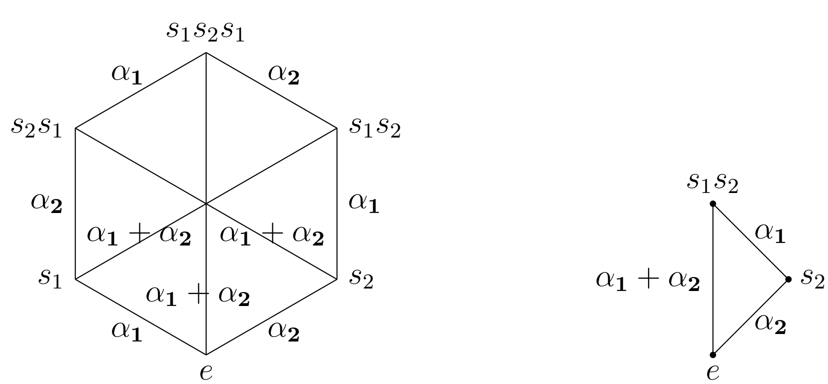


Figure 3.2: Moment graph of type A_2 (left) and type A_2/P_2 (right)

In the example of a root system of type A_2, take the subset $\Delta_P = \{\alpha_1\}$ of the simple roots. Then the set of minimal length representatives of the coset space W/W_P is $W^P = \{e, s_2, s_1 s_2\}$ and the corresponding moment graph is drawn in Figure 3.2.

3.2 Structure algebras

We first define the notion of a structure algebra over an abstract moment graph $\mathcal{G}$, then specialize it to define the structure algebra $\mathcal{Z}(\mathcal{G}^P)$ over the moment graph $\mathcal{G}^P$ associated with a pair (Φ, Φ_P) of a finite root system Φ and its subsystem Φ_P. After basic examples, we will describe two variations of the structure algebra $\mathcal{Z}(\mathcal{G}^P)$, namely, augmented and reduced structure algebras.

3.2.1 Definition

Let $\mathcal{G} = ((\mathcal{V}, \leq), \mathcal{E}, \mathcal{L})$ be a moment graph over a free R-module M, where R is a commutative ring. Set $\mathcal{S} := \mathrm{Sym}_R(M)$, the symmetric algebra of M over R, and $\mathcal{S}_\mathcal{V} := \bigoplus_{v \in \mathcal{V}} \mathcal{S}$, the free $\mathcal{S}$-module generated over the vertex set $\mathcal{V}$. The canonical $\mathcal{S}$-action on $\mathcal{S}_\mathcal{V}$ is written $f \cdot (z_v)_{v \in \mathcal{V}} = (f z_v)_{v \in \mathcal{V}}$ for $f \in \mathcal{S}$ and $(z_v)_{v \in \mathcal{V}} \in \mathcal{S}_\mathcal{V}$. The free $\mathcal{S}$-module $\mathcal{S}_\mathcal{V}$ forms a commutative $\mathcal{S}$-algebra under the coordinate-wise multiplication and addition. We now come to the definition of our main object of study.

Definition 3.2.1 (Structure algebra, [18, Section 3.2])**.** The *structure algebra* of $\mathcal{G}$ is the following $\mathcal{S}$-submodule of $\mathcal{S}_\mathcal{V}$:

$$\mathcal{Z}(\mathcal{G}) := \{(z_v)_{v \in \mathcal{V}} \in \mathcal{S}_\mathcal{V} \mid z_v - z_{v'} \in \mathcal{L}(v \to v')\mathcal{S} \text{ for all } v \to v' \in \mathcal{E}\}.$$

The following lemma is folklore.

Lemma 3.2.2. *$\mathcal{Z}(\mathcal{G})$ is an $\mathcal{S}$-subalgebra of $\mathcal{S}_\mathcal{V}$.*

Proof: It suffices to show that $\mathcal{Z}(\mathcal{G})$ is closed under multiplication. Let $(x_v)_{v \in \mathcal{V}}$, $(z_v)_{v \in \mathcal{V}} \in \mathcal{Z}(\mathcal{G})$ and $v \to v' \in \mathcal{E}$, labelled by some $m \in M \backslash \{0\}$. Then we have $x_v - x_{v'}, z_v - z_{v'} \in m\mathcal{S}$. It follows that $x_v z_v - x_{v'} z_{v'} = (x_v - x_{v'}) z_v + x_{v'}(z_v - z_{v'}) \in m\mathcal{S}$. Thus $(x_v z_v)_{v \in \mathcal{V}} \in \mathcal{Z}(\mathcal{G})$. ∎

Notation. An element $\xi \in \mathcal{Z}(\mathcal{G})$ admits two perspectives: either as an $|\mathcal{V}|$-tuple of $\mathcal{S}$-elements, or as a function $\mathcal{V} \to \mathcal{S}$. Using this second perspective, we will often denote the v-coordinate of ξ by $\xi(v)$.

We now adopt this notion to the moment graphs associated with root systems. Let Φ be a finite root system as defined in Definition 2.1.4, and let K_{min} be the minimal coefficient ring (see the paragraph after Definition 2.1.4 for definition). Let $K \subset \mathbb{R}$ be any subring containing K_{min}. Let U be a free K-module equipped with a basis and the associated dot product $(-,-)$, and let $\Phi \subset U$ be a K-realization of the root system (Definition 2.1.5). Let $\Delta \subset \Phi$ be a simple system, and let Δ_P be a (possibly empty) subset of Δ. Let $W = W(\Phi) \subset \mathrm{Aut}_K(U)$ be the Weyl group of Φ with respect to the K-realization. Let $\mathcal{G}^P$ be the moment graph associated with W^P as defined in Section 3.1.1. Then by setting $R = K$ and $\mathcal{S} = \mathrm{Sym}_K U$ in Definition 3.2.1, we obtain a structure algebra $\mathcal{Z}(\mathcal{G}^P)$.

Notice that in this construction, the polynomial algebra $\mathcal{S}$ depends on the K-realization of the root system $\Phi \subset U$. When the K-realization $\Phi \subset U$ is coming from a geometric K-representation $(\Phi, U)_K$ of a root datum (Section 2.1.1 after Definition 2.1.5), there is a canonical choice for $\mathcal{S}$ as follows. Let $\Phi \hookrightarrow \Lambda^\vee$ be a root datum with a simple system Δ and a (possibly empty) subset $\Delta_P \subset \Delta$. Let $K \subset \mathbb{R}$ be a (unital) subring. Let $(\Phi, U)_K$ be a geometric K-representation of the root datum (Definition 2.1.3). Set $U_\Lambda := \Lambda \otimes_{\mathbb{Z}} K$. By definition of $(\Phi, U)_K$, U contains U_Λ as a free K-submodule. We take $R = K$ and $\mathcal{S} = \mathrm{Sym}_K U_\Lambda$ in Definition 3.2.1 to define the structure algebra $\mathcal{Z}(\mathcal{G}^P)$.

Remark 3.2.3. (a) Viewed as an element of $\mathcal{S}$, each label $\alpha \in \Phi^+$ of the underlying moment graph $\mathcal{G}^P$ has degree one.

(b) After introducing the notion of Schubert classes in the next section, we will see that there is a natural inclusion of structure algebras $\mathcal{Z}(\mathcal{G}^P) \hookrightarrow \mathcal{Z}(\mathcal{G})$ (see Remark 3.3.17).

Remark 3.2.4. Let G be an adjoint semisimple linear algebraic group over $\mathbb{C}$ and let $\Phi \hookrightarrow \Lambda^\vee$ be its root datum. Let T be a maximal torus, B be a Borel subgroup containing T, and P be a parabolic subgroup. Let (Δ, Δ_P) be the pair of sets of simple roots corresponding to the pair (B, P), and let $\mathcal{G}^P$ be the associated moment graph as defined in Section 3.1.1. Then it is known that there is an $\mathcal{S}$-algebra isomorphism

$$H_T^\bullet(G/P, K) \simeq \mathcal{Z}(\mathcal{G}^P), \tag{3.2.1}$$

where $H_T^\bullet(G/P, K)$ denotes the T-equivariant cohomology ring of G/P with coefficients in K (cf. [10, Section 1]). The reader is referred to [21, Theorem 7.2] ($K = \mathbb{C}$) and [11, Theorem 8.11] ($K = \mathbb{Z}$) for the detail of the isomorphism (3.2.1).

3.2.2 Examples

Example 3.2.5. Let $K = \mathbb{Z}$, and let $\Phi \subset U$ be a K-realization of the root system of type A_1 with a simple root α_1 (see Definition 2.1.5), such that $U = \mathrm{span}_K \Phi$. Then $\mathcal{S} = \mathbb{Z}[\alpha_1]$ and

$$\mathcal{Z}(\mathcal{G}) = \{(z_e, z_{s_1}) \in \mathcal{S} \oplus \mathcal{S} \mid z_e - z_{s_1} \in \alpha_1 \mathcal{S}\}.$$

The underlying moment graph is shown in Figure 3.1.

Example 3.2.6. Let $K = \mathbb{Z}$, and let $\Phi \subset U$ be a K-realization of the root system of type A_2 with a simple system $\Delta = \{\alpha_1, \alpha_2\}$ and its subset $\Delta_P = \{\alpha_1\}$, such that $U = \operatorname{span}_K \Phi$. Then $\mathcal{S} = \mathbb{Z}[\alpha_1, \alpha_2]$ and

$$\mathcal{Z}(\mathcal{G}^P) = \Big\{(z_e, z_{s_2}, z_{s_1 s_2}) \in \mathcal{S} \oplus \mathcal{S} \oplus \mathcal{S} \mid z_e - z_{s_2} \in \alpha_2 \mathcal{S},$$
$$z_{s_2} - z_{s_1 s_2} \in \alpha_1 \mathcal{S}, z_e - z_{s_1 s_2} \in (\alpha_1 + \alpha_2)\mathcal{S}\Big\}.$$

The underlying moment graph is shown in Figure 3.2.

3.2.3 Augmented and reduced structure algebras

Let Φ be a finite root system with a simple system Δ and a (possibly empty) subset $\Delta_P \subset \Delta$. Let K_{min} be the minimal coefficient ring of Φ, and let $K \subset \mathbb{R}$ be a subring containing K_{min}. Let $\Phi \subset U$ be a K-realization of the root system (Definition 2.1.5). Let $\mathcal{G}^P$ be the moment graph associated with the pair (Δ, Δ_P) (Section 3.1.1), and let $\mathcal{Z}(\mathcal{G}^P)$ be the structure algebra defined in the paragraph after Lemma 3.2.2. Let $W \subset \mathrm{GL}_K(U)$ be the Weyl group of Φ with respect to the K-realization $\Phi \subset U$ (Definition 2.1.5). In this subsection, we introduce two derived structure algebras; the augmented structure algebra $\overline{\mathcal{Z}}(\mathcal{G}^P)$ and the reduced structure algebra $\widehat{\mathcal{Z}}(\mathcal{G}^P)$.

Definition 3.2.7 ([30, Section 6])**.** Let $a : \mathcal{S} \to K$ be the ring homomorphism given by $\gamma \mapsto 0$ for all $\gamma \in U$. We call this map the *augmentation map*. Let $\mathfrak{a}$ be the kernel of a, thus it is the ideal of $\mathcal{S}$ consisting of elements with trivial constant terms. The *augmented structure algebra* of $\mathcal{G}^P$ is the quotient $\mathcal{S}$-algebra of $\mathcal{Z}(\mathcal{G}^P)$ by the ideal $\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G}^P)$, and it is denoted by $\overline{\mathcal{Z}}(\mathcal{G}^P)$:

$$\overline{\mathcal{Z}}(\mathcal{G}^P) = \mathcal{Z}(\mathcal{G}^P)/\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G}^P).$$

We denote the canonical quotient map by

$$\overline{can} : \mathcal{Z}(\mathcal{G}^P) \twoheadrightarrow \overline{\mathcal{Z}}(\mathcal{G}^P).$$

Remark 3.2.8. Let us recall the setting of Remark 3.2.4. Then (3.2.1) restricts to the K-algebra isomorphism

$$H^\bullet(G/P, K) \simeq \overline{\mathcal{Z}}(\mathcal{G}^P) \tag{3.2.2}$$

where $H^\bullet(G/P, K)$ denotes the cohomology ring of G/P with coefficients in K. See [21, (1.2.4)] for the relation between $H^\bullet_T(G/P)$ and $H^\bullet(G/P)$. For a direct proof for the isomorphism (3.2.2) with $K = \mathbb{Z}$, the reader is referred to [27, Theorem 5.12 and Corollary 5.13].

Example 3.2.9 (Type A_1)**.** Let us recall the setting of Example 3.2.5. We have

$$\overline{\mathcal{Z}}(\mathcal{G}) \simeq \mathbb{Z}[x]/(x^2) \tag{3.2.3}$$

as $\mathbb{Z}$-algebras. In order to describe this isomorphism, the notion of Schubert classes is needed (see Definition 3.3.1, Example 3.3.6). Using the Schubert classes, one builds the $\mathbb{Z}$-module homomorphism $\mathcal{Z}(\mathcal{G}) \to \mathbb{Z}[x]/(x^2)$ by $\sigma^{(e)} \mapsto 1$ and $\sigma^{(s_1)} \mapsto x$, then shows that the kernel equals $\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G})$. In this example, the augmentation ideal $\mathfrak{a}$ is the ideal (α_1) of $\mathbb{Z}[\alpha_1]$. One can check by direct computation that $\sigma^{(e)}$ is the unit element of $\mathcal{Z}(\mathcal{G})$, and that $(\sigma^{(s_1)})^2$ belongs to $\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G})$.

Example 3.2.10 (Type A_2/P_2)**.** Let us recall the setting of Example 3.2.6. We have

$$\overline{\mathcal{Z}}(\mathcal{G}^P) \simeq \mathbb{Z}[x]/(x^3)$$

as $\mathbb{Z}$-algebras. Using the Schubert classes (see Definition 3.3.1, Example 3.3.8), one defines the $\mathbb{Z}$-module homomorphism $\mathcal{Z}(\mathcal{G}^P) \to \mathbb{Z}[x]/(x^3)$ by $\sigma^{(e)} \mapsto 1$, $\sigma^{(s_2)} \mapsto x$, and $\sigma^{(s_1 s_2)} \mapsto x^2$, then shows that the kernel equals $\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G}^P)$. In this example, the augmentation ideal $\mathfrak{a}$ is the ideal (α_1, α_2) of $\mathbb{Z}[\alpha_1, \alpha_2]$. By direct computation, one checks that $(\sigma^{(s_2)})^2 - \sigma^{(s_1 s_2)}$, $\sigma^{(s_2)}\sigma^{(s_1 s_2)}$, $(\sigma^{(s_1 s_2)})^2$ belong to $\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G}^P)$.

The W-action on U naturally extends to $\mathcal{S}$. Let $\mathcal{S}^W$ (resp. $\mathcal{S}^{W_P}$) denote the subring of $\mathcal{S}$ where every element is fixed under the W-action (resp. W_P-action). We have the canonical inclusions of rings $\mathcal{S}^W \subset \mathcal{S}^{W_P} \subset \mathcal{S}$. In particular, $\mathcal{S}$ and $\mathcal{S}^{W_P}$ are both $\mathcal{S}^W$-algebras.

Definition 3.2.11 (Borel map, [15, p. 570])**.** The *equivariant characteristic map* is the assignment $c : \mathcal{S}^{W_P} \to \mathcal{Z}(\mathcal{G}^P) : f \mapsto (w(f))_{w \in W^P}$ (cf. [15, p. 570]). Using this, we define the *Borel map* to be the map given by

$$\rho : \mathcal{S} \otimes_{\mathcal{S}^W} \mathcal{S}^{W_P} \to \mathcal{Z}(\mathcal{G}^P) : f \otimes g \mapsto f \cdot c(g). \tag{3.2.4}$$

When $P = \emptyset$, (3.2.4) coincides with Kaji's map (1.3) which appeared in Chapter 1. The interaction between $\overline{can}$ and the Borel map ρ is summarized in the following lemma.

Lemma 3.2.12 ([30, Section 6])**.** *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{S} \otimes_{\mathcal{S}^W} \mathcal{S}^{W_P} \arrow[r, "\rho"] \arrow[d, "a \otimes \mathrm{id}"] & \mathcal{Z}(\mathcal{G}^P) \arrow[d, "\overline{can}"] \\
K \otimes_{\mathcal{S}^W} \mathcal{S}^{W_P} \arrow[r, "\overline{\rho}"] & \overline{\mathcal{Z}}(\mathcal{G}^P)
\end{tikzcd}$$

where $\overline{\rho}$ is the restriction of ρ to $K \otimes_{\mathcal{S}^W} \mathcal{S}^{W_P}$, that is, $\overline{\rho}(f \otimes g) = f \cdot \mathrm{c}(g)$ for all $f \in K, g \in \mathcal{S}^{W^P}$.

Proof: This follows from the definition of each map: for $f \in \mathcal{S}, g \in \mathcal{S}^{W_P}$,

$$\overline{can}(\rho(f \otimes g)) = \overline{can}\Big(\big(fw(g)\big)_{w \in W^P}\Big) = \Big(a(f)w(g)\Big)_{w \in W^P} = \overline{\rho}\big(a(f) \otimes g\big).$$

∎

Combined with Remark 3.2.8, the restricted Borel map $\overline{\rho}$ describes the ordinary characteristic map $\mathcal{S} \to H^\bullet(G/B, \mathbb{Z})$, which was the original map used by Borel in [6] to prove the isomorphism $\mathcal{S}/\mathcal{S}^W \simeq H^\bullet(G/B, \mathbb{Q})$.

Besides the canonical $\mathcal{S}$-algebra structure, $\mathcal{Z}(\mathcal{G}^P)$ also has an $\mathcal{S}^{W_P}$-algebra structure in the following way: for $g \in \mathcal{S}^{W_P}$ and $(z_w)_{w \in W^P} \in \mathcal{Z}(\mathcal{G}^P)$

$$(z_w)_{w \in W^P \star} g := (z_w)_{w \in W^P} \, c(g) = \big(z_w w(g)\big)_{w \in W^P}. \tag{3.2.5}$$

The action is written on the right so as to distinguish it from the canonical $\mathcal{S}$-action on $\mathcal{Z}(\mathcal{G}^P)$. The canonical $\mathcal{S}$-action and the $\mathcal{S}^{W_P}$-action (3.2.5) commute, and together provide $\mathcal{Z}(\mathcal{G}^P)$ with the structure of an $\mathcal{S} \otimes_{\mathcal{S}^W} \mathcal{S}^{W_P}$-algebra as follows: for $f \otimes g \in \mathcal{S} \otimes_{\mathcal{S}^W} \mathcal{S}^{W_P}$ and $(z_w)_{w \in W^P} \in \mathcal{Z}(\mathcal{G}^P)$,

$$(f \otimes g)(z_w)_{w \in W^P} = \big(f \cdot (z_w)_{w \in W^P}\big)_\star g = \big(f z_w w(g)\big)_{w \in W^P}.$$

Definition 3.2.13 ([30, Section 6])**.** The *reduced structure algebra* of $\mathcal{G}^P$ is the quotient $\mathcal{S} \otimes_{\mathcal{S}^W} \mathcal{S}^{W_P}$-algebra of $\mathcal{Z}(\mathcal{G}^P)$ by the ideal $\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G}^P) + \mathcal{Z}(\mathcal{G}^P)_\star \mathfrak{a}^{W_P}$, and it is denoted by $\widehat{\mathcal{Z}}(\mathcal{G}^P)$:

$$\widehat{\mathcal{Z}}(\mathcal{G}^P) = \mathcal{Z}(\mathcal{G}^P)/(\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G}^P) + \mathcal{Z}(\mathcal{G}^P)_\star \mathfrak{a}^{W_P}).$$

We denote the canonical quotient map by

$$\widehat{can} : \mathcal{Z}(\mathcal{G}^P) \twoheadrightarrow \widehat{\mathcal{Z}}(\mathcal{G}^P). \tag{3.2.6}$$

Remark 3.2.14. (a) We have $\widehat{\mathcal{Z}}(\mathcal{G}^P) = \overline{\mathcal{Z}}(\mathcal{G}^P)/\overline{\mathcal{Z}}(\mathcal{G}^P)_\star \mathfrak{a}^{W_P}$.

(b) Let us recall the setting of Remark 3.2.4 and set $P = B$. Then it is known that there is an isomorphism

$$\mathrm{CH}^\bullet(G, K) \simeq \widehat{\mathcal{Z}}(\mathcal{G}). \tag{3.2.7}$$

where $\mathrm{CH}^\bullet(G, K)$ denotes the Chow ring of G with coefficients in K (cf. [10, Section 3]) The reader is referred to [10, Example 3.2] and [22, p. 21] for the detail of the isomorphism (3.2.7).

3.3 Combinatorial Schubert classes

We keep the setting and notation as described at the beginning of Section 3.2.3. In this section, we provide a self-contained exposition of the notion of *combinatorial Schubert classes* of a structure algebra $\mathcal{Z}(\mathcal{G}^P)$. The main references of this section are [1, Appendix D] and [26].

3.3.1 Definition

Definition 3.3.1 (Schubert classes, [1, Appendix D.3], [26, Corollary 4.11])**.** *The combinatorial Schubert class corresponding to* $w \in W$ *is a function* $\sigma^{(w)} : W \to \mathcal{S}$ given by the following formula: let $x \in W$ and let $x = s_1 s_2 \cdots s_r$ be a (not necessarily reduced) expression and we denote the simple root corresponding to s_i by α_i for each $1 \leq i \leq r$. Then we define if $w \nleq x$, $\sigma^{(w)}(x) = 0$; and if $w \leq x$,

$$\sigma^{(w)}(x) = \sum_{\mathfrak{i}} \prod_{\nu=1}^{\ell} s_1 s_2 \cdots s_{i_\nu - 1}(\alpha_{i_\nu}) \tag{3.3.1}$$

where $\ell = l(w)$, $\mathfrak{i} = (i_1, \cdots, i_\ell) \in \mathbb{Z}^\ell$ such that $1 \leq i_1 < i_2 < \cdots < i_\ell \leq r$ and $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. By convention, we set $\sigma^{(e)}(x) = 1$ for all $x \in W$.

Basic examples of combinatorial Schubert classes will be given in Section 3.3.2. The following result is well-known (cf. [3, Section 4] for the case where the underlying root system Φ is crystallographic). Here we include a direct proof for the convenience of the reader.

Lemma 3.3.2. *The formula* (3.3.1) *does not depend on the choice of the expression* $x = s_1 s_2 \cdots s_r$.

Proof: Let $w, x \in W$ with $w \neq e$ and $w \leq x$. We write $\ell = l(w)$. We will prove the statement by induction on r, the length of the expression of x under consideration.

Base Case: $r = 1$. In this case, the statement is trivially true.

Induction Step: We divide the set of indices $\mathfrak{i}$ appearing in (3.3.1) into two kinds. Let us denote by $\mathfrak{k}$ an $(\ell-1)$-tuple $\mathfrak{k} = (k_1, \cdots, k_{\ell-1}) \in \mathbb{Z}^{\ell-1}$ such that $1 \leq k_1 < \cdots < k_{\ell-1} \leq r-1$ and $w = s_{k_1} \cdots s_{k_{\ell-1}} s_r$ (a reduced expression ending with s_r). Let us denote by $\mathfrak{j}$ an ℓ-tuple $\mathfrak{j} = (j_1, \cdots, j_\ell) \in \mathbb{Z}^\ell$ such that $1 \leq j_1 < \cdots < j_\ell \leq r-1$ and $w = s_{j_1} \cdots s_{j_\ell}$ (a reduced expression not ending with s_r). Then (3.3.1) becomes

$$\begin{aligned}\sigma^{(w)}(x) =& \Big(\sum_{\mathfrak{k}} \prod_{\nu=1}^{\ell-1} s_1 s_2 \cdots s_{k_\nu - 1}(\alpha_{k_\nu}) \Big) s_1 s_2 \cdots s_{r-1}(\alpha_r) \\ &+ \sum_{\mathfrak{j}} \prod_{\nu=1}^{\ell} s_1 s_2 \cdots s_{j_\nu - 1}(\alpha_{j_\nu}).\end{aligned}$$

Let us write $v = ws_r$ and $y = s_1 \cdots s_{r-1}$. Then in the above expression,

$$\sum_{\mathfrak{k}} \prod_{\nu=1}^{\ell-1} s_1 s_2 \cdots s_{k_\nu - 1}(\alpha_{k_\nu}) = \sigma^{(v)}(y) \quad \text{with } y = s_1 \cdots s_{r-1},$$

$$\sum_{\mathfrak{j}} \prod_{\nu=1}^{\ell} s_1 s_2 \cdots s_{j_\nu - 1}(\alpha_{j_\nu}) = \sigma^{(w)}(y) \quad \text{with } y = s_1 \cdots s_{r-1}.$$

By the induction hypothesis, they do not depend on the choice of expression $y = s_1 \cdots s_{r-1}$. It remains to show that $s_1 \cdots s_{r-1}(\alpha_r)$ also does not depend on the expression of x. In order to do this, we make an observation that the only way to change the expression $x = s_1 \cdots s_r$ without moving the last simple reflection s_r and changing the number of simple reflections involved is to apply the braid moves (see Section 2.2.8) to the sequence $s_1 \cdots s_{r-1}$. This operation keeps the group element $y = s_1 \cdots s_{r-1}$ unchanged. Hence we may write $y(\alpha_r)$ in place of $s_1 \cdots s_{r-1}(\alpha_r)$. ∎

Remark 3.3.3. In the case where Φ is crystallographic, the combinatorial Schubert class $\sigma^{(w)}$ defined above is the function denoted by η^w in [1, Appendix D].

By Definition 3.3.1, it is clear that the Schubert class $\sigma^{(w)}$ belongs to $\bigoplus_{x \in W} \mathcal{S}$ for each $w \in W$. Shortly, it will be shown in Proposition 3.3.10 that $\sigma^{(w)}$ actually belongs to the structure algebra $\mathcal{Z}(\mathcal{G})$, and that when $w \in W^P$, the restriction $\sigma^{(w)}|_{W^P}$ belongs to $\mathcal{Z}(\mathcal{G}^P)$. Moreover, the set of all Schubert classes $\{\sigma^{(w)} \mid w \in W\}$ will be shown to form an $\mathcal{S}$-module basis for $\mathcal{Z}(\mathcal{G})$ (Proposition 3.3.15).

Before giving a few examples of Schubert classes, here are some most basic properties of $\sigma^{(w)}$.

Proposition 3.3.4 (Properties of $\sigma^{(w)}$, [1, Appendix D.1])**.** *Let $w \in W$.*

(a) *For each $x \in W$, $\sigma^{(w)}(x)$ is homogeneous of degree $l(w)$.*

(b) $\sigma^{(w)}(w) = \prod_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^-} \alpha$.

(c) *If $w \le x$, then $\sigma^{(w)}(x) \neq 0$.*

Proof: (a) follows directly from the formula (3.3.1).
(b) By Lemma 3.3.2, we may work with a reduced expression $w = s_1 s_2 \cdots s_\ell$ of our choice, where $\ell = l(w)$. Then the formula (3.3.1) gives us

$$\begin{aligned}\sigma^{(w)}(w) &= \prod_{j=1}^{\ell} s_1 s_2 \cdots s_{j-1}(\alpha_j) \\ &= \alpha_1 \cdot s_1(\alpha_1) \cdot s_1 s_2(\alpha_3) \cdot \cdots \cdot (s_1 s_2 \cdots s_{\ell-1})(\alpha_\ell).\end{aligned}$$

For each j, let us write $v = s_1 \cdots s_{j-1}$. Since $l(vs_j) > l(v)$, by Proposition 2.2.15, we have $v(\alpha_j) \in \Phi^+$.

To discuss the next part, let β denote the positive root $v(\alpha_j)$. Then the corresponding reflection is $s_\beta = vs_jv^{-1}$. We aim to compare $l(s_\beta w)$ and $l(w)$. Since

$$\begin{aligned} s_\beta w = (vs_jv^{-1})w &= (s_1 \cdots s_{j-1})(s_j s_{j-1} \cdots s_1)(s_1 s_2 \cdots s_\ell) \\ &= s_1 \cdots s_{j-1} \hat{s_j} s_{j+1} \cdots s_\ell \quad (\text{omission of } s_j), \end{aligned}$$

it follows that $l(s_\beta w) \leq \ell - 1 < l(w)$. By Proposition 2.2.15, we obtain $w^{-1}(\beta) \in \Phi^-$. Thus we have proven (b).
(c) Let $x = s_1 s_2 \cdots s_r$ be a reduced expression where each s_i denotes a simple reflection with the corresponding simple root α_i. By the definition of $\sigma^{(w)}$, we have

$$\sigma^{(w)}(x) = \sum_{\mathfrak{i}} \prod_{\nu=1}^{\ell} s_1 s_2 \cdots s_{i_\nu - 1}(\alpha_{i_\nu})$$

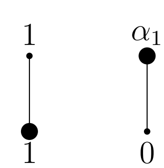
where $\ell = l(w)$, $\mathfrak{i} = (i_1, \cdots, i_\ell) \in \mathbb{Z}^\ell$ such that $1 \leq i_1 < i_2 < \cdots < i_\ell \leq r$ and $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is reduced. Since the expression $s_1 s_2 \cdots s_{i_\nu}$ is reduced for each $\mathfrak{i}$ and ν, by the same argument as we made for the first part of (b), it follows that $s_1 s_2 \cdots s_{i_\nu - 1}(\alpha_{i_\nu}) > 0$. Therefore, from the above expression of $\sigma^{(w)}(x)$, we have $\sigma^{(w)}(x) \neq 0$. ∎

Remark 3.3.5. By (a) of the above proposition, we may call the common homogeneous degree of $\sigma^{(w)}(x)$ $(x \in W)$ the *degree of* $\sigma^{(w)}$, and we will denote it by $\deg \sigma^{(w)}$. By Proposition 3.3.4 (b) and Proposition 2.2.15, we have

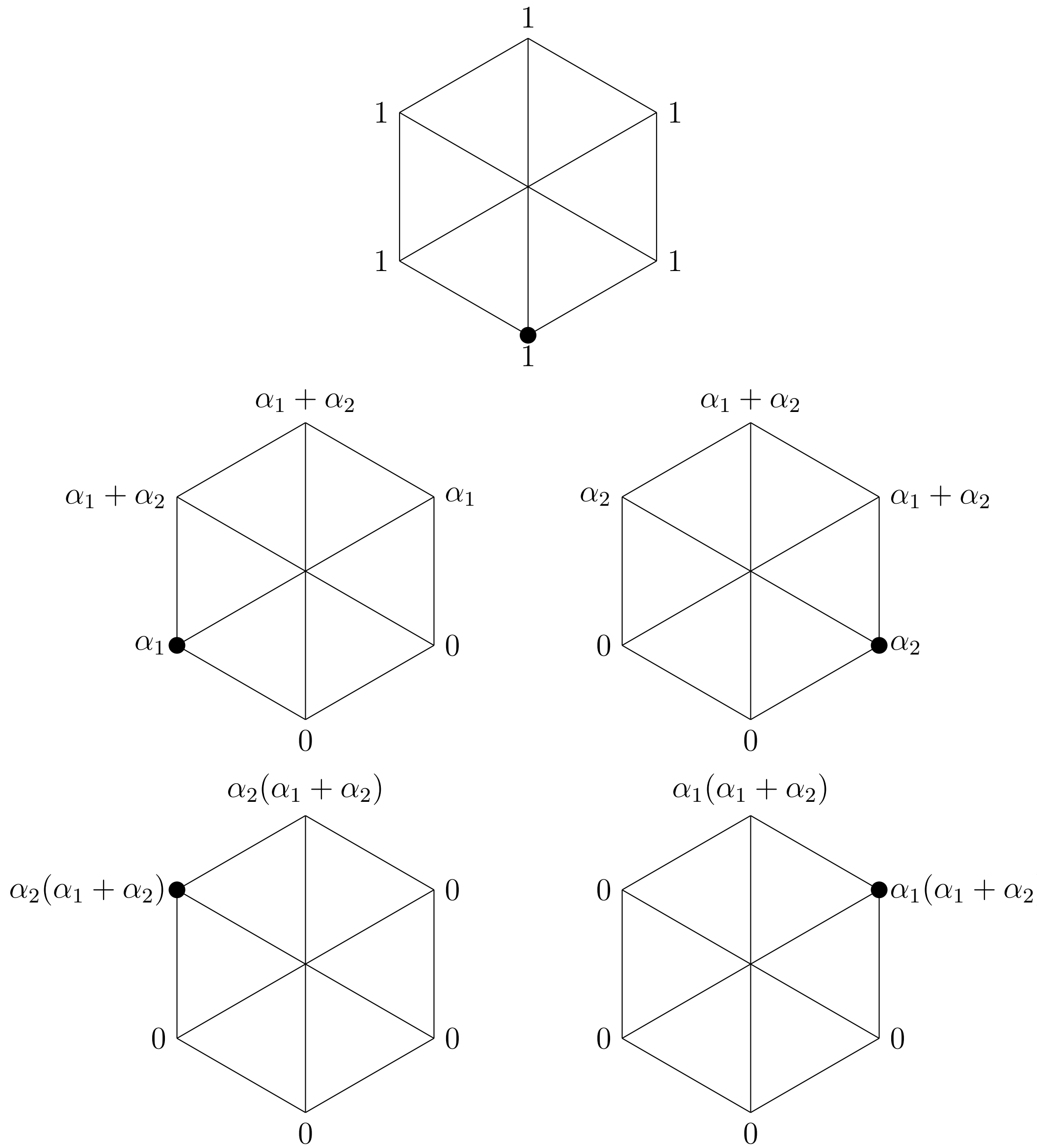
$$\begin{aligned} \deg \sigma^{(w)} = l(w) &= |\{\alpha \in \Phi^+ \mid l(s_\alpha w) < l(w)\}| \\ &= (\text{the number of incoming edges to } w). \end{aligned}$$

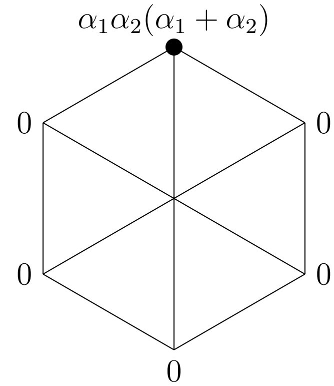
3.3.2 Examples

Example 3.3.6 (Type A_1)**.** The two Schubert classes of the structure algebra of the moment graph of type A_1 are presented below. Their underlying moment graph can be found in Figure 3.1. From left to right, the diagrams depict the Schubert classes of $w = e$ and $w = s_1$ in this order. The thick black dots in the picture emphasize the position of these group elements.

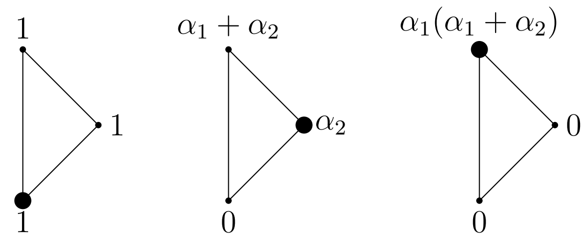


Example 3.3.7 (Type A_2)**.** The six Schubert classes of the structure algebra of the moment graph of type A_2 are presented below. Their underlying moment graph can be found in Figure 3.2. The position of the group element $w \in W$ corresponding to each Schubert class is emphasized by the thick dot in the picture. Hence starting from the top, the diagrams depict the Schubert classes of $w = e$, of $w = s_1$ and $w = s_2$, of $w = s_2 s_1$ and $w = s_1 s_2$, and finally of $w = s_1 s_2 s_1$ at the bottom.





Example 3.3.8 (Type A_2/P_2)**.** The three Schubert classes of the structure algebra of the moment graph of type A_2/P_2 are presented below. Their underlying moment graph can be found in Figure 3.2. From left to right, the diagrams depict the Schubert classes of $w = e$, $w = s_2$, and $w = s_1s_2$ in this order. The thick black dots in the picture emphasize the position of these group elements.



3.3.3 Further properties

We will now prove that the Schubert classes defined in Definition 3.3.1 belong to the structure algebra $\mathcal{Z}(\mathcal{G})$ and indeed form an $\mathcal{S}$-module basis. The following proposition is the key ingredient in proving both statements.

Proposition 3.3.9 ([1, Appendix D.2])**.** *Let $x \in W$ and let α be a simple root.*

$$\sigma^{(w)}(xs_\alpha) = \begin{cases} \sigma^{(w)}(x), & \text{if } l(ws_\alpha) > l(w) \\ \sigma^{(w)}(x) + x(\alpha)\sigma^{(ws_\alpha)}(x), & \text{if } l(ws_\alpha) < l(w) \end{cases} \tag{3.3.2}$$

Proof: If $l(ws_\alpha) > l(w)$, then no reduced expression of w ends with s_α. Hence for $xs_\alpha = s_1s_2\cdots s_rs_\alpha$, the parameter index $\mathfrak{i}$ appearing in the formula (3.3.1) remains the same as for $x = s_1\cdots s_r$, and for each $\mathfrak{i}$, we have $i_\ell \leq r$. Consequently, we obtain $\sigma^{(xs_\alpha)} = \sigma^{(w)}(x)$.

If $l(ws_\alpha) < l(w)$, then w has a reduced expression that ends with s_α. Let us devide the set of parameter indices $\mathfrak{i}$ in formula (3.3.1) into two parts: let $\underline{\mathfrak{i}} = (\underline{i}_1, \cdots, \underline{i}_\ell)$ denote those indices such that $s_{\underline{i}_1}\cdots s_{\underline{i}_\ell}$ is a reduced expression of w ending with s_α, and let $\bar{\mathfrak{i}} = (\bar{i}_1, \cdots, \bar{i}_\ell)$ denote those indices such that $s_{\bar{i}_1} \cdots s_{\bar{i}_\ell}$ is a reduced expression of w which does not end with s_α.

Let us first work with the $\underline{\mathfrak{i}}$'s. Here $s_{\underline{i}_1} \cdots s_{\underline{i}_{\ell-1}}$ is a reduced expression of ws_α and $s_{\underline{i}_\ell} = s_\alpha$ is fixed. We obtain

$$\begin{aligned}\sum_{\underline{\mathfrak{i}}} \prod_{\nu=1}^{\ell} s_1 s_2 \cdots s_{\underline{i}_\nu - 1}(\alpha_{\underline{i}_\nu}) &= \sum_{\underline{\mathfrak{i}}} s_1 s_2 \cdots s_r(\alpha) \prod_{\nu=1}^{\ell-1} s_1 s_2 \cdots s_{\underline{i}_\nu - 1}(\alpha_{\underline{i}_\nu}) \\ &= x(\alpha) \sum_{\underline{\mathfrak{i}}} \prod_{\nu=1}^{\ell-1} s_1 s_2 \cdots s_{\underline{i}_\ell - 1}(\alpha_{\underline{i}_\nu}) \\ &= x(\alpha)\sigma^{(ws_\alpha)}(x).\end{aligned}$$

Next, for the $\bar{\mathfrak{i}}$'s, the reduced expressions $w = s_{\bar{i}_1} \cdots s_{\bar{i}_\ell}$ appear as a subexpression of $x = s_1 s_2 \cdots s_r$ (the presence of s_α is irrelevant). Hence, we obtain

$$\sum_{\bar{\mathfrak{i}}} \prod_{\mu=1}^{\ell} s_1 s_2 \cdots s_{\bar{i}_\nu - 1}(\alpha_{\bar{i}_\nu}) = \sigma^{(w)}(x).$$

Finally, by formula (3.3.1), we obtain

$$\sigma^{(w)}(xs_\alpha) = \sigma^{(w)}(x) + x(\alpha)\sigma^{(ws_\alpha)}(x).$$

∎

Using Proposition 3.3.9, we now establish that the Schubert classes belong to the structure algebra $\mathcal{Z}(\mathcal{G})$.

Proposition 3.3.10 ([1, Lemma D.4])**.** *Let $x \in W$ and $\alpha \in \Phi^+$. Then*

$$\sigma^{(w)}(s_\alpha x) - \sigma^{(w)}(x) \in \alpha\mathcal{S}. \tag{3.3.3}$$

Proof: We will prove this by downward induction on $l(w)$ starting from $w = w_0$.
Base Case: $w = w_0$. In this case, we have

$$\sigma^{(w_0)}(w_0) = \prod_{\alpha \in \Phi^+} \alpha,$$

and $\sigma^{(w_0)}(x) = 0$ for all $x \neq w_0$. Thus the relation (3.3.3) holds for all $x \in W$ and $\alpha \in \Phi^+$.
Induction Step: Let $w \neq w_0$ and choose $\beta \in \Delta$ so that $ws_\beta > w$. Let $x \in W$. First, let us consider the case where $x(\beta) = \pm\alpha$, that is, $s_\alpha = xs_\beta x^{-1}$, yielding $s_\alpha x = xs_\beta$. Hence $\sigma^{(w)}(s_\alpha x) = \sigma^{(w)}(xs_\beta) = \sigma^{(w)}(x)$ by Proposition 3.3.9.

Now suppose $x(\beta) \neq \pm\alpha$ (hence $s_\alpha x(\beta) \neq \pm\alpha$). Since each positive root is a prime element of $\mathcal{S}$ and none of them is a multiple of another, neither $x(\beta)$ nor $s_\alpha x(\beta)$ is divisible by α. Hence the condition (3.3.3) is equivalent to

$$\sigma^{(w)}(s_\alpha x) - \sigma^{(w)}(x) \in \alpha\mathcal{S}[x(\beta)^{-1}, s_\alpha x(\beta)^{-1}].$$

Set $v = ws_\beta$. Note that $w = vs_\beta < v$. Then from the second line of (3.3.2), we have

$$\sigma^{(w)}(x) = \frac{\sigma^{(v)}(xs_\beta) - \sigma^{(v)}(x)}{x(\beta)}.$$

Using this, we may compute

$$\begin{aligned}\sigma^{(w)}(s_\alpha x) - \sigma^{(w)}(x) &= \frac{\sigma^{(v)}\big((s_\alpha x)s_\beta\big) - \sigma^{(v)}(s_\alpha x)}{s_\alpha x(\beta)} - \frac{\sigma^{(v)}(xs_\beta) - \sigma^{(v)}(x)}{x(\beta)} \\ &\equiv \frac{\sigma^{(v)}(xs_\beta) - \sigma^{(v)}(x)}{s_\alpha x(\beta)} - \frac{\sigma^{(v)}(xs_\beta) - \sigma^{(v)}(x)}{x(\beta)} \pmod{\alpha} \\ &= \big(\sigma^{(v)}(xs_\beta) - \sigma^{(v)}(x)\big)\Big(\frac{1}{s_\alpha x(\beta)} - \frac{1}{x(\beta)}\Big).\end{aligned}$$

Between the first line and the second, the induction hypothesis is used for v. Finally,

$$\begin{aligned}\frac{1}{s_\alpha x(\beta)} - \frac{1}{x(\beta)} &= \frac{1}{s_\alpha x(\beta)\cdot x(\beta)}\big(x(\beta) - s_\alpha x(\beta)\big) \\ &= \frac{1}{s_\alpha x(\beta)\cdot x(\beta)}\cdot 2B(x(\beta),\alpha)\alpha \in \alpha\mathcal{S}[x(\beta)^{-1}, s_\alpha x(\beta)^{-1}].\end{aligned}$$

∎

As a corollary, we see that the Schubert classes defined in Definition 3.3.1 belong to the structure algebra.

Corollary 3.3.11. *For each $w \in W$, the Schubert class $\sigma^{(w)}$ belongs to $\mathcal{Z}(\mathcal{G})$.*

Next, we establish that if $w \in W^P$, then the restriction $\sigma^{(w)}|_{W^P}$ belongs to $\mathcal{Z}(\mathcal{G}^P)$. We require a lemma.

Lemma 3.3.12. *Let $w \in W^P$, $x \in W$ and $y \in W_P$. Then $\sigma^{(w)}(xy) = \sigma^{(w)}(x)$.*

Proof: Let $y = s_1 s_2 \cdots s_\ell$ be a reduced expression and denote the simple reflection associated with s_i by α_i for each $1 \leq i \leq \ell$. Note that each α_i belongs to Δ_P. Since $w \in W^P$, we have $l(ws_i) > l(w)$ for each $1 \leq i \leq \ell$. Hence by applying Proposition 3.3.9 repeatedly, we obtain

$$\sigma^{(w)}(x) = \sigma^{(w)}(xs_1) = \sigma^{(w)}(xs_1s_2) = \cdots\sigma^{(w)}(xs_1s_2\cdots s_\ell) = \sigma^{(w)}(xy).$$

∎

Proposition 3.3.13. *Let $w, x \in W^P$ and $\alpha \in \Phi^+$. Then*

$$\sigma^{(w)}(x) - \sigma^{(w)}(\overline{s_\alpha x}) \in \alpha\mathcal{S}.$$

Proof: We have $s_\alpha x = \overline{s_\alpha x} y$ for some $y \in W_P$. Applying Lemma 3.3.12, we obtain

$$\sigma^{(w)}(x) - \sigma^{(w)}(\overline{s_\alpha x}) = \sigma^{(w)}(xy^{-1}) - \sigma^{(w)}(s_\alpha xy^{-1}).$$

By Proposition 3.3.10, this expression belongs to $\alpha\mathcal{S}$. ∎

Corollary 3.3.14. *For each $w \in W^P$, the restriction $\sigma^{(w)}|_{W^P}$ belongs to $\mathcal{Z}(\mathcal{G}^P)$.*

When we speak of a Schubert class corresponding to $w \in W^P$ as an element of $\mathcal{Z}(\mathcal{G}^P)$, we mean the restriction $\sigma^{(w)}|_{W^P}$. However, for the notational convenience, we will abuse notation, and denote $\sigma^{(w)}|_{W^P}$ also by $\sigma^{(w)}$. We will be able to tell them apart by paying attention to the ambient structure algebras.

The next proposition shows that the set of Schubert classes forms an $\mathcal{S}$-module basis for $\mathcal{Z}(\mathcal{G}^P)$.

Proposition 3.3.15 ([1, Appendix D.5])**.** *The set $\{\sigma^{(w)} \mid w \in W^P\}$ forms an $\mathcal{S}$-module basis for $\mathcal{Z}(\mathcal{G}^P)$.*

Proof: The main focus of the proof is to show that the set $\{\sigma^{(w)} \mid w \in W^P\}$ generates the structure algebra $\mathcal{Z}(\mathcal{G}^P)$. Let us choose a total order on the set W^P which is compatible with the Bruhat order. Let $\xi = (z_{w'})_{w' \in W^P} \in \mathcal{Z}(\mathcal{G}^P)$ with $\xi \neq 0$ and let $w \in W$ be the smallest element (in the total order) such that $z_w \neq 0$. We will use induction (from above) with respect to the total order to show that $\xi \in \mathrm{span}_{\mathcal{S}}\{\sigma^{(w')} \mid w' \in W^P\}$.

Base Case: $w = \overline{w_0}$, the unique longest element of W^P (see Section 2.2.6 after Proposition 2.2.17 for definition). We have $s_\alpha w < w$ for all $\alpha \in \Phi^+$ with $w^{-1}(\alpha) < 0$. Hence by the assumption on w, we have $z_{s_\alpha w} = 0$ and $z_w \equiv 0 \pmod{\alpha\mathcal{S}}$ for all $\alpha \in \Phi^+$ with $w^{-1}(\alpha) < 0$. Since each positive root α is a prime element of $\mathcal{S}$, and none of them is a multiple of another, it follows that z_w is divisible by $\prod_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^-} \alpha$. Thus $z_w = c \cdot \prod_{\alpha \in \Phi^+, w^{-1}(\alpha) < 0} \alpha = c \cdot \sigma^{(w)}(w)$ for some nonzero $c \in \mathcal{S}$. Since z_w is the only nonzero entry of ξ in this case, it follows that $\xi = c \cdot \sigma^{(w)}$.

Induction Step: We have $s_\alpha w < w$ for all $\alpha \in \Phi^+$ with $w^{-1}(\alpha) < 0$. Hence as before, we have $z_{s_\alpha w} = 0$ and $z_w \equiv 0 \pmod{\alpha\mathcal{S}}$ for all $\alpha \in \Phi^+$ with $w^{-1}(\alpha) < 0$. Thus $z_w = c \cdot \prod_{\alpha \in \Phi^+, w^{-1}(\alpha) < 0} \alpha$ for some nonzero $c \in \mathcal{S}$. Set $\eta := \xi - c \cdot \sigma^{(w)}$, then the induction hypothesis applies to η and $\xi = \eta + c \cdot \sigma^{(w)} \in \mathrm{span}_{\mathcal{S}}\{\sigma^{(w')} \mid w' \in W^P\}$. ∎

Definition 3.3.16. Let $\xi \in \mathcal{Z}(\mathcal{G}^P)$ and write $\xi = \sum_{w \in W^P} f_w \sigma^{(w)}$ for some $f_w \in \mathcal{S}$. The *support of* ξ is defined to be the set $\{w \in W^P \mid f_w \neq 0\}$ and denoted by $\mathrm{supp}(\xi)$.

Remark 3.3.17. Using the Schubert classes as $\mathcal{S}$-module bases, we have a natural inclusion of the structure algebras $\mathcal{Z}(\mathcal{G}^P) \hookrightarrow \mathcal{Z}(\mathcal{G})$ by the assignment $\sigma^{(w)} \mapsto \sigma^{(w)}$ for each $w \in W^P$.

Lemma 3.3.18 (cf. [26, Proposition 4.9])**.** *Let* $w \in W^P$ *and* $\xi := (x_{w'})_{w' \in W^P} \in \mathcal{Z}(\mathcal{G}^P)$*. The following are equivalent:*

(a) ξ *is homogeneous of degree* $l(w)$ *such that* $x_w \neq 0$ *and* $x_{w'} = 0$ *for all* $w' \ngeq w$*,*

(b) $\xi = c \cdot \sigma^{(w)}$ *for some nonzero constant* $c \in K$*.*

Proof: We give a proof that (a) implies (b) (the other direction is clear). We first treat the special case, namely when $w = e$. If $\xi \in \mathcal{Z}(\mathcal{G}^P)$ is homogeneous of degree $l(e) = 0$ with $c := x_e \neq 0$, then $x_{w'} = c$ for all $w' \in W^P$ by applying the edge relations of $\mathcal{Z}(\mathcal{G}^P)$ inductively on $l(w')$. Thus we obtain $\xi = c \cdot \sigma^{(e)}$.

We now treat the case where $w > e$. Since $x_{w'} = 0$ for all $w' < w$, for x_w ($\neq 0$) to satisfy all the relations $x_{w'} - x_w \in \mathcal{L}(w' \to w)\mathcal{S}$, it must be divisible by the product $\prod_{w'<w} \mathcal{L}(w' \to w)$. Notice that $\deg(\prod_{w'<w} \mathcal{L}(w' \to w)) = \sum_{w'<w} \deg(\mathcal{L}(w' \to w)) =$ (the number of edges $w' \to w$ in $\mathcal{G}^P$) $= l(w)$ (see Remarks 3.2.3 and 3.3.5). Thus the only possibility for x_w to have degree $l(w)$ is

$$x_w = c \cdot \prod_{w'<w} \mathcal{L}(w' \to w)$$

for some nonzero constant $c \in K$.

Take the difference $\xi - c \cdot \sigma^{(w)} := (y_{w'})_{w' \in W^P} \in \mathcal{Z}(\mathcal{G}^P)$. Observe that $y_w = 0$. We also have $y_{w'} = 0$ for all $w' \ngeq w$ by the assumption on ξ as well as Proposition 3.3.4 (c). We claim that $y_{w'} = 0$ for all $w' > w$ as well. Let $v \in W^P$ be minimal among such w'. We have $l(v) = l(w) + 1$. Then for all $w' < v$, we obtain $y_{w'} = 0$ since we have $l(w') \leq l(w)$, $y_{w'} = 0$ for all $w' \ngeq w$, and $y_w = 0$. Hence for y_v to satisfy all the relations $y_{w'} - y_v \in \mathcal{L}(w' \to v)\mathcal{S}$, it must be divisible by the product $\prod_{w'<v} \mathcal{L}(w' \to v)$, which has degree $l(v)$ (see the observation we made for x_w above). Since $\deg(\xi - c \cdot \sigma^{(w)}) = l(w) < l(v)$, $y_v = 0$ follows.

We may repeat the argument with w replaced by v and obtain $y_{w'} = 0$ for all $w' > v$ and $l(w') = l(v) + 1$. We repeat this process until we finally obtain $y_{\overline{w_0}} = 0$ for the longest element $\overline{w_0}$ of W^P.

Thus we obtain $\xi - c \cdot \sigma^{(w)} = 0$. ∎

Chapter 4

Twisted quadratic foldings of root systems

In this chapter, we discuss the notion of a *twisted quadratic folding* of a finite crystallographic root system following [30]. We will first give the construction (Sections 4.1–4.2), list key examples (Section 4.3), and then discuss the induced folding map between the structure algebras (Section 4.4).

4.1 Setup of a twisted folding

The goal of this section is to define the notion of a *folded representation* of a root datum (Definition 4.1.10), which lies at the heart of a twisted quadratic folding construction. We will first introduce the necessary concepts, a τ-form and a τ-operator.

4.1.1 The general setup

Let $K \subset \mathbb{R}$ be a (unital) subring and let L be a free quadratic K-algebra. Here, by the term *quadratic*, we mean that the underlying free K-module is of rank 2. Then $L = K[x]/(p(x))$, where $p(x) = x^2 - c_1x + c_2$ is some monic polynomial over K, and $(p(x))$ denotes the ideal of the polynomial ring $K[x]$ generated by $p(x)$. We assume that L is *separable* over K. Equivalently, the discriminant $D = {c_1}^2 - 4c_2$ is invertible in K (cf. [23, Appendix C]). The trivial quadratic K-algebra $K \times K$ can be obtained by taking $p(x) = x^2 - 1$. We will refer to this case as *split case*. We will refer to the case where L is not isomorphic to $K \times K$ as *non-split case*. When L is non-split, we further assume $D > 0$ so that L can be identified with a subring of $\mathbb{R}$.

It is known that $p(x)$ splits over L. Let us denote the two distinct roots of $p(x)$ in L by τ and σ, so that $p(x) = (x - \tau)(x - \sigma)$ over L. In the split case, we choose $\tau = [1, -1]$ and $\sigma = [-1, 1]$ in the basis $\{[1, 0], [0, 1]\}$ of $K \times K$. In the non-split case, we will always assume $\tau > \sigma$. In what follows, we fix a K-basis of L as $\{1, \tau\}$. For our later convenience, we record the multiplication formula of L-elements with respect to this basis: for $x = y + \tau\hat{y}, x' = y' + \tau\hat{y}' \in L$,

$$xx' = (y + \tau\hat{y})(y' + \tau\hat{y}') = (yy' - c_2\hat{y}\hat{y}') + \tau(y\hat{y}' + \hat{y}y' + c_1\hat{y}\hat{y}'). \tag{4.1.1}$$

4.1.2 τ-form and τ-operator

Let $\mathcal{U}$ be a free L-module of rank n equipped with a basis. Let $R_{L/K}(\mathcal{U})$ denote the free K-module of rank $2n$ obtained from $\mathcal{U}$ by restriction of scalars from L to K. Thus $\mathcal{U}$ and $R_{L/K}(\mathcal{U})$ agree as abelian groups, but their module structures are different. With respect to a basis, each $\tilde{u} \in \mathcal{U}$ has an L-coordinate description $\tilde{u} = (x_1, \cdots, x_n)$ where $x_i \in L$. For each i, by writing $x_i = y_i + \tau\hat{y}_i$ for some $y_i, \hat{y}_i \in K$, we obtain the following unique element $R_{L/K}(\tilde{u}) \in R_{L/K}(\mathcal{U})$:

$$R_{L/K}(\tilde{u}) = ((y_1, \hat{y_1}), \cdots, (y_n, \hat{y_n}))$$

where y_i, $\hat{y}_i \in K$. Conversely, given an element $u = (y_1, \cdots, y_{2n}) \in R_{L/K}(\mathcal{U})$ where $y_i \in K$, we obtain a unique element $u_L \in \mathcal{U}$ as follows:

$$u_L = ((y_1 + \tau y_2), \cdots, (y_{2n-1} + \tau y_{2n})). \tag{4.1.2}$$

With the above convention in mind, we set $U = R_{L/K}(\mathcal{U})$. This is to avoid the notational heaviness in the upcoming construction of foldings of root systems. We first define a K-bilinear form and a K-linear operator on U.

Definition 4.1.1 (τ-form, [30, Section 2])**.** The L-module $\mathcal{U}$ is an inner product space with respect to a basis and the associated L-bilinear dot product $(- \cdot -)$. Let $\mathrm{pr}_\tau : L \to K$ be the K-linear projection given by $\mathrm{pr}_\tau(y + \tau\hat{y}) = y$ where $y, \hat{y} \in K$. The *τ-form* is a symmetric K-bilinear form $(-,-)_\tau$ defined on U as follows: for each pair $(u, u') \in U \times U$, define

$$(u, u')_\tau = \mathrm{pr}_\tau(u_L \cdot u'_L). \tag{4.1.3}$$

In terms of the K-coordinates $u = (y_i, \hat{y}_i)_{i=1}^n$, $u' = (y'_i, \hat{y}'_i)_{i=1}^n$, by (4.1.1), we have

$$(u, u')_\tau = \mathrm{pr}_\tau\left((y_i + \tau\hat{y}_i)_{i=1}^n \cdot (y'_i + \tau\hat{y}'_i)_{i=1}^n\right) = \sum_{i=1}^n (y_i y'_i - c_2\hat{y}_i\hat{y}'_i).$$

Remark 4.1.2. From this formula, it follows that the τ-form $(-,-)_\tau$ is positive definite if and only if $c_2 \leq 0$. In particular, $(-,-)_\tau$ coincides with the K-bilinear dot product on U precisely when $c_2 = -1$. This will be used in Definition 4.1.10.

Definition 4.1.3 (τ-operator, [30, Definition 2.4])**.** The *τ-operator* is the K-linear automorphism $\mathcal{T} : U \to U$ defined by the following property:

$$(\mathcal{T}u)_L = \tau \cdot u_L, \tag{4.1.4}$$

where on the right-hand side, the element $u_L \in \mathcal{U}$ is multiplied by the scalar τ.

In terms of the K-coordinates, for $u = (y_i, \hat{y}_i)_{i=1}^n \in U$,

$$\mathcal{T}u = (-c_2\hat{y}_i, y_i + c_1\hat{y}_i)_{i=1}^n. \tag{4.1.5}$$

Example 4.1.4. In the split case, we have $\tau(y + \tau\hat{y}) = \hat{y} + \tau y$ for $y, \hat{y} \in K$. Hence the τ-operator $\mathcal{T}_L$ is the following switch involution:

$$\mathcal{T}\big((y_i, \hat{y}_i)_{i=1}^n\big) = (\hat{y}_i, y_i)_{i=1}^n.$$

for each $(y_i, \hat{y}_i)_{i=1}^n \in U$.

Lemma 4.1.5 ([30, Lemma 2.8])**.** *The τ-operator is adjoint with respect to the τ-form, that is, $(u, \mathcal{T}u')_\tau = (\mathcal{T}u, u')_\tau$ holds for all $u, u' \in U$.*

Proof: Let $u, u' \in U$. Then by the definition of the τ-form and (4.1.4), we have $(u, \mathcal{T}u')_\tau = \mathrm{pr}_\tau(u_L \cdot (\tau(u')_L)) = \mathrm{pr}_\tau((\tau u_L) \cdot (u')_L) = (\mathcal{T}u, u')_\tau$. ∎

Extending the scalars of U from K to L gives us a new free L-module $U_L := U \otimes_K L \simeq \mathcal{U} \times \mathcal{U}$. Let us extend the K-linear operator $\mathcal{T} : U \to U$ linearly to U_L, and denote the new L-linear operator by $\mathcal{T}_L : U_L \to U_L$. We may also extend the τ-form $(-,-)_\tau$ bilinearly to U_L. By abuse of notation, we denote the τ-form on U_L also by $(-,-)_\tau$. We now discuss the eigenspaces of $\mathcal{T}_L$ and the resulting decomposition of U_L.

By the defining relation of $\mathcal{T}$ (see (4.1.4)), its minimal polynomial over K is precisely $p(x) = x^2 - c_1x + c_2$. Thus $p(x)$ is the minimal polynomial of $\mathcal{T}_L$ over L, and τ, σ are its eigenvalues in L. Let us denote the eigenspaces of $\mathcal{T}_L$ corresponding to τ and σ by $U_L^{(\tau)}$ and $U_L^{(\sigma)}$ respectively. They are found to be

$$\begin{aligned} U_L^{(\tau)} &= \{(-\sigma z_i, z_i)_{i=1}^n \mid z_i \in L\}, \\ U_L^{(\sigma)} &= \{(-\tau z_i, z_i)_{i=1}^n \mid z_i \in L\}. \end{aligned}$$

Lemma 4.1.6 ([30, Lemma 2.5, Proposition 2.6 and Lemma 2.9])**.**

(a) *The L-module U_L admits a direct sum decomposition $U_L = U_L^{(\tau)} \oplus U_L^{(\sigma)}$ orthogonal with respect to $(-,-)_\tau$.*

(b) *By viewing $U \subset U_L$ as a K-submodule of U_L, we have*

$$U_L^{(\tau)} = (\mathcal{T} - \sigma I)(U), \quad U_L^{(\sigma)} = (\mathcal{T} - \tau I)(U).$$

If $u = u_\tau + u_\sigma$ is the decomposition for $u \in U$ given in (a), then

$$u_\tau = \tfrac{1}{\tau-\sigma}(\mathcal{T} - \sigma I)(u), \quad u_\sigma = \tfrac{1}{\sigma-\tau}(\mathcal{T} - \tau I)(u).$$

Proof: (a) The minimal polynomial of $\mathcal{T}_L$ is split and has two distinct roots τ and σ. Hence U_L decomposes into the direct sum of eigenspaces, that is, $U_L = U_L^{(\tau)} \oplus U_L^{(\sigma)}$. The sum is orthogonal with respect to $(-,-)_\tau$ since $\mathcal{T}_L$ is adjoint.
(b) follows from the fact that $\mathcal{T}_L$ is a diagonalizable L-linear operator on U_L. ∎

Remark 4.1.7 (Split case)**.** Let us consider the K-linear projection $p : L \to K$ given by $\tau \mapsto 1$ so that we have $p(y + \tau\hat{y}) = y + \hat{y}$ for each element $y + \tau\hat{y} \in L$. In the split case, under the induced map $p : U_L \to U$ given by $p\big((x_i)_{i=1}^{2n}\big) = \big(p(x_i)\big)_{i=1}^{2n}$ for $x_i \in L$, the eigenspace decomposition $U_L = U_L^{(\tau)} \oplus U_L^{(\sigma)}$ of the switch involution $\mathcal{T}$ (see Example 4.1.4) is mapped onto the eigenspace decomposition over K

$$U = p(U_L^{(\tau)}) \oplus p(U_L^{(\sigma)}) \tag{4.1.6}$$

where each $(y_i, \hat{y}_i)_{i=1}^n \in U$ decomposes as $\frac{1}{2}(y_i + \hat{y}_i, \hat{y}_i + y_i)_{i=1}^n + \frac{1}{2}(y_i - \hat{y}_i, \hat{y}_i - y_i)_{i=1}^n$. Observe that $p(U_L^{(\tau)})$ and $p(U_L^{(\sigma)})$ are the eigenspaces of $\mathcal{T}$ corresponding to eigenvalues 1 and -1 respectively. From now on, whenever we discuss the split case, we replace the eigenspace decomposition $U_L = U_L^{(\tau)} \oplus U_L^{(\sigma)}$ by (4.1.6) under the map p.

We now define a K-linear map π_τ as the composite $\pi_\tau : U \to U_L \to U_L^{(\tau)}$, where the first map is the canonical inclusion and the second is the orthogonal projection onto the τ-eigenspace $U_L^{(\tau)}$. By Lemma 4.1.6 (b), its formula is given by

$$\pi_\tau(u) = \tfrac{1}{\tau-\sigma}(\mathcal{T} - \sigma I)(u) \tag{4.1.7}$$

where $u \in U$. Using the K-coordinates $u = (y_i, \hat{y}_i)_{i=1}^n$, we have

$$\pi_\tau(u) = \tfrac{1}{\tau-\sigma}\big(-\sigma(y_i + \tau\hat{y}_i), y_i + \tau\hat{y}_i\big)_{i=1}^n. \tag{4.1.8}$$

In parallel to $\pi_\tau : U \to U_L^{(\tau)}$, let us define an L-linear map $\Pi_\tau : \mathcal{U} \to U_L^{(\tau)}$ as follows. Let $\tilde{u} \in \mathcal{U}$ and let $\tilde{u} = (x_i)_{i=1}^n$ be its L-coordinates. Then define

$$\Pi_\tau(\tilde{u}) = \tfrac{1}{\tau-\sigma}(-\sigma x_i, x_i)_{i=1}^n.$$

This is an isomorphism of L-modules, and by comparing to (4.1.8), we have

$$\pi_\tau \circ R_{L/K}(\tilde{u}) = \Pi_\tau(\tilde{u})$$

for all $\tilde{u} \in \mathcal{U}$. In particular, since the map $R_{L/K} : \mathcal{U} \to U$ is bijective, it follows that $\pi_\tau : U \to U_L^{(\tau)}$ is also bijective as a map of sets (notice that U is a K-module while $U_L^{(\tau)}$ is an L-module). The following formula is recorded for later use: for $u, u' \in U$,

$$(\pi_\tau(u), \pi_\tau(u'))_\tau = \tfrac{\sigma^2 - c_2}{D}(u_L \cdot u'_L), \tag{4.1.9}$$

where D and c_2 are as defined in Section 4.1.1.

Definition 4.1.8 ([30, Definition 2.11])**.** A nonzero element $u \in U$ is called *τ-rational* if $(u_L \cdot u_L) \in K$.

The following properties will be important.

Lemma 4.1.9 ([30, Lemma 2.10])**.** *Let $u \in U$ be τ-rational. Then the following hold:*

(a) $(u, \mathcal{T}u)_\tau = 0$,

(b) $(\mathcal{T}u, \mathcal{T}u)_\tau = -c_2(u, u)_\tau = -c_2(u_L \cdot u_L)$,

(c) *If* $(u_L \cdot u_L) \neq 0$, *then* $u \neq \mathcal{T}u$.

Proof: (a) By the definition of the τ-form, we have $(u, \mathcal{T}u)_\tau = \mathrm{pr}_\tau(u_L \cdot (\mathcal{T}u)_L) = \mathrm{pr}_\tau(u_L \cdot \tau u_L) = \mathrm{pr}_\tau(\tau(u_L \cdot u_L)) = 0$.
(b) We use the adjoint property of $\mathcal{T}$ (see Lemma 4.1.5) to obtain $(\mathcal{T}u, \mathcal{T}u)_\tau = (u, \mathcal{T}^2 u)_\tau = (u, (c_1\mathcal{T} - c_2 I)(u))_\tau = c_1(u, \mathcal{T}u)_\tau - c_2(u, u)_\tau = -c_2(u, u)_\tau = -c_2\big(\mathrm{pr}_\tau(u_L \cdot u_L)\big) = -c_2(u_L \cdot u_L)$.
(c) Let $u = \mathcal{T}u$. By (a), we have $0 = (u, \mathcal{T}u)_\tau = (u, u)_\tau = \mathrm{pr}_\tau(u_L \cdot u_L) = u_L \cdot u_L$. ∎

We now arrive at the following key preparatory notion of a twisted quadratic folding.

Definition 4.1.10 (Folded representation, [30, Definition 3.1])**.** Let $\Phi \hookrightarrow \Lambda^\vee$ be a root datum with a simple system $\Delta \subset \Phi$ specified. Let $K \subset \mathbb{R}$ be a unital subring, and let U be a free K-module of some even rank $2n$ equipped with a basis and the dot product $(-,-)$. Suppose $(\Phi, U)_K$ is a geometric K-representation of the root datum $\Phi \hookrightarrow \Lambda^\vee$ (see Definition 2.1.3). The pair $(\Phi, U)_K$ is called a *folded K-representation* if the following conditions are satisfied:

(F1) There exists a free quadratic separable K-algebra L and a free L-module $\mathcal{U}$ of rank n such that $R_{L/K}(\mathcal{U}) = U$. [Then fix the two roots $\tau, \sigma \in L$ with $\tau > \sigma$, the τ-operator $\mathcal{T}$, and the τ-form $(-,-)_\tau$ on U.]

(F2) The τ-form $(-,-)_\tau$ coincides with the dot product $(-,-)$ on U.

(F3) $\mathcal{T}(\Lambda) = \Lambda$ and $\mathcal{T}(\Lambda_r) = \Lambda_r$.

(F4) The simple system Δ admits a partition $\Delta = \Delta^{\mathcal{T}} \sqcup \Delta_{rat} \sqcup \mathcal{T}(\Delta_{rat})$, where $\Delta^{\mathcal{T}} := \{\alpha \in \Delta \mid \mathcal{T}\alpha = \alpha\}$ ($\mathcal{T}$-invariant simple roots) and Δ_{rat} consists of simple roots that are τ-rational.

Remark 4.1.11. (a) The condition (F2) is equivalent to $c_2 = -1$ (see Remark 4.1.2). In particular, the defining equation $\tau(\tau - c_1) = 1$ shows that τ is invertible in L.

(b) (F2) allows us to identify the τ-form $(-,-)_\tau$ with the dot product $(-,-)$ on U. Henceforth, we may freely switch between the two perspectives according to the aspect of the bilinear form we need to use.

(c) In the split case, all simple roots that are not fixed under $\mathcal{T}$ are τ-rational. Hence we divide these simple roots into $\Delta_{rat} \sqcup \mathcal{T}(\Delta_{rat})$ by making a choice.

(d) Since $\mathcal{T}$ is an automorphism of U, $\Delta_{rat} \cap \mathcal{T}(\Delta_{rat}) = \emptyset$ implies $|\Delta_{rat}| = |\mathcal{T}(\Delta_{rat})|$. In the next section, we will see that when L/K is non-split, we have $\Delta^{\mathcal{T}} = \emptyset$.

4.2 Construction of a twisted folding

Let us recall the setting and notation of Section 4.1.1. Let $(\Phi, U)_K$ be a folded K-representation of a root datum $\Phi \hookrightarrow \Lambda^\vee$ (Definition 4.1.10), and let $W = W(\Phi)$ be the corresponding Weyl group (Definition 2.1.3). All the notation is as introduced in Section 4.1. In this section, we discuss the construction of a twisted folding of the K-realized root system $\Phi \subset U$ (Definition 2.1.5). The outcome will be an L-realization of a new root system $\Phi_\tau \subset U_L^{(\tau)}$ (Section 4.2.2).

4.2.1 Associated reflections in $U_L^{(\tau)}$

With each element $\alpha \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat} \subset U$, we will associate a map which will turn out to be a reflection on $U_L^{(\tau)}$. We will write $\overline{\alpha} := \pi_\tau(\alpha)$. The procedure is divided into two cases.

Case 1: Let $\alpha \in \Delta_{rat}$. In this case, by Lemma 4.1.9 (a), we have $(\alpha, \mathcal{T}\alpha)_\tau = 0$, whence s_α and $s_{\mathcal{T}\alpha}$ commute. Set $\mathcal{R}^{(\alpha)} := s_{\mathcal{T}\alpha} s_\alpha$. Its action on $u \in U_L^{(\tau)}$ is given by

$$\begin{aligned}\mathcal{R}^{(\alpha)}(u) &= s_{\mathcal{T}\alpha}(u - \alpha^\vee(u)\alpha) \\ &= u - \alpha^\vee(u) - (\mathcal{T}\alpha)^\vee(u - \alpha^\vee(u)\alpha)\mathcal{T}\alpha \\ &= u - \alpha^\vee(u)\alpha - (\mathcal{T}\alpha)^\vee(u)\mathcal{T}\alpha. \end{aligned} \tag{4.2.1}$$

Lemma 4.2.1 ([30, Lemma 3.4])**.** $\mathcal{R}^{(\alpha)}\mathcal{T} = \mathcal{T}\mathcal{R}^{(\alpha)}$

Proof: For the notational convenience, we shall write $c := (\alpha,\alpha)_\tau = (\mathcal{T}\alpha,\mathcal{T}\alpha)_\tau$ (see Lemma 4.1.9 (b)). Let $u \in U$.

$$\begin{aligned}
\mathcal{T}\big(\mathcal{R}^{(\alpha)}u\big) &= \mathcal{T}\big(u - \tfrac{2(u,\alpha)_\tau}{c}\alpha - \tfrac{2(u,\mathcal{T}\alpha)_\tau}{c}\mathcal{T}\alpha\big) \quad \text{(by (4.2.1))}\\
&= \mathcal{T}u - \tfrac{2(u,\alpha)_\tau}{c}\mathcal{T}\alpha - \tfrac{2(u,\mathcal{T}\alpha)_\tau}{c}(c_1\mathcal{T}+I)\alpha\\
&= \mathcal{T}u - \tfrac{2(u,\mathcal{T}\alpha)_\tau}{c}\alpha - \tfrac{2(u,c_1\mathcal{T}\alpha+\alpha)_\tau}{c}\mathcal{T}\alpha\\
&= \mathcal{T}u - \tfrac{2(\mathcal{T}u,\alpha)_\tau}{c}\alpha - \tfrac{2(\mathcal{T}u,\mathcal{T}\alpha)_\tau}{c}\mathcal{T}\alpha\\
&= \mathcal{R}^{(\alpha)}\big(\mathcal{T}u\big).
\end{aligned}$$

In the second last equality, the adjoint property of $\mathcal{T}$ is used. ∎

We now extend $\mathcal{R}^{(\alpha)} : U \to U$ linearly to U_L, and denote the resulting L-linear operator by $\mathcal{R}^{(\alpha)}_L : U_L \to U_L$.

Proposition 4.2.2 ([30, Corollary 3.5 and Lemma 3.6])**.**

(a) $\mathcal{R}^{(\alpha)}_L$ *stabilizes the eigenspaces of* $\mathcal{T}$*, that is,* $\mathcal{R}^{(\alpha)}_L(U^{(\tau)}_L) = U^{(\tau)}_L$, $\mathcal{R}^{(\alpha)}_L(U^{(\sigma)}_L) = U^{(\sigma)}_L$,

(b) $\mathcal{R}^{(\alpha)}_L(\overline{\alpha}) = -\overline{\alpha}$,

(c) $\mathcal{R}^{(\alpha)}_L(u) = u$ *for all* $u \in U^{(\tau)}_L$ *such that* $(u,\overline{\alpha})_\tau = 0$.

Proof: (a) follows from Lemma 4.1.6 and Lemma 4.2.1.
(b) Using the formula (4.1.7) and Lemma 4.2.1, we obtain

$$\begin{aligned}
\mathcal{R}^{(\alpha)}_L(\pi_\tau(\alpha)) &= \mathcal{R}^{(\alpha)}_L\big(\tfrac{1}{\sigma-\tau}(\mathcal{T}\alpha - \sigma\alpha)\big)\\
&= \tfrac{1}{\sigma-\tau}\big(\mathcal{T}\mathcal{R}^{(\alpha)}_L(\alpha) - \sigma\mathcal{R}^{(\alpha)}_L(\alpha)\big)\\
&= \tfrac{1}{\sigma-\tau}\big(\mathcal{T}(-\alpha) - \sigma(-\alpha)\big)\\
&= -\tfrac{1}{\sigma-\tau}(\mathcal{T}\alpha - \sigma\alpha) = -\pi_\tau(\alpha).
\end{aligned}$$

(c) Let $u \in U^{(\tau)}_L$ such that $(u,\overline{\alpha})_\tau = 0$. Then $(u,\mathcal{T}\alpha - \sigma\alpha)_\tau = 0$, which is equivalent to $(u,\mathcal{T}\alpha)_\tau = \sigma(u,\alpha)_\tau$. By (4.2.1) we obtain

$$\begin{aligned}
\mathcal{R}^{(\alpha)}_L(u) &= u - \tfrac{2(u,\alpha)_\tau}{c}\alpha - \tfrac{2(u,\mathcal{T}\alpha)_\tau}{c}\mathcal{T}\alpha\\
&= u - \tfrac{2(u,\alpha)_\tau}{c}\alpha - \tfrac{2\sigma(u,\alpha)_\tau}{c}\mathcal{T}\alpha\\
&= u - \tfrac{2(u,\alpha)_\tau}{c}(\alpha + \sigma\mathcal{T}\alpha).
\end{aligned}$$

By observing that $\tau(\alpha+\sigma\mathcal{T}\alpha)=-(\mathcal{T}-\tau I)(\alpha)\in U_L^{(\sigma)}$ (see Lemma 4.1.6 (b)), and since $u\in U_L^{(\tau)}$ and $\mathcal{R}_L^{(\alpha)}(u)\in U_L^{(\tau)}$ (by (a)), we obtain $\alpha+\sigma\mathcal{T}\alpha\in U_L^{(\tau)}\cap U_L^{(\sigma)}=\{0\}$. Hence $\mathcal{R}_L^{(\alpha)}(u)=u$ follows. ∎

Therefore, we see that the L-linear operator $\mathcal{R}_L^{(\alpha)}:U_L\to U_L$ restricts to $U_L^{(\tau)}\to U_L^{(\tau)}$, and that the restriction $\mathcal{R}_L^{(\alpha)}|_{U_L^{(\tau)}}$ behaves as the reflection on $U_L^{(\tau)}$ corresponding to the element $\overline{\alpha}\in U_L^{(\tau)}$. We will denote the L-linear operator $\mathcal{R}_L^{(\alpha)}|_{U_L^{(\tau)}}$ by $R_{\overline{\alpha}}$.

Case 2: Let $\alpha\in\Delta^{\mathcal{T}}$. Using the K-coordinates $\alpha=(y_i,\hat{y}_i)_{i=1}^n$ and formula (4.1.5), the equation $\mathcal{T}\alpha=\alpha$ becomes

$$(\hat{y}_i,y_i+c_1\hat{y}_i)_{i=1}^n=(y_i,\hat{y}_i)_{i=1}^n.$$

By solving this set of equations, we obtain $y_i=\hat{y}_i$ for all $1\le i\le n$ and $c_1=0$. Thus $\tau=1$, $\sigma=-1$, and it follows that $\overline{\alpha}=\pi_\tau(\alpha)=\alpha$ (see (4.1.7)). In particular, $\alpha\in U_L^{(\tau)}$. This has a consequence that if we extend the simple reflection $s_\alpha:U\to U$ linearly to $U_L\to U_L$ and denote it by $(s_\alpha)_L$, then $(s_\alpha)_L$ stabilizes $U_L^{(\tau)}$: $(s_\alpha)_L(u)=u-\frac{2(x,\alpha)_\tau}{(\alpha,\alpha)_\tau}\alpha\in U_L^{(\tau)}$ for all $u\in U_L^{(\tau)}$. So the L-linear operator $(s_\alpha)_L$ restricts to $U_L^{(\tau)}\to U_L^{(\tau)}$. We denote this new operator by R_α. We have

Lemma 4.2.3 ([30, Lemma 3.8])**.** $R_\alpha\mathcal{T}=\mathcal{T}R_\alpha$.

Proof: Let $u\in U_L^{(\tau)}$. By using $\mathcal{T}\alpha=\alpha$ and the adjoint property of $\mathcal{T}$, we obtain

$$\begin{aligned}\mathcal{T}(R_\alpha(u))&=\mathcal{T}\big(u-\tfrac{2(u,\alpha)_\tau}{(\alpha,\alpha)_\tau}\alpha\big)\\&=\mathcal{T}u-\tfrac{2(u,\mathcal{T}\alpha)_\tau}{(\alpha,\alpha)_\tau}\alpha\\&=\mathcal{T}u-\tfrac{2(\mathcal{T}u,\alpha)_\tau}{(\alpha,\alpha)_\tau}\alpha=R_\alpha(\mathcal{T}u).\end{aligned}$$

∎

In both cases, the reflection $R_{\overline{\alpha}}$ ($R_{\overline{\alpha}}=R_\alpha$ for $\alpha\in\Delta^{\mathcal{T}}$) stabilizes $\pi_\tau(\Phi)\subset U_L^{(\tau)}$ as shown in the next proposition.

Proposition 4.2.4 ([30, Proposition 3.9])**.** *For each* $\alpha\in\Delta^{\mathcal{T}}\sqcup\Delta_{rat}$,

$$R_{\overline{\alpha}}(\pi_\tau(\Phi))=\pi_\tau(\Phi).$$

Proof: Lemma 4.2.1 and Lemma 4.2.3 together with the formula (4.1.7) imply that the K-linear map π_τ commutes with $R_{\overline{\alpha}}$: $R_{\overline{\alpha}}\circ\pi_\tau=\pi_\tau\circ(s_{\mathcal{T}\alpha}s_\alpha)$ for $\alpha\in\Delta_{rat}$ and $R_\alpha\circ\pi_\tau=\pi_\tau\circ s_\alpha$ for $\alpha\in\Delta^{\mathcal{T}}$. The result then follows from $s_{\mathcal{T}\alpha}s_\alpha(\Phi)\subset\Phi$, $s_\alpha(\Phi)\subset\Phi$, and the fact that $R_{\overline{\alpha}}$ is an automorphism of $U_L^{(\tau)}$ in both cases. ∎

4.2.2 The group W_τ and its root system Φ_τ

We now aim to generate a group out of the above reflections and find its underlying root system. Set $\Delta_\tau := \pi_\tau(\Delta^{\mathcal{T}} \sqcup \Delta_{rat})$.

Lemma 4.2.5. *The elements of the set Δ_τ are L-linearly independent.*

Proof: Recall the notation $\overline{\alpha} = \pi_\tau(\alpha)$ for $\alpha \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$. Let $\sum_{\alpha\in\Delta^{\mathcal{T}}\sqcup\Delta_{rat}} c_\alpha\overline{\alpha} = 0$ for some $c_\alpha \in L$. We have

$$0 = \sum_{\alpha\in\Delta^{\mathcal{T}}} c_\alpha\overline{\alpha} + \sum_{\alpha\in\Delta_{rat}} c_\alpha\overline{\alpha} = \sum_{\alpha\in\Delta^{\mathcal{T}}} \tfrac{c_\alpha}{\tau-\sigma}(\mathcal{T}-\sigma I)\alpha + \sum_{\alpha\in\Delta_{rat}} \tfrac{c_\alpha}{\tau-\sigma}(\mathcal{T}-\sigma I)\alpha$$
$$= \tfrac{1}{\tau-\sigma}\sum_{\alpha\in\Delta^{\mathcal{T}}}(c_\alpha - \sigma c_\alpha)\alpha + \tfrac{1}{\tau-\sigma}\sum_{\alpha\in\Delta_{rat}}\big(c_\alpha(\mathcal{T}\alpha) - \sigma c_\alpha\alpha\big).$$

Since the elements of $\Delta = \Delta^{\mathcal{T}} \sqcup \Delta_{rat} \sqcup \mathcal{T}(\Delta_{rat})$ are K-linearly independent, we obtain $c_\alpha - \sigma c_\alpha = 0$ for all $\alpha \in \Delta^{\mathcal{T}}$ and $c_\alpha = 0$ for all $c_\alpha \in \Delta_{rat}$. Since $\sigma > 0$, we have $c_\alpha = 0$ for all $\alpha \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$. ∎

Let us generate a subgroup W' of W by $\{s_\alpha, s_\beta s_{\mathcal{T}\beta} \mid \alpha \in \Delta^{\mathcal{T}}, \beta \in \Delta_{rat}\}$. We define W_τ to be the group obtained from W' by extending the base ring from K to L:

$$W_\tau := \langle (s_\alpha)_L, (s_\beta s_{\mathcal{T}\beta})_L \mid \alpha \in \Delta^{\mathcal{T}}, \beta \in \Delta_{rat}\rangle = \langle R_{\overline{\alpha}} \mid \overline{\alpha} \in \Delta_\tau\rangle.$$

Now define $\Phi_\tau := \{u(\overline{\alpha}) \mid \overline{\alpha} \in \Delta_\tau, u \in W_\tau\} \subset U_L^{(\tau)}$. Since $\Delta_\tau \subset \pi_\tau(\Phi)$ and $\pi_\tau(\Phi)$ is stabilized under W_τ (Proposition 4.2.4), it follows that $\Phi_\tau \subset \pi_\tau(\Phi)$. In particular, Φ_τ is finite. Then as a finite subset of $U_L^{(\tau)}$ obtained from an L-linearly independent subset Δ_τ under the action of the L-reflection group W_τ, the set $\Phi_\tau \subset U_L^{(\tau)}$ is a finite L-realized root system in the sense of Definition 2.1.5 (not necessarily crystallographic, see Example 4.3.6).

Definition 4.2.6. The root system Φ_τ is called the *τ-folded root system*, and the procedure of passing from Φ to Φ_τ is called the *τ-folding*, denoted by $\Phi \rightsquigarrow \Phi_\tau$.

Lemma 4.2.7 ([30, Lemma 4.2])**.** *The set Δ_τ forms a simple system of the root system Φ_τ.*

Proof: We shall show that $\pi_\tau(\Phi) \subset \operatorname{span}_L \Delta_\tau$ and that in the L-linear expansion of an element of $\pi_\tau(\Phi)$, the coefficients are either all ≥ 0 or all ≤ 0. Let $\gamma \in \Phi$ and write $\gamma = \sum_{\alpha\in\Delta} c_\alpha \alpha$ where $c_\alpha \in K$ and either all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$. Then we have

$$\begin{aligned}\pi_\tau(\gamma) &= \sum_{\alpha\in\Delta} c_\alpha \pi_\tau(\alpha) \\ &= \sum_{\alpha\in\Delta^{\mathcal{T}}} c_\alpha \pi_\tau(\alpha) + \sum_{\alpha\in\Delta_{rat}} (c_\alpha \pi_\tau(\alpha) + c_{\mathcal{T}\alpha}\pi_\tau(\mathcal{T}\alpha)) \\ &= \sum_{\alpha\in\Delta^{\mathcal{T}}} c_\alpha \alpha + \sum_{\alpha\in\Delta_{rat}} (c_\alpha \overline{\alpha} + c_{\mathcal{T}\alpha}\tau\overline{\alpha}),\end{aligned}$$

which is a L-linear combination of Δ_τ-elements where the coefficients are either all nonnegative or nonpositive (recall $\tau > 0$). ∎

Let us define $S_\tau := \{R_{\overline{\alpha}} \mid \alpha \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}\}$. Then by Proposition 2.2.13 and Lemma 4.2.7, the pair (W_τ, S_τ) is a finite Coxeter system.

The following group embedding is essential to the current work. The equivalent embedding is discussed in [30, Section 5].

Proposition 4.2.8. *The assignment $R_{\overline{\alpha}} \mapsto s_\alpha$ if $\alpha \in \Delta^{\mathcal{T}}$ and $R_{\overline{\alpha}} \mapsto s_\alpha s_{\mathcal{T}\alpha}$ if $\alpha \in \Delta_{rat}$ extends to a group embedding $\varepsilon : W_\tau \hookrightarrow W$.*

Proof: Observe that the assignment extends to the group isomorphism $W_\tau \to W'$, where W' is the group defined at the beginning of Section 4.2.2, and that W' is a subgroup of W. ∎

Using this embedding $\varepsilon : W_\tau \hookrightarrow W$, we may formulate the equivariant property of the projection map $\pi_\tau : U \to U_L^{(\tau)}$ as follows.

Proposition 4.2.9. $u(\pi_\tau(x)) = \pi_\tau(\varepsilon(u)(x))$ *for all* $u \in W_\tau$, $x \in U$.

Proof: Let $x \in U$. By Lemma 4.2.1 and 4.2.3, we have

$$R_{\overline{\alpha}}\big((\mathcal{T}_L - \sigma I_L)(x)\big) = (\mathcal{T}_L - \sigma I_L)\big(R_{\overline{\alpha}}(x)\big).$$

As we restrict the operator $\mathcal{T}_L - \sigma I_L : U_L \to U_L$ to $\pi_\tau : U \to U_L^{(\tau)}$, we obtain

$$R_{\overline{\alpha}}(\pi_\tau(x)) = \begin{cases} \pi_\tau(s_\alpha(x)) & \text{if } \alpha \in \Delta^{\mathcal{T}}, \\ \pi_\tau(s_{\mathcal{T}\alpha} s_\alpha(x)) & \text{if } \alpha \in \Delta_{rat}. \end{cases}$$

Hence $u\big(\pi_\tau(x)\big) = \pi_\tau\big(\varepsilon(u)(x)\big)$ for all $u \in W_\tau$. ∎

4.2.3 Folding of a subsystem

Let $(\Phi, U)_K$ be a folded K-representation of a root datum $\Phi \hookrightarrow \Lambda^\vee$ with a chosen simple system $\Delta \subset \Phi$. Let Δ_P be a (possibly empty) subset of Δ and further assume that the K-submodule of U spanned by Δ_P is stable under $\mathcal{T}$; that is, $\mathcal{T}(\mathrm{span}_K(\Delta_P)) = \mathrm{span}_K(\Delta_P)$. When this latter condition holds, we say that $\mathcal{T}$ *preserves* Δ_P. In this case, the partition $\Delta = \Delta^{\mathcal{T}} \sqcup \Delta_{rat} \sqcup \mathcal{T}(\Delta_{rat})$ restricts to the partition $\Delta_P = \Delta_P^{\mathcal{T}} \sqcup (\Delta_P)_{rat} \sqcup \mathcal{T}((\Delta_P)_{rat})$. In particular, we may apply the construction of Sections 4.2.1 and 4.2.2 to the parabolic subgroup W_P and obtain the L-linear reflection group $(W_P)_\tau$ acting on $U_L^{(\tau)}$. In what follows, we assume that $\mathcal{T}$ preserves Δ_P.

Lemma 4.2.10. $\varepsilon^{-1}(W_P) = (W_P)_\tau$.

Proof: By construction, we have $\varepsilon\big((W_P)_\tau\big) \subset W_P$. Hence $(W_P)_\tau \subset \varepsilon^{-1}(W_P)$. To see the other containment, let $u \in \varepsilon^{-1}(W_P)$ and let $u = R_{j_1} \cdots R_{j_l}$ be a W_τ-reduced expression. By assumption, $\varepsilon(u) \in W_P$. If we denote the simple root of W_τ corresponding to R_{j_p} by $\beta_{j_p} \in \Delta_\tau$ and set $\alpha_{j_p} := \pi_\tau^{-1}(\beta_{j_p}) \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$, then

$$\varepsilon(R_{j_p}) = \begin{cases} s_{\alpha_{j_p}} & \text{if } \alpha_{j_p} \in \Delta^{\mathcal{T}}, \\ s_{\mathcal{T}\alpha_{j_p}} s_{\alpha_{j_p}} & \text{if } \alpha_{j_p} \in \Delta_{rat}. \end{cases}$$

So $\varepsilon(u) \in W_P$ implies $\alpha_{j_p} \in (\Delta_P)^{\mathcal{T}} \sqcup (\Delta_P)_{rat} \subset \Delta_P$ (then $\mathcal{T}\alpha_{j_p} \in \Delta_P$ automatically follows). Hence $\beta_{j_p} = \pi_\tau(\alpha_{j_p}) \in (\Delta_P)_\tau$ and thus $u \in (W_P)_\tau$. ∎

Let $\theta \in U$ be a dominant vector with respect to Δ such that its stabilizer subgroup in W is precisely W_P. Then in $U_L^{(\tau)}$, the following holds:

Lemma 4.2.11 ([30, Lemma 5.2])**.** *$\mu := \pi_\tau(\theta) \in U_L^{(\tau)}$ is a dominant vector with respect to Δ_τ such that its stabilizer subgroup in W_τ is precisely $(W_P)_\tau$.*

Proof: Let $\alpha \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$. By the formula (4.1.7), we have

$$\begin{aligned}
(\pi_\tau(\theta), \pi_\tau(\alpha))_\tau &= \tfrac{1}{(\tau-\sigma)^2}((\mathcal{T} - \sigma I)\theta, (\mathcal{T} - \sigma I)\alpha)_\tau \\
&= \tfrac{1}{(\tau-\sigma)^2}\Big((\mathcal{T}\theta, \mathcal{T}\alpha)_\tau - 2\sigma(\mathcal{T}\theta, \alpha)_\tau + \sigma^2(\theta, \alpha)_\tau\Big) \\
&= \tfrac{1}{(\tau-\sigma)^2}\Big((\theta, (c_1\mathcal{T} + I)\alpha)_\tau - 2\sigma(\theta, \mathcal{T}\alpha)_\tau + \sigma^2(\theta, \alpha)_\tau\Big) \\
&= \tfrac{1}{(\tau-\sigma)^2}\Big((\sigma^2 + 1)(\theta, \alpha)_\tau + (c_1 - 2\sigma)(\theta, \mathcal{T}\alpha)_\tau\Big) \\
&= \tfrac{1}{(\tau-\sigma)^2}\Big((\sigma^2 + 1)(\theta, \alpha)_\tau + (\tau - \sigma)(\theta, \mathcal{T}\alpha)_\tau\Big).
\end{aligned}$$

Since $\theta \in U_l$ is dominant with respect to Δ, and since $\mathcal{T}\alpha \in \Delta$ (use $\alpha \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$) we have $(\theta, \alpha)_\tau \geq 0$ and $(\theta, \mathcal{T}\alpha)_\tau \geq 0$. Thus $(\mu, \overline{\alpha})_\tau \geq 0$ and this proves $\mu = \pi_\tau(\theta)$ is dominant with respect to Δ_τ.

If $(\mu, \overline{\alpha})_\tau = 0$, then by the formula (4.1.9), we have $\theta_L \cdot \alpha_L = 0$. Thus by the definition of the τ-form (see (4.1.3)), we obtain $(\theta, \alpha)_\tau = 0$. Conversely, suppose $(\theta, \alpha)_\tau = 0$, which implies $\alpha \in \Delta_P$. Then $\mathcal{T}\alpha \in \Delta_P$. If $\alpha \in \Delta^{\mathcal{T}}$, this is clear. If $\alpha \in \Delta_{rat}$, then $\mathcal{T}\alpha \in \mathcal{T}((\Delta_P)_{rat})$. Hence $(\theta, \mathcal{T}\alpha)_\tau = 0$ and the above equality gives us $(\mu, \overline{\alpha})_\tau = 0$. This proves $\mathrm{Stab}_{W_\tau}(\mu) = (W_P)_\tau$. ∎

Let us denote by W_τ^P the unique set of minimal length representatives for the left coset space $W_\tau/(W_P)_\tau$. By the above lemma, we have a poset isomorphism $(W_\tau^P, \leq) \simeq (W_\tau\mu, \leq)$ under the orbit map $u \mapsto u\mu$ (see Proposition 2.2.21).

Proposition 4.2.12. *The embedding $\varepsilon : W_\tau \hookrightarrow W$ restricts to $W_\tau^P \hookrightarrow W^P$.*

Proof: By Lemma 4.2.10, the embedding $\varepsilon : W_\tau \hookrightarrow W$ induces a well-defined injective map $W_\tau/(W_P)_\tau \hookrightarrow W/W_P : u(W_P)_\tau \mapsto \varepsilon(u)W_P$. Since $\mathrm{Stab}_{W_\tau}(\mu) = (W_P)_\tau$ (Lemma 4.2.11) and $\mathrm{Stab}_W(\theta) = W_P$, through the canonical isomorphisms

$$W_\tau/(W_P)_\tau \simeq W\mu \simeq W_\tau^P, \quad W/W_P \simeq W\theta \simeq W^P$$

we further obtain the injective map $W_\tau^P \to W^P : u \mapsto \varepsilon(u)$, which is indeed the restriction of $\varepsilon : W_\tau \hookrightarrow W$. ∎

4.3 Examples

In this section, we list examples of a twisted quadratic folding, which include all three types of classical foldings and the Lusztig folding mentioned in Chapter 1. The reader is strongly recommended to consult Appendix A while reading this section, where the Coxeter diagram of each folding along with the numbering of simple reflections can be found. The description of the folding from type E_6 to F_4 as a twisted quadratic folding (Example 4.3.4) is a new contribution to the literature.

4.3.1 Classical cases

The list of classical foldings of finite crystallographic root systems can be found in [34, Section 10.3], for example. In this section, we present each classical folding except for the cubic folding of a root system of type D_4 (see Remark 4.3.5) in the framework of a τ-folding as defined in Definition 4.2.6. Our main task here is to find a folded K-representation (Definition 4.1.10) of each given root datum for some choice of K.

Example 4.3.1 (Type A_{2n-1} to type C_n, [30, Example 4.3])**.** Let $K = \mathbb{Z}[\frac{1}{2}]$ and $L = K \times K$. We have $p(x) = x^2 - 1$ and $\tau = 1 \in K$ (see Remark 4.1.7). Let V be a free K-module of rank $2n$ equipped with a basis $\{e_i \mid 1 \leq i \leq 2n\}$ and the dot product $(-,-)$. We realize the root system Φ of type A_{2n-1} in V as in Example 2.1.6. We have a linear involution $T : V \to V$ defined by

$$T(e_i) = -e_{2n-i+1} \quad (1 \leq i \leq 2n),$$

which acts on the simple roots by $T(\alpha_i) = \alpha_{2n-i}$ for $1 \leq i \leq 2n-1$.

Let $\mathcal{U}$ be a free L-module of rank n. Then $U := \mathcal{R}_{L/K}(\mathcal{U})$ is a free K-module of rank $2n$ equipped with the τ-form $(-,-)_\tau$ and the τ-operator $\mathcal{T} : U \to U$ (switch involution, Example 4.1.4). Let us identify U with V under the following K-linear isomorphism $\iota : V \simeq U$:

$$\iota : (x_1, x_2, \cdots, x_{2n}) \mapsto \big((-x_{2n}, x_1), (-x_{2n-1}, x_2), \cdots, (-x_{n+1}, x_n)\big).$$

The map ι is designed so that it is a K-isometry with respect to the dot product $(-,-)$ on V and the τ-form on U, and the involution T defined above corresponds to the τ-operator $\mathcal{T}$ on U (i.e. $\iota \circ T = \mathcal{T} \circ \iota$).

The coordinates of the simple roots in the new K-realization $\Phi \subset U$ are:

$$\begin{aligned}
\iota(\alpha_1) &= \big((0,1), (0,-1), (0,0) \cdots, (0,0)\big),\\
\iota(\alpha_2) &= \big((0,0), (0,1), (0,-1), \cdots, (0,0)\big),\\
&\cdots\\
\iota(\alpha_{n-1}) &= \big((0,0), \cdots, (0,0), (0,1), (0,-1)\big),\\
\iota(\alpha_n) &= \big((0,0), \cdots, (0,0), (0,0), (1,1)\big),\\
\iota(\alpha_{n+1}) &= \big((0,0), \cdots, (0,0), (1,0), (-1,0)\big),\\
&\cdots\\
\iota(\alpha_{2n-1}) &= \big((1,0), (-1,0), (0,0) \cdots, (0,0)\big).
\end{aligned}$$

We have

$$\mathcal{T}(\iota(\alpha_i)) = \iota(\alpha_{2n-i}) \quad (1 \leq i \leq 2n-1).$$

In particular, the K-realized root system $\Phi \subset U$ is invariant under $\mathcal{T}$, and we obtain a partition $\Delta^{\mathcal{T}} \sqcup \Delta_{rat} \sqcup \mathcal{T}(\Delta_{rat})$ of the simple roots by taking

$$\Delta^{\mathcal{T}} = \{\iota(\alpha_n)\}, \quad \Delta_{rat} = \{\iota(\alpha_1), \cdots, \iota(\alpha_{n-1})\}, \quad \mathcal{T}(\Delta_{rat}) = \{\iota(\alpha_{n+1}), \cdots, \iota(\alpha_{2n-1})\}.$$

Thus $(\Phi, U)_K$ is a folded K-representation of the root datum $\Phi \hookrightarrow \Lambda^\vee$. We have

$$\Delta_\tau = \pi_\tau\big(\Delta^{\mathcal{T}} \sqcup \Delta_{rat}\big) = \{\overline{\iota(\alpha_1)}, \cdots, \overline{\iota(\alpha_{n-1})}, \overline{\iota(\alpha_n)}\}.$$

The projection operator $\pi_\tau : U \to U_L^{(\tau)}$ is $\pi_\tau(u) = \frac{1}{2}(\mathcal{T}u + u)$ for $u \in U$. Using the formula (4.1.9), we obtain the following table of values for $(\overline{\alpha_i}, \overline{\alpha_j})_\tau$ for $\alpha_i, \alpha_j \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$. In the table below, the image $\iota(\alpha_i)$ $(1 \leq i \leq 6)$ is simply denoted by α_i.

$(-,-)_\tau$	$\overline{\alpha_1}$	$\overline{\alpha_2}$	$\cdots$	$\overline{\alpha_{n-1}}$	$\overline{\alpha_n}$
$\overline{\alpha_1}$	1	$-\frac{1}{2}$	$\cdots$	0	0
$\overline{\alpha_2}$	$-\frac{1}{2}$	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\overline{\alpha_{n-1}}$	0	0	$\cdots$	1	-1
$\overline{\alpha_n}$	0	0	$\cdots$	-1	2

Hence the resulting root system Φ_τ is of type C_n.

Remark 4.3.2. It is recalled that in the construction of a classical folding of a root system, any two distinct simple roots in the orbit of the Coxeter diagram automorphism must be orthogonal ([34, Section 10.3.1]). It is for this reason that a root system of type $A_{2n} (n \geq 1)$ does not appear in the list of foldings.

Example 4.3.3 (Type D_{n+1} to type B_n, [30, Example 4.4])**.** Let $K = \mathbb{Z}[\frac{1}{2}]$ and $L = K \times K$. We have $p(x) = x^2 - 1$ and $\tau = 1 \in K$ (see Remark 4.1.7). Let V be a free K-module of rank $n+1$ equipped with a basis $\{e_i \mid 1 \leq i \leq n+1\}$ and the dot product $(-,-)$. We realize the root system Φ of type D_{n+1} in V as in Example 2.1.7. We have a linear involution $T : V \to V$ defined by

$$T(e_i) = \begin{cases} e_i, & \text{if } 1 \leq i \leq n, \\ -e_i, & \text{if } i = n+1, \end{cases}$$

which acts on the simple roots $\alpha_i = e_i - e_{i+1} (1 \leq i \leq n-1)$ by $T(\alpha_i) = \alpha_i$, on $\alpha_n = e_n - e_{n+1}$ by $T(\alpha_n) = \alpha_{n+1}$, and on $\alpha_{n+1} = e_n + e_{n+1}$ by $T(\alpha_{n+1}) = \alpha_n$.

Let $\mathcal{U}$ be a free L-module of rank n. Then $U = \mathcal{R}_{L/K}(\mathcal{U})$ is a free K-module of rank $2n$ equipped with the τ-form $(-,-)_\tau$ and the τ-operator $\mathcal{T} : U \to U$ (switch involution, see Example 4.1.4). Let us embed V into U under the following K-linear embedding $\iota : V \hookrightarrow U$:

$$\begin{aligned} e_1 &\mapsto \big((1,1), (0,0), \cdots, (0,0)\big), \\ e_2 &\mapsto \big((0,0), (1,1), \cdots, (0,0)\big), \\ &\cdots \\ e_n &\mapsto \big((0,0), \cdots, (0,0), (1,1)\big), \\ e_{n+1} &\mapsto \big((0,0), \cdots, (0,0), (1,-1)\big). \end{aligned}$$

The map ι is designed so that it is a K-isometry with respect to the dot product $(-,-)$ on V and the τ-form on U, and the τ-form on U, and the involution T defined above corresponds to the τ-operator $\mathcal{T}$ on U (i.e. $\iota \circ T = \mathcal{T} \circ \iota$). The coordinates of the simple roots in the new K-realization $\Phi \subset U$ are:

$$\begin{aligned}
\iota(\alpha_1) &= \big((1,1),(-1,-1),(0,0)\cdots,(0,0)\big),\\
\iota(\alpha_2) &= \big((0,0),(1,1),(-1,-1),\cdots,(0,0)\big),\\
&\cdots\\
\iota(\alpha_{n-1}) &= \big((0,0),\cdots,(0,0),(1,1),(-1,-1)\big),\\
\iota(\alpha_n) &= \big((0,0),\cdots,(0,0),(0,0),(0,2)\big),\\
\iota(\alpha_{n+1}) &= \big((0,0),\cdots,(0,0),(0,0),(2,0)\big).
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{T}(\iota(\alpha_i)) &= \iota(\alpha_i)\ (1 \le i \le n-1),\\
\mathcal{T}(\iota(\alpha_n)) &= \iota(\alpha_{n+1}),\\
\mathcal{T}(\iota(\alpha_{n+1})) &= \iota(\alpha_n).
\end{aligned}$$

In particular, the K-realized root system $\Phi \subset U$ is invariant under $\mathcal{T}$. We obtain the following partition of the simple system Δ:

$$\Delta^{\mathcal{T}} = \{\iota(\alpha_i) \mid 1 \le i \le n-1\}, \quad \Delta_{rat} = \{\iota(\alpha_n)\}, \quad \mathcal{T}(\Delta_{rat}) = \{\iota(\alpha_{n+1})\}.$$

Thus $(\Phi, U)_K$ is a folded K-representation of the root datum $\Phi \hookrightarrow \Lambda^\vee$. We have

$$\Delta_\tau = \pi_\tau(\Delta^{\mathcal{T}} \sqcup \Delta_{rat}) = \{\overline{\iota(\alpha_1)}, \cdots, \overline{\iota(\alpha_n)}\}.$$

The projection operator $\pi_\tau : U \to U_L^{(\tau)}$ is $\pi_\tau(u) = \frac{1}{2}(\mathcal{T}u + u)$ for $u \in U$. Using the formula (4.1.9), we obtain the following table of values for $(\overline{\alpha}_i, \overline{\alpha}_j)_\tau$ for $\alpha_i, \alpha_j \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$. In the table below, the image $\iota(\alpha_i)\,(1 \le i \le 6)$ is simply denoted by α_i.

$(-,-)_\tau$	$\overline{\alpha_1}$	$\overline{\alpha_2}$	$\cdots$	$\overline{\alpha_{n-1}}$	$\overline{\alpha_n}$
$\overline{\alpha_1}$	4	-2	$\cdots$	0	0
$\overline{\alpha_2}$	-2	4	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\overline{\alpha_{n-1}}$	0	0	$\cdots$	4	-2
$\overline{\alpha_n}$	0	0	$\cdots$	-2	2

Hence the resulting root system Φ_τ is of type B_n.

Example 4.3.4 (Type E_6 to type F_4)**.** Let $K = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$ and $L = K \times K$. We have $p(x) = x^2 - 1$ and $\tau = 1 \in K$ (see Remark 4.1.7). Let V be a free K-module of rank 9, equipped with a basis $\{e_i \mid 1 \le i \le 9\}$ and the dot product $(-,-)$. We realize the root system Φ of type E_6 in V by setting the coordinates of simple roots as follows:

$$\begin{aligned}
\alpha_1 &= (0,0,0;0,0,0;0,1,-1),\\
\alpha_2 &= (0,1,-1;0,0,0;0,0,0),\\
\alpha_3 &= (0,0,0;0,0,0;1,-1,0),\\
\alpha_4 &= (\tfrac{1}{3},-\tfrac{2}{3},\tfrac{1}{3};-\tfrac{2}{3},\tfrac{1}{3},\tfrac{1}{3};-\tfrac{2}{3},\tfrac{1}{3},\tfrac{1}{3}),\\
\alpha_5 &= (0,0,0;1,-1,0;0,0,0),\\
\alpha_6 &= (0,0,0;0,1,-1;0,0,0).
\end{aligned}$$

The reader can readily check that $\{\alpha_i \mid 1 \le i \le 6\}$ forms the Coxeter diagram of type E_6 enumerated as in [8, Section 4.12].

Let $\mathcal{U}$ be a free L-module of rank 6 and let $U = R_{L/K}(\mathcal{U})$, so that it is a free K-module of rank 12 equipped with the τ-form $(-,-)_\tau$ and the τ-operator $\mathcal{T} : U \to U$. We consider the following K-linear embedding ι of V into U: each $(x_1,x_2,x_3;x_4,x_5,x_6;x_7,x_8,x_9)$ in V maps to

$$\big((\tfrac{x_1}{\sqrt{2}},\tfrac{x_1}{\sqrt{2}}),(\tfrac{x_2}{\sqrt{2}},\tfrac{x_2}{\sqrt{2}}),(\tfrac{x_3}{\sqrt{2}},\tfrac{x_3}{\sqrt{2}});(x_4,x_7),(x_5,x_8),(x_6,x_9)\big).$$

The map ι is designed so that it is a K-isometry with respect to the dot product $(-,-)$ on V and the τ-form on U, and the τ-operator $\mathcal{T}$ switches $\iota(\alpha_1)$ and $\iota(\alpha_6)$, $\iota(\alpha_3)$ and $\iota(\alpha_5)$, and fixes $\iota(\alpha_2)$ and $\iota(\alpha_4)$. The coordinates of the images of the six simple roots are:

$$\begin{aligned}
\iota(\alpha_1) &= \big((0,0),(0,0),(0,0);(0,0),(0,1),(0,-1)\big),\\
\iota(\alpha_2) &= \big((0,0),(\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}),(-\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}});(0,0),(0,0),(0,0)\big),\\
\iota(\alpha_3) &= \big((0,0),(0,0),(0,0);(0,1),(0,-1),(0,0)\big),\\
\iota(\alpha_4) &= \big((\tfrac{1}{3\sqrt{2}},\tfrac{1}{3\sqrt{2}}),(-\tfrac{\sqrt{2}}{3},-\tfrac{\sqrt{2}}{3}),(\tfrac{1}{3\sqrt{2}},\tfrac{1}{3\sqrt{2}});(-\tfrac{2}{3},-\tfrac{2}{3}),(\tfrac{1}{3},\tfrac{1}{3}),(\tfrac{1}{3},\tfrac{1}{3})\big),\\
\iota(\alpha_5) &= \big((0,0),(0,0),(0,0);(1,0),(-1,0),(0,0)\big),\\
\iota(\alpha_6) &= \big((0,0),(0,0),(0,0);(0,0),(1,0),(-1,0)\big).
\end{aligned}$$

Let $\Phi \subset U$ denote the K-realized root system (Definition 2.1.5) determined by $\Delta := \{\iota(\alpha_i) \mid 1 \le i \le 6\}$. Let Λ be the free $\mathbb{Z}$-submodule of U generated by the basis Δ. Let us define $\alpha^\vee(x) = \frac{2(\alpha,x)_\tau}{(\alpha,\alpha)_\tau}$ for $\alpha \in \Phi, x \in U$. Since Φ is of type E_6 and hence crystallographic, the assignment $\Phi \hookrightarrow \Lambda^\vee : \alpha \mapsto \alpha^\vee$ is well-defined and satisfies (RD1)–(RD3) of Definition 2.1.1.

Since the τ-operator $\mathcal{T} : U \to U$ sends each element $\big((y_1,\hat{y}_1),(y_2,\hat{y}_2),\cdots,(y_6,\hat{y}_6)\big)$ in U to $\big((\hat{y}_1,y_1),(\hat{y}_2,y_2),\cdots,(\hat{y}_6,y_6)\big)$ (switch involution, Example 4.1.4), we observe the following relations among the simple roots:

$$\mathcal{T}(\iota(\alpha_1)) = \iota(\alpha_6), \quad \mathcal{T}(\iota(\alpha_6)) = \iota(\alpha_1), \quad \mathcal{T}(\iota(\alpha_3)) = \iota(\alpha_5), \quad \mathcal{T}(\iota(\alpha_5)) = \iota(\alpha_3),$$
$$\mathcal{T}(\iota(\alpha_2)) = \iota(\alpha_2), \quad \mathcal{T}(\iota(\alpha_4)) = \iota(\alpha_4).$$

In particular, the K-realized root system $\Phi \subset U$ is invariant under $\mathcal{T}$. We obtain the following partition of the simple system Δ:

$$\Delta^{\mathcal{T}} = \{\iota(\alpha_2), \iota(\alpha_4)\}, \quad \Delta_{rat} = \{\iota(\alpha_1), \iota(\alpha_3)\}, \quad \mathcal{T}(\Delta_{rat}) = \{\iota(\alpha_6), \iota(\alpha_5)\}.$$

Thus $(\Phi, U)_K$ is a folded K-representation of the root datum $\Phi \hookrightarrow \Lambda^\vee$.

The projection operator $\pi_\tau : U \to U_L^{(\tau)}$ is $\pi_\tau(u) = \frac{1}{2}(\mathcal{T}u + u)$ for $u \in U$. Using the formula (4.1.9), we obtain the following table of values for $(\overline{\alpha}_i, \overline{\alpha}_j)_\tau$ for $\alpha_i, \alpha_j \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$. In the table below, the image $\iota(\alpha_i)$ $(1 \leq i \leq 6)$ is simply denoted by α_i.

$(-,-)_\tau$	$\overline{\alpha_2}$	$\overline{\alpha_4}$	$\overline{\alpha_3}$	$\overline{\alpha_1}$
$\overline{\alpha_2}$	2	-1	0	0
$\overline{\alpha_4}$	-1	2	-1	0
$\overline{\alpha_3}$	0	-1	1	$-\frac{1}{2}$
$\overline{\alpha_1}$	0	0	$-\frac{1}{2}$	1

Hence the resulting root system Φ_τ is of type F_4.

Remark 4.3.5. Among the list of classical foldings (e.g. [34, Section 10.3]), the folding of a root system of type D_4 to a root system of type G_2 cannot be presented as a τ-folding in the sense of of Definition 4.2.6. This is because the automorphism of the Coxeter diagram of type D_4 used in this folding has order three.

4.3.2 Twisted cases

This section treats the so-called Lusztig folding of the root system of type E_8 (Example 4.3.6) and its two subcases (Examples 4.3.7–4.3.8). Once again, our main task is to find a folded K-representation of each given root datum for some choice of K.

Example 4.3.6 (Type E_8 to type H_4, [30, Example 4.5])**.** Let $K = \mathbb{Z}[\frac{1}{5}]$ and $L = \mathbb{Z}[\frac{1}{5}][\tau]$ where $\tau = \frac{1+\sqrt{5}}{2}$ (the golden ratio) is the positive root of the polynomial $p(x) = x^2 - x - 1$. Let $\mathcal{U}$ be a free L-module of rank 4, and let $U := \mathcal{R}_{L/K}(\mathcal{U})$, so that it is a free K-module of rank 8 equipped with the τ-form $(-,-)_\tau$ and the τ-operator $\mathcal{T} : U \to U$. We realize the root system of type E_8 in U by setting the coordinates of simple roots as follows (cf. [32, Figure 1]):

$$\begin{aligned}
\alpha_1 &= \big((-1,1),(0,-1),(0,0),(-1,0)\big) = (-\sigma,-\tau,0,-1),\\
\alpha_2 &= \big((0,0),(-1,1),(0,-1),(1,0)\big) = (0,-\sigma,-\tau,1),\\
\alpha_3 &= \big((0,0),(1,0),(-1,1),(0,-1)\big) = (0,1,-\sigma,-\tau),\\
\alpha_4 &= \big((0,0),(0,-1),(1,0),(1,1)\big) = (0,-\tau,1,\tau^2),\\
\alpha_5 &= \big((0,0),(0,1),(1,0),(-1,-1)\big) = (0,\tau,1,-\tau^2),\\
\alpha_6 &= \big((0,0),(1,0),(-1,-1),(0,1)\big) = (0,1,-\tau^2,\tau),\\
\alpha_7 &= \big((1,0),(-1,-1),(0,0),(0,-1)\big) = (1,-\tau^2,0,-\tau),\\
\alpha_8 &= \big((0,0),(-1,0),(-1,1),(0,1)\big) = (0,-1,-\sigma,\tau).
\end{aligned}$$

For each α_i, the L-coordinates of $(\alpha_i)_L$ (see (4.1.2) for the notation) is written on the right side. It is noted that in the folding of a root system of type E_8, following the existing literature on this particular folding (see [32] or [12] for example), we use a numbering of the simple roots that is different from the standard one used in [8]. The current numbering can be found in the illustration in Figure A.4 of Appendix A.

Let $\Phi \subset U$ denote the K-realized root system determined by $\Delta := \{\alpha_i \mid 1 \leq i \leq 8\}$. Let Λ be the free $\mathbb{Z}$-submodule of U generated by the basis Δ. Let us define $\alpha^\vee(x) = \frac{2(\alpha,x)_\tau}{(\alpha,\alpha)_\tau}$ for $\alpha \in \Phi, x \in U$. Since Φ is of type E_8 and hence crystallographic, the assignment $\Phi \hookrightarrow \Lambda^\vee : \alpha \mapsto \alpha^\vee$ is well-defined and satisfies (RD1)–(RD3) of Definition 2.1.1.

By Definition 4.1.3 and the L-coordinate descriptions of $(\alpha_i)_L$ $(1 \leq i \leq 8)$ found above, we observe the following relations among the simple roots (cf. [32, (4.2)]):

$$\begin{aligned}
\mathcal{T}\alpha_1 &= \alpha_7, & \mathcal{T}(\alpha_7) &= \alpha_1+\alpha_7,\\
\mathcal{T}\alpha_2 &= \alpha_6, & \mathcal{T}(\alpha_6) &= \alpha_2+\alpha_6,\\
\mathcal{T}\alpha_3 &= \alpha_5, & \mathcal{T}(\alpha_5) &= \alpha_3+\alpha_5,\\
\mathcal{T}\alpha_8 &= \alpha_4, & \mathcal{T}(\alpha_4) &= \alpha_4+\alpha_8.
\end{aligned}$$

In particular, the operator $\mathcal{T}$ stabilizes the root lattice Λ_r of this root datum. By computing the dot product $(\alpha_i)_L \cdot (\alpha_i)_L$ for $1 \leq i \leq 8$, we obtain the following partition of the simple system Δ:

$$\Delta^{\mathcal{T}} = \emptyset, \qquad \Delta_{rat} = \{\alpha_1,\alpha_2,\alpha_3,\alpha_8\}, \quad \mathcal{T}(\Delta_{rat}) = \{\alpha_7,\alpha_6,\alpha_5,\alpha_4\}.$$

Thus $(\Phi, U)_K$ is a folded K-representation of the root datum $\Phi \hookrightarrow \Lambda^\vee$.

The projection operator $\pi_\tau : U \to U_L^{(\tau)}$ is $\pi_\tau(u) = \frac{1}{\sqrt{5}}(\mathcal{T}u + \frac{1-\sqrt{5}}{2}u)$ for $u \in U$. Using the formula (4.1.9), we obtain the following table of values for $(\overline{\alpha}_i, \overline{\alpha}_j)_\tau$ for $\alpha_i, \alpha_j \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$.

$(-,-)_\tau$	$\overline{\alpha_1}$	$\overline{\alpha_2}$	$\overline{\alpha_3}$	$\overline{\alpha_8}$
$\overline{\alpha_1}$	$\frac{2(5-\sqrt{5})}{5}$	$\frac{-5+\sqrt{5}}{5}$	0	0
$\overline{\alpha_2}$	$\frac{-5+\sqrt{5}}{5}$	$\frac{2(5-\sqrt{5})}{5}$	$\frac{-5+\sqrt{5}}{5}$	0
$\overline{\alpha_3}$	0	$\frac{-5+\sqrt{5}}{5}$	$\frac{2(5-\sqrt{5})}{5}$	$\frac{(5-\sqrt{5})(-1-\sqrt{5})}{10}$
$\overline{\alpha_8}$	0	0	$\frac{(5-\sqrt{5})(-1-\sqrt{5})}{10}$	$\frac{2(5-\sqrt{5})}{5}$

Hence, the resulting root system Φ_τ is of type H_4.

The above folding of the root system of type E_8 induces the folding of the root system of type D_6 and of type A_4 as subcases.

Example 4.3.7 (Type D_6 to type H_3)**.** In the folding of the root system of type E_8, the subset $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}$ forms a simple system of type D_6, and it folds to a simple system $\{\beta_2, \beta_3, \beta_4\}$ of type H_3. See Figure A.4 of Appendix A for the Coxeter diagram of the folding, where the simple reflections are renumbered to match the standard numbering in [8, Plate IV].

Example 4.3.8 (Type A_4 to type H_2)**.** In the folding of the root system of type E_8, the subset $\{\alpha_3, \alpha_4, \alpha_5, \alpha_8\}$ forms a simple system of type A_4, and it folds to a simple system $\{\beta_3, \beta_4\}$ of type H_2. See Figure A.4 of Appendix A for the Coxeter diagram of the folding, where the simple reflections are renumbered to match the standard numbering in [8, Plate I].

4.4 Foldings and structure algebras

Let us recall the setting and notation of Section 4.1.1. Let $(\Phi, U)_K$ be a folded K-representation of a root datum $\Phi \hookrightarrow \Lambda^\vee$ (Definition 4.1.10) with a chosen simple system Δ. Let $\Delta_P \subset \Delta$ be a (possibly empty) subset preserved under $\mathcal{T} : U \to U$. Let $\Phi \rightsquigarrow \Phi_\tau$ be the τ-folding of Φ (Definition 4.2.6). We keep the notation of Sections 4.1 and 4.2. Let us denote the moment graph of the original Bruhat poset $(W^P, \leq)$ by $\mathcal{G}^P$ and the one of the folded Bruhat poset $(W_\tau^P, \leq)$ by $\mathcal{G}_\tau^P$. Recall the structure algebra $\mathcal{Z}(\mathcal{G}^P)$ defined in Section 3.2 (the second paragraph after Lemma 3.2.2). In this section, we introduce the ring homomorphism $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ between the structure algebras induced by the embedding map $\varepsilon : W_\tau^P \hookrightarrow W^P$ discussed in Section 4.2.3.

4.4.1 Induced map

We first define the folded structure algebra $\mathcal{Z}(\mathcal{G}_\tau^P)$. Recall that by Definition 2.1.3, U contains $U_\Lambda := \Lambda \otimes_{\mathbb{Z}} K$ as a free K-submodule. Let $\mathcal{S}_\tau := \mathrm{Sym}_L\, \pi_\tau(U_\Lambda)$. We take $R = L$ and $\mathcal{S} = \mathcal{S}_\tau$ in Definition 3.2.1 to obtain the structure algebra $\mathcal{Z}(\mathcal{G}_\tau^P)$. We will show below that $\pi_\tau(U_\Lambda)$ is indeed a free L-module.

Let Θ be a $\mathbb{Z}$-basis for Λ. Since $\mathcal{T}(\Lambda) = \Lambda$ (Definition 4.1.10 (F3)), the set Θ admits a partition $\Theta = \Theta^{\mathcal{T}} \sqcup \Theta_0 \sqcup \mathcal{T}(\Theta_0)$ for some $\Theta_0 \subset \Theta$, where $\Theta^{\mathcal{T}}$ denotes the $\mathcal{T}$-invariant elements. Under the canonical inclusion $\Lambda \subset U_\Lambda \subset (U_\Lambda)_L := U_\Lambda \otimes_K L$, Θ forms an L-basis for $(U_\Lambda)_L$. Let $\mathbf{p} : U_L \twoheadrightarrow U_L^{(\tau)}$ be the L-linear projection onto the τ-eigenspace.

Claim 1. The set $\mathbf{p}(\Theta^{\mathcal{T}} \sqcup \Theta_0) \subset U_L^{(\tau)}$ is L-linearly independent.

Proof: Let us consider an L-linear relation $\sum_{\alpha \in \Theta^{\mathcal{T}} \sqcup \Theta_0} c_\alpha \mathbf{p}(\alpha) = 0$ where $c_\alpha \in L$. We have

$$\begin{aligned} 0 = \sum_{\alpha \in \Theta^{\mathcal{T}} \sqcup \Theta_0} c_\alpha \mathbf{p}(\alpha) &= \sum_{\alpha \in \Theta^{\mathcal{T}} \sqcup \Theta_0} \tfrac{c_\alpha}{\tau - \sigma} (\mathcal{T}_L - \sigma I_L)(\alpha) \\ &= \sum_{\alpha \in \Theta^{\mathcal{T}} \sqcup \Theta_0} \tfrac{-\sigma c_\alpha}{\tau - \sigma} \alpha + \sum_{\beta \in \mathcal{T}(\Theta_0)} \tfrac{d_\beta}{\tau - \sigma} \beta \end{aligned}$$

where $d_\beta = c_\alpha$ if $\beta = \mathcal{T}\alpha, \alpha \in \Theta_0$. Since Θ is L-linearly independent, it follows that $\frac{-\sigma c_\alpha}{\tau - \sigma} = 0$ and $\frac{d_\beta}{\tau - \sigma} = 0$ for all $\alpha \in \Theta^{\mathcal{T}} \sqcup \Theta_0$ and $\beta \in \mathcal{T}(\Theta_0)$. Hence $c_\alpha = 0$ for all $\alpha \in \Theta^{\mathcal{T}} \sqcup \Theta_0$. ∎

On the other hand, we have

$$\pi_\tau(U_\Lambda) = \mathbf{p}\big((U_\Lambda)_L\big) = \mathbf{p}(\mathrm{span}_L\, \Theta) = \mathrm{span}_L\, \mathbf{p}(\Theta).$$

Since $\pi_\tau(\mathcal{T}x) = \tau \cdot \pi_\tau(x)$ for all $x \in \Lambda$, the set $\mathbf{p}(\mathcal{T}(\Theta_0))$ is L-linearly dependent to $\mathbf{p}(\Theta^{\mathcal{T}} \sqcup \Theta_0)$. Hence $\pi_\tau(U_\Lambda) = \operatorname{span}_L \mathbf{p}(\Theta) = \operatorname{span}_L \mathbf{p}(\Theta^{\mathcal{T}} \sqcup \Theta_0)$. Now the above claim guarantees that $\pi_\tau(U_\Lambda)$ is a free L-module.

We now define the induced map ε^* between the ambient free modules of $\mathcal{Z}(\mathcal{G}^P)$ and $\mathcal{Z}(\mathcal{G}^P_\tau)$. Let us recall that $\mathcal{S} = \operatorname{Sym}_K U_\Lambda$ in the original structure algebra $\mathcal{Z}(\mathcal{G}^P)$ (see the paragraphs after Definition 3.2.1). We define the map by:

$$\varepsilon^* : \bigoplus_{v \in W^P} \mathcal{S} \to \bigoplus_{u \in W^P_\tau} \mathcal{S} \to \bigoplus_{u \in W^P_\tau} \mathcal{S}_\tau$$

$$(z_v)_{v \in W^P} \mapsto (\tilde{z}_u)_{u \in W^P_\tau} \mapsto (\pi_\tau(\tilde{z}_u))_{u \in W^P_\tau} \tag{4.4.1}$$

where $\tilde{z}_u = z_{\varepsilon(u)}$ for each $u \in W^P_\tau$, and $\pi_\tau : \mathcal{S} \to \mathcal{S}_\tau$ is the morphism of graded K-algebras canonically induced from $\pi_\tau|_{U_\Lambda} : U_\Lambda \to \pi_\tau(U_\Lambda)$. For the later purpose, we make the following remark.

Remark 4.4.1. Let us recall that the K-linear map $\pi_\tau : U \to U^{(\tau)}_L$ is bijective as a map of sets (see Section 4.1.2 after Lemma 4.1.6 for definition). In particular, its restriction $\pi_\tau|_{U_\Lambda} : U_\Lambda \to \pi_\tau(U_\Lambda)$ is injective. Hence the induced map $\pi_\tau : \mathcal{S} \to \mathcal{S}_\tau$ is also injective.

We have the following:

Proposition 4.4.2 ([30, Theorem 6.2])**.** *The map ε^* restricts to $\mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}^P_\tau)$.*

Proof: Let $(z_v)_{v \in W^P} \in \mathcal{Z}(\mathcal{G}^P)$. Our aim is to show $(\pi_\tau(\tilde{z}_u))_{u \in W^P_\tau} \in \mathcal{Z}(\mathcal{G}^P_\tau)$. Let $u \to u'$ be an edge in the folded moment graph $\mathcal{G}^P_\tau$ labelled by some positive root $\beta \in \Phi^+_\tau$ (thus explicitly $u' = \overline{R_\beta u}$). We will prove the following relation between the two coordinates:

$$\pi_\tau(\tilde{z}_u) - \pi_\tau(\tilde{z}_{u'}) = \pi_\tau(z_{\varepsilon(u)} - z_{\varepsilon(u')}) \in \beta \mathcal{S}_\tau.$$

By Proposition 4.2.12, we have $\varepsilon(u') = \varepsilon(\overline{R_\beta u}) = \overline{\varepsilon(R_\beta u)} = \overline{\varepsilon(R_\beta)\varepsilon(u)}$. We have $\beta = w(\overline{\alpha})$ for some $\alpha \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$ and $w \in W_\tau$, whence $R_\beta = w R_{\overline{\alpha}} w^{-1}$. Now we consider two cases.

Case 1: If $\alpha \in \Delta^{\mathcal{T}}$, then we have $\varepsilon(R_\beta) = \varepsilon(w R_{\overline{\alpha}} w^{-1}) = \varepsilon(w) s_\alpha \varepsilon(w)^{-1} = s_{\varepsilon(w)(\alpha)}$, which is a reflection in W. Hence in the original structure algebra $\mathcal{Z}(\mathcal{G}^P)$ we have

$$z_{\varepsilon(u)} - z_{\varepsilon(u')} = z_{\varepsilon(u)} - z_{\overline{s_{\varepsilon(w)(\alpha)}\varepsilon(u)}} \in \varepsilon(w)(\alpha)\mathcal{S}.$$

By applying the map π_τ, we obtain

$$\pi_\tau(z_\varepsilon - z_{\varepsilon u'}) \in \pi_\tau(\varepsilon(w)(\alpha))\pi_\tau(\mathcal{S}) = w(\overline{\alpha})\mathcal{S}_\tau = \beta \mathcal{S}_\tau.$$

Here we used the property $\pi_\tau(\varepsilon(w)(\alpha)) = w(\pi_\tau(\alpha))$.

Case 2: If $\alpha \in \Delta_{rat}$, then we have $\varepsilon(R_\beta) = \varepsilon(w)s_\alpha s_{\mathcal{T}\alpha}\varepsilon(w)^{-1} = s_{\varepsilon(w)(\alpha)}s_{\varepsilon(w)(\mathcal{T}\alpha)}$. Put the intermediate vertex $v'' := \overline{s_{\varepsilon(w)(\mathcal{T}\alpha)}\varepsilon(u)}$. In the original structure algebra $\mathcal{Z}(\mathcal{G}^P)$, we have

$$z_{\varepsilon(u)} - z_{v''} \in \varepsilon(w)(\mathcal{T}\alpha)\mathcal{S},$$
$$z_{v''} - z_{\varepsilon(u')} = z_{v''} - z_{\overline{s_{\varepsilon(w)(\alpha)}v''}} \in \varepsilon(w)(\alpha)\mathcal{S}.$$

By applying the map π_τ, we obtain

$$\pi_\tau(z_{\varepsilon(u)} - z_{v''}) \in w(\pi_\tau(\mathcal{T}\alpha))\mathcal{S}_\tau = w(\tau\overline{\alpha})\mathcal{S}_\tau = w(\overline{\alpha})\tau\mathcal{S}_\tau = \beta\mathcal{S}_\tau,$$
$$\pi_\tau(z_{v''} - z_{\varepsilon(u')}) \in w(\overline{\alpha})\mathcal{S}_\tau = \beta\mathcal{S}_\tau.$$

This yields $\pi_\tau(z_{\varepsilon(u)}) - \pi_\tau(z_{\varepsilon(u')}) \in \beta\mathcal{S}_\tau$. ∎

By construction, the induced map ε^* is a ring homomorphism. It also commutes with the canonical $\mathcal{S}$-action, that is, for $\xi \in \mathcal{Z}(\mathcal{G}^P)$ and $f \in \mathcal{S}$, we have

$$\varepsilon^*(f \cdot \xi) = \pi_\tau(f) \cdot \big(\varepsilon^*(\xi)\big).$$

Furthermore, ε^* commutes also with the $\mathcal{S}^{W_P}$-action discussed in Section 3.2.3 after Lemma 3.2.12. That is, for $\xi \in \mathcal{Z}(\mathcal{G}^P)$ and $g \in \mathcal{S}^{W_P}$, we have

$$\varepsilon^*(\xi_\star g) = \varepsilon^*(\xi)_\star\big(\pi_\tau(g)\big).$$

Definition 4.4.3. The induced ring homomorphism $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ is called the *folding map.*

The reader will find examples of the folding map ε^* in Section 5.2.

Let us recall the notion of a Borel map from Definition 3.2.11. Let us denote the Borel map of the folded structure algebra $\mathcal{Z}(\mathcal{G}_\tau^P)$ by ρ_τ. The folding map ε^* commutes with the two Borel maps ρ and ρ_τ as in the following lemma.

Lemma 4.4.4 ([30, Lemma 6.4])**.** *We have the following commutative diagram:*

$$\begin{array}{ccc}
\mathcal{S} \otimes_{\mathcal{S}^W} \mathcal{S}^{W_P} & \xrightarrow{\rho} & \mathcal{Z}(\mathcal{G}^P) \\
\Big\downarrow{\scriptstyle \pi_\tau\otimes\pi_\tau} & & \Big\downarrow{\scriptstyle \varepsilon^*} \\
\mathcal{S}_\tau \otimes_{\mathcal{S}_\tau^{W_\tau}} \mathcal{S}_\tau^{(W_P)_\tau} & \xrightarrow{\rho_\tau} & \mathcal{Z}(\mathcal{G}_\tau^P)
\end{array}$$

Proof: Let $f \otimes g \in \mathcal{S} \otimes_{\mathcal{S}^W} \mathcal{S}^{W_P}$. By the definition of each map, we have

$$\varepsilon^*\big(\rho(f\otimes g)\big) = \varepsilon^*\big((f\cdot w(g))_{w\in W^P}\big) = \big(\pi_\tau(f\cdot\varepsilon(u)(g))\big)_{u\in W_\tau^P},$$
$$\rho_\tau\big(\pi_\tau(f)\otimes\pi_\tau(g)\big) = \big(\pi_\tau(f)\cdot u(\pi_\tau(g))\big)_{u\in W_\tau^P} = \big(\pi_\tau(f\cdot\varepsilon(u)(g))\big)_{u\in W_\tau^P}.$$

∎

4.4.2 Restriction to the augmented and reduced structure algebras

Let us recall the augmented structure algebra $\overline{\mathcal{Z}}(\mathcal{G}^P)$ and the reduced structure algebra $\widehat{\mathcal{Z}}(\mathcal{G}^P)$ from Section 3.2.3. The folding map ε^* restricts to these structure algebras as shown in the following two propositions.

Proposition 4.4.5 ([30, Theorem 6.2])**.** *The folding map ε^* restricts to the augmented structure algebras, that is, for each $\xi \in \mathcal{Z}(\mathcal{G}^P)$, the assignment*

$$\overline{\varepsilon}^* : \overline{\mathcal{Z}}(\mathcal{G}^P) \to \overline{\mathcal{Z}}(\mathcal{G}^P_\tau) : \overline{can}(\xi) \mapsto \overline{can}\big(\varepsilon^*(\xi)\big)$$

is well-defined.

Proof: Since the map $\pi_\tau : \mathcal{S} \to \mathcal{S}_\tau$ is surjective on the positive degrees by construction, we have $\pi_\tau(\mathfrak{a}) = \mathfrak{a}_\tau$. Hence $\varepsilon^*\big(\mathfrak{a} \cdot \mathcal{Z}(\mathcal{G}^P)\big) \subset \mathfrak{a}_\tau \cdot \mathcal{Z}(\mathcal{G}^P_\tau)$, and the folding map ε^* restricts to the quotients $\overline{\varepsilon}^* : \overline{\mathcal{Z}}(\mathcal{G}^P) \to \overline{\mathcal{Z}}(\mathcal{G}^P_\tau)$. ∎

Proposition 4.4.6 ([30, Corollary 6.7])**.** *The folding map ε^* restricts to the reduced structure algebras, that is, for each $\xi \in \mathcal{Z}(\mathcal{G}^P)$, the assignment*

$$\hat{\varepsilon}^* : \widehat{\mathcal{Z}}(\mathcal{G}^P) \to \widehat{\mathcal{Z}}(\mathcal{G}^P_\tau) : \widehat{can}(\xi) \mapsto \widehat{can}\big(\varepsilon^*(\xi)\big)$$

is well-defined.

Proof: What needs to be shown is $\overline{\varepsilon}^*\big(\overline{\mathcal{Z}}(\mathcal{G}^P)_\star \mathfrak{a}^{W_P}\big) \subset \overline{\mathcal{Z}}(\mathcal{G}^P_\tau)_\star \mathfrak{a}^{(W_P)_\tau}$. We recall the observation

$$\overline{\mathcal{Z}}(\mathcal{G}^P)_\star \mathfrak{a}^{W_P} = \overline{\rho}(K \otimes_{\mathcal{S}^W} \mathfrak{a}^{W_P}), \quad \overline{\mathcal{Z}}(\mathcal{G}^P_\tau)_\star \mathfrak{a}_\tau^{(W_P)_\tau} = \overline{\rho}_\tau(L \otimes_{\mathcal{S}_\tau^{W_\tau}} \mathfrak{a}_\tau^{(W_P)_\tau}).$$

from Remark 3.2.14. By the following commutative diagram

$$\begin{array}{ccc} K \otimes_{\mathcal{S}^W} \mathcal{S}^{W^P} & \xrightarrow{\overline{\rho}} & \overline{\mathcal{Z}}(\mathcal{G}^P) \\ \big\downarrow{\scriptstyle \pi_\tau \otimes \pi_\tau} & & \big\downarrow{\scriptstyle \overline{\varepsilon}^*} \\ L \otimes_{\mathcal{S}_\tau^{W_\tau}} \mathcal{S}_\tau^{(W_P)_\tau} & \xrightarrow{\overline{\rho}_\tau} & \overline{\mathcal{Z}}(\mathcal{G}^P_\tau) \end{array}$$

and the containment $(\pi_\tau \otimes \pi_\tau)(K \otimes_{\mathcal{S}^W} \mathfrak{a}^{W_P}) \subset L \otimes_{\mathcal{S}^{W_\tau}} \mathfrak{a}_\tau^{(W_P)_\tau}$, we obtain

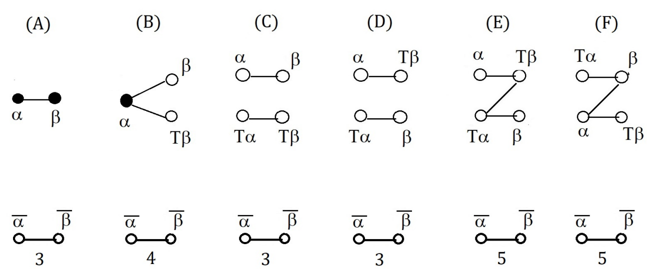
$$\overline{\varepsilon}^*\big(\overline{\rho}(K \otimes_{\mathcal{S}^W} \mathfrak{a}^{W_P})\big) = \overline{\rho}_\tau\big((\pi_\tau \otimes \pi_\tau)(K \otimes_{\mathcal{S}^W} \mathfrak{a}^{W_P})\big) \subset \overline{\rho}_\tau(L \otimes_{\mathcal{S}_\tau^{W_\tau}} \mathfrak{a}_\tau^{(W_P)_\tau}).$$

∎

4.5 Local description of a folding

Let us recall the setting of Section 4.1.1, and let $\Phi \rightsquigarrow \Phi_\tau$ be a τ-folding of a simply laced crystallographic root system Φ (Definition 4.2.6). We keep the notation of Sections 4.1 and 4.2. In this section, we give a local description of the projection map $\pi_\tau|_\Delta : \Delta \to \pi_\tau(\Delta)$ (Section 4.1.1), which will be essential in the main discussion of Chapter 5.

Proposition 4.5.1. *Let $\alpha, \beta \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$ such that $m := m(R_{\overline{\alpha}}, R_{\overline{\beta}}) \geq 3$. Then the possible configuration of the full subgraph $(\pi_\tau|_\Delta)^{-1}(\{\overline{\alpha}, \overline{\beta}\})$ is one of the following up to symmetry of α and β:*



Proof: We may assume, without loss of generality, $(\alpha, \alpha)_\tau = 1$ for all $\alpha \in \Delta$. Let us write $\alpha_L \cdot \beta_L = a + b\tau$ for some $a, b \in K$. Let us recall the following formulas from (4.1.3) and (4.1.9):

$$\begin{aligned}(\alpha, \beta)_\tau &= \mathrm{pr}_\tau(\alpha_L \cdot \beta_L), \\ (\overline{\alpha}, \overline{\beta})_\tau &= \tfrac{\sigma^2+1}{D}(\alpha_L \cdot \beta_L).\end{aligned}$$

Since $m \geq 3$, we have $(\overline{\alpha}, \overline{\beta})_\tau \neq 0$, and hence $\alpha_L \cdot \beta_L \neq 0$. Hence either $a \neq 0$ or $b \neq 0$ (or both). We know $(\alpha, \mathcal{T}\alpha)_\tau = 0$, $(\beta, \mathcal{T}\beta)_\tau = 0$ (see Proposition 4.1.9 (a)), and $(\alpha, \mathcal{T}\beta)_\tau = (\mathcal{T}\alpha, \beta)_\tau$ (the adjoint property of $\mathcal{T}$, see Lemma 4.1.5). Let us compute the remaining combinations below:

$$\begin{aligned}(\alpha, \beta)_\tau &= a, \\ (\alpha, \mathcal{T}\beta)_\tau &= (\mathcal{T}\alpha, \beta)_\tau = \mathrm{pr}_\tau(\tau(a + b\tau)) = \mathrm{pr}_\tau(b + (a + c_1 b)\tau) = b, \\ (\mathcal{T}\alpha, \mathcal{T}\beta)_\tau &= \mathrm{pr}_\tau(\tau^2(a + b\tau)) = \mathrm{pr}_\tau(a + c_1 b + (c_1^2 b + c_1 a + b)\tau) = a + c_1 b.\end{aligned}$$

Since the original root system Φ is finite crystallographic, in particular, its Coxeter diagram contains no circuit. Hence at least one of the above three values must be zero. Since Φ is taken to be simply laced, $a \neq 0$ (resp. $b \neq 0$) implies $a = (\alpha, \beta)_\tau = -\frac{1}{2}$ (resp. $b = (\alpha, \mathcal{T}\beta)_\tau = -\frac{1}{2}$).

Let us also note that we have the following:

$$\alpha_L \cdot \alpha_L = \begin{cases} 1+\tau & \text{if } \alpha \in \Delta^{\mathcal{T}}, \\ 1 & \text{if } \alpha \in \Delta_{rat}. \end{cases}$$

If $\alpha \in \Delta_{rat}$, this is a consequence of Lemma 4.1.9 (b). If $\alpha \in \Delta^{\mathcal{T}}$, then write $\alpha_L \cdot \alpha_L = c + d\tau$ for some $c, d \in K$. Then combine $c = (\alpha,\alpha)_\tau = (\alpha, \mathcal{T}\alpha)_\tau = d$ and $(\alpha,\alpha)_\tau = 1$.

We will denote the angle between $\overline{\alpha}$ and $\overline{\beta}$ by θ.

Case 1: $c_1 = 0$ (split case)
In this case, we have $(\alpha,\beta)_\tau = (\mathcal{T}\alpha, \mathcal{T}\beta)_\tau = a$. Suppose $\alpha = \mathcal{T}\alpha$. Then we also have $b = (\mathcal{T}\alpha, \beta)_\tau = (\alpha,\beta)_\tau = a$, and hence $a \neq 0$ and $b \neq 0$. We have

$$\begin{aligned} (\overline{\alpha}, \overline{\beta})_\tau &= \tfrac{1}{2}(\alpha_L \cdot \beta_L) = \tfrac{1}{2}(-\tfrac{1}{2} - \tfrac{1}{2}\tau), \\ (\overline{\alpha}, \overline{\alpha})_\tau &= \tfrac{1}{2}(\alpha_L \cdot \alpha_L) = \tfrac{1}{2}(1+\tau), \\ (\overline{\beta}, \overline{\beta})_\tau &= \tfrac{1}{2}(\beta_L \cdot \beta_L) = \begin{cases} \tfrac{1}{2}(1+\tau) & \text{if } \beta \in \Delta^{\mathcal{T}}, \\ \tfrac{1}{2} & \text{if } \beta \in \Delta_{rat}. \end{cases} \end{aligned}$$

Hence if $\beta \in \Delta^{\mathcal{T}}$, we obtain

$$-\cos\theta = \frac{-\frac{1}{4}(1+\tau)}{\frac{1}{2}(1+\tau)} = -\tfrac{1}{2},$$

yielding the diagram (A). If $\beta \in \Delta_{rat}$, we obtain

$$-\cos\theta = \frac{-\frac{1}{4}(1+\tau)}{\sqrt{\frac{1}{2}(1+\tau)\cdot\frac{1}{2}}} = -\tfrac{1}{2}\sqrt{1+\tau} = -\tfrac{1}{\sqrt{2}},$$

yielding the diagram (B) (in the last equality, $\tau = 1$ was used).

Now suppose $\alpha, \beta \notin \Delta^{\mathcal{T}}$. If $a = 0$, then $b \neq 0$. We have

$$\begin{aligned} (\overline{\alpha}, \overline{\beta})_\tau &= \tfrac{1}{2}(\alpha_L \cdot \beta_L) = \tfrac{1}{2}(-\tfrac{1}{2}\tau), \\ (\overline{\alpha}, \overline{\alpha})_\tau &= \tfrac{1}{2}(\alpha_L \cdot \alpha_L) = \tfrac{1}{2}, \\ (\overline{\beta}, \overline{\beta})_\tau &= \tfrac{1}{2}(\beta_L \cdot \beta_L) = \tfrac{1}{2}, \end{aligned}$$

and we obtain

$$-\cos\theta = \frac{-\frac{1}{4}\tau}{\frac{1}{2}} = -\tfrac{1}{2}\tau = -\tfrac{1}{2},$$

yielding the diagram (D). If $b = 0$, then $a \neq 0$. We have

$$\begin{aligned} (\overline{\alpha}, \overline{\beta})_\tau &= \tfrac{1}{2}(\alpha_L \cdot \beta_L) = \tfrac{1}{2}(-\tfrac{1}{2}), \\ (\overline{\alpha}, \overline{\alpha})_\tau &= \tfrac{1}{2}(\alpha_L \cdot \alpha_L) = \tfrac{1}{2}, \\ (\overline{\beta}, \overline{\beta})_\tau &= \tfrac{1}{2}(\beta_L \cdot \beta_L) = \tfrac{1}{2}, \end{aligned}$$

and we obtain

$$-\cos\theta = \frac{-\frac{1}{4}}{\frac{1}{2}} = -\frac{1}{2},$$

yielding the diagram (C).

Case 2: $c_1 \neq 0$ (non-split case)

If $a = 0$, then $b \neq 0$. The equality $(\mathcal{T}\alpha, \mathcal{T}\beta)_\tau = -\frac{1}{2}c_1 = -\frac{1}{2}$ enforces $c_1 = 1$, and $\tau = \frac{1+\sqrt{5}}{2}$ (the golden ratio). We have

$$(\overline{\alpha}, \overline{\beta})_\tau = \tfrac{\sigma^2+1}{5}(\alpha_L \cdot \beta_L) = \tfrac{\sigma^2+1}{5}(-\tfrac{1}{2}\tau),$$
$$(\overline{\alpha}, \overline{\alpha})_\tau = (\overline{\beta}, \overline{\beta})_\tau = \tfrac{\sigma^2+1}{5}(\beta_L \cdot \beta_L) = \tfrac{\sigma^2+1}{5},$$

and we obtain

$$-\cos\theta = \frac{\frac{\sigma^2+1}{5}(-\frac{1}{2}\tau)}{\frac{\sigma^2+1}{5}} = -\frac{1}{2}\tau = \frac{-1-\sqrt{5}}{4},$$

yielding the diagram (E).

If $b = 0$, then $a \neq 0$. We have

$$(\overline{\alpha}, \overline{\beta})_\tau = \tfrac{\sigma^2+1}{5}(\alpha_L \cdot \beta_L) = \tfrac{\sigma^2+1}{5}(-\tfrac{1}{2}),$$
$$(\overline{\alpha}, \overline{\alpha})_\tau = (\overline{\beta}, \overline{\beta})_\tau = \tfrac{\sigma^2+1}{5}(\beta_L \cdot \beta_L) = \tfrac{\sigma^2+1}{5},$$

and we obtain

$$-\cos\theta = \frac{\frac{\sigma^2+1}{5}(-\frac{1}{2})}{\frac{\sigma^2+1}{5}} = -\frac{1}{2},$$

yielding the diagram (C).

If $a + c_1 b = 0$, then both $a \neq 0$ and $b \neq 0$. The equality $-\frac{1}{2} - \frac{1}{2}c_1 = 0$ enforces $c_1 = -1$, and $\tau = \frac{-1+\sqrt{5}}{2}$. We have

$$(\overline{\alpha}, \overline{\beta})_\tau = \tfrac{\sigma^2+1}{5}(\alpha_L \cdot \beta_L) = \tfrac{\sigma^2+1}{5}(-\tfrac{1}{2} - \tfrac{1}{2}\tau),$$
$$(\overline{\alpha}, \overline{\alpha})_\tau = (\overline{\beta}, \overline{\beta})_\tau = \tfrac{\sigma^2+1}{5}(\beta_L \cdot \beta_L) = \tfrac{\sigma^2+1}{5},$$

and we obtain

$$-\cos\theta = \frac{\frac{\sigma^2+1}{5}(-\frac{1}{2}-\frac{1}{2}\tau)}{\frac{\sigma^2+1}{5}} = -\frac{1}{2} - \frac{1}{2}\tau = \frac{-1-\sqrt{5}}{4},$$

yielding the diagram (F). ∎

As a corollary of the above proof, we have the following.

Corollary 4.5.2. *In a twisted quadratic folding of a simply laced finite root system, the value of c_1 in the polynomial $p(x) = x^2 - c_1 x + c_2$ is either* 0, 1, *or* -1.

It is recalled that we have $c_2 = -1$ by Definition 4.1.10.

Remark 4.5.3. The situation of diagram (A) appears in the folding of type $D_{n+1}(n \geq 3)$ to type B_n (Example 4.3.3) and type E_6 to type F_4 (Example 4.3.4). The situation of diagram (B) appears in all three classical (quadratic) foldings (Examples 4.3.1, 4.3.3, 4.3.4). The situation of diagram (C) appears in all examples of twisted quadratic foldings in Section 4.3 except for the folding of type A_4 to type H_2 (Example 4.3.8). The situation of diagram (E) appears in the folding of type E_8 to type H_4 and its subfoldings (Examples 4.3.6–4.3.8).

The situation of diagram (D) can be realized as follows: for example, consider the twisted quadratic folding of type A_{2n-1} to type C_n as described in Example 4.3.1. Suppose n is odd and take $\Delta_{rat} = \{\alpha_1, \alpha_{2n-2}, \alpha_3, \alpha_{2n-4}, \cdots, \alpha_{n+1}\}$. Then $\mathcal{T}(\Delta_{rat}) = \{\alpha_{2n-1}, \alpha_2, \alpha_{2n-3}, \alpha_4, \cdots, \alpha_{n-1}\}$ and we still have the partition $\Delta = \Delta^{\mathcal{T}} \sqcup \Delta_{rat} \sqcup \mathcal{T}(\Delta_{rat})$. This alternative choice of Δ_{rat} is possible because all simple roots except the ones in $\Delta^{\mathcal{T}}$ are τ-rational in a split folding. Note also that the subset Δ_{rat} is not required to be connected in the Coxeter diagram in Definition 4.1.10.

Finally, the situation of diagram (F) can be realized as follows: consider the folding of type A_4 to type H_2 as described in Example 4.3.8. Let $\tau' > \sigma'$ be the two roots of the polynomial $p(x) = x^2 + x - 1$. Let $K = \mathbb{Z}[\frac{1}{5}]$ and $L = \mathbb{Z}[\frac{1}{5}][\tau']$. Let $\mathcal{U}$ be a free L-module of rank 4 and let $U = R_{L/K}(\mathcal{U})$. Then U is a free K-module of rank 8 equipped with the τ'-form $(-,-)_{\tau'}$ and the τ'-operator $\mathcal{T}'$. Let us K-realize the root system Φ of type A_4 in U by taking the coordinates of simple roots as follows:

$$\begin{aligned}
\alpha_1 &= \big((0,0),(0,1),(-1,0),(1,-1)\big) = (0,\tau',-1,(\tau')^2),\\
\alpha_2 &= \big((0,0),(-1,0),(1,1),(0,1)\big) = (0,-1,-\sigma',\tau'),\\
\alpha_3 &= \big((0,0),(1,0),(-1,-1),(0,1)\big) = (0,1,\sigma',\tau'),\\
\alpha_4 &= \big((0,0),(0,-1),(1,0),(1,-1)\big) = (0,-\tau',1,(\tau')^2).
\end{aligned}$$

For each α_i, the L-coordinates of $(\alpha_i)_L$ is written on the right side. Then α_2 and α_3 are τ'-rational, and we have the relation

$$\mathcal{T}'(\alpha_1) = \alpha_3 - \alpha_1, \quad \mathcal{T}'(\alpha_2) = \alpha_4, \quad \mathcal{T}'(\alpha_3) = \alpha_1, \quad \mathcal{T}'(\alpha_4) = \alpha_2 - \alpha_4,$$

yielding the partition

$$\Delta^{\mathcal{T}'} = \emptyset, \quad \Delta_{rat} = \{\alpha_2, \alpha_3\}, \quad \mathcal{T}'(\Delta_{rat}) = \{\alpha_1, \alpha_4\}.$$

In this way, we obtain the situation of diagram (F).

Chapter 5

Schubert classes and a folding map

Let $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}^P_\tau)$ be the folding map defined in (4.4.1). The aim of this chapter is to study the relationship between the combinatorial Schubert classes (Definition 3.3.1) of the original structure algebra $\mathcal{Z}(\mathcal{G}^P)$ and those of the folded structure algebra $\mathcal{Z}(\mathcal{G}^P_\tau)$ under the folding map ε^*.

5.1 Statement of the problem

Let us recall the setting and notation of Sections 4.1.1, 4.2, and 4.4. In particular, we have the embedding of the finite reflection groups $W_\tau \hookrightarrow W$ (Proposition 4.2.8), its restriction $\varepsilon : W^P_\tau \hookrightarrow W^P$ (Proposition 4.2.12), and the induced folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}^P_\tau)$ on the structure algebras (see (4.4.1)). We will denote the Schubert classes of $\mathcal{Z}(\mathcal{G}^P)$ by $\{\sigma^{(w)} \mid w \in W^P\}$, and those of $\mathcal{Z}(\mathcal{G}^P_\tau)$ by $\{\sigma^{(u)}_\tau \mid u \in W^P_\tau\}$ (Definition 3.3.1).

In this section, we list the questions about the folding map ε^* that we address in this thesis. The first question is the surjectivity of ε^*:

Question 1. *For which twisted quadratic foldings is the folding map ε^* surjective? Are there any conditions?*

By turning our attention to the Schubert bases of the structure algebras (Definition 3.3.1, Proposition 3.3.15), we may extend Question 1 and ask for the description of the ε^*-preimage of each Schubert class of the folded structure algebra $\mathcal{Z}(\mathcal{G}^P_\tau)$:

Question 2. *For each Schubert class $\sigma^{(u)}_\tau$ in $\mathcal{Z}(\mathcal{G}^P_\tau)$, what is the description of the preimage $(\varepsilon^*)^{-1}(\sigma^{(u)}_\tau)$ in $\mathcal{Z}(\mathcal{G}^P)$?*

As a special case, we postulate the following question:

Question 3. *Which Schubert class $\sigma^{(u)}_\tau \in \mathcal{Z}(\mathcal{G}^P_\tau)$ admits an element $w \in W^P$ such that $\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma^{(u)}_\tau$ for some $c \in L^\times$?*

Answering Question 3 provides a partial understanding of the ε^*-preimage of $\sigma_\tau^{(u)}$ over L, and hence a partial answer to Question 2. Let us remark also that since the Schubert classes $\{\sigma_\tau^{(u)} \mid u \in W_\tau^P\}$ forms a $\mathcal{S}_\tau$-module basis of $\mathcal{Z}(\mathcal{G}_\tau^P)$ (Proposition 3.3.15), if every Schubert class $\sigma_\tau^{(u)}(u \in W_\tau^P)$ admits $w \in W^P$ such that $\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)}$ for some $c \in L^\times$, then the map ε^* over L is surjective.

In this thesis, we give a complete answer to Questions 1 and 3, and thus a partial answer to Question 2. In Section 5.4, we will provide a concrete combinatorial criterion for an element $u \in W_\tau^P$ so that there exists $w \in W^P$ with $\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)}$ for some $c \in L^\times$ (Questions 3). In Section 5.6, we will show that the induced folding map ε^* is surjective for any twisted quadratic folding after extending the base ring to L (Question 1).

In order to investigate Question 3, we define the following key notion of this chapter.

Definition 5.1.1. Let $u \in W_\tau^P$. We say that the Schubert class $\sigma_\tau^{(u)}$ *is liftable* or *admits a lifting* if there exists some $w \in W^P$ and a nonzero $c \in L$ such that $\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)}$. For any such $w \in W^P$, the Schubert class $\sigma^{(w)}$ is called a *lifting* of $\sigma_\tau^{(u)}$, and we say that $\sigma_\tau^{(u)}$ *lifts to* $\sigma^{(w)}$. The folding map ε^* is said to have the *lifting property* if the Schubert class $\sigma_\tau^{(u)}$ admits a lifting for every $u \in W_\tau^P$.

Remark 5.1.2. The nonzero constant $c \in L$ appearing in Definition 5.1.1 is unique. This can be traced down to the fact that $\mathcal{S}_\tau$ (see Section 4.4.1 for the notation) is an integral domain.

5.2 Small examples

In this section, we present the computation of the images of the Schubert classes of $\mathcal{Z}(\mathcal{G}^P)$ under the folding map ε^* for some small structure algebras. What we observe in the computation serves as motivation for Questions 1–3 (Section 5.1) that we treat in this thesis. For the notation used for parabolic subgroups, see notation after Proposition 2.2.18.

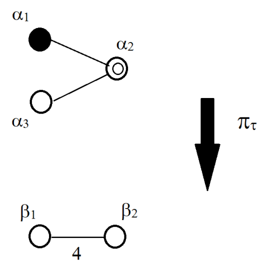
5.2.1 Classical cases

Let $\Phi \rightsquigarrow \Phi_\tau$ be the folding of the root system of type A_3 to the root system of type C_2 as described in Example 4.3.1. In particular, $K = \mathbb{Z}[\frac{1}{2}]$ and $L = K \times K$. The partition of the original simple roots is

$$\Delta^{\mathcal{T}} = \{\alpha_2\}, \quad \Delta_{rat} = \{\alpha_1\}, \quad \mathcal{T}(\Delta_{rat}) = \{\alpha_3\},$$

and the simple roots of the folded root system Φ_τ are $\Delta_\tau = \{\overline{\alpha_1}, \overline{\alpha_2}\}$. We set $\beta_1 := \overline{\alpha_1}$ and $\beta_2 := \overline{\alpha_2}$. The simple reflection of W corresponding to α_i is denoted by s_i for

$i = 1, 2, 3$ and the simple reflection of W_τ corresponding to β_j is denoted by R_j for $j = 1, 2$. The Coxeter diagram of this folding is depicted below.



Example 5.2.1 (Type A_3/P_2 to type C_2/P_2)**.** The moment graph associated with the Bruhat poset $(W^P, \leq)$ of type A_3 with $W_P = \langle s_1, s_3 \rangle$ is given in Table B.2 of Appendix B. For each $w \in W^P$, the image $\varepsilon^*(\sigma^{(w)})$ is computed in Table 5.1.

deg	$\sigma^{(w)} \in \mathcal{Z}(\mathcal{G}^P)$	$\varepsilon^*(\sigma^{(w)}) \in \mathcal{Z}(\mathcal{G}^P_\tau)$
0	$\sigma^{(e)}$	$\sigma_\tau^{(e)}$
1	$\sigma^{(s_2)}$	$\sigma_\tau^{(R_2)}$
2	$\sigma^{(s_1 s_2)}$ $\sigma^{(s_3 s_2)}$	$\sigma_\tau^{(R_1 R_2)}$ $\sigma_\tau^{(R_1 R_2)}$
3	$\sigma^{(s_3 s_1 s_2)}$	$\beta_1 \sigma_\tau^{(R_1 R_2)} + \sigma_\tau^{(R_2 R_1 R_2)}$
4	$\sigma^{(s_2 s_3 s_1 s_2)}$	$(\beta_1 + \beta_2)\sigma_\tau^{(R_2 R_1 R_2)}$

Table 5.1: ε^*-images of Schubert classes: type A_3/P_2 to type C_2/P_2

We have $W_\tau^P = \{e, R_2, R_1R_2, R_2R_1R_2\}$. Observe the following relations:

$$\begin{aligned}
\varepsilon^*(\sigma^{(e)}) &= \sigma_\tau^{(e)}, \\
\varepsilon^*(\sigma^{(s_2)}) &= \sigma_\tau^{(R_2)}, \\
\varepsilon^*(\sigma^{(s_1 s_2)}) = \varepsilon^*(\sigma^{(s_3 s_2)}) &= \sigma_\tau^{(R_1 R_2)}.
\end{aligned}$$

Regarding $\sigma_\tau^{(R_2 R_1 R_2)}$, we may compute

$$\sigma_\tau^{(R_2 R_1 R_2)} = \varepsilon^*(\sigma^{(s_3 s_1 s_2)}) - \beta_1 \sigma_\tau^{(R_1 R_2)} = \varepsilon^*(\sigma^{(s_3 s_1 s_2)} - \alpha_1 \sigma^{(s_1 s_2)})$$

using the last two lines of the table. Since $\{\sigma_\tau^{(u)} \mid u \in W_\tau^P\}$ generates $\mathcal{Z}(\mathcal{G}_\tau^P)$ (Proposition 3.3.15), it follows that ε^* over K is surjective.

Example 5.2.2 (Type $A_3/P_{1,3}$ to type C_2/P_1)**.** The moment graph associated with the Bruhat poset $(W^P, \leq)$ of type A_3 with $W_P = \langle s_2 \rangle$ is given in Table B.3 of Appendix B. For each $w \in W^P$, the image $\varepsilon^*(\sigma^{(w)})$ is computed in Table 5.2.

deg	$\sigma^{(w)} \in \mathcal{Z}(\mathcal{G}^P)$	$\varepsilon^*(\sigma^{(w)}) \in \mathcal{Z}(\mathcal{G}^P_\tau)$
0	$\sigma^{(e)}$	$\sigma_\tau^{(e)}$
1	$\sigma^{(s_1)}$	$\sigma_\tau^{(R_1)}$
	$\sigma^{(s_3)}$	$\sigma_\tau^{(R_1)}$
2	$\sigma^{(s_2s_1)}$	$\sigma_\tau^{(R_2R_1)}$
	$\sigma^{(s_2s_3)}$	$\sigma_\tau^{(R_2R_1)}$
	$\sigma^{(s_3s_1)}$	$\beta_1\sigma_\tau^{(R_1)} + \sigma_\tau^{(R_2R_1)}$
3	$\sigma^{(s_3s_2s_1)}$	$\sigma_\tau^{(R_1R_2R_1)}$
	$\sigma^{(s_1s_2s_3)}$	$\sigma_\tau^{(R_1R_2R_1)}$
	$\sigma^{(s_2s_3s_1)}$	$(\beta_1+\beta_2)\sigma_\tau^{(R_2R_1)}$
4	$\sigma^{(s_3s_2s_3s_1)}$	$(\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1)}$
	$\sigma^{(s_1s_2s_3s_1)}$	$(\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1)}$
5	$\sigma^{(s_3s_1s_2s_3s_1)}$	$\beta_1(\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1)}$

Table 5.2: ε^*-images of Schubert classes: type $A_3/P_{1,3}$ to type C_2/P_1

We have $W_\tau^P = \{e, R_1, R_2R_1, R_1R_2R_1\}$. Observe the following relations:

$$\begin{aligned}
\varepsilon^*(\sigma^{(e)}) &= \sigma_\tau^{(e)},\\
\varepsilon^*(\sigma^{(s_1)}) = \varepsilon^*(\sigma^{(s_3)}) &= \sigma_\tau^{(R_1)},\\
\varepsilon^*(\sigma^{(s_2s_1)}) = \varepsilon^*(\sigma^{(s_2s_3)}) &= \sigma_\tau^{(R_2R_1)},\\
\varepsilon^*(\sigma^{(s_1s_2s_3)}) = \varepsilon^*(\sigma^{(s_3s_2s_1)}) &= \sigma_\tau^{(R_1R_2R_1)}.
\end{aligned}$$

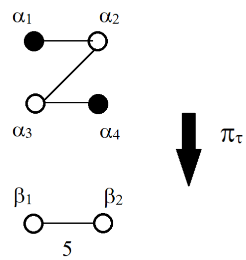
Thus in this case, ε^* has the lifting property. In particular, since $\{\sigma_\tau^{(u)} \mid u \in W_\tau^P\}$ generates $\mathcal{Z}(\mathcal{G}_\tau^P)$, ε^* over K is surjective.

5.2.2 Twisted cases

Let $\Phi \rightsquigarrow \Phi_\tau$ be the folding of the root system of type A_4 to the root system of type H_2 as described in Example 4.3.8. In particular, $K = \mathbb{Z}[\frac{1}{5}]$ and $L = K[\tau]$ where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The partition of the original simple roots is

$$\Delta_{rat} = \{\alpha_1, \alpha_4\}, \quad \mathcal{T}(\Delta_{rat}) = \{\alpha_3, \alpha_2\},$$

and the simple roots of the folded root system Φ_τ are $\Delta_\tau = \{\overline{\alpha_1}, \overline{\alpha_4}\}$. We set $\beta_1 := \overline{\alpha_1}$ and $\beta_2 := \overline{\alpha_4}$. The simple reflection of W corresponding to α_i is denoted by s_i for $i = 1, 2, 3, 4$ and the simple reflection of W_τ corresponding to β_j is denoted by R_j for $j = 1, 2$. The Coxeter diagram of this folding is depicted below.



Example 5.2.3 (Type $A_4/P_{2,4}$ to type H_2/P_2)**.** The moment graphs associated with the Bruhat poset $(W^P, \leq)$ of type A_4 with $W_P = \langle s_1, s_3\rangle$ and the Bruhat poset $(W_\tau^P, \leq)$ of type H_2 with $(W_P)_\tau = \langle R_1\rangle$ are given in Table B.5 and Table B.7 of Appendix B respectively. Table 5.4 computes the image $\varepsilon^*(\sigma^{(w)})$ for each $w \in W^P$. Due to space constraints, the elements $w \in W^P$ are described using the one-line notation (see Example 2.2.7 for definition) of the symmetric group $\mathfrak{S}_5$ in Table 5.4. Table 5.3 translates the one-line notation of $\mathfrak{S}_5$ into a product of simple reflections.

We have $W_\tau^P = \{e, R_2, R_1R_2, R_2R_1R_2, R_1R_2R_1R_2\}$. Observe the following:

$$\begin{aligned}
&\varepsilon^*\big(\sigma^{(12345)}\big) = \sigma_\tau^{(e)},\\
&\varepsilon^*\big(\sigma^{(13245)}\big) = \varepsilon^*\big(\sigma^{(12354)}\big) = \sigma_\tau^{(R_2)},\\
&\varepsilon^*\big(\sigma^{(23145)}\big) = \varepsilon^*\big(\sigma^{(14235)}\big) = \varepsilon^*\big(\sigma^{(12453)}\big) = \tau\sigma_\tau^{(R_1R_2)},\\
&\varepsilon^*\big(\sigma^{(15234)}\big) = \varepsilon^*\big(\sigma^{(13452)}\big) = \tau^2\sigma_\tau^{(R_2R_1R_2)},\\
&\varepsilon^*\big(\sigma^{(23451)}\big) = \tau^2\sigma_\tau^{(R_1R_2R_1R_2)}.
\end{aligned}$$

Thus in this case, ε^* has the lifting property. In particular, since $\{\sigma_\tau^{(u)} \mid u \in W_\tau^P\}$ generates $\mathcal{Z}(\mathcal{G}_\tau^P)$ and τ is invertible in L, ε^* over $L = K[\tau]$ is surjective.

$l(w)$	w in one-line notation	w as a product of simple reflections
0	12345	e
1	13245	s_2
	12354	s_4
2	23145	s_1s_2
	14235	s_3s_2
	13254	s_4s_2
	12453	s_3s_4
3	24135	$s_3s_1s_2$
	23154	$s_1s_4s_2$
	15234	$s_4s_3s_2$
	14253	$s_3s_4s_2$
	13452	$s_2s_3s_4$
4	34125	$s_2s_3s_1s_2$
	25134	$s_4s_3s_1s_2$
	24153	$s_3s_1s_4s_2$
	15243	$s_4s_3s_4s_2$
	14352	$s_2s_3s_4s_2$
	23451	$s_1s_2s_3s_4$
5	35124	$s_4s_2s_3s_1s_2$
	25143	$s_4s_3s_1s_4s_2$
	34152	$s_2s_3s_1s_4s_2$
	15342	$s_4s_2s_3s_4s_2$
	24351	$s_1s_2s_3s_4s_2$
6	45123	$s_3s_4s_2s_3s_1s_2$
	35142	$s_4s_2s_3s_1s_4s_2$
	34251	$s_1s_2s_3s_1s_4s_2$
	25341	$s_1s_4s_2s_3s_4s_2$
7	45132	$s_2s_3s_4s_2s_3s_1s_2$
	35241	$s_1s_4s_2s_3s_1s_4s_2$
8	45231	$s_1s_2s_3s_4s_2s_3s_1s_2$

Table 5.3: Translation of the one-line notation of $w \in W^P$ into a product of simple reflections (type $A_4/P_{2,4}$)

Table 5.3 supplements Table 5.4 by translating the one-line notation of $w \in W^P$ into a product of simple reflections. See the second column of Table 5.4.

deg	$\sigma^{(w)} \in \mathcal{Z}(\mathcal{G}^P)$	$\varepsilon^*(\sigma^{(w)}) \in \mathcal{Z}(\mathcal{G}^P_\tau)$
0	$\sigma^{(12345)}$	$\sigma_\tau^{(e)}$
1	$\sigma^{(13245)}$	$\sigma_\tau^{(R_2)}$
	$\sigma^{(12354)}$	$\sigma_\tau^{(R_2)}$
2	$\sigma^{(23145)}$	$\tau\sigma_\tau^{(R_1R_2)}$
	$\sigma^{(14235)}$	$\tau^2\sigma_\tau^{(R_1R_2)}$
	$\sigma^{(13254)}$	$\tau\beta_2\sigma_\tau^{(R_2)} + \tau^2\sigma_\tau^{(R_1R_2)}$
	$\sigma^{(12453)}$	$\tau\sigma_\tau^{(R_1R_2)}$
3	$\sigma^{(24135)}$	$\tau^2\beta_1\sigma_\tau^{(R_1R_2)} + \tau^3\sigma_\tau^{(R_2R_1R_2)}$
	$\sigma^{(23154)}$	$\tau(\tau\beta_1+\beta_2)\sigma_\tau^{(R_1R_2)} + \tau^2\sigma_\tau^{(R_2R_1R_2)}$
	$\sigma^{(15234)}$	$\tau^2\sigma_\tau^{(R_2R_1R_2)}$
	$\sigma^{(14253)}$	$\tau^2(\tau\beta_1+\beta_2)\sigma_\tau^{(R_1R_2)} + \tau\sigma_\tau^{(R_2R_1R_2)}$
	$\sigma^{(13452)}$	$\tau^2\sigma_\tau^{(R_2R_1R_2)}$
4	$\sigma^{(34125)}$	$\tau^3(\beta_1+\tau\beta_2)\sigma_\tau^{(R_2R_1R_2)} + \tau^2\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(25134)}$	$\tau^2(\beta_1+\tau\beta_2)\sigma_\tau^{(R_2R_1R_2)} + \tau^3\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(24153)}$	$\tau^2\beta_1(\tau\beta_1+\beta_2)\sigma_\tau^{(R_1R_2)} + \tau^3\big(\beta_1+(\beta_1+\tau\beta_2)\big)\sigma_\tau^{(R_2R_1R_2)}$
	$\sigma^{(15243)}$	$\tau^2\cdot\tau(\beta_1+\beta_2)\sigma_\tau^{(R_2R_1R_2)} + \tau^2\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(14352)}$	$\tau^3\cdot\tau(\beta_1+\beta_2)\sigma_\tau^{(R_2R_1R_2)} + \tau\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(23451)}$	$\tau^2\sigma_\tau^{(R_1R_2R_1R_2)}$
5	$\sigma^{(35124)}$	$\tau^3\beta_2(\beta_1+\tau\beta_2)\sigma_\tau^{(R_2R_1R_2)} + \tau^4\big(\beta_2+(\tau\beta_1+\beta_2)\big)\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(25143)}$	$\tau^2(\beta_1+\tau\beta_2)\tau(\beta_1+\beta_2)\sigma_\tau^{(R_2R_1R_2)} + \tau^4\big(\beta_1+(\beta_1+\tau\beta_2)\big)\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(34152)}$	$\tau^3(\beta_1+\tau\beta_2)\tau(\beta_1+\beta_2)\sigma_\tau^{(R_2R_1R_2)}$
	$\sigma^{(15342)}$	$\tau^3\beta_2\cdot\tau(\beta_1+\beta_2)\sigma_\tau^{(R_2R_1R_2)} + \tau^3\big(\beta_2+(\tau\beta_1+\beta_2)\big)\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(24351)}$	$\tau^2\big(\tau(\beta_1+\beta_2)+(\tau\beta_1+\beta_2)\big)\sigma_\tau^{(R_1R_2R_1R_2)}$
6	$\sigma^{(45123)}$	$\tau^4(\tau\beta_1+\beta_2)\tau(\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(35142)}$	$\tau^3(\beta_1+\tau\beta_2)(\tau\beta_1+\beta_2)\tau(\beta_1+\beta_2)\sigma_\tau^{(R_2R_1R_2)}$ $+\tau^4(\beta_1+\tau\beta_2)\tau(\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(34251)}$	$\tau^3(\beta_1+\tau\beta_2)\tau(\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(25341)}$	$\tau^3(\beta_1+\tau\beta_2)(\tau\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1R_2)}$
7	$\sigma^{(45132)}$	$\tau^4(\beta_1+\tau\beta_2)(\tau\beta_1+\beta_2)\tau(\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1R_2)}$
	$\sigma^{(35241)}$	$\tau^3(\beta_1+\tau\beta_2)(\tau\beta_1+\beta_2)\tau(\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1R_2)}$
8	$\sigma^{(45231)}$	$\tau^4\beta_1(\beta_1+\tau\beta_2)(\tau\beta_1+\beta_2)\tau(\beta_1+\beta_2)\sigma_\tau^{(R_1R_2R_1R_2)}$

Table 5.4: ε^*-images of Schubert classes: type $A_4/P_{2,4}$ to type H_2/P_2 (Example 5.2.3)

5.3 Combinatorial tools and basic properties of ε^*

Let the setting and notation be as stated in Section 5.1. This section is preparatory to Sections 5.4–5.6. In Section 5.3.1, we develop the properties of a combinatorial counterpart (see (5.3.1)) of the folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}^P_\tau)$ by focusing on the embedding $\varepsilon : W_\tau \hookrightarrow W$ of Coxeter groups (Proposition 4.2.8). Section 5.3.2 establishes the first basic properties of the folding map ε^*.

5.3.1 Combinatorial folding map φ on the Coxeter generating sets

As part of the setting, we have a folded K-representation $(\Phi, U)_K$ of a root datum $\Phi \hookrightarrow \Lambda^\vee$ with a chosen simple system Δ (Definition 4.1.10), and the L-realized τ-folded root system $\Phi_\tau \subset U_L^{(\tau)}$ (Definition 4.2.6). Let (W, S) denote the Coxeter system associated with the K-reflection group $W(\Phi)$ where $S = \{s_\alpha \mid \alpha \in \Delta\}$, and let (W_τ, S_τ) denote the Coxeter system associated with the L-reflection group W_τ where $S_\tau = \{R_\beta \mid \beta \in \Delta_\tau\}$ (see the paragraph after Lemma 4.2.7).

In the investigation of Question 3 (Sections 5.4–5.5), it will be essential for us to perceive the folding $\Phi \rightsquigarrow \Phi_\tau$ on the level of the corresponding Coxeter generating sets S and S_τ. We define a set-theoretic map $\varphi : S \to S_\tau$ by

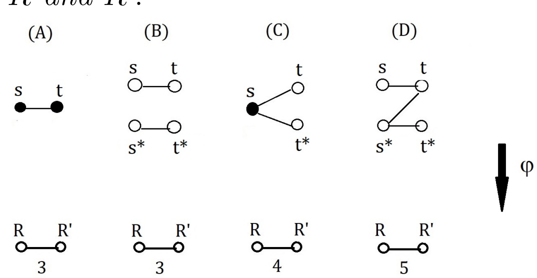
$$\varphi : S \to S_\tau : s_\alpha, s_{\mathcal{T}\alpha} \mapsto R_{\overline{\alpha}}, \tag{5.3.1}$$

for each $\alpha \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$ (see Definition 4.1.10 for notation). It is noted that φ is surjective due to the surjectivity of $\pi_\tau|_\Delta : \Delta \to \Delta_\tau$.

Definition 5.3.1 (opposite simple reflection)**.** Let $R \in S_\tau$ be a simple reflection such that the fibre $\varphi^{-1}(R)$ has exactly two elements. If s is one of them, we denote the other one by s^* and call it the *opposite simple reflection* of s.

Using φ, the local description of twisted quadratic foldings obtained in Proposition 4.5.1 can be rewritten as follows.

Proposition 5.3.2. *Let $R, R' \in W_\tau$ be simple reflections such that $m(R, R') \geq 3$. Then the possible configuration of the full subgraph $\varphi^{-1}(\{R, R'\})$ is one of the following, up to symmetry of R and R'.*



Proof: This is a direct translation of Proposition 4.5.1. Notice that (C) and (D) of Proposition 4.5.1 are merged to (C) of this proposition, and (E) and (F) of Proposition 4.5.1 to (D) of this proposition. ∎

For the φ-preimage of a pair (R, R') of simple reflections in W_τ such that $m(R, R') = 2$ (i.e., R and R' are disconnected in the Coxeter diagram), we have the following.

Proposition 5.3.3. *Let $R, R' \in W_\tau$ be simple reflections such that $m(R, R') = 2$. Let $s \in \varphi^{-1}(R)$ and $t \in \varphi^{-1}(R')$. Then $m(s, t) = 2$.*

Proof: Let us write $R = R_{\overline{\alpha}}$ and $R' = R_{\overline{\beta}}$ for some $\alpha, \beta \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$. The condition $m(R, R') = 2$ is equivalent to $(\overline{\alpha}, \overline{\beta})_\tau = 0$. By formula (4.1.9), this implies $(\alpha_L \cdot \beta_L) = 0$. Hence we have

$$\begin{aligned}
(\alpha, \beta)_\tau &= pr_\tau(\alpha_L \cdot \beta_L) = 0,\\
(\mathcal{T}\alpha, \beta)_\tau &= pr_\tau(\tau\alpha_L \cdot \beta_L) = pr_\tau(\tau(\alpha_L \cdot \beta_L)) = 0,\\
(\alpha, \mathcal{T}\beta)_\tau &= (\mathcal{T}\alpha, \beta)_\tau = 0 \text{ (the adjoint property of } \mathcal{T}),\\
(\mathcal{T}\alpha, \mathcal{T}\beta)_\tau &= pr_\tau(\tau\alpha_L \cdot \tau\beta_L) = pr_\tau(\tau^2(\alpha_L \cdot \beta_L)) = 0.
\end{aligned}$$

Thus for all possible choices of $s \in \varphi^{-1}(R_{\overline{\alpha}})$ and $t \in \varphi^{-1}(R_{\overline{\beta}})$, we have $m(s, t) = 2$. ∎

Let us recall that for each simple reflection R of W_τ, $\varepsilon(R) = s$ (if $\varphi^{-1}(R) = \{s\}$) or ss^* (if $\varphi^{-1}(R) = \{s, s^*\}$). The map φ has the following property.

Proposition 5.3.4. *Let $s_{i_1} \cdots s_{i_\ell}$ be a reduced expression in W, and let $R_{j_1} \cdots R_{j_k}$ be a reduced expression in W_τ. Suppose $\varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}})$ for all $1 \leq p \leq \ell - 1$. Then $s_{i_1} \cdots s_{i_\ell}$ is a subexpression of $\varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_k})$ if and only if $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ is a subexpression of $R_{j_1} \cdots R_{j_k}$.*

Proof: Let $R_j \in W_\tau$ be a simple reflection. Then a simple reflection $s_i \in W$ is a subexpression of $\varepsilon(R_j)$ if and only if $\varphi(s_i) = R_j$. This follows from the surjectivity of φ and the property $\varepsilon(\varphi(s_i)) = s_i$ (if $\alpha_i \in \Delta^{\mathcal{T}}$) or $s_i s_i^*$ (if $\alpha_i \in \Delta_{rat} \sqcup \mathcal{T}(\Delta_{rat})$).

Let $s_{i_1} \cdots s_{i_\ell}$ be a subexpression of $\varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_k})$. Then for each $1 \leq p \leq \ell$, s_{i_p} is a subexpression of $\varepsilon(R_{j_{q(p)}})$ for some $1 \leq q(p) \leq k$ such that $q(p) \leq q(p+1)$ and we have $\varphi(s_{i_p}) = R_{j_{q(p)}}$. Now the condition $\varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}})$ for all $1 \leq p \leq \ell - 1$ guarantees that $q(p) < q(p+1)$ for each p. Therefore, $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ is a subexpression of $R_{j_1} \cdots R_{j_k}$.

Conversely, let $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ be a subexpression of $R_{j_1} \cdots R_{j_k}$. Then for each $1 \leq p \leq \ell$, $\varphi(s_{i_p}) = R_{j_{q(p)}}$ for some $1 \leq q(1) < \cdots < q(\ell) \leq k$. Each s_{i_p} is a subexpression of $\varepsilon(R_{j_{q(p)}})$. Therefore, $s_{i_1} \cdots s_{i_\ell}$ is a subexpression of $\varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_k})$. ∎

In order to yield further properties of the map $\varphi : S \to S_\tau$ and the group embedding $\varepsilon : W_\tau \hookrightarrow W$, we introduce the notion of a *folding branch*.

Definition 5.3.5. Let Γ and Γ_τ denote the Coxeter graph of W and W_τ respectively. A *folding branch* of Γ with respect to $\varphi : S \to S_\tau$ is a full subgraph Γ' of Γ obtained by choosing one vertex from $\varphi^{-1}(R)$ for each vertex $R \in \Gamma_\tau$ such that $m(s, s') = 3$ for all vertices $s, s' \in \Gamma'$ with $m(\varphi(s), \varphi(s')) \geq 3$. A *collapsing part* of Γ with respect to φ is the full subgraph of $\varphi^{-1}(\{R, R'\})$ for a pair $R, R' \in S_\tau$ with $m(R, R') \geq 4$.

This definition remains valid when Γ is not connected. However, in all the examples we treat in this thesis, Γ is always connected.

Lemma 5.3.6. *A folding branch exists.*

Proof: Let Γ and Γ_τ be as in Definition 5.3.5. By Proposition 5.3.2, for every pair $R, R' \in \Gamma_\tau$ with $m(R, R') \geq 3$, it is always possible to find a pair $s, s' \in \Gamma$ such that $\varphi(s) = R$, $\varphi(s') = R'$ and $m(s, s') = 3$. This proves the claim. ∎

Example 5.3.7 (Folding branches)**.** We describe folding branches for the quadratic foldings listed in Section 4.3. For the Coxeter diagram of each folding, see Appendix A.

(1) **Type A_{2n-1} to C_n.** We have $S = \{s_1, \cdots, s_{2n-1}\}$ and $S_\tau = \{R_1, \cdots R_n\}$. The map $\varphi : S \to S_\tau$ sends s_i, s_{2n-i} to R_i for $1 \leq i \leq n-1$ and s_n to R_n. Then the Coxeter graph of the original group admits two folding branches: $\Gamma_1 = \{s_1, s_2, \cdots, s_n\}$ and $\Gamma_2 = \{s_{2n-1}, s_{2n-2}, \cdots, s_n\}$. It has a collapsing part $\Gamma' = \{s_{n-1}, s_n, s_{n+1}\}$.

(2) **Type D_{n+1} to B_n.** We have $S = \{s_1, \cdots, s_{n+1}\}$ and $S_\tau = \{R_1, \cdots, R_n\}$. The map $\varphi : S \to S_\tau$ sends s_i to R_i for $1 \leq i \leq n-1$ and s_n, s_{n+1} to R_n. Then the Coxeter graph of the original group admits two folding branches: $\Gamma_1 = \{s_1, \cdots, s_{n-1}, s_n\}$ and $\Gamma_2 = \{s_1, \cdots, s_{n-1}, s_{n+1}\}$. It has a collapsing part $\Gamma' = \{s_{n-1}, s_n, s_{n+1}\}$.

(3) **Type E_6 to F_4.** We have $S = \{s_1, \cdots, s_6\}$ and $S_\tau = \{R_1, \cdots, R_4\}$. The map $\varphi : S \to S_\tau$ sends s_2 to R_1, s_4 to R_2, s_3, s_5 to R_3, and s_1, s_6 to R_4. Then the Coxeter graph of the original group admits two folding branches: $\Gamma_1 = \{s_2, s_4, s_3, s_1\}$ and $\Gamma_2 = \{s_2, s_4, s_5, s_6\}$. It has a collapsing part $\Gamma' = \{s_3, s_4, s_5\}$.

(4) **Type E_8 to H_4.** We have $S = \{s_1, \cdots, s_8\}$ and $S_\tau = \{R_1, \cdots, R_4\}$. The map $\varphi : S \to S_\tau$ sends s_1, s_7 to R_1, s_2, s_6 to R_2, s_3, s_5 to R_3, and s_4, s_8 to R_4. Then the Coxeter graph of the original group admits three folding branches: $\Gamma_1 = \{s_1, s_2, s_3, s_4\}$, $\Gamma_2 = \{s_7, s_6, s_5, s_4\}$, and $\Gamma_3 = \{s_7, s_6, s_5, s_8\}$. It has a collapsing part $\Gamma' = \{s_3, s_4, s_5, s_8\}$.

Using φ and the notion of a folding branch, we can prove the following essential property of the embedding $\varepsilon : W_\tau \hookrightarrow W$.

Proposition 5.3.8. *If $R_{j_1} \cdots R_{j_\ell}$ is a reduced expression in W_τ, then $\varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_\ell})$ is a reduced expression in W.*

Proof: We break down the map ε into two steps. First, for each $1 \le p \le \ell$, let us choose $s_{i_p} \in \varphi^{-1}(R_{j_p})$ so that $s_{i_1}, \cdots, s_{i_\ell}$ all belong to a single folding branch. This is possible, since each folding branch Γ_k of the Coxeter diagram Γ of (W, S) is mapped under φ onto the Coxeter diagram Γ_τ of (W_τ, S_τ). We denote the simple root corresponding to s_{i_p} by α_{i_p}.

Let us observe that for those $1 \le p \le \ell - 1$ such that R_{j_p} and $R_{j_{p+1}}$ do not belong to the φ-image of a collapsing part of Γ, we have $m(s_{i_p}, s_{i_{p+1}}) = m(R_{j_p}, R_{j_{p+1}})$. Hence the only situation where the word $s_{i_1} \cdots s_{i_\ell}$ is not reduced is when $R_{j_1} \cdots R_{j_\ell}$ contains a consecutive subword of the form $[RR']_{m(R,R')}$ (see Section 2.2.8 for notation) where $m(R, R') \ge 4$, that is, equal to either 4 or 5 by Proposition 5.3.2.

Next, for each $1 \le p \le \ell$, we conduct the following operation.

$$\begin{cases} \text{if } |\varphi^{-1}(R_{i_p})| = 2, & \text{then insert } s^*_{i_p} \text{ on the right of } s_{i_p}, \\ \text{if } |\varphi^{-1}(R_{i_p})| = 1, & \text{then do nothing.} \end{cases}$$

When $|\varphi^{-1}(R_{i_p})| = 1$, we adopt the convention $s^*_{i_p} = s_{i_p}$.

We observe that $s^*_{i_1}, \cdots, s^*_{i_\ell}$ also belong to a single folding branch different from the first one. In particular, the observation of the second paragraph applies to the word $s^*_{i_1} \cdots s^*_{i_\ell}$ as well. Also, notice that if $s^*_{i_p} \ne s_{i_p}$ and $s^*_{i_p}$ does not belong to a collapsing part of Γ, then it commutes with all $s_{i_1}, \cdots, s_{i_\ell}$. By Proposition 5.3.2, each $s \in \Gamma$ not belonging to a collapsing part is disconnected from any folding branch not containing s.

Based on these observations, in order to determine if the W-word $\varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_\ell})$ is reduced or not, it suffices to look at the subwords consisting of elements of a collapsing part of Γ. More precisely, for $R, R' \in S_\tau$ with $m(R, R') \in \{4, 5\}$, we will study the word of the form $[\varepsilon(R)\varepsilon(R')]_m$ where $4 \le m \le m(R, R')$.

First, let us look at the case where $m(R, R') = 4$. The full subgraph of the preimage $\varphi^{-1}(\{R, R'\})$ is depicted in (C) of Proposition 5.3.2. We use the naming of the vertices given in the diagram (C). Then

$$\begin{aligned} \varepsilon(R)\varepsilon(R')\varepsilon(R)\varepsilon(R') = ss^*tss^*t &\sim s^*sts\boxed{s^*}t \\ &\sim ss^*ts^*\boxed{s}t \end{aligned}$$

where $\sim$ denotes the braid relation on $F(S)$ defined in Section 2.2.8. Notice that in the top right of the first line, the second s^* (boxed) is playing the role of a separator between sts and t, preventing the occurrence of $stst$. In the second line, the second s

(boxed) is separating s^*ts^* and t, preventing the occurrence of s^*ts^*t. One also checks (by direct manipulation of the word) that $tsts$ and ts^*ts^* cannot occur. Thus the word ss^*tss^*t is reduced.

Next, we look at the case where $m(R, R') = 5$. The full subgraph of the preimage $\varphi^{-1}(\{R, R'\})$ is depicted in (D) of Proposition 5.3.2. We use the naming of the vertices given in the diagram (D). Then

$$\varepsilon(R)\varepsilon(R')\varepsilon(R)\varepsilon(R') = ss^*tt^*ss^*tt^* \sim s^*t^*sts\boxed{s^*}tt^*$$

Observe that in the word on the right, the second s^* (boxed) is separating sts and t, preventing the occurrence of $stst$. By looking at all other braid equivalent words, one checks (by a similar separation phenomenon) that $tsts$, s^*ts^*t, ts^*ts^*, $s^*t^*s^*t^*$, or $t^*s^*t^*s^*$ cannot occur either. Thus the word $ss^*tt^*ss^*tt^*$ is reduced. In exactly the same way, we verify that $\varepsilon(R')\varepsilon(R)\varepsilon(R')\varepsilon(R)$ is reduced. Finally,

$$\varepsilon(R)\varepsilon(R')\varepsilon(R)\varepsilon(R')\varepsilon(R) = ss^*tt^*ss^*tt^*ss^* \sim s^*t^*sts\boxed{s^*}tst^*s^*$$

Observe that in the word on the right, once again s^* (boxed) is separating sts and ts preventing the occurrence of either $stst$ or $ststs$. By looking at all other braid equivalent words, one checks (by a similar separation phenomenon) that $tsts$, s^*ts^*t, ts^*ts^*, $s^*t^*s^*t^*$, $t^*s^*t^*s^*$, $tstst$, $s^*ts^*ts^*$, ts^*ts^*t, $s^*t^*s^*t^*s^*$, or $t^*s^*t^*s^*t^*$ cannot occur either. Thus the word $ss^*tt^*ss^*tt^*ss^*$ is reduced. ∎

As a corollary, we obtain the following.

Corollary 5.3.9. *If $u' \leq u$ in W_τ, then $\varepsilon(u') \leq \varepsilon(u)$ in W.*

Proof: Let $u = R_{j_1} \cdots R_{j_\ell}$ be a W_τ-reduced expression. Since $u' \leq u$, there is a sequence $1 \leq a_1 \leq \cdots \leq a_r \leq \ell$ so that $u' = R_{j_{a_1}} \cdots R_{j_{a_r}}$ is a W_τ-reduced expression (Proposition 2.2.19). By Proposition 5.3.8, $\varepsilon(u) = \varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_\ell})$ and $\varepsilon(u') = \varepsilon(R_{j_{a_1}}) \cdots \varepsilon(R_{j_{a_r}})$ are both W-reduced expressions. Since the latter is a subexpression of the former, by Proposition 2.2.19, $\varepsilon(u') \leq \varepsilon(u)$ follows. ∎

Let us recall the set of reduced expressions $\mathcal{R}(w)$ for an element $w \in W$ and $\mathcal{R}(W) = \cup_{w \in W} \mathcal{R}(w)$ (see notation at the end of Section 2.2.8). Let $F(S_\tau)$ be the free monoid generated on the Coxeter generating set S_τ (cf. after Definition 2.2.1). The map $\varphi : S \to S_\tau$ induces a new map $\widehat{\varphi} : \mathcal{R}(W) \to F(S_\tau)$ in the following way: let $\mathbf{i} := s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(W)$. Then apply φ to each s_{i_p}, and whenever $\varphi(s_{i_p}) = \varphi(s_{i_{p+1}})$ happens, replace the word $\varphi(s_{i_p})\varphi(s_{i_{p+1}})$ by $\varphi(s_{i_p})$. We introduce one notation to facilitate the following arguments.

Notation. Given $\varphi : S \to S_\tau$ associated with a twisted quadratic folding $\Phi \rightsquigarrow \Phi_\tau$ and an element $w \in W$, the set $\mathcal{R}_\varphi(w)$ is defined to be the subset of $\mathcal{R}(w)$ consisting of elements $\mathbf{i}$ such that the number of *adjacency of opposite simple reflections* appearing in $\mathbf{i}$ is maximal among all the elements of $\mathcal{R}(w)$. We define

$$\mathcal{R}_\varphi(W) := \bigcup_{w\in W} \mathcal{R}_\varphi(w).$$

For example, in the folding from type A_3 to C_2 (see Section 5.2.1), consider the group element $w = s_2s_1s_2s_3$. Then we have

$$\mathcal{R}(w) = \{s_2s_1s_2s_3, s_1s_2[s_1s_3], s_1s_2[s_3s_1]\},$$
$$\mathcal{R}_\varphi(w) = \{s_1s_2[s_1s_3], s_1s_2[s_3s_1]\},$$

where the maximal number of adjacency of opposite simple reflections is one (the adjacent opposite simple reflections are square-bracketed).

The following statement about $\widehat{\varphi}$ will be the foundation for the proof of Proposition 5.4.7 in Section 5.4.2.

Proposition 5.3.10. *Let $w \in W$ and let $\mathbf{i} = s_{i_1}\cdots s_{i_\ell} \in \mathcal{R}_\varphi(w)$. Then $\widehat{\varphi}(\mathbf{i}) \in \mathcal{R}(W_\tau)$.*

Proof: First, we prove the following claim:

Claim 2. Let w and $\mathbf{i}$ be as in the statement. Let $\mathbf{j} := \widehat{\varphi}(\mathbf{i}) = R_{j_1}\cdots R_{j_k}$ where $1 \le k \le \ell$ and each R_{j_q} denotes a simple reflection in W_τ. Then for $\mathbf{j}' := R_{j_1}\cdots R_{j_{k-1}}$, there exists $w' \in W$ and $\mathbf{i}' \in \mathcal{R}_\varphi(w')$ and $\widehat{\varphi}(\mathbf{i}') = \mathbf{j}'$.

This claim is proved as follows. There are $1 = a_1 < \cdots < a_k \le \ell$ such that $\varphi(s_{i_{a_p}}) = R_{j_p}$ for each $1 \le p \le k$. Take $w' = s_{i_1}s_{i_2}\cdots s_{i_{a_{k-1}}}$. The word $\mathbf{i}' := s_{i_1}s_{i_2}\cdots s_{i_{a_{k-1}}}$ belongs to $\mathcal{R}(w')$ since it is a consecutive subword of the reduced word $\mathbf{i}$. Moreover, since $\mathbf{i}'$ is obtained from $\mathbf{i} \in \mathcal{R}_\varphi(W)$ by removing either s_{a_k} or the word $s_{a_k}s^*_{a_k}$ (or $s^*_{a_k}s_{a_k}$) from the right end, $\mathbf{i}' \in \mathcal{R}_\varphi(W)$ must be the case (otherwise, $\mathbf{i} \notin \mathcal{R}_\varphi(W)$). This proves the claim.

Since by construction we have $R_{j_q} \ne R_{j_{q+1}}$ for all $1 \le q \le k-1$, in order for the sequence $R_{j_1}\cdots R_{j_k}$ not to be reduced, the only possibility is that it contains a pattern $[RR']_m$ (see Section 2.2.8) where $R, R' \in S_\tau$ and $m > m(R, R')$. We now argue that this situation cannot occur.

Let us consider all possible patterns of the preimage of each pair of simple reflections in W_τ such that $m(R, R') \ge 3$. There are four distinct patterns as in the figure of Proposition 5.3.2. In pattern (A), there is nothing to prove. In pattern (B), let us list all reduced words $\mathbf{i} \in \mathcal{R}_\varphi(W)$ such that $\widehat{\varphi}(\mathbf{i}) = RR'R$. Up to symmetry between

s and s^* (resp. t and t^*), they are:

$$\begin{aligned}
&[ss^*][tt^*][ss^*]\\
&[ss^*][tt^*]s\\
&s[tt^*][ss^*]\\
&s[tt^*]s\\
&s[tt^*]s^*\\
&sts
\end{aligned}$$

where a pair of opposite simple reflections are enclosed by brackets. By direct computation, we observe that adding t or t^* to the right of each of the above reduced word will result in either a non-reduced word or a reduced word not in $\mathcal{R}_\varphi(W)$. We give two examples below to describe these two points.

$$\begin{aligned}
[ss^*][tt^*][ss^*]t &\sim \underline{stst}s^*t^*s^*\\
stst^* &\sim s[tt^*]s
\end{aligned}$$

The first line shows a case where the word is not reduced (see the underlined part), and the second line shows a case where the word is reduced but not in $\mathcal{R}_\varphi(W)$.

In pattern (C), let us list all reduced words $\mathbf{i} \in \mathcal{R}_\varphi(W)$ such that $\widehat{\varphi}(\mathbf{i}) = RR'RR'$:

$$\begin{aligned}
&[ss^*]t[ss^*]t\\
&st[ss^*]t\\
&s^*t[ss^*]t
\end{aligned}$$

By direct computation, we observe that adding s or s^* to the right end of each of the above word will result in either a non-reduced word or a reduced word not in $\mathcal{R}_\varphi(W)$:

$$\begin{aligned}
[ss^*]t[ss^*]ts &\sim s\underline{s^*ts^*t}st\\
[ss^*]t[ss^*]ts^* &\sim s^*\underline{stst}s^*t\\
st[ss^*]ts &\sim [ss^*]t[ss^*]t\\
st[ss^*]ts^* &\sim \underline{stst}s^*t\\
s^*t[ss^*]ts &\sim \underline{s^*ts^*t}st\\
s^*t[ss^*]ts^* &\sim [ss^*]t[ss^*]t
\end{aligned}$$

The underlined parts indicate that the word is not reduced. The third and sixth lines are examples where the word is reduced but not in $\mathcal{R}_\varphi(W)$.

In pattern (D), let us list all reduced words $\mathbf{i} \in \mathcal{R}_\varphi(W)$ such that $\widehat{\varphi}(\mathbf{i}) = RR'RR'R$:

$[ss^*][tt^*][ss^*][tt^*][ss^*]$	$[ss^*][tt^*]s^*[tt^*]s$	$s[tt^*][ss^*]ts$
$s[tt^*][ss^*][tt^*][ss^*]$	$s[tt^*][ss^*][tt^*]s$	$st[ss^*][tt^*]s^*$
$s^*[tt^*][ss^*][tt^*][ss^*]$	$s[tt^*][ss^*][tt^*]s^*$	$s^*[tt^*][ss^*]ts$
$[ss^*][tt^*][ss^*][tt^*]s$	$s^*[tt^*][ss^*][tt^*]s$	$s^*[tt^*]s^*[tt^*]s$
$[ss^*][tt^*][ss^*][tt^*]s^*$	$s^*[tt^*][ss^*][tt^*]s^*$	$s[tt^*]s^*[tt^*]s^*$
$st[ss^*][tt^*][ss^*]$	$sts^*[tt^*][ss^*]$	$[ss^*]t^*s^*t[ss^*]$
$[ss^*][tt^*][ss^*]ts$	$s^*t^*[ss^*]t[ss^*]$	$[ss^*]ts^*t^*[ss^*]$
$s[tt^*]s^*[tt^*][ss^*]$	$st[ss^*][tt^*]s$	

By direct computation, we observe that adding t or t^* to the right end of each of the above word will result in either a non-reduced word, or a reduced word not in $\mathcal{R}_\varphi(W)$. Given below are two examples to illustrate these points. The first is an example of a non-reduced word (see the underlined part). The second is an example of a reduced word not in $\mathcal{R}_\varphi(W)$ (compare with the last line).

$$\begin{aligned}
[ss^*][tt^*][ss^*][tt^*][ss^*]t &\sim ss^*tt^*s^*stst^*s^* \\
&\sim ss^*t^*ts^*tst^*ts^*t \\
&\sim ss^*t^*s^*ts^*t^*s^*sts^* \\
&\sim st^*s^*\underline{t^*tt^*}s^*t^*sts^* \\
s[tt^*][ss^*][tt^*][ss^*]t &\sim t^*tsts^*tt^*ss^*t \\
&\sim tt^*ss^*ts^*t^*s^*st \\
&\sim [tt^*][ss^*][tt^*][ss^*][tt^*]
\end{aligned}$$

Finally, by Claim 2, we may discard the possibility of the appearance of the word $[RR']_m$ in $R_{j_1}\cdots R_{j_k}$ where $R, R' \in S_\tau$ for *all* $m > m(R, R')$. ∎

5.3.2 Basic properties of ε^*

We keep the setting as stated in the beginning of Section 5.3. We state below the first basic properties of ε^* in relation to the Schubert classes of $\mathcal{Z}(\mathcal{G}^P)$.

Proposition 5.3.11. *The following properties hold for ε^*.*

(a) $\varepsilon^*(\sigma^{(e)}) = \sigma_\tau^{(e)}$.

(b) $\varepsilon^*(\sigma^{(s_\alpha)}) = \varepsilon^*(\sigma^{(s_{\mathcal{T}\alpha})}) = \sigma_\tau^{(R_{\overline{\alpha}})}$ *for each* $\alpha \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$.

(c) $\mathrm{supp}(\varepsilon^*(\sigma^{(w)}) \subset \{u \in W_\tau^P \mid l_\tau(u) \leq l(w), \varepsilon(u) \geq w\}$ *(see Definition 3.3.16 for the notation).*

In terms of moment graphs, (c) tells us that an element $u \in \operatorname{supp}\big(\varepsilon^*(\sigma^{(w)})\big) \subset W_\tau^P$ embedded in W^P is located above w in the Bruhat order, but its length $l(\varepsilon(u))$ is capped by the other condition $l_\tau(u) \leq l(w)$.
Proof: (a) For each $x \in W_\tau^P$, we have $\varepsilon^*(\sigma^{(e)})(x) = \pi_\tau\big(\sigma^{(e)}(\varepsilon(x))\big) = \pi_\tau(1) = 1$.

(b) We have $\sigma^{(s_\alpha)}(e) = 0$ and $\sigma^{(s_\alpha)}(s_\alpha) = \alpha$. Further, if $\alpha \in \Delta_{rat}$, we have $\sigma^{(s_\alpha)}(s_{\mathcal{T}\alpha}) = 0$ and $\sigma^{(s_\alpha)}(s_\alpha s_{\mathcal{T}\alpha}) = \alpha$. The second equality follows from the fact that there are precisely two edges coming into the vertex $s_\alpha s_{\mathcal{T}\alpha}$, namely, $s_\alpha \to s_\alpha s_{\mathcal{T}\alpha}$ and $s_{\mathcal{T}\alpha} \to s_\alpha s_{\mathcal{T}\alpha}$, labelled by $\mathcal{T}\alpha, \alpha$ respectively. Since $\sigma^{(s_\alpha)}(s_\alpha) = \alpha$ and $\sigma^{(s_\alpha)}(s_{\mathcal{T}\alpha}) = 0$, by the divisibility conditions along the two edges, $\sigma^{(s_\alpha)}(s_\alpha s_{\mathcal{T}\alpha}) = \alpha$ follows. Using this information, we compute

$$\begin{aligned}
\varepsilon^*(\sigma^{(s_\alpha)})(e) &= \pi_\tau\big(\sigma^{(s_\alpha)}(\varepsilon(e))\big) = \pi_\tau(0) = 0,\\
\varepsilon^*(\sigma^{(s_\alpha)})(R_{\overline{\alpha}}) &= \pi_\tau\big(\sigma^{(s_\alpha)}(\varepsilon(R_{\overline{\alpha}}))\big)\\
&= \begin{cases} \pi_\tau\big(\sigma^{(s_\alpha)}(s_\alpha)\big) = \pi_\tau(\alpha) = \overline{\alpha} \ \text{ (if } \alpha \in \Delta^{\mathcal{T}}),\\ \pi_\tau\big(\sigma^{(s_\alpha)}(s_\alpha s_{\mathcal{T}\alpha})\big) = \pi_\tau(\alpha) = \overline{\alpha} \ \text{ (if } \alpha \in \Delta_{rat}). \end{cases}
\end{aligned}$$

Since $\sigma^{(s_\alpha)}$ is homogeneous of degree 1, so is $\varepsilon^*(\sigma^{(s_\alpha)})$ (ε^* preserves the degree of each coordinate). Since $\sigma_\tau^{((R_{\overline{\alpha}}))}$ is the unique element of $\mathcal{Z}(\mathcal{G}_\tau^P)$ of homogeneous degree 1 whose $R_{\overline{\alpha}}$-coordinate is $\overline{\alpha}$ (see Lemma 3.3.18), we conclude that $\varepsilon^*(\sigma^{(s_\alpha)}) = \sigma_\tau^{(R_{\overline{\alpha}})}$. A similar argument will give us $\varepsilon^*(\sigma^{(s_{\mathcal{T}\alpha})}) = \sigma_\tau^{(R_{\overline{\alpha}})}$.

(c) The claim is true for $w = e$ by (a). Hence suppose $w \neq e$. Let us express $\varepsilon^*(\sigma^{(w)})$ as a linear combination of the Schubert classes in $\mathcal{Z}(\mathcal{G}_\tau^P)$ as follows:

$$\varepsilon^*(\sigma^{(w)}) = \sum_{u \in W_\tau^P} f_u \sigma_\tau^{(u)}$$

where each f_u takes value in $\mathcal{S}_\tau$. We aim to show the following claim: if $l_\tau(u) > l(w)$ or $\varepsilon(u) \ngeq w$, then $f_u = 0$.

Let $u \in W_\tau^P$ such that $l_\tau(u) > l(w)$. Suppose $f_u \neq 0$. Since $\varepsilon^*(\sigma^{(w)})$ is homogeneous of $\deg \varepsilon^*(\sigma^{(w)}) = \deg \sigma^{(w)} = l(w)$, we have

$$\deg(f_u \sigma_\tau^{(u)}) = \deg f_u + \deg \sigma_\tau^{(u)} = \deg f_u + l_\tau(u) = l(w),$$

which implies $l_\tau(u) = l(w) - \deg f_u \leq l(w)$, a contradiction. Hence $f_u = 0$.

Now let $u \in W_\tau^P$ so that $\varepsilon(u) \ngeq w$. We will show $f_u = 0$ by induction on $l_\tau(u)$.
Base Case: If $u = e$, then we have

$$\varepsilon^*(\sigma^{(w)})(e) = f_e \cdot \sigma_\tau^{(e)}(e) = f_e.$$

Since $\varepsilon^*(\sigma^{(w)})(e) = \pi_\tau\big(\sigma^{(w)}(e)\big) = 0$, we obtain $f_e = 0$.
Induction Step: Let $l_\tau(u) > 1$. Suppose $f_{u'} = 0$ for all $u' \in W_\tau^P$ with $\varepsilon(u') \ngeq w$ and

$l_\tau(u') < l_\tau(u)$. We have

$$\varepsilon^*(\sigma^{(w)})(u) = \sum_{u' \le u} f_{u'} \cdot \sigma_\tau^{(u')}(u) = f_u \cdot \sigma_\tau^{(u)}(u) + \sum_{u' < u} f_{u'} \cdot \sigma_\tau^{(u')}(u).$$

Here we used $\sigma_\tau^{(u')}(u) = 0$ for all $u' \not\leq u$ (Definition 3.3.1) to refine the sum. For those u' with $u' < u$, we have $\varepsilon(u') < \varepsilon(u)$ (see Proposition 5.3.8), hence $\varepsilon(u') \not\geq w$ follows. Since we also have $l_\tau(u') < l_\tau(u)$, the induction hypothesis applies to these u', and we obtain $f_{u'} = 0$ for all $u' < u$. Thus the above equation becomes $\varepsilon^*(\sigma^{(w)})(u) = f_u \cdot \sigma_\tau^{(u)}(u)$. Since $\varepsilon(u) \not\geq w$, by Definition 3.3.1, we have $\sigma^{(w)}(\varepsilon(u)) = 0$. Hence

$$\varepsilon^*(\sigma^{(w)})(u) = \pi_\tau\big(\sigma^{(w)}(\varepsilon(u))\big) = 0.$$

Since $\sigma_\tau^{(u)}(u) \neq 0$ (Proposition 3.3.4 (b)) and $\mathcal{S}_\tau$ is an integral domain, we obtain $f_u = 0$. ∎

5.4 Lifting criterion

Let the setting and notation be as stated in Section 5.1. In Sections 5.4.1 and 5.4.2, we will answer Question 3 of Section 5.1 by providing a concrete criterion for a Schubert class $\sigma_\tau^{(u)}$ of $\mathcal{Z}(\mathcal{G}_\tau^P)$ to admit a lifting $\sigma^{(w)}$ in $\mathcal{Z}(\mathcal{G}^P)$ (Propositions 5.4.1 and 5.4.8). In Section 5.4.3, we will discuss a way to detect nonliftable Schubert classes in $\mathcal{Z}(\mathcal{G}_\tau^P)$ (Corollary 5.4.21).

5.4.1 Strategy

Let $u \in W_\tau^P$ with $l_\tau(u) = \ell$ so that $\deg \sigma_\tau^{(u)} = \ell$. Motivated by Lemma 3.3.18, in order to find a lifting of $\sigma_\tau^{(u)}$, we look for an element $w \in W^P$ with the following properties:

(C1) $l(w) = \ell$,

(C2) $w \le \varepsilon(u)$,

(C3) $w \not\leq \varepsilon(u')$ for all $u' \in W_\tau^P$ such that $u \not\leq u'$.

Proposition 5.4.1. *Let $u \in W_\tau^P$ with $l_\tau(u) = \ell$ and $w \in W^P$. Then $\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)}$ for some $0 \neq c \in L$ if and only if w satisfies the above conditions* (C1)–(C3).

Proof: ($\Rightarrow$) Let $w \in W^P$ such that $\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)}$ for some $0 \neq c \in L$. Then we have $l(w) = \deg \sigma^{(w)} = \deg \varepsilon^*(\sigma^{(w)}) = \deg \sigma_\tau^{(u)} = l_\tau(u) = \ell$. Thus (C1) is verified.

Since $0 \neq c \cdot \sigma_\tau^{(u)}(u) = \varepsilon^*(\sigma^{(w)})(u) = \pi_\tau\big(\sigma^{(w)}(\varepsilon(u))\big)$, we have $\sigma^{(w)}(\varepsilon(u)) \neq 0$. By Definition 3.3.1, $w \leq \varepsilon(u)$ follows. Thus (C2) is verified. Finally, let $u' \in W_\tau^P$ with $u \not\leq u'$. We have $0 = c \cdot \sigma_\tau^{(u)}(u') = \varepsilon^*(\sigma^{(w)})(u') = \pi_\tau\big(\sigma^{(w)}(\varepsilon(u'))\big)$. Since π_τ is injective (see Remark 4.4.1), we obtain $\sigma^{(w)}(\varepsilon(u')) = 0$. Now Proposition 3.3.4 (c) gives us $w \not\leq \varepsilon(u')$, verifying (C3).

($\Leftarrow$) Let $w \in W^P$ satisfy (C1)–(C3). We aim to show that $\varepsilon^*(\sigma^{(w)})$ is homogeneous of degree $l_\tau(u) = \ell$ with properties $\varepsilon^*(\sigma^{(w)})(u) \neq 0$ and $\varepsilon^*(\sigma^{(w)})(u') = 0$ for all $u' \not\geq u$, and then apply Lemma 3.3.18.

Since ε^* is degree-preserving, we have $\deg \varepsilon^*(\sigma^{(w)}) = \deg \sigma^{(w)} = l(w) = \ell$ by (C1). Let $u' \not\geq u$. Then by (C3), we have $\varepsilon(u') \not\geq w$ which implies $\sigma^{(w)}(\varepsilon(u')) = 0$ by Definition 3.3.1. Thus $\varepsilon^*(\sigma^{(w)})(u') = \pi_\tau\big(\sigma^{(w)}(\varepsilon(u'))\big) = \pi_\tau(0) = 0$. Finally, we have $\varepsilon^*(\sigma^{(w)})(u) = \pi_\tau\big(\sigma^{(w)}(\varepsilon(u))\big)$. By (C2), we have $\varepsilon(u) \geq w$, and hence $\sigma^{(w)}(\varepsilon(u)) \neq 0$ by Proposition 3.3.4 (c). Since π_τ is injective, we obtain $\varepsilon^*(\sigma^{(w)})(u) \neq 0$. ∎

Proposition 5.4.1 states the condition for a Schubert class $\sigma_\tau^{(u)}$ of $\mathcal{Z}(\mathcal{G}_\tau^P)$ to admit a lifting $\sigma^{(w)}$ in $\mathcal{Z}(\mathcal{G}^P)$ in terms of the relationship between the group elements $u \in W_\tau^P$ and $w \in W^P$. This provides the first answer to our question about the liftability of Schubert classes (Question 3). However, verifying (C3) is not an easy task even for the smallest examples of foldings we saw in Section 5.2 since it requires the full description of the moment graphs involved. In the next section, we look for a restatement of the conditions (C1)–(C3) using a combinatorial language of Coxeter groups.

5.4.2 Lifting criterion

Let us recall the map $\varphi : S \to S_\tau$ defined in (5.3.1). Using φ, we now define a subset $W^{\mathcal{F}} \subset W$ which will later be shown to control the liftability of Schubert classes of $\mathcal{Z}(\mathcal{G}_\tau^P)$.

Definition 5.4.2. The *folding subset* of W, denoted by $W^{\mathcal{F}}$, is a subset of W consisting of the identity element $e \in W$ and elements $w \in W$ satisfying the following.

(FS1) If $s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w)$, then $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \in \mathcal{R}(W_\tau)$.

(FS2) If $s_{i_1} \cdots s_{i_\ell}, s_{i'_1} \cdots s_{i'_\ell} \in \mathcal{R}(w)$, then $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}), \varphi(s_{i'_1}) \cdots \varphi(s_{i'_\ell}) \in \mathcal{R}(u)$ for some common $u \in W_\tau$.

Remark 5.4.3.

(a) Let $w \in W^{\mathcal{F}}$ and $s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w)$. Then for any $1 \leq p < q \leq \ell$, the consecutive subexpression $w' = s_{i_p} \cdots s_{i_q}$ satisfies (FS1) and (FS2), and hence $w' \in W^{\mathcal{F}}$.

(b) Let us recall the map $\widehat{\varphi} : \mathcal{R}(W) \to F(S_\tau)$ defined before Proposition 5.3.10. By (FS1), the map $\widehat{\varphi} : \mathcal{R}(W) \to F(S_\tau)$ restricted to $\cup_{w \in W^{\mathcal{F}}} \mathcal{R}(w)$ is the same as applying $\varphi : S \to S_\tau$ to each simple reflection in $s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w)$.

(c) Let $w \in W$ with a unique reduced expression $s_{i_1} \cdots s_{i_\ell}$. Then $w \in W^{\mathcal{F}}$ by the following argument. By (b) and Proposition 5.3.10, $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \in \mathcal{R}(W_\tau)$, verifying (FS1). Since $\mathcal{R}(w)$ is a singleton, (FS2) trivially holds.

Example 5.4.4 ($W^{\mathcal{F}}$ for the folding of type A_3 to type C_2)**.** Let $\Phi \rightsquigarrow \Phi_\tau$ be the folding of the root system of type A_3 as discussed at the beginning of Section 5.2.1. Table 5.5 shows the complete list of elements of $W^{\mathcal{F}}$.

length	Elements of $W^{\mathcal{F}}$
0	e
1	$s_1,\ s_2,\ s_3$
2	$s_1s_2,\ s_2s_1,\ s_2s_3,\ s_3s_2$
3	$s_1s_2s_3,\ s_3s_2s_1$

Table 5.5: Elements of $W^{\mathcal{F}}$: type A_3 to type C_2

Example 5.4.5 ($W^{\mathcal{F}}$ for the folding of type A_4 to type H_2)**.** Let $\Phi \rightsquigarrow \Phi_\tau$ be the folding of the root system of type A_4 as discussed at the beginning of Section 5.2.2. Table 5.6 shows the complete list of elements of $W^{\mathcal{F}}$.

length	Elements of $W^{\mathcal{F}}$
0	e
1	$s_1,\ s_2,\ s_3,\ s_4$
2	$s_1s_2,\ s_2s_1,\ s_2s_3,\ s_3s_2,\ s_3s_4,\ s_4s_3$
3	$s_1s_2s_3,\ s_2s_3s_4,\ s_3s_2s_1,\ s_4s_3s_2$
4	$s_1s_2s_3s_4,\ s_4s_3s_2s_1$

Table 5.6: Elements of $W^{\mathcal{F}}$: type A_4 to type H_2

By definition of the folding subset $W^{\mathcal{F}}$, the map $\varphi : S \to S_\tau$ extends to

$$\begin{aligned} \overline{\varphi} : W^{\mathcal{F}} \longrightarrow W_\tau : w = e &\mapsto e \\ w = s_{i_1} \cdots s_{i_\ell} &\mapsto \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) \end{aligned} \tag{5.4.1}$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is any reduced expression. The condition (FS2) guarantees that $\overline{\varphi}$ is well-defined, and (FS1) guarantees that it preserves the length, that is,

$l_\tau(\overline{\varphi}(w)) = l(w)$ for $w \in W$. Our first goal is to reformulate the lifting conditions (C1)–(C3) solely in terms of this map $\overline{\varphi}$ (Theorem 5.4.8).

Proposition 5.4.6. *Let $u \in W_\tau$ with $l_\tau(u) = \ell$. Suppose $w \in W$ satisfies* (C1), (C2) and $w \in W^{\mathcal{F}}$. Then w satisfies (C3).

Proof: Let $w \in W^{\mathcal{F}}$ and let $u = R_{j_1} \cdots R_{j_\ell}$ be a reduced expression. By applying the group homomorphism ε, we obtain

$$\varepsilon(u) = \varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_k}), \tag{5.4.2}$$

which is a reduced expression in W by Proposition 5.3.8. Since $w \leq \varepsilon(u)$, by Proposition 2.2.19, w admits some reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ which is a subexpression of (5.4.2). Since $w \in W^{\mathcal{F}}$, we have $\varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}})$ for all $1 \leq p \leq \ell - 1$. By Proposition 5.3.4, the sequence $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ is a subexpression of $u = R_{j_1} \cdots R_{j_\ell}$. Since the numbers of simple reflections involved in the two sequences are the same, we obtain $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = R_{j_1} \cdots R_{j_\ell} = u$.

Let $u' \in W_\tau$ such that $w \leq \varepsilon(u')$. We aim to show $u \leq u'$. Let $u' = R_{j'_1} \cdots R_{j'_k}$ be a reduced expression. Then

$$\varepsilon(u') = \varepsilon(R_{j'_1}) \cdots \varepsilon(R_{j'_k}) \tag{5.4.3}$$

is a reduced expression in W. Since $w \leq \varepsilon(u')$, by Proposition 2.2.19, w admits some reduced expression $w = s_{i'_1} \cdots s_{i'_\ell}$ which is a subexpression of (5.4.3). Since $w \in W^{\mathcal{F}}$, we have $\varphi(s_{i'_p}) \neq \varphi(s_{i'_{p+1}})$ for all $1 \leq p \leq \ell - 1$. By Proposition 5.3.4, it follows that $\varphi(s_{i'_1}) \cdots \varphi(s_{i'_\ell})$ is a subexpression of $u' = R_{j'_1} \cdots R_{j'_k}$. Since $w \in W^{\mathcal{F}}$, we have $\varphi(s_{i'_1}) \cdots \varphi(s_{i'_\ell}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = u$. Hence, by Proposition 2.2.19, $u \leq u'$ follows. ∎

Proposition 5.4.7. *Let $u \in W_\tau$ with $l_\tau(u) = \ell$. Suppose $w \in W$ satisfies the conditions* (C1)–(C3). Then $w \in W^{\mathcal{F}}$.

Proof: Our aim is to establish the defining properties (FS1), (FS2) of $W^{\mathcal{F}}$. First, we prove the following claim, which is apparently weaker than (FS1):

Claim 3. Every reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ satisfies $\varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}})$ for all $1 \leq p \leq \ell - 1$.

By the definition of the set $\mathcal{R}_\varphi(w)$ (see before Proposition 5.3.10), it suffices to prove the statement for the elements of $\mathcal{R}_\varphi(w)$. Let $s_{i'_1} \cdots s_{i'_\ell} \in \mathcal{R}_\varphi(w)$ and recall the map $\widehat{\varphi} : \mathcal{R}(W) \to F(S_\tau)$ (see before Proposition 5.3.10).

By Proposition 5.3.10, $\widehat{\varphi}(s_{i'_1} \cdots s_{i'_\ell})$ is W_τ-reduced. Write the W_τ-reduced word so obtained as $R_{j'_1} \cdots R_{j'_k}$, and let $u' \in W_\tau$ be the element determined by this reduced word, that is, $u' = R_{j'_1} \cdots R_{j'_k}$. Suppose $\varphi(s_{i'_p}) = \varphi(s_{i'_{p+1}})$ for some $1 \leq p \leq \ell - 1$. Then by construction, $k < l$, that is, $l_\tau(u') < l_\tau(u)$. Since $w = s_{i'_1} \cdots s_{i'_\ell}$ is a subexpression of $\varepsilon(u') = \varepsilon(R_{j'_1}) \cdots \varepsilon(R_{j'_k})$ by construction, by Proposition 2.2.19, we obtain $w \leq \varepsilon(u')$ where $u \not\leq u'$, a contradiction to (C3). Hence $\varphi(s_{i'_p}) \neq \varphi(s_{i'_{p+1}})$ for all $1 \leq p \leq \ell - 1$.

We now proceed to the proof of (FS1). Knowing that every element of $\mathcal{R}(w)$ has no adjacency of pairs of opposite simple reflections has two consequences. First, Lemma 5.3.10 applies to each $s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w)$. Second, the map $\widehat{\varphi}$ coincides with the operation of applying φ to each s_{i_p}. Thus, for each $s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w)$, Lemma 5.3.10 implies that $\widehat{\varphi}(s_{i_1} \cdots s_{i_\ell}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ is W_τ-reduced.

Finally, we prove (FS2). Let $u = R_{j_1} \cdots R_{j_\ell}$ be a reduced expression. Since $w \leq \varepsilon(u)$, by Proposition 2.2.19, w has a reduced expression $s_{i_1} \cdots s_{i_\ell}$ which is a subexpression of $\varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_\ell})$. Since $\varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}})$ holds for all $1 \leq p \leq \ell - 1$, by Proposition 5.3.4, $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ is a subexpression of $R_{j_1} \cdots R_{j_\ell}$. Since the number of simple reflections involved in the two sequences is the same, we obtain $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = R_{j_1} \cdots R_{j_\ell}$.

Suppose that there is $s_{i'_1} \cdots s_{i'_\ell} \in \mathcal{R}(w)$ with $\varphi(s_{i'_1}) \cdots \varphi(s_{i'_\ell}) \in \mathcal{R}(u')$ for some $u' \neq u$ in W_τ. By construction, $s_{i'_1} \cdots s_{i'_\ell}$ is a subexpression of $\varepsilon\big(\varphi(s_{i'_1})\big) \cdots \varepsilon\big(\varphi(s_{i'_\ell})\big)$. Since $u' = \varphi(s_{i'_1}) \cdots \varphi(s_{i'_\ell})$ is a W_τ-reduced expression, $\varepsilon(u') = \varepsilon(\varphi(s_{i'_1})) \cdots \varepsilon(\varphi(s_{i'_\ell}))$ is a W-reduced expression (Proposition 5.3.8). By Proposition 2.2.19, we obtain $w = s_{i'_1} \cdots s_{i'_\ell} \leq \varepsilon(u')$ where $l_\tau(u') = l_\tau(u)$ and $u' \neq u$ (hence $u \not\leq u'$). This is a contradiction to (C3). ∎

Let us recall the map $\overline{\varphi} : W^{\mathcal{F}} \to W_\tau$ defined in (5.4.1). We can now state and prove our first main result.

Theorem 5.4.8. *Let $u \in W_\tau$ with $l_\tau(u) = \ell$ and $w \in W$. Then w satisfies* (C1)–(C3) if and only if $w \in W^{\mathcal{F}}$ and $u = \overline{\varphi}(w)$.

Proof: Suppose w satisfies (C1)–(C3). Then by Proposition 5.4.6, $w \in W^{\mathcal{F}}$. Let $u = R_{j_1} \cdots R_{j_\ell}$ be a reduced expression. Since $w \leq \varepsilon(u)$, by Proposition 2.2.19, w has a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ which is a subexpression of $\varepsilon(R_{j_1}) \cdots \varepsilon(R_{j_\ell})$. Since we have $\varphi(s_{i_p}) \neq \varphi(s_{i_{p+1}})$ for all $1 \leq p \leq \ell - 1$ thanks to $w \in W^{\mathcal{F}}$, by Proposition 5.3.4, it follows that $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ is a subexpression of $R_{j_1} \cdots R_{j_\ell}$. Since the number of simple reflections involved in the two sequences is the same, we obtain $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = R_{j_1} \cdots R_{j_\ell}$, that is, $\overline{\varphi}(w) = u$.

Conversely, suppose $w \in W^{\mathcal{F}}$ and $u = \overline{\varphi}(w)$. Since $\overline{\varphi}$ is length-preserving, $l(w) = \ell$ must hold, establishing (C1). Next, $u = \overline{\varphi}(w)$ means that for some reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, we have $u = \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell})$ which is reduced. Then $s_{i_1} \cdots s_{i_\ell}$ is a subexpression of $\varepsilon(\varphi(s_{i_1})) \cdots \varepsilon(\varphi(s_{i_\ell}))$. Hence by Proposition 2.2.19, $w \leq \varepsilon(u)$, establishing (C2). Finally, by Proposition 5.4.6, $w \in W^{\mathcal{F}}$ implies (C3). ∎

Corollary 5.4.9. *Let $u \in W_\tau^P$. Then the Schubert class $\sigma_\tau^{(u)}$ admits a lifting to $\mathcal{Z}(\mathcal{G}^P)$ if and only if $u \in \overline{\varphi}(W^{\mathcal{F}} \cap W^P)$.*

Proof: This is a direct application of Proposition 5.4.1 and Theorem 5.4.8. ∎

Corollary 5.4.10. *Let $u \in W_\tau^P$ with $l_\tau(u) = \ell$. If there exists $w \in W^P$ with a unique reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ and if $\varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = u$, then $\sigma_\tau^{(u)}$ lifts to $\sigma^{(w)}$.*

Proof: By Remark 5.4.3 (c), w belongs to $W^{\mathcal{F}}$, and we have $\overline{\varphi}(w) = u$. Hence by Corollary 5.4.9, $\sigma_\tau^{(u)}$ admits a lifting, namely $\sigma^{(w)}$. ∎

Remark 5.4.11. As will be seen in Examples 5.4.14 and 5.4.15, the reformulated lifting criterion of Corollary 5.4.9 is much easier to work with compared to the original (C1)–(C3) given in Proposition 5.4.1. One does not need to draw a moment graph to verify the conditions of Theorem 5.4.8. A question remains as to how easy it is to list all the elements of the folding subset $W^{\mathcal{F}}$. By (C1), we know that the length of an element $w \in W^{\mathcal{F}}$ cannot exceed $l_\tau(u_0)$ where u_0 is the longest element of W_τ. For each $w \in W$ with $l(w) \leq l_\tau(u_0)$, we write down all the elements of $\mathcal{R}(w)$. If $\mathcal{R}(w)$ contains a reduced expression where the number of adjacency of opposite simple reflections (see before Proposition 5.3.10) is ≥ 1, then $w \notin W^{\mathcal{F}}$ by (FS1). Otherwise, we continue to check (FS2) for all the elements of $\mathcal{R}(w)$. This last step involves certain pattern avoidance of each reduced expression in addition to avoiding the adjacency of opposite simple reflections (cf. Corollary 5.4.21, Proposition 5.4.22). This is easy to verify in case $\mathcal{R}(w)$ is relatively small. For more general cases, a restatement of (FS2) in terms of an exhaustive list of pattern avoidance of reduced expressions of $w \in W$ is needed for the full computability of $W^{\mathcal{F}}$.

When a Schubert class $\sigma_\tau^{(u)} \in \mathcal{Z}(\mathcal{G}_\tau^P)$ lifts to $\sigma^{(w)} \in \mathcal{Z}(\mathcal{G}^P)$, we know explicitly the nonzero constant $c \in L$ appearing in the Definition 5.1.1.

Proposition 5.4.12. *Let $\sigma^{(w)} \in \mathcal{Z}(\mathcal{G}^P)$ be a lifting of $\sigma_\tau^{(u)} \in \mathcal{Z}(\mathcal{G}_\tau^P)$. Then*

$$\varepsilon^*(\sigma^{(w)}) = \tau^m \cdot \sigma_\tau^{(u)}$$

for some unique $m \in \mathbb{Z}_{\geq 0}$.

The value of m is determined at the end of the proof.

Proof: We have $\varepsilon^*(\sigma^{(w)}) = c \cdot \sigma_\tau^{(u)}$ for some nonzero $c \in L$. In particular, by looking at the coordinate at $\varepsilon(u)$, this implies $\pi_\tau\big(\sigma^{(w)}(\varepsilon(u))\big) = c \cdot \sigma_\tau^{(u)}(u)$ (for the notation, see Remark 3.3.3). Let $w = s_1 \cdots s_\ell$ be a W-reduced expression, and write $\varphi(s_i) =: R_i$. Then $u = \overline{\varphi}(w) = R_1 \cdots R_l$ is a W_τ-reduced expression (since $w \in W^{\mathcal{F}}$ by Theorem 5.4.8). Apply ε to this expression and write the W-reduced expression so obtained as $\varepsilon(u) = s'_1 \cdots s'_L$ for appropriate L. By construction, this expression contains $w = s_1 \cdots s_\ell$ as a subexpression. Denote this subexpression by $s'_{i_1} \cdots s'_{i_\ell}$ where $1 \leq i_1 < \cdots < i_\ell \leq L$. We have $s'_{i_p} = s_p$ and $\varphi(s'_{i_p}) = \varphi(s_p) = R_p$ for each $1 \leq p \leq \ell$. Let α_p denote the simple root corresponding to s'_{i_p}, and let β_p denote the simple root corresponding to R_p. By Definition 3.3.1, we compute

$$\sigma^{(w)}\big(\varepsilon(u)\big) = r_{i_1}(\alpha_1) \cdot r_{i_2}(\alpha_2) \cdot \cdots \cdot r_{i_\ell}(\alpha_\ell)$$

where $r_{i_p}(\alpha_p) = s'_1 s'_2 \cdots s'_{i_p - 1}(\alpha_p)$. If $\alpha_p \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$, then we have

$$\pi_\tau\big(r_{i_p}(\alpha_p)\big) = \pi_\tau\big(\varepsilon(R_1 \cdots R_{p-1})(\alpha_p)\big) = R_1 \cdots R_{p-1}(\beta_p)$$

If $\alpha_p \in \mathcal{T}(\Delta_{rat})$, then we have

$$\pi_\tau\big(r_{i_p}(\alpha_p)\big) = \pi_\tau\big(\varepsilon(R_1 \cdots R_{p-1})(\alpha_p)\big) = R_1 \cdots R_{p-1}(\tau\beta_p)$$

Hence we obtain

$$\pi_\tau\big(\sigma^{(w)}(\varepsilon(u))\big) = \tau^m \prod_{p=1}^{\ell} R_1 \cdots R_{p-1}(\beta_p) = \tau^m \cdot \sigma_\tau^{(u)}(u)$$

where m is the number of τ-rational roots appearing among α_p's. Thus $c = \tau^m$. ∎

Since $\{\sigma_\tau^{(u)} \mid u \in W_\tau^P\}$ generates $\mathcal{Z}(\mathcal{G}_\tau^P)$ and τ is invertible in L, if ε^* has the lifting property, then Proposition 5.4.12 implies that ε^* over L is surjective.

Corollary 5.4.9 motivates and justifies the following terminology.

Definition 5.4.13. Let $u \in W_\tau^P$. We say that u is *liftable* or *admits a lifting* if there exists an element $w \in W^P$ such that $w \in W^{\mathcal{F}}$ and $\overline{\varphi}(w) = u$. Any such element $w \in W^P$ is called a *lifting* of u and we say that u *lifts to* w.

Using this terminology, Corollary 5.4.9 says that a Schubert class $\sigma_\tau^{(u)} \in \mathcal{Z}(\mathcal{G}_\tau^P)$ lifts to $\sigma^{(w)} \in \mathcal{Z}(\mathcal{G}^P)$ if and only if u lifts to w.

Let us now study some examples of liftings of Schubert classes below.

Example 5.4.14 (Type A_4 to type H_2)**.** Let $\Phi \rightsquigarrow \Phi_\tau$ be the folding of the root system of type A_4 discussed at the beginning of Section 5.2.2. The folded Coxeter group W_τ is the dihedral group D_{10} and has 10 elements. For each $u \in W_\tau$ with $0 \leq l_\tau(u) \leq 4$, Table 5.7 shows a complete list of liftings $w \in W^{\mathcal{F}}$.

$l_\tau(u)$	$u \in W_\tau$	Liftings $w \in W^{\mathcal{F}}$
4	$R_1R_2R_1R_2$	$s_1s_2s_3s_4$
4	$R_2R_1R_2R_1$	$s_4s_3s_2s_1$
3	$R_1R_2R_1$	$s_1s_2s_3, \quad s_3s_2s_1$
3	$R_2R_1R_2$	$s_2s_3s_4, \quad s_4s_3s_2$
2	R_1R_2	$s_1s_2, \quad s_3s_2, \quad s_3s_4$
2	R_2R_1	$s_2s_1, \quad s_2s_3, \quad s_4s_3$
1	R_1	$s_1, \quad s_3$
1	R_2	$s_2, \quad s_4$
0	e	e

Table 5.7: Liftings of W_τ-elements: type A_4 to type H_2

In this case, all the liftings $w \in W^{\mathcal{F}}$ have a unique reduced expression. There is precisely one element $u \in W_\tau$ with length 5, namely the longest element $u = R_1R_2R_1R_2R_1 = R_2R_1R_2R_1R_2$. We will later see in Lemma 5.4.22 (VI)(a) that this u does not admit a lifting $w \in W$.

Example 5.4.15 (Type D_6/A_4 to type H_3/H_2)**.** Let $\Phi \rightsquigarrow \Phi_\tau$ be the folding of the root system of type D_6 as described in Example 4.3.7. The partition of the original simple roots is

$$\Delta_{rat} = \{\alpha_1, \alpha_2, \alpha_6\}, \quad \mathcal{T}(\Delta_{rat}) = \{\alpha_5, \alpha_4, \alpha_3\},$$

and the simple roots of the folded root system Φ_τ are $\Delta_\tau = \{\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_6}\}$. We set $\beta_1 := \overline{\alpha_1}$, $\beta_2 := \overline{\alpha_2}$, and $\beta_3 := \overline{\alpha_6}$. The simple reflection of W corresponding to α_i is denoted by s_i for $1 \le i \le 6$ and the simple reflection of W_τ corresponding to β_j is denoted by R_j for $j = 1, 2, 3$. We take the parabolic subgroup $W_P := \langle s_2, s_3, s_4, s_6 \rangle$, which is of type A_4. We have $\varphi(s_1) = \varphi(s_5) = R_1$, $\varphi(s_2) = \varphi(s_4) = R_2$, and $\varphi(s_3) = \varphi(s_6) = R_3$. Hence $(W_P)_\tau = \langle R_2, R_3 \rangle$, and it forms a parabolic subgroup of W_τ of type H_2. The set W_τ^P of unique minimal length representatives of the coset space $W_\tau/(W_P)_\tau$ has 12 elements. For each $u \in W_\tau^P$, Table 5.8 shows a complete list of liftings $w \in W^{\mathcal{F}} \cap W^P$.

In Table 5.8, observe first that each $w = s_{i_1} \cdots s_{i_\ell}$ (reduced expression) satisfies $\varphi(s_{i_p}) = R_{j_p}$ for each $1 \le p \le \ell$, where $u = R_{j_1} \cdots R_{j_\ell}$ is the prescribed reduced expression of u. For those $w \in W^P$ with $0 \le l(w) \le 5$, w has a unique reduced expression. Hence by Corollary 5.4.10, $\sigma_\tau^{(u)}$ lifts to $\sigma^{(w)}$. For those $w \in W^P$ with $6 \le l(w) \le 10$, observe that each w has precisely two distinct reduced expressions by commuting s_5 and s_6, and that the two reduced expressions obtained by applying φ to each $s_{i_p}(1 \le p \le l(w))$ give rise to the same element $u \in W_\tau$. Hence $w \in W^{\mathcal{F}} \cap W^P$, and by Corollary 5.4.9, $\sigma_\tau^{(u)}$ lifts to $\sigma^{(w)}$ for $w \in W$ given in Table 5.8.

$l_\tau(u)$	$u \in W_\tau$	Liftings $w \in W^{\mathcal{F}} \cap W^P$
10	$R_1R_2R_3R_2R_1R_3R_2R_3R_2R_1$	$s_1s_2s_3s_4s_5s_6s_4s_3s_2s_1$
9	$R_2R_3R_2R_1R_3R_2R_3R_2R_1$	$s_2s_3s_4s_5s_6s_4s_3s_2s_1$
8	$R_3R_2R_1R_3R_2R_3R_2R_1$	$s_3s_4s_5s_6s_4s_3s_2s_1$
7	$R_2R_1R_3R_2R_3R_2R_1$	$s_4s_5s_6s_4s_3s_2s_1$
6	$R_1R_3R_2R_3R_2R_1$	$s_5s_6s_4s_3s_2s_1$
5	$R_3R_2R_3R_2R_1$	$s_6s_4s_3s_2s_1$
5	$R_1R_2R_3R_2R_1$	$s_5s_4s_3s_2s_1, \quad s_1s_2s_3s_4s_5$
4	$R_2R_3R_2R_1$	$s_4s_3s_2s_1, \quad s_2s_3s_4s_5$
3	$R_3R_2R_1$	$s_3s_2s_1, \quad s_3s_4s_5, \quad s_6s_4s_5$
2	R_2R_1	$s_2s_1, \quad s_4s_5$
1	R_1	$s_1, \quad s_5$
0	e	e

Table 5.8: Liftings of W_τ-elements: type D_6/A_4 to type H_3/H_2

5.4.3 Nonliftable Schubert classes

In this section, we discuss a way to detect nonliftable elements of W_τ.

Lemma 5.4.16. *Let $u \in W_\tau$ with $l_\tau(u) = \ell$ and let $w \in W^{\mathcal{F}}$ be a lifting of u. Let $R_{j_1} \cdots R_{j_\ell} \in \mathcal{R}(u)$ and suppose that there exists $s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w)$ such that $\widehat{\varphi}(s_{i_1} \cdots s_{i_\ell}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_\ell}) = R_{j_1} \cdots R_{j_\ell}$ as elements in $\mathcal{R}(W_\tau)$. Then for any $1 \le p \le q \le \ell$, the consecutive subexpression $u' := R_{j_p}R_{j_{p+1}} \cdots R_{j_q}$ lifts to $w' := s_{i_p}s_{i_{p+1}} \cdots s_{i_q}$.*

Proof: Since $w' = s_{i_p} \cdots s_{i_q}$ is a consecutive subexpression of $w = s_{i_1} \cdots s_{i_\ell} \in W^{\mathcal{F}}$, w' also belongs to $W^{\mathcal{F}}$, and we have $\overline{\varphi}(w') = u'$. Hence w' is a lifting of u'. ∎

Lemma 5.4.17. *Let $R, R' \in W_\tau$ be simple reflections.*

(a) *If $m(R, R') = 4$, then $u = RR'RR'$ is not liftable.*

(b) *If $m(R, R') = 5$, then $u = RR'RR'R$ is not liftable.*

Remark 5.4.18. If $m(R, R') = 3$, then $u = RR'R$ is always liftable (by the existence of a folding branch, see Lemma 5.3.6).

Proof: (a) The set $\mathcal{R}(u)$ consists of precisely two elements: $RR'RR'$ and $R'RR'R$. By Proposition 5.3.2, in the Coxeter diagram, the configuration of the full subgraph $\varphi^{-1}(\{R, R'\})$ is (C) (up to symmetry of R and R'). Let $w \in W^{\mathcal{F}}$ be a lifting of u.

Case 1: $RR'RR' \in \widehat{\varphi}\big(\mathcal{R}(w)\big)$. There exists $\mathbf{i} := s_{i_1} \cdots s_{i_4} \in \mathcal{R}(w)$ such that

$$\widehat{\varphi}(\mathbf{i}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_4}) = RR'RR'.$$

By Lemma 5.4.16, $w' = s_{i_1}s_{i_2}s_{i_3}$ is a lifting of $u' = RR'R$. Since $w' \in W^{\mathcal{F}}$, the only possibilities are $s_{i_1}s_{i_2}s_{i_3} = sts^*$, or s^*ts. On the other hand, $\varphi(s_{i_4}) = R'$ forces $s_{i_4} = t$. However, both $w = sts^*t$ and $w = s^*tst$ violate (FS2) of Definition 5.4.2.

Case 2: $R'RR'R \in \widehat{\varphi}\big(\mathcal{R}(w)\big)$. There exists $\mathbf{i} := s_{i_1} \cdots s_{i_4} \in \mathcal{R}(w)$ such that

$$\widehat{\varphi}(\mathbf{i}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_4}) = R'RR'R.$$

By Lemma 5.4.16, $w' = s_{i_2}s_{i_3}s_{i_4}$ is a lifting of $u' = RR'R$. Since $w' \in W^{\mathcal{F}}$, the only possibilities are $s_{i_2}s_{i_3}s_{i_4} = sts^*$, or s^*ts. On the other hand, $\varphi(s_{i_1}) = R'$ forces $s_{i_1} = t$. However, $w = tsts^*$ and $w = ts^*ts$ both violate (FS2) of Definition 5.4.2.

(b) The set $\mathcal{R}(u)$ consists of precisely two elements: $RR'RR'R$ and $R'RR'RR'$. By Proposition 5.3.2, in the Coxeter diagram, the configuration of the full subgraph $\varphi^{-1}(\{R, R'\})$ is (D) (up to symmetry of R and R'). Let $w \in W^{\mathcal{F}}$ be a lifting of u.

Case 1: $RR'RR'R \in \widehat{\varphi}\big(\mathcal{R}(w)\big)$. There exists $\mathbf{i} := s_{i_1} \cdots s_{i_5} \in \mathcal{R}(w)$ such that

$$\widehat{\varphi}(\mathbf{i}) = \varphi(s_{i_1}) \cdots \varphi(s_{i_5}) = RR'RR'R.$$

By Lemma 5.4.16, $w' = s_{i_1}s_{i_2}s_{i_3}s_{i_4}$ is a lifting of $u' = RR'RR'$. Since $w' \in W^{\mathcal{F}}$, the only possibility is $s_{i_1}s_{i_2}s_{i_3}s_{i_4} = sts^*t^*$. On the other hand, $\varphi(s_{i_5}) = R$ forces $s_{i_5} = s$ or s^*. However, $w = sts^*t^*s$ and $w = sts^*t^*s^*$ both violate (FS2) of Definition 5.4.2.

Case 2: $R'RR'RR' \in \widehat{\varphi}\big(\mathcal{R}(w)\big)$. We may argue in exactly the same way as in Case 1. ∎

Proposition 5.4.19. *Let $u \in W_\tau$ with $l_\tau(u) = \ell$ and let $w \in W^{\mathcal{F}}$ be a lifting of u. Then for every $R_{j_1} \cdots R_{j_\ell} \in \mathcal{R}(u)$, there exists $s_{i_1} \cdots s_{i_\ell} \in \mathcal{R}(w)$ such that $\varphi(s_{i_p}) = R_{j_p}$ for each $1 \leq p \leq \ell$.*

Remark 5.4.20. Using the map $\widehat{\varphi}$ (see the paragraph before Lemma 5.3.10 for definition), by Remark 5.4.3 (a), the claim is written as $\widehat{\varphi}\big(\mathcal{R}(w)\big) = \mathcal{R}(u)$. One containment $\widehat{\varphi}\big(\mathcal{R}(w)\big) \subset \mathcal{R}(u)$ is already established by $w \in W^{\mathcal{F}}$ and $\overline{\varphi}(w) = u$.

Proof: Let $\mathbf{i} \in \mathcal{R}(w)$. Assume $\widehat{\varphi}\big(\mathcal{R}(w)\big) \subsetneq \mathcal{R}(u)$ and let $\mathbf{j} \in \mathcal{R}(u) \backslash \widehat{\varphi}\big(\mathcal{R}(w)\big)$. Since both $\widehat{\varphi}(\mathbf{i})$ and $\mathbf{j}$ belong to $\mathcal{R}(u)$, there exists a finite sequence of W_τ-braid moves that transforms $\widehat{\varphi}(\mathbf{i})$ into $\mathbf{j}$, and the corresponding sequence $\widehat{\varphi}(\mathbf{i}) = \mathbf{j}_0 \to \mathbf{j}_1 \to \cdots \to \mathbf{j}_r = \mathbf{j}$ of elements of $\mathcal{R}(u)$. Since $\widehat{\varphi}(\mathbf{i}) \in \widehat{\varphi}\big(\mathcal{R}(w)\big)$ and $\mathbf{j} \notin \widehat{\varphi}\big(\mathcal{R}(w)\big)$, there exists $0 \leq k \leq r-1$ such that $\mathbf{j}_k \in \widehat{\varphi}\big(\mathcal{R}(w)\big)$ and $\mathbf{j}_{k+1} \notin \widehat{\varphi}\big(\mathcal{R}(w)\big)$. Let us write $\mathbf{j}_k = \widehat{\varphi}(\mathbf{i}')$ for some $\mathbf{i}' = s_{i'_1} \cdots s_{i'_\ell} \in \mathcal{R}(w)$, and set $R_{j'_q} := \varphi(s_{i'_q})$ for each $1 \leq q \leq \ell$.

We shall now argue that in the W_τ-braid move $[R_{j'_p} R_{j'_{p+1}}]_m = [R_{j'_{p+1}} R_{j'_p}]_m$ that transforms $\mathbf{j}_k$ into $\mathbf{j}_{k+1}$, m cannot be equal to 2 or 3. After that, we will establish that $m \geq 4$ is also not possible, yielding a contradiction to our hypothesis.

The case of $m = 2$ is discarded as follows: by Lemma 5.3.3, $m = 2$ would imply that any $s \in \varphi^{-1}(R_{j'_p})$ and $t \in \varphi^{-1}(R_{j'_{p+1}})$ commute. Hence, there is a corresponding W-braid move $s_{i'_p} s_{i'_{p+1}} = s_{i'_{p+1}} s_{i'_p}$, which transforms $\mathbf{i}'$ to another element $\mathbf{i}'' \in \mathcal{R}(w)$ such that $\widehat{\varphi}(\mathbf{i}'') = \mathbf{j}_{k+1}$. This contradicts $\mathbf{j}_{k+1} \notin \widehat{\varphi}\big(\mathcal{R}(w)\big)$.

The case of $m = 3$ is discarded as follows: Set $R = R_{j'_p}$ and $R' = R_{j'_{p+1}}$. By Proposition 5.3.2, there are two possible configurations of the full subgraph $\varphi^{-1}(\{R, R'\})$, namely (A) and (B). In (A), there is the corresponding W-braid move $[s_{i'_p} s_{i'_{p+1}}]_3 = [s_{i'_{p+1}} s_{i'_p}]_3$, which transforms $\mathbf{i}'$ to another element $\mathbf{i}'' \in \mathcal{R}(w)$ such that $\widehat{\varphi}(\mathbf{i}'') = \mathbf{j}_{k+1}$, contradicting $\mathbf{j}_{k+1} \notin \widehat{\varphi}\big(\mathcal{R}(w)\big)$. In (B), the possible $\widehat{\varphi}$-preimages of the word $[RR']_3$ consisting of 3 simple reflections are: $sts, sts^*, st^*s^*, s^*t^*s^*, s^*t^*s, s^*ts$, among which only sts and $s^*t^*s^*$ are compatible with $w \in W^{\mathcal{F}}$. Hence, there is a corresponding W-braid move $[s_{i'_p} s_{i'_{p+1}}]_3 = [s_{i'_{p+1}} s_{i'_p}]_3$, which transforms $\mathbf{i}'$ to another element $\mathbf{i}'' \in \mathcal{R}(w)$ such that $\widehat{\varphi}(\mathbf{i}'') = \mathbf{j}_{k+1}$. This contradicts $\mathbf{j}_{k+1} \notin \widehat{\varphi}\big(\mathcal{R}(w)\big)$.

As for the case $m \geq 4$, we argue as follows: Set $R = R_{j'_p}$ and $R' = R_{j'_{p+1}}$. If $m = 4$, by Lemma 5.4.17 (a), $u' := [RR']_4$ is not liftable. Since $\mathbf{j}_k \in \widehat{\varphi}(\mathcal{R}(w))$, this is a contradiction to Lemma 5.4.16. If $m = 5$, by Lemma 5.4.17 (b), $u'' := [RR']_5$ is not liftable. Since $\mathbf{j}_k \in \widehat{\varphi}(\mathcal{R}(w))$, this is a contradiction to Lemma 5.4.16.

By Proposition 5.3.2, the cases $m \geq 6$ do not exist. This completes the proof. ∎

Corollary 5.4.21. *Let $u \in W_\tau$ with $l_\tau(u) = \ell$. If u admits a W_τ-reduced expression $u = R_{j_1} \cdots R_{j_\ell}$ which has a consecutive subexpression $u' := R_{j_p} R_{j_{p+1}} \cdots R_{j_q}$ $(1 \leq p \leq q \leq \ell)$ such that u' is not liftable, then u is not liftable.*

Proof: This is a direct application of Lemma 5.4.16 and Proposition 5.4.19. ∎

Before providing examples of nonliftable elements, we establish a list of basic nonliftable elements for each twisted quadratic folding described in Section 4.3. See Appendix A for the corresponding Coxeter diagrams and the numbering of the simple reflections.

Proposition 5.4.22 (Basic nonliftable elements)**.** *The following elements of W_τ are not liftable:*

(I) *Type A_{2n-1} to type C_n*

  (a) $R_n R_{n-1} R_n$

  (b) $R_{n-1} R_{n-2} R_{n-1} R_n R_{n-1}$

(c) $R_{n-1}R_nR_{n-1}R_n$

(II) *Type D_{n+1} to type B_n*

(a) $R_{n-1}R_nR_{n-1}$

(b) $R_{n-1}R_nR_{n-1}R_n$

(III) *Type E_6 to type F_4*

(a) $R_2R_3R_2$

(b) $R_3R_4R_3R_2R_3$

(c) $R_2R_3R_2R_3$

(IV) *Type E_8 to type H_4*

(a) $R_3R_2R_3R_4R_3$

(b) $R_3R_4R_3R_4R_3$

(V) *Type D_6 to type H_3*

(a) $R_2R_1R_2R_3R_2$

(b) $R_2R_3R_2R_3R_2$

(VI) *Type A_4 to type H_2*

(a) $R_1R_2R_1R_2R_1$

Proof: (I)(a), (II)(a), (III)(a): Proof for these three cases follows the same argument as one can see from the folding diagrams in Appendix A. We present here a proof for (I)(a). Set $u := R_nR_{n-1}R_n$. If u is liftable, then there exists some $w = s_{i_1}s_{i_2}s_{i_3} \in W^{\mathcal{F}}$ such that $\varphi(s_{i_1}) = R_n$, $\varphi(s_{i_2}) = R_{n-1}$ and $\varphi(s_{i_3}) = R_n$. There are precisely two candidates: $w_1 = s_ns_{n-1}s_n$ and $w_2 = s_ns_{n+1}s_n$. Observe that $s_ns_{n-1}s_n = s_{n-1}s_ns_{n-1}$ in W, but

$$\varphi(s_n)\varphi(s_{n-1})\varphi(s_n) = R_nR_{n-1}R_n \neq R_{n-1}R_nR_{n-1} = \varphi(s_{n-1})\varphi(s_n)\varphi(s_{n-1}).$$

Hence $w_1 \notin W^{\mathcal{F}}$. Similarly, $w_2 \notin W^{\mathcal{F}}$.

To obtain a proof for (II)(a), interchange the roles of R_{n-1} and R_n, and the roles of s_{n-1} and s_n. To obtain a proof for (III)(a), replace R_{n-1} by R_3, R_n by R_2, s_{n-1} by s_3, s_n by s_4, and s_{n+1} by s_5.

(I)(b) and (III)(b): Proof for these two cases follows the same argument. We present here a proof for (I)(b). Set $u := R_{n-1}R_{n-2}R_{n-1}R_nR_{n-1}$. If u is liftable, then by Proposition 5.4.19, there exists some $w = s_{i_1}\cdots s_{i_5} \in W^{\mathcal{F}}$ such that $\varphi(s_{i_p}) = R_{j_p}$ for each $1 \leq p \leq 5$. By Lemma 5.4.16, $s_{i_1}s_{i_2}$ is a lifting of $u_1 = R_{n-1}R_{n-2}$ and $s_{i_3}s_{i_4}s_{i_5}$

is a lifting of $u_2 = R_{n-1}R_nR_{n-1}$. The element u_1 has two liftings $w_1 = s_{n-1}s_{n-2}$ and $w_1' = s_{n+1}s_{n+2}$, and u_2 also has two liftings $w_2 = s_{n-1}s_ns_{n+1}$ and $w_2' = s_{n+1}s_ns_{n-1}$. Thus there are four candidates for w:

$$\begin{aligned}
v_1 &= w_1w_2 = s_{n-1}s_{n-2}s_{n-1}s_ns_{n+1} = s_{n-2}s_{n-1}s_ns_{n-2}s_{n+1} \\
v_2 &= w_1w_2' = s_{n-1}s_{n-2}s_{n+1}s_ns_{n-1} \\
v_3 &= w_1'w_2 = s_{n+1}s_{n+2}s_{n-1}s_ns_{n+1} \\
v_4 &= w_1'w_2' = s_{n+1}s_{n+2}s_{n+1}s_ns_{n-1} = s_{n+2}s_{n+1}s_ns_{n+2}s_{n-1}
\end{aligned}$$

Notice that a reduced expression of v_1 and v_2 contains a consecutive subexpression $s_{n-2}s_{n+1}$ and observe that $s_{n-2}s_{n+1} = s_{n+1}s_{n-2}$ in W but $R_{n-2}R_{n-1} \neq R_{n-1}R_{n-2}$ in W_τ. Hence $v_1, v_2 \notin W^{\mathcal{F}}$. A similar argument shows $v_3, v_4 \notin W^{\mathcal{F}}$ (look at the subexpression $s_{n+2}s_{n-1}$). Hence by Corollary 5.4.21, u is not liftable.

To obtain a proof for (III)(b), replace R_{n-2} by R_4, R_{n-1} by R_3, R_n by R_2, s_{n-2} by s_1, s_{n-1} by s_3, s_n by s_4, s_{n+1} by s_5, and s_{n+2} by s_6.
(I)(c), (II)(b), (III)(c) follow from Lemma 5.4.17 (a).
(IV)(a) and (V)(a): Proof for these two cases follows the same argument as one can see from the folding diagrams in Appendix A. We present here a proof for (IV)(a). Set $u := R_3R_2R_3R_4R_3$. If u is liftable, then by Proposition 5.4.19, there exists some $w = s_{i_1}\cdots s_{i_5} \in W^{\mathcal{F}}$ such that $\varphi(s_{i_p}) = R_{j_p}$ for each $1 \leq p \leq 5$. By Lemma 5.4.16, $s_{i_1}s_{i_2}$ is a lifting of $u_1 = R_3R_2$ and $s_{i_3}s_{i_4}s_{i_5}$ is a lifting of $u_2 = R_3R_4R_3$. The element u_1 has two liftings $w_1 = s_3s_2$ and $w_1' = s_5s_6$, and u_2 also has two liftings $w_2 = s_3s_4s_5$ and $w_2' = s_5s_4s_3$. Thus there are four candidates for w:

$$\begin{aligned}
v_1 &= w_1w_2 = s_3s_2s_3s_4s_5 = s_2s_3s_4s_2s_5 \\
v_2 &= w_1w_2' = s_3s_2s_5s_4s_3 \\
v_3 &= w_1'w_2 = s_5s_6s_3s_4s_5 \\
v_4 &= w_1'w_2' = s_5s_6s_5s_4s_3 = s_6s_5s_4s_6s_3
\end{aligned}$$

Notice that a reduced expression of v_1 and v_2 contains a consecutive subexpression s_2s_5 and observe that $s_2s_5 = s_5s_2$ in W but $R_2R_3 \neq R_3R_2$ in W_τ. Hence $v_1, v_2 \notin W^{\mathcal{F}}$. A similar argument shows $v_3, v_4 \notin W^{\mathcal{F}}$ (look at the subexpression s_6s_3). Hence by Corollary 5.4.21, u is not liftable.
(IV)(b), (V)(b) and (VI)(a) follow from Lemma 5.4.17 (b). ∎

Using Corollary 5.4.21 and Proposition 5.4.22, we can detect less trivial non-liftable elements of W_τ as in the following example.

Example 5.4.23 (Type E_8/D_6 to type H_4/H_3)**.** We take the parabolic subgroup $W_P := \langle s_2, s_3, s_4, s_5, s_6, s_8\rangle$, which is of type D_6. The map $\varphi : S \to S_\tau$ sends s_1, s_7 to R_1, s_2, s_6 to R_2, s_3, s_5 to R_3, and s_4, s_8 to R_4. Then $(W_P)_\tau = \langle R_2, R_3, R_4\rangle$ forms a parabolic subgroup of W_τ of type H_3. We will now give an example of an element of W_τ^P (of the smallest length) which does not admit a lifting. Let us look at

$$u := R_3R_2R_3R_4R_3R_1R_2R_4R_3R_4R_3R_2R_1 \in W_\tau^P.$$

This reduced expression contains the consecutive subexpression $u' := R_3R_2R_3R_4R_3$, for which there exists no $w' \in W^{\mathcal{F}}$ such that $\overline{\varphi}(w') = u'$ (by Lemma 5.4.22 (IV)(a)). Hence by Corollary 5.4.21, it follows that u admits no lifting.

More examples of nonliftable elements of W_τ will be seen in the next section (Propositions 5.5.1–5.5.6).

5.5 Lifting property: case studies

In this section, we provide a complete list of parabolic subgroups for each quadratic folding presented in Section 4.3 so that the induced map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ has the lifting property (Definition 5.1.1). We use $P \subset S$ to denote a parabolic subset for the original Coxeter group W, and $P_\tau \subset S_\tau$ to denote a parabolic subset for the folded Coxeter group W_τ. The notational convention of parabolic subgroups is explained before Proposition 2.2.17. It is also recalled that by construction of a folding of a subsystem Φ_P, the simple system Δ_P corresponding to P must be preserved by $\mathcal{T}$ (see the beginning of Section 4.2.3). Throughout this section, the reader is strongly recommended to consult Appendix A, where the Coxeter diagram of each folding along with the numbering of simple reflections can be found.

Proposition 5.5.1. *In the folding of type A_{2n-1} to type C_n, the induced map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ has the lifting property if and only if $P = P_{1,2n-1}$ or S.*

Proof: ($\Rightarrow$) Suppose $P_{1,2n-1} \not\subset P$, or equivalently, $(P_\tau)_1 \not\subset P_\tau$. We will find an element $u \in W_\tau^P$ that is not liftable. There exists some $2 \le i \le n$ such that $R_i \notin P_\tau$. If $R_n \notin P_\tau$, then $R_nR_{n-1}R_n \in W_\tau^P$, which is not liftable (Lemma 5.4.22 (I)(a)). If $R_{n-1} \notin P_\tau$, then $R_{n-1}R_{n-2}R_{n-1}R_nR_{n-1} \in W_\tau^P$, which is not liftable (Lemma 5.4.22 (I)(b)). For $2 \le i \le n-2$, set

$$u := R_{n-1}R_{n-2}R_{n-1}R_nR_{n-1}(R_{n-3}R_{n-2})(R_{n-4}R_{n-3})\cdots(R_{i-1}R_i).$$

Then $u \in W_\tau^P$. Since the above reduced expression contains a consecutive subexpression $u' := R_{n-1}R_{n-2}R_{n-1}R_nR_{n-1}$, which is not liftable, Corollary 5.4.21 implies that u is not liftable. Thus, $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ does not have the lifting property.
($\Leftarrow$) Let $P = P_{1,2n-1}$, or equivalently, $P_\tau = (P_\tau)_1$. An arbitrary element $u \in W_\tau^P$ with $l_\tau(u) \ge 1$ has a reduced expression either

$$u = R_jR_{j-1}\cdots R_1$$

for some $1 \leq j \leq n$ or

$$u = R_k R_{k+1} \cdots R_{n-2} R_{n-1} R_n R_{n-1} R_{n-2} \cdots R_2 R_1$$

for some $1 \leq k \leq n-1$. In the former case, u has the following two liftings in W^P:

$$w_1 = s_j s_{j-1} \cdots s_2 s_1$$
$$w_2 = s_{2n-j} s_{2n-j+1} \cdots s_{2n-2} s_{2n-1}.$$

Observe that these are unique reduced expressions for w_1 and w_2 respectively, hence belong to $W^{\mathcal{F}}$ indeed. In the latter case, u has the following two liftings in W^P:

$$w_3 = s_k s_{k+1} \cdots s_{n-2} s_{n-1} s_n s_{n+1} s_{n+2} \cdots s_{2n} s_{2n-1}$$
$$w_4 = s_{2n-k} s_{2n-k-1} \cdots s_{n+2} s_{n+1} s_n s_{n-1} s_{n-2} \cdots s_2 s_1.$$

Again, these are unique reduced expressions for w_3 and w_4 respectively, hence belong to $W^{\mathcal{F}}$. Thus ε^* has the lifting property. For $P = S$, the statement is clear. ∎

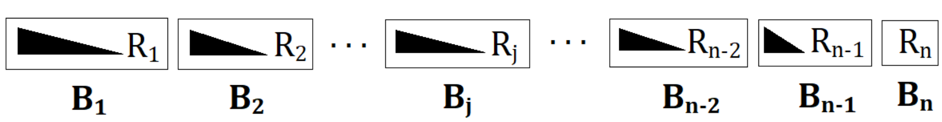
The idea of the proof comes from a close observation of the Coxeter diagram in Figure A.1. In the second direction, we first represent each element of W_τ^P by a suitable reduced expression, then look for a way to lift it to some reduced expression in W which travels along the edge of the Coxeter diagram in a most rigid way possible. The same idea is used for all the following propositions.

Proposition 5.5.2. *In the folding of type D_{n+1} to type B_n, the induced map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ has the lifting property if and only if $P = P_{n,n+1}$ or S.*

Proof: $(\Rightarrow)$ Suppose $P_{n,n+1} \not\subset P$, or equivalently, $(P_\tau)_n \not\subset P_\tau$. Then there exists $1 \leq i \leq n-1$ such that $R_i \notin P_\tau$. Set

$$u := R_{n-1} R_n R_{n-1} R_{n-2} \cdots R_i.$$

Then $u \in W_\tau^P$. Since the above reduced expression contains a consecutive subexpression $u' := R_{n-1} R_n R_{n-1}$, which is not liftable (Lemma 5.4.22 (II)(a)), Corollary 5.4.21 implies that u is not liftable. Hence ε^* does not have the lifting property.
$(\Leftarrow)$ Let $P = P_{n,n+1}$, or equivalently, $P_\tau = (P_\tau)_n$. First, we establish that each element of W_τ^P has a reduced expression of the following form:



where each block $\mathbf{B_j}$ is either empty ($\emptyset$) or a word of the form $R_kR_{k-1}\cdots R_j$ for some $j \le k \le n$ (the wedge within each block $\mathbf{B_j}$ in the diagram indicates that the subscripts of the simple reflections are decreasing), satisfying the following conditions: if $\mathbf{B_j} \ne \emptyset$, then $\mathbf{B_{j'}} \ne \emptyset$ for all $j' > j$; if the block $\mathbf{B_j}$ starts from R_n, then every bock $\mathbf{B_{j'}}$ with $j' > j$ also starts from R_n; if the block $\mathbf{B_j}$ does not start from R_n, then $0 \le |\mathbf{B_1}| \le |\mathbf{B_2}| \le \cdots \le |\mathbf{B_j}|$, where each $|\mathbf{B_i}|$ denotes the size of the block $\mathbf{B_i}$.

By construction, the above word is reduced, and every reduced word that is braid equivalent to it ends with R_n, hence the group element represented by the above reduced word belongs to W_τ^P. It remains to show that the reduced expressions of the above form exhaust all the elements of W_τ^P. We will use a counting argument.

We denote $b_i := |\mathbf{B_i}|$ for each $1 \le i \le n$. Let j be the smallest number such that block $\mathbf{B_j}$ begins with R_n. Then we have

$$0 \le b_1 \le b_2 \le \cdots \le b_{j-1} \le b_j = n - (j-1). \tag{5.5.1}$$

We first take care of two extreme cases, namely, $j = 1, n+1$. When $j = 1$, every block begins with R_n, yielding

$$(R_nR_{n-1}\cdots R_1)(R_nR_{n-1}\cdots R_2)\cdots(R_nR_{n-1})R_n.$$

When $j = n+1$, no block begins with R_n, yielding the empty word corresponding to $e \in W_\tau$. Now suppose $2 \le j \le n$. We will count the $(j-1)$-tuble $(b_1, \cdots, b_{j-1})$ satisfying (5.5.1).

For $x \in \mathbb{Q}$, let us denote the largest integer no greater than x by $\lceil x \rceil$. We need to distribute the numbers $\{0, 1, 2, \cdots, n-(j-1)\}$ among $b_1, b_2, \cdots, b_{j-1}$ in the weakly ascending order. Let $0 \le k \le \min\{j-2, n-(j-1)\}$. We place k separation walls between $j-1$ objects, and in the resulting $k+1$ separated sections, we distribute $k+1$ numbers in strictly ascending order from left to right. The number of choices we have for such distribution is

$$\binom{j-2}{k} \cdot \binom{n-(j-1)+1}{k+1}.$$

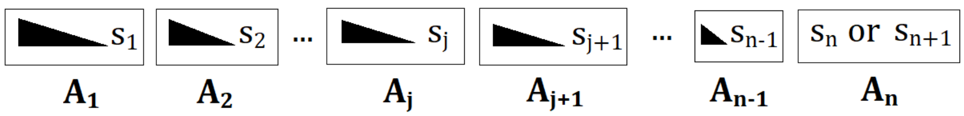
By varying $1 \le j \le n+1$, the total number of choices is found to be

$$\begin{aligned} F(n) := 2 + \sum_{j=2}^{\lceil \frac{n+3}{2} \rceil} \sum_{k=0}^{j-2} \binom{j-2}{k} \cdot \binom{n-(j-1)+1}{k+1} \\ + \sum_{j=\lceil \frac{n+3}{2} \rceil + 1}^{n} \sum_{k=0}^{n-(j-1)} \binom{j-2}{k} \cdot \binom{n-(j-1)+1}{k+1}. \end{aligned}$$

On the other hand, we have $|W_\tau^P| = \frac{|W_\tau|}{|(W_P)_\tau|} = \frac{|Bn|}{|A_{n-1}|} = \frac{2^n \cdot n\ !}{n\ !} = 2^n$.

Claim 4. We have $F(n) = 2^n$ for all $n \geq 2$. (Proof postponed at the end.)

Let $\mu = \mathbf{B_1 B_2} \cdots \mathbf{B_n}$ be a W_τ-reduced word of the form described above, and let $u \in W_\tau^P$ be the group element represented by μ. We now build a W-reduced word λ of the following form:



where each block $\mathbf{A_j}$ is obtained from block $\mathbf{B_j}$ as follows: if $\mathbf{B_j} = \emptyset$, then $\mathbf{A_j} = \emptyset$; if $\mathbf{B_j} = R_k R_{k-1} \cdots R_j$, then $\mathbf{A_j} = s s_{k-1} \cdots s_j$ where $s = s_k$ if $k \neq n$, and $s = s_n$ or s_{n+1} if $k = n$ subject to the condition: if $\mathbf{A_j}$ begins with s_n (resp. s_{n+1}), then $\mathbf{A_{j+1}}$ begins with s_{n+1} (resp. s_n).

Claim 5. The group element $w \in W$ represented by a W-reduced word λ of the above form belongs to $W^{\mathcal{F}}$.

We verify the conditions (FS1) and (FS2) of Definition 5.4.2. By definition of w via λ, no element of $\mathcal{R}(w)$ contains the consecutive subword $s_n s_{n+1}$ or $s_{n+1} s_n$. Hence Proposition 5.3.10 applies to λ, and we obtain that $\widehat{\varphi}(\lambda)$ is W_τ-reduced. Moreover, since s_n and s_{n+1} are not adjacent in λ, $\widehat{\varphi}(\lambda)$ coincides with applying φ to each simple reflection of λ. Thus (FS1) is verified.

Next, we observe from the Coxeter diagrams in Figure A.2 that among the W-braid moves $[st]_{m(s,t)} = [ts]_{m(s,t)}$, the only ones for which the corresponding W_τ-moves $[\varphi(s)\varphi(t)]_{m(s,t)} = [\varphi(t)\varphi(s)]_{m(s,t)}$ are false are the following:

$$s_n s_{n-1} s_n = s_{n-1} s_n s_{n-1}, \quad s_{n+1} s_{n-1} s_{n+1} = s_{n-1} s_{n+1} s_{n-1},$$

namely, the long braid moves within the collapsing part. By definition of w via λ, we can directly check that no element of $\mathcal{R}(w)$ contains the consecutive subword $s_n s_{n-1} s_n$, $s_{n-1} s_n s_{n-1}$, $s_{n+1} s_{n-1} s_{n+1}$ or $s_{n-1} s_{n+1} s_{n-1}$. Hence, for every W-braid move $[st]_{m(s,t)} = [ts]_{m(s,t)}$ within $\mathcal{R}(w)$, there is the corresponding W_τ-braid move $[\varphi(s)\varphi(t)]_{m(s,t)} = [\varphi(t)\varphi(s)]_{m(s,t)}$. Thus $\widehat{\varphi}\big(\mathcal{R}(w)\big) \subset \mathcal{R}(u)$. This verifies (FS2).

Hence $w \in W^{\mathcal{F}}$ and we have $\overline{\varphi}(w) = u$ as desired.

Proof of Claim 4: Set

$$f(n,j) := \begin{cases} \sum_{k=0}^{j-1} \begin{pmatrix} j-1 \\ k \end{pmatrix} \begin{pmatrix} n-j+1 \\ k+1 \end{pmatrix} & \text{if } 1 \leq j \leq \lceil \frac{n+1}{2} \rceil, \\ \sum_{k=0}^{n-j} \begin{pmatrix} j-1 \\ k \end{pmatrix} \begin{pmatrix} n-j+1 \\ k+1 \end{pmatrix} & \text{if } \lceil \frac{n+1}{2} \rceil + 1 \leq j \leq n-1. \end{cases}$$

We aim to show $f(n,j) = \binom{n}{j}$ for all $1 \le j \le n-1$. Once we have this,

$$F(n) = 2 + \sum_{j=1}^{n-1} \binom{n}{j} = \sum_{k=0}^{n} \binom{n}{j} = 2^n$$

will follow. We prove by induction on $n \ge 2$. If $n = 2$, then $j = 1$ and we have $f(2,1) = 2 = \binom{2}{1}$. Now assume $f(n,j) = \binom{n}{j}$ for $n \ge 2$. Let $1 \le j \le \lceil \frac{n+1}{2} \rceil$ (the same argument will apply to the case $\lceil \frac{n+1}{2} \rceil + 1 \le j \le n-1$). It suffices to show $f(n+1,j) - f(n,j) = \binom{n}{j-1}$. We have

$$\begin{aligned} f(n+1,j) - f(n,j) &= \sum_{k=0}^{j-1} \binom{j-1}{k} \binom{n-j+1}{k} \\ &= \sum_{k=0}^{j-1} \binom{j-1}{k} \binom{n-j}{k} + \sum_{k=1}^{j-1} \binom{j-1}{k} \binom{n-j}{k-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \binom{n}{j-1} = \tfrac{j}{n-j+1} \binom{n}{j} = \tfrac{j}{n-j+1} f(n,j) &= \sum_{k=0}^{j-1} \binom{j}{k+1} \binom{n-j}{k} \\ = \sum_{k=0}^{j-1} \binom{j-1}{k} \binom{n-j}{k} &+ \sum_{k=0}^{j-2} \binom{j-1}{k+1} \binom{n-j}{k}. \end{aligned}$$

Thus $f(n+1,j) - f(n,j) = \binom{n}{j-1}$ holds and we finish the proof. ∎

Proposition 5.5.3. *In the folding of type E_6 to type F_4, the induced map ε^* : $\mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}^P_\tau)$ has the lifting property if and only if $P = S$.*

Proof: Suppose $P_\tau \ne S_\tau$, or equivalently, $P \ne S$. If $R_2 \notin P_\tau$, then $R_2R_3R_2 \in W^P_\tau$, which is not liftable (Lemma 5.4.22(III)(a)). If $R_1 \notin P_\tau$, $R_2R_3R_2R_1 \in W^P_\tau$, which is not liftable since the reduced expression contains $R_2R_3R_2$ as a consecutive subexpression (Corollary 5.4.21). Similarly, if $R_3 \notin P_\tau$, then

$$R_3R_4R_3R_2R_3 \in W^P_\tau,$$

which is not liftable (Lemma 5.4.22 (III)(b)). Finally, if $R_4 \notin P_\tau$, then

$$R_2R_3R_2R_1R_4R_3R_2R_3R_4 \in W^P_\tau,$$

which is not liftable since it contains the consecutive subexpression $u' := R_2R_3R_2$, which is not liftable (Corollary 5.4.21). Thus ε^* does not have the lifting property. The other implication is clear. ∎

Proposition 5.5.4. *In the folding of type A_4 to type H_2, the induced map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ has the lifting property if and only if $P = P_{2,4}$, $P_{1,3}$ or S.*

Proof: In Example 5.4.14, it was seen that among all the ten elements of W_τ, $u := R_1R_2R_1R_2R_1$ was the only element which did not admit a lifting. We have $u \notin W_\tau^P$ if and only if $P_\tau \neq \emptyset$, or equivalently, $P \neq \emptyset$. Hence it follows that ε^* has the lifting property if and only if $P \neq \emptyset$, i.e., $P = P_{2,4}$, $P_{1,3}$ or S. ∎

Proposition 5.5.5. *In the folding of type D_6 to type H_3, the induced map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ has the lifting property if and only if $P = P_{1,5}$ or S.*

Proof: ($\Rightarrow$) Suppose $P_{1,5} \not\subset P$, or equivalently, $(P_\tau)_1 \not\subset P_\tau$. If $R_2 \notin P_\tau$, then $R_2R_1R_2R_3R_2 \in W_\tau^P$, which is not liftable (cf. Lemma 5.4.22 (IV)(a)). If $R_3 \notin P_\tau$, then $R_2R_1R_2R_3R_2R_3 \in W_\tau^P$, which is not liftable since the reduced expression contains a consecutive subexpression $R_2R_1R_2R_3R_2$, which is not liftable (Corollary 5.4.21). Thus the induced map ε^* does not have the lifting property.
($\Leftarrow$) This direction is proven in Example 5.4.15 for $P = P_{1,5}$ and it is trivial for $P = S$. ∎

Proposition 5.5.6. *In the folding of type E_8 to type H_4, the induced map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ has the lifting property if and only if $P = S$.*

Proof: Suppose $P_\tau \neq S_\tau$, or equivalently, $P \neq S$.

If $R_3 \notin P_\tau$, then $u_3 := R_3R_2R_3R_4R_3 \in W_\tau^P$, which is not liftable. If $R_1 \notin P_\tau$, then

$$u_1 := R_3R_2R_3R_4R_3R_1R_2R_4R_3R_4R_3R_2R_1 \in W_\tau^P,$$

which is not liftable (see the argument in Example 5.4.23). Hence $R_1 \in P_\tau$. Similarly, if $R_2 \notin P_\tau$, then

$$u_2 := R_3R_2R_3R_4R_3R_1R_2R_4R_3R_4R_3R_2 \in W_\tau^P,$$

which is not liftable. Finally, if $R_4 \notin P_\tau$, then

$$R_3R_2R_3R_4R_3R_4 \in W_\tau^P,$$

which is not liftable. Thus ε^* does not have the lifting property. The other implication is clear. ∎

5.6 Surjectivity of the folding map

Let the setting and notation be as stated in Section 5.1. In this section, we answer the question of surjectivity of the folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \to \mathcal{Z}(\mathcal{G}_\tau^P)$ (Question 1 of Section 5.1). We will first prove surjectivity of ε^* over L for the case $\Delta_P = \emptyset$ (Theorem 5.6.2). Then the corresponding result for $\Delta_P \neq \emptyset$ is obtained as a corollary.

5.6.1 Surjectivity of ε^* over $\mathbb{R}$

Our first statement toward the surjectivity of the folding map ε^* is its surjectivity over $\mathbb{R}$. We will elaborate on the argument made in [30, Remark 6.6].

Proposition 5.6.1 ([30, Remark 6.6])**.** *The folding map*

$$\varepsilon^* : \mathcal{Z}(\mathcal{G}) \otimes_K \mathbb{R} \to \mathcal{Z}(\mathcal{G}_\tau) \otimes_L \mathbb{R}$$

is surjective.

Proof: We extend the commutative diagram of Lemma 4.4.4 to $\mathbb{R}$.

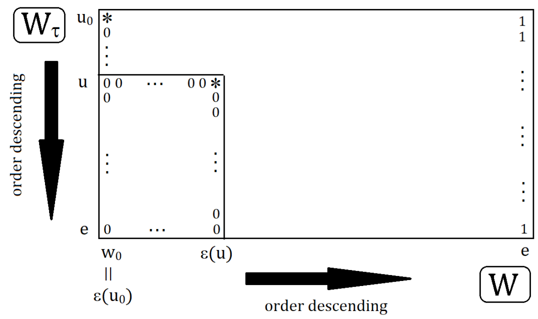
$$\begin{array}{ccc} \mathcal{S}_\mathbb{R} \otimes_{\mathcal{S}_\mathbb{R}^W} \mathcal{S}_\mathbb{R}^{W_P} & \xrightarrow{\rho} & \mathcal{Z}(\mathcal{G}^P) \otimes_K \mathbb{R} \\ \Big\downarrow{\scriptstyle \pi_\tau \otimes \pi_\tau} & & \Big\downarrow{\scriptstyle \varepsilon^*} \\ (\mathcal{S}_\tau)_\mathbb{R} \otimes_{(\mathcal{S}_\tau)_\mathbb{R}^{W_\tau}} (\mathcal{S}_\tau)_\mathbb{R}^{(W_P)_\tau} & \xrightarrow{\rho_\tau} & \mathcal{Z}(\mathcal{G}_\tau^P) \otimes_L \mathbb{R} \end{array}$$

where $\mathcal{S}_\mathbb{R} = \mathcal{S} \otimes_K \mathbb{R}$ and $(\mathcal{S}_\tau)_\mathbb{R} = \mathcal{S}_\tau \otimes_L \mathbb{R}$. By [26, Corollary 4.10], the two Borel maps ρ and ρ_τ in the diagram are isomorphisms. By construction, the map $\pi_\tau : \mathcal{S} \to \mathcal{S}_\tau$ over L is surjective (cf. Remark 4.4.1), and hence over $\mathbb{R}$. Combining these two facts, the above commutative diagram implies that ε^* over $\mathbb{R}$ is surjective. ∎

5.6.2 Proof for surjectivity of ε^*

Theorem 5.6.2. *The folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}) \otimes_K L \to \mathcal{Z}(\mathcal{G}_\tau)$ is surjective.*

Proof: Let us choose total orders on W and W_τ which are compatible with the respective Bruhat orders. Let us also choose the Schubert basis $\{\sigma^{(w)} \mid w \in W\}$ as an $\mathcal{S}$-basis for $\mathcal{Z}(\mathcal{G})$, and the standard basis $\{\delta_u \mid u \in W_\tau\}$ as an $\mathcal{S}_\tau$-basis for the ambient free module $\bigoplus_{u \in W_\tau} \mathcal{S}_\tau$ of $\mathcal{Z}(\mathcal{G}_\tau)$, where δ_u denotes the Kronecker delta. With respect to these bases and the total orders on W and W_τ, we may write the transformation matrix of ε^* as follows.



Call this matrix $\mathfrak{E} = (e_{u,w})$ where $u \in W_\tau, w \in W$. Let us note that the column vector $\mathbf{e}_w$ corresponding to $w \in W$ is of the form

$$\mathbf{e}_w = \Big(\varepsilon^*\big(\sigma^{(w)}\big)(u)\Big)_{u \in W_\tau} = \Big(\pi_\tau\big(\sigma^{(w)}(\varepsilon(u))\big)\Big)_{u \in W_\tau}.$$

This matrix is upper triangular since $\sigma^{(w)}(x) = 0$ for all $w \not\leq x$ (Definition 3.3.1), and the leading term of each row is located at the position $(u, \varepsilon(u))$. We now proceed to compute this leading term explicitly, using the coordinate formula of $\sigma^{(w)}$ (Definition 3.3.1).

Let $u = R_1 R_2 \cdots R_l$ be a W_τ-reduced expression, where each R_j denotes a simple reflection of W_τ. We denote the simple root corresponding to R_j by β_j. Let us recall the map $\varphi : S \to S_\tau$ (see (5.3.1)) and that $|\varphi^{-1}(R_j)|$ is equal to either 1 or 2. In the former case, we denote the unique preimage by s_j and the corresponding simple root by $\alpha_j (\in \Delta^{\mathcal{T}})$. In the latter case, we denote the two distinct preimages by s_j and s_j^* (cf. Definition 5.3.1). We adopt the convention that the simple root corresponding to s_j belongs to Δ_{rat} and we denote it by α_j. Then the simple root corresponding to s_j^* is $\mathcal{T}\alpha_j$. For each j, we have $\pi_\tau(\alpha_j) = \beta_j$.

As we apply ε to each R_j, two things can happen:

$$\varepsilon(R_j) = \begin{cases} s_j, & \text{if } |\varphi^{-1}(R_j)| = 1 \\ s_j s_j^*, & \text{if } |\varphi^{-1}(R_j)| = 2. \end{cases}$$

Let us write the W-reduced expression $\varepsilon(u) = \varepsilon(R_1) \cdots \varepsilon(R_l)$ as $\varepsilon(u) = s'_1 s'_2 \cdots s'_L$ for appropriate $L (\geq l)$, where each s'_i denotes a simple reflection of W, and denote the simple root corresponding to s'_i by α'_i. By the coordinate formula of Definition 3.3.1, we obtain

$$\sigma^{(\varepsilon(u))}(\varepsilon(u)) = \mathbf{r}_1(\alpha'_1) \cdot \mathbf{r}_2(\alpha'_2) \cdot \cdots \cdot \mathbf{r}_L(\alpha'_L)$$

where $\mathbf{r}_i(\alpha'_i) := s'_1 s'_2 \cdots s'_{i-1}(\alpha'_i)$. We will now apply π_τ to each $\mathbf{r}_i(\alpha'_i)$. Let us say $s'_i \in \varphi^{-1}(R_j)$. There are two cases to consider. If $\alpha'_i \in \Delta^{\mathcal{T}} \sqcup \Delta_{rat}$, then

$$\pi_\tau(\mathbf{r}_i(\alpha'_i)) = \pi_\tau\big(\varepsilon(R_1 \cdots R_{j-1})(\alpha_j)\big) = R_1 \cdots R_{j-1}(\beta_j).$$

If $\alpha'_i \in \mathcal{T}(\Delta_{rat})$, then

$$\pi_\tau(\mathbf{r}_i(\alpha'_i)) = \pi_\tau\big(\varepsilon(R_1 \cdots R_{j-1})s_j(\mathcal{T}\alpha_j)\big) = R_1 \cdots R_{j-1}(\tau\beta_j).$$

In both cases, Proposition 4.2.9 is used, and in the latter case, we also use $s_j(\mathcal{T}\alpha_j) = \mathcal{T}\alpha_j$ and $\pi_\tau(\mathcal{T}\alpha_j) = \tau\beta_j$. From this observation, it follows that

$$\pi_\tau\Big(\sigma^{(\varepsilon(u))}\big(\varepsilon(u)\big)\Big) = \tau^m \prod_j \big(R_1 \cdots R_{j-1}(\beta_j)\big)^{\epsilon_j} \tag{5.6.1}$$

where $\epsilon_j = 1$ if $|\varphi^{-1}(R_j)| = 1$ and $\epsilon_j = 2$ if $|\varphi^{-1}(R_j)| = 2$, and m is the number of Δ_{rat}-elements among $\alpha'_i (1 \leq i \leq L)$. Notice that the content of the product consists of $\beta \in \Phi_\tau^+$ and that τ is invertible in L.

Let us write $\mathcal{S}_L := \mathcal{S} \otimes_K L$ and $\mathcal{S}_\mathbb{R} := \mathcal{S} \otimes_K \mathbb{R}$ respectively. For a given $\eta \in \mathcal{Z}(\mathcal{G}_\tau)$, let us consider the equation $\varepsilon^*(\xi) = \eta$ where $\xi \in \mathcal{Z}(\mathcal{G})$. Let $\mathbf{x}$ denote the column vector expression of ξ with respect to the Schubert basis of $\mathcal{Z}(\mathcal{G})$, and let $\mathbf{h}$ denote the column vector expression of η with respect to the standard basis of $\bigoplus_{u \in W_\tau} \mathcal{S}_\tau$. Then the above equation is written as the following system of $\mathcal{S}$-linear equations:

$$\mathfrak{E}\mathbf{x} = \mathbf{h}.$$

By the fact that the matrix $\mathfrak{E}$ is upper triangular and the observation made after (5.6.1), this equation admits a solution $\mathbf{x}$ with each coordinate in $\mathcal{S}_L[(\Phi_\tau^+)^{-1}]$, where $\mathcal{S}_L[(\Phi_\tau^+)^{-1}]$ denotes the ring $\mathcal{S}_L$ adjoined with the inverses of elements of Φ_τ^+. On the other hand, by Proposition 5.6.1, the coordinates of the solution $\mathbf{x}$ are known to take values in $\mathcal{S}_\mathbb{R}$. Since $\mathcal{S}_L[(\Phi_\tau^+)^{-1}] \cap \mathcal{S}_\mathbb{R} = \mathcal{S}_L$, the coordinates of $\mathbf{x}$ indeed take values in $\mathcal{S}_L$. This proves that $\varepsilon^* : \mathcal{Z}(\mathcal{G}) \otimes_K L \to \mathcal{Z}(\mathcal{G}_\tau)$ is surjective. ∎

Corollary 5.6.3. *The folding map $\varepsilon^* : \mathcal{Z}(\mathcal{G}^P) \otimes_K L \to \mathcal{Z}(\mathcal{G}_\tau^P)$ is surjective.*

Proof: This follows from Theorem 5.6.2 and the fact that for each $u \in W_\tau^P$, the Schubert class $\sigma_\tau^{(u)} \in \mathcal{Z}(\mathcal{G}_\tau^P)$ is the restriction $\sigma_\tau^{(u)}|_{W_\tau^P}$ of the Schubert class $\sigma_\tau^{(u)} \in \mathcal{Z}(\mathcal{G}_\tau)$ (see before Proposition 3.3.15). ∎

Remark 5.6.4. Let us recall the folding from type E_8 to H_4 described in Example 4.3.6. Let us denote the structure algebra of the original moment graph by $\mathcal{Z}(E_8, K)$ and the folded structure algebra by $\mathcal{Z}(H_4, L)$. At the end of [30, Section 6], the authors provide a conjectual answer for a minimal ring presentation (cf. [16, Definition 1.1]) of the reduced structure algebra $\widehat{\mathcal{Z}}(H_4, L)$ by taking the $\widehat{\varepsilon}^*$-image of the minimal ring presentation of $\widehat{\mathcal{Z}}(E_8, K)$ known by [16, Corollary 6.2]. The surjectivity of ε^* over L is one of the two assumptions required to verify their conjectual presentation of $\widehat{\mathcal{Z}}(H_4, L)$.

Appendix A

Folding diagrams

In this appendix, we collect the Coxeter diagrams of all the examples of quadratic foldings discussed in Section 4.3. The number at each vertex is the index of each simple reflection that the vertex represents. In the Coxeter diagram associated with the original root system (drawn first in each folding), the simple reflections represented by black dots correspond to the elements of Δ_{rat}, those represented by white dots correspond to the elements of $\mathcal{T}(\Delta_{rat})$, and those represented by double white dots corresponds to the elements of $\Delta^{\mathcal{T}}$ respectively.

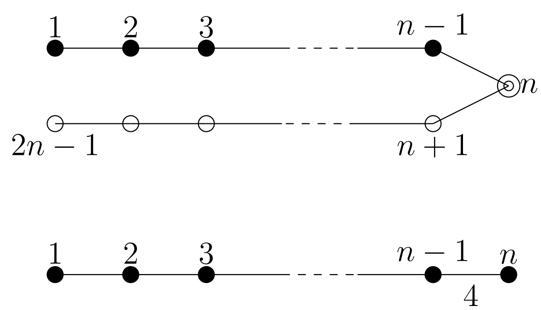


Figure A.1: Type A_{2n-1} to type C_n

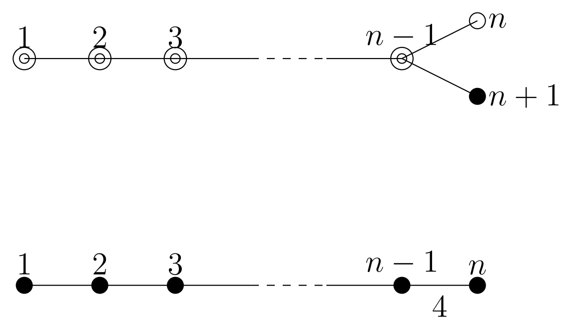


Figure A.2: Type D_{n+1} to type B_n

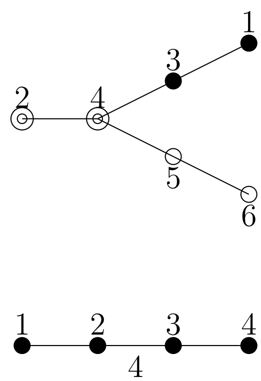


Figure A.3: Type E_6 to type F_4

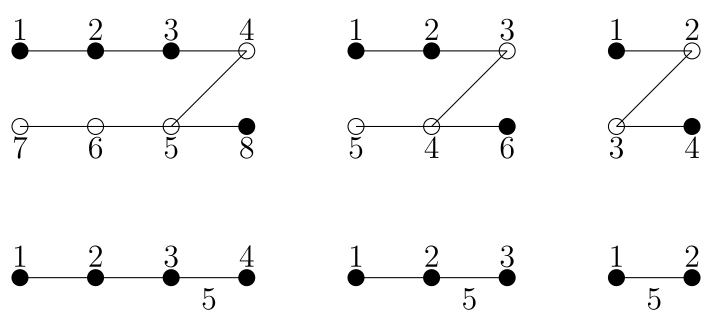


Figure A.4: Type E_8 to type H_4, type D_6 to type H_3, type A_4 to type H_2 (in this order from left to right)

Appendix B

Moment graphs

In this appendix, we list the moment graphs discussed in Section 5.2.

Folding from type A_3 to C_2 (cf. Section 5.2.1)

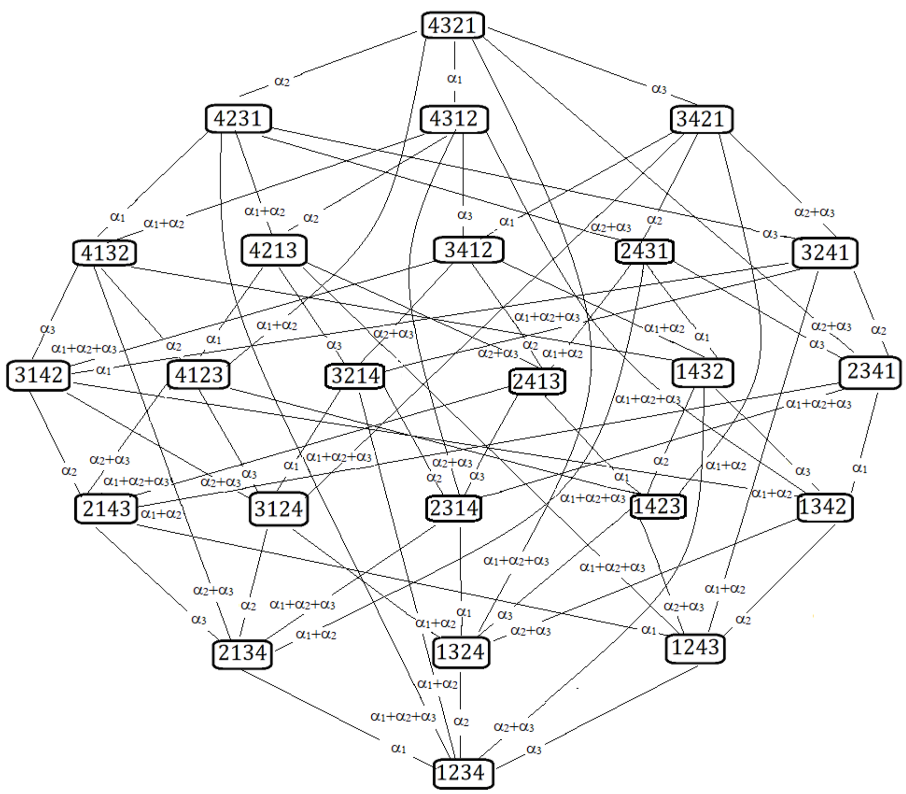


Figure B.1: Moment graph of type A_3

In Figure B.1, each element of the symmetric group $\mathfrak{S}_4$ is written in one-line notation (see the notation after Example 2.2.7). Table B.1 translates the one-line notation into a product of simple reflections. In Figures B.1–B.3, each α_i denotes the simple root corresponding to the simple reflection s_i.

$l(w)$	w in one-line notation	w as a product of simple reflections
0	1234	e
1	2134	s_1
	1324	s_2
	1243	s_3
2	2143	s_3s_1
	3124	s_2s_1
	2314	s_1s_2
	1423	s_3s_2
	1342	s_2s_3
3	3142	$s_2s_3s_1$
	4123	$s_3s_2s_1$
	3214	$s_1s_2s_1$
	2413	$s_3s_1s_2$
	1432	$s_2s_3s_2$
	2341	$s_1s_2s_3$
4	4132	$s_3s_2s_3s_1$
	4213	$s_3s_1s_2s_1$
	3412	$s_2s_3s_1s_2$
	2431	$s_3s_1s_2s_3$
	3241	$s_1s_2s_3s_1$
5	4231	$s_3s_1s_2s_3s_1$
	4312	$s_2s_3s_1s_2s_1$
	3421	$s_1s_2s_3s_1s_2$
6	4321	$s_3s_1s_2s_3s_1s_2$

Table B.1: Translation of the one-line notation of $w \in W$ into a product of simple reflections (type A_3)

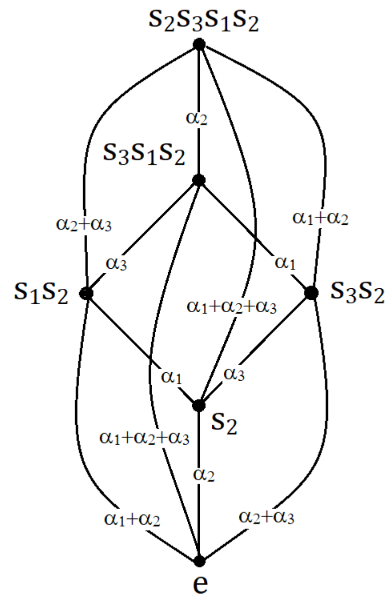


Figure B.2: Moment graph of type A_3/P_2

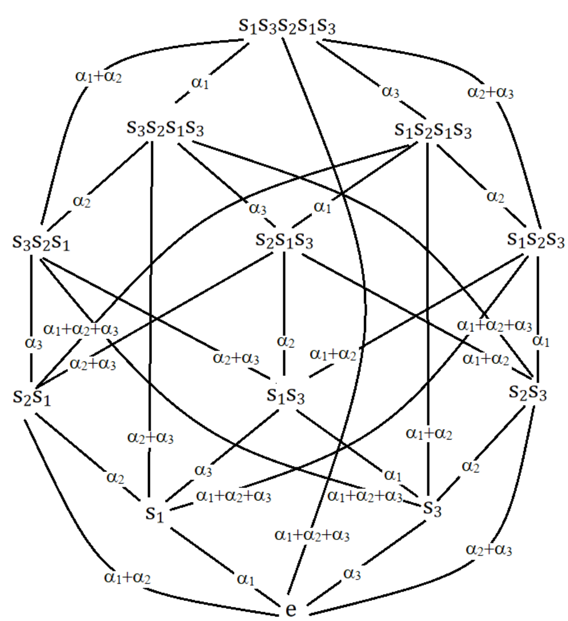


Figure B.3: Moment graph of type $A_3/P_{1,3}$

In Figure B.4, each β_j denotes the simple root corresponding to the simple reflection R_j. Under the folding from type A_3 to C_2, we have

$$\beta_1 = \pi_\tau(\alpha_1) = \pi_\tau(\alpha_3), \quad \beta_2 = \pi_\tau(\alpha_2).$$

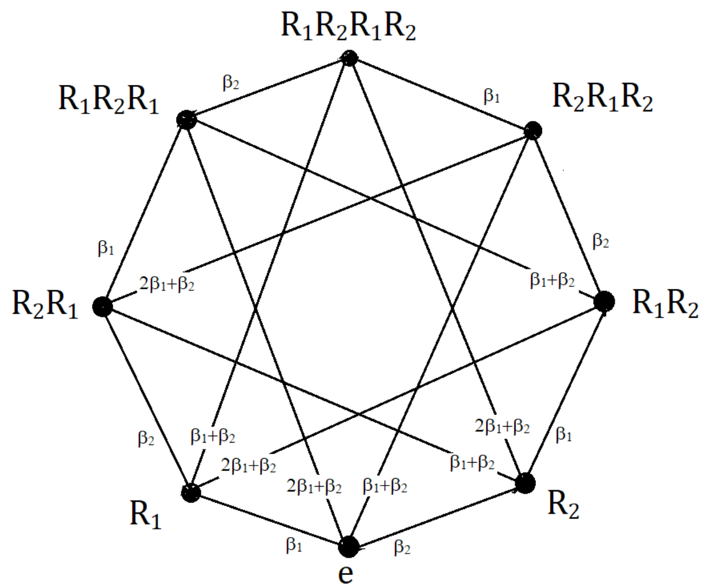


Figure B.4: Moment graph of type C_2

Folding from type A_4 to H_2 (cf. Section 5.2.2)

In Figure B.5, each element of the symmetric group $\mathfrak{S}_5$ is written in one-line notation (see the notation after Example 2.2.7). See Table 5.3 for translation of the one-line notation into a product of simple reflections. In Figure B.5, each α_i denotes the simple root corresponding to the simple reflection s_i. In Figures B.6 and B.7, each β_j denotes the simple root corresponding to the simple reflection R_j. Under the folding from type A_4 to H_2, we have

$$\beta_1 = \pi_\tau(\alpha_1), \quad \beta_2 = \pi_\tau(\alpha_4), \quad \tau\beta_1 = \pi_\tau(\alpha_3), \quad \tau\beta_2 = \pi_\tau(\alpha_2).$$

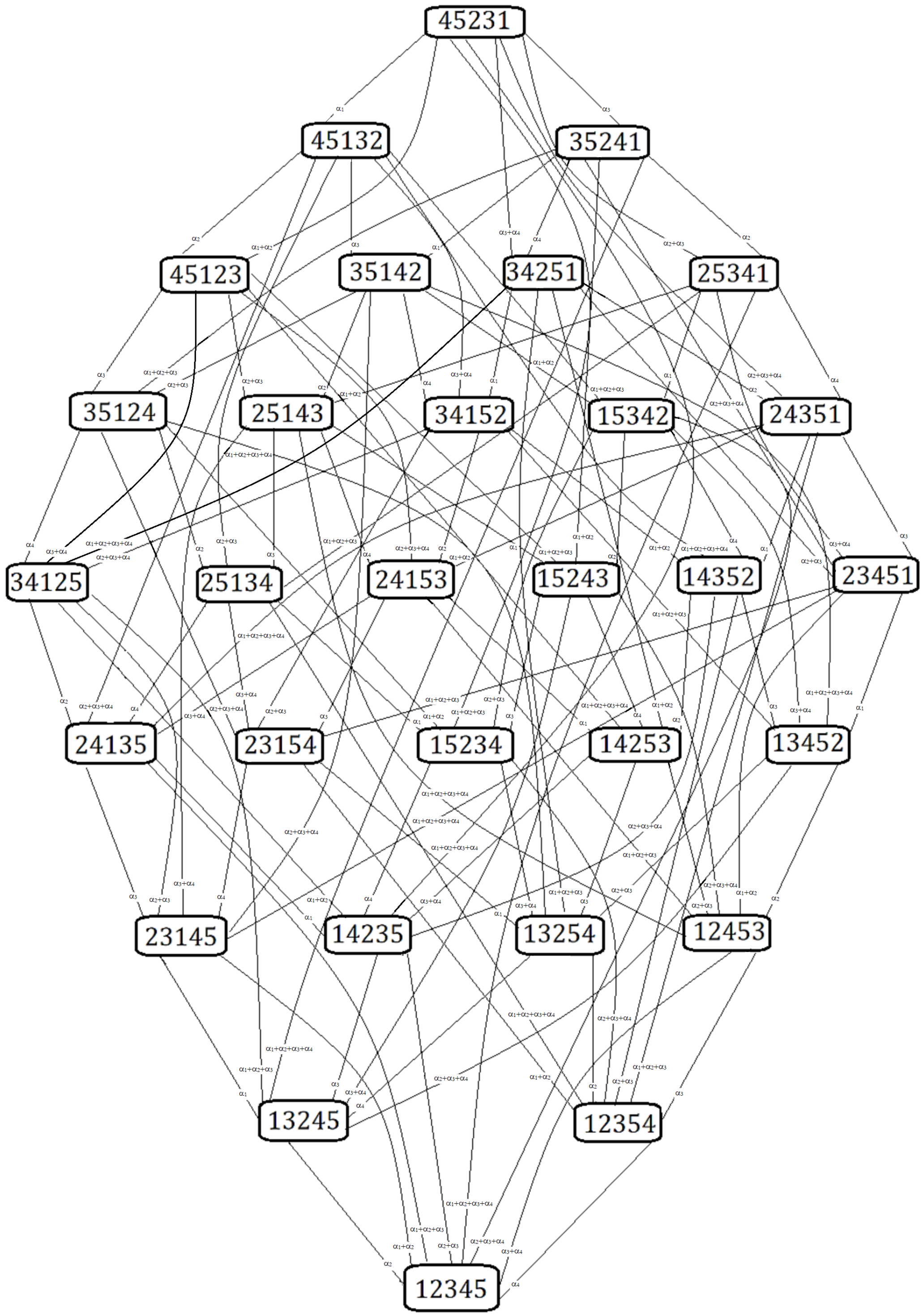


Figure B.5: Moment graph of type $A_4/P_{2,4}$

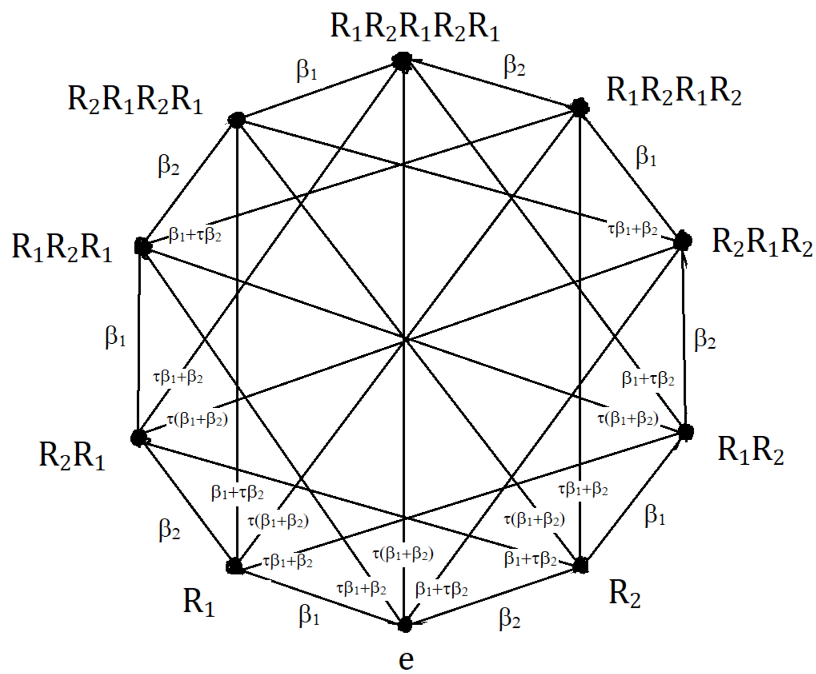


Figure B.6: Moment graph of type H_2

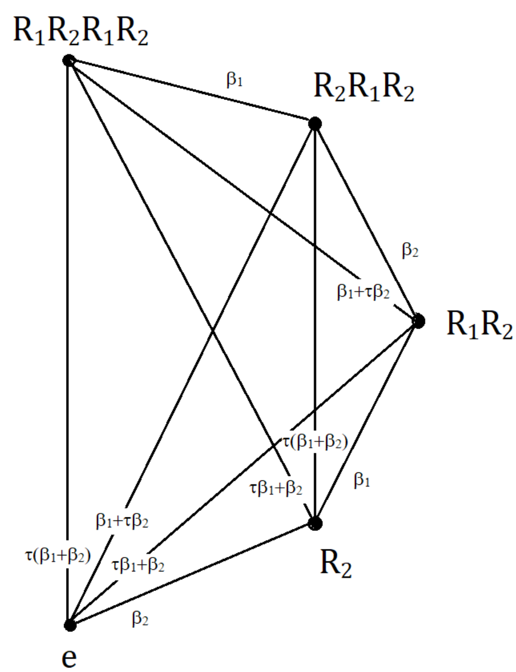


Figure B.7: Moment graph of type H_2/P_2

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List of Symbols

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