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CONNECTEDNESS PROPERTIES OF SELF-SIMILAR GRAPHS

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Abstract

This thesis is broadly concerned with two problems: obtaining the mathematical model of the specific infinite self-similar graph, and investigating the connectedness of the tree-like graph in order to show its relation to the associated hyperbolic space. Our main result concerning the former problem is that, in a variety of situations, the self-similar infinite structure obtained by using our method as the graph product of a disconnected finite graph and regular rooted tree can be connected (i.e. have the hyperbolic metric space associated to it). This addresses a question about the existence of the optimal depth for the breadth-first search algorithm and also has possible applications to the recent research topics in Psychological and Brain Sciences. We approach the connectedness problem by showing the similarity of obtained geometric structures to well known algebraic structures such as groupoid and pseudogroup. One of our main results is that, under the assumption that the emerged geometric self-similar structure is connected, it is naturally associated to the hyperbolic metric space. Thus, the variety of well known methods can be applied in further study. We also show that the connectedness of our structure can be reached in the finite number of steps or can not be reached at all. This gives the grounds for the optimal application of the breadth-first search algorithm.

Acknowledgements

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Finally, I would like to thank the many people who have helped and befriended me along the way. The list of names is so long that, were I to attempt to write it, I would inevitably leave someone out and suffer a pang of regret upon realizing my mistake. I hope, therefore, that those whom I would like to thank know who they are and know that they have my gratitude.

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Chapter 1

Introduction

As is typical for much of contemporary mathematics, this thesis touches upon a number of interconnected areas, in particular graph theory, group theory, and fractal (self-similar) sets theory. The fundamental object of interest that ties our work together is the finitely generated pseudogroup and the connectedness of the self-similar graph.

Self-similar sets are among the most fundamental structures in fractal analysis. Although, there are no objects in nature that exactly resemble a self-similar set, the study of such sets is still important. Despite their relatively simple construction, these sets still preserve many properties of more complex fractal structures. For example, many properties of the Cantor set can be effectively related to the boundary analysis of more complex fractals.

There is a number of motivations for this work. One of which comes from the recent research progress accomplished in Psychological and Brain Sciences. As various research results suggest, the connection patterns of the cerebral cortex consist of pathways linking neuronal populations across multiple levels of scale, from whole brain regions to local minicolumns [ASW06]. This nested interconnectivity suggests the hypothesis that cortical connections are arranged in fractal or self-similar patterns. These connection patterns were examined by calculating a broad range of structural measures, including small-world attributes and motif composition, as well as some global measures of functional connectivity, including complexity [AG03]. As fractal patterns vary, the strongly correlated changes were found in several structural and functional measures, suggesting that they emerge together and are mutually linked. Measures obtained from some modeled fractal patterns closely resemble those of real neuroanatomical data sets, supporting the original hypothesis. For example Fig. 1.1 illustrates the approach taken in [Spo06] to generate fractal or self-similar connection matrices that resemble those of brain networks. The process begins by generating an isolated elementary "group" consisting of $2m$ fully connected units (hierarchical level $k = 0$). Then, two of these groups are linked by generating connections between them at a given density ($k = 1$). Then this network is again duplicated and connections are generated between the two resulting subnets ($k = 2$). This process is repeated until a desired network size $N = 2n$ is reached. At each step the connection density is decreased, resulting in progressively sparser interconnectivity at higher hierarchical levels. Following this procedure, the resulting connection matrices exhibit self-similar (fractal) properties according to Olaf Sporns [Spo06].

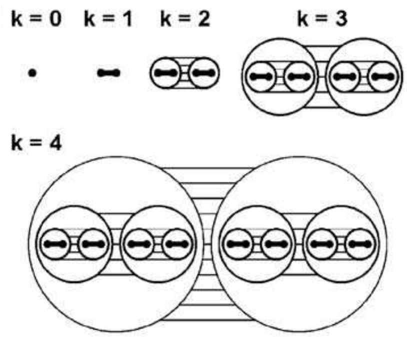


Figure 1.1: Self-similar connection matrices used for modeling of brain networks. From [Spo06]

Self-similar maze structures. Another motivation comes from the variety of well known "Fractal Maze" problems [Wol03], which, so far, have been mostly solved by the means of computer modeling. We briefly describe the general formulation here. Given a self similar graph connectedness of which is unknown, the reader is asked to establish if some vertices (states) can be connected. Below we give just a few illustrations of such problems (Fig. 1.2).

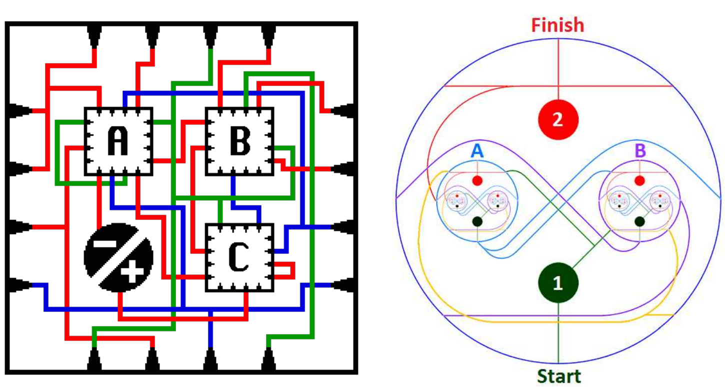


Figure 1.2: Areas marked A, B, C on the left hand side, and A, B on the right hand side are the exact copies of the original graph itself.

As it becomes clear from these examples, the only way to connect chosen vertices is by going through one or more copies of the initial graph. It is also obvious that taking the "wrong turn" somewhere along the way means "getting lost" in the infinitely spanning copies of the maze. While most of these solutions do not go "too deep", as the one in Fig. 1.3, and can be effectively sold using breadth-first search algorithm (see [Ski08] for details), there are still a number of open questions. First of all, the search method is described as being *complete* if it is guaranteed to find a goal state *if it exists*. In this regard it is unclear how many levels of the maze does the search algorithm need to pass through before a conclusion can be made that the desired state *does not exist*? To our best knowledge, there is no effective method presented or comprehensive analysis performed to answer this question. Thus, using the breadth-first search algorithm on an implicit representation of an infinite self-similar graph, the researcher can only hope that the desired state can be effectively reached within reasonable time limit. However, if the working time of such algorithm becomes quite extended and the researcher makes a decision to stop the algorithm, it is still unclear if the solution exists but algorithm "requires more time" to reach the desired state.

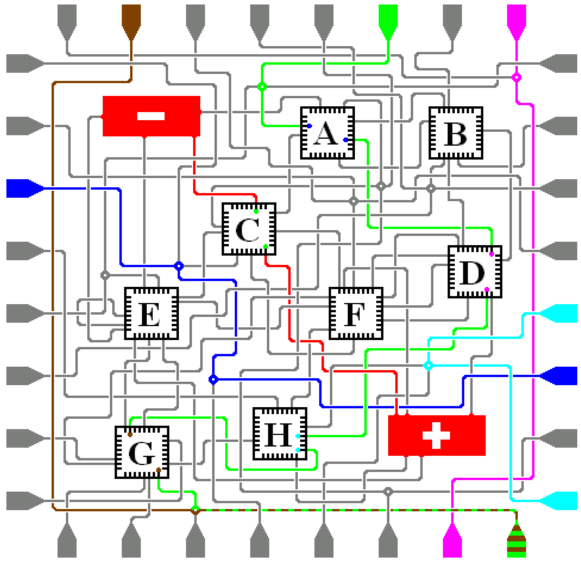


Figure 1.3: Fractal Maze Solution marked by colored paths. The solution only requires going two levels "down the maze"

Note the similarity in diagrams obtained by O. Sporns and Fractal Maze illustrations.

Gromov's Hyperbolic Groups and Spaces. Finally, and most importantly, we find one more motivation for the research presented in this work in Gromov's theory of Hyperbolic Groups and related Hyperbolic Spaces. It is known that the most natural discrete analogue of the hyperbolic plane is a finitely generated free group F_n of rank $n > 1$, which becomes a hyperbolic space upon identifying it with its Cayley graph, the regular tree.

Mathematicians also commonly use Cayley graphs associated to free semigroups and free monoids in their research (see for example [Kai09]). We provide more detailed report on free semi-groups and free monoids in section

2.2 of this thesis. Briefly speaking, a free monoid is an algebraic structure which emerges upon endowing the word space $\mathcal{W}^*$ with the concatenation $(v, w) \mapsto vw$ $(v, w \in \mathcal{W}^*)$. Where $\mathcal{W}^*$ is the space of all finite words generated from elements of a finite set $\mathcal{A} = \{a_1, a_2, ..., a_n\}$, together with the empty word $\{\emptyset\}$.

Below, is the regular rooted tree (Cayley graph) associated to the free monoid F_2^+, with generating set $\mathcal{A} = \{a, b\}$. (Fig.1.4)

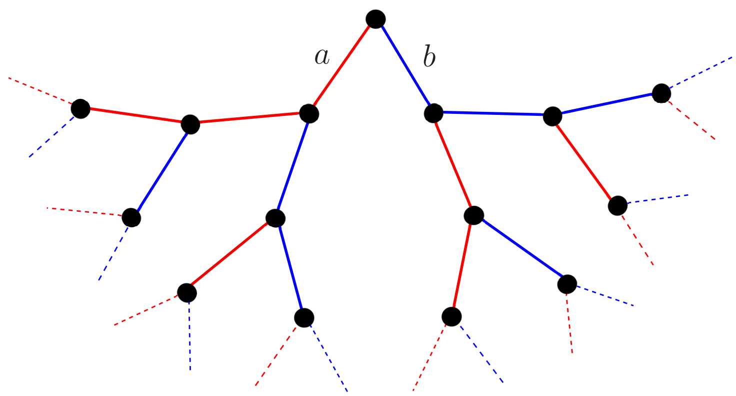


Figure 1.4: Cayley graph $T(\mathcal{A})$ of a free monoid F_2^+, $\mathcal{A} = \{a, b\}$

Vadim Kaimanovich has extended this construction by introducing the *Augmented Rooted Tree* ([Kai02]), which can be obtained by adding the set of *horizontal* edges $\mathcal{E}^h$ to the rooted tree, following the rule:

$$(x, y) \in \mathcal{E}^h \Longrightarrow |x| = |y|,\ (x^-, y^-) \in \mathcal{E}^h$$

where x^-, y^- are the predecessors of x, y.

The graph T^* with the vertex set $\mathcal{V}(T^*) = \mathcal{V}(T)$ and the edge set $\mathcal{E}(T^*) = \mathcal{E}(T) \cup (\mathcal{E}^h \setminus \text{diag})$ is called an *augmented rooted tree*. For full details see [Kai02].

It was proven in the above cited paper that *The Sierpinski graph (see section 2.5) is an augmented rooted tree whose underlying tree is the Cayley graph of the free monoid* $\mathcal{W}^*$.([Kai02], Prop. 3.18) and, consequently the Sierpinski graph $\mathcal{G}$ is Gromov hyperbolic. Reader can find more details on Sierpinski graph in section 2.5 of this thesis.

Figure 1.5 provides an example of an *augmented rooted tree*

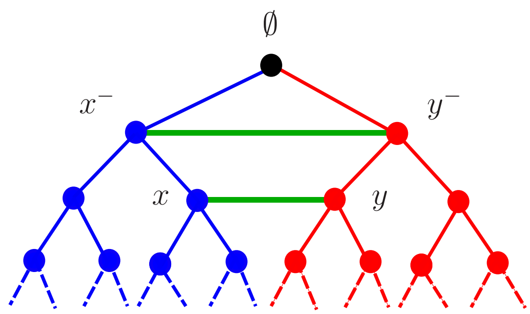


Figure 1.5: Augmented Rooted Tree

Taking similar approach, we shall use free monoid structure and its properties to build our model. More precisely, using the Cayley graph T of a free monoid F_n^+ we can define its Cartesian product $\mathcal{L} = \Gamma \Box T$ ([SJS12]) with any finite connected graph. The result of such operation is an infinite self similar "tree-like" graph. Figure 1.6 provides an example of the Cartesian product of graph Γ and binary rooted tree $T(\mathcal{A})$, with $\mathcal{A} = \{a, b\}$. Moreover, $\mathcal{L}$ is quasi-isometric to the underlying Cayley graph and thus Λ also a hyperbolic space. One algebraic structure that can be associated to $\mathcal{L}$ is a relation-free finitely generated pseudogroup on the graph Γ.

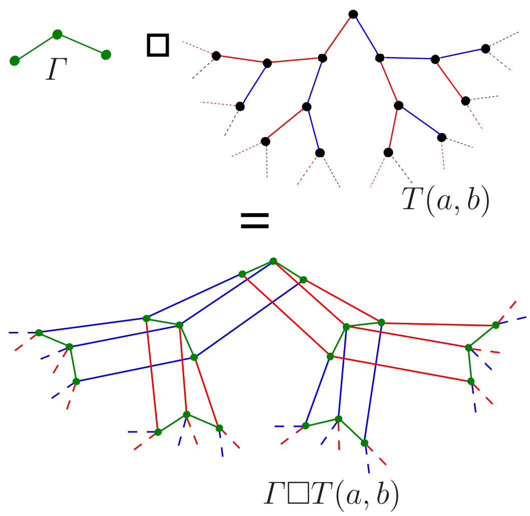


Figure 1.6: Graph $\mathcal{L} = \Gamma \Box T(a,b)$

Clearly, if the graph Γ is disconnected, then the resulting graph $\mathcal{L} = \mathcal{L}(\Gamma, T)$ is also disconnected (Fig. 1.7).

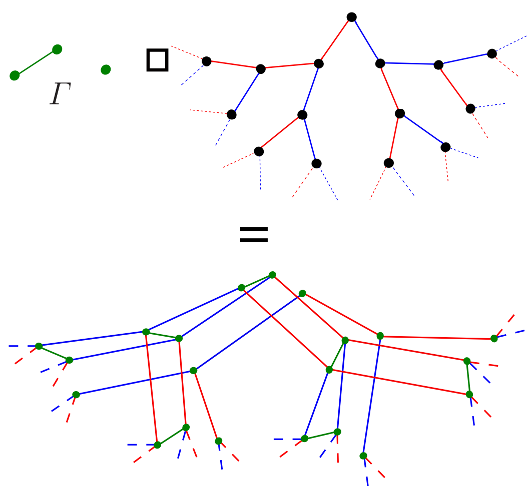


Figure 1.7: Graph $\mathcal{L} = \Gamma \Box T(a,b)$ (with Γ being disconnected)

Note that the graph $\mathcal{L} = \mathcal{L}(\Gamma, T)$ has two types of edges. First type of edges we shall call *horizontal*. These are the edges of the initial graph Γ and all its copies. All *horizontal* edges are marked in "green" in Fig. 1.6. The second type of edges, which we call *vertical* edges, are the edges that connect different copies of the initial graph Γ. All vertical edges are marked in "red" and "blue" in Fig. 1.6. We need to mark vertical edges in two different colors to reflect correspondence with two elements of the generating set $\mathcal{A} = \{a, b\}$. Interesting situation arises when one changes the set of *vertical* edges of the graph $\mathcal{L} = \mathcal{L}(\Gamma, T)$ in the following way. Some of the *vertical* edges are deleted. For some of the remaining vertical edges (without changing the edge color) one can choose one or both of their end-points to be in different vertices as shown in Fig.1.8. Note that one repeats this same choice for all levels of the graph $\mathcal{L}(\Gamma, T(a, b))$. It turns out that for certain choices of new endpoints for vertical edges, the new graph $\mathcal{L}^*$ becomes connected even with the initial graph Γ being disconnected. Furthermore, the connected graph $\mathcal{L}^*$, although it lacks some local regularity, is still quasi-isometric to the rooted regular tree. Whence the space Λ^* associated to $\mathcal{L}^*$ is a geodesic hyperbolic space.

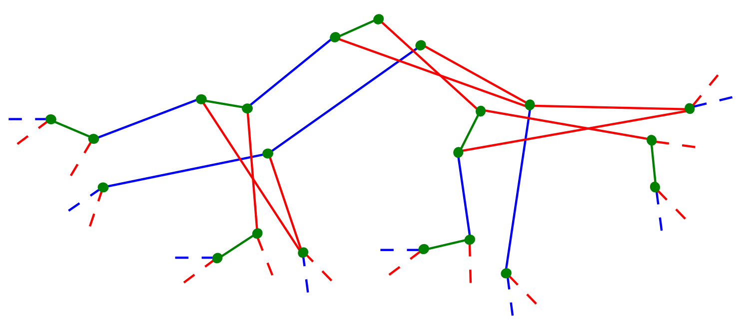


Figure 1.8: Modified graph $\mathcal{L}^*$, obtained from $\mathcal{L}$ in fig.1.7

Interestingly enough, the obtained graph $\mathcal{L}^*$ can be used as a mathematical model of a Fractal Maze with copies of the graph Γ being the "body" of the maze and vertical edges representing connections between levels. It is also quite possible that this mathematical model may be of some use in modeling the cortical connections patterns.

The main result of this thesis suggests that for the connected graph $\mathcal{L}^*(\mathit{\Gamma}, T(\mathcal{A}))$ there is a finite limit for the number of levels through which connectedness can be established. Thus, using for example breadth-first search method, one can be certain that the desired state, if it exists, can be only reached through the finite number of iterations of the algorithm. Theorem 4.2.1 shows that for any finite disconnected graph $\mathit{\Gamma}$, with N connected components, and regular rooted tree $T(\mathcal{A})$, with $\mathcal{A} = \{a_1, a_2, ..., a_m\}$, the number of levels through which graph $\mathcal{L}^*(\mathit{\Gamma}, T(\mathcal{A}))$ can be effectively connected (if the connectedness is possible) does not exceed $\binom{N-1}{2} + 1$.

This thesis is organized as follows. The first chapter is devoted to the overview of the background materials needed in order to build the desired mathematical model and perform the analysis on it. So we point out main facts from fractal analysis on graphs for further motivation [Kig01], [Man82], [Fal03] *etc.* The free monoid which naturally arises from the free semigroup by adding the group identity is viewed in relation with its Cayley graph - the regular rooted tree $T(\mathcal{A})$, with $\mathcal{A}$ being a finite generating set of a free semigroup [Cay78], [Kai09], [MK77], [Lot97]. Further, we explore the iterated function systems (IFS) that satisfy the open set condition with examples and properties of those [Fal03], [vK04]. We then introduce the class of self-similar groups that act on regular rooted trees ([Nek05], [Kai04], [Kai05], [GZ02]) which will help us better understand the self-similar properties of our model. We recall the notion of Gromov's hyperbolic groups and associated spaces [dLH87], [GdLH90] and further use this material to show that our model can be associated to Gromov hyperbolic space under certain conditions. In the next two sections, we bring up several facts from the theory of groupoids and pseudogroups, explaining how it can be applied to our model. The last section of Chapter 1 contains the material on different types of graph products used in the present work.

Chapter 2 of this thesis gives the detailed explanation on the actual method we use to obtain geometric model and further relate it to several algebraic structures. We then prove that the obtained *maze-graph* is associated to a metric geodesic Gromov hyperbolic space. Propositions 3.3.1, 3.3.2 and 3.3.3 give us necessary conditions for the connectedness of the *maze-graph*. We further explore two different cases and give (Th.4.1.1, Th.4.2.1) uniform bounds on the diameter of any *zero-orbit*(see section 4.2.1, def. 4.2.9) for these two viewed cases.

Chapter 2

Background

2.1 Fractal Analysis and Graphs

Fractals are relatively new and very interesting mathematical objects. They can be described as mathematical sets with a high degree of geometrical complexity. Fractals can be used to model many natural phenomena and objects. As objects that defy conventional measures such as length, fractals are most often characterised by their *fractal dimension*. Many examples of fractals, like the Sierpinski gasket, the Koch curve and the Cantor set, were already known to mathematicians early in the twentieth century. Those sets were originally pathological (or exceptional) counterexamples. For instance, the Cantor set is an example of an uncountable perfect set with zero Lebesgue measure. Consequently, they were thought of as purely mathematical objects. These sets had never been associated with any objects in nature.

This situation had not changed until Mandelbrot proposed the notion of fractals in the 1970s. In [Man82] he claimed that many objects in nature are not collections of smooth components. In mathematics, a new area called fractal geometry developed quickly on the foundation of geometric measure theory, harmonic analysis, dynamical systems and ergodic theory. Fractal geometry treats the properties of fractal sets and measures on them, like the Hausdorff dimension and the Hausdorff measure.

For simplicity of analysis, we will start with self-similar sets.

To quote J. Kigami [Kig01]: "Self-similar sets are a special class of fractals and there are no objects in nature which have the exact structures of self-similar sets. The reason is that self-similar sets are perhaps the simplest and the most basic structures in the theory of fractals. They should give us much information on what would happen in the general case of fractals".

Specifically, we will use graphs as approximation of fractals. See, for example, Figure 2.1.

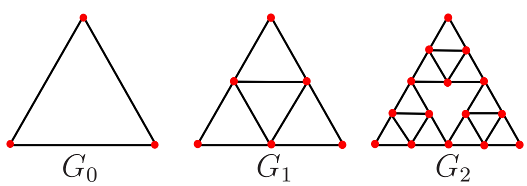


Figure 2.1: Approximation of Sierpinski Triangle by graphs

2.2 Free Monoid. Free Semigroup

Let $\mathcal{A}$ be a set, called an alphabet, with its elements called letters. A word of length n over an alphabet $\mathcal{A}$ is a sequence of elements $a_1 a_2 ... a_n$ with $a_i \in \mathcal{A}$. An empty word is denoted by $\emptyset$. We denote by $\mathcal{A}^*$, the set of all words over an alphabet $\mathcal{A}$ together with the empty word. The set $\mathcal{A}^*$ is naturally equipped with the associative binary operation, which is simply the concatenation of two sequences

$$a_1a_2a_3...a_n \cdot b_1b_2b_3...b_n = a_1a_2a_3...a_nb_1b_2b_3...b_n$$

The empty word $\emptyset$ is the neutral element for the concatenation operation, so for any word $w = a_1a_2a_3...$ from $\mathcal{A}^*$ we have

$$\emptyset w = w\emptyset = w$$

Definition 2.2.1. e.g.,[Lot97] A *monoid* is a set M with an associative binary operation and a neutral element $id = 1_M$.

Obviously, the set $\mathcal{A}^*$ defined above is a *monoid*.

Definition 2.2.2. A *morphism* of a monoid M into a monoid N is a mapping $\psi : M \to N$, compatible with binary operations of M and N

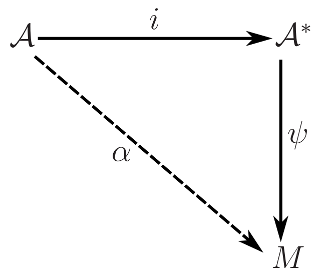
$$\psi(uv) = \psi(u)\psi(v);\ u, v \in M$$

and such that $\psi(1_M) = 1_N$.

Proposition 2.2.1. *e.g.,[Lot97] The set $\mathcal{A}^*$ defined as above satisfies the universal property, i.e. for any mapping $\alpha : \mathcal{A} \to M$ of $\mathcal{A}$ into a monoid M, there exists a unique monoid homomorphism $\psi : \mathcal{A}^* \to M$, such that the following diagram is commutative. (where i - is the natural injection of $\mathcal{A}$ into $\mathcal{A}^*$)*

Proof. Given $\alpha : \mathcal{A} \to M$, we define $\psi : \mathcal{A}^* \to M$ as follows:

First, define ψ on $\mathcal{A}_0$ (the empty word) as mapping to the identity element of M.



Second, define ψ on $\mathcal{A}_1 = \mathcal{A}$ to be equal to α.

Third, having defined ψ on $\mathcal{A}_j$, define ψ on $\mathcal{A}_{j+1}$ as follows: given $uv \in \mathcal{A}_{j+1}$, with $u \in \mathcal{A}_j$ and $v \in \mathcal{A}$, define $\psi(uv) = \psi(u)\alpha(v)$.

This defines ψ on all of $\mathcal{A}^*$. Because morphism ψ defined as above is clearly compatible with monoid binary operation of $\mathcal{A}^*$ and M, this is a monoid homomorphism.

Consider any other homomorphism $\tau : \mathcal{A}^* \to M$ which agrees with ψ on $\mathcal{A}_1$. Note that τ and ψ also must agree on $\mathcal{A}$. If τ and ψ agree on $\mathcal{A}_j$, then they will agree on $\mathcal{A}_{j+1}$, since $\tau(uv) = \tau(u)\alpha(v) = \psi(u)\alpha(v) = \psi(uv)$.

Thus, for any monoid M and mapping $\alpha : \mathcal{A} \to M$, there is a unique monoid homomorphism $\psi : \mathcal{A}^* \to M$ such that $\psi(a) = i(a)$ for all $a \in \mathcal{A}$ and $\alpha = \psi \circ i$. $\square$

Note: ψ is onto if and only if $\langle \alpha(\mathcal{A}) \rangle = M$.

Since $\mathcal{A}^*$ satisfies the *universal property* it is a free monoid over the set $\mathcal{A}$.

We end this subsection with the definition of a *free semigroup* as a special subset of $\mathcal{A}^*$.

Definition 2.2.3. [Lot97] The set of all nonempty words over the set $\mathcal{A}$ denoted by $\mathcal{A}^+$ is called a *free semigroup* over $\mathcal{A}$. We have that $\mathcal{A}^+ = \mathcal{A}^* \setminus \emptyset$.

The *length* $|w|$ of a word $w = a_1a_2...a_n$ ($a_i \in \mathcal{A}$) is the number n of the letters in w.

The mapping $w \mapsto |w|$ is a morphism of the free monoid $\mathcal{A}^*$ into the additive monoid $\mathbb{N}$ of positive integers.

2.3 Cayley Graphs

The Cayley graphs were introduced by Cayley in [Cay78]. The idea behind the Cayley graph is the following: given a group G, one would like, without imposing any assumptions on G, to associate to it a geometric object. Moreover, this object should be a true geometric realization of our group (i.e. a space upon which G acts freely and transitively). It turns out that such geometric realization of a group is precisely a Cayley graph of a group G, and it is defined as follows.

Definition 2.3.1. (Cayley graph) Let G be a group and $\mathcal{A} = \{a_i : i \in I\}$ a symmetric finite generating set for G. The Cayley graph of G constructed with respect to $\mathcal{A}$ is the graph Γ whose vertex set is identified with G and such that two elements g and g' are connected with an edge directed from g to g' and labeled with the generator $a_i \in \mathcal{A}$ if and only if $ga_i = g'$.

Note that the Cayley graph of a group G is not unique. The definition of such graph depends on the choice of generating set $\mathcal{A}$ of the group G. If G is finitely generated, then any two Cayley graphs of G constructed with respect to finite generating sets will be quasi - isometric.

The Cayley graph of a given group G is a space on which G acts freely and transitively.

See Figure 2.2 for the Cayley graph defined by the group $\mathbb{Z}^2$ with respect to generators $(1, 0)$, $(0, 1)$.

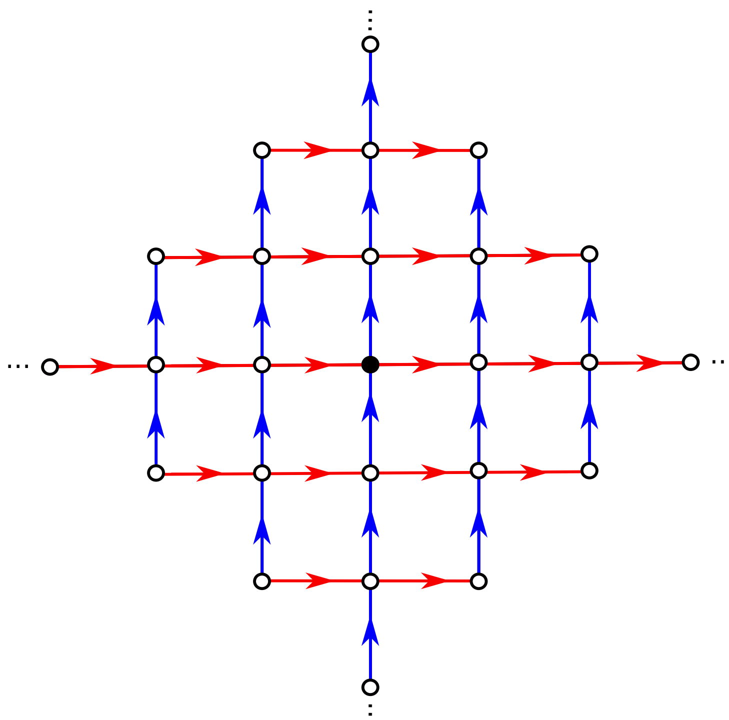


Figure 2.2: Cayley Graph of the group $\mathbb{Z}^2$ built w.r.t. generators $(1, 0)$ and $(0, 1)$.

2.4 Free groups

Elements of a generating set M - *generators* of a group G may have *relations* defined for some or all of them. These *relations* may be *trivial* (i.e. following from definitions), such as $xx^{-1} = e$ or *non-trivial*. One can define a group by a *presentation* specifying the set of *generators* M and the set of *relations* R among these *generators*. For example, the *dihedral group*, which is a group of symmetries of a regular n-sided polygon, has n rotation symmetries and n reflection symmetries. Thus, it is a group of order $2n$ with the most commonly used notation D_n. The *presentation* of D_n is:

$$D_n = \langle a, b \mid a^n = e; b^2 = e; (ab)^2 = e \rangle,$$

where a represents the rotation by $\frac{2\pi}{n}$, and b - the reflection across n lines through the origin, making angles of multiples of $\frac{\pi}{n}$ with each other. If n is odd, each axis of symmetry connects the midpoint of one side to the opposite vertex. If n is even, there are $n/2$ axes of symmetry connecting the midpoints of opposite sides and $n/2$ axes of symmetry connecting opposite vertices.

It turns out that some groups only accept *trivial relations* among their *generators* in certain generating set M. To construct such a group, one can do the following.

Fix two sets of symbols

$$X \cong \{x_i | i \in I\} \text{ and } X^{-1} \cong \{x_i^{-1} | i \in I\}$$

A word in an alphabet $\bar{X} = X \cup X^{-1}$ is an empty or finite sequence of symbols from $X \cup X^{-1}$. The number of elements of this sequence is called the length of the word. Any word that contains a subword of a form $x_i^{\varepsilon} x_i^{-\varepsilon}$ can be *reduced* by removing this subword. If a word does not contain any parts of the form defined above, this word is *irreducible*. Two words $x, y \in X$ are equivalent (denote $x \sim y$) if and only if y can be obtained from x through the finite number of inserts and cancellations of words of the form $x_i^{\varepsilon} x_i^{-\varepsilon}$.

Definition 2.4.1. [MK77] Let $X \cong \{x_i\}_{i \in I}$ be an alphabet. Let $F(X)$ be the set of all equivalence classes on X. Define a product operation on $F(X)$ as

$[x][y] = [xy]$ for any two equivalence classes $[x], [y] \in F(X)$. This definition does not depend on the choice of equivalence class representatives. We shall consider $F(X)$, a group with respect to the product operation defined above. The group $F(X)$ is called the free group with the generating set X.

The cardinality of the set X is called the *rank* of the group $F(X)$. The following proposition and its proof can be found in various forms in a great number of sources. See, for example [MK77] (Th.14.1.5).

Proposition 2.4.1. *Any finitely generated group can be presented as a factor group of a finitely generated free group.*

We finish this section with the illustration (Figure 2.3) showing the Cayley graph of a free group with two free generators.

2.4.1 Cayley Graph of a Free Monoid

The Rooted homogeneous tree is usually considered the most fundamental self-similar structure (e.g., see [Kai09]). An example of such tree arises from a geometric description of an iterated function system (IFS) satisfying the open set condition. Formally, let $\mathcal{A} \cong \{1, 2, ..., D\}$ be a finite alphabet. One can denote by $\mathcal{A}^*$, the set of all finite words in $\mathcal{A}$ together with the empty word $\{\emptyset\}$. Then $\mathcal{A}^*$ is a free semigroup with identity element, generated by the set $\mathcal{A}$ with composition defined as $(a, a') \mapsto aa'$. The rooted homogeneous tree

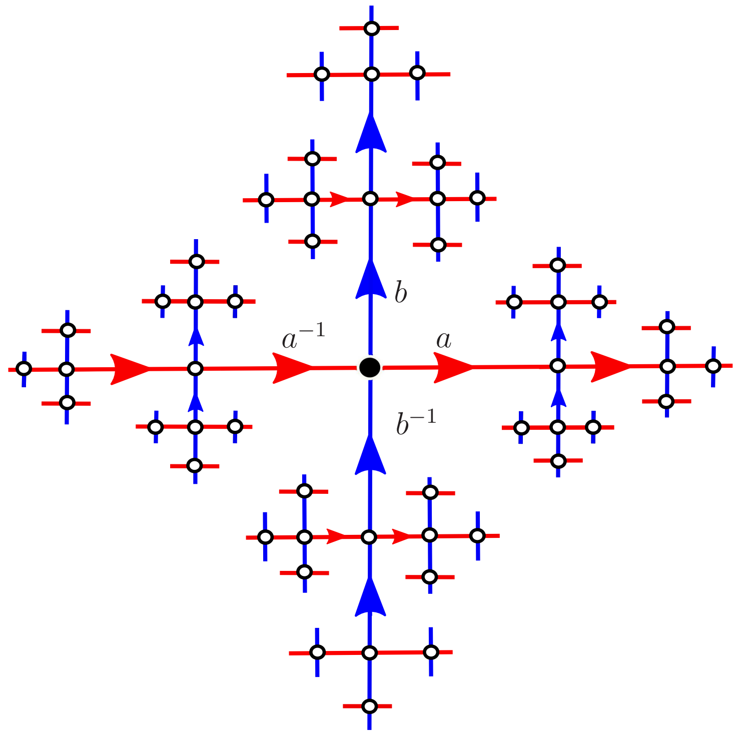


Figure 2.3: The Cayley Graph of a free group with two free generators a, b and $X = \{a, b\}$, $X^{-1} = \{a^{-1}, b^{-1}\}$

associated with this free monoid is the right Cayley graph of $\mathcal{A}^*$. We connect elements of this tree $T(\mathcal{A})$ as follows: connect a to ax with an edge for all $a \in \mathcal{A}^*$, $x \in \mathcal{A}$. One can distinguish levels $T_n \cong \mathcal{A}^n$ in this tree $T(\mathcal{A}) \cong \mathcal{A}^*$. Each level represents the set of all words of length n in $\mathcal{A}^*$. Level T_0 consists of only one word $\emptyset$ - the root of tree T. Every vertex $a \in T(\mathcal{A}) \cong \mathcal{A}^*$ is also the root of the corresponding subtree T_a, which consists of all words starting with a. Identification of the tree T and any of it's subtrees T_a can be obtained by the map $a' \mapsto aa'$.

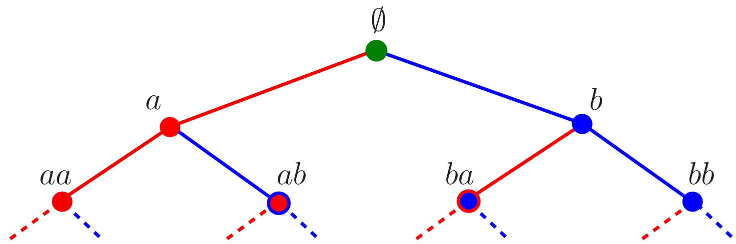


Figure 2.4: Tree T of $\mathcal{A} \cong \{a, b\}$

2.5 Iterated Function Systems (IFS) and Self-similar Sets

Many fractals consist of parts that are, in some way, similar to the whole. For example, the middle third Cantor set is the union of two similar copies of itself. These self-similarities are not only properties of the fractals: they may actually be used to define them. Iterated function systems do this in a unified way and, moreover, often lead to a simple way of finding dimensions.

Formally, we define self-similar set as follows.

Definition 2.5.1. [Fal03] A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *contractive similitude* if

$$|f(x) - f(y)| = c|x - y|;\ x, y \in \mathbb{R}^n$$

with contraction ratio $0 < c < 1$.

Definition 2.5.2. [Fal03] Let $f_1, ..., f_m$ be the set of contractive similitudes on $\mathbb{R}^n$. A compact nonempty set S satisfying $S = \bigcup_{i=1}^{m} f_i(S)$ is called the self-similar set of $f_1, ..., f_m$.

2.5.1 Examples and Some Properties of Self-similar Sets

Example 2.5.1. *Middle thirds Cantor set* is constructed by starting with the closed unit interval, $[0,1]$. Then setting $C_1 = [0,1] \setminus (1/3, 2/3)$ that is, removing the open middle third of $[0,1]$. Set

$$C_2 = C_1 \setminus \{(1/9, 2/9) \cup (7/9, 8/9)\}.$$

So to form C_2 we remove an open middle third from each subinterval in C_1. Continue this process by induction, so that C_n is $C_{n-1} - I_n$ where I_n is the collection of open middle thirds of subintervals in C_{n-1}. Now the standard middle third Cantor set, C, is the intersection of all C_n. Thus

$$C = \bigcap_{n=1}^{\infty} C_n$$

Now one sets $f_1 = \dfrac{1}{3}x$ and $f_2 = \dfrac{1}{3}x + \dfrac{2}{3}$ and gets $C = \bigcup_{i=1}^{2} f_i(C)$. Functions f_1, f_2 are clearly contractive similitudes with contraction ratio $1/3$. Thus C is the self-similar set defined by the IFS $\{f_1, f_2\}$ that satisfies the open set condition. See [Fal03] for more details.

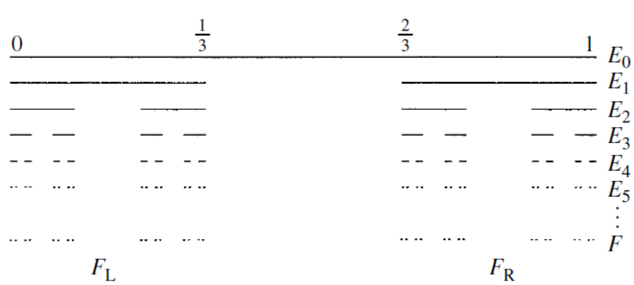


Figure 2.5: Standard Middle Thirds Cantor Set

Example 2.5.2. *Sierpinski triangle* Λ is constructed as follows. Take an equilateral triangle of side length equal to one, remove the inverted equilateral triangle of half length having the same center, then repeat this process for the remaining triangles infnitely many times. Sierpinski Triangle is constructed by using contractive similitudes f_1, f_2, f_3 with contraction ratio 1/2. We clearly have that $\Lambda = \bigcup_{i=1}^{3} f_i(\Lambda)$.

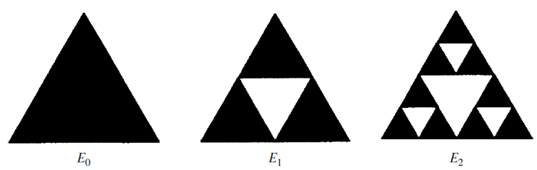


Figure 2.6: First three stages of Sierpinski Triangle construction

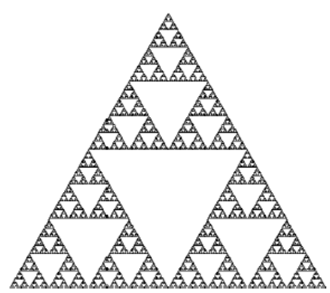


Figure 2.7: Sierpinski Triangle

The Sierpinski Triangle is defined by an iterated function system (IFS).

Example 2.5.3. *The Koch curve* is one of the earliest fractal curves to have been described. It first appeared in a 1904 paper [vK04]. The Koch curve can be constructed by starting with the line segment, then recursively altering it according to the following proceedure. First, divide the line segment into three segments of equal length. Second, draw an equilateral triangle that has the middle segment from step 1 as its base and points outward. Finally, remove the line segment that is the base of the triangle from step 2. The Koch curve is the limit approached as the above steps are followed over and over again.

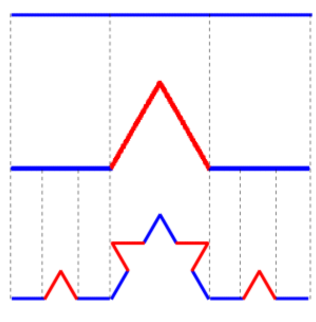


Figure 2.8: First three stages of construction of the Koch curve

Koch curve can be constructed as IFS by using contractive similitudes with contraction ratio 1/3 ([Ban91]).

$$f_1(x) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} x; \quad f_2(x) = \begin{bmatrix} \frac{1}{6} & \frac{-\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{1}{6} \end{bmatrix} x + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$$

$$f_3(x) = \begin{bmatrix} \frac{1}{6} & \frac{\sqrt{3}}{6} \\ \frac{-\sqrt{3}}{6} & \frac{1}{6} \end{bmatrix} x + \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{bmatrix}; \quad f_4(x) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} x + \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix}$$

The fixed attractor of this IFS is the Koch curve. Three copies of the Koch curve placed outward around the three sides of an equilateral triangle form a simple closed curve that forms the boundary of the Koch snowflake.

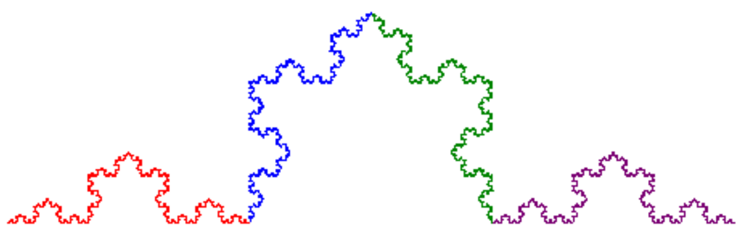


Figure 2.9: Koch Curve

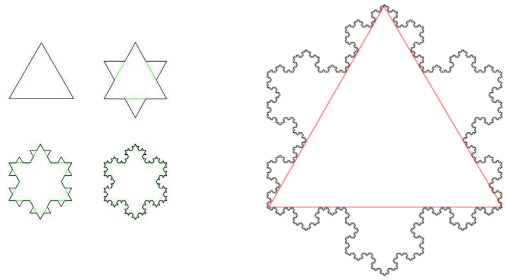


Figure 2.10: Koch Snowflake

2.6 Self-similar Groups

We now introduce a class of groups that act on regular rooted trees. Recalling the setup for the Cayley graph of a free monoid from the section 2.4.1, define the structure of rooted D-regular tree as follows. Let $\mathcal{A} \cong \{1, 2, ..., D\}$ be a finite alphabet. Denote by $\mathcal{A}^*$, the set of all finite words in $\mathcal{A}$ together with the empty word $\{\emptyset\}$. Then, the set $\mathcal{A}^*$ can be associated with the structure of a rooted D-regular tree as follows. The vertices of the tree are the words in $\mathcal{A}^*$, the root is the empty word $\emptyset$, the level T_n is the set $\mathcal{A}^n$ of words of length n over the alphabet $\mathcal{A}$. Level T_0 consists of only one word $\emptyset$ - the root of tree T. The "children" of each vertex $v \in \mathcal{A}^*$ are the D vertices of the form va, for $a \in \mathcal{A}$. Every vertex $a \in T(\mathcal{A}) \cong \mathcal{A}^*$ is also the root of the corresponding subtree T_a, which consists of all words starting with a. For details, see [Kai09] and [Nek05].

The group $\mathfrak{G} = \mathbf{Aut}(T)$ of automorphisms of the tree T will preserve each level of T. Thus, there is a homomorphism

$$g \mapsto \sigma = \sigma^g \text{ from } \mathfrak{G} \text{ to } \mathbf{Sym}(A),$$

where $\mathbf{Sym}(\mathcal{A})$ is the permutation group on $\mathcal{A}$. According to [Kai09], we can also assign to any automorphism $g \in \mathfrak{G}$, a set $\{g_a : a \in \mathcal{A}\}$ of automorphisms, indexed by the alphabet $\mathcal{A}$, by setting $b = \sigma^g(a)$. Defined like this, g can be used to set a one-to-one correspondence between subtrees T_a and T_b with respective roots in a and b. Considering the fact that both subtrees T_a and T_b are isomorphic to the initial tree $T(\mathcal{A})$, we can conclude that the map $g : T_a \to T_b$ is identified with an automorphism g_a of T.

The automorphism g_a is called the section of g at a [RG15]. It represents the action of g on the tails of words that start with a. Every automorphism g is uniquely determined by its root permutation σ^g and the D sections at the first level g_a, for $a \in \mathcal{A}$. For each $v \in \mathcal{A}^*$ and $a \in \mathcal{A}$ we have [RG15]:

$$g(av) = \sigma^g(a)g_a(v)$$

A set $Q \subseteq \mathbf{Aut}(\mathcal{A}^*)$ is a self-similar set of tree automorphisms if every section of every element of Q is itself in the set Q. For every word $w \in \mathcal{A}^*$, the action of each automorphism $q \in Q$ on tails of words that start with w, is exactly the action of some element of Q. A group $G \leq \mathbf{Aut}(\mathcal{A}^*)$ is a self-similar group of tree automorphisms if it is self-similar as a set [RG15]. For all tree automorphisms g_i and g_j, and all words $w \in \mathcal{A}^*$ we have

$$(g_i g_j)_w = g_i^{g_j(w)} g_j(w); \quad (g^{-1})_w = (g^{g^{-1}(w)})^{-1}$$

Arguably the most efficient way to define self similar group is the following [Kai09]. First note that a permutation $\sigma \in \mathbf{Sym}(\mathcal{A})$ and a set $\{g_a : g_a \in \mathfrak{G}; a \in \mathcal{A}\}$ determine the associated automorphism $g \in \mathfrak{G}$ in a unique way. This means that the automorphism $g \mapsto (\sigma^g; \{g_a\})$ is an isomorphism of the group $\mathfrak{G}$ and the semidirect product $\mathbf{Sym}(\mathcal{A}) \leftthreetimes \mathfrak{G}^{\mathcal{A}}$. One can use means of generalized permutation matrices to represent this structure. See [Kai09] for full details.

Namely, we shall assign to an automorphism $\mathcal{M}$ the generalized permutation matrix:

$$\mathcal{M}^{\sigma}_{ab} = \begin{cases} g_a & ; \quad \text{if } b = \sigma^g(a) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Example 2.6.1. The following Figure 2.11 represents the automorphism g with the permutation matrix

$$M^g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

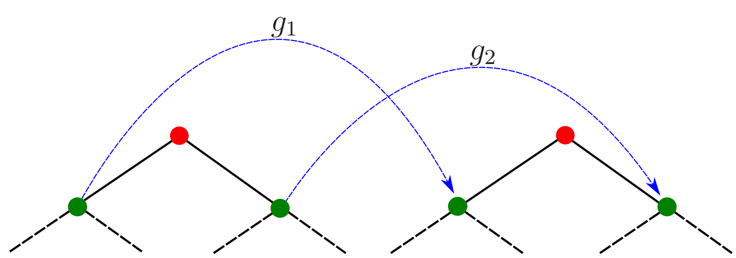


Figure 2.11:

Thus, for arbitrary group G and alphabet $\mathcal{A}$, one denotes the group of generalized permutation matrices of order $|\mathcal{A}|$ as $\mathbf{Sym}(\mathcal{A}; G) = \mathbf{Sym}(\mathcal{A}) \leftthreetimes G^{\mathcal{A}}$. These matrices contain only non-zero entries from the group G. The products of matrix elements are performed according to the group law of G and the group operation is a matrix multiplication. If we replace all group elements in the generalized permutation matrix with 1, we shall get the usual permutation matrix which corresponds to the natural homomorphism of $\mathbf{Sym}(\mathcal{A}; G)$ onto $\mathbf{Sym}(\mathcal{A})$. It follows that there is a natural isomorphism of the group

$\mathbf{Aut}(T) = \mathfrak{G}$ of all automorphisms of the rooted tree $T(\mathcal{A})$ and generalized permutation group $\mathbf{Sym}(\mathcal{A}; \mathfrak{G})$.

We now give definition of a self similar group, which can be found in [Kai09]. The following definition is actually equivalent to the definition given by V. Nekrashevych in [Nek05].

Definition 2.6.1. A countable subgroup $G \subset \mathfrak{G}$ is a self-similar group if all the non-zero elements of the matrices M^g $(g \in G)$, belong to G (or equivalently, if the restriction of the isomorphism $\mathfrak{G} \rightarrow \mathbf{Sym}(\mathcal{A}; \mathfrak{G})$ to G induces an embedding $G \hookrightarrow \mathbf{Sym}(\mathcal{A}; G)$).

Example 2.6.2. Consider the transformation $\beta : z \mapsto z + 1$ on the ring $\mathbb{Z}_2$ of 2-adic integers $\varepsilon_0 + \varepsilon_1 \cdot 2 + ... + \varepsilon_n \cdot 2^n + ...$, where $\varepsilon_i = 0$ or 1. This transformation acts as follows.

$$\beta(0w) = 1w$$

$$\beta(1w) = 0\beta(w)$$

Which, for instance, can be presented as:

$$\begin{array}{lllllllllll} 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & ... & \mapsto & 1 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & ... \;, \\ 1 & 0 & \varepsilon_2 & \varepsilon_3 & ... & \mapsto & 0 & 1 & \varepsilon_2 & \varepsilon_3 & ... \;, \\ 1 & 1 & 0 & \varepsilon_3 & ... & \mapsto & 0 & 0 & 1 & \varepsilon_3 & ... \;, \\ ... \end{array}$$

Now consider the binary rooted tree $T = T(\mathcal{A})$ with alphabet $\mathcal{A} = \{0; 1\}$. Clearly, the sequences $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ from the above, can be used to represent the boundary of T. Then, the transformation β, presented above can be also extended to an automorphism of T with the associated generalized permutation matrix

$$M^{\beta} = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$$

Whence, we can conclude that the infinite cyclic group $\langle\beta\rangle$, generated by the transformation β is a self-similar group, as a subgroup of the full group of automorphisms $\mathfrak{G} = \mathbf{Aut}(T)$ of the tree T. For full details on this example, see [Kai09].

2.7 Hyperbolic Groups and Spaces

Let G be finitely generated group with finite generating set $\mathcal{A}$. Let $\mathcal{A}$ be symmetric, meaning that if $a \in \mathcal{A}$, then $a^{-1} \in \mathcal{A}$. And let $\mathcal{A}$ not contain the identity of the group, ε. For any element $g \in G$ we call the minimal number of elements of $\mathcal{A}$ needed to write g - the length of g, and denote $|g|$. For any two elements $g_1, g_2 \in G$, we shall denote distance between g_1 and g_2 as $|g_2 - g_1| = |g_1^{-1}g_2|$.

Note: The distance function defined as above always returns an integer value. We shall use the notion of Cayley graph to fix this inconvenience. As we know, the vertices of the classical Cayley graph $\Gamma(G, \mathcal{A})$ naturally associated to the elements of the group G. Two vertices x and y of the Cayley graph are connected with an edge if and only if $|y - x| = 1$, equivalently $x^{-1}y \in \mathcal{A}$. Obviously, left shifts by elements of the group G set up the group action on the Cayley graph of the group G. Further, we can endow the set of the graph edges with a metric which will make all edges isometric to the closed interval $[0, 1]$. Then the distance between any two points of $\Gamma(G, \mathcal{A})$

is the lower length limit of paths that connect these two points. Thus, the Cayley graph of the group G becomes linear-connected metric space. Moreover, the natural embedding $(G, |\cdot|) \hookrightarrow \mathit{\Gamma}(G, \mathcal{A})$ is the isometry. This way we get a good idea of the geometric image of the group G.

The distance function defined above also depends on the choice of the generating set. To avoid this inconvenience, we shall define the *quasi-isometry* of metric spaces. The *quasi-isometry* notion is the mathematical reflection of the fact that two metric spaces are very similar when viewed from (large) distance. We use the definition from [GdLH90].

Definition 2.7.1. Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be two metric spaces. We shall say that X and Y are *quasi-isometric* if there exist two maps $f : X \to Y$ and $g : Y \to X$, and also two constants $\rho > 0$ and $C \geq 0$, such that

$$|f(x_2) - f(x_1)|_Y \leq \rho|x_2 - x_1|_X + C;\ \forall x_1, x_2 \in X$$
$$|g(y_2) - g(y_1)|_X \leq \rho|y_2 - y_1|_Y + C;\ \forall y_1, y_2 \in Y$$
$$|x - g(f(x))|_X \leq C;\ \forall x \in X$$
$$|y - f(g(y))|_Y \leq C;\ \forall y \in Y$$

This definition implies that (for large distances) images f and g satisfy the Lipschitz condition with the constant ρ and also that f and g are almost inverse of each-other.

We just need to state a few more definitions before we can define a Hyperbolic group. ([GdLH90])

Definition 2.7.2. Let $(X, |\cdot|)$ be a metric space. A *parametrized geodesic segment* that connects two points $x, y \in X$ ($|y - x| = d$) is an isometric embedding $f : [0, d] \to X$, such that $f(0) = x$, $f(d) = y$.

The image of f is called a *geodesic segment*. A metric space X is called *geodesic* if for any $x, y \in X$, there exists a *geodesic segment* that connects x and y.

Geodesic triangle in X with its vertices $x, y, z \in X$ is the union of three *geodesic segments* that connect x, y, z pairwise. The uniqueness of such geodesic segment is not required.

We shall denote $[x, y]$ one of the geodesic segments that connect x and y.

The restricted isometry property (RIP) for metric geodesic space X means that for any geodesic triangle $T = [x, y] \cup [y, z] \cup [z, x]$ in X and any point $a \in [y, z]$ we have

$$|[x, y] \cup [z, x] - a| \leq \delta$$

with restricted isometry constant (RIC) $\delta \geq 0$ (see Figure 2.12).

Definition 2.7.3. If there exists a finite number $\delta \geq 0$ such that geodesic metric space X satisfies the RIP with the RIC δ, then this space X is *hyperbolic*.

We are finally ready to define a *hyperbolic group*.

Definition 2.7.4. Finitely generated group G is *hyperbolic* if its Cayley graph $\mathit{\Gamma}(G, \mathcal{A})$ (for some finite generating set $\mathcal{A}$) is a *hyperbolic space*.

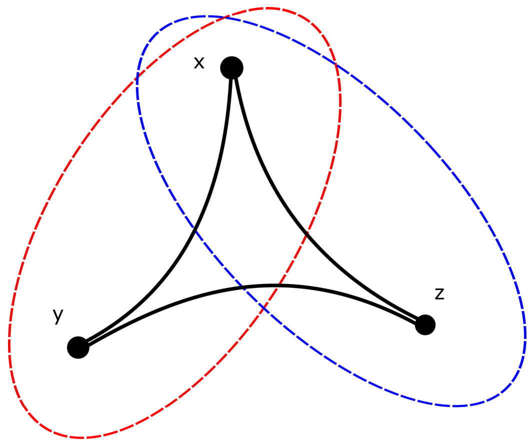


Figure 2.12: Restricted Isometry Property

Note that if the Cayley graph $\Gamma(G, \mathcal{A})$ is hyperbolic for some fixed finite generating set $\mathcal{A}_0$, it is also hyperbolic for any other finite generating set. It is also worth noting that if two groups G_1 and G_2 are quasi-isometric and one of them is hyperbolic, then the other one is also hyperbolic (Theorem 29, [GdLH90]).

2.8 Groupoids

A groupoid $\mathbf{G}$ is a small category in which each morphism is an isomorphism, so that it is determined by the set of objects $Obj(\mathbf{G}) = \mathbf{G}^{(0)}$ and a set of morphisms $Mor(\mathbf{G}) \cong \mathbf{G}$ endowed with the source and target maps $s, t : \mathbf{G} \to \mathbf{G}^{(0)}$.

Now denote by $\mathbf{G}^{(2)}$ the set of all pairs (g_1, g_2) from $\mathbf{G}$ that can be composed in the sense that $s(\mathrm{g}_2) = t(\mathrm{g}_1)$ (map g_1 applied first).

Formally:

$$\mathbf{G}^{(2)} = \{(g_1, g_2) \in \mathbf{G} \times \mathbf{G} : s(\mathrm{g}_2) = t(\mathrm{g}_1)\}$$

with composition being a map

$$\mathbf{G}^{(2)} \to \mathbf{G},\ (g_1, g_2) \to \mathsf{g}_1\mathsf{g}_2$$

and $s(\mathsf{g}_1\mathsf{g}_2) = s(\mathsf{g}_1)$, $t(\mathsf{g}_1\mathsf{g}_2) = t(\mathsf{g}_2)$.

The composition defined in this way, has following properties:

(i) There is an embedding $\varepsilon : \mathbf{G}^{(0)} \to \mathbf{G}$ which sets correspondence between elements $x \in \mathbf{G}^{(0)}$ and identical automorphisms ε_x in a following way.

$$s(\varepsilon_x) = t(\varepsilon_x) = x, \text{ for all } x \in \mathbf{G}^{(0)}$$

(ii) Any element g of the set of morphisms $\mathbf{G}$ has unique inverse morphism g^{-1} such that $s(\mathsf{g}^{-1}) = t(\mathsf{g})$ and $t(\mathsf{g}^{-1}) = s(\mathsf{g})$ with the following properties:

$$\mathsf{g}\mathsf{g}^{-1} = \varepsilon_{s(\mathsf{g})} \text{ and } \mathsf{g}^{-1}\mathsf{g} = \varepsilon_{t(\mathsf{g})}$$

(iii) The composition is associative whenever it is defined.

We shall say that two elements x, y of the object set $\mathbf{G}^{(0)}$ belong to the same *orbit* if they are isomorphic as objects of the category, or equivalently, if there exists a morphism $g \in \mathbf{G}$ such that $s(g) = x$ and $t(g) = y$. It follows that the relation of two elements belonging to the same *orbit* is an equivalence relation.

Example 2.8.1. If we think of the set of objects $\mathbf{G}^{(0)}$ as of the set of vertices of some graph Γ, then each pair $(s(\mathsf{g}), t(\mathsf{g}))$ can be geometrically represented as directed edge connecting two elements of $\mathbf{G}^{(0)}$. The beginning of this edge is at $s(\mathsf{g})$ and the end is at $t(\mathsf{g})$. Clearly, any existing edge of the graph can be thought of as the pair of opposit-directed edges and a groupoid can be generated on any connected graph. However, for future reference, it's the new directed edges that drawn to connect components of disconnected graph that we shall mostly look at. Figure 2.13 illustrates morphisms $g_1, g_2, g_3 \in \mathbf{G}$, associated to vertices of Γ (elements of $\mathcal{V}(\Gamma) \cong \mathbf{G}^{(0)}$) through the use of maps s and t. It becomes obvious from the illustration that $(g_2, g_3) \in \mathbf{G}^{(2)}$.

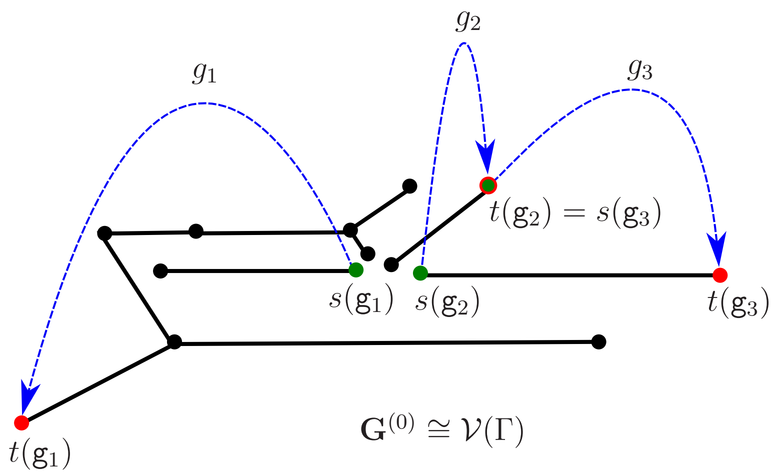


Figure 2.13: Graph Γ with the set of vertices $\mathbf{G}^{(0)}$ and morphisms g_1, g_2, g_3, associated to some elements of $\mathbf{G}^{(0)}$

2.9 Pseudogroups

The notion of a pseudogroup is a generalization of the concept of group of transformations. In contrast to the abstract algebraic approach, the concept of pseudogroup emerged from the geometric approach of Sophus Lie. Élie Cartan is broadly considered to be one of the first developers of the theory of pseudogroups due to his publications in early 1900s. See [Car04], [Car09]. Arguably, the notion of pseudogroup first appeared in the paper by Veblen and Whitehead [VW32] in the context of geometric objects studied in differential geometry. The first formal definition was given by Golab [Gol39]. However, the definition that we shall use (given in [Wal04]) appeared later and it avoids some complexities of generating a pseudogroup caused by the earlier definition. See [Wal04] for more details.

Recall what was the idea of a group of transformations of a space. Consider a space X and a group G of transformations of X. Then G consists of maps $g \in G$ defined globally on X, with $g : X \to X$ being bijections. Also, if $g \in G$ then $g^{-1} \in G$. And for $g_1, g_2 \in G$ we have $g_1 g_2 \in G$.

We now define pseudogroup as follows. Let X be a topological space. Denote by $Hom(X)$ the family of all homeomorphisms between open subsets of X. For any $f \in Hom(X)$, we denote the domain of f as D_f and the range of f as $R_f = f(D_f)$. Consider a subfamily $\mathcal{F}$ of $Hom(X)$. If $\mathcal{F}$ is closed under composition, inversion, and restriction to open subdomains and unions, it is called a *pseudogroup*. To give a formal definition, we refer to [Wal04](p.1).

Definition 2.9.1. A subfamily $\mathcal{F}$ of $Hom(X)$ is called a *pseudogroup* if it satisfies following conditions.

(1) $f, g \in \mathcal{F} \Rightarrow fg \in \mathcal{F}$

(2) $f \in \mathcal{F} \Rightarrow f^{-1} \in \mathcal{F}$

(3) if $f \in \mathcal{F}$ and $S \subset D_f$ is open, then $f|S \in \mathcal{F}$

(4) if $f \in Hom(X)$, D^* is an open cover of D_f and $f|S \in \mathcal{F}$ for any $S \subset D^*$, then $f \in \mathcal{F}$.

(5) $\underset{f \in \mathcal{F}}{\cup} D_f = X$

For the composition fg ($f, g \in \mathcal{F}$) to be defined it is necessary that the intersection of the range of f and the domain of g is not empty $R_f \cap D_g \neq \emptyset$. In this case the composition fg is defined as follows.

$$fg : f^{-1}(R_f \cap D_g) \to g(R_f \cap D_g)$$

Let A be some subset of $Hom(X)$. If A satisfies the condition that

$$\bigcup_{a \in A} (D_a \cup R_a) = X$$

then there exists a unique smallest pseudogroup $\mathcal{F}(A)$ that contains A. Further, $f \in \mathcal{F}(A)$ if and only if $f \in Hom(X)$ and for any $x \in D_f$ there exist a set of maps $\{a_i : i = 1, 2, .., n\} \subset A$, a set of exponents $\{\varepsilon_i\}$ $(\varepsilon_i \in \{\pm 1\}, \forall i)$, and an open neighborhood S of x $(S \subset D_f)$, such that f can be presented as a composition of elements of A with restriction to the open set S.

$$f|S = a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_n^{\varepsilon_n} |S$$

We say that $\mathcal{F}(A)$ is generated by A.
One can always generate pseudogroup by taking $A' = A \cup \{\mathrm{id}_X\}$ as the generating set.

Definition 2.9.2. A pseudogroup $\mathcal{F}(A)$ generated by a set of homeomorphisms $A \subset Hom(X)$ is called *finitely generated*, if $A = \{a_1, a_2, ..., a_n\}$ is a finite set.

Any finitely generated pseudogroup $\mathcal{F}(A)$ admits a finite *symmetric* generating set $A^* = A \cup A^{-1} \cup \{\mathrm{id}_X\}$.

Example 2.9.1. Of course, $Hom(X)$ is the largest pseudogroup on the space X, and $\mathcal{F}(\mathrm{id}_X)$ is the smallest pseudogroup on X.

If (X, d) is an arbitrary metric space, then all local isometries of X generate a pseudogroup $\mathrm{Iso}(X)$ on X. Because of the condition (3) in the definition 2.9.1, it is not necessary that any $f \in \mathrm{Iso}(X)$ maps D_f into R_f isometrically. Instead, it is sufficient that any $x \in D_f$ has a neighborhood S, such that $f|S : S \to f(S)$ is an isometry.

2.10 Graph Products

We now review some definitions and properties of graph products. For the purpose of this section we define a graph $\Gamma = (V(\Gamma), E(\Gamma))$ as an ordered pair of a vertex set V and an edge set E that consists of 2-element subsets of V.

For each edge $(x, y) \in E$, x and y are called adjacent vertices. A graph $\Gamma' = (V', E')$ is a subgraph of $\Gamma = (V(\Gamma), E(\Gamma))$, written as $\Gamma' \subseteq \Gamma$, if $V' \subseteq V$ and $E' \subseteq E$. A subgraph $\Gamma' = (V', E')$ is a spanning subgraph of $\Gamma = (V, E)$, if $V' = V$.

First we define the *Cartesian Product* of two graphs as this type of graph product operation will be of the most interest to us in further work.

Definition 2.10.1. [SJS12] A *Cartesian product* $\Gamma_\square = \Gamma_1 \square \Gamma_2$ of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is defined as follows:

$$V_\square = V_1 \times V_2 = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2\}$$

$$E_\square = \{(v_1 u_2; v_1 v_2) | v_1 \in V_1, (u_2, v_2) \in E_2\} \cup$$
$$\{(u_1 v_2; v_1 v_2) | (u_1, v_1) \in E_1, v_2 \in V_2\}$$

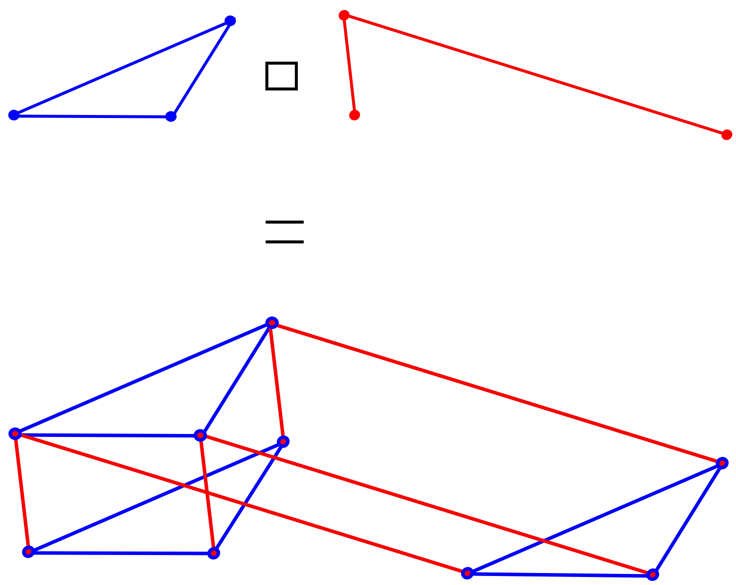


Figure 2.14: Two factors and their Cartesian product.

Some properties of Cartesian product of graphs are [Imr92]:

(i) Cartesian product operation is commutative as an operation on isomorphism classes of graphs (and graphs $A\square B$ and $B\square A$ are naturally isomorphic). However, it is not commutative as an operation on labeled graphs.

(ii) Cartesian product operation is associative in a sense that graphs $(A\square B)\square C$ and $A\square(B\square C)$ are naturally isomorphic.

For comparison, we also give definitions of *Tensor product* and *Strong (or Lexicographic) product* of graphs.

Definition 2.10.2. The *Tensor product* $\Gamma_\times = \Gamma_1 \times \Gamma_2$ of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is defined as follows:

$$V_\times = V_1 \times V_2 = \{(v_1, v_2)|v_1 \in V_1, v_2 \in V_2\}$$

$$E_\times = \{(u_1u_2; v_1v_2)|(u_1, v_1) \in E_1, (u_2, v_2) \in E_2\}$$

Definition 2.10.3. The *Strong product* $\Gamma_\boxtimes = \Gamma_1 \boxtimes \Gamma_2$ of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is defined as follows:

$$V_\boxtimes = V_1 \times V_2 = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2\}$$

$$E_\boxtimes = E_\square \cup E_\times$$

It follows from the definition of Cartesian product and Strong product that $\Gamma_\square$ is a spanning subgraph of $\Gamma_\boxtimes$. The edges that $\Gamma_\square$ shares with $\Gamma_\boxtimes$ are called *Cartesian edges*.

Figure 2.15 is a graphical representation of the rules that one needs to follow in order to produce sets of vertices and edges corresponding to the three different types of graph products described above.

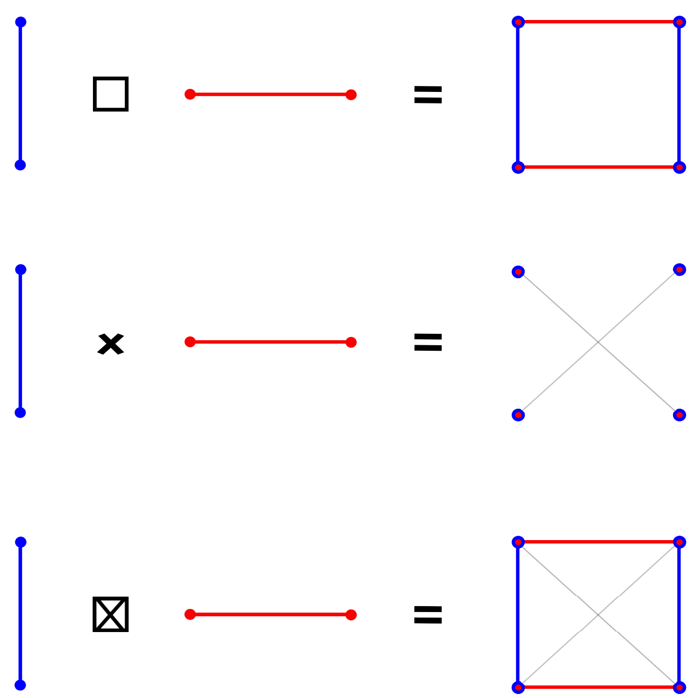


Figure 2.15: Cartesian product, Tensor product and Strong product of two graphs marked by colours

Chapter 3

Preliminaries

3.1 Initial Setup

Let $\mathit{\Gamma} = (\mathcal{V}(\mathit{\Gamma}), \mathcal{E}(\mathit{\Gamma}))$ be a finite graph with the vertex set $\mathcal{V}(\mathit{\Gamma})$ and the edge set $\mathcal{E}(\mathit{\Gamma}) \subset \mathcal{V}(\mathit{\Gamma}) \times \mathcal{V}(\mathit{\Gamma})$

Similarly to the section 2.4.1, define a finite set $\mathcal{A}$ to be an alphabet and set $\mathcal{W}^n = \{a_1 a_2 a_3 ... a_n : a_i \in \mathcal{A}\}$ to be the set of all words of length n in alphabet $\mathcal{A}$. Then the set

$$\mathcal{W}^* = \bigcup_{0 \leq n < \infty} \mathcal{W}^n$$

is a set of all finite words, with $\mathcal{W}^0 = \emptyset$ being an empty word.

If we define composition on $\mathcal{W}^*$ to be concatenation $(\mathbf{v}, \mathbf{w}) \to \mathbf{vw}$, then the set $\mathcal{W}^*$ together with this composition becomes a free monoid with associated rooted tree $T = T(\mathcal{A})$ being right Cayley graph with T_m corresponding to the set of all words of length m in $\mathcal{W}^*$.

The Cartesian product operation of the form $\Gamma \Box T$ gives the following graph (see section 2.10)

$$\mathcal{L} = (\mathcal{V}(\mathcal{L}), \mathcal{E}(\mathcal{L}))$$

with its set of vertices being

$$\mathcal{V}(\mathcal{L}) = \{(u, \mathbf{w}) : u \in \Gamma, \mathbf{w} \in \mathcal{W}^*\}, \text{ where } \mathbf{w} = a_1 a_2 ... a_k \text{ is a finite word.}$$

and set of edges $\mathcal{E}(\mathcal{L})$, that consists of edges of two types, *vertical* and *horizontal*. *Vertical* edges are formed by all pairs of the form

$$((u, \mathbf{w}), (u, \mathbf{w}a)), \text{ where } u \in \mathcal{V}(\Gamma); \mathbf{w} \in \mathcal{W}^*; a \in \mathcal{A}$$

Horizontal edges are formed by pairs of the form

$$((u, \mathbf{w})(v, \mathbf{w})) \text{ if and only if } (u, v) \in \mathcal{E}(\Gamma).$$

Now, to every word $\mathbf{w} = a_1 ... a_k \in \mathcal{W}^*$ of length k, we can associate a unique copy of the graph Γ

$$\Gamma_{\mathbf{w}} = \Gamma_{a_1 ... a_k} = \Gamma \times \mathbf{w}.$$

Then the set of copies of the graph Γ

$$\{\Gamma_{a_1 a_2 ... a_k} : a_1 a_2 ... a_k \in T_k\} = \Gamma \Box T_k$$

corresponds to the k'th level T_k of rooted tree $T = T(\mathcal{A})$. We shall also call these sets *levels* and denote

$$\mathcal{L}_k = \{\Gamma_{a_1 a_2 ... a_k} : a_1 ... a_k \in T_k\}$$

where $\mathcal{L}_0$ is just the initial graph Γ.

3.2 Hyperbolicity

3.2.1 Connected initial graph Γ

If the original graph Γ is a connected graph, we can endow the graph obtained as a result of the Cartesian product $\Gamma \Box T$ with the graph metric which will make all edges isometric to the closed interval $[0;1]$. Then $\mathcal{L}$ is a metric space. Moreover, since any two points of $\mathcal{L}$ can be connected by a geodesic segment, it follows that $\mathcal{L}$ or, more precisely, the union of all edges of this structure, is a geodesic metric space. Then the resulting structure $\mathcal{L} = \Gamma \Box T$ is quasi-isometric to the underlying rooted, M regular tree $T = T(\mathcal{A})$ $(|\mathcal{A}| = M)$. Consequently, we have that $\mathcal{L}$ is δ- hyperbolic with $\delta = d(\Gamma)$, where $d(\Gamma)$ is the diameter of the graph Γ.

To see the quasi-isometry, split the graph product $\mathcal{L} = \Gamma \Box T$ into equivalence classes as follows. We say that two *vertical edges* ε, ε' are equivalent if they connect two copies $\Gamma_{\mathbf{w}}$ and $\Gamma_{\mathbf{w}a}$ of the initial graph $\Gamma \in \mathcal{L}$, where the initial graph Γ can be considered the "root" of the graph product structure $\mathcal{L} = \Gamma \Box T$. We shall denote these equivalence classes $[\varepsilon]$. Now consider two metric geodesic spaces. First, X - isomorphic to the union of all edges $\cup(\epsilon)$ of regular rooted tree $T(\mathcal{A})$. Second, Y - isomorphic to the union of all edges $\cup(\varepsilon)$ of the graph product $\mathcal{L} = \Gamma \Box T$ for finite, connected graph Γ. Set two maps $f : X \to Y$ and $g : Y \to X$ to be

$$f : \epsilon \to [\varepsilon];\, \epsilon \in X,\ [\varepsilon] \in Y$$
$$g : [\varepsilon] \to \epsilon;\, \epsilon \in X,\ [\varepsilon] \in Y$$

Thus, the map f sets one-to-one correspondence between edge ϵ of regular rooted tree $T(\mathcal{A})$ and equivalence class $[\varepsilon]$ of edges in $\mathcal{L}$. The map g sets the inverse one-to-one correspondence of equivalence class $[\varepsilon]$ of vertical edges in Y and edge ϵ in X.

Note that the set of horizontal edges of each copy $\Gamma_{\mathbf{w}}$ of the initial graph Γ in $\mathcal{L}$ is merged by the map g into a single point (which corresponds to the vertex $\mathbf{w} \in T(\mathcal{A})$). Obviously, all conditions of the Definition 2.7.1:

$$|f(x_2) - f(x_1)|_Y \leq \rho|x_2 - x_1|_X + C; \, \forall x_1, x_2 \in X$$
$$|g(y_2) - g(y_1)|_X \leq \rho|y_2 - y_1|_Y + C; \, \forall y_1, y_2 \in Y$$
$$|x - g(f(x))|_X \leq C; \, \forall x \in X$$
$$|y - f(g(y))|_Y \leq C; \, \forall y \in Y$$

are satisfied for the above setup, if one takes constants $\rho = 1$ and $C = d(\Gamma)$ (where $d(\Gamma)$ is the diameter of the initial graph Γ) and choose to use the same metric for these two spaces X, Y that makes all edges isometric to a closed interval $[0, 1]$. Since the regular rooted tree is a hyperbolic space, it follows from ([GdLH90], *Theorem 29*) that our graph product $\mathcal{L} = \Gamma \Box T$ (with connected finite graph Γ) is δ-hyperbolic, with $\delta = d(\Gamma)$.

3.2.2 Disconnected Initial Graph Γ

However, if the initial graph Γ is disconnected, we can't talk about quasi-isometry between $\mathcal{L}$ and $T(\mathcal{A})$ any more because the resulting structure $\mathcal{L} = \Gamma \Box T(\mathcal{A})$ can not be defined as a metric space due to its disconnectedness. Thus, the connectedness of the initial graph Γ is a key property in order for $\mathcal{L} = \Gamma \Box T(\mathcal{A})$ to be considered a hyperbolic space.

One of the ways to overcome this obstacle is to try to *connect* the initial graph Γ (and the whole structure $\mathcal{L}$) without changing the initial graph Γ itself. For instance, we can not simply add several edges to the graph Γ to enlarge its connected components. Instead, one can go the following way. Suppose there exists a set of partial bijections between the sets of vertices of the graph Γ

$$\{f_a : a \in \mathcal{A}\}, \quad f_a : D_a \to R_a$$

where

$$D_a, R_a \subset \mathcal{V}(\Gamma), \forall a \in \mathcal{A}$$

Now consider the graph $\mathcal{L} = \Gamma \Box T$ and its levels, the set of graphs $\{\Gamma_{\mathbf{w}}\}$ ($\mathbf{w} \in \mathcal{W}^*$). Clearly, any mapping f_a can be used to establish one to one correspondence between $D_a \subset \mathcal{V}(\Gamma)$ on the level $\mathcal{L}_0 \cong \Gamma$ and $R_a \subset \mathcal{V}(\Gamma_a)$ on the level $\mathcal{L}_1 \supset \Gamma_a$.

Obviously, we have the same subsets D_a, R_a in each $\mathcal{V}(\Gamma_{\mathbf{w}})$. Then for every $\Gamma_{a_1 a_2 \ldots a_k}$ there exists a unique sequence $f_{a_1} f_{a_2} \ldots f_{a_k}$. Under certain conditions (i.e. sets D_a and R_a have non-empty intersections), this sequence can be associated to a path form Γ (the "root" of our structure) to the graph $\Gamma_{a_1 \ldots a_k}$.

We shall now "erase" all vertical edges of graph $\mathcal{L}$ and "draw" new vertical *directed edges* formed by all pairs

$$((u, \mathbf{w}); (f_a(u), \mathbf{w}a)),$$

where

$$u \in D_a \subset \mathcal{V}(\Gamma_{\mathbf{w}}),\ f_a(u) \in R_a \subset \mathcal{V}(\Gamma_{\mathbf{w}a})$$

This way we obtain a new graph $\mathcal{L}^*$ formed by the modified product $\Gamma\tilde{\Box}T(\mathcal{A})$ (which is the $\Gamma\Box T(\mathcal{A})$ with removed vertical edges) and the set of functions $\{f_a : a \in \mathcal{A}\}$ (See Fig.3.1)

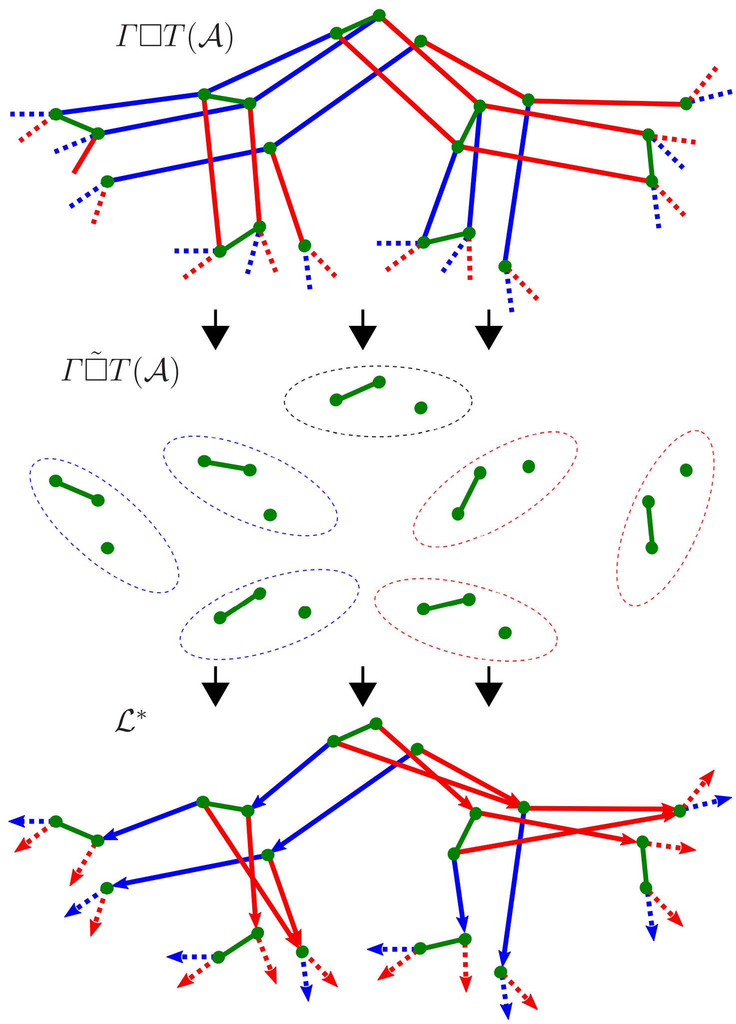


Figure 3.1: Graph $\mathcal{L}^*$ formed by the modified product $\Gamma\tilde{\Box}T(\mathcal{A})$ and the set of functions $\{f_a : a \in \mathcal{A}\}$

Example 3.2.1. Let the graph Γ be defined by setting

$$\mathcal{V}(\Gamma) = \{1, 2, 3, 4, 5\},\ \mathcal{E}(\Gamma) = \{(1, 2); (2, 1); (2, 3); (3, 2); (4, 5); (5, 4)\}.$$

The alphabet is $\mathcal{A} = \{a, b\}$, consequently: $T(\mathcal{A})$ is a binary rooted tree. The Cartesian product $\Gamma \Box T$ gives a graph isomorphic to the one shown in Figure 3.2 (for the first three levels). Define the set $\{f_i : i \in \{a, b\}\}$ to be as follows.

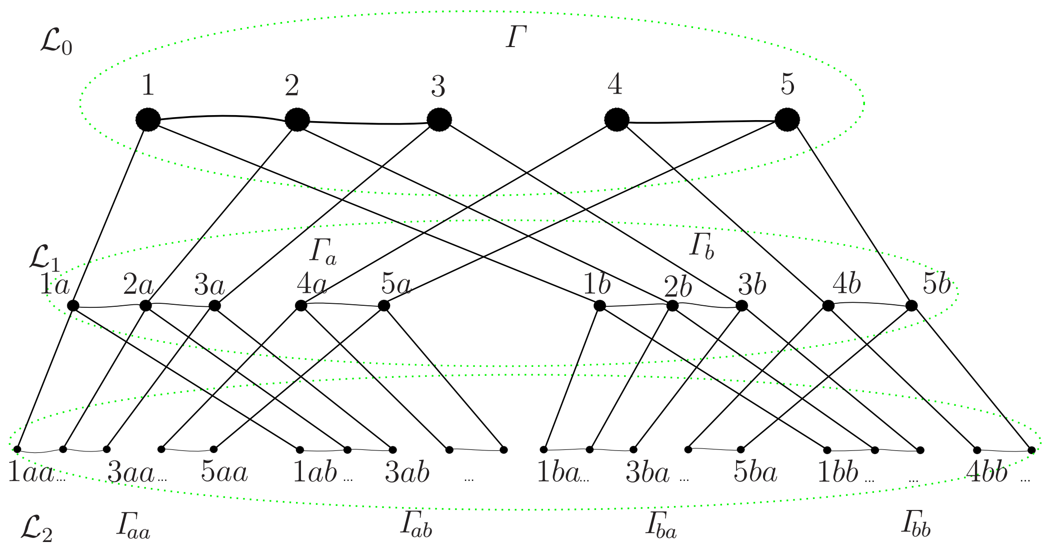


Figure 3.2: First three levels $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ of graph $\mathcal{L} = \Gamma \Box T$

$$f_a : \{1, 2\} \to \{4, 5\};\ f_b : \{3, 5\} \to \{1, 4\}$$

Let $f_a(1) = 4,\ f_a(2) = 5,\ f_b(3) = 1,\ f_b(5) = 4$. Further, we erase all vertical edges and draw the new directed edges. We have

$$D_a = \{1, 2\},\ D_b = \{3, 5\},\ R_a = \{4a, 5a\},\ R_b = \{1b, 4b\}.\ f_a(1) = 4a,$$
$$f_a(2) = 5a,\ f_b(3) = 1b,\ f_b(5) = 4b.$$

The new graph $\mathcal{L}^*$ is isomorphic to the one shown in Figure 3.3 (for the first three levels).

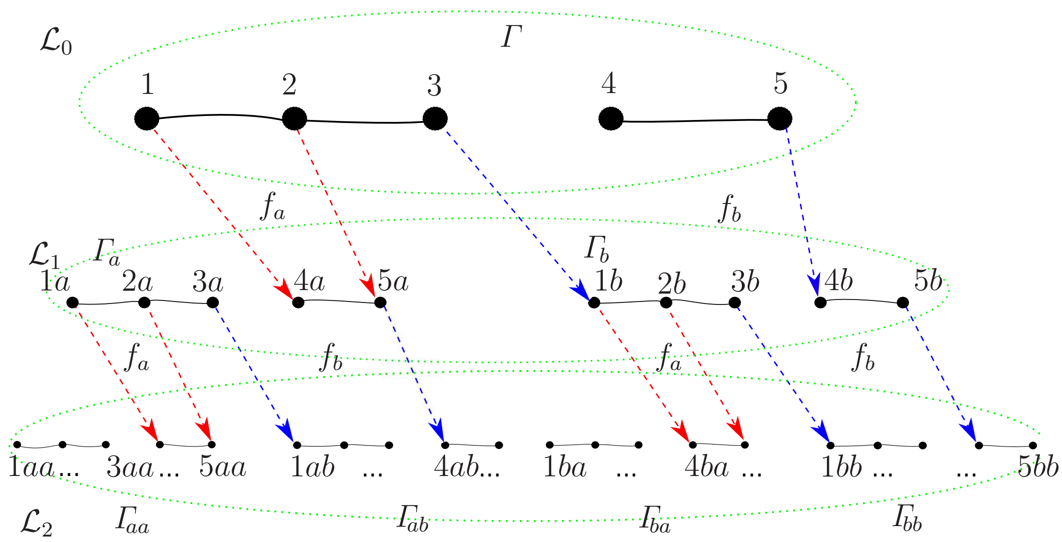


Figure 3.3: Demonstrates the first three levels of the resulting structure $\mathcal{L}^*$ (with new set of vertical edges).

In this new structure we can write a path that connects the vertex 1 and the vertex 3 in one of the following ways:

$$(1, 2), (2, 3)$$

or

$$(1, f_a(1)), (4a, 5a), (5a, f_a^{-1}(5a)), (2, 3).$$

However, there is no path that connects vertex 1 to vertex 5.

3.2.3 Maze-Graph

We shall call this new graph $\mathcal{L}^*$ formed by the modified product $\Gamma\tilde{\square}T(\mathcal{A})$ (which is the $\Gamma\square T(\mathcal{A})$ with removed vertical edges) and the set of functions $\{f_a : a \in \mathcal{A}\}$ a *maze-graph* and denote

$$\mathcal{L}^* = \mathcal{L}^*(\Gamma, T(\mathcal{A}), \{f_a : a \in \mathcal{A}\})$$

The name is due to the fact that connecting two vertices of a disconnected initial graph $\Gamma = \mathcal{L}_0$ in $\mathcal{L}^*$ would be very similar to finding a path that connects two nodes of a fractal maze as discussed in the introduction.

It is worth noting that since each sequence $f_{a_1} f_{a_2} ... f_{a_k}$ has a unique copy $\Gamma_{a_1 a_2 ... a_k}$ associated to it, we can only move "up" and "down" between two fixed graphs $\Gamma_{a_1 a_2 ... a_k} \in \mathcal{L}_k$ and $\Gamma_{a_1 a_2 ... a_k a_{k+1}} \in \mathcal{L}_{k+1}$ using *vertical edges* uniquely defined by $f_{a_{k+1}}$.

If the obtained *maze-graph* $\mathcal{L}^* = \mathcal{L}^*(\Gamma; T(\mathcal{A}); \{f_a : a \in \mathcal{A}\})$ is connected, it is quasi-isometric to the underlying tree $T(\mathcal{A})$. This can be shown using the same consideration as in Section 3.2.1. It follows that the associated space Λ^* (which is the union $\cup(\varepsilon)$ of all edges of $\mathcal{L}^*$) is a δ-hyperbolic geodesic metric space (with $\delta = d(\Gamma)$).

3.3 Necessary Conditions of Connectedness

Consider the *maze-graph* $\mathcal{L}^* = \mathcal{L}^*(\Gamma, T(\mathcal{A}), \{f_a : a \in \mathcal{A}\})$ described above. Let the initial graph Γ be finite and disconnected. Depending on the choice of maps $f_a : D_a \to R_a$ $(D_a, R_a \subset \mathcal{V}(\mathcal{L}^*))$ one obtains either connected or disconnected graph $\mathcal{L}^*$. Furthermore, $\mathcal{L}^*$ is the infinite self-similar graph by construction, as well as its underlying rooted tree $T(\mathcal{A})$.

The easy way to ensure that the infinite graph $\mathcal{L}^*$ is connected is to show that the initial disconnected graph $\Gamma = \mathcal{L}_0$ can be connected using the set of partial bijections $\{f_a\}$ as edges. Then, by self similarity of $\mathcal{L}^*$, any graph $\Gamma_{\mathbf{w}}$ $(\mathbf{w} \in \mathcal{W}^*)$ at any level $\mathcal{L}_i$ $(i = |\mathbf{w}|)$ is the root of a *maze-graph* $\mathcal{L}^*_{\mathbf{w}}$ which is isomorphic to the whole graph $\mathcal{L}^*$.

First we need to establish when two components of the disconnected initial graph Γ become connected in $\mathcal{L}^*$

Definition 3.3.1. We shall say that a function $f_a : D_a \to R_a$ $(D_a, R_a \subset \mathcal{V}(\mathcal{L}^*))$ connects two components C and C' of the initial disconnected graph Γ in the *maze-graph* $\mathcal{L}^*$, if there is a pair of vertices

$$x, y \in D_a$$

such that

$$x \in C,\ y \in C'$$

and

$$f_a(x), f_a(y) \in C^{(1)} \subset \Gamma_a$$

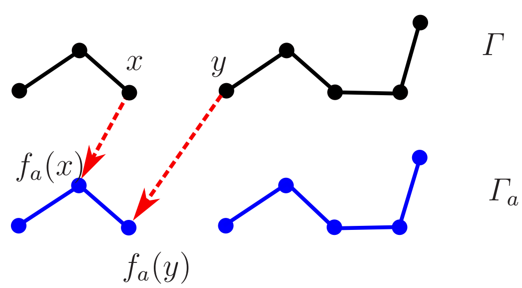


Figure 3.4: Illustrates two components of the initial graph Γ being connected in $\mathcal{L}^*$ by edges $(x, f_a(x))$ and $(y, f_a(y))$.

It follows that the initial disconnected graph Γ can not be connected in the *maze-graph* $\mathcal{L}^*$ if none of the functions in the set $\{f_a : a \in \mathcal{A}\}$ connects some of the components of the graph Γ in the sense of definition 3.3.1. Clearly, it's easier to see the conditions on the set of functions $f_a : D_a \to R_a$ under which the initial graph $\mathit{\Gamma}$ can not be connected.

Suppose that the disconnected graph $\mathit{\Gamma}$ has N connected components $\{C_i : i = 1, .., N\}$. Consider the *maze-graph* $\mathcal{L}^*(\mathit{\Gamma}, T(\mathcal{A}), \{f_a : a \in \mathcal{A}\})$. We claim that $\mathit{\Gamma}$ can not be connected in $\mathcal{L}^*$ if at least one of the following conditions is satisfied.
Keeping the above notation, we have

Proposition 3.3.1. *If at least one connected component of the initial disconnected graph $\mathit{\Gamma}$ in the maze-graph $\mathcal{L}^*$ has no common elements with any domain D_a ($\forall a \in \mathcal{A}$), such that $f_a : D_a \to R_a$ for the function set $\{f_a : a \in \mathcal{A}\}$, then the initial disconnected graph $\mathit{\Gamma}$ can not be connected in $\mathcal{L}^*$.*
Formally: If there is at least one

$$C_{i_0} \in \{C_i | i = 1, ..., N\}$$

such that

$$C_{i_0} \cap \{D_a | a \in \mathcal{A}\} = \emptyset.$$

Then the initial disconnected graph $\mathit{\Gamma}$ can not be connected in $\mathcal{L}^*$.

Proof. According to the Definition 3.3.1, in order to connect C_{i_0} to some other connected component C of $\mathit{\Gamma}$, we need a pair of vertices $x \in C_{i_0}$, $y \in C$, such that $f_a(x), f_a(y) \in C^{(1)} \subset \mathit{\Gamma}_a$ for some $a \in \mathcal{A}$. But this is impossible since $C_{i_0} \cap \{D_a | a \in \mathcal{A}\} = \emptyset$. Thus the initial disconnected graph $\mathit{\Gamma}$ can not be connected in $\mathcal{L}^*$. □

Proposition 3.3.2. *If there is at least one function $f_{a_0} \in \{f_a : a \in \mathcal{A}\}$ with its domain completely in one and only one connected component C_{i_0} of the*

initial disconnected graph $\mathit{\Gamma}$ (in the maze-graph $\mathcal{L}^*$) and no other than f_{a_0} function from $\{f_a\}$ has its domain intersecting with C_{i_0}, then the graph $\mathit{\Gamma}$ can not be connected in $\mathcal{L}^*$.
Formally: If there exist

$$D_{a_0} \in \{D_a | a \in \mathcal{A}\}$$

and there exist

$$C_{i_0} \in \{C_i | i = 1, ..., N\}$$

such that

$$D_{a_0} \subseteq C_{i_0}$$

and

$$(\{D_a | a \in \mathcal{A}\} \setminus D_{a_0}) \cap C_{i_0} = \emptyset \text{ and } (\{C_i | i = 1, ..., N\} \setminus C_{i_0}) \cap D_{a_0} = \emptyset$$

then the initial disconnected graph $\mathit{\Gamma}$ can not be connected in $\mathcal{L}^*$.

Proof. The proof again follows from definition 3.3.1. In fact, this condition is equivalent to proposition 3.3.1 formulated for the *maze graph* $\mathcal{L}^*(\mathit{\Gamma}, T(\mathcal{A}), \{f_a : a \in \mathcal{A}\} \setminus f_{a_0})$ □

Proposition 3.3.3. *If no function from $\{f_a : a \in \mathcal{A}\}$ in the maze-graph $\mathcal{L}^*$ connects any of the connected components of the initial disconnected graph $\mathit{\Gamma}$ in the sense of definition 3.3.1, then the graph $\mathit{\Gamma}$ can not be connected in $\mathcal{L}^*$.*
Formally: If there exists no $f_i \in \{f_a | a \in \mathcal{A}\}$, such that for some

$$x, y \in D_i \ (x \in C, \ y \in C')$$

we have

$$f_i(x), f_i(y) \in C^{(1)} \subset \mathit{\Gamma}_i,$$

then the initial disconnected graph $\mathit{\Gamma}$ can not be connected in $\mathcal{L}^*$.

Proof. Clearly, if the above is true, then definition 3.3.1 is not applicable to any two connected components C and C' of $\mathit{\Gamma}$. Thus, the initial disconnected graph $\mathit{\Gamma}$ can not be connected in $\mathcal{L}^*$. □

As it has been mentioned before, if at least one of the conditions formulated here is true, the maze-graph con not be connected. However, non of the conditions being true is not enough to claim that the maze-graph can be connected. Thus, we need further analysis.

We shall look separately at two cases. First, we take the alphabet $\mathcal{A}$ to be just one element set, $\mathcal{A} = \{a\}$. Second, the finite alphabet $\mathcal{A}$ consists of two or more elements, $\mathcal{A} = \{a_i\}_{i=1}^{M}$ ($M \geq 2$). As it will be shown later, the structure which emerges in one element case can be related to a *groupoid*, and the resulting structure in multi-element case is closely related to a *relation free pseudogroup*.

Chapter 4

Connectedness of the Maze-Graph

In this chapter we shall use the relation of our mathematical model to groupoid and pseudogrup, explaining the differences and performing the analysis on the connectedness problem. We split our model into two cases and explain why it's our belief that this approach is relevant to our work.

4.1 Single Element Alphabet and The Related Structure

Viewing the case when the generating set $\mathcal{A}$ consists of a single element separately allows us to set the ground for future consideration and lets us better understand the relation of the obtained infinite self similar graph to a certain finitely generated groupoid. The material presented in the three following sections will also be helpful in building an algebraic structure that is related to our geometric model and is similar to that of a relation free finitely generated pseudogroup. Another reason why we view this case separately is that the actual result is slightly different from the case when the finite generating set $\mathcal{A}$ consists of more than one element.

4.1.1 Groupoid or Pseudogroup?

Why is the case when the alphabet $\mathcal{A}$ consists of a single element $\mathcal{A} = \{a\}$ different from the case when $\mathcal{A} = \{a_1, a_2, ..., a_M\}$? While it is true that both cases could be viewed together and the arising structure is closely related to a pseudogroup of transformations, we have to keep in mind the set goal to establish whether initial disconnected finite graph Γ can be connected in $\mathcal{L}^*$ using a set of mappings between subsets of vertices of the maze-graph $\mathcal{L}^*$. In this context, the case with a single element alphabet tends to be simpler and could provide a good background for further analysis. Furthermore, the resulting structure generated in this case is more conveniently related to a groupoid of morphisms than to a pseudogroup with a single generating function. This is due to the fact that using a groupoid structure allows us to view any connected component of the disconnected graph Γ as a single element of groupoid's object set and concentrate on connectivity issues. The same approach is not possible in the case when the alphabet $\mathcal{A}$ consists of $1 < M \leq \infty$ elements and consequently, the set of mappings $\{f_a : a \in \mathcal{A}\}$ consists of M elements. This is because any attempt to switch to a groupoid structure leads to considering equivalence classes and simplifies the case to the single element case.

We shall use the following example to set the grounds for further consideration.

Example 4.1.1. Let $\mathcal{A}$ be a single element set $\mathcal{A} = \{a\}$, consequently the word space $\mathcal{W}^*$ is simply the set of all finite strings of the form $aa...a$ together with $\emptyset$, and the set of functions $\{f_a\}$ consists of a single partial bijection $f : D_f \to R_f$. Let Γ be a finite graph with three connected components. See Figure 4.1 below. The connected components are: $C_1 = \{1, 2; (1, 2); (2, 1)\}$, $C_2 = \{3, 4; (3, 4); (4, 3)\}$, $C_3 = \{5, 6; (5, 6); (6, 5)\}$.

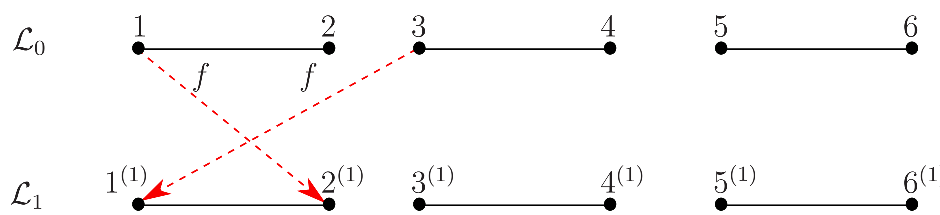


Figure 4.1: Single mapping case. $\mathcal{A} = \{a\}$

Let the domain of f be: $D_f = \{1, 3, 4, 5\}$ and some of the values of the range of f are $f(1) = 2^{(1)}$, $f(3) = 1^{(1)}$ (with $x^{(1)} \in \mathcal{L}_1$). Note that each pair of vertices $x, f(x) \in \mathcal{V}(\mathcal{L}^*)$ can be viewed as a directed edge $(x, f(x)) \in \mathcal{E}(\mathcal{L}^*)$. It's easy to see that any two vertices $x \in C_1$ and $y \in C_2$ can be connected by a path that contains $f(1)$ and $f^{-1}(1^{(1)})$.

$$(2, 1), (1, f(1)), (2^{(1)}, 1^{(1)}), (1^{(1)}, f^{-1}(1^{(1)})), (3, 4).$$

So the edges created by the function f connect the two components C_1 and C_2 at the level $\mathcal{L}_1 = \Gamma_a$. We shall denote the new connected component, obtained from the union $C_1^{(1)} \cup C_1 \cup C_2$ with the help of the edges created by function f, as C^f. Say we want to complete the range of the function f and define two remaining values of R_f to enlarge these two connected components. We only have three options relatively to C^f and C_3.

First, we can have (Figure 4.2)

$$f(5) \in C^f$$

In this case the initial graph can be connected using the second iteration

$$ff = f^2 \text{ and } \mathcal{L}_2 = \Gamma_{aa}, \text{ not depending on the value of } f(4).$$

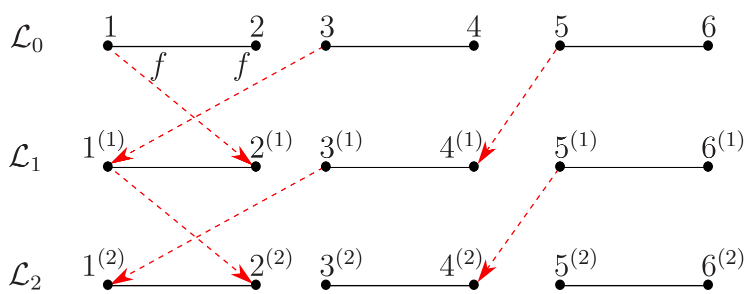


Figure 4.2:

Second, we can have (Figure 4.3)

$$f(5) \in C_3 \text{ and } f(4) \in C^f$$

It is easy to see that in this case any number of iterations of the function $f...f = f^p$, $p > 0$ will keep its values in different connected components and neither of C^f or C_3 can be enlarged.

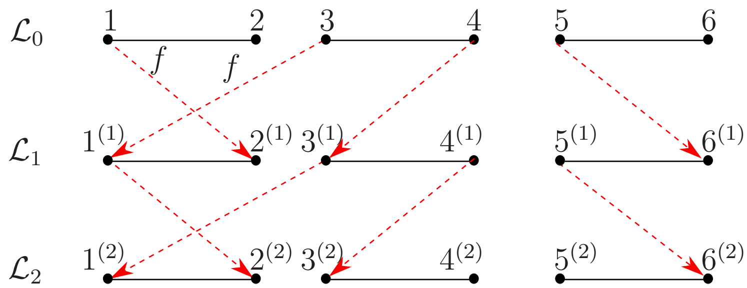


Figure 4.3:

And third, we can have (Figure 4.4)

$$f(5) \in C_3 \text{ and } f(4) \in C_3.$$

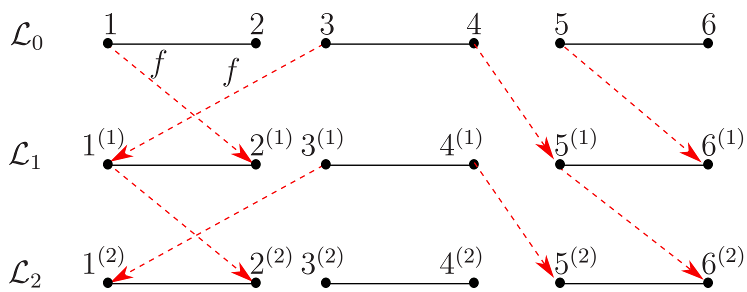


Figure 4.4:

It is obvious that in this case both of the connected components can be enlarged by using first iteration of the function f and initial graph can be connected. We can see that a graph that consists of three connected components can be connected using two or less iterations of the function f chosen in a certain way.

4.1.2 Relation to Groupoids

The initial setup for the *maze-graph* $\mathcal{L}^*$ which arises from a disconnected finite graph $\mathit{\Gamma}$ and a single function f (as the result of the process, described in section 3.1) can be related to a certain groupoid $\mathbf{G}$ as follows. Let the set of objects $\mathbf{G}^{(0)}$ be isomorphic to the set $\{C_i : i = 1, 2, ..., N\}$ of connected components of $\mathit{\Gamma}$. Having a function $f : D_f \to R_f$,($D_f, R_f \subset \mathcal{V}(\mathit{\Gamma})$), we shall say that two maps $x \mapsto f(x)$ and $y \mapsto f(y)$ are *equivalent* and denote $[x, f(x)]$, if $x, y \in C_n$ and $f(x), f(y) \in C_m$ for some $C_n, C_m \in \{C_i : i = 1, 2, ..., N\}$.

Now we clearly can set-up two mappings $s, t : \mathbf{G} \to \mathbf{G}^{(0)}$, where the set $\mathbf{G}$ of morphisms is precisely the set of all equivalence classes of maps between connected components of the disconnected graph $\mathit{\Gamma}$.

Thus, for instance, two elements $x \neq y \in D_f$ will correspond to the same point $s(\mathsf{g})$ in $\mathbf{G}^{(0)}$ *if and only if* x and y are in the same connected component of the graph $\mathit{\Gamma}$.

One can picture the set of morphisms $\mathbf{G} = \{g_i\}$ as the set of *arrows* g_i associated to each equivalence class $[x_i,\ f(x_i)]$. Obviously, each element of the object set $\mathbf{G}^{(0)}$ corresponds to the morphism $gg^{-1} \in \mathbf{G}$, thus the set of morphisms $\mathbf{G}$ contains the identity map.

Example 4.1.2. We return to Example 4.1.1 to illustrate the resulting structure and its relation to a groupoid. Given a graph $\mathit{\Gamma}$ as in Example 4.1.1 and the function f_a, we get the following structure (see Figure 4.5) and associated groupoid by letting the set of connected components of the graph $\mathit{\Gamma}$ to be the object set $\mathbf{G}^{(0)}$ of the groupoid $\mathbf{G}$.

$$\mathbf{G}^{(0)} \cong \{C_1, C_2, C_3\}$$

Set of morphisms $\mathbf{G} \cong \{g_1, g_2, g_3\}$. Maps $s, t : \mathbf{G} \to \mathbf{G}^{(0)}$, defined by the function f are

$$s(\mathsf{g}_1) = C_1,\ s(\mathsf{g}_2) = C_2,\ s(\mathsf{g}_3) = C_3 \text{ and } t(\mathsf{g}_1) = t(\mathsf{g}_2) = C_1,\ t(\mathsf{g}_3) = C_2$$

Clearly, in this case the set $\mathbf{G}^{(2)}$ of all composable pairs is generated by: $(g_1, g_1); (g_2, g_1); (g_3, g_2)$. It becomes obvious from figure 4.5 that C_1 and C_2 can be connected through the level $\mathcal{L}_1$ and C_3 can be connected with C_1 or C_2 through the level $\mathcal{L}_2$ in this example.

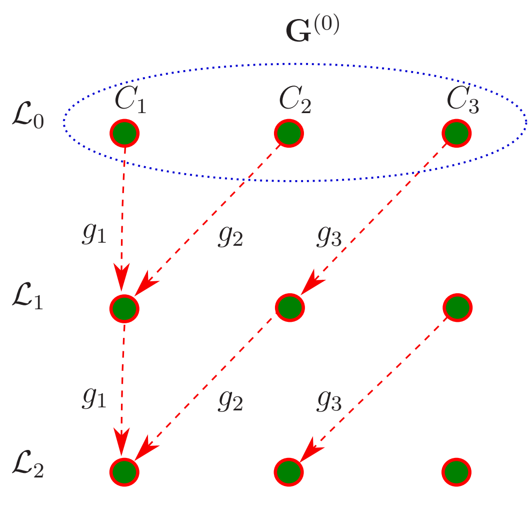


Figure 4.5: Illustrates the structure $\mathcal{L}$ related to the groupoid $\mathbf{G}$ and composable pairs which form the set $\mathbf{G}^{(2)}$

We can now reformulate definition 3.3.1 in the following way.

Definition 4.1.1. Two objects $x \neq y \in \mathbf{G}^{(0)}$ of the groupoid $\mathbf{G}$ are *linked* if there exist two morphisms $f = f_1 f_2 ... f_k, g = g_1 g_2 ... g_k \in \mathbf{G}$ such that

$$s(f) \neq s(g) \text{ and } t(f) = t(g)$$

with $t(f_i) \neq t(g_i)$ for all $1 \leq i < k$.

Definition 3.3.1 and definition 4.1.1 are equivalent if we have

$$\mathbf{G}^{(0)} \cong \{C_i\}, \text{ where } \sqcup\{C_i\} = \Gamma$$

4.1.3 Connectedness

Two *linked* objects obviously belong to the same *orbit* in the sense of the classical definition(see section 2.8). However, due to our specific goal, we shall only be interested in *linked* objects as defined above. To see the difference, note that for any two linked objects $x, y \in \mathbf{G}^{(0)}$, according to the classical definition, there exists a morphism $\mathsf{g} \in \mathbf{G}$ such that $s(\mathsf{g}) = x$ and $t(\mathsf{g}) = y$. However, in our case the morphism $g \in \mathbf{G}$ must be of the form:

$$\mathsf{g} = f_1^{\varepsilon_1} f_2^{\varepsilon_2} ... f_k^{\varepsilon_k} g_k^{\varepsilon_{k+1}} g_{k-1}^{\varepsilon_{k+2}} ... g_1^{\varepsilon_{2k}}, \text{ where } g_1, ..., g_k, f_1, ..., f_k \in \mathbf{G};$$
$$\varepsilon_1, ..., \varepsilon_n \in \{\pm 1\}$$

with the property that

$$\sum_{i=1}^{2m} \varepsilon_i = 0$$

for any subsequence $\mathsf{g}_m = f_1^{\varepsilon_1} f_2^{\varepsilon_2} ... f_m^{\varepsilon_m} g_m^{\varepsilon_{m+1}} g_{m-1}^{\varepsilon_{m+2}} ... g_1^{\varepsilon_{2m}}$, $m \leq k$.

Definition 4.1.2. We shall say that two *linked* objects form the *V-link*.

Define the length of any finite sequence $g = g_1g_2...g_n$ of composable morphisms $g_i \in \mathbf{G}$ as the number of elements in it.

$$|g| = n$$

Definition 4.1.3. The distance between two objects $x, y \in \mathbf{G}^{(0)}$ that belong to the same orbit is the length of a minimal sequence $g = g_1g_2...g_n \in \mathbf{G}$, such that $s(g) = x$ and $t(g) = y$

Note that for any two linked objects x, y we can distinguish two sequences of the same length.

$$g^+ = g_1g_2...g_k \text{ and } g^- = g_{k+1}^{-1}g_{k+2}^{-1}...g_n^{-1};\ (n = 2k).$$

Replacing each element of g^- with the inverse morphism and reversing the order of elements, we receive a new sequence $(g^-)^{-1} = h_1h_2...h_k$ with $h_i = g_{n-i+1}^{-1}$. Reassigning terms, we get the new pair $\gamma = (\gamma_1, \gamma_2)$, with $\gamma_1 = g^+$ and $\gamma_2 = (g^-)^{-1}$

We shall call this pair of sequences $\gamma = (\gamma_1, \gamma_2)$ - a *link-pair*

Definition 4.1.4. The object set $\mathbf{G}^{(0)}$ is *connected* if any two objects from $\mathbf{G}^{(0)}$ are *linked*.

Consider the groupoid $\mathbf{G} = \{\mathbf{G}^{(0)}, \mathbf{G}, s, t\}$, related to the *maze-graph* $\mathcal{L}^* = \{\mathit{\Gamma}, T(\mathcal{A}), \{f_a : a \in \mathcal{A}\}\}$, as shown above, with $|\mathbf{G}^{(0)}| = N$ and $|\mathcal{A}| = 1$.

Theorem 4.1.1. *The length of any V-link does not exceed* $2(N - 1)$.

Proof. According to *Proposition* 3.3.3 formulated above, in order to *link* objects from $\mathbf{G}^{(0)}$ we must have at least one pair of morphisms $g_1 \neq g_2 \in \mathbf{G}$

such that $s(g_1) \neq s(g_2)$ and $t(g_1) = t(g_2)$. Then, for certain objects $x, y \in \mathbf{G}^{(0)}$, there must exist a *link-pair* $\gamma = (\gamma_1, \gamma_2)$ such that $s(\gamma_1) \neq s(\gamma_2)$ and $t(\gamma_1) = t(\gamma_2)$. The sequences $\gamma_1 = g_1 g_2 ... g_n$ and $\gamma_2 = h_1 h_2 ... h_n$ of composable morphisms from $\mathbf{G}$ must satisfy the obvious condition that

$$t(g_i) \neq t(h_i), \forall (1 \leq i < n)$$

Further, considering the fact that a *link-pair* emerges from any pair of morphisms (g_i, h_j) $(1 \leq i, j \leq n)$ with the property that $s(g_i) \neq s(h_j)$ and $t(g_i) = t(h_j)$, each time the target destination ($t(g_i)$ or $t(h_j)$) in $\mathbf{G}^{(0)}$ is assigned to certain element (starting at certain point $s(g_i)$, $s(h_j)$), this same destination can't be chosen again starting at any different point, in order to avoid creating a new *link-pair* (see Figure 4.6). Following this pattern,

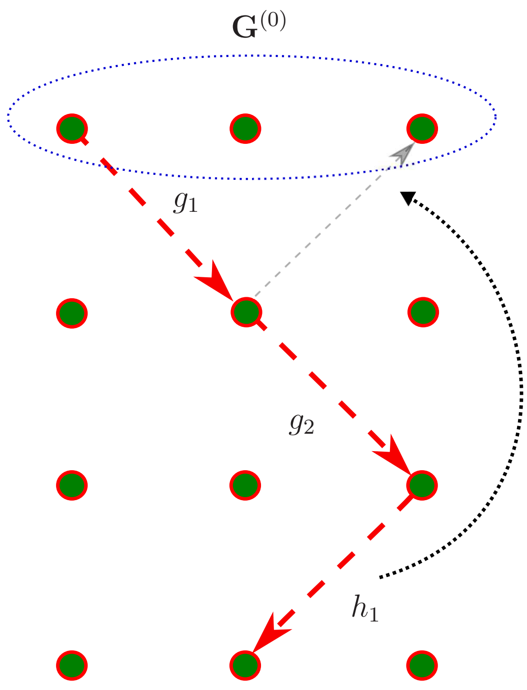


Figure 4.6:

one can deduce that the maximal length of a sequence γ_i of the viewed *link-*

pair $\gamma = (\gamma_1, \gamma_2)$ is $|\gamma_i| = N - 1$ when two sequences $\gamma_1 = g_1 g_2 ... g_n$ and $\gamma_2 = h_1 h_2 ... h_n$ satisfy the following conditions:

$$s(g_i) = t(g_i),\ \forall i$$

$$s(h_i) \neq t(h_i),\ \forall i$$

Thus, the diameter of any *zero-orbit* of any object $x \in \mathbf{G}^{(0)}$ with $|\mathbf{G}^{(0)}| = N$ can not exceed $2(N-1)$ □

Thus, besides the groupoid that can be naturally generated on the set of vertices and existing edges of any graph, we can generate another groupoid, that does not change the initial graph and has a very useful property that it connects any finite disconnected graph (with finite number of connected components) in a finite number of steps.

4.2 Multi-element Alphabet

We now move on to the general case. Consider an alphabet $\mathcal{A}$ of cardinality $M \geq 2$. Let $\mathit{\Gamma}$ be a finite disconnected graph with N connected components $\{C_j | j = 1, 2, 3, ..., N\}$ and set $\{f_a : a \in \mathcal{A}\}$ a set of partial bijections between subsets of $\mathcal{V}(\mathit{\Gamma})$. Further, we perform the Cartesian product operation $\mathit{\Gamma} \square T(\mathcal{A})$ to get an infinite (self-similar) graph $\mathcal{L}$. Finally, we delete all vertical edges from the set $\mathcal{E}(\mathcal{L})$ and insert new directed edges (arrows), formed by all pairs

$$((v, \mathbf{w}); (f_a(v), \mathbf{w}a)),$$

$$v \in D_a \subset \mathcal{V}(\mathit{\Gamma}_{\mathbf{w}}),\ f_a(v) \in R_a \subset \mathcal{V}(\mathit{\Gamma}_{\mathbf{w}a}),\ \mathbf{w} \in \mathcal{W}^*,\ a \in \mathcal{A}$$

Again, as a result of this process, we obtain a new structure which we call the *maze-graph*

$$\mathcal{L}^* = \mathcal{L}^*(\Gamma; T(\mathcal{A}); \{f_a : a \in \mathcal{A}\})$$

This new structure $\mathcal{L}^*$ is itself an infinite self-similar graph. We shall call the modified product $\Gamma \tilde{\Box} T(\mathcal{A})$ (which is $\Gamma \Box T(\mathcal{A})$ with all its *vertical* edges removed) - the *base graph set*, and the set of functions $\{f_a : a \in \mathcal{A}\}$ - *the generating function set* of the *maze-graph* $\mathcal{L}^*$.

4.2.1 Relation to Pseudogroups

This new structure can be related to the pseudogroup of morphisms generated by the set $\{f_a\} \cup \{\mathrm{id}_\Gamma\}$ on sets of vertices $\{D_a, R_a\}$ $(\forall a \in \mathcal{A})$ of the original graph Γ.

Here, each partial bijection f_a maps two subsets of $\mathcal{V}(\Gamma)$ as follows

$$f_a : D_a \to R_a$$

Classical definition of the pseudogroup implies that two morphisms, $f : D_f \to R_f$ and $g : D_g \to R_g$, are composable if $R_f \cap D_g \neq \emptyset$ and the composition is

$$f \circ g : f^{-1}(R_f \cap D_g) \to g(R_f \cap D_g)$$

However, we shall use a weaker condition for two morphisms to be composable. We do not need the intersection $R_f \cap D_g$ to be non-empty. It is sufficient for us that some of the elements of R_f and D_g belong to the same connected component. This is due to the geometric nature of our structure

$\mathcal{L}^*$ and the set goal to establish whether graph Γ can be connected using the set $\{f_a : a \in \mathcal{A}\}$.

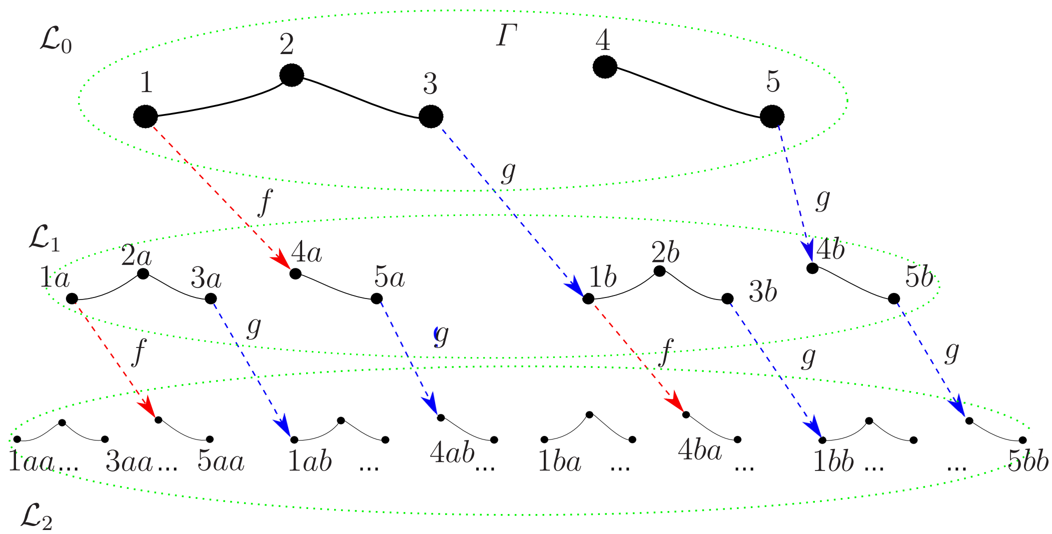


Figure 4.7: Structure allows composable pairs (f, g); (g, g) and (g, f)

For example, the structure presented in Figure 4.7 allows for the "composition" $f \circ g$ even though $R_f \cap D_g = \emptyset$. Clearly, the "composition" $f \circ g$ is possible because the elements from the range of function f simply belong to the same connected component with the elements that form the domain of the function g.

4.2.2 Transportation Map

We shall be more interested in *reachability* of the set of vertices D_g from R_f rather than the exact intersection $R_f \cap D_g$. The *reachability* in our case can be defined similarly to the one used in Discrete Time Markov Chains. Following [Nor97], we shall say that the vertex $y \in \mathcal{V}(\Gamma)$ is reachable from the vertex $x \in \mathcal{V}(\Gamma)$ if there exist a finite sequence of vertices

$$x = v_0, v_1, ..., v_n = y$$
$$\text{such that } v_i \in \mathcal{V}(\Gamma),\ \forall 0 \leq i \leq n$$
$$\text{and } (v_{i-1}, v_i); (v_i, v_{i+1}) \in \mathcal{E}(\Gamma),\ \forall 1 \leq i \leq n.$$

Since our initial graph Γ is finite and disconnected, it is obvious that for any two vertices $x, y \in \mathcal{V}(\Gamma)$, we have that y is always reachable from x if they both belong to the same connected component, and y is not reachable from x otherwise.
Obviously y being reachable from x is an equivalence relation.

This situation can be thought of as a certain transportation map. Imagine, we need to get to Washington, DC from Ottawa, ON. Say there are no direct flights on the desired date of departure. Thus, we first take a flight from Ottawa (YOW) to Toronto (YTZ). Even-though there are many departure spots within the same airport (YTZ), we have to move around the YTZ and find the departure spot to Washington (IAD), then take a second flight. In this regard, tree airports YOW, YTZ and IAD present three copies of the same object (initial graph Γ). Whence, it is possible for us to "compose" two flights due to the reachability of the departure spot to the Washington (IAD) from the arrival spot in Toronto (YTZ), and despite the fact that our arrival spot at YTZ does not coincide with the departure spot to IAD.

Remark. Although functions in the set $\{f_a : a \in \mathcal{A}\}$ are defined as partial bijections, their compositions can't be considered as functions. Instead, the composition $f \circ g$ can be described as a *partial binary operation*

$$f \circ g : \Gamma \times \Gamma \to \Gamma,$$

which has its domain and co-domain in D_f and D_g and generates the result under the condition that D_g is reachable from R_f.

We now need to give a few definitions to formalize this observation.

Definition 4.2.1. For any vertex x of the graph Γ, $x \in \mathcal{V}(\Gamma)$, its *neighborhood* $\mathcal{N}(x)$ is the equivalence class $[x]$ of vertices in Γ (or equivalently, the set $\{y_i\}$ of all vertices of Γ, reachable from x).

$$\forall x \in \mathcal{V}(\Gamma),\ [x] = \mathcal{N}(x)$$

Definition 4.2.2. For a set of vertices $X = \{x_1, x_2, ..., x_n\} \in \mathcal{V}(\Gamma)$, the *neighborhood* of X is the union of all *neighborhoods* $\mathcal{N}(x_i)$ $(1 \leq i \leq n)$

$$\mathcal{N}(X) = \bigcup_{i=1}^{n} \mathcal{N}(x_i)$$

Thus, in our case, we shall say that two functions f and g are composable if the neighborhoods of R_f and D_g have nonempty intersection.

Definition 4.2.3. For two functions $f, g \in \mathcal{L}^*$, such that $f : D_f \to R_f$ and $g : D_g \to R_g$ the composition $f \circ g$ exists if $\mathcal{N}(R_f) \cap \mathcal{N}(D_g) \neq \emptyset$, with the composition $f \circ g : X \times Y \to Z$ being a *partial binary operation*, where

$$f^{-1}(X) \subset D_f;\ Y \subset D_g;\ Z \subset R_g,$$

with each map $f \circ g : (x, y) \mapsto z$ defined under condition that y is *reachable* from x.

From now on, the defined above composition is the one we are talking about and not the composition of two functions in its classical meaning. It is also worth noting that writing a sequence of composable functions $f_1 f_2 ... f_k$ we mean that any pair $f_i \circ f_{i+1}$ $(1 \leq i < k)$ is defined as the partial binary operation in the sense of Definition 4.2.3.

Since connectedness of the *maze-graph* $\mathcal{L}^*$ is of the greatest interest to us, we first need to set some rules and define when two components of the original disconnected graph Γ are considered connected in $\mathcal{L}^*$. To avoid confusion, we shall call this type of setting up connection between two components/*neighborhoods* - *linking*. We now give the definition of *direct-linking* and *indirect-linking* of two *neighborhoods*.

Definition 4.2.4. A function $f \in \{f_a : a \in \mathcal{A}\}$ is *direct-linking* two *neighborhoods* $\mathcal{N}(x) \neq \mathcal{N}(y)$ $(x, y \in D_f)$ if $\mathcal{N}(f(x)) = \mathcal{N}(f(y))$. The *neighborhoods* $\mathcal{N}(x)$ and $\mathcal{N}(y)$ are said to be *direct-linked* by the function f.

If one views each pair $(x, f(x))$ as the directed edge of the structure $\mathcal{L}^*$, then *direct-linking* means writing a path

$$x \to f(x) \to ... \to f(y) \to y$$

that connects two components C_x, C_y $(x \in C_x, y \in C_y)$ of the initial graph Γ.

The following definition naturally arises from the previous one.

Definition 4.2.5. Two *neighborhoods* $\mathcal{N}(x) \neq \mathcal{N}(y)$ are *indirect-linked* if there exist two sequences, $\phi = f_{a_1} f_{a_2} ... f_{a_n}$ and $\psi = f_{a_1} f_{a_2} ... f_{a_n}$ of composable functions from $\{f_a : a \in \mathcal{A}\}$ such that (***i***) $x, y \in D_{f_{a_1}}$

(***ii***) $\mathcal{N}(\phi(x)) = \mathcal{N}(\psi(y))$

(***iii***) for any two subsequences $\phi_k = f_{a_1} f_{a_2} ... f_{a_k}$ and $\psi_k = f_{a_1} f_{a_2} ... f_{a_k}$ $(k < n)$, we have $\mathcal{N}(\phi_k(x)) \neq \mathcal{N}(\psi_k(y))$

A pair of sequences (ϕ, ψ) is *indirect-linking* the *neighborhoods* $\mathcal{N}(x)$ and $\mathcal{N}(y)$. We shall call such pair of sequences - a *link-pair*.

Clearly, it is possible that two *neighborhoods* $\mathcal{N}_1, \mathcal{N}_2$ are neither *direct-linked* nor *indirect-linked*. However, for such two *neighborhoods* there may exist a third *neighborhood* $\mathcal{N}_3$ that is *direct/indirect-linked* to each of the first two. In this case, we shall call *neighborhoods* $\mathcal{N}_1, \mathcal{N}_2$ - *co-linked*.

Definition 4.2.6. Two *neighborhoods* $\mathcal{N}_i \neq \mathcal{N}_j$ are called *co-linked* if there exists a *neighborhood* $\mathcal{N}_t$ that is *linked* to both *neighborhoods*, $\mathcal{N}_i$ and $\mathcal{N}_j$.

Note: Further, we shall simply write *linked* to refer to the case when two *neighborhoods* $\mathcal{N}_i \neq \mathcal{N}_j$ are *direct-linked*, *indirect-linked* or *co-linked* if the exact type of *linking* need not be specified.

Now having defined *the generating function set* $\{f_a : a \in \mathcal{A}\}$ of the structure $\mathcal{L}^*$ in such a way that allows to link any two *neighborhoods* from the set $\bigsqcup_{i=1}^{N}(\mathcal{N}_i) = \bigsqcup_{i=1}^{N}(C_i) = \Gamma$ means connecting the graph Γ and, consequently, connecting the whole *maze-graph* $\mathcal{L}^*$.

Definition 4.2.7. The set of *neighborhoods* $\{\mathcal{N}_i\}_{i=1}^{n}$ is *totally linked* if any two *neighborhoods* $\mathcal{N}_k, \mathcal{N}_j \in \{\mathcal{N}_i\}_{i=1}^{n}$ $(1 \leq k, j \leq n)$ are *linked*.

The length of any sequence $\phi = f_{a_1} ... f_{a_k}$ of composable functions from $\{f_a : a \in \mathcal{A}\}$ is simply defined as the number of elements in this sequence

$$\phi = f_{a_1}...f_{a_k} \Rightarrow |\phi| = k$$

Definition 4.2.8. The distance between two *linked* neighborhoods $\mathcal{N}_1$ and $\mathcal{N}_2$ is equal to the length of the shortest sequence $f = f_1 f_2 ... f_n$ of composable functions, such that $f : x \mapsto y$ for some $x \in \mathcal{N}_1$, $y \in \mathcal{N}_2$

For two *linked* neighborhoods, the sequence f is of the form $f = f_1^{\varepsilon_1} f_2^{\varepsilon_2} ... f_n^{\varepsilon_n}$, where $f_1 ... f_n \in \{f_a : a \in \mathcal{A}\}$ and $\varepsilon_1 ... \varepsilon_n \in \{\pm 1\}$, with the property that $\sum_{i=1}^{n} \varepsilon_i = 0$.
Moreover, for *direct/indirect-linked* neighborhoods, the sequence is of the form $f = f_1 f_2 ... f_k f_k^{-1} ... f_2^{-1} f_1^{-1}$ (with $|f| = n = 2k$). Two subsequences $f^+ = f_1 f_2 ... f_k$ and $(f^-)^{-1} = f_1 f_2 ... f_k$ form a *link-pair.*
The length of the *link-pair* (ϕ, ψ) will also be defined as $|(\phi, \psi)| = 2k$ for the pair of sequences $\phi = f_{a_1}...f_{a_k}$, $\psi = f_{a_1}...f_{a_k}$.

We shall say that the pair (ϕ_k, ψ_k) is a *sub-pair* of (ϕ, ψ) if ϕ_k and ψ_k are respective subsequences of ϕ and ψ.
For the purpose of this chapter, we shall also define the distance between two vertices x and y to be zero if these vertices belong to the same *neighborhood* (or equivalently, if $[y] = [x]$)

$$\mathcal{N}(x) = \mathcal{N}(y) \Leftrightarrow |y - x| = 0$$

If $\mathcal{N}(x) \neq \mathcal{N}(y)$ and $\mathcal{N}(x)$ is *direct/indirect linked* to $\mathcal{N}(y)$, then the distance between x and y is the length of a shortest *link-pair* $(\phi(x), \psi(y))$.
If two *neighborhoods* $\mathcal{N}(x)$ and $\mathcal{N}(z)$ are *co-linked* by the *neighborhood* $\mathcal{N}(y)$, then the distance between x and z is equal to the sum of distances $|z - x| = |y - x| + |z - y|$.

Definition 4.2.9. Two neighborhoods belong to a *V-link* if they are *direct-linked* or *indirect-linked*

4.2.3 Connectedness

It turns out that if the set of *neighborhoods* $\{\mathcal{N}_i\}_{i=1}^N$ in the described setup for the *maze-graph* $\mathcal{L}^*$ can be *totally linked* then the length of any *V-link* does not exceed $2(\binom{N-1}{2}+1)$, where N is the number of connected components in the initial disconnected graph Γ.

Theorem 4.2.1. *The length of any V-link in the connected maze-graph* $\mathcal{L}^* = \{\Gamma; T(\mathcal{A}); \{f_a : a \in \mathcal{A}\}\}$ *does not exceed* $2(\binom{N-1}{2}+1)$.

Proof. First note that by Proposition 3.3.3 there must be at least one function $f \in \{f_a\}$ which *direct-links* two *neighborhoods* from $\{\mathcal{N}_i\}_{i=1}^N$. Without loss of generality, assume that $\mathcal{N}_1$ and $\mathcal{N}_2$ are *direct-linked* by the function f_1. Suppose that no other function from $\{f_a\}$ is *direct-linking* any two *neighborhoods*. Also suppose that the function f_1 is only *direct-linking* $\mathcal{N}_1$ and $\mathcal{N}_2$ and no other two *neighborhoods*. Then, for any $\mathcal{N}_{i_0}, \mathcal{N}_{j_0}$ $(1 \leq i_0, j_0 \leq N)$, we must have the *link-pair* (ϕ, ψ) with

$$\phi = f_{a_1} f_{a_2} ... f_{a_{k-1}} f_1 \text{ and } \psi = f_{a_1} f_{a_2} ... f_{a_{k-1}} f_1$$

that is *indirect-linking* $\mathcal{N}_{i_0}$ and $\mathcal{N}_{j_0}$.

Note that we can define a sequence of neighborhoods $\mathcal{N}_{i_1}\mathcal{N}_{i_2}...\mathcal{N}_{i_{k-1}}\mathcal{N}_{i_k}$ for each sequence of composable functions $f_{a_1} f_{a_2} ... f_{a_{k-1}} f_k$, with the property that $f_{a_m}(x) \in \mathcal{N}_{i_m}$.

As it has been shown before (Section 3.2.1, Section 3.2.3), the maze-graph $\mathcal{L}^* = \mathcal{L}^*(\Gamma; T(\mathcal{A}); \{f_a : a \in \mathcal{A}\})$ is quasi-isometric to the underlying regular rooted tree $T(\mathcal{A})$. One of the properties of $T(\mathcal{A})$ is that any subtree T_x (with vertex $x \in T(\mathcal{A})$ being its root) is isomorphic to the whole tree $T(\mathcal{A})$. Setting up the map $\mathbf{w'} \mapsto \mathbf{ww'}$ we obtain the identification of our maze-graph $\mathcal{L}^*$ with its subgraph $\mathcal{L}^*_{\Gamma_{\mathbf{w}}}$. Consequently, any subgraph $\mathcal{L}^*_{\Gamma_{\mathbf{w}}}$ (with $\Gamma_{\mathbf{w}}$ being its "root") of the maze graph $\mathcal{L}^*$ is isomorphic to the whole maze-graph.

Consequently, the following is true for the link-pair (ϕ, ψ):

If one replaces two sequences

$$\phi = f_{a_1} f_{a_2} ... f_{a_{k-1}} f_1 \text{ and } \psi = f_{a_1} f_{a_2} ... f_{a_{k-1}} f_1$$

with the corresponding sequences of neighborhoods

$$\phi^* = \mathcal{N}_{i_1} \mathcal{N}_{i_2} ... \mathcal{N}_{i_{k-1}} \mathcal{N}_{i_k}$$
$$\psi^* = \mathcal{N}_{j_1} \mathcal{N}_{j_2} ... \mathcal{N}_{j_{k-1}} \mathcal{N}_{j_k}$$

and finds that $\mathcal{N}_{i_n} = \mathcal{N}_{i_m}$, and at the same time $\mathcal{N}_{j_n} = \mathcal{N}_{j_m}$, one can replace a pair $(\mathcal{N}_{i_n},\ \mathcal{N}_{j_n})$ in sequences $\phi^*,\ \psi^*$ with the pair $(\mathcal{N}_{i_m},\ \mathcal{N}_{j_m})$ and delete all elements of both sequences $\phi^*,\ \psi^*$ between those pairs of neighborhoods. Thus, one can reduce both sequences ϕ and ψ by all elements between $f_{a_{n-1}}$ and $f_{a_{m-1}}$ without changing the fact that the reduced link pair (ϕ', ψ') is indirect-linking two above specified neighborhoods $\mathcal{N}_{i_0},\ \mathcal{N}_{j_0}$. This is due to the above described property that the subgraph $\mathcal{L}^*_{\Gamma_{\mathbf{w}}}$ is isomorphic to the whole maze-graph $\mathcal{L}^*$ and also to the fact that our sequences $\phi,\ \psi$ were built upon the reachability principle rather than classical function composition principle.

It follows that two sequences ϕ^* and ψ^* can only contain non-repeating pairs of elements $(\mathcal{N}_{i_n},\ \mathcal{N}_{j_n})$ to be irreducible. Clearly, that is $\binom{N-1}{2}+1$ elements, where N is the number of connected components in the initial disconnected graph Γ.

To see this, note that we have $\binom{N-1}{2}$ choices for the initial neighborhoods $\mathcal{N}_{i_0},\ \mathcal{N}_{j_0}$ which reflects the above convention that two neighborhoods $\mathcal{N}_1,\ \mathcal{N}_2$ were direct-linked and thus can not be chosen. Also, once we exhausted all possible choices of pairs of neighborhoods after making $\binom{N-1}{2}$ steps, we still have to include f_1 into our sequence due to the above convention that it is the only function from the set $\{f_a : a \in \mathcal{A}\}$ that is direct linking two neighborhoods.

It follows that the corresponding sequences $\phi,\ \psi$ of the link-pair (ϕ, ψ) can also consist of at most $\binom{N-1}{2}+1$ elements each, to be irreducible.

Whence, the diameter of any *zero-orbit* in the connected *maze-graph* $\mathcal{L}^*$ does not exceed $2(\binom{N-1}{2}+1)$.

□

4.2.4 Conclusions and Future Directions

Obviously, the connectedness of the structure $\mathcal{L}^*$ greatly depends on the particular choice of functions in the set $\{f_a : a \in \mathcal{A}\}$. However, it is worth noting that the conditions formulated above in the section 3.3, are in-line with those defined for pseudogroup (definition 2.9.1).

Theorem 4.2.1 above shows that for a certain choice of functions f_a, $(a \in \mathcal{A})$ the connectedness is either reached in a finite number of steps or is impossible to reach. This is due to the fact that $\mathcal{L}^*$ is a self-similar graph for which any subgraph that starts at $\Gamma_{\mathbf{w}}$ ($\mathbf{w} \in \mathcal{W}^*$) is isomorphic to the whole structure.

The exact algebraic structure of the obtained *maze-graph* $\mathcal{L}^*$ is the one to be defined. As it was shown, although it is very close to both, groupoid and pseudogroup structures, it can not be fully associated to neither of those.

Another topic that was of great interest but was not included into this thesis is to define a class of self similar groups that act on *maze-graph* and study the properties of this group action. It is our intention to use methods developed in [Nek05] , [Kai09], [Kai04] for the future study of the obtained structure.

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