Aspects of Isotropy in Small Categories

Sakif Khan

Thesis submitted to the Faculty of Graduate and Postdoctoral Studies in partial fulfillment of the requirements for the degree of Master of Science in Mathematics[1]

Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

[1]The M.Sc. program is a joint program with Carleton University, administered by the Ottawa-Carleton Institute of Mathematics and Statistics

Abstract

In the paper [FHS12], the authors announce the discovery of an invariant for Grothendieck toposes which they call the isotropy group of a topos. Roughly speaking, the isotropy group of a topos carries algebraic data in a way reminiscent of how the subobject classifier carries spatial data. Much as we like to compute invariants of spaces in algebraic topology, we would like to have tools to calculate invariants of toposes in category theory. More precisely, we wish to be in possession of theorems which tell us how to go about computing (higher) isotropy groups of various toposes. As it turns out, computation of isotropy groups in toposes can often be reduced to questions at the level of small categories and it is therefore interesting to try and see how isotropy behaves with respect to standard constructions on categories. We aim to provide a summary of progress made towards this goal, including results on various commutation properties of higher isotropy quotients with colimits and the way isotropy quotients interact with categories collaged together via certain nice kinds of profunctors. The latter should be thought of as an analogy for the Seifert-van Kampen theorem, which allows computation of fundamental groups of spaces in terms of fundamental groups of smaller subspaces.

Dedications

To my parents and to all those teachers to whom justice cannot be rendered in words.

Acknowledgement

The task of determining the perhaps unique combination of people and circumstances which results in the genesis of a document like this one is a non-trivial task for a mind such as the one writing this sentence. However, for any given enterprise, there are candidates who more obviously bear the onus of bringing to fruition the project at hand.

Foremost among these is my advisor Dr. Pieter Hofstra without whom this thesis would have neither started, ended nor been endlessly procrastinated upon. I am deeply grateful for his countless ideas and the generosity with which he shared both his thoughts and his time. I am equally indebted to him for his remarkable moral and intellectual support in the course of my mathematical maturation since I was a callow undergraduate. In particular, I am thankful for his encouragement in terms of letting me explore my own ideas and presenting my work at conferences.

Also important to my nascent scholarship are the insights imparted to me at various points by various professors at the mathematics department. Chief among them were the category theorists who were all too willing to give freely of their time and intellect and who were always ready to advise, encourage and recommend. Additionally, I would like to express my gratitude toward the members of my thesis committee for dedicating time to reading, critiquing and improving my thesis.

My most heartfelt thanks must of course go towards my wonderful family who have unfailingly supported me over the sometimes long years of being far from home. My parents have given me all that I needed and more while my uncle and grandmother generously shared their home with me during my early years abroad.

Lastly, I must mention the many friends and companions who provided far too many opportunities to eat, drink, make merry, commiserate and who have borne the brunt of my buckwheater bloviations on philosophy and politics. Disappointingly enough for some of them, I still adhere to elements of the cosmogony of Borges' Masked Dyer of Merv.

Contents

Preface

Among the many types of mathematical objects studied in contemporary mathematics, few are as fascinating or as intricate as Grothendieck toposes. A large part of what makes toposes compelling is the fact that a topos is a multi-faceted mathematical object possessing simultaneously spatial properties and algebraic properties. It is therefore natural to ask how we might be able to get our hands on the spatial and algebraic data contained in a given topos. More precisely, given a topos $\mathcal{E}$, is it possible to assign to $\mathcal{E}$ invariants which carry the relevant mathematical information? In the spatial case, the answer is well-known: take the subobject classifier of $\mathcal{E}$. For the algebraic case, on the other hand, the question was only given a clear, affirmative answer in the recent paper [FHS12], in which the authors announce the discovery of an invariant for Grothendieck toposes which they call the isotropy group of a topos. The obvious step succeeding the discovery of a new invariant for a class of mathematical objects is the development of some generally applicable technology which allows computation of the invariant in cases of interest. The aim of the current thesis is to provide some results in this direction. Lastly, we mention that while the proper context for the work carried out is topos theory, all the mathematics done here can be gone through without any mention of toposes whatsoever! Indeed, computation of isotropy groups in (presheaf) toposes can often be reduced to questions at the level of small categories and the notion of isotropy (for a presheaf topos) can be recast in a straightforward way at the level of small categories. This also explains why we seem to only try to see how isotropy behaves with respect to standard constructions on small categories. More precisely, we include results on various commutation properties of higher isotropy quotients with colimits and the way isotropy quotients interact with categories collaged together via nice sorts of profunctors. The latter should be thought of as an analogy for the Seifert-van Kampen theorem, which allows computation of fundamental groups of spaces in terms of fundamental groups of smaller subspaces.

Chapter 1

Introduction and Motivation

The present chapter begins with a brief recap of the category theoretic minimum required for the work carried out in this thesis. After that, we introduce the more specialized concepts involved in our work and provide motivation for the project we undertake via an informal conceptual description of the broader background against which the mathematical content of this document resides.

1.1 Basic Categorical Notions

In this introductory section, we briefly review the concepts from category theory used throughout this document. Here and elsewhere, the standard material is drawn from [Mac98] and the reader is invited to consult this source for more nuanced descriptions and examples of the ideas under discussion. Recall that a **category** $\mathbb{C}$ consists of a class $\mathrm{Ob}(\mathbb{C})$ of objects and a class $\mathrm{Mor}(\mathbb{C})$ of morphisms between these objects. Each morphism has unique source and target objects and we can compose morphisms that have matching source and target. Furthermore, this composition is associative and each object in the category has corresponding to it an identity morphism that composes trivially with any morphism having the object in question as either source or target. Given a category $\mathbb{C}$, we may immediately define a (possibly) new category $\mathbb{C}^{op}$ called the **opposite category** that has the same object class but its morphism class consists of arrows in $\mathbb{C}$ going backwards; that is, a morphism in $\mathbb{C}^{op}$ is obtained by interchanging source and target of an element of $\mathrm{Mor}(\mathbb{C})$. Given objects $C, D \in \mathrm{Ob}(\mathbb{C})$, we denote by $\mathrm{Hom}_{\mathbb{C}}(C, D)$ the **hom-set**, or the set of all morphisms, from C to D. A category is said to be **small** if both $\mathrm{Ob}(\mathbb{C})$ and $\mathrm{Mor}(\mathbb{C})$ are sets and not proper classes. It is said to be **locally small** if each hom-set is a set and not a proper class.

For categories $\mathbb{C}$ and $\mathbb{D}$, a **functor** $F : \mathbb{C} \to \mathbb{D}$ is specified by a pair of functions: an object function that maps objects of $\mathbb{C}$ to objects of $\mathbb{D}$ and an arrow function that behaves similarly on morphisms. Furthermore, the functor is required to respect identities and compositions of morphisms, i.e., for any $C \in \text{Ob}(\mathbb{C})$, $F(1_C) = 1_{F(C)}$ and given $f, g \in \text{Mor}(\mathbb{C})$ such that the composite $g \circ f$ is defined, $F(g \circ f) = F(g) \circ F(f)$. Functors are thus morphisms between categories. Of special interest to us will be functors of the form $F : \mathbb{C}^{op} \to \mathbf{Set}$, where **Set** denotes the locally small category whose objects are (small) sets and whose morphisms are functions. Such a functor is called a **presheaf** on the category $\mathbb{C}$.

Central also to the sequel are ideas of **congruence** on a small category and **quotient categories**. Given $\mathbb{C}$ a small category, let R be a function that associates to each pair $C, D \in \text{Ob}(\mathbb{C})$ a binary relation $R_{C,D}$ on the hom-set $\text{Hom}_{\mathbb{C}}(C, D)$. R is a congruence if each $R_{C,D}$ is actually an equivalence relation and R also satisfies the condition that if $f, f' : C \to D$ are such that $fR_{C,D}f'$, then for all $g : C' \to D$ and $h : D' \to D$ we obtain $(hfg)R_{C,D}(hf'g)$. We can then define the quotient category $\mathbb{C}/R$ to be just the category with $\text{Ob}(\mathbb{C}/R) = \text{Ob}(\mathbb{C})$ and taking the hom-set $\text{Hom}_{\mathbb{C}/R}(C, D)$ to be the set-theoretic quotient $\text{Hom}_{\mathbb{C}}(C, D)/R_{C,D}$. The category $\mathbb{C}/R$ then possesses the obvious universal property and has the typical characteristics expected of a quotient gadget (see [Mac98]).

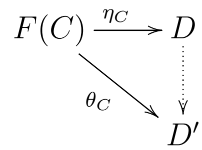
Next, we require the notions of **natural transformation**, **cocone** and **colimit** and we define these in that order. Just as functors are morphisms of categories, so are natural transformations morphisms of functors. Given functors $F, G : \mathbb{C} \to \mathbb{D}$, a natural transformation $\eta : F \Rightarrow G$ is specified by an assignment to each object $C \in \text{Ob}(\mathbb{C})$ an arrow $\eta_C : F(C) \to G(C)$ in $\mathbb{D}$ such that given any morphism $f : C \to C'$ in $\mathbb{C}$, the diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\eta_{C'}]{} & G(C') \end{array}$$

commutes. We say then that η is **natural** in C and that η_C is the **component** of η at C. Moreover, we may compose two natural transformations by simply defining composition to be composition on each component arrow. A cocone is an instance of a natural transformation in which the target functor G is a constant functor ΔD for some $D \in \text{Ob}(\mathbb{D})$ (the constant functor at D just maps every object in $\mathbb{C}$ to D and maps every morphism in $\mathbb{C}$ to 1_D). Unwrapping this, a cocone under F is given by arrows $\eta_C : F(C) \to D$ such that the triangles

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & D \\ {\scriptstyle F(f)}\downarrow & \nearrow_{\eta_{C'}} & \\ F(C') & & \end{array}$$

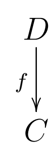
commute for all $f \in \text{Mor}(\mathbb{C})$. A colimit under F is, if it exists and denoted $\varinjlim F$, a universal such cocone. That is, a colimit is a cocone $\eta : F \to \Delta D$ such that given any other cocone $\theta : F \to \Delta D'$, there exists a unique arrow $D \to D'$ (really a natural transformation $\Delta D \Rightarrow \Delta D'$) for which we obtain



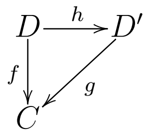
Colimits are unique up to (unique) isomorphism whenever they exist and the question of existence is subject to the category $\mathbb{D}$ at hand. Criteria for existence and further details regarding colimits in general may be found in [Mac98]. For our purposes, it will be enough to know that all small colimits (i.e., colimits under functors which have small domain categories) exist in both **Set** and **Cat** where the latter denotes the category consisting of all small categories and functors between them (in fact, if we add in natural transformations, **Cat** becomes an example, indeed the primordial example, of what is known as a 2-category but this would take us too far afield).

Now that we are acquainted with functors and natural transformations, we can introduce **functor categories**. We have already seen that the opposite category construction gives us a way of constructing new categories from old. Here is one other way to do so. Fix two categories $\mathbb{C}$ and $\mathbb{D}$. We may consider the collection of all functors going from $\mathbb{C}$ to $\mathbb{D}$ and, since natural transformations provide a suitable notion of morphism between functors, we can form the category $\mathbb{D}^{\mathbb{C}}$ obtained by taking a functor from $\mathbb{C}$ to $\mathbb{D}$ to be an object and taking a natural transformation to be a morphism. This construction produces the functor category from $\mathbb{C}$ to $\mathbb{D}$. Especially important examples of functor categories are **presheaf categories** or categories of the form $\mathbf{Set}^{\mathbb{C}^{op}}$ for some small category $\mathbb{C}$. Presheaf categories belong to an important class of categories called **toposes** that are now central to several areas of modern mathematics and as a matter of fact, the article [FHS12] that inspired the current project arose from the study of topos theory. The book [MM94] provides a relatively accessible introduction to this particular branch of category theory.

In closing this section, we discuss an elementary construction important to both topos theory and category theory in general. Take any category $\mathbb{C}$ and fix $C \in \mathrm{Ob}(\mathbb{C})$. Define the **slice category** or **under-category**, denoted $\mathbb{C}/C$, of $\mathbb{C}$ by C as follows: objects are arrows



in the original category $\mathbb{C}$ and given two such objects $f : D \to C$, $g : D' \to C$ a morphism is an arrow $h : D \to D'$ in $\mathbb{C}$ making the triangle



commute. Observe that a slice category comes equipped with a projection functor $P : \mathbb{C}/C \to \mathbb{C}$ that maps an object $f : D \to C$ to the domain object D and does nothing to the morphisms.

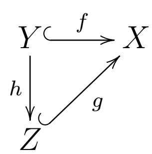
1.2 Some Motivational Remarks

The exiguousness of any topos theoretic material in the preceding introductory section is, apart from a symptom of the author's dearth of knowledge on the subject, indicative of the level of material present in the rest of this document. However, we feel that it will be useful to record here a few observations for the cognoscenti. The reader, novice or otherwise, is free to ignore these remarks if they so wish.

Take a small category $\mathcal{E}$ and consider some object $X \in \mathcal{E}$. Recall that by a **subobject** of X we mean an isomorphism class of monics into X. The set of all subobjects of X forms a poset under the ordering defined by

$$(f : Y \to X) \leq (g : Z \to X)$$

if and only if there exists $h : Y \to Z$ such that the diagram



commutes (note that we are abusing graphics here; each arrow is really a class of arrows). Thus, to each object of $\mathcal{E}$, we can associate a poset of subobjects. Moreover, this association is functorial,i.e., we obtain a functor $\mathrm{Sub}(-) : \mathcal{E} \to \mathbf{Pos}$ defined in the obvious way. Two questions naturally arise when we examine the subobject functor. First, can we extend $\mathrm{Sub}(-)$ to a presheaf on $\mathcal{E}$ and second, is the functor representable? The answer to the former is yes provided that $\mathcal{E}$ is equipped with pullbacks. It is an easy and useful fact that pullbacks of monics are monics and assigning pullbacks of subobjects provides a contravariant functor on $\mathcal{E}$. The answer to the latter question is also contingent on the category at hand but having a positive answer now means that the category possesses a much richer structure. Indeed, for a (locally small) category with finite limits, representability of the subobject functor is equivalent to the category having a subobject classifier. Therefore, toposes represent one class of categories for which there is a representable functor $\mathrm{Sub}(-) : \mathcal{E}^{op} \to \mathbf{Pos}$ (we are abusing notation yet again) defined using subobjects. Incidentally, a poset of subobjects for a topos is not just a poset but also a Heyting algebra.

At this point, it will be useful(?) to make some pseudo-sensical remarks about toposes in general. Toposes of the Grothendieck variety arose in connection with cohomology theories as studied by the French school of algebraic geometry in the middle of the 20^{th} century. Grothendieck toposes naturally possess both spatial and algebraic aspects and an aphorism communicated to the author goes as follows: "the concept of a topos is the pushout of the concepts of space and algebra". The following quote from [Joh02] may shed further light on how a topos

> "...resembles a space in that it possesses 'localisations' and 'coverings', but which has an additional feature, not present in topology, that its 'points' have nontrivial automorphism groups which form part of the structure of the 'space'."

The subobject classifier of a topos captures information about the spatial or "localic" aspects of the topos and we see thereby that representability of the subobject functor is deeply intertwined with gauging how "spatial" a topos is. Furthermore, the representing object in this case is a locale (an object which lives in the category $\mathbf{Frm}^{op}$) internal to the topos. It now behoves us to ask whether we can form a similar narrative for the algebraic aspects of a topos. Just as presheaves into **Pos** carry spatial information, presheaves into **Grp** carry algebraic information and fashioning an algebraic narrative amounts to producing a canonical solution to the following problem:

> Find a representable presheaf $\mathcal{Z} : \mathcal{E}^{op} \to \mathbf{Grp}$ such that the representing object of $\mathcal{Z}$ is a group internal to $\mathcal{E}$.

[FHS12] produces exactly an answer to this quandary and the algebraic analogue of the subobject classifier is what the authors call the **isotropy group** of a topos.

All the comments so far are meant to provide a rough sketch of the broader context within which the current project was undertaken. The present document purports only to make a small exploration into the algebraic aspect of toposes by examining a particular kind of interesting phenomenon that arises once we start thinking about isotropy in certain concrete instances. As mentioned at the beginning of this section, the motivation provided is directed mostly at those with interest in and some knowledge of topos theory. However, it is also possible to provide motivation at a more elementary level and it is hoped that the reader lacking any familiarity with topos theory will find adequate incentive in the following section.

1.3 Isotropy and Isotropy Quotients

Consider a small category G in which every morphism is invertible. Such a category is called a **groupoid** and it has the following interesting property: suppose we take $A, B\,\mathrm{Ob}(G)$, an (auto)morphism $\alpha : A \to A$ and a morphism $f : B \to A$. Then we can always solve the following 'lifting" problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ B & \dashrightarrow & B \end{array}$$

That is, we can find an automorphism $\beta : B \to B$ that makes the diagram commute and this is achieved just by conjugation: define $\beta = f^{-1}\alpha f$. Moreover, this lifting is natural in the sense that if we stack diagrams

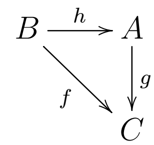
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ B & \xrightarrow{\beta} & B \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ C & \xrightarrow{\gamma} & C \end{array}$$

where β is the lift of α along f and γ is the lift of β along g, then γ is the lift of α along $f \circ g$. This can be verified directly by just using the equational expression for β and γ. Now, the lifts as described exist in any small groupoid. It is clear that the same property is not enjoyed by all small categories however. In what follows, we will describe how we can begin searching for categories that *do* come equipped with a nice system of lifts and much of our project is dedicated to producing explicit examples of such categories. Part of the point we wish to get across through our examples is that the categories which possess lifts are not in any way pathological or required to satisfy some stringent conditions (such as invertibility of all morphisms).

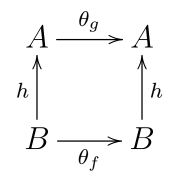
In the paper [FHS12], the authors present their discovery of the fact that every Grothendieck topos contains within it a group object that arises in a canonical way. This canonical group object is called the **isotropy group** of the topos and the article produces, among many other things, an explicit description of what this group object is. However, in the present document, we will not go into the theory developed there and indeed, we will not even say what a Grothendieck topos or a group object is. The interested reader may readily find the required definitions in [MM94] and, for the sake of this reader, we will only go so far as to say that the isotropy group for the topos of presheaves $\mathbf{Set}^{\mathbb{C}^{op}}$ on a small category $\mathbb{C}$ is the functor $Z : \mathbb{C}^{op} \to \mathbf{Set}$ defined by

$$Z(C) = \{\text{automorphisms of } \mathbb{C}/C \to \mathbb{C}\}$$

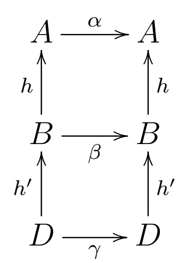
Here, we mean by $\mathbb{C}/C \to \mathbb{C}$ the projection functor described in the previous section and an automorphism of this functor refers to a natural transformation $\theta : P \Rightarrow P$ such that each component is an isomorphism. Explicitly, we obtain automorphisms (*inside* $\mathbb{C}$ this time) $\theta_g : A \to A$ where the index is some arrow $g : A \to C$. Naturality means that given any $f : B \to C$ and an $h : B \to A$ with



the square



is commutative. Let us fix $\theta_g := \alpha$ for a moment. Then we may think of θ producing for us "lifts" of α along any arrow with codomain D'. In fact, θ specifies a system of lifts for each of its automorphic components. Moreover, the lifts have to be assigned in a consistent manner, i.e., if we stack squares



with γ being the lift of β along h' and β being the lift of α along h, then γ needs to be the lift of α along the composite $h \circ h'$. This completes the description of an element of $Z(C)$ and since Z is the isotropy group of our preseheaf topos, we give a special name to the automorphisms in $\mathbb{C}$ that arise in this way:

Definition 1.3.1. For a small category $\mathbb{C}$, an **isotropy map** in $\mathbb{C}$ is an automorphism in $\mathbb{C}$ that is in the image of the obvious projection $Z(C) \to \mathrm{Hom}_{\mathbb{C}}(C, C)$. We denote the collection of all isotropy maps by $\mathcal{I}$.

It follows from a few remarks in [FHS12] that the objects in $\mathbb{C}$ together with the isotropy maps form a subcategory of $\mathbb{C}$. Our purpose is to observe what happens when we try to 'kill' isotropy in a category. To this end, we let $\mathbb{C}/\mathcal{I}$ be the category obtained when we quotient out by the least congruence on $\mathbb{C}$ that makes the identification $\alpha \sim 1$

whenever $\alpha \in \mathcal{I}$. We might expect at first that the category $\mathbb{C}/\mathcal{I}$ will have no isotropy whatsoever; however, as we shall see, almost exactly the opposite of this is true.

Before proceeding, it will be useful to state and prove some technical facts concerning the quotient category $\mathbb{C}/\mathcal{I}$ and the quotient map $\pi : \mathbb{C} \to \mathbb{C}/\mathcal{I}$.

Lemma 1.3.2. *Let $f, g : A \to B$ be parallel maps in $\mathbb{C}$. Then*

$$\pi(f) = \pi(g) \Leftrightarrow \exists \alpha \in \mathcal{I} \text{ such that } f = g\alpha$$

Proof: ($\Leftarrow$) If we have $\alpha \in \mathcal{I}$ such that $f = g\alpha$, then $\pi(f) = \pi(g\alpha) = \pi(g)\pi(\alpha) = \pi(g)1 = \pi(g)$.

($\Rightarrow$) Observe first that the congruence is generated by arrows $\alpha \in \mathcal{I}$ and the stated condition holds for these generating cases. $\mathcal{I}$ being a subcategory of $\mathbb{C}$ implies that the relation is an equivalence relation. Take now $f, g, f', g' \in \mathrm{Mor}(\mathbb{C})$ such that $\pi(f) = \pi(g)$ and $\pi(f') = \pi(g')$. We wish to prove that $\pi(f'f) = \pi(g'g)$. By the inductive hypothesis, we obtain that $\exists \alpha\alpha' \in \mathcal{I}$ such that $f = g\alpha$ and $f' = g'\alpha'$. Note that α lifts along g to, say, $\alpha'' \in \mathcal{I}$ and this produces the equation $f'f = g'\alpha' g\alpha$. As the congruence on $\mathbb{C}$ is generated by the arrows $\alpha \in \mathcal{I}$, we are done. ∎

Lemma 1.3.3. *The quotient map π is conservative.*

Proof: Let $f : A \to B$ in $\mathbb{C}$ be a map such that $\pi(f)$ is an isomorphism. As π is full, the inverse of $\pi(f)$ is of the form $\pi(g)$ for some $g : B \to A$. Now, $\pi(f)\pi(g) = 1 \Rightarrow \pi(fg) = 1 \Rightarrow fg = \beta$ for $\beta \in \mathcal{I}$ by Lemma 1.1. Similarly, $gf = \alpha$ for some $\alpha \in \mathcal{I}$. Note that α is an automorphism on A and β is an automorphism on B. So, we obtain $\alpha^{-1}gf = 1$ and also that $f\alpha^{-1}g = 1$. ∎

Lemma 1.3.4. *For $\alpha \in \mathbb{C}$, $\pi(\alpha) = 1 \Leftrightarrow \alpha \in \mathcal{I}$.*

Proof: ($\Rightarrow$) If $\pi(\alpha) = 1$, then $\pi(\alpha) = \pi(1)$ and lemma 1.1 implies that there exists $\beta \in \mathcal{I}$ such that $\alpha = 1 \circ \beta = \beta \in \mathcal{I}$.

($\Leftarrow$) This follows trivially from the definitions. ∎

Before moving on, we would like to point out that lemmas 1.3.2 and 1.3.4 lie at the heart of many results contained in this chapter and the ubiquity of the usage of these two statements is such that we will rarely mention it when invoking them.

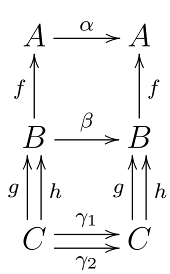
1.4 Higher Order Isotropy

We have already alluded to the fact that isotropy behaves in a rather counter-intuitive way, i.e., that an attempt to kill off isotropy maps in a category may not necessarily result in a category with no isotropy maps. We now demonstrate this claim by constructing an explicit example of a category, which we denote $\mathbb{X}[2]$ for reasons which will become apparent in the sequel, which will be a non-full subcategory of **Set**, such that the quotient $\mathbb{X}[2]/\mathcal{I}$ has non-trivial isotropy.

We define the category $\mathbb{X}[2]$ as follows:

- $\text{Ob}(\mathbb{X}[2]) = \{A, B, C\}$ where
  - $A = \{0, 1\}$
  - $B = \{a, b, c, d\}$
  - $C = \{p, q, r, s\}$
- $\text{Mor}(\mathbb{X}[2])$ shall consist of the following list of functions together with the identities and all possible composites:
  - $\alpha : A \to A$ the permutation (01)
  - $\beta : B \to B$ the permutation $(ac)(bd)$
  - $\gamma_1 : C \to C$ the permutation $(pr)(qs)$
  - $\gamma_2 : C \to C$ the permutation $(ps)(qr)$
  - $f : B \to A$ defined by $f(a) = f(b) = 0, f(c) = f(d) = 1$
  - $g : C \to B$ defined by $g(p) = a, g(q) = b, g(r) = c, g(s) = d$
  - $h : C \to B$ defined by $h(p) = b, h(q) = a, h(r) = c, h(s) = d$

It may be useful to have the following picture of $\mathbb{X}[2]$ in mind



We will explicate $\text{Mor}(\mathbb{X}[2])$ but we need to make a few observations before doing so:

- $fg = fh$

- $\alpha f = f\beta$ so that β is the unique lift of α along f
- $\beta g = g\gamma_1$ and $\beta g \neq g\gamma_2$ so that γ_1 is the unique lift of β along g
- $\beta h = h\gamma_2$ and $\beta h \neq h\gamma_1$ so that γ_2 is the unique lift of β along h

The last two points imply that α does not have a consistent lift along fg and therefore, it is not isotropy in $\mathbb{X}[2]$.

We now compute the arrows in $\mathbb{X}[2]$:

- $\mathrm{Hom}_{\mathbb{X}[2]}(A, B) = \mathrm{Hom}_{\mathbb{C}}(A, C) = \mathrm{Hom}_{\mathbb{C}}(B, C) = \emptyset$ by the very definition of $\mathbb{X}[2]$.
- $\mathrm{Hom}_{\mathbb{X}[2]}(A, A) = \{1, \alpha\}$: This is trivial.
- $\mathrm{Hom}_{\mathbb{X}[2]}(B, B) = \{1, \beta\}$: This is similarly obvious.
- $\mathrm{Hom}_{\mathbb{X}[2]}(C, C) = \{1, \gamma_1, \gamma_2, \gamma_1\gamma_2\}$: Note that $\gamma_1\gamma_2 = \gamma_2\gamma_1$, $\gamma_1^2 = 1$ and $\gamma_2^2 = 1$. We also have $\gamma_1\gamma_2\gamma_1 = \gamma_1 = \gamma_2\gamma_1\gamma_2$ and so, $\mathrm{Hom}_{\mathbb{X}[2]}(C, C)$ is just the Klein 4-group.
- $\mathrm{Hom}_{\mathbb{X}[2]}(B, A) = \{f, \alpha f\}$: A map from B to A is a composite of an arrow in $\mathrm{Hom}_{\mathbb{X}[2]}(B, B)$ followed by f followed by an arrow in $\mathrm{Hom}_{\mathbb{X}[2]}(A, A)$. The only possible maps are thus f, αf, βf and $\alpha f \beta$. The identity $\alpha f = f\beta$ implies that f and αf are the only distinct maps.
- $\mathrm{Hom}_{\mathbb{X}[2]}(C, A) = \{fg, \alpha fg\}$: We need to look at compositions which consist of arrows in $\mathrm{Hom}_{\mathbb{X}[2]}(C, C)$ followed by arrows in g or h followed by arrows in $\mathrm{Hom}_{\mathbb{X}[2]}(B, A)$:

$$
\begin{aligned}
fg1_c &= fg \\
fg\gamma_1 &= f\beta g = \alpha fg \\
fg\gamma_2 &= fh\gamma_2 = f\beta h = \alpha fh = \alpha fg \\
fg\gamma_1\gamma_2 &= \alpha fg\gamma_2 = \alpha^2 fg = fg \\
fh1_c &= fh = fg \\
fh\gamma_1 &= fg\gamma_1 = \alpha fg \\
fh\gamma_2 &= fg\gamma_2 = \alpha fg \\
fh\gamma_1\gamma_2 &= fg\gamma_1\gamma_2 = fg \\
\alpha fg1_c &= \alpha fg \\
\alpha fg\gamma_1 &= \alpha f\beta g = \alpha^2 fg = fg \\
\alpha fg\gamma_2 &= \alpha fh\gamma_2 = \alpha f\beta h = \alpha^2 fh = \alpha^2 fg = fg \\
\alpha fg\gamma_1\gamma_2 &= \alpha^2 fg\gamma_2 = \alpha^3 fg = \alpha fg \\
\alpha fh1_c &= \alpha fh = \alpha fg
\end{aligned}
$$

$$\alpha f h \gamma_1 = \alpha f g \gamma_1 = \alpha^2 f g = f g$$
$$\alpha f h \gamma_2 = \alpha f g \gamma_2 = \alpha^2 f g = f g$$
$$\alpha f h \gamma_1 \gamma_2 = \alpha f g \gamma_1 \gamma_2 = \alpha f g$$

This exhausts all the possibilities.

- $\mathrm{Hom}_{\mathbb{X}[2]}(C, B) = \{g, h, g\gamma_1, g\gamma_2, g\gamma_1\gamma_2, h\gamma_1, h\gamma_2, h\gamma_1\gamma_2\}$: A map from C to B consists of a map in $\mathrm{Hom}_{\mathbb{X}[2]}(C, C)$ followed by g or h followed by a map in $\mathrm{Hom}_{\mathbb{X}[2]}(B, B)$. We clearly get all the maps listed but we get no more since $\beta g = g\gamma_1$ and $\beta h = h\gamma_2$. That is, post-composing any of the listed maps with β does not produce any new arrows.

This completes our enumeration of $\mathrm{Mor}(\mathbb{X}[2])$. Identifying the isotropy maps in $\mathbb{X}[2]$ will now enable us to similarly describe $\mathrm{Mor}(\mathbb{X}[2]/\mathcal{I})$.

Firstly, note that all endomorphisms on C are automorphisms and indeed, $\mathrm{Mor}_{\mathbb{X}[2]}(C, C)$ is a group. So, all elements of $\mathrm{Mor}_{\mathbb{X}[2]}(C, C)$ belong to $\mathcal{I}$. Since β has unique lifts along g and h, β is also isotropy in $\mathbb{X}[2]$ and $\mathrm{Mor}_{\mathbb{X}[2]}(B, B) \subseteq \mathcal{I}$. Finally, as already indicated, α is *not* isotropy in $\mathbb{C}$. Hence,

$$\mathcal{I} = \{1_A\} \cup \mathrm{Mor}_{\mathbb{X}[2]}(B, B) \cup \mathrm{Mor}_{\mathbb{X}[2]}(C, C).$$

Using the functoriality of π, we can easily compute $\mathrm{Mor}(\mathbb{X}[2]/\mathcal{I})$. Note that $\mathrm{Ob}(\mathbb{X}[2]/\mathcal{I}) = \mathrm{Ob}(\mathbb{X}[2])$.

Here is the complete description of the quotient $\mathbb{X}[2]/\mathcal{I}$:

- $\mathrm{Ob}(\mathbb{X}[2]/\mathcal{I}) = \{A, B, C\}$ where
  - $A = \{0, 1\}$
  - $B = \{a, b, c, d\}$
  - $C = \{p, q, r, s\}$
- $\mathrm{Mor}(\mathbb{X}[2]/\mathcal{I})$ is given by the hom-sets:
  - $\mathrm{Hom}_{\mathbb{X}[2]/\mathcal{I}}(A, B) = \mathrm{Hom}_{\mathbb{C}/\mathcal{I}}(A, C) = \mathrm{Hom}_{\mathbb{C}/\mathcal{I}}(B, C) = \emptyset$
  - $\mathrm{Hom}_{\mathbb{X}[2]/\mathcal{I}}(A, A) = \{1, \pi(\alpha)\}$
  - $\mathrm{Hom}_{\mathbb{X}[2]/\mathcal{I}}(B, B) = \{1\}$
  - $\mathrm{Hom}_{\mathbb{X}[2]/\mathcal{I}}(C, C) = \{1\}$
  - $\mathrm{Hom}_{\mathbb{X}[2]/\mathcal{I}}(B, A) = \{\pi(f)\}$: Note that $\alpha f = f\beta$ and so, $\pi(\alpha f) = \pi(f\beta) = \pi(f)\pi(\beta) = \pi(f)$ as $\pi(\beta) = 1$.
  - $\mathrm{Hom}_{\mathbb{X}[2]/\mathcal{I}}(C, A) = \{\pi(f)\pi(g)\}$: Observe that $\pi(\alpha f g) = \pi(\alpha f)\pi(g) = \pi(f)\pi(g)$

– $\mathrm{Hom}_{\mathbb{X}[2]/\mathcal{I}}(C, B) = \{\pi(g), \pi(h)\}$

The main thing to observe here is that we have a (unique) non-identity map from A to itself. Since π is conservative and α an automorphism, $\pi(\alpha)$ is an automorphism in $\mathbb{X}[2]/\mathcal{I}$. Moreover, it is isotropy in $\mathbb{X}[2]/\mathcal{I}$ since it lifts in a trivial fashion along $\pi(f)$ and $\pi(fg)$. Hence, the quotient $\mathbb{X}[2]/\mathcal{I}$ has non-trivial isotropy. We say that $\mathbb{X}[2]$ has *second-order isotropy*. More generally, we have the following

Definition 1.4.1. For a small category $\mathbb{C}$, we can keep taking isotropy quotients to get a sequence

$$\mathbb{C} \twoheadrightarrow \mathbb{C}/\mathcal{I} \twoheadrightarrow \mathbb{C}/\mathcal{I}^2 \twoheadrightarrow \cdots$$

which eventually stabilizes and where, for a limit ordinal μ, $\mathbb{C}/\mathcal{I}^\mu$ is the colimit $\varinjlim_{\alpha<\mu} \mathbb{C}/\mathcal{I}^\alpha$. We say that $\mathbb{C}$ **has λ^{th}-order isotropy** if the chain stabilizes at stage λ. (In particular, a category with non-trivial isotropy maps has 1^{st}-order isotropy and so on.)

Observe that the chain of categories in the preceding definition stabilizes for the simple reason that a quotient functor cannot create new arrows and the cardinality of the morphism class in a quotient category is at most the cardinality of the morphism class of the original category. Thus, the cardinality of morphism classes in each category further down the sequence is either smaller, in which case the congruence is non-trivial, or the same as that of a predecessor, in which case the congruence is trivial and the chain has stabilized. Of course, we must ask whether there actually exist categories with λ^{th}-order isotropy for any ordinal λ. This naturally leads to the following questions:

- How can we actually tell when a (presheaf topos on a) category possesses higher-order isotropy?
- Turning this question around, how might we build a category with a desired **isotropy rank** of some (ordinal) order?
- What can we say about isotropy groups of categories assuming knowledge of isotropy ranks?
- Answering the first question answers: how do we compute isotropy groups of categories?
- Answering the second question answers: how do we build categories with desired isotropy groups? – much the same way Eilenberg-MacLane spaces or Moore spaces are constructed to have certain homotopy/homology groups.

In trying to answer these questions, it will be helpful to tighten our focus on one question: for a given ordinal λ, how can we build a category which has λ^{th}-order isotropy? The rough answer is: for finite-order isotropy, repeatedly take the collage of certain simple profunctors; these fit together into a "nice" sequential diagram, the colimit of which gives ω^{th}-order isotropy; repeat for higher successor and limit ordinals. This motivates the need to develop some technology for manipulating/building (isotropy) automorphisms in small categories and this is precisely what we do in the sequel. We remark that the technical results contained in the present chapter are due to P. Hofstra and while some of the initial lemmas in chapter 2 may have appeared elsewhere in different form, the main results are novel. Additionally, everything in chapter 3 past the expository material on profunctors is new as is the entirety of chapter 4. Finally, chapter 5 is meant as a space to speculate on further developments arising from the results contained in this thesis.

Chapter 2

Sequential Colimits and Isotropy

As mentioned at the end of the introductory chapter, we require some techniques to work with and eventually compute isotropy groups of presheaf toposes generated by small categories. The purpose of the current chapter is to explore isotropy groups of presheaf toposes associated to categories which arise as colimits of sequences of other categories. Along the way, we shall introduce the crucial concepts of higher-order isotropy and isotropy ranks for categories and we will then explore their preservation properties by certain classes of functors. It goes without saying that we shall heavily employ the technology of colimits under sequential diagrams in **Cat** and we refer the reader to [Awo10] for a refresher on the construction and behaviour of such colimits.

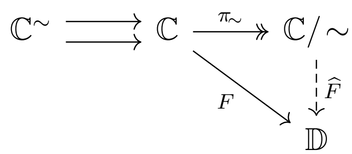
2.1 Commutation of Isotropy Quotients and Colimits

We now move on to explicating a few results stemming from an attempt to answer the following question: given a functor $F : J \to \mathbf{Cat}$, under what conditions on F does taking the colimit under F commute with taking the isotropy quotient of F? The reason we are interested in this question is that, as stated at the end of the last chapter, constructing categories with higher-order isotropy requires that we understand how to control isotropy (ranks) in certain kinds of colimits. One way to control isotropy in a colimit is to make sure that the isotropy in the colimit depends in a straightforward way on the isotropy of the categories under which we take the colimit.

We start with some small lemmas. We mention as well that the notation for congruences and congruence categories is borrowed from [Awo10] (in particular, we use $\mathbb{C}^{\sim}$ to indicate the congruence category, which has the same objects as $\mathbb{C}$ but whose morphisms are pairs of congruent maps).

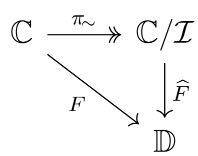
Lemma 2.1.1. *Let* $\mathbb{C}$ *be a category which is endowed with a congruence* $\sim$ *and suppose* $F : \mathbb{C} \to \mathbb{D}$ *is a functor. Then the following are equivalent:*

(i) There exists a faithful functor $\widehat{F} : \mathbb{C}/\sim \to \mathbb{D}$ *as in the diagram*



(ii) The functor $F : \mathbb{C} \to \mathbb{D}$ *satisfies the following condition: given a pair of morphisms* $f, g \in \mathbb{C}$, $F(f) = F(g) \Leftrightarrow f \sim g$.

Proof: $(i) \Rightarrow (ii)$: We have the commutative triangle



and thus,

$$\begin{aligned} F(f) = F(g) &\Leftrightarrow \widehat{F} \circ \pi_{\sim}(f) = \widehat{F} \circ \pi_{\sim}(g) \\ &\Leftrightarrow \pi_{\sim}(f) = \pi_{\sim}(g) \\ &\Leftrightarrow f \sim g \end{aligned}$$

where we have used the faithfulness of $\widehat{F}$ for the second equivalence.

$(ii) \Rightarrow (i)$: Since $f \sim g \Rightarrow F(f) = F(g)$, the universal property of $\mathbb{C}/\sim$ implies existence of $\widehat{F}$. Now take a parallel pair of morphisms $s, t \in \mathbb{C}/\sim$; as $\pi_{\sim}$ is full and bijective on objects, there exists a parallel pair of morphisms $f, g \in \mathbb{C}$ such that $\pi_{\sim}(f) = s$ and $\pi_{\sim}(g) = t$. Observe that

$$\begin{aligned} \widehat{F}(s) = \widehat{F}(t) &\Leftrightarrow \widehat{F}(\pi_{\sim}(f)) = \widehat{F}(\pi_{\sim}(g)) \\ &\Leftrightarrow F(f) = F(g) \\ &\Leftrightarrow f \sim g \\ &\Leftrightarrow \pi_{\sim}(f) = \pi_{\sim}(g) \\ &\Leftrightarrow s = t \end{aligned}$$

and $\widehat{F}$ is therefore faithful. ∎

Lemma 2.1.2. *For any functor $F : \mathbb{C} \to \mathbb{D}$, there is at most one functor $F/\mathcal{I} : \mathbb{C}/\mathcal{I} \to \mathbb{D}/\mathcal{I}$ making the square*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{D} \\ \downarrow & & \downarrow \\ \mathbb{C}/\mathcal{I} & \xrightarrow[F/\mathcal{I}]{} & \mathbb{D}/\mathcal{I} \end{array}$$

commute, where the vertical arrows are isotropy quotient functors.

Proof: Suppose we had functors $G, G' : \mathbb{C}/\mathcal{I} \to \mathbb{D}/\mathcal{I}$ making the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{D} \\ \downarrow & & \downarrow \\ \mathbb{C}/\mathcal{I} & \overset{G}{\underset{G'}{\rightrightarrows}} & \mathbb{D}/\mathcal{I} \end{array}$$

commute. If $f \in \mathbb{C}$ is a morphism, then chasing f around either square, we get

$$\begin{array}{ccc} f & \longmapsto & F(f) \\ \downarrow & & \downarrow \\ [f] & \longmapsto & [F(f)] \end{array}$$

and we conclude that $G(f) = [F(f)] = G'(f)$. ∎

As well, we establish some terminology which will become useful shortly.

Definition 2.1.3. If $F : \mathbb{C} \to \mathbb{D}$ is a functor, we say that F is an **embedding** if it is full and injective on morphisms. Also, F **preserves isotropy** in case, for an arrow $f \in \mathbb{C}$, $F(f)$ gets trivialised in $\mathbb{D}/\mathcal{I}$ whenever f gets trivialised in $\mathbb{C}/\mathcal{I}$. Lastly, F **reflects isotropy** if the converse holds, i.e., $F(f)$ being trivial in $\mathbb{D}/\mathcal{I}$ implies f is trivial in $\mathbb{C}/\mathcal{I}$.

We now prove a few technical results which will be required further on.

Lemma 2.1.4. *Embeddings reflect isotropy. More precisely, let $H : \mathbb{C} \to \mathbb{D}$ be a functor which is full and injective on morphisms. If $t : C \to C$ is an automorphism in $\mathbb{C}$ such that $Ht \sim 1$ in $\mathbb{D}$, then $t \sim 1$ in $\mathbb{C}$.*

Proof: Suppose we are given $t : C \to C$ such that $Ht \sim 1$ and that $m : C' \to C$ is a morphism in $\mathbb{C}$. We want to be able to lift t along this m. Consider $H(m)$:

$H(C') \to H(C)$; by hypothesis, $H(t)$ lifts along $H(m)$ as in the diagram

$$\begin{array}{ccc} H(C) & \xrightarrow{H(t)} & H(C) \\ {\scriptstyle H(m)}\uparrow & & \uparrow{\scriptstyle H(m)} \\ H(C') & \xrightarrow[s']{} & H(C') \end{array}$$

Since H is full, $s' = H(s)$ for some $s : C' \to C'$ in $\mathcal{C}$ and as H is injective on morphisms, the diagram

$$\begin{array}{ccc} H(C) & \xrightarrow{H(t)} & H(C) \\ {\scriptstyle H(m)}\uparrow & & \uparrow{\scriptstyle H(m)} \\ H(C') & \xrightarrow[H(s)]{} & H(C') \end{array}$$

can be reflected back into $\mathbb{C}$ as

$$\begin{array}{ccc} C & \xrightarrow{t} & C \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle m} \\ C' & \xrightarrow[s]{} & C' \end{array}$$

so that f lifts along m to s. We now check consistency of lifts. Let $n : C'' \to C'$ be a morphism in $\mathbb{C}$ and assume that f lifts along $m \circ n$ to $r : C'' \to C''$

$$\begin{array}{ccc} C & \xrightarrow{t} & C \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle m} \\ C' & & C' \\ {\scriptstyle n}\uparrow & & \uparrow{\scriptstyle n} \\ C'' & \xrightarrow[r]{} & C'' \end{array}$$

Take again the image

$$\begin{array}{ccc} H(C) & \xrightarrow{H(t)} & H(C) \\ {\scriptstyle H(m)}\uparrow & & \uparrow{\scriptstyle H(m)} \\ H(C') & & H(C') \\ {\scriptstyle H(n)}\uparrow & & \uparrow{\scriptstyle H(n)} \\ H(C'') & \xrightarrow[H(r)]{} & C'' \end{array}$$

Since $H(t)$ lifts along $H(m)$ to $H(s)$, $H(t)$ lifts along $H(m \circ n) = H(m) \circ H(n)$ to $H(r)$ and $H(t)$ lifts consistently, $H(s)$ must lift along $H(n)$ to $H(r)$

$$\begin{array}{ccc} H(C') & \xrightarrow{H(s)} & H(C') \\ {\scriptstyle H(n)}\uparrow & & \uparrow{\scriptstyle H(n)} \\ H(C'') & \xrightarrow[H(r)]{} & C'' \end{array}$$

Reflecting this last diagram into $\mathbb{C}$, we get that s lifts along n to r. ∎

Lemma 2.1.5. *Let $F : \mathbb{C} \to \mathbb{D}$ be an isotropy-preserving functor which is also an embedding. The induced functor $F/\mathcal{I} : \mathbb{C}/\mathcal{I} \to \mathbb{D}/\mathcal{I}$ is injective on morphisms.*

Proof: Note that the quotient maps $\pi_{\mathbb{C}}$ and $\pi_{\mathbb{D}}$ are identities on objects. Take any $f, g \in \mathrm{Mor}(\mathbb{C})$; then

$$\begin{aligned} F/\mathcal{I}([f]) = F/\mathcal{I}([g]) &\Leftrightarrow F/\mathcal{I} \circ \pi_{\mathbb{C}}(f) = F/\mathcal{I} \circ \pi_{\mathbb{C}}(f) \\ &\Leftrightarrow \pi_{\mathbb{D}} \circ F(f) = \pi_{\mathbb{D}} \circ F(g) \\ &\Leftrightarrow [Ff] = [Fg] \\ &\Leftrightarrow \exists \psi \text{ such that } \psi \sim 1 \text{ and } Ff \circ \psi = Fg. \end{aligned}$$

As F is an embedding and the (co)domain of ψ, which must be the domain of Ff, is of the form FC for some $C \in \mathbb{C}$, $\psi = F(\varphi)$ for some φ in $\mathbb{C}$. Using lemma 2.1.4, we deduce that φ is an isotropy map in $\mathbb{C}$ and since F is fully faithful, $Ff \circ F\varphi = Fg$ implies that $f\varphi = g$. Hence, $[f] = [g]$. ∎

Now we can focus on colimits of categories. It will be helpful once more to employ a specialized argot.

Definition 2.1.6. A **sequential diagram (of categories)** is a functor of the form $\mathbb{C}[-] : \lambda \to \mathbf{Cat}$, where λ is the chain category of ordinals less than λ. Given ordinals $\alpha \leq \beta < \lambda$, the functor $\mathbb{C}_\alpha^\beta : \mathbb{C}[\alpha] \to \mathbb{C}[\beta]$ is called a **transition map (from α to β)** in case $\mathbb{C}_\alpha^\beta$ is an embedding. (Co)limits under sequential diagrams are called **sequential (co)limits** while a natural transformation $\eta : \mathbb{C}[-] \Rightarrow \mathbb{D}[-]$ between two sequential diagrams $\mathbb{C}[-], \mathbb{D}[-] : \lambda \to \mathbf{Cat}$ is said to be a **sequential transformation**.

We will be concerned with functors $\mathbb{C}[-] : \lambda \to \mathbf{Cat}$ such that given ordinals

$\alpha \leq \beta < \lambda$, the functor $\mathbb{C}_\alpha^\beta : \mathbb{C}[\alpha] \to \mathbb{C}[\beta]$ lifts as in the diagram

$$\begin{array}{ccc} \mathbb{C}[\alpha] & \xrightarrow{\mathbb{C}_\alpha^\beta} & \mathbb{C}[\beta] \\ {\scriptstyle \pi_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\beta} \\ \mathbb{C}[\alpha]/\mathcal{I} & \underset{\mathbb{C}_\alpha^\beta/\mathcal{I}}{\dashrightarrow} & \mathbb{C}[\beta]/\mathcal{I} \end{array}$$

where the vertical arrows are isotropy quotient functors.. We thus derive a second functor $\mathbb{C}[-]/\mathcal{I} : \lambda \to \mathbf{Cat}$ from $\mathbb{C}[-]$ and, writing $\mathbb{C}[\lambda]$ for the colimit $\varinjlim_{\alpha<\lambda} \mathbb{C}[\alpha]$ and $(\mathbb{C}[-]/\mathcal{I})(\lambda)$ for the colimit $\varinjlim_{\alpha<\lambda}(\mathbb{C}[\alpha]/\mathcal{I})$, we wish to obtain sufficient conditions for an equivalence

$$(\mathbb{C}[-]/\mathcal{I})(\lambda) \cong \mathbb{C}[\lambda]/\mathcal{I}.$$

One way we might go about finding a candidate for such an equivalence is as follows. First, denote by $\eta : \mathbb{C}[-] \Rightarrow \Delta\mathbb{C}[\lambda]$ the universal cocone under $\mathbb{C}[-]$ and by $\varepsilon : \mathbb{C}[-]/\mathcal{I} \Rightarrow \Delta((\mathbb{C}[-]/\mathcal{I})(\lambda))$ the universal cocone under $\mathbb{C}[-]/\mathcal{I}$. Next, notice that the canonical quotient functors $\pi_\alpha : \mathbb{C}[\alpha] \to \mathbb{C}[\alpha]/\mathcal{I}$ form the components of a natural transformation $\pi : \mathbb{C}[-] \Rightarrow \mathbb{C}[-]/\mathcal{I}$ (this follows from the very definition of the functor $\mathbb{C}[-]/\mathcal{I}$) and therefore, we have an induced functor $\rho : \mathbb{C}[\lambda] \to (\mathbb{C}[-]/\mathcal{I})(\lambda)$ such that the diagram

$$\begin{array}{ccc} \mathbb{C}[-] & \overset{\eta}{\Longrightarrow} & \Delta\mathbb{C}[\lambda] \\ {\scriptstyle \pi}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \Delta\rho} \\ \mathbb{C}[-]/\mathcal{I} & \underset{\varepsilon}{\Longrightarrow} & \Delta(\mathbb{C}[-]/\mathcal{I})(\lambda) \end{array}$$

is commutative. Now, the isotropy quotient of $\mathbb{C}[\lambda]$ can be described as a co-equaliser

$$\mathbb{C}[\lambda]^{\mathcal{I}} \rightrightarrows \mathbb{C}[\lambda] \xrightarrow{\pi_\lambda} \mathbb{C}[\lambda]/\mathcal{I}$$

where $\mathbb{C}[\lambda]^{\mathcal{I}}$ is the congruence category under the isotropy relation and the parallel arrows are the obvious projections as explained in [Awo10]. Given that we already have a functor $\rho : \mathbb{C}[\lambda] \to (\mathbb{C}[-]/\mathcal{I})(\lambda)$, our initial question is answered once we produce a solution to the following problem: under what hypotheses on ρ do we obtain a faithful surjective-on-morphisms functor $\widehat{\rho} : \mathbb{C}[\lambda]/\mathcal{I} \to (\mathbb{C}[-]/\mathcal{I})(\lambda)$ fitting into the diagram

$$\begin{array}{cccc} \mathbb{C}[\lambda]^{\sim} \rightrightarrows & \mathbb{C}[\lambda] & \xrightarrow{\pi_\lambda} & \mathbb{C}[\lambda]/\mathcal{I} \\ & & {\scriptstyle \rho}\searrow & \big\downarrow{\scriptstyle \widehat{\rho}} \\ & & & (\mathbb{C}[-]/\mathcal{I})(\lambda) \end{array}$$

We can apply the following bit of abstract nonsense in our situation.

Lemma 2.1.7. *Let $\mathbb{C}[-] : \lambda \to$ **Cat** be a sequential diagram of categories. Then the following are equivalent.*

(i) There exists a faithful surjective-on-morphisms functor $\widehat{\rho} : \mathbb{C}[\lambda]/\mathcal{I} \to (\mathbb{C}[-]/\mathcal{I})(\lambda)$ as in the diagram

$$\begin{array}{ccccc} \mathbb{C}[\lambda]^{\mathcal{I}} & \rightrightarrows & \mathbb{C}[\lambda] & \xrightarrow{\pi_\lambda} & \mathbb{C}[\lambda]/\mathcal{I} \\ & & & \searrow^{\rho} & \downarrow{\scriptstyle \widehat{\rho}} \\ & & & & (\mathbb{C}[-]/\mathcal{I})(\lambda) \end{array}$$

(ii) The functor $\rho : \mathbb{C}[\lambda] \to (\mathbb{C}[-]/\mathcal{I})(\lambda)$ satisfies the following condition: given a pair of morphisms $f, g \in \mathbb{C}[\lambda]$, $\rho(f) = \rho(g) \Leftrightarrow f \sim g$.

Proof: By lemma 2.1.1, we only need to show that the second hypothesis implies that $\widehat{\rho}$ is surjective on morphisms. Take any morphism $m \in (\mathbb{C}[-]/\mathcal{I})(\lambda)$; as $(\mathbb{C}[-]/\mathcal{I})(\lambda) = \varinjlim_{\alpha<\lambda}(\mathbb{C}[\alpha]/\mathcal{I})$, $m = \varepsilon_\alpha(T)$ for some $\alpha < \lambda$ and a morphism $T \in \mathbb{C}[\alpha]/\mathcal{I}$. Since π_α is full and bijective on objects, $\mathrm{dom}(T)$ and $\mathrm{cod}(T)$ are objects in $\mathbb{C}[\alpha]$ and there exists an arrow $t : \mathrm{dom}(T) \to \mathrm{cod}(T)$ in $\mathbb{C}[\alpha]$ satisfying $T = \pi_\alpha(t)$. Chasing it around the diagram

$$\begin{array}{ccc} \mathbb{C}[\alpha] & \xrightarrow{\eta_\alpha} & \mathbb{C}[\lambda] \\ {\scriptstyle \pi_\alpha}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathbb{C}[\alpha]/\mathcal{I} & \xrightarrow[\varepsilon_\alpha]{} & (\mathbb{C}[-]/\mathcal{I})(\lambda) \end{array}$$

yields

$$\begin{array}{ccc} t & \longmapsto & \eta_\alpha(t) \\ \downarrow & & \downarrow \\ T & \longmapsto & m \end{array}$$

Hence,

$$m = \rho\eta_\alpha(t) = \widehat{\rho}(\pi_\lambda\eta_\alpha(t))$$

and $\widehat{\rho}$ is surjective on morphisms. ∎

We have thus further reduced the problem to finding conditions on $\mathbb{C}[-]$ which imply statement (ii) in lemma 2.1.7. We will do this in two steps. First, we will produce hypotheses on the universal cocones η and ε which are sufficient for (ii) and then we will explicate conditions which in turn imply these hypotheses.

Proposition 2.1.8. *Let $\mathbb{C}[-] : \lambda \to$ **Cat** be a sequential diagram of categories and, using the same notation as in the preceding, assume that the following conditions are satisfied.*

(i) For all $\alpha < \lambda$, η_α is an embedding.
(ii) For all $\alpha < \lambda$, η_α preserves and reflects isotropy.
(iii) For all $\alpha < \lambda$, ε_α is injective on morphisms.

Then given a pair of morphisms $s, t \in \mathbb{C}[\lambda]$, $\rho(s) = \rho(t) \Leftrightarrow s \sim t$.

Proof: Take s, t as in the statement of the lemma. There exist $\alpha, \beta < \lambda$ and $f : X \to Y \in \mathbb{C}[\alpha]$, $g : X' \to Y' \in \mathbb{C}[\beta]$ such that $s = \eta_\alpha(f)$ and $t = \eta_\beta(g)$. Without loss of generality, we may assume that $\alpha \leq \beta < \lambda$ and that we therefore have the commutative triangle

$$\begin{array}{ccc} \mathbb{C}[\alpha] & \xrightarrow{\mathbb{C}_\alpha^\beta} & \mathbb{C}[\beta] \\ & \searrow^{\eta_\alpha} & \downarrow{\eta_\beta} \\ & & \mathbb{C}[\lambda] \end{array}$$

where $\mathbb{C}_\alpha^\beta$ denotes the image under $\mathbb{C}$ of the unique arrow $\alpha \to \beta$ in λ. We can thus rewrite

$$s = \eta_\alpha(f) = \eta_\beta \mathbb{C}_\alpha^\beta(f)$$

and we deduce

$$\begin{aligned} s \sim t &\Leftrightarrow \eta_\alpha(f) \sim \eta_\beta(g) \\ &\Leftrightarrow \eta_\beta \mathbb{C}_\alpha^\beta(f) \sim \eta_\beta(g) \\ &\Leftrightarrow \eta_\beta \mathbb{C}_\alpha^\beta(f) = \eta_\beta(g) H \quad \text{for some } H \sim 1 \text{ in } \mathbb{C}[\lambda]. \end{aligned}$$

Now, H is by definition an automorphism and it must have the same domain as $\eta_\beta \mathbb{C}_\alpha^\beta(f)$ so that

$$\text{dom}(H) = \text{cod}(H) = \text{dom}(\eta_\beta \mathbb{C}_\alpha^\beta(f)) = \eta_\beta \mathbb{C}_\alpha^\beta(X) = \eta_\alpha(X).$$

By hypothesis (i), η_α is fully faithful and $H = \eta_\alpha(h)$ for some unique automorphism $h : X \to X$ in $\mathbb{C}[\alpha]$. Hypothesis (ii) implies that this h must also be isotropy in $\mathbb{C}[\alpha]$ and since $\mathbb{C}_\alpha^\beta$ maps isotropy elements to isotropy elements, $\mathbb{C}_\alpha^\beta(h)$ is isotropy in $\mathbb{C}[\beta]$. The above chain of equivalences can thus be continued as follows

$$\begin{aligned} &\eta_\beta \mathbb{C}_\alpha^\beta(f) = \eta_\beta(g) H \quad \text{for some } H \sim 1 \text{ in } \mathbb{C}[\lambda] \\ &\Leftrightarrow \eta_\beta(\mathbb{C}_\alpha^\beta f) = \eta_\beta(g) \eta_\beta(\mathbb{C}_\alpha^\beta h) \quad \text{for some } h \sim 1 \text{ in } \mathbb{C}[\beta] \\ &\Leftrightarrow \eta_\beta(\mathbb{C}_\alpha^\beta f) = \eta_\beta(g \circ \mathbb{C}_\alpha^\beta h) \quad \text{for some } h \sim 1 \text{ in } \mathbb{C}[\beta] \\ &\Leftrightarrow \mathbb{C}_\alpha^\beta f = g \circ \mathbb{C}_\alpha^\beta h \quad \text{using injectivity of } \eta_\beta \text{ on morphisms} \\ &\Leftrightarrow \mathbb{C}_\alpha^\beta f \sim g \quad \text{in } \mathbb{C}[\beta] \\ &\Leftrightarrow \pi_\beta(\mathbb{C}_\alpha^\beta f) = \pi_\beta(g) \end{aligned}$$

In the fourth line, we have used the fact that $\mathbb{C}_\alpha^\beta$ reflects and preserves isotropy if both η_α and η_β do. To see this, simply observe that

$$\mathbb{C}_\alpha^\beta h \sim 1 \Leftrightarrow \eta_\beta(\mathbb{C}_\alpha^\beta h) \sim 1 \Leftrightarrow \eta_\alpha(h) \sim 1 \Leftrightarrow h \sim 1.$$

Put succinctly, conditions (i) and (ii) together imply that

$$s \sim t \Leftrightarrow \pi_\beta(\mathbb{C}_\alpha^\beta f) = \pi_\beta(g).$$

Recall now the commutativity of the square

$$\begin{array}{ccc} \mathbb{C}[\beta] & \xrightarrow{\eta_\beta} & \mathbb{C}[\lambda] \\ {\scriptstyle \pi_\beta}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathbb{C}[\beta]/\mathcal{I} & \xrightarrow[\varepsilon_\beta]{} & (\mathbb{C}[-]/\mathcal{I})(\lambda) \end{array}$$

Combining this with hypothesis (iii), we get

$$\begin{aligned} \rho(s) = \rho(t) &\Leftrightarrow \rho\eta_\alpha(s) = \rho\eta_\beta(g) \\ &\Leftrightarrow \rho\eta_\beta(\mathbb{C}_\alpha^\beta f) = \rho\eta_\beta(g) \\ &\Leftrightarrow \varepsilon_\beta\pi_\beta(\mathbb{C}_\alpha^\beta f) = \varepsilon_\beta\pi_\beta(g) \\ &\Leftrightarrow \pi_\beta(F_\alpha^\beta f) = \pi_\beta(g). \end{aligned}$$

Hence,

$$s \sim t \Leftrightarrow \pi_\beta(\mathbb{C}_\alpha^\beta f) = \pi_\beta(g) \Leftrightarrow \rho(s) = \rho(t).$$

∎

Observe that conditions (i) and (iii) in proposition 2.1.8 are purely categorical in nature and we should expect only abstract-nonsensical hypotheses on $\mathbb{C}[-]$ to suffice. This is indeed the case.

Lemma 2.1.9. *Let $\mathbb{C}[-] : \lambda \to$ **Cat** be a sequential diagram of categories and suppose each of the functors $\mathbb{C}_\alpha^\beta : \mathbb{C}[\alpha] \to \mathbb{C}[\beta]$ is injective on morphisms. Then each of the colimiting functors $\eta_\alpha : \mathbb{C}(\alpha) \to \mathbb{C}[\lambda]$ is injective on morphisms as well. Furthermore, if each $\mathbb{C}_\alpha^\beta$ is full, then so is each η_α.*

Proof: Fix an $\alpha < \lambda$ and consider $\eta_\alpha : \mathbb{C}[\alpha] \to \mathbb{C}[\lambda]$. Suppose we have morphisms $f, g \in \mathbb{C}[\alpha]$ such that $\eta_\alpha(f) = \eta_\alpha(g)$. We claim that $f = g$. Assume, towards a contradiction, that $f \neq g$. Note that $\eta_\alpha(f) = \eta_\alpha(g)$ means that f and g get identified in the colimit and we deduce from this that there must exist an ordinal β satisfying $\alpha < \beta < \lambda$ such that $\mathbb{C}_\alpha^\beta(f) = \mathbb{C}_\alpha^\beta(g)$. Hence, $\mathbb{C}_\alpha^\beta$ cannot be injective on morphisms: contradiction. Therefore, η_α must be injective on morphisms.

Suppose moreover that each $\mathbb{C}_\alpha^\beta$ is full. Given $X, Y \in \mathbb{C}[\alpha]$, take any morphism $t : \eta_\alpha(X) \to \eta_\alpha(Y)$ in $\mathbb{C}[\lambda]$. There exists $\beta < \lambda$ such that $t = \eta_\beta(f)$ for some $f : X' \to Y'$ in $\mathbb{C}[\beta]$. If $\beta \leq \alpha$, then we can rewrite $t = \eta_\beta(f) = \eta_\alpha(\mathbb{C}_\beta^\alpha f)$ and $\mathbb{C}_\beta^\alpha f$ is then a pre-image in $\mathrm{Hom}_{\mathbb{C}[\alpha]}(X, Y)$ for $t \in \mathrm{Hom}_{\mathbb{C}[\lambda]}(\eta_\alpha X, \eta_\alpha Y)$. Assume then that $\alpha \leq \beta$; rewrite $\eta_\alpha(X) = \eta_\beta(\mathbb{C}_\alpha^\beta X)$ and similarly for Y. So, $t = \eta_\beta f$ is a morphism from $\eta_\beta(\mathbb{C}_\alpha^\beta X)$ to $\eta_\beta(\mathbb{C}_\alpha^\beta Y)$. As η_β is injective on morphisms, it is in particular injective on objects and

$$\begin{aligned} \mathrm{dom}(t) = \mathrm{dom}(t) &\Leftrightarrow \mathrm{dom}(\eta_\beta f) = \mathrm{dom}(\eta_\beta f) \\ &\Leftrightarrow \eta_\beta(X') = \eta_\beta(\mathbb{C}_\alpha^\beta X) \\ &\Leftrightarrow X' = \mathbb{C}_\alpha^\beta X. \end{aligned}$$

Analogously, $Y' = \mathbb{C}_\alpha^\beta Y$ and $f : \mathbb{C}_\alpha^\beta X \to \mathbb{C}_\alpha^\beta Y$ is an element of $\mathrm{Hom}_{\mathbb{C}[\beta]}(\mathbb{C}_\alpha^\beta X, \mathbb{C}_\alpha^\beta Y)$. Since $\mathbb{C}_\alpha^\beta$ is full, there exists an arrow $g : X \to Y$ in $\mathbb{C}[\alpha]$ such that $f = \mathbb{C}_\alpha^\beta(g)$. Hence,

$$t = \eta_\beta(f) = \eta_\beta(\mathbb{C}_\alpha^\beta g) = \eta_\alpha(g)$$

and g is a pre-image in $\mathrm{Hom}_{\mathbb{C}[\alpha]}(X, Y)$ for $t \in \mathrm{Hom}_{\mathbb{C}[\lambda]}(\eta_\alpha X, \eta_\alpha Y)$. We conclude that η_α is full. ∎

Remark 2.1.10. Although we shall not need it now, observe that the only property of $\mathbb{C}[-]$ we really needed in the first part was the existence of factorisations

$$\begin{array}{ccc} \mathbb{C}[\alpha] & \xrightarrow{\mathbb{C}_\alpha^\beta} & \mathbb{C}[\beta] \\ & \searrow^{\eta_\alpha} & \downarrow{\scriptstyle \eta_\beta} \\ & & \mathbb{C}[\lambda] \end{array}$$

Thus, the proof of lemma 2.1.9 goes through, *mutatis mutandis*, for any connected pre-order. For the second part of the lemma, we require that this pre-order also be a directed set containing upper bounds of sets with appropriate cardinality.

The last piece we need to solve our problem is to find sufficient conditions on $\mathbb{C}$ which imply condition (ii) of proposition 2.1.8. We can use lemma 2.1.4 to guide the search.

Lemma 2.1.11. *Let $\mathbb{C}[-] : \lambda \to \mathbf{Cat}$ be a sequential diagram of categories such that each $\mathbb{C}_\alpha^\beta : \mathbb{C}[\alpha] \to \mathbb{C}[\beta]$ is an embedding. Suppose that each $\mathbb{C}_\alpha^\beta$ also preserves isotropy, i.e., given any morphism $f \in \mathbb{C}[\alpha]$,*

$$f \sim 1 \Rightarrow \mathbb{C}_\alpha^\beta f \sim 1.$$

Then the components of the colimit $\eta_\alpha : \mathbb{C}[\alpha] \to \mathbb{C}[\lambda]$ inherit the property of preserving isotropy.

Proof: Fix an ordinal $\alpha < \lambda$ and let $f : X \to X$ be a morphism which is isotropy in $\mathbb{C}[\alpha]$. We wish to show that $\eta_\alpha f : \eta_\alpha(X) \to \eta_\alpha(X)$ is isotropy in $\mathbb{C}[\lambda]$. First, define β to be the ordinal

$$\beta = \sup\{\gamma < \lambda \mid \exists Y \in \mathrm{Ob}(\mathbb{C}[\gamma]), t \in \mathrm{Mor}(\mathbb{C}[\lambda]) \text{ such that } \mathrm{dom}(t) = \eta_\gamma(Y), \mathrm{cod}(t) = \eta_\alpha(X)\}.$$

Observe that β satisfies $\alpha \leq \beta < \lambda$ and that if $m' : Y' \to \eta_\alpha(X)$ is any morphism in $\mathbb{C}[\lambda]$, then we can re-write this as $\eta_\beta(m) : \eta_\beta(Y) \to \eta_\beta(\mathbb{C}^\beta_\alpha X)$ for some object $Y \in \mathbb{C}[\beta]$ and some morphism $m \in \mathbb{C}[\beta]$. Note that f being isotropy in $\mathbb{C}[\alpha]$ implies that $\mathbb{C}^\beta_\alpha f : \mathbb{C}^\beta_\alpha(X) \to \mathbb{C}^\beta_\alpha(X)$ is isotropy in $\mathbb{C}[\beta]$ since $\mathbb{C}^\beta_\alpha$ is assumed to preserve isotropy. This now provides a way to lift $\eta_\alpha f : \eta_\alpha(X) \to \eta_\alpha(X)$ along a given $m' : Y' \to \eta_\alpha(X)$ as follows. Re-write m' as already indicated and note that we have a lift

$$\begin{array}{ccc}
\mathbb{C}^\beta_\alpha(X) & \xrightarrow{\mathbb{C}^\beta_\alpha f} & \mathbb{C}^\beta_\alpha(X) \\
{\scriptstyle m}\uparrow & & \uparrow{\scriptstyle m} \\
Y & \xrightarrow[\mathbb{C}^\beta_\alpha f|_m]{} & Y
\end{array}$$

and this diagram faithfully and fully embeds into $\mathbb{C}[\lambda]$

$$\begin{array}{ccc}
\eta_\alpha(X) & \xrightarrow{\eta_\alpha f} & \eta_\alpha(X) \\
{\scriptstyle m'}\uparrow & & \uparrow{\scriptstyle m'} \\
Y' & \xrightarrow[\eta_\beta(\mathbb{C}^\beta_\alpha f|_m)]{} & Y'
\end{array}$$

so that $\eta_\alpha f$ lifts to $\eta_\beta(\mathbb{C}^\beta_\alpha f|_m)$ along m'. Lastly, we have to check that such lifts are consistent. So suppose we have an arrow $n' : Z' \to Y'$ in $\mathbb{C}[\lambda]$ and $\eta_\alpha f$ lifts to a map $s' : Z' \to Z'$ along $m' \circ n'$

$$\begin{array}{ccc}
\eta_\alpha(X) & \xrightarrow{\eta_\alpha f} & \eta_\alpha(X) \\
{\scriptstyle m'}\uparrow & & \uparrow{\scriptstyle m'} \\
Y' & & Y' \\
{\scriptstyle n'}\uparrow & & \uparrow{\scriptstyle n'} \\
Z' & \xrightarrow[s']{} & Z'
\end{array}$$

Note that $\mathrm{dom}(m' \circ n') = Z' = \eta_\gamma(Z)$ for some $\gamma < \lambda$ and $Z \in \mathbb{C}[\gamma]$. Obviously, $\gamma \leq \beta$ and we can re-write $Z' = \eta_\beta(\mathbb{C}^\beta_\gamma Z)$ and analogously, $s' = \eta_\gamma(s) = \eta_\beta(\mathbb{C}^\beta_\gamma s)$ for some $s : Z \to Z$ in $\mathbb{C}[\gamma]$ and $n' = \eta_\beta(n)$ for some $n : \mathbb{C}^\beta_\gamma(Z) \to Y$ in $\mathbb{C}[\beta]$. Therefore, we can re-draw the above diagram as

$$\begin{CD}
\eta_\beta(\mathbb{C}^\beta_\alpha X) @>{\eta_\beta(\mathbb{C}^\beta_\alpha f)}>> \eta_\beta(\mathbb{C}^\beta_\alpha X) \\
@A{\eta_\beta(m)}AA @AA{\eta_\beta(m)}A \\
\eta_\beta(Y) @. \eta_\beta(Y) \\
@A{\eta_\beta(n)}AA @AA{\eta_\beta(n)}A \\
\eta_\beta(\mathbb{C}^\beta_\gamma s) @>>{\eta_\beta(\mathbb{C}^\beta_\gamma s)}> \eta_\beta(\mathbb{C}^\beta_\gamma s)
\end{CD}$$

and reflecting this back into $\mathbb{C}[\beta]$, we obtain

$$\begin{CD}
\mathbb{C}^\beta_\alpha(X) @>{\mathbb{C}^\beta_\alpha f}>> \mathbb{C}^\beta_\alpha(X) \\
@A{m}AA @AA{m}A \\
Y @. Y \\
@A{n}AA @AA{n}A \\
\mathbb{C}^\beta_\gamma(Z) @>>{\mathbb{C}^\beta_\gamma s}> \mathbb{C}^\beta_\gamma(Z)
\end{CD}$$

We deduce that $\mathbb{C}^\beta_\alpha f|_{m\circ n} = \mathbb{C}^\beta_\gamma s$; since $\mathbb{C}^\beta_\alpha f$ has consistent lifts in $\mathbb{C}[\beta]$, we also have

$$(\mathbb{C}^\beta_\alpha f|_m)|_n = \mathbb{C}^\beta_\alpha f|_{m\circ n} = \mathbb{C}^\beta_\gamma s$$

$$\begin{CD}
\mathbb{C}^\beta_\alpha(X) @>{\mathbb{C}^\beta_\alpha f}>> \mathbb{C}^\beta_\alpha(X) \\
@A{m}AA @AA{m}A \\
Y @>{\mathbb{C}^\beta_\alpha f|_m}>> Y \\
@A{n}AA @AA{n}A \\
\mathbb{C}^\beta_\gamma(Z) @>>{\mathbb{C}^\beta_\gamma s}> \mathbb{C}^\beta_\gamma(Z)
\end{CD}$$

and embedding this diagram into $\mathbb{C}[\lambda]$ via η_β yields

$$\begin{CD}
\eta_\beta(\mathbb{C}^\beta_\alpha X) @>{\eta_\beta(\mathbb{C}^\beta_\alpha f)}>> \eta_\beta(\mathbb{C}^\beta_\alpha X) \\
@A{\eta_\beta(m)}AA @AA{\eta_\beta(m)}A \\
\eta_\beta(Y) @>{\eta_\beta(\mathbb{C}^\beta_\alpha f|_m)}>> \eta_\beta(Y) \\
@A{\eta_\beta(n)}AA @AA{\eta_\beta(n)}A \\
\eta_\beta(\mathbb{C}^\beta_\gamma s) @>>{\eta_\beta(\mathbb{C}^\beta_\gamma s)}> \eta_\beta(\mathbb{C}^\beta_\gamma s)
\end{CD}$$

Re-drawing this diagram finally gives us

$$\begin{array}{ccc} \eta_\alpha(X) & \xrightarrow{\eta_\alpha f} & \eta_\alpha(X) \\ {\scriptstyle m'}\uparrow & & \uparrow{\scriptstyle m'} \\ Y' & \xrightarrow{\eta_\beta(\mathbb{C}_\alpha^\beta f|_m)} & Y' \\ {\scriptstyle n'}\uparrow & & \uparrow{\scriptstyle n'} \\ Z' & \xrightarrow[s']{} & Z' \end{array}$$

and we conclude

$$s' = \eta_\alpha f|_{m' \circ n'} = \eta_\beta(\mathbb{C}_\alpha^\beta f|_m)|_{n'} = (\eta_\alpha f|_{m'})|_{n'}$$

∎

We now summarise our main result.

Proposition 2.1.12. *If $\mathbb{C}[-] : \lambda \to$ **Cat** is a sequential diagram of categories, then the isotropy quotient $\mathbb{C}[\lambda]/\mathcal{I}$ of the colimit is equivalent to the colimit $(\mathbb{C}[-]/\mathcal{I})(\lambda)$ of the isotropy quotients in case all of the following conditions are satisfied:*

(i) For all $\alpha \leq \beta < \lambda$, the functor $\mathbb{C}_\alpha^\beta : \mathbb{C}[\alpha] \to \mathbb{C}[\beta]$ preserves isotropy, i.e., there is an induced functor

$$\begin{array}{ccc} \mathbb{C}[\alpha] & \xrightarrow{\mathbb{C}_\alpha^\beta} & \mathbb{C}[\beta] \\ {\scriptstyle \pi_\alpha}\downarrow & & \downarrow{\scriptstyle \pi_\beta} \\ \mathbb{C}[\alpha]/\mathcal{I} & \underset{\mathbb{C}_\alpha^\beta/\mathcal{I}}{\dashrightarrow} & \mathbb{C}[\beta]/\mathcal{I} \end{array}$$

(ii) For all $\alpha \leq \beta < \lambda$, the functor $\mathbb{C}_\alpha^\beta : \mathbb{C}[\alpha] \to \mathbb{C}[\beta]$ is an embedding.

(iii) For all $\alpha \leq \beta < \lambda$, the induced functor $\mathbb{C}_\alpha^\beta/\mathcal{I} : \mathbb{C}[\alpha]/\mathcal{I} \to \mathbb{C}[\beta]/\mathcal{I}$ is injective on morphisms.

Proof: We just check that all the hypotheses of proposition 2.1.8 are satisfied. By lemma 2.1.9, conditions (ii) and (iii) of our statement imply, respectively, hypotheses (i) and (iii) of proposition 2.2. Conditions (i) and (ii) together imply hypothesis (ii) in the proposition by application of lemmas 2.1.4 and 2.1.11. ∎

Note that the most trivial case to which our proposition applies is when the functor F assigns to each $\alpha < \lambda$ the α^{th} isotropy quotient of a fixed category $\mathbb{C}$; the conclusion of our proposition (re)captures the fact that if β is a limit ordinal less than λ, then the β^{th} isotropy quotient of $\mathbb{C}$ is the colimit of all the preceding isotropy quotients.

We conclude this section with pithy restatements of the key results.

Theorem 2.1.13. *Sequential colimits under diagrams with isotropy-preserving transition maps commute with isotropy quotients.*

Corollary 2.1.14. *Sequential colimits under diagrams with isotropy-preserving inclusions commute with isotropy quotients.*

2.2 Isotropy Rank

We will want to put corollary 2.1.14 to work to carry out a construction in the sequel. However, we will also require some additional technology to provide a convenient conceptual setting for said construction. In keeping with the leitmotif of most of our results, the notions we delineate will be preserved under colimits in the appropriate sense.

Definition 2.2.1. Let $\mathbb{C}$ be a small category and $\varphi : C \to C$ an automorphism in $\mathbb{C}$. The **isotropy rank** of φ, denoted $||\varphi||_{\mathbb{C}}$, is defined by

$$||\varphi||_{\mathbb{C}} = \begin{cases} 0 & \text{if } \varphi = 1_C \\ \bigwedge\{\lambda \mid \pi^{\lambda}_{\mathcal{I}}(\varphi) = 1_C\} & \text{if } \exists \lambda > 0 \text{ such that } \pi^{\lambda}_{\mathcal{I}}(\varphi) = 1_C \\ -\infty & \text{otherwise.} \end{cases}$$

The **isotropy rank of the category** $\mathbb{C}$ is, if it exists, the supremum

$$||\mathbb{C}||_{\mathcal{I}} := \bigvee\{||\varphi||_{\mathbb{C}} \mid \varphi \text{ is an automorphism in } \mathbb{C}\} \geq 0.$$

A functor $F : \mathbb{C} \to \mathbb{D}$ **preserves isotropy up to rank** λ in case $||F(\varphi)||_{\mathbb{D}} = ||\varphi||_{\mathbb{C}}$ for all automorphisms $\varphi \in \mathrm{Mor}(\mathbb{C})$ with $||\varphi||_{\mathbb{C}} \leq \lambda$. If we also have that $||\mathbb{C}||_{\mathcal{I}} \leq ||\mathbb{D}||_{\mathcal{I}}$ and F preserves isotropy up to rank $||\mathbb{C}||_{\mathcal{I}}$, then F is said to simply **preserve isotropy ranks**.

Observe that isotropy-preserving functors are precisely the functors which preserve isotropy up to rank 1. More generally, a functor which preserves isotropy up to rank λ is one which can be lifted along isotropy quotient maps as indicated in the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{D} \\ {\scriptstyle \pi^1_{\mathcal{I}}}\downarrow & & \downarrow{\scriptstyle \pi^1_{\mathcal{I}}} \\ \mathbb{C}/\mathcal{I} & \overset{F/\mathcal{I}^2}{\dashrightarrow} & \mathbb{D}/\mathcal{I} \\ {\scriptstyle \pi^2_{\mathcal{I}}}\downarrow & & \downarrow{\scriptstyle \pi^2_{\mathcal{I}}} \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathbb{C}/\mathcal{I}^{\lambda} & \overset{F/\mathcal{I}^{\lambda}}{\dashrightarrow} & \mathbb{D}/\mathcal{I}^{\lambda} \end{array}$$

This follows from the fact that F preserves isotropy at each successor stage while the induced arrow at the limit stage is just the arrow induced by the universal property of the isotropy quotient.

Lemma 2.2.2. *For a category* $\mathbb{C}$*,*

$$||\mathbb{C}||_{\mathcal{I}} = \bigwedge\{\lambda \mid \mathbb{C}/\mathcal{I}^{\lambda} = \mathbb{C} \text{ for all } \mu > \lambda\}.$$

Proof: There are two cases to consider: either $||\mathbb{C}||_{\mathcal{I}} = 0$ or $||\mathbb{C}||_{\mathcal{I}} > 0$. In the former case, any non-identity automorphism $\varphi : C \to C$ in $\mathbb{C}$ satisfies $||\varphi||_{\mathbb{C}} = -\infty$ and there is thus no $\lambda \geq 0$ such that $\pi^{\lambda}_{\mathcal{I}}(\varphi) = 1_C$; this implies immediately that the congruence on $\mathbb{C}$ induced by the isotropy relation is trivial and the quotient functor is therefore the identity on $\mathbb{C}$. On the other hand, if $||\mathbb{C}||_{\mathcal{I}} > 0$, then there exists a non-trivial automorphism φ with isotropy rank greater than 0 and the RHS of the equation in the statement of the lemma is necessarily greater than 0 since killing off φ requires that we quotient at least once. Moreover, by definition of $||\varphi||_{\mathbb{C}}$, $\pi^{||\varphi||_{\mathbb{C}}}_{\mathcal{I}}(\mathbb{C}) \neq \mathbb{C}$ since $\pi^{||\varphi||_{\mathbb{C}}}_{\mathcal{I}}(\varphi) = 1_C \neq \varphi$ and the RHS of our equation is at least $||\varphi||_{\mathbb{C}}$. We conclude from this that the RHS is an upper bound for all the $||\varphi||_{\mathbb{C}}$ and hence that

$$||\mathbb{C}||_{\mathcal{I}} \leq \bigwedge\{\lambda \mid \pi^{\lambda}_{\mathcal{I}}(\mathbb{C}) = \mathbb{C} \text{ for all } \mu > \lambda\}.$$

The converse inequality follows from the observation that the only non-trivial automorphisms in $\pi^{||\mathbb{C}||_{\mathcal{I}}}_{\mathcal{I}}(\mathbb{C})$ necessarily have isotropy rank $-\infty$ and the isotropy congruence is trivial beyond the $||\mathbb{C}||^{th}_{\mathcal{I}}$ stage. ∎

The next lemma is used implicitly throughout the sequel.

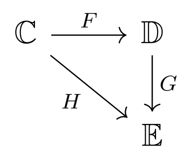
Lemma 2.2.3. *Given a category* $\mathbb{C}$ *and an automorphism* $\varphi : C \to C$ *in* $\mathbb{C}$*,* $||\varphi||_{\mathbb{C}}$ *cannot be a limit ordinal.*

Proof: We offer the following small observations. First, taking successive isotropy quotients provides a sequential diagram

$$\mathbb{C} \xrightarrow{\pi^1_{\mathcal{I}}} \mathbb{C}/\mathcal{I} \xrightarrow{\pi^2_{\mathcal{I}}} \mathbb{C}/\mathcal{I}^2 \xrightarrow{\pi^3_{\mathcal{I}}} \cdots \xrightarrow{\pi^{\lambda}_{\mathcal{I}}} \mathbb{C}/\mathcal{I}^{\lambda}$$

of categories where each functor is full and all categories have the same class of objects. Second, if we look at the proof of lemma 2.1.9, we see that we have actually shown the following: given a sequential diagram of categories where each functor in the diagram is injective on objects and full, each of the colimiting arrows is injective on objects and full. This statement certainly applies to the specific case of isotropy quotients. Putting these two facts together allows us to deduce that, in case λ as above is a limit ordinal, $\mathbb{C}/\mathcal{I}^{\lambda}$ has the same objects as $\mathbb{C}$ and each colimiting functor $\mathbb{C}/\mathcal{I}^{\mu} \to \mathbb{C}/\mathcal{I}^{\lambda}$ is full. This means that a commutative square in $\mathbb{C}/\mathcal{I}^{\lambda}$ can be lifted to a commutative square $\mathbb{C}/\mathcal{I}^{\mu}$ for sufficiently large μ such that $\mu < \lambda$ and μ is a successor ordinal. Hence, if an automorphism φ in $\mathbb{C}$ has consistent lifts in $\mathbb{C}/\mathcal{I}^{\lambda}$, it must already have those consistent lifts at a successor stage in the sequential diagram. ∎

Lemma 2.2.4. *Suppose we are given a diagram of functors*



If G preserves isotropy up to rank λ and so does at least one of F and H, then all three functors preserve isotropy up to rank λ. If F and H preserve isotropy up to rank λ and F is surjective on automorphisms which have isotropy rank at most λ, then G also preserves isotropy up to rank λ.

Proof: Let φ be an automorphism in $\mathbb{C}$ with $||\varphi||_{\mathbb{C}} \leq \lambda$. If F and G preserve isotropy up to rank λ, then

$$\begin{aligned} ||H(\varphi)||_{\mathbb{E}} &= ||GF(\varphi)||_{\mathbb{E}} \\ &= ||G(F(\varphi))||_{\mathbb{E}} \\ &= ||F(\varphi)||_{\mathbb{D}} \\ &= ||\varphi||_{\mathbb{C}}. \end{aligned}$$

If H and G preserve isotropy up to rank λ, then

$$\begin{aligned} ||F(\varphi)||_{\mathbb{D}} &= ||G(F(\varphi))||_{\mathbb{E}} \\ &= ||GF(\varphi)||_{\mathbb{E}} \\ &= ||H(\varphi)||_{\mathbb{E}} \\ &= ||\varphi||_{\mathbb{C}}. \end{aligned}$$

If F is surjective on automorphisms as described, then every automorphism ψ with $||\psi||_{\mathbb{D}} \leq \lambda$ is of the form $F(\varphi)$ for some φ in $\mathbb{C}$; if also F and H preserve isotropy up to rank λ, then

$$\begin{aligned} ||G(\psi)||_{\mathbb{E}} &= ||G(F(\varphi))||_{\mathbb{E}} \\ &= ||GF(\varphi)||_{\mathbb{E}} \\ &= ||H(\varphi)||_{\mathbb{E}} \\ &= ||\varphi||_{\mathbb{C}} \end{aligned}$$

$$= ||F(\varphi)||_{\mathbb{D}}$$
$$= |||\psi||_{\mathbb{D}}.$$

∎

In some instances where we use this lemma, we shall say that functors which preserve isotropy ranks enjoy an "almost 2-out-of-3 property".

Lemma 2.2.5. *Let $\mathbb{C}[-] : \lambda \to$ **Cat** be a sequential diagram of categories equipped with transition maps which preserve isotropy up to rank μ. If $\eta : \mathbb{C}[-] \Rightarrow \mathbb{C}[\lambda]$ denotes the sequential colimit under F, then each of the arrows $\eta_\alpha : \mathbb{C}[\alpha] \to \mathbb{C}[\lambda]$ preserves isotropy up to rank μ.*

Proof: Take an $\alpha < \lambda$ and consider the functor $\eta_\alpha : \mathbb{C}[\alpha] \to \mathbb{C}[\lambda]$; suppose we have an automorphism $\varphi : C \to C$ in $\mathbb{C}[\alpha]$ such that $\nu := ||\varphi||_{\mathbb{C}[\alpha]} \leq \mu$. If $\nu = 0$, the proof is trivial and we assume that $\nu = -\infty$ or $0 < \nu \leq \mu$. Using theorem 2.1.13 and our observation that maps which preserve isotropy ranks can be lifted along isotropy quotient maps, we may picture the situation via the following diagram

$$\begin{array}{ccccccccc}
\mathbb{C}[0] & \longrightarrow & \mathbb{C}[1] & \longrightarrow & \mathbb{C}[2] & \longrightarrow & \cdots & \longrightarrow & \mathbb{C}[\lambda] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{C}[0]/\mathcal{I} & \longrightarrow & \mathbb{C}[1]/\mathcal{I} & \longrightarrow & \mathbb{C}[2]/\mathcal{I} & \longrightarrow & \cdots & \longrightarrow & \mathbb{C}[\lambda]/\mathcal{I} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{C}[0]/\mathcal{I}^2 & \longrightarrow & \mathbb{C}[1]/\mathcal{I}^2 & \longrightarrow & \mathbb{C}[2]/\mathcal{I}^2 & \longrightarrow & \cdots & \longrightarrow & \mathbb{C}[\lambda]/\mathcal{I}^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{C}[0]/\mathcal{I}^\sigma & \longrightarrow & \mathbb{C}[1]/\mathcal{I}^\sigma & \longrightarrow & \mathbb{C}[2]/\mathcal{I}^\sigma & \longrightarrow & \cdots & \longrightarrow & \mathbb{C}[\lambda]/\mathcal{I}^\sigma
\end{array}$$

where vertical maps are quotient maps and horizontal maps are arrows in a sequential diagram. By lemmas 2.1.5 and 2.1.9, each of the horizontal arrows and each of the colimiting arrows in a given level is an embedding. Now suppose that $\eta_\alpha(\varphi)$ gets mapped to an identity along some vertical path starting at $\mathbb{C}[\lambda]$; assume that this path ends at $\mathbb{C}[\lambda]/\mathcal{I}^\tau$. By commutativity of the (sub)diagram(s) above, we deduce that there is a path in the diagram which starts at $\mathbb{C}[\alpha]$, goes straight downwards to $\mathbb{C}[\alpha]/\mathcal{I}^\tau$ and then takes a right turn towards $\mathbb{C}[\lambda]/\mathcal{I}^\tau$. But since each of the colimiting arrows under the sequential diagram at the τ^{th} floor is an embedding, the image of φ in $\mathbb{C}[\alpha]/\mathcal{I}^\tau$ must be an identity. If $0 < \nu \leq \mu$, the highest floor at which this can occur is $\tau = \nu$ and since each of the horizontal arrows cannot map non-identity arrows to non-identity arrows, we are done. Lastly, if $\nu = -\infty$, then the argument just provided gives a proof by contradiction. ∎

Lastly, we can consider the interaction between two chains of categories

$$\mathbb{C}[0] \longrightarrow \mathbb{C}[1] \longrightarrow \mathbb{C}[2] \longrightarrow \cdots$$

and

$$\mathbb{D}[0] \longrightarrow \mathbb{D}[1] \longrightarrow \mathbb{D}[2] \longrightarrow \cdots$$

A morphism from the first chain to the second is of course given by a sequence of functors $\theta_\alpha : \mathbb{C}[\alpha] \to \mathbb{D}[\alpha]$ and we can ask for conditions under which the induced functor on colimits, indicated by the dashed arrow in the diagram

$$\begin{array}{ccccccccc} \mathbb{C}[0] & \longrightarrow & \mathbb{C}[1] & \longrightarrow & \mathbb{C}[2] & \longrightarrow & \cdots & \longrightarrow & \mathbb{C}[\lambda] \\ \downarrow{\scriptstyle\theta_0} & & \downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\theta_2} & & & & \downarrow{\scriptstyle\theta_\lambda} \\ \mathbb{D}[0] & \longrightarrow & \mathbb{D}[1] & \longrightarrow & \mathbb{D}[2] & \longrightarrow & \cdots & \longrightarrow & \mathbb{D}[\lambda] \end{array}$$

is possessed of desired properties with respect to isotropy. The motivation for the result below goes back to the tower we considered just before proving lemma 2.2.2.

Proposition 2.2.6. *Let $\mathbb{C}[-], \mathbb{D}[-] : \lambda \to$ **Cat** be sequential diagrams of categories and suppose $\theta : \mathbb{C}[-] \Rightarrow \mathbb{D}[-]$ is a sequential transformation. Denote by $\eta : \mathbb{C}[-] \Rightarrow \Delta\mathbb{C}[\lambda]$, $\varepsilon : \mathbb{D}[-] \Rightarrow \Delta\mathbb{D}[\lambda]$ the obvious universal cocones and assume that the following conditions hold.*

(i) For all $\alpha < \lambda$, $||\mathbb{C}[\alpha]||_{\mathcal{I}} \leq ||\mathbb{D}[\alpha]||_{\mathcal{I}}$.
(ii) For all $\alpha < \lambda$, each of θ_α, $\mathbb{C}_\alpha^\beta$ and $\mathbb{D}_\alpha^\beta$ preserves isotropy rank.
(iii) For all $\alpha \leq \beta < \lambda$, each of $\mathbb{C}_\alpha^\beta, \mathbb{D}_\alpha^\beta$ is a transition map.

If $\theta_\lambda : \mathbb{C}[\lambda] \to \mathbb{D}[\lambda]$ is the functor induced by the arrows $\varepsilon_\alpha \circ \theta_\alpha : \mathbb{C}[\alpha] \to \mathbb{D}[\lambda]$, then θ_λ preserves isotropy ranks.

Proof: From lemma 2.2.5, we have that η_α, ε_α preserve isotropy ranks and the almost 2-out-of-3 property of lemma 2.2.4 can be applied to the diagram

$$\begin{array}{ccc} \mathbb{C}[\alpha] & \xrightarrow{\eta_\alpha} & \mathbb{C}[\lambda] \\ {\scriptstyle\theta_\alpha}\downarrow & & \downarrow{\scriptstyle\widetilde{\theta}} \\ \mathbb{D}[\alpha] & \xrightarrow[\varepsilon_\alpha]{} & \mathbb{D}[\lambda] \end{array}$$

∎

We see therefore that given a nice sequential diagram, we can deduce isotropy ranks of the colimit from the isotropy ranks of the categories present in the diagram. Our task now is therefore to develop some tools which allow us to work with isotropy ranks within categories that fit into such nice sequential diagrams.

Chapter 3

Profunctors and Isotropy

In the last chapter, we produced some results concerning isotropy within categories which can be realised as certain kinds of colimits. In this chapter, we attempt to do the same for categories which arise as collages of certain kinds of profunctors. Since profunctors are not yet part of the canon of basic category theory, we start with a quick review of the calculus of profunctors, an account of which may readily be found in [B0́0] and [Rie10]. We mention in passing that profunctors are one of those mathematical objects whose name is a function of geography; those of a Continental persuasion often refer to profunctors as **distributors** while the Australian tendency is to call profunctors **(bi)modules**.

3.1 A Brief Introduction to Profunctors and Fibrations

In developing the theory of profunctors, it will be helpful to start out by examining the motivating example of a profunctor. For a small category $\mathbb{C}$, we can assign to each object $c \in \mathbb{C}$, the contravariant functor or representable presheaf $\mathrm{Hom}_{\mathbb{C}}(-, c) : \mathbb{C}^{op} \to \mathbf{Set}$. This functor is of course just the image of c under the Yoneda embedding $\mathbb{C} \to \widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{op}}$ and Cartesian closedness of **Cat** allows us to uncurry the Yoneda embedding into a bifunctor $\mathbb{C}^{op} \times \mathbb{C} \to \mathbf{Set}$ which acts on objects via $\langle c', c \rangle \mapsto \mathrm{Hom}_{\mathbb{C}}(c', c)$. If we are given a functor $F : \mathbb{C} \to \mathbb{D}$, i.e., F is an arrow in **Cat**, we can either compose in the first variable of the uncurried Yoneda to obtain the functor $\mathbb{D}^{op} \times \mathbb{C} \to \mathbf{Set}$ defined on objects by $\langle d, c \rangle \mapsto \mathrm{Hom}_{\mathbb{D}}(d, F(c))$ or we compose in the second variable to get a functor $\mathbb{C}^{op} \times \mathbb{D} \to \mathbf{Set}$ defined on objects by $\langle c, d \rangle \mapsto \mathrm{Hom}_{\mathbb{D}}(F(c), d)$. In the first case, we can curry to get a functor $\mathbb{C} \to \widehat{\mathbb{D}}$ while in the second case, we obtain $\mathbb{D} \to \widehat{\mathbb{C}}$. The former we call the **representable profunctor** induced by F and the latter is the **corepresentable profunctor** on F. But this provides only the archetypical example.

Given categories $\mathbb{C}$ and $\mathbb{D}$, a **profunctor from** $\mathbb{C}$ **to** $\mathbb{D}$, denoted $H : \mathbb{C} \nrightarrow \mathbb{D}$, is simply a functor $H : \mathbb{D}^{op} \times \mathbb{C} \to \mathbf{Set}$. Given objects $d \in \mathbb{D}$, $c \in \mathbb{C}$, we obtain a set $H(d, c)$, an element of which is called a **heteromorphism from** d **to** c. Observe that, since **Cat** is Cartesian closed, we have a natural bijection of sets

$$\mathbf{Cat}(\mathbb{D}^{op} \times \mathbb{C}, \mathbf{Set}) \cong \mathbf{Cat}(\mathbb{C}, \mathbf{Set}^{\mathbb{D}^{op}}) = \mathbf{Cat}(\mathbb{C}, \widehat{\mathbb{D}})$$

and a profunctor from $\mathbb{C}$ to $\mathbb{D}$ can thus be conceived of as a functor which maps $\mathbb{C}$ into the presheaf topos generated by $\mathbb{D}$. Viewing profunctors from this perspective has the advantage of making transparent what it means to compose two profunctors (or, indeed, that profunctors *can* be composed). Given $H_1 : \mathbb{C} \nrightarrow \mathbb{D}$, $H_2 : \mathbb{D} \nrightarrow \mathbb{E}$, we define the composite $H_2 \circ H_1 : \mathbb{C} \nrightarrow \mathbb{E}$ as the map $\mathbb{E}^{op} \times \mathbb{C} \to \mathbf{Set}$ corresponding to the composite

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{H_1} & \widehat{\mathbb{D}} \\ & \searrow & \downarrow{\scriptstyle \widehat{H_2}} \\ & & \widehat{\mathbb{E}} \end{array}$$

where $\widehat{H_2}$ is the colimit-preserving functor induced by the universal property of $\widehat{D}$ as the free cocompletion of $\mathbb{D}$

$$\begin{array}{ccc} \mathbb{D} & \xhookrightarrow{\mathbf{y}} & \widehat{\mathbb{D}} \\ & {\scriptstyle H_2}\searrow & \downarrow{\scriptstyle \widehat{H_2}} \\ & & \widehat{\mathbb{E}} \end{array}$$

Moreover, this immediately provides a candidate identity profunctor $\mathbb{C} \nrightarrow \mathbb{C}$ on a category $\mathbb{C}$: it is the (map corresponding to the) Yoneda embedding $\mathbf{y} : \mathbb{C} \to \widehat{\mathbb{C}}$.

The above discussion suggests that we can organise categories and profunctors between them into a category. Unfortunately, this is not quite true; composition of profunctors fails to be associative. The underlying reason for this is that functors induced by the universal property of a presheaf topos are only unique up to isomorphism and the second triangle above is only guaranteed to commute up to natural isomorphism. However, what we *do* get is a **bi**category **Prof** consisting of categories as objects, profunctors as 1-morphisms and natural transformation 2-morphisms which carry extra "associativity data". We will not delve too deeply into the bicategorical aspects here except to mention that **Prof** is conceptually important as the prototypical example of a bicategory and that profunctors are also the archetypical example of a **proarrow equipment**, which provides a powerful technical tool for studying bicategories and their 2-categorical cousins such as **double categories** (see [Woo82] and [Woo85]). We also remark that while proarrow equipments are a Canadian produce, they are mostly consumed as an Australian import, much of which this author finds indigestible. Convenient for us then that we only need to be able to deal with categories “one at a time”, i.e., instead of having to deal with *all* profunctors, we will only need to work with profunctors $\mathbb{D}^{op} \times \mathbb{C} \to \mathbf{Set}$ for fixed categories $\mathbb{C}$ and $\mathbb{D}$. Indeed, the main result we will need is the following, where $\mathbf{DFib}(\mathbb{C}, \mathbb{D})$ indicates the category of **discrete fibrations** which we define later.

Theorem 3.1.1. *There exist equivalences of categories*

$$\mathbf{DFib}(\mathbb{C}, \mathbb{D}) \simeq [\mathbb{D}^{op} \times \mathbb{C}, \mathbf{Set}]$$

pseudo-natural in $\mathbb{C}$ *and* $\mathbb{D}$.

Unravelling the statement of this theorem requires us to take a detour through the theory of (Grothendieck) fibrations. For the moment, we encourage the reader to think of the above result as a “representation” or “modelling” theorem which says that profunctors can be modelled as objects in the category $\mathbf{DFib}(\mathbb{C}, \mathbb{D})$.

Prior to proceeding with the general theory, we would like to point out that an excellent way of finding conceptual purchase on profunctors is to think of them as categorified versions of set-theoretic relations. Recall that a bare set X can be thought as a discrete category with object set X and no non-identity morphisms. Let us work out what data is encoded by a profunctor $R : X \nrightarrow Y$ – since we have already given away the punchline, we may as well use suggestive notation – for discrete categories X and Y. The profunctor associates for each pair $\langle y, x\rangle$ of elements a set $R\langle y, x\rangle$ and since there are no non-identity arrows in the product category $Y^{op} \times X = Y \times X$, there is no further data. Now, for a given pair $\langle y, x\rangle$, the set $R\langle y, x\rangle$ is either empty or not. In the former case, we think of this as saying that y is not related to x and in the latter case that y is related to x by R. Note, however, that there are no restrictions on what the non-empty sets $R\langle y, x\rangle$ can be! There is thus a seeming conceptual gap here: if we were to attempt a naive generalization of relations to the level of categories, we would perhaps come up with functors $Y \times X \to \mathbf{Set}$ assigning either the empty set – the von Neumann ordinal 0 – or the singleton set – the von Neumann ordinal 1 – to each $R\langle y, x\rangle$ in order to categorically emulate the characteristic functions which carve out relations as subsets of the Cartesian product $Y \times X$. What is absent from the classical notion of relation but is present in the categorified version is *proof-relevance.* That is, the data of a profunctor is not merely a specification of which elements in Y are related to what elements in X but also *how* a given element of Y is related to some other element of X. Each distinct element inside a non-empty $R\langle y, x\rangle$ can be thought of as giving a distinct *proof* that y is related to x. There is some exploration of this theme and its applications in the computer science literature (for example, [Win05]).

Next, let us see that profunctor composition corresponds to relation composition. Suppose we have profunctors $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$ for discrete categories X, Y and Z. Recall that the composite $S \circ R$ is defined to be the functor $Z^{op} \times X \to \mathbf{Set}$

corresponding to the composite

$$\begin{array}{ccc} X & \xrightarrow{R} & \widehat{Y} \\ & \searrow^{S\circ R} & \downarrow{\scriptstyle \widehat{S}} \\ & & \widehat{Z} \end{array}$$

What this means is that for given $\langle z, x\rangle \in Z^{op} \times X = Z \times X$, we have the equality

$$(S \circ R)\langle z, x\rangle = \widehat{S}(R\langle -, x\rangle)(z)$$

and using a bit of abstract nonsense, as in [Awo10] for instance, gives us the conclusion that the set on the right-hand side of the equation is non-empty precisely when there exists a $y \in Y$ such that $R\langle y, x\rangle$ and $S\langle z, y\rangle$ are non-empty. In other words, z is $(S \circ R)$-related to x if and only if there exists y which is R-related to x and z is S-related to y. Hence, we have recovered the usual notion of relation composition. Again, the profunctor composition is a little bit more than the classical notion since it is sensitive to proof relevance.

With this bit of (hopefully) clarifying exposition out of the way, we move on to the study of profunctors in the abstract. We note that the organization of the succeeding exposition is an emulation of the first half of [Rie10], with simplifying technicalities drawn from [BW90] and [Str14]. Before discussing categorical fibrations of any sort, it will be helpful to recall some basic algebraic topology (see [Hat01]).

Definition 3.1.2. Given a topological space B, a **covering space** of B is given a continuous surjective map $p : E \to B$ such that for each $b \in B$, there is an open neighbourhood U of b where $p^{-1}(U)$ is a union of disjoint open sets in E, each of which is mapped homeomorphically onto U by p.

One of the main results which makes covering spaces so useful in computations is the famous **path lifting property**.

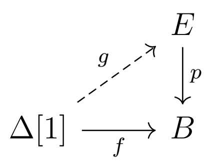
Theorem 3.1.3. *If $p : E \to B$ is a covering space, e is a point in E, $f : [0, 1] \to B$ is a path starting at $b' \in B$ and ending at $pe \in B$, then there is a unique path $g : [0, 1] \to E$ ending at e. That is, there is a unique filler*

$$\begin{array}{ccc} & & E \\ & \overset{g}{\nearrow} & \downarrow{\scriptstyle p} \\ [0,1] & \xrightarrow[f]{} & B \end{array}$$

making the triangle commute.

Using the old French trick of turning a theorem into a definition and replacing spaces by categories and continuous maps by functors produces the simplest kind of categorical fibration.

Definition 3.1.4. A functor $p : E \to B$ is a **discrete fibration** if for each object $e \in E$ and arrow $f : b' \to pe$, there exists a unique morphism $g : e' \to e$ such that $pg = f$. Equivalently, using $\Delta[1]$ to denote the walking arrow $\bullet \to \bullet$, there is a unique filler



making the diagram commute.

Before moving on, let us clarify what we mean by a **fiber** for a functor like $p : E \to B$; for an object $b \in B$, note that there is actually a *category* formed by all those objects in E which get mapped to b. Unsurprisingly, we denote this category $p^{-1}(b)$, its morphisms are exactly the arrows in E which get mapped to the identity on b and we call it the **fiber of** p **over** b. As the name suggests, discrete fibrations have discrete fibers; that is, each $p^{-1}(b)$ has no non-identity morphisms as a category. This is so because, assuming $p^{-1}(b)$ is non-empty, the identity arrow $b \to b$ is required to have a unique lift and p already maps identities to identities. Now, just as the notion of discrete fibration provides the simplest type of categorical fibration, there is corresponding to it a simplest type of "representation theorem".

Theorem 3.1.5. *There is an isomorphism of categories*

$$\mathbf{DFib}(\mathbb{C}) \cong [\mathbb{C}^{op}, \mathbf{Set}] = \widehat{\mathbb{C}},$$

where $\mathbf{DFib}(\mathbb{C})$ *is the category of discrete fibrations considered as a full subcategory of the slice category* $\mathbf{Cat}/\mathbb{C}$*.*

Since the verification of this equivalence is not overly difficult, it is instructive to go through it in order to get a feel for how fibrations correspond to functors in a natural way.

Proof: Suppose we start with a discrete fibration $p : E \to \mathbb{C}$ and we wish to create canonically a presheaf $P : \mathbb{C}^{op} \to \mathbf{Set}$ from the data of p. To each object $c \in \mathbb{C}$, we of course have the discrete fiber $p^{-1}(c)$ and we define P on objects by letting $P(c)$ be the object set of the category $p^{-1}(c)$. Given a morphism $f : c' \to c$, we must now define a set map $P(f) : P(c) \to P(c')$, i.e., we need to map objects of $p^{-1}(c)$ to those in $p^{-1}(c')$. If $p^{-1}(c)$ is empty, there is nothing to do here. Otherwise, for $e \in E$ with $pe = c$, we can associate a unique map $g : e' \to e$ such that $pg = f$ owing to the fact that f is an arrow in $\mathbb{C}$ with codomain pe. The obvious thing to do here is to set $P(f)(e) = e'$ and functoriality of P obtains from the uniqueness of the lifts.

Conversely, assume that we are given the presheaf P and we wish to construct from it a discrete fibration $p : E \to \mathbb{C}$; recall we can always slice a functor under the functor picking out the terminal set in order to obtain the **category of elements** $*/P$. Observe that an object of $*/P$ is specified by a pair $\langle c \in \mathbb{C}, x \in P(c)\rangle$ and a map $\langle c, x\rangle \to \langle c', x'\rangle$ is a morphism $f : c' \to c$ satisfying $P(f)(x) = x'$. Taking $p : */P \to \mathbb{C}$ to be the projection onto the first coordinate gives a discrete fibration with an arrow $f : c' \to p\langle c, x\rangle$ lifting to $\langle c', P(f)(x)\rangle \to \langle c, x\rangle$. The uniqueness of the lift is obvious when we note that any $g : \langle c'', x''\rangle \to \langle c, x\rangle$ with $pg = f$ implies

$$c'' = p\langle c'', x''\rangle = p(\mathrm{dom}(g)) = \mathrm{dom}(pg) = \mathrm{dom}(f) = c'$$

and $x'' = P(f)(x)$. ∎

In the proof above, we used the category of elements to construct a fibration. We would like to point out that there is a beautiful exploration of the noumenal form of the category of elements contained in [Dug98] and the author's exposition makes clear why the category of elements is the natural candidate for a canonical discrete fibration over $\mathbb{C}$.

Corollary 3.1.6. *There is an isomorphism of categories*

$$\mathbf{DFib}(\mathbb{D} \times \mathbb{C}^{op}) \cong [\mathbb{D}^{op} \times \mathbb{C}, \mathbf{Set}].$$

Proof: Observe that $(\mathbb{D}^{op} \times \mathbb{C})^{op} \cong \mathbb{D} \times \mathbb{C}^{op}$; now apply the theorem just proved. ∎

Hence, a profunctor $\mathbb{C} \nrightarrow \mathbb{D}$ is modelled by a discrete fibration over $\mathbb{D} \times \mathbb{C}^{op}$. Such fibrations in turn can be re-expressed in a form more amenable to our work.

Definition 3.1.7. Let $\mathbb{C}$ be a category with a subcategory $\mathbb{D}$. We say that $\mathbb{D}$ is a **sieve** if, for every arrow $f \in \mathbb{C}$, $\mathrm{cod}(f) \in \mathbb{D}$ implies $f \in \mathbb{D}$. Dually, $\mathbb{D}$ is a **cosieve** in case $\mathrm{dom}(f) \in \mathbb{D}$ implies $f \in \mathbb{D}$.

Definition 3.1.8. For categories $\mathbb{C}$ and $\mathbb{D}$, a **two-sided codiscrete cofibration from $\mathbb{C}$ to $\mathbb{D}$** is a pair of functors

$$\mathbb{C} \xrightarrow{\;i\;} K \xleftarrow{\;j\;} \mathbb{D}$$

where i is a cosieve inclusion, j is a sieve inclusion and i and j are jointly bijective on objects, i.e., an object in K is either of the form $i(c)$ or the form $j(d)$ with $c \in \mathbb{C}$ and $d \in \mathbb{D}$.

Theorem 3.1.9. *There is an equivalence of categories*

$$\mathbf{DFib}(\mathbb{D} \times \mathbb{C}^{op}) \simeq \mathbf{CoDCoFib}(\mathbb{C}, \mathbb{D}),$$

where $\mathbf{CoDCoFib}(\mathbb{C}, \mathbb{D})$ *is the category of two-sided codiscrete cofibrations from* $\mathbb{C}$ *to* $\mathbb{D}$ *considered as a full subcategory of* $\mathbf{CoSpan}(\mathbb{C}, \mathbb{D})$.

Proof: Suppose we are given a discrete fibration $p : E \to \mathbb{D} \times \mathbb{C}^{op}$. Define a category K whose object set is the disjoint union of the object sets of $\mathbb{C}$ and $\mathbb{D}$. For the morphisms of K, we let

$$\begin{aligned}
\mathrm{Hom}_K(c, c') &= \mathrm{Hom}_{\mathbb{C}}(c, c') \\
\mathrm{Hom}_K(d, d') &= \mathrm{Hom}_{\mathbb{D}}(d, d') \\
\mathrm{Hom}_K(d, c) &= \mathrm{Ob}(p^{-1}(d, c)) \\
\mathrm{Hom}_K(c, d) &= \emptyset.
\end{aligned}$$

We obtain obvious inclusions $i : \mathbb{C} \to K$, $j : \mathbb{D} \to K$ and the definition of the hom-sets in K makes clear that i is a cosieve inclusion while j is a sieve inclusion.

In the other direction, assume we start with a two-sided codiscrete cofibration

$$\mathbb{C} \xrightarrow{\;i\;} K \xleftarrow{\;j\;} \mathbb{D}$$

Let E be the category which has objects $\langle d \in \mathbb{D}, c \in \mathbb{C}, x \in \mathrm{Hom}_K(j(d), i(c))\rangle$ and an arrow $\langle d', c', x'\rangle \to \langle d, c, x\rangle$ in E is a pair of arrows $\langle f : d' \to d \in \mathbb{D}, g : c \to c' \in \mathbb{C}\rangle$ such that $x' = i(g) \circ x \circ j(f)$ in E. Take $p : E \to \mathbb{D} \times \mathbb{C}^{op}$ to be the projection on to the first two variables. The lift of an arrow $\langle d', c'\rangle \to p\langle d, c, x\rangle$ is given by $\langle d', c', i(g) \circ x \circ j(f)\rangle \to \langle d, c, x\rangle$. ∎

Corollary 3.1.10. *A profunctor* $\mathbb{C} \nrightarrow \mathbb{D}$ *is represented uniquely up to isomorphism by a cospan of functors, called the* ***collage of the profunctor****,*

$$\mathbb{C} \xrightarrow{\;i\;} K \xleftarrow{\;j\;} \mathbb{D}$$

where i *is a cosieve inclusion and* j *is a sieve inclusion.*

Lemma 3.1.11. *A two-sided codiscrete cofibration is corresponds uniquely up to isomorphism to a functor* $K \to \Delta[1]$.

Proof: If

$$\mathbb{C} \xrightarrow{\;i\;} K \xleftarrow{\;j\;} \mathbb{D}$$

is a two-sided codiscrete cofibration, then we define $F : K \to \Delta[1]$ so that the triangles

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{i} & K \\ & \searrow^{0} & \downarrow F \\ & & \Delta[1] \end{array} \qquad \begin{array}{ccc} \mathbb{D} & \xrightarrow{j} & K \\ & \searrow^{1} & \downarrow F \\ & & \Delta[1] \end{array}$$

commute; that is, we map everything in $\mathbb{C}$ to the domain of the walking arrow, we map everything in $\mathbb{D}$ to the codomain of the walking arrow and an arrow $j(d) \to i(c)$ in K is mapped to the walking arrow itself.

Conversely, given a functor $F : K \to \Delta[1]$, take $\mathbb{D}$ to be the fibre over 0, $\mathbb{C}$ to be the fibre over 1 and i and j to be the obvious inclusions of the fibres into K. Since every object in K belongs to at least one of the fibres, i and j are jointly bijective. Functoriality of F implies that any arrow in K which gets mapped to the walking arrow has domain of the form $j(d)$ and codomain of the form $i(c)$ and this is enough to ensure that i is a cosieve inclusion and j is a sieve inclusion. ∎

Hence, a profunctor $\mathbb{C} \nrightarrow \mathbb{D}$ is the same thing as a functor $p : K \to \Delta[1]$ satisfying $p^{-1}(0) = \mathbb{D}$ and $p^{-1}(1) = \mathbb{C}$. What is interesting is that p might be a functor such that the hom-sets $\mathrm{Hom}_K(d, c)$ are non-empty and we thus have **heteromorphisms** joining $\mathbb{D}$ to $\mathbb{C}$ with $\mathbb{C}$ being "stacked" on top of $\mathbb{D}$ according to rules contained in the data of p or, equivalently, the data contained in the profunctor $\mathbb{C} \nrightarrow \mathbb{D}$. In particular, it makes sense to ask whether, given a heteromorphism $d \to c$, we can lift automorphisms on c to automorphisms on d along this heteromorphism. This suggests that profunctors allow isotropy elements in $\mathbb{D}$ and $\mathbb{C}$ to interact in non-trivial ways. This is one of the themes we pursue in the sequel.

3.2 Isotropy Group of a Collage

Our immediate aim is to answer the following question: given categories $\mathbb{C}$ and $\mathbb{D}$ collaged together by a profunctor $H : \mathbb{C} \nrightarrow \mathbb{D}$, what can we say about $\mathcal{Z}_K$ in terms of $\mathcal{Z}_{\mathbb{C}}$ and $\mathcal{Z}_{\mathbb{D}}$?

Lemma 3.2.1. *Let $H : \mathbb{C} \nrightarrow \mathbb{D}$ be a profunctor with*

$$\begin{array}{ccccc} \mathbb{C} & & & & \mathbb{D} \\ & \searrow^{\phi_1} & & \swarrow^{\phi_2} & \\ & & K & & \end{array}$$

the associated two-sided codiscrete cofibration. The triangle

$$\begin{array}{ccc} \mathbb{D}^{op} & \xrightarrow{\phi_2^{op}} & K^{op} \\ & \searrow_{\mathcal{Z}_{\mathbb{D}}} & \downarrow{\scriptstyle \mathcal{Z}_K} \\ & & \mathbf{Grp} \end{array}$$

commutes up to isomorphism.

Proof: We define a natural isomorphism $\varepsilon : \mathcal{Z}_{\mathbb{D}} \Rightarrow \mathcal{Z}_K \circ \phi_2^{op}$ on components $\varepsilon_d : \mathcal{Z}_{\mathbb{D}}(d) \to \mathcal{Z}_K(d)$; we denote by $\pi^{\mathbb{D},d} : \mathbb{D}/d \to \mathbb{D}$ the canonical forgetful functor. Given an automorphism $\alpha : \pi^{\mathbb{D},d} \Rightarrow \pi^{\mathbb{D},d}$, we automatically have an automorphism $\alpha : \pi^{K,d} \Rightarrow \pi^{K,d}$ simply owing to the fact that $K(x,d) \subseteq \mathbb{D}(x,d)$. Clearly, α is an automorphism of $\pi^{K,d}$ and naturality just obtains from naturality of the given α. ∎

It is clear therefore that the isotropy rank of a morphism in $\mathbb{D}$ is equal to the isotropy rank when we consider that morphism as an arrow in K. But we wish to also have some control over the isotropy ranks of morphisms in K which come from $\mathbb{C}$. As might be expected, making this problem amenable requires that we restrict the kinds of profunctors, and thus collages, we work with.

Definition 3.2.2. Suppose we are given the following data:

- Two small categories $\mathbb{C}$ and $\mathbb{D}$.
- For each pair $\langle d \in \mathbb{D}, c \in \mathbb{C} \rangle$ of objects, a set $M(d,c)$.
- For each element $m \in M(d,c)$, a group homomorphism $\phi_m : \mathrm{Aut}_{\mathbb{C}}(c) \to \mathrm{Aut}_{\mathbb{D}}(d)$.

We generate a profunctor according to the following recipe:

- The **free profunctor** on M: Define $H' : \mathbb{D}^{op} \times \mathbb{C} \to \mathbf{Set}$ on objects by

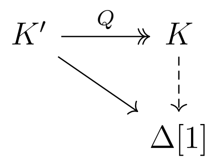
  $$H'(d,c) = \{(f,m,g) \mid \mathrm{dom}(f) = d, \mathrm{cod}(g) = c, m \in M(\mathrm{cod}(f), \mathrm{dom}(g))\}$$

  and on arrows $\langle x : d' \to d \in \mathbb{D}, y : c \to c' \in \mathbb{C} \rangle$ by

  $$(f,m,g) \mapsto (fx, m, yg).$$

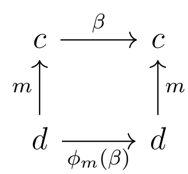
  Let the collage of H' be denoted by K'. Note that each $m \in M(d,c)$ can be considered to exist as the arrow $(\mathrm{id}, m, \mathrm{id}) : d \to c$ in K'.

- The **profunctor generated by data** of M and ϕ: Impose a congruence on the collage K' by declaring two morphisms $(f, m, g), (f', m', g') : d \to c$ equivalent if and only if $m = m'$ and there exists an automorphism $\beta : \mathrm{cod}(m) \to \mathrm{cod}(m)$ such that $f' = \phi_m(\beta) \circ f$ and $g = g' \circ \beta$. This gives a quotient functor $Q : K' \to K$; since all arrows of the form $(f, m, g) \in K'$ are mapped to $0 \to 1$ by the functor $K' \to \Delta[1]$, the universal property of Q induces a functor as shown



  The functor $K \to \Delta[1]$ corresponds to a unique profunctor $H : \mathbb{D}^{op} \times \mathbb{C} \to \mathbf{Set}$ and this is the profunctor we want.

A few remarks on the intuition behind this construction. We think of the set $M(d, c)$ as a set of "generating heteromorphisms" and the function ϕ_m is thought of as assigning lifts to automorphisms on c along m



One of the advantages of profunctors generated by data is that we have an explicit description of the isotropy elements in the collage.

Definition 3.2.3. Let $K \to \Delta[1]$ be the collage of a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ. Given an isotropy element $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$ of $\mathbb{C}$ and a family of isotropy elements $\langle \tau^{m,g} : \pi^{\mathbb{D},d} \Rightarrow \pi^{\mathbb{D},d} \mid [(\mathrm{id}, m, g)] : d \to c \rangle$ we say that σ is **ϕ-compatible** with $\langle \tau^{m,g} \rangle$ if $\tau^{m,g}_{\mathrm{id}:d \to d} = \phi_m(\sigma_g)$ whenever the notation makes sense.

We mention in passing that the definition of ϕ-compatibility can be expanded.

Lemma 3.2.4. *An isotropy element $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$ is ϕ-compatible with a family $\langle \tau^{m,g} : \pi^{\mathbb{D},d} \Rightarrow \pi^{\mathbb{D},d} \mid [(\mathrm{id}, m, g)] : d \to c \rangle$ of isotropy elements if and only if $\phi_m(\sigma_g)|_f = \tau^{m,g}_{f:d' \to d}$.*

Proof: One direction is trivial. For the other direction, observe that naturality of τ implies $\tau^{,gm}_{\mathrm{id}}|_f = \tau^{m,g}_f$ and so,

$$\phi_m(\sigma_g)|_f = \tau^{m,g}_{\mathrm{id}}|_f = \tau^{m,g}_f.$$

∎

Proposition 3.2.5. *Let $K \to \Delta[1]$ be the collage corresponding to a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ. The isotropy elements $\pi^{K,c} \Rightarrow \pi^{K,c}$ in K are precisely the ϕ-compatible families.*

Proof: First, suppose we are given an isotropy element $\alpha : \pi^{K,c} \Rightarrow \pi^{K,c}$; define $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$ simply as the image of α under the projection $\mathcal{Z}_K(c) \to \mathcal{Z}_{\mathbb{C}}(c)$ of isotropy groups onto the full subcategory $\mathbb{C}$ of K. Next, pick out those $d \in \mathbb{D}$ such that there exists a heteromorphism $[(\mathrm{id}, m, g)] : d \to c$ in K; define $\tau^{m,g}$ by setting

$$\tau^{m,g}_{f:d' \to d} = \alpha_{[(f,m,g)]}.$$

We check the naturality of $\tau^{m,g}$ as follows. Suppose we have a morphism

$$\begin{array}{ccc} d'' & \xrightarrow{h} & d' \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle f} & \\ & d & \end{array}$$

in the slice category $\mathbb{D}/d$ and we wish to show that the square

$$\begin{array}{ccc} d'' & \xrightarrow{\tau^{m,g}_{f'}} & d'' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ d' & \xrightarrow[\tau^{m,g}_{f}]{} & d' \end{array}$$

commutes. Observe that we also have the commutative triangle

$$\begin{array}{ccc} d'' & \xrightarrow{h} & d' \\ & {\scriptstyle [(f',m,g)]}\searrow \quad \swarrow{\scriptstyle [(f,m,g)]} & \\ & c & \end{array}$$

and naturality of α gives commutativity of the square

$$\begin{array}{ccc} d'' & \xrightarrow{\alpha_{[(f',m,g)]}} & d'' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ d' & \xrightarrow[\alpha_{[(f,m,g)]}]{} & d' \end{array}$$

Hence,

$$\tau^{m,g}_{f} \circ h = \alpha_{[(f,m,g)]} \circ h = h \circ \alpha_{[(f',m,g)]} = h\tau^{m,g}_{f'}.$$

As for ϕ-compatibility, consider the tower

$$\begin{array}{ccc} c & \xrightarrow{\alpha_{\mathrm{id}}} & c \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ c' & \xrightarrow{\alpha_g} & c' \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle m} \\ d & \xrightarrow{\phi_m(\alpha_g)} & d \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ d' & \xrightarrow{\alpha_{[(f,m,g)]}} & d' \end{array}$$

in K. The top square commutes by naturality of α as does the periphery of the rectangle consisting of the bottom two squares. Thus, since the middle square commutes by definition of arrows in K, the bottom square must commute, which implies

$$\phi_m(\sigma_g)f = \phi_m(\alpha_g)f = f\alpha_{[(f,m,g)]} = f\tau_f^{m,g}$$

and so, $\phi_m(\sigma_g)|_f = \tau_f^{m,g}$.

Conversely, suppose we start with a ϕ-compatible family $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$ and $\langle \tau^{m,g} : \pi^{\mathbb{D},d} \Rightarrow \pi^{\mathbb{D},d} \mid \exists[(\mathrm{id}, m, g)] : d \to c\rangle$. Define $\alpha : \pi^{K,c} \Rightarrow \pi^{K,c}$ on components by

$$\alpha_{g:c'\to c} = \sigma_g$$

and

$$\alpha_{[(f,m,g)]:d\to c} = \tau^{m,g}_{f:d\to d'} = \tau^{m,g}_{\mathrm{id}}|_f.$$

Note that naturality of α reduces to naturality of σ and the $\tau^{m,g}$ and that ϕ-compatibility implies

$$\alpha_{[(f,m,g)]:d\to c} = \phi_m(\sigma_g)|_f$$

. Let us check that the component $\alpha_{[(f,m,g)]}$ is well-defined; suppose we have $(f, m, g), (f', m', g') : d \to c$, with $f : d \to d'$, $m : d' \to c'$ and $g : c' \to c$ in K' such that $[(f, m, g)] = [(f', m', g')]$. Then $m = m'$ and there exists an automorphism $\beta : c' \to c'$ with $f' = \phi_m(\beta)f$ and $g = g'\beta$. Observe that $g = g'\beta$ means that β is a morphism

$$\begin{array}{ccc} c' & \xrightarrow{\beta} & c' \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle g'} & \\ & c & \end{array}$$

in the slice category $\mathbb{C}/c$ and we thus have a naturality square

$$\begin{array}{ccc} c' & \xrightarrow{\sigma_g} & c' \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ c' & \xrightarrow[\sigma_{g'}]{} & c' \end{array}$$

We deduce from this the identity

$$\beta^{-1}\sigma_{g'}\beta = \sigma_g$$

and applying the group homomorphism ϕ_m to both sides of the equation yields

$$\phi_m(\beta)^{-1}\phi_m(\sigma_{g'})\phi_m(\beta) = \phi_m(\sigma_g).$$

This last equation can be re-arranged to give the identity

$$\phi_m(\sigma_{g'})\phi_m(\beta) = \phi_m(\beta)\phi_m(\sigma_g),$$

which can be interpreted as the commutative square

$$\begin{array}{ccc} d' & \xrightarrow{\phi_m(\sigma_{g'})} & d' \\ {\scriptstyle \phi_m(\beta)}\uparrow & & \uparrow{\scriptstyle \phi_m(\beta)} \\ d' & \xrightarrow[\phi_m(\sigma_g)]{} & d' \end{array}$$

Hence,

$$\phi_m(\sigma_{g'})|_{\phi_m(\beta)} = \phi_m(\sigma_g)$$

and we have

$$\alpha_{[(f',m',g')]} = \phi_{m'}(\sigma_{g'})|_{f'} = \phi_m(\sigma_{g'})|_{\phi_m(\beta)f} = (\phi_m(\sigma_{g'})|_{\phi_m(\beta)})|_f = \phi_m(\sigma_g)|_f = \alpha_{[(f,m,g)]}.$$

We conclude that α is a well-defined automorphism of $\pi^{K,c}$. ∎

Note that in showing well-definedness, we have actually shown that for equivalent arrows (f,m,g), (f',m',g'), $\phi_m(\sigma_g)|_f = \tau_f^{m,g}$ if and only if $\phi_{m'}(\sigma_{g'}) = \tau_{f'}^{m',g'}$.

Before we proceed, we remark that a heteromorphism $d \to c$ in the collage K is precisely an arrow in the slice category $\mathbb{D}/c$ when we consider $\mathbb{D}$ as a subcategory of K and c as an object in K. There is thus a forgetful functor $\mathcal{U} : \mathbb{D}/c \to \mathbb{D}$ and this induces a functor $\mathcal{U}^{op} : (\mathbb{D}/c)^{op} \to \mathbb{D}^{op}$ on opposite categories. The latter can be post-composed with the isotropy functor $\mathcal{Z}_{\mathbb{D}} : \mathbb{D}^{op} \to \mathbf{Grp}$ to give a **Grp**-valued functor $\mathcal{Z}_{\mathbb{D}}\mathcal{U}^{op} : (\mathbb{D}/c)^{op} \to \mathbf{Grp}$. We shall denote by H_c the limit $\varprojlim \mathcal{Z}_{\mathbb{D}}\mathcal{U}^{op}$ in **Grp**. Moreover, we observe that the universal cone $\eta : \Delta H_c \Rightarrow \mathcal{Z}_{\mathbb{D}}\mathcal{U}^{op}$ comes equipped with components $\eta_{[(\mathrm{id},m,g)]:d\to c} : H_c \to \mathcal{Z}_{\mathbb{D}}(d)$ and hence, there is a group homomorphism

$$\mathcal{P}_c : H_c \to \prod_{[(\mathrm{id},m,g)]} \mathcal{Z}_{\mathbb{D}}(d)$$

induced by the universal property of the product in the codomain. Also, for each $d \in \mathbb{D}$, there is an arrow $\mathcal{Z}_{\mathbb{D}}(d) \to \mathrm{Aut}_{\mathbb{D}}(d)$ which takes an isotropy element τ :

$\pi^{\mathbb{D},d} \Rightarrow \pi^{\mathbb{D},d}$ and outputs the automorphism $\tau_{\mathrm{id}} : d \to d$. So, there is a further group homomorphism

$$\mathfrak{I}_c : \prod_{[(\mathrm{id},m,g)]} \mathcal{Z}_{\mathbb{D}}(d) \to \prod_{[(\mathrm{id},m,g)]} \mathrm{Aut}_{\mathbb{D}}(d).$$

Next, we have an arrow

$$\mathcal{Q}_c : \mathcal{Z}_{\mathbb{C}}(c) \to \prod_{g:c' \to c} \mathrm{Aut}_{\mathbb{C}}(c')$$

in **Grp** which splays an isotropy element $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$ into its components $\langle \sigma_g \mid g : c' \to c \rangle$. Also, for given $m : d \to c'$, applying ϕ_m to each component of $\langle \sigma_g \rangle$ gives a homomorphism

$$\mathrm{Aut}_{\mathbb{C}}(c') \to \prod_{[(\mathrm{id},m,g)]} \mathrm{Aut}_{\mathbb{D}}(d)$$

and this induces an arrow

$$\mathcal{J}_c : \prod_{g:c' \to c} \mathrm{Aut}_{\mathbb{C}}(c') \to \prod_{[(\mathrm{id},m,g)]} \mathrm{Aut}_{\mathbb{D}}(d).$$

Now, there is a actually a cone $\varepsilon : \Delta \mathcal{Z}_K(c) \Rightarrow \mathcal{Z}_{\mathbb{D}} \mathcal{U}^{op}$ given on components

$$\varepsilon_{[(f,m,g)]:d \to c} : \mathcal{Z}_K(c) \to \mathcal{Z}_{\mathbb{D}}(d)$$

by

$$\langle \sigma, \langle \tau^{n,h} \rangle \rangle \mapsto \tau^{m,g}$$

and the universal property of H_c induces an arrow $p_c : \mathcal{Z}_K(c) \to H_c$. Lastly, there is an obvious projection

$$q_c : \mathcal{Z}_K(c) \to \mathcal{Z}_{\mathbb{C}}(c)$$

defined by

$$\langle \sigma, \langle \tau^{n,h} \rangle \rangle \mapsto \sigma.$$

With all of this preliminary material in place, we can now state

Theorem 3.2.6. *Let $K \to \Delta[1]$ be the collage corresponding to a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ. For an object $c \in \mathbb{C}$, $\mathcal{Z}_K(c)$ is the pullback*

$$\begin{array}{ccccc}
\mathcal{Z}_K(c) & & \xrightarrow{\quad q_c \quad} & & \mathcal{Z}_{\mathbb{C}}(c) \\
 & & & & \big\downarrow \mathcal{Q}_c \\
\big\downarrow p_c & & & & \prod_{g:c' \to c} \mathrm{Aut}_{\mathbb{C}}(c') \\
 & & & & \big\downarrow \mathcal{J}_c \\
H_c & \xrightarrow[\mathcal{P}_c]{} & \prod_{[(\mathrm{id},m,g)]} \mathcal{Z}_{\mathbb{D}}(d) & \xrightarrow[\mathfrak{I}_c]{} & \prod_{[(\mathrm{id},m,g)]} \mathrm{Aut}_{\mathbb{D}}(d)
\end{array}$$

Proof: First, the square in the diagram of the theorem statement commutes since chasing elements gives

$$\begin{array}{ccccc} \langle \sigma, \tau^{n,h} \rangle & & \longmapsto & & \sigma \\ \downarrow & & & & \downarrow \\ & & & & \langle \sigma_g \rangle \\ & & & & \downarrow \\ \langle \tau^{m,g} \rangle & \longmapsto & \langle \tau^{m,g}_{\mathrm{id}} \rangle & \longmapsto & \langle \tau^{m,g}_{\mathrm{id}} \rangle = \langle \phi_m(\sigma_g) \rangle \end{array}$$

and the equation in the bottom-right corner is nothing but ϕ-compatibility. Suppose then that we are given a group G together with group homomorphisms $\rho : G \to H_c$, $\chi : G \to \mathcal{Z}_{\mathbb{C}}(c)$ satisfying

$$\mathfrak{I}_c \circ \mathcal{P}_c \circ \rho = \mathcal{J}_c \circ \mathcal{Q}_c \circ \chi.$$

Given an element $a \in G$, we claim that the pair $\langle \chi(a), \mathcal{P}_c\rho(a) \rangle$ is a ϕ-compatible pair of isotropy elements. This is easy to see since chasing a around the diagram yields the equation

$$\langle (\widehat{\rho}_{[(\mathrm{id},m,g)]})_{\mathrm{id}} \rangle = \langle \phi_m(\chi(a)_g) \rangle,$$

where $\widehat{\rho}$ is the unique cone $\Delta G \Rightarrow \mathcal{Z}_{\mathbb{D}}\mathcal{U}^{op}$ corresponding to ρ. It is clear that we can uniquely define a group homomorphism $\langle \rho, \chi \rangle : G \to \mathcal{Z}_K(c)$ by $a \mapsto \langle \chi(a), \mathcal{P}_c\rho(a) \rangle$ which fulfils the necessary commutativity conditions. ∎

Hence, isotropy in the collage of a profunctor generated by data is computed via pullback involving the isotropy groups of the categories used to generate the collage. This is similar in spirit to the way the fundamental group of a space created by joining together two smaller spaces can be computed by a pushout involving the fundamental groups of the smaller constituent spaces. We may thus consider the above result a sort of Seifert-van Kampen Theorem for isotropy groups of toposes.

Collages generated by data also have the pleasant property that isotropy quotients of such collages are again collages generated by data. Before stating this fact as a proposition, we remark that while the full isotropy congruence on $\mathbb{C}$ is induced by the inlcusion $\mathcal{Z}_{\mathbb{C}}(c) \to \mathrm{Aut}_{\mathbb{C}}(c)$, when $\mathbb{C}$ is considered as a subcategory of the collage K, there is an intermediate congruence induced by $\mathcal{Z}_K(c) \to \mathcal{Z}_{\mathbb{C}}(c) \to \mathrm{Aut}_{\mathbb{C}}(c)$ where the first map is projection onto isotropy group of a full subcategory.

Proposition 3.2.7. *Let K be the collage corresponding to a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ. Denote by $\mathbb{C}/\!\sim$ the congruence induced on $\mathbb{C}$ by trivialising ϕ-compatible automorphisms and let $\phi/\mathcal{I}$ be the collection of group homomorphisms $(\phi/\mathcal{I})_m : \mathrm{Aut}_{\mathbb{C}/\sim}(c) \to \mathrm{Aut}_{\mathbb{D}/\mathcal{I}}(d)$ defined by $[\beta] \mapsto [\phi_m(\beta)]$. Then $K/\mathcal{I}$ is the collage corresponding to the profunctor generated by data of $\mathbb{C}/\!\sim$, $\mathbb{D}/\mathcal{I}$, M and $\phi/\mathcal{I}$.*

Proof: Note that functor $K \to \Delta[1]$ gives rise to an obvious functor $K/\mathcal{I} \to \Delta[1]$ via the universal property of the quotient

$$\begin{array}{ccc} K & \twoheadrightarrow & K/\mathcal{I} \\ & \searrow & \downarrow \\ & & \Delta[1] \end{array}$$

Observe that, for generating heteromorphisms $m, m' : d \to c$ in K, $(\mathrm{id}, m, \mathrm{id}) \sim (\mathrm{id}, m', \mathrm{id})$ precisely when $m = m'$ so that the data of M in the isotropy quotient $K/\mathcal{I}$ persists without modification. Let us recall the span of functors

$$\mathbb{C} \xrightarrow{\ i\ } K \xleftarrow{\ j\ } \mathbb{D}$$

with the collage at the apex. The Seifert-van Kampen Theorem for isotropy groups tells us that the automorphisms in K which are killed off by taking the isotropy quotient $K/\mathcal{I}$ are precisely either the isotropy elements in $\mathbb{D}$ or isotropy elements in $\mathbb{C}$ which can be made ϕ-compatible. Hence, the isotropy-preserving cosieve inclusion $j : \mathbb{D} \to K$ induces an isotropy-preserving cosieve inclusion $j/\mathcal{I} : \mathbb{D}/\mathcal{I} \to K/\mathcal{I}$ and a sieve inclusion $i/\sim: \mathbb{C}/\sim \to K/\mathcal{I}$. Obviously, $j/\mathcal{I}$ and $i/\sim$ are jointly bijective on objects of K since taking congruences do not affect objects in a category. Finally, the definition of $\phi/\mathcal{I}$ makes it clear that the necessary commutation properties are satisfied. ∎

The following diagram, where the left horizontal arrows are sieve inclusions, the right horizontal arrows are cosieve inclusions and the vertical arrows are quotient functors, exhibits the persistence of "generated by data" as a property of collages

$$\begin{array}{ccccc} \mathbb{C} & \longrightarrow & K & \longleftarrow & \mathbb{D} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}/\sim & \longrightarrow & K/\mathcal{I} & \longleftarrow & \mathbb{D}/\mathcal{I} \end{array}$$

Note that the quotient $\mathbb{C}/\sim$ is an intermediate quotient in the sense that we have a commutative triangle

$$\begin{array}{ccc} \mathbb{C} & \twoheadrightarrow & \mathbb{C}/\sim \\ & \searrow & \downarrow \\ & & \mathbb{C}/\mathcal{I} \end{array}$$

where the diagonal arrow is the isotropy quotient functor and the vertical arrow is induced by the universal property of the quotient $\mathbb{C}/\sim$. It is clear that any finite isotropy quotient of a collage generated by data is also a collage generated by data. However, "generated by data" as a property actually persists across all isotropy quotients. Note that for any ordinal λ, there are colimit-preserving functors $\mathbb{D}/\mathcal{I}^{(-)} : \lambda \to \mathbf{Cat}$, $K/\mathcal{I}^{(-)} : \lambda \to \mathbf{Cat}$ and $\mathbb{C}/\sim^{(-)}: \lambda \to \mathbf{Cat}$ corresponding to each of the columns of the tower

$$\begin{array}{ccccc}
\mathbb{D} & \longrightarrow & K & \longleftarrow & \mathbb{C} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{D}/\mathcal{I} & \longrightarrow & K/\mathcal{I} & \longleftarrow & \mathbb{C}/\sim \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{D}/\mathcal{I}^2 & \longrightarrow & K/\mathcal{I}^2 & \longleftarrow & \mathbb{C}/\sim^2 \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{D}/\mathcal{I}^\lambda & \longrightarrow & K/\mathcal{I}^\lambda & \longleftarrow & \mathbb{C}/\sim^\lambda
\end{array}$$

where the quotients $\mathbb{C}/\sim^\mu$ are generated iteratively and $\mathbb{C}/\sim^\mu$ is defined to be the colimit $\varinjlim_{\nu<\mu} \mathbb{C}/\sim^\nu$ in case μ is a limit ordinal. Furthermore, each ϕ_m induces a tower

$$\begin{array}{ccc}
\mathrm{Aut}_{\mathbb{C}}(c) & \xrightarrow{\phi_m} & \mathrm{Aut}_{\mathbb{D}}(d) \\
\downarrow & & \downarrow \\
\mathrm{Aut}_{\mathbb{C}/\sim}(c) & \xrightarrow{\phi_m/\mathcal{I}} & \mathrm{Aut}_{\mathbb{D}/\mathcal{I}}(d) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{Aut}_{\mathbb{C}/\sim^\lambda}(c) & \xrightarrow{\phi_m/\mathcal{I}^\lambda} & \mathrm{Aut}_{\mathbb{D}/\mathcal{I}^\lambda}(d)
\end{array}$$

in **Grp** where, for limit ordinal μ, $\mathrm{Aut}_{\mathbb{C}/\sim^\mu}(c)$, $\mathrm{Aut}_{\mathbb{D}/\mathcal{I}^\mu}(d)$ and $\phi_m/\mathcal{I}^\mu$ are defined using colimits again.

Proposition 3.2.8. *Let K be the collage corresponding to a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ and λ an ordinal. Given any ordinal λ, $K/\mathcal{I}^\lambda$ is the collage generated by data of $\mathbb{C}/\sim_\lambda$, $\mathbb{D}/\mathcal{I}^\lambda$, M and $\phi/\mathcal{I}^\lambda$.*

Proof: By transfinite induction.

- **Base case**: This is trivial just by definition of K as the collage generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ.

- **Successor case**: Assume that $\lambda = \mu+1$ for some ordinal μ and that $K/\mathcal{I}^\mu$ is the collage generated by data of $\mathbb{C}/\sim^\mu$, $\mathbb{D}/\mathcal{I}^\mu$, M and $\phi/\mathcal{I}^\mu$. Then, by proposition 3.2.7, $(K/\mathcal{I}^\mu)/\mathcal{I} = K/\mathcal{I}^\lambda$ is the collage generated by data of $(\mathbb{C}/\sim^\mu)/\sim = \mathbb{C}/\sim^\lambda$, $(\mathbb{D}/\mathcal{I}^\mu)/\mathcal{I} = \mathbb{D}/\mathcal{I}^\lambda$, M and $(\phi/\mathcal{I}^\mu)/\mathcal{I} = \phi/\mathcal{I}^\lambda$.

- **Limit case**: Assume that λ is a limit ordinal and that for all $\mu < \lambda$, $K/\mathcal{I}^\mu$ is the collage generated by data of $\mathbb{C}/\sim^\mu$, $\mathbb{D}/\mathcal{I}^\mu$, M and $\phi/\mathcal{I}^\mu$. Let $\Gamma : \mathbb{C}/\sim^{(-)} \Rightarrow \Delta\mathbb{C}/\sim^\lambda$ be the universal cocone for $\mathbb{C}/\sim^{(-)}$, $\Theta : \mathbb{D}/\mathcal{I}^{(-)} \Rightarrow \Delta\mathbb{D}/\mathcal{I}^\lambda$ the universal cocone for $\mathbb{D}/\mathcal{I}^{(-)}$ and $\Lambda : K/\mathcal{I}^{(-)} \Rightarrow \Delta K/\mathcal{I}^\lambda$ the universal cocone for $K/\mathcal{I}^\lambda$. Observe that for each $\mu < \lambda$, we have an induced cosieve inclusion $i/\sim^\mu$: $\mathbb{C}/\sim^\mu \to K/\mathcal{I}^\mu$ and an induced sieve inclusion $j/\mathcal{I}^\mu : \mathbb{D}/\mathcal{I}^\mu \to K/\mathcal{I}^\mu$. Thus we can define cocones $\widehat{\Gamma} : \mathbb{C}/\sim^{(-)} \Rightarrow \Delta(K/\mathcal{I}^\lambda)$ and $\widehat{\Theta} : \mathbb{D}/\mathcal{I}^{(-)} \Rightarrow \Delta(k/\mathcal{I}^\lambda)$ respectively on components by $\widehat{\Gamma}_\mu := \Lambda_\mu \circ (i/\mathcal{I}^\mu)$ and $\widehat{\Theta}_\mu := \Lambda_\mu \circ (j/\mathcal{I}^\mu)$. This then induces functors $i/\mathcal{I}^\lambda : \mathbb{C}/\sim^\lambda \to K/\mathcal{I}^\lambda$ and $j/\mathcal{I}^\lambda : \mathbb{D}/\mathcal{I}^\lambda \to K/\mathcal{I}^\lambda$. Note also that each category $\mathbb{C}/\sim^\mu$ has the same object set as $\mathbb{C}$ and hence, the category $\mathbb{C}/\sim^\lambda$ will retain exactly the objects of $\mathbb{C}$; analogous statements apply to $\mathbb{D}/\mathcal{I}^\lambda$ and $K/\mathcal{I}^\lambda$. Given that we have a commutative square

$$\begin{array}{ccc} \mathbb{C}/\sim^\mu & \xrightarrow{i/\mathcal{I}^\mu} & K/\mathcal{I}^\mu \\ {\scriptstyle \Gamma_\mu}\big\downarrow & & \big\downarrow{\scriptstyle \Lambda_\mu} \\ \mathbb{C}/\sim^\lambda & \xrightarrow[i/\mathcal{I}^\lambda]{} & K/\mathcal{I}^\lambda \end{array}$$

  where the vertical arrows are identities on objects and the top horizontal arrow is a sieve inclusion, the bottom horizontal arrow is necessarily a sieve inclusion. Similarly, we can deduce that $j/\mathcal{I}^\lambda$ is a cosieve inclusion and $K/\mathcal{I}^\lambda$ is therefore a collage of the categories $\mathbb{C}/\sim^\lambda$ and $\mathbb{D}/\mathcal{I}^\lambda$. Let us now reason about heteromorphisms $d \to c$ in $K/\mathcal{I}^\lambda$; given that $K/\mathcal{I}^\lambda$ is the apex of a sequential colimit, for a morphism $h : d \to c$ in the category, there is a least $\mu < \lambda$ such that $h = \Lambda_\mu(\widehat{h})$. Since $\widehat{h}$ is an arrow in the μ^{th} isotropy quotient of K, there must have been an original heteromorphism $[(f, m, g)] : d \to c$ which descends to $\widehat{h}$. Hence, $h = \Lambda_0([(f, m, g)])$ where $f : d \to d'$ is a morphism in $\mathbb{D}$, $g : c' \to c$ is a morphism in $\mathbb{C}$ and $m \in M(d, c)$. Recall that composition of arrows in K is inherited from composition in the free profunctor generated by M and by definition of the free profunctor, we can decompose the arrow (f, m, g) in the free collage as $g \circ (\mathrm{id}, m, \mathrm{id}) \circ f$. Therefore, $[(f, m, g)] = [g] \circ ([\mathrm{id}, m, \mathrm{id}]) \circ [f]$ and we obtain the equation

$$\Lambda_0([(f, m, g)]) = \Lambda_0([g]) \circ \Lambda_0([(\mathrm{id}, m, \mathrm{id})]) \circ \Lambda_0([f]).$$

But we also have the commutative squares

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{i} & K \\ {\scriptstyle \Gamma_0}\downarrow & & \downarrow{\scriptstyle \Lambda_0} \\ \mathbb{C}/\sim^\lambda & \xrightarrow[i/\mathcal{I}^\lambda]{} & K/\mathcal{I}^\lambda \end{array} \qquad \begin{array}{ccc} \mathbb{D} & \xrightarrow{j} & K \\ {\scriptstyle \Theta_0}\downarrow & & \downarrow{\scriptstyle \Lambda_0} \\ \mathbb{D}/\mathcal{I}^\lambda & \xrightarrow[j/\mathcal{I}^\lambda]{} & K/\mathcal{I}^\lambda \end{array}$$

where all horizontal arrows are subcategory inclusions. So, we have

$$\Lambda_0([(f, m, g)]) = \Gamma_0(g) \circ \Lambda_0([(\mathrm{id}, m, \mathrm{id})]) \circ \Theta_0(f).$$

We conclude from this that a heteromorphism $d \to c$ in $K/\mathcal{I}^\lambda$ is one of the form $G \circ \Lambda_0([(\mathrm{id}, m, \mathrm{id})]) \circ F$ for some $F : c' \to c$in $\mathbb{C}/\sim^\lambda$ and $G : d \to d'$ in $\mathbb{D}/\mathcal{I}^\lambda$. Since the data of M remains unchanged across all isotropy quotients of K, this is enough to deduce that $K/\mathcal{I}^\lambda$ is a quotient of the free collage generated by $\mathbb{C}/\sim^\lambda$, $\mathbb{D}/\mathcal{I}^\lambda$ and M. To see that $K/\mathcal{I}^\lambda$ is in fact the quotient induced by $\phi/\mathcal{I}^\lambda$, we need only remember the commutative squares (one for each $m \in M(d, c)$)

$$\begin{array}{ccc} \mathrm{Aut}_{\mathbb{C}}(c) & \xrightarrow{\phi_m} & \mathrm{Aut}_{\mathbb{D}}(d) \\ {\scriptstyle \Gamma_0(-)}\downarrow & & \downarrow{\scriptstyle \Theta_0(-)} \\ \mathrm{Aut}_{\mathbb{C}/\sim^\lambda}(c) & \xrightarrow[\phi_m/\mathcal{I}^\lambda]{} & \mathrm{Aut}_{\mathbb{D}/\mathcal{I}^\lambda}(d) \end{array}$$

∎

Lastly, if we take a collage generated by data, restricting to a full subcategory in the domain or codomain of the profunctor $\mathbb{C} \nrightarrow \mathbb{D}$ corresponding to it gives a collage generated by data again.

Proposition 3.2.9. *Let K be the collage corresponding to a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ. Suppose $\widetilde{\mathbb{C}}$ is a full subcategory of $\mathbb{C}$ and $\widetilde{\mathbb{D}}$ is a full subcategory of $\mathbb{D}$. Then we obtain a collage $\widetilde{K}$ generated by data of $\widetilde{\mathbb{C}}$, $\widetilde{\mathbb{D}}$, $\widetilde{M}$ and $\widetilde{\phi}$, where $\widetilde{M}$ is the subset*

$$\{M(d, c) \mid d \in \widetilde{\mathbb{D}}, c \in \widetilde{\mathbb{C}}\} \subseteq \{M(d, c) \mid d \in \mathbb{D}, c \in \mathbb{C}\}$$

and $\widetilde{\phi}$ gives, for each $\langle d, c\rangle \in \widetilde{\mathbb{D}}^{op} \times \widetilde{\mathbb{C}}$, the original collection of functions $\{\phi_m \mid m \in M(d, c)\}$ but ignores any collection $\{\phi_m \mid m \in M(d, c)\}$ if $\langle d, c\rangle \notin \widetilde{\mathbb{D}}^{op} \times \widetilde{\mathbb{C}}$.

Proof: The free collage $\widetilde{K'}$ generated on $\widetilde{\mathbb{C}}$, $\widetilde{\mathbb{D}}$ and $\widetilde{M}$ is obviously a full subcategory of the free collage generated on $\mathbb{C}$, $\mathbb{D}$ and M and the way we have defined $\widetilde{\phi}$ ensures that $\widetilde{K}$ is nothing but the image of $\widetilde{K'}$ under the composite

$$\begin{array}{ccccc} \widetilde{K'} & \hookrightarrow & K' & \overset{Q}{\twoheadrightarrow} & K \\ & \searrow & & & \downarrow \\ & & & & \Delta[1] \end{array}$$

∎

Hence, we have discovered a class of small categories which are closed under taking full subcategories and arbitrary isotropy quotients and which obey the Seifert-van Kampen Theorem for isotropy groups (in the sense that the presheaf toposes generated by these categories have isotropy groups which are computed via the pullback recipe).

3.3 Isotropy Ranks in a Collage

The Seifert-van Kampen Theorem we proved gives us a macro level grasp of isotropy in a collage. What we desire now is a more granular approach since, at the end, we wish to be able to compute isotropy ranks of actual automorphisms in the collage. The main obstacle in this task is that, in general, it is difficult to give precise statements about the intermediate quotients $\mathbb{C}/\sim$. Achieving a granular level of control on isotropy ranks is greatly simplified if we know that either $\mathbb{C}/\sim=\mathbb{C}$ or $\mathbb{C}/\sim=\mathbb{C}/\mathcal{I}$. This motivates finding sufficient conditions under which these equalities obtain.

Lemma 3.3.1. *Let K be the collage of a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ. Suppose that for each isotropy element $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$, there exists a heteromorphism $[(\mathrm{id}, m, g)] : d \to c$ such that*

$$||\phi_m(\sigma_g)||_{\mathbb{D}} > 1.$$

Then $\mathbb{C}/\sim=\mathbb{C}$.

Proof: If we take a heteromorphism $[(\mathrm{id}, m, g)] : d \to c$ and an isotropy element $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$, it is impossible to make σ ϕ-compatible since such compatibility requires $\phi_m(\sigma_g)$ to coincide with some isotropy element in $\mathbb{D}$ and our hypothesis on the isotropy rank of lifts ensures that such a condition fails. Hence, there are no ϕ-compatible pairs in the collage K and the congruence induced on $\mathbb{C}$ is necessarily trivial. ∎

Lemma 3.3.2. *Let K be the collage of a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ. Suppose that for each $c \in \mathbb{C}$ which is the target of a heteromorphism and each isotropy element $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$, all heteromorphisms $[(\mathrm{id}, m, g)] : d \to c$ satisfy*

$$0 \leq ||\phi_m(\sigma_g)||_{\mathbb{D}} \leq 1.$$

Then $\mathbb{C}/\sim = \mathbb{C}/\mathcal{I}$.

Proof: In this case, *all* isotropy elements in $\mathbb{C}$ are isotropy elements in K. This is so since the hypothesis on the isotropy ranks of the lifts $\phi_m(\sigma_g)$ implies that each σ_g lifts to either an identity morphism or a morphism which is isotropy in $\mathbb{D}$. Hence, the automorphisms of $\mathbb{C}$ which get killed in $K/\mathcal{I}$ are precisely those which get killed in $\mathbb{C}/\mathcal{I}$. ∎

Now, the last two results we proved are still "macro" in the sense that they tell us something about isotropy quotients (or congruences) of categories as a whole and do not immediately yield results on isotropy ranks. However, imposing sufficiently generous conditions on the collage provides a way to bridge the gap. Before proceeding, it will be helpful to note that for a collage K generated by data, we can associate a family of functions

$$\{\mathcal{S}_c^K : \mathrm{Aut}_{\mathbb{C}}(c) \to ||\mathbb{D}||_{\mathcal{I}} + ||\mathbb{C}||_{\mathcal{I}} + 1 \mid c \in \mathbb{C} \text{ such that } \exists d \in \mathbb{D} \text{ with } M(d,c) \neq \emptyset\},$$

where $||\mathbb{D}||_I + ||\mathbb{C}||_I + 1$ is the set of all ordinals up to and including $||\mathbb{D}||_I + ||\mathbb{C}||_{\mathcal{I}}$, defined by

$$\mathcal{S}_c^K(\beta) = \bigvee_d \{||\phi_m(\beta)||_{\mathbb{D}} \mid m : d \to c\}.$$

That is, for a $c \in \mathbb{C}$ which is the target of a heteromorphism in K and an automorphism $\beta : c \to c$, we look at the commutative squares

$$\begin{array}{ccc} c & \xrightarrow{\beta} & c \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle m} \\ d & \xrightarrow[\phi_m(\beta)]{} & d \end{array}$$

and record in $\mathcal{S}_c^K(\beta)$ the highest isotropy rank obtained in $\mathbb{D}$ by the lifts $\phi_m(\beta)$ across all generating heteromorphisms m.

Definition 3.3.3. Let I be an index set, X and Y be sets and $\{f_i : X \to Y \mid i \in I\}$ a family of functions with common domain and codomain. We say that the family $\{f_i \mid i \in I\}$ is **jointly injective** if given any distinct pair of elements $x, x' \in X$, there exists an $i \in I$ such that $f_i(x) \neq f_i(x')$.

Lemma 3.3.4. *Let K be the collage corresponding to a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ. In addition, suppose the following conditions hold.*

1. *Every automorphism in $\mathbb{D}$ has non-negative isotropy rank.*
2. *Every object in $\mathbb{C}$ becomes the target of some heteromorphism in K.*
3. *There is a unique $c_0 \in \mathbb{C}$ such that $M(d, c_0) \neq \emptyset$ for any $d \in \mathbb{D}$.*
4. *The family $\{\phi_m \mid m \in M(d, c_0)\}$ of group homomorphisms is jointly injective for each $d \in \mathbb{D}$.*
5. *For any two non-identity automorphisms $\beta, \beta' : c_0 \to c_0$, $\mathcal{S}^K_{c_0}(\beta) = \mathcal{S}^K_{c_0}(\beta')$.*
6. *For each $g : c_0 \to c$ in $\mathbb{C}$ and each non-trivial isotropy element $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$, σ_g is a non-identity automorphism.*

Denote by S the ordinal $\mathcal{S}^K_{c_0}(\beta)$ for a non-trivial automorphism $\beta : c_0 \to c_0$. Given any isotropy element $\varsigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$, we have

$$||\varsigma||_K = \begin{cases} S & \text{if } S \text{ is a successor ordinal,} \\ S+1 & \text{if } S \text{ is a limit ordinal.} \end{cases}$$

Proof: Observe that since all automorphisms in $\mathbb{D}$ have non-negative isotropy rank, S is necessarily non-negative. Moreover, the second and third hypotheses together imply that every heteromorphism in K factors through c_0, i.e., every heteromorphism is of the form $[(\mathrm{id}, m, g)]$ for some $m : d_0 \to c_0$. Next, if $\beta : c_0 \to c_0$ is a non-identity automorphism, joint injectivity of $\{\phi_m \min M(d_0, c_0)\}$ gives existence of an $m \in M(d_0, c_0)$ such that $\phi_m(\beta)$ is a non-identity automorphism in $\mathbb{D}$. We now proceed by transfinite induction on S.

- **Base case**: The base case occurs for $S = 1$. This implies that for each $c \in \mathbb{C}$ which is the target of a heteromorphism and each isotropy element $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$, all heteromorphisms $[(\mathrm{id}, m, g)] : d \to c$ satisfy

$$0 \leq ||\phi_m(\sigma_g)||_{\mathbb{D}} \leq 1$$

  and applying lemma 3.3.2 tells us that all isotropy elements in $\mathbb{C}$ become trivialised in $K/\mathcal{I}$. Hence, if $\varsigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$ is a non-trivial isotropy element, then ς_{id} has isotropy rank 1 in K and we obtain $||\varsigma_{\mathrm{id}}||_K = 1 = S$.

- **Successor case**: Suppose that S is a successor ordinal with predecessor P, i.e., $S = P + 1$ for an ordinal $P > 0$. We show that for each isotropy element $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$, there exists a heteromorphism $[(\mathrm{id}, m, g)] : d \to c$ such that $||\phi_m(\sigma_g)||_{\mathbb{D}} > 1$. Firstly, there exists a a heteromorphism $[(\mathrm{id}, m, g)] : d \to c$

by the second hypothesis; secondly, the third hypothesis allows us to assume that $m : d_0 \to c_0$ and that σ_g is not the identity. Using the joint injectivity of the fourth assumption gives existence of an $n \in M(d_0, c_0)$ such that $\phi_n(\sigma_g)$ is not the identity and so, $\mathcal{S}^K_{c_0}(\sigma_g) \geq 1$. But the fifth assumption then says that $\mathcal{S}^K_{c_0}(\sigma_g) = S > 1$. As S is a successor ordinal, there must be an actual automorphism of the form $\phi_k(\sigma_h)$ with $||\phi_k(\sigma_h)||_{\mathbb{D}} = S > 1$ as required. Since the argument just carried out holds for an arbitrary $\sigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$, we may now apply lemma 3.3.1 together with proposition 3.2.7 deduce that $K/\mathcal{I}$ is the collage generated by data of $\mathbb{C}$, $\mathbb{D}/\mathcal{I}$, M and $\phi/\mathcal{I}$. As no automorphism in a category can actually obtain an isotropy rank equal to a limit ordinal and isotropy ranks of individual automorphisms decrease by 1 in the isotropy quotient, we may deduce that, for a fixed isotropy element $\varsigma : \pi^{\mathbb{C},c} \Rightarrow \pi^{\mathbb{C},c}$, $\mathcal{S}^{K/\mathcal{I}}_{c_0}(\varsigma_{\text{id}}) = P$. By the inductive hypothesis, we obtain $||\varsigma_{\text{id}}||_{K/\mathcal{I}} = P$ and hence $||\varsigma_{\text{id}}||_K = P + 1 = S$.

- **Limit case**: Assume now that S is a limit ordinal. Then $S \geq \omega > 0$ and we can use exactly the same argument as in the successor case to establish that $\mathbb{C}/\sim$ is equal to $\mathbb{C}$. Now apply the inductive hypothesis for each ordinal $\mu < S$ to deduce that $||\varsigma||_{K/\mathcal{I}^\mu} > 1$. Hence, S is a lower bound for $||\varsigma||_K$ and as an automorphism cannot have limit ordinal isotropy rank, we in fact have the sharper lower bound $S + 1 \leq ||\varsigma||_K$. Lastly, note that $||\varsigma||_K$ cannot exceed $S + 1$ since, by definition of S, there can be no automorphisms in $\mathbb{D}$ of the form $\phi_m(\varsigma_g)$ satisfying $||\phi_m(\varsigma_g)||_{\mathbb{D}} > 1$ and so, $||\varsigma_{\text{id}}||_K$ is exactly $S + 1$ in this case.

∎

On inspecting the statement of the preceding lemma, it may seem that the hypotheses imposed on the collage are too artificial, too restrictive or both. To this somewhat valid point, we remark that it is surprisingly difficult to establish even a lower bound for isotropy ranks for an arbitrary collage generated by data and that the main impetus underlying the result on isotropy ranks we *did* prove is a construction carried out in the sequel.

Proposition 3.3.5. *Let K be the collage corresponding to a profunctor generated by data of $\mathbb{C}$, $\mathbb{D}$, M and ϕ and suppose that all the hypotheses of lemma 3.3.4 are satisfied. In addition, assume that $S = ||\mathbb{D}||_{\mathcal{I}}$. Given an object $c \in \mathbb{C}$ and a non-identity automorphism $\beta : c \to c$ with non-negative isotropy rank, we have*

$$||\beta||_K = \begin{cases} ||\mathbb{D}||_{\mathcal{I}} + ||\beta||_{\mathbb{C}} - 1 & \text{if } ||\mathbb{D}||_{\mathcal{I}} \text{ is a successor ordinal,} \\ ||\mathbb{D}||_{\mathcal{I}} + ||\beta||_{\mathbb{C}} & \text{if } ||\mathbb{D}||_{\mathcal{I}} \text{ is a limit ordinal.} \end{cases}$$

Proof: By lemma 3.3.4, the isotropy automorphisms ς_{id} in $\mathbb{C}$ satisfy the formula

$$||\varsigma_{\mathrm{id}}||_K = \begin{cases} ||\mathbb{D}||_{\mathcal{I}} & \text{if } ||\mathbb{D}||_{\mathcal{I}} \text{ is a successor ordinal,} \\ ||\mathbb{D}||_{\mathcal{I}} + 1 & \text{if } ||\mathbb{D}||_{\mathcal{I}} \text{ is a limit ordinal.} \end{cases}$$

Therefore, the tower of isotropy quotients of K has an initial segment which looks like

$$\begin{array}{ccccc} \mathbb{D} & \longrightarrow & K & \longleftarrow & \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}/\mathcal{I} & \longrightarrow & K/\mathcal{I} & \longleftarrow & \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}/\mathcal{I}^2 & \longrightarrow & K/\mathcal{I}^2 & \longleftarrow & \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}/\mathcal{I}^{||\mathbb{D}||_{\mathcal{I}}} & \longrightarrow & K/\mathcal{I}^{||\mathbb{D}||_{\mathcal{I}}} & \longleftarrow & \mathbb{C}/\mathcal{I} \end{array}$$

if $||\mathbb{D}||_{\mathcal{I}}$ is a successor ordinal and is of the form

$$\begin{array}{ccccc} \mathbb{D} & \longrightarrow & K & \longleftarrow & \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}/\mathcal{I} & \longrightarrow & K/\mathcal{I} & \longleftarrow & \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}/\mathcal{I}^2 & \longrightarrow & K/\mathcal{I}^2 & \longleftarrow & \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}/\mathcal{I}^{||\mathbb{D}||_{\mathcal{I}}} & \longrightarrow & K/\mathcal{I}^{||\mathbb{D}||_{\mathcal{I}}} & \longleftarrow & \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}/\mathcal{I}^{||\mathbb{D}||_{\mathcal{I}}+1} & \longrightarrow & K/\mathcal{I}^{||\mathbb{D}||_{\mathcal{I}}+1} & \longleftarrow & \mathbb{C}/\mathcal{I} \end{array}$$

when $||\mathbb{D}||_{\mathcal{I}}$ is a limit ordinal. In either case, since every automorphism in $\mathbb{D}$ has non-negative isotropy rank, the chain of isotropy quotients for $\mathbb{D}$ has stabilised and extending the tower downwards only modifies the rightmost column of the tower. Moreover, there can be no non-identity automorphisms of the form $\phi/\mathcal{I}^{\lambda}(\varsigma_g)$ in K beyond this point and the intermediate congruences induced on $\mathbb{C}$ are therefore the iterated isotropy quotients. The desired result now obtains easily by a sufficient downward extension of the tower(s) portrayed above. ∎

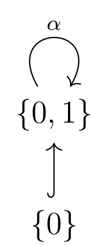
Chapter 4

Models of Higher-Order Isotropy

This last short chapter is devoted to an application of the theory we have built up thus far. Recall that one of our original motivations, as explicated in the introductory chapter, is to build, for an arbitrary ordinal λ, a category $\mathbb{X}[\lambda]$ such that $||\mathbb{X}[\lambda]||_{\mathcal{I}} = \lambda$. We remind the reader also of the rough analogy with Eilenberg-Moore spaces in homology and Eilenberg-MacLane spaces in homotopy which are built via inductive recipes to have (reduced) homology or homotopy group equal to the coefficient group in dimension n and trivial groups in all other dimensions.

4.1 Construction of Models

It is rather easy to give examples of categories with isotropy rank $-\infty$, 0 and 1. We let $\mathbb{X}[-\infty]$ be the category



where α is the unique permutation on the set $\{0, 1\}$ and there are no other non-identity arrows. The vertical arrow is just subset inclusion and α cannot lift to the identity along this arrow. Therefore, the unique non-identity automorphism in the category fails to be an isotropy element and all iterated isotropy congruences are trivial. We shall now employ the notation from the section on higher-order isotropy in the first chapter. Define $\mathbb{X}[0]$ to be the category with one object C and no non-identity morphisms and let $\mathbb{X}[1]$ be the category with unique object C and generating automorphisms γ_1 and γ_2 so that $\mathrm{Aut}_{\mathbb{X}[1]}(C) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Now, we already constructed the category $\mathbb{X}[2]$ when we gave an example of a category with second-order isotropy.

We will not repeat here the details of creating $\mathbb{X}[2]$ as a subcategory of **Set** and we only recall the picture

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ B & \xrightarrow{\beta} & B \\ {\scriptstyle g}\uparrow\uparrow{\scriptstyle h} & & {\scriptstyle g}\uparrow\uparrow{\scriptstyle h} \\ C & \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} & C \end{array}$$

where the horizontal arrows are generating idempotent automorphisms. For later use, we will also carve out the subcategory $\mathbb{X}[1\frac{1}{2}]$

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ B & \xrightarrow[\beta]{} & B \end{array}$$

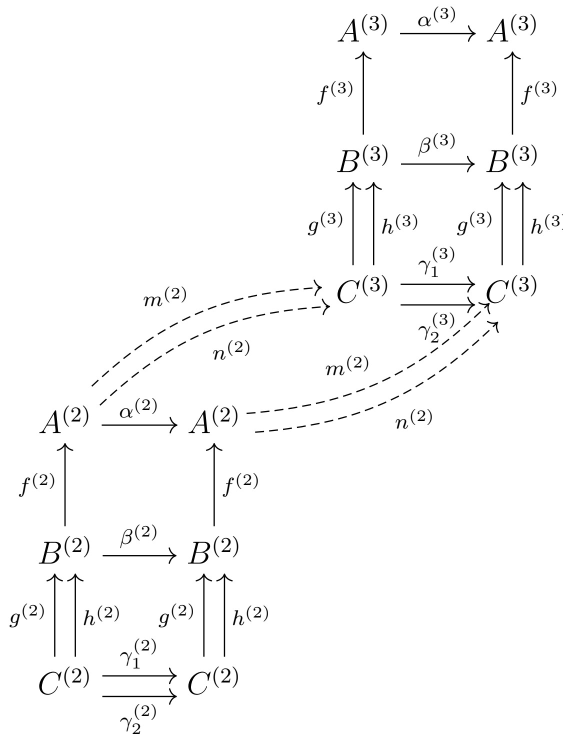
of $\mathbb{X}[2]$.

We will now define $\mathbb{X}[\lambda]$ for $\lambda > 2$ as collages of profunctors generated by data. We remark that the construction we give is of a technical nature and we confess that much of the notation we employ is somewhat recondite. Observe also that all the categories we construct are subcategories of **Set**.

Definition 4.1.1. Construction by transfinite recursion.

- **Base case**: For $\lambda = 3$, the categories involved are two copies of $\mathbb{X}[2]$. The objects of the first copy shall be labelled as $x^{(2)}$ and morphisms shall be labelled $s^{(2)}$. Analogously, the objects in the second copy will have label $x^{(3)}$ and morphisms will be denoted $s^{(3)}$. The rest of the data is as follows: $M[3]$ is specified by the single set
$$M[3](A^{(2)}, C^{(3)}) = \{m^{(2)}, n^{(2)}\}$$
and $\phi[3]$ thus needs to be a pair of group homomorphisms
$$\{\phi[3]_{m^{(2)}}, \phi[3]_{n^{(2)}} : \mathrm{Aut}_{\mathbb{X}[2]}(C^{(3)}) \to \mathrm{Aut}_{\mathbb{X}[2]}(A^{(2)})\}.$$
We let
$$\begin{aligned} \phi[3]_{m^{(2)}}(\gamma_1^{(3)}) &= \alpha^{(2)} \\ \phi[3]_{m^{(2)}}(\gamma_2^{(3)}) &= \mathrm{id} \\ \phi[3]_{n^{(2)}}(\gamma_1^{(3)}) &= \mathrm{id} \\ \phi[3]_{n^{(2)}}(\gamma_2^{(3)}) &= \alpha^{(2)} \end{aligned}$$

and extend by composition. The category $\mathbb{X}[3]$ is the collage of the profunctor generated by data of $\mathbb{X}[2]$, $\mathbb{X}[2]$, $M[3]$ and $\phi[3]$. The picture to have in mind is something like the following.



where the dashed arrows indicate the heteromorphisms coming from the profunctor.

- **Successor case**: If λ is a successor ordinal, then we can break down λ as a sum $\lambda = L + F$ where L is 0 or a limit ordinal and $F > 0$ is a natural number. We now split the construction into cases. Before proceeding, we remark that the same notation, $M[\lambda]$ and $\phi[\lambda]$, indicates different things in different cases. Also, in what follows, μ will denote the unique ordinal $\mu \geq 3$ such that $\lambda = \mu + 1$.

  1. $F = 1$ **and** L **is a limit ordinal**: We note that $L = \mu$ here and that μ is thus a limit ordinal. In this case, $\mathbb{X}[\lambda]$ is the collage generated by data of $\mathbb{X}[1]$, $\mathbb{X}[\mu]$, $M[\lambda]$ and $\phi[\lambda]$. Objects in the copy of $\mathbb{X}[1]$ will be labelled $x^{(\mu+2)}$, or equivalently $x^{(L+2)}$, and similarly for its arrows. Now, either $\mu = \omega$ or there must be a limit ordinal μ' such that $\mu' < \mu$ and for any other limit ordinal $\mu'' < \mu$, $\mu'' \leq \mu'$. With the implicit assumption that $\mu' = 0$ in case $\mu = \omega$, we let $M[\lambda]$ assign to each ordinal ν such that $\mu' + 2 \leq \nu < \mu$ the set

$$M[\lambda](A^{(\nu)}, C^{(\mu+2)}) = \{u^{(\nu)}, v^{(\nu)}\}$$

and $\phi[\lambda]$ associates for each such ordinal ν the pair of group homomorphisms

$$\{\phi[\lambda]_{u^{(\nu)}}, \phi[\lambda]_{v^{(\nu)}} : \mathrm{Aut}_{\mathbb{X}[1]}(C^{\mu+2}) \to \mathrm{Aut}_{\mathbb{X}[\mu]}(A^{(\nu)})\}$$

satisfying

$$\begin{aligned}
\phi[\lambda]_{u^{(\nu)}}(\gamma_1^{(\mu+2)}) &= \alpha^{(\nu)} \\
\phi[\lambda]_{u^{(\nu)}}(\gamma_2^{(\mu+2)}) &= \mathrm{id} \\
\phi[\lambda]_{v^{(\nu)}}(\gamma_1^{(\mu+2)}) &= \mathrm{id} \\
\phi[\lambda]_{v^{(\nu)}}(\gamma_2^{(\mu+2)}) &= \alpha^{(\nu)}.
\end{aligned}$$

2. $F = 2$ **and** L **is a limit ordinal**: In this case, $\mathbb{X}[\lambda]$ is the collage generated by $\mathbb{X}[1\frac{1}{2}]$, $\mathbb{X}[\mu]$, $M[\lambda]$ and $\phi[\lambda]$. The objects of $\mathbb{X}[1\frac{1}{2}]$ are given labels $x^{(\lambda)}$, or equivalently $x^{(L+2)}$, and analogously for morphisms. Now let $M[\lambda]$ specify the single set

$$M[\lambda](C^{(L+2)}, B^{(L+2)}) = \{g^{(L+2)}, h^{(L+2)}\}$$

and let $\phi[\lambda]$ specify the group homomorphisms

$$\{\phi[\lambda]_{g^{(L+2)}}, \phi[\lambda]_{h^{(L+2)}} : \mathrm{Aut}_{\mathbb{X}[1\frac{1}{2}]}(B^{(L+2)}) \to \mathrm{Aut}_{\mathbb{X}[\mu]}(C^{(L+2)})\}$$

which satisfy

$$\begin{aligned}
\phi[\lambda]_{g^{(L+2)}}(\beta^{(L+2)}) &= \gamma_1^{(L+2)} \\
\phi[\lambda]_{h^{(L+2)}}(\beta^{(L+2)}) &= \gamma_2^{(L+2)}
\end{aligned}$$

3. $F \geq 3$: This last case is almost exactly like the base case with the intuition being that we are "stacking" a copy of $\mathbb{X}[2]$ atop $\mathbb{X}[\mu]$ via two generating heteromorphisms. We define $\mathbb{X}[\lambda]$ to be the collage of $\mathbb{X}[2]$, $\mathbb{X}[\mu]$, $M[\lambda]$ and $\phi[\lambda]$. The objects of the copy of $\mathbb{X}[2]$ involved here will be marked $x^{(\lambda)}$ and similarly for its morphisms. Now let $M[\lambda]$ specify the single set

$$M[\lambda](A^{(\mu)}, C^{(\lambda)}) = \{m^{(\mu)}, n^{(\mu)}\}$$

and we take $\phi[\lambda]$ to be the collection of group homomorphisms

$$\{\phi[\lambda]_{m^{(\mu)}}, \phi[\lambda]_{n^{(\mu)}} : \mathrm{Aut}_{\mathbb{X}[2]}(C^{(\lambda)}) \to \mathrm{Aut}_{\mathbb{X}[\mu]}(A^{(\mu)})\}$$

which satisfy

$$\phi[\lambda]_{m^{(\mu)}}(\gamma_1^{(\lambda)}) = \alpha^{(\mu)}$$

$$\phi[\lambda]_{m^{(\mu)}}(\gamma_2^{(\lambda)}) = \mathrm{id}$$
$$\phi[\lambda]_{n^{(\mu)}}(\gamma_1^{(\lambda)}) = \mathrm{id}$$
$$\phi[\lambda]_{n^{(\mu)}}(\gamma_2^{(\lambda)}) = \alpha^{(\mu)}.$$

- **Limit case**: If λ is a limit ordinal, take $\mathbb{X}[\lambda]$ to be the colimit $\varinjlim_{\mu<\lambda} \mathbb{X}[\mu]$ in **Cat**.

4.2 Proof of Higher-Order Isotropy

The contents of this section are just as advertised on the tin.

Lemma 4.2.1. *Let μ be a limit ordinal. Assume that*

- $||\mathbb{X}[\mu]||_{\mathcal{I}} = \mu$*;*
- *every automorphism in $\mathbb{X}[\mu]$ has non-negative isotropy rank;*
- *the automorphism $\alpha^{(\nu)} : A^{(\nu)} \to A^{(\nu)}$ in $\mathbb{X}[\mu]$ has isotropy rank equal to ν for any ordinal ν such that $\nu < \mu$.*

Then

$$||\gamma_1^{(\mu+2)}||_{\mathbb{X}[\mu+1]} = ||\gamma_1^{(\mu+2)}||_{\mathbb{X}[\mu+1]} = ||\gamma_1^{(\mu+2)}\gamma_2^{(\mu+2)}||_{\mathbb{X}[\mu+1]} = \mu + 1$$

and

- $||\mathbb{X}[\mu+1]||_{\mathcal{I}} = \mu + 1$*;*
- *every automorphism in $\mathbb{X}[\mu]$ has non-negative isotropy rank.*

Proof: We verify that the hypotheses of lemma 3.3.4 hold. Recall that $\mathbb{X}[\mu+1]$ is collaged out of $\mathbb{X}[\mu]$ and $\mathbb{X}[1]$.

1. By assumption, all automorphisms in $\mathbb{X}[\mu]$ have non-negative isotropy rank.
2. The only object in $\mathbb{X}[1]$ is $C^{(\mu+2)}$ and it all generating heteromorphisms have it as target.
3. As mentioned in the preceding, $C^{(\mu+2)}$ is the unique object in the copy of $\mathbb{X}[1]$.
4. For a fixed $\nu < \mu$, the pair of group homomorphisms $\phi[\mu+1]_{u^\nu}$, $\phi[\mu+1]_{v^{(\nu)}}$ is seen to be jointly injective upon examination of the values they take.

$$\begin{array}{ll}
\phi[\mu+1]_{u^{(\nu)}}(\mathrm{id}_{C^{(\mu+2)}}) = \mathrm{id}_{A^{(\nu)}} & \phi[\mu+1]_{v^{(\nu)}}(\mathrm{id}_{C^{(\mu+2)}}) = \mathrm{id}_{A^{(\nu)}} \\
\phi[\mu+1]_{u^{(\nu)}}(\gamma_1^{(\mu+2)}) = \alpha^{(\nu)} & \phi[\mu+1]_{v^{(\nu)}}(\gamma_1^{(\mu+2)}) = \mathrm{id}_{A^{(\nu)}} \\
\phi[\mu+1]_{u^{(\nu)}}(\gamma_2^{(\mu+2)}) = \mathrm{id}_{A^{(\nu)}} & \phi[\mu+1]_{v^{(\nu)}}(\gamma_2^{(\mu+2)}) = \alpha^{(\nu)} \\
\phi[\mu+1]_{u^{(\nu)}}(\gamma_1^{(\mu+2)}\gamma_2^{(\mu+2)}) = \alpha^{(\nu)} & \phi[\mu+1]_{v^{(\nu)}}(\gamma_1^{(\mu+2)}\gamma_2^{(\mu+2)}) = \alpha^{(\nu)}
\end{array}$$

5. Observe that each $\gamma_i^{(\mu+2)}$ and $\gamma_1^{(\mu+2)}\gamma_2^{(\mu+2)}$ lifts to $\alpha^{(\nu)}$ along at least one of the generating heteromorphisms $u^{(\nu)}$ or $v^{(\nu)}$. Since this holds for all $\nu < \mu$ and $||\alpha^{(\nu)}||_{\mathbb{X}[\mu]} = \nu$, we have that

$$\mathcal{S}^{\mathbb{X}[\mu+1]}_{C^{(\mu+2)}}(\gamma_1^{(\mu+2)}) = \mathcal{S}^{\mathbb{X}[\mu+1]}_{C^{(\mu+2)}}(\gamma_2^{(\mu+2)}) = \mathcal{S}^{\mathbb{X}[\mu+1]}_{C^{(\mu+2)}}(\gamma_1^{(\mu+2)}\gamma_2^{(\mu+2)}) = \mu = ||\mathbb{X}[\mu]||_{\mathcal{I}}.$$

6. Since $\mathbb{X}[1]$ has a unique object, all automorphisms are isotropy elements and they lift non-trivially along other non-identity maps.

Now we can use proposition 3.3.5 to compute

$$\begin{aligned} ||\zeta||_{\mathbb{X}[\mu+1]} &= ||\mathbb{X}[\mu]||_{\mathcal{I}} + ||\zeta||_{\mathbb{X}[1]} \\ &= \mu + 1 \end{aligned}$$

whenever $\zeta \in \{\gamma_1^{(\mu+2)}, \gamma_2^{(\mu+2)}, \gamma_1^{(\mu+2)}\gamma_2^{(\mu+2)}\}$. The rest of the conclusions in the lemma statement are immediate. ∎

Lemma 4.2.2. *Let μ be a limit ordinal. Assume that*

- $||\mathbb{X}[\mu+1]||_{\mathcal{I}} = \mu + 1$*;*
- *every automorphism in $\mathbb{X}[\mu+1]$ has non-negative isotropy rank;*
- *we have the equalities*

$$||\gamma_1^{(\mu+2)}||_{\mathbb{X}[\mu+1]} = ||\gamma_1^{(\mu+2)}||_{\mathbb{X}[\mu+1]} = ||\gamma_1^{(\mu+2)}\gamma_2^{(\mu+2)}||_{\mathbb{X}[\mu+1]} = \mu + 1.$$

Then, for an automorphism ζ in $\mathbb{X}[\mu+2]$,

$$||\zeta||_{\mathbb{X}[\mu+2]} = \begin{cases} \mu+1 & \text{if } \zeta = \beta^{(\mu+2)} \\ \mu+2 & \text{if } \zeta = \alpha^{(\mu+2)} \end{cases}$$

and

- $||\mathbb{X}[\mu+2]||_{\mathcal{I}} = \mu + 2$*;*
- *every automorphism in $\mathbb{X}[\mu+2]$ has non-negative isotropy rank.*

Proof: We check that the hypotheses of lemma 3.3.4 apply in our case. Recall that $\mathbb{X}[\mu+2]$ is collaged together from $\mathbb{X}[\mu+1]$ and $\mathbb{X}[1\frac{1}{2}]$.

1. By assumption, all automorphisms in $\mathbb{X}[\mu+1]$ have non-negative isotropy rank.

2. The only objects in $\mathbb{X}[1\frac{1}{2}]$ are $B^{(\mu+2)}$ and $A^{(\mu+2)}$ with $B^{(\mu+2)}$ being a target of some generating heteromorphism. Since $B^{(\mu+2)}$ connects to $A^{(\mu+2)}$ via $f^{(\mu+2)}$, $A^{(\mu+2)}$ is also the target of a heteromorphism.

3. By construction, $B^{(\mu+2)}$ is the sole target of all generating heteromorphisms.

4. As $\beta^{(\mu+2)}$ is the unique non-trivial automorphism on $B^{(\mu+2)}$ and it has at least one non-trivial lift along a generating heteromorphism, the group homomorphisms $\phi[\mu+2]_{g^{(\mu+2)}}$, $\phi[\mu+2]_{h^{(\mu+2)}}$ are jointly injective.

5. Since $\beta^{(\mu+2)}$ is the only non-identity automorphism on $B^{(\mu+2)}$, the fifth hypothesis of lemma 3.3.4 is vacuously true.

6. The only non-identity automorphisms in $\mathbb{X}[1\frac{1}{2}]$ are $\alpha^{(\mu+2)}$ and $\beta^{(\mu+2)}$ and $\alpha^{(\mu+2)}$ lifts to $\beta^{(\mu+2)}$ along $f^{(\mu+2)}$.

Observe that $\beta^{(\mu+2)}$ lifts along $g^{(\mu+2)}$ to $\gamma_1^{(\mu+2)}$. Since $\gamma_1^{(\mu+2)} = ||\mathbb{X}[\mu+1]||_{\mathcal{I}}$, it is clear that $\mathcal{S}^{\mathbb{X}[\mu+2]}_{B^{(\mu+2)}}(\beta^{(\mu+2)}) = ||\mathbb{X}[\mu+1]||_{\mathcal{I}}$ and we may apply proposition 3.3.5 to deduce that

$$
\begin{aligned}
||\zeta||_{\mathbb{X}[\mu+2]} &= \begin{cases} ||\mathbb{X}[\mu+1]||_{\mathcal{I}} + ||\beta^{(\mu+2)}||_{\mathbb{X}[1\frac{1}{2}]} - 1 & \text{if } \zeta = \beta^{(\mu+2)} \\ ||\mathbb{X}[\mu+1]||_{\mathcal{I}} + ||\alpha^{(\mu+2)}||_{\mathbb{X}[1\frac{1}{2}]} - 1 & \text{if } \zeta = \alpha^{(\mu+2)} \end{cases} \\
&= \begin{cases} \mu+1+1-1 & \text{if } \zeta = \beta^{(\mu+2)} \\ \mu+1+2-1 & \text{if } \zeta = \alpha^{(\mu+2)} \end{cases} \\
&= \begin{cases} \mu+1 & \text{if } \zeta = \beta^{(\mu+2)} \\ \mu+2 & \text{if } \zeta = \alpha^{(\mu+2)}. \end{cases}
\end{aligned}
$$

The other two conclusions in the lemma statement now obtain in a straightforward way from this calculation. ∎

Lemma 4.2.3. *Let L be an ordinal equal to either 0 or a limit ordinal and suppose F is a natural number such that $F \geq 3$. Assume that*

- $||\mathbb{X}[L+F-1]||_{\mathcal{I}} = L+F-1$;
- *every automorphism in $\mathbb{X}[L+F-1]$ has non-negative isotropy rank;*
- *the automorphism $\alpha^{(L+F-1)} : A^{(L+F-1)} \to A^{(L+F-1)}$ in $\mathbb{X}[L+F-1]$ has isotropy rank equal to $||\mathbb{X}[L+F-1]||_{\mathcal{I}}$.*

Then, for an automorphism ζ in $\mathbb{X}[L+F]$,

$$||\zeta||_{\mathbb{X}[L+F]} = \begin{cases} L+F-1 & \text{if } \zeta \in \{\gamma_1^{(L+F)}, \gamma_2^{(L+F)}, \gamma_1^{(L+F)}\gamma_2^{(L+F)}, \beta^{(L+F)}\} \\ L+F & \text{if } \zeta = \alpha^{(L+F)} \end{cases}$$

and

- $||\mathbb{X}[L+F]||_{\mathcal{I}} = L+F$;
- *every automorphism in $\mathbb{X}[L+F]$ has non-negative isotropy rank;*
- *the automorphism $\alpha^{(L+F)} : A^{(L+F)} \to A^{(L+F)}$ in $\mathbb{X}[L+F]$ has isotropy rank equal to $||\mathbb{X}[L+F]||_{\mathcal{I}}$.*

Proof: We shall apply the isotropy rank formula developed in the last chapter. Recall that $\mathbb{X}[L+F]$ is collaged out of the categories $\mathbb{X}[2]$ and $\mathbb{X}[L+F-1]$. We verify that all the hypotheses in the statement of lemma 3.3.4 are satisfied by the collage $\mathbb{X}[L+F]$.

1. All automorphisms in $\mathbb{X}[L+F-1]$ have non-negative isotropy rank by assumption.

2. Examining the third clause of the successor case in definition 4.1.1, we see that every object in the copy of $\mathbb{X}[2]$ has a morphism coming into it from $C^{(L+F)}$ and that $C^{(L+F)}$ is the target of a generating heteromorphism.

3. It is clear from our construction of $\mathbb{X}[L+F]$ that $C^{(L+F)}$ is the only object in $\mathbb{X}[L+F]$ which is a target of any generating heteromorphism.

4. The group homomorphisms $\phi[L+F]_{m^{(L+F-1)}}, \phi[L+F]_{n^{(L+F-1)}}$ are seen to be jointly injective upon an exhaustive listing of the values each homomorphism takes

$$\begin{array}{ll} \phi[L+F]_{m^{(L+F-1)}}(\mathrm{id}_{C^{(L+F)}}) = \mathrm{id}_{A^{(L+F-1)}} & \phi[L+F]_{n^{(L+F-1)}}(\mathrm{id}_{C^{(L+F)}}) = \mathrm{id}_{A^{(L+F-1)}} \\ \phi[L+F]_{m^{(L+F-1)}}(\gamma_1^{(L+F)}) = \alpha^{(L+F-1)} & \phi[L+F]_{n^{(L+F-1)}}(\gamma_1^{(L+F)}) = \mathrm{id}_{A^{(L+F-1)}} \\ \phi[L+F]_{m^{(L+F-1)}}(\gamma_2^{(L+F)}) = \mathrm{id}_{A^{(L+F-1)}} & \phi[L+F]_{n^{(L+F-1)}}(\gamma_2^{(L+F)}) = \alpha^{(L+F-1)} \\ \phi[L+F]_{m^{(L+F-1)}}(\gamma_1^{(L+F)}\gamma_2^{(L+F)}) = \alpha^{(L+F-1)} & \phi[L+F]_{n^{(L+F-1)}}(\gamma_1^{(L+F)}\gamma_2^{(L+F)}) = \alpha^{(L+F-1)} \end{array}$$

5. Observe that each of $\gamma_1^{(L+F)}$, $\gamma_2^{(L+F)}$ and $\gamma_1^{(L+F)}\gamma_2^{(L+F)}$ lift to $\alpha^{(L+F-1)}$ along some generating heteromorphism and since

$$||\alpha^{(L+F-1)}||_{\mathbb{X}[L+F-1]} = ||\mathbb{X}[L+F-1]||_{\mathcal{I}}$$

, we obtain

$$\mathcal{S}^{\mathbb{X}[L+F]}_{C^{(L+F)}}(\gamma_1^{(L+F)}) = ||\mathbb{X}[L+F-1]||_{\mathcal{I}}$$

$$\mathcal{S}^{\mathbb{X}[L+F]}_{C^{(L+F)}}(\gamma_2^{(L+F)}) = ||\mathbb{X}[L+F-1]||_{\mathcal{I}}$$
$$\mathcal{S}^{\mathbb{X}[L+F]}_{C^{(L+F)}}(\gamma_1^{(L+F)}\gamma_2^{(L+F)}) = ||\mathbb{X}[L+F-1]||_{\mathcal{I}}.$$

6. The isotropy elements in the copy of $\mathbb{X}[2]$ are $\beta^{(L+F)}$, $\gamma_1^{(L+F)}$, $\gamma_2^{(L+F)}$ and $\gamma_1^{(L+F)}\gamma_2^{(L+F)}$ and by our computations from chapter 1 again, each has a non-trivial lift along a non-identity morphism in $\mathbb{X}[2]$.

Proposition 3.3.5 implies that

$$||\zeta||_{\mathbb{X}[L+F]} = ||\mathbb{X}[L+F-1]||_{\mathcal{I}} + ||\zeta||_{\mathbb{X}[2]} - 1 = L+F-1+||\zeta||_{\mathbb{X}[2]} - 1.$$

Hence,

$$\begin{aligned} ||\zeta||_{\mathbb{X}[L+F]} &= \begin{cases} L+F-1+1-1 & \text{if } \zeta \in \{\gamma_1^{(L+F)}, \gamma_2^{(L+F)}, \gamma_1^{(L+F)}\gamma_2^{(L+F)}, \beta^{(L+F)}\} \\ L+F-1+2-1 & \text{if } \zeta = \alpha^{(L+F)} \end{cases} \\ &= \begin{cases} L+F-1 & \text{if } \zeta \in \{\gamma_1^{(L+F)}, \gamma_2^{(L+F)}, \gamma_1^{(L+F)}\gamma_2^{(L+F)}, \beta^{(L+F)}\} \\ L+F & \text{if } \zeta = \alpha^{(L+F)}. \end{cases} \end{aligned}$$

The claims in the conclusion of the lemma statement are now seen to follow easily. ∎

Lemma 4.2.4. *Let μ be a limit ordinal. If $||\mathbb{X}[\nu]||_{\mathcal{I}} = \nu$ for all ordinals $\nu < \mu$, then $||\mathbb{X}[\mu]||_{\mathcal{I}} = \mu$.*

Proof: Observe that our construction of the $\mathbb{X}[\lambda]$ is such that there is an inclusion $\mathbb{X}[\lambda] \hookrightarrow \mathbb{X}[\lambda+1]$ and proposition 3.2.8 implies that such an inclusion in fact preserves isotropy of all ranks. Hence, we obtain a sequence of categories

$$\mathbb{X}[0] \hookrightarrow \mathbb{X}[1] \hookrightarrow \mathbb{X}[2] \hookrightarrow \cdots \hookrightarrow \mathbb{X}[\lambda]$$

which, by virtue of preservation of isotropy ranks and corollary 2.1.14, can be propagated vertically to produce

$$\begin{array}{ccccccccc}
\mathbb{X}[0] & \hookrightarrow & \mathbb{X}[1] & \hookrightarrow & \mathbb{X}[2] & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{X}[\lambda] \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathbb{X}[0]/\mathcal{I} & \hookrightarrow & \mathbb{X}[1]/\mathcal{I} & \hookrightarrow & \mathbb{X}[2]/\mathcal{I} & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{X}[\lambda]/\mathcal{I} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathbb{X}[0]/\mathcal{I}^2 & \hookrightarrow & \mathbb{X}[1]/\mathcal{I}^2 & \hookrightarrow & \mathbb{X}[2]/\mathcal{I}^2 & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{X}[\lambda]/\mathcal{I}^2 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathbb{X}[0]/\mathcal{I}^\mu & \hookrightarrow & \mathbb{X}[1]/\mathcal{I}^\mu & \hookrightarrow & \mathbb{X}[2]/\mathcal{I}^\mu & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{X}[\lambda]/\mathcal{I}^\mu
\end{array}$$

where all horizontal arrows are inclusions which preserve isotropy ranks and all vertical arrows are isotropy quotient functors. Note that the above diagram applies to any λ and in particular to $\lambda = \mu$. If ν is any ordinal such that $\nu < \mu$, then the assumption that $||\mathbb{X}[\nu]||_\mathcal{I} = \nu$ implies that the final segment of the chain of categories at the ν^{th} "floor" of the diagram is still non-trivial and therefore also that the ν^{th} isotropy quotient of $\mathbb{X}[\mu]$ is non-trivial. However, at the μ^{th} "floor", there are no categories in the chain which have automorphisms with positive isotropy rank. Hence, the μ^{th} quotient of $\mathbb{X}[\mu]$ has no automorphisms with positive isotropy rank. As all automorphisms in all categories involved have non-negative isotropy ranks, we deduce that the isotropy chain for $\mathbb{X}[\mu]$ stabilizes precisely at stage μ and thus, $||\mathbb{X}[\mu]||_\mathcal{I} = \mu$. ∎

Finally, we put everything together.

Theorem 4.2.5. *If λ is either $-\infty$ or an ordinal, then $||\mathbb{X}[\lambda]||_\mathcal{I} = \lambda$.*

Proof: As stated earlier in the chapter, it is trivial to show that $||\mathbb{X}[\lambda]||_\mathcal{I} = \lambda$ for the cases $\lambda = -\infty, 0, 1$. For $\lambda \geq 2$, we proceed by transfinite induction on λ.

- **Base case**: The base case occurs when $\lambda = 2$ and we have already demonstrated in chapter 1 that $\mathbb{X}[2]$ has isotropy rank 2.

- **Successor case**: Since our construction of $\mathbb{X}[\lambda]$ has three cases, the proof also has three cases. As in the construction, we first split λ as $L + F$ for L either 0 or a limit ordinal and F a natural number such that $F \geq 1$.

  1. $F = 1$ **and** L **is a limit ordinal**: Apply lemma 4.2.1.
  2. $F = 2$ **and** L **is a limit ordinal**: Apply lemma 4.2.2.

3. $F \geq 3$: Apply lemma 4.2.3.

- **Limit case**: Apply lemma 4.2.4.

∎

Chapter 5

Conclusion and Future Work

In closing this thesis, we offer some remarks on possible avenues of exploration suggested by our results and we try to situate our work within a broader context. To briefly recap what we have done, we started out with the idea that the class of small categories have an invariant called isotropy associated to them in the form of presheaves of groups. More precisely, as with topological spaces, there is a functorial association of categories with the algebraic gadgets $\mathbf{Grp}^{\mathbb{C}^{op}}$ and, for given $\mathbb{C}$, this has a nice low-tech description in terms of the morphisms in $\mathbb{C}$. We then pointed out that isotropy has a rather intriguing persistence property wherein we can form the sequence

$$\mathbb{C} \twoheadrightarrow \mathbb{C}/\mathcal{I} \twoheadrightarrow \mathbb{C}/\mathcal{I}^2 \twoheadrightarrow \cdots$$

of iterated isotropy quotients and the sequence in general is not trivial. The lodestar of this thesis has been the (constructive) existence of categories which witness this non-triviality. Prior to pursuing the results contained here, we had already constructed categories with n^{th}-order isotropy for arbitrary but finite n, subsequent to which we were able to produce a category with ω^{th}-order isotropy. However, the construction carried out initially was purely combinatorial in nature and extension to the λ^{th}-order case for $\lambda > \omega$ was both non-obvious and would apparently involve even more Daedalean combinatorics than in the finite case. This inspired an effort to "dissolve" the combinatorics by way of theory and this intent is perhaps most pronounced in the second chapter, the key result of which is essentially that the limit ordinal case of our construction follows from generalised abstract nonsense.

Having thus freed ourselves to focus on the successor case, the goal was to construct the nice sorts of maps for which the main theorem of chapter 2 applied. Fortunately, there was even more abstract nonsense – namely, profunctors and the maps they induce on categories via collages – which happened to have the right properties vis-a-vis isotropy. The highlight of chapter 3 is the Seifert-van Kampen Theorem for isotropy groups, which allows us to hope that there may be a calculus of invariants for isotropy groups in analogy with calculi which exist and are well-known for invariants in algebraic topology. Our main regret is that chapter 3 does not provide the most parsimonious presentation of the key concepts involved. As a result, the distinction between what is truly essential and what is only a technicality required to obtain the necessary results are not always clear. However, it is our belief that this shortcoming has since been fixed in our paper [FHK]. Apart from the much needed separation between conceptual generalities and sedulous technicalities, the paper explicates a more general mechanism, called the **isotropy span**, for comparing isotropy groups across functors $F : \mathbb{C} \to \mathbb{D}$. Additionally, a portion of the paper is dedicated to rightfully emplacing isotropy rank within the theory of Grothendieck toposes.

Consequent to the work carried out in the last 3 chapters, we believe we have now earned some right for speculation. One question that naturally arose in the course of our investigation was to what extent there exists "test objects" or "cells" for higher-order isotropy. The analogy is with the notion of spheres as cells in algebraic topology. Does there exist a class of categories $\{\mathbb{S}_k\}$ which give information about isotropy of $\mathbb{C}$ when we map into $\mathbb{C}$ via functors $\mathbb{S}_k \to \mathbb{C}$? Although we have not mentioned it here, there are not only pullback diagrams involving isotropy groups of categories but there are actually pullback diagrams involving the categories themselves which fall out as a result of using profunctors for constructing bigger categories out of smaller ones (see [FHK] for more on this). Recall that we often attach cells, such as spheres and disks, to spaces via pullback diagrams in **Top**. The fact that we have theorems for categories which are filtered in a "nice" way and that this filtration is obtained by means of iterative construction via pullbacks which "attach cells" $\mathbb{X}[2]$ makes the analogy with CW-complexes irresistible. Finally, it is our fervent hope that this thesis represents a starting point for a calculus of invariants for isotropy groups. While the existence, let alone the form, of such a calculus is far from obvious, we look forward to some systematic attempt to explore what such a calculus might entail. Given how general categories are as mathematical objects, it would be very exciting indeed if we were able to deconstruct and reconstruct categories the way we do so with spaces and representations.

Bibliography

[Awo10] Steve Awodey. *Category Theory.* Number 52 in Oxford Logic Guides. Oxford University Press, second edition, 2010.

[B0́0] Jean Bénabou. Distributors at Work, June 2000. Available at http://www.mathematik.tu-darmstadt.de/ streicher/FIBR/DiWo.pdf.

[BW90] Michael Barr and Charles Wells. *Category Theory for Computing Science.* Number S-567 in Prentice-Hall International Series in Computer Science. Prentice-Hall, 1990. Available at http://www.math.mcgill.ca/triples/Barr-Wells-ctcs.pdf.

[Dug98] Daniel Dugger. Sheaves and Homotopy Theory, 1998. Available at http://math.mit.edu/ dspivak/files/cech.pdf.

[FHK] Jonathon Funk, Pieter Hofstra, and Sakif Khan. Higher Isotropy. Submitted for publication.

[FHS12] Jonathon Funk, Pieter Hofstra, and Benjamin Steinberg. Isotropy and Crossed Toposes. *Theory and Applications of Categories*, 26(24):660–709, November 2012. Available at http://www.emis.de/journals/TAC/volumes/26/24/26-24abs.html.

[Hat01] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2001. Available at https://www.math.cornell.edu/ hatcher/AT/AT.pdf.

[Joh02] Peter Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* Clarendon Press, September 2002.

[Mac98] Saunders MacLane. *Categories for the Working Mathematician.* Number 5 in Graduate Texts in Mathematics. Springer, second edition, September 1998.

[MM94] Saunders MacLane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Universitext. Springer, October 1994.

[Rie10] Emily Riehl. Two-sided Discrete Fibrations in 2-Categories and Bicategories, 2010. Available at http://www.math.jhu.edu/ eriehl/fibrations.pdf.

[Str14] Thomas Streicher. Fibred Categories à la Jean Bénabou, 2014. Available at http://www.mathematik.tu-darmstadt.de/ streicher/FIBR/FibLec.pdf.

[Win05] Glynn Winskel. Relations in Concurrency. In *Proceedings of the 20th Annual IEEE Symposium on Logic in Computer Science*, LICS '05, pages 2–11, Washington, DC, USA, 2005. IEEE Computer Society.

[Woo82] R. J. Wood. Abstract Pro Arrows I. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 23(3):279–290, 1982.

[Woo85] R. J. Wood. Proarrows II. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 26(2):135–168, 1985.