The Isotropy Group for the Topos of Continuous G-Sets

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Abstract

The objective of this thesis is to provide a detailed analysis of a new invariant for Grothendieck topoi in the special case of the topos of continuous G-sets and continuous G-equivariant maps. We use a well-known site to present the isotropy group in elementary terms, as systems of right cosets of open subgroups of G. We establish properties of the the isotropy group for an arbitrary topological group and use the developed theory to compute the isotropy group for the Schanuel topos.

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Chapter 1

Introduction

It was shown by Freyd that there exists a boolean topos $\mathcal{B}$ such that for every Grothendieck Topos $\mathcal{E}$, there is an internal locale L of $\mathcal{B}$ such that $\mathcal{E}$ is equivalent to a subcategory of L-sheaves. In fact, he showed we can take $\mathbb{B}$ to be the category of continuous $Aut(\mathbb{N})$ sets, $\mathbf{Cont}(Aut(\mathbb{N}))$ [?]. The motivation for Freyd's development was to create a bridge between independence proofs in topos-theory and their set-theoretic analogs [?]. Freyd's representation theorem for Grothendieck topoi can be broken into three steps: Picking a topos $\mathbf{Cont}(G)$ for some topological group G, picking a locale L internal to $\mathbf{Cont}(G)$ to form the topos of L-sheaves, and lastly constructing a subcategory of the L-sheaves. In fact, Freyd showed only one iteration of this process is necessary [?]. For the case of sheaves, isotropy is always trivial [?].

A more recent development for Grothendieck topoi due to Funk, Hofstra and Steinberg is the theory of Isotropy for Grothendieck topoi. It was shown that for any Grothendieck topos $\mathcal{E}$ there exists an internal group object Z which makes each object E of $\mathcal{E}$ a Z-object [?]. Intuitively the isotropy group can be thought of as encoding the algebraic information of a topos.

Isotropy for a topos is defined as a presheaf on groups

$$\mathcal{Z} : \mathcal{E}^{op} \to \mathbf{Grp}; \quad Z(E) = Aut(\mathcal{E}/E \xrightarrow{\pi_!} \mathcal{E}).$$

Here the functor $\pi_!$ is the geometric morphism induced by base change along the terminal arrow $E \to *$. It was shown that for a Grothendieck topos $\mathcal{E}$, this functor is representable. We call the representing object $Z_{\mathcal{E}}$ the isotropy group of $\mathcal{E}$.

The main objective of this thesis is to compute the isotropy group for the topos $\mathbf{Cont}(G)$ of continuous G-sets and continuous G-equivariant functions for an arbitrary topological group G. Concretely, the main results consist of the following.

1. Towards the analysis of the isotropy group we present the topos of continuous G-sets as a topos on a site constructed from open subgroups. (Corollary **??**)

2. Using this presentation of $\mathbf{Cont}(G)$, we provide a representation of the isotropy group as families of right cosets. (Theorem **??**)

3. An important step in the process is a technical lemma that allows us to simplify the computation in terms of a wide subcategory of the site. (Lemma **??**)

4. We identify an important subgroup of the isotropy group which satisfies a uniformity principle. (Definition **??**)

5. We identify this subgroup with the commutator subgroup. (Theorem **??**)

6. We relate this subgroup across a well known geometric morphism to the isotropy group of discrete G-sets. (Corollary **??**)

7. Lastly, using all this machinery, we compute the isotropy group for the category of continuous $Aut(\mathbb{N})$-Sets and continuous $Aut(\mathbb{N})$-equivariant maps. (Corollary **??**)

The thesis is structured in the following way. In Chapter 2 we give the background information on Topological groups. We include results that allow us to assume that G is a nearly discrete. Chapter 3 presents the relevant background on Grothendieck topoi. Here we give the important aspects of the theory such as subobject classifiers, exponentials and generating sets. Chapter 4 is a detailed look at our main example of a Grothendieck topos, $\mathbf{Cont}(G)$. We also introduce a well-known site $(\mathbb{T}_G, J)$ and describe the equivalence $\mathbf{Cont}(G) \cong \mathbf{Sh}(\mathbb{T}_G, J)$. This result is not new, for instance see [**?**], but we work out the example in more detail. Chapter 5 begins by introducing the basics of isotropy for categories. Next we work towards the characterization of isotropy for $\mathbf{Cont}(G)$ in terms of families of right cosets of open subgroups of G, and we also introduce the notion of uniform isotropy elements. Finally, Chapter 6 computes isotropy for $\mathbf{Cont}(Aut(X))$ where X is an infinite set.

1.0.1 Main Contributions

The first 3 chapters of this thesis review the various prerequisite material needed for computing the isotropy group.

While many of the results are well-known, we filled in details where we could not find them in the literature. In particular, details surrounding nearly discreteness. (Definition 2.3.17 - Theorem 2.3.22)

The main contributions of the thesis are found in Chapter 5 and 6.

1. We provide a technical lemma which allows us to cut down the site and provide a simple description of the isotropy group. (Section 5.4)

2. We identify a subgroup of the isotropy group $\mathcal{H}$ whose elements are called *uniform elements of isotropy.* (Section 5.6)

3. We show that for each open subgroup U, $\mathcal{H}(U)$ is isomorphic to the commutator subgroup $C(U) = \{g \in G \mid \forall u \in U, gu = ug\}$. (Theorem 5.6.4/5)

4. Lastly, we use the machinery we develop to compute the isotropy group for $\mathbf{Cont}(Aut(\mathbb{N}))$. (Chapter 6)

Chapter 2

Topological Groups

Topological groups are well-behaved structures in many senses as we shall see. They are an example of homogeneous topological spaces, the group topology can be completely determined by the filter of open sets around the identity element, and the interaction between the topology and algebra ensures that even a small amount of assumed separation provides us with a Hausdorff space.

One important aspect of topological groups comes from the fact that a locally compact Hausdorff group comes with the Haar measure [**?**].

While important, this is not particularly needed for this work. Our interest in topological groups lies in their usefulness in the representation of Grothendieck topos described by Freyd in [**?**]. It is important to note that most authors assume atleast T_0 in the defintion of topological groups. We don't need this assumption but we will see that we can restrict the class of topological groups we are looking at to those whose topology is *nearly discrete*.

In this chapter we will describe the important aspects of topological groups needed for this work. We assume the reader is familiar with point-set topology and group theory covered in the standard texts [**?**, **?**]. This chapter has been adapted from various sources on the topic of Topological Groups [**?**, **?**, **?**].

2.1 Basic Concepts

Definition 2.1.1. A *topological group* is a tuple $(G, \cdot, \tau)$ where G is a set, $(G, \cdot)$ is a group, (G, τ) is a topological space and the maps $m : G \times G \to G$ and $i : G \to G$ written as $m(g, h) = gh$ and $i(g) = g^{-1}$ are continuous with respect to τ.

For simplification, we will simply talk about a topological group G unless we need the full tuple specified to avoid confusion.

Examples of topological groups are the following.

Example 2.1.2.

1. Let G be any group, then G is a topological group when given the discrete or indiscrete topology.
2. The familiar groups $(\mathbb{C}, +)$, $(\mathbb{R}, +)$, $(\mathbb{R} - \{0\}, \cdot)$ with their standard topologies.
3. Any subgroup of a topological group is again a topological group with respect to the subspace topology.

Definition 2.1.3. Let G, H be topological groups. A *topological group morphism* $f : G \to H$ is a homomorphism of groups, which is also continuous with respect to the topologies on G and H. A morphism is called an isomorphism if it is both an isomorphism of groups and a homeomorphism of spaces.

Definition 2.1.4. Topological groups and continuous group homomorphisms form a category we denote as **TopGrp**.

Recall that for any topological space X we can apply the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ to get a set $U(X)$. For any set A there are two ways to construct a topological space that are related to the forgetful functor U. The first, $D : \mathbf{Set} \to \mathbf{Top}$ equips A with the discrete topology, the second, $I : \mathbf{Set} \to \mathbf{Top}$, equips A with the indiscrete topology.

These three functors provide us with an adjunction $I \vdash U \vdash D$.

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "U"] & \mathbf{Set} \arrow[l, bend right, "D"'] \arrow[l, bend left, "I"]
\end{tikzcd}$$

That the above three functors form an adjunction is a consequence of fact that any function whose domain is a discrete space is continuous, and any function whose codomain is an indiscrete space is continuous. This adjunction between **Top** and **Set** lifts to an adjunction between **TopGrp** and **Grp** such that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "U"] & \mathbf{Set} \arrow[l, bend right, "D"'] \arrow[l, bend left, "I"] \\
\mathbf{TopGrp} \arrow[u, "U"] \arrow[r, "U"] & \mathbf{Grp} \arrow[u, "U"'] \arrow[l, bend right, "D"'] \arrow[l, bend left, "I"]
\end{tikzcd}$$

In general, homomorphisms of groups are not always continuous as the following example illustrates.

Example 2.1.5. Let $\mathbb{Z}_i$ denote the set of integers with the indiscrete topology, and $\mathbb{Z}_d$ the integers with the discrete topology. The identity morphism $id : \mathbb{Z}_i \to \mathbb{Z}_d$ is an isomorphism of the groups under addition. Since the only open sets of $\mathbb{Z}_i$ are the empty set $\emptyset$ and the whole set $\mathbb{Z}_i$, the function id is not even continuous and hence not a homeomorphism.

We give an alternative characterization for being a topological group.

Proposition 2.1.6. *A group G equipped with a topology is a topological group if and only if the map $f : G \times G \to G$ defined by $f(x, y) = xy^{-1}$ is continuous.*

Proof: If G is a topological group, then $m(x, y) = xy$ and $i(y) = y^{-1}$ are continuous. Write $f = m \circ \langle id_G, i\rangle$ then f is the composition of continuous functions, hence continuous.

Conversely, suppose f is continuous. Let $c_e : G \to G$ be the constant function on G which maps everything to the identity element. Constant functions are always continuous. We can then write i as

$$i = f \circ \langle c_e, id_G \rangle. \tag{2.1.1}$$

Thus i is the composition of continuous functions, hence continuous. Similarly, we can write m as

$$m = f \circ \langle id_G, i \rangle. \tag{2.1.2}$$

Since i was shown to be continuous, and f is assumed continuous again we have a composition of continuous functions. Thus m is also continuous. By definition this makes G a topological group. ∎

For any topological group G we have many familiar homeomophisms between G and itself.

Proposition 2.1.7. *Let G be a topological group and $g \in G$ an element. The following maps are all homeomorphisms:*

1. *$l_g : G \to G$ defined by $l_g(x) = gx$.*
2. *$r_g : G \to G$ defined by $r_g(x) = xg$.*
3. *$j_g : G \to G$ defined by $j_g(x) = g^{-1}xg$.*
4. *for any $h, g \in G$ the function $t_g^h : G \to G$ defined by $t_g^h(x) = gxh^{-1}$.*

Proof: Firstly, all 4 maps can be expressed as the composition of the inverse and multiplication continuous maps, hence they are continuous. To see they are homeomorphisms note that $l_{g^{-1}}, r_{g^{-1}}, j_{g^{-1}}, t_{g^{-1}}^{h^{-1}}$ are the respective continuous inverses. ∎

We are particularly interested in the conjugation map j_g which is also a homomorphism.

Proposition 2.1.8. *Let G be a topological group and $H \subseteq G$ a subgroup of G. Then H is open in G if and only if there exists an open neighbourhood V of e contained in H.*

Proof: If H is open, then by definition, H is an open neighbourhood of e and we are done.

Suppose there exists an open neighbourhood V of e contained in H. We want to show that for each $h \in H$ we can find an open neighbourhood $U \subseteq H$ containing h.

Consider the left coset hV in the quotient space G/V. Since multiplication on the left is a homeomorphism, hV is an open set. Furthermore V is contained in H and H is a subgroup. Thus H is closed under the group operation, and therefore $hV \subseteq H$. Finally, V contains the identity element and hence h is an element of hV. We can do this for any $h \in H$ so it follows that H is open. ∎

Proposition 2.1.9. *Let G be a topological group. If H is an open subgroup of G then H is closed.*

Proof: Form the coset space G/H. Since H is open by assumption and from Proposition [**??**] the right and left multiplication maps are homeomorphisms, it follows that the complement of H in G is open. Hence H is closed. ∎

Topological groups are particularly nice spaces to work with, this is reflected in a number of properties. We are particularly interested in two properties of topological groups. Firstly, if we assume a small amount of separation, the cooperation between the topological and algebraic structure forces our topological group to be a regular Hausdorff space. [**?**]

Theorem 2.1.10. *Let G be a T_0 topological group. Then G is*

1. T_1
2. *Hausdorff*
3. *Regular Hausdorff.*

2.2 Neighbourhood Filter

The second property we are interested in is that the topology for a topological group is completely determined by the open neighbourhoods of the identity. In preperation for the proof of this fact we first note the following:

Proposition 2.2.1. *Let G be a topological group and $g, h \in G$ two points. There exists a homeomorphism $f : G \to G$ which maps g to h.*

Proof: Take f to be defined as $f(x) = g^{-1}xh$. Since multiplication is continuous, f will be a continuous function. The inverse is defined as $f^{-1}(x) = gxh^{-1}$ and again is continuous since multiplication is continuous. ∎

We use filters to compare how a topological space "looks" at a point.

Definition 2.2.2. A subset F of a partially ordered set $(S, \leq)$ is called a *filter* if the following hold.

1. F is nonempty.

2. For each $x, y \in F$ there is a $w \in F$ such that $w \leq x$ and $w \leq y$.

3. For each $x \in F$ and $y \in S$, if $x \leq y$ then $y \in F$.

Definition 2.2.3. Let X be a topological space and $x \in X$ a point. We denote by $N(x) = \{U \subseteq X \mid x \in U\}$ the neighbourhood filter at the point x.

Recall that a filter on a set X is a non-empty set of subsets of X which does not contain the empty-set, is closed under finite intersections and is upward closed.

Proposition 2.2.4. *$N(x)$ is a filter.*

Proof: Firstly, the set $N(x)$ consists of all open subsets of a space X which contain the point x. Thus we can not have the empty set as an element of $N(x)$. Secondly, a finite intersection of two open neighbourhoods of x is open and contains x, hence in $N(x)$. Lastly, any open set containing an element of $N(x)$ is in particular an open neighbourhood of x, hence in $N(x)$. Therefore $N(x)$ is a filter. ∎

Proposition 2.2.5. *Let G be a topological group and $g, h \in G$ two elements. Then there exists a bijection between $N(g)$ and $N(h)$.*

Proof: By Proposition **??** (d), for each pair of elements $g, h \in G$, we have a topological group isomorphism $f = t_g^h : G \to G$ defined by $f(x) = gxh^{-1}$ which maps h to g. Since f is a homeomorphism of spaces, it is an open map, hence for each $U \in N(g)$ we have $f(U) \in N(h)$. Conversely, the inverse f^{-1} maps g to h and is also a homeomorphism. So each $V \in N(h)$ gets mapped to $f^{-1}(V) \in N(g)$. ∎

Proposition **??** allows us to focus on open sets which contain the identity element.

Proposition 2.2.6. *Let H and G be two topological groups and $f : G \to H$ a homomorphism of groups. Then f is continuous if and only if f is continuous at e.*

Proof: Clearly if f is continuous then it is continuous at e. Suppose that f is continuous at e and U is an open subset of H. Let x be an element of G such that $f(x) \in U$. The preimage $f^{-1}(U)$ contains x, and is open if and only if $x^{-1}f^{-1}(U)$ is an open neighbourhood of the identity.

Let $g \in f^{-1}(U)$. Then $f(x^{-1}g) = f(x)^{-1}f(g)$ since f is a homomorphism of groups. By our choice of g, $f(g) \in U$, so $f(x)^{-1}f(g) \in f(x)^{-1}U$. Since left and right multiplication is a homeomorphism, and $f(x) \in U$, $f(x)^{-1}U$ is an open neighbourhood of the identity.

The preimage $f^{-1}(f(x)^{-1}U)$ contains the point $f(x)^{-1}g$ and is open since f is continuous at the identity. Since g is an arbitrary point of $f^{-1}(U)$ we just need to show $f^{-1}(f(x)^{-1}U) \subseteq x^{-1}f^{-1}(U)$.

Let $z \in f^{-1}(f(x)^{-1}U)$ then $f(z) = f(x)^{-1}u$ for some $u \in U$. Since f is a homomorphism, we get $f(xz) = f(x)f(z) = yy^{-1}u = u \in U$. Hence $z \in x^{-1}f^{-1}(U)$. ∎

Proposition 2.2.7. *Let N be the neighbourhood filter at the identity. Let $U \in N$. Then the following hold:*

1. *There exists a $V \in N$ with $VV \subseteq U$*
2. *There exists a $V \in N$ with $V^{-1} \subseteq U$*
3. *There exists a $V \in N$ and $a \in G$ such that $aVa^{-1} \subseteq U$*

Proof: Since multiplication is continuous $m^{-1}(U) \subseteq G \times G$ is open. The set U is a neighbourhood of e and therefore $m^{-1}(U)$ contains the tuple (e, e). There is a basis element $W_1 \times W_2 \subseteq G \times G$ with W_1 and W_2 open sets of G, such that $(e, e) \in W_1 \times W_2 \subseteq m^{-1}(U)$. Let $V = W_1 \cap W_2$. Then we have $(e, e) \in V \times V$ and $V \times V \subseteq m^{-1}(U)$. Hence $m(V \times V) = VV \subseteq U$. The other proofs are the same except we are using continuity of the inverse function $i : G \to G$ and conjugation by an element $a \in G$ respectively. ∎

Definition 2.2.8. Let G be a topological group and $V \subseteq G$. We say V is *symmetric* if $V = V^{-1}$.

Proposition 2.2.9. *Let U be any neighbourhood of e, then there exists a symmetric neighbourhood V of e with the property that $VV \subseteq U$.*

Proof: Let Z be a neighbourhood of e. Then by the above there exists a neighbourhood Y of e with $YY \subseteq Z$. Let $V = Y \cap Y^{-1}$. Since $y \in Y$ if and only if $y^{-1} \in Y^{-1}$ and $y \in Y^{-1}$ if and only if $y^{-1} \in Y$, elements of V are precisely those elements which are both in Y and Y^{-1}. It follows that $V = V^{-1}$. Moreover, $VV \subseteq YY \subseteq U$ as we wanted. ∎

Lemma 2.2.10. *Let U be any neighbourhood of e. There exists a neighbourhood V of e such that $VV^{-1} \subseteq U$.*

Proof: Let U be any neighbourhood of e. By property (1) of Proposition **??** there exists a neighbourhood W of e with $WW \subseteq U$. By property (2) of Proposition **??** there exists a neighbourhood X of e with $X^{-1} \subseteq W$. Take $V = X \cap W$. Then $V \subseteq X$, $V \subseteq W$, and $V^{-1} \subseteq W$. Therefore $VV^{-1} \subseteq WW \subseteq U$. ∎

Proposition 2.2.11. *Let G be a group and F a filter of subsets of G. Suppose that F satisfies the properties in Proposition* **??**. *Then there exists a unique topology on G where F is the filter of neighbourhoods at the identity.*

Proof: Suppose $U \in F$ then by (1) of Proposition **??** there is a $V \in F$ with $VV \subseteq U$. Moreover, by (2) there is a $W \in F$ with $W^{-1} \subseteq V$. Since $VV \subseteq U$ we have $WW^{-1} \subseteq U$ hence U contains the identity element.

Define a set of subsets of G by

$$\tau = \{U \subseteq G \mid \forall a \in U \; \exists W \in F, \; aW \subseteq U\}. \tag{2.2.1}$$

We claim this is a topology on G making G a topological group. Clearly X is in τ. Since $\emptyset$ contains no elements the condition to be in τ holds.

Let $\cup_{i \in I} U_i$ be an arbitrary union of sets from τ. Let $a \in \cup_{i \in I} U_i$, then $a \in U_i$ for some $i \in I$. Thus there exists a $W_i \in F$ with $aW_i \subseteq U_i$ and hence $aW_i \subseteq \cup_{i \in I} U_i$.

Let $\cap_{i \in I} U_i$ be a finite intersection of sets $U_i \in \tau$ and take some element $a \in \cap_{i \in I} U_i$. For each $i \in I$ there is some $W_i \in F$ with $aW_i \subseteq U_i$. Consider the set

$$W = \cap_{i \in I} W_i. \tag{2.2.2}$$

Since F is a filter, $W \in F$. We get $aW \subseteq aW_i \subseteq U_i$ for each i. Thus $aW \subseteq \cap_{i \in I} U_i$. Thus τ is a topology on G. The last thing we need to check is that G is a topological group.

Let $x, y \in G$ and U a neighbourhood of e. By property (3) of Proposition **??** there exists a neighbourhood V of e with $yVy^{-1} \subseteq U$ or equivalently, $Vy^{-1} \subseteq y^{-1}U$. By Lemma **??** there exists a neighbourhood W of e with $WW^{-1} \subseteq U$. Now xW and yW are both open neighbourhoods of x and y respectively. Thus $xW \times yW$ is an open neighbourhood of (x, y) in $G \times G$. Furthermore we have $f(xW \times yW) = xWW^{-1}y^{-1} \subseteq xVy^{-1} \subseteq xy^{-1}U$. It follows that $xW \times yW$ is contained in $f^{-1}(xy^{-1}U)$. Since every neighbourhood of xy^{-1} will be of the form $xy^{-1}U$ for some $U \in F$ by definition of the topology, this gives us that f is continuous, hence G is a topological group. ∎

Proposition 2.2.12. *Let B be a non-empty set of neighbourhoods of e satisfying the properties:*

1. *If $X, Y \in B$ then there exists a $W \in B$ with $W \subseteq X \cap Y$.*
2. *$\emptyset \notin B$.*

Define $F = \{U \mid \exists X \in B,\ X \subseteq U\}$. Then F is a filter.

Proof: Since each set $X \in B$ is contained in itself we have $B \subseteq F$. Since B is non-empty, F must also be non-empty. Moreover each set $X \in B$ is a subset of G so $G \in F$.

Let $X, Y \in F$ then there are $U, V \in B$ with $U \subseteq X$ and $V \subseteq Y$. We then have $U \cap V \subseteq X \cap Y$. By (1) there exists an $A \in B$ with $A \subseteq U \cap V \subset X \cap Y$. Thus $X \cap Y \in F$.

Lastly, if $X \in F$ then there is an $A \in B$ with $A \subseteq X$. For any Y with $X \subseteq Y$ we have $A \subseteq Y$, hence $Y \in F$.

It follows that F is a filter. ∎

Sets of the form in Proposition **??** are called *filter bases.* By Proposition **??** and Proposition **??** we can construct a topology from a filter base.

Example 2.2.13. For any metric space M recall that we have a neighbourhood base at $x \in M$ defined by $N(x) = \{B_{1/n}(x) \mid n \in \mathbb{N}\}$. For the group $(\mathbb{R}, +)$ of real numbers under addition, we have $B_{1/n}(x) = (-1/n, 1/n)$.

2.3 Continuous Group Actions

An important tool in the study of groups is to consider their action on other objects. Our main interest will be on how topological groups act on discrete sets.

Definition 2.3.1. Let G be a topological group. A G-set is a set X equipped with an action map $\alpha : X \times G \to X$ such that the following diagrams commute.

$$\begin{tikzcd} X \times G \times G \arrow[r, "\alpha \times id_G"] \arrow[d, "id \times m"'] & X \times G \arrow[d, "\alpha"] \\ X \times G \arrow[r, "\alpha"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \times 1 \arrow[r, "id_X \times e"] \arrow[dr, "\cong"'] & X \times G \arrow[d, "\alpha"] \\ & X \end{tikzcd}$$

The map $e : 1 \to G$ picks out the identity element of G.

If α is continuous with respect to the discrete topology on X, then we say X is a *continuous G-set.*

For a continuous G-set X we will often write $x \cdot g$ for the action instead of $\alpha(x, g)$. We are mainly interested in looking at the category of continuous G-sets.

Definition 2.3.2. Let (X, α) be a G-set and $x \in X$ an element of X. The *stabilizer subgroup* at x is the subgroup of G defined by $G_x = \{g \in G \mid x \bullet g = x\}$.

Proposition 2.3.3. *Let G be a topological group, and X a continuous G-set with action map $\alpha : X \times G \to X$. The action α is continuous if and only if for each $x \in X$ the stabilizer subgroup G_x is open in G.*

Proof: Suppose $\alpha : X \times G \to X$ is a continuous action. Let $\overline{\alpha}$ be the restriction of α to the set $\{x\} \times G$. Since α, is continuous, $\overline{\alpha}$ is continuous, hence $\overline{\alpha}^{-1}(\{x\})$ is open. But this is just $\{x\} \times G_x$, and therefore we must have G_x open.

Conversely, suppose for each $x \in X$, the stabilizer subgroup G_x is open in G. From Proposition **??** we know that left multiplication is a homeomorphism, so for every $g \in G$, gG_x is also open. Suppose $(y, g) \in \alpha^{-1}(x)$. By definition of the product topology $\{y\} \times gG_x$ is open in $X \times G$. Suppose $h \in gG_x$, then $h = gs$ for some $s \in G_x$ and therefore $y \bullet h = y \bullet gs = x \bullet s = x$. This gives us $\{y\} \times gG_x \subseteq \alpha^{-1}(x)$, hence $\alpha^{-1}(x)$ is open. ∎

Corollary 2.3.4. *Let G be a topological group and X a continuous G-Set. If G has the indiscrete topology, then the action is trivial.*

Proof: By Proposition [**??**], for each $x \in X$ the stabilizer subgroup G_x is open in G. Since G has the indiscrete topology there are only two open subsets: the $\emptyset$ and G. Since $x \bullet e = x$, we know $e \in G_x$. Therefore we must have $G_x = G$ for each $x \in X$. Hence for each $x \in X$ and $g \in G$ we have $x \bullet g = x$, and thus the action is trivial. ∎

When looking at actions of a topological group on a discrete set we can work with a restricted class of topological groups. By Proposition **??** every open subgroup is closed. If G is connected, then the only open and closed subsets of G are G and the empty set. If X is a continuous G-set, Proposition **??** tells us that for every element $x \in X$ the stabilizer subgroup G_x is open. By definition of an action, we must have that $x \bullet e = x$. Therefore G_x is not empty. So we must have $G_x = G$. Hence the action of G on X is trivial.

Now suppose H is a subgroup of a topological group G. For any continuous G-set X with action map α, we can define a new action $\beta : X \times H \to X$ by $\beta(x, h) = \alpha(x, h)$. This is a continuous action of H on X, making X into a continuous H-set.

We denote by C the connected component of G containing the identity element. We will refer to C as the *identity component*.

Proposition 2.3.5. *The identity component C is a closed normal subgroup.*

Proof: Firstly, C is closed since connected components are always closed subsets. If X, Y are two connected spaces, then $X \times Y$ is again a connected space, and continuous images of connected spaces are connected. Thus $C \times C$ is a connected subspace of $G \times G$, and since $(e, e) \in C \times C$, $m(C \times C)$ is a connected subset of G containing the identity. Moreover, for any $x \in C$, $m(e, x) = x$ hence the multiplication of G restricts to a multiplication on C. Similarly, the inversion map also restricts to inversion on C. Hence C is a subgroup of G.

Recall from Proposition **??** that for any $g \in G$ we have the map $\phi_g : G \to G$ defined as $\phi_g(x) = gxg^{-1}$, and this map is a topological group isomorphism of G. It follows that $\phi_g(C)$ is a connected set containing the identity. It follows that $\phi_g(C) = C$, Hence C is a normal subgroup of G. ∎

Since C is a closed normal subgroup of G, any continuous G-set provides us with a continuous C-set. Since C is connected, this action will be trivial.

Recall that for any normal subgroup N of G we can form the quotient group G/N, consisting of the right (or left) cosets Nx of N in G. For a continuous G-set X, it is not obvious how to define a continuous action of G/N on X. Generally, the naive action $x \bullet Ng = x \bullet g$ is not well-defined since $x \bullet ng \neq x \bullet n'g$ for all $ng, n'g \in Ng$.

In the case where $N = C$, the identity component of G, the connectedness of C ensures that we do have $x \bullet c = x \bullet c'$ for all $c, c' \in C$.

We will show that G/C is a topological group, and the action is indeed a continuous action. First we give some propositions about topological groups and quotient maps.

Proposition 2.3.6. *Let U and H be subgroups of G with U being open. Then $HU = \{hu \mid h \in H, u \in U\}$ is an open set of G.*

Proof: We can write $HU = \cup_{h \in H} hU$. By Proposition **??** left multiplication is a homeomorphism, hence hU is open for each $h \in H$. Since HU is the union of open sets, it is open. ∎

Recall that a continuous function $f : X \to Y$ between topological spaces X and Y is called *open* if for any open set U of X, $f(U)$ is open in Y. By definition of the quotient topology, a set $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open in X.

Proposition 2.3.7. *Let G/N have the quotient topology. Then $q : G \to G/N$ is an open map.*

Proof: Suppose U is an open set of G. Define the set $NU = \{nu \in G \mid n \in N, u \in U\} = \cup_{n \in N} nU$. For each $n \in N$, nU is open, hence NU is an arbitrary union of open sets and therefore also open. We want to show $q^{-1}(q(U)) = NU$. Let $x \in q^{-1}(q(U))$

then $q(x) = Nx \in q(U)$. Therefore $Nx = Nu$ for some $u \in U$ and we can write $x = hu$ for some $h \in N$, hence $x \in NU$. Conversely, if $x \in NU$ then $x = nu$ for some $n \in N$ and $u \in U$. So we have $q(x) = Nx = Nnu = Nu \in q(U)$. Thus $x \in q^{-1}(q(U))$ as needed. ∎

Proposition 2.3.8. *Let $q : G \to G/N$ be the quotient map. Then the map $q \times q : G \times G \to G/N \times G/N$ defined as $(q \times q)(x, y) = (q(x), q(y))$ is also a quotient map.*

Proof: First notice that for any pair $(Nx, Ny) \in G/N \times G/N$ we have $(q \times q)(x, y) = (q(x), q(y)) = (Nx, Ny)$, hence $q \times q$ is surjective. To show $q \times q$ is a quotient map we just need to show it is an open map. Let $V \subseteq G \times G$ be an open subset. By definition of the product topology $V = \cup_{i \in I} U_i \times H_i$ where U_i and H_i are open subsets of G. Then $q \times q(V) = q \times q(\cup_{i \in I} U_i \times H_i) = \cup_{i \in I} q \times q(U_i \times H_i) = \cup_{i \in I} q(U_i) \times q(H_i)$. Since q is an open map, this is again an open set. Hence $q \times q$ is an open map. Thus $q \times q$ is a quotient map. ∎

Using the above proposition, we can show that quotient groups of topological groups are still topological groups.

Proposition 2.3.9. *Let G be a topological group and N a normal subgroup. Then G/N is a topological group.*

Proof: The multiplication map $\overline{m} : G/N \times G/N \to G/N$ and inversion map $\overline{i} : G/N \to G/N$ are defined as

$$\overline{m}(Nx, Ny) = Nxy, \ \overline{i}(Nx) = Nx^{-1}. \tag{2.3.1}$$

Consider the quotient map $q : G \to G/N$ and the quotient map $\overline{q} : G \times G \to G/N \times G/N$ defined in Proposition **??**. We can construct the following commutative diagram.

$$\begin{tikzcd} G \times G \arrow[r, "\overline{q}"] \arrow[d, "m"'] & G/N \times G/N \arrow[d, "\overline{m}"] \\ G \arrow[r, "m"] & G/N \end{tikzcd}$$

Since both m and q are continuous, it follows that the composition $\overline{mq}$ is continuous.

Let U be an open subset of G/N. Since $\overline{q}$ is a quotient map, the subset $\overline{m}^{-1}(U)$ of $G/N \times G/N$ is open if and only if $\overline{q}^{-1}(\overline{m}^{-1}(U))$ is open. This is equivalent to $(\overline{mq})^{-1}(U)$ being open, which follows from $\overline{mq}$ being continuous. Hence $\overline{m}$ is continuous. Similarly, notice that $\overline{i}q = qi$. By a similar argument we get $\overline{i}$ is continuous.

Therefore G/N is a topological group. ∎

An immediate consequence of Proposition **??** is that the group G/C is a topological group.

Corollary 2.3.10. *The group G/C is a topological group.*

Lastly, any continuous G-set, is also a continuous G/C-set.

Theorem 2.3.11. *Let X be a continuous G-set, then X is a continuous G/C-set with action map $\alpha : X \times G/C \to X$ defined by*

$$\alpha(x, Cg) = x \bullet g. \tag{2.3.2}$$

Proof: Suppose we have $Cg = Ch$. Then we can write $g = ch$ for some $c \in C$. We have

$$\begin{aligned} x \bullet Cg &= x \bullet g & (2.3.3)\\ &= x \bullet ch & (2.3.4)\\ &= (x \bullet c) \bullet h & (2.3.5)\\ &= x \bullet h & (2.3.6)\\ &= x \bullet Ch. & (2.3.7) \end{aligned}$$

Therefore α is well-defined. The identity of G/C is the coset $C = Ce$, so $x \bullet C = x$. Lastly for $Cg, Ch \in G/C$ we have

$$\begin{aligned} (x \bullet Cg) \bullet Ch &= (x \bullet g) \bullet h & (2.3.8)\\ &= x \bullet gh & (2.3.9)\\ &= x \bullet Cgh. & (2.3.10) \end{aligned}$$

It follows that α is indeed an action of G/C on X. Let $\overline{G}_x$ be the stabilizer subgroup of the element $x \in X$ in G/C. To see that $\overline{G}_x$ is an open subgroup, notice that since C is connected, and as we have noted, C acts trivially on X, C will be contained in every stabilizer subgroup for the action of G on X. If $q : G \to G/C$ is the quotient map, we have $q(G_x) = \overline{G}_x$. Since q is an open map, and G_x is open, $\overline{G}_x$ is open in G/C. Thus X is a continuous G/C-set. ∎

Proposition 2.3.12. *Let G be a topological group. Then G/C is totally disconnected.*

Proof: Let X be a connected subset of G/C. Suppose that Cx, Cy are elements of X such that $Cx \neq Cy$. Since Cx and Cy are connected components of G, $q^{-1}(X)$ contains at least two connected components, and is therefore not connected. Take U, V to be non-empty open subsets of $q^{-1}(X)$ such that $q^{-1}(X) = U \cup V$, and $U \cap V$ is empty. Applying q we have $X = q(U) \cup q(V)$. Since q is open $q(U)$ and $q(V)$ are open, and by our assumption on U and V they are also non-empty and have empty intersection. This contradicts the assumption that X is connected. Therefore we must have that the only connected components of G/C are the singleton sets. ∎

Corollary 2.3.13. *Let G be a topological group and C the connected component of the identity of G. Then we have an isomorphism of categories* $\mathbf{Cont}(G) \cong \mathbf{Cont}(G/C)$.

Proof: By Proposition **??** we have a functor $\mathcal{N} : \mathbf{Cont}(G) \to \mathbf{Cont}(G/C)$. Conversely, we can define a functor $F : \mathbf{Cont}(G/C) \to \mathbf{Cont}(G)$ by $F(X, \alpha) = (X, \mu)$, where we define μ by precomposing α with the quotient map $q : G \to G/C$

$$\mu = X \times G \xrightarrow{id_X \times q} X \times G/C \xrightarrow{\alpha} X.$$

Since $q(e) = Ce$ we have $\mu(x, e) = \alpha(x, Ce) = x$. Moreover

$$\begin{aligned} \mu(\mu(x, g), h) &= \alpha(\alpha(x, Cg), Ch) \\ &= \alpha(x, Cgh) \\ &= \mu(x, gh). \end{aligned}$$

Therefore X is a G-set. Notice that $G_x = q^{-1}((G/C)_x)$. Since α is continuous $(G/C)_x$ is open and therefore so is G_x.

Let (X, α) be a continuous G-set, $N(X, \alpha) = (X, \gamma)$, and $F(X, \gamma) = (X, \mu)$. Then we have

$$\mu(x, g) = \gamma(x, Cg) = \alpha(x, g). \tag{2.3.11}$$

Since x and g are arbitrary we get $\mu = \alpha$. Therefore $F \circ N = Id_{\mathbf{Cont}(G)}$. Similarly we get $N \circ F = Id_{\mathbf{Cont}(G/C)}$. Thus $\mathbf{Cont}(G)$ and $\mathbf{Cont}(G/C)$ are isomorphic categories. ∎

Proposition 2.3.14. *Let G be a connected topological group. Then if H is an open subgroup of G, $H = G$.*

Proof: Suppose H is open in G and $H \neq G$. By Proposition **??** H is also closed in G. Since G is connected, the only clopen subsets of G are the empty set and G itself. A contradiction. ∎

Example 2.3.15. Let $G = (\mathbb{R}, +)$ be the group of reals under addition. With the standard topology on $\mathbb{R}$, G is a connected topological group [**?**]. Since the connected component of 0 is the whole group, any continuous action $\alpha : \mathbb{R} \times X \to X$ is the trivial action. This follows from the above identification of $\mathbf{Cont}(G)$ and $\mathbf{Cont}(G/C)$ since $G/C \cong \{0\}$. Alternatively, recall that the action α is continuous if and only if G_x is an open subgroup of $\mathbb{R}$ for each $x \in X$. Since $0 \in G_x$ for every $x \in X$, and from Proposition **??** it follows that $G_x = \mathbb{R}$ for every $x \in X$.

Example 2.3.16. Let $G = (GL(n, \mathbb{R}), \cdot)$, the general linear group over the real numbers with matrix multiplication. The elements of G are all matrices with non-zero determinant. The determinant $det : GL(n, \mathbb{R}) \to \mathbb{R}$ is a continuous function [**?**], and hence the images of the connected components of $GL(n, \mathbb{R})$ are connected in $\mathbb{R}$. Since G contains only matrices which have non-zero determinant, the image of det in $\mathbb{R}$ is $\mathbb{R}/\{0\}$. Hence $GL(n, \mathbb{R})$ can not be connected, since it's image is not connected. Moreover, the inverse images $det^{-1}((-\infty, 0))$ and $det^{-1}((0, \infty))$ correspond with the connected components of $GL(n, \mathbb{R})$ [**?**]. Thus we have $G/C_0 \cong \mathbb{Z}_2$ and from the above results, $\mathbf{Cont}(G) \cong \mathbf{Cont}(\mathbb{Z}_2)$.

For the theory of continuous G-sets, we can assume that G is a totally disconnected Hausdorff space. We can cut down the class of groups a little further by considering nearly discrete topological groups.

Definition 2.3.17. A topological group is called *nearly discrete* if the intersection of all open subgroups is the trivial group.

Proposition 2.3.18. *Let I be the intersection of all open subgroups of a topological group G. Then I is a closed normal subgroup of G.*

Proof: Consider an element $g \in I$. By Proposition **??** conjugation is a homeomorphism and therefore for every open subgroup U, $h^{-1}Uh$ is also an open subgroup. For any element $h \in G$ $h^{-1}gh \in h^{-1}Uh$ for every open subgroup U. In particular this holds for the open subgroup hUh^{-1}. Thus we have $h^{-1}gh \in h^{-1}(hUh^{-1})h = U$ for every open subgroup U. Hence $h^{-1}gh \in I$.

By Proposition **??**, since every open subgroup is also closed, and arbitrary intersections of closed sets is closed, it follows that I is also closed. ∎

Proposition 2.3.19. *The quotient group G/I is nearly discrete.*

Proof: For any quotient group G/N, subgroups of G/N are in a one-to-one correspondence with subgroups of G containing N [**?**]. Since I is the intersection of all open subgroups, this gives us a one-to-one correspondence between open subgroups of G/I and G. Taking the intersection of all open subgroups gives us I, the identity element of G/I. ∎

Proposition 2.3.20. *Let (X, μ) be a continuous G-set. Define a map $\alpha : X \times G/I \to X$ by $\alpha(x, Ig) = \mu(x, g)$. Then (X, α) is a continuous G/I-set.*

Proof: Consider an element $h \in I$. By definition of I, h is an element of every open subgroup of G. Since X is a continuous G-set, for each element $x \in X$ the stabilizer subgroup G_x is an open subgroup of G. Hence $h \in G_x$ for each $x \in X$. Thus we have $\alpha(x, I) = x$. It also follows from this observation that the action is well-defined. We also have $\alpha(x, Igh) = \mu(\mu(x, Ig), Ih) = \alpha(\alpha(x, g), h)$.

Let H_x be the stabilizer subgroup of G/I for the element $x \in X$. Denote by $q : G \to G/I$ the canonical quotient map which is also an open map [**?**]. Then we have

$$\begin{aligned} q(G_x) &= \{Ig \mid g \in G_x\} \\ &= \{Ig \mid \mu(x, g) = x\} \\ &= \{Ig \mid \alpha(x, Ig) = x\} \\ &= H_x. \end{aligned}$$

Thus H_x is open making α continuous by Proposition **??**. ∎

Proposition **??** provides us with a functor $\mathcal{I} : \mathbf{Cont}(G) \to \mathbf{Cont}(G/I)$. Using the quotient map $q : G \to G/I$, for any G/I-set (X, α_X) we have the functor $F : \mathbf{Cont}(G/I) \to \mathbf{Cont}(G)$ defined by $F(X, \alpha_X) = (X, \mu_X)$ where μ_X is is the arrow

$$X \times G \xrightarrow{\langle id_X, q \rangle} X \times G/I \xrightarrow{\alpha_X} X.$$

Proposition 2.3.21. *For any topological group G the following holds.*

- $\mathcal{I} \circ F = id_{\mathbf{Cont}(G/I)}$.
- $F \circ \mathcal{I} = id_{\mathbf{Cont}(G)}$.

Proof: Suppose (X, α_X) is a continuous G/I-set, $F(X, \alpha_X) = (X, \mu_X)$ and $\mathcal{I}((X, \mu_X)) = (X, \gamma_X)$. By definition of F and $\mathcal{I}$ we have

$$\begin{aligned} \gamma_X(x, Ig) &= \mu_X(x, g) \\ &= \alpha_X(x, Ig). \end{aligned}$$

Thus $\mathcal{I} \circ F = id_{\mathbf{Cont}(G/I)}$.

Suppose (X, μ_X) is a continuous G-set, $\mathcal{I}((X, \mu_X)) = (X, \alpha_X)$ and $F((X, \alpha_X)) = (X, \zeta_X)$. Similar to above, we have

$$\begin{aligned} \zeta_X(x, g) &= \alpha_X(x, Ig) \\ &= \mu_X(x, g) \end{aligned}$$

Thus $F \circ \mathcal{I} = id_{\mathbf{Cont}(G)}$. ∎

Corollary 2.3.22. *For a topological group G we have an isomorphism of categories $\mathbf{Cont}(G) \cong \mathbf{Cont}(G/I)$.*

Proof: This follows directly from Proposition **??** ∎

The above results allow us to cut down the class of topological groups we are looking at to consider those groups that are Hausdorff and nearly discrete.

Chapter 3

Topos Theory

Topos theory can be thought as a generalization of set theory. One of the strengths and certainly interesting aspects of topos theory is the many ways the subject can be approached. The many approaches to topos theory are covered in the text “Sketches of an Elephant” by Johnstone [?]. Our main interest will be in the more geometric flavor of topos theory.

Topoi provide arenas where we can carry out many of the well known constructions in the category **Sets** and we can reason about the objects of the topos in much the same way we reason about sets. Additionally, some topoi can also be thought of as generalized spaces. In this section we introduce the main ideas of topos theory and some examples. We call a category $\mathcal{E}$ an Elementary topos if $\mathcal{E}$ has all finite limits, exponentials, and a subobject classifier. Here we restrict our view to a smaller class of topoi called Grothendieck topoi, which have a more geometric flavour.

3.1 Basic Concepts

One familiar notion from set theory is the concept of subsets of a set. We can generalize the idea of subset of a set to subobject of an object for an arbitrary category.

3.1.1 Subobjects

Definition 3.1.1. Let $\mathbb{C}$ be any category, Z an object of $\mathbb{C}$ and $s : X \to Z$ and $t : Y \to Z$ monomorphisms in $\mathbb{C}$. Define a preorder on monomorphisms into Z by $s \leq t$ if and only if there exists an arrow $u : X \to Y$ such that $s = t \circ u$.

The first thing to note about this definition is if $a \leq b$ then there is at most one morphism between a and b and such a morphism is necessarily a monomorphism.

Proposition 3.1.2. *If $a : A \to Z$ and $b : B \to Z$ are monomorphisms and $s : A \to B$ is a morphism such that $a = b \circ s$ then s is unique and s is also a monomorphism.*

Proof: Suppose s is not unique, then $a = b \circ t = b \circ s$. Since b is a monomorphism we have $s = t$. Next suppose f, g are morphisms such that $s \circ f = s \circ g$. Then we have

$$\begin{aligned} s \circ f &= s \circ g \\ b \circ s \circ f &= b \circ s \circ g \\ a \circ f &= a \circ g. \end{aligned}$$

Since a is a monomorphism, this gives us $f = g$. Thus s is also a monomorphism. ∎

If we have two monomorphisms a, b into Z with $a \leq b$ and $b \leq a$ then we get an isomorphism between the domain of a and the domain of b.

Proposition 3.1.3. *Suppose $a : A \to Z$ and $b : B \to Z$ are monomorphisms and $a \leq b$ and $b \leq a$, then A and B are isomorphic.*

Proof: Since $a \leq b$ and $b \leq a$ there exists morphisms $s : A \to B$ and $t : B \to A$ such that $a = b \circ s$ and $b = a \circ t$. Substituting these equations into one another we get

$$a = a \circ t \circ s,$$
$$b = b \circ s \circ t.$$

Since both a and b are monomorphisms and $a = a \circ id_A$ and $b = b \circ id_B$ we have $t \circ s = id_A$ and $t \circ s = id_B$. Thus s, t are isomorphisms. ∎

We can define a relation $\sim$ on the preorder of monomorphisms by $a \sim b$ if and only if $a \leq b$ and $b \leq a$.

Proposition 3.1.4. *The relation $\sim$ on the preorder of monomorphisms into Z is an equivalence relation.*

Proof: Let $a : A \to Z$, $b : B \to Z$, and $c : C \to Z$ be monomorphisms. Notice that $a = a \circ id_A$ and therefore $a \leq a$, hence $a \sim a$. If $a \sim b$ then $a \leq b$ and $b \leq a$. Equivalently, $b \leq a$ and $a \leq b$. Thus $b \sim a$. If $a \sim b$ and $b \sim c$ then we have $a \leq b$, $b \leq a$, $b \leq c$ and $c \leq b$. Using transitivity of $\leq$ we have $a \leq c$ and $c \leq a$, hence $a \sim c$. Thus $\sim$ is an equivalence relation. ∎

We call the equivalence classes of monomorphisms *subobjects* of Z, and denote the collection of all subobjects of Z $Sub_{\mathbb{C}}(Z)$. Notice that the original preorder on monomorphisms into Z defines a partial order on $Sub_{\mathbb{C}}(Z)$.

3.1.2 Subobject Classifier

Recall from set theory that there is a one-to-one correspondence between the collection of subsets of a set X and the collection of functions of the form $\chi : X \to \{0, 1\}$. Given a subset Y of X, we call $\chi_Y : X \to \{0, 1\}$ the characteristic function for Y, and it is defined as follows.

$$\chi_Y(x) = \begin{cases} 1 & : x \in Y \\ 0 & : \text{otherwise.} \end{cases} \tag{3.1.1}$$

Conversely, given a function $\chi : X \to \{0, 1\}$ then $\chi^{-1}(1)$ defines a subset of X.

Now since Y is a subset of X, we have the inclusion monomorphism $i : Y \to X$. We can form the following commutative square of sets, where $true : \mathbb{1} \to \{0, 1\}$ is the function which picks out $1 \in \{0, 1\}$.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "i"'] & \mathbb{1} \arrow[d, "true"] \\ X \arrow[r, "\chi_Y"] & \{0,1\} \end{tikzcd}$$

The top arrow $Y \to \mathbb{1}$ is the unique arrow that exists since $\mathbb{1}$ is a terminal object in the category **Set**. In fact, the above diagram is a pullback square. Let U be any set with arrows $\phi : U \to \mathbb{1}$ and $\psi : U \to X$ such that the following diagram commutes.

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & \mathbb{1} \arrow[d, "true"] \\ X \arrow[r, "\chi_Y"] & \{0,1\} \end{tikzcd}$$

Commutativity of the above diagram tells us that each $\phi(u)$ is actually an element of Y in X, so we can regard ϕ as a function $\phi : U \to Y$. This function is unique up to isomorphism such that the following diagram commutes.

$$\begin{tikzcd} U \arrow[dr, dashed, "\phi"] \arrow[drr, bend left] \arrow[ddr, bend right, "\phi"'] & & \\ & Y \arrow[r] \arrow[d, "i"'] & \mathbb{1} \arrow[d, "true"] \\ & X \arrow[r, "\chi_Y"] & \{0,1\} \end{tikzcd}$$

Each subset of a set X corresponds to a unique arrow from X to $\{0, 1\}$. In this sense, $\{0, 1\}$ classifies the subobjects of a set X. We can generalize this notion to categories with finite limits.

Definition 3.1.5. Let $\mathbb{C}$ be a category with finite limits. A *subobject classifier* is a monomorphism $t : \mathbb{1} \to \Omega$ such that for any monomorphism $S \to X$ there exists a unique map $\phi : X \to \Omega$ such that the following diagram is a pullback.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathbb{1} \arrow[d, "t"] \\ X \arrow[r, dashed, "\phi"] & \Omega \end{tikzcd}$$

The subobject classifier can be thought of as "picking out true" from an object of possible truth values which we denote as Ω. For **Set** this is the two point set $\{true, false\}$ or $\{0, 1\}$.

Given a set X we can define the powerset $P(X)$ which consists of all subsets of X. Similarly, for X an object in a Grothendieck topos $\mathcal{E}$, we have the set $Sub(X)$ of subobjects of X. In fact we have a functor $Sub(-) : \mathcal{E}^{op} \to \mathbf{Set}$ which assigns to each object a collection of subobjects. In fact $Sub(X)$ is a Heyting Algebra [**?**]

For an arrow $f : A \to B$ the function $Sub(f) : Sub(B) \to Sub(A)$ is defined by taking the pullback along f. This is well-defined since monomorphisms are preserved by pullbacks and the composition of pullback squares is again a pullback square [**?**].

Most importantly, we can determine when a category $\mathbb{C}$ has a subobject classifier by looking at the functor $Sub(-) : \mathcal{E}^{op} \to \mathbf{Set}$.

Theorem 3.1.6. *There exists a subobject classifier if and only if the functor $Sub(-)$ is representable [**?**].*

$$Sub(-) \cong Hom_{\mathcal{E}}(-, \Omega). \tag{3.1.2}$$

It is important to note that having a subobject classifier is a strong property. We have already seen what the subobject classifier is for the category **Sets**, but many well-known categories don't have subobject classifiers. For instance the categories, **Top**, **Grp**, and **Rng** don't have subobject classifiers. We will see a couple more examples of subobject classifiers later in the chapter.

3.1.3 Exponentials

For two sets X and Y we denote by Y^X the set of all functions from X to Y. The set Y^X comes with a canonical map $e : Y^X \times Y \to X$ defined by $e(f, x) = f(x)$. For any function between sets $f : Z \times X \to Y$ we have a corresponding function $\overline{f} : Z \to Y^X$ defined as $\overline{f}(z)(x) = f(z, x)$.

For any category with finite products we can axiomatize the construction.

Definition 3.1.7. Let $\mathbb{C}$ be a category with finite products, and A, B objects of $\mathbb{C}$. An *exponential object* is an object A^B along with a map $e : A^B \times B \to A$ called the *evaluation map* which is universal in the following sense: Given an object Z and map $f : Z \times A \to B$ there exists a unique map $g : Z \to A^B$ such that the following diagram commutes.

$$\begin{tikzcd}
Z \times B \arrow[rr, "g \times id_B"] \arrow[dr, "f"'] & & A^B \times B \arrow[dl, "e"] \\
& A &
\end{tikzcd}$$

3.1.4 Generating Families

When working with sets we have the luxury that sets are completely determined by their points. The terminal object of **Set** is the singleton set $\{*\} = \mathbb{1}$. Given any point $x \in X$ we can define a function $\overline{x} : \mathbb{1} \to X$ by $\overline{x}(*) = x$.

Lemma 3.1.8. *For any set X there is a bijection $X \cong Hom_{\mathbf{Set}}(\mathbb{1}, X)$.*

Lemma 3.1.9. *For any functions $f, g : X \to Y$ between sets X, Y, $f = g$ if and only if $f(x) = g(x)$ for all $x \in X$.*

We can diagrammatically represent the latter by saying for any function $x : \mathbb{1} \to X$, $f = g$ if and only if for every $x \in X$ the following diagram commutes.

$$\mathbb{1} \xrightarrow{x} X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

This is not the situation for every category. For example in the category **Grp** of groups and homomorphisms there is only one function $e : \mathbb{1} \to G$ for any group. Namely, the homomorphism that "picks out" the identity element of G.

Definition 3.1.10. Let $\mathbb{C}$ be a category with terminal object $\mathbb{1}$ and let C any object of $\mathbb{C}$. A *global element* of C is a map $x : \mathbb{1} \to C$.

In general global elements do not distinguish maps. Therefore we can not talk about elements similarly to how we do for a set X. This situation is remedied with the concept of generalized elements.

Definition 3.1.11. For any category $\mathbb{C}$ and object C, a *generalized element* is a map $x : U \to C$. We say x is a U-point and denote it $x \in_U C$ when the domain is relevant.

Lemma 3.1.12. *For any category $\mathbb{C}$ and maps $f, g : C \to D$, $f = g$ if and only if $fx = gx$ for every $x \in_U C$ and every object U.*

Proof: If $f = g$, clearly this holds. Conversely, take the point $id_C : C \to C$. ∎

Due to this we will sometimes use the notation $f(x)$ for the composition of f with a generalized element x. In general testing for equality can not be done by focusing just on global elements. In a given category we can still hope to exhibit a nice family of "test objects", maps out of which determine equality of maps.

Definition 3.1.13. Let $\mathbb{C}$ be a category. We call a family of objects $\mathcal{S} = \{S_\alpha \mid \alpha \in I\}$ a *generating family* when, given a pair of arrows $f, g : A \to B$, if $f \circ t_\alpha = g \circ t_\alpha$ for every arrow $t_\alpha : S_\alpha \to A$ then $f = g$. The objects in the generating family are called *generating objects*. If the generating family is a set, we will refer to it as *generating set*

Example 3.1.14. The archetypal example of a Grothendieck topos is the category **Set**. We have seen that the subobject classifier is given by $\Omega = \{0, 1\}$ with $true : 1 \to \Omega$ being the function that picks out 1. Exponentials in **Set** are the sets Y^X of functions of the form $f : X \to Y$. Lastly, the family consisting of only the terminal object is a generating family for the category of sets. Recall that maps from the terminal object determine the points of sets, and functions are completely determined by how they act on points of a set. Generating objects generalizes this idea. Instead of looking at any generalized element to determine if two arrows in a category are equivalent, we can simply look at the generalized elements whose domain is contained in the generating family.

3.2 Sheaves on a Topological Space

As mentioned earlier, we are mainly interested in Grothendieck topoi. To get a feel for these objects we will first look at an important example of sheaves on a topological space, or equivalently, the collection of local homeomorphisms over a topological space.

Definition 3.2.1. Let X be a topological space, $\mathcal{O}(X)$ the poset of open subsets of X partially ordered by subset inclusion. A *sheaf* on X is a presheaf $P : \mathcal{O}(X)^{op} \to$ **Set** such that for any open covering $\{U_i\}$ of an open set U we have

1. If $s, t \in P(U)$ such that for all i $P(U_i \to U)(s) = P(U_i \to U)(t)$, then $s = t$.
2. If for each U_i we have an element $s_i \in P(U_i)$, and for any two $s_j \in P(U_j), s_i \in P(U_i)$ we have $P(U_i \cap U_j \to U_j)(s_j) = P(U_i \cap U_j \to U_i)(s_i)$ then there exists an $s \in P(U)$ such that for each U_i, $P(U_i \to U)(s) = s_i$.

The intuition behind this becomes clear once we look at an example.

Example 3.2.2. Let X, Y be topological spaces. There is a functor $C(-, Y) : \mathcal{O}(X)^{op} \to$ **Set** which assigns to each open set U of X the set of continuous functions $C(U, Y) = \{f : U \to Y \mid f \text{ continuous}\}$. A map $i : V \to U$ in $\mathcal{O}(X)$ exists if and only if $V \subseteq U$. On functions $C(i, Y) : C(U, Y) \to C(V, Y)$ is precomposition $f \mapsto f \circ i$. In other words, this is the restriction of f to the subset V, $f|_V$. It follows that $C(-, Y)$ is a presheaf on X.

Now suppose we are given an open subset U of X, an open covering $\{U_i\}$ of U, and two functions $f, g \in C(U, Y)$ such that for each i, $f|_{U_i} = g|_{U_i}$. If $x \in U$, then $x \in U_i$ for some i and thus $f(x) = f|_{U_i}(x) = g|_{U_i}(x) = g(x)$.

Lastly, suppose we have continuous functions $f_i : U_i \to Y$ for each U_i such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ any i, j. Then we can create a new function $f : U \to Y$ defined by $f(x) = f_i(x)$ where i is such that $x \in U_i$. This is well-defined since whenever $x \in U_i \cap U_j$ $f(x) = f_i(x) = f_j(x)$. The function f is also continuous and has the property that $f|_{U_i} = f_i$. This makes $C(-, Y)$ a sheaf on X.

Due to the connection with continuous functions, when talking about presheaves we often use as notation $P(U_i \to U)(s) = s|_{U_i}$. From the definition and the above example, we can view sheaves as a collection of objects with a common property P such that P is preserved when we "glue" the objects together. In the above example the property P is continuity. For an example of a presheaf that is not a sheaf, consider the presheaf $B : \mathcal{O}(X)^{op} \to \mathbf{Set}$ where for an open set U of X, $B(U)$ is the set continuous functions $f : U \to \mathbb{R}$ which are bounded. This is a presheaf, but not a sheaf since for the identity $f(x) = x$ and the cover $U_i = (-i, i)$, clearly each $f|_{U_i}$ is bounded, but f is not.

Definition 3.2.3. We denote by $\mathbf{Sh}(X)$ the subcategory of $\mathbf{Set}^{\mathcal{O}(X)^{op}}$ of sheaves and natural transformations.

Definition 3.2.4. Let E and X be topological spaces, and $p : E \to X$ a function between them. A *local section* of p at an open subset U of X is a function $s : U \to Y$ such that the composition $fs = i$ is the inclusion of U into Y.

We denote by $\Gamma_p(U)$ the collection of all local sections of p at the open subset U of X. For a map $p : E \to X$, $\Gamma_p : \mathcal{O}(X)^{op} \to \mathbf{Set}$ is a functor. For an arrow $U \to V$ in $\mathcal{O}(X)$ and a section $s : V \to E$, we have $\Gamma_p(U \to V)(s) = s|_U$ is the restriction of s to U. Moreover we actually have a functor $\Gamma : \mathbf{Top/X} \to \mathbf{Sets}^{\mathcal{O}(X)^{op}}$ which associates each map $p : E \to X$ in $\mathbf{Top/X}$ to the functor Γ_p.

The functor Γ forms the right adjoint of an adjunction between $\mathbf{Top/X}$ and $\mathbf{Set}^{\mathcal{O}(X)^{op}}$. This is captured in the following theorem [**?**].

Theorem 3.2.5. *For any topological space X there is an adjunction*

$$\mathbf{Set}^{\mathcal{O}(X)^{op}} \rightleftarrows \mathbf{Top}/X. \tag{3.2.1}$$

This adjunction cuts down to an equivalence $\mathbf{Sh}(X) \simeq LH/X$*, where LH/X is the category of local homeomorphisms over X.*

For another example of a subobject classifier we can look at $\mathbf{Sh}(X)$. For an open set U of X the functor $\Omega : \mathcal{O}(X)^{op} \to \mathbf{Set}$ at U is $\Omega(U) = \{W \in \mathcal{O}(X) \mid W \subseteq U\}$, the collection of open subsets of U. For an arrow $i : U \to V$, we get the function $\Omega(i)(W) = U \cap W$. The arrow $\mathit{true} : \mathbb{1} \to \Omega$ at an open set U, takes the point of $\mathbb{1}(U)$ and sends it to the element $U \in \Omega(U)$.

For a sheaf P and a subsheaf S, the "characteristic function" $\phi : P \to \Omega$ at U, takes each $x \in P(U)$ and maps it to the union W of all open subsets $W_i \subseteq U$ such that $x|_{W_i} \in S(W_i)$. The sheaf condition for S ensures us then that $x \in S(W)$ [**?**].

3.3 Grothendieck Topoi

We can generalize these concepts to sheaves on small categories. These are precisely the Grothendieck topoi.

Definition 3.3.1. Let $\mathbb{C}$ be a small category, and C an object of $\mathbb{C}$. A *Sieve* over C is a set of arrows S, such that for each arrow $f \in S$ we have $cod(f) = C$ and for any arrow g in $\mathbb{C}$ with $cod(g) = dom(f)$, $f \circ g \in S$.

If S is a sieve on C, $h : D \to C$ an arrow in $\mathbb{C}$, then $h^*(S) = \{f \mid cod(f) = D,\ h \circ f \in S\}$ is a sieve on D.

Definition 3.3.2. A *Grothendieck Topology* is a function J which associates to each object C of $\mathbb{C}$ a collection of sieves $J(C)$ such that

1. The maximal sieve τ_C is in $J(C)$
2. For any map $h : C' \to C$ and any sieve $S \in J(C)$, $h^*(S) \in J(C')$.
3. Let S and T be two sieves with $S \in J(C)$ and T such that for any $h : C' \to C$ in S we have $h^*(T) \in J(C')$ then $T \in J(C)$.

Similarly to how a topology on a space determines the open subsets of the space, the sieves that J picks out at an object C are called *covering sieves*.

Definition 3.3.3. Let $\mathbb{C}$ be a small category, and J a Grothendieck topology on $\mathbb{C}$. We call the pair $(\mathbb{C}, J)$ a *site*.

Example 3.3.4. Let $\mathbb{C}$ be any small category. We can define a topology J by $S \in J(U)$ if and only if $S = \tau_C$. Clearly $J(C)$ contains the maximal sieve. For any map $h : C' \to C$. $h \in S$ since S is maximal. For any arrow f with $h \circ f$ defined we must have $h \circ f \in S$ and therefore $h^*(S)$ is the maximal sieve on $dom(h)$. Lastly, if T is a sieve such that for any arrow h with $cod(h) = C$, $h^*(T)$ is maximal, in particular, $id_C^*(T)$ is maximal. Hence T is the maximal sieve on C.

Example 3.3.5. Let X be a topological space and consider the partially ordered set of opens $\mathcal{O}(X)$ as a category. Each arrow $i : V \to U$ denotes the relation $V \subseteq U$ between the open sets V and U. Let S be a sieve on U. Then we can regard S as a downward closed collection of open subsets of U.

For an open set U define the topology J by $S \in J(U)$ if and only if $\bigcup S = U$.

The maximal sieve on U is $\mathcal{O}(U)$, the collection of all open sets of U with the subspace topology inherited from X. This is clearly in $J(U)$. Denote by $S^*(V)$ the set $i^*(S)$ where $i : V \to U$ is an arrow in $\mathcal{O}(X)$. Then $S^*(V) = \{W \in \mathcal{O}(X) \mid W \subseteq V,\ W \in S\}$ and for any element $x \in \bigcup S^*(V)$ we must to have $x \in V$, since for each $W \in S^*(V)$ we have $W \subseteq V$. Conversely, for any $x \in V$ there is a $W' \in S$ with

$x \in W'$. Let $W = V \cap W'$. Then $W \subseteq W'$ gives us $W \in S$ therefore $W \in S^*(V)$. This gives us that $\bigcup S^*(V) = V$ and hence $S^*(V) \in J(V)$.

Lastly, suppose S, T are sieves on U where S is a covering sieve, and T has the property that for each $V \subseteq U$, $T^*(V) \in J(V)$. By the definition of $T^*(V)$, for each $x \in \cup T^*(U)$ we get $x \in U$. Conversely, consider the element $x \in U$ and let V be an open neighborhood of x. Then $x \in \cap T^*(V) = V$ and therefore there exists a $W \in T$ such that $x \in W$. This shows that J is a Grothendieck topology on $\mathcal{O}(X)$.

An important example that we will be using later is the *Atomic Topology* which we will often refer to simply as At when denoting a site with the atomic topology.

Example 3.3.6. Let $\mathbb{C}$ be a category such that for any two arrows $f : B \to C$,$g : A \to C$ there exists an object O that completes the following commutative diagram.

$$\begin{tikzcd} O \arrow[r, dashed] \arrow[d, dashed] & B \arrow[d, "f"] \\ A \arrow[r, "g"'] & C \end{tikzcd}$$

We define a Grothendieck topology At as

$$S \in At \Leftrightarrow S \text{ is not empty.} \tag{3.3.1}$$

The maximal sieve τ_C contains at least the identity element, and hence is in At. If $h : D \to C$ is a function, for any map $f : A \to C$ in S there exists an object O and arrows $\alpha : O \to D$ and $\beta : O \to A$ such that $h\alpha = f\beta$ and since $f \in S$ it follows that $h\alpha \in S$. Lastly, Let S, T be sieves, with $S \in J(C)$, and for each $h : A \to C \in S$, $h^*(T) \in J(A)$. Since $h^*(T)$ is non-empty, for any $f \in h^*(T)$ we have $ht \in T$ hence, T is non-empty. So $T \in J(C)$.

Definition 3.3.7. Let S be a sieve over an object C. A *matching family* for S is a function which assigns to each $f : B \to C$ in S, an element $x_f \in P(B)$ such that for any morphism $g : A \to B$ we have $P(g)(x_f) = x_{fg}$.

For a matching family $(x_f)_{f \in S}$, an *amalgamation* is an element $x \in P(C)$ such that $P(f)(x) = x_f$ for every $f \in S$.

Definition 3.3.8. Let $(\mathbb{C}, J)$ be a site and P a presheaf on $\mathbb{C}$. We say P is a *sheaf* if for any sieve S, and any matching family for S, there exists a unique amalgamation.

The above definition is equivalent to the following diagram being an equalizer [?].

$$P(C) \xrightarrow{e} \prod_{f \in S} P(dom(f)) \underset{a}{\overset{p}{\rightrightarrows}} \prod_{\substack{f,g \ f \in S \\ dom(f)=cod(g)}} P(dom(g)).$$

Here the equalizer is given by $e(x) = \{x_f\}$, p is defined as $p(\{x_f\})_{f,g} = x_{fg}$ and a as $a(\{x_f\})_{f,g} = P(g)(x_f)$. For a site with the atomic topology $(\mathbb{C}, At)$ we can classify the sheaves as follows.

Proposition 3.3.9. *A presheaf P is a sheaf for $(\mathbb{C}, At)$ if and only if for any morphism $f : A \to C$ and any $y \in P(A)$, if for every diagram*

$$E \underset{g}{\overset{h}{\rightrightarrows}} A \xrightarrow{f} C,$$

we have $P(g)(y) = P(h)(y)$ then there exists a unique $x \in P(C)$ such that $P(f)(x) = y$.

Proof: [?] ∎

Because of Proposition **??**, we think of every arrow in $\mathbb{C}$ as a cover of its codomain [?].

Corollary 3.3.10. *Let P be a sheaf for the atomic topology on the small category $\mathbb{C}$. Then for any map $f : A \to B$ we have $P(f)$ is monic.*

Proof: Let $f : A \to B$ and $P(f)(x) = P(f)(y) = z$. Then we have $z \in P(A)$. If we have the commutative diagram

$$E \underset{g}{\overset{h}{\rightrightarrows}} A \xrightarrow{f} B,$$

then $f \circ g = f \circ h$ hence $P(h)(z) = P(h)P(f)(x) = P(f \circ h)(x) = P(f \circ g)(x) = P(g)P(f)(x) = P(g)(z)$. Since P is a sheaf, Proposition **??** gives us that there exists a unique $u \in P(B)$ such that $P(f)(u) = z$. But $z = P(f)(x) = P(f)(y)$ hence we must have $x = y = u$. Thus $P(f)$ is injective. ∎

Given any site $(\mathbb{C}, J)$ and an object C of $\mathbb{C}$ a presheaf of the form $\mathbb{C}(-, C)$ is called *representable*. We will be particularly interested in Grothendieck topologies where these presheaves are sheaves.

Definition 3.3.11. For any site $(\mathbb{C}, J)$ we say J is subcanonical if every representable presheaf is a sheaf.

Definition 3.3.12. A *Grothendieck Topos* is a category $\mathcal{E}$ such that $\mathcal{E}$ is equivalent to the category $Sh(\mathbb{C}, J)$ of sheaves on some site $(\mathbb{C}, J)$.

The following result is called the *Fundamental Theorem of Topos Theory.*

Theorem 3.3.13. *[?] Let $\mathcal{E}$ be a (Grothendieck) topos, and E an object of $\mathcal{E}$. Then the slice category $\mathcal{E}/E$ is a (Grothendieck) topos.*

Example 3.3.14. For any small category $\mathbb{C}$ the category $\mathbf{Set}^{\mathbb{C}^{op}}$ is a Grothendieck topos. Consider the topology J where each $S \in J(C)$ is the maximal sieve. Equivalently, our sieves are all representable functors $Hom_{\mathbb{C}}(-, C)$. This makes every presheaf P a sheaf in $\mathbf{Set}^{\mathbb{C}^{op}}$.

3.4 Geometric Morphisms

We also have a notion of maps between topoi.

Definition 3.4.1. Let $\mathcal{E}, \mathcal{F}$ be Grothendieck topoi. A *geometric morphism* is a pair of adjoint functors (f^*, f_*) such that the left adjoint, f^* preserves finite limits. We will often refer to a geometric morphism simply by $f : \mathcal{E} \to \mathcal{F}$. The right adjoint, f_*, is called the direct image, while the left adjoint, f^*, is called the inverse image.

Example 3.4.2. Let $\mathcal{E}$ be a topos, and $f : A \to B$ any arrow in $\mathcal{E}$ then we can define a geometric morphism between the slice categories $f : \mathcal{E}/A \to \mathcal{E}/B$. The inverse image functor $f^* : \mathcal{E}/B \to \mathcal{E}/A$ is defined by taking the pullback of arrows along $f : A \to B$, while its left adjoint $f_! : \mathcal{E}/A \to \mathcal{E}/B$ is defined by post composition with f. The inverse image functor f^* has a right adjoint f_* [?], and since it is right adjoint to $f_!$ it preserves limits [?]. Thus we have a geometric morphism.

Example 3.4.3. In the case of Example **??** suppose that $f : A \to B$ is the terminal arrow $f : A \to *$. Then we have a geometric morphism $f : \mathcal{E}/A \to \mathcal{E}$. Precomposition with the terminal arrow gives another terminal arrow. Since these arrows are in one-to-one correspondence with the objects of $\mathcal{E}$, $f_!$ returns the domain of the object of $\mathcal{E}/A$. The pullback of a terminal arrow $A \to *$ along another terminal arrow $E \to *$ gives us the projection $\pi_E : A \times E \to E$.

Since f is a geometric morphisms we have to have the bijection

$$Hom(f^*(C), B \xrightarrow{g} A) \cong Hom(C, f_*(B \xrightarrow{g} A)).$$

Suppose we have the following arrow

$$\begin{tikzcd} C \times A \arrow[r, "h"] \arrow[dr, "\pi_A"'] & B \arrow[d, "g"] \\ & A. \end{tikzcd}$$

We can transpose these arrows to get the following commutative diagram.

$$\begin{tikzcd} C \arrow[r, "\overline{h}"] \arrow[d] \arrow[dr, "\overline{\pi_A}"] & B^A \arrow[d, "g^A"] \\ * \arrow[r, "i"] & A^A. \end{tikzcd}$$

The map i comes from taking the transpose of the identity on A. We define $f_*(B \xrightarrow{g} A)$ to be the pullback in the following diagram

$$\begin{tikzcd} f_*(B \xrightarrow{g} A) \arrow[r] \arrow[d] & B^A \arrow[d, "g^A"] \\ * \arrow[r, "i"] & A^A. \end{tikzcd}$$

Thus for any map $h \in Hom(f^*(C), B \xrightarrow{g} A)$ we get the map $\overline{h} : C \to B^A$ and since $f_*(B \xrightarrow{g} A)$ is a pullback we get a unique arrow $h' : C \to f_*(B \xrightarrow{g} A)$ making the following diagram commute

$$\begin{tikzcd} C \arrow[dr, dashed, "h'"] \arrow[drr, bend left, "\overline{h}"] \arrow[ddr, bend right] & & \\ & f_*(B \xrightarrow{g} A) \arrow[r] \arrow[d] & B^A \arrow[d] \\ & * \arrow[r] & A^A \end{tikzcd}$$

Definition 3.4.4. Let $\phi : \mathcal{E} \to \mathcal{G}$ and $\nu : \mathcal{G} \to \mathcal{F}$ be two geometric morphisms between Grothendieck topos $\mathcal{E}, \mathcal{G}$, and $\mathcal{F}$. Then we can form the composition $\nu \circ \phi : \mathcal{E} \to \mathcal{F}$ where the direct image $(\nu \circ \phi)_* = (\nu_* \circ \phi_*)$ and the inverse image is given by $(\nu \circ \phi)^* = (\nu^* \circ \phi^*)$.

Proposition 3.4.5. *With this definition of composition, we get a category **GTopos** of Grothendieck topos and geometric morphisms between them.*

Proof: For any Grothendieck topos we have an identity geometric morphism $id_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ where the direct and inverse image are both the identity functor on $\mathcal{E}$. Associativity of composition follows from the associativity of functor composition and the fact that for functors F, G, H, T with $F \dashv H, G \dashv T$ we have $F \circ G \dashv H \circ T$. ∎

3.5 Freyd's Representation of Grothendieck Topoi

One of the main motivations for computing the isotropy group of the topos of continuous G sets for an arbitrary G-set is Freyd's representation theorem. [?]

Definition 3.5.1. Let $\mathcal{E}$ be a Grothendieck Topos. A subcategory $\mathbb{A}$ of $\mathcal{E}$ is called an *exponential variety* if $\mathbb{A}$ is closed under cartesian products, taking subobjects, and taking power objects.

The statement from the original paper is as follows. [?]

Theorem 3.5.2. *There exists a boolean topos $\mathcal{B}$ such that for every Grothendieck topos $\mathcal{A}$ there exists a locale $\mathbb{L}$ in $\mathcal{B}$ such that $\mathcal{A}$ appears as an exponential variety in the topos of $\mathbb{L}$-sheaves over $\mathcal{B}$. $\mathbb{L}$ is boolean if and only if $\mathcal{A}$ is.*

Freyd's representation theorem showed that every Grothendieck topos can be obtained from a single iteration of 3 constructions.

1. Taking the topos of continuous G sets for some topological group G.
2. Taking a sheaves on some internal locale $\mathbb{L}$ of $\mathbf{Cont}(G)$.
3. Taking an exponential variety $\mathbb{A}$ of $Sh_{\mathbf{Cont}(G)}(L)$.

Moreover, it was also shown that we can take G to be the group $Aut(\mathbb{N})$ of bijections of the natural numbers, or the group $Aut(\mathbb{Q})$ of order preserving bijections of the rational numbers, each equipped with the topology generated by point-wise stabilizers of finite sets.

Chapter 4

Continuous G-sets

For any group G, the category of G-sets can be identified with the category of presheaves $\mathbf{Set}^{G^{op}}$. Each presheaf P determines a set $P(*) = X$. Each arrow is a group element $g \in G$. Thus we define a G-set $X = P(*)$ with action $x \bullet g = P(g)(x)$. Conversely, each G-set determines a presheaf on G.

For a topological group G, we have the category of continuous G-sets, which we denote $\mathbf{Cont}(G)$. Since each continuous G-set is given the discrete topology, $\mathbf{Cont}(G)$ is a full subcategory of $\mathbf{Set}^{G^{op}}$. We have an inclusion functor

$$i : \mathbf{Cont}(G) \to \mathbf{Set}^{G^{op}}.$$

Proposition 4.0.1. *The inclusion* $i : \mathbf{Cont}(G) \to \mathbf{Set}^{G^{op}}$ *has a right adjoint* $\rho : \mathbf{Set}^{G^{op}} \to \mathbf{Cont}(G)$ *defined as* $\rho(X) = \{x \in X \mid G_x \text{ is open}\}$.

Proof: Let X be a continuous G-set and Y a G-set. Suppose we have a map $f : X \to \rho(Y)$ in $\mathbf{Cont}(G)$. For any $x \in X$ and $g \in G$ we have $f(x \bullet g) = f(x) \bullet g$. In particular, if $g \in G_x$ then $f(x) \bullet g = f(x)$. Thus $G_x \subseteq G_{f(x)}$ and therefore by Proposition **??** $G_{f(x)}$ is open. This tells us that the image of f has to be in $\rho(Y)$. So we can extend the codomain to all of Y. Thus we have a new map $\overline{f} : X \to Y$ in $\mathbf{Set}^{G^{op}}$. By the same argument, we can restrict the codomain of a map $f : X \to Y$ in $\mathbf{Set}^{G^{op}}$ to $f : X \to \rho(Y)$. This establishes a bijection between the hom-sets $\mathbf{Set}^{G^{op}}(X, Y)$ and $\mathbf{Cont}(G)(X, \rho(Y))$ and therefore giving us that ρ is the right adjoint to the inclusion functor. ∎

4.1 Subobject Classifier

Recall that the subobject classifier for the category of sets is the two element set $\{0, 1\}$, and the function $true : \mathbb{1} \to \{0, 1\}$ picks out the element 1.

The same object equipped with the trivial action by G works as the subobject classifier for the category of G-Sets, and $\mathbf{Cont}(G)$. Recall that a subobject S of X is an equivalence class of monic arrows $m : S \to X$ such that there is a unique map $\chi_S : X \to \Omega$ making the following diagram a pullback.

$$\begin{tikzcd} S \arrow[r] \arrow[d, "m"] & \mathbb{1} \arrow[d, "true"] \\ X \arrow[r, "\chi_S"] & \Omega \end{tikzcd}$$

Regarding this as a diagram in $\mathbf{Cont}(G)$, each arrow is now a G-equivariant continuous function. If α_S and α_X denote the action maps for S and X respectively. We have the following diagram.

$$\begin{tikzcd} S \times G \arrow[r, "\alpha_S"] \arrow[d, "m \times id_G"] & S \arrow[d, "m"] \\ X \times G \arrow[r, "\alpha_X"] & X. \end{tikzcd}$$

Therefore S can be recognized as a subset of X that is closed under the action of G on X. Adding the additional requirement that all maps be continuous does not change Ω. So the subobject classifier for $\mathbf{Cont}(G)$ is the arrow $true : \mathbb{1} \to \{0, 1\}$.

4.2 Exponential

The exponential Y^X in $\mathbf{Set}^{G^{op}}$ is the set of all functions $f : X \to Y$ with the action of an element $g \in G$ on f giving us a new map $f \bullet g : X \to Y$ defined as $(f \bullet g)(x) = f(x \bullet g^{-1}) \bullet g$.

Using the adjunction $i \vdash \rho$ in Proposition **??** and that $\rho \circ i(X) = X$ for every continuous G-set X, we have the following equivalences.

$$\begin{aligned} Hom(Z, \rho_* \rho^*(Y)^{\rho^*(X)}) &\cong Hom(\rho^*(Z), \rho^*(Y)^{\rho^*(X)}) && (4.2.1)\\ &\cong Hom(\rho^*(Z) \times \rho^*(X), \rho^*(Y)) && (4.2.2)\\ &\cong Hom(\rho^*(Z \times X), \rho^*(Y)) && (4.2.3)\\ &\cong Hom(Z \times X, \rho_* \rho^*(Y)) && (4.2.4)\\ &\cong Hom(Z \times X, Y). && (4.2.5) \end{aligned}$$

Thus the exponential for $\mathbf{Cont}(G)$ is given by $\rho i(Y)^{i(X)}$.

4.3 Generating Set

We will now exhibit a generating family for the category of G-Sets. Recall that that every G-set can be written as the union of orbits of its elements.

Definition 4.3.1. Let X be a G-set. We say X is *transitive* if for any pair of elements $x, y \in X$ there exists an element $g \in G$ such that $x \bullet g = y$.

Given a G-set X and an element $x \in X$ the orbit of x, $\mathcal{O}_x$, consists of all $y \in X$ such that there exists a group element $g \in G$ with $x \bullet g = y$. The orbits of a G-set form a partition, allowing each G-set to be written as the coproduct of its orbits. Each orbit of an element $x \in X$ is a transitive set. Given two elements $z, y \in \mathcal{O}_x$ of the orbit of $x \in X$ there exists elements $g, h \in G$ such that $x \bullet g = y$ and $x \bullet h = z$. Rearranging we have $y \bullet g^{-1}h = z$. Hence the orbits are transitive.

For a group G, and subgroup H, we can construct a G-set out of the set of right cosets G/H. The action of G on G/H is defined by right multiplication, hence $Hx \bullet g = Hxg$.

Proposition 4.3.2. *For any group G and subgroup H, the set of right cosets of H is a transitive G-set.*

Proof: Let Hg and Ht be two right cosets. The representative $g, t \in G$ are group elements, hence we have $Hg \bullet g^{-1} = Hgg^{-1} = H$. Thus $Hg \bullet g^{-1}t = Ht$, making G/H a transitive G-set. ∎

The collection of transitive G-sets is determined by the coset spaces for each subgroup of G.

Proposition 4.3.3. *A G-set X is transitive if and only if X is isomorphic to G/H for some subgroup H of G*

Proof: Suppose X is transitive and take any element $x \in X$. Let $H = G_x$, the stabilizer subgroup of x in G. Now define a map $\phi : G/H \to X$ by $\phi(Hg) = x \bullet g$. The map ϕ is well-defined since $Hg = Ht$ gives us $t = hg$ for some $h \in H$. Thus $x \bullet t = x \bullet hg = x \bullet g$, where the second equality comes from $H = G_x$. Next suppose $\phi(Hg) = \phi(Ht)$. Then we have $x \bullet g = x \bullet t$ which gives us $x \bullet gt^{-1} = x$, hence $gt^{-1} \in H$. Thus $H = Hgt^{-1}$ and equivalently $Ht = Hg$. Therefore ϕ is injective. Now by the transitivity of X, for any $y \in X$ there exists a $g \in G$ with $x \bullet g = y$. Therefore $\phi(Hg) = x \bullet g = y$, hence ϕ is surjective. Lastly, for any $s \in G$ we have

$$\begin{aligned}
\phi(Hg \bullet s) &= \phi(Hgs) \\
&= x \bullet gs \\
&= (x \bullet g) \bullet s
\end{aligned}$$

$$= \phi(Hg) \bullet s.$$

Thus ϕ is an isomorphism of G sets.

Conversely, suppose X is isomorphic to G/H as G sets. Then there exists an equivariant bijection $\phi : G/H \to X$. Thus for any $y \in X$ there is a $g \in G$ such that $y = phi(Hg)$. By the definition of the action on G/H and equivariance of ϕ we have

$$\begin{aligned} y &= \phi(Hg) \\ &= \phi(H \bullet g) \\ &= \phi(H) \bullet g \\ &= x \bullet g \end{aligned}$$

Since x and y are arbitrary elements, this gives us that for each pair of elements $x, y \in X$ there exists a $g \in G$ such that $x \bullet g = y$. Thus X is transitive. ∎

Proposition 4.3.4. *The family $(G/H \mid H$ is a subgroup of $G)$ is a generating set for the category of G-sets.*

Proof: Firstly, this is indeed a set since the collection of subgroups of G is a set (subset of the power set of G).

As we noted, every G-set is the disjoint union of its orbits which are transitive G-sets. Let $f, g : X \to Y$ be two equivariant maps, and O_x the orbit of $x \in X$. From Proposition **??** we have an isomorphism $\phi : G/H \to O_x$ for some subgroup H of G. Composing this with the inclusion of O_x in X we have the following commutative diagram.

$$G/H \xrightarrow{i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Commutativity tells us that for each coset Hg we have $f(x \bullet g) = g(x \bullet g)$. This holds for every orbit $O_x \in X$. Thus $f = g$. ∎

The above results also works if we go through with the additional restriction that our subgroups are open. Thus we have the following result.

Proposition 4.3.5. *The family $\{G/H \mid H$ is an open subgroup of $G\}$ is a generating set for the category* $\mathbf{Cont}(G)$.

4.4 A Site for Cont(G)

In this section we will describe a site for the category $\mathbf{Cont}(G)$ of continuous G-sets and equivariant maps. We denote by $\mathbb{T}_G$ the category where

- Objects of $\mathbb{T}_G$ are open subgroups of G.
- For two open subgroups V and U a map $[V,g,U] : V \to U$ is a coset Ug of G/U such that $gV \subseteq Ug$.
- Composition of two maps $[V,g,U], [W,h,V]$ is given by $[W,h,V] \circ [V,g,U] = [W,gh,U]$. This operation is justified by $ghW \subseteq gVh \subseteq Ugh$.

Proposition 4.4.1. *With the above definition $\mathbb{T}_G$ is a category.*

Proof: The identity on U is given by $[U,e,U]$. For any map $[V,g,U]$ we have the composition $[U,e,U] \circ [V,g,U]$ corresponds to $egV \subseteq eUg$ which is equivalent to $gV \subseteq Ug$ or the map $[V,g,U]$. Similarly we get $[V,g,U] \circ [V,e,V] = [V,g,U]$.

Given three maps $[V,g,U], [U,h,W], [W,s,X]$. Associativity of composition holds since $hgV \subseteq Whg$ gives us $shgV \subseteq Xshg$, and if $shU \subseteq Xsh$ then $gV \subseteq Ug$ implies $shgV \subseteq Xshg$. It follows from this that $[W,s,X] \circ ([U,h,W] \circ [V,g,U]) = ([W,s,X] \circ [U,h,W]) \circ [V,g,U]$. ∎

The maps of $\mathbb{T}_G$ are particularly nice maps.

Proposition 4.4.2. *Let $[V,h,U] : V \to U$ be a morphism in $\mathbb{T}_G$. Then:*

1. *$[V,h,U]$ is an epimorphism.*
2. *$[V,h,U]$ is an isomorphism if and only if $hV = Uh$*
3. *For an isomorphism $[V,h,U]$, the inverse is the arrow $[U,h^{-1},V]$.*
4. *$[V,h,U]$ can be decomposed as $[hVh^{-1},e,U] \circ [V,h,hVh^{-1}]$.*

$$\begin{tikzcd} V \arrow[rr, "{[V,h,hVh^{-1}]}"] \arrow[dr, "{[V,h,U]}"'] & & hVh^{-1} \arrow[dl, "{[hVh^{-1},e,U]}"] \\ & U & \end{tikzcd}$$

5. *$[V,h,U]$ can be decomposed as $[h^{-1}Uh,h,U] \circ [V,e,h^{-1}Uh]$.*

$$\begin{tikzcd} V \arrow[rr, "{[V,h,h^{-1}Uh]}"] \arrow[dr, "{[V,h,U]}"'] & & h^{-1}Uh \arrow[dl, "{[h^{-1}Uh,h,U]}"] \\ & U & \end{tikzcd}$$

Proof:

1. Suppose we have maps $t, s : U \to W$ such that $th = sh$. This is equivalent to saying $Wth = Wsh$ and hence $Ws = Wt$. Hence as maps in $\mathbb{T}_G$ $s = t$.

2. By (3) if $h : V \to U$ is an isomorphism then we have an inverse $h^{-1} : U \to V$. By definition this gives us: $hV \subseteq Uh$ and $h^{-1}U \subseteq Vh^{-1}$, or $Uh \subseteq hV$ and hence $Uh = hV$. Conversely, if $Uh = hV$ then we have $hV \subseteq Uh$ and $Uh \subseteq hV$ or $h^{-1}U \subseteq Vh^{-1}$.

3. If h is an isomorphism then there is some $g : U \to V$ such that $hg = e_U : U \to U$ (the identity on U) and $gh = e_V : V \to V$ (the identity on V). This is equivalent to saying that $Uhg = U$ and $Vgh = V$, hence $Uh = Ug^{-1}$ and $Vg = Vh^{-1}$. Therefore the inverse of $h : V \to U$ is $h^{-1} : U \to V$.

4. By definition $h : V \to U$ gives us $hV \subseteq Uh$ and therefore $hVh^{-1} \subseteq U$. Thus we have a map $[hVh{-}1, e, U] : hVh^{-1} \to U$. Notice we also have the equality $hV = (hVh^{-1})h$, and hence an arrow $[V, h, hVh^{-1}]$ which has an inverse $[hVh^{-1}, h^{-1}, V]$ and is therefore an isomorphism. Taking the composition we get $[hVh^{-1}, e, U] \circ [V, h, hVh^{-1}] = [V, h, U]$ as we wanted.

5. Similarly to (4), since $hV \subseteq Uh$ we have $V \subseteq h^{-1}Uh$. By a similar argument as in (4) we can write $[V, h, U] = [h^{-1}Uh, h, U] \circ [V, e, h^{-1}Uh]$.

∎

We claim that $\mathbb{T}_G$ with the atomic topology At defined in Example **??** is a site for $\mathbf{Cont}(G)$. Towards this result we first verify that the atomic topology on $\mathbb{T}_G$ is well-defined.

Proposition 4.4.3. *Given open subgroups U, V, W and maps $[U, g, W] : U \to W$ and $[V, h, W]$ there exists an object O and maps completing the following diagram*

$$\begin{tikzcd} O \arrow[r, dashed] \arrow[d, dashed] & U \arrow[d, "{[U,g,W]}"] \\ V \arrow[r, "{[V,g,W]}"] & W \end{tikzcd}$$

Proof: The given maps are equivalent to having $gU \subseteq Wg$, $hV \subseteq Wh$. We can rewrite these as $gUg^{-1} \subseteq W$, $hVh^{-1} \subseteq W$. Take any open subgroup $O \subseteq gUg^{-1} \cap hVh^{-1}$. Such an open subgroup exists since conjugation is a topological group isomorphism, and therefore $hVh^{-1}$and gUg^{-1} are open subgroups. The open subgroup O has the property that $g^{-1}O \subseteq g^{-1}gUg^{-1} = Ug^{-1}$, and $h^{-1}U \subseteq h^{-1}hVh^{-1} = Vh^{-1}$ hence we have maps $[O, g^{-1}, U]$, and $[O, h^{-1}, V]$ which make the above diagram commute. ∎

Let U be an open subgroup of G. Then G acts continuously on the coset space G/U by the action defined by $Ux \bullet g = Uxg$. If $[V, g, U] : V \to U$ is a map in $\mathbb{T}_G$, then we can define a map $G/V \to G/U$ between coset spaces by $Vx \mapsto Ugx$. This is well defined since for any two maps $[V, g, U], [V, h, U] : V \to U$ in $\mathbb{T}_G$, if $Vx = Vy$ then $Ugx = Ugvy$. Since $gv \in gV \subseteq Ug$ there is an element $u \in U$ such that $Ugvy = Uugy = Ugy$.

Proposition 4.4.4. *Let X be a continuous G-set. Then $\Gamma(X) = Hom_{\mathbf{Cont}(G)}(G/(-), X) : \mathbb{T}_G^{op} \to \mathbf{Set}$ is a functor.*

Proof: Since G/U is a continuous G-set for each open subgroup U, the set $Hom_{\mathbf{Cont}(G)}(G/U, X)$ is always defined. For any $\phi \in Hom_{\mathbf{Cont}(G)}(G/U, X)$, ϕ is equivariant. By the definition of the action of G on G/U we have $\phi(Ug) = \phi(U \bullet g) = \phi(U) \bullet g$. It follows that $\phi(U)$ is an element $x \in X$ such that $x \bullet u = x$ for every $u \in U$.

Conversely, take any element $x \in X$ such that $x \bullet u = x$ for every $u \in U$. Define $\phi : G/U \to X$ by $\phi(Ug) = x \bullet g$. This is well-defined since $Ug = Uh$ gives us $gh^{-1} \in U$ and thus $x \bullet gh^{-1} = x$, hence $x \bullet g = x \bullet h$. Hence $Hom_{\mathbf{Cont}(G)}(G/U, X)$ is the set of U fixed points of X.

Now given any map $[V, Ug, U] : V \to U$ between two open subgroups V and U. If x is a U fixed point of X, then for any $v \in V$ we have $gvg^{-1} \in U$ by definition of an arrow of $\mathbb{T}_G$. Since x is a fix point of elements of U, this gives us $x \bullet gvg^{-1} = x$ or $x \bullet gv = x \bullet g$. Hence $x \bullet g$ is a V fixed point of X. If $Ug = Uh$, again we have $gh^{-1} \in U$ so $x \bullet g = x \bullet h$.

So $Hom_{\mathbf{Cont}(G)}(G/(-), X)$ is well-defined. The identity morphisms $[U, Ue, U] : U \to U$ then gives us the identity morphism on the U fixed points of X, $x \mapsto x \bullet e = x$. For two morphisms $[V, Ug, U] : V \to U$, $[W, Vh, V] : W \to V$ the composition $[W, Ugh, U] : W \to U$ gives us the morphism $x \mapsto x \bullet gh = (x \bullet g) \bullet h$ by associativity of the action.

It follows that $Hom_{\mathbf{Cont}(G)}(G/(-), X)$ preserves composition of maps and the identity, and is therefore a functor. ∎

Consider a category $\mathbb{C}$ which has a subcategory $\mathbb{E}$ whose maps are all epimorphisms. Suppose that for any arrow f in $\mathbb{C}$ we can factor f as

$$f = e_f \circ i_f, \tag{4.4.1}$$

where e_f is an epimorphism in $\mathbb{E}$, and i_f is an isomorphism in $\mathbb{C}$. Additionally, suppose we have the following diagram in $\mathbb{C}$.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd} \tag{4.4.2}$$

Using the factorization **??** we can rewrite Diagram **??** as

$$\begin{tikzcd} & A' \arrow[dr, "e_f"] & & & \\ A \arrow[ur, "i_f"] \arrow[dr, "i_g"'] & & B \arrow[r, "i_h"] & B' \arrow[r, "e_h"] & C. \\ & A'' \arrow[ur, "e_g"'] & & & \end{tikzcd}$$

Each of the i's are isomorphisms, so we can rewrite this starting from A' as

$$\begin{tikzcd} A' \arrow[r, shift left, "e_f"] \arrow[r, shift right, "g \circ i_f^{-1}"'] & B \arrow[r, "i_h"] & B' \arrow[r, "e_h"] & C \end{tikzcd}$$

By our hypothesis on $\mathbb{C}$ the function $i_h \circ e_f = e'_f \circ i'_h$ so we can again rewrite the above starting at A''' as

$$\begin{tikzcd} A''' \arrow[r, shift left, "e'_f"] \arrow[r, shift right, "i_h \circ g \circ i_f \circ i'^{-1}_h"'] & B' \arrow[r, "e_h"] & C. \end{tikzcd}$$

Here A''' is isomorphic to A, and B' is isomorphic to B. Thus using the factorization in $\mathbb{C}$, for a diagram of the form **??** we can assume f and h are epimorphisms. We apply this in the following Proposition.

Proposition 4.4.5. *For any continuous G-set,* $\Gamma(X)$ *is a sheaf.*

Proof: We will use the classification of sheaves for the atomic topology given in Proposition **??**. By Proposition **??** and the above discussion we only have to consider diagrams of the form

$$\begin{tikzcd} V \arrow[r, shift left, "{[V,e,U]}"] \arrow[r, shift right, "{[V,g,U]}"'] & U \arrow[r, "{[U,e,W]}"] & W. \end{tikzcd}$$

By definition of composition and the arrows of $\mathbb{T}_G$ we have $Wg = W$ and hence $g \in W$. Suppose $y \in \Gamma(X)(U)$ such that $\Gamma(X)([V,e,U])(y) = \Gamma(X)([V,g,U])(y)$ for every such diagram. By definition this is equivalent to $y \centerdot e = y \centerdot g$. Hence g is an element of W which fixes y.

Now consider any element $w \in W$, and open subgroup $V \subseteq U$. Take $V' = V \cap w^{-1}Vw$. Then we have $V' \subseteq V \subseteq U \subseteq W$, and $wV' \subseteq ww^{-1}Vw = Vw \subseteq Uw \subseteq W$. This gives us a diagram of the form above.

Since we can do this for any $w \in W$ we have $y \centerdot w = y$ for all $w \in W$ and hence $y \in \Gamma(X)(W)$. The function $\Gamma(X)([U,e,W])(y) = y \centerdot e = y$ and this is clearly the unique element. By Proposition **??** $\Gamma(X)$ is a sheaf on $\mathbb{T}_G$. ∎

Definition 4.4.6. Let $\Lambda : \mathbf{Sh}(\mathbb{T}_G, At) \to \mathbf{Set}^{G^{op}}$ be the functor defined by $\Lambda(P) = \underset{U \in \Omega(G)}{\underrightarrow{Lim}}\ P(U)$ and for any natural transformation $\alpha : P \Rightarrow Q$, $\Lambda(\alpha)([x,U]) = [\alpha_U(x), U]$ where $x \in P(U)$.

Lemma 4.4.7. *If $P : \mathbb{T}_G^{op} \to \mathbf{Set}$ is a sheaf with respect to the atomic topology on $\mathbb{T}_G$ then ΛP is a G-set.*

Proof: Let $P : \mathbb{T}_G^{op} \to \mathbf{Set}$ be a sheaf, and $\Lambda(P) = \underset{U}{\underrightarrow{lim}} PU$ taken over the full subcategory of objects of $\mathbb{T}_G$ and morphisms of the form $[U,e,V] : U \subseteq V$. Equip $\Lambda(P)$ with the action $[x,U] \centerdot g = [P(g^{-1}Ug \xrightarrow{g} U)(x), g^{-1}Ug]$. Then we have a functor $\Lambda : \mathbf{Sh}(\mathbb{T}_G, At) \to \mathbf{Set}^{G^{op}}$.

If $\alpha : P \Rightarrow Q$ is a map of sheaves, then $\Lambda(\alpha) : \Lambda(P) \to \Lambda(Q)$ defined by $[x,U] \mapsto [\alpha_U(x), U]$. If $[x,U] = [y,V]$ then by definition there exists a $W \subseteq U \cap V$ such that $P(W \xrightarrow{e} U)(x) = P(W \xrightarrow{e} V)(y)$. Since α is a natural transformation we have $\alpha_W(P(W \xrightarrow{e} U)(x)) = Q(W \xrightarrow{e} U)(\alpha_U(x))$ and $\alpha_W(P(W \xrightarrow{e} V)(y)) = Q(W \xrightarrow{e} V)(\alpha_V(y))$. Therefore $[\alpha_U(x), U] = [\alpha_V(y), V]$ and Λ is well-defined.

Next we want to show that there is a group action on $\Lambda(P)$ defined by

$$[x,U] \centerdot g = [P(g^{-1}Ug \xrightarrow{g} U)(x), g^{-1}Ug].$$

Observe that since $[U,e,U]$ is the identity on U we have

$$[x,U] \centerdot e = [P(U \xrightarrow{e} U)(x)] = [x,U].$$

Lastly for $g,h \in G$ we have the following equalities.

$$([x,U] \centerdot g) \centerdot h = [P(g^{-1}Ug \xrightarrow{g} U)(x), g^{-1}Ug] \centerdot h$$

$$
\begin{aligned}
&= [P(h^{-1}g^{-1}Ugh \xrightarrow{gh} U)(P(g^{-1}Ug \xrightarrow{g} U)(x)), h^{-1}g^{-1}Ugh] \\
&= [P(h^{-1}g^{-1}Ugh \xrightarrow{gh} U)(x), h^{-1}g^{-1}Ugh] \\
&= [x, U] \bullet gh.
\end{aligned}
$$

∎

Proposition 4.4.8. *For any sheaf $P : \mathbb{T}_G^{op} \to \mathbf{Set}$, $\Lambda(P)$ is a continuous G-set.*

Proof: By definition of the action of G on $\Lambda(P)$ for any element $u \in U$ we have

$$
\begin{aligned}
[x, U] \bullet u &= [P(u^{-1}Uu \xrightarrow{u} U)(x), u^{-1}Uu] \\
&= [P(U \xrightarrow{e} U)(x), U] \\
&= [x, U].
\end{aligned}
$$

Therefore $U \subseteq G_{[x,U]}$, from which it follows from Proposition **??** that $G_{[x,U]}$ is an open subgroup. ∎

In light of Proposition **??** we have the following factorization of $\Lambda : \mathbf{Sh}(\mathbb{T}_G, At) \to \mathbf{Set}^{G^{op}}$.

$$
\begin{tikzcd}
\mathbf{Sh}(\mathbb{T}_G, At) \arrow[rr, "\Lambda"] \arrow[dr] & & \mathbf{Set}^{G^{op}} \\
& \mathbf{Cont}(G) \arrow[ur, "i"'] &
\end{tikzcd}
$$

Proposition 4.4.9. *There exists an adjunction $\Lambda \dashv \Gamma$*

Proof: Given an arrow $f : \Lambda(P) \to X$ of G-sets and an open subgroup U, define an arrow $f_{*U} : P(U) \to \Gamma(X)(U)$ by $f_{*U}(x) = f([x, U])$. Consider the composition $\Gamma(X)([V, g, U]) \circ f_{*U}$. Since f is equivariant, when applied to an element $x \in P(U)$ we have

$$
\begin{aligned}
(\Gamma(X)([V, g, U]) \circ f_{*U})(x) &= f([x, U]) \bullet g \\
&= f([x, U] \bullet g).
\end{aligned}
$$

By definition of the action and the elements of the equivalence classes this gives us

$$
\begin{aligned}
&= f([P(g^{-1}Ug \xrightarrow{g} U)(x), g^{-1}Ug]) \\
&= f([P(g^{-1}Ug \xrightarrow{g} U)(x)|_V, V]) \\
&= f([P([V, g, U])(x), V]).
\end{aligned}
$$

Finally by definition of f_* we have that that this is equal to

$$(f_{*V} \circ P([V,g,U]))(x). \tag{4.4.3}$$

Thus $f_* : P \Rightarrow \Gamma(X)$ is a natural transformation.

Now suppose that $\alpha : P \Rightarrow \Gamma(X)$ is a natural transformation of sheaves on $\mathbb{T}_G$. Define a new map $\alpha^* : \Lambda(P) \to X$ by $\alpha^*([x,U]) = \alpha_U(x)$. By definition of the equivalence relation if $[x,U] = [y,V]$ then there exists a subgroup $W \subseteq U \cap V$ such that $y|_W = x|_W$.

Since α is a natural transformation we have the following commutative diagram.

$$\begin{tikzcd}
P(U) \arrow[r, "\alpha_U"] \arrow[d, "P([W,e,U])"'] & \Gamma(X)(U) \arrow[d, "(-)\bullet e"] \\
P(W) \arrow[r, "\alpha_W"] & \Gamma(X)(W) \\
P(V) \arrow[r, "\alpha_V"] \arrow[u, "P([W,e,V]"] & \Gamma(X)(V). \arrow[u, "(-)\bullet e"']
\end{tikzcd}$$

Since $x|_W = y|_W$ we have $\alpha_W(x|_W) = \alpha_W(y|_W)$. Thus it follows from the commutativity of the above diagram that we also have $\alpha_U(x) = \alpha_W(x|_W) = \alpha_W(y|_W) = \alpha_V(y)$, and hence α^* is well defined. Moreover, for any $x \in P(U)$ and by naturality of α we have:

$$\begin{aligned}
\alpha^*([x]\bullet g) &= \alpha^*([P(g^{-1}Ug \xrightarrow{g} U)(x)]) \\
&= \alpha_{g^{-1}Ug}(P(g^{-1}Ug \xrightarrow{g} U)(x)) \\
&= \alpha_U(x)\bullet g \\
&= \alpha^*([x])\bullet g
\end{aligned}$$

Therefore α^* is equivariant.

For any arrow $f : \Lambda(P) \to X$ we have the following equalities.

$$\begin{aligned}
(f_*)^*([x,U]) &= f_{*U}(x) \\
&= f([x,U]).
\end{aligned}$$

Similarly for any natural transformation $\alpha : P \Rightarrow \Gamma(X)$.

$$\begin{aligned}
(\alpha^*)_{*U}(x) &= \alpha^*([x,U]) \\
&= \alpha_U(x)
\end{aligned}$$

Thus we have a bijection between the hom-sets $\mathbf{Cont}(G)(\Lambda(P), X) \cong \mathbf{Sh}(\mathbb{T}_G, At)(P, \Gamma(X))$ and hence Λ is right adjoint to Γ. ∎

The unit of the adjunction is given by the collection of morphisms $\eta_P : P \Rightarrow \Gamma\Lambda(P)$ where $\eta_{P(U)} : P(U) \to \Gamma\Lambda(P)(U)$ is defined by $\eta_{P(U)}(x) = [x, U]$.

The counit of the adjunction is given by the natural transformation $\epsilon : \Lambda\Gamma \Rightarrow ID_{\mathbf{Cont}(G)}$ given by $\epsilon_X([x, U]) = id_{\Gamma(X)(U)}(x) = x$.

Lemma 4.4.10. *The counit is a natural isomorphism.*

Proof: Suppose $\epsilon([x, U]) = \epsilon([y, V])$ then we have $x = y$ and hence $[x, U] = [y, V]$. Since X is a continuous G-set, the stabilizer subgroup G_x is an open subgroup of G. Therefore $x \in \Gamma(X)(G_x)$, $[x, G_x] \in \Lambda\Gamma(X)$, and $\epsilon([x, G_x]) = x$. ∎

Lemma 4.4.11. *The unit is an isomorphism of sheaves.*

Proof: By Corollary **??** each function $P([W, e, U])$ is injective in **Set**. Therefore if we have $[x, U] = [y, U]$ then we must have $x = y$. This gives us injectivity of $\eta_{P(U)}$. For any element $[x, U] \in \Gamma\Lambda(P)(U)$ the definition of the equivalence classes gives us $x \in P(U)$. Hence this is surjective at each U and therefore an isomorphism at each U. Since P is arbitrary, η_P is a natural isomorphism for each P.

Suppose we have a natural transformation between sheaves $\alpha : P \Rightarrow Q$. For any open subgroup U we want to show that the following square commutes.

$$\begin{tikzcd} P(U) \arrow[r, "\eta_{P(U)}"] \arrow[d, "\alpha_U"'] & \Gamma\Lambda(P)(U) \arrow[d, "\Gamma\Lambda(\alpha)"] \\ Q(U) \arrow[r, "\eta_{Q}(U)"] & \Gamma\Lambda(Q)(U) \end{tikzcd}$$

For any $x \in P(U)$ we have the following equalities.

$$\begin{aligned} (\Gamma\Lambda(\alpha)(U) \circ \eta_{P(U)})(x) &= \Gamma\Lambda(\alpha)(U)([x, U]) \\ &= [\alpha_U(x), U] \\ &= (\eta_{Q(U)} \circ \alpha_U)(x). \end{aligned}$$

Thus η_P is natural in P.

∎

Corollary 4.4.12. *We have the equivalence* $\mathbf{Cont}(G) \cong \mathbf{Sh}(\mathbb{T}_G, At)$.

Proof: This follows from Proposition **??** and Lemmas **??** and **??**. ∎

We have shown that the category $\mathbf{Cont}(G)$ of continuous G-sets and equivariant functions is equivalent to the category of sheaves on the site $(\mathbb{T}_G, At)$, and thus a topos by Definition **??**. Recall that in the construction of $\mathbb{T}_G$, the objects consist only of the open subgroups of G.

Additionally, if we let $\mathcal{F}_G$ be the collection of open subgroups of G, since the intersection of open subgroups is again an open subgroup, and the empty set is not an open subgroup, $\mathcal{F}_G$ is a filter base. Thus we can generate a new topology τ_F on G; call this new topological group G'.

Since $\mathbb{T}_G$ only takes into consideration the open subgroups of G, $\mathbb{T}_{G'}$ will be the same category. Hence $\mathbf{Cont}(G) \cong \mathbf{Cont}(G')$. Thus when considering the category of continuous G-sets, we can assume without any loss of generality, that the topology on G is generated by a filter base of open subgroups.

Chapter 5

Isotropy

The Isotropy group of Grothendieck topoi was introduced by Hofstra, Funk and Steinberg in [?]. In loc. cit. it was shown that every Grothendieck topos has an internal group associated with it, called its isotropy group. Moreover, there is a a canonical action of the isotropy group on each object of the topos. Every map is equivariant with respect to this action, and the action of the isotropy group on itself is the conjugation action. In a sense, the isotropy group is viewed as encoding the internal symmetries of the topos or the algebraic data, in an analogous way to how the subobject classifier encodes spatial data.

5.1 Internal Groups

We begin by looking at the categorical notion of a group object in a category. For any category with finite products we can talk about internal groups.

Definition 5.1.1. Let $\mathbb{C}$ be a category with finite products and G an object of $\mathbb{C}$. We say G is a group object if there exists maps $m : G\times G \to G$, $i : G \to G$, $e : \mathbb{1} \to G$ such that the following diagrams hold.

$$\begin{tikzcd} G\times G\times G \arrow[r, "id_G\times m"] \arrow[d, "m\times id_G"'] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "!"'] & G\times G \arrow[r, "i\times id_G"] & G\times G \arrow[d, "m"] \\ \mathbb{1} \arrow[rr, "e"] & & G \\ G \arrow[r, "\Delta"] \arrow[u, "!"] & G\times G \arrow[r, "id_G\times i"] & G\times G \arrow[u, "m"'] \end{tikzcd}$$

For the category **Set**, the definition of a group object coincides with the axiomatic definition of a group. Next, we may internalize the notion of an action of a group.

Definition 5.1.2. For a group object G of a topos $\mathcal{E}$, an object X of $\mathcal{E}$ is called a right G-object if there exists a map $\alpha_X : X \times G \to X$, called the *action map*, such that the following diagrams commute.

$$\begin{tikzcd} X\times G\times G \arrow[r, "id_X\times m"] \arrow[d, "\alpha_X\times id_G"'] & X\times G \arrow[d, "\alpha_X"] \\ X\times G \arrow[r, "\alpha_X"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\cong"] \arrow[rrd, "id_X"] & X\times \mathbb{1} \arrow[r, "id_X\times e"] & X\times G \arrow[d, "\alpha_X"] \\ & & X \end{tikzcd}$$

Definition 5.1.3. Let G be a group, and X, Y right G-objects. Then an arrow $f : X \to Y$ is called G-equivariant if the following diagram commutes.

$$\begin{tikzcd} X\times G \arrow[r, "\alpha_X"] \arrow[d, "id_X\times f"'] & X \arrow[d, "f"] \\ Y\times G \arrow[r, "\alpha_Y"] & Y \end{tikzcd}$$

We can look at what group objects in familiar categories are.

Example 5.1.4. The following are examples of group objects for various well-known categories.

- In **Set**, the category of sets and functions, the group objects are groups.
- In **Top**, the category of topological spaces and continuous functions, the group objects are Topological groups.
- In **Grp**, the category of groups and homomorphisms, the group objects are Abelian groups.

One important example is the topos of presheaves for a category $\mathbb{C}$.

Example 5.1.5. Let $G : \mathbb{C}^{op} \to \mathbf{Set}$ be a group object in $\mathbf{Set}^{\mathbb{C}^{op}}$. By definition this provides us with the following natural transformations which make the diagrams in Definition **??** commute.

$$m : G \times G \Rightarrow G,\ i : G \Rightarrow G,\ e : \mathbb{1} \to G.$$

Fixing an object C of $\mathbb{C}$, $G(C)$ is an object of **Set** and we have the following functions.

$$m_C : G(C) \times G(C) \to G(C),\ i_C : G(C) \to G(C),\ e_C : 1 \to G(C).$$

These functions also make the diagrams of Definition **??** commute, hence $G(C)$ is a group in **Set**. So in fact we can consider G as a presheaf of groups, $G : \mathbb{C}^{op} \to \mathbf{Grp}$.

5.2 Isotropy for Categories

We now want to describe isotropy for a small category $\mathbb{C}$.

Definition 5.2.1. Let $\mathbb{C}$ be a small category. The *isotropy group* for $\mathbb{C}$ is the presheaf of groups $\mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to \mathbf{Grp}$ defined by

$$\mathcal{Z}_{\mathbb{C}}(C) = Aut(\mathbb{C}/C \xrightarrow{\pi_C} \mathbb{C}).$$

When no confusion arises we will drop the subscript $\mathbb{C}$ and simply refer to $\mathcal{Z}$ as the isotropy group.

An element $\tau \in \mathcal{Z}(C)$ is a natural isomorphism $\tau : \pi_{\mathbb{C}} \Rightarrow \pi_{\mathbb{C}}$ where π is the functor which takes an object $f : A \to C$ of $\mathbb{C}/C$ to its domain C in $\mathbb{C}$. For an arrow $h : A \to B$ between arrows $f : A \to C$ and $g : B \to C$ in $\mathbb{C}/C$ the naturality condition for τ means that the following diagram commutes (here, we write τ_C for the component of τ at the identity $1 : C \to C$).

$$\begin{tikzcd} A \arrow[r, "\tau_f"] \arrow[d, "h"'] \arrow[dd, bend right, "f"'] & A \arrow[d, "h"] \arrow[dd, bend left, "f"] \\ B \arrow[r, "\tau_g"] \arrow[d, "g"'] & B \arrow[d, "g"] \\ C \arrow[r, "\tau_C"] & C \end{tikzcd}$$

Here the horizontal maps are automorphisms. It is important to note that when f is an isomorphism, the automorphisms are determined by conjugation of τ_C. Hence, if $\mathbb{C}$ is a groupoid, we necessarily have $\tau_f = f^{-1}\tau_C f$ for all f, and we find that $\mathcal{Z}(C) = Aut(C)$, the automorphism group of C. This group is sometimes called the isotropy group at C of the groupoid, which explains the terminology. (See Example **??** for a more detailed calculation of the isotropy group of a group.) However, the assignment $C \mapsto Aut(C)$ is, for a general category $\mathbb{C}$, not functorial, and part of what the isotropy group of $\mathbb{C}$ does is remedy this defect, in the sense that an element of isotropy gives us a way of "conjugating" an automorphism by general maps in $\mathbb{C}$. Indeed, there is a canonical comparison map

$$\mathcal{Z}(C) \to Aut(C)$$

which sends an element of isotropy τ to the component $\tau_C : C \to C$ at the identity of C. In general, this map (which is a group homomorphism) is neither injective nor surjective.

5.3 Isotropy Group of a Topos

We can repeat the definition of the isotropy group of a category in the particular case when that category is a topos:

Definition 5.3.1. Let $\mathcal{E}$ be a Grothendieck topos. The *Isotropy* functor $\mathcal{Z}_{\mathcal{E}} : \mathcal{E}^{op} \to$ **Grp** is defined as

$$\mathcal{Z}_{\mathcal{E}}(E) = Aut(\mathcal{E}/E \to \mathcal{E}). \tag{5.3.1}$$

This time, an element $\tau \in \mathcal{Z}_{\mathcal{E}}(E)$ is a natural isomorphism $\tau : \pi_! \Rightarrow \pi_!$ where π is the geometric morphism induced by a base change along the terminal arrow $E \to *$. We may consider such a natural isomorphism as a natural transformation between the left adjoints $\pi_!$, which takes an object $f : A \to E$ of $\mathcal{E}/E$ to its domain A in $\mathcal{E}$.

One of the main results from [**?**] is that $\mathcal{Z}_{\mathcal{E}}$ is a representable functor in the case that $\mathcal{E}$ is a Grothendieck topos.

Theorem 5.3.2. *For any Grothendieck topos $\mathcal{E}$ there exists a group object $Z_{\mathcal{E}}$ such that*

$$\mathcal{Z}(A) \cong Hom_{\mathcal{E}}(A, Z_{\mathcal{E}}). \tag{5.3.2}$$

We call $Z_{\mathcal{E}}$ the isotropy group of $\mathcal{E}$.

Again, when no confusion arises, we will drop the subscript and simply refer to the isotropy group of a topos as Z. To obtain Z, precompose the isotropy functor with the (opposite of) the Yoneda embedding

$$\mathbb{C}^{op} \to \mathbf{Sh}(\mathbb{C}, J)^{op} \simeq \mathcal{E}^{op} \to \mathbf{Grp}$$

(where $\mathbb{C}$ is a small generating subcategory of $\mathcal{E}$); then this composite is actually an object of $\mathcal{E}$. Note that it also follows that the isotropy functor Z actually takes values in small groups (i.e. the group of natural automorphisms of π is a small group).

When we have a site with a subcanonical Grothendieck topology for a Grothendieck topos $\mathcal{E}$, we can in fact describe isotropy of $\mathcal{E}$ in terms of the site [?].

Theorem 5.3.3. *Let $\mathcal{E} \cong \mathbf{Sh}(\mathbb{C}, J)$ be a Grothendieck topos. If J is a subcanonical Grothendieck topology then there exists an isomorphism of isotropy groups $Z_{\mathcal{E}} \cong Z_{\mathbb{C}}$.*

Let α be the natural transformation which corresponds to the identity $Z \to Z$ in $Hom_{\mathcal{E}}(Z, Z)$. For any general element of Z, $x : X \to Z$, under the bijection we have the following commutative diagram from Theorem **??**.

$$\begin{tikzcd}
\mathcal{E}(Z, Z) \arrow[r] \arrow[d, "x\circ -"'] & \mathcal{Z}(Z) \arrow[d, "\overline{x}"] \\
\mathcal{E}(X, Z) \arrow[r] & \mathcal{Z}(X)
\end{tikzcd}$$

The map $\overline{x}$ takes a natural isomorphism $t \in \mathcal{Z}(Z)$ to the morphism $\overline{t}(\alpha)$ defined as

$$X \times X \xrightarrow{id_X \times x} X \times Z \xrightarrow{\alpha_X} X. \tag{5.3.3}$$

Let $(f, t) : U \to X \times Z$ be a general element of $X \times Z$. That is $f : U \to X$, and $t : U \to Z$. By Theorem **??** we can regard t as an element of $Aut(\mathcal{E}/U \to \mathcal{E})$. For an object U of $\mathcal{E}$ we then define $\alpha_E : E \times Z \to E$ to be $\alpha_E(f, t) = f \bullet t : X \to E$ where for a general element $x : U \to X$ we have $(f \bullet t)(x) = t_E(f(x), x)$.

It is shown in [?] that this is indeed an action of Z on E.

Proposition 5.3.4. *For each object E of $\mathcal{E}$, α_E makes E a Z-object internal to $\mathcal{E}$.*

Moreover, every arrow is Z-equivariant.

Proposition 5.3.5. *Every arrow in $\mathcal{E}$ is equivariant with respect to the action $\alpha_X : X \times Z \to X$.*

Proof: Let $(f, t) : X \to E \times Z$ be a general element of $E \times Z$. By Theorem **??** we can view t as an element of $Z(X)$. In particular a natural isomorphism where the component at E is $t_E : E \times X \to E \times X$.

Let $m : E \to B$ be any map in $\mathcal{E}$. For a general element $x \in X$, by definition $\alpha_E(f,t)(x) = t_E(f(x),x)$. Composing with m we have $m \circ t_E(f(x),x) = t_B((m \circ f)(x),x)$ by naturality of t. Again, by definition of α we have $t_B((m \circ f)(x),x) = \alpha_B(m \circ f,t)(x)$. This holds for any generalized elements f,t,x, hence $m \circ \alpha_E = \alpha_B \circ (m \times id_Z)$. This gives us that every map between objects in $\mathcal{E}$ is equivariant with respect to the action of Z. ∎

Before we move on to computing the isotropy group for $\mathbf{Cont}(G)$ we give some basic examples.

Example 5.3.6. We can view the topos of presheaves as sheaves on a site where every presheaf is a sheaf. This is subcanonical and therefore by Theorem **??** the isotropy group of the presheaf category $\mathbf{Set}^{\mathbb{C}^{op}}$ is given by

$$Z(C) = Aut(\mathbb{C}/C \to \mathbb{C}). \tag{5.3.4}$$

Example 5.3.7. In the case where $\mathbb{C}$ is a group G viewed as a one object category with every morphism an isomorphism, we have that the isotropy group is given by

$$Z(*) = Aut(G/* \to G). \tag{5.3.5}$$

Since every morphism in G is an isomorphism, each component of $\tau \in Z(*)$ is determined by τ_e as $\tau_g = g^{-1}\tau_e g$.

Proposition 5.3.8. *The isotropy group Z of the topos of G-sets for a discrete group G is isomorphic to $(G, conj)$, the group acting on itself by conjugation.*

Proof: Consider the function $\phi : Z(*) \to (G, conj)$ defined by $\phi(\tau) = \tau_e$. Then we have $(\tau \circ \alpha)_e = \tau_e \circ \alpha_e$. Composition is determined by group multiplication so $\tau_e \circ \alpha_e = \tau_e\alpha_e = \phi(\tau)\phi(\alpha)$. The identity element of $Z(*)$ is given by $\epsilon_g = e$ for every $g \in G$. In particular $\phi(\epsilon) = e$. Thus ϕ is a homomorphism.

If $\phi(\tau) = \phi(\nu)$ then $\tau_e = \nu_e$. For each $h \in G$, $\tau_h = h^{-1}\tau_e h = h^{-1}\nu_e h = \nu_h$. This holds for every $h \in G$ and hence ϕ is injective.

Given any $g \in G$ define a map $\tau_h = h^{-1}gh$. For any elements $x, y, z \in G$ with $x = yz$, we have

$$\begin{aligned}\tau_x &= x^{-1}\tau_e x \\ &= z^{-1}y^{-1}\tau_e yz \\ &= z^{-1}\tau_y z.\end{aligned}$$

So for an arbitrary arrow in $G/*$, the following diagram commutes.

$$\begin{tikzcd}
* \arrow[r, "\tau_x"] \arrow[d, "z"'] & * \arrow[d, "z"] \\
* \arrow[r, "\tau_y"] \arrow[d, "y"'] & * \arrow[d, "y"] \\
* \arrow[r, "\tau_e"] & *.
\end{tikzcd}$$

Therefore $\tau \in Z(*)$. Lastly, the action of G on $Z(*)$ is given by $\tau \bullet g = Z(g)(\tau)$. This is defined by

$$Z(g)(\tau)_h = \tau_{gh} = h^{-1}\tau_g h.$$

$\blacksquare$

For another example we look at sheaves on a topological space.

Example 5.3.9. Recall that $\mathbf{Sh}(X)$ is a subcategory of $\mathbf{Set}^{\mathcal{O}(X)^{op}}$. Moreover, it is also subcanonical [**?**]. As we will see in Theorem **??** the isotropy groups of $\mathbf{Sh}(X)$ and $\mathcal{O}(X)$ coincide. Since $\mathcal{O}(X)$ is a partially ordered set, the only isomorphisms are the trivial isomorphisms. Thus the isotropy group for $\mathbf{Sh}(X)$ is the trivial group.

It is important to note that for all Grothendieck Topoi, the action of the isotropy group Z on the subobject classifier Ω is trivial [**?**].

5.4 A Technical Lemma

In this section we establish a technical result which will be of help in determining the isotropy group of the topos of continuous G-sets. It allows us to reduce the problem of finding the isotropy group of a small category to that of a certain type of wide subcategory. In the next section we will apply this result to the category $\mathbb{T}_G$.

We begin with two definitions.

Definition 5.4.1. A subcategory $\mathbb{A}$ of $\mathbb{C}$ is called *wide* if the $Obj(\mathbb{A}) = Obj(\mathbb{C})$.

Definition 5.4.2. A subcategory $\mathbb{A}$ of $\mathbb{C}$ is called *full on automorphisms* if the inclusion $i : \mathbb{A} \to \mathbb{C}$ induces an isomorphism $Aut_{\mathbb{A}}(A) \cong Aut_{\mathbb{C}}(iA)$ for every object A of $\mathbb{A}$.

Given a subcategory $\mathbb{A}$ of $\mathbb{C}$, we have the inclusion functor $i : \mathbb{A} \hookrightarrow \mathbb{C}$. The functor i induces another functor $i^* : \mathbf{Set}^{\mathbb{C}^{op}} \to \mathbf{Set}^{\mathbb{A}^{op}}$ defined by

$$i^*(P) = P \circ i^{op},\ i^*(\alpha) = \alpha_{i^{op}}.$$

Now fix an object C of $\mathbb{A}$ and define a function $\phi_C : i^*Z_{\mathbb{C}}(C) = Z_{\mathbb{C}}(iC) \to Z_{\mathbb{A}}(C)$ by

$$\phi_C(\tau) = i^*\tau\ ; (i^*\tau)_f = \tau_{i(f)}.$$

Note that this is well-defined precisely when $Aut_{\mathbb{A}}(C) = Aut_{\mathbb{C}}(C)$. Since each arrow in $\mathbb{A}$ can be considered as an arrow in $\mathbb{C}$, τ_i is still an element of isotropy.

Lemma 5.4.3. *Let $\mathbb{A}$ be a subcategory of $\mathbb{C}$ that is full on automorphisms and let C an object of $\mathbb{A}$. Then ϕ_C as defined above is a homomorphism of groups; moreover, the ϕ_C are natural in C.*

Proof: To prove naturality we need to check that, for an arrow $f : C \to D$, the following diagram commutes.

$$\begin{tikzcd}
i^*Z_{\mathbb{C}}(D) \arrow[r, "\phi_D"] \arrow[d, "\overline{i(f)}"'] & Z_{\mathbb{A}}(D) \arrow[d, "\overline{f}"] \\
i^*Z_{\mathbb{C}}(C) \arrow[r, "\phi_C"] & Z_{\mathbb{A}}(C).
\end{tikzcd} \tag{5.4.1}$$

Here $\overline{f}$ is the functor which assigns to each $\tau \in Z_{\mathbb{A}}(D)$ an element of isotropy $\tau_f \in Z_{\mathbb{A}}(C)$. For an object $g : X \to C$ in $\mathbb{A}/C$ $(\tau_f)_g = \tau_{fg}$.

Commutativity of Diagram **??** comes down to showing $(\tau_f)_i = \tau_f$. This follows from the fact that for any arrow $g : X \to C$, $i(g) = g$ in $\mathbb{C}$. Hence for any arrow $g : X \to C$ in $\mathbb{A}/C$ we have:

$$((\tau_f)_i)_g = (\tau_f)_{i(g)}$$

$$
\begin{aligned}
&= (\tau_f)_g \\
&= \tau_{fg}.
\end{aligned}
$$

Fixing an object C of $\mathbb{A}$. Let $f : X \to C$ be an object of $\mathbb{A}/C$. Then for elements $\alpha, \tau \in i^* Z_{\mathbb{C}}(D)$ we get:

$$
\begin{aligned}
(\phi_C(\tau) \circ \phi_C(\alpha))_f &= (\tau_i \circ \alpha_i)_f \\
&= \tau_f \circ \alpha_f \\
&= (\tau \circ \alpha)_f \\
&= (\tau \circ \alpha)_i(f) \\
&= \phi_C(\tau \circ \alpha)_f.
\end{aligned}
$$

Thus for each object C of $\mathbb{A}$, ϕ_C is a homomorphism. Hence ϕ is a natural homomorphism.

∎

Lemma 5.4.4. *Let $\mathbb{A}$ be a full-on-automorphisms subcategory of $\mathbb{C}$ and C an object of $\mathbb{A}$. Additionally, suppose that every arrow f in $\mathbb{C}$ can be factored as $f = f_a \circ f_i$ where f_i is an isomorphism, and f_a is an arrow in $\mathbb{A}$. Then ϕ_C is injective.*

Proof: Suppose we have two elements $\tau, \alpha \in i^* Z_{\mathbb{C}}(C)$ and that $\phi_C(\tau) = \phi_C(\alpha)$. Hence $\tau_i = \alpha_i$. By our assumption on $\mathbb{C}$, for any arrow $f : X \to C$ we can factor $f = f_a \circ f_i$ where $f_a : X' \to C$ is an arrow in $\mathbb{A}$ and $f_i : X \to X'$ is an isomorphism in $\mathbb{C}$. Since τ and α are elements of isotropy, we have the following diagram.

$$
\begin{tikzcd}
X \arrow[r, bend left, "\tau_f"] \arrow[r, bend right, "\alpha_f"'] \arrow[d, "f_i"] & X \arrow[d, "f_i"] \\
X' \arrow[r, "\tau_{f_a} = \alpha_{f_a}"] \arrow[d, "f_a"] & X' \arrow[d, "f_a"] \\
C \arrow[r, "\tau_C"] & C.
\end{tikzcd}
$$

By definition of isotropy, the diagram commutes when we consider just τ_f and α_f as the top horizontal arrow. We want to show that $\tau_f = \alpha_f$. Since f_i is an isomorphism, we have

$$\tau_f = f_i^{-1} \circ \tau_{f_a} f_i.$$

Similarly we also have

$$\alpha_f = f^{-1} \circ \alpha_{f_a} \circ f_i$$

$$\begin{aligned} &= f^{-1} \circ \tau_{f_a} \circ f_i \\ &= \tau_f. \end{aligned}$$

Since $f : X \to C$ is an arbitrary arrow of $\mathbb{C}$, we have $\alpha = \tau$ and hence ϕ_C is injective. ∎

Note that the condition that every arrow factors as an isomorphism followed by an arrow in the subcategory means that in the slice category $\mathbb{C}/C$, each object is isomorphic to an object from the sub-slice category $\mathbb{A}/C$. Thus, informally speaking, the isotropy of $\mathbb{C}$ should be determined by that of $\mathbb{A}$. To make this formal, let us consider the following subset of $Z_{\mathbb{A}}(C)$.

$$S = \{\tau \in Z_{\mathbb{A}}(C) \mid \forall f, g \in \mathbb{A}/C, h : f \to g \text{ is an isomorphism} \Rightarrow \tau_f = h^{-1}\tau_g h\}$$

Then we have:

Lemma 5.4.5. *Let $\mathbb{A}$ be a wide subcategory of $\mathbb{C}$ that is full on automorphisms, and C an object of $\mathbb{A}$. Additionally, suppose that every arrow f in $\mathbb{C}$ can be factored as $f = f_a \circ f_i$ where f_i is an isomorphism, and f_a is an arrow in $\mathbb{A}$. Then we have*

$$Img\ \phi_C = S.$$

Proof: First we want to show that $Img\ \phi_C \subseteq S$. Suppose $\tau \in Img\ \phi_C$. Then there exists a $\alpha \in Z_{\mathbb{C}}(C)$ such that $\tau = \alpha_i$. Suppose we have two arrows $f : X \to C, g : Y \to C$ in $\mathbb{A}$ such that there exists an isomorphism $h : X \to Y$ in $\mathbb{C}$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "i(f)"'] & Y \arrow[d, "i(g)"] \\ & C. \end{tikzcd}$$

Since $\alpha \in Z_{\mathbb{C}}(C)$, the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\alpha_{i(f)}"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "\alpha_{i(g)}"] \arrow[d, "i(g)"'] & Y \arrow[d, "i(g)"] \\ C \arrow[r, "\alpha_C"] & C \end{tikzcd}$$

By Commutativity of the above diagram and the definition of τ we have:

$$\tau_f = (\alpha_i)_f$$

$$
\begin{aligned}
&= \alpha_{i(f)} \\
&= h^{-1} \circ \alpha_{i(g)} \circ h \\
&= h^{-1} \circ (\alpha_i)_g \circ h \\
&= h^{-1} \circ \tau_g \circ h.
\end{aligned}
$$

Now we want to show the opposite inclusion. Suppose $\tau \in S$. We want to show there exists $\alpha \in Z_{\mathbb{C}}(C)$ such that $\phi_C(\alpha) = \alpha_i = \tau$. Let $f = f_a \circ f_i$ be an object in $\mathbb{C}/C$ and define $\alpha_f = f_i^{-1} \tau_{f_a} f_i$. Suppose we have the following arrow in $\mathbb{C}/C$.

$$
\begin{tikzcd}
X \arrow[r, "h"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \\
& C.
\end{tikzcd}
$$

To show that α is natural, we need to show the following diagram commutes.

$$
\begin{tikzcd}
X \arrow[r, "\alpha_f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\
Y \arrow[r, "\alpha_g"] \arrow[d, "g"'] & Y \arrow[d, "g"] \\
C \arrow[r, "\alpha_C"] & Y
\end{tikzcd}
\tag{5.4.2}
$$

By our assumption on $\mathbb{C}$ we can write $g = g_a \circ g_i$. So the bottom square of Diagram **??** can be redrawn as follows.

$$
\begin{tikzcd}
Y \arrow[r, "\alpha_g"] \arrow[d, "g_i"'] & Y \arrow[d, "g"] \\
Y' \arrow[r, "\alpha_{g_a}"] \arrow[d, "g_a"'] & Y' \arrow[d, "g_a"] \\
C \arrow[r, "\alpha_C"] & Y
\end{tikzcd}
$$

By definition of α we have $\alpha_{g_a} = \tau_{g_a}$ and $\alpha_g = g_i^{-1} \tau_{g_a} g_i$, which makes the bottom square commute. Using the factorization of f we can redraw the top square of Diagram **??** in a similar way.

$$\begin{tikzcd}
X \arrow[r, "\alpha_f"] \arrow[d, "h_i"'] & Y \arrow[d, "h_i"] \\
X' \arrow[r, "\alpha_{gh_a}"] \arrow[d, "h_a"'] & Y' \arrow[d, "h_a"] \\
Y \arrow[r, "\alpha_g"] & Y
\end{tikzcd}$$

By our definition of α we have that $\alpha_f = f_i^{-1}\tau_{f_a}f_i$. To show naturality for the top square we need to have $\alpha_f = h_i^{-1}\circ\alpha_{gh_a}\circ h_i$ and to show that $h_a\circ\alpha_{gh_a} = \alpha_g\circ h_a$. Suppose that we define a map $s = g_i h_a$. Then we have $gh_a = g_a g_i h_a = g_a s$. Hence $\alpha_{gh_a} = \alpha_{g_a s}$. Using the factorization in $\mathbb{C}$ we also have $s = s_a s_i$ and therefore $g_a s = g_a s_a s_i$ and hence $\alpha_{gh_a} = \alpha_{g_a s} = s_i^{-1} \circ \tau_{g_a s_a} \circ s_i$.

Since h is an arrow between f and g in $\mathbb{C}/C$ we have:

$$\begin{aligned} f &= f_a f_i = gh \\ &= g_a g_i h_a h_i \\ &= g_a s_a s_i h_i. \end{aligned}$$

Thus we can write

$$f_a = g_a s_a s_i h_i f_i^{-1}.$$

Since $\tau \in S$ this gives us

$$\tau_{f_a} = f_i h_i^{-1} s_i^{-1} \circ \tau_{g_a s_a} \circ s_i h_i f_i^{-1}.$$

Using this we get the following equality.

$$\begin{aligned} \alpha_f &= f_i^{-1} \circ \tau_{f_a} \circ f_i \\ &= f_i^{-1} \circ f_i h_i^{-1} s_i^{-1} \circ \tau_{g_a s_a} \circ s_i h_i f_i^{-1} \circ f_i \\ &= h_i^{-1} s_i^{-1} \circ \tau_{g_a s_a} \circ s_i h_i \\ &= h_i^{-1} \alpha_{g_a s} h_i \\ &= h_i^{-1} \alpha_{gh_a} h_i. \end{aligned}$$

Lastly we have

$$\begin{aligned} \alpha_g \circ h_a &= g_i^{-1}\tau_{g_a} g_i \circ h_a \\ &= g_i^{-1}\tau_{g_a} s_a s_i \\ &= g_i^{-1} s_a \tau_{g_a s_a} s_i \\ &= h_a s_i^{-1} \tau_{g_a s_a} s_i \end{aligned}$$

$$= h_a \circ \alpha_{g_a s}$$
$$= h_a \circ \alpha_{g h_a}.$$

Thus $\alpha \in Z_{\mathbb{C}}(C)$ is such that $\phi_C(\alpha) = \tau$. ∎

5.5 The Isotropy Group of Cont(G)

We will use the results from the previous section to compute the isotropy group for a general topological group. As we noted in Corollary **??** we can assume that G is nearly discrete. In order to apply the results about wide subcategories and isotropy, we need to identify a suitable wide subcategory of $\mathbb{T}_G$.

Definition 5.5.1. Let $\mathbb{W}$ be the subcategory of $\mathbb{T}_G$ defined as follows.

Objects : Open subgroups of G.

Morphisms : Arrows of $\mathbb{T}_G$ generated by automorphisms of the form $[V, g, V] : V \to V$ and the inclusions $[V, e, U] : V \to U$

The category $\mathbb{W}$ contains all arrows that are formed by the composition of arrows of the form $[V, g, V] : V \to V$ and $[V, e, U] : V \to U$. This excludes isomorphisms of the form $[V, g, U] : V \to U$.

Since each object is also an object of $\mathbb{T}_G$, $\mathbb{W}$ is a wide subcategory of $\mathbb{T}_G$. Moreover each automorphism of $\mathbb{T}_G$ is also in $\mathbb{W}$, hence Lemma **??** applies. We have reduced the problem of computing isotropy of **Cont**(G) to computing isotropy for $Z_{\mathbb{W}}$ since we have the following.

$$Z_{\mathbf{Cont}(G)} \cong Z_{\mathbb{T}_G} \subseteq Z_{\mathbb{W}}.$$

From Proposition **??** we know that each component of isotropy represents an element of $N(V)/V$ where V is an open subgroup of U. Hence we can regard each $\tau \in Z_{\mathbb{W}}$ as a collection $\tau = (Vh_V) \in \prod_{V \subseteq U} N(V)/V$. As we have seen in Lemma **??**, restriction along an isomorphism is conjugation and hence we can focus on the maps of $\mathbb{W}$ which are of the form $[V, e, U] : V \to U$. This gives us the following representation of the isotropy group of $\mathbb{W}$,

$$Z_{\mathbb{W}}(U) = \{(Vh_V)_{V \subseteq U} \in \prod_{V \subseteq U} \frac{N(V)}{V} \mid W \subseteq V \Rightarrow Vh_W = Vh_V\}.$$

Hence we find:

Theorem 5.5.2. *Let G be a topological group and Z denote the isotropy group of* $\mathbf{Cont}(G)$*. At any open subgroup U of G, $Z(U)$ is equivalent to the following set.*

$$\{(Vh_V)_{V\subseteq U} \in \prod_{V\subseteq U} \frac{N(V)}{V} \mid \forall W, V \subseteq U, g \in G\ (gWg^{-1} \subseteq V \to Vh_V = Vgh_Wg^{-1})\}.$$

Proof: This follows from the definition of $Z_{\mathbb{W}}(U)$ and the definition of $Img\ \phi_U$ from Lemma **??**. ∎

Unless otherwise specified, we will refer to the isotropy group of $\mathbf{Cont}(G)$ simply as Z.

Example 5.5.3. Let G be a group with the discrete topology and $\langle Vh_V\rangle \in Z(U)$ an element of isotropy. Since every subgroup is open in G, the trivial subgroup is open. By our definition of Z, an automorphism of $\{e\}$ is a coset $N(\{e\})/\{e\}$. Every group element normalizes the trivial group, so we get that $Z(\{e\})$ is isomorphic to G.

Consider Z as an object of $\mathbf{Cont}(G)$. Then by the definition elements are equivalence classes $[\langle Vh_V\rangle, U]$ where if $[\langle Vh_V\rangle, U] = [\langle Vg_V\rangle, T]$ then there exists an open subgroup $W \subseteq U \cap T$ such that

$$Z([W, e, U])(\langle Vh_V\rangle) = Z([W, e, V])(\langle Vg_V\rangle).$$

This is equivalent to saying that for every open subgroup X of W, we have $Xg_X = Xh_X$. Since the trivial group is open, we can take $W = \{e\}$ to be the trivial group. Hence we will get that Z in $\mathbf{Cont}(G)$ is isomorphic to G.

Now recall that we have an action on Z in $\mathbf{Cont}(G)$ for a sheaf on $\mathbb{T}_G$ defined by

$$[x, U] \bullet g = [Z(g^{-1}Ug \xrightarrow{Ug} U)(x), g^{-1}Ug].$$

For $x = \langle Vh_V\rangle$ recall that we get

$$Z(g^{-1}Ug \xrightarrow{Ug} U)(\langle Vh_V\rangle) = \langle g^{-1}Vg(g^{-1}h_Vg)\rangle.$$

In particular since each element $[\langle Vh_V\rangle, U] = [\langle\{e\}g, \{e\}]$ for some $g \in G$. This action is precisely the conjugation action on G. Thus we have $Z = (G, conj)$ as expected.

5.6 Uniform Isotropy Elements

Given our description of the isotropy group for the topos $\mathbf{Cont}(G)$ we now look at a subgroup of Z defined as

$$\mathcal{H}(U) = \{\langle Vh_V\rangle \in Z(U) \mid Wh_V = Wh_W,\ \forall W \subseteq V\}.$$

Note the difference in orientation: the compatibility condition for an element $\langle Vh_V\rangle$ is that for all $W \subseteq V$ we have $Vh_V = Vh_W$.

Definition 5.6.1. An element of isotropy $\tau \in \mathcal{H}(U)$ is called *uniform*.

The name uniform is justified by the following lemma.

Lemma 5.6.2. *Let G be a nearly discrete group and $\langle V h_V \rangle \in \mathcal{H}(U)$ a uniform element of isotropy. Then for every $V \subseteq U$, $h_V = h_U$.*

Proof: Suppose V is an open subgroup of U. By definition of $Z(U)$ we have $Uh_U = Uh_V$. By definition of $H(U)$ we have $Vh_U = Vh_V$. Thus there exists a $u \in U$ and $v \in V$ such that the following equalities hold:

$$\begin{aligned} h_U &= uh_V \\ &= vh_V. \end{aligned}$$

Cancelling h_V, we thus we get $u = v$, and therefore in particular $u \in V$. Since V is arbitrary this holds for every open subgroup of U. Therefore, we find that $u \in \bigcap_{V \subseteq U} V$, and since G is nearly discrete we must have $\bigcap_{V \subseteq U} V$ is the trivial subgroup. This forces $u = e$. Thus we have deduced that $h_U = h_V$ for every open subgroup V of U. ∎

In light of Lemma **??** we will drop the subscripts when talking about elements of $\mathcal{H}$.

Lemma 5.6.3. *Let G be a nearly discrete topological group and $\langle Vh \rangle$ and $\langle Vg \rangle$ be two elements of $\mathcal{H}(U)$. If $\langle Vh \rangle = \langle Vg \rangle$ then $g = h$.*

Proof: Let V be any open subgroup of U. Since we have equality of isotropy elements, we get equality of the components, $Uh = Ug$ and $Vh = Vg$. This gives us the following pair of equations for h.

$$\begin{aligned} h &= ug \\ &= vg \end{aligned}$$

Similar to Lemma **??**, we get $h = g$. ∎

We now aim to provide an alternative description of the group $\mathcal{H}$, in terms of centralizer subgroups. Recall that for an element $h \in G$, and subgroup U of G, the centralizer subgroup of h and of U are defined as

$$\begin{aligned} C(h) &= \{g \in G \mid gh = hg\} \\ C(U) &= \{g \in G \mid \forall u \in U,\ gu = ug\}. \end{aligned}$$

If we consider a subgroup $V \subseteq U$, then we get an inclusion map $C(V \subseteq U) : C(U) \to C(V)$. (This is well-known; in fact C forms a Galois correspondence from the lattice of subgroups of G to itself: we have $U \subseteq C(V)$ if and only if $V \subseteq C(U)$.) However, we may also view C as a functor $C : \mathbb{T}_G^{op} \to \mathbf{Grp}$. To do this, we merely note that an isomorphism $U \to gUg^{-1}$ in $\mathbb{T}_G$ induces an isomorphism between the associated commutator subgroups $C(U) \to C(gUg^{-1})$. In fact, this isomorphism is simply given by conjugation with g, since we have $C(gUg^{-1}) \cong gC(U)g^{-1}$.

The fact that C is a sheaf, follows from its relation to $\mathcal{H}$ that we describe below.

As we saw in Proposition **??** the isotropy group of $\mathbf{Set}^{G^{op}}$ is the G-set $(G, conj)$ where $conj : G \times G \to G$ is defined to be the conjugation action.

Recall from the equivalence between $\mathbf{Cont}(G)$ and $\mathbf{Sh}(\mathbb{T}_G, At)$ that each continuous G-set, X, determines a sheaf on $\mathbb{T}_G$. We will denote this as $X(U) = \{x \in X \mid U \subseteq G_x\}$ for each open subgroup U of G. We can apply this to the particular continuous G-set $\rho((G, conj))$ as a sheaf. Then we have

$$\begin{aligned}
\rho((G, conj))(U) &= \{h \in G \mid U \subseteq G_h\} \\
&= \{h \in G \mid U \subseteq C(h)\} \\
&= \{h \in G \mid h \in C(U)\} \\
&= C(U).
\end{aligned}$$

Next we describe the relationship between $\mathcal{H}$ and C.

Theorem 5.6.4. *Let G be a nearly discrete topological group. Then $\mathcal{H} \cong C$.*

Proof: Let U be an open subgroup of G. Define the map $\psi_U : C(U) \to \mathcal{Z}(U)$ by $\psi_U(g) = \langle Vg \rangle$. Since each coset representative is equal, we have $Img\ \phi_U \subseteq \mathcal{H}(U)$. If $\phi_U(g) = \phi_U(h)$, then $\langle Vg \rangle = \langle Vh \rangle$. By Lemma **??** we have $g = h$ and therefore ϕ_U is injective. For a uniform element $\langle Vg \rangle \in \mathcal{H}(U)$ notice that we have

$$\begin{aligned}
\langle Vg \rangle &= \mathcal{H}(U \xrightarrow{Ue} U)(\langle Vg \rangle) \\
&= \mathcal{H}(U \xrightarrow{Uu} U)(\langle Vg \rangle) \\
&= \langle Vu^{-1}gu \rangle.
\end{aligned}$$

Again by Lemma **??** we have $g = u^{-1}gu$ and therefore $g \in C(U)$, giving us surjectivity of ϕ_U. This is certainly an isomorphism since

$$\phi_U(gh) = \langle Vgh \rangle = \langle Vg \rangle \langle Vg \rangle = \phi_U(g)\phi_U(h).$$

Suppose we have an arrow $[V, e, U] : V \to U$ in $\mathbb{T}_G$. Then $\mathcal{H}([V, e, U])(\langle Wg \rangle) = \langle Wg \rangle$ where W now varies over all open subgroups of V. For the commutator subgroup we have $C([V, e, U])(g) = g$ since $C(U) \subseteq C(V)$. Thus we have the following commutative diagram.

$$\begin{tikzcd}
C(U) \arrow[r, "\psi_U"] \arrow[d, "C([U,e,V])"'] & \mathcal{H}(U) \arrow[d, "\mathcal{H}([U,e,V])"] \\
C(V) \arrow[r, "\psi_V"] & \mathcal{H}(V).
\end{tikzcd}$$

Therefore ψ is a natural isomorphism between C and $\mathcal{H}$. ∎

As a consequence we have the following.

Corollary 5.6.5. *For G a nearly discrete topological group*

$$\rho((G, conj)) \cong \mathcal{H}$$

Chapter 6

Automorphisms of $\mathbb{N}$

6.1 The Isotropy group of **Cont**($Aut(\mathbb{N})$)

As an example we will compute the isotropy group for **Cont**(G) when G is the group of permutations of the natural numbers. This topos is of particular interest since it has been shown that every Grothendieck topos arises as an exponential variety in the category of sheaves on a locale internal to **Cont**($Aut(\mathbb{N})$) [**?**]. First we look at a slightly more general case than $\mathbb{N}$.

Let X be any infinite set. We can form the group $(Aut(X), \circ)$ of automorphisms (self isomorphisms) of X under composition. Let $P_<(X)$ be the set of subsets of X which have cardinality strictly less than that of X. We can define a filter base for $Aut(X)$ by $\mathcal{U} = \{U_K | K \in P_<(X)\}$where U_K is the point-wise stabilizer,

$$U_K = \{\alpha \in Aut(X) \mid \alpha(x) = x,\ \forall x \in K\}. \tag{6.1.1}$$

Proposition 6.1.1. *The collection $\mathcal{U}$ has the following properties:*

1. *$\mathcal{U}$ is closed under finite intersection*
2. *$\mathcal{U}$ doesn't contain the empty set*

Proof: Consider two sets $K, L \in P_<(X)$. If $\alpha \in U_K \cap U_L$ then for each $x \in K$ we have $\alpha(x) = x$ and for each $x \in L$, $\alpha(x) = x$. Hence $x \in K \cup L$ then $\alpha(x) = x$, thus $\alpha \in U_{K\cup L}$.

By properties of cardinal arithmetic $|K \cup L| < |X|$. So $K \cup L \in P_<(X)$, and consequently $U_{K\cup L} \in \mathcal{U}$.

Now suppose that $\emptyset \in \mathcal{U}$, then we have that for some $K \in P_<(X)$, $U_K = \emptyset$. In particular, this means that there are no automorphisms of X which fix all elements of K. Clearly, this fails since for any subset $K \subseteq X$ we have that the identity fixes all elements of X, hence all elements of K. Thus the empty set $\emptyset$ is not an element of $\mathcal{U}$. ∎

Proposition **??** tells us that $\mathcal{U}$ is a filter base. As we saw in Chapter 1, filter bases generate topologies.

Corollary 6.1.2. *The collection $\mathcal{U}$ defines a topology on $Aut(X)$.*

Proof: Firstly, Proposition **??** gives us that $\mathcal{U}$ satisfies the requirements in Proposition **??**, hence we can define a filter F from $\mathcal{U}$, and therefore by Proposition **??**, a topology on $Aut(X)$. ∎

Recall that the topology is given by

$$\tau_{\mathcal{U}} = \{V \subseteq Aut(X) \mid \exists K \in P_<(X),\ \beta \in Aut(X),\ \beta U_K \subseteq V\}. \tag{6.1.2}$$

One thing to note is that this filter base contains all open subgroups.

Proposition 6.1.3. *For each $K \in P_<(X)$ we have U_K is a subgroup of $Aut(X)$.*

Proof: We noted in the proof of Proposition **??** that every U_K contains the identity of $Aut(X)$. If $\alpha, \beta \in U_K$ then $\alpha(x) = x$ for each $x \in K$ by definition. Applying α^{-1} gives us $\alpha^{-1}(x) = x$ for each $x \in K$. Additionally, $\alpha\beta(x) = \alpha(x) = x = \beta(x) = \beta\alpha(x)$ for each $x \in K$, hence we have that the composition $\alpha\beta$ is also in U_K. Since K was an arbitrary element of $P_<(X)$ we have our desired result. ∎

Proposition 6.1.4. *The 3-tuple $(Aut(X), \tau_{\mathcal{U}}, \circ)$ defines a topological group.*

Proof: By Proposition **??** and **??** we only have to show that the map $f(\alpha, \beta) = \alpha\beta^{-1}$ is continuous at the identity. If $U \subseteq V$ then we have $U^{-1} \subseteq V^{-1}$ hence if V is open about the identity, there exists a $U_K \subseteq V$. By Proposition **??** we have that $U_K^{-1} = U_K$ hence $U_K \subseteq V^{-1}$. Note also that the function $(-)^{-1}$ is its own inverse. So it follows from this observation that taking inverses is continuous.

Next, suppose V is open and that there exists $ab \in V$. By definition of the topology there exists $\alpha U_K \subseteq V$ such that $ab \in \alpha U_K$. We can take $\alpha U_K = abU_K$.

Define the set $L \subseteq X$ to be $b(K)$. Since b is an automorphism we have $|L| = |K|$, so $L \in P_<(X)$. So the set U_L is a basic open subgroup of $Aut(X)$. Consider the set $aU_L \times bU_K \subseteq Aut(X) \times Aut(X)$, this is a basic open set in the product topology on $Aut(X) \times Aut(X)$.

For any element (au, bu') of $aU_L \times bU_K$ we have for every $x \in K$, we have $bu'(x) = b(x)$ since $u' \in U_K$, and $u(b(x)) = b(x)$ since $u \in U_L$ and $L = \alpha(K)$. Thus $aubu'(x) = ab(x)$ for each $x \in K$. This gives us $aubu' \in abU_K$ and thus $aU_K \times bU_L \subseteq m^{-1}(abU_K) \subseteq m^{-1}(V)$. Hence $m^{-1}(V)$ is open, and $(Aut(X), \tau_{\mathcal{U}}, \circ)$ is a topological group. ∎

We can given an alternate characterization of the elements of the coset αU_K as follows.

Lemma 6.1.5. *For each $\alpha \in Aut(X)$ the set $\alpha U_K = \{\beta \in Aut(X) \mid \beta|_K = \alpha|_K\}$.*

Proof: If $\alpha u \in \alpha U_K$ then for every $x \in K$ we have $\alpha u(x) = \alpha(x)$ since u fixes every x.

Conversely, suppose $\alpha|_K = \beta|_K$. Define ν and γ as

$$\nu(x) = \begin{cases} \beta(x) & : x \in K \\ x & : x \notin K \end{cases} \tag{6.1.3}$$

$$\gamma(x) = \begin{cases} x & : x \in K \\ \beta(x) & : x \notin K. \end{cases} \tag{6.1.4}$$

Notice that we can decompose β as $\beta = \nu \circ \gamma$. In fact, since $\alpha|_K = \beta|_K$ we can rewrite ν as

$$\nu(x) = \begin{cases} \alpha(x) & : x \in K \\ x & : x \notin K. \end{cases} \tag{6.1.5}$$

Notice also, that γ fixes the elements of K, hence $\gamma \in U_K$. We can do a similar decomposition for α, say $\alpha = \nu \circ \lambda$ where $\lambda \in U_K$. Hence we have $\alpha U_K = \nu U_K = \beta U_K$ giving us $\beta \in \alpha U_K$. ∎

The normalizers of each basic open subgroup U_K can be described as all those elements which permute K.

Proposition 6.1.6. *Let U_K be a basic open subgroup, and $N(U_K)$ its normalizer. Then $N(U_K) = \{\alpha \in Aut(X) \mid \alpha(K) = K\}$.*

Proof: By definition of the normalizer, for each $\theta \in U_K$, we have $\alpha^{-1}\theta\alpha \in U_K$. Equivalently, for each $x \in K$ $\alpha^{-1}\theta\alpha(x) = x$. Applying α to both sides gives us $\theta\alpha(x) = \alpha(x)$. Since this must hold for every $\theta \in U_K$ it holds for any such θ with the property that $\theta(x) = x$ if and only if $x \in K$. Thus we must have $\alpha(x) \in K$, and therefore $\alpha(K) = K$. ∎

With the description given in Proposition **??** each normalizer of the basic open subgroups U_K contains all the functions that permute K, but fix everything else.

Definition 6.1.7. Let $K \in P_<(X)$, and α an automorphism of X. We say α is a K*-permutation* if for each x not in K, $\alpha(x) = x$.

These K-permutations are precisely the elements of the commutator subgroup $C(U_K)$ for each $K \in P_<(X)$.

Proposition 6.1.8. *An element $\alpha \in G$ is a K-permutation if and only if $\alpha \in C(U_K)$.*

Proof: Firstly, suppose α is a K-permutation and let $\phi \in U_K$. If $x \in K$ then $\alpha(x) \in K$ hence $\phi(\alpha(x)) = \alpha(x)$ and therefore $\alpha^{-1}\phi\alpha(x) = x = \phi(x)$. If $x \notin K$ then $\alpha(x) = x = \alpha^{-1}(x)$, and $\phi(x) \notin K$. Hence $\alpha^{-1}\phi\alpha(x) = \phi(x)$.

For the converse, consider an element $\phi \in U_K$ such that $\phi(x) = x$ if and only if $x \in K$. For any $\alpha \in C(U_K)$ we have, for each $x \in K$, $\phi\alpha(x) = \alpha(\phi(x)) = \alpha(x)$. By our assumption on ϕ this is equivalent to $\alpha(x) \in K$, hence $\alpha(K) = K$. Now we just want to show that for any $x \notin K$, $\alpha(x) = x$. Consider a subset $L \subseteq \mathbb{N}$ such that $K \subseteq L$. Then if $\phi \in U_L$ for each element $x \in K$ we have $\phi(x) = x$, hence $\phi \in U_K$ and $U_L \subseteq U_K$. Additionally, we have $C(U_K) \subseteq C(U_L)$. Take any element $\alpha \in C(U_K)$ then by the above we have $\alpha(K) = K$ and $\alpha(L) = L$ whenever $K \subseteq L$. Consider any element $l \notin K$ and let $L = K \cup \{l\}$. If $\alpha(l) \in K$ then since $\alpha^{-1}(K) = K$ also, we must have $l \in K$, a contradiction. Thus $\alpha(l) \notin K$ and since $\alpha(L) = L$ we must have $\alpha(l) = l$. Since our choice of l is arbitrary we can do this for any $x \notin K$. Thus for each $x \notin K$, $\alpha(x) = x$. Thus α is a K-permutation.

∎

From Lemma **??** and Proposition **??** we get the following consequence.

Corollary 6.1.9. *For every element of the normalizer $\alpha \in N(U_K)$, there exists a K-permutation, β such that $\alpha U_K = \beta U_K$.*

Proof: In the proof of Lemma **??** any map $\alpha \in N(U_K)$ can be decomposed as $\alpha = \nu \circ \gamma$, where ν restricts to α on K and the identity elsewhere, and γ restricts to the identity on K and α elsewhere. Since α is assumed to be in the normalizer of U_K, by Proposition **??** $\alpha(K) = K$, hence $\nu(K) = K$. By how ν is defined we have that $\nu(x) = x$ for each x not in K. Thus ν is a K-permutation, and the decomposition of α gives us that $\alpha \in \nu U_K$, so $\nu U_K = \alpha U_K$. ∎

Corollary 6.1.10. *For each basic open subgroup U_K we have $N(U_K)/U_K \cong Sym(K)$.*

Proof: Define the map $\phi_{U_K} : N(U_K)/U_K \to Sym(K)$ by $\phi_{U_K}(\alpha U_K) = \alpha|_K$. By Proposition **??** every coset representative restricts to the same permutation on K, hence this is well-defined. Moreover, by the same Proposition we get injectivity, since $\beta \in \alpha U_K$ if and only if $\beta|_K = \alpha|_K$. For any permutation $\delta \in Sym(K)$ we have the extension $\overline{\delta}$ defined as

$$\overline{\delta}(x) = \begin{cases} \delta(x) & : x \in K \\ x & : x \notin K \end{cases}$$

Hence $\phi_{U_K}(\overline{\delta} U_K) = \delta$. So ϕ_{U_K} is surjective. Lastly, the identity element of $N(U_K)/U_K$ corresponds to the identity symmetry of K and

$$\begin{aligned} \phi_{U_K}(\alpha U_K \beta U_K) &= \phi_{U_K}(\alpha\beta U_K) \\ &= \alpha\beta|_K \\ &= \alpha|_K \beta|_K \\ &= \phi_{U_K}(\alpha)\phi_{U_K}(\beta). \end{aligned}$$

The equality $\alpha\beta|_K = \alpha|_K\beta|_K$ holds because we have $\alpha(K) = K$ and $\beta(K) = K$. Therefore ϕ_{U_K} is an isomorphism of groups.

Finally, by Corollary **??** we can assume each representative is a K-permutation. By Proposition **??** each K-permutation is in the commutator subgroup $C(U_K)$ hence also contained in $C(U_L)$ for any $U_L \subseteq U_K$. This gives us a function $N(U_K)/U_K \to N(U_L)/U_L$ whenever $U_L \subseteq U_K$. This corresponds to extending a symmetry of K to a symmetry of L by the identity. This gives us the following commutative diagram.

$$\begin{array}{ccc} N(U_K)/U_K & \xrightarrow{\phi_{U_K}} & Sym(K) \\ \downarrow & & \downarrow \\ N(U_L)/U_L & \xrightarrow{\phi_{U_L}} & Sym(L) \end{array}$$

Thus ϕ_U is a natural isomorphism. ∎

Recall the definition of the sheaf of uniform elements $\mathcal{H}(U) = \{\langle Vh_V \rangle \mid Wh_V = Wh_W,\ \forall W \subseteq V\}$.

Theorem 6.1.11. *Let X be an infinite set and $G = Aut(X)$ a topological group with the topology generated by the point-wise stabilizers of sets $K \in P_<(X)$. Then $\mathcal{H} \cong Z$.*

Proof: It is sufficient to show that each isotropy element of $Z(U)$ is actually a uniform element. Suppose $\langle U_L \beta_L \rangle \in Z(U_K)$. This gives us the following commutative diagram.

$$\begin{array}{ccc} U_K & \xrightarrow{U_K\beta_K} & U_K \\ \uparrow & & \uparrow \\ U_L & \xrightarrow{U_L\beta_L} & U_L \end{array}$$

By Corollary **??** we know that $U_K\beta_K = U_K\alpha$ where α is a K-permutation.

Let $x \in L - K$, and define a new set $N = K \cup \{x\}$. We get the following commutative diagrams.

$$\begin{array}{ccc} U_K & \xrightarrow{U_K\beta_K} & U_K \\ \uparrow & & \uparrow \\ U_N & \xrightarrow{U_N\beta_N} & U_N \end{array} \qquad\qquad \begin{array}{ccc} U_N & \xrightarrow{U_N\beta_N} & U_N \\ \uparrow & & \uparrow \\ U_L & \xrightarrow{U_L\beta_L} & U_L \end{array}$$

From the diagram on the left, we have $\beta_N(K) = \alpha(K) = K$. Thus $\beta_N(x) = x$ otherwise $\beta_N(x) \in K$, but since $\beta_N(K) = K$ we must have $x \in K$, a contradiction. From the diagram on the right we get $\beta_L(N) = \beta_N(N) = N$. Thus $\beta_L(x) = x$ otherwise, $\beta_L(x) \in K$ and from the first commutative diagram, $\beta_L(K) = \alpha(K) = K$. So again we would have $x \in K$, a contradiction. We can do this for any $x \in L - K$. Hence β_L is a K-permutation with the property that $\beta_L(K) = \alpha(K)$. Thus we must have $\beta_L = \alpha$. ∎

As a direct consequence we have the following.

Corollary 6.1.12. *Let* $G = Aut(\mathbb{N})$ *with the topology generated by the point-wise stabilizers of finite sets. Then* $\mathcal{H} \cong Z$*.*

Proof: Substitute $\mathbb{N}$ for X in Theorem **??**. ∎

We point out that the above result could have been obtained in a more elementary manner, namely by investigating the isotropy group of another site for the topos $\mathbf{Cont}(Aut(\mathbb{N}))$. This site has a more logical flavour: it is the opposite of the category of finite sets and injective functions. By direct calculation, one may then find the isotropy group of this category. However, this of course bypasses most of the informative and more generally applicable group-theoretic machinery developed in this thesis.

Chapter 7

Conclusion

In this thesis we have provided a way of computing the isotropy group of $\mathbf{Cont}(G)$ for an arbitrary topological group G. Moreover, our presentation of elements of isotropy relies only on the topology of G. This was done through looking at the the comparison maps in the following diagram.

$$\begin{tikzcd} Z_{\mathbb{W}}(U) & Z_{\mathbb{T}_G}(U) \arrow[l, hook', "w"] \\ & \mathcal{H}(U) \arrow[u, hook, "h"] \end{tikzcd}$$

7.1 Further Work

Freyd's representation theorem provides us with a recipe for constructing a Grothendieck Topos. In this work we look at computing isotropy for a topos of the form $\mathbf{Cont}(G)$ for some topological group G. The case for a topos that is equivalent to sheaves on a locale was covered in the original paper [**?**]. So an obvious extension to this work would be to look at how isotropy could be computed for an exponential variety.

An interesting question is whether it is the case that for a general topological group every element of isotropy is uniform.

Freyd's represnetaiton theorem also mentioned that using the group of order-preserving automorphisms of $\mathbb{Q}$ would also be sufficient in the construction. So computing the isotropy group for $\mathbf{Cont}(G)$ when G is the order-preserving bijections of $\mathbb{Q}$ could also be illuminating.

Much of our construction used the fact that the groups being considered are assumed to be nearly discrete. Analysts use the terminology "Topologically Non-Archimedian" to describe the same property. One interesting result from the literature states that every nearly discrete topological group is isomorphic to a subgroup of $Homeo(X)$ for some stone space X[**?**]. While we didn't get a chance to go too deep into the literature, there may be some interesting connections to explore here.

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