Universal monoidal categories with duals

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Thesis submitted to the Faculty of Science in partial fulfillment of the requirements
for the degree of
Master of Science Mathematics and Statistics[1]

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[1]The M.Sc. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

String diagrams form a diagrammatic notation used in many domains. To understand the simplest diagrams that express some properties, we can look at universal categories. These satisfy universal properties and can be described by presentations in terms of generators and relations. In this thesis, we examine some examples of universal categories, namely the (oriented) Temperley-Lieb and (oriented) Brauer categories. These are respectively the free linear monoidal category on a self-dual object or pair of dual objects, and the free linear symmetric monoidal category on a symmetrically self-dual object or pair of dual objects.

Then, to make precise the connection between presentations and universal properties, we exhibit an adjoint functor from a category of generators and relations to the category of linear monoidal categories. We also suggest a general recipe to find a presentation of the category satisfying a specific universal property.

Our main goal is to better understand the links between string diagrams, representation theory, generators and relations, and universal properties.

Résumé

Les diagrammes de cordes constituent une notation utilisée dans plusieurs domaines. Pour comprendre les diagrammes les plus simples exprimant certaines propriétés, nous pouvons considérer les catégories universelles. Ces catégories satisfont des propriétés universelles et peuvent être décrites à l'aide de présentations consistant en générateurs et relations. Dans cette thèse, nous examinons quelques exemples de catégories universelles: les catégories (orientées ou non) de Temperley-Lieb et de Brauer. Il s'agit respectivement de la catégorie linéaire monoidale libre sur une paire d'objets duaux ou sur un objet autodual, et de la catégorie linéaire symétrique monoidal libre sur une paire d'objets duaux ou sur un objet symétriquement autodual.

Ensuite, pour rendre précis le lien entre présentations et propriétés universelles, nous décrivons un foncteur adjoint allant d'une catégorie de générateurs et relations vers la catégorie des catégories linéaires monoidales. Nous suggérons aussi une méthode générale pour trouver une présentation d'une catégorie satisfaisant une propriété universelle donnée.

Notre objectif principal est de mieux comprendre les liens entre diagrammes de corde, théorie des représentations, présentations et propriétés universelles.

Acknowledgments

I would like to thank first my supervisor, Professor Alistair Savage. When I asked about your research, it took you less than five minutes to convince me that I had found the right subject area. Then, your project suggestion interested me from the very beginning and continued to motivate me all along. You opened to me the world of mathematical research, and I'll always be grateful for that.

The Department of Mathematics and Statistics has always been very supportive to me. I owe my mathematical skills and enthusiasm mainly to Professors M. Nevins, P.-E. Parent, T. Giordano, A. Broadbent, P. Scott, V. Leblanc and M. Newman. Special mentions to Professor B. Dionne, for administrative and academic support, and to Professor D. Roy, who was the first to introduce me to mathematical research. I also want to thank the NSERC and OGS for financial support.

For all the great conversations, about maths, politics and life, sincere thanks to Jérémy, Dene, Charms, Maria, Trinity, Joey, Jane, Masoomeh, Maiko, Adèle, Gage, Khalil, Jules, Samuel, Raj, Alex and Jérémie. After we needed to social distance, I realized that working and procrastinating alone is much more boring than working and procrastinating together.

The examiners, Professors Hadi Salmasian and Colin Ingals, provided valuable feedback that helped improve the quality of this thesis. Also, special thanks to Sébastien Lord and Youssef Mousaaid for comments on an earlier draft, and to Professor Richard Blute for his help with Chapter 5.

During the 2020 Fall Western Sectional Meeting of the AMS, I had the chance to attend the *Special Session on Monoidal Categories in Representation Theory*. I am grateful to the organizers and presenters for helping me understand the broader context of my work.

Moreover, I want to thank everyone who helped me become who I am. In particular, thanks to the *Grande-Rivière* High School, to the scout movement, to the "family mass" parish group, to Fr. Maxime Allard for being my mentor in all life blessings and challenges, and to Fr. John O'Brien for helping me discern my next steps.

Finally, I want to thank all my extended family; you know I love you all. I am particularly grateful to my parents and grandparents, who encouraged me to undertake mathematical studies and always surrounded me with warmth and affection.

Contents

Chapter 1

Introduction

1.1 Aim of the thesis

This thesis considers how to relate universal categories, string diagrams and some algebras used in representation theory. In the examples that interest us — the Temperley-Lieb and Brauer categories, and their oriented counterparts — such concepts can be linked through the notion of generators and relations. Hence, an important figure to keep in mind while reading this thesis is the following, which includes the main fields of mathematics involved in this work.

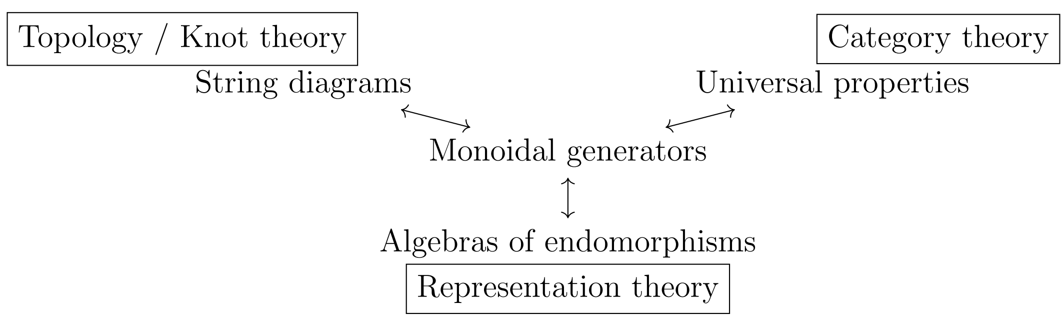


Figure 1.1: Central concepts in this thesis

These concepts are defined and related in the following ways:

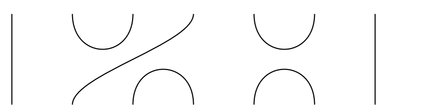
- Algebras of endomorphisms are defined in Definition 2.1.5,
- String diagrams are defined in § 2.3,
- Universal properties are defined in § 5.1,
- Monoidal generators and monoidal presentations are defined in Definition 5.2.5.
- The arrow "Monoidal generators → Universal properties" is described in § 5.2.

- The arrow "Universal properties → Monoidal generators" is partially described in § 5.3
- The arrow "Algebras of endomorphisms → Monoidal generators" is treated in [Eas20, § 2], in the sense that he shows how to go from generators of the algebras to generators of the monoidal category. However this treatment considers only monoidal categories generated by a single object, which we call the "unoriented case."
- The remaining arrows are partially illustrated in the next section and in Chapter 4.

1.2 A tale about generators: topology, representation theory, and category theory

Here is a story involving a topologist, a representation theorist, and a category theorist, each arriving at the same point but from different backgrounds.

The topologist is interested in braids, tangles and knot theory. One of the questions she asks herself is "What would happen if I considered string diagrams like



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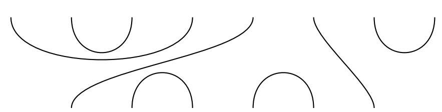
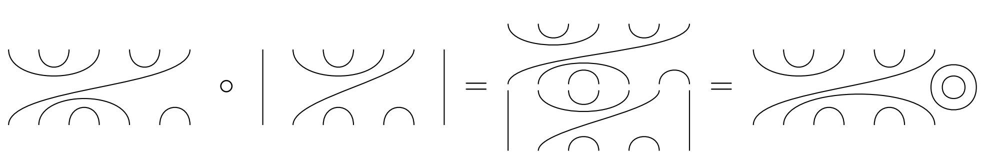
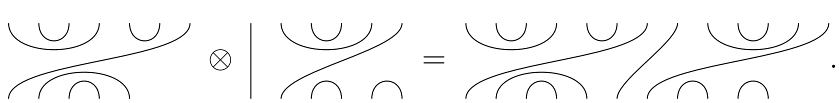


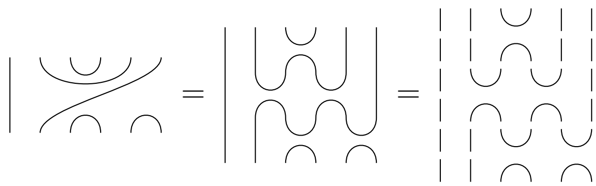
and glued them one on top of the other?" More precisely, she could define some type of diagram, which we will call "Temperley-Lieb diagram from n to m", as the diagrams such that n endpoints at the bottom and m endpoints at the top are linked pairwise by non-crossing strands. Then she could say that we compose diagrams by gluing them together vertically when the number of endpoint matches, as in



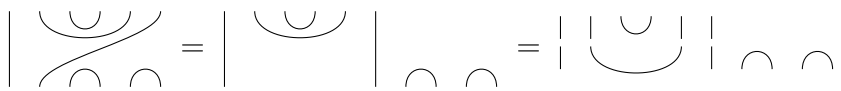
She could also allow horizontal juxtaposition, which we denote by the symbol $\otimes$, as in



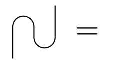
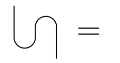
A natural question is to know if there exist some basic building blocks generating all Temperley-Lieb diagrams. For instance, is it true that we can form all of them with just $|$, $\cup$ and $\cap$? If our topologist agrees that two Temperley-Lieb diagrams are the same whenever the same pairs of endpoints are linked — independently of some continuous deformation — the answer is yes. Indeed, through such continuous deformations we get for instance



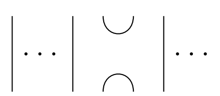
or simply



depending on taste. There are obviously some technicalities to check — see Proposition 4.2.9 for precise statement and reference — but this topologist is pleased to know that all the diagrams she was considering are generated by $|$, $\cup$, $\cap$ and the relations

 and .

The representation theorist is interested in the representations of the orthogonal group O_n. He learns that the Temperley-Lieb algebras can prove useful for this (see §4.6). Nowadays, it is common to describe the *Temperley-Lieb algebra on n strands* $\mathcal{TL}_n(\delta)$ as the vector space (or free module) whose basis consists of Temperley-Lieb diagrams from n to n, where δ is some chosen number. Composition is same as above, with the rule that each bubble is replaced by a factor of δ. To better understand these algebras, he looks for generators, and sees that in modern notation we can describe the generators of $\mathcal{TL}_n(\delta)$ as



(n strands with the cup and cap linking the positions i and $i+1$) with some relations. After understanding that he must work with n generators and $(n^2+3n-2)/2$ relations for every $n \in \mathbb{N}$, he learns that the Temperley-Lieb category exists. In the *Temperley-Lieb category* $\mathcal{TL}(\delta)$, we consider the vector space having as basis the Temperley-Lieb diagrams from n to m, *for every* $n, m \in \mathbb{N}$. With this, we can use the same horizontal juxtaposition as the topologist. Such a category seems at first more complicated, but our representation theorist is pleased to learn that these extra structures reduce the number of generators needed. It is well-known that all the generators and relations in Temperley-Lieb algebras can be obtained from only $|$, $\cup$, $\cap$, $\cap\cup = |$, $\cup\cap = |$

and $\bigcirc = \delta$. In fact, [Eas20, § 2] provides a method for going from generators of an algebra to generators of a category with operation $\otimes$ in many cases.

Finally, a category theorist is interested in generalizing results and structures from linear algebra. In particular, she likes monoidal categories and dual objects, which are respectively the generalization of the tensor product of vector spaces and of dual vector spaces (these notions are detailed in Definition 2.2.10 and Definition 2.5.1). She also knows that free structures and universal properties can often give valuable insights (standard examples are in § 5.1, with a technical definition in Definition 5.1.1). Hence, she asks herself "what is the universal monoidal category on a self-dual object of dimension δ?" As the others, she looks for generators. Her guess is that if she takes the minimum number of generators and relations imposed by the property "being a self-dual object of dimension δ", this will give her the right universal/free category. By Definition 2.5.2, an object $|$ is self-dual if and only if there exists maps $\cup$ and $\cap$ satisfying $\cap\!\cup = |$, $\cup\!\cap = |$, and by Definition 2.9.4 it has dimension δ if and only if $\bigcirc = \delta$. Hence our category theorist guesses that the generators of her universal category are an object $|$ with morphisms $\cup$ and $\cap$, subject to the relations $\cap\!\cup = |$, $\cup\!\cap = |$ and $\bigcirc = \delta$. It takes some verifications to make sure this works — see [Abr08, Proposition 1.1] or our Proposition 4.2.10 — but her guess proves right.

Therefore, we have three mathematicians with different goals who found the same generators, and are hence considering the same mathematical structure, called the *Temperley-Lieb category*. The morale of the story is that the various elements in Figure 1.1 can often be related by generators and relations, to give an unified understanding of the categories considered.

1.3 Contributions to knowledge

The concept of universal categories and related notions are now standard tools for a handful of experts. For many mathematicians, however, the concepts we introduce in this thesis are unfamiliar. This work aims to fill this gap, by providing an introduction to universal monoidal categories. We do this by collecting relevant results, comparing the main universal categories in use, and filling some gaps. As far as we know, the following results and definitions did not appear elsewhere: 4.3.6–4.3.11, 4.4.11, 4.5.11, 5.2.5–5.2.8, and 5.3.1–5.3.3. The aim is both to provide a precise reference to help future works, and to assist the beginner in developing an intuition about this subject.

1.4 Organization of this thesis

In Chapters 2 and 3 we cover preliminaries, including monoidal and linear categories, string diagrams, dual objects, self-dual objects, pivotal categories, dimensions, and symmetric monoidal categories. In Chapter 4 we consider the Temperley-Lieb, oriented Temperley-Lieb, Brauer, and oriented Brauer categories. For each we discuss their string diagrams, generators, universal property, and some of their representation theory. In Chapter 5 we examine how to go from generators to universal properties and vice-versa; we use adjoint functors to obtain general results.

As with many mathematical theses, no previous knowledge is assumed from the reader, except for mathematical maturity and some linear algebra. However, sections 2.1 and 2.2 can be rather dry for readers who do not know category theory. Such readers may prefer to skip directly to § 2.3, and accept that there exist "monoidal categories" such that string diagrams form a precise mathematical notation.

Chapter 2

Monoidal categories and dual objects

In this chapter and the following, we set the stage for Chapters 4 by introducing string diagrams and various notions related to monoidal categories. We believe that the material described in these preliminaries is interesting in its own right, so we motivate the definitions of monoidal categories and dual objects by using the category of vector spaces. We also illustrate some of the definitions with categories of sets and of representations of groups. We refer the reader to [ML98] and [Sel11] for further reading about this chapter and the next.

2.1 Categories

In category theory, the approach is to consider morphisms — also called maps, arrows, or sometimes functions. These morphisms should go from one "object" (set or space) to another, we should be able to compose them in an associative way, and we want identity morphisms.

Definition 2.1.1 (Category)**.** A *category* $\mathscr{C}$ consists of

- Objects A, B ...
- Morphisms f, g, h ...
- Domain and codomain assignments: we denote $f\colon A \to B$ to say $\mathrm{dom}(f) = A$, $\mathrm{cod}(f) = B$,
- Composition: a map $\circ$ that assigns to each pair of morphisms f, g satisfying $\mathrm{cod}(f) = \mathrm{dom}(g)$ a morphism $g \circ f\colon \mathrm{dom}(f) \to \mathrm{cod}(g)$,

such that

- Identity: For each object A there exists a morphism $1_A\colon A \to A$ such that
$$f \circ 1_A = f = 1_B \circ f \qquad \text{for any } f\colon A \to B,$$
- Associativity $(h \circ g) \circ f = h \circ (g \circ f)$ for all morphisms f, g, h with $\mathrm{cod}(f) = \mathrm{dom}(g)$, $\mathrm{cod}(g) = \mathrm{dom}(h)$.

We write $\mathrm{ob}\,(\mathscr{C}) = \{A \text{ object in } \mathscr{C}\}$ and $\mathrm{Hom}_{\mathscr{C}}(A, B) = \{f\colon A \to B \text{ in } \mathscr{C}\}$, calling $\mathrm{Hom}_{\mathscr{C}}(A, B)$ a *hom-set*. We sometimes abuse notation by writing $A \in \mathscr{C}$ instead of $A \in \mathrm{ob}\,(\mathscr{C})$.

When a morphism $f\colon A \to B$ has an inverse $g\colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$, we say that f is an *isomorphism* and that A and B are *isomorphic*.

Example 2.1.2. There are plenty of standard examples of categories. Some of them are

- **Set** where objects are sets and morphisms are functions,
- $\mathbf{Vect}_{\Bbbk}$ where objects are vector spaces and morphisms are linear maps (for some fixed field $\Bbbk$),
- **Top** where objects are topological spaces and morphisms are continuous maps,
- manifolds with smooth functions,
- the categories **Grp**, **Ring**, R–**mod** and R–**Alg** of groups/rings/R-modules/R-algebras and their respective homomorphisms.

In this chapter we will mainly focus on $\mathbf{Vect}_{\Bbbk}$, since this category has all the structures we will be considering.

When considering only morphisms and the operation of composition, a lot can be done. However we can have more tools if we add structures to our categories. A first interesting structure mimics the fact that in $\mathbf{Vect}_{\Bbbk}$ we can add linear maps and multiply them by scalars, so that $\mathrm{Hom}(A, B)$ is itself a vector space.

Throughout this document, R is a commutative ring with unity.

Definition 2.1.3 (Linear category)**.** An *R-linear category* is a category such that

- $\mathrm{Hom}(A, B)$ is an R-module for all objects A, B,
- composition $\circ$ is R-bilinear.

Example 2.1.4. The categories $\mathbf{Vect}_{\Bbbk}$, R–**mod** and R–**Alg** are all linear. This means that we can add morphisms or multiply by scalars, so it makes sense to consider
$$2f + 9g, \qquad (3f + 4g) \circ (5h) = 15f \circ h + 20g \circ h, \qquad 0\,(f \circ g \circ h) = 0, \quad \text{etc.}$$

Definition 2.1.5 (Algebras of endomorphisms)**.** Let $\mathscr{C}$ be a linear category. Then for any object $A \in \mathrm{ob}\,\mathscr{C}$, the hom-set $\mathrm{End}_{\mathscr{C}}(A) := \mathrm{Hom}_{\mathscr{C}}(A, A)$ is an algebra when endowed with the R-linear structure and the composition. This is called an *algebra of endomorphisms*.

2.2 Monoidal categories

Linearity is nice, but it considers each hom-set separately. In many of our examples there exists operations between different hom-sets, for example tensor products, direct sums, disjoint unions... We examine these examples to axiomatize such an extra structure.

Example 2.2.1. Consider the category of vector spaces with the tensor product $\otimes$. The tensor product on objects is associative up to isomorphism,

$$(U \otimes V) \otimes W \simeq U \otimes (V \otimes W) \quad \text{for all vector spaces } U, V, W. \tag{2.2.1}$$

Moreover, the field $\Bbbk$ acts here as an identity element, since

$$\Bbbk \otimes V \simeq V \simeq V \otimes \Bbbk \quad \text{for each vector space } V. \tag{2.2.2}$$

Hence we see that the triple $(\mathbf{Vect}_\Bbbk, \otimes, \Bbbk)$ almost forms a monoid, by replacing equalities with isomorphisms.

Definition 2.2.2 (Monoid)**.** A *monoid* is a triple $(M, \bullet, e)$ with a set M, an operation $\bullet\colon M \times M \to M$ and an *identity element* e such that

$$(a \bullet b) \bullet c = a \bullet (b \bullet c) \quad \text{and}$$
$$a \bullet e = a = e \bullet a\ .$$

In category theory, it is often acceptable to replace equalities by isomorphisms, if we have "nice" isomorphisms, which are called *natural isomorphisms*. For vector spaces saying that an isomorphism is "nice" often amounts to saying that it is basis-independent, but in full generality we first need the concept of functor.

Definition 2.2.3 (Functor)**.** For categories $\mathscr{C}$ and $\mathcal{D}$, a (covariant) *functor* $F\colon \mathscr{C} \to \mathcal{D}$ is composed of maps

$$F\colon \operatorname{ob}(\mathscr{C}) \to \operatorname{ob}(\mathcal{D}) \quad \text{and}$$

$$F\colon \operatorname{Hom}_{\mathscr{C}}(A, B) \to \operatorname{Hom}_{\mathcal{D}}(F(A), F(B)) \quad \forall A, B \in \operatorname{ob}\mathscr{C},$$

satisfying $F(1_A) = 1_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$.

We often write $FA = F(A)$ and $Ff = F(f)$.

Contravariant functors are defined in the same way, except that F exchanges domains and codomains and $F(g \circ f) = F(f) \circ F(g)$. In this thesis, by "functor" we will mean "covariant functor".

Definition 2.2.4 (Natural transformation)**.** For $\mathscr{C}$ and $\mathcal{D}$ categories and $F\colon \mathscr{C} \to \mathcal{D}$, $G\colon \mathscr{C} \to \mathcal{D}$, a *natural transformation* η from F to G, written $\eta\colon F \to G$, is a family of morphisms that satisfy the following two requirements.

1. The natural transformation must associate every object A in $\mathscr{C}$ to a morphism $\eta_A\colon F(A) \to G(A)$ between objects of $\mathcal{D}$. The morphism η_A is called the *component* of η at A.
2. Components must be such that for every morphism $f\colon A \to A'$ in $\mathscr{C}$ we have

$$\eta_{A'} \circ F(f) = G(f) \circ \eta_A \,,$$

which means that we have the commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(A') & \xrightarrow{\eta_{A'}} & G(A') \end{array} .$$

We can also say that the family of morphisms $\eta_A\colon F(A) \to G(A)$ is natural in A.

A natural transformation η is called a *natural isomorphism* if each component η_A is an isomorphism in $\mathcal{D}$.

Example 2.2.5. We continue Example 2.2.1 by showing that the isomorphisms (2.2.1) and (2.2.2) are natural. We first look at the isomorphism $\Bbbk \otimes V \simeq V$. To show that we have a natural isomorphism, we identify the functors F and G where

$$F\colon \mathbf{Vect}_\Bbbk \to \mathbf{Vect}_\Bbbk, \quad V \mapsto \Bbbk \otimes V, \quad (f\colon V \to V') \mapsto (1_\Bbbk \otimes f\colon \Bbbk \otimes V \to \Bbbk \otimes V')\,,$$

while $G\colon \mathbf{Vect}_\Bbbk \to \mathbf{Vect}_\Bbbk$ is the identity functor. Then, by defining

$$\eta_V\colon \Bbbk \otimes V \to V, \quad a \otimes v \mapsto av\,,$$

we satisfy the two conditions of Definition 2.2.4.

1. For every $V \in \mathrm{ob}\,(\mathbf{Vect}_\Bbbk)$, we have a linear map $\eta_V\colon F(V) \to G(V)$.
2. For every linear map $f\colon V \to V'$ we have

$$\begin{aligned}(\eta'_V \circ F(f))(a \otimes v) &= \eta'_V(1_\Bbbk \otimes f(a \otimes v)) = \eta'_V(a \otimes f(v)) = af(v) \\ &= f(av) = f(\eta_V(a \otimes v)) = (G(f) \circ \eta_V)(a \otimes v)\,.\end{aligned}$$

Similarly, the isomorphism $V \simeq V \otimes \Bbbk$ is natural in V with

$$\eta_V\colon V \to V \otimes \Bbbk, \quad v \otimes a \mapsto av\,,$$

and the isomorphism $(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$ is natural in U, V and W via

$$\eta_{U,V,W}\colon (U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w).$$

We can therefore see that we have a "monoid" $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$, where equalities are replaced by *natural isomorphisms*. Even better, we can relate composition $\circ$ and tensor product $\otimes$, in a manner analogous to distributivity $a(b+c) = ab+ac$ in a field (although distributivity as such does not work here).

Take morphisms $f_i \colon U_i \to V_i, g_i \colon V_i \to W_i$, with $v_i \in U_i$, $i = 1, 2$. We have the equality

$$\begin{aligned}(g_1 \otimes g_2) \circ (f_1 \otimes f_2)(v_1 \otimes v_2) &= (g_1 \otimes g_2)(f_1(v_1) \otimes f_2(v_2)) \\ &= g_1(f_1(v_1)) \otimes g_2(f_2(v_2)) \\ &= (g_1 \circ f_1)(v_1) \otimes (g_2 \circ f_2)(v_2) \\ &= (g_1 \circ f_1) \otimes (g_2 \circ f_2)(v_1 \otimes v_2)\,,\end{aligned}$$

which implies the following *interchange law*:

$$(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)\,. \tag{2.2.3}$$

A second, equivalent form, of the interchange law is

$$(f \otimes 1) \circ (1 \otimes g) = f \otimes g = (1 \otimes g) \circ (f \otimes 1). \tag{2.2.4}$$

Before collecting properties of $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$ together in a definition, we check that these properties hold for another standard example.

Example 2.2.6 (Cartesian product on **Set**). Consider the triple $(\mathbf{Set}, \times, \{*\})$, where $\{*\}$ is any singleton (all singletons are isomorphic), with the isomorphisms

$$\begin{gathered}(A \times B) \times C \simeq A \times (B \times C) \\ ((a, b), c) \mapsto (a, (b, c))\end{gathered}$$

and

$$\begin{gathered}\{*\} \times A \simeq A \simeq A \times \{*\} \\ (*, a) \mapsto a \mapsto (a, *)\end{gathered}$$

for all sets A, B and C.

We can show with straightforward computations that these isomorphisms are natural in A, B and C, and that the interchange law

$$(g_1 \times g_2) \circ (f_1 \times f_2) = (g_1 \circ f_1) \times (g_2 \circ f_2)$$

holds.

We are now almost ready to define monoidal categories; we introduce for that the notions of product category and bifunctor.

Definition 2.2.7 (Product category,[Awo10, §1.6.1])**.** For two categories $\mathscr{C}$ and $\mathcal{D}$, their product $\mathscr{C} \times \mathcal{D}$ has objects of the form (C, D) for $C \in \operatorname{ob}(C)$ and $D \in \operatorname{ob}(D)$, and morphisms of the form $(f, g)\colon (C, D) \to (C', D')$ for $f\colon C \to C'$ and $g\colon D \to D'$. Composition and units are defined componentwise; that is

$$(f', g') \circ (f, g) = (f' \circ f, g' \circ g) \quad \text{and} \quad 1_{(C,D)} = (1_C, 1_D).$$

Definition 2.2.8 (Bifunctor)**.** A bifunctor is a functor $F\colon \mathscr{C} \times \mathcal{D} \to \mathcal{E}$ where the domain category is a product category.

Lemma 2.2.9 ([Awo10, Lemma 7.14])**.** *An assignment of objects and morphisms* $-\otimes-\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ *is a bifunctor if and only if*

- $C \otimes -\colon \mathscr{C} \to \mathscr{C}$ *and* $-\otimes C\colon \mathscr{C} \to \mathscr{C}$ *are functors for all objects* C *in* $\mathscr{C}$*, and*
- *the interchange law* (2.2.4) *is satisfied for all morphisms* $f_i\colon A_i \to B_i$ *and* $g_i\colon B_i \to C_i$ *in* $\mathscr{C}$*,* $i = 1, 2$*.*

In that case we also have $1_{A\otimes B} = 1_A \otimes 1_B$*.*

Definition 2.2.10 (Monoidal category)**.** A *monoidal category* $(\mathscr{C}, \otimes, \mathbb{1})$ is a category $\mathscr{C}$ equipped with

- a bifunctor $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the *tensor product* or *monoidal product*, and
- a unit object $\mathbb{1}$,

such that

- $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$ for all objects A, B, C,
- $\mathbb{1} \otimes A \simeq A \simeq A \otimes \mathbb{1}$ for each object A,

where the isomorphisms are *natural* and satisfy the following *coherence conditions*:

1. both ways of going from $((A \otimes B) \otimes C) \otimes D$ to $A \otimes (B \otimes (C \otimes D))$ are equal,
2. both ways of going from $(A \otimes 1) \otimes B$ to $A \otimes B$ are equal.

The coherence conditions are traditionally made precise through the "pentagon" and "triangle" commutative diagrams. These can be found at Definition A.1.

How do we know which coherence conditions to choose? The two conditions above are standard because they are necessary and sufficient to prove Mac Lane's coherence theorem [ML98, Theorem VII.2.1]. This theorem states that any other similar "coherence condition" will also hold, see Theorem A.2 in the appendix.

Examples 2.2.11. There exist many examples of monoidal categories. It is an interesting exercise to check that each of the following satisfy the definition (the first two are Examples 2.2.5 and 2.2.6). Note that with sets we can use the disjoint union but not the normal union, since the union is only an operation on the sets, not on functions.

$$\begin{array}{ll} (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk) & (\mathbf{Set}, \times, \{*\}) \\ (\mathbf{Vect}_{\Bbbk}, \oplus, 0) & (R\text{–}\mathbf{Mod}, \otimes, R) \\ (\mathbf{Set}, \amalg, \emptyset) & (R\text{–}\mathbf{Alg}, \otimes, R) \\ (\mathbf{Ab}, \otimes, \mathbb{Z}) & \ldots \end{array}$$

When considering examples that arise naturally, as in Examples 2.2.11, §2.6 and §2.7, we must use the notion of monoidal categories in its full generality. However, when constructing examples—as in Chapter 4—we are free to impose equalities in place of isomorphisms. In this case we can forget about the coherence conditions entirely, and the monoidal categories we get are called *strict*.

Definition 2.2.12 (Strict monoidal category)**.** A *strict monoidal category* $(\mathscr{C}, \otimes, \mathbb{1})$ is a category $\mathscr{C}$ equipped with

- a bifunctor $\otimes \colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, and
- a unit object $\mathbb{1}$,

such that

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all objects A, B, C,
- $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$ for each object A,
- $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ for all morphisms f, g, h,
- $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$ for each morphism f.

Strict monoidal categories and general monoidal categories are related in a nice way, which involves the concepts of equivalence and monoidal functors.

Definition 2.2.13 (Monoidal functor)**.** If $\mathscr{C}$ and $\mathcal{D}$ are monoidal categories, a functor $F \colon \mathscr{C} \to \mathcal{D}$ is *strict monoidal* if it satisfies

$$F(A \otimes B) = FA \otimes FB, \quad F(f \otimes g) = Ff \otimes Fg \quad \text{and} \quad F\mathbb{1}_{\mathscr{C}} = \mathbb{1}_{\mathcal{D}}.$$

for all objects A, B and morphisms f, g.

There is also a notion of *strong monoidal functor*, where equalities are replaced by natural isomorphisms plus coherence axioms, which can be found in Definition A.3. Note that any strict functor is also strong.

Definition 2.2.14 (Monoidal natural transformation)**.** If F and G are two strict monoidal functors, a natural transformation $\eta\colon F \to G$ is monoidal if $\eta_{\mathbb{1}} = 1_{\mathbb{1}}$ and $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ for all objects A and B.

When F and G are strong monoidal functors, see Definition A.4

Definition 2.2.15 (Equivalence of categories)**.** Categories $\mathscr{C}$ and $\mathcal{D}$ are equivalent if there exist functors $F\colon \mathscr{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathscr{C}$ with natural isomorphisms $\eta_1\colon 1_{\mathscr{C}} \to F \circ G$ and $\eta_2\colon 1_{\mathcal{D}} \to G \circ F$.

Monoidal categories $\mathscr{C}$ and $\mathcal{D}$ are monoidally equivalent if there exist strong monoidal functors $F\colon \mathscr{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathscr{C}$ with monoidal natural isomorphisms $\eta_1\colon 1_{\mathcal{D}} \to F \circ G$ and $\eta_2\colon 1_{\mathscr{C}} \to G \circ F$.

Proposition 2.2.16 ([ML98, Theorem XI.3.1])**.** *Every monoidal category is equivalent to a strict monoidal category via strong monoidal functors.*

In addition to relating monoidal and strict monoidal categories, the concept of monoidal equivalences is used in this thesis to identify distinct objects in a given category. This is explained in Remark 4.1.7 and is used in Propositions 4.3.11 and 4.5.11.

It is interesting to contrast equivalences with isomorphisms of categories. When categories are isomorphic they have all the same categorical properties, which is not the case for equivalent categories.

Definition 2.2.17. Two categories $\mathscr{C}$ and $\mathcal{D}$ are *isomorphic* if there exists functors $F\colon \mathscr{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathscr{C}$ such that $F \circ G = 1_{\mathcal{D}}$ and $G \circ F = 1_{\mathscr{C}}$.

We can combine the concepts of linear category and monoidal category.

Definition 2.2.18 (Linear monoidal category)**.** An *R-linear monoidal category* is a category that is both R-linear and monoidal, and such that $\otimes$ is R-bilinear on morphisms. In such a category, for $\delta \in R$ we will often write δ to mean the morphism $\delta \cdot 1_{\mathbb{1}}$.

A *linear functor* between linear categories is a functor that is linear on morphisms. A *linear monoidal functor* is a functor that is both linear and monoidal (strong or strict).

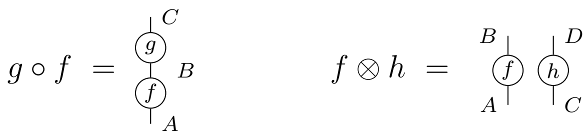
2.3 String diagrams

In monoidal categories we have two operations, composition $\circ$ and monoidal product $\otimes$, which are related through the interchange law (2.2.3) that indicates they both have the same importance. This suggests that we should use two dimensions to write morphisms in these categories, hence use diagrams. We depict a morphism $f\colon A \to B$

by a circle on a line with the letter f inside — which we call a "coupon" — and the identity as a line without coupon, which is to say

(2.3.1)

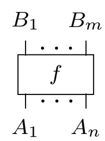
We can indicate composition by stacking morphisms vertically, and monoidal product by juxtaposing horizontally. Note that we read diagrams from bottom to top.



(2.3.2)

We can also represent $f\colon A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_m$ by using multiple strands for the multiple objects, as in

$$f\colon A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_m \;=$$



. (2.3.3)

By the axiom $\mathbb{1} \otimes A \simeq A \simeq A \otimes \mathbb{1}$ and the fact that these isomorphisms are natural, we see that $1_{\mathbb{1}}$ does not change anything, neither when composing nor when tensoring. In fact we can consider that $\mathbb{1} \otimes A = A = A \otimes \mathbb{1}$ and $1_{\mathbb{1}} \otimes f = f = f \otimes 1_{\mathbb{1}}$, by Proposition 2.2.16. Hence we usually leave $\mathbb{1}$ and $1_{\mathbb{1}}$ invisible, or use dotted lines if needed:

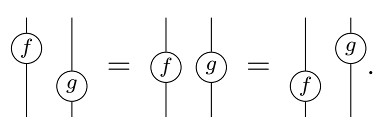
$$1_{\mathbb{1}} = \vdots \text{ or invisible.} \tag{2.3.4}$$

All the diagrams formed from morphism symbols, vertical stacking and horizontal juxtaposition are called *string diagrams*. It is frequent to give specific symbols to important morphisms when such symbols help geometric intuition.

An important reason for the success of string diagrams is the interchange law (2.2.4), which is

$$(f \otimes 1) \circ (1 \otimes g) = f \otimes g = (1 \otimes g) \circ (f \otimes 1).$$

When interpreting this equation in string diagrams, it simply states that we can slide morphism tokens up and down as needed:



(2.3.5)

A second factor that makes string diagrams useful is that they hide the isomorphisms $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$ and $\mathbb{1} \otimes A \simeq A \simeq A \otimes \mathbb{1}$ from Definition 2.2.10.

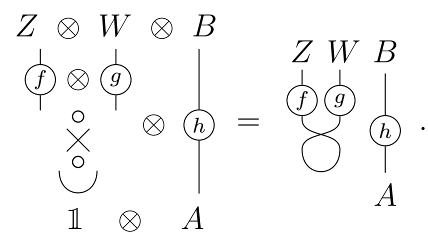
By Mac Lane's coherence theorem (Theorem A.2) and by Theorem 2.2.16, we do not need to mention explicitly these isomorphisms. Hence, string diagrams help us concentrate on what is important.

Example 2.3.1. Suppose we have morphisms

$$\eta\colon \mathbb{1} \to Y \otimes X, \qquad s\colon Y \otimes X \to X \otimes Y, \qquad f\colon X \to Z,$$
$$g\colon Y \to W \qquad \text{and} \qquad h\colon A \to B,$$

and want to combine them as $((f \otimes g) \circ s \circ \eta) \otimes h$. We can remove all the parentheses and operation signs by using string diagrams. Moreover, we can decide to assign some particular symbols when it helps visualize the domains and codomains, for instance with $\eta = \cup$ and $s = \times$. We can then combine all of it and remove the symbols $\circ$ and $\otimes$ to obtain

$$((f \otimes g) \circ s \circ \eta) \otimes h \;=$$



Note that we do not need parenthesis anymore due to the interchange law and Mac Lane's coherence theorem; any way of putting parenthesis in such a diagram would give the same morphism.

Remark 2.3.2. The author of this thesis gained much understanding of string diagrams from the article [Sav21], which gives an overview of various usage of string diagrams for categorification. Another source of inspiration for this thesis is [Sav19], which considers briefly the four main examples of Chapter 4. For more information on string diagrams, a standard reference is [TV17]. An interesting application of string diagrams to the world of quantum information can be found in [CK17].

2.4 Dual vector spaces

One of the most interesting structures we can add to a category is duality. This derives directly from the dual of a vector space, and can be nicely represented with string diagrams.

In this section we work in $\mathbf{FinVect}_{\Bbbk}$, the category of finite-dimensional vector spaces. Fix $V \in \mathbf{FinVect}_{\Bbbk}$ with a basis B. The dual vector space is $V^* = \mathrm{Hom}(V, \Bbbk)$ and it has dual basis $\{\delta_v \colon v \in B\}$, with δ_v being the linear map defined by

$$\delta_v(v') = \begin{cases} 1 & v = v', \\ 0 & v \neq v', \end{cases} \qquad \text{for any } v' \in B.$$

To transform duality into a categorical property, we need to obtain equations of morphisms instead of conditions on elements. Indeed, it is possible to obtain two equations involving the δ_v's, one to get 1_V and the other to get 1_{V^*}.

First, for any $w \in V$, we have $\sum_{v\in B} \delta_v(w)v = w$, which means

$$\left(w \mapsto \sum_{v\in B} \delta_v(w)v \right) = 1_V. \tag{2.4.1}$$

Indeed, by expanding $w = \sum_{v'\in B} a_{v'}v'$ in the basis B and by applying the definition of δ_v, we get

$$\sum_{v\in B} \delta_v \left(\sum_{v'\in B} a_{v'}v' \right) v = \sum_{v,v'\in B} a_{v'}\delta_v(v')v = \sum_{v'\in B} a_{v'}v'. \tag{2.4.2}$$

Moreover, for any $f \in V^*$ we have $\sum_{v\in B} f(v)\delta_v = f$, which means

$$\left(f \mapsto \sum_{v\in B} f(v)\delta_v \right) = 1_{V^*}. \tag{2.4.3}$$

We can see this by evaluating each side on the arbitrary element $\sum_{v'\in B} a_{v'}v' \in V$:

$$\left(\sum_{v\in B} f(v)\delta_v \right) \left(\sum_{v'\in B} a_{v'}v' \right) = \sum_{v'\in B} a_{v'}f(v') = f \left(\sum_{v'\in B} a_{v'}v' \right).$$

The next step is to express the morphisms $w \mapsto \sum_{v\in B} \delta_v(w)v$ and $f \mapsto \sum_{v\in B} f(v)\delta_v$ as some compositions. Define

$$\begin{aligned} &\eta\colon \Bbbk \to V^* \otimes V, \quad 1 \mapsto \sum_{v\in B} \delta_v \otimes v, \quad \text{and} \\ &\epsilon\colon V \otimes V^* \to \Bbbk, \quad v \otimes f \mapsto f(v). \end{aligned} \tag{2.4.4}$$

We then have

$$\begin{gathered} V \simeq V \otimes \Bbbk \xrightarrow{1_V \otimes \eta} V \otimes V^* \otimes V \xrightarrow{\epsilon \otimes 1_V} \Bbbk \otimes V \simeq V, \\ w \mapsto w \otimes 1 \mapsto \sum_{v\in B} w \otimes \delta_v \otimes v \mapsto \sum_{v\in B} \delta_v(w) \otimes v \mapsto \sum_{v\in B} \delta_v(w)v = w, \end{gathered} \tag{2.4.5}$$

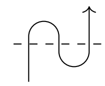
which means $(\epsilon \otimes 1_V) \circ (1_V \otimes \eta)$ can be identified with 1_V.

This is an equation involving only morphisms as desired, and can be turn into a string diagrams equation. If we put $\uparrow = V$ and $\downarrow = V^*$ — and knowing that we don't draw $\Bbbk$ in our string diagrams — it makes sense to relabel

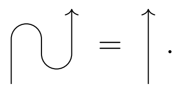
 $= \eta\colon \Bbbk \to \downarrow \otimes \uparrow,$  $= \epsilon\colon \uparrow \otimes \downarrow \to \Bbbk\,.$

We then obtain

$$(\epsilon \otimes 1_V) \circ (1_V \otimes \eta) \;=\; (\frown\!\!\searrow \otimes \uparrow) \circ (\uparrow \otimes \smile\!\!\nearrow) \;=\; \tag{2.4.6}$$



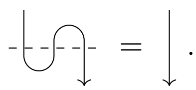
Hence the equation we found is



Using the same $\smile\!\!\nearrow$ and $\frown\!\!\searrow$, we can use the equation $(\, f \mapsto \sum_{v\in B} f(v)\, \delta_v \,) = 1_{V^*}$, to obtain

$$\begin{gathered} V^* \simeq \Bbbk \otimes V^* \xrightarrow{\smile\!\nearrow \otimes 1_{V^*}} V^* \otimes V \otimes V^* \xrightarrow{1_{V^*} \otimes \frown\!\searrow} V^* \otimes \Bbbk \simeq V^*, \\ f \mapsto 1 \otimes f \mapsto \sum_{v\in B} \delta_v \otimes v \otimes f \mapsto \sum_{v\in B} \delta_v \otimes f(v) \mapsto \sum_{v\in B} f(v)\, \delta_v = f. \end{gathered} \tag{2.4.7}$$

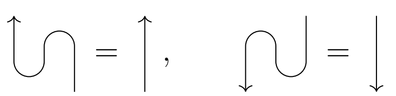
This means that $(1_{V^*} \otimes \frown\!\!\searrow) \circ (\smile\!\!\nearrow \otimes 1_{V^*})$ is the identity on V^*, so



Finally, we can ask ourselves if there is any reason why our zigzags are going from left to right. We find that the reverse is very similar, so we define

$$\begin{aligned} \nwarrow\!\!\smile &: \Bbbk \to V \otimes V^*, \quad 1 \mapsto \sum_{v\in B} v \otimes \delta_v, \quad \text{and} \\ \swarrow\!\!\frown &: V^* \otimes V \to \Bbbk, \quad f \otimes v \mapsto f(v), \end{aligned} \tag{2.4.8}$$

to get



To make sure that these constructions behave nicely, it is pleasing to know that the maps we defined are basis-independent.

Lemma 2.4.1. *The map* $\smile\!\!\nearrow\colon \Bbbk \to V^* \otimes V,\ 1 \mapsto \sum_{v\in B} \delta_v \otimes v$ *is independent of the basis* B.

Proof: It is well-known that $V^* \otimes V$ is isomorphic to $\mathrm{End}(V)$, for instance through

$$\begin{aligned} V^* \otimes V &\xrightarrow{\simeq} \mathrm{End}(V) \\ f \otimes v &\longmapsto (\ w \mapsto f(w) v\ )\ . \end{aligned} \tag{2.4.9}$$

Using this isomorphism, we get that $\sum_{v\in B} \delta_v \otimes v$ corresponds to

$$w \mapsto \sum_{v\in B} \delta_v(w) \otimes v = w$$

which is the identity (by equation (2.4.2)), and this is true for any basis B. ∎

2.5 Dual and self-dual objects

From the linear algebra definition of a dual vector space, we derived equations that could hold in an arbitrary monoidal category, and can be expressed with string diagrams. We can therefore generalize the notion of a "dual object".

Definition 2.5.1. Let $\uparrow$ and $\downarrow$ be two objects in a monoidal category. We say that $\downarrow$ is *right dual* to $\uparrow$ (and $\uparrow$ is *left dual* to $\downarrow$) if there exists morphisms

$$\cup\!\!\nearrow : \mathbb{1} \to \downarrow \otimes \uparrow \quad \text{and} \quad \cap\!\!\searrow : \uparrow \otimes \downarrow \to \mathbb{1}$$

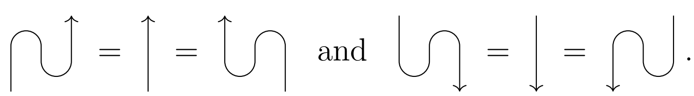
satisfying the *zigzag equations*


(2.5.1)

We then say that $\cup\!\!\nearrow$ is the unit and $\cap\!\!\searrow$ is the counit of this duality

Hence $\uparrow$ and $\downarrow$ are duals on both sides if there also exist morphisms

$$\nwarrow\!\!\cup : \mathbb{1} \to \uparrow \otimes \downarrow \quad \text{and} \quad \swarrow\!\!\cap : \downarrow \otimes \uparrow \to \mathbb{1} \quad \text{such that}$$

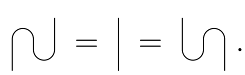

(2.5.2)

In this case, we simply say that $\downarrow$ is the *dual* — or "two-sided dual" — of $\uparrow$. We then say that $\cup\!\!\nearrow$ and $\cap\!\!\searrow$ are the *left* unit and counit, while $\nwarrow\!\!\cup$ and $\swarrow\!\!\cap$ are the *right* unit and counit of $\uparrow$. We can also use the word *cup* for a unit and *cap* for a counit.

When an object is equal to its dual, we do not need to distinguish $\uparrow$ from $\downarrow$, so we can remove orientations and we often denote the object by $|$.

Definition 2.5.2. An object $|$ in a monoidal category is called *self-dual* if it is its own dual, that is if there exists

$$\cup : \mathbb{1} \to | \otimes | \quad \text{and} \quad \cap : | \otimes | \to \mathbb{1} \quad \text{such that}$$


(2.5.3)

Example 2.5.3. Suppose that $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, and consider a finite-dimensional vector space $V \in \operatorname{ob}(\mathbf{Vect}_{\Bbbk})$. We can choose an inner product $\langle \cdot, \cdot \rangle$ on V, and an orthonormal basis B of V. We then see that V is self-dual, since

$$\sum_{v \in B} \langle w, v \rangle v = w. \tag{2.5.4}$$

Indeed, define

$$\cup : \Bbbk \to V \otimes V,\ 1 \mapsto \sum_{v \in B} v \otimes v \quad \text{and} \quad \cap : V \otimes V \to \Bbbk,\ w \otimes v \mapsto \langle w, v \rangle \qquad (2.5.5)$$

to obtain
$$V \cong V \otimes \Bbbk \xrightarrow{1_V \otimes \cup} V \otimes V \otimes V \xrightarrow{\cap \otimes 1_V} \Bbbk \otimes V \cong V, \qquad (2.5.6)$$
$$w \mapsto w \otimes 1 \mapsto \sum_{v \in B} w \otimes v \otimes v \mapsto \sum_{v \in B} \langle w, v \rangle \otimes v \mapsto \sum_{v \in B} \langle w, v \rangle v = w.$$

Here we are using the fact that $\langle v, \cdot \rangle = \delta_v$ since B is orthonormal, which permits us to identify V with its dual. We have an analogue of Lemma 2.4.1.

Lemma 2.5.4. *The map* $\cup : \Bbbk \to V \otimes V,\ 1 \mapsto \sum_{v \in B} v \otimes v$ *is independent of the orthonormal basis* B.

Proof: The proof is the same as for Lemma 2.4.1, with the isomorphism

$$V \otimes V \xrightarrow{\simeq} \mathrm{End}(V) \quad v_1 \otimes v_2 \longmapsto \langle \,\cdot\, , v_1 \rangle v_2\,. \qquad \blacksquare$$

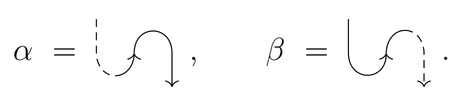
Remark 2.5.5. Even though the category of all $\Bbbk$-vector spaces is a monoidal category, for infinite-dimensional vector spaces we cannot do the same analysis, since

$$\cup\!\!\nearrow : \Bbbk \to V^* \otimes V, \quad 1 \mapsto \sum_{v \in B} \delta_v \otimes v$$

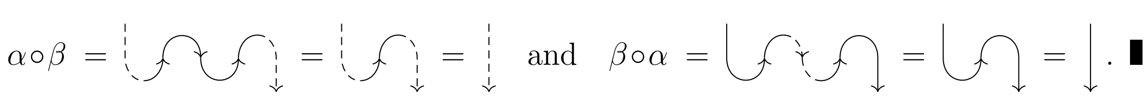
would not be a finite sum and would not be a well-defined element of the tensor product.

Proposition 2.5.6 (Unicity of the dual)**.** *If an object has two right duals (respectively two left duals), these duals are isomorphic.*

Proof: Suppose that $\uparrow$ possesses two right duals $\downarrow$ and $\downarrow'$, with respective cups and caps $\cup\!\!\nearrow$, $\cap\!\!\searrow$, $\cup'\!\!\nearrow$ and $\cap\!\!\searrow'$. Define the maps $\alpha : \downarrow \to \downarrow'$ and $\beta : \downarrow' \to \downarrow$ by



Then α and β are isomorphisms by the zigzag equations of $\downarrow$ and $\downarrow'$:



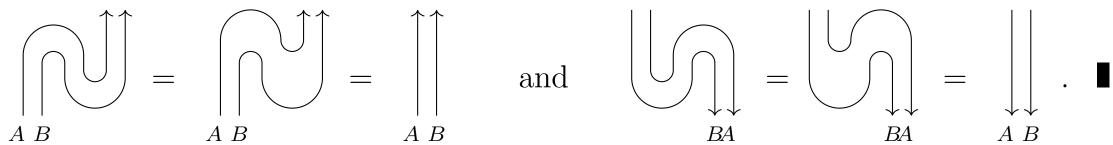
Concerning unicity, we cannot be more precise than "unique up to isomorphism". The next proposition and remark show one instance where more than one choice of dual can be interesting.

Proposition 2.5.7 (Tensor product of duals)**.** *Suppose that A has right dual A^* and B has right dual B^*, with respective cups and caps , , and .*

We can choose $(A \otimes B)^* = B^* \otimes A^*$, with cup and cap

(2.5.7)

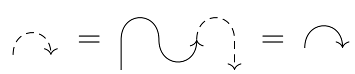
Proof: By the interchange law and the zigzag equations,



Remark 2.5.8. We could also define

and then choose $(A \otimes B) = A^* \otimes B^*$, but this is possible only in a symmetric monoidal category (see Definition 3.2.1). If the category is symmetric monoidal we do have that $B^* \otimes A^*$ is isomorphic to $A^* \otimes B^*$, so this is consistent with Proposition 2.5.6.

Remark 2.5.9. Even when a pair of dual objects is specified, we can have multiple choices of units and counits. For instance, if we are in a linear category and $c \in R$ is invertible, then we can replace , by c and c^{-1}, and they still satisfy the zigzag equations. However, if we choose a unit, then the counit is uniquely determined, and vice-versa. Indeed, if is a chosen unit and , are two counits, then by their respective zigzag equations we get



2.6 Example: Representations

Fix a group G and a field $\Bbbk$. The category of all finite-dimensional $\Bbbk$-representations of G forms a monoidal category, and every representation has a dual representation in that category.

Recall that for V a $\Bbbk$-vector space, the *general linear group* $\mathrm{GL}(V)$ consists of all the invertible linear maps going from V to V,

$$\mathrm{GL}(V) = \{f \in \mathrm{Hom}(V, V) \mid f \text{ invertible}\}. \tag{2.6.1}$$

Definition 2.6.1. A $\Bbbk$-*representation* of a group G is a $\Bbbk$-vector space V with a group homomorphism from G to $\mathrm{GL}(V)$. Hence it is a pair (V,ρ) where $\rho\colon G\to\mathrm{GL}(V)$ satisfies $\rho(g_1g_2)=\rho(g_1)\rho(g_2)\ \forall g_1,g_2\in G$.

For (V_1,ρ_1) and (V_2,ρ_2) representations of G, a morphism $T\colon(V_1,\rho_1)\to(V_2,\rho_2)$ is a linear map $T\colon V_1\to V_2$ satisfying

$$\forall g\in G,\quad \rho_2(g)\circ T=T\circ\rho_1(g).$$

Such a map is called an *intertwiner*.

The automorphism $\rho(g)$ is sometimes called the *action* of g, and we can write $g\cdot v=\rho(g)v$ if there is no risk of confusion between various actions.

It is clear that the identity map $1_V\colon V\to V$ is an intertwiner $(V,\rho)\to(V,\rho)$ for any ρ. If we have two intertwiners $T\colon(V_1,\rho_1)\to(V_2,\rho_2)$ and $S\colon(V_2,\rho_2)\to(V_3,\rho_3)$, the usual composition of functions $S\circ T$ is indeed an intertwiner $(V_1,\rho_1)\to(V_3,\rho_3)$, since for any $g\in G$

$$\rho_3(g)\circ(S\circ T)=S\circ\rho_2(g)\circ T=S\circ T\circ\rho_1(g).$$

Definition 2.6.2. For a fixed group G and field $\Bbbk$, we denote by $\mathrm{Rep}_{\Bbbk}(G)$ the category where objects are finite-dimensional $\Bbbk$-representations of G and morphisms are intertwiners; using the usual composition of linear maps and identity.

On this category we can define a tensor product

$$(V_1,\rho_1)\otimes(V_2,\rho_2)=(V_1\otimes V_2,\rho_1\otimes\rho_2)$$

with

$$(\rho_1\otimes\rho_2)(g)(v_1\otimes v_2)=\rho_1(g)v_1\otimes\rho_2(g)v_2\quad\forall g\in G, v_i\in V_i.$$

We also have a unit object $(\Bbbk,\phi)$ called the *trivial representation*, which sends every group element to the identity on $\Bbbk$: for any $g\in G$, $\phi(g)=1_{\Bbbk}$.

Lemma 2.6.3. $(\mathrm{Rep}_{\Bbbk}(G),\otimes,(\Bbbk,\phi))$ *is a monoidal category.*

Proof: The interchange law is still verified pointwise. We can use the same isomorphisms as for vector spaces,

$$\big((V_1,\rho_1)\otimes(V_2,\rho_2)\big)\otimes(V_3,\rho_3)\simeq(V_1,\rho_1)\otimes\big((V_2,\rho_2)\otimes(V_3,\rho_3)\big)$$
$$(v_1\otimes v_2)\otimes v_3\mapsto v_1\otimes(v_2\otimes v_3)$$

$$(\Bbbk,\phi)\otimes(V,\rho)\simeq(V,\rho)\simeq(V,\rho)\otimes(\Bbbk,\phi)\ .$$
$$1\otimes a\mapsto a\mapsto a\otimes 1$$

It is only necessary to check that these isomorphisms are intertwiners and are natural, and that they satisfy the coherence conditions. These are straightforward computations. ∎

We can now define dual representations; this is where we need the representations to have finite dimension.

Definition 2.6.4. If $(V, \rho) \in \mathrm{Rep}_{\Bbbk}(G)$, the *dual representation* (V^*, ρ^*) consists in $V^* = \mathrm{Hom}(V, \Bbbk)$ and for any $g \in G$,

$$(\rho^*(g)f)(v) = f(\rho(g)^{-1}v) \quad \forall f \in V^*, v \in V.$$

The inverse is necessary for this to be a representation: for any $g_1, g_2 \in G$ we have $\rho^*(g_1)\rho^*(g_2) = \rho^*(g_1 g_2)$ since

$$\begin{aligned}(\rho^*(g_1)\rho^*(g_2)f)(v) &= (\rho^*(g_2)f)(\rho(g_1)^{-1}v) = f(\rho(g_2)^{-1}\rho(g_1)^{-1}v) \\ &= f(\rho(g_1 g_2)^{-1}v) = (\rho^*(g_1 g_2)f)(v) \quad \forall f \in V^*, v \in V.\end{aligned}$$

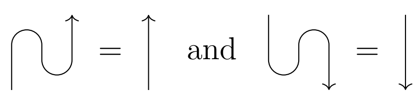
Proposition 2.6.5. *For any finite-dimensional representation (V, ρ) of G, the dual representation (V^*, ρ^*) is left and right dual to (V, ρ).*

Proof: Let us consider the same unit and counit as before,

 $: (\Bbbk, \phi) \to (V^*, \rho^*) \otimes (V, \rho), \quad 1 \mapsto \sum_{v \in B} \delta_v \otimes v,$ and

 $: (V, \rho) \otimes (V^*, \rho^*) \to (\Bbbk, \phi), \quad v \otimes f \mapsto f(v),$

for B a basis of V. If these are intertwiners, then we know by §2.4 that they do satisfy



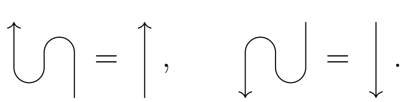
for $\uparrow = (V, \rho)$, $\downarrow = (V^*, \rho^*)$.

Similarly, if

 $: (\Bbbk, \phi) \to (V, \rho) \otimes (V^*, \rho^*), \quad 1 \mapsto \sum_{v \in B} v \otimes \delta_v,$ and

 $: (V^*, \rho^*) \otimes (V, \rho) \to (\Bbbk, \phi), \quad f \otimes v \mapsto f(v),$

are intertwiners, we get



We show that the first two are intertwiners to establish that (V^*, ρ^*) is indeed the dual of (V, ρ); the last two can be treated similarly. It is useful to use basis independence from Lemma 2.4.1.

Let $g \in G$. To show that $\curvearrowright$ is an intertwiner, we want to show that

$$\curvearrowright \circ (\rho \otimes \rho^*)(g) = \phi(g) \circ \curvearrowright$$

(ϕ is the trivial representation $g \mapsto 1_{\Bbbk}$). The left-hand side is

$$v \otimes f \mapsto \rho(g)v \otimes \rho^*(g)f = \rho(g)v \otimes (w \mapsto f(\rho(g)^{-1}w)) \mapsto f(\rho^{-1}(g)\rho(g)v) = f(v),$$

while the right-hand side is

$$v \otimes f \mapsto f(v) \mapsto \phi(g)f(v) = f(v),$$

so it is indeed an intertwiner.

To show that $\smile\!\!\nearrow$ is an intertwiner, we want to show that

$$\smile\!\!\nearrow \circ \, \phi(g) = (\rho^* \otimes \rho)(g) \circ \smile\!\!\nearrow.$$

The left-hand side is

$$1 \mapsto \phi(g)1 = 1 \mapsto \sum_{v \in B} \delta_v \otimes v,$$

while the right-hand side is

$$1 \mapsto \sum_{v \in B} \delta_v \otimes v \mapsto \sum_{v \in B} \rho^*(g)\delta_v \otimes \rho(g)v \mapsto \sum_{v \in B} (w \mapsto \delta_v(\rho(g)^{-1}w)) \otimes \rho(g)v.$$

These maps are the same since

$$\sum_{v \in B} \delta_v \otimes v = \sum_{v \in B} (w \mapsto \delta_v(\rho(g)^{-1}w)) \otimes \rho(g)v,$$

which can be seen by applying to each side the isomorphism (2.4.9) between $V^* \otimes V$ and $\mathrm{End}(V)$. Since $\rho(g)$ is invertible, we know that $B' = \{\rho(g)v \mid v \in B\}$ is also a basis of V. We can therefore write any arbitrary $w \in V$ as $w = \sum_{v' \in B} a_{\rho(g)v'} \rho(g)v'$, and then this isomorphism applied to $\sum_{v \in B} (w \mapsto \delta_v(\rho(g)^{-1}w)) \otimes \rho(g)v$ gives

$$\begin{aligned} w \mapsto & \sum_{v \in B} \delta_v(\rho(g)^{-1}w)\rho(g)v \\ =& \sum_{v \in B} \delta_v(\rho(g)^{-1} \sum_{v' \in B} a_{\rho(g)v'} \rho(g)v')\rho(g)v \\ =& \sum_{v \in B} a_{\rho(g)v} \rho(g)v \\ =& w, \end{aligned}$$

which is the identity (hence the same as the isomorphism applied to $\sum_{v\in B}\delta_v\otimes v$). This concludes the proof that (V^*,ρ^*) is indeed the dual of (V,ρ). ∎

Proposition 2.6.6. *Let (V,ρ) be a finite-dimensional real or complex representation of G, where V is endowed with an inner product $\langle\cdot,\cdot\rangle$. Suppose that ρ is a* unitary representation*, meaning that it preserves the inner product:*

$$\forall g\in G, \forall v,w\in V,\ \langle\rho(g)w,\rho(g)v\rangle=\langle w,v\rangle.$$

Then (V,ρ) is self-dual.

Proof: We use the same unit and counit as in Example 2.5.3, with $\Bbbk=\mathbb{R}$ or $\mathscr{C}$,

$$\smile : \Bbbk\to V\otimes V,\ 1\mapsto\sum_{v\in B}v\otimes v \quad\text{and}\quad \frown : V\otimes V\to\Bbbk,\ w\otimes v\mapsto\langle w,v\rangle.$$

As before we only need to show that these two maps are intertwiners, and this will prove that (V,ρ) is self-dual, since we know by Example 2.5.3 that the zigzag equations are satisfied.

For the counit, we want to show that for any $g\in G$,

$$\phi(g)\circ\frown=\frown\circ(\rho\otimes\rho)(g).$$

(ϕ is the trivial representation $g\mapsto 1_{\Bbbk}$). The left-hand side is

$$w\otimes v\mapsto\langle w,v\rangle\mapsto\langle w,v\rangle,$$

while the right-hand side is

$$w\otimes v\mapsto\rho(g)w\otimes\rho(g)v\mapsto\langle\rho(g)w,\rho(g)v\rangle=\langle w,v\rangle$$

since ρ preserves the inner product.

For the unit, we want to show that for any $g\in G$,

$$\smile\circ\phi(g)=(\rho\otimes\rho)(g)\circ\smile.$$

The left-hand side is

$$1\mapsto 1\mapsto\sum_{v\in B}v\otimes v,$$

while the right-hand side is

$$1\mapsto\sum_{v\in B}v\otimes v\mapsto\sum_{v\in B}\rho(g)v\otimes\rho(g)v.$$

We conclude by noting that $\sum_{v\in B}v\otimes v=\sum_{v\in B}\rho(g)v\otimes\rho(g)v$ by Lemma 2.5.4, since here $B'=\{\rho(g)v\mid v\in B\}$ is an orthonormal basis of V, by unitarity of ρ. ∎

Example 2.6.7. Consider the general linear group $\mathrm{GL}_n(\Bbbk)$ that consists of all invertible $n \times n$ matrices A with entries in $\Bbbk$. Take its defining representation $V = \Bbbk^n$, with action $A \cdot v = Av$. Its dual is $V^* = \mathrm{Hom}(\Bbbk^n, \Bbbk)$ with action:

$$\text{for } f \in V^*, A \in \mathrm{GL}_n(\Bbbk), \quad (A \cdot f)(v) = f(A^{-1}v)\ \forall v \in V.$$

The orthogonal group is

$$\mathrm{O}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^T A = AA^T = I\} \tag{2.6.2}$$
$$= \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \langle Av, Aw\rangle = \langle v, w\rangle\ \forall v, w \in \mathbb{R}^n\},$$

where $\langle v, w\rangle = v^T w$ is the standard inner product on $\mathbb{R}^n$. Since by definition it consists of matrices preserving the inner product, its defining representation $V = \mathbb{R}^n$ (with action $A \cdot v = Av$) is self-dual.

2.7 Example: Relations

It is possible to have categories with the same objects but with very different properties, by only changing the morphisms. For instance, in **Set** there are no dual objects (see Lemma 2.7.3 below). However, if we take our objects to be sets but our morphisms to be relations instead of functions, we then get that every set is self-dual. To understand the composition of relations, first recall the definition of functions.
Recall: A function $f\colon A \to B$ is a subset $f \subseteq A \times B$ satisfying

$$\forall a \in A\ \exists!\, b \in B \text{ such that } (a, b) \in f\ . \tag{2.7.1}$$

For $f \subseteq A \times B$, $g \subseteq B \times C$, the composite $g \circ f \subseteq A \times C$ is defined by

$$(a, c) \in g \circ f \text{ iff there exists } b \in B \text{ such that } (a, b) \in f, (b, c) \in g. \tag{2.7.2}$$

The function notation usually consists in writing $b = f(a)$ or $f\colon a \mapsto b$ instead of $(a, b) \in f$, in which case the composition condition is

$$c = g(f(a)) \text{ iff } \exists\, b \in B \text{ such that } b = f(a), g(b) = c.$$

However for relations we will keep the subset notation.

Definition 2.7.1 (Category of relations)**.** The category **Rel** consists of:

- Objects: Sets A, B, $C, \dots,$
- Morphisms: Relations $R \subseteq A \times B$,

- Composition: For $R \subseteq A \times B$ and $S \subseteq B \times C$, the composite $S \circ R \subseteq A \times C$ satisfies

$$(a, c) \in S \circ R \text{ iff there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S,$$

- Identity as with functions: $1_A \subseteq A \times A$ satisfies $(a, b) \in 1_A$ iff $a = b$.

Lemma 2.7.2. *The category* **Rel** *is monoidal when using the cartesian product* $\times$*, with unit object being the singleton* $\{*\}$*.*

Proof: It suffices to verify each axiom of Definition 2.2.10, using the isomorphisms in Example 2.2.6 and straightforward computations. ∎

Lemma 2.7.3. *Let A and B be sets. Suppose that B is right dual to A in the monoidal category* $(\mathbf{Set}, \times, \{*\})$*. Then A and B are singletons.*

Proof: Duality in the category **Set** means there are functions $\cup^{A} \colon \{*\} \to B \otimes A$ and ${}_{A}\cap \colon A \otimes B \to \{*\}$ such that ${}_{A}\cap\cup = {}_{A}|$. Since these are functions, we must have $\cup^{A}(*) = (b_0, a_0)$ for some fixed $b_0 \in B, a_0 \in A$, and ${}_{A}\cap(a, b) = *$ for all $(a, b) \in A \times B$. Hence, for any $a \in A$, we have

$${}_{A}\cap\cup \colon a \mapsto (a, *) \mapsto (a, b_0, a_0) \mapsto (*, a_0) \mapsto a_0.$$

When the zigzag is equal to 1_A, this means that $a = a_0$ for all $a \in A$, so A is a singleton. The same argument with $\cup\cap = |$ shows that B is a singleton. ∎

Proposition 2.7.4. *In* **Rel***, every set is self-dual.*

Proof: For A a set, define

$$\cup \subseteq \{*\} \times (A \times A), \qquad \Big(*, (a, a)\Big) \in \cup \quad \forall a \in A,$$

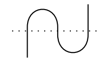
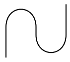
$$\cap \subseteq (A \times A) \times \{*\}, \qquad \Big((a, a), *\Big) \in \cap \quad \forall a \in A.$$

Then

$$(a, a') \in \cap\cup$$

$$\Leftrightarrow \ \exists\, (b, c, d) \in A \times A \times A \text{ such that}$$

$$\Big((a,*),(b,c,d)\Big) \in \mid\cup \quad \text{and} \quad \Big((b,c,d),(*,a')\Big) \in \cap\mid,$$
$$\Leftrightarrow\ a=b,\ c=d \quad \text{and} \quad b=c,\ d=a'.$$

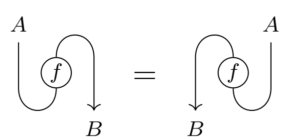
This implies $a=b=c=d=a'$. Since $(a,a') \in$  $\Leftrightarrow$ $a=a'$, we proved  $= \big|$. The other equation is proven similarly. ∎

Remark 2.7.5. The zigzag equation in the category of finite-dimensional Hilbert spaces can be used to describe *quantum teleportation* ([BBC$^+$93]), a physical phenomenon important in quantum information. When considering the zigzag equation in the category **Rel**, we can obtain a description of the *one-time pad* which is a cryptographical primitive. These applications of duality are described, for instance, in [CK17, p. 137-140].

2.8 Pivotal categories

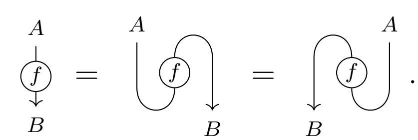
In this section and the next, objects and morphisms are taken in a strict monoidal category.

Definition 2.8.1. For any $f\colon A \to B$, if A and B have duals and the condition



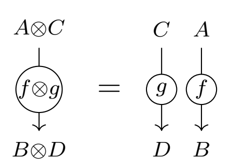
(2.8.1)

is satisfied, we say that the *mate* of f is well-defined and is

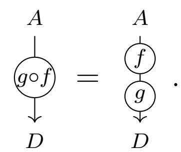


A monoidal category is *strictly pivotal* if each object A has a dual A^* such that $A^{**}=A$, $\mathbb{1}^*=\mathbb{1}$, $(A\otimes B)^* = B^*\otimes A^*$ with cups and caps satisfying (2.5.7), and every morphism f satisfies the condition (2.8.1).

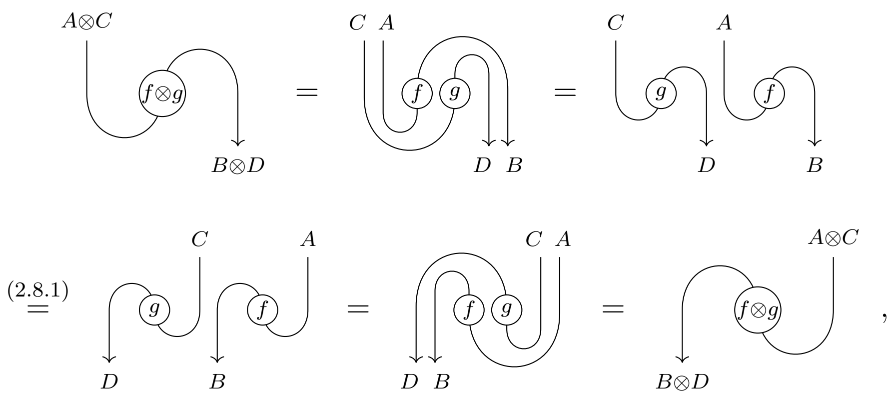
Proposition 2.8.2 (Contravariance of the mate)**.** *Suppose that $f\colon A \to B$ and $g\colon C \to D$ have well-defined mates. Assume that the cups and caps satisfy* (2.5.7)*. Then the mate $(f\otimes g)^*$ is well-defined and $(f\otimes g)^* = g^*\otimes f^*$, that is*



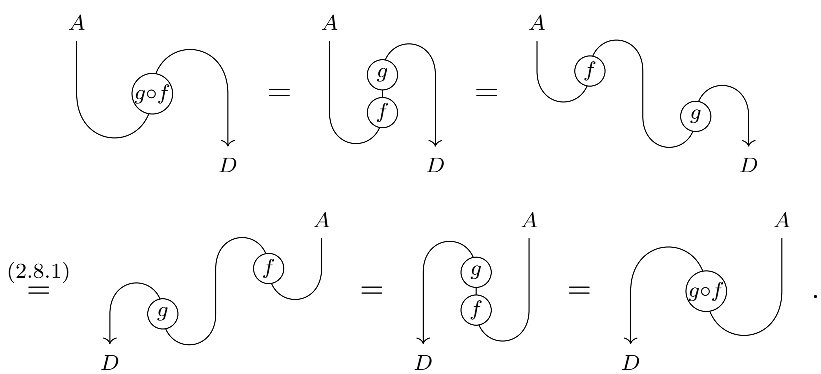
If moreover $B = C$, then the mate $(g \circ f)^*$ is well-defined, and $(f \circ g)^* = g^* \circ f^*$, that is



Proof: We have, using interchange and zigzag,

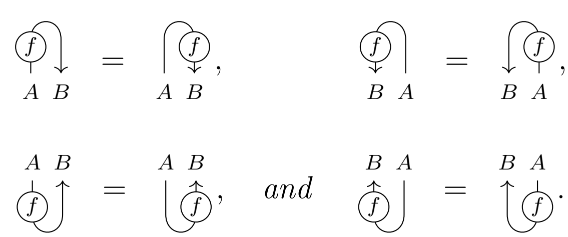


and

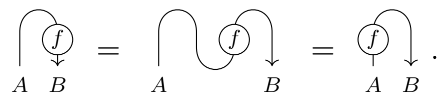


This gives the required equalities $(f \otimes g)^* = g^* \otimes f^*$ and $(g \circ f)^* = f^* \circ g^*$. ∎

Lemma 2.8.3. *Whenever the mate of $f : A \to B$ is well-defined, we have*



Proof: This directly follows from the definition of the mate and the zigzag equation. For instance

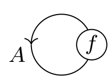
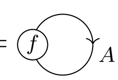


∎

2.9 Dimension and trace

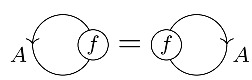
We generalize the notion of dimension of a vector space, by using the fact that the dimension is the trace of the identity. This definition takes advantage of duality, but a more general notion of trace exists, as in [Sel11, p.32].

Definition 2.9.1. Let $f\colon A \to A$, and suppose that A possesses a dual. We define the *left trace* and the *right trace* of f by

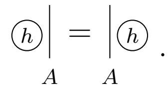
$\mathrm{tr}_\ell(f) =$  , $\mathrm{tr}_r(f) =$  .

If these traces are equal, the *trace* of f is $\mathrm{tr}(f) = \mathrm{tr}_\ell(f) = \mathrm{tr}_r(f)$.

Definition 2.9.2 ([Sel11, p.25])**.** A category is *spatial pivotal* if it is pivotal, if every $f\colon A \to A$ satisfies

 (2.9.1)

(all traces are well-defined), and if all endomorphisms of $\mathbb{1}$ are *central* in the sense that for any $h\colon \mathbb{1} \to \mathbb{1}$ and object A,

 (2.9.2)

In a spatial pivotal category we can think of diagrams as moving in a three-dimensional space, hence the name.

Proposition 2.9.3. *Let $f\colon A \to A$, $g\colon B \to B$. Assume that A and B have duals, and the traces of f and g are well-defined. If the category is R-linear, then for any $a \in R$*

$$\mathrm{tr}(af) = a\,\mathrm{tr}(f) \quad \textit{and} \quad \mathrm{tr}(f+g) = \mathrm{tr}(f) + \mathrm{tr}(g).$$

If A and B satisfy the equation (2.9.2) *for all $h\colon \mathbb{1} \to \mathbb{1}$, then*

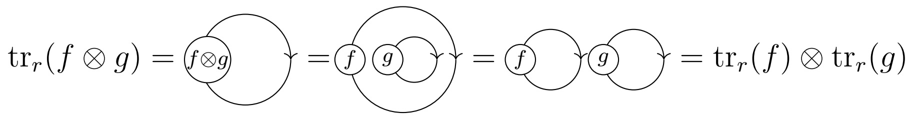
$$\mathrm{tr}(f \otimes g) = \mathrm{tr}(f) \otimes \mathrm{tr}(g).$$

Let $f'\colon A \to B$ and $g'\colon B \to A$. If the mate of f is well-defined, then

$$\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f).$$

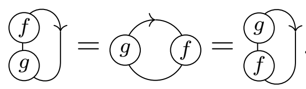
All of this is also true for tr_ℓ and tr_r.

Proof: Linearity of the trace comes directly from linearity of the tensor product and composition. If A satisfy (2.9.2) (endomorphisms of $\mathbb{1}$ commute with 1_A), then



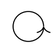
and similarly for tr_ℓ.

If the mate of f is well-defined, we can use Lemma 2.8.3 to show

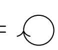


Hence $\mathrm{tr}_r(f \circ g) = \mathrm{tr}_r(g \circ f)$, and similarly for tr_ℓ. ∎

Definition 2.9.4. Let $\uparrow$ be dual to $\downarrow$. Suppose that  = , and  $|_A = |_A$  for any object A. Then the *dimension* of $\uparrow$ and $\downarrow$ is

$$\dim \uparrow = \dim \downarrow =$$  .

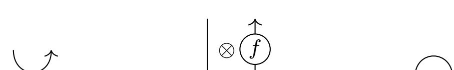
Remark 2.9.5. In an R-linear monoidal category, the dimension of $\uparrow$ is well-defined whenever  =  $= \delta \cdot 1_{\mathbb{1}}$ for some $\delta \in R$, since $1_{\mathbb{1}}$ is central. In that case we write $\dim\uparrow = \delta$, in accordance with the convention in Definition 2.2.18. In a spatial pivotal category, the dimension of all objects is defined.

Example 2.9.6. For finite-dimensional vector spaces, these definitions coincide with the usual definitions of dimension and trace. Letting $\dim_{\Bbbk} V$ denote the dimension of V as a $\Bbbk$-vector space, and with the notation of §2.4, we have  =  $= \dim_{\Bbbk} V$, since

$$\Bbbk \longrightarrow V^* \otimes V \longrightarrow \Bbbk \qquad \text{and} \qquad \Bbbk \longrightarrow V \otimes V^* \longrightarrow \Bbbk$$

$$1 \mapsto \sum_{v \in B} \delta_v \otimes v \mapsto \sum_{v \in B} \delta_v(v) = \dim_{\Bbbk} V \qquad 1 \mapsto \sum_{v \in B} v \otimes \delta_v \mapsto \sum_{v \in B} \delta_v(v) = \dim_{\Bbbk} V.$$

More generally, for $f \colon V \to V$, we have $\mathrm{tr}_\ell(f) = \sum_{v \in B} \delta_v(f(v)) = \mathrm{tr}_r(f)$ since



$$\Bbbk \longrightarrow V^* \otimes V \longrightarrow V^* \otimes V \longrightarrow \Bbbk,$$

$$1 \longmapsto \sum_{v \in B} \delta_v \otimes v \longmapsto \sum_{v \in B} \delta_v \otimes f(v) \longmapsto \sum_{v \in B} \delta_v(f(v)).$$

Example 2.9.7. By the proof of Proposition 2.6.5, we know that the units and counits of V are also units and counits of (V,ρ), for any group homomorphism $\rho\colon G \to \mathrm{GL}(V)$. Hence, $\dim(V,\rho) = \dim V$ for any finite-dimensional representation. Moreover, if $f\colon (V,\rho) \to (V,\rho)$ is an intertwiner, its trace as a morphism of representations is equal to its trace as a morphism of vector spaces.

Remark 2.9.8. When $\circlearrowright \neq \circlearrowleft$, [TV17, p.48] defines a left dimension $\dim_\ell(\uparrow) = \circlearrowright$ and a right dimension $\dim_r(\uparrow) = \circlearrowleft$.

An example of a category where the hypotheses of Definition 2.9.4 are not satisfied for all objects is the Heisenberg category (see [Kho14, §2.1] and [MS18, Definition 2.1]). Indeed, for the generating object $\uparrow$ in that category, we have

$$\circlearrowleft\uparrow = \uparrow\circlearrowleft + \uparrow \quad \text{and} \quad \circlearrowright = 1 \neq \circlearrowleft.$$

Chapter 3

Symmetric monoidal categories

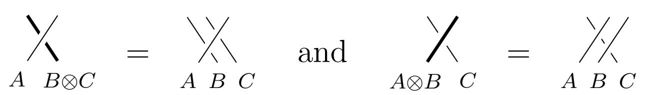
We collect here some properties of braided and symmetric monoidal categories, with emphasis on string diagramatics and categories of representations. References for the first two sections are [ML98, Chapter XI] and [Sel11, §3.3 and §3.5].

3.1 Braided monoidal categories

Definition 3.1.1 ([Sel11, p.14])**.** A *braiding* on a monoidal category is a natural family of isomorphisms $c_{A,B}\colon A \otimes B \to B \otimes A$, represented by 

with inverse 

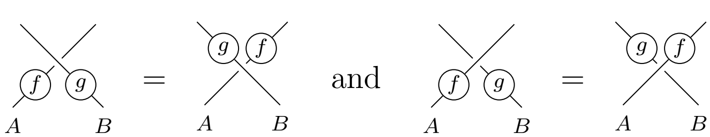
, and satisfying the "hexagon equations"



(3.1.1)

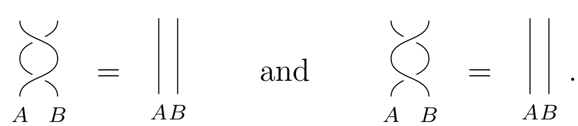
If a monoidal category admits a braiding, it is called *braided*.

Saying that the braidings are natural means that



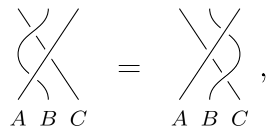
for any f, g. (3.1.2)

Saying that they are isomorphisms means that



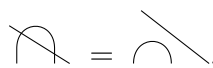
(3.1.3)

When combining naturality with the hexagon equations, by using $f =$  we obtain the *Yang-Baxter equation*



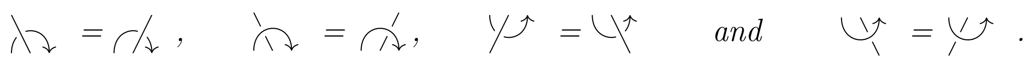
(3.1.4)

and when f or g is a cup or cap we obtain equations like

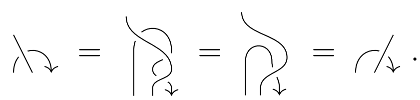


(3.1.5)

Lemma 3.1.2. *Let $\mathscr{C}$ be a braided monoidal category. Suppose that $\uparrow \in \mathscr{C}$ is left dual of $\downarrow \in \mathscr{C}$, with cup and cap $\cup\!\!\nearrow$ and $\cap\!\!\searrow$. Then for any object $|$ we have*



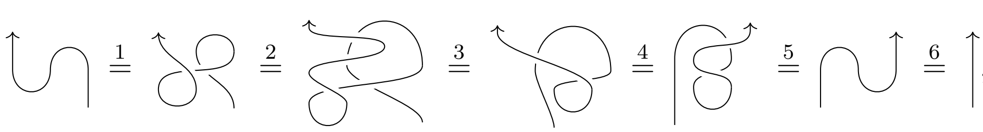
Proof: Each of these equalities is shown by first introducing double crossing, and then moving the cap or cup by naturality. For instance, the first equality is



∎

Proposition 3.1.3. *Let $\mathscr{C}$ be a braided monoidal category. Suppose that $\uparrow \in \mathscr{C}$ is left dual of $\downarrow \in \mathscr{C}$, with cup and cap $\cup\!\!\nearrow$ and $\cap\!\!\searrow$. Then $\uparrow$ is also right dual of $\downarrow$, with cup and cap*

Proof: We show $\uparrow\!\!\cup\!\cap = \uparrow$, the equation $\swarrow\!\!\cap\!\cup = \downarrow$ is similar. The graphical proof is



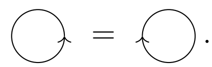
where (1) is the definition of $\nwarrow\!\!\cup$ and $\swarrow\!\!\cap$, (2) introduces inverse braidings, (3) is by naturality of the braiding (and the hexagonal equation), (4) is Lemma 3.1.2, (5) removes inverse braidings, and (6) comes from left duality. ∎

Corollary 3.1.4. *Let $\mathscr{C}$ be a braided monoidal category, and suppose that $\uparrow$ has a right dual $\downarrow$ and a left dual $\downarrow$ (dashed). Then $\downarrow$ and $\downarrow$ (dashed) are isomorphic*

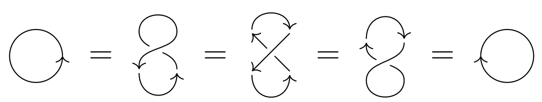
Proof: By Proposition 3.1.3, $\downarrow$ is left dual of $\uparrow$. Since $\uparrow$ has two left duals, these are isomorphic by Proposition 2.5.6. ∎

Proposition 3.1.5. *Let $\mathscr{C}$ be a braided monoidal category. Suppose that $\uparrow \in \mathscr{C}$ is dual to $\downarrow \in \mathscr{C}$, with left unit and counit*  *and* *. If the right unit satisfies*

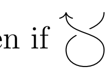
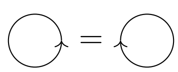
then we have



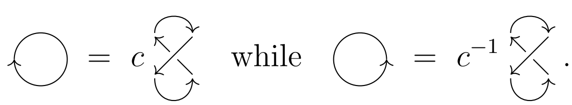
Proof: We have  by unicity of the counit (Remark 2.5.9), so

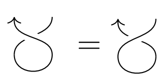
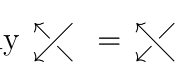


∎

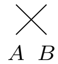
Remark 3.1.6. Consider $\mathscr{C}$, $\uparrow$, $\downarrow$,  and  as in Proposition 3.1.5. Even if  = , we don't always have . For instance, if

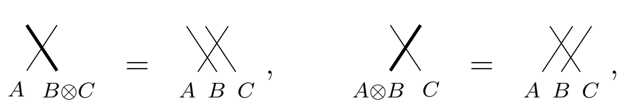
for some $c \in R$ not self-inverse, then these still satisfy the zigzag equations but



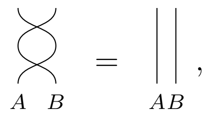
Note also that the equation  does not imply . This can be seen for instance in the category of tangles (see [Tur90, Definition 1.1]).

3.2 Symmetric monoidal categories

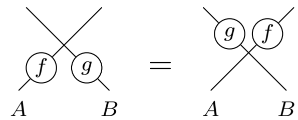
Definition 3.2.1. A monoidal category is *symmetric* if for every objects A, B there is a *symmetry*  $: A \otimes B \to B \otimes A$ such that

 (3.2.1)

(these are called the *hexagon equations*),



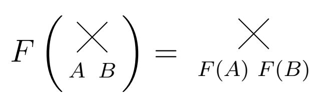
, and (3.2.2)



for any f, g. (3.2.3)

Every symmetric monoidal category is braided. The equation 3.2.3 means that the symmetries are natural in both objects.

Definition 3.2.2. Symmetric monoidal functors are monoidal functors F sending symmetries to symmetries, so satisfying



$\forall$ objects A, B.

Example 3.2.3. Consider the monoidal category of vector spaces $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$. This is a symmetric monoidal category, where for any vector spaces V, W the symmetry is

$$: V \otimes W \to W \otimes V, \quad \sum v_i \otimes w_i \mapsto \sum w_i \otimes v_i \ .$$

Indeed, for any vectors $u \in U, v \in V, w \in W$ and linear maps $f\colon V \to V'$, and $g\colon W \to W'$, the hexagon equations (3.2.1) are verified since both sides of the left equality correspond to the map

$$u \otimes v \otimes w \mapsto v \otimes w \otimes u$$

and both sides of the right equality correspond to the map

$$u \otimes v \otimes w \mapsto w \otimes u \otimes v \ ;$$

the self-inverse equation (3.2.2) is verified by

$$v \otimes w \mapsto w \otimes v \mapsto v \otimes w \ ;$$

and the naturality equation (3.2.3) is verified since both sides correspond to

$$v \otimes w \mapsto g(w) \otimes f(v) \ .$$

Example 3.2.4. The monoidal category $\mathrm{Rep}_{\Bbbk}(G)$ of $\Bbbk$-representations of G (with tensor product as defined on p. 21) is also a symmetric monoidal category. Let $(V, \rho), (W, \sigma) \in \mathrm{Rep}_{\Bbbk}(G)$. We use the same symmetries

$$: V \otimes W \to W \otimes V, \quad \sum v_i \otimes w_i \mapsto \sum w_i \otimes v_i$$

as in the previous example. We only need to show that this is indeed an intertwiner $(V,\rho)\otimes(W,\sigma)\to(W,\sigma)\otimes(V,\rho)$. For $g\in G, v\in V, w\in W$, this amounts to

$$\left((\sigma\otimes\rho)(g)\circ \right.$$  $$\left.\right)(v\otimes w) = \left(\right.$$  $$\left.\circ(\rho\otimes\sigma)(g)\right)(v\otimes w)\ ,$$

which is true since both sides are equal to $\sigma(g)w\otimes\rho(g)v$.

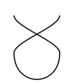
Remark 3.2.5. The fact that $\mathrm{Rep}_{\Bbbk}(G)$ is symmetric monoidal only applies when G is a group and the tensor product is defined as on page 21. However, this does not hold in some other contexts; see for instance the category of representations of $U_q(\mathfrak{g})$ examined in §4.6.2.

Remark 3.2.6. When a symmetric monoidal category is also pivotal, it is often called a *compact closed category*, as in [Sel11, 4.8].

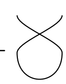
3.3 Symmetrically self-dual and Schurian objects

When a self-dual object $|$ is in a symmetric monoidal category, we have two cups automatically arising. The normal cup  arising from the self-duality, and the cup  from Proposition 3.1.3. Equality of these two cups is related to objects satisfying an abstract "Schur's lemma".

Definition 3.3.1. A self-dual object $|$ in a symmetric monoidal category is *symmetrically self-dual* if

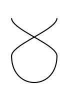
 $=$  ,

and is *antisymmetrically self-dual* if

 $= -$  .

Proposition 3.3.2. *Let $(V,\rho)\in\mathrm{Rep}_{\Bbbk}(G)$ for $\Bbbk=\mathbb{R}$ or $\Bbbk=\mathbb{C}$, and suppose that ρ is a unitary representation. Then (V,ρ) is symmetrically self-dual.*

Proof: When we take the unit from the proof of Proposition 2.6.6, and compose it with the symmetry of Example 3.2.4, we obtain

 $$= \ 1\mapsto\sum_{v\in B}v\otimes v\mapsto\sum_{v\in B}v\otimes v \ =$$  . ∎

Example 3.3.3. The standard representation of the orthogonal group $O_n(\mathbb{R})$ is symmetrically self-dual, since it preserves the inner product by definition (see Example 2.6.7).

An example of antisymmetrically self-dual object is related to the symplectic group. This group is defined analogously to the orthogonal group, but with a different bilinear pairing instead of the inner product. Define $\Omega = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$,

$$\omega(w, v) = \langle w, \Omega v\rangle \quad \forall w, v, \in \mathbb{R}^n, \quad \text{and} \tag{3.3.1}$$

$$\mathrm{Sp}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \omega(Aw, Av) = \omega(w, v) \;\; \forall w, v \in \mathbb{R}^n\}. \tag{3.3.2}$$

This is a group, so $\mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}_n(\mathbb{R}))$ is a symmetric monoidal category by Example 3.2.4.

Consider any representation (V, ρ) in $\mathrm{Rep}_{\mathbb{R}}(\mathrm{Sp}_n(\mathbb{R}))$, for instance the defining representation $V = \mathbb{R}^n$ with action $\rho(A)v = Av$. To show that this is an antisymmetrically self-dual representation, we want to mimic the calculations in Example 2.5.3, by finding an analogue of equation (2.5.4). By that equation, for an orthonormal basis B of V, we have

$$w = \sum_{v \in B} \langle w, v\rangle v = \sum_{v \in B} \langle w, \Omega(\Omega^{-1}v)\rangle \Omega(\Omega^{-1}v) = \sum_{v' \in B} \omega(w, v')\Omega v'. \tag{3.3.3}$$

Therefore, we can define

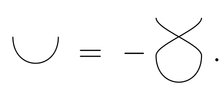
$$\cup : \Bbbk \to V \otimes V,\; 1 \mapsto \sum_{v \in B} v \otimes \Omega v \quad \text{and} \quad \cap : V \otimes V \to \Bbbk,\; w \otimes v \mapsto \omega(w, v) \tag{3.3.4}$$

to verify the zigzag equations as in (2.5.6). We can then check that $\cup$ and $\cap$ are intertwiners, as in Proposition 2.6.6, to obtain that (V, ρ) is self-dual. It is interesting to note that here the zigzag $\cup\!\cap = |$ does not follow solely from the proof of $\cap\!\cup = |$. We also need to use $\Omega^{-1} = -\Omega$ and $\omega(w, v) = -\omega(v, w)$ to deduce $w = \sum_{v \in B} \omega(\Omega v, w)v$ from equation (3.3.3).

Finally, we can conclude that (V, ρ) is antisymmetrically self-dual by noting that

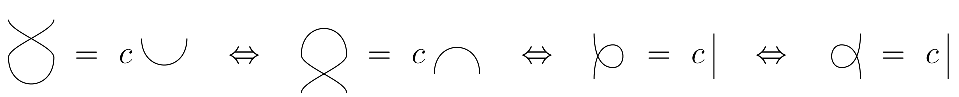
$$\sum_{v \in B} \Omega v \otimes v = \sum_{v' \in B} v' \otimes \Omega^{-1}v' = -\sum_{v' \in B} v' \otimes \Omega v',$$

and therefore

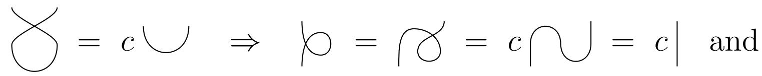


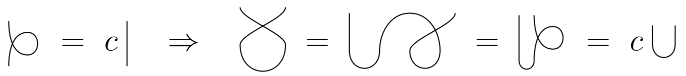
Definition 3.3.4. An object A in an R-linear category is *Schurian* if $\mathrm{End}(A) = R$.

Proposition 3.3.5. *In a R-linear symmetric monoidal category, for any scalar $c \in R$ we have*

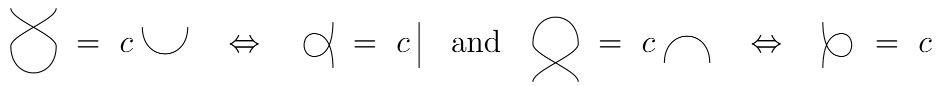


Proof: By lemma 3.1.2,



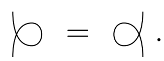


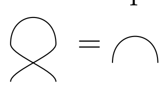
The proofs of



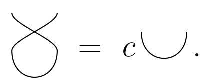
are by horizontal or vertical reflection of the above. ∎

Corollary 3.3.6. *Let | be a self-dual object in a linear symmetric monoidal category. If this object is Schurian, then*

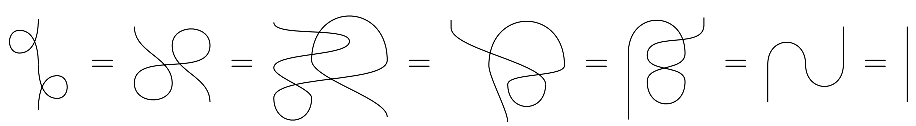


By Proposition 3.3.5, we could equivalently have defined a symmetrically self-dual object with the equality 

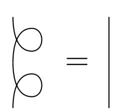
Proposition 3.3.7. *Let | be a self-dual object in a R-linear symmetric monoidal category. If this object is Schurian, then there exists a square root of unity c such that*



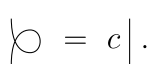
Proof: By self-duality and the axioms of symmetric monoidal category,



By Corollary 3.3.6, this means that

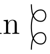


Since | is Schurian, there exists a square root of unity c such that



This concludes by Proposition 3.3.5. ∎

Corollary 3.3.8. *If the only square roots of unity in R are ± 1 (for instance in an integral domain) and the object $|$ is Schurian, then it is either symmetrically self-dual or anti-symmetrically self-dual.*

Remark 3.3.9. To deduce  $= | \Rightarrow$  $= \pm |$ we need the hypothesis that $|$ is Schurian. However we don't need linear structures to obtain  $= |$; this is one of the axioms of a *tortile category* or *ribbon category*, see Definition 4.28 of [Sel11].

Chapter 4

Examples of universal categories with duals

In this chapter, we consider four examples of universal categories. Each of them is defined by some type of diagrams and described by a presentation of generators and relations. We also prove a universal property for each of them — hence the name "universal category" — which in §4.6 is applied to some categories of representations.

4.1 Free categories and presentations of categories

Definition 4.1.1. For A a set, the free monoid on A is denoted by $\langle A\rangle$ and consists of all finite words in elements of A. Formally,

$$\langle A\rangle = \{a_1^{\epsilon_1}\cdots a_n^{\epsilon_n} \mid n\in\mathbb{N}, a_i\in A, \epsilon_i\in\mathbb{N}\setminus\{0\},\ \text{and}\ \forall i\ a_i\neq a_{i+1}\}, \tag{4.1.1}$$

with neutral element being when $n=0$, and product being concatenation of words ($a_1^{\epsilon_1}\cdots a_n^{\epsilon_n}\bullet b_1^{\epsilon'_1}\cdots b_m^{\epsilon'_m} = a_1^{\epsilon_1}\cdots a_n^{\epsilon_n}b_1^{\epsilon'_1}\cdots b_m^{\epsilon'_m}$).

Recall that in this thesis R is a commutative ring. The free R-module on A consists of all formal R-linear combinations of elements of A.

Definition 4.1.2. A *quiver* (or *directed graph*) $Q=(Q_0,Q_1,s,t)$ is composed of a set of vertices Q_0 and a set of edges Q_1 with two maps $s,t\colon Q_1\to Q_0$ assigning to each edge e its source $s(e)$ and its target $t(e)$. We usually denote it by $Q=(Q_0,Q_1)$, leaving s and t implied. If $s(e)=a$ and $t(e)=b$, we often write $e\colon a\to b$. A quiver morphism is a pair $\phi=(\phi_0,\phi_1)\colon(Q_0,Q_1)\to(Q'_0,Q'_1)$ such that $\phi_0\colon Q_0\to Q'_0$ and $\phi_1\colon Q_1\to Q'_1$ preserve sources and targets, in the sense that $\phi_0\circ s = s\circ\phi_1$ and $\phi_0\circ t = t\circ\phi_1$.

Following [CM17, Definition 5.3], we define a *monoidal quiver* to be (Q_0,Q_1,s,t) composed of a set of vertices Q_0 and a set of edges Q_1, with source and target maps $s,t\colon Q_1\to\langle Q_0\rangle$. Hence it is a tuple (Q_0,Q_1,s,t) such that $(\langle Q_0\rangle,Q_1,s,t)$ is a quiver.

A morphism of monoidal quivers is a pair $(\phi_0, \phi_1)\colon (Q_0, Q_1) \to (Q'_0, Q'_1)$ such that $(\phi_0, \phi_1)\colon (\langle Q_0\rangle, Q_1) \to (\langle Q'_0\rangle, Q'_1)$ is a morphism of quivers, and $\phi_0\colon \langle Q_0\rangle \to \langle Q'_0\rangle$ is a morphism of monoids. With this we obtain the category **MonQuiv** of monoidal quivers, with composition by components and $1_{(Q_0,Q_1)} = (1_{Q_0}, 1_{Q_1})$.

Definition 4.1.3 (see [Eas20, §2.2] and [CM17, Lemma 5.4])**.** Let $Q = (Q_0, Q_1)$ be a monoidal quiver. The *free monoidal category* on Q is the category C where objects are words in $a \in Q_0$, and morphisms are built from the $f \in Q_1$ through composition and tensor products. Formally, $\mathrm{ob}\,(C) = \langle Q_0\rangle$ and morphisms in C are built by applying recursively the following:

1. For any object $c \in \mathrm{ob}\,(C)$, there is a morphism $1_c\colon c \to c$;
2. Any $f\colon a \to b$ in Q_1 is a morphism;
3. If $f\colon a \to b$ and $g\colon b \to c$ are morphisms, $g \circ f\colon a \to c$ is a morphism;
4. If $f\colon a \to b$ and $g\colon c \to d$ are morphisms, $f \otimes g\colon a \otimes c \to b \otimes d$ is a morphism.

Two morphisms are equal if they are related by a series of equalities of the form $f \circ 1_a = f = 1_b \circ f$ or $(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)$ (the interchange law). The identity on an object c is 1_c, while $\circ$ and $\otimes$ are defined by points 3 and 4.

To define the *free R-linear monoidal category* on Q, let us denote by F_1Q the free monoidal category on Q, and by F_2Q the free R-linear monoidal category on Q. We then have same objects $\mathrm{ob}\,(F_2Q) = \mathrm{ob}\,(F_1Q)$, and for any objects a, b we define $\mathrm{Hom}_{F_2Q}(a, b)$ as the free R-module on $\mathrm{Hom}_{F_1Q}(a, b)$. We often talk of "free linear monoidal categories" without mentioning the ring R.

In the previous definition, we define composition and tensor products formally. A more explicit construction of the free monoidal category can be found in [CM17, Lemma 5.4], with details and linearity discussed in [Liu18, §4].

Example 4.1.4. When $Q_0 = Q_1 = \emptyset$, the free monoidal category on Q contains only $\mathbb{1}$ and $1_{\mathbb{1}}$. Hence, the free R-linear monoidal category on Q is isomorphic to R.

If $Q_0 = \{|\}$ is any singleton and Q_1 is empty, then in the free monoidal category the objects are of the form $|^{\otimes n}$ for any $n \in \mathbb{N}$, and the only morphisms are identities.

If $Q_0 = \{|\}$ and $Q_1 = \{f\}$ with $f\colon | \to |$, then in the free monoidal category the object are again $|^{\otimes n}$ for $n \in \mathbb{N}$. By composing we obtain morphisms f^m for any $m \in \mathbb{N}$, and therefore after tensoring the morphisms are all those of the form $f^{m_1} \otimes \cdots \otimes f^{m_n}$ for $m_i, n \in \mathbb{N}$.

Definition 4.1.5. Let C be a monoidal category. A *relation on C* is a function $\sim$ that assigns to any pair of objects $a, b \in C$ a binary relation on the set $\mathrm{Hom}(a, b)$. It is called an *equivalence* if each binary relation is an equivalence relation on the hom-set.

Following [Eas20, p. 4], we say that a relation $\boldsymbol{\sim}$ on C is a *monoidal congruence* if it is an equivalence that is compatible with composition and tensor product, in the sense that for all morphisms $f \boldsymbol{\sim} f'$, $g \boldsymbol{\sim} g'$ and $h \boldsymbol{\sim} h'$ in C the following holds whenever $\mathrm{dom}(g) = \mathrm{cod}(f)$:

$$g \circ f \boldsymbol{\sim} g' \circ f' \quad \text{and} \quad f \otimes h \boldsymbol{\sim} f' \otimes h'. \tag{4.1.2}$$

This simply generalizes [ML98, §II.8] to the monoidal setting.

For $f\colon a \to b$ in C and $\sim$ a relation on C, the *equivalence class* of f is

$$[f]_\sim = \{g\colon a \to b \mid g \boldsymbol{\sim} f\}, \tag{4.1.3}$$

for $\boldsymbol{\sim}$ for the smallest monoidal congruence containing $\sim$. We may write $[f]$ instead of $[f]_\sim$ if the relation $\sim$ is clear from the context. In many cases we write f instead of $[f]$, so we may write $f = g$ to indicate $f \sim g$.

The *quotient* of C by $\sim$ is the category $C/\!\sim$ that has same objects as in C, and where morphisms are equivalence relations $[f]_\sim$ of morphisms f in C. This is a monoidal category with $[g] \circ [f] = [g \circ f]$ and $[f \otimes g] = [f] \otimes [g]$ (well-defined since the equivalence classes use a monoidal congruence).

Example 4.1.6. Consider the free monoidal category on $Q = (\{|\}, \{f\colon | \to |\})$, as in Example 4.1.4. If we make f nilpotent with the relation $f^k \sim 1_|$ for some $k \in \mathbb{N}$, then when quotienting by $\sim$ we obtain all morphisms of the form $f^{m_1} \otimes \cdots \otimes f^{m_n}$ for $n \in \mathbb{N}$ and $m_i < k$. This is similar to what happens in $\mathbb{Z}_k \times \cdots \times \mathbb{Z}_k$.

Remark 4.1.7. How can we identify objects? One could be tempted to "equate" objects by "quotienting" through a similar definition, but this will not work easily. The problem is that if we simply declare objects to be equal, we don't know what happens with morphisms. These will need to be also equated in some way that respects composition, but there could be more than one such choice. One solution is to use coherence conditions and rewriting systems. Using this strategy, we find in [CM17, §2.2 and §5] a "presentation modulo" that permit us to quotient objects. The approach we use instead in Propositions 4.3.11 and 4.5.11 is to add isomorphisms between the objects, and see if we can obtain a monoidal equivalence to a category with fewer objects. In Definition 2.2.15 some natural transformations are involved, and it is naturality that forces morphisms to be equal in the right manner.

Definition 4.1.8. For Q a monoidal quiver, denote by FQ the free linear monoidal category on Q. If $\sim$ is a relation on FQ, we say that the category $FQ/\!\sim$ has *linear monoidal presentation* $(Q, \sim)$. Elements of Q_0 are its generating objects, and elements of Q_1 are its generating morphisms.

Example 4.1.9. Let C be a linear monoidal category, and suppose that its objects form a free monoid, in the sense that there exists a set Q_0 such that $\mathrm{ob}\,(C) = \langle Q_0 \rangle$.

Then C always has a trivial presentation where Q_1 consists of all morphisms, and for any f and g in the free linear monoidal category on Q, we take $f \sim g$ iff f and g are equal in C.

Remark 4.1.10. In this thesis, we often talk of the *free linear monoidal category* with some property; these can also be called *universal categories*. This concept is slightly more general than simply taking the free linear monoidal category FQ on some monoidal quiver Q; it can involve quotienting by some relations. When we say that a category is the free linear (symmetric) monoidal category on some self-dual or dual objects, we mean that it satisfies some universal property, in the sense of Definition 5.1.1 (from [ML98, §III.1]). Defining in full generality the *free linear monoidal category with some property* would get quite technical. Instead, in each of Propositions 4.2.10, 4.3.10, 4.4.9 and 4.5.10 we specify the relevant universal property, which can be considered as the definition of the free category mentioned.

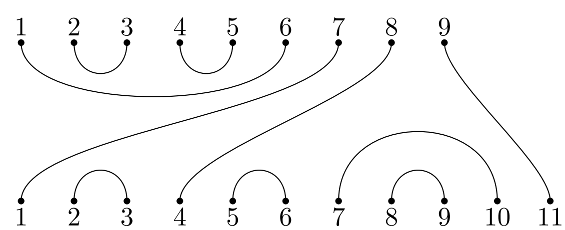
4.2 Temperley-Lieb category

In this section, we describe in detail the Temperley-Lieb category mentioned in §1.2.

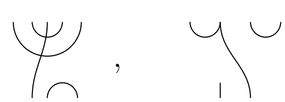
Definition 4.2.1 (Temperley-Lieb diagrams)**.** A *Temperley-Lieb diagram* consists of two horizontal rows of dots, one on top of the other, such that the dots are linked pairwise by strings that do not cross themselves. Temperley-Lieb diagrams are identified up to *planar isotopy*, meaning that any continuous deformation of the strings gives the same diagram, as long as the endpoints are the same and no crossings are introduced.

We say that a diagram goes from n to m endpoints if there are n dots in the bottom row and m dots in the top row. The empty diagram is a Temperley-Lieb diagram from zero to zero.

Example 4.2.2. Here is an example of Temperley-Lieb diagram from $n = 11$ to $m = 9$ endpoints. The dots are usually omitted, depending on the author.



The following diagrams are not Temperley-Lieb diagrams, since the first contains crossings, and the second contains three endpoints connected at once and one endpoint unconnected.



Remark 4.2.3. Note that for every n and m, there exists only a finite number of ways to join the endpoints, so a finite number of Temperley-Lieb diagrams. Moreover, there exist Temperley-Lieb diagrams from n to m if and only if n and m have the same parity.

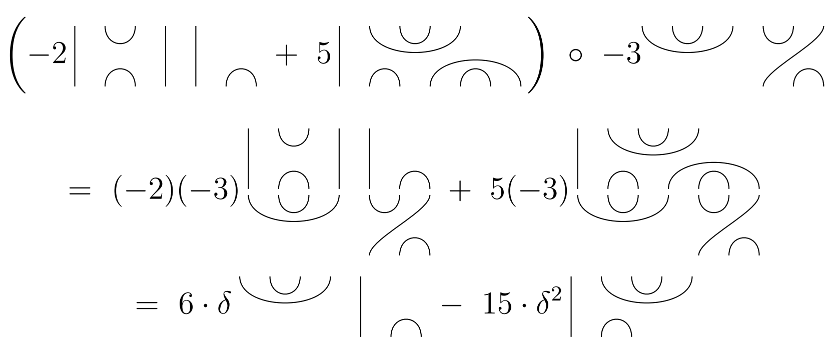
Definition 4.2.4 (Temperley-Lieb category)**.** Let $\delta \in R$. In the *Temperley-Lieb category* $\mathcal{TL}(\delta)$, objects consist of formal tensors $|^{\otimes n}$ for $n \in \mathbb{N}$, corresponding to small parallel vertical lines. For any $n, m \in \mathbb{N}$, the hom-set $\mathrm{Hom}(|^{\otimes n}, |^{\otimes m})$ is the free R-module on the set of Temperley-Lieb diagrams from n to m endpoints.

Composition is done by vertical concatenation of diagrams (extended linearly), in which Temperley-Lieb diagrams are stacked on top of each other and corresponding endpoints are glued together. We read diagrams from bottom to top; $f \circ g$ consists in putting f on top of g. Any closed loop thus formed is then replaced by a factor of δ.

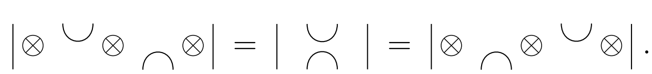
In $\mathrm{Hom}(|^{\otimes n}, |^{\otimes n})$, the identity is the diagram composed of n vertical strings $|\,|\cdots|$.

We define a tensor product operation $\otimes$ by horizontal juxtaposition of diagrams, and with $|^{\otimes n} \otimes |^{\otimes m} = |^{\otimes n+m}$.

Example 4.2.5. We have, for instance, composition and tensor products



and



Lemma 4.2.6. *The Temperley-Lieb category is a strict monoidal category, when using* $\mathbb{1} = |^{\otimes 0}$ *(the identity* $1_{\mathbb{1}}$ *is the empty diagram).*

Proof: Associativity of composition and the identity property hold since diagrams are defined up to planar isotopy. Hence $\mathcal{TL}(\delta)$ is indeed a category. The reader can verify each axiom of Definition 2.2.12, which holds either by properties of the free monoid $\langle | \rangle$, or by the properties of horizontal juxtaposition of diagrams. ∎

Definition 4.2.7 (Temperley-Lieb algebra)**.** The *Temperley-Lieb algebra on* n *strands* $TL_n(\delta)$ is the hom-set $\mathrm{End}_{\mathcal{TL}}(\delta)(|^{\otimes n})$.

Remark 4.2.8. There are multiple equivalent ways to define Temperley-Lieb diagrams. Another common definition is in terms of partition diagrams as in [Eas20, §3.1], and Abramsky gave an algebraic characterization of planarity in [Abr08, §6]. Historically, Temperley-Lieb algebras were first introduced by Temperley and Lieb through generators and relations in [TL71], while the Temperley-Lieb categoy was first used in [GL98].

Turaev used the name *skein category* for the Temperley-Lieb category in [Tur10, §2.1].

Many authors use the numbers $n \in \mathbb{N}$ as the objects in $\mathcal{TL}(\delta)$, instead of $|^{\otimes n}$. Although this simplifies notation, we do not follow this convention since it does not extend well to the oriented case.

The Temperley-Lieb category and Temperley-Lieb algebra are used in many aspects of math. One can find a list of usage of Temperley-Lieb algebras and category in [AM07a, p. 1] and [AM07b, p. 2], with many references; [Eas20, §1] also gives a lot of background reading.

Proposition 4.2.9. *As a linear monoidal category, $\mathcal{TL}(\delta)$ has a presentation with generating object $|$, generating morphisms $\cup$ and $\cap$, and relations*

$$\cap\!\cup = | = \cup\!\cap, \quad \bigcirc = \delta. \tag{4.2.1}$$

Proof: This is [Eas20, Theorem 3.20]. Alternatively, one could follow the proof of Proposition 4.3.6 below, removing all mentions of orientation. ∎

Proposition 4.2.10 (Universal property)**.** *The Temperley-Lieb category is the free linear monoidal category on a self-dual object of dimension δ.*

This means that, for any linear monoidal category C containing a self-dual object V of dimension δ with unit ξ, there exists a unique monoidal functor $\psi\colon \mathcal{TL}(\delta) \to C$ satisfying

$$| \mapsto V, \ \cup \mapsto \xi. \tag{4.2.2}$$

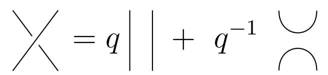
Proof: Existence: We must verify that for each relation $f = f'$ in Proposition 4.2.9, we have $\psi(f) = \psi(f')$. This is true since V is self-dual with $\dim(V) = \delta$; see Definitions 2.5.2 and 2.9.4.

Unicity: By Remark 2.5.9, there exists a unique counit ζ associated to the unit ξ. Since ψ is monoidal, it must send $\cap$ to a counit of ξ, hence to ζ. Therefore ψ is specified on every monoidal generator of $\mathcal{TL}(\delta)$, so is unique. ∎

Proposition 4.2.11. *$\mathcal{TL}(\delta)$ is a strict spatial pivotal category, and each object is self-dual.*

Proof: [Zha17, thm. 4.3] tells us that $\mathcal{TL}(\delta)$ is a strict pivotal category, and [Zha17, p. 42] explains that the left and right trace of any morphism is the same, hence equation (2.9.1) is satisfied. Since any endomorphism of $\mathbb{1} = |^{\otimes 0}$ is a scalar, equation (2.9.2) is also safisfied. ∎

Proposition 4.2.12 ([BSA18, §2.2])**.** *If there exists $q \in R$ invertible such that $\delta = -q^2 - q^{-2}$, then $\mathcal{TL}(\delta)$ is braided with braiding determined by*



(4.2.3)

If $\delta = -2$, then $\mathcal{TL}(\delta)$ is a symmetric monoidal category, with same braiding.

This braiding is related to the Jones polynomial ([Jon05]). See [Abr08, §2.5] for a short discussion.

We can extend these properties to any category with dual objects, by using the universal property Proposition 4.2.10.

Corollary 4.2.13. *Let C be a linear monoidal category, and $V \in C$ be a self-dual object of dimension δ with unit ζ and counit ξ. Suppose that all objects of C are of the form $V^{\otimes n}$ for some $n \in \mathbb{N}$, and that morphisms are generated by ζ and ξ. Then C is a spatial pivotal category with braiding, where every object is self-dual.*

Proof: All of these properties are preserved by monoidal functors, so the functor $\psi\colon \mathcal{TL}(\delta) \to C$ preserve these properties, and C is the image of ψ. ∎

4.3 Oriented Temperley-Lieb category

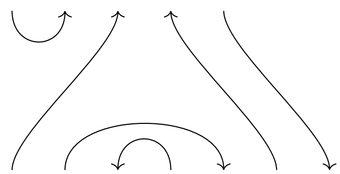
The oriented Temperley-Lieb category is the free linear monoidal category on a pair of dual objects of dimension δ.

In the oriented Temperley-Lieb category, the objects will be words in the symbols $\uparrow$ and $\downarrow$, which we can write as $\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow$ or $\uparrow^{\otimes 2}\downarrow\uparrow^{\otimes 3}\downarrow^{\otimes 2}$, as needed. Hence we will be using the free monoid $\langle \uparrow, \downarrow \rangle$.

Definition 4.3.1. An *oriented Temperley-Lieb diagram* consists of two horizontal rows of dots, one on top of the other, which are linked pairwise by non-crossing oriented strings. Oriented Temperley-Lieb diagrams are identified up to *planar isotopy*, meaning that any deformation of the strings gives the same diagram, as long as the endpoints are the same and no crossings are introduced.

For $a, b \in \langle \uparrow, \downarrow \rangle$, we say that a diagram goes from a to b if the orientation of strings matches a at the bottom and b at the top. The empty diagram is an oriented Temperley-Lieb diagram from the empty word to the empty word.

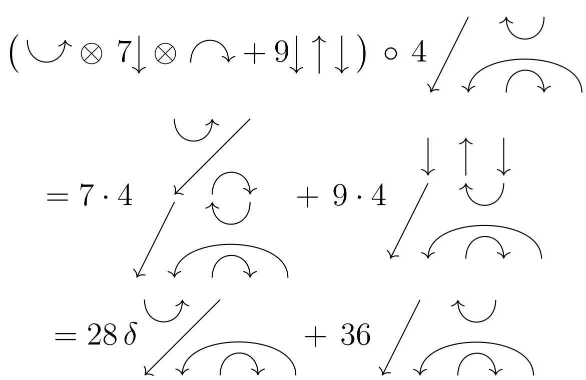
Example 4.3.2. The following is an oriented Temperley-Lieb diagram from $\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$ to $\downarrow\uparrow\uparrow\uparrow\downarrow$.



Definition 4.3.3. Let $\delta \in R$. In the *oriented Temperley-Lieb category* $\mathcal{OTL}(\delta)$, objects are elements of $\langle \uparrow, \downarrow \rangle$ (words in $\uparrow$ and $\downarrow$). For any $a, b \in \langle \uparrow, \downarrow \rangle$, the hom-set $\mathrm{Hom}_{\mathcal{OB}(\delta)}(a, b)$ is the free R-module on the set of oriented Temperley-Lieb diagrams from a to b. Composition is done by vertical concatenation, extended linearly. Any closed loop thus formed is then replaced by a factor of δ. In $\mathrm{Hom}(a, a)$ the identity is the diagram composed of non-crossing vertical strings.

We define a tensor product $\otimes$ by concatenation of words and horizontal juxtaposition of diagrams.

Example 4.3.4.

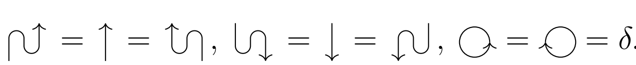


Remark 4.3.5. The oriented Temperley-Lieb category is defined as an analogue of the Temperley-Lieb category and the oriented Brauer category, but is not often studied by itself. It is mentioned quickly in [MM13] and at the very end of [Abr08], and [Die04, p. 136] defines it under the name $T^o A$.

Proposition 4.3.6. *As a linear monoidal category, $\mathcal{OTL}(\delta)$ has a presentation with generating object $\uparrow$ and $\downarrow$, generating morphisms*

(4.3.1)

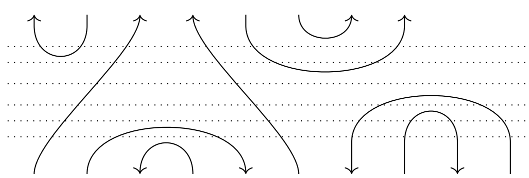
and relations



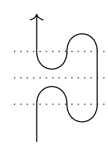
(4.3.2)

To prove this, we could rely on the presentation of the category of oriented tangles in [Tur90, Theorem 3.2], by considering the oriented Temperley-Lieb category as a subcategory of the category of oriented tangles. We choose instead to prove it directly, to give the reader a taste of how such proof goes. By removing all mentions of orientations, the proof below can also be used for the presentation of the Temperley-Lieb category (Proposition 4.2.9). Our proof shares similarities with the proof of [LZ15, Theorem 2.6], but we can simplify arguments since there are fewer morphisms in the oriented Temperley-Lieb category than in the Brauer category (we do not need to deal with crossings).

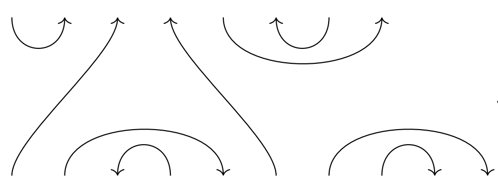
Proof: First, let us show that any oriented Temperley-Lieb diagram can be generated (through $\circ$ and $\otimes$) by $\cup\!\!\nearrow$, $\frown\!\!\searrow$, $\nwarrow\!\!\cup$ and $\swarrow\!\!\frown$; this is sufficient to show that the category $\mathcal{OTL}(\delta)$ is generated by these, since oriented Temperley-Lieb diagrams form bases of the hom-sets. For the sake of this argument, we say that an oriented Temperley-Lieb diagram is drawn in *general position* if all the strings are smooth curves such that no two critical points have the same vertical coordinate. When an oriented Temperley-Lieb diagram is drawn in general position, it is possible to draw horizontal lines such that there is one critical point between any two consecutive lines, and the horizontal lines do not touch any string. For instance, the diagrams



and



are drawn in general position, but it is not the case for the diagram



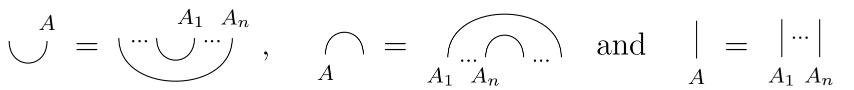
Now, since oriented Temperley-Lieb diagrams are defined up to planar isotopy, any such diagram D can be drawn in general position. Note that this is similar to what is done in [Tur10, Lemma 3.1.1]. Since there is only one critical point between two successive horizontal lines, this means that $D = D_1 \circ D_2 \circ \cdots \circ D_n$, where every D_i is of the form

$$D_i = 1_A \otimes f \otimes 1_B \quad A, B \in \langle \uparrow, \downarrow \rangle,$$

for f one of the generators $\cup\!\!\nearrow$, $\frown\!\!\searrow$, $\nwarrow\!\!\cup$, $\swarrow\!\!\frown$.

This shows that, as a linear monoidal category, $\mathcal{OTL}(\delta)$ has a presentation with generating object $\uparrow$ and $\downarrow$ and generating morphisms (4.3.1). We now turn to the relations. It is clear that two diagrams related by the relations $\cap\!\cup\!\!\uparrow \;=\; \uparrow \;=\; \uparrow\!\!\cup\!\cap$, $\cup\!\cap\!\!\downarrow \;=\; \downarrow \;=\; \downarrow\!\!\cap\!\cup$ will be isotopic, hence equal. Now, suppose that f and g are two morphisms generated (through $\circ$ and $\otimes$) by the above-mentioned generators. Our task is to show that if f and g are planar isotopic as diagrams, then they are equal through the relations (4.3.2). For this, we use two lemmas which permit us to deform our diagrams until they are in a kind of canonical form; an illustration of the lemmas can be found after the proof.

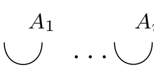
As in Proposition 2.5.7, for any $A = A_1 \dots A_n \in \langle \uparrow, \downarrow \rangle$ we have



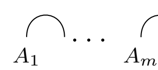
(4.3.3)

Lemma 4.3.7. *Let f be a morphism built from the generators* (4.3.1)*.*

1. *If $f\colon \mathbb{1} \to \mathbb{1}$, then $f = \delta^k$ for some $k \in \mathbb{N}$;*
2. *if $f\colon A \to A$ for a generator $A \in \{\uparrow, \downarrow\}$, then $f = \delta^k\big|_A$ for some $k \in \mathbb{N}$;*
3. *if $f\colon \mathbb{1} \to A$, then there exist objects $A_1, \dots, A_m$ and $k \in \mathbb{N}$ such that*

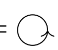
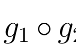
   $f = \delta^k$  ;

4. *if $f\colon A \to \mathbb{1}$, then there exist objects $A_1, \dots, A_m$ and $k \in \mathbb{N}$ such that*

   $f = \delta^k$  ;

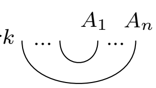
where in each case the mentioned equality uses only the relations (4.3.2)*.*

Proof of lemma: We proceed by strong induction on the number of generators. Note that if two generators g_1, g_2 touch at one endpoint, then (since there are no crossings) they must be part of one of the zigzags in (4.3.2), hence we can remove g_1 and g_2 to obtain an identity in their place.

1. Assume $f\colon \mathbb{1} \to \mathbb{1}$. Base case: if there are zero generators in f, then $f = 1_{\mathbb{1}} = \delta^0$. Assume that the statement is true for all f built from n or fewer generators, and consider f built from $n+1$ generators. If there are generators g_1 and g_2 touching at exactly one endpoint, then we can remove them with a zigzag to obtain that f is built from $n-1$ generators. Otherwise, every pair of generators touch at zero or two endpoints. Since $f\colon \mathbb{1} \to \mathbb{1}$ and there are no crossings, there must be some generators g_1 and g_2 such that $g_1 \circ g_2 =$  or $g_1 \circ g_2 =$ , with no morphism inside of it. Hence, $g_1 \circ g_2 = \delta$ and $f = \delta f'$ for f' built from $n-1$ generators, so we can apply the induction hypothesis.

2. Assume $f\colon A \to A$ for A a generator. Base case: if there are zero generators, then $f = |_A$. Consider f built from $n+1$ generators. If there are generators g_1 and g_2 touching at exactly one endpoint, then we can remove them with a zigzag and f can be built from $n-1$ generators, so we apply induction. Otherwise, every generator must be part of a subdiagram going from $\mathbb{1}$ to $\mathbb{1}$ (we cannot have "isolated generators" touching nothing else, since f goes from one endpoint to one endpoint). Hence, by point 1 we obtain that $f = \delta^k |_A$.
3. Assume $f\colon \mathbb{1} \to A$. Base case: if there is one generator, then $f = \smile\!\!\uparrow$ or $f = \uparrow\!\!\smile$. If there are generators touching at exactly one endpoint, we can remove them with a zigzag and apply induction. If there are pairs of generators touching at both endpoints, then they are part of a subdiagram from $\mathbb{1}$ to $\mathbb{1}$, and by point 1 we can replace such a subdiagram by δ^k. The remaining generators have no endpoints in common with other generators. If it is possible to write $f = f_1 \otimes \smile^{A'} \otimes f_2$ for some $A' \in \{\uparrow, \downarrow\}$, then f_1 and f_2 have fewer generators so we conclude by induction. Otherwise, since there are no crossings, we must have

$$f = \delta^k$$



which is of the right form.

The proof for $f\colon A \to \mathbb{1}$ (case 4) is similar as the proof for $f\colon \mathbb{1} \to A$ (case 3). ∎

Lemma 4.3.8. *Let f be a morphism built from the generators* (4.3.1). *By using only the relations* (4.3.2), *we can write it as*

$$f = \delta^k f_1 \otimes \cdots \otimes f_n, \tag{4.3.4}$$

where for every i either $f_i = \smile^{A_i}$ or $f_i = {}_{A_i}\frown$ or $f_i = |_{A_i}$ $(k \in \mathbb{N}, A_i \in \langle \uparrow, \downarrow \rangle)$.

Moreover, this writing is unique if we ask that no two successive f_i are identities, and that the cups are positioned at the left of the caps when possible. Technically, these requirements read as

$$f_i = \Big|_{A_i} \quad \Rightarrow \quad f_{i+1} \textit{ is not of type } \Big|_{A_{i+1}}, \quad \textit{and} \tag{4.3.5}$$

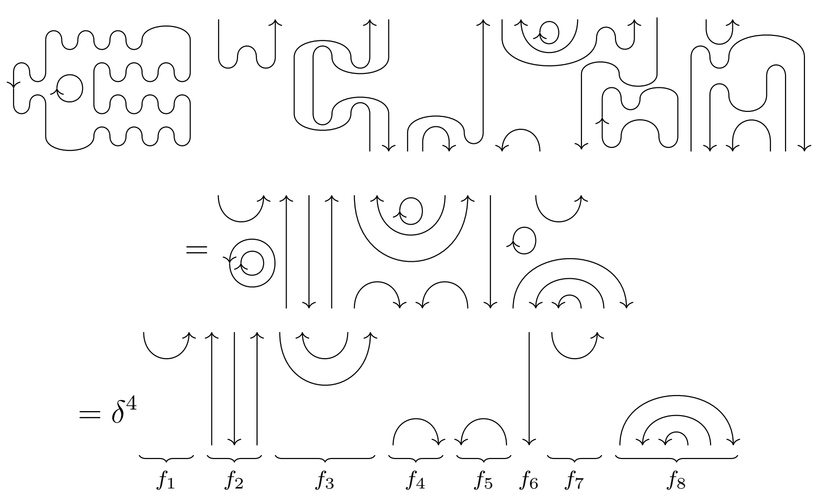
$$f_i = \smile^{A_i}, \ f_j = {}_{A_j}\frown \quad \Rightarrow \quad i < j \textit{ or there exists } j < k < i \textit{ st } f_k = \Big|_{A_i}. \tag{4.3.6}$$

Proof of lemma: First, consider "through strands", by which we mean morphisms $A_1 \to A_1$ for some generating object A_1. For adjacent through strands, we can use point 2 of Lemma 4.3.7 to write them as $|_A$ for some $A \in \langle\uparrow,\downarrow\rangle$. Between two successive $|_A$ and $|_B$ (with no through strands between them), since there are no crossings, we have a morphism of the form $f' \circ f''$ for $f' \colon \mathbb{1} \to A'$ and $f'' \colon A'' \to \mathbb{1}$. Write $f' \circ f'' = (f' \otimes 1_{\mathbb{1}}) \circ (1_{\mathbb{1}} \otimes f'') = f' \otimes f''$ to obtain, by points 3 and 4 of Lemma 4.3.7, that $f' \circ f'' = \delta^k g_1 \otimes \cdots \otimes g_m$ for $g_i = \cup^{A_i}$ or ${}_{A_i}\cap$. Since everything can be arranged as some $|_A$ with such $f' \circ f''$ in between, f is of the right form.

Unicity: When $f = \delta^k f_1 \otimes \cdots \otimes f_n$ is of the form mentioned in the lemma, it is not possible to apply any of the relations (4.3.2) and remain of that form. Hence, the only modifications we can make are vertical or horizontal translations (by using the interchange law and $1_{\mathbb{1}} \otimes g = g = g \otimes 1_{\mathbb{1}}$) and grouping differently the morphisms as f_i. Grouping differently is forbidden by (4.3.5), horizontal translations are stopped by (4.3.6), and vertical translations do not affect the type of f_i. Therefore, we have unicity. ∎

End of the proof of Proposition 4.3.6: Suppose that the morphisms f and g are planar isotopic as diagrams. Then they are both equal to the same form as in Lemma 4.3.8 (by unicity and since two distinct diagrams of that form are not isotopic). Hence $f = g$. This concludes the proof of the proposition. ∎

The following calculation illustrates how to apply Lemmas 4.3.7 and 4.3.8 to obtain the mentioned canonical form.



Remark 4.3.9. We can summarize the preceding proof as follows: denote by C the category having presentation from Proposition 4.3.6, and by $F \colon C \to \mathcal{TL}(\delta)$ the functor that sends a morphism built from generators to the corresponding oriented

Temperley-Lieb diagram. Our goal is then to show that F is an isomorphism of monoidal categories. This functor is well-defined since diagrams related by the relations (4.3.2) will be isotopic. It is surjective on objects and morphisms because of the argument involving diagrams in general position at the beginning of the proof. It is injective on objects and morphisms because of the argument using the two lemmas to obtain a canonical form. Then, it is straightforward to show that F is a strict monoidal functor and an isomorphism of categories.

The same analysis can be applied to the proof of [LZ15, Theorem 2.6] for the Brauer category, although injectivity is proved differently when there are crossings.

Proposition 4.3.10 (Universal property)**.** *The oriented Temperley-Lieb category is the free linear monoidal category on a pair of dual objects of dimension δ.*

This means that, for any linear monoidal category C containing a pair of dual objects V, V^* of dimension δ with left unit ξ_1 and right unit ξ_2, there exists a unique $\psi\colon \mathcal{OTL}(\delta) \to C$ satisfying

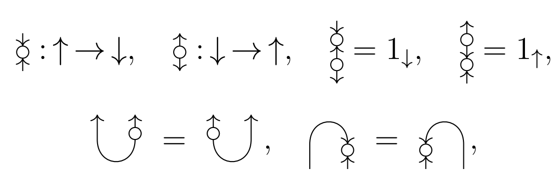
$$\uparrow \mapsto V,\ \downarrow \mapsto V^*,\ \cup^{\nearrow} \mapsto \xi_1,\ {}^{\nwarrow}\cup \mapsto \xi_2. \tag{4.3.7}$$

Proof: Existence: We must verify that for each relation $f = f'$ in Proposition 4.3.6, we have $\psi(f) = \psi(f')$. This is true since V and V^* are dual with $\dim(V) = \delta$; see Definitions 2.5.1 and 2.9.4.

Unicity: By Remark 2.5.9, there exist unique counits ζ_1 and ζ_2 associated respectively to the units ξ_1 and ξ_2. Since ψ is monoidal, it must send counits to counits, hence to $\cap_{\searrow} \mapsto \zeta_1$ and ${}_{\swarrow}\cap \mapsto \zeta_2$. Therefore ψ is specified on every monoidal generator of $\mathcal{OTL}(\delta)$, so is unique. ∎

Recall that the definition of monoidal equivalence is at Definition 2.2.15.

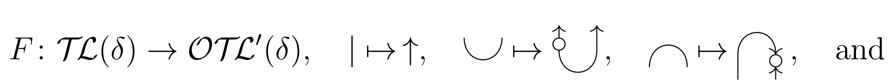
Proposition 4.3.11. *Denote by $\mathcal{OTL}'(\delta)$ the category $\mathcal{OTL}(\delta)$ to which we add two morphisms and four relations*



(4.3.8)

thus making $\uparrow$ isomorphic to $\downarrow$, and ensuring that the new morphisms are their own mate. Then the category $\mathcal{OTL}'(\delta)$ is monoidally equivalent to $\mathcal{TL}(\delta)$.

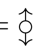
Proof: We use the monoidal functors



$$G\colon \mathcal{OTL}'(\delta) \to \mathcal{TL}(\delta), \quad \uparrow \mapsto |, \quad \downarrow \mapsto |,$$

It is clear that $G \circ F = 1_{\mathcal{TL}(\delta)}$. Define the transformation

$$\tau\colon 1_{\mathcal{OTL}'(\delta)} \to F \circ G \quad \text{by}$$

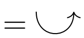
$\tau_{\uparrow} = 1_{\uparrow} : \uparrow \to \uparrow, \quad \tau_{\downarrow} =$  $: \downarrow \to \uparrow,$

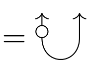
and the fact that monoidal natural transformations must satisfy $\tau_{A\otimes B} = \tau_A \otimes \tau_B$ for all objects A and B. Since $\tau_{\uparrow}$ and $\tau_{\downarrow}$ are isomorphisms, we only need to show that τ is natural to complete the proof that $\mathcal{OTL}'(\delta)$ and $\mathcal{TL}(\delta)$ are monoidally equivalent.

We must show that for any $f\colon A \to A'$ in $\mathcal{OTL}'(\delta)$, we have

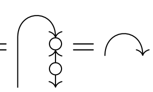
$$(F \circ G)(f) \circ \tau_A = \tau_{A'} \circ 1_{\mathcal{OTL}'(\delta)}(f) = \tau_{A'} \circ f$$

(see Definition 2.2.4). Since τ is monoidal, it is enough to show this for each generating morphism of $\mathcal{OTL}'(\delta)$. Note that we must have $\tau_{\mathbb{1}} = 1_{\mathbb{1}}$ since τ is monoidal.

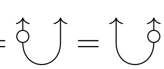
If $f =$  $: \mathbb{1} \to \downarrow\uparrow$,

$F(G(f)) \circ \tau_{\mathbb{1}} = F(G(f)) =$  $= \tau_{\downarrow\uparrow} \circ f.$

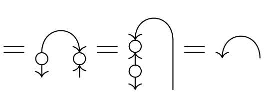
If $f =$  $: \uparrow\downarrow \to \mathbb{1}$,

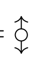
$F(G(f) \circ \tau_{\uparrow\downarrow} =$  $= \tau_{\mathbb{1}} \circ f.$

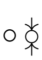
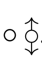
If $f =$  $: \mathbb{1} \to \uparrow\downarrow$, we must use the first equality of (4.3.8) to get

$F(G(f)) \circ \tau_{\mathbb{1}} = F(G(f)) =$  $= \tau_{\uparrow\downarrow} \circ f.$

If $f =$  $: \downarrow\uparrow \to \mathbb{1}$, we must use the second equality of (4.3.8) to get

$F(G(f)) \circ \tau_{\downarrow\uparrow} =$  $= \tau_{\mathbb{1}} \circ f.$

For $f =$  and $f =$ , $F(G(f)) = 1_{\uparrow}$, so the calculation boils down to

$1_{\uparrow} \circ \tau_{\uparrow} = \tau_{\downarrow} \circ$  and $1_{\uparrow} \circ \tau_{\downarrow} = \tau_{\uparrow} \circ$ . ∎

This illustrates well the concept of monoidal equivalence. With this technique, one can identify objects by adding isomorphisms and a few equations; see Remark 4.1.7. Note also that we cannot have an isomorphism of categories between $\mathcal{OTL}'(\delta)$ and $\mathcal{TL}(\delta)$, since in $\mathcal{TL}(\delta)$ all objects are self-dual, which is not the case for $\mathcal{OTL}'(\delta)$.

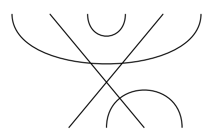
4.4 Brauer category

The Brauer category is the free linear symmetric monoidal category on a self-dual object of dimension δ.

Definition 4.4.1. A *Brauer diagram* consists of two horizontal rows of dots, one on top of the other, such that the dots are linked pairwise. Brauer diagrams are identified up to *isotopy*, meaning that any deformation of the strings gives the same diagram, as long as the endpoints are the same.

We say that a diagram goes from n to m endpoints if there are n dots in the bottom row and m dots in the top row. The empty diagram is a Brauer diagram from zero to zero.

Example 4.4.2. Here is a Brauer diagram from $n = 4$ to $m = 6$ endpoints.



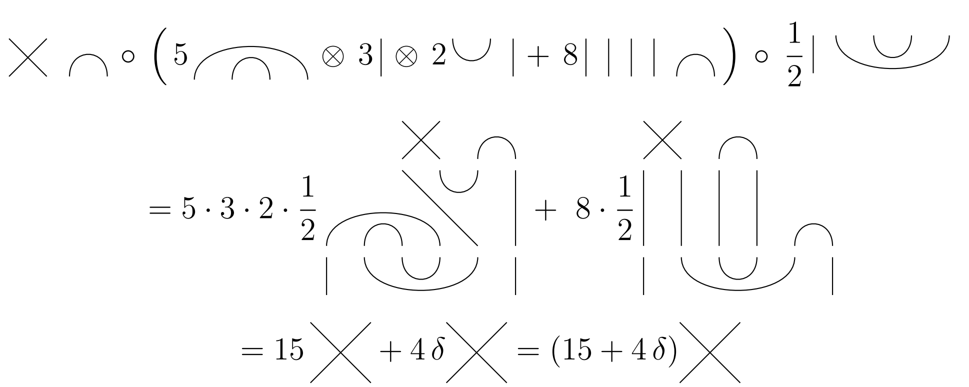
Definition 4.4.3. Let $\delta \in R$. In the *Brauer category* $\mathcal{B}(\delta)$, objects consist of formal tensors $|^{\otimes n}$ for $n \in \mathbb{N}$, corresponding to small parallel vertical lines. For any $n, m \in \mathbb{N}$, the hom-set $\mathrm{Hom}(|^{\otimes n}, |^{\otimes m})$ is the free R-module on the set of Brauer diagrams from n to m endpoints.

Composition is done by vertical concatenation of diagrams (extended linearly), in which Brauer diagrams are stacked on top of each other and corresponding endpoints are glued together. Any closed loop thus formed is then replaced by a factor of δ.

In $\mathrm{Hom}(|^{\otimes n}, |^{\otimes n})$, the identity is the diagram composed of n vertical strings $|\,|\cdots|$.

We define a tensor product operation $\otimes$ by horizontal juxtaposition of diagrams, and with $|^{\otimes n} \otimes |^{\otimes m} = |^{\otimes n+m}$.

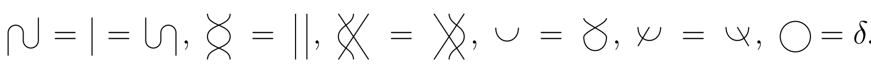
Example 4.4.4.



Definition 4.4.5. The *Brauer algebra on* n *strands* $B_n(\delta)$ is the hom-set $\mathrm{End}_{\mathcal{B}(\delta)}(|^{\otimes n})$.

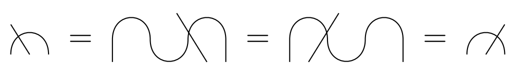
Remark 4.4.6. For more information on the Brauer category and Brauer algebras, a good reference is [LZ15]. Brauer algebras have been used in representation theory for a long time, starting with [Bra37].

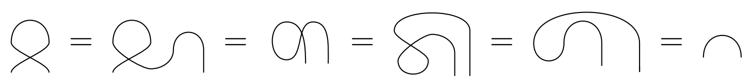
Proposition 4.4.7 ([LZ15, Theorem 2.6])**.** *As a linear monoidal category, $\mathcal{B}(\delta)$ has a presentation with generating object $|$, generating morphisms $\cup$, $\cap$ and $\times$, and relations*


(4.4.1)

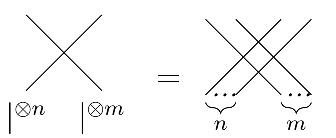
Proof: In the presentation indicated in [LZ15, Theorem 2.6], they include the horizontal and vertical reflection of the above relations. Hence, their presentation include also

These two relations follow from our smaller presentation, by computing

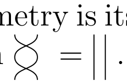

and


∎

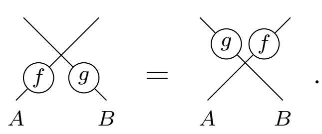
Lemma 4.4.8. *The Brauer category is symmetric monoidal, with symmetries*


(4.4.2)

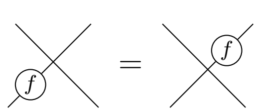
Proof: The hexagon equations (3.2.1) are satisfied by definition, while the inverse and naturality conditions are satisfied since isotopic diagrams are equal.

Alternatively, we can prove the lemma by using only the presentation.The fact that a symmetry is its own inverse (equation (3.2.2)) is verified by repeatedly applying the relation  .

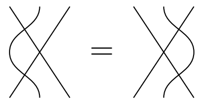
We must show naturality (3.2.3), meaning that for any $f: A \to A'$, $g: B \to B'$,



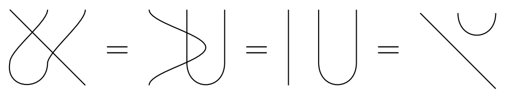
For that, we note that this equation is true when f is a generator. Indeed, if $f = \times$, the equation



is precisely the relation



If $f = \cup$ then we obtain



and similarly when $f = \cap$. The naturality equation is therefore proved by moving one after the other the generators contained in f and g. ∎

Proposition 4.4.9 (Universal property)**.** *The Brauer category is the free linear symmetric monoidal category on a symmetrically self-dual object of dimension δ.*

This means that, for any linear symmetric monoidal category C containing a symmetrically self-dual object V of dimension δ with unit ξ, there exists a unique symmetric monoidal functor $\psi\colon \mathcal{B}(\delta) \to C$ satisfying

$$| \mapsto V, \ \cup \mapsto \xi. \tag{4.4.3}$$

Proof: Existence: We must verify that for each relation $f = f'$ in Proposition 4.4.7, we have $\psi(f) = \psi(f')$. This is true since V is symmetrically self-dual with $\dim(V) = \delta$, and C is symmetric momoidal, so V and C satisfy Definitions 2.5.2, 2.9.4, 3.2.1, and 3.3.1, and Lemma 3.1.2.

Unicity: By Remark 2.5.9, there exists a unique counit ζ associated to the unit ξ. Since ψ is monoidal, it must send $\cap$ to a counit of ξ, hence to ζ. Since ψ is symmetric, it must send symmetries to symmetries. Therefore ψ is specified on every monoidal generator of $\mathcal{B}(\delta)$, so is unique. ∎

Proposition 4.4.10. *In the category $\mathcal{B}(\delta)$, the object $|$ is Schurian.*

Proof: There is only one Brauer diagram from 1 to 1, which is the identity $|$. This means that $\mathrm{End}(|)$ has a basis with a single element, so $\mathrm{End}(|) = R$. See Definition 3.3.4 for the definition of a Schurian object. ∎

If an object A is Schurian and self-dual in a linear symmetric monoidal category, by Proposition 3.3.7 there is a square root of unity c such that $^A = c \cup^A$.

Let's denote by $\mathcal{B}^c(\delta)$ the category that is identical to $\mathcal{B}(\delta)$, except for the relation  $= c\cup$ replacing  $= \cup$. Then this category is monoidally isomorphic to $\mathcal{B}(\delta)$ through the isomorphism

$$\begin{aligned} \mathcal{B}(\delta) &\simeq \mathcal{B}^c(\delta)\ . \\ \cup &\mapsto \cup \\ \cap &\mapsto \cap \\ \times &\mapsto c\times \end{aligned} \tag{4.4.4}$$

We thus obtain that the Brauer category is the free linear symmetric monoidal category on a Schurian self-dual object, but the unique monoidal functor involved might not be a symmetric monoidal functor.

Proposition 4.4.11. *For any linear symmetric monoidal category C containing a Schurian self-dual object V of dimension δ with unit ξ, there exists a unique monoidal functor $\psi \colon \mathcal{B}(\delta) \to C$ satisfying $\mid \mapsto V$, $\smile \mapsto \xi$.*

Remark 4.4.12. It is interesting to note that in Corollary 4.2.13, the properties of the Temperley-Lieb category hold in any monoidal category generated by a self-dual object. However, we need a distinct universal property when dealing with Schurian objects, Proposition 4.4.11 does not follow from Proposition 4.4.9. This is because the property of being Schurian is not defined by equations between morphisms, and therefore is not preserved by functors.

4.5 Oriented Brauer category

The oriented Brauer category is the free linear symmetric monoidal category on a pair of dual objects of dimension δ. Its morphisms are sums of oriented diagrams with crossings.

Definition 4.5.1. An *oriented Brauer diagram* consists of two horizontal rows of dots, one on top of the other, which are linked pairwise by oriented strings. Oriented Brauer diagrams are identified up to *isotopy*, meaning that any continuous deformation of the strings gives the same diagram, as long as the endpoints are fixed.

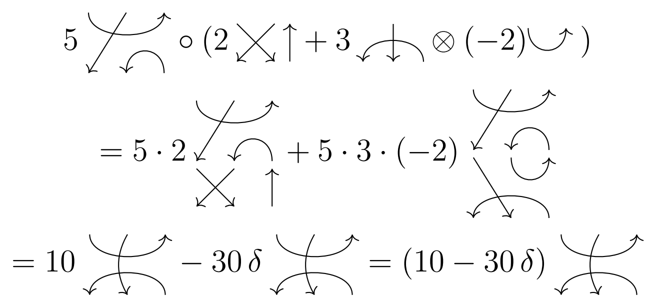
For $a, b \in \langle \uparrow, \downarrow \rangle$, we say that a diagram goes from a to b if the orientation of strings matches a at the bottom and b at the top. The empty diagram is an oriented Brauer diagram from the empty word to the empty word.

Example 4.5.2. The following is an oriented Brauer diagram from $\downarrow\downarrow\uparrow\uparrow\downarrow$ to $\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow$.

Definition 4.5.3. Let $\delta \in R$. In the *oriented Brauer category* $\mathcal{OB}(\delta)$, objects are elements of $\langle\uparrow,\downarrow\rangle$ (words in $\uparrow$ and $\downarrow$). For any $a, b \in \langle\uparrow,\downarrow\rangle$, the hom-set $\mathrm{Hom}_{\mathcal{OB}(\delta)}(a, b)$ is the free R-module on the set of oriented Brauer diagrams from a to b. Composition is done by vertical concatenation, extended linearly. Any closed loop thus formed is then replaced by a factor of δ. In $\mathrm{Hom}(a, a)$ the identity is the diagram composed of non-crossing vertical strings.

We define a tensor product $\otimes$ by concatenation of words and horizontal juxtaposition of diagrams.

Example 4.5.4.



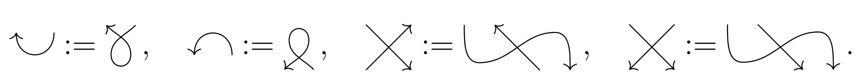
Lemma 4.5.5. *The oriented Brauer category is a strict monoidal category, with* $\mathbb{1}$ *being the empty word and* $1_{\mathbb{1}}$ *the empty diagram.*

Proof: Composition is associative, and vertical strings without crossings form the identity for each hom-set, since diagrams are defined up to isotopy. Hence $\mathcal{OB}(\delta)$ is indeed a category. The reader can verify each axiom of Definition 2.2.12, which hold either by properties of the free monoid $\langle\uparrow,\downarrow\rangle$, or by the properties of horizontal juxtaposition of diagrams. ∎

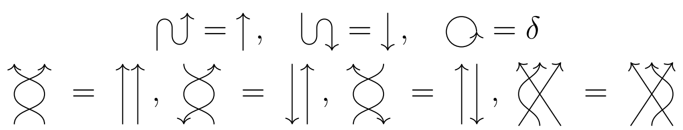
Definition 4.5.6. The endomorphism algebra $B_{r,s}(\delta) = \mathrm{End}_{\mathcal{OB}(\delta)}(\uparrow^{\otimes r}\downarrow^{\otimes s})$ is called the *walled Brauer algebra*. It has been used for many years to study Schur-Weyl duality, before the appearance of the oriented Brauer category. See for instance [CDVDM08] and [BCNR17, p. 1]. Note that any object a in $\mathcal{OB}(\delta)$ will be isomorphic to an object of the form $\uparrow^{\otimes r}\downarrow^{\otimes s}$.

Remark 4.5.7. Various variants of the oriented Brauer category appear in the literature. For instance, [BCNR17] define the affine oriented Brauer category $\mathcal{AOB}$, the cyclotomic oriented Brauer category $\mathcal{OB}^f$, and graded versions of these. The category $\mathcal{OB}(\delta)$ has also been studied extensively in [Rey15], especially concerning its representations.

Proposition 4.5.8 ([BCNR17, Theorem 1.1])**.** *Consider the generating objects* $\uparrow$ *and* $\downarrow$*, and the generating morphisms* ◡↗, ↙◠, ⤧ *and* ⤨*. Define*


(4.5.1)

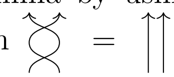
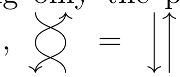
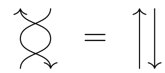
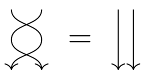
As a linear monoidal category, $\mathcal{OB}(\delta)$ *has a presentation with generators* $\uparrow$, $\downarrow$, , ,  *and* *, and relations*

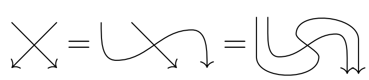


(4.5.2)

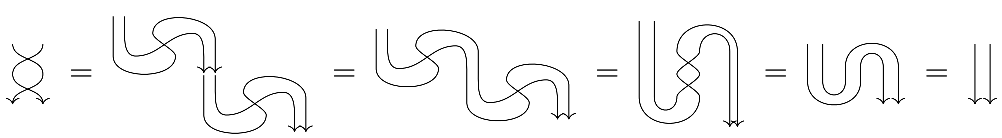
Lemma 4.5.9. *The oriented Brauer category is symmetric monoidal, with symmetries determined by* *,* *,* *,*  *and the hexagon equations* (3.2.1)*.*

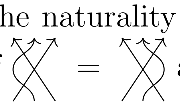
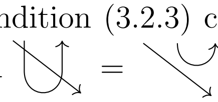
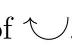
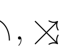
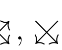
Proof: The hexagon equations are satisfied by definition, while the inverse and naturality conditions are satisfied since isotopic diagrams are equal.

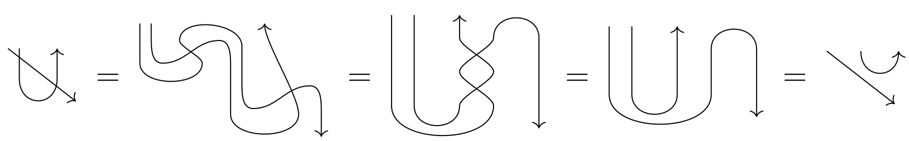
Alternatively, we can prove the lemma by using only the presentation. The condition on inverses (3.2.2) comes from , ,  and , where the last equation comes from noticing that

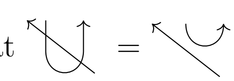
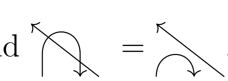


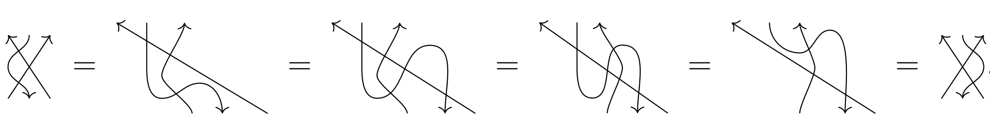
so



The naturality condition (3.2.3) comes from the fact that we can prove all variants of  and . The reader can check all relevant relations (there are eight of each of these two types) by using the definitions of , , , , and by using mates to "rotate equations by 180°". It is preferable to start by proving the equations involving cups or caps, and then use them to prove the equations with three crossings. For instance, we can directly prove



and other variants. Similarly, one can show that  and , and then observe that

 ∎

Proposition 4.5.10 (Universal property)**.** *The oriented Brauer category is the free linear symmetric monoidal category on a pair of dual objects of dimension δ.*

This means that, if C is any linear symmetric monoidal category containing a pair of dual objects $V, V^* \in C$ of dimension δ with left unit ξ, then there exists a unique symmetric monoidal functor $\psi\colon \mathcal{OB}(\delta) \to C$ satisfying

$$\uparrow \mapsto V,\ \downarrow \mapsto V^*,\ \circlearrowright\!\cup \mapsto \xi. \tag{4.5.3}$$

Proof: First, note that since $\mathcal{OB}(\delta)$ is symmetric monoidal, the equations $\cap\!\!\cup\!\!\uparrow = \uparrow$, $\cup\!\!\cap\!\!\downarrow = \downarrow$ and $\bigcirc = \delta$ are enough to ensure that $\uparrow$ and $\downarrow$ are dual on both sides with dimension δ (see Propositions 3.1.3 and 3.1.5).

Unicity: Suppose that such a ψ exists. By Remark 2.5.9, there exists a unique counit ζ associated to the unit ξ. Since ψ is symmetric monoidal, it must send $\curvearrowright$ to a counit of ξ, hence to ζ, and it must send symmetries to symmetries. Hence ψ is specified on every monoidal generator of $\mathcal{OB}(\delta)$, so is unique if it exists.

Existence: We must verify that for each relation $f = f'$ in Proposition 4.5.8, we have $\psi(f) = \psi(f')$. This is true since V and V^* are dual objects, their dimension is δ, and C is symmetric monoidal, so V and C satisfy Definitions 2.5.1, 2.9.4 and 3.2.1. ∎

Proposition 4.5.11. *Denote by $\mathcal{OB}'(\delta)$ the category $\mathcal{OB}(\delta)$ to which we add two morphisms and three relations*

$$\updownarrow\!\circ : \uparrow \to \downarrow, \quad \circ\!\updownarrow : \downarrow \to \uparrow, \quad \substack{\circ\\\circ} = 1_\downarrow, \quad \substack{\circ\\\circ} = 1_\uparrow,$$

$$\times\!\circ = \times\!\circ \tag{4.5.4}$$

thus making $\uparrow$ isomorphic to $\downarrow$, and ensuring naturality of the symmetries. Then the category $\mathcal{OB}'(\delta)$ is monoidally equivalent to $\mathcal{B}(\delta)$.

Proof: Recall that the definition of a monoidal equivalence is at Definition 2.2.15. We use the monoidal functors

$$F\colon \mathcal{B}(\delta) \to \mathcal{OB}'(\delta), \quad | \mapsto \uparrow, \quad \cup \mapsto \circ\!\cup\!\!\uparrow, \quad \cap \mapsto \downarrow\!\cap\!\circ, \quad \times \mapsto \times, \quad \text{and}$$

$$G\colon \mathcal{OB}'(\delta) \to \mathcal{B}(\delta), \quad \uparrow \mapsto |, \quad \downarrow \mapsto |,$$

$$\updownarrow\!\circ \mapsto 1_|, \quad \circ\!\updownarrow \mapsto 1_|, \quad \circlearrowright\!\cup \mapsto \cup, \quad \curvearrowright \mapsto \cap, \quad \times \mapsto \times, \quad \times \mapsto \times.$$

It is clear that $G \circ F = 1_{\mathcal{B}(\delta)}$. As in the proof of Proposition 4.3.11, define the monoidal transformation

$$\tau\colon 1_{\mathcal{OB}'(\delta)} \to F \circ G,$$

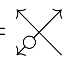
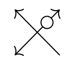
$$\tau_\uparrow = 1_\uparrow : \uparrow \to \uparrow, \quad \tau_\downarrow = \circ\!\updownarrow : \downarrow \to \uparrow.$$

We must show that τ is natural, which by Definition 2.2.4 amounts to checking

$$(F \circ G)(f) \circ \tau_A = \tau_{A'} \circ f$$

for every generator $f \colon A \to A'$.

If f is ∪↗, ∩↘, ↓↑ or ↑↓, the calculation is the same as in the proof of Proposition 4.3.11. For $f = \times$, this is automatic since $F(G(f)) = f$ and $\tau_{\uparrow\uparrow} = 1_{\uparrow\uparrow}$. When $f = \times \colon \downarrow\uparrow \to \uparrow\downarrow$, we use equation (4.5.4) to get

$$F(G(f)) \circ \tau_{\downarrow\uparrow} =$$  $$=$$  $$= \tau_{\uparrow\downarrow} \circ f. \qquad \blacksquare$$

4.6 Representation theory of some groups and quantum groups

We now have four universal categories that are related by the following functors:

$$\begin{array}{ccc} \mathcal{OTL}(\delta) & \longrightarrow & \mathcal{TL}(\delta) \\ \downarrow & & \downarrow \\ \mathcal{OB}(\delta) & \longrightarrow & \mathcal{B}(\delta) \end{array} \tag{4.6.1}$$

These functors are unique if we impose

$$\begin{array}{ccccc} \mathcal{OTL}(\delta) & \to & \mathcal{TL}(\delta) & \to & \mathcal{B}(\delta) \\ \uparrow & \mapsto & | & \mapsto & | \\ \downarrow & \mapsto & | & \mapsto & | \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathcal{OTL}(\delta) & \to & \mathcal{OB}(\delta) & \to & \mathcal{B}(\delta) \\ \uparrow & \mapsto & \uparrow & \mapsto & | \\ \downarrow & \mapsto & \downarrow & \mapsto & | \end{array} \tag{4.6.2}$$

We can think of $\mathcal{TL}(\delta)$ as a subcategory of $\mathcal{B}(\delta)$, and $\mathcal{OTL}(\delta)$ as a subcategory of $\mathcal{OB}(\delta)$, where we only keep morphisms without crossings. Moreover, by Proposition 4.3.11, when we add an isomorphism $\uparrow \simeq \downarrow$ to $\mathcal{OTL}(\delta)$ we get a category monoidally equivalent to $\mathcal{TL}(\delta)$, and by Proposition 4.5.11 the same is true for $\mathcal{B}(\delta)$ and $\mathcal{OB}(\delta)$.

Hence, we have four tools which can be used to analyze representations in various contexts.

4.6.1 Representations of groups

Let G be a group and $(V, \rho) \in \mathrm{Rep}_{\Bbbk}(G)$ be a finite-dimensional representation of G (the monoidal category $\mathrm{Rep}_{\Bbbk}(G)$ is defined in Definition 2.6.2). Let (V^*, ρ^*) be the dual representation of (V, ρ) (as in Definition 2.6.4), and denote by δ the dimension of (V, ρ) (which is the same as the dimension of V by Example 2.9.7). By Example 3.2.4

we know that $\mathrm{Rep}_{\Bbbk}(G)$ is symmetric monoidal. Hence, by Proposition 4.5.10, we know that there is a symmetric monoidal functor $\psi\colon \mathcal{OB}(\delta) \to \mathrm{Rep}_{\Bbbk}(G)$ such that $\psi(\uparrow) = (V,\rho)$ and $\psi(\downarrow) = (V^*,\rho^*)$. Moreover, this functor ψ is uniquely determined by where it sends the unit $\smile\!\!\uparrow$. Standard choices for the left unit and left counit are

$$\xi\colon V_0 \to V^* \otimes V, \quad 1 \mapsto \sum_{v\in B} \delta_v \otimes v, \quad \text{and} \quad \zeta\colon V \otimes V^* \to V_0, \quad v \otimes f \mapsto f(v), \tag{4.6.3}$$

for B some basis of $\Bbbk^n$, and V_0 the trivial representation $(\Bbbk,\phi)$, where $\phi(A)c = c$ for all $A \in G$, $c \in \Bbbk$. The standard choices for the right unit and counit are similar. These are intertwiners by the proof of Proposition 2.6.5 (note that ξ is independent of the basis chosen by Lemma 2.4.1).

Suppose that $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$, and that V is an inner product space. If ρ is a unitary representation (which means that $\langle \rho(g)w, \rho(g)v\rangle = \langle w, v\rangle$ for any $g \in G$, $v, w \in V$), by Proposition 3.3.2 we know that (V,ρ) is symmetrically self-dual. Hence, by Proposition 4.4.9, we know that there is a symmetric monoidal functor $\psi\colon \mathcal{B}(\delta) \to \mathrm{Rep}_{\Bbbk}(G)$ such that $\psi(|) = (V,\rho)$, where δ is the dimension of (V,ρ). Moreover, this functor ψ is uniquely determined by where it sends the unit $\smile$. Standard choices for the unit and counit are

$$\xi\colon V_0 \to V \otimes V,\ 1 \mapsto \sum_{v\in B} v \otimes v \quad \text{and} \quad \zeta\colon V \otimes V \to V_0,\ w \otimes v \mapsto \langle w, v\rangle \tag{4.6.4}$$

for B some orthonormal basis of $\Bbbk^n$. These are intertwiners by the proof of Proposition 2.6.6, and ξ is independent of the basis chosen by Lemma 2.5.4).

4.6.2 Representations of quantum groups

For representations of groups, we can also use $\mathcal{OTL}(\delta)$ and $\mathcal{TL}(\delta)$ in the same way. However, if the category of representations is monoidal but not symmetric monoidal, then we cannot use the Brauer categories and must use the Temperley-Lieb categories instead.

To give examples of categories of representations that are not symmetric monoidal, it is interesting to consider Hopf algebras and quantum enveloping algebras (also called *quantum groups*). We refer the reader to Chapters III and XVI of [Kas95].

Take any Hopf algebra H with antipode S. Then the category $\mathrm{Rep}_{\Bbbk}(H)$ of finite-dimensional $\Bbbk$-representations (or $\Bbbk$-modules) of H is a monoidal category, where every representation V of H has a dual representation $V^* = \mathrm{Hom}_{\Bbbk}(V,\Bbbk)$ with action

$$(hf)(v) = f(S(h)v) \quad \forall h \in H, f \in V^*, v \in V$$

(see [Kas95, §III.5]). Hence, by Proposition 4.3.10, there is a functor $\psi\colon \mathcal{OTL}(n) \to \mathrm{Rep}_{\Bbbk}(H)$ such that $\psi(\uparrow) = V$ and $\psi(\downarrow) = V^*$ (where $n = \dim V$), and this functor is uniquely determined by where it sends $\smile\!\!\uparrow$ and $\uparrow\!\!\smile$.

We can have non-commutative and non-cocommutative Hopf algebras. For instance, one can consider a Lie algebra $\mathfrak{g}$ and then take the quantum enveloping algebra $H = U_q(\mathfrak{g})$. For such Hopf algebras, the category $\mathrm{Rep}_{\Bbbk}(U_q(\mathfrak{g}))$ will not be symmetric monoidal, hence it would not be possible to use the oriented Brauer category. However, the category $\mathrm{Rep}_{\Bbbk}(U_q(\mathfrak{g}))$ will be braided in general, and this leads to quantum analogues of the Brauer categories.

When $U_q(\mathfrak{g})$ has a self-dual representation, we can use the Temperley-Lieb category. For instance, the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ (defined in [Kas95, Chapter VI]) has a self-dual representation of dimension two. Hence, there is a functor $\mathcal{TL}(2) \to \mathrm{Rep}(U_q(\mathfrak{sl}_2))$, see [FK97, §1.3] for details.

4.6.3 Deligne's category

In some cases, we can do more that having a functor from a Brauer or Temperley-Lieb category to a category of representations. We can get all finite-dimensional representations by adding direct sums and direct summands, which can be done by using the additive Karoubi envelope. For C a category, we denote by $\mathrm{Kar}(C)$ the additive Karoubi envelope, which consists in taking the additive completion (see for instance [Che14, Definition 2.4.8]) and then its Karoubi envelope (also *idempotent completion*, see for instance [BS01, Definition 1.2]).

Consider the general linear group $G = \mathrm{GL}_n(\Bbbk)$ that consists of all invertible $n \times n$ matrices with entries in $\Bbbk$. Denote by $V_1 = (\Bbbk^n, \rho)$ the defining representation. By the above discussion, there is a unique symmetric functor $\psi\colon \mathcal{OB}(n) \to \mathrm{Rep}(\mathrm{GL}_n(\Bbbk))$ such that $\uparrow \mapsto V_1$, $\downarrow \mapsto V_1^*$ and $\smile\!\!\nearrow \mapsto \xi$.

The image of ψ contains all representations which are tensor products of V_1 and V_1^*. To get all finite-dimensional representations, we can use Deligne's category. In [Del07, Définition 10.2], he defines a category $\underline{\mathrm{Rep}}_0(\mathrm{GL}(n), \Bbbk)$, which is equivalent to $\mathcal{OB}(n)$. Then, *loc. cit.* defines $\underline{\mathrm{Rep}}(\mathrm{GL}(n), \Bbbk) = \mathrm{Kar}(\underline{\mathrm{Rep}}_0(\mathrm{GL}(n), \Bbbk))$. He then proves in Théorème 10.4 that his category $\underline{\mathrm{Rep}}(\mathrm{GL}(n), \Bbbk)$ is equivalent to $\mathrm{Rep}(\mathrm{GL}_n(\Bbbk))$ (to be precise, it is $\underline{\mathrm{Rep}}(\mathrm{GL}(n), \Bbbk)/(\text{negligible morphisms})$ which is equivalent to the category of finite-dimensional representations of $\mathrm{GL}_n(\Bbbk)$, where a morphism $f\colon A \to B$ is negligible if for any $u\colon B \to A$ we have $\mathrm{tr}(fu) = 0)$).

Similarly, consider the orthogonal group

$$O_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \langle Av, Aw \rangle = \langle v, w \rangle \ \forall v, w \in \mathbb{R}^n\},$$

where $\langle v, w \rangle = v^T w$ is the standard inner product on $\mathbb{R}^n$. (We could also take $O_n(\mathbb{C})$ with the same definition, in which case this bilinear pairing would not be an inner product but everything would still follow.) Denote by $V_1 = (\mathbb{R}^n, \rho)$ the defining representation, where

$$\rho(A)v = Av \quad \forall A \in O_n(\mathbb{R}),\ v \in \mathbb{R}^n.$$

We then have a unique symmetric functor $\psi\colon \mathcal{B}(n) \to \mathrm{Rep}(O_n(\mathbb{R}))$ such that $| \mapsto V_1$ and $\smile \mapsto \xi$.

We can also use Deligne's category here. In [Del07, Définition 9.2], he uses the notation $\mathrm{Rep}_0(O(n), \Bbbk)$ for what we call the Brauer category. Then, in paragraph 9.3 he uses the additive Karoubi envelope to define defines $\underline{\mathrm{Re}}\mathrm{p}(O(n), \Bbbk) = \mathrm{Kar}(\underline{\mathrm{Re}}\mathrm{p}_0(O(n), \Bbbk))$. [Del07, Théorème 9.6] then says that his category $\underline{\mathrm{Re}}\mathrm{p}(O(n), \mathbb{R})$ (quotiented by negligible morphisms) is equivalent to $\mathrm{Rep}(O_n(\mathbb{R}))$. It is interesting that his Théorème 9.6 also applies to the symplectic and orthosymplectic groups, whose representations can likewise be studied using the Brauer category. This is related to Example 3.3.3.

There is a similar phenomenon with the Temperley-Lieb category and $U_q sl_2$. $\mathrm{Kar}(\mathcal{TL}(2))$ is equivalent to $\mathrm{Rep}(U_q(\mathfrak{sl}_2)$, see for instance [Mor07, §3.3.1].

For the oriented Temperley-Lieb category, we need to be careful since $\mathcal{OTL}(\delta)$ is not braided, while $\mathrm{Rep}(U_q(\mathfrak{g}))$ is braided for many $\mathfrak{g}$. We cannot expect $\mathrm{Kar}(\mathcal{OTL}(\delta))$ to be equivalent to a braided category, which might be one reason why the oriented Temperley-Lieb category is less used in the literature.

Note that it is an active area of research to find some category $\mathscr{C}$ such that $\mathrm{Kar}(\mathscr{C})$ be equivalent to $\mathrm{Rep}(\mathfrak{g})$, for various Lie algebras $\mathfrak{g}$. This is done in [Kup96] for rank 2 Lie algebras, with categories $\mathscr{C}$ of "spiders and webs", while [Mor07] and [CKM14] treat the cases $\mathfrak{g} = \mathfrak{sl}_n$. An example of recent work is [RT20], which considers $\mathfrak{g} = \mathfrak{sp}_6$.

Chapter 5

Adjoint functors and presentations of linear monoidal categories

In Chapter 4, we saw four examples of categories that have a presentation with generators and relations, and they each satisfied a universal property. It is a general phenomenon that presentations and universal properties are related. In this chapter, we consider the arrow "Monoidal generators $\leftrightarrow$ Universal properties" from Figure 1.1. In one direction, we show that every category satisfies a universal property whenever it has a presentation. For the converse, we offer an algorithm that finds a presentation for some free categories satisfying specific universal properties.

To achieve these goals, our main tool consists in adjoint functors. This has a very different flavour as the other chapters, and we will not be using string diagrams here (although string diagrams can be used when dealing with adjoints in some cases).

5.1 Adjoint functors

Adjoint functors are special pairs of functors that arise in many areas of mathematics. There are three equivalent definitions of adjunctions, related to universal properties, hom-sets, and duality. In this work, we use the characterization using universal properties.

Definition 5.1.1 (Universal property, [ML98, §III.1])**.** Consider categories $\mathscr{C}$ and $\mathcal{D}$, a functor $U\colon \mathcal{D} \to \mathscr{C}$, and an object $X \in \operatorname{ob}\mathscr{C}$. A pair (A, η), for A an object of $\mathcal{D}$ and $\eta\colon X \to UA$, is said to satisfy a *universal property* if for any $A' \in \operatorname{ob}\mathcal{D}$ and any morphism $f\colon X \to UA'$ there exists a unique $g\colon A \to A'$ such that $f = Ug \circ \eta$. In that case the pair (A, η) is said to be a *universal morphism* from X to U.

The following definition of adjunction says that F is left adjoint to U if for any object $X \in \mathscr{C}$, the pair (FX, η_X) is a universal morphism from X to U, and if η is natural.

Definition 5.1.2 ([Awo10, def. 9.1])**.** Given categories and functors $F\colon \mathscr{C} \to \mathcal{D}$ and $U\colon \mathcal{D} \to \mathscr{C}$, F is *left adjoint* to U (and U is *right adjoint* to F) if there is a natural transformation $\eta\colon 1_{\mathscr{C}} \to U \circ F$ (so $\forall\, f\colon C \to C'$, $\eta_{C'} \circ f = UFf \circ \eta_C$) such that

$$\forall\, C \in \mathscr{C}, D \in \mathcal{D}, f\colon C \to UD, \quad \exists!\, g\colon FC \to D \text{ st } f = Ug \circ \eta_C.$$

In this case we can say that (F, U, η) is an *adjunction*, and η is called the *unit*.

It is customary to represent this as a commutative diagram.

$$FC \xrightarrow{\;g\;} D \qquad \begin{array}{ccc} UFC & \xrightarrow{\;Ug\;} & UD \\ {\scriptstyle \eta_C}\uparrow & \nearrow{\scriptstyle f} & \\ C & & \end{array} \tag{5.1.1}$$

Proposition 5.1.3. *Let $F\colon \mathscr{C} \to \mathcal{D}$ and $U\colon \mathcal{D} \to \mathscr{C}$ be functors.*

1. *F is left adjoint to U if and only if there is an isomorphism*

$$\mathrm{Hom}_{\mathscr{C}}(FA, B) \cong \mathrm{Hom}_{\mathcal{D}}(A, UB)$$

   which is natural in $A \in \mathscr{C}, B \in \mathcal{D}$.

2. *Moreover, F is left adjoint to U if and only if there exist natural transformations $\eta\colon 1_{\mathscr{C}} \to U \circ F$ (called the* unit*) and $\epsilon\colon F \circ U \to 1_{\mathcal{D}}$ (called the* counit*) such that for each $A \in \mathscr{C}, B \in \mathcal{D}$,*

$$1_{FB} = \epsilon_{FB} \circ F(\eta_B) \quad \textit{and} \quad 1_{UA} = U(\epsilon_A) \circ \eta_{UA}.$$

Proof: [ML98, p. 80] defines adjunctions with the hom-set condition, as in point 1 of our Proposition 5.1.3. Then [ML98, Theorems IV.1.1 and IV.1.2] prove that this is equivalent to the unit/counit characterization (point 2 of Proposition 5.1.3) and to the universal property characterization (Definition 5.1.2). ∎

Remark 5.1.4. The definition using Hom-sets is analogous to the condition for adjoint linear maps, and this is where the name "adjoint functor" comes from. The "unit-counit definition" corresponds to saying that F is left-dual to U in the category where objects are functors and morphisms are natural transformations. However to state this precisely we need to use 2-categories instead of monoidal categories, since the "product" of functors is the composition $F \otimes G = F \circ G$, which is not always defined.

Example 5.1.5 (Vector spaces). Let $\mathscr{C} = \mathbf{Set}$, and let $\mathcal{D} = \mathbf{Vect}_{\Bbbk}$ the category of $\Bbbk$-vector spaces, for $\Bbbk$ a field. Let $U\colon \mathbf{Vect}_{\Bbbk} \to \mathbf{Set}$ be the forgetful functor that assigns to each vector space its underlying set, and to each linear map its underlying set function. For any set X, let $FX \in \mathbf{Vect}_{\Bbbk}$ be the free vector space over X (so the vector space having X as a basis). Then we all know that for any other vector space $V \in \mathbf{Vect}_{\Bbbk}$, if we define a linear map on the basis X then we are automatically defining it on all of FX. In terms of morphisms, this means that if we define some set map $f\colon X \to UV$, then there exists a unique linear map $g\colon FX \to V$ satisfying $f(x) = g(x)\ \forall x \in X$. We can express this as $f = Ug \circ \eta$ if we take $\eta_X\colon X \to UFX$ to be the inclusion map sending x to x.

Hence, from the fact "a linear map is uniquely determined by what it does on a basis", we arrive at the conclusion that FX, the free vector space on X, always satisfies a universal property. From this, when we define the functor $F\colon \mathbf{Set} \to \mathbf{Vect}_{\Bbbk}$ as sending X to FX and sending a function $f\colon X \to Y$ to the function

$$Ff\colon FX \to FY,\ \sum a_i x_i \mapsto \sum a_i f(x_i),$$

we get that the free vector space functor F is left adjoint to the forgetful functor U. It is straightforward to check that this η is natural: for any $h\colon X \to Y$, it is clear that applying $x \mapsto x$ before or after h does not change anything, so $\eta_Y \circ h = UFh \circ \eta_X$.

Example 5.1.6 (Polynomials). Any polynomial ring satisfies a universal property, so is in the range of an adjoint functor. To see this, we again take $\mathscr{C} = \mathbf{Set}$, but $\mathcal{D}$ will be $\mathbf{ComutAlg}_R$, the category of commutative R-algebras, for R a commutative ring. The forgetful functor here is $U\colon \mathbf{ComutAlg}_R \to \mathbf{Set}$ that sends an algebra to its underlying set, and an algebra homomorphism to its underlying set function. For a set $X = \{x_1, x_2, \dots\}$, the free commutative R-algebra on X is the polynomial ring $FX = R[x_1, x_2, \dots]$. We have a clear inclusion $\eta_X\colon \{x_1, x_2, \dots\} \to U(R[x_1, x_2, \dots])$ that sends x_i to x_i.

To get the adjunction, we simply need to recall that if an algebra homomorphism has as domain a polynomial ring, it is entirely determined when we know what happens to the variables x_i. This means that for any $A \in \mathbf{ComutAlg}_R$ and any $f\colon \{x_1, x_2, \dots\} \to UA$, there exists a unique $g\colon R[x_1, x_2, \dots] \to A$ satisfying $f(x_i) = g(x_i)$ for each i — namely, $g(\sum a_i x_i^{\alpha_i}) = \sum a_i f(x_i)^{\alpha_i}$. This in turns is precisely the statement that $f = Ug \circ \eta$.

Hence, if we define a free functor $F\colon \mathbf{Set} \to \mathbf{ComutAlg}_R$ as sending $X \to FX$ and $f\colon X \to Y$ to

$$Ff\colon FX \to FY,\ \sum a_i x_i^{\alpha_i} \mapsto \sum a_i f(x_i)^{\alpha_i},$$

we get that F is left adjoint to the forgetful functor U. As in the previous example, η is natural since for any $h\colon X \to Y$, $\eta_Y \circ h = UFh \circ \eta_X$.

Example 5.1.7 (Discrete and trivial topologies). For topological spaces, the forgetful functor has both a left and a right adjoint. Define **Top** to be the category whose objects are topological spaces $(X, \mathcal{T})$ (for $\mathcal{T}$ a collection of open subsets of X) and morphisms $f\colon (X, \mathcal{T}) \to (Y, \mathcal{T}')$ are continuous maps (for any $U \in \mathcal{T}'$, we have $f^{-1}(U) \in \mathcal{T}$). Define the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ that sends $(X, \mathcal{T})$ to its underlying set X and $f\colon (X, \mathcal{T}) \to (Y, \mathcal{T}')$ to the underlying function. Define also the functors sending a set to its discrete and its trivial topologies,

$$\mathbf{discrete}\colon \mathbf{Set} \to \mathbf{Top},\ X \mapsto (X, \mathcal{P}(X)),$$

$$\mathbf{trivial}\colon \mathbf{Set} \to \mathbf{Top},\ X \mapsto (X, \{\emptyset, X\}).$$

In both cases, for a morphism $f\colon X \to Y$ we can take the same morphism for $\mathbf{discrete}(f)$ and $\mathbf{trivial}(f)$, since anything going from a discrete topology to something is continuous (since clearly $f^{-1}(U) \in \mathcal{P}(X)$) and anything going from something to a trivial topology is continuous (since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$).

We then have that **discrete** is left adjoint to U, and that **trivial** is right adjoint to U. We can directly check the definition. For **discrete**, since $U(\mathbf{discrete}(X)) = X$, we can take η_X to be the identity (which is clearly natural), and we just need to check that

$$\forall X \in \mathbf{Set},\ (Y, \mathcal{T}) \in \mathbf{Top}, f\colon X \to Y,\ \exists!\, g\colon (X, \mathcal{P}(X)) \to (Y, \mathcal{T}) \text{ st } f = Ug \circ \eta_X.$$

This is asking that $f(x) = g(x)$ for all $x \in X$, so we have unicity. Proving existence amounts to proving that $g\colon x \mapsto f(x)$ is continuous, but this is clear since for all $U \in \mathcal{T}$ we have $g^{-1}(U) \in \mathcal{P}(X)$.

To show that **trivial** is right adjoint to U, we first define for any topological space $(X, \mathcal{T})$ the map

$$\eta_{(X,\mathcal{T})}\colon (X, \mathcal{T}) \to (X, \{\emptyset, X\}), \quad x \mapsto x\,;$$

this map is necessarily continuous since its codomain has the trivial topology. Then, we directly have

$$\forall (X, \mathcal{T}) \in \mathbf{Top}, Y \in \mathbf{Set}, f\colon (X, \mathcal{T}) \to (Y, \{\emptyset, Y\}),$$

$$\exists!\, g\colon X \to Y \text{ such that } f = \mathbf{trivial}(g) \circ \eta_{(X,\mathcal{T})},$$

simply by taking $g(x) = f(x)$ for all $x \in X$. It is interesting to notice that in this last example the existence and unicity of g do not depend on the topology chosen, but the continuity of $\eta_{(X,\mathcal{T})}$ require that we use the trivial topology.

Example 5.1.8. If we consider the functor $F\colon \mathbf{Set} \to \mathbf{Grp}$ that sends a set X to the free group on X, and the functor $U\colon \mathbf{Grp} \to \mathbf{Set}$ that sends a group to its underlying set, we obtain that F is left adjoint to U. See Appendix B for the details of the verification.

Other standard examples can be found in [ML98, IV.2].

Proposition 5.1.9 ([Awo10, prop. 9.9]). *Adjoints are unique up to isomorphism. This means that if $F\colon \mathscr{C} \to \mathcal{D}$ is left adjoint to $U\colon \mathcal{D} \to \mathscr{C}$ and $U'\colon \mathcal{D} \to \mathscr{C}$, then U and U' are naturally isomorphic. Similarly if it has two right adjoints V and V' then V and V' are naturally isomorphic.*

5.2 Presentation functor of linear monoidal categories

When dealing with a specific linear monoidal category, as in Chapter 4, it is often the case that we can find generators and relations to describe the morphisms. Whenever we have such a presentation, there is a universal property associated to it. Our new Theorem 5.2.6 makes this precise, by stating that the functor that sends generators and relations to the category generated by them is a left adjoint functor.

In Appendix B, we prove a similar theorem for presentations of groups in full details. This appendix serves as a warm-up for the present section; the readers who are not used to the linear monoidal categories and their quotients could benefit from reading it first.

Definition 5.2.1. A category is *small* if its objects and morphisms form sets (as opposed to proper classes). We denote by **LinMonCat** the category where objects are small R-linear monoidal categories, and morphisms are functors that are both strong monoidal and linear.

Definition 5.2.2. The *forgetful functor* $U\colon$ **LinMonCat** $\to$ **MonQuiv** sends a linear monoidal category $\mathscr{C}$ to its underlying quiver (Q_0, Q_1, s, t) where $Q_0 = \operatorname{ob}\mathscr{C}$, Q_1 is the set of all morphisms in $\mathscr{C}$, s sends a morphism to its domain, and t sends a morphism to its codomain (forgetting the data of composition, identities and linear structure). U sends a linear monoidal functor ψ to its underlying monoidal quiver morphism $U(\psi)$.

Definition 5.2.3. The *free functor* $F\colon$ **MonQuiv** $\to$ **LinMonCat** sends a monoidal quiver (Q_0, Q_1) to the free linear monoidal category on (Q_0, Q_1), as defined in Definition 4.1.3. F sends a morphism of monoidal quivers $\phi\colon (Q_0, Q_1) \to (Q_0', Q_1')$ to the linear monoidal functor determined by $F\phi(a \otimes b) = \phi(a) \otimes \phi(b)$, $F\phi(f \otimes g) = \phi(f) \otimes \phi(g)$ and $F\phi(f \circ g) = \phi(f) \circ \phi(g)$, for $a, b \in Q_0$, $f, g \in Q_1$.

Proposition 5.2.4. *The free functor $F\colon$ **MonQuiv** $\to$ **LinMonCat** is left adjoint to the forgetful functor $U\colon$ **LinMonCat** $\to$ **MonQuiv**. Its unit is $\tau\colon 1_{\mathbf{MonQuiv}} \to U \circ F$ where $\tau_Q\colon Q \to UFQ$ sends $a \in Q_0$ to $a \in \langle Q_0 \rangle$ and $f \in Q_1$ to f in FQ.*

Proof: By [Liu18, thm 4.1], we know that for any $Q \in \mathbf{MonQuiv}$, any $C \in \mathbf{LinMonCat}$ and any $\phi\colon Q \to UC$, there exists a unique $\psi\colon FQ \to C$ such that $U\psi \circ \tau_Q = \phi$. Moreover it is straightforward to check that τ is natural, hence we have an adjonction. ∎

Definition 5.2.5. Recall that the definition of quotients by relations is at Definition 4.1.5, and presentations are defined at Definition 4.1.8. The category **Pres** is the category having as objects all pairs $(Q, \sim)$ for $Q \in \mathbf{MonQuiv}$ and $\sim$ a relation on FQ. A morphism $\phi\colon (Q, \sim) \to (Q', \sim')$ is a monoidal quiver morphism $\phi\colon Q \to Q'$ that satisfies the condition

$$\forall f, g\colon a \to b \text{ in } FQ,\ f \sim g \Rightarrow F\phi(f) \sim' F\phi(g). \tag{5.2.1}$$

Composition and identities are the same as in **MonQuiv**.

The *presentation functor of linear monoidal categories* is

$$\begin{aligned} P\colon \mathbf{Pres} &\to \mathbf{LinMonCat}, \\ (Q,\sim) &\mapsto FQ/{\sim} \\ \phi\colon (Q,\sim) \to (Q',\sim') \quad &\mapsto \quad P\phi\colon FQ/{\sim} \to FQ'/{\sim'}, \\ &\qquad\qquad a \mapsto F\phi(a) \\ &\qquad\qquad [f]_\sim \mapsto [F\phi(f)]_{\sim'} \end{aligned} \tag{5.2.2}$$

The forgetful functor from **LinMonCat** to **Pres** is

$$\begin{aligned} \mathcal{U}\colon \mathbf{LinMonCat} &\to \mathbf{Pres}, \\ C &\mapsto (UC, =) \\ \psi\colon C \to C' \quad &\mapsto \quad \mathcal{U}\psi = U\psi : (UC,=) \to (UC',=). \end{aligned} \tag{5.2.3}$$

Finally, we have the transformation $\eta\colon 1_{\mathbf{LMCRel}} \to \mathcal{U} \circ P$ defined by

$$\forall (Q,\sim) \in \mathbf{Pres}, \quad \eta_{(Q,\sim)} = U\pi_{(FQ,\sim)} \circ \tau_Q. \tag{5.2.4}$$

Note that this $P\phi$ is well-defined by the condition (5.2.1), and that $\mathcal{U}\psi$ is indeed a morphism in **Pres** since the condition "$\forall f, g\colon a \to b \in FUC,\ f = g \ \Rightarrow\ F\mathcal{U}\psi(f) = F\mathcal{U}\psi(g)$" is trivially satisfied.

Theorem 5.2.6. *The presentation functor $P\colon \mathbf{Pres} \to \mathbf{LinMonCat}$ is left adjoint to the forgetful functor $\mathcal{U}\colon \mathbf{LinMonCat} \to \mathbf{Pres}$ with unit η. This means that*

$$\begin{gathered} \forall (Q,\sim) \in \mathbf{Pres},\ C \in \mathbf{LinMonCat},\ \phi\colon (Q,\sim) \to (UC,=), \\ \exists!\, \psi\colon FQ/{\sim} \to C \text{ such that } \phi = \mathcal{U}\psi \circ \eta_{(Q,\sim)}. \end{gathered} \tag{5.2.5}$$

Moreover, $\eta_{(Q,\sim)}$ is injective for any $(Q,\sim) \in \mathbf{Pres}$ satisfying

$$\forall \text{ distinct } f, f'\colon a \to b \text{ in } Q, \quad \tau_Q(f) \not\sim \tau_Q(f'). \tag{5.2.6}$$

The condition (5.2.6) is a "non-degeneracy" requirement, to avoid that some generating morphisms be equated by $\sim$. For instance, if in the Temperley-Lieb category we add a generator $\widetilde{\cup}$ and then impose $\cup = \widetilde{\cup}$, we are not getting anything new. It seems as if we have more generators, but in fact we will get an isomorphic category, with the obvious isomorphism $[\cup]_\sim \mapsto [\cup]_\sim$, $[\cap]_\sim \mapsto [\cap]_\sim$. Hence the only effect of this change is to make η non-injective, since $\eta(\cup) = [\cup]_\sim = \eta(\widetilde{\cup})$.

In order to prove Theorem 5.2.6, we now introduce some technical tools: the category **LMCRel** and the functors Π and E.

Definition 5.2.7. The category **LMCRel** is the category having as objects all pairs $(C, \sim)$ for $C \in$ **LinMonCat** and $\sim$ a relation on C. A morphism $\psi\colon (C,\sim) \to (C', \sim')$ is a linear monoidal functor $\psi\colon C \to C'$ that satisfies the condition

$$\forall f, g\colon a \to b \text{ in } C, \quad f \boldsymbol{\sim} g \Rightarrow \psi(f) \boldsymbol{\sim}' \psi(g), \tag{5.2.7}$$

for $\boldsymbol{\sim}$ and $\boldsymbol{\sim}'$ the smallest congruences containing $\sim$ and $\sim'$ respectively.

The *quotient functor* is

$$\begin{aligned} \Pi\colon \mathbf{LMCRel} &\to \mathbf{LinMonCat}, \\ (C,\sim) &\mapsto C/\sim \\ \psi\colon (C,\sim) \to (C',\sim') \quad &\mapsto \quad \Pi\psi\colon \langle C\rangle/\sim \to \langle C'\rangle/\sim', \\ &\qquad [f]_\sim \mapsto [\psi(f)]_{\sim'} \end{aligned} \tag{5.2.8}$$

This $\Pi(\psi)$ is well-defined by the condition (5.2.7).

We use the functor

$$\begin{aligned} E\colon \mathbf{LinMonCat} &\to \mathbf{LMCRel}, \\ C &\mapsto (C, =) \\ \psi\colon C \to C' \quad &\mapsto \quad E\psi = \psi : (C,=) \to (C,=), \end{aligned} \tag{5.2.9}$$

by defining the relation $=$ on C as the relation where only equal morphisms are related.

Finally, we have the transformation $\pi\colon 1_{\mathbf{LMCRel}} \to E \circ \Pi$ defined by

$$\pi_{(C,\sim)}\colon (C,\sim) \to (C/\sim, =), \quad f \mapsto [f]_\sim. \tag{5.2.10}$$

Lemma 5.2.8. *The functor* Π *is left adjoint to* E *with unit* π*.*

Proof: By Definition 5.1.2, we must show that for any pair $(C,\sim) \in$ **LMCRel**, $C' \in$ **LinMonCat** and $\psi\colon (C,\sim) \to (C',=)$, there exists a unique $\psi'\colon C/\sim \to C'$ such that

$$\psi = \psi' \circ \pi_{(C,\sim)}.$$

Since $\psi\colon (C,\sim) \to (C',=)$ means that

$$\forall f,g\colon a \to b \text{ in } C, \quad f \sim g \Rightarrow \psi(f) = \psi(g),$$

we obtain existence and unicity of ψ' by [ML98, prop. II.8.1].

Moreover, we have that π is natural since

$$\forall \psi\colon (C,\sim) \to (C',\sim'), \quad \pi_{(C',\sim')} \circ \psi = E\,\Pi\psi \circ \pi_{(C,\sim)}.$$

Indeed, both sides are precisely $f \mapsto [\psi(f)]_{\sim'}$. ∎

The proof of Theorem 5.2.6 boils down to composing the adjunctions (F,U,τ) and (Π,E,π), to obtain the adjunction $(P,\mathcal{U},\eta)$. To do this, we need the following proposition.

Proposition 5.2.9 ([ML98, prop. IV.8.1])**.** *Suppose that $F\colon A_1 \to A_2$ is left adjoint to $U\colon A_2 \to A_1$ with unit η, and $\bar{F}\colon A_2 \to A_3$ is left adjoint to $\bar{U}\colon A_3 \to A_2$ wth unit $\bar{\eta}$. Then $\bar{F} \circ F\colon A_1 \to A_3$ is left adjoint to $G \circ \bar{G}\colon A_3 \to A_1$ with unit $G(\bar{\eta}_F) \circ \eta$ — this last natural transformation having component $G(\bar{\eta}_{F(X)}) \circ \eta_X$ for any $X \in \mathscr{C}$.*

Proof of Theorem 5.2.6: To apply Proposition 5.2.9 to F, U and τ, we need to modify slightly the domains and codomains by defining F', U' and τ' that act only on the group parts of the pairs in **Pres**.

$$F'\colon \mathbf{Pres} \to \mathbf{LMCRel}, \quad (Q,\sim) \mapsto (FQ,\sim),$$
$$\phi\colon (Q,\sim) \to (Q',\sim') \quad \mapsto \quad F\phi\colon (FQ,\sim) \to (FQ',\sim'),$$

$$U'\colon \mathbf{LMCRel} \to \mathbf{Pres}, \quad (C,\sim) \mapsto (UC,\sim),$$
$$\psi\colon (C,\sim) \to (C',\sim') \quad \mapsto \quad U\psi\colon (UC,\sim) \to (UC',\sim),$$

$$\tau'\colon 1_{\mathbf{Pres}} \to U' \circ F',\ \tau'_{(Q,\sim)} = \tau_Q\colon (Q,\sim) \to (UFQ,\sim).$$

Conditions (5.2.7) and (5.2.1) for $F'\phi$ and $U'\psi$ translate into tautologies. We can see that τ' is still natural in the same way as τ, and that F' and U' satisfy the condition

$$\forall (Q,\sim) \in \mathbf{Pres}, (C,\sim') \in \mathbf{LMCRel}, \phi\colon (Q,\sim) \to (UC,\sim'),$$
$$\exists!\, \psi\colon (FQ,\sim) \to (C,\sim') \text{ st } \phi = U'\psi \circ \tau_{(Q,\sim)}.$$

Indeed $\phi\colon (Q,\sim) \to (UC,\sim')$ implies $\phi\colon Q \to UC$ so by Proposition 5.2.4 we have a unique $\psi\colon FQ \to C$ satisfying $\phi = U\psi \circ \tau_Q$ (and therefore satisfying $\phi = U'\psi \circ \tau'_{(Q,\sim)}$). Knowing that $\psi(a) = F\phi(a)$, we only need to check condition (5.2.7) for the morphism $\psi\colon (FQ,\sim) \to (C,\sim')$:

$$\forall f,g\colon a \to b \text{ in } FQ \quad f \sim g \Rightarrow F\phi(f) \sim' F\phi(g) \Rightarrow \psi(f) \sim' \psi(g),$$

by using condition (5.2.1) for ϕ. Hence F' is left adjoint to U' with unit τ'.

We remark that

$$P = \Pi \circ F' \colon \mathbf{Pres} \to \mathbf{LMCRel} \to \mathbf{LinMonCat},$$

$$\mathcal{U} = U' \circ E \colon \mathbf{LinMonCat} \to \mathbf{LMCRel} \to \mathbf{Pres}, \quad \text{and}$$

$$\eta = U'\pi_{F'} \circ \tau' \colon 1_{\mathbf{Pres}} \to \mathcal{U} \circ P.$$

Hence we can conclude that P is left adjoint to $\mathcal{U}$ with unit η by applying Proposition 5.2.9.

Finally, suppose that $(Q, \sim) \in \mathbf{Pres}$ is such that condition (5.2.6) is satisfied. Since $\eta_{(Q,\sim)} = U\pi_{(FQ,\sim)} \circ \tau_Q$,

$$f \neq f' \;\Rightarrow\; \tau_Q(f) \not\sim \tau_Q(f')$$

$$\Rightarrow\; \eta_{(Q,\sim)}(f) = U\pi_{(FQ,\sim)}(\tau_Q(f)) = [\tau_Q(f)] \neq [\tau_Q(f')] = U\pi_{(FQ,\sim)}(\tau_Q(f')) = \eta_{(Q,\sim)}(f').$$

Hence $\eta_{(Q,\sim)}$ is injective. ∎

The preceding proof can be visualized as follow: we have six functors

$F' \colon \mathbf{Pres} \to \mathbf{LMCRel},\ (Q,\sim) \mapsto (FQ,\sim)$ $\quad U' \colon \mathbf{LMCRel} \to \mathbf{Pres},\ (C,\sim) \mapsto (UC,\sim)$

$\Pi \colon \mathbf{LMCRel} \to \mathbf{LinMonCat},\ (C,\sim) \mapsto C/{\sim}$ $\quad E \colon \mathbf{LinMonCat} \to \mathbf{LMCRel},\ C \mapsto (C,=)$

$P \colon \mathbf{Pres} \to \mathbf{LinMonCat},\ (Q,\sim) \mapsto FQ/{\sim}$ $\quad \mathcal{U} \colon \mathbf{LinMonCat} \to \mathbf{Pres},\ C \mapsto (UC,=)$

(omitting their effect on morphisms). For a given $\phi \colon (Q,\sim) \to (UC,=)$, we use the adjonction of F' and U' to get a unique $\psi_1 \colon F'(Q,\sim) \to (C,=)$ such that

$$\begin{array}{ccc} U'F'(Q,\sim) & \xrightarrow{U\psi_1} & U'(C,=) \\ {\scriptstyle \tau'_Q}\uparrow & \nearrow_{\phi} & \\ (Q,\sim) & & \end{array}$$

commutes, and then by adjunction of Π and E we get a unique $\psi \colon \Pi F'(Q,\sim) \to C$ such that

$$\begin{array}{ccc} E\,\Pi F'(Q,\sim) & \xrightarrow{E\psi} & EC \\ {\scriptstyle \pi_{F'(Q,\sim)}}\uparrow & \nearrow_{\psi_1} & \\ F'(Q,\sim) & & \end{array}$$

commutes. This ψ is indeed the one we want, since by using $P = \Pi \circ F'$, $\mathcal{U} = U' \circ E$, and $\eta_{(Q,\sim)} = U'\pi_{F'(Q,\sim)} \circ \tau'_Q$, we obtain commutativity of

$$\begin{array}{ccc} \mathcal{U}P(Q,\sim) & \xrightarrow{\mathcal{U}\psi} & \mathcal{U}C\ . \\ {\scriptstyle \eta_{(Q,\sim)}}\uparrow & \nearrow_{\phi} & \\ (Q,\sim) & & \end{array}$$

In fact ψ can be described with, for $a, b \in Q_0$, $f, g \in Q_1$, $\psi(a \otimes b) = \phi(a) \otimes \phi(b)$, $\psi[f \otimes g]_\sim = [\phi(f)]_\sim \otimes [\phi(g)]_\sim$ and $\psi(f \circ g) = [\phi(f)]_\sim \circ [\phi(g)]_\sim$.

Remark 5.2.10. Most algebraic constructions admit free functors, see [Awo10, ex. 9.37, prop. 9.38] and [ML98, p. 124-125]. Whenever quotients exist, we could expect to have analogue adjoint functors of quotients Π and E, and then analogue adjoint functors of presentations P and $\mathcal{U}$. For instance, it should be possible to get an analogue to Theorem 5.2.6 for presentations of rings, algebras, modules... In particular Theorem B.13 is the analogue for groups, and the discussion in this appendix can be adapted easily for presentations of monoids.

5.3 Constructing universal categories with desired properties

We wish to describe the arrow "Universal properties → Monoidal generators" from Figure 1.1. One way to treat this is to consider generators and relations, to which we add some *local properties* (attributed to the generating objects or morphisms) and *global properties* (attributed to the whole category). We would like to know which presentation gives us the "free category on these generators with these properties".

In this thesis, we are particularly interested in objects being self-dual or symmetrically self-dual, possessing a dual, or having a dimension $\dim(A) \in R$; and the category being symmetric monoidal. To encode this, we use three different sets of objects: O_{sd}, O_{ssd} and O_d, which respectively contain the objects we want to be self-dual, the objects we want to be symmetrically self-dual, and the object to which we want to add a dual. We also specify a dimension function $\dim\colon O_{sd} \cup O_{ssd} \cup O_d \to R$, and we use a boolean value sym = true if we want the category to be symmetric monoidal, and sym = false if we do not want symmetries. We must have sym = true if some objects are symmetrically self-dual.

In what follows, when we instruct to "add" an element a to a set Q, we mean that we replace the set Q by the disjoint union $Q \cup \{a\}$.

Algorithm 5.3.1.
Input: Disjoint finite sets of objects O_{sd}, O_{ssd} and O_d with respective cardinalities n_1, n_2 and n_3, function $\dim\colon O_{sd} \cup O_{ssd} \cup O_d \to R$. Boolean value $sym \in \{\text{true}, \text{false}\}$, which must be true if $O_{ssd} \neq \emptyset$.

1. Initialize: set the monoidal quiver $(Q_0, Q_1) = (O_{sd} \cup O_{ssd} \cup O_d, \emptyset)$.
   (a) For every object $A \in O_{sd} \cup O_{ssd}$, add to Q_1 morphisms $\cup^A \colon \mathbb{1} \to A \otimes A$ and $\cap_A \colon A \otimes A \to \mathbb{1}$.
   (b) If *sym* = false, for every $\uparrow_A \in O_d$, add to Q_0 an object $\downarrow_A$ and add to Q_1 morphisms

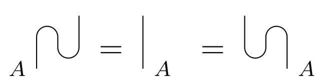
$$^A\!\circlearrowleft\colon \mathbb{1} \to \downarrow_A \otimes \uparrow_A, \quad {}_A\!\curvearrowright\colon \uparrow_A \otimes \downarrow_A \to \mathbb{1},$$

$$\circlearrowright^A\colon \mathbb{1} \to \uparrow_A \otimes \downarrow_A \quad \text{and} \quad \curvearrowleft_A \downarrow_A \otimes \uparrow_A \to \mathbb{1}.$$

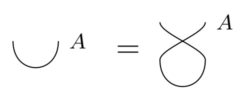
   (c) If *sym* = true, add to Q_1 morphisms $\underset{A\ B}{\times} : A \otimes B \to B \otimes A$ for every $A, B \in O_{sd} \cup O_{ssd} \cup O_d$.
   For each $\uparrow_A \in O_d$, add to Q_0 an object $\downarrow_A$ and add to Q_1 morphisms $^A\!\circlearrowleft$ and ${}_A\!\curvearrowright$, and define $\circlearrowright^A :=$ (crossed cup)^A, $\curvearrowleft_A :=$ (crossed cap)$_A$.

   This gives a monoidal quiver of generators $Q = (Q_0, Q_1)$.

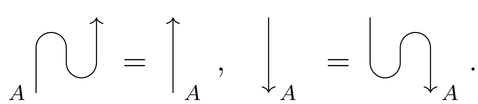
2. Initialize: set the relation $\sim = \emptyset$.
   (a) For every $A \in O_{sd} \cup O_{ssd}$, add to $\sim$ relations



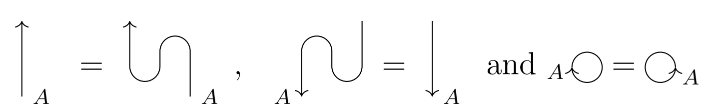
   and if $A \in O_{ssd}$ add



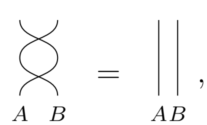
   Add also $\bigcirc_A = \dim(A)$ if A has dimension $\dim(A) \in R$.
   (b) For every $A \in O_d$, add to $\sim$ relations

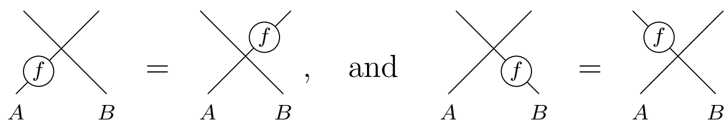


   Add also $\circlearrowright_A = \dim_A$ if A has dimension $\dim(A) \in R$.
   (c) If *sym* = false, for every $A \in O_d$ add to $\sim$ relations



   (d) If *sym* = true, add to $\sim$ relations





(5.3.1)

for all $A, B \in Q_0$, $f \in Q_1$.

This gives a relation $\sim$.

Output: The linear monoidal presentation $(Q, \sim)$.

Remark 5.3.2. More generally, we should be able to treat in the same way any local property defined by equations between morphisms. For instance, we could consider morphisms that are invertible or have trace $\mathrm{tr}(f) \in R$. It could be interesting to consider Schurian objects, since these are not defined by such equations. We could also consider the global properties of being braided, pivotal, traced, having a twist or a dagger, or other properties from [Sel11]. Moreover, the functor $P\colon \mathbf{Pres} \to \mathbf{LinMonCat}$ used here automatically adds linearity, but it is straightforward to devise a similar presentation functor of monoidal categories $\mathbf{Pres} \to \mathbf{MonCat}$.

A different approach to reach the same goal could be to follow [KL80]. They consider how to create free symmetric monoidal categories on some quiver where all objects have duals; so our algorithm generalizes part of their paper. They also give indications on how to generalize their results, see §4 and §10.

Theorem 5.3.3. *Consider* n_1*,* n_2*,* n_3*,* dim *and sym as in the input of Algorithm 5.3.1,* $(Q, \sim)$ *its output, and* $C = P(Q, \sim) = FQ/\sim$ *the category having* $(Q, \sim)$ *as presentation.*

If sym = false, C *is the free linear monoidal category on* n_1 *self-dual objects,* n_2 *symmetrically self-dual objects, and* n_3 *objects with a dual, each having specified dimension.*

If sym = true, C *is the free linear symmetric monoidal category on* n_1 *self-dual objects,* n_2 *symmetrically self-dual objects, and* n_3 *objects with a dual, each having specified dimension.*

Proof: To show that C is such free category, we must show that it satisfies the following universal property:

Let D be any linear monoidal (resp. symmetric monoidal) category containing objects with similar properties, in the sense that there exists $\phi\colon Q \to UD$ such that $\phi(A)$ is self-dual for any $A \in O_{sd}$, symmetrically self-dual for any $A \in O_{ssd}$ and has a dual for any $A \in O_d$, and satisfying also $\dim(\phi(A)) = \dim(A)$ for every $A \in O_{sd} \cup O_{ssd} \cup O_d$. Then we must show that there exists a unique monoidal functor $\psi\colon C \to D$ such that $\phi = \mathcal{U}\psi \circ \eta_{(Q,\sim)}$.

First, we remark that when *sym* = true the category C is symmetric monoidal. The hexagon equations (3.2.1) are satisfied by definition, while the invertibility (3.2.2) and naturality (3.2.3) axioms are satisfied for generators and then extended to all objects. Moreover, by construction the generating objects satisfy the required properties of duality and dimension. Note that when sym = true the relations in step 2b are enough to ensure that $\uparrow_A$ and $\downarrow_A$ are dual on both sides with specified dimensions, by Propositions 3.1.3 and 3.1.5.

We apply Theorem 5.2.6 to C. Since the objects $\phi(A)$ satisfy the same properties as the objects A, ϕ satisfies

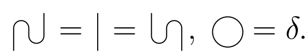
$$\forall f, g\colon a \to b \text{ in } FQ',\ f \sim g \Rightarrow F\phi(f) = F\phi(g)$$

which is condition (5.2.1). Therefore, we have $\phi\colon (Q, \sim) \to (UD, =)$. By Theorem 5.2.6, there thus exists a unique $\psi\colon P(Q, \sim) \to D$ such that $\phi = \mathcal{U}\psi \circ \eta_{(Q,\sim)}$. ∎

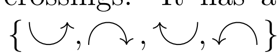
Note that when sym = true, the functor $\psi\colon C \to D$ is a symmetric functor, since it must preserve the symmetries in Q, and therefore all symmetries.

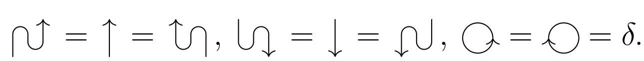
Examples 5.3.4. This algorithm can be illustrated with the four categories in Chapter 4. For each, bases of the hom-sets consist in diagrams that link dots pairwise in two horizontal lines; with or without orientation and with or without crossings.

- §4.2: The Temperley-Lieb category $\mathcal{TL}(\delta)$ consists of non-oriented diagrams without crossings. It has a presentation with $Q_0 = \{\,|\,\}$, $Q_1 = \{\cup, \cap\}$, and relations

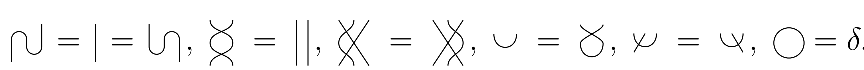
  Hence, by Algorithm 5.3.1 and Theorem 5.3.3, it is the free linear monoidal category on a self-dual object of dimension δ.

- §4.3: The oriented Temperley-Lieb category $\mathcal{OTL}(\delta)$ consists of oriented diagrams without crossings. It has a presentation with objects $Q_0 = \{\uparrow, \downarrow\}$, morphisms $Q_1 =$ , and relations

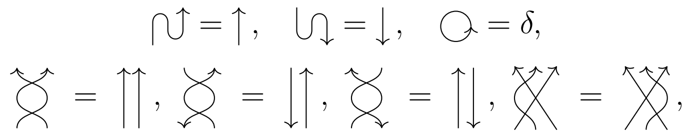
  Hence it is the free linear monoidal category on a pair of dual objects of dimension δ.

- §4.4: The Brauer category $\mathcal{B}(\delta)$ consists of non-oriented diagrams with crossings. It has a presentation with $Q_0 = \{\,|\,\}$, $Q_1 = \{\cup, \cap, \times\}$, and relations

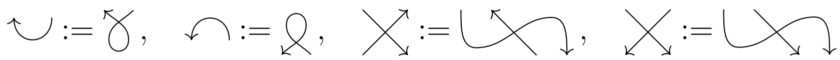
  

  Hence it is the free linear symmetric monoidal category on a symmetrically self-dual object of dimension δ.

- §4.5: The oriented Brauer category $\mathcal{OB}(\delta)$ consists of oriented diagrams with crossings. It has a presentation with $Q_0 = \{\uparrow, \downarrow\}$, $Q_1 = \{$$\}$ and relations

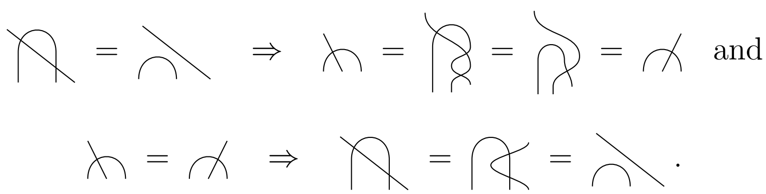


using the notation



Hence, it is the free linear symmetric monoidal category on a pair of dual objects of dimension δ.

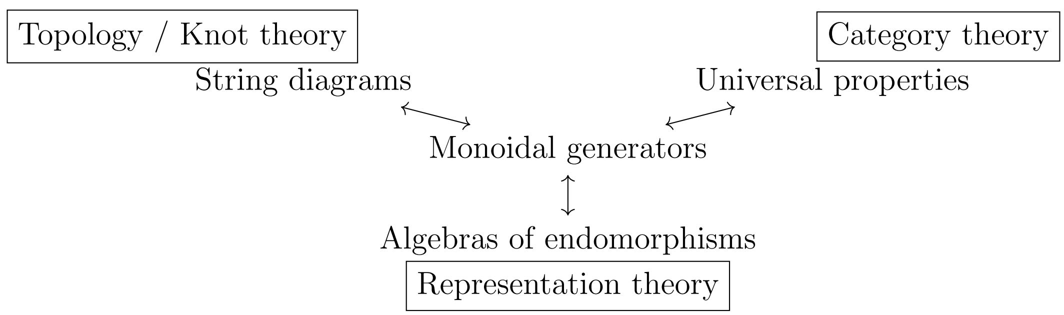
It is interesting to consider the relations (5.3.1) in step 2d of the algorithm, and compare them with the presentations of the Brauer and oriented Brauer categories. If $f =$ , we get the Yang-Baxter equations (3.1.4). If f is a cup or cap, we get relations equivalent to , which we could call the "pitchfork relations". For instance, if $f =$ ,



Chapter 6

Conclusion and future directions

We conclude as we started, with the figure summarizing our work.



In this thesis, we constantly tried to increase our understanding of this figure, either with results and theorems, or by using examples. The relation between string diagrams and generators has been exemplified by many small calculations in Chapters 2 and 3. Every time we see string diagram computations involving specific morphisms, we can think of them as if the morphisms were generators, and deforming the diagrams corresponds to using relations. The link representations has been evoked in §2.6 and Chapter 3. In §5.2 we described completely how to go from generators to universal properties, while §5.3 considered how to go from universal properties to generators in some particular cases.

To put together the six arrows indicated in our figure, we considered four examples of universal categories in Chapter 4. For the Temperley-Lieb, oriented Temperley-Lieb, Brauer and oriented Brauer categories, we described their morphisms through string diagrams, gave a presentation, and wrote their universal property. We have that $\mathcal{TL}(\delta)$ is the free linear monoidal category on a self-dual object of dimension δ, $\mathcal{OTL}(\delta)$ is the free linear monoidal category on a pair of dual objects of dimension δ, $\mathcal{B}(\delta)$ is the free linear symmetric monoidal category on a symmetrically self-dual object of dimension δ, and $\mathcal{OB}(\delta)$ is the free linear symmetric monoidal category on a pair of dual objects of dimension δ.

This exploration leads naturally to many generalizations. Here are five suggestions of future directions.

- The categories considered in Chapter 4 have only one generating object or one dual pair of generating objects, and they have no generating morphisms except those required for duality. It would be interesting to consider examples with more generators, maybe something like the free linear monoidal category on a self-dual object and a pair of dual objects, related by two generating morphisms. In such situations, we would probably need to use different colours or thicknesses to distinguish the strings associated to different morphisms.
- This thesis considered only universal categories with an interesting universal property, but there are plenty of diagrammatic categories that interest people involved in categorification with applications in representation theory. It would be a worthy task to assemble and compare these in a survey. Examples of such categories include various types of Heisenberg, wreath product, partition and Brauer categories, and their quantum, Frobenius, affine and super analogues.
- Presentations by generators and relations arise in many areas of mathematics, and it is interesting to compare the use of presentations in various settings. For instance, when considering the category of groups as in Appendix B, we can remember the *word problem*, which asks for an algorithm to know when two words in the generators are equal, given some group presentation. We can similarly define the word problem for monoidal categories: for a given presentation of a monoidal category, and morphisms built from the generators through tensors and composition, we want an algorithm testing whether two such morphisms are equal. The question is then to know if this word problem is interesting and leads to nice results. For diagrammatic categories as those considered in this thesis, the word problem can at first glance appear to be easy, since the relations then boil down to some sort of isotopy. However, further reflection shows that we could get some depth even for simple examples, since knot theory teaches us that it is non-trivial to know when two knots are isotopic.
- In §5.3, the algorithm is focused on a few properties. As mentioned in Remark 5.3.2, it could easily be generalized. It seems possible to add in this algorithm any property of generators that can be described by equations between morphisms. Moreover, we could consider many global properties of the category, in addition to being linear, monoidal or symmetric monoidal. When considering the arrow "Universal properties $\to$ Monoidal generators", the ultimate goal could be to get an algorithm that takes as input a given universal property, and outputs a presentation for a category satisfying this universal property. This goal does not seem easy, but it is always fun to dream large.

- Finally, one could investigate in more detail the arrows “String diagrams $\leftrightarrow$ Monoidal generators $\leftrightarrow$ Algebras of endomorphisms” of our figure. For instance, we could ask the following questions. Which sets of generators and relations lead to intuitive string diagrams manipulations? If we have a description of a diagrammatic category, is there some recipe to find a presentation for it? From a description of the algebras of endomorphisms, can we find a monoidal presentation of the category? If we know only the presentation of the category, is there some ways to easily compute properties of the endomorphism algebras and the other categories on which we act?

Much is known about this among experts, but much remains also to be discovered and formalized. The author is proud of having contributed to the grand adventure of mathematical research, and is profoundly grateful to all the mathematicians who worked and continue to work towards a greater understanding of the universe.

Appendix A

Commutative diagrams for monoidal categories

When monoidal categories or functors are not strict, there are many natural isomorphisms that we want to be well-behaved. To make precise what we mean by "well-behaved", it is standard to use commutative diagrams. We say that a diagram commutes when we get the same morphisms by composing along different paths sharing the same endpoints. The first definition add precision to Definition 2.2.10, while the last two definitions generalize Definitions 2.2.13 and 2.2.14.

The definitions used here can be found in [Sel11, p. 9 and 13]. However, for monoidal natural transformations Selinger mentions two diagrams but only writes one, hence we use [Bae04, Definition 11] to complete it.

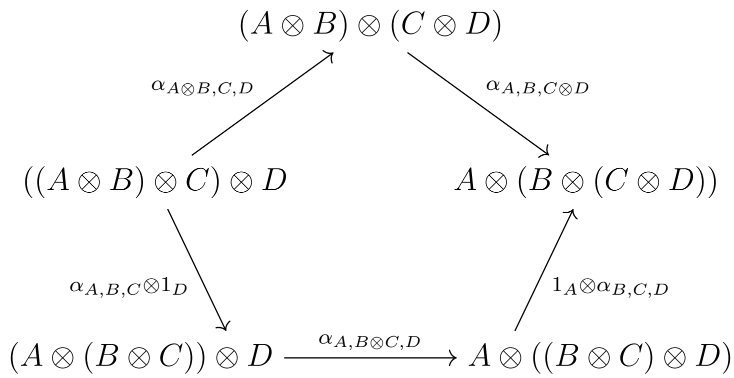
Definition A.1 (Monoidal category)**.** A *monoidal category* $(\mathscr{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a category $\mathscr{C}$ with

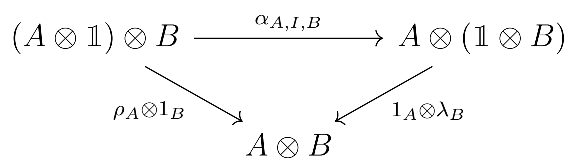
- a bifunctor $\otimes \colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ (meaning $1_{A\otimes B} = 1_A \otimes 1_B$ and it satisfies the *interchange law* (2.2.3)), and
- a unit object $\mathbb{1}$,

such that we have natural isomorphisms

$$\alpha_{A,B,C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C) \,,$$
$$\lambda_A \colon \mathbb{1} \otimes A \to A \quad \text{and}$$
$$\rho_A \colon A \otimes \mathbb{1} \to A,$$

which satisfy the "pentagon axiom" and "triangle axiom".



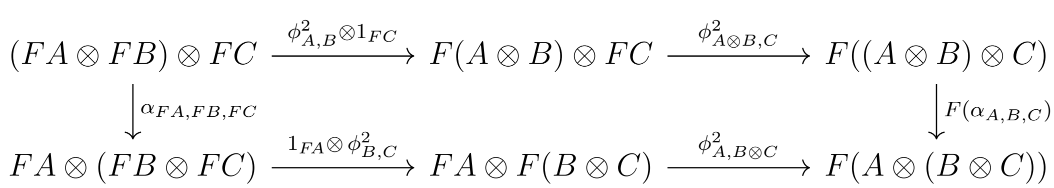


The natural isomorphism α is often called the *associator*, while λ and ρ are sometimes called the *left* and *right unitors*.

Mac Lane's coherence theorem tells us that any diagram composed of associators and unitors will commute. The statement of [ML98, Theorem VII.2.1] is quite abstract, so we use the formulation of [EGNO15, Theorem 2.9.2].

Theorem A.2 (Mac Lane's coherence). *Let $X_1, \dots, X_n$ be objects in a monoidal category $\mathscr{C}$. Let P_1, P_2 be any two parenthesized products of $X_1, \dots, X_n$ (in this order) with aribrary insertions of the unit object $\mathbb{1}$. Lef $f, g\colon P_1 \to P_2$ be two isomorphisms, obtained by composing associators and unitors and their inverses possibly tensored with identity morphisms. Then $f = g$.*

Definition A.3 (Strong monoidal functor). If $\mathscr{C}$ and $\mathcal{D}$ are monoidal categories, a *strong monoidal functor* is a functor $F\colon \mathscr{C} \to \mathcal{D}$ accompanied by a natural isomorphism $\phi^2_{A,B}\colon FA \otimes FB \to F(A \otimes B)$ and an isomorphism $\phi^0\colon \mathbb{1}_{\mathcal{D}} \to F\mathbb{1}_{\mathscr{C}}$, such that the three following diagrams commute.



$$\begin{array}{ccc} FA\otimes \mathbb{1} & \xrightarrow{\rho_{FA}} & FA \\ {\scriptstyle 1_{FA}\otimes\phi^0}\downarrow & & \uparrow{\scriptstyle F(\rho_A)} \\ FA\otimes F\mathbb{1} & \xrightarrow{\phi^2_{A,\mathbb{1}}} & F(A\otimes\mathbb{1}) \end{array} \qquad \begin{array}{ccc} \mathbb{1}\otimes FA & \xrightarrow{\lambda_{FA}} & FA \\ {\scriptstyle \phi^0\otimes 1_{FA}}\downarrow & & \uparrow{\scriptstyle F(\lambda_A)} \\ F\mathbb{1}\otimes FA & \xrightarrow{\phi^2_{\mathbb{1},A}} & F(\mathbb{1}\otimes A) \end{array}$$

Definition A.4 (Monoidal natural transformation)**.** Let $\mathscr{C}$ and $\mathcal{D}$ be two monoidal categories, and $(F\phi^2, \phi^0)$, (G, ψ^2, ψ^0) two strong functors from $\mathscr{C}$ to $\mathcal{D}$. Then a natural transformation $\eta\colon F \to G$ is *monoidal* if the two following diagrams commute.

$$\begin{array}{ccc} FA\otimes FB & \xrightarrow{\phi^2_{A,B}} & F(A\otimes B) \\ {\scriptstyle \eta_A\otimes\eta_B}\downarrow & & \downarrow{\scriptstyle \eta_{A,B}} \\ GA\otimes GB & \xrightarrow{\psi^2_{A,B}} & G(A\otimes B) \end{array} \qquad \begin{array}{ccc} \mathbb{1} & & \\ {\scriptstyle \phi^0}\downarrow & \searrow^{\psi^0} & \\ F\mathbb{1} & \xrightarrow[\eta_{\mathbb{1}}]{} & G\mathbb{1} \end{array}$$

Appendix B

Presentations of groups

In this appendix we define a functor sending generators and relations to the corresponding presentation of a group, and we show that this functor has a right adjoint. For this, we define the category **Gener** where the objects are pairs $(X, \sim)$ of a set and a relation, and an intermediary category **GrpRel** where the objects are pairs $(G, \sim)$ of a group and a relation. This is analogous to §5.2. Most definitions and results are standard, but the author did not see B.9, B.10, B.12 or B.13 appear elsewhere.

Definition B.1. A *group* is a triple $(G, \bullet, e_G)$ with G a set, $\bullet\colon G \times G \to G$ an operation on G and $e_G \in G$, such that for all $a, b, c \in G$,

1. $(a \bullet b) \bullet c) = a \bullet (b \bullet c)$ (associativity),
2. $a \bullet e_G = a = e_G \bullet a$ (neutral element),
3. $\exists\, a^{-1} \in G$ such that $a \bullet a^{-1} = e_G = a^{-1} \bullet a$ (existence of the inverse).

We usually denote the group by G, and the operation symbol $\bullet$ is often omitted.

A *group homomorphism* $f\colon G \to G'$ is a function from G to G' such that for all $a, b \in G$, $f(ab) = f(a)f(b)$.

These form the category **Grp** of groups with group homomorphisms.

Definition B.2 (Free groups)**.** Let X be a set. We denote by $\langle X\rangle_{\mathrm{G}}$ the *free group* on X, which is created by taking all formal inverses and products of elements of X. We use exponent notation to simplify the writing. In details,

$$\langle X\rangle_{\mathrm{G}} = \{a_1^{\epsilon_1} \cdots a_n^{\epsilon_n} \mid n \in \mathbb{N}, a_i \in X, \epsilon_i \in \mathbb{Z}\backslash\{0\},\ \text{and } \forall i\ a_i \neq a_{i+1}\},$$

where the conditions $a_i \neq a_{i+1}$ and $\epsilon_i \neq 0$ are to make sure that the expression is in a reduced form. This is a group where the product is concatenation of the expressions, followed by adding the exponents and removing elements with zero exponent, and where the neutral element is when $n = 0$.

For any function $f\colon X \to Y$, we define the corresponding group homomorphism

$$\overline{f}\colon \langle X\rangle_{\mathrm{G}} \to \langle Y\rangle_{\mathrm{G}}, \quad \overline{f}(a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}) = f(a_1)^{\epsilon_1} \cdots f(a_n)^{\epsilon_n}.$$

The *free functor of groups* is

$$F\colon \mathbf{Set} \to \mathbf{Grp},\ X \mapsto \langle X\rangle_{\mathrm{G}},\ f \mapsto \overline{f}. \tag{B.1}$$

Definition B.3. The *forgetful functor* $U\colon \mathbf{Grp} \to \mathbf{Set}$ sends a group G to its underlying set $U(G)$ (forgetting the operation and neutral element), and a group homomorphism $g\colon G \to H$ to the underlying function.

Proposition B.4. *The functor F is left adjoint to U with unit $\tau\colon 1_{\mathbf{Set}} \to U \circ F$, for $\tau_X\colon X \to U(\langle X\rangle_{\mathrm{G}}),\ x \mapsto x$.*

Proof: To satisfy Definition 5.1.2, we must show that for any $X \in \mathbf{Set}$, $G \in \mathbf{Grp}$ and $f\colon X \to U(G)$, there exists a unique $g\colon \langle X\rangle_{\mathrm{G}} \to G$ such that $f = U(g) \circ \tau_X$. We also need to show that τ is natural.

Saying that $f = U(f) \circ \tau_X$ means that $f(x) = g(x)$ for any $x \in X$, and therefore

$$g(a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}) = g(a_1)^{\epsilon_1} \cdots g(a_n)^{\epsilon_n} = f(a_1)^{\epsilon_1} \cdots f(a_n)^{\epsilon_n} = \overline{f}(a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}).$$

Hence the unique group homorphism that works is $g(a) = \overline{f}(a)\ \forall a \in \langle X\rangle_{\mathrm{G}}$. (Note that $g \neq \overline{f}$ as homomorphisms, since the codomains do not match).

Finally, we have that τ is natural since

$$\forall h\colon X \to Y,\ \tau_Y \circ h = U(\overline{h}) \circ \tau_X.$$

Indeed, both sides are precisely $x \mapsto h(x)$. ∎

Example B.5. The free group on one generator is isomorphic to $\mathbb{Z}$. If $X = \{x\}$, then $\langle X\rangle_{\mathrm{G}} = \{x^\epsilon \mid \epsilon \in \mathbb{Z}\}$, so we have the isomorphism $\mathbb{Z} \to \langle X\rangle_{\mathrm{G}}, n \mapsto x^n$.

Since $\mathbb{Z} = \langle 1\rangle_{\mathrm{G}}$ is the free group on one generator, it embeds in any group G, and the embedding is uniquely determined by the choice of where to send 1; choosing where to send 1 is the same a choosing a $f\colon \{1\} \to U(G)$.

Example B.6. In $\langle\{a,b,c\}\rangle_{\mathrm{G}}$, we find expresssions like $ab,\ c^3a^{-1}c,\ a^{-1}ba, abcb^5c^{-7}, \ldots$ There are products such as

$$ab \bullet b^{-1}c = abb^{-1}c = ac, \qquad abab \bullet ca^{-8} = ababca^{-8},$$

$$cab^2 \bullet b^{-2}a^{-1}c^{-1} \bullet a^9 = cab^2b^{-2}a^{-1}c^{-1} \bullet a^9 = e_{\langle\{a,b,c\}\rangle_{\mathrm{G}}} \bullet a^9 = a^9\,.$$

For the map $f\colon \{a,b,c\} \to \{1\},\ a \mapsto 1, b \mapsto 1, c \mapsto 1$, the corresponding group homomorphism $\overline{f}\colon \langle\{a,b,c\}\rangle_{\mathrm{G}} \to \mathbb{Z}$ sends an expression to the number $n \in \mathbb{Z}$ that is the sum of the exponents.

If we want to define a map from $\langle\{a,b,c\}\rangle_{\mathrm{G}}$ to a group G, it suffices to choose where to send a, b and c, since this determines a function $f\colon \{a,b,c\} \to U(G)$.

Definition B.7. Let G be a group. A *congruence* on G is an equivalence relation $\boldsymbol{\sim}$ on G such that

$$\forall a_1, a_2, b_1, b_2 \in G \quad a_1 \boldsymbol{\sim} a_2 \text{ and } b_1 \boldsymbol{\sim} b_2 \Rightarrow a_1 b_1 \boldsymbol{\sim} a_2 b_2.$$

If $\sim$ is any relation on G (meaning $\sim \subseteq G \times G$), we take $\boldsymbol{\sim}$ to be the smallest congruence containing $\sim$, and the *equivalence class* of an element $a \in G$ is

$$[a]_\sim = \{b \in G \mid a \boldsymbol{\sim} b\}.$$

The *quotient* of G by a relation $\sim$ is the group

$$G/\!\sim = \{[a]_\sim \mid a \in G\},$$

with product $[a]_\sim [b]_\sim = [ab]_\sim$ (well-defined since $\boldsymbol{\sim}$ is a congruence) and neutral element $e_{G/\sim} = [e_G]_\sim$.

We define the *quotient map* $\pi_{(G,\sim)} \colon G \to G/\!\sim,\ a \mapsto [a]_\sim$.

We may write $[a]$ instead of $[a]_\sim$ if the relation $\sim$ is clear from the context.

Remark B.8. The quotient groups are often constructed through normal subgroups instead of congruences. These definitions are equivalent and give the same universal properties, but quotienting by a relation is easier to generalize to other algebraic and categorical situations.

The category **GrpRel** and the functor Π and E combine groups and relations; they are technical tools used to prove Theorem B.13.

Definition B.9. The category **GrpRel** is the category having as objects all pairs $(G, \sim)$ for $G \in \mathbf{Grp}$ and $\sim$ a relation on G. A morphism $f \colon (G, \sim) \to (G', \sim')$ is a group homomorphism $f \colon G \to G'$ that satisfies the condition

$$\forall a, b \in G \quad a \boldsymbol{\sim} b \Rightarrow f(a) \boldsymbol{\sim}' f(b), \tag{B.2}$$

for $\boldsymbol{\sim}$ and $\boldsymbol{\sim}'$ the smallest congruences containing $\sim$ and $\sim'$ respectively.

The *quotient functor* is

$$\begin{aligned} \Pi \colon \mathbf{GrpRel} &\to \mathbf{Grp}, \\ (G, \sim) &\mapsto G/\!\sim \\ f \colon (G, \sim) \to (G', \sim') \quad &\mapsto \quad \Pi(f) \colon \langle G \rangle_{\mathrm{G}}/\!\sim \to \langle G' \rangle_{\mathrm{G}}/\!\sim', \\ & \qquad\qquad [a]_\sim \mapsto [f(a)]_{\sim'} \end{aligned} \tag{B.3}$$

This $\Pi(f)$ is well-defined by the condition (B.2).

We use the functor

$$\begin{aligned} E \colon \mathbf{Grp} &\to \mathbf{GrpRel}, \\ G &\mapsto (G, =) \\ g \colon G \to G' \quad &\mapsto \quad g : (G, =) \to (G', =). \end{aligned} \tag{B.4}$$

This morphism $E(g) = g$ is indeed in **GrpRel**, since the corresponding condition "$\forall a, b \in G,\ a = b \ \Rightarrow\ g(a) = g(b)$" is trivially satisfied.

Proposition B.10. *The functor* Π *is left adjoint to* E*, with unit* π *sending* $(G, \sim)$ *to the quotient map* $\pi_{(G,\sim)}$.

Proof: By Definition 5.1.2, we must show that for any pair $(G, \sim) \in \mathbf{GrpRel}$, $H \in \mathbf{Grp}$ and $f\colon (G, \sim) \to (H, =)$, there exists a unique $g\colon G/{\sim} \to H$ such that

$$f = g \circ \pi_{(G,\sim)}.$$

This condition means that $f(a) = g([a])$ for all $a \in G$, so we have unicity. We only need to show that the map $g\colon [a] \mapsto f(a)$ is well-defined. For this, notice that $f\colon (G, \sim) \to (H, =)$ means that

$$\forall a, b \in G \; a \boldsymbol{\sim} b \Rightarrow f(a) = f(b).$$

Hence, we have

$$[a] = [b] \Longrightarrow a \boldsymbol{\sim} b \Longrightarrow f(a) = f(b) \Longrightarrow g([a]) = g([b]),$$

so g is well-defined.

Finally, we have that π is natural since

$$\forall h\colon (G, \sim) \to (G', \sim'),\; \pi_{(G',\sim')} \circ h = E(\Pi(h)) \circ \pi_{(G,\sim)}.$$

Indeed, both sides are precisely $a \mapsto [h(a)]_{\sim'}$. ∎

We now define the functor of presentations, and then conclude with two proofs of our main result. The first one is a straightforward calculation combining the proofs of Propositions B.4 and B.10. The second one helps to understand the proof of Theorem 5.2.6, and relies on Proposition 5.2.9 to compose adjoints.

Definition B.11. Let X be a set, $\sim$ a relation on $\langle X\rangle_{\mathrm{G}}$. We say that the pair $(X, \sim)$ is a *presentation* of the group $\langle X\rangle_{\mathrm{G}}/{\sim}$, and that X is the *set of generators* of this group.

Definition B.12. The category **Gener** is the category having as objects all pairs $(X, \sim)$ for $X \in \mathbf{Set}$ and $\sim$ a relation on $\langle X\rangle_{\mathrm{G}}$. A morphism $f\colon (X, \sim) \to (Y, \sim')$ is a function $f\colon X \to Y$ that satisfies the condition

$$\forall a, b \in \langle X\rangle_{\mathrm{G}} \quad a \boldsymbol{\sim} b \Rightarrow \overline{f}(a) \boldsymbol{\sim}' \; \overline{f}(b), \tag{B.5}$$

for $\boldsymbol{\sim}$ and $\boldsymbol{\sim}'$ the smallest congruences containing $\sim$ and $\sim'$ respectively.

The *presentation functor* is

$$\begin{aligned} P\colon \mathbf{Gener} &\to \mathbf{Grp}\,. \\ (X, \sim) &\mapsto \langle X\rangle_{\mathrm{G}}/{\sim} \\ f\colon (X, \sim) \to (Y, \sim') \quad &\mapsto \quad Pf\colon \langle X\rangle_{\mathrm{G}}/{\sim} \to \langle Y\rangle_{\mathrm{G}}/{\sim'} \\ &\qquad [a]_{\sim} \mapsto [\,\overline{f}(a)]_{\sim'} \end{aligned} \tag{B.6}$$

This $P(f)$ is well-defined by the condition (B.5).

We use the following forgetful functor, for U as in Definition B.3:

$$\begin{aligned} \mathcal{U}\colon \mathbf{Grp} &\to \mathbf{Gener}, \\ G &\mapsto (U(G), =) \\ g\colon G \to G' \quad &\mapsto \quad \mathcal{U}g = U(g) : (U(G), =) \to (U(G'), =). \end{aligned} \tag{B.7}$$

The condition "$\forall a, b \in \langle U(G)\rangle_{\mathrm{G}},\ a = b \ \Rightarrow\ \overline{\mathcal{U}(g)}(a) = \overline{\mathcal{U}(g)}(b)$" is trivially satisfied.

Finally, we define the transformation $\eta\colon 1_{\mathbf{Gener}} \to \mathcal{U} \circ P$ where

$$\eta_{(X,\sim)}\colon (X, \sim) \to (U(\langle X\rangle_{\mathrm{G}}/\!\sim), =),\ x \mapsto [x]_\sim. \tag{B.8}$$

Theorem B.13. *The functor P is left adjoint to $\mathcal{U}$ with unit η.*

Proof: (**Direct version**)
We must show that, for any $(X, \sim) \in \mathbf{Gener}$, $G \in \mathbf{Grp}$ and $f\colon (X, \sim) \to (U(G), =)$, there exists a unique $g\colon \langle X\rangle_{\mathrm{G}}/\!\sim \to G$ such that $f = \mathcal{U}(g) \circ \eta_{(X,\sim)}$. We combine the proofs of the two previous results. Since $f = \mathcal{U}(g) \circ \eta_{(X,\sim)}$ means $f(x) = g([x])$ for all $x \in X$, for $a_1^{\epsilon_1} \cdots a_n^{\epsilon_n} \in \langle X\rangle_{\mathrm{G}}$, $a_i \in X$ we have

$$g([a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}]) = g([a_1])^{\epsilon_1} \cdots g([a_n])^{\epsilon_n} = f(a_1)^{\epsilon_1} \cdots f(a_n)^{\epsilon_n} = \overline{f}(a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}).$$

Hence the only possibility is $g([a]) = \overline{f}(a)\ \forall a \in \langle X\rangle_{\mathrm{G}}$.

This is well-defined since $f\colon (X, \sim) \to (U(G), =)$ means that

$$\forall a, b \in \langle X\rangle_{\mathrm{G}}\ a \sim b \Rightarrow \overline{f}(a) = \overline{f}(b),$$

so

$$[a] = [b] \Longrightarrow a \sim b \Longrightarrow \overline{f}(a) = \overline{f}(b) \Longrightarrow g([a]) = g([b]).$$

Finally, we prove that η is natural, meaning that

$$\forall h\colon (X, \sim) \to (Y, \sim'),\ \eta_{(Y,\sim')} \circ h = \mathcal{U}(P(h)) \circ \eta_{(X,\sim)}.$$

We evaluate both sides on $x \in X$ and obtain the equality of these two morphisms:

$$\eta_{(Y,\sim')}(h(x)) = [h(x)]_{\sim'} \quad \text{and}$$

$$((U \circ P)h)(\eta_{(X,\sim)}(x)) = ((U \circ P)h)([x]_\sim) = [h(x)]_{\sim'}.$$

∎

For the second proof, the idea is to "glue together" the functors F and Π to obtain P. However, the domains and codomains do not match, which justifies defining the variants F', U' and τ'.

Proof: (**Composing adjoints**)
To apply Proposition 5.2.9 to F, U and τ, we need to modify slightly the domains and codomains by defining F', U' and τ' that act only on the group parts of the pairs in **Gener**.

$$F' \colon \mathbf{Gener} \to \mathbf{GrpRel}, \quad (X, \sim) \mapsto (\langle X \rangle_{\mathrm{G}}, \sim),$$
$$f \colon (X, \sim) \to (Y, \sim') \quad \mapsto \quad \overline{f} \colon (\langle X \rangle_{\mathrm{G}}, \sim) \to (\langle Y \rangle_{\mathrm{G}}, \sim'),$$

$$U' \colon \mathbf{GrpRel} \to \mathbf{Gener}, \quad (G, \sim) \mapsto (U(G), \sim),$$
$$g \colon (G, \sim) \to (G', \sim') \quad \mapsto \quad U(g) \colon (U(G), \sim) \to (U(G'), \sim)$$

$$\tau' \colon 1_{\mathbf{Gener}} \to U' \circ F', \ \tau'_{(X,\sim)} = \tau_X \colon (X, \sim) \to (U(\langle X \rangle_{\mathrm{G}}), \sim).$$

Conditions (B.2) and (B.5) for $F'(f)$ and $U'(g)$ translate into tautologies. For τ', we have that τ_X is indeed a morphism from $(X, \sim)$ to $(U(\langle X \rangle_{\mathrm{G}}), \sim)$ — satisfying (B.5) — since $\overline{\tau}_X(a) = a$ for all $a \in \langle X \rangle_{\mathrm{G}}$.

We can see that τ' is still natural in the same way as τ, and that F' and U' satisfy the condition

$$\forall (X, \sim) \in \mathbf{Gener}, (G, \sim') \in \mathbf{GrpRel}, f \colon (X, \sim) \to (U(G)), \sim'),$$
$$\exists!\, g \colon (\langle X \rangle_{\mathrm{G}}, \sim) \to (G, \sim') \text{ st } f = U'(g) \circ \tau_{(X,\sim)}.$$

Indeed $f \colon (X, \sim) \to (U(G)), \sim')$ implies $f \colon X \to U(G)$ so by Proposition B.4 we have a unique $g \colon \langle X \rangle_{\mathrm{G}} \to G$ satisfying $f = U(g) \circ \tau_X$ (and therefore satisfying $f = U'(g) \circ \tau'_{(X,\sim)}$). Knowing that $g(a) = \overline{f}(a)$, we only need to check condition (B.2) for $g \colon (\langle X \rangle_{\mathrm{G}}, \sim) \to (G, \sim')$:

$$\forall a, b \in \langle X \rangle_{\mathrm{G}} \quad a \sim b \Rightarrow \overline{f}(a) \sim' \overline{f}(b) \Rightarrow g(a) \sim' g(b),$$

by using condition (B.5) for f. Hence F' is still left adjoint to U' with unit τ'.

We remark that

$$P = \Pi \circ F' \colon \mathbf{Gener} \to \mathbf{GrpRelGrp},$$

$$\mathcal{U} = U' \circ E \colon \mathbf{Grp} \to \mathbf{Gener}, \quad \text{and}$$

$$\eta_{(X,\sim)} = U\pi_{(\langle X \rangle_{\mathrm{G}},\sim)} \circ \tau_X \colon (X, \sim) \to (U(\langle X \rangle_{\mathrm{G}}/\!\sim, =)$$
$$\text{so} \quad \eta = U'\pi_{F'} \circ \tau' \colon 1_{\mathbf{Gener}} \to \mathcal{U} \circ P.$$

Hence we can conclude that P is left adjoint to $\mathcal{U}$ with unit η by applying Proposition 5.2.9. ∎

Example B.14. Let us consider the additive group $\mathbb{Z}_n$. Since this group is cyclic, we can use $X = \{1\}$ and $n \sim 0$ to get $\mathbb{Z}_n \cong \langle X \rangle_{\mathrm{G}}/\sim$. We could also replace 1 by any element having a multiplicative inverse (hence having also order n), so the presentation is not unique.

Note that we could also use $X = U(\mathbb{Z}_n)$ with $\sim$ being the entire table of Cayley of $(\mathbb{Z}_n, +)$, but it is unnecessarily large.

Example B.15. When we consider the symmetric group S_n (group of permutations over n elements), we can have a presentation with generators $X = \{\sigma_1, \ldots, \sigma_n\}$ (the generator σ_i corresponds to swapping positions i and $i+1$), and relations

- $\sigma_i^2 \sim 1$,
- $\sigma_i \sigma_j \sim \sigma_j \sigma_i$ for $|i - j| > 1$,
- $(\sigma_i \sigma_{i+1})^3 \sim 1$.

Remark B.16. We could also define presentations of monoids (recall that *monoids* are sets with an associative operation and a neutral element, without necessarily inverses). Recall that the definition of the free monoid is

$$\langle X \rangle = \{a_1^{\epsilon_1} \cdots a_n^{\epsilon_n} \mid n \in \mathbb{N}, a_i \in X, \epsilon_i \in \mathbb{N}, \text{ and } \forall i \; a_i \neq a_{i+1}\}$$

(the difference is $\epsilon_i \in \mathbb{N}$ instead of in $\mathbb{Z}$), with product being concatenation and adding the exponents (but no cancellation of inverses). The quotients and presentations would be defined in the same way, resulting in the same universal properties. In that case, the presentations would be related by the fact that for any set X,

$$\langle X \rangle = \langle X \cup \{x^{-1} \mid x \in X\} \rangle_{\mathrm{G}}/\sim \quad \text{for } xx^{-1} \sim e, \; x^{-1}x \sim e.$$

In this context, it is common to say that the set X is an *alphabet*, and the elements of the free monoid $\langle X \rangle$ are *words* in the symbols of that alphabet.

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