The Bala-Carter classification of nilpotent orbits of semisimple Lie algebras

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Abstract

Conjugacy classes of nilpotent elements in complex semisimple Lie algebras are classified using the Bala-Carter theory. In this theory, nilpotent orbits in $\mathfrak{g}$ are parametrized by the conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ of Levi subalgebras of $\mathfrak{g}$ and distinguished parabolic subalgebras of $[\mathfrak{l}, \mathfrak{l}]$. In this thesis we present this theory and use it to give a list of representatives for nilpotent orbits in $\mathfrak{so}(8)$ and from there we give a partition-type parametrization of them.

Résumé

Les classes de conjugaison des éléments nilpotents dans les algèbres de Lie semisimples complexes sont classifiées en utilisant la theorie de Bala-Carter. Dans cette théorie, les orbites nilpotentes dans $\mathfrak{g}$ sont paramétrées par les classes de conjugaison de paires $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ de sous-algèbres de Levi de $\mathfrak{g}$ et de sous-algèbres paraboliques distinguées de $[\mathfrak{l}, \mathfrak{l}]$. Dans cette thèse, nous présentons cette théorie et l'utilisons pour donner une liste de représentants des orbites nilpotentes dans $\mathfrak{so}(8)$ et à partir de là nous leur donnons une paramétrisation de type partition.

Dedications

To Carolann Sweeney and to Mom: this humble work of mine does not match all your expectations but I would never have accomplished it without your love and care and your smiles.

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Chapter 1

Introduction

Nilpotent orbits in a complex semisimple Lie algebra $\mathfrak{g}$ are used, in the Springer correspondence, to produce representations of the Weyl group $\mathcal{W}$ of the Cartan type of $\mathfrak{g}$, see Theorem [6, Theorem 10.1.4]. The irreducible representations of $\mathcal{W}$, in turn, are closely connected to the representations of $\mathfrak{g}$, via the theory of primitive ideals of the universal enveloping algebra $U(\mathfrak{g})$. These are not the only motivations to classify nilpotent orbits, but also the fact that they have a symplectic structure.

Dynkin and Kostant partially solved the problem by means of the weighted Dynkin diagrams of the Cartan type of $\mathfrak{g}$ and proved that there are only a finite number of nilpotent orbits: each nilpotent orbit of $\mathfrak{g}$ is uniquely labelled by a weighted Dynkin diagram. However, not all weighted Dynkin diagrams are labels of a nilpotent orbit.

On the other hand, using the theory of the Jordan canonical form, we know that the nilpotent orbits in $\mathfrak{sl}_n$ are in bijection with the partitions of n. By using a refined apparatus, Springer, Steinberg and Gerstenhaber described recipes to obtain a partition-type classification of the nilpotent orbits of the classical simple Lie algebras over $\mathbb{C}$.

It was not until Bala and Carter that a clear and complete parametrization of nilpotent orbits of the exceptional algebras was obtained. Bala and Carter slightly modified a mysterious attempt of Dynkin of a classification making use of reductive subalgebras of $\mathfrak{g}$, which nonetheless, but not without mistakes, gave the weighted Dynkin diagrams of the exceptional orbits. In fact, the Bala-Carter theory of nilpotent orbits applies for any complex semisimple Lie algebra.

However, the Bala-Carter classification, to the best of our knowledge, seems to appeal substantially to some few fundamental properties of Lie algebras and endomorphisms over the complex field. What are the sinews of the proof of the Bala-Carter theorem which make those properties necessary? And is the Bala-Carter correspondence, which is very abstract, reasonably applicable to some concrete case?

Obviously, one needs to have a fair knowledge and command of the general structure theory of semisimple Lie algebras, which is laid out in Chapter 2. This second chapter is essentially an enumeration of the definitions and properties concerning semisimple Lie algebras that we will use throughout the thesis. We give as well just enough details about the construction of Chevalley groups in order to define nilpotent orbits.

In Chapter 3, we deal with the representation theory of $\mathfrak{sl}_2$. We have the impression that the representations of $\mathfrak{sl}_2$ are always present in some sense in the proofs we give in Chapter 4.

The fourth chapter will try to give an answer to our first question. We follow closely the treatment of Collingwood and McGovern in [6, Chapter 8] in so doing. Our humble contribution is to give the details of almost all the proofs in this chapter. General facts on toral subalgebras and their centralizers are stated and proved. Then we give a proof of the Jacobson-Morozov theorem, which is different from the book's. In the last section of the chapter, we give all the necessary results, and the proofs of most of them, which lead to Bala-Carter's theorem, Theorem 4.3.12.

In the last chapter, we apply the Bala-Carter classification to the Lie algebra $\mathfrak{so}_8$. We give in the two first sections the basic structure theory of $\mathfrak{so}_8$ and a conjugation theorem for Levi subalgebras, Theorem 5.2.1. The third section, applying Theorem 5.2.1, is mainly computational: we list the conjugacy classes of Levi subalgebras of $\mathfrak{so}_8$ using a case-by-case treatment. In the fourth section, a detailed list of the distinguished parabolic subalgebras for each conjugacy class of Levi is provided. Three lemmas are therein stated, which are but particular cases of the fact that the only distinguished parabolic of $\mathfrak{sl}_n$ is the Borel subalgebra. The distinguished parabolic subalgebras in $\mathfrak{l} = \mathfrak{g}$ are in the next section. In the last section are computed the representatives of each nilpotent orbit and a partition classification of these orbits is given, which turns out to be the same as the result by Springer and Steinberg for our particular case.

Chapter 2

Preliminaries

Throughout this work, unless otherwise stated, k denotes a field of characteristic zero; for some results, we additionally require k to be algebraically closed. Our main references are [7] and [1]. The results cited here are those needed for subsequent chapters.

2.1 Lie algebras : definitions and key examples

Definition 2.1.1 (Lie algebra)**.** *A* Lie algebra over *k is a vector space $\mathfrak{g}$ endowed with a bilinear application*

$$\begin{aligned} [\ \ ,\ \ ]\colon \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (X, Y) &\mapsto [X, Y] \end{aligned}$$

with the following properties:

1. *For all $X \in \mathfrak{g}$, $[X, X] = 0$.*

2. *For all* $X, Y, Z \in \mathfrak{g}$

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

Property 2 is called the Jacobi identity*.*

Example 2.1.2. Any vector space $\mathfrak{v}$ over k can be made into a Lie algebra with the Lie bracket defined by $[X,Y]=0$ for all X and Y in $\mathfrak{v}$. We say that such a Lie bracket is *trivial*. A Lie algebra whose Lie bracket is trivial is called *abelian*.

Example 2.1.3. The set of all $n \times n$ matrices over k together with the Lie bracket defined by $[X,Y]=XY-YX$ and denoted $\mathfrak{gl}(n)$ is a Lie algebra.

Example 2.1.4. Let $\mathfrak{sl}(n)$ denote the subspace of $\mathfrak{gl}(n)$ consisting of matrices M whose trace $\mathrm{Tr}(M)$ is equal to zero. Endowed with the restriction of the Lie bracket of $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$ is a Lie algebra.

Remark 2.1.5. Let V be a n–dimensional vector space over k and denote $\mathfrak{gl}(V)$ the set of all k–endomorphisms of V. Then we can identify $\mathfrak{gl}(V)$ with $\mathfrak{gl}(n)$ by choosing a basis of V. We then define $\mathfrak{sl}(V)$ as the subset of $\mathfrak{gl}(V)$ consisting of all the endomorphisms of V with trace equal to zero, and this subspace is obviously isomorphic to $\mathfrak{sl}(n)$.

The following definitions are from [7, p. 1] and [7, p. 6].

Definition 2.1.6 (Subalgebra and ideal)**.** *A subspace* $\mathfrak{s}$ *of* $\mathfrak{g}$ *is a* subalgebra *if* $[X,Y]\in\mathfrak{s}$ *for all* $X, Y \in \mathfrak{s}$*. A subspace* $\mathfrak{i}$ *of* $\mathfrak{g}$ *is an* ideal $\mathfrak{g}$ *if* $X \in \mathfrak{g}$ *and* $Y \in \mathfrak{i}$ *together imply* $[X,Y]\in\mathfrak{i}$.

In addition to $\mathfrak{sl}(n)$, the Lie algebra of principal interest to this thesis is the *(special) orthogonal Lie algebra* of rank n, $\mathfrak{so}(2n)$, which is defined as follows, using the choices in [3, Ch.VIII, §13, no. 4].

Let V denote a vector space over k, of dimension $\dim V = 2n$. Let J denote the matrix

$$J = \begin{pmatrix} 0 & S_n \\ S_n & 0 \end{pmatrix},$$

where S_n is the square matrix of order n with ones on the anti-diagonal and zeros everywhere else. For example

$$S_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let ψ be the nondegenerate symmetric bilinear form on V defined by the matrix J; that is, for all $v, w \in V$, $\psi(v, w) = v^t J w$. By labeling the basis of V by $\{e_1, e_2, e_3, \cdots, e_n, e_{-n}, \cdots, e_{-3}, e_{-2}, e_{-1}\}$, we have $\psi(e_i, e_j) = \delta_{i,-j}$, where $\delta_{a,b} = 1$ if $a = b$ and 0 otherwise.

Remark 2.1.7. The above basis is called a *Witt basis* for V (associated to ψ).

Definition 2.1.8 (The Lie algebra $\mathfrak{so}(2n)$)**.** *The Lie algebra $\mathfrak{so}(2n)$ is the Lie subalgebra of $\mathfrak{gl}(2n)$ consisting of all $X \in \mathfrak{gl}(2n)$ such that*

$$\psi(Xv, w) = -\psi(v, Xw), \quad \textit{for all } v, w \in V.$$

To justify this definition, we need to prove that $\mathfrak{so}(2n)$ is a Lie algebra with the Lie bracket $[,]$ defined by

$$[X, Y] = XY - YX.$$

Namely, suppose $X, Y \in \mathfrak{so}(2n)$. Then, since

$$\begin{aligned}\psi([X,Y]v,w) &= \psi(XYv - YXv, w) = \psi(X(Yv),w) - \psi(Y(Xv),w) \\ &= -\psi(Yv, Xw) - (-\psi(Xv, Yw)) = \psi(v, YXw) - \psi(v, XYw) \\ &= \psi(v, YXw - XYw) \\ &= -\psi(v,[X,Y]w),\end{aligned}$$

we have that $[X,Y] \in \mathfrak{so}(2n)$. Therefore $\mathfrak{so}(2n)$ is a Lie subalgebra of $\mathfrak{gl}(2n)$.

We claim that the algebra $\mathfrak{so}(2n)$ is not an ideal of $\mathfrak{gl}(2n)$. Namely, let H be the linear transformation of V defined by $He_i = e_i$ and $He_{-i} = -e_{-i}$ for all $1 \le i \le n$; we see directly that $H \in \mathfrak{so}(2n)$. Let $Y \in \mathfrak{gl}(2n)$ be defined by $Ye_1 = e_{-1}$, and $Ye_j = 0$ for all other elements of the Witt basis; then $Y \notin \mathfrak{so}(2n)$. We have that $[H,Y] = 2Y$, which does not lie in $\mathfrak{so}(2n)$. So $\mathfrak{so}(2n)$ is not an ideal of $\mathfrak{gl}(2n)$.

Remark 2.1.9. In terms of matrices with respect to the Witt basis of V above, the elements X of $\mathfrak{so}(2n)$ satisfy the relation $X^tJ = -JX$. If we write X in block form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then we easily compute that the four matrices A, B, C, D of order n satisfy the relations

$$S_nC = -C^tS_n, \quad S_nB = -B^tS_n, \quad \text{and} \quad S_nD = -A^tS_n.$$

In particular, the matrices B and C are antisymmetric with respect to the anti-diagonal; for example, if $n = 3$ such a matrix has the form

$$\begin{pmatrix} a & b & 0 \\ c & 0 & -b \\ 0 & -c & -a \end{pmatrix}.$$

Similarly, since $D = -S_n A^t S_n$, the pair (A, D) of matrices has the form (for $n = 3$)

$$\left(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \begin{pmatrix} -i & -f & -c \\ -h & -e & -b \\ -g & -d & -a \end{pmatrix} \right).$$

2.2 Subalgebras of centralizers

Definition 2.2.1 (Center of a Lie algebra)**.** *Let $\mathfrak{g}$ be a Lie algebra. Then the set*

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g};\ [X, Y] = 0 \quad \textit{for all} \quad Y \in \mathfrak{g}\}$$

is an ideal of $\mathfrak{g}$ called the center *of $\mathfrak{g}$.*

Example 2.2.2. The center of $\mathfrak{gl}(n)$ consists of all scalar matrices, and is an abelian ideal. The center of $\mathfrak{sl}(n)$ is trivial. If $n \geq 2$, then the center of $\mathfrak{so}(2n)$ is also trivial; but $Z(\mathfrak{so}(2)) = \mathfrak{so}(2)$ since $\mathfrak{so}(2)$ is abelian.

Definition 2.2.3 (Centralizer and normalizer of a subset)**.** *Let $\mathfrak{s}$ be subset of a Lie algebra $\mathfrak{g}$. The* centralizer *of $\mathfrak{s}$ in $\mathfrak{g}$ is the subalgebra*

$$\mathfrak{g}^{\mathfrak{s}} := \{A \in \mathfrak{g} \colon [A, S] = 0 \textit{ for all } S \in \mathfrak{s}\},$$

and the normalizer *of* $\mathfrak{s}$ *in* $\mathfrak{g}$ *is the subalgebra*

$$N_{\mathfrak{g}}(\mathfrak{s}) := \{N \in \mathfrak{g} \colon [N, S] \in \mathfrak{s} \textit{ for all } S \in \mathfrak{s}\}.$$

Remark 2.2.4. That these two sets above are subalgebras of $\mathfrak{g}$ is a consequence of the Jacobi identity.

Remark 2.2.5. Instead of writing $\mathfrak{g}^{\{X\}}$, we write $\mathfrak{g}^X$.

2.3 Semisimple and reductive Lie algebras

Definition 2.3.1. *A representation of a Lie algebra* $\mathfrak{g}$ *is a linear map* $\varphi \colon \mathfrak{g} \to \mathfrak{gl}(V)$ *for some vector space* V *such that for all* $X, Y \in \mathfrak{g}$, $[\varphi(X), \varphi(Y)] = \varphi([X, Y])$. *The* adjoint representation *is the map* $ad \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *given on each* $X \in \mathfrak{g}$ *by* $(ad(X))(Y) = [X, Y]$ *for all* $Y \in \mathfrak{g}$.

Remark 2.3.2. That *ad* is a representation of $\mathfrak{g}$ is a restatement of the Jacobi identity (Property 2).

The following definitions of a simple, semisimple, and reductive Lie algebra were adapted from [1, §6, No. 1,4,6].

Definition 2.3.3 (Simple, semisimple, and reductive Lie algebras)**.** *A nonzero Lie algebra* $\mathfrak{g}$ *is* simple *if it is nonabelian and contains no nontrivial ideals. It is* semisimple *if it is a direct sum of simple ideals; equivalently, if its only abelian ideal is the zero ideal* $\mathbf{0}$*. A Lie algebra* $\mathfrak{g}$ *is reductive if it is abelian or if* $ad(\mathfrak{g})$ *is semisimple.*

Example 2.3.4. The Lie algebras $\mathfrak{sl}(n)$ for $n \geq 2$ and $\mathfrak{so}(2n)$, for $n \geq 3$, are simple, and therefore semisimple and reductive. The Lie algebra $\mathfrak{gl}(n)$ is reductive since $ad(\mathfrak{gl}(n))$ is isomorphic to $\mathfrak{sl}(n)$, but it is not semisimple since its center is nontrivial.

Since $[\mathfrak{g},\mathfrak{i}]=\mathfrak{i}$ for any simple ideal $\mathfrak{i}$, it follows from this definition that if $\mathfrak{g}$ is reductive, then $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]\oplus Z(\mathfrak{g})$, with $[\mathfrak{g},\mathfrak{g}]$ semisimple. In fact, by [1, §6, No. 6], all reductive Lie algebras have this form.

We end this section by giving a useful criterion to know whether a Lie algebra is semisimple, or if a subalgebra is reductive. First we need to introduce a particular bilinear form on a Lie algebra.

Definition 2.3.5 (The Killing form)**.** *The* Killing form *is the bilinear form* $\kappa\colon \mathfrak{g}\times\mathfrak{g}\to k$, $(X,Y)\mapsto \kappa(X,Y):=\mathrm{Tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y)$.

Lemma 2.3.6. *The Killing form is* $\mathrm{ad}(\mathfrak{g})$*-invariant, that is, for all* $X,Y,Z\in\mathfrak{g}$*, we have*

$$\kappa(\mathrm{ad}(X)Y,Z)+\kappa(Y,\mathrm{ad}(X)Z)=0.$$

Proof: Since ad is a representation of $\mathfrak{g}$, we have $\mathrm{ad}[X,Y]=[\mathrm{ad}\,X,\mathrm{ad}\,Y]=\mathrm{ad}\,X\,\mathrm{ad}\,Y-\mathrm{ad}\,Y\,\mathrm{ad}\,X$. Using also the property that $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$ for two linear maps A and B, we have

$$\begin{aligned}\kappa(\mathrm{ad}(X)Y,Z)&=\mathrm{Tr}(\mathrm{ad}[X,Y]\,\mathrm{ad}\,Z)=\mathrm{Tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y\,\mathrm{ad}\,Z-\mathrm{ad}\,Y\,\mathrm{ad}\,X\,\mathrm{ad}\,Z)\\&=\mathrm{Tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y\,\mathrm{ad}\,Z)-\mathrm{Tr}(\mathrm{ad}\,Y\,\mathrm{ad}\,X\,\mathrm{ad}\,Z)\\&=\mathrm{Tr}(\mathrm{ad}\,Y\,\mathrm{ad}\,Z\,\mathrm{ad}\,X)-\mathrm{Tr}(\mathrm{ad}\,Y\,\mathrm{ad}\,X\,\mathrm{ad}\,Z)\\&=\mathrm{Tr}(\mathrm{ad}\,Y[\mathrm{ad}\,Z,\mathrm{ad}\,X])=-\mathrm{Tr}(\mathrm{ad}\,Y\,\mathrm{ad}(X)Z)\\&=-\kappa(Y,\mathrm{ad}(X)Z)\end{aligned}$$

as required. ∎

The following is from [1, §6, No.1, Theorem 1].

Theorem 2.3.7. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form κ is nondegenerate.*

2.4 Nilpotent Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. Define the sequence of subalgebras $\{\mathfrak{g}^n | n \geq 0\}$ as follows. Let $\mathfrak{g}^0 = \mathfrak{g}$, and for all $i > 0$, $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$. Then $\mathfrak{g}^i$ is an ideal of $\mathfrak{g}$ for all $i \geq 0$. This sequence is called the *descending central series* of $\mathfrak{g}$.

Definition 2.4.1 (Nilpotent Lie algebra)**.** *A Lie algebra $\mathfrak{g}$ is called* nilpotent *if in the descending central series of $\mathfrak{g}$ there exists some $m \geq 0$ such that $\mathfrak{g}^m = 0$.*

Example 2.4.2. Any abelian Lie algebra is nilpotent. Let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{gl}(3)$ consisting of all strictly upper triangular matrices. Then $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ consists of all matrices which are zero except possibly in position $(1, 3)$ and $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] = \{0\}$. Thus $\mathfrak{g}$ is a nilpotent Lie algebra.

The next theorem is from [7, p. 12].

Theorem 2.4.3 (Engel)**.** *A Lie algebra is nilpotent if and only if* $\operatorname{ad} X$ *is a nilpotent endomorphism for all $X \in \mathfrak{g}$.*

2.5 The Jordan-Chevalley decomposition

There are nilpotent and semisimple Lie algebras; but there is a separate and distinct notion of nilpotent and semisimple elements of a reductive Lie algebra, which we will use throughout this thesis. We begin with the Jordan-Chevalley decomposition.

The properties of the Jordan-Chevalley decomposition is used with efficacy in many proofs hence we give here a minimal presentation of it.

Definition 2.5.1 (Abstract Jordan-Chevalley decomposition)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic 0. An element X of $\mathfrak{g}$ is said to have* an abstract Jordan-Chevalley decomposition *if there exists a unique couple (X_s, X_n) respectively of semi-simple and nilpotent elements of $\mathfrak{g}$ such that $[X_s, X_n] = 0$, $X = X_s + X_n$ and for any finite-dimensional representation π of $\mathfrak{g}$, the Jordan-Chevalley decomposition of $\pi(X)$ is $\pi(X_s) + \pi(X_n)$. The elements X_s and X_n are called respectively the* semisimple *and the* nilpotent parts *of X.*

In the case where k is a field of characteristic 0, which is general enough for our purposes, Cagliero and Szechtman [4, Theorem 2] give the following result.

Theorem 2.5.2. *Let k be a field of characteristic 0 and $\mathfrak{g}$ be a Lie algebra over k. An element X of $\mathfrak{g}$ has an abstract Jordan-Chevalley decomposition if and only if X belongs to the derived algebra $[\mathfrak{g}, \mathfrak{g}]$, in which case the semisimple and nilpotent parts of X also belong to $[\mathfrak{g}, \mathfrak{g}]$.*

Thus, for any semisimple Lie algebra, nilpotent and semisimple elements are well-defined. For the purposes of this thesis, it is convenient to extend the notion of nilpotent and semisimple elements to a reductive Lie algebra. The following is from [6, Chapter.1, §1.1].

Definition 2.5.3 (Semisimple and Nilpotent elements for reductive Lie algebras)**.** *Let $\mathfrak{g}$ be a reductive Lie algebra. We define $X \in \mathfrak{g}$ to be* semisimple *if $\operatorname{ad} X$ is a semisimple endomorphism of $\mathfrak{g}$, that is, diagonalizable over an algebraic closure of k. We say X is* nilpotent *if $X \in [\mathfrak{g}, \mathfrak{g}]$ and $\operatorname{ad} X$ is a nilpotent endomorphism of $\mathfrak{g}$.*

It is clear that this definition specializes to the one above if $\mathfrak{g}$ is semisimple. Moreover, the only element which is at once semisimple and nilpotent under this definition is the zero element. Note that the elements of $Z(\mathfrak{g})$, which is the kernel of ad, are defined to be semisimple, since for each $X \in Z(\mathfrak{g})$.

The following example illustrates that the abstract Jordan decomposition does not exist in a general reductive Lie algebra.

Example 2.5.4. Suppose $\mathfrak{g} = k$, an abelian Lie algebra. Then $\varphi_1 = id\colon k \to \mathfrak{gl}(1)$ is a representation of $\mathfrak{g}$ such that $\varphi(X)$ is semisimple for each $X \in \mathfrak{g}$. On the other hand, the map $\varphi_2\colon k \to \mathfrak{gl}(2)$ given by

$$\varphi_2(X) = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$$

is a representation with respect to which $\varphi_2(X)$ is nilpotent for each $X \in \mathfrak{g}$. Therefore the elements of $\mathfrak{g}$ do not have an abstract Jordan decomposition.

2.6 Cartan and toral subalgebras

We now define a class of subalgebras which are important to the structure theory of reductive Lie algebras. The following definition is of [3, Chapter VII, §2, No.1, Definition 1].

Definition 2.6.1 (Cartan subalgebras)**.** *A* Cartan subalgebra *of a Lie algebra* $\mathfrak{g}$ *is a subalgebra which is nilpotent and self-normalizing in* $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ over k admits a nonzero Cartan subalgebra $\mathfrak{h}$. See [3, Chapter VII, §2, No.3, Corollary 1]. For example, one can show that the abelian subalgebra of diagonal matrices in $\mathfrak{gl}(n)$ is its own normalizer, hence it is a Cartan subalgebra of $\mathfrak{gl}(n)$.

In the particular case where $\mathfrak{g}$ is nilpotent, we know from [3, Chapter VII, §2, No.1, Example 1] that the only Cartan subalgebra of $\mathfrak{g}$ is $\mathfrak{g}$ itself.

Definition 2.6.2 (Toral subalgebras)**.** *Let $\mathfrak{g}$ be a reductive Lie algebra. A* toral subalgebra *of $\mathfrak{g}$ is a nonzero subalgebra which consists of semisimple elements.*

The following elementary lemma is from [7, Chapter II, §8.1, Lemma], and will be handy in future proofs. Even though the author specifies that the Lie algebra be semisimple, our definition of a semisimple element allows us to extend his proof in the reductive case.

Lemma 2.6.3. *Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field k. Then a toral subalgebra of $\mathfrak{g}$ is abelian.*

Example 2.6.4. The subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}(n)$ of diagonal matrices D such that $\mathrm{Tr}(D) = 0$ is a toral subalgebra.

The following theorem is from [7, 15.3, Corollary and 8.1, Lemma].

Theorem 2.6.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra over k. Then the Cartan subalgebras of $\mathfrak{g}$ are exactly the maximal toral subalgebras of $\mathfrak{g}$. In particular Cartan subalgebras are abelian.*

To extend this to the reductive case, first note the following lemma, from [3, Chapter VII, §2, Proposition 2].

Lemma 2.6.6. *Let $(\mathfrak{g}_i)_{i\in I}$ be a finite family of Lie algebras over k and $\mathfrak{g} = \oplus_{i\in I}\mathfrak{g}_i$. The Cartan subalgebras of $\mathfrak{g}$ are the subalgebras of the form $\prod_{i\in I}\mathfrak{h}_i$ where $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$.*

Combining this result with the fact that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$ when $\mathfrak{g}$ is reductive and knowing that $Z(\mathfrak{g})$ is the only Cartan subalgebra of the center of $\mathfrak{g}$, we have the following result.

Corollary 2.6.7. *Let $\mathfrak{g}$ be a reductive Lie algebra. Then the Cartan subalgebras of $\mathfrak{g}$ are the subalgebras of the form $\mathfrak{h} \oplus Z(\mathfrak{g})$, where $\mathfrak{h}$ is a Cartan subalgebra of $[\mathfrak{g}, \mathfrak{g}]$.*

We now prove the following lemma.

Lemma 2.6.8. *Suppose $\mathfrak{g}$ is a reductive Lie algebra and $\mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, where $\mathfrak{h}$ is a Cartan subalgebra of $[\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}$, and all maximal toral subalgebras of $\mathfrak{g}$ are of this form.*

Proof: With our definition of nilpotent and semisimple elements of a reductive Lie algebra, elements of $Z(\mathfrak{g})$ are all semisimple. Thus since $\mathfrak{h}$ is toral, $\mathfrak{t}$ is an abelian Lie algebra consisting of semisimple elements, that is, toral.

Suppose $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}$. Then since $Z(\mathfrak{g}) \oplus \mathfrak{t}$ is a subalgebra consisting of semisimple elements, it is toral, so by maximality equals $\mathfrak{t}$. Thus $Z(\mathfrak{g}) \subseteq \mathfrak{t}$. Write an element $T \in \mathfrak{T}$ as $T = Z + Y$ for some $Z \in Z(\mathfrak{g})$ and some $Y \in [\mathfrak{g}, \mathfrak{g}]$. Then $T - Z \in \mathfrak{t}$ so $Y \in \mathfrak{h} := \mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]$. Thus $\mathfrak{t} = Z(\mathfrak{g}) + \mathfrak{h}$. Now $\mathfrak{h}$ is maximal toral in $[\mathfrak{g}, \mathfrak{g}]$ since $\mathfrak{t}$ is maximal toral in $\mathfrak{g}$, so $\mathfrak{h}$ is a Cartan subalgebra of $[\mathfrak{g}, \mathfrak{g}]$, as required. ∎

We remark that maximal toral subalgebras of a non-reductive subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ do not need to coincide with Cartan subalgebras of $\mathfrak{r}$. For example, if $\mathfrak{r}$ is an abelian subalgebra consisting of nilpotent elements, then it is its own Cartan subalgebra but has no toral subalgebras.

2.7 Structure of semisimple Lie algebras

Throughout this section, we assume $\mathfrak{g}$ is a semisimple Lie algebra over $k = \mathbb{C}$. The results in this section are standard for semisimple Lie algebras and extend to the reductive case directly.

Fix a choice of Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then the elements of $\mathfrak{h}$ acts semisimply on $\mathfrak{g}$ under the adjoint action. Since $\mathfrak{h}$ is abelian, this action is simultaneously diagonalizable. This leads to the following definition.

Definition 2.7.1 (roots, root space decomposition)**.** *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, let $\mathfrak{g}_\alpha$ be the subspace*

$$\{X \in \mathfrak{g} |\, [H, X] = \alpha(H)X \quad \textit{for all} \quad H \in \mathfrak{h}\}$$

and let Φ be the subset of $\mathfrak{h}^* \setminus \{0\}$ such that $\mathfrak{g}_\alpha$ is nonzero. The elements of Φ are called the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$*, the spaces $(\mathfrak{g}_\alpha)_{\alpha\in\Phi}$ are* the root spaces of $\mathfrak{g}$ relative to $\mathfrak{h}$*. The* root space decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$, or the Cartan decomposition of $\mathfrak{g}$ *is*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

Remark 2.7.2. It is possible for a semisimple Lie algebra over a non-algebraically closed field k to admit a Cartan decomposition; in this case, we say the Lie algebra *splits* over k.

Example 2.7.3. Let $\mathfrak{g} = \mathfrak{sl}(n)$ and fix $\mathfrak{h}$ as the Cartan subalgebra of diagonal matrices in $\mathfrak{g}$. For $H = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \in \mathfrak{h}$, let $\alpha_{i,j}(H) := (\epsilon_i - \epsilon_j)(H) = \lambda_i - \lambda_j$ for $1 \le i, j \le n$. The roots of $\mathfrak{sl}(n)$ are exactly the $\alpha_{i,j}$'s for $1 \le i \ne j \le n$ and the root spaces are given by $\mathfrak{g}_{\alpha_{i,j}} = \{\lambda E_{i,j} | \lambda \in k\}$, where $E_{i,j}$ is the elementary $n \times n$ matrix having 1 at the entry (i, j) and 0 elsewhere.

Definition 2.7.4 (simple and positive roots)**.** *[7, p.47] Given $(\mathfrak{g}, \mathfrak{h}, \Phi)$ as in 2.7.1, a* base *is a subset Δ of Φ such that:*

- *Δ is a basis of $\mathrm{span}_{\mathbb{C}}(\Phi)$; and*

- *any* $\beta \in \Phi$ *can be written as a sum* $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ *such that* $k_\alpha \in \mathbb{Z}$ *and moreover, either all* $k_\alpha \geq 0$ *for all* $\alpha \in \Delta$ *or else* $k_\alpha \leq 0$ *for all* $\alpha \in \Delta$.

An element of Δ *is called a* simple root. *A* positive root *is a* $\beta \in \Phi$ *such that all the coefficients* k_α *in the previous decomposition of* β *as sum of simple roots are positive; the set of all such is denoted* Φ^+ *and depends on the choice of* Δ.

One can prove that base of simple roots exists, see for instance [7].

Example 2.7.5. Let $\mathfrak{g} = \mathfrak{sl}(n)$, with $\mathfrak{h}$ and $\Phi = \{\alpha_{i,j} = \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$ as before. Then one can see directly that a base for Φ is $\Delta = \{\alpha_{i,i+1} \mid 1 \leq i < n\}$, with respect to which $\Phi^+ = \{\alpha_{i,j} = \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}$.

The sets Φ arising from semisimple Lie algebras are examples of *root systems*, in the following sense.

Definition 2.7.6 (Root sytem, Weyl group)**.** *[7, p.42] Given a real euclidean space* $(E, (\cdot,\cdot))$*, a* (crystallographic) root system *in* E *is a subset* Φ *of* E *satisfying:*

1. Φ *is finite.*
2. *If* $\alpha \in \Phi$*, the only multiples of* α *in* Φ *are* α *and* $-\alpha$.
3. *If* $\alpha \in \Phi$*, then* Φ *is invariant by the reflection* σ_α *defined by* $\sigma_\alpha(\beta) := \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$.
4. *If* $\alpha, \beta \in \Phi$*, then* $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \mathbb{Z}$.

The Weyl group of a root system Φ *is the subroup of* $GL(E)$ *generated by the reflections* σ_α *for* $\alpha \in \Phi$.

This is related to root systems in Lie algebras as follows. Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ and let Φ denote the root system of $\mathfrak{g}$ relative to

$\mathfrak{h}$. By [7, §8.1, p.36, Corollary], the restriction of the Killing form to $\mathfrak{h}$ is nondegenerate. Therefore the application $\mathfrak{h} \to \mathfrak{h}^*$ defined by $H \mapsto \kappa(H, \cdot)$ is an isomorphism of vector spaces. Given any two roots $\alpha, \beta \in \Phi$, we define $(\alpha, \beta) := \kappa(T_\alpha, T_\beta)$, where T_α and T_β are respectively the inverse images of α and β via the aforementioned isomorphism. Then the $\mathbb{R}$–vector space $E = \mathrm{span}_{\mathbb{R}}(\Phi) \subset \mathfrak{h}^*$ endowed with $(\cdot, \cdot)$ is an euclidean space and Φ is a root system in E. One can prove that the roots Φ of a semisimple Lie algebra relative to a Cartan subalgebra $\mathfrak{h}$ spans $\mathfrak{h}^*$, see [7, p.37, §8.3, Proposition], that is, $\mathrm{span}_{\mathbb{C}}(E) = \mathfrak{h}^*$.

Example 2.7.7. It can be shown by direct calculation that for $\mathfrak{g} = \mathfrak{sl}(n)$, the Killing form is given by $\kappa(X, Y) = 2n\mathrm{Tr}(XY)$. It is more convenient to replace κ by its scalar multiple, the trace form; then setting $H_{i,j}$ to be the diagonal matrix $E_{ii} - E_{jj}$, we have $\alpha_{i,j}(H) = \mathrm{Tr}(H_{i,j}H)$. (We would have $T_{\alpha_{i,j}} = \frac{1}{2n} H_{i,j}$.) Under this identification, the reflection $\sigma_{i,i+1}$ acts on $H \in \mathfrak{h}$ by permuting the i and $i+1$ entries; thus the Weyl group is isomorphic to the symmetric group S_n.

Now let $n_{\alpha\beta} := \frac{4(\alpha,\beta)^2}{(\alpha,\beta)(\alpha,\beta)} = 4\cos^2\theta_{\alpha\beta}$, where $\theta_{\alpha\beta}$ is the angle between α and β for $\alpha, \beta \in \Delta$. Since $n_{\alpha\beta} \in \mathbb{Z}$, we can prove that $n_{\alpha\beta}$ can only take the values 0, 1, 2, 3. When $n_{\alpha\beta} = 1$ then α and β have the same length. In the cases where $n_{\alpha\beta} = 2$ or 3, one of α and β is respectively $\sqrt{2}$ or $\sqrt{3}$ times longer than the other. When $n_{\alpha\beta} = 0$, nothing can be said about the proportions of the lengths of α and β.

Definition 2.7.8 (Dynkin diagram)**.** *The* Dynkin diagram *of a root system Φ is a directed graph with $l = |\Delta|$ nodes, each labelled by one simple root of Φ such that two nodes α and β are connected by an edge of multiplicity $n_{\alpha\beta}$ which, in the case that $n_{\alpha\beta} > 1$, is directed toward the shorter root.*

Example 2.7.9. Let $\mathfrak{g} = \mathfrak{sl}(n)$, with Δ as above. If $1 \le i < j < n$, then we can see that $\mathrm{Tr}(H_{i,i+1}H_{j,j+1}) = \delta_{i+1,j}$ and $\mathrm{Tr}(H_{i,i+1}H_{i,i+1}) = 2$, so $n_{\alpha_{i,i+1}\alpha_{j,j+1}} = \delta_{i+1,j}$. This yields the

Dynkin diagram of $\mathfrak{sl}(n)$ as follows, where the simple roots from left to right are $\alpha_{1,2}, \ldots, \alpha_{n-1,n}$:

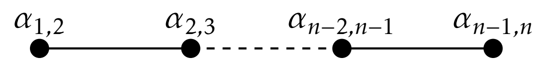


Figure 2.1: Dynkin diagram of type A_n

In fact, from the Dynkin diagram of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ one can reconstruct $\mathfrak{g}$ uniquely up to isomorphism. We direct the reader to [7, 8.5, Theorem], [7, 11.4, Theorem] and [7, 12.1, Theorem].

Remark 2.7.10. We will present the root system for the Lie algebra $\mathfrak{so}(2n)$ in a later chapter.

2.8 Parabolic and Levi subalgebras

In this section, we will define the most used subalgebras in this work: Borel, Levi, and parabolic subalgebras. Those definitions are related. The following definition is from [7, §16.3, p.83].

Definition 2.8.1 (Borel subalgebra)**.** *A* Borel subalgebra *in a Lie algebra $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$.*

Definition 2.8.2 (Levi subalgebra and parabolic subalgebra)**.** *A* Levi subalgebra of *$\mathfrak{g}$ is a subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ which contains a Cartan subalgebra $\mathfrak{h}$ and such that there exists a subset $\Delta' \subseteq \Delta \subset \Phi$ of the simple roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ satisfying*

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Delta' \rangle} \mathfrak{g}_\alpha,$$

where $\langle \Delta' \rangle$ denotes the intersection of the $\mathbb{Z}$–span of Δ' in $\mathfrak{h}^*$ and Φ.

A parabolic subalgebra $\mathfrak{p}$ *is a subalgebra of* $\mathfrak{g}$ *such that there exists a Levi subalgebra* $\mathfrak{l}(\Delta') = \mathfrak{h} \oplus \bigoplus_{\alpha\in\langle\Delta'\rangle} \mathfrak{g}_\alpha$ *of* $\mathfrak{g}$ *for which*

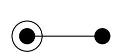
$$\mathfrak{p} = \mathfrak{l} \oplus \bigoplus_{\alpha\in\Phi^+\setminus\langle\Delta'\rangle} \mathfrak{g}_\alpha.$$

Alternately, parabolic subalgebras can be characterized as subalgebras which contain a Borel subalgebra; they are proven to decompose as a direct sum of their unipotent radical and a reductive subalgebra, which is a Levi subalgebra, see [6, Chapter 3, §3.8, Lemma 3.8.1 (iv)]

Example 2.8.3. If $\mathfrak{g} = \mathfrak{sl}(n)$, then with respect to our standard choices, we have the Borel subalgebra $\mathfrak{b}$ consisting of all upper triangular matrices in $\mathfrak{g}$, which is also a parabolic subalgebra relative to $\Delta' = \emptyset$. If $n = 3$ and $\Delta = \{\alpha_{1,2}\}$, then the corresponding standard Levi subalgebra and parabolic subalgebra are

$$\mathfrak{l} = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & -a-d \end{pmatrix} \mid a,b,c,d \in k \right\} \subset \mathfrak{p} = \left\{ \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & -a-d \end{pmatrix} \mid a,b,c,d,e,f \in k \right\}.$$

We note that one can parametrize the standard Levi subalgebras of $\mathfrak{g}$ by colouring the vertices of the Dynkin diagram corresponding to the simple roots in Δ'. For example, the colored Dynkin diagram corresponding to the Levi subalgebra $\mathfrak{l}$ is



2.9 Chevalley algebras and groups

The material of this section is mainly from [5]. We offer a short summary of how to construct a matrix group over any field K from a given complex Lie algebra $\mathfrak{g}$, called its *Chevalley group.*

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be its Cartan decomposition. For $\alpha \in \Phi$, let $T_\alpha \in \mathfrak{h}$ such that $\alpha(H) = \kappa(T_\alpha, H)$ for all $H \in \mathfrak{h}$ and let $H_\alpha = \frac{2T_\alpha}{\kappa(T_\alpha,T_\alpha)}$. It is a key result [5, p.56, Theorem 4.2.1] that there exists a choice of basis E_α of each root space $\mathfrak{g}_\alpha$ such that for all $\alpha, \beta \in \Phi$

- $[H_\alpha, H_\beta] = 0$,
- $[H_\beta, E_\alpha] = M_{\alpha\beta}E_\alpha$ for some $M_{\alpha\beta} \in \mathbb{Z}$,
- $[E_\alpha, E_{-\alpha}] = H_\alpha$,
- $[E_\alpha, E_\beta] = 0$ if $\alpha + \beta \notin \Phi$,
- $[E_\alpha, E_\beta] = A_{\alpha\beta}E_{\alpha+\beta}$ for some $A_{\alpha\beta} \in \mathbb{Z}$, if $\alpha + \beta \in \Phi$.

In other words there exists a basis of $\mathfrak{g}$ such that the structure constants of $\mathfrak{g}$ according to that basis are all integers. Such a basis $\mathcal{C} = \{H_\alpha, \alpha \in \Delta; E_\beta, \beta \in \Phi\}$ is called a *Chevalley basis of* $\mathfrak{g}$. Now, given a Chevalley basis $\mathcal{C}$ of $\mathfrak{g}$ as above let $\mathfrak{g}_\mathbb{Z}$ be the $\mathbb{Z}$–module with basis $\mathcal{C}$. If K is an arbitrary field, then $\mathfrak{g}_K := K \otimes_\mathbb{Z} \mathfrak{g}_\mathbb{Z}$ is a vector space over K with basis $\mathcal{C}' = \{1_K \otimes H_\alpha, \alpha \in \Delta; 1_K \otimes E_\beta, \beta \in \Phi\}$.

In turn we endow $\mathfrak{g}_K$ with the bilinear operator $[,]$ defined by $[1_K \otimes X, 1_K \otimes Y] := 1_K \otimes [X, Y]$ for all X, Y in $\mathcal{C}$, which when extended linearly makes $\mathfrak{g}_K$ into a Lie algebra over K, called a *Chevalley algebra*.

On the other hand, since $\operatorname{ad} E_\alpha$ is a nilpotent derivation of $\mathfrak{g}$ (see [5, §4.3] for a detail of the proof) for each $\alpha \in \Phi$, for any $t \in K$ the exponential map

$$X_\alpha(t) := \exp(t \operatorname{ad} E_\alpha) := \sum_{n=0}^{\infty} \frac{(t \operatorname{ad} E_\alpha)^n}{n!}$$

is a finite sum, hence a well-defined endomorphism of $\mathfrak{g}$. One proves that its inverse is $X_\alpha(-t)$, and that it is an automorphism of $\mathfrak{g}$, see [5, §4.3, Lemma 4.3.1.]. Moreover, it is shown that the image of an element of the Chevalley basis by $X_\alpha(t)$ is a linear combination of elements of the basis with coefficients in K.

Remark 2.9.1. In fact, this also works if K is a field of characteristic p. Let $\bar{X}_\alpha(t)$ be the endomorphism of $\mathfrak{g}_K$ whose matrix with respect to the basis $\mathcal{C}'$ is obtained from the matrix of $X_\alpha(t)$ (with respect to $\mathcal{C}$) by reducing modulo the prime field of K its coefficients. It is proven in [5, §4.4, Proposition 4.4.2] that $\bar{X}_\alpha(t)$ is a $K-$automorphism of the Lie algebra $\mathfrak{g}_K$ for all $\alpha \in \Phi$ and $t \in K$.

Definition 2.9.2. *The* Chevalley group of $\mathfrak{g}$ over K *is the subgroup* $G_{\mathrm{ad}}(K)$ *of the* K*–automorphisms of* $\mathfrak{g}_K$ *generated by the elements* $X_\alpha(t)$ *for all* $t \in K$ *and* $\alpha \in \Phi$.

As one might expect, the group $G_{\mathrm{ad}}(K)$ is uniquely determined up to isomorphism by the Lie algebra $\mathfrak{g}$ and the field K, see [5, §4.4, Proposition 4.4.3]. If $\mathfrak{g}$ is a Lie algebra over K, the Chevalley group of $\mathfrak{g}$ over K will be simply denoted G_{ad}.

If we fix $\alpha \in \Phi$ then the one-parameter subgroup $U_\alpha = \{X_\alpha(t) \mid t \in K\}$ is called a *root subgroup* of G_{ad}. There exists also a subgroup H of G_{ad}, called the Cartan subgroup corresponding to $\mathfrak{h}$, whose action on $\mathfrak{g}_K$ preserves the Cartan decomposition of $\mathfrak{g}_K$. For more details, the reader may see [5, Chapters 5, 7].

Let us apply the notations of Section 2.8. Then the subgroup B generated by H together with the root subgroups U_α, for $\alpha \in \Phi^+$, is the Borel subgroup of G_{ad}

corresponding to $\mathfrak{b}$, and we may similarly define $L = \langle H, U_\alpha \mid \alpha \in \langle \Delta' \rangle \rangle$ the Levi subgroup of G_{ad} corresponding to $\mathfrak{l}$, and $P = \langle L, B \rangle$ as the parabolic subgroup corresponding to $\mathfrak{p}$.

Finally, if $\mathfrak{g}$ is a semisimple Lie algebra and G_{ad} its adjoint Chevalley group, we define the *adjoint representation of* G_{ad} to be the natural action of elements of G_{ad} on $\mathfrak{g}$, expressed as

$$\begin{aligned} \mathrm{Ad}\colon G_{\text{ad}} &\to \mathrm{Aut}(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}(g)\colon X \mapsto g \cdot X. \end{aligned}$$

Remark 2.9.3. $\mathrm{Aut}(\mathfrak{g})$ can be identified as a quotient of a subgroup of $GL(n^2)$ if $\mathfrak{g} \subseteq \mathfrak{gl}(n)$, and in this case, $g \cdot X$ is $g \cdot X \cdot g^{-1}$ where $\cdot$ is here matrix multiplication.

The following theorem defines the key relationship between $\mathfrak{g}$ and G_{ad} when $k = \mathbb{C}$. It is proven, for example, in [7, §16].

Proposition 2.9.4 (G_{ad}–Conjugacy of some subalgebras of $\mathfrak{g}$ over an algebraically closed field)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with associated Chevalley group G_{ad}, which acts on $\mathfrak{g}$ via* Ad*. Then all Cartan subalgebras of $\mathfrak{g}$ are conjugate via* Ad*. Moreover, all Borel subalgebras, Levi subalgebras, and parabolic subalgebras are conjugate via* Ad *to standard such subalgebras.*

Now we can define the central theme of this work.

Definition 2.9.5 (Nilpotent orbit)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over k and G_{ad} its adjoint group (over k). A* nilpotent orbit *in $\mathfrak{g}$ is a subset of $\mathfrak{g}$ of the form $G_{\text{ad}} \cdot X$, where X is a nilpotent element of $\mathfrak{g}$ and $G_{\text{ad}} \cdot X = \{\mathrm{Ad}(g)X \mid g \in G_{\text{ad}}\}$.*

We denote by $\mathcal{O}_X$ the set $G_{\text{ad}} \cdot X$.

Finally, when $\mathfrak{g}$ is a complex semisimple Lie algebra, we have the following conjugacy theorem, from [7, p. 84, 16.4, Theorem].

Theorem 2.9.6. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A Borel subalgebra $\mathfrak{b}$ is G_{ad}–conjugated to the standard Borel subalgebra*

$$\mathfrak{h} \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and Φ^+ is a choice of positive roots of $\mathfrak{h}$.

Chapter 3

Representation theory of $\mathfrak{sl}_2(k)$

The purpose of this chapter is to give a brief summary of some necessary results regarding finite-dimensional $\mathfrak{sl}_2(k)$–modules, also called representations of $\mathfrak{sl}_2(k)$ over k. We classify the irreducible modules and prove that for any positive integer n, we can construct an irreducible representation of dimension n. At the same time, we give an explicit basis for each irreducible representation which is opposite to the one usually given in the literature, but which is more convenient for our purposes. Our main reference is [7]; the choice of basis proven here is also used in [6].

3.1 The Lie algebra $\mathfrak{sl}_2(k)$

In this section we give some basic facts about the Lie algebra $\mathfrak{sl}_2(k)$. Let k be a field of characteristic zero. Recall that $\mathfrak{sl}_2(k)$ is the Lie algebra of 2×2 matrices over k

with trace 0. A basis for $\mathfrak{sl}_2(k)$ is given by $\{X, H, Y\}$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying the relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad \text{and} \quad [X, Y] = H. \tag{3.1.1}$$

More generally, in a Lie algebra $\mathfrak{g}$, any three elements $H, X, Y \in \mathfrak{g}$ satisfying the above relations are called an $\mathfrak{sl}_2$–triple, and they span a subalgebra isomorphic to $\mathfrak{sl}_2(k)$.

Lemma 3.1.1. *In the Lie algebra $\mathfrak{sl}_2(k)$, H is semisimple, and X, Y are nilpotent.*

Proof: We apply Definition 2.5.3 to these elements of the simple, hence reductive, Lie algebra $\mathfrak{sl}_2(k)$. The matrices of $\operatorname{ad} H$, $\operatorname{ad} X$ and $\operatorname{ad} Y$ with respect to the basis $\{X, H, Y\}$ are respectively

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Since the matrix of $\operatorname{ad} H$ is diagonal, H is semisimple. The matrices of $\operatorname{ad} X$ and $\operatorname{ad} Y$ are nilpotent; furthermore, from the relations $[\frac{1}{2}H, X] = X$ and $[-\frac{1}{2}H, Y] = Y$, we deduce that X and Y are in $[\mathfrak{sl}_2(k), \mathfrak{sl}_2(k)] = \mathfrak{sl}_2(k)$. Therefore they are nilpotent elements of $\mathfrak{sl}_2(k)$. ∎

As remarked in Section 2.5 this result implies that the image of H in any representation of $\mathfrak{sl}_2(k)$ is also semisimple.

In this chapter, we are interested in representations of $\mathfrak{sl}_2(k)$ over k. Such a representation is a vector space V over k equipped with an action of $\mathfrak{sl}_2(k)$, in other words a Lie algebra homomorphism

$$\phi\colon \mathfrak{sl}_2(k) \to \mathfrak{gl}(V), \quad A \mapsto \phi(A)\colon V \to V, \quad v \mapsto A \cdot v := \phi(A)v.$$

More precisely, for any $A \in \mathfrak{sl}_2(k)$, we have a linear map

$$\begin{aligned} V &\to V \\ v &\mapsto A \cdot v \end{aligned}$$

such that $[A, B] \cdot v = A \cdot (B \cdot v) - B \cdot (A \cdot v)$ for all $v \in V$ and $A, B \in \mathfrak{sl}_2(k)$.

Two representations V and W of $\mathfrak{sl}_2(k)$ are isomorphic if there exists an isomorphism $T\colon V \to W$ of vector spaces such that for each $A \in \mathfrak{sl}_2(k)$ and every $v \in V$, we have

$$A \cdot T(v) = T(A \cdot v).$$

We can also say that T *intertwines* the action of A on the two modules.

We state a key result of the representation theory of semisimple Lie algebras, without proof; see for example [7, p. 28, §6.3, Theorem].

Theorem 3.1.2 (Weyl's Theorem)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field k of characteristic zero. Then every finite-dimensional module of $\mathfrak{g}$ over k decomposes as a direct sum of simple modules.*

Therefore, it suffices to classify the simple modules, also called irreducible representations, of $\mathfrak{sl}_2(k)$.

3.2 Simple $\mathfrak{sl}_2(k)$-modules

Let V be a finite-dimensional representation of $\mathfrak{sl}_2(k)$ over k; then as remarked, the image of H in $\mathfrak{gl}(V)$ is semisimple.

Suppose first that all the eigenvalues λ of H lie in k. Then V can be written as the direct sum of the eigenspaces V_λ under the action of H, that is

$$V = \bigoplus_\lambda V_\lambda,$$

where $V_\lambda = \{v \in V \mid H \cdot v = \lambda v\}$. The eigenvalues of the action of H are called the *weights of* H. When V_λ is not zero, then it is called a *weight space* and nonzero element of V_λ is called *a weight vector* (of weight λ).

Lemma 3.2.1. *If $v \in V$ is a vector of weight λ then $X \cdot v$ is a vector of weight $\lambda + 2$ and $Y \cdot v$ is a vector of weight $\lambda - 2$.*

Proof: We have $H \cdot (X \cdot v) = [H, X] \cdot v + X \cdot H \cdot v = 2X \cdot v + X \cdot (\lambda v) = (\lambda + 2)X \cdot v$. The same argument gives $H \cdot (Y \cdot v) = (\lambda - 2)v$. ∎

Since V is a direct sum of a finite number of weight spaces, it follows from 3.2.1 that there exists a weight λ and a nonzero $w_0 \in V_\lambda$ such that $Y \cdot w_0 = 0$. Such an element w_0 is called a *lowest weight vector* of V.

Proposition 3.2.2. *Suppose that V is an $\mathfrak{sl}_2(k)$-module such that all the weights of H lie in k. Let w_0 be a lowest weight vector of weight $-\lambda$. Set $w_i = X \cdot w_{i-1}$ for $i \geq 1$. Then the span of $\{w_0, w_1, \ldots\}$ is invariant under the action of $\mathfrak{sl}_2(k)$, with action given by*

1. *$H \cdot w_i = (-\lambda + 2i)w_i$ for all $i \geq 0$;*

2. *$Y \cdot w_i = \mu_i w_{i-1}$, where $\mu_i = i(\lambda - i + 1)$, for all $i \geq 0$; and*

3. *$X \cdot w_i = w_{i+1}$, for all $i \geq 0$.*

Furthermore, if V is irreducible and finite-dimensional, then $\dim(V) = \lambda+1$; in particular, $\lambda \in \mathbb{N}$ and all the weights of H on V are integers.

Proof: Showing that the three formulae hold will imply that the span of these vectors is invariant under the action of $\mathfrak{sl}_2(k)$. The formula (3) is by definition, and the formula (1) follows from Lemma 3.2.1, as follows. Since the weight of w_0 is $-\lambda$, Lemma 3.2.1 implies that the weight of w_1 is $(-\lambda + 2)$. Thus by induction, we have that the weight of w_i is $-\lambda + 2i$ for all $i \geq 0$. Hence $H \cdot w_i = (\lambda + 2i)w_i$, for all $i \geq 0$.

We prove formula (2) by induction. Since $0(\lambda - 0 + 1) = 0$ and $Y \cdot w_0 = 0$, it holds for $i = 0$. Now assume it is true for some $i \geq 0$. We have

$$\begin{aligned}
Y \cdot w_{i+1} &= Y \cdot X \cdot w_i \\
&= X \cdot Y \cdot w_i - [X, Y] \cdot w_i \\
&= X \cdot (i(\lambda - i + 1))w_{i-1} - H \cdot w_i, \text{ by the inductive hypothesis} \\
&= i(\lambda - i + 1)w_i - (-\lambda + 2i)w_i, \text{ using formulae (1) and (3)} \\
&= (i+1)(\lambda - i)w_i,
\end{aligned}$$

as required. Thus the formula (2) holds for all $i \geq 0$.

Now suppose that V is irreducible and finite-dimensional. If each w_i were nonzero, then the infinite set $\{w_0, w_1, \ldots\}$ would be linearly independent, since it consists of eigenvectors of the action of H corresponding to distinct eigenvalues; this is impossible. Therefore there exists a minimal $m \in \mathbb{N}$ such that $w_{m+1} = 0$. By (2), $0 = Y \cdot w_{m+1} = \mu_{m+1} w_m$; since $w_m \neq 0$, we must have $0 = \mu_{m+1} = (m+1)(\lambda - m)$, which implies (since $m \geq 0$) that $\lambda = m \in \mathbb{N}$. Moreover, since V is irreducible, the submodule $\{w_0, w_1, \cdots, w_m\}$ must equal all of V, whence $\dim(V) = m + 1 = \lambda + 1$. The rest of the proposition follows. ∎

Lemma 3.2.3. *If V is an irreducible finite-dimensional $\mathfrak{sl}_2(k)$-module over k, then all of the eigenvalues of the action of H on V lie in k, so V has the form given in the previous lemma.*

Proof: Since H is semisimple, construct a finite extension field k' of k containing the eigenvalues $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ the action of H on V. Set $V_{k'} = V \otimes_k k'$. Concretely, if $\{v_0, \ldots, v_n\}$ is any basis of V, then $\{v_0 \otimes 1, \ldots, v_n \otimes 1\}$ is a basis of $V_{k'}$ over k'; therefore we can and do identify these two bases. Similarly, we can identify the basis $\{H, X, Y\}$ of $\mathfrak{sl}_2(k)$ with the basis $\{H \otimes 1, X \otimes 1, Y \otimes 1\}$ of $\mathfrak{sl}_2(k') = \mathfrak{sl}_2(k) \otimes_k k'$.

The $\mathfrak{sl}_2(k)$-module structure on V uniquely defines an $\mathfrak{sl}_2(k')$-module structure on $V_{k'}$, as follows. For each $A \in \mathfrak{sl}_2(k)$, $v \in V$ and $s, t \in k'$, set

$$(A \otimes t) \cdot (v \otimes s) = (A \cdot v) \otimes ts$$

and extend this to $V_{k'}$ by linearity.

Now suppose V is an irreducible finite-dimensional $\mathfrak{sl}_2(k)$ module. Then $V_{k'}$ is a finite-dimensional $\mathfrak{sl}_2(k')$ module, hence completely reducible by Weyl's theorem. Each submodule of $V_{k'}$ is invariant under H, hence is a direct sum of weight spaces; by construction these weights lie in $\Lambda \subset k'$. Thus Proposition 3.2.2 applies over k', and implies that all the weights of these submodules are in fact integers. Thus $\Lambda \subset \mathbb{Z} \subset k$. ∎

Lemma 3.2.4. *There exists an irreducible $\mathfrak{sl}_2(k)$ module of dimension n, for every positive* *integer n, and it is unique up to isomorphism of modules.*

Proof: To prove existence, we verify that if one sets $\lambda = m = n-1$ and chooses a basis $\{w_0, w_1, \cdots, w_m\}$ of k^n, then the formulae (1), (2), (3) define a Lie algebra homomorphism of $\mathfrak{sl}_2(k)$ into $\mathfrak{gl}(n,k)$. It suffices by linearity to verify that for each $i \in \{0,1,\ldots,m\}$, and for each pair U, U' in a basis of $\mathfrak{sl}_2(k)$, that

$$[U, U'] \cdot w_i = U \cdot U' \cdot w_i - U' \cdot U \cdot w_i.$$

We must thus show that the relations of (3.1.1) are preserved under this action. These are straightforward to show and so we show only one. For example, if $i > 0$ then

$$X \cdot Y \cdot w_i - Y \cdot X \cot w_i = X \cdot (\mu_i w_{i-1}) - Y \cdot (w_{i+1}) = (\mu_i - \mu_{i+1}) w_i.$$

Since $\mu_i - \mu_{i+1} = i(\lambda - i + 1) - (i+1)(\lambda - i) = i - (\lambda - i) = 2i - \lambda$, we conclude that

$$X \cdot Y \cdot w_i - Y \cdot X \cot w_i = (2i - \lambda) w_i = H \cdot w_i = [X, Y] \cdot w_i$$

as required. If $i = 0$, then $Y \cdot w_0 = 0$, and $Y \cdot (X \cdot w_0) = Y \cdot w_1 = \lambda w_0$, so again the equality is seen to follow.

If V and W are two irreducible representations of $\mathfrak{sl}_2(k)$ over k of the same dimension $n \geq 1$, then by Proposition 3.2.2, they each admit a basis of weight vectors satisfying the relations (1), (2), (3), and therefore the linear isomorphism identifying these bases intertwines the two representations of $\mathfrak{sl}_2(k)$. ∎

These propositions give us a very explicit realization of the simple modules of V. In the next section, we derive some minor results about this realization that we can use later to quickly decompose an $\mathfrak{sl}_2(k)$-module into simple modules.

3.3 Decompositions of $\mathfrak{sl}_2(k)$-modules

Suppose V is an $\mathfrak{sl}_2(k)$-module of dimension n and choose bases for its simple submodules as per Proposition 3.2.2. Then we note that the matrix of the action of X is the Jordan canonical form of a nilpotent element in $\mathfrak{gl}(n,k)$, with partitions corresponding to the dimensions of the simple submodules. For example, if $V = W \oplus W'$ with $\dim(W) = 3$ and $\dim(W') = 2$, then

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, the matrix of H is diagonal, with entries decreasing in steps of 2 and symmetric about zero on each simple summand. In the above example, the diagonal entries of H would be $(2,0,-2,1,-1)$.

We can infer much more about the decomposition of V into irreducible subrepresentations from consideration of these weights.

Lemma 3.3.1. *Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_2(k)$ and let V_λ denote the λ-weight space of V, so that $V = \oplus_\lambda V_\lambda$. Then*

1. *the number of irreducible summands of V is* $\dim(V_0) + \dim(V_1)$*;*
2. *if* $\dim(V_1) = 0$*, then all submodules of V have odd dimension;*
3. *if* $\dim(V_0) = \dim(V_2)$ *then V does not contain the trivial (1-dimensional) module as a submodule;*

4. *if $m \geq 2$, then $X\colon V_{m-2} \to V_m$ is a surjective map which is a bijection if and only if V does not contain any submodules of dimension $m-1$.*

Proof: Decompose V as a direct sum of irreducible subrepresentations. If W is one such, then by Proposition 3.2.2 it is a direct sum

$$W_{-m} \oplus W_{-m+2} \oplus \cdots \oplus W_{m-2} \oplus W_m$$

of weight spaces, corresponding to the weights $\omega = \{-m, -m+2, \cdots, m-2, m\}$, for some $m \geq 0$. Since all pairs of weights in W differ by an even integer, not both of 0 and 1 can occur in ω. If m is even, then $0 \in \omega$; if m is odd, then $1 \in \omega$. Therefore $\dim(W_0) + \dim(W_1) = 1$.

Adding these over all irreducible summands of V yields the first two statements.

The third statement is a special case of the last. We note that if $m \geq 2$ occurs as a weight in ω for a simple submodule W, then $m-2 \geq 0$ and so in particular also lies in ω. Any m-weight vector in W is therefore the image under X of an $m-2$-weight vector in W. The surjectivity of the map now follows by linearity. Suppose now that W is a simple submodule containing a vector of weight $m-2$. Then if $m-2$ is not a highest weight of ω, X maps this $m-2$ weight space bijectively onto the m-weight space; but if $m-2$ is a highest weight space, then X acts by 0. Therefore the map is bijective if and only if there are no highest weight vectors in V of weight $m-2$, that is, if V contains no simple submodules of dimension $m-1$. ∎

We use these ideas in Chapter 5 to infer the decomposition into irreducible subrepresentations of $\mathfrak{sl}_2(k)$, based largely on the action of the element H of an $\mathfrak{sl}_2(k)$ triple.

Chapter 4

Bala-Carter classification of nilpotent orbits

In this chapter we will give an exposé of the Bala-Carter method for the classification of nilpotent orbits of a semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field k of characteristic 0.

4.1 Centralizers of toral and Levi subalgebras.

We will give some general facts relating to toral subalgebras and Levi subalgebras. The following results are about the centre of Levi subalgebras and the centralizers of Levi and toral subalgebras. Throughout, $\mathfrak{h}$ will denote a Cartan sublgebra of a $\mathfrak{g}$ and Φ the associated root system with basis Δ. Let $\mathfrak{a}$ be a subalgebra of a Lie algebra $\mathfrak{g}$ and let

$$\mathfrak{g}^{\mathfrak{a}} := \{B \in \mathfrak{g} \colon [A, B] = 0 \text{ for all } A \in \mathfrak{a}\}$$

denote the *centralizer* of $\mathfrak{a}$ in $\mathfrak{g}$ and let $Z(\mathfrak{a})$ denote the center of $\mathfrak{a}$. Using the Jacobi identity, it is easy to prove that the bracket of two elements of $\mathfrak{g}^{\mathfrak{a}}$ is still an element of $\mathfrak{g}^{\mathfrak{a}}$, so it is easily seen that $\mathfrak{g}^{\mathfrak{a}}$ is a subalgebra of $\mathfrak{g}$. Let us start with an elementary observation.

Lemma 4.1.1. *We have* $\mathfrak{a} \subseteq \mathfrak{g}^{Z(\mathfrak{a})}$ *and* $Z(\mathfrak{a}) \subseteq \mathfrak{g}^{\mathfrak{a}}$.

Proof: These inclusions are immediate applications of the definitions. ∎

We will prove that in the case where $\mathfrak{a}$ is a Levi subalgebra, these inclusions become equalities. For that end, we need the following proposition and lemma.

Proposition 4.1.2. *Let* $\mathfrak{l} = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Delta' \rangle} \mathfrak{g}_\alpha$ *be a Levi subalgebra. Then the center of* $\mathfrak{l}$ *is given by*

$$Z(\mathfrak{l}) = \bigcap_{\alpha \in \Delta'} \ker \alpha \subseteq \mathfrak{h}.$$

In particular, the center of a Levi subalgebra is a toral subalgebra.

Proof: Indeed, let $C \in Z(\mathfrak{l})$. Then $[C, H] = 0$ for all $H \in \mathfrak{h}$. Therefore $C \in \mathfrak{g}^{\mathfrak{h}}$. But we know that the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is a subalgebra of its normalizer in $\mathfrak{g}$, which is $\mathfrak{h}$ itself, since $\mathfrak{h}$ is a Cartan subalgebra. We deduce that $C \in \mathfrak{h}$. Next note that $[C, \mathfrak{l}] = 0$. In particular, $[C, X_\alpha] = 0 = \alpha(C) X_\alpha$ for all $X_\alpha \in \mathfrak{g}_\alpha$ and for all $\alpha \in \Delta'$. Thus we see that $\alpha(C) = 0$ for all $\alpha \in \Delta'$ and

$$Z(\mathfrak{l}) \subseteq \bigcap_{\alpha \in \Delta'} \ker \alpha.$$

For the inverse inclusion, let $X \in \bigcap_{\alpha \in \Delta'} \ker \alpha$. Since we have

$$\bigcap_{\alpha \in \Delta'} \ker \alpha \subseteq \mathfrak{h} \subseteq \mathfrak{l}$$

and $\mathfrak{h}$ is abelian, therefore $[X,\mathfrak{h}] = 0$ and $[X,X_\alpha] = \alpha(X)X_\alpha = 0$ for all $\alpha \in \langle\Delta'\rangle$ and for all $X_\alpha \in \mathfrak{g}_\alpha$. Hence $X \in Z(\mathfrak{l})$. ∎

Lemma 4.1.3. *Let V be a finite-dimensional vector space over a field k and let R be a finite subset of V^*, the space of linear forms on V, such that $\mathfrak{c} = \cap_{\alpha\in R}\ker\alpha \neq 0$. For all nontrivial $\beta \in V^* \setminus \operatorname{span}(R)$ there exists a $X \in \mathfrak{c}$ such that $\beta(X) \neq 0$.*

Proof: We can assume without loss of generality that R is linearly independent. Then let $R = \{e^1,\cdots,e^k\}$ and complete R by $e^{k+1},\cdots,e^n$ to form a basis $\mathcal{B}' = \{e^1,\cdots,e^n\}$ of V^*. Let $\mathcal{B} = \{e_1,\cdots,e_n\}$ be the dual basis to $\mathcal{B}'$. Since $\beta \in V^* \setminus \operatorname{span}(\mathcal{B}')$, we can write $\beta = \sum_{i=1}^n b_i e^i$, such that there exists some index $i_0 > k$ such that $b_{i_0} \neq 0$. Set $X = e_{i_0}$. Then $\beta(X) = b_{i_0} \neq 0$. Moreover, for each $1 \leq i \leq k$, we have $e^i(X) = 0$, so $X \in \mathfrak{c}$. ∎

Lemma 4.1.4. *Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{h}$. Then $\mathfrak{g}^{\mathfrak{a}}$ is $\mathfrak{h}$–invariant (under the adjoint representation).*

Proof: Let $C \in \mathfrak{g}^{\mathfrak{a}}$ and $H \in \mathfrak{h}$. We need to prove that $[H,C] \in \mathfrak{g}^{\mathfrak{a}}$. In other words, we need to show that

$$[[H,C],D] = 0$$

for all $D \in \mathfrak{a}$. On the one hand, we have $[C,D] = 0$ for all $C \in \mathfrak{g}^{\mathfrak{a}}$ and for all $D \in \mathfrak{a}$. On the other hand, $[D,H] = 0$ for all $D \in \mathfrak{a}$ and all $H \in \mathfrak{h}$ since $\mathfrak{a} \subseteq \mathfrak{h}$ and $\mathfrak{h}$ is abelian. Finally by the Jacobi identity, we have $[[H,C],D]+[H,[C,D]]+[C,[D,H]] = 0$, thus the equality we needed to prove. ∎

In particular, this lemma implies that when $\mathfrak{a} \subset \mathfrak{h}$ then $\mathfrak{g}^{\mathfrak{a}}$ admits a root space decomposition relative to $\mathrm{ad}(\mathfrak{h})$. Since $\mathfrak{h}$ is abelian it is contained in $\mathfrak{g}^{\mathfrak{a}}$. Moreover, since $\mathfrak{g}$ is semisimple, each root space is one-dimensional. So this decomposition has the form

$$\mathfrak{g}^{\mathfrak{a}} = \mathfrak{h} \oplus \sum_{\alpha \in S} \mathfrak{g}_\alpha$$

for some subset $S \subseteq \Phi$.

As promised, we will now observe that there is a "double commutant law", via the "centralizer operator" between a Levi subalgebra and its center.

Proposition 4.1.5. *Let* $\mathfrak{l} = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Delta' \rangle} \mathfrak{g}_\alpha$ *be a Levi subalgebra and* $\mathfrak{c}$ *its center. Then we have*

$$\mathfrak{g}^{\mathfrak{c}} = \mathfrak{l} \quad \text{and} \quad \mathfrak{g}^{\mathfrak{l}} = \mathfrak{c}.$$

Proof: For the first equality, since $\mathfrak{c}$ is the centre of $\mathfrak{l}$, thus we have $\mathfrak{l} \subseteq \mathfrak{g}^{\mathfrak{c}}$, by Lemma 4.1.1. Now, by Proposition 4.1.2, $\mathfrak{c}$ is a subalgebra of $\mathfrak{h}$. Thus by Lemma 4.1.4, $\mathfrak{g}^{\mathfrak{c}}$ is an $\mathfrak{h}$–submodule of $\mathfrak{g}$. But $\mathfrak{l}$ is also $\mathfrak{h}$–invariant. So if $\mathfrak{l} \neq \mathfrak{g}^{\mathfrak{c}}$, we can write

$$\mathfrak{g}^{\mathfrak{c}} = \mathfrak{l} \oplus \sum_{\beta \in S'} \mathfrak{g}_\beta$$

for some subset $S' \subseteq \Phi \setminus \langle \Delta' \rangle$.

Let $\beta \in \Phi \setminus \langle \Delta' \rangle$. Thus $\beta \in \mathfrak{h}^*$. Applying Lemma 4.1.3 to $R = \Delta'$ and $V = \mathfrak{h}$, we deduce there exists a $C \in \mathfrak{c} = \cap_{\alpha \in \Delta'} \ker \alpha$ such that $\beta(C) \neq 0$. Let $X \in \mathfrak{g}_\beta \cap \mathfrak{g}^{\mathfrak{c}}$. Then $[C, X] = 0$ since $X \in \mathfrak{g}^{\mathfrak{c}}$. On the other hand

$$[C, X] = \beta(C)X$$

since $X \in \mathfrak{g}_\beta$. Since $\beta(C) \neq 0$ we have $X = 0$. We deduce that $\mathfrak{g}_\beta \cap \mathfrak{g}^{\mathfrak{c}} = 0$ for all $\beta \in \Phi \setminus \langle \Delta' \rangle$. Therefore $S' = \emptyset$ and $\mathfrak{g}^{\mathfrak{c}} = \mathfrak{l}$.

For the second equality, we have $\mathfrak{c} \subseteq \mathfrak{l}$, therefore $\mathfrak{g}^{\mathfrak{l}} \subseteq \mathfrak{g}^{\mathfrak{c}}$. And $\mathfrak{g}^{\mathfrak{c}} = \mathfrak{l}$, by the first equality. Therefore, $\mathfrak{g}^{\mathfrak{l}}$ consists of all elements of $\mathfrak{l}$ which commute with $\mathfrak{l}$, that is the center of $\mathfrak{l}$. So $\mathfrak{g}^{\mathfrak{l}} = \mathfrak{c}$.

∎

Now we shall prove that the centralizer in a Lie algebra $\mathfrak{g}$ of a toral subalgebra $\mathfrak{t}$ is a Levi subalgebra of $\mathfrak{g}$. The following proposition is a key to the proof, from [2, Chapter VI, §1, Proposition 24].

Lemma 4.1.6. *Let Φ be a root system of a vector space V, Φ' the intersection of Φ with a subspace of V, so that Φ' is a root system in the subspace* $\mathrm{span}\{\Phi'\}$. *Let Δ' be a basis of Φ'.*

- *There exists a basis of Φ containing Δ'.*
- *Φ' is the set of the elements of Φ which are linear combinations of the elements of Δ'.*

Then we have the desired result.

Proposition 4.1.7. *Let $\mathfrak{t}$ be a toral subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Then the centralizer $\mathfrak{g}^{\mathfrak{t}}$ of $\mathfrak{t}$ is a Levi subalgebra of $\mathfrak{g}$.*

Proof: There exists a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which contains $\mathfrak{t}$, hence by Theorem 2.6.5, $\mathfrak{h}$ is a Cartan subalgebra. Let Φ be the root system associated to $\mathfrak{h}$ and denote $\Phi(\mathfrak{t})$ the subset of Φ which consists of all α satisfying $\mathfrak{t} \subseteq \ker \alpha$. By Lemma 4.1.4, we have the following root space decomposition:

$$\mathfrak{g}^{\mathfrak{t}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{t})} \mathfrak{g}_{\alpha}. \tag{4.1.1}$$

By Lemma 4.1.6, given a basis Δ' of $\Phi(\mathfrak{t})$ there is a basis Δ of Φ which contains Δ'. Hence

$$\mathfrak{g}^{\mathfrak{t}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Delta' \rangle} \mathfrak{g}_\alpha. \tag{4.1.2}$$

Therefore $\mathfrak{g}^{\mathfrak{t}}$ is a Levi subalgebra of $\mathfrak{g}$, which is standard relative to the potentially different choice of basis Δ. ∎

Lemma 4.1.8. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{t}$ and $\mathfrak{s}$ subalgebras of $\mathfrak{g}$ such that $\mathfrak{t} \subseteq \mathfrak{s}$. Then we have*

$$\mathfrak{g}^{\mathfrak{s}} \subseteq \mathfrak{g}^{\mathfrak{t}}.$$

Proof: Indeed, let $A \in \mathfrak{g}^{\mathfrak{s}}$. Thus, $[A, S] = 0$ for all $S \in \mathfrak{s}$. In particular, the previous identity is true for all $T \in \mathfrak{t}$. Hence, $A \in \mathfrak{g}^{\mathfrak{t}}$. ∎

Corollary 4.1.9. *Let $\mathfrak{g}$ be a semismiple Lie algebra and X an element of $\mathfrak{g}$. The map*

$$\mathfrak{t} \mapsto \mathfrak{g}^{\mathfrak{t}}$$

is a decreasing map from the set of toral subalgebras of $\mathfrak{g}^X$ and the set of Levi subalgebras of $\mathfrak{g}$ containing X.

Proof: First we show this map is well defined. By the previous proposition, $\mathfrak{g}^{\mathfrak{t}}$ is indeed a Levi subalgebra of $\mathfrak{g}$. Moreover, if $\mathfrak{t} \subseteq \mathfrak{g}^X$ it is obvious that $X \in \mathfrak{g}^{\mathfrak{t}}$. Secondly, the previous lemma shows it is a decreasing map. ∎

Lemma 4.1.10. *Let $\mathfrak{g}$ be a semisimple Lie algebra and suppose $X \in \mathfrak{g}$. If $\mathfrak{s}$ is a maximal toral subalgebra of $\mathfrak{g}^X$ then we have $\mathfrak{s} = Z(\mathfrak{g}^{\mathfrak{s}})$.*

Proof: Let $\mathfrak{s}$ be a maximal toral subalgebra of $\mathfrak{g}^X$. Since $\mathfrak{s}$ is abelian, $\mathfrak{s} \subseteq \mathfrak{g}^{\mathfrak{s}}$. Now let $Z \in \mathfrak{g}^{\mathfrak{s}}$. Then for all $S \in \mathfrak{s}$ we have $[S, Z] = 0$, so $\mathfrak{s} \subseteq Z(\mathfrak{g}^{\mathfrak{s}})$.

By Proposition 4.1.7, $\mathfrak{g}^{\mathfrak{s}}$ is a Levi subalgebra, and thus by Proposition 4.1.2, $Z(\mathfrak{g}^{\mathfrak{s}})$ is a toral subalgebra. Since $\mathfrak{s} \subseteq \mathfrak{g}^X$, we have $X \in \mathfrak{g}^{\mathfrak{s}}$, so in particular everything in the center of $\mathfrak{g}^{\mathfrak{s}}$ commutes with X, so $Z(\mathfrak{g}^{\mathfrak{s}}) \subseteq \mathfrak{g}^X$. By the maximality of $\mathfrak{s}$, we conclude that $\mathfrak{s} = Z(\mathfrak{g}^{\mathfrak{s}})$. ∎

The following is a key result which describes the set of minimal Levi subalgebras containing X.

Corollary 4.1.11. *Let $\mathfrak{g}$ be a semisimple Lie algebra and X a nilpotent element in $\mathfrak{g}$. There is a one-to-one correspondence between the set of maximal toral subalgebras of $\mathfrak{g}^X$ and the set of minimal Levi subalgebras of $\mathfrak{g}$ containing X given by*

$$\mathfrak{t} \mapsto \mathfrak{g}^{\mathfrak{t}}, \tag{4.1.3}$$

and the inverse application of (4.1.3) *is given by*

$$\mathfrak{l} \mapsto Z(\mathfrak{l}). \tag{4.1.4}$$

Proof: *Surjectivity*. Let $\mathfrak{l}$ be a minimal Levi subalgebra of $\mathfrak{g}$ such that $X \in \mathfrak{l}$. Let $\mathfrak{c} = Z(\mathfrak{l})$. Then we know that $\mathfrak{c}$ is a toral subalgebra, by Proposition 4.1.2. And $\mathfrak{c} \subseteq \mathfrak{g}^X$ since $[X, \mathfrak{c}] = 0$. Let $\mathfrak{t}$ be a maximal toral subalgebra of $\mathfrak{g}^X$ containing $\mathfrak{c}$. Then by Lemma 4.1.8 we have that $\mathfrak{g}^{\mathfrak{t}} \subseteq \mathfrak{g}^{\mathfrak{c}}$. By Proposition 4.1.5, $\mathfrak{g}^{\mathfrak{c}} = \mathfrak{l}$ and, by Proposition 4.1.7, $\mathfrak{g}^{\mathfrak{t}}$ is a Levi subalgebra. Since $\mathfrak{t} \subseteq \mathfrak{g}^X$, we have $X \in \mathfrak{g}^{\mathfrak{t}}$. Therefore by minimality of $\mathfrak{l}$, $\mathfrak{g}^{\mathfrak{t}} = \mathfrak{l}$. Thus we have shown that there exists a maximal toral subalgebra of $\mathfrak{g}^X$, $\mathfrak{t}$, such that $\mathfrak{g}^{\mathfrak{t}} = \mathfrak{l}$.

Injectivity. Let $\mathfrak{s}$ and $\mathfrak{t}$ be two maximal toral subalgebras of $\mathfrak{g}^X$ such that $\mathfrak{g}^{\mathfrak{s}} = \mathfrak{g}^{\mathfrak{t}}$. Thus we have $Z(\mathfrak{g}^{\mathfrak{s}}) = Z(\mathfrak{g}^{\mathfrak{t}})$ and, according to Lemma 4.1.10, we have $\mathfrak{s} = \mathfrak{t}$. We have thus shown that the map (4.1.3) is a bijection. Moreover, we have shown in proving the injectivity that the map (4.1.4) is a left inverse of the map (4.1.3), thus these two maps are inverses of one another. ∎

Next we want to show that maximal toral subalgebras of a Lie algebra $\mathfrak{g}$ are conjugate under the Lie group of $\mathfrak{g}$.

Lemma 4.1.12. *If* $\mathfrak{t}$ *is a maximal toral subalgebra of a Lie algebra* $\mathfrak{r}$ *and if* $Z \in \mathfrak{r}^{\mathfrak{t}}$ *admits an abstract Jordan composition* $Z = Z_s + Z_n$*, then the semisimple part* Z_s *of* Z *belongs to* $\mathfrak{t}$.

Proof: Since Z commutes with $\mathfrak{t}$, by the abstract Jordan decomposition we know that its semisimple part Z_s also commutes with $\mathfrak{t}$. Hence the subspace $\mathfrak{t} + \langle Z_s \rangle$ is a subalgebra of $\mathfrak{r}$ and its elements are semisimple, thus it is a toral subalgebra of $\mathfrak{r}$ containing $\mathfrak{t}$. By maximality of $\mathfrak{t}$, we deduce $\mathfrak{t} + \langle Z_s \rangle = \mathfrak{t}$, therefore $Z_s \in \mathfrak{t}$. ∎

Corollary 4.1.13. *Let* $\mathfrak{r}$ *be subalgebra of* $\mathfrak{g}$ *and let* $\mathfrak{t}$ *be a maximal toral subalgebra of* $\mathfrak{r}$. *Then* $\mathfrak{r}^{\mathfrak{t}}$ *is a nilpotent subalgebra.*

Proof: Let $Z = Z_s + Z_n$ be the abstract Jordan decoposition of $Z \in mathfrakr^{\mathfrak{t}}$. Since Z commutes with $\mathfrak{t}$, so does Z_n, and so $Z_n \in mathfrakr^{\mathfrak{t}}$. The adjoint action of $\mathfrak{t}$ on $\mathfrak{r}^{\mathfrak{t}}$ is trivial. By Lemma 4.1.12, $Z_s \in \mathfrak{t}$ so $\mathrm{ad}_{Z_s} = 0$ on $\mathfrak{r}^{\mathfrak{t}}$. Therefore on $\mathfrak{r}^{\mathfrak{t}}$ we have $\mathrm{ad}_Z = \mathrm{ad}_{Z_n}$. By definition of a nilpotent element of a reductive Lie algebra, $\mathrm{ad}\, Z_n$ is a nilpotent endomorphism of $\mathfrak{r}^{\mathfrak{t}}$. Therefore, ad_Z is a nilpotent endomorphism of $\mathfrak{r}^{\mathfrak{t}}$ for all $Z \in \mathfrak{r}^{\mathfrak{t}}$. By Engel's theorem, $\mathfrak{r}^{\mathfrak{t}}$ is nilpotent. ∎

Lemma 4.1.14. *Let* $\mathfrak{r}$ *be a subalgebra of* $\mathfrak{g}$ *and let* $\mathfrak{t}$ *be a maximal toral subalgebra of* $\mathfrak{r}$*. Then* $\mathfrak{r}^{\mathfrak{t}}$ *is a Cartan subalgebra of* $\mathfrak{r}$*, and* $\mathfrak{t}$ *consists exactly of the semisimple elements in* $\mathfrak{r}^{t}$*.*

Proof: By Corollary 4.1.13, we know $\mathfrak{r}^{\mathfrak{t}}$ is nilpotent. Let us show that $\mathfrak{r}^{\mathfrak{t}}$ is self-normalizing in $\mathfrak{r}$. Let $\mathfrak{n} = \{Z \in \mathfrak{r} \mid [Z, \mathfrak{r}^{\mathfrak{t}}] \subseteq \mathfrak{r}^{\mathfrak{t}}\}$ denote the normalizer of $\mathfrak{r}^{\mathfrak{t}}$ in $\mathfrak{r}$. This algebra is stable by the adjoint action of $\mathfrak{t}$, therefore it is the direct sum of the $\mathfrak{t}-$weight spaces under the adjoint representation, and $\mathfrak{r}^{\mathfrak{t}}$ is exactly the 0-weight space.

Suppose to the contrary that there exists $Z \in \mathfrak{n}$, $T \in \mathfrak{t}$ and a nonzero $\lambda \in \mathbb{C}$ such that $[T, Z] = \lambda Z$. Then $[\mathfrak{t}, Z] \neq 0$, so $Z \notin \mathfrak{r}^{\mathfrak{t}}$. But $Z = [T, Z] \in [\mathfrak{r}^{\mathfrak{t}}, Z] \subseteq \mathfrak{r}^{\mathfrak{t}}$, a contradiction.

Therefore $\mathfrak{n} = \mathfrak{r}^{\mathfrak{t}}$ and $\mathfrak{r}^{\mathfrak{t}}$ is a Cartan subalgebra of $\mathfrak{r}$. By Lemma 4.1.12, if $Z = Z_s$ is a semisimple element of $\mathfrak{r}^{\mathfrak{t}}$, then $Z_s \in \mathfrak{t}$. ∎

Proposition 4.1.15. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be two subalgebras of* $\mathfrak{g}$*. If* $g \in G_{\mathrm{ad}}$ *satisfies* $\mathrm{Ad}(g)\mathfrak{m} = \mathfrak{n}$ *then* $\mathrm{Ad}(g)\mathfrak{g}^{\mathfrak{m}} = \mathfrak{g}^{\mathfrak{n}}$*.*

Proof: Let $Z \in \mathfrak{g}^{\mathfrak{n}}$. Then $[Z, N] = 0$ for all $N \in \mathfrak{n}$. But each $N \in \mathfrak{n}$ can be writen as $\mathrm{Ad}(g)M$ for some $M \in \mathfrak{m}$. Thus $[Z, \mathrm{Ad}(g)M] = 0$ for all $M \in \mathfrak{m}$. Equivalently, $\mathrm{Ad}(g)[\mathrm{Ad}(g^{-1})Z, M] = 0$, which is equivalent to $[\mathrm{Ad}(g^{-1})Z, M] = 0$, for all $M \in \mathfrak{m}$. We deduce that $Z \in \mathfrak{g}^{\mathfrak{n}}$ if and only if $\mathrm{Ad}(g^{-1})Z \in \mathfrak{g}^{\mathfrak{m}}$, and this implies that $\mathfrak{g}^{\mathfrak{n}} = \mathrm{Ad}(g)\mathfrak{g}^{\mathfrak{m}}$. ∎

We note that since $\mathfrak{r}$ is not a semisimple Lie subalgebra of $\mathfrak{g}$, the Cartan subalgebras and toral subalgebras of $\mathfrak{r}$ need not coincide; in fact, Lemma 4.1.14 shows how they may differ. The following results show that they are nonetheless conjugate by an element of the adjoint group of $\mathfrak{r}$.

Corollary 4.1.16. *In the setting of Lemma 4.1.14, let R be the subgroup of G_{ad} generated by* $\{\exp(\mathrm{ad}(Z)) \mid Z \in \mathfrak{r}\}$*. Then R acts transitively on the set of all maximal toral subalgebras of* $\mathfrak{r}$.

Proof: By [7, §16], all Cartan subalgebras of a Lie algebra $\mathfrak{r}$ are conjugate via elements of R. Therefore if $\mathfrak{s}$ and $\mathfrak{t}$ are two maximal toral subalgebras of $\mathfrak{r}$, then by Lemma 4.1.14, $\mathfrak{r}^{\mathfrak{s}}$ and $\mathfrak{r}^{\mathfrak{t}}$ are Cartan subalgebras of $\mathfrak{r}$, thus conjugate by some element $g \in R$. Since R acts by automorphisms of $\mathfrak{g}$, it preserves the abstract Jordan decomposition. Since by Lemma 4.1.14 we can recover $\mathfrak{s}$ and $\mathfrak{t}$ as the semisimple elements of $\mathfrak{r}^{\mathfrak{s}}$ and $\mathfrak{r}^{\mathfrak{t}}$, respectively, it follows that $\mathfrak{s}$ and $\mathfrak{t}$ are conjugate by g. ∎

Theorem 4.1.17. *Let X be a nilpotent element of $\mathfrak{g}$. Then two minimal Levi subalgebras containing X are G_{ad}^{X}–conjugate.*

Proof: If $\mathfrak{k}$ and $\mathfrak{l}$ are two minimal Levi subalgebras containing X, then by Corollary 4.1.11, $\mathfrak{m} = Z(\mathfrak{k})$ and $\mathfrak{n} = Z(\mathfrak{l})$ are two maximal toral subalgebras of $\mathfrak{g}^X$, which, by Corollary 4.1.16, are conjugate by an element $g \in G_{\mathrm{ad}}^X = \{g \in G_{\mathrm{ad}} \mid g \cdot X = X\}$. By Corollary 4.1.11, we have $\mathfrak{g}^{\mathfrak{m}} = \mathfrak{k}$ and $\mathfrak{g}^{\mathfrak{n}} = \mathfrak{l}$. Therefore, applying Proposition 4.1.15 to $Z(\mathfrak{k})$ and $Z(\mathfrak{l})$, we have that $\mathfrak{k}$ and $\mathfrak{l}$ are conjugate by g.

∎

4.2 The Jacobson-Morozov theorem

In this section, we just need a field k of characteristic zero.

Definition 4.2.1 (Standard triple)**.** *A* standard triple *(or a* $\mathfrak{sl}_2$–triple*) in a Lie algebra* $\mathfrak{g}$ *is a triple* (H, X, Y) *which generates a subalgebra* $\mathfrak{a}$ *of* $\mathfrak{g}$ *isomorphic to* $\mathfrak{sl}_2$*. The elements* H, X, Y *are called respectively the* neutral, nilpositive*, and* nilnegative elements *of the triple.*

We will prove in this section that if X is a nilpotent element in a semisimple Lie algebra $\mathfrak{g}$, then there exist $H, Y \in \mathfrak{g}$ such that (H, X, Y) is a standard triple in which X is the nilpositive element. The lemmas and the theorems of this section are from [3, Chapter VIII, §11, No. 2], unless otherwise stated.

Let $\mathfrak{a}$ be an associative algebra over k, such as the algebra of endomorphisms of a finite-dimensional vector space. One can define on $\mathfrak{a}$ a Lie algebra structure by setting $[A, B] := AB - BA$ for all A and B in $\mathfrak{a}$.

Lemma 4.2.2. *Let* V *be a finite-dimensional vector space over* k *and let* A *and* B *be endomorphisms of* V*. Suppose* A *is nilpotent and* $[A, [A, B]] = 0$*. Then* AB *is nilpotent.*

Proof: Denote $C = [A, B]$. Then $[A, C] = 0$, so $AC = CA$. We prove by induction that $[A, BC^p] = C^{p+1}$ for all $p \geq 0$. Suppose $p \geq 0$ and that $[A, BC^{p-1}] = C^p$. Then

$$\begin{aligned}
[A, BC^p] &= ABC^p - BC^pA \\
&= ABC^{p-1}C - BC^{p-1}AC && \text{since } AC = CA \\
&= (ABC^{p-1} - BC^{p-1}A)C \\
&= [A, BC^{p-1}]C \\
&= C^pC && \text{by induction} \\
&= C^{p+1}.
\end{aligned}$$

Therefore for each $p \geq 1$, C^p can be written as a commutator. Since $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ this gives that $\mathrm{Tr}(C^p) = 0$ for all $p \geq 1$. Thus all eigenvalues of C are 0 and C is nilpotent.

Let us now prove that for all positive integers p, we have $[B, A^p] = p[B,A]A^{p-1}$. This is true for $p = 0$. Let $p \geq 1$ and assume the result to be true with p replaced by $p-1$. But then we have, for $p \geq 1$,

$$\begin{aligned}
[B, A^p] &= BA^p - A^p B \\
&= BAA^{p-1} - ABA^{p-1} + ABA^{p-1} - A^p B \\
&= [B,A]A^{p-1} + A[B, A^{p-1}] \\
&= [B,A]A^{p-1} + A((p-1)[B,A]A^{p-2}) && \text{by induction} \\
&= [B,A]A^{p-1} + (p-1)[B,A]A^{p-1} && \text{since } [A,[A,B]] = 0 \\
&= p[B,A]A^{p-1}.
\end{aligned}$$

Now let $\overline{k}$ be an algebraic closure of k and let $\lambda \in \overline{k}$ be any eigenvalue of AB. Let $x \in V \otimes_k \overline{k}$ be such that $ABx = \lambda x$ and $x \neq 0$. By hypothesis A is nilpotent. Let r be the least positive integer satisfying $A^r x = 0$. We have

$$\lambda A^{r-1}x = A^{r-1}ABx = A^r Bx = BA^r x - [B, A^r]x = -r[B,A]A^{r-1}x.$$

Since $A^{r-1}x \neq 0$ and $r > 0$ this shows $A^{r-1}x$ is an eigenvector of $[B,A]$ with eigenvalue $\lambda/(-r)$. Since $[B,A]$ is nilpotent, we deduce that $\lambda = 0$. Therefore, all the eigenvalues of AB are equal to zero, thus AB is nilpotent. ∎

The following lemma is technical but is a classic computation on associative algebras.

Lemma 4.2.3. *From [3, Chapter VIII, § 1, No. 1, Lemma 1] Let $\mathfrak{a}$ be an associative algebra over k. Let H, X and Z be elements of $\mathfrak{a}$ such that $[H,X] = 2X$ and $[Z,X] = H$. Then we* *have*

$$[Z, X^n] = nX^{n-1}(H+n-1) = n(H-n+1)X^{n-1} \text{ for all } n \in \mathbb{N}, \tag{4.2.1}$$

where we abbreviate $H+n-1 = H+(n-1)I$, *where* I *is the identity element of* $\mathfrak{a}$.

Proof: Let us first show by induction that

$$[Z, X^n] = nX^{n-1}(H+n-1) \text{ for all } n \geq 1. \tag{4.2.2}$$

For $n = 1$, (4.2.2) is valid since we have $[Z, X] = H$. Suppose (4.2.2) holds with $n = p \geq 1$; then

$$\begin{aligned}
[Z, X^{p+1}] &= ZX^{p+1} - X^{p+1}Z \\
&= (ZX^p)X - X(X^pZ) \\
&= (X^pZ + [Z, X^p])X - X(X^pZ) \\
&= X^pZX - X(X^pZ) + [Z, X^p]X \\
&= X^p(ZX) - X^p(XZ) + [Z, X^p]X \\
&= X^p[Z, X] + (pX^{p-1}(H+p-1))X && \text{by induction} \\
&= X^pH + pX^{p-1}(HX + pX - X) && \text{since } [Z, X] = H \\
&= X^pH + pX^{p-1}(2X + XH + pX - X) && \text{since } [H, X] = 2X] \\
&= (p+1)X^p(H+p),
\end{aligned}$$

hence the result. Secondly, we need to prove that

$$X^n(H+n) = (H-n)X^n \text{ for all } n \geq 0. \tag{4.2.3}$$

The equality is obviously true for $n = 0$. Furthermore, if $X^p(H+p) = (H-p)X^p$ for some $p \geq 0$, then

$$\begin{aligned} X^{p+1}(H+p) &= (XH - pX)X^p \\ &= (HX - 2X - pX)X^p \\ &= (H - (p+2))X^{p+1} \end{aligned}$$

Therefore, $X^{p+1}(H+p+1) = (H-p-1)X^{p+1}$. Thus we have (4.2.1). ∎

Now we return to the setting that $\mathfrak{g}$ is a semisimple Lie algebra over k, and $X \in \mathfrak{g}$.

Lemma 4.2.4. *Let $H, X \in \mathfrak{g}$ such that X is nilpotent, $[H,X] = 2X$ and $H \in [X, \mathfrak{g}]$. Then there exists $Y \in \mathfrak{g}$ such that (H,X,Y) is either $(0,0,0)$ or an $\mathfrak{sl}_2$–triple.*

Proof: Suppose $X \neq 0$. Let $\mathfrak{n} := \ker(\operatorname{ad}_{\mathfrak{g}} X)$. First, since $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism, and since $[H,X] = 2X$, we have that $[\operatorname{ad} H, \operatorname{ad} X] = 2 \operatorname{ad} X$. Hence $\operatorname{ad} X \circ \operatorname{ad} H(\mathfrak{n}) = \operatorname{ad} H \circ \operatorname{ad} X(\mathfrak{n}) - 2 \operatorname{ad} X(\mathfrak{n}) = 0$. We deduce that $\operatorname{ad} H(\mathfrak{n}) \subset \mathfrak{n}$.

Now, let $Z \in \mathfrak{g}$ such that $H = -[X,Z]$. Our strategy will be to find some $Z' \in \mathfrak{n}$ such that $\operatorname{ad} H(Z' - Z) = -2(Z' - Z)$; then $Y = Z' - Z$ will also satisfy $[X,Y] = [X, -Z] = H$, as required.

We have $[\operatorname{ad} Z, \operatorname{ad} X] = \operatorname{ad} H$. By Lemma 4.2.3, applied to the images under ad of H, Z, and X in the associative algebra $\operatorname{End}(\mathfrak{g})$, we have

$$[\operatorname{ad} Z, (\operatorname{ad} X)^n] = n(\operatorname{ad} H - n + 1)(\operatorname{ad} X)^{n-1}, \text{ for all } n \geq 1.$$

Let $\mathfrak{g}_n := (\operatorname{ad} X)^n(\mathfrak{g})$ for all $n \in \mathbb{N}$. Let $n \geq 1$. If $U \in \mathfrak{g}_{n-1}$, then there exists $U' \in \mathfrak{g}$ such that $U = (\operatorname{ad} X)^{n-1}(U')$, and thus $[\operatorname{ad} Z, (\operatorname{ad} X)^n](U') = n(\operatorname{ad} H - n + 1)(U)$.Thus

$$\operatorname{ad} Z \circ (\operatorname{ad} X)^n(U') - (\operatorname{ad} X)^n \circ \operatorname{ad} Z(U') = n(\operatorname{ad} H - n + 1)(U)$$

or, equivalently,

$$\operatorname{ad} Z \circ \operatorname{ad} X(U) - (\operatorname{ad} X)^n(\operatorname{ad} Z(U')) = n(\operatorname{ad} H - n + 1)(U).$$

Hence we have $(\operatorname{ad} H - n + 1)(U) \in \operatorname{ad} Z \circ \operatorname{ad} X(U) + \mathfrak{g}_n$ for all $U \in \mathfrak{g}_{n-1}$. Moreover, if $\operatorname{ad} X(U) = 0$, meaning $U \in \mathfrak{n} \cap \mathfrak{g}_{n-1}$, then $(\operatorname{ad} H - n + 1)(U) \in \mathfrak{g}_n$. Thus, since $\operatorname{ad} H(\mathfrak{n}) \subset \mathfrak{n}$, we have

$$(\operatorname{ad} H - n + 1)(\mathfrak{n} \cap \mathfrak{g}_{n-1}) \subset \mathfrak{n} \cap \mathfrak{g}_n, \text{ for all } n \geq 1. \tag{4.2.4}$$

We next show that -2 is not an eigenvalue of $\operatorname{ad} H$ acting on $\mathfrak{n}$, as follows. Since $\operatorname{ad} X$ is nilpotent, we can find $N \in \mathbb{N}$ such that $\mathfrak{g}_{N-1} \neq 0$ and $\mathfrak{g}_N = 0$. Suppose $\lambda \in k$ is an eigenvalue of $\operatorname{ad} H$ on $\mathfrak{n}$ and let $V \in \mathfrak{n}$ be an eigenvector. According to (4.2.4), we have $\operatorname{ad} H(V) \in \mathfrak{n} \cap \mathfrak{g}_1$. Hence $\lambda V \in \mathfrak{n} \cap \mathfrak{g}_1$. Applying the same relation for λV, we have $\lambda(\lambda - 1)V \in \mathfrak{n} \cap \mathfrak{g}_2$. Iterating the same argument for $n = 3, \cdots, N$, we have that $\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-N+1)V = 0$, since $\mathfrak{g}_N = 0$. We deduce that $\lambda \in \mathbb{N}$. Therefore -2 is not an eigenvalue of $\operatorname{ad} H$ in $\mathfrak{n}$.

Hence $(\operatorname{ad} H + 2)$ is invertible in $\operatorname{End}(\mathfrak{n})$. Moreover

$$\begin{aligned}
[X, [H, Z] + 2Z] &= [X, [H, Z]] + 2[X, Z] \\
&= [[X, H], Z] + [H, [X, Z]] + 2[X, Z] \\
&= [-2X, Z] + [H, -H] + 2[X, Z] \\
&= 0.
\end{aligned}$$

We deduce that $[H,Z]+2Z \in \mathfrak{n}$. By invertibility, there exists $Z' \in \mathfrak{n}$ such that $(\operatorname{ad} H+2)(Z') = [H,Z]+2Z$. So we have $[H,Z']+2Z' = [H,Z]+2Z$, which implies $[H,Z'-Z] = -2(Z'-Z)$. Therefore if we set $Y = Z'-Z$, we have $[H,Y] = -2Y$ and $[X,Y] = [X,Z'] - [X,Z] = H$. ∎

Theorem 4.2.5 (Jacobson-Morozov theorem)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra. Let X be a nonzero nilpotent element in $\mathfrak{g}$. Then there exists $H, Y \in \mathfrak{g}$ such that (H,X,Y) is a $\mathfrak{sl}_2$–triple.*

Proof: Let $\mathfrak{n} = \ker(\operatorname{ad} X)^2$. If $Z \in \mathfrak{n}$, we have $[X,[X,Z]] = 0$, and so, as we have argued in the proof of Lemma 4.2.4, $[\operatorname{ad} X, [\operatorname{ad} X, \operatorname{ad} Z]] = 0$. Since $\operatorname{ad} X$ is nilpotent, we have by Lemma 4.2.2, that $\operatorname{ad} X \circ \operatorname{ad} Z$ is nilpotent. Hence $0 = \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Z) = \kappa(X,Z)$, for all $Z \in \mathfrak{n}$, where κ is the (nondegenerate) Killing form of $\mathfrak{g}$. Thus $X \in \mathfrak{n}^{\perp}$ belongs to the orthogonal complement of $\mathfrak{n}$ with respect to κ. In particular, $\mathfrak{n} \neq \mathfrak{g}$ so $(\operatorname{ad} X)^2 \neq 0$.

Now let Y be a nonzero vector in $\mathfrak{g}$ such that $Y = (\operatorname{ad} X)^2 Y'$ for some $Y' \in \mathfrak{g}$. For all $Z \in \mathfrak{n}$ we have, by the invariance property of the Killing form, that

$$\kappa(Y,Z) = \kappa((\operatorname{ad})^2 Y', Z) = \kappa(Y', (\operatorname{ad} X)^2 Z) = \kappa(Y', 0) = 0.$$

Therefore $Y \in \mathfrak{n}^{\perp}$ (and so in particular, $Y \notin \mathfrak{n}$) for all nonzero $Y \in (\operatorname{ad} X)^2 \mathfrak{g}$. Now by the rank-nullity theorem applied to the endomorphism $(\operatorname{ad} X)^2$ on $\mathfrak{g}$, we have that $\dim((\operatorname{ad} X)^2 \mathfrak{g}) + \dim(\mathfrak{n}) = \dim(\mathfrak{g})$, so the fact that $(\operatorname{ad} X)^2 \mathfrak{g} \subseteq \mathfrak{n}^{\perp}$ implies that $\mathfrak{n}^{\perp} = (\operatorname{ad} X)^2 \mathfrak{g}$.

Thus since $X \in \mathfrak{n}^{\perp}$, there exists Y'' in $\mathfrak{g}$ such that $X = (\operatorname{ad} X)^2 Y''$. Set $H = -2[X, Y'']$.

We have

$$[H,X] = [-2[X,Y''],X] = -2[[X,Y''],X] = 2[X,[X,Y'']] = 2X,$$

and $H \in (\operatorname{ad} X)\mathfrak{g}$.

We conclude the existence of a Y_0 such that (H,X,Y_0) is a standard triple in $\mathfrak{g}$ by applying Lemma 4.2.4. ∎

We end this section by mentioning important and useful results on conjugacy of standard triples and the subalgebras in $\mathfrak{g}$ they generate. The first theorem is from [6, §3.4, Theorem 3.4.10].

Theorem 4.2.6 (Kostant). *Let $\{H,X,Y\}$ and $\{H',X,Y'\}$ be two standard triples in $\mathfrak{g}$, over an algebraically closed field k, with the same nilpositive element X. Then there exists $x \in G_{ad}$ such that $x \cdot H = H'$, $x \cdot X = X$ and $x \cdot Y = Y'$.*

The second result is from [3, Chapter VIII, §11, Proposition 1]

Proposition 4.2.7. *Let A be a group of automorphims of $\mathfrak{g}$ containing G_{ad}. Let (H,X,Y) and (H',X',Y') be standard triples in $\mathfrak{g}$. Moreover, let $\mathfrak{a} = kH + kX + kY$ and $\mathfrak{a}' = kH' + kX' + kY'$. Consider the following conditions:*

- *X and X' are A–conjugates.*
- *(H,X,Y) and (H',X',Y') are A–conjugates: there exists a $g \in A$ such that $g \cdot H = H'$, $g \cdot X = X'$ and $g \cdot Y = Y'$.*
- *$\mathfrak{a}$ and $\mathfrak{a}'$ are A–conjugates.*

Then the first two conditions are equivalent and the second implies the third in general. When the base field k is algebraically closed, then the three conditions are equivalent.

These results establish that, for $k = \mathbb{C}$, the classification of G_{ad}-nilpotent orbits of $\mathfrak{g}$ is equivalent to the classification of G_{ad}-conjugacy classes of $\mathfrak{sl}_2(k)$-triples in $\mathfrak{g}$.

4.3 Distinguished orbits and Jacobson-Morozov parabolic subalgebras.

We have concluded in Section 4.1 that to each nilpotent element X we can assign a unique G_{ad}−conjugacy class $[\mathfrak{l}]$ of minimal Levi subalgebras containing X. It follows that $X \in [\mathfrak{l}, \mathfrak{l}]$ so we can talk of the nilpotent orbit of X in the semisimple Lie algebra $[\mathfrak{l}, \mathfrak{l}]$. Our basic strategy in this section is to produce a nilpotent orbit in $\mathfrak{g}$ from a nilpotent orbit in $[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}$. Furthermore the minimal Levi subalgebra of $\mathfrak{l}$ containing X is $\mathfrak{l}$ itself, whence we have a motivation for the following definition.

Definition 4.3.1 (Distinguished nilpotent element and orbit)**.** *From [6, § 8.2, p. 121]. Let $\mathfrak{g}$ be a reductive Lie algebra. A nilpotent element $X \in \mathfrak{g}$ is* distinguished *if the minimal Levi subalgebra of $\mathfrak{g}$ containing X is $\mathfrak{g}$. A nilpotent orbit $\mathcal{O}_X$ is called* distinguished *if it is the orbit of a distinguished nilpotent element X.*

There is a device to characterize distinguished orbits in terms of the dimensions of some subalgebras attached to them, which is rather combinatorial. Recall that when X is a nilpotent element of a semisimple Lie algebra of $\mathfrak{g}$, then, by the Jacobson-Morozov theorem, there exists a $\mathfrak{sl}_2$−triple in $\mathfrak{g}$ having X as its nilpositive element. In other words, there exists $H, Y \in \mathfrak{g}$ such that (H, X, Y) forms a basis of subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. We know from the representation theory of $\mathfrak{sl}_2$ that the weights of $\mathfrak{g}$ as $\mathfrak{a}$−module are all integers and that $\mathfrak{g}$ is the direct sum of the weight spaces of H: $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$.

Definition 4.3.2 (Jacobson-Morozov parabolic subalgebra of a nilpotent element)**.**

With the previous notations, the Jacobson-Morozov parabolic algebra *of the nilpotent element X is the subalgebra $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$.*

That $\mathfrak{p}$ is parabolic follows from the fact that we can choose a Cartan subalgebra $\mathfrak{h}$ containing H and a set of positive roots Φ^+ such that $\alpha(H)$ is a nonnegative integer for all $\alpha \in \Phi^+$. Hence $\mathfrak{p}$ contains a Borel subalgebra. We know also that $\mathfrak{g}_0$ is a Levi subalgebra by using Proposition 4.1.7, since $\mathfrak{g}_0$ is the centralizer of the toral subalgebra kH. Thus $\bigoplus_{i>0} \mathfrak{g}_i$ is the nilradical of $\mathfrak{p}$. We will denote them respectively by $\mathfrak{l}_X$ and $\mathfrak{u}_X$.

Now we know that the centralizer $\mathfrak{g}^X$ of X is ad_H –stable, by using the Jacobi identity. Therefore we have

$$\mathfrak{g}^X = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i^X,$$

where $\mathfrak{g}_i^X := \mathfrak{g}^X \cap \mathfrak{g}_i$. Moreover, a vector V in $\mathfrak{g}$ killed by X is a linear combination of maximal weight vectors of each irreducible representation in the decomposition of $\mathfrak{g}$, thus V has a nonnegative weight. Hence

$$\mathfrak{g}^X = \bigoplus_{i \geq 0} \mathfrak{g}_i^X.$$

We want to give one more result concerning the structure of the centralizer in $\mathfrak{g}$ of a nonzero nilpotent element X, in terms of the centralizer of the Lie algebra $\mathfrak{a} = kH + kX + kY$. We need the following lemma, from [6, Chapter 3, §3.4, Lemma 3.4.5].

Lemma 4.3.3. *Let X be a nonzero nilpotent element of a semisimple Lie algebra $\mathfrak{g}$. Then $\mathfrak{u}^X := \mathfrak{g}^X \cap [\mathfrak{g}, X]$ is an $\mathrm{ad}\, H$-invariant nilpotent ideal of $\mathfrak{g}^X$. Moreover, we have*

$$\mathfrak{u}^X = \bigoplus_{i>0} \mathfrak{g}_i^X.$$

Proof: The proof is a rather instructive and interesting application of the representation theory of $\mathfrak{sl}_2$.

Since $[H,X]=2X$, then by the Jacobi identity we can see that both $\mathfrak{g}^X$ and $[\mathfrak{g},X]$ are $\operatorname{ad} H$-invariant subalgebras of $\mathfrak{g}$. We now show that their intersection is an ideal of $\mathfrak{g}^X$.

Let $Z\in\mathfrak{u}^X$ and $A\in\mathfrak{g}^X$. We have $[X,[A,Z]]=[[Z,X],A]+[[X,A],Z]=0$ because Z and A are in $\mathfrak{g}^X$. Furthermore, there exists some $Z'\in\mathfrak{g}$ such that $Z=[Z',X]$, so $[A,Z]=[A,[Z',X]]=[[A,Z'],X]+[[X,A],Z']=[[A,Z'],X]\in[\mathfrak{g},X]$. Therefore, $\mathfrak{u}^X$ is an $\operatorname{ad} H$-invariant ideal of $\mathfrak{g}^X$.

Recall from Lemma 3.3.1 that if $i>0$ then $[\mathfrak{g}_{i-2},X]=\mathfrak{g}_i$. If $i=0$, then $\mathfrak{g}_0^X$ consists of all the highest weight vectors of $\mathfrak{g}$ with respect to (H,X,Y) of highest weight zero, that is, it is a direct sum of all the copies of the trivial representation inside of $\mathfrak{g}$. In particular, $\mathfrak{g}_0^X\cap[\mathfrak{g},X]=\{0\}$. We deduce that $\mathfrak{u}^X=\mathfrak{g}^X\cap[\mathfrak{g},X]=\bigoplus_{i>0}\mathfrak{g}_i^X$.

Finally, and again, from the representation theory of $\mathfrak{sl}_2$ we have $[\mathfrak{g}_i^X,\mathfrak{g}_j^X]\subset\mathfrak{g}_{i+j}^X$ for all $i,j\in\mathbb{Z}$. Since $\mathfrak{g}$ is finite dimensional, for a large i, we have $\mathfrak{g}_i^X=0$. Hence the descending central series of $\mathfrak{u}^X$ is 0 for a large i and we conclude that $\mathfrak{u}^X$ is nilpotent. ∎

Keeping with the previous notations, let $\phi\colon\mathfrak{sl}_2\to\mathfrak{g}$ be the Lie algebra homomorphism sending the standard basis of $\mathfrak{sl}_2$ to the standard triple (H,X,Y). The centralizer of the subalgebra $\mathfrak{a}=kH+kX+kY$ is therefore the centralizer of the image of $\mathfrak{sl}_2$ by ϕ, so we can denote $\mathfrak{g}^{\mathfrak{a}}$ by $\mathfrak{g}^{\phi}$ interchangeably. We appeal once more to the representation theory of $\mathfrak{sl}_2$ to establish the structure of $\mathfrak{g}^X$. We know $\phi(\mathfrak{sl}_2)$ is an irreducible representation of $\mathfrak{sl}_2$ of highest weight 2, that $\mathfrak{g}^Y=\bigoplus_{i\le 0}\mathfrak{g}_i^Y$, where, as before, $\mathfrak{g}_i^Y=\mathfrak{g}^Y\cap\mathfrak{g}_i$ and that $\mathfrak{g}^H=\mathfrak{g}_0$. Hence $\mathfrak{g}^{\phi}$ is a submodule of $\mathfrak{g}_0^X\cap\mathfrak{g}_0^Y$, on which $\mathfrak{sl}_2$ acts trivially and irreducibly. Thus $\mathfrak{g}^\phi = \mathfrak{g}^X \cap \mathfrak{g}^Y = \mathfrak{g}^X_0$ and

$$\mathfrak{g}^X = \mathfrak{u}^X \oplus \mathfrak{g}^\phi. \tag{4.3.1}$$

Finally, we will prove the following lemma.

Lemma 4.3.4. *With the above notations, $\mathfrak{g}^\phi$ is a reductive Lie algebra.*

Proof: We claim that the restriction of the Killing form of $\mathfrak{g}$ on $\mathfrak{g}^\phi$ is nondegenerate, which is enough to prove that $\mathfrak{g}^\phi$ is reductive according to Theorem 2.3.7. Here again, we use the fact that the Killing form is an invariant bilinear form on $\mathfrak{g}$: we have $\kappa(Z,[B,X]) = \kappa([Z,B],X)$ for all B,X,Z in $\mathfrak{g}$. In particular, if $Z \in \mathfrak{g}^X$, we have $\kappa(Z,[B,X]) = -\kappa(Z,[X,B]) = -\kappa([Z,X],B) = 0$. And since $\mathfrak{g}$ is semisimple, the Killing form is nondegenerate, and so, as we have already argued before, that the orthogonal complement of $\mathfrak{g}^X$ is $[\mathfrak{g},X]$ with respect to κ. Hence we have that κ restricts to a nondegenerate bilinear form on $\mathfrak{g}^X/(\mathfrak{g}^X \cap [\mathfrak{g},X]) = \mathfrak{g}^X/\mathfrak{u}^X$, which module, according to equation (4.3.1), is isomorphic to $\mathfrak{g}^\phi$. ∎

The following definitions are given here because they can be used to realize the Bala-Carter correspondence; they are used in proofs that we do not include here, and we refer to them in the statement of Theorem 4.3.12.

Definition 4.3.5 (Induced nilpotent orbit, Richardson orbit)**.** *Let $\mathfrak{g}$ be a reductive Lie algebra and let $\mathfrak{l}$ be Levi subalgebra of $\mathfrak{g}$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$. If $\mathcal{O}_\mathfrak{l}$ is a nilpotent orbit in $\mathfrak{l}$, the* nilpotent orbit of $\mathfrak{g}$ induced from $\mathcal{O}_\mathfrak{l}$, *denoted* $\mathrm{Ind}^\mathfrak{g}_\mathfrak{p}(\mathcal{O}_\mathfrak{l})$, *is the unique orbit $\mathcal{O}_\mathfrak{g}$ which intersects $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ in an open dense set. A* Richardson orbit *of $\mathfrak{g}$ is a nilpotent orbit induced from the zero orbit of a Levi subalgebra.*

The reader interested in the proof of the existence and the uniqueness of $\mathrm{Ind}^\mathfrak{g}_\mathfrak{l}(\mathcal{O}_\mathfrak{l})$ might consult [6, Chapter 7, §7.1, Theorem 7.1.1]. It is mentioned there that $\mathcal{O}_\mathfrak{g} \cap$

$(\mathcal{O}_{\mathfrak{l}} + \mathfrak{n})$ is a single P_{ad}–orbit, P_{ad} being the adjoint group of the parabolic $\mathfrak{p}$ having $\mathfrak{l}$ as a factor. Furthermore, it is proved in [6, Chapter 7, §7.1, Theorem 7.1.3] that two parabolic subalgebras of $\mathfrak{g}$ with the same Levi subalgebra $\mathfrak{l}$ will induce the same nilpotent orbit from $\mathcal{O}_{\mathfrak{l}}$.

Our aim now is to prove that when X is distinguished then $\mathfrak{g}_{2k+1} = 0$ for all $k \geq 0$. This of course is equivalent to the fact that $\mathfrak{g}_{2k+1} = 0$ for all k in $\mathbb{Z}$, by application of the representation theory of $\mathfrak{sl}_2$, which, in turn, is equivalent to saying that $\mathfrak{g}_1 = 0$. This motivates the following definition.

Definition 4.3.6 (Even nilpotent element and orbit, even standard triple)**.** *Let X be a nilpotent element of a reductive Lie algebra $\mathfrak{g}$, (H, X, Y) a standard triple in $\mathfrak{g}$ containing X, generating a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, and let $\mathcal{O}_X$ be the orbit of X in $\mathfrak{g}$. If $\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ is the the decomposition of $\mathfrak{g}$ as a $\mathfrak{a}$–module, we say that X, $\mathcal{O}_X$, (H, X, Y) respectively are* even nilpotent element, even nilpotent orbit *and* even standard triple *if* $\mathfrak{g}_1 = 0$.

Proposition 4.3.7. *Let X be a nilpotent element of $\mathfrak{g}$ and $\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}_i$ its Jacobson-Morozov parabolic subalgebra. Then X is distinguished if and only if* $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_2$.

Proof: First assume $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_2$. By Lemma 3.3.1, we therefore know that $\mathfrak{g}$ has no trivial $\mathfrak{sl}_2$–submodules. Hence, from (4.3.1), we have $\mathfrak{g}^X = \mathfrak{u}^X$, since $\mathfrak{g}^\phi$ is the sum of all trivial subrepresentations of $\mathfrak{sl}_2$ in $\mathfrak{g}$. Therefore by Lemma 4.3.3, $\mathfrak{g}^X$ is nilpotent and moreover $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Hence $\mathfrak{g}^X$ has no nonzero semisimple elements, hence no nonzero toral subalgebras, and so the minimal Levi subalgebra containing X is $\mathfrak{g}$ according to Corollary 4.1.11.

Conversely, assume X is distinguished, so that the minimal Levi subalgebra containing X is $\mathfrak{g}$. By Corollary 4.1.11 again, $\mathfrak{g}^X$ contains no nonzero toral subalgebras. Therefore in particular $\mathfrak{g}^\phi$ contains no nonzero semisimple elements. But $\mathfrak{g}^\phi$ is reductive, so its center consists of semisimple elements; we conclude that $Z(\mathfrak{g}^\phi) = 0$, so that $\mathfrak{g}^{\phi}$ is in fact a semisimple Lie algebra, which cannot consist entirely of nilpotent elements unless it is zero. Thus $\mathfrak{g}^{\phi} = \{0\}$. Therefore $\mathfrak{g}$ contains no trivial submodules, and so $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_2)$ by Lemma 3.3.1. ∎

In fact, in [6, Chapter 8, §8.2, Theorem 8.2.3], Collingwood and McGovern prove that every distinguished nilpotent orbit is even. The proof of this result uses the fact that if Q is the subgroup of G_{ad} corresponding to a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$, and the orbit $Q \cdot Z$ is dense in $\mathfrak{u}$ for some Z, then $[\mathfrak{q}, Z] = \mathfrak{u}$, which is beyond the scope of this thesis.

Thus it remains only to discern distinguished orbits among even ones. The following theorem is from [6, Chapter 8, §8.2, Theorem 8.2.6].

Theorem 4.3.8. *An even nilpotent orbit $\mathcal{O}_X$ is distinguished if and only if its Jacobson-Morozov parabolic subalgebra $\mathfrak{p} = \mathfrak{l}_X \oplus \mathfrak{u}_X$ satisfies $\dim \mathfrak{l}_X = \dim \mathfrak{u}_X/[\mathfrak{u}_X, \mathfrak{u}_X]$.*

Proof: Let $\mathcal{O}_X$ be an even nilpotent orbit. Let $\mathfrak{u}' := \bigoplus_{i \geq 4} \mathfrak{g}_i$ (using the previous notations). By definition of an even orbit, we have that $\mathfrak{g}_1 = 0$, whence $\mathfrak{g}_3 = 0$ as well by $\mathfrak{sl}_2$ theory. Therefore we have $\mathfrak{g}_0 = \mathfrak{l}_X$ and since $\mathfrak{u}_X = \bigoplus_{i \geq 2} \mathfrak{g}_i$, we have $\dim \mathfrak{g}_2 = \dim \mathfrak{u}_X - \dim \mathfrak{u}'$. Moreover, $\mathfrak{u}_X = \bigoplus_{i \geq 2} \mathfrak{g}_i$ implies $[\mathfrak{u}_X, \mathfrak{u}_X] \subseteq \mathfrak{u}'$. Since $X \in \mathfrak{u}_X$, this map is surjective by Lemma 3.3.1 and equality holds. Hence, $\dim \mathfrak{g}_2 = \dim \mathfrak{u}_X/[\mathfrak{u}_X, \mathfrak{u}_X]$.

It now follows that $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_2$ if and only if $\dim \mathfrak{l}_X = \dim \mathfrak{u}_X/[\mathfrak{u}_X, \mathfrak{u}_X]$. By Proposition 4.3.7, this holds if and only if X is distinguished. ∎

We call a parabolic subalgebra of $\mathfrak{g}$ satisfying the conditions in Theorem 4.3.8 about the dimensions of its Levi subalgebra and its nilradical a *distinguished parabolic* *subalgebra.*

Example 4.3.9. Consider the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Then $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is spanned by the images of the simple root vectors. Since $\mathfrak{g}$ is semisimple, the dimension of its Cartan subalgebra is equal to the number of simple roots, so it follows that the Borel subalgebra is distinguished.

The Borel subalgebra is moreover the Jacobson-Morozov parabolic subalgebra corresponding to $X = \sum_{\alpha \in \Delta} X_\alpha$, where $X_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ for each $\alpha \in \Delta$. To see this, choose $H \in \mathfrak{h}$ such that $[H, X] = 2X$ and complete this to an $\mathfrak{sl}_2(k)$ triple. Then $[H, X_\alpha] > 0$ for each simple root α, so it follows that $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{n} = \oplus_{i>0} \mathfrak{g}_i$, as required.

Our final intermediate result, whose proof again uses a relationship between the adjoint group and the Lie algebra which is deeper than the scope of this thesis, states that this example is representative of the general case. It is from [6, Chapter 8, §8.2, Theorem 8.2.2].

Theorem 4.3.10. *Any distinguished parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is the Jacobson-Morozov parabolic subalgebra of a distinguished nilptotent element X. In particular, the map*

$$\mathcal{O}_X \mapsto [\mathfrak{p}]$$

from the set of distinguished orbits in $\mathfrak{g}$ to the set of $G_{\mathrm{ad}}-$conjugacy classes of distinguished parabolic subalgebras of $\mathfrak{g}$, where [q] is the conjugacy class of the Jacobson-Morozov parabolic subalgebras of X, is bijective. Moreover, the orbit $\mathcal{O}_X$ is the Richardson orbit attached to $\mathfrak{p}$.

Before the final theorem of this chapter, we want to prove the following lemma, which will be used in the proof of the theorem of Bala-Carter.

Lemma 4.3.11. *Let X be a nilpotent element of a semisimple Lie algebra $\mathfrak{g}$ and let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$. Then X is distinguished in $[\mathfrak{l}, \mathfrak{l}]$ if and only if $\mathfrak{l}$ is a minimal Levi subalgebra of $\mathfrak{g}$ containing X.*

Proof: Suppose first that $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are two Levi subalgebras of $\mathfrak{g}$. Each is a Levi subalgebra of a parabolic subalgebra, and all parabolic subalgebras are conjugate to standard parabolic subalgebras, so we may assume that

$$\mathfrak{l}_i = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Delta_i \rangle} \mathfrak{g}_\alpha$$

for some subsets $\Delta_i \subseteq \Delta$. Considering the finite list of possibilities, we conclude that $\mathfrak{l}_1 \subseteq \mathfrak{l}_2$ if and only if $\Delta_1 \subseteq \Delta_2$, and $\mathfrak{l}_1 \subsetneq \mathfrak{l}_2$ if and only if $\Delta_1 \subsetneq \Delta_2$.

Notice also that if $\mathfrak{l}_i$ is as above, then

$$[\mathfrak{l}_i, \mathfrak{l}_i] = \operatorname{span}\{H_\alpha \mid \alpha \in \Delta_i\} \oplus \bigoplus_{\alpha \in \langle \Delta_i \rangle} \mathfrak{g}_\alpha.$$

Therefore if $\mathfrak{l}_1 \subseteq \mathfrak{l}_2$ then $[\mathfrak{l}_1, \mathfrak{l}_1] \subseteq [\mathfrak{l}_2, \mathfrak{l}_2]$, and the subalgebra

$$\mathfrak{m} = \operatorname{span}\{H_\alpha \mid \alpha \in \Delta_2 \setminus \Delta_1\} \oplus [\mathfrak{l}_1, \mathfrak{l}_1]$$

is a standard Levi subalgebra of $[\mathfrak{l}_2, \mathfrak{l}_2]$ with respect to its Cartan subalgebra $\mathfrak{h}_2 = \operatorname{span}\{H_\alpha \mid \alpha \in \Delta_2\}$ and the root system $\Phi_2 = \langle \Delta_2 \rangle$. Notice that

$$[\mathfrak{l}_1, \mathfrak{l}_1] \subseteq \mathfrak{m} \subseteq \mathfrak{l}_1$$

where at each inclusion we may be increasing the dimension of the algebra, but strictly by adding more vectors from $\mathfrak{h}$.

Conversely, suppose that $\mathfrak{m}$ is a Levi subalgebra of $[\mathfrak{l}_2, \mathfrak{l}_2]$. Conjugating by an element of the adjoint group of $[\mathfrak{l}_2, \mathfrak{l}_2]$ as necessary, we may assume that $\mathfrak{m}$ is a standard Levi subalgebra, hence of the form

$$\mathfrak{m} = \mathfrak{h}_2 \oplus \bigoplus_{\alpha \in \langle \Delta_1 \rangle} [\mathfrak{l}_2, \mathfrak{l}_2]_\alpha$$

for some $\Delta_1 \subseteq \Delta_2$. The spaces $[\mathfrak{l}_2, \mathfrak{l}_2]_\alpha$, being one-dimensional and non-zero by construction, are equal to $\mathfrak{g}_\alpha$ for each $\alpha \in \Phi_2$. It follows that

$$\mathfrak{l}_1 = \operatorname{span}\{H_\alpha \mid \alpha \in \Delta \setminus \Delta_2\} \oplus \mathfrak{m}$$

is equal to

$$\mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Delta_1 \rangle} \mathfrak{g}_\alpha$$

whence it is a standard Levi of $\mathfrak{g}$. Moreover, since $\mathfrak{m} \subseteq [\mathfrak{l}_2, \mathfrak{l}_2] \subseteq \mathfrak{l}_2$ and $\mathfrak{h} \subseteq \mathfrak{l}_2$, we have that $\mathfrak{l}_1 \subseteq \mathfrak{l}_2$.

Therefore there is a one-to-one correspondence between standard Levi subalgebras $\mathfrak{m}$ of $[\mathfrak{l}_2, \mathfrak{l}_2]$ and standard Levi subalgebras $\mathfrak{l}_1$ contained in $\mathfrak{l}_2$.

Now suppose that X is a nilpotent element of $\mathfrak{g}$. Without loss of generality assume it is part of a standard triple $\{H, X, Y\}$ such that $H \in \mathfrak{h}$. Let $\mathfrak{l}_2$ be a minimal standard Levi subalgebra containing X. Then $X = [H, X] \in [\mathfrak{l}_2, \mathfrak{l}_2]$ and $ad(X)$ is nilpotent on $\mathfrak{g}$ hence on $[\mathfrak{l}_2, \mathfrak{l}_2]$, so X is nilpotent in both $\mathfrak{l}_2$ and its derived algebra.
If X were not distinguished in $[\mathfrak{l}_2, \mathfrak{l}_2]$ then there would be a Levi subalgebra $\mathfrak{m}$ of $[\mathfrak{l}_2, \mathfrak{l}_2]$ containing X, which up to conjugacy by an element of the subgroup of G_{ad} generated by the roots in $\mathfrak{l}_2$ (which is the adjoint group of $[\mathfrak{l}_2, \mathfrak{l}_2]$) we may assume is a standard Levi. Then the associated standard Levi $\mathfrak{l}_1$ is a strictly smaller Levi subalgebra of $\mathfrak{g}$, since it has a smaller root system. Since $X \in \mathfrak{m} \subseteq \mathfrak{l}_1$, this is a contradiction.

Conversely, if $X \in [\mathfrak{l}_2, \mathfrak{l}_2]$ is nilpotent, then by the abstract Jordan decomposition

it is nilpotent in any representation, in particular, the adjoint representation on $\mathfrak{g}$. Thus X is nilpotent in $\mathfrak{g}$.
Suppose that X is a distinguished nilpotent element of $[\mathfrak{l}_2, \mathfrak{l}_2]$, meaning that it does not lie in any proper Levi subalgebra $\mathfrak{m}$ of $[\mathfrak{l}_2, \mathfrak{l}_2]$. By the above correspondence of Levi subalgebras, it therefore cannot Lie in any $\mathfrak{l}_1$, where $\mathfrak{l}_1$ is a standard proper Levi subalgebra of $\mathfrak{g}$ contained in $\mathfrak{l}_2$. So $\mathfrak{l}_2$ is a minimal standard Levi subalgebra containing X.

Therefore $\mathfrak{l}_2$ is a minimal standard Levi subalgebra containing X if and only if X is a distinguished nilpotent element of $[\mathfrak{l}_2, \mathfrak{l}_2]$. Replacing $\mathfrak{l}_2$ by a G_{ad}^X−conjugate as necessary, we infer the general result. ∎

We are now able to give a parametrization of the nilpotent orbits of a reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ in terms of its Levi subalgebras and the distinguished parabolic subalgebras of each of them. This theorem is rephrased from a pair of results which can be found in [6, §8.2, page 125, Theorem 8.2.12].

Theorem 4.3.12 (Bala-Carter)**.** *Let X be a nilpotent element of $\mathfrak{g}$, distinguished in a Levi subalgebra $\mathfrak{l}$ and let $\mathfrak{p}_{\mathfrak{l}}$ be the Jacobson-Morozov parabolic subalgebra in $[\mathfrak{l}, \mathfrak{l}]$ corresponding to a standard triple containing X. The map*

$$\mathcal{O}_X \mapsto [(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})] \tag{4.3.2}$$

between the nilpotent G_{ad}−orbits of $\mathfrak{g}$ and the G_{ad}−conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ is a bijection whose inverse is

$$[(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})] \mapsto G_{\mathrm{ad}} \cdot \mathrm{Ind}_{\mathfrak{p}_{\mathfrak{l}}}^{\mathfrak{l}}(0). \tag{4.3.3}$$

Proof: We need first to prove that the map (4.3.2) is well defined. Let X and X' be nilpotent elements of $\mathfrak{g}$ that are G_{ad}−conjugate, in other words $\mathcal{O}_X = \mathcal{O}_{X'}$. We want to prove that the pairs $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ and $(\mathfrak{l}', \mathfrak{p}_{\mathfrak{l}'})$ corresponding to X and X' respectively via (4.3.2) are G_{ad}−conjugate. Since $X' \in \mathcal{O}_X$, there exists $g \in G_{\text{ad}}$ such that $g \cdot X' = X$. Then $g \cdot \mathfrak{l}'$ is a minimal Levi subalgebra containing $g \cdot X' = X$. Hence $g \cdot \mathfrak{l}'$ and $\mathfrak{l}$ are conjugate by an element g' of G_{ad}^X by Theorem 4.1.17. Therefore $g'g \cdot \mathfrak{p}_{\mathfrak{l}'}$ is the Jacobson-Morozov parabolic of $g'g \cdot X' = X$ in $[g'g \cdot \mathfrak{l}', g'g \cdot \mathfrak{l}'] = [\mathfrak{l}, \mathfrak{l}]$. By Theorem 4.3.10, they are G_{ad}−conjugate. Hence the map (4.3.2) is well defined.

The map (4.3.2) is injective by Theorem 4.3.10. To show surjectivity, let $(\mathfrak{l}, \mathfrak{p}_{\mathfrak{l}})$ be a pair of Levi subalgebra of $\mathfrak{g}$ and a distinguished parabolic subalgebra of $[\mathfrak{l}, \mathfrak{l}]$. Then $\mathfrak{p}_{\mathfrak{l}}$ is the Jacobson-Morozov parabolic subalgebra of a distinguished nilpotent element X of $[\mathfrak{l}, \mathfrak{l}]$. Since $[\mathfrak{l}, \mathfrak{l}]$ is semisimple and X is nilpotent in $[\mathfrak{l}, \mathfrak{l}]$, its image in $\mathfrak{g}$ by the canonical injection is also nilpotent. Moreover $\mathfrak{l}$ is a minimal Levi subalgebra in $\mathfrak{g}$ which contains X. Therefore X is distinguished in $\mathfrak{l}$. ∎

Chapter 5

Nilpotent orbits of $\mathfrak{so}(8)$

5.1 Basic structure of $\mathfrak{so}(2n)$

Let $n \geq 2$. We use our notation for the matrix form of the Lie algebra $\mathfrak{g} = \mathfrak{so}(2n)$ over the field $k = \mathbb{C}$ as defined in Section 2.1. In this section we provide the necessary details of its structure theory to allow us to elaborate on the Bala-Carter classification of its nilpotent orbits in the case that $n = 4$. These results are standard; for example, see [3, Ch.VIII, §13, no. 4].

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the subalgebra of diagonal matrices

$$\mathfrak{h} = \{H = \mathrm{diag}(a_1, a_2, \cdots, a_n, -a_n, -a_{n-1}, \cdots, -a_1) \mid a_i \in \mathbb{C}\}.$$

Given $1 \leq i \leq n$, define $\varepsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(H) = a_i$ for $H \in \mathfrak{h}$ as above. Then the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are

$$\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq n\}.$$

Given $\alpha \in \Phi$, one can check that a basis for the corresponding root space $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is given by X_α, where X_α is given in matrix form by

$$X_\alpha = \begin{cases} E_{i,j} - E_{-j,-i} & \text{if } \alpha = \varepsilon_i - \varepsilon_j; \\ E_{i,-j} - E_{j,-i} & \text{if } \alpha = \varepsilon_i + \varepsilon_j; \\ E_{-i,j} - E_{-j,i} & \text{if } \alpha = -\varepsilon_i - \varepsilon_j, \end{cases} \tag{5.1.1}$$

for each $1 \leq i \neq j \leq n$.

A choice of positive system is $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$, with respect to which the Borel subalgebra is the subalgebra of all upper triangular matrices of $\mathfrak{g}$. A simple system is

$$\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-2} - \varepsilon_{n-1}, \varepsilon_{n-1} + \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\};$$

when needed, we abbreviate these roots, in order, as $\alpha_1, \alpha_2, \cdots, \alpha_n$.

To each simple root α we attach a basis element of $\mathfrak{h}$ by setting

$$H_\alpha = [X_\alpha, X_{-\alpha}].$$

This is given by

$$H_{\alpha_i} = \begin{cases} E_{ii} - E_{i+1,i+1} - E_{-i,-i} + E_{-(i+1),-(i+1)} & \text{if } 1 \leq i < n \\ E_{n-1,n-1} + E_{n,n} - E_{-(n-1),-(n-1)} - E_{-n,-n} & \text{if } i = n. \end{cases} \tag{5.1.2}$$

Then for each $\alpha \in \Delta$, $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ is an $\mathfrak{sl}(2)$–triple.

5.2 The Lie algebra $\mathfrak{so}(8)$

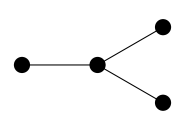
Let us now specialize to the case that $n = 4$, so that $\mathfrak{g} = \mathfrak{so}(8)$. Then $|\Delta| = 4$ and $\dim(\mathfrak{g}) = 16$.

It is useful to note the following relations, which also suffice to prove that Δ is a simple system for Φ in this case:

$$\begin{aligned}
\alpha_1 &= \varepsilon_1 - \varepsilon_2 & \alpha_2 + \alpha_4 &= \varepsilon_2 + \varepsilon_4 \\
\alpha_2 &= \varepsilon_2 - \varepsilon_3 & \alpha_1 + \alpha_2 + \alpha_3 &= \varepsilon_1 - \varepsilon_4 \\
\alpha_3 &= \varepsilon_3 - \varepsilon_4 & \alpha_1 + \alpha_2 + \alpha_4 &= \varepsilon_1 + \varepsilon_4 \\
\alpha_4 &= \varepsilon_3 + \varepsilon_4 & \alpha_2 + \alpha_3 + \alpha_4 &= \varepsilon_2 + \varepsilon_3 \\
\alpha_1 + \alpha_2 &= \varepsilon_1 - \varepsilon_3 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= \varepsilon_1 + \varepsilon_3 \\
\alpha_2 + \alpha_3 &= \varepsilon_2 + \varepsilon_4 & \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 &= \varepsilon_1 + \varepsilon_2
\end{aligned} \tag{5.2.1}$$

We note that this set is preserved under any permutation of the roots $\{\alpha_1, \alpha_3, \alpha_4\}$ (but is asymmetric with respect to α_2).

The inner product on $\mathfrak{h}^*$ induced by the Killing form restricted to $\mathfrak{h}$ is proportional to the form satisfying $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, whence it follows that the Dynkin diagram for $\mathfrak{g}$ is



In [7, §12.1], it is shown that the Weyl group of $\mathfrak{so}(8)$ is generated by the set of permutations on the symbols

$$\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$$

together with an even number of sign changes, that is, maps of the form

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mapsto (-\varepsilon_1, -\varepsilon_2, \varepsilon_3, \varepsilon_4).$$

One can verify that Φ is preserved under these transformations. The following general theorem is stated in [6, Lemma 3.8.1].

Theorem 5.2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathcal{W}$ denote its Weyl group. Then two Levi subalgebras $\mathfrak{l}_\Delta$ and $\mathfrak{l}'_\Delta$ defined respectively by the subsets of simple roots Δ and Δ' are G_{ad}–conjugate if and only if Δ and Δ' are $\mathcal{W}$–conjugate.*

The Bala-Carter theorem states that the nilpotent orbits of $\mathfrak{g}$ under G_{ad} are in one-to-one correspondence with G_{ad}-conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_\mathfrak{l})$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_\mathfrak{l}$ is a distinguished parabolic subalgebra of the semisimple Lie algebra $[\mathfrak{l}, \mathfrak{l}]$.

Therefore, to enumerate the nilpotent orbits of G_{ad} in $\mathfrak{g}$, using the Bala-Carter theorem, we first need to generate a list of all Levi subalgebras of $\mathfrak{g}$ up to G_{ad}-conjugacy. By Proposition 2.9.4, it suffices to enumerate the Levi subalgebras of standard parabolic subalgebras of $\mathfrak{g}$ (see Section 5.7) and then determine, using Theorem 5.2.1, which of these are G_{ad} conjugate. We do so in Section 5.3.

Moreover, for each conjugacy class of Levi subalgebras, we will determine the isomorphism class of the derived algebra $[\mathfrak{l}, \mathfrak{l}]$. We will see that, except for the Levi subalgebra $\mathfrak{l} = \mathfrak{g}$, the derived algebra is isomorphic to one of $\mathfrak{sl}(4)$, $\mathfrak{sl}(3)$, or a direct sum of copies of $\mathfrak{sl}(2)$. We will prove in Section 5.4 that the only distinguished parabolic subalgebra of such a Lie algebra is the Borel subalgebra.

When $\mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] = \mathfrak{g}$, however, the situation is more interesting. Therefore, at the same time that we enumerate the Levi subalgebras of $\mathfrak{g}$ in Section 5.3, we also test the corresponding parabolic subalgebras of $\mathfrak{g}$ to identify those which are distinguished. Again by Proposition 2.9.4, this will allow us to subsequently enumerate all the Bala-Carter pairs of the form $(\mathfrak{g}, \mathfrak{p}_{\mathfrak{g}})$, up to G_{ad}-conjugacy.

5.3 Levi subalgebras of $\mathfrak{so}(8)$

Using our standard notation, we write $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ for the Levi decomposition of a standard parabolic subalgebra. If the parabolic subalgebra $\mathfrak{p}$ is defined by the a set of simple roots Δ' and the positive roots Φ^+, we let $S = \Phi^+ \setminus \langle \Delta' \rangle$. Let $S' \subset S$ be the subset of roots of S which **cannot** be written as a sum of two roots in S; then $\dim(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]) = |S'|$. Therefore, the parabolic subalgebra $\mathfrak{p}$ is distinguished if and only if $\dim(\mathfrak{l}) = |S'|$.

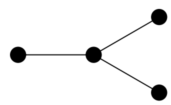
In the following, we enumerate all standard Levi and parabolic subalgebras by enumerating all possible subsets Δ' of Δ.

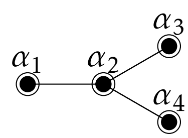
5.3.1 Cases: $|\Delta'| = 0$ or $|\Delta'| = 4$

If $\Delta' = \emptyset$, then $\mathfrak{l}$ is equal to $\mathfrak{h}$, the Cartan subalgebra, whose derived algebra $[\mathfrak{l}, \mathfrak{l}] = \{0\}$. The associated standard parabolic subalgebra $\mathfrak{p}$ is the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. Thus all simple roots lie in $S = \Phi^+ \setminus \Delta' = \Phi^+$, and by definition these are the only positive roots which cannot be written as a sum of two other roots in Φ^+, so $S' = \Delta$. Therefore $\dim \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}] = |\Delta| = 4 = \dim(\mathfrak{l})$, so $\mathfrak{p}$ is a distinguished parabolic subalgebra of $\mathfrak{g}$.

At the other extreme, if $\Delta' = \Delta$, then $\mathfrak{l} = \mathfrak{g}$, which is equal to its derived algebra. The associated parabolic subalgebra is $\mathfrak{p} = \mathfrak{l}$ and so $\mathfrak{u} = \{0\}$. It follows that $\mathfrak{p}$ is not distinguished in $\mathfrak{g}$.

Each of these two Levi subalgebras represent a unique G_{ad}-conjugacy class of Levi subalgebras of $\mathfrak{g}$. They are identified by the following Dynkin diagrams, respectively:





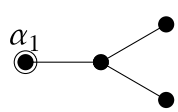
5.3.2 Case: $|\Delta'| = 1$

There are four distinct standard parabolic subalgebras of $\mathfrak{g}$, each corresponding to a choice of simple root α. In each case, we have

$$\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha},$$

so that $\dim(\mathfrak{l}) = 6$ and $[\mathfrak{l}, \mathfrak{l}] \cong \mathfrak{sl}(2)$.

To determine which of these are G_{ad}-conjugate, we note that a standard result about root systems is that all long roots are conjugate via the Weyl group (see, for example, [7, §10 Lemma C]). In $\mathfrak{g}$, all roots have the same length; therefore by Theorem 5.2.1, all Levi subalgebras of rank 1 are G_{ad} conjugate. Thus for the purposes of the Bala-Carter theorem, it suffices to consider the Levi subalgebra corresponding to $\Delta' = \{\alpha_1\}$, whose Dynkin diagram is as follows:

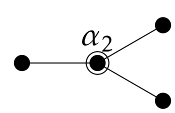


The parabolic subalgebras corresponding to different choices of Δ' are not conjugate; each is a standard parabolic subalgebra. We wish to determine which of them, if any, are distinguished.

Let $\Delta' = \{\alpha_1\}$, and consider $S = \Phi^+ \setminus \{\alpha_1\}$. Using (5.2.1), we see that the set of roots in S which cannot be written as a sum of two other roots in S is $S' = \{\alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2\}$, so $\dim(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]) = 4 \neq \dim(\mathfrak{l})$, and $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is not distinguished.

By symmetry of the set of roots under a cyclic permutation of the set $\{\alpha_1, \alpha_3, \alpha_4\}$, we infer that the parabolic subalgebras corresponding to $\Delta' = \{\alpha_3\}$ and $\Delta' = \{\alpha_4\}$ are also not distinguished.

If $\Delta' = \{\alpha_2\}$, on the other hand, then the subset of $S = \Phi^+ \setminus \{\alpha_2\}$ which cannot be written as a sum of elements of S is $S' = \{\alpha_1, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4\}$, which has order 6. It follows that $\dim(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]) = 6 = \dim \mathfrak{l}$ and thus that this parabolic subalgebra is distinguished:



5.3.3 Case: $|\Delta'| = 2$

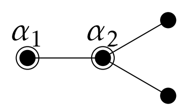
There are $\binom{4}{2} = 6$ choices of subset $\Delta' = \Delta_{i,j}$ of Δ with two elements. We consider each in turn, and then establish which corresponding Levi subalgebras are conjugate.

5.3.3.1 Subcase: $\Delta' \in \{\Delta'_{1,2}, \Delta'_{2,3}, \Delta'_{2,4}\}$

When $\Delta_{1,2} = \{\alpha_1, \alpha_2\}$, we have $\langle \Delta_{1,2} \rangle = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$. Therefore

$$\mathfrak{l} = \mathfrak{h} \oplus \oplus_{\alpha \in \langle \Delta_{1,2} \rangle} \mathfrak{g}_\alpha$$

has dimension $4 = 6 = 10$, and $[\mathfrak{l}, \mathfrak{l}] \cong \mathfrak{sl}(3)$. Its Dynkin diagram is



In this case, $S = \Phi^+ \setminus \langle \Delta' \rangle$ has 9 elements, most of which cannot be written as a sum of elements of S, namely

$$S' = \{\alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4\}.$$

Therefore $\dim(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]) = 6 < \dim(\mathfrak{l}) = 10$, and the associated parabolic subalgebra is not distinguished.

By the symmetry of the roots under cyclical permutation of the subset $\{\alpha_1, \alpha_3, \alpha_4\}$, we deduce that the parabolic subalgebras corresponding to $\Delta_{2,3}$ and to $\Delta_{2,4}$ are also not distinguished.

Lemma 5.3.1. *The subsets of Φ generated by $\Delta_{1,2}$, $\Delta_{2,3}$ and $\Delta_{2,4}$ are conjugate by the Weyl group.*

Proof: The permutation

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mapsto (\varepsilon_4, \varepsilon_2, \varepsilon_3, \varepsilon_1)$$

lies in the Weyl group and sends $\langle \Delta_{1,2} \rangle = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}$ to $\langle \Delta_{2,3} \rangle =$

$\{\pm(\varepsilon_4-\varepsilon_2), \pm(\varepsilon_2-\varepsilon_3), \pm(\varepsilon_4-\varepsilon_3)\}$. On the other hand, the signed permutation

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mapsto (-\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_4)$$

sends $\langle\Delta_{2,3}\rangle = \{\pm(\varepsilon_2-\varepsilon_3), \pm(\varepsilon_3-\varepsilon_4), \pm(\varepsilon_2-\varepsilon_4)\}$ to $\langle\Delta_{2,4}\rangle = \{\pm(\varepsilon_2-\varepsilon_3), \pm(\varepsilon_3+\varepsilon_4), \pm(\varepsilon_2+\varepsilon_4)\}$. ∎

It follows from this lemma, by Theorem 5.2.1, the corresponding Levi subalgebras are G_{ad} conjugate.

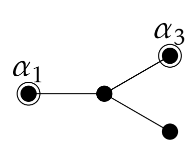
5.3.3.2 Subcase: $\Delta' \in \{\Delta_{1,3}, \Delta_{1,4}, \Delta_{3,4}\}$

Now consider the three remaining standard parabolic subalgebras corresponding to $\Delta_{1,3}$, $\Delta_{1,4}$ and $\Delta_{3,4}$. In all three cases, we have that

$$\langle\Delta_{i,j}\rangle = \{\pm\alpha_i, \pm\alpha_j\}$$

since $\alpha_i + \alpha_j$ is not a root. Therefore $\mathfrak{l} = \mathfrak{h} \oplus \oplus_{\alpha\in\{\pm\alpha_i,\pm\alpha_j\}}\mathfrak{g}_\alpha$ is a Levi subalgebra of dimension $4+4=8$. The derived algebra in each case is isomorphic to two copies of $\mathfrak{sl}(2)$.

Consider now the case that $\Delta' = \Delta_{1,3}$, whose Dynkin diagram we picture as follows:



Then $S = \Phi^+ \setminus \{\alpha_1, \alpha_3\}$ and so the only roots in S which are not sums of other roots of S are

$$S' = \{\alpha_2, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}.$$

Therefore $\dim(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]) = 4 < \dim(\mathfrak{l}) = 8$, and the corresponding parabolic subalgebra is not distinguished.

Using the symmetry of Φ^+ under the cyclical permutation of the set $\{\varepsilon_1, \varepsilon_3, \varepsilon_4\}$, we conclude that none of these three parabolic subalgebras are distinguished.

We next claim that the three Levi subalgebras are not conjugate under G_{ad}. We first prove the following lemma.

Lemma 5.3.2. *The sets $\langle\Delta_{1,3}\rangle, \langle\Delta_{1,4}\rangle$ and $\langle\Delta_{3,4}\rangle$ lie in distinct orbits under the action of the Weyl group.*

Proof: The three sets of roots are

$$\langle\Delta_{1,3}\rangle = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_3 - \varepsilon_4)\}$$
$$\langle\Delta_{3,4}\rangle = \{\pm(\varepsilon_3 - \varepsilon_4), \pm(\varepsilon_3 + \varepsilon_4)\}$$
$$\langle\Delta_{1,4}\rangle = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_3 + \varepsilon_4)\}.$$

First note that applying any signed permutation in the Weyl group to the set $\langle\Delta_{3,4}\rangle$ would again give a set of four roots which are linear combinations of only two basis elements $\{\varepsilon_i, \varepsilon_j\}$. So this set is not conjugate to either of the two others via the Weyl group.

Now we show there cannot exist an element of the Weyl group sending $\langle\Delta_{1,3}\rangle$ to $\langle\Delta_{1,4}\rangle$. If we apply an unsigned permutation, or a permutation changing all four signs, to $\langle\Delta_{1,3}\rangle$, then both $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_3 - \varepsilon_4$ are sent to roots of the form $\varepsilon_i - \varepsilon_j$; therefore no such transformation can take $\langle\Delta_{1,3}\rangle$ to $\langle\Delta_{1,4}\rangle$. If we apply a signed permutation that changes exactly two signs, then it either sends both $\varepsilon_1 - \varepsilon_2$ and

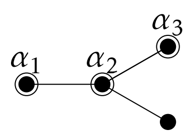
$\varepsilon_3 - \varepsilon_4$ to roots of the form $\varepsilon_i - \varepsilon_j$, or else sends both to roots of the form $\pm(\varepsilon_i + \varepsilon_j)$. In either case, the image cannot be $\langle \Delta_{1,4} \rangle$. Therefore none of these three subroot-systems are conjugate under the Weyl group. ∎

It now follows from Theorem 5.2.1 that the Levi subalgebras corresponding to $\Delta_{1,3}, \Delta_{3,4}$ and $\Delta_{1,4}$ are all non-conjugate under G_{ad}.

5.3.4 Case $|\Delta'| = 3$

It remains to consider the four possible Levi subalgebras of rank 3.

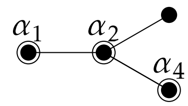
5.3.4.1 $\Delta' = \{\alpha_1, \alpha_2, \alpha_3\}$



The first corresponds to $\Delta' = \{\alpha_1, \alpha_2, \alpha_3\}$. Here, $\langle \Delta' \rangle = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq 4\}$, which is the root system for $\mathfrak{sl}(4)$. It follows that $\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Delta' \rangle} \mathfrak{g}_\alpha$ has dimension $4+6+6 = 16$, and that the derived algebra $[\mathfrak{l}, \mathfrak{l}]$ is isomorphic to $\mathfrak{sl}(4)$.

We have $S = \Phi^+ \setminus \langle \Delta' \rangle = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq 4\}$, none of which can be written as a sum of other roots in S, so $S' = S$. Therefore $\dim(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]) = 6 < \dim(\mathfrak{l})$, so the corresponding parabolic subalgebra is not distinguished.

5.3.4.2 $\Delta' = \{\alpha_1, \alpha_2, \alpha_4\}$



The next corresponds to $\Delta' = \{\alpha_1, \alpha_2, \alpha_4\}$. Here,

$$\langle \Delta' \rangle = \{\pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_4) \mid 1 \leq i \neq j \leq 3\},$$

which as a root system is the set $\{\pm\alpha_1, \pm\alpha_2, \pm\alpha_4, \pm(\alpha_1 + \alpha_2), \pm(\alpha_2 + \alpha_4), \pm(\alpha_1 + \alpha_2 + \alpha + 4)\}$. Thus this root system is abstractly isomorphic to that of $\mathfrak{sl}(4)$.

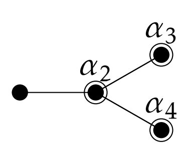
It follows that the corresponding Levi subalgebra is $4+6+6 = 16$-dimensional, and that the derived algebra $[\mathfrak{l}, \mathfrak{l}] \cong \mathfrak{sl}(4)$.

The set $S = \Phi^+ \setminus \langle \Delta' \rangle$ is

$$\{\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha + 4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\};$$

since this set is characterized as the set of all positive roots whose expression as a linear combination of simple roots contains α_3 with coefficient equal to 1, none of these are linear combinations of the others, and again $S = S'$. Therefore $\dim(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]) = 6$ and again the corresponding parabolic subalgebra is not distinguished.

5.3.4.3 $\Delta' = \{\alpha_2, \alpha_3, \alpha_4\}$



We now consider $\Delta' = \{\alpha_2, \alpha_3, \alpha_4\}$. The subrootsystem it generates is

$$\langle \Delta' \rangle = \{\pm\varepsilon_i \pm \varepsilon_j \mid 2 \leq i \neq j \leq 4\}$$

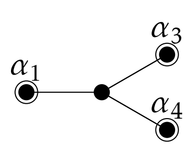
which is the root system of $\mathfrak{so}(6)$. It can also be characterized as the set

$$\{\pm\alpha_3, \pm\alpha_2, \pm\alpha_4, \pm(\alpha_3 + \alpha_2), \pm(\alpha_2 + \alpha_4), \pm(\alpha_3 + \alpha_2 + \alpha_4)\}$$

whence it is also the root system of $\mathfrak{sl}(4)$. From either point of view, we see that $\mathfrak{l} = \mathfrak{h} \oplus \oplus_{\alpha \in \langle \Delta' \rangle}$ is 16-dimensional, with derived algebra isomorphic to $\mathfrak{so}(6) \cong \mathfrak{sl}(4)$. By similar arguments to the preceding, we can show that the associated parabolic subalgebra is not distinguished.

At this point, we note that the arguments of Lemma 5.3.2 can be applied to show that these three subrootsystems of the subcases 5.3.4.1, 5.3.4.2, and 5.3.4.3 cannot be conjugate via an element of the Weyl group, and therefore that these three Levi subalgebras are not conjugate via G_{ad}.

5.3.4.4 $\Delta' = \{\alpha_1, \alpha_3, \alpha_4\}$



The final Levi subalgebra of rank 3 to consider is generated by $\Delta' = \{\alpha_1, \alpha_3, \alpha_4\}$. Then

$$\langle \Delta' \rangle = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_3 - \varepsilon_4), \pm(\varepsilon_3 + \varepsilon_4)\}$$

is a subrootsystem with only six elements, so that the associated Levi subalgebra has $\dim(\mathfrak{l}) = 4 + 6 = 10$. It follows that $\mathfrak{l}$ is not isomorphic to any of the three preceding Levi subalgebras of rank 3, and therefore in particular is not G_{ad}-conjugate to any of them.

In fact, each pair of roots above generates a subalgebra isomorphic to $\mathfrak{sl}(2)$ inside $\mathfrak{l}$, and it follows that

$$[\mathfrak{l},\mathfrak{l}] \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2).$$

The set $S = \Phi^+ \setminus \langle \Delta' \rangle$ can be characterized as the subset of all positive roots which when expressed as a linear combination of simple roots, include α_2 with a positive coefficient. Of the nine such roots, only one can be written as a linear combination of the other two, hence $\dim(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]) = 8 < \dim(\mathfrak{l})$, and so the corresponding parabolic subalgebra is not distinguished.

5.4 Distinguished parabolic subalgebras of $[\mathfrak{l},\mathfrak{l}]$ when $\mathfrak{l} \neq \mathfrak{g}$

Our goal in this section is to enumerate all possible distinguished parabolic subalgebras of a Lie algebra of one of the following types, which represent the isomorphism classes of proper Levi subalgebras of $\mathfrak{g}$:

$$\mathfrak{sl}(2),\ \mathfrak{sl}(2) \oplus \mathfrak{sl}(2),\ \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2),\ \mathfrak{sl}(3),\ \mathfrak{sl}(4). \tag{5.4.1}$$

We saw that the Borel subalgebra is always a distinguished parabolic subalgebra, and that $\mathfrak{g}$ is not distinguished in itself. Therefore for the Lie algebra $\mathfrak{sl}(2)$ there is nothing left to prove, since it has only two parabolic subalgebras, the Borel subalgebra and itself.

Lemma 5.4.1. *The only distinguished parabolic subalgebra of a direct sum of copies of*

$\mathfrak{sl}(2)$ *is the Borel subalgebra.*

Proof: The root system for a direct sum of $k \geq 2$ copies of $\mathfrak{sl}(2)$ has the form

$$\Phi = \{\pm\beta_1, \pm\beta_2, \ldots, \pm\beta_k\},$$

with $\Delta = \Phi^+ = \{\beta_i \mid 1 \leq i \leq k\}$. Since no positive roots are sums of other positive roots, it follows that for any $\Delta' \subset \Delta$, $S' = \Phi^+ \setminus \Delta'$. Therefore $|S'| = k - |\Delta'|$. On the other hand, $\dim(\mathfrak{l}_{\Delta'}) = \dim(\mathfrak{h}) + |\langle\Delta'\rangle|$, which is equal to $k + 2|\Delta'|$. Therefore the parabolic is distinguished if and only if $\Delta' = \emptyset$, which corresponds to the Borel subalgebra. ∎

Lemma 5.4.2. *The only distinguished parabolic subalgebra of* $\mathfrak{sl}(3)$ *is the Borel subalgebra (up to conjugacy).*

Proof: The Lie algebra $\mathfrak{sl}(3)$ has a two-dimensional Cartan subalgebra $\mathfrak{h}$ and a root system of the form $\Phi = \{\pm\beta_1, \pm\beta_2, \pm(\beta_1 + \beta_2)\}$ where $\Delta = \{\beta_1, \beta_2\}$.

If $\Delta' = \{\beta_1\}$, then $S = \Phi^+ \setminus \{\beta_1\}$ has only 2 elements, so it follows trivially that $S' = S$. Therefore $\dim(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]) = 2$, whereas $\dim(\mathfrak{l}) = \dim(\mathfrak{h}) + |\{\pm\beta_1\}| = 4$. Therefore the corresponding parabolic subalgebra is not distinguished.

The case for $\Delta' = \{\beta_2\}$ is equivalent, by the symmetry of the root system, and the cases $\Delta' = \emptyset$ and $\Delta' = \Delta$ were treated above. Therefore, the Borel subalgebra is the only distinguished parabolic subalgebra of $\mathfrak{sl}(3)$. ∎

Lemma 5.4.3. *The only distinguished parabolic subalgebra of* $\mathfrak{sl}(4)$ *is the Borel subalgebra (up to conjugacy).*

Proof: The Lie algebra $\mathfrak{sl}(4)$ has a three-dimensional Cartan subalgebra $\mathfrak{h}$ and a root system with positive roots of the form

$$\Phi^+ = \{\beta_1, \beta_2, \beta_3, \beta_1 + \beta_2, \beta_2 + \beta_3, \beta_1 + \beta_2 + \beta_3\}$$

corresponding to $\Delta = \{\beta_1, \beta_2, \beta_3\}$. Eliminating $\Delta' = \emptyset$ (which gives the distinguished Borel subalgebra) and $\Delta' = \Delta$ (which gives $\mathfrak{sl}(4)$, which is not distinguished) leaves the following cases.

If $\Delta' = \{\beta_1\}$ then the subset of $S = \Phi^+ \setminus \Delta'$ consisting of elements that are not sums of elements of S is $S' = \{\alpha_2, \alpha_3, \alpha_1 + \alpha_2\}$. Thus $\dim(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]) = 3$ whereas $\dim(\mathfrak{l}) = \dim(\mathfrak{h}) + 2 = 5$, so the corresponding parabolic subalgebra is not distinguished. By symmetry, the same result holds for $\Delta' = \{\beta_3\}$.

If $\Delta' = \{\beta_2\}$, then $S = \{\beta_1, \beta_3, \beta_1 + \beta_2, \beta_2 + \beta_3, \beta_1 + \beta_2 + \beta_3\}$ has five elements and we see that $S' = S \setminus \{\beta_1 + \beta_2 + \beta_3\}$ has four elements. Thus $\dim(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]) = 4$ whereas $\dim(\mathfrak{l}) = \dim(\mathfrak{h}) + 2 = 5$. So the corresponding parabolic subalgebra is not distinguished.

If $\Delta' = \{\beta_1, \beta_2\}$, then $|\langle\Delta\rangle| = 6$, so then $\dim(\mathfrak{l}) = \dim(\mathfrak{h}) + 6 = 9$. Since this is larger than $|\Phi^+|$ (and $S' \subset \Phi^+$), we immediately deduce that the corresponding parabolic subalgebra is not distinguished. By symmetry, the same holds for $\Delta' = \{\beta_2, \beta_3\}$.

Finally, if $\Delta' = \{\beta_1, \beta_3\}$ then $|\langle\Delta'\rangle| = 4$ so $\dim(\mathfrak{l}) = 7$, which again exceeds $|\Phi^+|$, so the corresponding parabolic subalgebra is not distinguished. ∎

5.5 Summary

In Section 5.3, we saw that in all, there are 11 conjugacy classes of Levi subalgebras of $\mathfrak{g}$, enumerated in Figure 5.1. When $\mathfrak{l}$ is a proper Levi subalgebra, we have that

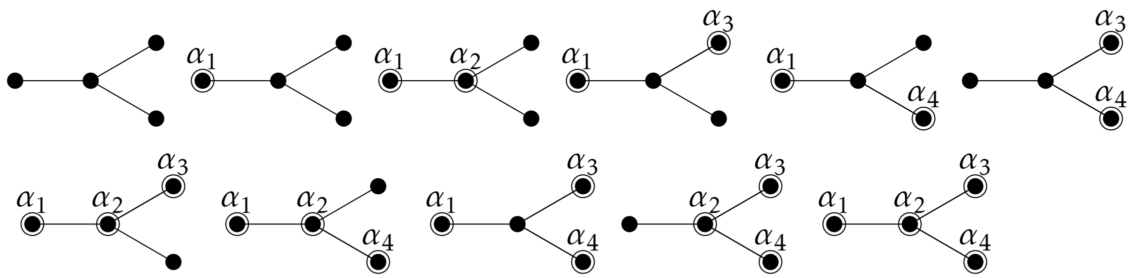


Figure 5.1: Representatives of the 11 G_{ad}-conjugacy classes of Levi subalgebras of $\mathfrak{so}(8)$.

$[\mathfrak{l},\mathfrak{l}]$ is one of the subalgebras from the list (5.4.1), and we showed in Lemmas 5.4.1, 5.4.2, and 5.4.3 that each of these contain a unique distinguished parabolic subalgebra, which is the Borel subalgebra.

Therefore, to each proper Levi subalgebra we attach exactly one pair $(\mathfrak{l},\mathfrak{p}_{\mathfrak{l}})$ up to G_{ad}−conjugacy in the Bala-Carter Theorem.

When $\mathfrak{l}=\mathfrak{g}$, then $\mathfrak{l}$ has exactly two distinguished standard parabolic subalgebras: the Borel subalgebra $\mathfrak{p}=\mathfrak{b}$, and the parabolic subalgebra corresponding to $\Delta'=\{\alpha_2\}$, which we can denote $\mathfrak{p}_{\alpha_2}$. Therefore to $\mathfrak{l}$ we attach two pairs

$$(\mathfrak{g},\mathfrak{b}) \quad \text{and} \quad (\mathfrak{g},\mathfrak{p}_{\alpha_2})$$

in the Bala-Carter classification.

Therefore, by the Bala-Carter Theorem, there are a total of 12 nilpotent G_{ad}−orbits in $\mathfrak{g}$.

In the next section, we use the Bala-Carter theory to produce representatives of each of these orbits and compare these with the Steinberg classification of nilpotent orbits by partitions of 8.

5.6 Representatives of Nilpotent Orbits of $\mathfrak{so}(8)$

We saw that if the distinguished parabolic subalgebra $\mathfrak{p}$ is the Borel subalgebra, then a representative for the distinguished nilpotent orbit whose Jacobson-Morozov parabolic is given by $X = \sum_{\alpha \in \Delta'} X_\alpha$.

For each of the 11 Bala-Carter pairs $(\mathfrak{l}, \mathfrak{p})$ determined above such that $\mathfrak{p}$ is a Borel subalgebra, we determine an element $H \in \mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}]$ such that $[H, X] = 2X$. Then by the Jacobson-Morozov theorem 4.2.5, there exists $Y \in \mathfrak{g}$ such that $\{H, X, Y\}$ is an $\mathfrak{sl}(2)$–triple. The eigenvalues of the action of H on the standard representation $V \cong \mathbb{C}^8$ then determine its decomposition into irreducible subrepresentations. The resulting decomposition defines a partition of 8, which we record.

$\Delta' = \emptyset : X = 0$. This is the degenerate case of the zero orbit, to which we attach the trivial partition $[1^8]$.

$\Delta' = \{\alpha_1\} : X = X_{\alpha_1}$ and $H = H_{\alpha_1}$. Since $H_{\alpha_1} = E_{11} - E_{22} - E_{-1,-1} + E_{-2,-2}$ is a diagonal matrix, we see that its eigenvalues, counted with multiplicity, are $(1, -1, 0, 0, 0, 0, 1, -1)$. These are the eigenvalues of the decomposition of V into two copies of the irreducible subrepresentation of dimension 2, and four copies of the irreducible subrepresentation of dimension 1. Therefore, we attach to this orbit the partition $[2^2, 1^4]$.

$\Delta' = \{\alpha_1, \alpha_2\} : X = X_{\alpha_1} + X_{\alpha_2}$. A diagonal element

$$H = \mathrm{diag}(a_1, a_2, a_3, a_4, -a_4, -a_3, -a_2, -a_1)$$

acts on X by $[H, X] = \alpha_1(H)X_{\alpha_1} + \alpha_2(H)X_{\alpha_2}$. We require that $[H, X] = 2X$, whence $a_1 - a_2 = 2$ and $a_2 - a_3 = 2$. Moreover, $H \in [\mathfrak{l}, \mathfrak{l}]$ implies that $H \in \mathrm{span}\{H_{\alpha_1}, H_{\alpha_2}\}$. This leads to the choice

$$H = 2H_{\alpha_1} + 2H_{\alpha_2} = \mathrm{diag}(2, 0, -2, 0, 0, 2, 0, -2).$$

These are the weights of a direct sum of two copies of a three-dimensional representation, and two copies of a trivial representation, so the associated partition is $[3^2, 1^2]$.

$\Delta' = \{\alpha_1, \alpha_3\} : X = X_{\alpha_1} + X_{\alpha_3}$. Since $\alpha_i(H_{\alpha_j}) = 0$ for $\{i, j\} = \{1, 3\}$, we can take

$$H = H_{\alpha_1} + H_{\alpha_3} = \mathrm{diag}(1, -1, 1, -1, 1, -1, 1, -1)$$

whence the weights are those of four copies of a 2-dimensional irreducible representation, whence the partition is $[2^4]$.

$\Delta' = \{\alpha_1, \alpha_4\}$: $X = X_{\alpha_1} + X_{\alpha_4}$. As above, we can take

$$H = H_{\alpha_1} + H_{\alpha_4} = \mathrm{diag}(1, -1, 1, 1, -1, -1, 1, -1)$$

which gives the same partition $[2^4]$.

$\Delta' = \{\alpha_3, \alpha_4\}$: $X = X_{\alpha_3} + X_{\alpha_4}$. As above, we can take

$$H = H_{\alpha_3} + H_{\alpha_4} = \mathrm{diag}(0, 0, 2, 0, 0, -2, 0, 0)$$

which corresponds to the partition $[3, 1^5]$.

$\Delta' = \{\alpha_1, \alpha_2, \alpha_3\} : X = X_{\alpha_1} + X_{\alpha_2} + X_{\alpha_3}$. Since for an

$$H = \mathrm{diag}(a_1, a_2, a_3, a_4, -a_4, -a_3, -a_2, -a_1)$$

we have

$$[H, X] = (a_1 - a_2)X_{\alpha_1} + (a_2 - a_3)X_{\alpha_2} + (a_3 - a_4)X_{\alpha_3}$$

it follows that we should choose $a_1 - a_2 = a_2 - a_3 = a_3 - a_4 = 2$. With this condition and $H \in \mathrm{span}\{H_\alpha \mid \alpha \in \Delta'\}$, we have

$$H = 3H_{\varepsilon_1 - \varepsilon_2} + 4H_{\varepsilon_2 - \varepsilon_3} + 3H_{\varepsilon_3 - \varepsilon_4} = \mathrm{diag}(3, 1, -1, -3, 3, 1, -1, -3).$$

Thus the corresponding partition is $[4^2]$.

$\Delta' = \{\alpha_1, \alpha_2, \alpha_4\} : X = X_{\alpha_1} + X_{\alpha_2} + X_{\alpha_4}$. Since for an arbitrary $H \in \mathfrak{h}$ as above we have

$$[H, X] = (a_1 - a_2)X_{\alpha_1} + (a_2 - a_3)X_{\alpha_2} + (a_3 + a_4)X_{\alpha_4}$$

we need $a_1 - a_2 = a_2 - a_3 = a_3 + a_4 = 2$, when $(a_1, a_2, a_3, a_4) = (a+4, a+2, a, 2-a)$ for some choice of $a = a_3$. Since H should lie in the span of $\{H_\alpha \mid \alpha \in \Delta'\}$, it follows that the vector $(a+4, a+2, a, 2-a)$ should be in the span of the vectors $(1, -1, 0, 0)$, $(0, 1, -1, 0)$, and $(0, 0, 1, 1)$ (which correspond to $H_{\alpha_1}, H_{\alpha_2}$ and H_{α_4} respectively). This yields $a = -1$, or

$$H = 3H_{\alpha_1} + 4H_{\alpha_2} + 3H_{\alpha_4} = \mathrm{diag}(3, 1, -1, 3, -3, 1, -1, -3)$$

which again corresponds to the partition $[4^2]$.

$\Delta' = \{\alpha_2, \alpha_3, \alpha_4\} : X = X_{\alpha_2} + X_{\alpha_3} + X_{\alpha_4}$. Then the condition that $\alpha(H) = 2$ for each $\alpha \in \Delta'$ gives $a_2 - a_3 = 2$ and $a_3 \pm a_4 = 2$. Thus $a_4 = 0, a_3 = 2$ and $a_2 = 4$. We have that

$a_1 = 0$ since $H \in [\mathfrak{l}, \mathfrak{l}]$. Therefore,

$$H = \mathrm{diag}(0, 4, 2, 0, 0, -2, -4, 0) = 4H_{\alpha_2} + 3H_{\alpha_3} + 3H_{\alpha_4}$$

which gives the partition $[5, 1^3]$.

$\Delta' = \{\alpha_1, \alpha_3, \alpha_4\} : X = X_{\alpha_1} + X_{\alpha_3} + X_{\alpha_4}$. Here, $[H, X] = 2$ implies that $a_1 - a_2 = 2$ while $a_3 \pm a_4 = 2$. This yields $a_4 = 0$ and $a_3 = 2$. The condition that $H \in [\mathfrak{l}, \mathfrak{l}]$ gives $a_1 = -a_2$, whence

$$H = \mathrm{diag}(1, -1, 2, 0, 0, -2, 1, -1) = H_{\alpha_1} + H_{\alpha_3} + H_{\alpha_4}$$

and the partition is $[3, 2^2, 1]$.

$\Delta' = \Delta : X = X_{\alpha_1} + X_{\alpha_2} + X_{\alpha_3} + X_{\alpha_4}$, the sum of all the simple root vectors. The relation $[H, X] = 2X$ implies

$$a_1 - a_2 = 2, a_2 - a_3 = 2, a_3 - a_4 = 2, a_3 + a_4 = 2$$

whence

$$H = \mathrm{diag}(6, 4, 2, 0, 0, -2, -4, -6) = 6H_{\alpha_1} + 10H_{\alpha_2} + 6H_{\alpha_3} + 6H_{\alpha_4}$$

which corresponds to the partition $[7, 1]$.

Finally, we need to determine a representative for the nilpotent orbit corresponding to the pair $(\mathfrak{g}, \mathfrak{p}_{\alpha_2})$. This orbit is characterized as the one which is even and whose corresponding Jacobson-Morozov parabolic subalgebra is $\mathfrak{p}_{\alpha_2}$. If $\{Y, H, X\}$ is an $\mathfrak{sl}(2)$–triple for this orbit, then we may deduce the following properties.

To be an even orbit implies that $\alpha(H)$ is even for all $\alpha \in \Phi$. Since all roots are sums or differences of pairs of elements on the diagonal of H, we deduce that all the diagonal elements of H are even. Decomposing $\mathfrak{g}$ as a direct sum of weight spaces $\mathfrak{g}_i$ under H, we have $\mathfrak{l} = \mathfrak{h} = \mathfrak{g}_0$, hence $\alpha_i \geq 2$ for each simple root α_i. The set $\mathfrak{g}_2$ is the complement of $[\mathfrak{u}, \mathfrak{u}]$ in $\mathfrak{u}$. Recall by our calculations in Section 5.3 that the set $S' = \{\alpha_1, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4\}$ parametrizes a basis for $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$, hence $\mathfrak{g}_2$. The constraint that $\alpha(H) = 2$ for each $\alpha \in S'$ uniquely determines H:

$$H = \mathrm{diag}(4, 2, 2, 0, 0, -2, -2, -4) = 4H_{\alpha_1} + 6H_{\alpha_2} + 4H_{\alpha_3} + 4H_{\alpha_4} \tag{5.6.1}$$

and thus the partition $[5, 3]$.

Determining a choice of X is more difficult. By the statement of Theorem 4.3.12, and in particular the definition of an induced orbit, there is a dense set of representatives of this orbit in $\mathfrak{g}_2$, but this is insufficient information to choose a representative. Instead we consider the following.

Since $\mathfrak{p}$ is the Jacobson-Morozov parabolic subalgebra corresponding to the $\mathfrak{sl}_2$–triple $\{H, X, Y\}$, we have $X \in \mathfrak{g}_2$ and $Y \in \mathfrak{g}_{-2}$. Furthermore, these two elements satisfy $[X, Y] = H$. Conversely, given $\{H, X, Y\}$ satisfying these three conditions, it follows that $\{H, X, Y\}$ is an $\mathfrak{sl}_2$–triple and its Jacobson-Morozov parabolic subalgebra, which is determined by H, is equal to $\mathfrak{p}$.

So consider $X = \sum_{\alpha \in S'} c_\alpha X_\alpha$ and $Y = \sum_{\alpha \in S'} d_\alpha X_{-\alpha}$. Using the explicit matrix expressions (5.1.1) for these root vectors, we compute

$$\begin{aligned}[X,Y] = &(-c_{\alpha_1+\alpha_2}d_{\alpha_1} + c_{\alpha_2+\alpha_3}d_{\alpha_3} + c_{\alpha_2+\alpha_4}d_{\alpha_4})X_{\alpha_2} \\ &+ (-c_{\alpha_1}d_{\alpha_1+\alpha_2} + c_{\alpha_3}d_{\alpha_2+\alpha_3} + c_{\alpha_4}d_{\alpha_2+\alpha_4})X_{-\alpha_2} \\ &+ (c_{\alpha_1}d_{\alpha_1} + c_{\alpha_1+\alpha_2}d_{\alpha_1+\alpha_2})H_{\alpha_1} \\ &+ (c_{\alpha_1+\alpha_2}d_{\alpha_1+\alpha_2} + c_{\alpha_2+\alpha_3}d_{\alpha_2+\alpha_3} + c_{\alpha_2+\alpha_4}d_{\alpha_2+\alpha_4})H_{\alpha_2} \\ &+ (c_{\alpha_3}d_{\alpha_3} + c_{\alpha_2+\alpha_3}d_{\alpha_2+\alpha_3})H_{\alpha_3} \\ &+ (c_{\alpha_4}d_{\alpha_4} + c_{\alpha_2+\alpha_4}d_{\alpha_2+\alpha_4})H_{\alpha_4}.\end{aligned}$$

Comparing this with (5.6.1) gives an underdetermined system of six quadratic equations in the twelve variables $\{c_\alpha, d_\alpha \mid \alpha \in S'\}$, whose solution set is the dense set of orbit representatives. To choose one such solution, we observe that the system remains consistent if one of $(c_{\alpha_2+\alpha_3}, d_{\alpha_2+\alpha_4})$ or $(c_{\alpha_2+\alpha_4}, d_{\alpha_2+\alpha_3})$ is the zero vector, in which case each choice of $(c_{\alpha_1}, c_{\alpha_3}, c_{\alpha_4}, c_{\alpha_1+\alpha_2})$ determines a unique solution. Thus one choice of orbit representative is

$$X = X_{\alpha_1} + X_{\alpha_3} + 2X_{\alpha_4} + X_{\alpha_1+\alpha_2} - X_{\alpha_2+\alpha_4}$$

and the corresponding element Y completing the $\mathfrak{sl}_2$–triple is

$$Y = -2X_{-\alpha_1} + 4X_{-\alpha_3} + 2X_{-\alpha_4} + 6X_{-\alpha_1-\alpha_2} + 6X_{-\alpha_2-\alpha_3}.$$

Summary

In the previous section, we explicitly established the case corresponding to $\mathfrak{so}(8)$ of the following theorem due to Springer and Steinberg, using the Bala-Carter classification [6, Theorem 5.1.4].

Theorem 5.6.1 (Springer and Steinberg)**.** *The G_{ad} nilpotent orbits of $\mathfrak{so}(2n)$ are parametrized by partitions of $2n$ in which even parts occur with even multiplicity, with the exception that to each partition which is composed only of even parts there correspond two distinct orbits.*

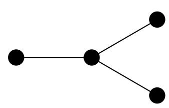
In the case that $n = 4$, this theorem gives the parametrizing set

$$[7,1],\ [5,3],\ [5,1^3],\ [4^2]\times 2,\ [3^2,1^2],\ [3,2^2,1],\ [3,1^5],\ [2^4]\times 2,\ [2^2,1^4],\ [1^8],$$

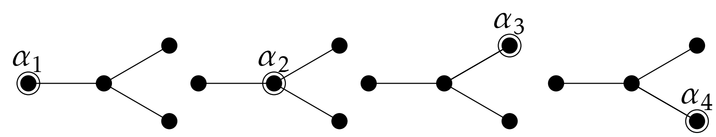
which is exactly the collection of partitions obtained in the previous section.

5.7 Dynkin diagrams of the Levi subalgebras of $\mathfrak{so}(8)$

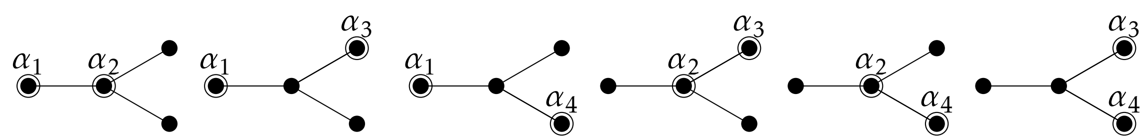
$\Delta' = \emptyset$



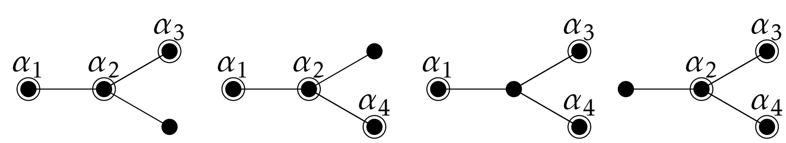
$|\Delta'| = 1$



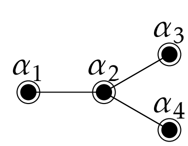
$|\Delta'| = 2$



$|\Delta'| = 3$



$|\Delta'| = 4$



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