# On the Restriction of Supercuspidal Representations: An In-Depth Exploration of the Data

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# Abstract

Let $\mathbb{G}$ be a connected reductive group defined over a $p$-adic field $F$ which splits over a tamely ramified extension of $F$, and let $G = \mathbb{G}(F)$. We also assume that the residual characteristic of $F$ does not divide the order of the Weyl group of $\mathbb{G}$.

Following J.K. Yu's construction, the irreducible supercuspidal representation constructed from the $G$-datum $\Psi$ is denoted $\pi_G(\Psi)$. The datum $\Psi$ contains an irreducible depth-zero supercuspidal representation, which we refer to as the depth-zero part of the datum. Under our hypotheses, the J.K. Yu Construction is exhaustive.

Given a connected reductive $F$-subgroup $\mathbb{H}$ that contains the derived subgroup of $\mathbb{G}$, we study the restriction $\pi_G(\Psi)|_H$ and obtain a description of its decomposition into irreducible components along with their multiplicities. We achieve this by first describing a natural restriction process from which we construct $H$-data from the $G$-datum $\Psi$. We then show that the obtained $H$-data, and conjugates thereof, construct the components of $\pi_G(\Psi)|_H$, thus providing a very precise description of the restriction. Analogously, we also describe an extension process that allows to construct $G$-data from an $H$-datum $\Psi_H$. Using Frobenius Reciprocity, we obtain a description for the components of $\text{c-Ind}_H^G \pi_H(\Psi_H)$.

From the obtained description of $\pi_G(\Psi)|_H$, we prove that the multiplicity in $\pi_G(\Psi)|_H$ is entirely determined by the multiplicity in the restriction of the depth-zero piece of the datum. Furthermore, we use Clifford theory to obtain a formula for the multiplicity of each component in $\pi_G(\Psi)|_H$. As a particular case, we take a look at the regular depth-zero supercuspidal representations and obtain a condition for a multiplicity free restriction.

Finally, we show that our methods can also be used to define a restriction of Kim-Yu types, allowing to study the restriction of irreducible representations which are not supercuspidal.

# Dedications

I dedicate this thesis to my parents, Nicole and Zoël Bourgeois, and to my maternal grandparents, Marie-Jeanne and Robert Poirier, who have consistently encouraged my academic and non-academic pursuits. I hope I can continue to make you proud.

# Acknowledgement

First and foremost, I would like to thank my supervisor Dr. Monica Nevins. Coming from a Master's in applied mathematics, the learning curve to studying the representation theory of $p$-adic groups was quite substantial, and I am forever indebted to her for the time that she invested in me. I am incredibly grateful for the guidance and support provided during the last four years, and also for the freedom to express my ideas and lead the direction of my research project. Thank you for going above and beyond as a mentor and allowing me to grow into the mathematician that I am today.

The learning curve I was faced with was made a lot more pleasant alongside my classmates. I learned all about reductive groups through a study group with fellow PhD students Cameron Ruether and Maiko Serizawa. Thank you for keeping me motivated to learn, for being genuinely interested in my work over the years, and for the friendships along the way.

Initially this project was centered around restricting a $G$-datum to $H$-data, but thanks to Dr. Nicolas Arancibia's curiosity during one of my first seminar presentations, the question of inducing $G$-data from an $H$-datum was brought forward. It is because of this intervention that the story of data that we present in this thesis can come to a natural close.

Many experts in the field have provided feedback on my research, without whom this thesis would not be complete. In particular, I would like to thank Dr. Peter Latham for the many discussions on types and other adjacent topics, and Dr. Jeffrey Adler for the many email correspondences regarding regular depth-zero supercuspidal representations and his multiplicity results. I would also like to thank the participants of the session "Representation Theory of Groups Defined Over Local Fields" during the 2019 CMS Summer Meeting in Regina for the feedback on my presentation. Namely, the various discussions with Dr. Loren Spice, Dr. Joshua Lansky, Dr. Jerrod Smith and Dr. David Rowe were very fruitful. I am happy to be part of such an open and supportive mathematical community.

I would also like to thank my examiners for their reviews of this work. In particular, I highly appreciated the comments from my external examiner, Dr. Fiona Murnaghan. It was truly rewarding for the value of my work to be acknowledged by such a great expert in this field of research, and for her to share her knowledge with me. I am also very grateful to the members of my thesis advisor committee, Dr. Hadi Salmasian and Dr. Paul Mezo, who have followed my progress since the early stages of my thesis, consistently attending my seminar talks and asking questions to help me further my research.

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# Preface

While I was working on my PhD thesis, I participated in the french version of the 3-Minute Thesis Competition (3MT®). 3MT® is a national competition open to graduate students from all disciplines of academia in which the participants must communicate the key concepts and significance of their research to a non-specialized audience in no more than 3 minutes. Being a pure mathematician, I saw this as a great and challenging opportunity to work on my science communication skills. There were different rounds to the competition. Students started by competing at the faculty level, and then at the university level. The judges from the university level competition would then select their two favorite candidates to compete in the regional, and eventually the national competitions.

Because most of the audience was likely not going to be familiar with the language of mathematics, I decided that a good relatable analogy was necessary if I wanted any chance of advancing into the competition. With the help of my husband, I came up with the idea of comparing my supercuspidal representations to IKEA furniture.

Almost certainly everyone (at least in Ottawa) has experienced the frustrations and pride that come with successfully putting together a piece of IKEA furniture, so this analogy was promising to be very relatable. We think of the $G$-datum as being our assembly pieces, and the J.K. Yu construction as the manual of instructions that tell you how to assemble the pieces together in order to obtain your piece of furniture, or in this case your supercuspidal representation. This analogy will be used a few times in the thesis to give the reader an intuition behind the ideas that we use.

With this analogy, I broadly explained my research project and competed at the university level. I received great praise from the judges for successfully communicating an abstract research project, but unfortunately I was not selected in the top two from our campus that would advance to the next level. This was however a great experience and I thoroughly enjoyed the challenge. As a consequence of this experience, all my non-mathematician friends and family now think that my last four years have been dedicated to studying IKEA furniture and becoming an expert on how to build it.

As a reader of this thesis, I am sure that you will appreciate the depth (pun intended) behind this work and look beyond the pieces of IKEA furniture.

# Chapter 1

# Introduction

Representation theory is an area of pure mathematics which studies abstract algebraic structures by mapping their elements into matrix spaces. This area of mathematics has many applications. In particular, the representation theory of *p-adic groups* is of special interest, as it bridges to number theory via the Langlands correspondence.

Given a $p$-adic group $G$, that is the $F$-points of a (connected) reductive group $\mathbb{G}$ with $F$ a $p$-adic field, we let $p$ be the residual characteristic of $F$. Here, when we say $p$-adic field, we actually mean a non-archimedean local field. Throughout this work, we will also be assuming that $\mathbb{G}$ splits over a tamely ramified extension $E$ of $F$ and that $p$ does not divide the order of the Weyl group of $\mathbb{G}$. The building blocks for representation theory are what we call the irreducible representations. Given an irreducible representation $\pi$ of $G$, Jacquet's Subrepresentation Theorem [Jac71] tells us that $\pi$ is either *supercuspidal* or it is a subrepresentation of the *parabolic induction* of a supercuspidal representation. This means that irreducible supercuspidal representations form an important class of representations that needs to be studied in the context of $p$-adic groups.

The first $p$-adic group that was studied was $G = \mathrm{GL}_n(F)$. Many authors contributed to the development of the theory of supercuspidal representations for $\mathrm{GL}_n(F)$ [How77, Car79, Moy86, BK93, BK98, BK99], and Fintzen provides a good historical summary in her introduction of [Fin18]. For classical groups, an exhaustive construction for supercuspidal representations was provided by [Ste08] for $p \neq 2$. If we want to be looking at an arbitrary $p$-adic group $G$ however, we must turn to Moy and Prasad for a deeper classification of the supercuspidal representations.

In the 1990s, Moy and Prasad associated a *depth* to each irreducible admissible representation of $G$ using Bruhat-Tits theory [MP94], splitting the class of irreducible supercuspidal representations into two categories: those of depth zero and those of positive depth. The levels of classification for irreducible representations of $G$ is summarized in Figure 1.1. Moy and Prasad also described an exhaustive construction for irreducible depth-zero supercuspidal representations [MP96].

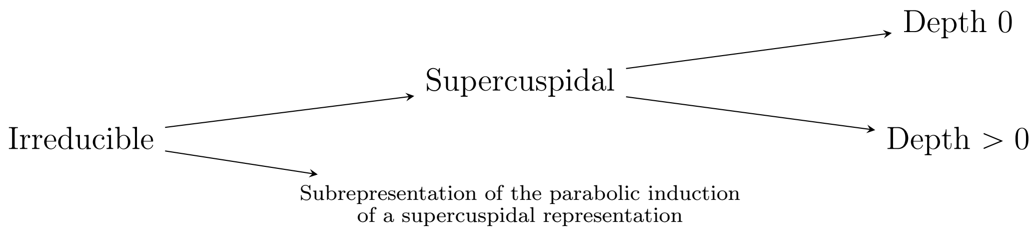



Figure 1.1: Classification of irreducible representations of $G$.

A construction for irreducible positive-depth supercusidal representations of arbitrary tamely ramified reductive groups came a few years later with what is now referred to as the *J.K. Yu Construction* [Yu01]. The J.K. Yu Construction builds on the work done by Adler [Adl98], and uses what is referred to as a *$G$-datum*. A $G$-datum $\Psi$ can be represented in particular by a 4-tuple $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$, where $\vec{\mathbb{G}} = (\mathbb{G}^0, \ldots, \mathbb{G}^d)$ is a *tamely ramified twisted Levi sequence*, $\rho$ is an irreducible representation of $G^0_{[y]}$ that induces irreducibly to a depth-zero supercuspidal representation of $G^0$ and $\vec{\phi} = (\phi^0, \ldots, \phi^d)$ is a sequence of quasicharacters that satisfy a specific *genericity* condition. Given that a $G$-datum contains the information of a depth-zero supercuspidal representation, the Moy-Prasad Construction can be viewed as a special case of the J.K. Yu Construction. It was shown that, under the hypothesis that $p$ does not divide the order of the Weyl group of $\mathbb{G}$, the J.K. Yu Construction is exhaustive [Kim07, Fin18], meaning that every irreducible supercuspidal representation of $G$ arises from a $G$-datum.

Given a $G$-datum $\Psi$, we let $\pi_G(\Psi)$ denote the corresponding irreducible supercuspidal representation obtained via the J.K. Yu Construction. It is possible for different $G$-data to produce the same irreducible supercuspidal representation. This leads to the notion of *equivalence* of $G$-data, placing the $G$-data $\Psi$ and $\dot{\Psi}$ in the same class whenever $\pi_G(\Psi) \simeq \pi_G(\dot{\Psi})$. Hakim and Murnaghan gave a very precise description of these equivalence classes of $G$-data in [HM08]. As a consequence, the irreducible supercuspidal representations are uniquely determined by the equivalence classes of their corresponding $G$-data.

Now, one of the fundamental questions of representation theory is to study how a completely reducible representation decomposes into irreducible subrepresentations (or components), and to what multiplicity these components appear. In particular, in this thesis, we will be studying the restriction of an irreducible supercuspidal representation $\pi_G(\Psi)$ to $H$, where $H = \mathbb{H}(F)$ with $\mathbb{H}$ being a closed connected subgroup of $\mathbb{G}$ that contains the derived group $\mathbb{G}_{\mathrm{der}} = [\mathbb{G}, \mathbb{G}]$. The goals are to describe explicitly the components of $\pi_G(\Psi)|_H$ and to determine a formula for their multiplicity. We will also be able to determine the conditions under which such a restriction is *multiplicity free*, or equivalently, decomposes with multiplicity one. This is a well-posed question since $\pi_G(\Psi)|_H$ is completely reducible, as per the following theorem.

**Theorem 1** ([Tad92, Lemma 2.1]). *Let $\mathbb{G}$ be a reductive group defined over a locally compact non-archimedean field $F$, let $\mathbb{H}$ be a reductive $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$, $G = \mathbb{G}(F)$ and $H = \mathbb{H}(F)$. Let $\pi$ be a smooth irreducible representation of $G$. Then $\pi|_H$ is a finite direct sum of irreducible representations of $H$.*

The reason for studying such a restriction is motivated by the following conjecture from Adler and Prasad.

**Conjecture 2** ([AP06, Conjecture 2.6]). *Let $\mathbb{H}$ be a quasi-split reductive algebraic group (not necessarily connected) over a local field $F$. Let $\mathbb{G}$ be a reductive algebraic group (not necessarily connected) containing $\mathbb{H}$ such that the derived groups of $\mathbb{H}$ and $\mathbb{G}$ are the same, and such that $\mathbb{G}/\mathbb{H}$ is connected. Then multiplicity one holds for restriction of irreducible admissible representations of $\mathbb{G}(F)$ to $\mathbb{H}(F)$.*

Adler and Prasad have successfully proven this conjecture for various pairs $(\mathbb{G}, \mathbb{H})$, such as $(\mathrm{GL}(n), \mathrm{SL}(n))$ [AP06, Theorem 1.3], $(\mathrm{GO}(n), \mathrm{O}(n))$ and $(\mathrm{GSp}(2n), \mathrm{Sp}(2n))$ [AP06, Theorem 1.4], as well as $(\mathrm{GU}(n), \mathrm{U}(n))$ [AP19, Theorem 9]. The conjecture was also shown for the pair $(\mathrm{GSO}(n), \mathrm{SO}(n))$ [GT19]. Adler and Prasad actually discovered their conjecture to be false by finding a counterexample of a depth-zero supercuspidal representation of $\mathrm{GU}(n)$ whose restriction to $\mathrm{SU}(n)$ restricts with multiplicity two [AP19, Theorem 10]. They discuss in [AP19, Remark 11] that the failure of the multiplicity one result is likely due to the fact that the components of the restriction are not *regular* in the sense of [Kal19].

Note that Adler and Prasad's condition that $\mathbb{G}$ and $\mathbb{H}$ have the same derived subgroup is equivalent to our condition that $\mathbb{H}$ contains $\mathbb{G}_{\mathrm{der}}$. The hypotheses of the conjecture require $\mathbb{H}$ to be quasi-split over $F$, but $\mathbb{G}$ and $\mathbb{H}$ need not be connected. In our setting, $\mathbb{H}$ does not need to be quasi-split over $F$, but both $\mathbb{G}$ and $\mathbb{H}$ will be connected. Furthermore, we will be restricting our attention to irreducible supercuspidal representations, which are a subclass of the irreducible admissible representations. This is because we have lots of information on how the irreducible supercuspidal representations are constructed.

Now, given a $G$-datum $\Psi$, the components of $\pi_G(\Psi)|_H$ are irreducible supercuspidal representations of $H$, which means they can be obtained via the J.K. Yu Construction using some $H$-data, under the exhaustion hypothesis. This means that in order to understand the restriction of $\pi_G(\Psi)$, we must understand the data. More specifically, one can ask how the $H$-data associated to the components of $\pi_G(\Psi)|_H$ are related to the initial $G$-datum $\Psi$.

In [Nev15], Nevins described the relationship between the $H$-data and $\Psi$ in the very specific case where $H = G_{\mathrm{der}}$ and $\Psi$ is a *toral datum of length one*. We will be generalizing her results significantly in this thesis. Indeed, one of the first main results that we present (Theorem 3.3.1) is showing that there is a very natural restriction process that we can apply to $\Psi$ in order to construct a family of $H$-data, $\mathrm{Res}(\Psi)$. Indeed, it suffices to restrict each element of the datum to its analog in $H$.

Conversely, this thesis also studies the novel question of extending an $H$-datum to $G$-data. Given a *normalized* $H$-datum $\Psi_H$, we show that there is a natural induction process that allows us to construct a family of $G$-data, $\mathrm{Ext}(\Psi_H)$, whose restrictions contain $\Psi_H$ (Theorem 3.5.5). Furthermore, every $H$-datum is $H$-equivalent to an $H$-datum that can be extended (Corollary 3.5.6).

The next follow-up question to this restriction process of data is whether the data from $\mathrm{Res}(\Psi)$ construct the various components that appear in the restriction of $\pi_G(\Psi)$ to $H$. This feels like a very natural connection to make, yet in order to answer this question, one must really dive into all the details and technicalities of the J.K. Yu Construction. This turns out to be a highly non-trivial task. One of our main contributions, which is stated in Theorem 4.4.1 can be summarized as follows:

**Theorem 3** (Theorem 4.4.1)**.** *Let $\Psi$ be a $G$-datum and $\pi_G(\Psi)$ be its corresponding irreducible supercuspidal representation. Let $C$ be a set of representatives of the double cosets of $H\backslash G/K$, where $K$ is the subgroup associated with $\Psi$ as per the J.K. Yu Construction. Then, the supercuspidal representations constructed from $\{{}^t\,\mathrm{Res}(\Psi) : t \in C\}$ exhaust all the components of $\pi_G(\Psi)|_H$.*

Frobenius Reciprocity allows us to obtain an analogous result with respect to the extension of $H$-data (Theorem 4.5.1):

**Theorem 4** (Theorem 4.5.1)**.** *Let $\Psi_H$ be an $H$-datum and $\pi_H(\Psi_H)$ be its corresponding irreducible supercuspidal representation. Let $C$ be a set of representatives of the double cosets of $H\backslash G/K$, where $K$ is the subgroup associated with $\Psi$ as per the J.K. Yu Construction. Then, the supercuspidal representations constructed from $\{{}^{t^{-1}}\,\mathrm{Ext}(\Psi_H) : t \in C\}$ exhaust all the irreducible supercuspidal subrepresentations of* $\text{c-Ind}_H^G\,\pi_H(\Psi_H)$.

Because we are able to obtain such a precise description of $\pi_G(\Psi)|_H$, we can then ask questions about the multiplicity of the various components. Another major contribution of this thesis is showing that the multiplicity in $\pi_G(\Psi)|_H$ is solely determined by the multiplicity in the depth-zero part, $\rho$, of the datum $\Psi$. More specifically, the multiplicity in $\pi_G(\Psi)|_H$ is determined by the multiplicity in $\rho|_{H^0_{[y]}}$, where $H^0 = G^0 \cap H$ (Theorem 5.1.1). In particular, $\pi_G(\Psi)|_H$ is multiplicity free if and only if $\rho|_{H^0_{[y]}}$ is multiplicity free. Furthermore, we establish that every component of $\pi_G(\Psi)|_H$ has the same multiplicity and that this multiplicity can be expressed in terms of the dimensions of $\rho$ and one of the components of $\rho|_{H^0_{[y]}}$ (Corollary 5.2.11).

Even though we are able to write a formula for the multiplicity, it can still be difficult to compute in practice. We consider the special case where the representation

$\rho$ induces to a depth-zero supercuspidal representation which is *regular* in the sense of [Kal19] to provide an example for when we can determine the components of $\rho|_{H^0_{[y]}}$.

When $\rho$ induces to a regular depth-zero supercuspidal representation, it can be parametrized by a pair $(\mathbb{S}, \theta)$, where $\mathbb{S}$ is an *elliptic maximally unramified* $F$-torus and $\theta$ is a *regular* depth-zero character of $S = \mathbb{S}(F)$. This means that we have a very specific description for $\rho$ which allows us to study the restriction $\rho|_{H^0_{[y]}}$ in great detail. We can then obtain the conditions for a multiplicity free restriction. Our multiplicity free result for $\rho|_{H^0_{[y]}}$ is stated in Corollary 6.4.10 and confirms Adler and Prasad's observations that some notion of regularity is useful. We obtain as a corollary (Corollary 6.4.11) a multiplicity free result for the restriction of a regular depth-zero supercuspidal representation, which is also shown by Adler and Mishra in a recent pre-print [AM19, Theorem 5.3]. The approach we provide here is more explicit in the sense that we describe more precisely the components in the restriction.

The thesis will be divided as follows. In Chapter 2, we provide the necessary structure theory of $p$-adic groups required for constructing supercuspidal representations, and we also discuss some basic representation theory results. While most of this chapter serves as a literature review, we show lots of new relations between the *toral, parabolic, Levi and Moy-Prasad filtration subgroups* of $\mathbb{G}$ and those of $\mathbb{H}$ (Sections 2.1.2 and 2.2.2.3).

In Chapter 3, we discuss everything related to data. It is in this chapter that we describe our natural restriction process that allows us to construct $H$-data from a $G$-datum (Theorem 3.3.1), and our natural induction process that allows us to construct $G$-data from an $H$-datum (Theorem 3.5.5). We also prove that these restriction and induction processes that we apply on data are consistent with Hakim and Murnaghan's notion of equivalence. That is, equivalent $G$-data can restrict to equivalent $H$-data and equivalent $H$-data can extend to equivalent $G$-data (Propositions 3.4.3 and 3.6.3).

Chapter 4 is where we dive into the J.K. Yu Construction and study the restriction $\pi_G(\Psi)|_H$ in great detail. In the process of proving Theorem 3 above, we show that restriction commutes with the various steps of the J.K. Yu Construction (Theorem 4.2.7 and Proposition 4.3.4). As such, this thesis confirms that while this construction is seemingly complicated, it still behaves nicely. One can then use Frobenius Reciprocity to obtain Theorem 4.

It is in Chapter 5 that we tackle the multiplicity in the restriction $\pi_G(\Psi)|_H$. We start by showing that the multiplicity is entirely determined by the restriction of $\rho$ to $H^0_{[y]}$. Then, we apply results from Clifford theory on $\rho|_{H^0_{[y]}}$ to obtain a precise description of $\rho|_{H^0_{[y]}}$ as well as a formula for the multiplicity of each component.

In Chapter 6, we change our focus to the regular depth-zero supercuspidal representations. The construction of such representations is closely related to the theory of *finite groups of Lie type* and *Deligne-Lusztig cuspidal* *representations*. Indeed, the pair $(\mathbb{S}, \theta)$ that parametrizes the regular depth-zero supercuspidal representation (or equivalently $\rho$) determines a Deligne-Lusztig cuspidal representation that is necessary in its construction. The Deligne-Lusztig cuspidal representations have a very precise character formula. We have not found any references in the literature that explicitly compute the restriction of these character formulas. As such, we conduct the calculations in this thesis and show that the restriction of a Deligne-Lusztig cuspidal representation is again a Deligne-Lusztig cuspidal representation under some notion of *general position* (Theorem 6.2.8). This restriction formula is one of the tools that we use, along with Kaletha's method for constructing regular depth-zero supercuspidal representations, to obtain our condition for a multiplicity free restriction (Corollary 6.4.10) that aligns with Adler and Prasad's discussion.

Finally, in Chapter 7, we discuss the impact and consequences of our work. It is clear that we bring contributions to the theory of irreducible supercuspidal representations. Not only do our results apply to the J.K. Yu Construction, but they can also apply to other types of constructions that use different data, such as Kaletha's construction for regular depth-zero supercuspidal representations (Section 7.2.2) and Hakim's construction for supercuspidal representations (Section 7.2.3). Furthermore, we show how our results on the restriction of data can be transferred to define a restriction of *types*. The theory of types is useful when studying irreducible representations which are not supercuspidal. In particular, the restriction of types that we define (Theorem 7.3.5) allows us to obtain information on the restriction of an arbitrary irreducible representation of $G$. As such, the implications of our work go beyond just the irreducible supercuspidal representations.

In conclusion, while the main motivation for this thesis was to obtain a multiplicity formula for the restriction of irreducible supercuspidal representations, we have developed many pertinent results in the process, thus contributing to the representation theory of both $p$-adic groups and finite groups of Lie type. Even though the J.K. Yu Construction has been described in many works, we hope that this thesis can further demystify it and serve as a reference for the newcomers into the research area of representation theory of $p$-adic groups.

# Chapter 2

# Structure Theory of $p$-adic Groups

A $p$-adic group $G$ is the $F$-points of a (connected) reductive algebraic group $\mathbb{G}$ with $F$ a $p$-adic field $F$. The structure theory of these groups plays an important role in their representation theory. In particular, one can associate a Bruhat-Tits building to the group $G = \mathbb{G}(F)$ which allows to keep track of all its important compact open subgroups, namely the Moy-Prasad filtration subgroups, which are necessary in the construction of supercuspidal representations.

In this chapter, we will start by reviewing the important concepts related to the description of (connected) reductive groups in Section 2.1. Section 2.2 will build up to defining the Bruhat-Tits building of $G$, as well as the Moy-Prasad filtration subgroups. Those two notions come from what is called an affine apartment of $G$, which is what will be presented first in the section. Given a (connected) reductive $F$-subgroup $\mathbb{H}$ of $\mathbb{G}$ which contains $\mathbb{G}_{\text{der}}$, the main goal of this thesis is to understand the restriction to $H$ of supercuspidal representations of $G$. As such, we also need to understand the structure theory of $\mathbb{H}$ and how it relates to that of $\mathbb{G}$. We will be making the connections that exist between $G$ and $H$ while we progress through the first two sections of this chapter.

The last two sections will be dedicated to the representation theory of $p$-adic groups. In Section 2.3, we recall some important results from representation theory and define supercuspidal and cuspidal representations, as well as explain why these types of representations are of particular interest. We will see that the supercuspidal representations are divided into two categories based on the notion of depth. The last section of this chapter provides a summary of the construction for supercuspidal representations of depth zero and touches on the J.K. Yu Construction very briefly.

## 2.1 Reductive Groups

### 2.1.1 Tori, Root Subgroups, Levi and Parabolic Subgroups

Let $K$ be an algebraically closed field and let $\mathbb{G}$ be a reductive (algebraic) group. Following Humphreys' definition, a reductive group will mean a $K$-variety endowed with the structure of a group which has trivial unipotent radical and is *connected* [Hum75]. We write $K^\times$ for $\mathbb{G}_m = \mathbb{G}_m(K)$ and $K$ for $\mathbb{G}_a = \mathbb{G}_a(K)$. When working with reductive groups, there is a very nice presentation in terms of generators, which is analogous to the Lie algebra root space decomposition of Lie($\mathbb{G}$) = $\mathfrak{g}$. This description relies on special types of subgroups of $\mathbb{G}$, which we introduce below.

**Definition 2.1.1.** *A torus $\mathbb{T}$ of $\mathbb{G}$ is a subgroup which is isomorphic to $(K^\times)^n$ for some $n > 0$.*

*A morphism (of algebraic groups) $\mathbb{T} \to K^\times$ is called a (algebraic) character of $\mathbb{T}$ and the group of (algebraic) characters of $\mathbb{T}$ is denoted by $X^*(\mathbb{T})$.*

**Remark 2.1.2.** By convention, the character group $X^*(\mathbb{T})$ will be expressed in additive notation. That is, given $\alpha, \beta \in X^*(\mathbb{T})$, $\alpha + \beta$ and $-\alpha$ are the maps defined by $(\alpha + \beta)(t) = \alpha(t)\beta(t)$ and $-\alpha(t) = \alpha(t^{-1})$, respectively, for all $t \in \mathbb{T}$.

**Definition 2.1.3.** *Let* $\mathfrak{g}$= Lie($\mathbb{G}$). *The root system $\Phi(\mathbb{G}, \mathbb{T})$ associated to $\mathbb{T}$ is the set*

$$\Phi(\mathbb{G}, \mathbb{T}) = \{\alpha \in X^*(\mathbb{T})\backslash\{0\} : \mathfrak{g}_\alpha \neq \{0\}\},$$

*where*

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{Ad}(t)X = \alpha(t)X \textit{ for all } t \in \mathbb{T}\}.$$

When the context is obvious, we shall simply write $\Phi$ to denote $\Phi(\mathbb{G}, \mathbb{T})$. Because $\mathbb{G}$ is reductive, we have that $\Phi$ is a crystallographic root system over the vector space $X^*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$. The inner product $(\cdot, \cdot)$ on this vector space is induced by the Killing form of the Lie algebra Lie($\mathbb{T}$) = $\mathfrak{t}$ and is such that $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi(\mathbb{G}, \mathbb{T})$. This root system gives us a decomposition of the Lie algebra $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where the subalgebras $\mathfrak{g}_\alpha$ are one-dimensional for all $\alpha \in \Phi$ [Hum75, Corollary 26.2B]. When the characteristic of $K$ is zero, this is precisely the classical Cartan decomposition of the Lie algebra $\mathfrak{g}$ [Hum75, Section 16.4]. The following theorem provides us with an analogous description for the group $\mathbb{G}$ when it is reductive and $\mathbb{T}$ is maximal, meaning that it is not contained in a larger torus of $\mathbb{G}$.

**Theorem 2.1.4** ([Hum75, Theorem 26.3])**.** *Let $\mathbb{G}$ be a reductive group over the algebraically closed field $K$, let $\mathbb{T}$ be a maximal torus with associated root system $\Phi$ and let $\alpha \in \Phi$.*

*a) There exists a unique $\mathbb{T}$-stable subgroup $\mathbb{U}_\alpha$ having Lie algebra $\mathfrak{g}_\alpha$.*

*b) There exists an isomorphism $\epsilon_\alpha : K \to \mathbb{U}_\alpha$ such that for all $t \in \mathbb{T}, x \in K$,*

$$t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x).$$

*c) The group $\mathbb{G}$ is generated by $\mathbb{T}$ and the groups $\mathbb{U}_\alpha, \alpha \in \Phi$, that is*

$$\mathbb{G} = \langle \mathbb{T}, \mathbb{U}_\alpha : \alpha \in \Phi \rangle.$$

**Remark 2.1.5.** The subgroups $\mathbb{U}_\alpha, \alpha \in \Phi$, are referred to as the root subgroups of $\mathbb{G}$ (associated to $\mathbb{T}$). By part b) of the previous theorem, we have that the root subgroups are normalized by $\mathbb{T}$. It follows that every element $x \in \mathbb{G}$ can be expressed in the form $x = tu$, where $t \in \mathbb{T}$ and $u \in \langle \mathbb{U}_\alpha : \alpha \in \Phi \rangle$.

Before proceeding to some examples, we recall some additional facts about reductive groups which will be useful.

**Proposition 2.1.6.** *Let $\mathbb{G}$ be a reductive group, $\mathbb{T}$ a maximal torus and $\Phi = \Phi(\mathbb{G}, \mathbb{T})$. Let $Z(\mathbb{G})$ and $\mathbb{G}_{\text{der}}$ denote the center and the derived subgroup of $\mathbb{G}$, respectively.*

*a) Let $Z(\mathbb{G})^\circ$ denote the identity component of $Z(\mathbb{G})$. The intersection $Z(\mathbb{G})^\circ \cap \mathbb{G}_{\text{der}}$ is finite and $\mathbb{G} = Z(\mathbb{G})^\circ \mathbb{G}_{\text{der}}$* [Bor91, Proposition 14.2]. *We refer to this decomposition as the central isogeny of $\mathbb{G}$ with $\mathbb{G}_{\text{der}}$.*

*b) The center $Z(\mathbb{G})$ is in $\mathbb{T}$. In particular, $Z(\mathbb{G}) = \bigcap_{\alpha\in\Phi} (\ker \alpha)^\circ$* [Hum75, Corollary 26.2B].

**Corollary 2.1.7.** *Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$. Then $\mathbb{T} = Z(\mathbb{G})^\circ(\mathbb{T} \cap \mathbb{G}_{\text{der}})$.*

**Proof:** The groups $Z(\mathbb{G})^\circ$ and $(\mathbb{T} \cap \mathbb{G}_{\text{der}})$ are both subgroups of $\mathbb{T}$, and therefore $Z(\mathbb{G})^\circ(\mathbb{T} \cap \mathbb{G}_{\text{der}}) \subset \mathbb{T}$. For the converse, take $t \in \mathbb{T}$. By the central isogeny of $\mathbb{G}$ with $\mathbb{G}_{\text{der}}$, we have that $t = zg$ for some $z \in Z(\mathbb{G})^\circ$, $g \in \mathbb{G}_{\text{der}}$. It follows that $g = z^{-1}t \in \mathbb{T} \cap \mathbb{G}_{\text{der}}$. Hence, the result follows. ∎

Let us now look at some examples on how to calculate the root systems associated to maximal tori of $\mathbb{G}$ and determining the corresponding root subgroups.

**Example 2.1.8.** Consider the reductive group $\mathbb{G} = \mathrm{SL}_2(K)$ of $2 \times 2$ matrices with determinant 1. Then $\mathfrak{g} = \mathfrak{sl}_2(K)$ is the set of $2 \times 2$ matrices of trace 0. The subgroup of diagonal matrices, $\mathbb{T} = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in K^\times \right\}$, is a maximal torus of $\mathbb{G}$, and the maps

$$\chi_1 : \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mapsto a, \chi_2 : \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mapsto a^{-1}$$

are the most natural characters one can think of in $X^*(\mathbb{T})$. The Lie algebra of $\mathbb{T}$ is given by $\mathfrak{t} = \left\{ \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} : x \in K \right\}$. We evaluate the adjoint action of $\mathbb{T}$ on an element of $\mathfrak{g}$:

$$\mathrm{Ad}\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) \begin{bmatrix} x & y \\ z & -x \end{bmatrix} = \begin{bmatrix} x & a^2 y \\ (a^{-1})^2 z & -x \end{bmatrix}.$$

Using additive notation, as per Remark 2.1.2, one then sees that $\Phi = \{\alpha, -\alpha\}$, where $\alpha = \chi_1 - \chi_2$, $\mathfrak{g}_\alpha = \left\{ \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} : y \in K \right\}$ and $\mathfrak{g}_{-\alpha} = \left\{ \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix} : z \in K \right\}$.

Note that this provides us with a root space decomposition for $\mathfrak{sl}_2(K)$,

$$\mathfrak{sl}_2(K) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$

One can then deduce that the root subgroups are given by

$$\mathbb{U}_\alpha = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in K \right\}, \mathbb{U}_{-\alpha} = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} : x \in K \right\}.$$

∎

**Example 2.1.9.** When considering $\mathbb{G} = \mathrm{SL}_3(K)$ and its diagonal subgroup as the maximal torus $\mathbb{T}$, we obtain $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$, where $\alpha = \chi_1 - \chi_2$ and $\beta = \chi_2 - \chi_3$. The root subgroups are given by

$$\mathbb{U}_\alpha = \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbb{U}_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \mathbb{U}_{\alpha+\beta} = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\mathbb{U}_{-\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbb{U}_{-\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{bmatrix}, \mathbb{U}_{-\alpha-\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{bmatrix},$$

where the symbol $*$ indicates that any element of $K$ can be chosen.

This root system is of type $A_3$ and is represented geometrically by Figure 2.1.

∎

Every (abstract) root system has what we call a simple system [Hum90, Theorem 1.3], that is a set $\Delta \subset \Phi$ which is a basis for $\mathrm{span}_{\mathbb{R}} \Phi$ and such that every root in $\Phi$ is an integer linear combination of $\Delta$ with coefficients all of the same sign. In the root system of type $A_3$ discussed above, $\Delta = \{\alpha, \beta\}$. We can also define the set of

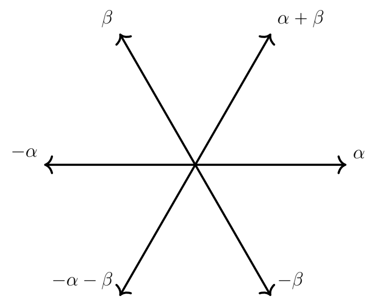



Figure 2.1: Root system of type $A_3$.

positive roots $\Phi^+$ to be the roots for which the coefficients in the linear combination are all positive. For a reductive group $\mathbb{G}$, certain subgroups can be created from $\Delta$ and $\Phi^+$, and play an important role in the representation theory.

**Definition 2.1.10.** *Let $\mathbb{G}$ be a reductive group defined over an algebraically closed field $K$, and $\mathbb{T}$ a maximal torus with associated root system $\Phi$. Let $\Delta$ be a simple system in $\Phi$. For $\delta \subset \Delta$, we let $\Phi_\delta$ denote the sub-root system generated by $\delta$, that is the set of integer linear combinations of $\delta$ which are roots. The standard Levi subgroup associated to $\delta$ is*

$$\mathbb{M}_\delta = \langle \mathbb{T}, \mathbb{U}_\alpha : \alpha \in \Phi_\delta \rangle.$$

*The standard parabolic subgroup associated to $\delta$ is*

$$\mathbb{P}_\delta = \langle \mathbb{T}, \mathbb{U}_\alpha : \alpha \in \Phi_\delta \cup \Phi^+ \rangle.$$

Set $\mathbb{N}_\delta = \langle \mathbb{U}_\alpha : \alpha \in \Phi^+ \backslash \Phi_\delta \rangle$. This is a unipotent subgroup and is normal in $\mathbb{P}_\delta$. The parabolic subgroup $\mathbb{P}_\delta$ can be expressed as a semidirect product $\mathbb{P}_\delta = \mathbb{M}_\delta \ltimes \mathbb{N}_\delta$. This is called the *Levi decomposition* of $\mathbb{P}_\delta$. When $\delta = \varnothing$, the obtained standard parabolic subgroup $\mathbb{P}_\varnothing$ is called a Borel subgroup, and the corresponding Levi subgroup $\mathbb{M}_\varnothing$ is simply the maximal torus $\mathbb{T}$.

**Example 2.1.11.** Recall the group structure of $\mathbb{G} = \mathrm{SL}_3(K)$ with respect to the diagonal maximal torus $\mathbb{T}$. The simple system consists of two elements, $\Delta = \{\alpha, \beta\}$, and the set of positive roots is $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$. It follows that $\mathbb{G}$ has three proper standard parabolic subgroups with respect to $\mathbb{T}$: they are

$$\mathbb{P}_\varnothing = \langle \mathbb{T}, \mathbb{U}_\gamma : \gamma \in \Phi^+ \rangle = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix},$$

$$\mathbb{P}_\alpha = \langle \mathbb{T}, \mathbb{U}_\gamma : \gamma \in \Phi^+ \cup \{\pm\alpha\} \rangle = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix},$$

$$\mathbb{P}_\beta = \langle \mathbb{T}, \mathbb{U}_\gamma : \gamma \in \Phi^+ \cup \{\pm\beta\} \rangle = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

The symbol $*$ indicates that any element of $K$ can be chosen, as long as the resulting matrix lies in $\mathrm{SL}_3(K)$. ∎

**Remark 2.1.12.** This definition of standard Levi and standard parabolic subgroups depends on the maximal torus that was chosen. Since all maximal tori of $\mathbb{G}$ are conjugate [Hum75, Corollary 21.3A], we can produce other Levi and parabolic subgroups by conjugating the standard subgroups described above.

A Levi subgroup $\mathbb{M} \subset \mathbb{G}$ is itself a reductive group [Hum75, Section 30.2]. In the following lemmas, we ensure that our definition of $\mathbb{M}$ stated above is robust in the sense that it coincides with its description in terms of a maximal torus of $\mathbb{M}$ and corresponding root subgroups.

**Lemma 2.1.13.** *Let $\mathbb{M}$ be a Levi subgroup of $\mathbb{G}$. Then, maximal tori of $\mathbb{M}$ are also maximal tori of $\mathbb{G}$.*

**Proof:** Assume without loss of generality that $\mathbb{M}$ is a standard Levi subgroup relative to a maximal torus $\mathbb{T}$. This ensures that $\mathbb{T}$ is a maximal torus of $\mathbb{M}$. Indeed, if $\mathbb{T}$ was not a maximal torus of $\mathbb{M}$, then we could find a bigger torus $\mathbb{T}'$ such that $\mathbb{T} \subsetneq \mathbb{T}' \subset \mathbb{M} \subset \mathbb{G}$, which is a contradiction with the maximality of $\mathbb{T}$ in $\mathbb{G}$. Now, given a maximal torus $\mathbb{S}$ of $\mathbb{M}$, $\mathbb{S}$ will be conjugate to $\mathbb{T}$ by an element of $\mathbb{M} \subset \mathbb{G}$. Hence, $\mathbb{S}$ is a maximal torus of $\mathbb{G}$. ∎

**Lemma 2.1.14.** *Let $\mathbb{M}$ be a Levi subgroup of $\mathbb{G}$, $\mathbb{T}$ be a maximal torus of $\mathbb{M}$, $\Phi_\mathbb{M} = \Phi(\mathbb{M}, \mathbb{T})$ and $\mathfrak{m} = \mathrm{Lie}(\mathbb{M})$. Then $\Phi_\mathbb{M}$ is a sub-root system of $\Phi(\mathbb{G}, \mathbb{T})$ and $\mathfrak{m}_\alpha = \mathfrak{g}_\alpha$ for all $\alpha \in \Phi_\mathbb{M}$.*

**Proof:** From Lemma 2.1.13, we know that $\mathbb{T}$ is a maximal torus of $\mathbb{G}$. Given $\alpha \in \Phi_\mathbb{M}$,

$$\mathfrak{m}_\alpha = \{X \in \mathfrak{m} : \mathrm{Ad}(t)X = \alpha(t)X \text{ for all } t \in \mathbb{T}\} \neq \{0\}.$$

Since $\mathfrak{m} \subset \mathfrak{g}$, $\mathfrak{m}_\alpha \subset \{X \in \mathfrak{g} : \mathrm{Ad}(t)X = \alpha(t)X \text{ for all } t \in \mathbb{T}\} = \mathfrak{g}_\alpha$. It follows that $\mathfrak{g}_\alpha \neq \{0\}$, so $\alpha \in \Phi$. Furthermore, since $\mathfrak{m}_\alpha$ and $\mathfrak{g}_\alpha$ are both one-dimensional, we must have $\mathfrak{m}_\alpha = \mathfrak{g}_\alpha$. ∎

**Corollary 2.1.15.** *Let $\mathbb{M}$ be a Levi subgroup of $\mathbb{G}$ and $\mathbb{T}$ be a maximal torus of $\mathbb{M}$. Then the root subgroups of $\mathbb{M}$ with respect to $\mathbb{T}$ are also root subgroups of $\mathbb{G}$ with respect to $\mathbb{T}$.*

With this last corollary, we are now convinced that Definition 2.1.10 for a standard Levi subgroup coincides with its description in terms of generators with respect to a maximal torus.

### 2.1.2 Structure of Subgroups that Contain $\mathbb{G}_{\text{der}}$

Let $\mathbb{G}_{\text{der}}$ denote the derived subgroup of $\mathbb{G}$, that is the subgroup which is generated by the commutators of $\mathbb{G}$. This subgroup $\mathbb{G}_{\text{der}}$, which is also denoted $[\mathbb{G}, \mathbb{G}]$, is a semisimple reductive group. As such, it has its tori, root subgroups, Levi and parabolic subgroups.

**Proposition 2.1.16** ([Hum75, Exercise 26.10]). *Let $\mathbb{G}$ be a reductive group, $\mathbb{T}$ a maximal torus and $\Phi = \Phi(\mathbb{G}, \mathbb{T})$. Then $\mathbb{G}_{\text{der}} = \langle \mathbb{U}_\alpha : \alpha \in \Phi \rangle$.*

**Proof:** Let $\alpha \in \Phi$. We start by showing that $\mathbb{U}_\alpha \subset \mathbb{G}_{\text{der}}$, that is every $u \in \mathbb{U}_\alpha$ can be written as a commutator. By part b) of Theorem 2.1.4, we have that $t\epsilon_\alpha(x)t^{-1}\epsilon_\alpha(x)^{-1} = \epsilon_\alpha(\alpha(t)x)\epsilon_\alpha(-x)$ for all $t \in \mathbb{T}, x \in K$. Since $\epsilon_\alpha$ is a morphism, this is equal to $\epsilon_\alpha((\alpha(t) - 1)x)$. Thus, to show that $u \in \mathbb{G}_{\text{der}}$, it suffices to show that $u = \epsilon_\alpha((\alpha(t) - 1)x)$ for some $t \in \mathbb{T}, x \in K$. We know that $u = \epsilon_\alpha(y)$ for some $y \in K$. Take $t \in \mathbb{T}$, $t \notin \ker \alpha$. Note that we can find such an element as roots are nontrivial characters by definition. Then $\alpha(t) - 1 \neq 0$, and therefore $\alpha(t) - 1 \in K^\times$. Letting $x = y(\alpha(t) - 1)^{-1}$, it follows that $u = \epsilon_\alpha((\alpha(t) - 1)x)$. Hence $u \in \mathbb{G}_{\text{der}}$, and $\mathbb{U}_\alpha \subset \mathbb{G}_{\text{der}}$ for all $\alpha \in \Phi$. Therefore $\langle \mathbb{U}_\alpha : \alpha \in \Phi \rangle \subset \mathbb{G}_{\text{der}}$.

For the converse, let $x \in \mathbb{G}_{\text{der}}$. Without loss of generality, assume that $x = aba^{-1}b^{-1}$ for some $a, b \in \mathbb{G}$. By Remark 2.1.5, we have that $a = tu$ and $b = sv$ for some $u, v \in \langle \mathbb{U}_\alpha : \alpha \in \Phi \rangle$, $t, s \in \mathbb{T}$. Then $x = tu(sv)(tu)^{-1}(sv)^{-1}$. Using the fact that root subgroups are normalized by $\mathbb{T}$, we can write $x = tst^{-1}s^{-1}\bar{u}$ for some $\bar{u} \in \langle \mathbb{U}_\alpha : \alpha \in \Phi \rangle$. Since $\mathbb{T}$ is isomorphic to copies of $K^\times$, it is abelian, and so $tst^{-1}s^{-1} = 1$ meaning that $x \in \langle \mathbb{U}_\alpha : \alpha \in \Phi \rangle$. ∎

**Remark 2.1.17.** The previous proposition, along with Remark 2.1.5, allows us to write $\mathbb{G} = \mathbb{T}\mathbb{G}_{\text{der}}$. This is different from the central isogeny $\mathbb{G} = Z(\mathbb{G})^\circ\mathbb{G}_{\text{der}}$ since $\mathbb{T}$ is a larger subgroup that does not necessarily have finite intersection with $\mathbb{G}_{\text{der}}$. In some cases, it will be more convenient to use the central isogeny of $\mathbb{G}$ with $\mathbb{G}_{\text{der}}$, whereas for others, the decomposition of $\mathbb{G}$ as $\mathbb{T}\mathbb{G}_{\text{der}}$ will suffice.

Let $\mathbb{H}$ be a closed connected subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\text{der}}$. It follows that $\mathbb{H}$ is a reductive group. It will be in our interest to understand the relationships between the tori, Levi subgroups and parabolic subgroups of $\mathbb{H}$ and those of $\mathbb{G}$. The remaining results of this section will establish how the subgroups of $\mathbb{H}$ are related to those of $\mathbb{G}$. Note that $\mathbb{G}_{\text{der}} \subset \mathbb{H}$ implies that $\mathbb{H}$ is a normal subgroup of $\mathbb{G}$. Indeed, for all $g \in \mathbb{G}, h \in \mathbb{H}$, $ghg^{-1}h^{-1} \in \mathbb{G}_{\text{der}} \subset \mathbb{H}$ and therefore $ghg^{-1} \in \mathbb{H}$ for all $g \in \mathbb{G}, h \in \mathbb{H}$.

**Proposition 2.1.18.** *Let $\mathbb{H}$ be a closed subgroup of $\mathbb{G}$ and $\mathbb{T}$ a (not necessarily maximal) torus of $\mathbb{G}$. Then $(\mathbb{T} \cap \mathbb{H})^\circ$ is a torus of $\mathbb{H}$.*

**Proof:** Since $\mathbb{T}$ is a torus, $\mathbb{T} \simeq (K^\times)^n \simeq \mathbb{D}(n, K)$ for some $n$, where $\mathbb{D}(n, K) \subset \mathrm{GL}(n, K)$ denotes the subgroup of $n \times n$ diagonal matrices with entries in $K$ and nonzero determinant. By [Hum75, Theorem 16.2], a closed and connected subgroup of $\mathbb{D}(n, K)$ is a torus. Since $\mathbb{T} \cap \mathbb{H}$ is closed in $\mathbb{T}$, it follows that $\mathbb{T} \cap \mathbb{H}$ is isomorphic to a closed subgroup of $\mathbb{D}(n, K)$. By [Hum75, Proposition 7.3], $(\mathbb{T} \cap \mathbb{H})^\circ$ is closed in $\mathbb{T} \cap \mathbb{H}$, and therefore $(\mathbb{T} \cap \mathbb{H})^\circ$ is a closed and connected subgroup of $\mathbb{D}(n, K)$. ∎

**Proposition 2.1.19.** *Let $\mathbb{H}$ be a closed normal subgroup of $\mathbb{G}$, $\mathbb{T}$ a maximal torus of $\mathbb{G}$. Then $(\mathbb{T} \cap \mathbb{H})^\circ$ is a maximal torus of $\mathbb{H}$. Furthermore, every maximal torus of $\mathbb{H}$ is of the form $(\mathbb{T} \cap \mathbb{H})^\circ$ for some maximal torus $\mathbb{T}$ of $\mathbb{G}$.*

**Proof:** Let $\mathbb{S}$ be a maximal torus of $\mathbb{H}$. Then $\mathbb{S}$ is contained in some maximal torus of $\mathbb{G}$. It follows that $\mathbb{S} \subset g\mathbb{T}g^{-1}$ for some $g \in \mathbb{G}$, as all maximal tori are conjugate. So, $\mathbb{S} \subset g\mathbb{T}g^{-1} \cap \mathbb{H}$. Since $\mathbb{S}$ is connected, we have $\mathbb{S} \subset (g\mathbb{T}g^{-1} \cap \mathbb{H})^\circ$. By Proposition 2.1.18, $(g\mathbb{T}g^{-1} \cap \mathbb{H})^\circ$ is a torus of $\mathbb{H}$. It follows from maximality of $\mathbb{S}$ that $\mathbb{S} = (g\mathbb{T}g^{-1} \cap \mathbb{H})^\circ$. Normality of $\mathbb{H}$ also implies that $(\mathbb{T} \cap \mathbb{H})^\circ$ is a maximal torus of $\mathbb{H}$. ∎

**Remark 2.1.20.** Let $\mathbb{H}$ and $\mathbb{T}$ be as in the previous proposition. If we add the hypothesis that $\mathbb{H}$ is connected, then $\mathbb{T} \cap \mathbb{H}$ will be a maximal torus of $\mathbb{H}$. This is because $C_{\mathbb{G}}(\mathbb{T}) \cap \mathbb{H}$ will be connected [Con14, Example 2.2.6], where $C_{\mathbb{G}}(\mathbb{T})$ denotes the centralizer of $\mathbb{T}$ in $\mathbb{G}$. Since $\mathbb{T}$ is a maximal torus, we have that $C_{\mathbb{G}}(\mathbb{T}) = \mathbb{T}$ [Hum75, Corollary 26.2A], and therefore $\mathbb{T} \cap \mathbb{H}$ is connected.

**Remark 2.1.21.** As $\mathbb{G}_{\text{der}}$ is connected [Hum75, Corollary 7.5] and closed [Hum75, Proposition 17.2B], we can set $\mathbb{H} = \mathbb{G}_{\text{der}}$ in the previous remark as well as all the subsequent results in this section.

**Corollary 2.1.22.** *Let $\mathbb{H}$ be a closed connected subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$. Let $\mathbb{S}$ be a maximal torus of $\mathbb{H}$. Then there exists a unique maximal torus $\mathbb{T}$ of $\mathbb{G}$ such that $\mathbb{S} = \mathbb{T} \cap \mathbb{H}$.*

**Proof:** By the previous proposition and remark, $\mathbb{S} = \mathbb{T} \cap \mathbb{H}$, with $\mathbb{T}$ a maximal torus in $\mathbb{G}$. It remains to show uniqueness. Assume $\mathbb{S} = \mathbb{T} \cap \mathbb{H}$ and $\mathbb{S} = \overline{\mathbb{T}} \cap \mathbb{H}$, where $\mathbb{T}$ and $\overline{\mathbb{T}}$ are maximal tori of $\mathbb{G}$. Then $\overline{\mathbb{T}} = {}^g\mathbb{T}$ for some $g \in \mathbb{G}$ since maximal tori of $\mathbb{G}$ are conjugate. The normality of $\mathbb{H}$ in $\mathbb{G}$ implies that $\mathbb{S} = \mathbb{T} \cap \mathbb{H} = {}^g\mathbb{T} \cap \mathbb{H} = {}^g(\mathbb{T} \cap \mathbb{H}) = {}^g\mathbb{S}$. It follows that $g$ normalizes $\mathbb{S}$. We show that $g$ must also normalize $\mathbb{T}$. Indeed, take $t \in \mathbb{T}$. By Corollary 2.1.7, $t = zs$ for some $z \in Z(\mathbb{G})^\circ$ and $s \in \mathbb{T} \cap \mathbb{G}_{\text{der}} \subset \mathbb{S}$. Then $gtg^{-1} = gzsg^{-1} = z(gsg^{-1}) \in Z(\mathbb{G})^\circ\mathbb{S} \subset \mathbb{T}$. We conclude that $g$ normalizes $\mathbb{T}$ and $\mathbb{T} = \overline{\mathbb{T}}$. ∎

**Proposition 2.1.23.** *Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$, $\mathbb{H}$ a closed connected subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$ and $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$ a maximal torus of $\mathbb{H}$. Then the intersection of $\mathbb{H}$ with the normalizer of $\mathbb{T}$ in $\mathbb{G}$, $N_{\mathbb{G}}(\mathbb{T})$, is equal to the normalizer of $\mathbb{T}_{\mathbb{H}}$ in $\mathbb{H}$, $N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}})$.*

**Proof:** The proof of Corollary 2.1.22 along with the normality of $\mathbb{H}$ shows that $N_{\mathbb{G}}(\mathbb{T}_{\mathbb{H}}) = N_{\mathbb{G}}(\mathbb{T})$. By intersecting both sides of the equality with $\mathbb{H}$, we obtain that $N_{\mathbb{G}}(\mathbb{T}) \cap \mathbb{H} = N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}})$ as stated. ∎

**Proposition 2.1.24.** *Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$, $\mathbb{H}$ a closed connected subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$ and $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$. Then $N_{\mathbb{G}}(\mathbb{T})/\mathbb{T} \simeq N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}})/\mathbb{T}_{\mathbb{H}}$.*

**Proof:** By the previous proposition, $N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}}) \subset N_{\mathbb{G}}(\mathbb{T})$, so we can consider the morphism

$$\begin{aligned} f : N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}})/\mathbb{T}_{\mathbb{H}} &\to N_{\mathbb{G}}(\mathbb{T})/\mathbb{T} \\ h\mathbb{T}_{\mathbb{H}} &\mapsto h\mathbb{T}. \end{aligned}$$

One can verify easily that this map is well-defined. The kernel of this map is

$$\ker f = \{h \in N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}}) : h\mathbb{T} = \mathbb{T}\} = \{h \in N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}}) \cap \mathbb{T}\}.$$

Since $N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}}) = N_{\mathbb{G}}(\mathbb{T}) \cap \mathbb{H}$, we conclude that the map is injective.

Now, given $g \in N_{\mathbb{G}}(\mathbb{T})$, the central isogeny of $\mathbb{G}$ with $\mathbb{G}_{\text{der}}$ allows us to write $g = zh$ for some $z \in Z(\mathbb{G})^\circ \subset \mathbb{T}, h \in \mathbb{G}_{\text{der}} \subset \mathbb{H}$ (Proposition 2.1.6). It follows that $g\mathbb{T} = h\mathbb{T}$, where $h = z^{-1}g \in N_{\mathbb{G}}(\mathbb{T}) \cap \mathbb{H} = N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}})$ (Proposition 2.1.23). Therefore, we conclude that the map is surjective. ∎

**Remark 2.1.25.** The groups $W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ and $W_H = N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}})/\mathbb{T}_{\mathbb{H}}$ are called the *Weyl groups* of $\mathbb{G}$ and $\mathbb{H}$, respectively. Therefore, by the previous proposition, $\mathbb{G}$ and $\mathbb{H}$ have the same Weyl group.

We let $\mathbb{H}$ remain a closed connected subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\text{der}}$. In Proposition 2.1.16, we provided a description of $\mathbb{G}_{\text{der}}$ in terms of the root subgroups of $\mathbb{G}$ with respect to the maximal torus $\mathbb{T}$. Now, we know that $\mathbb{T} \cap \mathbb{H}$ is a maximal torus of $\mathbb{H}$, which means it admits a root system and corresponding root subgroups. In what follows, we work to verify that the root subgroups $\mathbb{U}_\alpha, \alpha \in \Phi(\mathbb{G}, \mathbb{T})$ are also root subgroups of $\mathbb{H}$ with respect to the maximal torus $\mathbb{T} \cap \mathbb{H}$. This will allow us to intersect Levi subgroups and parabolic subgroups of $\mathbb{G}$ with $\mathbb{H}$ in a very natural way.

**Lemma 2.1.26.** *Let $\mathbb{G}$ be a reductive group, $\mathbb{T}$ a maximal torus and $\Phi = \Phi(\mathbb{G}, \mathbb{T})$. Let $\mathbb{H}$ be a closed connected subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$. Let $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$ and $\Phi_{\mathbb{H}} = \Phi(\mathbb{H}, \mathbb{T}_{\mathbb{H}})$. Then for all $\alpha \in \Phi, \alpha|_{\mathbb{T}_{\mathbb{H}}} \in \Phi_{\mathbb{H}}$ and $|\Phi| = |\Phi_{\mathbb{H}}|$.*

**Proof:** Let $\alpha \in \Phi$. First note that $\text{Lie}(\mathbb{G}_{\text{der}}) = [\mathfrak{g}, \mathfrak{g}] \subset \text{Lie}(\mathbb{H}) = \mathfrak{h}$. Since $\mathbb{U}_\alpha \subset \mathbb{G}_{\text{der}}$ (Proposition 2.1.16), it follows that $\mathfrak{g}_\alpha \subset \mathfrak{h}$. Therefore,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{h} : \text{Ad}(t)X = \alpha(t)X \text{ for all } t \in \mathbb{T}\}.$$

Since $\mathbb{T}_{\mathbb{H}} \subset \mathbb{T}$,

$$\mathfrak{g}_\alpha \subset \{X \in \mathfrak{h} : \text{Ad}(t)X = \alpha(t)X \text{ for all } t \in \mathbb{T}_{\mathbb{H}}\} = \mathfrak{h}_{\alpha|_{\mathbb{T}_{\mathbb{H}}}}.$$

This forces $\mathfrak{h}_{\alpha|_{\mathbb{T}_{\mathbb{H}}}} \neq \{0\}$ and therefore $\alpha|_{\mathbb{T}_{\mathbb{H}}} \in \Phi_{\mathbb{H}}$.

Next, assume that $\alpha|_{\mathbb{T}_{\mathbb{H}}} = \beta|_{\mathbb{T}_{\mathbb{H}}}$ for $\alpha, \beta \in \Phi$. We show that this implies $\alpha = \beta$. Indeed, recall that Corollary 2.1.7 allows us to write every $t \in \mathbb{T}$ in the form $t = zs$ for some $z \in Z(\mathbb{G})^\circ, s \in \mathbb{T} \cap \mathbb{G}_{\text{der}} \subset \mathbb{T}_{\mathbb{H}}$. In particular, $z \in \bigcap_{\alpha\in\Phi} (\ker \alpha)^\circ$ (Proposition 2.1.6). Therefore, $(\alpha - \beta)(t) = (\alpha - \beta)(zs) = (\alpha - \beta)(s) = 0$. Hence, $\alpha = \beta$ and we conclude that $|\Phi| = |\Phi_{\mathbb{H}}|$. ∎

**Lemma 2.1.27.** *A Lie algebra decomposition of $\mathfrak{h}$ is given by*

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{H}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

*where* $\mathfrak{t}_{\mathbb{H}} = \text{Lie}(\mathbb{T}_{\mathbb{H}})$.

**Proof:** As discussed in Section 2.1.1,

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{H}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{h}_{\alpha|_{\mathbb{T}_{\mathbb{H}}}}.$$

By the previous lemma, we have that $\mathfrak{g}_\alpha \subset \mathfrak{h}_{\alpha|_{\mathbb{T}_{\mathbb{H}}}}$, which implies equality as both subalgebras are one-dimensional. The conclusion follows. ∎

**Corollary 2.1.28.** *Let $\mathbb{G}, \mathbb{T}$, $\mathbb{H}$ and $\mathbb{T}_{\mathbb{H}}$ be as in Lemma 2.1.26. The root subgroups $\mathbb{U}_\alpha, \alpha \in \Phi(\mathbb{G}, \mathbb{T})$ of $\mathbb{G}$ are precisely the root subgroups of $\mathbb{H}$ with respect to the maximal torus $\mathbb{T}_{\mathbb{H}}$ of $\mathbb{H}$.*

**Remark 2.1.29.** The previous corollary allows us to write $\mathbb{H} = \langle \mathbb{T}_{\mathbb{H}}, \mathbb{U}_\alpha : \alpha \in \Phi \rangle$ to be the presentation of $\mathbb{H}$ in terms of root subgroups of $\mathbb{H}$. By Proposition 2.1.16, we then have $[\mathbb{H}, \mathbb{H}] = \langle \mathbb{U}_\alpha : \alpha \in \Phi \rangle = \mathbb{G}_{\text{der}}$, where $[\mathbb{H}, \mathbb{H}]$ is the derived subgroup of $\mathbb{H}$.

The previous corollary also allows us to define standard Levi and parabolic subgroups of $\mathbb{H}$ by replacing $\mathbb{T}$ with $\mathbb{T}_{\mathbb{H}}$ in Definition 2.1.10. Furthermore, we can establish a clear relationship between the Levi and parabolic subgroups of $\mathbb{G}$ with the Levi and parabolic subgroups of $\mathbb{H}$ via the following proposition.

**Proposition 2.1.30.** *Let $\mathbb{G}$ be a reductive group and $\mathbb{H}$ be a closed connected subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$. Then given a Levi subgroup $\mathbb{M}$ of $\mathbb{G}$, $\mathbb{M} \cap \mathbb{H}$ is a Levi subgroup of $\mathbb{H}$. Furthermore, every Levi subgroup of $\mathbb{H}$ is of the form $\mathbb{M} \cap \mathbb{H}$ for some unique Levi subgroup $\mathbb{M}$ of $\mathbb{G}$.*

**Proof:** Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$ with associated root system $\Phi$. Without loss of generality, we may assume that $\mathbb{M}$ is a standard Levi subgroup. It follows that for some $\delta \subset \Delta$, we have

$$\mathbb{M} = \langle \mathbb{T}, \mathbb{U}_\gamma : \gamma \in \Phi_\delta \rangle.$$

We show that

$$\mathbb{M} \cap \mathbb{H} = \langle \mathbb{T}_{\mathbb{H}}, \mathbb{U}_\gamma : \gamma \in \Phi_\delta \rangle,$$

implying that the intersection is a standard Levi subgroup of $\mathbb{H}$. Clearly, we have $\langle \mathbb{T}_{\mathbb{H}}, \mathbb{U}_\gamma : \gamma \in \Phi_\delta \rangle \subset \mathbb{M} \cap \mathbb{H}$. Conversely, assume $x \in \mathbb{M} \cap \mathbb{H}$. By Remark 2.1.5, we may write $x = tu$ for some $t \in \mathbb{T}$, $u \in \langle \mathbb{U}_\gamma : \gamma \in \Phi_\delta \rangle$. By Proposition 2.1.16, $u \in \mathbb{G}_{\text{der}} \subset \mathbb{H}$, which implies that $t = xu^{-1} \in \mathbb{T} \cap \mathbb{H} = \mathbb{T}_{\mathbb{H}}$. The conclusion follows

That all Levi subgroup of $\mathbb{H}$ arise uniquely in this way follows from Corollary 2.1.22 and Corollary 2.1.28. ∎

**Remark 2.1.31.** The previous proposition remains true if we replace the word "Levi" by "parabolic", and the proof is essentially the same.

We end this section by proving some results regarding centralizers, which will be useful in Section 6.1.3.

**Lemma 2.1.32.** *Let $\mathbb{G}$ be a reductive group, and let $\mathbb{H}$ be a closed connected subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$. Then, given a semisimple element $s \in \mathbb{H}$, $C_{\mathbb{G}}(s)^\circ \cap \mathbb{H} = C_{\mathbb{H}}(s)^\circ$, where $C_{\mathbb{G}}(s)$ denotes the centralizer of $s$ in $\mathbb{G}$.*

**Proof:** Since $s$ is semisimple, it follows from [Hum75, Corollary 19.3] that $s \in \mathbb{T}$ for some maximal torus $\mathbb{T}$ of $\mathbb{G}$. By [Car93, Theorem 3.5.3], we have that

$$C_{\mathbb{G}}(s)^\circ = \langle \mathbb{T}, \mathbb{U}_\alpha : \alpha \in \Phi(\mathbb{G}, \mathbb{T}) \text{ such that } \alpha(s) = 1 \rangle.$$

Using a similar argument to Proposition 2.1.30,

$$C_{\mathbb{G}}(s)^\circ \cap \mathbb{H} = \langle \mathbb{T}_{\mathbb{H}}, \mathbb{U}_\alpha : \alpha \in \Phi(\mathbb{G}, \mathbb{T}) \text{ such that } \alpha(s) = 1 \rangle,$$

where $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$. We then use the result of Lemma 2.1.26 to identify $\Phi(\mathbb{G}, \mathbb{T})$ with $\Phi(\mathbb{H}, \mathbb{T}_{\mathbb{H}})$ and obtain

$$\begin{aligned} C_{\mathbb{G}}(s)^\circ \cap \mathbb{H} &= \langle \mathbb{T}_{\mathbb{H}}, \mathbb{U}_\alpha : \alpha \in \Phi(\mathbb{H}, \mathbb{T}_{\mathbb{H}}) \text{ such that } \alpha(s) = 1 \rangle \\ &= C_{\mathbb{H}}(s)^\circ. \end{aligned}$$

∎

**Remark 2.1.33.** Note that groups of the form $C_{\mathbb{G}}(s)^\circ$ for $s \in \mathbb{G}$ semisimple correspond to what we call *pseudo-Levi subgroups* of $\mathbb{G}$ [Som98, Proposition 2]. So the previous lemma shows that intersecting a pseudo-Levi subgroup of $\mathbb{G}$ with $\mathbb{H}$ produces a pseudo-Levi subgroup of $\mathbb{H}$ when $s \in \mathbb{H}$.

**Lemma 2.1.34.** *Let $\mathbb{G}$ be a reductive group, and let $\mathbb{H}$ be a closed connected subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$. Let $s$ be a semisimple element of $\mathbb{H}$. Then $C_{\mathbb{G}}(s)^\circ = Z(C_{\mathbb{G}}(s)^\circ)C_{\mathbb{H}}(s)^\circ$.*

**Proof:** It is clear that $C_{\mathbb{G}}(s)^\circ \supset Z(C_{\mathbb{G}}(s)^\circ)C_{\mathbb{H}}(s)^\circ$, so we show the other inclusion. For the converse, we use the central isogeny of $\mathbb{G}$ with $\mathbb{G}_{\text{der}}$. Given $g \in C_{\mathbb{G}}(s)^\circ$, $g = zh$ for some $z \in Z(\mathbb{G})^\circ$ and $h \in \mathbb{G}_{\text{der}} \subset \mathbb{H}$. Since $Z(\mathbb{G})^\circ \subset C_{\mathbb{G}}(s)^\circ$ it follows that $z \in Z(C_{\mathbb{G}}(s)^\circ)$. Furthermore, $h = z^{-1}g \in \mathbb{H} \cap C_{\mathbb{G}}(s)^\circ = C_{\mathbb{H}}(s)^\circ$ (Lemma 2.1.32). ∎

**Lemma 2.1.35.** *Let $\mathbb{G}$ be a reductive group, and let $\mathbb{H}$ be a closed connected subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$. Let $s$ be a semisimple element of $\mathbb{H}$. Then $Z(C_{\mathbb{H}}(s)^\circ) = Z(C_{\mathbb{G}}(s)^\circ) \cap \mathbb{H}$.*

**Proof:** From the definitions, we have that $Z(C_{\mathbb{G}}(s)^\circ) \cap \mathbb{H} \subset Z(C_{\mathbb{H}}(s)^\circ)$. For the converse, let $g \in Z(C_{\mathbb{H}}(s)^\circ)$. Then $g$ is an element of $C_{\mathbb{H}}(s)^\circ = C_{\mathbb{G}}(s)^\circ \cap \mathbb{H}$ (Lemma 2.1.32) that commutes with $C_{\mathbb{H}}(s)^\circ$. By Lemma 2.1.34, we have that $g$ must also commute with $C_{\mathbb{G}}(s)^\circ$. Therefore $g \in Z(C_{\mathbb{G}}(s)^\circ) \cap \mathbb{H}$, which concludes the proof. ∎

**Lemma 2.1.36.** *Let $\mathbb{G}$ be a reductive group, and let $\mathbb{H}$ be a closed connected subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$. Let $s$ be a semisimple element of $\mathbb{H}$. Then $C_{\mathbb{G}}(s)^\circ/Z(C_{\mathbb{G}}(s)^\circ) \simeq C_{\mathbb{H}}(s)^\circ/Z(C_{\mathbb{H}}(s)^\circ)$.*

**Proof:** We use Lemma 2.1.34 to rewrite $C_{\mathbb{G}}(s)^\circ/Z(C_{\mathbb{G}}(s)^\circ)$ as $C_{\mathbb{H}}(s)^\circ Z(C_{\mathbb{G}}(s)^\circ)/Z(C_{\mathbb{G}}(s)^\circ)$, which is isomorphic to $C_{\mathbb{H}}(s)^\circ/\left(C_{\mathbb{H}}(s)^\circ \cap Z(C_{\mathbb{G}}(s)^\circ)\right)$ by the second isomorphism theorem. Using Lemma 2.1.32, we have that $C_{\mathbb{H}}(s)^\circ \cap Z(C_{\mathbb{G}}(s)^\circ) = \mathbb{H} \cap Z(C_{\mathbb{G}}(s)^\circ)$, which in turn is equal to $Z(C_{\mathbb{H}}(s)^\circ)$ by Lemma 2.1.35. The result follows. ∎

### 2.1.3 Rational Points of Reductive Groups

In the previous section, we were discussing the structure of reductive groups with $K$ an algebraically closed field. However, in this thesis, we want to be working over $p$-adic fields, which are never algebraically closed, and their residue fields which are finite. It is possible to define reductive groups over fields which are not algebraically closed. When $\mathbb{G}$ is a variety in the affine space $\mathbb{A}^n = K^n$ for some $n > 0$, then for a subfield $k \subset K$, taking the $k$-rational points consists of intersecting with $k^n$. We denote this intersection by $\mathbb{G}(k)$. Note that this intersection could be empty. When $\mathbb{G}$, along with group multiplication and inversion, are defined over $k$, then $\mathbb{G}$ is said to be a $k$-group and $\mathbb{G}(k)$ is a nonempty subgroup of $\mathbb{G}$. For matrix groups, this consists of restricting the entries in the matrices to come from the subfield $k$, or by intersecting with $\mathrm{GL}_n(k)$. In this language, $\mathbb{G}$ is then defined over $K$, so it is convenient to identify the algebraic group with its group of $K$-rational points. In particular, we write $\mathbb{G}$ for $\mathbb{G}(K)$.

Given a $k$-torus $\mathbb{T}$ of $\mathbb{G}$, that is a torus which is defined over $k$, it is not necessarily true that $\mathbb{T}(k)$ is isomorphic to $(k^\times)^n$ for some $n > 0$. So, when going to the $k$-rational points, there is a notion of being $k$-split which we define below.

**Definition 2.1.37.** *Let $\mathbb{T}$ be a $k$-torus of $\mathbb{G}$.*

*a) The torus $\mathbb{T}$ is said to be be $k$-split if and only if $\mathbb{T}(k) \simeq (k^\times)^n$ for some $n > 0$.*

*b) The torus $\mathbb{T}$ is said to be $k$-minisotropic if and only if all $k$-split subtori of $\mathbb{T}$ are in $Z(\mathbb{G})$.*

c) *The torus $\mathbb{T}$ is said to be $k$-anisotropic if and only if it contains no $k$-split subtori.*

**Remark 2.1.38.** Some references prefer to use the word *elliptic* when referring to a minisotropic torus.

**Remark 2.1.39.** When using the adjectives $k$-split, $k$-minisotropic and $k$-anisotropic to describe a torus $\mathbb{T}$, it will be implicit that $\mathbb{T}$ is a $k$-torus. Furthermore, we shall refer to a maximal $k$-split torus as a torus which is $k$-split and maximal with respect to this property. This is not to be confused with a $k$-split maximal torus, which is a maximal torus that is also $k$-split. Similar to having the maximal tori of $\mathbb{G}$ conjugate under $\mathbb{G}$, we have that the maximal $k$-split tori of $\mathbb{G}$ are conjugate under $\mathbb{G}(k)$ [Hum75, Theorem 34.5A].

When a reductive group is defined over $k$, one can talk about its $k$-rank. To calculate the $k$-rank of $\mathbb{G}$, one calculates the dimension of a maximal $k$-split torus of $\mathbb{G}$. Since all such tori are $\mathbb{G}(k)$-conjugate, the $k$-rank of $\mathbb{G}$, denoted by $r_k(\mathbb{G})$, is independent of the choice of maximal $k$-split torus. Note that if $\mathbb{S}$ is a maximal $k$-split torus of $\mathbb{G}$, then $r_k(\mathbb{G}) = r_k(\mathbb{S})$.

The $k$-group $\mathbb{G}$ is said to be $k$-split if it has a maximal torus $\mathbb{T}$ that is $k$-split. When the maximal torus $\mathbb{T}$ is $k$-split, the isomorphisms $\epsilon_\alpha : K \to \mathbb{U}_\alpha$ can be taken to be $k$-isomorphisms for all $\alpha \in \Phi$ [Hum75, Section 34.4]. This means the root subgroups $\mathbb{U}_\alpha$ are defined over $k$ and our presentation given in Theorem 2.1.4 transfers over to the $k$-rational points, that is

$$\mathbb{G}(k) = \langle \mathbb{T}(k), \mathbb{U}_\alpha(k) : \alpha \in \Phi \rangle.$$

When the maximal torus $\mathbb{T}$ with which we have chosen to construct our root system is not $k$-split, the root subgroups may not be defined over $k$, in which case we cannot simply think of the presentation given by Theorem 2.1.4 to describe $\mathbb{G}(k)$. Let us see how we can obtain a description of $\mathbb{G}(k)$ in terms of generators when the maximal torus $\mathbb{T}$ chosen is not $k$-split through an example.

**Example 2.1.40.** Let $\mathbb{G} = \mathrm{SL}_3(K)$. Let $k \subset K$ be a subfield which is not algebraically closed and let $\zeta \in k$ be an element which is not a square in $k$. Consider

$$\mathbb{T} = \left\{ \begin{bmatrix} a & b & 0 \\ b\zeta & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, b \in K, a^2 - b^2\zeta \neq 0, c = (a^2 - b^2\zeta)^{-1} \right\}.$$

Then $\mathbb{T}$ is a $k$-minisotropic torus. We can show that this torus is split over the field $\bar{k} = k[\sqrt{\zeta}]$. Indeed, commuting operators are simultaneously diagonalizable so we can compute the eigenvalues of $\begin{bmatrix} a & b & 0 \\ b\zeta & a & 0 \\ 0 & 0 & (a^2-b^2\zeta)^{-1} \end{bmatrix}$, which are given by $\lambda = a \pm b\sqrt{\zeta}$ and $\lambda = (a^2 - b^2\zeta)^{-1}$. Let

$$\overline{\mathbb{T}} = \left\{ \begin{bmatrix} a+b\sqrt{\zeta} & 0 & 0 \\ 0 & a-b\sqrt{\zeta} & 0 \\ 0 & 0 & (a^2-b^2\zeta)^{-1} \end{bmatrix} : a, b \in K \right\}.$$

Computing the associated eigenspaces to each of the eigenvalues above, we deduce that $\mathbb{T} = Q\overline{\mathbb{T}}Q^{-1}$, where $Q = \begin{bmatrix} 1 & -1/2(\sqrt{\zeta})^{-1} & 0 \\ \sqrt{\zeta} & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We show that $\mathbb{T}$ is split over $\bar{k}$ by showing that $\overline{\mathbb{T}}$ is $\bar{k}$-split, as conjugation is an isomorphism. One can easily verify that

$$\overline{\mathbb{T}}(\bar{k}) \to (\bar{k}^\times)^2$$

$$\begin{bmatrix} a+b\sqrt{\zeta} & 0 & 0 \\ 0 & a-b\sqrt{\zeta} & 0 \\ 0 & 0 & (a^2-b^2\zeta)^{-1} \end{bmatrix} \mapsto (a+b\sqrt{\zeta}, a-b\sqrt{\zeta})$$

is an isomorphism, and therefore $\mathbb{T}(\bar{k}) \simeq (\bar{k}^\times)^2$.

It follows from Example 2.1.9 that the root subgroups associated to $\overline{\mathbb{T}}(\bar{k})$ are given by

$$\overline{\mathbb{U}}_\alpha(\bar{k}) = \begin{bmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \overline{\mathbb{U}}_\beta(\bar{k}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \overline{\mathbb{U}}_{\alpha+\beta}(\bar{k}) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\overline{\mathbb{U}}_{-\alpha}(\bar{k}) = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \overline{\mathbb{U}}_{-\beta}(\bar{k}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{bmatrix}, \overline{\mathbb{U}}_{-\alpha-\beta}(\bar{k}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{bmatrix},$$

where the symbol $*$ indicates that any element of $\bar{k}$ can be chosen. Conjugating these root subgroups by $Q$, we obtain the root subgroups associated to $\mathbb{T}(\bar{k})$:

$$\mathbb{U}_\alpha(\bar{k}) = Q\overline{\mathbb{U}}_\alpha(\bar{k})Q^{-1} = \left\{ \begin{bmatrix} 1-x\sqrt{\zeta} & x & 0 \\ -x\zeta & x\sqrt{\zeta}+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : x \in \bar{k} \right\},$$
$$\mathbb{U}_\beta(\bar{k}) = Q\overline{\mathbb{U}}_\beta(\bar{k})Q^{-1} = \left\{ \begin{bmatrix} 1 & 0 & -\frac{1}{2}x(\sqrt{\zeta})^{-1} \\ 0 & 1 & 1/2x \\ 0 & 0 & 1 \end{bmatrix} : x \in \bar{k} \right\},$$

$$\mathbb{U}_{\alpha+\beta}(\bar{k}) = Q\overline{\mathbb{U}}_{\alpha+\beta}(\bar{k})Q^{-1} = \left\{ \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & x\sqrt{\zeta} \\ 0 & 0 & 1 \end{bmatrix} : x \in \bar{k} \right\},$$

$$\mathbb{U}_{-\alpha}(\bar{k}) = Q\overline{\mathbb{U}}_{-\alpha}(\bar{k})Q^{-1} = \left\{ \begin{bmatrix} 1 - \frac{1}{4}x(\sqrt{\zeta})^{-1} & -\frac{1}{4}x\zeta^{-1} & 0 \\ \frac{1}{4}x & 1 + \frac{1}{4}x(\sqrt{\zeta})^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} : x \in \bar{k} \right\},$$

$$\mathbb{U}_{-\beta}(\bar{k}) = Q\overline{\mathbb{U}}_{-\beta}(\bar{k})Q^{-1} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x\sqrt{\zeta} & x & 1 \end{bmatrix} : x \in \bar{k} \right\},$$

$$\mathbb{U}_{-\alpha-\beta}(\bar{k}) = Q\overline{\mathbb{U}}_{-\alpha-\beta}(\bar{k})Q^{-1} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}x & \frac{1}{2}x(\sqrt{\zeta})^{-1} & 1 \end{bmatrix} : x \in \bar{k} \right\}.$$

These root subgroups are clearly not defined over $k$ and so we cannot take the $k$-rational points. This is because we cannot have $x \in k$ and $x\sqrt{\zeta} \in k$ for $x \neq 0$. However, after combining some of the root subgroups together, one gets a subgroup that is defined over $k$ and $k$-rational points can be taken. Indeed, one can calculate that

$$\langle \mathbb{U}_{\alpha}(\bar{k}), \mathbb{U}_{-\alpha}(\bar{k}) \rangle = \begin{bmatrix} \mathrm{SL}_2(\bar{k}) & 0 \\ 0 & 1 \end{bmatrix},$$

$$\langle \mathbb{U}_{\beta}(\bar{k}), \mathbb{U}_{\alpha+\beta}(\bar{k}) \rangle = \left\{ \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b \in \bar{k} \right\},$$

$$\langle \mathbb{U}_{-\beta}(\bar{k}), \mathbb{U}_{-\alpha-\beta}(\bar{k}) \rangle = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix} : a, b \in \bar{k} \right\},$$

which are all defined over $k$.

On the other hand, the maximal $k$-split subtorus of $\mathbb{T}$ is $\mathbb{S} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{bmatrix} : a \in K^{\times} \right\}$. We compute the root spaces with respect to this torus $\mathbb{S}$:

$$\mathfrak{g}_{\lambda} = \{ g \in \mathfrak{sl}_3(K) : \mathrm{Ad}(s)(g) = \lambda(s)g \text{ for all } s \in \mathbb{S} \}.$$

For $s = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{bmatrix} \in \mathbb{S}, g = \begin{bmatrix} x & y & z \\ r & s & t \\ u & v & -x-s \end{bmatrix} \in \mathfrak{sl}_3(K),$

$$\mathrm{Ad}(s)(g) = \begin{bmatrix} x & y & a^3 z \\ r & s & a^3 t \\ a^{-3}u & a^{-3}v & -x-s \end{bmatrix}.$$

Letting $\lambda = \epsilon_1 - \epsilon_3 = \epsilon_2 - \epsilon_3$, we then have

$$\mathfrak{g}_\lambda = \left\{ \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} : z, t \in K \right\}, \mathfrak{g}_{-\lambda} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u & v & 0 \end{bmatrix} : u, v \in K \right\},$$

giving us a Lie algebra decomposition $\mathfrak{sl}_3(K) = \mathrm{Lie}(C_{\mathbb{G}}(\mathbb{S})) \oplus \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}$ with $C_{\mathbb{G}}(\mathbb{S})$ defined over $k$ [Hum75, Theorem 34.4]. The root subgroups are then $\mathbb{U}_\lambda = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} : z, t \in K \right\}, \mathbb{U}_{-\lambda} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{bmatrix} : u, v \in K \right\}$ and $\mathbb{G} = \langle C_{\mathbb{G}}(\mathbb{S}), \mathbb{U}_\lambda, \mathbb{U}_{-\lambda} \rangle$. Notice that $\mathbb{U}_\lambda = \langle \mathbb{U}_\beta, \mathbb{U}_{\alpha+\beta} \rangle$, and $\beta|_{\mathbb{S}} = (\alpha + \beta)|_{\mathbb{S}}$. Similarly, $\mathbb{U}_{-\lambda} = \langle \mathbb{U}_{-\beta}, \mathbb{U}_{-\alpha-\beta} \rangle$ and $-\beta|_{\mathbb{S}} = (-\alpha - \beta)|_{\mathbb{S}}$. The roots $\alpha$ and $-\alpha$ of $\mathbb{T}$ are trivial on $\mathbb{S}$ and do not appear in our root subgroups with respect to $\mathbb{S}$, but instead appear in $C_{\mathbb{G}}(\mathbb{S}) = \langle \mathbb{T}, \mathbb{U}_\alpha, \mathbb{U}_{-\alpha} \rangle$. Furthermore, since $C_{\mathbb{G}}(\mathbb{S}), \mathbb{U}_\lambda$ and $\mathbb{U}_{-\lambda}$ are all defined over $k$, we have that $\mathbb{G}(k) = \langle C_{\mathbb{G}}(\mathbb{S})(k), \mathbb{U}_\lambda(k), \mathbb{U}_{-\lambda}(k) \rangle$. ∎

Essentially, when we take the $k$-rational points of our reductive $k$-group $\mathbb{G}$ with maximal $k$-torus $\mathbb{T}$, we can still obtain a presentation in terms of root subgroups corresponding to a maximal $k$-split subtorus $\mathbb{S} \subset \mathbb{T}$. The root system $\Phi(\mathbb{G}, \mathbb{S})$ in this case is equivalent to restricting to $\mathbb{S}$ the elements of $\Phi(\mathbb{G}, \mathbb{T})$ which are nontrivial on $\mathbb{S}$. This root system may be nonreduced. The root subgroups with respect to $\mathbb{S}$ are combinations of some of the root subgroups with respect to $\mathbb{T}$, those of roots that restrict identically to $\mathbb{S}$. For a more detailed description regarding the root subgroups associated to a maximal $k$-split torus, one can consult the summary provided in [Mur05, Section 7] or [Bor91, Chapter 21].

The parabolic subgroups which are defined over $k$ are referred to as $k$-parabolic subgroups. Similarly, we have $k$-Levi subgroups. A detailed description of these subgroups is provided in [Bor91, Chapter 21]. For our purposes, it will suffice to think of the $k$-parabolic subgroups and $k$-Levi subgroups over a field for which they split, so that we can retrieve the descriptions from Section 2.1.1. Keep in mind however that taking the $k$-rational points might not preserve these descriptions, as illustrated in the previous example.

An important thing to note is that taking $k$-rational points does not commute with different types of group operations. For example, one cannot always say that the $k$-rational points of a product (quotient) is the product (quotient) of the $k$-rational points, which leads to the following remark.

**Remark 2.1.41.** If $\mathbb{G}$ is not $k$-split, $[\mathbb{G}(k), \mathbb{G}(k)] \subsetneq \mathbb{G}_{\mathrm{der}}(k)$ in general. To get a sense of why this is true, $\mathbb{G}_{\mathrm{der}}(k)$ is like taking all the commutators of $\mathbb{G}$ and only keeping those that are defined over $k$, which ought to be a larger group than $[\mathbb{G}(k), \mathbb{G}(k)]$.

This subtlety will be important when discussing the representation theory of $p$-adic groups. This means, for example, that the restriction of characters of $\mathbb{G}(k)$ to $\mathbb{G}_{\text{der}}(k)$ need not be trivial.

Furthermore, our central isogeny of $\mathbb{G}$ with $\mathbb{G}_{\text{der}}$ cannot always be applied at the level of the $k$-points. However, this isogeny can be in some sense carried over to maximal $k$-split tori as outlined in the following proposition.

**Proposition 2.1.42.** *Let $\mathbb{S}$ be a maximal $k$-split torus of $\mathbb{G}$. Then $\mathbb{S} = (\mathbb{S} \cap Z(\mathbb{G})^\circ)^\circ (\mathbb{S} \cap \mathbb{G}_{\text{der}})^\circ$, where $(\mathbb{S} \cap Z(\mathbb{G})^\circ)^\circ$ is a maximal $k$-split torus of $Z(\mathbb{G})^\circ$ and $(\mathbb{S} \cap \mathbb{G}_{\text{der}})^\circ$ is a maximal $k$-split torus of $\mathbb{G}_{\text{der}}$. Furthermore, $r_k(\mathbb{G}) = r_k(Z(\mathbb{G})^\circ) + r_k(\mathbb{G}_{\text{der}})$.*

**Proof:** The result is provided by [BT65, Proposition 4.27], where we take the isogeny of $\mathbb{G}$ with $\mathbb{G}_{\text{der}}$. ∎

**Remark 2.1.43.** If $\mathbb{H}$ is a closed connected $k$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\text{der}}$ and $\mathbb{S}$ is a maximal $k$-split torus of $\mathbb{G}$, then $(\mathbb{S} \cap \mathbb{H})^\circ$ is a maximal $k$-split torus of $\mathbb{H}$. Indeed, since maximal $k$-split tori are conjugate under $\mathbb{G}(k)$, we can apply the argument from Proposition 2.1.19.

**Corollary 2.1.44.** *Let $\mathbb{H}$ be a closed connected $k$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\text{der}}$. Let $\mathbb{S}$ be a maximal $k$-split torus of $\mathbb{G}$, and let $C_{\mathbb{G}}(\mathbb{S})$ denote its centralizer in $\mathbb{G}$. Then $C_{\mathbb{G}}(\mathbb{S}) \cap \mathbb{H} = C_{\mathbb{H}}((\mathbb{S} \cap \mathbb{H})^\circ)$.*

**Proof:** It follows from the definitions that $C_{\mathbb{G}}(\mathbb{S}) \cap \mathbb{H} \subset C_{\mathbb{H}}((\mathbb{S} \cap \mathbb{H})^\circ)$. Conversely, let $g \in C_{\mathbb{H}}((\mathbb{S} \cap \mathbb{H})^\circ)$. We must show that $g$ commutes with every element of $\mathbb{S}$. By what precedes, $\mathbb{S}$ is the product of $(\mathbb{S} \cap Z(\mathbb{G})^\circ)^\circ$ and $(\mathbb{S} \cap \mathbb{G}_{\text{der}})^\circ$, where $(\mathbb{S} \cap Z(\mathbb{G})^\circ)^\circ \subset Z(\mathbb{G})$ and $(\mathbb{S} \cap \mathbb{G}_{\text{der}})^\circ \subset (\mathbb{S} \cap \mathbb{H})^\circ$. So, given $t \in \mathbb{S}$, $t = zs$ for some $z \in Z(\mathbb{G}), s \in (\mathbb{S} \cap \mathbb{H})^\circ$. Since $g$ commutes with both $z$ and $s$, it commutes with $t$. Hence, $g \in C_{\mathbb{G}}(\mathbb{S}) \cap \mathbb{H}$. ∎

In Chapter 6, we will require the following result regarding $k$-ranks.

**Proposition 2.1.45.** *Let $\mathbb{G}$ be a reductive $k$-group, $\mathbb{T} \subset \mathbb{G}$ a maximal $k$-torus. Let $\mathbb{H}$ be a closed connected $k$-subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$, and let $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$ be a maximal $k$-torus of $\mathbb{H}$. Then $r_k(\mathbb{G}) - r_k(\mathbb{T}) = r_k(\mathbb{H}) - r_k(\mathbb{T}_{\mathbb{H}})$.*

**Proof:** From Proposition 2.1.42, we have that $r_k(\mathbb{G}) = r_k(\mathbb{G}_{\text{der}}) + r_k(Z(\mathbb{G})^\circ)$ and $r_k(\mathbb{H}) = r_k([\mathbb{H}, \mathbb{H}]) + r_k(Z(\mathbb{H})^\circ)$. We recall that $[\mathbb{H}, \mathbb{H}] = \mathbb{G}_{\text{der}}$ (Remark 2.1.29). It follows that

$$r_k(\mathbb{G}) - r_k(\mathbb{H}) = r_k(Z(\mathbb{G})^\circ) - r_k(Z(\mathbb{H})^\circ).$$

It then suffices to show that $r_k(\mathbb{T}) - r_k(\mathbb{T}_{\mathbb{H}}) = r_k(Z(\mathbb{G})^\circ) - r_k(Z(\mathbb{H})^\circ)$.

Using the isogeny $\mathbb{T} = Z(\mathbb{G})^\circ \mathbb{T}_{\text{der}}$ of $\mathbb{T}$ with $\mathbb{T}_{\text{der}} = \mathbb{T} \cap \mathbb{G}_{\text{der}}$ from Corollary 2.1.7, we obtain a result analogous to Proposition 2.1.42. In particular, $r_k(\mathbb{T}) = r_k(Z(\mathbb{G})^\circ) + r_k(\mathbb{T}_{\text{der}})$. Similarly, we have that $r_k(\mathbb{T}_{\mathbb{H}}) = r_k(Z(\mathbb{H})^\circ) + r_k(\mathbb{T}_{\mathbb{H}} \cap [\mathbb{H}, \mathbb{H}])$.

Using again the fact that $\mathbb{G}_{\text{der}} = [\mathbb{H}, \mathbb{H}]$, we see that $\mathbb{T}_{\text{der}} = \mathbb{T}_{\mathbb{H}} \cap [\mathbb{H}, \mathbb{H}]$. Therefore, we conclude that $r_k(\mathbb{T}) - r_k(\mathbb{T}_{\mathbb{H}}) = r_k(Z(\mathbb{G})^\circ) - r_k(Z(\mathbb{H})^\circ)$, which concludes the proof. ∎

## 2.2 $p$-adic Groups

From this point forward, we will mostly be working with $p$-adic groups, that is the $F$-points of reductive groups, where $F$ is a $p$-adic field. By $p$-adic field, we mean any algebraic extension of the $p$-adic numbers $\mathbb{Q}_p$ or the Laurent series $\mathbb{F}_q((x))$ of some finite field $\mathbb{F}_q$ [Cas19, Section 4].

Given a $p$-adic field $F$, we denote by $|\cdot|$ its (normalized) norm, $\text{val}(\cdot)$ its (normalized) valuation which maps into $\mathbb{Z}$, $\mathcal{O}_F$ its ring of integers, and by $\mathfrak{p}_F$ the unique maximal ideal of $\mathcal{O}_F$. The ring of integers $\mathcal{O}_F$ consists of all elements with valuation greater than or equal to zero, whereas $\mathfrak{p}_F$ consists of all elements with strictly positive valuation. We fix a generator of $\mathfrak{p}_F$, called the uniformizer, and denote it by $\varpi_F$. The uniformizer satisfies $\text{val}(\varpi_F) = 1$. Finally, we let $\mathfrak{f} = \mathcal{O}_F / \mathfrak{p}_F$ be the *residue field* of $F$, which is a finite field of prime characteristic $p$ . Some of the algebraic groups we will be encountering will be defined over this residue field $\mathfrak{f}$.

Given an algebraic field extension $E$ of $F$, one can extend the valuation and norm to $E$ . The extended valuation on $E$ is again denoted $\text{val}(\cdot)$. We also have the inclusion $\mathcal{O}_F \subset \mathcal{O}_E$ and $\mathfrak{p}_F \subset \mathfrak{p}_E$. If $e_{E/F}$ denotes the *ramification index* of $E/F$ [Ogg10, Section 7.3], we have $\text{val}(\varpi_E^{e_{E/F}}) = 1$, or equivalently, the valuation over $E$ maps into $\frac{1}{e_{E/F}}\mathbb{Z}$. We say that the extension is *unramified* when $e_{E/F} = 1$ and *tamely ramified* when $e_{E/F}$ is prime to $p$. We do not expand further on the properties of $p$-adic fields in this work. The reader that is not familiar with $p$-adic fields is invited to consult the following references as a starting point [Ogg10, Cas19, Bak20].

**Remark 2.2.1.** The topology on the field $F$ induces a topology on the $p$-adic group $G = \mathbb{G}(F)$. In particular, because every open subgroup of $F$ is closed, every open subgroup of $G$ is also closed. This is not the same as the Zariski topology which is applied on $\mathbb{G}$, but rather a finer topology, meaning that $G$ has more open and closed sets than $\mathbb{G}$.

It is now known that the representation theory of $p$-adic groups relies on some geometric objects, called the apartment and the building, that were described by

Bruhat and Tits [BT72, BT84]. Those objects allow us to keep track of an important family of open and compact subgroups of $G$, known as the Moy-Prasad filtration subgroups [MP94, MP96]. Those subgroups play a vital role in the construction of the supercuspidal representations. The supercuspidal representations will be introduced in Section 2.3.2. In this section, we start by defining the apartment of a $p$-adic group, to then proceed to the Moy-Prasad filtrations subgroups and the Bruhat-Tits building. The descriptions of these objects rely heavily on the structure theory of reductive groups that we have just explored in the previous sections.

### 2.2.1 The Affine Apartment

Let $\mathbb{G}$ be a reductive group which is defined over the $p$-adic field $F$. Let $G = \mathbb{G}(F)$, which is a $p$-adic group. The purpose of this section is to provide the reader with a general intuition of what the apartment is. For that reason, we will only be providing a high level description. The theory of apartments is provided in great detail in [BT72, BT84]. Rabinoff provides a good introduction for the case where $G$ is semisimple and simply connected in [Rab03]. We recommend this last reference as a starting point for the reader that wants to learn more about apartments beyond what is mentioned in this work.

#### 2.2.1.1 Coroots

Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$. In Section 2.1.1, we defined the notion of root system with respect to $\mathbb{T}$ as a particular subset of $X^*(\mathbb{T})$. In a dual manner, we can define a set of coroots associated to $\mathbb{T}$ from the set of its (algebraic) cocharacters.

**Definition 2.2.2.** *A morphism $K^\times \to \mathbb{T}$ is called a cocharacter of $\mathbb{T}$ and the group of cocharacters is denoted by $X_*(\mathbb{T})$.*

The group of characters $X^*(\mathbb{T})$ and the group of cocharacters $X_*(\mathbb{T})$ are dual $\mathbb{Z}$-modules. Indeed, composing a character with a cocharacter yields an element of $X^*(K^\times)$, and $X^*(K^\times) \simeq \mathbb{Z}$ [Hum75, Section 16.1]. This allows us to define a pairing as follows:

$$\begin{aligned} \langle \cdot, \cdot \rangle : X^*(\mathbb{T}) \times X_*(\mathbb{T}) &\to \mathbb{Z} \\ (\chi, \gamma) &\mapsto m, \end{aligned}$$

when $\chi(\gamma(t)) = t^m$ for all $t \in K^\times$.

With the above pairing, one can then define a set of coroots.

**Definition 2.2.3.** *Let $\Phi = \Phi(\mathbb{G}, \mathbb{T})$. For all $\alpha \in \Phi$, we define $\breve{\alpha}$ as the element of $X_*(\mathbb{T})$ which satisfies $\langle \beta, \breve{\alpha} \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ for all $\beta \in \Phi$, where $(\cdot, \cdot)$ denotes the inner product over the vector space $X^*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ which is induced by the Killing form of the Lie algebra $\mathfrak{t}$.*

The cocharacter $\breve{\alpha}$ is called a coroot and is uniquely determined by the above characterization. The set $\breve{\Phi} = \{\breve{\alpha} : \alpha \in \Phi\}$ is called the set of coroots of $\mathbb{T}$. This set is also a root system and lives in the vector space $X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, which is dual to $X^*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$. We refer to $\breve{\Phi}$ as the dual root system associated to $\mathbb{T}$.

**Remark 2.2.4.** If we identify the two dual vector spaces via their pairing, then $\breve{\alpha}$ coincides with $\frac{2\alpha}{(\alpha,\alpha)}$. Hence, one can simply think of the coroot $\breve{\alpha}$ as being a scalar multiple of $\alpha$, making it easy to represent our set of coroots geometrically.

Recall that for all $\alpha \in \Phi$, $\alpha|_{\mathbb{T}_{\mathbb{H}}} \in \Phi(\mathbb{H}, \mathbb{T}_{\mathbb{H}})$ (Lemma 2.1.26). Since the coroots are uniquely determined by the pairing, we have that $\breve{\alpha} = \overbrace{\alpha|_{\mathbb{T}_{\mathbb{H}}}}$, that is $\breve{\alpha} \in X_*(\mathbb{T}_{\mathbb{H}}) \subset X_*(\mathbb{T})$.

**Example 2.2.5.** Let us continue our example of $\mathbb{G} = \mathrm{SL}_3(K)$, and compute the set of coroots $\breve{\Phi}$. Recall that $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ is a root system of type $A_3$ (see Figure 2.1), where $\alpha = \epsilon_1 - \epsilon_2$ and $\beta = \epsilon_2 - \epsilon_3$. Computing inner products, we have that the coroot $\breve{\alpha}$ must satisfy the following relations

$$\langle \alpha, \breve{\alpha} \rangle = 2, \langle \beta, \breve{\alpha} \rangle = -1, \langle \alpha + \beta, \breve{\alpha} \rangle = 1.$$

That is, we are looking for the cocharacter $\breve{\alpha}$ such that $\alpha(\breve{\alpha}(t)) = t^2$, $\beta(\breve{\alpha}(t)) = t^{-1}$, and $(\alpha+\beta)(\breve{\alpha}(t)) = t$ for all $t \in K^\times$. It follows that

$$\breve{\alpha} : t \mapsto \begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, we obtain that

$$\breve{\beta} : t \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{bmatrix}.$$

The other coroots are given by $\overbrace{\alpha+\beta} = \breve{\alpha} + \breve{\beta}, -\breve{\alpha}, -\breve{\beta}, -(\overbrace{\alpha+\beta}) = -\breve{\alpha} - \breve{\beta}$. Note that we only have $\overbrace{\alpha+\beta} = \breve{\alpha} + \breve{\beta}$ since the root system and dual root system are both of the same type $A_3$. When the root system and its dual are of different types (for example $B_n$ with $C_n$), we do not have $\overbrace{\alpha+\beta} = \breve{\alpha} + \breve{\beta}$ for all $\alpha, \beta \in \Phi$. ∎

#### 2.2.1.2 The Case where $\mathbb{G}$ is $F$-split

In order to define the apartment of $G = \mathbb{G}(F)$, we consider the set $X_*(\mathbb{T}, F)$, which consists of the cocharacters of $\mathbb{T}$ that are defined over the field $F$. In the case that $\mathbb{G}$ is split over $F$, and $\mathbb{T}$ is $F$-split, then $X_*(\mathbb{T}, F) = X_*(\mathbb{T})$.

**Definition 2.2.6.** *The affine apartment of $G = \mathbb{G}(F)$ associated to $\mathbb{T}$, denoted by $\mathcal{A}(\mathbb{G}, \mathbb{T}, F)$, is the vector space $X_*(\mathbb{T}, F) \otimes_{\mathbb{Z}} \mathbb{R}$.*

The pairing $\langle \cdot, \cdot \rangle$ extends linearly to $\mathcal{A}(\mathbb{G}, \mathbb{T}, F)$. Given our root system $\Phi = \Phi(\mathbb{G}, \mathbb{T})$, we can define a set of *affine roots*

$$\Phi^{\text{aff}} = \{\alpha + n : \alpha \in \Phi, n \in \mathbb{Z}\}.$$

Every $\varphi = \alpha + n \in \Phi^{\text{aff}}$ is an affine function on $\mathcal{A}(\mathbb{G}, \mathbb{T}, F)$, via $\langle \varphi, x \rangle = \langle \alpha, x \rangle + n$ for all $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F)$

From these affine roots, we can define corresponding affine hyperplanes over the apartment, which produce a tiling of the vector space. For all $\varphi \in \Phi^{\text{aff}}$, the associated hyperplane $H_\varphi$ is defined as

$$H_\varphi = \{x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F) : \langle \varphi, x \rangle = 0\}.$$

For each $\alpha \in \Phi$, the coroot $\check{\alpha}$ is such that $\langle \alpha, \check{\alpha} \rangle = 2$, which means we can locate the hyperplane $H_{\alpha-2}$. From there, we can draw all other hyperplanes and tile our space, as illustrated in Figures 2.2 and 2.3.

In general, we have that $\mathcal{A}(\mathbb{G}, \mathbb{T}, F) = \mathcal{A}(\mathbb{G}_{\text{der}}, \mathbb{T}_{\text{der}}, F) \oplus (X_*(Z(\mathbb{G}), F) \otimes_{\mathbb{Z}} \mathbb{R})$. The apartment of $G_{\text{der}}$ in this orthogonal decomposition is called the *reduced apartment* of $G$, and we write $\mathcal{A}^{\text{red}}(\mathbb{G}, \mathbb{T}, F)$ for $\mathcal{A}(\mathbb{G}_{\text{der}}, \mathbb{T}_{\text{der}}, F)$. The examples we have considered thus far are groups $\mathbb{G}$ for which $\mathbb{G} = \mathbb{G}_{\text{der}}$. Such groups are called *perfect*. As a consequence, the decomposition of the apartment in terms of the reduced apartment described above has not yet been made apparent. Let us illustrate this decomposition on an important example of a group that is not perfect.

**Example 2.2.7.** Consider the Levi subgroup $\mathbb{M}_\alpha(F)$ of $\mathrm{SL}_3(F)$. Since $\mathrm{SL}_3(K)$ is $F$-split, it follows from Definition 2.1.10 that

$$\mathbb{M}_\alpha(F) = \langle \mathbb{T}(F), \mathbb{U}_\alpha(F), \mathbb{U}_{-\alpha}(F) \rangle = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix},$$

where the symbol $*$ indicates that any element of $F$ can be chosen, as long as the resulting matrix lies in $\mathrm{SL}_3(F)$. The apartment of $\mathbb{M}_\alpha(F)$ will be the same as for $\mathrm{SL}_3(F)$, since we are considering the same maximal torus, and therefore the

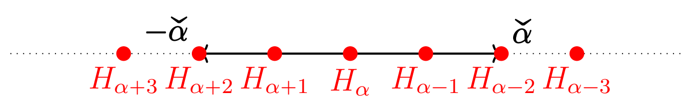



Figure 2.2: Tiling of the apartment $\mathcal{A}(\mathbb{G}, \mathbb{T}, F)$, where $G = \mathrm{SL}_2(F)$ and $\mathbb{T}$ is the diagonal torus.

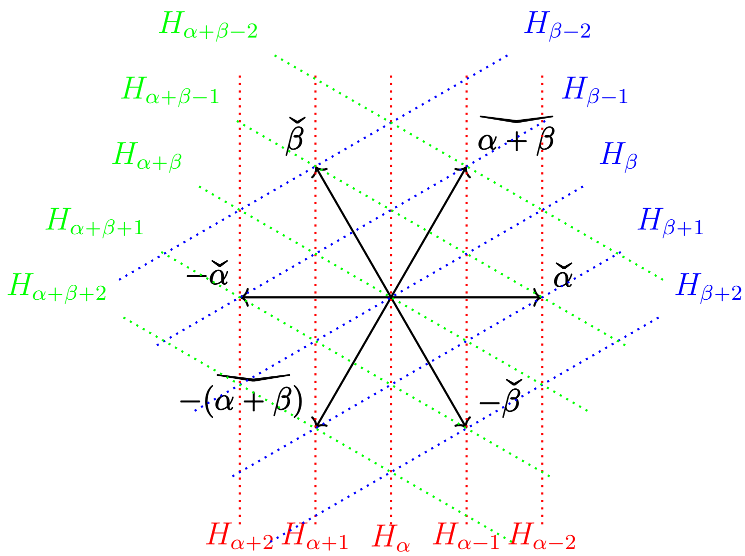



Figure 2.3: Tiling of the apartment $\mathcal{A}(\mathbb{G}, \mathbb{T}, F)$, where $G = \mathrm{SL}_3(F)$ and $\mathbb{T}$ is the diagonal torus.

same group of cocharacters. However, the hyperplane structure will be different. Indeed, we remove the roots $\pm\beta, \pm(\alpha+\beta)$ and their corresponding hyperplanes from Figure 2.3, which gives us the apartment for $\mathbb{M}_\alpha(F) \simeq \mathrm{GL}_2(F)$. Setting $\mathbb{G}(F) = \mathbb{M}_\alpha(F) = \mathrm{GL}_2(F), \mathbb{G}_{\mathrm{der}}(F) = \mathrm{SL}_2(F)$ and $\overline{\mathbb{T}}$ is the diagonal torus of $\mathbb{G}$, the direct sum decomposition $\mathcal{A}(\mathbb{G}, \overline{\mathbb{T}}, F) = \mathcal{A}(\mathbb{G}_{\mathrm{der}}, \overline{\mathbb{T}}_{\mathrm{der}}, F) \oplus X_*(Z(\mathrm{GL}_2(F)))$ is now apparent, as illustrated in Figure 2.4.

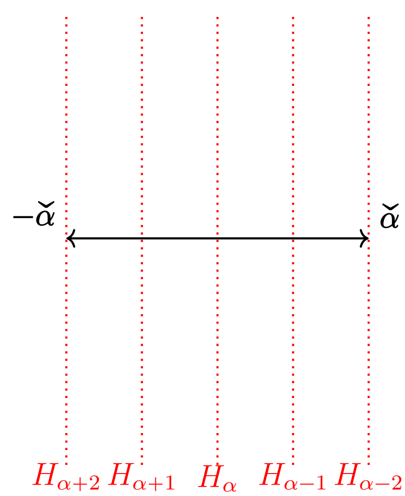



Figure 2.4: Tiling of the apartment $\mathcal{A}(\mathbb{G}, \overline{\mathbb{T}}, F)$, where $G = GL_2(F)$ and $\overline{\mathbb{T}}$ is the diagonal torus.

#### 2.2.1.3 The Case where $\mathbb{G}$ is not $F$-split

In the case that $\mathbb{G}$ is not $F$-split, we define the apartment with respect to a maximal $F$-split torus, say $\mathbb{S}$. The apartment associated to $\mathbb{S}$ is given by $\mathcal{A}(\mathbb{G}, \mathbb{S}, F) = X_*(\mathbb{S}, F) \otimes_{\mathbb{Z}} \mathbb{R}$. The hyperplane structure will be provided by the affine hyperplanes which are associated to the root system $\Phi(\mathbb{G}, \mathbb{S})$.

Assume that $\mathbb{G}$ splits over a finite Galois extension $E$ of $F$ and that $\mathbb{S} \subset \mathbb{T}$, where $\mathbb{T}$ is a maximal torus of $\mathbb{G}$ which is $E$-split. Then $\mathrm{Gal}(E/F)$ acts on the apartment $\mathcal{A}(\mathbb{G}, \mathbb{T}, E)$ as follows. Given $\chi \in X_*(\mathbb{T}, E)$ and $\gamma \in \mathrm{Gal}(E/F)$, $\gamma \cdot \chi$ is the element of $X_*(\mathbb{T}, E)$ defined by $\gamma \cdot \chi(x) = \gamma(\chi(\gamma^{-1}(x)))$ for all $x \in E^\times$. We then extend this action linearly over $X_*(\mathbb{T}, E) \otimes_{\mathbb{Z}} \mathbb{R}$. In the case that $E$ is a tamely ramified extension of $F$, one has $\mathcal{A}(\mathbb{G}, \mathbb{T}, E)^{\mathrm{Gal}(E/F)} = \mathcal{A}(\mathbb{G}, \mathbb{S}, F)$. This is mentioned for example in [HM08, Section 2.5] and is one of the reasons for which the J.K. Yu Construction for supercuspidal representations of $G$ requires that $\mathbb{G}$ splits over a tamely ramified extension $E$ of $F$ (Section 3.1).

### 2.2.2 Moy-Prasad Filtrations

The reason we are interested in the apartment of $G$ is because this geometric object allows us to keep track of important families of open and compact subgroups of $G$. Let us first describe these subgroups in the case that $\mathbb{G}$ is split over $F$.

#### 2.2.2.1 The Case where $\mathbb{G}$ is $F$-split

Let $\mathbb{G}$ be a reductive $F$-group and $\mathbb{T}$ an $F$-split maximal torus, $T = \mathbb{T}(F)$. Set $T_0 = \mathbb{T}(\mathcal{O}_F)$, and for $r > 0$ let $T_r = \mathbb{T}(1 + \mathfrak{p}_F^{\lceil r \rceil})$ and $T_{r^+} = \bigcup_{s > r} T_s$. This provides us with a filtration of $T$. The subgroup $T_0$ coincides with the maximal bounded subgroup of $T$. Using the set of affine roots, we can define a family of subgroups of the root subgroups. For all $\varphi = \alpha + n \in \Phi^{\mathrm{aff}}$, we set $U_\varphi = \mathbb{U}_\alpha(\mathfrak{p}_F^n)$. We refer to these groups as *affine root subgroups.*

**Remark 2.2.8.** Note that the subgroups $\mathbb{T}$ and $\mathbb{U}_\alpha$ are actually *schemes*, so they can be defined over commutative rings and not just fields.

**Definition 2.2.9.** [Rab03, Definition 3.1] *Let $\mathbb{G}$ and $\mathbb{T}$ be as above. For every point $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F)$, we define the associated parahoric subgroup $G_{x,0}$ by*

$$G_{x,0} = \langle T_0, U_\varphi : \varphi \in \Phi^{\mathrm{aff}}, \langle \varphi, x \rangle \geqslant 0 \rangle.$$

*We also define*

$$G_{x,0^+} = \langle T_{0^+}, U_\varphi : \varphi \in \Phi^{\mathrm{aff}}, \langle \varphi, x \rangle > 0 \rangle.$$

*The group $G_{x,0^+}$ is called the pro-unipotent radical of $G_{x,0}$.*

In some cases, it can be handy to have another equivalent definition for the parahoric subgroup $G_{x,0}$ and its pro-unipotent radical, which we provide in the next lemma.

**Lemma 2.2.10.** *Let $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F)$. Then*

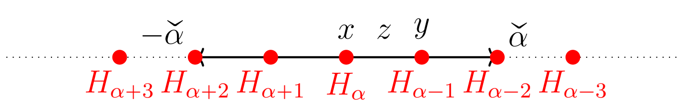
$$G_{x,0} = \langle \mathbb{T}(\mathcal{O}_F), \mathbb{U}_\alpha(\mathfrak{p}_F^{-\lfloor\langle\alpha,x\rangle\rfloor}) : \alpha \in \Phi\rangle, \text{ and}$$
$$G_{x,0^+} = \langle \mathbb{T}(1+\mathfrak{p}_F), \mathbb{U}_\alpha(\mathfrak{p}_F^{1-\lceil\langle\alpha,x\rangle\rceil}) : \alpha \in \Phi\rangle.$$

**Proof:** Let $\alpha \in \Phi$. Given $n \in \mathbb{Z}$, $\varphi = \alpha + n \in \Phi^{\text{aff}}$ and $U_\varphi = \mathbb{U}_\alpha(\mathfrak{p}_F^n)$. Since $\mathfrak{p}_F^m \subset \mathfrak{p}_F^n$ for all $m > n$, we will have $U_{\alpha+m} \subset U_\varphi$ for all $m > n$. Hence, to obtain a more concise description of $G_{x,0}$ and $G_{x,0^+}$ in terms of the root subgroups $\mathbb{U}_\alpha$, it suffices to find the smallest integers $n^*$ and $n^{**}$ which satisfy $\langle \alpha + n*, x\rangle \geqslant 0$ and $\langle \alpha + n^{**}, x\rangle > 0$, respectively.

Requiring $\langle \alpha + n, x\rangle \geqslant 0$ is equivalent to $n \geqslant -\langle\alpha, x\rangle$. The smallest integer $n^*$ to satisfy this inequality is $n^* = -\lfloor\langle\alpha, x\rangle\rfloor$. Similarly, requiring $\langle\alpha + n, x\rangle > 0$ is equivalent to $n > -\langle\alpha, x\rangle$. The smallest integer $n^{**}$ to satisfy this inequality is $n^{**} = 1 - \lceil\langle\alpha, x\rangle\rceil$. Finally, $T_{0^+} = \bigcup_{r>0} T_r = \mathbb{T}(1 + \mathfrak{p}_F)$. Thus, the result follows. ∎

Given $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F)$, the last lemma provides a simpler description of the parahoric subgroup $G_{x,0}$ and its pro-unipotent radical by determining how each root evaluates on $x$ through the pairing $\langle\cdot,\cdot\rangle$. Let us look at an example with $\mathrm{SL}_2(F)$.

**Example 2.2.11.** Let $x, y, z$ be the three points in $\mathcal{A}(\mathbb{G}, \mathbb{T}, F)$ illustrated in the diagram below, where $G = \mathrm{SL}_2(F)$, and $\mathbb{T}$ is the diagonal torus.




For the point $x$, we have that $\langle\alpha, x\rangle = 0 = \langle-\alpha, x\rangle$, so both the floor and ceiling functions of these evaluate to zero. One can then show that

$$G_{x,0} = \langle \mathbb{T}(\mathcal{O}_F), \mathbb{U}_\alpha(\mathcal{O}_F), \mathbb{U}_{-\alpha}(\mathcal{O}_F)\rangle = \begin{bmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathcal{O}_F & \mathcal{O}_F \end{bmatrix},$$
$$G_{x,0^+} = \langle \mathbb{T}(1+\mathfrak{p}_F), \mathbb{U}_\alpha(\mathfrak{p}_F), \mathbb{U}_{-\alpha}(\mathfrak{p}_F)\rangle = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix},$$

as subgroups of $\mathrm{SL}_2(F)$.

For the point $y$, $\langle\alpha, x\rangle = 1$ and $\langle-\alpha, x\rangle = -1$, and therefore,

$$G_{y,0} = \langle \mathbb{T}(\mathcal{O}_F), \mathbb{U}_\alpha(\mathfrak{p}_F^{-1}), \mathbb{U}_{-\alpha}(\mathfrak{p}_F)\rangle = \begin{bmatrix} \mathcal{O}_F & \mathfrak{p}_F^{-1} \\ \mathfrak{p}_F & \mathcal{O}_F \end{bmatrix},$$

$$G_{y,0^+} = \langle \mathbb{T}(1+\mathfrak{p}_F), \mathbb{U}_\alpha(\mathcal{O}_F), \mathbb{U}_{-\alpha}(\mathfrak{p}_F^2)\rangle = \begin{bmatrix} 1+\mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F^2 & 1+\mathfrak{p}_F \end{bmatrix},$$

as subgroups of $\mathrm{SL}_2(F)$.

For the point $z$, $0 < \langle \alpha, z\rangle < 1$, which means that the floor and ceiling function evaluate to 0 and 1, respectively. Similarly, $-1 < \langle -\alpha, z\rangle < 0$. Therefore,

$$G_{z,0} = \langle \mathbb{T}(\mathcal{O}_F), \mathbb{U}_\alpha(\mathcal{O}_F), \mathbb{U}_{-\alpha}(\mathfrak{p}_F)\rangle = \begin{bmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{bmatrix},$$
$$G_{z,0^+} = \langle \mathbb{T}(1+\mathfrak{p}_F), \mathbb{U}_\alpha(\mathcal{O}_F), \mathbb{U}_{-\alpha}(\mathfrak{p}_F)\rangle = \begin{bmatrix} 1+\mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix},$$

as subgroups of $\mathrm{SL}_2(F)$. Note that all points lying between the hyperplanes $H_\alpha$ and $H_{\alpha-1}$ will produce the same parahoric subgroup and pro-unipotent radical. ∎

In general, the hyperplane structure divides the apartment into various connected components, which are referred to as *facets*. The parahoric subgroup and its pro-unipotent radical only depend on the facet containing the point, so it is common to talk about the parahoric subgroup associated to a facet of the apartment rather than the parahoric subgroup associated to a point of the apartment.

Once we have all our parahoric subgroups and pro-unipotent radicals, we can proceed to defining their filtrations [MP94]. It is important to note that unlike the parahoric subgroup, the filtration subgroups depend exactly on the point $x$ and not just on the facet containing $x$.

**Definition 2.2.12.** *For any nonnegative real number $r$ and $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F)$, we define*

$$G_{x,r} = \langle T_r, U_\varphi : \varphi \in \Phi^{\mathrm{aff}}, \langle \varphi, x\rangle \geqslant r\rangle, \text{ and } G_{x,r^+} = \bigcup_{s>r} G_{x,s}.$$

*The subgroups $\{G_{x,r} : r > 0\}$ are called the Moy-Prasad filtration subgroups of $G_{x,0}$.*

**Remark 2.2.13.** Similarly to Lemma 2.2.10, we can express the Moy-Prasad filtration subgroups in terms of how the point in the apartment evaluates on each root. Indeed, for any nonnegative real number $r$ and $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F)$, we have

$$G_{x,r} = \langle \mathbb{T}(1+\mathfrak{p}_F^{\lceil r\rceil}), \mathbb{U}_\alpha(\mathfrak{p}_F^{-\lfloor\langle \alpha, x\rangle - r\rfloor}) : \alpha \in \Phi\rangle$$
$$G_{x,r^+} = \langle \mathbb{T}(1+\mathfrak{p}_F^{\lfloor r\rfloor+1}), \mathbb{U}_\alpha(\mathfrak{p}_F^{1-\lceil\langle \alpha, x\rangle - r\rceil}) : \alpha \in \Phi\rangle.$$

The subgroups

$$\mathbb{U}_\alpha(F)_{x,r} := \langle U_{\alpha+n} : \langle \alpha+n, x\rangle \geqslant r\rangle = \mathbb{U}_\alpha(\mathfrak{p}_F^{-\lfloor\langle \alpha, x\rangle - r\rfloor})$$

and

$$\mathbb{U}_\alpha(F)_{x,r^+} := \langle U_{\alpha+n} : \langle \alpha+n, x\rangle > r\rangle = \mathbb{U}_\alpha(\mathfrak{p}_F^{1-\lceil\langle\alpha,x\rangle-r\rceil})$$

are compact subgroups for $r \geqslant 0$ and form a filtration of $U_\alpha = U_{\alpha+0}$.

From the description of $G_{x,r}$ and $G_{x,r^+}$ provided in the previous remark, one sees that there will be a certain interval $I \subset \mathbb{R}$ for which $G_{x,r} = G_{x,s}$ for all $r, s \in I$. This means that while the Moy-Prasad filtration subgroups are indexed by the real numbers, the values of $r$ at which they change is a discrete subset of $\mathbb{R}$.

**Example 2.2.14.** Recall the points $x, y, z$ in $\mathcal{A}(\mathbb{G}, \mathbb{T}, F)$ from Example 2.2.11, where $G = \mathrm{SL}_2(F)$ and $\mathbb{T}$ is the diagonal torus. By the description above,

$$G_{x,r} = \langle \mathbb{T}(1+\mathfrak{p}_F^{\lceil r\rceil}), \mathbb{U}_\alpha(\mathfrak{p}_F^{-\lfloor -r\rfloor}), \mathbb{U}_\alpha(\mathfrak{p}_F^{-\lfloor -r\rfloor})\rangle.$$

The filtration subgroups of $G_{x,0}$ are given by

$$G_{x,r} = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix}, 0 < r \leqslant 1,$$
$$G_{x,r} = \begin{bmatrix} 1+\mathfrak{p}_F^2 & \mathfrak{p}_F^2 \\ \mathfrak{p}_F^2 & 1+\mathfrak{p}_F^2 \end{bmatrix}, 1 < r \leqslant 2,$$
$$G_{x,r} = \begin{bmatrix} 1+\mathfrak{p}_F^3 & \mathfrak{p}_F^3 \\ \mathfrak{p}_F^3 & 1+\mathfrak{p}_F^3 \end{bmatrix}, 2 < r \leqslant 3, \text{etc.}$$

Our filtration subgroups for $G_{x,0}$ can then be indexed by the integers, $G_{x,0} \supset G_{x,1} \supset G_{x,2} \supset G_{x,3} \supset \cdots$. Similarly, the filtration of $G_{y,0}$ will be indexed by integers.

Assuming that $\langle \alpha, z\rangle = 1/2$, we have

$$G_{z,r} = \langle \mathbb{T}(1+\mathfrak{p}_F^{\lceil r\rceil}), \mathbb{U}_\alpha(\mathfrak{p}_F^{-\lfloor 1/2-r\rfloor}), \mathbb{U}_{-\alpha}(\mathfrak{p}_F^{-\lfloor -1/2-r\rfloor})\rangle.$$

It follows that

$$G_{z,r} = \begin{bmatrix} 1+\mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & 1+\mathfrak{p}_F \end{bmatrix}, 0 < r \leqslant 1/2,$$
$$G_{z,r} = \begin{bmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & 1+\mathfrak{p}_F \end{bmatrix}, 1/2 < r \leqslant 1,$$
$$G_{z,r} = \begin{bmatrix} 1+\mathfrak{p}_F^2 & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & 1+\mathfrak{p}_F^2 \end{bmatrix}, 1 < r \leqslant 3/2, \text{etc.}$$

Our filtration subgroups for $G_{z,0}$ can then be indexed by half integers, $G_{z,0} \supset G_{z,1/2} \supset G_{z,1} \supset G_{z,3/2} \supset \cdots$. Note that if $z$ had not been chosen to be such that $\langle \alpha, z\rangle = 1/2$, the intervals on which the filtration subgroups are the same would not all be of the same length. ∎

We now list some properties of the Moy-Prasad filtration subgroups. The groups $G_{x,r}, G_{x,r^+}$ are open, closed and compact for all $r \geqslant 0$ [Rab03, Proposition 3.4]. We also see from the definitions that $G_{x,s} \subset G_{x,r}$ whenever $s > r$ and that there exists $r' \geqslant r$ such that $G_{x,r'} = G_{x,r^+}$. Furthermore, for all $r, s \geqslant 0, [G_{x,r}, G_{x,s}] \subset G_{x,r+s}$ [MP94, Section 2.6]. As a consequence of this last property, we have the following proposition.

**Proposition 2.2.15.** *1) Let $r \geqslant 0$. For every $s > r$, we have that $G_{x,s}$ is a normal subgroup of $G_{x,r}$. In particular, $G_{x,r^+}$ is a normal subgroup of $G_{x,r}$.*

*2) The quotient $G_{x,r}/G_{x,r^+}$ is abelian for all $r > 0$.*

**Proof:**

1) Since $[G_{x,r}, G_{x,s}] \subset G_{x,r+s}$, $hgh^{-1}g^{-1} \in G_{x,r+s} \subset G_{x,s}$ for all $h \in G_{x,r}, g \in G_{x,s}$. It follows that $hgh^{-1} \in G_{x,s}$ for all $h \in G_{x,r}, g \in G_{x,s}$. Therefore, $G_{x,s}$ is a normal subgroup of $G_{x,r}$.

2) We recall from basic group theory that given a group $J$, if $J'$ is subgroup of $J$ that contains $[J, J]$, then $J/J'$ is abelian. Since $[G_{x,r}, G_{x,r}] \subset G_{x,2r} \subset G_{x,r^+}$ for all $r > 0$ the result follows.

∎

We can define filtrations at the level of the Lie algebra $\mathfrak{g}$ analogously. For $r \geqslant 0$, the subsets $\mathfrak{g}_{x,r}$ and $\mathfrak{g}_{x,r^+}$ are $\mathcal{O}_F$-submodules of $\mathfrak{g}$ [MP94, Section 3.2]. Following the notation from [Yu01] and [HM08], we use colons to abbreviate quotients of filtration subgroups. For example, for $s > r$, $G_{x,r:s} = G_{x,r}/G_{x,s}$ and $\mathfrak{g}_{x,r:s} = \mathfrak{g}_{x,r} / \mathfrak{g}_{x,s}$. When $r > 0$, the quotient $G_{x,r:r^+}$ is isomorphic to the corresponding Lie algebra quotient $\mathfrak{g}_{x,r:r^+}$. We provide the details of this isomorphism in Appendix A.2. The quotient of a parahoric subgroup $G_{x,0}$ with its pro-unipotent radical is even more interesting, as it results in the $\mathfrak{f}$-points of a reductive group. We will denote this reductive group by $\mathcal{G}_x$ and refer to it as the *reductive quotient of* $\mathbb{G}$ *at* $x$. In other words, $\mathcal{G}_x$ is the reductive group such that $\mathcal{G}_x(\mathfrak{f}) = G_{x,0:0^+}$. The following theorem provides a description of this reductive quotient in the case that $\mathbb{G}$ is $F$-split.

**Theorem 2.2.16** ([Rab03, Theorem 3.17])**.** *Suppose $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, F)$. Let $\Phi_x = \{\alpha \in \Phi : \langle \alpha, x\rangle \in \mathbb{Z}\}$. Then $\mathcal{G}_x(\mathfrak{f}) \simeq \langle \mathbb{T}(\mathfrak{f}), \mathbb{U}_\alpha(\mathfrak{f}) : \alpha \in \Phi_x\rangle$.*

**Remark 2.2.17.** The root system $\Phi_x$ can also be described as

$$\{\alpha \in \Phi : \text{ there exists } \alpha + n \in \Phi^{\text{aff}} \text{ such that } \langle \alpha + n, x\rangle = 0\}.$$

**Example 2.2.18.** Let $x, y, z$ be the points in the apartment of $\mathrm{SL}_2(F)$ with respect to the diagonal torus from Example 2.2.11. Then $G_{x,0:0^+} = \mathrm{SL}_2(\mathfrak{f}) = G_{y,0:0^+}$ and $G_{z,0:0^+} = \mathbb{T}(\mathfrak{f})$. ∎

To understand the description of $\mathcal{G}_x(\mathfrak{f})$ provided by Theorem 2.2.16, we sketch part of the proof. Given $\alpha \in \Phi_x$, $s = -\langle \alpha, x \rangle$ is an integer, meaning that $\mathbb{U}_\alpha(\mathfrak{p}_F^s)$ is the affine root subgroup in $G_{x,0}$ and $\mathbb{U}_\alpha(\mathfrak{p}_F^{s+1})$ is the affine root subgroup in $G_{x,0^+}$. The subgroup $\mathbb{U}_\alpha(\mathfrak{f})$ is isomorphic to the projection of $\mathbb{U}_\alpha(\mathfrak{p}_F^s)$ in the quotient $G_{x,0:0^+}$. Indeed, to obtain this isomorphism, we map $\epsilon_\alpha(t)G_{x,0^+}$ to $\epsilon_\alpha(\bar{t})$ for every $t \in \mathfrak{p}_F^s$, where $\bar{t}$ is the image of $t$ in $\mathfrak{p}_F^s / \mathfrak{p}_F^{s+1} \simeq \mathfrak{f}$. For $\beta \notin \Phi_x$, we have $-\lfloor \langle \beta, x \rangle \rfloor = 1 - \lceil \langle \beta, x \rangle \rceil$ and therefore $\mathbb{U}_\beta(\mathfrak{p}_F^{-\lfloor \langle \beta, x \rangle \rfloor}) \subset G_{x,0^+}$, which is why those root subgroups do not appear in the description provided by Theorem 2.2.16. In other words, the root subgroups of $G_{x,0:0^+}$ are simply the projection of the affine root subgroups of $G_{x,0}$ into the quotient.

#### 2.2.2.2 The Case where $\mathbb{G}$ is not $F$-split

In the case that the group $\mathbb{G}$ is not $F$-split, the notions of parahoric subgroup and Moy-Prasad filtrations are still defined, but their descriptions are not as simple as in the split case.

We consider a maximal unramified extension of $F$, which we denote by $F^{\mathrm{un}}$. It is well known that the residue field of $F^{\mathrm{un}}$ is an algebraic closure of $\mathfrak{f}$, so we denote the residue field of $F^{\mathrm{un}}$ by $\bar{\mathfrak{f}}$. Furthermore, $\mathbb{G}$ is quasisplit over $F^{\mathrm{un}}$, meaning that it contains a Borel subgroup which is defined over $F^{\mathrm{un}}$. In [MP94], Moy and Prasad first proceed to describe the parahoric subgroups and filtration subgroups of $\mathbb{G}(F^{\mathrm{un}})$. Fintzen provides a summary of these descriptions in [Fin17, Section 2.4]. We follow her notation here, as the notation from [MP94] is no longer used in most recent works.

Let $\mathbb{S}$ be a maximal $F^{\mathrm{un}}$-split torus. Because $\mathbb{G}$ is quasisplit over $F^{\mathrm{un}}$, we have that $C_{\mathbb{G}}(\mathbb{S}) = \mathbb{T}$ is a maximal $F^{\mathrm{un}}$-torus of $\mathbb{G}$ [Bor91, Proposition 20.6]. Given $x \in \mathcal{A}(\mathbb{G}, \mathbb{S}, F^{\mathrm{un}})$, Bruhat and Tits [BT84, Section 5.2] associated a parahoric group scheme over $\mathcal{O}_{F^{\mathrm{un}}}$, which Fintzen denotes by $\mathbb{P}_x$. This scheme $\mathbb{P}_x$ is such that its generic fiber is isomorphic to $\mathbb{G}(F^{\mathrm{un}})$. The parahoric subgroup of $\mathbb{G}(F^{\mathrm{un}})$ associated to $x$ is then given by $\mathbb{G}(F^{\mathrm{un}})_{x,0} = \mathbb{P}_x(\mathcal{O}_{F^{\mathrm{un}}})$.

To define the Moy-Prasad filtration subgroups of $\mathbb{G}(F^{\mathrm{un}})_{x,0}$, one starts by defining a filtration of the torus $\mathbb{T}(F^{\mathrm{un}})$. Let

$$\mathbb{T}(F^{\mathrm{un}})_b = \{t \in \mathbb{T}(F^{\mathrm{un}}) : \mathrm{val}(\chi(t)) = 0 \text{ for all } \chi \in X^*(\mathbb{T})\}.$$

This subgroup is the maximal bounded subgroup of $\mathbb{T}(F^{\mathrm{un}})$. When $\mathbb{T}$ is $F^{\mathrm{un}}$-split, this is equivalent to the subgroup $\mathbb{T}(F^{\mathrm{un}})_0$ defined earlier. However, when $\mathbb{T}$ is not $F^{\mathrm{un}}$-split, $\mathbb{T}(F^{\mathrm{un}})_b$ and $\mathbb{T}(F^{\mathrm{un}})_0$ are different. Indeed, Bruhat and Tits describe $\mathbb{T}(F^{\mathrm{un}})_0$ as being the connected component of $\mathbb{T}(F^{\mathrm{un}})_b$ [BT84, Section 5.2].

This description of $\mathbb{T}(F^{\mathrm{un}})_0$ is not ideal for working with intersections, and in particular, we will be discussing the intersections of Moy-Prasad filtrations subgroups of $\mathbb{G}(F^{\mathrm{un}})$ with $\mathbb{H}(F^{\mathrm{un}})$ for some closed connected $F^{\mathrm{un}}$-subgroup $\mathbb{H}$ containing $\mathbb{G}_{\mathrm{der}}$ in Section 2.2.2.3. Instead, we will use what is referred to as the Kottwitz map to describe $\mathbb{T}(F^{\mathrm{un}})_0$. The Kottwitz map of $\mathbb{T}(F^{\mathrm{un}})$, which we denote by $\mathrm{Kot}_{\mathbb{T}(F^{\mathrm{un}})}$, is a surjective homomorphism from $\mathbb{T}(F^{\mathrm{un}})$ that maps into a subgroup of $X_*(\mathbb{T})$. As the Kottwitz map is not the main focus of this section, we provide the details in Appendix A.3. With this map, $\mathbb{T}(F^{\mathrm{un}})_0$ then corresponds to $\ker(\mathrm{Kot}_{\mathbb{T}(F^{\mathrm{un}})})$ [PR08, Appendix (Proposition 3)].

Similarly to the previous section, we can also define a filtration of $\mathbb{T}(F^{\mathrm{un}})$. For $r > 0$, we set

$$\mathbb{T}(F^{\mathrm{un}})_r = \{t \in \mathbb{T}(F^{\mathrm{un}})_0 : \mathrm{val}(\chi(t) - 1) \geqslant r \text{ for all } \chi \in X^*(\mathbb{T})\}.$$

When $\mathbb{T}$ is $F^{\mathrm{un}}$-split, this definition coincides with the filtration of the torus that was provided earlier.

Let $\Phi_{F^{\mathrm{un}}} = \Phi(\mathbb{G}, \mathbb{S})$. The set $\Phi_{F^{\mathrm{un}}}$ consists of the restriction to $\mathbb{S}$ of the roots from $\Phi = \Phi(\mathbb{G}, \mathbb{T})$ which are nontrivial on $\mathbb{S}$. This root system may be nonreduced. In [Fin17, Sections 2.3 and 2.4], Fintzen defines a set of affine roots $\Phi^{\mathrm{aff}}_{F^{\mathrm{un}}}$ and associates some affine root subgroup $\mathbb{U}_\varphi(F^{\mathrm{un}})$ to each $\varphi \in \Phi^{\mathrm{aff}}_{F^{\mathrm{un}}}$. The affine roots are of the form $\lambda + k$, where $\lambda \in \Phi_{F^{\mathrm{un}}}$, and $k \in \mathbb{R}$. Note that $k$ need not be an integer as was the case for the definition of $\Phi^{\mathrm{aff}}$ in the previous section. Recall from Example 2.1.40 that

$$\mathbb{U}_\lambda = \langle \mathbb{U}_\alpha : \alpha \in \Phi(\mathbb{G}, \mathbb{T}), \alpha|_{\mathbb{S}} = \lambda \rangle,$$

so $\mathbb{U}_\lambda$ is likely not one-dimensional. Similarly to the previous section, the subgroup $\mathbb{U}_{\lambda+k}(F^{\mathrm{un}})$ is a compact subgroup of $\mathbb{U}_\lambda(F^{\mathrm{un}})$.

For $r \geqslant 0$, the Moy-Prasad filtration subgroups are

$$\mathbb{G}(F^{\mathrm{un}})_{x,r} = \langle \mathbb{T}(F^{\mathrm{un}})_r, \mathbb{U}_\varphi(F^{\mathrm{un}}) : \varphi \in \Phi^{\mathrm{aff}}_{F^{\mathrm{un}}}, \langle \varphi, x \rangle \geqslant r \rangle, \text{ and } \mathbb{G}(F^{\mathrm{un}})_{x,r^+} = \bigcup_{s>r} \mathbb{G}(F^{\mathrm{un}})_{x,s}.$$

For our purposes, it is not necessary to know the precise description of the subgroups $\mathbb{U}_\varphi(F^{\mathrm{un}})$ in as much detail as is provided in [Fin17]. It suffices for us to know that they are subgroups of $\mathbb{G}_{\mathrm{der}}$, and that every $\mathbb{U}_\varphi(F^{\mathrm{un}}) \subset \mathbb{G}(F^{\mathrm{un}})_{x,r}$ is normalized by $\mathbb{T}(F^{\mathrm{un}})_r$ [MP94, Section 2.4].

Recall from Section 2.2.1.3 that taking the $\mathrm{Gal}(F^{\mathrm{un}}/F)$-fixed points of $\mathcal{A}(\mathbb{G}, \mathbb{S}, F^{\mathrm{un}})$ gives us an apartment of $G$. So, given a point $x \in \mathcal{A}(\mathbb{G}, \mathbb{S}, F^{\mathrm{un}})$ that is fixed under $\mathrm{Gal}(F^{\mathrm{un}}/F)$, $\mathbb{G}(F^{\mathrm{un}})_{x,r}$ is defined over $F$ and $G_{x,r} = \mathbb{G}(F^{\mathrm{un}})_{x,r} \cap G$, $r \geqslant 0$ [MP94, Sections 2.1 and 3.8]. Similarly, we can describe the subgroups $G_{x,r^+}$, $r \geqslant 0$. This allows us to define the parahoric subgroups and Moy-Prasad filtration subgroups of $G$. The properties listed in Section 2.2.2.1 all hold in this more general setting. We re-enumerate these properties below for the reader's convenience.

1) For all $r, s \geqslant 0$, $[G_{x,r}, G_{x,s}] \subset G_{x,r+s}$.

2) Let $r, s \geqslant 0$. $G_{x,s}$ is a normal subgroup of $G_{x,r}$ whenever $s > r$.

3) For all $r \geqslant 0$, there exists $r' \geqslant r$ such that $G_{x,r'} = G_{x,r^+}$

4) The quotient $G_{x,r:r^+}$ is abelian for all $r > 0$.

When $\mathbb{G}$ splits over a tamely ramified extension of $F$, we have an isomorphism between $G_{x,r:r^+}$ and $\mathfrak{g}_{x,r:r^+}$ for $r > 0$ (see Appendix A.2).

Furthermore, the quotient $G_{x,0:0^+}$ is the $\mathfrak{f}$-points of a reductive group that we denote by $\mathcal{G}_x$, that is $G_{x,0:0^+} = \mathcal{G}_x(\mathfrak{f})$ and $\mathcal{G}_x(\bar{\mathfrak{f}}) = \mathbb{G}(F^{\mathrm{un}})_{x,0:0^+}$. A description of this group in terms of the parahoric scheme $\mathbb{P}_x$ is provided in [Fin17, Section 2.4]. It is known that the Galois group $\mathrm{Gal}(F^{\mathrm{un}}/F)$ has a natural identification with the Galois group $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$. As a result, $G_{x,0:0^+}$ is the subgroup of $\mathbb{G}(F^{\mathrm{un}})_{x,0:0^+}$ which is fixed by $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$. Since $\bar{\mathfrak{f}}$ is an algebraic closure of $\mathfrak{f}$, $\mathcal{G}_x$ has to be split over $\bar{\mathfrak{f}}$. We can then identify the algebraic reductive group $\mathcal{G}_x$ with $\mathcal{G}_x(\bar{\mathfrak{f}})$.

Similarly to Theorem 2.2.16, the root system $\Phi_x = \Phi(\mathcal{G}_x)$ consists of only certain roots of $\Phi = \Phi(\mathbb{G}, \mathbb{T})$. Fintzen describes this set $\Phi_x$ in [Fin17, Lemma 2.4.1] as

$$\{\alpha \in \Phi : \text{ there exists } \alpha|_{\mathbb{S}} + k \in \Phi^{\mathrm{aff}}_{F^{\mathrm{un}}} \text{ such that } \langle \alpha|_{\mathbb{S}} + k, x\rangle = 0\}.$$

The root subgroups of $\mathcal{G}_x$ are obtained by projecting the affine root subgroups of $\mathbb{G}(F^{\mathrm{un}})_{x,0}$ into the quotient $\mathbb{G}(F^{\mathrm{un}})_{x,0:0^+}$ [Fin17, Lemma 2.5.1].

#### 2.2.2.3 Going from $G$ to $H$

We saw with Proposition 2.1.30 that intersecting an $F$-parabolic subgroup or an $F$-Levi subgroup of $\mathbb{G}$ with a closed connected $F$-subgroup $\mathbb{H}$ containing $\mathbb{G}_{\mathrm{der}}$ yields an $F$-parabolic subgroup or an $F$-Levi subgroup of $\mathbb{H}$, respectively. The same can be asked about the parahoric subgroups and Moy-Prasad filtration subgroups.

Let $\mathbb{S}$ be a maximal $F$-split torus of $\mathbb{G}$. Given that $\mathcal{A}(\mathbb{G}, \mathbb{S}, F) = \mathcal{A}^{\mathrm{red}}(\mathbb{G}, \mathbb{S}, F) \oplus (X_*(Z(\mathbb{G}), F) \otimes_{\mathbb{Z}} \mathbb{R})$, every $x \in \mathcal{A}(\mathbb{G}, \mathbb{S}, F)$ is of the form $\overline{x} + z$ for some unique $\overline{x} \in \mathcal{A}^{\mathrm{red}}(\mathbb{G}, \mathbb{S}, F), z \in X_*(Z(\mathbb{G}), F) \otimes_{\mathbb{Z}} \mathbb{R}$. We let $[x] = \{\overline{x} + z : z \in X_*(Z(\mathbb{G}), F) \otimes_{\mathbb{Z}} \mathbb{R}\}$. Note that $[x]$ is considered a point of $\mathcal{A}^{\mathrm{red}}(\mathbb{G}, \mathbb{S}, F)$ when $\mathcal{A}^{\mathrm{red}}(\mathbb{G}, \mathbb{S}, F)$ is viewed as a quotient of $\mathcal{A}(\mathbb{G}, \mathbb{S}, F)$.

**Lemma 2.2.19.** *Let $x$ and $\overline{x}$ be as above. Then, $G_{x,r} = G_{\overline{x},r}$ for all $r \geqslant 0$.*

**Proof:** We have that $x = \overline{x} + z$ for some $z \in X_*(Z(\mathbb{G}), F) \otimes_{\mathbb{Z}} \mathbb{R}$. Given $\alpha \in \Phi(\mathbb{G}, \mathbb{S})$, we have $Z(\mathbb{G}) \subset \ker \alpha$ (Proposition 2.1.6), implying in particular that $\alpha(z(t)) = 1 = t^0$ for all $t \in F^\times$. It follows that $\langle \alpha, z\rangle = 0$, and $\langle \alpha, \overline{x} + z\rangle = \langle \alpha, \overline{x}\rangle$. Given that the Moy-Prasad filtration subgroups at $x$ depend on how the roots evaluate on $x$, we conclude that $G_{x,r} = G_{\overline{x},r}$ for all $r \geqslant 0$. ∎

**Remark 2.2.20.** Given a closed connected $F$-subgroup $\mathbb{H}$ that contains $\mathbb{G}_{\mathrm{der}}$, we have that $\mathcal{A}^{\mathrm{red}}(\mathbb{G},\mathbb{S},F) = \mathcal{A}^{\mathrm{red}}(\mathbb{H},\mathbb{S}\cap\mathbb{H},F)$ as $\mathbb{G}_{\mathrm{der}} = [\mathbb{H},\mathbb{H}]$ (Remark 2.1.29). Since

$$\mathcal{A}(\mathbb{G},\mathbb{S},F) = \mathcal{A}^{\mathrm{red}}(\mathbb{G},\mathbb{S},F) \oplus (X_*(Z(\mathbb{G}),F)\otimes_{\mathbb{Z}}\mathbb{R}) \text{ and}$$

$$\mathcal{A}(\mathbb{H},\mathbb{S}\cap\mathbb{H},F) = \mathcal{A}^{\mathrm{red}}(\mathbb{H},\mathbb{S}\cap\mathbb{H},F) \oplus (X_*(Z(\mathbb{H}),F)\otimes_{\mathbb{Z}}\mathbb{R}),$$

we have a natural embedding

$$\mathcal{A}^{\mathrm{red}}(\mathbb{G},\mathbb{S},F) \hookrightarrow \mathcal{A}(\mathbb{H},\mathbb{S}\cap\mathbb{H},F) \hookrightarrow \mathcal{A}(\mathbb{G},\mathbb{S},F)$$

via the inclusion $X_*(Z(\mathbb{H}),F) \subset X_*(Z(\mathbb{G}),F)$. So, given a point $x \in \mathcal{A}(\mathbb{G},\mathbb{S},F)$, one can write things like $H_{x,r}$ and $H_{x,r^+}$, $r \geqslant 0$, without loss of generality from the previous lemma. That is, we can think of $x$ as being the corresponding point $\overline{x}$ which is in $\mathcal{A}^{\mathrm{red}}(\mathbb{G},\mathbb{S},F)$, as illustrated in Figure 2.5.

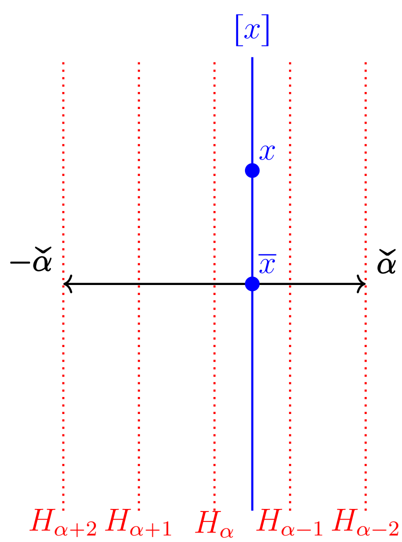



Figure 2.5: Visual representation of $[x]$ and $\overline{x}$ given $x$ in $\mathcal{A}(\mathbb{G},\mathbb{T},F)$, where $G = \mathrm{GL}_2(F)$ and $\mathbb{T}$ is the diagonal torus. It is clear from this picture that $G_{x,r} = G_{\bar{x},r}$ for all $r \geqslant 0$. Note that this visualization relies on a choice of location for the origin in the affine space.

Let $\mathbb{H}$ denote a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$. In the case where $\mathbb{G}$ is $F$-split, it is easy to see that, for all $r \geqslant 0$, $G_{x,r} \cap H = H_{x,r}$ and $G_{x,r^+} \cap H = H_{x,r^+}$. Indeed, this is because the affine root subgroups are normalized by the toral part of the filtration subgroup, so the argument from Proposition 2.1.30 is applied. In the case that $\mathbb{G}$ is not $F$-split, a bit more work is needed.

Let $\mathbb{S}$ be a maximal $F^{\mathrm{un}}$-split torus and $\mathbb{T} = C_{\mathbb{G}}(\mathbb{S})$ be a maximal $F^{\mathrm{un}}$-torus as in Section 2.2.2.2. We have that $(\mathbb{S}\cap\mathbb{H})^\circ$ is a maximal $F^{\mathrm{un}}$-split torus of $\mathbb{H}$ (Remark 2.1.43) and $C_{\mathbb{H}}((\mathbb{S}\cap\mathbb{H})^\circ) = C_{\mathbb{G}}(\mathbb{S}) \cap \mathbb{H} = \mathbb{T}\cap\mathbb{H} = \mathbb{T}_{\mathbb{H}}$ Corollary 2.1.44.

**Lemma 2.2.21.** *Let $\mathbb{T}$ and $\mathbb{H}$ be as above and let $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$. For all $r \geqslant 0$, we have $\mathbb{T}_{\mathbb{H}}(F^{\mathrm{un}})_r = \mathbb{T}(F^{\mathrm{un}})_r \cap \mathbb{H}(F^{\mathrm{un}})$.*

**Proof:** First, we have that $\mathbb{T}(F^{\mathrm{un}})_0 \cap \mathbb{H}(F^{\mathrm{un}}) = \mathbb{T}_{\mathbb{H}}(F^{\mathrm{un}})_0$. Indeed, recalling that $\mathbb{T}(F^{\mathrm{un}})_0 = \ker(\mathrm{Kot}_{\mathbb{T}(F^{\mathrm{un}})})$, one can show that $\ker(\mathrm{Kot}_{\mathbb{T}(F^{\mathrm{un}})}) \cap \mathbb{T}_{\mathbb{H}}(F^{\mathrm{un}}) = \ker(\mathrm{Kot}_{\mathbb{T}_{\mathbb{H}}(F^{\mathrm{un}})})$ (Proposition A.3.1).

For $r > 0$, we have that

$$\begin{aligned}\mathbb{T}(F^{\mathrm{un}})_r \cap \mathbb{H}(F^{\mathrm{un}}) &= \{t \in \mathbb{T}(F^{\mathrm{un}})_0 : \mathrm{val}(\chi(t) - 1) \geqslant r \text{ for all } \chi \in X^*(\mathbb{T})\} \cap \mathbb{H}(F^{\mathrm{un}}) \\ &= \{t \in \mathbb{T}_{\mathbb{H}}(F^{\mathrm{un}})_0 : \mathrm{val}(\chi|_{\mathbb{T}_{\mathbb{H}}}(t) - 1) \geqslant \chi \text{ for all } \chi \in X^*(\mathbb{T})\}\end{aligned}$$

by what precedes. Using the fact that restricting characters of $\mathbb{T}$ to $\mathbb{T}_{\mathbb{H}}$ produces characters of $\mathbb{T}_{\mathbb{H}}$ and that all characters of $\mathbb{T}_{\mathbb{H}}$ arise this way [Bor91, Proposition 8.2], we conclude that

$$\begin{aligned}\mathbb{T}(F^{\mathrm{un}})_r \cap \mathbb{H}(F^{\mathrm{un}}) &= \{t \in \mathbb{T}_{\mathbb{H}}(F^{\mathrm{un}})_0 : \mathrm{val}(\chi(t) - 1) \geqslant r \text{ for all } \chi \in X^*(\mathbb{T}_{\mathbb{H}})\} \\ &= \mathbb{T}_{\mathbb{H}}(F^{\mathrm{un}})_r.\end{aligned}$$

∎

**Proposition 2.2.22.** *Let $\mathbb{H}$ be as above. For all $r \geqslant 0$, we have $H_{x,r} = G_{x,r} \cap H$ and $H_{x,r^+} = G_{x,r^+} \cap H$.*

**Proof:** Let $r \geqslant 0$. Since $\mathbb{G}$ and $\mathbb{H}$ have the same root subgroups (Corollary 2.1.28), we have

$$\mathbb{H}(F^{\mathrm{un}})_{x,0} = \langle \mathbb{T}_{\mathbb{H}}(F^{\mathrm{un}})_r, \mathbb{U}_{\varphi}(F^{\mathrm{un}}) : \varphi \in \Phi^{\mathrm{aff}}_{F^{\mathrm{un}}}, \langle \varphi, x\rangle \geqslant r\rangle.$$

Since the affine root subgroups are in $\mathbb{G}_{\mathrm{der}}(F^{\mathrm{un}}) \subset \mathbb{H}(F^{\mathrm{un}})$ and $\mathbb{T}(F^{\mathrm{un}})_r$ normalizes the affine root subgroups, the argument from Proposition 2.1.30 implies

$$\mathbb{G}(F^{\mathrm{un}})_{x,r} \cap \mathbb{H}(F^{\mathrm{un}}) = \langle \mathbb{T}(F^{\mathrm{un}})_r \cap \mathbb{H}(F^{\mathrm{un}}), \mathbb{U}_{\varphi}(F^{\mathrm{un}}) : \varphi \in \Phi^{\mathrm{aff}}_{F^{\mathrm{un}}}, \langle \varphi, x\rangle \geqslant r\rangle.$$

Since $\mathbb{T}(F^{\mathrm{un}})_r \cap \mathbb{H}(F^{\mathrm{un}}) = \mathbb{T}_{\mathbb{H}}(F^{\mathrm{un}})_r$ (Lemma 2.2.21), we conclude that $\mathbb{G}(F^{\mathrm{un}})_{x,r} \cap \mathbb{H}(F^{\mathrm{un}}) = \mathbb{H}(F^{\mathrm{un}})_{x,r}$. Therefore, $G_{x,r} \cap H = H_{x,r}$. Since $G_{x,r^+} = G_{x,r'}$ for some $r' \geqslant r$, the other equality follows. ∎

**Remark 2.2.23.** When $\mathbb{G}$ splits over a tamely ramified extension $E$ of $F$ and $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, E)^{\mathrm{Gal}(E/F)}$, where $\mathbb{T}$ is an $E$-split maximal torus of $\mathbb{G}$, then Hakim and Murnaghan point out that $G_{x,r} = \mathbb{G}(E)^{\mathrm{Gal}(E/F)}_{x,r} = \mathbb{G}(E)_{x,r} \cap G$ for $r > 0$ [HM08, Section 2.5]. Hence, this provides us with a quick alternative argument that shows $G_{x,r} \cap H = H_{x,r}$ for $r > 0$. However, this argument cannot be applied for $r = 0$ as they also mention that the parahoric subgroup $G_{x,0}$ may not be equal to $\mathbb{G}(E)_{x,0} \cap G$.

Given a closed connected $F$-subgroup $\mathbb{H}$ that contains $\mathbb{G}_{\text{der}}$, we wish to discuss the relationships between the reductive quotients $G_{x,0:0^+}$ and $H_{x,0:0^+}$, as this will prove to be useful in future sections. Using similar notation as in the previous section, $G_{x,0:0^+}$ and $H_{x,0:0^+}$ are the $\mathfrak{f}$-points of reductive groups which we denote by $\mathcal{G}_x$ and $\mathcal{H}_x$, respectively. While $\mathcal{H}_x$ is not technically a subgroup of $\mathcal{G}_x$, it can be identified with one via an isomorphism, as per the following lemma.

**Lemma 2.2.24.** *The quotient $\mathbb{H}(F^{\text{un}})_{x,0}\mathbb{G}(F^{\text{un}})_{x,0^+}/\mathbb{G}(F^{\text{un}})_{x,0^+}$ can be identified with the $\bar{\mathfrak{f}}$-points of a reductive group $\binom{\mathcal{H}}{\mathcal{G}}_x \subset \mathcal{G}_x$. Furthermore, there is an isomorphism $\iota$ between the algebraic groups $\mathcal{H}_x$ and $\binom{\mathcal{H}}{\mathcal{G}}_x$ which is defined over $\mathfrak{f}$.*

**Proof:** By the second isomorphism theorem, we have

$$\begin{aligned}\mathbb{H}(F^{\text{un}})_{x,0}/\mathbb{H}(F^{\text{un}})_{x,0^+} &= \mathbb{H}(F^{\text{un}})_{x,0}/\left(\mathbb{H}(F^{\text{un}})_{x,0} \cap \mathbb{G}(F^{\text{un}})_{x,0^+}\right)\\ &\simeq \mathbb{H}(F^{\text{un}})_{x,0}\mathbb{G}(F^{\text{un}})_{x,0^+}/\mathbb{G}(F^{\text{un}})_{x,0^+}.\end{aligned}$$

Indeed, this isomorphism is given by

$$\begin{aligned}\iota : \mathbb{H}(F^{\text{un}})_{x,0:0^+} &\to \mathbb{H}(F^{\text{un}})_{x,0}\mathbb{G}(F^{\text{un}})_{x,0^+}/\mathbb{G}(F^{\text{un}})_{x,0^+}\\ h\mathbb{H}(F^{\text{un}})_{x,0^+} &\mapsto h\mathbb{G}(F^{\text{un}})_{x,0^+}.\end{aligned}$$

Since $\mathbb{H}(F^{\text{un}})_{x,0:0^+}$ is a reductive group, we conclude that $\mathbb{H}(F^{\text{un}})_{x,0}\mathbb{G}(F^{\text{un}})_{x,0^+}/\mathbb{G}(F^{\text{un}})_{x,0^+} = \binom{\mathcal{H}}{\mathcal{G}}_x(\bar{\mathfrak{f}})$ for some reductive group $\binom{\mathcal{H}}{\mathcal{G}}_x$. It is clear from the definitions that $\binom{\mathcal{H}}{\mathcal{G}}_x(\bar{\mathfrak{f}}) \subset \mathcal{G}_x(\bar{\mathfrak{f}})$ and therefore $\binom{\mathcal{H}}{\mathcal{G}}_x \subset \mathcal{G}_x$. Note that the isomorphism $\iota$ restricts to an isomorphism between $\mathcal{H}_x(\mathfrak{f})$ and $\binom{\mathcal{H}}{\mathcal{G}}_x(\mathfrak{f})$ upon replacing $F^{\text{un}}$ by $F$, that is $\iota$ is defined over $\mathfrak{f}$. ∎

**Remark 2.2.25.** We can also think of $H_{x,r:r^+}$ as a subgroup of $G_{x,r:r^+}$ for all $r > 0$ by establishing similar isomorphisms between $H_{x,r:r^+}$ and $H_{x,r}G_{x,r^+}/G_{x,r^+}$. However, these quotients do not correspond to reductive groups.

**Lemma 2.2.26.** *The subgroup $\binom{\mathcal{H}}{\mathcal{G}}_x$ is a closed connected $\mathfrak{f}$-subgroup of $\mathcal{G}_x$ that contains $[\mathcal{G}_x, \mathcal{G}_x]$.*

**Proof:** Since $\binom{\mathcal{H}}{\mathcal{G}}_x$ is a reductive group, it is closed and connected. We identify the algebraic groups with their $\bar{\mathfrak{f}}$-points. We have that

$$[\mathcal{G}_x(\bar{\mathfrak{f}}), \mathcal{G}_x(\bar{\mathfrak{f}})] = [\mathbb{G}(F^{\text{un}})_{x,0}, \mathbb{G}(F^{\text{un}})_{x,0}]\mathbb{G}(F^{\text{un}})_{x,0^+}/\mathbb{G}(F^{\text{un}})_{x,0^+}.$$

The subgroup $[\mathbb{G}(F^{\text{un}})_{x,0}, \mathbb{G}(F^{\text{un}})_{x,0}]$ is contained in both $\mathbb{G}(F^{\text{un}})_{x,0}$ and $\mathbb{G}_{\text{der}}(F^{\text{un}}) \subset \mathbb{H}(F^{\text{un}})$. Therefore, $[\mathbb{G}(F^{\text{un}})_{x,0}, \mathbb{G}(F^{\text{un}})_{x,0}] \subset \mathbb{H}(F^{\text{un}})_{x,0}$ (Proposition 2.2.22), and $[\mathcal{G}_x(\bar{\mathfrak{f}}), \mathcal{G}_x(\bar{\mathfrak{f}})] \subset \binom{\mathcal{H}}{\mathcal{G}}_x(\bar{\mathfrak{f}})$. ∎

### 2.2.3 The Bruhat-Tits Building

When constructing an apartment for our $p$-adic group $G = \mathbb{G}(F)$, we had to choose a maximal $F$-split torus of $\mathbb{G}$. There are many choices for such tori. Indeed, the maximal $F$-split tori of $\mathbb{G}$ are all $G$-conjugate. Given another maximal $F$-split torus, one could describe the apartment of $G$, the parahoric subgroups and the Moy-Prasad filtration subgroups with respect to this different torus. In order to keep track of all the apartments and families of open compact subgroups of $G$ at once, we use the Bruhat-Tits building [BT72, BT84]. In very broad terms, the Bruhat-Tits building is a geometric structure that glues together all the apartments of $G$ in a consistent and well-defined manner. As such, a point from the building automatically belongs to some affine apartment. Since this structure does not depend on a choice of torus, we denote it by $\mathcal{B}(\mathbb{G}, F)$. Essentially, $\mathcal{B}(\mathbb{G}, F)$ is the Cartesian product $G \times \mathcal{A}(\mathbb{G}, \mathbb{T}, F)$ modulo some equivalence relation, where $\mathbb{T}$ is a choice of maximal $F$-split torus. Then the left action of $G$ on the first factor induces an action of $G$ on $\mathcal{B}(\mathbb{G}, F)$. Given a point $x$ in $\mathcal{B}(\mathbb{G}, F)$, one can then define the stabilizer of the point $x$ with respect to this $G$-action, which is denoted by $G_x$. When $\mathbb{G}$ is semisimple and simply connected, the stabilizer of $x$ coincides with the parahoric subgroup of $G$ at $x$, that is $G_x = G_{x,0}$ [Rab03, Lemma 4.5]. For a more general reductive group $\mathbb{G}$, we have the inclusion $G_{x,0} \subset G_x$.

We list a few facts about the apartments and the Bruhat-Tits building, some of which can be found in [Rab03].

1) For all $g \in G$, the apartment associated to ${}^g\mathbb{T}$ is $g \cdot \mathcal{A}(\mathbb{G}, \mathbb{T}, F)$.

2) For any two elements of $\mathcal{B}(\mathbb{G}, F)$, there is an apartment containing both of them.

3) For every $g \in G, r \geqslant 0$ and $x \in \mathcal{B}(\mathbb{G}, F)$, we have ${}^gG_{x,r} = G_{g\cdot x,r}$ and ${}^gG_{x,r^+} = G_{g\cdot x,r^+}$.

4) We have that $\mathcal{B}(\mathbb{G}, F) = \mathcal{B}(\mathbb{G}_{\mathrm{der}}, F) \times (X_*(Z(\mathbb{G}), F) \otimes_{\mathbb{Z}} \mathbb{R})$ [BT84, Section 4.2.16].

5) If $F \subset E$ is an extension of $F$ which is tamely ramified, we have $\mathcal{B}(\mathbb{G}, F) = \mathcal{B}(\mathbb{G}, E)^{\mathrm{Gal}(E/F)}$ [Pra01].

**Remark 2.2.27.** Given property (4) listed above, we call $\mathcal{B}(\mathbb{G}, F)$ the *enlarged* building of $G$, whereas $\mathcal{B}(\mathbb{G}_{\mathrm{der}}, F)$, which can also be denoted $\mathcal{B}^{\mathrm{red}}(\mathbb{G}, F)$, is called the *reduced* building of $G$.

Because of property (5), when $\mathbb{G}$ splits over a tamely ramified extension $E$ of $F$, one can carry all calculations relative to the group's structure theory at the level of $E$ and use the Galois fixed-points to reduce back down to $F$. This strategy is used a lot by Yu in his construction for supercuspidal representations, which is why we must require our $p$-adic group to split over a tamely ramified extension.

Given a group $G$ for which each apartment is one-dimensional, one can visualize the Bruhat-Tits building. For example, when $G = \mathrm{SL}_2(F)$, and $F = \mathbb{Q}_2$, the building is illustrated in Figure 2.6. Each path in the graph corresponds to an apartment. Each vertex in the figure represents a hyperplane for an apartment, and the edges are the maximal facets in the apartments. To see how many edges are connected to a vertex, we calculate quotients of stabilizers. For instance, if we recall the points $x$ and $z$ in the apartment of $\mathrm{SL}_2(F)$ from Example 2.2.11, we can calculate the number of edges coming out from vertex $x$ by computing the size of $G_x/G_z$. Indeed, this quotient represents the elements of $G$ that fix the point $x$ but do not fix the facet containing $z$. By counting the number of representatives of this quotient, we are counting the number of edges attached to $x$ in the building. Since $\mathrm{SL}_2(\mathrm{F})$ is semisimple and simply connected, $G_x = G_{x,0}$ and $G_z = G_{z,0}$. Using [Rab03, Theorem 3.17 and Proposition 3.22], we have $G_{x,0}/G_{x,0^+} \simeq \mathrm{SL}_2(\mathfrak{f})$ and $G_{z,0}/G_{x,0^+} \simeq \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a, b \in \mathfrak{f} \right\}$. To calculate $|G_{x,0}/G_{z,0}|$ one can then calculate $|G_{x,0}/G_{x,0^+}|$ and $|G_{z,0}/G_{x,0^+}|$ as $G_{x,0}/G_{z,0}$ is isomorphic to the quotient of $G_{x,0}/G_{x,0^+}$ by $G_{z,0}/G_{x,0^+}$. One works out that $|G_{x,0}/G_{x,0^+}| = q(q^2-1)$ and $|G_{z,0}/G_{x,0^+}| = q(q-1)$, where $q = |\mathfrak{f}|$. Therefore, $|G_{x,0}/G_{z,0}| = q+1$. In particular, when $F = \mathbb{Q}_2$, $\mathfrak{f} = \mathbb{Z}/2\mathbb{Z}$ and $q = 2$, we obtain that 3 edges must be connected to each vertex.

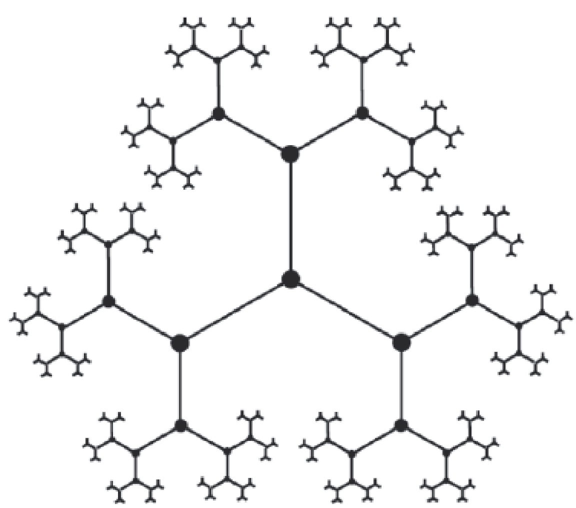


Figure 2.6: Image of the Bruhat-Tits building for $\mathrm{SL}_2(\mathbb{Q}_2)$ by 부 고. The illustration is licensed under a Creative Commons Attribution-Share Alike 4.0 International license and is available at https://commons.wikimedia.org/wiki/File:Bruhat-Tits-tree-for-Q-2.png.

For a higher dimensional apartment, it becomes extremely difficult if not impossible to visualize the complete Bruhat-Tits building. The pictures in Figure 2.7

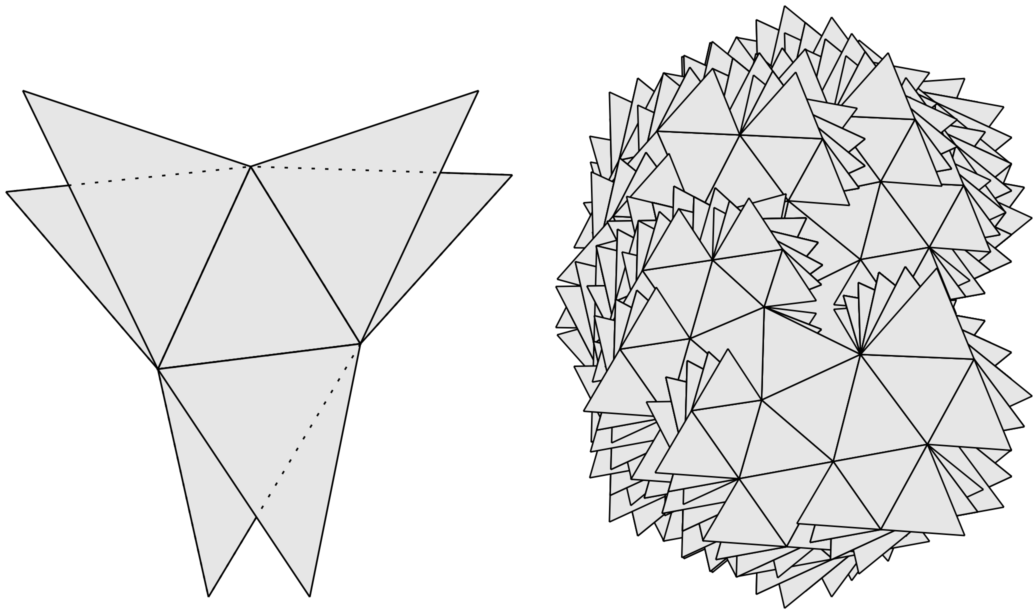


Figure 2.7: "A small fragment of a building" (left) and "A bigger fragment of a building" (right) by Paul Garrett. Both illustrations are licensed under a Creative Commons Attribution 3.0 Unported License and are available at http://www-users.math.umn.edu/∼garrett/pix/. The descriptions of the original illustrations have been cropped out.

show fragments of buildings for which the apartments are two-dimensional, as is the case with $\mathrm{SL}_3(F)$ for example.

Since the structure theory of the Bruhat-Tits building is not the main focus of this thesis, we do not go into more details in this work. For the eager reader, Rabinoff provides a good introduction to the Bruhat-Tits building for semisimple simply connected reductive groups in [Rab03].

## 2.3 Representation Theory of $p$-adic Groups

In this section, we introduce some of the important elements of the representation theory for $p$-adic groups. In particular, we will define what the supercuspidal representations are and explain why they are the building blocks for all representations of $p$-adic groups. Furthermore, we will see that the supercuspidal representations can be divided into two categories via the notion of depth, those of depth zero and those of positive depth. Finally, we will end this section with a description of the Moy-Prasad Construction for depth-zero supercuspidal representations and comment briefly on the J.K. Yu Construction for positive-depth supercuspidal representations.

### 2.3.1 Basics of Representation Theory

In this section, $G$ denotes a $p$-adic group. Let us recall some of the basic concepts of representation theory. We assume that the reader is already familiar with these concepts, and only provide a short summary. In Appendix A.1, the reader can find the proofs for various results which we use in this thesis. Those results should not be surprising to someone that is familiar with representation theory. This appendix should not be read on its own, but rather serve as a reference, if needed, as results from it are called.

**Definition 2.3.1.** *A representation $(\pi, V)$ of $G$ is a group homomorphism $\pi : G \to \mathrm{End}(V)$, where $V$ is a complex vector space. The dimension of the representation $(\pi, V)$ is the dimension of the vector space $V$. We say that $(\pi, V)$ is smooth if the set $\mathrm{stab}_G(v) = \{g \in G : \pi(g)v = v\}$ is open for all $v \in V$. Furthermore, we say that $(\pi, V)$ is admissible if it is smooth and $V^K = \{v \in V : \pi(k)v = v \text{ for all } k \in K\}$ is finite-dimensional for all compact subgroups $K \subset G$.*

**Remark 2.3.2.** A smooth representation $(\pi, \mathbb{C})$ of $G$ is referred to as a *(complex smooth) character* of $G$. It will be implicit that the word *character* refers to a complex smooth character when in the context of representations. In some references, the word *character* is reserved for unitary representations, and the word *quasicharacter* is then used for complex smooth one-dimensional representations which are not required to be unitary. For our purposes, both of these terms will be used interchangeably when talking about one-dimensional complex smooth representations.

When referring to a representation $(\pi, V)$, we may simply write $\pi$ or $V$. Given representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ of $G$, one can define a representation $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ as follows. Given $g \in G$, $(\pi_1 \oplus \pi_2)(g)(v_1 + v_2) = \pi_1(g)v_1 + \pi_2(g)v_2$ for all $v_1 \in V_1, v_2 \in V_2$. Similarly, one can also define a representation $(\pi_1 \otimes \pi_2, V_1 \otimes_{\mathbb{C}} V_2)$ of $G$, by viewing $G$ as the subgroup $\{(g, g) : g \in G\}$ of $G \times G$.

**Definition 2.3.3.** *We say that $W \subset V$ is a $G$-invariant subspace if $\pi(g)w \in W$ for all $g \in G, w \in W$, that is, $W$ is preserved by the action of $G$ induced by the representation. The representation $(\pi, V)$ is said to be irreducible if it has no proper nontrivial $G$-invariant subspaces.*

**Remark 2.3.4.** If a complex representation is smooth and irreducible, then it is admissible [Blo05, Proposition 2].

Having a $G$-invariant subspace $W \subset V$ induces a *subrepresentation* $(\pi, W)$ of $G$ and a *quotient representation* $(\pi, V/W)$. Given two representations $\pi_1, \pi_2$ of $G$, we may write $\pi_2 \subset \pi_1$ to signify that $\pi_2$ is a subrepresentation of $\pi_1$.

**Definition 2.3.5.** *A representation $\pi$ is said to be completely reducible if it can be written as a direct sum of irreducible subrepresentations.*

When a representation is completely reducible, we may refer to its irreducible subrepresentations as *components*. When a group $G$ is finite or compact, every finite-dimensional representation is completely reducible, and every irreducible representation is finite-dimensional.

**Definition 2.3.6.** *Given two representations $(\pi_1, V)$ and $(\pi_2, W)$ of $G$, an intertwining map between $\pi_1$ and $\pi_2$ is a linear map $f : V \to W$ such that $f \circ \pi_1(g) = \pi_2(g) \circ f$ for all $g \in G$. The set of all intertwining maps between $\pi_1$ and $\pi_2$ is denoted* $\mathrm{Hom}_G(\pi_1, \pi_2)$.

When it is nontrivial, the space $\mathrm{Hom}_G(\pi_1, \pi_2)$ allows us to determine relationships between the representations $\pi_1$ and $\pi_2$. Indeed, if $\pi_1$ is irreducible, then $\pi_1$ is a subrepresentation of $\pi_2$ if and only if $\mathrm{Hom}_G(\pi_1, \pi_2) \neq \{0\}$; and if $\pi_2$ is irreducible, then $\pi_2$ is a quotient representation of $\pi_1$ if and only if $\mathrm{Hom}_G(\pi_1, \pi_2) \neq \{0\}$ (Proposition A.1.3). When a representation is completely reducible, quotient representations are also subrepresentations.

**Definition 2.3.7.** *Let $\pi_1$ and $\pi_2$ be representations of $G$. If $\mathrm{Hom}_G(\pi_1, \pi_2)$ contains an invertible element, $\pi_1$ and $\pi_2$ are said to be isomorphic representations, and we write $\pi_1 \simeq \pi_2$.*

**Definition 2.3.8.** *Let $\pi$ be a completely reducible representation of $G$, and $\pi'$ be an irreducible representation of $G$. If each component of $\pi$ is isomorphic to $\pi'$, then we say that $\pi$ is $\pi'$-isotypic.*

Given a group $G$ and a subgroup $G' \subset G$, one can construct representations of $G'$ from representations of $G$ and vice versa. Indeed, given a representation $(\pi, V)$ of $G$, we can define its *restriction to $G'$*, denoted by $\mathrm{Res}_{G'}^{G} \pi$ or $\pi|_{G'}$, as

$$\begin{aligned} \pi|_{G'} : G' &\to \mathrm{End}(V) \\ g' &\mapsto \pi(g'). \end{aligned}$$

When given a representation $(\pi', V')$ of $G'$, we can produce a representation of $G$ from $\pi'$ by considering the vector space

$$\mathrm{Ind}_{G'}^{G} V' = \{f : G \to V' \text{ locally constant } | f(gg') = \pi'(g')^{-1} f(g)\, \forall g' \in G'\}.$$

The representation of $G$ *induced* by $\pi'$, denoted by $\mathrm{Ind}_{G'}^{G}\pi'$, is defined by the left-action of $G$ on the space of functions $\mathrm{Ind}_{G'}^{G} V'$. More precisely, given $f \in \mathrm{Ind}_{G'}^{G} V', k \in G$, $\mathrm{Ind}_{G'}^{G}\pi'(k)f$ is the function defined by $\mathrm{Ind}_{G'}^{G}\pi'(k)f(g) = f(k^{-1}g)$ for all $g \in G$. The compact induction of $\pi'$, denoted $\text{c-Ind}_{G'}^{G} \pi'$, is the left-action of $G$ on the subspace

$\text{c-Ind}_{G'}^{G} V'$ of $\text{Ind}_{G'}^{G} V'$ consisting of locally constant functions with compact support modulo $G'$. We have more control over the subspace $\text{c-Ind}_{G'}^{G} V'$, as functions in this space are nonzero on only finitely many cosets of $G/G'$ when $G'$ is open. In the case where $G/G'$ is compact, $\text{Ind}_{G'}^{G} V' = \text{c-Ind}_{G'}^{G} V'$. This happens for example when $G'$ is of finite index in $G$, or when $G'$ is a parabolic subgroup of $G$. When the context is obvious, we may use $\text{c-Ind}_{G'}^{G} \pi'$ to refer to the vector space $\text{c-Ind}_{G'}^{G} V'$.

We now recall two important results that involve both restriction and induction, Frobenius Reciprocity and the Mackey Decomposition. The statement of Frobenius Reciprocity can be found for example in [Mur09, Section 5].

**Theorem 2.3.9** (Frobenius Reciprocity)**.** *Let $G' \subset G$ be a subgroup, $\pi$ be a smooth representation of $G$ and $\pi'$ be a smooth representation of $G'$. Then*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_{G'}^{G} \pi') \simeq \mathrm{Hom}_{G'}(\mathrm{Res}_{G'}^{G} \pi, \pi'),$$

*or equivalently induction is right adjoint to restriction. Furthermore, if $G'$ is open, then*

$$\mathrm{Hom}_G(\text{c-Ind}_{G'}^{G} \pi', \pi) \simeq \mathrm{Hom}_{G'}(\pi', \mathrm{Res}_{G'}^{G} \pi),$$

*or equivalently compact induction is left adjoint to restriction.*

**Remark 2.3.10.** As a consequence of Frobenius Reciprocity, we have that $\mathrm{Hom}_G(\text{c-Ind}_{G'}^{G} \pi', \pi) \neq \{0\}$ if and only if $\mathrm{Hom}_{G'}(\pi', \mathrm{Res}_{G'}^{G} \pi) \neq \{0\}$. Furthermore, when $\pi$ is irreducible, having $\mathrm{Hom}_G(\pi, \text{c-Ind}_{G'}^{G} \pi') \neq \{0\}$ implies that $\mathrm{Hom}_{G'}(\mathrm{Res}_{G'}^{G} \pi, \pi') \neq \{0\}$. Indeed, this is because $\pi \subset \text{c-Ind}_{G'}^{G} \pi' \subset \mathrm{Ind}_{G'}^{G} \pi'$ and therefore $\mathrm{Hom}_G(\pi, \mathrm{Ind}_{G'}^{G} \pi') \neq \{0\}$.

**Theorem 2.3.11** (Mackey Decomposition [Kut77])**.** *Let $B$ be a closed subgroup of $G$ and $K$ be a subgroup of $G$ that contains an open compact subgroup. Let $C$ be the set of representatives of the double cosets of $B\backslash G/K$ and let $\pi$ be a finite-dimensional smooth representation of $K$. Define ${}^cK := cKc^{-1}$ for all $c \in C$ and ${}^c\pi(k) := \pi(c^{-1}kc)$ for all $k \in {}^cK$. Then*

$$\mathrm{Res}_{B}^{G} \text{c-Ind}_{K}^{G} \pi = \bigoplus_{c \in C} \text{c-Ind}_{B \cap {}^cK}^{B} \mathrm{Res}_{B \cap {}^cK}^{{}^cK} {}^c\pi.$$

**Remark 2.3.12.** In the case where $N$ is a closed normal subgroup of $G$ that contains an open compact subgroup, and $\pi$ is a finite-dimensional smooth representation of $N$, the Mackey Decomposition allows us to write the following:

$$\mathrm{Res}_{N}^{G} \text{c-Ind}_{N}^{G} \pi = \bigoplus_{c \in C} \text{c-Ind}_{N \cap {}^cN}^{N} \mathrm{Res}_{N \cap {}^cN}^{{}^cN} {}^c\pi,$$

where $C$ is the set of coset representatives of $N\backslash G/N = G/N$. Since $N$ is normal in $G$, we have that ${}^cN = N$ for all $c \in C$, and that ${}^c\pi$ is a representation of $N$. It then follows that

$$\mathrm{Res}_{N}^{G} \text{c-Ind}_{N}^{G} \pi = \bigoplus_{c \in C} {}^c\pi.$$

**Corollary 2.3.13.** *Let $N$ be an open normal subgroup of $G$, $\pi$ be a smooth representation of $G$ and $\pi'$ be an irreducible finite-dimensional smooth representation of $N$ contained in $\operatorname{Res}_N^G \pi$. Then $\pi|_N = \bigoplus_{c \in C'} {}^c\pi'$, where $C'$ is a subset of a set of coset representatives of $G/N$.*

**Proof:** Let us first note from Remark 2.2.1 that $N$ is also closed. Furthermore, since $G$ has a neighborhood basis of compact open subgroups [Cas19], $N$ contains a compact open subgroup.

We have that $\pi' \subset \operatorname{Res}_N^G \pi$ which implies that $\operatorname{Hom}_N\left(\pi', \operatorname{Res}_N^G \pi\right) \neq \{0\}$. By Frobenius Reciprocity, this is equivalent to $\operatorname{Hom}_G\left(\text{c-Ind}_N^G \pi', \pi\right) \neq \{0\}$, meaning that $\pi$ is a quotient of $\text{c-Ind}_N^G \pi'$. Taking the restriction to $N$ on both sides yields $\pi|_N$ as a quotient of $\operatorname{Res}_N^G \text{c-Ind}_N^G \pi'$. But the Mackey Decomposition tells us that $\operatorname{Res}_N^G \text{c-Ind}_N^G \pi' = \bigoplus_{c \in C} {}^c\pi'$, where $C$ is a set of coset representatives of $G/N$. Hence, $\operatorname{Res}_N^G \text{c-Ind}_N^G \pi'$ is completely reducible as $\pi'$ is irreducible, which means that $\pi|_N$ is a subrepresentation of $\bigoplus_{c \in C} {}^c\pi'$. The result follows. ∎

When $(\pi, V)$ is a representation of $G$ which is trivial on a normal subgroup $N \subset G$, then one can *factor the representation through to the quotient* $G/N$ by setting $\pi(gN) = \pi(g)$ for all $g \in G$. Similarly, given a representation $(\pi, G/N)$, one can *pull it back* to a representation of $G$. Note that the representation of a quotient group is irreducible if and only if its pullback is irreducible. In this work, we will often identify representations of quotient groups with their pullbacks.

### 2.3.2 Supercuspidal and Cuspidal Representations

The goal of representation theory is often to understand how a completely reducible representation decomposes into irreducible components, meaning that the irreducible representations form an important class of representations. In the case of $p$-adic group, another class of representations, called the supercuspidal representations, is of particular interest. The reason for studying this class of representations is due to a Theorem by Jacquet, which we state below. For the remainder of this chapter, unless otherwise specified, $G = \mathbb{G}(F)$, where $\mathbb{G}$ is a reductive group and $F$ is a $p$-adic field.

**Theorem 2.3.14** (Jacquet's Subrepresentation Theorem [Jac71])**.** *Let $\pi$ be an irreducible smooth representation of $G$. Then there exists a parabolic subgroup $P = M \ltimes N$ and an irreducible supercuspidal representation $\sigma$ of $M$ such that $\pi$ is a subrepresentation of* $\text{c-Ind}_P^G \sigma$.

**Remark 2.3.15.** Jacquet's Subrepresentation Theorem tells us that given an irreducible smooth representation $\pi$ of $G$, $\pi$ is either supercuspidal, or it is a subrepresentation of $\text{c-Ind}_P^G \sigma$ for some proper parabolic subgroup $P \subset G$ and supercuspidal representation $\sigma$. Equivalently, $\pi$ is supercuspidal if and only if $\mathrm{Hom}_G(\pi, \text{c-Ind}_P^G \sigma) = \{0\}$ for all proper parabolic subgroups $P \subset G$ and supercuspidal representations $\sigma$.

This last theorem implies that the supercuspidal representations of $G$ are the building blocks for all representations of $G$. Now that we have the motivation for studying supercuspidal representations, let us define what they are.

**Definition 2.3.16.** *Let $(\pi, V)$ be a smooth representation of $G$. For each parabolic subgroup $P = M \ltimes N$ of $G$, we define $V(N) := \mathrm{span}\{\pi(n)v - v : n \in N, v \in V\}$, and $V_N = V/V(N)$. The representation $(\pi, V)$ is said to be supercuspidal if $V_N = \{0\}$ for all proper parabolic subgroups $P = M \ltimes N$.*

**Remark 2.3.17.** When $F$ is a finite field, the condition that $V = V(N)$ is equivalent to saying that the space of $N$-fixed vectors, $V^N$, is trivial. If $(\pi, V)$ is an *irreducible* representation of $G$ such that $V^N = \{0\}$ for all proper parabolic subgroups $P = M \ltimes N$, then we say that $\pi$ is *cuspidal*. As such, when $F$ is a finite field, we can replace the word "supercuspidal" with "cuspidal" in the statements of Theorem 2.3.14 and Remark 2.3.15.

Note that cuspidal representations are finite-dimensional, as algebraic groups over finite fields are finite groups and irreducible representations of finite groups are finite-dimensional. We also point out that to determine if a representation is (super)cuspidal, it suffices to look at the proper standard parabolic subgroups.

**Example 2.3.18.** If $G = T$ is a torus, then it does not have any roots, and therefore does not have any proper parabolic subgroups. Therefore, when $F$ is a $p$-adic field, all representations of $T$ are supercuspidal, and when $F$ is a finite field, all *irreducible* representations of $T$ are cuspidal. ∎

In general, it can be difficult to determine whether a representation is supercuspidal, and even more difficult to construct such representations. The strategy for constructing supercuspidal representations relies on Mautner's Theorem, which we state below. All existing constructions for supercuspidal representations are simply applications of this theorem.

**Theorem 2.3.19** (Mautner's Theorem [Mau64])**.** *Let $(\pi, V)$ be a smooth representation of $K \subset G$. Assume that $K$ is open and compact modulo the center of $G$. If $\text{c-Ind}_K^G \pi$ is irreducible, then it is supercuspidal (and admissible).*

When we know that a representation is (super)cuspidal, we can obtain information about its irreducible subrepresentations.

**Proposition 2.3.20.** *Let $\mathbb{G}$ be a reductive group defined over a field $F$ which is either finite or p-adic, $G = \mathbb{G}(F)$, $(\pi, V)$ a completely reducible representation of $G$. Let $P = M \ltimes N$ be a parabolic subgroup of $G$ and $U$ be a subrepresentation of $V$. If $V_N = \{0\}$, then $U_N = \{0\}$.*

**Proof:** Assume that $V_N = \{0\}$. By the complete reducibility of $\pi$, we can write $V = U \oplus U'$, where $U'$ is a complement of $U$. Letting $\pi_U$ and $\pi_{U'}$ denote the corresponding subrepresentations, we can then write $\pi = \pi_U \oplus \pi_{U'}$. We show that

$$U = U(N) = \operatorname{span}\{\pi_U(n)u - u : n \in N, u \in U\}.$$

By hypothesis, we have that $V = V(N) = \operatorname{span}\{\pi(n)v - v : n \in N, v \in V\}$. Let $u \in U$. In particular, $u \in V$, so $u = \sum_j a_j(\pi(n_j)v_j - v_j)$ for some $n_j \in N, v_j \in V$. Since $v_j \in V$, it has a unique decomposition $v_j = u_j + u'_j$, where $u_j \in U, u'_j \in U'$. Then

$$\begin{aligned}
u &= \sum_j a_j(\pi(n_j)v_j - v_j) \\
&= \sum_j a_j(\pi(n_j)(u_j + u'_j) - (u_j + u'_j)) \\
&= \sum_j a_j(\pi_U(n_j)(u_j) + \pi_{U'}(n_j)u'_j - u_j - u'_j) \\
&= \sum_j a_j(\pi_U(n_j)u_j - u_j) + \sum_j a_j(\pi_{U'}(n_j)u'_j - u'_j),
\end{aligned}$$

where $\sum_j a_j(\pi_U(n_j)u_j - u_j) \in U$ and $\sum_j a_j(\pi_{U'}(n_j)u'_j - u'_j) \in U'$. By uniqueness of the expression of $u$, it follows that $u = \sum_j a_j(\pi_U(n_j)u_j - u_j) \in \operatorname{span}\{\pi_U(n)u - u : n \in N, u \in U\}$. Hence, we conclude that $U = U(N)$, and therefore $U_N = \{0\}$. ∎

**Remark 2.3.21.** The previous proposition implies that such a representation $\pi$ decomposes as a sum of cuspidal representations when $F$ is a finite field. When $F$ is a $p$-adic field, it means that a completely reducible supercuspidal representation decomposes as a direct sum of irreducible supercuspidal representations.

**Proposition 2.3.22.** *Let $\mathbb{G}$ be a reductive group defined over a p-adic (finite) field $F$, $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\mathrm{der}}$, $G = \mathbb{G}(F)$ and $H = \mathbb{H}(F)$. Let $(\pi, V)$ be a supercuspidal (cuspidal) representation of $G$. Then $\operatorname{Res}_H^G \pi$ is a supercuspidal representation (decomposes as a sum of cuspidal representations) of $H$.*

**Proof:** Let $\mathbb{P} = \mathbb{M} \ltimes \mathbb{N}$ be an $F$-parabolic subgroup of $\mathbb{H}$, $P = \mathbb{P}(F), M = \mathbb{M}(F), N = \mathbb{N}(F)$. We note that $P = M \ltimes N$. Assume without loss of generality that $\mathbb{P}$ is a proper standard parabolic subgroup. It follows that $\mathbb{N}$ is completely generated by root subgroups. Since root subgroups of $\mathbb{H}$ are also root subgroups of $\mathbb{G}$ (Corollary 2.1.28), $\mathbb{N}$ is equal to the unipotent part of a standard parabolic subgroup of $\mathbb{G}$. Given a supercuspidal (cuspidal) representation $\pi$ of $G$, we have that $V = V(N)$, and therefore $\pi$ is also supercuspidal (decomposes as a direct sum of cuspidal representations) as a representation of $H$. ∎

**Proposition 2.3.23.** *Let $\mathbb{G}$ be a reductive group defined over a finite field $F$, $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ containing $\mathbb{G}_{\text{der}}$, $G = \mathbb{G}(F)$ and $H = \mathbb{H}(F)$. Let $\pi_H$ be a cuspidal representation of $H$ and $\pi_G$ be an irreducible subrepresentation of $\text{c-Ind}_H^G \pi_H$. Then $\pi_G$ is a cuspidal representation of $G$.*

**Proof:** We first note that $\pi_G$ contains $\pi_H$ upon restriction to $H$. Indeed, $\pi_G \subset \text{c-Ind}_H^G \pi_H$ implies that $\text{Hom}_G(\pi_G, \text{c-Ind}_H^G \pi_H) \neq \{0\}$, which in turn implies $\text{Hom}(\text{Res}_H^G \pi_G, \pi_H) \neq \{0\}$ by Frobenius Reciprocity (Remark 2.3.10). It follows that $\pi_H$ is a quotient $\text{Res}_H^G \pi_G$. But, Theorem 1 tells us that $\text{Res}_H^G \pi_G$ is completely reducible so we conclude that $\pi_H \subset \text{Res}_H^G \pi_G$.

Now, we proceed by contradiction and assume that $\pi_G$ is not cuspidal. Jacquet's Subrepresentation Theorem (Theorem 2.3.14) implies that there exists a proper parabolic subgroup $P$ of $G$, and a cuspidal representation $\sigma$ such that $\pi_G \subset \text{c-Ind}_P^G \sigma$. It follows that $\pi_H$ is an irreducible subrepresentation of $\text{Res}_H^G \text{c-Ind}_P^G \sigma$ and $\text{Hom}_H(\pi_H, \text{Res}_H^G \text{c-Ind}_P^G \sigma) \neq \{0\}$. Applying the Mackey formula to this restriction, we obtain

$$\text{Res}_H^G \text{c-Ind}_P^G \sigma = \bigoplus_{c \in C} \text{c-Ind}_{H \cap {}^cP}^H \text{Res}_{H \cap {}^cP}^{{}^cP} {}^c\sigma,$$

where $C$ is the set of representatives of $H \backslash G / P$. For all $c \in C$, we have that $H \cap {}^cP$ is a proper parabolic subgroup of $H$ by Proposition 2.1.30. Furthermore $\text{Res}_{H \cap {}^cP}^{{}^cP} {}^c\sigma$ decomposes as a sum of cuspidal representations by Proposition 2.3.22 and Remark 2.3.21. Say this decomposition is given by $\sigma|_{P \cap H} = \bigoplus_{j \in J} \sigma_j$, where $J$ is some index set. It follows that

$$\text{Res}_H^G \text{c-Ind}_P^G \sigma = \bigoplus_{\substack{c \in C \\ j \in J}} \text{c-Ind}_{H \cap {}^cP}^H {}^c\sigma_j.$$

On the other hand, because $\pi_H$ is cuspidal, we have that $\text{Hom}_H(\pi_H, \text{c-Ind}_{H \cap {}^cP}^H {}^c\sigma_j) = \{0\}$ for all $c \in C, j \in J$, (Remark 2.3.15), and therefore $\text{Hom}_H(\pi_H, \bigoplus_{\substack{c \in C \\ j \in J}} \text{c-Ind}_{H \cap {}^cP}^H {}^c\sigma_j) = \{0\}$. But this contradicts with the fact that $\text{Hom}_H(\pi_H, \text{Res}_H^G \text{c-Ind}_P^G \sigma) \neq \{0\}$ above. Hence, $\pi_G$ is a cuspidal representation.

∎

### 2.3.3 Depth of a Representation

Using Bruhat-Tits theory, Moy and Prasad defined the notion of depth for the irreducible admissible representations of $G$ as follows.

**Theorem 2.3.24** ([MP94, Theorem 5.2]). *If $(\pi, V)$ is an irreducible admissible representation of $G$, then there exists a nonnegative rational number $r$ with the property that $r$ is the minimal number such that there exists $x \in \mathcal{B}(\mathbb{G}, F)$ satisfying $V^{G_{x,r^+}} \neq \{0\}$. This minimal number $r$ is called the* depth *of $\pi$.*

**Remark 2.3.25.** In particular, the notion of depth can be applied to irreducible supercuspidal representations and to characters of $G$ (Remark 2.3.4), dividing these classes of representations into those of depth zero and those of positive depth.

One can easily interpret the notion of depth when the representation at hand is a character. Indeed, in the case where $\pi$ is a character, $\mathbb{C}$ has only $\{0\}$ and $\mathbb{C}$ as subspaces. Then by definition, $r$ is the smallest nonnegative rational number such that $\mathbb{C}^{G_{x,r^+}} \neq \{0\}$ for some $x \in \mathcal{B}(\mathbb{G}, F)$. Therefore, $r$ is the smallest nonnegative rational number such that $\mathbb{C}^{G_{x,r^+}} = V$ for some $x \in \mathcal{B}(\mathbb{G}, F)$, that is $\pi|_{G_{x,r^+}} = 1$. This is equivalent to $\pi|_{G_{x,r^+}} = 1$ for all $x \in \mathcal{B}(\mathbb{G}, F)$ [HM08, Definition 2.46]. Hence, for a nontrivial character $\pi$, the depth is the number $r$ such that $\pi|_{G_{x,r}} \neq 1$ and $\pi|_{G_{x,r^+}} = 1$ for all $x \in \mathcal{B}(\mathbb{G}, F)$. By definition, the trivial character has depth zero.

### 2.3.4 Construction of Supercuspidal Representations

The concept of depth divides the irreducible supercuspidal representations into two categories, those of depth zero and those of positive depth. As mentioned previously, all known constructions of supercuspidal representations are applications of Mautner's Theorem.

Moy and Prasad described how to construct depth-zero supercuspidal representations in [MP96]. The Moy-Prasad Construction is as follows: one starts with a maximal parahoric subgroup $G_{x,0}$. The reductive quotient $G_{x,0:0^+}$ corresponds to the $\mathfrak{f}$-points of a reductive group which we denoted by $\mathcal{G}_x$ in Section 2.2.2.2. Given a cuspidal representation $\sigma$ of this quotient, one can pull it back to $G_{x,0}$. One then considers the group $N_G(G_{x,0})$, which is open and compact modulo the center of $G$, and $\epsilon_G(\sigma)$, the set of irreducible representations of $N_G(G_{x,0})$ which contain $\sigma$ on restriction to $G_{x,0}$. Then, given $\rho \in \epsilon_G(\sigma)$, the representation $\text{c-Ind}_{N_G(G_{x,0})}^G \rho$ is irreducible and supercuspidal of depth zero [MP96, Proposition 6.6]. Furthermore this construction is exhaustive, as per the following theorem.

**Theorem 2.3.26** ([MP96, Proposition 6.8]). *Every irreducible depth-zero supercuspidal representation of $G$ has the form $\text{c-Ind}_{N_G(G_{x,0})}^G \rho$ for some maximal parahoric subgroup $G_{x,0}$ of $G$, cuspidal representation $\sigma$ of $G_{x,0:0^+}$ pulled back to $G_{x,0}$, and $\rho \in \epsilon_G(\sigma)$.*

Note that the Moy-Prasad Construction is not explicit, as it does not describe how to construct the representation $\rho$. However, we have the following relationship between $\rho$ and $\sigma$.

**Lemma 2.3.27.** *Let $\sigma$ be a cuspidal representation of $G_{x,0:0^+}$ and $\rho \in \epsilon_G(\sigma)$. Then*

$$\rho|_{G_{x,0}} = \bigoplus_{c \in C'} {}^c\sigma,$$

*where $C'$ is a subset of a set of coset representatives of $N_G(G_{x,0})/G_{x,0}$.*

**Proof:** By definition of $\epsilon_G(\sigma)$, we have that $\sigma \subset \rho|_{G_{x,0}}$. Furthermore, $G_{x,0}$ is normal in $N_G(G_{x,0})$ and $\sigma$ is a finite-dimensional representation. The result thus follows from Corollary 2.3.13. ∎

The construction for positive-depth supercuspidal representations is more tedious, and will be discussed in detail later in this work (Section 4.1). The J.K. Yu Construction for positive-depth supercuspidal representations has the Moy-Prasad Construction for depth-zero supercuspidal representations as a special case. So, in a sense, it is a generalization that provides an algorithm for constructing any irreducible supercuspidal representation of $G$ when $\mathbb{G}$ splits over a tamely ramified extension $E$ of $F$ and the characteristic $p$ of $\mathfrak{f}$ is large enough.

Very briefly, constructing a supercuspidal representation of positive depth of $G$ is like assembling a piece of IKEA furniture. The J.K. Yu Construction starts with a (generic cuspidal) $G$-datum, which is like the assembly pieces. The various steps of the J.K. Yu Construction then describe how to combine these pieces together like an instruction manual.

Under certain hypotheses, the J.K. Yu Construction is exhaustive. The first exhaustion result came from Kim in 2007 [Kim07], and holds if $F$ is of characteristic zero and $p$ is very large. In 2018, Fintzen was able to generalize the exhaustion result to non-archimedean local fields of arbitrary characteristic, with a much less restrictive condition on $p$.

**Theorem 2.3.28** ([Fin18, Theorem 8.1])**.** *Assume that $\mathbb{G}$ splits over a tame extension of $F$ and $p \nmid |W|$, where $W$ denotes the Weyl group of $\mathbb{G}$. Every smooth irreducible supercuspidal representation of $G$ arises from the J.K. Yu Construction.*

**Remark 2.3.29.** For all irreducible root systems, one can calculate directly $|W|$ as illustrated in Table 2.1. The condition that $p \nmid |W|$ implies that $p$ is neither a *bad* nor *torsion* prime [Fin18, Lemma 2.2], two notions that appear in various parts of the literature. It also implies that $p$ does not divide the order of the *fundamental group* of $\mathbb{G}_{\text{der}}$ [Kal19, Section 3.7.4]. As such, when $p \nmid |W|$, we are assured that the structure theory of the group behaves nicely.

We will be assuming that $p \nmid |W|$ in this work. Not only does this hypothesis ensure the exhaustion of irreducible supercuspidal representations via the J.K. Yu Construction, it also simplifies some aspects of the construction (to be discussed in Section 7.1).

| Type | $\lvert W\rvert$ |
|---|---|
| $A_n(n \geqslant 1)$ | $(n+1)!$ |
| $B_n(n \geqslant 3)$ | $2^n n!$ |
| $C_n(n \geqslant 2)$ | $2^n n!$ |
| $D_n(n \geqslant 3)$ | $2^{n-1} n!$ |
| $E_6$ | $2^7 3^4 5$ |
| $E_7$ | $2^{10} 3^4 5 \cdot 7$ |
| $E_8$ | $2^{14} 3^5 5^2 7$ |
| $F_4$ | $2^7 3^2$ |
| $G_2$ | $2^2 3$ |

Table 2.1: Part of [Fin18, Table 1] describing the order of the Weyl groups for irreducible root systems.

# Chapter 3

# Data for Supercuspidal Representations

Let $\mathbb{G}$ be a reductive group defined over a $p$-adic field $F$, $G = \mathbb{G}(F)$, and let $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$. We assume that $\mathbb{G}$ splits over a tamely ramified extension $E$ of $F$ and that $p$ does not divide the order of the Weyl group of $\mathbb{G}$.

Given an irreducible supercuspidal representation $\pi$, it is obtained from the J.K. Yu construction, and therefore was constructed from some $G$-datum $\Psi$. When studying the restriction of $\pi$ to $H$, the result decomposes into a finite direct sum of irreducible supercuspidal representations of $H$ (Theorem 1, Proposition 2.3.22 and Remark 2.3.21), say $\pi|_H = \bigoplus_{1 \leqslant i \leqslant m} \pi_i$ for some $m > 0$. Each of the components is constructed from some $H$-datum under the hypothesis on $p$, as $\mathbb{G}$ and $\mathbb{H}$ have the same Weyl groups (Remark 2.1.25). We illustrate this in Figure 3.1. Based on this

$$\begin{tikzcd}
\pi|_H \arrow[d, dash, dashed] & = & \pi_1 \arrow[d, dash, dashed] & \oplus & \pi_2 \arrow[d, dash, dashed] & \oplus & \cdots & \oplus & \pi_m \arrow[d, dash, dashed] \\
\Psi & & \Psi_1 & & \Psi_2 & & \cdots & & \Psi_m
\end{tikzcd}$$

Figure 3.1: Illustration of the $H$-data that appear in the restriction $\pi|_H$. $\Psi_i$ denotes the $H$-datum required for the construction of $\pi_i$, $1 \leqslant i \leqslant m$.

diagram, some natural questions arise with respect to the relationship between the $G$-datum $\Psi$ and the $H$-data $\Psi_i$, $1 \leqslant i \leqslant m$.

In this chapter, we start by describing what a $G$-datum is in Section 3.1 as per [Yu01, HM08]. We then proceed to discussing equivalence classes of $G$-data in Section 3.2, that is families of different data that produce the same supercuspidal representation [HM08]. When we think of our analogy of a supercuspidal representation being a piece of IKEA furniture, it makes sense for it to be possible to obtain a supercuspidal representation from different $G$-data, the same way that a piece of furniture could be constructed with different types of building materials.

We explain how one can construct data for $H$ starting from a $G$-datum through a natural restriction process in Section 3.3, and discuss the equivalence classes of these $H$-data in relation to the equivalence class of the initial $G$-datum in Section 3.4. One can also consider the converse problem, one of retrieving $G$-data from an $H$-datum, which will be done in Section 3.5. Finally, we discuss the equivalence classes of these $G$-data in relation to the equivalence class of the $H$-datum in Section 3.6.

## 3.1 Definition of $G$-datum

The datum for constructing supercuspidal representations is composed of five elements, each satisfying interrelated conditions [Yu01, HM08]. We begin by stating these conditions as axioms and will provide definitions and expand on some of the more technical axioms thereafter.

**Definition 3.1.1.** *A sequence* $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ *is a (generic cuspidal) G-datum if and only if:*

*D1)* $\vec{\mathbb{G}}$ *is a tamely ramified twisted Levi sequence* $\vec{\mathbb{G}} = (\mathbb{G}^0, \mathbb{G}^1, \dots, \mathbb{G}^d)$ *in* $\mathbb{G}$*, where* $\mathbb{G}^d = \mathbb{G}$ *and such that* $Z(\mathbb{G}^0)/Z(\mathbb{G})$ *is F-anisotropic;*

*D2)* $y$ *is a point in* $\mathcal{B}(\mathbb{G}, F) \cap \mathcal{A}(\mathbb{G}, \mathbb{T}, E)$*, where* $\mathbb{T}$ *is a maximal torus of* $\mathbb{G}^0$ *(and maximal in all* $\mathbb{G}^i$*), and E is a Galois tamely ramified splitting field of* $\mathbb{T}$ *(hence of* $\vec{\mathbb{G}}$*);*

*D3)* $\vec{r} = (r_0, r_1, \dots, r_d)$ *is a sequence of real numbers satisfying* $0 < r_0 < r_1 < \cdots < r_{d-1} \leqslant r_d$ *if* $d > 0$*,* $0 \leqslant r_0$ *if* $d = 0$*;*

*D4)* $\rho$ *is an irreducible representation of* $K^0 = G^0_{[y]}$ *such that* $\rho|_{G^0_{y,0+}}$ *is 1-isotypic and* $\text{c-Ind}_{K^0}^{G^0} \rho$ *is irreducible supercuspidal;*

*D5)* $\vec{\phi} = (\phi^0, \phi^1, \dots, \phi^d)$ *is a sequence of quasicharacters, where* $\phi^i$ *is a* $G^{i+1}$*-generic character of* $G^i$ *(relative to y) of depth* $r_i$ *for* $0 \leqslant i \leqslant d-1$*. If* $r_{d-1} < r_d$*, we assume* $\phi^d$ *is of depth* $r_d$*, otherwise* $\phi^d = 1$*.*

From the $G$-datum $\Psi$, one can use the J.K. Yu construction to produce a sequence of representations $\vec{\pi} = (\pi^0, \dots, \pi^d)$, where $\pi^i$ is a supercuspidal representation of $G^i$ of depth $r_i$. To be consistent with this superscript notation, we let $\pi^{-1} := \text{c-Ind}_{K^0}^{G^0} \rho$. Our main interest will lie with $\pi^d$, the supercuspidal representation of $G$ of depth $r_d$, which we will denote by $\pi_G(\Psi)$ hereafter. The details behind the construction of $\pi_G(\Psi)$ will be outlined in Section 4.1. Axioms D1, D4 and D5 are the more technical of the five, so we expand on their definitions and some of their properties below.

**Axiom D1** Saying that $\vec{\mathbb{G}} = (\mathbb{G}^0, \mathbb{G}^1, \ldots, \mathbb{G}^d)$ is a tamely ramified twisted Levi sequence means that $\mathbb{G}^i \subset \mathbb{G}^{i+1}, 0 \leqslant i \leqslant d-1$, and $\mathbb{G}^i$ is a tamely ramified twisted Levi subgroup for all $0 \leqslant i \leqslant d$. As per [HM08, Definition 2.40], the adjectives tamely ramified refer to the extension $F \subset E$, whereas twisted means $\mathbb{G}^i$ is an $E$-Levi $F$-subgroup of $\mathbb{G}$. That is, $\mathbb{G}^i(E)$ can be described in terms of generators as per Definition 2.1.10 for Levi subgroups, whereas $\mathbb{G}^i(F)$ is defined but might not be described as nicely.

The following lemma ensures the quotient $Z(\mathbb{G}^0)/Z(\mathbb{G})$ is well-defined.

**Lemma 3.1.2.** *For all $0 \leqslant i < j \leqslant d$, we have $Z(\mathbb{G}^j) \subset Z(\mathbb{G}^i)$.*

**Proof:** We have that $Z(\mathbb{G}^j) \subset \mathbb{T} \subset \mathbb{G}^i$ (Proposition 2.1.6). Furthermore, $\mathbb{G}^i \subset \mathbb{G}^j$, so $Z(\mathbb{G}^j)$ centralizes $\mathbb{G}^i$. Thus, $Z(\mathbb{G}^j) \subset Z(\mathbb{G}^i)$. ∎

The condition that $Z(\mathbb{G}^0)/Z(\mathbb{G})$ is $F$-anisotropic implies that

$$X_*(Z(\mathbb{G}^i), F) \otimes \mathbb{R} = X_*(Z(\mathbb{G}), F) \otimes \mathbb{R}$$

for $0 \leqslant i \leqslant d$, which properly defines the embeddings $\mathcal{B}(\mathbb{G}^i, F) \hookrightarrow \mathcal{B}(\mathbb{G}, F)$ [Yu01, Remark 3.4].

**Axiom D4** Here, $[y]$ denotes the projection of $y$ in the reduced building of $G$, which is a vertex in the reduced building of $G^0$, and as such $G^0_{y,0}$ is a maximal parahoric subgroup of $G^0$ [Yu01, Remark 3.7]. In other words, $[y] = \{(y, v) : v \in X_*(Z(\mathbb{G}), F) \otimes \mathbb{R}\}$, as introduced in Section 2.2.2.3. Define $K^0 = G^0_{[y]}$ to be the set of elements in $G^0$ that fix $[y]$. We have that $K^0 = N_{G^0}(G^0_{y,0})$ [Yu01, Lemma 3.3], and so this axiom describes precisely the construction of a depth-zero supercuspidal representation as per Theorem 2.3.26. That is, $\rho \in \epsilon_{G^0}(\sigma)$ for some cuspidal $\sigma$ of $G^0_{y,0:0+}$. The following two propositions enumerate other properties related to this axiom that will be useful later in this work.

**Proposition 3.1.3.** *Let $Z^0$ denote the center of $G^0$. The index $[K^0 : Z^0 G^0_{y,0}]$ is finite.*

**Proof:** The subgroup $K^0$ is compact modulo the center of $G^0$, which means $K^0/Z^0$ is compact. Let $p : K^0 \to K^0/Z^0$ be the projection map. By definition of the quotient topology, $G^0_{y,0}$ is open in $K^0$ if and only if $p(G^0_{y,0}) = Z^0 G^0_{y,0}/Z^0$ is open in $K^0/Z^0$. Hence $Z^0 G^0_{y,0}/Z^0$ is open and $\bigcup\limits_{k \in K^0/Z^0} k\left(Z^0 G^0_{y,0}/Z^0\right)$ is an open covering of $K^0/Z^0$. Since $K^0/Z^0$ is compact, there exists a finite subcover, thus $\left[K^0/Z^0 : Z^0 G^0_{y,0}/Z^0\right]$ is finite. We conclude that $[K^0 : Z^0 G^0_{y,0}]$ is finite as $(K^0/Z^0) \Big/ \left(Z^0 G^0_{y,0}/Z^0\right) \simeq K^0/Z^0 G^0_{y,0}$. ∎

**Remark 3.1.4.** Note that the proof of the previous proposition remains the same if we replace $G^0_{y,0}$ by any open subgroup of $K^0$, meaning that $[K^0 : Z^0\tilde{K}]$ is finite for all open subgroups $\tilde{K} \subset K^0$.

**Proposition 3.1.5.** *The representation $\rho$ of $K^0$ is finite-dimensional.*

**Proof:** Let $\sigma$ be a cuspidal of $G^0_{y,0:0+}$ for which the pullback to $G^0_{y,0}$ is contained in $\rho|_{G^0_{y,0}}$. We know that irreducible representations of compact groups are finite-dimensional. Therefore, $\sigma$ is a finite-dimensional representation of $G^0_{y,0}$. We can extend $\sigma$ to a representation $\sigma'$ of $Z^0G^0_{y,0}$ as follows. For all $z \in Z^0, g \in G^0_{y,0}$, $\sigma'(zg) := \rho(z)\sigma(g)$. This is well-defined because $\rho|_{Z^0\cap G^0_{y,0}} = \bigoplus_{c\in C'} \sigma|_{Z^0\cap G^0_{y,0}}$, where $C'$ is a subset of a set of coset representatives of $K^0/G^0_{y,0}$ (Lemma 2.3.27). This representation $\sigma'$ acts on the same vector space as $\sigma$ because $Z^0$ acts by scalar multiplication via $\rho$ (Lemma A.1.4).

Since $\rho \in \epsilon_{G^0}(\sigma)$, we have that $\mathrm{Hom}_{G^0_{y,0}}(\sigma, \mathrm{Res}^{K^0}_{G^0_{y,0}} \rho) \neq \{0\}$. It follows that $\mathrm{Hom}_{Z^0G^0_{y,0}}(\sigma', \mathrm{Res}^{K^0}_{Z^0G^0_{y,0}} \rho) \neq \{0\}$, which is equivalent to $\mathrm{Hom}_{K^0}(\text{c-Ind}^{K^0}_{Z^0G^0_{y,0}} \sigma', \rho) \neq \{0\}$ by Frobenius Reciprocity. Since $[K^0 : Z^0G^0_{y,0}]$ is finite (Proposition 3.1.3), and $\sigma'$ is a finite-dimensional representation, it follows from Proposition A.1.8 that $\text{c-Ind}^{K^0}_{Z^0G^0_{y,0}} \sigma'$ is also a finite-dimensional representation, along with all its subrepresentations and quotients. We then conclude that $\rho$ is a finite-dimensional representation. ∎

**Axiom D5** We must build up to the notion of genericity for the quasicharacters. We follow the notation from [HM08, Section 3.1]. Let $\mathbf{t} = \mathrm{Lie}(\mathbb{T})$ and $\mathfrak{t} = \mathbf{t}(F)$. For all $0 \leqslant i \leqslant d-1$ let $\boldsymbol{\mathfrak{z}}^i$ denote the center of $\boldsymbol{\mathfrak{g}}^i = \mathrm{Lie}(\mathbb{G}^i)$, and $\boldsymbol{\mathfrak{z}}^{i,*}$ its dual. We set $\mathfrak{z}^i = \boldsymbol{\mathfrak{z}}^i(F)$ and $\mathfrak{z}^{i,*} = \boldsymbol{\mathfrak{z}}^{i,*}(F)$. We also have $\mathfrak{z}^i_{r_i} = \mathfrak{z}^i \cap \mathfrak{t}_{r_i}$ and $\mathfrak{z}^i_{r_i^+} = \mathfrak{z}^i \cap \mathfrak{t}_{r_i^+}$ and define

$$\mathfrak{z}^{i,*}_{-r_i} = \{X^* \in \mathfrak{z}^{i,*} : X^*(Y) \in \mathfrak{p}_F \text{ for all } Y \in \mathfrak{z}^i_{r_i^+}\}.$$

Given an element from $\mathfrak{z}^{i,*}$, one can view it as an element of $\mathfrak{g}^{i,*}$ by extending it trivially on $\mathfrak{g}^{i'}$, where $\mathfrak{g}^{i'} = [\boldsymbol{\mathfrak{g}}^i,\boldsymbol{\mathfrak{g}}^i](F)$. This follows from the fact that $\mathfrak{g}^i = \mathfrak{z}^i \oplus \mathfrak{g}^{i'}$ [AR00, Proposition 3.1]. We can also extend the scalars of our Lie algebras to $E$, meaning that elements of $\mathfrak{z}^{i,*}$ can evaluate on $\boldsymbol{\mathfrak{g}}^i(E)$.

**Definition 3.1.6.** *Let $X^* \in \mathfrak{z}^{i,*}_{-r_i}$. For all $a \in \Phi(\mathbb{G},\mathbb{T})$, let $H_a = d\check{a}(1)$. If $\mathrm{val}(X^*(H_a)) = -r_i$ for all $a \in \Phi(\mathbb{G}^{i+1},\mathbb{T})\backslash\Phi(\mathbb{G}^i,\mathbb{T})$, then $X^*$ is said to be a $G^{i+1}$-generic element of depth $-r_i$.*

**Remark 3.1.7.** The valuation condition stated in the previous definition is what Yu calls condition GE1) in [Yu01]. In [Yu01, HM08], a second condition must be satisfied in order for an element to be generic, referred to as GE2). However, given our underlying hypothesis on $p$, condition GE1) implies condition GE2) [Yu01, Lemma 8.1], which allows us to simplify Yu's original definition of genericity.

**Lemma 3.1.8.** *For all $a \in \Phi = \Phi(\mathbb{G}, \mathbb{T})$, we have $H_a \in \mathfrak{t}(E)_0$.*

**Proof:** Because $\mathbb{G}$ is split over $E$, we have that

$$\mathbb{T}(E)_0 = \{t \in \mathbb{T}(E) : \mathrm{val}(\chi(t)) = 0 \text{ for all } \chi \in X^*(\mathbb{T})\},$$

as mentioned in Section 2.2.2.2. Given $a \in \Phi$, $\breve{a}(1) \in \mathbb{T}(E)$ and $\chi(\breve{a}(1)) = 1$ as $\chi$ and $\breve{a}$ are homomorphisms. It follows that $\breve{a}(1) \in \mathbb{T}(E)_0$, and therefore $H_a \in \mathfrak{t}(E)_0$. ∎

Given a quasicharacter $\chi^i$ of $G^i$ of depth $r_i$, it is trivial on $G^i_{y,r_i^+}$ and can thus factor through to a character of $G^i_{y,r_i:r_i^+}$. The notion of genericity allows to express $\chi^i$ as a character of $G^i_{y,r_i:r_i^+}$ in terms of a $G^{i+1}$-generic element, as per the following definition.

**Definition 3.1.9** ([HM08, Definition 3.9])**.** *Fix a character $\psi$ of $F$ which is nontrivial on $\mathcal{O}_F$ and trivial on $\mathfrak{p}_F$. A quasicharacter $\chi^i$ of $G^i$ is said to be $G^{i+1}$-generic (relative to $y$) of depth $r_i$ if $\chi^i$ is trivial on $G^i_{y,r_i^+}$ and nontrivial on $G^i_{y,r_i}$, and there exists a $G^{i+1}$-generic element $X^* \in \mathfrak{z}^{i,*}_{-r_i}$ of depth $-r_i$ such that $\chi^i(e(Y + \mathfrak{g}^i_{y,r_i^+})) = \psi(X^*(Y))$ for all $Y \in \mathfrak{g}^i_{y,r_i}$. The map $e$ is the isomorphism between $\mathfrak{g}^i_{y,r_i:r_i^+}$ and $G^i_{y,r_i:r_i^+}$ described in Appendix A.2. We say that $X^*$ realizes the restriction of $\chi^i$ to $G^i_{y,r_i}$.*

**Remark 3.1.10.** Under our assumption on $p$, the character $\chi^i$ of $G^i$ is $G^{i+1}$-generic relative to $y$ if and only if it is $G^{i+1}$-generic relative to $x$ for all $x \in \mathcal{B}(\mathbb{G}^i, F)$ [Mur11, Lemma 4.7].

**Remark 3.1.11.** The notion of genericity is very technical. One interpretation we can give comes from [Mur11, Remark 6.4] and is the following. If $\chi^i$ is a quasicharacter of $G^i$ which is $G^{i+1}$-generic of depth $r_i$, this means that $\chi^i|_{G^i_{y,r_i}}$ is not the restriction to $G^i_{y,r_i}$ of a quasicharacter of a twisted Levi subgroup $\dot{G}$ of $G^{i+1}$ such that $G^i \subset \dot{G}$. So in a sense, genericity means that we cannot obtain the restriction to $G^i_{y,r_i}$ from a bigger twisted Levi subgroup.

Let us show that one actually does not need the statement that $\chi^i$ is nontrivial on $G^i_{y,r_i}$ in the above definition, as it follows from the two other properties listed.

**Proposition 3.1.12.** *Let $\chi^i$ be a quasicharacter of $G^i$ such that $\chi^i|_{G^i_{y,r_i^+}} = 1$. If there exists a $G^{i+1}$-generic element $X^* \in \mathfrak{z}^{i,*}_{-r_i}$ of depth $-r_i$ that realizes the restriction of $\chi^i$ to $G^i_{y,r_i}$, then $\chi^i$ is nontrivial on $G^i_{y,r_i}$, that is, $\chi^i$ is $G^{i+1}$-generic (relative to $y$) of depth $r_i$.*

**Proof:** By hypothesis, we have that $\chi^i(e(Y + \mathfrak{g}^i_{y,r_i^+})) = \psi(X^*(Y))$ for all $Y \in \mathfrak{g}^i_{y,r_i}$. Showing that $\chi^i|_{G^i_{y,r_i}}$ is nontrivial is equivalent to showing that there exists $Y \in \mathfrak{g}^i_{y,r_i}$ such that $X^*(Y) \notin \ker\psi$.

Let's proceed by contradiction and assume that $X^*(Y) \in \ker\psi$ for all $Y \in \mathfrak{g}^i_{y,r_i}$. We claim that this implies $X^*(Y) \in \mathfrak{p}_F$ for all $Y \in \mathfrak{g}^i_{y,r_i}$. Indeed, assume there exists $Z \in \mathfrak{g}^i_{y,r_i}$ such that $b = X^*(Z) \in \ker\psi \backslash \mathfrak{p}_F$, and choose $b' \in \mathcal{O}_F^\times$ such that $b' \notin \ker\psi$. Such a $b'$ exists by definition of $\psi$. We have that $b^{-1}b' \in \mathcal{O}_F$ since $b$ has valuation less than or equal to zero. Since $\mathfrak{g}^i_{y,r_i}$ is an $\mathcal{O}_F$-submodule, we have that $b^{-1}b'Z \in \mathfrak{g}^i_{y,r_i}$. But $X^*(b^{-1}b'Z) = b^{-1}b'X^*(Z) = b' \notin \ker\psi$, which is a contradiction. Hence, under the assumption that $X^*(Y) \in \ker\psi$ for all $Y \in \mathfrak{g}^i_{y,r_i}$, we must also have $X^*(Y) \in \mathfrak{p}_F$ for all $Y \in \mathfrak{g}^i_{y,r_i}$. In particular, $X^*(Y) \in \mathfrak{p}_F$ for all $Y \in \mathfrak{t}_{r_i}$, which implies $X^*(Y) \in \mathfrak{p}_E$ for all $Y \in \mathfrak{t}(E)_{r_i}$ by extending the scalars to $\mathcal{O}_E$. Letting $e_{E/F}$ denote the ramification index of $E/F$, we have that $\varpi_E^{e_{E/F}r_i} H_a \in \mathfrak{t}(E)_{r_i}$, as $H_a \in \mathfrak{t}(E)_0$ (Lemma 3.1.8). It follows that $X^*(\varpi_E^{e_{E/F}r_i} H_a) \in \mathfrak{p}_E$ for all $a \in \Phi(\mathbb{G}^{i+1}, \mathbb{T}) \backslash \Phi(\mathbb{G}^i, \mathbb{T})$, which implies that $\mathrm{val}(\varpi_E^{e_{E/F}r_i} X^*(H_a)) > 0$. This is a contradiction with genericity, as GE1) ensures $\mathrm{val}(X^*(H_a)) = -r_i$ for all $a \in \Phi(\mathbb{G}^{i+1}, \mathbb{T}) \backslash \Phi(\mathbb{G}^i, \mathbb{T})$. ∎

**Remark 3.1.13.** The initial statement of Axiom D5 provided in [Yu01] does not state genericity of the quasicharacters. However, it was shown that the genericity condition is in fact required to guarantee that the construction results in a supercuspidal representation [Yu01, Theorem 15.1]. The definition we have provided for the $G$-datum is what the authors in [HM08] refer to as a generic cuspidal $G$-datum.

**Remark 3.1.14.** The notion of genericity means that the quasicharacter in question is realized by a very particular central element. In general though, given a quasicharacter $\phi$ of $G$ of depth $r > 0$ and a point $y \in \mathcal{B}(\mathbb{G}, F)$, we have that $\phi|_{G_{y,(r/2)^+}}$ is realized by an element of $\mathfrak{z}^*_{-r}$ given our underlying hypothesis on $p$ [Kal19, Lemma 3.5.2]. This is what Hakim and Murnaghan referred to as hypothesis $C(G)$ in [HM08].

We end the discussion on genericity with the following lemma, which shows that one has control over the depth of a product of characters when one of the characters is generic.

**Lemma 3.1.15.** *Let $\mathbb{G}'$ be a twisted Levi subgroup of $\mathbb{G}$, $\phi'$ be a $G$-generic quasicharacter of $G'$ (relative to $y$) of depth $r'$ and $\phi$ be a quasicharacter of $G$ of* *depth $r \leqslant r'$. Then $\phi'\phi$ is a $G$-generic quasicharacter of $G'$ (relative to $y$) of depth $r'$. (By $\phi'\phi$, it is implied that we mean $\phi'(\phi|_{G'})$.)*

**Proof:** Let us consider two cases, $r < r'$ and $r = r'$. If $r < r'$, then $\phi|_{G'_{y,r'}} = 1$, and therefore $\phi'\phi|_{G'_{y,r'}} = \phi'|_{G'_{y,r'}}$. The conclusion thus follows by genericity of $\phi'$.

Now let us assume $r = r'$. We first note that $(\phi'\phi)|_{G'_{y,r'+}} = 1$, so to show genericity, we must show that $\phi'\phi|_{G'_{y,r'}}$ is realized by a $G$-generic element of depth $-r'$ as per Proposition 3.1.12. Murnaghan showed in [Mur11, Lemma 4.9] that $\phi|_{G'}$ and $\phi$ must be of the same depth $r'$. Then, from Remark 3.1.14, we have that $\phi|_{G'_{y,(r'/2)+}}$ is realized by an element of $\mathfrak{z}'^{,*}_{-r'}$. In particular, since $G'_{y,r'} \subset G'_{y,(r'/2)+}$, it follows that $\phi|_{G'_{y,r'}}$ is realized by an element of $\mathfrak{z}'^{,*}_{-r'}$, say $Z^*$. If we let $X^*$ be the $G$-generic element in $\mathfrak{z}'^{,*}_{-r'}$ of depth $-r'$ that realizes $\phi'|_{G'_{y,r'}}$, then $X^* + Z^*$ is a $G$-generic element of depth $-r'$ [HM08, Lemma 4.21]. Indeed, this is because $H_a = d\check{\alpha}(1)$ lies in the lie algebra of $\mathbb{G}_{\text{der}}$ (Section 2.2.1.1), and elements of $\mathfrak{z}^*$ are trivial there. Therefore $\text{val}((X^* + Z^*)(H_a)) = \text{val}(X^*(H_a)) = -r'$ for all $a \in \Phi\backslash\Phi'$ . Furthermore, one can easily see that $X^* + Z^*$ realizes $\phi'\phi|_{G'_{y,r'}}$. Indeed, for all $Y \in \mathfrak{g}'_{y,r'}$, we have

$$\phi'\phi(e(Y + \mathfrak{g}'_{y,r'+})) = \phi'(e(Y + \mathfrak{g}'_{y,r'+}))\phi(e(Y + \mathfrak{g}'_{y,r'+})),$$

where $e$ denotes the isomorphism between $\mathfrak{g}'_{y,r':r'+}$ and $G'_{y,r':r'+}$ described in Appendix A.2. Since $\phi'|_{G'_{y,r'}}$ is realized by $X^*$ and $\phi|_{G'_{y,r'}}$ is realized by $Z^*$, we conclude that

$$\phi'\phi(e(Y + \mathfrak{g}'_{y,r'+})) = \psi(X^*(Y))\psi(Z^*(Y)) = \psi((X^* + Z^*)(Y)).$$

Hence, $\phi'\phi$ is a $G$-generic character of $G'$ of depth $r'$. ∎

**Remark 3.1.16.** In general, when multiplying two characters of the same depth, there is a possibility that the product results in a character of smaller depth. For example, if we multiply a character of positive depth $\phi$ with the character $\phi'$ defined by $\phi'(g) = \phi(g^{-1})$, then we obtain the trivial character, which is of depth zero.

**Tuples of Data** While the $G$-datum is a 5-tuple, not all elements need to be explicitly mentioned. For example, $\vec{r}$ refers to the depths of the quasicharacters in the sequence $\vec{\phi}$, so we can omit it and have it be implied in the 4-tuple $(\vec{\mathbb{G}}, y, \rho, \vec{\phi})$. The datum can even be further reduced to a triple. Indeed, [Yu01, Theorem 15.7] states that the equivalence class of the representation constructed from the $G$-datum only depends on the triple $(\vec{\mathbb{G}}, \pi^{-1}, \vec{\phi})$, where $\pi^{-1} = \text{c-Ind}_{K^0}^{G^0} \rho$. This triple is referred to as a reduced datum in [HM08]. Conversely, given a depth-zero supercuspidal representation $\pi^{-1}$, one can associate to it a vertex $y$ and a representation $\rho$ via the Moy-Prasad construction from Theorem 2.3.26, recovering a 5-tuple from a reduced datum.

**Remark 3.1.17.** When $\vec{\mathbb{G}} = \mathbb{G}$ and $\vec{\phi} = 1$, then the corresponding $G$-datum is simply $\Psi = (\mathbb{G}, y, \rho, 1)$, while the reduced datum is the irreducible supercuspidal representation of depth zero $\pi^{-1} = \text{c-Ind}_{G_{[y]}}^{G} \rho$. In this case, we have that $\pi_G(\Psi) = \pi^{-1}$. This will be made apparent once we review the J.K. Yu construction in Section 4.1. This means that the Moy-Prasad construction for irreducible depth-zero supercuspidal representations is a special case of the J.K. Yu construction.

One can restrict axioms D1-D5 further to produce specific types of $G$-data. For example, one can consider what is called a *normalized* $G$-datum, introduced by Kaletha, which is defined below.

**Definition 3.1.18** ([Kal19, Definition 3.7.1])**.** *Let $\vec{\mathbb{G}} = (\mathbb{G}^0, \ldots, \mathbb{G}^d)$ be a twisted Levi sequence of $\mathbb{G}$. For all $0 \leqslant i \leqslant d$, let $\mathbb{G}^i_{\mathrm{sc}}$ denote the simply connected (or universal) cover* [Ste68, p.88] *of* $[\mathbb{G}^i, \mathbb{G}^i]$ *and set $G^i_{\mathrm{sc}} = \mathbb{G}^i_{\mathrm{sc}}(F)$. Let $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ be a $G$-datum, where $\vec{\phi} = (\phi^0, \ldots, \phi^d)$. The datum $\Psi$ is said to be normalized if the pullback of $\phi^i$ to $G^i_{\mathrm{sc}}$ is trivial for all $0 \leqslant i \leqslant d$.*

We will see in the next section that any arbitrary $G$-datum is somehow related to a normalized $G$-datum (Lemma 3.2.2).

## 3.2 Equivalence Classes of $G$-data

Let $\Psi$ and $\dot{\Psi}$ be two $G$-data. Even when these two data are different, it is possible for $\Psi$ and $\dot{\Psi}$ to generate the same supercuspidal representation, that is $\Psi \neq \dot{\Psi}$ and $\pi_G(\Psi) \simeq \pi_G(\dot{\Psi})$. In [HM08], Hakim and Murnaghan showed that the equivalence class of the representation $\pi_G(\Psi)$ depends on what they called the $G$-equivalence class of $\Psi$.

To define the notion of $G$-equivalence for data, Hakim and Murnaghan introduce three ways of altering a $G$-datum $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$ into a different $G$-datum $\dot{\Psi}$. The first alteration is called $G$-conjugation, in which $\dot{\Psi} = {}^g\Psi = ({}^g\vec{\mathbb{G}}, g \cdot y, {}^g\rho, {}^g\vec{\phi})$. The datum $\dot{\Psi} = {}^g\Psi$ is such that $\pi_G(\Psi) \simeq \pi_G(\dot{\Psi})$ [HM08, Section 5.1.1]. The second alteration is called an elementary transformation [HM08, Definition 5.2], in which $\dot{\Psi} = (\vec{\mathbb{G}}, \dot{y}, \dot{\rho}, \vec{\phi})$, where $[\dot{y}] = [y]$ and $\dot{\rho} \simeq \rho$. Again, in this case, we have $\pi_G(\Psi) \simeq \pi_G(\dot{\Psi})$ because the corresponding reduced data for $\Psi$ and $\dot{\Psi}$ are the same. The third alteration is what Hakim and Murnaghan call a refactorization. This notion is a bit more technical and is defined as follows.

**Definition 3.2.1** ([HM08, Definition 4.19]). *Let $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$ and $\dot{\Psi} = (\vec{\mathbb{G}}, \dot{y}, \dot{\rho}, \vec{\dot{\phi}})$. For all $0 \leqslant i \leqslant d$, we define a quasicharacter $\chi^i$ of $G^i$ as follows. For all $g \in G^i$,*

$$\chi^i(g) = \prod_{j=i}^{d} \phi^j(g)\dot{\phi}^j(g)^{-1}.$$

*If $(\dot{\rho}, \vec{\dot{\phi}})$ satisfies the conditions*

*F0) if $\phi^d = 1$ then $\dot{\phi}^d = 1$,*

*F1) $\dot{\phi}^i|_{G^i_{y,r^+_{i-1}}} = \phi^i\chi^{i+1}|_{G^i_{y,r^+_{i-1}}}$ for all $i$ where $r_{-1} = 0$ and $\chi^{d+1} = 1$ (in other words $\chi^i|_{G^i_{y,r^+_{i-1}}} = 1$),*

*F2) $\dot{\rho} = \rho \otimes (\chi^0|_{K^0})$,*

*then we say that $\dot{\Psi}$ is a refactorization of $\Psi$.*

Now, two $G$-data $\Psi$ and $\dot{\Psi}$ are said to be $G$-equivalent if $\dot{\Psi}$ can be obtained from $\Psi$ by a finite sequence of refactorizations, $G$-conjugations and elementary transformations [HM08, Definition 6.1]. The normalized data that we described in Definition 3.1.18 are actually representatives of the $G$-equivalence classes of data, as per the following lemma.

**Lemma 3.2.2** ([Kal19, Lemma 3.7.2]). *Every $G$-datum is $G$-equivalent to a normalized $G$-datum when $p$ does not divide the order of the Weyl group of $\mathbb{G}$.*

**Remark 3.2.3.** In Kaletha's original statement of the lemma, the hypothesis required is that $p$ does not divide the order of the fundamental group of $\mathbb{G}_{\mathrm{der}}$. However, as we have pointed out in Remark 2.3.29, this hypothesis is satisfied when $p \nmid |W|$, where $W$ is the absolute Weyl group of $\mathbb{G}$.

The concept of $G$-equivalence defined by Hakim and Murnaghan is precisely what establishes the equivalence class of an irreducible supercuspidal representation obtained via the J.K. Yu construction, as per the following theorem.

**Theorem 3.2.4** ([HM08, Theorems 6.6, 6.7]). *Suppose $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$ and $\dot{\Psi} = (\vec{\dot{\mathbb{G}}}, \dot{y}, \dot{\rho}, \vec{\dot{\phi}})$ are $G$-data. Set $\phi = \prod_{i=0}^{d} \phi^i|_{G^0}$, $\dot{\phi} = \prod_{i=0}^{d} \dot{\phi}^i|_{\dot{G}^0}$. Then the following are equivalent:*

*1) $\pi_G(\Psi) \simeq \pi_G(\dot{\Psi})$.*

*2) $\Psi$ and $\dot{\Psi}$ are $G$-equivalent.*

*3) There exists $g \in G$ such that $K^0 = {}^g\dot{K}^0$, $\vec{\mathbb{G}} = {}^g\dot{\vec{\mathbb{G}}}$ and $\rho \otimes \phi \simeq {}^g(\dot{\rho} \otimes \dot{\phi})$.*

**Remark 3.2.5.** The equality $K^0 = \dot{K}^0$ implies that $[y] = [\dot{y}]$. The explanation in [HM08, Proof of Theorem 6.7] is as follows. We have that $[y]$ and $[\dot{y}]$ are vertices of the reduced building of $G^0$. If $[y] \neq [\dot{y}]$, one can find an element of $G^0$ that fixes $[y]$, but that does not fix $[\dot{y}]$, which would imply $K^0 \neq \dot{K}^0$, as these are the subgroups that fix $[y]$ and $[\dot{y}]$, respectively.

**Remark 3.2.6.** In the original statements of [HM08, Theorems 6.6, 6.7] which we have partly recalled above, Hakim and Murnaghan have an additional hypothesis, denoted $C(\vec{G})$. Kaletha showed that this hypothesis is not necessary [Kal19, Corollary 3.5.5], which is why we have omitted it in our statement.

**Corollary 3.2.7.** *Suppose that $\Psi = (G, y, \rho, 1)$ and $\dot{\Psi} = (G, \dot{y}, \dot{\rho}, 1)$ are $G$-data for depth-zero supercuspidal representations. Assume that $\pi_G(\Psi) \simeq \pi_G(\dot{\Psi})$. Then there exists $g \in G$ such that $G_{y,0} = {}^gG_{\dot{y},0}$ and $\rho \simeq {}^g\dot{\rho}$.*

**Proof:** By Theorem 3.2.4, there exists $g \in G$ such that $G_{[y]} = {}^gG_{[\dot{y}]}$ and $\rho \simeq {}^g\rho$. Remark 3.2.5 implies that $[y] = g \cdot [\dot{y}]$, and therefore $G_{y,0} = G_{g\cdot\dot{y},0} = {}^gG_{\dot{y},0}$ by Lemma 2.2.19. ∎

The previous corollary was actually first due to Moy and Prasad [MP96, Theorem 6.11]. Again, we come to a realization that the J.K. Yu construction has as a special case the Moy-Prasad construction for irreducible depth-zero supercuspidal representations.

## 3.3 Constructing $H$-data from a $G$-datum

Let $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ be a $G$-datum. In this section, we will discuss how one can construct data for $H$ from the $G$-datum $\Psi$.

Since $H$ is a subgroup of $G$, the most natural thing to do is to restrict all the elements of the datum $\Psi$ to the smaller subgroup $H$. However there is a small subtlety that we must account for with the quasicharacter sequence. Indeed, when restricting quasicharacters, depths can potentially be reduced unpredictably, whereas the definition of datum requires that the depths in the sequence are strictly increasing, with the exception of the last quasicharacter, which is allowed to be trivial (Axiom D5). For the generic quasicharacters $\phi^0, \ldots, \phi^{d-1}$, we expect the depths to be preserved upon restriction since the genericity condition relies on coroots (Definition 3.1.9), and $\mathbb{G}$ and $\mathbb{H}$ have the same root system (Lemma 2.1.26). As for the last character, we cannot simply restrict $\phi^d$ to $H$ if the depth of this restriction is less than or equal to $r_{d-1}$. Instead, in this case, we fold in the character $\phi^d|_H$ earlier in the sequence. The main result of this section is the construction of $H$-data from the $G$-datum $\Psi$ via this natural restriction process, which is stated by the following theorem and illustrated in Figure 3.2.

**Theorem 3.3.1.** *Let* $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ *be a* $G$*-datum. Let* $\mathbb{H}^i = \mathbb{G}^i \cap \mathbb{H}$ *and* $\phi_H^i = \phi^i|_{H^i}$, $0 \leqslant i \leqslant d$*. Let* $\tilde{r}$ *denote the depth of* $\phi_H^d$*.*

*1) If* $\tilde{r} > r_{d-1}$*, set* $\vec{\phi_H} = (\phi_H^0, \dots, \phi_H^{d-1}, \phi_H^d)$ *and* $\vec{\tilde{r}} = (r_0, \dots, r_{d-1}, \tilde{r})$*.*

*2) If* $\tilde{r} \leqslant r_{d-1}$*, set* $\vec{\phi_H} = (\phi_H^0, \cdots, \phi_H^{d-1}\phi_H^d, 1)$ *and* $\vec{\tilde{r}} = (r_0, \dots, r_{d-1}, r_{d-1})$*.*

*Then, for each irreducible subrepresentation* $\rho_\ell$ *of* $\rho_H := \rho|_{K_H^0}$ *with* $K_H^0 = H_{[y]}^0$*,* $\Psi_\ell = (\vec{\mathbb{H}}, y, \vec{\tilde{r}}, \rho_\ell, \vec{\phi_H})$ *is an* $H$*-datum, where* $\vec{\mathbb{H}} = (\mathbb{H}^0, \dots, \mathbb{H}^d)$*.*

$$\begin{tikzcd}[column sep=small]
(\mathbb{G}^0, \arrow[d, "\cap\mathbb{H}"] & \mathbb{G}^1, \arrow[d, "\cap\mathbb{H}"] & \dots, & \mathbb{G}^d), \arrow[d, "\cap\mathbb{H}"] & \rho, \arrow[d, dashed] & (\phi^0, \arrow[d, "\mathrm{Res}^{G^0}_{H^0}"'] & \dots, & \phi^{d-1}, \arrow[d, "\mathrm{Res}^{G^{d-1}}_{H^{d-1}}"'] & \phi^d) \arrow[d, "\mathrm{Res}^{G^d}_{H^d}"] \\
(\mathbb{H}^0, & \mathbb{H}^1, & \dots, & \mathbb{H}^d), & \rho_\ell, & (\phi_H^0, & \dots, & \phi_H^{d-1}, & \phi_H^d) \arrow[l, bend left, "\tilde{r} \leqslant r_{d-1}"]
\end{tikzcd}$$

Figure 3.2: Summary of the construction of $H$-data from a $G$-datum $\Psi$, where $\rho_\ell$ is any irreducible subrepresentation of $\rho_H$.

**Remark 3.3.2.** To avoid having to consider cases when applying Theorem 3.3.1, one can assume $\phi^d = 1$. This can be done without loss of generality (Lemma 4.1.2), which will become clear once we have reviewed the J.K. Yu construction in Section 4.1

Theorem 3.3.1 is a generalization of the results in [Nev15], in which Nevins treated the case of a *toral datum of length one* and considered the restriction to $\mathbb{H} = \mathbb{G}_{\text{der}}$. A toral datum of length one is a very simple case for which $\vec{\mathbb{G}} = (\mathbb{T}, \mathbb{G})$, where $\mathbb{T}$ is an $F$-minisotropic maximal torus of $\mathbb{G}$. In a toral datum of length one, it can be assumed without loss of generality that $\rho = 1$. Indeed, in this case, we have $K^0 = T$, which means that $\rho$ is a character since $T$ is abelian and $\rho$ is assumed to be irreducible. It is then clear from Definition 3.2.1 that $\dot{\Psi} = ((\mathbb{T}, \mathbb{G}), y, \vec{r}, \rho, (\phi^0, \phi^1))$ is a refactorization of $\Psi = ((\mathbb{T}, \mathbb{G}), y, \vec{r}, 1, (\rho\phi^0, \phi^1))$ and therefore $\pi_G(\dot{\Psi}) \simeq \pi_G(\Psi)$ (Theorem 3.2.4). Not only is the datum considered in [Nev15] simpler, but restricting to $G_{\text{der}}$ also allows for some simplifications. Indeed, every character of $G_{\text{der}}$ is of depth zero when the fundamental group of $\mathbb{G}_{\text{der}}$ has order prime to $p$ [Kal19, Lemma 3.5.1], which is satisfied with our underlying condition that $p$ does not divide $|W|$ (see Remark 2.3.29). As a consequence, we know that the depth of $\phi^1|_{G_{\text{der}}}$ will be smaller than $r$. We state Nevins' result as a corollary of our theorem.

**Corollary 3.3.3** ([Nev15, Proposition 4.5]). *Let $\Psi = ((\mathbb{T}, \mathbb{G}), y, r, 1, (\phi^0, \phi^1))$ be a toral datum of length one for $G$, where $r$ denotes the depth of the quasicharacter $\phi^0$ of $T$ and $\phi^1$ is a quasicharacter of $G$ which is either trivial, or of depth $\tilde{r} > r$. Let $\mathbb{T}_{\mathrm{der}} = \mathbb{T} \cap \mathbb{G}_{\mathrm{der}}$. Set $\phi^0_{\mathrm{der}} = \phi^0|_{T_{\mathrm{der}}}$ and $\phi^1_{\mathrm{der}} = \phi^1|_{G_{\mathrm{der}}}$. Define*

$$\Psi_{\mathrm{der}} = ((\mathbb{T}_{\mathrm{der}}, \mathbb{G}_{\mathrm{der}}), y, 1, r, (\phi^0_{\mathrm{der}}\phi^1_{\mathrm{der}}, 1)).$$

*Then $\Psi_{\mathrm{der}}$ is a toral datum of length one for $G_{\mathrm{der}}$.*

Notice that in the case of the toral datum of length one, only one $G_{\mathrm{der}}$-datum was obtained by this restriction process. In our more general case, we see that multiple data may be produced, depending on how $\rho_H$ decomposes into irreducible subrepresentations.

In order to prove Theorem 3.3.1, we must show that Axioms D1 to D5 hold in the context of $H$, that is,

D1) $\vec{\mathbb{H}}$ is a tamely ramified twisted Levi sequence and $Z(\mathbb{H}^0)/Z(\mathbb{H})$ is $F$-anisotropic;

D2) $y$ is a point in $\mathcal{B}(\mathbb{H}, F) \cap \mathcal{A}(\mathbb{H}, \mathbb{T}_{\mathbb{H}}, E)$, where $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$ is a maximal torus of $\mathbb{H}^0$ (and maximal in all $\mathbb{H}^i$), and $E$ is a Galois tamely ramified splitting field of $\mathbb{T}_{\mathbb{H}}$ (hence of $\vec{\mathbb{H}}$);

D3) $\vec{r}$ is a sequence of real numbers satisfying $0 < r_0 < r_1 < \cdots < r_{d-1}$ if $d > 0$, $0 \leqslant r_0$ if $d = 0$;

D4) $\rho_\ell$ is an irreducible representation of $K^0_H = H_{[y]}$ such that $\rho_\ell|_{H^0_{y,0^+}}$ is 1-isotypic and $\pi_\ell^{-1} = \text{c-Ind}_{K^0_H}^{H^0} \rho_\ell$ is irreducible supercuspidal;

D5) $\vec{\phi}_H$ is a sequence of quasicharacters, where the $i$th quasicharacter of the sequence is $H^{i+1}$-generic (relative to $y$) of depth $r_i$ for $0 \leqslant i \leqslant d-1$.

We have divided this proof into three different subsections, proving Axiom D1, Axioms D2 and D4, and Axioms D3 and D5, respectively.

### 3.3.1 Proving Axiom D1

The sequence $\vec{\mathbb{H}}$ is obtained by intersecting $\vec{\mathbb{G}}$ with $\mathbb{H}$, and therefore it suffices to apply arguments from Section 2.1.2 in which we had discussed precisely this type of intersection.

**Lemma 3.3.4.** *The sequence $\vec{\mathbb{H}}$ is a tamely ramified twisted Levi sequence in $\mathbb{H}$.*

**Proof:** Since $\mathbb{G}^i(E)$ is a Levi subgroup of $\mathbb{G}(E)$ for $0 \leqslant i \leqslant d$, it follows from Proposition 2.1.30 that $\mathbb{H}^i(E) = \mathbb{G}^i(E) \cap \mathbb{H}(E)$ is a Levi subgroup of $\mathbb{H}(E)$. Therefore $\mathbb{G}^i \cap \mathbb{H}$ is an $E$-Levi $F$-subgroup of $\mathbb{H}$. We conclude that $\vec{\mathbb{H}}$ is a tamely ramified twisted Levi sequence in $\mathbb{H}$. ∎

**Remark 3.3.5.** For all $0 \leqslant i \leqslant d$ we have that $\mathbb{H}^i$ is a closed connected subgroup of $\mathbb{G}^i$ that contains $[\mathbb{G}^i, \mathbb{G}^i]$. Indeed, $[\mathbb{G}^i, \mathbb{G}^i] \subset \mathbb{G}^i \cap [\mathbb{G}, \mathbb{G}] \subset \mathbb{G}^i \cap \mathbb{H} = \mathbb{H}^i$.

The previous lemma shows that the first part of Axiom D1 holds. It remains to show that $Z(\mathbb{H}^0)/Z(\mathbb{H})$ is $F$-anisotropic. In order to prove this result, we establish some useful lemmas.

**Lemma 3.3.6.** *For all $0 \leqslant i \leqslant d$, we have $Z(\mathbb{H}^i) = Z(\mathbb{G}^i) \cap \mathbb{H}$.*

**Proof:** Suppose $g \in Z(\mathbb{G}^i) \cap \mathbb{H}$. Then $g \in \mathbb{G}^i \cap \mathbb{H} = \mathbb{H}^i$ and commutes with each element of $\mathbb{G}^i$. In particular, $g$ commutes with each element of $\mathbb{H}^i$ and therefore lies in $Z(\mathbb{H}^i)$.

For the converse, recall from Proposition 2.1.30 that $\mathbb{G}^i$ and $\mathbb{H}^i$ only differ in their generators by their tori, $\mathbb{T}$ and $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$. It then suffices to show that $Z(\mathbb{H}^i)$ centralizes $\mathbb{T}$. Since $Z(\mathbb{H}^i) \subset \mathbb{T}_{\mathbb{H}} \subset \mathbb{T}$ (Proposition 2.1.6), and $\mathbb{T}$ is abelian, the result follows. ∎

**Lemma 3.3.7.** *For all $0 \leqslant i \leqslant d$, we have $\mathbb{G}^i = Z(\mathbb{G})^\circ \mathbb{H}^i$.*

**Proof:** Recall the central isogeny (Proposition 2.1.6) $\mathbb{G} = Z(\mathbb{G})^\circ \mathbb{G}_{\mathrm{der}}$. Since $\mathbb{G}_{\mathrm{der}} \subset \mathbb{H}$, we have $\mathbb{G} = Z(\mathbb{G})^\circ \mathbb{H}$. Then $\mathbb{G}^i = \mathbb{G}^i \cap Z(\mathbb{G})^\circ \mathbb{H}$, Since $Z(\mathbb{G})^\circ \subset \mathbb{T} \subset \mathbb{G}^i$, it follows that

$$\mathbb{G}^i \cap Z(\mathbb{G})^\circ \mathbb{H} = Z(\mathbb{G})^\circ (\mathbb{G}^i \cap \mathbb{H}) = Z(\mathbb{G})^\circ \mathbb{H}^i.$$

∎

**Lemma 3.3.8.** *For all $0 \leqslant i \leqslant d$, we have $Z(\mathbb{G}^i) = Z(\mathbb{G})^\circ Z(\mathbb{H}^i)$.*

**Proof:** Take $x \in Z(\mathbb{G}^i) \subset \mathbb{G}^i$. Then by the previous lemma, we have $x = zy$ for some $z \in Z(\mathbb{G})^\circ, y \in \mathbb{H}^i$. For all $z' \in Z(\mathbb{G})^\circ, y' \in \mathbb{H}^i$, we have $x(z'y') = (z'y')x$ since $x$ is central in $\mathbb{G}^i$. Since $z$ and $z'$ are central in $\mathbb{G}$, it follows that $y'y = yy'$, and therefore $y \in Z(\mathbb{H}^i)$. We conclude that $Z(\mathbb{G}^i) \subset Z(\mathbb{G})^\circ Z(\mathbb{H}^i)$.

For the converse, we apply Lemmas 3.1.2 and 3.3.6. ∎

**Proposition 3.3.9.** *For all $0 \leqslant i \leqslant d$, we have $Z(\mathbb{G}^i)/Z(\mathbb{G}) \simeq Z(\mathbb{H}^i)/Z(\mathbb{H})$.*

**Proof:** Using the previous lemma, we rewrite the quotient on the left-hand side as

$$Z(\mathbb{G}^i)/Z(\mathbb{G}) = Z(\mathbb{G})^\circ Z(\mathbb{H}^i)/Z(\mathbb{G})^\circ Z(\mathbb{H}).$$

One can then easily check that $Z(\mathbb{G})^\circ Z(\mathbb{H}^i)/Z(\mathbb{G})^\circ Z(\mathbb{H}) \simeq Z(\mathbb{H}^i)/Z(\mathbb{H})$ by considering that map that sends $zZ(\mathbb{G})^\circ Z(\mathbb{H})$ to $zZ(\mathbb{H})$ for all $z \in Z(\mathbb{H}^i)$. ∎

**Corollary 3.3.10.** *The quotient $Z(\mathbb{G}^0)/Z(\mathbb{G})$ is $F$-anisotropic if and only if $Z(\mathbb{H}^0)/Z(\mathbb{H})$ is $F$-anisotropic.*

The last corollary completes the proof that Axiom D1 is satisfied.

### 3.3.2 Proving Axioms D2 and D4

Given the point $y$ in $\mathcal{B}(\mathbb{G}, F) \cap \mathcal{A}(\mathbb{G}, \mathbb{T}, E)$, we can assume without loss of generality that $y$ is in $\mathcal{B}(\mathbb{H}, F) \cap \mathcal{A}(\mathbb{H}, \mathbb{T}_{\mathbb{H}}, E)$ by Remark 2.2.20. Furthermore, $\mathbb{T}_{\mathbb{H}}$ is a maximal torus of $\mathbb{H}^0$ (and maximal in all $\mathbb{H}^i$), and $E$ is a Galois tamely ramified splitting field of $\mathbb{T}_{\mathbb{H}}$ (hence of $\vec{\mathbb{H}}$). Thus, Axiom D2 holds.

According to [Yu01, Remark 3.7], $[y]$ is a vertex of the reduced building of $G^0$. Even though $G^0$ and $H^0$ are different, they have the same reduced buildings, as $[\mathbb{G}^0, \mathbb{G}^0] = [\mathbb{H}^0, \mathbb{H}^0]$ (Remarks 2.1.29 and 3.3.5). So, we have that $[y]$ is a vertex of $\mathcal{B}^{\mathrm{red}}(\mathbb{H}^0, F)$. Therefore, $H^0_{y,0}$ is a maximal parahoric subgroup. By [Yu01, Lemma 3.3], $N_{H^0}(H^0_{y,0}) = H^0_{[y]}$. Furthermore,

$$H^0_{[y]} = \{g \in G^0 \cap H : g \cdot [y] = [y]\} = G^0_{[y]} \cap H.$$

To maintain consistent notation, we set $K^0_H = H^0_{[y]}$, so that $K^0_H = K^0 \cap H$.

**Lemma 3.3.11.** *Let $\rho_H := \rho|_{K^0_H}$. Then $\rho_H$ decomposes as a finite sum of irreducible representations.*

**Proof:** We have that $\rho_H$ is a finite-dimensional representation as $\rho$ is finite-dimensional (Proposition 3.1.5). Since $H^0_{y,0}$ is a compact group, it follows that $\rho_H|_{H^0_{y,0}}$ is completely reducible. If $Z^0_H$ denotes the center of $H^0$, Lemma A.1.4 has $Z^0_H$ acting by scalar multiplication. It follows that any $H^0_{y,0}$-invariant subspace is also a $Z^0_H H^0_{y,0}$-invariant subspace, which implies that $\rho_H|_{Z^0_H H^0_{y,0}}$ is completely reducible. Since $Z^0_H H^0_{y,0}$ has finite index in $K^0_H$ (Proposition 3.1.3), we conclude that $\rho_H$ is completely reducible by [BH06, Lemma 2.7]. ∎

Given the representation $\rho_H := \rho|_{K^0_H}$, we write its decomposition into (irreducible) components as $\rho_H = \bigoplus_{\ell \in L} \rho_\ell$, where $L$ is some finite set.

**Remark 3.3.12.** We actually have a more precise description of $\rho_H$. Indeed, because $K_H^0$ is a normal subgroup of $K^0$, we can apply Corollary 2.3.13. Therefore, if $\rho'$ denotes a component of $\rho_H$, the index set $L$ is a finite subset of a set of coset representatives of $K^0/K_H^0$ and for all $\ell \in L$ $\rho_\ell = {}^\ell\rho'$. This more refined description of $\rho_H$ will be used in Section 5.2.2.

In order to prove that Axiom D4 holds, we must show that each representation $\rho_\ell$ gives rise to a depth-zero supercuspidal representation of $H^0$, or equivalently that its restriction to $H^0_{y,0:0^+}$ contains a cuspidal representation as per Section 2.3.4. Using the description of $\rho|_{G^0_{y,0}}$, we can obtain information on $\rho_\ell|_{H^0_{y,0}}$. Indeed, since c-$\mathrm{Ind}_{K^0}^{G^0}\rho$ is a supercuspidal representation of depth zero, we have that $\rho \in \epsilon_{G^0}(\sigma)$ for some cuspidal representation $\sigma$ of $G^0_{y,0:0^+}$. We expect that, upon restricting $\rho$ further down to $H^0_{y,0}$, we can find cuspidal representations, as will be shown via the following two lemmas. Note that we will identify representations of quotient groups with their pullbacks, as explained at the end of Section 2.3.

**Lemma 3.3.13.** *Let $\sigma$ denote a cuspidal representation of $G^0_{y,0:0^+}$ which is contained in $\rho|_{G^0_{y,0}}$ and let $\sigma_H := \sigma|_{H^0_{y,0}}$. Then $\sigma_H$ decomposes as a direct sum of cuspidal representations of $H^0_{y,0:0^+}$.*

**Proof:** Recall from Remark 3.3.5 that $\mathbb{H}^0$ is a closed connected subgroup of $\mathbb{G}^0$ that contains $[\mathbb{G}^0, \mathbb{G}^0]$. Let us also recall the notation from Section 2.2.2.3 and let $\mathcal{G}_y^0$ and $\mathcal{H}_y^0$ denote the reductive quotients of $\mathbb{G}^0$ and $\mathbb{H}^0$ at $y$, respectively. We also let $\binom{\mathcal{H}^0}{\mathcal{G}^0}_y$ be the reductive group whose $\mathfrak{f}$-points identifies with $H^0_{y,0}G^0_{y,0^+}/G^0_{y,0^+}$.

We have that $\sigma$ is a cuspidal representation of $\mathcal{G}_y^0(\mathfrak{f})$. Because $[\mathcal{G}_y^0, \mathcal{G}_y^0] \subset \binom{\mathcal{H}^0}{\mathcal{G}^0}_y$ (Lemma 2.2.26), it follows from Proposition 2.3.22 that $\sigma|_{\binom{\mathcal{H}^0}{\mathcal{G}^0}_y(\mathfrak{f})}$ decomposes as a direct sum of cuspidal representations of $\binom{\mathcal{H}^0}{\mathcal{G}^0}_y(\mathfrak{f})$, say $\bigoplus_{j\in J}\sigma_j$ for some finite set $J$. Those representations can be pulled back to $H^0_{y,0}$, and so $\sigma_H = \sigma|_{H^0_{y,0}} = \bigoplus_{j\in J}\sigma_j$.

Now, since $\sigma|_{H^0_{y,0^+}} = 1$, each $\sigma_j$ factors through to a representation of $\mathcal{H}_y^0(\mathfrak{f})$. Furthermore, $\sigma_j$ has to be cuspidal because $\binom{\mathcal{H}^0}{\mathcal{G}^0}_y(\mathfrak{f}) \simeq \mathcal{H}_y^0(\mathfrak{f})$ (Lemma 2.2.24).

∎

**Lemma 3.3.14.** *Let $\rho_\ell$, $\ell \in L$, be a component of $\rho_H$. Then $\rho_\ell|_{H^0_{y,0}}$ contains a cuspidal representation of $H^0_{y,0:0^+}$.*

**Proof:** Let $\sigma_H = \bigoplus_{j\in J}\sigma_j$ be the decomposition into irreducible representations of $\sigma_H$ provided by the previous Lemma, where $J$ is some finite set. Elements of $K^0$ normalize

$G^0_{y,0}$ as well as $H$, and therefore normalize $H^0_{y,0} = G^0_{y,0} \cap H$ (Proposition 2.2.22). Therefore, by Lemma 2.3.27, we have that

$$\rho|_{H^0_{y,0}} \subset \bigoplus_{c \in K^0/G^0_{y,0}} {}^c\sigma|_{H^0_{y,0}} = \bigoplus_{c \in K^0/G^0_{y,0}} {}^c\left(\bigoplus_{j \in J} \sigma_j\right).$$

The right-hand side is a representation of the finite group $H^0_{y,0:0^+}$ and therefore has a unique decomposition into irreducible representations up to isomorphism. On the left-hand side, we have $\rho|_{H^0_{y,0}} = \rho_H|_{H^0_{y,0}} = \bigoplus_{\ell \in L} \rho_\ell|_{H^0_{y,0}}$. Since $\rho|_{H^0_{y,0}}$ is completely reducible, being a finite-dimensional representation (Proposition 3.1.5) of a compact group, $\rho_\ell|_{H^0_{y,0}}$ is also completely reducible (Proposition A.1.1) and its decomposition into irreducible representations also has to be unique up to isomorphism. Hence, we conclude that $\rho_\ell|_{H^0_{y,0}}$ must contain one of the cuspidal representations (up to isomorphism), and therefore $\rho_\ell \in \epsilon_{H^0}({}^c\sigma_j)$ for some $j \in J$, $c \in K^0/G^0_{y,0}$. ∎

**Theorem 3.3.15** (Axiom D4)**.** *For all* $\ell \in L$, $\rho_\ell|_{H^0_{y,0^+}}$ *is* 1*-isotypic and* $\pi_\ell^{-1} := \text{c-Ind}_{K^0_H}^{H^0} \rho_\ell$ *is an irreducible supercuspidal representation of depth zero.*

**Proof:** Given $\ell \in L$, the previous lemma tells us that $\rho_\ell \in \epsilon_{H^0}(\sigma^*)$ for some cuspidal representation $\sigma^*$ of $H^0_{y,0:0^+}$. It follows from Section 2.3.4 that $\pi_\ell^{-1} = \text{c-Ind}_{K^0_H}^{H^0} \rho_\ell$ is an irreducible supercuspidal representation of depth zero. That $\rho_\ell|_{H^0_{y,0^+}}$ is 1-isotypic follows from the fact that $\rho|_{G^0_{y,0^+}}$ is 1-isotypic. ∎

### 3.3.3 Proving Axioms D3 and D5

Given the definition of $\vec{r}$, it clearly satisfies Axiom D3. In order to prove that Axiom D5 holds, we need to show that the quasicharacters $\phi^i_H$ are $H^{i+1}$-generic of depth $r_i$, $0 \leqslant i \leqslant d-1$. As mentioned earlier in this chapter, this ought to be true as genericity is defined partly in terms of coroots (Definition 3.1.9), and $\mathbb{G}$ and $\mathbb{H}$ have the same root system (Lemma 2.1.26). One must simply go through all the technicalities of the definition, which we will do via some lemmas. Before stating these intermediate results, we introduce the notation that we will be using in this section.

Recall the notation from [HM08] that we introduced when discussing Axiom D5 in Section 3.1. Analogously, let $\mathfrak{t}_{\mathbb{H}} = \text{Lie}(\mathbb{T}_{\mathbb{H}})$, $\mathfrak{t}_H = \mathfrak{t}_{\mathbb{H}}(F)$ and $T_H = \mathbb{T}_{\mathbb{H}}(F)$. For $0 \leqslant i \leqslant d-1$, let $\mathfrak{z}^i_{\mathbb{H}}$ denote the center of $\mathfrak{h}^i = \text{Lie}(\mathbb{H}^i)$, and $\mathfrak{z}^{i,*}_{\mathbb{H}}$ its dual. We set $\mathfrak{z}^i_H = \mathfrak{z}^i_{\mathbb{H}}(F)$ and $\mathfrak{z}^{i,*}_H = \mathfrak{z}^{i,*}_{\mathbb{H}}(F)$. We also have $(\mathfrak{z}^i_H)_{r_i} = \mathfrak{z}^i_H \cap (\mathfrak{t}_H)_{r_i}$ and $(\mathfrak{z}^i_H)_{r_i^+} = \mathfrak{z}^i_H \cap (\mathfrak{t}_H)_{r_i^+}$ and define

$$(\mathfrak{z}^{i,*}_H)_{-r_i} = \{X^* \in \mathfrak{z}^{i,*}_H : X^*(Y) \in \mathfrak{p}_F \text{ for all } Y \in (\mathfrak{z}^i_H)_{r_i^+}\}.$$

Given an element from $\mathfrak{z}^{i,*}$, recall that one can view it as an element of $\mathfrak{g}^{i,*}$ by extending it trivially on $\mathfrak{g}^{i\prime}$, where $\mathfrak{g}^{i\prime} = [\mathfrak{g}^i,\mathfrak{g}^i](F) = [\mathfrak{h}^i,\mathfrak{h}^i](F)$. This follows from the fact that $\mathfrak{g}^i = \mathfrak{z}^i \oplus \mathfrak{g}^{i\prime}$ [AR00, Proposition 3.1].

**Lemma 3.3.16.** *Let $0 \leqslant i \leqslant d-1$. Suppose $X^* \in \mathfrak{z}^{i,*}_{-r_i}$ is a $G^{i+1}$-generic element of depth $-r_i$. Then $X^*|_{\mathfrak{h}^i}$ is an $H^{i+1}$-generic element of depth $-r_i$.*

**Proof:** We must show that $X^*|_{\mathfrak{h}^i}$ is an element of $(\mathfrak{z}^{i,*}_H)_{-r_i}$ and that $\mathrm{val}(X^*|_{\mathfrak{h}^i}(H_a)) = -r_i$ for all $a \in \Phi(\mathbb{H}^{i+1}, \mathbb{T}_{\mathbb{H}})\backslash\Phi(\mathbb{H}^i, \mathbb{T}_{\mathbb{H}})$.

We view $X^*$ as an element of $\mathfrak{g}^{i,*}$. It follows that $X^*|_{\mathfrak{h}^i}$ is an element of $\mathfrak{z}^{i,*}_H$ viewed as an element of $\mathfrak{h}^{i,*}$. Since $X^* \in \mathfrak{z}^{i,*}_{-r_i}$, we have that $X^*(Y) \in \mathfrak{p}_F$ for all $Y \in \mathfrak{z}^i_{r_i^+}$. In particular, $X^*(Y) \in \mathfrak{p}_F$ for all $Y \in (\mathfrak{z}^i_H)_{r_i^+}$. Hence, $X^*|_{\mathfrak{h}^i} \in (\mathfrak{z}^{i,*}_H)_{-r_i}$. Finally, we show that $X^*|_{\mathfrak{h}^i}$ is $H^{i+1}$-generic of depth $-r_i$. Recall that the roots of $\mathbb{H}^{i+1}$ are simply the restriction to $\mathbb{T}_{\mathbb{H}}$ of the roots of $\mathbb{G}^{i+1}$ (Lemma 2.1.26) and that for all $a \in \Phi(\mathbb{G}^{i+1}, \mathbb{T})$, $\breve{a} = \widebreve{a|_{\mathbb{T}_{\mathbb{H}}}}$, meaning that $H_a = H_{a|_{\mathbb{T}_{\mathbb{H}}}}$ (Section 2.2.1). Since $X^*$ is $G^{i+1}$-generic of depth $-r_i$, we have that for all $a \in \Phi(\mathbb{G}^{i+1}, \mathbb{T})\backslash\Phi(\mathbb{G}^i, \mathbb{T})$, $\mathrm{val}(X^*(H_a)) = -r_i$. It follows that $\mathrm{val}(X^*|_{\mathfrak{h}^i}(H_a)) = -r_i$ for all $a \in \Phi(\mathbb{H}^{i+1}, \mathbb{T}_{\mathbb{H}})\backslash\Phi(\mathbb{H}^i, \mathbb{T}_{\mathbb{H}})$. ∎

**Lemma 3.3.17.** *Let $0 \leqslant i \leqslant d-1$. The restriction of the $G^{i+1}$-generic quasicharacter $\phi^i$ of $G^i$ to $H^i$, $\phi^i_H$, is $H^{i+1}$-generic.*

**Proof:** Let $X^*$ be a $G^{i+1}$-generic element of depth $-r_i$ that realizes the quasicharacter $\phi^i$, and let $e$ and $e_H$ be the isomorphisms between $\mathfrak{g}^i_{y,r_i:r_i^+}$ and $G^i_{y,r_i:r_i^+}$, and $\mathfrak{h}^i_{y,r_i:r_i^+}$ and $H^i_{y,r_i:r_i^+}$, respectively (Appendix A.2). By Lemma 3.3.16, $X^*|_{\mathfrak{h}^i}$ is an $H^{i+1}$-generic element of depth $-r_i$. We show that $X^*|_{\mathfrak{h}^i}$ realizes $\phi^i_H$, that is for all $Y \in \mathfrak{h}^i_{y,r_i}$, $\phi^i_H(e_H(Y + \mathfrak{h}^i_{y,r_i^+})) = \psi(X^*|_{\mathfrak{h}^i}(Y))$. Take $Y \in \mathfrak{h}^i_{y,r_i}$. Since $\phi^i$ is realized by $X^*$, we have

$$\begin{aligned}\psi(X^*|_{\mathfrak{h}^i}(Y)) &= \psi(X^*(Y)) \\ &= \phi^i(e(Y + \mathfrak{g}^i_{y,r_i^+})).\end{aligned}$$

The isomorphism $e$ can be viewed as a map from $\mathfrak{g}^i_{y,r_i}$ to $G^i_{y,r_i:r_i^+}$ with kernel $\mathfrak{g}^i_{y,r_i^+}$. Letting $\iota$ denote the isomorphism from $H^i_{y,r:r_i^+}$ to $H^i_{y,r_i}G^i_{y,r_i^+}/G^i_{y,r_i^+}$ given by the second isomorphism theorem, we have that $e|_{\mathfrak{h}^i_{y,r_i}} = \iota \circ e_H$ (Proposition A.2.1). This allows us to obtain the following:

$$e(Y + \mathfrak{g}^i_{y,r_i^+}) = e(Y) = e|_{\mathfrak{h}^i_{y,r_i}}(Y) = \iota \circ e_H(Y) = \iota \circ e_H(Y + \mathfrak{h}^i_{y,r_i^+}).$$

Combining this with what precedes, it follows that

$$\psi(X^*|_{\mathfrak{h}^i}(Y)) = \phi^i \circ \iota \circ e_H(Y + \mathfrak{h}^i_{y,r_i^+}).$$

One can then easily verify that $\phi^i \circ \iota = \phi^i_H$. Indeed, we know that $\phi^i$ is trivial on $G^i_{y,r_i^+}$, which means that it factors through to a character of $G^i_{y,r_i:r_i^+}$. Then, for all $h \in H^i_{y,r_i}$, $\phi^i \circ \iota(h) = \phi^i \circ \iota(hH^i_{y,r_i^+}) = \phi^i(hG^i_{y,r_i^+}) = \phi^i_H(h)$. Hence, we conclude that

$$\psi(X^*|_{\mathfrak{h}^i}(Y)) = \phi^i_H(e_H(Y + \mathfrak{h}^i_{y,r_i^+})).$$

∎

With this last lemma, we now have all the tools to prove the following theorem.

**Theorem 3.3.18.** *The sequence $\vec{\phi}_H$ of quasicharacters of $\vec{\mathbb{H}}$ is such that the ith quasicharacter of the sequence is $H^{i+1}$-generic of depth $r_i$ for all $0 \leqslant i \leqslant d-1$.*

**Proof:** By the previous lemma, we know that the characters $\phi^i_H$ are $H^{i+1}$-generic of depth $r_i$ for all $0 \leqslant i \leqslant d-1$. Our definition of $\vec{\phi}_H$ depends on the depth $\tilde{r}$ of $\phi^d_H$, so there are two cases to consider.

1) If $\tilde{r} > r_{d-1}$, we have that $\vec{\phi}_H = (\phi^0_H, \ldots, \phi^{d-1}_H, \phi^d_H)$, and the conclusion follows by what precedes.

2) If $\tilde{r} \leqslant r_{d-1}$, we have that $\vec{\phi}_H = (\phi^0_H, \ldots, \phi^{d-1}_H\phi^d_H, 1)$. Lemma 3.1.15 tells us that $\phi^{d-1}_H\phi^d_H$ is $H$-generic of depth $r_{d-1}$, and the conclusion follows.

∎

## 3.4 Equivalence Classes of Constructed $H$-data

Given that we have just seen how to produce $H$-data from $G$-data in Section 3.2, one might wonder what can be said about their corresponding equivalence classes. Let us start this section by first defining the notion of *restriction* in relation to the $H$-data produced from a $G$-datum $\Psi$.

**Definition 3.4.1.** *Let $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ be a G-datum, and assume $\rho_H$ decomposes into irreducible representations as $\rho_H = \bigoplus_{\ell \in L} \rho_\ell$ for some finite set L. For all $\ell \in L$, let $\Psi_\ell = (\vec{\mathbb{H}}, y, \vec{\tilde{r}}, \rho_\ell, \vec{\phi}_H)$ as per Theorem 3.3.1. The set* $\mathrm{Res}(\Psi) = \{\Psi_\ell : \ell \in L\}$ *is called the restriction of $\Psi$. Given two G-data $\Psi$ and $\dot{\Psi}$ of the same length, we will say that* $\mathrm{Res}(\Psi) \simeq \mathrm{Res}(\dot{\Psi})$ *if every element of* $\mathrm{Res}(\Psi)$ *is H-equivalent to an element of* $\mathrm{Res}(\dot{\Psi})$ *and vice versa.*

Now, if $\Psi$ and $\dot{\Psi}$ are $G$-equivalent $G$-data, a natural question to ask is whether $\mathrm{Res}(\Psi) \simeq \mathrm{Res}(\dot{\Psi})$. Given that $H$ is normal in $G$, one can talk about the $G$-conjugation of $H$-data and can verify easily from the definitions that $\mathrm{Res}({}^g\Psi) = {}^g\,\mathrm{Res}(\Psi)$ for all $g \in G$. However, when $g \notin H$, we cannot expect to have $\mathrm{Res}(\Psi) \simeq \mathrm{Res}({}^g\Psi)$ since equivalence between restrictions is defined at the level of $H$-equivalence (and therefore $H$-conjugation). Let us illustrate this with an example.

**Example 3.4.2.** Let $G = \mathrm{GL}_2(F)$ and let $\eta$ be a fixed nonsquare in $\mathcal{O}_F^\times$. We consider $\Psi$ to be a toral datum of length one with $\vec{\mathbb{G}} = (\mathbb{T}, \mathbb{G})$, where

$$\mathbb{T}(E) = \left\{ \begin{bmatrix} a & b \\ b\eta & a \end{bmatrix} : a, b \in E \text{ such that } a^2 - b^2\eta \neq 0 \right\}.$$

For $g = \begin{bmatrix} 1 & 0 \\ 0 & \varpi_F \end{bmatrix} \in G$, we have that

$${}^g\mathbb{T}(E) = \left\{ \begin{bmatrix} a & b\varpi_F^{-1} \\ b\eta\varpi_F & a \end{bmatrix} : a, b \in E \text{ such that } a^2 - b^2\eta \neq 0 \right\}.$$

Letting $\mathbb{H} = \mathbb{G}_{\mathrm{der}}$, we have

$$\mathbb{T}_{\mathbb{H}}(E) = \left\{ \begin{bmatrix} a & b \\ b\eta & a \end{bmatrix} : a, b \in E \text{ such that } a^2 - b^2\eta = 1 \right\} \text{ and}$$

$${}^g\mathbb{T}_{\mathbb{H}}(E) = \left\{ \begin{bmatrix} a & b\varpi_F^{-1} \\ b\eta\varpi_F & a \end{bmatrix} : a, b \in E \text{ such that } a^2 - b^2\eta = 1 \right\}.$$

Note that $\mathbb{T}_{\mathbb{H}}$ and ${}^g\mathbb{T}_{\mathbb{H}}$ are maximal non-split tori of $\mathbb{H}$ that are not $H$-conjugate [Nev13, Table 1]. Therefore, there is no element of $H$ that conjugates $(\mathbb{T}_{\mathbb{H}}, \mathbb{H})$ to $({}^g\mathbb{T}_{\mathbb{H}}, \mathbb{H})$, hence $\mathrm{Res}(\Psi) \not\simeq \mathrm{Res}({}^g\Psi)$ by Theorem 3.2.4. ∎

Any other kind of conjugation, elementary transformation or refactorization transforming $\Psi$ into $\dot{\Psi}$ will lead to isomorphic restrictions, as shown by the following proposition.

**Proposition 3.4.3.** *Let $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ and $\dot{\Psi} = (\vec{\dot{\mathbb{G}}}, \dot{y}, \vec{r}, \dot{\rho}, \vec{\dot{\phi}})$ be two $G$-data. Assume one of the three following conditions hold*

1) *$\dot{\Psi}$ is an $H$-conjugate of $\Psi$,*

2) *$\dot{\Psi}$ is an elementary transformation of $\Psi$,*

3) *$\dot{\Psi}$ is a refactorization of $\Psi$,*

*then* $\mathrm{Res}(\Psi) \simeq \mathrm{Res}(\dot{\Psi})$.

**Proof:**

1) It is immediate from the definition that $\mathrm{Res}(\Psi)$ is stable under $H$-conjugation.

2) By definition of elementary transformation (Section 3.2), we have that $\vec{\mathbb{G}} = \dot{\vec{\mathbb{G}}}, \vec{\phi} = \dot{\vec{\phi}}$, $[y] = [\dot{y}]$ and $\rho \simeq \dot{\rho}$. Since $\rho \simeq \dot{\rho}$, their decompositions into irreducible subrepresentations upon restriction to $K_H^0$ are equivalent, that is $\bigoplus_{\ell \in L} \rho_\ell \simeq \bigoplus_{\ell \in \dot{L}} \dot{\rho}_\ell$ and $L = \dot{L}$. Then, for all $i \in L$, $\rho_i \simeq \dot{\rho}_j$ for some $j \in L$, meaning that $\dot{\Psi}_j$ is an elementary transformation of $\Psi_i$. The result follows.

3) By definition of refactorization (Definition 3.2.1), we have that $(\dot{\rho}, \dot{\vec{\phi}})$ satisfies the conditions

   F0) if $\phi_d = 1$ then $\dot{\phi}_d = 1$,

   F1) $\dot{\phi}^i|_{G^i_{y,r^+_{i-1}}} = \phi^i \chi^{i+1}|_{G^i_{y,r^+_{i-1}}}$ for all $i$ where $r_{-1} = 0$ and $\chi^{d+1} = 1$ (in other words $\chi^i|_{G^i_{r^+_{i-1}}} = 1$),

   F2) $\dot{\rho} = \rho \otimes (\chi^0|_{K^0})$,

   where $\chi^i(g) = \prod_{j=i}^{d} \phi^j(g) \dot{\phi}^j(g)^{-1}$ for all $g \in G^i$.

Letting $\bar{\chi}^i(g) = \prod_{j=i}^{d} \phi^j_H(g) \dot{\phi}^j_H(g)^{-1}$ for all $g \in H^i$, one can easily verify that $\bar{\chi}^i = \chi^i|_{H^i}$. We can then take the restrictions of statements F1) and F2) to the corresponding groups in $H$ and obtain conditions of similar nature. From those conditions, it would then follow that for all $i \in L$, there exists $j \in L$ such that $\dot{\Psi}_i$ is $H$-equivalent to $\Psi_j$. ∎

The previous proposition shows that the restriction of data that we described is robust, as it agrees with equivalence classes in a way that we would expect.

## 3.5 Constructing $G$-data from an $H$-datum

We have shown that given a $G$-datum $\Psi$, one can construct various data for $H$, $\mathrm{Res}(\Psi)$, with Theorem 3.3.1. It is a natural question to ask whether this restriction exhausts all possible $H$-data. In other words, one can ask whether an $H$-datum $\Psi_H$ can be lifted to a $G$-datum such that $\Psi_H \in \mathrm{Res}(\Psi)$.

In order to do so, if $\Psi_H = (\vec{\mathbb{H}}, y, \vec{r}, \rho', \vec{\phi_H})$, then we must be able to extend the twisted Levi sequence $\vec{\mathbb{H}}$, the irreducible representation $\rho'$ and the quasicharacters in $\vec{\phi_H}$ to have $\Psi_H \in \mathrm{Res}(\Psi)$ for some $G$-datum $\Psi$. The lifting of the Levi sequence is given by the following lemma.

**Lemma 3.5.1.** *Let $\vec{\mathbb{H}} = (\mathbb{H}^0, \ldots, \mathbb{H}^d)$ be a twisted Levi sequence of $\mathbb{H}$. Then there exists a unique twisted Levi sequence $\vec{\mathbb{G}} = (\mathbb{G}^0, \ldots, \mathbb{G}^d)$ of $\mathbb{G}$ such that $\mathbb{G}^i \cap \mathbb{H} = \mathbb{H}^i$ for all $0 \leqslant i \leqslant d$ and $Z(\mathbb{G}^0)/Z(\mathbb{G})$ is $F$-anisotropic.*

**Proof:** By definition, we have that $\mathbb{H}^i$ is a Levi subgroup of $\mathbb{H}$. It follows from Proposition 2.1.30 that there exists a unique Levi subgroup $\mathbb{G}^i$ of $\mathbb{G}$ such that $\mathbb{H}^i = \mathbb{G}^i \cap \mathbb{H}$. Furthermore, $\mathbb{G}^i$ is an $E$-Levi $F$-subgroup, and is therefore a twisted Levi subgroup. That the quotient $Z(\mathbb{G}^0)/Z(\mathbb{G})$ is $F$-anisotropic follows from Corollary 3.3.10. ∎

The extension of $\rho'$ is given by the following proposition.

**Proposition 3.5.2.** *Let $\rho'$ be an irreducible representation of $K_H^0$ such that $\rho'|_{H^0_{y,0^+}}$ is 1-isotypic and $\pi_H^{-1} = \text{c-Ind}_{K_H^0}^{H^0} \rho'$ is irreducible supercuspidal. Let $\rho$ be an irreducible subrepresentation of $\text{c-Ind}_{K_H^0}^{K^0} \rho'$. Then $\rho$ is an irreducible representation of $K^0$ such that $\rho|_{G^0_{y,0^+}}$ is 1-isotypic, $\pi^{-1} = \text{c-Ind}_{K^0}^{G^0} \rho$ is irreducible supercuspidal, and $\rho'$ is a component of $\rho|_{K_H^0}$.*

We expand on the proof of this proposition in Section 3.5.1.

At first glance, there seems to be an obstacle when trying to extend the quasicharacters in $\vec{\phi_H}$. Indeed, given a sequence of quasicharacters $\vec{\phi_H} = (\phi_H^0, \ldots, \phi_H^d), 0 \leqslant i \leqslant d-1$ that satisfies Axiom D5) for an $H$-datum, there is no guarantee for each $\phi_H^i$, $0 \leqslant i \leqslant d-1$, to be trivial on $[G^i, G^i] \subset H^i$. This is because there can be differences between the groups $[G^i, G^i]$ and $[H^i, H^i]$. When $\phi_H^i, 0 \leqslant i \leqslant d-1$, is not trivial on $[G^i, G^i]$, it is not possible to extend it to a quasicharacter of $G^i$, as quasicharacters of $G^i$ are by definition trivial on $[G^i, G^i]$. However, if we remember the notion of normalized $H$-data (Definition 3.1.18) and recall Lemma 3.2.2, we see that every $H$-datum is $H$-equivalent to a normalized $H$-datum. Furthermore, normalized $H$-data satisfy the condition on the quasicharacters described above that we require, as shown in the next lemma.

**Lemma 3.5.3.** *Let $\mathbb{H} = (\mathbb{G}^0 \cap \mathbb{H}, \ldots, \mathbb{G}^d \cap \mathbb{H})$, where $\vec{\mathbb{G}} = (\mathbb{G}^0, \ldots, \mathbb{G}^d)$ is a twisted Levi sequence of $\mathbb{G}$ (Lemma 3.5.1). Let $\Psi_H = (\vec{\mathbb{H}}, y, \vec{r}, \vec{\phi_H})$, where $\vec{\phi_H} = (\phi_H^0, \ldots, \phi_H^d)$, be a normalized $H$-datum. Then $\phi_H^i|_{[G^i, G^i]} = 1$ for all $0 \leqslant i \leqslant d$.*

**Proof:** Since $\Psi_H$ is assumed normalized, the pullback of $\phi^i_H$ is trivial on $H^i_{\mathrm{sc}}$ for all $0 \leqslant i \leqslant d$. Now, we have that $H^i_{\mathrm{sc}} = \mathbb{H}^i_{\mathrm{sc}}(F)$ contains $[\mathbb{H}^i, \mathbb{H}^i](F) = [\mathbb{G}^i, \mathbb{G}^i](F)$ (Remark 2.1.29). It follows that $\phi^i_H$ is trivial on $[\mathbb{G}^i, \mathbb{G}^i](F) \supset [G^i, G^i]$ for all $0 \leqslant i \leqslant d$. ∎

Now that we know that every $H$-datum will be $H$-equivalent to a datum for which $\phi^i_H|_{[G^i,G^i]} = 1, 0 \leqslant i \leqslant d$, we state the proposition that allows us to extend quasicharacters.

**Proposition 3.5.4.** *Let $\vec{\phi_H} = (\phi^0_H, \ldots, \phi^d_H)$ be the quasicharacter sequence of a normalized $H$-datum. Then for all $0 \leqslant i \leqslant d$, there exists a quasicharacter $\phi^i$ of $G^i$ of depth $r_i$ such that $\phi^i|_{H^i} = \phi^i_H$, with the exception that we set $\phi^d = 1$ if $\phi^d_H = 1$. Furthermore, for $0 \leqslant i \leqslant d-1$, this character $\phi^i$ is $G^{i+1}$-generic of depth $r_i$.*

We expand on the proof of this last proposition in Section 3.5.2. The statement of this last proposition leads us to the main theorem of this section.

**Theorem 3.5.5.** *Let $\Psi_H = (\vec{\mathbb{H}}, y, \vec{r}, \rho', \vec{\phi_H})$ be a normalized $H$-datum, where $\vec{\mathbb{H}} = (\mathbb{H}^0, \ldots, \mathbb{H}^d)$. Then there exists a (possibly multiple) $G$-datum $\Psi$ such that $\Psi_H \in \mathrm{Res}(\Psi)$.*

**Proof:** Lemma 3.5.1 gives us a twisted Levi sequence $\vec{G} = (\mathbb{G}^0, \ldots, \mathbb{G}^d)$ such that $\mathbb{H}^i = \mathbb{G}^i \cap \mathbb{H}$ for all $0 \leqslant i \leqslant d$. Proposition 3.5.4 tells us that any irreducible subrepresentation of $\text{c-Ind}_{K^0_H}^{K^0} \rho'$ satisfies Axiom D4) for a $G$-datum. By Lemma 3.5.3, we can apply Proposition 3.5.2 which allows us to extend $\phi^i_H, 0 \leqslant i \leqslant d-1$, to $G^{i+1}$-generic quasicharacters of depth $r_i$ of $G^i$, and $\phi^d_H$ to a quasicharacter of $G$ of depth $r_d$. ∎

**Corollary 3.5.6.** *Every $H$-datum is $H$-equivalent to an $H$-datum which is contained in* $\mathrm{Res}(\Psi)$ *for some $G$-datum $\Psi$.*

### 3.5.1 Extending the representation $\rho'$

The goal of this subsection is to prove Proposition 3.5.2. Given the level of technical detail required, we split the proof of the proposition into various lemmas.

By definition, the restriction of $\rho'$ to $H^0_{y,0}$ contains a cuspidal representation of $H^0_{y,0:0^+}$, say $\sigma'$. Let $\overline{\sigma'}$ be the representation of $H^0_{y,0}G^0_{y,0^+}/G^0_{y,0^+}$ defined by $\overline{\sigma'} = \sigma' \circ \iota^{-1}$, where $\iota$ is the isomorphism from $H^0_{y,0:0^+}$ to $H^0_{y,0}G^0_{y,0^+}/G^0_{y,0^+}$ (Lemma 2.2.24). Note that the pullbacks of $\overline{\sigma'}$ and $\sigma'$ to $H^0_{y,0}$ are the same.

**Lemma 3.5.7.** *Let $G' = G^0_{y,0:0^+}$ and $H' = H^0_{y,0}G^0_{y,0^+}/G^0_{y,0^+}$ and $\overline{\sigma'}$ be defined as previously. Then $\overline{\sigma} = \text{c-Ind}_{H'}^{G'}\overline{\sigma'}$ decomposes as a sum of cuspidal representations of $G'$.*

**Proof:** Setting $\mathbb{G} = \mathbb{G}^0$ and $\mathbb{H} = \mathbb{H}^0$ in Section 2.2.2.3, we have that $G' = \mathcal{G}_y(\mathfrak{f})$ and $H' = \binom{\mathcal{H}}{\mathcal{G}}_y(\mathfrak{f})$. Since $[\mathcal{G}_y, \mathcal{G}_y] \subset \binom{\mathcal{H}}{\mathcal{G}}_y$ (Lemma 2.2.26), it follows that all irreducible subrepresentations of $\overline{\sigma}$ are cuspidal (Proposition 2.3.23). ∎

**Corollary 3.5.8.** *Let $\sigma'$ be a cuspidal representation of $H^0_{y,0:0^+}$ contained in $\rho'|_{H^0_{y,0}}$. Then $\text{c-Ind}_{H^0_{y,0}}^{G^0_{y,0}}\sigma'$ decomposes as a direct sum of pullbacks of cuspidal representations of $G^0_{y,0:0^+}$.*

**Proof:** We have that induction commutes with pullback, as per shown in Proposition A.1.10. Using this result in combination with the previous lemma completes the proof. ∎

**Lemma 3.5.9.** *Let $\rho$ be an irreducible subrepresentation of $\text{c-Ind}_{K^0_H}^{K^0}\rho'$. Then the restriction of $\rho$ to $G^0_{y,0}$ contains a cuspidal representation of $G^0_{y,0:0^+}$.*

**Proof:** Let us restrict $\text{c-Ind}_{K^0_H}^{K^0}\rho'$ to $G^0_{y,0}$. From the Mackey Decomposition, we obtain

$$\text{Res}^{K^0}_{G^0_{y,0}}\text{c-Ind}^{K^0}_{K^0_H}\rho' = \bigoplus_{c\in C}\text{c-Ind}^{G^0_{y,0}}_{G^0_{y,0}\cap{}^cK^0_H}\text{Res}^{K^0_H}_{G^0_{y,0}\cap{}^cK^0_H}{}^c\rho',$$

where $C$ is a set of coset representatives of $G^0_{y,0}\backslash K^0/K^0_H$. Since $K^0_H = K^0 \cap H$ is normal in $K^0$, we have that ${}^cK^0_H = K^0_H$ for all $c \in C$. This implies that $G^0_{y,0} \cap {}^cK^0_H = G^0_{y,0} \cap K^0 \cap H = H^0_{y,0}$. Therefore,

$$\text{Res}^{K^0}_{G^0_{y,0}}\text{c-Ind}^{K^0}_{K^0_H}\rho' = \bigoplus_{c\in C}\text{c-Ind}^{G^0_{y,0}}_{H^0_{y,0}}\text{Res}^{K^0_H}_{H^0_{y,0}}{}^c\rho'.$$

Recall that $\text{Res}^{K^0_H}_{H^0_{y,0}}\rho' = \bigoplus_{d\in D}{}^d\sigma'$ where $D$ is a subset of a set of coset representatives of $H^0_{[y]}/H^0_{y,0}$ (Lemma 2.3.27). Hence, our Mackey decomposition can be rewritten as

$$\text{Res}^{K^0}_{G^0_{y,0}}\text{c-Ind}^{K^0}_{K^0_H}\rho' = \bigoplus_{\substack{c\in C\\ d\in D}}\text{c-Ind}^{G^0_{y,0}}_{H^0_{y,0}}{}^{cd}\sigma'.$$

Since $\text{c-Ind}^{G^0_{y,0}}_{H^0_{y,0}}{}^{cd}\sigma'$ decomposes as a direct sum of cuspidal representations of $G^0_{y,0:0^+}$ (Corollary 3.5.8), we have that

$$\text{Res}^{K^0}_{G^0_{y,0}}\text{c-Ind}^{K^0}_{K^0_H}\rho' = \bigoplus_{j\in J}\sigma_j,$$

for some cuspidal representations $\sigma_j$ of $G^0_{y,0:0^+}$ and some index set $J$. Hence, given the component $\rho$ of $\text{c-Ind}_{K^0_H}^{K^0} \rho'$, we conclude that $\rho|_{G^0_{y,0}}$ must contain some cuspidal representation of $G^0_{y,0:0^+}$. ∎

Since $\rho|_{G^0_{y,0}}$ contains a cuspidal representation of $G^0_{y,0:0^+}$, we have that $\text{c-Ind}_{K^0}^{G^0} \rho$ is an irreducible depth-zero supercuspidal representation, as per the description of Moy and Prasad which we reviewed in Section 2.3.4. Furthermore, $\rho'$ is a component of $\rho|_{K^0_H}$ by construction, which means we have completed the proof of Proposition 3.5.2.

### 3.5.2 Extending the Quasicharacters $\phi^i_H$

The goal of this subsection is to prove the Proposition 3.5.4. Let $0 \leqslant i \leqslant d$. Due to the level of technical detail required, the proof will be split into various lemmas. We start by extending our quasicharacter $\phi^i_H$ to a quasicharacter of $G^i$ of depth $r_i$.

**Lemma 3.5.10.** *Let $\phi^i_H$ be a quasicharacter of $H^i$ which is trivial on $[G^i, G^i]$. Then any irreducible subrepresentation of* $\text{c-Ind}_{H^i}^{G^i} \phi^i_H$ *is a quasicharacter of $G^i$ that restricts to $\phi^i_H$.*

**Proof:** From the Mackey Decomposition, we have that

$$\text{Res}^{G^i}_{[G^i,G^i]} \text{c-Ind}_{H^i}^{G^i} \phi^i_H = \bigoplus_{c \in C} \text{c-Ind}^{[G^i,G^i]}_{{}^cH^i \cap [G^i,G^i]} \text{Res}^{{}^cH^i}_{{}^cH^i \cap [G^i,G^i]} {}^c\phi^i_H,$$

where $C$ is a set of double coset representatives of $[G^i, G^i]\backslash G^i/H^i$. We can simplify this expression. Indeed, $G^i$ normalizes both $G^i$ and $H$, and therefore $H^i$. Furthermore, we have that $[G^i, G^i] \subset H^i$. Thus,

$$\text{Res}^{G^i}_{[G^i,G^i]} \text{c-Ind}_{H^i}^{G^i} \phi^i_H = \bigoplus_{c \in C} \text{Res}^{H^i}_{[G^i,G^i]} {}^c\phi^i_H = \bigoplus_{c \in C} 1.$$

It follows that every irreducible subrepresentation of $\text{c-Ind}_{H^i}^{G^i} \phi^i_H$ factors through $[G^i, G^i]$. Since $G^i/[G^i, G^i]$ is abelian, all irreducible representations are 1-dimensional, and therefore any irreducible subrepresentation of $\text{c-Ind}_{H^i}^{G^i} \phi^i_H$ is a quasicharacter of $G^i$ that restricts to $\phi^i_H$. ∎

**Lemma 3.5.11.** *Let $\phi^i_H$ be a quasicharacter of $H^i$ which is trivial on $[G^i, G^i]$. Then at least one of the irreducible subrepresentations of* $\text{c-Ind}_{H^i}^{G^i} \phi^i_H$ *has depth $r_i$.*

**Proof:** Since $\phi^i_H$ has depth $r_i$, we know that $\phi^i_H|_{H^i_{y,r_i}} \neq 1$ and $\phi^i_H|_{H^i_{y,r_i+}} = 1$ (Remark 2.3.25). Given an irreducible subrepresentation $\phi$ of $\text{c-Ind}_{H^i}^{G^i} \phi^i_H$, we have that

$\phi$ is a quasicharacter of $G^i$ and $\phi|_{G^i_{y,r_i}} \neq 1$ since $\phi|_{H^i_{y,r_i}} = \phi^i_H|_{H^i_{y,r_i}} \neq 1$ (Lemma 3.5.10). So, any irreducible subrepresentation of c-$\mathrm{Ind}_{H^i}^{G^i} \phi^i_H$ has depth greater than or equal to $r_i$. Using Mackey theory, we have

$$\mathrm{Res}^{G^i}_{G^i_{y,r_i+}} \text{c-}\mathrm{Ind}^{G^i}_{H^i} \phi^i_H = \bigoplus_{c \in C} \text{c-}\mathrm{Ind}^{G^i_{y,r_i+}}_{G^i_{y,r_i+} \cap {}^c H^i} \mathrm{Res}^{{}^c H^i}_{G^i_{y,r_i+} \cap {}^c H^i} {}^c\phi^i_H,$$

where $C$ is the set of double coset representatives of $G^i_{y,r_i+} \backslash G^i / H^i$. Since $H^i$ is normal in $G^i$, $G^i_{y,r_i^+} \cap {}^c H^i = H^i_{y,r_i^+}$ (Proposition 2.2.22). That and the fact that $\phi^i_H|_{H^i_{y,r_i^+}} = 1$ allows us to simplify the Mackey formula to

$$\mathrm{Res}^{G^i}_{G^i_{y,r_i+}} \text{c-}\mathrm{Ind}^{G^i}_{H^i} \phi^i_H = \bigoplus_{c \in C} \text{c-}\mathrm{Ind}^{G^i_{y,r_i+}}_{H^i_{y,r_i+}} 1.$$

Since the trivial representation is clearly a subrepresentation of this last decomposition, we conclude that at least one of the irreducible subrepresentations of c-$\mathrm{Ind}^{G^i}_{H^i} \phi^i_H$ has depth $r_i$. ∎

For all $0 \leqslant i \leqslant d$, let $\mathcal{E}_i$ denote the set of characters of $G^i$ of depth $r_i$ that restrict to $\phi^i_H$. By what precedes, this set is nonempty because we are assuming that the character sequence $\vec{\phi}_H$ is from a normalized datum in Proposition 3.5.4 which implies that $\phi^i_H$ is trivial on $[G^i, G^i]$ for all $0 \leqslant i \leqslant d$ (Lemma 3.5.3).

We will now proceed in various steps to show that every quasicharacter $\phi^i \in \mathcal{E}_i$ is $G^{i+1}$-generic of depth $r_i$, $0 \leqslant i \leqslant d-1$. Given $\phi^i \in \mathcal{E}_i$, the idea is to extend the $H^{i+1}$-generic element of depth $-r_i$ of $(\mathfrak{z}^{i,*}_H)_{-r_i}$ that realizes $\phi^i_H|_{H^i_{y,r_i}}$ to a $G^{i+1}$-generic element of depth $-r_i$ of $\mathfrak{z}^{i,*}_{-r_i}$ that realizes $\phi^i|_{G^i_{y,r_i}}$.

From [AR00, Proposition 3.1], we have the following Lie algebra decomposition: $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}](F)$ is the Lie algebra of $G_{\mathrm{der}}$ and $\mathfrak{z}$ is the $F$-points of the center of $\mathfrak{g}$. As a consequence, $\mathfrak{g}^i = (\mathfrak{g}^i \cap \mathfrak{g}') \oplus \mathfrak{z}$, meaning that every element of $\mathfrak{g}^{i,*}$ (or $\mathfrak{z}^{i,*}$) is viewed as having two parts, one from $\mathfrak{z}^*$ and one from $(\mathfrak{g}^i \cap \mathfrak{g}')^*$ (or $(\mathfrak{z}^i \cap \mathfrak{g}')^*$), where $\mathfrak{z}^i \cap \mathfrak{g}'$ corresponds the $F$-points of the center of $\mathfrak{g}^i \cap \mathfrak{g}'$ (Lemma 3.3.6). We define $(\mathfrak{z}^i \cap \mathfrak{g}')_{r_i}$ and $(\mathfrak{z}^i \cap \mathfrak{g}')^*_{-r_i}$ analogously to Section 3.3.3. The previous discussion is summarized by the following lemma.

**Lemma 3.5.12.** *For all $0 \leqslant i \leqslant d-1$ we have $\mathfrak{z}^i = (\mathfrak{z}^i \cap \mathfrak{g}') \oplus \mathfrak{z}$ and $\mathfrak{g}^i_{y,r_i} = (\mathfrak{g}^i \cap \mathfrak{g}')_{y,r_i} \oplus \mathfrak{z}_{r_i}$. Furthermore, elements of $(\mathfrak{z}^i \cap \mathfrak{g}')^*$ and $\mathfrak{z}^*$ can be extended trivially over $\mathfrak{z}$ and $\mathfrak{z}^i \cap \mathfrak{g}'$, respectively, to elements of $\mathfrak{z}^{i,*}$.*

By the genericity of $\phi^i_H$, we know that $\phi^i_H|_{H^i_{y,r_i}}$ is realized by an $H^{i+1}$-generic element of depth $-r_i$, say $X^* \in (\mathfrak{z}^{i,*}_H)_{-r_i}$. The results from Section 3.3.3 (in which we set $\mathbb{G} = \mathbb{H}, \mathbb{H} = \mathbb{G}_{\mathrm{der}}$ and note that $\mathbb{G}^i \cap \mathbb{G}_{\mathrm{der}} = \mathbb{H}^i \cap \mathbb{G}_{\mathrm{der}}$) tell us that $\phi^i_H|_{G^i \cap G_{\mathrm{der}}}$ is $(G^{i+1} \cap G_{\mathrm{der}})$-generic of depth $r_i$, that it is realized by the $(G^{i+1} \cap G_{\mathrm{der}})$-generic element of depth $-r_i$ $X^*|_{\mathfrak{g}^i\cap\mathfrak{g}'} \in (\mathfrak{z}^i \cap \mathfrak{g}')^*_{-r_i}$ and that $\psi(X^*|_{\mathfrak{g}^i\cap\mathfrak{g}'}(Y)) = \phi^i(e(Y + \mathfrak{g}^i_{y,r_i^+}))$ for all $Y \in (\mathfrak{g}^i \cap \mathfrak{g}')_{y,r_i}$ (Lemma 3.3.17). As a consequence, for all $Y \in (\mathfrak{g}^i \cap \mathfrak{g}')_{y,r_i}, Z \in \mathfrak{z}_{r_i}$,

$$\begin{aligned}\phi(e(Y + Z + \mathfrak{g}^i_{y,r_i^+})) &= \phi(e(Y + \mathfrak{g}^i_{y,r_i^+}))\phi(e(Z + g^i_{y,r_i^+}))\\ &= \psi(X^*|_{\mathfrak{g}^i\cap\mathfrak{g}'}(Y))\phi(e(Z + g^i_{y,r_i^+})).\end{aligned}$$

Therefore, to get that $\phi^i$ is $G^{i+1}$-generic of depth $r_i$, our goal is now to find an element $\bar{X}^* \in \mathfrak{z}^*_{-r_i}$ such that $\phi(e(Z + \mathfrak{g}^i_{y,r_i^+})) = \psi(\bar{X}^*(Z))$ for all $Z \in \mathfrak{z}_{r_i}$ and $\mathcal{X}^* := X^*|_{\mathfrak{g}^i\cap\mathfrak{g}'} + \bar{X}^* \in \mathfrak{z}^{i,*}_{-r_i}$ is $G^{i+1}$-generic of depth $-r_i$. We prove the existence of such an element $\bar{X}^*$ in the following lemma.

**Lemma 3.5.13.** *Set $0 \leqslant i \leqslant d-1$ and let $\phi^i \in \mathcal{E}_i$. Then there exists $\bar{X}^* \in \mathfrak{z}^*_{-r_i}$ such that $\phi^i(e(Z + \mathfrak{g}^i_{y,r_i^+})) = \psi(\bar{X}^*(Z))$ for all $Z \in \mathfrak{z}_{r_i}$, where $e$ is the isomorphism between $\mathfrak{g}^i_{y,r_i:r_i^+}$ and $G^i_{y,r_i:r_i^+}$ from Appendix A.2. Furthermore, $\mathcal{X}^* = X^*|_{\mathfrak{g}^i\cap\mathfrak{g}'} + \bar{X}^* \in \mathfrak{z}^{i,*}_{-r_i}$ is $G^{i+1}$-generic of depth $-r_i$. Note that we view $X^*|_{\mathfrak{g}^i\cap\mathfrak{g}'}$ and $\bar{X}^*$ as elements of $\mathfrak{z}^{i,*}$ via Lemma 3.5.12.*

**Proof:** Since $\phi^i$ is trivial on $G^i_{y,r_i+}$, we have that $\phi^i(e(\cdot + \mathfrak{g}^i_{y,r_i+}))$ is a representation of $\mathfrak{z}_{r_i}$ which factors through $\mathfrak{z}_{r_i+}$. Therefore, $\phi^i(e(\cdot + \mathfrak{g}^i_{y,r_i+}))$ is a representation of $\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+}$. Now, for all $Y^* \in (\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+})^*$, we have that $\psi \circ Y^*$ is a character of $\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+}$. Note that $\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+}$ is a vector space over $\mathfrak{f}$, so elements of $(\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+})^*$ map $\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+}$ into $\mathfrak{f}$. Since $\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+}$ is a finite abelian group, the number of characters is equal to the order of the group.

Now, given $Y_1^*, Y_2^* \in (\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+})^*$, we show that $\psi \circ Y_1^* \neq \psi \circ Y_2^*$ when $Y_1^* \neq Y_2^*$. Indeed, $\psi \circ Y_1^* = \psi \circ Y_2^*$ implies that $(Y_1^* - Y_2^*)(Z) \in \ker\psi$ for all $Z \in \mathfrak{z}_{r_i}/\mathfrak{z}_{r_i^+}$. If we assume that $Y_1^* - Y_2^* \neq 0$, then $(Y_1^* - Y_2^*)(Z') = \alpha \neq 0$ for some $Z' \in \mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+}$ and $\alpha \in \mathfrak{f}$. Take $\beta \in \mathfrak{f}$ such that $\beta \notin \ker\psi$. Such a $\beta$ exists since we know $\psi$ is not trivial on $\mathcal{O}_F$. Then $\beta\alpha^{-1}Z' \in \mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+}$, but $(Y_1^* - Y_2^*)(\beta\alpha^{-1}Z') = \beta \notin \ker\psi$, which is a contradiction with what precedes. Hence, if $\psi \circ Y_1^* = \psi \circ Y_2^*$, we must have $Y_1^* = Y_2^*$.

We conclude that

$$\{\psi \circ Y^* : Y^* \in (\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+})^*\}$$

is the set of all characters of $\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+}$. Therefore, there exists $\bar{X}^* \in (\mathfrak{z}_{r_i}/\mathfrak{z}_{r_i+})^*$ such that $\psi \circ \bar{X}^* = \phi^i \circ e|_{\mathfrak{z}_{r_i}}$. This element $\bar{X}^*$ can be pulled back to an element in $(\mathfrak{z}_{r_i})^*$. We have that $\mathfrak{z}_{r_i} = \mathfrak{z}_{\bar{r}_i+}$ for some $\bar{r}_i \leqslant r_i$ (Section 2.2.2) meaning that $(\mathfrak{z}_{r_i})^* = \mathfrak{z}^*_{-\bar{r}_i}$ [Yu01, Section 5]. In particular, $\mathfrak{z}^*_{-\bar{r}_i} \subset \mathfrak{z}^*_{-r_i}$ and $\bar{X}^* \in \mathfrak{z}^*_{-r_i}$. Furthermore, for all $Z \in \mathfrak{z}_{r_i}$, we have

$$\Psi(\bar{X}^*(Z)) = \phi^i(e(Z + \mathfrak{g}^i_{y,r_i+})).$$

Finally, we show that $\mathcal{X}^* = X^*|_{\mathfrak{g}^i\cap\mathfrak{g}'} + \bar{X}^* \in \mathfrak{z}^{i,*}_{-r_i}$ is $G^{i+1}$-generic of depth $-r_i$. For all $Z \in \mathfrak{z}^i_{r_i^+}$, we have that $Z = Z_1 + Z_2$ for some $Z_1 \in (\mathfrak{z}^i \cap \mathfrak{g}')_{r_i^+}, Z_2 \in \mathfrak{z}_{r_i^+}$ by

Lemma 3.5.12. It follows that $\mathcal{X}^*(Z) = X^*|_{\mathfrak{g}^i \cap \mathfrak{g}'}(Z_1) + \bar{X}^*(Z_2)$. This result lies in $\mathfrak{p}_F$, as $X^*|_{\mathfrak{g}^i \cap \mathfrak{g}'} \in (\mathfrak{z}^i \cap \mathfrak{g}')^*_{-r_i}$ and $\bar{X}^* \in \mathfrak{z}^*_{-r_i}$. The genericity of $\mathcal{X}^*$ follows from the genericity of $X^*|_{\mathfrak{g}^i \cap \mathfrak{g}'}$ since $H_a \in \mathfrak{g}'$ for all $a \in \Phi(\mathbb{G}^{i+1}, \mathbb{T}) \backslash \Phi(\mathbb{G}^i, \mathbb{T})$. ∎

By construction, the $G^{i+1}$-generic element of depth $-r_i$ $\mathcal{X}^*$ provided by the previous lemma realizes $\phi^i|_{G^i_{y,r_i}}$ and therefore $\phi^i$ is $G^{i+1}$-generic of depth $r_i$. We have thus proven the following proposition, which in turn completes the proof of Proposition 3.5.4.

**Proposition 3.5.14.** *Set $0 \leqslant i \leqslant d-1$ and let $\phi^i \in \mathcal{E}_i$. Then $\phi^i$ is a $G^{i+1}$-generic quasicharacter of $G^i$ of depth $r_i$.*

## 3.6 Equivalence Classes of Constructed $G$-data

Let $\mathbb{G}$ and $\mathbb{H}$ be as before in this chapter. We just saw in Section 3.5 how to go backwards from an $H$-datum to $G$-data. Similarly to Section 3.4, we can ask the following question. If $\Psi_H$ and $\dot{\Psi}_H$ are normalized $H$-data which are $H$-equivalent, are the $G$-data produced from $\Psi_H$ and $\dot{\Psi}_H$ via Theorem 3.5.5 $G$-equivalent? This is what this section will aim to answer. Before we proceed to exploring this question in more detail, let us first define one notion in relation to the $G$-data constructed from a normalized $H$-datum $\Psi_H$.

**Definition 3.6.1.** *Let $\Psi_H = (\vec{\mathbb{H}}, y, \vec{r}, \rho', \vec{\phi_H})$ be a normalized $H$-datum. Let $\mathcal{E}_{\rho'}$ denote the set of irreducible subrepresentations of $\text{c-Ind}_{K_H^0}^{K^0} \rho'$ and $\mathcal{E}_i$ denote the set of irreducible subrepresentations of $\text{c-Ind}_{H^i}^{G^i} \phi_H^i$ with depth $r_i, 0 \leqslant i \leqslant d$. Given $\rho \in \mathcal{E}_{\rho'}, \phi^i \in \mathcal{E}_i, 0 \leqslant i \leqslant d$, we have that $\Psi = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ with $\vec{\phi} = (\phi^0, \ldots, \phi^d)$ is a $G$-datum such that $\Psi_H \in \mathrm{Res}(\Psi)$ as per Theorem 3.5.5. The set $\mathrm{Ext}(\Psi_H) = \{(\vec{G}, y, \vec{r}, \rho, \vec{\phi}) : \rho \in \mathcal{E}_{\rho'}, \phi^i \in \mathcal{E}_i, 0 \leqslant i \leqslant d\}$ is called the extension of $\Psi_H$. Given two normalized $H$-data $\Psi_H$ and $\dot{\Psi}_H$ of the same length, we will say that $\mathrm{Ext}(\Psi_H) \simeq \mathrm{Ext}(\dot{\Psi}_H)$ if every element of $\mathrm{Ext}(\Psi_H)$ is $G$-equivalent to an element of $\mathrm{Ext}(\dot{\Psi}_H)$ and vice versa.*

**Remark 3.6.2.** Note that the notion of extension is only defined for a normalized $H$-data. However, given an $H$-datum $\Psi_H$, we may as well assume it is normalized as it is $H$-equivalent to a normalized $H$-datum (Lemma 3.2.2), say $\dot{\Psi}_H$.

We expect $H$-equivalent normalized $H$-data to produce isomorphic extensions since $H \subset G$. But here, we can allow for conjugation outside of $H$ on our $H$-data since the equivalence of the extensions is at the level of $G$. The following proposition is analogous to Proposition 3.4.3, but for extensions of $H$-data.

**Proposition 3.6.3.** *Let* $\Psi_H = (\vec{\mathbb{H}}, y, \vec{r}, \rho', \vec{\phi}_H)$ *and* $\dot{\Psi}_H = (\dot{\vec{\mathbb{H}}}, \dot{y}, \vec{r}, \dot{\rho}', \dot{\vec{\phi}}_H)$ *be two normalized* $H$*-data. Assume one of the three following conditions hold*

*1)* $\dot{\Psi}_H$ *is a* $G$*-conjugate of* $\Psi_H$*,*

*2)* $\dot{\Psi}_H$ *is an elementary transformation of* $\Psi_H$*,*

*3)* $\dot{\Psi}_H$ *is a refactorization of* $\Psi_H$*,*

*then* $\mathrm{Ext}(\Psi_H) \simeq \mathrm{Ext}(\dot{\Psi}_H)$.

**Proof:**

1) It is immediate from the definition that $\mathrm{Ext}(\Psi_H)$ is stable under $G$-conjugation.

2) By definition of elementary transformation (Section 3.2), we have that $\vec{\mathbb{H}} = \dot{\vec{\mathbb{H}}}$, $[y] = [\dot{y}]$ and $\rho' \simeq \dot{\rho}'$. It follows that $\text{c-Ind}_{K_H^0}^{K^0} \rho' \simeq \text{c-Ind}_{K_H^0}^{K^0} \dot{\rho}'$, and therefore every irreducible subrepresentation $\rho$ of $\text{c-Ind}_{K_H^0}^{K^0} \rho'$ is isomorphic to an irreducible subrepresentation of $\text{c-Ind}_{K_H^0}^{K^0} \dot{\rho}'$, say $\dot{\rho}$. Therefore, if $\vec{\phi}$ denotes a sequence of quasicharacters that extend those of $\vec{\phi}_H$, we have that $(\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi}) \in \mathrm{Ext}(\Psi_H)$ is an elementary transformation of $(\vec{\mathbb{G}}, \dot{y}, \vec{r}, \dot{\rho}, \vec{\phi}) \in \mathrm{Ext}(\dot{\Psi}_H)$.

3) By definition of refactorization (Definition 3.2.1), we have that $(\dot{\rho}', \dot{\vec{\phi}}_H)$ satisfies the conditions

   F0) if $\phi_H^d = 1$ then $\dot{\phi}_H^d = 1$,

   F1) $\phi_H^i|_{H^i_{y,r_{i-1}^+}} = \dot{\phi}_H^i \chi^{i+1}|_{H^i_{y,r_{i-1}^+}}$ for all $i$ where $r_{-1} = 0$ and $\chi^{d+1} = 1$ (in other words $\chi^i|_{H^i_{y,r_{i-1}^+}} = 1$),

   F2) $\rho' = \dot{\rho}' \otimes (\chi^0|_{K_H^0})$,

   where $\chi^i(h) = \prod_{j=i}^{d} \dot{\phi}_H^j(h) \phi_H^j(h)^{-1}$ for all $h \in H^i$.

Let $\vec{\phi}$ be a sequence of quasicharacters that extends $\vec{\phi}_H$ and let $\rho$ be an extension of $\rho'$. Recall that we set $\phi^d = 1$ if $\phi_H^d = 1$. We must show the existence of an extension $\dot{\vec{\phi}}$ of $\dot{\vec{\phi}}_H$ and an extension $\dot{\rho}$ of $\dot{\rho}'$ so that $(\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ is $G$-equivalent to $(\vec{\mathbb{G}}, \dot{y}, \vec{r}, \dot{\rho}, \dot{\vec{\phi}})$. Let us first find candidates for $\dot{\phi}^{d-i}, 0 \leqslant i \leqslant d$ that will satisfy F0) and F1). To do so, we proceed by induction on $i$.

**i=0:** If $\phi^d = 1$, then we set $\dot{\phi}^d = 1$ and we are done. If $\phi^d \neq 1$, note that the depth of $\chi^d$ is at most $r_{d-1}$ by F1) above. Extend $\chi^d$ to a character $Y^d$ of $G$ of the same depth. Note that we can do this because $\chi^d$ is trivial on $[G, G]$, so the existence of such an extension follows from Lemmas 3.5.10 and 3.5.11. Set $\dot{\phi}^d := Y^d \phi^d$. Then, for all $h \in H^d$

$$\begin{aligned}\dot{\phi}^d|_{H^d}(h) &= Y^d \phi^d|_{H^d}(h)\\ &= \dot{\phi}^d_H(h)\phi^d_H(h)^{-1}\phi^d_H(h)\\ &= \dot{\phi}^d_H(h).\end{aligned}$$

So, $\dot{\phi}^d$ is a character of $G^d$ that restricts to $\dot{\phi}^d_H$. Since $Y^d$ is of depth at most $r_{d-1}$, we have $\dot{\phi}^d|_{G_{y,r_d}} = \phi^d|_{G_{y,r_d}}$. Thus, $\dot{\phi}^d$ is of depth $r_d$ and in particular $\phi^d|_{G_{y,r^+_{d-1}}} = \dot{\phi}^d|_{G_{y,r^+_{d-1}}}$.

**i > 0:** Assume that we have $\dot{\phi}^{d-k}$ for $0 \leqslant k < i$ that satisfy F1) stated in terms of $G$ and let's proceed to finding $\dot{\phi}^{d-i}$. Set $\overline{\chi}^{d-i+1}$ and $(\overline{\chi}^{d-i+1})^{-1}$ to be the characters of $G^{d-i+1}$ defined by $\overline{\chi}^{d-i+1}(g) = \prod_{j=d-i+1}^{d} \dot{\phi}^j(g)\phi^j(g)^{-1}$ and $(\overline{\chi}^{d-i+1})^{-1}(g) = \overline{\chi}^{d-i+1}(g^{-1})$ for all $g \in G^{d-i+1}$, respectively. We proceed similarly to above and extend $\chi^{d-i}$ to a character $Y^{d-i}$ of $G^{d-i}$ of depth at most $r_{d-i-1}$. Set $\dot{\phi}^{d-i} := Y^{d-i}\phi^{d-i}(\overline{\chi}^{d-i+1})^{-1}$. Then, for all $h \in H^{d-i-1}$ we have

$$(\overline{\chi}^{d-i+1})^{-1}(h) = \prod_{j=d-i+1}^{d} \dot{\phi}^j_H(h)^{-1}\phi^j_H(h)$$

and

$$Y^{d-i}(h) = \chi^{d-i}(h) = \dot{\phi}^{d-i}_H(h)\phi^{d-i}_H(h)^{-1} \prod_{j=d-i+1}^{d} \dot{\phi}^j_H(h)\phi^j_H(h)^{-1},$$

and therefore $\dot{\phi}^{d-i}|_{H^{d-i}}(h) = \dot{\phi}^{d-i}_H(h)$. So, $\dot{\phi}^{d-i}$ is a character of $G^{d-i}$ that restricts to $\dot{\phi}^{d-i}_H$. Since $\dot{\phi}^{d-i}_H$ is of depth $r_{d-i}$, $\dot{\phi}^{d-i}$ is nontrivial on $G^{d-i}_{y,r_{d-i}}$. Furthermore, since $Y^{d-i}$ is of depth at most $r_{d-i-1}$ and $\overline{\chi}^{d-i+1}$ is of depth at most $r_{d-i}$, we have that $\dot{\phi}^{d-i}|_{G^{d-i}_{y,r^+_{d-i}}} = 1$. Thus, $\dot{\phi}^{d-i}$ is an element of $\mathcal{E}_{d-i}$, and is therefore a $G^{d-i+1}$-generic quasicharacter of depth $r_{d-i}$ by Proposition 3.5.14. Finally, we see that $\dot{\phi}^{d-i}\overline{\chi}^{d-i+1}|_{G^{d-i}_{y,r^+_{d-i-1}}} = \phi^{d-i}|_{G^{d-i}_{y,r^+_{d-i-1}}}$.

Thus, we are able to find a sequence of quasicharacters $\vec{\dot{\phi}}$ that extends $\vec{\dot{\phi}}_H$ and satisfies F1) stated in terms of $G$.

From F2), we have that $\text{c-Ind}_{K_H^0}^{K^0} \rho' = \text{c-Ind}_{K_H^0}^{K^0} \left(\dot{\rho}' \otimes \left(\chi^0|_{K_H^0}\right)\right)$. Using the fact that $\chi^0|_{K_H^0} = \overline{\chi}^0|_{K_H^0}$ and Proposition A.1.11, we obtain $\text{c-Ind}_{K_H^0}^{K^0} \rho' = \text{c-Ind}_{K_H^0}^{K^0} \dot{\rho}' \otimes (\overline{\chi}^0|_{K^0})$. So, given an irreducible subrepresentation $\rho$ of $\text{c-Ind}_{K_H^0}^{K^0} \rho'$, it is isomorphic to $\dot{\rho} \otimes (\overline{\chi}^0|_{K^0})$ for some $\dot{\rho} \in \text{c-Ind}_{K_H^0}^{K^0} \dot{\rho}'$. Hence, the data $(\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi}) \in \text{Ext}(\Psi_H)$ and $(\vec{\mathbb{G}}, \dot{y}, \vec{r}, \dot{\rho}, \dot{\vec{\phi}}) \in \text{Ext}(\dot{\Psi}_H)$ are $G$-equivalent. ∎

The previous proposition shows that the extension of data that we described is robust, as it agrees with equivalence classes.

# Chapter 4

# Restricting Supercuspidal Representations

In this chapter $\mathbb{G}$ still denotes a reductive group defined over a $p$-adic field $F$, $G = \mathbb{G}(F)$, and $\mathbb{H}$ is a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\text{der}}$. We also assume that $\mathbb{G}$ splits over a tamely ramified extension $E$ of $F$ and that $p$ does not divide the order of the Weyl group of $\mathbb{G}$. In Chapter 3, we showed how to construct various $H$-data from a $G$-datum $\Psi$. Now it's time to link the data with their corresponding supercuspidal representations.

Recall that our initial question is to study the restriction to $H$ of an irreducible supercuspidal representation $\pi_G(\Psi)$. Since each component of this restriction is an irreducible supercuspidal representation of $H$, it can be constructed from some $H$-datum, as we had illustrated in Figure 3.1. If we think of the supercuspidal $\pi_G(\Psi)$ as being our piece of IKEA furniture as before, the assembly pieces required to construct the different components of $\pi_G(\Psi)$ ought to have been included in the assembly pieces of the representation itself. For this reason, for all $\Psi_H \in \mathrm{Res}(\Psi)$, we expect to see $\pi_H(\Psi_H)$ appear as a component of $\pi_G(\Psi)|_H$ (Figure 4.1). However, we do not expect the set $\mathrm{Res}(\Psi)$ to exhaust the $H$-data that construct the components of the restriction. Indeed, this is because $\mathrm{Res}(\Psi)$ is not stable under $G$-conjugacy, as discussed in Section 3.4, yet $\pi_G(\Psi) \simeq \pi_G({}^g\Psi)$ for all $g \in G$. For this reason, we also expect to see the $H$-data from $\mathrm{Res}({}^g\Psi)$ to give rise to components of $\pi_G(\Psi)|_H$.

The main result that we will prove in this chapter is that $\pi_G(\Psi)|_H = \bigoplus_{t \in C} \left( \bigoplus_{\ell \in L} \pi_H({}^t\Psi_\ell) \right)$ (Theorem 4.4.1), where $C$ is a set of coset representatives of $H\backslash G/K$ and $K$ is some specific open subgroup which is compact modulo the center of $G$.

In order to study the restriction of the representation $\pi_G(\Psi)$ for an arbitrary $G$-datum $\Psi$, we need to go through the steps of the J.K Yu Construction. We start this chapter by providing a summary of this construction in Section 4.1. In subsequent

$$
\begin{tikzcd}
 & \pi_G(\Psi) \arrow[d, dash, dashed] & & \\
 & \left(\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi}\right) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
 \left(\vec{\mathbb{H}}, y, \vec{r}, \rho_{\ell_1}, \vec{\phi_H}\right) \arrow[d, dash, dashed] & \left(\vec{\mathbb{H}}, y, \vec{r}, \rho_{\ell_2}, \vec{\phi_H}\right) \arrow[d, dash, dashed] & \cdots & \left(\vec{\mathbb{H}}, y, \vec{r}, \rho_{\ell_{|L|}}, \vec{\phi_H}\right) \arrow[d, dash, dashed] \\
 \pi_H(\Psi_{\ell_1}) & \pi_H(\Psi_{\ell_2}) & \cdots & \pi_H(\Psi_{\ell_{|L|}})
\end{tikzcd}
$$

Figure 4.1: Restriction of the $G$-datum $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ and corresponding supercuspidal representations.

sections of this Chapter (Section 4.2, 4.3), we will explore each step of the J.K. Yu Construction into more depth and see that restriction commutes in a sense with all those steps, with the restriction of $\pi_G(\Psi)$ finally being computed in Section 4.4. Having the description of this restriction will allow us to describe the irreducible supercuspidal subrepresentations of $\text{c-Ind}_H^G \pi_H(\Psi_H)$ for an $H$-datum $\Psi_H$ via Frobenius Reciprocity in Section 4.5.

## 4.1 Summary of the J.K. Yu Construction

Recall that the J.K. Yu Construction produces a supercuspidal representation $\pi_G(\Psi)$ from a $G$-datum $\Psi = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$. In order to obtain $\pi_G(\Psi)$, Yu constructs an open subgroup, $K^d$, which is compact modulo the center of $G$, and a representation $\kappa_G(\Psi)$ of $K^d$. The construction of the representation $\kappa_G(\Psi)$ uses $\rho$ and $\vec{\phi}$, and is such that $\text{c-Ind}_{K^d}^G \kappa_G(\Psi)$ is irreducible. The result of this induction is then supercuspidal by Mautner's Theorem, and corresponds to our representation $\pi_G(\Psi)$. Furthermore $\pi_G(\Psi)$ is of depth $r_d$.

**Remark 4.1.1.** Fintzen recently mentioned in [Fin19] that the original proof from [Yu01] that shows $\pi_G(\Psi)$ is a supercuspidal representation was referring to a misprinted statement. Fintzen provides a proof that $\pi_G(\Psi)$ is in fact irreducible and supercuspidal in [Fin19, Theorem 3.1] using alternative methods, confirming that the J.K. Yu Construction is in fact valid.

We give a brief summary for the construction of $\kappa_G(\Psi)$. First, Yu defines the subgroups

$$\begin{aligned}
K^0 &= G^0_{[y]}, \\
K^0_+ &= G^0_{y,0^+}, \\
K^{i+1} &= K^0 G^1_{y,s_0} \cdots G^{i+1}_{y,s_i}, \\
K^{i+1}_+ &= K^0_+ G^1_{y,s_0^+} \cdots G^{i+1}_{y,s_i^+}
\end{aligned}$$

for all $0 \leqslant i \leqslant d-1$, where $s_i = r_i/2$. For all $0 \leqslant i \leqslant d$, $K^i$ is an open subgroup of $G$ and is compact modulo the center of $G^i$. We view $\phi^i$ as a character of $K^i$ by restriction. As a first step, Yu extends the character $\phi^i$ to a (possibly higher-dimensional) representation $\phi^{i\prime}$ of $K^{i+1}$ for all $0 \leqslant i \leqslant d-1$. This extension is nontrivial and requires the theory of Heisenberg groups and Weil representations in most cases. We will give a summary for the construction of this extension in Section 4.2.1. The representations $\phi^{i\prime}, 0 \leqslant i \leqslant d-2$, are then extended further to representations of $K^d$ by a process called inflation, which we describe in Section 4.3.1. The resulting representations are denoted by $\kappa^i, 0 \leqslant i \leqslant d-2$. For consistency of notation, we let $\kappa^{d-1} = \phi^{d-1\prime}$ and $\kappa^d = \phi^d$. The representation $\rho$ can also be inflated to $K^d$, and we denote its inflation by $\kappa^{-1}$, treating $\rho$ as the element of index $-1$ in the sequence $(\rho, \vec{\phi})$. We then have $\kappa_G(\Psi) = \kappa^{-1} \otimes \kappa^0 \otimes \cdots \otimes \kappa^d$. Figure 4.2 below illustrates this construction of $\kappa_G(\Psi)$. When $\Psi = (\mathbb{G}, y, \rho, 1)$, we see that $\kappa_G(\Psi) = \rho$ and therefore $\pi_G(\Psi) = \text{c-Ind}_{G_{[y]}}^G \rho$ is a depth-zero supercuspidal representation (as previously mentioned in Remark 3.1.17).

$$\begin{tikzcd}
 & (\phi^0, K^0) \arrow[d, "\text{extend}"] & \cdots & \left(\phi^{d-2}, K^{d-2}\right) \arrow[d, "\text{extend}"] & \left(\phi^{d-1}, K^{d-1}\right) \arrow[d, "\text{extend}"] & \left(\phi^d, K^d\right) \arrow[d, "="] \\
(\rho, K^0) \arrow[d, "\text{inflate}"] & \left(\phi^{0\prime}, K^1\right) \arrow[d, "\text{inflate}"] & \cdots & \left(\phi^{d-2\prime}, K^{d-1}\right) \arrow[d, "\text{inflate}"] & \left(\phi^{d-1\prime}, K^d\right) \arrow[d, "="] & \left(\phi^{d\prime}, K^d\right) \arrow[d, "="] \\
\left(\kappa^{-1}, K^d\right) & \left(\kappa^0, K^d\right) & \cdots & \left(\kappa^{d-2}, K^d\right) & \left(\kappa^{d-1}, K^d\right) & \left(\kappa^d, K^d\right)
\end{tikzcd}$$

$$\kappa_G(\Psi) = \kappa^{-1} \otimes \kappa^0 \otimes \kappa^1 \otimes \kappa^2 \cdots \otimes \kappa^{d-2} \otimes \kappa^{d-1} \otimes \kappa^d$$

Figure 4.2: Summary for the construction of $\kappa_G(\Psi)$ in the J.K. Yu Construction.

In Remark 3.3.2, we mentioned that the last quasicharacter of $\vec{\phi}$ could be assumed trivial without loss of generality and simplify the application of Theorem 3.3.1. Now that we described the J.K. Yu Construction, we can fully justify this remark with the following lemma.

**Lemma 4.1.2.** *Let* $\Psi = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ *be a* $G$*-datum, where* $\vec{\phi} = (\phi^0, \ldots, \phi^{d-1}, \phi^d)$. *Set* $\Psi' = (\vec{G}, y, \vec{r}, \rho, \vec{\phi'})$, *where* $\vec{\phi'} = (\phi^0, \ldots, \phi^{d-1}, 1)$. *Then* $\pi_G(\Psi) \simeq \pi_G(\Psi') \otimes \phi^d$.

**Proof:** We can see from Figure 4.2 that $\kappa_G(\Psi) = \kappa_G(\Psi') \otimes \phi^d$. It follows that

$$\pi_G(\Psi) = \text{c-Ind}_{K^d}^G \kappa_G(\Psi) = \text{c-Ind}_{K^d}^G \left(\kappa_G(\Psi') \otimes \phi^d|_{K^d}\right).$$

Proposition A.1.11 allows us to take out the character $\phi^d$ from the induction, so we obtain

$$\pi_G(\Psi) \simeq \text{c-Ind}_{K^d}^G \kappa_G(\Psi') \otimes \phi^d = \pi_G(\Psi') \otimes \phi^d.$$

∎

Now, let $\Psi = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ be a $G$-datum and let us study $\pi_G(\Psi)|_H$. Recall that $\rho_H = \rho|_{K_H^0}$ decomposes as a finite sum of irreducible representations (Lemma 3.3.11), and that we wrote this decomposition as $\rho_H = \bigoplus_{\ell \in L} \rho_\ell$ for some finite set $L$. Theorem 3.3.1 then produced a family of $H$-data, $\text{Res}(\Psi) = \{\Psi_\ell : \ell \in L\}$. For each $\ell \in L$, we establish analogous notation for the construction of the supercuspidal representation $\pi_H(\Psi_\ell)$ of $H$. We let

$$\begin{aligned} K_H^0 &= H_{[y]}^0, \\ K_{H+}^0 &= H_{y,0^+}^0, \\ K_H^{i+1} &= K_H^0 H_{y,s_0}^1 \cdots H_{y,s_i}^{i+1}, \\ K_{H+}^{i+1} &= K_{H+}^0 H_{y,s_0^+}^1 \cdots H_{y,s_i^+}^{i+1} \end{aligned}$$

for all $0 \leqslant i \leqslant d-1$.

**Proposition 4.1.3.** *For all* $-1 \leqslant i \leqslant d-1$, $K_H^{i+1} = K^{i+1} \cap H$.

**Proof:** We have already established that $K_H^0 = K^0 \cap H$ in Section 3.3.2. Now, let $0 \leqslant i \leqslant d-1$. We start by showing that

$$K^{i+1} = K^0 H_{y,s_0}^1 \cdots H_{y,s_i}^{i+1}.$$

Since $H_{y,s_i}^{i+1} \subset G_{y,s_i}^{i+1}$ for all $0 \leqslant i \leqslant d-1$, we have $K^0 H_{y,s_0}^1 \cdots H_{y,s_i}^{i+1} \subset K^{i+1}$. For the converse, let $Z^i = Z(\mathbb{G}^i)^\circ(F)$ denote the $F$-points of the identity component of the center of $\mathbb{G}^i$ and let $\overline{G^i} = [\mathbb{G}^i, \mathbb{G}^i](F) \subset H^i$. Then $Z_r^i \overline{G^i}_{y,r} = G_{y,r}^i$ for all $r > 0$ [DR09, Lemma B.7.2]. Applying this result, we obtain

$$K^{i+1} = K^0 G_{y,s_0}^1 \cdots G_{y,s_i}^{i+1} = K^0 Z_{s_0}^1 \overline{G^1}_{y,s_0} \cdots Z_{s_i}^{i+1} \overline{G^{i+1}}_{y,s_i}.$$

We note that $Z^{i+1}_{s_i} \subset Z(\mathbb{G}^{i+1})(F) \subset Z(\mathbb{G}^0)(F)$ for all $0 \leqslant i \leqslant d-1$ (Lemma 3.1.2). Furthermore, $Z(\mathbb{G}^0)(F) \subset K^0$ as $K^0 = N_{G^0}(G^0_{y,0})$ (Section 3.1). It follows that

$$K^{i+1} = K^0 \overline{G^1}_{y,s_0} \cdots \overline{G^{i+1}}_{y,s_i} \subset K^0 H^1_{y,s_0} \cdots H^{i+1}_{y,s_i},$$

and therefore

$$K^{i+1} = K^0 H^1_{y,s_0} \cdots H^{i+1}_{y,s_i}.$$

Now, we clearly have $K^{i+1}_H \subset K^{i+1} \cap H$. Conversely, given an element $k'$ of $K^{i+1} \cap H$, it is in particular an element of $K^{i+1}$ and is thus in the form $k' = kh$ for some $k \in K^0, h \in H^1_{y,s_0} \cdots H^{i+1}_{y,s_i} \subset H$ by what precedes. It follows that $k = k'h^{-1} \in K^0 \cap H = K^0_H$. Hence we conclude that $k' \in K^{i+1}_H$. ∎

Analogously to Figure 4.2, we denote by ${\phi^i_H}'$ the extension of $\phi^i_H$ to $K^{i+1}_H$ for all $0 \leqslant i \leqslant d-1$, and by $\kappa^i_H$ the inflation of ${\phi^i_H}'$ to $K^d_H$ for all $0 \leqslant i \leqslant d-2$. We denote the inflation to $K^d_H$ of $\rho_\ell$ by $\kappa_\ell^{-1}$. Thus, in the case where the depth $\tilde{r}$ of $\phi^d_H$ is greater than $r_{d-1}$, we have the following diagram summarizing the J.K. Yu Construction.

$$\begin{tikzcd}
 & (\phi^0_H, K^0_H) \arrow[d, "\text{extend}"] & \cdots & \left(\phi^{d-2}_H, K^{d-2}_H\right) \arrow[d, "\text{extend}"] & \left(\phi^{d-1}_H, K^{d-1}_H\right) \arrow[d, "\text{extend}"] & \left(\phi^d_H, K^d_H\right) \arrow[d, "="] \\
(\rho_\ell, K^0_H) \arrow[d, "\text{inflate}"] & \left({\phi^0_H}', K^1_H\right) \arrow[d, "\text{inflate}"] & \cdots & \left(\phi^{d-2\prime}_H, K^{d-1}_H\right) \arrow[d, "\text{inflate}"] & \left(\phi^{d-1\prime}_H, K^d_H\right) \arrow[d, "="] & \left({\phi^d_H}', K^d_H\right) \arrow[d, "="] \\
\left(\kappa_\ell^{-1}, K^d_H\right) & \left(\kappa^0_H, K^d_H\right) & \cdots & \left(\kappa^{d-2}_H, K^d_H\right) & \left(\kappa^{d-1}_H, K^d_H\right) & \left(\kappa^d_H, K^d_H\right)
\end{tikzcd}$$

$$\kappa_H(\Psi_\ell) = \kappa_\ell^{-1} \otimes \kappa^0_H \otimes \kappa^1_H \otimes \cdots \otimes \kappa^{d-2}_H \otimes \kappa^{d-1}_H \otimes \kappa^d_H$$

Figure 4.3: Summary for the construction of $\kappa_H(\Psi_\ell)$ in the J.K. Yu Construction in the case where $\tilde{r} > r_{d-1}$.

In the case where $\tilde{r} \leqslant r_{d-1}$, then $(\phi^{d-1}_H \phi^d_H, 1)$ are the last two characters of $\vec{\phi}_H$, and therefore $\kappa_H(\Psi_\ell) = \kappa_\ell^{-1} \otimes \kappa^0_H \otimes \kappa^1_H \otimes \cdots \otimes \kappa^{d-2}_H \otimes \kappa^{d-1,d}_H \otimes 1$, where $\kappa^{d-1,d}_H = (\phi^{d-1}_H \phi^d_H)'$ is the extension of $\phi^{d-1}_H \phi^d_H$ to $K^d_H$.

In order to properly understand the restriction $\pi_G(\Psi)|_H$, we must understand the extension and inflation processes from the J.K. Yu Construction in more detail. Since the characters $\phi^i_H, 0 \leqslant i \leqslant d-1$, are simply restricted from the characters $\phi^i, 0 \leqslant i \leqslant d-1$, it is natural to expect that we have ${\phi^i_H}' = {\phi^i}'|_{K^{i+1}_H}$ and $\kappa^i_H = \kappa^i|_{K^d_H}$ for all $0 \leqslant i \leqslant d-1$. Indeed, Section 4.2 is dedicated to showing that ${\phi^i_H}' = {\phi^i}'|_{K^{i+1}_H}$ for all $0 \leqslant i \leqslant d-1$, whereas Section 4.3 shows the analogous result for the representations

$\kappa^i$, $0 \leqslant i \leqslant d-1$. This means that the restriction to $H$ of $\pi_G(\Psi)$ commutes in a sense with the various steps of the J.K. Yu Construction, allowing us to connect the diagrams from Figures 4.2 and 4.3, as illustrated in Figure 4.4.

$$\begin{tikzcd}
(\rho, K^0) \arrow[r, "\text{restrict}"] \arrow[d, "\text{inflate}"'] \arrow[dr, phantom, "\circlearrowleft"] & \left(\bigoplus\limits_{\ell\in L} \rho_\ell, K_H^0\right) \arrow[d, "\text{inflate}"] \\
\left(\kappa^{-1}, K^d\right) \arrow[r, "\text{restrict}"'] & \left(\bigoplus\limits_{\ell\in L} \kappa_\ell^{-1}, K_H^d\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\phi^i, K^i) \arrow[r, "\text{restrict}"] \arrow[d, "\text{extend}"'] \arrow[dr, phantom, "\circlearrowleft"] & (\phi_H^i, K_H^i) \arrow[d, "\text{extend}"] \\
\left(\phi^{i\prime}, K^{i+1}\right) \arrow[r, "\text{restrict}"'] \arrow[d, "\text{inflate}"'] \arrow[dr, phantom, "\circlearrowleft"] & \left({\phi_H^i}', K_H^{i+1}\right) \arrow[d, "\text{inflate}"] \\
\left(\kappa^i, K^d\right) \arrow[r, "\text{restrict}"'] & \left(\kappa_H^i, K_H^d\right)
\end{tikzcd}$$

Figure 4.4: Commutativity of the restriction with extension and inflation for $\phi^i$, $0 \leqslant i \leqslant d-1$, and $\rho$.

From Figure 4.4, we will show in Section 4.3 that $\kappa_G(\Psi)|_{K_H^d} = \bigoplus\limits_{\ell\in L} \kappa_H(\Psi_\ell)$, which will lead us to finally being able to compute the restriction $\pi_G(\Psi)|_H$ in Section 4.4.

## 4.2 Restricting the Representations $\phi^{i\prime}$

The main goal of this section is to show that $\phi^{i\prime}|_{K_H^{i+1}} = {\phi_H^i}'$ for all $0 \leqslant i \leqslant d-1$ (Theorem 4.2.7). To achieve this, we first review the specifics behind the construction of the extension $\phi^{i\prime}$ of $\phi^i$.

### 4.2.1 Construction of $\phi^{i\prime}$

Let $0 \leqslant i \leqslant d-1$. To extend the character $\phi^i$ to a representation of $K^{i+1}$, Yu introduces subgroups of $\mathbb{G}(E)$, which are denoted $J^{i+1}(E)$ and $J_+^{i+1}(E)$, and are defined as follows:

$$J^{i+1}(E) = \langle \mathbb{T}(E)_{r_i}, \mathbb{U}_a(E)_{y,r_i}, \mathbb{U}_b(E)_{y,s_i} : a \in \Phi(\mathbb{G}^i, \mathbb{T}), b \in \Phi(\mathbb{G}^{i+1}, \mathbb{T}) \backslash \Phi(\mathbb{G}^i, \mathbb{T})\rangle,$$
$$J_+^{i+1}(E) = \langle \mathbb{T}(E)_{r_i}, \mathbb{U}_a(E)_{y,r_i}, \mathbb{U}_b(E)_{y,s_i^+} : a \in \Phi(\mathbb{G}^i, \mathbb{T}), b \in \Phi(\mathbb{G}^{i+1}, \mathbb{T}) \backslash \Phi(\mathbb{G}^i, \mathbb{T})\rangle,$$

where the subgroup $\mathbb{U}_\alpha(E)_{y,r}$ denotes a filtration subgroup of $\mathbb{U}_\alpha(E)$, as per Remark 2.2.13.

We write $J^{i+1}$ and $J_+^{i+1}$ for $J^{i+1}(E) \cap G^{i+1}$ and $J_+^{i+1}(E) \cap G^{i+1}$, respectively. Yu also used the notation $(\mathbb{G}^i(E), \mathbb{G}^{i+1}(E))_{y,(r_i,s_i)}$ for $J^{i+1}(E)$, and $(G^i, G^{i+1})_{y,(r_i,s_i)}$ for

$J^{i+1}$. Similarly, $J_+^{i+1} = (G^i, G^{i+1})_{y,(r_i,s_i^+)}$. We have that $K^{i+1} = K^i J^{i+1} = K^i G_{y,s_i}^{i+1}$ and $K_+^{i+1} = K_+^i J_+^{i+1} = K_+^i G_{y,s_i^+}^{i+1}$ for all $0 \leqslant i \leqslant d-1$ [HM08, Section 3.1].

Given the previous group descriptions, in order to extend the character $\phi^i$ of $K^i$ to $K^{i+1}$, we must find a way to extend it over $J^{i+1}$, or $G_{y,s_i}^{i+1}$. The process of extending $\phi^i$ over $G_{y,s_i^+}^{i+1}$ is relatively simple. As $\phi^i$ is of depth $r_i$, it is trivial on $G_{y,r_i^+}^i$ which means that $\phi^i|_{G_{y,s_i^+}^i}$ can be viewed as a character of $G_{y,s_i^+:r_i^+}^i$. The choice to set $s_i = r_i/2$ is not arbitrary. Indeed, this is the smallest value so that $[G_{y,s_i^+}^i, G_{y,s_i^+}^i] \subset G_{y,r_i^+}^i$ (by the property (1) of the Moy-Prasad filtrations discussed in 2.2.2.2), ensuring that $G_{y,s_i^+:r_i^+}^i$ is an abelian group. Therefore, by the representation theory of abelian groups, $\phi^i|_{G_{y,s_i^+}^i}$ can be extended easily to a character of $G_{y,s_i^+:r_i^+}^{i+1}$, therefore extending $\phi^i|_{G_{y,s_i^+}^i}$ to a character $\overline{\phi^i}$ of $G_{y,s_i^+}^{i+1}$. Yu considers the simplest such extension in [Yu01, Section 4] by going to the Lie algebra and extending trivially.

One can then extend this character $\overline{\phi^i}$ uniquely to a character $\widehat{\phi^i}$ of $K^i G_{y,s_i^+}^{i+1}$, which is defined by $\widehat{\phi^i}(g\overline{g}) = \phi^i(g)\overline{\phi^i}(\overline{g})$ for all $g \in K^i, \overline{g} \in G_{y,s_i^+}^{i+1}$. This is a well-defined character since $K^i \cap G_{y,s_i^+}^{i+1} = G_{y,s_i^+}^i$ and $\phi^i|_{G_{y,s_i^+}^i} = \overline{\phi^i}|_{G_{y,s_i^+}^i}$. An equivalent description for $\widehat{\phi^i}$ is the (unique) character of $K^i G_{y,s_i^+}^{i+1}$ that agrees with $\phi^i$ on $K^i$ and is trivial on $(G^i, G^{i+1})_{y,(r_i^+,s_i^+)}$ [HM08, Section 3.1]. We summarize the construction of $\widehat{\phi^i}$ in Figure 4.5.

**Remark 4.2.1.** For the reader who is familiar with the J.K. Yu Construction, the description of $\widehat{\phi^i}$ that we provide above is not exactly the same as in [HM08, Section 3.1] and [Yu01]. Indeed, in [HM08, Yu01], the character produced is for a much larger group than necessary, $K^0 G_{y,0}^i G_{y,s_i^+}$. The description we provide above produces the same character of $K^i G_{y,s_i^+}^{i+1}$.

We have just extended $\phi^i$ to a character of $K^i G_{y,s_i^+}^{i+1} = K^i J_+^{i+1}$, but we need to extend it further to $K^{i+1} = K^i J^{i+1}$. There are two cases to consider.

*Case 1:* When $J^{i+1} = J_+^{i+1}$, we have obtained the desired extension to $K^{i+1}$, which is defined by $\phi^{i\prime}(kj) = \phi^i(k)\widehat{\phi^i}(j)$ for all $k \in K^i, j \in J^{i+1}$. Note that this is simply another way of expressing the character $\widehat{\phi^i}$ described above, and highlights that the work required resulted in a character of $J_+^{i+1}$.

*Case 2:* When $J^{i+1} \neq J_+^{i+1}$, we cannot simply extend the quasicharacter $\phi^i$ to be a quasicharacter of $K^i G_{y,s_i}^{i+1} = K^i J^{i+1}$. Indeed, consider the following lemma.

$$\begin{tikzcd}
 & (\phi^i, G^i) \arrow[dl, "\text{restrict}"'] \arrow[d, "\text{restrict}"] & \\
(\phi^i, K^i) \arrow[dr] & \left(\phi^i, G^i_{y,s_i^+}\right) \arrow[r, "\text{extend}"] & \left(\overline{\phi^i}, G^{i+1}_{y,s_i^+}\right) \arrow[dl] \\
 & \left(\widehat{\phi}^i, K^i G^{i+1}_{y,s_i^+}\right) &
\end{tikzcd}$$

Figure 4.5: Summary of the construction of the character $\widehat{\phi}^i$.

**Lemma 4.2.2.** *Assume that $J^{i+1} \neq J^{i+1}_+$ and let $\phi^i$ be a $G^{i+1}$-generic character of $G^i$ of depth $r_i$. Then $\phi^i|_{G^i_{y,s_i}}$ cannot be extended to a character of $G^{i+1}_{y,s_i}$.*

**Proof:** Let us proceed by contradiction and assume we can extend $\phi^i$ to $G^{i+1}_{y,s_i}$. This means $\phi^i$ needs to be trivial on $[G^{i+1}_{y,s_i}, G^{i+1}_{y,s_i}] \cap G^i_{y,r_i}$. If $X^*$ denotes the $G^{i+1}$-generic element that realizes $\phi^i$, then we have that $X^*(Y) \in \ker\psi$ for all $Y \in [\mathfrak{g}^{i+1}_{y,s_i}, \mathfrak{g}^{i+1}_{y,s_i}] \cap \mathfrak{g}^i_{y,r_i}$. Using an argument similar to the one in the proof of Proposition 3.1.12, we must have $X^*(Y) \in \mathfrak{p}_F$ for all $Y \in [\mathfrak{g}^{i+1}_{y,s_i}, \mathfrak{g}^{i+1}_{y,s_i}] \cap \mathfrak{g}^i_{y,r_i}$. Extending the scalars to $\mathcal{O}_E$, we have $X^*(Y) \in \mathfrak{p}_E$ for all $Y \in [\mathfrak{g}^{i+1}(E)_{y,s_i}, \mathfrak{g}^{i+1}(E)_{y,s_i}] \cap \mathfrak{g}^i(E)_{y,r_i}$.

As we are in the case where $J^{i+1} \neq J^{i+1}_+$, we also have $J^{i+1}(E) \neq J^{i+1}_+(E)$. Therefore, there exists $\beta \in \Phi(\mathbb{G}^{i+1}, \mathbb{T}) \backslash \Phi(\mathbb{G}, \mathbb{T})$ such that $\mathbb{U}_\beta(E)_{y,s_i} \neq \mathbb{U}_\beta(E)_{y,s_i^+}$. From Remark 2.2.13, we can write

$$\mathbb{U}_\beta(E)_{y,s_i} = \mathbb{U}_\beta(\mathfrak{p}_E^{-\lfloor e_{E/F}(\beta(y)-s_i)\rfloor}), \text{ and } \mathbb{U}_\beta(E)_{y,s_i^+} = \mathbb{U}_\beta(\mathfrak{p}_E^{1-\lceil e_{E/F}(\beta(y)-s_i)\rceil}).$$

Note that the ramification index $e_{E/F}$ of $E$ over $F$ appears because the affine roots over $E$ will be in terms of the integer multiples of $\frac{1}{e_{E/F}}$ given that the valuation over $E$ maps into $\frac{1}{e_{E/F}}\mathbb{Z}$. Therefore, $\mathbb{U}_\beta(E)_{y,s_i} \neq \mathbb{U}_\beta(E)_{y,s_i^+}$ implies that $-\lfloor e_{E/F}(\beta(y) - s_i)\rfloor \neq 1 - \lceil e_{E/F}(\beta(y) - s_i)\rceil$, or equivalently that $e_{E/F}(\beta(y) - s_i)$ is an integer. The expression $e_{E/F}(-\beta(y) - s_i)$ must also be in $\mathbb{Z}$, as $e_{E/F}(-\beta(y) - s_i) + e_{E/F}(\beta(y) - s_i) = -e_{E/F}r_i$, and $-r_i \in \frac{1}{e_{E/F}}\mathbb{Z}$ since it's the result of a valuation over $E$. Next, we have that $d\epsilon_\beta(\varpi_E^{-e(\beta(y)-s_i)}) \in \mathfrak{g}^{i+1}(E)_{y,s_i}$ and $[d\epsilon_\beta(\varpi_E^{-e(\beta(y)-s_i)}), d\epsilon_\beta(\varpi_E^{-e(\beta(y)-s_i)})] = \varpi_E^{e_{E/F}} H_\beta$ [Yu01, Section 6]. Therefore, $\varpi^{e_{E/F}r_i} H_\beta \in [\mathfrak{g}^{i+1}(E)_{y,s_i}, \mathfrak{g}^{i+1}(E)_{y,s_i}] \cap \mathfrak{g}^i(E)_{y,r_i}$, and under the hypothesis that $\phi^i$ extends to $G^{i+1}_{y,s_i}$, we should have $X^*(\varpi^{e_{E/F}r_i} H_\beta) \in \mathfrak{p}_E$. But this is a contradiction with the genericity of $\phi^i$, as we must have $\mathrm{val}(X^*(H_\beta)) = -r_i$. ∎

This means that we cannot extend our character $\phi^i$ across $G^{i+1}_{y,s_i}$, or $J^{i+1}$. We must then construct a higher dimensional representation. Yu does this using the theory of Heisenberg groups and Weil representations. Since the theory of Heisenberg groups and Weil representations has been discussed in many previous works (see for example [Yu01, HM08, Nev15]), we do not go into all the details in this work. Rather, we provide an overall summary and we invite the curious reader to consult these references for more information.

Yu shows that the quotient $W^i = J^{i+1}/J^{i+1}_+$ is a symplectic vector space. Let $\xi^i = \widehat{\phi}^i|_{J^{i+1}_+}$ and set $N^i = \ker(\xi^i)$. The quotient $\mathcal{H}^i = J^{i+1}/N^i$ is a Heisenberg $p$-group with center $Z^i = J^{i+1}_+/N^i$ and such that $\mathcal{H}^i/Z^i \simeq W^i$. Furthermore, $\mathcal{H}^i \simeq W^i \boxtimes Z^i$, where $W^i \boxtimes Z^i$ denotes couples $(w, z), w \in W^i, z \in Z^i$ under a specific multiplication rule. Since $G^i_{y,r_i} \subset J^{i+1}_+$ and $\phi^i$ is of depth $r_i$, this means that $\xi^i$ is a nontrivial central character. The version of the Stone-von Neumann Theorem as stated in [McN12, Theorem 3] says that, up to isomorphism, there is a unique irreducible representation $\eta_{\xi^i}$ of the Heisenberg $p$-group $\mathcal{H}^i$ having $\xi^i$ as its central character.

Now, let $\mathrm{Sp}(W^i)$ denote the symplectic group of $W^i$ and set $\mathrm{Sp}(\mathcal{H}^i)$ to be the set of automorphisms of $\mathcal{H}^i$ that act trivially on $Z^i$. The group $\mathrm{Sp}(W^i)$ can be viewed as a subgroup of $\mathrm{Sp}(\mathcal{H}^i)$ as $\mathcal{H}^i \simeq W^i \boxtimes Z^i$. Given the representation $\eta_{\xi^i}$ of $\mathcal{H}^i$ above, the Stone-von Neumann Theorem tells us that ${}^g\eta_{\xi^i} \simeq \eta_{\xi^i}$ for all $g \in \mathrm{Sp}(W^i)$ since ${}^g\eta_{\xi^i}$ and $\eta_{\xi^i}$ both have $\xi^i$ as their central character. So, for all $g \in \mathrm{Sp}(W^i)$, there exists an intertwining map $w(g)$ which intertwines ${}^g\eta_{\xi^i}$ and $\eta_{\xi^i}$.

It turns out that the maps $w(g), g \in \mathrm{Sp}(W^i)$, can be chosen compatibly to make a representation of $\mathrm{Sp}(W^i)$, which is called the *Weil representation*. This means that the representation $\eta_{\xi^i}$ can be naturally extended to a representation $\widehat{\eta_{\xi^i}}$ of $\mathrm{Sp}(W^i) \ltimes \mathcal{H}^i$, where $w = \widehat{\eta_{\xi^i}}|_{\mathrm{Sp}(W^i)}$ and $\eta_{\xi^i} = \widehat{\eta_{\xi^i}}|_{\mathcal{H}^i}$.

The representation $\widehat{\eta_{\xi^i}}$ is called the *Heisenberg-Weil lift* of $\eta_{\xi^i}$. When $p \neq 3$ and $\dim W^i > 2$, the Heisenberg-Weil lift is unique. When $p = 3$ and $\dim W^i = 2$, three lifts are possible, but Hakim and Murnaghan single out one of them as being the optimal choice [HM08, Definition 2.17].

Using the action by conjugation of $K^i$ on $J^{i+1}$, one can think of $K^i$ as a subgroup of $\mathrm{Sp}(W^i) \subset \mathrm{Sp}(\mathcal{H}^i)$ up to homomorphism. Hence, composing this homomorphism with the Heisenberg-Weil lift obtained from the central character $\xi^i$, we obtain a new representation $\omega^i$ of $K^i \ltimes \mathcal{H}^i$.

The representation $\phi^{i\prime}$ is then defined as $\phi^{i\prime}(kj) = \phi^i(k)\omega^i(k, jN^i)$ for all $k \in K^i, j \in J^{i+1}$. Note that $\phi^{i\prime}$ is no longer a character in this case, but rather a representation whose dimension is related to the size of the quotient $J^{i+1}/J^{i+1}_+$.

### 4.2.2 Calculating the Restriction of $\phi^{i\prime}$

Before we discuss the restriction of the representation $\phi^{i\prime}$, recall the groups $K^{i+1}_H$ and $K^{i+1}_{H+}$ for all $-1 \leqslant i \leqslant d-1$ required in the construction of the supercuspidal $\pi_H(\Psi_\ell)$, and the notations for the extensions of the characters $\phi_H^i$ that appear in Figure 4.3. Analogously to the notation of the previous section, we define the subgroups $J_H^{i+1}(E)$ and $J_{H+}^{i+1}(E)$ of $\mathbb{H}(E)$ as follows:

$$J_H^{i+1}(E) = \langle \mathbb{T}_{\mathbb{H}}(E)_{r_i}, \mathbb{U}_a(E)_{y,r_i}, \mathbb{U}_b(E)_{y,s_i} : a \in \Phi(\mathbb{G}^i, \mathbb{T}), b \in \Phi(\mathbb{G}^{i+1}, \mathbb{T})\backslash\Phi(\mathbb{G}^i, \mathbb{T})\rangle,$$
$$J_{H+}^{i+1}(E) = \langle \mathbb{T}_{\mathbb{H}}(E)_{r_i}, \mathbb{U}_a(E)_{y,r_i}, \mathbb{U}_b(E)_{y,s_i^+} : a \in \Phi(\mathbb{G}^i, \mathbb{T}), b \in \Phi(\mathbb{G}^{i+1}, \mathbb{T})\backslash\Phi(\mathbb{G}^i, \mathbb{T})\rangle.$$

We write $J_H^{i+1}$ and $J_{H+}^{i+1}$ for $J_H^{i+1}(E) \cap H^{i+1}$ and $J_{H+}^{i+1}(E) \cap H^{i+1}$, respectively. We have that $J_H^{i+1} = (H^i, H^{i+1})_{y,(r_i,s_i)}$, $J_{H+}^{i+1} = (H^i, H^{i+1})_{y,(r_i,s_i^+)}$, $K_H^{i+1} = K_H^i J_H^{i+1}$ and $K_{H+}^{i+1} = K_{H+}^i J_{H+}^{i+1}$ for all $0 \leqslant i \leqslant d-1$.

**Lemma 4.2.3.** *For all $0 \leqslant i \leqslant d-1$, $J_H^{i+1} = J^{i+1} \cap H^{i+1}$ and $J_{H+}^{i+1} = J_+^{i+1} \cap H^{i+1}$.*

**Proof:** We prove the first equality, and the second equality can be obtained using similar arguments.

From the definition of the groups, we have that $J_H^{i+1}(E) = J^{i+1}(E) \cap \mathbb{H}(E)$. Indeed, the filtration subgroups of the root subgroups lie in $\mathbb{H}(E)$ and are normalized by $\mathbb{T}(E)_{r_i}$, so we apply the argument from Proposition 2.1.30 which essentially says it suffices to intersect $\mathbb{T}(E)_{r_i}$ with $\mathbb{H}(E)$. Since $J^{i+1}(E) \subset \mathbb{G}^{i+1}(E)$, we have $J_H^{i+1}(E) = J^{i+1}(E) \cap \mathbb{H}^{i+1}(E)$. Thus,

$$J_H^{i+1} = J_H^{i+1}(E) \cap H^{i+1} = J^{i+1}(E) \cap H^{i+1} = J^{i+1} \cap H^{i+1}.$$

∎

**Proposition 4.2.4.** *For all $0 \leqslant i \leqslant d-1$, $J^{i+1} = J_H^{i+1} J_+^{i+1}$.*

**Proof:** First, recall that $\mathbb{T}(E)_{r_i}$ normalizes the filtrations of the root subgroups that partly generate $J^{i+1}(E)$. So, given $g \in J^{i+1}(E)$, we can write $g = ut$ for some $t \in \mathbb{T}(E)_{r_i} \subset J_+^{i+1}(E)$ and

$$u \in \langle \mathbb{U}_a(E)_{y,r_i}, \mathbb{U}_b(E)_{y,s_i} : a \in \Phi(\mathbb{G}^i, \mathbb{T}), b \in \Phi(\mathbb{G}^{i+1}, \mathbb{T})\backslash\Phi(\mathbb{G}^i, \mathbb{T})\rangle \subset J_H^{i+1}(E).$$

Therefore, we have $J^{i+1}(E) = J_H^{i+1}(E) J_+^{i+1}(E)$. In order to prove the equality also holds over the $F$-points, we require results from [Yu01].

We proceed similarly to the proof of [Yu01, Lemma 2.10]. Take $g \in J^{i+1} = J^{i+1}(E) \cap G^{i+1}$. Then $g$ is fixed by $\mathrm{Gal}(E/F)$. By what precedes, $g = ut$ for some $u \in J_H^{i+1}(E)$, $t \in J_+^{i+1}(E)$. It follows that the image of $g$ in $J^{i+1}(E)/J_+^{i+1}(E) = J_H^{i+1}(E)J_+^{i+1}(E)/J_+^{i+1}(E)$, given by $uJ_+^{i+1}(E)$, is fixed by $\mathrm{Gal}(E/F)$. But the second isomorphism theorem tells us $J_H^{i+1}(E)J_+^{i+1}(E)/J_+^{i+1}(E) \simeq J_H^{i+1}(E)/J_H^{i+1}(E) \cap J_+^{i+1}(E)$, where

$J_H^{i+1}(E) \cap J_+^{i+1}(E) = J_{H+}^{i+1}(E)$ as a consequence of Lemma 4.2.3. Hence, $uJ_{H+}^{i+1}(E)$ is fixed by $\mathrm{Gal}(E/F)$.

Next, [Yu01, Corollary 2.3] states that the natural homomorphism

$$J_H^{i+1} \to \left(J_H^{i+1}(E)/J_{H+}^{i+1}(E)\right)^{\mathrm{Gal}(E/F)}$$

is a surjection. This implies that there exists $g_0 \in J_H^{i+1}$ such that $g_0 J_{H+}^{i+1}(E) = uJ_{H+}^{i+1}(E)$, that is $g_0^{-1}u \in J_{H+}^{i+1}(E)$. Hence, $u \in J_H^{i+1} J_{H+}^{i+1}(E)$. We deduce that $g = g'\overline{g}$ for some $g' \in J_H^{i+1}, \overline{g} \in J_+^{i+1}(E)$. Furthermore, we have that $\overline{g} = g'^{-1}g \in G^{i+1}$, and therefore $\overline{g} \in J_+^{i+1}$. The result follows. ∎

**Corollary 4.2.5.** *Let $\beta : J_H^{i+1}/J_{H+}^{i+1} \to J^{i+1}/J_+^{i+1}$ be defined by $\beta(jJ_{H+}^{i+1}) = jJ_+^{i+1}$ for all $j \in J_H^{i+1}$. Then $\beta$ is an isomorphism.*

**Proof:** Well-definedness and injectivity of the map are easily verified. Surjectivity follows from the previous proposition. ∎

**Corollary 4.2.6.** *For all $0 \leqslant i \leqslant d-1$, $J^{i+1} = J_+^{i+1}$ if and only if $J_H^{i+1} = J_{H+}^{i+1}$.*

This last corollary is important, as it implies that the theory of Weil representations is always applied simultaneously in the construction of $\phi^{i\prime}$ and $\phi_H^{i\,\prime}$. This is crucial for us to be able to relate those two extensions. Indeed, if it was possible for $J_H^{i+1} = J_{H+}^{i+1}$, but $J^{i+1} \neq J_+^{i+1}$, then talking about restricting $\phi^{i\prime}$, a representation of dimension greater than 1, would not result in the character $\phi_H^{i\,\prime}$. We are now ready to state the main theorem of this section.

**Theorem 4.2.7.** *For all $0 \leqslant i \leqslant d-1$, $\phi_H^{i\,\prime} = \phi^{i\prime}|_{K_H^{i+1}}$.*

**Proof:** For a fixed $0 \leqslant i \leqslant d-1$, we have two cases to consider.

*Case 1 ($J^{i+1} = J_+^{i+1}$):* In this case, we have $J_H^{i+1} = J_{H+}^{i+1}$ by Corollary 4.2.6. It follows that $\phi_H^{i\,\prime}(kj) = \phi_H^i(k)\widehat{\phi_H^i}(j)$ for all $k \in K_H^i, j \in J_H^{i+1}$, where $\widehat{\phi_H^i}$ is the (unique) quasicharacter of $K_H^i H_{y,s_i^+}^{i+1}$ that agrees with $\phi_H^i$ on $K_H^i$ and is trivial on $(H^i, H^{i+1})_{y,(r_i^+,s_i^+)}$.

Clearly, we have that $\widehat{\phi}^i|_{K_H^i H_{y,s_i+}^{i+1}}$ agrees with $\phi_H^i$ on $K_H^i$, as it agrees with $\phi^i$ on $K^i$, and is trivial on $(H^i, H^{i+1})_{y,(r_i^+,s_i^+)}$, as it is trivial on $(G^i, G^{i+1})_{y,(r_i^+,s_i^+)}$. These restrictions are illustrated in Figure 4.6. Therefore, by uniqueness, we have $\widehat{\phi}^i|_{K_H^i H_{y,s_i+}^{i+1}} = \widehat{\phi_H^i}$.

So, for all $k \in K_H^i, j \in J_H^{i+1}$

$$\phi^{i\prime}|_{K_H^{i+1}}(kj) = \phi^i(k)\widehat{\phi}^i(j) = \phi_H^i(k)\widehat{\phi_H^i}(j) = \phi_H^{i\,\prime}(kj).$$

$$\begin{tikzcd}
 & (\phi^i_H, K^i_H) & \\
 & (\phi^i, K^i) \arrow[u] & \\
\left(\widehat{\phi}^i, K^i G^{i+1}_{y,s_i^+}\right) \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \left(\widehat{\phi^i_H}, K^i_H H^{i+1}_{y,s_i^+}\right) \arrow[uul] \arrow[ddl] \\
 & \left(1, (G^i, G^{i+1})_{y,(r_i^+,s_i^+)}\right) \arrow[d] & \\
 & \left(1, (H^i, H^{i+1})_{y,(r_i^+,s_i^+)}\right) &
\end{tikzcd}$$

Figure 4.6: Diagram of restrictions for $\widehat{\phi}^i$ and $\widehat{\phi^i_H}$.

*Case 2 ($J^{i+1} \neq J^{i+1}_+$):* In this case, we have that $\phi^{i\prime}$ is constructed using the Heisenberg-Weil lift $\omega^i$, which is a representation of $K^i \ltimes J^{i+1}/\ker(\xi^i)$ constructed from $\xi^i = \widehat{\phi}^i|_{J^{i+1}_+}$. Since $J^{i+1}_H \neq J^{i+1}_{H+}$, we require a Heisenberg-Weil lift to extend $\widehat{\phi^i_H}$, which we denote by $\omega^i_H$. Note that $\omega^i_H$ is a representation of $K^i_H \ltimes J^{i+1}_H/\ker(\xi^i_H)$, where $\xi^i_H = \widehat{\phi^i_H}|_{J^{i+1}_{H+}} = \widehat{\phi}^i|_{J^{i+1}_{H+}}$. In the Appendix, we show that $\omega^i(k, j\ker(\xi^i)) = \omega^i_H(k, j\ker(\xi^i_H))$ for all $k \in K^i_H, j \in J^{i+1}_H$ (Theorem A.4.1). Hence, for all $k \in K^i_H, j \in J^{i+1}_H$,

$$\phi^{i\prime}|_{K^{i+1}_H}(kj) = \widehat{\phi}^i(k)\omega^i(k, j\ker(\xi^i)) = \widehat{\phi^i_H}(k)\omega^i_H(k, j\ker(\xi^i_H)) = {\phi^i_H}'(kj).$$

∎

## 4.3 Restricting $\kappa_G(\Psi)$

The main goal of this section is to show that $\kappa_G(\Psi)|_{K^d_H} \simeq \bigoplus_{\ell \in L} \kappa_H(\Psi_\ell)$ (Theorem 4.3.6). To achieve this, we must first understand how the representations $\kappa^i, 0 \leqslant i \leqslant d$, of $K^d$ are constructed. In the J.K. Yu Construction, those representations are obtained from the representations $\phi^{i\prime}$ of the smaller groups $K^{i+1}$ via a process called inflation.

We start this section by defining this process in general and exploring some of its properties.

### 4.3.1 Inflation

**Lemma 4.3.1.** [HM08, Section 3.4] *If $\mu$ is a representation of $A$ which is 1-isotypic on $A \cap B$, then there exists a unique extension of $\mu$ to $AB$, denoted $\inf_A^{AB} \mu$, which is 1-isotypic on $B$. This extension is called the* inflation *of $\mu$ to $AB$.*

**Proof:** We define $\inf_A^{AB} \mu$ as follows.

$$\inf_A^{AB} \mu(ab) = \mu(a) \text{ for all } a \in A, b \in B.$$

As $\mu$ is 1-isotypic on $A \cap B$, this map is well-defined. Indeed, if $a'b' = ab$, then $a^{-1}a' = bb'^{-1}$. It follows that $a^{-1}a' \in A \cap B$, so $\mu(a^{-1}a') = 1$. Thus, $\mu(a') = \mu(a)$, and $\inf_A^{AB} \mu(a'b') = \inf_A^{AB} \mu(ab)$. Furthermore, this extension is 1-isotypic on $B$ by definition.

To prove uniqueness, let $\gamma$ be a representation of $AB$ which is 1-isotypic on $B$ and such that $\gamma|_A = \mu$. Then for all $a \in A, b \in B$,

$$\gamma(ab) = \gamma(a)\gamma(b) = \gamma(a) = \mu(a) = \inf_A^{AB} \mu(ab).$$

Hence, $\gamma = \inf_A^{AB} \mu$. ∎

**Lemma 4.3.2.** *Let $\chi = \chi_1 \oplus \cdots \oplus \chi_n$ be a representation of $A$ which is 1-isotypic on $A \cap B$. Then, $\inf_A^{AB} \chi = \inf_A^{AB} \chi_1 \oplus \cdots \oplus \inf_A^{AB} \chi_n$, that is inflation commutes with direct sum.*

**Proof:**

Since $\chi$ is 1-isotypic on $A \cap B$, we have that $\chi_k|_{A\cap B} = 1$ for all $1 \leqslant k \leqslant n$, so we have inflations $\inf_A^{AB} \chi_k, 1 \leqslant k \leqslant n$. For all $ab \in AB$,

$$\inf_A^{AB} \chi(ab) = \chi(a) = \chi_1(a) \oplus \cdots \oplus \chi_n(a) = \inf_A^{AB} \chi_1(ab) \oplus \cdots \oplus \inf_A^{AB} \chi_n(ab).$$

∎

**Lemma 4.3.3.** *Let $\mu$ be a representation of $A$ which is 1-isotypic on $A \cap B$. Then*

$$\left(\inf_A^{AB} \mu\right)|_{(A\cap C)(B\cap C)} = \inf_{A\cap C}^{(A\cap C)(B\cap C)} \left(\mu|_{A\cap C}\right).$$

**Proof:** Let $a \in A \cap C, b \in B \cap C$. Then

$$\left(\inf_A^{AB} \mu\right)|_{(A\cap C)(B\cap C)}(ab) = \mu(a) = \mu|_{A\cap C}(a) = \inf_{A\cap C}^{(A\cap C)(B\cap C)} (\mu|_{A\cap C})(ab).$$

∎

In the context of the J.K. Yu Construction, the representations $\phi^{i'}$ get inflated to $K^d$ for all $0 \leqslant i \leqslant d-2$. Indeed, $\phi^{i'}$ is a representation of $K^{i+1}$ which is 1-isotypic on $K^{i+1} \cap J^{i+2} = G^{i+1}_{y,r_{i+1}}$ for all $0 \leqslant i \leqslant d-2$. Therefore, they can be inflated to $K^{i+2} = K^{i+1}J^{i+2}$. As explained in [HM08, Section 3.4], this previous inflation will be 1-isotypic on $K^{i+2} \cap J^{i+3} = G^{i+2}_{y,r_{i+2}} \subset G^{i+2}_{y,r_{i+1}} \subset J^{i+2}$. We may then repeat the inflation process inductively to obtain an inflation to $K^d$. Furthermore, since $K^d = K^{i+1}J^{i+2}\cdots J^d$, this inflation will be defined as follows:

$$\inf_{K^{i+1}}^{K^d} \phi^{i'}(kj) = \phi^{i'}(k) \text{ for all } k \in K^{i+1}, j \in J^{i+2}\cdots J^d.$$

Similarly, we can inflate the representation $\rho$ to $K^d$ for the same reason. The following notation is used to denote the inflations:

$$\kappa^{-1} = \inf_{K^0}^{K^d} \rho, \kappa^i = \inf_{K^{i+1}}^{K^d} \phi^{i'}, 0 \leqslant i \leqslant d-2.$$

We also have $\kappa^{d-1} = \phi^{d-1'}, \kappa^d = \phi^d|_{K^d}$. Analogous notation is used for the construction of the supercuspidal representations of $H$, as illustrated in Figure 4.3.

### 4.3.2 Calculating the Restriction of $\kappa_G(\Psi)$

**Proposition 4.3.4.** *For all $0 \leqslant i \leqslant d-1, \kappa_H^i = \kappa^i|_{K_H^d}$.*

**Proof:** For $0 \leqslant i \leqslant d-1$, we have that ${\phi_H^i}' = \phi^{i'}|_{K_H^{i+1}}$ (Theorem 4.2.7), which implies that $\inf_{K_H^{i+1}}^{K_H^d} {\phi_H^i}' = \inf_{K_H^{i+1}}^{K_H^d}(\phi^{i'}|_{K_H^{i+1}})$. Furthermore, since $K^{i+1} \cap H = K_H^{i+1}$ for all $0 \leqslant i \leqslant d-1$ (Proposition 4.1.3), Lemma 4.3.3 tells us that

$$\inf_{K_H^{i+1}}^{K_H^d}(\phi^{i'}|_{K_H^{i+1}}) = \left(\inf_{K^{i+1}}^{K^d} \phi^{i'}\right)|_{K_H^d}.$$

Therefore, $\kappa_H^i = \inf_{K_H^{i+1}}^{K_H^d}(\phi^{i'}|_{K_H^{i+1}}) = \left(\inf_{K^{i+1}}^{K^d} \phi^{i'}\right)|_{K_H^d} = \kappa^i|_{K_H^d}$.

∎

**Lemma 4.3.5.** *The restriction of $\kappa^{-1}$ to $K_H^d$ decomposes as $\kappa^{-1}|_{K_H^d} = \bigoplus_{\ell\in L} \kappa_\ell^{-1}$.*

**Proof:** Recall the decomposition into irreducible representations of $\rho|_{K_H^0} = \bigoplus_{\ell\in L}\rho_\ell$, where $L$ is some finite set. We have

$$\kappa^{-1}|_{K_H^d} = (\inf_{K^0}^{K^d}\rho)|_{K_H^d}.$$

Lemma 4.3.3 allows us to write

$$\kappa^{-1}|_{K_H^d} = \inf_{K_H^0}^{K_H^d}(\rho|_{K_H^0}).$$

Since $\rho|_{K_H^0} = \bigoplus_{\ell\in L}\rho_\ell$ and that inflation commutes with direct sum (Lemma 4.3.2), we obtain

$$\begin{aligned}\kappa^{-1}|_{K_H^d} &= \bigoplus_{\ell\in L}\inf_{K_H^0}^{K_H^d}\rho_\ell \\ &= \bigoplus_{\ell\in L}\kappa_\ell^{-1}.\end{aligned}$$

∎

We are now ready to prove the main result of this section.

**Theorem 4.3.6.** *Given the G-datum $\Psi$, $\kappa_G(\Psi)|_{K_H^d} \simeq \bigoplus_{\ell\in L}\kappa_H(\Psi_\ell)$.*

**Proof:** We have that

$$\kappa_G(\Psi)|_{K_H^d} = \kappa^{-1}|_{K_H^d} \otimes \bigotimes_{0\leqslant i\leqslant d}\kappa^i|_{K_H^d}.$$

From Proposition 4.3.4 and Lemma 4.3.5, we can rewrite the right-hand side as

$$\kappa_G(\Psi)|_{K_H^d} = \left(\bigoplus_{\ell\in L}\kappa_\ell^{-1}\right)\otimes\bigotimes_{0\leqslant i\leqslant d}\kappa_H^i.$$

Finally, Lemma A.1.6 allows us to take out the direct sum and obtain

$$\kappa_G(\Psi)|_{K_H^d} = \bigoplus_{\ell\in L}\left(\kappa_\ell^{-1}\otimes\bigotimes_{0\leqslant i\leqslant d}\kappa_H^i\right).$$

In the case where the depth $\tilde{r}$ of $\phi_H^d$ is greater than $r_{d-1}$, then $\kappa_H(\Psi_\ell) = \kappa_\ell^{-1}\otimes\bigotimes_{0\leqslant i\leqslant d}\kappa_H^i$ (Figure 4.3) and we obtain our result. When $\tilde{r}\leqslant r_{d-1}$, we must find a way to compare $\kappa_H^{d-1}\otimes\kappa_H^d$ with $\kappa_H^{d-1,d}\otimes 1$, where $\kappa_H^{d-1,d} = (\phi_H^{d-1}\phi_H^d)'$.

This requires going into the technicalities of the Heisenberg-Weil lift. We provide the details in Appendix A.4.2, in which we show that $\kappa_H^{d-1} \otimes \kappa^d \simeq \kappa_H^{d-1,d} \otimes 1$. Thus,

$$\kappa_\ell^{-1} \otimes \bigotimes_{0 \leqslant i \leqslant d} \kappa_H^i \simeq \kappa_\ell^{-1} \otimes \kappa_H^0 \otimes \cdots \otimes \kappa_H^{d-2} \otimes \kappa_H^{d-1,d} \otimes 1 = \kappa_H(\Psi_\ell),$$

and the result follows.

∎

## 4.4 Restricting $\pi_G(\Psi)$

We are now ready to calculate the restriction of $\pi_G(\Psi)$ to $H$.

**Theorem 4.4.1.** *Given the datum $\Psi$, $\pi_G(\Psi)|_H \simeq \bigoplus_{t \in C} \left( \bigoplus_{\ell \in L} \pi_H({}^t\Psi_\ell) \right)$, where $C$ is a set of representatives of $H \backslash G / K^d$.*

**Proof:** From the Mackey Decomposition, we have that

$$\mathrm{Res}_H^G \pi_G(\Psi) = \mathrm{Res}_H^G \text{c-Ind}_{K^d}^G \kappa_G(\Psi) = \bigoplus_{t \in C} \text{c-Ind}_{{}^t(K^d \cap H)}^H \mathrm{Res}_{{}^t(K^d \cap H)}^{{}^t(K^d)} {}^t\kappa_G(\Psi).$$

We have that ${}^t\kappa_G(\Psi) = \kappa_G({}^t\Psi)$ [HM08, Section 5.1.1], which implies that

$$\mathrm{Res}_H^G \pi_G(\Psi) = \bigoplus_{t \in C} \text{c-Ind}_{{}^tK_H^d}^H \mathrm{Res}_{{}^tK_H^d}^{{}^tK^d} \kappa_G({}^t\Psi).$$

Applying Theorem 4.3.6 to the datum ${}^t\Psi$, we have that $\mathrm{Res}_{{}^tK_H^d}^{{}^tK^d} \kappa_G({}^t\Psi) \simeq \bigoplus_{\ell \in L} \kappa_H(({}^t\Psi)_\ell)$. From the definition of conjugate $G$-datum (Section 3.2), it is clear that $({}^t\Psi)_\ell = {}^t\Psi_\ell$. Finally, given that induction commutes with direct sum (Proposition A.1.9), we obtain

$$\mathrm{Res}_H^G \pi_G(\Psi) \simeq \bigoplus_{t \in C} \left( \bigoplus_{\ell \in L} \text{c-Ind}_{{}^tK_H^d}^H \kappa_H({}^t\Psi_\ell) \right) = \bigoplus_{t \in C} \left( \bigoplus_{\ell \in L} \pi_H({}^t\Psi_\ell) \right).$$

∎

On top of giving a complete description of the restriction of irreducible positive-depth supercuspidal representations, Theorem 4.4.1 also provides a description for the restriction of depth-zero supercuspidal representations, as per the following corollary.

**Corollary 4.4.2.** *Let $\Psi = (\vec{G}, y, \rho, 1)$ be a $G$-datum associated to an irreducible depth-zero supercuspidal representation. Then*

$$\mathrm{Res}_H^G \pi_G(\Psi) = \bigoplus_{t\in C} \left( \bigoplus_{\ell\in L} \pi_H({}^t\Psi_\ell) \right),$$

*or equivalently,*

$$\mathrm{Res}_H^G \text{c-Ind}_{G_{[y]}}^G \rho = \bigoplus_{t\in C} \left( \bigoplus_{\ell\in L} \text{c-Ind}_{{}^tH_{[y]}}^H {}^t\rho_\ell \right),$$

*where $C$ is a set of representatives of $H\backslash G/H_{[y]}$.*

## 4.5 Frobenius Reciprocity for Inducing $\pi_H(\Psi_H)$

Recall that in Section 3.5, we also studied how to go backwards from an $H$-datum $\Psi_H$ to $G$-data. Analogously, we ought to be able to say something about the irreducible subrepresentations of $\text{c-Ind}_H^G \pi_H(\Psi_H)$. Recall the notation $\mathrm{Ext}(\Psi_H)$ for the set of $G$-data that extend $\Psi_H$.

**Theorem 4.5.1.** *Let $\Psi_H$ be a normalized $H$-datum. Every irreducible supercuspidal subrepresentation of $\text{c-Ind}_H^G \pi_H(\Psi_H)$ is of the form $\pi_G(\Psi)$, where $\Psi \in {}^{t^{-1}}\mathrm{Ext}(\Psi_H)$ for some $t \in C$, $C$ a set of coset representatives of $H\backslash G/K^d$.*

**Proof:** Let $\pi$ be an irreducible supercuspidal subrepresentation of $\text{c-Ind}_H^G \pi_H(\Psi_H)$. Therefore $\pi = \pi_G(\Psi)$ for some $G$-datum $\Psi$ and $\mathrm{Hom}_G(\pi_G(\Psi), \text{c-Ind}_H^G \pi_H(\Psi_H)) \neq \{0\}$. This implies that $\mathrm{Hom}_H(\pi_G(\Psi)|_H, \pi_H(\Psi_H)) \neq \{0\}$ by Frobenius Reciprocity, and $\pi_H(\Psi_H)$ is a component of $\pi_G(\Psi)|_H$ as $\pi_G(\Psi)|_H$ is completely reducible (Theorem 1). Theorem 4.4.1 implies that $\Psi_H \in {}^t\mathrm{Res}(\Psi)$, up to $H$-equivalence, for some $t \in C$, where $C$ is a set of coset representatives of $H\backslash G/K^d$. Or, equivalently, ${}^{t^{-1}}\Psi_H \in \mathrm{Res}(\Psi)$, which implies that $\Psi \in \mathrm{Ext}({}^{t^{-1}}\Psi_H) = {}^{t^{-1}}\mathrm{Ext}(\Psi_H)$. ∎

# Chapter 5

# Multiplicity in the Restriction

Recall that $\mathbb{G}$ is a reductive group defined over a $p$-adic field $F$, which splits over a tamely ramified extension, and that the residual characteristic of $F$ does not divide the order of the Weyl group of $\mathbb{G}$. We also maintain the notation $G = \mathbb{G}(F)$, and $\mathbb{H}$ is a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$. Given a supercuspidal representation $\pi_G(\Psi)$, we saw how $\pi_G(\Psi)|_H$ decomposes as a sum of irreducible supercuspidal representations in the previous chapter (Theorem 4.4.1). The next natural question to ask is to what multiplicities do the (irreducible) components of this decomposition appear.

Recall the decomposition of $\rho_H = \rho|_{K_H^0}$ into irreducible components, $\rho_H = \bigoplus_{\ell \in L} \rho_\ell$ for some finite set $L$. It is clear that when the summands of $\rho_H$ are not distinct, we cannot have a multiplicity free decomposition. Indeed, this would mean that $\rho_i \simeq \rho_j$ for some $i \neq j \in L$, which would lead to having $\Psi_i$ being an elementary transformation of $\Psi_j$ (Section 3.2), implying that $\pi_H(\Psi_i) \simeq \pi_H(\Psi_j)$ (Theorem 3.2.4). One can then ask as a first question whether this is the only possible source of multiplicity. It turns out that it is. Indeed, we will show in the first section of this chapter that the multiplicity of $\pi_H({}^t\Psi_\ell)$ in $\pi_G(\Psi)|_H$ is equal to the multiplicity of $\rho_\ell$ in $\rho_H$ (Theorem 5.1.1). This in turn implies that the question of multiplicity for the restriction to $H$ of an irreducible supercuspidal representation reduces to studying the restriction of the depth-zero part of the datum.

Because $\rho$ and $\rho_H$ are finite-dimensional representations, we will be able to apply results from Clifford theory in order to get a formula for the multiplicity in $\rho_H$. As such, we devote Section 5.2.1 to summarize the important results from Clifford theory that we will be using. In Section 5.2.2, we will see how we can apply these results to $\rho_H$ even though $K_H^0$ might not be of finite index in $K^0$. We will see that every component of $\rho_H$ must appear with the same multiplicity, and we will provide a formula for this multiplicity at the end of this chapter (Proposition 5.2.10). Finally, we obtain as a corollary of this multiplicity formula a refinement of the decomposition of $\pi_G(\Psi)|_H$ in Corollary 5.2.11.

## 5.1 Source of Multiplicity in the Restriction

Let $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$ be a $G$-datum. Recall that $\rho_H$ is the restriction to $K_H^0$ of $\rho$ and that we have a decomposition into components $\rho_H = \bigoplus_{\ell \in L} \rho_\ell$ for some finite set $L$. Furthermore, $\Psi_\ell$ denotes the $H$-datum in $\operatorname{Res}(\Psi)$ that corresponds to $\rho_\ell$ as per Theorem 3.3.1, and by Theorem 4.4.1, we have that $\pi_G(\Psi)|_H \simeq \bigoplus_{t \in C} \left( \bigoplus_{\ell \in L} \pi_H({}^t\Psi_\ell) \right)$, where $C$ is a set of representatives of the double cosets of $H \backslash G / K^d$. The goal of this section is to prove the following theorem.

**Theorem 5.1.1.** *Let $C$ be a set of representatives of the double cosets of $H \backslash G / K^d$ and let $\ell \in L$. Then for all $t \in C$, the multiplicity of the component $\pi_H({}^t\Psi_\ell)$ in $\pi_G(\Psi)|_H$ is equal to the multiplicity of $\rho_\ell$ in $\rho_H$. In particular, $\pi_G(\Psi)|_H$ is multiplicity free if and only if $\rho_H$ is multiplicity free.*

The proof of this theorem will be divided into two lemmas (Lemmas 5.1.6 and 5.1.7). However, before we can proceed to those lemmas, we need two results from [Mur11].

**Proposition 5.1.2** ([Mur11, Proposition 5.4])**.** *Let $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$ be a $G$-datum, $\phi$ be the character of $G^0$ defined by $\phi = \prod_{i=0}^{d} \phi^i|_{G^0}$ and set $G^0_{r^+} = \bigcup_{x \in \mathcal{B}(\mathbb{G}^0, F)} G^0_{x,r^+}$ for all $r \geqslant 0$. Suppose there exists some element $g$ such that $\mathbb{G} = {}^g\mathbb{G}$ and $\phi|_{G^0_{0^+}} = {}^g\phi|_{G^0_{0^+}}$. Then, setting $r_{-1} = 0$, we have $\prod_{j=i}^{d} \phi^j|_{G^i_{r^+_{i-1}}} = \prod_{j=i}^{d} {}^g\phi^j|_{G^i_{r^+_{i-1}}}$ for $0 \leqslant i \leqslant d$.*

**Remark 5.1.3.** The element $g$ in the last proposition can be taken to be from a larger group $\tilde{G} \supset G$.

**Lemma 5.1.4.** *Let $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$ be a $G$-datum, $\vec{\mathbb{H}}$ and $\vec{\phi_H}$ be the Levi and quasicharacter sequences of $H$ constructed from Theorem 3.3.1, and $\phi_H$ be the character of $H^0$ defined by $\phi_H = \prod_{i=0}^{d} \phi^i_H|_{H^0}$. Suppose that there exists $g \in G$ such that $\vec{\mathbb{H}} = {}^g\vec{\mathbb{H}}$ and $\phi_H|_{H^0_{0^+}} = {}^g\phi_H|_{H^0_{0^+}}$. Then $g \in G^0$.*

**Proof:** This is an adaptation of [Mur11, Lemma 5.6], so we proceed similarly to Murnaghan's proof therein.

By Proposition 5.1.2, we have $\prod_{j=i}^{d} \phi^j_H|_{H^i_{r^+_{i-1}}} = \prod_{j=i}^{d} {}^g\phi^j_H|_{H^i_{r^+_{i-1}}}$ for $0 \leqslant i \leqslant d$. Letting $Z^i = Z(\mathbb{G}^i)^\circ(F)$ and $\overline{G^i} = [\mathbb{G}^i, \mathbb{G}^i](F) \subset H^i$, we have that $G^i_{r^+} = Z^i_{r^+}\overline{G^i}_{r^+}$ for all

$r \geqslant 0$ [DR09, Lemma B.7.2] and therefore $G^i_{r^+} = Z^i_{r^+} H^i_{r^+}$. As a consequence, we have that $G^i_{r^+_{i-1}} = {}^g G^i_{r^+_{i-1}}$ and

$$\prod_{j=i}^{d} \phi^j|_{G^i_{r^+_{i-1}}} = \prod_{j=i}^{d} {}^g\phi^j|_{G^i_{r^+_{i-1}}} \quad \text{for } 0 \leqslant i \leqslant d. \tag{5.1.1}$$

In particular, setting $i = d-1$ in (5.1.1), we have $\phi^{d-1}\phi^d|_{G^{d-1}_{r^+_{d-2}}} = {}^g(\phi^{d-1}\phi^d)|_{G^{d-1}_{r^+_{d-2}}}$. Because $\phi^d$ is a character of $G$, we have that $\phi^d = {}^g\phi^d$ and therefore $\phi^{d-1}|_{G^{d-1}_{r^+_{d-2}}} = {}^g\phi^{d-1}|_{G^{d-1}_{r^+_{d-2}}}$. Since $r_{d-1} > r_{d-2}$, it follows that $\phi^{d-1}|_{G^{d-1}_{r_{d-1}}} = {}^g\phi^{d-1}|_{G^{d-1}_{r_{d-1}}}$. This last equality implies that $g \in G^{d-1}$ [Mur11, Lemmas 4.2 and 4.4], as explained in [Mur11, Lemma 5.6].

Using the fact that $g \in G^{d-1}$ implies $\phi^{d-1}\phi^d = {}^g(\phi^{d-1}\phi^d)$, we put $i = d-2$ in (5.1.1) and repeat the same argument as above to conclude that $g \in G^{d-2}$. We proceed recursively to conclude that we must have $g \in G^0$. ∎

**Remark 5.1.5.** Note that the statements above are modifications of [Mur11, Proposition 5.4 and Lemma 5.6] to suit our needs. The initial statements are more general and apply to a wider range of characters, namely those that are *$G$-factorizable* in the sense of [Mur11, Definition 5.1].

Also, the initial statements of [Mur11, Proposition 5.4 and Lemma 5.6] contain an extra hypothesis, called $C(G')^+_w$. However, Kaletha's work [Kal19] allows us to omit this hypothesis, as was previously mentioned in Remark 3.2.6.

We are now ready to proceed to the two lemmas that will prove Theorem 5.1.1.

**Lemma 5.1.6.** *Let $C$ be a set of representatives of the double cosets of $H\backslash G/K^d$, and let $s, t \in C$. Assume that $s \neq t$. Then for all $\ell, k \in L$, $\pi_H({}^s\Psi_\ell) \not\simeq \pi_H({}^t\Psi_k)$.*

**Proof:** We show the contrapositive statement. Assume that $\pi_H({}^s\Psi_\ell) \simeq \pi_H({}^t\Psi_k)$. From Theorem 3.2.4, there exists $h \in H$ such that ${}^sK^0_H = {}^{ht}K^0_H$, ${}^s\vec{\mathbb{H}} = {}^{ht}\vec{\mathbb{H}}$ and

$${}^s(\rho_\ell \otimes \phi_H) \simeq {}^{ht}(\rho_k \otimes \phi_H),$$

where $\phi_H = \prod_{i=0}^{d} \phi^i_H|_{H^0}$. Equivalently, $K^0_H = {}^{s^{-1}ht}K^0_H$, $\vec{\mathbb{H}} = {}^{s^{-1}ht}\vec{\mathbb{H}}$ and

$$\rho_\ell \otimes \phi_H \simeq {}^{s^{-1}ht}(\rho_k \otimes \phi_H) \text{ (Lemma A.1.7).}$$

Since $\rho_j|_{H^0_{y,0^+}}$ is 1-isotypic for $j = \ell, k$, it follows that

$$\phi_H|_{H^0_{y,0^+}} = ({}^{s^{-1}ht}\phi_H)|_{H^0_{y,0^+}}.$$

The equivalence becomes an equality when we have characters, as intertwining maps simply correspond to scalar multiplication. Given that the depth of a character does not depend on the point of the building that describes the Moy-Prasad filtration subgroups (Remark 2.3.25), we have that

$$\phi_H|_{H^0_{0+}} = ({}^{s^{-1}ht}\phi_H)|_{H^0_{0+}}.$$

Applying Lemma 5.1.4, we have that $s^{-1}ht \in G^0$.

Recall from Remark 3.2.5 that ${}^{s^{-1}ht}K^0_H = K^0_H$ implies $s^{-1}ht \cdot [y] = [y]$, implying that $s^{-1}ht \in G^0_{[y]} = K^0 \subset K^d$. Hence, $t \in HsK^d$, meaning that $s$ and $t$ belong to the same double coset of $H\backslash G/K^d$. Because $s$ and $t$ are coset representatives, we conclude that they must be equal. ∎

The previous lemma implies that the multiplicity in the restriction $\pi_G(\Psi)|_H$ must come from within $\bigoplus_{\ell \in L} \pi_H(\Psi_\ell)$. The next step is then to determine the condition under which $\pi_H(\Psi_\ell) \simeq \pi_H(\Psi_k)$, $\ell, k \in L$, which is provided by the following lemma.

**Lemma 5.1.7.** *Let $\ell, k \in L$. Then $\pi_H(\Psi_\ell) \simeq \pi_H(\Psi_k)$ if and only if $\rho_\ell \simeq \rho_k$.*

**Proof:** Assume that $\rho_\ell \simeq \rho_k$. Then $\Psi_k$ is an elementary transformation of $\Psi_\ell$ (Section 3.2) and $\pi_H(\Psi_\ell) \simeq \pi_H(\Psi_k)$ (Theorem 3.2.4).

Conversely, if $\pi_H(\Psi_\ell) \simeq \pi_H(\Psi_k)$, we know from Theorem 3.2.4 that there exists $h \in H$ such that $\vec{\mathbb{H}} = {}^h\vec{\mathbb{H}}, K^0_H = {}^hK^0_H$, and $\rho_\ell \otimes \phi_H \simeq {}^h(\rho_k \otimes \phi_H)$, where $\phi_H = \prod_{i=0}^d \phi^i_H|_{H^0}$. Proceeding similarly to the proof of Lemma 5.1.6, we have that $h \in K^0 \cap H = K^0_H$. Because $h \in H^0$, we have $\phi_H = {}^h\phi_H$. We then conclude that $\rho_\ell \simeq {}^h\rho_k$, which implies that $\rho_\ell \simeq \rho_k$ as $h \in K^0_H$ (Lemma A.1.5). ∎

Combining Lemmas 5.1.6 and 5.1.7 together gives us the proof of Theorem 5.1.1.

**Proof of Theorem 5.1.1:**

Let $s \in C$ and $j \in L$. By Lemma 5.1.7, the multiplicity of $\rho_j$ in $\rho_H$ corresponds to the multiplicity of $\pi_H({}^s\Psi_j)$ in $\bigoplus_{\ell \in L} \pi_H({}^s\Psi_\ell)$. Since $\pi_H({}^s\Psi_j) \not\simeq \pi_H({}^t\Psi_k)$ when $s \neq t$ regardless of $k \in L$ (Lemma 5.1.6), this multiplicity corresponds to the multiplicity of $\pi_H({}^s\Psi_j)$ in $\bigoplus_{t \in C} \left( \bigoplus_{\ell \in L} \pi_H({}^t\Psi_\ell) \right)$. ∎

Theorems 4.4.1 and 5.1.1 generalize [Nev15, Theorem 5.2] in which $H = G_{\mathrm{der}}$ and $\Psi$ is a toral datum of length one. In this case $\rho$ is assumed trivial and $\mathrm{Res}(\Psi)$ consists of only one element (Corollary 3.3.3), meaning that the restriction $\pi_G(\Psi)|_H$ is automatically multiplicity free as per the following corollary.

**Corollary 5.1.8** ([Nev15, Theorem 5.2]). *Let $\Psi = ((\mathbb{T},\mathbb{G}), y, r, 1, (\phi^0,\phi^1))$ be a toral datum of length one. Let $\Psi_{\mathrm{der}}$ denote the restriction to $G_{\mathrm{der}}$ of $\Psi$ as per Corollary 3.3.3. Then $\pi_G(\Psi)$ decomposes with multiplicity one as*

$$\pi_G(\Psi)|_H \simeq \bigoplus_{t\in C} \pi_{G_{\mathrm{der}}}({}^t\Psi_{\mathrm{der}}),$$

*where $C$ is a set of coset representatives of $G_{\mathrm{der}}\backslash G/K^1$.*

We end this section by mentioning that Theorem 5.1.1 applies to both positive-depth and depth-zero supercuspidal representations as they are all produced from the J.K. Yu Construction. This allows us to restate our result as follows.

**Corollary 5.1.9.** *Let $\ell \in L$. Then the multiplicity of $\pi_H(\Psi_\ell)$ in $\pi_G(\Psi)|_H$ is equal to the multiplicity of $\pi_\ell^{-1}$ in $\pi^{-1}|_{H^0}$, where $\pi^{-1} := \text{c-Ind}_{K^0}^{G^0}\rho$ and $\pi_\ell^{-1} := \text{c-Ind}_{K_H^0}^{H^0}\rho_\ell$. In particular, $\pi_G(\Psi)|_H$ is multiplicity free if and only if $\pi^{-1}|_{H^0}$ is multiplicity free.*

**Proof:** We know from Theorem 5.1.1 that the multiplicity of $\pi_H(\Psi_\ell)$ in $\pi_G(\Psi)|_H$ is equal to the multiplicity of $\rho_\ell$ in $\rho_H$. On the other hand, if we consider the datum $\Psi^0 = (G^0, y, \rho, 1)$, we have $\pi_{G^0}(\Psi^0) = \pi^{-1}$. Then, Theorem 5.1.1 tells us that the multiplicity of $\pi_{H^0}(\Psi_\ell^0) = \pi_\ell^{-1}$ in $\pi_{G^0}(\Psi^0)|_{H^0}$ is equal to the multiplicity of $\rho_\ell$ in $\rho_H$. The result thus follows. ∎

The statement of Corollary 5.1.9 makes it clear that the multiplicity upon restriction of a positive-depth supercuspidal representation really depends on the depth-zero supercuspidal representation of the reduced datum. Therefore, the question of determining multiplicities in the restriction of an irreducible supercuspidal representation reduces to an equivalent question for irreducible depth-zero supercuspidal representations.

## 5.2 Determining the Multiplicity in the Restriction

In this section, our goal is to obtain a more precise decomposition of $\rho_H$ and know the multiplicity of each component. By Theorem 5.1.1, this will provide us with the multiplicity in $\pi_G(\Psi)|_H$. In order to achieve this, we will need some results from Clifford theory. We start this section by reviewing those results.

### 5.2.1 Some Clifford Theory

In this subsection, $G$ simply denotes a group, and $N$ is a normal subgroup of $G$. When $G$ and $N$ are finite groups, Clifford theory allows us to describe the relation between representations of $G$ and those of $N$. In particular, if $\pi$ is an irreducible representation of $G$, this theory offers a very precise description of $\pi|_N$. Indeed, every two components of $\pi|_N$ are conjugate and every component appears with the same multiplicity. Furthermore, this multiplicity can be explicitly calculated. A summary of Clifford theory can for example be found in [CSST09].

We can adapt the proofs from [CSST09] for groups $G$ and $N$ which are not finite. We will proceed to do so in this section, but require a few hypotheses. For the remainder of this section, we assume that $N$ has finite index $[G : N]$ in $G$. We also need $N$ to be open, closed and containing an open compact subgroup so that we may apply Mackey theory and Frobenius Reciprocity. Furthermore, we assume that $\pi$ is an irreducible finite-dimensional representation of $G$ and that $\pi|_N$ is completely reducible. The main result that we will take away from this section is Proposition 5.2.7, but we must establish some new notation and a few intermediate results before we get there.

**Definition 5.2.1.** *Let $\pi'$ be an irreducible representation of $N$. The set $I_G(\pi') = \{g \in G : {}^g\pi' \simeq \pi'\}$ is called the normalizer of $\pi'$ in $G$, or the inertial subgroup of $\pi'$ (in $G$).*

A consequence of Mackey theory is that we can obtain a condition for the induction of a representation to be irreducible, known as the Mackey Irreducibility Criterion. In particular, when $\pi'$ is an irreducible representation of $N$, the condition for $\text{c-Ind}_N^G \pi'$ to be irreducible is phrased in terms of the inertial subgroup of $\pi'$. Indeed, we have that $\text{c-Ind}_N^G \pi'$ is irreducible if and only if $I_G(\pi') = N$.

**Definition 5.2.2.** *Let $\pi'$ be an irreducible representation of $N$, and let $I = I_G(\pi')$. We define $\widehat{I}(\pi')$ to be the set of irreducible representations of $I$ which are contained in $\text{c-Ind}_N^I \pi'$.*

**Lemma 5.2.3.** *Let $\pi'$ be an irreducible representation of $N$ and let $I = I_G(\pi')$. Then $\text{Res}_N^I \text{c-Ind}_N^I \pi'$ is $\pi'$-isotypic. In particular, for all $\Pi \in \widehat{I}(\pi')$, we have that $\Pi|_N$ is $\pi'$-isotypic.*

**Proof:** From the Mackey Decomposition, we have that $\text{Res}_N^I \text{c-Ind}_N^I \pi' = \bigoplus_{c \in C} {}^c\pi'$, where $C$ is a set of coset representatives of $I/N$. In particular, every element $c \in C$ lies in $I$, meaning that ${}^c\pi' \simeq \pi'$. The result thus follows. ∎

**Lemma 5.2.4.** *Let $\pi'$ be an irreducible representation of $N$ and let $I = I_G(\pi')$. Then $\text{c-Ind}_I^G \Pi$ is irreducible for all $\Pi \in \widehat{I}(\pi')$.*

**Proof:** By the Mackey Irreducibility Criterion, showing that $\text{c-Ind}_I^G \Pi$ is irreducible is equivalent to showing that the normalizer of $\Pi$ in $G$ is equal to $I$. Take $g \in G$ such that ${}^g\Pi \simeq \Pi$. We must show that $g \in I$. Since $N$ is a normal subgroup, we have that ${}^g(\Pi|_N) \simeq \Pi|_N$. Since $\Pi|_N$ is $\pi'$-isotypic (Lemma 5.2.3), we conclude that ${}^g\pi' \simeq \pi'$ and that $g \in I$. ∎

**Lemma 5.2.5.** *Let $\pi'$ be an irreducible representation of $N$, $I = I_G(\pi')$ and $\pi$ be an irreducible representation of $G$ such that $\pi|_N$ contains $\pi'$. Then $\pi = \text{c-Ind}_I^G \Pi$ for some $\Pi \in \widehat{I}(\pi')$.*

**Proof:** Since $\pi' \subset \pi|_N$, Frobenius Reciprocity tells us that $\pi \subset \text{c-Ind}_N^G \pi'$. Since $I$ is a subgroup that lies between $G$ and $N$, we have that $\text{c-Ind}_N^G \pi' = \text{c-Ind}_I^G(\text{c-Ind}_N^I \pi')$. According to Lemma 5.2.4, every component of $\text{c-Ind}_N^I \pi'$ (or equivalently every element of $\widehat{I}(\pi')$) induces irreducibly to $G$. This means that the components of $\text{c-Ind}_N^G \pi'$ are of the form $\text{c-Ind}_I^G \Pi$ for some $\Pi \in \widehat{I}(\pi')$. ∎

**Lemma 5.2.6.** *Let $\pi'$ be an irreducible representation of $N$, $I = I_G(\pi')$ and $\pi$ be a representation of $G$ such that $\pi|_N$ contains $\pi'$. Let $\Pi$ be an element of $\widehat{I}(\pi')$ such that $\pi = \text{c-Ind}_I^G \Pi$. Then $\Pi|_N = m\pi'$, where*

$$m = \frac{\dim(\pi)}{[G : I]\dim(\pi')},$$

*and $\dim(\pi)$ and $\dim(\pi')$ denote the dimensions of $\pi$ and $\pi'$, respectively.*

**Proof:** Recall from Lemma 5.2.3 that $\Pi|_N$ is $\pi'$-isotypic. Therefore $\Pi|_N = m\pi'$ for some $m > 0$, and $\dim(\Pi|_N) = m\dim(\pi')$. Since $\Pi$ and $\Pi|_N$ act on the same vector space, their dimensions are the same, implying that

$$m = \frac{\dim(\Pi)}{\dim(\pi')}.$$

On the other hand, we have that $\pi = \text{c-Ind}_I^G \Pi$, meaning that $\dim(\pi) = [G : I]\dim(\Pi)$ (Proposition A.1.8), or equivalently

$$\dim(\Pi) = \frac{\dim(\pi)}{[G : I]}.$$

The result then follows by combining this equation for $\dim(\Pi)$ with the above equation for $m$.

∎

Finally, we arrive to the main result of this section.

**Proposition 5.2.7.** *Let $\pi'$ be an irreducible representation of $N$, $I = I_G(\pi')$ and $\pi$ be a representation of $G$ such that $\pi|_N$ contains $\pi'$. Then*

$$\operatorname{Res}_N^G \pi = m\left(\bigoplus_{c\in C} {}^c\pi'\right),$$

*where*

$$m = \frac{\dim(\pi)}{[G:I]\dim(\pi')},$$

*and $C$ is a set of coset representatives of $N\backslash G/I$. Furthermore, every component of $\bigoplus_{c\in C} {}^c\pi'$ is distinct, meaning that every component of $\operatorname{Res}_N^G \pi$ has multiplicity $m$.*

**Proof:** By Lemma 5.2.5, we have that $\pi = \text{c-Ind}_I^G \Pi$ for some $\Pi \in \widehat{I}(\pi')$. Using the Mackey Decomposition, we obtain

$$\operatorname{Res}_N^G \pi = \operatorname{Res}_N^G \text{c-Ind}_I^G \Pi = \bigoplus_{c\in C} \text{c-Ind}_{N\cap {}^cI}^N \operatorname{Res}_{N\cap {}^cI}^{{}^cI} {}^c\Pi,$$

where $C$ is a set of coset representatives of $N\backslash G/I$. Because $N$ is a normal subgroup of $G$ and $N \subset I$, this simplifies as

$$\operatorname{Res}_N^G \pi = \bigoplus_{c\in C} \operatorname{Res}_N^{{}^cI} {}^c\Pi.$$

By Lemma 5.2.6, we have that $\operatorname{Res}_N^I \Pi = m\pi'$, where

$$m = \frac{\dim(\pi)}{[G:I]\dim(\pi')}.$$

Therefore, we obtain the stated decomposition of $\operatorname{Res}_N^G \pi$.

Furthermore, we note that having ${}^c\pi' \simeq {}^d\pi'$, $c, d \in C$ implies that $d^{-1}c \in I$ and therefore $c$ and $d$ lie in the same double coset. This means that every component of $\bigoplus_{c\in C} {}^c\pi'$ is distinct. ∎

### 5.2.2 Formula for the Multiplicity

We now wish to apply the results of the previous section in order to obtain a description of $\rho_H = \rho|_{K_H^0}$, where $K_H^0 = H_{[y]}^0$. The quotient $K^0/K_H^0$ might not be of finite index, so we cannot necessarily apply Clifford theory directly by setting $G = K^0$ and $N = K_H^0$. However, because $K^0$ is compact modulo the center of $G^0, Z^0$, the index $[K^0 : Z^0K_H^0]$ is finite (Remark 3.1.4). Furthermore, $Z^0K_H^0$ is open (and closed) and contains a compact open subgroup. Therefore, we can apply the Clifford theory from the previous section to $\rho|_{Z^0K_H^0}$ by setting $G = K^0$ and $N = Z^0K_H^0$.

Let us first recall the more refined description of $\rho_H$ from Remark 3.3.12. Fix a component $(\rho', V')$ of $(\rho_H, V)$. Then the set $L$ is a finite subset of a set of coset representatives of $K^0/K_H^0$ and, for all $\ell \in L$, $\rho_\ell = {}^\ell\rho'$ so that

$$\rho_H = \bigoplus_{\ell \in L} \rho_\ell = \bigoplus_{\ell \in L} {}^\ell\rho'.$$

**Lemma 5.2.8.** *Let $(\rho', V')$ be a component of $(\rho_H, V)$. Then $(\overline{\rho'}, V')$ defined by $\overline{\rho'}(zh) = \rho(z)\rho'(h)$ for all $z \in Z^0, h \in K_H^0$ is a component of $(\rho|_{Z^0K_H^0}, V)$.*

**Proof:** Firstly, the representation $\overline{\rho'}$ is well-defined. Indeed, we see from the above description that $\rho|_{K_H^0 \cap Z^0} = \bigoplus_{\ell \in L} \rho'|_{K_H^0 \cap Z^0}$. Therefore, $\rho(z)|_{V'} = \rho'(z)$ for all $z \in K_H^0 \cap Z^0$.

It is clear that $\overline{\rho'}$ is a subrepresentation of $\rho|_{Z^0K_H^0}$ by definition. Furthermore, because $Z^0$ acts by scalar multiplication through $\rho$ (Lemma A.1.4), any $Z^0K_H^0$-invariant subspace of $V'$ is also a $K_H^0$-invariant subspace of $V'$. This means that $\overline{\rho'}$ is irreducible, and is therefore a component of $\rho|_{Z^0K_H^0}$. ∎

**Lemma 5.2.9.** *Let $(\rho', V')$ and $(\overline{\rho'}, V')$ be as per the previous lemma. Then $I_{K^0}(\overline{\rho'}) = I_{K^0}(\rho')$.*

**Proof:** Let $g \in K^0$. We must show that ${}^g\overline{\rho'} \simeq \overline{\rho'}$ if and only if ${}^g\rho' \simeq \rho'$. It is clear that if ${}^g\overline{\rho'} \simeq \overline{\rho'}$, then ${}^g\rho' \simeq \rho'$ since $\rho' = \overline{\rho'}|_{K_H^0}$.

For the converse, assume that $\rho' \simeq {}^g\rho'$ and let $f$ be an invertible intertwining map between ${}^g\rho'$ and $\rho'$. Then, for all $h \in K_H^0, v \in V'$, ${}^g\rho'(h) \circ f(v) = f \circ \rho'(h)(v)$. One then sees that for every constant $k \in \mathbb{C}$,

$$k\,{}^g\rho'(h)(f(v)) = f(k\rho'(h)(v)).$$

Because $Z^0$ acts by scalar multiplication, for all $z \in Z^0$, we have that $\rho(z) = c_z \mathrm{Id}_V$ for some $c_z \in \mathbb{C}$, where $\mathrm{Id}_V$ is the identity map on $V$. Therefore, for all $z \in Z^0$ we obtain that

$$c_z\,{}^g\rho'(h)(f(v)) = f(c_z\rho'(h)(v)),$$

or equivalently,

$${}^g\overline{\rho'}(zh)(f(v)) = f(\overline{\rho'}(zh)(v)).$$

Hence $f$ is an invertible intertwining map between ${}^g\overline{\rho'}$ and $\overline{\rho'}$. The result follows. ∎

**Proposition 5.2.10.** *Let $\rho'$ be a component of $\rho_H = \rho|_{K_H^0}$, $I = I_{K^0}(\rho')$ and*

$$m = \frac{\dim(\rho)}{[K^0 : I]\dim(\rho')}.$$

*Then*

$$\rho_H = m\left(\bigoplus_{d\in D} {}^d\rho'\right),$$

*where $D$ is a set of coset representatives of $K_H^0\backslash K^0/I$. In other words, each component of $\rho_H$ has multiplicity $m$.*

**Proof:** Let $\overline{\rho'}$ be the extension of $\rho'$ to $Z^0K_H^0$ as per Lemma 5.2.8, which is a component of $\rho|_{Z^0K_H^0}$. Since $Z^0K_H^0$ is of finite index in $K^0$ and that all representations at play are finite-dimensional, we apply Proposition 5.2.7 and obtain

$$\rho|_{Z^0K_H^0} = m\left(\bigoplus_{d\in D} {}^d\overline{\rho'}\right),$$

where

$$m = \frac{\dim(\rho)}{[K^0 : I_{K^0}(\overline{\rho'})]\dim(\overline{\rho'})},$$

and $D$ is a set of coset representatives of $Z^0K_H^0\backslash K^0/I_{K^0}(\overline{\rho'})$. Because $Z^0 \subset I_{K^0}(\overline{\rho'})$, we note that $Z^0K_H^0\backslash K^0/I_{K^0}(\overline{\rho'}) = K_H^0\backslash K^0/I_{K^0}(\overline{\rho'})$. Since $I_{K^0}(\overline{\rho'}) = I$ (Lemma 5.2.9) and $\dim(\overline{\rho'}) = \dim(\rho')$, $D$ is a set of coset representatives of $K_H^0\backslash K^0/I$ and

$$m = \frac{\dim(\rho)}{[K^0 : I]\dim(\rho')}.$$

Restricting $\rho|_{Z^0K_H^0}$ to $K_H^0$, we conclude that

$$\rho_H = m\left(\bigoplus_{d\in D} {}^d\rho'\right).$$

∎

As a corollary of Proposition 5.2.10 and Theorem 5.1.1, we can refine our description of $\pi_G(\Psi)|_H$.

**Corollary 5.2.11.** *Let $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$ be a $G$-datum, $\rho'$ be a component of $\rho_H$ and $I = I_{K^0}(\rho')$. Let $D$ be a set of coset representatives of $K_H^0\backslash K^0/I$, and for all $d \in D$ let $\Psi_d$ be the $H$-datum associated to ${}^d\rho'$. Then*

$$\pi_G(\Psi)|_H = m\left(\bigoplus_{t\in C, d\in D} \pi_H({}^t\Psi_d)\right),$$

*where*

$$m = \frac{\dim(\rho)}{[K^0 : I]\dim(\rho')},$$

*and $C$ is a set of coset representatives of $H\backslash G/K^d$. Furthermore, every component of $\bigoplus_{t\in C, d\in D} \pi_H({}^t\Psi_d)$ is distinct, and every component of $\pi_G(\Psi)|_H$ has multiplicity $m$.*

**Proof:** Given the decomposition $\rho_H = \bigoplus_{\ell\in L} \rho_\ell$, it follows from Theorem 4.4.1 that

$$\pi_G(\Psi)|_H \simeq \bigoplus_{t\in C}\left(\bigoplus_{\ell\in L}\pi_H({}^t\Psi_\ell)\right),$$

where $C$ is a set of representatives of the double cosets of $H\backslash G/K^d$. Therefore, using the decomposition of $\rho_H$ provided by Proposition 5.2.10, we can rewrite the restriction as

$$\pi_G(\Psi)|_H = \bigoplus_{t\in C}\left(\bigoplus_{d\in D} m\pi_H({}^t\Psi_d)\right) = m\left(\bigoplus_{t\in C, d\in D}\pi_H({}^t\Psi_d)\right),$$

where

$$m = \frac{\dim(\rho)}{[K^0 : I]\dim(\rho')}.$$

Furthermore, it follows from Theorem 5.1.1 that each component of $\bigoplus_{t\in C, d\in D}\pi_H({}^t\Psi_d)$ is distinct. ∎

**Remark 5.2.12.** Because $\overline{\rho'} \subset \rho|_{Z^0K_H^0}$, we know from Frobenius Reciprocity that $\rho \subset \text{c-Ind}_{Z^0K_H^0}^{K^0}\overline{\rho'}$. This means that $\dim(\rho) \leqslant [K^0 : Z^0K_H^0]\dim(\rho')$, or equivalently

$$\frac{\dim(\rho)}{\dim(\rho')} \leqslant [K^0 : Z^0K_H^0].$$

It follows that

$$m \leqslant \frac{[K^0 : Z^0K_H^0]}{[K^0 : I]} = [I : Z^0K_H^0],$$

as $Z^0K_H^0 \subset I$. Therefore, the value of the multiplicity is directly related to how close $I$ is from being equal to $Z^0K_H^0$. In particular, when $G^0 = Z^0H^0$ (or the central isogeny for $\mathbb{G}^0$ is defined over $F$), we have $m = 1$.

# Chapter 6

# Regularity for a Multiplicity Free Restriction

Given a $G$-datum $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$, we saw in Chapter 5 that the multiplicity in $\pi_G(\Psi)|_H$ is related to the multiplicity in $\rho_H = \rho|_{K_H^0}$. In particular, if $\rho'$ is a component of $\rho_H$, we saw that the mutiplicity of each component of $\pi_G(\Psi)|_H$ is given by

$$m = \frac{\dim(\rho)}{[K^0 : I]\dim(\rho')},$$

where $I = I_{K^0}(\rho')$ is the normalizer of $\rho'$ in $K^0$ (Corollary 5.2.11). Therefore, when one can single out a component of $\rho_H$ and compute its normalizer in $K^0$, the multiplicity $m$ can be explicitly calculated. The goal of this chapter is to provide an example of a class of representations $\rho$ for which it is possible to explicitly isolate components $\rho'$ in $\rho_H$ and derive conditions for a multiplicity free restriction.

Because $\mathbb{H}^0$ is such that $[\mathbb{G}^0, \mathbb{G}^0] \subset \mathbb{H}^0$, we can alleviate notation and assume that $\rho$ is an irreducible representation of $G_{[y]}$ that induces irreducibly to a depth-zero supercuspidal representation of $G$ that we wish to restrict to $H_{[y]}$ (that is replace $\mathbb{G}^0$ by $\mathbb{G}$ and $\mathbb{H}^0$ by $\mathbb{H}$). What we know about $\rho$ is that $\rho|_{G_{y,0}}$ contains the pullback of a cuspidal representation $\sigma$ of $G_{y,0:0^+}$ (Theorem 2.3.26), where $G_{y,0:0^+} = \mathcal{G}_y(\mathfrak{f})$ for some reductive group $\mathcal{G}_y$ (Section 2.2.2.2). However, we need more information to compute its dimension and isolate one of the components of $\rho_H = \rho|_{H_{[y]}}$ to compute $m$.

In this chapter, we restrict our attention to when $\rho$ induces to a *regular* depth-zero supercuspidal representation in the sense of [Kal19, Definition 3.4.19]. The regular depth-zero supercuspidal representations are a very important class. Indeed, Kaletha points out that most depth-zero supercuspidal representations are regular (paragraph following [Kal19, Definition 3.7.3]).

In particular, having $\text{c-Ind}_{G_{[y]}}^{G} \rho$ be regular means that $\sigma$ is a *Deligne-Lusztig cuspidal representation.* The definition of a Deligne-Lusztig cuspidal representation requires the notion of a *Deligne-Lusztig virtual character.* Therefore, we will start by defining these kinds of characters, along with some of their properties, in Section 6.1, before discussing the Deligne-Lusztig cuspidal representations in Section 6.2.

Kaletha provides a method for constructing regular irreducible supercuspidal representations of depth zero in [Kal19], which we recall in Section 6.3. As a consequence of this construction, we will obtain a specific description of $\rho$, that gives us more information than just its restriction to $G_{y,0}$. In particular, that description of $\rho$ will allow us to describe the components of $\rho_H = \rho|_{H_{[y]}}$ (Proposition 6.4.6) and derive a condition for a multiplicity free restriction (Corollary 6.4.10).

Finally, we end this chapter with a few remarks regarding the statement of our multiplicity free result and some remaining open questions related to computing multiplicities in Section 6.5.

## 6.1 Deligne-Lusztig Virtual Characters

In this section, we temporarily change our focus and let $\mathbb{G}$ simply denote a reductive group over an algebraically closed field of characteristic $p$. Let $\mathrm{Fr} : \mathbb{G} \to \mathbb{G}$ be a Frobenius map (as per [Car93, Section 1.17]) and $\mathbb{G}^{\mathrm{Fr}} = \{g \in \mathbb{G} : \mathrm{Fr}(g) = g\}$. For our purposes, it suffices to know that the map Fr is a surjective homomorphism for which $\mathbb{G}^{\mathrm{Fr}}$ is finite. The group $\mathbb{G}^{\mathrm{Fr}}$ is called a *finite group of Lie type.*

**Remark 6.1.1.** If $\mathbb{G}$ is a reductive group defined over the finite field $\mathfrak{f}$, then $\mathbb{G}(\mathfrak{f})$ is a finite group of Lie type. Indeed, one can view $\mathbb{G}(\mathfrak{f})$ as the $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$-fixed points of $\mathbb{G}$. In this case, the Galois group of every finite extension $\mathfrak{f}'$ of $\mathfrak{f}$ such that $\mathfrak{f} \subset \mathfrak{f}' \subset \bar{\mathfrak{f}}$ is generated by a single element, which is a Frobenius map of $\mathbb{G}$. Taking the inverse limit of all those maps gives us a Frobenius map Fr of $\mathbb{G}$ which consists of taking the $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$-points of $\mathbb{G}$. This means that $\mathbb{G}^{\mathrm{Fr}} = \mathbb{G}(\mathfrak{f})$ is a finite group of Lie type.

Furthermore, there exists a class of finite groups of Lie type which do not correspond to taking rational points of a reductive group, called the Suzuki and Ree groups [Car93, Section 1.19]. So, while one of our underlying goals in this chapter is to talk about (Deligne-Lusztig) virtual characters of $\mathbb{G}(\mathfrak{f}) = \mathcal{G}_y(\mathfrak{f})$, the theory of such characters applies in the more general context of finite groups of Lie type. As such, we will formulate our results in this more general setting to remain faithful to the literature on the topic.

Deligne and Lusztig defined a class of generalized (or virtual) characters of the group $\mathbb{G}^{\mathrm{Fr}}$ in [DL76]. The theory of such characters was reviewed in great detail in [Car93]. The main goal of this section is to obtain the formula for the restriction of one of their virtual characters (Theorem 6.1.9) as this will prove useful when restricting $\rho$. Since we did not find such a formula in the existing literature, we provide all the details in this section, starting from the definition of a virtual character to doing explicit calculations with the character formulas.

### 6.1.1 Formulas for Virtual Characters

Given a maximal torus $\mathbb{T}$ of $\mathbb{G}$ which is Fr-stable and a complex character $\theta$ of $\mathbb{T}^{\mathrm{Fr}}$, we can construct a *virtual character* $R^{\mathbb{G}}_{\mathbb{T},\theta}$ which maps $\mathbb{G}^{\mathrm{Fr}}$ into $\mathbb{C}$. If the context is obvious, we simply write $R_{\mathbb{T},\theta}$ for $R^{\mathbb{G}}_{\mathbb{T},\theta}$.

The original description of $R_{\mathbb{T},\theta}$ provided in [DL76] relies on $l$-adic cohomology groups with compact support of a variety $\tilde{X}$, where $l$ is a prime number different from $p$. These $l$-adic cohomology groups are denoted by $H^i_c(\tilde{X}, \overline{\mathbb{Q}}_l)$, $i \geqslant 0$. We will not provide the definition for these cohomology groups here, as it turns out that we will be able to bypass this notion shortly. The curious reader is invited to consult [DL76, Car93] for more details on these cohomology groups.

To define the variety $\tilde{X}$, we choose a Borel group $\mathbb{B}$ that contains $\mathbb{T}$. Note that this Borel subgroup need not be Fr-stable. We have a Levi decomposition of $\mathbb{B}$, $\mathbb{B} = \mathbb{U}\mathbb{T}$, where $\mathbb{U} \subset \mathbb{G}_{\mathrm{der}}$ is some unipotent subgroup. Then, we define the variety $\tilde{X}$ as $\tilde{X} := L^{-1}_{\mathbb{G}}(\mathbb{U})$, where $L_{\mathbb{G}}$ is the Lang map of $\mathbb{G}$ defined by

$$\begin{aligned} L_{\mathbb{G}} : \mathbb{G} &\to \mathbb{G} \\ g &\mapsto g^{-1}\,\mathrm{Fr}(g). \end{aligned}$$

Note that the Lang-Steinberg Theorem states that this map is surjective.

The groups $\mathbb{G}^{\mathrm{Fr}}$ and $\mathbb{T}^{\mathrm{Fr}}$ act on $\tilde{X}$ by left multiplication and right multiplication, respectively [Car93, Section 7.2]. Those actions then induce actions on the cohomology groups $H^i_c(\tilde{X}, \overline{\mathbb{Q}}_l)$. The original formula for the virtual character $R_{\mathbb{T},\theta}$ provided in [DL76, Section 1.20] is based on the action of $\mathbb{G}^{\mathrm{Fr}}$ and is given as follows:

$$R_{\mathbb{T},\theta}(g) = \sum_{i \geqslant 0} (-1)^i \mathrm{trace}(g, H^i_c(\tilde{X}, \overline{\mathbb{Q}}_l)_\theta) \text{ for all } g \in \mathbb{G}^{\mathrm{Fr}},$$

where $H^i_c(\tilde{X}, \overline{\mathbb{Q}}_l)_\theta$ is the $\mathbb{T}^{\mathrm{Fr}}$-submodule of $H^i_c(\tilde{X}, \overline{\mathbb{Q}}_l)$ on which $\mathbb{T}^{\mathrm{Fr}}$ acts by the character $\theta$.

**Remark 6.1.2.** For a unipotent element $u \in \mathbb{G}^{\mathrm{Fr}}$, we set

$$Q^{\mathbb{G}}_{\mathbb{T}}(u) = R_{\mathbb{T},1}(u).$$

The map $Q^{\mathbb{G}}_{\mathbb{T}}$ from the set of unipotent elements is called a *Green function*. The Green function $Q^{\mathbb{G}}_{\mathbb{T}}(u)$ is integer-valued [DL76, Section 4].

The first thing that we might think when looking at the formula for $R_{\mathbb{T},\theta}$ is that it depends on the choice of Borel subgroup $\mathbb{B}$. However, it was shown that it actually does not [DL76, Corollary 4.3],[Car93, Proposition 7.3.6]. The second thing that we might think is that this formula for $R_{\mathbb{T},\theta}$ does not seem easy to work with at first glance. Luckily, there is a way to bypass the $l$-adic cohomology subgroups and obtain an alternate formula for $R_{\mathbb{T},\theta}$.

By what precedes above, for all $g \in \mathbb{G}^{\mathrm{Fr}}, t \in \mathbb{T}^{\mathrm{Fr}}$, we can view $(g,t)$ as an automorphism of $\tilde{X}$, defined by $(g,t)\tilde{x} = g\tilde{x}t$ for all $\tilde{x} \in \tilde{X}$. By [Car93, Property 7.1.3], this automorphism induces a non-singular map of $H^i_c(\tilde{X}, \overline{\mathbb{Q}}_l)$ into itself. This action allows us to define what we call the *Lefschetz number* of $(g,t)$ on $\tilde{X}$,

$$\mathscr{L}_{\mathbb{G}}((g,t),\tilde{X}) = \sum_{i\geqslant 0}(-1)^i \mathrm{trace}((g,t), H^i_c(\tilde{X}, \overline{\mathbb{Q}}_l)).$$

Note that the Lefschetz number is an integer that does not depend on the choice of the prime number $l$ ($l \neq p$) [Car93, Property 7.1.4].

Now, we have that the Frobenius map commutes with $(g,t)$ as maps of $\tilde{X}$. Indeed, given that $\mathrm{Fr}(g) = g$ and $\mathrm{Fr}(t) = t$, we obtain that for all $x \in \tilde{X}$

$$(\mathrm{Fr} \circ (g,t))(x) = \mathrm{Fr}(gxt) = g\,\mathrm{Fr}(x)t = ((g,t) \circ \mathrm{Fr})(x).$$

Following [Car93, Appendix (h)], an application of Grothendieck's trace formula allows us to obtain

$$\mathscr{L}_{\mathbb{G}}((g,t),\tilde{X}) = \lim_{s\to\infty} \overline{R}_{\mathbb{G}}(s),$$

where $\overline{R}_{\mathbb{G}}(s)$ is the rational function with power series expansion $-\sum_{n=1}^{\infty}\left|\tilde{X}^{\mathrm{Fr}^n \circ (g,t)^{-1}}\right| s^n$. Here $\tilde{X}^{\mathrm{Fr}^n \circ (g,t)^{-1}}$ denotes the elements of $\tilde{X}$ that are fixed by the map $\mathrm{Fr}^n \circ (g,t)^{-1}$, that is $\tilde{X}^{\mathrm{Fr}^n \circ (g,t)^{-1}} = \{\tilde{x} \in \tilde{X} : (\mathrm{Fr}^n \circ (g,t)^{-1})(\tilde{x}) = \tilde{x}\}$. The Lefschetz number is what allows us to obtain an alternate formula for $R_{\mathbb{T},\theta}$ that is no longer phrased in terms of $l$-adic cohomology subgroups, as per the following proposition.

**Proposition 6.1.3** ([Car93, Proposition 7.2.3])**.** *For all $g \in \mathbb{G}^{\mathrm{Fr}}$,*

$$R_{\mathbb{T},\theta}(g) = \frac{1}{|\mathbb{T}^{\mathrm{Fr}}|}\sum_{t\in\mathbb{T}^{\mathrm{Fr}}} \theta(t^{-1})\mathscr{L}_{\mathbb{G}}((g,t),\tilde{X}),$$

*where $\mathscr{L}_{\mathbb{G}}((g,t),\tilde{X})$ is the Lefschetz number of $(g,t)$ on $\tilde{X}$.*

This formula will be of great use to us in Section 6.1.2. There is a second alternate formula for $R_{\mathbb{T},\theta}$ that is also accessible for the computations we will do in Section 6.1.3. This formula makes use of the Green function defined above.

**Theorem 6.1.4** ([DL76, Theorem 4.2],[Car93, Theorem 7.2.8])**.** *Let $g \in \mathbb{G}^{\mathrm{Fr}}$ and assume its Jordan decomposition is given by $g = su = us$, where $s$ is semisimple and $u$ is unipotent. Then*

$$R_{\mathbb{T},\theta}(g) = \frac{1}{|(C_{\mathbb{G}}(s)^\circ)^{\mathrm{Fr}}|}\sum_{\substack{x\in\mathbb{G}^{\mathrm{Fr}}\\ x^{-1}sx\in\mathbb{T}^{\mathrm{Fr}}}} \theta(x^{-1}sx)Q^{C_{\mathbb{G}}(s)^\circ}_{x\mathbb{T}x^{-1}}(u).$$

### 6.1.2 Virtual Characters and Isomorphisms

One natural question that can be asked about the virtual character $R_{\mathbb{T},\theta}$ is whether or not it commutes in a sense with isomorphisms of algebraic groups. For instance, if $\mathbb{G}'$ is a reductive group such that there is an isomorphism of algebraic groups $f : \mathbb{G} \to \mathbb{G}'$, then $\mathrm{Fr}_f = f \circ \mathrm{Fr} \circ f^{-1}$ is a Frobenius map of $\mathbb{G}'$. Note that $f$ will map $\mathbb{G}^{\mathrm{Fr}}$ to $(\mathbb{G}')^{\mathrm{Fr}_f}$. Indeed, for all $g \in \mathbb{G}^{\mathrm{Fr}}$, $\mathrm{Fr}_f(f(g)) = f(\mathrm{Fr}(g)) = f(g)$, and therefore $f(g) \in (\mathbb{G}')^{\mathrm{Fr}_f}$. Furthermore, $f$ maps the maximal Fr-stable torus $\mathbb{T}$ of $\mathbb{G}$ to a maximal $\mathrm{Fr}_f$-stable torus $\mathbb{T}'$ of $\mathbb{G}'$. A natural question that one might ask is if for all $g \in \mathbb{G}^{\mathrm{Fr}}$, $R_{\mathbb{T},\theta}(g) = R_{\mathbb{T}',\theta\circ f^{-1}}(f(g))$. While it is not hard to believe that this last equality is certainly true, no reference in the literature was found in these regards. Therefore, we will prove this statement in Corollary 6.1.8 of this section, to which we build up with a few lemmas. We will see that the result is simply an application of the various definitions we have seen in the last section.

Since $f$ is an isomorphism of algebraic groups, $\mathbb{B}' = f(\mathbb{B})$ is a Borel subgroup of $\mathbb{G}'$ that contains $\mathbb{T}'$, with $\mathbb{B}' = \mathbb{U}'\mathbb{T}'$, where $\mathbb{U}' = f(\mathbb{U})$. Let $\tilde{X}_f = L_{\mathbb{G}'}^{-1}(\mathbb{U}')$, where

$$
\begin{aligned}
L_{\mathbb{G}'} : \mathbb{G}' &\to \mathbb{G}' \\
g &\mapsto g^{-1}\,\mathrm{Fr}_f(g)
\end{aligned}
$$

is the Lang map of $\mathbb{G}'$.

**Lemma 6.1.5.** *The variety $\tilde{X}_f$ is equal to $f(\tilde{X})$.*

**Proof:** We have that $x_f \in \tilde{X}_f$ if and only if $x_f^{-1}\,\mathrm{Fr}_f(x_f) = u'$ for some $u' \in \mathbb{U}'$. Applying $f^{-1}$ and simplifying, we obtain an equivalent condition which is that of $(f^{-1}(x_f))^{-1}\,\mathrm{Fr}(f^{-1}(x_f)) = f^{-1}(u')$. This last condition is satisfied if and only if $f^{-1}(x_f) \in \tilde{X}$, which is equivalent to $x_f \in f(\tilde{X})$.

∎

Given $g \in \mathbb{G}^{\mathrm{Fr}}, t \in \mathbb{T}^{\mathrm{Fr}}$, $(f(g), f(t))$ is an automorphism of $\tilde{X}_f = f(\tilde{X})$. We also have that $(f(g), f(t))$ commutes with $\mathrm{Fr}_f$, as $(g,t)$ commutes with Fr. The Lefschetz number of $(f(g), f(t))$ on $\tilde{X}_f$, which we will denote $\mathscr{L}_{\mathbb{G}'}((f(g), f(t)), \tilde{X}_f)$, is then the limit of the rational function $\overline{R}_{\mathbb{G}'}(s)$ with power series expansion $-\sum_{n=1}^{\infty} \left|\tilde{X}_f^{\mathrm{Fr}_f^n \circ (f(g),f(t))^{-1}}\right| s^n$.

**Lemma 6.1.6.** *Let $\tilde{X} = L_{\mathbb{G}}^{-1}(\mathbb{U})$ and $\tilde{X}_f = L_{\mathbb{G}'}^{-1}(\mathbb{U}')$ be the varieties from before. Then $\tilde{X}_f^{\mathrm{Fr}_f^n \circ (f(g),f(t))^{-1}} = f(\tilde{X}^{\mathrm{Fr}^n \circ (g,t)^{-1}})$.*

**Proof:** We have that $x_f \in \tilde{X}_f^{\mathrm{Fr}_f^n \circ (f(g),f(t))^{-1}}$ if and only if $x_f = \left(\mathrm{Fr}_f^n \circ (f(g), f(t))^{-1}\right)(x_f)$. We can apply $f^{-1}$ on both sides and simplify as follows to obtain an equivalent condition:

$$\begin{aligned} f^{-1}(x_f) &= f^{-1}\big(f \circ \mathrm{Fr}^n \circ f^{-1}\left(f(g^{-1})x_f f(t^{-1})\right)\big) \\ &= \mathrm{Fr}^n \left(g^{-1} f^{-1}(x_f) t^{-1}\right) \\ &= \left(\mathrm{Fr}^n \circ (g,t)^{-1}\right)\left(f^{-1}(x_f)\right). \end{aligned}$$

This last condition happens if and only if $f^{-1}(x_f) \in \tilde{X}^{\mathrm{Fr}^n \circ (g,t)^{-1}}$, or equivalently if $x_f \in f(\tilde{X}^{\mathrm{Fr}^n \circ (g,t)^{-1}})$.

∎

**Corollary 6.1.7.** *The Lefschetz number of $(g,t)$ on $\tilde{X}$ is equal to the Lefschetz number of $(f(g), f(t))$ on $\tilde{X}_f$.*

**Proof:** By Lemma 6.1.6, we have that $\left|\tilde{X}^{\mathrm{Fr}^n \circ (g,t)^{-1}}\right| = \left|\tilde{X}_f^{\mathrm{Fr}_f^n \circ (f(g), f(t))^{-1}}\right|$, and therefore $\overline{R}_{\mathbb{G}}(s) = \overline{R}_{\mathbb{G}'}(s)$. Thus, $\mathscr{L}_{\mathbb{G}}((g,t), \tilde{X}) = \mathscr{L}_{\mathbb{G}'}((f(g), f(t)), \tilde{X}_f)$. ∎

**Corollary 6.1.8.** *Let $\mathbb{G}$ be a reductive group with Frobenius map* Fr *and $\mathbb{T}$ be a maximal* Fr*-stable torus of $\mathbb{G}$. Let $f : \mathbb{G} \to \mathbb{G}'$ be an isomorphism of algebraic groups and let $\mathbb{T}'$ be the maximal* $\mathrm{Fr}_f$*-stable torus of $\mathbb{G}'$ to which $\mathbb{T}$ maps, with* $\mathrm{Fr}_f = f \circ \mathrm{Fr} \circ f^{-1}$*. Then, for all $g \in \mathbb{G}^{\mathrm{Fr}}$ we have*

$$R_{\mathbb{T},\theta}(g) = R_{\mathbb{T}',\theta \circ f^{-1}}(f(g)).$$

**Proof:** Given $g \in \mathbb{G}^{\mathrm{Fr}}$, recall from Proposition 6.1.3 that

$$R_{\mathbb{T},\theta}(g) = \frac{1}{|\mathbb{T}^{\mathrm{Fr}}|} \sum_{t \in \mathbb{T}^{\mathrm{Fr}}} \theta(t^{-1}) \mathscr{L}_{\mathbb{G}}((g,t), \tilde{X}).$$

Since $f : \mathbb{T} \to \mathbb{T}'$ is an isomorphism, we have that $f(\mathbb{T}^{\mathrm{Fr}}) = (\mathbb{T}')^{\mathrm{Fr}_f}$, for the same reason we had $f(\mathbb{G}^{\mathrm{Fr}}) = (\mathbb{G}')^{\mathrm{Fr}_f}$ at the beginning of this section. It follows that every element $t \in \mathbb{T}^{\mathrm{Fr}}$ is of the form $t = f^{-1}(t')$ for some $t' \in (\mathbb{T}')^{\mathrm{Fr}_f}$. Then, for every $g \in \mathbb{G}^{\mathrm{Fr}}$,

$$R_{\mathbb{T},\theta}(g) = \frac{1}{|(\mathbb{T}')^{\mathrm{Fr}_f}|} \sum_{t' \in (\mathbb{T}')^{\mathrm{Fr}_f}} \theta \circ f^{-1}(t'^{-1}) \mathscr{L}_{\mathbb{G}}((g,t), \tilde{X}).$$

Since $\mathscr{L}_{\mathbb{G}}((g,t), \tilde{X}) = \mathscr{L}_{\mathbb{G}'}((f(g), f(t)), \tilde{X}_f)$ by Corollary 6.1.7, the result follows. ∎

### 6.1.3 Restricting the Virtual Characters

The main goal of this section will be to prove the following theorem.

**Theorem 6.1.9.** *Let $\mathbb{G}$ be a reductive group defined over an algebraically closed field of characteristic $p$ and $\mathrm{Fr} : \mathbb{G} \to \mathbb{G}$ be a Frobenius map. Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$ and $\mathbb{H}$ be a subgroup of $\mathbb{G}$ which is closed, connected and contains $\mathbb{G}_{\mathrm{der}}$. Furthermore, we assume that $\mathbb{T}$ and $\mathbb{H}$ are $\mathrm{Fr}$-stable. Then $R_{\mathbb{T},\theta}|_{\mathbb{H}^{\mathrm{Fr}}} = R_{\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}}}$, where $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$ and $\theta_{\mathbb{H}} = \theta|_{\mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}}$.*

We note that $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$ is a maximal torus of $\mathbb{H}$ by Remark 2.1.20. The formula provided by Theorem 6.1.4 for the virtual character $R_{\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}}}$ at an element $g \in \mathbb{H}^{\mathrm{Fr}}$ with Jordan decomposition $g = su$ is

$$R_{\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}}}(g) = \frac{1}{|(C_{\mathbb{H}}(s)^{\circ})^{\mathrm{Fr}}|} \sum_{\substack{x \in \mathbb{H}^{\mathrm{Fr}} \\ x^{-1}sx \in \mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}}} \theta_{\mathbb{H}}(x^{-1}sx) Q^{C_{\mathbb{H}}(s)^{\circ}}_{x\mathbb{T}_{\mathbb{H}}x^{-1}}(u).$$

To see that this expression will equal to $R_{\mathbb{T},\theta}(g)$, we must establish some relations between the various groups at play that appear in the formula above and the analogous formula for $R_{\mathbb{T},\theta}(g)$ coming from Theorem 6.1.4.

An immediate consequence of the Lang-Steinberg Theorem is that if $\mathbb{H}_1, \mathbb{H}_2$ are Fr-stable closed subgroups of $\mathbb{G}$ such that $\mathbb{H}_2$ is normal in $\mathbb{H}_1$ and $\mathbb{H}_2$ is connected, then we have $\mathbb{H}_1^{\mathrm{Fr}}/\mathbb{H}_2^{\mathrm{Fr}} \simeq (\mathbb{H}_1/\mathbb{H}_2)^{\mathrm{Fr}}$ [Car93, Section 1.17]. This corollary of the Lang-Steinberg Theorem leads us to the following statement.

**Proposition 6.1.10.** *Let $\mathbb{G}, \mathbb{T}$ and $\mathbb{H}$ be as in Theorem 6.1.9. Then $\mathbb{G}^{\mathrm{Fr}} = \mathbb{H}^{\mathrm{Fr}}\mathbb{T}^{\mathrm{Fr}}$.*

**Proof:** We have that $\mathbb{G} = \mathbb{T}\mathbb{G}_{\mathrm{der}}$ (Remark 2.1.17). Therefore, $\mathbb{G} = \mathbb{T}\mathbb{H}$ as $\mathbb{G}_{\mathrm{der}} \subset \mathbb{H}$. It follows that $\mathbb{G}/\mathbb{H} = \mathbb{T}\mathbb{H}/\mathbb{H}$, which is isomorphic to $\mathbb{T}/\mathbb{T}_{\mathbb{H}}$ by the second isomorphism theorem, where $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$. Since $\mathbb{H}$ and $\mathbb{T}_{\mathbb{H}}$ are connected, it follows from the Lang-Steinberg Theorem that $\mathbb{T}^{\mathrm{Fr}}/\mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}} \simeq \mathbb{G}^{\mathrm{Fr}}/\mathbb{H}^{\mathrm{Fr}}$, and therefore $\mathbb{T}^{\mathrm{Fr}}\mathbb{H}^{\mathrm{Fr}}/\mathbb{H}^{\mathrm{Fr}} \simeq \mathbb{G}^{\mathrm{Fr}}/\mathbb{H}^{\mathrm{Fr}}$. We conclude that $\mathbb{G}^{\mathrm{Fr}} = \mathbb{T}^{\mathrm{Fr}}\mathbb{H}^{\mathrm{Fr}}$, or equivalently $\mathbb{G}^{\mathrm{Fr}} = \mathbb{H}^{\mathrm{Fr}}\mathbb{T}^{\mathrm{Fr}}$ by normality of $\mathbb{H}$. ∎

**Corollary 6.1.11.** *Let $\mathbb{G}, \mathbb{T}$ and $\mathbb{H}$ be as in Theorem 6.1.9. Let $s$ be a semisimple element of $\mathbb{H}^{\mathrm{Fr}}$ and let $g \in \mathbb{G}^{\mathrm{Fr}}$ be such that $s \in {}^{g}\mathbb{T}$. Then $[\mathbb{T}^{\mathrm{Fr}} : \mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}] = [(C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}} : (C_{\mathbb{H}}(s)^{\circ})^{\mathrm{Fr}}]$, where $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$.*

**Proof:** We have that ${}^{g}\mathbb{T}$ is a maximal torus which is Fr-stable and that ${}^{g}\mathbb{T} \simeq \mathbb{T}$ is an isomorphism. By the Lang-Steinberg Theorem, we have that $({}^{g}\mathbb{T})^{\mathrm{Fr}}/({}^{g}\mathbb{T}_{\mathbb{H}})^{\mathrm{Fr}} \simeq \mathbb{T}^{\mathrm{Fr}}/\mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}$. Since $s \in ({}^{g}\mathbb{T})^{\mathrm{Fr}}$, $({}^{g}\mathbb{T})^{\mathrm{Fr}} \subset (C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}}$. It follows that

$$({}^{g}\mathbb{T})^{\mathrm{Fr}}/({}^{g}\mathbb{T}_{\mathbb{H}})^{\mathrm{Fr}} \simeq ({}^{g}\mathbb{T})^{\mathrm{Fr}}\mathbb{H}^{\mathrm{Fr}}/\mathbb{H}^{\mathrm{Fr}} \subset (C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}}\mathbb{H}^{\mathrm{Fr}}/\mathbb{H}^{\mathrm{Fr}} \subset \mathbb{G}^{\mathrm{Fr}}/\mathbb{H}^{\mathrm{Fr}}.$$

Since $\mathbb{T}^{\mathrm{Fr}}/\mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}} \simeq \mathbb{G}^{\mathrm{Fr}}/\mathbb{H}^{\mathrm{Fr}}$ (Proposition 6.1.10), we must have $\mathbb{T}^{\mathrm{Fr}}/\mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}} \simeq (C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}}\mathbb{H}^{\mathrm{Fr}}/\mathbb{H}^{\mathrm{Fr}}$, which is isomorphic to $(C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}}/\big((C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}} \cap \mathbb{H}^{\mathrm{Fr}}\big)$ by the second isomorphism theorem. Since $(C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}} \cap \mathbb{H}^{\mathrm{Fr}} = (C_{\mathbb{H}}(s)^{\circ})^{\mathrm{Fr}}$ (Lemma 2.1.32), the result follows. ∎

**Proof of Theorem 6.1.9:**
Let $g = su$ be the Jordan decomposition of $g \in \mathbb{H}^{\mathrm{Fr}}$. Since the Jordan decomposition is unique, this must also correspond to the decomposition of $g$ in $\mathbb{G}^{\mathrm{Fr}}$. We have that $\mathbb{G}^{\mathrm{Fr}} = \mathbb{H}^{\mathrm{Fr}}\mathbb{T}^{\mathrm{Fr}}$ (Proposition 6.1.10). Furthermore, since $s \in \mathbb{H}^{\mathrm{Fr}}$ and $\mathbb{H}^{\mathrm{Fr}}$ is normal in $\mathbb{G}^{\mathrm{Fr}}$, requiring $x^{-1}sx \in \mathbb{T}^{\mathrm{Fr}}$ is equivalent to requiring $x^{-1}sx \in \mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}$. Therefore, we can start modifying the formula for the restricted virtual character as follows:

$$\begin{aligned} R_{\mathbb{T},\theta}|_{\mathbb{H}^{\mathrm{Fr}}}(g) &= \frac{1}{|(C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}}|} \sum_{\substack{x \in \mathbb{G}^{\mathrm{Fr}} \\ x^{-1}sx \in \mathbb{T}^{\mathrm{Fr}}}} \theta(x^{-1}sx) Q_{x\mathbb{T}x^{-1}}^{C_{\mathbb{G}}(s)^{\circ}}(u) \\ &= \frac{1}{|(C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}}|} \sum_{\substack{x \in \mathbb{H}^{\mathrm{Fr}}\mathbb{T}^{\mathrm{Fr}} \\ x^{-1}sx \in \mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}}} \theta_{\mathbb{H}}(x^{-1}sx) Q_{x\mathbb{T}x^{-1}}^{C_{\mathbb{G}}(s)^{\circ}}(u). \end{aligned}$$

Now, let $\{t_1, \ldots, t_m\}$ be a set of coset representatives of $\mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}\backslash\mathbb{T}^{\mathrm{Fr}} \simeq \mathbb{H}^{\mathrm{Fr}}\backslash\mathbb{H}^{\mathrm{Fr}}\mathbb{T}^{\mathrm{Fr}}$. Then $\mathbb{H}^{\mathrm{Fr}}\mathbb{T}^{\mathrm{Fr}} = \mathbb{H}^{\mathrm{Fr}}t_1 \cup \cdots \cup \mathbb{H}^{\mathrm{Fr}}t_m$. Given $x \in \mathbb{H}^{\mathrm{Fr}}\mathbb{T}^{\mathrm{Fr}}$, we have $x = \bar{g}t_i$ for some $1 \leqslant i \leqslant m$, $\bar{g} \in \mathbb{H}^{\mathrm{Fr}}$. Note that the right coset notation is necessary to have our elements of $\mathbb{H}^{\mathrm{Fr}}$ expressed in this particular way to ensure the following simplifications. Since $t_i \in \mathbb{T}^{\mathrm{Fr}}$, we have that $x\mathbb{T}x^{-1} = \bar{g}\mathbb{T}\bar{g}^{-1}$ and $\theta(x^{-1}sx) = \theta(t_i^{-1}\bar{g}^{-1}s\bar{g}t_i) = \theta(\bar{g}^{-1}s\bar{g})$. This allows us to rewrite the restricted virtual character as follows:

$$\begin{aligned} R_{\mathbb{T},\theta}|_{\mathbb{H}^{\mathrm{Fr}}}(g) &= \frac{1}{|(C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}}|} \sum_{\substack{\bar{g} \in \mathbb{H}^{\mathrm{Fr}} \\ 1 \leqslant i \leqslant m \\ \bar{g}^{-1}s\bar{g} \in \mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}}} \theta_{\mathbb{H}}(\bar{g}^{-1}s\bar{g}) Q_{\bar{g}\mathbb{T}\bar{g}^{-1}}^{C_{\mathbb{G}}(s)^{\circ}}(u) \\ &= \frac{m}{|(C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}}|} \sum_{\substack{\bar{g} \in \mathbb{H}^{\mathrm{Fr}} \\ \bar{g}^{-1}s\bar{g} \in \mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}}} \theta_{\mathbb{H}}(\bar{g}^{-1}s\bar{g}) Q_{\bar{g}\mathbb{T}\bar{g}^{-1}}^{C_{\mathbb{G}}(s)^{\circ}}(u). \end{aligned}$$

We have that $m = [\mathbb{T}^{\mathrm{Fr}} : \mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}]$, but this is equal to $[(C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}} : (C_{\mathbb{H}}(s)^{\circ})^{\mathrm{Fr}}]$ by Proposition 6.1.11. It follows that $\frac{m}{|(C_{\mathbb{G}}(s)^{\circ})^{\mathrm{Fr}}|} = \frac{1}{|(C_{\mathbb{H}}(s)^{\circ})^{\mathrm{Fr}}|}$. Therefore,

$$R_{\mathbb{T},\theta}|_{\mathbb{H}^{\mathrm{Fr}}}(g) = \frac{1}{|(C_{\mathbb{H}}(s)^{\circ})^{\mathrm{Fr}}|} \sum_{\substack{\bar{g} \in \mathbb{H}^{\mathrm{Fr}} \\ \bar{g}^{-1}s\bar{g} \in \mathbb{T}_{\mathbb{H}}^{\mathrm{Fr}}}} \theta_{\mathbb{H}}(\bar{g}^{-1}s\bar{g}) Q_{\bar{g}\mathbb{T}\bar{g}^{-1}}^{C_{\mathbb{G}}(s)^{\circ}}(u).$$

It remains to show that $Q_{\bar{g}\mathbb{T}\bar{g}^{-1}}^{C_{\mathbb{G}}(s)^{\circ}}(u) = Q_{\bar{g}\mathbb{T}_{\mathbb{H}}\bar{g}^{-1}}^{C_{\mathbb{H}}(s)^{\circ}}(u)$ for all $\bar{g} \in \mathbb{H}^{\mathrm{Fr}}$ to complete the proof.

Let $\mathrm{ad} : C_{\mathbb{G}}(s)^\circ \to C_{\mathbb{G}}(s)^\circ/Z(C_{\mathbb{G}}(s)^\circ)$ be the projection map of $C_{\mathbb{G}}(s)^\circ$ into the adjoint group $\mathrm{ad}(C_{\mathbb{G}}(s)^\circ) = C_{\mathbb{G}}(s)^\circ/Z(C_{\mathbb{G}}(s)^\circ)$. Similarly, we let $\mathrm{ad}_{\mathbb{H}}$ be the projection map of $C_{\mathbb{H}}(s)^\circ$ into the adjoint group $\mathrm{ad}_{\mathbb{H}}(C_{\mathbb{H}}(s)^\circ) = C_{\mathbb{H}}(s)^\circ/Z(C_{\mathbb{H}}(s)^\circ)$. According to [DL76, Formula 4.1.1], we have that $Q^{C_{\mathbb{G}}(s)^\circ}_{\bar{g}\mathbb{T}\bar{g}^{-1}}(u) = Q^{\mathrm{ad}(C_{\mathbb{G}}(s)^\circ)}_{\mathrm{ad}(\bar{g}\mathbb{T}\bar{g}^{-1})}(\mathrm{ad}(u))$ and $Q^{C_{\mathbb{H}}(s)^\circ}_{\bar{g}\mathbb{T}_{\mathbb{H}}\bar{g}^{-1}}(u) = Q^{\mathrm{ad}_{\mathbb{H}}(C_{\mathbb{H}}(s)^\circ)}_{\mathrm{ad}_{\mathbb{H}}(\bar{g}\mathbb{T}_{\mathbb{H}}\bar{g}^{-1})}(\mathrm{ad}_{\mathbb{H}}(u))$. We showed in Lemma 2.1.36 that $\mathrm{ad}_{\mathbb{H}}(C_{\mathbb{H}}(s)^\circ) \simeq \mathrm{ad}(C_{\mathbb{G}}(s)^\circ)$. Indeed, this isomorphism sends $cZ(C_{\mathbb{H}}(s)^\circ)$ to $cZ(C_{\mathbb{G}}(s)^\circ)$ for all $c \in C_{\mathbb{H}}(s)^\circ$. Furthermore, this isomorphism maps $\mathrm{ad}_{\mathbb{H}}(\bar{g}\mathbb{T}_{\mathbb{H}}\bar{g}^{-1})$ to $\bar{g}\mathbb{T}_{\mathbb{H}}\bar{g}^{-1}Z(C_{\mathbb{G}}(s)^\circ)/Z(C_{\mathbb{G}}(s)^\circ)$. The central isogeny restricted to tori (Corollary 2.1.7) gives us that $\bar{g}\mathbb{T}_{\mathbb{H}}\bar{g}^{-1}Z(C_{\mathbb{G}}(s)^\circ)/Z(C_{\mathbb{G}}(s)^\circ) = \mathrm{ad}(\bar{g}\mathbb{T}\bar{g}^{-1})$. It then follows from Corollary 6.1.8 that $R^{\mathrm{ad}_{\mathbb{H}}(C_{\mathbb{H}}(s)^\circ)}_{\mathrm{ad}_{\mathbb{H}}(\bar{g}\mathbb{T}_{\mathbb{H}}\bar{g}^{-1}),1}(\mathrm{ad}_{\mathbb{H}}(u)) = R^{\mathrm{ad}(C_{\mathbb{G}}(s)^\circ)}_{\mathrm{ad}(\bar{g}\mathbb{T}\bar{g}^{-1}),1}(\mathrm{ad}(u))$, that is, $Q^{\mathrm{ad}_{\mathbb{H}}(C_{\mathbb{H}}(s)^\circ)}_{\mathrm{ad}_{\mathbb{H}}(\bar{g}\mathbb{T}_{\mathbb{H}}\bar{g}^{-1})}(\mathrm{ad}_{\mathbb{H}}(u)) = Q^{\mathrm{ad}(C_{\mathbb{G}}(s)^\circ)}_{\mathrm{ad}(\bar{g}\mathbb{T}\bar{g}^{-1})}(\mathrm{ad}(u))$, and therefore $Q^{C_{\mathbb{G}}(s)^\circ}_{\bar{g}\mathbb{T}\bar{g}^{-1}}(u) = Q^{C_{\mathbb{H}}(s)^\circ}_{\bar{g}\mathbb{T}_{\mathbb{H}}\bar{g}^{-1}}(u)$. ∎

## 6.2 Deligne-Lusztig Cuspidal Representations

Let us now return to the world of algebraic groups defined over finite fields. In this section, $\mathbb{G}$ denotes a reductive group defined over the finite field $\mathfrak{f}$. The goal of this section is to define the concept of a Deligne-Lusztig cuspidal representation and use the information we know about Deligne-Lusztig virtual characters from the previous section to compute the restriction of such cuspidal representations.

Recall that $\mathbb{G}(\mathfrak{f})$ is a finite group of Lie type, as per Remark 6.1.1, where the Frobenius map Fr is the action of the Galois group $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$. Because $\mathbb{G}(\mathfrak{f})$ is a finite group, its irreducible representations are all finite-dimensional and can be identified with their characters. (Recall that a character $\chi_\phi$ of a finite-dimensional representation $\phi$ of a group $G$ is defined by $\chi_\phi(g) = \mathrm{Trace}(\phi(g))$ for all $g \in G$. The finite-dimensional representations of $G$ are uniquely determined up to isomorphism by their characters, which means we identify $\phi$ with $\chi_\phi$.)

Given a maximal torus $\mathbb{T}$ which is defined over $\mathfrak{f}$ (or Fr-stable) and a character $\theta$ of $\mathbb{T}(\mathfrak{f})$, we saw that we could create a virtual character of $\mathbb{G}(\mathfrak{f})$, denoted $R_{\mathbb{T},\theta}$. In order to talk about when $R_{\mathbb{T},\theta}$ corresponds to an actual irreducible representation of $\mathbb{G}(\mathfrak{f})$, we must first introduce the notion of general position for $\theta$ as per [DL76].

**Definition 6.2.1** ([DL76, Definition 5.15])**.** *Let $\mathbb{T}$ be a maximal $\mathfrak{f}$-torus of $\mathbb{G}$ and $\theta$ be a character of $\mathbb{T}(\mathfrak{f})$. The character $\theta$ is said to be in general position if its normalizer (or stabilizer) in $(N_{\mathbb{G}}(\mathbb{T})/\mathbb{T})(\mathfrak{f})$ is trivial.*

**Remark 6.2.2.** Note that $(N_{\mathbb{G}}(\mathbb{T})/\mathbb{T})(\mathfrak{f}) \simeq N_{\mathbb{G}}(\mathbb{T})(\mathfrak{f})/\mathbb{T}(\mathfrak{f})$ as a consequence of the Lang-Steinberg Theorem (Section 6.1.3).

Let us consider an example of a character which is in general position.

**Example 6.2.3.** Let $\mathfrak{f}$ be a finite field of characteristic $p$ of order $q$. Set $\mathbb{G}(\mathfrak{f}) = \mathrm{GL}_2(\mathfrak{f})$ and $\mathbb{T}$ to be the diagonal torus of $\mathbb{G}$ which is maximal and $\mathfrak{f}$-split. That is,

$$\mathbb{T}(\mathfrak{f}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b, \in \mathfrak{f}^\times \right\}.$$

Because $\mathbb{T}$ is $\mathfrak{f}$-split, $(N_{\mathbb{G}}(\mathbb{T})/\mathbb{T})(\mathfrak{f})$ can be thought of as the absolute Weyl group, which has representatives, $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Every element in the equivalence class of $w$ conjugates $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ to $\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$, $a, b \in \mathfrak{f}^\times$. Now, let us define the following character of $\mathbb{T}(\mathfrak{f})$:

$$\begin{aligned} \theta : \mathbb{T}(\mathfrak{f}) &\to \mathfrak{f}^\times \\ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} &\mapsto a^{\frac{q-1}{2}} \end{aligned}$$

Note that this is an algebraic character, but it becomes a complex smooth character after we embed $\mathfrak{f}^\times$ into $\mathbb{C}^\times$. Such an embedding exists because $\mathfrak{f}^\times$ is a cyclic group of order $q-1$ and is therefore isomorphic to the group of $(q-1)$st roots of unity in $\mathbb{C}^\times$. We will work with the algebraic character for computations, but keep in mind that we are viewing it as a smooth complex character.

We claim that $\theta$ is in regular position, or equivalently that it is not fixed by any element in the equivalence class of $w$. This is because there are some values of $a$ and $b$ in $\mathfrak{f}^\times$ for which $\theta\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = a^{\frac{q-1}{2}}$ and $\theta\left(\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}\right) = b^{\frac{q-1}{2}}$ are not equal. Indeed, if this was not the case, then $\mathfrak{f}^\times$ would be a group of order $\frac{q-1}{2}$. This means that no nontrivial element of the Weyl group stabilizes $\theta$, and therefore $\theta$ is in general position. ∎

Using this notion of general position, Deligne and Lusztig established the following theorem.

**Theorem 6.2.4** ([DL76, Proposition 7.4 and Theorem 8.3])**.** *Let $\mathbb{T}$ be a maximal $\mathfrak{f}$-torus of $\mathbb{G}$, $\theta$ be a character of $\mathbb{T}(\mathfrak{f})$, and let $r_{\mathfrak{f}}(\mathbb{G})$ and $r_{\mathfrak{f}}(\mathbb{T})$ denote the $\mathfrak{f}$-split ranks of $\mathbb{G}$ and $\mathbb{T}$, respectively.*

1) *If $\theta$ is in general position, then $(-1)^{r_{\mathfrak{f}}(\mathbb{G})-r_{\mathfrak{f}}(\mathbb{T})}R_{\mathbb{T},\theta}$ is the character of an irreducible representation of $\mathbb{G}(\mathfrak{f})$.*

2) *If $\theta$ is in general position and $\mathbb{T}$ is elliptic, that is not contained in any proper $\mathfrak{f}$-parabolic subgroup, then $(-1)^{r_{\mathfrak{f}}(\mathbb{G})-r_{\mathfrak{f}}(\mathbb{T})}R_{\mathbb{T},\theta}$ is the character of a cuspidal representation of $\mathbb{G}(\mathfrak{f})$.*

**Remark 6.2.5.** The definition of an elliptic torus provided by the above theorem is equivalent to the previously defined notion of an $\mathfrak{f}$-minisotropic torus (Definition 2.1.37).

In Example 6.2.3, the torus $\mathbb{T}$ we considered is not elliptic, so while the character $\theta$ described therein is in general position, the virtual character associated to the pair $(\mathbb{T}, \theta)$ would not correspond to a cuspidal representation of $\mathrm{GL}_2(\mathfrak{f})$. Let us illustrate with an example the notion of general position when $\mathbb{T}$ is elliptic.

**Example 6.2.6.** Let $\mathfrak{f}$ be a finite field of characteristic $p$ of order $q$. Set $\mathbb{G}(\mathfrak{f}) = \mathrm{GL}_n(\mathfrak{f})$. Let $\mathfrak{f}_n$ denote the unique (up to isomorphism) degree $n$ extension of $\mathfrak{f}$, and fix a basis $B$ of $\mathfrak{f}_n$ over $\mathfrak{f}$. One can view $\mathfrak{f}_n^\times$ as a subgroup of $\mathbb{G}(\mathfrak{f})$ via the following homomorphism:

$$\begin{aligned} \mathfrak{f}_n^\times &\to \mathbb{G}(\mathfrak{f}) \\ y &\mapsto g_y, \end{aligned}$$

where $g_y$ is the matrix corresponding to left multiplication by $y$ in $\mathfrak{f}_n$ with respect to the chosen basis $B$. This embedding of $\mathfrak{f}_n^\times$ into $\mathbb{G}(\mathfrak{f})$ corresponds to an elliptic maximal torus $\mathbb{T}$ of $\mathbb{G}$, that is $\mathfrak{f}_n^\times \simeq \mathbb{T}(\mathfrak{f})$. In fact, it is known that every elliptic maximal torus of $\mathbb{G}$ is obtained this way.

Now, since $\mathbb{T}(\mathfrak{f})$ is abelian and semisimple, its elements are simultaneously diagonalizable so that there exists $Q \in \mathbb{G}(\mathfrak{f}_n)$ such that $Q^{-1} g_y Q$ is a diagonal matrix for all $y \in \mathfrak{f}_n^\times$. Fix an eigenvector of $g_y$ and let $\lambda_y$ denote its corresponding eigenvalue. Then, one sees that $\sigma(\lambda_y)$ is also an eigenvalue of $g_y$ for all $\sigma \in \mathrm{Gal}(\mathfrak{f}_n / \mathfrak{f})$ so that all eigenvalues of $g_y$ are Galois conjugate.

Given an element $n \in N_{\mathbb{G}}(\mathbb{T})(\mathfrak{f})$, the eigenvalues of $n^{-1} g_y n$ are the same as those of $g_y$, but this conjugation may permute their order. Since we know that all eigenvalues of $g_y$ are Galois conjugate, we can identify $n$ with an element of $\mathrm{Gal}(\mathfrak{f}_n / \mathfrak{f})$, inducing an isomorphism $N_{\mathbb{G}}(\mathbb{T})(\mathfrak{f}) / \mathbb{T}(\mathfrak{f}) \simeq \mathrm{Gal}(\mathfrak{f}_n / \mathfrak{f})$. It follows that $N_{\mathbb{G}}(\mathbb{T})(\mathfrak{f}) / \mathbb{T}(\mathfrak{f})$ is the cyclic group of order $n$, whereas the absolute Weyl group of $\mathbb{G}$, $N_{\mathbb{G}}(\mathbb{T}) / \mathbb{T}$, is the symmetric group $S_n$.

Given the above discussion, one then concludes that a character of $\mathbb{T}(\mathfrak{f})$ is in general position if and only if it is $\mathrm{Gal}(\mathfrak{f}_n / \mathfrak{f})$-invariant when $\mathbb{T}(\mathfrak{f})$ is identified with $\mathfrak{f}_n^\times$. ∎

Based on the previous theorem, a cuspidal representation whose character formula is given by $(-1)^{r_{\mathfrak{f}}(\mathbb{G}) - r_{\mathfrak{f}}(\mathbb{T})} R_{\mathbb{T},\theta}$ for some elliptic maximal $\mathfrak{f}$-torus $\mathbb{T}$ and character $\theta$ in general position will be referred to as a *Deligne-Lusztig cuspidal representation.* Note that not all cuspidal representations are of the Deligne-Lusztig type in general. Since we know how to restrict virtual characters of the form $R_{\mathbb{T},\theta}$, this means that we also know how to restrict Deligne-Lusztig cuspidal representations.

Recall from Theorem 6.1.9 that if $\mathbb{H}$ is a closed connected $\mathfrak{f}$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$, then $R_{\mathbb{T},\theta}|_{\mathbb{H}(\mathfrak{f})} = R_{\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}}}$, where $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$ and $\theta_{\mathbb{H}} = \theta|_{T_H}$. When the center of $\mathbb{H}$ is connected, the notion of general position is equivalent to a weaker notion that Deligne and Lusztig call *nonsingular* [DL76, Theorem 5.16]. We do not expand on this last notion here, but rather point out that a nonsingular character restricts to a nonsingular character, and therefore when the center of $\mathbb{H}$ is connected, $\theta_{\mathbb{H}}$ is guaranteed to be in general position. However, when the center of $\mathbb{H}$ is not connected and $\theta$ is in general position, this does not necessarily imply that $\theta_{\mathbb{H}}$ is also in general position. Let us illustrate this with an example.

**Example 6.2.7.** Let $\mathfrak{f}$ be a finite field of characteristic $p$ of order $q$ as before. Set $\mathbb{G}(\mathfrak{f}) = \mathrm{GL}_2(\mathfrak{f})$, $\mathbb{T}$ to be the diagonal torus of $\mathbb{G}$ which is maximal and $\mathfrak{f}$-split and $\mathbb{H}(\mathfrak{f}) = \mathrm{SL}_2(\mathfrak{f})$. Then $\mathbb{T}_{\mathbb{H}}$ is the diagonal torus in $\mathbb{H}$ and is also $\mathfrak{f}$-split. That is,

$$\mathbb{T}(\mathfrak{f}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b, \in \mathfrak{f}^\times \right\}, \mathbb{T}_{\mathbb{H}}(\mathfrak{f}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathfrak{f}^\times \right\}.$$

Because $\mathbb{T}$ and $\mathbb{T}_{\mathbb{H}}$ are $\mathfrak{f}$-split, $(N_{\mathbb{G}}(\mathbb{T})/\mathbb{T})(\mathfrak{f})$ and $(N_{\mathbb{H}}(\mathbb{T}_{\mathbb{H}})/\mathbb{T}_{\mathbb{H}})(\mathfrak{f})$ can be thought of as the absolute Weyl groups, which have the same representatives, $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Let $\theta$ be the character of $\mathbb{T}(\mathfrak{f})$ from Example 6.2.3, which we showed is in general position. The character $\theta_{\mathbb{H}}$ is then given by

$$\begin{aligned} \theta_{\mathbb{H}} : \mathbb{T}_{\mathbb{H}}(\mathfrak{f}) &\to \mathfrak{f}^\times \\ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} &\mapsto a^{\frac{q-1}{2}}. \end{aligned}$$

Note that we are viewing $\theta_{\mathbb{H}}$ as a complex smooth character by embedding $\mathfrak{f}^\times$ into $\mathbb{C}^\times$ as per Example 6.2.3.

We claim that $\theta_{\mathbb{H}}$ is not in general position. Indeed, because $\mathfrak{f}^\times$ is cyclic of order $q-1$, we have that $a^{\frac{q-1}{2}} = (a^{\frac{q-1}{2}})^{-1} = (a^{-1})^{\frac{q-1}{2}}$ for all $a \in \mathfrak{f}^\times$. It follows that

$${}^{w}\theta_{\mathbb{H}}\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) = \theta_{\mathbb{H}}\left(\begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix}\right) = \theta_{\mathbb{H}}\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right)$$

for all $a \in \mathfrak{f}^\times$. This means that $w$ is a nontrivial element of $(N_{\mathbb{G}}(\mathbb{T})/\mathbb{T})(\mathfrak{f})$ that stabilizes $\theta_{\mathbb{H}}$, and therefore $\theta_{\mathbb{H}}$ is not in general position. ∎

We now state the following theorem for the restriction of a Deligne-Lusztig cuspidal representation.

**Theorem 6.2.8.** *Let $\mathbb{G}, \mathbb{T}$ and $\theta$ satisfy the hypotheses of Theorem 6.2.4(2). Let $\mathbb{H}$ be a closed, connected $\mathfrak{f}$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$, $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$ and $\theta_{\mathbb{H}} = \theta|_{\mathbb{T}_{\mathbb{H}}(\mathfrak{f})}$. Then $(-1)^{r_{\mathfrak{f}}(\mathbb{G})-r_{\mathfrak{f}}(\mathbb{T})} R_{\mathbb{T},\theta}|_{\mathbb{H}(\mathfrak{f})} = (-1)^{r_{\mathfrak{f}}(\mathbb{H})-r_{\mathfrak{f}}(\mathbb{T}_{\mathbb{H}})} R_{\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}}}$, and is the character of a cuspidal representation of $\mathbb{H}(\mathfrak{f})$ if $\theta_{\mathbb{H}}$ is in general position.*

**Proof:** We have that $\mathbb{T}_{\mathbb{H}}$ is a maximal torus of $\mathbb{H}$ (Remark 2.1.20). Furthermore, $\mathbb{T}_{\mathbb{H}}$ cannot be contained in any proper $\mathfrak{f}$-parabolic subgroup by Proposition 2.1.30, and is therefore elliptic. So, given the cuspidal representation $(-1)^{r_{\mathfrak{f}}(\mathbb{G})-r_{\mathfrak{f}}(\mathbb{T})} R_{\mathbb{T},\theta}$, we have that $(-1)^{r_{\mathfrak{f}}(\mathbb{H})-r_{\mathfrak{f}}(\mathbb{T}_{\mathbb{H}})} R_{\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}}}$ is also cuspidal under the assumption that $\theta_{\mathbb{H}}$ is in general position. Since $r_{\mathfrak{f}}(\mathbb{H}) - r_{\mathfrak{f}}(\mathbb{T}_{\mathbb{H}}) = r_{\mathfrak{f}}(\mathbb{G}) - r_{\mathfrak{f}}(\mathbb{T})$ (Proposition 2.1.45) and $R_{\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}}} = R_{\mathbb{T},\theta}|_{\mathbb{H}(\mathfrak{f})}$ (Theorem 6.1.9), the result follows. ∎

**Remark 6.2.9.** For the remainder of this chapter, whenever we are calling a Deligne-Lusztig cuspidal representation, we will represent it by its character and write $\pm R_{\mathbb{T},\theta}$ as a shortcut for $(-1)^{r_{\mathfrak{f}}(\mathbb{G})-r_{\mathfrak{f}}(\mathbb{T})} R_{\mathbb{T},\theta}$.

We end this section with the following remark.

**Remark 6.2.10.** When $\theta_{\mathbb{H}}$ is not in general position, we know that the restriction $\pm R_{\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}}}$ will decompose as a finite direct sum of cuspidal representations by Proposition 2.3.20. However, we do not know whether these cuspidal components are of the Deligne-Lusztig type as the character of any irreducible representation of $\mathbb{H}(\mathfrak{f})$ appears in some Deligne-Lusztig virtual character of $\mathbb{H}(\mathfrak{f})$ [Car93, Corollary 7.5.8]. This means that we require $\theta_{\mathbb{H}}$ to be in general position if we want to ensure any kind of control over the restriction of $\pm R_{\mathbb{T},\theta}$.

## 6.3 Regular Supercuspidal Representations

In this section, we return to the notation $\mathbb{G}$ for a reductive group which is defined over a $p$-adic field $F$, $G = \mathbb{G}(F)$. We also assume that $\mathbb{G}$ splits over a tamely ramified extension of $F$ and that $p$ does not divide the order of the Weyl group of $\mathbb{G}$. The goal of this section is to study Kaletha's construction for a regular depth-zero supercuspidal representation of $G$, $\pi$, in order to retrieve a more specific description of the inducing representation $\rho$ of $G_{[y]}$ that satisfies $\pi = \text{c-Ind}_{G_{[y]}}^{G} \rho$.

We let $F^s$ denote a separable closure of $F$ and $F^{\mathrm{un}}$ be the maximal unramified extension of $F$ within $F^s$. Recall from Section 2.2.2.2 that the residue field of $F^{\mathrm{un}}$ is an algebraic closure of $\mathfrak{f}$, so we denote it by $\bar{\mathfrak{f}}$.

Given a point $y \in \mathcal{B}(\mathbb{G}, F)$, we let $\mathcal{G}_y$ denote the reductive quotient of $\mathbb{G}$ at $y$, that is the algebraic group whose $\mathfrak{f}$-points corresponds to the quotient $G_{y,0:0^+}$. We let $\mathcal{S}$ be a maximal torus of $\mathcal{G}_y$ which is defined over $\mathfrak{f}$ and not contained in any proper $\mathfrak{f}$-parabolic subgroups of $\mathcal{G}_y$. We use the word *elliptic* to refer to this last property. The regular depth-zero supercuspidal representations are defined as follows.

**Definition 6.3.1** ([Kal19, Definition 3.4.19])**.** *Let $\pi$ be an irreducible depth-zero supercuspidal representation of $G$. Then $\pi$ is said to be regular if the cuspidal representation $\sigma$ contained in $\pi|_{G_{y,0}}$ as per Theorem 2.3.26 is a Deligne-Lusztig cuspidal representation $\pm R_{\mathcal{S},\overline{\theta}}$, where $\mathcal{S}$ is an elliptic maximal $\mathfrak{f}$-torus of $\mathcal{G}_y$ and $\overline{\theta} : \mathcal{S}(\mathfrak{f}) \to \mathbb{C}^\times$ is a regular character.*

To explain what it means for a character of $\mathcal{S}(\mathfrak{f})$ to be regular, one first needs the notion of a *maximally unramified* torus of $\mathbb{G}$.

**Definition 6.3.2** ([Kal19, Definition 3.4.2])**.** *A maximal torus $\mathbb{S}$ of $\mathbb{G}$ is said to be maximally unramified if it is equal to the centralizer of its maximal unramified subtorus.*

Since the definition of maximally unramified only applies to a maximal torus, the word maximal will be implied when using the adjective maximally unramified, that is a maximally unramified maximal torus will simply be referred to as maximally unramified torus should no confusion arise.

**Remark 6.3.3.** By [Kal19, Fact 3.4.1], a torus $\mathbb{S}$ of $\mathbb{G}$ is maximally unramified if and only if the action of the inertia group of $\mathrm{Gal}(F^s/F)$ on $\Phi(\mathbb{S}, \mathbb{G})$ preserves a set of positive roots.

**Remark 6.3.4.** Given a maximally unramified $F$-torus $\mathbb{S}$, one can associate to $\mathbb{S}$ a vertex $[y]$ of the reduced building $\mathcal{B}^{\mathrm{red}}(\mathbb{G}, F)$ [Kal19, Lemma 3.4.3]. In particular, Kaletha explains that this point $[y]$ belongs to $\mathcal{A}^{\mathrm{red}}(\mathbb{G}, \mathbb{S}, F^{\mathrm{un}})$ and is its unique $\mathrm{Gal}(F^{\mathrm{un}}/F)$-fixed point.

The reason for being interested in elliptic maximally unramified $F$-tori of $\mathbb{G}$ is because there is a correspondence between the $G_{y,0^+}$-conjugacy classes of elliptic maximally unramified $F$-tori of $\mathbb{G}$ associated to $[y]$ and the elliptic maximal $\mathfrak{f}$-tori of $\mathcal{G}_y$, as per the following lemma.

**Lemma 6.3.5** ([Kal19, Lemmas 3.4.4 - 3.4.6])**.** *Let $\mathbb{S}$ be an elliptic maximally unramified $F$-torus of $\mathbb{G}$, and let $[y]$ be its associated vertex in $\mathcal{B}^{\mathrm{red}}(\mathbb{G}, F)$. Then the following statements hold.*

1) *The image of $S_0$ in $G_{y,0:0^+}$ is the $\mathfrak{f}$-points of an elliptic maximal $\mathfrak{f}$-torus $\mathcal{S}$ of $\mathcal{G}_y$.*

2) *For every unramified extension $F'$ of $F$, the image of $\mathbb{S}(F')_0$ in $\mathbb{G}(F')_{y,0:0^+}$ is the $\mathfrak{f}'$-points of $\mathcal{S}$, where $\mathfrak{f}'$ is the residue field of $F'$, and every elliptic maximal $\mathfrak{f}$-torus of $\mathcal{G}_y$ arises this way.*

3) *If $\mathbb{S}_1, \mathbb{S}_2$ are two elliptic maximally unramified $F$-tori with associated vertex $[y]$ such that $\mathbb{S}_1(F^{\mathrm{un}})_0$ and $\mathbb{S}_2(F^{\mathrm{un}})_0$ have the same image in $\mathcal{G}_y(\bar{\mathfrak{f}})$, then $\mathbb{S}_1$ and $\mathbb{S}_2$ are $G_{y,0^+}$-conjugate.*

The following lemma is not explicitly shown in [Kal19], but is implied throughout the paper.

**Lemma 6.3.6.** *Let $\mathbb{S}$ be an elliptic maximally unramified $F$-torus with associated vertex $[y]$. Then $SG_{y,0}$ is a normal subgroup of $G_{[y]}$.*

**Proof:** Kaletha shows in [Kal19, Lemma 3.4.10 (1)] that $N_{\mathbb{G}}(\mathbb{S})(F) \subset G_{[y]}$. This is because the action of $N_{\mathbb{G}}(\mathbb{S})(F)$ on $\mathcal{B}^{\mathrm{red}}(\mathbb{S}, F^{\mathrm{un}})$ preserves $\mathcal{A}^{\mathrm{red}}(\mathbb{G}, \mathbb{S}, F^{\mathrm{un}})$ and commutes with $\mathrm{Gal}(F^{\mathrm{un}}/F)$. So, for $n \in N_{\mathbb{G}}(\mathbb{S})(F)$, $\gamma \in \mathrm{Gal}(F^{\mathrm{un}}/F)$, $n \cdot [y] = n \cdot (\gamma([y])) = \gamma(n \cdot [y])$. This implies $n \cdot [y] = [y]$ because $[y]$ in the unique $\mathrm{Gal}(F^{\mathrm{un}}/F)$-fixed point of $\mathcal{A}^{\mathrm{red}}(\mathbb{G}, \mathbb{S}, F^{\mathrm{un}})$. In particular, this means that $S \subset G_{[y]}$. Therefore, $S$ normalizes $G_{y,0}$ and $SG_{y,0}$ is a subgroup of $G_{[y]}$.

Now, take $g \in G_{[y]}$. The torus ${}^g\mathbb{S}$ is an elliptic maximally unramified $F$-torus associated to the vertex $g \cdot [y] = [y]$. So, $\mathbb{S}$ and ${}^g\mathbb{S}$ are two elliptic maximally unramified $F$-tori associated to the same vertex $[y]$. Furthermore, one sees that $\mathbb{S}(F^{\mathrm{un}})_0$ and ${}^g\mathbb{S}(F^{\mathrm{un}})_0$ have the same projection in $\mathcal{G}_y(\bar{\mathfrak{f}})$ meaning that $S$ and ${}^gS$ are $G_{y,0^+}$-conjugate (Lemma 6.3.5). So, given $s \in S, h \in G_{y,0}$ we compute the conjugation of $sh$ by $g$

$$gshg^{-1} = (gsg^{-1})(ghg^{-1}),$$

where $ghg^{-1} \in G_{y,0}$. We claim that $gsg^{-1} \in SG_{y,0}$ and that therefore $SG_{y,0}$ is normalized by $g$. Indeed, by what precedes, we have that $gsg^{-1} = \bar{g}\bar{s}\bar{g}^{-1}$ for some $\bar{s} \in S, \bar{g} \in G_{y,0^+}$. We rewrite this last expression as $gsg^{-1} = \bar{s}(\bar{s}^{-1}\bar{g}\bar{s})\bar{g}^{-1}$. Since $S \subset G_{[y]}, \bar{s}^{-1}\bar{g}\bar{s} \in G_{y,0}$ and therefore $gsg^{-1} \in SG_{y,0}$. We conclude that $SG_{y,0}$ is a normal subgroup of $G_{[y]}$. ∎

We are now ready to define what it means for a character to be regular.

**Definition 6.3.7** ([Kal19, Definition 3.4.16])**.** *Let $\mathcal{S}$ be an elliptic maximal $\mathfrak{f}$-torus of $\mathcal{G}_y$ and let $\mathbb{S}$ be a corresponding elliptic maximally unramified $F$-torus of $\mathbb{G}$ as per Lemma 6.3.5. A character $\overline{\theta}$ of $\mathcal{S}(\mathfrak{f})$ is said to be regular if its normalizer (or stabilizer) in $N_{\mathbb{G}}(\mathbb{S})(F)/\mathbb{S}(F)$ is trivial.*

**Remark 6.3.8.** The character $\overline{\theta}$ in the previous definition is thought of as being the restriction of a depth-zero character of $S$, which we denote by $\theta$. In other words, $\theta$ is a character of $S$ and $\theta|_{S_0}$ is the pullback of $\overline{\theta}$. We say that $\theta$ is a regular character of $S$ whenever $\overline{\theta}$ is a regular character of $\mathcal{S}(\mathfrak{f})$ in the sense of the above definition. Note that having $\theta$ be regular is equivalent to $I_{N_{\mathbb{G}}(\mathbb{S})(F)}(\theta|_{S_0}) = S$.

Now, given an elliptic maximal $\mathfrak{f}$-torus $\mathcal{S}$ of $\mathcal{G}_y$ and a regular character $\overline{\theta}$ of $\mathcal{S}(\mathfrak{f})$, $\pm R_{\mathcal{S},\overline{\theta}}$ is a Deligne-Lusztig cuspidal representation of $\mathcal{G}_y(\mathfrak{f})$. Indeed, this is because the notion of regularity is stronger than that of general position, as per the following lemma.

**Lemma 6.3.9** ([Kal19, Fact 3.4.18]). *Let $\mathcal{S}$ be an elliptic maximal $\mathfrak{f}$-torus of $\mathcal{G}_y$ and let $\overline{\theta}$ be a regular character of $\mathcal{S}(\mathfrak{f})$. Then $\overline{\theta}$ is in general position.*

**Proof:** Since $N_{G_{y,0}}(\mathbb{S}) \subset N_{\mathbb{G}}(\mathbb{S})(F)$, the stabilizer of $\overline{\theta}$ in $N_{\mathbb{G}}(\mathbb{S})(F)/\mathbb{S}(F)$ being trivial implies that its stabilizer in $N_{G_{y,0}}(\mathbb{S})/\mathbb{S}(F)_0$ is also trivial. Furthermore, Kaletha shows that the natural quotient map $N_{G_{y,0}(\mathbb{S})}/\mathbb{S}(F)_0 \to N_{\mathcal{G}_y}(\mathcal{S})(\mathfrak{f})/\mathcal{S}(\mathfrak{f})$ is bijective [Kal19, Lemma 3.4.10(2)]. Therefore, if $\overline{\theta}$ is regular, then its stabilizer in $N_{\mathcal{G}_y}(\mathcal{S})(\mathfrak{f})/\mathcal{S}(\mathfrak{f})$ is trivial. But $N_{\mathcal{G}_y}(\mathcal{S})(\mathfrak{f})/\mathcal{S}(\mathfrak{f}) \simeq (N_{\mathcal{G}_y}(\mathcal{S})/\mathcal{S})(\mathfrak{f})$ (Remark 6.2.2), and therefore the stabilizer of $\overline{\theta}$ in $(N_{\mathcal{G}_y}(\mathcal{S})/\mathcal{S})(\mathfrak{f})$ is trivial, or equivalently, $\overline{\theta}$ is in general position. ∎

Let $\mathbb{S}$ be an elliptic maximally unramified $F$-torus of $\mathbb{G}$ with associated vertex $[y]$ and let $\theta$ be a regular depth-zero character of $S$. From the pair $(\mathbb{S},\theta)$, Kaletha constructs a representation $\pi_{(\mathbb{S},\theta)}$, which we summarize in the following three steps [Kal19]:

1) Let $\mathcal{S}$ be the elliptic maximal $\mathfrak{f}$-torus of $\mathcal{G}_y$ associated to $\mathcal{S}$ as per Lemma 6.3.5 and $\overline{\theta}$ be the regular character of $\mathcal{S}(\mathfrak{f})$ obtained from $\theta$ by restricting to $S_0$ and construct the corresponding Deligne-Lusztig cuspidal representation $\pm R_{\mathcal{S},\overline{\theta}}$.

2) Let $\kappa_{(\mathcal{S},\overline{\theta})}$ denote the pullback of $\pm R_{\mathcal{S},\overline{\theta}}$ to $G_{y,0}$. Extend $\kappa_{(\mathcal{S},\overline{\theta})}$ to an irreducible representation (acting on the same vector space) $\kappa_{(\mathbb{S},\theta)}$ of $SG_{y,0}$. An explicit construction is provided in [Kal19, Section 3.4.4]. For our purposes, it suffices to know that such an extension exists.

3) Set $\pi_{(\mathbb{S},\theta)} = \text{c-Ind}_{SG_{y,0}}^{G} \kappa_{(\mathbb{S},\theta)}$.

**Remark 6.3.10.** The regularity of $\overline{\theta}$ implies that the normalizer of the representation $\kappa_{(\mathbb{S},\theta)}$ in $G_{[y]}$, that is the set $\{g \in G_{[y]} : {}^{g}\kappa_{(\mathbb{S},\theta)} \simeq \kappa_{(\mathbb{S},\theta)}\}$, is equal to $SG_{y,0}$ [Kal19, Proof of Lemma 3.4.20].

Indeed, Kaletha uses [DL76, Theorem 6.8] and [Kal19, Lemma 3.4.5] to show that having $g \in G_{[y]}$ such that $\kappa_{(\mathbb{S},\theta)} \simeq {}^{g}\kappa_{(\mathbb{S},\theta)}$ implies that there exists $g' \in G_{y,0}$ such that $g'g \in N_{\mathbb{G}}(\mathbb{S})(F)$ and ${}^{g'g}(\theta|_{S_0}) = \theta|_{S_0}$, or equivalently, $g'g \in I_{N_{\mathbb{G}}(\mathbb{S})(F)}(\theta|_{S_0})$. The regularity of $\theta$ implies that $g'g \in S$ and therefore $g \in SG_{y,0}$.

**Proposition 6.3.11.** *Let $\mathbb{S}, \theta$ and $\pi_{(\mathbb{S},\theta)}$ be as above. Then $\pi_{(\mathbb{S},\theta)}$ is an irreducible regular depth-zero supercuspidal representation of $G$.*

**Proof:** By Remark 6.3.10, the normalizer in $G_{[y]}$ of $\mathsf{\kappa}_{(\mathbb{S},\theta)}$ is the normal subgroup $SG_{y,0}$ (Lemma 6.3.6). From the Mackey Irreducibility Criterion, we have that $\rho := \text{c-Ind}_{SG_{y,0}}^{G_{[y]}} \mathsf{\kappa}_{(\mathbb{S},\theta)}$ is irreducible. Therefore, $\rho$ is an irreducible representation of $G_{[y]}$ whose restriction to $G_{y,0}$ contains the Deligne-Lusztig cuspidal representation $\pm R_{\mathcal{S},\bar{\theta}}$. Thus, it follows from the Moy-Prasad Construction (Theorem 2.3.26) that $\text{c-Ind}_{G_{[y]}}^{G} \rho = \text{c-Ind}_{SG_{y,0}}^{G} \mathsf{\kappa}_{(\mathbb{S},\theta)}$ is an irreducible supercuspidal representation of depth zero. Furthermore, this supercuspidal representation is regular in the sense of Definition 6.3.1 by construction. ∎

We summarize Kaletha's construction of $\pi_{(\mathbb{S},\theta)}$ in Figure 6.1. Not only does this construction produce irreducible regular depth-zero supercuspidal representations of $G$, it is exhaustive as per the following proposition.

$$\begin{tikzcd}
 & & & \left(\rho, G_{[y]}\right) \arrow[dr, "\text{induce}"] & \\
\left(\pm R_{\mathcal{S},\bar{\theta}}, \mathcal{G}_y(\mathfrak{f})\right) \arrow[r, "\text{pullback}"] & \left(\mathsf{\kappa}_{(\mathcal{S},\bar{\theta})}, G_{y,0}\right) \arrow[r, "\text{extend}"] & \left(\mathsf{\kappa}_{(\mathbb{S},\theta)}, SG_{y,0}\right) \arrow[ur, "\text{induce}"] \arrow[rr, "\text{induce}"] & & \left(\pi_{(\mathbb{S},\theta)}, G\right)
\end{tikzcd}$$

Figure 6.1: Summary of Kaletha's construction for regular depth-zero supercuspidal representations of $G$.

**Proposition 6.3.12** ([Kal19, Proposition 3.4.27])**.** *Every irreducible regular depth-zero supercuspidal representation of $G$ is of the form $\pi_{(\mathbb{S},\theta)}$ for some elliptic maximally unramified $F$-torus $\mathbb{S}$ and regular depth-zero character $\theta : S \to \mathbb{C}^\times$.*

As a consequence of the previous proposition, whenever $\rho$ induces irreducibly to a regular depth-zero supercuspidal representation, we can assume without loss of generality that $\rho = \text{c-Ind}_{SG_{y,0}}^{G_{[y]}} \mathsf{\kappa}_{(\mathbb{S},\theta)}$ for some elliptic maximally unramified $F$-torus $\mathbb{S}$ and regular depth-zero character $\theta$ of $S$.

## 6.4 A Condition for a Multiplicity Free Restriction

We maintain the same notation as in the previous section.

Let $\rho$ be an irreducible representation of $G_{[y]}$ such that $\pi = \text{c-Ind}_{G_{[y]}}^{G} \rho$ is a regular supercuspidal representation of depth zero of $G$. By what precedes in

Section 6.3, $\pi = \pi_{(\mathbb{S},\theta)}$ for some elliptic maximally unramified $F$-torus $\mathbb{S}$ of $\mathbb{G}$ and regular depth-zero character $\theta$ of $S$. This elliptic maximally unramified torus is associated to a vertex $[y]$ of $\mathcal{B}^{\mathrm{red}}(\mathbb{G}, F)$ and corresponds to an elliptic maximal $\mathfrak{f}$-torus $\mathcal{S}$ of $\mathcal{G}_y$. Furthermore, we can assume without loss of generality that $\rho = \text{c-Ind}_{SG_{y,0}}^{G_{[y]}} \kappa_{(\mathbb{S},\theta)}$, where $\kappa_{(\mathbb{S},\theta)}$ is an extension to $SG_{y,0}$ of the Deligne-Lusztig cuspidal representation $\kappa_{(\mathcal{S},\bar{\theta})} = \pm R_{\mathcal{S},\bar{\theta}}$.

Let $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$. In this section we will be focusing on the restriction of $\rho$ to the subgroup $H_{[y]}$ and determining under what condition this restriction is multiplicity free.

We let $\mathcal{H}_y$ denote the reductive quotient of $\mathbb{H}$ at $y$, that is the algebraic group whose $\mathfrak{f}$-points corresponds to the quotient $H_{y,0:0^+}$. We also let $\binom{\mathcal{H}}{\mathcal{G}}_y$ denote the reductive group whose $\mathfrak{f}$-points corresponds to $H_{y,0}G_{y,0^+}/G_{y,0^+}$. Recall from Lemma 2.2.24 that we have an isomorphism of algebraic groups $\iota : \mathcal{H}_y(\bar{\mathfrak{f}}) \to \binom{\mathcal{H}}{\mathcal{G}}_y(\bar{\mathfrak{f}})$ given by $\iota(h\mathbb{H}(F^{\mathrm{un}})_{y,0^+}) = h\mathbb{G}(F^{\mathrm{un}})_{y,0^+}$ for all $h \in \mathbb{H}(F^{\mathrm{un}})_{y,0}$. Note that we have identified the algebraic groups with their $\bar{\mathfrak{f}}$-points.

**Lemma 6.4.1.** *Let $\mathbb{S}$ be an elliptic maximally unramified $F$-torus of $\mathbb{G}$ with associated vertex $[y]$ in $\mathcal{B}^{\mathrm{red}}(\mathbb{G}, F)$, and let $\mathcal{S}$ be its corresponding elliptic maximal $\mathfrak{f}$-torus of $\mathcal{G}_y$ as per Lemma 6.3.5. Then $\mathbb{S}_{\mathbb{H}} = \mathbb{S} \cap \mathbb{H}$ is an elliptic maximally unramified $F$-torus of $\mathbb{H}$ with associated vertex $[y]$ in $\mathcal{B}^{\mathrm{red}}(\mathbb{H}, F)$ and $\mathcal{S} \cap \binom{\mathcal{H}}{\mathcal{G}}_y$ is an elliptic maximal $\mathfrak{f}$-torus of $\binom{\mathcal{H}}{\mathcal{G}}_y$.*

**Proof:** Because $[\mathcal{G}_y, \mathcal{G}_y] \subset \binom{\mathcal{H}}{\mathcal{G}}_y \subset \mathcal{G}_y$ (Lemma 2.2.26), we can apply the results from Section 2.1.2 (namely Remark 2.1.20 and Proposition 2.1.30) to conclude that $\mathcal{S} \cap \binom{\mathcal{H}}{\mathcal{G}}_y$ is an elliptic maximal $\mathfrak{f}$-torus of $\binom{\mathcal{H}}{\mathcal{G}}_y$.

The same argument shows that $\mathbb{S}_{\mathbb{H}}$ is an elliptic maximal $F$-torus of $\mathbb{H}$. We also have that $\mathbb{S}_{\mathbb{H}}$ is maximally unramified. Indeed, Lemma 2.1.26 allows us to identify $\Phi(\mathbb{S}_{\mathbb{H}}, \mathbb{H})$ and $\Phi(\mathbb{S}, \mathbb{G})$. Since the action of the inertia group of $\mathrm{Gal}(F^s/F)$ on $\Phi(\mathbb{S}, \mathbb{G})$ preserves a set of positive roots (Remark 6.3.3), its action on $\Phi(\mathbb{S}_{\mathbb{H}}, \mathbb{H})$ must do the same.

We note that $\mathcal{B}^{\mathrm{red}}(\mathbb{G}, F) = \mathcal{B}^{\mathrm{red}}(\mathbb{H}, F)$ as $\mathbb{G}$ and $\mathbb{H}$ have the same derived subgroup (Remark 2.1.29), meaning that $[y]$ is a vertex of $\mathcal{B}^{\mathrm{red}}(\mathbb{H}, F)$. For the same reason, we have that $\mathcal{A}^{\mathrm{red}}(\mathbb{G}, \mathbb{S}, F^{\mathrm{un}}) = \mathcal{A}^{\mathrm{red}}(\mathbb{H}, \mathbb{S}_{\mathbb{H}}, F^{\mathrm{un}})$. By Remark 6.3.4, $[y]$ is thus the unique $\mathrm{Gal}(F^{\mathrm{un}}/F)$-fixed point of $\mathcal{A}^{\mathrm{red}}(\mathbb{H}, \mathbb{S}_{\mathbb{H}}, F^{\mathrm{un}})$ and is therefore the vertex associated to $\mathbb{S}_{\mathbb{H}}$. ∎

**Lemma 6.4.2.** *Let $\mathbb{S}$ be an elliptic maximally unramified $F$-torus of $\mathbb{G}$ with associated vertex $[y]$ in $\mathcal{B}^{\mathrm{red}}(\mathbb{G}, F)$, and let $\mathcal{S}$ be its corresponding elliptic maximal $\mathfrak{f}$-torus of $\mathcal{G}_y$ as per Lemma 6.3.5. Set $\mathbb{S}_{\mathbb{H}} = \mathbb{S} \cap \mathbb{H}$ and let $\mathcal{S}_{\mathcal{H}}$ denote its corresponding elliptic* *maximal $\mathfrak{f}$-torus of $\mathcal{H}_y$. Then $\iota(\mathcal{S}_{\mathcal{H}}) = \mathcal{S} \cap \binom{\mathcal{H}}{\mathcal{G}}_y$, where $\iota$ is the isomorphism between $\mathcal{H}_y$ and $\binom{\mathcal{H}}{\mathcal{G}}_y$.*

**Proof:** By Lemma 6.3.5, we have that $\mathcal{S}(\bar{\mathfrak{f}}) = (\mathbb{S}(F^{\mathrm{un}})_0\mathbb{G}(F^{\mathrm{un}})_{y,0^+})/\mathbb{G}(F^{\mathrm{un}})_{y,0^+}$ and $\mathcal{S}_{\mathcal{H}}(\bar{\mathfrak{f}}) = (\mathbb{S}_{\mathbb{H}}(F^{\mathrm{un}})_0\mathbb{H}(F^{\mathrm{un}})_{y,0^+})/\mathbb{H}(F^{\mathrm{un}})_{y,0^+}$. It follows that $\iota(\mathcal{S}_{\mathcal{H}}(\bar{\mathfrak{f}})) = (\mathbb{S}_{\mathbb{H}}(F^{\mathrm{un}})_0\mathbb{G}(F^{\mathrm{un}})_{y,0^+})/\mathbb{G}(F^{\mathrm{un}})_{y,0^+}$ is an elliptic maximal $\mathfrak{f}$-torus of $\binom{\mathcal{H}}{\mathcal{G}}_y$. Since $\mathbb{S}_{\mathbb{H}}(F^{\mathrm{un}})_0$ is contained in both $\mathbb{S}(F^{\mathrm{un}})_0$ and $\mathbb{H}(F^{\mathrm{un}})_{y,0}$, $\iota(\mathcal{S}_{\mathcal{H}}(\bar{\mathfrak{f}})) \subset \left(\mathcal{S} \cap \binom{\mathcal{H}}{\mathcal{G}}_y\right)(\bar{\mathfrak{f}})$. The maximality of both tori implies that the inclusion is an equality. ∎

The result of Lemma 6.4.2 is summarized in the diagram of Figure 6.2.

$$\begin{tikzcd}
\mathbb{S} \subset \mathbb{G} \arrow[rr, "\text{intersect}"] \arrow[d, leftrightarrow] & & \mathbb{S}_{\mathbb{H}} \subset \mathbb{H} \arrow[d, leftrightarrow] \\
\mathcal{S} \subset \mathcal{G}_y \arrow[r, "\text{intersect}"] & \mathcal{S} \cap \binom{\mathcal{H}}{\mathcal{G}}_y \arrow[r, "\iota^{-1}"] & \mathcal{S}_{\mathcal{H}} \subset \mathcal{H}_y
\end{tikzcd}$$

Figure 6.2: Correspondence between the elliptic maximally unramified $F$-torus $\mathbb{S}_{\mathbb{H}}$ with the elliptic maximal $\mathfrak{f}$-torus of $\mathcal{S}_{\mathcal{H}}$ given the correspondence between $\mathbb{S}$ and $\mathcal{S}$.

**Proposition 6.4.3.** *Let $\mathbb{S}$ be an elliptic maximally unramified $F$-torus of $\mathbb{G}$ associated to the vertex $[y]$ and let $\mathcal{S}$ be its corresponding elliptic maximal $\mathfrak{f}$-torus of $\mathcal{G}_y$ as per Lemma 6.3.5. Denote by $\mathcal{S}_{\mathcal{H}}$ the elliptic maximal $\mathfrak{f}$-torus of $\mathcal{H}_y$ corresponding to the elliptic maximally unramified $F$-torus $\mathbb{S}_{\mathbb{H}} = \mathbb{S} \cap \mathbb{H}$ of $\mathbb{H}$. Let $\overline{\theta}$ be a regular character of $\mathcal{S}(\mathfrak{f})$ and assume that $\overline{\theta}_{\mathcal{H}} = \overline{\theta}|_{\mathcal{S}\cap\binom{\mathcal{H}}{\mathcal{G}}_y(\mathfrak{f})} \circ \iota$ is also regular. Then*

$$\kappa_{(\mathcal{S},\overline{\theta})}|_{H_{y,0}} = \kappa_{(\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}})},$$

*where $\kappa_{(\mathcal{S},\overline{\theta})}$ and $\kappa_{(\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}})}$ denote the pullbacks of $\pm R_{\mathcal{S},\overline{\theta}}$ and $\pm R_{\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}}}$ to $G_{y,0}$ and $H_{y,0}$, respectively. That is, we have the diagram illustrated in Figure 6.3.*

**Proof:** For all $h \in H_{y,0}$, we have

$$\kappa_{(\mathcal{S},\overline{\theta})}(h) = \pm R_{\mathcal{S},\overline{\theta}}(hG_{y,0^+}) = \pm R_{\mathcal{S},\overline{\theta}}|_{\binom{\mathcal{H}}{\mathcal{G}}_y(\mathfrak{f})}(hG_{y,0^+}).$$

$$
\begin{tikzcd}
\left(\pm R_{\mathcal{S},\overline{\theta}}, \mathcal{G}_y(\mathfrak{f})\right) \arrow[d, "\text{pullback}"] & \left(\pm R_{\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}}}, \mathcal{H}_y(\mathfrak{f})\right) \arrow[d, "\text{pullback}"] \\
\left(\kappa_{(\mathcal{S},\overline{\theta})}, G_{y,0}\right) \arrow[r, "\text{restrict}"] & \left(\kappa_{(\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}})}, H_{y,0}\right)
\end{tikzcd}
$$

Figure 6.3: Diagram summarizing the restriction of $\kappa_{(\mathcal{S},\overline{\theta})}$ to $H_{y,0}$ as per Proposition 6.4.3.

Since $[\mathcal{G}_y, \mathcal{G}_y] \subset \binom{\mathcal{H}}{\mathcal{G}}_y$ (Lemma 2.2.26), it follows from Theorem 6.2.8 that $\pm R_{\mathcal{S},\overline{\theta}}|_{\binom{\mathcal{H}}{\mathcal{G}}_y(\mathfrak{f})} = \pm R_{\mathcal{S}\cap\binom{\mathcal{H}}{\mathcal{G}}_y,\overline{\theta}_{\binom{\mathcal{H}}{\mathcal{G}}_y}}$, where $\overline{\theta}_{\binom{\mathcal{H}}{\mathcal{G}}_y}$ denotes the restriction of $\overline{\theta}$ to $\left(\mathcal{S}\cap\binom{\mathcal{H}}{\mathcal{G}}_y\right)(\mathfrak{f})$. Therefore

$$\kappa_{(\mathcal{S},\overline{\theta})}(h) = \pm R_{\mathcal{S}\cap\binom{\mathcal{H}}{\mathcal{G}}_y,\overline{\theta}_{\binom{\mathcal{H}}{\mathcal{G}}_y}}(hG_{y,0^+}).$$

We can rewrite the right-hand side as $\pm R_{\iota(\mathcal{S}_{\mathcal{H}}),\overline{\theta}_{\mathcal{H}}\circ\iota^{-1}}(\iota(hH_{y,0^+}))$ by Lemma 6.4.2. But Corollary 6.1.8 tells us this last expression is equal to $\pm R_{\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}}}(hH_{y,0^+})$. Therefore, we obtain

$$\kappa_{(\mathcal{S},\overline{\theta})}(h) = \pm R_{\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}}}(hH_{y,0^+}) = \kappa_{(\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}})}(h).$$

∎

**Remark 6.4.4.** There is no guarantee for $\overline{\theta}_{\mathcal{H}}$ to be regular when $\overline{\theta}$ is regular, since we have seen that a character in general position does not necessarily restrict to another character in general position (Example 6.2.7). However, under the hypothesis that $\overline{\theta}_{\mathcal{H}}$ is also regular, we know that $\pm R_{\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}}}$ is cuspidal and that the normalizer in $H_{[y]}$ of an extension to $S_H H_{y,0}$ of $\kappa_{(\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}})}$ is equal to $S_H H_{y,0}$ as discussed in Section 6.3.

We note the following relationship between the groups $SG_{y,0}$ and $S_H H_{y,0}$.

**Lemma 6.4.5.** *Let $\mathbb{S}$ be an elliptic maximally unramified $F$-torus of $\mathbb{G}$ with associated vertex $[y]$ and let $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$. Then $SG_{y,0} \cap H = S_H H_{y,0}$, where $S_H = S \cap H$.*

**Proof:** It is clear that $S_H H_{y,0} \subset SG_{y,0} \cap H$.

For the converse, take $h \in SG_{y,0} \cap H$. Then $h = sg$ for some $s \in S, g \in G_{y,0}$. We claim that $G_{y,0} = S_0 H_{y,0} G_{y,0^+}$. Indeed, let $\mathcal{S}$ be the elliptic maximal $\mathfrak{f}$-torus of $\mathcal{G}_y$

associated to $\mathbb{S}$ as per Lemma 6.3.5. Because $\binom{\mathcal{H}}{\mathcal{G}}_y$ contains $[\mathcal{G}_y, \mathcal{G}_y]$ (Lemma 2.2.26), we can apply Proposition 6.1.10 and obtain

$$\mathcal{G}_y(\mathfrak{f}) = \mathcal{S}(\mathfrak{f})\binom{\mathcal{H}}{\mathcal{G}}_y(\mathfrak{f}),$$

or equivalently

$$G_{y,0}/G_{y,0^+} = S_0 G_{y,0^+}/G_{y,0^+} \cdot H_{y,0}G_{y,0^+}/G_{y,0^+}.$$

Therefore, $G_{y,0} = S_0 H_{y,0} G_{y,0^+}$ as claimed.

This means that we can rewrite $g = s'h'g'$ for some $s' \in S_0$, $h' \in H_{y,0}$, $g' \in G_{y,0^+}$ so that $h = ss'h'g'$. Furthermore, [DR09, Lemma B.7.2] allows us to write $G_{y,0^+} = Z_{0^+}\overline{G}_{y,0^+}$ where $Z = Z(\mathbb{G})^\circ(F) \subset S$ and $\overline{G} = \mathbb{G}_{\mathrm{der}}(F) \subset H$. This means that we can write $g' = z\overline{g}$ for some $z \in Z \subset S, \overline{g} \in \overline{G}_{y,0^+} \subset H_{y,0^+}$, meaning that

$$h = ss'h'z\overline{g} = ss'zh'\overline{g}.$$

But $ss'z = h(h'\overline{g})^{-1} \in S \cap H = S_H$ and $h'\overline{g} \in H_{y,0}$. Thus $h \in S_H H_{y,0}$ which concludes the proof. ∎

What we have pointed out in Remark 6.4.4 motivates us to impose the additional condition that $\overline{\theta}_{\mathcal{H}}$ be regular in the following theorem. Note that this character $\overline{\theta}_{\mathcal{H}}$ corresponds to the character $\theta|_{S_H}$ of $S_H$.

**Proposition 6.4.6.** *Let $\mathbb{S}$ be an elliptic maximally unramified $F$-torus of $\mathbb{G}$ with associated vertex $[y]$. Let $\theta$ be a regular depth-zero character of $S$ and set $\rho = \text{c-Ind}_{SG_{y,0}}^{G_{[y]}} \kappa_{(\mathbb{S},\theta)}$. Let $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$ and set $\mathbb{S}_{\mathbb{H}} = \mathbb{S} \cap \mathbb{H}$. Assume that $\theta|_{S_H}$ is also regular. Then $\rho' = \text{c-Ind}_{S_H H_{y,0}}^{H_{[y]}}\left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right)$ is a component of $\rho_H = \rho|_{H_{[y]}}$. Furthermore, the multiplicity of $\rho'$ in $\rho_H$ is equal to $m = [I_{G_{[y]}}(\rho') : H_{[y]}SG_{y,0}]$.*

**Proof:** The Mackey decomposition allows us to compute the restriction of $\rho$ to $H_{[y]}$ as follows:

$$\begin{aligned}\rho|_{H_{[y]}} &= \mathrm{Res}_{H_{[y]}}^{G_{[y]}} \text{c-Ind}_{SG_{y,0}}^{G_{[y]}} \kappa_{(\mathbb{S},\theta)} \\ &= \bigoplus_{c \in C} \text{c-Ind}_{H_{[y]} \cap {}^c(SG_{y,0})}^{H_{[y]}} \mathrm{Res}_{H_{[y]} \cap {}^c(SG_{y,0})}^{{}^c(SG_{y,0})} \kappa_{(\mathbb{S},\theta)},\end{aligned}$$

where $C$ is a set of coset representatives of $H_{[y]}\backslash G_{[y]}/SG_{y,0}$. In particular, because $SG_{y,0} \cap H_{[y]} = SG_{y,0} \cap H = S_H H_{y,0}$ (Lemma 6.4.5), we see that $\text{c-Ind}_{S_H H_{y,0}}^{H_{[y]}}\left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right)$ is a subrepresentation of $\rho_H$. By Proposition 6.4.3, we see that $\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}$ is an extension to $S_H H_{y,0}$ of $\kappa_{(\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}})}$. By hypothesis, we have assumed that $\theta|_{S_H}$ (and therefore $\overline{\theta}_{\mathcal{H}}$) is regular, meaning that the normalizer in $H_{[y]}$ of $\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}$ is equal to $S_H H_{y,0}$ as discussed in Remark 6.4.4. It is then a consequence of the Mackey Irreducibility Criterion that $\rho' = \text{c-Ind}_{S_H H_{y,0}}^{H_{[y]}} \left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right)$ is irreducible, meaning that $\rho'$ is a component of $\rho_H$.

Now, for the multiplicity, we know from Proposition 5.2.10 that the multiplicity of $\rho'$ in $\rho_H$ is given by

$$m = \frac{\dim(\rho)}{[G_{[y]} : I_{G_{[y]}}(\rho')] \dim(\rho')},$$

where $I_{G_{[y]}}(\rho')$ is the normalizer in $G_{[y]}$ of $\rho'$. We observe that $\dim(\rho) = [G_{[y]} : SG_{y,0}] \dim(\kappa_{(\mathbb{S},\theta)})$ and $\dim(\rho') = [H_{[y]} : S_H H_{y,0}] \dim(\kappa_{(\mathbb{S},\theta)})$ by Proposition A.1.8, which implies that

$$m = \frac{[G_{[y]} : SG_{y,0}]}{[G_{[y]} : I_{G_{[y]}}(\rho')][H_{[y]} : S_H H_{y,0}]}.$$

But $[H_{[y]} : S_H H_{y,0}] = [H_{[y]} SG_{y,0} : SG_{y,0}]$ as $S_H H_{y,0} = SG_{y,0} \cap H_{[y]}$ (Lemma 6.4.5), implying that $H_{[y]}/S_H H_{y,0} \simeq H_{[y]} SG_{y,0}/SG_{y,0}$ by the second isomorphism theorem. Therefore,

$$m = \frac{[G_{[y]} : SG_{y,0}]}{[G_{[y]} : I_{G_{[y]}}(\rho')][H_{[y]} SG_{y,0} : SG_{y,0}]},$$

which simplifies to $m = [I_{G_{[y]}}(\rho') : H_{[y]} SG_{y,0}]$ as $H_{[y]} SG_{y,0} \subset I_{G_{[y]}}(\rho') \subset G_{[y]}$. ∎

**Remark 6.4.7.** Without the hypothesis that $\theta|_{S_H}$ is also regular, one would need to decompose $\text{c-Ind}_{S_H H_{y,0}}^{H_{[y]}} \left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right)$ in order to find a component $\rho'$ of $\rho_H$. Furthermore, this decomposition could potentially lead to additional multiplicity in $\rho_H$.

If $\rho$ satisfies the hypotheses of Proposition 6.4.6 and that $I_{G_{[y]}}(\rho') = H_{[y]} SG_{y,0}$, then the restriction $\rho_H$ will be multiplicity free. For this to happen, we need to impose a slightly stronger condition on the character $\theta$. Following the language of [AM19], we have the following definition.[1]

**Definition 6.4.8.** *Let $\theta'$ be a character of $S_H$. We say that $\theta'$ is regular in $G$ if $I_{N_{\mathbb{G}}(\mathbb{S})(F)}(\theta'|_{(S_H)_0}) = S$.*

**Remark 6.4.9.** It is easy to see that if $\theta'$ is a character of $S_H$ which is regular in $G$, then it is a regular in the sense previously defined (Remark 6.3.8). Indeed, it suffices to take the equality $I_{N_{\mathbb{G}}(\mathbb{S})(F)}\left(\theta'|_{(S_H)_0}\right) = S$, intersect both sides with $H$ and use the fact that $N_{\mathbb{G}}(\mathbb{S}) \cap \mathbb{H} = N_{\mathbb{H}}(\mathbb{S}_{\mathbb{H}})$ (Proposition 2.1.23) to obtain $I_{N_{\mathbb{H}}(\mathbb{S}_{\mathbb{H}})(F)}\left(\theta'|_{(S_H)_0}\right) = S_H$.

[1] The version of the pre-print [AM19] cited here contains a typo in its Definition 5.2, which was confirmed by one of the authors through a private correspondence.

**Corollary 6.4.10.** *Let $\mathbb{S}, \mathbb{H}, \mathbb{S}_{\mathbb{H}}, \theta, \rho$ and $\rho'$ be as in Proposition 6.4.6. Assume furthermore that $\theta|_{S_H}$ is regular in $G$. Then the restriction $\rho_H = \rho|_{H_{[y]}}$ is multiplicity free.*

**Proof:** By the previous proposition, it suffices to show that $I_{G_{[y]}}(\rho') \subset H_{[y]}SG_{y,0}$ to obtain equality between the two sets.

Let $g \in I_{G_{[y]}}(\rho')$. Then ${}^{g}\,\text{c-Ind}_{S_H H_{y,0}}^{H_{[y]}}\left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right) \simeq \text{c-Ind}_{S_H H_{y,0}}^{H_{[y]}}\left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right)$. From the Mackey Decomposition (Remark 2.3.12), it follows that ${}^{g}\left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right) \simeq {}^{h}\left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right)$ for some $h \in H_{[y]}$, or equivalently ${}^{h^{-1}g}\left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right) \simeq \left(\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}\right)$. Using Kaletha's argument in [Kal19, Lemma 3.4.20] which was briefly summarized in Remark 6.3.10, there exists $g' \in H_{y,0}$ such that $g'h^{-1}g \in N_{\mathbb{G}}(\mathbb{S}_{\mathbb{H}})(F)$ and ${}^{g'h^{-1}g}\left(\theta|_{(S_H)_0}\right) = \theta|_{(S_H)_0}$. From Proposition 2.1.23, one sees that $N_{\mathbb{G}}(\mathbb{S}_{\mathbb{H}}) = N_{\mathbb{G}}(\mathbb{S})$, and therefore $g'h^{-1}g \in I_{N_{\mathbb{G}}(\mathbb{S})(F)}\left(\theta|_{(S_H)_0}\right)$. Therefore, $g'h^{-1}g \in S$ and $g \in H_{[y]}SG_{y,0}$. ∎

Using Theorem 5.1.1, we restate the previous result in terms of the supercuspidal representation. The statement we provide below has also been proved by Adler and Mishra in a recent preprint [AM19, Theorem 5.3]. While their proof is very concise, our method provides a more explicit description of the restriction.

**Corollary 6.4.11.** *Let $\pi_{(\mathbb{S},\theta)}$ be a regular depth-zero supercuspidal representation of $G$, where $\mathbb{S}$ is an elliptic maximally unramified $F$-torus and $\theta$ is a regular depth-zero character of $S$. Let $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\text{der}}$ and set $\mathbb{S}_{\mathbb{H}} = \mathbb{S} \cap \mathbb{H}$. Assume furthermore that $\theta|_{S_H}$ is regular in $G$. Then $\pi_{(\mathbb{S},\theta)}|_H$ is multiplicity free.*

Finally, using Corollary 5.1.9, this allows us to write a statement for a multiplicity free restriction of certain irreducible supercuspidal representations of arbitrary depth.

**Corollary 6.4.12.** *Let $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$ be a $G$-datum. Assume that $\pi^{-1} = \text{c-Ind}_{G^0_{[y]}}^{G^0}\,\rho$ is a regular depth-zero supercuspidal representation that satisfies the hypotheses of Corollary 6.4.11 when setting $\mathbb{G} = \mathbb{G}^0$ and $\mathbb{H} = \mathbb{H}^0$. Then $\pi_G(\Psi)|_H$ is multiplicity free.*

## 6.5 Remarks and Future Directions

Recall from the introduction (Chapter 1) that our motivation for studying the multiplicity problem for the restriction of supercuspidal representations, apart from it being a fundamental representation theory question, came from Conjecture 2 by Adler and Prasad [AP06]. This conjecture stated that the irreducible admissible representations of $\mathbb{G}$ should restrict with multiplicity one to subgroups $\mathbb{H}$ that contain $\mathbb{G}_{\text{der}}$, along with a few additional underlying hypotheses.

We also discussed how they later found a counterexample of an irreducible representation of $\mathrm{GU}(n)$ whose restriction to $\mathrm{SU}(n)$ decomposes with multiplicity two [AP19, Theorem 10]. More precisely, this representation of $\mathrm{GU}(n)$ that they constructed was a regular depth-zero supercuspidal representation $\pi_{(\mathbb{S},\theta)}$ and such that $\theta|_{S\cap\mathrm{SU(n)}}$ was not regular. However, simply having regularity in the restriction is not enough to guarantee a multiplicity free restriction as seen with Proposition 6.4.6 and Corollary 6.4.12. So, while the failure of regularity is a potential source for obtaining higher multiplicity, as stated in [AP19, Remark 11], other factors also come into play.

The main reason for wanting to study the regular supercuspidal representations in this chapter was to obtain a more concise description of the representation $\rho$ in order to isolate a component $\rho'$ of $\rho_H = \rho|_{H_{[y]}}$. There was a lot of work involved in order to get to Proposition 6.4.6, which means that even though we have a formula for the multiplicity in the restriction of $\rho$, it is not necessarily easy to compute in practice.

While Kaletha gives as a remark that most depth-zero supercuspidal representations are regular (paragraph following [Kal19, Definition 3.7.3]), more work is required in order to say something about the multiplicities in the restriction of a representation $\rho$ that does not satisfy the hypotheses of Proposition 6.4.6. For instance, when $\theta|_{S_H}$ is not regular, we have not provided a description for the restriction of $\kappa_{(\mathbb{S},\theta)}$ which is key for understanding $\rho_H = \rho|_{H_{[y]}}$. We do not pursue this question further in this thesis, but rather leave it as an open problem.

# Chapter 7

# Digging Deeper

In this final chapter of the thesis, we discuss some future directions of research and the impact that this work has on the representation theory of $p$-adic groups. As a first question, one can ask whether the results of this thesis have been stated in the greatest possible generality. This is what we discuss in Section 7.1. In Section 7.2, we talk about more recent kinds of data that can be used to construct representations, and how the main idea of this thesis, namely the restriction of data to obtain a description of the restriction of a representation, can be applied in other contexts. Finally, we end with a discussion on the theory of types in Section 7.3. The theory of types can be used to answer questions about non-supercuspidal representations. In particular, we show that our results for a restriction of data can be transferred to define a restriction of types allowing us to obtain information about the restriction of any kind of irreducible representation of $G$. As such, the implications of our work go beyond just the irreducible supercuspidal representations.

## 7.1 Underlying Assumption on $p$

Throughout this thesis, we have assumed that the residual characteristic $p$ of the $p$-adic field $F$ does not divide the order of the Weyl group of $\mathbb{G}$. This assumption is not extremely restrictive, as shown in Table 2.1.

Without this hypothesis on $p$, the J.K. Yu Construction is still valid and produces irreducible supercuspidal representations, though it might not produce all such representations (Theorem 2.3.28). In order to check whether our restriction of data (Theorem 3.3.1) and restriction formula (Theorem 4.4.1) are still valid when there is no restriction on $p$, some additional work would be required.

Indeed, in the original definition for genericity of the quasicharacters in a $G$-datum, there are two conditions that need to be satisfied, which Yu refers to as conditions GE1) and GE2). As mentioned in Remark 3.1.7, with our underlying assumption on $p$, condition GE1) automatically implies condition GE2). For this reason, we were able to completely omit condition GE2) in this thesis. Removing the hypothesis on $p$ means that there are additional verifications to be made in the restriction (extension) of generic characters in Section 3.3.3 (Section 3.5.2). This assumption on $p$ also allowed to assume that all characters can be realized by a central element when restricted to a certain depth (Remark 3.1.14), which was used in Lemmas 3.1.15 and A.4.7. Therefore, those statements would also need to be reworked if the hypothesis on $p$ were to be removed.

We note that if Theorems 3.3.1 and 4.4.1 hold without the assumption on $p$, then all our subsequent results on multiplicities follow.

## 7.2 Restriction of Data

The main theme of this thesis was to take a J.K. Yu $G$-datum $\Psi$, and see how the restriction of such a datum allows us to describe the restriction of the corresponding supercuspidal representation $\pi_G(\Psi)$. The same idea can be applied to different types of data, and is not limited to supercuspidal representations of $p$-adic groups. We have come across other examples in the thesis, but they were not highlighted as such. Indeed, we were implicitly working with different types of data in the study of Deligne-Lusztig virtual characters and regular depth-zero supercuspidal representations, which we discuss in Sections 7.2.1 and 7.2.2, respectively. We also mention another type of data that can be used for the construction of supercuspidal representations in Section 7.2.3, namely the data described by Hakim in [Hak18].

### 7.2.1 Data for Deligne-Lusztig Characters and Cuspidal Representations

Recall the Deligne-Lusztig virtual characters from Section 6.1. Given a reductive group $\mathbb{G}$ defined over a finite field $\mathfrak{f}$ (or a finite group of Lie type for the more general context), we saw that the Deligne-Lusztig virtual characters are parametrized by pairs $(\mathbb{T},\theta)$, where $\mathbb{T}$ is a maximal $\mathfrak{f}$-torus of $\mathbb{G}$ and $\theta$ is a character of $\mathbb{T}(\mathfrak{f})$. Therefore, such a pair could be thought of as a datum for the construction of the Deligne-Lusztig virtual character $R_{\mathbb{T},\theta}$. Given a closed connected subgroup $\mathbb{H}$ of $\mathbb{G}$ containing $\mathbb{G}_{\mathrm{der}}$, the restriction of the datum $(\mathbb{T},\theta)$ to $\mathbb{H}(\mathfrak{f})$ would be given by $(\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}})$, where $\mathbb{T}_{\mathbb{H}}=\mathbb{T}\cap\mathbb{H}$ and $\theta_{\mathbb{H}}=\theta|_{\mathbb{T}_{\mathbb{H}}(\mathfrak{f})}$. Furthermore, we saw that $R_{\mathbb{T},\theta}|_{\mathbb{H}(\mathfrak{f})}=R_{\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}}}$ (Theorem 6.1.9). This aligns exactly with what we observed for supercuspidal representations. That is, the restriction to $\mathbb{H}(\mathfrak{f})$ is constructed from the restricted data. Unlike the supercuspidal representations, the virtual character $R_{\mathbb{T},\theta}$ is not obtained from an induction, so the restriction to $\mathbb{H}(\mathfrak{f})$ is more direct and we do not have any conjugates that appear.

When the torus $\mathbb{T}$ is also elliptic, and $\theta$ is in general position, then $(\mathbb{T},\theta)$ can be thought of as a datum for a Deligne-Lusztig cuspidal representation. However, the character $\theta_{\mathbb{H}}$ need not be in general position (Example 6.2.7), meaning that the restriction of $(\mathbb{T},\theta)$ to $\mathbb{H}(\mathfrak{f})$ does not necessarily produce a datum for a Deligne-Lusztig cuspidal representation of $\mathbb{H}(\mathfrak{f})$. This means that the restriction of $(\mathbb{T},\theta)$ to $\mathbb{H}$ does not necessarily produce a datum of the same type. However, when $\theta_{\mathbb{H}}$ is also in general position, then the restriction of $\pm R_{\mathbb{T},\theta}$ gives us precisely the Deligne-Lusztig cuspidal representation of $\mathbb{H}(\mathfrak{f})$ constructed from the restricted datum $(\mathbb{T}_{\mathbb{H}},\theta_{\mathbb{H}})$ (Theorem 6.2.8). So, when the restriction of a datum produces data of the same type, the components of the restriction are constructed from the restricted data, as was the case for supercuspidal representations.

### 7.2.2 Data for Regular Supercuspidal Representations

Given a reductive group $\mathbb{G}$ which is defined over a $p$-adic field $F$, we saw in Section 6.3 that Kaletha's regular depth-zero supercuspidal representations of $G$ can be parametrized by pairs $(\mathbb{S},\theta)$, where $\mathbb{S}$ is an elliptic maximally unramified $F$-torus of $\mathbb{G}$ and $\theta$ is a regular depth-zero character of $S=\mathbb{S}(F)$. Therefore, these pairs can be viewed as another type of datum. Similarly to the above discussion, restricting the pair $(\mathbb{S},\theta)$ to a reductive $F$-subgroup $\mathbb{H}$ that contains $\mathbb{G}_{\mathrm{der}}$ does not necessarily provide us with another such pair. Indeed, this is because the restriction of this pair would be given by $(\mathbb{S}_{\mathbb{H}},\theta_{\mathbb{H}})$, where $\mathbb{S}_{\mathbb{H}}=\mathbb{S}\cap\mathbb{H}$ and $\theta_{\mathbb{H}}=\theta|_{S_H}$, and $\theta_{\mathbb{H}}$ might not be regular. However, in the case that $\theta_{\mathbb{H}}$ is also regular, then we have a valid restriction of datum, and obtain a formula for the restriction $\pi_{(\mathbb{S},\theta)}|_H$ in terms of the restricted datum $(\mathbb{S}_{\mathbb{H}},\theta_{\mathbb{H}})$ as follows.

**Theorem 7.2.1.** *Let $\mathbb{S}$ be an elliptic maximally unramified $F$-torus of $\mathbb{G}$ with associated vertex $[y]$ and let $\theta$ be a regular depth-zero character of $S$. Let $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\mathrm{der}}$. Set $\mathbb{S}_{\mathbb{H}}=\mathbb{S}\cap\mathbb{H}$ and assume that $\theta_{\mathbb{H}}=\theta|_{S_H}$ is also regular. Then*

$$\pi_{(\mathbb{S},\theta)}|_H=\bigoplus_{c\in C}\pi_{({}^c\mathbb{S}_{\mathbb{H}},{}^c\theta_{\mathbb{H}})},$$

*where $C$ is a set of coset representatives of $H\backslash G/SG_{y,0}$.*

**Proof:** We use the Mackey decomposition formula to compute the restriction:

$$\begin{aligned}\pi_{(\mathbb{S},\theta)}|_H&=\mathrm{Res}_H^G\,\text{c-Ind}_{SG_{y,0}}^G\,\kappa_{(\mathbb{S},\theta)}\\&=\bigoplus_{c\in C}\text{c-Ind}_{H\cap{}^c(SG_{y,0})}^H\,\mathrm{Res}_{H\cap{}^c(SG_{y,0})}^{{}^c(SG_{y,0})}\,{}^c\kappa_{(\mathbb{S},\theta)},\end{aligned}$$

where $C$ is a set of coset representatives of $H\backslash G/SG_{y,0}$. Because $H\cap SG_{y,0}=S_HH_{y,0}$ (Lemma 6.4.5), we can rewrite the restriction as

$$\pi_{(\mathbb{S},\theta)}|_H=\bigoplus_{c\in C}\text{c-Ind}_{{}^c(S_HH_{y,0})}^H\,{}^c\kappa_{(\mathbb{S},\theta)}|_{{}^c(S_HH_{y,0})}.$$

We also know that $\kappa_{(\mathbb{S},\theta)}|_{S_H H_{y,0}}$ induces irreducibly to $H$ as $\pi_{(\mathbb{S}_{\mathbb{H}},\theta_{\mathbb{H}})}$, being an extension of the cuspidal representation $\kappa_{(\mathcal{S}_{\mathcal{H}},\overline{\theta}_{\mathcal{H}})}$ (Propositions 6.3.11 and 6.4.3). One can also check that ${}^c\kappa_{(\mathbb{S},\theta)} = \kappa_{({}^c\mathbb{S},{}^c\theta)}$, meaning that $\text{c-Ind}^{H}_{{}^c(S_H H_{y,0})}\, {}^c\kappa_{(\mathbb{S},\theta)}|_{{}^c(S_H H_{y,0})} = \pi_{({}^c\mathbb{S}_{\mathbb{H}},{}^c\theta_{\mathbb{H}})}$ and the result thus follows. ∎

**Remark 7.2.2.** When considering the corresponding J.K. Yu datum, one can apply our key decomposition theorem, Theorem 4.4.1, which describes the restriction of the supercuspidal representation in terms of the restricted data. The result of this decomposition formula is the same as what we just described above. Indeed, when we apply Theorem 4.4.1, this gives us a double sum, one over a set $C$ of coset representatives of $H\backslash G/G_{[y]}$, and the second one over a set $D$ of coset representatives of $H_{[y]}\backslash G_{[y]}/SG_{y,0}$ coming from the restriction to $H_{[y]}$ of $\rho = \text{c-Ind}^{G_{[y]}}_{SG_{y,0}}\, \kappa_{(\mathbb{S},\theta)}$. One can then verify that the multiplication of the sets $C$ and $D$ give a set of coset representatives of $H\backslash G/SG_{y,0}$.

In [Kal19], Kaletha also talks about *regular* supercuspidal representations of arbitrary depth. A regular supercuspidal representation is defined as a supercuspidal representation whose depth-zero piece of its reduced datum is a regular depth-zero supercuspidal representation. Such representations are parametrized by *tame elliptic regular pairs* $(\mathbb{S},\theta)$. Indeed, Kaletha shows that every reduced J.K. Yu datum for a regular supercuspidal representation corresponds to a tame elliptic regular pair $(\mathbb{S},\theta)$, and that every tame elliptic regular pair can be extended into a reduced J.K. Yu datum [Kal19, Proposition 3.7.8]. The correspondence between the two types of data is given as follows.

Given a reduced $G$-datum $\Psi = (\vec{\mathbb{G}}, \pi^{-1}, \vec{\phi})$ for a regular supercuspidal representation, we have that $\pi^{-1} = \pi_{(\mathbb{S},\phi^{-1})}$ for some elliptic maximally unramified $F$-torus $\mathbb{S}$ of $\mathbb{G}^0$ and regular depth-zero character $\phi^{-1}$ of $S$. From the datum, we set $\theta = \prod_{i=-1}^{d} \phi^i|_S$ and obtain a tame elliptic regular pair $(\mathbb{S},\theta)$. Conversely, from a tame elliptic regular pair $(\mathbb{S},\theta)$, a regular reduced datum $\Psi = (\vec{\mathbb{G}}, \pi_{(\mathbb{S},\phi^{-1})}, \vec{\phi})$ can be constructed by obtaining a *Howe factorization* $(\phi^{-1}, \vec{\phi})$ of $\theta$ satisfying $\theta = \prod_{i=-1}^{d} \phi^i|_S$.

Under the assumption that $\phi_H^{-1} := \phi^{-1}|_{S_H}$ is also regular, one easily sees that the datum $(\vec{\mathbb{H}}, \pi_{(\mathbb{S}_{\mathbb{H}},\phi_H^{-1})}, \vec{\phi})$ from $\text{Res}(\Psi)$ corresponds to the pair $(\mathbb{S}_{\mathbb{H}}, \theta_H)$, where $\theta_H = \theta|_{S_H}$. Therefore, the restriction of tame elliptic regular pairs (under an additional regularity hypothesis) commutes with the restriction of regular reduced J.K. Yu data, as illustrated in Figure 7.1.

So, Kaletha's tame elliptic regular pairs serve as a different kind of datum for the regular supercuspidal representations. Because we have a clear picture of the

$$\begin{tikzcd}
(\mathbb{S},\theta) \arrow[r, "\text{restrict}"] \arrow[d, leftrightarrow] & (\mathbb{S}_{\mathbb{H}},\theta_H) \arrow[d, leftrightarrow] \\
(\vec{\mathbb{G}},\pi_{(\mathbb{S},\phi^{-1})},\vec{\phi}) \arrow[r, "\text{"restrict"}"] & (\vec{\mathbb{H}},\pi_{(\mathbb{S}_{\mathbb{H}},\phi_H^{-1})},\vec{\phi_H})
\end{tikzcd}$$

Figure 7.1: Commutativity of the restriction of tame elliptic pairs and regular reduced J.K. Yu data.

restriction to $H$ of supercuspidal representations from the perspective of J.K. Yu data, the results easily transfer over to the tame elliptic regular pairs as per the following corollary. Note however, that we are a bit more confined with the tame elliptic regular pairs as we need some regularity condition (namely $\phi_H^{-1}$ to be regular) in order for the restriction of such a pair to remain of the same type.

**Corollary 7.2.3.** *Let $(\mathbb{S},\theta)$ be a tame elliptic pair with associated vertex $[y]$ and let $\pi_{(\mathbb{S},\theta)}$ be the corresponding regular supercuspidal representation. Let $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\text{der}}$, set $\mathbb{S}_{\mathbb{H}} = \mathbb{S}\cap\mathbb{H}$ and $\theta_H = \theta|_{S_H}$. Assume that $(\mathbb{S}_{\mathbb{H}},\theta_H)$ is also a tame elliptic regular pair. Then*

$$\pi_{(\mathbb{S},\theta)}|_H = \bigoplus_{t\in C,\ell\in L}\pi_{({}^{t\ell}\mathbb{S}_{\mathbb{H}},{}^{t\ell}\theta_H)},$$

*where $C$ and $L$ are sets of coset representatives of $H\backslash G/K^d$ and $K_H^0\backslash K^0/SG_{y,0}^0$, respectively.*

**Proof:** Let $(\vec{\mathbb{G}},\pi_{(\mathbb{S},\phi^{-1})},\vec{\phi})$ be the regular reduced datum obtained from $(\mathbb{S},\theta)$ as described above. We have that $\pi_{(\mathbb{S},\phi^{-1})} = \text{c-Ind}_{K^0}^{G^0}\rho$, where $\rho = \text{c-Ind}_{SG_{y,0}^0}^{K^0}\kappa_{(\mathbb{S},\phi^{-1})}$ (Section 6.3). Therefore, $\pi_{(\mathbb{S},\theta)} = \pi_G(\Psi)$, where $\Psi = (\vec{\mathbb{G}},y,\rho,\vec{\phi})$. By Proposition 6.4.6, we have that

$$\rho|_{K_H^0} = \bigoplus_{\ell\in L}\text{c-Ind}_{{}^{\ell}(S_H H_{y,0}^0)}^{K_H^0}\kappa_{({}^{\ell}\mathbb{S}_{\mathbb{H}},{}^{\ell}\phi_H^{-1})},$$

where $\phi_H^{-1} = \phi^{-1}|_{S_H}$ and $L$ is a set of coset representatives of $K_H^0\backslash K^0/SG_{y,0}^0$.

Setting $\rho_\ell = \text{c-Ind}_{{}^{\ell}(S_H H_{y,0}^0)}^{K_H^0}\kappa_{({}^{\ell}\mathbb{S}_{\mathbb{H}},{}^{\ell}\phi_H^{-1})}$, $\ell\in L$, it follows from Theorem 4.4.1 that

$$\pi_{(\mathbb{S},\theta)}|_H = \pi_G(\Psi)|_H = \bigoplus_{t\in C}\left(\bigoplus_{\ell\in L}\pi_H({}^{t}\Psi_\ell)\right),$$

where $C$ is a set of coset representatives of $H\backslash G/K^d$. We have that the reduced datum associated to $\Psi_\ell$ is $(\vec{\mathbb{H}},\pi_{({}^{\ell}\mathbb{S}_{\mathbb{H}},{}^{\ell}\phi_H^{-1})},\vec{\phi_H})$ meaning that $\pi_H(\Psi_\ell) = \pi_{({}^{\ell}\mathbb{S}_{\mathbb{H}},{}^{\ell}\phi_H^{-1})}$. The conclusion follows. ∎

### 7.2.3 Hakim's Data

In Section 3.2, we discussed the notion of $G$-equivalence of $G$-data established by Hakim and Murnaghan that determines the equivalence class of a supercuspidal representation. Given a $G$-datum $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$, we see from Theorem 3.2.4(3) that the equivalence class of $\pi_G(\Psi)$ is essentially determined by $\rho \otimes \prod_{i=0}^{d} \phi^i|_{G^0_{[y]}}$. This means that in a sense, only the information on $G^0$ was necessary for the construction of the supercuspidal representation.

From this observation, Hakim constructs a new kind of datum in [Hak18], even though he might not refer to it directly as such. Hakim's data consist of triples $(\mathbb{G}', [y], \phi)$, where $\mathbb{G}'$ is a tamely ramified twisted Levi subgroup of $\mathbb{G}$, $[y]$ is a vertex in $\mathcal{B}^{\mathrm{red}}(\mathbb{G}', F)$ and $\phi$ is a *permissible* representation of $G'_{[y]}$ [Hak18, Definition 2.1.1]. We shall refer to such a triple as a *Hakim-datum*. From the permissible representation $\phi$, Hakim describes the algorithm that produces an irreducible supercuspidal representation $\pi(\phi)$ of $G$.

To connect this with the J.K. Yu Construction, given a $G$-datum $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$, the representation $\phi := \rho \otimes \prod_{i=0}^{d} \phi^i|_{G^0_{[y]}}$ is a permissible representation of $G^0_{[y]}$. This means that from the J.K. Yu datum $\Psi$, we have a corresponding Hakim-datum $(\mathbb{G}^0, [y], \phi)$. Furthermore, Hakim shows that his representation $\pi(\phi)$ is isomorphic to $\pi_G(\Psi)$ [Hak18, Lemma 4.0.1]. With our underlying hypothesis on $p$, the J.K. Yu Construction is exhaustive, so every Hakim-datum has to come from a $G$-datum. From this correspondence, we obtain a commutative diagram (Figure 7.2) for the restriction to $H$ of $G$-data and Hakim-data. This means that we can translate our results for the restriction of a supercuspidal representation in terms of the Hakim-data using Figure 7.2.

$$\begin{tikzcd}
(\vec{\mathbb{G}}, y, \rho, \vec{\phi}) \arrow[r, "\text{restrict}"] \arrow[d, leftrightarrow] & \{(\vec{\mathbb{H}}, y, \rho_\ell, \vec{\phi_H}), \ell \in L\} \arrow[d, leftrightarrow] \\
\left(\mathbb{G}^0, [y], \rho \otimes \prod_{i=0}^{d} \phi^i|_{G^0_{[y]}}\right) \arrow[r, "\text{restrict}"] & \left\{\left(\mathbb{H}^0, [y], \rho_\ell \otimes \prod_{i=0}^{d} \phi^i_H|_{H^0_{[y]}}\right), \ell \in L\right\}
\end{tikzcd}$$

Figure 7.2: Commutativity of the restriction of a $G$-datum and corresponding Hakim-datum. $L$ is the index set for the components of $\rho_H = \rho|_{H^0_{[y]}}$.

When our underlying hypothesis on $p$ is not satisfied (and that the J.K. Yu Construction is not necessarily exhaustive), one can still obtain a Hakim-datum from a $G$-datum in the same way. However, there is a possibility for Hakim's construction to produce supercuspidal representations that are not captured by Yu's construction [Hak18, Remark 3.3.6], meaning that not all Hakim-data come from $G$-data. Given a Hakim-datum $(\mathbb{G}', [y], \phi)$ for $G$, if the components of $\phi|_{G'\cap H}$ are also permissible, then we have a restriction process for Hakim-data and can obtain a result similar to Theorem 4.4.1. Future work would then include determining conditions under which the components of the restriction of a permissible representation remain permissible.

## 7.3 Implications for the Theory of Types

This thesis focused primarily on the study of irreducible supercuspidal representations of the $p$-adic group $G = \mathbb{G}(F)$. This is only a fraction of the irreducible representations of $G$. Indeed, recall from Jacquet's Subrepresentation Theorem (Theorem 2.3.14) that the irreducible representations of $G$ which are not supercuspidal are obtained as subrepresentations of the parabolic induction of supercuspidal representations. The task of studying the irreducible representations of $G$ which are not supercuspidal is a whole different problem. Unlike the supercuspidal representations, irreducible non-supercuspidal representations are not all obtained by compact induction from an open compact-mod-center subgroup. That is there is no equivalent of Mautner's Theorem (Theorem 2.3.19) for such representations. Rather, one can turn to the theory of types to obtain information, which applies to all representations of $G$.

The theory of types was introduced by Bushnell and Kutzko in [BK98]. This theory allows to describe representations of $G$ in terms of representations of certain compact open subgroups. To define what a type is, one needs the *Bernstein decomposition* [Ber84] for the category $\mathcal{R}(G)$ of smooth complex representations of $G$. This decomposition lets us write $\mathcal{R}(G)$ as a direct product of subcategories over some index set $\mathfrak{B}$

$$\mathcal{R}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}} \mathcal{R}^{\mathfrak{s}}(G).$$

Elements of $\mathfrak{B}$ correspond to certain equivalence classes of pairs $(M, \sigma)$, where $M$ is a Levi subgroup (of a parabolic subgroup) of $G$ and $\sigma$ is an irreducible supercuspidal representation of $M$. The subcategories $\mathcal{R}^{\mathfrak{s}}(G), \mathfrak{s} \in \mathfrak{B}$, are called *Bernstein blocks*. Types give an explicit description of each Bernstein block as per the following definition.

**Definition 7.3.1.** *Let $\mathfrak{s} \in \mathfrak{B}$, $J$ be a compact open subgroup of $G$ and $\lambda$ be a smooth irreducible representation of $J$. We say that $(J, \lambda)$ is an $\mathfrak{s}$-type if for every smooth irreducible representation $\pi$ of $G$, we have that $\pi \in \mathcal{R}^{\mathfrak{s}}(G)$ if and only if $\pi|_J$ contains $\lambda$.*

*We say that an irreducible smooth representation $\pi$ of $G$ contains a type if there exists an $\mathfrak{s}$-type $(J, \lambda)$ for some $\mathfrak{s} \in \mathfrak{B}$ such that $\pi|_J$ contains $\lambda$.*

Thus, we can obtain a description of $\mathcal{R}(G)$ entirely in terms of representations of compact open groups via the construction of an $\mathfrak{s}$-type for each $\mathfrak{s} \in \mathfrak{B}$.

### 7.3.1 Restriction of Types

In the case where $\pi = \pi_G(\Psi)$ is an irreducible supercuspidal representation of $G$ constructed from the $G$-datum $\Psi$, one can easily construct a type which is contained in $\pi$ [BK98, Proposition 5.4]. Indeed, recall that $\pi = \text{c-Ind}_{K^d}^G \kappa_G(\Psi)$, where $K^d = G^0_{[y]} G^1_{y,s_0} \cdots G^d_{y,s_{d-1}}$. The group $K^d$ is an open subgroup of $G$ which is compact modulo the center of $G$. The maximal compact open subgroup of $K^d$ is $J_{\vec{\mathbb{G}}} = G^0_y G^1_{y,s_0} \cdots G^d_{y,s_{d-1}}$, where $G^0_y$ is the stabilizer in $G^0$ of $y$. Letting $\lambda_G(\Psi)$ be a component of $\kappa_G(\Psi)|_{J_{\vec{\mathbb{G}}}}$, we have that $(J_{\vec{\mathbb{G}}}, \lambda_G(\Psi))$ is an $\mathfrak{s}$-type contained in $\pi_G(\Psi)$, where $\mathfrak{s}$ corresponds to the pair $(G, \pi_G(\Psi))$. Essentially, one obtains a type contained in $\pi_G(\Psi)$ by restricting the pair $(\kappa_G(\Psi), K^d)$ obtained from the datum $\Psi$ that induces the representation.

The restriction of a $G$-datum to $H$-data from Section 3.3 induces a restriction of types contained in $\pi_G(\Psi)$ to types contained in components of $\pi_G(\Psi)|_H$. Indeed, given a $G$-datum $\Psi$, Theorem 3.3.1 allows us to construct a set of $H$-data $\operatorname{Res}(\Psi) = \{\Psi_\ell, \ell \in L\}$. We define $J_{\vec{\mathbb{H}}} = H^0_y H^1_{y,s_0} \cdots H^d_{y,s_{d-1}}$ which is compact open and contained in $K^d_H$, and for each $\ell \in L$, we let $\lambda_H(\Psi_\ell)$ be a component of $\kappa_H(\Psi_\ell)|_{J_{\vec{\mathbb{H}}}}$. Then $(J_{\vec{\mathbb{H}}}, \lambda_H(\Psi_\ell))$ is a type contained in $\pi_H(\Psi_\ell)$. Furthermore, as a consequence of Theorem 4.3.6, which described the restriction $\kappa_G(\Psi)|_{K^d_H}$, we have that

$$\kappa_G(\Psi)|_{J_{\vec{\mathbb{H}}}} = \bigoplus_{\ell \in L} \kappa_H(\Psi_\ell)|_{J_{\vec{\mathbb{H}}}}.$$

Therefore, for each component $\lambda_G(\Psi) \subset \kappa_G(\Psi)|_{J_{\vec{\mathbb{G}}}}$, $\lambda_G(\Psi)|_{J_{\vec{\mathbb{H}}}}$ is a direct sum of components of $\kappa_H(\Psi_\ell)|_{J_{\vec{\mathbb{H}}}}$. Hence, when we restrict the type $(J_{\vec{\mathbb{G}}}, \lambda_G(\Psi))$, we obtain types contained in components of $\pi_G(\Psi)|_H$. We illustrate this restriction of types in Figure 7.3.

$$\begin{tikzcd}
\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi}) \arrow[r, "\text{restrict}"] \arrow[d, "\text{type}"'] & \{\Psi_\ell = (\vec{\mathbb{H}}, y, \rho_\ell, \vec{\phi_H}), \ell \in L\} \arrow[d, "\text{type}"'] \\
\{(J_{\vec{\mathbb{G}}}, \lambda_G(\Psi)), \lambda_G(\Psi) \subset \kappa_G(\Psi)|_{J_{\vec{\mathbb{G}}}}\} \arrow[r, "\text{restrict}"] & \left\{(J_{\vec{\mathbb{H}}}, \lambda_H(\Psi_\ell)), \lambda_H(\Psi_\ell) \subset \kappa_H(\Psi_\ell)|_{J_{\vec{\mathbb{H}}}}, \ell \in L\right\}
\end{tikzcd}$$

Figure 7.3: Restriction of the types $(J_{\vec{\mathbb{G}}}, \lambda_G(\Psi))$ contained in $\pi_G(\Psi)$ to types contained in $\pi_H(\Psi_\ell), \ell \in L$.

**Remark 7.3.2.** From the restriction of data, we can only obtain types that are contained in $\pi_H(\Psi_\ell), \ell \in L$, meaning that our restriction of types did not construct types for each component of $\pi_G(\Psi)|_H$. Because we know precisely the restriction $\pi_G(\Psi)|_H$ (Theorem 4.4.1), we can deduce the types that are contained in the other components of the restriction. Indeed, to obtain a type contained in $\pi_H({}^t\Psi_\ell), t \in G$, we simply conjugate one of the types contained in $\pi_H(\Psi_\ell)$ by $t$.

For an irreducible representation $\pi$ which is not supercuspidal, one can still obtain types contained in $\pi$ via a construction described by Kim and Yu in [KY17]. The Kim-Yu Construction for types is very similar to Yu's construction of irreducible supercuspidal representations. Indeed, to construct a type, one starts from a 5-tuple, which we will refer to as a *type-datum*. This type-datum is a modified version of a $G$-datum and is defined as follows [KY17, Section 7.2].

**Definition 7.3.3.** *A sequence $\Sigma = ((\vec{\mathbb{G}}, \mathbb{M}^0), (y, \{I\}), \vec{r}, (M_y^0, \tilde{\rho}), \vec{\phi})$ is a type-datum for $G$ if and only if:*

*TD1)* *$\vec{\mathbb{G}}$ is a tamely ramified twisted Levi sequence $\vec{\mathbb{G}} = (\mathbb{G}^0, \dots, \mathbb{G}^d)$ in $\mathbb{G}$, and $\mathbb{M}^0$ is a Levi subgroup of $\mathbb{G}^0$;*

*TD2)* *$y$ is a point in $\mathcal{B}(\mathbb{M}^0, F)$ and $\{I\}$ is a commutative diagram of embeddings of buildings which is $\vec{s}$-generic relative to $y$ in the sense of* [KY17, Section 3]*, where $\vec{s} = (0, r_0/2, \dots, r_d/2)$;*

*TD3)* *$\vec{r} = (r_0, \dots, r_d)$ is a sequence of real numbers satisfying $0 < r_0 < \cdots < r_{d-1} \leqslant r_d$ if $d > 0$, $0 \leqslant r_0$ if $d = 0$;*

*TD4)* *$M_{y,0}^0$ is a maximal parahoric subgroup of $M^0$ and $M_y^0$ is a compact open subgroup of $M^0$ containing $M_{y,0}^0$ as a normal subgroup, and $\tilde{\rho}$ is an irreducible smooth representation of $M_y^0$ such that $\tilde{\rho}|_{M_{y,0}^0}$ contains a cuspidal representation of $M_{y,0:0+}^0$;*

*TD5)* *$\vec{\phi} = (\phi^0, \dots, \phi^d)$ is a sequence of quasicharacters, where $\phi^i$ is a $G^{i+1}$-generic character of $G^i$ of depth $r_i$ relative to $x$ for all $x \in \mathcal{B}(\mathbb{G}^i, F)$, $0 \leqslant i \leqslant d-1$. If $r_{d-1} < r_d$, we assume $\phi^d$ is of depth $r_d$, otherwise $\phi^d = 1$.*

Without loss of generality, one can assume that the point $y$ belongs to the reduced building of $\mathbb{M}^0$ (Remark 2.2.20). It is clear from the definition of type-datum that we can adapt the statement of Theorem 3.3.1 to obtain a restriction of type-datum as follows.

**Theorem 7.3.4.** *Let $\Sigma = ((\vec{\mathbb{G}}, \mathbb{M}^0), (y, \{I\}), \vec{r}, (M_y^0, \tilde{\rho}), \vec{\phi})$ be a type-datum for $G$. Let $\mathbb{M}_{\mathbb{H}}^0 = \mathbb{M}^0 \cap \mathbb{H}$, $\mathbb{H}^i = \mathbb{G}^i \cap \mathbb{H}$ and $\phi_H^i = \phi^i|_{H^i}$ for all $0 \leqslant i \leqslant d$. Let $\tilde{r}$ denote the depth of $\phi_H^d$.*

1) *If $\tilde{r} > r_{d-1}$, set $\vec{\phi}_H = (\phi_H^0, \ldots, \phi_H^{d-1}, \phi_H^d)$ and $\vec{\tilde{r}} = (r_0, \ldots, r_{d-1}, \tilde{r})$.*

2) *If $\tilde{r} \leqslant r_{d-1}$, set $\vec{\phi}_H = (\phi_H^0, \cdots, \phi_H^{d-1}\phi_H^d, 1)$ and $\vec{\tilde{r}} = (r_0, \ldots, r_{d-1})$.*

*Then, for all irreducible subrepresentation $\tilde{\rho}_\ell$ of $\tilde{\rho}|_{(M_H^0)_y}$, $\Sigma_\ell = ((\vec{\mathbb{H}}, \mathbb{M}_{\mathbb{H}}^0), (y, \{I\}), \vec{\tilde{r}}, ((M_H^0)_y, \tilde{\rho}_\ell), \vec{\phi}_H)$ is a type-datum for $H$, where $\vec{\mathbb{H}} = (\mathbb{H}^0, \ldots, \mathbb{H}^d)$.*

Given a type-datum $\Sigma$, let $J_{(\vec{\mathbb{G}}, \mathbb{M}^0)} = M_y^0 G_{y,s_0}^1 \cdots G_{y,s_{d-1}}^d$, which is an open compact subgroup of $G$. A process completely analogous to the J.K. Yu Construction results in an irreducible representation $\lambda_G(\Sigma)$ of $J_{(\vec{\mathbb{G}}, \mathbb{M}^0)}$. The pair $(J_{(\vec{\mathbb{G}}, \mathbb{M}^0)}, \lambda_G(\Sigma))$ is an $\mathfrak{s}$-type for some $\mathfrak{s} \in \mathfrak{B}$ [KY17, Theorem 7.5]. Such a type will be called a *Kim-Yu type*. Furthermore, we can adapt the result of Theorem 4.3.6, which described the restriction $\kappa_G(\Psi)|_{K_H^d}$, to obtain the following.

**Theorem 7.3.5.** *Let $\Sigma = ((\vec{\mathbb{G}}, \mathbb{M}^0), (y, \{I\}), \vec{r}, (M_y^0, \rho), \vec{\phi})$ be a type-datum for $G$. Assume that the decomposition into components of $\rho|_{(M_H^0)_y}$ is given by $\rho|_{(M_H^0)_y} = \bigoplus_{\ell \in L} \rho_\ell$ for some index set $L$. For each $\ell \in L$, let $\Sigma_\ell$ be the type-datum for $H$ as per Theorem 7.3.4, $J_{(\vec{\mathbb{H}}, \mathbb{M}_{\mathbb{H}}^0)} = (M_H^0)_y H_{y,s_0}^1 \cdots H_{y,s_{d-1}}^d$ and $\lambda_H(\Sigma_\ell)$ be the irreducible representation of $J_{(\vec{\mathbb{H}}, \mathbb{M}_{\mathbb{H}}^0)}$ constructed from $\Sigma_\ell$. Then*

$$\lambda_G(\Sigma)|_{J_{(\vec{\mathbb{H}}, \mathbb{M}_{\mathbb{H}}^0)}} = \bigoplus_{\ell \in L} \lambda_H(\Sigma_\ell).$$

Note that we can also apply the results of Section 3.5 and define an extension of types to go from a type-datum for $H$ to type-data for $G$.

**Remark 7.3.6.** Given a $G$-datum $\Psi = (\vec{\mathbb{G}}, y, \vec{r}, \rho, \vec{\phi})$, the type $(J_\Psi, \lambda_G(\Psi))$ is a Kim-Yu type by setting $\Sigma = ((\vec{\mathbb{G}}, \mathbb{G}^0), y, \vec{r}, (G_y^0, \tilde{\rho}), \vec{\phi})$, where $\tilde{\rho}$ is a component of $\rho|_{G_y^0}$.

### 7.3.2 Consequences and Conclusion

Fintzen showed that every (smooth) irreducible representation of $G$ contains a Kim-Yu type when $p$ does not divide the order of the Weyl group of $\mathbb{G}$ [Fin18, Theorem 7.12]. This is a refinement of the exhaustion result obtained by Kim and Yu [KY17, Theorem 9.1]. This means that we are able to obtain information on the restriction of any irreducible representation $\pi$ of $G$ via our restriction of types (Theorems 7.3.4 and 7.3.5) as per the following corollary.

**Corollary 7.3.7.** *Let $\pi$ be an irreducible representation of $G$ and let $(J_\Sigma, \lambda_G(\Sigma))$ be a Kim-Yu type contained in $\pi$. Let $\{\Sigma_\ell, \ell \in L\}$ be the type-data for $H$ obtained from $\Sigma$ as per Theorem 7.3.4. Then, for all $\ell \in L$, $(J_{(\vec{\mathbb{H}}, \mathbb{M}_{\mathbb{H}}^0)}, \lambda_H(\Sigma_\ell))$ is a Kim-Yu type which is contained in a component of $\pi|_H$.*

By Remark 7.3.2, we do not expect our restriction of types to yield a type for each component of $\pi|_H$. Furthermore, at this stage, we do not know whether we can expect a decomposition formula like that of Theorem 4.4.1 for an irreducible representation $\pi$ which is not supercuspidal, but future work could investigate the direct implications of the restriction of Kim-Yu types on the decomposition of $\pi|_H$.

By what precedes, we see that the consequences of this thesis go beyond just the supercuspidal representations. While this was the main focus of our work, by transferring our results to the theory of types, we have a larger impact and offer future directions for the general study of irreducible representations of $p$-adic groups and the category of smooth complex representation $\mathcal{R}(G)$ of $G$.

# Appendix A

# The Devil is in the Details

This appendix contains results that were needed in the thesis, but that did not quite fit in the natural flow of the narrative presented due to the significant amount of additional structure and notation required.

Section A.1 contains statements and proofs related to representation theory, which is meant to serve as a reference for the reader that is not very familiar with representations.

Sections A.2 to A.4 contain proofs to more technical statements. The results that we present are very naturally expected to hold, but we did not find proofs in the existing literature, which is why we present them here. Some of these technical results also require heavy notation, and presenting them within the course of the thesis would have caused some digressions from the main storyline. We reassure the reader that the proofs we present in this appendix are straightforward once the structures and notations have been introduced.

## A.1 Representations

In this section we present a series of proofs for results regarding representations that was used throughout the thesis. Most of these results should be known, or at least seem intuitive, to the reader that is familiar with representation theory. This appendix should not be read on its own, but rather serve as a reference, when needed, as results from it are called within the thesis.

**Proposition A.1.1.** *Let $(\pi, V)$ be a completely reducible representation of $G$. Then any subrepresentation of $(\pi, V)$ is also completely reducible.*

**Proof:** It is well known that a representation $V$ is completely reducible if and only if every subrepresentation of $V$ has a $G$-invariant complement in $V$. So, given a subrepresentation $W \subset V$, we must show that every subrepresentation of $W$ has a $G$-invariant complement in $W$.

Let $U$ be a subrepresentation of $W$, that is a $G$-invariant subspace of $W$. In particular, $U \subset V$ and by complete reducibility of $V$, there exists a $G$-invariant subspace $U' \subset V$ such that $U \oplus U' = V$. It then follows that $U' \cap W$ is a $G$-invariant subspace of $W$ and $U \oplus (U' \cap W) = W$. Therefore $U$ has a $G$-invariant complement in $W$ and $W$ is completely reducible.

∎

**Lemma A.1.2.** *Let $(\pi, V), (\rho, W)$ be two representations of $G$ such that $\mathrm{Hom}_G(\pi, \rho) \neq \{0\}$. Then for all $f \in \mathrm{Hom}_G(\pi, \rho)$,*

*a)* $\ker f$ *is a $G$-invariant subspace of $V$ (with respect to the action induced by $\pi$);*

*b)* $\mathrm{Im}\, f$ *is a $G$-invariant subspace of $W$ (with respect to the action induced by $\rho$).*

**Proof:** By definition, $f : V \to W$ is an intertwining operator, meaning that $\rho(g) \circ f = f \circ \pi(g)$ for all $g \in G$.

a) Let $v \in \ker f$. We must show that $\pi(g)v \in \ker f$ for all $g \in G$. Given $g \in G$, $f(\pi(g)v) = \rho(g)(f(v))$. Since $v \in \ker f$, it follows that $f(\pi(g)v) = \rho(g)(0) = 0$. Thus, $\pi(g)v \in \ker f$.

b) Let $w \in \mathrm{Im}\, f$. We must show that $\rho(g)w \in \mathrm{Im}\, f$ for all $g \in G$. We have that $w = f(v)$ for some $v \in V$, so given $g \in G$, we have that $\rho(g)w = \rho(g)(f(v))$. Since $f$ is an intertwining operator, it follows that $\rho(g)w = f(\pi(g)v)$ and therefore $\rho(g)w \in \mathrm{Im}\, f$.

∎

**Proposition A.1.3.** *Let $(\pi, V), (\rho, W)$ be two representations of $G$ such that $\mathrm{Hom}_G(\pi, \rho) \neq \{0\}$.*

*a) If $\pi$ is irreducible, then $V$ is a subrepresentation of $W$.*

*b) If $\rho$ is irreducible, then $W$ is a quotient of $V$.*

*Conversely, if $\pi$ is a subrepresentation of $\rho$, or if $\rho$ is a quotient of $\pi$, then $\mathrm{Hom}_G(\pi, \rho) \neq \{0\}$.*

**Proof:** Let $0 \neq f \in \mathrm{Hom}_G(\pi, \rho)$. We have by Lemma A.1.2 that $\ker f$ and $\mathrm{Im}\, f$ are $G$-invariant subspaces with respect to $\pi$ and $\rho$, respectively.

a) Since $\ker f$ is a $G$-invariant subspace of $V$ and $(\pi, V)$ is irreducible, we must have $\ker f = \{0\}$ or $\ker f = V$. But $f \neq 0$ by hypothesis, which forces $\ker f = \{0\}$. This means that $\operatorname{Im} f \neq \{0\}$ is a $G$-invariant subspace of $W$. By the first isomorphism theorem, we have that $\operatorname{Im} f \simeq {}^{V}/_{\ker f} = V$. Therefore, the subrepresentation $(\rho, \operatorname{Im} f)$ of $(\rho, W)$ is isomorphic to $(\pi, V)$.

b) Since $\operatorname{Im} f$ is a $G$-invariant subspace of $W$ and $(\rho, W)$ is irreducible, we must have $\operatorname{Im} f = \{0\}$ or $\operatorname{Im} f = W$. Since $f \neq 0$, this forces $W = \operatorname{Im} f$. But, $\operatorname{Im} f \simeq {}^{V}/_{\ker f}$, where $\ker f$ is a $G$-invariant subspace of $V$. Hence, we have that $(\rho, W)$ is isomorphic to a quotient representation of $(\pi, V)$.

If $\pi$ is a subrepresentation of $\rho$, then $V \subset W$ and $\pi(g)v = \rho(g)v$ for all $v \in V, g \in G$. It then suffices to take the inclusion map $f : V \to W \in \operatorname{Hom}_G(\pi, \rho)$. If $\rho$ is a quotient of $\pi$, then $W = V/U$ for some $G$-invariant subspace $U \subset V$ and $\rho(g)(v + U) = \pi(g)(v) + U$ for all $v \in V, g \in G$. It then suffices to take the projection map $p : V \to V/U \in \operatorname{Hom}_G(\pi, \rho)$. ∎

**Lemma A.1.4.** *Let $Z$ denote the center of the group $G$ and let $G' \subset G$ be a subgroup that contains $Z$. Then for any irreducible representation of $G'$, the center $Z$ acts by scalar multiplication.*

**Proof:** Let $(\pi, V)$ be an irreducible representation of $G'$. For all $g' \in G'$, $z \in Z$,$v \in V$,

$$\pi(z)\pi(g')v = \pi(g')\pi(z)v.$$

It follows that for all $z \in Z$, $\pi(z)$ is an intertwining map. Since $\pi$ is irreducible, it follows from Schur's lemma that $\pi(z) = c_z \mathrm{Id}_V$ for some constant $c_z$, where $\mathrm{Id}_{\mathrm{V}}$ is the identity map on $V$. Hence, the center acts by scalar multiplication. ∎

**Lemma A.1.5.** *Let $\pi$ be a representation of $G$. For all $g \in G$, we have ${}^{g}\pi \simeq \pi$.*

**Proof:** Let $g \in G$. It suffices to show that $\pi(g)$ is an intertwining map between $\pi$ and ${}^{g}\pi$. For all $g' \in G$, $\pi(g) \circ {}^{g}\pi(g') = \pi(g)\pi(g^{-1}g'g) = \pi(g'g) = \pi(g') \circ \pi(g)$. ∎

**Lemma A.1.6.** *Let $(\pi, U), (\rho, V), (\psi, W)$ be representations of $G$. Then $(\pi \oplus \rho) \otimes \psi = (\pi \otimes \psi) \oplus (\rho \otimes \psi)$.*

**Proof:** We proceed by direct computation. For all $g \in G, u \in U, v \in V, w \in W$

$$(\pi \oplus \rho) \otimes \psi(g)((u + v) \otimes w) = (\pi \oplus \rho)(g)(u + v) \otimes \psi(g)w$$

$$\begin{aligned}
&= (\pi(g)u + \rho(g)v) \otimes \psi(g)w \\
&= (\pi(g)u \otimes \psi(g)w) + (\rho(g)v \otimes \psi(g)w) \\
&= (\pi \otimes \psi)(g)(u \otimes w) + (\rho \otimes \psi)(g)(v \otimes w) \\
&= (\pi \otimes \psi) \oplus (\rho \otimes \psi)(g)((u \otimes w) \oplus (v \otimes w)).
\end{aligned}$$

∎

**Lemma A.1.7.** *Let $G' \subset G$ be a subgroup. Let $(\pi, V)$ be a representation of $G'$. Take $g \in G$ and let $(\rho, W)$ be a representation of ${}^gG'$. Then ${}^g\pi \simeq \rho$ if and only if $\pi \simeq {}^{g^{-1}}\rho$.*

**Proof:** We have that ${}^g\pi \simeq \rho$ if and only if there exists an invertible intertwining map $f : V \to W$ such that $f \circ {}^g\pi(\bar{g}') = \rho(\bar{g}') \circ f$, or equivalently $f \circ \pi(g^{-1}\bar{g}'g) = \rho(\bar{g}') \circ f$, for all $\bar{g}' \in {}^gG'$. Every $\bar{g}' \in {}^gG'$ is of the form $gg'g^{-1}$ for some $g' \in G'$ and therefore $f \circ \pi(g^{-1}\bar{g}'g) = \rho(\bar{g}') \circ f$ for all $\bar{g}' \in {}^gG'$ if and only if $f \circ \pi(g') = \rho(gg'g^{-1}) \circ f$ for all $g' \in G'$. Thus, the bijective map $f$ intertwines ${}^g\pi$ and $\rho$ if and only if it intertwines $\pi$ and ${}^{g^{-1}}\rho$. ∎

**Proposition A.1.8.** *Let $N \subset G$ be a subgroup of $G$ of finite index and let $(\pi, V)$ be a finite-dimensional representation of $N$. Then $\text{c-Ind}_N^G \pi$ is a finite-dimensional representation of $G$ of dimension $[G : N] \dim(V)$.*

**Proof:** Note here that because $N$ is of finite index, $G/N$ is compact and therefore induction and compact induction are the same.

We must show that the vector space

$$\text{c-Ind}_N^G V = \{f : G \to V : f(gn) = \pi(n^{-1})f(g) \text{ for all } n \in N\}$$

is finite-dimensional. Let $\{v_j : j \in J\}$ be a basis of $V$, and $\{g_i, i \in I\}$ be a set of coset representatives of $G/N$, where $I$ and $J$ are finite index sets. Define $f_{ij} : G \to V$ as follows. For all $n \in N$,

$$f_{ij}(g_\ell n) = \begin{cases} \pi(n^{-1})v_j \text{ if } \ell = i, \\ 0 \text{ if } \ell \neq i. \end{cases}$$

We claim that $\{f_{ij} : i \in I, j \in J\}$ is a basis for the vector space $\text{c-Ind}_N^G V$, and therefore $\text{c-Ind}_N^G V$ is of finite dimension $|I||J| = [G : N] \dim(V)$.

Indeed, we start by showing that $f_{ij} \in \text{c-Ind}_N^G V$ for all $i \in I, j \in J$. Given $g \in G$, $g = g_\ell n$ for some $\ell \in I, n \in N$. So, for all $n' \in N$, we have

$$f_{ij}(gn') = f_{ij}(g_\ell nn') = \begin{cases} \pi((nn')^{-1})v_j \text{ if } \ell = i, \\ 0 \text{ if } \ell \neq i. \end{cases}$$

Since $\pi((nn')^{-1}) = \pi(n'^{-1})\pi(n^{-1})$, it follows that $f_{ij}(gn') = \pi(n'^{-1})f_{ij}(g)$. Hence, $f_{ij} \in \text{c-Ind}_N^G V$ for all $i \in I, j \in J$.

Next, we show that $\{f_{ij} : i \in I, j \in J\}$ spans $\text{c-Ind}_N^G V$. Let $f \in \text{c-Ind}_N^G V$. Then for all $i \in I$ $f(g_i) \in V$, which means that $f(g_i) = \sum_{j \in J} c_{ij} v_j$ for some scalars $c_{ij}$. We then have that

$$f = \sum_{i \in I} \sum_{j \in J} c_{ij} f_{ij}.$$

Indeed, for all $\ell \in I, n \in N$,

$$\begin{aligned} f(g_\ell n) &= \pi(n^{-1}) f(g_\ell) \\ &= \pi(n^{-1}) \sum_{j \in J} c_{\ell j} v_j \\ &= \sum_{j \in J} c_{\ell j} \pi(n^{-1}) v_j \\ &= \sum_{j \in J} c_{\ell j} f_{\ell, j}(g_\ell n). \end{aligned}$$

Since $f_{ij}(g_\ell n) = 0$ for all $i \neq \ell$, it follows that $f(g_\ell n) = \sum_{i \in I} \sum_{j \in J} c_{ij} f_{ij}(g_\ell n)$.

Finally, we verify that $\{f_{ij} : i \in I, j \in J\}$ is a set of linearly independent vectors by showing that $\sum_{i \in I} \sum_{j \in J} a_{ij} f_{ij} = 0$ implies that $a_{ij} = 0$ for all $i \in I, j \in J$. Indeed, if $\sum_{i \in I} \sum_{j \in J} a_{ij} f_{ij} = 0$, then $\sum_{i \in I} \sum_{j \in J} a_{ij} f_{ij}(g_\ell n) = 0$ for all $\ell \in I, n \in N$, which simplifies to $\sum_{j \in J} a_{\ell j} f_{\ell j}(g_\ell n) = 0$ for all $\ell \in I, n \in N$. But,

$$\sum_{j \in J} a_{\ell j} f_{\ell j}(g_\ell n) = \sum_{j \in J} a_{\ell j} \pi(n^{-1}) v_j = \pi(n^{-1}) \left( \sum_{j \in J} a_{\ell j} v_j \right)$$

is trivial for all $\ell \in I$ if and only if $\sum_{j \in J} a_{\ell j} v_j = 0$ for all $\ell \in I$. Since $\{v_j : j \in J\}$ is a set of linearly independent vectors, being a basis for $V$, we conclude that $a_{\ell j} = 0$ for all $\ell \in I, j \in J$.

Thus, we have shown that $\{f_{ij}, i \in I, j \in J\}$ is a basis for $\text{c-Ind}_N^G V$ and therefore the dimension of $\text{c-Ind}_N^G V$ is equal to $|I||J| = [G : N] \dim(V)$ as claimed. ∎

**Proposition A.1.9.** *Let $G' \subset G$ be a subgroup of $G$ and let $(\pi, V), (\rho, W)$ be two representations of $G'$. Then $\text{c-Ind}_{G'}^G(\pi \oplus \rho) = \text{c-Ind}_{G'}^G \pi \oplus \text{c-Ind}_{G'}^G \rho$, that is induction commutes with direct sum.*

**Proof:** We show that the vector spaces $\text{c-Ind}_{G'}^{G}(V \oplus W)$ and $\text{c-Ind}_{G'}^{G} V \oplus \text{c-Ind}_{G'}^{G} W$ are the same.

Let $f \in \text{c-Ind}_{G'}^{G} V \oplus \text{c-Ind}_{G'}^{G} W$. Then $f = f_V + f_W$ for some $f_V \in \text{c-Ind}_{G'}^{G} V$, $f_W \in \text{c-Ind}_{G'}^{G} W$. For all $g' \in G', g \in G$,

$$\begin{aligned} f(gg') &= f_V(gg') + f_W(gg') \\ &= \pi(g'^{-1})f_V(g) + \rho(g'^{-1})f_W(g) \\ &= (\pi \oplus \rho)(g'^{-1})(f_V(g) + f_W(g)) \\ &= (\pi \oplus \rho)(g'^{-1})f(g). \end{aligned}$$

Therefore, $\text{c-Ind}_{G'}^{G} V \oplus \text{c-Ind}_{G'}^{G} W \subset \text{c-Ind}_{G'}^{G}(V \oplus W)$.

Conversely, take $f \in \text{c-Ind}_{G'}^{G}(V \oplus W)$. Then for all $g \in G$, $f(g) = f_V(g) + f_W(g)$ for some $f_V(g) \in V$, $f_W(g) \in W$. These projections define maps $f_V : G \to V$, $f_W : G \to W$. Then, for all $g' \in G'$, we have $f(gg') = f_V(gg') + f_W(gg')$. On the other hand,

$$\begin{aligned} f(gg') &= (\pi \oplus \rho)(g'^{-1})f(g) \\ &= (\pi \oplus \rho)(g'^{-1})(f_V(g) + f_W(g)) \\ &= \pi(g'^{-1})f_V(g) + \rho(g'^{-1})f_W(g). \end{aligned}$$

By uniqueness of the direct sum decomposition of $V \oplus W$, it follows that $f_V(gg') = \pi(g'^{-1})f_V(g)$ and $f_W(gg') = \rho(g'^{-1})f_W(g)$ for all $g' \in G', g \in G$. That is, $f_V \in \text{c-Ind}_{G'}^{G} V$ and $f_W \in \text{c-Ind}_{G'}^{G} W$. Hence, $\text{c-Ind}_{G'}^{G}(V \oplus W) \subset \text{c-Ind}_{G'}^{G} V \oplus \text{c-Ind}_{G'}^{G} W$. ∎

**Proposition A.1.10.** *Let $N \subset G' \subset G$ be groups, with $N$ normal in both $G'$ and $G$. Let $(\pi, V)$ be a representation of $G'/N$ and identify it with its pullback to $G'$. Then,*

$$\text{c-Ind}_{G'/N}^{G/N} \pi \simeq \text{c-Ind}_{G'}^{G} \pi,$$

*where $\text{c-Ind}_{G'/N}^{G/N} \pi$ is identified with its pullback to $G$. That is, the pullback of an induction is the induction of a pullback.*

**Proof:** We must find an invertible linear map $T : \text{c-Ind}_{G'/N}^{G/N} V \to \text{c-Ind}_{G'}^{G} V$ that intertwines $\text{c-Ind}_{G'/N}^{G/N} \pi$ and $\text{c-Ind}_{G'}^{G} \pi$, where

$$\begin{aligned} \text{c-Ind}_{G'/N}^{G/N} V &= \{f : G/N \to V : f(gNg'N) = \pi(g'^{-1}N)f(gN) \text{ for all } g'N \in G'/N\}, \\ \text{c-Ind}_{G'}^{G} V &= \{f : G \to V : f(gg') = \pi(g'^{-1})f(g) \text{ for all } g' \in G'\}. \end{aligned}$$

Given $f \in \text{c-Ind}_{G'/N}^{G/N} V$, the map $\bar{f} : G \to V$ defined by $\bar{f}(g) = f(gN)$ for all $g \in G$ is an element of $\text{c-Ind}_{G'}^{G} V$. Indeed, for all $g' \in G', g \in G$,

$$\begin{aligned}\bar{f}(gg') &= f(gg'N)\\ &= f(gNg'N)\\ &= \pi(g'^{-1}N)f(gN)\\ &= \pi(g'^{-1})\bar{f}(g).\end{aligned}$$

Therefore, we define $T$ as follows:

$$\begin{aligned}T : \text{c-Ind}_{G'/N}^{G/N} V &\to \text{c-Ind}_{G'}^{G} V\\ f &\mapsto \bar{f},\end{aligned}$$

where $\bar{f}$ is defined as previously.

We start by showing that $T$ is bijective. It is surely injective because $T(f_1) = T(f_2)$ implies that $f_1(gN) = f_2(gN)$ for all $g \in G$, and therefore $f_1 = f_2$. Given $\bar{f} \in \text{c-Ind}_{G'}^{G} V$, define

$$\begin{aligned}f : G/N &\to V\\ gN &\mapsto \bar{f}(g).\end{aligned}$$

This map is well-defined because $\pi$ is trivial on $N$ (as a representation of $G'$) by definition, and so $\bar{f}(n) = 1$ for all $n \in N$. As a consequence, $\bar{f}(g) = \bar{f}(h)$ when $gN = hN$. Next, for all $g' \in G', g \in G$, $f(gNg'N) = \bar{f}(gg') = \pi(g'^{-1})\bar{f}(g) = \pi(g'^{-1}N)f(gN)$, and therefore $f \in \text{c-Ind}_{G'/N}^{G/N} V$. Furthermore, $T(f) = \bar{f}$, which implies that $T$ is surjective.

It remains to show that $T$ is an intertwining map, that is the following diagram commutes for all $\tilde{g} \in G$.

$$\begin{tikzcd}
\text{c-Ind}_{G'/N}^{G/N} V \arrow[r, "\text{c-Ind}_{G'/N}^{G/N} \pi(\tilde{g})"] \arrow[d, "T"'] & \text{c-Ind}_{G'/N}^{G/N} V \arrow[d, "T"] \\
\text{c-Ind}_{G'}^{G} V \arrow[r, "\text{c-Ind}_{G'}^{G} \pi(\tilde{g})"] & \text{c-Ind}_{G'}^{G} V
\end{tikzcd}$$

Take $f \in \text{c-Ind}_{G'/N}^{G/N} V$. Then $\text{c-Ind}_{G'/N}^{G/N} \pi(\tilde{g})(f)$ maps $gN$ to $f(\tilde{g}^{-1}NgN)$ for all $g \in G$, and therefore $T \circ \text{c-Ind}_{G'/N}^{G/N} \pi(\tilde{g})(f)$ maps $gN$ to $\bar{f}(\tilde{g}^{-1}g)$ for all $g \in G$. On the other hand, $T(f)$ maps $gN$ to $\bar{f}(g)$ for all $g \in G$, which implies that $\text{c-Ind}_{G'}^{G} \pi(\tilde{g}) \circ T(f)$ maps $gN$ to $\bar{f}(\tilde{g}^{-1}g)$ for all $g \in G$. Hence, $T \circ \text{c-Ind}_{G'/N}^{G/N} \pi(\tilde{g}) = \text{c-Ind}_{G'}^{G} \pi(\tilde{g}) \circ T$. ∎

**Proposition A.1.11.** *Let $G' \subset G$ be a subgroup of $G$, $(\pi, V)$ be a representation of $G'$ and $\chi$ be a character of $G$. Then* $\text{c-Ind}_{G'}^{G}(\pi \otimes \chi|_{G'}) \simeq \text{c-Ind}_{G'}^{G} \pi \otimes \chi$.

**Proof:** We must find an invertible linear map $T : \text{c-Ind}_{G'}^{G}(V \otimes_{\mathbb{C}} \mathbb{C}) \to \text{c-Ind}_{G'}^{G} V \otimes_{\mathbb{C}} \mathbb{C}$ that intertwines $\text{c-Ind}_{G'}^{G}(\pi \otimes \chi|_{G'})$ and $\text{c-Ind}_{G'}^{G} \pi \otimes \chi$, where

$$\text{c-Ind}_{G'}^{G}(V \otimes_{\mathbb{C}} \mathbb{C}) = \{f : G \to V : f(gg') = (\pi \otimes \chi)(g'^{-1})f(g) \text{ for all } g' \in G', g \in G\},$$

$$\text{c-Ind}_{G'}^{G} V \otimes_{\mathbb{C}} \mathbb{C} = \{f : G \to V : f(gg') = \pi(g'^{-1})f(g) \text{ for all } g' \in G', g \in G\} \otimes_{\mathbb{C}} \mathbb{C}.$$

For all $f \in \text{c-Ind}_{G'}^{G}(V \otimes_{\mathbb{C}} \mathbb{C})$, let $\bar{f}$ be the map that sends $g$ to $f(g)\chi(g) \otimes 1$ for all $g \in G$. Then $\bar{f} \in \text{c-Ind}_{G'}^{G} V \otimes_{\mathbb{C}} \mathbb{C}$ since the map $g \mapsto f(g)\chi(g)$ lies in $\text{c-Ind}_{G'}^{G} V$. Indeed, for all $g' \in G', g \in G$ $f(gg')\chi(gg') = (\pi \otimes \chi)(g'^{-1})f(g)\chi(g)\chi(g') = \pi(g'^{-1})f(g)\chi(g)$. Therefore, we define $T$ as follows:

$$\begin{aligned} T : \text{c-Ind}_{G'}^{G}(V \otimes_{\mathbb{C}} \mathbb{C}) &\to \text{c-Ind}_{G'}^{G} V \otimes_{\mathbb{C}} \mathbb{C} \\ f &\mapsto \bar{f}, \end{aligned}$$

where $\bar{f}$ is defined as previously.

We start by showing that $T$ is bijective. It is surely injective because $T(f_1) = T(f_2)$ implies that $f_1(g)\chi(g) = f_2(g)\chi(g)$ for all $g \in G$, and therefore $f_1 = f_2$. Given $\bar{f} \in \text{c-Ind}_{G'}^{G} V$, let $f$ be the map that sends $g$ to $\bar{f}(g)\chi(g^{-1})$ for all $g \in G$. Then one can verify that $f \in \text{c-Ind}_{G'}^{G}(V \otimes_{\mathbb{C}} \mathbb{C})$ and $T(f) = \bar{f} \otimes 1$, which implies $T$ is surjective.

It remains to show that $T$ is an intertwining map, that is the following diagram commutes for all $\tilde{g} \in G$.

$$\begin{tikzcd} \text{c-Ind}_{G'}^{G}(V \otimes_{\mathbb{C}} \mathbb{C}) \arrow[r, "\text{c-Ind}_{G'}^{G}(\pi \otimes \chi|_{G'})(\tilde{g})"] \arrow[d, "T"'] & \text{c-Ind}_{G'}^{G}(V \otimes_{\mathbb{C}} \mathbb{C}) \arrow[d, "T"] \\ \text{c-Ind}_{G'}^{G} V \otimes_{\mathbb{C}} \mathbb{C} \arrow[r, "\left(\text{c-Ind}_{G'}^{G} \pi \otimes \chi\right)(\tilde{g})"] & \text{c-Ind}_{G'}^{G} V \otimes_{\mathbb{C}} \mathbb{C} \end{tikzcd}$$

Take $f \in \text{c-Ind}_{G'}^{G}(V \otimes_{\mathbb{C}} \mathbb{C})$. Then $\text{c-Ind}_{G'}^{G}(\pi \otimes \chi|_{G'})(\tilde{g})(f)$ maps $g$ to $f(\tilde{g}^{-1}g)$ for all $g \in G$, and therefore $T \circ \text{c-Ind}_{G'}^{G}(\pi \otimes \chi|_{G'})(\tilde{g})(f)$ maps $g$ to $f(\tilde{g}^{-1}g)\chi(g) \otimes 1$ for all $g \in G$. On the other hand, $T(f)$ maps $g$ to $f(g)\chi(g) \otimes 1$ for all $g \in G$, which implies that $\left(\text{c-Ind}_{G'}^{G} \pi \otimes \chi\right)(\tilde{g}) \circ T(f)$ maps $g$ to $f(\tilde{g}^{-1}g)\chi(\tilde{g}^{-1}g) \otimes \chi(\tilde{g}) = f(\tilde{g}^{-1}g)\chi(g) \otimes 1$ for all $g \in G$. We conclude that $T \circ \text{c-Ind}_{G'}^{G}(\pi \otimes \chi|_{G'}) = (\tilde{g}) \left(\text{c-Ind}_{G'}^{G} \pi \otimes \chi\right)(\tilde{g}) \circ T$ for all $\tilde{g} \in G$. ∎

## A.2 On the Mock Exponential Map

Let $\mathbb{G}$ be a reductive group defined over $F$, and let $G = \mathbb{G}(F)$. Assume that $\mathbb{G}$ splits over a tamely ramified extension $E$ of $F$. Let $\mathbb{H}$ be a closed connected $F$-subgroup of $\mathbb{G}$ that contains $\mathbb{G}_{\text{der}}$. Set $x \in \mathcal{B}(\mathbb{G}, F)$ and $r > 0$. The goal of this section is to show that the isomorphism $e_H : \mathfrak{h}_{x,r:r^+} \to H_{x,r:r^+}$ defined by Adler and used in the J.K. Yu Construction can essentially be viewed as a restriction of Adler's isomorphism $e : \mathfrak{g}_{x,r:r^+} \to G_{x,r:r^+}$, a result needed in the proof of Lemma 3.3.17.

Let $\mathbb{T}$ be a maximal $F$-torus that splits over $E$ and let $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, E)$. Set a $\mathbb{Z}$-basis of $X^*(\mathbb{T})$, $\{\chi_1, \ldots, \chi_n\}$. The isomorphism $e$ between $\mathfrak{g}_{x,r:r^+}$ and $G_{x,r:r^+}$ defined by Adler is constructed via the theory of mock exponential maps [Adl98, Section 1.5]. Hakim provides a summary of this construction in [Hak18, Section 3.4] which we quickly recall here.

One starts by defining a mock exponential map on $\mathfrak{t}(E)_r$,

$$\mathcal{M}\text{-exp} : \mathfrak{t}(E)_r \to \mathbb{T}(E)_r.$$

This map is defined by the implicit condition $\chi_i(\mathcal{M}\text{-exp}(X)) = 1 + d\chi_i(X)$ for all $X \in \mathfrak{t}(E)_r, 1 \leqslant i \leqslant n$. For each $\alpha \in \Phi = \Phi(\mathbb{G}, \mathbb{T})$, we have an exponential map $\exp_\alpha : \mathfrak{g}_\alpha(E)_{x,r} \to \mathbb{U}_\alpha(E)_{x,r}$ because elements of $\mathfrak{g}_\alpha(E)$ are nilpotent. Choosing an ordering of the roots, we can combine the maps $\exp_\alpha, \alpha \in \Phi$, and $\mathcal{M}$-exp to obtain a new map

$$\mathcal{M} : \mathfrak{t}(E)_r \times \prod_{\alpha\in\Phi} \mathfrak{g}_\alpha(E)_{x,r} \to \mathbb{T}(E)_r \times \prod_{\alpha\in\Phi} \mathbb{U}_\alpha(E)_{x,r}.$$

The multiplication map

$$\mu : \mathbb{T}(E)_r \times \prod_{\alpha\in\Phi} \mathbb{U}_\alpha(E)_{x,r} \to \mathbb{G}(E)_{x,r}$$

is bijective [BT72, Proposition 6.4.48], and so is its corresponding Lie algebra map, which is denoted $\nu$. The mock exponential map $e$ of $\mathfrak{g}(E)_{x,r}$ can then be represented by the commutative diagram in Figure A.1. Adler showed that this map induces an isomorphism of the quotient groups [Adl98, Section 1.5], which we also denote by $e$. This isomorphism is independent of the choice of basis for $X^*(\mathbb{T})$ [Adl98, Lemma 1.5.1] and is actually independent of the choice of torus $\mathbb{T}$ such that $x \in \mathcal{A}(\mathbb{G}, \mathbb{T}, E)$ [Adl98, Corollary 1.6.6]. Furthermore, the quotients $\mathfrak{g}(E)_{x,r:r^+}$ and $\mathbb{G}(E)_{x,r:r^+}$ are abelian (Section 2.2.2), meaning that any choice of ordering of the roots produces the same map. This isomorphism $e$ is illustrated by the commutative diagram in Figure A.2.

Given the $F$-subgroup $\mathbb{H}$ defined above, let $\mathbb{T}_{\mathbb{H}} = \mathbb{T} \cap \mathbb{H}$, $\mathfrak{h} = \text{Lie}(\mathbb{H})$ and $\mathfrak{t}_{\mathbb{H}} = \text{Lie}(\mathbb{T}_{\mathbb{H}})$. We define a mock exponential map $e_H$ for $\mathfrak{h}(E)_{x,r}$ analogously (Figure A.3), by first constructing a mock exponential map $(\mathcal{M}\text{-exp})_H$ from $\mathfrak{t}_{\mathbb{H}}(E)_r$ to $\mathbb{T}_{\mathbb{H}}(E)_r$ and extending it to a map $\mathcal{M}_H$ via the same exponential maps on the

$$\begin{tikzcd}
\mathfrak{g}(E)_{x,r} \arrow[r, "e"] \arrow[d, "\nu^{-1}"'] & \mathbb{G}(E)_{x,r} \\
\mathfrak{t}(E)_r \times \prod_{\alpha\in\Phi} \mathfrak{g}_\alpha(E)_{x,r} \arrow[r, "\mathcal{M}"] & \mathbb{T}(E)_r \times \prod_{\alpha\in\Phi} \mathbb{U}_\alpha(E)_{x,r} \arrow[u, "\mu"']
\end{tikzcd}$$

Figure A.1: Commutative diagram describing the mock exponential map $e$ of $\mathfrak{g}(E)_{x,r}$ as illustrated in [Hak18, Section 3.4].

$$\begin{tikzcd}
\mathfrak{g}(E)_{x,r:r^+} \arrow[r, "e"] \arrow[d, "\nu^{-1}"'] & \mathbb{G}(E)_{x,r:r^+} \\
\mathfrak{t}(E)_{r:r^+} \times \prod_{\alpha\in\Phi} \mathfrak{g}_\alpha(E)_{x,r:r^+} \arrow[r, "\mathcal{M}"] & \mathbb{T}(E)_{r:r^+} \times \prod_{\alpha\in\Phi} \mathbb{U}_\alpha(E)_{x,r:r^+} \arrow[u, "\mu"']
\end{tikzcd}$$

Figure A.2: Commutative diagram describing the isomorphism $e$ from $\mathfrak{g}(E)_{x,r:r^+}$ to $\mathbb{G}(E)_{y,r:r^+}$ as illustrated in [Hak18, Section 3.4].

filtrations of the root subgroups. To do this, we restrict the $\mathbb{Z}$-basis $\{\chi_1, \ldots, \chi_n\}$ of $X^*(\mathbb{T})$ to $\mathbb{T}_{\mathbb{H}}$. This will produce a spanning set for $X^*(\mathbb{T}_{\mathbb{H}})$ from which we can extract a $\mathbb{Z}$-basis. Indeed, this is because every character of $\mathbb{T}_{\mathbb{H}}$ is the restriction of a character of $\mathbb{T}$ [Bor91, Proposition 8.2]. By definition, we set the map $(\mathcal{M}\text{-exp})_H$ to satisfy $\chi_i|_{\mathbb{T}_{\mathbb{H}}}((\mathcal{M}\text{-exp})_H(X)) = 1 + d\chi_i|_{\mathbb{T}_{\mathbb{H}}}(X)$ for all $X \in \mathfrak{t}_{\mathbb{H}}(E)_r, 1 \leqslant i \leqslant n$ so that $(\mathcal{M}\text{-exp})_H = \mathcal{M}\text{-exp}|_{\mathfrak{t}_{\mathbb{H}}(E)_r}$ and therefore $\mathcal{M}_H = \mathcal{M}|_{\mathfrak{t}_{\mathbb{H}}(E)_r \times \prod_{\alpha\in\Phi} \mathfrak{g}_\alpha(E)_{x,r}}$.

$$\begin{tikzcd}
\mathfrak{h}(E)_{x,r} \arrow[r, "e_H"] \arrow[d, "\nu_H^{-1}"'] & \mathbb{H}(E)_{x,r} \\
\mathfrak{t}_{\mathbb{H}}(E)_r \times \prod_{\alpha\in\Phi} \mathfrak{g}_\alpha(E)_{x,r} \arrow[r, "\mathcal{M}_H"] & \mathbb{T}_{\mathbb{H}}(E)_r \times \prod_{\alpha\in\Phi} \mathbb{U}_\alpha(E)_{x,r} \arrow[u, "\mu_H"']
\end{tikzcd}$$

Figure A.3: Commutative diagram describing the mock exponential map $e_H$ of $\mathfrak{h}(E)_{x,r}$.

**Proposition A.2.1.** *Let $e_H : \mathfrak{h}(E)_{x,r:r^+} \to \mathbb{H}(E)_{x,r:r^+}$ be the mock exponential map of $\mathfrak{h}(E)_{x,r}$. Then $e|_{\mathfrak{h}(E)_{x,r}} = \iota \circ e_H$, where $\iota$ is the isomorphism from $\mathbb{H}(E)_{x,r:r^+}$ to $\mathbb{H}(E)_{x,r}\mathbb{G}(E)_{x,r:r^+}/\mathbb{G}(E)_{x,r^+}$ given by the second isomorphism theorem.*

**Proof:** We clearly have that $\nu_H^{-1} = \nu^{-1}|_{\mathfrak{h}(E)_{x,r}}$ and $\mu_H = \mu|_{\mathbb{H}(E)_{x,r}}$. Given $h \in \mathfrak{h}(E)_{x,r}$, let $\nu^{-1}(h) = t \times \prod_{\alpha\in\Phi} g_\alpha$. Then, by the previous lemma, $e|_{\mathfrak{h}(E)_{x,r}}(h) = \mathcal{M}_H(t \times \prod_{\alpha\in\Phi} g_\alpha)\mathbb{G}(E)_{x,r^+}$. On the other hand, $e_H(h) = \mathcal{M}_H(t \times \prod_{\alpha\in\Phi} g_\alpha)\mathbb{H}(E)_{x,r^+}$. The conclusion follows since $\iota$ maps $h\mathbb{H}(E)_{x,r^+}$ to $h\mathbb{G}(E)_{x,r^+}$ for all $h \in \mathbb{H}(E)_{x,r}$. ∎

By taking the Gal$(E/F)$-fixed points of $\mathfrak{g}(E)_{x,r:r^+}$ and $\mathbb{G}(E)_{x,r:r^+}$, $e$ induces an isomorphism $e : \mathfrak{g}_{x,r:r^+} \to G_{x,r:r^+}$ [Yu01, Corollary 2.4]. Similarly, $e_H$ induces an isomorphism between $\mathfrak{h}_{x,r:r^+}$ and $H_{x,r:r^+}$, and the equality $e|_{\mathfrak{h}_{x,r}} = \iota \circ e_H$ holds at the level of $F$-points, where $\iota : H_{x,r:r^+} \to H_{x,r}G_{x,r^+}/G_{x,r^+}$.

## A.3 On the Kottwitz Map

In Section 2.2.2.2, we used the Kottwitz map to describe the filtration subgroups of a torus. The goal of this section is to show that the restriction of a Kottwitz map to a subtorus is again a Kottwitz map, a result used in Lemma 2.2.21 to discuss the intersection of Moy-Prasad filtration subgroups of $G$ with $H$.

Let $\mathbb{G}$ be a reductive group defined over a $p$-adic field $L$, and let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$. Let val$(\cdot)$ be the valuation map on $L$, which we scale to take values in $\mathbb{Z}$. Let $\bar{L}$ be an algebraic closure of $L$ and $I = \mathrm{Gal}(\bar{L}/L)$ be the absolute Galois group of $L$. We denote by $X^*(\mathbb{T})^I$ the subgroup of $X^*(\mathbb{T})$ fixed by $I$ and by $X_*(\mathbb{T})_I$ the largest quotient of $X_*(\mathbb{T})$ on which $I$ acts trivially. Kottwitz explains how there is a natural surjection

$$q : X_*(\mathbb{T})_I \to \mathrm{Hom}(X^*(\mathbb{T})^I, \mathbb{Z})$$

and an obvious functorial map

$$v : \mathbb{T}(L) \to \mathrm{Hom}(X^*(\mathbb{T})^I, \mathbb{Z})$$

in [Kot97, Section 7.2]. Indeed, $q$ is the map that sends $\mu \in X_*(\mathbb{T})_I$ to the homomorphism $\lambda \mapsto \langle\lambda, \mu\rangle$, where $\langle\cdot,\cdot\rangle$ is the canonical pairing between $X^*(\mathbb{T})$ and $X_*(\mathbb{T})$ discussed in Section 2.2.1, and $v$ maps $t \in \mathbb{T}(L)$ to the homomorphism $\lambda \mapsto \mathrm{val}(\lambda(t))$. Kottwitz then proceeds to construct a surjective homomorphism

$$\mathrm{Kot}_{\mathbb{T}(L)} : \mathbb{T}(L) \to X_*(\mathbb{T})_I$$

that satisfies $q \circ \mathrm{Kot}_{\mathbb{T}(L)} = v$. We refer to $\mathrm{Kot}_{\mathbb{T}(L)}$ as the Kottwitz map of $\mathbb{T}(L)$.

**Proposition A.3.1.** *Let $\mathbb{S}$ be a subtorus of $\mathbb{T}$, and let $\mathrm{Kot}_{\mathbb{T}(L)}$ and $\mathrm{Kot}_{\mathbb{S}(L)}$ be the Kottwitz maps for $\mathbb{T}(L)$ and $\mathbb{S}(L)$, respectively. Then $\ker(\mathrm{Kot}_{\mathbb{T}(L)}) \cap \mathbb{S}(L) = \ker(\mathrm{Kot}_{\mathbb{S}(L)})$.*

**Proof:** Let $\mathbb{S} \hookrightarrow \mathbb{T}$ be the inclusion map. This map induces an inclusion $X_*(\mathbb{S})_I \hookrightarrow X_*(\mathbb{T})_I$. Kottwitz then tells us that the diagram

$$\begin{tikzcd} \mathbb{T}(L) \arrow[r, "\mathrm{Kot}_{\mathbb{T}(L)}"] & X_*(\mathbb{T})_I \\ \mathbb{S}(L) \arrow[u, hook] \arrow[r, "\mathrm{Kot}_{\mathbb{S}(L)}"] & X_*(\mathbb{S})_I \arrow[u, hook] \end{tikzcd}$$

commutes [Kot97, (7.2.7)]. This means that $\mathrm{Kot}_{\mathbb{T}(L)}|_{\mathbb{S}(L)} = \mathrm{Kot}_{\mathbb{S}(L)}$, and the result follows. ∎

## A.4 On the Heisenberg-Weil Lift

This section contains all the proofs related to claims we have made about the Heisenger-Weil lifts. In particular, we discuss the restriction of a Heisenberg-Weil lift in Section A.4.1 which was needed to restrict the representations $\phi^{i\prime}$ in Theorem 4.2.7. We also discuss in Section A.4.2 the relationship between $(\phi_H^{d-1}\phi_H^d)'$ and $\phi_H^{d-1\prime}\phi_H^d$, a result needed in Theorem 4.3.6.

There are many details behind the construction of the Heisenber-Weil lifts which we have not discussed in this thesis. To make this section as concise as possible, we provide references for the concepts that we have not defined rather than repeating another summary of the whole construction. The reader that would like to know all the details of the theory of Heisenberg-Weil lifts is invited to consult [Yu01, HM08, Nev15]. In particular, we recommend [HM08, Section 2.3].

### A.4.1 Restricting the Heisenberg-Weil Lift

Set $0 \leqslant i \leqslant d-1$. Let us recall the notation used for the theory of Heisenberg-Weil lifts to extend $\widehat{\phi}^i$ in Section 4.2.1. We also establish analogous notation for $H$.

$$\begin{aligned}
&\xi^i = \widehat{\phi}^i|_{J_+^{i+1}}, \xi_H^i = \widehat{\phi_H^i}|_{J_{H+}^{i+1}} \\
&N^i = \ker(\xi^i), N_H^i = \ker(\xi_H^i) \\
&\mathcal{H}^i = J^{i+1}/N^i, \mathcal{H}_H^i = J_H^{i+1}/N_H^i \\
&Z^i = J_+^{i+1}/N^i, Z_H^i = J_{H+}^{i+1}/N_H^i \\
&W^i = J^{i+1}/J_+^{i+1}, W_H^i = J_H^{i+1}/J_{H+}^{i+1}
\end{aligned}$$

$$W^{i\#} = W^i \boxtimes \mathbb{F}_p, W_H^{i\ \#} = W_H^i \boxtimes \mathbb{F}_p$$

The groups $\mathcal{H}^i$ and $\mathcal{H}_H^i$ are Heisenberg $p$-groups associated to the symplectic vector spaces $W^i$ and $W_H^i$, respectively. Recall also that $Z^i$ and $Z_H^i$ are the centers of $\mathcal{H}^i$ and $\mathcal{H}_H^i$, respectively, which are both isomorphic to $\mathbb{F}_p$. This means that characters of $Z^i$ and $Z_H^i$ are isomorphisms into $\mu_p$, the set of $p$th roots of unity in $\mathbb{C}$. From the central characters $\xi^i$ and $\xi_H^i$, we construct Heisenberg-Weil lifts $\omega^i$ and $\omega_H^i$ which are representations of $K^i \ltimes \mathcal{H}^i$ and $K_H^i \ltimes \mathcal{H}_H^i$, respectively.

The goal of this section is to prove the following theorem, which was necessary when restricting the extensions $\phi^{i'}$ (Theorem 4.2.7).

**Theorem A.4.1.** *For all $k \in K_H^i, j \in J_H^{i+1}$, $\omega_H^i(k, jN_H^i) = \omega^i(k, jN^i)$.*

We will achieve this with the help of the following Proposition.

**Proposition A.4.2** ([Nev15, Proposition 3.2])**.** *For $i \in \{1, 2\}$ let $H_i$ be a Heisenberg group with center $Z_i$. Fix a nontrivial character $\phi_i$ of $Z_i$ and a corresponding special isomorphism $\nu_i : H_i \to W_i^\#$ in the sense of* [HM08, Definition 2.29, Remark 2.33]*, where $W_i = H_i/Z_i$. Let $T_i$ be a group and suppose further that $\nu_i$ is relevant for a homomorphism $f_i : T_i \to Sp(H_i)$ in the sense of* [HM08, Definition 3.17]*. Let $(f^i)_{\nu_i}$ be the map that sends $t \in T_i$ to $\nu_i \circ f^i(t) \circ \nu_i^{-1}$ for all $t \in T_i, i = 1, 2$. Suppose we have a group isomorphism $\alpha : H_1 \to H_2$, inducing $\bar{\alpha} : W_1 \to W_2$, and a group homomorphism $\delta : T_1 \to T_2$ such that the following diagrams commute.*

$$\begin{tikzcd} H_1 \arrow[r, "\nu_1"] \arrow[d, "\alpha"'] & W_1^\# \arrow[d, "\bar{\alpha} \times 1"] \\ H_2 \arrow[r, "\nu_2"] & W_2^\# \end{tikzcd} \qquad \begin{tikzcd} T_1 \arrow[r, "(f_1)_{\nu_1}"] \arrow[d, "\delta"'] & Sp(W_1) \arrow[d, "\mathrm{inn}(\bar{\alpha})"] \\ T_2 \arrow[r, "(f_2)_{\nu_2}"] & Sp(W_2) \end{tikzcd}$$

*Then we have $\phi_2 \circ \alpha = \phi_1$ on $Z_1$ and $\bar{\alpha}$ is a symplectic isomorphism, whence the maps in the diagram above are well-defined homomorphisms. Let $(\tau_2, V)$ be a Heisenberg representation of $H_2$ with central character $\phi_2$. Then $\tau_1 = \tau_2 \circ \alpha$ is a Heisenberg representation of $H_1$ with central character $\phi_1$. Let $\omega_i$ denote the pullback of the Heisenberg-Weil lift corresponding to $\nu_i$ of $\tau_i$ to $T_i \ltimes H_i$. Then for all $t \in T_1$,$h \in H_1$, we have*

$$\omega_2(\delta(t), \alpha(h)) = \omega_1(t, h),$$

*that is, the lifts coincide.*

Let $f^i : K^i \to \mathrm{Sp}(\mathcal{H}^i)$ and $f^i_H : K^i_H \to \mathrm{Sp}(\mathcal{H}^i_H)$ be the homomorphisms coming from the actions by conjugation of $K^i$ on $J^{i+1}$ and of $K^i_H$ on $J^{i+1}_H$, respectively. Let $\nu^i$ and $\nu^i_H$ be the special isomorphisms arising from the split polarizations of $\mathcal{H}^i$ and $\mathcal{H}^i_H$, respectively, as per [HM08, Lemma 2.35]. The maps $\nu^i$ and $\nu^i_H$ are relevant for $f^i$ and $f^i_H$, respectively. The strategy to proving Theorem A.4.1 is to start by showing that the homomorphism $\alpha : \mathcal{H}^i_H \to \mathcal{H}^i$ which maps $jN^i_H$ to $jN^i$ for all $j \in J^{i+1}_H$ is an isomorphism that induces the symplectic isomorphism $\beta : W^i_H \to W^i$ from Corollary 4.2.5, then to consider the inclusion map $\delta : K^i_H \to K^i$ and to show that the following diagrams commute.

$$\begin{tikzcd} \mathcal{H}^i_H \arrow[r, "\nu^i_H"] \arrow[d, "\alpha"'] & W^{i\,\#}_H \arrow[d, "\beta \times 1"] \\ \mathcal{H}^i \arrow[r, "\nu^i"] & W^{i\#} \end{tikzcd} \qquad \begin{tikzcd} K^i_H \arrow[r, "(f^i_H)_{\nu^i_H}"] \arrow[d, "\delta"'] & Sp(W^i_H) \arrow[d, "\mathrm{inn}(\beta)"] \\ K^i \arrow[r, "(f^i)_{\nu^i}"] & Sp(W^i) \end{tikzcd}$$

Figure A.4: Commutative diagrams for the restriction of the Heisenberg-Weil lift.

#### A.4.1.1 Proving that $\alpha$ is an Isomorphism

We start by establishing the relationship between $N^i$ and $N^i_H$ with the following lemma.

**Lemma A.4.3.** *Let $N^i_H$ and $N^i$ be defined as above. Then $N^i_H = N^i \cap J^{i+1}_{H+} = N^i \cap J^{i+1}_H$.*

**Proof:** Let $j : J^{i+1}_{H+} \to J^{i+1}_+$ be the inclusion map. We have that $\xi^i_H = \xi^i \circ j$ which imples that $\ker \xi^i_H = \ker(\xi^i \circ j) = \ker \xi^i \cap J^{i+1}_{H+}$. Therefore $N^i_H = N^i \cap J^{i+1}_{H+}$.

Next, we show that $N^i \cap J^{i+1}_{H+} = N^i \cap J^{i+1}_H$. Clearly, we have $N^i \cap J^{i+1}_{H+} \subset N^i \cap J^{i+1}_H$. To show the converse, let $x \in N^i \cap J^{i+1}_H \subset J^{i+1}_+ \cap J^{i+1}_H$. Since $J^{i+1}_H = J^{i+1} \cap H^{i+1}$ and $J^{i+1}_{H+} = J^{i+1}_+ \cap H^{i+1}$ (Lemma 4.2.3), $J^{i+1}_+ \cap J^{i+1}_H = J^{i+1}_{H+}$ and therefore $x \in N^i \cap J^{i+1}_{H+}$. ∎

**Proposition A.4.4.** *Let $\alpha : \mathcal{H}^i_H \to \mathcal{H}^i$ be the map defined by $\alpha(jN^i_H) = jN^i$ for all $j \in J^{i+1}_H$. Then $\alpha$ is an isomorphism and $\alpha$ induces the isomorphism $\beta$.*

**Proof:** We start by showing that the map $\alpha$ is well-defined. Given $jN^i_H = j'N^i_H$, it follows that $j^{-1}j' \in N^i_H \subset N^i$. Therefore, $jN^i = j'N^i$ and the map $\alpha$ is well-defined.

The map $\alpha$ is injective. Indeed, by Lemma A.4.3 the kernel of $\alpha$ is $\ker \alpha = \{jN^i_H : j \in J^{i+1}_H \cap N^i\} = \{N^i_H\}$. Furthermore, it is clear from the definition that $\alpha$ maps $Z^i_H$ into $Z^i$. Since both $Z^i_H$ and $Z^i$ are cyclic groups of order $p$, it follows that $\alpha(Z^i_H) = Z^i$.

Let pr and $\mathrm{pr}_H$ denote the projections of $\mathcal{H}^i$ into $W^i$ and $\mathcal{H}^i_H$ into $W^i_H$ via the isomorphisms $W^i \simeq \mathcal{H}^i/Z^i$ and $W^i_H \simeq \mathcal{H}^i_H/Z^i_H$, respectively. Then $\beta^{-1} \circ \mathrm{pr}_H = \mathrm{pr} \circ \alpha$. It follows that $\mathrm{pr} \circ \alpha$ is surjective, as $\beta^{-1}$ and $\mathrm{pr}_H$ are both surjective. Now, let $y \in \mathcal{H}^i$ and consider its projection into $W^i$, $yZ^i$. Since $\mathrm{pr} \circ \alpha$ is surjective, there exists $x \in \mathcal{H}^i_H$ such that $\mathrm{pr} \circ \alpha(x) = yZ^i$. It follows that $\alpha(x)Z^i = yZ^i$, or equivalently, $\alpha(x)y^{-1} \in Z^i$. Since $\alpha|_{Z^i_H}$ is surjective, there exists $z \in Z^i_H$ such that $\alpha(z) = \alpha(x)y^{-1}$, implying that $y = \alpha(x)\alpha(z)^{-1} = \alpha(xz^{-1})$. Hence, we conclude that $\alpha$ is surjective, and therefore is an isomorphism. Furthermore, the isomorphism that $\alpha$ induces between $W^i_H$ and $W^i$ is clearly equal to $\beta$. ∎

#### A.4.1.2 Showing the Commutativity of the Diagrams in Figure A.4

**Proposition A.4.5.** *Let $\alpha$ and $\beta$ be the isomorphisms as defined previously, and let $\nu^i$ and $\nu^i_H$ be the special isomorphisms arising from the split polarizations of $\mathcal{H}^i$ and $\mathcal{H}^i_H$, respectively, as per* [HM08, Lemma 2.35]. *Then the left diagram of Figure A.4 commutes, that is $\nu^i \circ \alpha = (\beta \times 1) \circ \nu^i_H$.*

**Proof:** Let $\mathcal{H}^i_H(+), \mathcal{H}^i_H(-)$ be Yu's *split polarization* of $\mathcal{H}^i_H$([Yu01, Proposition 11.4], [HM08, Section 3.3]). This means that every element of $hN^i_H$ can then be written in the form $(h_+N^i_H)(h_-N^i_H)(zN^i_H)$ for some $h_+N^i_H \in \mathcal{H}^i_H(+)$, $h_-N^i_H \in \mathcal{H}^i_H(-)$, $zN^i_H \in Z^i_H$. Furthermore, the associated special isomorphism to this split polarization is $\nu^i_H : \mathcal{H}^i_H \to {W^i_H}^{\#}$ defined by

$$\nu^i_H((h_+N^i_H)(h_-N^i_H)(zN^i_H)) = ((h_+J^{i+1}_{H+})(h_-J^{i+1}_{H+}), z[h_+, h_-]^{p+1/2}),$$

where $z[h_+, h_-]^{p+1/2}$ is viewed as an element of $\mathbb{F}_p$ [HM08, Lemma 2.35].

From the description provided in [HM08, Section 3.3], one sees that $\mathcal{H}^i(+) := \alpha(\mathcal{H}^i_H(+))$, $\mathcal{H}^i(-) := \alpha(\mathcal{H}^i_H(-))$ is Yu's split polarization of $\mathcal{H}^i$, and $\nu^i$ is the associated special isomorphism as per [HM08, Lemma 2.35]. For all $(h_+N^i_H)(h_-N^i_H)(zN^i_H) \in \mathcal{H}^i_H$ we have that

$$\begin{aligned}\nu^i \circ \alpha((h_+N^i_H)(h_-N^i_H)(zN^i_H)) &= \nu_i((h_+N^i)(h_-N^i)(zN^i)) \\ &= ((h_+J^{i+1}_+)(h_-J^{i+1}_+), z[h_+, h_-]^{p+1/2}),\end{aligned}$$

where $z[h_+, h_-]^{p+1/2}$ is viewed as an element of $\mathbb{F}_p$ .

On the other hand,

$$(\beta \times 1) \circ \nu^i_H((h_+N^i_H)(h_-N^i_H)(zN^i_H)) = (\beta \times 1)((h_+J^{i+1}_{H+})(h_-J^{i+1}_{H+}), z[h_+, h_-]^{p+1/2})$$

$$= ((h_+ J_+^{i+1})(h_- J_+^{i+1}), z[h_+, h_-]^{p+1/2}).$$

Therefore, we conclude that $\nu^i \circ \alpha = (\beta \times 1) \circ \nu_H^i$. ∎

**Proposition A.4.6.** *Let $\beta, \nu^i$ and $\nu_H^i$ be as in the previous proposition and let $f^i : K^i \to \mathrm{Sp}(\mathcal{H}^i)$ and $f_H^i : K_H^i \to \mathrm{Sp}(\mathcal{H}_H^i)$ be the homomorphisms coming from the actions by conjugation of $K^i$ on $J^{i+1}$ and of $K_H^i$ on $J_{H+}^{i+1}$, respectively. Let $\delta : K_H^i \to K^i$ be the inclusion map. Then the right diagram of Figure A.4 commutes, that is $\mathrm{inn}(\beta) \circ (f_H^i)_{\nu_H^i} = (f^i)_{\nu^i} \circ \delta$, where $(f^i)_{\nu^i}$ is the map that sends $k \in K^i$ to $\nu^i \circ f^i(k) \circ {\nu^i}^{-1}$ for all $k \in K^i$.*

**Proof:** First note that we are viewing $\mathrm{Sp}(W^i)$ and $\mathrm{Sp}(W_H^i)$ as subgroups of $\mathrm{Sp}(\mathcal{H}^i)$ and $\mathrm{Sp}(\mathcal{H}_H^i)$, respectively, via the inclusion $g \mapsto g \times 1$.

For all $k \in K_H^i$, using the definitions of our maps, we have that

$$\begin{aligned}(\mathrm{inn}(\beta) \circ (f_H^i)_{\nu_H^i})(k) &= \mathrm{inn}(\beta)(\nu_H^i \circ f_H^i(k) \circ {\nu_H^i}^{-1}) \\ &= (\beta \times 1) \circ \nu_H^i \circ f_H^i(k) \circ {\nu_H^i}^{-1} \circ (\beta^{-1} \times 1).\end{aligned}$$

By Proposition A.4.5, we have that $(\beta \times 1) \circ \nu_H^i = \nu^i \circ \alpha$, which allows us to rewrite the previous expression as

$$(\mathrm{inn}(\beta) \circ (f_H^i)_{\nu_H^i})(k) = (\nu^i \circ \alpha) \circ f_H^i(k) \circ (\alpha^{-1} \circ {\nu^i}^{-1}).$$

It remains to show that $(\nu^i \circ \alpha) \circ f_H^i(k) \circ (\alpha^{-1} \circ {\nu^i}^{-1}) = (f^i)_{\nu^i} \circ \delta(k)$, where $(f^i)_{\nu^i} \circ \delta(k) = \nu^i \circ f^i(k) \circ {\nu^i}^{-1}$ is an element of $\mathrm{Sp}(W^{i\#})$. Take $(hZ^i, z) \in W^{i\#}$, and let $jN^i = {\nu^i}^{-1}(hZ^i, z)$ with $j \in J_H^{i+1}$. Note that we can choose $j \in J_H^{i+1}$ by surjectivity of $\alpha$. Then,

$$(\alpha^{-1} \circ {\nu^i}^{-1})(hZ^i, z) = jN_H^i.$$

Applying $f_H^i(k)$ to this last expression, it follows that

$$f_H^i(k) \circ (\alpha^{-1} \circ {\nu^i}^{-1})(hZ^i, z) = kjk^{-1}N_H^i,$$

and so

$$\alpha \circ f_H^i(k) \circ (\alpha^{-1} \circ {\nu^i}^{-1})(hZ^i, z) = kjk^{-1}N^i.$$

We then have

$$(\nu^i \circ \alpha) \circ f_H^i(k) \circ (\alpha^{-1} \circ {\nu^i}^{-1})(hZ^i, z) = \nu^i(kjk^{-1}N^i).$$

But,

$$\nu^i(kjk^{-1}N^i) = \nu^i \circ f^i(k)(jN^i) = \nu^i \circ f^i(k) \circ {\nu^i}^{-1}(hZ^i, z).$$

Hence, we that conclude that $(\nu^i \circ \alpha) \circ f_H^i(k) \circ (\alpha^{-1} \circ {\nu^i}^{-1}) = (f^i)_{\nu^i} \circ \delta(k)$ and therefore $\mathrm{inn}(\beta) \circ (f_H^i)_{\nu_H^i} = (f^i)_{\nu^i} \circ \delta$. ∎

### A.4.2 Proving that $\kappa_H^{d-1,d} \otimes 1 \simeq \kappa_H^{d-1} \otimes \kappa_H^d$

Let $\Psi = (\vec{\mathbb{G}}, y, \rho, \vec{\phi})$ be a $G$-datum. When the depth $\tilde{r}$ of $\phi_H^d$ is less than or equal to $r_{d-1}$, the last two quasicharacters in the character sequence $\vec{\phi_H}$ are $(\phi_H^{d-1}\phi_H^d, 1)$. The goal of this section is to show that $\kappa_H^{d-1,d} \otimes 1 \simeq \kappa_H^{d-1} \otimes \kappa_H^d$, or equivalently $(\phi_H^{d-1}\phi_H^d)' \simeq (\phi_H^{d-1})'\phi_H^d$, which is required in the proof of Theorem 4.3.6.

In [HM08, Proposition 4.24], Hakim and Murnaghan show that if $\dot{\Psi}_H$ is a refactorization of $\Psi_H$, then $\kappa_H(\Psi_H) \simeq \kappa_H(\dot{\Psi}_H)$. When looking at the details of their proof, we only need $\vec{\phi}_H$ and $\dot{\vec{\phi}}_H$ to satisfy condition F1) from the definition of refactorization (Definition 3.2.1) for this result to hold, and we do not even require the last quasicharacters of the sequences to be of largest depth. This means that we can set $\vec{\phi}_H = (\phi_H^0, \ldots, \phi_H^{d-1}, \phi_H^d)$ and $\dot{\vec{\phi}}_H = (\phi_H^0, \ldots, \phi_H^{d-1}\phi_H^d, 1)$ and obtain the wanted result. Because this is a very specific case in which the result applies, we recall the proof in this section, only written specifically for our case with less notation and with a bit more detail. But first, let us proceed to the following lemma.

**Lemma A.4.7.** *The quasicharacter $\widehat{\phi_H^{d-1}\phi_H^d}$ is equal to $\widehat{\phi_H^{d-1}}\phi_H^d$.*

**Proof:** Recall that $\widehat{\phi_H^{d-1}\phi_H^d}$ is the (unique) quasicharacter of $K_H^{d-1}H_{y,r_{d-1}^+}$ that agrees with $\phi_H^{d-1}\phi_H^d$ on $K_H^{d-1}$ and is trivial on $(H^{d-1}, H)_{y,(r_{d-1}^+, s_{d-1}^+)}$, and that we have an analogous description for $\widehat{\phi_H^{d-1}}$. Therefore, to show the wanted equality, it suffices to show that $\phi_H^d$ is trivial on $(H^{d-1}, H)_{y,(r_{d-1}^+, s_{d-1}^+)}$.

Set $\tilde{s} = \tilde{r}/2$. Since $\phi_H^d$ is of depth $\tilde{r}$, Remark 3.1.14 tells us that $\phi_H^d|_{H_{y,\tilde{s}^+}}$ is realized by an element $Z^*$ of $(\mathfrak{z}_H^*)_{-\tilde{r}}$, where $\mathfrak{z}_H^*$ denotes the $F$-points of the dual of the center of Lie($\mathbb{H}$) as in Section 3.3.3. If we let $\psi^E$ denote a character of $E$ that coincides with $\psi$ on $F$, then we can define a quasicharacter $(\phi_H^d)^E$ of $\mathbb{H}(E)_{y,\tilde{s}^+:\tilde{r}^+}$ that agrees with $\phi_H^d$ as follows. For all $Y \in \mathfrak{h}(E)_{y,\tilde{s}^+}$,

$$(\phi_H^d)^E(e(Y + \mathfrak{h}(E)_{y,\tilde{r}^+})) = \psi^E(Z^*(Y)),$$

where $e$ denotes the isomorphism between $\mathfrak{h}(E)_{y,\tilde{s}^+:\tilde{r}^+}$ and $\mathbb{H}(E)_{y,\tilde{s}^+:\tilde{r}^+}$ as per Appendix A.2.

The character $(\phi_H^d)^E$ is trivial on $(\mathbb{H}^{d-1}(E), \mathbb{H}(E))_{y,(r_{d-1}^+, s_{d-1}^+)}$. Indeed, $(\phi_H^d)^E$ is trivial on $\mathbb{H}^{d-1}(E)_{y,r_{d-1}^+}$ being of depth $\tilde{r} \leqslant r_{d-1}$. Furthermore, $Z^*$ is trivial on elements of $[\mathfrak{h}, \mathfrak{h}](E)$, and therefore $(\phi_H^d)^E$ is trivial on $\mathbb{U}_\beta(E)_{y,s_{d-1}^+}$ for all $\beta \in \Phi \backslash \Phi^{d-1}$. Thus, we conclude that $\phi_H^d$ is trivial on $(H^{d-1}, H)_{y,(r_{d-1}^+, s_{d-1}^+)}$. ∎

Let us now establish the notation that we will be using for the Heisenberg-Weil lifts. Let $W = J_H^d/J_{H+}^d$ be the symplectic group we are considering for the

Heisenberg-Weil lifts of $\phi_H^{d-1}\phi_H^d$ and $\phi_H^{d-1}$.

$$\begin{aligned}
&\xi = \widehat{\phi_H^{d-1}}|_{J_{H+}^d}, \dot{\xi} = \widehat{\phi_H^{d-1}\phi_H^d}|_{J_{H+}^d}\\
&N = \ker(\xi), \dot{N} = \ker(\dot{\xi})\\
&\mathcal{H} = J_H^d/N, \dot{\mathcal{H}} = J_H^d/\dot{N}\\
&Z = J_{H+}^d/N, \dot{Z} = J_{H+}^d/\dot{N}
\end{aligned}$$

Recall that $Z$ and $\dot{Z}$ are isomorphic to $\mathbb{F}_p$ and therefore characters of $Z$ and $\dot{Z}$ are isomorphisms into $\mu_p$, the set of $p$th roots of unity in $\mathbb{C}$. Let $\mathcal{H}^\natural = W \boxtimes \mu_p, \tau^\natural$ be the Heisenberg representation of $\mathcal{H}^\natural$ associated to the identity map on $\mu_p$, and let $\overline{\tau^\natural}$ be the corresponding Weil representation. Letting $\mathcal{H}^\# = W \boxtimes Z$ and $\dot{\mathcal{H}}^\# = W \boxtimes \dot{Z}$, we have that $\tau^\# = \tau^\natural \circ (1 \times \xi)$ and $\dot{\tau}^\# = \tau^\natural \circ (1 \times \dot{\xi})$ are the Heisenberg representations of $\mathcal{H}^\#$ and $\dot{\mathcal{H}}^\#$ with central characters $\xi$ and $\dot{\xi}$, respectively. This means that $\tau^\#, \dot{\tau}^\#$ and $\tau^\natural$ all act on the same vector space, and they all give rise to the same Weil representation of $\mathrm{Sp}(W)$. Indeed, this is because a map intertwines $\tau^\natural$ and ${}^g\tau^\natural$ if and only if it intertwines $\tau^\#$ and ${}^g\tau^\#$, and $\dot{\tau}^\#$ and ${}^g\dot{\tau}^\#$.

Now, to obtain Heisenberg representations $\tau$ and $\dot{\tau}$ of $\mathcal{H}$ and $\dot{\mathcal{H}}$, we use Yu's special isomorphisms $\nu : \mathcal{H} \to \mathcal{H}^\#$ and $\dot{\nu} : \dot{\mathcal{H}} \to \dot{\mathcal{H}}^\#$ to pull back $\tau^\#$ and $\dot{\tau}^\#$, respectively. That is, $\tau = \tau^\# \circ \nu$ and $\dot{\tau} = \dot{\tau}^\# \circ \dot{\nu}$. For all $h \in J_H^d$, we have that $\nu(h) = (hJ_{H+}^d, \mu(h))$ for some map $\mu : J_H^d \to Z$ such that $\mu(h) = hN$ for all $h \in J_{H+}^d$ [HM08, Remark 2.30]. Similarly, there is a map $\dot{\mu} : J_H^d \to \dot{Z}$ associated to the definition $\dot{\mu}$. We are now in a position to compare the maps $\tau$ and $\dot{\tau}$, which we may view as representations of $J_H^d$.

**Lemma A.4.8.** *For all $h \in J_H^d$, $\dot{\tau}(h) = \tau(h)\dot{\xi}(\dot{\mu}(h))\xi(\mu(h))^{-1}$.*

**Proof:** For all $h \in J_H^d$, we expand the definition of $\tau$ as follows

$$\begin{aligned}
\tau(h) &= \tau^\# \circ \nu(h)\\
&= \tau^\natural \circ (1 \times \xi) \circ \nu(h)\\
&= \tau^\natural \circ (1 \times \xi)(hJ_{H+}^d, \mu(h))\\
&= \tau^\natural(hJ_{H+}^d, \xi(\mu(h))).
\end{aligned}$$

Since the central character of $\tau^\natural$ is the identity map on $\mu_p$, we obtain that $\tau(h) = \tau^\natural(hJ_{H+}^d, 1)\xi(\mu(h))$. Similarly, for all $h \in J_H^d$, we have $\dot{\tau}(h) = \tau^\natural(hJ_{H+}^d, 1)\dot{\xi}(\dot{\mu}(h))$. The conclusion follows. ∎

The Heisenberg representations $\tau$ and $\dot{\tau}$ can be extended to representations $\widehat{\tau}$ and $\widehat{\dot{\tau}}$ of $\mathrm{Sp}(W) \ltimes \mathcal{H}$ and $\mathrm{Sp}(W) \ltimes \dot{\mathcal{H}}$, respectively, where $\widehat{\tau}|_{\mathrm{Sp}(W)} = \overline{\tau^\natural} = \widehat{\dot{\tau}}|_{\mathrm{Sp}(W)}$,

$\widehat{\tau}|_{\mathcal{H}} = \tau$ and $\widehat{\dot{\tau}}|_{\dot{\mathcal{H}}} = \dot{\tau}$. Now, given the homomorphism $f : K_H^{d-1} \to \mathrm{Sp}(W)$ induced by action of $K_H^{d-1}$ by conjugation on $J_H^d$, we obtain Heisenberg-Weil lifts $\omega$ and $\dot{\omega}$, which are the representations of $K_H^{d-1} \ltimes J_H^d$ defined by

$$\omega(k,j) = \overline{\tau^\natural}(f(k))\tau(j) \text{ and } \dot{\omega}(k,j) = \overline{\tau^\natural}(f(k))\dot{\tau}(j)$$

for all $k \in K_H^{d-1}, j \in J_H^d$. Using these definitions and Lemma A.4.8, we see that $\dot{\omega}(k,j) = \omega(k,j)\dot{\xi}(\dot{\nu}(j))\xi(\mu(j))^{-1}$ for all $k \in K_H^{d-1}, j \in J_H^d$.

Finally, the representations $(\phi_H^{d-1}\phi_H^d)'$ and $(\phi_H^{d-1})'$ of $K_H^d = K_H^{d-1}J_H^d$ are defined as

$$(\phi_H^{d-1}\phi_H^d)'(kj) = (\phi_H^{d-1}\phi_H^d)(k)\dot{\omega}(k,j) \text{ and } (\phi_H^{d-1})'(kj) = \phi_H^{d-1}(k)\omega(k,j)$$

for all $k \in K_H^{d-1}, j \in J_H^d$. With what precedes, we can now establish a relationship between the two extensions.

**Lemma A.4.9.** *For all $k \in K_H^{d-1}, j \in J_H^d$, $(\phi_H^{d-1}\phi_H^d)'(kj) = (\phi_H^{d-1})'(kj)\eta(kj)$, where $\eta(kj) = \phi_H^d(k)\dot{\xi}(\dot{\mu}(j))\xi(\mu(j))^{-1}$.*

**Proof:** Let's first show that $\eta$ is a well-defined map. For all $j \in J_{H+}^d$ we have that $\dot{\xi}(\dot{\mu}(j))\xi(\mu(j))^{-1} = \widehat{\phi_H^{d-1}}(j)\phi_H^d(j)\widehat{\phi_H^{d-1}}(j)^{-1} = \phi_H^d(j)$. In particular, $\phi_H^d(j) = \dot{\xi}(\dot{\mu}(j))\xi(\mu(j))^{-1}$ on $K_H^{d-1} \cap J_H^d = H_{y,r_{d-1}}^{d-1} \subset J_{H+}^d$.

Next, we proceed by direct computation to show the equality. For all $k \in K_H^{d-1}, j \in J_H^d$,

$$\begin{aligned}(\phi_H^{d-1}\phi_H^d)'(kj) &= \phi_H^{d-1}(k)\phi_H^d(k)\dot{\omega}(k,j)\\ &= \phi_H^{d-1}(k)\omega(k,j)\phi_H^d(k)\dot{\xi}(\dot{\mu}(j))\xi(\mu(j))^{-1}\\ &= (\phi_H^{d-1})'(kj)\eta(kj).\end{aligned}$$

∎

This last lemma implies that in order to show that $(\phi_H^{d-1}\phi_H^d)' \simeq (\phi_H^{d-1})'\phi_H^d$, it is equivalent to show that $(\phi_H^{d-1})'\phi_H^d \simeq (\phi_H^{d-1})'\eta$, or equivalently, $(\phi_H^{d-1})' \simeq (\phi_H^{d-1})'\overline{\eta}$, where $\overline{\eta}(k) = \eta(k)\phi_H^d(k)^{-1}$ for all $k \in K_H^d$. We note that $\overline{\eta}$ is trivial on $K_H^{d-1}J_{H+}^d$ by definition.

Since we are working with finite-dimensional representations, it suffices to show that the characters of $(\phi_H^{d-1})'$ and $(\phi_H^{d-1})'\overline{\eta}$ are the same. (Recall that the character $\chi_\phi$ of a finite-dimensional representation $\phi$ of $G$ is given by $\chi_\phi(g) = \mathrm{Trace}(\phi(g))$ for all $g \in G$.) Note that the character of a representation is constant on conjugacy classes.

**Lemma A.4.10.** *Let $\chi_{(\phi_H^{d-1})'}$ and $\chi_{(\phi_H^{d-1})'\overline{\eta}}$ denote the characters of $(\phi_H^{d-1})'$ and $(\phi_H^{d-1})'\overline{\eta}$, respectively. Then $\chi_{(\phi_H^{d-1})'} = \chi_{(\phi_H^{d-1})'\overline{\eta}}$, that is $(\phi_H^{d-1})' \simeq (\phi_H^{d-1})'\overline{\eta}$.*

**Proof:** Recall that $(\phi_H^{d-1})'(kj) = \phi_H^{d-1}(k)\omega(k,j) = \phi_H^{d-1}(k)\overline{\tau^\natural}(f(k))\tau(j)$ for all $k \in K_H^{d-1}, j \in J_H^d$, where $\tau$ is a Heisenberg representation. Since the character of a Heisenberg representation has support in the center [Yu01, Lemma 12.1], the support of the character of $\tau$ lies in $J_{H+}^d/N$. Therefore, $\chi_{(\phi_H^{d-1})'}$ is only nontrivial on elements of $K_H^{d-1}J_H^d$ that are conjugate in $K^d$ to elements of $K_H^{d-1}J_{H+}^d$. Since $\overline{\eta}$ is trivial on $K_H^{d-1}J_{H+}^d$, we conclude that $\chi_{(\phi_H^{d-1})'\overline{\eta}} = \chi_{(\phi_H^{d-1})'}\overline{\eta} = \chi_{(\phi_H^{d-1})'}$. ∎

With this last lemma and the discussion preceding it, we can conclude that $(\phi_H^{d-1}\phi_H^d)' \simeq (\phi_H^{d-1})'\phi_H^d$, and therefore $\kappa_H^{d-1,d} \otimes 1 \simeq \kappa_H^{d-1} \otimes \kappa_H^d$.

# Bibliography


[Adl98] J. D. Adler, *Refined anisotropic $K$-types and supercuspidal representations*, Pacific J. Math. **185**(1), 1–32 (1998).

[AM19] J. D. Adler and M. Mishra, *Regular Bernstein Blocks*, ArXiv e-prints (2019), arXiv:1909.09966v2.

[AP06] J. D. Adler and D. Prasad, *On certain multiplicity one theorems*, Israel J. Math. **153**, 221–245 (2006).

[AP19] J. D. Adler and D. Prasad, *Multiplicity upon restriction to the derived subgroup*, Pacific J. Math. **301**(1), 1–14 (2019).

[AR00] J. D. Adler and A. Roche, *An intertwining result for $p$-adic groups*, Canad. J. Math. **52**(3), 449–467 (2000).

[Bak20] A. Baker, An Introduction to p-adic Numbers and p-adic Analysis, `https://www.ams.org/open-math-notes/omn-view-listing?listingId=110818`, 2020, [Online course notes; last accessed May 2020].

[Ber84] J. N. Bernstein, Le "centre" de Bernstein, in *Representations of reductive groups over a local field*, Travaux en Cours, pages 1–32, Hermann, Paris, 1984, Edited by P. Deligne.

[BH06] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* $\mathrm{GL}(2)$, volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 2006.

[BK93] C. J. Bushnell and P. C. Kutzko, *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, volume 129 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, 1993.

[BK98] C. J. Bushnell and P. C. Kutzko, *Smooth representations of reductive $p$-adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77**(3), 582–634 (1998).

[BK99] C. J. Bushnell and P. C. Kutzko, *Semisimple types in* $\mathrm{GL}_n$, Compositio Math. **119**(1), 53–97 (1999).

[Blo05] C. Blondel, *Quelques propriétés des paires couvrantes*, Math. Ann. **331**(2), 243–257 (2005).

[Bor91] A. Borel, *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, second edition, 1991.

[BT65] A. Borel and J. Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. (27), 55–150 (1965).

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (41), 5–251 (1972).

[BT84] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (60), 197–376 (1984).

[Car79] H. Carayol, *Représentations supercuspidales de* $\mathrm{GL}_n$, C. R. Acad. Sci. Paris Sér. A-B **288**(1), A17–A20 (1979).

[Car93] R. W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester, 1993, Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[Cas19] W. Casselman, Analysis on profinite groups, `http://www.math.ubc.ca/~cass/research/essays.html`, 2019, [Online course notes; last accessed May 2020].

[Con14] B. Conrad, Reductive group schemes, in *Autour des schémas en groupes. Vol. I*, volume 42/43 of *Panor. Synthèses*, pages 93–444, Soc. Math. France, Paris, 2014.

[CSST09] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, Clifford theory and applications, volume 156, pages 29–43, 2009, Functional analysis.

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103**(1), 103–161 (1976).

[DR09] S. DeBacker and M. Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169**(3), 795–901 (2009).

[Fin17] J. Fintzen, *On the Moy-Prasad filtration*, ArXiv e-prints (November 2017), arXiv:1511.00726.

[Fin18] J. Fintzen, *Types for tame p-adic groups*, ArXiv e-prints (October 2018), arXiv:1810.04198.

[Fin19] J. Fintzen, *Tame cuspidal representations in non-defining characteristics*, ArXiv e-prints (2019), arXiv:1905.06374.

[GT19] T. Gee and O. Taïbi, *Arthur's multiplicity formula for* $\mathbf{GSp}_4$ *and restriction to* $\mathbf{Sp}_4$, J. Éc. polytech. Math. **6**, 469–535 (2019).

[Hak18] J. Hakim, *Constructing tame supercuspidal representations*, Represent. Theory **22**, 45–86 (2018).

[HM08] J. Hakim and F. Murnaghan, *Distinguished tame supercuspidal representations*, Int. Math. Res. Pap. IMRP (2), Art. ID rpn005, 166 (2008).

[How77] R. E. Howe, *Tamely ramified supercuspidal representations of* $\mathrm{Gl}_n$, Pacific J. Math. **73**(2), 437–460 (1977).

[Hum75] J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York-Heidelberg, 1975, Graduate Texts in Mathematics, No. 21.

[Hum90] J. E. Humphreys, *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1990.

[Jac71] H. Jacquet, Représentations des groupes linéaires $p$-adiques, in *Theory of group representations and Fourier analysis (Centro Internaz. Mat. Estivo (C.I.M.E.), II Ciclo, Montecatini Terme, 1970)*, pages 119–220, 1971.

[Kal19] T. Kaletha, *Regular supercuspidal representations*, J. Amer. Math. Soc. **32**(4), 1071–1170 (2019).

[Kim07] J.-L. Kim, *Supercuspidal representations: an exhaustion theorem*, J. Amer. Math. Soc. **20**(2), 273–320 (2007).

[Kot97] R. E. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. **109**(3), 255–339 (1997).

[Kut77] P. C. Kutzko, *Mackey's theorem for nonunitary representations*, Proc. Amer. Math. Soc. **64**(1), 173–175 (1977).

[KY17] J.-L. Kim and J.-K. Yu, Construction of tame types, in *Representation theory, number theory, and invariant theory*, volume 323 of *Progr. Math.*, pages 337–357, Birkhäuser/Springer, Cham, 2017.

[Mau64] F. I. Mautner, *Spherical functions over $\mathfrak{b}$-adic fields. II*, Amer. J. Math. **86**, 171–200 (1964).

[McN12] P. J. McNamara, Principal series representations of metaplectic groups over local fields, in *Multiple Dirichlet series, L-functions and automorphic forms*, volume 300 of *Progr. Math.*, pages 299–327, Birkhäuser/Springer, New York, 2012.

[Moy86] A. Moy, *Local constants and the tame Langlands correspondence*, Amer. J. Math. **108**(4), 863–930 (1986).

[MP94] A. Moy and G. Prasad, *Unrefined minimal $K$-types for $p$-adic groups*, Invent. Math. **116**(1-3), 393–408 (1994).

[MP96] A. Moy and G. Prasad, *Jacquet functors and unrefined minimal $K$-types*, Comment. Math. Helv. **71**(1), 98–121 (1996).

[Mur05] F. Murnaghan, Linear algebraic groups, in *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 379–391, Amer. Math. Soc., Providence, RI, 2005.

[Mur09] F. Murnaghan, Representations of Reductive $p$-adic Groups, `http://www.math.toronto.edu/murnaghan/course.html`, 2009, [Online course notes; last accessed May 2020].

[Mur11] F. Murnaghan, Parametrization of tame supercuspidal representations, in *On certain L-functions*, volume 13 of *Clay Math. Proc.*, pages 439–469, Amer. Math. Soc., Providence, RI, 2011.

[Nev13] M. Nevins, *Branching rules for supercuspidal representations of* $\mathrm{SL}_2(k)$*, for $k$ a $p$-adic field*, J. Algebra **377**, 204–231 (2013).

[Nev15] M. Nevins, *Restricting toral supercuspidal representations to the derived group, and applications*, J. Pure Appl. Algebra **219**(8), 3337–3354 (2015).

[Ogg10] F. Oggier, Introduction to Algebraic Number Theory, `http://www1.spms.ntu.edu.sg/~frederique/ANT10.pdf`, 2010, [Online course notes; accessed May 2020].

[PR08] G. Pappas and M. Rapoport, *Twisted loop groups and their affine flag varieties*, Adv. Math. **219**(1), 118–198 (2008), With an appendix by Thomas J. Haines and Micheal Rapoport.

[Pra01] G. Prasad, *Galois-fixed points in the Bruhat-Tits building of a reductive group*, Bull. Soc. Math. France **129**(2), 169–174 (2001).

[Rab03] J. Rabinoff, The Bruhat-Tits building of a p-adic Chevalley group and an application to representation theory, 2003, Senior Honors Thesis, Department of Mathematics of Harvard University.

[Som98] E. Sommers, *A generalization of the Bala-Carter theorem for nilpotent orbits*, Internat. Math. Res. Notices (11), 539–562 (1998).

[Ste68] R. Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968, Notes prepared by John Faulkner and Robert Wilson.

[Ste08] S. Stevens, *The supercuspidal representations of $p$-adic classical groups*, Invent. Math. **172**(2), 289–352 (2008).

[Tad92] M. Tadić, *Notes on representations of non-Archimedean* $\mathrm{SL}(n)$, Pacific J. Math. **152**(2), 375–396 (1992).

[Yu01] J.-K. Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14**(3), 579–622 (2001).

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