Soficity and other dynamical aspects of groupoids and inverse semigroups

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Abstract

This thesis is divided into four chapters. In the first one, all the pre-requisite theory of semigroups and groupoids is introduced, as well as a few new results - such as a short study of ∨-ideals and quotients in distributive semigroups and a non-commutative Loomis-Sikorski Theorem.

In the second chapter, we motivate and describe the sofic property for probability measure-preserving groupoids and prove several permanence properties for the class of sofic groupoids. This provides a common ground for similar results in the particular cases of groups ([44]) and equivalence relations ([41, 42]). In particular, we prove that soficity is preserved under finite index extensions of groupoids. We also prove that soficity can be determined in terms of the full group alone, answering a question by Conley, Kechris and Tucker-Drob.

In the third chapter we turn to the classical problem of reconstructing a topological space from a suitable structure on the space of continuous functions. We prove that a locally compact Hausdorff space can be recovered from classes of functions with values on a Hausdorff space together with an appropriate notion of disjointness, as long as some natural regularity hypotheses are satisfied. This allows us to recover (and even generalize) classical theorem by Kaplansky, Milgram, Banach-Stone, among others, as well as recent results of the similar nature, and obtain new consequences as well. Furthermore, we extend the techniques used here to obtain structural theorems related to topological groupoids.

In the fourth and final chapter, we study dynamical aspects of partial actions of inverse semigroups, and in particular how to construct groupoids of germs and (partial) crossed products and how do they relate to each other. This chapter is based on joint work with Viviane Beuter.
Acknowledgements

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<td>Borel semigroup of $\mathcal{G}$</td>
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<tr>
<td>$\mathcal{G}_x, \mathcal{G}_y, \mathcal{G}_z$</td>
<td>Arrows with specified source and range</td>
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<tr>
<td>$\mathcal{G}_x^e$</td>
<td>Isotropy group at $x$</td>
<td>5</td>
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<td>$\mathcal{G}_F(S)$</td>
<td>Groupoids of ultrafilters on $S$</td>
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<tr>
<td>$KB(\mathcal{G})$</td>
<td>Ample semigroup of a groupoid $\mathcal{G}$</td>
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<tr>
<td>$R(\alpha)$</td>
<td>Orbit groupoid of a group action $\alpha$</td>
<td>5</td>
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<tr>
<td>$\mathcal{R}(\mathcal{G})$</td>
<td>Orbit equivalence relation of $\mathcal{G}$</td>
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<tr>
<td>$\text{Meas}(\mathcal{G}), \text{Meas}(\mathcal{G}, \mu)$</td>
<td>Measured semigroup of a $(\mathcal{G}, \mu)$</td>
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<td>$\sqcup_i \mathcal{G}_i$</td>
<td>Coproduct of groupoids</td>
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<td>$\sum_n t_n \mathcal{G}_n$</td>
<td>Convex combination groupoid</td>
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<td>$|f|_{r, \tau}$</td>
<td>$(I, \tau)$-norm</td>
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<td>$G \ltimes_\theta X$</td>
<td>Transformation groupoid</td>
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## Ordered sets

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<tr>
<td>$0$</td>
<td>(in a poset) The minimum of a poset</td>
<td>10</td>
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<tr>
<td>$\vee F$</td>
<td>$a \lor b$ Joins in a poset</td>
<td>10</td>
</tr>
<tr>
<td>$\wedge F$</td>
<td>$a \land b$ Meets in a poset</td>
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<td>$\in$</td>
<td>Compact containment</td>
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## Partial actions

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<td>Partial action $\theta$ on $X$</td>
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<td>$A \ltimes_\alpha S$</td>
<td>Crossed product</td>
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\begin{itemize}
\item \(S \ast_{\theta} X = S \ast X\) \hspace{1cm} 190
\item \(S \ltimes_{\theta} X\) Groupoid of germs \hspace{1cm} 191
\item \(S_x\) \hspace{1cm} 232
\end{itemize}

### Semigroups

- \(0\) (in an inverse semigroup) The absorbing element of an inverse semigroup \hspace{1cm} 17
- \(1\) Unit of a monoid \hspace{1cm} 10
- \([s]\) Basic open set of \(\mathcal{G}_P(S)\) \hspace{1cm} 35
- \(\text{Bor}(\mathcal{G})\) Borel semigroup of \(\mathcal{G}\) \hspace{1cm} 49
- \(\text{Fix}(s)\) Fixed point idempotent \(s\) \hspace{1cm} 57
- \(\mathcal{G}_P(S)\) Groupoids of ultrafilters on \(S\) \hspace{1cm} 34
- \(\leq\) Order in an inverse semigroup \hspace{1cm} 16
- \(\text{Bis}(\mathcal{G})\) Semigroups of bisections of a groupoid \(\mathcal{G}\) \hspace{1cm} 12
- \(\mathcal{G}(S)\) Maximal group homomorphic image of \(S\) \hspace{1cm} 227
- \(\text{KB}(\mathcal{G})\) Ample semigroup of a groupoid \(\mathcal{G}\) \hspace{1cm} 33
- \(\mathcal{I}(X)\) Semigroup of partial bijections of \(X\) \hspace{1cm} 11
- \(\mathcal{I}_s\) Set of fixed idempotents of \(s\) \hspace{1cm} 57
- \(\text{Meas}(\mathcal{G}), \text{Meas}(\mathcal{G}, \mu)\) Measured semigroup of a \((\mathcal{G}, \mu)\) \hspace{1cm} 55
- \(\text{B}(\mathcal{G})\) Semigroups of open bisections of \(\mathcal{G}\) \hspace{1cm} 187
- \(\text{tr}_\mu(s)\) Trace of \(s\) \hspace{1cm} 63
- \(\prod_{\mu} S, \mu, \mathcal{G}\) Ultrafilter product of Boolean inverse monoids with invariant means \hspace{1cm} 86
- \(\text{supp}(s)\) Support of \(s\) \hspace{1cm} 104
- \(\tilde{a}\) The class of an element \(a\) in a quotient semigroup \hspace{1cm} 44
- \(d\) Uniform metric \hspace{1cm} 60
- \(\mathcal{E}(S)\) Idempotent lattice of \(S\) \hspace{1cm} 13
- \(F \cdot G\) Product of two filters \hspace{1cm} 34
- \(F^*\) \hspace{1cm} 11
- \(F^{-1} = F^*\) Inverse of a filter \hspace{1cm} 34
- \(FG\) \hspace{1cm} 11
- \(S/I\) Quotient of \(I\) by \(S\) \hspace{1cm} 44
- \(s^*\) Inverse element \hspace{1cm} 10
- \(t \setminus s\) Relative complement of \(s\) in \(t\) \hspace{1cm} 24
- \(t \setminus s\) Relative complement of \(s\) in \(t\) \hspace{1cm} 32
- \(tF, Ft\) \hspace{1cm} 11

### Other symbols

- \((x_i)_i\) An element of a product space \hspace{1cm} 75
- \(1_A\) Characteristic function of a set \(A\) \hspace{1cm} 203
- \([f \neq \theta], [f = \theta]\) \hspace{1cm} 129
- \(#X\) Cardinality of a finite \(X\) \hspace{1cm} 21
- \(\text{id}_A\) Identity function of a set \(A\) \hspace{1cm} 13
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<td>Measure algebra of (X, µ)</td>
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<td>MAlg(X, µ)</td>
<td>Measure algebra of (X, µ)</td>
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<td>S¹</td>
<td>Circle group</td>
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<td>I(U)</td>
<td>⊥-ideal associated to an open set U</td>
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<tr>
<td>U(I)</td>
<td>Open set associated to a ⊥-ideal I</td>
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<tr>
<td>P(I)</td>
<td>Power set of I</td>
<td>75</td>
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<tr>
<td>Gn</td>
<td>Permutation group on n elements</td>
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</tr>
<tr>
<td>µ#</td>
<td>Normalized counting measure on a finite set</td>
<td>88</td>
</tr>
<tr>
<td>dom(f), ran(f)</td>
<td>Domain and range of a partial bijection f.</td>
<td>11</td>
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<tr>
<td>graph(f)</td>
<td>Graph of a function f</td>
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<tr>
<td>Iso⁺(R, S)</td>
<td>Set of additive isomorphisms.</td>
<td>212</td>
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<tr>
<td>∏ₘᵢ Mᵢ</td>
<td>Ultraproduct metric space</td>
<td>80</td>
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<td>σ(f), σθ(f)</td>
<td>Interior of supp(f)</td>
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<td>supp(f), suppθ(f)</td>
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<td>A△B</td>
<td>Symmetric difference</td>
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<td>Aᵣ(G)</td>
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<td>C(X, H)</td>
<td>Set of continuous functions</td>
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<td>Cc(X), Cc(X, θ)</td>
<td>Set of compactly supported functions</td>
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<td>d#</td>
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<td>f ⊥ g</td>
<td>Disjointness</td>
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</tr>
<tr>
<td>f ⊥ ⊥ g</td>
<td>Strong disjointness</td>
<td>130</td>
</tr>
<tr>
<td>f ⊆ g</td>
<td></td>
<td>130</td>
</tr>
<tr>
<td>f ⊆ g</td>
<td></td>
<td>130</td>
</tr>
<tr>
<td>Z(f), Zθ(f)</td>
<td>Complement of supp(f)</td>
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Preface

Semigroups and groupoids are intrinsically related to dynamical systems, and the oldest motivation comes from the theory of differential equations: Suppose \( \Omega \) is an open subset of \( \mathbb{R}^n \) and \( F : \Omega \to \mathbb{R}^n \) is a Lipschitz vector field. The Picard-Lindelöf Theorem guarantees that the time-independent initial value problem

\[
y'(t) = F(y(t)), \quad y(0) = u, \quad (u \in \Omega)
\]

always has a unique solution \( y^u \) defined on some maximal interval around 0. The flow of \( F \) is the function given by \( Y(u, t) = y^u(t) \), and which is defined on some open subset of \( \Omega \times \mathbb{R} \).

Given \( t \in \mathbb{R} \), the equation \( \theta_t(u) = Y(u, t) = y^u(t) \) defines a diffeomorphism \( \theta_t : U_t \to V_t \) between open (possibly empty) subsets of \( \Omega \), and uniqueness of the solutions of (P1) implies that

\[
\theta_s(\theta_t(u)) = \theta_{s+t}(u)
\]

whenever the left-hand side is defined. In modern language, \( t \mapsto \theta_t \) is a partial action of the additive group of \( \mathbb{R} \) by partial diffeomorphisms of \( \Omega \).

Groupoids, on the other hand, provide a geometric counterpart to inverse semigroups and partial actions: Still considering the dynamical system provided by (P1), suppose \( \theta_s(\theta_t(u)) \) is defined, where \( s, t \geq 0 \), and consider partial trajectories \( y^u|_{[0,t]} : [0,t] \to \Omega \) and \( y^{\theta_t(u)}|_{[0,s]} : [0,s] \to \Omega \). Equation (P2) then states that the concatenation of \( y^u|_{[0,t]} \) with \( y^{\theta_t(u)}|_{[0,s]} \) coincides with \( y^u|_{[0,s+t]} \):

\[
\begin{array}{cc}
\bullet & y^u|_{[0,t]} \\
u & \theta_t(u) & y^{\theta_t(u)}|_{[0,s]} & \theta_{s+t}(u)
\end{array}
\]

This construction can be generalized to a transformation groupoid, and it retains most local information about the dynamical system.

The main goal of this thesis is to study the dynamical properties of inverse semigroups and groupoids and how they relate to each other. In the first chapter we introduce groupoids and inverse semigroups, including all classical theorems which
will be used throughout the next chapters. A precise relationship between groupoids and distributive inverse semigroups is provided by non-commutative Stone duality (based on [113]) - and a measurable analogue by the non-commutative Loomis-Sikorski Theorem - and provides the picture that will be maintained throughout this thesis: Inverse semigroups should usually be thought of as semigroups of bisections of some groupoid. Even if this is not the case, the Vagner-Preston Theorem still allows us to regard inverse semigroups as semigroups of partial bijections on some set. We also develop a short theory of V-ideals for Boolean inverse semigroups and their quotients, which proves to be extremely manageable and adequate to the study of non-faithful invariant means on Boolean inverse semigroups.

In the second chapter we consider the sofic property for groupoids. Soficity was initially considered by Gromov, as a generalization of both amenability and residual finiteness for groups ([76]), and is a general property of approximability by finite groups. We adopt the same point of view as [140], by introducing soficity as a property regarding the existence of appropriate finite models, in the sense of continuous logic, for our groupoid. This approach significantly simplifies several approximation arguments and is in accordance to the more algebraic approach taken throughout this thesis. The first course of action is then to prove that finite groupoids are sofic in the usual sense ([19, 134, 31]) and use this to provide a slightly modified, however equivalent and more manageable, description of the sofic property. Moreover, we study a notion of finite index for probability-measure preserving groupoids which restricts to the usual notions of finite index for subgroups and ergodic sub-equivalence relations. We then proceed to prove several permanence properties for the class of sofic groupoids, which can be used to provide an elementary proof that hyperfinite groupoids are sofic. We finish the chapter by proving that, under weak conditions, soficity can be determined in terms of the full group of a probability measure-preserving groupoid alone, which provides an answer to a question in [31].

The third chapter starts by introducing two notions of disjointness for general functions between Hausdorff spaces, and we apply a result from the first chapter to obtain a general recovery theorem for locally compact spaces. Using this we are able to obtain recovery theorems and classifications of isomorphisms for several different algebraic structures on families of continuous functions - including new results of this sort. In particular, we prove that a Stone space $X$ can be recovered from its group of circle-valued maps and the subset of those maps which attain 1 at some point of $X$. To prove the necessity of the latter condition, at the end of the thesis we added an Appendix with a proof, kindly provided by professor Vladimir Pestov, that there are non-homeomorphic spaces with isomorphic groups of circle-valued functions.

In the last chapter, which is mostly based on joint work with Viviane Beuter, we deal with partial actions of inverse semigroups as considered by Buss and Exel in [22]. We then define groupoids of germs - which generalize transformation groupoids - and crossed products associated to partial actions. In the main theorem of this chapter,
Theorem 4.4.32, we describe the Steinberg algebra of an ample Hausdorff groupoid of germs as a crossed product algebra, which generalizes previously known results. In fact, we also employ the theory from the previous chapter to determine all diagonal-preserving isomorphisms between Steinberg algebras of ample Hausdorff groupoids. We proceed to describe relationships between global and partial actions of associated groups and semigroups, and again obtain new results related to canonical isomorphisms between the associated groupoids or algebras. We finish this chapter with a description and study of orbit equivalence for partial actions of inverse semigroups.
Chapter 1

Groupoids and inverse semigroups

In this chapter we introduce all terms and notation that will be used throughout the thesis. For the sake of completeness, proofs are provided for most results, except those which are trivial consequences of previous ones or have essentially the same proof as in more well-known cases (for example, for groups). The introductory parts are somewhat based on [135] and [88], although the order of presentation of the topics (and consequently some proofs) is substantially different from either.

There are a few parts which are new to the theory, or at least not completely standardized, such as non-commutative versions of the Loomis-Sikorski Theorems (Theorems 1.8.17 and 1.8.18) and the study of quotients of distributive semigroups by ∨-ideals (Section 1.5). These abstract results will then be applied in subsequent chapters, and they will to simplify significantly some computations.

We start by introducing groupoids in a graph-theoretical fashion, a manner similar to how categories are described in [120]. Although this is a somewhat convoluted way of defining them, it should make the overall structure and dynamical nature of groupoids clearer. However after this we will give the algebraic definition of groupoids as well which will be used more often throughout this document for its succinctness.

1.1 Groupoids

1.1.1 Groupoids as directed graphs

Definition 1.1.1 ([135]). A directed graph is a tuple $G = (G^{(0)}, G^{(1)}, s, r)$, where $G^{(0)}$ is a set whose elements are called vertices, $G^{(1)}$ is a set whose elements are called edges or arrows, and $s$ and $r$ are maps from $G^{(1)}$ to $G^{(0)}$, called respectively the source and range maps.

Directed graphs can be depicted as points, representing the vertices, and arrows,
representing the edges, pointing from its source to its range.

\[ \xymatrix{ s(a) \ar@{<->}[r]^a & v(a) } \]

It should be noted that the directed graphs we consider here are sometimes called multigraphs or pseudographs, since we allow loops and multiple edges between two vertices.

**Example 1.1.2.** Let \( X \) be a set and \( R \) any binary relation on \( X \), i.e., a subset of \( X^2 \). Define a directed graph \( G \) by letting \( G(0) = X \), \( G(1) = R \), \( s(y, x) = x \) and \( v(y, x) = y \) for all \((y, x) \in R\).

As a sub-example, let \( X = \{a, b, c\} \) and \( R = \{(b, a), (c, b), (a, c)\} \). The associated graph can be depicted as

\[ \xymatrix{ b \ar@<1ex>[r] & a \ar@<1ex>[l] & c \ar@<1ex>[l] \ar@<1ex>[r] } \]

**Example 1.1.3.** Let \( G^{(0)} = G^{(1)} = \mathbb{Z} \), \( s(n) = n \) and \( v(n) = n + 1 \). This directed graph can be depicted as

\[ \cdots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow \cdots \rightarrow n \rightarrow n + 1 \rightarrow \cdots \]

**Example 1.1.4.** Generally, a discrete-time dynamical system consists of a set \( X \) and a collection \( \mathcal{F} \) of partial functions of \( X \), that is, functions \( f : \text{dom}(f) \rightarrow X \) where \( \text{dom}(f) \subseteq X \). Given such data, we define a directed graph \( G \) by letting \( G^{(0)} = X \), \( G^{(1)} = \{(f, x) : f \in \mathcal{F}, x \in \text{dom}(f)\} \), \( s(f, x) = x \) and \( v(f, x) = f(x) \).

This directed graph \( G \) is depicted by drawing elements of \( X \) as points and arrows from points of \( X \) to their image under elements of \( \mathcal{F} \), whenever defined.

As a sub-example, let \( X = \mathbb{Z} \), and \( f_i(n) = n + i \) for \( i = 0, 1, 2 \). The associated graph is depicted as

\[ \cdots \rightarrow -2 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow \cdots \rightarrow n \rightarrow n + 1 \rightarrow \cdots \]

In order to obtain a suitable notion for the composition of arrows, we need to introduce the concept of a category.

**Definition 1.1.5.** Let \( G = (G^{(0)}, G^{(1)}, s, v) \) be a directed graph. Given vertices \( x, y \in G^{(0)} \), denote by

- \( G_x = s^{-1}(x) \), the set of arrows leaving from \( x \);
\begin{itemize}
  \item $G^y = r^{-1}(y)$, the set of arrows arriving at $y$;
  \item $G^y_x = G^y \cap G_x$, the set of arrows from $x$ to $y$.
\end{itemize}

**Definition 1.1.6.** A *small category* consists of a directed graph $C = (C^{(0)}, C^{(1)}, s, r)$ together with the set $C^{(2)} = \{(a, b) \in C^{(1)} \times C^{(1)} : s(a) = r(b)\}$ and a *composition* map $C^{(2)} \to C^{(1)}$, $(a, b) \mapsto ab$, satisfying the following properties:

(i) If $(a, b)$ and $(b, c) \in C^{(2)}$, then $(a, bc), (ab, c) \in C^{(2)}$ and $a(bc) = (ab)c$.

(ii) For every $x \in C^{(0)}$, there exists $id_x \in C^{x_x}$ such that if $a \in C_x$ and $b \in C^x$, then $a \ id_x = a$ and $id_x b = b$.

The set $C^{(2)}$ is called the set of *composable pairs*.

General categories are defined in the same manner, however we allow $C^{(0)}$ and $C^{(1)}$ to be general classes instead of sets, however we will restrict our study to sets.

Note that $C^{(2)} = \bigcup_{x \in C^{(0)}} C_x \times C^x$, and so the pair $(a, b)$ of arrows in a category $C$ is *composable* if they meet “tip-to-tail”:

\[
\begin{array}{ccc}
  & b & \\
  \circ \ar{b}{r(b) = s(a)} & \circ \ar{a}{r(a)}
\end{array}
\]

The arrow $id_x \in C^{x_x}$ in item (ii) is unique for each $x \in C^{(0)}$, and it is called the *unit* at $x$. It is customary to identify each $x \in C^{(0)}$ with its respective unit, thus viewing $C^{(0)}$ as a subset of $C^{(1)}$. This allows us, moreover, to identify a category with its set of arrows.

Moreover, from item (i) we obtain, whenever $(a, b) \in C^{(2)}$, $(ab, id_{s(b)}) \in C^{(2)}$, and so $s(ab) = r(id_{s(b)}) = s(b)$ and similarly $r(ab) = r(a)$.

**Definition 1.1.7.** A *functor* between categories $C$ and $D$ is a map $\phi : C \to D$ satisfying:

(i) $\phi(C^{(0)}) \subseteq D^{(0)}$.

(ii) If $(a, b) \in C^{(2)}$, then $(\phi(a), \phi(b)) \in D^{(2)}$ and $\phi(ab) = \phi(a) \phi(b)$.

A functor $\phi$ is an *isomorphism* if it is bijective and $\phi^{-1}$ is also a functor.

**Example 1.1.8.** Let $X$ be a set and $\leq$ be a preorder on $X$ (i.e., a reflexive and transitive relation), and $R = \{(y, x) : x \leq y\}$ the associated subset of $X^2$. Let $G$ be the directed graph associated with this pair $(X, R)$ as in Example 1.1.2. In this case, we have $G^{(2)} = \{((z, y), (y, x)) : (y, x), (z, y) \in R\}$, so defining composition by $(z, y)(y, x) = (z, x)$ we obtain a category structure on $G$. 

In fact, since \( R \) is the set of arrows of \( \mathcal{G} \) we identify \( \mathcal{G} \) and \( R \), and moreover, \( R^{(0)} = \{(x,x): x \in X\} \) can be obviously identified with \( X \) via \( x \mapsto (x,x) \).

The categories \( \mathcal{C} \) obtained in this manner (up to isomorphism) are precisely those such that, for every \( x, y \in \mathcal{C}^{(0)} \), \( \mathcal{C}^y_x \) has at most one arrow.

**Example 1.1.9.** Let \( M \) be a monoid (see Definition 1.2.1). Let \( M^{(0)} = \{\ast\} \) be a singleton set, \( M^{(1)} = M \) and \( s(m) = r(m) = \ast \) for every \( m \in M \). In this case \( M^{(2)} = M \times M \), and define composition as the monoid operation of \( M \). This allows us to see \( M \) as a category.

Conversely, if \( \mathcal{C} \) is any category and \( x \in \mathcal{C}^{(0)} \), then \( \mathcal{C}^x_x \) is a monoid. If \( \mathcal{C}^{(0)} = \{x\} \) is a singleton, then \( \mathcal{C} \) is the category associated with the monoid \( \mathcal{C}^x_x \) as above.

**Example 1.1.10.** Let \( \theta \) be a global action (see Definition 4.1.5) of a monoid \( M \) on a set \( X \), denoted by concatenation. We define the category \( M \ltimes \theta X \) whose underlying graph is similar to the one as in Example 1.1.4.

Namely, let \( (M \ltimes \theta X)^{(0)} = X \), \( (M \ltimes \theta X)^{(1)} = M \times X \), \( s(m,x) = x \), \( r(m,x) = mx \). In this case, \( (M \ltimes \theta X)^{(2)} = \{(m,nx),(n,x): x \in X, m,n \in M\} \), so we define the composition as \( (m,nx)(n,x) = (mn,x) \).

Note that if \( X \) is a singleton \( X = \{\ast\} \) and \( M \) acts on \( X \) trivially, we recover Example 1.1.9.

We say that an arrow \( a \) of a category \( \mathcal{C} \) is *invertible* if there exists another arrow \( b \) such that \( (a,b),(b,a) \in \mathcal{C}^{(2)} \), \( ba = s(a) \) and \( ab = r(a) \). In this case, the arrow \( b \) is unique and it is denoted by \( a^{-1} \), we call \( a^{-1} \) the *inverse* of \( a \), and \( a \) is called *invertible*.

It is standard fact that \( (a^{-1})^{-1} = a \) and \( (ab)^{-1} = b^{-1}a^{-1} \) whenever \( a \) and \( b \) are invertible and \( (a,b) \in \mathcal{C}^{(2)} \).

**Definition 1.1.11** ([135]). A *groupoid* \( \mathcal{G} \) is a small category in which all arrows are invertible.

**Example 1.1.12.** Every group is a groupoid, in the same way as Example 1.1.9.

**Example 1.1.13.** If \( \theta \) is an action of a group \( G \) on a set \( X \) (as usual, denoted by concatenation), then the category \( G \ltimes \theta X \) (as in Example 1.1.10) is a groupoid, which we call a *transformation groupoid*. The inverse of an arrow \( (g,x) \in G \ltimes \theta X \) is \( (g^{-1},gx) \).

\[
\begin{align*}
x & \xrightarrow{g} gx & \xrightarrow{h} (hg)x \\
\end{align*}
\]

When there is no risk of confusion, we will drop the index \( \theta \) and denote this groupoid simply as \( G \ltimes X \).

**Example 1.1.14.** A preorder \( R \) on a set \( X \), regarded as a category as in Example 1.1.8 is a groupoid if and only if \( R \) is an equivalence relation. (See 1.1.17 below.)
Example 1.1.15. If $G$ is a groupoid, then the image of the map $(r, s) : G \to G^{(0)} \times G^{(0)}$, $(r, s)(a) = (r(a), s(a))$, is an equivalence relation on $G^{(0)}$, called the orbit equivalence relation, which we will denote by $\mathcal{R}(G)$. An $\mathcal{R}(G)$-equivalence class is called an orbit.

- If $\theta$ is an action of a group $G$ on a space $X$, then $\mathcal{R}(G \ltimes_\theta X)$ is the usual orbit equivalence relation of the action $\theta$, so orbits of $G \ltimes_\theta X$ coincide with orbits of $\theta$. We will abuse notation and denote, in this case, the orbit groupoid as $\mathcal{R}(\theta) = \mathcal{R}(G \ltimes_\theta X)$.

- If $R$ is an equivalence relation on a set $X$, then the orbit of an element $x \in X$ is the $R$-equivalence class of $x$.

In fact, it is possible to determine the groupoids which are isomorphic to an equivalence relation, in the sense above, and those are called the principal groupoids.

Definition 1.1.16. If $G$ is a groupoid and $x \in G^{(0)}$, we call the set $G_x^x$ the isotropy group of $G$ at $x$.

Note that $G_x^x$ is indeed a group with the operation inherited from $G$.

Definition 1.1.17. A groupoid $G$ is principal if $G_x^x$ is a trivial group for all $x \in G^{(0)}$.

Example 1.1.18. If $\theta$ is an action of a group $G$ on a set $X$, then the transformation groupoid $G \ltimes_\theta X$ is principal if, and only if, the action $\theta$ is free: Indeed, for all $x \in X$, let $G_x = \{g \in G : gx = x\}$ be the stabilizer of $x$, so the map

$$G_x \to (G \ltimes_\theta X)_x^x, \quad g \mapsto (g, x)$$

is a group isomorphism. Thus $G \ltimes_\theta X$ is principal if and only if $G_x$ is trivial for all $x \in X$, i.e., $\theta$ is free.

Proposition 1.1.19. A groupoid $G$ is principal if and only if there exists an equivalence relation $R$ on a set $X$ which is isomorphic to $G$, as a groupoid.

Proof. Every equivalence relation, regarded as a groupoid, is principal, and conversely if $G$ is a principal groupoid then the map $(r, s) : G \to \mathcal{R}(G)$ is a groupoid isomorphism. □

In the following sections and chapters, we will consider topological versions of these constructions, and generalize the transformation groupoids to groupoids of germs of partial actions of inverse semigroups.
1.1.2 Groupoids as algebraic structures

A groupoid can be alternatively (algebraically) defined as a set $G$ endowed with a partial binary map $G^{(2)} \to G$, $(a, b) \mapsto ab$, where $G^{(2)}$ is a subset of the product $G \times G$, satisfying

\begin{enumerate}[(G1)]
\item If $(a, b)$ and $(b, c) \in G^{(2)}$, then $(ab, c) \in G^{(2)}$ and $(ab)c = a(bc)$;
\item For all $a \in G$, there is $p \in G$ such that $(p, a) \in G^{(2)}$ and if $(a, b) \in G^{(2)}$, then $(pa)b = b$;
\item For all $a \in G$, there is $q \in G$ such that $(a, q) \in G^{(2)}$ and if $(c, a) \in G^{(2)}$, then $c(aq) = c$.
\end{enumerate}

(note that the equations at the end of (G2) and (G3) make sense by (G1)).

Indeed, any groupoid, in the categorical sense, already comes with the structure described above. Conversely, if $G$ is a groupoid as described above, the elements $p$ and $q$ in (G2) and (G3) are unique with those properties, and in fact they are equal so we denote $p = q = a^{-1}$. Define the source and range maps as $s(a) = a^{-1}a$ and $r(a) = aa^{-1}$, and let $G^{(0)} = r(G) = s(G)$, $G^{(1)} = G$. This gives us all the remaining graph-theoretical structure.

Therefore, in order to describe a groupoid $G$, it is enough to simply describe a partial binary operation on $G$, which we identify with $G$ as a set, satisfying (G1)-(G3) above, so the remaining structure is uniquely determined.

**Definition 1.1.20.** A map $\phi : G \to H$ between two groupoids $G$ and $H$ is a morphism if $(a, b) \in G^{(2)}$ implies $(\phi(a), \phi(b)) \in H^{(2)}$, and $\phi(ab) = \phi(a)\phi(b)$.

**Proposition 1.1.21.** Let $\phi : G \to H$ be a groupoid morphism. Then

\begin{enumerate}[(a)]
\item $\phi(a^{-1}) = \phi(a)^{-1}$ for all $a \in G$;
\item $\phi(G^{(0)}) \subseteq H^{(0)}$;
\item $\phi(s(a)) = s(\phi(a))$ and $\phi(r(a)) = r(\phi(a))$ for all $a \in G$;
\item If $\phi$ is invertible then $\phi^{-1}$ is also a morphism.
\end{enumerate}

(In particular groupoids morphisms are precisely their functors.)

**Proof.** (a) Since $\phi(a) = \phi(a)\phi(a^{-1})\phi(a)$, we can multiply both sides on the left and on the right by $\phi(a)^{-1}$ and obtain $\phi(a)^{-1} = \phi(a^{-1})$.

(b) and (c) follow from (a).
(d) Suppose \( \phi \) is invertible and \((\phi(a), \phi(b)) \in H^{(2)}\). From (c),
\[
\phi(s(a)) = s(\phi(a)) = r(\phi(b)) = \phi(r(b))
\]
so \( s(a) = r(b) \) and thus \( \phi^{-1}(\phi(a) \phi(b)) = \phi^{-1}(\phi(ab)) = ab = \phi^{-1}(\phi(a)) \phi^{-1}(\phi(b)) \), therefore \( \phi^{-1} \) is a morphism. \( \square \)

**Example 1.1.22.** Every set \( X \) can be regarded as a groupoid by setting \( X^{(2)} = \{(x, x) : x \in X\} \) and the product \( xx = x \) for all \( x \in X \).

Alternatively, let \( X^{(2)} \) be the identity equivalence relation on \( X \), regarded as a groupoid as in 1.1.14, or let \( \{1\} \) be the trivial group acting (trivially) on \( X \) and consider the transformation groupoid \( \{1\} \ltimes X \) as in 1.1.13. Any of the two bijections
\[
X \to X^{(2)} \quad \text{or} \quad X \to \{1\} \ltimes X
\]
induces the same groupoid structure on \( X \) as above.

In any case, the groupoids constructed in this manner are precisely the groupoids \( G \) satisfying \( G^{(0)} = G \), and these are called the **unit groupoids**.

The next two examples show that the groupoid structure of a transformation groupoid is not enough to recover the initial group action.

**Example 1.1.23.** Let \( G \) be a group acting on two sets \( X \) and \( Y \). We say that the \( G \)-spaces \( X \) and \( Y \) are **conjugate** if there exist a bijection \( F : X \to Y \) satisfying \( F(gx) = gF(x) \) for all \( g \in G \) and \( x \in X \).

Clearly, if \( X \) and \( Y \) are conjugate, then the groupoids \( G \ltimes X \) and \( G \ltimes Y \) are isomorphic – namely, \( (g, x) \mapsto (g, F(x)) \) is an isomorphism – however the converse is not true.

Let \( G = (\mathbb{Z}/2\mathbb{Z})^2 = \langle a, b | a^2 = b^2 = [a, b] = 1 \rangle \) and \( X = \{1, 2, 3, 4\} \). We define two actions \( \alpha \) and \( \beta \) of \( G \) on \( X \) on the generators in the following way:
\[
\alpha(a) = (1, 2), \quad \alpha(b) = (3, 4), \quad \beta(a) = (1, 2)(3, 4) \quad \text{and} \quad \beta(b) = \text{id}_X,
\]
where \((i_1 \cdots i_n)\) is denotes the cycle taking \( i_j \mapsto i_{j+1} \). These actions are not conjugate, because \( \alpha(a) \) is faithful but \( \beta \) is not (even more strongly, there is no automorphism \( \zeta \) of \( G \) for which \( \alpha \) is conjugate to \( \beta \zeta \)), but the respective transformation groupoids are isomorphic, and the isomorphism can be deduced from the usual depictions of these groupoids:
Example 1.1.24. Following the previous example, suppose that two groups $G_1$ and $G_2$ act on spaces $X_1$ and $X_2$, respectively, and that the groupoids $G_1 \ltimes X_1$ and $G_2 \ltimes X_2$ are isomorphic. This clearly implies that $X_1$ and $X_2$ are in bijection, since they are the unit spaces of the respective groupoids, but it is not necessary that $G_1$ and $G_2$ are isomorphic.

Let $X = \{x, y\}$, 

$$G_1 = \mathbb{Z}/4\mathbb{Z} = \langle g \mid g^4 = 1 \rangle \quad \text{and} \quad G_2 = (\mathbb{Z}/2\mathbb{Z})^2 = \langle a, b \mid a^2 = b^2 = [a, b] = 1 \rangle,$$

and consider actions of $\alpha_1$ of $G_1$ and $\alpha_2$ of $G_2$ on $X$ as 

$$\alpha_1(g) = (x, y), \quad \alpha_2(a) = (x, y), \quad \alpha_2(b) = \text{id}_X.$$

The map $\Phi : G_1 \ltimes X \to G_2 \ltimes X$ given by 

$$\Phi(1, x) = (1, x), \quad \Phi(g, x) = (a, x), \quad \Phi(g^2, x) = (b, x), \quad \Phi(g^3, x) = (ab, x),$$

$$\Phi(1, y) = (1, y), \quad \Phi(g, y) = (a, y), \quad \Phi(g^2, y) = (b, y), \quad \Phi(g^3, y) = (ab, y)$$

is an isomorphism, even though $G_1$ and $G_2$ are not isomorphic.

1.1.3 Constructions with groupoids

Definition 1.1.25. Let $\{G_i : i \in I\}$ be a family of groupoids. We endow the disjoint union $G = \sqcup G_i$ with the set of composable pairs $G^{(2)} = \sqcup G_i^{(2)}$ and the operations which restrict to the initial operation in each $G_i$. We call this the coproduct of the groupoids $G_i$.

From the definition of the coproduct we can readily see that $\sqcup G_i$ satisfies the following universal property: There are canonical morphisms $\iota_i : G_i \to G$ (namely, the inclusion maps) such that for every groupoid $\mathcal{H}$ and every family of morphisms $\phi_i : G_i \to \mathcal{H}$, there exists a unique morphism $\Phi : G \to \mathcal{H}$ for which $\phi_i = \Phi \circ \iota_i$ for all $i$.

Definition 1.1.26. The product of a family $\{G_i\}$ of groupoids is the usual Cartesian product $\prod_i G_i$ endowed with the partial operation $(a_i)_{i \in I} (b_i)_{i \in I} = (a_ib_i)_{i \in I}$, which is defined when every term $a_ib_i$ is defined.

The product $\prod_i G_i$ also satisfies the usual universal property: There are canonical morphisms $\pi_j : \prod_i G_i \to G_j$ (namely, the $j$-th projection map) such that for every groupoid $\mathcal{H}$ and family of morphisms $\phi_j : \mathcal{H} \to G_j$, there exists a unique morphism $\Phi : \mathcal{H} \to \prod_i G_i$ such that $\phi_j = \pi_j \circ \Phi$ for all $j$. 
Definition 1.1.27 ([120 176]). A groupoid is connected or transitive if it has only one orbit (Example 1.1.15), or equivalently if the map \((r, s) : G \to G^{(0)} \times G^{(0)}\) is surjective.

Example 1.1.28. The only connected equivalence relation on a set \(X\) is \(X^2\).

Example 1.1.29. If \(\theta\) is an action of a group \(G\) on a set \(X\), then \(G \ltimes \theta X\) is connected if and only if \(\theta\) is transitive.

Proposition 1.1.30. Every groupoid \(G\) is a coproduct of connected groupoids.

Proof. Let us denote by \(\text{orb}(G)\) the collection of orbits of elements of \(G^{(0)}\). If \(A \in \text{orb}(G)\) then the set \(G^A := s^{-1}(A) \cap r^{-1}(A)\) is a connected subgroupoid of \(G\), and it should be clear that \(G = \bigsqcup_{A \in \text{orb}(G)} G^A\).

The next theorem shows that transformation groupoids - or groups and principal groupoids - can be seen as the “building blocks” for general groupoids.

Theorem 1.1.31. (a) Every connected groupoid \(G\) is isomorphic to one of the form \(G \times (X^2)\), where \(G\) is a group, \(X\) is a set and \(X^2\) is the coarsest equivalence relation on \(X\) as in Example 1.1.14.

(b) Every connected groupoid \(G\) is isomorphic to a transformation groupoid (of a transitive group action).

(c) Every groupoid \(G\) is isomorphic to a coproduct of transformation groupoids (of transitive group actions).

Proof. (a) Let \(v \in G^{(0)}\) be an arbitrary element, and let \(G = G_v\), which is a group with the operation endowed from \(G\). For every \(x \in G^{(0)}\), choose \(t_x \in G^x_v\).

The map \(G \times (G^{(0)})^2 \to G\) given by \((g, (y, x)) \mapsto t_y g t_x^{-1}\) is a groupoid isomorphism, and its inverse is \(a \mapsto (t^{-1}_{\tau(a)} a t{s(a)}, (\tau(a), s(a)))\).

(b) By item (a) we just need to consider a groupoid of the form \(G = G \times (X^2)\), where \(G\) is a group and \(X\) is a set. Endow \(X\) with a group structure and consider the action of \(G \times X\) on \(X\) as \((g, x) y = x y\).

Then the map \((G \times X) \ltimes X \to G \times (X^2), ((g, x), y) \mapsto (g, (x y, y))\) is a groupoid isomorphism, and its inverse is \((g, (x, y)) \mapsto ((g, x y^{-1}), y)\).

(c) follows immediately from item (b) and the previous proposition.  

---

The statement that every set \(X\) can be endowed with a group structure (in fact, abelian) is equivalent to the Axiom of Choice. If \(X\) is finite this is clear enough, and if \(X\) is infinite then it has the same cardinality as the set of its finite subsets, which is an abelian group under symmetric difference. For the converse, see [79].
1. GROUPOIDS AND INVERSE SEMIGROUPOIDS

1.2 Inverse semigroups

Nomenclature and notation

Before we start this section, let us fix some nomenclature that will be used throughout the text: Let \((L, \leq)\) be a poset:

1. The *join* of a subset \(F \subseteq L\) is its supremum, and we denote it by \(\bigvee F\) if it exists. If \(F = \{a_1, \ldots, a_n\}\) is finite, we also denote \(\bigvee F = a_1 \lor \cdots \lor a_n\).

2. The *meet* of a subset \(F \subseteq L\) is its infimum, and we denote it by \(\bigwedge F\) if it exists. If \(F = \{a_1, \ldots, a_n\}\) is finite, we also denote \(\bigwedge F = a_1 \land \cdots \land a_n\).

We say that \((L, \leq)\) is a

3. \(^\wedge\)-semilattice (read "meet semilattice") if it admits arbitrary finite meets.

4. lattice, if it admits arbitrary finite meets and joins.

We say that the poset \(L\) has a *zero* if it has a minimum, denoted by \(0 = \bigwedge L\).

1.2.1 Inverse semigroups

**Definition 1.2.1.** A *semigroup* is a set \(S\) endowed with an associative binary operation \((s, t) \mapsto st\). A semigroup is

1. *regular* if for every \(s \in S\) there exists an element \(t \in S\), called an *inverse* of \(s\), satisfying \(sts = s\) and \(tst = t\).

2. an *inverse semigroup* if it is regular and every \(s \in S\) admits a unique inverse. In this case we denote it \(s^*\).

3. a *monoid* if it has an *identity* or *unit*, that is, an element \(1 \in S\) (necessarily unique) satisfying \(1s = s1 = s\) for all \(s \in S\).

A *subsemigroup* of a semigroup \(S\) is a subset \(T \subseteq S\) which is closed under the semigroup operation. A *sub-inverse semigroup* of an inverse semigroup is a sub-semigroup which is closed under inverses.

**Definition 1.2.2.** A *morphism* of semigroups is a map \(\theta : S \to T\) satisfying \(\theta(s_1s_2) = \theta(s_1)\theta(s_2)\) for all \(s_1, s_2 \in S\). Moreover, \(\theta\) is an *isomorphism* if it is invertible and \(\theta^{-1}\) is also a morphism.
As usual, we extend the product notation of elements of a semigroup $S$ to subsets of $S$, namely

$$FG = \{fg : (f, g) \in F \times G\}, \quad tF = \{tf : f \in F\} \quad \text{and} \quad Ft = \{ft : f \in F\}$$

whenever $F, G \subseteq S$ and $t \in S$. If $S$ is an inverse semigroup, we also denote $F^* = \{f^* : f \in F\}$.

**Remark.** If $S$ is an inverse semigroup and $T$ is a subsemigroup which is regular, then every element $t \in T$ admits an inverse $t' \in T$. However since inverses are unique in $S$ then $t' = t^*$. Therefore regular sub-semigroups of inverse semigroups are sub-inverse semigroups.

**Example 1.2.3.** Every $\wedge$-semilattice is an inverse semigroup with the meet as the operation: $xy = x \wedge y$. In this case, every element is its own inverse: $x^* = x$.

We will always regard $\wedge$-semilattices as inverse semigroups with this operation.

**Example 1.2.4.** If $S$ is an inverse semigroup and $s, t \in S$, then the equation $sts = s$ might be satisfied even though $t$ is not the inverse of $s$.

As a particular case of the previous example, let $X$ be a nonempty set and consider the power set $\mathcal{P}(X)$ of $X$, ordered by set inclusion. Then $\mathcal{P}(X)$ is a $\wedge$-semilattice, and hence an inverse semigroup, under set intersection: $AB = A \wedge B = A \cap B$ for all $A, B \in \mathcal{P}(X)$.

However, $\emptyset X \emptyset = \emptyset$, even though $X$ is not the inverse of $\emptyset$.

**Example 1.2.5.** Not every regular semigroup is an inverse semigroup. For example, if $S$ is any set we endow it with the operation $st = s$ for all $s, t \in S$, which makes it regular but not an inverse semigroup if $|S| \geq 2$. In fact, any two elements are inverses of each other in this semigroup.

**Example 1.2.6.** Let $X$ be a set. Let $\mathcal{I}(X)$ be the set of all partial bijections of $X$, i.e., the set of all bijections $f : \text{dom}(f) \to \text{ran}(f)$, where $\text{dom}(f)$ and $\text{ran}(f)$ are subsets of $X$. We make it a semigroup with the canonical partial composition of maps, namely, given $f, g \in \mathcal{I}(X)$,

1. $\text{dom}(gf) = f^{-1}(\text{dom}(g) \cap \text{ran}(f))$;
2. $\text{ran}(gf) = g(\text{dom}(g) \cap \text{ran}(f))$;
3. $(gf)(x) = g(f(x))$ for all $x \in \text{dom}(gf)$.

This makes $\mathcal{I}(X)$ an inverse semigroup. Namely, the inverse of $f \in \mathcal{I}(X)$ is the usual inverse function $f^* = f^{-1} : \text{ran}(f) \to \text{dom}(f)$.

$\mathcal{I}(X)$ serves as an analogue to permutation groups, and in fact a version of Cayley's Theorem, called the *Vagner-Preston Theorem* (see Theorem 1.2.26) holds: every inverse semigroup is isomorphic to a sub-inverse semigroup of some $\mathcal{I}(X)$. 


1. GROUPOIDS AND INVERSE SEMIGROUPS

The semigroups we will be most interested in are the semigroup of bisections of a groupoid, which we now introduce.

**Definition 1.2.7.** A *bisection* in a groupoid $G$ is a subset $A \subseteq G$ such that the source and range maps are injective on $A$.

**Example 1.2.8.** A subset of a group is a bisection if and only if it is either empty or a singleton.

If $A$ and $B$ are subsets of a groupoid $G$, we denote their product by

$$AB = \{ab : (a,b) \in (A \times B) \cap G(2)\}$$

and

$$A^{-1} = \{a^{-1} : a \in A\}$$

**Example 1.2.9.** The collection $\text{Bis}(G)$ of bisections of a groupoid $G$ is an inverse semigroup under the product and inverse of subsets of $G$.

Let us briefly recall the notion of the graph of a function.

**Definition 1.2.10.** If $f : X \to Y$ is a function between sets $X$ and $Y$, the *graph* of $f$ is the set

$$\text{graph}(f) = \{(f(x), x) : x \in X\}.$$ 

**Remark.** Usually (see [93, p. 11] or [81, Section 8]), the graph of a function is instead defined as the set of pairs $(x, f(x))$ for $x \in X$. However, this convention is more consistent with the convention, adopted throughout the whole thesis, that functions “act on the left”.

Recall that a relation (between sets $X$ and $Y$) is a subset $A \subseteq Y \times X$. The composition of relations $A \subseteq Y \times X$ and $B \subseteq Z \times Y$ is the relation

$$B \circ A = \{(z,x) : \text{there exists } y \in Y \text{ such that } (z,y) \in B \text{ and } (y,x) \in A\}.$$ 

(Note that if $R$ is an equivalence relation, regarded as a groupoid, and $A, B \subseteq R \subseteq X \times X$, then the product $BA$ coincides with the composition $B \circ A$.)

If $f : X \to Y$ and $g : Y \to Z$ are functions, then $\text{graph}(f) \subseteq Y \times X$ and $\text{graph}(g) \subseteq Z \times Y$ are relations, and

$$\text{graph}(g \circ f) = \text{graph}(g) \circ \text{graph}(f)$$

i.e., graph preserves composition. In particular, the map $f \mapsto \text{graph}(f)$ defines an injective semigroup morphism between the semigroup $X^X$ of functions from a set $X$ to itself (under function composition) to the semigroup $\mathcal{P}(X \times X)$ of binary relations on $X$ (under relation composition).
Example 1.2.11. If $R$ is an equivalence relation on a set $X$ (seen as a principal groupoid), then $\text{Bis}(R)$ can be identified with the subsemigroup of $\mathcal{I}(X)$

$$\mathcal{J} = \{ f \in \mathcal{I}(X) : \forall x \in \text{dom}(f), (f(x), x) \in R \}.$$ 

Indeed, the map $\theta : \text{Bis}(R) \to \mathcal{J}$, given by

$$\theta_A = r \circ (s|_A)^{-1} : s(A) \to r(A)$$

is a semigroup morphism with inverse $f \mapsto \text{graph}(f)$. Whenever necessary, we will identify $\text{Bis}(R)$ with $\mathcal{J}$ in this manner.

In particular, if $R = X \times X$ then $\mathcal{J} = \mathcal{I}(X)$, so $\text{Bis}(X \times X)$ is isomorphic to $\mathcal{I}(X)$.

Example 1.2.12. If $\theta$ is an action of a group $G$ on a set $X$, then the map $g \mapsto \{g\} \times X$ is an injective morphism (Definition 1.2.2) of $G$ into $\text{Bis}(G \ltimes \theta X)$.

The set of idempotents in a semigroup plays several important roles in their representation theory (Chapter 4), so we introduce a notation for it.

Definition 1.2.13. Given a semigroup $S$, denote by $E(S) = \{ e \in S : e^2 = e \}$ the set of idempotents of $S$.

Note that if $s$ and $t$ are inverse elements in a semigroup $S$ then $st \in E(S)$, and conversely every $e \in E(S)$ is an inverse of itself. In particular, if $S$ is an inverse semigroup then $E(S) = \{ s^* s : s \in S \}$. If $S$ is a monoid then $1 = 1^2 \in E(S)$.

Example 1.2.14. Given a set $A$, let $\text{id}_A : A \to A$ be the identity function of $A$. Then if $X$ is a set, the idempotent set of $\mathcal{I}(X)$ is

$$E(\mathcal{I}(X)) = \{ \text{id}_A : A \subseteq X \}.$$ 

Also note that $\text{id}_{A \cap B} = \text{id}_A \text{id}_B$ whenever $A, B \subseteq X$, so by identifying subsets of $X$ with their respective identity functions, the semigroup operation of $\mathcal{I}(X)$ corresponds to intersection of sets. Moreover, $f^* f = \text{id}_{\text{dom}(f)}$ and $f f^* = \text{id}_{\text{ran}(f)}$, so $f^* f$ corresponds to the domain of $f$ and $f f^*$ corresponds to the range of $f$.

We now prove that $E(S)$ is a commutative sub-inverse semigroup of an inverse semigroup $S$, and this is an important fact for calculations which will be performed throughout the next sections and chapters.

Theorem 1.2.15 ([88, Theorem 5.1.2], [108, Theorem 1.1.3]). A regular semigroup $S$ is inverse if and only if the elements of $E(S)$ commute. In this case, $E(S)$ is a sub-inverse semigroup of $S$.
Proof. First assume that $S$ is inverse and let $e, f \in E(S)$. Let $x = (ef)^*$ be the unique inverse of $ef$. Then

$$(ef)(fxe)(ef) = (ef)x(ef) = ef \quad \text{and} \quad (fxe)(ef)(fxe) = f(xef)e = fxe$$

so $(fxe) = (ef)^* = x$. Then we obtain

$$x^2 = (fxe)(fxe) = f(xef)e = fxe = x$$

so $x$ is idempotent, and therefore it is its own inverse, that is, $x = fxe = ef$ is idempotent. Similarly, $fe$ is idempotent, so

$$fe = (fe)^2 = f(ef)e = fxe = x = ef$$

Conversely, assume elements of $E(S)$ commute and let $t_1$ and $t_2$ be inverses of some $s \in S$. Since $st_1, st_2, t_1s$ and $t_2s$ are idempotents, then

$$t_1 = t_1st_1 = t_1(st_2)(st_1) = t_1(st_1)(st_2) = t_1st_2 = (t_1s)(t_2s)t_2 = (t_2s)(t_1s)t_2$$

$$= t_2st_2 = t_2$$

so $S$ is inverse.

If elements of $E(S)$ commute, it is easy to see $E(S)$ is closed under product, and since every element of $E(S)$ is its own inverse then $E(S)$ is a sub-inverse semigroup of $S$. \hfill \square

An easy application of this result allows us to conclude that inverses in inverse semigroups respect the usual laws of products.

**Corollary 1.2.16.** Let $S$ be an inverse semigroup. Then

(a) $(s^*)^* = s$ for all $s \in S$;

(b) $(st)^* = t^*s^*$ for all $s \in S$;

(c) If $s \in S$ and $e \in E(S)$ then $ses^* \in E(S)$.

Inverse semigroups have several applications in the theory of C*-algebras: for example, important classes of C*-algebras are generated by semigroups of partial isometries, such as those of Cuntz-Krieger, graph and more generally ultragraph C*-algebras. For more details, we refer to [51, 124, 170]. In fact, every regular semigroup of partial isometries of a C*-algebra is an inverse semigroup, as we prove now. (Compare with [135, Proposition 2.1.4].)

**Proposition 1.2.17.** Let $S$ be a multiplicative semigroup of partial isometries of a C*-algebra $A$. The following are equivalent:
1. GROUPOIDS AND INVERSE SEMIGROUPS

(1) $S$ is closed under adjoints;

(2) $S$ is regular;

(3) $S$ is an inverse semigroup.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3): Let us first prove that $E(S) = \{ p \in S : p$ is a projection $\}$. If $p \in E(S)$, then in particular $p$ is a partial isometry and $(p^*)^2 = p^*$, so

$$(p - p^*)^3 = p^3 - p^2p - pp^*p + p(p^*)^2 - p^*p^2 + p^*pp^* + (p^*)^2p - (p^*)^3$$

$$= p - pp^* - p + pp^* - p^*p + p^* + p^*p - p^* = 0$$

and since $p - p^*$ is a normal element of $A$, the functional calculus ([133, Theorem 2.1.3]) implies that $p = p^*$, that is $p$ is a projection.

Now let us prove that $E(S)$ is commutative: if $p, q \in E(S)$, then $pq \in S$ is a partial isometry and $p$ and $q$ are projections, so

$$pq = (pq)(pq)^*(pq) = pq^*pq = (pq)^2$$

and similarly $(qp)^2 = qp$, and thus

$$(pq - qp)^3 = (pq)^3 - (pq)^2(qp) - (pq)(qp)(pq) + (pq)(qp)^2$$

$$- (qp)(pq)^2 + (qp)(pq)(qp) + (qp)^2(pq) - (qp)^3$$

$$= pq - ppq - pq + ppq + pq + ppq - qp = 0.$$}

and again, because $p, q$ are projections then $pq - qp$ is normal in $A$ and the functional calculus implies $pq = qp$. Therefore elements of $E(S)$ commute and $S$ is inverse by Theorem [1.2.15].

(3) $\Rightarrow$ (1): Let $u \in S$, $v$ be the inverse of $u$ in $S$, and $u^*$ be the adjoint of $u$ in $A$. Without loss of generality, we can assume $A$ is contained in the algebra $B(H)$ of bounded operators of some Hilbert space $H$. In this case, a map $u \in A$ is a partial isometry if and only if $u$ is isometric on ker$(u)^\perp$, and in this case $u(\ker(u)^\perp) = u(H) = \ker(u^*)^\perp$ ([133, Theorem 2.3.3]). Then for all $x \in (\ker u)^\perp$,

$$\|x\| = \|u(x)\| = \|u(v(u(x)))\| \leq \|v(u(x))\| \leq \|u(x)\| \leq \|x\|$$

(1.2.1)

and this implies that $\|v(u(x))\| = \|u(x)\|$, so $u(x) \in (\ker v)^\perp$ because $v$ is a partial isometry. In other words, since $u$ is a partial isometry,

$$(\ker u^*)^\perp = u((\ker u)^\perp) \subseteq (\ker v)^\perp$$

Similarly, from $v^*u^*v^* = (uv^*)^* = v^*$, we conclude that $(\ker v)^\perp \subseteq (\ker u^*)^\perp$, and therefore $(\ker u^*)^\perp = (\ker v)^\perp$. Since $u^*$ and $v$ are partial isometries, it is therefore enough to check that $u^* = v$ on $(\ker u^*)^\perp$.
If $y \in (\ker u^*)^\perp$, then $x := u^*(y) \in (\ker u)^\perp$, and equation (1.2.1) implies $\|v(u(x))\| = \|x\|$. But since $S$ is regular, the proof of $(2) \Rightarrow (3)$ also applies and shows that $vu$ is a projection, and thus
\[ u^*(y) = x = v(u(x)) = v(u(u^*(y))) = v(y). \]

The converse also holds: every inverse semigroup is isomorphic to a semigroup of partial isometries of some C*-algebra. See the discussion after Theorem 1.2.26.

We define a partial order on an inverse semigroup $S$ with any of the following equivalent statements.
\[ s \leq t \iff \exists e \in E(S) \ (te = s) \iff \exists e \in E(S) \ (et = s) \iff s = ts^*s \iff s = ss^*t \]
(the equivalence of these statements can be proven using Theorem 1.2.15).

**Example 1.2.18.** If $S$ is an inverse semigroup, the order on $E(S)$ is described by $e \leq f \iff ef = e$ for $e, f \in E(S)$, and then $E(S)$ becomes a $\land$-semilattice, where for any two elements $e, f \in E(S), ef = e \land f$.

**Example 1.2.19.** In $\mathcal{I}(X)$, $f \leq g$ if and only if $g$ is an extension of $f$, that is, $\text{dom}(f) \subseteq \text{dom}(g)$ and $g|_{\text{dom}(f)} = f$.

**Example 1.2.20.** If $G$ is a group then $g \leq h$ in $G$ if and only if $g = h$.

**Example 1.2.21.** If $G$ is a groupoid and $A, B \in \text{Bis}(G)$, then $A \leq B \iff A \subseteq B$.

Using the different descriptions of the order of $S$ and commutativity of $E(S)$ the following theorem can be proven without problems.

**Theorem 1.2.22** (See [88, Section 5.2]). Let $S$ be an inverse semigroup. Then

(a) $s \leq t$ if and only if $s^* \leq t^*$; 

(b) If $s \leq t$ and $z \in S$ then $sz \leq tz$ and $zs \leq zt$; 

(c) If $s \in S$, $e \in E(S)$ and $s \leq e$ then $s \in E(S)$. 

(d) If $s \leq t$ then $s^*s = t^*t$ and $ss^* = tt^*$.

(e) If $s \leq t$ and $s^*s = t^*t$ then $s = t$.

The next simple proposition gives a simple characterization of zeros in inverse semigroups as absorbing elements.

**Proposition 1.2.23.** An element $p$ of an inverse semigroup $S$ is a zero (minimum) of $S$ if and only if it is absorbing (that is, $ps = p$ for all $s \in S$).
Proof. If \( p \) is absorbing, then for all \( s \), \( p = ps^*s \), so \( p \leq s \) and \( p = \bigwedge S \);

In the other direction, if \( p = \bigwedge S \) then \( p \leq p^*p \), which is idempotent, and so \( p \in E(S) \) by Theorem \([1.2.22](c)\) and in particular \( p = p^* \). For all \( s \in S \), \( p = sp^*p = sp \), so \( p \) is absorbing.

Therefore, we denote any absorbing element of an inverse semigroup \( S \) by \( 0 \) and call it a \textit{zero}, which agrees with the poset definition at the beginning of this section.

**Example 1.2.24.** In general, a zero for \( E(S) \) is not necessarily a zero for \( S \) when \( S \) is an inverse semigroup. For example, if \( G \) is a nontrivial group then \( E(G) = \{1\} \), and 1 is the minimum of \( E(G) \) but not of \( G \).

We will now prove a generalization of Cayley’s Theorem in group theory, which allows us to think of \( I(X) \) as a sort of “canonical” inverse semigroup. The next proposition summarizes some basic properties of morphisms of semigroups which will be used throughout the text.

**Proposition 1.2.25.** Let \( \theta : S \to T \) be a morphism of semigroups.

(a) \( \theta(S) \) is a subsemigroup of \( T \);

(b) If \( \theta \) is invertible then it is immediately an isomorphism;

(c) If \( S \) is regular then \( \theta(S) \) is regular;

(d) If \( S \) is regular and \( T \) is an inverse semigroup then \( \theta(S) \) is a sub-inverse semigroup of \( T \).

(e) If \( S \) is inverse and \( \theta \) is surjective, then \( \theta(E(S)) = E(T) \), and \( T \) is an inverse semigroup;

(f) If \( S \) and \( T \) are inverse semigroups then \( \theta(s^*) = \theta(s)^* \) for all \( s \in S \).

**Proof.** (a) and (b) can be proven in the same manner as the analogous statements for groups. (c) is clear and (d) follows from Remark \([1.2.1]\) and the previous items, and (f) follows easily from the uniqueness of inverses in inverse semigroups. The only item we need to prove then is (e), so suppose \( S \) is inverse, \( \theta \) is surjective and let \( u = \theta(s) \in E(T) \), where \( s \in S \). Letting \( e = (ss^*)(s^*s) \in E(S) \), we obtain

\[
\phi(e) = u\phi(s^*s)u = u^2\phi(s^*s)u^2 = \phi(s^2(s^2)^2s^2) = \phi(s^2) = u^2 = u
\]

so \( u = \phi(e) \in \phi(E(S)) \). In particular, \( E(T) \) is commutative and \( T = \theta(S) \) is regular, so \( T \) is an inverse semigroup by Theorem \([1.2.15]\).

**Theorem 1.2.26** (Vagner-Preston Theorem, \([17]\) \([41]\)). Every inverse semigroup \( S \) is isomorphic to a sub-inverse semigroup of \( I(X) \) for some set \( X \).
Proof. For every \( s \in S \), let \( D_{s^*} = \{ t \in S : tt^* \leq s^*s \} \), and define a map \( \alpha_s : D_{s^*} \rightarrow D_s \) by \( \alpha_s(z) = sz \). To check that \( \alpha_s \) is well-defined, note that for all \( z \in D_{s^*} \),

\[
\alpha_s(z)\alpha_s(z)^* = (sz)(sz)^* = szz^*s^* \leq ss^*,
\]

so \( \alpha_s(z) = sz \in D_s \). The verification that \( \alpha_{s^*} = \alpha_s^{-1} \) follows easily from the definition of the order on \( S \).

In order to prove that \( \alpha_s \circ \alpha_t = \alpha_{st} \), we first compare their domains, that is, we show that

\[
\alpha_t^{-1}(D_t \cap D_{s^*}) = D_{(st)^*}.
\]

If \( z \in D_{(st)^*} \), then \( zz^* \leq (st)^*(st) = t^*s^*st \leq t^*t \) (because \( t^*s^*s \leq t^* \)), so \( z \in D_{t^*} \), and

\[
(\alpha_t(z))(\alpha_t(z))^* = (tz)(tz)^* = tzz^*t^* \leq t(st)^*(st)t^* = tt^*stt^* \leq s^*s
\]

that is, \( z \in \alpha_t^{-1}(D_t \cap D_{s^*}) \).

Conversely, if \( z \in \alpha_t^{-1}(D_t \cap D_{s^*}) \), then \( \alpha_t(z) \in D_{s^*} \) which implies \( \alpha_t(z) = s^*s\alpha_t(z) \), so

\[
zz^*(st)^*(st) = zz^*t^*s^*st = z\alpha_t(z)^*s^*st = z\alpha_t(z)t = zz^*t^*t = zz^*
\]

and therefore \( z \in D_{(st)^*} \). It is then clear that \( \alpha_{st} = \alpha_s \circ \alpha_t \).

Finally, for the injectivity of \( \alpha \), simply note that if \( \alpha_s = \alpha_t \) then \( s = \alpha_s(s^*s) = \alpha_t(s^*s) = ts^*s \) so \( s \leq t \) and symmetrically, \( t \leq s \), so \( \alpha : S \rightarrow I(S) \) is an isomorphism from \( S \) to a sub-inverse semigroup of \( I(S) \).

The map \( \alpha \) will be used in later chapters, so we give it a name.

**Definition 1.2.27.** The map \( \alpha : S \rightarrow I(S) \) is called the *canonical left action of \( S \) on itself*.

**Remark.** For inverse semigroups with zero, it is natural to ask that morphisms preserve zeros, however this is not the case for the canonical left action. Namely, if \( S \) has a zero and \( \alpha : S \rightarrow I(S) \) is the canonical left action, then \( \theta(0) = \text{id}_{\{0\}} \), which is not the empty function. However, this is easily remedied by noticing that \( 0 \) is fixed by \( \alpha_s \) for all \( s \in S \), and so we can restrict each \( \alpha_s \) to the complement of \( \{0\} \) and obtain again an injective morphism \( \alpha_0 : S \rightarrow I(S \setminus \{0\}) \), \( \alpha_0(s) = \alpha_s|_{S \setminus \{0\}} \), which in this case will satisfy \( \alpha_0(0) = \emptyset \).

We can use the Wagner-Preston Theorem in order to prove that all inverse semigroups are isomorphic to some semigroup of partial isometries of some C*-algebra (this was first proved in [37]). Indeed, it is enough to prove this for \( I(X) \), where \( X \) is a set. Consider the Hilbert space \( H = \ell^2(X) \) with the standard basis \( \{ \delta_x : x \in X \} \). Given \( f \in I(X) \), we define \( T_f \in B(H) \) by

\[
T_f(\delta_x) = \begin{cases} 
\delta_{f(x)}, & \text{if } x \in \text{dom}(f), \\
0, & \text{otherwise}, 
\end{cases}
\]

and the map \( f \mapsto T_f \) is an injective semigroup morphism.
1.2.2 Distributive semigroups

Let $X$ be a set and $f, g$ two partial bijections of $X$. Oftentimes we are interested in considering a common upper bound of $f$ and $g$, that is, a new function which extends both $f$ and $g$. This is possible, for example, if $f$ and $g$ coincide on the intersection of their domains, and $f^*$ and $g^*$ also coincide on the intersection of their domains. We will deal with this problem in the more abstract setting of general inverse semigroups (as opposed to $I(X)$).

**Proposition 1.2.28.** Suppose $S$ is an inverse semigroup and $s, t \in S$ admit a common upper bound (that is, there is some $z \in S$ with $s \leq z$ and $t \leq z$). Then $s^*t \in E(S)$ and $st^* \in E(S)$.

**Proof.** Let $z \in S$ be a common upper bound of $s$ and $t$. Then $s^*t \leq z^*z$, which is an idempotent and thus $s^*t \in E(S)$. The same argument with $s^*$ and $t^*$ in place of $s$ and $t$ yields $st^* \in E(S)$ as well. □

**Definition 1.2.29.** Two elements $s, t \in S$ are compatible if $s^*t$ and $st^*$ belong to $E(S)$. A subset $F \subseteq S$ is called compatible if any two elements of $F$ are compatible.

**Proposition 1.2.30.** Let $S$ be an inverse semigroup, $s, t \in S$ two compatible elements and $z \in S$. Then

(a) $s^*$ and $t^*$ are compatible;

(b) $zs$ and $zt$ are compatible.

(c) $s \wedge t$ exists, and $s \wedge t = st^*t = tt^*s = ts^*s = ss^*t$.

**Proof.** (a) should be clear enough.

(b) We have

$$(zs)^*(zt) = (s^*z^*z)t \leq s^*t \quad \text{and} \quad (zs)(zt)^* = z(st^*)z^*$$

so $(zs)^*(zt) \in E(S)$ by Theorem 1.2.22(c), and $(zs)(zt)^* \in E(S)$ by Theorem 1.2.16(c), so $zs$ and $zt$ are compatible.

(c) We just prove that $s \wedge t = st^*t$, and the other equalities are similar. Since, $st^* \in E(S)$, then $st^*t \leq t$, and also $st^*t \leq s$, so $st^*t$ is a lower bound of $s$ and $t$. If $p \leq s, t$, then

$$p = pp^*p \leq st^*t$$

so $st^*t$ is the largest lower bound of $s$ and $t$, that is, their meet. □
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Example 1.2.31. If \( X \) is a set and \( f, g \in \mathcal{I}(X) \), then \( f^*g \in E(\mathcal{I}(X)) \) if and only if \( f(x) = g(x) \) whenever \( x \in \text{dom}(f) \cap \text{dom}(g) \), and similarly \( fg^* \in E(\mathcal{I}(X)) \) if and only if \( f^{-1}(x) = g^{-1}(x) \) whenever \( x \in \text{ran}(f) \cap \text{ran}(g) \). Thus, if \( f \) and \( g \) are compatible we may construct the partial bijection \( f \lor g \), defined by

- \( \text{dom}(f \lor g) = \text{dom}(f) \cup \text{dom}(g) \); \( \text{ran}(f \lor g) = \text{ran}(f) \cup \text{ran}(g) \);
- \( (f \lor g)(x) = f(x) \) if \( x \in \text{dom}(f) \); \( (f \lor g)(x) = g(x) \) if \( x \in \text{ran}(g) \).

We will be interested in semigroups which satisfy a distributive property for the product with respect to joins. Actually, distributivity of the product over meets is always satisfied, but there are inverse semigroups such that the product does not distribute over joins.

Proposition 1.2.32. Suppose \( S \) is an inverse semigroup, \( t \in S \), \( F \) is a nonempty subset of \( S \) and that \( \text{\lor} F \) exists. Then \( \text{\lor}(tF) \) and \( \text{\lor}(Ft) \) exist, and \( \text{\lor}(tF) = t(\text{\lor} F) \) and \( \text{\lor}(Ft) = (\text{\lor} F)t \).

Proof. For all \( f \in F \), \( t(\text{\lor} F) \leq tf \), so \( t(\text{\lor} F) \) is a lower bound of \( tF \). To prove it is the largest one, assume \( h \) is another lower bound of \( tF \). Then \( t^*h \leq t^*tf \leq f \) for all \( f \in F \), which implies \( t^*h \leq \text{\lor} F \).

Letting \( p \in F \) be arbitrary, we have \( h \leq tp \) and so \( hh^* \leq tpp^*t^* \leq tt^* \), therefore

\[ h = hh^*h \leq tt^*h \leq t(\text{\lor} F). \]

The proof that \( \text{\lor}(Ft) = (\text{\lor} F)t \) is similar. \( \square \)

Definition 1.2.33 ([113]. A **distributive semigroup** is an inverse semigroup \( S \) such that any two compatible elements \( s, t \in S \) admit a join \( s \lor t \), and for every \( z \in S \), \( z(s \lor t) = (zs) \lor (zt) \).

Note that in the definition we assume, in principle, only left distributivity of the product over joins, however right distributivity follows as a consequence, as we will see in Proposition 1.2.39.

Example 1.2.34. The simplest non-distributive inverse semigroup is the diamond lattice \( M_3 \):

```
1
\rightarrow
s \quad z \quad t
\leftarrow
0
```

Here an arrow \( x \rightarrow y \) denotes \( x \leq y \), and the semigroup operation is the meet. \( M_3 \) admits binary joins, but

\[ z(s \lor t) = z1 = z \quad \text{however} \quad (zs) \lor (zt) = 0 \lor 0 = 0. \]
Example 1.2.35. The semigroup $\text{Bis}(G)$ of bisections of a groupoid $G$ is distributive. More generally, if $\mathcal{F}$ is any (possibly infinite) compatible subset of $\text{Bis}(G)$, then the join of $\mathcal{F}$ exists and coincides with the union of its elements:

$$\bigvee \mathcal{F} = \bigcup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$$

Moreover, in this case we have $A(\bigvee \mathcal{F}) = \bigvee (A \mathcal{F})$ for any $A \in \text{Bis}(G)$. We say that $\text{Bis}(G)$ is \textit{infinitely distributive} ([150]). (Cf. Definition 1.2.40.)

Example 1.2.36. If $X$ is a set then, by Example 1.2.11, $\mathcal{I}(X)$ is isomorphic to $\text{Bis}(X \times X)$, and thus it is (infinitely) distributive by Example 1.2.35 above. If $\mathcal{F}$ is any compatible subset of $\mathcal{I}(X)$, then the join of $\mathcal{F}$ is the function given by

$$\bigvee \mathcal{F} : \bigcup_{f \in \mathcal{F}} \text{dom}(f) \to \bigcup_{f \in \mathcal{F}} \text{ran}(f), \quad \left(\bigvee \mathcal{F}\right)(x) = f(x),$$

where $f \in \mathcal{F}$ is chosen such that $x \in \text{dom}(f)$ (“dom” and “ran” stand for the domain and range of an element of $\mathcal{I}(X)$).

Proposition 1.2.37. Let $S$ be an inverse semigroup.

(a) If $F \subseteq E(S)$ admits a join, then $\bigvee F \in E(S)$.

(b) If $F \subseteq S$ admits a join, then $F^*$ also admits a join, namely $\bigvee (F^*) = (\bigvee F)^*$.

Proof. (a) Let $e = \bigvee F$. For every $f \in F$, $f = f^*f \leq e^*e$, so the definition of join implies $e \leq e^*e \in E(S)$, therefore $e \in E(S)$;

(b) Should be clear from 1.2.22(a) and 1.2.30(a). \(\square\)

In the remainder of this subsection we will consider distributive properties for sets with more than two elements. We differentiate between finite and infinite sets for the sake of clarity and to obtain appropriate notation in Chapter 2, and specifically Definition 2.2.1.

Definition 1.2.38. Given a set $X$, we denote by $|X|$ the cardinality of $X$. In the particular case that $X$ is finite, we will use instead the symbol $\#X$ to denote its cardinality (number of elements).

Proposition 1.2.39. If $S$ is a distributive semigroup, then every finite nonempty compatible set $F \subseteq S$ admits a join, and for every $t \in S$,

$$\bigvee (tF) = t \left(\bigvee F\right) \quad \text{and} \quad \bigvee (Ft) = \left(\bigvee F\right) t.$$
Proof. This follows from induction on the cardinality \( \#F \): If \( \#F = 1 \) this is trivial, so let us just assume this holds for every compatible set with up to \( n \) elements, and \( \#F = n + 1 \). Let \( z \in F \) be arbitrary. Then the distributive property implies
\[
z^* \left( \bigvee (F \setminus \{z\}) \right) = \bigvee \{z^* s : s \in F \setminus \{z\}\},
\]
a join of idempotents and hence an idempotent. Similarly, \( z \left( \bigvee (F \setminus \{z\}) \right)^* \in E(S) \), so \( z \) and \( \bigvee (F \setminus \{z\}) \) are compatible and thus admit a join. From general order theory it should be clear that
\[
z \lor \left( \bigvee (F \setminus \{z\}) \right) = \bigvee F.
\]
Given any \( t \in S \), we have
\[
t \left( \bigvee F \right) = t \left( z \lor \left( \bigvee (F \setminus \{z\}) \right) \right) = tz \lor \bigvee (tF \setminus \{tz\}) = \bigvee (tF).
\]
To obtain \( (\bigvee F) t = \bigvee (Ft) \), we use the previous case with \( t^* \) and \( F^* \) and obtain
\[
t^* \left( \bigvee F^* \right) = \left( \bigvee t^* F^* \right)
\]
and taking inverses and applying Proposition 1.2.37(b) yields \( (\bigvee F) t = \bigvee (Ft) \). \( \square \)

To finish, we prove that every distributive inverse semigroup is also distributive with respect to joins and meets. We perform this in a more general setting by considering inverse semigroups which are distributive with respect to a given cardinality. For more information on cardinals, see \[93\, Chapter 3\]. We will not make assumptions about regularity of cardinals.

**Definition 1.2.40** ([93, p. 82]). Let \( \kappa \) be a cardinal. An inverse semigroup \( S \) is \( \kappa \)-distributive if for every compatible set \( F \subseteq S \) with \( |F| < \kappa \), the supremum \( \bigvee F \) exists and for all \( z \in S \),
\[
z \left( \bigvee F \right) = \bigvee (zF).
\]
Note that \( |zF| \leq |F| \) and \( zF \) is compatible by Proposition 1.2.30 and that \( (\bigvee F) z = \bigvee (Fz) \) also holds by Proposition 1.2.37(b).

**Proposition 1.2.39** states that distributive inverse semigroups (as in Definition 1.2.33) are precisely the \( \aleph_0 \)-distributive semigroups, where \( \aleph_0 = |\mathbb{N}| \) is the first infinite cardinal.

The following proposition provides a sharper version (with a significantly different proof) of the main result of [150].

**Proposition 1.2.41.** Let \( S \) be a \( \kappa \)-distributive inverse semigroup, and suppose \( F \subseteq S \) is compatible, \( |F| < \kappa \). If \( z \in S \) is such that \( z \land f \) exists for all \( f \in F \), then \( z \land (\bigvee F) \) exists and
\[
z \land (\bigvee F) = \bigvee (z \land F), \quad \text{where } z \land F = \{z \land f : f \in F\}.
\]
Proof. Since $z \wedge F$ is bounded above by $\bigvee F$, then $z \wedge F$ is compatible. Clearly, $\bigvee (z \wedge F)$ is a lower bound of $\{z, \bigvee F\}$, so we need to prove it is the largest one: Suppose $p$ is a lower bound of $\{z, \bigvee F\}$. Then

$$p = (\bigvee F)p^*p = \bigvee (Fp^*p)$$

Given $f \in F$, we have $f, p \leq \bigvee F$, so $f$ and $p$ are compatible and thus $fp^* \in E(S)$, so $fp^*p \leq p \leq z$. But of course we also have $fp^*p \leq f \wedge z$ for all $f \in F$. Therefore

$$p = \bigvee (Fp^*p) \leq \bigvee (F \wedge z).$$

This proves that $\bigvee (F \wedge z)$ is the largest lower bound of $\{z, \bigvee F\}$, that is, its meet. \qed

The finitary case of this proposition becomes:

**Corollary 1.2.42.** Let $S$ be a distributive inverse semigroup. If $s \vee t$, $z \wedge s$ and $z \wedge t$ exist, then $z \wedge (s \vee t)$ exists and

$$z \wedge (s \vee t) = (z \wedge s) \vee (z \wedge t)$$

We will need to deal with maps that preserve more than just semigroup operations, so we define morphisms which respect the order of semigroups in a stronger manner:

**Definition 1.2.43.** A $\vee$-morphism of inverse semigroups is a semigroup morphism $\theta : S \rightarrow T$ such that whenever $s, t \in S$ are such that the join $s \vee t$ exists, then the join $\theta(s) \vee \theta(t)$ exists as well and $\theta(s \vee t) = \theta(s) \vee \theta(t)$. A $\wedge$-morphism is defined in a similar manner.

**Remark.** In general, the canonical left action of a distributive inverse semigroup (which is used in the Wagner-Preston Theorem) is not a $\vee$-morphism.

For example, suppose that $S$ is any distributive semigroup which admits two compatible but non-comparable elements $s$ and $t$, i.e., $s \vee t$ exists and is different from both $s$ and $t$. Let $\alpha : S \rightarrow \mathcal{I}(S)$ be the canonical left action of $S$ (Definition 1.2.27). Then $(s \vee t)^*(s \vee t)$ belongs to the domain of $\alpha_{s \vee t}$, but it does not belong to the domain of $\alpha_s \vee \alpha_t$, so $\alpha_{s \vee t}$ is strictly larger than $\alpha_s \vee \alpha_t$.

On the other hand, every distributive inverse semigroup $S$ admits a canonical injective $\vee$-morphism $S \rightarrow \mathcal{I}(X)$ for some set $X$. See [173, Section 3.3].

### 1.2.3 Relative complements

The final structural property we will have to deal with are relative complements, which allow us to look at semigroup-theoretic analogues of set differences.
Definition 1.2.44. Let $S$ be a distributive semigroup with zero and $s \leq t \in S$. A relative complement of $s$ in $t$ is an element $c \leq t$ such that $s \wedge c = 0$ and $t = s \vee c$ (in particular, $s$ and $c$ are compatible).

Lemma 1.2.45. If $S$ is a distributive inverse semigroup, $s \leq t$ in $S$ and $c, d \in S$ are relative complements of $s$ in $t$, then $c = d$.

Proof. Since $s \vee c = t$ exists, then $s$ and $c$ are compatible and by Proposition 1.2.30(c), $sc^*c = s \wedge c = 0$, thus

$$c = tc^*c = (d \vee s)c^*c = dc^*c \vee sc^*c = dc^*c$$

so $c \leq d$. The reverse inequality is similar.

Definition 1.2.46. If it exists, we denote the relative complement of an element $s$ in $t$ ($s \leq t$), in a distributive inverse semigroup, by $t \setminus s$.

Lemma 1.2.47. If $S$ is an inverse semigroup, $s, t \in S$, $s \wedge t$ exists and $z \leq t$, then $s \wedge z$ exists and $s \wedge z = (s \wedge t)z^*z$

Proof. We already know that

$$(s \wedge t)z^*z = (sz^*z) \wedge (tz^*z) = (sz^*z) \wedge z,$$

which is a lower bound of $s$ and $z$. It is readily checked that it is the largest one.

In the next proposition, we initially assume that the existence of $t \setminus s$ implies $s \leq t$, however this condition is not necessary in general when we extend the notion of relative complements in Boolean inverse semigroups (Section 1.4).

Proposition 1.2.48. Let $S$ be a distributive inverse semigroup and $s, t, z, a \in S$.

(a) If $(t \setminus s)$ exists, then $(zt) \setminus (zs)$ exists and is equal to $z(t \setminus s)$. Similarly, $(tz) \setminus (sz)$ exists and is equal to $(t \setminus s)z$.

(b) If $t \wedge z$, $s \wedge z$ and $(t \setminus s)$ exist, then

$$(t \wedge z) \setminus (s \wedge z) = (t \setminus s) \wedge z$$

in the sense that these elements exist and are equal.

(c) If $a \leq s \vee t$, and $a \setminus s$ exists, then $a \setminus s \leq t$.

(d) If $a \setminus s$ and $a \setminus t$ exist, and $s \leq t$, then $a \setminus t \leq a \setminus s$. 
**Proof.** (a) Let us verify that $z(t \setminus s)$ satisfies the properties of the relative complement of $zs$ in $zt$. If $p \leq z(t \setminus s)$, then $z^*p \leq (t \setminus s) \land s = 0$, so $p = zz^*p = 0$. On the other hand, 
$$zt = z(s \lor (t \setminus s)) = zs \lor z(t \setminus s)$$
which is precisely what we need.

(b) Since $t \land z$ exists and $t \setminus s \leq t$, then by the previous lemma, $(t \setminus s) \land z$ exists. Let us show it satisfies the properties of the relative complement of $(s \land z)$ in $(t \land z)$: On one hand, if $p \leq (t \setminus s) \land z$ and $p \leq (s \land z)$ then $p \leq (t \setminus s) \land s = 0$, so 
$$[(t \setminus s) \land z] \land (s \land z) = 0.$$ 
Moreover, using the previous lemma, 
$$t \land z = ((t \setminus s) \lor s) \land z = ((t \setminus s) \land z) \lor (s \land z)$$
therefore $(t \setminus s) \land z$ is the relative complement of $(s \land z)$ in $(t \land z)$.

(c) Since $a, s, t \leq s \lor t$, then $a, s$ and $t$ (or any smaller elements) are compatible and Proposition 1.2.42 allows us to compute 
$$a \setminus s = (a \setminus s) \land a = (a \setminus s) \land (s \lor t) = ((a \setminus s) \land s) \lor ((a \setminus s) \land t)$$
$$= 0 \lor ((a \setminus s) \land t) \leq t$$

(d) We are assuming $s \leq t \leq a$, so we have 
$$a = s \lor (a \setminus s) \leq t \lor (a \setminus s)$$
which implies $a \setminus t \leq a \setminus s$ by item (c). \hfill \Box

### 1.3 Topological groupoids

**Definition 1.3.1.** A **topological groupoid** is a groupoid $G$ endowed with a topology which makes the product map $G^{(2)} \to G$ and the inversion $G \to G$ continuous (where we consider the product topology on $G^{(2)}$).

**Example 1.3.2.** Any topological group, seen as a groupoid, is a topological groupoid.

**Example 1.3.3.** If $R$ is an equivalence relation on a topological space $X$, then $R$ is a topological groupoid when endowed with the topology induced by the product topology of $X \times X$, however in general this is not the most suitable topology (see Example 1.3.9).
Example 1.3.4. Let $G$ be a topological group acting continuously on a topological space $X$ (that is, the action map $G \times X \to X$, $(g, x) \mapsto gx$ is continuous). Then the transformation groupoid $G \ltimes X$ endowed with the product topology of $G \times X$ is a topological groupoid.

Example 1.3.5. Let $G$ be any groupoid and assume $G^{(0)}$ is endowed with some (arbitrary) topology. Define a topology on $G$ where $A \subseteq G$ is open if and only if $A \cap G^{(0)}$ is open in $G^{(0)}$. This makes $G$ a topological groupoid. Indeed, $A^{-1} \cap G^{(0)} = A \cap G^{(0)}$ for all $A \subseteq G$, which implies that the inversion is continuous, so let us check the product is continuous: Let $(a, b) \in G^{(2)}$ and $U$ any open set containing $ab$. There are a few possibilities:

- If $a, b \notin G^{(0)}$, then $\{a\}$ and $\{b\}$ are open and $\{a\} \{b\} = \{ab\} \subseteq U$;
- If $a \notin G^{(0)}$ but $b \in G^{(0)}$, then $\{a\}$ and $G^{(0)}$ are open, respectively, and $\{a\} G^{(0)} = \{ab\} \subseteq U$;
- The case $a \in G^{(0)}$ and $b \notin G^{(0)}$ is similar to the previous one;
- If both $a, b \in G^{(0)}$, then $a = b \in U \cap G^{(0)}$ and $(U \cap G^{(0)})(U \cap G^{(0)}) = U \cap G^{(0)} \subseteq U$;

In any case there are open sets $A$ and $B$ containing $a$ and $b$, respectively, such that $AB \subseteq U$, which proves that the product is continuous.

Definition 1.3.6. A groupoid is étale if the source $s : G \to G^{(0)}$ map is a local homeomorphism.

Étale groupoids were first considered in depth by Renault in [146], in connection with $C^*$-algebras, as those groupoids which have open unit space and admit a left Haar system. We initially adopt the current (equivalent) working definition, see Definition 2.6 and Proposition 2.8 of [146], and Theorem 1.3.11 below.

Example 1.3.7. Every topological space, seen as a unit groupoid (Example 1.1.22), is étale.

Example 1.3.8. If $G$ is a topological group acting continuously on a topological space $X$, then the transformation groupoid $G \ltimes X$ (endowed with the product topology) is étale if and only if $G$ is discrete.

Example 1.3.9. Let $R$ be an equivalence relation on a topological space $(X, \nu)$. In general, the product topology makes $R$ a non-étale topological groupoid, the simplest example being $R = X \times X$ when $X$ has any non-isolated point (see Proposition 1.3.10).

On the other hand, suppose that $(X, \nu)$ is Hausdorff and $\tau$ is any topology on $R$ for which:
(i) \((R, \nu)\) is a compact étale topological groupoid;

(ii) The range map \(r : (R, \tau) \to (X, \nu)\) is continuous.

Then \(\tau\) is induced from the product topology of \((X, \nu) \times (X, \nu)\) (see [68, Proposition 3.2]). In particular, in this case the topology \(\nu\) of \(X\), identified with \(R^{(0)}\), is induced from \(\tau\). (Compare this with Example 1.6.15 in the Borel setting.)

The initial problem we will deal with is showing that \(G^{(0)}\) is open in \(G\) whenever \(G\) is étale. This is not immediate from the étale condition, which deals, in principle, only with the relative topology of \(G^{(0)}\) induced by that of \(G\).

Note that if \(G\) is a topological groupoid and \(A\) is open in \(G\), then \(A^{-1}\) is immediately open (since the inversion \(a \mapsto a^{-1}\) is a homeomorphism), however it is not necessarily true that \(AB\) is open when \(A\) and \(B\) are open (e.g. in Example 1.3.5).

**Proposition 1.3.10.** If \(G\) is an étale groupoid, then the open bisections of \(G\) form a basis for its topology. Moreover, if \(A\) is any open bisection of an étale groupoid \(G\) then the source map restricts to a homeomorphism \(s|_A : A \to s(A)\) (and similarly for the range map). In particular, \(s^{-1}(x)\) is discrete for all \(x \in G^{(0)}\).

**Proof.** If \(I : G \to G\), \(I(a) = a^{-1}\) is the inversion map, then \(r = s \circ I\), so both \(s\) and \(r\) are local homeomorphisms and it follows that open bisections form a basis for the topology of \(G\). Since the source map restricts to a bijective local homeomorphism on any open bisection it actually restricts to a homeomorphism, as desired.

The last statement follows from the fact that if \(A\) is a bisection and \(x \in G^{(0)}\) then \(s^{-1}(x) \cap A\) is either empty or a singleton. \(\square\)

**Theorem 1.3.11** ([151, Theorem 5.18]). Let \(G\) be a topological groupoid and let \(m : G^{(2)} \to G\), \(m(a, b) = ab\) be the product map. The following are equivalent:

1. \(G\) is étale;
2. \(G^{(0)}\) is open and \(m\) is a local homeomorphism;
3. \(G^{(0)}\) is open in \(G\) and \(m\) is an open map (or equivalently, the product of open sets in \(G\) is open);
4. \(G^{(0)}\) is open and the source map \(s : G \to G^{(0)}\) is an open map.

**Proof.** (1) \(\Rightarrow\) (2): Assume \(G\) is étale, and let \(\pi : G^{(2)} \to G\), \(\pi(a, b) = b\), which is immediately continuous. Let us show that \(\pi\) is a local homeomorphism: If \(A\) and \(B\) are open bisections in \(G\), then \(\pi((A \times B) \cap G^{(2)}) = B \cap r_{|B}^{-1}(s(A) \cap r(B))\). Since \(r(B)\) and \(s(A)\) are open in \(G^{(0)}\) and \(r_{|B} : B \to r(B)\) is a homeomorphism we conclude that \(\pi\) is open, and since it is injective on \((A \times B) \cap G^{(2)}\), then \(\pi\) is a local homeomorphism, as we wanted. We now have \(s \circ m = s \circ \pi\), and both \(s\) and \(\pi\) are local homeomorphisms,
so $m$ is a local homeomorphism as well. In particular, if $A$ is a bisection then $s(A) = m((A \times A^{-1}) \cap G^{(2)})$ is open in $G$, so since open bisections cover $G$ and $G^{(0)} = s(G)$, $G^{(0)}$ is open in $G$.

The implication $(2) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (4)$: Assuming $(3)$, if $A$ is open then $s(A) = (A^{-1}A) \cap G^{(0)}$ is open as well.

$(4) \Rightarrow (1)$: Let $a \in G$. The map

$$f: \{(g, h) \in G : s(g) = s(h)\} \to G, \quad f(g, h) = gh^{-1}$$

is continuous, since $G$ is a topological groupoid. We have $f(a, a) = r(a) \in G^{(0)}$, and $G^{(0)}$ is open. Thus there exist open sets $C, D \subseteq G$ containing $a$ such that

$$CD^{-1} = f \left( \{(c, d) \in C \times D : s(c) = s(d)\} \right) \subseteq G^{(0)}.$$

This implies that the source map is injective on the neighbourhood $C \cap D$ of $a$. Therefore, the source map is continuous (again, since $G$ is a topological groupoid), open, and locally injective, hence a local homeomorphism.

The next examples prove that a topological groupoid may have open unit space but the product of two open sets might not be open and vice-versa.

**Example 1.3.12.** Given a topological group $G$, seen as a topological groupoid, the product of open sets of $G$ is always open, however the unit space is the singleton $\{1\}$, so $G$ is étale if and only if it is discrete.

**Example 1.3.13.** Given an infinite set $P$ of integers, we consider the topology on $\mathbb{Z}$ generated by the sets $p\mathbb{Z} + a$, where $a \in \mathbb{Z}$ and $p \in P$. This makes $\mathbb{Z}$ a non-discrete (i.e., non-étale), countable, second-countable, zero-dimensional, non-locally compact, Hausdorff topological group.

**Example 1.3.14.** Let $G$ be a groupoid such that $G^{(0)}$ is endowed with some topology, and consider $G$ with the topology induced by $G^{(0)}$ as in example 1.3.5. The unit space $G^{(0)}$ is open, so $G$ is étale if and only if the product of open sets is open. This happens if and only if for every $a \in G \setminus G^{(0)}$, $s(a)$ is isolated in $G^{(0)}$.

**Example 1.3.15.** As a particular case of the previous example, let $\mathbb{Z}_2 = \{0, 1\}$ be the group with two elements, and consider the interval $[0, 1]$ as a unit groupoid. Endow the (algebraic) product groupoid $\mathbb{Z}_2 \times [0, 1]$ with the topology whose open sets are those $A \subseteq \mathbb{Z}_2 \times [0, 1]$ such that $\{x \in [0, 1] : (0, x) \in A\}$ is open in $[0, 1]$. Then $\mathbb{Z}_2 \times [0, 1]$ is a non-étale, Hausdorff topological groupoid with open unit space. Indeed, $\{(1, 0)\}$ is open in $\mathbb{Z}_2 \times [0, 1]$, but $\{(1, 0)\} \{\{(1, 0)\} = \{(0, 0)\}$ is not.

Finally, whenever dealing with topological structures we will be mostly interested in maps which preserve the topology up to the correct degree, so we define precisely what we mean by topological groupoid morphisms below.

**Definition 1.3.16.** A topological groupoid morphism is a continuous groupoid morphism of topological groupoids. A topological groupoid isomorphism is a groupoid morphism which is also a homeomorphism.
1.4 Stone duality for ample Hausdorff groupoids

In [163], Marshall Stone first studied the representation theory of abstract (unital) Boolean algebras of sets, which culminates in a duality between the categories of unital Boolean algebras and of now called Stone spaces, which allows us to use both algebraic or geometric methods to study either of these types of structures. In fact, further improvements ([165]) of this theory lead to a more general duality between distributive lattices and coherent spaces ([96 II.3.4]).

In this section we will start by reviewing the classical version of Stone duality and the outline of its proof. Similar ideas will serve as a motivation for a more general (“non-commutative”) version which will be proven in details. The proof presented here is loosely based on [109] and [113]. We should note that even more general dualities between classes of inverse semigroups and topological groupoids have been obtained in recent years (for example, see [14] and [113]), however the version presented here will be sufficient for all our goals.

1.4.1 Classical Stone duality

Let $(X, \tau)$ be a topological space, that is, $\tau$ is the lattice of open sets of $X$. The question we would like to study is “how much of $X$ can one recover by looking solely at $\tau$, or actually any sublattice of it, as a poset”?

**Example 1.4.1.** Let $(X, \tau)$ be a topological space and $x, y \in X$ be such that every neighbourhood of $x$ contains $y$ and vice-versa. If we let $(Y, \tau_Y)$ be the quotient space obtained by identifying $x$ and $y$, endowed with the quotient topology, then the lattices of open sets $\tau_X$ and $\tau_Y$ are isomorphic. More precisely, the quotient map $\pi : X \to Y$ is continuous, so it induces a “preimage map” $\pi^{-1} : \tau_Y \to \tau_X$ which is in fact a lattice isomorphism.

This example shows that, in order to be able to recover $X$ from a lattice of open sets of $X$, we should start by restricting our study to $T_0$ spaces. The example above also points us how one could try to recover a point $x$ of a $T_0$ space $X$ from the lattice of open sets: By identifying $x$ with the collection $\tau_x = \{ U \in \tau : x \in U \}$. In fact, the $T_0$ property states exactly that $x \mapsto \tau_x$ is an injective map. The problem then becomes to find topological conditions which allow us to characterize the subsets $\tau_x$ of $\tau$. When dealing with the whole topology $\tau$, these are the completely prime filters. For details in this direction we refer to [96 II.3.3 and 3.4].

However, we will be mostly interested in Hausdorff, locally compact spaces, endowed with bases of open subsets which are not necessarily closed under arbitrary unions (such as bases of compact-open sets, or regular open sets). In this setting, it is possible to obtain a duality between a certain class of topological spaces, called Stone spaces, and Boolean algebras.
Definition 1.4.2. A Stone space is a zero-dimensional, compact Hausdorff topological space.

Given a Stone space $X$, the collection $\text{Cl}(X)$ of clopen subsets of $X$ is a unital Boolean algebra (Definition 1.4.9). The direction which associates a topological space to a Boolean algebra uses the notion of ultrafilter.

Definition 1.4.3. Let $P$ be a set and $\prec$ a transitive relation on $P$. A nonempty subset $F \subseteq P$ is said to be:

1. **upwards (respectively, downwards) closed** if $x \in F$ and $x \prec y$ (respectively, $y \prec x$) imply $y \in F$ for all $x, y \in P$;

2. **downwards directed** if for all $x, y \in F$, there is $z \in F$ with $z \prec x$ and $z \prec y$;

3. a $\prec$-filter if it is nonempty, upwards closed and downwards directed;

4. a $\prec$-ultralfilter if it is a maximal proper $\prec$-filter.

In the case that $(P, \leq)$ is an ordered set and there is no risk of confusion, we will call $\leq$-filters and $\leq$-ultrafilters simply filters and ultrafilters.

Remark. If $P$ is a $\wedge$-semilattice, then a subset $F \subseteq P$ is a filter if and only if $s, t \in F$ implies $s \wedge t \in F$. If $P$ is a poset with zero, then a filter $F \subseteq P$ is proper if and only if $0 \notin F$.

Given a unital Boolean algebra $B$ we associate the collection $\hat{B}$ of ultrafilters on $B$, and endow it with the topology generated by sets $[a] = \{F \in \hat{B} : a \in F\}$, where $a \in B$, which makes it a Stone space (see [164]). These two constructions can be formalized as functors and yield the following duality:

Theorem 1.4.4 (Classical Stone duality). The category $\text{Bool}$ of unital Boolean algebras and their morphisms is dually equivalent to the category $\text{Ston}$ of Stone spaces and continuous functions.

In the rest of this section we will use these ideas in the more general setting of groupoids and inverse semigroups.

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2In Stone’s original work he considers *ideals* instead of filters. These two types of sets are dual to each other and in natural bijection, see Chapters 3 and 5 of [163].
1.4.2 Recovering a locally compact Hausdorff space from its open sets

Let \((X, \tau)\) be a locally compact Hausdorff space, \(B\) a basis of open sets of \(X\) and \(x \in X\). Using the ideas above, we can retrieve a point \(x \in X\) from the collection \(B_x = \{U \in B : x \in U\}\) by taking their intersection: \(\{x\} = \bigcap B_x\), so the problem becomes to determine, in order-theoretic terms, which subsets of \(B\) have the form \(B_x\) for a certain \(x \in X\).

If the basis \(B\) is not closed under unions this is not always possible (the ideas in Theorem 3.1.17 can be adapted to provide an example of this). One possible way to solve this is to use an auxiliary relation given by “compact containment”.

**Definition 1.4.5.** Given \(A, B \in \tau\), the compact containment relation \(\sqsubseteq\) is defined by

\[
A \sqsubseteq B \iff \overline{A} \text{ is compact and } \overline{A} \subseteq B
\]

**Example 1.4.6.** If \(X\) is a zero-dimensional, locally compact Hausdorff space and \(B\) is a basis of compact-open subsets of \(X\), then \(\sqsubseteq\) coincides with \(\subseteq\) on \(B\).

This relation was first studied by Shirota in [158], where he managed to characterize bases consisting of all regular open subsets of compact Hausdorff spaces, as Boolean algebras endowed with an extra auxiliary relation which encodes the properties of “compact containment”.

Similarly to classical Stone duality, we define \(\hat{B}^{\sqsubseteq}\) as the set of all \(\sqsubseteq\)-ultrafilters on \(B\). The following result will be useful in the remaining of this section as well as in Chapter 3. (Compare with [158] Lemma 3. See also [14] Proposition 3.4.)

**Theorem 1.4.7.** Let \(B\) be a basis of relatively compact subsets of a locally compact Hausdorff space \(X\). Then the map \(\pi : X \to \hat{B}^{\sqsubseteq}\) defined by

\[
\pi(x) = B_x = \{U \in B : x \in U\}
\]

is a homeomorphism.

**Proof.** First, we prove that \(\pi\) is well-defined, that is, that \(B_x\) is indeed a \(\sqsubseteq\)-ultrafilter of \(B\). It is immediate to see that \(B_x\) is a proper \(\sqsubseteq\)-filter.

Let us prove that every proper \(\sqsubseteq\)-filter \(F\) of \(B\) is contained in some \(B_x\). Indeed, since \(F\) is downwards directed with respect to \(\sqsubseteq\), this means that for all \(n \geq 1\) and \(A_1, \ldots, A_n \in F\) there is some \(U \in F\) with \(\overline{U}\) compact and \(\overline{U} \subseteq A_1 \cap \cdots \cap A_n\). Then:

(i) Since \(F\) is nonempty, at least one element of \(F\) has compact closure.

(ii) \(\bigcap_{U \in F} \overline{U} = \bigcap_{A \in F} A\).

(iii) The family \(\{A : A \in F\}\) has the finite intersection property, because \(F\) is proper.
By Cantor’s Intersection Theorem, \( \bigcap_{A \in F} A = \bigcap_{U \in F} \overline{U} \) is nonempty. If we take any \( x \in \bigcap_{A \in F} A \) we obtain \( F \subseteq B_x \).

This fact now implies that \( B_x \) is indeed a \( \equiv \)-ultrafilter: If \( F \) is any proper \( \equiv \)-filter containing \( B_x \), then, as seen above, there exists \( y \in X \) with \( B_x \subseteq F \subseteq B_y \). But \( X \) is Hausdorff, so actually \( x = y \) and \( B_x = F \). Therefore \( \pi \) is well-defined.

The same fact implies that \( \pi \) is surjective: If \( F \) is a \( \equiv \)-ultrafilter, we can choose \( x \) such that \( F \subseteq B_x \), and maximality of \( F \) implies \( F = B_x \).

Finally, it is clear that \( \pi(A) = [A] \) for all \( A \in B \), so \( \pi \) is a bijection between topological spaces which sends basic open sets to (sub-)basic open sets, hence a homeomorphism. \( \square \)

### 1.4.3 Non-commutative Stone duality for Boolean semigroups and ample Hausdorff groupoids.

We can now describe a duality theorem for ample Hausdorff groupoids and Boolean semigroups. This was first proved by Lawson and later generalized to non-Hausdorff groupoids in a joint work with Lenz, see [109] and [113] using techniques from the theory of locales. Even more recently, Bice and Starling ([14]) have obtained a duality for groupoids which do not necessarily have bases of compact-open-sets, but which are locally Hausdorff, using a version of Theorem 1.4.7. It is important to note that these two results require distinct assumptions on the classes of groupoids considered, and are non-comparable at the present moment. The definitions below are adapted from [113], however we will refer to generalized Boolean algebras (initially defined in [162]) and unital Boolean algebras explicitly, and we also adopt the more conventional nomenclature of ample instead of Boolean groupoids throughout this work. Most of the results here appear, in some form, in [109], but we will not assume the groupoids under consideration to have compact unit space.

**Definition 1.4.8.** An étale groupoid \( G \) is **ample** if it admits a basis of compact-open bisections.

**Definition 1.4.9.** A **generalized Boolean algebra** is a lattice with zero which is distributive (as a semigroup under meets) and which admits relative complements of any two (comparable) elements. A **unital Boolean algebra** is a generalized Boolean algebra admitting a maximum (in particular, it is a monoid).

**Definition 1.4.10 ([113 3.4]).** A distributive semigroup with zero \( S \) is **weakly Boolean** if \( E(S) \) is a generalized Boolean algebra. A weakly Boolean semigroup which is also a \( \land \)-semilattice is called a **Boolean** semigroup.

If \( B \) is a Boolean inverse semigroup, and \( s, t \in B \), we define the **relative complement** of \( s \) in \( t \) as

\[
t \setminus s = t \setminus (s \land t),
\]

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which extends the notion of relative complement given in Definition 1.2.44 to non-compatible elements (this is the same procedure as in Notation 3.1.11). The same results as in Proposition 1.2.48 hold with easy proofs.

An immediate application of Proposition 1.2.48(a), which allows us to distribute products over relative complements, is the following:

**Proposition 1.4.11.** If $S$ is a weakly Boolean inverse semigroup and $t \leq s$ in $S$, then $s \setminus t$ exists, and $s \setminus t = s(s^*s \setminus tt)$.

First let us prove that every ample Hausdorff groupoid gives rise to a Boolean inverse semigroup.

**Proposition 1.4.12** ([109, Proposition 2.18(7)]). If $G$ is an ample groupoid and $G(0)$ is Hausdorff then the product of compact-open bisections is compact-open.

**Proof.** Suppose $A$ and $B$ are compact-open bisections, and $AB \subseteq \bigcup_{i \in I} C_i$, where $C_i$ are also compact-open bisections. Then $A^{-1}A = s(A)$ and $BB^{-1} = r(B)$ are compact-open subsets of $G(0)$, $A^{-1}ABB^{-1} = AA^{-1} \cap BB^{-1}$ is also open and compact, since compact subsets of $G(0)$ are closed, and $A^{-1}ABB^{-1} \subseteq \bigcup_{i \in I} A^{-1}C_iB^{-1}$. We can then find a finite subset $\{C_1, \ldots, C_n\} \subseteq \{C_i : i \in I\}$ such that $A^{-1}ABB^{-1} \subseteq \bigcup_{i=1}^n A^{-1}C_iB^{-1}$, and thus

$$AB = AA^{-1}ABB^{-1}B \subseteq A \left(\bigcup_{i=1}^n A^{-1}C_iB^{-1}\right) B = \bigcup_{i=1}^n AA^{-1}C_iB^{-1}B \subseteq \bigcup_{i=1}^n C_i.$$  

**Example 1.4.13.** It is not necessarily true that the product of two compact-open bisections of an étale, non-Hausdorff groupoid, is compact-open as well. For example, let $X = \mathbb{N} \cup \{\infty_1, \infty_2\}$, where $\infty_1 \neq \infty_2$ are two “points at infinity”, be the following “bug-eyed” compactification of $\mathbb{N}$: a subset $A \subseteq X$ is open if and only if it is either cofinite or contained in $\mathbb{N}$.

If we see $X$ as a compact, ample, non-Hausdorff unit groupoid (Example 1.1.22), so bisections of $X$ are simply its subsets and the product of bisections is their intersection. Then $A = \mathbb{N} \cup \{\infty_1\}$ and $B = \mathbb{N} \cup \{\infty_2\}$ are compact-open subsets of $X$, however their intersection $A \cap B = \mathbb{N}$ is not compact.

**Definition 1.4.14.** If $G$ is an ample Hausdorff groupoid, we denote by $KB(G)$ the semigroup of compact-open bisections of $G$, and call it the ample semigroup of $G$.

**Proposition 1.4.15** ([109, Proposition 2.18(8)]). If $G$ is an ample Hausdorff groupoid then $KB(G)$ is a Boolean inverse semigroup.

**Proof.** Note that $E(KB(G)) = KB(G(0))$, and this is precisely the collection of compact-open subsets of $G(0)$. Since $G(0)$ is Hausdorff it should be clear that this is a generalized Boolean algebra. Moreover, $G$ is Hausdorff, so $KB(G)$ is closed under intersections and it is therefore a $\Lambda$-semigroup as well. Distributivity is easy enough to verify.
The map $G \mapsto \mathbf{KB}(G)$ will yield one of the directions of the duality we desire. Now let us work in the other direction, constructing an ample Hausdorff groupoid from a Boolean inverse semigroup $S$.

Let $S$ be a Boolean inverse semigroup. Given any two (possibly non-proper) filters $F$ and $G$ on $S$, define their product as

$$F \cdot G = \{ u \in S : u \geq fg \text{ for some } f \in F, g \in G \}$$

(this is the upwards closure of $FG = \{ fg : f \in F, g \in G \}$), and define

$$F^{-1} = F^* = \{ f^* : f \in F \}.$$

The inversion map $I : s \mapsto s^*$ is an order isomorphism of $S$, so $I(F) = F^{-1}$ is a filter (respectively, proper filter, ultrafilter) on $S$ if and only if $F$ is a filter (respectively, proper filter, ultrafilter) on $S$.

**Definition 1.4.16.** Given a Boolean inverse semigroup $S$, denote by $\mathcal{G}_P(S)$ the set of ultrafilters on $S$ (with respect to the usual order of $S$).

We also define

$$\mathcal{G}_P(S)^{(2)} = \{ (F, G) \in \mathcal{G}_P(S) \times \mathcal{G}_P(S) : F \cdot G \neq S \}.$$

Note that, by the definition of $F \cdot G$, we have $0 \in F \cdot G$ if and only if there are $f \in F$ and $g \in G$ such that $fg = 0$. Therefore,

$$\mathcal{G}_P(S)^{(2)} = \{ (F, G) \in \mathcal{G}_P(S) \times \mathcal{G}_P(S) : fg \neq 0 \text{ for all } f \in F \text{ and } g \in G \}.$$

**Proposition 1.4.17** (Compare with [109, Proposition 2.13]). Let $S$ be a Boolean inverse semigroup.

(a) If $F, G$ are filters, then $F \cdot G$ is also a filter;

(b) If $F, G, H$ are filters on $S$, then $F \cdot (G \cdot H) = (F \cdot G) \cdot H$ (and as usual we denote this common filter by $F \cdot G \cdot H$);

(c) If $F \in \mathcal{G}_P(S)$ then $F^{-1} \in \mathcal{G}_P(S)$;

(d) If $F \in \mathcal{G}_P(S)$ and $G$ is a filter such that $fg \neq 0$ for all $f \in F$ and $g \in G$, then $F \cdot G \cdot G^{-1} = F$.

(e) If $F \in \mathcal{G}_P(S)$ and $G$ is a filter such that $fg \neq 0$ for all $f \in F$ and $g \in G$, then $F \cdot G \in \mathcal{G}_P(S)$;

(f) $\mathcal{G}_P(S)$ is a groupoid with the product of ultrafilters restricted to $\mathcal{G}_P(S)^{(2)}$. 

Proof. (a) As $F \cdot G$ is upwards closed, we simply need to prove it is directed below:
Suppose $u_1, u_2 \in F \cdot G$, and choose $f_1, f_2 \in F$, $g_1, g_2 \in G$ with $f_i g_i \leq u_i$. Then
$$ (f_1 \land f_2)(g_1 \land g_2) \leq u_1 \land u_2 $$
so $u_1 \land u_2 \in F \cdot G$. Therefore $F \cdot G$ is a filter.

(b) and (c) are clear.

(d) Suppose $F$ and $G$ satisfy the given properties. If $f \in F$ and $g \in G$ then $f gg^* \leq f$, so $f \in F \cdot G \cdot G^{-1}$. This proves that $F \subseteq F \cdot G \cdot G^{-1}$. If we show that $F \cdot G \cdot G^{-1}$ is a proper filter then maximality of $F$ implies that $F = F \cdot G \cdot G^{-1}$.

Given $u \in F \cdot G \cdot G^{-1}$, choose $f \in F$ and $g_1, g_2 \in G$ with $f g_1 g_2^{-1} \leq u$. Let $g = g_1 \land g_2 \in G$. Since $0 \neq f g = f g g^{-1} g$, we have $0 \neq f g g^{-1} \leq u$, thus $F \cdot G \cdot G^{-1}$ is proper.

(e) Suppose $P$ is any proper filter containing $F \cdot G$. Let us prove that $P \cdot G^{-1}$ is proper as well: Given $p \in P$ and $g \in G$, let $f$ be any element of $F$. Choose $q \in P$ with $q \leq f g, p$, so $0 \leq q = q q^* q \leq p g^* f^* q$. In particular, $p g^* \not= 0$.

We then obtain $F = F \cdot G \cdot G^{-1} \subseteq P \cdot G^{-1}$, so $F = P \cdot G^{-1}$ because $F$ is maximal. Therefore $P \subseteq P \cdot G^{-1} \cdot G = F \cdot G$.

(f) follows from items (b), (c), (d), (e), and the simple fact that $(F, F^{-1}) \in \mathcal{G}_P(S)^{\langle 2 \rangle}$ for all $F \in \mathcal{G}_P(S)$. (Item (e) in fact implies that $F \cdot G \in \mathcal{G}_P(S)$ whenever $(F, G) \in \mathcal{G}_P(S)^{\langle 2 \rangle}$.)

Now, we endow $\mathcal{G}_P(S)$ with the topology generated by the family of subsets of the form
$$ [s] = \{ F \in \mathcal{G}_P(S) : s \in F \}, \quad s \in S. $$

Our next goal is to prove that this topology makes $\mathcal{G}_P(S)$ an ample Hausdorff groupoid, but for this we need a lemma, which is well-known and widely used when dealing with Boolean algebras, and in fact the proof is essentially the same.

**Lemma 14.18.** A proper filter $F$ on a Boolean semigroup $S$ is an ultrafilter if and only if it is prime: If $s \lor t \in F$, then $s \in F$ or $t \in F$.

**Proof.** Suppose $F$ is an ultrafilter and $s \lor t \in F$. There are two possibilities:

**Case 1:** There is some $f \in F$ with $f \land t = 0$.

Then $f \land (s \lor t) = (f \land s) \lor (f \land t) = f \land s \in F$, which implies $s \in F$;

**Case 2:** For every $f \in F$, $f \land t \not= 0$.

Then the set $\{ u \in S : u \geq f \land t \text{ for some } f \in F \}$ is a proper filter containing $F$, so maximality implies it is equal to $F$. In particular, if $f \in F$ is arbitrary then $t \geq f \land t$ and therefore $t \in F$;
In the other directions, suppose $F$ is prime and $G$ is another proper filter containing $F$. Let $g \in G$ and $f \in F$. Then $f = (f \land g) \lor (f \setminus g)$. Since $g \land (f \setminus g) = 0$, $f \setminus g$ does not belong to $G$ and in particular does not belong to $F$, so primeness of $F$ implies $f \land g \in F$ and therefore $g \in F$. This proves $F = G$ and therefore $F$ is an ultrafilter.

Recall from definition 1.2.29 that two elements $s, t$ of an inverse semigroup $S$ are compatible if $s \ast t, st \ast \in E(S)$.

**Lemma 1.4.19** ([109, Lemma 2.21(2)-(8)]). Let $S$ be a Boolean inverse semigroup. Then for all $s, t \in S$,

(a) $[st] = [s][t]$;

(b) $s \leq t$ if and only if $[s] \subseteq [t]$;

(c) $[s \land t] = [s] \cap [t]$;

(d) If $s$ and $t$ are compatible then $[s \lor t] = [s] \cup [t]$;

(e) $[s^*] = [s]^{-1}$ for all $s \in S$;

(f) If $s, t \in S$ and $[s] \cup [t]$ is a bisection then $s$ and $t$ are compatible.

**Proof.**

(a) The inclusion $[s][t] \subseteq [st]$ is clear from the definition of the product on $\mathcal{G}_P(S)$. Suppose then that $P \in [st]$, and let $Q = \{ q \in S : q \geq t^* \}$. Then $Q$ is a filter and $pq \neq 0$ for any $p \in P$ and $q \in Q$, so Proposition 1.4.17(e) implies that $F = P \cdot Q \in \mathcal{G}_P(S)$. Again, one readily verifies that $f^*p \neq 0$ whenever $f \in F$ and $p \in P$, so $G = F^{-1} \cdot P \in \mathcal{G}_P(S)$ again by Proposition 1.4.17(e) and (c). Using Proposition 1.4.17(d) and (c) we get

$$F \cdot G = F \cdot F^{-1}P = P$$

and in particular $(F, G) \in \mathcal{G}_P(S)^{(2)}$.

(b) Since filters are upwards closed then $s \leq t$ implies $[s] \subseteq [t]$. Conversely, if $s$ is not smaller than $t$ this means that $s \setminus t \neq 0$. Any ultrafilter $F$ containing $s \setminus t$ (which exists by Zorn’s lemma) will be an element of $[s] \setminus [t]$.

(c) The inclusion $\subseteq$ follows from item (b), whereas inclusion $\supseteq$ follows from Remark 1.4.1.

(d) Inclusion $\supseteq$ follows from item (b), and inclusion $\subseteq$ follows from Lemma 1.4.18.

(e) is an easy calculation using the definition of the inverse of ultrafilters:

$$[s]^{-1} = \{ F^{-1} : F \in [s] \} = \{ F^{-1} : s \in F \in \mathcal{G}_P(S) \} = \{ G : s^* \in G \in \mathcal{G}_P(S) \} = [s^*]$$
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(f) Suppose $[s] \cup [t]$ is a bisection. Then

$$[s^*t] = [s]^{-1}[t] \subseteq ([s] \cup [t])^{-1}([s] \cup [t]) = s ([s] \cup [t]) \subseteq \mathcal{G}_P(S)^{(0)}$$

and from this we conclude that $[s^*t]$ itself is a bisection, so (a) and (e) imply

$$[s^*t] = [s^*t]^{-1}[s^*t] = [(s^*t)^*(s^*t)]$$

and item (b) implies $s^*t = (s^*t)^*(s^*t)$, therefore $s^*t \in E(S)$ (Theorem $1.2.22(c)$).

The same argument with the bisection $([s] \cup [t])^{-1} = [s^*] \cup [t^*]$ implies $st^* \in E(S)$ as well, and therefore $s$ and $t$ are compatible. \qed

**Proposition 1.4.20 (1.109, Lemma 2.21 (12))**. $\mathcal{G}_P(S)$ is an ample Hausdorff groupoid and $\text{KB} (\mathcal{G}_P(S)) = \{ [s] : s \in S \}$.

**Proof.** The inversion on $\mathcal{G}_P(S)$ is continuous, since $[s]^{-1} = [s^*]$ for all $s \in S$. To prove that the product is continuous, suppose $(F, G) \in \mathcal{G}_P(S)^{(2)}$ and $[s], s \in S$, is a basic neighbourhood of $F \cdot G$. This means that $s \in F \cdot G$, so that there are $f \in F$ and $g \in G$ with $fg \leq s$. Therefore $[f]$ and $[g]$ are basic neighbourhoods of $F$ and $G$, respectively, and $[f][g] = [fg] \subseteq [s]$ by Lemma $1.4.19(a)$ and (b).

In order to prove that $\mathcal{G}_P(S)$ is étale, we first show that

$$\mathcal{G}_P(S)^{(0)} = \{ P \in \mathcal{G}_P(S) : P \cap E(S) \neq \emptyset \}$$

From this we obtain $\mathcal{G}_P(S)^{(0)} = \bigcup_{e \in E(S)} [e]$, which is open.

Suppose $P \in \mathcal{G}_P(S)^{(0)}$, which means that $P = P^{-1} \cdot P$. Letting $s \in P$ be arbitrary, we have $s^*s \in (P^{-1} \cdot P) \cap E(S) = P \cap E(S)$.

Conversely, suppose $P \in \mathcal{G}_P(S)$ and $P \cap E(S) \neq \emptyset$. Let $e \in P \cap E(S)$ be arbitrary. For all $p \in P$, we have $p \wedge e \in E(S) \cap P$, so

$$(p \wedge e)^*(p \wedge e) = p \wedge e \leq p$$

which proves that $p \in P^{-1} \cdot P$, that is, $P \subseteq P^{-1}P$. From maximality of $P$ we obtain $P = P^{-1} \cdot P \in \mathcal{G}_P(S)^{(0)}$.

To conclude that $\mathcal{G}_P(S)$ is étale, we notice that the product of open sets is open by Lemma $1.4.19(a)$, and since $\mathcal{G}_P(S)^{(0)}$ then $\mathcal{G}_P(S)$ is étale by Theorem $1.3.11$.

Now to show that $\mathcal{G}_P(S)$ is ample, we prove that every basic open set $[s], s \in S$, is compact: Suppose $[s] = \bigcup_{r \in R} [r]$. By taking intersections and applying Lemma $1.4.19(c)$ if necessary, we can assume $r \leq s$ for all $r \in R$, and in particular any two elements of $R$ are compatible. We will show that there is a finite subset $F \subseteq R$ such that $s = \sqrt{F}$.

Suppose this was not the case. Then the family $\mathcal{B} = \{ s \setminus \sqrt{F} : F \subseteq R \text{ finite} \}$ is a filter basis, that is, it is closed under meets and does not contain 0, so its upwards closure

$$\mathcal{B}^\uparrow = \{ t \in S : \exists b \in \mathcal{B} (b \leq t) \}$$
is a proper filter. Using Zorn’s Lemma, let \( F \) be any ultrafilter containing \( \mathcal{B} \). Then \( F \) is an ultrafilter containing \( s \) which does not contain any \( r \in R \), that is, \( F \in [s] \setminus \bigcup_{r \in R} [r] \), a contradiction.

Therefore there is a finite subset \( F \subseteq R \) with \( s = \bigvee F \), so \( [s] = \bigcup_{r \in F} [r] \) by Lemma 1.4.19(d), that is, \( \{ [r] : r \in F \} \) is a finite subcover of \( \{ [s] : r \in R \} \), and we conclude that \( [s] \) is compact. All of this proves that \( \mathcal{G}_P(S) \) is ample.

In order to prove that \( [s] \) is a bisection let us just show that the source map \( F \mapsto F^{-1} \cdot F \) is injective on \( [s] \), since the range is dealt with similarly: Suppose \( F, G \in [s] \) and \( F^{-1} \cdot F = G^{-1} \cdot G \). Given \( f \in F \), let \( z = f \wedge s \in F \), so \( z^*z \in F^{-1} \cdot F = G^{-1} \cdot G \) and we can choose \( g \in G \) with \( z^*z \geq g^*g \), thus

\[
f \geq z = sz^*z \geq sg^*g \in G \cdot G^{-1} \cdot G = G
\]

so \( F \subseteq G \) and therefore \( F = G \) by maximality.

Now let us show that \( \mathcal{G}_P(S) \) is Hausdorff: Suppose \( F \neq G \) in \( \mathcal{G}_P(S) \). By symmetry, we may assume \( F \setminus G \neq \emptyset \), so let \( s \in F \setminus G \) and \( g \in G \) be an arbitrary element.

On one hand, \( s \wedge g \not\in G \), but \( g = (s \wedge g) \lor (g \setminus s) \), so since \( \mathcal{G} \) is prime this implies \( g \setminus s \in \mathcal{G} \). Therefore \( [s] \) and \( [g \setminus s] \) are two disjoint open sets containing \( F \) and \( G \), respectively.

Finally, we just need to prove that \( KB(\mathcal{G}_P(S)) = \{ [s] : s \in S \} \). Any element of \( KB(\mathcal{G}_P(S)) \) can be written as a finite union \( \bigcup_{i=1}^n [s_i] \) of basic open sets, so applying Lemma 1.4.19(f) we conclude that the elements \( s_i \) are pairwise compatible rewrite it as \( \bigcup_{i=1}^n [s_i] = \bigvee_{i=1}^n s_i \) by item (d) of the same lemma. 

Given an ample Hausdorff groupoid \( \mathcal{G} \), let \( \kappa_\mathcal{G} : \mathcal{G} \to \mathcal{G}_P(KB(\mathcal{G})) \) be given by

\[
\kappa_\mathcal{G}(a) = \{ A \in KB(\mathcal{G}) : a \in A \}
\]

(note this is well-defined by Lemma 1.4.18 because \( \kappa_\mathcal{G}(g) \) is a prime filter) and given a Boolean semigroup \( S \), let \( \zeta_S : S \to KB(\mathcal{G}_P(S)) \) be given by

\[
\zeta_S(s) = \{ F \in \mathcal{G}_P(S) : s \in F \}
\]

**Proposition 1.4.21 ([109], Proposition 2.23).** If \( S \) is a Boolean inverse semigroup then \( \zeta_S \) is an inverse semigroup isomorphism. If \( \mathcal{G} \) is an ample Hausdorff groupoid, then \( \kappa_\mathcal{G} \) is a topological groupoid isomorphism.

**Proof.** We already know that \( \zeta_S \) is an inverse semigroup isomorphism by Proposition 1.4.20 and Lemma 1.4.19(a).

On the other hand, \( \kappa_\mathcal{G} \) is a homeomorphism by Theorem 1.4.7. To see that it is a groupoid morphism, let \((a, b) \in \mathcal{G}^{(2)}\). If \( A \in \kappa_\mathcal{G}(a) \) and \( B \in \kappa_\mathcal{G}(b) \) then \( ab \in AB \), so in particular \( AB \neq \emptyset \) which proves, by Proposition 1.4.17(e) that \( (\kappa_\mathcal{G}(a), \kappa_\mathcal{G}(b)) \in \mathcal{G}_P(KB(\mathcal{G}))^{(2)} \).

The inclusion \( \kappa_\mathcal{G}(a) \cdot \kappa_\mathcal{G}(b) \subseteq \kappa_\mathcal{G}(ab) \) is also clear from the argument above, so the fact that both \( \kappa_\mathcal{G}(a) \cdot \kappa_\mathcal{G}(b) \) and \( \kappa_\mathcal{G}(ab) \) are ultrafilters on \( KB(\mathcal{G}) \) implies that \( \kappa_\mathcal{G}(a) \cdot \kappa_\mathcal{G}(b) = \kappa_\mathcal{G}(ab) \). 

\[ \square \]
The only part that is left is to define the appropriate morphisms of Boolean inverse semigroups and ample Hausdorff groupoids.

**Definition 1.4.22.** A morphism of Boolean inverse semigroups is an inverse semigroup morphism \( \theta : S \rightarrow T \) which is both a \( \wedge \) and a \( \vee \)-morphism, and which satisfies \( \theta(0) = 0 \). We say that \( \theta \) is proper if for all \( t \in T \), there are \( t_1, \ldots, t_n \in T \) and \( s_1, \ldots, s_n \in S \) such that

\[
 t = \bigvee_{i=1}^{n} t_i \quad \text{and} \quad t_i \leq \theta(s_i) \quad \text{for all } i.
\]

(Note that semigroup isomorphisms of Boolean inverse semigroups are proper morphisms of Boolean inverse semigroups.)

**Remark.** Every morphism of Boolean inverse semigroups \( \theta : S \rightarrow T \) preserves relative complements, since if \( s, t \in S \) then

\[
 \theta(s) \leq \theta(t \vee (s \setminus t)) = \theta(t) \vee \theta(s \setminus t)
\]

and

\[
 \theta(t) \wedge \theta(s \setminus t) \theta(t \wedge (s \setminus t)) = \theta(0) = 0
\]

and this means precisely that \( \theta(s \setminus t) = \theta(s) \setminus \theta(t) \).

**Example 1.4.23.** A morphism of Boolean inverse monoids \( \theta : S \rightarrow T \) is proper if and only if \( \theta(1) = 1 \).

**Definition 1.4.24.** A groupoid morphism \( \phi : \mathcal{G} \rightarrow \mathcal{H} \) is called

1. **star-injective** if \( s(a) = s(b) \) and \( \phi(a) = \phi(b) \) implies \( a = b \);
2. **star-surjective** if whenever \( s(h) = \phi(x) \) where \( x \in \mathcal{G}^{(0)} \), there is \( a \in \mathcal{G} \) with \( s(a) = x \) and \( \phi(a) = h \).
3. a **covering** if it is both star-injective and surjective.

(Note that groupoid isomorphisms are coverings.)

Recall that a map \( \phi : X \rightarrow Y \) between topological spaces \( X \) and \( Y \) is proper if whenever a subset \( K \subseteq Y \) is compact, the preimage \( \phi^{-1}(K) \subseteq X \) is compact as well.

**Proposition 1.4.25** ([109] Propositions 2.20 and 2.22). (a) If \( \phi : \mathcal{G} \rightarrow \mathcal{H} \) is a proper continuous covering morphism of ample Hausdorff groupoids, then

\[
 KB(\phi) : KB(\mathcal{H}) \rightarrow KB(\mathcal{G}), \quad A \mapsto \phi^{-1}(A),
\]

is a proper morphism of Boolean inverse semigroups.
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(b) If \( \theta : S \rightarrow T \) is a proper morphism of Boolean inverse semigroups, then

\[
\mathcal{G}_P(\theta) : \mathcal{G}_P(T) \rightarrow \mathcal{G}_P(S), \quad F \mapsto \theta^{-1}(F),
\]

is a proper continuous covering morphism of topological groupoids.

Proof. (a) If \( A \in \mathbf{KB}(\mathcal{H}) \) then \( \phi^{-1}(A) \) is open, since \( \phi \) is continuous, and compact, since \( \phi \) is proper, so in order to prove that \( \mathbf{KB}(\phi) \) is well-defined it remains to prove that \( \phi^{-1}(A) \) is a bisection.

Suppose \( g, h \in \phi^{-1}(A) \) with \( s(g) = s(h) \). Then \( \phi(g), \phi(h) \in A \), and

\[
s(\phi(g)) = \phi(s(g)) = \phi(s(h)) = s(\phi(h)),
\]

so \( \phi(g) = \phi(h) \) because \( A \) is a bisection, and then \( g = h \) because \( \phi \) is star-injective. This prove that the source map is injective on \( A \), and the range map is dealt with similarly, therefore \( \phi^{-1}(A) \in \mathbf{KB}(\mathcal{G}) \).

To prove that \( \mathbf{KB}(\phi) \) is a morphism of inverse semigroups, let \( A, B \in \mathbf{KB}(\mathcal{H}) \). Suppose \( g \in \phi^{-1}(AB) \), so \( \phi(g) \in A \) and we can write \( \phi(g) = ab \) for certain \( a \in A \) and \( b \in B \). Since \( \phi \) is star-surjective, we have \( b = \phi(h) \) for some \( h \in \mathcal{G} \) with \( s(h) = s(g) \), and so the product \( gh^{-1} \) is well-defined and \( a = \phi(gh^{-1}) \). This proves that \( g = (gh^{-1})h \), where \( gh^{-1} \in \phi^{-1}(A) \) and \( h \in \phi^{-1}(B) \). We conclude that \( \phi^{-1}(AB) \subseteq \phi^{-1}(A)\phi^{-1}(B) \). The converse inclusion is trivial because \( \phi \) is a morphism of groupoids. Moreover, \( \mathbf{KB}(\phi) \) preserves empty sets (the zeroes of the semigroups), unions and intersections and therefore it is a Boolean inverse semigroup morphism.

It remains only to check that \( \mathbf{KB}(\phi) \) is proper: If \( C \in \mathbf{KB}(\mathcal{G}) \), then \( \phi(C) \) is a compact subset of \( \mathcal{H} \), so we can find finitely many \( A_1, \ldots, A_n \in \mathbf{KB}(\mathcal{H}) \) such that \( \phi(C) \subseteq \bigcup_{i=1}^{n} A_i \), that is, \( C \subseteq \bigcup_{i=1}^{n} \phi^{-1}(A_i) \). We already know that \( \phi^{-1}(A_i) \in \mathbf{KB}(\mathcal{G}) \), so taking \( C_i = C \cap \phi^{-1}(A_i) \in \mathbf{KB}(\mathcal{G}) \) we conclude that

\[
C = \bigcup_{i=1}^{n} C_i \quad \text{and} \quad C_i \subseteq \phi^{-1}(A_i) = \mathbf{KB}(\phi)(A_i) \quad \text{for all } i
\]

which means that \( \mathbf{KB}(\phi) \) is proper.

(b) To prove that \( \mathcal{G}_P(\theta) \) is well-defined, suppose that \( F \) is a prime filter in \( T \). First of all, let us prove that \( \theta^{-1}(F) \) is nonempty. Letting \( t \in F \) be arbitrary, choose \( t_1, \ldots, t_n \in T \) and \( s_1, \ldots, s_n \in S \) with

\[
t = \bigvee_{i=1}^{n} t_i \quad \text{and} \quad t_i \leq \theta(s_i) \quad \text{for all } i
\]
which is possible because $\theta$ is proper. For some $i$ we have $t_i \in F$ and so $\theta(s_i) \in F$, that is, $s_i \in \theta^{-1}(F)$, which is therefore nonempty. The properties of Boolean inverse semigroup morphisms then imply, quite readily, that $\theta^{-1}(F)$ is a prime filter in $S$ whenever $F$ is a prime filter in $T$, so $\mathcal{G}_\theta(\theta)$ is well-defined. Moreover, for all $s \in S$,

$$
\mathcal{G}_\theta(\theta)^{-1}([s]) = \left\{ F \in \mathcal{G}_\theta(T) : \theta^{-1}(F) \in [s] \right\} \\
= \left\{ F \in \mathcal{G}_\theta(T) : s \in \theta^{-1}(F) \right\} \\
= \left\{ F \in \mathcal{G}_\theta(T) : \theta(s) \in F \right\} = [\theta(s)],
$$

so the preimage of basic compact-open sets of $\mathcal{G}_\theta(S)$ is open in $\mathcal{G}_\theta(T)$ and $\mathcal{G}_\theta(\theta)$ is continuous and proper.

Suppose now that $(F, G) \in \mathcal{G}_\theta(T)^{(2)}$, and let us prove that $(\theta^{-1}(F), \theta^{-1}(G)) \in \mathcal{G}_\theta(S)^{(2)}$, or equivalently that whenever $t \in \theta^{-1}(F)$ and $r \in \theta^{-1}(G)$, we have $tr \neq 0$. Indeed, $\theta(tr) = \theta(t)\theta(r) \in FG$, which does not contain $0$, and so $tr \neq 0$ because $\theta(0) = 0$. The inclusion $\theta^{-1}(F) \cdot \theta^{-1}(G) \subseteq \theta^{-1}(FG)$ is then clear, and since both sides are ultrafilters we conclude that $\theta^{-1}(F) \cdot \theta^{-1}(G) = \theta^{-1}(FG)$, so $\mathcal{G}_\theta(\theta)$ is a groupoid morphism.

In order to prove that $\mathcal{G}_\theta(\phi)$ is star-injective, suppose $F, G \in \mathcal{G}_\theta(T)$ are such that $s(F) = s(G)$ and $\theta^{-1}(F) = \theta^{-1}(G)$. Given an arbitrary element $t \in F$, properness of $\theta$ again yields $t_i \leq t$ and $s_i \in S$ such that $t_i \leq \theta(s_i)$, so $s_i \in \theta^{-1}(F) = \theta^{-1}(G)$, so $\theta(s_i) \in G \cap F$. Letting $z = \theta(s_i) \wedge t$, we have $z^*z \in s(F) = s(G)$, so

$$
z = \theta(s_i)z^*z \in F \cap G
$$

so $t \geq z \in G$. Therefore $F \subseteq G$ and maximality implies $F = G$.

It remains only to prove that $\mathcal{G}_\theta(\phi)$ is star-surjective. Suppose $E \in \mathcal{G}_\theta(T)^{(0)}$, $H \in \mathcal{G}_\theta(S)$ and $s(H) = \theta^{-1}(E)$. If $h \in H$ and $e \in E$, then $\theta(h)^*\theta(h) = \theta(h^*h)\sin \theta(s(H)) = E$, which implies $\theta(h)^*\theta(h)e \neq 0$ and in particular $\theta(h)e \neq 0$. Letting

$$
H' = \{ u \in T : u \geq \theta(h) \text{ for some } h \in H \}
$$

we obtain, by Proposition 1.4.17(e), that $H' \cdot E \in \mathcal{G}_\theta(T)$. Since $s(H' \cdot E) \supseteq s(E) = E$, then maximality of $E$ implies $s(H' \cdot E) = E$. Again using $s(H) = \theta^{-1}(E)$ and maximality of $H$ we obtain $H = \theta^{-1}(H' \cdot E)$.  

Finally, it is straightforward to verify that covering groupoid morphisms are closed under composition, and similarly for proper morphisms of Boolean inverse semigroups, so we define $\textbf{AmpBo}$ as the category of ample Hausdorff groupoids and proper continuous covering morphisms, and $\textbf{InvBo}$ the category of Boolean inverse semigroups and their proper morphisms.
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Theorem 1.4.26 ([13, Theorem 3.25]). $\text{Amp}_H$ and $\text{Inv}_{\text{Bo}}$ are dually equivalent.

Proof. This is done by noticing that the functors $\text{KB} : \text{Amp}_H \to \text{Inv}_{\text{Bo}}$ and $\mathcal{G}_p : \text{Inv}_{\text{Bo}} \to \text{Amp}_H$ along with the natural isomorphisms $\kappa$ and $\zeta$ of Proposition 1.4.21 describe a duality, which is an easy calculation: For example, if $\phi : G \to G$ is a proper continuous morphism of topological groupoids and $a \in G$ then

$$\mathcal{G}_p(\text{KB}(\phi))(\kappa_G(a)) = \text{KB}(\phi)^{-1}(\kappa_G(a))$$

$$= \{ A \in \text{KB}(H) : \text{KB}(\phi)(A) \in \kappa_G(a) \}$$

$$= \{ A \in \text{KB}(H) : \phi^{-1}(A) \in \kappa_G(a) \}$$

$$= \{ A \in \text{KB}(H) : a \in \phi^{-1}(A) \}$$

$$= \{ A \in \text{KB}(H) : \phi(a) \in A \} = \kappa_H(\phi(g))$$

which means that the diagram

$$\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\phi} & \mathcal{H} \\
\kappa_G \downarrow & & \kappa_H \\
\mathcal{G}_p(\text{KB}(G)) & \xrightarrow{\mathcal{G}_p(\text{KB}(\phi))} & \mathcal{G}_p(\text{KB}(H))
\end{array}$$

commutes, and this proves that $\kappa$ is a natural isomorphism. $\zeta$ is dealt with similarly.

1.5 Quotients of distributive semigroups

Our goal in this Section is to define quotients of Boolean inverse monoids. In general, an ideal of an inverse semigroup $S$ is a subset $I \subseteq S$ which satisfies $SI \cup IS \subseteq I$. The quotient $S/I$ could be defined as the quotient semigroup of $S$ by the smallest congruence which makes all elements of $I$ equivalent, in which case we obtain the Rees factor of $S$ by $I$ ([145]) and the congruence is given by collapsing all elements of $I$ into a zero element, and nothing else:

$$s \sim t \iff s = t \text{ or } \{s, t\} \subseteq I$$

However, the Rees factor does not preserve the order-theoretic operations of inverse semigroups.

Example 1.5.1. Let $X = \{a, b\}$ be a set with two elements and $S = \mathcal{P}(X)$ the lattice of subsets of $X$. Consider the ideal $I = \{\{a\}, \emptyset\}$, and let $T$ be the Rees factor of $S$ by $I$ and $\pi : S \to T$ the quotient map.

Then $\{a\} \lor \{b\} = X$, however $\pi(\{a\}) \lor \pi(\{b\}) = \pi(\{\{b\}\})$, which is not $\pi(X)$. 
Therefore we need to consider more specific classes of ideals and a more suitable congruence. We will be interested in $\lor$-ideals, as defined in [106]. (These are called $\lor$-closed ideals in [105] [111] [112], tightly closed in [110] and additive in [173].)

**Definition 1.5.2.** A $\lor$-ideal (read join ideal) of a distributive inverse semigroup $S$ is a nonempty subset $I \subseteq S$ satisfying

(i) For all $s \in S$ and $i \in I$, $si \in I$ and $is \in I$ (i.e., $I$ is absorbing);

(ii) For any two compatible elements $i, j \in I$, we have $i \lor j \in I$.

Quotients of (weakly) Boolean inverse semigroups, under additive congruences, are described in [173, Section 3.4], and in particular their quotients by $\lor$-ideals in [173, Proposition 3.4.6]. Although this is the case of most interest, and in which most proofs can be significantly simplified, we initially consider the larger class of distributive inverse semigroups.

Given a $\lor$-ideal $I$ in $S$, define the relation $\sim_I$ on $S$ by

$$a \sim_I b \iff \exists s \in S, \exists i, j \in I (a = s \lor i \text{ and } b = s \lor j)$$

**Lemma 1.5.3.** If $S$ is a $\lor$-ideal of a distributive inverse semigroup $S$, then

(a) If $S$ has a zero then $0 \in I$;

(b) $i \in I$ if and only if $i^* \in I$;

(c) If $i \in I$ and $j \leq i$ then $j \in I$;

(d) $a \sim_I b$ if and only if $a^* \sim_I b^*$;

(e) If $a = b \lor i$ and $i \in I$ then $a \sim_I b$.

**Proof.**

(a) Letting $i \in I$ be arbitrary (because $I \neq \varnothing$), we have $0 = 0i \in I$.

(b) If $i \in I$ then $i^* = ii^* \in I$.

(c) If $i \in I$ and $j \leq i$ then $j = ij^*j \in I$, because $I$ is absorbing.

(d) If $a \sim_I b$, choose $s \in S$ and $i, j \in I$ with $a = s \lor i$, $b = s \lor j$, so that $a^* = s^* \lor i^*$ and $b^* = s^* \lor j^*$, and thus $a^* \sim_I b^*$ by item (b).

(e) If $a = b \lor i$ with $i \in I$, then since $b = b \lor bi^*i$ and $bi^*i \in I$ we conclude that $a \sim_I b$.

**Theorem 1.5.4.** If $I$ is a $\lor$-ideal of a distributive inverse semigroup $S$, then $\sim_I$ is a congruence.
Proof. To see that the relation $\sim_I$ is reflexive, take $i \in I$ an arbitrary element, so that for any $a \in S$ we have $a = a \lor ai \lor i$ and $ai \lor i \in I$, therefore $a \sim_I a$. Moreover $\sim_I$ is immediately symmetric from its definition, so in order to check that it is an equivalence relation, suppose $a \sim_I b$ and $b \sim_I c$, so there are $s, t \in S$ and $i, j, k, l \in I$ with

$$a = s \lor i, \quad b = s \lor j = t \lor k, \quad c = t \lor l.$$  

In particular, $s, t \leq b$ so $s$ and $t$ are compatible and $s \land t = st \lor t$. Also, $i, sk \lor k \leq a$, so $i$ and $sk \lor k$ are compatible and belong to $I$ (since it is an ideal) and hence $i \land sk \lor k \in I$.

Using distributivity, we obtain

$$a = s \lor i = sb \lor i = s(t \lor k \lor i) \lor i = (s \lor t) \lor (sk \lor k \lor i)$$

and similarly, $l \lor tj \lor j \in I$ and $c = (s \lor t) \lor (l \lor tj \lor j)$, which proves $a \sim_I c$ and so $\sim_I$ is an equivalence relation.

To check that $\sim_I$ is a congruence, first note that if $a \sim_I b$ and $c \in S$, then we can write

$$a = s \lor i, \quad b = s \lor j$$

for certain $s \in S$, $i, j \in I$, so by distributivity,

$$ac = sc \lor ic \quad \text{and} \quad bc = sc \lor jc$$

and therefore $ac \sim_I bc$, and similarly $ca \sim_I ba$. This fact and transitivity of $\sim_I$ imply that if $a \sim_I b$ and $c \sim_I d$, then $ac \sim_I bc \sim_I bd$, so $\sim_I$ is a congruence. \qed

Definition 1.5.5. If $I$ is a $\lor$-ideal of a distributive inverse semigroup $S$, we denote by $S/I$ the quotient semigroup of $S$ by $\sim_I$. The $\sim_I$-class of an element $s \in S$ will be denoted $\tilde{s}^I$. We denote by $\pi_I : S \to S/I$ the quotient map.

Theorem 1.5.6. Let $I$ be a $\lor$-ideal of a distributive inverse semigroup $S$. Then $S/I$ is a distributive semigroup and $(\tilde{a}^I)^* = (a^*)^I$ for all $a \in S$. The quotient map $\pi_I : S \to S/I$ is a $\lor$ and a $\land$-morphism. Moreover, $S/I$ has a zero if and only if $I$ is downwards directed, in which case $I = I^{-1}(0)$.

Proof. Since $\pi_I$ is a surjective semigroup morphism, then Proposition 1.2.25 implies that $S/I$ is an inverse semigroup and $(\tilde{a}^I)^* = (a^*)^I$ for all $a \in S$.

Suppose $a, b \in S$ are compatible. Then $(a \lor b)^I$ is an upper bound of both $\tilde{a}^I$ and $\tilde{b}^I$, so let us prove it is the smallest one. If $\tilde{p}^I \geq \tilde{a}^I, \tilde{b}^I$, where $p \in S$, then we can find $e \in E(S)$, $s \in S$, and $i_1, i_2 \in I$ satisfying

$$pe = s \lor i_1, \quad a = s \lor i_2,$$

and in particular $s \leq p$. Similarly, we can find $t \in S$ and $j_2 \in I$ with $t \leq p$ and $b = t \lor j_2$. Thus $s$ and $t$ are compatible and $s \lor t \leq p$, so

$$a \lor b = s \lor i_2 \lor t \lor j_2 = (s \lor t) \lor (i_2 \lor j_2)$$

and so

$$a \lor b = s \lor i_2 \lor t \lor j_2 = (s \lor t) \lor (i_2 \lor j_2)$$
which proves \((a \vee b)^I = (s \vee t)^I \leq \tilde{p}^I\).

If \(a \wedge b\) exists, then \((a \wedge b)^I\) is a lower bound of both \(\tilde{a}^I\) and \(\tilde{b}^I\), so to prove it is the largest one suppose \(\tilde{p}^I \leq \tilde{a}^I, \tilde{b}^I\). There are idempotents \(e, f \in E(S)\), elements \(s, t \in S\) and \(i_1, i_2, j_1, j_2 \in I\) with

\[
ae = s \vee i_1, \quad p = s \vee i_2, \quad bf = t \vee j_1, \quad p = t \vee j_2.
\]

Moreover, all of \(s, t, i_2, j_2\) are bounded by \(p\) and are thus compatible and admit pairwise meets and joins, and combinations thereof, so we can apply Proposition 1.2.42 to distribute meets over joins and obtain

\[
p = (s \vee i_2) \wedge (t \vee j_2) = (s \wedge t) \vee k
\]

for an appropriate \(k \in I\). This implies \(\tilde{p}^I = (s \wedge t)^I \leq (a \wedge b)^I\).

If \(I\) is downwards directed, then any two elements of \(I\) are \(\sim_I\)-equivalent (to any common lower bound), and since \(I\) is an ideal this implies that the class \(\tilde{i}^I\) of any \(i \in I\) is the zero of \(S/I\). This in fact proves that, in this case, \(I \subseteq \pi_I^{-1}(0)\).

Conversely, suppose \(\tilde{o}^I\) is a zero in \(S/I\), where we may assume \(o \in E(S)\), and let \(a, b \in I\). Since \((ao)^I = \tilde{o}^I = (bo)^I\), there are \(s, t \in S\) and \(i_1, i_2, j_1, j_2 \in I\) with

\[
ao = s \vee i_1, \quad o = s \vee i_2, \quad bo = t \vee j_1, \quad \text{and } o = t \vee j_2.
\]

In particular, \(s \leq ao\) so \(s \in I\). From \(s, t \leq o\) we see that \(s\) and \(t\) are compatible, \(s \wedge t = st^*t \in I\) and \(s \wedge t\) is a lower bound of both \(a\) and \(b\), and so \(I\) is downwards directed. Moreover, \(o = s \vee i_2 \in I\), which proves \(\pi_I^{-1}(0) \subseteq I\) in this case. \(\square\)

**Example 1.5.7.** \(S/I\) is not necessarily a distributive inverse semigroup. Let \(X = \{1, 2, 3, 4, 5, 6, 7\}\), \(a, b \in \mathcal{I}(X)\) be the partial bijections of \(X\) defined as

\[
a(1) = 2, \quad a(4) = 5, \quad b(1) = 3, \quad b(6) = 7
\]

and then consider the subsemigroup \(S \subseteq \mathcal{I}(X)\) of partial bijections \(f : \text{dom}(f) \to \text{ran}(f)\) such that

- if \(4 \in \text{dom}(f)\) then \(a \leq f\) or \(a^*a \leq f\);
- if \(5 \in \text{dom}(f)\) then \(a^* \leq f\) or \(aa^* \leq f\);
- if \(6 \in \text{dom}(f)\) then \(b \leq f\) or \(b^*b \leq f\);
Let us prove that this is a sub-inverse semigroup of $\mathcal{I}(X)$: Suppose $f, g \in S$, and that $4 \in \text{dom}(fg)$. Then $4 \in \text{dom}(g)$. There are two possibilities:

- If $a \leq g$, then $g(4) = a(4) = 5 \in \text{dom}(f)$, and there are two sub-possibilities:
  - if $a^* \leq f$, then $a^* a \leq fg$;
  - if $aa^* \leq f$, then $a = aa^* a \leq fg$;

- If $aa^* \leq g$, then $g(4) = 4 \in \text{dom}(f)$, and there are two sub-possibilities:
  - If $a \leq f$, then $a = aa^* a \leq fg$;
  - If $a^* a \leq f$, then $a^* a = aa^* a \leq fg$;

In any case, either $a \leq fg$ or $a^* a \leq fg$ whenever $4 \in \text{dom}(fg)$. The other cases are dealt with similarly. Also, note that $S$ is closed under inverses of $\mathcal{I}(X)$.

Now we can prove that $S$ is closed under joins of $\mathcal{I}(X)$. Suppose that $f, g \in S$ are compatible, so the join $f \vee g$ exists in $\mathcal{I}(X)$. Again, suppose that $4 \in \text{dom}(f \vee g)$, so without loss of generality let us assume further that $4 \in \text{dom}(f)$. Then either $a \leq f \leq f \vee g$ or $a^* a \leq f \leq f \vee g$. The other cases are dealt with similarly.

Let $I = \{f \in S : \text{dom}(f) \subseteq \{1, 2, 3\}\}$. The restriction map $\theta : S \rightarrow \mathcal{I}(\{4, 5, 6, 7\})$, $\theta(f) = f|_{\text{dom}(f) \cap \{4, 5, 6, 7\}}$, is a morphism and $\theta(f) = \theta(g) \iff f \sim_I g$, so $\theta$ factors through an injective morphism of $S/I$ into $\mathcal{I}(\{4, 5, 6, 7\})$.

However, $\theta(a)$ and $\theta(b)$ are compatible but do not have a common join in $\theta(S)$. Indeed, suppose that $\theta(f) = \theta(a) \vee \theta(b)$, where $f \in S$. Then $f(4) = a(4) = 5$, which implies, by the definition of $S$, that $a \leq f$. However similarly, $f(6) = b(6) = 7$ which implies $b \leq f$, but $a$ and $b$ are not compatible in $S$, a contradiction.

This proves that $\theta(S)$ - and therefore $S/I$ - is not a distributive inverse semigroup.

On the other hand, if $S$ is (weakly) Boolean then $S/I$ is also (weakly) Boolean. In the previous example, note that the restriction $a_{\{1\}}$ does not admit a relative complement in $a$.

**Theorem 1.5.8.** If $S$ is a weakly Boolean inverse semigroup and $I$ is a $\lor$-ideal of $S$, then for all pair of compatible elements $\alpha, \beta \in S/I$ there are compatible elements $a, b \in S$ such that $\tilde{a}^l = \alpha$ and $\tilde{b}^l = \beta$.

**Proof.** Suppose $\alpha = \tilde{a}^l$ and $\beta = \tilde{b}^l$ are compatible in $S/I$. By Proposition [1.2.25](e), we can find $e \in E(S)$ and $i \in I$ such that $b^* a = e \lor i$. Note that $bi \leq bb^* a \leq a$, so using Proposition [1.4.11], we can consider the element $a_0 = a \setminus bi$.

Similarly, there are $f \in E(S)$ and $j \in I$ such that $ba^* = f \lor j$, and we can consider the element $b_0 = b \setminus ja$. Let us prove that $a_0$ and $b_0$ are compatible:

$b_0^* a_0 \leq b^* (a \setminus bi) = (b^* a) \setminus (b^* bi)$
but since \( i \leq b^* a \) then \( b^* bi = i \). Applying Proposition \[1.2.48(\text{c})\],
\[
b^*_0 a_0 \leq b^* a \setminus i \leq e
\]
so \( b^*_0 a_0 \in E(S) \). Similarly, \( b_0 a^*_0 \in E(S) \), so \( a_0 \) and \( b_0 \) are compatible. Moreover, \( a = a_0 \lor bi \) and \( b = b_0 \lor ja \), so from \[1.5.3(\text{e})\] we have
\[
\alpha = \tilde{a}^I = \tilde{a}_0^I \quad \text{and} \quad \beta = \tilde{b}^I = \tilde{b}_0^I \]

An immediate consequence of this and Theorem \[1.5.6\] follows.

**Theorem 1.5.9.** If \( S \) is a weakly Boolean inverse semigroup and \( I \) is a \( \lor \)-ideal of \( S \), then \( S/I \) is weakly Boolean, and the quotient map \( \pi_I : S \to S/I \) is a \( \lor \)-morphism. If \( S \) is Boolean then \( S/I \) is Boolean and the quotient map \( \pi_I : S \to S/I \) is a morphism of Boolean inverse semigroups.

**Proposition 1.5.10.** If \( S \) is a Boolean inverse semigroup and \( I \subseteq S \) is a \( \lor \)-ideal, then \( a \sim_I b \) if and only if \( a \setminus b, b \setminus a \in I \).

**Proof.** On one hand, if \( a \sim_I b \) then take \( s \in S \) and \( i, j \in I \) with \( a = s \lor i \), \( b = s \lor j \). Then by Proposition \[1.2.48(\text{c})\] and (d),
\[
a \setminus b \leq a \setminus s \leq i
\]
so \( a \setminus b \in I \) by Lemma \[1.5.3(\text{c})\], and similarly \( b \setminus a \in I \).

Conversely, if \( a \setminus b, b \setminus a \in I \) then \( a = (a \land b) \lor (a \setminus b) \) and \( b = (a \land b) \lor (b \setminus a) \) implies \( a \sim_I b \). □

To finish this section, we describe \( \lor \)-ideals of distributive inverse semigroups as the zero sets of semigroup morphisms, and prove an Isomorphism Theorem for \( \lor \)-ideals of Boolean inverse semigroups. The simpler direction of the following proposition was already noticed in \[1.4.11\] Proposition 2.10, without proof.

**Proposition 1.5.11.** A nonempty subset \( I \) of a distributive inverse semigroup \( S \) with zero is a \( \lor \)-ideal if and only if there is an inverse semigroup \( T \) with 0 and a \( \lor \)-morphism \( \theta : S \to T \) with \( I = \theta^{-1}(0) \).

**Proof.** It is easy enough to verify that \( \theta^{-1}(0) \) is a \( \lor \)-ideal, under the assumptions above, and the converse direction follows using the canonical morphism \( S \to S/I \) and Theorem \[1.5.6\]. □

**Theorem 1.5.12** (Isomorphism Theorem for Boolean inverse semigroups). Let \( \theta : S \to T \) be a morphism of Boolean inverse semigroups (as in Definition \[1.4.22\]) and \( I \subseteq \theta^{-1}(0) \) a \( \lor \)-ideal. Then
(a) $\theta$ factors uniquely to a Boolean inverse semigroup morphism $\theta_I : S/I \to T$;

(b) $I = \theta^{-1}(0)$ if and only if $\theta_I$ is injective.

**Proof.** First note that if $I \subseteq \theta^{-1}(0)$ then $\theta(0) = 0$ because $0 \in I$.

(a) We just need to prove that $a \sim_I b$ implies $\theta(a) = \theta(b)$. If $a \sim_I b$ then choose $s \in S$ and $i, j \in I$ with

$$a = s \lor i \quad \text{and} \quad b = s \lor j$$

so

$$\theta(a) = \theta(s) \lor \theta(i) = \theta(s) \lor 0 = \theta(s) \lor \theta(j) = \theta(b)$$

so $\theta$ factors through a map $\theta_I : S/I \to T$, and the factor map is unique because the quotient map $S \to S/I$ is surjective. To see that $\theta_I$ is a morphism of Boolean inverse semigroups, apply Theorem 1.5.6 and the fact that compatible elements in $S/I$ are images of compatible elements in $S$ (by Theorem 1.5.8).

(b) Let us prove that if $I = \theta^{-1}(0)$ then $\theta_I$ is injective. If $\theta(a) = \theta(b)$, then

$$\theta(a \setminus b) = \theta(a) \setminus \theta(b) = \theta(a) \setminus \theta(a) = 0$$

so $a \setminus b \in \theta^{-1}(0) = I$, and similarly $b \setminus a \in I$, which means by 1.5.10 that $a \sim_I b$. This means that $\theta_I$ is injective.

Conversely, if $\theta_I$ is injective and $a \in \theta^{-1}(0)$ then

$$\theta_I(\tilde{a}^I) = \theta(a) = 0 = \theta_I(\tilde{0}^I)$$

so $a \sim_I 0$, which means that $a \in I$. \hfill $\Box$

### 1.6 Probability measure-preserving groupoids

In principle, we will consider more general Borel groupoids than in most usual references - namely, we will not make any assumptions about standardness or analyticity of the Borel structure at hand. See Mackey ([121, 122]), Ramsay ([144]) and Hahn ([77, 78]).

**Definition 1.6.1.** A **Borel groupoid** consists of a groupoid $\mathcal{G}$ endowed with a Borel (measurable) structure (i.e., a $\sigma$-algebra) such that the product and inverse maps are Borel (where $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(2)}$ are endowed with the induced Borel structures from $\mathcal{G}$ and $\mathcal{G} \times \mathcal{G}$, respectively).

Note that we are not assuming that $\mathcal{G}^{(0)}$ is Borel in $\mathcal{G}$, nor that $\mathcal{G}^{(2)}$ is Borel in $\mathcal{G} \times \mathcal{G}$.
Example 1.6.2. Let $X$ be any uncountable set, endowed with the $\sigma$-algebra of countable or co-countable subsets. Then the diagonal $\Delta_X = \{(x,x) \in X \times X : x \in X\}$ is not Borel in $X \times X$.

- As a unit groupoid, $X$ is a Borel groupoid, but $X^{(2)} = \Delta_X$ is not Borel in $X \times X$;
- Let $\mathcal{G} = X \times X$, the coarsest equivalence relation on $X$, endowed with the product $\sigma$-algebra. Then $\mathcal{G}$ is a Borel groupoid, but $\mathcal{G}^{(0)} = \Delta_X$ is not Borel in $\mathcal{G}$ (and $\mathcal{G}^{(2)}$ is also not Borel in $\mathcal{G} \times \mathcal{G}$).

Example 1.6.3. Every étale groupoid with the Borel $\sigma$-algebra (generated by the topology) is a Borel groupoid.

Example 1.6.4. If $\mathcal{G}$ is a $\sigma$-compact ample Hausdorff groupoid, then $\mathcal{G}$ is a Borel groupoid when endowed with the Baire $\sigma$-algebra (generated by compact $G$\textsubscript{\textdelta} sets, or equivalently by $\text{KB}(\mathcal{G})$ – see remark below). Indeed, given $A \in \text{KB}(\mathcal{G})$, the inverse $A^{-1}$ also belongs to $\text{KB}(\mathcal{G})$ and so the inversion is measurable. To check that the product map $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is measurable, let $A \in \text{KB}(\mathcal{G})$. For any pair $C, D \in \text{KB}(\mathcal{G})$, the set $m^{-1}(A) \cap (C \times D)$ is a compact-open subset of $\mathcal{G}^{(2)}$, and so it can be written as a finite union

$$m^{-1}(A) \cap (C \times D) = \bigcup_{i=1}^{n} (A_i \times B_i) \cap \mathcal{G}^{(2)}$$

for some $A_i, B_i \in \text{KB}(\mathcal{G})$. In particular, $m^{-1}(A) \cap (C \times D)$ is measurable in $\mathcal{G}^{(2)}$, for any $C, D \in \text{KB}(\mathcal{G})$, and $\sigma$-compactness of $\mathcal{G}$ allows us to conclude that $m^{-1}(A)$ is measurable in $\mathcal{G}^{(2)}$.

Remark. If $X$ is any zero-dimensional, locally compact Hausdorff topological space and $\mathcal{B}$ is a basis for $X$ whose elements are compact-open, then the Baire $\sigma$-algebra of $X$ is generated by $\mathcal{B}$. Indeed, every element of $\mathcal{B}$ is a compact $G$\textsubscript{\textdelta}, so it is a Baire set. Conversely, if $K \subseteq X$ is compact, $U \subseteq X$ is open and $K \subseteq U$, there are $A_1, \ldots, A_n \in \mathcal{B}$ such that $K \subseteq \bigcup_{i=1}^{n} A_i \subseteq U$. If $K$ is $G$\textsubscript{\textdelta} we conclude that $K$ is a countable intersection of finite unions of element of $\mathcal{B}$, and therefore every Baire set is in the $\sigma$-algebra generated by $\mathcal{B}$.

Borel groupoids are the measurable analogues of topological groupoids, and - in the same manner as in the topological setting - we will be interested in semigroups of bisections which encode their Borel structure.

Definition 1.6.5. Given a Borel groupoid $\mathcal{G}$, define $\text{Bor}(\mathcal{G})$ as the set of Borel subsets of $\mathcal{G}$ for which the source and range maps are Borel isomorphisms onto Borel subsets of $\mathcal{G}^{(0)}$ (given the induced structure from $\mathcal{G}$). We call it the Borel semigroup of $\mathcal{G}$. 
Note that if \( A \in \text{Bor}(\mathcal{G}) \) then \( A^* \in \text{Bor}(\mathcal{G}) \) as well.

**Proposition 1.6.6.** \( \text{Bor}(\mathcal{G}) \) is closed under products: If \( A, B \in \text{Bor}(\mathcal{G}) \) then \( AB = \{ ab : (a, b) \in (A \times B) \cap \mathcal{G}^{(2)} \} \in \text{Bor}(\mathcal{G}) \).

**Proof.** If \( A \in \text{Bor}(\mathcal{G}) \) and \( D \subseteq \mathcal{G} \) is Borel, then \( AD \) consists of the elements of \( \mathcal{G} \) such that when multiplied by an appropriate element of \( A^* \) on the left yield an element of \( D \), i.e.,
\[
AD = \{ c \in r^{-1}(\tau(A)) : r|_{A^*}^{-1}(\tau(c))^{-1}c \in D \}
\]
is Borel, because the range map \( \tau \) restricts to a Borel isomorphism from \( A \) to the Borel subset \( \tau(A) \) of \( \mathcal{G}^{(0)} \), and the product and inverse maps are Borel. Similarly, \( DA \) is Borel.

If \( A, B \in \text{Bor}(\mathcal{G}) \), then the argument above shows that \( AB \) is Borel. If \( D \subseteq AB \) is Borel, then \( \tau(D) = DB^* \) is Borel, because \( B^* \in \text{Bor}(\mathcal{G}) \). In particular \( \tau(AB) \) is Borel in \( \mathcal{G}^{(0)} \) and this also proves that the range map is a Borel isomorphism from \( AB \) to its image. The source map is dealt with similarly, so \( AB \in \text{Bor}(\mathcal{G}) \). \( \square \)

In general, \( \text{Bor}(\mathcal{G}) \) is not equal to the set of bisections which are Borel subsets of \( \mathcal{G} \), and \( \mathcal{G} \) cannot be covered by elements of \( \text{Bor}(\mathcal{G}) \). Therefore we will restrain from mentioning “Borel bisections”, except when these two notions coincide.

**Example 1.6.7.** Let \( \omega_1 \) be the first uncountable ordinal\(^3\) viewed as a unit groupoid, and let \( F \) be any nontrivial countable group with unit \( 1_F \). Let \( \mathcal{G} = \omega_1 \sqcup F \) be the coproduct groupoid. We endow \( \mathcal{G} \) with the \( \sigma \)-algebra generated by sets of the forms
\[
[0, \alpha] = \{ \beta \in \omega_1 : \beta \leq \alpha \}, \quad \alpha \in \omega_1
\]
and
\[
\{ a \}, \quad a \in F \setminus \{ 1_F \}.
\]
Let us check that \( \mathcal{G} \) is a Borel groupoid: given \( \alpha \in \omega_1 \), \([0, \alpha]^{-1} = [0, \alpha] \) is Borel, and if \( a \in F \setminus \{ 1_F \} \) then \( \{ a \}^{-1} = \{ a^{-1} \} \) is also Borel, because \( a^{-1} \neq 1_F \), so the inversion map is Borel.

Note, moreover, that \( \mathcal{G}^{(0)} = \omega_1 \cup \{ 1_F \} = \mathcal{G} \setminus \bigcup_{a \in F \setminus \{ 1_F \}} \{ a \} \) is Borel in \( \mathcal{G} \) (and can be identified with the successor ordinal \( \omega_1 + 1 \)).

Let \( m : \mathcal{G}^{(2)} \to \mathcal{G} \) be the product map. If \( \alpha \in \omega_1 \), then
\[
m^{-1}([0, \alpha]) = ([0, \alpha] \times [0, \alpha]) \cap \mathcal{G}^{(2)}
\]
\(^3\)Up to order isomorphism, this is the only uncountable well-ordered set such that every bounded subset is countable. See [93, Chapter 2] and [177, 1.19] for more details.
is Borel in $G^{(2)}$, and if $a \in F \setminus \{1_F\}$, then
\[
m^{-1}(\{a\}) = \bigcup_{g \in F \setminus \{1_F, a\}} \left[ (\{g\} \times \{g^{-1}a\}) \cup (G^{(0)} \times \{a\}) \cup (\{a\} \cup G^{(0)}) \right] \cap G^{(2)}
\]
is also Borel in $G^{(2)}$. Therefore $G$ is a Borel groupoid.

Now note that every Borel set in $G$ either contains a set of the form
\[
(\alpha, \omega_1) \cup \{1_F\} = \{\beta \in \omega_1 : \alpha < \beta\} \cup \{1_F\}, \quad \alpha \in \omega_1
\]
or its complement does. In particular, $\{1_F\}$ is not Borel in $G$, and this implies that no $a \in F \setminus \{1_F\}$ is contained in any element of $\text{Bor}(G)$, even though $\{a\}$ is a bisection which is a Borel subset of $G$.

**Proposition 1.6.8.** Let $G$ be a Borel groupoid for which there exists a countable family $C \subseteq \text{Bor}(G)$ such that $G \subseteq \bigcup C$. Then every bisection of $G$ which is a Borel subset belongs to $\text{Bor}(G)$.

**Proof.** If $C \in C$ and $A$ is a Borel subset of $C$, then of course $A \in \text{Bor}(G)$, since the source and range maps on $A$ are simply the restrictions of the source and range maps on $C$ and so are Borel isomorphism onto their (Borel) images.

In general, if $A$ is a bisection of $G$ which is a Borel subset, then $A = \bigcup_{C \in C} (A \cap C)$, and each term $A \cap C$ belongs to $\text{Bor}(G)$, hence $A \in \text{Bor}(G)$.

Recall that a Borel space $X$ is standard if there is a separable (equivalently, second-countable) completely metrizable topology on $X$ which induces the initial Borel structure. Standard Borel spaces have good analytical properties and are more manageable than general Borel spaces. For a detailed discussion of standard Borel spaces (and more general classes of Borel spaces), see [99].

We recall the reader that the graph of a function $f : X \to Y$ is $\text{graph}(f) = \{(f(x), x) : x \in X\}$ (see Definition 1.2.10 and the remark succeeding it).

**Theorem 1.6.9 ([99 Theorems 14.12 and 15.2]).** Let $X,Y$ be standard Borel spaces and $f : X \to Y$. Then $f$ is Borel if and only if $\text{graph}(f)$ is Borel in $Y \times X$, and in this case, if $A \subseteq X$ is Borel and $f|_A$ is injective then $f(A)$ is Borel.

Let $\pi_X$ and $\pi_Y$ be the projection maps of $X \times Y$:
\[
\pi_X(x,y) = x, \quad \pi_Y(x,y) = y.
\]
A subset $P \subseteq X \times Y$ is called a graph if $\pi_Y$ is injective on $P$. One important result in the study of standard Borel spaces is the Lusin-Novikov (Uniformization) Theorem.
Theorem 1.6.10 ([99, Theorem 18.10]). Let \(X, Y\) be standard Borel spaces and \(P \subseteq X \times Y\) a Borel subset such that for all \(x \in X\), the section
\[
P_y = \{x \in X : (x, y) \in P\}
\]
is (at most) countable. Then \(P\) is a countable union of Borel graphs.

We will be interested in the following equivalent form of the Lusin-Novikov Theorem (compare with Exercises 18.14 and 18.15 of [99]):

Theorem 1.6.11. Let \(X\) and \(Y\) be two standard Borel spaces and \(f : X \to Y\) a countable-to-one function, that is, \(f^{-1}(y)\) is (at most) countable for all \(y \in Y\). Then there is a Borel partition \(\{X_n\}_n\) of \(X\) such that the restriction \(f|_{X_n} : X_n \to f(X_n)\) is a Borel isomorphism — that is, \(f|_{X_n}\) is bijective with Borel inverse — for all \(n\) (and \(f(X_n)\) is Borel in \(Y\) for all \(n\)).

In particular if \(A \subseteq X\) is Borel then \(f(A)\) is Borel in \(Y\).

Proof. Let \(P = \{(x, f(x)) \in X \times Y : x \in X\}\), which is Borel since it is the opposite of graph(f), and all sections \(P_y (y \in Y)\) are countable (in fact, \(P_y = f^{-1}(y) \times \{y\}\)). By the Lusin-Novikov Theorem, there are Borel graphs \(P_n \subseteq P\) such that \(P = \bigcup_{n=1}^{\infty} P_n\), and of course we can assume the \(P_n\) are pairwise disjoint, by substituting \(P_n\) by \(P_n \setminus \bigcup_{i=1}^{n-1} P_i\) for \(n \geq 2\), if necessary.

The projection \(\pi_X : P \to X\) is a bijection, so we can take \(X_n = \pi_X(P_n)\). Theorem 1.6.9 implies both that each of the sets \(X_n\) and \(f(X_n) = \pi_Y(P_n)\) are Borel in \(X\) and \(Y\), respectively, and that \(f|_{X_n}\) is a Borel isomorphism. \(\square \)

Remark. The usual form of the Lusin-Novikov Theorem follows from the one above, since if \(P\) has countable sections \(P_y (y \in Y)\) then the projection \(\pi_Y\) is countable-to-one on \(P\), and the partition \(\{P_n\}_n\) yielded by Theorem 1.6.11 will be a partition by Borel graphs.

Corollary 1.6.12. If \(X\) and \(Y\) are standard Borel spaces and \(f : X \to Y\) is a countable-to-one surjective Borel map, then there is a Borel map \(g : Y \to X\) such that \(fg = 1_Y\), that is, \(f\) admits a Borel section (equivalently, there is a Borel subset \(X'\) of \(X\) such that the restriction \(f|_{X'}\) is a Borel isomorphism onto \(Y\)).

Proof. Let \(\{X_n\}_n\) be a countable Borel partition of \(X\) such that \(f|_{X_n}\) is a Borel isomorphism onto its image for all \(n\). In particular, \(Y = \bigcup_n f(X_n)\), so we define \(g = f|_{X_1}^{-1}\) on \(f(X_1)\), and \(g = f|_{X_n}^{-1}\) on \(X_n \setminus \bigcup_{i=1}^{n-1} f(X_i)\) for all \(n \geq 2\). \(\square \)

Definition 1.6.13 ([5, 6]). A groupoid \(G\) is \(v\)-discrete if \(G^x = v^{-1}(x)\) is (at most) countable for all \(x \in G^{(0)}\) (equivalently, \(G_x = s^{-1}(x)\) is countable for all \(x \in G^{(0)}\)).
Remark. In [146, Definition 2.6], Renault uses \( r \)-discreteness in the topological setting to mean that the unit space \( G \) of a locally compact Hausdorff groupoid is open. If \( G \) is second-countable and \( G^{(0)} \) is open then \( G_x \) is countable for all \( x \in X \), but the converse is not true even in the case of a second-countable groupoid (see Example 1.3.13). The definition we adopt is of more common usage in the measurable setting (such as in [6]).

An immediate consequence of the Lusin-Novikov Theorem (1.6.11) and Proposition 1.6.8 is the following:

**Proposition 1.6.14.** If \( G \) is a standard \( r \)-discrete Borel groupoid, then \( G \) is covered by countably many elements of \( \text{Bor}(G) \), and every Borel subset of \( G \) which is a bisection belongs to \( \text{Bor}(G) \).

Remark. If \( X \) is any (possibly non-standard) Borel space, seen as a Borel unit groupoid, then \( \text{Bor}(G) \) is simply the family of Borel subsets of \( X \), and \( X \) is covered by \( \{X\} \). Thus standardness of a Borel groupoid is not in general a consequence of \( G \) being covered by countably many elements of \( \text{Bor}(G) \).

**Example 1.6.15.** If \( X \) is a standard Borel space and \( R \) is a \( r \)-discrete equivalence relation on \( X \), then the only standard groupoid Borel structure on \( R \) which induces the initial Borel structure of \( X = R^{(0)} \) is the one coming from the product \( X \times X \). This follows from the Lusin-Novikov Theorem applied to the inclusion map \( R \hookrightarrow X \times X \).

In this case, \( R \) is a Borel subset of \( X \times X \), and under the usual identification of bisections of \( R \) as partial maps on \( X \) preserving \( R \)-equivalence classes (Example 1.2.11), elements of \( \text{Bor}(R) \) corresponds partial Borel automorphisms \( f : A \to B \), where \( A, B \subseteq X \) are Borel.

**Example 1.6.16.** Every countable group is standard and \( r \)-discrete as a groupoid, with the discrete \( \sigma \)-algebra.

**Example 1.6.17.** More generally, if \( G \) is a countable group acting via Borel automorphisms on a standard Borel space \( X \), then the transformation groupoid \( G \ltimes X \) and the orbit groupoid \( R(G \ltimes X) \) are standard and \( r \)-discrete (where we endow \( G \) with the discrete \( \sigma \)-algebra).

**Theorem 1.6.18.** Let \( G \) be an \( r \)-discrete standard Borel groupoid. Let \( R = R(G) = (r, s)(G) = \{(r(a), s(a)) : a \in G\} \) be the equivalence relation on \( G^{(0)} \), endowed with the \( \sigma \)-algebra coming from the product \( \sigma \)-algebra of \( G^{(0)} \times G^{(0)} \). Then \( R \) is also \( r \)-discrete and standard Borel, hence an \( r \)-discrete Borel groupoid on its own right. Moreover, there is a map \( \rho : \text{Bor}(R) \to \text{Bor}(G) \) such that \( (r, s) \circ \rho = \text{id}_{\text{Bor}(R)} \).

**Proof.** As usual, we identify \( G^{(0)} \) and \( R^{(0)} \). If we denote by \( s_G \) and \( s_R \) the source maps and on \( G \) and \( R \), respectively, and similarly for the range maps, then for all \( x \in G^{(0)} \),

\[
\tau_{R}^{-1}(x) = (t_G, s_G)((t_G)^{-1}(x))
\]
so $R$ is $\tau$-discrete. The structure maps of $R$ are all Borel, and $R$ is a Borel image of the standard Borel space $G$ so $R$ is standard as well.

Since $(t_G, s_g) : G \to R$ is surjective, let $g : R \to G$ be any Borel section, by Corollary 1.6.12 so that $g$ is injective and induces, by Theorem 1.6.9 a map $\rho : \text{Bor}(R) \to \text{Bor}(G)$, $\rho(A) = g(A)$. □

The following is also an important result, which proves that every principal standard discrete groupoid is actually the orbit groupoid of some Borel group action.

**Corollary 1.6.19** (Feldman-Moore, [56]). If $X$ is a standard Borel space and $R \subseteq X \times X$ is an $\tau$-discrete Borel equivalence relation (that is $R$ is Borel in $X \times X$), then there is a countable group $G$ and an action of $G$ on $X$ by Borel automorphisms such that $R = \{(gx, x) : g \in G\}$

**Proof.** Since $X$ is standard, we endow it with a completely metrizable separable, and so second-countable and Hausdorff, topology which induces the Borel structure, and so the set $\{(x, y) \in X \times X : x \neq y\}$ is open and can be written as a countable union of open rectangles sets $E_n = A_n \times B_n$, where $A_n, B_n \subseteq X$ are open and $A_n \cap B_n = \emptyset$.

By the Lusin-Novikov Theorem 1.6.10, we can partition $R$ into subsets $R_n$ for which the source and range maps $s, t : R_n \to X$, $s(y, x) = x$, $t(y, x) = y$, are Borel isomorphisms onto their images. For every pair of positive integers $m, n$, set

$$\gamma_{m,n}(x) = \begin{cases} t(s |_{R_n}^{-1}(x)), & \text{if } x \in A_m \cap s(R_n), \\ s(t |_{R_n}^{-1}(x)), & \text{if } x \in B_m \cap t(R_n), \\ x, & \text{otherwise.} \end{cases}$$

In other words, $\gamma_{m,n}(x) = y$ if $(x, y)$ belongs to $R_n \cap ((A_m \times B_m) \cup (B_m \times A_m))$ and $\gamma(x) = x$ if there is no such $y$. Then $\gamma_{m,n}$ is a Borel automorphism of $X$. Letting $G$ be the group generated by $\gamma_{m,n}$, endowed with the canonical action on $X$ $(gx = g(x))$ we obtain the desired result. □

**Remark.** Since the generators $\gamma_{m,n}$ of the group above satisfy $\gamma_{m,n}^2 = \text{id}$, $G$ is a homomorphic image of a countable free product of the group $\mathbb{Z}/(2\mathbb{Z})$ of order 2.

**Definition 1.6.20.** Let $G$ be a Borel groupoid and $\mu$ a Borel measure on $G^{(0)}$. We say that $\mu$ is invariant if $\mu(s(A)) = \mu(t(A))$ for all $A \in \text{Bor}(G)$. A probability measure-preserving groupoid is a Borel groupoid endowed with an invariant probability measure $\mu$ on $G^{(0)}$, which we will sometimes denote as $(G, \mu)$ whenever $\mu$ needs to be written explicitly.

**Remark.** If $G$ is covered by countably many elements of $\text{Bor}(G)$, then $\mu$ is an invariant measure if and only if

$$\int_{G^{(0)}} \#(A \cap G^x) \, d\mu(x) = \int_{G^{(0)}} \#(A \cap G_\varnothing) \, d\mu(x)$$
for every Borel $A \subseteq \mathcal{G}^{(0)}$. The proof that these integrals are well-defined follows
the same lines as in the proof of [50, Theorem 2]. However, we will only need the
invariance described in the definition above.

We recall from Propositions 1.6.8 and 1.6.14 that if $\mathcal{G}$ is a $\tau$-discrete standard
Borel groupoid then $\text{Bor}(\mathcal{G})$ coincides with the set of Borel subsets of $\mathcal{G}$ which are
also bisections.

**Proposition 1.6.21.** If $\mathcal{G}$ is a $\tau$-discrete standard Borel groupoid and $\mu$ is a measure
on $\mathcal{G}^{(0)}$, then $\mu$ is invariant for $\mathcal{G}$ if and only if $\mu$ is invariant for the orbit relation
$\mathcal{R}(\mathcal{G})$.

**Proof.** By Proposition 1.6.14

$$\text{Bor}(\mathcal{R}(\mathcal{G})) = (r, s)(\text{Bor}(\mathcal{G}))$$

and if $A \in \text{Bor}(\mathcal{G})$, and $F = (r, s)(A)$, then

$$s(A) = s(F) \quad \text{and} \quad r(A) = r(F)$$

from which the desired result readily follows.

For the case of group actions, see Proposition 2.3.2

Let $(\mathcal{G}, \mu)$ be a probability measure-preserving groupoid. We define the null ideal
of $(\mathcal{G}, \mu)$ as

$$\text{Null}(\mathcal{G}, \mu) = \{ A \in \text{Bor}(\mathcal{G}) : \mu(s(\theta)) = 0 \}.$$ 

Note that $\text{Null}(\mathcal{G}, \mu)$ is a $\lor$-ideal of $\text{Bor}(\mathcal{G})$.

**Definition 1.6.22.** The measured semigroup of a probability measure-preserving
groupoid $(\mathcal{G}, \mu)$ is the semigroup quotient

$$\text{Meas}(\mathcal{G}, \mu) = \text{Bor}(\mathcal{G})/\text{Null}(\mathcal{G}, \mu)$$

which we simply denote by $\text{Meas}(\mathcal{G})$ when the measure $\mu$ is implicit and there is no
risk of confusion.

By Theorem 1.5.9 $\text{Meas}(\mathcal{G}, \mu)$ is a Boolean inverse semigroup, and in Section
1.8.2 we will see that it is actually $\sigma$-complete.

When there is no risk of confusion, we will not usually make a distinction between
$\text{Bor}(\mathcal{G})$ and $\text{Meas}(\mathcal{G}, \mu)$, in the same manner that one usually sees elements of an $L^1$
space on a probability space $(X, \mu)$ as functions on $X$, instead of classes of functions
which agree $\mu$-a.e.

**Example 1.6.23.** If $R$ is a $\tau$-discrete, probability measure-preserving Borel equivalence
relation on a a standard probability space $(X, \mu)$ (see Example 1.6.15) then
$\text{Meas}(R)$ is isomorphic to the semigroup of partial Borel automorphisms of $X$
preerving $R$-equivalence classes, modulo identifying functions which are defined and
coincide a.e. on the union of their domains.
1.7 Invariant means

In this section we will consider the algebraic analogues of invariant measures on groupoids, which are called invariant means. These were originally studied in [106] in the context of paradoxical decompositions, and their connections with C*-algebras were described in [159].

We will be interested in the analytical structure that such an invariant mean will induce on a Boolean inverse monoid, and this will be revisited when we introduce sofic groupoids and study them in the next chapter. This is also a setting we will consider in the next section, on which we will prove a non-commutative version of the Loomis-Sikorski Theorem and determine Boolean semigroups which are measured semigroups of probability measure-preserving groupoids.

Although we will introduce the concepts as in [159], we will reword some definitions to fit better to the nomenclature and notation we have used so far. It should be noted that Boolean inverse monoids of [159] correspond to weakly Boolean inverse monoids as in Definition 1.4.10.

1.7.1 Covers and fixed idempotents

Definition 1.7.1. Let $L$ be a poset with zero. A cover of an element $x \in L$ is a family $C \subseteq L$ such that whenever $e \in L$ satisfies $c \wedge e = 0$ for all $c \in C$, then $x \wedge e = 0$ as well. A cover of a subset $X \subseteq L$ is a family $C \subseteq L$ which is a cover of every element of $X$.

Proposition 1.7.2. If $L$ is a generalized Boolean algebra, $X \subseteq L$ is downwards closed, and $C$ is a finite subset of $L$, then the following are equivalent:

(1) $C$ is a cover of $X$;

(2) For every nonzero $x \in X$ there is some $c \in C$ with $x \wedge c \neq 0$.

(3) $\bigvee C$ is an upper bound of $X$;

Proof. (1)$\Rightarrow$(2): If $x \in X$ satisfies $x \wedge c = 0$ for all $c \in C$, then the definition of cover implies $x = x \wedge x = 0$.

(2)$\Rightarrow$(3): If $x \in X$, then the element $y = x \setminus \bigvee C$ of $X$ satisfies $y \wedge c = 0$ for all $c \in C$, so $y = 0$ by property (2) and thus $x \leq y \vee \bigvee C = \bigvee C$.

(3)$\Rightarrow$(1): If $e \in L$ satisfies $e \wedge c = 0$ for all $c \in C$, then for all $x \in X$,

$$e \wedge x \leq e \wedge \bigvee_{c \in C} c = e \wedge \bigvee_{c \in C} c = 0$$

We need now to consider fixed idempotents of elements of inverse semigroups. These can be thought of as the set of fixed points of a function, as described in Example 1.7.6.
Definition 1.7.3 ([159, Definition 3.6]). If $S$ is an inverse semigroup, we say that an idempotent $e \in E(S)$ is fixed by an element $s \in S$ if $se = e$. We denote by $\mathcal{I}_s = \{ e \in E(S) : se = e \}$ the set of fixed idempotents of $s$.

The following lemma is immediate from the facts that if $e$ is idempotent then $e^*e = e$, and that $E(S)$ is downwards directed.

Lemma 1.7.4. If $S$ is an inverse semigroup, then for all $s \in S$, 

$$\mathcal{I}_s = \{ e \in E(S) : e \leq s \} = \{ e \in S : e \leq s \text{ and } e \leq s^*s \}.$$ 

In particular, $\mathcal{I}_s$ is downwards closed and $\mathcal{I}_{s^*} = \mathcal{I}_s$.

Definition 1.7.5 ([155, Definition 1.7]). If $S$ is an inverse semigroup, $s \in S$ and the meet $s \land s^*s$ exists, we denote it by $\text{Fix}(s) = s \land s^*s$ and call $\text{Fix}(s)$ the fixed point idempotent of $s$.

Since, by Lemma 1.7.4, $\mathcal{I}_s$ is the set of lower bounds of $\{ s, s^*s \}$, then

$$\text{Fix}(s) = \bigvee \mathcal{I}_s$$

in the sense that one of these elements exists if and only if the other exists, in which case they are equal and $\text{Fix}(s)$ is the maximal idempotent fixed by $s$.

Example 1.7.6. If $f \in \mathcal{I}(X)$ for some set $X$, then $\text{Fix}(f)$ is the identity function of the set of fixed points of $f$, $\{ x \in \text{dom}(f) : f(x) = x \}$, and so under the usual interpretation that $E(\mathcal{I}(X))$ consists of subsets of $X$, $\text{Fix}(f)$ is the set of fixed points of $f$. (See Example 1.2.14.)

Example 1.7.7. If $A \in \text{KB}(\mathcal{G})$ for some ample Hausdorff groupoid $\mathcal{G}$, then $\text{Fix}(A) = A \cap \mathcal{G}^{(0)}$.

In [155], Leech studied the fixed point operator $\text{Fix}$ for an inverse monoid $S$, and in particular proved that $\text{Fix}(s)$ is defined for all $s \in S$ if and only if $S$ admits meets (with respect to its usual order). Lemma 1.7.8(b) below describes the same fact for inverse semigroups.

Lemma 1.7.8. Suppose that $S$ is an inverse semigroup and $\text{Fix}(s)$ exists for all $s \in S$. Then whenever $s, t \in S$,

(a) $\text{Fix}(s) = \text{Fix}(s^*)$;

(b) $s \land t$ exists and $s \text{Fix}(s^*t) = t \text{Fix}(s^*t) = s \land t$;

(c) $\text{Fix}(s^*t)$ is a lower bound of $\{ s^*s, t^*t \}$. 


(d) \( \text{Fix}(s^*t) = (s \land t)^*(s \land t) \);

(e) If \( z \in S \) then \( \text{Fix}(s^*t) \text{Fix}(t^*z) \leq \text{Fix}(s^*z) \);

(f) \( \text{Fix}(st^*) = t \text{Fix}(s^*t)t^* = s \text{Fix}(s^*t)t^* \);

Proof. (a) By 1.7.4, \( \text{Fix}(s) = \bigvee I_s = \bigvee I_{s^*} = \text{Fix}(s^*) \).

(b) Since \( s \text{Fix}(s^*t) \leq s \) and \( s \text{Fix}(s^*t) \leq ss^*t \leq t \), then \( s \text{Fix}(s^*t) \) is a lower bound of \( \{s, t\} \). If \( p \) is any other lower bound of \( \{s, t\} \), then \( p^*p \) is an idempotent and \( p^*p \leq s^*t \), so \( p^*p \leq \text{Fix}(s^*t) \), which implies \( p = pp^*p \leq s \text{Fix}(s^*t) \). This proves \( s \text{Fix}(s^*t) = s \land t \), and the other equality is similar.

(c) follows from item (a) and 1.7.4.

(d) follows from (b) and (c).

(e) Since \( \text{Fix}(s^*t) \text{Fix}(t^*z) \in E(S) \), then applying Lemma 1.7.4 we obtain

\[
\text{Fix}(s^*t) \text{Fix}(t^*z) \leq s^*tt^*z \leq s^*z
\]

that is, \( \text{Fix}(s^*t) \text{Fix}(t^*z) \in I_{s^*z} \) and so \( \text{Fix}(s^*t) \text{Fix}(t^*z) \leq \bigvee I_{s^*z} = \text{Fix}(s^*z) \)

(f) By (b) and (d),

\[
t \text{Fix}(s^*t)t^* = t \text{Fix}(s^*t) \text{Fix}(s^*t)t^* = (t \text{Fix}(s^*t))(t \text{Fix}(s^*t))^*
= (s \land t)(s \land t)^*(s^* \land t^*)^*(s^* \land t^*) = \text{Fix}(st^*),
\]

and the other equality is similar. \( \square \)

Corollary 1.7.9. If \( S \) is a Boolean inverse semigroup and \( s, t \in S \) then \( t \setminus s = t(t^*t \setminus \text{Fix}(s^*t)) \).

Proof. Using the previous lemma and Proposition 1.2.48(a),

\[
t \setminus s = t \setminus (s \land t) = t(t^*t \setminus (s \land t)^*(s \land t)) = t(t^*t \setminus \text{Fix}(s^*t)) \quad \square
\]

Note that, by Proposition 1.7.2, the condition in item (1) below is precisely condition (H) considered in [159] when we restrict our study to weakly Boolean inverse semigroups (which are the semigroups of interest there).

Proposition 1.7.10. Given a weakly Boolean inverse semigroup \( S \), the following are equivalent:

(1) For every \( s \in S \), there is a finite subset \( C \subseteq I_s \) which is a cover of \( I_s \);

(2) For all \( s \in S \), \( \bigvee I_s \) exists;
(3) For all $s \in S$, $\text{Fix}(s)$ exists;

(4) $S$ is a Boolean inverse semigroup.

Proof. (1)$\Rightarrow$(2): Suppose (1) is valid, $s \in S$ and $C \subseteq \mathcal{I}_s$ is a finite cover $C$ of $\mathcal{I}_s$. Since $\mathcal{I}_s$ is a compatible set, $\bigvee C$ exists and it is an upper bound for $\mathcal{I}_s$ by Proposition 1.7.2. This obviously implies $\bigvee C = \bigvee \mathcal{I}_s$.

(2)$\iff$(3) is clear from the comment below Definition 1.7.5.

(3)$\Rightarrow$(4) follows from Lemma 1.7.8(b).

(4)$\Rightarrow$(1): If $S$ is Boolean then $\text{Fix}(s)$ is the maximum of $\mathcal{I}_s$ and $\{\text{Fix}(s)\}$ is a finite cover of $\mathcal{I}_s$ (by Proposition 1.7.2).

1.7.2 Invariant means and uniform metrics

Definition 1.7.11 ([106]). A mean on a weakly Boolean inverse semigroup $S$ is a map $\mu : E(S) \to [0, \infty)$ satisfying

(i) $\mu(e \lor f) = \mu(e) + \mu(f)$ whenever $e, f \in E(S)$, $ef = 0$.

and a mean $\mu$ is called invariant if

(ii) $\mu(s^*s) = \mu(ss^*)$ for all $s \in S$;

A mean $\mu$ on $S$ will be called faithful if $\mu(e) = 0$ implies $e = 0$, and if $S$ is a monoid and $\mu(1) = 1$, we say that $\mu$ is normalized.

Example 1.7.12. If $X$ is a set, and again by identifying $E(\mathcal{I}(X))$ with subsets of $X$, a mean on $\mathcal{I}(X)$ corresponds to a finite, finitely additive measure on $X$ (endowed with the discrete $\sigma$-algebra).

Example 1.7.13. Let $S$ be a Boolean inverse semigroup. By non-commutative Stone duality (Theorem 1.4.26), there exists an ample Hausdorff groupoid $\mathcal{G}$ such that $S = \text{KB}(\mathcal{G})$.

Identifying $E(S)$ with the collection of compact-open subsets of $\mathcal{G}^{(0)}$, an invariant mean $\mu$ on $E(S)$ corresponds to a pre-measure, also denoted $\mu$, on the ring of compact-open subsets of $\mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)}$ is zero-dimensional, we can extend $\mu$ uniquely to a regular, locally finite Borel measure on $\mathcal{G}^{(0)}$. This is a consequence of a theorem of Mařík ([123]), although a simple application of the Riesz-Markov-Kakutani Theorem also yields an elementary proof in the case at hand (see the Remark below). This interpretation is sometimes useful because we have more freedom to take unions in $\text{KB}(\mathcal{G})$, even though we do not end up with elements of $\text{KB}(\mathcal{G})$. 
Remark. If $X$ is a zero-dimensional, locally compact Hausdorff space, then every finitely additive pre-measure $\mu$ on $\mathbf{KB}(X)$ (the collection of compact-open subsets of $X$) extends uniquely to a regular Borel measure on $X$. Indeed, $\mu$ is in fact $\sigma$-additive because all sets of $\mathbf{KB}(X)$ are compact-open, so if $A = \bigcup_n A_n$ where $A, A_n \in \mathbf{KB}(X)$ and the sets $A_n$ are pairwise disjoint, then all except finitely many $A_n$ are empty, from which $\mu(A) = \sum_n \mu(A_n)$ follows.

Let $A$ be the complex span of all characteristic functions of elements of $\mathbf{KB}(X)$, and consider the functional $\tau : A \rightarrow \mathbb{C}$, $\tau(f) = \int f d\mu$.

Then $A$ is dense in $C_c(X, \mathbb{C})$ (the vector space of complex-valued, compactly supported continuous functions on $X$) with uniform norm, so the continuous positive functional $\tau$ extends uniquely to a positive functional on $C_c(X, \mathbb{C})$.

By the Riesz-Markov-Kakutani Representation Theorem ([155, Theorem 2.14]), $\tau$ corresponds to a unique regular Borel measure $\nu$ on $X$, which extends $\mu$ because $\nu(A) = \tau(1_A) = \mu(A)$ for all $A \in \mathbf{KB}(X)$.

Either using Example 1.7.13 or proceeding in a manner similar to that of measure theory, the following properties of invariant means follow easily:

Proposition 1.7.14. Suppose $\mu$ is a mean on a weakly Boolean semigroup $S$. Then

(a) $\mu(0) = 0$;
(b) $\mu(e \lor f) = \mu(e) + \mu(f) - \mu(ef)$ for all $e, f \in E(S)$.
(c) If $e \leq f$ in $E(S)$ then $\mu(f) = \mu(e) + \mu(f \setminus e)$.

Invariant means induce a metric structure on the corresponding semigroup in a natural manner.

Definition 1.7.15. If $\mu$ is an invariant mean on a Boolean inverse semigroup $S$, we define the uniform metric (induced by $\mu$) as

$$d_\mu(s, t) = \mu((s^*s \lor t^*t) \setminus \text{Fix}(s^*t))$$

and whenever $S$ is endowed with an invariant mean, this is the metric structure we will consider (see 1.7.17(b)).

Example 1.7.16. If $S = \mathbf{KB}(G)$ and $\mu$ is an invariant measure on $G^{(0)}$, regarded as an invariant mean on $E(S)$ as in Example 1.7.13 then $d_\mu(A, B) = \mu(s(A \Delta B))$, where $A \Delta B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of $A$ and $B$.

Proposition 1.7.17. Suppose $\mu$ is an invariant mean on a Boolean semigroup $S$. Then if $s, t, z \in S$,

---

\[4\]See the beginning of Subsection 3.5.1 for the definition of regular measure.
(a) \( d_\mu(s, t) = \mu((s \setminus t)^*(s \setminus t) \lor (t \setminus s)^*(t \setminus s)) \);

(b) \( d_\mu \) is a pseudometric on \( S \);

(c) \( d_\mu(s^{-1}, t^{-1}) = d_\mu(s, t) \);

(d) \( d_\mu(zs, zt) \leq d_\mu(s, t) \) and \( d_\mu(sz, tz) \leq d_\mu(s, t) \);

(e) \( d_\mu(sz, st) \leq d_\mu(s, t) \) and \( d_\mu(sz, tz) \leq d_\mu(s, t) \);

(f) \( d_\mu(s_1s_2, t_1t_2) \leq d_\mu(s_1, t_2) + d_\mu(s_2, t_2) \);

(g) \( d_\mu(s \lor t, z \lor w) \leq d_\mu(s \lor t, z \lor w) + d_\mu(s \lor t, z \lor w) \) where the second inequality follows from item (a).

Proof. Items (a) and (b) actually follow from Examples 1.7.13 and 1.7.16 but we can give a purely semigroup-theoretic proof of them:

(a) follows from Lemma 1.7.8(d) and (a);

(b) Let \( s, t, z \in S \). Since \( s \setminus t = ((s \setminus t) \setminus z) \lor ((s \setminus z) \setminus t) \), then

\[
(s \setminus t)^*(s \setminus t) \leq (s \setminus z)^*(s \setminus z) \lor (z \setminus t)^*(z \setminus t)
\]

and similarly with the roles of \( s \) and \( t \) interchanged, so simply apply (a) to obtain the result.

(c) follows from (a) and property (i) in the definition of invariant mean 1.7.11.

(d) follows from distributivity of the product over relative complements and items (a) and (c);

(e) By (b) and (d),

\[
d_\mu(s_1s_2, t_1t_2) \leq d_\mu(s_1s_2, t_1s_2) + d_\mu(t_1s_2, t_1t_2) \leq d_\mu(s_1, t_1) + d_\mu(s_2, t_2).
\]

(f) Just note that

\[
d_\mu(s \setminus t, z \setminus w) \leq d_\mu(s \setminus t, s \setminus w) + d_\mu(s \setminus w, z \setminus w) \leq d_\mu(t, w) + d_\mu(s, z)
\]

where the second inequality follows from item (a).

(g) Using the distributive properties, we obtain

\[
(s \lor t) \setminus (z \lor w) = (s \setminus (z \lor w)) \lor (t \setminus (z \lor w)) \leq (s \setminus z) \lor (t \setminus w)
\]

and similarly \( (z \lor w) \setminus (s \setminus t) \leq (z \setminus s) \lor (w \setminus t) \), then the result follows from item (a).
(h) If $S$ is a unital Boolean algebra, and if we denote the complement of an element $x \in S$ by $x^c = 1 \setminus x$, then

$$(s \setminus t) \setminus (z \setminus w) = st^c(zw)^c = st^c(z^c \lor w) = st^c z^c \lor st^c w$$

$$= s(t \lor w)^c \lor (sw)t^c = (s \setminus (t \lor w)) \lor ((sw) \setminus t)$$

For a general Boolean inverse monoid, the same calculation on the unital Boolean algebra $\{x \in S : x \leq s\}$ (and intersecting all elements with $s$, if necessary) will also yield

$$(s \setminus t) \setminus (z \setminus w) = (s \setminus (t \lor w)) \lor ((s \land w) \setminus t)$$

and similarly $(z \setminus w) \setminus (s \setminus t) = (z \setminus (w \lor t)) \lor ((z \land t) \setminus w)$. These equalities and item (a) yield the desired inequality.

\[\square\]

Corollary 1.7.18. If $\mu$ is an invariant mean on a Boolean inverse semigroup $S$, then

$$\mathcal{N}(\mu) = \{n \in S : \mu(n^*n) = 0\} = \{n \in S : d_\mu(0, n) = 0\}$$

is a $\lor$-ideal of $S$, and $s \sim_{\mathcal{N}(\mu)} t$ if and only if $d_\mu(s, t) = 0$. Moreover, $\mu$ factors to a faithful invariant mean on the quotient $S/\mathcal{N}(\mu)$, whose associated metric is the quotient metric of $d_\mu$.

We will still denote the invariant mean on $S/\mathcal{N}(\mu)$ by $\mu$ and the respective quotient metric by $d_\mu$.

Example 1.7.19. If $(G, \mu)$ is a probability measure-preserving groupoid, and we identify $\mu$ with an invariant mean on $\text{Bor}(G)$ in the usual manner, then $\text{Meas}(G, \mu) = \text{Bor}(G)/\mathcal{N}(\mu)$.

Proposition 1.7.20. Given an invariant mean $\mu$ on a Boolean inverse semigroup $S$, $d_\mu$ is a metric if and only if $\mu$ is faithful.

Proof. Suppose $d_\mu$ is a metric and $\mu(e) = 0$. Then $d_\mu(e, 0) = \mu(e^*e) = \mu(e) = 0$, so $e = 0$ and $\mu$ is faithful.

Conversely, suppose $\mu$ is faithful and $d_\mu(s, t) = 0$. This means that

$$\mu(s^*s \lor t^*t \setminus \text{Fix}(s^*t)) = 0,$$

so $(s^*s \lor t^*t) \setminus \text{Fix}(s^*t) = 0$. But Lemma 1.7.8(c) implies $\text{Fix}(s^*t) \leq s^*s \lor t^*t$, therefore $s^*s = t^*t = \text{Fix}(s^*t)$ by faithfulness of $\mu$, thus Lemma 1.7.8(b) yields

$$s = ss^*s = s \text{Fix}(s^*t) = t \text{Fix}(s^*t) = tt^*t = t,$$

so $d_\mu$ is a metric.
The last function we will associate to an invariant mean \( \mu \) on a Boolean inverse semigroup is the *trace*, which is a simple extension of \( \mu \) to all of \( S \) and which encodes \( d_\mu \) in a simpler manner.

**Definition 1.7.21.** If \( S \) is a Boolean inverse semigroup endowed with an invariant mean \( \mu \) and \( s \in S \), define the *trace* of \( s \) as \( \text{tr}(s) = \mu(\text{Fix}(s)) \) (or \( \text{tr}_\mu \) if we need to make \( \mu \) explicit).

**Example 1.7.22.** Let \((X, \mu)\) be a standard probability space and \( S \) the inverse semigroup of measure-preserving Borel automorphisms of \( X \). By identifying \( E(S) \) with the set of Borel subsets of \( X \) and \( \mu \) with an invariant mean on \( S \), the trace of \( f \in S \) is \( \text{tr}_\mu(f) = \mu(\{x \in X : f(x) = x\}) \).

**Example 1.7.23.** As a particular case of the previous example, let \( X \) be a finite set with \( n \) elements (seen as a discrete Borel space) and \( \mu_\# \) the normalized counting measure on \( X \): \( \mu_\#(A) = \#A/n \) for all \( A \subseteq X \), where \( \#A \) stands for the cardinality of \( A \). Then \( \mathcal{I}(X) \) is precisely the set of measure-preserving Borel automorphisms of \((X, \mu_\#)\).

Let \( S \) be the inverse semigroup of \( n \times n \) partial permutation matrices, that is, \( n \times n \) matrices such that every row and every column has entries in \( \{0, 1\} \), and at most one entry in each row and in each column is nonzero. Then \( S \) and \( \mathcal{I}(X) \) are isomorphic via the map \( \theta : \mathcal{I}(X) \rightarrow S, f \mapsto [f_{ij}]_{ij} \) given by

\[
f_{ij} = \begin{cases} 
1, & \text{if } j \in \text{dom}(f) \text{ and } f(j) = i, \\
0, & \text{otherwise}.
\end{cases}
\]

Composing the trace on \( \mathcal{I}(X) \) (induced by \( \mu_\# \)) with \( \theta^{-1} \), we obtain the usual normalized matrix trace on \( S \): \( \text{tr}([m_{ij}]_{ij}) = (1/n) \sum_{i=1}^{n} m_{ii} \).

**Example 1.7.24.** If \( S = KB(G) \) and \( \mu \) is an invariant mean on \( S \), then \( \text{tr}(A) = \mu(A \cap G^{(0)}) \) for all \( A \in S \).

**Theorem 1.7.25.** Let \( S, T \) be Boolean inverse monoids with faithful and normalized invariant means \( \mu \) and \( \nu \), respectively. Then a semigroup morphism \( \theta : S \rightarrow T \) preserves traces if and only if it is isometric with respect to \( d_\mu \) and \( d_\nu \). In this case, \( \theta \) is a morphism of Boolean inverse monoids satisfying \( \theta(1) = 1 \) (i.e., \( \theta \) is proper).

**Proof.** First suppose \( \theta \) preserves the trace. Then the definitions of \( d_\mu \) and \( \text{tr}_\mu \), and Proposition 1.7.14(c) imply

\[
d_\mu(s, t) = \text{tr}_\mu(s^* s) + \text{tr}_\mu(t^* t) - \text{tr}_\mu(s^* s^* t) - \text{tr}_\mu(s^* t)
\]

and similarly for \( d_\nu \), and so it is clear that \( \theta \) is isometric.
Conversely, suppose \( \theta \) is isometric. First we prove that \( \theta(0) = 0 \) and \( \theta(1) = 1 \). Since \( 0 \leq 1 \) then \( \theta(0) \leq \theta(1) \), so
\[
1 = d_{\mu}(0, 1) = d_{\nu}(\theta(0), \theta(1)) = \nu(((\theta(0) \lor \theta(1)) \setminus (\theta(0) \theta(1)))) = \nu(\theta(1) \setminus \theta(0))
\]
and faithfulness of \( \nu \) implies \( \theta(1) \setminus \theta(0) = 1 \), so \( \theta(1) = 1 \) and \( \theta(0) = 0 \). Now note that for all \( s \in S \),
\[
tr_{\mu}(s) = \mu(Fix(s)) = \mu(s^*s) - \mu(s^*s \setminus Fix(s)) = d_{\mu}(s^*s, 0) - d_{\mu}(s^*s, s)
\]
and similarly for \( tr_{\nu} \). Therefore \( \theta \) preserves traces.

In this case, we have already verified that \( \theta(0) = 0 \) and \( \theta(1) = 1 \). In order to verify that \( \theta \) preserves meets, we first verify that \( \theta(Fix(s)) = Fix(\theta(s)) \) for all \( s \in S \): Indeed, \( Fix(s) \) is an idempotent fixed by \( s \), so \( \theta(Fix(s)) \) is an idempotent fixed by \( \theta(s) \) and so \( \theta(Fix(s)) \leq \sqrt{I_{\theta(s)}} = Fix(\theta(s)) \) (from Definition 1.7.3 and the paragraph following it). As
\[
\nu(\theta(Fix(s))) = tr_{\nu}(\theta(Fix(s))) = tr_{\mu}(Fix(s)) = \mu(Fix(s)) = tr_{\mu}(s) = tr_{\nu}(\theta(s)) = \nu(Fix(\theta(s))),
\]
faithfulness of \( \nu \) implies \( \theta(Fix(s)) = Fix(\theta(s)) \).

Given \( s, t \in S \), since \( s \land t = s Fix(s^*t) \) (Lemma 1.7.8(b)), it follows from the previous argument that \( \theta(s \land t) = \theta(s) \land \theta(t) \).

Similarly, in order to prove that \( \theta \) is a \( \lor \)-morphism we first assume \( e, f \in E(S) \). Then \( \theta(e \lor f) \) is an upper bound of \( \{ \theta(e), \theta(f) \} \), and so \( \theta(e) \lor \theta(f) \leq \theta(e \lor f) \). But on the other hand,
\[
\nu(\theta(e) \lor \theta(f)) = tr_{\nu}(\theta(e)) + tr_{\nu}(\theta(f)) - tr_{\nu}(\theta(ef)) = tr_{\mu}(e) + tr_{\mu}(f) - tr_{\mu}(ef) = tr_{\nu}(e \lor f) = tr_{\nu}(\theta(e \lor f)) = \nu(\theta(e \lor f))
\]
and again faithfulness of \( \nu \) implies \( \theta(e) \lor \theta(f) = \theta(ef) \).

Given compatible elements \( s, t \in S \), \( \theta(s) \) and \( \theta(t) \) are compatible in \( T \), and \( \theta(s \lor \theta(t) \leq \theta(s \lor t) \). But
\[
(\theta(s \lor \theta(t))^* (\theta(s) \lor \theta(t)) = \theta(s)^* \theta(s) \lor \theta(t)^* \theta(t) = (s^*s \lor t^*t) = \theta(s^*s \lor t^*t) = \theta((s \lor t)^*(s \lor t)) = \theta(s \lor t)^* \theta(s \lor t)
\]
so \( \theta(s \lor \theta(t) = \theta(s \lor t) \) (Theorem 1.2.22(e)).
1.8 Non-commutative Loomis-Sikorski Theorem

The goal of this section is to characterize all Boolean semigroups which are measured semigroups of probability measure-preserving groupoids, or more generally quotients of Borel semigroups by (σ-)ideals of “null sets”.

We need to assume one additional condition on such semigroups – namely that the corresponding groupoid under non-commutative Stone duality is σ-compact. This condition also guarantees that the Borel structure of the groupoid is sufficiently manageable, and is always satisfied by unital Boolean algebras, so we obtain a true generalization of the classical Loomis-Sikorski theorem - which we briefly recall.

1.8.1 Classical Loomis-Sikorski Theorem

Definition 1.8.1 ([58, 314A]). A generalized Boolean algebra $B$ is said to be $\sigma$-complete if it admits joins of countable subsets.

Definition 1.8.2 ([58, 322A]). A probability algebra is a pair $(B, \mu)$, where $B$ is a $\sigma$-complete unital Boolean algebra and $\mu : B \to [0, 1]$ is a function, called a probability measure, satisfying:

(i) $\mu(0) = 0$;

(ii) $\mu(e) > 0$ whenever $e \neq 0$;

(iii) $\mu(1) = 1$;

(iv) If $\{e_n\}$ is a sequence of disjoint elements of $B$, then

$$\mu\left(\bigvee_{n \in \mathbb{N}} a_n\right) = \sum_{n \in \mathbb{N}} \mu(a_n).$$

In other terms, $\mu$ is a normalized, faithful invariant mean on $B$ (regarded as an inverse semigroup) satisfying the countable additivity described in (iv) (as opposed to only finite additivity).

Alternatively, conditions (i) and (ii) could be substituted by the single condition $\mu^{-1}(0) = \{0\}$.

Example 1.8.3. Let $X$ be a space endowed with a $\sigma$-algebra $\mathcal{A}$ and $\mathcal{I}$ an ideal of $\mathcal{A}$ closed under countable unions. Then $\mathcal{A}/\mathcal{I}$ is a $\sigma$-complete unital Boolean algebra.

If $\mu$ is a probability measure defined on $\mathcal{A}$ and $\mathcal{I}$ is the ideal of null sets of $\mathcal{A}$, then $\mu$ factors to a probability measure on the quotient $\mathcal{A}/\mathcal{I}$. In this case we denote $\mathcal{A}/\mathcal{I}$ endowed with the invariant mean $\mu$ by $\text{MAlg}(X, \mu)$, and call it the measure algebra of $(X, \mu)$. 
Definition 1.8.4. An isomorphism between two probability algebras \((B, \mu)\) and \((C, \nu)\) is a Boolean algebra isomorphism \(\phi : B \to C\) such that \(\nu(\phi(a)) = \mu(a)\) for all \(a \in B\).

There are two versions of interest of the classical Loomis-Sikorski theorem, which read as follows:

Theorem 1.8.5 ([119]). If \(B\) is a \(\sigma\)-complete unital Boolean algebra, then there is a set \(X\), a \(\sigma\)-algebra \(A\) on \(X\) and an ideal \(I\) of \(A\) closed under countable unions such that \(B\) is isomorphic to \(A/I\).

Theorem 1.8.6 ([15]). Every probability algebra \((B, \mu)\) is isomorphic to \(\text{MAlg}(X, \mu)\) for some probability space \(X\).

In fact, the second version of the theorem follows from the first one, since if \(\mu\) is a probability measure on \(A/I\) (in the sense of Definition 1.8.2), then \(\mu\) induces a probability measure (in the usual sense) on \((X, A)\), via composition with the quotient map \(A \to A/I\), and \(I\) is the ideal of null sets.

1.8.2 Non-commutative Loomis-Sikorski Theorem

We are ready to prove the main theorem of this section. Its proof is an adaptation of the original proof by Loomis [119]. Although all theorems and proofs could be stated in more general terms (for algebras which are complete with respect to some cardinal similarly to Definition 1.2.40), we will only consider countable joins and meets, which is the usual setting of measure theory.

Definition 1.8.7. A Boolean inverse semigroup is \(\sigma\)-complete if every countable compatible subset admits a joint.

Note that, in principle, we do not assume \(\mathfrak{N}_1\)-distributivity in the sense of Definition 1.2.40 (where \(\mathfrak{N}_1\) is the first uncountable cardinal), however an appropriate usage of relative complements yields this, see Proposition 1.8.11(a).

Example 1.8.8. If \(\mathcal{G}\) is a Borel groupoid, then \(\text{Bor}(\mathcal{G})\) is a \(\sigma\)-complete Boolean inverse monoid.

We first prove a general form of de Morgan’s laws, which allows us to work with infinite meets and joins in Boolean algebras which are not necessary complete nor (infinitely) distributive. We temporarily denote the complement of an element \(e\) of a unital Boolean algebra \(B\) by \(e^c = 1 \setminus e\), and if \(F \subseteq B\), \(F^c = \{e^c : e \in F\}\).

Lemma 1.8.9. Let \(B\) be a unital Boolean algebra and \(F\) a subset of \(B\) for which \(\bigvee F\) exists. Then \(\bigwedge (F^c)\) also exists, and \(\bigwedge (F^c) = (\bigvee F)^c\). Dually, \(\bigvee F\) exists if and only if \(\bigwedge (F^c)\) exists and \((\bigwedge F)^c = \bigvee (F^c)\).
Proof. If \( e \in F \), then \( e \leq \bigvee F \), so \( (\bigvee F)^c \leq e^c \). This shows that \((\bigvee F)^c\) is a lower bound of \(F^c\). If \( p \) is any other lower bound of \(F^c\), then \( p^c \) is an upper bound of \(F\), so \( \bigvee F \leq p^c \), or dually \( p \leq (\bigvee F)^c \). This is the desired result.

Lemma 1.8.10. Let \( S \) be a weakly Boolean inverse semigroup and \( g \in S \). We define \( G \) to be the semigroup with underlying set

\[
G = \{g\}^\downarrow = \{ s \in S : s \leq g \}
\]

and whose operation is the meet of \( S : G^2 \to G, (s, t) \mapsto s \wedge t \) (which is well-defined by Proposition 1.2.30(c)).

Then \( G \) is a unital Boolean algebra. The order of \( S \) restricts to the order of \( G \), and arbitrary joins and meets in \( S \) coincide with arbitrary joins and meets in \( G \). If \( S \) is \( \sigma \)-distributive then \( G \) is \( \sigma \)-complete.

Proof. By definition, \( G \) is a \( \wedge \)-semilattice and the operation in \( G \), so it is an inverse semigroup. The fact that the order and all joins and meets in \( G \) coincide with the same notions in \( S \) is trivial, so \( G \) is a lattice with zero, and \( \sigma \)-completeness of \( G \) will also follow immediately from \( \sigma \)-distributivity of \( S \).

The distributivity of \( G \) follows from Proposition 1.2.42 and \( G \) admits relative complements of comparable elements by Proposition 1.4.11. Thus \( G \) is a generalized Boolean algebra, and clearly \( g \) is the unit of \( G \).

We are now ready to prove that \( \sigma \)-complete Boolean algebras are \( \aleph_1 \)-distributive.

Proposition 1.8.11. Any \( \sigma \)-complete Boolean inverse semigroup \( S \) admits meets of countable compatible subsets \( F \subseteq S \). Moreover, if \( F \subseteq S \) is a countable compatible subset and \( z \in S \) then

\[
\begin{align*}
(a) \quad z (\bigvee F) &= \bigvee (zF) \quad \text{and} \quad (\bigvee F) z = \bigvee (Fz); \quad \text{Thus \( S \) is} \ \aleph_1\text{-distributive;} \\
(b) \quad z \wedge (\bigvee F) &= \bigvee_{f \in F} (z \wedge f); \\
(c) \quad (\bigvee F) \setminus z &= \bigvee_{f \in F} (f \setminus z); \\
(d) \quad z \setminus (\bigvee F) &= \bigwedge_{f \in F} (z \setminus f).
\end{align*}
\]

Proof. First assume \( F \subseteq S \) is countable subset, and take \( g \in F \) arbitrary. By Lemma 1.8.10, the set \( \{g\}^\downarrow = \{ t \in S : t \leq g \} \) is a \( \sigma \)-complete unital Boolean algebra with the order endowed from \( S \), so the meet \( \bigwedge_{f \in F} (f \wedge g) \) exists and is the same in both \( \{g\}^\downarrow \) and \( S \), and clearly it is the meet of \( F \) in \( S \).

(a) Of course, it is enough to prove just one of the equalities. On one hand, \( zf \leq z \bigvee F \) whenever \( f \in F \), so \( \bigvee (zF) \leq z \bigvee F \).
For the other inequality, let us first assume that \( ff^* \leq z^*z \) for all \( f \in F \), and under this assumption let us prove that \( z \bigvee F \) is the smallest upper bound of \( zF \). If \( p \) is any upper bound of \( zF \), then for all \( f \in F \)

\[
p \geq zf \implies z^*p \geq z^*zf \geq ff^*f = f
\]

and so \( \bigvee F \leq z^*p \) and therefore \( z \bigvee F \leq zz^*p \leq p \).

In the general case, for every \( f \in F \), we have

\[
f = ff^*f = ((ff^* \setminus z^*z) \lor (z^*zff^*)) f = (ff^* \setminus z^*z) f \lor (z^*zf)
\]

Note, moreover, that \( \bigvee_{f \in F} (ff^* \setminus z^*z) f \) and \( \bigvee_{f \in F} (z^*zf) \) are both bounded by \( \bigvee F \), and thus are compatible and we can conclude that

\[
\bigvee F \leq \left( \bigvee_{f \in F} (ff^* \setminus z^*z) f \right) \lor \left( \bigvee_{f \in F} (z^*zf) \right)
\]

Multiplying both sides by \( z \) on the left, we obtain

\[
z \left( \bigvee F \right) \leq \left[ z \left( \bigvee_{f \in F} (ff^* \setminus z^*z) f \right) \right] \lor \left[ z \left( \bigvee_{f \in F} (z^*zf) \right) \right]
\]

Now, for all \( f \in F \), \( (ff^*) \setminus (z^*z) \leq [(\bigvee F) (\bigvee F)^*] \setminus (z^*z) \), so

\[
z \left( \bigvee_{f \in F} (ff^* \setminus z^*z) f \right) \leq (z^*z) \left[ (\bigvee F) (\bigvee F)^* \setminus (z^*z) \right] = z0 = 0
\]

where the next-to-last equality follows from the definition of relative complements and since products coincide with meets in \( E(S) \). Therefore

\[
z \left( \bigvee F \right) \leq z \left( \bigvee_{f \in F} (z^*zf) \right)
\]

Using the first case, we can distribute the product by \( z \) over \( \bigvee_{f \in F} (z^*zf) \), that is,

\[
z \left( \bigvee F \right) \leq \left( \bigvee_{f \in F} z(z^*zf) \right) = \bigvee zF.
\]

(b) This is a particular case of Proposition 1.2.41.
From item (b) we obtain \( \bigvee_{f \in F} (f \setminus z) \cap z = 0 \), and by usual properties of the join

\[
\bigvee F = \bigvee_{f \in F} f = \bigvee_{f \in F} (f \setminus z) \vee (z \land f) \leq \bigvee_{f \in F} \left( (f \setminus z) \vee (z \land \bigvee F) \right)
\]

\[
= \left( \bigvee_{f \in F} (f \setminus z) \right) \vee (z \land \bigvee F)
\]

so \( \bigvee_{f \in F} (f \setminus z) \) satisfies the defining properties of \( (\bigvee F) \setminus z \).

(d) This follows from item (b) and Lemma 1.8.9 applied on the Boolean algebra \( \{ t \in S : t \leq z \} \).

**Definition 1.8.12.** A \( \sigma \)-ideal of a \( \sigma \)-complete Boolean inverse semigroup \( S \) is an ideal \( I \) of \( S \) such that if \( F \subseteq S \) is countable and compatible, then \( \bigvee F \in I \).

If \( I \) is a \( \lor \)-ideal of a Boolean inverse semigroup \( S \) and \( s \in S \), recall from Section 1.5 that \( \bar{s}^I \) denotes the image of \( s \) under the canonical quotient map \( S \to S/I \).

**Lemma 1.8.13.** Let \( S \) be a \( \sigma \)-complete Boolean inverse semigroup, \( I \) a \( \sigma \)-ideal of \( S \) and \( \pi : S \to S/I \) the quotient map.

(a) If \( F \subseteq S \) is countable and \( \bar{f}^I = \bar{g}^I \) for all \( f, g \in F \), then \( \bar{f}^I = (\bigwedge F)^\sim I \) for all \( f \in F \);

(b) If \( \mathcal{A} \subseteq S/I \) is countable and compatible, then there is a countable compatible subset \( F \subseteq S \) such that \( \pi(F) = \mathcal{A} \).

**Proof.** (a) Under the given assumption, if \( f \in F \) then \( f \setminus (\bigwedge F) = \bigvee_{g \in F} (f \setminus g) \in I \), by Propositions 1.5.10 and 1.8.11(d), which is what we want.

(b) For every pair \( (\alpha_1, \alpha_2) \in \mathcal{A} \), choose, by 1.5.8 compatible elements \( C_1(\alpha_1, \alpha_2) \) and \( C_2(\alpha_1, \alpha_2) \in S \) such that \( C_1(\alpha_1, \alpha_2)^\sim I = \alpha_1 \), and then let \( D(\alpha) = \bigwedge_{\beta \in \mathcal{A}} C_1(\alpha, \beta) \land C_2(\beta, \alpha) \). By item (a), \( D(\alpha)^\sim I = \alpha \) for all \( \alpha \).

Given \( \alpha, \beta \in \mathcal{A} \), we have \( D(\alpha) \leq C_1(\alpha, \beta) \) and \( D(\beta) \leq C_2(\alpha, \beta) \), so \( D(\alpha) \) and \( D(\beta) \) are compatible for all \( \alpha, \beta \in \mathcal{A} \). Therefore \( F = \{ D(\alpha) : \alpha \in A \} \) has the required properties.

**Theorem 1.8.14.** If \( S \) is a \( \sigma \)-complete Boolean inverse semigroup and \( I \) is a \( \sigma \)-ideal of \( S \), then \( S/I \) is a \( \sigma \)-complete Boolean inverse semigroup as well. If \( \pi : S \to S/I \) is the quotient map, then for all countable compatible subsets \( F \subseteq S \), \( \pi(\bigvee F) = \bigvee \pi(F) \).
Proof. By item (b) of the lemma above, we just need to prove the last part. Let $F \subseteq S$ be countable and compatible. Of course, $\pi(\bigvee F) = (\bigvee F)^\sim_I$ is an upper bound for $\pi(F)$, so we need to prove it is the smallest one. Let $p \in S$ such that $\tilde{p}^I$ is another upper bound for $\pi(F)$. Then for all $f \in F$,

$$\tilde{f}^I = \tilde{f}^I \land \tilde{p}^I = (f \land p)^\sim_I$$

because $\pi$ is a morphism of Boolean inverse semigroups (Theorem 1.5.6). This means that $f \setminus p \in I$ for all $f \in F$, and so

$$\bigvee F \setminus \left( (\bigvee F) \land p \right) = \bigvee_{f \in F} (f \setminus p) \in I$$

hence

$$(\bigvee F)^\sim_I = (\bigvee F \land p)^\sim_I \leq \tilde{p}^I.$$

For our version of the Loomis-Sikorski theorem, we need the semigroups to satisfy the following condition:

**Definition 1.8.15.** We say that a Boolean inverse semigroup satisfies condition (SC) if there is a subset $C \subseteq S$ satisfying

- (SC1) $C$ is countable;
- (SC2) For every $s \in S$, there is a finite subset $F \subseteq C$ such that $s = \bigvee_{f \in F} (s \land f)$.

**Remark.** Every $\sigma$-complete Boolean inverse semigroup satisfying (SC) is actually a monoid, namely if $C \subseteq S$ satisfies (SC1) and (SC2) then $U = \bigvee_{c \in C} \text{Fix}(c)$ is the unit for $S$. Indeed, for all $e \in E(S)$,

$$e \leq \bigvee_{c \in C} (e \land c) \leq \bigvee_{c \in C} \text{Fix}(c) = U$$

so $U$ is the unit of $E(S)$ and thus the unit of $S$.

Note that every unital Boolean algebra $B$ satisfies (SC), namely $C = \{1\}$ satisfies (SC1) and (SC2).

**Proposition 1.8.16.** If $G$ is an ample Hausdorff groupoid and $S = \text{KB}(G)$, then $S$ satisfies condition (SC) if and only if $G$ is $\sigma$-compact. More precisely, a subset $C \subseteq \text{KB}(G)$ satisfies (SC1)-(SC2) if and only if $G \subseteq \bigcup C$.

**Proof.** If $G$ is $\sigma$-compact, then we have $G = \bigcup_n K_n$ for some sequence of compact open sets $K_n \subseteq G$. Since $\text{KB}(G)$ is an open cover of $G$, for each $n$ there is a finite subset $C_n \subseteq \text{KB}(G)$ satisfying $K_n \subseteq \bigcup C_n$. Then $C = \bigcup_n C_n$ is countable, which is condition (SC1). Since elements of $\text{KB}(G)$ are compact and $C$ is an open cover of $G$, condition (SC2) is satisfied by $C$.

Conversely, if $C \subseteq \text{KB}(G)$ satisfies (SC1)-(SC2), then $C$ is a countable cover of $G$ by compact(-open) sets, so $G$ is $\sigma$-compact. \qed
Theorem 1.8.17 (Non-commutative Loomis-Sikorski). Let $S$ be a $\sigma$-complete Boolean inverse semigroup satisfying (SC). Then there exists a $\tau$-discrete Borel groupoid $G$ and an ideal $I \subseteq \text{Bor}(G)$ such that $S$ is isomorphic to $\text{Bor}(G)/I$.

Proof. Using non-commutative Stone duality, more precisely Proposition 1.4.21, we let $G = G_P(S)$, which is an ample Hausdorff groupoid, and $\sigma$-compact since $S$ satisfies (SC) (Proposition 1.8.16). Endowing $G$ with the Baire $\sigma$-algebra (generated by $KB(G)$) we obtain a Borel groupoid (Example 1.6.4). Moreover, $\text{Bor}(G)$ coincides with the set of bisections which are Borel subsets of $G$ (Proposition 1.6.8).

Note that, even though $KB(G) \cong S$ is $\sigma$-complete, countable joins in $KB(G)$ will not be given by unions in general, and similarly for meets.

Let $\zeta = \zeta_S : S \to KB(G) \subseteq \text{Bor}(G)$ be the canonical isomorphism (Proposition 1.4.21). We fix a countable family $\{c_n\}_{n \in \mathbb{N}}$ in $S$ satisfying properties (SC1) and (SC2), and let $C_n = \zeta(c_n)$, so that $G \subseteq \bigcup_n C_n$.

Define
\[
\mathcal{N} = \left\{ \bigcap_{n=1}^{\infty} \zeta(s_n) : s_1, s_2, \ldots \in S \text{ and } \bigwedge_{n=1}^{\infty} s_n = 0 \right\}
\]
and let $\mathcal{I}$ be the collection of those $A \in \text{Bor}(G)$ which are contained in some countable union $\bigcup_{m=1}^{\infty} N_m$, where $N_m \in \mathcal{N}$. Then $\mathcal{I}$ is clearly closed under countable unions. To check it is an ideal, first suppose $K = \bigcap_n \zeta(s_n) \in \mathcal{N}$, where $\bigwedge_n s_n = 0$. For $A \in \text{Bor}(G)$,
\[
AK \subseteq \bigcup_{m=1}^{\infty} C_m \bigcap_{n=1}^{\infty} \zeta(s_n) = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \zeta(c_m s_n)
\]
and for all $m$, $\bigwedge_n c_m s_n = c_m \bigwedge_n s_n = 0$, which proves that $AK \in \mathcal{I}$. Similarly, $KA \in \mathcal{I}$, so $\mathcal{I}$ is a $\sigma$-ideal.

Let $\theta : S \to \text{Bor}(G)/\mathcal{I}$, $\theta(s) = \widetilde{\zeta(s)}^{-1}$, the composition of $\zeta$ with the quotient map $\text{Bor}(G) \to \text{Bor}(G)/\mathcal{I}$. We will prove that $\theta$ is an isomorphism, and we do this in several steps, the first one being a weaker form of injectivity.

If $s \neq 0$ then $\zeta(s) \not\in \mathcal{I}$: Indeed, suppose $N_1, N_2, \ldots \in \mathcal{N}$, where $N_n = \bigcap_m \zeta(a^n_m)$ ($a^n_m \in S$) are such that $\bigwedge_m a^n_m = 0$ for each $n$.

Since $s \neq 0$ and
\[
0 \neq s = s \setminus 0 = s \setminus \bigcap_m a^1_m = \bigcup_m (s \setminus a^1_m)
\]
then there exists a number $k(1)$ such that $s \setminus a^1_{k(1)} \neq 0$. By induction on $N$, we can find a sequence $k(1), k(2), \ldots$ such that $s \setminus (a^1_{k(1)} \lor \cdots \lor a^N_{k(N)}) \neq 0$ for all $n$. This means that $\zeta(s) \setminus \bigcup_{n=1}^{N} \zeta(a^n_{k(n)}) \neq \emptyset$ for all $N \in \mathbb{N}$, that is, $\zeta(s)$ is not contained in any finite union $\bigcup_{n=1}^{N} \zeta(a^n_{k(n)})$, $N \in \mathbb{N}$. Since $\zeta(s)$ is
compact, Cantor’s Intersection Theorem implies that \( \zeta(s) \) is not contained in \( \bigcup_{n=1}^{\infty} \zeta(a_n) \), and in particular \( \zeta(s) \) is also not contained in \( \bigcup_{n=1}^{\infty} N_n \). This proves that \( \zeta(s) \notin I \) whenever \( s \neq 0 \).

\( \theta \) is injective: If \( a \neq b \) in \( S \), then either \( a \setminus b \) or \( b \setminus a \neq 0 \), which implies that either \( \zeta(a) \setminus \zeta(b) \) or \( \zeta(b) \setminus \zeta(a) \notin I \), but in any case we obtain \( \theta(a) \neq \theta(b) \), so \( \theta \) is injective.

\( \theta \) preserves countable joins: If \( s_1, s_2, \ldots \in S \) are compatible, then \( \bigvee_{j=1}^{\infty} s_j \) is an upper bound of all \( s_i \), and so \( \bigcup_{i=1}^{\infty} \zeta(s_i) \subseteq \zeta \left( \bigvee_{j=1}^{\infty} s_j \right) \). Since

\[
\zeta \left( \bigvee_{j=1}^{\infty} s_j \right) \setminus \bigcup_{i=1}^{\infty} \zeta(s_i) = \bigcap_{i=1}^{\infty} \left( \bigvee_{j=1}^{\infty} s_j \setminus s_i \right)
\]

and

\[
\bigwedge_{i=1}^{\infty} \left( \bigvee_{j=1}^{\infty} s_j \setminus s_i \right) = \bigvee_{j=1}^{\infty} s_j \setminus \bigvee_{i=1}^{\infty} s_i = 0,
\]

we conclude that \( \zeta \left( \bigvee_{j=1}^{\infty} s_j \right) \setminus \bigcup_{i=1}^{\infty} \zeta(s_i) \in I \) and we are done.

Consider again, the fixed family of elements \( C_n = \zeta(c_n) \) with \( \mathcal{G} = \bigcup_n C_n \), and let \( p \in \mathbb{N} \) be temporarily fixed. In order to prove that \( \theta \) is surjective we first prove a weaker version. Recall from Section 1.5 that \( A \sim_I B \) means that \( A^I = B^I \).

If \( A \subseteq C_p \) is Borel, then \( A \sim_I \zeta(s) \) for some \( s \leq c_p \): The collection of Borel sub-

sets of \( C_p \) is generated, as a \( \sigma \)-algebra, by the family

\[
\{ A \in \text{KB}(\mathcal{G}) : A \subseteq C_p \} = \{ \zeta(a) : a \leq c_p \},
\]

so we need to prove that the collection \( \mathcal{B} \) of those Borel \( A \subseteq C_p \) which are \( I \)-equivalent to some \( \zeta(s) \), where \( s \leq c_p \), forms a \( \sigma \)-algebra on \( C_p \). If \( A \sim_I \zeta(s) \), then \( C_p \setminus A \sim_I \zeta(c_p \setminus s) \), so \( \mathcal{B} \) is closed under complements (in \( C_p \)). If \( A_1, A_2, \ldots \in \mathcal{B} \), choose \( s_1, s_2, \ldots \leq c_p \) with \( A_i \sim_I \zeta(s_i) \). In particular the \( s_i \) are pairwise compatible, and we have already verified that \( \zeta(\bigvee_i s_i) \) is \( I \)-equivalent to \( \bigcup_i \zeta(s_i) \), and thus \( I \)-equivalent to \( \bigcup_i A_i \) as well (Theorem 1.8.14).

\( \theta \) is surjective: Let \( A \in \text{Bor}(\mathcal{G}) \) be arbitrary. Again, decompose \( A = \bigcup_n A_n \) where \( A \subseteq C_n \). Choose \( s_n \in S \) with \( \zeta(s_n) \sim_I A_n \). The \( A_n \) are pairwise compatible, so \( \theta(s_n) \) are also pairwise compatible. Since \( \theta \) is injective, the \( s_n \) are pairwise compatible. Again using the fact that \( \theta \) preserves countable compatible joins, we see that \( A \sim_I \zeta(\bigvee_n s_n) \).
We can thus obtain a version characterizing inverse Boolean monoids satisfying (SC) and endowed with countably additive invariant means.

**Corollary 1.8.18.** Suppose $S$ is a $\sigma$-complete Boolean inverse monoid satisfying (SC), endowed with an invariant mean $\mu$ which is countably additive (i.e., a probability on $E(S)$). Then there exists a $\tau$-discrete probability measure-preserving groupoid $(G, \mu')$ and an isomorphism from $S$ to $\text{Meas}(G, \mu')$ which is isometric with respect to the uniform metrics.

**Remark.** We could have defined Borel groupoids as those endowed with $\sigma$-rings of sets, and not $\sigma$-algebras, assuming the structural maps are Borel in the appropriate sense (see Halmos, [80]). Then we would obtain, essentially with the same proofs, a version of the non-commutative Loomis-Sikorski theorems for Boolean inverse semigroup not necessarily satisfying (SC).
Chapter 2

Sofic groupoids

The notion of soficity for groups was introduced by Gromov ([76]) in his work on the Gottschalk Surjunctivity Conjecture in Symbolic Dynamics, as a common generalization of both amenable and LEF (“locally embeddable into finite”) groups. The terminology “sofic” was later coined by Weiss ([174]), and it comes from the Hebrew word for “finite”. This class of groups quickly showed its importance, due to both its tractability as well as its generality - In fact, at the moment no group has been shown to be non-sofic.

Sofic groups can be thought, in a broad sense, as those which obey all formulas which are “almost true” for finite groups. This allows one to prove, with the appropriate care, that several open conjectures for general countable groups actually hold in the sofic case: To list a few:

- The Gottschalk Surjunctivity Conjecture ([76, 174]);
- Connes’ Embedding Conjecture for their von Neumann algebras ([139]);
- Kaplansky’s Direct Finiteness Conjecture ([42]);
- Algebraic Eigenvalues Conjecture ([139])

For a more detailed introduction to soficity and the closely related class of hyperlinear groups, see [25].

More recently, Lewis Bowen ([17, 18]) introduced the notion of sofic entropy, later improved by David Kerr and Hanfeng Li ([102, 101, 103]), and which can be used to prove that large classes of Bernoulli shifts are non-isomorphic. This direction of work also points to interesting dynamical properties that sofic groups should have, and in particular a precise definition of a soficity for dynamical systems becomes of interest.

In 2010, Elek and Lippner [41] introduced the notion of soficity for equivalence relations in the same spirit as Gromov’s original definition – namely, if α is an action
of the free group $\mathbb{F}_\infty$ (on infinitely many generators) by measure-preserving automorphisms of a standard probability space $(X, \mu)$, then the orbit equivalence relation $R(\alpha)$ is sofic if the Schreier graph of the $\mathbb{F}_\infty$-space $X$ can be approximated, in a suitable sense, by Schreier graphs of finite $\mathbb{F}_\infty$-spaces.

Alternative definitions by Ozawa ([134]) and Păunescu ([142]) describe soficity at the level of the measured semigroup of $R$ or in terms of the action of the full group of $R$ on the measure algebra of the base space. These formulations have the advantages of being easily generalized to probability measure-preserving groupoids, and not depending on group actions. (See Definition 2.3.1.)

The main goal of this chapter is to describe soficity for probability measure-preserving groupoids in full details, closely relating it to how finite models can be used to describe the given sofic groupoid. We will prove several closure properties for the class of sofic groupoids in Section 2.3. The main results of this chapter (Theorems 2.4.16 and 2.4.18) provide a weakening in the definition of soficity, and answer a question posed in [31], using ideas similar to those in Dye’s recovery theorem of an equivalence relation from its full group ([39]). In the last section we study soficity of transformation groupoids of probability measure-preserving actions of countable groups.

2.1 Ultraproducts and continuous logic

2.1.1 Ultrafilters and ultralimits

Given a set $I$ and a collection of sets $\{X_i : i \in I\}$, an element of the product space $\prod_i X_i$ will be denoted by $(x_i)_{i \in I}$ or simply $(x_i)_i$, corresponding to a function $i \mapsto x_i$. In particular, an element of a function space $X^I$ will also be denoted as $(x_i)_i$.

By filters and ultrafilters of sets on $I$ we mean filters and ultrafilters, according to Definition 1.4.3 on the power set $\mathcal{P}(I)$ of $I$, which is a poset under inclusion (more precisely, a unital Boolean algebra under intersection), so let us briefly recall these notions in this special case: a filter of sets on $I$ is a nonempty collection $\mathcal{U}$ of subsets of $I$ satisfying

(i) If $F, G \in \mathcal{U}$, then $F \cap G \in \mathcal{U}$;

(ii) If $F \in \mathcal{U}$ and $F \subseteq G$ then $G \in \mathcal{U}$;

and we say $\mathcal{U}$ is proper if $\emptyset \notin \mathcal{U}$, and an ultrafilter of sets if it is a maximal proper filter of sets. Equivalently, a filter of sets $\mathcal{U}$ is an ultrafilter if and only if it is prime:

(iii) If $F \cup G \in \mathcal{U}$ then $F \in \mathcal{U}$ or $G \in \mathcal{U}$.

An ultrafilter $\mathcal{U}$ (of sets) on $I$ is free if $\bigcap \mathcal{U} = \emptyset$, and principal otherwise. Every principal ultrafilter $\mathcal{U}$ on $I$ is of the form $I_x := \{A \subseteq I : x \in A\}$ for some $x \in I$. 
(indeed, just choose \( x \in \bigcap \mathcal{U} \), so maximality of \( \mathcal{U} \) implies \( \mathcal{U} = I_x \)). In particular, an ultrafilter \( \mathcal{U} \) is principal if and only if it contains some finite subset of \( I \).

We can think of filters of sets as providing a notion of “largeness” for subsets of \( I \), that is, let \( \mathcal{U} \) be a fixed filter on \( I \), and let us temporarily call its elements large. The conditions above can be rewritten as

(i') If \( F \) and \( G \) are large then \( F \cap G \) is large as well;

(ii') If \( F \) is large and \( F \subseteq G \) then \( G \) is large as well.

This idea leads to an intuitive notion of convergence: \((x_i)\) converges to \( x \in X \) if the set of all \( i \in I \) for which \( x_i \) is sufficiently close to \( x \) is large. This can be formalized as follows:

**Definition 2.1.1.** Let \( I \) be a set, \( X \) a topological space, and \( \mathcal{U} \) a filter of sets on \( I \). We say that a family \((x_i) \in X^I\) converges to a point \( x \in X \) along \( \mathcal{U} \) if for every neighbourhood \( U \) of \( x \), the set \( \{i \in I : x_i \in U\} \) belongs to \( \mathcal{U} \). We call \( x \) a limit point of \((x_i)\) along \( \mathcal{U} \), and if \( \mathcal{U} \) is an ultrafilter of sets we call \( x \) a ultralimit.

This usage of filters can be thought of as a common generalization of both nets and the filters of subsets of a topological space \( X \) (see \[177\], section 12):

1. If \( \Lambda \) is a directed set and \((x_\lambda)\) is a net in \( X \), then convergence of this net corresponds to convergence of \((x_\lambda)\) along the filter of nonempty upwards closed subsets of \( \Lambda \);

2. If \( \mathcal{U} \) is a filter of subsets of \( X \), we take our index set \( I = X \) and consider the identity thread \((x)_x \in X^X \). Then convergence of \( \mathcal{U} \) ([177 12.3]) coincides with the notion above of convergence of \((x)_x \) along \( \mathcal{U} \).

As such, topological properties of \( X \) can be characterized in terms of convergence of functions along ultrafilters. The following characterizations of compactness and the Hausdorff property in terms of ultralimits are analogues to those in terms of nets (see [177 13.7 and 17.4]).

**Theorem 2.1.2.** \( X \) is compact if and only if for every set \( I \) and every ultrafilter of sets \( \mathcal{U} \) on \( I \), every function \((x_i) \in X^I\) converges along \( \mathcal{U} \).

**Proof.** First suppose that some function \((x_i)_i \) does not converge along \( \mathcal{U} \). Using compactness, we can find a finite open cover \( U_1, \ldots, U_n \) of \( X \) such that for \( 1 \leq k \leq n \), the set \( F_k = \{ i \in I : x_i \in U_k \} \) does not belong to \( \mathcal{U} \). But \( I = F_1 \cup \cdots \cup F_n \), which contradicts \( \mathcal{U} \) being an ultrafilter.

Conversely, suppose every function \((x_i)_i \in X^I\) converges along every ultrafilter on \( I \). We will use the correspondence above between nets and filters of sets to prove that every net on \( X \) has a cluster point, which is equivalent to \( X \) being compact.
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([177, Theorem 17.4]). So suppose \((\Lambda, \leq)\) is a directed set and \((x_\lambda)_{\lambda \in \Lambda}\) is a net in \(X\). Let \(U\) be any ultrafilter of sets on \(\Lambda\) containing all nonempty, upwards closed subsets of \(\Lambda\), which exists by Zorn’s Lemma. Then \((x_\lambda)_\lambda\) converges along \(U\) to some \(x \in X\), that is, if \(U\) is a neighbourhood of \(x\) and \(\rho \in \Lambda\), then the set \(\{\lambda \geq \rho : x_\lambda \in U\}\) belongs to \(U\) and is therefore nonempty, and this means that \((x_\lambda)_\lambda\) clusters at \(x\).

Theorem 2.1.3. \(X\) is Hausdorff if and only if ultralimits are unique.

Proof. First assume \(X\) is Hausdorff, and suppose a function \((x_i)_i\) converges to \(x\) and \(y\) along an ultrafilter \(U\) on \(I\). For \(z = x, y\), let \(U_z\) be an open neighbourhood of \(z\), so that \(F_z = \{i \in I : x_i \in U_z\} \in U\). Then \(F_x \cap F_y \in U\), and in particular there exists some \(j \in F_x \cap F_y\), that is \(x_j \in U_x \cap U_y \neq \emptyset\). This proves that every neighbourhood of \(x\) intersects every neighbourhood of \(y\), which means that \(x = y\) by the Hausdorff property.

Conversely, suppose that ultralimits are unique. To prove that \(X\) is Hausdorff, we simply need to prove that nets on \(X\) have at most one limit point ([177, Theorem 13.7]). Suppose that a net \((x_\lambda)_{\lambda \in \Lambda}\) converges to two points \(x\) and \(y\) in \(X\). Again, choose an ultrafilter \(U\) of sets on \(\Lambda\) containing all nonempty upwards closed subsets of \(\Lambda\). Then \((x_\lambda)_\lambda\) converges along \(U\) to both \(x\) and \(y\), hence \(x = y\).

Definition 2.1.4. If a family \((x_i)_i\) has a unique ultralimit \(x\) along an ultrafilter \(U\), we denote \(x = \lim_{i \to U} x_i\), or simply \(x = \lim_U x_i\).

2.1.2 Metric signatures and structures

In this section we will introduce ultraproducts, whose main motivation comes from Model Theory. A classical and still very interesting reference for algebraic ultraproducts in this setting is [10]. One way to describe modelatibility in the algebraic setting is the following:

Let \(B\) be an algebra (or a ring, group, Boolean algebra, etc. . . ) and let \(\mathcal{A}\) be a collection of algebras (or rings, groups, etc. . . ). We say that \(B\) is modelled by \(\mathcal{A}\) if for every finite subset \(F \subseteq B\), there is \(A \in \mathcal{A}\) and a map \(\phi : B \to A\) which is injective on \(F\), and which satisfies the usual properties of morphisms however only for elements of \(F\). Ultraproducts are then certain algebras which allow us to verify modelability by \(\mathcal{A}\) in terms of the existence of injective morphisms into ultraproducts.

We should note, however, that ultraproducts are not unique up to isomorphism, and there are other algebras (groups, etc. . . ) which can be used to check for modelability by \(\mathcal{A}\). (In the metric case, see [139], 3.11-3.13 for examples related to hyperlinear groups, and Section 2 of [43] for examples related to sofic groups.)

However, we will be interested in analytical, or more precisely metric structures. These are similar to classical (algebraic) structures in the sense that they have constants and functions, however predicates become continuous functions as well, with
values in $[0,1]$, instead of truth functions with values in $\{0,1\}$. This is actually of
great interest, since it allows us to ask how “far” is a statement from being the truth.
For example, two elements of a metric space $(M,d)$ are equal if and only if $d(x,y) = 0$,
and more generally they are precise at distance $d(x,y)$ from each other, so the dis-
tance map tell us how far are they from being the equal. With this, we can introduce
a notion of “metric modelability” similar to the one (algebraic) above and construct
ultraproducts with the analogue property. For more details, we refer to [29, 11].

Convention: If $(M,d^M)$ and $(N,d^N)$ are metric spaces, the product space $(M \times
N,d^{M \times N})$ will be considered with the supremum metric: $d^{M \times N}((m_1,n_1),(m_2,n_2)) = \max(d^M(m_1,m_2),d^N(n_1,n_2))$.

Recall the definition of a modulus of (uniform) continuity.

**Definition 2.1.5.** Let $(M,d^M)$ and $(N,d^N)$ be (pseudo-)metric spaces and $f : M \to
N$ a function. A modulus of (uniform) continuity for a $f$ is a function $\Delta : (0,\infty) \to
(0,\infty)$ which satisfies

$$
\forall \epsilon > 0, \text{ if } x,y \in M \text{ and } d^M(x,y) < \Delta(\epsilon) \text{ then } d^N(f(x),f(y)) \leq \epsilon
$$

In other words, a modulus of continuity for a function $f$ described “how close
two points in $M$ have to be in order for their images under $f$ to be close in $N$”. The
usage of strict and non-strict inequalities in the definition is necessary for the results
that follow.

**Definition 2.1.6.** A metric signature is a collection $\sigma = (\mathcal{P},\mathcal{F},\mathcal{C})$, where

- $\mathcal{P}$ is a collection of triples $(P,n(P),\Delta_P)$, where $P$ is a predicate symbol, and
  $n(P)$ is a positive integer, called its arity, and $\Delta_P$ is a function $\Delta_P : (0,\infty) \to
  (0,\infty)$;

- $\mathcal{F}$ is a collection of triples $(f,n(f),\Delta_f)$, where $f$ is a function symbol, $n(f)$ is a
  positive integer, called the arity of $f$, and $\Delta_f$ is a function $\Delta_f : (0,\infty) \to (0,\infty)$;

- $\mathcal{C}$ is a collection of constant symbols.

We will usually identify elements of $\mathcal{P}$ and $\mathcal{F}$ with their predicate and function
symbols, respectively.

**Definition 2.1.7.** A $\sigma$-structure consists of a metric space $L$ of diameter at most $1$
and an interpretation function, which associates:

- For every predicate symbol $P \in \mathcal{P}$, a function $P^L : L^{n(P)} \to [0,1]$ which has
  $\Delta_P$ as a modulus of continuity;

- For every function symbol $f \in \mathcal{F}$, a function $f^L : L^{n(f)} \to L$ which has $\Delta_f$ as
  a modulus of continuity;
• For every constant symbol $c \in \mathcal{C}$, a distinguished element $c^L \in L$.

We will always assume that the metric structure $\sigma$ has a binary predicate $d$, with modulus of continuity the identity function of $(0, \infty)$, and whose interpretation is the metric $d^M$ of the $\sigma$-structure $M$. We also assume we are given an infinite set $\mathcal{V}_L$ of variables, and that $\mathcal{P}, \mathcal{F}, \mathcal{C}, \mathcal{V}$ are pairwise disjoint.

Definition 2.1.8. An embedding of $\sigma$-structures $L$ and $M$ is a map $\phi : L \to M$ satisfying

- $P^M(\phi(x)) = P^L(x)$ for all $x \in L$ and $P \in \mathcal{P}$;
- $f^M(\phi(x_1), \ldots, \phi(x_n(f))) = \phi(f^L(x_1, \ldots, x_n(f)))$ for all $x_1, \ldots, x_n(f) \in L$ and $f \in \mathcal{F}$;
- $c^M = \phi(c^L)$ for all $c \in \mathcal{C}$.

(in particular, embeddings are always isometric since $d$ is a predicate interpreted as the metric).

Example 2.1.9. (1) A bounded metric space is a structure with the trivial signature (containing only the distance symbol $d$);

(2) Every classical structure, in the sense of first order logic ([86]), is a metric structure when endowed with the discrete metric: $d(x, y) = 0$ if $x = y$ and $1$ otherwise. The modulus of continuity for all function and predicate symbols is taken as $\Delta(\epsilon) = 0$ for all $\epsilon > 0$.

(3) A group $G$ with a bi-invariant, bounded metric $d$ is a structure for the signature containing one binary function symbol (the product) and one unary function symbol (the inverse) and appropriate moduli of continuity.

Example 2.1.10. A Boolean inverse monoid $S$ with a normalized, faithful invariant mean $\mu$ is a metric structure for the signature containing binary function symbols $\land, \lnot$ (for the semigroup operation), a unary function symbol $(\cdot)^*$ (and appropriate moduli of continuity) and constant symbols $0, 1$. We can also consider a unary predicate $E$ determining idempotents via $E(s) = d_{\mu}(s, s^*s)$.

One slightly more problematic operation is the “join”, which is only defined for compatible elements. One way to remedy this is to define a new operation $\hat{\vee}$ on $S$ as

$$s \hat{\vee} t = s \lor ((tt^* \setminus ss^*)t(t^*t \setminus s^*s)),$$

so that $s \hat{\vee} t = s \lor t$ whenever $s$ and $t$ are compatible. This way we can add the symbol $\lor$ to the signature of Boolean inverse monoids, and interpret it as $\hat{\vee}$ in $S$. 

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Remark. The metric theory of unbounded metric spaces is also of interest in functional analysis (see [83] for example), however some modifications need to be made, usually either considering pointed metric spaces, or by considering many-sorted structures, which we will not do here.

Definition 2.1.11. Let $\sigma$ be a metric structure. The collection of terms of $\sigma$ is the smallest set containing all constant symbols and variables of $\sigma$, and such that if $t_1, \ldots, t_n$ are terms and $f$ is an $n$-ary function symbol then $f(t_1, \ldots, t_n)$ is also a term. In the same manner as classical model theory ([86]), we will write $t(v_1, \ldots, v_n)$ to denote that the variables of $t$ are among $v_1, \ldots, v_n$.

2.1.3 Ultraproducts of metric structures and modelability

We will now construct ultraproducts of metric structures of a given signature following [11]. Let $\{(M_i, d_i) : i \in I\}$ be a collection of metric spaces of diameter at most $1$, and let $\mathcal{U}$ be an ultrafilter of sets on $I$. We first consider the product space $\prod_i(M_i, d_i)$, and define a pseudometric on it by setting

$$d^\mathcal{U}(\langle x_i \rangle_i, \langle y_i \rangle_i) = \lim_{i \to \mathcal{U}} d_i(x_i, y_i)$$

Let $\prod_{\mathcal{U}} M_i$ (or $\prod_{\mathcal{U}}(M_i, d_i)$ if we need to make the metrics $d_i$ explicit) be the quotient metric space, whose metric we also denote, $d^\mathcal{U}$.

Definition 2.1.12. We call the metric space $(\prod_{\mathcal{U}} M_i, d^\mathcal{U})$ the ultraproduct of the metric spaces $(M_i, d_i)$. We denote the class of an element $(x_i)_i \in \prod_i M_i$ in $\prod_{\mathcal{U}} M_i$ by $(x_i)_{\mathcal{U}}$.

Example 2.1.13. If $\mathcal{U} = \{A \subseteq I : i_0 \in A\}$ for some $i_0 \in I$, then $\prod_{\mathcal{U}} M_i$ is isometric to $M_{i_0}$. More precisely, the map

$$\prod_{\mathcal{U}} M_i \to M_{i_0}, \quad (x_i)_{\mathcal{U}} \mapsto x_{i_0}$$

is a (well-defined) isometry.

The following proposition can be proven along the same lines as when one constructs a completion of a metric space. See [154, Exercise 3.24].

Proposition 2.1.14. If $\mathcal{U}$ is a free ultrafilter, then $\prod_{\mathcal{U}} M_i$ is complete.

---

The same theory can be developed, almost word-by-word, for any other choice of uniform bound for the diameters. Alternatively, every bounded metric space can be normalized to diameter 1.
The following “distributivity” of ultraproducts over products will be necessary, and it follows quite easily from the fact that if $(a_i)_i, (b_i)_i$ are bounded functions of real numbers on a set $I$ and $\mathcal{U}$ is an ultrafilter of sets on $I$, then
\[
\lim_{\mathcal{U}} (\max(a_i, b_i)) = \max(\lim_{\mathcal{U}} a_i, \lim_{\mathcal{U}} b_i)
\]
because, indeed, either the set $\{i \in I : a_i > b_i\}$ or its complement belongs to $\mathcal{U}$.

**Proposition 2.1.15.** If $\{(M_i, d_i^M) : i \in I\}$ and $\{(N_i, d_i^N) : i \in I\}$ are collections of metric spaces (of diameter at most 1) and $\mathcal{U}$ is an ultrafilter of sets on $I$, then the map
\[
\prod_{i \in I} (M_i \times N_i) \to \left( \prod_{i \in I} M_i \right) \times \left( \prod_{i \in I} N_i \right), \quad (m_i, n_i)_{\mathcal{U}} \mapsto ((m_i)_{\mathcal{U}}, (n_i)_{\mathcal{U}})
\]
is a well-defined isometry.

The next obvious course of work is to define a natural $\sigma$-structure on an ultraproduct of $\sigma$-structures, where $\sigma$ is a metric signature.

**Proposition 2.1.16.** Let $\{(M_i, d_i^M) : i \in I\}$ and $\{(N_i, d_i^N) : i \in I\}$ be two families of metric spaces of uniformly bounded diameters, over the same set $I$, and let $\mathcal{U}$ be an ultrafilter on $I$. If $\{f^i : M_i \to N_i : i \in I\}$ is a family of functions admitting a common modulus of continuity $\Delta$, then there exists a unique function $f^\mathcal{U} : \prod_{i \in I} M_i \to \prod_{i \in I} N_i$ satisfying $f^\mathcal{U}((x_i)_{\mathcal{U}}) = (f^i(x_i))_{\mathcal{U}}$ for every $(x_i)_{\mathcal{U}} \in \prod_{i \in I} M_i$. Moreover, $\Delta$ is also a modulus of continuity for $f^\mathcal{U}$.

**Proof.** Let $f = \prod_{i \in I} f_i : \prod_{i \in I} M_i \to \prod_{i \in I} N_i$ be given by $f((x_i)_i) = (f_i(x_i))_i$. Given $\epsilon > 0$, suppose $d^\mathcal{U}((x_i)_i, (y_i)_i) \leq \Delta(\epsilon)$. This implies that $\{i \in I : d_i^M(x_i, y_i) < \Delta(\epsilon)\} \subseteq \mathcal{U}$, and since $\Delta$ is a common modulus of continuity for all $f_i$, then
\[
\{i \in I : d_i^M(x_i, y_i) < \Delta(\epsilon)\} \subseteq \{i \in I : d_i^N(f_i(x_i), f_i(y_i)) \leq \epsilon\} \subseteq \mathcal{U},
\]
which implies $d^\mathcal{U}(f((x_i)_i), f((y_i)_i)) \leq \epsilon$. In other words, $\Delta$ is a modulus of continuity for the function $f$ with respect to the respective pseudometrics $d^\mathcal{U}$, and in particular if $d^\mathcal{U}((x_i)_i, (y_i)_i) = 0$ then $d^\mathcal{U}(f((x_i)_i), f((y_i)_i)) = 0$, so $f$ factors uniquely to a function $f^\mathcal{U} : \prod_{i \in I} M_i \to \prod_{i \in I} N_i$ with all the desired properties.

**Proposition 2.1.17.** If $\{(M_i, d_i^M) : i \in I\}$ is a family of $\sigma$-structures and $\mathcal{U}$ is an ultrafilter of sets on $I$, then the ultrapower $\prod_{i \in I} M_i$ is also a $\sigma$-structure, where the interpretation is given by
\[
f^\mathcal{U}((m_i^1)_{\mathcal{U}}, \ldots, (m_i^n_{(f)})_{\mathcal{U}}) = ((f_{M_i}^1(m_i^1), \ldots, f_{M_i}^n(m_i^n_{(f)}))_{\mathcal{U}}
\]
\[ P^\mathcal{U} \left( \left( m_1^i \right)_\mathcal{U}, \ldots, \left( m_n^{P(i)} \right)_\mathcal{U} \right) = \lim_\mathcal{U} P^M \left( m_1^i, \ldots, m_n^{P(i)} \right) \]
\[ c^\mathcal{U} = (c^M)_\mathcal{U} \]
for every function symbol \( f \), every predicate symbol \( P \) and every constant symbol \( c \).

**Proof.** The only non-trivial parts are to prove that the interpretation of functions and predicates have the correct moduli of continuity. This statement for functions follows from Propositions 2.1.15 and 2.1.16.

To prove that \( P^\mathcal{U} \) has modulus of continuity \( \Delta_P \) (given by \( \sigma \)), where \( P \) is a predicate symbol, first note that the function \( \lim_\mathcal{U} \prod_\mathcal{U} [0, 1] \to [0, 1] \), which associates any \( (t_i)_\mathcal{U} \) to \( \lim_\mathcal{U} t_i \), has the identity \( \epsilon \mapsto \epsilon \) as a modulus of continuity. Using this fact with Propositions 2.1.15 and 2.1.16 again, we see that the given interpretation \( P^\mathcal{U} \) has \( \Delta_P \) as a modulus of continuity. \( \square \)

**Remark.** If \( \{(M_i, d^{M_i}) : i \in I\} \) is a family of \( \sigma \)-structures and \( \mathcal{U}, \mathcal{V} \) are distinct free ultrafilters of sets on \( I \), the question of whether the ultraproducts \( \prod_\mathcal{U} M_i \) and \( \prod_\mathcal{V} M_i \) are isomorphic or not may depend on set-theoretic statements.

For example, if \( \mathfrak{M} \) is a separable \( \Pi_1^1 \) factor and \( M_i \) is the unitary group of \( \mathfrak{M} \) for all \( i \) (endowed with the \( L^2 \)-metric induced by the trace of \( M \)), then the Continuum Hypothesis holds if and only if any two ultraproducts over free ultrafilters are isomorphic – See [55, Theorem 3.1].

A similar result, for ultrapowers of the real numbers, was proven in the classical setting in [152] (however in this case there exist non-isomorphic ultrapowers under the Continuum Hypothesis). For results not dealing with the Continuum Hypothesis, see Exercises 6.1.19 and 6.5.35 of [28].

The following theorem then allows us to obtain a precise relationship between embeddings into ultraproducts of metric structures and “approximate embeddings” into certain metric structures.

**Theorem 2.1.18.** Let \( \sigma = (\mathcal{P}, \mathcal{F}, \mathcal{C}) \) be a metric signature, let \( \mathcal{M} \) be a collection of \( \sigma \)-structures and \( L \) another \( \sigma \)-structure. Then the following are equivalent:

1. \( L \) embeds into an ultraproduct of elements of \( \mathcal{M} \);

2. For every collection of finite subsets \( A \subseteq L, P \subseteq \mathcal{P}, F \subseteq \mathcal{F} \) and \( C \subseteq \mathcal{C} \) and every \( \epsilon > 0 \), there exists \( M \in \mathcal{M} \) and a map \( \phi : L \to M \) such that
   \[ \sup_{a_i \in A, P \in P} |P^L(a_1, \ldots, a_n(P)) - P^M(\phi(a_1), \ldots, \phi(a_n(P)))| < \epsilon \]  \hspace{1cm} (2.1.1)
   \[ \sup_{a_i \in A, f \in F} d^M(\phi(f^L(a_1, \ldots, a_n(f))), f^M(\phi(a_1), \ldots, \phi(a_n(f)))) < \epsilon. \] \hspace{1cm} (2.1.2)
   \[ \sup_{c \in C} d^M(\phi(c^L), c^M) < \epsilon \] \hspace{1cm} (2.1.3)
Proof. (1)⇒(2): Assume such θ exists. Let \( \pi : \prod_i M_i \to \prod_{U_i} M_i \) be the quotient map and take a section \( \Theta = (\Theta_i) : L \to \prod_i M_i, \Theta(x) = (\Theta_i(x))_i \), of \( \theta \) along \( \pi \):

\[
\begin{array}{c}
L \xrightarrow{\Theta} \prod_i M \\
\downarrow \theta \quad \downarrow \pi \\
\prod_{U_i} M_i
\end{array}
\]

Note that \( \Theta \) is not necessarily a \( \sigma \)-morphism. However, if \( P \) is a finite set of predicate symbols and \( A \subseteq L \) is finite, then

\[
P^L(x_1, \ldots, x_{n(P)}) = \lim_{U \to \infty} P^{M_i}(\Phi^i_1(x_1), \ldots, \Phi^i_{n(P)}(x_{n(P)}))
\]

for all \( x_1, \ldots, x_n \in A \) and \( P \in P \), and similarly for function and constant symbols. But the definition of ultralimits, and the fact that \( U \) is closed under intersections, means that, given \( (A, P, F, C, \epsilon) \) as in (2), the set of all \( i \) for which \([2.1.1], [2.1.2] \) and \([2.1.3] \) holds belongs to \( U \), and in particular it holds for any element of this set.

(2)⇒(1): Let \( I \) be the collection of tuples \( \bar{p} = (A, P, F, C, \epsilon) \) as in (2) above. We order \( I \) by

\[
(A, P, F, C, \epsilon) \leq (A', P', F', C', \epsilon') \iff A \subseteq A', P \subseteq P', F \subseteq F', C \subseteq C' \text{ and } \epsilon \geq \epsilon'
\]

(note that the order is reversed in the last entry), which makes \( I \) a directed set.

For every \( \bar{p} \in I \), take a map \( \phi_{\bar{p}} : L \to M_{\bar{p}} \) satisfying the conditions \([2.1.1], [2.1.2] \) and \([2.1.3] \) for that given \( \bar{p} \). Then for every predicate symbol \( P \) and every \( x_1, \ldots, x_{n(P)} \in M_{\bar{p}} \),

\[
\lim_{\bar{p} \to \infty} P^{M_{\bar{p}}}(\phi_{\bar{p}}(x_1), \ldots, \phi_{\bar{p}}(x_{n(P)})) = P^L(x_1, \ldots, \phi(x_{n(P)})) \quad (2.1.4)
\]

and similarly for function and constant symbols, so letting \( U \) be an ultrafilter on \( I \) (as a set) containing all upwards closed subsets of \( I \), the same equation as in \([2.1.4] \) with ultralimits along \( U \) instead of limits along \( I \) and the function and constant symbol analogues still hold, which means that by composing \( \Phi = \prod_{\bar{p}} : L \to \prod_{\bar{p}} M_{\bar{p}} \) with the quotient map onto \( \prod_{U_i} M_i \) gives us a \( \sigma \)-morphism.

The theorem above shows us that ultraproducts can be used to give a precise notion of modelability by a class of structures (of a given signature), so it is natural to ask whether all statements which are “true”, in a suitable sense, in each of the factors \( M_i \), will also be true in any ultraproduct \( \prod_{U_i} M_i \), and consequently on any structure modelled by those \( M_i \).

We will now prove that ultraproducts can be used as models of its terms, in a suitable sense. More precisely, the (truth) value of any sentence in the ultraproduct will be the same as the limit (along the given ultrafilter) of the (truth) values of the same sentence in each component. This is the content of Łoś’ Theorem \([2.1.22] \).
Definition 2.1.19. An atomic formula of $\sigma$ is an expression (string of symbols) of the form $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms and $P$ is an $n$-ary predicate symbol. The collection of formulas of $\sigma$ is the smallest which satisfies the following properties:

(i) All atomic formulas are formulas;
(ii) If $u : [0, 1]^n \to [0, 1]$ is a continuous function and $\varphi_1, \ldots, \varphi_n$ are $L$-formulas, then $u(\varphi_1, \ldots, \varphi_n)$ is also a formula;
(iii) If $\varphi$ is a formula and $v$ is a variable then $\sup_v \varphi$ and $\inf_v \varphi$ are formulas.

Free and bound variables are defined the same way as in classical logic: A variable $v$ is bound if all occurrences of $v$ appear within $\sup_v \varphi$ or $\inf_v \varphi$, and $v$ is free otherwise. We write $\varphi(v_1, \ldots, v_n)$ to denote that $v_1, \ldots, v_n$ are the free variables of $\varphi$. A sentence is a formula without free variables.

We will now define interpretations of terms and formulas in a $\sigma$-structure $M$. Intuitively, we simply interpret all predicates, constant and function symbols in $M$ with the given interpretation function of $M$, all continuous functions $[0, 1]^n \to [0, 1]$ as themselves, and the free variables as unspecified elements of $M$.

Definition 2.1.20. The interpretation $t$ of a term $t(v_1, \ldots, v_n)$ (i.e., $v_1, \ldots, v_n$ are the variables appearing in $t$) in a $\sigma$-structure $M$ is a function $t^M : M^n \to M$, which is defined inductively on the complexity of $t$:

- Constants keep their original interpretation; The interpretation of a variable $v$ is simply the identity function: $v^M = \text{id}_M$;
- If $f$ is an $n$-ary function symbol and $t_1, \ldots, t_n$ are terms, all of whose variables are among $v_1, \ldots, v_k$, the interpretation of $f(t_1, \ldots, t_n)$ is the function

$$M^k \to M, \quad \overline{x} \mapsto f^M(t_1^M(\overline{x}), \ldots, t_n^M(\overline{x}))$$

where $t_i^M(\overline{x}) = t_i^M(\overline{x}_i)$, $\overline{x}_i$ being obtained from $\overline{x}$ by disregarding any entries corresponding to variables not appearing in $t$.

In other words, the interpretation of a term $t(v_1, \ldots, v_n)$ at $(x_1, \ldots, x_n) \in M^n$ is the element of $M$ obtained by substituting each variable $v_i$ by the corresponding element $x_i$, and interpreting any function or constant symbol in $M$.

Remark. By writing a term as $t(v_1, \ldots, v_n)$, we are considering a non-canonical ordering on the set of variables, so a more precise way of defining it would be as a function $t^M : M^{\mathcal{V}(t)} \to M$, where $\mathcal{V}(t)$ is the (unordered) set of variables in $t$, in a manner similar to above.

However, we will keep using the more common approach from [11] and [80].
An analogous comment for formulas, considered below, also hold.

**Definition 2.1.21.** The interpretation of a formula \( \varphi(v_1, \ldots, v_k) \) is a function \( \varphi^M : M^k \to [0, 1] \). If \( \varphi \) does not have free variables (i.e., is a sentence), we regard \( M^0 \) as a singleton so \( \varphi^M \) is given by a single number, called the value of \( \varphi \) in \( M \).

- The interpretation of an atomic formula \( P(t_1, \ldots, t_n) \) at a tuple \( \overline{x} \in M^k \) is \( P^M(t_1(\overline{x}), \ldots, t_n(\overline{x})) \).
- If \( \varphi_1, \ldots, \varphi_n \) are formulas and \( u : [0, 1]^n \to [0, 1] \) is a continuous function, the value of \( u(\varphi_1, \ldots, \varphi_n) \) at \( \overline{x} \in M^k \) is \( u(\varphi_1^M(\overline{x}), \ldots, \varphi_n^M(\overline{x})) \).
- If \( \varphi(v, v_1, \ldots, v_n) \) is a formula, then the values of \( \sup_v \varphi \) and \( \inf_v \varphi \) at \( \overline{x} \in M^k \) are \( \sup_{n \in M} \varphi(n, \overline{x}) \) and \( \inf_{n \in M} \varphi(n, \overline{x}) \), respectively.

(Again, \( t(\overline{x}) \) is obtained by disregarding entries corresponding to variables not appearing in \( t \), and similarly for subformulas.)

**Theorem 2.1.22 ([11], Theorem 5.4], Continuous Łoś Theorem).**

If \( \prod_M M_i \) is an ultraproduct of \( \sigma \)-structures, then for every formula \( \varphi(v_1, \ldots, v_n) \), and every \( (x_i^1)_i, \ldots, (x_i^n)_i \in \prod_i M_i \),

\[
\varphi^U((x_i^1)_U, \ldots, (x_i^n)_U) = \lim_{U} \varphi^M_i(x_i^1, \ldots, x_i^n)
\]

The proof can be done on the complexity of formulas (or more precisely, by proving that the collection of formulas for which this holds is closed under (i)-(iii) of Definition 2.1.19), however the notation during the proof gets somewhat cumbersome and so we simply sketch it: Property (i) is immediate from the interpretation we use on ultraproducts. Property (ii) uses the fact that whenever \( u : [0, 1]^n \to [0, 1] \) is a continuous function and \( (x_i)_i \in ([0, 1]^n)^I \),

\[
\lim_{i \to U} u(x_i) = u(\lim_{i \to U} x_i),
\]

which can be proven just as with the usual notion of limits.

For property (iii) we need to use the fact that

\[
\sup_{(n_i)_i \in \prod_i M_i} \left( \lim_{i \to U} f^i(n_i) \right) = \lim_{i \to U} \left( \sup_{n_i \in M_i} f^i(n_i) \right)
\]

whenever we are given a collection of functions \( f^i : M_i \to [0, 1] \). This can be verified with the properties of ultralimits and real suprema. The case of infima is analogous.

### 2.1.4 Ultraproducts of Boolean inverse monoids

In this subsection, we review two constructions of ultraproducts of Boolean inverse monoids endowed with invariant means.
First construction

Let \( \{ (S_i, \mu_i) : i \in I \} \) be a collection of Boolean inverse monoids with faithful, normalized invariant means \( \mu_i \), and let \( d^i \) be the corresponding uniform metric on \( S_i \), that is,

\[
d^i(s, t) = \mu_i((s \setminus t)^*(s \setminus t) \lor (t \setminus s)^*(t \setminus s)).
\]

Then \( \prod_i S_i \) is also a Boolean inverse monoid with pointwise operations, and all the order-theoretical structure (the order itself, joins, meets, fixed idempotents, etc...) are also the pointwise ones.

Example 2.1.23. We can interpret finite products as follow: If \( G \) and \( H \) are ample Hausdorff groupoids and \( G \sqcup H \) is the coproduct groupoid with coproduct topology, then \( KB(G) \times KB(H) \) is isomorphic to \( KB(G \sqcup H) \), via

\[
KB(G \sqcup H) \to KB(G) \times KB(H), \quad A \mapsto (A \cap G, A \cap H).
\]

This is a particular instance of products being mapped to coproducts and vice-versa under categorical equivalences.

Considering again the family \( \{ (S_i, \mu_i) : i \in I \} \) as above, define a normalized mean \( \mu^U : E(\prod_i S_i) \to [0, 1] \)

\[
\mu^U((e_i)_i) = \lim_U \mu_i(e_i), \quad \text{for all } (e_i)_i \in E\left( \prod_{i \in I} S_i \right) = \prod_{i \in I} E(S_i),
\]

so the uniform pseudometric (Definition 1.7.15) \( d^U \) associated to \( \mu^U \) is precisely the ultralimit of the uniform metrics \( d^i \) associated to \( \mu_i \). Hence considering the ideal

\[
\mathcal{N} = \left\{ (s_i)_i \in \prod_i S_i : \mu_U((s_i)_i) = 0 \right\},
\]

the ultraproduct \( \prod_U S_i \) coincides, as a metric space, with the Boolean inverse monoid quotient \( (\prod_i S_i) / \mathcal{N} \), endowed with the induced invariant mean and metric from \( \mu_U \) (see 1.7.18).

Second construction

Let \( \{ (S_i, \mu_i) : i \in I \} \) and \( d^i \) be as above. As in 2.1.10 for each \( i \in I \), \( d^i \) defines a metric structure on \( S_i \), over the signature containing binary function symbols \( \lor, \land, \setminus \) and \( \cdot \) (where \( \lor \) is interpreted appropriately), a unary function symbol \( ( )^* \) and constant symbols 0, 1. The moduli of continuity for these symbols can be obtained from Proposition 1.7.17. The ultraproduct \( \prod_U S_i \) is therefore also a \( \sigma \)-structure, and one can verify that it is a Boolean inverse semigroup by repeated applications of Łoś' Theorem. For example:
• Each $S_i$ is associative, which means that the sentence

$$\sup_x \sup_y \sup_z d((xy)z, x(yz))$$

has value zero in all $S_i$, so the same sentence has the same value on $\prod \mathcal{U} S_i$, and this is precisely saying that it is associative;

Some other properties can be proven similarly, however conditional statement usually require more care. For example, if we want to prove that inverses are unique, this means that for all $s, t$, either $d(s, sts) > 0$, $d(t, tst) > 0$ or $d(t, s^*) = 0$. However, the limit of positive numbers (along an ultrafilter) could be zero. Instead, we need to perform the following analysis:

**Lemma 2.1.24.** $\prod \mathcal{U} S_i$ is an inverse semigroup.

*Proof.* For all $s, t$ in a Boolean inverse semigroup $S$ with an invariant mean $\mu$, we have

$$s^* \setminus t = s^* \setminus (s^* stss^*) = s^* (s \setminus sts) s^*$$

and similarly for $t \setminus s^*$, and from this we can conclude that

$$d_\mu(s^*, t) = \mu((s^* \setminus t)^*(s^* \setminus t) \lor (t \setminus s^*)^*(t \setminus s^*)) \leq d_\mu(s, sts) + d_\mu(t, tst)$$

Thus the formula $d(x^*, y)$ is always smaller than the formula $d_\mu(x, xyx) + d(y, xyx)$ (divide them by 2, if you want to guarantee all numbers are between 0 and 1) in each of the components $S_i$, so by Łos’ Theorem the same is valid in $\prod \mathcal{U} S_i$ as well: For every $s, t \mathcal{U} \in \prod \mathcal{U} S_i$,

$$d^\mathcal{U}(s^*, t) \leq d^\mathcal{U}(s, sts) + d^\mathcal{U}(t, tst)$$

and this guarantees that the only inverse of $s$ will be $s^*$.

To verify that the interpretations of $\land$, $\lor$ and $\setminus$ in the ultraproduct $\prod \mathcal{U} S_i$ are indeed those of meets, joins (of compatible elements) and relative complements of inverse semigroups, a similar analysis is needed in principle. However, by construction, the structure of $\prod \mathcal{U} S_i$ is the same as the semigroup structure given in the first construction above, so we omit these proofs.

**Remark.** Alternatively, Boolean inverse semigroups can be axiomatized by equations (i.e., without conditional statements) in terms of “biases” (see [173, Section 3.4]). Using this axiomatization instead, we can easily conclude, by Łos’ Theorem, that an ultraproduct of Boolean inverse semigroups is also a Boolean inverse semigroup.

We can restate the condition of modelability by a certain class of semigroups as follows:
Theorem 2.1.25. Let \((S, \mu)\) be a Boolean inverse monoid with an invariant mean, and let \(\mathcal{T}\) be another collection of Boolean inverse semigroups endowed with invariant means. Then the following are equivalent:

1. There exists a trace-preserving (equivalently, isometric) semigroup morphism 
   \(\theta : S \to \prod_{U(T_i, \mu_i)}\), where \((T_i, \mu_i) \in \mathcal{T}\).
2. For every finite subset \(F\) of \(S\) and every \(\epsilon > 0\), there exists \((T, \nu) \in \mathcal{T}\) and a map \(\phi : S \to T\) satisfying
   \[\sup_{s \in F} |\text{tr}(s) - \text{tr}(\phi(s))| \leq \epsilon \quad \text{and} \quad \sup_{s, t \in S} d_{\nu}(\phi(st), \phi(s)\phi(t)) \leq \epsilon\]

Moreover, a map \(\theta\) as in (1) is necessarily a morphism of Boolean inverse monoids. A map \(\phi\) as in (2) is called a \((F, \epsilon)\)-approximate morphism.

Proof. The equivalence of trace-preserving and isometric properties for \(\theta\), as well as it being a morphism of Boolean inverse monoid, follow from Theorem 1.7.25 (and if \(\mu\) is not faithful). The equivalence is then just a particular case of Theorem 2.1.18.

2.2 Finite probability measure-preserving groupoids

Whenever \(\{(S_i, \mu_i) : i \in I\}\) is a collection of Boolean inverse monoids with invariant means and \(\mathcal{U}\) is an ultrafilter of sets on \(I\), the ultraproduct \(\prod_{\mathcal{U}} S_i\) will be considered with the usual structure described in Subsection 2.1.4. Moreover, if \((G, \mu)\) is a probability measure-preserving groupoid (Definition 1.6.20), we will not make a distinction between \(\text{Bor}(G, \mu)\) and \(\text{Meas}(G, \mu)\), unless strictly necessary.

The same way as the properties of a measure space \((X, \mu)\) are usually described in terms of its Borel sets and the measure, the dynamical properties of a probability measure-preserving groupoid \((G, \mu)\) will be described in terms of the semigroup \(\text{Meas}(G, \mu)\). Since soficity is a property which describes the groupoids which can be modelled by finite groupoids, in the sense of the previous section, we start by describing the latter class.

Definition 2.2.1. If \(X\) is a finite set, we denote by \(#X\) the cardinality of \(X\), and we let \(\mu_\#\) be the normalized counting measure on \(X\) (as a discrete Borel space):
\[\mu_\#(A) = \#A/#X \quad \text{for all} \quad A \subseteq X.\]

Proposition 2.2.2. If \(G\) is a connected finite groupoid, then the only invariant probability measure on \(G^{(0)}\) (as a discrete Borel space) is the normalized counting measure \(\mu_\#\) on \(G^{(0)}\).
Proof. If \( \mu \) is an invariant measure on \( \mathcal{G}(0) \), then for all \( x, y \in \mathcal{G}(0) \) we can choose \( a \in \mathcal{G} \) with \( s(a) = x \) and \( r(a) = y \). In particular, \( A = \{a\} \) is a bisection of \( \mathcal{G} \), so
\[
\mu(\{x\}) = \mu(A^*A) = \mu(\mathcal{A}A^*) = \mu(\mathcal{A})
\]
(where \( A^* = \{a^{-1} : a \in A\} \) is the inverse of \( A \)), and so \( \mu \) is a multiple of the counting measure. If \( \mu \) is a probability measure then \( \mu = \mu^\# \).

In particular, if \( X \) is a finite set and we consider the finest equivalence relation \( X^2 \) on \( X \) as a (discrete) probability measure-preserving groupoid, the only invariant measure on \( X \), identified as the unit space \( (X^2)^{(0)} \) is the normalized counting measure \( \mu^\# \) on \( X \).

Whenever necessary, we will identify \( \mu^\# \) with a normalized, faithful invariant mean on \( \text{Bor}(X^2) \) (as in Example 1.7.13) and so it induces a metric \( d^\# \) on \( \text{Bor}(X^2) \), which is usually called the normalized Hamming distance, and the associated trace \( \text{tr}^\# \) (Definition 1.7.21).

Since \( X^2 \) is an equivalence relation, we identify \( \text{Bor}(X^2) = \mathcal{I}(X) \), as in Example 1.2.11 in which case the trace and distance become
\[
\text{tr}^\#(f) = \mu^\# \{ x \in \text{dom}(f) : f(x) = x \}
\]
\[
d^\#(f, g) = \mu^\#(\text{dom}(f) \triangle \text{dom}(g)) + \mu^\# \{ x \in \text{dom}(f) \cap \text{dom}(g) : f(x) \neq g(x) \}
\]

Following the notation introduced in Definition 1.6.22 we will denote this Boolean inverse semigroup endowed with this invariant mean and the associated metric and trace as \( \text{Meas}(X^2, \mu^\#) \) or simply \( \text{Meas}(X^2) \). We will prove that these groupoids actually serve as models for arbitrary finite probability measure-preserving groupoids.

**Proposition 2.2.3.** (a) If \((\mathcal{G}, \mu)\) is a connected finite probability measure-preserving groupoid, then there exists a finite set \( X \) and an isometric semigroup embedding
\[
\theta : \text{Meas}(\mathcal{G}, \mu) \to \text{Meas}(X^2, \mu^\#).
\]

(b) If \( \mathcal{G} \) is a finite probability measure-preserving groupoid, then there are finite sets \( X_n, n \in \mathbb{N} \) and an isometric semigroup embedding \( \Theta : \text{Meas}(\mathcal{G}) \to \prod_U \text{Meas}(X_n^2) \), where \( U \) is any free ultrafilter on \( \mathbb{N} \).

Before proceeding with the proof, we introduce some notation which will be useful here as well as later in this chapter.

**Definition 2.2.4.** If \( \{(\mathcal{G}_n, \mu_n)\}_{n} \) is a sequence of probability measure-preserving groupoids and \( t_n \) are non-negative numbers such that \( \sum_n t_n = 1 \), we construct the convex combination groupoid \( \mathcal{G} = \sum t_n \mathcal{G}_n \) as follows: As groupoid, \( \mathcal{G} \) is the coproduct \( \mathcal{G} = \sqcup_n \mathcal{G}_n \), endowed with the Borel structure generated by the Borel structures of each \( \mathcal{G}_n \), and the measure \( \mu \) on \( \mathcal{G} \) is given by \( \mu(A) = \sum_n t_n \mu_n(A \cap \mathcal{G}_n) \).
Proof of 2.2.3. (a) Using 1.1.31(a), suppose \( \mathcal{G} = G \times Y^2 \), where \( G \) is a finite group, \( Y \) is a finite set, and \( Y^2 \) is the coarsest equivalence relation on \( Y \).

Moreover, since \( \mathcal{G} \) is connected then the invariant probability measure on \( \mathcal{G}^{(0)} = Y \) is the normalized counting measure on \( Y \). Let \( X = G \times Y \) and set \( \pi : \text{Meas}(G \times Y^2) \to \text{Meas}(X^2) \) by

\[
\theta(A) = \{((gh, y), (h, x)) : (g, (y, x)) \in A, h \in G\}
\]

which is easy to verify to be an isometric embedding.

(b) First suppose \( \mathcal{G} = t\mathcal{H} + (1 - t)\mathcal{K} \), where \( \mathcal{H}, \mathcal{K} \) are connected finite probability measure-preserving groupoids. If \( t = 0 \) then the natural inclusion \( \text{Bor}(\mathcal{K}) \subseteq \text{Bor}(\mathcal{G}) \) factors to an isometric isomorphism \( \text{Meas}(\mathcal{K}) \to \text{Meas}(\mathcal{G}) \), and similarly if \( t = 1 \), so we can assume \( t \neq 0, 1 \). Suppose further that \( t \) is rational, say \( t = p/q \) where, \( p, q \in \mathbb{N} \), \( 0 < p < q \).

By (a), take finite sets \( X \) and \( Y \) and isometric embeddings \( \pi_\mathcal{H} : \text{Meas}(\mathcal{H}) \to \text{Meas}(X^2) \) and \( \pi_\mathcal{K} : \text{Meas}(\mathcal{K}) \to \text{Meas}(Y^2) \). Let \( [q] = \{0, 1, \ldots, q - 1\} \) be a finite set with \( q \) elements, and set \( Z = [q] \times X \times Y \) and \( \theta : \text{Meas}(\mathcal{G}) \to \text{Meas}(Z^2) \) by

\[
\theta(A) = \{((j, x_2, y), (j, x_1, y)) : 0 \leq j \leq p - 1, y \in Y, (x_2, x_1) \in \pi_\mathcal{H}(A \cap \mathcal{H})\} \\
\cup \{((j, x_2, y_2), (j, x, y_1)) : p \leq j \leq q, x \in X, (y_2, y_1) \in \pi_\mathcal{K}(A \cap \mathcal{K})\}
\]

so that \( \theta \) is an isometric embedding.

Iterating the argument above, if \( \mathcal{G} \) is a rational convex combination of connected groupoids then \( \text{Meas}(\mathcal{G}) \) embeds isometrically into some measured semigroup \( \text{Meas}(Z^2) \) for some finite set \( Z \).

Now given an arbitrary finite probability measure-preserving groupoid \( (\mathcal{G}, \mu) \), Proposition 1.1.30 guarantees that \( \mathcal{G} \) is a coproduct of finite connected groupoids, say \( \mathcal{G} = \bigsqcup_{i=1}^k \mathcal{G}_i \). As above, we can assume \( \mu(\mathcal{G}_i^{(0)}) \neq 0 \) for all \( i \), so the map \( A \mapsto \mu(A)/\mu(\mathcal{G}_i^{(0)}) \) is an invariant measure on \( \mathcal{G}_i \), thus it coincides with the normalized counting measure. This implies that \( \mathcal{G} \) is actually the convex combination

\[
\mathcal{G} = \sum_{i=1}^k \mu(\mathcal{G}_i^{(0)}) \mathcal{G}_i.
\]

Taking a sequence of rational tuples \( (t_1^n, \ldots, t_k^n) \), \( n = 1, 2, \ldots \) such that

\[
\sum_{i=1}^k t_i^n = 1 \quad \text{for all } n, \quad \text{and} \quad t_i^n \xrightarrow{n \to \infty} \mu(\mathcal{G}_i^{(0)}) \quad \text{for all } i,
\]
we obtain isometric semigroup morphisms \( \theta_n : \text{Meas}(\sum t_n G_i^{(0)}) \to \text{Meas}(X_n^2) \)

for certain finite sets \( X_n \). Define a semigroup morphism

\[
\Theta : \text{Meas}(G) \to \prod_{\mathcal{U}} \text{Meas}(X_{\mathcal{U}}^2), \quad \Theta(A) = (\theta_n(A))_{\mathcal{U}},
\]

which is isometric with respect to \( d_\mu \).

\[\square\]

### 2.3 Sofic groupoids

**Definition 2.3.1.** A probability measure-preserving groupoid \( G \) is **sofic** if there exists a collection \( \{ G_i : i \in I \} \) of finite probability measure-preserving groupoids and an isometric (equivalently, trace-preserving) semigroup morphism \( \Theta : \text{Meas}(G) \to \prod_{\mathcal{U}} \text{Meas}(G_i^2) \) for some ultrafilter \( \mathcal{U} \) on \( I \). We call \( \theta \) a **sofic embedding** of \( G \).

Originally (see [134] or [31]), sofic groupoids were defined as those whose measured semigroup embeds isometrically into an ultraproduct of semigroups of the form \( \text{Meas}(X^2) \), where \( X \) is a finite set, however Proposition 2.2.3 and Theorem 2.1.25 imply that this coincides with the definition given above.

The definition we use also allows us to make constructions with finite groupoids more freely, whereas if we restricted ourselves to those of the form \( \text{Meas}(X^2) \) we would have to perform arguments as in the proof of 2.2.3 repeatedly.

Currently, no example of non-sofic groupoid exists.

**Remark.** (i) For every positive integer \( n \), let \( Y_n = \{0, \ldots, n-1\} \) be a set with \( n \) elements. Suppose \( p, q, n \) and \( r \) are non-negative integers \( p = qn + r \), where \( 0 \leq r < n \). Define \( \phi_{p,n} : \text{Meas}(Y_n^2) \to \text{Meas}(Y_p^2) \) by

\[
\phi_{p,n}(A) = \{(tn+j, tn+i) : 0 \leq t \leq q-1, \quad (j, i) \in A\}
\]

Then \( \phi_{p,n} \) is a semigroup embedding, and for all \( A, B \in \text{Meas}(Y_n^2) \),

\[
|d_\#(\phi_{p,n}(A), \phi_{p,n}(B)) - d_\#(A, B)| = \left( \frac{nq}{nq + r} - 1 \right) d_\#(A, B) < \frac{1}{p},
\]

which converges to 0 as \( p \to \infty \).

(ii) If \( (G, \mu) \) is a standard \( r \)-discrete probability measure-preserving groupoid, then \( \text{Meas}(G, \mu) \) is separable. Indeed, we can define a measure on \( G \) by

\[
\nu(A) = \int_{G^{(0)}} \#(G \cap A) d\mu(x)
\]

which is \( \sigma \)-finite because \( G \) can be covered by countably many Borel bisections, all of which have measure at most 1. The associated pseudometric \( d_\nu(A, B) = \)
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\[ \nu(A \Delta B) \] on the collection \( B \) of Borel subsets of \( G \) is then separable, because \( G \) is standard \( \text{[99, Exercise 17.43]} \), and any \( d_\nu \)-dense countable subset of \( \text{Bor}(G) \) will also be dense in \( \text{Meas}(G, \mu) \) with respect to \( d_\mu \).

Using facts (i) and (ii) above, a simple adaptation of the proof of Theorem 2.1.25 gives us the following:

**Proposition 2.3.2.** If \((G, \mu)\) is a standard \( \tau \)-discrete probability measure-preserving groupoid, then the following are equivalent:

1. \((G, \mu)\) is sofic;
2. For any sequence of finite sets \( Y_n \) such that \( \#Y_n \to \infty \) and for any free ultrafilter \( U \) on \( \mathbb{N} \), \( G \) admits an isometric semigroup embedding into \( \prod_U \text{Meas}(Y_n^2) \);
3. For any sequence of finite sets \( Y_n \) such that \( \#Y_n \to \infty \), there exists a sequence of maps \( \theta_n : \text{Meas}(G) \to \text{Meas}(Y_n^2) \) such that for any \( A, B \in \text{Meas}(G) \),
   \[ \lim_{n \to \infty} d_\#(\theta_n(A)\theta_n(B), \theta_n(AB)) = 0 \quad \text{and} \quad \lim_{n \to \infty} \text{tr}_\#(\theta_n(A)) = \text{tr}_\mu(A). \]

(The third item can be compared to the notion of sofic approximation given in \( \text{[19]} \).)

**Example 2.3.3.** Let \( G \) be a countable group, seen as a probability measure-preserving groupoid endowed with the point-mass measure at 1 and the discrete Borel structure. Then \( G \) is sofic as a groupoid, if and only if it is sofic as a group in the sense of Gromov \( \text{[76]} \). See also \( \text{[42, 44, 139, 174]} \) and Lemma 2.4.13.

We will analyze this particular case further in Section 2.5.

We now study permanence properties of the class of sofic groupoids. We will simply say that an invariant measure \( \mu \) on a Borel groupoid \( G \) is sofic if \((G, \mu)\) is a sofic (probability measure-preserving) groupoid. Given a non-null Borel subgroupoid \( H \) of \( G \) (that is, \( \mu(H^{(0)}) \neq 0 \)), denote by \( \mu_H \) the normalized measure on \( H^{(0)} \), i.e., \( \mu_H(A) = \mu(A) / \mu(H^{(0)}) \) for all Borel \( A \subseteq H^{(0)} \), and by \( \text{tr}_H \) for the corresponding trace on \( \text{Meas}(H, \mu_H) \).

A similar class of results for sofic groups is given in Theorem 2.5.20.

**Theorem 2.3.4.** Let \((G, \mu)\) be a probability measure-preserving groupoid.

(a) If \( \mu \) is a strong limit\(^2\) of sofic measures on \( G \), then \( \mu \) is sofic as well.

(b) Any convex combination (finite or countably infinite) of sofic measures is sofic.

\(^2\)Recall that a net \( \{\mu_i\}_{i \in I} \) of measures on a measurable space \((X, B)\) converges strongly to a measure \( \mu \) if \( \mu_i(A) \to \mu(A) \) for all \( A \in B \).
(c) If $\mu$ has a disintegration of the form $\mu = \int_X p_x d\nu(x)$, where $(X, \nu)$ is a probability space and $\nu$-a.e. $p_x$ is a probability measure on $G^{(0)}$ such that $(G, p_x)$ is sofic, then $(G, \mu)$ is also sofic.

(d) If $(G, \mu)$ is sofic and $H$ is a non-null subgroupoid of $G$ then $(H, \mu_H)$ is sofic.

(e) If $\{H_n\}_n$ is a countable Borel partition of $G$ in non-null subgroupoids, then $G$ is sofic if and only if each $H_n$ is sofic.

(f) If $\nu \ll \mu$, both $\nu$ and $\mu$ being invariant probability measures on $G^{(0)}$, and $\mu$ is sofic, then $\nu$ is sofic as well.

Recall that we are identifying $\text{Meas}(G)$ with $\text{Bor}(G)$ whenever necessary. Moreover, we will use the following notation: If $x$ and $y$ are (complex) numbers and $\epsilon > 0$, we write $x = y \pm \epsilon$ whenever $|x - y| \leq \epsilon$.

**Proof of Theorem 2.3.4**

- (a) is clear since soficity is an approximation property for the measure (or more precisely, by Theorem 2.1.25).

- (b) Suppose $\nu, \rho$ are sofic measures and $\mu = t\nu + (1 - t)\rho$, $0 < t < 1$. Take sofic embeddings $\phi : \text{Meas}(G, \nu) \to \prod_U \text{Meas}(G_i)$ and $\psi : \text{Meas}(G, \rho) \to \prod_V \text{Meas}(H_j)$, where $U$ and $V$ are ultrafilters on index sets $I$ and $J$, respectively.

  Given $i$ and $j$, $\text{Meas}(G_i)$ naturally embeds into $\text{Meas}(tG_i + (1 - t)H_j)$ as a semigroup (via the map induced by the inclusion $G_i \subseteq tG_i + (1 - t)H_j$), however the trace is modified by a multiplicative factor of $t$. Similarly, $\text{Meas}(H_j)$ also embeds into $\text{Meas}(tG_i + (1 - t)H_j)$ as a semigroup, and the images of both $\text{Meas}(G_i)$ and $\text{Meas}(H_j)$ have disjoint ranges and sources, so their elements are pairwise compatible.

  Let $W$ be any ultrafilter on $I \times J$ containing all $F \times G$, where $F \in U$ and $G \in V$. Using the argument above, we can embed $\prod_U \text{Meas}(G_i)$ and $\prod_V \text{Meas}(H_j)$ into $\prod_W \text{Meas}(tG_i + (1 - t)H_j)$ as subsemigroups, in a way that modifies the traces by multiplicative factors of $t$ and $(1 - t)$, respectively. Let $\Phi$ and $\Psi$ be the compositions of $\phi$ and $\psi$, respectively, with respect to these embeddings.

  Define $\Theta : \text{Meas}(G, \mu) \to \prod_W \text{Meas}(tG_i + (1 - t)H_j)$ by

  $$\Theta(A) = \Phi(A) \lor \Psi(A), \quad \text{for all } A \in \text{Meas}(G, \mu).$$

  Then $\Theta$ is a trace-preserving semigroup morphism, that is, a sofic embedding. The finite case follows by iteration, and the countable infinite case follows from (a).
(c) From the previous items it suffices to check that \( \mu \) is a limit of convex combinations of sofic \( p_x \). Let \( K \) be a finite collection of Borel subsets of \( \mathcal{G}^{(0)} \) and \( \epsilon > 0 \). For each \( A \in K \), consider the Borel map

\[
p^A : X \to [0, 1], \quad p^A(x) = p_x(A).
\]

Partitioning \([0, 1]\) into intervals of diameter at most \( \epsilon \) and taking preimages under \( p^A \) for each \( A \in K \), we can find a finite partition \( \{X_j\}_{j=1}^N \) of \( X \) for which

\[
|p_x(A) - p_y(A)| < \epsilon \quad \text{for all } A \in K \quad \text{whenever } x \text{ and } y \text{ belong to the same set } X_j.
\]

We can moreover assume all \( X_j \) are non-null, so choose \( x(j) \in X_j \) with \( p_x(j) \) sofic. Then for \( A \in K \),

\[
\mu(A) = \int_X p_x(A)d\nu(x) = \sum_j \left( \int_{X_j} p_{x(j)}(A) \pm \epsilon d\nu(x) \right)
\]

\[
= \sum_j \left( \int_{X_j} p_{x(j)}(A)d\nu(x) \pm \nu(X_j)\epsilon \right)
\]

\[
= \sum_j \left( \int_{X_j} p_{x(j)}(A)d\nu(x) \right) \pm \epsilon
\]

\[
= \left( \sum_j \nu(X_j)p_{x(j)} \right)(A) \pm \epsilon.
\]

as we expected.

(d) Let \( K \subseteq \text{Meas} (\mathcal{H}, \mu_{\mathcal{H}}) \) be a finite subset and \( \epsilon > 0 \). Let \( \theta : \text{Meas} (\mathcal{G}, \mu) \to \prod_i \text{Meas}(\mathcal{G}_i, \mu_i) \) be a sofic embedding of \( \mathcal{G} \).

Consider maps \( \Theta_i : \text{Meas} (\mathcal{G}, \mu) \to \text{Meas}(\mathcal{G}_i, \mu_i) \) such that \( \theta(A) = (\Theta_i(A))_i \) for all \( A \in \text{Meas}(\mathcal{G}, \mu) \), and let \( \Theta = (\Theta_i)_i : \text{Meas}(\mathcal{G}, \mu) \to \prod_i \text{Meas}(\mathcal{G}_i, \mu_i) \).

Even though \( \Theta \) is not necessarily a semigroup morphism, we can assume that

\[
\Theta(E(\text{Meas}(\mathcal{G}, \mu))) \subseteq \prod_i E(\text{Meas}(\mathcal{G}_i, \mu_i)) \quad \text{(Proposition 1.2.25(e))}.
\]

Note that \( \text{Meas}(\mathcal{H}, \mu_{\mathcal{H}}) \) is contained in \( \text{Meas}(\mathcal{G}, \mu) \) as a semigroup, but with a different metric. Moreover, for every \( A \in \text{Meas}(\mathcal{H}, \mu_{\mathcal{H}}) \), \( A = \mathcal{H}^{(0)}A\mathcal{H}^{(0)} \), so substituting \( \Theta(A) \) by \( \Theta(\mathcal{H}^{(0)})\Theta(A)\Theta(\mathcal{H}^{(0)}) \) if necessary, we still have a section \( \Theta \) of \( \theta \), but now we have, for all \( i \),

\[
\Theta_i(A) \subseteq \Theta_i(\mathcal{H}^{(0)})\mathcal{G}_i\Theta_i(\mathcal{H}^{(0)}).
\]

Consider then \( \mathcal{H}_i = \Theta_i(\mathcal{H}^{(0)})\mathcal{G}_i\Theta_i(\mathcal{H}^{(0)}) \), which is a subgroupoid of \( \mathcal{G}_i \) and endow it with the subgroupoid measure. The distance in \( \text{Meas}(\mathcal{H}_i) \) is the same as the distance in \( \text{Meas}(\mathcal{G}_i, \mu_i) \), up to a multiplicative factor of

\[
\mu_i(\mathcal{H}_i)^{-1} = \text{tr}_{\mu_i}(\Theta_i(\mathcal{H}^{(0)}))^{-1}
\]
which converges to $\text{tr}_\mu(\mathcal{H}^{(0)})^{-1}$ along $\mathcal{U}$. This is the same multiplicative factor which relates the metrics in $\text{Meas}(\mathcal{H}, \mu_\mathcal{H})$ and $\text{Meas}(\mathcal{G}, \mu)$, so we obtain a sofic embedding

$$\text{Meas}(\mathcal{H}, \mu_\mathcal{H}) \to \prod_{i} \text{Meas}(\mathcal{H}_i), \quad A \mapsto (\Theta_i(A))_i.$$  

(e) Apply items (d) and (b) with the fact that $G = \sum_j \mu(H_j)H_j$.

(f) Let $f = dv/d\mu$. Since both $\mu$ and $\nu$ are invariant measures then $f$ is an invariant function, in the sense that there is a $\mu$-co-null subset $Y$ of $\mathcal{G}^{(0)}$ such that $f(s(a)) = f(r(a))$ whenever $s(a) \in Y$. By modifying $f$ on a null set, we may assume it is indeed invariant (i.e., $Y = \mathcal{G}^{(0)}$).

Let $\epsilon > 0$. Partitioning $[0, \infty)$ into countably many intervals of diameter at most $\epsilon$ and taking their preimages under $f$ and obtain a partition $X_1, X_2, \ldots$ of $\mathcal{G}^{(0)}$ by invariant subsets (that is, $r(s^{-1}(X_j)) = X_j$ for all $j$; See Definition 2.3.5 below) such that $|f(x) - f(y)| < \epsilon$ whenever $x$ and $y$ belong to the same $X_j$. Fix points $x(j) \in X_j$. Then for all $A \subseteq X$,

$$\nu(A) = \sum_j \int_{X_j \cap A} f(x(j))d\mu(x) \pm \epsilon = \sum_j f(x(j))\mu(X_j)\mu_j(A) \pm \epsilon,$$

where $\mu_j$ is the normalized measure on $X_j$: $\mu_j(A) = \mu(A \cap X_j)/\mu(X_j)$ (as usual, we can assume all $X_j$ to be non-null). Since $1 = \nu(X) = \sum_j f(x_j)\mu(X_j) \pm \epsilon$, we can obtain

$$\nu(A) = \sum_j \left( \frac{f(x(j))\mu(X_j)}{\sum_i f(x(i))\mu(X_i)} \right) \mu_j(A) \pm 2 \frac{\epsilon}{1 \pm \epsilon}.$$

Each $\mu_j$ is sofic by item (d), since it is the normalized measure of the subgroupoid $X_j \mathcal{G}X_j$, and since $\mathcal{G} = \bigsqcup_j X_j \mathcal{G}X_j$, then items (a), (b) and (e) imply that $\nu$ is sofic. \hfill \qed

**Definition 2.3.5.** If $\mathcal{G}$ is a probability measure-preserving groupoid and $A \subseteq \mathcal{G}^{(0)}$, we say that $A$ is **invariant** if $r(s^{-1}(A)) = A$. We say that $\mathcal{G}$ is **ergodic** if its only invariant Borel subsets are either null or conull.

If $\mathcal{G}$ is a standard $r$-discrete probability measure-preserving groupoid, then there exists a standard probability space $(Y, \nu)$ and a family $\{p_y : y \in Y\}$ of invariant probability measures on $\mathcal{G}^{(0)}$ such that

(i) Each measure $p_y$ is **ergodic**: If $A \subseteq \mathcal{G}^{(0)}$ is a Borel subset satisfying $s(r^{-1}(A)) = A$ (we say that such $A$ is **invariant**), then $A$ is either $p_y$-null or $p_y$-conull;
(ii) For every Borel \( B \subseteq G^{(0)} \), the map \( y \mapsto p_y(B) \) is Borel on \( Y \) and
\[
\mu(B) = \int_Y p_y(B) d\nu(y)
\]
(this follows from Proposition 1.6.21, the Feldman-Moore result 1.6.19 and [75, Theorem 1.1]). This is called an ergodic decomposition of \((G, \mu)\), and each probability measure-preserving groupoid \((G, p_y)\) is called an ergodic component of \((G, \mu)\). Item (b) of the theorem above then implies that if every ergodic component of \( G \) is soc, so is \( G \). In particular, the question of whether every (probability measure-preserving) groupoid is soc is equivalent to the question of whether every ergodic groupoid is soc.

The previous theorem is more related to the question of how is soficity preserved under taking subgroupoids. We will now deal with the opposite direction, and to this end we introduce a notion of finite index. The index for probability measure preserving groupoids was initially introduced and studied by Kida in [104]. We will use a slight variation of his notion, using Dye’s full group \([G]\).

**Definition 2.3.6.** The measured full group \([G, \mu]\), or simply \([G]\), of a probability measure-preserving groupoid \((G, \mu)\) is the group of \( A \in \Meas(G, \mu) \) such that \( A^*A = G^{(0)} \), or equivalently those \( A \in \Meas(G, \mu) \) for which \( \mu(A^*A) = 1 \).

**Example 2.3.7.** If \( G \) is a discrete group, endowed with the point-mass measure at 1, the Borel (and the measured) semigroup of \( G \) is \( \{\{g\} : g \in G\} \cup \{\emptyset\} \), that is, \( G \) and a zero. The full group of \( G \) is isomorphic to \( G \).

**Example 2.3.8.** If \( R \) is a \( r \)-discrete, probability measure-preserving Borel equivalence relation on a a standard probability space \((X, \mu)\), and we identify \( \Meas(R) \) with the semigroup of partial Borel automorphisms of \( X \) (modulo functions which differ on null sets) which preserve \( R \)-equivalence classes, then the full group \([R]\) consists of the elements of \( \Meas(R) \) with conull domain.

**Definition 2.3.9.** A subgroupoid \( \mathcal{H} \) of a probability measure-preserving groupoid \( G \) has finite index in \( G \) if there exist \( \psi_1, \ldots, \psi_k \in [G] \) such that \( \{\psi_iH : i = 1 \ldots k\} \) is a partition of \( G \) up to null sets, that is,
\[
(i) \quad \psi_i\mathcal{H} \cap \psi_j\mathcal{H} = \emptyset \text{ whenever } i \neq j;
(ii) \quad s(\mathcal{G} \setminus \bigcup_{i=1}^k \psi_iH) \text{ is null in } G^{(0)}.
\]
We call \( \psi_1, \ldots, \psi_k \) left transversals of \( \mathcal{H} \) in \( G \).

---

3These are sometimes called the invertible elements of a monoid. Whenever we use this nomenclature we will recall this definition.
Note that if $\mathcal{H}$ has finite index in $\mathcal{G}$ then $\mathcal{H}^{(0)} = \mathcal{G}^{(0)}$ (again, up to null sets).

A general question is how much of the structure of a probability measure-preserving groupoid $\mathcal{G}$ can be described by the structures of the orbit equivalence relation $\mathcal{R}(\mathcal{G})$ and the isotropy groups $\mathcal{G}_x^x$, $x \in \mathcal{G}^{(0)}$; for example, $\mathcal{G}$ is amenable if and only if $\mathcal{R}(\mathcal{G})$ is amenable and a.e. isotropy group $\mathcal{G}_x^x$ is amenable (see [6, Theorem 4.2.7]). This naturally leads to the question: “If $\mathcal{H}$ is a subgroupoid of a probability measure-preserving groupoid $\mathcal{G}$ with $\mathcal{H}^{(0)} = \mathcal{G}^{(0)}$, how does finite index of $\mathcal{H} \subseteq \mathcal{G}$ relates to the index of the relations $\mathcal{R}(\mathcal{H})$ (in the sense of Feldman-Sutherland-Zimmer, [57]), and the index of the isotropy groups $\mathcal{H}_x^x \subseteq \mathcal{G}_x^x$?” We answer this question in the ergodic case.

**Theorem 2.3.10.** Suppose $\mathcal{H}$ is an ergodic subgroupoid of a standard $\tau$-discrete probability measure-preserving groupoid $(\mathcal{G}, \mu)$ with $\mathcal{H}^{(0)} = \mathcal{G}^{(0)}$ (up to null sets). Then $\mathcal{H}$ has finite index in $\mathcal{G}$ if and only if $\mathcal{R}(\mathcal{H})$ has finite index in $\mathcal{R}(\mathcal{G})$ and $\mathcal{H}_x^x$ has finite index in $\mathcal{G}_x^x$ for $\mu$-a.e. $x \in \mathcal{G}^{(0)}$.

**Proof.** First suppose $\mathcal{H}$ has finite index in $\mathcal{G}$, and let $x \in \mathcal{H}^{(0)}$. Let $\psi_1, \ldots, \psi_k$ be left transversals for $\mathcal{H} \subseteq \mathcal{G}$. Given $x \in X$, let us prove that every $\mathcal{R}(\mathcal{G})$-class is a finite union of $\mathcal{R}(\mathcal{H})$-classes. For each $i$, choose, if possible, $p_i \in \psi_i$ with $\tau(p_i) = x$, which is in fact unique with this property. If $(x, y) \in \mathcal{R}(\mathcal{G})$, then using $\mathcal{G} = \bigcup_{i=1}^k \psi_i \mathcal{H}$, we can choose $i \in \{1, \ldots, k\}$ and $h \in \mathcal{H}$ such that $x = \tau(p_i h) = \tau(p_i)$, and $s(p_i h) = y$. In particular, $(s(p_i), y) \in \mathcal{R}(\mathcal{H})$, so the $\mathcal{R}(\mathcal{G})$-class of $x$ is the union of the finitely many $\mathcal{R}(\mathcal{H})$-classes of $s(p_i)$, $i = 1, \ldots, k$.

To prove that $\mathcal{H}_x^x$ has finite index in $\mathcal{G}_x^x$, consider, for each $i$, $p_i \in \psi_i$ with $\tau(p_i) = x$ and, whenever possible, choose $h_i \in \mathcal{H}$ such that $s(h_i) = x$ and $\tau(h_i) = s(p_i)$. Let $a \in \mathcal{G}_x^x$. We can write $a = p_i h$ for some $i$ and some $h \in \mathcal{H}$, and in particular $h_i$ is defined and $a = p_i h_i (h_i^{-1} h)$, where $h_i^{-1} h \in \mathcal{H}_x^x$, so the elements $p_i h_i$ form a complete set of representatives of $\mathcal{H}_x^x$ in $\mathcal{G}_x^x$.

Conversely, suppose the latter condition is satisfied, and take invertible choice functions $f_1, \ldots, f_k$ for $\mathcal{R}(\mathcal{H}) \subseteq \mathcal{R}(\mathcal{G})$ [57, Lemma 1.3]: These are a.e.-defined Borel automorphisms of $\mathcal{G}^{(0)}$ such that for a.e. $x \in \mathcal{G}^{(0)}$, the $\mathcal{R}(\mathcal{G})$-class of $x$ is partitioned by the $\mathcal{R}(\mathcal{H})$-classes of $f_i(x)$, $i = 1, \ldots, k$.

Using Theorem 1.6.18 consider elements $\phi_1, \ldots, \phi_k \in [\mathcal{G}]$ satisfying $(\tau, s)(\phi_i) = \{(f_i(x), x) : x \in \mathcal{G}^{(0)}\}$.

The index map $x \mapsto [\mathcal{G}_x^x : \mathcal{H}_x^x]$ is Borel and $\mathcal{H}$-invariant, so it is equal a.e. to a number $m$. Consider the isotropy subgroupoids

$$
\text{Iso}(\mathcal{G}) = \{g \in \mathcal{G} : s(g) = \tau(g)\}, \quad \text{Iso}(\mathcal{H}) = \{h \in \mathcal{H} : s(h) = \tau(h)\}. \quad (2.3.1)
$$

Let $\theta_1 = \mathcal{H}^{(0)}$. If $\theta_1, \ldots, \theta_k$ are defined and $k < m$, let $\theta_{k+1}$ be a subset of $\text{Iso}(\mathcal{G}) \backslash \bigcup_{i=1}^k \theta_i \text{Iso}(\mathcal{H})$ such that the source map restricts to a Borel isomorphism of $\theta_{k+1}$ onto $\mathcal{H}^{(0)}$ (again, up to null sets). Such $\theta_{k+1}$ exists by the last part of Corollary 1.6.12.
Thus we obtain elements $\theta_1, \ldots, \theta_m \in [\mathcal{G}]$ such that $s(a) = r(a)$ for all $a \in \theta_j$, and such that for a.e. $x \in \mathcal{G}^{(0)}$, $\{\theta_j^x\mathcal{H}_x^z\}$ is a partition of $\mathcal{G}_x$. We can now check that the elements $\phi_i^{-1}\theta_j$ are left transversal for $\mathcal{H}$ in $\mathcal{G}$:

- If $a \in \mathcal{G}$, there is some $i$ such that $s(a)$ belongs to the same $\mathcal{R}(\mathcal{H})$-class as $f_i(r(a))$, so there is $h \in \mathcal{H}$ such that $(r(h), s(h)) = (s(a), f_i(r(a)))$. Let $p_i \in \phi_i$ such that $(r(p_i), s(p_i)) = (f_i(r(a)), r(a))$.

\[
s(a) \xrightarrow{o} r(a) \xrightarrow{t_j} f_i(r(a))
\]

Then $p_iah \in \mathcal{G}^{r(a)}$. Up to $r(a)$ belonging to a null set, we can find $j$ and $t_j \in \theta_j$ such that $p_iah \in t_j\mathcal{H}^{r(a)}$, so $a \in p_i^{-1}t_j \mathcal{H} \subseteq \phi_i^{-1}\theta_j \mathcal{H}$.

- To check that the sets $\phi_i^{-1}\theta_j \mathcal{H}$ are pairwise disjoint, suppose

\[
p_1^{-1}t_1h_1 = p_2^{-1}t_2h_2
\]

where $p_k \in \phi_{i_k}$, $t_k \in \theta_{j_k}$, for certain indices $i_1, i_2, j_1, j_2$, and $h_k \in \mathcal{H}$. Let $x = s(h_1) = s(h_2)$ and $y = s(p_1) = s(p_2)$.

\[
t_1
\]

\[
t_2
\]

Then $r(p_1) = s(p_1^{-1}) = r(t_1) = s(t_1) = r(h_1)$, so

\[
(f_{i_1}(y), x) = (r(s|_{\phi_i}^{-1}(y)), s(h_1)) = (r(p_1), s(h_1)) = (r, s)(h_1) \in \mathcal{R}(\mathcal{H})
\]

and similarly $(f_{i_2}(y), x) \in \mathcal{R}(\mathcal{H})$. Since $\{f_i\}$ are choice functions, we obtain $i_1 = i_2$. Then $\phi_{i_1} = \phi_{i_2}$ is a bisection containing $p_1, p_2$, and $s(p_1) = y = s(p_2)$, so $p_1 = p_2$. Therefore

\[
t_1h_1 = t_2h_2
\]

that is, $t_1 = t_2h_2h_1^{-1} \in \theta_{j_2}\mathcal{H}_x^z$, where $z = f_{i_1}(y)$, and this implies $j_1 = j_2$ by our choice of $\theta_j$. □
We now relate the notion of finite index introduced above with Kida’s original definition (\([114]\)). If \(\mathcal{H}\) is a subgroupoid of a groupoid \(\mathcal{G}\), we define an equivalence relation \(\sim_{\mathcal{H}}\) by
\[
g \sim_{\mathcal{H}} h \iff \tau(g) = \tau(h) \quad \text{and} \quad g^{-1}h \in \mathcal{H}.
\]
Note that each \(\sim_{\mathcal{H}}\)-equivalence class is contained in \(\mathcal{G}^x\) for some \(x \in \mathcal{G}^{(0)}\). For every \(x \in \mathcal{G}^{(0)}\), let \(I(x)\) be the number (possibly infinite) of \(\sim_{\mathcal{H}}\)-equivalence classes contained in \(\mathcal{G}^x\). We call \(I : \mathcal{G}^{(0)} \to \mathbb{N} \cup \{\infty\}\) the \textit{index map}, and \(I(x)\) is the \textit{index of \(\mathcal{H}\) in \(\mathcal{G}\) at \(x\)}.

**Corollary 2.3.11.** Suppose \(\mathcal{H}\) is an ergodic subgroupoid of a standard \(\tau\)-discrete probability measure-preserving groupoid \((\mathcal{G}, \mu)\) with \(\mathcal{H}^{(0)} = \mathcal{G}^{(0)}\) (up to null sets). Then \(\mathcal{H}\) has finite index in \(\mathcal{G}\) if and only if the equivalence relation \(\sim_{\mathcal{H}}\) on \(\mathcal{G}\) is periodic (i.e., a.e. equivalence class is finite). Moreover, if \(\psi_1, \ldots, \psi_k\) are left transversals of \(\mathcal{H}\) in \(\mathcal{G}\), then the index map \(I : \mathcal{G}^{(0)} \rightarrow \mathbb{N} \cup \{\infty\}\) is a.e.-constant and equal to \(k\).

**Proof.** First assume \(\sim_{\mathcal{H}}\) is periodic. Given \(x \in \mathcal{G}^{(0)}\), let \(\{g_1, \ldots, g_k\}\) be representatives of all the \(\sim_{\mathcal{H}}\)-equivalence classes contained in \(\mathcal{G}^x\). If \((s(g_i), x) \in \mathcal{R}(\mathcal{H})\), choose \(h_i \in \mathcal{H}\) with \(s(h_i) = s(g_i)\) and \(\tau(h_i) = x\). First we will prove that the set
\[
\{h_i g_i^{-1} : h_i \text{ is defined}\}
\]
is a full set of representatives of \(\mathcal{H}_x^\times\) in \(\mathcal{G}_x^x\). Indeed, if \(g \in \mathcal{G}_x^x\), then there is \(i\) such that \(g_i^{-1} g \in \mathcal{H}\), so \((s(g_i), x) = (\tau, s)(g_i^{-1} g) \in \mathcal{R}(\mathcal{H})\) and thus \(h_i\) is defined and we have \(h_i g_i^{-1} g \in \mathcal{H}\). Therefore \(\mathcal{H}_x^\times\) has finite index in \(\mathcal{G}_x^x\). A similar argument proves that the \(\mathcal{R}(\mathcal{G})\)-equivalence class of \(x\) is the union of the \(\mathcal{R}(\mathcal{H})\)-equivalence classes of \(s(g_1), \ldots, s(g_k)\), so \(\mathcal{R}(\mathcal{H})\) has finite index in \(\mathcal{R}(\mathcal{G})\). By Theorem 2.3.10 \(\mathcal{H}\) has finite index in \(\mathcal{G}\).

Now we assume \(\psi_1, \ldots, \psi_k \in [\mathcal{G}]\) are left transversals of \(\mathcal{H}\) in \(\mathcal{G}\), i.e., the collection \(\{\psi_i \mathcal{H} : i = 1, \ldots, k\}\) is a partition of \(\mathcal{G}\) (up to null sets). It immediately follows that for a.e. \(x \in \mathcal{G}^{(0)}\), we can choose \(p_i \in \psi_i\) such that \(\tau(p_i) = x\) (\(i = 1, \ldots, k\)) and \(\{p_1, \ldots, p_k\}\) is a complete set of representatives of the \(\sim_{\mathcal{H}}\)-classes contained in \(\mathcal{G}^x\). This proves that \(\sim_{\mathcal{H}}\) is periodic and the index map \(I\) is a.e. equal to \(k\). \(\square\)

**Remark.** As a consequence of Corollary 2.3.11 if \(\mathcal{H}\) is an ergodic finite index-subgroupoid of \(\mathcal{G}\) then any two sets of left transversals of \(\mathcal{H}\) in \(\mathcal{G}\) have the same number of elements, which we thus call the \textit{index of \(\mathcal{H}\) in \(\mathcal{G}\)} and denote by \([\mathcal{G} : \mathcal{H}]\).

Following the proof of Theorem 2.3.10 the map \(x \mapsto [\mathcal{G}_x^x : \mathcal{H}_x^x]\) is a.e.-constant to the index \([\text{Iso}(\mathcal{G}) : \text{Iso}(\mathcal{H})]\) of the respective isotropy subgroupoids (Equation 2.3.1), and moreover \([\mathcal{G} : \mathcal{H}] = [\text{Iso}(\mathcal{G}) : \text{Iso}(\mathcal{H})][\mathcal{R}(\mathcal{G}) : \mathcal{R}(\mathcal{H})]\).

We now prove that finite index extensions of sofic groupoids preserve soficity. This can be compared with the similar fact that sofic-by-amenable extensions of groups are also sofic (see 44. Theorem 1) or Theorem 2.5.20(d)).
Theorem 2.3.12. Suppose $\mathcal{H} \subseteq \mathcal{G}$ is of finite index. If $\mathcal{H}$ is sofic, so is $\mathcal{G}$.

Proof. Suppose that $\psi_1, \ldots, \psi_N$ are left transversals for $\mathcal{H} \subseteq \mathcal{G}$. For each $A \in \operatorname{Meas}(\mathcal{G})$, let $A_{i,j} = \psi_i^{-1}A\psi_j \cap H$. Note that $A_{i,j} \in \operatorname{Meas}(\mathcal{H})$ and that $A_{i,j}^{-1} = A_{j,i}$.

Moreover, $A_{i,j}A_{k,l} = \emptyset$ whenever $j \neq l$. Let $Y$ be a set with $N$ elements. Given $(i,j) \in Y^2$, let $E_{i,j} = \{(i,j)\} \subseteq \operatorname{Meas}(Y^2)$. (As a partial transformation on $Y$, $E_{i,j}$ is simply defined by $E_{i,j}(j) = i$).

Let $\Phi : \operatorname{Meas}(\mathcal{G}) \to \prod_{\mathcal{U}} \operatorname{Meas}(\mathcal{G}_n)$ be a sofic embedding of $\mathcal{G}$.

Recall that $Y$ is identified with $(Y^2)^{(0)}$, the unit of $\operatorname{Meas}(Y^2)$. For each $n$, the map $I_n : \operatorname{Meas}(\mathcal{G}_n) \to \operatorname{Meas}(\mathcal{G}_n \times Y^2)$, $C \mapsto C \times Y$, is an isometric semigroup morphism, so Proposition 2.1.16 allows us to induce a map $I : \prod_{\mathcal{U}} \operatorname{Meas}(\mathcal{G}_n) \to \prod_{\mathcal{U}} \operatorname{Meas}(\mathcal{G}_n \times Y^2)$. Let $J : \operatorname{Meas}(Y^2) \to \prod_{\mathcal{U}} \operatorname{Meas}(\mathcal{G}_n \times Y^2)$ be given by $J(E) = (\mathcal{G}_n^{(0)} \times E)_{\mathcal{U}}$, which is also an isometric semigroup embedding.

Given $U = (U_n)_{\mathcal{U}} \in \prod_{\mathcal{U}} \operatorname{Meas}(\mathcal{G}_n)$ and $E \in \operatorname{Meas}(Y^2)$, we denote

$$U \times E = I(U)J(E) = (U_n \times E)_{\mathcal{U}} \in \prod_{\mathcal{U}} \operatorname{Meas}(\mathcal{G}_n \times Y^2).$$

Define $\Xi : \operatorname{Meas}(\mathcal{G}) \to \prod_{\mathcal{U}} \operatorname{Meas}(\mathcal{G}_n \times Y^2)$ by

$$\Xi(A) = \bigvee_{i,j=1}^N \Phi(A_{i,j}) \times E_{i,j}.$$

First we prove that $\Xi$ is well-defined, or more precisely that the terms in the right-hand side have disjoint sources and ranges: Let $(i,j)$ and $(k,l)$ be given. Then

$$(\Phi(A_{i,j}) \times E_{i,j}) (\Phi(A_{k,l}) \times E_{k,l})^{-1} = \Phi(A_{i,j}A_{l,k}) \times (E_{i,j}E_{l,k}).$$

If $j \neq l$ the right-hand side is empty, so assume $j = l$. Then

$$A_{i,j}A_{j,k} = (\psi_i^{-1}A\psi_j \cap H)(\psi_j^{-1}A\psi_k \cap H). \quad (2.3.2)$$

If this product is nonempty, then we have $p_i \in \psi_i$, $p_j, q_j \in \psi_j$, $q_k \in \psi_k$ and $g, h \in A$ such that the product $(p_i^{-1}gp_j)(q_j^{-1}hq_k)$ is defined, and both terms belong to $\mathcal{H}$. But in particular $s(p_j) = t(q_j^{-1}) = s(q_j)$, so $p_j = q_j$. Similarly $g = h$, and so $p_i^{-1}q_k \in \mathcal{H}$, thus $q_k \in \psi_l\mathcal{H} \cap \psi_k\mathcal{H}$, which implies $i = k$.

This proves that the ranges of the terms in the definition of $\Xi(A)$ are disjoint. The sources are dealt with similarly, and so $\Xi$ is well-defined.

Now we need to show that $\Xi$ is a morphism. Suppose $A, B \in \operatorname{Meas}(\mathcal{G})$. We have

$$\Xi(A)\Xi(B) = \bigvee_{i,j,k,l} \Phi(A_{i,j}B_{k,l}) \times (E_{i,j}E_{k,l}) = \bigvee_{i,j,l} \Phi(A_{i,j}B_{j,l}) \times E_{i,l}.$$
On the other hand \( \Xi(AB) = \bigvee_{i,j} \Phi((AB)_{i,j}) \times E_{i,l} \), so we are done if we show that for given \( i, l \),
\[
\bigcup_j A_{i,j} B_{j,l} = (AB)_{i,l}.
\]

The inclusion \( \subseteq \) is quite straightforward, using a similar argument to the one right after equation 2.3.2 above. For the converse, suppose \( p_i^{-1} abp_l \in (AB)_{i,l} \), where \( p_i \in \psi_i, p_l \in \psi_l, a \in A \) and \( b \in B \). Choose \( j \) such that \( bp_l \in \psi_j H \), so there is a unique \( p_j \in \psi_j \) such that the product \( p_j^{-1} bp_l \) is defined and in \( \mathcal{H} \). Therefore
\[
p_i^{-1} abp_l = (p_i^{-1} ap_j)(p_j^{-1} bp_l) \in A_{i,j} B_{j,l}.
\]

We need to show that \( \Xi \) is trace-preserving. Note that
\[
\text{tr} \Xi(A) = \frac{1}{N} \sum_{i=1}^{N} \text{tr}(A_{i,i}),
\]
so we are done if we prove that \( \text{tr}(A_{i,i}) = \text{tr}(A) \). To this end, let us prove that
\[
A_{i,i} \cap \mathcal{G}^{(0)} = s \{ g \in \psi_i : r(g) \in A \cap \mathcal{G}^{(0)} \}
\]
An element of \( A_{i,i} \cap \mathcal{G}^{(0)} \) has the form \( x = p_i^{-1} ap_i \) for \( a \in A \) and \( p_i \in \psi_i \). It follows that \( x = s(p_i) \), so \( r(p_i) = a \in A \cap \mathcal{G}^{(0)} \). Conversely, if \( x = s(g) \), where \( g \in \psi_i \) and \( r(g) \in A \cap \mathcal{G}^{(0)} \), then \( x = g^{-1} r(g) g \in \psi_i^{-1} A \psi_i \cap \mathcal{G}^{(0)} \).

Finally, we obtain
\[
\text{tr}(A_{i,i}) = \mu(\psi_i^{-1} A \psi_i \cap \mathcal{G}^{(0)}) = \mu(s(\psi_i \cap r^{-1}(A \cap \mathcal{G}^{(0)}))) = \mu(r(\psi_i \cap r^{-1}(A \cap \mathcal{G}^{(0)}))) = \mu(A \cap \mathcal{G}^{(0)}) = \text{tr}(A),
\]
because \( r|_{\psi_i} : \psi_i \rightarrow \mathcal{G}^{(0)} \) is surjective (up to null sets).

With these results at hand, we can provide a few examples of sofic groupoids. The following definition is a natural generalization from the case of equivalence relations.

**Definition 2.3.13** ([100], p. 13, 18]). A probability measure-preserving groupoid \( \mathcal{G} \) is **periodic** if \( \mathcal{G}_x \) is finite for a.e. \( x \in \mathcal{G}^{(0)} \), and \( \mathcal{G} \) is **hyperfinite** if it is standard and an increasing union of periodic Borel subgroupoids (as usual, up to null sets). (In particular, every hyperfinite groupoid is \( r \)-discrete.)

Hyperfinite groupoids are the measured analogues of the AF groupoids introduced in [146]). The Borel case has also been studied, at least in the case of equivalence relations, in [91].

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*Periodic groupoids are also called *finite*, but this could cause confusion with finiteness in the set-theoretic sense.*
Corollary 2.3.14. Every (standard) hyperfinite groupoid is sofic.

Proof. First note that every measure space \((X,\mu)\), seen as a unit groupoid (i.e., \(X = X^{(0)}\)), is sofic. Indeed, \(\mu = \int \delta_x d\mu(x)\), where \(\delta_x\) is the point-mass measure on \(x\), and \((X,\delta_x)\), as a probability measure-preserving groupoid, is isomorphic to a singleton (unit groupoid), hence finite and sofic.

Suppose \(G\) is periodic. Let \(G_n = \{ a \in G : \#(G_{s(a)}) = n \}\). Then the \(G_n\) form a Borel partition by subgroups of \(G\), so it suffices to show that each \(G_n\) is sofic. Of course, every isotropy group \((G_n)^x\) is finite, as is every equivalence class of \(R(G_n)\), so the subgroupoid \((G_n)^{(0)}\) has finite index in \(G_n\), which is therefore sofic.

For a general hyperfinite groupoid \((G,\mu)\), write \(G\) as an increasing union of periodic subgroupoids, say \(G = \bigcup_{n=1}^{\infty} H_n\), where \(H_1 \subseteq H_2 \subseteq \cdots\) are periodic. Denoting by \(\mu_n\) the normalized measure on \(H_n\), regarded as a measure on \(G\), then \(\text{Meas}(G,\mu_n)\) is obviously isometrically isomorphic to \(\text{Meas}(H_n,\mu_n)\) and so \((G,\mu_n)\) is sofic. Since \(\mu\) is the strong limit of \(\mu_n\), then \((G,\mu)\) is sofic.

To finish this section, we prove that soficity is preserved under products. Given probability spaces \((X,\mu)\) and \((Y,\nu)\), we denote by \(\mu \times \nu\) the product probability measure on the product \(\sigma\)-algebra of \(X \times Y\).

Theorem 2.3.15. Two standard, \(r\)-discrete probability measure-preserving groupoids \((G,\mu)\) and \((H,\nu)\) are sofic if and only if \((G \times H,\mu \times \nu)\) is sofic.

Proof. One direction is clear, since \(\text{Meas}(G)\) embeds isometrically into \(\text{Meas}(G \times H)\) via \(A \mapsto A \times H^{(0)}\), and similarly for \(\text{Meas}(H)\).

Let \(\mathfrak{M}\) be the submonoid of \(\text{Meas}(G \times H)\) of elements of the form \(\bigcup_{i=1}^{n} A_i \times B_i\), where

- \(A_i \in \text{Meas}(G)\), \(B_i \in \text{Meas}(H)\);
- for all \(i \neq j\), \(s(A_i) \cap s(A_j) = \emptyset\) or \(s(B_i) \cap s(B_j) = \emptyset\); and
- for all \(i \neq j\), \(t(A_i) \cap t(A_j) = \emptyset\) or \(t(B_i) \cap t(B_j) = \emptyset\).

Let us show that \(\mathfrak{M}\) is dense in \(\text{Meas}(G \times H)\).

Let \(\phi \in \text{Meas}(G \times H)\) and \(\epsilon > 0\). From usual measure theory, we can take \(A_i \in \text{Meas}(G)\) and \(B_i \in \text{Meas}(H)\) such that

\[
(\mu \times \nu) \left( \phi \triangle \left( \bigcup_{ij} A_i \times B_i \right) \right) < \epsilon,
\]

and with the sets \(A_i \times B_i\) pairwise disjoint. For \(i \neq j\), let

\[
h_{i,j} = s \big|_{A_i \times B_i} s(A_j \times B_j) = A_i s(A_j) \times B_i s(B_j) \quad \text{and} \quad h_i = \bigcup_{j \neq i} h_{i,j}.
\]
Let $x \in h_i \cap \phi$, so $s(x) = s(a_j, b_j)$ for some $(a_j, b_j) \in A_j \times B_j$. If $(a_j, b_j) \in \phi$, we would obtain $(a_j, b_j) = x \in A_i \times A_i$, which happens only if $i = j$. This proves that

$$s \left( \bigcup_i (h_i \cap \phi) \right) \subseteq s \left( \bigcup_j (A_j \times B_j) \setminus \phi \right).$$

In particular, $d(\phi, \phi \setminus \bigcup_i h_i) < \epsilon$, from which follows that

$$d\left( \bigcup_i (A_i \times B_i \setminus h_i), \phi \right) = d\left( \left( \bigcup_i A_i \times B_i \right) \setminus \left( \bigcup_i h_i \right), \phi \right) < 2\epsilon.$$

Since each $h_i$ is a rectangle, we can rewrite $\bigcup_i (A_i \times B_i \setminus h_i)$ as a union of disjoint rectangles with the desired property for the source map. To deal with the range map one can apply the same argument to $\phi^{-1}$ and take intersections.

So given sofic embeddings $\Phi : \text{Meas}(G) \to \prod_u \text{Meas}(G_n)$ and $\Psi : \text{Meas}(H) \to \prod_u \text{Meas}(H_n)$ (which can be taken under the same ultrafilter on $\mathbb{N}$) set $\Phi \times \Psi : \mathcal{M} \to \text{Meas}(G_n \times H_n)$ by

$$(\Phi \times \Psi) \left( \bigcup_{ij} A_i \times B_i \right) = \bigcup_{ij} \Phi(A_i) \times \Psi(B_i),$$

where the $A_i$ and $B_i$ satisfy the condition in the definition of $\mathcal{M}$.

The element in the right-hand side is well-defined and does not depend on the choice of $A_i$ and $B_i$ since sofic embeddings preserve sources, ranges, and intersections. $\Phi \times \Psi$ is then an trace-preserving embedding of $\mathcal{M}$, and hence extends uniquely to an isometric embeddings of $\text{Meas}(G \times H)$.

\[\square\]

2.4 Soficity and the full group

**Convention:** During this Section, we will only consider standard, $\tau$-discrete probability measure-preserving groupoids and standard probability spaces.

In this Section we will prove (Theorem 2.4.18) that, under weak dynamical conditions, the full group of a probability measure-preserving groupoid, as a metric group, is enough to determine soficity. This answer, in this case, a question posed by Conley, Kechris and Tucker-Drob in [31].

If $(X, \mu)$ is a probability space, seen as a unit groupoid, then the measured semigroup $\text{Meas}(X, \mu)$ coincides with what is usually called its measure algebra, that is, the $\sigma$-complete Boolean algebra of Borel subsets of $X$ modulo null sets, endowed with its given probability measure. We will follow the usual convention and denote it by $\text{MAlg}(X, \mu)$ (or simply $\text{MAlg}(X)$) instead, in order to decrease the risk of
confusion later on. Given a probability measure-preserving groupoid \((\mathcal{G}, \mu)\), the set of idempotents of \(\text{Meas}(\mathcal{G}, \mu)\) can be identified with \(\text{MAlg}(\mathcal{G}^{(0)}, \mu)\).

Again, for each \(n \in \mathbb{N}\), fix \(Y_n\) a set with \(n\) elements and consider the full equivalence relation \(Y_n^2\), whose full group \([Y_n^2]\) can be identified as the permutation group \(\mathfrak{S}_n\) on \(n\) elements, as in Example [2.3.8] with the normalized Hamming distance

\[
d_\#(f, g) = \# \{ i \in Y_n : f(i) \neq g(i) \} / n.
\]

A well-known theorem of Dye [10] states that when \(R\) is an aperiodic equivalence relation, the full group \([R]\) completely determines \(R\). With this in mind, we prove that a probability measure-preserving groupoid \(\mathcal{G}\) is sofic if and only if \([\mathcal{G}]\) embeds isometrically into \(\prod_\mathcal{U} \mathfrak{S}_n\), as long as \(\mathcal{G}\) is “sufficiently dynamic”.

**Definition 2.4.1.** A group \(G\) with a bi-invariant metric \(d\) is metrically sofic if it embeds isometrically into an ultraproduct \(\prod_\mathcal{U} \mathfrak{S}_n\) of permutation groups with their normalized Hamming distances.

**Example 2.4.2.** It is easy to come up with non-metrically sofic groups, even of diameter one. For example, let \(G = \{1, i, -1, -i\}\) be the copy of the cyclic group of order 4 inside \(\mathbb{C}\), and endow it with the metric \(d_G(g, h) = |g - h|/2\), which gives it diameter 1. For every element \(f\) of a permutation group \(\mathfrak{S}_n\), \(d_\#(f^2, 1) \leq d(f, 1)\), so the same is valid for elements of ultraproducts of permutation groups by Šošić Theorem [2.1.22]. However, \(d_G(i^2, 1) > d(i, 1)\), so \((G, d_G)\) is not metrically sofic.

We will need a few technical lemmas relating the full group \([\mathcal{G}]\), the measure algebra \(\text{MAlg}(\mathcal{G}^{(0)})\) and the measured semigroup \(\text{Meas}(\mathcal{G})\). Before this, we do a quick detour back to general Boolean inverse monoid theory.

**Definition 2.4.3.** If \(S\) is a Boolean inverse monoid and \(s \in S\), the support of \(s\) is \(\text{supp}(s) = s^*s \setminus \text{Fix}(s)\).

**Example 2.4.4.** If \(S = \mathcal{I}(X)\), then \(\text{supp}(f)\) can be identified with its usual support, that is, \(\{ x \in \text{dom}(f) : f(x) \neq x \}\).

**Lemma 2.4.5.** Let \(S\) be a Boolean inverse monoid with a faithful normalized invariant mean \(\mu\), and suppose \(s, t \in S\), \(ss^* = s^*s = tt^* = t^*t = 1\). Then

\[
\begin{align*}
(a) & \quad d_{\mu}(s, 1) = \mu(\text{supp}(s)); \\
(b) & \quad d_{\mu}(s, t) = d_{\mu}(s^*t, 1); \\
(c) & \quad \text{supp}(s) = \text{Fix}(t) \text{ if and only if } d_{\mu}(s, t) = 1 \text{ and } \text{tr}_{\mu}(s) + \text{tr}_{\mu}(t) = 1; \\
(d) & \quad \text{supp}(s) \text{ supp}(t) = 0 \text{ if and only if } d_{\mu}(s, t) = d_{\mu}(1, s) + d_{\mu}(1, t). \text{ In this case, } \\
& \quad \text{supp}(s^*t) = \text{supp}(s) \lor \text{supp}(t);
\end{align*}
\]
(e) If \( \text{supp}(s) \text{supp}(t) = 0 \) then \( \text{supp}(st) = \text{supp}(s) \lor \text{supp}(t) \).

(f) \( \text{supp}(sts^*) = s \text{supp}(t)s^* \)

Proof. (a) This follows from the original definition of the uniform metric:

\[
d_\mu(s, 1) = \mu(s^*s \lor 1^*1 \setminus \text{Fix}(s^*1)) = \mu(1 \setminus \text{Fix}(s)) = 1 - \text{tr}_\mu(s).
\]

(b) Since \( t^*t = tt^* = 1 \),

\[
d_\mu(s, t) = d_\mu(st^*, t) \leq d_\mu(st^*, 1) = d_\mu(st^*, tt^*) \leq d_\mu(s, t).
\]

(c) First suppose \( d_\mu(s, t) = \text{tr}_\mu(s) + \text{tr}_\mu(t) = 1 \). Since \( \text{Fix}(s) \text{Fix}(t) \leq \text{Fix}(s^*t) \), then (a) and (b) imply \( d_\mu(s, t) \leq 1 - \mu(\text{Fix}(s) \text{Fix}(t)) \), so since \( \mu \) is faithful, we have \( \text{Fix}(s) \text{Fix}(t) = 0 \). From the second equality, we conclude that \( \mu(\text{Fix}(s) \lor \text{Fix}(t)) = 1 \), and therefore \( \text{Fix}(t) = 1 \setminus \text{Fix}(s) = \text{supp}(s) \).

In the other direction, suppose \( \text{supp}(s) = \text{Fix}(t) \). Then of course \( \text{tr}_\mu(s) + \text{tr}_\mu(t) = 1 \), and for the other equality, we simply prove that \( \text{Fix}(s^*t) = 0 \). Indeed, if \( e = e^2 \leq s^*t \), then

\[
\text{Fix}(t)e = s^*t \text{Fix}(t)e = s^* \text{Fix}(t)e
\]

which implies \( \text{Fix}(t)e \leq \text{Fix}(s^*) = \text{Fix}(s) = 1 \setminus \text{Fix}(t) \). But also \( \text{Fix}(t)e \leq \text{Fix}(t) \), so \( \text{Fix}(t)e = 0 \). Similarly, \( \text{Fix}(s)e = 0 \), and therefore \( e = e(\text{Fix}(s) \lor \text{Fix}(t)) = 0 \).

(d) Since \( \text{Fix}(s) \text{Fix}(t) \leq \text{Fix}(s^*t) \), then taking complements in \( E(S) \) yields

\[
\text{supp}(s^*t) \leq \text{supp}(s) \lor \text{supp}(t) \quad (2.4.1)
\]

and then

\[
d_\mu(s, t) = \mu(\text{supp}(s^*t)) \leq \mu(\text{supp}(s) \lor \text{supp}(t))
\leq \mu(\text{supp}(s)) + \mu(\text{supp}(t)) = d_\mu(s, 1) + d_\mu(t, 1). 
\quad (2.4.2)
\]

Note that the second inequality is an equality if and only if \( \text{supp}(s) \text{supp}(t) = 0 \). This proves that if \( d_\mu(s, 1) + d_\mu(t, 1) = d_\mu(s, t) \) then \( \text{supp}(s) \text{supp}(t) = 0 \).

Conversely, if \( \text{supp}(s) \text{supp}(t) = 0 \) then \( \text{Fix}(s) \lor \text{Fix}(t) = 1 \). Since

\[
\text{Fix}(t) \text{Fix}(s^*t) = \text{Fix}(s^*t) \text{Fix}(t) = s^*t \text{Fix}(s^*t) \text{Fix}(t)
\]

Then \( \text{Fix}(t) \text{Fix}(s^*t) \leq \text{Fix}(s^*) = \text{Fix}(s) \), therefore

\[
\text{Fix}(s^*t) = \text{Fix}(s) \text{Fix}(s^*t) \lor \text{Fix}(t) \text{Fix}(s^*t) \subseteq \text{Fix}(s)
\]

and similarly \( \text{Fix}(s^*t) \leq \text{Fix}(t) \). This proves \( \text{Fix}(s^*t) = \text{Fix}(s) \text{Fix}(t) \), so \( \text{supp}(s) \lor \text{supp}(t) = \text{supp}(s^*t) \), which is the opposite inclusion of Equation \( (2.4.1) \), and so all inequalities in Equation \( (2.4.2) \) are in fact equalities.
(e) This follows from (d) and the fact that \( \text{supp}(s) = \text{supp}(s^*) \), because \( s^*s = ss^* = 1 \).

(f) \( \text{supp}(sts^*) = 1 \setminus st^* = ss^* \setminus st^* = s(1 \setminus t)s^* = s \text{supp}(t)s^* \).

From items (c) and (d) of the lemma above, we immediately obtain:

**Corollary 2.4.6.** Let \( \theta : [G] \to \prod_{U} \mathcal{S}_n \) be an isometric embedding and \( \alpha, \beta \in [G] \).

(a) \( \text{supp}(\alpha) = \text{Fix}(\beta) \) if and only if \( \text{supp}(\theta(\alpha)) = \text{Fix}(\theta(\beta)) \).

(b) \( \text{supp}(\alpha) \cap \text{supp}(\beta) = \emptyset \) if and only if \( \text{supp}(\theta(\alpha)) \land \text{supp}(\theta(\beta)) = 0 \).

**Definition 2.4.7.** A probability measure-preserving groupoid \( G \) is aperiodic if \( G_x \) is infinite for a.e. \( x \in G^{(0)} \).

**Example 2.4.8.** A standard probability measure-preserving equivalence relation \( R \) on a standard probability space \( (X, \mu) \) is aperiodic if and only if the equivalence class of a.e. point of \( x \in X \) is infinite.

**Lemma 2.4.9.** Suppose \( (G, \mu) \) is an aperiodic probability measure-preserving groupoid. Then for all \( A \in \text{MAAlg}(G^{(0)}) \), there exists \( \alpha \in [G] \) such that \( \text{supp}(\alpha) = A \).

Recall that we only consider \( r \)-discrete and standard groupoids.

**Proof.** We can decompose \( G \) as \( G = H \sqcup K \), where \( H \) and \( K \) are subgroupoids, \( |r(G_x)| = \infty \) for all \( x \in H^{(0)} \), and \( K_x^* \) is infinite for all \( x \in K^{(0)} \), and we may assume that both \( H^{(0)} \) and \( K^{(0)} \) are non-null in \( G^{(0)} \) (in either case the proof can be adapted in an obvious manner).

The equivalence relation \( R(H) = (r, s)(H) \) on \( H^{(0)} \) is also aperiodic, so by [100, Lemma 4.10], there exists \( f \in [R(H)] \) such that \( \text{supp}(f) = A \cap H^{(0)} \), and by Theorem 1.6.18 there exists \( \alpha_H \) such that \( (r, s)(\alpha_H) = f \), so \( A \cap H^{(0)} = \text{supp}(\alpha_H) \).

Consider the subset of \( K \)

\[
Z = \{ k \in K : s(k) = r(k) \} \setminus A.
\]

The restriction of the source map to \( Z \) is countable-to-one and its image contains \( A \cap K[0] \), so by Corollary 1.6.12 implies that there exits \( \beta \subseteq Z \) such that \( s(\beta) = A \). Then \( \alpha_K = \beta \cup (K \setminus A) \) belongs to the full group \([K]\), and \( \text{supp}(\alpha_K) = A \cap K^{(0)} \).

To finish, let \( \alpha = \alpha_H \cup \alpha_K \), so \( \alpha \in [G] \) and \( \text{supp}(\alpha) = A \).

**Lemma 2.4.10.** Suppose that \( (G, \mu) \) is an aperiodic probability measure-preserving groupoid, \( \theta : [G] \to \prod_{U} \mathcal{S}_n \) is an isometric embedding, and let \( \alpha, \beta \in [G] \). Then \( \text{supp}(\alpha) = \text{supp}(\beta) \) if and only if \( \text{supp}(\theta(\alpha)) = \text{supp}(\theta(\beta)) \).
Proof. Assume $\text{supp}(\alpha) = \text{supp}(\beta)$. By Lemma 2.4.9, choose $\gamma \in [\mathcal{G}]$ such that $\text{supp}(\gamma) = X \setminus \text{supp}(\alpha)$, i.e., $\text{Fix}(\gamma) = \text{supp}(\alpha)$. Applying Corollary 2.4.6(a) twice,
\[
\text{supp}(\alpha) = \text{supp}(\beta) \iff \text{Fix}(\gamma) = \text{supp}(\theta(\beta)) \iff \text{Fix}(\theta(\gamma)) = \text{supp}(\theta(\beta)) \iff \text{supp}(\theta(\alpha)) = \text{supp}(\theta(\beta)).
\]

The next lemma will allow us to extend elements of $\text{Meas}(\mathcal{G})$ to elements of $[\mathcal{G}]$.

**Lemma 2.4.11.** If $\gamma \in \text{Meas}(\mathcal{G})$, then there exists $\tilde{\gamma} \in [\mathcal{G}]$ with $\gamma \subseteq \tilde{\gamma}$.

**Proof.** Let $\gamma_0 = \gamma$, and for all $n \geq 1$, set
\[
\gamma_n = \{ g \in \gamma^{-n} : s(g) \not\subseteq s(\gamma) \text{ and } r(g) \not\subseteq r(\gamma) \}.
\]
Using the fact that $\gamma$ is a bisection and by construction of the $\gamma_n$, we have
\[
s(\gamma_n) \cap s(\gamma_m) = r(\gamma_n) \cap r(\gamma_m) = \emptyset, \quad \text{whenever } n \neq m.
\]
The sets $\bigcup_n s(\gamma_n)$ and $\bigcup_n r(\gamma_n)$ are both contained $s(\gamma) \cup r(\gamma)$, so let us prove they are conull in there. Set
\[
B = r(\gamma) \setminus \bigcup_n s(\gamma_n) = \bigcap_n r(\gamma^n) \setminus s(\gamma),
\]
and for all $m \geq 0$, set
\[
B_m = r(\gamma^{-m}B) = \bigcap_n r(\gamma^n) \cap s(\gamma^m) \setminus s(\gamma^{m+1})
\]
These sets are pairwise disjoint, and since $\mu$ is invariant then they have the same measure, because
\[
B_{m+1} = r(\gamma^{-1}B_m), \quad \text{and } B_m = r(\gamma B_{m+1}), \quad \text{for all } m.
\]
But $\mu$ is finite, therefore $\mu(B) = 0$, and $\bigcup_n s(\gamma_n)$ is conull in $s(\gamma) \cup r(\gamma)$. Similarly,
\[
\bigcup_n r(\gamma_n) \text{ is conull in } s(\gamma) \cup r(\gamma).
\]
Therefore $\tilde{\gamma} = \bigcup_n \gamma_n \cup ([0] \setminus (s(\gamma) \cup r(\gamma)))$ has the desired properties. 

**Example 2.4.12.** If $R$ is a $\tau$-discrete probability-measure preserving Borel equivalence relation on a standard probability space $(X, \mu)$, then the previous lemma implies that every partial Borel automorphism $f : A \to B$ ($A, B \subseteq X$ Borel) with $\text{graph}(f) \subseteq R$ can be extended (up to a null subset of $A$) to an a.e.-defined Borel automorphism $\tilde{f} : X \to X$ with $\text{graph}(\tilde{f}) \subseteq R$. 

For the next lemma, again consider finite sets \( Y_n \) with \( \#Y_n = n \). We need to check that the group of invertible elements of \( \prod_{\mathcal{U}} \text{Meas}(Y_n^2) \) (that is, those elements \( \sigma \) for which \( \sigma \sigma^* = \sigma^* \sigma = 1 \)) coincides with \( \prod_{\mathcal{U}} \mathcal{S}_n \).

Recall that an element of the ultraproduct of metric structures \( \prod_{\mathcal{U}} M_n \) is denoted as \( (x_n)_{\mathcal{U}} \), where \( x_n \in M_n \) for all \( n \).

**Lemma 2.4.13.** If \( \sigma = (\sigma_n) \in \prod_{\mathcal{U}} \text{Meas}(Y_n^2) \) is invertible, in the sense that \( \sigma \sigma^* = (\text{id}_{Y_n})_{\mathcal{U}} \), then there are \( \tilde{\sigma}_n \in \mathcal{S}_n \) such that \( \sigma = (\tilde{\sigma}_n)_{\mathcal{U}} \).

**Proof.** Since \( d_\#(\text{id}_{Y_n}, \sigma_n \sigma_n^*) = 1 - \mu_\#(\sigma_n^* \sigma_n) \) converges to 0 along \( \mathcal{U} \), then \( \mu_\#(\sigma_n^* \sigma_n) \) converges to 1. Simply extend each \( \sigma_n \) arbitrarily to an element \( \tilde{\sigma}_n \in \mathcal{S}_n \), so that

\[
d_\#(\tilde{\sigma}_n, \sigma) = \mu(\tilde{\sigma}_n^* \tilde{\sigma}_n \setminus \sigma_n^* \sigma_n) = 1 - \mu_\#(\sigma_n^* \sigma_n)
\]

which, again, converges to 0 along \( \mathcal{U} \), and therefore \( (\sigma_n)_{\mathcal{U}} = (\tilde{\sigma}_n)_{\mathcal{U}} \). \( \square \)

This way, the group of invertible elements of \( \prod_{\mathcal{U}} \text{Meas}(Y_n^2) \) is naturally identified with \( \prod_{\mathcal{U}} \mathcal{S}_n \), whereas the idempotent lattice \( E(\prod_{\mathcal{U}} \text{Meas}(Y_n^2)) \) is identified with \( \prod_{\mathcal{U}} \text{MAlg}(Y_n) \) (because \( E(\prod_{\mathcal{U}} \text{Meas}(Y_n^2)) \) is simply the image of the idempotent lattice of \( \prod_{n \in \mathbb{N}} \text{Meas}(Y_n^2) \) under the canonical quotient map; see Subsection 2.1.4).

**Definition 2.4.14.** If \( G_1 \) and \( G_2 \) are groups acting on sets \( X_1 \) and \( X_2 \), respectively, \( \theta : G_1 \to G_2 \) is a morphism and \( \phi : X_1 \to X_2 \) is a function, we say that the pair \((\theta, \phi)\) is **covariant** if \( \phi(gx) = \theta(g)\phi(x) \) for all \( g \in G_1 \) and \( x \in X_1 \). If \( G_1 = G_2 = G \) and \( \theta = \text{id}_G \), we simply say that \( \phi \) is covariant.

**Remark.** Covariant pairs will be used also in Section 2.5 and in Chapter 4 (see Definition 4.3.12). In fact, covariant maps are a useful tool when constructing morphisms from structures which are “built from” both \( G \) and a \( G \)-space \( X \): In fact, the proofs of Theorems 2.4.16 and 4.4.8 follow similar general lines in different settings (however the fine details in each case differ substantially).

**Definition 2.4.15.** Let \( S \) be a Boolean inverse monoid and consider \( S^\times \) the group of \( g \in S \) such that \( gg^* = g^* g = 1 \). We define an action of \( S^\times \) on \( E(S) \) as \( g \cdot e = geg^* \). Note that if \( \mu \) is an invariant mean on \( S \) then for all \( g \in S^\times \) and \( e \in E(S) \),

\[
\mu(e) = \mu(eg^*ge) = \mu((ge)^*(ge)) = \mu((ge)(ge)^*) = \mu(ge)
\]

so this action is isometric and trace-preserving. In particular,

- If \( S = \text{Meas}(\mathcal{G}) \) for a probability measure-preserving groupoid \( \mathcal{G} \), then \( [\mathcal{G}] \) acts on \( \text{MAlg}(\mathcal{G}^{(0)}) \) via

\[
\gamma \cdot A = v(\gamma A), \quad \text{for all } \gamma \in [\mathcal{G}] \text{ and } A \in \text{MAlg}(\mathcal{G}^{(0)}).
\]
• If $S = \prod U \text{Meas}(Y^2_n)$ (where $Y_n$ are finite sets with $n$ elements), then $\prod U \mathcal{S}_n$ acts on $\prod U \mathcal{MAlg}(Y^2_n)$ via

$$(\sigma_n)_U \cdot (A_n)_U = (\sigma_n(A_n))_U.$$ 

**Theorem 2.4.16.** A standard, $r$-discrete, aperiodic probability measure-preserving groupoid $(G, \mu)$ is sofic if and only if its full group $[G]$ is metrically sofic. Moreover, every isometric group embedding of $[G]$ into an ultraproduct $\prod U \mathcal{S}_n$ extends uniquely to a sofic embedding of $G$.

Before starting, let us give an idea of the extension part of the proof (which is the hardest): Using the previous lemmas, every element of $\text{Meas}(G)$ can be written as $\alpha A$, where $\alpha \in [G]$ and $A \in \mathcal{MAlg}(G^{(0)})$, so if we have two semigroup embeddings $\theta : [G] \to \prod U \mathcal{S}_n$ and $\phi : \mathcal{MAlg}(G^{(0)}) \to \prod U \mathcal{MAlg}(Y_n)$, we may define $\Theta(\alpha A) = \theta(\alpha) \phi(A)$, which a priori might depend on the choice of $\alpha$ and $A$. To guarantee that $\Theta$ is a morphism, we $(\theta, \phi)$ to be covariant. To obtain this, we will define $\phi$ from $\theta$.

**Proof of 2.4.16.** If $G$ is sofic, then any sofic embedding $\Theta : \text{Meas}(G) \to \prod U \text{Meas}(Y^2_n)$ restricts to an isometric group embedding of $[G]$, which is therefore metrically sofic.

Let us prove the uniqueness first: If $\Theta : \text{Meas}(G) \to \prod U \text{Meas}(Y^2_n)$ is an isometric embedding, then for every $\gamma \in \text{Meas}(G)$ choose, by Lemmas 2.4.9 and 2.4.11, $\tilde{\gamma}$ and $\beta \in [G]$ with $\text{supp}(\beta) = s(\gamma)$ and $\gamma \subseteq \tilde{\gamma}$. By Theorem 1.7.25, $\Theta$ is a unital morphism of Boolean inverse semigroups and in particular it preserves supports, so

$$\Theta(\gamma) = \Theta(\tilde{\gamma}) \text{ supp } \Theta(\beta),$$

hence $\Theta$ is uniquely determined by its restriction to $[G]$.

Now suppose $\theta : [G] \to \prod U \mathcal{S}_n$ is an isometric embedding, and let us use the ideas above to extend it to $\text{Meas}(G)$.

We define $\phi : \mathcal{MAlg}(G^{(0)}) \to \prod U \mathcal{MAlg}(Y_n)$ as follows: given $A \in \mathcal{MAlg}(G^{(0)})$, choose, by Lemma 2.4.9, $\alpha \in [G]$ such that $\text{supp}(\alpha) = A$, and define $\phi(A) = \text{supp}(\theta(A))$. By Lemma 2.4.10, $\phi(A)$ does not depend on the choice of $\alpha$. We need several steps to finish this proof, namely,

1. $\phi$ preserves meets:

   Let $A, B \in \mathcal{MAlg}(G^{(0)})$, and consider $\alpha, \beta, \gamma \in [G]$ with

   $$\text{supp}(\gamma) = A \cap B, \quad \text{supp}(\alpha) = A \setminus B, \quad \text{and} \quad \text{supp}(\beta) = B \setminus A.$$ 

   By 2.4.6(b) and 2.4.5(e), the supports of $\theta(\gamma)$ and $\theta(\alpha)$ are disjoint and

   $$\text{supp}(\theta(\gamma\alpha)) = \text{supp}(\theta(\gamma)\theta(\alpha)) = \text{supp}(\theta(\gamma)) \lor \text{supp}(\theta(\alpha)).$$
and similarly for \( \gamma \) and \( \beta \). Lemma 2.4.5(e) also implies that
\[
\operatorname{supp}(\gamma \alpha) = A \quad \text{and} \quad \operatorname{supp}(\gamma \beta) = B,
\]
so again by 2.4.6(b), and using distributivity of meets and joins,
\[
\phi(A) \land \phi(B) = \operatorname{supp}(\theta(\gamma \alpha)) \land \operatorname{supp}(\theta(\gamma \beta))
= (\operatorname{supp}(\theta(\gamma)) \lor \operatorname{supp}(\theta(\alpha))) \land (\operatorname{supp}(\theta(\gamma)) \lor \operatorname{supp}(\theta(\beta)))
= \operatorname{supp}(\theta(\gamma)) = \phi(A \cap B).
\]

2. If \( \alpha \in \mathcal{G} \), then \( \phi(\operatorname{Fix}(\alpha)) = \operatorname{Fix}(\theta(\alpha)) \):
Choose again \( \beta \in \mathcal{G} \) with \( \operatorname{supp}(\beta) = \operatorname{Fix}(\alpha) \), so by 2.4.6(a),
\[
\phi(\operatorname{Fix}(\alpha)) = \phi(\operatorname{supp}(\beta)) = \operatorname{supp}(\theta(\beta)) = \operatorname{Fix}(\theta(\alpha)).
\]

3. \( \phi \) preserves the respective invariant means:
Since \( \theta \) is isometric then it is trace-preserving, so \( \phi \) preserves invariant means by item 2.

4. \((\theta, \phi)\) is covariant:
Let \( A \in \mathcal{MAlg}(\mathcal{G}(0)) \) and \( \alpha \in \mathcal{G} \). Take \( \beta \in \mathcal{G} \) with \( \operatorname{supp}(\beta) = A \). Using 2.4.5(f),
\[
\phi(\alpha \cdot A) = \phi(\operatorname{supp}(\alpha \beta \alpha^*) = \operatorname{supp}(\theta(\alpha \beta \alpha^*)) = (\theta(\alpha) \cdot \operatorname{supp}(\theta(\beta))) = \theta(\alpha) \cdot \phi(A).
\]
Recall that both \( \mathcal{G} \) and \( \mathcal{MAlg}(\mathcal{G}(0)) \) are subsemigroups of \( \mathcal{Meas}(\mathcal{G}) \), and similarly for \( \prod_{\mathcal{U}} \mathcal{G}_n \) and \( \prod_{\mathcal{U}} \mathcal{MAlg}(Y_n) \) inside \( \prod_{\mathcal{U}} \mathcal{Meas}(Y_n^2) \).

5. If \( \alpha, \beta \in \mathcal{G} \), \( A \in \mathcal{MAlg}(\mathcal{G}(0)) \) satisfy \( \alpha A = \beta A \), then \( \theta(\alpha) \phi(A) = \theta(\beta) \phi(A) \):
Items 1. and 2. above imply
\[
\phi(A) \subseteq \operatorname{Fix}(\theta(\alpha^* \beta)),
\]
which in turn yields
\[
\theta(\beta) \phi(B) = \theta(\alpha) \theta(\alpha^* \beta)(\phi(A)) = \theta(\alpha) \phi(A).
\]

Now we define \( \Theta : \mathcal{Meas}(\mathcal{G}) \rightarrow \prod_{\mathcal{U}} \mathcal{Meas}(Y_n^2) \) as \( \Theta(\alpha) = \theta(\tilde{\alpha}) \phi(\alpha^* \alpha) \), where \( \tilde{\alpha} \in \mathcal{G} \) is chosen, by Lemma 2.4.11 such that \( \alpha \subseteq \tilde{\alpha} \). Item 5. above implies that \( \Theta(\alpha) \) does not depend on the choice of \( \tilde{\alpha} \).
Note that \( \Theta \) extends both \( \theta \) and \( \phi \). Let us show that \( \Theta \) is a sofic embedding.

Suppose \( \alpha \subseteq \tilde{\alpha}, \beta \subseteq \tilde{\beta}, \) where \( \alpha, \beta \in \text{Meas}(G) \) and \( \tilde{\alpha}, \tilde{\beta} \in [G] \). Then
\[
\alpha \beta = \tilde{\alpha} \tilde{\beta} (\beta^* \beta \cap (\beta^* \cdot \alpha^* \alpha)) .
\]
In fact, this formula holds independently of \( G \), so it will hold with \( Y_n^2 \) in place of \( G \) and on \( \prod_{i} \text{Meas}(Y_n^2) \) by Łos’ Theorem \([2.1.22]\). This is enough to conclude that \( \Theta(\alpha \beta) = \Theta(\alpha) \Theta(\beta) \), since both \( (\theta, \phi) \) is a covariant pair of semigroup morphisms.

It remains only to check that \( \Theta \) is trace-preserving. Let \( \alpha \in \text{Meas}(G) \). If we show that \( \text{Fix}(\Theta(\alpha)) = \phi(\text{Fix} \alpha) \) we are done because \( \phi \) preserves invariant means.

Given \( \tilde{\alpha} \in [G] \) with \( \alpha \subseteq \tilde{\alpha} \), let \( A = \text{Fix}(\alpha) \) and \( B = \text{Fix}(\tilde{\alpha}) \setminus A \). Note that \( A = \text{Fix}(\tilde{\alpha}) \cap \alpha^* \alpha \), so
\[
\phi(A) = \text{Fix}(\theta(\tilde{\alpha})) \land (\phi(\alpha^* \alpha)).
\]
Since \( \Theta(\alpha) = \theta(\tilde{\alpha}) \phi(\alpha^* \alpha) \leq \theta(\tilde{\alpha}) \), then \( \Theta(\alpha)^* \Theta(\alpha) = \phi(\alpha^* \alpha) \), so
\[
\text{Fix}(\Theta(\alpha)) = \text{Fix}(\theta(\tilde{\alpha})) \land (\phi(\alpha^* \alpha)) = \phi(A) = \phi(\text{Fix}(\alpha)),
\]
and we conclude that \( \Theta \) is a sofic embedding of \( G \). \( \square \)

We will now extend this result allowing \( G \) to have periodic points, however we still need that \( G_x \) is not a singleton for a.e. \( x \in G^{(0)} \). Set
\[
\text{Per}_{\geq 2}(G) = \{ x \in G^{(0)} : 2 \leq \#(G_x) < \infty \}.
\]

**Lemma 2.4.17.** There exists \( \alpha \in [G] \) with \( \text{supp}(\alpha) = \text{Per}_{\geq 2}(G) \).

**Proof.** Let \( \mathcal{P} = \text{Per}_{\geq 2}(G) \). Substituting \( G \) by the subgroupoid \( \mathcal{P} G \mathcal{P} \) if necessary, we may assume \( \mathcal{P} = G^{(0)} \). As in the proof of Lemma \( 2.4.9 \) decompose \( G \) as \( G = \mathcal{H} \sqcup \mathcal{K} \), where \( \mathcal{H} \) and \( \mathcal{K} \) are subgroupoids of \( G \), \( \mathcal{H}_x \) is a is principal, and \( \#(\mathcal{K}_x^2) \geq 2 \) for a.e. \( x \in \mathcal{K}^{(0)} \).

The last part of Corollary \( 1.6.12 \) applied to the restriction of the source map
\[
s : \{ k \in \mathcal{K} : s(k) = r(k) \} \setminus \mathcal{K}^{(0)} \to \mathcal{K}^{(0)}
\]
provides \( \beta \in \text{Meas}(\mathcal{K}) \) with \( \text{supp}(\beta) = \mathcal{K}^{(0)} \).

Since \( \mathcal{H} \) is principal it is (isomorphic to) an equivalence relation on \( \mathcal{H}^{(0)} \). For each \( n \geq 2 \), let \( \mathcal{H}_n = \{ h \in \mathcal{H} : \#(\mathcal{H}_{\#(h)}) = n \} \). Using the selection theorem for periodic relations \([99] \) Theorem 12.16], decompose \( \mathcal{H}_n^{(0)} \) into a Borel partition \( X_1, \ldots, X_n \) such that each \( X_i \) contains exactly one representative of each \( \mathcal{H} \)-class. The Borel subset
\[
\gamma_n = (X_1 \times X_n) \cup \bigcup_{i=1}^{n-1} (X_{i+1} \times X_i)
\]
is a bisection of \( \mathcal{H} \). (Seen as a function, it maps each \( x \in X_i \) to the unique element of \( X_{i+1} \) (or \( X_1 \) if \( i = n \)) which is equivalent to \( x \).) Then \( \text{supp}(\gamma_n) = \mathcal{H}_n^{(0)} \).

Therefore \( \alpha = \beta \cup \bigcup_n \gamma_n \in \text{Meas}(G) \), and \( \text{supp}(\alpha) = \text{Per}_{\geq 2}(G) \). \( \square \)
Theorem 2.4.18. Suppose \((G, \mu)\) is a (standard, \(r\)-discrete) probability measure-preserving groupoid such that for all \(x \in G^{(0)}\), \(#G_x \geq 2\). Then \((G, \mu)\) is sofic if and only if \([G]\) is metrically sofic.

Proof. Let \(P = \text{Per}_{\geq 2}(G)\) and \(\text{Aper} = G^{(0)} \setminus P\), and consider the subgroupoid \(H = G \Delta \text{Aper}\) of \(G\). We have already seen that the subgroupoid \(G P\) is sofic, since it is periodic, and since \(G\) is a convex combination of \(G P\) and \(H\), it suffices to show that \([H, \mu_H]\) is metrically sofic. Fix any \(\rho \in G\) with \(\text{supp} \rho = P\). Let \(\theta : G \to \prod_{U} S_m\) be an isometric embedding.

Consider the embedding \(\iota : [H, \mu_H] \to [G, \mu], \alpha \mapsto \tilde{\alpha} = \alpha \cup \text{Aper}\). Then for all \(\alpha \in [H, \mu_H]\),

\[
\text{tr}_{\mu}(\iota(\alpha)) = \mu(\text{Aper}) \text{tr}_{H}(\alpha)
\]

By 2.4.6(a), \(\text{supp}(\theta(\iota(\alpha))) \subseteq \text{Fix}(\theta(\rho))\) whenever \(\alpha \in [H, \mu_H]\).

As in the proof of Theorem 2.3.4(d), we can “restrict” \(\theta(\iota(\alpha))\) to \(\text{Fix}(\theta(\rho))\) and obtain a new semigroup embedding \(\eta : [H] \to \prod_{U} S_{m_n}\) (where \(m_n \leq n\)), in such a way that for all \(\alpha \in [H, \mu_H]\),

\[
\text{tr}(\eta(\alpha)) = \text{tr}(\theta(\rho))^{-1} \text{tr}(\theta(\iota(\alpha))) = \text{tr}_{\mu}(\rho)^{-1} \text{tr}_{\mu}(\iota(\alpha)) = \mu(\text{Aper})^{-1} \mu(\text{Aper}) \text{tr}_{H}(\alpha)
\]

\[
= \text{tr}_{H}(\alpha) \mu(\theta(\rho), 1)^{-1} = d(\rho, 1)^{-1} = \mu(\text{Aper})^{-1}
\]

so \(\eta\) is in fact a sofic embedding of \(H\).

\(\Box\)

2.5 Sofic group actions

In this section we will study actions which induce sofic transformation groupoids. Again, we will focus only on \(r\)-discrete, standard probability-measure preserving groupoids.

Definition 2.5.1. Let \(G\) be a group and \(\theta\) and \((X, \mu)\) a measure space. An action \(\theta\) of \(G\) by Borel automorphisms of \(X\) is measure-preserving if \(\mu(\theta_g(A)) = \mu(A)\) for all Borel \(A \subseteq X\) and \(g \in G\).

Proposition 2.5.2. Let \(\theta\) be an action of a countable group \(G\) on a standard probability space \((X, \mu)\) by Borel automorphisms of \(X\). Then the following are equivalent:

1. \(\theta\) measure-preserving.
2. \(\mu\) is invariant for the Borel groupoid \(G \ltimes \theta X\);
3. \(\mu\) is invariant for the orbit equivalence relation \(\mathcal{R}(\theta)\).
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Proof. The equivalence \((2) \iff (3)\) is proven in Proposition \ref{proposition:equivalence1}. For every \(g \in G\) and Borel \(A \subseteq X\), the element \(\{g\} \times A\) of \(\text{Bor}(G \rtimes \theta A)\) satisfies
\[
\sigma(\{g\} \times A) = A, \quad \tau(\{g\} \times A) = \theta_g(A),
\]
and every element of \(\text{Bor}(G \rtimes \theta A)\) is in fact a countable disjoint union of these elements. The equivalence \((1) \iff (2)\) follows immediately.

Recall that equivalence relations correspond to principal groupoids (Proposition \ref{proposition:equiv_relation}), and that free actions are those which induce principal transformation groupoids (Example \ref{example:free_action}). Since we are working in the category of measured spaces, null sets can be essentially disregarded.

Definition 2.5.3. A probability measure-preserving action \(\theta\) of a countable group \(G\) on a measure space \((X, \mu)\) is (essentially) free if for all \(g \in G\), the set \(\text{Fix}(\theta_g) = \{x \in X : \theta_g(x) = x\}\) of points fixed by \(g\) is null.

Definition 2.5.4. A standard, \(\tau\)-discrete probability measure-preserving groupoid \((\mathcal{G}, \mu)\) is (essentially) principal if a.e. point of \(\mathcal{G}^{(0)}\) has trivial isotropy, i.e., if
\[
\mu \left( \{ x \in \mathcal{G}^{(0)} : \mathcal{G}^x = \{x\} \} \right) = 1.
\]

Remark. If \(\theta\) is an essentially free probability measure-preserving action of a countable group \(G\) on a standard probability space \((X, \mu)\), we can find a conull subset \(Y\) of \(X\) such that \(\theta_g(Y) \subseteq Y\) for all \(g \in G\), and \(\theta_g(x) \neq x\) for all \(x \in Y\), namely,
\[
Y = \bigcap_{g \in G} \text{supp}(\theta_g) = \{x \in X : \forall g \in G, \theta_g(x) \neq x\}.
\]
A similar comment holds for \(\tau\)-discrete, standard probability measure-preserving groupoids. Therefore our usage of “free” and “principal”, in the category of measure spaces, will not be contradictory with the usual one, in the category of sets.

Moreover, certain classes of topological groupoids are called “essentially principal”, \([146, 125, 126]\). To avoid any confusion, we will use “topologically principal” and “effective” instead for topological groupoids (see Definitions \ref{definition:topologically_principal} and \ref{definition:effective}).

The following can be proven with the same arguments of Example \ref{example:free_action} and the remark above.

Proposition 2.5.5. A probability measure-preserving action \(\theta\) of a countable group \(G\) on a probability space \((X, \mu)\) is free if and only if the transformation groupoid \(G \rtimes X\) is principal.
We will now define Bernoulli shifts. Let $G$ be any countable group and $X$ any set. We consider the action $L$ of left multiplication of $G$ on itself: $L_g : G \to G$, $L_g(h) = gh$ for all $h, g \in G$, and then dualize $L$ to an action $\beta$ on the space $X^G$ of functions from $G$ to $X$ by setting

$$\beta_g(f) = f \circ L_{g^{-1}} \quad \text{for all } f \in X^G \text{ and } g \in G.$$ 

If we use the product notation $X^G = \prod_{G} X = \{(x_h)_{h \in G} : x_h \in X\}$ instead, the action $\beta$ becomes

$$\beta_g((x_h)_{h \in G}) = (x_{g^{-1}h})_{h \in G}$$

for all $g \in G$ and all $(x_h)_{h \in G} \in X^G$.

**Definition 2.5.6.** The action $\beta$ above is called the Bernoulli shift of $G$ on $X$.

Bernoulli shifts are one of the basic examples in Symbolic Dynamics (see [32]). In fact, they can be used to model other actions, as the simple proposition below shows. (Recall Definition 2.4.14 for a covariant map.)

**Proposition 2.5.7.** Let $\theta$ be any action of a countable group $G$ on a set $X$, and let $\beta$ be the Bernoulli shift of $G$ and $X$. Then the map $F : X \to X^G$, $F(x) = (\theta_g^{-1}(x))_{g \in G}$ is injective and covariant.

**Proposition 2.5.8.** Let $G$ be a nontrivial countable group and $(X, \mu)$ a standard probability space. Endow $X^G$ with the product measure coming from $\mu$. Then the Bernoulli shift $\beta$ of $G$ on $X$ is a probability measure-preserving action of $G$, and it is free if and only if $(X, \mu)$ is non-atomic, or if $G$ is infinite and $(X, \mu)$ is not an atom.

**Proof.** The invariance of measure is immediate on cylinder sets (those of the form $\prod_{g \in G} X_g$, where $X_g \subseteq X$ is Borel), and since these generate the Borel structure of $X^G$ then $\beta$ is measure-preserving.

- First assume that $(X, \mu)$ is non-atomic. Then $(X, \mu)$ is isomorphic to the interval $[0, 1]$ with Lebesgue measure ([99], Theorem 17.41), and

  $$\text{Fix}(\theta_g) = \{(x_h)_h \in X^G : (x_{g^{-1}h})_h = (x_h)_h\} \subseteq \{(x_h)_h \in X^G : x_1 = x_g\}$$

  The latter set can be regarded as $\Delta \times X^G \setminus \{1 \cdot g^{-1}\}$, where $\Delta$ is the diagonal of $X \times X$, and so it has measure 0.

- Now assume that $G$ is infinite and $X$ is not an atom. Choose distinct elements $h_1, h_2, \ldots \in G$ such that all $h_1, h_2, \ldots, g^{-1}h_1, g^{-1}h_2, \ldots$ are pairwise distinct. Then

  $$\text{Fix}(\beta_g) \subseteq \{(x_h)_h \in X^G : x_{g^{-1}h_i} = x_{h_i} \text{ for all } i\},$$

  and similarly to the previous case above we can see the latter set as an infinite product of diagonals in $X \times X$ and copies of $X$. The diagonal in $X \times X$ has measure smaller than 1 (because $X$ is not an atom), so an infinite product of them will have measure 0.
If \((X, \mu)\) has an atom \(\{x_0\}\) and \(G\) is finite, then \(\{(x_0)_h\}\) is an atom of \(X^G\) and 
\(\beta_g((x_0)_h) = (x_0)_h\) for all \(g \in G\). If \(X = \{x_0\}\) then \(X^G = \{(x_0)_h\}\) is also an atom. In any of these cases, \(\beta\) is not free.

Remark. In fact, the classification, up to covariant isomorphisms, of Bernoulli shifts has led to the development of Entropy Theory for measure-preserving actions of groups. Initially, entropy theory was developed for actions of \(\mathbb{Z}\), then extended to actions of amenable groups, and recently a theory of entropy for sofic groups has been developed by Bowen [17], and subsequently improved by Kerr and Li [102]. We refer to the survey paper [59] for more details and the history of entropy theory.

We will now be interested in sofic actions.

Definition 2.5.9. A probability measure-preserving action \(\theta\) of a countable group \(G\) on a space \((X, \mu)\) is sofic if the probability measure-preserving groupoid \((G \ltimes X, \mu)\) is sofic.

For the remainder of this section, let us recall and fix some notation.

Conventions

- If \(X\) is a set and \(f \in \mathcal{I}(X)\), we regard \(\text{Fix}(f)\) and \(\text{supp}(f)\) as in Examples 1.7.6 and 2.4.4, i.e.,
  \[
  \text{Fix}(f) = \{x \in \text{dom}(f) : f(x) = x\}, \quad \text{supp}(f) = \{x \in \text{dom}(f) : f(x) \neq x\}.
  \]

- We will consider only countable groups, standard probability spaces and measure-preserving actions.

- We again consider finite sets \(Y_n\) with \(n\) elements, endowed with their normalized counting measures \(\mu_\#\), the permutation group \(S_n\) of \(Y_n\), the measure algebra \(\mathsf{MAlg}(Y_n)\) and the measured semigroups \(\mathsf{Meas}(Y^2_n) = \mathcal{I}(Y_n)\) endowed with their normalized Hamming distances \(d_\#\) and traces \(\text{tr}_\#\). We will also use the symbols \(\mu_\#, d_\#\) and \(\text{tr}_\#\) for corresponding functions on ultraproducts.

- \(U\) will always denote a ultrafilter of sets on \(\mathbb{N}\).

Let \(\theta\) be a measure-preserving action of \(G\) on a standard probability space \((X, \mu)\).

- We identify each \(g \in G\) with the element \(\{g\} \times X\) of the full group \([G \ltimes_\theta X]\) (compare with Example 1.2.12). We also identify \(E(\mathsf{Meas}(G \ltimes X))\) with \(\mathsf{MAlg}(X)\), so if \(g \in G\) and \(A \in \mathsf{MAlg}(X)\), the product \(gA\) corresponds to the element \(\{g\} \times A\) of \(\mathsf{Meas}(G \ltimes X)\).
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- We consider the actions of $G$ on $\text{MAlg}(X)$ and of $\prod_U \mathfrak{S}_{n_k}$ on $\prod_U \text{MAlg}(Y_{n_k})$ as in Definition 2.4.15:

  $$g \cdot A = \theta_g(A), \quad (\sigma_k)_U \cdot (A_k)_U = (\sigma_k(A_K))_U.$$ 

We can compare actions of countable groups with the notion of *weak containment* (an analogous notion for representations has been widely studied; see [9]): By Proposition 2.1.16, every probability measure-preserving action $\theta$ of $G$ on a probability space $(Y, \nu)$ induces an action $\theta^U$ of $G$ on $\prod_U \text{MAlg}(Y)$ by

$$\theta^U_g(A_n)_U = (\theta_g(A_n))_U.$$ 

**Definition 2.5.10 ([100, p. 64]).** Let $\alpha$ and $\theta$ be free probability measure-preserving actions of a countable group $G$ on spaces $(X, \mu)$ and $(Y, \nu)$, respectively. We say that $\alpha$ is *weakly contained* in $\theta$ if there exists an isometric morphism

$$\phi : \text{MAlg}(X) \to \prod_U \text{MAlg}(Y)$$

which is covariant with respect to $\alpha$ and $\theta^U$.

In other words, if an action $\alpha$ is weakly contained in $\theta$ then $\alpha$ can be modelled, in the sense of continuous logic, by $\theta$. Another important property of Bernoulli shifts, initially proved by Abért and Weiss in [2] and later extended by Seward in [157], is that Bernoulli shifts are weakly contained in every other free probability measure-preserving action.

**Theorem 2.5.11 ([2, Theorem 1.1]).** Let $\beta$ be the Bernoulli shift of a countable group $G$ on a standard probability space $(X, \mu)$ (where $X^G$ is endowed with the product measure). Then any free probability-measure-preserving action of $G$ on any standard probability space $(Y, \nu)$ weakly contains $\beta$.

We will now describe soficity of an action in terms of covariant morphisms.

**Lemma 2.5.12.** Suppose $X$ and $Y$ are Borel spaces with $Y$ standard, and $f, g : X \to Y$ are Borel maps with $f(x) \neq g(x)$ for all $x \in X$. Then there exists a countable partition $\{A_n\}$ of $X$ such that $f(A_n) \cap g(A_n) = \emptyset$ for all $n$.

Consequently, if $\mu$ is a probability measure on $X$, then for all $\epsilon > 0$ there exists a finite Borel partition $A_1, \ldots, A_n, C$ of $X$ with $f(A_i) \cap g(A_i) = \emptyset$ for $1 \leq i \leq n$ and $\mu(C) < \epsilon$.

**Proof.** Let $d$ be a metric on $Y$ compatible with the Borel structure and for which $(Y, d)$ is separable. Let $I_1 = \{x \in X : d(f(x), g(x)) > 1\}$ and for $k \geq 2$, consider the Borel set

$$I_k = \{x \in X : 1/k < d(f(x), g(x)) \leq 1/(k - 1)\}$$
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so \( \{I_k\}_k \) is a Borel partition of \( X \). Given \( k \), let \( \{B^n_k\}_n \) be a Borel partition of \( Y \) by sets of diameter at most \( 1/k \), and let \( A^n_k = f^{-1}(B^n_k) \cap I_k \). If \( x \in A^n_k \), then \( f(x) \in B^n_k \) and \( d(g(x), f(x)) > 1/k \geq \text{diam } B^n_k \), so \( g(x) \notin B^n_k \). This proves

\[
f(A^n_k) \cap g(A^n_k) \subseteq B^n_k \cap g(A^n_k) = \emptyset \quad \text{for all } k \text{ and } n,
\]

and \( \{A^n_k : k, n\} \) is a countable Borel partition of \( X \). \( \square \)

**Lemma 2.5.13.** Let \( \theta \) be a probability measure-preserving action of \( G \) on \( (X, \mu) \). Suppose \( \sigma : G \to \prod_U \mathfrak{S}_n \) and \( \phi : \text{MAAlg}(X) \to \prod_U \text{MAAlg}(Y_{n_k}) \) are covariant semigroup morphisms, with \( \phi \) isometric. Then for all \( g \in G \) and \( A \in \text{MAAlg}(X) \),

\[
\text{tr}(\sigma(g)\phi(A)) \leq \text{tr}(gA).
\]

**Proof.** Given \( g \in G \) and \( A \in \text{MAAlg}(X) \), let \( B = \text{supp } \theta_g \cap A \). Given \( \epsilon > 0 \), choose, by Lemma 2.5.12 a finite partition \( B_1, \ldots, B_n, C \) of \( B \) with \( \theta_g(B_i) \cap B_i = \emptyset \) for \( 1 \leq i \leq n \) and \( \mu(C) < \epsilon \).

Then the elements \( \phi(B_i) \) are pairwise disjoint, contained in \( \phi(A) \), and covariance of \( (\sigma, \phi) \) implies

\[
[\sigma(g) \cdot \phi(B_i)] \cap \phi(B_i) = \phi(g \cdot B_i) \cap \phi(B_i) = \phi(\theta_g(B_i) \cap B_i) = 0,
\]

and from this we obtain \( \phi(B_i) \subseteq \text{supp } (\sigma(g)) \cap \phi(A) = \text{supp}(\sigma(g)\phi(A)) \). It follows that

\[
\text{tr}(\sigma(g)\phi(A)) = \mu_\#(\phi(A)) - \mu_\#(\text{supp}(\theta_g\phi(A))) \leq \mu_\#(\phi(A)) - \sum_{i=1}^n \mu_\#(\phi(B_i))
\]

\[
= \mu(A) - \sum_{i=1}^n \mu(B_i) = \text{tr}(gA) + \mu(C) \leq \text{tr}(gA) + \epsilon.
\]

Since \( \epsilon \) is arbitrary, we obtain the result. \( \square \)

The next theorem formalizes the description of soficity used by Ozawa in [134].

**Theorem 2.5.14.** A free probability measure-preserving action \( \theta \) of \( G \) on \( (X, \mu) \) is sofic if and only if there are a group morphism \( \sigma : G \to \prod_U \mathfrak{S}_n \) and an isometric morphism \( \phi : \text{MAAlg}(X) \to \prod_U \text{MAAlg}(Y_{n_k}) \) such that \( (\sigma, \phi) \) is covariant with respect to the canonical actions.

Recall, from the conventions in page 115, that we identify \( G \) and \( \text{MAAlg}(X) \) with subsemigroups of the measured semigroup \( \text{Meas}(G \rtimes \theta X) \). The proof of this theorem is similar in parts to that of Theorem 2.4.16.
Proof of [2.5.14]. First, suppose \( \theta \) is sofic, and let \( \Phi : \text{Meas}(G \ltimes \theta X) \to \prod_U \text{Meas}(Y^2_{n_k}) \) be a sofic embedding. Restricting \( \Phi \) to \( G \) and to \( \text{MAlg}(X) \) we obtain the morphisms \( \sigma \) and \( \phi \), and \( \phi \) is isometric. If \( A \in \text{MAlg}(X) \) and \( g \in G \),

\[
\phi(g \cdot A) = \Phi(gAg^{-1}) = \sigma(g)\phi(A)\Phi(g)^{-1} = \sigma(g) \cdot \phi(A),
\]

so \( (\sigma, \phi) \) is covariant.

Conversely, suppose the covariant pair \( (\sigma, \phi) \) exists. We need a few steps in order to define a sofic embedding of \( G \ltimes \theta X \).

1. If \( g, h \in G \) and \( A, B \in \text{MAlg}(X, \mu) \) satisfy \( A \cap B = \theta_g(A) \cap \theta_h(B) = \emptyset \), then \( \sigma(g)\phi(A) \) and \( \sigma(h)\phi(B) \) are compatible in \( \prod_U \text{Meas}(Y^2_{n_k}) \).

By covariance, we have

\[
(\sigma(g)\phi(A))^*(\sigma(h)\phi(B)) = \phi(A)\sigma(g^{-1}h)\phi(B) = \phi(A \cap (g^{-1}h \cdot B))\sigma(g^{-1}h)
\]

and since

\[
A \cap (g^{-1}h \cdot B) = \theta_g^{-1}(\theta_g(A) \cap \theta_h(B)) = \emptyset,
\]

then \( (\sigma(g)\phi(A))^*(\sigma(h)\phi(B)) = 0 \), and similarly \( (\sigma(g)\phi(A))^*(\sigma(h)\phi(B))^* = 0 \), so \( \sigma(g)\phi(A) \) and \( \sigma(h)\phi(B) \) are compatible.

2. If \( A_1, \ldots, A_n \) are pairwise disjoint and \( g \in G \), then \( \sigma(g)\phi(\bigcup_{i=1}^n A_i) = \bigvee_{i=1}^n \sigma(g)\phi(A_i) \).

First note that \( \bigvee_{i=1}^n \sigma(g)\phi(A_i) \) is well-defined by item 1., so this follows from distributivity of \( \prod_U \text{Meas}(Y^2_{n_k}) \) and the fact that \( \phi \) preserves joins (Theorem 1.7.25).

Every element \( B \) of \( \text{Meas}(G \ltimes \theta X) \) can be written uniquely as a countable join \( B = \bigcup_{g \in G} gB_g \), where \( B_g := \sigma(B \cap (\{g\} \times X)) \). Let \( \mathfrak{M} \) be the submonoid of \( \text{Meas}(G \ltimes \theta X) \) consisting of those \( B \in \text{Meas}(G \ltimes \theta X) \) for which all except finitely many \( B_g \) are empty. Define

\[
\Theta : \mathfrak{M} \to \prod_U \text{Meas}(Y^2_{n_k}), \quad \Theta(B) = \bigvee_{g \in G} \sigma(g)\phi(B_g),
\]

which is well-defined by item 1. above. To verify that \( \Theta \) is a morphism, note that

\[
AB = \bigcup_{g, h \in G} gAhB_h = \bigcup_{g, h \in G} gh((h^{-1}A_g) \cap B_h) = \bigcup_{g \in G} k \left( \bigcup_{g \in G}((g^{-1}k)^{-1} \cdot A_g) \cap B_{g^{-1}k} \right),
\]

and similarly \( \Theta(A)\Theta(B) = \bigvee_{k \in G} \sigma(k) \left( \bigvee_{g \in G}((\sigma(g)^{-1}\sigma(k))^{-1} \cdot \phi(A_g)) \land \phi(B_{g^{-1}k}) \right) \), so the facts that \( (\sigma, \phi) \) is covariant and item 2. above imply that \( \Theta(AB) = \Theta(A)\Theta(B) \).

Let us prove that \( \Theta \) is trace-preserving.
Since $\theta$ is free then $\text{tr}(gB_g) = 0$ whenever $g \neq 1$, and Lemma 2.5.13 implies also that $\text{tr}(\sigma(g)\phi(B_g)) = 0$ when $g \neq 1$. Therefore

$$\text{tr}(\Phi(B)) = \sum_{g \in G} \text{tr}(\sigma(g)\phi(B_g)) = \mu(\phi(B)) = \mu(B) = \sum_{g \in G} \text{tr}(gB_g) = \text{tr}(B).$$

Therefore $\Theta$ is trace-preserving, and hence isometric on the dense submonoid $\mathcal{M}$ of $\text{Meas}(G \ltimes_\theta X)$, and thus extends uniquely to a sofic embedding of $G \ltimes_\theta X$. □

**Definition 2.5.15.** A covariant pair $(\sigma, \phi)$ as in the Theorem 2.5.14 will be called a covariant sofic embedding.

In terms of approximation properties, as in 2.1.18, this theorem can be rewritten as follows:

**Corollary 2.5.16.** A free probability measure-preserving action $\theta$ of $G$ on $(X, \mu)$ is sofic if and only if for every $\varepsilon > 0$ and for every pair of finite subsets $F \subseteq G$ and $A \subseteq \text{MAlg}(X)$, there are $n \in \mathbb{N}$ and maps $\sigma : G \to \mathfrak{S}_n$ and $\phi : \text{MAlg}(X) \to \text{MAlg}(Y_n)$ satisfying

1. $d_\#(\sigma(gh), \sigma(g)\sigma(h)) < \varepsilon$;
2. $d_\#(\phi(A \cap B), \phi(A) \cap \phi(B)) < \varepsilon$;
3. $|\mu_\#(\phi(A)) - \mu(A)| < \varepsilon$;
4. $d_\#(\phi(\theta_g(A)), \sigma(g) \cdot \phi(A)) < \varepsilon$

for all $g, h \in F$ and all $A, B \in A$.

**Definition 2.5.17.** A pair of maps $(\sigma, \phi)$ satisfying properties (i)-(iv) above is called an $(F, A, \varepsilon)$-almost covariant morphism.

As an immediate consequence of this description of soficity, we obtain:

**Corollary 2.5.18.** Soficity of free actions is preserved under weak containment: If $\alpha$ is a free sofic action which weakly contains a free action $\beta$, then $\beta$ is sofic as well.

Therefore, if an infinite countable group $G$ admits a free sofic action then any Bernoulli shift of $G$ is sofic by Theorem 2.5.11 so Bernoulli shifts should come up as a natural example of sofic actions. We can now present the elegant proof of soficity of Bernoulli shifts obtained by Ozawa in full details.

**Proposition 2.5.19** ([41, 134, 142]). Every Bernoulli shift of a sofic group is sofic.
**Proof (based on [137])**. Let \((X, \mu)\) be a (standard) probability space and consider the Bernoulli shift \(\beta\) of \(G\) on \(X^G\). If \(G\) is finite, then \(G \ltimes X^G\) is periodic and hence sofic (Corollary 2.3.14). Assume then that \(G\) is infinite so \(\beta\) is free.

We assume \(G\) is sofic. Let \(\theta = (\theta_k)_{\mathcal{U}} : G \to \prod_{\mathcal{U}} \mathcal{S}_{U_k}\) be a sofic embedding. Let \(\phi = (\phi_k)_{\mathcal{U}} : \operatorname{MAlg}(X) \to \prod_{\mathcal{U}} \operatorname{MAlg}(Y_{m_k})\) be an isometric semigroup embedding (Corollary 2.3.14). We may assume that for all \(k\), \(\phi_k(X) = Y_{m_k}\) and \(\phi_k(\emptyset) = \emptyset\).

For each \(k\), let \(Z_k = Y_{m_k}^\mathcal{U}\) be the set of functions \(f : Y_{m_k} \to Y_{m_k}\). We identify \(Y_{m_k} \times Z_k\) with \(Y_{m_k \times m_k}\) (since both sets have the same number of elements), and the group \(\mathcal{S}_{Y_{m_k} \times Z_k}\) of permutations on \(Y_{m_k} \times Z_k\) with \(\mathcal{S}_{m_k \times m_k}\). We will define a covariant sofic embedding \((\psi, \eta)\), given by maps \(\eta : G \to \prod_{\mathcal{U}} \mathcal{S}_{Y_{m_k} \times Z_k}\) and \(\psi : \operatorname{MAlg}(X^G) \to \prod_{\mathcal{U}} \operatorname{MAlg}(Y_{m_k} \times Z_k)\). First, we define \(\psi\) on cylinder sets of \(X^G\) (which generate its Borel structure).

Given \(A_1, \ldots, A_n \in \operatorname{MAlg}(X)\) and distinct \(g_1, \ldots, g_n \in G\), a **cylinder set** is one which can be written as

\[
C^{A_1, \ldots, A_n}_{g_1, \ldots, g_n} = \bigcap_{j=1}^n \{ f \in X^G : f(g_j) \in A_j \}.
\]

This representation of a nonempty cylinder set in terms of \(g_1, \ldots, g_n\) and \(A_1, \ldots, A_n\) is unique up to those \(A_j\) which are equal to \(X\) (and their order), and such a representation describes the empty set if and only if at least one of the \(A_j\) is empty. We write \(\mathfrak{C}\) for the collection of cylinder sets in \(X^G\). Note that \(\mathfrak{C}\) is closed under intersections.

For each \(k\), define \(\psi_k : \operatorname{MAlg}(X^G) \to \operatorname{MAlg}(Z_k)\) by

\[
\psi_k(C^{A_1, \ldots, A_n}_{g_1, \ldots, g_n}) = \bigcap_{j=1}^n \{ (i, f) \in Y_{m_k} \times Z_k : f(\theta_k(g_j)^{-1}(i)) \in \phi_k(A_j) \}.
\]

By uniqueness of the representation of cylinder sets and since \(\phi_k(X) = Y_{m_k}\) and \(\phi_k(\emptyset) = \emptyset\), the definition of \(\psi\) does not depend on the given choice (and order) of \(g_j\) and \(A_j\).

Therefore we obtain a map \(\psi = (\psi)_{\mathcal{U}} : \mathfrak{C} \to \prod_{\mathcal{U}} \operatorname{MAlg}(Y_{m_k} \times Z_k)\), which we later extend to all of \(\operatorname{MAlg}(X)\). For this, we first prove that \(\psi\) preserves meets (intersections).

Consider cylinders \(C^{A_1, \ldots, A_n}_{g_1, \ldots, g_n}\) and \(C^{B_1, \ldots, B_n}_{h_1, \ldots, h_m}\). By modifying the choice and order of \(g_i, h_i, A_i, B_i\), we can assume \(m = n\) and \(g_i = h_i\), so that

\[
C^{A_1, \ldots, A_n}_{g_1, \ldots, g_n} \cap C^{B_1, \ldots, B_n}_{g_1, \ldots, g_n} = C^{A_1 \cap B_1, \ldots, A_n \cap B_n}_{g_1, \ldots, g_n}.
\]

It follows that, for all \(k\),

\[
\psi_k(C^{A_1, \ldots, A_n}_{g_1, \ldots, g_n}) \cap \psi_k(C^{B_1, \ldots, B_n}_{g_1, \ldots, g_n})
\]
$$\bigcap_{j=1}^{n} \{(i, f) \in Y_{nk} \times Z_k : f(\theta_k(g_j)^{-1}(i)) \in \phi_k(A_j \cap \phi_k(B_j))\} \quad (2.5.1)$$

and

$$\psi_k(C^{A_1 \ldots A_n}_{g_1 \ldots g_n} \cap C^{B_1 \ldots B_n}_{g_1 \ldots g_n}) = \bigcap_{j=1}^{n} \{(i, f) \in Y_{nk} \times Z_k : f(\theta_k(g_j)^{-1}(i)) \in \phi_k(A_j \cap B_j)\} \quad (2.5.2)$$

Let $M_k = \max \{d_\#(\phi_k(A_j) \cap \phi_k(B_j), \phi_k(A_j \cap B_j)) : j = 1, \ldots, n\}$.

If $(i, f)$ is in the symmetric difference of the sets described in Equations (2.5.1) and (2.5.2), then there is $j$ such that

$$f(\theta_k(g_j)^{-1}(i)) \in (\phi_k(A_j) \cap \phi_k(B_k)) \Delta \phi_k(A_j \cap B_j).$$

For each $j$ and each $i \in Y_{nk}$, there are exactly

$$m_{nk}^{n_k-1} \cdot \#[(\phi_k(A_j) \cap \phi_k(B_k)) \Delta \phi_k(A_j \cap B_j)]$$

functions from $Y_{nk}$ to $Y_{mk}$ which map $\theta_k(g_j)(i)$ to an element of

$$(\phi_k(A_j) \cap \phi_k(B_k)) \Delta \phi_k(A_j \cap B_j).$$

Letting $j \in \{1, \ldots, n\}$ and $i \in Y_{nk}$ go through all possible values, we conclude that the (normalized Hamming) distance of the sets in Equations (2.5.1) and (2.5.2) is at most

$$\frac{1}{n_k m_{nk}^{n_k}} \cdot n_k \cdot \sum_{j=1}^{n} m_{nk}^{n_k-1} \#[(\phi_k(A_j) \cap \phi_k(B_k)) \Delta \phi_k(A_j \cap B_j)] =$$

$$\sum_{j=1}^{n} d_\#(\phi_k(A_j) \cap \phi_k(B_j), \phi_k(A_j, B_j))$$

which converges to 0 along $\mathcal{U}$, because $\phi = (\phi_k)_{U}$ is a sofic embedding. Therefore, $\psi$ satisfies $\psi(C_1 \cap C_2) = \psi(C_1) \wedge \psi(C_2)$ for all cylinders $C_1, C_2$, and moreover $\psi(\emptyset) = 0$. This allows us to extend $\psi$ to a semigroup embedding to the (Boolean) algebra of sets generated by cylinders, by setting

$$\psi(A) = \bigvee_k \psi(C_k)$$

where $\{C_k\}_k$ is any finite collection of pairwise disjoint cylinders whose union is $A$. This way, $\psi$ still preserves joins and meets.
Now we show $\psi$ is isometric, or equivalently that it preserves traces (measures) by Theorem 1.7.25. But again, since $\psi$ preserves joins, it is enough to prove that $\psi$ preserves the trace of cylinders, so consider a cylinder of the form
\[ C = C_{g_1^{A_1} \cdots g_n^{A_n}}. \]
For all $k \in \mathbb{N}$, let
\[ I_k = \{ i \in Y_{n_k} : \theta_{g_1}(i), \theta_{g_2}(i), \ldots, \theta_{g_n}(i) \text{ are pairwise distinct} \}. \]
Since $\theta$ is a sofic embedding, $\mu_\#(I_k)$ converges to 1 along $\mathcal{U}$.
If $i \in I_k$, then there are exactly
\[ m_k^{n_k-n} \cdot \prod_{j=1}^{n} \#(\phi_k(A_j)) \]
functions $f : Y_{n_k} \to Y_{m_k}$ for which $(i, f) \in \psi_k(C)$. Therefore,
\[
\begin{align*}
\mu_\#(\psi_k(C)) &= \mu_\#(\psi_k(C) \cap (I_k \times Z_k)) + \mu_\#(\psi_k(C) \setminus (I_k \times Z_k)) \\
&\leq \frac{\#(I_k)}{n_k} m_k^{n_k-n} \cdot \prod_{j=1}^{n} (\#(\phi_k(A_j))) + \mu_\#(\psi_k(C) \setminus (I_k \times Z_k)) \\
&= \mu_\#(I_k) \prod_{j=1}^{n} \mu_\#(A_j) + \mu_\#(\psi_k(C) \setminus (I_k \times Z_k)).
\end{align*}
\]
The first term above converges to $\prod_{j=1}^{n} \mu_\#(A_j)$, which is exactly the measure of $C$, whereas the second term is bounded by
\[
\mu_\#(\psi_k(C) \setminus (I_k \times Z_k)) \leq \mu_\#(Y_{n_k} \setminus Z_k) = 1 - \mu_\#(I_k)
\]
which converges to 0 along $\mathcal{U}$. Therefore $\psi$ is isometric, and since the algebra generated by cylinders is dense in $\text{MAAlg}(X^G)$, $\psi$ can be extended to an isometric semigroup embedding $\psi : \text{MAAlg}(X^G) \to \prod_{U} \text{MAAlg}(Y_{n_k} \times Z_k)$.

Now we let, for each $k$, $\eta_k : G \to \mathcal{S}_{Y_{n_k} \times Z_k}$ be defined by
\[ \eta_k(g)(i, f) = (\theta_g(i), f), \quad \text{for all } g \in G \text{ and } (i, f) \in Y_{n_k} \times Z_k. \]
Clearly,
\[ d_\#(\eta_k(g)\eta_k(h), \eta_k(gh)) = d_\#(\theta_k(g)\theta_k(h), \theta_k(gh)) \]
for all $g, h \in G$. Thus we define a morphism $\eta = (\eta_k)_{U} : G \to \prod_{U} \mathcal{S}_{Y_{n_k} \times Z_k}$.

We denote by $\beta$ the Bernoulli shift of $G$ on $X$. It remains only to show that $(\eta, \psi)$ is covariant, and again it is enough to prove this for cylinder sets. Let $g \in G$ and a cylinder $C_{g_1^{A_1} \cdots g_n^{A_n}}$. Then
2. SOFIC GROUPOIDS

• $\beta \overset{C^{A_1,\ldots,A_n}}{g_{i_1,\ldots,g_{n}}}$;

• $\eta_k(g)(\psi_k(C_{g_{i_1,\ldots,g_{n}}^{A_1,\ldots,A_n}})) = \bigcap_{j=1}^{n} \{(i, f) : f(\theta_k(g_j)^{-1}\theta_k(g)^{-1}(i)) \in \phi_k(A_j)\}$; and

• $\psi_k(C_{g_{i_1,\ldots,g_{n}}^{A_1,\ldots,A_n}}) = \bigcap_{j=1}^{n} \{(i, f) : f(\theta_k(g_j)^{-1}(i)) \in \phi_k(A_j)\}$.

If $(i, f)$ is in the symmetric difference $\eta_k(g)(\psi_k(C_{g_{i_1,\ldots,g_{n}}^{A_1,\ldots,A_n}})) \triangle \psi_k(C_{g_{i_1,\ldots,g_{n}}^{A_1,\ldots,A_n}})$, then there is $j$ for which

$$\theta_k(g_j)^{-1}(i) \neq \theta_k(g_j)^{-1}\theta_k(g)^{-1}(i)$$

and there are $m_k^{n_k}$ choices for $f \in Z_k$. Thus the distance between these two sets is at most

$$\frac{1}{n_km_k^{n_k}} \sum_{j=1}^{n} n_kd_\#(\theta_k(g_j)^{-1}, \theta_k(g_j)^{-1}\theta_k(g)^{-1}) \cdot m_k^{n_k} = \sum_{j=1}^{n} d_\#(\theta_k(g_j)^{-1}, \theta_k(g_j)^{-1}\theta_k(g)^{-1})$$

which converges to $0$ since $\theta$ is a sofic embedding. This proves that $\eta(g)(\psi(C)) = \psi(g \cdot C)$ for all $g \in G$ and all cylinders $C$, and hence for all $C \in \mathcal{MAlg}(X^G)$. Therefore $\beta$ is sofic by Theorem 2.5.14.

One important outstanding question in the area is whether all actions of sofic groups are sofic. However, we may study the related question of how large the class of groups with sofic actions is, by proving that it is closed under some group-theoretic constructions. First, let us quickly review, without proofs, the known closure properties of the class of sofic groups.

**Theorem 2.5.20.** The class of sofic groups is closed under:

(a) Products [44]: If $\{G_i\}_{i \in I}$ is any family of sofic groups, then $\prod_{i \in I} G_i$ is sofic;

(b) Subgroups [44]: If $G$ is sofic and $H \subseteq G$ is a subgroup, then $H$ is sofic;

(c) Direct limits [44]: If $(\phi_{j,i} : G_i \rightarrow G_j)_{i,j}$ is a direct net of sofic groups, then the direct limit $\lim_{i \rightarrow \infty} G_i$ is sofic;

(d) Amenable-by-sofic extensions [44]: If $N$ is a normal subgroup of $G$, $G/N$ is amenable and $N$ is sofic, then $G$ is sofic;

(e) Unrestricted wreath products by amenable groups [7]: If $G$ is sofic and $A$ is amenable then $G \wr A$ is amenable;

(f) Free products amalgamated over an amenable subgroup [14, 43]: If $G$ and $H$ are sofic and $A$ is a common amenable subgroup of $G$ and $H$, then $G \ast_A H$ is sofic.
(g) Wreath products [32]: If $G$ and $H$ are sofic then $G \wr H$ is sofic;

(h) Graph limits (see [33] for details).

**Definition 2.5.21** ([142]). We denote by $\mathcal{I}$ the class of groups whose probability measure-preserving actions are all sofic.

We will be interested in proving that the class $\mathcal{I}$ is closed under some of the constructions listed in 2.5.20.

**Theorem 2.5.22** ([142]). A group $G$ belongs to the class $\mathcal{I}$ if and only if every free probability-measure preserving action of $G$ is sofic.

**Proof.** Suppose that every free probability-measure preserving action of $G$ is sofic. Let $\alpha$ be any probability measure-preserving action of $G$ on $(X, \mu)$. Let $\beta$ be any free probability-measure preserving action of $G$ on a space $(Y, \mu)$ (e.g., a Bernoulli shift on a non-atomic space), and define a new free probability measure-preserving action $\Gamma$ of $G$ on $X \times Y$ by

$$\Gamma_g(x, y) = (\alpha_g(x), \beta_g(y))$$

by assumption, $\Gamma$ is sofic, so there exists a sofic embedding $\Theta : \text{Meas}(G \wr_\Gamma (X \times Y)) \to \prod_{i \in I} \text{Meas}(Y_{\mu_i}^\alpha)$. We can embed $\text{Meas}(G \rtimes_\alpha X)$ isometrically into $\text{Meas}(G \rtimes_\Gamma (X \times Y))$, via $A \mapsto A \times Y$, so composing this with $\Phi$ we obtain a sofic embedding of $G \rtimes_\alpha X$. 

**Theorem 2.5.23.** The class $\mathcal{I}$ is closed under

(a) Subgroups: If $G \in \mathcal{I}$ and $H \subseteq G$ is a subgroup, then $H \in \mathcal{I}$;

(b) Direct limits: If $(\phi_{j,i} : G_i \to G_j)$ is a direct net of groups in $\mathcal{I}$, then $\lim_{i \to \infty} G_i \in \mathcal{I}$;

(c) Co-amenable extensions: If $N$ is a co-amenable subgroup of $G$ (that is, the left action of $G$ on $G/N$ is amenable) and $N \in \mathcal{I}$, then $G \in \mathcal{I}$;

(d) Free products amalgamated over an amenable subgroup [142, Theorem 3.17]: If $G, H \in \mathcal{I}$ and $A$ is a common amenable subgroup of $G$ and $H$, then $G *_A H$ is sofic.

**Proof.** (a) Suppose $G \in \mathcal{I}$ and $H$ is a subgroup of $G$. We let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non-negative integers. To simplify the notation, let us assume that the index $[G : H]$ is infinite, although the same arguments work in the case of finite index.

Let $\{a_i : i \in \mathbb{N}_0\}$ be a complete set of left representatives of $H$ in $G$ with $a_0 = 1$. For each pair $(g, i) \in G \times \mathbb{N}_0$, let $J(g, i) \in \mathbb{N}_0$ be the unique number such that $ga_i \in a_{J(g, i)}H$. Then for all $g$, $a_i^{-1}g^{-1}a_{J(g, i)} \in H$.
Note that, if we identify \( N_0 \) with the quotient space \( G/H \) with via \( i \mapsto a_i H \), the map \( J : G \times N_0 \to N_0 \) corresponds to the action map of the usual left action of \( G \) on \( G/H \).

Let us show that any free probability measure-preserving action \( \alpha \) of \( H \) on a space \((X,\mu)\) is sofic. First, we construct a probability measure-preserving action \( \theta \) of \( G \) on \( X^{N_0} \) by setting

\[
[\theta_g(x)]_i = \alpha(a_i^{-1} g a_{J(g^{-1},i)}) (x_{J(g^{-1},i)}), \quad \text{for all } x = (x_i)_i \in X^{N_0}
\]

Then \( \theta \) is a measure-preserving action of \( G \) on \( X^{N_0} \), so we can take a sofic embedding \( \Theta : \text{Meas}(G \ltimes_\theta X^{N_0}) \to \prod \text{Meas}(Y^2) \). The map

\[
i : \text{Meas}(H \ltimes_\alpha X) \to \text{Meas}(G \ltimes_\theta X^{N_0})
\]

\( i(A) = \{ (g, (x_i)_i) \in G \times X^{N_0} : (g, x_0) \in A \} \)

is an isometric semigroup morphism, because \( \alpha_0 = 1 \), so \( \Theta \circ i \) is a sofic embedding for \( H \ltimes_\alpha X \).

(b) Suppose \( G = \lim \rightarrow G_i \), where \( G_i \in \mathcal{S} \). For each \( i \), let \( p_i : G_i \to G \) be the canonical morphism. Let \( \alpha \) be a free probability measure-preserving action of \( G \) on \((X,\mu)\).

Given a finite \( F \subseteq G \), choose \( i \) such that \( F \cup F^2 \subseteq p_i(G_i) \), and for each \( x \in F \cup F^2 \) choose \( x^i \in G_i \) such that \( p_i(x^i) = x \).

Given \( x, y \in F \), we do not necessarily have \( (x^i)(y^i) = (xy)^i \), but since

\[
p_i((xy)^i) = xy = p_i(x^i y^i)
\]

then there is \( j \geq i \) such that \( p_{j,i}(x^j y^j) = p_{j,i}((xy)^i) \), so substituting \( i \) by \( j \) and \( x^i \) by \( p_{j,i}(x^j) \) for all \( x \in F \cup F^2 \) if necessary, we may assume \( x^i y^i = (xy)^i \) for all \( x \in F \).

Let \( \alpha_i \) be the composition of \( \alpha \) with \( p_i \), which is a probability measure-preserving but not necessarily free. In any case, \( \alpha_i \) is sofic, so we can find a covariant pair of morphisms

\[
\eta : G_i \to \prod \mathcal{S}_{n_k} \quad \text{and} \quad \phi : \text{MAlg}(X) \to \prod \text{MAlg}(\text{MAlg}(Y_{n_k}))
\]

and defining \( \sigma(f) = \eta(f_i) \) we obtain, for all \( f \in F \) and \( A \subseteq \text{MAlg}(A) \),

\[
\phi(f \cdot A) = \phi(f_i \cdot A) = \eta(f_i) \cdot \phi(A) = \sigma(f) \cdot \phi(A)
\]

and

\[
\sigma(f g) = \eta((fg)^i) = \eta(f^i) \eta(g^i) = \sigma(f) \sigma(g)
\]

so using Theorem \( \ref{thm:almost_covariance} \) we can obtain almost covariant morphisms for \( F \) (and any finite subset of \( \text{MAlg}(X) \)), in the sense of Corollary \( \ref{cor:almost_covariance} \). This proves that \( \alpha \) is sofic.
(c) Let $\alpha$ be a free probability measure-preserving action of $G$ on $(X, \mu)$, $F \subseteq G$ finite, $\mathcal{A} \subseteq \text{MAAlg}(X)$ finite and $\epsilon > 0$. Using co-amenability of $N \subseteq G$, choose a finite subset $P \subseteq G/N$ such that
\[
\#((F \cdot P) \triangle P) \leq \epsilon \cdot \#(P)
\]
where $F \cdot P = \{(fg)N : f \in F, gN \in P\}$. Let $a \mapsto \pi$ be a section of the quotient map $G \rightarrow G/N$ i.e., $\pi N = a$ for all $a \in G/N$.

The restriction $\alpha|_N$ is free and probability measure-preserving, and for all $f \in F$ and $a \in P$,
\[
(\overline{ga})^{-1}g\overline{a} \in N.
\]
Let $F' = \{(\overline{ga})^{-1}g\overline{a} : g \in F^2 \cup F, a \in P\}$, which is a finite set. Similarly, let $\mathcal{A}' = \{\alpha(\overline{a}^{-1})(A) : a \in P, A \in \mathcal{A}\}$, which is also a finite subset of $\text{MAAlg}(X)$.

Since $N \in \mathcal{S}$ we can find $n \in \mathbb{N}$ and maps
\[
\sigma : N \rightarrow \mathcal{S}_n \quad \text{and} \quad \phi : \text{MAAlg}(X) \rightarrow \text{MAAlg}(Y_n)
\]
which form a $(F', \mathcal{A}', \epsilon)$-almost covariant morphism. Define\footnote{Again, we denote by $\mathcal{S}_Z$ the permutation group of a set $Z$.} $\pi : G \rightarrow \mathcal{S}_{Y_n \times P}$ by setting
\[
\pi(g)(i, a) = (\sigma((\overline{ga})^{-1}g\overline{a}))(i, ga),
\]
which is defined on a set of measure $\geq 1 - \epsilon$. Then for all $g, h \in F$ we have
\[
\pi(g)\pi(h)(i, a) \equiv \pi(g)(\sigma((\overline{ha})^{-1}h\overline{a}))(i, ha)
\]
\[
\equiv (\sigma((\overline{gha})^{-1}gh\overline{a}))\sigma((\overline{ha})^{-1}h\overline{a})(i, gh a)
\]
\[
\equiv (\sigma((\overline{gha})^{-1}gh\overline{a})(\overline{ha})^{-1}h\overline{a})(i, gh a)
\]
\[
= (\sigma((\overline{ha}^{-1}gh\overline{a}))(i, gh a) \equiv \pi(gh)(i, a)
\]
where the “$\equiv$” means that this equality is valid only up to a set of measure $\epsilon$. So all these terms are equal on a set of measure $\geq 1 - 5\epsilon$, which means that $d_\#(\pi(g)\pi(h), \pi(gh)) \leq 5\epsilon$. This corresponds to property (i) of an almost covariant morphism (Corollary 2.5.16).

We also define $\psi : \text{MAAlg}(X) \rightarrow \text{MAAlg}(Y_n \times P)$ by setting
\[
\psi(A) = \bigcup_{a \in P} \phi(\alpha(\overline{a}^{-1})(A)) \times \{a\}
\]
and easy computations, similar to the one above, prove that properties (ii) and (iii) of Corollary 2.5.16 are satisfied. We prove almost covariance of $\pi$ and $\psi$. Using the notation “$\equiv$” as above we have, for $g \in F$ and $A \in \mathcal{A}$,
\[
\pi(g)(\psi(A)) \equiv \bigcup_{a \in P} \sigma(\overline{ga}^{-1}g\overline{a})[\phi(\alpha(\overline{a}^{-1}))(A)] \times \{ga\}
\[ \phi(\alpha(g^{-1}a)(A)) \times \{ga\} \]
\[ \phi(\alpha(b^{-1}g)(A)) \times \{b\} \]
\[ \phi(\alpha(b^{-1})(\alpha(g)(A))) \times \{b\} \]
\[ \psi(\alpha(g)(A)) \]

and therefore \((\pi, \psi)\) is a \((F, A, 5\epsilon)\)-almost covariant morphism.

(d) We refer to [142, Theorem 3.17].
Chapter 3

Disjoint continuous functions

Since Stone’s seminal work [163], the relations between algebraic and analytical structures have been a topic of great interest in Algebra, Analysis and Logic. In fact, the non-commutative Stone duality theorem proved in the first chapter (1.4.26) is a prime example of this.

Now suppose that \( X \) is a compact, Hausdorff topological space, and let \( C(X) \) be the set of continuous real or complex-valued functions on \( X \). Then \( C(X) \) comes with several natural algebraic structures, induced by pointwise operations, and a natural question is how to recover the space \( X \) (up to homeomorphism) from \( C(X) \).

In general, the topological vector space structure of \( C(X) \) (with the topology coming from the uniform norm) is not sufficient to determine \( X \) up to homeomorphism. Thus, we need to consider additional structures on \( C(X) \) in order to recover \( X \), and in fact several results of this nature have been proven. Two compact, Hausdorff spaces \( X \) and \( Y \) are homeomorphic if and only if \( C(X) \) and \( C(Y) \) are isomorphic as:

- Banach spaces (under isometric linear isomorphisms); (Banach-Stone ‘32-37 [8, 164]);
- rings (Gelfand-Kolmogorov ‘39 [61, 60]);
- multiplicative semigroups (Milgram ‘40 [130]);
- C*-algebras (Gelfand-Naimark ‘43 [62]);
- ordered sets (Kaplansky ‘47 [98]);
- vector space with the disjointness relation (Jarosz ‘90 [92]).

\(^1\)This is a consequence of Milutin’s Theorem, which states that for any two uncountable compact metrizable spaces \( X \) and \( Y \), \( C(X) \) and \( C(Y) \), with their respective uniform norms, are isomorphic as topological vector spaces. See [95, Chapter 36, Theorem 2.1] or [131, 137].
3. DISJOINT CONTINUOUS FUNCTIONS

There are also more recent results by Kania-Rmoutil ([97]) and Li-Wong ([116]) which generalize the previous theorems to different degrees, and studies of spaces of non-scalar valued functions by Hernández-Ródenas ([84]) and Kaplansky ([98]). Our goal is to use a (stronger version of) the disjointness relation studied by Jarosz, but without the vector space structure, in order to obtain further generalizations of all of these results and new consequences.

This chapter is organized as follows: In the first section we introduce the terminology and the notation which will be used in this chapter, and we prove the two main theorems ([Theorems 3.1.12 and 3.1.13]). In the second section we introduce “disjointness ideals”, which can be used to provide alternative proofs of the main theorems of the first section, and which will play a role in the study of Steinberg algebras in the next chapter. In the third section, we study an important class maps, called “basic”, between spaces of functions, and which will appear in most applications of the previous results. The results of this section apply to different algebraic and analytical structures on sets of continuous functions. In the fourth section we recover the classical and recent results described above, and generalize some of them. In the last section we obtain new results: a classification of linear isomorphisms which are isometric with respect to $L^1$-norms ([Theorem 3.5.2]), classifications of classes of isomorphisms of algebras associated to groupoids ([Theorems 3.5.11 and 3.5.14]), and of classes of isomorphisms of groups of circle-valued functions ([Theorems 3.5.18 and 3.5.15]).

The theory developed here will also be used in Chapter 4 to obtain “recovery” results for topological partial actions from algebraic partial actions (Theorem 4.2.4 and Proposition 4.4.37), and to classify diagonal-preserving isomorphisms of certain Steinberg algebras (Theorem 4.4.27).

In Appendix [A] we include a proof of the existence of a topological group isomorphism between $C([0,1], S^1)$ and $C([0,1]^2, S^1)$. The original example was given by Vladimir Pestov in his referee report of Ahmed Al-Rawashdeh’s PhD thesis [3]. Because the report is not widely available, we reproduce the example with the kind permission of its author.

3.1 Disjointness, $\perp$ and $\updownarrow$-isomorphisms

Let $X$ be a locally compact Hausdorff space and $H$ be a Hausdorff space. We denote by $C(X, H)$ the set of continuous functions from $X$ to $H$. We also fix for the remainder of this section a function $\theta \in C(X, H)$. For two functions $f, g : X \to H$, we denote

$$[f \neq g] = \{x \in X : f(x) \neq g(x)\} \quad \text{and} \quad [f = g] = X \setminus [f \neq g].$$

\(^2\)It would be expected that this theorem is known, but no reference to it could be found at the time of writing.
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Definition 3.1.1. Given $f \in C(X, H)$, we define the $(\theta)$-support of $f$ as

$$\text{supp}^\theta(f) = [f \neq \emptyset].$$

We also denote by $\sigma^\theta(f)$ the interior of $\text{supp}^\theta(f)$, and $Z^\theta(f)$ the complement of $\text{supp}^\theta(f)$:

$$\sigma^\theta(f) = \text{int} \text{supp}^\theta(f) \quad \text{and} \quad Z^\theta(f) = X \setminus \text{supp}^\theta(f).$$

Let $C_c(X, \theta)$ be the set of continuous functions from $X$ to $H$ with compact $(\theta)$-support. Whenever there is no risk of confusion, we will drop $\theta$ from the notation and write simply $\text{supp}(f)$, $\sigma(f)$, $Z(f)$ and $C_c(X)$.

We will be interested in relations between $\subseteq, \overline{\in}, \perp$ and $\bot$.

Lemma 3.1.4. $f \perp g$ if and only if $\sigma(f) \cap \sigma(g) = \emptyset$.

Proof. By definition, $f \perp g \iff [f \neq \emptyset] \cap [g \neq \emptyset] = \emptyset$. Since both $[f \neq \emptyset]$ and $[g \neq \emptyset]$ are open (because $H$ is Hausdorff), we can take closures and interiors of $[f \neq \emptyset]$ and obtain

$$f \perp g \iff \sigma(f) \cap [g \neq \emptyset] = \emptyset,$$

and repeating the argument with $[g \neq \emptyset]$ yields the result.

Remark. In general, $\sigma(f)$ is the regularization\footnote{The \textit{regularization} of an open set $A$ in a topological space $X$ is $\text{int} \overline{A}$, and it is the smallest regular open set of $X$ containing $A$.} of $[f \neq \emptyset]$, and so $\sigma(f)$ may “forget” some points on which $f(x) = \theta(x)$. For example, if $X = H = [0,1]$, $\theta = 0$ and $f(x) = x$, then $\sigma(f) = X$ even though $f(0) = 0$.

Also, $Z(f)$ is the complement of $\sigma(f)$ in the lattice of regular open sets of $X$ (see \cite[Chapter 10]{70}], and note that $\overline{\sigma(f)} = \text{supp}(f)$.

Example 3.1.2. If $H = \mathbb{R}$ or $\mathbb{C}$, and $\theta = 0$ is the constant zero function, we obtain the usual notions of support, and $f \perp g$ if and only if $fg = 0$.

Example 3.1.3. If $H = X$ and $\theta = \text{id}_X$, then $\text{supp}(f) = \{x \in X : f(x) \neq x\}$.
We will, moreover, be interested in recovering a locally compact Hausdorff topological space $X$ not from the whole set $C_c(X)$, but instead from a subcollection $A \subseteq C_c(X)$. We will need to assume, however, that there are enough functions in $A$ in order to separate points of $X$.

**Definition 3.1.5.** Let $A \subseteq C_c(X)$ be a subset containing $\theta$. Denote $\sigma(A) = \{\sigma(f) : f \in A\}$. We say that $(X, \theta, A)$ (or simply $A$) is

1. weakly regular if $\sigma(A)$ is a basis for the topology of $X$.

2. regular if for every $x \in X$, every neighbourhood $U$ of $x$ and every $c \in H$ there is $f \in A$ with $f(x) = c$ and $\text{supp}(f) \subseteq U$.

**Definition 3.1.6.** Suppose $A \subseteq C_c(X)$ is weakly regular. A family $A \subseteq A$ is a cover of an element $b \in A$ if given $h \in A$, $h \perp a$ for all $a \in A$ implies $h \perp b$.

By Lemma 3.1.4, $A$ is a cover for $b$ if and only if $\{\sigma(f) : f \in A\}$ is a cover for $\sigma(b)$, in the sense of Definition 1.7.1.

**Lemma 3.1.7.** Suppose $A$ is weakly regular, and let $A \subseteq A$ and $b \in A$. The following are equivalent:

1. $A$ is a cover of $b$;

2. The closure of $\bigcup_{a \in A} [a \neq \theta]$ contains $\text{supp}(b)$.

**Proof.** (1)⇒(2): Let $x \in \text{supp}(b)$. Take an open neighbourhood of $x$ of the form $\sigma(h)$, $h \in A$. Since $\text{supp}(b) = \overline{\sigma(b)}$, the intersection $\sigma(h) \cap \sigma(b)$ is nonempty and thus $h$ and $b$ are not disjoint. From $A$ being a cover, $h$ is not disjoint to some $a \in A$, which means that $\sigma(h) \cap [a \neq \theta]$ is nonempty. Since $A$ is weakly regular then $x$ is in the closure of $\bigcup_{a \in A} [a \neq \theta]$.

(2)⇒(1): Suppose $h \in A$ is such that $h \perp a$ for all $a \in A$. This means that $(\bigcup_{a \in A} [a \neq \theta]) \cap [h \neq \theta] = \emptyset$. Taking the closure of the first term and using (2) we conclude that $[b \neq \theta] \cap [h \neq \theta] \subseteq \text{supp}(b) \cap [h \neq \theta] = \emptyset$, so $h \perp b$.

If $A \subseteq C_c(X)$ and $\theta \in A$, note that $\subseteq$ is a preorder on $A$, and the only infimum\(^4\) of $A$ is $\theta$. Alternatively, $\theta$ is the only element of $A$ such that $\theta \perp \theta$. This shows that the function $\theta$ is uniquely determined in terms of either $\perp$ or $\subseteq$.

**Theorem 3.1.8.** Suppose $A$ is weakly regular. If $f, g \in A$, then

(a) $f \subseteq g \iff \forall h(h \perp g \Rightarrow h \perp f)$;

\(^4\)Infima for preorders are defined the same way as for orders, namely if $A$ is a subset of a preordered space $(P, \leq)$, an element $x$ is called an infimum of $A$ if $x \leq a$ for all $a \in A$, and if for any other element $y$ such that $y \leq a$ for all $a \in A$, we have $y \leq x$. 

(b) \( f \perp g \iff \) The \( \subseteq \)-infimum of \( \{f, g\} \) is \( \theta \) (which is the \( \subseteq \)-infimum of \( \mathcal{A} \));

(c) \( f \subseteq g \iff \forall h(h \in f \Rightarrow h \in g) \);

(d) \( f \subseteq g \iff \forall h(h \perp g \Rightarrow h \perp f) \);

(e) \( f \perp g \iff \exists h_1, k_1, \ldots, h_n, k_n \in \mathcal{A} \) such that \( \{h_1, \ldots, h_n\} \) is a cover of \( f \), \( h_i \in k_i \) and \( k_i \perp g \);

(f) \( f \in g \iff \forall b \in \mathcal{A}, \exists h_1, \ldots, h_n \in \mathcal{A} \) such that \( \{h_1, \ldots, h_n, g\} \) is a cover of \( b \) and \( h_i \perp f \).

By items (a) and (b), \( \perp \) and \( \subseteq \) are equi-expressible (i.e., each one is completely determined by the other). By (c) and (d) one can recover \( \subseteq \) (and hence \( \perp \)) from either \( \in \) or \( \perp \), which in turn implies, from (e) and (f), that \( \in \) and \( \perp \) are also equi-expressible.

Proof.  

(a) \( \Rightarrow \): If \( f \subseteq g \) and \( h \perp g \), then \( \sigma(f) \cap \sigma(h) \subseteq \sigma(g) \cap \sigma(h) = \emptyset \), thus \( h \perp f \).

\( \Leftarrow \): Suppose \( \sigma(f) \) is not contained in \( \sigma(g) \). Since \( \sigma(f) = \text{int}(\{f \neq \emptyset\}) \) and \( \sigma(g) \) is regular, this means that \( \{f \neq \emptyset\} \) is not contained in \( \sigma(g) = \text{supp}(g) \), so there is an open set \( U \) such that \( f \neq \emptyset \) on \( U \) but \( g = \emptyset \) on \( U \). By weak regularity, there is some \( h \) such that \( \emptyset \neq [h \neq \emptyset] \subseteq U \), so \( f \) and \( h \) are not disjoint but \( g \) and \( h \) are.

(b) \( \Rightarrow \): If \( f \subseteq g \), then \( \sigma(h) \subseteq \sigma(f) \cap \sigma(g) = \emptyset \), so \( h = \emptyset \).

\( \Leftarrow \): If \( f \) and \( g \) are not disjoint, then there is some \( h \) with \( \emptyset \neq \sigma(h) \subseteq \sigma(f) \cap \sigma(g) \), by weak regularity, so \( h \subseteq f \) and \( h \neq \emptyset \).

(c) This is immediate since \( X \) is locally compact Hausdorff and \( \mathcal{A} \) is weakly regular, so \( \sigma(f) = \bigcup \{\sigma(h) : h \in f\} \) for all \( f \).

(d) The proof is essentially the same as (a), so we omit it.

(e) \( \Rightarrow \): Suppose \( f \perp g \). For every \( x \in \text{supp}(f) \), take \( h_x, k_x \in \mathcal{A} \) such that \( x \in \sigma(h_x) \), \( h_x \in k_x \) and \( k_x \perp g \). By compactness, we can take finitely many points \( x_1, \ldots, x_n \), and associated functions \( h_1, k_1, \ldots, h_n, k_n \), such that \( \{h_1, \ldots, h_n\} \) covers \( f \).

\( \Leftarrow \): Suppose such \( h_i, k_i \) exist. Then

\[
\text{supp}(f) \subseteq \bigcup_{i=1}^{n} \text{supp}(h_i) \subseteq \bigcup_{i=1}^{n} \sigma(k_i) \subseteq X \setminus \text{supp}(g)
\]

and so \( f \perp g \).
One should be careful with the connections between the pairs of relations $(\perp, \perp)$ and $(\subseteq, \subseteq)$. For example, $\perp$ and $\perp$ can coincide but $\subseteq$ and $\subseteq$ may not and vice-versa. See the example below.

**Example 3.1.9.** Let $X = H = \mathbb{R}$ and $\theta = 0$, so that we are dealing with the usual notion of support. Let $\{ (a_n, b_n) : n \in \mathbb{N} \} \ (a_n < b_n)$ be a basis of open intervals for the usual topology of $\mathbb{R}$ with $|b_n - a_n| \to 0$. Let $\{ p_n : n \in \mathbb{N} \}$ be an one-to-one enumeration of the prime numbers.

For each $n$, let $\widetilde{a}_n$ and $\widetilde{b}_n$ be, respectively, the largest and smallest rational numbers with denominators $p_n$ as reduced fractions, and which satisfy

$$\widetilde{a}_n \leq a_n < b_n \leq \widetilde{b}_n.$$ 

Namely,

$$\widetilde{a}_n = \frac{[a_n p_n]}{p_n} \quad \text{and} \quad \widetilde{b}_n = \frac{[b_n p_n]}{p_n}$$

where $[\cdot]$ and $[\cdot]$ are the usual floor and ceiling functions. In particular, $|\widetilde{a}_n - a_n| + |\widetilde{b}_n - b_n| \leq 2/p_n \to 0$ (since the enumeration of the primes is one-to-one), and thus the sets $U_n = (\widetilde{a}_n, \widetilde{b}_n)$ also form a basis of $\mathbb{R}$. 

**Remark.** One should be careful with the connections between the pairs of relations $(\perp, \perp)$ and $(\subseteq, \subseteq)$. For example, $\perp$ and $\perp$ can coincide but $\subseteq$ and $\subseteq$ may not and vice-versa. See the example below.
3. DISJOINT CONTINUOUS FUNCTIONS

For each \( n \), let \( f_n \in C_c(\mathbb{R}) \) with \( \sigma(f_n) = U_n \), e.g.

\[
f_n(x) = \max(0,(x-a_n)(b_n-x)),
\]

and let \( \mathcal{A} = \{ f_n : n \in \mathbb{N} \} \). Then \( \perp \) and \( \sqsubseteq \) coincide on \( \mathcal{A} \), as do \( \subseteq \) and \( \in \), since the boundaries of all \( U_n \) are pairwise disjoint.

Letting \( V = (\tilde{a}_1, \tilde{b}_1 + 1) \) and \( g_V \) be a continuous function with \( \sigma(g_V) = V \), then \( \perp \) and \( \sqsubseteq \) still coincide in \( \mathcal{A} \cup \{ g_V \} \), however \( \subseteq \) and \( \in \) do not, since \( f_1 \subseteq g_V \) but \( f_1 \not\subseteq g_V \).

Alternatively, set \( W = (\tilde{b}_1, \tilde{b}_1 + 1) \) and let \( g_W \) be any continuous function with \( \sigma(g_W) = W \). Then \( \subseteq \) and \( \in \) still coincide in \( \mathcal{A} \cup \{ g_W \} \), however \( \perp \) and \( \sqsubseteq \) do not, because \( f_1 \perp g \) but not \( f_1 \sqsubseteq g \).

**Example 3.1.10** (Kania-Rmoutil, [97]). Suppose that \( X \) is a Stone space (Definition \[1.4.2\]) and that \( H \) is a Hausdorff space with at least two points. Let \( \mathcal{A} \) consist of all continuous \( H \)-valued functions with finite range and \( \theta \) be any element of \( \mathcal{A} \). Regularity of \( \mathcal{A} \) is an immediate consequence of \( X \) being zero-dimensional (and \( H \) containing more than one point). Define the *compatibility ordering* on \( C(X) \) by

\[
f \preceq g \iff g|_{\text{supp}(f)} = f|_{\text{supp}(f)}.
\]

Then \( \perp \) coincides with \( \sqsubseteq \), and it can be described in terms of \( \preceq \) by

\[
f \perp g \iff f \sqsubseteq g \iff \inf(f,g) = \theta(= \inf \mathcal{A}) \text{ and } \{ f, g \} \text{ has a } \preceq \text{-upper bound.}
\]

We now state and prove the two main theorems of this chapter. Fix two locally compact Hausdorff spaces \( X \) and \( Y \), and for \( Z \in \{ X, Y \} \) a Hausdorff space \( H_Z \), a continuous map \( \theta_Z : Z \rightarrow H_Z \), and a subset \( \mathcal{A}(Z) \subseteq C_c(Z, \theta_Z) \).

**Definition 3.1.11.** We call a map \( T : \mathcal{A}(X) \rightarrow \mathcal{A}(Y) \) a \( \perp \)-morphism if \( f \perp g \) implies \( Tf \perp Tg \); \( T \) is a \( \perp \)-isomorphism if it is bijective and both \( T \) and \( T^{-1} \) are \( \perp \)-morphisms. \( \subseteq \) and \( \sqsubseteq \) and \( \preceq \)-isomorphisms are defined analogously.

**Theorem 3.1.12.** Suppose \( \mathcal{A}(X) \) and \( \mathcal{A}(Y) \) are weakly regular and \( T : \mathcal{A}(X) \rightarrow \mathcal{A}(Y) \) is a \( \perp \)-isomorphism. Let \( f, g \in \mathcal{A}(X) \). Then \( \sigma(f) \subseteq \sigma(g) \) if and only if \( \sigma(Tf) \subseteq \sigma(Tg) \). In particular, \( Z(f) = \emptyset \) if and only if \( Z(Tf) = \emptyset \).

**Proof.** Immediate from 3.1.8(a), because every \( \perp \)-morphism is a \( \subseteq \)-morphism (and vice-versa).

**Theorem 3.1.13.** If \( \mathcal{A}(X) \) and \( \mathcal{A}(Y) \) are weakly regular and \( T : \mathcal{A}(X) \rightarrow \mathcal{A}(Y) \) is a \( \perp \)-isomorphism then there is a unique homeomorphism \( \phi : Y \rightarrow X \) such that \( \phi(\text{supp}(Tf)) = \text{supp}(f) \) for all \( f \in \mathcal{A}(X) \) (equivalently, \( \phi(\sigma(Tf)) = \sigma(f) \), or \( \phi(Z(Tf)) = Z(f) \), for all \( f \in \mathcal{A}(X) \)).
Proof. By Theorem 3.1.8, \(\perp\)-isomorphisms coincide with \(\subseteq\)-isomorphisms, so the result is an easy application of Theorem 1.4.7. We add the details for the sake of completeness.

Let \(\Phi : \sigma(A(X)) \rightarrow \sigma(A(Y))\) be the map given by \(\Phi(\sigma(f)) = \sigma(Tf)\) for all \(f \in A(X)\), which is well-defined by 3.1.12. This is a \(\subseteq\)-isomorphism, so, using the same notation as in 1.4.7, we induce a homeomorphism \(\Phi^*: \sigma(A(Y)) \equiv \rightarrow \sigma(A(X)) \equiv\), \(\Phi^*(F) = \Phi^{-1}(F)\).

For \(Z \in \{X, Y\}\), let \(\kappa_Z : Z \rightarrow \sigma(A(Z))\) be the homeomorphism given by \(\kappa_Z(z) = \{\sigma(f) : z \in \sigma(f)\}\). We then define \(\phi = \kappa_X^{-1} \circ \Phi^* \circ \kappa_Y : Y \rightarrow X\), which is a homeomorphism. For all \(y \in Y\),

\[
\kappa_X(\phi(y)) = \Phi^{-1}(\kappa_Y(y)) = \{\sigma(f) : y \in \sigma(Tf)\}
\]

which means that for each \(f \in A(X)\), \(y \in \sigma(Tf)\) if and only if \(\phi(y) \in \sigma(f)\). Therefore \(\phi(\sigma(Tf)) = \sigma(f)\), or equivalently taking closures we obtain \(\phi(\text{supp}(Tf)) = \text{supp}(f)\).

\(\square\)

For an alternative proof, see the end of Section 3.2.

**Definition 3.1.14.** The unique homeomorphism \(\phi\) associated with \(T\) as in 3.1.13 will be called the \(T\)-homeomorphism.

We finish this section by proving that the hypothesis that \(T\) is a \(\perp\)-isomorphism in Theorem 3.1.13 cannot be weakened to a \(\perp\)-isomorphism in general (Corollaries 3.1.22 and 3.1.24).

Let \(X\) and \(Y\) be locally compact Hausdorff spaces. We will consider only real-valued functions and \(\theta_X\) and \(\theta_Y\) the respective zero functions on \(X\) and \(Y\), so that supports of continuous functions are the usual ones. By Theorem 3.1.12, any \(\perp\)-isomorphism \(T : C_c(X) \rightarrow C_c(Y)\) induces an order isomorphism between collections of regular open sets of \(X\) and \(Y\), respectively. So in order to find \(\perp\)-isomorphisms, we will look at spaces with isomorphic collections of regular open sets.

**Definition 3.1.15.** Let \(X\) be a topological space. We denote by \(\text{RO}(X)\) the collection of regular open subsets of \(X\), and by \(\text{RO}_K(X)\) the collection of regular open subsets of \(X\) with compact closure. Given \(A \in \text{RO}_K(X)\), we define \(\Sigma_X(A) = \{f \in C_c(X) : \sigma(f) = A\}\).

A generalized Boolean algebra is *conditionally complete* if every bounded family admits a join. In the case of unital Boolean algebras we just call these *complete* (see [63, Definition I-4.5]).

It is well-known that the family of regular open subsets of a topological space is a complete unital Boolean algebra (see [70, Theorem 10.1]). Similarly, one can
prove that $RO_K(X)$ is a conditionally complete generalized Boolean algebra for any Hausdorff space $X$.

In order to find non-homeomorphic spaces $X$ and $Y$ such that $C_c(X)$ and $C_c(Y)$ are $\bot$-isomorphic, we need the following lemma:

**Lemma 3.1.16.** Let $X$ be a separable, locally compact Hausdorff space and $A \in RO_K(x)$. If $A \neq \emptyset$ and the set $\Sigma_X(A) = \{ f \in C_c(X) : \sigma(f) = A \}$ is nonempty, then it has the cardinality of the continuum, $|\Sigma_X(A)| = 2^{\aleph_0}$.

**Proof.** If $f \in \Sigma_X(A)$ then $f \neq 0$, and the continuum-many functions $\lambda f$, $\lambda \in \mathbb{R}$, also belong to $\Sigma_X(A)$, so $|\Sigma_X(A)| \geq 2^{\aleph_0}$. Conversely, if $D \subseteq X$ is a dense countable subset of $X$ then the map $C(X) \to \mathbb{R}^D$, $f \mapsto f|_D$ is injective, so

$$|\Sigma_X(A)| \leq |C(X)| \leq |\mathbb{R}^D| = |\mathbb{R}|^{|D|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}. \qed$$

**Theorem 3.1.17.** Suppose that:

1. $X$ and $Y$ are separable, locally compact Hausdorff spaces;
2. For all nonempty $A \in RO_K(X)$ and $B \in RO_K(Y)$, $\Sigma_X(A)$ and $\Sigma_Y(B)$ are nonempty;
3. $\varphi : RO_K(X) \to RO_K(Y)$ is an order isomorphism (with respect to set inclusion).

Then $C_c(X)$ and $C_c(Y)$ are $\bot$-isomorphic.

**Proof.** Given $A \in RO_K(X)$, the sets $\Sigma_X(A)$ and $\Sigma_Y(\varphi(A))$ have the same cardinality, by Lemma 3.1.16 so consider any bijection $T_A : \Sigma_X(A) \to \Sigma_Y(\varphi(A))$. Then the map

$$T : C_c(X) \to C_c(Y), \quad T(f) = T_{\sigma(f)}(f)$$

is a $\bot$-isomorphism. $\Box$

For our first example we employ the theory of Stone spaces and Stone duality as in Section 1.4. Recall that

- Given a zero-dimensional, locally compact Hausdorff space $\mathfrak{C}$, $KB(\mathfrak{C})$ is the collection of compact-open subsets of $\mathfrak{C}$.
- Given a generalized Boolean algebra $B$, $\mathcal{G}_p(B)$ is the topological space (unit groupoid) of ultrafilters on $B$, with the topology generated by the sets $[a] = \{ F \in \mathcal{G}_p(B) : a \in F \}$, $a \in B$. 
Example 3.1.18. In general, arbitrary joins do not correspond to unions under Stone duality. Let $X$ be an infinite set, and consider the complete unital Boolean algebra $\mathcal{P}(X)$ of subsets of $X$, and $\mathcal{C} = \mathcal{G}_F(\mathcal{P}(X))$. By Proposition 1.4.21 the map $\mathcal{P}(X) \ni A \mapsto [A] \in \text{KB}(\mathcal{C})$ is an order isomorphism so $\text{KB}(\mathcal{C})$ is complete.

Given $a \in X$, let us denote by $\{a\}^\uparrow = \{A \subseteq X : a \in A\}$ the (principal) ultrafilter generated by the element $\{a\}$ of $\mathcal{P}(X)$.

Let $F \in \mathcal{C}$ be any ultrafilter which does not contain finite subsets of $X$, and whose existence is guaranteed by Zorn’s Lemma. Given a basic neighbourhood $[A]$ of $F$ in $\mathcal{C}$, where $A \subseteq X$, choose any $a \in A$ and note that $\{a\}^\uparrow \in [A] \setminus \{F\}$, and therefore $F$ is non-isolated in $\mathcal{C}$.

Since $\mathcal{C}$ is Hausdorff then $\bigcup \{[A] : F \notin [A]\} = \mathcal{C} \setminus \{F\}$, and as $F$ is non-isolated then this set is not clopen. Therefore the join of $\{[A] : F \notin [A]\}$ in $\text{KB}(\mathcal{C})$ is actually $\mathcal{C}$.

More explicitly, in $\mathcal{P}(X)$ we have

$$\bigvee \{A : F \notin [A]\} = \bigvee \{A : A \notin F\} \supseteq \bigvee_{x \in X} \{x\} = X,$$

and thus $\bigvee \{[A] : F \notin [A]\} = [X] = \mathcal{C}$ in $\text{KB}(\mathcal{C})$.

Lemma 3.1.19 (Compare with [87, Lemma 6.10]). Let $\mathcal{C}$ be a zero-dimensional, locally compact Hausdorff space and suppose that the Boolean algebra $\text{KB}(\mathcal{C})$ is conditionally complete. Then $\text{KB}(\mathcal{C}) = \text{RO}_K(\mathcal{C})$.

Proof. The inclusion $\text{KB}(\mathcal{C}) \subseteq \text{RO}_K(\mathcal{C})$ is immediate, so let us deal with the converse.

Let $A \in \text{RO}_K(\mathcal{C})$, so in particular $A$ is contained in some element of $\text{KB}(\mathcal{C})$ and the family $\{V \in \text{KB}(\mathcal{C}) : V \subseteq A\}$ is bounded. Let

$$U = \bigvee \{V \in \text{KB}(\mathcal{C}) : V \subseteq A\},$$

where the join is taken in $\text{KB}(\mathcal{C})$. We have

$$A = \bigcup \{V \in \text{KB}(A) : V \subseteq A\} \subseteq U.$$ 

If $W \in \text{KB}(\mathcal{C})$ and $W \subseteq U \setminus A$, then $A \subseteq U \setminus W$ and so

$$U = \bigvee \{V \in \text{KB}(\mathcal{C}) : V \subseteq A\} \subseteq U \setminus W,$$

hence $W = \emptyset$. This proves that $U \setminus A = \emptyset$ and so $U \subseteq A$. However, $U$ is clopen and $A$ is regular open, which imply $A = U \in \text{KB}(\mathcal{C})$.

Lemma 3.1.20. If $X$ is a separable locally compact Hausdorff space, then $\mathcal{C} = \mathcal{G}_F(\text{RO}_K(X))$ is separable.
Proof. Let \( \{ x_n : n \in \mathbb{N} \} \) be a countable dense subset of \( X \). For each \( n \), by Zorn’s Lemma there exists \( G_n \in \mathcal{C} \) such that

\[
\{ U \in \text{RO}_K(X) : x_n \in U \} \subseteq G_n.
\]

Let us prove that \( \{ G_n : n \in \mathbb{N} \} \) is dense in \( \mathcal{C} \). Given a basic open subset \([A]\) of \( \mathcal{C} \), where \( A \in \text{RO}_K(X) \), find \( n \in \mathbb{N} \) with \( x_n \in A \), so \( A \subseteq G_n \) and therefore \( G_n \in [A] \).

\[ \square \]

**Lemma 3.1.21.** If \( X \) is a second-countable locally compact Hausdorff space and \( A \in \text{RO}_K(X) \), then there is \( f \in C_c(X) \) such that \( \sigma(f) = A \).

Proof. First choose a countable family of compact subsets \( K_n \subseteq A \) such that \( \bigcup_n K_n = A \). For each \( n \) we can, by Urysohn’s Lemma and regularity of \( X \), find a continuous function \( f_n : X \to [0, 1] \) such that \( f_n(k) = 1 \) for all \( k \in K_n \) and \( \text{supp}(f_n) \subseteq A \). Letting \( f = \sum_{n=1}^{\infty} 2^{-n} f_n \) we obtain \( [f \neq 0] = \sigma(f) = A \), because \( A \) is regular. \[ \square \]

As a consequence of Lemmas 3.1.19, 3.1.20 and 3.1.21, and Theorem 3.1.17, we conclude:

**Corollary 3.1.22.** There exists a zero-dimensional, compact Hausdorff topological space \( \mathcal{C} \) (namely, \( \mathcal{C} = \mathcal{G}_P(\text{RO}_K([0, 1])) \)) which is, in particular, not homeomorphic to \([0, 1] \) such that \( \mathcal{C}(\mathcal{C}) \) and \( \mathcal{C}([0, 1]) \) are \( \perp \)-isomorphic.

Note that, in the corollary above, the Boolean algebra \( \text{RO}([0, 1]) \) is uncountable, thus \( \mathcal{C} = \mathcal{G}_P(\text{RO}_K([0, 1])) \) is not second-countable.

For our second example, we will consider only second-countable spaces. In the next lemma, we denote by \( \text{int}_Z \) and \( \text{cl}_Z \) the interior and closure operators on subsets of a topological space \( Z \).

**Lemma 3.1.23.** Let \( X \) be a topological space and \( U \) an open set of \( X \). Then

(a) If \( A \in \text{RO}(X) \) then \( A \cap U \in \text{RO}(U) \).

(b) The map

\[ \varphi_U : \text{RO}(X) \to \text{RO}(U), \quad \varphi_U(A) = A \cap U \]

is order-preserving and surjective; The map \( \zeta_U : A \mapsto \text{int}_X(\text{cl}_X(A)) \) is an order-preserving right inverse to \( \varphi_U \);

(c) \( \varphi_U \) is an order isomorphism if and only if \( U \) is dense in \( X \).

Proof. (a) Given \( A \in \text{RO}(X) \), since \( U \) is open, we have

\[ \text{int}_U(\text{cl}_U(A \cap U)) = \text{int}_U(\text{cl}_X(A) \cap U) = \text{int}_X(\text{cl}_X(A)) \cap U = A \cap U, \]

and this proves that \( A \cap U \in \text{RO}(U) \).
(b) The last statement is the only non-trivial part. If \( A \in \text{RO}(U) \), we again use the fact that \( U \) is open, as in item (a), to obtain

\[
\varphi_U(\zeta_U(A)) = \text{int}_X(\text{cl}_X(A)) \cap U = \text{int}_U(\text{cl}_X(A) \cap U) = \text{int}_U(\text{cl}_U(A)) = A,
\]
as desired.

(c) If \( U \) is not dense in \( X \), then there exists a nonempty set \( A \in \text{RO}(X) \) which is disjoint with \( U \), so

\[
\varphi_U(A) = \emptyset = \varphi_U(\emptyset),
\]
and thus \( \varphi_U \) is not injective.

Now assume that \( U \) is dense in \( X \), so let us prove that the map \( \zeta_U \) of item (b) is a left inverse of \( \varphi_U \). Given \( A \in \text{RO}(X) \), since \( U \) is dense in \( X \),

\[
\zeta_U(\varphi_U(A)) = \text{int}_X(\text{cl}_X(A \cap U)) = \text{int}_X(\text{cl}_X(A)) = A.
\]

Let \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \) be the complex unit circle.

**Corollary 3.1.24.** \( C([0,1]) \) and \( C(S^1) \) are \( \perp \)-isomorphic.

**Proof.** Let \( X = (0,1) \) and \( Y = S^1 \setminus \{1\} \). Then \( X \) and \( Y \) are homeomorphic, and two applications of Lemma 3.1.23 imply that \( \text{RO}([0,1]) \) and \( \text{RO}(S^1) \) are order-isomorphic. Lemmas 3.1.20 and 3.1.21 and Theorem 3.1.17 imply that \( C([0,1]) \) and \( C(S^1) \) are \( \perp \)-isomorphic.

### 3.2 \( \perp \)-ideals

Here we will introduce \( \perp \)-ideals, which will be used in relation to partial actions of inverse semigroups in the next chapter. As usual, we fix a locally compact Hausdorff space \( X \), a continuous function \( \theta \) from \( X \) to a Hausdorff space \( H \) and a weakly regular family \( \mathcal{A} \subseteq C_c(X, \theta) \).

**Definition 3.2.1.** A finite family \( B \subseteq \mathcal{A} \) is said to be a **strong cover** of an element \( a \in \mathcal{A} \) if there is another finite family \( \tilde{B} \subseteq \mathcal{A} \) such that:

(i) For all \( \tilde{b} \in \tilde{B} \), there is some \( b \in B \) with \( \tilde{b} \subseteq b \);

(ii) \( \tilde{B} \) is a cover of \( a \) (Definition 3.1.6).

The following proposition shows that strong covers encode information about the closures of sets. It is a direct consequence of the definition of \( \subseteq \) and Lemma 3.1.7.

**Proposition 3.2.2.** A finite family \( B \subseteq \mathcal{A} \) is a strong cover of \( a \in \mathcal{A} \) if and only if \( \text{supp}(a) \subseteq \bigcup_{b \in B} \sigma(b) \).
Definition 3.2.3. A \( \bot \)-ideal in \( \mathcal{A} \) is a subset \( I \subseteq A \) such that, for all \( a \in \mathcal{A} \),
\[
a \in I \iff \text{there is a finite subset } B \subseteq I \text{ which is a strong cover of } a.
\]

Example 3.2.4. Suppose \( \mathcal{A} \) is a vector sublattice of \( C_c(X, \mathbb{R}) \) or \( C(X, \mathbb{C}) \) (self-adjoint in the latter case), where \( X \) is a locally compact Hausdorff space. Then a subset \( I \subseteq \mathcal{A} \) is a \( \bot \)-ideal if and only if \( I \) is an additive subgroup and
\[
f \in I \iff \exists g \in I \text{ such that } f \in g.
\]

Indeed, first suppose \( I \) is a \( \bot \)-ideal. If \( f \in I \), then there is a finite strong cover \( B \subseteq I \) of \( f \). For every \( b \in B \), take another finite strong cover \( C(b) \subseteq I \) of \( b \). Then \( \bigcup_{b \in B} C(b) \) is a strong cover of \( g = \sum_{b \in B} |\Re(b)| + |\Im(b)| \), thus \( g \in I \) and \( f \in g \), which proves implication “\( \Rightarrow \)” of (3.2.1). Implication “\( \Leftarrow \)” is immediate since \( f \in g \) if and only if \( \{g\} \) is a strong cover of \( f \).

To prove that \( I \) is additive, let \( f_1, f_2 \in I \). The implication “\( \Rightarrow \)” of (3.2.1), yields \( g_1, g_2 \in I \) with \( f_i \in g_i \). Then \( \{g_1, g_2\} \) is a finite strong cover of \( f_1 + f_2 \), so \( f_1 + f_2 \in I \).

Now suppose that \( I \) is additive and satisfies property (3.2.1), and let us show that \( I \) is a \( \bot \)-ideal. Of course, if \( f \in I \) then \( f \) admits a finite cover by a single element of \( I \) (by the “\( \Rightarrow \)” part of (3.2.1)). Conversely, suppose \( f \) admits a finite strong cover \( B \subseteq I \). From (3.2.1) take, for all \( b \in B \), \( g_b \in I \) with \( b \in g_b \), so that \( \tilde{b} := |\Re(b)| + |\Im(b)| \in g_b \) as well. This in turn implies \( \tilde{b} \in I \), so \( g = \sum_{b \in B} \tilde{b} \in I \) and \( f \in g \), so we conclude that \( f \in I \) again from (3.2.1).

We will now prove that the lattice of open subsets of a space \( X \) is order-isomorphic to the lattice of \( \bot \)-ideals of a weakly regular tuple \((X, \theta, \mathcal{A})\).

Definition 3.2.5. Suppose \((X, \theta, \mathcal{A})\) is weakly regular. Given an open set \( U \subseteq X \), denote \( \mathbb{I}(U) = \{ f \in \mathcal{A} : \text{supp}(f) \subseteq U \} \). Given a \( \bot \)-ideal \( I \subseteq \mathcal{A} \), denote \( \mathbb{U}(I) = \bigcup_{f \in I} \sigma(f) \).

Note that \( \mathbb{I}(U) \) is a \( \bot \)-ideal of \( \mathcal{A} \): If \( f \in \mathbb{I}(U) \) then we use weak regularity of \( \mathcal{A} \) and compactness of \( \text{supp}(f) \) to find a finite collection \( B \subseteq \mathcal{A} \) such that
\[
\text{supp}(f) \subseteq \bigcup_{b \in B} \sigma(b) \subseteq U
\]
In particular, \( B \subseteq \mathbb{I}(U) \) and \( B \) is a strong cover of \( f \). This provides the implication “\( \Rightarrow \)” of Definition 3.2.3 whereas “\( \Leftarrow \)” is trivial.

Theorem 3.2.6. Suppose \((X, \theta, \mathcal{A})\) is weakly regular.

(a) For every \( \bot \)-ideal \( I \) of \( \mathcal{A} \), \( I = \mathbb{I}(\mathbb{U}(I)) \);

(b) For every open subset \( U \subseteq X \), \( U = \mathbb{U}(\mathbb{I}(U)) \);
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(c) The map $U \mapsto I(U)$ is an order isomorphism between the lattices of open sets of $X$ and $\perp$-ideals of $A$.

Proof. (a) Let $I$ be a $\perp$-ideal. The inclusion $I \subseteq I(U(I))$ follows easily from the property of $\perp$-ideals: if $f \in I$, then take a finite strong cover $B \subseteq I$ of $f$, so that $\text{supp}(f) \subseteq \bigcup_{b \in B} \sigma(b) \subseteq U(I)$.

Conversely, if $f \in I(U(I))$ we use compactness of $\text{supp}(f)$ to find a finite family $B \subseteq I$ with $\text{supp}(f) \subseteq \bigcup_{b \in B} \sigma(b)$, so $B$ is a strong cover of $f$ and therefore $f \in I$.

(b) Suppose $U \subseteq X$ is open. By weak regularity of $(X, \theta, A)$, we have

$$U = \bigcup_{f \in A} \sigma(f) = \bigcup_{f \in I(U)} \sigma(f) = U(I(U)).$$

(c) The previous items prove that $U \mapsto I(U)$ is a bijection, with inverse $I \mapsto U(I)$. It is clear that both maps are order-preserving.

Assume $(X, \theta, A)$ is weakly regular. Let $\hat{A}$ be the collection of maximal $\perp$-ideals of $A$, and endow it with the topology generated by the sets

$$U(f) = \left\{ I \in \hat{A} : \exists g \in f \text{ such that } g \not\in I \right\}, \quad f \in A.$$ 

By Theorem 3.2.6, we obtain a bijection $\xi : X \to \hat{A}$, $\xi(x) = I(X \setminus \{x\})$. Since for all $x \in X$ and $f \in A$,

$$x \in \sigma(f) \iff \exists g \in f \text{ such that } x \in \text{supp}(g) \iff \xi(x) \in U(f),$$

then $\xi(\sigma(f)) = U(f)$, which proves that $\xi$ is a homeomorphism. This provides another proof of Theorem 3.1.13.

3.3 Basic $\perp$-isomorphisms

In this section we will develop techniques to classify isomorphisms for spaces of functions with different algebraic structures. As in the preceding sections, we will be interested mostly in spaces of continuous functions between topological spaces, however the initial notions we will deal with can be defined in purely set-theoretical terms.

Let $X$ and $H_X$ be sets, and consider a class $A(X) \subseteq (H_X)^X$ of $H_X$-valued functions on $X$. Given a point $x \in X$, denote by $A(X)|_x$ the set of images of $x$ under elements of $A(X)$, i.e.

$$A(X)|_x = \{ f(x) : f \in A(X) \}.$$ (3.3.1)
If $Y$ is another set and $\phi : Y \to X$ is a map, denote by

$$Y \times_{(\phi, \mathcal{A}(X))} H_X = \bigcup_{y \in Y} \{y\} \times \mathcal{A}(X)|_{\phi(y)}.$$ 

Note that $Y \times_{(\phi, \mathcal{A}(X))} H_X$ is equal to $Y \times H_X$ if and only if the following property is satisfied: For every $y \in Y$ and every $c \in H_X$, there exists $f \in \mathcal{A}(X)$ such that $f(\phi(y)) = c$.

**Definition 3.3.1.** Let $X$, $H_X$, $Y$ and $H_Y$ be sets, $\phi : Y \to X$ be a function, and consider a class of functions $\mathcal{A}(X) \subseteq (H_X)^X$.

Given maps $\phi : Y \to X$ and $\chi : Y \times_{(\phi, \mathcal{A}(X))} H_X \to H_Y$, we define $T_{(\phi, \chi)} : \mathcal{A}(X) \to (H_Y)^Y$ by

$$(T_{(\phi, \chi)} f)(y) = \chi(y, f(\phi(y))), \quad \forall f \in \mathcal{A}(X), \quad \forall y \in Y. \quad (3.3.2)$$

**Definition 3.3.2.** Let $X$, $H_X$, $Y$ and $H_Y$ be sets, $\phi : Y \to X$ be a map, and consider classes of functions $\mathcal{A}(X) \subseteq (H_X)^X$ and $\mathcal{A}(Y) \subseteq (H_Y)^Y$. A map $T : \mathcal{A}(X) \to \mathcal{A}(Y)$ is called $\phi$-basic if there exists $\chi : Y \times_{(\phi, \mathcal{A}(X))} H_X \to H_Y$ such that $T = T_{(\phi, \chi)}$. We call such $\chi$ a $(\phi, T)$-transform.

**Remark.** In the definition above, we ignore the fact that the codomain of $T$ is $\mathcal{A}(Y)$, while the codomain of $T_{(\phi, \chi)}$ is $(H_Y)^Y$.

In simpler terms, a basic map is one that is induced naturally by the transformation $\phi : Y \to X$, and the field of partial transformations $\chi(y, \cdot) : \mathcal{A}(X)|_{\phi(y)} \subseteq H_X \to H_Y$ ($y \in Y$).

**Example 3.3.3.** Let $\phi : Y \to X$ and $\psi : H_X \to H_Y$ be functions. Then the map

$$T : (H_X)^X \to (H_Y)^Y, \quad Tf = \psi \circ f \circ \phi$$

is $\phi$-basic, and the $(\phi, T)$-transform $\chi$ is given by $\chi(y, z) = \psi(z)$.

The next example will appear, in some form, in most applications (Sections 3.4 and 3.5).

**Example 3.3.4.** Suppose that $X$ is a locally compact Hausdorff space, $H_X$ is a Hausdorff space, $\theta_X \in C(X, H_X)$ and $\mathcal{A}(X) \subseteq C_c(X, \theta_X)$. Let $Y$ and $H_Y$ be topological spaces, $\phi : Y \to X$ be a homeomorphism, and $\chi : Y \times H_X \to H_Y$ be a continuous map such that for every $y \in Y$, the section $\chi(y, \cdot) : H_X \to H_Y$ is a bijection.

For every $f \in \mathcal{A}(X)$, define $Tf \in C(Y, H_Y)$ as

$$Tf : Y \to H_Y, \quad Tf(y) = \chi(y, f(\phi(y))),$$

and let $\mathcal{A}(Y) = \{Tf : f \in \mathcal{A}(X)\}$. Also define $\theta_Y = T\theta_X$. Then
1. $T$ is $\phi$-basic, and the $(\phi, T)$-transform is the restriction of $\chi$ to $Y \times_{(\phi, A(X))} H_X$;

2. $(X, \theta_X, A(X))$ is (weakly) regular if and only if $(Y, \theta_Y, A(Y))$ is (weakly) regular. In this case, $T$ is a $\perp$-isomorphism and $\phi$ is the $T$-homeomorphism.

Note that not every $\perp$-isomorphism is given as in the previous example.

**Example 3.3.5.** Suppose that $X = Y$ is compact Hausdorff, $H_X = H_Y = \mathbb{R}$ and $\theta_X = \theta_Y = 0$, so we simply write $C(X) = C(X, \mathbb{R})$. Let $T : C(X) \to C(X)$ be any bijection satisfying $[f \neq 0] = [Tf \neq 0]$ for all $f \in C(X)$. Then $T$ is a $\perp$-isomorphism, and the $T$-homeomorphism is the identity $\text{id}_X : X \to X$. Let us look at two particular cases:

- Suppose that $X$ is not a singleton, $T(1) = 2$, $T(2) = 1$, and $Tf = f$ for every $f \neq 1, 2$. Then $T$ is non-basic (see Proposition 3.3.6(a)) and discontinuous with respect to either the topology of uniform convergence, or the topology of pointwise convergence.

- If $X = \{\ast\}$ is a singleton, we identify $C(X)$ with $\mathbb{R}$, so any self-bijection $T : \mathbb{R} \to \mathbb{R}$ preserving 0 is a basic $\perp$-automorphism. In this case, the $T$-transform $\chi$ coincides with $T$ (or more precisely $\chi(\ast, z) = T(z)$ for all $z \in \mathbb{R}$), and “most” (cardinality-wise) of these are discontinuous: indeed, there are $c^c = 2^{2^{\aleph_0}}$ self-bijections of $\mathbb{R} \setminus \{0\}$ (where $c = 2^{\aleph_0}$ is the continuum) but only $2^{\aleph_0} = c$ of these are continuous.

In the next proposition, we again consider only sets (without topologies).

**Proposition 3.3.6.** Let $A(X) \subseteq (H_X)^X$ and $A(Y) \subseteq (H_Y)^Y$, and consider maps $\phi : Y \to X$ and $T : A(X) \to A(Y)$. Then

(a) $T$ is $\phi$-basic if and only if for all $y \in Y$,

$$f(\phi(y)) = g(\phi(y)) \implies Tf(y) = Tg(y), \quad \forall f, g \in A(X). \quad (3.3.3)$$

In this case,

(b) the $(\phi, T)$-transform $\chi$ is unique.

(c) A section $\chi(y, \cdot)$ is injective if and only if

$$Tf(y) = Tg(y) \implies f(\phi(y)) = g(\phi(y)), \quad \forall f, g \in A(X). \quad (3.3.4)$$

(d) A section $\chi(y, \cdot)$ is surjective if and only if

$$H_Y = \{Tf(y) : f \in A(X)\}. \quad (3.3.5)$$
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Proof. If $T$ satisfies \textbf{3.3.3}, define $\chi$ by

$$\chi(y, t) = Tf(y),$$

whenever $f \in A(X)$ is any function satisfying $f(\phi(y)) = t$. Then $\chi(y, t)$ does not depend on the choice of $f$ by Equation \textbf{3.3.3}, and hence $T$ is $\phi$-basic.

The converse direction, as well as items (b), (c) and (d) are immediate from the formula $Tf(y) = \chi(y, f(\phi(y)))$, which holds for all $f \in A(X)$ and $y \in Y$. \hfill \Box

The next proposition proves that, under usual regularity hypotheses, $\perp$-isomorphisms can be basic only with respect to their corresponding homeomorphisms.

**Proposition 3.3.7.** Let $X$ and $Y$ be locally compact Hausdorff spaces, $H_X$ and $H_Y$ be Hausdorff spaces, $\theta_X \in C(X, H_X)$, $\theta_Y \in C(Y, H_Y)$ and $A(X) \subseteq C_c(X, \theta_X)$ and $A(Y) \subseteq C_c(Y, \theta_Y)$ be two regular classes of functions.

Suppose that $T : A(X) \to A(Y)$ is a $\perp$-isomorphism which is $\phi$-basic for some map $\phi : Y \to X$, and such that $T^{-1} : A(Y) \to A(X)$ is $\psi$-basic for some map $\psi : X \to Y$. Then $\phi$ is invertible, $\psi = \phi^{-1}$ and $\phi$ is the $T$-homeomorphism.

**Lemma 3.3.8.** Under the hypotheses of Proposition \textbf{3.3.7} $\phi : Y \to X$ is the only map for which $T$ is $\phi$-basic.

Proof. Let $\phi_1 = \phi$ and assume that $\phi_2 : Y \to X$ is another map such that $T$ is $\phi_2$-basic. Given $y \in Y$, we will prove that $\phi_1(y) = \phi_2(y)$. Assume that this is not true, and let us deduce a contradiction. Note, in particular, that $|Y| \geq 2$, so regularity of $A(Y)$ implies $|H_Y| \geq 2$. Let $\chi_1$ and $\chi_2$ be the $(\phi_i, T)$ and $(\phi_2, T)$-transforms, respectively.

Since $A(Y)$ is regular and $|H_Y| \geq 2$, there exists $c \in H_X$ such that

$$\chi_1(y, c) \neq \theta_Y(y). \quad (3.3.5)$$

Since we are assuming that $\phi_1(y) \neq \phi_2(y)$, regularity of $A(X)$ yields $f \in A(X)$ such that $f(\phi_1(y)) = c$ and $f(\phi_2(y)) = \theta_X(\phi_2(y))$. As $T$ is a $\perp$-isomorphism we have $T\theta_X = \theta_Y$, and the properties of the $(\phi_i, T)$-transforms $\chi_i$ ($i = 1, 2$) imply

$$\theta_Y(y) = T\theta_X(y) = \chi_2(y, \theta_X(\phi_2(y))) = \chi_2(y, f(\phi_2(y))) = Tf(y) = \chi_1(y, f(\phi_1(y))) = \chi_1(y, c),$$

contradicting inequality \textbf{3.3.5}. \hfill \Box

**Proof of Proposition 3.3.7** Since $T$ and $T^{-1}$ are $\phi$ and $\psi$-basic, respectively, then the identity map $\text{id}_{A(X)} = T^{-1} \circ T$ is $(\psi \circ \phi)$-basic (by Proposition \textbf{3.3.6}a)). However, $\text{id}_{A(X)}$ is $\psi$-basic, (again by Proposition \textbf{3.3.6}a); the $(\text{id}_Y, \text{id}_{A(X)})$-transform is given by $\chi(y, t) = t$. Lemma \textbf{3.3.8} applied to $T^{-1} \circ T$ then implies $\psi \circ \phi = \text{id}_Y$, and similarly $\phi \circ \psi = \text{id}_X$. This proves that $\phi^{-1} = \psi$. 

Now denote by $\rho : Y \to X$ the $T$-homeomorphism, so we need to prove that $\phi = \rho$. Proposition 3.3.6(a) applied to $T$ and $T^{-1}$ yields, for all $y \in Y$ and all $f \in \mathcal{A}(X)$,

$$Tf(y) = \theta_Y(y) \iff f(\phi(y)) = \theta_X(\phi(y)). \quad (3.3.6)$$

Given $y \in Y$, regularity of $\mathcal{A}(X)$ gives us

$$\{\phi(y)\} = \bigcap_{f(\phi(y)) \neq \theta_X(\phi(y))} [f \neq \theta_X]$$

and using the equivalence (3.3.6), the definition of $\sigma^{\theta_X}$ (Definition 3.1.1) and the characterization of the $T$-homeomorphism $\rho$ (Definition 3.1.14), we obtain

$$\{\phi(y)\} \subseteq \bigcap_{Tf(y) \neq \theta_Y(y)} \sigma^{\theta_X}(f) = \bigcap_{Tf(y) \neq \theta_Y(y)} \rho(\sigma^{\theta_Y}(Tf)) = \{\rho(y)\}$$

where the last equality follows as $\rho$ is injective and $\mathcal{A}(Y) = T(\mathcal{A}(X))$ is regular. We thus conclude that $\phi = \rho$ is the $T$-homeomorphism.

3.3.1 Algebraic signatures and basic maps

In our examples and consequences we will consider different algebraic structures on spaces of continuous functions. To this end, recall (see [86]) that an algebraic signature is a collection $\eta$ of pairs $(\ast, n)$, where $\ast$ is a (function) symbol and $n$ is a non-negative integer, called the arity of $\ast$. A model of $\eta$ consists of a set $H$ and a map associating to each $(\ast, n) \in \eta$ a function $\ast : H^n \to H$, $(c_1, \ldots, c_n) \mapsto c_1 \ast \cdots \ast c_n$. (We use the convention that $H^0$ is a singleton set, so that a 0-ary function symbol is the same as a constant.)

This can be seen as a particular case of a metric signature (Definition 2.1.6), where only discrete metric spaces (with the 0–1 metric) are considered. (See Example 2.1.9(2).)

If $H$ is a model of $\eta$ and $X$ is a set then the function space $H^X$ can also be regarded as a model of $\eta$ with the pointwise structure: $(f_1 \ast \cdots \ast f_n)(x) = f_1(x) \ast \cdots \ast f_n(x)$ for all $f_1, \ldots, f_n \in H^X$, all $x \in X$, and all $n$-ary function symbols $\ast$.

A morphism of two models $H_1$ and $H_2$ of a given signature $\eta$ is a map $m : H_1 \to H_2$ such that for any $n$-ary function symbol $\ast$ of $\eta$ and any $x_1, \ldots, x_n \in H_1$, we have $m(x_1 \ast \cdots \ast x_n) = m(x_1) \ast \cdots \ast m(x_n)$.

Finally, a submodel of a model $H$ of a signature $\eta$ is a subset $K \subseteq H$ such that for all $n$-ary symbols $\ast$ of $\eta$ and any $d_1, \ldots, d_n \in K$, $d_1 \ast \cdots \ast d_n \in K$, so that $K$ can be naturally regarded as a model of $\eta$.

In the topological setting, a continuous model $H$ of a signature $\eta$ is defined in the same manner, but we assume that all maps are continuous. In this case, if $X$ is a topological space then $C(X, H)$ is a submodel of $H^X$. 

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Example 3.3.9. The standard signature for groups consists of a binary (2-ary) operation \( \cdot \) (the group operation), a unary operation \( (\cdot)^{-1} \) (the inversion), and a constant (0-ary operation) 1 (the unit).

Proposition 3.3.10. Let \( X \) and \( Y \) be sets. Suppose that \( H_X \) and \( H_Y \) are models for an algebraic signature \( \eta \), and that \( \mathcal{A}(X) \) and \( \mathcal{A}(Y) \) are submodels of \( (H_X)^X \) and \( (H_Y)^Y \). Then for all \( x \in X \), \( \mathcal{A}(X)|_x \) is a submodel of \( H_X \), and similarly \( \mathcal{A}(Y)|_y \) is a submodel of \( H_Y \) for all \( y \in Y \). (See Equation 3.3.10.)

Let \( T : \mathcal{A}(X) \to \mathcal{A}(Y) \) be a basic map with respect to a function \( \phi : Y \to X \), and let \( \chi \) be the \( T \)-transform. Then \( T \) is a morphism (for \( \eta \)) if and only if every section \( \chi(y, \cdot) \) is a morphism (from \( \mathcal{A}(X)|_{\phi(y)} \) to \( \mathcal{A}(Y)|_y \)).

Proof. Given \( x \in X \), the evaluation map \( \pi_x : (H_X)^X \to H_X \), \( \pi_x(f) = f(x) \), is a morphism, and it follows that \( \mathcal{A}(X)|_x = \pi_x(\mathcal{A}(X)) \) is a submodel of \( H_X \).

Suppose that \( T \) as above is a morphism and let \( y \in Y \). Given an \( n \)-ary symbol * of \( \eta \), suppose \( t_1, \ldots, t_n \in \mathcal{A}(X)_{\phi(y)} \). Take functions \( f_1, \ldots, f_n \in \mathcal{A}(X) \) such that \( f_i(\phi(y)) = t_i \), so \( f_1 \cdots f_n \in \mathcal{A}(X) \) since \( \mathcal{A}(X) \) is a submodel of \( C(X, H_X) \). Then

\[
\chi(y, t_1 \cdots t_n) = T(f_1 \cdots f_n)(y) = Tf_1(y) \cdots Tf_n(y) = \chi(y, t_1) \cdots \chi(y, t_n).
\]

This proves that \( \chi(y, \cdot) \) is a morphism. The converse implication is similar. \( \Box \)

3.3.2 Group-valued maps

We first note a simple property for basic isomorphisms of groups of functions which will be used throughout the next section:

Proposition 3.3.11. Suppose that \( H_X \) and \( H_Y \) are groups, \( \mathcal{A}(X) \) and \( \mathcal{A}(Y) \) are subgroups of \( (H_X)^X \) and \( (H_Y)^Y \), respectively, \( T : \mathcal{A}(X) \to \mathcal{A}(Y) \) is a group isomorphism and \( \phi : Y \to X \) is a function. Then \( T \) is basic with respect to \( \phi \) if and only if for all \( y \in Y \),

\[
f(\phi(y)) = 1 \implies Tf(y) = 1, \quad \forall f \in \mathcal{A}(X).
\]

Proof. This follows directly from 3.3.6(a), that \( f(x) = g(x) \) if and only if \( (fg^{-1})(x) = 1 \) and the fact that \( T \) preserves products and inverses. \( \Box \)

In the topological setting, assume that \( H \) is a topological group, \( X \) is a locally compact Hausdorff space and \( \theta \in C(X, H) \). Then, \( C_c(X, \theta) \) and \( C_c(X, 1) \) are \( \sqcap \)-isomorphic, via \( f \mapsto f\theta^{-1} \), and in fact this same map can be used to show that \( C_c(X, \theta) \) is (weakly) regular if and only if \( C_c(X, 1) \) is (weakly) regular.

Moreover, if \( C_c(X, \theta) \) is a subgroup of \( C(X, H) \) then \( \theta = 1 \) outside of a compact set. Indeed, let \( f, g \in C_c(X, \theta) \) be arbitrary, and suppose that \( fg \in C_c(X, \theta) \) as well.
As \( f \) and \( fg \) coincide with \( \theta \) outside of a compact, then \( g = 1 \) outside of a compact. Therefore, in this case, we obtain \( C_c(X, \theta) = C_c(X, 1) \).

Since we will be interested in structure-preserving maps between submodels of \( C(X, H) \), in the case the codomain is a group we may always assume that \( \theta = 1 \). In the case that \( H = \mathbb{R} \) or \( \mathbb{C} \), as additive groups, we reobtain the usual notion of support.

### 3.3.3 Continuity

Now, we study continuity of basic \( \perp \)-isomorphisms and relate it to the continuity of its transform, and for this we need a few results from general topology.

**Proposition 3.3.12.** If \( F \) is an infinite subset of a regular Hausdorff space \( X \), then there exists a countable infinite subset \( \{ y_1, y_2, \ldots \} \subseteq F \) and open sets \( U_n \) such that \( y_n \in U_n \) for all \( n \) and \( U_n \cap U_m = \emptyset \) if \( n \neq m \).

**Proof.** First assume \( F \) has a cluster point \( y \). Let \( y_1 \in Y \setminus \{ y \} \) be arbitrary. Take disjoint neighbourhoods \( U_1 \) of \( y_1 \) and \( V_1 \) of \( y \), and then repeat the procedure in \( V_1 \). This way, we can construct sequences of points \( y_1, y_2, \ldots \in Y \) and of neighbourhoods \( U_n \) of \( y_n \) and \( V_n \) of \( y \) such that

- \( V_{n+1} \subseteq V_n \);
- \( U_n \cap V_n = \emptyset \);
- \( U_{n+1} \subseteq V_n \).

If \( n < m \), then \( U_m \cap U_n \subseteq V_{m-1} \cap U_n \subseteq V_n \cap U_n = \emptyset \), as wanted.

Now assume that \( F \) does not have a cluster point. Take any infinite countable subset \( \{ y_n : n \in \mathbb{N} \} \subseteq F \). Given \( k \), since \( y_k \) is a not a cluster point of \( F \), we can take neighbourhoods \( V_k \) such that \( V_k \cap \{ y_n : n \in \mathbb{N} \} = \{ y_k \} \). By regularity, take a neighbourhood \( U_1 \) of \( y_1 \) such that \( U_1 \subseteq V_1 \), and then inductively take, for a given \( n \), a neighbourhood \( U_{n+1} \) of \( y_{n+1} \) such that \( U_{n+1} \subseteq V_n \setminus \bigcup_{i=1}^{n} U_i \). \qed

For the next proposition, recall ([177, 27.4]) that a topological space \( H \) is locally path-connected if every point \( t \in H \) admits a neighbourhood basis consisting of path-connected subsets.

**Proposition 3.3.13.** Let \( X \) be a locally compact Hausdorff space, \( \{ x_n \} \) be a sequence of elements of \( X \), \( \{ U_n \} \) a sequence of open subsets of \( X \) such that
(i) \( x_n \in U_n \);

(ii) \( U_n \cap U_m = \emptyset \) for all \( n \neq m \).

Let \( H \) be a Hausdorff first-countable locally path-connected topological space and consider \( \{g_n : U_n \to H\} \) a family of continuous functions such that \( g_n(x_n) \) converges to some \( t \in H \). Then

(a) there exists a continuous function \( f : X \to H \) such that \( f(x_n) = g_n(x_n) \) for all sufficiently large \( n \), and \( f(x) = t \) for all \( x \notin \bigcup_n U_n \).

(b) if \( H = \mathbb{R} \), there is a continuous function \( f : X \to H \) such that \( f = g_n \) on a neighbourhood of \( x_n \) and \( f(x) = t \) for all \( x \notin \bigcup_n U_n \).

(Note that it is not necessary to take subsequences!)

Proof. (a) Let \( \{W_n\}_n \) be a decreasing basis of path-connected neighbourhoods of \( t \).

Disregarding any \( n \) such that \( g_n(x_n) \) does not belong to \( W_1 \), and repeating the sets \( W_k \) if necessary (i.e., considering a new sequence of neighbourhoods of \( t \) of the form

\[
W_1, W_1, \ldots, W_1, W_2, W_2, \ldots, W_2, \ldots,
\]

where each \( W_k \) is repeated finitely many times) we may assume that \( t_n := g_n(x_n) \in W_n \).

For each \( n \), take a continuous path \( \alpha_n : [0, 1] \to W_n \) such that \( \alpha_n(0) = t_n \) and \( \alpha_n(1) = t \). Now take continuous functions \( b_n : X \to [0, 1] \) such that \( b_n(x_n) = 0 \) and \( b_n = 1 \) outside \( U_n \).

Define \( f \) as \( \alpha_n \circ b_n \) on each \( U_n \), and as \( t \) on \( X \setminus \bigcup_n U_n \).

Let us show that \( f \) is continuous: If \( x \in U_n \) for some \( n \) then \( f = \alpha_n \circ b_n \) on a neighbourhood of \( x \) and so it is continuous at \( x \). Let us then assume that \( x \notin \bigcup U_n \) and in particular \( \alpha_n(b_n(x)) = f(x) = t \) for all \( n \).

Consider a neighbourhood \( W_N \) of \( t \), where \( N \) is an integer. Take a neighbourhood \( U \) of \( x \) such that if \( n \leq N \) and \( y \in U \), then \( \alpha_n(b_n(y)) \in W_N \) (e.g. \( U = \bigcap_{n=1}^N (\alpha_n \circ b_n)^{-1}(W_N) \)). If \( y \in U \), there are three cases:

- If \( y \in U_n \) for some \( n \leq N \), then \( f(y) = \alpha_n(b_n(y)) \in W_N \), by the choice of \( U \);
- If \( y \in U_n \) for some \( n > N \), then \( f(y) = \alpha_n(b_n(y)) \in W_n \subseteq W_N \), by the choice of \( \alpha_n \);
- If \( y \notin \bigcup U_n \) then \( f(y) = t \in W_N \).

In any case, we have \( f(U) \subseteq W_N \), which proves the continuity of \( f \).
(b) For each \( n \), choose an open neighbourhood \( V_n \) of \( x_n \) such that \( V_n \subseteq U_n \). By considering even smaller neighbourhoods if necessary we can assume \(|g_n(x) - g_n(x_n)| < 1/n\) for all \( x \in V_n \). Up to modifying \( g_n \) except on a neighbourhood of \( x_n \), we can assume \( g_n = t \) on \( U_n \setminus V_n \). Define \( f = g_n \) on each \( U_n \) and \( f = t \) on \( X \setminus \bigcup_n U_n \). The proof that \( f \) is continuous is similar to that of item (a). \( \square \)

**Theorem 3.3.14.** Let \( X \) and \( Y \) be locally compact Hausdorff and for \( Z \in \{X, Y\} \), \( H_Z \) a Hausdorff space and \( \theta_Z \in C(Z, H_Z) \) be given such that \((Z, \theta_Z, C_c(Z, \theta_Z))\) is regular.

Suppose that \( T : C_c(X, \theta_X) \to C_c(Y, \theta_Y) \) is a \( \sqsubseteq \)-isomorphism, that \( \phi : Y \to X \) is the \( T \)-homeomorphism \( \phi \), and that that \( T \) is \( \phi \)-basic. Let \( \chi : Y \times H_X \to H_Y \) be the corresponding \((\phi, T)\)-transform. Consider the following statements:

\begin{enumerate}
  \item \( \chi \) is continuous.
  \item Each section \( \chi(y, \cdot) \) is a continuous;
  \item \( T \) is continuous with respect to the topologies of pointwise convergence.
\end{enumerate}

Then the implications \((1) \Rightarrow (2) \iff (3)\) always hold.

If \( X \), \( Y \) and \( H_X \) are first countable, \( H_X \) is locally path-connected and \( \theta_X \) is constant, then \((2) \Rightarrow (1)\).

**Remark.** 1. In the last part of the theorem, if \( H_X \) admits any structure of topological group then the condition that \( \theta_X \) is constant can be dropped, by the discussion in Subsection 3.3.2

2. The domain of the \((\phi, T)\)-transform \( \chi \) is \( Y \times H_X \) because we assume that \( C_c(X, \theta_X) \) is regular.

**Proof.** The implication \((1) \Rightarrow (2)\) is trivial.

\((2) \Rightarrow (3)\): Suppose \( f_i \to f \) pointwise. Then for all \( y, \) the section \( \chi(y, \cdot) \) is continuous, thus

\[ T f_i(y) = \chi(y, f_i(\phi(y))) \to \chi(y, f(\phi(y))) = T f(y). \]

This proves that \( T f_i \to T f \) pointwise.

\((3) \Rightarrow (2)\): Assume that \( T \) is continuous with respect to pointwise convergence. Let \( y \in Y \) be fixed. Suppose that \( t_i \to t \) in \( H_X \), and let us prove that \( \chi(y, t_i) \to \chi(y, t) \). Choose any function \( f \in C_c(X, \theta_X) \) such that \( f(\phi(y)) = t \).

Let \( \text{Fin}(X) \) be the collection of finite subsets of \( X \), ordered by inclusion. We will construct a net \( \{f_{(F,i)}\}_{(F,i) \in \text{Fin}(X) \times I} \) of functions in \( C_c(X, \theta_X) \) such that

\begin{enumerate}
  \item \( f_{(F,i)} \to f \) pointwise;
\end{enumerate}
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(ii) \( f_{(F,i)}(\phi(y)) = t_i \).

Given \( F \in \operatorname{Fin}(X) \) and \( i \in I \), consider a family \( \{U_x : x \in F \cup \{\phi(y)\}\} \) of pairwise disjoint open sets such that \( x \in U_x \) for each \( x \).

Given \( x \in F \cup \{\phi(y)\} \) consider, by regularity of \( C_c(X, \theta_X) \), a function \( f_{(F,i,x)} \in C_c(X, \theta_X) \) such that \( \operatorname{supp}(f_{(F,i,x)}) \subseteq U_x \) and

\[
    f_{(F,i,x)}(x) = \begin{cases} 
        t_i, & \text{if } x = \phi(y) \\
        f(x), & \text{otherwise}. 
    \end{cases}
\]

We then define \( f_{(F,i)} : X \to H_X \) by

\[
    f_{(F,i)} = f_{(F,i,x)} \text{ on each set } U_x \text{ and } f_{(F,i)} = \theta_X \text{ on } X \setminus \bigcup_{x \in F \cup \{\phi(y)\}} U_x.
\]

This way we obtain \( f_{(F,i)} \in C_c(X, \theta_X) \). Properties (i) and (ii) above are immediate because \( t_i \to t \).

Since \( f_{(F,i)} \to f \) pointwise then \( Tf_{(F,i)} \to Tf \) pointwise. For each \( F \in \operatorname{Fin}(X) \) and \( i \in I \), we have

\[
    \chi(y, t_i) = \chi(y, f_{(F,i)}(\phi(y))) = Tf_{(F,i)}(y)
\]

so by considering \( i \) and \( F \) sufficiently large we see that \( \chi(y, t_i) \to Tf(y) = \chi(y, t) \) as \( i \to \infty \).

We now assume further that \( X, Y \) and \( H_X \) are first countable, \( H_X \) is locally path-connected and \( \theta_X \) is constant. Let \( c \in H_X \) such that \( \theta_X(x) = c \) for all \( x \in X \).

(2)\(\Rightarrow\)(1): Assume that each section \( \chi(y, \cdot) \) is continuous. In order to prove that \( \chi \) is continuous, we simply need to prove that for any converging sequence \( (y_n, t_n) \to (y, t) \) in \( Y \times H_X \), we can take a subsequence \( (y_{n'}, t_{n'}) \) such that \( \chi(y_{n'}, t_{n'}) \to \chi(y, t) \) as \( n' \to \infty \).

Given a converging sequence \( (y_n, t_n) \to (y, t) \), consider an open \( Y' \subseteq Y \) with compact closure such that \( y, y_i \in Y' \) for all \( i \).

We have two cases: If for a given \( z \in Y \) the set \( N(z) = \{n \in \mathbb{N} : y_n = z\} \) is infinite, then we necessarily have \( z = y \). Restricting the sequence \( (y_n, t_n) \) to \( N(y) \) and using continuity of the section \( \chi(y, \cdot) \), we obtain \( \chi(y_n, t_n) = \chi(y_n, t) \to \chi(y, t) \) as \( n \to \infty \), \( n \in N(y) \).

Now assume that none of the sets \( N(z) = \{n \in \mathbb{N} : y_n = z\} \) \( (z \in Y) \) is infinite. We may then take a subsequence and assume that all the elements \( y_n \) are distinct, and actually never equal to \( y \). Using Proposition 3.3.12 and
taking another subsequence if necessary, consider pairwise disjoint open subsets $U_n \subseteq \phi(Y')$ with $\phi(y_n) \in U_n$ and $\phi(y) \in X \setminus \bigcup_n U_n$. We then consider, by Proposition 3.3.13, a continuous function $f : \overline{\phi(Y')} \to H_X$ such that $f(\phi(y_n)) = t_n$ and $f = t$ on $X \setminus \bigcup_n U_n$. In particular, $f = t$ on the boundary $\partial(\phi(Y'))$.

We now need to extend $f$ to an element of $C_c(X, \theta_X)$ (this is where we use that $\theta_X = c$ is constant). We have two cases:

**Case 1:** $t$ is in the path-connected component of $c$:

Since $H_X$ is locally path-connected, there is a continuous path $\beta : [0, 1] \to H_X$ with $\beta(0) = t$ and $\beta(1) = c$. Let $g : X \to [0, 1]$ be a function with $g = 0$ on $\phi(Y')$ and $g = 1$ outside of a compact containing $\phi(Y')$. By defining $f = \beta \circ g$ outside of $\phi(Y')$, we obtain $f \in C_c(X, \theta_X)$. ($f$ is continuous because $f = t = \beta \circ g$ on $\partial(\phi(Y'))$.)

**Case 2:** $t$ is not in the path-connected component of $c$:

Since $H_X$ is locally path-connected, its path-connected components are clopen, and regularity of $C_c(X, \theta_X)$ then implies that $X$ (and thus also $Y = \phi(X)$) is zero-dimensional. In particular, we could have assumed at the beginning that $Y'$ is clopen, so simply set $f = c$ outside of $\phi(Y')$.

In any case, we obtain $f \in C_c(X, \theta_X)$ with $f(\phi(y)) = t$ and $f(\phi(y_n)) = t_n$, so

$$
\chi(y_n, t_n) = Tf(y_n) \to Tf(y) = \chi(y, t).
$$

### 3.3.4 Non-vanishing bijections

Let $X$ and $Y$ be compact Hausdorff spaces, $H_X$ and $H_Y$ Hausdorff spaces, $\theta_X \in C(X, H_X)$, $\theta_Y \in C(Y, H_Y)$ and $A(X)$ and $A(Y)$ regular subsets of $C_c(X, \theta_X)$ and $C_c(Y, \theta_Y)$, respectively.

**Definition 3.3.15** ([84]). We call a bijection $T : A(X) \to A(Y)$ **non-vanishing** if for every $f_1, \ldots, f_n \in A(X)$,

$$
\bigcap_{i=1}^n [f_i = \theta_X] = \emptyset \iff \bigcap_{i=1}^n [Tf_i = \theta_Y] = \emptyset.
$$

**Proposition 3.3.16.** If $T : A(X) \to A(Y)$ is a non-vanishing bijection, then $T$ is a $\sqcup$-isomorphism.

**Proof.** First note that

$$
f \perp g \iff [f \neq \theta_X] \cap [g \neq \theta_X] = \emptyset \iff [f = \theta_X] \cup [g = \theta_X] = X
$$

$$
\iff \text{every closed subset of } X \text{ intersects } [f = \theta_X] \text{ or } [g = \theta_X].
$$
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So from this and the fact that the sets \([h = \theta_X] \ (h \in \mathcal{A}(X))\) form a closed basis (and Cantor’s Intersection Theorem), we obtain

\[
f \perp g \iff \forall h_1, \ldots, h_n \in \mathcal{A}(X), \quad \text{if } \bigcap_{i=1}^{n} [h_i = \theta_X] \cap [f = \theta_X] = \bigcap_{i=1}^{n} [h_i = \theta_X] \cap [g = \theta_X] = \emptyset \quad \text{then } \bigcap_{i=1}^{n} [h_i = \theta_X] = \emptyset.
\]

The last condition is preserved under non-vanishing bijections, and so \(T\) is a \(\perp\)-isomorphism.

Now we show that \(f \perp g\) is equivalent to the following statement: There are finite families \(\{a_i\}, \{b_j\}\) and \(\{c_k\}\) in \(\mathcal{A}(X)\) such that

\[(i) \quad \bigcap_{i,j,k} [a_i = \theta_X] \cap [b_j = \theta_X] \cap [c_k = \theta_X] = \emptyset;\]

\[(ii) \quad a_i \perp b_j \text{ for all } i \text{ and } j;\]

\[(iii) \quad f \perp b_j, f \perp c_k, g \perp a_i, \text{ and } g \perp c_k \text{ for all } i, j \text{ and } k.\]

Indeed, if \(f \perp g\), by regularity and compactness one can take finite families \(\{a_i\}, \{b_j\}\) satisfying (ii) and such that \(\text{supp}(f) \subseteq \bigcup_i [a_i \neq \theta_X]\) and \(\text{supp}(g) \subseteq \bigcup_j [b_j \neq \theta_X]\). Then take a finite family \(\{c_k\}\) such that (iii) is satisfied and such that

\[X \setminus \left( \bigcup_{i,j} [a_i \neq \theta_X] \cup [b_j \neq \theta_X] \right) \subseteq \bigcup_k [c_k \neq \theta_X],\]

which implies (i).

Conversely, if such families \(\{a_i\}, \{b_j\}, \{c_k\}\) in \(\mathcal{A}(X)\) exist, suppose \(x \in \text{supp}(f) \cap \text{supp}(g)\). By item (i), there is at least one of the indices \(i, j\) or \(k\) such that \(c_k(x)\) is not equal to \(\theta_X(x)\). Since \(x \in \text{supp}(f)\), and \(f \perp b_j\) and \(f \perp c_k\) by (iii), the only possibility is \(a_i(x) \neq \theta_X(x)\) for some \(i\). The same argument with \(g\) in place of \(f\) yields \(b_j(x) \neq \theta_X(x)\) for some \(j\), contradicting (ii).

Therefore \(T\) is a \(\perp\)-isomorphism. \(\square\)

**Theorem 3.3.17.** For every non-vanishing bijection \(T : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)\) there is a unique homeomorphism \(\phi : Y \rightarrow X\) such that \([f = \theta_X] = \phi([Tf = \theta_Y])\) for all \(f \in \mathcal{A}(X)\).

**Proof.** By Proposition 3.3.16 we already know that \(T\) is a \(\perp\)-isomorphism, so let \(\phi\) be the \(T\)-homeomorphism. Recall (Definition 3.1.1) that \(\sigma^\theta_X(f) = \text{int}([f \neq \theta_X])\) and \(Z^\theta_X(f) = \text{int}([f = \theta_X])\) for all \(f \in \mathcal{A}(X)\). Let us prove that

\[f(x) = \theta_X(x) \iff \forall h_1, \ldots, h_n \in \mathcal{A}(X), \quad \text{if } \bigcap_{i=1}^{n} [h_i \neq \theta_X] \cap [f = \theta_X] = \bigcap_{i=1}^{n} [h_i = \theta_X] \cap [g = \theta_X] = \emptyset \quad \text{then } \bigcap_{i=1}^{n} [h_i = \theta_X] = \emptyset.
\]
if \( x \not\in \bigcup_{i=1}^{n} \sigma^0(h_i) \) then \( \bigcap_{i=1}^{n}[h_i = \theta_X] \cap [f = \theta_X] \neq \emptyset \), \hspace{1cm} (3.3.7)

Indeed, for the “\( \Rightarrow \)” direction, assume that \( f(x) = \theta_X(x) \) and \( h_1, \ldots, h_n \) are such that \( x \not\in \bigcup\sigma^0(h_i) \). Then \( x \in \bigcap_i[h_i = \theta_X] \cap [f = \theta_X] \), and this set is nonempty.

For the converse we prove the contrapositive. Assume that \( f(x) \neq \theta_X(x) \), so regularity and compactness give us \( h_1, \ldots, h_n \in A(X) \) such that \( x \in Z^0(h_i) \) for all \( i \) and \( X \setminus [f \neq \theta_X] \subseteq \bigcup_i[h_i \neq \theta_X] \), which negates the right-hand side of \( (3.3.7) \).

Since \( \phi(\sigma^0(h)) = \sigma^0(h) \) for all \( h \in A(X) \) and \( T \) is non-vanishing, the condition of \( (3.3.7) \) is preserved by \( T \) and therefore, \( \phi \) has the desired property. \( \square \)

### 3.4 Recovering known results

In this section we will recover several known results, some in greater generality than in their original statements, dealing with different algebraic structures on classes of continuous functions and their isomorphisms.

The general procedure we will use is the following: Suppose that \( X \) and \( Y \) are locally compact Hausdorff spaces, \( H_X \) and \( H_Y \) are Hausdorff spaces, \( \theta_X \in C(X, H_X) \), \( \theta_Y \in C(Y, H_Y) \) and \( A(X) \) and \( A(Y) \) regular subsets of \( C_c(X, \theta_X) \) and \( C_c(Y, \theta_Y) \), respectively.

1. Describe the relation \( \perp \) with the algebraic structure at hand. In general, we will first describe \( \perp \) and use it in order to describe \( \perp \).

2. Given an algebraic isomorphism \( T : A(X) \to A(Y) \) for appropriate classes of functions \( A(X) \) and \( A(Y) \), the first item ensures that \( T \) is a \( \perp \)-isomorphism, so let \( \phi : Y \to X \) be the \( T \)-homeomorphism.

3. Prove that \( T \) is \( \phi \)-basic. Let \( \chi \) be the \((\phi, T)\)-transform.

4. Items 1.-3. also apply to \( T^{-1} \), which is thus also a basic \( \perp \)-isomorphism.

5. Items 3. and 4. imply, by Proposition \( 3.3.6(a) \) and \( (b) \), that the sections \( \chi(y, \cdot) \) are injective and in the case that \( A(Y) \) is regular, item \( (d) \) implies that \( \chi(y, \cdot) \) is also surjective.

6. If we have a classification of algebraic isomorphisms between \( H_X \) and \( H_Y \), this classification will apply to each section \( \chi(y, \cdot) \) of the \((\phi, T)\)-transform, by Proposition \( 3.3.10 \). This will in turn describe \( T \) completely.
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3.4.1 Milgram’s Theorem

In this subsection, we will always assume that $X$ is a locally compact Hausdorff space, $H = \mathbb{R}$ and $\theta = 0$ (the zero map from $X$ to $\mathbb{R}$). We denote $C_c(X, \mathbb{R}) = C_c(X, 0)$ (as in Definition 3.1.1). If $Y$ is another locally compact space then $C_c(Y, \mathbb{R})$ is treated similarly. Note that $C_c(X, \mathbb{R})$ is regular by Urysohn’s Lemma.

First, we present a well-known classification of continuous multiplicative isomorphisms of $\mathbb{R}$ (see [130, Lemma 4.3], for example). We present its proof for the sake of completeness. Given $t \in \mathbb{R}$, $\text{sgn}(t)$ denotes the sign of $t$ (the sign of 0 is $\text{sgn}(0) = 0$).

**Proposition 3.4.1.** Let $\tau: \mathbb{R} \to \mathbb{R}$ be a multiplicative isomorphism. Then

(a) Given $x \in \mathbb{R}$, $x \geq 0$ if and only if $\tau(x) \geq 0$;

(b) $\tau(-x) = -\tau(x)$ for all $x \in \mathbb{R}$;

(c) The following are equivalent:

(1) $\tau$ is continuous;

(2) $\tau$ is continuous at 0;

(3) If $0 < x < 1$ then $0 < \tau(x) < 1$;

(4) $\tau$ is increasing;

(5) $\tau$ has the form $\tau(x) = \text{sgn}(x)|x|^p$ for some $p > 0$;

**Proof.**

(a) Simply note that $x \geq 0$ if and only if $x = y^2$ for some $y$.

(b) If $x = 0$ this is trivial. If $x \neq 0$, then $-x$ is the only number satisfying $(-x)^2 = x^2$ and $-x \neq x$.

(c) The implications (5) $\Rightarrow$ (1) $\Rightarrow$ (2) are trivial.

(2) $\Rightarrow$ (3): If $0 < x < 1$ then $x^n \to 0$, so $\tau(x)^n \to \tau(0) = 0$ which implies $\tau(x) < 1$.

(3) $\Rightarrow$ (4) is immediate from (a) and (b).

(4) $\Rightarrow$ (5): Letting $p = \log_2(\tau(2)) > 0$ (because $\tau(2) > \tau(1) = 1$), we have $\tau(2^q) = (2^q)^p$ for all $q \in \mathbb{Q}$. Thus the restriction of $\tau$ to $[0, \infty)$ is an increasing map with dense image, hence surjective and continuous. Moreover, $\tau$ coincides with $x \mapsto x^p$ on a dense set of $[0, \infty)$, thus $\tau(x) = x^p$ for all $x \geq 0$ and thus for all $x$ by (b).

We will now classify multiplicative isomorphisms from $C_c(X, \mathbb{R})$ and $C_c(Y, \mathbb{R})$. Following the procedure outlined in page 153, we first describe $\bot$ in multiplicative terms.
Lemma 3.4.2. If \( f, g \in C_c(X, \mathbb{R}) \), then
\[
  f \perp g \iff \exists h \in C_c(X, \mathbb{R}) \text{ such that } hf = f \text{ and } hg = 0 \tag{3.4.1}
\]

Proof. Indeed, first assume \( f \perp g \), i.e., \( \text{supp}(f) \cap \text{supp}(g) = \emptyset \). By Urysohn’s Lemma, there exists \( h \in C_c(X, \mathbb{R}) \) such that \( h = 1 \) on \( \text{supp}(f) \) and \( \text{supp}(h) \subseteq X \setminus \text{supp}(g) \). Then \( h \) satisfies the condition (3.4.1).

Conversely, if there is \( h \in C_c(X, \mathbb{R}) \) satisfying (3.4.1), then \( h = 1 \) on \( \text{supp}(f) \) and so \( f \in h \). As \( hg = 0 \) if and only if \( h \perp g \), we conclude that \( f \perp g \). \(
\)

Theorem 3.4.3 (Milgram’s Theorem, [130 Theorem A], for locally compact spaces). Let \( X \) and \( Y \) be locally compact Hausdorff spaces and let \( T : C_c(X, \mathbb{R}) \to C_c(Y, \mathbb{R}) \) be a multiplicative isomorphism. Then there exists a homeomorphism \( \phi : Y \to X \), a subset \( F \subseteq Y \) whose points are isolated in \( Y \) and such that \( F \cap K \) is finite for any compact \( K \subseteq Y \), and a continuous positive function \( p : Y \setminus F \to (0, \infty) \) satisfying
\[
  Tf(y) = \text{sgn}(f(\phi(y)))|f(\phi(y))|^p(y)
\]
for all \( f \in C_c(X, \mathbb{R}) \) and \( y \in Y \setminus F \).

Proof. By Lemma 3.4.2 \( T \) is a \( \perp \)-isomorphism, so let \( \phi \) be the \( T \)-homeomorphism. We prove that \( T \) is \( \phi \)-basic in a few steps:

1. If \( f, h \in C_c(X, \mathbb{R}) \), then \( f = 1 \) on \( \text{supp}(h) \) if and only if \( Tf = 1 \) on \( \text{supp}(Th) \):
   
   To verify this, simply note that \( f = 1 \) on \( \text{supp}(h) \) if and only if \( fh = h \), and similarly for \( Tf \). Since \( T \) is multiplicative we are done.

2. If \( f, h \in C_c(X, \mathbb{R}) \), then \( f \neq 0 \) on \( \text{supp}(h) \) if and only if \( Tf \neq 0 \) on \( \text{supp}(Th) \):
   
   If \( f \neq 0 \) on \( \text{supp}(h) \), we can find \( g \in C_c(X, \mathbb{R}) \) such that \( g = 1/f \) on \( \text{supp}(h) \).
   
   Item 1. implies that \( TfTg = 1 \) on \( \text{supp}(Th) \), and in particular \( Tf \neq 0 \) on \( \text{supp}(Th) \).

3. If \( f \in C_c(X, \mathbb{R}) \) and \( y \in Y \), then \( f(\phi(y)) \neq 0 \) if and only if \( Tf(y) \neq 0 \):

   Assume \( f(\phi(y)) \neq 0 \). Choose \( h \in C_c(X, \mathbb{R}) \) such that \( \phi(y) \in \text{supp}(h) \subseteq [f \neq 0] \).
   
   Then item 2. implies that \( Tf \neq 0 \) on \( \text{supp}(Th) \), which contains \( y \).

4. If \( f, g, h \in C_c(X, \mathbb{R}) \), then \( f \) and \( g \) coincide on \( Y \) if and only if \( Tf \) and \( Tg \) coincide and are nonzero on \( \text{supp}(h) \):

   This is an immediate consequence of item 1.

5. In particular, from 3., \( f(\phi(y)) = 0 \) if and only if \( Tf(y) = 0 \).
6. If \( f \in C_c(X, \mathbb{R}) \) and \( y \in Y \), then \( f(\phi(y)) = 1 \) if and only if \( Tf(y) = 1 \):

Suppose this was not the case, say \( f(\phi(y)) = 1 \) but \( Tf(y) \neq 1 \), and let us deduce a contradiction. Take a neighbourhood \( Y' \) of \( \phi(y) \) with compact closure. In particular from 1., \( f \) is not constant on any neighbourhood of \( \phi(y) \), so there is a sequence of distinct points \( y_n \in Y' \) and \( r > 0 \) such that

\[
\begin{align*}
(i) & \quad f(y_n)^n \to 1; \\
(ii) & \quad |Tf(y_n) - 1| > r \text{ for all } n.
\end{align*}
\]

By Proposition 3.3.13(b), we can consider a tail of the sequence \( \{y_n\}_n \) if necessary, and take a continuous function \( g : X \to \mathbb{R} \) which coincides with \( f^n \) on a neighbourhood of \( y_n \) for each \( n \). Property (i) allows us to consider another tail of the sequence \( \{y_n\}_n \), if necessary, and change \( g \) by \( \max(g, 1/2) \), so we may assume that \( g \neq 0 \) everywhere. Let \( u \in C_c(X, \mathbb{R}) \) be a function with \( u = 1 \) on \( Y' \), so \( ug \in C_c(X, \mathbb{R}) \) and \( ug = f^n \) on a neighbourhood of \( \phi(y_n) \) for all \( n \).

Let \( z \) be a cluster point of the sequence \( \{y_n\}_n \), so in particular \( T(ug)(z) \) is a cluster point of the sequence \( T(ug)(y_n) = Tf(y_n)^n \), where this equality follows from 4. Since \( |Tf(y_n) - 1| > r > 0 \), the only possibility is \( T(ug)(z) = 0 \), and by 5, this means that \( 0 = (ug)(\phi(z)) = g(\phi(z)) \) (because \( z \in Y' \)), contradicting the fact that \( g \) is nonzero.

We conclude, from 5. and 6. that \( T \) is \( \phi \)-basic. Let \( \chi : Y \times \mathbb{R} \to \mathbb{R} \) be the \( T \)-transform.

Let \( F = \{y \in Y : \chi(y, \cdot) \text{ is discontinuous}\} \). We will now prove that \( F \cap K \) is finite for any compact subset \( K \subseteq Y \). Otherwise there would be distinct \( y_1, y_2, \ldots \in K \) such that \( \chi(y_n, \cdot) \) is unbounded on every neighbourhood of 0, so we can consider a strictly decreasing sequence \( t_n \to 0 \) such that \( \chi(y_n, t_n) > n \). Going to a subsequence if necessary, we can construct, by Proposition 3.3.13, \( f \in C_c(X, \mathbb{R}) \) with \( f(\phi(y_n)) = t_n \), so \( Tf(y_n) > n \) for all \( n \), a contradiction to the boundedness of \( Tf \).

Now we show that \( F \) is open: If \( z \in F \), then there is \( t \in (0, 1) \) with \( \chi(z, t) > 1 \). Take \( f \in C_c(X, \mathbb{R}) \) such that \( f = t \) on a neighbourhood \( U \) of \( \phi(z) \). In particular \( Tf(z) > 2 \), so there is a neighbourhood \( W \) of \( z \) such that \( Tf > 2 \) on \( W \). Then for all \( y \in \phi^{-1}(U) \cap W \), \( \chi(y, t) = Tf(y) > 2 \), so \( y \in F \).

Therefore, \( F \) consists of isolated points, since \( Y \) is locally compact. For \( y \notin F \), Proposition 3.3.10 and Proposition 3.4.1 imply that \( \chi(y, \cdot) \) has the form \( \chi(y, t) = \text{sgn}(t)|t|^{p(y)} \) for some \( p(y) > 0 \). Let \( U \) be an open subset of \( Y \) not intersecting \( F \) and with compact closure. Take any function \( f \in C_c(X, \mathbb{R}) \) with \( f = 2 \) on \( \phi(U) \). Then for \( y \in U \),

\[
2^{p(y)} = \chi(y, 2) = \chi(y, f(\phi(y))) = Tf(y)
\]

so \( p(y) = \log_2(Tf(y)) \) for \( y \in U \), showing that \( p \) is continuous on \( U \). Since \( Y \setminus F \) is the union of such \( U \) we are done. 

\[ \square \]
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3.4.2 Group-Valued functions; Hernández-Ródenas Theorem

Given topological groups $G$ and $H$, denote by $\text{AbsIso}(G, H)$ the set of algebraic group isomorphisms from $G$ to $H$, and by $\text{TopIso}(G, H)$ the set of topological (i.e., homeomorphisms which are) group isomorphisms.

Let $X$ and $Y$ be compact Hausdorff spaces and $G$ a Hausdorff topological group. In [84, Theorem 3.7], Hernández and Ródenas classified non-vanishing group morphisms (note necessarily isomorphisms) $T : C(X, G) \to C(Y, G)$ which satisfy the following properties:

(i) There exists a continuous group morphism $\psi : G \to C(X, G)$, where $C(X, G)$ is endowed with the topology of pointwise convergence, such that for all $\alpha \in G$ and all $y \in Y$, $T(\psi(\alpha))(y) = \alpha$;

(ii) For every continuous endomorphism $\theta : G \to G$ and every $f \in C(X, G)$, $T(\theta \circ f) = \theta \circ (Tf)$.

If $T$ is a group isomorphism and $T^{-1}$ is continuous (with respect to uniform convergence) then condition (i) is immediately satisfied, however this is not true for (ii): For example, if $\text{TopIso}(G, G)$ is non-abelian and $\rho \in \text{TopIso}(G, G)$ is any non-central element, then the map $T : C(X, G) \to C(X, G)$ given by $Tf = \rho \circ f$ is a group isomorphism, and a self-homeomorphism of $C(X, G)$ with the topology of uniform convergence, which does not satisfy (ii).

In the next theorem we obtain the same type of classification as in [84, Theorem 3.7], without assuming condition (ii), however we consider only non-vanishing group isomorphisms. Given a compact Hausdorff space $X$, a topological group $G$ and $\alpha \in G$, denote by $\overline{\alpha}$ the constant function $X \to G$, $x \mapsto \alpha$. We endow $C(X, G)$ with the topology of pointwise convergence.

**Theorem 3.4.4.** Suppose that $G$ and $H$ are Hausdorff topological groups, and $X$ and $Y$ are compact Hausdorff spaces for which $(X, \overline{\text{G}}, C(X, G))$ and $(Y, \overline{\text{H}}, C(Y, H))$ are regular.

Let $T : C(X, G) \to C(Y, H)$ be a non-vanishing group isomorphism (Definition 3.3.15). Then there exist a homeomorphism $\phi : Y \to X$ and a map $w : Y \to \text{AbsIso}(H, G)$ such that $Tf(y) = w(y)(f(\phi(y)))$ for all $y \in Y$ and $f \in C(X, G)$.

If $T$ is continuous on the constant functions then each $w(y)$ is continuous and $T$ is continuous on $C(X, G)$. If both $T$ and $T^{-1}$ are continuous on the constant functions then $w(y) \in \text{TopIso}(H, G)$ and $T$ is a homeomorphism for the topologies of pointwise convergence.

**Proof.** By Theorem 3.3.17 and Proposition 3.3.11, there is a homeomorphism $\phi : Y \to X$ such that $T$ is $\phi$-basic, and the sections $\chi(y, \cdot) : G \to H$ of the $(\phi, T)$-transform $\chi$ are group morphisms by Proposition 3.3.10. Similar facts hold for $T^{-1}$, so Proposition...
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3.3.6 implies that each section $\chi(y, \cdot)$ is bijective. Letting $w(y) = \chi(y, \cdot)$ we are done with the first part.

Now note that for all $\alpha \in G$ and $y \in Y$,

$$w(y)(\alpha) = w(y)(\alpha(\phi(y))) = T\alpha(y)$$

which implies that every $w(y)$ is continuous if and only if $T$ is continuous on the constant functions (because the map

$$G \to \{\alpha : \alpha \in G\} \subseteq C(X, G), \quad \alpha \mapsto \alpha$$

is a homeomorphism). In this case, from the equality

$$T f(y) = w(y)(f(\phi(y)) \quad \text{for all} \quad f \in C(X, G) \quad \text{and} \quad y \in Y,$$

we can readily see that $T$ is continuous. The last part, assuming also that $T^{-1}$ is continuous on the constant functions, is similar, using $T^{-1}$ and $w(y)^{-1}$ in place of $T$ and $w(y)$.

3.4.3 Kaplansky’s Theorem

Let $R$ be a totally ordered set without supremum or infimum, considered as a topological space with the order topology, and let $X$ be a locally compact Hausdorff space. We consider the pointwise order on $C(X, R)$: $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x$, which makes $C(X, R)$ a lattice: for all $f, g \in C(X, R)$ and $x \in X$,

$$(f \lor g)(x) = \max(f(x), g(x)), \quad \text{and} \quad (f \land g)(x) = \min(f(x), g(x)).$$

Denote by $C_b(X, R)$ the sublattice of bounded continuous functions from $X$ to $R$.

We will consider sublattices $\mathcal{A}$ of $C_b(X, R)$ which satisfy

(L1) for all $f, g \in \mathcal{A}$, $[f \neq g]$ is compact;

(L2) for all $f \in \mathcal{A}$, every open set $U \subseteq X$, every $x \in U$ and $\alpha \in R$, there exists $g \in \mathcal{A}$ such that $g(x) = \alpha$ and $[g \neq f] \subseteq U$.

Example 3.4.5 (Kaplansky, [98]). Suppose that $X$ is compact and $\mathcal{A}$ is an $R$-normal sublattice of $C(X, R)$. Condition (L1) is trivial, so let us check that $\mathcal{A}$ satisfies (L2):

Suppose $f$, $U$, $x$ and $\alpha$ are as in that condition. For the sake of the argument we can assume $f(x) \leq \alpha$. Let $\beta$ be any lower bound of $f(X)$, and from $R$-normality find $h \in \mathcal{A}$ such that $h(x) = \alpha$ and $h = \beta$ outside $U$. Then $g = f \lor h$ has the desired properties.

Following [98], this means that for any pair of disjoint closed sets $F, G \subseteq X$ and $\alpha, \beta \in R$, there is $f \in \mathcal{A}$ which equals $\alpha$ on $F$ and $\beta$ on $G$. 
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Example 3.4.6 (Li–Wong, [116]). Suppose that $X$ is compact, $R = \mathbb{R}$, and $\mathcal{A}$ is a regular additive subgroup of $C(X, \mathbb{R})$, so (L1) is also trivial and again we need to verify condition (L2): Let $f, U, x$ and $\alpha$ be as in (L2). By regularity, take $h$ such that $\text{supp}(h) \subseteq U$ and $h(x) = \alpha - f(x)$. Then $g = f + h$ has the desired properties.

We will now recover the main result of [98]. The following lemma is based on [158].

Lemma 3.4.7. Let $\mathcal{A}$ be a sublattice of $C_b(X, R)$ satisfying (L1) and (L2), and let $f_0$ be any element of $\mathcal{A}$. Let $\mathcal{A}_{\geq f_0} = \{ f \in \mathcal{A} : f \geq f_0 \}$. Then $(X, f_0, \mathcal{A}_{\geq f_0})$ is weakly regular and for $f, g \in \mathcal{A}_{\geq f_0}$,

(a) $f \perp g \iff f \wedge g = f_0$ (which is the minimum of $\mathcal{A}_{\geq f_0}$);

(b) $f \preceq g$ is equivalent to the following statement:

"for every bounded subset $\mathcal{H} \subseteq \mathcal{A}$ such that $h \subseteq f$ for all $h \in \mathcal{H}$, there is an upper bound $k$ of $\mathcal{H}$ such that $k \subseteq g$."

(K)

Proof. Weak regularity is immediate from (L2) and the fact that $R$ does not have a supremum, and item (a) is trivial. Let us prove (b). First suppose $f \preceq g$ and $\mathcal{H} \subseteq \mathcal{A}$ is a bounded subset such that $h \subseteq f$ for all $h \in \mathcal{H}$. Let $\alpha \in R$ be an upper bound of $\bigcup_{h \in \mathcal{H}} h(X)$. From weak regularity and compactness of $\text{supp}(f)$, we can take finitely many functions $k_1, \ldots, k_n$ such that $k_i \subseteq g$, and for every $x \in \text{supp}(f)$ there is some $i$ with $k_i(x) > \alpha$. Letting $k = \bigvee_{i=1}^n k_i$ we obtain the desired properties.

Conversely, suppose that condition (K) holds. Let $\alpha$ be any upper bound of $f_0(X)$ and again take $\beta > \alpha$. Let $\mathcal{H} = \{ h \in \mathcal{A}_{\geq f_0} : h \leq \beta, h \subseteq f \}$. Let $k$ be an upper bound of $\mathcal{H}$ with $k \subseteq g$. By Property (L2), we have $\sigma(f) = \bigcup_{h \in \mathcal{H}} \sigma(h)$, so $k \geq \beta$ on $\sigma(f)$ and thus also on $\text{supp}(f)$, which implies $f \preceq k$. Since $k \subseteq g$ then $f \preceq g$.

As an immediate consequence of Lemma 3.4.7 and Theorem 3.1.13 we have the following generalization of Kaplansky’s Theorem:

Theorem 3.4.8 (Kaplansky [98]). Suppose $R$ has no supremum nor infimum, $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are sublattices of $C_b(X, R)$ and $C_b(Y, R)$, respectively, satisfying conditions (L1) and (L2), and $T: \mathcal{A}(X) \to \mathcal{A}(Y)$ is a lattice isomorphism. Then for any $f_0 \in \mathcal{A}$, $T$ restricts to a $\perp$-isomorphism of the regular sublattices $\mathcal{A}(X)_{\geq f_0}$ and $\mathcal{A}(Y)_{\geq Tf_0}$. In particular, $X$ and $Y$ are homeomorphic.

Our immediate goal is to prove that the homeomorphism between $X$ and $Y$ given by Theorem 3.4.8 does not depend on the choice of function $f_0$ in Lemma 3.4.7.
Lemma 3.4.9. Under the conditions of Theorem 3.4.8, let \( \phi : Y \rightarrow X \) be the \( T|_{\mathcal{A}(X)_{f_0}} \)-homeomorphism. If \( f, g \in \mathcal{A}(X)_{f_0} \) then \( \phi^{-1}(\text{int}([f = g])) = \text{int}([Tf = Tg]) \).

Proof. We will use the superscript “\( f_0 \)” as in Definition 3.1.1 that is, for all \( f \geq f_0 \)

\[
\sigma^{f_0}(f) = \text{int}([f \neq f_0]), \quad \text{and} \quad Z^{f_0}(f) = \text{int}([f = f_0]).
\]

and similarly with \( Tf_0 \) in place of \( f_0 \). Then \( \phi \) is the only homeomorphism satisfying \( \sigma^{f_0}(f) = \phi(\sigma^{Tf_0}(Tf)) \), or equivalently \( Z^{f_0}(f) = \phi(Z^{Tf_0}(Tf)) \), for all \( f \geq f_0 \).

First, assume that \( f \leq g \), and let \( U = \text{int}([f = g]) \). For all \( x \in [f < g] \), choose a function \( k_x \in \mathcal{A}(X)_{f_0} \) such that

- \( \sigma^{f_0}(k_x) \cap [f = g] = \emptyset \);
- \( k_x(x) = g(x) \);
- \( k_x \leq g \).

Then \( g = \sup \{ f, k_x : x \in [f < g] \} \), and since \( T \) is a lattice isomorphism we obtain \( Tg = \sup \{ Tf, Tk_x : x \in [f < g] \} \).

Let us prove that \( Tf(x) = Tg(x) \) for all \( x \in \phi^{-1}(U) \). Since \( U \subseteq \bigcap_{x \in [f < g]} Z^{f_0}(k_x) \), then \( \phi^{-1}(U) \subseteq \bigcap_{x \in [f < g]} Z^{Tf_0}(Tk_x) \).

Given \( y \in \phi^{-1}(U) \), use property (L2) to find \( h \in \mathcal{A}(Y) \) such that \( h(y) = Tf(y) \) and \( h = Tg \) outside of \( \phi^{-1}(U) \). Then \( h' = (Tf \lor h) \land Tg \) is an upper bound of \( \{ Tf, Tk_x : x \in [f < g] \} \), so it is also an upper bound of \( Tg \), and in particular,

\[
Tg(y) \leq ((h \lor Tf) \land Tg)(y) = Tf(y) \land Tg(y) \leq Tf(y)
\]

so \( Tg(y) = Tf(y) \).

In the general case, if \( f = g \) on an open set \( U \) then \( f = f \land g = g \) on \( U \), so the previous case implies that \( Tf \) and \( Tg \) coincide with \( T(f \land g) \) on \( \phi^{-1}(U) \).

Theorem 3.4.10. Under the conditions of Theorem 3.4.8, there exists a unique homeomorphism \( \phi : Y \rightarrow X \) such that \( \phi(\text{int}([Tf = Tg])) = \text{int}([f = g]) \) for all \( f, g \in \mathcal{A}(X) \). (In this case we will still call \( \phi \) the \( T \)-homeomorphism.)

Proof. For all \( f_0 \in \mathcal{A}(X) \), let \( \phi^{f_0} : Y \rightarrow X \) be the \( T|_{\mathcal{A}(X)_{f_0}} \)-homeomorphism. Given \( f_0, g_0 \in \mathcal{A}(X) \), Lemma 3.4.9 implies that \( \phi^{f_0 \lor g_0} \) satisfies the property of the \( T|_{\mathcal{A}(X)_{f_0}} \)-homeomorphism, which uniquely defines \( \phi^{f_0} \), and similarly for \( \phi^{g_0} \), so \( \phi^{f_0} = \phi^{f_0 \lor g_0} = \phi^{g_0} \). We are done by letting \( \phi = \phi^{f_0} \) for some arbitrary \( f_0 \in \mathcal{A}(X) \). \( \square \)
A natural goal now is to classify the lattice isomorphisms as given in Theorem 3.4.10. Although at the moment we cannot prove that all lattice isomorphisms as above are basic, this is true in some cases. A similar argument to the one used in the proof of Theorem 3.3.6 appears in [24], although in a different context (considering lattices of possibly unbounded real-valued continuous functions on complete metric spaces). See also [89, 90].

**Theorem 3.4.11.** Suppose that $X$ and $Y$ are first-countable (and locally compact Hausdorff), that $R = \mathbb{R}$ with the usual order, and that $T : C_c(X, 0) \to C_c(Y, 0)$ is a lattice isomorphism. Then there are a unique homeomorphism $\phi : Y \to X$ and a continuous function $\chi : Y \times \mathbb{R} \to \mathbb{R}$ such that

$$Tf(y) = \chi(y, f(\phi(y))) \quad \text{for all } y \in Y \text{ and } f \in C_c(X, 0) \quad (3.4.2)$$

and $\chi(y, \cdot) : \mathbb{R} \to \mathbb{R}$ is an increasing bijection for all $y \in Y$.

**Proof.** Suppose $x \in X$ and $f(x) = g(x)$. Using Propositions 3.3.12 and 3.3.13(b), we can find a sequence $\{x_n\}_n$ of points in $X$ converging to $x$ and a function $h \in C_c(X, 0)$ such that, for all $n$, $h = f$ on some neighbourhood of $x_{2n}$ and $h = g$ on some neighbourhood of $x_{2n+1}$. Therefore, $Tf$ will coincide with $Th$ on some neighbourhood of $\phi^{-1}(x_{2n})$ for all $n \in \mathbb{N}$,

$$Th(\phi^{-1}(x)) = \lim_{n \to \infty} Th(\phi^{-1}(x_{2n})) = \lim_{n \to \infty} Tf(\phi^{-1}(x_{2n})) = Tf(\phi^{-1}(x))$$

and similarly $Tg(\phi^{-1}(x)) = Th(\phi^{-1}(x)) = Tf(\phi^{-1}(x))$, which proves that $T$ is basic.

Therefore, there is a function $\chi : Y \times \mathbb{R} \to \mathbb{R}$ such that Equation (3.4.2) is satisfied. Every section $\chi(y, \cdot) : \mathbb{R} \to \mathbb{R}$ is surjective, since $C_c(Y, 0)$ is regular, and the same argument with $T^{-1}$, and Proposition 3.3.6 imply that $\chi(y, \cdot)$ is also injective, hence a bijection.

Proposition 3.3.10 applied to the signature of lattices (with the binary symbol “$\lor$” interpreted as “join”) implies that the sections $\chi(y, \cdot)$ are lattice isomorphisms of $\mathbb{R}$ for all $n$, and in particular homeomorphisms. Theorem 3.3.14 implies that $\chi$ is continuous (and $T$ is a homeomorphism for the topologies of pointwise convergence.)

**Example 3.4.12.** There are non-first countable spaces for which the conclusion of Theorem 3.3.6 holds.

Let $\Omega = \omega_1 \cup \{\omega_1\}$ be the successor of the first uncountable ordinal. We extend$^6$ the order of $\omega_1$ to $\Omega$ by setting $\alpha < \omega_1$ for all $\alpha \in \omega_1$, and $\Omega$ is a compact Hausdorff space with the order topology. If $Z$ is any first-countable space, then any continuous function $f : \Omega \to Z$ is constant on a neighbourhood of $\omega_1$: Indeed, for every $n \in \mathbb{N}$

---

$^6$The fact that $\omega_1 \notin \omega_1$ follows from the Axiom of Regularity, see [35], p. 3.
choose $\alpha_n < \omega_1$ such that $|f(\beta) - f(\omega_1)| < 1/n$ whenever $\beta \geq \alpha_n$. Letting $\alpha = \sup_n \alpha_n$, we have $\alpha < \omega_1$ and $f(\beta) = f(\omega_1)$ for all $\beta \in [\alpha, \omega_1]$

Now suppose that $R = \mathbb{R}$ and $T : C(\Omega) \to C(\Omega)$ is a lattice isomorphism, and let $\phi : \Omega \to \Omega$ be the $T$-homeomorphism. Since $\omega_1$ is the only non-$G_\delta$ point of $\Omega$, we have $\phi(\omega_1) = \omega_1$. The previous paragraph allows us to identify the lattices

$$C_c(\omega_1) \simeq \{ f \in C(\Omega) : f(\omega_1) = 0 \},$$

which then induces a lattice isomorphism $T|_{\omega_1} : C_c(\omega_1) \to C_c(\omega_1)$. In this case, note that $\phi|_{\omega_1}$ is the $T|_{\omega_1}$-homeomorphism. We can now prove that $T$ is basic with respect to $\phi$.

Let $f, g \in C(\Omega)$. If $f(\omega_1) = g(\omega_1)$, then the first paragraph implies that $f = g$ on some neighbourhood of $\omega_1$ and thus $Tf(\omega_1) = Tg(\omega_1)$. If $f(\alpha) = g(\alpha)$ for some $\alpha < \omega_1$, consider $\tilde{f} \in C_c(\omega_1, 0)$ given by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \leq \alpha, \\ 0, & \text{otherwise}, \end{cases}$$

and define $\tilde{g}$ similarly. Since $(-\infty, \alpha]$ is open in $\Omega$, then

$$Tf(\phi^{-1}(\alpha)) = T\tilde{f}(\phi^{-1}(\alpha)) \quad \text{and} \quad Tg(\phi^{-1}(\alpha)) = T\tilde{g}(\phi^{-1}(\alpha)),$$

and since $\omega_1$ is first-countable, we use the previous example to conclude that

$$T\tilde{f}(\phi^{-1}(\alpha)) = T\tilde{g}(\phi^{-1}(\alpha)),$$

so $Tf(\phi^{-1}(\alpha)) = Tg(\phi^{-1}(\alpha))$. Therefore $T$ is basic (with respect to $\phi$).

In the case additive lattice isomorphisms of spaces of real-valued functions, we do not require the first-countability hypothesis.

**Theorem 3.4.13.** Suppose $R = \mathbb{R}$, and $T : C_c(X) \to C_c(Y)$ is an additive lattice isomorphism. Then there are a unique homeomorphism $\phi : Y \to X$ and a unique positive continuous function $p : Y \to (0, \infty)$ such that $Tf(y) = p(y)f(\phi(y))$ for all $f \in C_c(X)$ and $y \in Y$.

**Proof.** First note that for all $f \in C_c(X)$, $|f| = (f \vee 0) - (f \wedge 0)$, so $T|f| = |Tf|$. Let $\phi : Y \to X$ be the $T$-homeomorphism, given by Theorem [3.4.10]

Now suppose $f(x) = 0$ but $Tf(\phi^{-1}(x)) \neq 0$. First take a compact neighbourhood $U$ of $x$ and $r > 0$ such that $|Tf| > r$ on $\phi^{-1}(U)$. Moreover, $f$ is not constant on any neighbourhood of $x$, so there is a sequence of distinct points $x_n \in U$ such that $|f(x_n)| < n^{-2}$. Using Propositions [3.3.12] and [3.3.13](b), we can take a subsequence if
necessary and consider \( g \in C_c(X) \) such that for all \( n \), \( g = nf \) on a neighbourhood of \( x_n \). Then \( Tg = nTf \) on a neighbourhood of \( \phi^{-1}(x_n) \), however

\[
nr < nTf(\phi^{-1}x_n) = nTg(\phi^{-1}x_n),
\]

which contradicts the fact that \( g \) is bounded.

Therefore \( T \) is basic with respect to \( \phi \), so let \( \chi \) be the \( T \)-transform. Each section \( \chi(y, \cdot) \) is an additive order-preserving bijection (Propositions 3.3.10 and 3.3.6) and hence has the form \( \chi(y, t) = p(y)t \) for some \( p(y) > 0 \). If \( Tf(y) \neq 0 \), then \( f(\phi(y)) \neq 0 \) as well and \( p = Tf/(f \circ \phi) \) on a neighbourhood of \( y \), thus \( p \) is continuous. \( \square \)

### 3.4.4 Li–Wong Theorem

In [116], Li and Wong proved Theorem 3.4.14 which can be seen as a generalization of Theorem 3.4.13. We will proceed in the opposite direction, i.e., by proving their result (or more precisely, the specific case where the domains are compact) instead as a consequence of the more general Theorem 3.4.8.

**Theorem 3.4.14** (Li–Wong [116]). Let \( X \) and \( Y \) be compact Hausdorff spaces, and \( A(X) \) and \( A(Y) \) be two regular vector sublattices of \( C(X, \mathbb{R}) \) and \( C(Y, \mathbb{R}) \), respectively. Suppose that \( T : A(X) \to A(Y) \) is a linear isomorphism which preserves non-vanishing functions, that is, for all \( f \in A(X) \),

\[
0 \in f(X) \iff 0 \in Tf(Y).
\]

Then there is a homeomorphism \( \phi : Y \to X \) and a continuous non-vanishing function \( p : Y \to \mathbb{R} \) such that \( Tf(y) = p(y)f(\phi(y)) \) for all \( f \in A(X) \) and \( y \in Y \).

**Proof.** In order to apply Theorem 3.4.8 we need to modify \( T \) to obtain a lattice isomorphism. Since \( A(X) \) is a sublattice, then for all \( f \in A(X) \),

\[
f^+ = \max(f, 0), \quad f^- = \max(-f, 0) \quad \text{and} \quad |f| = f^+ + f^- \quad \text{belong to } A(X).
\]

As \( A(X) \) is regular, we can take finitely many functions \( f_1, \ldots, f_n \in A(X) \) such that for all \( x \in X \), \( f_i(x) \neq 0 \) for some \( i \), and therefore \( F = \sum_{i=1}^n |f_i| \in A(X) \) and \( F \) is non-vanishing, so \( TF \) is also non-vanishing. We define new classes of functions

\[
\mathcal{B}(X) = \{ f/F : f \in A(X) \}, \quad \text{and} \quad \mathcal{B}(Y) = \{ f/TF : f \in A(Y) \}
\]

It is immediate to see that \( \mathcal{B}(X) \) and \( \mathcal{B}(Y) \) are regular, and that \( \mathcal{B}(X) \) is a sublattice of \( C(X, \mathbb{R}) \) because \( F > 0 \). However we need to prove that \( \mathcal{B}(Y) \) is a sublattice of \( C(Y, \mathbb{R}) \).

Let \( U = [TF > 0] \) and \( V = [TF < 0] \), which are complementary clopen sets in \( Y \), and \( 1_U \) and \( 1_V \) their respective characteristic functions. We first prove that for all \( f \in \mathcal{A}(Y) \), we have \( f1_U \in \mathcal{A}(Y) \). Let \( k = \sup_{y \in Y} |f(y)| \).
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Use regularity of $\mathcal{A}(Y)$, compactness of $U$ and the fact that $\mathcal{A}(Y)$ is a sublattice to find a function $u \in \mathcal{A}(X)$ such that $u > k$ on $U$ and $u = 0$ on $V$, and similarly $v \in \mathcal{A}(Y)$ with $v < -k$ on $V$ and $v = 0$ on $U$. Then

$$f 1_U = (f \land (u - v)) \lor 0 \in \mathcal{A}(Y)$$

It also follows that $f 1_V = f - f 1_U \in \mathcal{A}(Y)$. Therefore if $b \in \mathcal{B}(Y)$ then $b(TF)$, $b(TF)1_U$ and $b(TF)1_V$ belong to $\mathcal{A}(Y)$, thus

$$(b \lor 0) = \frac{(b(TF)1_U \lor 0)}{TF} + \frac{(b(TF)1_V \lor 0)}{TF} \in \mathcal{B}(Y).$$

In general, if $b, c \in \mathcal{B}(Y)$ then

$$b \lor c = b + ((c - b) \lor 0) \in \mathcal{B}(Y),$$

and therefore $\mathcal{B}(Y)$ is a sublattice of $C(Y, \mathbb{R})$.

Now define a linear isomorphism $S : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$, $S(f) = T(fF)/TF$, which preserves non-vanishing functions. Note that in this case, the constant functions at $1$ on $X$ and $Y$ belong to $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, and that $S(1) = 1$. Given a scalar $\lambda$, linearity and the non-vanishing property of $S$ imply that, for all $f \in \mathcal{B}(X)$,

$$\lambda \notin f(X) \iff f - \lambda \text{ is non-vanishing} \iff Sf - \lambda \text{ is non-vanishing} \iff \lambda \notin Sf(Y),$$

so $f(X) = Sf(Y)$ and in particular $f \geq 0$ if and only if $Sf \geq 0$. Therefore $S$ is a lattice isomorphism, and we can apply Kaplansky’s theorem (3.4.10) to find the $S$-homeomorphism $\phi : Y \rightarrow X$. Now we need to prove that $S$ is $\phi$-basic.

Suppose $f(x) \neq 0$ for a given $x \in X$, and let us assume, without loss of generality, that $f(x) > 0$. Then $f > 0$ on some neighbourhood $U$ of $x$. Again using compactness of $X \setminus U$ and regularity of the sublattice $\mathcal{B}(X)$ we can construct a function $g \in \mathcal{B}(X)$ such that $g = 0$ on some neighbourhood of $x$ and $g > 0$ on $X \setminus U$. Letting $\bar{f} = f \lor g$, we have $\bar{f} = f$ on some neighbourhood of $x$, so $S\bar{f} = Sf$ on some neighbourhood of $\phi^{-1}(x)$. But $\bar{f}$ is non-vanishing, so $S\bar{f}$ is also non-vanishing and in particular $Sf(\phi^{-1}(x)) \neq 0$. This proves that $S$ is basic with respect to $\phi$. Letting $\chi : Y \times \mathbb{R} \rightarrow \mathbb{R}$ be the $S$-transform, we have that all sections $\chi(y, \cdot)$ are linear and increasing (Theorem 3.3.10), hence of the form

$$\chi(y, t) = P(y)t$$

for a certain $P(y) > 0$. Denoting the constant function $x \mapsto 1$ by $1$ (either on $X$ or $Y$), then

$$P(y) = \chi(y, 1) = \chi(y, 1(\phi(y))) = S1(y) = 1(y) = 1.$$
that is, \( \chi(y, t) = t \) for all \( t \in \mathbb{R} \).

Finally, for all \( f \in A(X) \) and \( y \in Y \),

\[
Tf(y) = (TF)(y) \left[ S \left( \frac{f}{F} \right)(y) \right] = (TF)(y) \chi \left( y, \frac{f(\phi(y))}{F(\phi(y))} \right) = \frac{TF(y)}{F(\phi(y))} f(\phi(y)),
\]

as we wanted. \( \square \)

We can extend this result to the complex case as follows:

**Corollary 3.4.15.** Suppose \( T : C(X, \mathbb{C}) \to C(Y, \mathbb{C}) \) is a linear isomorphism which preserves non-vanishing functions. Then there is a homeomorphism \( \phi : Y \to X \) and a continuous non-vanishing function \( p : Y \to \mathbb{C} \) such that \( Tf(y) = p(y)f(\phi(y)) \) for all \( f \in C(X, \mathbb{C}) \) and \( y \in Y \).

**Proof.** Substituting \( T \) by \( S = T/T(1) \) if necessary, we may assume \( T(1) = 1 \). The same argument as in the proof above implies \( Tf(Y) = f(X) \) for all \( f \in C(X, \mathbb{C}) \), and in particular the restriction of \( T \) to \( C(X, \mathbb{R}) \) gives us a linear isomorphism \( T|_{C(X, \mathbb{R})} \) preserving non-vanishing functions. We apply Theorem 3.4.14 to this case to obtain \( p \) and \( \phi \) satisfying \( Tf = p \cdot (f \circ \phi) \) for all \( f \in C(X, \mathbb{R}) \). Since \( C(X, \mathbb{R}) \) generates \( C(X, \mathbb{C}) \) then the same formula is valid for all \( f \in C(X, \mathbb{C}) \). \( \square \)

### 3.4.5 Jarosz’ Theorem

Throughout this subsection, we fix \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). Given a locally compact Hausdorff space \( X \), we let \( C_c(X) \) be the Banach space of \( \mathbb{K} \)-valued, compactly supported, continuous function on \( X \), where supports are the usual ones, i.e., \( \text{supp}(f) = \{ f \neq 0 \} \).

**Theorem 3.4.16 (Jarosz [92]).** If \( T : C_c(X) \to C_c(Y) \) is a linear \( \perp \)-isomorphism, then there exist a homeomorphism \( \phi : Y \to X \) and a continuous non-vanishing function \( p : Y \to \mathbb{K} \) such that \( Tf(y) = p(y)f(\phi(y)) \) for all \( f \in C_c(X) \) and \( y \in Y \).

**Proof.** First assume that \( X \) and \( Y \) are compact, and let us show that \( f \neq 0 \) everywhere if and only if \( Tf \neq 0 \) everywhere. Suppose otherwise, say \( f(x) = 0 \), and we have two cases: first, if \( f \) is constant on a neighbourhood of \( x \), this means that \( Z(f) \neq \emptyset \), and Theorem 3.1.12 implies that \( Z(Tf) \neq \emptyset \), and in particular \( Tf(y) = 0 \) for any \( y \in Z(Tf) \).

In the second case, if \( f \) is not constant on any neighbourhood of \( x \), we proceed in the same manner as in Theorem 3.4.13: first find a sequence of distinct point \( x_n \in X \) such that \( 0 < |f(x_n)| < n^{-2} \). Propositions 3.3.12 and 3.3.13(b) allows us to take a subsequence if necessary and construct \( g \in C(X) \) which coincides with \( nf \) on a neighbourhood of \( x_n \) for all \( n \). However, linearity and the \( \perp \)-preserving property imply that \( Tg = nTf \) on some open set as well (Theorem 3.1.12), which contradicts \( Tf \) being bounded because \( \sup_{y \in Y} |Tf(y)| > 0 \).
The result follows in this case from the Li–Wong Theorem (Theorem 3.4.14 or Corollary 3.4.15).

**Now let X and Y be arbitrary locally compact Hausdorff.** Given \( b \in C_c(X) \), set \( T_b : C(\text{supp}(b)) \to C(\text{supp}(Tb)) \) as \( T_b f = (Tf')|_{\text{supp}(Tb)} \), where \( f' \) is any element of \( C_c(X) \) extending \( f \). Note that \( T_b f \) does not depend on the choice of \( f' \), since, for all \( f', g' \in C_c(X) \),

\[
|f'|_{\text{supp}(b)} = |g'|_{\text{supp}(b)} \iff \sigma(b) \subseteq [f' = g'] \iff \sigma(b) \subseteq \sigma(f' - g') \iff \sigma(b) \cap \sigma(f' - g') = \emptyset \iff b \perp (f' - g'),
\]

and the last condition is preserved by \( T \) since it is an additive \( \perp \)-isomorphism (Theorem 3.1.12).

Since \( f \perp g \) if and only if \( |f'|_{\text{supp}(b)} \perp |g'|_{\text{supp}(b)} \) for all \( b \), the previous case allows us to obtain functions \( p^b \) and \( \phi^b \) such that \( Tf(y) = p^b(y) f(\phi^b(y)) \) for all \( y \in \text{supp}(Tb) \). Clearly, if \( b \subseteq b' \) then \( |p'|_{\text{supp}(b)} = p^b \) and \( |\phi'|_{\text{supp}(b)} = \phi^b \). Thus defining \( p \) and \( \phi \) as \( p(y) = p^b(y) \) and \( \phi(y) = \phi^b(y) \) where \( b \in C_c(X) \) is such that \( y \in \text{supp}(b) \) we obtain the desired maps. \( \square \)

### 3.4.6 Banach-Stone Theorem

We use the same notation as in the previous subsection. Given a locally compact Hausdorff space \( X \), endow \( C_c(X) \) with the supremum norm: \( \|f\|_\infty = \sup_{x \in X} |f(x)| \).

Recall that, by the Riesz-Markov-Kakutani Representation Theorem ([146, Theorem 2.14](#)), continuous linear functionals on \( C_c(X) \) correspond to (integration with respect to) regular Borel measures on \( X \). As a consequence, the extremal points \( T \) of the unit ball of the dual of \( C_c(X) \) have the form \( T(f) = \lambda f(x) \) for some \( x \in X \) and \( |\lambda| = 1 \).

Given \( f \in C_c(X) \), denote by \( N(f) \) the set of extremal points \( T \) in the unit ball of the dual space \( C_c(X)^*_T \) such that \( T(f) \neq 0 \). From the previous paragraph we obtain

\[
f \perp g \iff N(f) \cap N(g) = \emptyset, \tag{BS}
\]

and the Banach-Stone Theorem is an immediate consequence of Jarosz’ Theorem.

**Theorem 3.4.17** (Banach-Stone [104]). Let \( X \) and \( Y \) be locally compact Hausdorff spaces and let \( T : C_c(X) \to C_c(Y) \) be an isometric linear isomorphism. Then there exists a homeomorphism \( \phi : Y \to X \) and a continuous function \( p : Y \to \mathbb{S}^1 \) for which

\[
Tf(y) = p(y) f(\phi(y)) \quad \forall f \in C(X), \forall y \in Y.
\]
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3.5 New results

3.5.1 $L^1$-spaces of locally compact spaces

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be fixed. Given a topological space $X$, $C_c(X)$ will denote the space of $\mathbb{K}$-valued compactly supported continuous functions on $X$ (the support of $f \in C(X, \mathbb{K})$ is the usual one: $\text{supp}(f) = \{f \neq 0\}$.)

We will say that a Borel measure $\mu$ on a locally compact Hausdorff space $X$ is regular (\cite{155}) if

- $\mu(K) < \infty$ for all compact $K \subseteq X$ (i.e., we always assume regular measures are locally finite);

- For every Borel $E \subseteq X$,
  $$\mu(E) = \inf \{\mu(V) : E \subseteq V, V \text{ open}\};$$

- For every open $U \subseteq X$ with $\mu(U) < \infty$;
  $$\mu(U) = \sup \{\mu(K) : K \subseteq U, K \text{ compact}\}.$$

and recall that the support of $\mu$ is the set of points $x \in X$ whose neighbourhoods always have positive measure. We say that $\mu$ is fully supported (on $X$) is the support of $\mu$ coincides with $X$, i.e., if every nonempty open subset has positive measure.

Lemma 3.5.1. Let $X$ be a locally compact Hausdorff space and $\mu$ a fully supported Borel measure on $X$ such that every compact subset of $X$ has finite measure. If $\|\cdot\|_1$ denotes the corresponding $L^1$-norm, then for all $f, g \in C_c(X)$, $f \perp g$ if and only if

$$\|Af +Bg\|_1 = |A|\|f\|_1 + |B|\|g\|_1 \quad \forall A, B \in \mathbb{K}$$

Proof. If $f \perp g$ then $|Af +Bg| = |A||f| + |B||g|$, so the condition described above is immediate.

Suppose $f$ and $g$ are not disjoint, say $x \in X$ is such that $f(x) \neq 0$ and $g(x) \neq 0$, so up to nonzero multiples we may assume $f(x) = g(x) = 1$. In this case, there is an open set $U$, and thus of nonzero measure under $\mu$, such that $|f(y) - g(y)| < f(y) - 1/2$ for all $y \in U$, so

$$\|f - g\|_1 \leq \int_{X\setminus U} |f - g|d\mu + \int_U |f|d\mu - \frac{\mu(U)}{2} < \|f\|_1 + \|g\|_1$$

Theorem 3.5.2. Let $X$ and $Y$ be locally compact Hausdorff spaces with fully supported regular Borel measures $\mu_X$ and $\mu_Y$, and let $T : C_c(X) \to C_c(Y)$ be a linear
isomorphism which is isometric with respect to the $L^1$-norms. Then there exists a homeomorphism $\phi : Y \to X$ and a continuous function $p : Y \to S^1$ such that

$$Tf(y) = p(y) \frac{d\mu_X}{d(\phi_*\mu_Y)}(\phi(y))f(\phi(y))$$

for all $f \in C_c(X)$ and $y \in Y$.

**Proof.** By the previous lemma, $T$ is a $\perp$-isomorphism, so Jarosz’ Theorem (3.4.16) implies that there is a homeomorphism $\phi : Y \to X$ and a non-vanishing function $P : Y \to \mathbb{C}$ such that

$$T(f)(y) = P(y)f(\phi(y)),$$

for all $f \in C_c(X)$ and $y \in Y$.

Now using the fact that $T$ is isometric, we have, for every $f \in C_c(X)$,

$$\int_X |f|d\mu_X = \int_Y |Tf|d\mu_Y = \int_Y |P||f \circ \phi|d\mu_Y = \int_X |P \circ \phi^{-1}||f|d(\phi_*\mu_Y)$$

and this means that $|P \circ \phi^{-1}|$ is a continuous instance of the Radon-Nikodym derivative $d\mu_X/d(\phi_*\mu_Y)$. Since $p = P/|P| : Y \to S^1$ is continuous, we obtain the result. \hfill \square

### 3.5.2 Measured groupoid convolution algebras

In the next two results, we will focus on convolution algebras of topological groupoids. First, we will consider measured groupoids in the sense of Hahn. See [77, 78, 146, 147, 156]. Note that throughout this section we consider only regular measures. Initially, given a locally compact Hausdorff topological groupoid $\mathcal{G}$, we consider $C_c(\mathcal{G})$, the space of real or complex-valued, compactly supported, continuous functions on $\mathcal{G}$, simply as a vector space (with pointwise operations).

**Definition 3.5.3** ([146] Definition 2.2]). A (continuous) **left Haar system** for a locally compact Hausdorff topological groupoid $\mathcal{G}$ is a collection of regular Borel measures $\lambda = \{\lambda^x : x \in \mathcal{G}^{(0)}\}$ on $\mathcal{G}$ such that

(i) For each $x \in \mathcal{G}^{(0)}$, $\lambda^x$ has support contained in $\mathcal{G}^x$;

(ii) (left invariance) For each $a \in \mathcal{G}$, $\lambda^{(a)}(aE) = \lambda^{(a)}(E)$ for every compact $E \subseteq \mathcal{G}^{(a)}$;

(iii) (continuity) For each $f \in C_c(\mathcal{G})$, the map

$$\mathcal{G}^{(0)} \to \mathbb{C}, \quad x \mapsto \int f d\lambda^x$$

is continuous.
We will not make any distinction of whether each $\lambda^x$ is considered as a measure on $G$ or as a measure on $G^x$. We say that $\lambda$ is \textit{fully supported} if the support of $\lambda^x$ is all of $G^x$ for all $x \in G(0)$.

**Remark.** It follows from condition (iii) above that if $K$ is a compact subset of $G$, then $\sup_{x \in G(0)} \lambda^x(K) < \infty$.

**Example 3.5.4.** If $G$ is a locally compact Hausdorff topological group, a left Haar system on $G$ is the same as a left Haar measure on $G$.

**Example 3.5.5.** If $X$ is a locally compact Hausdorff topological space, considered as a unit groupoid, a left Haar system on $X$ consists of measures $p_x$ supported on $\{x\}$ for $x \in X$, and so can be regarded as a continuous non-negative function $p : X \to [0, \infty)$.

**Example 3.5.6.** Let $G$ be a locally compact Hausdorff topological group, acting continuously on a locally compact Hausdorff topological space $X$. Let $\lambda$ be a left Haar measure for $G$, and $f : X \to [0, \infty)$ a continuous $G$-invariant function (i.e., $f(gx) = f(x)$ for all $x \in X$ and $g \in G$). We define a Haar system $(\lambda, f)$ on $G \ltimes X$ by setting

$$(\lambda, f)^x(A) = f(x)\lambda \left( \{ g \in G : (g, g^{-1}(x)) \in A \} \right)$$

for all $x \in X$ (regarded as the unit space $(G \ltimes X)^{(0)}$) and $A \subseteq G \ltimes X$.

**Example 3.5.7.** In contrast with the case of groups, we can have non-trivial Haar systems without full support on groupoids. Let $X = \{a, b\}$ be a set with two points and $G = X \times X$ the coarsest equivalence relation on $X$. For all $x \in X$ and all $A \subseteq G$, set $\lambda^x(A) = \#(A \cap \{(x, a)\})$. Then $\lambda = \{\lambda^a, \lambda^b\}$ is a Haar system on $G$ such that $\lambda^x$ is non-trivial and not fully supported for all $x \in X = G(0)$.

Left invariance of $\lambda$ implies that for all $a \in G$ and $f \in C_c(G^x)$

$$\int f(s)d\lambda^x(s) = \int f(at)d\lambda^x(t)$$

and we can endow $C_c(G)$ with convolution product

$$(fg)(a) = \int f(s)g(s^{-1}a)\lambda^x(s) = \int f(at)g(t^{-1})\lambda^x(t)$$

which makes $C_c(G)$ an algebra.

**Definition 3.5.8.** Let $G$ be a locally compact Hausdorff topological groupoid with a left Haar system $\lambda$. Given a regular Borel measure $\mu$ on $G(0)$, the measure \textit{induced} by $\mu$ and $\lambda$ is the unique regular measure $(\lambda \otimes \mu)$ on $G$ which satisfies

$$(\lambda \otimes \mu)(E) = \int_{G(0)} \lambda^x(E)d\mu(x)$$
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for every compact $E \subseteq \mathcal{G}$. Note that this is well-defined because we are only considering continuous Haar systems, and the existence of $\lambda \otimes \mu$ is guaranteed by the Riesz-Markov-Kakutani Representation Theorem ([146, Theorem 2.14]).

If $\mu$ is fully supported on $\mathcal{G}^{(0)}$ and $\lambda$ is a Haar system for a locally compact Hausdorff groupoid $\mathcal{G}$, then $\lambda \otimes \mu$ is fully supported on $\mathcal{G}$: Indeed, if $A \subseteq \mathcal{G}$ is open (and has compact closure, without loss of generality) then $r(A)$ is open and nonempty in $\mathcal{G}^{(0)}$ ([146, Proposition 2.4]), and for all $x \in r(A)$, $A \cap G^x$ is open and nonempty in $G^x$. Thus

$$\lambda \otimes \mu(A) = \int_{r(A)} \lambda^x(A \cap G^x) d\mu(x) > 0$$

because we’re integrating a strictly positive function on a set of positive measure.

The following lemma will allow us to verify if certain maps are groupoid morphisms.

**Lemma 3.5.9.** Given a topological groupoid $\mathcal{G}$ with $\mathcal{G}^{(0)}$ Hausdorff and $a, b \in \mathcal{G}$, we have $s(a) = r(b)$ if and only if for every pair of neighbourhoods $U$ of $a$ and $V$ of $b$ the product $UV$ is nonempty.

**Proof.** From the second condition one can construct two nets $(a_i)_i$ and $(b_i)_i$ (over the same ordered set) converging to $a$ and $b$, respectively, such that $s(a_i) = r(b_i)$, and so $s(a) = r(b)$ because $\mathcal{G}^{(0)}$ is Hausdorff. The reverse implication is trivial.

**Lemma 3.5.10.** If $\lambda$ and $\mu$ are continuous Haar systems on a locally compact Hausdorff topological groupoid $\mathcal{G}$ such that the Radon-Nikodym derivatives $D^x = \frac{d\lambda^x}{d\mu^x}$ exist for all $x \in \mathcal{G}^{(0)}$, then $D$ is invariant in the sense that for all $a \in \mathcal{G}$ and $\mu^{s(a)}$-almost every $g \in \mathcal{G}^{s(a)}$, $D^r(a)(ag) = D^s(a)(g)$.

**Proof.** Using invariance of $\mu$ and $\lambda$, we have, for every $f \in C_c(\mathcal{G}^{s(a)})$,

$$\int f(t)D^s(a)(at)d\mu^{s(a)}(t) = \int f(a^{-1}s)D^s(a)(s)d\mu^{s(a)}(s) = \int f(a^{-1}s)d\lambda^{s(a)}(s) = \int f(t)d\lambda^{s(a)}(t),$$

which is the defining characteristic of the Radon-Nikodym derivative.

Now we prove that the convolution algebra $C_c(\mathcal{G})$ together with the $L^1$-norm coming from $\lambda \otimes \mu$, where $\lambda$ is a (fully supported) Haar system on $\mathcal{G}$ and $\mu$ is a fully supported measure on $\mathcal{G}^{(0)}$ completely determines the triple $(\mathcal{G}, \lambda, \mu)$, up to isomorphism (compare to [136]). We denote by $S^1$ the circle group (of complex numbers with absolute value 1 under multiplication).
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**Theorem 3.5.11.** Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids. For each $Z \in \{\mathcal{G}, \mathcal{H}\}$, let $\lambda_Z$ be a fully supported Haar system on $Z$, and $\mu_Z$ a fully supported regular Borel measure on $Z^{(0)}$.

If $T : C_c(\mathcal{G}) \to C_c(\mathcal{H})$ is an algebra isomorphism which is isometric with respect to the $L^1$-norms of $\lambda_Z \otimes \mu_Z$ ($Z \in \{\mathcal{G}, \mathcal{H}\}$), then there is a (unique) topological groupoid isomorphism $\phi : \mathcal{H} \to \mathcal{G}$ and a continuous morphism $p : \mathcal{H} \to S^1$ such that

$$Tf(h) = p(h) D(\phi(h)) f(\phi(h))$$

where $D$ is a continuous instance of the Radon-Nikodym derivative

$$D(a) = \frac{d\lambda_{\mathcal{G}}^\phi(a)}{d(\phi_* \lambda_{\mathcal{H}}^\phi)^{-1}(s(a))}(a)$$

and in this case, $\mu_\mathcal{G} = \phi_* \mu_\mathcal{H}$.

**Proof.** Again applying Lemma 3.5.1 and Jarosz’ Theorem (3.4.16), we can find a homeomorphism $\phi : \mathcal{H} \to \mathcal{G}$ and a continuous non-vanishing scalar function $P$ such that

$$Tf(h) = P(h) f(\phi(h)) \quad \text{for all } f \in C_c(\mathcal{G}) \text{ and } h \in \mathcal{H}.$$ 

Let us check that $\phi$ is a groupoid morphism. Suppose $(a, b) \in \mathcal{H}^{(2)}$, but such that $\phi(ab)$ is not equal to $\phi(a)\phi(b)$, which might a priori not even be defined. By continuity of the product on $\mathcal{G}$ or 3.5.9 (depending on whether $\phi(a)\phi(b)$ is defined or not), find neighbourhoods $U$ and $V$ of $a$ and $b$, respectively, such that $\phi(ab) \notin \phi(U)\phi(V)$.

Choose non-negative functions $f_a, f_b \in C_c(\mathcal{H})$ such that

$$\text{supp}(f_a) \subseteq U, \quad \text{supp}(f_b) \subseteq V \quad \text{and} \quad f_a(a) = f_b(b) = 1.$$ 

Then $ab \in \text{supp}(f_a f_b)$, because $\lambda_\mathcal{H}$ has full support and so

$$\phi(ab) \in \text{supp}(T^{-1}(f_a f_b)) = \text{supp}(T^{-1}(f_a)T^{-1}(f_b)) \subseteq \text{supp}(T^{-1}(f_a)) \text{supp}(T^{-1}(f_b)) = \phi(\text{supp}(f_a))\phi(\text{supp}(f_b)) \subseteq \phi(U)\phi(V),$$

a contradiction. Therefore $\phi$ is a morphism and a homeomorphism, thus a topological groupoid isomorphism (Proposition 1.1.21(d)).

If $f, g \in C_c(\mathcal{G})$ and $c \in \mathcal{H}$, then on one hand

$$T(fg)(c) = (TfTg)(c) = \int_{\mathcal{G}^{(c)}} Tf(t)Tg(t^{-1}c) \lambda_{\mathcal{H}}^\phi(t) \, dt$$

$$= \int_{\mathcal{H}^{(c)}} P(t)f(\phi(t))P(t^{-1}c)g(\phi(t^{-1}c))d\lambda_{\mathcal{H}}^\phi(t)$$

$$= \int_{\mathcal{G}^{(c)}} P(\phi^{-1}(s))f(s)P(\phi^{-1}(s)^{-1}c)g(s^{-1}\phi(c))d(\phi_* \lambda_{\mathcal{H}}^\phi)(s).$$
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and on the other

\[ T(fg)(c) = P(c)(fg)(\phi(c)) = P(c)\int_{G^{\phi(t)}} f(t)g(t^{-1}\phi(c))d\lambda_{G}^{\phi(t)}(t) \]

Now let \( f \in C_c(G^{\phi(t)}) \) be an arbitrary non-negative function. Define \( g \in C_c(G^{\phi(s)}) \) by \( g(t) = f(\phi(c)t^{-1}) \). Extending \( f \) and \( g \) arbitrarily to elements of \( C_c(G) \), the equality above becomes

\[ \int_{G^{\phi(t)}} P(\phi^{-1}(s))P(\phi^{-1}(s)^{-1}c)|f(s)|^2d\phi\lambda_{R}^{\phi(t)}(s) = P(c)\int_{G^{\phi(t)}} |f(s)|^2d\lambda_{G}^{\phi(t)}(s) \]

for all non-negative \( f \in C_c(G^{\phi(t)}) \) and all \( c \in H \). Define \( D : G \to \mathbb{C} \) (or simply \( \mathbb{R} \) in the real case) by

\[ D(s) = \frac{P(\phi^{-1}(s))P(\phi^{-1}(s)^{-1})}{P(\phi^{-1}(t))}. \]

Using Equation (3.5.1) with \( y = r(c) \) in place of \( c \), we obtain

\[ \int_{G^{\phi(y)}} D(s)|f(s)|^2d(\phi\lambda_{H}^{y}) = \int_{G^{\phi(y)}} |f(s)|^2d\lambda_{G}^{y}(s) \]

for all \( f \in C_c(G^{\phi(y)}) \), thus \( D \) is a continuous instance of the Radon-Nikodym derivative

\[ D(s) = \frac{d\lambda_{G}^{\phi(y)}}{d(\phi\lambda_{H}^{y})}(s) \]

and since all functions involved are continuous, and \( \lambda_{G}^{\phi(t)} \) has full support, the same equality is actually valid for all \( s \in G^{\phi(t)} \). Equivalently, for all \( c \in H \) and all \( t \in H^{\phi} \), \( P(t)P(t^{-1}) = D(\phi(t))P(c) \). Together with Lemma 3.5.10, this implies that the map \( p = P/(D \circ \phi) \) is a continuous groupoid morphism from \( H \) to the group of non-zero scalars.

Now let us verify that \( \mu_G \) and \( \phi_* \mu_H \) are equivalent measures. Suppose \( K \subseteq G^{(0)} \) is a compact set with positive measure \( \mu_G \). For every \( x \in K \), choose any nonempty open set \( A_x \) in \( G^x \) with compact closure (although \( A_x \) is not necessarily open in \( G \)). Letting \( E = \bigcup_{x \in K} A_x \), we obtain a compact subset of \( G \) with positive measure \( \lambda_G \otimes \mu_G \). Using regularity of the measures, we can extend the formula \( T(f) = P \circ f \circ \phi \) to characteristic functions of compact sets, and the resulting map will also be an isometry. In particular,

\[ (\lambda_H \otimes \mu_H)(\phi^{-1}(E)) = (\lambda_G \otimes \mu_G)(E) > 0 \]
so \( r(\phi^{-1}(E)) = \phi^{-1}(K) \) has positive measure \( \mu_H \). By inner regularity of the measures, we conclude that \( \mu_G \) is absolutely continuous with respect to \( \phi_* \mu_H \), and the reverse is similar.

As for the last part, let us denote by \( \| \cdot \|_Z \) the \( L^1 \)-norm with respect to \( \lambda_Z \otimes \mu_Z \) when \( Z \in \{G, H\} \). For all \( f \in C_c(G) \) we have

\[
\|Tf\|_H = \int_{\mathcal{H}^{(0)}} \left( \int_{\mathcal{H}^V} |Tf| d\lambda^Z_H \right) d\mu_H(y) = \int_{\mathcal{H}^{(0)}} \left( \int_{\mathcal{H}^V} D[p(\phi)] d\lambda^Z_H \right) d\mu_H(y)
\]

\[
= \int_{\mathcal{G}^{(0)}} \left( \int_{\mathcal{G}^V} |(p \circ \phi^{-1}) f| d\lambda^G_H \right) d(\phi_* \mu_H)(x)
\]

\[
= \int_{\mathcal{G}^{(0)}} \left( \int_{\mathcal{G}^V} |(p \circ \phi^{-1}) f| d\lambda^G_H \right) d(\phi_* \mu_H)(x)
\]

\[
= \int_{\mathcal{G}^{(0)}} \left( \int_{\mathcal{G}^V} [(p \circ \phi^{-1})(s)] \left( \frac{d((\phi_* \mu_H)(x))}{d\mu_G} \right) |f(s)| d\lambda^Z_G(s) \right) d\mu_G(x)
\]

\[
= \int_{\mathcal{G}} |p \circ \phi^{-1}| \left( \frac{d((\phi_* \mu_H))}{d\mu_G} \circ r \right) |f(s)| d(\lambda_G \otimes \mu_G)
\]

and since

\[
\|Tf\|_H = \|f\|_G = \int_{\mathcal{G}} |f| d(\lambda_G \otimes \mu_G)
\]

we obtain

\[
|p \circ \phi^{-1}| = \frac{d\mu_G}{d((\phi_* \mu_H))} \circ r \quad (\lambda_G \otimes \mu_G)\text{-a.e.}
\]

(3.5.2)

Since \( p \) is a morphism then \( p(\mathcal{H}^{(0)}) = \{1\} \), which, along with Equation (3.5.2) and continuity of \( p \), yields \( |p| = |p \circ r| = 1 \) on \( \mathcal{H} \). The same Equation (3.5.2) then also implies \( \mu_G = \phi_* \mu_H \).

As an immediate consequence, when considering only locally compact Hausdorff groups, we obtain:

**Corollary 3.5.12.** Let \( G \) and \( H \) be locally compact Hausdorff groups with Haar measures \( \lambda_G \) and \( \lambda_H \), respectively, and \( C_c(G) \) and \( C_c(H) \) the respective (real or complex) convolution algebras with \( L^1 \) metrics induced by \( \lambda_G \) and \( \lambda_H \). If \( T : C_c(G) \rightarrow C_c(H) \) is an isometric algebra isomorphism, then there are a topological group isomorphism \( \phi : H \rightarrow G \) and a continuous character \( p \) on \( H \) such that

\[
T(h) = c p(h) \phi(h)
\]

for all \( h \in H \), where \( c \) is the constant such that \( \lambda_G = c \lambda(\phi_* \lambda_H) \).

**Remark.** A stronger version of the corollary above, for \( L^1 \)-algebras of locally compact Hausdorff groups, was first proven by Wendel in [175]. Further generalizations of Wendel’s Theorem were proven in [166] and [167], and closely results in [94] and [172].
3.5.3 \((I, r)\)-Groupoid convolution algebras

In the next result we will again use the convolution algebras of topological groupoids, however now we will consider another norm, which was already defined in the work of Hahn ([78]) and played an important role in Renault’s work ([146]). Let \( \mathcal{G} \) be a locally compact étale Hausdorff groupoid, \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) and \( \theta = 0 \), and let \( \lambda \) be a Haar system for \( \mathcal{G} \). Again, we will consider the convolution algebra \( C_c(\mathcal{G}) = C_c(\mathcal{G}, \mathbb{K}) \) as defined in the previous subsection.

Every fully supported left Haar system on an étale groupoid is essentially the counting measure ([146] 2.7), in the sense that for all \( a \in \mathcal{G} \), \( \lambda^{t(a)}(\{a\}) > 0 \) and the map \( a \mapsto \lambda^{t(a)}(\{a\}) \) is constant on each set \( \mathcal{G}_a^\circ \), where \( x, y \in \mathcal{G}^{(0)} \): this follows by computing

\[
\lambda^x(\{x\}) = \lambda^{s(a)}(\{s(a)\}) = \lambda^{t(a)}(a \{s(a)\}) = \lambda^{t(a)}(\{a\}).
\]

We define the \((I, r)\)-norm on \( C_c(\mathcal{G}) \) as

\[
\|f\|_{I,r} = \sup_{x \in \mathcal{G}^{(0)}} \int |f|d\lambda^x
\]

Note that this norm is always finite, by the remark succeeding Definition 3.5.3

The unit space \( \mathcal{G}^{(0)} \) of \( \mathcal{G} \) is a closed unit subgroupoid, hence (trivially) étale, Hausdorff and locally compact itself. The convolution product on \( C_c(\mathcal{G}^{(0)}) \) coincides with the pointwise product, and the \((I, r)\)-norm is the uniform one:

\[
\|f\|_{I,r} = \|f\|_{\infty} = \sup_{x \in \mathcal{G}^{(0)}} |f(x)|, \quad \forall f \in C_c(\mathcal{G}^{(0)}).
\]

Moreover, \( \mathcal{G}^{(0)} \) is also open in \( \mathcal{G} \) (because \( \mathcal{G} \) is étale), so we can identify \( C_c(\mathcal{G}^{(0)}) \) with the subalgebra \( \{f \in C_c(\mathcal{G}^{(0)}): \text{supp}(f) \subseteq \mathcal{G}^{(0)} \} \) of \( C_c(\mathcal{G}) \).

**Definition 3.5.13.** The algebra \( C_c(\mathcal{G}^{(0)}) \), identified as a subalgebra of \( C_c(\mathcal{G}) \), is called the diagonal subalgebra of \( C_c(\mathcal{G}) \). If \( \mathcal{G} \) and \( \mathcal{H} \) are locally compact étale Hausdorff groupoids, an isomorphism \( T : C_c(\mathcal{G}) \to C_c(\mathcal{H}) \) is called diagonal-preserving if \( T(C_c(\mathcal{G}^{(0)})) = C_c(\mathcal{H}^{(0)}) \).

**Theorem 3.5.14.** Let \( \mathcal{G} \) and \( \mathcal{H} \) be locally compact Hausdorff étale groupoids with continuous fully supported left Haar systems \( \lambda_\mathcal{G} \) and \( \lambda_\mathcal{H} \), respectively, and \( T : C_c(\mathcal{G}) \to C_c(\mathcal{H}) \) a diagonal-preserving algebra isomorphism, isometric with respect to the \((I, r)\)-norms. Then there is a (unique) topological groupoid isomorphism \( \phi : \mathcal{H} \to \mathcal{G} \) and a continuous morphism \( p : \mathcal{H} \to S^1 \) such that

\[
Tf(h) = p(h)D(\phi(h))f(\phi(h))
\]

where \( D \) is a continuous instance of the Radon-Nikodym derivative

\[
D(a) = \frac{d\lambda_\mathcal{G}^{t(a)}}{d(\phi_\ast \lambda_\mathcal{H}^{-1}(t(a)))}(a).
\]
3. DISJOINT CONTINUOUS FUNCTIONS

Proof. By the Banach-Stone Theorem (3.4.17), there is a homeomorphism \( \phi : \mathcal{H}^{(0)} \rightarrow \mathcal{G}^{(0)} \) and a continuous function \( P : \mathcal{H}^{(0)} \rightarrow \mathbb{S}^1 \) such that \( Tf(y) = P(y)f(\phi(y)) \) for all \( f \in C_c(\mathcal{G}^{(0)}) \) and \( y \in \mathcal{H}^{(0)} \). Since \( T \) is multiplicative we obtain \( P = 1 \). (The same conclusion can be obtained in a similar manner by Milgram’s or Jarosz’ Theorem.)

For each \( x \in \mathcal{G}^{(0)} \), let \( \{ a^x_i : i \in I_x \} \) be a net of functions in \( C_c(\mathcal{G}^{(0)}) \) satisfying:

(i) \( 0 \leq a^x_i \leq 1 \), and \( a^x_i(x) = 1 \);

(ii) \( \bigcap_i \text{supp}(a^x_i) = \{ x \} \);

(iii) If \( j \geq i \) then \( [a^x_j \neq 0] \subseteq [a^x_i \neq 0] \).

Items (ii) and (iii), and compactness of each \( \text{supp}(a^x_i) \) imply that \( \{ [a^x_i \neq 0] : i \in I_x \} \) is a neighbourhood basis at \( x \). For \( y \in \mathcal{H}^{(0)} \), let \( a^y_i = T(a^x_i) = a^x_i \circ \phi \), so that the net \( \{ a^y_i : i \in I_x \} \) satisfies (i)-(ii) as well.

Continuity of \( \lambda_G \) implies that for all \( x \in \mathcal{G}^{(0)} \) and \( f \in C_c(\mathcal{G}) \), \( \lim_{i \in I_x} \| a^x_i f \|_{I_x} = \int |f(y)|d\lambda^x(y) \), and similarly on \( \mathcal{H} \).

Given \( f, g \in C_c(\mathcal{G}) \), we use Lemma 3.5.1 to obtain

\[
\int \mathcal{H}^y |Tf| d\lambda^y_H = \int \mathcal{H}^y |p(t)(D \circ \phi)|f \circ \phi| d\lambda^y_H = \int_{\mathcal{G}^{(0)}(y)} |p \circ \phi^{-1}| |f| d\lambda^y_G
\]

and the last condition is preserved by \( T \), so by Jarosz’ Theorem \( T \) is of the form \( Tf(\alpha) = \overline{P}(\alpha)f(\overline{\phi}(\alpha)) \) for a certain homeomorphism \( \phi : \mathcal{H} \rightarrow \mathcal{G} \) and a non-vanishing continuous scalar function \( P \). We can readily see that \( \overline{P} \) and \( \overline{\phi} \) are extensions of \( P \) and \( \phi \), respectively, so instead let us instead denote \( \overline{\phi} = \phi \) and \( \overline{P} = P \).

The proof that \( \phi \) is a groupoid isomorphism, and that \( P \) can be decomposed as \( P = (D \circ \phi)p \) for the (continuous) Radon-Nikodym derivative \( D \) and some continuous morphism \( p : \mathcal{H} \rightarrow \mathbb{C} \setminus \{ 0 \} \) is the same as in Theorem 3.5.11 but the verification that \( |p| = 1 \) is different.

Given \( y \in \mathcal{H}^{(0)} \) and \( f \in C_c(\mathcal{G}) \),

\[
\int_{\mathcal{H}^y} |Tf| d\lambda^y_H = \int_{\mathcal{H}^y} |p(t)(D \circ \phi)|f \circ \phi| d\lambda^y_H = \int_{\mathcal{G}^{(0)}(y)} |p \circ \phi^{-1}| |f| d\lambda^y_G
\]

Considering again the functions \( a^y_i(\phi(y)) \) and \( a^y_i \), and the fact that \( T \) is isometric we obtain

\[
\int_{\mathcal{G}^{(0)}(y)} |p \circ \phi^{-1}| |f| d\lambda^y_G = \lim_{i \in I_\phi(y)} \| a^y_i Tf \|_{I_x} = \lim_{i \in I_\phi(y)} \| a^y_i f \|_{I_x} = \int |f| d\lambda^y_G
\]

for all \( f \in C_c(\mathcal{G}) \), which implies that \( |p| = 1 \) \( \lambda^y_H \)-a.e. Since \( p \) is continuous and \( \lambda_H \) is fully supported, we conclude that \( |p| = 1 \) on \( \mathcal{H} \).
3.5.4 Groups of circle-valued functions

A natural question in C*-algebra theory is whether we can extend isomorphisms of unitary groups of C*-algebras to isomorphisms (or anti/conjugate-isomorphisms) of the whole C*-algebras. Dye proved in [38] that this is always possible for continuous von Neumann factors, however this is not true in the general C*-algebraic case, even in the commutative case\footnote{Recall that the unitary group of a commutative C*-algebra $C(X)$, where $X$ is compact Hausdorff, is $C(X, S^1)$.} (see Appendix A). Therefore we should consider isomorphisms between unitary groups which preserve more structure than just the product, such as an analogue to that of Theorem 3.4.14.

**Theorem 3.5.15.** Let $X$ and $Y$ be two Stone spaces (Definition 1.4.2). Suppose that $T : C(X, S^1) \to C(Y, S^1)$ is a group isomorphism such that $1 \in f(X) \iff 1 \in Tf(X)$. Then there exist a homeomorphism $\phi : Y \to X$, a finite subset $F' \subseteq Y$ whose points are isolated (in $Y$) and a continuous function $p : Y \setminus F \to \{\pm 1\}$ satisfying $Tf(y) = f(\phi(y))^{p(y)}$ for all $y \in Y \setminus F$.

In particular, if $X$ (and/or $Y$) do not have isolated points then $F = \emptyset$.

The following lemma, based on [116], will be crucial to the proof of the theorem.

**Lemma 3.5.16.** Suppose that $X$ is a Stone space. For every pair of continuous functions $f, g : X \to S^1$ and for every finite subset $F \subseteq X$ such that $f(F) \cup g(F)$ does not contain $1$, there exists $h \in C(X, S^1)$ such that
\[
h(x) \notin \{f(x), g(x)\} \text{ for all } x \text{ and } h(F) = \{1\}.
\]

**Proof.** For every point $y \in F$, choose a clopen set $U_y$ containing $y$ such that $f(U_y) \cup g(U_y)$ does not contain $1$. For every other point $x \in X' := X \setminus \bigcup_{y \in F} U_y$, there is a clopen set $U \subseteq X'$ such that $f(U) \cup g(U) \neq S^1$. Using compactness of $X'$ and taking complements and intersections if necessary we can find a clopen partition $U_1, \ldots, U_n$ of $X'$ such that $f(U_i) \cup g(U_i) \neq S^1$ for all $i$. Simply choose $z_i \in S^1 \setminus (f(U_i) \cup g(U_i))$ and define $h = z_i$ on $U_i$, and $h = 1$ on $\bigcup_{f \in F} U_f$. \hfill $\Box$

**Proof of Theorem 3.5.15.** For the notion of support we will use (Definition 3.1.1), we take $\theta = 1$, the constant function at $1$, so regularity of $C(X, S^1)$ is immediate.

Suppose that $f \perp g$ but that $Tf \not\perp Tg$. By Lemma 3.5.16 there exists $H \in C(Y, S^1)$ such that $H \neq Tf, Tg$ everywhere, but that $1 \in H(Y)$. Let $h = T^{-1}H$. Then $Tf^{-1}h$ and $Tg^{-1}h$ do not attain $1$, which implies that $f^{-1}h$ and $g^{-1}h$ do not attain $1$ as well. Thus
\[
h^{-1}(1) = X \cap h^{-1}(1) = (f^{-1}(1) \cup g^{-1}(1)) \cap h^{-1}(1) = (g^{-1}(1) \cup h^{-1}(1)) \cup (f^{-1}(1) \cap h^{-1}(1)) \subseteq (g^{-1}h)^{-1}(1) \cup (f^{-1}h)^{-1}(1) = \emptyset.
\]
But \((Th)^{-1}(1) = H^{-1}(1)\) is nonempty, contradicting the given property of \(T\).

Therefore \(f \perp g\) implies \(Tf \perp Tg\), and the same argument yields the opposite implication, so \(T\) is a \(\perp\)-isomorphism. Let \(\mathcal{A}(X)\) and \(\mathcal{A}(Y)\) be the subgroups of order-2 elements of \(C(X,S^{1})\) and \(C(Y,S^{1})\), respectively (i.e., the groups of continuous functions with values in \([-1,1]\)).

\(\mathcal{A}(X)\) and \(\mathcal{A}(Y)\) are also regular, since \(X\) and \(Y\) are zero-dimensional, and the restriction \(T|_{\mathcal{A}(X)} : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)\) is a \(\perp\)-isomorphism, because \(\perp\) and \(\perp\) coincide on \(\mathcal{A}(X)\) and \(\mathcal{A}(Y)\). Let \(\phi : Y \rightarrow X\) be the corresponding \(T|_{\mathcal{A}(X)}\)-homeomorphism.

Let \(h \in C(X,S^{1})\) be arbitrary. Since \(\sigma(h) = \bigcup_{a \in \mathcal{A}(X), a \leq h} \sigma(a)\) and \(T\) is a \(\perp\)-isomorphism, we obtain by Theorem 3.12 that, for all \(h \in C(X,S^{1})\),

\[
\phi(\sigma(T h)) = \bigcup_{a \in \mathcal{A}(X) \, a \leq h} \phi(\sigma(T a)) = \bigcup_{a \in \mathcal{A}(X) \, a \leq h} \sigma(a) = \sigma(h)
\]

Since \(\phi\) is a homeomorphism it preserves closures, from which it follows that \(T\) is also a \(\perp\)-isomorphism, and \(\phi\) is also the \(T\)-homeomorphism.

**Claim:** \(f(\phi(y)) = 1 \Longleftrightarrow Tf(y) = 1\).

Suppose \(f(\phi(y)) \neq 1\). Choose a function \(g \in C(X,S^{1})\) which coincides with \(f\) on a neighbourhood of \(\phi(y)\) and such that \(1 \notin g(X)\). Then \(1 \notin T g(Y)\) and since \(Tf\) coincides with \(Tg\) on a neighbourhood of \(y\) then \(Tf(\phi(y)) = Tg(\phi(y)) \neq 1\). The other direction is analogous, and thus we have proved the claim.

Therefore \(T\) is basic. Let \(\chi\) be the \(T\)-transform, so that each section \(\chi(y, \cdot)\) is an automorphism of the circle. If \(\chi(y, \cdot)\) is continuous then it has the form \(\chi(y, z) = z^{p(y)}\) where \(p(y) \in \{\pm 1\}\). Let us prove that for all except finitely many \(y \in Y\), the section \(\chi(y, \cdot)\) is continuous. The following argument is adapted from [1].

Let \(F = \{y \in Y : \chi(y, \cdot)\) is discontinuous\}, and suppose that \(F\) were infinite. By Proposition 3.12 there are countably infinitely many distinct points \(y_{n} \in F\) \((n \in \mathbb{N})\), such that no \(y_{n}\) lies in the closure of the other ones. We can choose a sequence \(z_{n} \rightarrow 1\) such that \(\chi(y_{n}, z_{n})\) lies in the second or third quadrant\(^8\). Define \(f(\phi(y_{n})) = z_{n}, f = 1\)

---

\(^8\) To see this: Let \(\arg : S^{1} \rightarrow (-\pi, \pi]\) be a function such that \(z = e^{i \arg(z)}\) for all \(z \in S^{1}\). Suppose \(\tau\) is a discontinuous automorphism of the circle. Then there is a neighbourhood \(V\) of 1 such that for every neighbourhood \(V\) of 1, there is a point \(z \in V\) for which \(\tau(z) \notin U\).

Take an integer \(k > 1\) such that if \(t \notin U\) then \(|\arg t| > \pi/k\). Let \(V\) be any neighbourhood of 1 and \(z \in V\) such that \(\tau(z) \notin U\), so \(|\arg(\tau(z))| > \pi/k\). Choose a positive integer \(m\) such that

\[
\frac{\pi}{m+1} \leq |\arg(\tau(z))| \leq \frac{\pi}{m}, \text{ so in particular } m < k.
\]

Since \(m \geq 1\),

\[
\frac{\pi}{2} \leq \frac{m}{m+1} \pi \leq m|\arg(\tau(z))| = |\arg(\tau(z))| \leq \pi,
\]

and the equality in the middle is allowed because \(m < k\). Thus \(z^{m}\) is an element of \(V^{m} \subseteq V^{k}\) such that \(\tau(z^{m})\) is in the second or third quadrant. Since the sets \(V^{k}\) (where \(k\) depends solely on \(\tau\) and \(U\)) form a neighbourhood basis at the identity we are done.
on the boundary of \( \{ \phi(y_n) : n \in \mathbb{N} \} \) and extend \( f \) continuously to all of \( X \). Let \( y \) be an accumulation point of \( \{ y_n \} \), so that in particular \( f(\phi(y)) = 1 \). Then

\[
Tf(y) = \chi(y, 1), \quad Tf(y_n) = \chi(y_n, z_n)
\]

But \( y \) is an accumulation point of the \( y_n \), and \( Tf(y_n) \) lies in the second or third quadrant while \( Tf(y) = 1 \), a contradiction to the continuity of \( Tf \).

Therefore \( F \) is finite, so now we show that it is open in order to conclude that its points are isolated in \( Y \). Let \( y \in Y \) and choose \( z_0 \in \mathbb{S}^1 \) of the form \( z_0 = e^{it} \) where \(-\pi/4 \leq t \leq \pi/4\), but such that \( \chi(y, z_0) \) is in the second or third quadrant, so in particular it is not \( z_0 \) nor \( z_0^{-1} \). Denote by \( z_0 \) the constant function at \( z_0 \), we that

\[
T(z_0)(y) = \chi(y, z_0) \neq z_0, z_0^{-1}.
\]

Since \( T(z_0) \) is continuous, there is a neighbourhood \( U \) of \( y \) such that \( \chi(x, z_0) \neq z_0, z_0^{-1} \) for all \( x \in U \), so \( x \in F \).

Therefore \( Y' = Y \setminus F \) is also compact, and we already constructed the function \( p : Y' \to \{ \pm 1 \} \) with the desired property. To see that \( p \) is continuous, denote by \( i \) the constant function \( x \mapsto i \) and note that

\[
p^{-1}(1) = \{ y \in Y' : \chi(y, i) = i \} = \{ y \in Y' : T(i)(y) = i \} = T(i)^{-1}(i) \cap Y'
\]

and similarly \( p^{-1}(-1) = T(i)^{-1}(-i) \cap Y' \), so these two sets, which are complementary in \( Y' \), are closed and hence clopen.

**Example 3.5.17.** As an easy example where the subset \( F \subseteq Y \) in the previous theorem is nonempty, let \( X = Y = \{ \ast \} \) be (equal) singletons, and let \( t : \mathbb{S}^1 \to \mathbb{S}^1 \) be a discontinuous automorphism of \( \mathbb{S}^1 \).

Consider map \( T : C(X, \mathbb{S}^1) \to C(Y, \mathbb{S}^1) \), \( T(f)(\ast) = t(f(\ast)) \) (in other words, \( T \) is the function obtained from \( t \) by identifying \( C(X, \mathbb{S}^1) \) and \( C(Y, \mathbb{S}^1) \) with \( \mathbb{S}^1 \)). Then \( T \) satisfies the hypotheses of the previous theorem but \( F = Y \).

We now endow \( C(X, \mathbb{S}^1) \) with the uniform metric:

\[
d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|
\]

(which is the metric coming from the \( C^* \)-algebra \( C(X, \mathbb{C}) \)).

**Theorem 3.5.18.** If \( X \) and \( Y \) are as above and \( T : C(X, \mathbb{S}^1) \to C(Y, \mathbb{S}^1) \) is an isometric isomorphism, then there is a homeomorphism \( \phi : Y \to X \) and a continuous function \( p : Y \to \{ \pm 1 \} \) such that \( Tf(y) = f(\phi(y))^p(y) \) for all \( y \in Y \).

**Proof.** We identify each \( \lambda \in \mathbb{S}^1 \) with the corresponding constant map on \( X \) or \( Y \). The constant function \(-1\) is characterized by the following two properties:
• \((-1)^2 = 1;\)

• If \(g^3 = 1\), then \(d_\infty(-1, g) \in \{1, 2\} \).

Thus \(T(-1) = -1\). A function \(f\) does not attain 1 if and only if \(d_\infty(-1, f) < 1\), so \(T\) preserves functions not attaining 1, and we apply Theorem 3.5.15 (or more precisely its proof) in order to obtain a homeomorphism \(\phi : Y \to X\), a function \(\chi : Y \times S^1 \to S^1\) and a continuous function \(p : Y' \to \{-1, 1\}\), where \(Y' = \{y \in Y : \chi(y, \cdot)\text{ is continuous}\}\), such that

\[
Tf(y) = \chi(y, f(\phi(y)) \quad \text{and} \quad \chi(y', t) = p(y')
\]

for all \(y \in Y\), \(y' \in Y'\) and \(f \in C(X, S^1)\). It remains only to prove that \(Y' = Y\), i.e., every section \(\chi(y, \cdot)\) is continuous.

If \(\lambda_i \to \lambda\) in \(S^1\) then we also have uniform convergence of the corresponding constant functions, so

\[
\chi(y, \lambda_i) = T(\lambda_i)y \to T(\lambda)y = \chi(y, \lambda)
\]

thus \(\chi(y, \cdot)\) is continuous for all \(y\). \(\square\)
Chapter 4

Partial actions of inverse semigroups

Partial actions of groups on C*-algebras (initially defined for \( \mathbb{Z} \) in \( \cite{14} \) and for general discrete groups in \( \cite{127} \)), are a powerful tool in the study of C*-algebras associated to dynamical systems (\( \cite{16, 26, 52, 65, 71, 74} \)). In \( \cite{35} \) Dokuchaev and Exel introduced partial group actions in a purely algebraic context, and although this algebraic theory is not yet as well-developed as its C*-algebraic counterpart, it has attracted interest from researchers in the area due to important classes of algebras, such as graph and ultragraph Leavitt path algebras, being described as (partial) crossed products (\( \cite{72, 73} \)).

Groupoids are also being extensively used in order to study and classify similar classes of algebras (\( \cite{30} \)), since every partial group action on a topological space induces a transformation groupoid (\( \cite{1} \)), and every étale groupoids can be obtained in this manner (\( \cite{19} \) Proposition 5.4). In fact, a categorical relationship between the representation theories of groupoids and of inverse semigroups is described in \( \cite{23} \).

In the C*-algebraic case, the algebras of transformation groupoids coincide with crossed products of associated partial actions (\( \cite{1} \)). However, in the algebraic case only partial results of this nature had been obtained (\( \cite{13, 34} \)).

In this chapter we will study partial actions of inverse semigroups, as defined in \( \cite{22} \). They are a common generalization of both partial actions of groups and actions of inverse semigroups. We will be mostly interested in partial actions on topological spaces and algebras, although the same theory can be extended to other classes of algebraic or geometric spaces. These allows us to construct groupoids of germs as in \( \cite{49} \) and crossed products associated to partial actions of inverse semigroups. These constructions generalize transformation groupoids and the usual crossed products by groups, respectively.

In this general setting, we prove that the Steinberg algebra of a Hausdorff groupoid of germs is (naturally) isomorphic to a crossed product algebra for partial actions of inverse semigroups (Theorem \( \ref{thm:steinberg_groupoid} \)). This is a generalization of the aforementioned results of \( \cite{13, 34} \), and also an algebraic analogue of Abadie’s result.
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Moreover, there are canonical constructions which allow us to build new actions from partial actions of inverse semigroups ([49, 22]), and so we study the isomorphism of the associated groupoids of germs and crossed product algebras. In particular, we obtain algebraic analogues of some of the results of [129].

Orbit equivalence and full groups for actions of \( \mathbb{Z} \) were initially studied by Giordano, Putnam and Skau ([66, 67, 68]), and by Li in [118, 117] for partial actions of discrete groups. The notion of continuous orbit equivalence can be naturally extended to partial actions of inverse semigroups. We also introduce and study a natural notion of freeness for partial inverse semigroup actions (which is more restrictive than the one in [53]). This notion corresponds to the associated groupoids of germs being topologically principal. We also prove that two ample, topologically free partial inverse semigroup actions are continuously orbit equivalent if and only if their corresponding groupoids of germs are isomorphic.

4.1 Partial morphisms and partial actions

Definition 4.1.1. A partial morphism from an inverse semigroup \( S \) to a (not necessarily inverse) semigroup \( T \) is a map \( \theta : S \to T \) satisfying, for all \( s \) and \( t \) in \( S \),

\[
\begin{align*}
(PM1) \ & \theta(s)\theta(s^*)\theta(s) = \theta(s); \\
(PM2) \ & \theta(s)\theta(t)\theta(t^*) = \theta(st)\theta(t^*); \\
(PM3) \ & \theta(s^*)\theta(s)\theta(t) = \theta(s^*)\theta(st).
\end{align*}
\]

The following results can be found in [22], with different proofs.

Proposition 4.1.2. If \( \theta : S \to T \) is a partial morphism from an inverse semigroup \( S \) to a semigroup \( T \), then

(a) For all \( s, t \in S \), \( \theta(s)\theta(t) = \theta(st)\theta(t^*)\theta(t) = \theta(s)\theta(s^*)\theta(st) \);

(b) \( \theta(E(S)) \subseteq E(T) \);

(c) If \( e \in E(S) \) and \( s \in S \), then \( \theta(e)\theta(s) = \theta(es) \) and \( \theta(s)\theta(e) = \theta(se) \);

(d) \( \theta \) is a morphism if and only if for all \( s \in S \), \( \theta(s^*s) = \theta(s^*)\theta(s) \).

Proof. (a) We have

\[
\theta(s)\theta(t) \overset{(PM1)}{=} \theta(s)\theta(t)\theta(t^*)\theta(t) \overset{(PM2)}{=} \theta(st)\theta(t^*)\theta(t)
\]

and the other equality is similar.
(b) If $e = e^2 \in S$, then
\[
\theta(e) (PM1) \theta(e) \theta(e^*) \theta(e) (PM2) \theta(e) \theta(e^* e) = \theta(e) \theta(e),
\]
so $\theta(e) \in E(T)$.

(b) If $s \in S$ and $e \in E(S)$ then
\[
\theta(e) \theta(s) (b) = \theta(e) \theta(s) (PM3) \theta(e) \theta(es) (a) = \theta(e^2 s) \theta(es) \theta(es) (PM1) = \theta(es)
\]

(c) If $\theta$ is a partial morphism satisfying the latter condition, then for all $s, t \in S$,
\[
\theta(s) \theta(t) (a) = \theta(st) \theta(st^*) \theta(t) = \theta(st) \theta(t^*) (a) = \theta(st) \theta((st)^*) \theta(st^* t) (PM1) = \theta(st).
\]

Partial morphisms between inverse semigroups can be determined in terms of the order structure (page 16).

**Proposition 4.1.3.** A map $\theta : S \to T$ between inverse semigroups is a partial morphism if and only if for all $s, t \in S$,

(i) $\theta(s^*) = \theta(s)^*$;

(ii) $\theta(s) \theta(t) \leq \theta(st)$;

(iii) $\theta(s) \leq \theta(t)$ whenever $s \leq t$.

**Proof.** First assume that $\theta$ is a partial morphism. Property (i) follows from (PM1); (ii) follows from (i) and 4.1.2(a); and (iii) is a consequence of 4.1.2(b) and (c).

In the other direction, suppose (i), (ii) and (iii) are satisfied. Then (PM1) is immediate from (i). As for (PM2), we have
\[
\theta(s) \theta(t) \theta(t^*) (ii) \leq \theta(st) \theta(t^*)
\]
and conversely
\[
\theta(st) \theta(t^*) (i) = \theta(st) \theta(t^*) \theta(t) \theta(t^*) (ii) \leq \theta(stt^*) \theta(t) \theta(t^*) \leq \theta(s) \theta(t) \theta(t^*) .
\]

The easiest way to construct partial morphisms which are not morphisms is by restriction.

**Example 4.1.4.** Let $\theta : S \to T$ be a morphism of inverse semigroups, and let $u \in E(T)$. Define $\theta^u : S \to T$ by $\theta^u(s) = u \theta(s) u$. Then $\theta^u$ is a partial morphism, called the restriction or compression of $\theta$ to $u$. 

Definition 4.1.5 ([22]). A partial action of an inverse semigroup $S$ on a set $X$ is a partial morphism $\theta : S \to I(X)$. If $\theta$ is a morphism we call it a global action.

If we need to make the domain and codomains of each map $\theta_s$ explicit, we will write $\theta = (\theta_s : X_s^s \to X_s)_{s \in S}$.

The main difference between a partial and a global action is how the product and order of the semigroup are preserved. According to Proposition 4.1.3, an equivalent way to define a partial action $\theta$ on a set $X$ is by providing a collection $\{\theta_s\}_{s \in S}$ of partial maps on $X$ which satisfies

- If $\theta_s(x)$ is defined then $\theta_{s^*}(\theta_s(x))$ is defined and equals $x$;
- If $\theta_t(\theta_s(x))$ is defined, then $\theta_{ts}(x)$ is also defined and equals $\theta_t(\theta_s(x))$;
- If $\theta_s(x)$ is defined and $s \leq t$, then $\theta_t(x)$ is also defined and equals $\theta_s(x)$;

Convention: Whenever $\theta : S \to I(X)$ is a partial action of an inverse semigroup $S$ on a set $X$, we will denote $\theta(s) = \theta_s$, and by $X_s^s$, or simply $X_s$, when there is no risk of confusion, the domain of $\theta_s$.

Example 4.1.6. Let $\theta : S \to I(X)$ be a global action (that is, a semigroup morphism). If $A \subseteq X$, we define $\theta_A : S \to I(A)$ by letting $(\theta_A)_s$ be the restriction of $\theta_s$ to $A \cap \theta_s^{-1}(X_s \cap A)$. Then $\theta_A$ is a partial action of $S$ on $A$.

Note that if $\id_A$ is the identity function of $A$, then $(\theta_A)_s = \id_A \theta_s \id_A$, so enforcing our identification that idempotents of $I(X)$ are the subsets of $X$, $\theta_A$ is precisely the compression of $\theta$ to $A$ (Example 4.1.4).

There are also slightly different ways of describing partial actions of inverse semigroups (see Proposition 3.3 and 3.4 [22]), but we will mostly use the description of 4.1.3.

Proposition 4.1.7. If $\theta : S \to I(X)$ is a partial action, then for all $s, t \in S$,

(a) if $s \leq t$ then $X_s \subseteq X_t$;
(b) $\theta_s(X_s^s \cap X_t) = X_s \cap X_{st}$;
(c) $X_s \cap X_t = X_{tt^*s} \cap X_t$;
(d) $X_s \subseteq X_{s^*s}$.

Moreover, $\theta$ is a global action if and only if $X_s^s = X_{s^*s}$ for all $s$.

Proof. (a) If $s \leq t$ then $s^* \leq t^*$, so by Proposition 4.1.3 $\theta_{s^*} \leq \theta_t$, and in particular the domain $X_s$ of $\theta_{s^*}$ is contained in the domain $X_t$ of $\theta_{t^*}$.
(b) For all \(s, t \in S\), the domain of \(\theta_t \circ \theta_s\) is contained in \(X_{s^*}\), and also \(\theta_t \circ \theta_s \leq \theta_{ts}\), so comparing their domains we obtain

\[
\theta_s^{-1}(X_s \cap X_t^*) \subseteq X_{s^*} \cap X_{(ts)^*}.
\]

Using \(\theta_s^{-1} = \theta_{s^*}\) and substituting \(s\) by \(s^*\) and \(t\) by \(t^*\) in Equation (4.1.1) yields

\[
\theta_s(X_{s^*} \cap X_t) \subseteq X_s \cap X_{st^*}.
\]

Now substituting \(t\) by \((st)^*\) in Equation (4.1.1), and applying \(\theta_s\) on both sides yields

\[
X_s \cap X_{st} \subseteq \theta_s(X_{s^*} \cap X_{(t^*s)^*}) \subseteq \theta_s(X_{s^*} \cap X_t),
\]

where the second inclusion follows from item (a).

(c) Using (b), we compute

\[
X_s \cap X_t = \theta_t(\theta_t^*(X_t \cap X_s)) = \theta_t(X_{t^*} \cap X_{t^*s}) = X_t \cap X_{tt^*s}.
\]

(d) Using (b),

\[
X_{s^*} = \theta_{s^*}(X_s \cap X_s) = X_{s^*} \cap X_{s^*s} \subseteq X_{s^*s}.
\]

Suppose now \(X_{s^*} = X_{s^*s}\). Then for all \(x \in X_{s^*s} = X_{s^*}\),

\[
\theta_{s^*s}(x) = x = \theta_{s^*}\theta_s(x),
\]

that is, \(\theta_{s^*s} \leq \theta_s \circ \theta_s \leq \theta_{s^*s}\), so \(\theta\) is global by (4.1.2)(d).

In the cases where \(X\) has some extra structure (topological and/or algebraic) we will be mostly interested in partial actions that preserve this structure to some degree, which is what we will consider in subsequent sections.

4.2 Topological partial actions and groupoid of germs

4.2.1 Topological partial actions

**Definition 4.2.1.** A (non-degenerate) **topological partial action** of an inverse semigroup \(S\) is a partial action \(\theta\) of \(S\) on a topological space \(X\) which satisfies:

(i) For all \(s \in S\), \(X_s\) is an open subset of \(X\) and \(\theta_s : X_{s^*} \to X_s\) is a homeomorphism;

(ii) \(X = \bigcup_{s \in S} X_s\).

Since \(X_s \subseteq X_{ss^*}\) for all \(s\), (ii) can be substituted with

(iii') \(X = \bigcup_{e \in E(S)} X_e\);
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and we end up with an equivalent description of topological partial actions. Condition (ii) is usually called non-degeneracy, and it is more suitable to assume this as a necessary condition on all the partial actions we consider. If only (i) is satisfied then we can always substitute $X$ by $\bigcup_{s \in S} X_s$ and obtain a non-degenerate topological partial action of $S$.

The next theorem shows that partial actions on locally compact Hausdorff spaces can always be lifted to actions on zero-dimensional spaces. This is an equivariant version of the well-known fact that every compact metrizable Hausdorff space is the continuous image of the Cantor set ([4]). A proof of an analogue result for groups actions is given in [64].

**Theorem 4.2.2.** Let $X$ be a locally compact Hausdorff space and $\theta$ a topological partial action of an inverse semigroup $S$ on $X$. Then there exist

- a zero-dimensional, locally compact Hausdorff space $\mathcal{E}$;
- a topological partial action $\alpha = (\alpha_s : \mathcal{E}_s \to \mathcal{E}_s)_{s \in S}$ of $S$ on $\mathcal{E}$; and
- a continuous surjection $\pi : \mathcal{E} \to X$;

such that $\pi^{-1}(X_s) \subseteq \mathcal{E}_s$ and for all $c \in \pi^{-1}(X_s)$

$$
\pi(\alpha_s(c)) = \theta_s(\pi(c)).
$$

This is usually stated as $\theta$ being a factor of $\alpha$.

**Proof.** Let $\mathcal{B}$ be the generalized Boolean algebra of regular open subsets of $X$ with compact closure. Let $\mathcal{E} = \mathcal{F}_P(\mathcal{B})$, the topological space of ultrafilters on $\mathcal{B}$ (as in non-commutative Stone duality). Recall that a basic open subset of $\mathcal{E}$ has the form

$$
[U] = \{F \in \mathcal{E} : U \in F\}
$$

for some $U \in \mathcal{B}$. Also recall that joins and meets in $\mathcal{B}$ are given by

$$
U \lor V = \text{int}(U \cup V), \quad U \land V = U \cap V.
$$

For every ultrafilter $F \in \mathcal{E}$, the intersection $\bigcap_{U \in F} U$ is nonempty, because $F$ satisfies the finite intersection property, and is actually a singleton because $F$ is prime (see 1.4.18). Thus we can define a map $\pi : \mathcal{E} \to X$ satisfying $\{\pi(F)\} = \bigcap_{U \in F} U$.

We first note the following property:

$$
\text{If } F \in \mathcal{E} \text{ and } \pi(F) \in U \in \mathcal{B}, \text{ then } U \in F. \tag{4.2.1}
$$

Indeed, suppose $\pi(F) \in U$. Let $V \in F$ be arbitrary. Then

$$
V = (V \land U) \lor (V \setminus U),
$$
where “−” is the usual set difference. Since \( \pi(F) \) does not belong to \( (V \setminus U) \) then \( V \setminus U \notin F \), so primeness implies \( V \land U \in F \) and therefore \( U \in F \) as well since \( F \) is a filter.

If \( U \in \mathcal{B} \), then Property (4.2.1) implies that \( \pi^{-1}(U) = [U] \), which is open in \( \mathcal{C} \). Since \( \mathcal{B} \) is a basis for \( X \) then \( \pi \) is continuous.

If \( x \in X \), then the family \( \{U \in \mathcal{B} : x \in U\} \) is a filter on \( \mathcal{B} \) and therefore is contained in some ultrafilter \( F \in \mathcal{C} \), which necessarily satisfies \( \pi(F) = x \) because

\[
\{\pi(F)\} = \bigcap_{U \in F} U \subseteq \bigcap_{x \in U \in \mathcal{B}} U = \{x\},
\]

and therefore \( \pi \) is surjective.

Define a partial action \( \tilde{\theta} \) of \( S \) on \( \mathcal{B} \) by setting, for all \( s \in S \),

- \( \mathcal{B}_s = \{U \in \mathcal{B} : U \subseteq X_s\} \); and
- \( \tilde{\theta}_s : \mathcal{B}_s^* \to \mathcal{B}_s, \tilde{\theta}_s(U) = \theta_s(U) \).

Note that each set \( \mathcal{B}_s \) is an order ideal of \( \mathcal{B} \), and that \( \tilde{\theta}_s : \mathcal{B}_s^* \to \mathcal{B}_s \) is an order isomorphism.

Given \( s \in S \), set

- \( \mathcal{C}_s = \{F \in \mathcal{C} : F \cap \mathcal{B}_s \neq \emptyset\} \); and
- \( \alpha_s(F) = \left\{U \in \mathcal{B} : \tilde{\theta}_s(V) \subseteq U \text{ for some } V \in F \cap \mathcal{B}_s^*\right\} \) for all \( F \in \mathcal{C}_s^* \).

Note that \( \alpha_s(F) \) is a proper filter on \( \mathcal{B} \) for all \( F \in \mathcal{C}_s^* \), so let us verify that it is prime. Suppose \( U_1 \lor U_2 \in \alpha_s(F) \). Then there is \( V \in F \cap \mathcal{B}_s^* \) such that \( \tilde{\theta}_s(V) \subseteq U_1 \lor U_2 \), i.e.,

\[
(V \land \tilde{\theta}_s(U_1)) \lor (V \land \tilde{\theta}_s(U_2)) = V \in F,
\]

hence primeness of \( F \) implies that \( V_i := V \land \tilde{\theta}_s(U_i) \in F \) for some \( i \in \{1, 2\} \). Since \( V_i \subseteq V \) and \( \mathcal{B}_s^* \) is an order ideal then \( V_i \in F \cap \mathcal{B}_s^* \), and moreover \( \tilde{\theta}_s(V_i) \subseteq U_i \), so \( U_i \in \alpha_s(F) \). This proves that \( \alpha_s(F) \) is prime.

Therefore we obtain a partial action \( \alpha = (\alpha_s : \mathcal{C}_s^* \to \mathcal{C}_s)_{s \in S} \). Given \( s \in S \),

\[
\mathcal{C}_s = \bigcup_{U \in \mathcal{B}_s} [U]
\]

is open in \( \mathcal{C} \). Let us verify that \( \alpha_s \) is continuous for all \( s \in S \). Suppose \( F \in \mathcal{C}_s \). Given \( s \in S \), \( F \in \mathcal{C} \) and \( U \in \mathcal{B} \),

\[
\alpha_s(F) \in [U] \iff \exists V \in \mathcal{B} \text{ such that } V \in F, V \in \mathcal{B}_s^* \text{ and } \tilde{\theta}_s(V) \subseteq U
\]
which means that

$$\alpha_s^{-1}([U]) = \bigcup \left\{ [V] : V \in B_s^* \text{ and } \tilde{\theta}_s(V) \subseteq U \right\}$$

an open subset of $\mathcal{C}$, thus $\alpha_s$ is continuous. This proves that $\alpha$ is a topological partial action of $S$ on $\mathcal{C}$.

Property [4.2.1] also implies $\pi^{-1}(X_s) \subseteq \mathcal{C}$. If $F \in \pi^{-1}(X_s)$,

$$\{\pi(\alpha_s(F))\} = \bigcap_{U \in F} \overline{\alpha_s(U)} = \theta_s \left( \bigcap_{U \in F} U \right) = \{\theta_s(\pi(F))\}$$

which is the final desired property for $\alpha$.

\[ \square \]

**Example 4.2.3.** Let $\mathcal{G}$ be an étale, locally compact Hausdorff groupoid, and denote by $B(\mathcal{G})$ the semigroup of open bissections of $\mathcal{G}$.

The *canonical action* of $B(\mathcal{G})$ on $\mathcal{G}^{(0)}$ is defined as $\tau = (\tau_U : s(U) \rightarrow r(U))_{U \in B(\mathcal{G})}$, where $\tau_U : s(U) \rightarrow r(U)$ is the homeomorphism $\tau_U = r \circ s|_U^{-1}$. In simpler terms, $\tau_U(s(a)) = r(a)$ whenever $a$ is an arrow in $\mathcal{G}$ with $s(a) \in s(U)$.

Let us prove that $\tau$ is a global action, that is, that $\tau_{UV} = \tau_U \circ \tau_V$ for any $U, V \in B(\mathcal{G})$: We have

$$\tau_{UV}^{-1}(s(V)) = \left\{ x \in s(V) : r(s|_V^{-1}(x)) \in s(U) \right\} = \left\{ x \in s(V) : x \in s|_V(r^{-1}(s(U))) \right\}$$

$$= s(V \cap r^{-1}(s(U))) = s(UV)$$

In other words, the domains of $\tau_{UV}$ and of $\tau_U \circ \tau_V$ are the same. For every $x \in s(UV)$, take $a \in U$ and $b \in V$ such that $x = s(ab)$. Then

$$\tau_{UV}(x) = r(ab) = r(a) = \tau_U(s(a)) = \tau_U(r(b)) = \tau_U(\tau_V(s(b))) = \tau_U(\tau_V(x)).$$

### 4.2.2 $\perp$-preserving partial actions

**Theorem 4.2.4.** Suppose $(X, \theta, \mathcal{A})$ is weakly regular ([3.1.5]) and $\alpha = (\alpha_s : I_s^* \rightarrow I_s)_{s \in S}$ is a partial action of an inverse semigroup $S$ on $\mathcal{A}$ such that

(i) $I_s$ is a $\perp$-ideal of $\mathcal{A}$;

(ii) $\alpha_s : I_{s^*} \rightarrow I_s$ is a $\perp$-isomorphism.

Then there exists a unique collection of homeomorphisms $\theta_s : U_{s^*} \rightarrow U_s$ ($s \in S$) such that

1. For all $s \in S$, $U_s$ is open in $X$ and $\theta_s : U_{s^*} \rightarrow U_s$ is a homeomorphism;

2. $I_s = I(U_s) = \{ f \in \mathcal{A} : \text{supp}(f) \subseteq U_s \};$
(3) for all \( f \in I_{s^*} \), \( \theta_s(\text{supp}(f)) = \text{supp}(\alpha_s(f)) \).

In this case, \( \theta = (\theta_s)_{s \in S} \) is a topological partial action of \( S \) on \( X \). Moreover, \( \alpha \) is a global action if and only if \( \theta \) is global.

**Proof.** As in Theorem 3.2.6 we associate to every \( \bot \)-ideal \( I \) of \( \mathcal{A} \) the open set

\[
U(I) = \bigcup_{f \in I} \sigma(f) = \bigcup_{f \in I} \text{supp}(f),
\]

which yields an order isomorphism \( I \mapsto U(I) \) with inverse

\[
U \mapsto I(U) = \{ f \in \mathcal{A} : \text{supp}(f) \subseteq U \}.
\]

For every \( s \in S \), let \( U_s = U(I_s) \), so that \( I_s = I(U_s) \) is the \( \bot \)-ideal associated to \( U_s \). We identify \( I_s \) with the class

\[
\mathcal{A}(U_s) = \{ f|_{U_s} : \text{supp}(f) \subseteq U_s \}
\]

via the map \( I_s \to \mathcal{A}(U_s) \), \( f \mapsto f|_{U_s} \), which preserves \( \bot \). Moreover, \((U_s, \theta|_{U_s}, \mathcal{A}(U_s))\) is weakly regular, so \( \alpha_s \) can be seen as a \( \bot \)-isomorphism

\[
\alpha_s : \mathcal{A}(U_{s^*}) \simeq I_{s^*} \to I_s \simeq \mathcal{A}(U_s)
\]

and by Theorem 3.1.13 there exists a homeomorphism \( \theta_{s^*} : U_s \to U_{s^*} \) satisfying, for all \( f \in I_{s^*} \),

\[
\theta_{s^*}(\text{supp} \alpha_s(f)) = \text{supp}(f)
\]

Switching \( s \) by \( s^* \), and substituting \( f \) by \( \alpha_s(f) \) we obtain properties (1)-(3) for \( \theta \).

To verify that \( \theta \) is a partial action, we will need the following: for all \( s,t \in S \) and \( f \in \mathcal{A} \),

\[
\text{supp}(f) \subseteq U(\alpha_t^{-1}(I_{s^*} \cap I_t)) \iff f \in \alpha_t^{-1}(I_{s^*} \cap I_t)
\]

\[
\iff f \in I_{t^*} \quad \text{and} \quad \alpha_t(f) \in I_{s^*}
\]

\[
\iff \text{supp}(f) \subseteq U_{t^*} \quad \text{and} \quad \theta_t(\text{supp}(f)) = \text{supp}(\alpha_t(f)) \subseteq U_{s^*}
\]

\[
\iff \text{supp}(f) \subseteq \theta_t^{-1}(U_{s^*} \cap U_t),
\]

and therefore

\[
U(\alpha_t^{-1}(I_{s^*} \cap I_t)) = \theta_t^{-1}(U_{s^*} \cap U_t). \tag{4.2.2}
\]

It remains only to prove that \( \theta \) is a partial action. For this let us verify that the conditions of Definition 4.1.1 are satisfied.
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(i) Given \( s \in S \), the map \( \theta_s \circ \theta_s : U_s \to U_s \) is a homeomorphism, and for all \( f \in \mathcal{A}(U_s) \),
\[
\theta_s(\theta_s(supp(f))) = \theta_s(supp(\alpha_s(f))) = supp(\alpha_s \alpha_s(f)) = supp(f)
\]
so weak regularity of \( \mathcal{A}(U_s) \) implies that \( \theta_s \circ \theta_s = id_{U_s} \). This is enough to conclude that \( (\theta_s) = (\theta_s)^* \) (because these functions have opposite domains and codomains).

(ii) Let \( s, t \in S \). Since \( \alpha_{t}^{-1}(I_s \cap I_t) \subseteq I_{(st)^*} \), and by Equation (4.2.2) we have
\[
\theta_{t}^{-1}(U_s \cap U_t) \subseteq U(I_{(st)^*}) = U_{(st)^*},
\]
and by property (2) we obtain that, whenever \( supp(f) \subseteq \theta_{t}^{-1}(U_t \cap U_s) \),
\[
\theta_s(\theta_t(supp(f))) = \theta_{st}(supp(f)),
\]
so weak regularity again implies that \( \theta_s \theta_t \leq \theta_{st} \).

(iii) Suppose \( s \leq t \) in \( S \). Then \( I_s \subseteq I_t \) and since the map \( I \mapsto U(I) \) is an order isomorphism (Theorem 3.2.6(c)), \( U_s \subseteq U_t \). For all \( f \in I_s \),
\[
\theta_s(supp(f)) = supp(\alpha_s(f)) = supp(\alpha_t(f)) = \theta_t(supp(f))
\]
so weak regularity once again implies that \( \theta_s(x) = \theta_t(x) \) for all \( x \in U_s \), so \( \theta_s \leq \theta_t \).

Now note that \( \alpha \) is global if and only if for all \( s, t \), \( I_{(st)^*} \subseteq \alpha_{t}^{-1}(I_s \cap I_t) \). By Equation (4.2.2), this is equivalent to
\[
U_{(st)^*} \subseteq \theta_{t}^{-1}(U_s \cap U_t)
\]
which happens if and only if \( \theta \) is a global action. \( \square \)

4.2.3 Groupoid of germs

Groupoids of germs were already considered by Paterson ([135]) for localizations of inverse semigroups, and for canonical actions of pseudogroups by Renault ([148]). In [49], Exel defined groupoids of germs for arbitrary actions of inverse semigroups on topological spaces in a similar, albeit more general, manner than both previous definitions of groupoids of germs.

The objective of this section is to construct a groupoid of germs associated to any partial action of an inverse semigroup in a way that generalizes both groupoids of germs of inverse semigroup actions, and transformation groupoids of partial group actions.
Let $\theta$ be a topological partial action of an inverse semigroup $S$ on a topological space $X$. We denote by $S \ast_\theta X = S \ast X$ the subset of $S \times X$ given by

$$S \ast X = \{(s, x) \in S \times X : x \in X_s^\ast\}.$$  

and define the following equivalence relation $\sim$ on $S \ast X$: for every $(s, x)$ and $(t, y)$ in $S \ast X$

$$(s, x) \sim (t, y) \iff x = y \text{ and } \exists u \in S \text{ such that } u \leq s, t \text{ and } x \in X_{u^\ast}.$$  

(4.2.3)

We say that the equivalence class of $(s, x)$ is the germ of $s$ at $x$, and we denote it by $[s, x]$.

**Remark 4.2.5.** Notice that if $(s, x), (t, y) \in S \ast X$ then

$$(s, x) \sim (t, y) \iff x = y \text{ and } \exists e \in E(S) \text{ such that } x \in X_e, \text{ and } se = te.$$  

(4.2.4)

Indeed, if $u$ satisfies the condition in Equation (4.2.3), take $e = u^\ast u$. Then $x \in X_u \subseteq \{x \mid s,x \} \ast u^\ast u$ and $e$ satisfies the condition in Equation (4.2.4).

On the other hand, if $e$ satisfies (4.2.4), letting $u = se = te$ yields the desired properties of (4.2.3).

**Remark 4.2.6.** If $u \leq s$ in $S$ and $x \in X_{u^\ast}$, then $x \in X_{s^\ast}$ and $[s, x] = [u, x]$.

**Lemma 4.2.7.** Suppose $(s, x) \in S \ast X$. Then

(a) $(s^\ast, \theta_s(x)) \in S \ast X$;

(b) If $(t, y) \in S \ast X$ and $\theta_t(y) = x$, then $(st, y) \in S \ast X$.

**Proof.** (a) Since $\theta_s(x) \in X_s = X_{(s^\ast)^\ast}$ then $(s^\ast, \theta_s(x)) \in S \ast X$.

(b) By assumption, $y = \theta_t(x) \in \theta_t(X_t \cap X_s^\ast) \subseteq X_{(st)^\ast}$. 

**Lemma 4.2.8.** Let $(s_1, x), (s_2, x), (t_1, y), (t_2, y) \in S \ast X$ with $[s_1, x] = [s_2, x], [t_1, y] = [t_2, y]$ and $\theta_{t_1}(y) = x$. Then

(a) $\theta_{s_1}(x) = \theta_{s_2}(x)$ and $\theta_{t_2}(y) = \theta_{t_1}(y) = x$;

(b) $[s_1t_2, y] = [s_2t_2, y]$.

**Proof.** Take $u \leq s_1, s_2$ and $v \leq t_1, t_2$ with $x \in X_{u^\ast}$ and $y \in X_{v^\ast}$.

(a) Since $u \leq s_1, s_2$ and $x \in X_{u^\ast}$ then by Remark 4.2.6

$$\theta_{s_1}(x) = \theta_u(x) = \theta_{s_2}(x)$$

and similarly $\theta_{t_2}(y) = \theta_v(y) = \theta_{t_1}(y) = x$. 


(b) We have \(uv \leq s_1t_1, s_2t_2\), and since \(\theta_{v}(y) = \theta_{t_1}(y) = x \in X_{u^{*}}\), then
\[
y = \theta_{v^{*}}(x) \in \theta_{v^{*}}(X_{v} \cap X_{u^{*}}) \subseteq X_{(uv)^{*}},
\]
which proves that \((s_1t_1, y) \sim (s_2t_2, y)\).

\[\]

**Definition 4.2.9.** If \(\theta\) is a topological action of an inverse semigroup \(S\) on a topological space \(X\), we let
\[
S \ltimes_{\theta} X = \{[s, x] : s \in S, x \in X_s^{*}\} = (S \ast X)/\sim
\]
be the set of germs. We call \(S \ltimes_{\theta} X\) (or \(S \ltimes X\) for short) the groupoid of germs of \(\theta\).

To describe the groupoid structure of \(S \ltimes_{\theta} X\), we define the set \((S \ltimes X)^{(2)}\) of composable pairs as
\[
(S \ltimes X)^{(2)} = \{([s, x], [t, y]) : x = \theta_{t}(y)\},
\]
(note that \(\theta_{t}(y)\) depends only on the class \([t, y]\), by Lemma 4.2.8(a)). Given a composable pair \(([s, x], [t, y]) \in (S \ltimes X)^{(2)}\), define the product as
\[
[s, x][t, y] = [st, y],
\]
which is well-defined by Lemma 4.2.8(b), and it is routine to check that this operation defines a groupoid structure on \(S \ltimes X\). The inverse of \([s, x] \in S \ltimes_{\theta} X\) is
\[
[s, x]^{-1} = [s^{*}, \theta_{s}(x)]
\]
and the unit space of \(S \ltimes X\) is \(\{[e, x] : e \in E(S), x \in X_s\}\). We would now like to endow \(S \ltimes X\) with an appropriate topology.

**Lemma 4.2.10.** For every \(s \in S\) and \(U \subseteq X_{s^{*}}\), let
\[
[s, U] = \{[s, x] \in S \ltimes X : x \in U\}.
\]
The collection \(B_{\text{germ}}\) of all sets of the form \([s, U]\), where \(s \in S\) and \(U\) is an open subset of \(X_{s^{*}}\), is a basis for a topology on \(S \ltimes X\).

**Proof.** Using the definition of germs (Equation 4.2.3), we obtain
\[
[s, U] \cap [t, V] = \bigcup_{u \leq s \leq t} [u, U \cap V \cap X_{u^{*}}]
\]
whenever \(s, t \in S\), \(U \subseteq X_{s^{*}}\) and \(V \subseteq X_{t^{*}}\), hence the desired result. \(\square\)
Proposition 4.2.11. $S \ltimes X$ is a topological groupoid with the topology induced by the basis $\mathcal{B}_{\text{germ}}$.

Proof. Let $m : (S \ltimes X)^{(2)}$ be the product map, and suppose $u \in S, U \subseteq X_{v^*}$. Let us prove that

$$m^{-1}([u, U]) = \bigcup \{ ([s, X_{s^*}] \times [t, U \cap X_{t^*}] \cap (S \ltimes X)^{(2)} : s, t \in S \text{ and } st \leq u \}$$

(4.2.5)

Indeed the inclusion “⊇” is immediate from the definition of the product, so assume $([s, y], [t, x]) \in m^{-1}([u, U])$. This means that there exists $v \leq st, u$ such that $x \in X_{v^*}$ and $[st, x] = [v, x] = [u, x]$, and thus

$$([s, y], [t, x]) = ([s, y], [tv^*v, x])$$

which belongs to the set in the right-hand side of Equation (4.2.5). This proves the reverse inclusion and it follows that $m$ is continuous.

Now, note that whenever $s \in S$ and $U \subseteq X_{s^*}$,

$$[s, U]^{-1} = [s^*, \theta_s(U)]$$

so continuity of the inversion follows immediately. □

From now on, we always consider $S \ltimes X$ with the topology induced by the basis $\mathcal{B}_{\text{germ}}$ as in Lemma 4.2.10.

Proposition 4.2.12. The groupoid $S \ltimes X$ is étale, and each basic open set $[s, U]$, where $s \in S$ and $U \subseteq X_{s^*}$ is open, is a bisection of $S \ltimes X$.

Proof. Given $s \in S$ and $U \subseteq X_{s^*}$ an open set, the source map on $[s, U]$ is given by

$$s : [s, U] \to [s^*s, U], \quad [s, x] \mapsto [s^*s, x]$$

and injectivity is immediate by the definition of germs in Equation (4.2.3).

In particular, $s([s, U]) = [s^*s, U]$ is open in $(S \ltimes X)^{(0)}$. Any basic open subset of $[s, U]$ is of the form $[s, V]$ where $V \subseteq U$, so $s([s, V]) = [s^*s, V]$ is open in $(S \ltimes X)^{(0)}$. Therefore the source map is a local homeomorphism and $S \ltimes X$ is étale. The verification that the range map is a homeomorphism from $[s, U]$ to its image is similar. □

The unit space $(S \ltimes X)^{(0)}$ of $S \ltimes X$ can be naturally identified with $X$ under the correspondence

$$\phi : (S \ltimes X)^{(0)} \to X, \quad \phi([e, x]) = x \quad (e \in E(S))$$

(4.2.6)
To check that this map is injective, just note that if \( \phi([e, x]) = \phi([f, y]) \) \( (e, f \in E(S)) \), then \( x = y \) and \( x \in X_e \cap X_f \subseteq X_{ef} \), so \([e, x] = [ef, x] = [f, x]\) by the definition of germs. A basic open set of \((S \ltimes X)^{(0)}\) has the form \([e, U]\) for some \(e \in E(S)\) and \(U \subseteq X_e\) open, and
\[
\phi([e, U]) = U
\]
so \( \phi \) takes basic open sets of \((S \ltimes X)^{(0)}\) to basic open sets of \(X\), and is therefore a homeomorphism.

Since the source and range maps of \(S \ltimes X\) are given by \(s[s, x] = [s^*s, x]\) and \(t[s, x] = [ss^*, \theta_s(x)]\), then enforcing this identification we will write
\[
s[s, x] = x \quad \text{and} \quad t[s, x] = \theta_s(x).
\]

Moreover, if \(B\) is a basis for the topology of \(X\), then a basis for \(S \ltimes X\) consists of those sets of the form \([s, U]\) with \(U \in B\). Hence, if \(X\) is totally disconnected then the collection of sets of the form \([s, U]\) with \(U\) compact-open subset of \(X\), is a basis for \(S \ltimes X\).

**Corollary 4.2.13.** If \(X\) is a zero-dimensional, locally compact Hausdorff space then \(S \ltimes X\) is an ample groupoid.

**Example 4.2.14.** Following Paterson ([135]), a localization consists of an action \(\theta\) of an inverse semigroup \(S\) on a topological space \(X\) such that \(\{X_s : s \in S\}\) is a basis for the topology of \(S\). The groupoid of germs in the sense of Paterson coincides with the definition above of groupoids of germs.

**Example 4.2.15.** Let \(X\) be a locally compact Hausdorff space. The canonical action of \(\mathcal{I}(X)\) on \(X\) is the action \(\tau\) given by \(\tau_\phi = \phi\) for all \(\phi \in \mathcal{I}(X)\). A pseudogroup on \(X\) is a sub-inverse semigroup of \(\mathcal{I}(X)\) whose elements are homeomorphisms between open subsets of \(X\).

Let \(B\) be a basis for the topology of \(X\), and for each \(B \in B\) consider its identity function \(\text{id}_B : B \to B\).

Given a pseudogroup \(\mathcal{G}\) on \(X\), let \(\mathcal{GB}\) be the sub-inverse semigroup of \(\mathcal{I}(X)\) generated by \(\mathcal{G} \cup \{\text{id}_B : B \in B\}\), which is again a pseudogroup on \(X\), and in fact the canonical action of \(\mathcal{GB}\) on \(X\) is a localization.

The groupoid of germs of \(\mathcal{G}\) in the sense of Renault ([148]) coincides with the groupoid of germs \(\mathcal{GB} \ltimes X\) defined above.

The following are natural and well-known examples of constructions which are possible with groupoids of germs (and already appear in some form in [161]).

**Example 4.2.16 (Transformation groupoid).** In the case that \(S\) is a discrete group, the equivalence relation on \(S \ltimes X\) is trivial and the topology is the product topology, that is, \(S \ltimes X\) is the transformation groupoid, and in particular it is Hausdorff if and only if \(X\) is Hausdorff.
Example 4.2.17 (Maximal group image). An easy example is the case when \( X \) is a singleton set on which \( S \) acts trivially, that is, \( \theta_s \) is simply the identity on \( X \) for all \( s \in S \). It is then straightforward to see that \( S \ltimes X \) is the maximal group image \( G(S) \) of \( S \) (see \[133\] or Section 4.5 for the definition of \( G(S) \)): two elements of \( S \cong S \ast X \) have the same germ if and only if they have a common lower bound, which is how we define \( G(S) \).

Example 4.2.18 (Restricted product groupoid). Another example is the case \( X = E(S) \) with the discrete topology, and \( \theta \) is the Munn representation of \( S \) \([132]\): \( X_s = \{ e \in E(S) : e \leq ss^* \} \) and \( \theta_s(e) = ses^* \) for all \( e \in X_s \).

Now from \( S \) we can construct the restricted product groupoid \((S, \cdot)\) \([108]\), which is the same as \( S \) but the product \( s \cdot t = st \) is defined only when \( ss^* = tt^* \).

Then \( S \ltimes E(S) \) is a discrete groupoid, and the map

\[ S \ltimes E(S) \to (S, \cdot), \quad [s, e] \mapsto se \]

is an isomorphism of (discrete) groupoids with inverse \( s \mapsto [s, s^*s] \).

The following example is based on \[161\] Example 5.18 (compare with Example 1.4.13).

Example 4.2.19. Let \( S = \mathbb{N} \sqcup \{ \infty, z \} \) be a disjoint union of the lattice \( \mathbb{N} \) and a two-set element \( \{ \infty, z \} \), with product given, for \( m, n \in \mathbb{N} \),

\[
\begin{align*}
nm &= \min(n, m), & n\infty &= \infty n = n z = zn = n, \\
z\infty &= \infty z = z & z^2 &= \infty^2 = \infty.
\end{align*}
\]

In other words, \( S \) is the semigroup obtained by adjoining the lattice \( \mathbb{N} \) to the group or order 2 \( \{ z, \infty \} \), in a way that every element of \( \mathbb{N} \) is smaller than \( z \) and \( \infty \).

Let \( X = E(S) = \mathbb{N} \cup \{ \infty \} \), seen as the one-point compactification of the natural numbers, and \( \theta \) the Munn representation of \( S \), so that \( S \ltimes X = (S, \cdot) \), however with the topology whose open sets are either cofinite or contained in \( \mathbb{N} \). In particular, \( S \ltimes X \) is not Hausdorff.

Example 4.2.20 \([49\) Proposition 5.4]). Let \( \mathcal{G} \) be an \( \acute{e}tale \) groupoid, \( \tau \) the canonical action of the semigroup of open bisections \( B(\mathcal{G}) \) on \( \mathcal{G}^{(0)} \) (Example 4.2.3), and \( S \) any sub-inverse semigroup of \( B(\mathcal{G}) \) which covers \( \mathcal{G} \) (that is, \( \mathcal{G} = \bigcup_{A \in S} A \)), and which is closed under intersections. Then the map \( S \ltimes \mathcal{G}^{(0)} \to \mathcal{G} \), \([A, x] \mapsto s |^1_A(x)\), is an isomorphism of topological groupoids.

In particular, if \( \mathcal{G} \) is an ample Hausdorff groupoid, then the groupoid of germs \( KB(\mathcal{G}) \ltimes \mathcal{G}^{(0)} \) of the (restriction of the) canonical action on \( KB(\mathcal{G}) \) is isomorphic to the groupoid \( \mathcal{G} \).
We will be mostly interested in Hausdorff groupoids, and in particular conditions on inverse semigroups which guarantee that groupoids of germs are Hausdorff.

**Definition 4.2.21.** A poset \((L, \leq)\) is a \((\wedge, \cdot)\) weak semilattice if for all \(s, t \in L\) there exists a finite subset \(F \subseteq L\) such that

\[
\{x \in L : x \leq s \text{ and } x \leq t\} = \bigcup_{f \in F} \{x \in L : x \leq f\}
\]

**Example 4.2.22.** If \(G\) is an ample Hausdorff groupoid then \(\text{KB}(G)\) is a semilattice, and \(U \wedge V = U \cap V\).

The following relation between inverse semigroups which are weak semilattices and the topology of its groupoids of germs can be proven just as in [161, Theorem 5.17].

**Proposition 4.2.23.** [161, Theorem 5.17] An inverse semigroup \(S\) is a weak semilattice if and only if for any topological partial action \(\theta : S \to \mathcal{I}(X)\) such that \(X_s\) is clopen for all \(s \in S\), the groupoid of germs \(S \ltimes X\) is Hausdorff.

In particular, if \(S\) is a weak semilattice and \(X\) is totally disconnected, then the groupoid of germs \(S \ltimes X\) is an ample Hausdorff groupoid.

**Remark.** The hypothesis that the domains of the partial action are clopen is necessary. For example, even if \(G\) is an ample non-Hausdorff groupoid then \(B(G)\) is still a semilattice, however, as in Example 4.2.20, the groupoid of germs \(B(G) \ltimes G^{(0)} \simeq G\) is not Hausdorff.

A common example of semigroups which are weak semilattices are the \(E\)-unitary and \(E^*\)-unitary semigroups, which will be described in Section 4.5. See Example 4.5.4.

### 4.3 Algebraic partial actions and crossed products

**Definition 4.3.1** ([35]). If \(R\) is a commutative unital ring, \(S\) is an inverse semigroup and \(A\) is an associative algebra over \(R\), a (non-degenerate) algebraic partial action of \(S\) on \(A\) is a partial action \(\alpha : S \to \mathcal{I}(A)\) such that:

1. For all \(s \in S\), \(A_s\) is an ideal of \(A\) and \(\alpha_s : A_{s^*} \to A_s\) is an \(R\)-isomorphism;
2. \(A = \text{span} \bigcup_{s \in S} A_s\).

Partial actions on rings are defined in the same way (e.g. by regarding rings as \(\mathbb{Z}\)-algebras in the natural way). All partial actions on algebras will be considered algebraic, in the sense described here.
Similarly to the case of topological partial actions, Property (ii) can be substituted with

\[(\text{ii}^{'}) \quad A = \text{span} \bigcup_{e \in E(S)} A_e;\]

and this non-degeneracy condition will always be assumed to hold, since we can always substitute \(A\) with \(\text{span} \bigcup_{e \in E(S)} A_e\) if necessary.

Crossed products for partial actions of inverse semigroups are defined in the same way as crossed products of global actions (see [54]), as long as we take appropriate care when dealing with partial morphisms.

**Definition 4.3.2.** Let \(\alpha\) be a (algebraic) partial action of \(S\) on a \(R\)-algebra \(A\). We denote by \(\mathcal{L}(\alpha)\) the \(R\)-module of all finite formal sums

\[
\sum_{s \in S} a_s \delta_s, \quad \text{where} \quad a_s \in A_s \quad \text{and} \quad \delta_s \quad \text{is a formal symbol.}
\]

More precisely, \(\mathcal{L}(\alpha)\) is the free \(R\)-module generated by formal elements \(a_s \delta_s\), where \(s \in S\) and \(a_s \in A_s\), satisfying the relations

\[
(a_s \delta_s) + (b_s \delta_s) = (a_s + b_s) \delta_s \quad \text{and} \quad \lambda(a_s \delta_s) = (\lambda a_s) \delta_s
\]

for all \(s \in S\), \(a_s, b_s \in A_s\) and \(\lambda \in R\).

**Remark.** Alternatively, \(\mathcal{L}(\alpha)\) coincides with the \(R\)-module of all finitely supported functions \(f : S \rightarrow A\) which satisfy \(f(s) \in A_s\) for all \(s \in S\). One should be a somewhat careful with this interpretation, since we could think of \(\delta_s\) as the characteristic function of \(\{s\}\), which will not be an element of \(\mathcal{L}(\alpha)\) in general. The index \(s\) appearing in a term \(a_s \delta_s\) will be useful to keep track of which ideal the element \(a_s\) belongs to.

We define a product on the generators of \(\mathcal{L}(\alpha)\) by

\[
(a_s \delta_s)(b_t \delta_t) = a_s (\alpha_s^*(a_s) b_t) \delta_st.
\]

**Example 4.3.3.** Let \(\alpha\) be a global action of a group \(G\) on an \(R\)-algebra \(A\). Then, as a module, \(\mathcal{L}(\alpha)\) coincides with the group algebra \(A \rtimes_{\alpha} G\). Since all functions \(\alpha_s\) are automorphisms of \(A\), we can calculate the product on \(\mathcal{L}(\alpha)\) as

\[
(a_s \delta_s)(b_t \delta_t) = \alpha_s(\alpha_{s-1}(a_s) b_t) \delta_st = \alpha_s(\alpha_{s-1}(a_s)) \alpha_s(b_t) \delta_st = a_s \alpha_s(b_t) \delta_st
\]

so \(\mathcal{L}(\alpha)\) is in fact the same as the group algebra \(A \rtimes_{\alpha} G\).

In general, this product might make \(\mathcal{L}(\alpha)\) a non-associative algebra ([35, Example 3.5]). We will prove associativity in a particular case, which will always be satisfied for the algebras we will consider. For more general conditions which guarantee associativity of \(\mathcal{L}(\alpha)\), see 3.1 and 3.2 of [35].
Definition 4.3.4. We say that an algebra $A$

- is idempotent if for all $a \in A$, there are finitely many elements $b_1, c_1, \ldots, b_n, c_n$
  such that $a = \sum_i b_i c_i$;

- has local units if for every finite subset $F \subseteq A$, there exists $e \in A$ such that $e^2 = e$ and $ef = fe = f$ for all $f \in F$, and call such $e$ a local unit for $F$.

Proposition 4.3.5. Suppose that $\alpha = (\alpha_s: A_s \to A_s)_{s \in S}$ is an algebraic partial action of an inverse semigroup $S$ on an $R$-algebra $A$, and that every ideal $A_s$ is idempotent. Then $L(\alpha)$ is associative.

In particular, if every ideal $A_s$ has local units then $L(\alpha)$ is associative.

Proof. Of course, it is enough to prove that

$$(a_s \delta_s)((b_t \delta_t)(c_z \delta_z)) = ((a_s \delta_s)(b_t \delta_t))(c_z \delta_z)$$

for all $s, t, z \in S$ and $a_s \in A_s$, $b_t \in A_t$ and $c_z \in A_z$. Suppose initially that $b_t = b^1_t b^2_t$, where $b^1_t \in A_t$. Then

$$(a_s \delta_s)((b_t \delta_t)(c_z \delta_z)) = \alpha_s(\alpha_s^* (a_s) \alpha_t (\alpha_t^* (b_t) c_z)) \delta_{stz} = \alpha_s (\alpha_s^* (a_s) b^1_t \alpha_t (\alpha_t^* (b^2_t) c_z)) \delta_{stz}$$

Since $\alpha_s, b^1_t \in A_t$, we get

$$(a_s \delta_s)((b_t \delta_t)(c_z \delta_z)) = \alpha_s (\alpha_t \alpha_t^* (\alpha_s^* (a_s) b^1_t \alpha_t (\alpha_t^* (b^2_t) c_z))) \delta_{stz}$$

and this coincides precisely with the definition of $((a_s \delta_s)(b_t \delta_t))(c_z \delta_z)$. Since every element of $A_t$ is a sum of products of elements of $A_t$ then we are done. \[\square\]

Definition 4.3.6. Given a partial action $\alpha$ of an inverse semigroup $S$ on an algebra $A$, we let $N(\alpha)$ be the additive subgroup of $L(\alpha)$ generated by all elements of the form

$$a_r \delta_r - a_r \delta_s, \quad \text{where} \quad r \leq s \quad \text{and} \quad a_r \in A_r.$$ 

Proposition 4.3.7. If $\alpha$ is an action of $S$ on an algebra $A$, then $N(\alpha)$ is a two-sided ideal of $L(\alpha)$. 

Proof. If \( a_r \delta_r - a_r \delta_s \), where \( a \in A_r \) and \( r \leq s \) and \( \lambda \in R \), then
\[
\lambda(a_r \delta_r - a_r \delta_s) = (\lambda a_r) \delta_r - (\lambda a_r) \delta_s
\]
so \( \mathcal{N}(\alpha) \) is a submodule of \( \mathcal{L}(\alpha) \). If \( b_z \delta_z \in \mathcal{L} \), then
\[
(b_z \delta_z)(a_r \delta_r - a_r \delta_s) = \alpha_z(a_z(b_z) a_r) \delta_{rz} - \alpha_z(\alpha_z(b_z) a_r) \delta_{zs}
\]
and since \( zr \leq zs \) then \( \mathcal{N}(\alpha) \) is a left ideal of \( \mathcal{L}(\alpha) \), and similarly it is also a right ideal.

**Definition 4.3.8.** Given an (algebraic) partial action \( \alpha \) of \( S \) on an algebra \( A \), the (algebraic) crossed product \( A \rtimes_{\alpha} S \) is the quotient algebra
\[
A \rtimes_{\alpha} S = \mathcal{L}(\alpha)/\mathcal{N}(\alpha)
\]
and as usual, we will simply write \( \mathcal{L} \), \( \mathcal{N} \) and \( A \rtimes S \) when there is no risk of confusion.

The class of an element \( x \in \mathcal{L}(\alpha) \) in \( A \rtimes_{\alpha} S \) will be denoted by \( \overline{x} \).

**Remark 4.3.9.** The difference from \( \mathcal{L} \) to \( A \rtimes S \) is that \( A \rtimes S \) respects the order structure of \( S \): Using the presentation of \( \mathcal{L}(\alpha) \) given in Definition 4.3.2, we obtain the following presentation of \( A \rtimes S \): It is the free module generated by symbols \( \overline{a_s \delta_s} \) where \( s \in S \) and \( a_s \in A_s \), and satisfying the following relations:

- \( \overline{a_s \delta_s} + \overline{b_s \delta_s} = \overline{(a_s + b_s) \delta_s} \), whenever \( a_s, b_s \in A_s \);
- \( \lambda \overline{a_s \delta_s} = \overline{(\lambda a_s) \delta_s} \), whenever \( a_s \in A_s \) and \( \lambda \in R \);
- \( \overline{a_r \delta_r} = \overline{a_r \delta_s} \), whenever \( a_r \in A_r \) and \( r \leq s \).

These are sometimes called “partial crossed products” or “partial skew inverse semigroup algebras” (e.g. in [30] and [13], respectively), while the name “crossed product” (or “skew inverse semigroup algebra”) are reserved for global actions. Instead, we adopt the simplest nomenclature.

Note that, since \( \mathcal{N} \) is defined, in principle, simply as an additive subgroup of \( \mathcal{L} \), then every map \( \phi : \mathcal{L} \to B \), where \( B \) is any algebra (or even simply a group), which satisfies \( \phi(a_r \delta_r) = \phi(a_r \delta_s) \) whenever \( s \leq r \) factors through \( A \rtimes S \).

**Definition 4.3.10.** The diagonal subalgebra of a partial crossed product \( A \rtimes S \) is the subalgebra generated by elements of the form \( \overline{a \delta_e} \), where \( e \in E(S) \) and \( a \in A_e \).

**Proposition 4.3.11.** Suppose \( \alpha \) is a (non-degenerate), algebraic partial action of \( S \) on \( A \), and assume that

(i) \( A \) has local units; and
(ii) every ideal $A_e$, $e \in E(S)$, has local units.

Then the map $\iota : A \to A \rtimes S$, given by $a \mapsto \sum_{i=1}^{n} a_i \delta_{e_i}$, where $e_i \in E(S)$ and $a_i \in A_{e_i}$ are chosen such that $a = \sum_{i=1}^{n} a_i$, is a well-defined injective algebra morphism.

Proof. We need to prove that the map $\iota$ is well-defined, that is, if $e_1, \ldots, e_n$ are idempotents and $a_i \in A_{e_i}$ are such that $\sum_{i=1}^{n} a_i = 0$ in $A$, then $\sum_{i=1}^{n} a_i \delta_{e_i} = 0$ in $A \rtimes S$. We proceed by induction. If $n = 1$ this is trivial, so suppose that the statement is valid for sums up to $n$ elements.

Suppose that $a + \sum_{i=1}^{n} a_i = 0$, where $a \in A_e$ and $a_i \in A_{e_i}$ for certain $e, e_i \in E(S)$. Let $u \in A$ be a local unit for $\{a, a_1, \ldots, a_n\}$, and $v \in A$ be a local unit for $\{a\}$. Then

$$0 = (u - v)a = -\sum_{i=1}^{n} (u - v)a_i,$$

and by the induction hypothesis,

$$0 = \sum_{i=1}^{n} ((u - v)a_i) \delta_{e_i} = \sum_{i=1}^{n} a_i \delta_{e_i} - \sum_{i=1}^{n} (va_i) \delta_{e_i}. \quad (4.3.1)$$

However, $va_i \in A_{e_i} \cap A_e \subseteq A_{ee_i}$, so

$$\sum_{i=1}^{n} (va_i) \delta_{e_i} = \sum_{i=1}^{n} (va_i) \delta_{ee_i} = \sum_{i=1}^{n} (va_i) \delta_e = \left(v \sum_{i=1}^{n} va_i\right) \delta_e = -va\delta_e = -a\delta_e,$$

and the previous equation implies that $a\delta_e + \sum_{i=1}^{n} a_i \delta_{e_i} = 0$, as we wanted.

It is readily seen that $\iota$ is an module morphism, and to check that it is an algebra morphism, take generating elements of $A$ of the form $a \in A_e$, $b \in A_f$, where $e, f \in E(S)$. Then $ab \in A_{ef}$, and

$$\iota(a)\iota(b) = (a\delta_e)(b\delta_f) = \alpha_e(\alpha_e \cdot (a)b) \delta_{ef} = (ab) \delta_{ef} = \iota(ab).$$

Now consider the module morphism $\tau : A \rtimes S \to A$ satisfying $\tau(a \delta_A) = a$, which exists because of the presentation of $A \rtimes S$ given in Remark 4.3.9. Then as $\tau \circ \iota$ is the identity of $A$, $\iota$ is injective.

Remark. Note that the map $\tau$ above is not an algebra morphism in general. For example, let $S = \langle g | g^2 = 1 \rangle$ is the cyclic group of order 2 and $A = \mathbb{C}^2$ with the pointwise structure of $\mathbb{C}$-algebra. Define an action $\alpha$ of $S$ on $A$

$$\alpha_g(a, b) = (b, a), \quad \alpha_1(a, b) = (a, b)$$

for all $(a, b) \in A$. Let $x = (1, 0)\delta_g$ and $y = (0, 1)\delta_1$. Then $xy = x$, so

$$\tau(xy) = \tau(x) = (1, 0)$$

however,

$$\tau(x)\tau(y) = (1, 0)(0, 1) = (0, 0).$$
Lemma 4.3.14. Let \( \alpha \) and \( \theta \) be a covariant representation \((\alpha, \theta)\), where \( \theta : S \to B \) is a partial morphism of \( S \) to the multiplicative semigroup of \( B \), and \( \phi : A \to B \) is an algebra morphism, satisfying the covariance condition
\[
\phi(\alpha_s(a_{s^*})) = \theta(s)\phi(a_{s^*})\theta(s^*) \quad \text{for all } s \in S \text{ and } a_{s^*} \in A_{s^*}.
\]
We say that \((\theta, \phi)\) is non-degenerate if \( \theta(s)\theta(s^*) \in \phi(A_s) \) for all \( s \in S \) (and degenerate otherwise).

Example 4.3.13. Let \( X \) be a non-compact, locally compact Hausdorff space, and 
\( S \) be the semigroup under intersection of clopen subsets of \( X \). Let \( A = C_c(X) \) be the (real or complex) algebra of compactly supported continuous functions on \( X \). For each \( U \in S \), let \( A_U = \{ f \in C_c(X) : \text{supp}(f) \subseteq U \} \), and \( \alpha_U \) be the identity on \( A_U \). Then \( \alpha \) is a global action on \( A \).

Let \( B = C_b(X) \) be the algebra of continuous bounded functions on \( X \). Define a covariant representation \((\theta, \phi)\) of \( \alpha \) on \( B \) by letting \( \phi : A \hookrightarrow B \) be the inclusion, and \( \theta(U) \) be the characteristic function of \( U \in S \). Then \( (\theta, \phi) \) is degenerate, because \( \theta(X)\theta(X) = \theta(X) \) is the constant function at 1, which does not belong to \( \phi(A) \).

Lemma 4.3.14. Let \((\theta, \phi)\) be a covariant representation of an algebraic partial action \( \alpha : S \to I(A) \).

(a) If \( a_s \in A_s \) then
\[
\phi(a_s) = \theta(s)\theta(s^*)\phi(a_s) = \phi(a_s)\theta(s)\theta(s^*). \tag{4.3.2}
\]

(b) If \( s \leq t \) and \( a_s \in A_s \) then \( \phi(a_s)\theta(s) = \phi(a_s)\theta(t) \).

(c) If \( A_s \) has a unit \( 1_s \) and \((\theta, \phi)\) is non-degenerate, then
\[
\phi(1_s)\theta(s) = \theta(s) \quad \text{and} \quad \theta(s^*)\phi(1_s) = \theta(s^*).
\]

Proof. (a) As \( a_s = \alpha_s(\alpha_{s^*}(a_s)) \), by covariance we get
\[
\phi(a_s) = \theta(s)\theta(s^*)\phi(a_s)\theta(s)\theta(s^*).
\]

The partial morphism properties of \( \theta \) imply
\[
\theta(s)\theta(s^*)\theta(s)\theta(s^*) = \theta(s)\theta(s^*),
\]
and Equation \(4.3.2\) follows from these two equalities.
We have
\[
\begin{align*}
\phi(a_s)\theta(t) &= \phi(a_s)\theta(s)\theta(s^t)\theta(t) = \phi(a_s)\theta(s)s^t \theta(t) \\
&= \phi(a_s)\theta(s)\theta(s^ts) = \phi(a_s)\theta(s)\theta(s^t)\theta(s) = \phi(a_s)\theta(s).
\end{align*}
\]

(c) Since \((\theta, \phi)\) is non-degenerate, then \(\theta(s)\theta(s^t) \in \phi(A_s)\), so
\[
\theta(s) = \theta(s)\theta(s^t)\theta(s) = \phi(1_s)\theta(s)\theta(s^t)\theta(s) = \phi(1_s)\theta(s)
\]
and the other equality is similar. 

\begin{theorem}
Suppose that \(\alpha\) is an algebraic partial action of an inverse semigroup \(S\) on an algebra \(A\) with local units, and such that \(A_s\) has a unit \(1_s\) for all \(s \in S\). Define
\[
\sigma : S \to A \times S \quad \text{by} \quad \sigma(s) = \overline{1_s \delta_s}.
\]
Let \(\iota : A \to A \times S\) be the embedding of \(A\) as the diagonal subalgebra of \(A \times S\). Then \((\sigma, \iota)\) is a universal non-degenerate covariant representations of \(\alpha\) in the following sense:
\begin{enumerate}
\item[(i)] \((\sigma, \iota)\) is a non-degenerate covariant representation;
\item[(ii)] If \((\theta, \phi)\) is any other non-degenerate covariant representation of \(\alpha\) on an algebra \(B\), then there exists a unique algebra morphism \(\Phi : A \times S \to B\) such that
\[
\phi = \Phi \circ \iota \quad \text{and} \quad \theta = \Phi \circ \sigma.
\]
\end{enumerate}
\end{theorem}

\begin{proof}
First we prove that \(\sigma\) is a partial morphism. Given \(s, t \in S\), we have \(\alpha_s(1_s) = 1_s\) because \(\alpha_s : A_{s^*} \to A_s\) is an algebra isomorphism, and it follows that
\[
\sigma(s)\sigma(s^*) = (1_s)\delta_s(1_s)\delta_{s^*} = \overline{1_s \delta_{ss^*}}. \tag{4.3.3}
\]

Using Equation (4.3.3), we compute
\[
\sigma(s)\sigma(s^*)\sigma(s) = \overline{1_s \delta_s 1_{s^*} \delta_{ss^*}} = \overline{1_s \delta_{ss^*} 1_s} = \sigma(s).
\]

We have
\[
\sigma(s)\sigma(t)\sigma(t^*) = \overline{1_s 1_{s^*} 1_t} \delta_{s{s^*}t^*} \quad \text{and} \quad \sigma(st)\sigma(t^*) = \overline{1_{st(1_{st})} 1_{t^*}} \delta_{s{s^*}t^*}.
\]

On one hand, \(1_s, 1_t\) is the unit of \(A_{s^*} \cap A_t\), so \(\alpha_s(1_s, 1_t)\) is the unit of \(\alpha_s(A_{s^*} \cap A_t)\).

On the other hand, \(1_{st(1_{st})}, 1_{t^*}\) is the unit of \(A_{st(1_{st})^*} \cap A_{t^*}\), so \(\alpha_s(1_{st(1_{st})}, 1_{t^*})\) is the unit of \(\alpha_s(A_{st(1_{st})^*} \cap A_{t^*})\). Proposition (4.1.7)(b) and (c) imply
\[
\begin{align*}
&\alpha_s(A_{s^*} \cap A_t) \overset{4.1.7(b)}{=} A_s \cap A_{s^*} \quad (4.1.7(c)) A_{(st)(st)^*} \cap A_{st} = A_{(st)^*} \cap A_{st} \\
&\quad \overset{4.1.7(b)}{=} \alpha_s(A_{st(1_{st})} \cap A_{t^*}),
\end{align*}
\]
and we conclude that \(\sigma(s)\sigma(t)\sigma(t^*) = \sigma(st)\sigma(t^*)\).
As in the previous item, we compute
\[ \sigma(s^*)\sigma(s)\sigma(t) = 1_s^*1_t\delta_{s^*st} \quad \text{and} \quad \sigma(s^*)\sigma(st) = \alpha_{s^*}(1_s1_{st})\delta_{s^*st} \]
and note that both coefficients are the units of
\[ \alpha_{s^*}(A_s \cap A_{st}) = A_{s^*} \cap A_{s^*st} = A_{s^*} \cap A_t \]
again using Proposition [4.1.7(b)] and (c).

We already know that \( \iota \) is an algebra morphism. Using the definition of the product in \( A \times S \) and the fact that \( \alpha_s(1_s^*) = 1_s \) we immediately obtain covariance of \((\sigma, \iota)\), and non-degeneracy follows from Equation [4.3.3].

As for the universal property, suppose \((\theta, \phi)\) is a covariant representation of \( \alpha \) on an algebra \( B \). Using the presentation of \( A \times S \) in Remark [4.3.9] and Lemma [4.3.14(b)], we can define a module morphism \( \Phi : A \times S \to B \) satisfying \( \Phi(a_s\delta_s) = \phi(a_s)\theta(s) \) for all \( a_s \in A_s \) \( (s \in S) \).

Let us check that \( \Phi \) is an algebra morphism: Let \( a_s\delta_s, b_t\delta_t \) be generators of \( A \times S \). Then
\[ \Phi((a_s\delta_s)(b_t\delta_t)) = \Phi((a_s\alpha_{s^*}(a_s)b_t)\delta_{st}) = \phi(a_s(\alpha_{s^*}(a_s)b_t)\theta(st)) \]
and using covariance and \( \phi \) being multiplicative,
\[ \Phi((a_s\delta_s)(b_t\delta_t)) = \theta(s)\theta(s^*)\phi(a_s)\theta(s)\phi(b_t)\theta(s^*)\theta(st) \quad (4.3.4) \]
On the other hand
\[ \Phi(a_s\delta_s)\Phi(b_t\delta_t) = \phi(a_s)\theta(s)\phi(b_t)\theta(t) \quad (4.3.5) \]
so we will modify the right-hand term of Equation [4.3.4] to obtain the right-hand term of [4.3.5].

From Lemma [4.3.14(a)]
\[ \theta(s)\theta(s^*)\phi(a_s) = \phi(a_s) \quad (4.3.6) \]
By Lemma [4.3.14(c)] and (a), (and the inclusion \( 1_s^*b_t \in A_{s^*} \)), we obtain
\[ \theta(s)\phi(b_t)\theta(s^*)\theta(st) = \theta(s)\phi(1_s^*b_t)\theta(s^*)\theta(s)\theta(t) = \theta(s)\phi(1_s^*b_t)\theta(t) \]
\[ = \theta(s)\phi(b_t)\theta(t) \quad (4.3.7) \]
Using Equations [4.3.6], [4.3.7], we conclude from [4.3.4] and [4.3.5] that \( \Phi \) is an algebra morphism.

Given \( e \in E(S) \) and \( a_e \in A_e \), Lemma [4.3.14(a)] and \( \theta(e) \) being idempotent imply that
\[ \Phi(\iota(e_e)) = \phi(a_e)\theta(e) = \phi(a_e)\theta(e^s) = \phi(a_e) \]
and so \( \Phi \circ \iota = \phi \) on \( A_e \) and thus on all of \( A \). The fact that \( \Phi \circ \sigma = \theta \) follows from Lemma [4.3.14(c)].

Finally, uniqueness of \( \Phi \) with the properties in (ii) is a consequence of \( a_s\delta_s \) being equal to \( \iota(a_s)\sigma(\delta_s) \) for all \( s \in S \) and \( a_s \in A_s \). \( \square \)
4.4 Steinberg algebras

4.4.1 Definitions and notation

Steinberg algebras were independently introduced in [161] and [30], as algebraic analogues of groupoid C*-algebras and generalizations of Leavitt path algebras and universal inverse semigroup algebras. We refer to [13] and [27] for more details.

In this section, $R$ is a fixed commutative ring with unit. Given an ample Hausdorff groupoid $G$, we denote by $R^G$ the $R$-module of $R$-valued functions on $G$. Given $A \subseteq G$, we define $1_A$ as the characteristic function of $A$ (with values in $R$).

**Definition 4.4.1** ([161, Definition 4.1]). Given an ample Hausdorff groupoid $G$, $A_R(G)$ is the $R$-submodule of $R^G$ generated by the characteristic functions of compact-open bisections of $G$.

The support of $f \in R^G$ is defined as $\text{supp}(f) = \{a \in G : f(a) \neq 0\}$.

**Proposition 4.4.2** ([30, Lemma 3.3]). If $G$ is an ample Hausdorff groupoid, then $A_R(G)$ coincides with the $R$-module of locally constant compactly supported $R$-valued functions on $G$. Every $f \in A_R(G)$ can be written as a finite linear combination

$$f = \sum_{i=1}^{n} r_i 1_{A_i}$$

where $r_i \in R$ and $\{A_i : i = 1, \ldots, n\}$ is a family of pairwise disjoint compact-open bisections of $G$, and $\text{supp}(f) = \{a \in G : f(a) \neq 0\}$.

Recall that $KB(G)$ is the semigroup of compact-open bisections of $G$ (Definition 1.4.14).

**Proof.** Since $G$ is Hausdorff, then compact-open bisections are clopen and their characteristic functions are locally constant. The same is true of their linear combinations.

Conversely, if $f$ is a locally constant, compactly supported function on $G$, $f^{-1}(0)$ is clopen, hence $\text{supp}(f) = G \setminus f^{-1}(0) = G \setminus f^{-1}(0)$ is clopen as well.

Using compactness of $\text{supp}(f)$ and since $KB(G)$ is a basis for the topology of $G$, we can find $A_1, \ldots, A_n \in KB(G)$ and $r_1, \ldots, r_n$ such that $\text{supp}(f) = \bigcup_{i=1}^{n} A_i$ and $f = r_i$ on $A_i$. Substituting $A_j$ by $A_j \setminus \bigcup_{i=1}^{j-1} A_i$ for all $j \geq 2$, if necessary, the family $\{A_i : i = 1, \ldots, n\}$ forms a partition of $\text{supp}(f)$, and the equality

$$f = \sum_{i=1}^{n} r_i 1_{A_i}$$

holds. In particular, $f$ is a linear combination of characteristic functions of compact-open bisections, and therefore $f \in A_R(G)$. \hfill $\square$
4. PARTIAL ACTIONS OF INVERSE SEMIGROUPS

Remark 4.4.3. Endowing $R$ with the discrete topology, $A_R(\mathcal{G})$ coincides with the set of compactly supported functions from $\mathcal{G}$ to $R$.

For every $f \in A_R(\mathcal{G})$ (ample Hausdorff) and every $x \in \mathcal{G}^{(0)}$, $\text{supp}(f) \cap \mathcal{G}_x$ is compact in the discrete space $\mathcal{G}_x$, and hence finite. Similarly, $\text{supp}(f) \cap \mathcal{G}^x$ is finite, and so we can define the convolution product

$$(fg)(a) = \sum_{xy=a} f(x)g(y) = \sum_{c \in \mathcal{G}^{(a)}} f(c)g(c^{-1}a) = \sum_{d \in \mathcal{G}_{a(a)}} f(ad^{-1})g(d). \quad (4.4.1)$$

With this product, $A_R(\mathcal{G})$ is an associative algebra.

Definition 4.4.4 ([161, 30]). The $R$-module $A_R(\mathcal{G})$, endowed with the convolution product of Equation 4.4.1 is called the Steinberg algebra $\mathcal{G}$ (or $(\mathcal{G}, R)$, to make $R$ explicit).

From the convolution formula we obtain, for all $f, g \in A_R(\mathcal{G})$.

$$\text{supp}(fg) \subseteq \{ab : (a, b) \in (\text{supp}(f) \times \text{supp}(g)) \cap \mathcal{G}^{(2)} \} = \text{supp}(f) \text{supp}(g).$$

We will now prove a universal property for Steinberg algebras, and for this we need the notion of a Boolean representation. These are the semigroup morphisms of a Boolean inverse semigroup $S$ into the multiplicative semigroup of an algebra $B$ which respect the order structure of $S$.

Definition 4.4.5 ([30, Definition 3.9]). A Boolean representation of a Boolean inverse semigroup $S$ on an algebra $B$ is a map morphism $\pi : S \to B$ of $S$ in the multiplicative semigroup of $B$ which satisfies

$$\pi(s \lor t) = \pi(s) + \pi(t)$$

whenever $s$ and $t$ are compatible and $s \land t = 0$.

Example 4.4.6. If $\mathcal{G}$ is an ample Hausdorff groupoid, then the map $\text{KB}(\mathcal{G}) \to A_R(\mathcal{G}), \ U \mapsto 1_U$, is a Boolean representation of $\text{KB}(\mathcal{G})$.

This notion of Boolean representation coincides with [159, Definition 3.1] – except for the fact that we do not assume that the algebra is involutive – as we prove below:

Proposition 4.4.7. A semigroup morphism $\pi : S \to B$ from a Boolean inverse semigroup $S$ to the multiplicative semigroup of an algebra $B$ is a Boolean representation if and only if

(i) $\pi(0) = 0$;

(ii) $\pi(s \lor t) = \pi(s) + \pi(t) - \pi(s \land t)$ whenever $s, t \in S$ are compatible;
(Recall from 1.2.30(c) that if s and t are compatible in S then s \wedge t = st^*t.)

**Proof.** Of course if \( \pi \) satisfies (i) and (ii) then \( \pi \) is a Boolean representation, so we prove the converse direction.

Assume \( \pi \) is a Boolean representation. Then
\[
\pi(0) = \pi(0 \lor 0) = \pi(0) + \pi(0)
\]
so \( \pi(0) = 0 \). If \( s, t \in S \) are compatible, then \( s \land (t \setminus s) = 0 \), so
\[
\pi(s \lor t) = \pi(s \lor (t \setminus s)) = \pi(s) + \pi(t \setminus s)
\]
But since \( t = (t \land s) \lor (t \setminus s) \), then
\[
\pi(t) = \pi(s \land t) + \pi(t \setminus s)
\]
and Property (ii) follows. \( \square \)

Now we describe the universal property of \( A_R(G) \), which was given in the case of complex algebras in [30, Theorem 3.10]. Essentially the same proof works in the case of a general commutative unital ring \( R \).

**Theorem 4.4.8.** Let \( R \) be a commutative ring with unit, and \( G \) an ample Hausdorff groupoid. Then \( A_R(G) \) is universal for Boolean representations of \( KB(G) \), i.e., if \( \pi : KB(G) \to B \) is a Boolean representation of \( KB(G) \) in an \( R \)-algebra \( B \), there exists a unique \( R \)-algebra morphism \( \Phi : A_R(G) \to B \) such that \( \Phi(1_U) = \pi(U) \) for all \( U \in KB(G) \).

**Proof.** Uniqueness of such \( \Phi \) is immediate as \( A_R(G) \) is generated by the family \( \{1_U : U \in KB(G)\} \). Given \( f \in A_R(G) \), write \( f \) as a linear combination \( f = \sum_{i=1}^{n} r_i 1_{U_i} \) for certain \( r_i \in R \) and a pairwise disjoint collection of compact-open bisections \( U_i \in KB(G) \). Define \( \Phi : A_R(G) \to B \) by
\[
\Phi(f) = \Phi \left( \sum_{i=1}^{n} r_i 1_{U_i} \right) = \sum_{i=1}^{n} r_i \pi(U_i).
\]
We first need to prove that \( \Phi \) does not depend on the choice of \( r_i \) and \( U_i \).

Suppose \( \sum_{i=1}^{n} r_i 1_{U_i} = \sum_{j=1}^{m} s_j 1_{V_j} \), where \( s_j \in R \) and \( \{V_j\}_j \) is a pairwise disjoint collection in \( KB(G) \). Then \( \sum_{i,j} r_i 1_{U_i \cap V_j} = f = \sum_{i,j} s_j 1_{U_i \cap V_j} \). First of all, let us prove that for all pair of indices \( i, j \), we have
\[
r_i \pi(U_i \cap V_j) = s_j \pi(U_i \cap V_j).
\] (4.4.2)
There are two possibilities:
**Case 1:** $U_i \cap V_j = \emptyset$.

Then $\pi(U_i \cap V_j) = 0$ so Equation (4.4.2) is trivially valid.

**Case 2:** $U_i \cap V_j \neq \emptyset$.

In this case, choose $a \in U_i \cap V_j$. Then $r_i = f(a) = s_j$, so of course $r_i \pi(U_i \cap V_j) = s_j \pi(U_i \cap V_j)$.

We can partition each $U_i$ as $U_i = \bigcup_j (U_i \cap V_j)$, and similarly we can partition each $V_j$ as $V_j = \bigcup_i (U_i \cap V_j)$, so using Equation (4.4.2) and the properties of Boolean representations,

$$
\sum_{i=1}^{n} r_i \pi(U_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} r_i \pi(U_i \cap V_j) = \sum_{j=1}^{m} \sum_{i=1}^{n} s_j \pi(U_i \cap V_j) = \sum_{j=1}^{m} s_j \pi(V_j)
$$

therefore $\Phi$ is well-defined.

In order to prove that $\Phi$ is an $R$-module morphism, we apply an argument similar to the one above: If $f, g \in A_R(\mathcal{G})$ have representations $f = \sum_{i=1}^{n} r_i 1_{A_i}$ and $g = \sum_{j=1}^{m} s_j 1_{B_j}$ as above, we can, if necessary, add terms of the form $0 \cdot 1_{B_j \setminus \text{supp}(f)}$ to the representation of $f$, and similarly for $g$, and assume that $\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} B_j$. Therefore we may rewrite, for all $\lambda \in R$,

$$
f = \sum_{i,j} r_i 1_{A_i \cap B_j}, \quad g = \sum_{i,j} s_j 1_{A_i \cap B_j}, \quad \text{and} \quad (f + \lambda g) = \sum_{i,j} (r_i + \lambda s_j) 1_{A_i \cap B_j},
$$

and the definition of $\Phi$ readily implies $\Phi(f + \lambda g) = \Phi(f) + \lambda \Phi(g)$, so $\Phi$ is an $R$-module morphism. If $U, V \in \mathbf{KB}(\mathcal{G})$, then

$$
\Phi(1_U 1_V) = \Phi(1_U V) = \pi(UV) = \pi(U) \pi(V) = \Phi(1_U) \Phi(1_V),
$$

and since $\{1_U : U \in \mathbf{KB}(\mathcal{G})\}$ generates $A_R(\mathcal{G})$ then $\Phi$ is an algebra morphism. \qed

**Definition 4.4.9.** The diagonal subalgebra of $A_R(\mathcal{G})$ is

$$
D_R(\mathcal{G}) = \{ f \in A_R(\mathcal{G}) : \text{supp}(f) \subseteq \mathcal{G}^{(0)} \}.
$$

Since $\text{supp}(fg) \subseteq \text{supp}(f) \cup \text{supp}(g)$ and $\text{supp}(f + \lambda g) \subseteq \text{supp}(f) \cup \text{supp}(g)$ for all $f, g \in A_R(\mathcal{G})$ and $\lambda \in R$, it follows that $D_R(\mathcal{G})$ is indeed a subalgebra of $A_R(\mathcal{G})$.

Moreover, the convolution product on $D_R(\mathcal{G})$ coincides with the pointwise product: for all $f, g \in D_R(\mathcal{G})$ and $x \in \mathcal{G}^{(0)}$, we have $\mathcal{G}^x \cap \mathcal{G}^{(0)} = \{ x \}$, so

$$
(fg)(x) = \sum_{a \in \mathcal{G}^x \cap \text{supp}(f)} f(a)g(a^{-1}x) = f(x)g(x^{-1}x) = f(x)g(x).
$$

In particular, $D_R(\mathcal{G})$ is commutative, since we assume the ring $R$ is commutative.

Moreover, $D_R(\mathcal{G})$ is isomorphic to $A_R(\mathcal{G}^{(0)})$ via the map

$$
E : D_R(\mathcal{G}) \to A_R(\mathcal{G}^{(0)}), \quad f \mapsto f|_{\mathcal{G}^{(0)}}.
$$

whose inverse is given as follows: If $f \in A_R(\mathcal{G}^{(0)})$, then $E^{-1}(f)$ is the extension of $f$ to $\mathcal{G}$ by setting $E^{-1}(f) = 0$ on $\mathcal{G} \setminus \mathcal{G}^{(0)}$. 


4.4.2 Diagonal-preserving isomorphisms of Steinberg Algebras

The goal of this section is to prove that the Steinberg algebra of an ample Hausdorff groupoid \( \mathcal{G} \) together with its diagonal algebra completely characterize \( \mathcal{G} \). Although the main theorem of this subsection (Theorem 4.4.27) is partially stated and proven (for more general graded Steinberg algebras) in [27, Corollary 3.14], we can obtain a precise classification of the diagonal-preserving isomorphisms of Steinberg algebras, as described in Theorem 4.4.27 and Corollary 4.4.28.

We will need to recover the bisections of \( \mathcal{G} \) from \( A_R(\mathcal{G}) \), and in particular the compact-open subsets of \( \mathcal{G}^{(0)} \). The main idea is, again, to identify subsets of \( \mathcal{G}^{(0)} \) with their characteristic functions, and these are precisely the functions which attain only the values 0 and 1. We thus need to assume an extra condition on the ring \( R \).

**Definition 4.4.10 ([107, X.7]).** A (nontrivial) commutative unital ring \( R \) is indecomposable if its only idempotents are 0 and 1. Equivalently, \( R \) is indecomposable if it cannot be written as a direct sum \( R \cong R_1 \oplus R_2 \), where \( R_1 \) and \( R_2 \) are nontrivial rings.

A subset \( A \) of a groupoid \( \mathcal{G} \) is a bisection if and only if \( AA^{-1} \cup A^{-1}A \subseteq \mathcal{G}^{(0)} \). A similar type of condition will be used to recover an ample Hausdorff groupoid \( \mathcal{G} \) from the pair \( (A_R(\mathcal{G}), D_R(\mathcal{G})) \).

**Definition 4.4.11.** A normalizer of \( D_R(\mathcal{G}) \) is an element \( f \in A_R(\mathcal{G}) \) for which there exists \( g \in A_R(\mathcal{G}) \) such that

(i) \( fD_R(\mathcal{G})g \subseteq D_R(\mathcal{G}) \) and \( gD_R(\mathcal{G})f \subseteq D_R(\mathcal{G}) \);

(ii) \( fgf = f \) and \( gfg = g \).

We denote by \( N_R(\mathcal{G}) \) the set of normalizers of \( D_R(\mathcal{G}) \). An element \( g \) satisfying (i) and (ii) above will be called and inverse of \( f \) relative to \( D_R(\mathcal{G}) \).

**Example 4.4.12.** If \( A \in \text{KB}(\mathcal{G}) \) then \( 1_A \in N_R(\mathcal{G}) \). More generally, if \( \lambda_1, \ldots, \lambda_n \) are invertible elements in \( R \) and \( U_1, \ldots, U_n \) are compatible disjoint compact-open bisections (that is, \( \bigcup_i U_i \) is also a bisection), then \( f = \sum_i \lambda_i 1_{U_i} \) is a normalizer of \( D_R(\mathcal{G}) \). The unique inverse of \( f \) relative to \( D_R(\mathcal{G}) \) is given by \( f^* = \sum_i \lambda_i^{-1} 1_{U_i^{-1}} \), that is, \( f^*(a) = f(a^{-1})^{-1} \) for all \( a \in \text{supp}(f)^{-1} \).

In order to recover \( \text{KB}(\mathcal{G}) \) from \( (A_R(\mathcal{G}), D_R(\mathcal{G})) \), we need that all normalizers of \( D_R(\mathcal{G}) \) have the form described in the example above, so additional conditions will have to be assumed on the groupoids we consider. The next few results show us that \( N_R(\mathcal{G}) \) encodes some of the semigroup structure of \( \text{KB}(\mathcal{G}) \).

**Lemma 4.4.13.** Suppose that \( f \in N_R(\mathcal{G}) \).
(a) If \( g \) is an inverse of \( f \) relative to \( D_R(G) \), then \( fg, gf \in D_R(G) \).

(b) The inverse of \( f \) relative to \( D_R(G) \) is unique.

**Proof.**

(a) We have \( fg = f1_{\text{supp}(f)}g \in D_R(G) \) and similarly \( gf = g1_{\text{supp}(f)}f \in D_R(G) \).

(b) Let \( g_1, g_2 \) be inverses of \( f \) relative to \( D_R(G) \). Then

\[
g_1 = g_1fg_1 = g_1(fg_2f)g_1 = (g_1f)(g_2f)g_1.
\]

But since \( g_1f \) and \( g_2f \in D_R(G) \), which is a commutative subalgebra, then

\[
g_1 = (g_1f)(g_2f)g_1 = (g_2f)(g_1f)g_1 = g_2fg_1g_2 = g_2(fg_2f)g_1 = g_2(fg_2)(fg_1),
\]

and similarly, since \( fg_2, fg_1 \in D_R(G) \),

\[
g_1 = g_2(fg_2)(fg_1) = g_2(fg_1)fg_2 = g_2(fg_1)g_2 = g_2fg_2 = g_2.
\]

\[\square\]

**Definition 4.4.14.** We denote the inverse of \( f \in N_R(G) \) relative to \( D_R(G) \) by \( f^* \).

**Proposition 4.4.15.** \( N_R(G) \) is an inverse semigroup under products, and the inverse of \( f \in N_R(G) \) is precisely the inverse relative to \( D_R(G) \).

**Proof.** Since \( D_R(G) \) is commutative, we can use Lemma 4.4.13(a) for \( f, g \in N_R(G) \) to obtain

\[
(fg)(g^*f^*)(fg) = f(gg^*)(f^*f)g = f(f^*f)(gg^*)g = fg
\]

and similarly \((g^*f^*)(gf)(g^*f^*) = g^*f^* \). Moreover,

\[
(fg)D_R(G)(g^*f^*) = f(gD_R(G)g^*)f^* \subseteq fD_R(G)f^* \subseteq D_R(G)
\]

and similarly \((g^*f^*)D_R(G)(fg) \subseteq D_R(G) \). This means that \( fg \in N_R(G) \), with inverse relative to \( D_R(G) \) equal to \( g^*f^* \). Therefore \( N_R(G) \) is a regular semigroup.

We will prove that \( N_R(G) \) is an inverse semigroup by proving that its idempotents commute. Let \( f \in E(N_R(G)) \), that is, \( f^2 = f \). Then since \( ff^*, f^*f \in D_R(G) \) (Lemma 4.4.13(a)),

\[
f^* = f^*ff^* = (f^*f)(ff^*) = (ff^*)(f^*f) = f(f^*f)(f^*f)f = f^2 = f
\]

and thus \( f = f^*f \in D_R(G) \). This proves \( E(N_R(G)) \subseteq D_R(G) \), so idempotents in \( N_R(G) \) commute and \( N_R(G) \) is an inverse semigroup.

\[\square\]

The following property was considered in [160], when working on the same recovery problem.
Definition 4.4.16. If $\mathcal{G}$ is an ample Hausdorff groupoid and $R$ is an indecomposable (commutative, unital) ring, we say that $(\mathcal{G}, R)$ satisfies the local bisection hypothesis if $\text{supp}(f)$ is a bisection for all $f \in N_R(\mathcal{G})$.

The following stronger condition was considered in [27], and is more easily checked than the one above.

Definition 4.4.17. If $\mathcal{G}$ is an ample Hausdorff groupoid and $R$ is an indecomposable (commutative, unital) ring, we say that $(\mathcal{G}, R)$ satisfies condition (S) if the set of all $x \in G^{(0)}$ such that the group ring $RG_x$ has only trivial units is dense in $G^{(0)}$.

The property of a group-ring $RG$ (where $G$ is a group and $R$ is a ring) having only trivial units has been studied, for example, in [85]. A group $G$ is indexed if there exists a non-trivial group morphism from $G$ to $\mathbb{Z}$, and indicable throughout if every nontrivial finitely generated subgroup of $G$ is indexed. (Note that if $G$ is indicable throughout then $G$ is torsion-free.)

Theorem 4.4.18 ([85, Theorem 13]). If $G$ is indicable throughout and $R$ is an integral domain, then $RG$ has only trivial units.

Every free group, and every torsion-free abelian group is indicable throughout. The class of indicable throughout groups is closed under products, free products and extensions (see [85]).

In [27, Lemma 3.5], the authors proved that condition (S) implies the We now prove that condition (S) implies the local bisection hypothesis, as first proved in [27, Lemma 3.5].

Remark. In [27] the authors consider only Steinberg algebras over integral domains, and moreover a

Lemma 4.4.19 ([27, Lemma 3.5]). Suppose $\mathcal{G}$ is an ample Hausdorff groupoid, $R$ is an indecomposable ring and $f \in N_R(\mathcal{G})$.

(a) $ff^* = 1_{\mathcal{G}(\text{supp}(f^*))} = 1_{\mathcal{G}(\text{supp}(f))}$ and similarly $f^*f = 1_{\mathcal{G}(\text{supp}(f))} = 1_{\mathcal{G}(\text{supp}(f^*))}$;

(b) For all $U \subseteq \text{supp}(f)$, $f1_Uf^* = 1_{\mathcal{G}(\text{supp}(f) \cap \text{supp}(f^*)^{-1}(U))} = 1_{\mathcal{G}(\text{supp}(f^*) \cap \text{supp}(f)^{-1}(U))}$;

(c) If $\text{supp}(f)$ is a bisection, then for all $a \in \text{supp}(f)$ we have $f^*(a^{-1}) = f(a)^{-1}$.

Proof. (a) $ff^* \in D_R(\mathcal{G})$ and $(ff^*)^2 = ff^*$, so since $R$ has only trivial idempotents we obtain $ff^* = 1_U$ for some $U \in \mathcal{KB}(G^{(0)})$. Note that $1_Uf = ff^*f = f$ and $f^*1_U = f^*ff^* = f$.

---

If $G$ is a group and $R$ is a unital ring, a trivial unit of $RG$ is an element of the form $ug$ where $u$ is invertible in $R$ and $g \in G$.
so \( \tau(\text{supp}(f)) \cup \sigma(\text{supp}(f^*)) \subseteq U \). On the other hand,

\[
U = \text{supp}(ff^*) = \sigma(\text{supp}(f f^*)) \subseteq \sigma(\text{supp}(f)) \sigma(\text{supp}(f^*)),
\]

therefore \( U = \sigma(\text{supp}(f^*)) \), and similarly \( U = \tau(\text{supp}(f)) \).

(b) Let \( U \) be a compact-open subset of \( \sigma(\text{supp}(f)) \). In particular, \( 1_U \in N_R(\mathcal{G}) \) and so \( f 1_U \in N_R(\mathcal{G}) \) as well, by Proposition 4.4.15. By (a),

\[
f 1_U f^* = (f 1_U)(1_U f^*) = 1_{\tau(\text{supp}(f 1_U))} = 1_{\tau(\text{supp}(f) \cap \sigma^{-1}(U))}
\]

and similarly \( f 1_U f^* = 1_{\sigma(\text{supp}(f^*) \cap \tau^{-1}(U))} \).

(c) If \( a \in \text{supp}(f) \), then by item (a),

\[
1 = f^* f(\sigma(a)) = \sum_{cd=\sigma(a)} f^*(c) f(d).
\]

Since \( \text{supp}(f) \) is a bisection, the only element \( d \in \text{supp}(f) \) with \( \sigma(d) = \sigma(a) \) is \( d = a \) itself, that is,

\[
1 = f^*(a^{-1}) f(a),
\]

which means that \( f^*(a^{-1}) = f(a)^{-1} \). \( \square \)

**Proposition 4.4.20.** Suppose that \( \mathcal{G} \) is an ample Hausdorff groupoid, \( R \) is an indecomposable ring and that \( (\mathcal{G}, R) \) satisfies condition \((S)\). Then \( (\mathcal{G}, R) \) satisfies the local bisection hypothesis.

**Proof.** Given \( f \in N_R(\mathcal{G}) \), we need to prove that \( \text{supp}(f) \) is a bisection.

Let \( a_0 \in \text{supp}(f) \), \( x = \sigma(a_0) \) and \( y = \tau(a_0) \). First, assume that the group ring \( RG_x^z \) has only trivial units. Let us show that \( \text{supp}(f) \cap \mathcal{G}_x \) has only trivial units. Let us show that \( \text{supp}(f) \cap \mathcal{G}_x = \{a_0\} \).

Since \( \mathcal{G}_x \) is discrete and \( \text{supp}(f) \) is compact then \( \text{supp}(f) \cap \mathcal{G}_x \) is finite, so the set \( \tau(\text{supp}(f) \cap \mathcal{G}_x) \) is also finite and contains \( y \). Choose a compact-open subset \( U \subseteq \mathcal{G}^{(0)} \) such that

\[
\tau(\text{supp}(f) \cap \mathcal{G}_x) \cap U = \{y\}.
\]

Given \( b \in \mathcal{G}_x \), we have

\[
f^* 1_U f(b) = \sum_{\substack{cd=b \\\ \tau(d) \in U}} f^*(c) f(d).
\]

Of course, in the sum above we just need to consider \( d \in \text{supp}(f) \), so if \( \tau(d) \in U \) as well we will obtain \( \tau(d) = y \). Thus we can rewrite

\[
f^* 1_U f(b) = \sum_{\substack{cd=b \\\ \tau(d) = y}} f^*(c) f(d). \tag{4.4.3}
\]
Now by Lemma 4.4.19(b), $f^*1_U f = 1_{s(supp(f) \cap r^{-1}(U))}$. But since $s(supp(f) \cap r^{-1}(U))$ is contained in $G^{(0)}$ and contains $s(a) = x$, then for all $b \in \mathcal{G}_x$,

$$f^*1_U f(b) = \begin{cases} 1, & f b = x, \\ 0, & \text{otherwise}. \end{cases} \quad (4.4.4)$$

Consider the elements of the group ring $R\mathcal{G}_x$

$$\alpha = \sum_{d \in \mathcal{G}_x} f(d)a_0^{-1}d, \quad \beta = \sum_{c \in \mathcal{G}_x} f^*(c)ca_0.$$

Then using Equations (4.4.3) and (4.4.4),

$$\beta \alpha = \sum_{b \in \mathcal{G}_x} \left( \sum_{c \in \mathcal{G}_x} f^*(c)f(d) \right) b = \sum_{c \in \mathcal{G}_x} f^*(c)f(d)x = x, \quad (4.4.5)$$

and similarly $\alpha \beta = x$ (using the facts that $R$ is commutative and $cd = x$ if and only if $a_0^{-1}dca_0 = x$). (Recall that $x$ is the unit of $R\mathcal{G}_x$.)

Therefore $\alpha$ is invertible in $R\mathcal{G}_x$, hence it is a trivial invertible, which means that $supp(f) \cap \mathcal{G}_x$ is a singleton. Since $a_0$ is by construction in this set, we are done with this first part. Moreover, Equation (4.4.5) implies that $f^*(a_0^{-1}) = f(a_0)^{-1}$.

Now we show that $supp(f) \cap \mathcal{G}_x = \{a_0\}$. Suppose that $b_0 \in supp(f) \cap \mathcal{G}_x \setminus \{a_0\}$. The same way as above, take $V \in KB(G^{(0)})$ such that $V \cap r(supp(f)) = \{y, r(b_0)\}$. We already know from above that $\{b_0\} = supp(f) \cap \mathcal{G}_x^{r(b_0)}$ and that $f^*(b_0^{-1}) = f(b_0)^{-1}$, and similarly for $a_0$, so Lemma 4.4.19(b) yields

$$1 = 1_{s(r^{-1}(V) \cap supp(f))}(x) = f^*1_V f(x) = \sum_{cd=x, r(d)=y} f^*(c)f(d) = \sum_{cd=x, r(d)=y \text{ or } r(b_0)} f^*(c)f(d) = f^*(a_0^{-1})f(a_0) + f^*(b_0^{-1})f(b_0) = 2,$$

a contradiction to $R$ being a non-trivial ring.

Now to the general case, suppose $a, b \in supp(f)$ with $s(a) = s(b)$. Take any two bisections $U$ and $V$ containing $a$ and $b$, respectively. Then $s(U) \cap s(V)$ is a nonempty open set, so there is $x \in s(U) \cap s(V)$ satisfying the condition described in (S). This implies that $supp(f) \cap \mathcal{G}_x$ is a singleton intersecting both $U$ and $V$, and therefore is contained in both $U$ and $V$ simultaneously. Since $G$ is Hausdorff, this implies that $a = b$, so the source map is injective on $supp(f)$. The range is dealt with similarly. □

An important class of groupoids consists of the topologically principal ones, whose associated algebras have been extensively studied (see, for example, [20, 50, 143, 149]). In fact it is possible to classify $C^*$-algebras which come from them (see [148]).
Definition 4.4.21. A topological groupoid \( \mathcal{G} \) is \textit{topologically principal} if the set of all \( x \in X \) whose isotropy group \( \mathcal{G}^x \) is trivial is dense in \( \mathcal{G}(0) \).

As an immediate consequence of the previous propositions, we obtain the following:

Corollary 4.4.22. If \( \mathcal{G} \) is an ample Hausdorff topologically principal groupoid and \( R \) is an indecomposable ring, then \((\mathcal{G}, R)\) satisfies the local bisection hypothesis.

We are ready to classify diagonal-preserving isomorphisms of Steinberg algebras of groupoids and rings satisfying the local bisection hypothesis. For this, let us first define the class of maps of interest:

Definition 4.4.23. Let \( R \) and \( S \) be rings and \( \mathcal{G} \) be a groupoid. Denote by \( \text{Iso}_+(R, S) \) the set of additive isomorphisms from \( R \) to \( S \). A map \( \chi : \mathcal{G} \to \text{Iso}_+(R, S) \) satisfying \( \chi(ab)(rs) = \chi(a)(r)\chi(b)(s) \) for all \((a, b) \in \mathcal{G}^{(2)} \) and \( r, s \in R \) will be called a cocycle.

Example 4.4.24. Consider \( C_2 = \{1, g\} \), the group of order 2, acting on itself by left multiplication and consider the transformation groupoid \( \mathcal{G} = C_2 \rtimes C_2 \).

Let \( R = S = \mathbb{Z} \). If we define \( \chi : \mathcal{G} \to \text{Iso}_+(R, S) \) by \( \chi(1, y)(r) = r \) and \( \chi(g, r) = -r \), then \( \chi \) is a cocycle. Note that \( \chi(g, 1) \) is not a ring isomorphism.

Example 4.4.25. Suppose \( R \) is a unital ring and \( \chi : \mathcal{G} \to \text{Iso}_+(R, R) \) is a cocycle. Then \( \chi \) is a morphism from the groupoid \( \mathcal{G} \) to the group (under composition) \( \text{Iso}_+(R, R) \) if, and only if, \( \chi(x) = \text{id}_R \) for all \( x \in \mathcal{G}(0) \).

Indeed, assume \( \chi(x) = \text{id}_R \) for all \( x \in \mathcal{G}(0) \). If \((a, b) \in \mathcal{G}^{(2)} \) and \( r \in R \), then the cocycle condition yields

\[
\chi(a)(\chi(b)(r)) = \chi(a s(a))\chi(1)(\chi(b)(r)) = \chi(a)(1)\chi(s(a))(\chi(b)(r)),
\]

and since \( \chi(s(a)) = \text{id}_R \),

\[
\chi(a)(\chi(b)(r)) = \chi(a)(1)\chi(b)(r) = \chi(ab)(r)
\]

where the last equality follows from the cocycle condition again. Thus \( \chi(ab) = \chi(a) \circ \chi(b) \), so \( \chi \) is a morphism.

The converse is trivial, since groupoid morphisms map units to units, and the only unit of the group \( \text{Iso}_+(R, R) \) is the identity map of \( R \).

Proposition 4.4.26. Let \( R \) and \( S \) be commutative unital rings, \( \mathcal{G} \) a groupoid and \( \chi : \mathcal{G} \to \text{Iso}_+(R, S) \) a cocycle. Then

(a) For all \( x \in \mathcal{G}(0) \), \( \chi(x) \) is a ring isomorphism;

(b) For all \( a \in \mathcal{G} \), if \( u \in R \) is invertible then \( \chi(a)(u) \) is invertible in \( S \), and \( \chi(a)(u)^{-1} = \chi(a^{-1})(u) \).
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(c) For all $a \in \mathcal{G}$, $\chi(s(a)) = \chi(t(a))$. In other words, the restriction of $\chi$ to $\mathcal{G}^{(0)}$ is invariant.

Proof. The cocycle condition states that $\chi(ab)(rs) = \chi(a)(r)\chi(b)(s)$ for all $a, b, r, s$. Taking $a = b = x$ yields (a). Taking $b = a^{-1}$, $r = u$ and $s = u^{-1}$ yields (b), and for item (c) we use commutativity of $S$:

$$\chi(s(a))(r) = \chi(a^{-1}a)(1r) = \chi(a^{-1})(1)\chi(a)(r) = \chi(a)(r)\chi(a^{-1})(1) = \chi(t(a))(r).$$

We endow $\text{Iso}_+(R, S)$ with the topology of pointwise convergence, so that a map $\chi$ from a topological space $X$ to $\text{Iso}_+(R, S)$ is continuous if and only if for every $r \in R$, the map $X \ni x \mapsto \chi(x)(r) \in S$ is continuous, that is, locally constant.

Theorem 4.4.27. Let $\mathcal{G}$ and $\mathcal{H}$ be ample Hausdorff groupoids. Let $R$ and $S$ be two indecomposable (commutative, unital) rings such that $(\mathcal{G}, R)$ and $(\mathcal{H}, S)$ satisfy the local bisection hypothesis. Let $T : A_R(\mathcal{G}) \to A_S(\mathcal{H})$ be a diagonal-preserving ring isomorphism, that is, $T(D_R(\mathcal{G})) = D_S(\mathcal{H})$.

Then there exists a unique topological groupoid isomorphism $\phi : \mathcal{H} \to \mathcal{G}$ and a continuous cocycle $\chi : \mathcal{H} \to \text{Iso}_+(R, S)$ such that $Tf(a) = \chi(a)(\phi(a))$ for all $a \in \mathcal{H}$ and $f \in A_R(\mathcal{G})$.

To prove this theorem we will use the theory of disjoint functions from the previous chapter. Recall that if $f \in A_R(\mathcal{G})$ then $\text{supp}(f) = \{a \in \mathcal{G} : f(x) \neq 0\}$ is compact-open, that $A_R(\mathcal{G})$ coincides with $C_c(\mathcal{G})$ and that $A_R(\mathcal{G})$ is regular (in the sense of Definition 3.1.5), since $\mathcal{G}$ is zero-dimensional, and similarly for $\mathcal{H}$ and $S$. We refer to Section 3.1 for the notation and nomenclature.

Proof. Since $T$ preserves the respective diagonal algebras, it also preserves their normalizers, i.e., $T(N_R(\mathcal{G})) = N_S(\mathcal{H})$. Let us describe disjointness first for elements in $N_R(\mathcal{G})$. The local bisection hypothesis implies, by Lemma 4.4.19(c), that an element $f$ of $N_R(\mathcal{G})$ has the form

$$f = \sum_{i=1}^{n} \lambda_i 1_{U_i},$$

where $\lambda_1, \ldots, \lambda_n$ are invertible elements in $R$ and $U_1, \ldots, U_n$ are disjoint compact-open bisections of $\mathcal{G}$ such that $\bigcup_{i=1}^{n} U_i = \text{supp}(f)$ is also a compact-open bisection. A similar statement holds for $N_S(\mathcal{H})$.

If $f, g \in N_R(\mathcal{G})$, then $f \subseteq g$ if and only if $f = gp$ for some $p \in D_R(\mathcal{G})$: Indeed,

- If $f = gp$ then $\text{supp}(f) \subseteq \text{supp}(g) \text{supp}(p) \subseteq \text{supp}(g)$;
- Conversely, if $\text{supp}(f) \subseteq \text{supp}(g)$ take $p = g^* f$. Then

$$\text{supp}(p) \subseteq \text{supp}(g^*) \text{supp}(f) \subseteq (\text{supp}(g))^{-1} \text{supp}(g) \subseteq \mathcal{G}^{(0)}$$

where the last inclusion follows from $\text{supp}(g)$ being a bisection. The equality $f = gp$ then follows from Lemma 4.4.19(a).
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Therefore $T$ preserves inclusion of normalizers. Since $N_R(G)$ contains the collection $\{1_U : U \in KB(G)\}$ then it is regular (Definition 3.1.15), because $KB(G)$ is a basis for the topology of $G$. Hence $T$ also preserver disjointness of normalizers (Theorem 3.1.8).

To prove that $T$ preserves disjointness in all of $A_R(G)$, we decompose elements of $A_R(G)$ in terms of elements of $N_R(G)$ and $D_R(G)$: if $f, g \in A_R(G)$, then $f \perp g$ if and only if there are finite collections of normalizers $f_i, g_j \in N_R(G)$ and elements $\tilde{f}_i, \tilde{g}_j \in D_R(G)$ ($1 \leq i \leq n$, $1 \leq j \leq m$) such that

$$f = \sum_i f_i \tilde{f}_i, \quad g = \sum_j g_j \tilde{g}_j$$

Indeed, if there are such $f_i, g_j, \tilde{f}_i, \tilde{g}_j$ then $\text{supp}(f) \subseteq \bigcup_i \text{supp}(f_i)$ and $\text{supp}(g) \subseteq \bigcup_j \text{supp}(g_j)$, and the latter sets are disjoint.

Conversely, we write $f = \sum_i \lambda_i 1_{A_i}$, where the $A_i$ are pairwise disjoint compact-open bisections and $\lambda_i \neq 0$, and take $f_i = 1_{A_i}$ and $\tilde{f}_i = \lambda_i 1_{\phi(A)}$, so that $\text{supp}(f) = \bigcup_{i=1}^n \text{supp}(f_i)$. Similarly, writing $g = \sum_j \tilde{g}_j g_j$ where $g_j \in N_R(G)$ and $\text{supp}(g) = \bigcup_j \text{supp}(g_j)$, then $f \perp g$ implies $f_i \perp g_j$ for all $i$ and $j$.

Therefore, $T$ is a $\perp$-isomorphism. Note that $\perp$ and $\perp$ coincide in $A_R(G)$, since its elements are locally constant (similarly to Example 3.1.10). Then $T$ is a $\perp$-isomorphism, so let $\phi : \mathcal{H} \to G$ be the $T$-homeomorphism.

The verification that $\phi$ is a groupoid isomorphism is similar to that of Theorem 3.5.11. Assuming $(a, b) \in \mathcal{H}^{(2)}$, but such that $\phi(ab)$ is not equal to $\phi(a)\phi(b)$ (which might not even be defined), we can use continuity of the multiplication or Lemma 3.5.9 to find compact-open bisections $U$ and $V$ containing $a$ and $b$, respectively, such that $\phi(ab) \notin \phi(U)\phi(V)$. Then $ab \in \text{supp}(1_U)$, so

$$\phi(ab) \in \text{supp}(T1_U) = \text{supp}((T1_U)(T1_V)) \subseteq \text{supp}(T1_U) \text{supp}(T1_V) = \phi(U)\phi(V),$$

a contradiction.

Since elements of $A_R(G)$ (and $A_S(\mathcal{H})$) are locally constant,

$$f(\phi(a)) = 0 \iff \phi(a) \in Z(f) \iff x \in Z(Tf) \iff Tf(a) = 0,$$

and therefore $T$ is basic (by additivity of $T$ and Proposition 3.3.11). Let $\chi$ be the $T$-transform. Since $T$ is additive with the pointwise operations, each section $\chi(a) = (\chi(\alpha, \cdot))$ is additive (by Proposition 3.3.10). This yields a map $\chi : G \to \text{Iso}_+(R, S)$ such that

$$Tf(a) = \chi(a)(f(\phi(a))), \quad \text{for all } f \in A_R(G) \text{ and } a \in \mathcal{H},$$

and we need now to verify that $\chi$ is a cocycle.

If $(a, b) \in \mathcal{H}^{(2)}$ and $r, s \in R$, choose compact-open bisections $U, V$ of $G$ containing $\phi(a)$ and $\phi(b)$, respectively. Then using multiplicativity of $T$ we obtain

$$\chi(ab)(rs) = \chi(ab)((r1_U)(s1_V)(\phi(ab))) = T((r1_U)(s1_V))(ab)$$
cles

\[ T(r_{1_U})T(s_{1_V})(ab) = \sum_{cd=ab} T(r_{1_U})(c)T(s_{1_V})(d) \]

\[ = \sum_{cd=ab} \chi(c)(r_{1_U}(\phi(c)))\chi(d)(s_{1_V}(\phi(d))) \]

If \( cd = ab \) then \( \tau(c) = \tau(a) \), so \( \tau(\phi(c)) = \tau(\phi(a)) \). If also \( \phi(c) \in \text{supp}(1_U) \) we have \( \phi(c) = \phi(a) \), because \( U \) is a bisection, so \( c = a \) and \( d = c^{-1}ab = b \). Therefore

\[ \chi(ab)(rs) = \chi(a)(r)b(s). \]

so \( \chi \) is a cocycle.

It remains only to prove that \( \chi \) is continuous: Let \( r \in R \) be fixed, \( a \in \mathcal{H} \) and \( U \) any compact-open bisection containing \( \phi(a) \). For all \( b \in \phi^{-1}(U) \),

\[ \chi(b)(r) = \chi(b)(r_{1_U}(\phi(b))) = T(r_{1_U})(b) \]

which means that the map \( b \mapsto \chi(b)(r) \) coincides with \( T(r_{1_U}) \) on \( \phi^{-1}(U) \) and thus it is continuous.

We should note that according to \[160\], the local bisection hypothesis is preserved by diagonal-preserving isomorphisms, so the same result is valid if we assume, in principle, that only \((\mathcal{G}, R)\) satisfies this condition.

From this we can immediately classify the group of diagonal-preserving automorphisms of Steinberg algebras satisfying the local bisection hypothesis. Let \( \mathcal{G} \) be a groupoid and \( R \) a ring. Denote by \( C(\mathcal{G}, R) \) the set of all continuous cocycles \( \chi : \mathcal{G} \to \text{Iso}(R, R) \), which is a group with the canonical (pointwise) structure: \((\chi \rho)(a) = \chi(a) \circ \rho(a)\) for all \( \chi, \rho \in C(\mathcal{G}, R) \) and \( a \in \mathcal{G} \), where \( \circ \) denotes composition.

Let \( \text{Aut}(\mathcal{G}) \) be the group of topological groupoid automorphisms of \( \mathcal{G} \). Then \( \text{Aut}(\mathcal{G}) \) acts on \( C(\mathcal{G}, R) \) in the usual (dual) manner: for \( \phi \in \text{Aut}(\mathcal{G}) \), \( \chi \in C(\mathcal{G}, R) \) and \( a \in \mathcal{G} \) set \( (\phi \chi)(a) = \chi(\phi^{-1}a) \).

Denote by \( \text{Aut}(A_R(\mathcal{G}), D_R(\mathcal{G})) \) the group of diagonal-preserving ring automorphisms of \( A_R(\mathcal{G}) \).

**Corollary 4.4.28.** If \((\mathcal{G}, R)\) satisfies the local bisection hypothesis, then the group \( \text{Aut}(A_R(\mathcal{G}), D_R(\mathcal{G})) \) is isomorphic to the semidirect product \( C(\mathcal{G}, R) \rtimes \text{Aut}(\mathcal{G}) \).

**Proof.** Given \((\chi, \phi) \in C(\mathcal{G}, R) \rtimes \text{Aut}(\mathcal{G})\), set \( T_{(\chi, \phi)} \in \text{Aut}(A_R(\mathcal{G}), D_R(\mathcal{G})) \) by

\[ T_{(\chi, \phi)}f(a) = \chi(a)f(\phi^{-1}(a)). \]

The map \((\chi, \phi) \mapsto T_{(\chi, \phi)}\) is a group morphism and it is surjective by the previous theorem. As for injectivity, note that \( \phi \) and \( \chi \) are the homeomorphism and transform associated to the \( \|\|\)-isomorphism \( T_{(\chi, \phi^{-1})} \), and since these are unique we conclude that \((\chi, \phi) \mapsto T_{(\chi, \phi)}\) is an isomorphism.

\[ \square \]
4.4.3 Steinberg algebras as crossed products

Again, we consider a commutative ring with unit $R$. Given a zero-dimensional, locally compact Hausdorff space $X$, the Steinberg algebra of $X$, as a unit groupoid, is the algebra $A_R(X)$ of locally constant, compactly supported $R$-valued functions on $X$, with pointwise operations.

In [13], Beuter and Gonçalves showed that any Steinberg algebra of a transformation groupoid, given by a partial action of a group, $A_R(G \rtimes X)$, is isomorphic to a crossed product $A_R(G^0) \rtimes KB(G)$ (see [13] Theorem 3.2). Similarly, in [34], Demeneghi proved that any Steinberg algebra of a Hausdorff groupoid of germs associated to an ample action of an inverse semigroup $S$ is isomorphic to a crossed product $A_R(G(0)) \rtimes KB(G)$ (see [34, Theorem 2.3.6]). Note that these results are non-comparable, since [13] deals with partial actions of groups whereas [34] considers actions of semigroups.

Our current objective is to generalize both results above. More precisely, let $\theta = (\theta_s : X_s^* \to X_s)_{s \in S}$ be a topological partial action of an inverse semigroup $S$ on a zero-dimensional, locally compact Hausdorff space $X$. Define, for each $s \in S$,

$$D_s = \{ f \in A_R(X) : \text{supp } f \subseteq X_s \}.$$  

We will always identify $D_s$ with $A_R(X_s)$, by extending elements of $A_R(X_s)$ to all of $X$ by zero on $X \setminus X_s$. Define

$$\alpha_s : D_{s^*} \to D_s$$
$$f \mapsto f \circ \theta_s^{-1}$$

It is routine to check that $\alpha$ is an algebraic partial action of $S$ on $A_R(X)$.

**Definition 4.4.29.** $\alpha$ as above is called the partial action *induced* by $\theta$.

We will prove that, as long as the groupoid of germs $S \ltimes X$ is Hausdorff, the Steinberg algebra $A_R(S \ltimes X)$ is isomorphic to the crossed product algebra $A_R(X) \rtimes \alpha S$. In order to prove such an isomorphism, we need some preliminary lemmas. Following the notation of Subsection 4.2.3, recall that if $s \in S$ and $U \subseteq X_{s^*}$, we denote $[s,U] = \{ [s,x] : x \in U \}$, and that we identify $s([s,x])$ with $x$.

**Lemma 4.4.30.** Given $s \in S$ any subset $B$ of $[s,X_{s^*}]$ is of the form $B = [s,s(B)]$.

**Proof.** Since $[s,X_{s^*}]$ is a bisection (Proposition 4.2.12), then for any subset $B \subseteq [s,X_{s^*}]$ we have

$$B = s^{-1}(s(B)) \cap [s,X_{s^*}] = [s,s(B)].$$  

\qed
Lemma 4.4.31. Let \( S \bowtie X \) be a Hausdorff groupoid of germs, and consider two finite collections \( \{ [s_i, U_i] : i = 1, \ldots, n \} \) and \( \{ [t_j, V_j] : j = 1, \ldots, m \} \) of pairwise disjoint basic compact-open bisections of \( S \bowtie X \) such that

\[
\bigcup_{i=1}^{n} [s_i, U_i] = \bigcup_{j=1}^{m} [t_j, V_j].
\]

Then for each pair \( i, j \), there is a finite set \( \{ [u_{ki}^{ij}, W_{ki}^{ij}] \}_{k=1}^{l_{ij}} \) of pairwise disjoint basic neighbourhoods of \( S \bowtie X \) such that

\[
[s_i, U_i] \cap [t_j, V_j] = \bigcup_{k=1}^{l_{ij}} [u_{ki}^{ij}, W_{ki}^{ij}] \quad \text{and} \quad u_{ki}^{ij} \leq s_i, t_j.
\]

Proof. Given \( i \) and \( j \), let \( b \) be an element of \( [s_i, U_i] \cap [t_j, V_j] \). Then \( [s_i, s(b)] = [t_j, s(b)] \), so there is \( u_b \in S \) such that \( s(b) \in X_{u_b} \) and \( u_b \leq s_i, t_j \), so

\[
b = [u_b, s(b)] \in [u_b, X_{u_b} \cap U_i \cap V_j] \subseteq [s_i, U_i] \cap [t_j, V_j].
\]

Using compactness of \( [s_i, U_i] \cap [t_j, V_j] \), we find a finite cover for this set with elements of the form \( [u_{ki}^{ij}, W_{ki}^{ij}] \) for certain \( u_{ki}^{ij} \leq s_i, t_j \) and \( W_{ki}^{ij} \subseteq U_i \cap V_j \), that is,

\[
[s_i, U_i] \cap [t_j, V_j] = \bigcup_{k=1}^{l_{ij}} [u_{ki}^{ij}, W_{ki}^{ij}] .
\]

For \( k \geq 2 \), we can substitute \( [u_{ki}^{ij}, W_{ki}^{ij}] \) by \( [u_{ki}^{ij}, W_{ki}^{ij}] \setminus \bigcup_{p=1}^{k-1} [u_{ki}^{ij}, W_{ki}^{ij}] \) and use Lemma 4.4.30 to rewrite this set as \( [u_{ki}^{ij}, W_{ki}^{ij}] \) for appropriate \( W_{ki}^{ij} \), to obtain the desired partition of \( [s_i, U_i] \cap [t_j, V_j] \).

We now prove another main theorem of this section. We refer to the notations of Definitions 4.3.2, 4.3.6 and 4.3.8.

Theorem 4.4.32. Let \( \theta = (\theta_s : X_s^* \to X_s)_{s \in S} \) be a topological partial action of an inverse semigroup \( S \) on a zero-dimensional, locally compact Hausdorff space \( X \), and \( \alpha = (\alpha_s : D_s^* \to D_s)_{s \in S} \) the partial action induced by \( \theta \) (Definition 4.4.29).

If the groupoid of germs \( S \bowtie \theta X \) is Hausdorff then the Steinberg \( A_R(S \bowtie \theta X) \) is isomorphic, as an \( R \)-algebra, to the crossed product \( A_R(X) \bowtie \alpha S \).

Note that we cannot use the universal property of crossed products described in Theorem 4.3.15 because the algebra \( D_s \) is not unital if \( X_s \) is not compact.

The main idea of this proof is to identify a symbol \( \delta_s \), as in Definition 4.3.2 with the characteristic function of the bisection \( [s, X_s^*] \), and to use the inclusions \( D_s \subseteq A_R(X) \) and the identification of \( A_R(X) \) as the diagonal subalgebra of \( A_R(S \bowtie X) \) as in Definition 4.4.9 and the discussion following it.
Proof of 4.4.32. Given a generating element \( f_s\delta_s \) of \( \mathcal{L}(\alpha) \), let \( \phi(f_s\delta_s) : S \rtimes X \to R \) be given by
\[
\phi(f_s\delta_s)(a) = \begin{cases} 
  f_s(r(a)), & \text{if } a \in [s, X_s^*], \\
  0, & \text{otherwise.}
\end{cases}
\]

We first prove that \( \phi(f_s\delta_s) \in A_R(S \rtimes X) \), i.e., that \( \phi(f_s\delta_s) \) is locally constant and has compact support. Notice that the definition of \( \phi(f_s\delta_s) \) implies
\[
\text{supp}(\phi(f_s\delta_s)) = [s, \theta_s^{-1}(\text{supp}(f))],
\]
which is compact-open and in particular clopen because \( S \rtimes X \) is Hausdorff.

Of course, \( \phi(f_s\delta_s) \) is constant equal to 0 on the complement
\[
(S \rtimes X) \setminus \text{supp}(\phi(f_s\delta_s)),
\]
and since \( \phi(f_s\delta_s) \) coincides with the composition \( f_s \circ r \) on \( \text{supp}(\phi(f_s\delta_s)) \) and \( f_s \) is locally constant, then \( \phi(f_s\delta_s) \) is also locally constant on \( \text{supp}(\phi(f_s\delta_s)) \). We conclude that \( \phi(f_s\delta_s) \) is locally constant on complementary clopen subsets of \( S \rtimes X \), so \( \phi(f_s\delta_s) \) is locally constant.

Using the presentation of \( \mathcal{L}(\alpha) \) in Definition 4.3.2 we extend \( \phi \) linearly to an \( R \)-module morphism \( \phi : \mathcal{L}(\alpha) \to A_R(S \rtimes X) \). Let us prove that \( \phi \) vanishes all elements of form \( f_s\delta_s - f_s\delta_t \), where \( s \leq t \) and \( f_s \in D_s \). Given \( a \in S \rtimes X \), we have three possibilities:

**Case 1:** If \( a \in [s, X_s^*] \) then \( a \in [t, X_{t^*}] \), and
\[
\phi(f_s\delta_s - f_s\delta_t)(a) = f_s(r(a)) - f_t(r(a)) = 0;
\]

**Case 2:** if \( a \in [t, X_{t^*}] \setminus [s, X_s^*] \) then \( r(a) \notin X_s \), because \( r \) is injective on \( [t, X_{t^*}] \), and \( f_s(r(a)) = 0 \) because \( f_s \in D_s \). Thus
\[
\phi(f_s\delta_s - f_s\delta_t)(a) = 0 - f_t(r(a)) = 0.
\]

**Case 3:** if \( a \notin [t, X_{t^*}] \) then \( a \notin [s, X_s^*] \) as well, so
\[
\phi(f_s\delta_s - f_s\delta_t)(a) = 0 - 0 = 0;
\]

Therefore, \( \phi \) vanishes on the ideal \( \mathcal{M}(\alpha) \) and hence factors through the quotient \( \mathcal{L}(\alpha)/\mathcal{M}(\alpha) = A_R(X) \rtimes S \) to an \( R \)-module morphism \( \Phi : A_R(X) \rtimes S \to A_R(S \rtimes X) \). At this point, we have not checked that \( \Phi \) is an algebra morphism.

We will now construct the inverse of \( \Phi \) using the universal property of Steinberg algebras, Theorem 4.4.8
Given $U \in \mathbf{KB}(S \times X)$, decompose $U$ as a disjoint union $U = \bigcup_i [s_i, U_i]$ for certain $s_i \in S$ and $U_i \subseteq X_{s_i}$. Define

$$\psi(U) = \sum_{i=1}^{n} 1_{\theta_{s_i}(U_i)} \delta_{s_i} \in A_R(X) \rtimes S$$

We need to prove that $\psi$ does not depend on the given decomposition of $U$. Suppose $U = \bigcup_i [s_i, U_i] = \bigcup_j [t_j, V_j]$, both unions being disjoint. By Lemma 4.4.31 we can find collections of elements $u_k^{ij} \leq s_i, t_j$ and $W_k^{ij} \subseteq U_i \cap V_j$ such that for all $i$ and $j$,

$$[s_i, U_i] \cap [t_j, V_j] = \bigcup_{k=1}^{l^{ij}} [u_k^{ij}, W_k^{ij}]$$

For a fixed $i$, we have

$$[s_i, U_i] = \bigcup_{j=1}^{m} [s_i, U_i] \cap [t_j, V_j] = \bigcup_{j=1}^{m} \bigcup_{k=1}^{l^{ij}} [u_k^{ij}, W_k^{ij}]$$

as a disjoint union, so comparing their ranges we obtain

$$\theta_{s_i}(U_i) = \bigcup_{j=1}^{m} \bigcup_{k=1}^{l^{ij}} \theta_{u_k^{ij}}(W_k^{ij})$$

and this union is also disjoint. Therefore

$$1_{\theta_{s_i}(U_i)} = \sum_{j=1}^{m} \sum_{k=1}^{l^{ij}} 1_{\theta_{u_k^{ij}}(W_k^{ij})}$$

and similarly, for a fixed $j$, $1_{\theta_{t_j}(V_j)} = \sum_{i=1}^{n} \sum_{k=1}^{l^{ij}} 1_{\theta_{u_k^{ij}}(u_k^{ij})}$.

Now we can compute

$$\sum_{i=1}^{n} 1_{\theta_{s_i}(U_i)} \delta_{s_i} = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l^{ij}} 1_{\theta_{u_k^{ij}}(W_k^{ij})} \delta_{s_i} = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l^{ij}} \sum_{k=1}^{l^{ij}} 1_{\theta_{u_k^{ij}}(W_k^{ij})} \delta_{u_k^{ij}} \delta_{s_i}$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{l^{ij}} 1_{\theta_{u_k^{ij}}(W_k^{ij})} \delta_{t_j} = \sum_{j=1}^{m} \sum_{i=1}^{n} 1_{\theta_{t_j}(V_j)} \delta_{t_j}$$

where the equalities marked by (*) follow because $1_{\theta_{u_k^{ij}}(W_k^{ij})} \in D_{u_k^{ij}}$ and $u_k^{ij} \leq s_i, t_j$.

Therefore $\psi(U)$ does not depend on the initial decomposition of $U$. We still need to prove it is a Boolean representation.
First, suppose \([s, U], [t, V]\) are basic bisections in \(\text{KB}(S \times X)\). Then
\[
[s, U][t, V] = [st, V \cap \theta_t^{-1}(U \cap X_t)],
\]
so
\[
\psi([s, U][t, V]) = \underbrace{1_{\theta_s(t \cup \theta_t^{-1}(U \cap X_t))}}_{\delta_{st}}.
\]
But \(V \cap \theta_t^{-1}(U \cap X_t)\) is contained in \(X_t^* \cap \theta_t^{-1}(X_{s*} \cap X_t)\), which is the domain of \(\theta_s \circ \theta_t\), so we may compute
\[
\theta_{st}(V \cap \theta_t^{-1}(U \cap X_t)) = \theta_s(\theta_t(V \cap \theta_t^{-1}(U \cap X_t)))
\]
\[
= \theta_s(\theta_t(V) \cap U \cap X_t) = \theta_s(\theta_t(V) \cap U)
\]
and therefore
\[
\psi([s, U][t, V]) = \underbrace{1_{\theta_s(\theta_t(V) \cap U)}}_{\delta_{st}}
\] (4.4.6)
On the other hand,
\[
\psi([s, U])\psi([t, V]) = (1_{\theta_s(U)}\delta_s)(1_{\theta_t(V)}\delta_t) = \alpha_s(\alpha_s^*(1_{\theta_s(U)}1_{\theta_t(V)})\delta_{st}.
\]
Since
\[
\alpha_s^*(1_{\theta_s(U)}) = 1_{\theta_s(U)} \circ \theta_s = 1_U
\]
then
\[
\alpha_s(\alpha_s^*(1_{\theta_s(U)})1_{\theta_t(V)}) = \alpha_s(1_U1_{\theta_t(V)}) = \alpha_s(1_U\theta_t(V)) = 1_{U \cap \theta_t(V)} \circ \theta_s = 1_{\theta_s(\theta_t(V))}
\]
and thus
\[
\psi([s, U])\psi([t, V]) = \underbrace{1_{\theta_s(\theta_t(V) \cap U)}}_{\delta_{st}}
\] (4.4.7)
so Equations (4.4.6) and (4.4.7) imply that \(\psi([s, U][t, V]) = \psi([s, U])\psi([t, V])\). The definition of \(\psi\) then readily implies that \(\psi(AB) = \psi(A)\psi(B)\) for all compact-open bisections \(A, B \subseteq S \times X\), i.e., \(\psi\) is a semigroup morphism.

The definition of \(\psi\) also makes it clear that \(\psi(A \cup B) = \psi(A) + \psi(B)\) whenever \(A\) and \(B\) are compatible compact-open bisections of \(S \times X\) with \(A \cap B = \emptyset\), so \(\psi\) is a Boolean representation of \(\text{KB}(S \times X)\).

Then by the universal property \(A_R(G)\) (Theorem 4.4.8), there exists a unique \(R\)-algebra morphism \(\Psi : A_R(S \times X) \to A_R(X) \times S\) satisfying \(\Psi(1_U) = \psi(U)\) for all \(U \in \text{KB}(G)\).

It remains to check that \(\Psi\) is the inverse of \(\Phi\). Suppose \(s \in S\) and \(U \subseteq X_{s*}\). Then
\[
\Phi(\Psi(1_{[s, U]})) = \Phi(1_{\theta_s(U)}\delta_s) = \begin{cases} 1_{\theta_s(U)} \circ r & \text{on } [s, X_{s*}] \\ 0 & \text{everywhere else} \end{cases}
\]
\[
= 1_{[s, U]}.
\]
4. PARTIAL ACTIONS OF INVERSE SEMIGROUPS

Since the family \( \{ 1_{[s,U]} : s \in S, U \subseteq S^2 \} \) generates \( A_R(S \times X) \), as an \( R \)-module, then \( \Phi(\Psi(f)) = f \) for all \( f \in A_R(S \times X) \).

In the other direction, if \( s \in S \) and \( U \subseteq X_s \), by the definition of \( \phi \) we have

\[
\Phi(1_U \delta_s) = 1_{[s, \theta_s^{-1}(U)]},
\]
so

\[
\Psi(\Phi(1_U \delta_s)) = \Psi(1_{[s, \theta_s^{-1}(U)]}) = 1_{\theta_s(\theta_s^{-1}(U))} \delta_s = 1_U \delta_s.
\]

Since \( \{ 1_U \delta_s : s \in S, U \subseteq X_s \} \) generates \( A_R(X) \rtimes S \) as an \( R \)-module, we conclude that \( \Psi(\Phi(x)) = x \) for all \( x \in A_R(X) \rtimes S \). Therefore, \( \Phi = \Psi^{-1} \), and \( \Psi \) is an algebra isomorphism.

**Remark 4.4.33.** Note that the isomorphism \( \Phi : A_R(X) \rtimes S \to A_R(S \times X) \) obtained in the proof of Theorem 4.4.32 above maps the diagonal subalgebra of \( A_R(X) \rtimes S \) (as in Definition 4.3.10) to the diagonal subalgebra of \( A_R(S \times X) \) (as in Definition 4.4.9). Under the usual identifications of both of these diagonal subalgebras as \( A_R(X) \), the isomorphism \( \Phi \) restricts to the identity of \( A_R(X) \).

**Corollary 4.4.34.** Let \( \mathcal{G} \) be an ample Hausdorff groupoid. Then the Steinberg Algebra \( A_R(\mathcal{G}) \) is isomorphic to the crossed products \( A_R(\mathcal{G}^{(0)}) \rtimes_\mu B(\mathcal{G}) \) and \( A_R(\mathcal{G}^{(0)}) \rtimes_\eta KB(\mathcal{G}) \), where \( \mu \) and \( \eta \) are induced by the canonical actions of \( B(\mathcal{G}) \) and \( KB(\mathcal{G}) \) on \( \mathcal{G}^{(0)} \) (Example 4.2.20).

**Proof.** By Example 4.2.20, \( \mathcal{G} \) is isomorphic to the groupoids of germs \( B(\mathcal{G}) \rtimes \mathcal{G}^{(0)} \) and \( KB(\mathcal{G}) \rtimes \mathcal{G}^{(0)} \), given by the respective canonical actions of \( B(\mathcal{G}) \) and \( KB(\mathcal{G}) \) on \( \mathcal{G}^{(0)} \). Then the desired result follows from Theorem 4.4.32.

It is interesting to note that the crossed products \( A_R(\mathcal{G}^{(0)}) \rtimes B(\mathcal{G}) \) and \( A_R(\mathcal{G}^{(0)}) \rtimes KB(\mathcal{G}) \) arise from global actions and not simply partial actions. Further, using Theorem 4.4.32 and Corollary 4.4.34 for the groupoid of germs of a partial action, we obtain

\[
A_R(X) \rtimes S \simeq A_R(S \times X) \simeq A_R(X) \rtimes_\eta KB(S \times X),
\]
where \( \eta \) is induced by the canonical action of \( KB(S \times X) \) on \( (S \times X)^{(0)} \simeq X \).

### 4.4.4 Constructing a Steinberg algebra from a crossed product \( A_R(X) \rtimes S \)

In the previous section we saw that the Steinberg algebra of an ample Hausdorff groupoid of germs can be seen as a crossed product of its diagonal subalgebra. In this section we will be interested in the opposite direction, that is, to characterize the crossed products of the form \( A_R(X) \rtimes_\theta S \) which can be realized as Steinberg algebras \( A_R(S \rtimes_\theta X) \) in such a way that \( \alpha \) is the partial action induced by \( \theta \). To do this, we
need to perform a “point-realization” of algebraic partial actions on algebras of the form $A_R(X)$.

Some results of this nature are already known. For example, in the C*-algebraic case, if $X$ is a locally compact Hausdorff space, then every closed ideal of $C_0(X)$ is of the form $C_0(U) \cong \{ f \in C(X) : f = 0 \text{ on } X \setminus U \}$ for some (unique) open set $U \subseteq X$ ([168 Exercise 3.2.3(5)]), and the Gelfand-Naimark Theorem ([133 Theorem 2.1.10]) implies that every $\ast$-isomorphism $T : C_0(U) \to C_0(V)$ between two such ideals is of the form $T(f) = f \circ \phi$ for some (unique) homeomorphism $\phi : V \to U$. This gives a one-to-one correspondence between algebraic partial actions of an inverse semigroup $S$ on $C_0(X)$ (by $\ast$-automorphisms of closed ideals) and topological partial actions of $S$ on $X$.

In [12], an analogous result in the purely algebraic setting was proven. Namely, let $K$ be a field, $X$ be a set, and denote by $F_0(X)$ the $K$-algebra of $K$-valued functions on $X$ with finite support, endowed with the pointwise operations. Then ideals of $F_0(X)$ and their isomorphisms correspond to subsets and partial bijections of $X$, in a manner similar to the one above. This also yields a one-to-one correspondence between the algebraic partial actions of an inverse semigroup $S$ on $X$ and the topological partial actions of $S$ on $F_0(X)$.

In this section, we will show that the same occurs with partial actions of inverse semigroups on non-discrete spaces. In order to obtain a bijective correspondence between partial actions of $S$ on $X$ and their induced partial actions on $A_R(X)$, we need a few preliminary results. We will use the terminology and notation of Chapter 3.

**Proposition 4.4.35.** Let $R$ be an indecomposable ring and let $X$ and $Y$ be zerodimensional, locally compact Hausdorff spaces. A map $T : A_R(X) \to A_R(Y)$ is an isomorphism of $R$-algebras if and only if there exists a (necessarily unique) homeomorphism $\phi : Y \to X$ such that $T(f) = f \circ \phi$ for all $f \in A_R(X)$.

**Proof.** The “if” part is straightforward, so suppose that $T : A_R(X) \to A_R(Y)$ is an isomorphism of $R$-algebras. As unit groupoids, $X$ and $Y$ are principal, and their Steinberg algebras $A_R(X)$ and $A_R(Y)$ coincide with the respective diagonals $D_R(X)$ and $D_R(Y)$.

By Theorem 4.4.27, we have $Tf(y) = \chi(y)(f(\phi(y)))$ for some homeomorphism $\phi : Y \to X$ and a cocycle $\chi$ (Definition 4.4.23). Moreover, the map

$$Y \times R \to R, \quad (y, r) \mapsto \chi(y)(r)$$

is the $T$-transform (Definition 3.3.1). Since all operations are pointwise, each section $\chi(y)$ of the $T$-transform is an $R$-algebra automorphism of $R$ (Theorem 3.3.10), so it is necessarily the identity automorphism. This means that $Tf = f \circ \phi$ for all $f \in A_R(X)$. $\square$
We now classify ideals with local units of the $R$-algebra $A_R(X)$, as in Theorem 3.2.6. Using the same notation of Section 3.2, note that if $U$ is an open subset of $X$, then

$$I(U) := \{f \in A_R(X) : \text{supp}(f) \subseteq U\}$$

is an ideal of $A_R(X)$ with local units. Indeed, if $f_1, \ldots, f_n \in I(U)$ then the characteristic function $1_K$ of $K = \bigcup_{i=1}^n \text{supp}(f_i)$ is a local unit for $\{f_1, \ldots, f_n\}$. Also note that $U$ is compact if and only if $I(U)$ has a unit (namely, the characteristic function $1_U$ is its identity).

**Proposition 4.4.36.** Let $R$ be an indecomposable ring and $X$ a zero-dimensional, locally compact Hausdorff space. Then the map $U \mapsto I(U)$ is an order isomorphism between the lattices of open subsets of $X$ and ideals with local units of $A_R(X)$. The inverse map is given by $I \mapsto U(I) := \bigcup_{f \in I} \text{supp}(f)$.

**Proof.** We will prove that ideals with local units of $A_R(X)$ are precisely the $\perp$-ideals (Definition 3.2.3). By Theorem 3.2.6 and the discussion before this proposition, every $\perp$-ideal of $A_R(X)$ is an ideal with local units of the $R$-algebra $A_R(X)$.

Assume now that $I$ is an ideal with local units of $A_R(X)$. We need to prove that an element $a \in A_R(X)$ belongs to $I$ if and only if it admits a finite strong cover (Definition 3.2.1) by elements of $I$. On one hand, if $f \in I$ then $\{f\}$ is a finite strong cover of $f$ contained in $I$.

Conversely, assume $f \in A_R(X)$ admits a finite strong cover $B \subseteq I$, so $\text{supp}(f) \subseteq \bigcup_{b \in B} \text{supp}(b)$ (Proposition 3.2.2). Let $u \in I$ be a local unit for $B$. In particular $u$ is an idempotent, so $u = 1_A$ for some compact-open subset $A \subseteq X$, and for all $b \in B$ and all $x \in \text{supp}(b)$,

$$u(x)b(x) = b(x) \neq 0$$

so $u(x) = 1$. This proves that $u = 1$ on $\bigcup_{b \in B} \text{supp}(b)$, which contains $\text{supp}(f)$, and thus $f = uf \in I$.

Therefore, ideals with local units of the $R$-algebra $A_R(X)$ are the same as $\perp$-ideals, so the result follows from Theorem 3.2.6. \qed

**Proposition 4.4.37.** Let $R$ be an indecomposable ring, $X$ a zero-dimensional, locally compact Hausdorff space and $S$ an inverse semigroup. If $\alpha = (\alpha_s : D_s^* \to D_s)_{s \in S}$ is an algebraic partial action of $S$ on the $R$-algebra $A_R(X)$, such that each ideal $D_s$ has local units, then there is a topological partial action $\theta = (\theta_s : X^* \to X_s)_{s \in S}$ of $S$ on $X$ which induces $\alpha$ (see Definition 4.4.29).

**Proof.** The proof is similar to that of Theorem 4.2.4.

Given $\alpha$ as above, for each $s \in S$ let $X_s = U(D_s)$ be the open subset of $X$ associated to $D_s$. Under the usual identification of $D_s = I(X_s)$ with $A_R(X_s)$, we see each map $\alpha_s$ as an $R$-algebra isomorphism

$$\alpha_s : D_s^* \simeq A_R(X_s^*) \to A_R(X_s) \simeq D_s$$
By Proposition 4.4.35, for each isomorphism \( \alpha_s : A_R(X_{s^*}) \to A_R(X_s) \), there is a unique homeomorphism \( \theta_{s^*} : X_s \to X_{s^*} \) such that
\[
\alpha_s(f) = f \circ \theta_{s^*} \quad \text{for all } f \in A_R(X_{s^*}) \simeq D_{s^*}.
\]
The homeomorphisms \( \theta_s \) satisfy properties (1)-(3) of Theorem 4.2.4, so \( \theta : s \mapsto \theta_s \) is a partial action of \( S \) on \( X \) which induces \( \alpha \).

So from an algebraic partial action \( \alpha \) on \( A_R(X) \) we managed to obtain an appropriate topological partial action \( \theta \) on \( X \). In order to identify the crossed product algebra of \( \alpha \) with the Steinberg algebra of the groupoid of germs of \( \theta \), we need to guarantee that the conditions of Theorem 4.4.32 are satisfied.

First we describe the ideals associated with clopen sets algebraically. The following definition is an algebraic version of [21, Definition 1.5.9].

**Definition 4.4.38.** A conditional expectation of a \( R \)-algebra \( A \) onto a subalgebra \( B \) is an \( R \)-module map \( E : A \to B \) such that

1. \( E(b) = b \) for all \( b \in B \) (i.e., \( E \) is a projection onto \( B \));
2. For all \( a \in A \) and \( b \in B \), \( E(ba) = bE(a) \) and \( E(ab) = E(a)b \) (i.e., \( E \) is a \( B \)-bimodule morphism).

**Remark.** If \( B \) has local units then condition (ii) above can be substituted by

(ii)' For all \( a \in A \) and \( b, b' \in B \), \( E(bab') = bE(a)b' \).

Indeed, if \( a \in A \) and \( b \in B \), let \( u \in B \) be a unit for \( \{E(ab), b, E(a)\} \). Then
\[
E(ab) = uE(ab)u \overset{\text{(ii)'} }= E(uabu) = E(uab) = uE(a)b = E(a)b,
\]
and similarly \( E(ba) = bE(a) \), so (ii) is satisfied.

**Example 4.4.39.** Let \( R \) be any (nontrivial) commutative unital ring. We consider the subalgebras \( B \subseteq A \) of the \( 2 \times 2 \) matrix algebra \( M_2(R) \)
\[
A = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in R \right\} \quad \text{and} \quad B = \left\{ \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} : a \in R \right\}.
\]

Let \( E : A \to B \) be given by \( E\left( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \). Then \( E \) is a projection of \( A \) onto \( B \) and satisfies (ii)', because \( BAB = 0 \). However, \( E \) is not a left (nor right) \( B \)-module morphism, because
\[
\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} E \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = 0, \quad \text{but} \quad E \left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\]
Lemma 4.4.40. Let $R$ be an indecomposable ring, $X$ a zero-dimensional, locally compact Hausdorff space and $U$ an open subset of $X$. Then the following are equivalent:

1. $U$ is clopen;

2. There exists a conditional expectation of $A_R(X)$ onto $I(U)$;

3. There exists an ideal $J$ of $A_R(X)$ such that $A_R(X) = I(U) \oplus J$ (as $R$-modules).

Proof. (1)⇒(2): If $U$ is clopen, then $1_U$ is continuous and the map $E : A_R(X) \to I(U)$, $E(f) = f 1_U$ is a conditional expectation. (Even though that $1_U$ does not belong to $I(U)$ is not compact.)

(2)⇒(3): Suppose $E : A_R(X) \to I(U)$ is a conditional expectation, and define $J = \{ f - E(f) : f \in A \}$. Then $J$ is an $R$-submodule of $A_R(X)$ such that $A_R(X) = I(U) \oplus J$, since $E$ is a projection onto $I(U)$. Let us check that $J$ is an ideal: Suppose $f \in A_R(X)$ and $g \in I(U)$. Let $u \in I(U)$ be a local unit for $\{E(fg), f\}$. Then since $E$ is a right $I(U)$-morphism, and $I(U)$ is an ideal:

$$E(fg) = E(fg)u = E(fgu) = E(f)gu = E(f)g$$

so

$$(f - E(f))g = fg - E(f)g = fg - E(fg) \in J$$

and therefore $J$ is a right ideal. Similarly it is a left ideal.

(3)⇒(1): Suppose that $A_R(X) = I(U) \oplus J$ for some ideal $J$. Let us show that $J \subseteq I(X \setminus \overline{U})$. Indeed, if $f \in J$ and $x \in \overline{U}$, then $f$ is constant on a neighbourhood $V$ of $x$. Choose $y \in U \cap V$, and $W$ a compact-open neighbourhood satisfying $y \in W \subseteq U \cap V$. Since $I(U)$ and $J$ are complementary ideals, then $f 1_W = 0$ and in particular

$$f(x) = f(y) = (f 1_W)(y) = 0.$$

Therefore $J \subseteq I(X \setminus \overline{U})$. We can now prove that $U$ is clopen.

Given $x \in \overline{U}$, let $V$ be any compact-open neighbourhood of $x$. By hypothesis, we may write $1_V = f + g$ for some $f \in I(U)$ and $g \in J \subseteq I(X \setminus \overline{U})$, so $1 = 1_V(x) = f(x) + g(x) = f(x)$. In particular $x \in supp(f) \subseteq U$. Therefore $U$ is clopen.

Remark. From Equation (4.4.8) it follows that the conditional expectation $E : A_R(X) \to I(U)$ is an $R$-algebra morphism.

By Lemma 4.4.40 and Theorems 4.2.23 and 4.4.32 we conclude the following:

Theorem 4.4.41. Let $S$ be an inverse semigroup which is a weak semilattice, $R$ be an indecomposable ring and $X$ a zero-dimensional, locally compact Hausdorff space. Let $\alpha = (\alpha_s : D_s \to D_s)_{s \in S}$ be an algebraic partial action of $S$ on $A_R(X)$ where each ideal $D_s$ has local units and satisfies any condition of Lemma 4.4.40.

Then $A_R(X) \ltimes \alpha X$ is isomorphic to a Steinberg algebra $A_R(S \ltimes \theta X)$ where $\theta$ is a topological partial action of $S$ on $X$ which induces $\alpha$. 
4.5 Partial actions from associated groups and inverse semigroups

4.5.1 $E$ and $E^*$-unitary inverse semigroups

We will now describe how to construct partial actions of groups from actions of inverse semigroups and vice-versa. The class of inverse semigroups which allows us to do this in a more precise manner is that of $E$-unitary inverse semigroups.

Definition 4.5.1. An inverse semigroup $S$ is

1. $E$-unitary if whenever $s \in S$ and $e \in E(S)$, if $e \leq s$ then $s \in E(S)$;

2. $E^*$-unitary (sometimes called $0$-$E$-unitary) if it has a zero and whenever $s \in S$ and $e \in E(S) \setminus \{0\}$, if $e \leq s$ then $s \in E(S)$.

Note that an inverse semigroup $S$ with $0$ is $E$-unitary if and only if $S = E(S)$, which is a somewhat trivial case. Moreover, we can always adjoin a zero (that is, an absorbing) element $0$ to any semigroup $S$, and the semigroup $S_0$ obtained is $E^*$-unitary if and only if $S$ is $E$-unitary.

Example 4.5.2. Every group is $E$-unitary.

Example 4.5.3. Every $\wedge$-semilattice is $E$-unitary.

Example 4.5.4. If $S$ is $E^*$-unitary then it is a semilattice: Indeed, given $s, t \in S$, if $\{s, t\}$ does not admit any nonzero lower bound then $s \wedge t = 0$. If $\{s, t\}$ admits a nonzero lower bound, then $s$ and $t$ are compatible, so $s \wedge t = ts^*s$.

As a consequence, every $E$-unitary inverse semigroup $S$ is a weak semilattice: We can embed $S$ into an $E^*$-unitary semigroup $S_0$ by adjoining a $0$. Given $s, t \in S$, let $F = \{s \wedge t\} \setminus \{0\}$, which is either empty or equal to $\{s \wedge t\}$, but in any case a finite subset of $S$, so that $\{x \in S : x \leq s, t\} = \bigcup_{f \in F} \{x \in S : x \leq f\}$.

Let us rewrite the $E$ and $E^*$-unitary condition in terms of the compatibility relation.

Lemma 4.5.5. [108, Theorem 2.4.6]

(a) An inverse semigroup $S$ with $0$ is $E^*$-unitary if and only if whenever $s, t \in S$ have a non-zero common lower bound then $s$ and $t$ are compatible.

(b) An inverse semigroup $S$ is $E$-unitary if and only if whenever $s, t \in S$ have a common lower bound then $s$ and $t$ are compatible.
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Proof. (a) First assume $S$ is $E^*$-unitary and $s, t \in S$ have a common nonzero lower bound $u \in S \setminus \{0\}$. Then $u^* u \leq s^* t, t^* s$ and $uu^* \neq 0$, so both $s^* t$ and $st^*$ are idempotent and $s$ and $t$ are compatible.

Conversely, suppose the latter condition is satisfied, $e \leq s$ and $e \in E(S) \setminus \{0\}$. Then $e \leq s, s^* s$, which implies that $s$ and $s^* s$ are compatible and thus $s = s(s^* s)^*$ is an idempotent.

Item (b) follows from (a) by adjoining a 0 to $S$.

4.5.2 Maximal group image

To each inverse semigroup $S$ we can naturally associate a group $G(S)$ in the following manner: define a relation in $S$ by

$$s \sim t \iff \exists u \in S \text{ such that } u \leq s, t.$$  \hfill (4.5.1)

Alternatively, $s \sim t$ if and only if there exists $e \in E(S)$ such that $es = et$ (and the proof of this equivalence is similar to that in Remark 4.2.5).

Using the fact that the order of $S$ is preserved under products and inverses, it is easy enough to see that $\sim$ is in fact a congruence, so we endow the quotient set $S/\sim$ with the quotient quotient semigroup structure. Given $s \in S$, we denote by $[s]$ the equivalence class of $s$ with respect to the relation (4.5.1).

Proposition 4.5.6 ([135 Proposition 2.1.2]). Let $S$ an inverse semigroup. The quotient

$$G(S) := S/\sim$$

is a group. Furthermore, $G(S)$ is the maximal group homomorphic image of $S$ in the sense that every semigroup morphism $\theta : S \to G$ from $S$ to a group $G$ factors through $G(S)$.

Proof. Let $e \in E(S)$ be arbitrary and set $u = [e]$. If $f \in E(S)$ is any other element, then $ef \leq e, f$, so $[f] = [e] = u$.

If $s \in S$, then

$$[s]u = [s][s^* s] = [ss^* s] = [s]$$

and similarly $u[s] = [s]$, so $u$ is a unit for $G(S)$, and since

$$[s][s^*] = [ss^*] = u$$

then $[s^*]$ is the inverse of $[s]$, so $G(S)$ is a group.

If $G$ is a group and $\theta : S \to G$ is a morphism, then suppose $s \sim t$, that is, there is $u \in S$ with $u \leq s, t$, so $\theta(u) \leq \theta(s), \theta(t)$. But since the order on $G$ is equality, we have $\theta(s) = \theta(u) = \theta(t)$, so $\theta$ factors through a morphism $G(S) \to G$. 

Example 4.5.7. If $G$ is a group then $G(G)$ is isomorphic to $G$.

Example 4.5.8. If $E$ is a $\wedge$-semilattice then $G(E) = \{1\}$ is the trivial group.

Example 4.5.9. If $S$ is an inverse semigroup with a zero, then $G(S) = \{1\}$ is the trivial group.

We will now describe how partial actions of $E$-unitary inverse semigroups induce partial actions of their maximal group homomorphic images. The following lemma will be necessary:

Lemma 4.5.10. Suppose $\theta$ is a partial action of an inverse semigroup $S$ on a set $X$. If $s,t \in S$ are compatible, then $\theta_s, \theta_t \in I(X)$ are compatible.

Proof. If $s$ and $t$ are compatible then $s^*t \in E(S)$, so $\theta_s t \in E(I(X))$. Since $\theta_s^* \theta_t \leq \theta_{s^* t}$, then $\theta_s^* \theta_t$ is an idempotent, and similarly $\theta_s \theta_t^*$ is also idempotent, so $\theta_s$ and $\theta_t$ are compatible.

Theorem 4.5.11. Let $\theta = (\theta_s : X_s^* \rightarrow X_s)_{s \in S}$ be a topological partial action of an $E$-unitary inverse semigroup $S$ on a topological space $X$.

Then there is a unique topological partial action $\tilde{\theta} = (\tilde{\theta}_\gamma : \tilde{X}_\gamma^{-1} \rightarrow \tilde{X}_\gamma)_{\gamma \in G(S)}$ of $G(S)$ on $X$ such that

(i) $\tilde{X}_\gamma = \bigcup_{[s] = \gamma} X_s$ for all $\gamma \in G(S)$;

(ii) $\tilde{\theta}_{\gamma s}(x) = \theta_s(x)$ for all $s \in S$ and $x \in X_s^*$;

(in other words, $\tilde{\theta}_\gamma$ is the join of $\{\theta_s : [s] = \gamma\}$ in $I(X)$).

Remark. If one allows degenerate partial actions, then item (i) implies that $\theta$ is non-degenerate if and only if $\tilde{\theta}$ is non-degenerate.

Proof of Theorem 4.5.11. Given $s,t \in S$, if $[s] = [t]$ then $s$ and $t$ admit a common lower bound, and since $S$ is $E$-unitary, this implies that $s$ and $t$ are compatible (Lemma 4.5.5), so $\theta_s$ and $\theta_t$ are compatible (Lemma 4.5.10).

Thus, given $\gamma \in G(S)$, the family $\{\theta_s : s \in \gamma\}$ is compatible in $I(X)$, so we may define (as in Example 1.2.36) $\tilde{\theta}_\gamma = \bigvee_{s \in \gamma} \theta_s$. Letting $\tilde{X}_\gamma$ be given as in item (i), note that $\tilde{X}_\gamma^{-1}$ and $\tilde{X}_\gamma$ are, respectively, the domain and range of $\tilde{\theta}_\gamma$. By definition, we have $\tilde{\theta}_\gamma^{-1} = \tilde{\theta}_\gamma^{-1}$ for all $\gamma \in G(S)$.

Using infinite distributivity of $I(X)$ (Example 1.2.36) we obtain, for all $\gamma, \delta \in G(S)$,

$$\tilde{\theta}_\gamma \circ \tilde{\theta}_\delta = \bigvee_{s \in \gamma} \theta_s \circ \theta_t \leq \bigvee_{s \in \gamma, t \in \delta} \theta_{st} \leq \bigvee_{z \in \gamma \delta} \theta_z = \tilde{\theta}_{\gamma \delta}.$$
Hence \( \tilde{\theta} = \left( \tilde{\theta}_\gamma : \tilde{X}_{\gamma^{-1}} \to \tilde{X}_\gamma \right)_{\gamma \in G(S)} \) is a partial action, by Proposition 4.1.3. Note that for each \( \gamma \in G(S) \), \( X_\gamma \) is a union of open sets, hence open, and \( \theta_\gamma \) is a join of continuous functions on open sets, hence continuous. Therefore \( \tilde{\theta} \) is a topological partial action satisfying (i) and (ii).

Proposition 4.5.12. Let \( \theta \) be a topological partial action of an \( E \)-unitary inverse semigroup \( S \) on a space \( X \) and \( \tilde{\theta} \) be the induced partial action on \( G(S) \). Then

\[
S \ltimes_{\theta} X \cong G(S) \ltimes_{\tilde{\theta}} X
\]

Proof. Consider the map \([s, x] \mapsto ([s], x)\). Since \( S \) is \( E \)-unitary this map is well-defined and is clearly a morphism, and surjectivity follows since \( \tilde{\theta}_\gamma = \bigsqcup_{[s] = \gamma} \theta_s \). As for injectivity, suppose \( ([s], x) = ([t], y) \), so \( x = y \) and \( [s] = [t] \). We can in fact assume \( x \in X_{s^*} \cap X_{t^*} \). Then \( s \) and \( t \) are compatible, which implies \( s \wedge t = st^*t \). Since

\[
x \in X_{s^*} \cap X_{t^*} \subseteq X_{s^*} \cap X_{t^*t} = \theta_{t^*t}(X_{s^*} \cap X_{t^*t}) \subseteq X_{t^*t^*} = X_{(s \wedge t)^*}
\]

we conclude that \([s, x] = [t, y]\). \( \Box \)

4.5.3 Universal inverse semigroups

At this point, we have described a strong relationship between partial actions of an \( E \)-unitary inverse semigroup and partial actions of the associated (maximal homomorphic image) group. Here we will be interested in the other direction, that is, to a group \( G \) we want to associate a semigroup \( S \) with a map \( G \to S \) such that every partial action of \( G \) factors through a partial action of \( S \). Of course, we could simply take \( S = G \), so instead we will look for a semigroup \( S \) such that partial action of \( G \) factor through global actions of \( S \).

This construction was initially done in [48], where Exel defines the universal inverse semigroup \( S(G) \) of a group \( G \), with the property that partial actions of \( G \) correspond to global actions of \( S(G) \). This was further generalized in [22], where Buss and Exel extended this theory from groups to inverse semigroups. However, we will deal only with the group case, as explained above.

Given a group \( G \), let \( S(G) \) be the universal semigroup generated by symbols of the form \([t], t \in G\), modulo the relations

(i) \([s^{-1}][s][t] = [s^{-1}][st]\);

(ii) \([s][t][t^{-1}] = [st][t^{-1}]\);

(iii) \([s][1] = [s]\);

(iv) \([1][s] = [s]\);
For every $t \in G$, denote by $\epsilon_t$ the element $\epsilon_t = [t][t^{-1}]$. We call $S(G)$ the universal semigroup of $G$. The next theorem lists all the necessary properties of $S(G)$ we will need.

**Theorem 4.5.13** ([IS] Propositions 2.3, 2.5 and 3.2; Theorems 3.4 and 4.2). Let $G$ be a group. Then $S(G)$ is an inverse semigroup such that

(a) For all $s \in G$, $[s]^* = [s^{-1}]$;

(b) For every $\gamma \in S(G)$, there are a unique $n \geq 0$ and distinct elements $r_1, \ldots, r_n, s$ in $G$ such that

\[
(i) \quad \gamma = \epsilon_{r_1} \cdots \epsilon_{r_n} [s]; \\
(ii) \quad r_i \neq 1 \text{ and } r_i \neq s \text{ for all } i; \\
(iii) \quad r_i \neq r_j \text{ for } i \neq j;
\]

We call such a decomposition $\gamma = \epsilon_{r_1} \cdots \epsilon_{r_n} [s]$ the standard form of $\gamma$, which is unique up to the order of $r_1, \ldots, r_n$.

The description of $S(G)$ with generators and relations also immediately implies the following:

**Theorem 4.5.14** ([IS] Theorem 4.2). Let $G$ be a group. If $\theta$ is a topological partial action of $G$ on a topological space $X$, then there is a unique topological global action $\tilde{\theta}$ of $S(G)$ on $X$ such that $\tilde{\theta}_t = \theta_t$ for all $t \in G$.

**Proposition 4.5.15.** Let $G$ be a group and $S(G)$ its universal inverse semigroup. Then

(a) $S(G)$ is $E$-unitary (see [IS] Remark 3.5); \\
(b) The map $G \to G(S(G))$, $g \mapsto [[g]]$, is an isomorphism.

**Proof.** (a) Suppose $\epsilon \leq \alpha$ where $\epsilon \in E(S(G))$. Writing $\alpha$ and $\epsilon$ in standard form, we obtain

\[
\alpha = \epsilon_{s_1} \cdots \epsilon_{s_n} [s] \quad \text{and} \quad \epsilon = \epsilon_{\epsilon_1} \cdots \epsilon_{\epsilon_m} [1]
\]

Since $\epsilon \leq \alpha$ and $[1]$ is a unit of $S(G)$, then

\[
\epsilon_{\epsilon_1} \cdots \epsilon_{\epsilon_m} [1] = \epsilon = \epsilon\alpha = \epsilon_{\epsilon_1} \cdots \epsilon_{\epsilon_m} \epsilon_{s_1} \cdots \epsilon_{s_n} [s]
\]

From the uniqueness of the standard form of $\epsilon$ we conclude that $s = 1$ and $\alpha$ is an idempotent.
(b) First note that for all \( s, t \in G \),
\[
[s][t] = [st][t^{-1}] = [st] \epsilon_{it},
\]
so the map \( G \to S(G), \ g \mapsto [g] \), is a partial morphism, and the map \( S(G) \to G(S(G)) \), \( \alpha \mapsto [\alpha] \) is a morphism. So \( g \mapsto [[g]] \) is a partial morphism between groups, hence a morphism.

Given \( \alpha \in S(G) \), since \( \alpha = \epsilon_{s_1} \cdots \epsilon_{s_n} [s] \) for certain \( s, s_1, \ldots, s_n \in G \), we get \( [\alpha] = [[s]] \), so \( g \mapsto [[g]] \) is surjective.

If \( [[g]] = 1 = [[1]] \), then there is an idempotent \( \epsilon = \epsilon_{e_1} \cdots \epsilon_{e_n} [1] \) for which
\[
[g] \epsilon_{e_1} \cdots \epsilon_{e_n} = [1] \epsilon_{e_1} \cdots \epsilon_{e_n}
\]
and the uniqueness of the standard form implies \( g = 1 \).

Therefore, the map \( S \leftrightarrow S(G) \) is a left inverse to the map \( G \leftrightarrow S(G) \), and in particular every group is the maximal group image of some \( E \)-unitary inverse semigroup.

Suppose \( \theta \) is a partial action of \( G \) on a set \( X \), \( \tilde{\theta} \) is the corresponding global action of \( S(G) \) and \( \gamma \) is the induced partial action on \( G(S(G)) \). Every \( s \in S(G) \) is smaller than a unique element of the form \( [g], \ g \in G \), so \( [s] = [[s]] \). Moreover, \( [g] \) is the maximum element of \( [[g]] \), which yields \( \gamma_{[[g]]} = \theta_g \) for all \( g \in G \).

This means that the isomorphism of Proposition 4.5.15 also preserves partial actions of \( G \) appropriately, so we obtain as an immediate consequence:

**Corollary 4.5.16.** Let \( \theta \) be a topological partial action of a group \( G \) on \( X \) and \( \tilde{\theta} \) be the induced global action of \( S(G) \). Then
\[
G \ltimes_{\tilde{\theta}} X \simeq S(G) \ltimes_{\tilde{g}} X.
\]

The following interesting corollary shows that for these universal semigroup, partial actions always extend to global actions:

**Corollary 4.5.17.** Let \( G \) be a group, \( S = S(G) \) and \( \theta \) a topological partial action of \( S \) on \( X \). Then there exists a global action \( \alpha \) of \( S \) on \( X \) such that \( \theta_s \leq \alpha_s \) for all \( s \in S \) and \( S \ltimes_{\theta} X \simeq S \ltimes_{\alpha} X \).

**Proof.** Let

1. \( \gamma \) be the partial action of \( G(S) \) induced by \( \theta \);
2. \( \gamma' \) be the composition of \( \gamma \) with the isomorphism \( G \to G(S(G)), \ g \mapsto [[g]] \);
3. \( \alpha \) be the action of \( S = S(G) \) induced by \( \gamma' \).

Then for all \( s \in S \),
\[
\theta_{[s]} \leq \gamma_{[[s]]} = \gamma'_{s} = \alpha_{[s]}
\]
and
\[
S \ltimes_{\theta} X \simeq G(S) \ltimes_{\gamma} X \simeq G \ltimes_{\gamma'} X \simeq S(G) \ltimes_{\alpha} X = S \ltimes_{\alpha} X.
\]
4.6 Topologically free partial actions

In this section we introduce a notion of topological freeness for partial actions of inverse semigroups which will be used later in our study of continuous orbit equivalence. We use this notion in Theorem 4.6.15 to describe $E$-unitary inverse semigroups in terms of the existence of certain topologically free partial actions.

**Definition 4.6.1.** The isotropy subgroupoid of a groupoid $G$ is the subgroupoid

$$\text{Iso}(G) = \bigcup_{x \in G(0)} G^x_x = \{a \in G : s(a) = r(a)\}.$$ 

**Definition 4.6.2 ([148]).** A topological groupoid $G$ is effective if the interior of $\text{Iso}(G)$ coincides with the unit space $G(0)$.

Proposition 3.6 of [148] shows that every Hausdorff topologically principal étale Hausdorff groupoid (Definition 4.4.21) is effective. The converse is true when we add the assumptions that $G$ is second countable and that its unit space $G(0)$ has the Baire property.

If $\theta$ is a partial action of an inverse semigroup $S$ on a set $X$, we will denote by $S_x$ the subset $\{s \in S : x \in X_s\}$ of $S$.

**Definition 4.6.3 ([53] Definition 4.1).** Suppose $\theta$ is a partial action of an inverse semigroup $S$ on a topological space $X$. Given $x \in X$ and $s \in S_x$, we say that

1. $x$ is fixed by $s$ if $\theta_s(x) = x$;
2. $x$ is trivially fixed by $s$ if there is $e \in E(S)$ such that $e \leq s$ and $x \in X_e$.

We also define the set

$$\Lambda(\theta) = \{x \in X : \text{if } s \in S_x \text{ and } x \text{ is fixed by } s \text{ then } x \text{ is trivially fixed by } s\}.$$ 

(4.6.1)

We should mention, however, that partial actions which correspond to effective groupoids of germs were defined under the name “topologically free” in [53], so in order to avoid confusion throughout this paper, we will therefore call the class of actions defined in [53] effective. This way, topologically free partial actions will correspond to topologically principal groupoids of germs, whereas effective partial actions will correspond to effective groupoids of germs.

**Definition 4.6.4 ([53] Definition 4.1]).** A partial action $\theta = (\theta_s : X_s \to X_s)_{s \in S}$ of an inverse semigroup $S$ on topological space $X$ is effective if for all $s \in S$, the interior of the set of points fixed by $s$ coincides with the set of points trivially fixed by $s$. 
**Proposition 4.6.5** ([53, Theorem 4.7]). Given a partial action \( \theta \) of an inverse semigroup \( S \) on a locally compact Hausdorff space \( X \), the corresponding groupoid of germs \( S \times X \) is effective if and only if \( \theta \) is effective.

**Definition 4.6.6.** Let \( \theta = (\theta_s : X_s^* \to X_s)_{s \in S} \) be a partial action of an inverse semigroup \( S \) on a topological space \( X \). We say that \( \theta \) is topologically free if \( \Lambda(\theta) \) is dense in \( X \) (Equation (4.6.1)).

Topological freeness of a topological partial action \( \theta \) of a discrete countable group \( G \) on a locally compact, Hausdorff, and second countable topological space \( X \) was first defined in [117, Definition 2.3] as follows: For every \( g \in G \setminus \{1\} \), the set \( \text{Fix}(\theta_g) \) has empty interior. This condition is formally weaker than the one we adopt, but the countability hypotheses on \( G \) and \( X \) assure that, in this case, this is equivalent to Definition 4.6.6. An application of Baire’s Category Theorem, as in [117, Lemma 2.4], gives the following result:

**Proposition 4.6.7.** Let \( S \) be a countable inverse semigroup and let \( X \) be a locally compact, Hausdorff, and second countable topological space. Then a partial action \( \theta = (\theta_s : X_s^* \to X_s)_{s \in S} \) of \( S \) on \( X \) is topologically free if and only if for all \( s \in S \), the set

\[
\{ x \in X_s^* : \text{if } \theta_s(x) = x \text{ then there is } e \in E(S) \text{ with } e \leq s \text{ and } x \in X_e \}
\]

is dense in \( X_s^* \).

By a free partial action we mean a topologically free partial action on a discrete space (that is, a set). It is interesting to note that freeness of a partial action implies that the associated groupoid of germs is Hausdorff, as the next proposition shows, however this is not true for topologically free partial actions, as in Example 4.6.9.

**Proposition 4.6.8.** If \( \theta = (\theta_s : X_s^* \to X_s)_{s \in S} \) is a free partial action of an inverse semigroup \( S \) on a locally compact Hausdorff space \( X \) then \( S \times X \) is Hausdorff.

**Proof.** Suppose \( [s,x] \neq [t,y] \) are elements of \( S \times X \). If \( x \neq y \), choose disjoint neighbourhoods \( U,V \) of \( x \) and \( y \) in \( X \), respectively. Clearly, \( [s,U \cap X_s^*] \) and \( [t,V \cap X_t^*] \) are disjoint neighbourhoods of \( [s,x] \) and \( [t,y] \), respectively.

Next, assume \( x = y \). By freeness of \( \theta \), we have that \( \theta_s(x) = \theta_t(x) \) if and only if there is \( u \in S \), such that \( u \leq s,t \) and \( x \in X_u^* \), or equivalently \( [s,x] = [t,x] \). Hence, if \( [s,x] \neq [t,x] \) then \( \theta_s(x) \neq \theta_t(x) \). Since \( X \) is Hausdorff, there are disjoint neighbourhoods \( U \) and \( V \) of \( \theta_s(x) \) and \( \theta_t(x) \), respectively. It is easy to see that \( [s,\theta_s(U)] \) and \( [t,\theta_t(V)] \) are disjoint neighbourhoods of \( [s,x] \) and \( [t,x] \), respectively.

**Remark.** The proof above is a combination of the facts that free partial actions on (discrete) sets correspond to (discrete) principal groupoids of germs, and that every principal topological groupoid with Hausdorff unit space is itself Hausdorff.
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Example 4.6.9. As in Example 4.2.19, let $S = \mathbb{N} \cup \{\infty, z\}$ be the inverse semigroup obtained by adjoining to the lattice $\mathbb{N}$ the group $\{\infty, z\}$ of order 2 (where $\infty$ represents the unit), such that for all $n \in \mathbb{N}$,

\[ n\infty = \infty n = nz = zn = n. \]

Again, consider $\theta$ the Munn representation of $S$ on $X = E(S) = \mathbb{N} \cup \{\infty\}$, endowed with the same topology as the one-point compactification of $\mathbb{N}$. This is a topologically free partial action, since $\Lambda(\theta) = \mathbb{N}$ is dense in $X$, however the associated groupoid of germs $S \ltimes X$ is not Hausdorff (see Example 4.2.19).

If $G$ is a group, by Definition 4.6.6, a partial action of $G$ is free if for all $x \in X$ (and for all $g \in G_x$), one has that $\theta_g(x) = x$ implies $g = 1$, where 1 is the identity of $G$, which is the usual notion of freeness for partial group actions.

The following proposition is a useful rephrasing of freeness, and the proof is similar to the argument in Remark 4.2.5.

Proposition 4.6.10. Let $\theta = (\theta_s : X_s^* \to X_s)_{s \in S}$ be a topological partial action of $S$ on $X$. The $\Lambda(\theta)$ (Equation (4.6.1)) coincides with

\[ \{x \in X : \forall s, t \in S_x (\text{if } \theta_s(x) = \theta_t(x) \text{ then there exists } \exists u \leq s, t \text{ and } x \in X_u)\}. \]

In particular, $\theta$ is topologically free if and only if the set above is dense in $X$.

We will now reword topological freeness of a partial action in terms of the groupoid of germs $S \ltimes X$.

Proposition 4.6.11. Suppose that $\theta = (\theta_s : X_s^* \to X_s)_{s \in S}$ is a partial action of an inverse semigroup $S$ on a locally compact Hausdorff space $X$. Then the groupoid of germs $S \ltimes X$ is topologically principal if and only if the action $\theta$ is topologically free.

Proof. As usual, we may assume the action $\theta$ is non-degenerate and identify $X$ with $(S \ltimes X)^{(0)}$. Then it is enough to prove that, under this identification, $\Lambda(\theta)$ consists of all $x \in X$ with trivial isotropy, that is,

\[ \Lambda(\theta) = \{x \in X : (S \ltimes X)^x = \{x\}\}. \]

Let $x \in X$ be given. First suppose $x \in \Lambda(\theta)$ and $[s, x] \in (S \ltimes X)^x$. This means that $x = t[s, x] = \theta_s(x)$, so there is $e \in E(S) \cap S_x$, $e \leq s$, which implies $[s, x] = [e, x] = x$.

Conversely suppose $(S \ltimes X)^x = \{x\}$ and let $s \in S_x$ with $\theta_s(x) = x$. This means that $[s, x] \in (S \ltimes X)^x$, and so $[s, x] = [e, x]$ for some idempotent $e \in S_x$. By definition of the groupoid of germs, we can find another idempotent $f \in S_x$ with $se = ef$, so in particular $ef$ is an idempotent, $ef \leq s$, and $x \in X_{ef}$. This proves $x \in \Lambda(\theta)$. \[\square\]

We finish this section by describing how $E$-unitary inverse semigroups can be characterized in terms of their partial actions.
Let \( \theta = (\theta_s : X_s^* \to X_s)_{s \in S} \) be a topological partial action of an \( E \)-unitary inverse semigroup \( S \) on a topological space \( X \) and \( \tilde{\theta} = \left( \tilde{\theta}_\gamma : \tilde{X}_{\gamma^{-1}} \to \tilde{X}_\gamma \right)_{\gamma \in G(S)} \) be the partial action of \( G(S) \) on \( X \) induced by \( \theta \), as in Theorem 4.5.11. Then \( \theta \) is topologically free if and only if \( \tilde{\theta} \) is topologically free.

**Proof.** We just need to prove that \( \Lambda(\theta) = \Lambda(\tilde{\theta}) \). Suppose \( x \in \Lambda(\theta) \), and that \( \gamma \in G(S) \) fixes \( x \). Choose \( s \in S \) such that \( [s] = \gamma \) and \( \theta_s(x) = \tilde{\theta}_\gamma(x) = x \). Since \( x \in \Lambda(\theta) \) then there is \( e \in E(S) \), \( e \leq s \) with \( x \in X_e \). As \( S \) is \( E \)-unitary, we have \( s \in E(S) \) and \( \gamma = [s] \) is the identity of \( G(S) \). This proves that \( \Lambda(\theta) \subseteq \Lambda(\tilde{\theta}) \).

Conversely, if \( x \in \Lambda(\tilde{\theta}) \) and \( s \in S \) fixes \( x \), then \( \tilde{\theta}_{[s]}(x) = \theta_s(x) = x \). By hypothesis, this implies that \( [s] \) is the unit of \( G(S) \), i.e., \( [s] = [s^*s] \). Since \( S \) is \( E \)-unitary then \( s \in E(S) \).

**Proposition 4.6.13.** Let \( S \) be an \( E \)-unitary inverse semigroup and \( \theta \) be a topologically free partial action of \( S \) on \( X \) with \( X_s \neq \emptyset \) for all \( s \). Then \( E(S) = \{ s : \theta_s \text{ is idempotent} \} \).

**Proof.** If \( \theta_s \) is an idempotent, just choose \( x \in X_s \cap \Lambda(\theta) \), so the equality \( \theta_s(x) = x \) is valid and implies \( e \leq s \) for some \( e \in E(S) \), so \( s \) is idempotent because \( S \) is \( E \)-unitary. \( \square \)

In [114], a morphism \( \theta : S \to T \) of inverse semigroups is called **idempotent-pure** if \( \theta^{-1}(E(T)) = E(S) \), and in [22] these were called **essentially injective** morphisms. The same notion could be used for partial morphisms and actions.

**Proposition 4.6.14.** Let \( S \) be an inverse semigroup and \( \theta = (\theta_s : X_s^* \to X_s)_{s \in S} \) be a partial action of \( S \) a space \( X \) such that

1. \( \theta \) factors through \( G(S) \) – there is a partial action \( \tilde{\theta} = \left( \tilde{\theta}_\gamma : \tilde{X}_{\gamma^{-1}} \to \tilde{X}_\gamma \right)_{\gamma \in G(S)} \) such that \( \tilde{\theta}_{[s]}(x) = \theta_s(x) \) for all \( x \in X_s^* \) (in particular \( X_s \subseteq \tilde{X}_{[s]} \) for all \( s \in S \));
2. \( \{ s \in S : \theta_s \text{ is idempotent} \} = E(S) \)

Then \( S \) is \( E \)-unitary.

**Proof.** Suppose \( e \in E(S) \), \( e \leq s \). We need to prove that \( s \), or equivalently by (ii) that \( \theta_s \), is idempotent. Since \( e \leq s \), we have \( 1 = [e] = [s] \), thus for all \( x \in X_s^* \), \( \theta_s(x) = \tilde{\theta}_{[s]}(x) = x \), so \( \theta_s \) is idempotent and \( s \) is idempotent by (ii). \( \square \)

Given an inverse semigroup \( S \), consider the canonical left action \( \alpha \) of \( S \) on itself (Definition 1.2.27): \( \alpha_s(t) = st \) whenever \( t^*t \leq s^*s \) \( (s,t \in S) \).

**Theorem 4.6.15.** \( S \) is \( E \)-unitary if and only if it admits a topologically free partial action satisfying (i) and (ii) of the Proposition 4.6.14.
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Proof. Proposition 4.6.14 proves one direction. Assume then that $S$ is $E$-unitary, and let us prove that the canonical left action $\alpha$ of $S$ is free: Suppose $st = t$, where $tt^* \leq s^*s$. Then $s \geq s^*t^*t$, which is idempotent, so $s$ is idempotent itself. This clearly implies that the action $\alpha$ is free. Condition (i) is satisfied by Theorem 4.5.11 and condition (ii) by Proposition 4.6.13.

An immediate consequence of the above proof is:

Corollary 4.6.16. $S$ is $E$-unitary if and only if the canonical left action $S \to I(S)$ is free.

4.7 Continuous orbit equivalence

In [117], Li characterized continuous orbit equivalence of topologically free partial group actions in terms of diagonal-preserving isomorphisms of the associated $C^*$-crossed products. In this section, we will extend the notion of continuous orbit equivalence to partial actions of inverse semigroups and characterize orbit equivalence of topologically free systems in terms of diagonal-preserving isomorphisms of the associated skew inverse semigroup rings.

Recall that if $\theta = (\theta_s : X_s^* \to X_s)_{s \in S}$ is a partial action of an inverse semigroup $S$ on a topological space $X$, we write $S \ast X = \{(s, x) \in S \times X : x \in X_s^*\}$.

Definition 4.7.1. Let $\theta = (\theta_s : X_s^* \to X_s)_{s \in S}$ and $\gamma = (\gamma_t : Y_t^* \to Y_t)_{t \in T}$ be topological partial actions on topological spaces $X$ and $Y$, respectively. We say that $\theta$ and $\gamma$ are continuously orbit equivalent if there exists a homeomorphism

$$\varphi : X \to Y$$

and continuous maps

$$a : S \ast X \to T \quad \text{and} \quad b : T \ast Y \to S$$

such that for all $x \in X$, $s \in S_x$, $y \in Y$ and $t \in T_y$,

(i) $\varphi(\theta_s(x)) = \gamma_{a(s,x)}(\varphi(x))$;

(ii) $\varphi^{-1}(\gamma_t(y)) = \theta_{b(t,y)}(\varphi^{-1}(y))$.

Implicitly, we require that $a(g, x) \in T_{\varphi(x)}$ and $b(t, y) \in S_{\varphi^{-1}(y)}$. We will call the triple $(\varphi, a, b)$ a continuous orbit equivalence between $\theta$ and $\gamma$.

Our next goal is to prove, under appropriate conditions, that continuous orbit equivalence of topologically free partial actions is equivalent to isomorphism of the
Lemma 4.7.2. Let $S$ of this, we need to prove some identities related to how $(\varphi, a, b)$ do not belong to a common category that we have previously defined.

Given a continuous orbit equivalence $(\varphi, a, b)$ between partial actions $\theta$ and $\gamma$ as in Definition [4.7.1] we will factor the map $a : S \rtimes_\theta X \to T$ to a product-preserving map from the groupoid of germs $S \rtimes_\theta X$ to $T$, and similarly for the map $b : T \rtimes_\gamma Y \to S$. For this, we need to prove some identities related to how $(\varphi, a, b)$ respects the structure of $S$ and $T$.

**Lemma 4.7.2.** Let $\theta = (\theta_s : X_s^\ast \to X_s)_{s \in S}$ and $\gamma = (\gamma_t : Y_t^\ast \to Y_t)_{t \in T}$ be topologically free partial actions, and $(\varphi, a, b)$ is a continuous orbit equivalence between $\theta$ and $\gamma$. Assume that the groupoids of germs $S \rtimes_\theta X$ and $T \rtimes_\gamma Y$ are Hausdorff. Then

(a) $[s_1, x] = [s_2, x]$ implies that $[a(s_1, x), \varphi(x)] = [a(s_2, x), \varphi(x)]$, for all $x \in X$ and $s_1, s_2 \in S_x$.

(b) $[a(s_1 s_2, x), \varphi(x)] = [a(s_1, \theta_{s_2}(x)) a(s_2, x), \varphi(x)]$ for all $x \in X$ and $s_2 \in S_x$.

(c) $[b(a(s, x), \varphi(x)), x] = [s, x]$, for all $x \in X$ and $s \in S_x$.

**Proof.** (a) Let $x \in X$ and $s_1, s_2 \in S_x$. Choose an open neighbourhood $U$ of $x \in X$ such that

$$a(s_1, x) = a(s_1, x) \quad \text{and} \quad a(s_2, x) = a(s_2, x)$$

whenever $\bar{x} \in U$. Then for all $\bar{x} \in U \cap \varphi^{-1}(\Lambda(\gamma))$,

$$[s_1, \bar{x}] = [s_2, \bar{x}] \implies \theta_{s_1}(\bar{x}) = \theta_{s_2}(\bar{x}) \implies \varphi(\theta_{s_1}(\bar{x})) = \varphi(\theta_{s_2}(\bar{x}))$$

$$\implies \gamma_{a(s_1, \bar{x})}(\varphi(\bar{x})) = \gamma_{a(s_2, \bar{x})}(\varphi(\bar{x})).$$

so the definition of $\Lambda(\gamma)$ and the given property of $U$ imply $[a(s_1, x), \varphi(\bar{x})] = [a(s_2, x), \varphi(\bar{x})]$. Since $\gamma$ is topologically free, $\Lambda(\gamma)$ is dense in $Y$, so $U \cap \varphi^{-1}(\Lambda(\gamma))$ is dense in $U$. Taking the limit $\bar{x} \to x$ we conclude that $[a(s_1, x), \varphi(x)] = [a(s_2, x), \varphi(x)]$ (we are using the fact that $T \rtimes_\gamma Y$ is Hausdorff, so limits are unique).

(b) Choose an open neighbourhood $U$ of $x \in X$ such that

$$a(s_1 s_2, \bar{x}) = a(s_1 s_2, x), \quad a(s_1, \theta_{s_2}(\bar{x})) = a(s_1, \theta_{s_2}(x)) \quad \text{and} \quad a(s_2, \bar{x}) = a(s_2, x)$$

for all $\bar{x} \in U$. Then for all $\bar{x} \in U \cap \varphi^{-1}(\Lambda(\gamma))$

$$\gamma_{a(s_1 s_2, \bar{x})}(\varphi(\bar{x})) = \varphi(\theta_{s_1 s_2}(\bar{x})) = \varphi(\theta_{s_1}(\theta_{s_2}(\bar{x})) = \gamma_{a(s_1, \theta_{s_2}(\bar{x}))}(\varphi(\theta_{s_2}(\bar{x})))$$

\[\text{We refrain from using the word "morphism" in this case since groupoids and inverse semigroups do not belong to a common category that we have previously defined.}\]
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\[ \gamma a(s_1, \theta_{s_2}(\bar{x})) (\gamma a(s_2, \bar{x})(\varphi(\bar{x}))) = \gamma a(s_1, \theta_{s_2}(\bar{x})) a(s_2, \bar{x})(\varphi(\bar{x})) \]

so, as in item (a), the given property of \( U \) and the definition of \( \Lambda(\gamma) \) imply

\[ [a(s_1 s_2, x), \varphi(\bar{x})] = [a(s_1, \theta_{s_2}(x)) a(s_2, x), \varphi(\bar{x})]. \]

Since \( \varphi^{-1}(\Lambda(\gamma)) \cap U \) is dense in \( U \) we conclude that

\[ [a(s_1 s_2, x), \varphi(x)] = [a(s_1, \theta_{s_2}(x)) a(s_2, x), \varphi(x)]. \]

(c) Similarly to the previous items, take neighbourhoods \( U \) of \( x \) and \( V \) of \( \varphi(x) \) such that

\[ a(s, \bar{x}) = a(s, x) \quad \text{and} \quad b(a(s, x), \tilde{y}) = b(a(s, x), \varphi(x)) \]

whenever \( \bar{x} \in U \) and \( \tilde{y} \in V \). Then for all \( \bar{x} \in U \cap \varphi^{-1}(V) \cap \Lambda(\theta) \),

\[ \theta b(a(s, \bar{x}), \varphi(\bar{x}))(\bar{x}) = \varphi^{-1}(\gamma a(s, \bar{x})(\varphi(\bar{x}))) = \varphi^{-1}(\varphi(\theta_s(\bar{x}))) = \theta_s(\bar{x}) \]

so the properties of \( U, V \) and \( \Lambda(\theta) \) yield \( [b(a(s, x), \varphi(x)), \bar{x}] = [s, \bar{x}] \) and again taking \( \bar{x} \to x \) gives us the desired result.

**Theorem 4.7.3.** Let \( \theta = (\theta_s : X_s \to X_s)_{s \in S} \) and \( \gamma = (\gamma_t : Y_t \to Y_t)_{t \in T} \) be topologically free, continuously orbit equivalent partial actions, and suppose that the groupoids of germs \( S \bowtie_\theta X \) and \( T \bowtie_\gamma Y \) are Hausdorff. Then \( S \bowtie_\theta X \) and \( T \bowtie_\gamma Y \) are isomorphic as topological groupoids.

**Proof.** Let \((\varphi, a, b)\) be a continuous orbit equivalence between \( \theta \) and \( \gamma \). By Lemma 4.7.2(a) we can define

\[ \Phi : S \bowtie_\theta X \to T \bowtie_\gamma Y, \quad \Phi([s, x]) = [a(s, x), \varphi(x)] \]

and by Lemma 4.7.2(b) it is a groupoid morphism. Since \( a \) and \( \varphi \) are continuous then \( \Phi \) is continuous. Similarly, the map

\[ \Psi : T \bowtie_\gamma Y \to S \bowtie_\theta X, \quad \Psi([t, y]) = [b(t, y), \varphi^{-1}(y)] \]

is a continuous groupoid morphism as well. \( \Phi \) and \( \Psi \) are inverses of each other due to Lemma 4.7.2(c).

Note that the continuous maps \( a \) and \( b \) in the definition of continuous orbit equivalence take values in discrete spaces (namely, the corresponding semigroups), and so \( X \) and \( Y \) are required to have sufficiently many clopen sets in order for a continuous orbit equivalence between the corresponding partial actions to exist. Since we will now be interested in constructing an orbit equivalence for two actions from an isomorphism of the corresponding groupoids of germs, we will need to concentrate on spaces which have sufficiently many clopen sets and partial actions which respect this structure.
Definition 4.7.4 ([161] Definition 5.2]). A topological partial action \( \theta \) of an inverse semigroup \( S \) on a topological space \( X \) is *ample* if

(i) \( X \) is locally compact, Hausdorff and zero-dimensional;

(ii) \( X_s \) is a compact-open subset of \( X \) for all \( s \in S \).

**Lemma 4.7.5.** Let \( \theta = (\theta_s : X_s \to X_s)_{s \in S} \) and \( \gamma = (\gamma_t : Y_t \to Y_t)_{t \in T} \) be ample partial actions of inverse semigroups \( S \) and \( T \) on \( X \) and \( Y \), respectively, such that the groupoids of germs \( S \ltimes_\theta X \) and \( T \ltimes_\gamma Y \) are Hausdorff. Let \( \Phi : S \ltimes_\theta X \to T \ltimes_\gamma Y \) be a topological isomorphism, and consider the restriction \( \varphi := \Phi|_X : X \to Y \). Then for all \( s \) in \( S \) there are elements \( t_1, \ldots, t_n \) in \( T \) and there are disjoint compact-open subsets \( K_1, \ldots, K_n \) of \( Y \) such that:

(a) \( K_i \subseteq Y_{t_i} \);

(b) \( \varphi(X_s) = \bigcup_{i=1}^n K_i \);

(c) \( \{ \varphi^{-1}(K_i) : i = 1, \ldots, n \} \) is a partition of \( X_s \);

(d) For all \( i \in \{1, \ldots, n\} \) and for all \( x \in \varphi^{-1}(K_i) \), one has that \( \Phi([s, x]) = [t_i, \varphi(x)] \).

**Proof.** Since \( [s, X_s] \) is a compact-open bisection, then \( \Phi([s, X_s]) \) is a compact-open bisection in \( T \ltimes_\gamma Y \), so there are elements \( t_1, \ldots, t_n \) of \( T \) and disjoint compact-open subsets \( K_1, \ldots, K_n \) of \( Y \) with \( K_i \subseteq Y_{t_i} \) such that

\[
\Phi([s, X_s]) = \bigcup_{i=1}^n [t_i, K_i]. \tag{4.7.1}
\]

Then item (a) is trivially satisfied. Item (b) follows by taking sources on both sides of Equation \((4.7.1)\), and (c) follows from (b) and the fact that \( \varphi \) is injective.

Let us prove (d). Consider \( i \in \{1, \ldots, n\} \) and \( x \in \varphi^{-1}(K_i) \). Then Equation \((4.7.1)\) implies that \( \Phi([s, x]) \in [t_j, K_j] \), for some \( j \in \{i, \ldots, n\} \), and in particular \( s([s, x]) \in K_j \). As \( K_1, \ldots, K_n \) are pairwise disjoint and

\[
s(\Phi([s, x])) = \Phi(s([s, x])) = \varphi(x) \in K_i,
\]

then \( K_j = K_i \). Therefore, \( \Phi([s, x]) = [t_i, \varphi(x)] \). \( \square \)

We are now ready to prove that topological isomorphisms between Hausdorff groupoids of germs yield a continuous orbit equivalence between the respective partial actions.

**Theorem 4.7.6.** Let \( \theta = (\theta_s : X_s \to X_s)_{s \in S} \) and \( \gamma = (\gamma_t : Y_t \to Y_t)_{t \in T} \) be ample partial actions of \( S \) and \( T \) on \( X \) and \( Y \), respectively, and suppose that the groupoids of germs \( S \ltimes_\theta X \) and \( T \ltimes_\gamma Y \) are isomorphic and Hausdorff. Then \( \theta \) and \( \gamma \) are continuously orbit equivalent.
Proof. Let $\Phi : S \ltimes_\theta X \to T \ltimes_\gamma Y$ be an isomorphism of topological groupoids. Then

$$\varphi := \Phi|_X : X \to Y$$

is a homeomorphism.

Given $s \in S$, and choose $t_1, \ldots, t_n \in T$ and compact-open subsets $K_1, \ldots, K_n \subseteq Y$ satisfying properties (a)-(d) of Lemma 4.7.5. Define $a(s, x) = t_i$ whenever $x \in \varphi^{-1}(K_i)$, so that $a$ is a continuous map on $\bigcup_{s \in S} \{s\} \times X_{s^*}$. This way, we define a continuous function $a$ on all of $S \star X = \bigcup_{s \in S} \{s\} \times X_{s^*}$. Let us show that $a$ satisfies the desired property for a continuous orbit equivalence between $\theta$ and $\gamma$: Given $(s, x) \in S \star X$, let $t = a(s, x)$, so the definition of $a(s, x)$ implies

$$\gamma_{a(s, x)}(\varphi(x)) = r([a(s, x), \varphi(x)]) = r([t, \varphi(x)]) = r(\Phi[s, x]) = \Phi(r[s, x]) = \varphi(\theta_s(x)),$$

as desired.

Proceeding similarly with $\Phi^{-1}$ in place of $\Phi$, we construct a function $b : T \star Y \to S$ with analogous properties, so that $a$ and $b$ describe a continuous orbit equivalence between $\theta$ and $\gamma$.

Corollary 4.7.7. Let $S$ be an inverse semigroup which is a weak semilattice and $\theta$ be an ample partial action of $S$ on $X$. Let $\tau$ be the canonical action of $KB(S \ltimes_\theta X)$ on $X$ (see Example 4.2.3). Then $\theta$ and $\gamma$ are continuously orbit equivalent.

Recall that in the definition of ample partial action we assume that the domains are compact-open. The conclusion of Theorem 4.7.6 is not true in general without this assumption, i.e., there are non-continuously orbit equivalent topologically free partial actions with isomorphic groupoids of germs.

Example 4.7.8. Let $\omega_1$ be the first uncountable ordinal endowed with the order topology, making it zero-dimensional, locally compact and Hausdorff (see 1.19 and 17.2(c) of [177]). Let $S = \{1\}$ be the trivial group, acting trivially on $\omega_1$.

Consider $T = \omega_1$ as a lattice, and the action $\gamma$ of $T$ on $\omega_1$ in which $\gamma_t$ is the identity function of $\{\alpha \in \omega_1 : \alpha < t\}$ for all $t \in T$.

The transformation groupoids $S \ltimes_\omega_1$ and $T \ltimes \omega_1$ are both unit groupoids and isomorphic to $\omega_1$. However, there is no continuous orbit equivalence between the trivial action of $S$ and $\gamma$.

Indeed, first we identify $S \star \omega_1 = \{1\} \times \omega_1$ with $\omega_1$ in the obvious manner. Let $\varphi : \omega_1 \to \omega_1$ be a homeomorphism and $a : \omega_1 \to T$ a continuous function. The condition in Definition 4.7.1 is that $\varphi(\alpha)$ belongs to the domain of $\gamma_{a(\alpha)}$ for all $\alpha$, that is,

$$\varphi(\alpha) < a(\alpha) \quad \text{for all } \alpha \in \omega_1. \quad (4.7.2)$$

In other words, the continuous function $f : \omega_1 \to \omega_1$, $f(\alpha) = a(\varphi^{-1}(\alpha))$ satisfies $\alpha < f(\alpha)$ for all $\alpha \in \omega_1$, so let us prove that this leads to a contradiction (the
argument is similar to that of [177, Example 20.11]: Let \( \alpha_0 \in \omega_1 \) be arbitrary and for all \( n \geq 0 \) set \( \alpha_{n+1} = f(\alpha_n) \). Then \( \beta = \sup_n \alpha_n \) satisfies \( \beta = \lim_n \alpha_n \), and since \( f \) is continuous then
\[
\beta = \lim_n f(\alpha_n) = \lim_n \alpha_{n+1} = \beta,
\]
a contradiction.

Similar ideas are used in [177, Example 20.11] to prove that \( \omega_1 \) is not paracompact, which suggests that some property regarding paracompactness of the domains of the partial action needs to be satisfied in order for Theorem 4.7.6 to be valid. The general condition that allows the theorem to be proven is ultraparacompactness: a zero-dimensional, locally compact, Hausdorff space is ultraparacompact if every open cover can be refined to a clopen partition. See [46] and the references therein for distinct characterizations and more information about this property.

For example, every second-countable (or more generally every Lindelöf), zero-dimensional, locally compact Hausdorff space is ultraparacompact. The proof of (the analogue of) Theorem 4.7.6 with ultraparacompact domains instead of compact-open is essentially the same, but it is preferable to leave the theorem stated with more standard - and elegant - assumptions.

### 4.8 Isomorphism theorems

In this short section we connect all the isomorphism theorems we have proved. But first, let us consider another semigroup associated to a groupoid.

We will use the terminology of [125]. For each compact-open bisection \( U \) of an ample groupoid \( G \), we denote by \( \tau_U \) the homeomorphism given by the canonical action of \( \mathrm{KB}(G) \) on \( G^{(0)} \), namely \( \tau_U = r \circ (s | U^{-1}) : s(U) \to r(U) \). Recall from Example 4.2.3 that \( U \mapsto \tau_U \) is a morphism from \( \mathrm{KB}(G) \) to \( I(G^{(0)}) \).

**Definition 4.8.1.** The topological full pseudogroup of an ample groupoid is the semigroup
\[
[[G]] = \{ \tau_U : U \in \mathrm{KB}(G) \}.
\]

**Example 4.8.2.** If \( \theta \) is an action of an inverse semigroup \( S \) on a zero-dimensional, locally compact Hausdorff space \( X \), then the topological full pseudogroup \( [[S \ltimes \theta X]] \) is the set of all partial homeomorphisms \( \varphi : U \to V \) for which there are \( s_1, \ldots, s_n \in S \) and compact-open \( U_1, \ldots, U_n \) such that

(i) \( U = \bigcup_{i=1}^n U_i \);

(ii) \( U_i \subseteq X_{s_i^*} \) for all \( i \); and

(iii) \( \varphi|_{U_i} = \theta_{s_i}|_{U_i} \).
The analogue to the theorem below was proven in [148, Corollary 3.3] when one considers open bisections instead of compact-open ones. In any case, we provide a short and direct proof of it.

**Proposition 4.8.3.** Suppose \( \mathcal{G} \) is an ample (possibly non-Hausdorff) groupoid. Then the morphism \( \tau : \text{KB}(\mathcal{G}) \to [[\mathcal{G}]] \) is an isomorphism if and only if \( \mathcal{G} \) is effective.

**Proof.** First assume that \( \mathcal{G} \) is effective, that is, \( \mathcal{G}^{(0)} = \text{int} (\text{Iso}(\mathcal{G})) \), and assume \( \tau_U = \tau_V \). Then

\[ \tau_{V^*U} = \tau_{V^*} \circ \tau_U = \text{id}_{s(V)}, \]

(4.8.1)

which means that \( V^*U \subseteq \text{Iso}(\mathcal{G}) \). Since \( V^*U \) is open then \( V^*U \subseteq \mathcal{G}^{(0)} \). Using this and comparing the domains in Equation (4.8.1) we obtain

\[ V^*U = s(V^*U) = s(V), \]

thus \( V = VV^*U \subseteq U \), and symmetrically we obtain \( U \subseteq V \).

Conversely, suppose \( \text{int}(\text{Iso}(\mathcal{G})) \neq \mathcal{G}^{(0)} \). Take any nonempty compact-open bisection \( U \subseteq \text{int}(\text{Iso}(\mathcal{G})) \) which is not contained in \( \mathcal{G}^{(0)} \). Then \( U \neq s(U) \) but \( \tau_U = \tau_{s(U)} \), so \( \tau \) is not injective. \( \square \)

We can now describe the equivalence between continuous orbit equivalence for partial actions, isomorphisms of groupoids of germs, isomorphisms of topological full pseudogroups and diagonal-preserving isomorphisms of Steinberg algebras, consequently diagonal-preserving isomorphisms of crossed products.

**Theorem 4.8.4.** Let \( R \) be an indecomposable ring, \( \theta \) and \( \gamma \) ample, topologically free partial actions of inverse semigroups \( S \) and \( T \) on \( X \) and \( Y \), respectively, and suppose that the groupoids of germs \( S \ltimes_\theta X \) and \( T \ltimes_\gamma Y \) are Hausdorff. Then the following are equivalent:

1. The partial actions \( \theta \) and \( \gamma \) are continuously orbit equivalent;
2. The groupoids of germs \( S \ltimes_\theta X \) and \( T \ltimes_\gamma Y \) are topologically isomorphic;
3. The inverse semigroups \( \text{KB}(S \ltimes_\theta X) \) and \( \text{KB}(T \ltimes_\gamma Y) \) are isomorphic;
4. The inverse semigroups \( [[S \ltimes_\theta X]] \) and \( [[T \ltimes_\gamma Y]] \) are isomorphic;
5. There exists a diagonal-preserving algebra (or ring) isomorphism between the Steinberg algebras \( A_R(S \ltimes_\theta X) \) and \( A_R(T \ltimes_\gamma Y) \);
6. There exists a diagonal-preserving algebra (or ring) isomorphism between the crossed products \( A_R(X) \rtimes S \) and \( A_R(Y) \rtimes T \) (with respect to the algebraic partial actions induced by \( \theta \) and \( \gamma \)).
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Proof. (1) $\iff$ (2) follows from Theorems 4.7.3 and 4.7.6.
(2) $\iff$ (3) follows from non-commutative Stone duality (Theorem 1.4.26).
(3) $\iff$ (4) follows from Proposition 4.8.3.
(2) $\iff$ (5) follows from Theorem 4.4.27.
(5) $\iff$ (6) follows from Theorem 4.4.32. \qed
Appendix A

A group isomorphism between $C(X, \mathbb{S}^1)$ and $C(Y, \mathbb{S}^1)$ for non-homeomorphic $X$ and $Y$

Let $X$ be a compact Hausdorff space, and $G$ be either $\mathbb{R}$ or $\mathbb{S}^1$, which we write as an additive abelian topological group. We endow $C(X, G)$ with the pointwise operations and supremum metric, as usual, which induces the compact-open topology on $C(X, G)$ and makes it a topological group. Given $x_0 \in X$, denote $C(X, x_0, G) = \{ f \in C(X, G) : f(x_0) = 0 \}$.

Note that $C(X, G)$ is isomorphic, as a topological group, to $G \oplus C(X, x_0, G)$ for any $x_0 \in G$, via the map

$\phi_{x_0} : C(X, G) \to G \oplus C(X, x_0, G), \quad \phi_{x_0}(f) = (f(x_0), f - f(x_0)) \quad (A.1)$

Lemma. If $X$ and $Y$ are compact Hausdorff spaces and $\psi : C(Y, \mathbb{R}) \to C(X, \mathbb{R})$ is a bounded linear isomorphism, then there is a bounded linear isomorphism $\phi : C(Y, G) \to C(X, G)$ such that $\phi(1) = 1$ (where 1 denotes the constant function $x \mapsto 1$ on either $X$ or $Y$).

Proof. Given $k \neq 0$, the map $t \mapsto kt$ is an isomorphism of $\mathbb{R}$. Using that and the isomorphism $\phi_{x_0}$ and this, we see that the map $\phi_k : C(X, \mathbb{R}) \to C(X, \mathbb{R}), \phi_k(f) = f + (k - 1)f(x_0)$, is a linear isomorphism.

Since $\psi(1_Y) \neq 0$, choose any $x_0$ such that $\psi(1_Y)(x_0) \neq 0$, and take $k = 2 \frac{\|\psi(1_Y)\|}{\psi(1_Y)(x_0)} + 1$. Then $k \neq 0$ and for all $x \in X$, $\phi_k(\psi(1_Y))(x) = \psi(1_Y)(x) + 2\|\psi(1_Y)\| > 0$. Therefore $\Phi(f) = \phi_k(f)/\phi_k(\psi(1_Y))$ satisfies the given conditions. \hfill $\square$

Lemma. Suppose $X$ and $Y$ are compact Hausdorff and $\psi : C(X, \mathbb{R}) \to C(Y, \mathbb{R})$ is a bounded linear isomorphism. Then for any $x_0 \in X$ and any $y_0 \in Y$, $C(X, x_0, \mathbb{R})$ and $C(Y, y_0, \mathbb{R})$ are isomorphic as topological vector spaces.
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Proof. By the previous lemma, take an isomorphism $\psi : C(X, \mathbb{R}) \to C(Y, \mathbb{R})$ with $\psi(1) = 1$. Considering the chain of isomorphisms

$$\mathbb{R} \oplus C(X, x_0, \mathbb{R}) \xrightarrow{\phi_{x_0}^{-1}} C(X, \mathbb{R}) \xrightarrow{\psi} C(Y, \mathbb{R}) \xrightarrow{\phi_{y_0}} \mathbb{R} \oplus C(Y, y_0, \mathbb{R})$$

we can conclude that the map $\psi_0 : C(X, x_0, \mathbb{R}) \to C(Y, y_0, \mathbb{R})$, $\psi_0(f) = \psi(f) - \psi(f)(y_0)$, is surjective, so we just need to check that it is injective. If $\psi_0(f) = 0$, then

$$\phi_{y_0} \psi \phi_{x_0}^{-1}(-\psi(f)(y_0), f) = 0$$

which implies that $f = 0$ since $\phi_{y_0} \psi \phi_{x_0}^{-1}$ is an isomorphism. 

Suppose $X$ and $Y$ are metrizable, compact, Hausdorff, path-connected and contractible. Then their first cohomotopy groups are trivial (indeed, this follows from any two maps on $X$ and $Y$ being homotopic; see [138] for a reference), and Milutin’s Theorem ([95, Chapter 36, Theorem 2.1], or [131, 137]) states that $C(X, \mathbb{R})$ and $C(Y, \mathbb{R})$ are isomorphic as topological vector spaces.

By [138, Proposition 13], $C(X, x_0, \mathbb{R})$ and $C(X, x_0, S^1)$ are isomorphic as topological groups, and similarly with $Y$ in place of $X$, hence the previous lemma and equation A.1 imply that if $x_0 \in X$ and $y_0 \in Y$,

$$C(X, S^1) \simeq S^1 \oplus C(X, x_0, \mathbb{R}) \simeq S^1 \oplus C(Y, y_0, \mathbb{R}) \simeq C(Y, S^1).$$

Therefore, by taking $X = [0, 1]$ and $Y = [0, 1]^2$, we obtain two non-homeomorphic spaces with $C(X, S^1)$ and $C(Y, S^1)$ isomorphic as topological groups, or in other terms we obtain the existence of non-isomorphic commutative C*-algebras, namely $C([0, 1], \mathbb{C})$ and $C([0, 1]^2, \mathbb{C})$ with topologically isomorphic unitary groups.
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