Topics in many-valued and quantum algebraic logic

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Abstract

Introduced by C.C. Chang in the 1950s, MV algebras are to many-valued (Łukasiewicz) logics what boolean algebras are to two-valued logic. More recently, effect algebras were introduced by physicists to describe quantum logic.

In this thesis, we begin by investigating how these two structures, introduced decades apart for wildly different reasons, are intimately related in a mathematically precise way.

We survey some connections between MV/effect algebras and more traditional algebraic structures. Then, we look at the categorical structure of effect algebras in depth, and in particular see how the partiality of their operations cause things to be vastly more complicated than their totally defined classical analogues.

In the final chapter, we discuss coordinatization of MV algebras and prove some new theorems and construct some new concrete examples, connecting these structures up (requiring a detour through the world of effect algebras!) to boolean inverse semigroups.

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Summary of new contributions

Corrections and details

These results are not new per se, but in each of the following, we correct nontrivial errors or fill in gaps in the existing literature.

- Propositions 1.1.17 and 1.2.11

  (Major) — These classifications of the MV and effect maps $[0,1] \to [0,1]$ and $[0,1]^2 \to [0,1]$ are folklore, presumably known to researchers in the field, but neither proof nor statement is to be found anywhere.

- Theorem 1.4.3 and Corollary 1.4.6

  (Major) — This is an important result about the categorical isomorphism between MV algebras and a certain subclass of effect algebras, first discovered in 1997 and which is frequently cited. However, no correct and complete proof of the correct statement is to be found anywhere to date. See remarks following the theorem for more details.

- Definition/Proposition 2.1.2

  (Minor) — Routine proof of how to obtain an interval effect algebra from partially ordered abelian groups which has not been explicitly written out.

- Theorem 2.1.5

  (Minor) — An oversight in a uniqueness clause in a definition of the universal group of an effect algebra, which causes the original proof of its existence/uniqueness to have some erroneously leaps in logic near the end.

New results

These are original results, constructions, and insights of the author.

- Examples 3.2.14, 3.2.15, and 3.2.16

  These examples shed light on the intuition behind the abstract construction of coequalizers of effect algebras, and may be useful towards finding a general cleaner description.

- Proposition 3.3.1, Lemma 3.3.6, Theorem 3.3.7

  This section gives a classification of the monomorphisms and regular monomorphisms of effect algebras. The current widespread definition of "monomorphism" in the present literature fails to coincide with the categorical notion, and is therefore confusing at best and wrong at worst.

- Definition 4.3.1 and Theorem 4.3.4

  This is a direct and intuitive coordinatization of the MV algebra of rationals in $[0,1]$, answering a problem left open in [LS14].

- Theorems 4.4.1 and 4.4.2

  Decomposition theorems for the coordinatization of MV algebras, which are likely to be useful for future concrete examples of coordinatization.

- Lemmas 4.5.1, 4.5.2, and Theorem 4.5.4

  We give a first example of a coordinatization of an MV algebra which does not embed into $[0,1]$, the Chang algebra, and highlight in particular the usefulness of the decomposition theorems in the thought process that led to the idea.

Contents

Chapter 0

A brief history of many-valued logics

Since the days of Aristotle, it has been questioned whether traditional two-valued logic is truly sufficient. The father of logic himself considered statements such as "There will be a sea battle tomorrow." What is the truth value of such a statement? There could well be two navies preparing to clash on the high seas, but what if there were a nontrivial probability that a hurricane wipes out the ships before they can engage? One could then neither say that the statement is true nor that it is false!

Many-valued logics originally arose in the early twentieth century. They were introduced by the Polish logician Jan Łukasiewicz, initially as a three-valued logic (which can already answer Aristotle's problem with a perhaps unsatisfactory third truth value of "I dunno.") This was then generalized to n-valued finite logics, before being, as it nearly always happens in mathematics, generalized to the infinite case.

If we allow truth values in the entire unit interval $[0, 1]$, and a weather forecasting station that we believe is reputable says that the chance of the hurricane happening tomorrow is 27%, and we assume this is the only thing standing in the way of the silly humans who have nothing better to do than kill each other, then we can assign a truth value of 0.73 to the statement.

A particularly interesting application of finite n-valued logic is *Ulam's game* with lies or errors. In the traditional version of this two-player game, player A chooses a particular number x_0 in some search space S (some fixed set of numbers), and player B must try to guess the number by asking only yes-or-no questions. If player A is always telling the truth, then the truth value associated to any $x \in S$ at a particular state of the game is 0 if any of the answers player A has given player B would rule out the possibility that $x_0 = x$; otherwise, the truth value is 1. Note a truth value of 1 does not mean that x *is* x_0, just that it is still *possible* given the current information.

Now suppose that player A is a finitely-charged Pinocchio who may lie up to n times for some fixed n. It is now necessary for player B to receive a total of $n+1$ answers from player A that rule out the possibility that $x = x_0$ before he can really rule it out — since the first n times could just be player A using up his lies. Hence, we are now working in a logic with $n+2$ truth values $\{0, \frac{1}{n+1}, \ldots, \frac{n}{n+1}, 1\}$. The truth value now measures how *close* we are to being able to rule out a particular number. The interested reader is encouraged to consult [CDM00, Chapter 5] for further details.

In classical many-valued logic, there are two key, but subtly different interpretations — the truth values could be expressing either *probabilities* of truth (as the example of the weather forecast), or *degrees* of truth (as the example with Ulam's game). The difference is that in the former, there is an uncertainty to the information which is not present in the latter. For the reader who is interested in more on this, or other topics on the pure logic side of the coin, we recommend [HÓ0].

Logics with infinitely many truth values have recently become of great interest due to the rapidly exploding fields of quantum physics/mechanics/computing. Since the qubit is not binary and exists as a superposition of 0s and 1s, it is natural again to consider truth values in $[0, 1]$. The twist, when compared to classical many-valued logics, is that the operations in question become partial — in order to account for the quantum notion that some pairs of events are not simultaneously observable.

In this thesis, we study MV algebras and effect algebras, which are to classical and quantum many-valued logics what boolean algebras are to traditional two-valued logic. MV algebras were introduced in the 1950s by C.C. Chang to directly generalize boolean algebras to the many-valued setting, whilst effect algebras were introduced in the 1990s by M. Bennett and D. Foulis and arose from quantum measurement theory and quantum probability theory.

Chapter 1

Overview of effect algebras and MV algebras

It was not immediately evident that MV algebras and effect algebras, which were introduced in different eras by researchers in different fields, had anything to do with each other. It turns out, however, that they are intimately linked in the sense that MV algebras are just a particular subclass of effect algebras! The main goal of this chapter is to give an introduction and overview of both types of algebras, culminating in the theorems that tie them together.

Rather than follow the standard historical ordering, we follow the standard mathematical ordering and begin with the most general structures, and then proceed to the specialized cases.

Notation: When dealing with partial operations, we write "$A\downarrow$" to mean "A is defined", and "$A\uparrow$" to mean "A is undefined". Equations involving partial operations implicitly assume the terms involved are defined, even if not explicitly specified.

1.1 Effect algebras

We begin by giving two equivalent formulations of the definition of an effect algebra via a more general construction, and then give some first examples of effect algebras. We then define effect algebra homomorphisms and the category of effect algebras.

Definition 1.1.1 (Partial commutative monoid)**.** A *partial commutative monoid*, or PCM, is a triple $(M, \widetilde{\oplus}, 0)$ wherein M is a set, $\widetilde{\oplus} : M \times M \to M$ is a partial binary operation, and $0 \in M$ is a distinguished element, such that the following hold for all $a, b, c \in M$:

(PCM 1) *Commutativity*: if $a \mathbin{\widetilde{\oplus}} b \downarrow$, then $b \mathbin{\widetilde{\oplus}} a \downarrow$, and $a \mathbin{\widetilde{\oplus}} b = b \mathbin{\widetilde{\oplus}} a$.

(PCM 2) *Associativity*: if $b \mathbin{\widetilde{\oplus}} c \downarrow$ and $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) \downarrow$, then $a \mathbin{\widetilde{\oplus}} b \downarrow$, $(a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c \downarrow$, and $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) = (a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c$.

(PCM 3) *Zero law*: $a \mathbin{\widetilde{\oplus}} 0 \downarrow$ and $a \mathbin{\widetilde{\oplus}} 0 = a$.

We define a binary relation $a \leq b$ to mean there exists some x such that $a \mathbin{\widetilde{\oplus}} x = b$.

Remark 1.1.2. It may appear at first glance from our definition that the definedness of triple sums for associativity is asymmetric. It follows easily from commutativity, however, that it is in fact strict — that is, if $(a \mathbin{\widetilde{\oplus}} b) \downarrow$ and $(a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c \downarrow$, then also $b \mathbin{\widetilde{\oplus}} c \downarrow$, $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) \downarrow$, and $(a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c = a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c)$. Alternatively, one can bake symmetry into the definition by using Kleene directed equality (which declares the left-hand side is defined if and only if the right-hand side is defined, and when this is the case then they are equal).

Lemma 1.1.3. *If $(M, \widetilde{\oplus}, 0)$ is a partial commutative monoid, then $\leq$ induces a pre-order on M.*

Proof. Reflexivity follows immediately from (PCM 3). Now suppose $a \leq b$ and $b \leq c$. Let x, y be elements such that $a \mathbin{\widetilde{\oplus}} x = b$ and $b \mathbin{\widetilde{\oplus}} y = c$. Then, $(a \mathbin{\widetilde{\oplus}} x) \mathbin{\widetilde{\oplus}} y = c$ and transitivity follows from (PCM 1)-(PCM 2). □

Definition 1.1.4 (Generalized effect algebra)**.** A *generalized effect algebra* is a partial commutative monoid $(G, \widetilde{\oplus}, 0)$ where the following hold for all $a, b, c \in G$:

(GEA 1) Cancellation law: If $a \mathbin{\widetilde{\oplus}} b = a \mathbin{\widetilde{\oplus}} c$, then $b = c$.

(GEA 2) Positivity law: If $a \mathbin{\widetilde{\oplus}} b = 0$ then $a = b = 0$.

Whenever $a \leq b$, we define $b \,\widetilde{\ominus}\, a = c$ where c is the unique element such that $b = a \,\widetilde{\oplus}\, c$. Note that $1 \,\widetilde{\ominus}\, a = a^{\perp}$.

Lemma 1.1.5. *If $(G, \widetilde{\oplus}, 0)$ is a generalized effect algebra, then $\leq$ induces a partial order on M.*

Proof. We have reflexivity and transitivity from Lemma 1.1.3. Suppose now that $a \leq b$ and $b \leq a$ and let x, y be the elements such that $a \,\widetilde{\oplus}\, x = b$ and $b \,\widetilde{\oplus}\, y = a$. Then $a \,\widetilde{\oplus}\, 0 = a = b \,\widetilde{\oplus}\, y = (a \,\widetilde{\oplus}\, x) \,\widetilde{\oplus}\, y = a \,\widetilde{\oplus}\, (x \,\widetilde{\oplus}\, y)$, whence by (GEA 1) we have $x \,\widetilde{\oplus}\, y = 0$ and so by (GEA 2) $x = y = 0$. Thus $a = a \,\widetilde{\oplus}\, 0 = b$. □

Definition 1.1.6 (Effect algebra)**.** An *effect algebra* is a quadruple $(E, \widetilde{\oplus}, (-)^{\perp}, 0)$ wherein $(E, \widetilde{\oplus}, 0)$ is a partial commutative monoid and $(-)^{\perp} \colon E \to E$ is a (total) unary operation called *orthocomplement* (we write $0^{\perp} = 1$, call 0 the *zero* of E, and call 1 the *unit* of E), such that the following hold for all $a \in E$:

(EA 1) Orthocomplement law: $a^{\perp}$ is the unique element satisfying $a \,\widetilde{\oplus}\, a^{\perp} = 1$.

(EA 2) Zero-one law: if $a \,\widetilde{\oplus}\, 1 \downarrow$, then $a = 0$.

Remark 1.1.7. The uniqueness of (EA 1) together with commutativity yield $a = (a^{\perp})^{\perp}$.

Proposition 1.1.8. *A partial commutative monoid $(M, \widetilde{\oplus}, 0)$ is an effect algebra if and only if it is a generalized effect algebra together with a greatest element $1 = 0^{\perp}$.*

Proof. Suppose $(M, \widetilde{\oplus}, (-)^{\perp}, 0)$ is an effect algebra. Then:

(a) If $a \,\widetilde{\oplus}\, b = a \,\widetilde{\oplus}\, c$, then $(a \,\widetilde{\oplus}\, c) \,\widetilde{\oplus}\, (a \,\widetilde{\oplus}\, b)^{\perp} = (a \,\widetilde{\oplus}\, b) \,\widetilde{\oplus}\, (a \,\widetilde{\oplus}\, b)^{\perp} = 1$. It follows that $c \,\widetilde{\oplus}\, (a \,\widetilde{\oplus}\, (a \,\widetilde{\oplus}\, b)^{\perp}) = 1$ so by the uniqueness of (EA 1) we get $c^{\perp} = (a \,\widetilde{\oplus}\, (a \,\widetilde{\oplus}\, b)^{\perp})$. Similarly, $b \,\widetilde{\oplus}\, (a \,\widetilde{\oplus}\, (a \,\widetilde{\oplus}\, b)^{\perp}) = 1$ so $b^{\perp} = (a \,\widetilde{\oplus}\, (a \,\widetilde{\oplus}\, b)^{\perp})$. Thus, $b = c$ and the cancellation law holds.

(b) If $a \,\widetilde{\oplus}\, b = 0$, then $a \,\widetilde{\oplus}\, b \,\widetilde{\oplus}\, 1 \downarrow$ and since $a \,\widetilde{\oplus}\, b \downarrow$ by assumption, then (PCM 1)-(PCM 2) give us $a \,\widetilde{\oplus}\, 1 \downarrow$ and $b \,\widetilde{\oplus}\, 1 \downarrow$. Then from (EA 2) we get $a = b = 0$, so the positivity law holds.

So $(M, \widetilde{\oplus}, 0)$ is a generalized effect algebra. Furthermore we have for all $a \in M$ that $a \mathbin{\widetilde{\oplus}} a^\perp = 1$, so $a \leq 1$ and 1 is indeed the greatest element.

Now suppose $(M, \widetilde{\oplus}, 0)$ is generalized effect algebra with greatest element 1. We define $a^\perp = 1 \mathbin{\widetilde{\ominus}} a$ for all $a \in M$. Then:

(a) By definition, $a^\perp$ is the unique element satisfying $a \mathbin{\widetilde{\oplus}} a^\perp = 1$, so the orthocomplement law holds.

(b) If $a \mathbin{\widetilde{\oplus}} 1 \downarrow$, then $1 \leq a \mathbin{\widetilde{\oplus}} 1$. But 1 is the greatest element so it follows from $\leq$ being a partial order that $0 \mathbin{\widetilde{\oplus}} 1 = 1 = a \mathbin{\widetilde{\oplus}} 1$, so by (GEA 1) we have $a = 0$. Thus, the zero-one law holds.

Therefore, $(M, \widetilde{\oplus}, (-)^\perp, 0)$ is an effect algebra. In particular, whenever dealing with an effect algebra, we now may also assume (GEA 1)-(GEA 2). □

Definition 1.1.9 (Sub-effect algebra)**.** Let $(E, \widetilde{\oplus}, (-)^\perp, 0)$ be an effect algebra. Then, a subset $F \subseteq E$ is a *sub-effect algebra* of E if the following conditions hold for all $a, b \in E$.

(SEA 1) $0, 1 \in F$.

(SEA 2) If $a \in F$, then $a^\perp \in F$.

(SEA 3) If $a, b \in F$ and $a \mathbin{\widetilde{\oplus}} b \downarrow$, then $a \mathbin{\widetilde{\oplus}} b \in F$.

Lemma 1.1.10. *Let E be an effect algebra. For all $a, b, c \in E$, the following hold.*

(a) $a \mathbin{\widetilde{\oplus}} b \downarrow$ if and only if $a \leq b^\perp$.

(b) When defined, $(a \mathbin{\widetilde{\ominus}} b) \mathbin{\widetilde{\ominus}} c = a \mathbin{\widetilde{\ominus}} (b \mathbin{\widetilde{\oplus}} c)$.

(c) When defined, $a \mathbin{\widetilde{\ominus}} (b \mathbin{\widetilde{\ominus}} c) = (a \mathbin{\widetilde{\ominus}} b) \mathbin{\widetilde{\oplus}} c$.

(d) When defined, $(a \mathbin{\widetilde{\oplus}} b)^\perp = a^\perp \mathbin{\widetilde{\ominus}} b = b^\perp \mathbin{\widetilde{\ominus}} a$.

(e) $a \leq b$ if and only if $b^\perp \leq a^\perp$.

Proof. (a) Suppose $a \mathbin{\widetilde{\oplus}} b\downarrow$. Then, $b \mathbin{\widetilde{\oplus}} (a \mathbin{\widetilde{\oplus}} (a \mathbin{\widetilde{\oplus}} b)^{\perp}) = (a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} (a \mathbin{\widetilde{\oplus}} b)^{\perp} = 1$. By (EA 1), $b^{\perp} = a \mathbin{\widetilde{\oplus}} (a \mathbin{\widetilde{\oplus}} b)^{\perp}$, whence $a \leq b^{\perp}$.

Now suppose that $a \leq b^{\perp}$. Then there is $x \in E$ such that $a \mathbin{\widetilde{\oplus}} x = b^{\perp}$. So then, $(a \mathbin{\widetilde{\oplus}} x) \mathbin{\widetilde{\oplus}} b = 1$. From (PCM 1)-(PCM 2), we obtain $a \mathbin{\widetilde{\oplus}} b\downarrow$.

(b) By definition, $(a \mathbin{\widetilde{\ominus}} b) \mathbin{\widetilde{\ominus}} c$ is the unique element d satisfying

$$a \mathbin{\widetilde{\ominus}} b = c \mathbin{\widetilde{\oplus}} d,$$

and $a \mathbin{\widetilde{\ominus}} (b \mathbin{\widetilde{\oplus}} c)$ is the unique element d' satisfying

$$a = (b \mathbin{\widetilde{\oplus}} c) \mathbin{\widetilde{\oplus}} d'.$$

But also, $a \mathbin{\widetilde{\ominus}} b$ is the unique element e satisfying

$$a = b \mathbin{\widetilde{\oplus}} e.$$

So then $a = b \mathbin{\widetilde{\oplus}} e = b \mathbin{\widetilde{\oplus}} (c \mathbin{\widetilde{\oplus}} d')$. By (GEA 1), we have $e = c \mathbin{\widetilde{\oplus}} d'$. But also $e = a \ominus b = c \mathbin{\widetilde{\oplus}} d$. So (GEA 1) again gives us $d = d'$.

(c) Similar calculation as part (b).

(d) Recall that we can write $a^{\perp} = 1 \mathbin{\widetilde{\ominus}} a$. Thus, using part (b),

$$(a \mathbin{\widetilde{\oplus}} b)^{\perp} = 1 \mathbin{\widetilde{\ominus}} (a \mathbin{\widetilde{\oplus}} b) = (1 \mathbin{\widetilde{\ominus}} a) \mathbin{\widetilde{\ominus}} b = a^{\perp} \mathbin{\widetilde{\ominus}} b.$$

Similarly, $(a \mathbin{\widetilde{\oplus}} b)^{\perp} = (b \mathbin{\widetilde{\oplus}} a)^{\perp} = b^{\perp} \mathbin{\widetilde{\ominus}} a$.

(e) Suppose $a \leq b$. So there is $x \in E$ satisfying $a \mathbin{\widetilde{\oplus}} x = b$. Then, using part (d),

$$b^{\perp} = (a \mathbin{\widetilde{\oplus}} x)^{\perp} = a^{\perp} \mathbin{\widetilde{\ominus}} x.$$

This is equivalent to $b^{\perp} \mathbin{\widetilde{\oplus}} x = a^{\perp}$, whence $b^{\perp} \leq a^{\perp}$.

The proof of the converse is analogous.

□

Example 1.1.11. The closed interval $[0,1] \subseteq \mathbb{R}$, with $\widetilde{\oplus}$ being the usual addition of real numbers and $0, 1$ in their usual roles, is an effect algebra. We have $a \mathbin{\widetilde{\oplus}} b \downarrow$ if and only if $a + b \leq 1$ and the orthocomplement given by $a^\perp = 1 - a$. It is clear that the effect algebra order $\leq$ coincides with the standard ordering on $[0,1]$.

Considering instead the interval $[0,1)$ with the same operation and $a \mathbin{\widetilde{\oplus}} b \downarrow$ if and only if $a + b < 1$, we obtain, by dropping the existence of a top element and orthocomplementation, a generalized effect algebra (which is not an effect algebra).

In [Boo54, *p*.66], while discussing the operation $+$, George Boole wrote *"... the classes or things added together in thought should be mutually exclusive. The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common."* It would seem that according to Boole, disjunction *should* be a partial operation, defined only if the operands are indeed disjoint. This leads us to our next examples.

Example 1.1.12. Let X be a set. Then its power set $\mathcal{P}(X)$ is an effect algebra with $\widetilde{\oplus}$ being union of sets, and $A \mathbin{\widetilde{\oplus}} B \downarrow$ if and only if $A \cap B = \varnothing$. We have $\varnothing$ as 0, the full set X as 1, and $A^\perp = X \backslash A$.

Example 1.1.13. Effect algebras can be seen as generalizations of boolean algebras as follows, following the same idea as Example 1.1.12. Given a boolean algebra B, we obtain an effect algebra with $p \mathbin{\widetilde{\oplus}} q \downarrow$ if and only if $p \wedge q = \bot$, in which case $p \mathbin{\widetilde{\oplus}} q$ is $p \vee q$. Then, $p^\perp = \neg p$, $\bot$ is 0, and $\top$ is 1.

Example 1.1.14. Denote SA_n the $n \times n$ self-adjoint complex matrices, and $\langle -, - \rangle$ the standard inner product on $\mathbb{C}^n$. Define an order on SA_n by

$$M \leq N \Leftrightarrow \langle x, Mx \rangle \leq \langle x, Nx \rangle \text{ for all } x \in \mathbb{C}^n.$$

Then, *effects* are the self-adjoint matrices

$$\mathcal{E}_n = \{M \in \mathrm{SA}_n \mid 0 \leq A \leq I_n\},$$

which form an effect algebra with $\widetilde{\oplus}$ given by matrix addition and defined precisely when the sum is in $\mathcal{E}_n$. The origin of the definition of an effect algebra is due to Foulis and Bennett, as an abstraction from this example.

Definition 1.1.15 (Homomorphism (of effect algebras))**.** Let E, F be effect algebras. A function $f\colon E \to F$ is called an *effect algebra homomorphism* if it satisfies the following.

(EH 1) $f(1_E) = 1_F$.

(EH 2) If $a, b \in E$ and $a \mathbin{\widetilde{\oplus}} b \downarrow$, then $f(a) \mathbin{\widetilde{\oplus}} f(b) \downarrow$ and $f(a \mathbin{\widetilde{\oplus}} b) = f(a) \mathbin{\widetilde{\oplus}} f(b)$.

We denote the *category of effect algebras*, with objects effect algebras and arrows effect algebra homomorphisms, by **EA**.

Lemma 1.1.16. *Let $f\colon E \to F$ be an effect algebra homomorphism. Then:*

(a) f also preserves the bottom element; i.e. $f(0_E) = 0_F$.

(b) f preserves $\leq$; if $a, b \in E$ and $a \leq b$, then $f(a) \leq f(b)$.

(c) f preserves orthocomplements; if $a \in E$, then $f(a^\perp) = f(a)^\perp$.

Proof. (a) By (PCM 3), we know $1_E \mathbin{\widetilde{\oplus}} 0_E \downarrow$, and $1_E \mathbin{\widetilde{\oplus}} 0_E = 1_E$. Then by (EH 1)-(EH 2),

$$1_F = f(1_E) = f(1_E \mathbin{\widetilde{\oplus}} 0_E) = f(1_E) \mathbin{\widetilde{\oplus}} f(0_E) = 1_F \mathbin{\widetilde{\oplus}} f(0_E).$$

Then, (EA 2) forces $f(0_E) = 0_F$.

(b) Suppose $a \leq b$. Then, there exists c such that $a \mathbin{\widetilde{\oplus}} c = b$. Then by (EH 2), $f(a) \mathbin{\widetilde{\oplus}} f(c) \downarrow$, and $f(a) \mathbin{\widetilde{\oplus}} f(c) = f(a \mathbin{\widetilde{\oplus}} c) = f(b)$, so $f(a) \leq f(b)$.

(c) We have $f(a) \mathbin{\widetilde{\oplus}} f(a^\perp) = f(a \mathbin{\widetilde{\oplus}} a^\perp) = f(1_A) = 1_B$. By uniqueness of orthocomplements, $f(a^\perp) = f(a)^\perp$.

□

Proposition 1.1.17. *The effect algebra homomorphisms from $[0,1] \to [0,1]$ and $[0,1]^2 \to [0,1]$ (where we define the operation on $[0,1]^2$ pointwise; i.e. $(a,b) \mathbin{\widetilde{\oplus}} (c,d) \downarrow$ precisely when $a + c \leq 1$ and $b + d \leq 1$, in which case $(a,b) \mathbin{\widetilde{\oplus}} (c,d) = (a+c, b+d)$) are precisely as follows.*

(a) The only effect algebra homomorphism $[0,1] \to [0,1]$ is the identity.

(b) There are a continuum of effect algebra homomorphisms $[0,1]^2 \to [0,1]$, given, for $a \in [0,1]$, by

$$f_a(x,y) = ax + (1-a)y.$$

Proof. (a) Let $f\colon [0,1] \to [0,1]$ be an effect algebra homomorphism. We already know $f(0) = 0$ and $f(1) = 1$. We have for every $n \in \mathbb{N}^+$,

$$\begin{aligned} 1 &= f(1) \\ &= f(\underbrace{1/n \mathbin{\widetilde{\oplus}} 1/n \mathbin{\widetilde{\oplus}} \ldots \mathbin{\widetilde{\oplus}} 1/n}_{\text{n times}}) \\ &= \underbrace{f(1/n) \mathbin{\widetilde{\oplus}} f(1/n) \mathbin{\widetilde{\oplus}} \ldots \mathbin{\widetilde{\oplus}} f(1/n)}_{\text{n times}} \\ &= \underbrace{f(1/n) + f(1/n) + \ldots + f(1/n)}_{\text{n times}} \\ &= nf(1/n), \end{aligned}$$

so $f(1/n) = 1/n$.

Now, if $m < n$, then

$$\begin{aligned} f(m/n) &= f(\underbrace{1/n + 1/n + \ldots + 1/n}_{\text{m times}}) \\ &= f(\underbrace{1/n \mathbin{\widetilde{\oplus}} 1/n \mathbin{\widetilde{\oplus}} \ldots \mathbin{\widetilde{\oplus}} 1/n}_{\text{m times}}) \\ &= \underbrace{f(1/n) \mathbin{\widetilde{\oplus}} f(1/n) \mathbin{\widetilde{\oplus}} \ldots \mathbin{\widetilde{\oplus}} f(1/n)}_{\text{m times}} \\ &= \underbrace{1/n + 1/n + \ldots + 1/n}_{\text{m times}} \\ &= m/n. \end{aligned}$$

Thus, every rational number in $[0,1]$ is a fixed point of f.

Now let $a \in (0,1)$ be arbitrary. Let $\varepsilon > 0$ such that $B(a,\varepsilon) \subseteq [0,1]$. Since the rationals are dense in the reals, there must exist $q_1, q_2 \in \mathbb{Q}$ such that

$$a - \varepsilon < q_1 \le a \le q_2 < a + \varepsilon.$$

By Lemma 1.1.16(b), we have $f(q_1) \leq f(a) \leq f(q_2)$. But since q_1 and q_2 are fixed points of f, we get $q_1 \leq f(a) \leq q_2$ from which it follows that $a - \varepsilon < f(a) < a + \varepsilon$.

This holds for arbitrarily small ε, so we must have $f(a) \in \bigcap_{\varepsilon > 0} B(a, \varepsilon) = \{a\}$, so $f(a) = a$. Hence, f is the identity map.

(b) Let $a \in [0,1]$. We have

$$f_a(1,0) = a(1) + (1-a)(0) = a,$$

and so each f_a is a distinct map.

We now verify that each f_a lands in $[0,1]$. Indeed, since $f_a(x,y) = ax + (1-a)y$ and each of $a, x, (1-a), y$ is nonnegative, then for any given a, f_a never takes values less than 0. Moreover, f_a takes its greatest value when $x = y = 1$, whence we have $f_a(1,1) = a + 1 - a = 1$. So indeed, $f_a([0,1]^2) \subseteq [0,1]$. We have also shown that f_a satisfies (EH 1).

For (EH 2), suppose that $(x,y) \mathbin{\widetilde{\oplus}} (z,w) \downarrow$. This means $(x \mathbin{\widetilde{\oplus}} z, y \mathbin{\widetilde{\oplus}} w) \downarrow$, so $x + z \leq 1$ and $y + w \leq 1$. We have

$$ax + (1-a)y + az + (1-a)w = a(x+z) + (1-a)(y+w). \tag{1.1.1}$$

The expression (1.1.1) takes its greatest value when $x + z = 1$ and $y + w = 1$, in which case we obtain $a(1) + (1-a)(1) = 1$. So whenever $x + z \leq 1$ and $y + w \leq 1$, (1.1.1) is no greater than 1, so indeed

$$\begin{aligned} f_a(x,y) \mathbin{\widetilde{\oplus}} f_a(z,w) &= (ax + (1-a)y) \mathbin{\widetilde{\oplus}} (az + (1-a)(y+w)) \\ &= ax + (1-a)y + az + (1-a)(y+w) \\ &= a(x+z) + (1-a)(y+w) \\ &= f_a(x+z, y+w) \\ &= f_a((x,y) \mathbin{\widetilde{\oplus}} (z,w)). \end{aligned}$$

It remains to show that every effect algebra homomorphism $[0,1]^2 \to [0,1]$ is one of these f_a. Let $g\colon [0,1]^2 \to [0,1]$ be an effect algebra homomorphism, and write $g(1,0) = a$. We will show that $g = f_a$ using a slight generalization of the ε-ball trick in part (a).

For all $n \in \mathbb{N}^+$ we have

$$\begin{aligned} a &= g(1,0) \\ &= g(\underbrace{(1/n,0) \,\widetilde{\oplus}\, (1/n,0) \,\widetilde{\oplus}\, \ldots \,\widetilde{\oplus}\, (1/n,0)}_{\text{n times}}) \\ &= \underbrace{g(1/n,0) \,\widetilde{\oplus}\, g(1/n,0) \,\widetilde{\oplus}\, \ldots \,\widetilde{\oplus}\, g(1/n,0)}_{\text{n times}} \\ &= ng(1/n), \end{aligned}$$

so $g(1/n,0) = \frac{1}{n}(a)$.

It follows, using a similar argument as in part (a), that if $m < n$, then $g(m/n,0) = \frac{m}{n}(a)$. So for every rational $q \in [0,1]$, we have $g(q,0) = qa$.

Now let $x \in [0,1]$ be arbitrary. Let $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq [0,1]$. Since the rationals are dense in the reals, we have rational numbers q_1 and q_2 such that

$$x - \varepsilon < q_1 \leq x \leq q_2 < x + \varepsilon.$$

In particular, we have $(q_1,0) \leq (x,0) \leq (q_2,0)$. Using Lemma 1.1.16(b), we get $aq_1 = g(q_1,0) \leq g(x,0) \leq g(q_2,0) = aq_2$. But since $x - \varepsilon < q_1$ and $q_2 < x + \varepsilon$, this means we have

$$ax - a\varepsilon = a(x-\varepsilon) < aq_1 \leq g(x,0) \leq aq_2 < a(x+\varepsilon) = ax + a\varepsilon.$$

This holds for arbitrarily small ε, so we have we have $g(x,0) \in \bigcap_{\varepsilon>0} B(ax, a\varepsilon)$. But a is just a fixed constant, so that ε can be made arbitrary small implies that $a\varepsilon$ can also be made arbitrary small, whence $g(x,0) \in \bigcap_{\varepsilon>0} B(ax,\varepsilon) = \{ax\}$. Hence, $g(x,0) = ax$.

Now, since $(1,0) \,\widetilde{\oplus}\, (0,1) \downarrow$, we have

$$a \,\widetilde{\oplus}\, g(0,1) = g(1,0) \,\widetilde{\oplus}\, g(0,1) = g((1,0) \,\widetilde{\oplus}\, (0,1)) = g(1,1) = 1,$$

so we must have $g(0,1) = 1 - a$. By a similar argument as above, we have that $g(0,y) = (1-a)y$ for all $y \in [0,1]$. Then, for any $(x,y) \in [0,1]^2$, we must have

$$g(x,y) = g((x,0) \mathbin{\widetilde{\oplus}} (0,y)) = g(x,0) \mathbin{\widetilde{\oplus}} g(0,y) = ax + (1-a)y,$$

so $g = f_a$ as desired.

□

1.2 MV algebras

Definition 1.2.1 (MV algebra)**.** An *MV algebra* is a quadruple $(A, \oplus, \neg, 0)$ wherein A is a set, $\oplus\colon A \times A \to A$ is a (total) binary operation, $\neg\colon A \to A$ is a (total) unary operation, and $0 \in A$ is a distinguished element, such that the following hold for all $a, b, c \in A$:

(MV 1) *Associativity*: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(MV 2) *Commutativity*: $a \oplus b = b \oplus a$.

(MV 3) *Zero law*: $a \oplus 0 = a$.

(MV 4) *Involution*: $\neg\neg a = a$.

(MV 5) *Absorption law*: $a \oplus \neg 0 = \neg 0$.

(MV 6) *Łukasiewicz axiom*: $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$.

We define also

- $1 = \neg 0$.
- $a \otimes b = \neg(\neg a \oplus \neg b)$.
- $a \ominus b = a \otimes \neg b = \neg(\neg a \oplus b)$.
- $a \multimap b = \neg a \oplus b$.

Lemma 1.2.2. *In any MV algebra, we have:*

(MV 1') $\neg 1 = 0$.

(MV 2') $a \oplus b = \neg(\neg a \otimes \neg b) = \neg(\neg a \ominus b)$.

(MV 3') $a \oplus 1 = 1$.

(MV 4') $(a \ominus b) \oplus b = (b \ominus a) \oplus a$.

(MV 5') $a \oplus \neg a = 1$.

Proof. The statement (MV 1') follows immediately from applying (MV 4) to the defining equation. Similarly for (MV 2') we have

$$\neg(\neg a \ominus b) = \neg(\neg a \otimes \neg b) = \neg(\neg(\neg\neg a \oplus \neg\neg b)) = a \oplus b.$$

Then, (MV 3') is just a rewrite of (MV 5). To obtain (MV 4'), we rewrite (MV 6) as follows:

$$\begin{aligned}(a \ominus b) \oplus b &= (a \otimes \neg b) \oplus b \\ &= \neg(\neg a \oplus b) \oplus b \\ &= \neg(\neg b \oplus a) \oplus a \\ &= (b \otimes \neg a) \oplus a \\ &= (b \ominus a) \oplus a.\end{aligned}$$

Finally, by taking $y = 1$ in (MV 6), we obtain (MV 5') as follows:

$$x \oplus \neg x = \neg x \oplus x = \neg(0 \oplus x) \oplus x = \neg(\neg 1 \oplus x) \oplus x = \neg(\neg x \oplus 1) \oplus 1 = 1. \qquad \square$$

We define, as for effect algebras, a binary relation $a \leq b$ to mean there exists an x such that $a \oplus x = b$.

An MV algebra for which the induced order is total is called an *MV chain*.

Lemma 1.2.3. *Let A be an MV algebra, and $a, b \in A$. The following are equivalent.*

(a) $a \leq b$.

(b) $\neg a \oplus b = 1$.

(c) $a \otimes \neg b = 0$.

(d) $b = a \oplus (b \ominus a)$.

Proof. (a) $\implies$ (b): Let $x \in A$ such that $x \oplus a = b$. Then, using (MV 5') we obtain

$$\neg a \oplus b = \neg a \oplus (x \oplus a) = (\neg a \oplus a) \oplus x = 1 \oplus x = 1.$$

(b) $\implies$ (c): Using (MV 4) and (MV 1'), we obtain

$$a \otimes \neg b = \neg(\neg a \oplus b) = \neg 1 = 0.$$

(c) $\implies$ (d): Using (MV 3) and (MV 4'), we have

$$a \oplus (b \ominus a) = (b \ominus a) \oplus a = (a \ominus b) \oplus b = (a \otimes \neg b) \oplus b = 0 \oplus b = b.$$

(d) $\implies$ (a): Immediate. $\square$

Lemma 1.2.4. *If $(A, \oplus, \neg, 0)$ is an MV algebra, then $\leq$ induces a partial order on A.*

Proof. Reflexivity and transitivity are analogous to the proof of Lemma 1.1.3.

For antisymmetry, suppose $a \leq b$ and $b \leq a$. By Lemma 1.2.3 (c), $a \leq b$ is equivalent to $a \otimes \neg b = 0$. Then, $b \leq a$ is equivalent, by Lemma 1.2.3 (d) and the above, to

$$a = b \oplus (a \ominus b) = b \oplus (a \otimes \neg b) = b \oplus 0 = b.$$ $\square$

Example 1.2.5. The unit interval $[0, 1]$ with $\oplus$ as the cutoff addition, $a \oplus b = \min(1, a + b)$, and $\neg a = 1 - a$, is an MV algebra. Note that this is similar to $[0, 1]$ as an effect algebra, except that when the sum $a + b$ exceeds 1, we take $a \oplus b = 1$ instead of undefined.

It follows from routine calculations that $a \ominus b = \max(0, a - b)$ and $a \otimes b = \max(0, a + b - 1)$. It is clear that the MV algebra order coincides with the standard ordering on $[0, 1]$.

Example 1.2.6. Any boolean algebra X can be viewed as an MV algebra by taking $p \oplus q = p \vee q$, $\neg p = \neg p$, and 0 as the bottom element of X. The mysterious-looking Łukasiewicz axiom (MV 6) becomes

$$\neg(\neg a \vee b) \vee b = \neg(\neg b \vee a) \vee a.$$

This is logically equivalent

$$(a \to b) \to b = (b \to a) \to a,$$

which is (classically) a tautology!

Note that in such an MV algebra, every element is idempotent. Conversely, it is known that all MV algebras in which all elements are idempotent are in fact boolean algebras. For more details, see [CDM00, pp.25–26]

Example 1.2.7. Let $\mathcal{C}$ be the set consisting of the formal symbols

$$0, c, c+c, c+c+c, \ldots,$$

$$1, 1-c, 1-c-c, 1-c-c-c, \ldots.$$

For short, we write (for $n \in \mathbb{N}^+$), $0 \cdot c = 0$, $n \cdot c = \underbrace{c + c + \ldots + c}_{n \text{ factors of } c}$, $1 - 0 \cdot c = 1$, and $1 - n \cdot c = 1 - \underbrace{c - c - \ldots - c}_{n \text{ factors of } c}$.

We define addition as follows:

- If $x = n \cdot c$ and $y = m \cdot c$, then $x \oplus y = (n+m) \cdot c$.
- If $x = 1 - n \cdot c$ and $y = 1 - m \cdot c$, then $x \oplus y = 1$.
- If $x = n \cdot c$ and $y = 1 - m \cdot c$, or if $x = 1 - m \cdot c$ and $y = n \cdot c$, then

$$x \oplus y = \begin{cases} 1 - (m-n) \cdot c & \text{if } n < m, \\ 1 & \text{if } m \leq n. \end{cases}$$

We define complementation by

$$\neg x = \begin{cases} 1 - n \cdot c & \text{if } x = n \cdot c, \\ n \cdot c & \text{if } x = 1 - n \cdot c. \end{cases}$$

Then, $(\mathcal{C}, \oplus, \neg, 0)$ is a MV algebra known as *Chang's MV algebra*, which was first introduced in [Cha58].

Lemma 1.2.8. *Let A be an MV algebra. The natural order $\leq$ satisfies, for all $a, b, c \in A$,*

(a) $a \leq b$ if and only if $\neg b \leq \neg a$.

(b) If $a \leq b$, then for all $x \in A$, $a \oplus x \leq b \oplus x$, and $a \otimes x \leq b \otimes x$.

(c) $a \otimes b \leq c$ if and only if $a \leq \neg b \oplus c$.

(d) $a \ominus b \leq c$ if and only if $a \leq b \oplus c$.

Proof. (a) By Lemma 1.2.3 (b), we have

$$a \leq b \Leftrightarrow \neg a \oplus b = 1 \Leftrightarrow \neg\neg b \oplus \neg a = 1 \Leftrightarrow \neg b \leq \neg a.$$

(b) If $a \leq b$, there is $y \in A$ such that $a \oplus y = b$. So,

$$(a \oplus x) \oplus y = (a \oplus y) \oplus x = b \oplus x,$$

whence $a \oplus x \leq b \oplus x$.

Then applying (a), we have $\neg b \leq \neg a$, so by the above, $\neg b \oplus \neg x \leq \neg a \oplus \neg x$. Applying (a) again yields

$$a \otimes x = \neg(\neg a \oplus \neg x) \leq \neg(\neg b \oplus \neg x) = b \otimes x.$$

(c) By Lemma 1.2.3 (b), we have

$$a \otimes b \leq c \Leftrightarrow 1 = \neg(a \otimes b) \oplus c = \neg a \oplus \neg b \oplus c \Leftrightarrow a \leq \neg b \oplus c.$$

(d) Using (c), we have

$$a \ominus b \leq c \Leftrightarrow a \otimes \neg b \leq c \Leftrightarrow a \leq \neg\neg b \oplus c \Leftrightarrow a \leq b \oplus c.$$

Note that (c) and (d) say that, regarding A as a poset category, $(-) \otimes b$ is left adjoint to $\neg b \oplus (-)$, and $(-) \ominus b$ is left adjoint to $b \oplus (-)$. □

Definition 1.2.9 (Homomorphism (of MV algebras))**.** Let A, B be MV algebras. A function $f\colon A \to B$ is called an *MV algebra homomorphism* if it satisfies the following for all $a, b \in A$.

(MH 1) $f(0_A) = 0_B$.

(MH 2) $f(a \oplus b) = f(a) \oplus f(b)$.

(MH 3) $f(\neg a) = \neg f(a)$.

We denote the *category of MV algebras*, with objects MV algebras and arrows MV algebra homomorphisms, by **MV**.

Lemma 1.2.10. *Let $f\colon A \to B$ be an MV algebra homomorphism. Then:*

(a) f also preserves the top element; i.e. $f(1_A) = 1_B$.

(b) f preserves $\leq$; if $a, b \in A$ and $a \leq b$, then $f(a) \leq f(b)$.

Proof. (a) We have

$$f(1_A) = f(\neg 0_A) = \neg f(0_A) = \neg 0_B = 1_B.$$

(b) Analogous to Lemma 1.1.16 (b).

□

Proposition 1.2.11. *The MV algebra homomorphisms from $[0,1] \to [0,1]$ and $[0,1]^2 \to [0,1]$ (where we define the operation on $[0,1]^2$ pointwise; i.e. $(a,b) \oplus (c,d) = (\max(a + c, 1), \max(b + d, 1))$) are given precisely as follows.*

(a) The only MV algebra homomorphism $[0,1] \to [0,1]$ is the identity.

(b) *There are exactly two MV algebra homomorphisms* $[0,1]^2 \to [0,1]$ *; namely, the projections* π_1 *and* π_2*.*

Proof. (a) Let $f\colon [0,1] \to [0,1]$ be a MV algebra homomorphism. We already know $f(0) = 0$ and $f(1) = 1$. We have

$$f(1/2) = f(\neg 1/2) = \neg f(1/2) = 1 - f(1/2),$$

so $f(1/2) = 1/2$. For every $n \in \mathbb{N}^+$, we have

$$\begin{aligned} 1/2 &= f(1/2) \\ &= f(\underbrace{1/2n \oplus 1/2n \oplus \ldots \oplus 1/2n}_{\text{n times}}) \\ &= \underbrace{f(1/2n) \oplus f(1/2n) \oplus \ldots \oplus f(1/2n)}_{\text{n times}} \\ &= \underbrace{f(1/2n) + f(1/2n) + \ldots + f(1/2n)}_{\text{n times}} \\ &= nf(1/2n), \end{aligned}$$

(note that from the third to the fourth line, we need to use the fact that the sum is strictly less than 1; this is why the exact same proof for the case of effect algebras does not work here) so $f(1/2n) = 1/2n$. Then,

$$f(1/n) = f(1/2n \oplus 1/2n) = f(1/2n) \oplus f(1/2n) = 1/2n \oplus 1/2n = 1/n.$$

The remainder of the proof follows along the same lines (after concluding $f(1/n) = 1/n$) as Proposition 1.1.17(a).

(b) Let $g\colon [0,1]^2 \to [0,1]$ be a MV algebra homomorphism. Similarly to Proposotion 1.1.17(b), we will first show that g must be of the form

$$f_a(x, y) = ax + (1 - a)y$$

for some $a \in [0,1]$. We will then argue that only $a = 0$ and $a = 1$ yield MV algebra homomorphisms, and that these are the projections π_2 and π_1, respectively.

Write $g(1/2,0) = \frac{1}{2}(a)$. By a similar calculation to part (a), we have for $n \in \mathbb{N}^+$ that

$$\frac{1}{2}(a) = ng(1/2n, 0),$$

so $g(1/2n, 0) = \frac{1}{2n}(a)$, and it follows that

$$g(1/n, 0) = g(1/2n, 0) \oplus g(1/2n, 0) = \frac{1}{2n}(a) + \frac{1}{2n}(a) = \frac{1}{n}(a).$$

Following similar reasoning as in Proposition 1.1.17(b) (after deducing $g(1/n, 0) = \frac{1}{n}(a)$), we conclude $g(x, 0) = ax$ for all $x \in [0, 1]$.

We have also

$$g(1/2, 1/2) = g(\neg(1/2, 1/2)) = \neg g(1/2, 1/2) = 1 - g(1/2, 1/2),$$

so $g(1/2, 1/2) = 1/2$. Then,

$$\frac{1}{2}(a) \oplus g(0, 1/2) = g(1/2, 0) \oplus g(0, 1/2) = g(1/2, 1/2) = 1/2,$$

so $g(0, 1/2) = \frac{1}{2}(1 - a)$, from which it similarly follows that $g(0, y) = (1 - a)y$ for all $y \in [0, 1]$. Thus,

$$g(x, y) = g((x, 0) \oplus (0, y)) = g(x, 0) \oplus g(0, y) = ax + (1 - a)y.$$

Hence, any MV algebra homomorphism $[0, 1]^2 \to [0, 1]$ is of the form f_a for some $a \in [0, 1]$.

If $a = 0$, we get

$$f_0(x, y) = 0x + (1 - 0)y = y = \pi_2(x, y),$$

which is easily seen to be a MV algebra homomorphism.

Now suppose $a \neq 0$. We require, for all (x, y),$(z, w) \in [0, 1]^2$,

$$f_a(x, y) \oplus f_a(z, w) = f_a((x, y) \oplus (z, w)).$$

On the one hand,

$$\begin{aligned} f_a(x,y) \oplus f_a(z,w) &= \min(1, f_a(x,y) + f_a(z,w)) \\ &= \min(1, ax + (1-a)y + az + (1-a)w) \\ &= \min(1, a(x+z) + (1-a)(y+w)). \end{aligned} \tag{1.2.1}$$

On the other hand,

$$\begin{aligned} f_a((x,y) \oplus (z,w)) &= f_a(x \oplus z, y \oplus w) \\ &= f_a(\min(1, x+z), \min(1, y+w)) \\ &= a \cdot \min(1, x+z) + (1-a) \cdot \min(1, y+w) \\ &= \min(a, a(x+z)) + \min(1-a, (1-a)(y+w)). \end{aligned} \tag{1.2.2}$$

In particular, take $(x,y) = (\frac{1}{4}, \frac{3}{4}) = (z,w)$. We see that (1.2.1) yields

$$\min\left(1, \frac{a}{2} + (1-a)\frac{3}{2}\right) = \min\left(1, \frac{a}{2} + \frac{3}{2} - (a)\frac{3}{2}\right) = \min\left(1, \frac{3}{2} - a\right). \tag{1.2.3}$$

On the other hand, we have that (1.2.2) yields

$$\min\left(a, \frac{a}{2}\right) + \min\left(1-a, (1-a)\frac{3}{2}\right) = \frac{a}{2} + 1 - a = 1 - \frac{a}{2}. \tag{1.2.4}$$

Since a is assumed to be nonzero, $1 - \frac{a}{2} \neq 1$, so the only way for (1.2.3) and (1.2.4) to be equal is if $\min(1, \frac{3}{2} - a) = \frac{3}{2} - a$. Then, equating the two yields

$$\begin{aligned} 1 - \frac{a}{2} &= \frac{3}{2} - a, \\ \frac{a}{2} &= \frac{1}{2}, \\ a &= 1, \end{aligned}$$

so the only other MV algebra homomorphism possible is

$$f_1(x,y) = (1)x + (1-1)y = x = \pi_1(x,y).$$

□

1.3 Lattice ordered algebras

We now turn our attention to the question of when MV and effect algebras are lattice ordered (that is, the condition that their natural partial orders endow them with the structure of a lattice), and how this relates the two.

Proposition 1.3.1. *In every MV algebra A, the natural partial order $\leq$ gives A the structure of a lattice. In particular,*

(a) Joins are given by $a \vee b = (a \ominus b) \oplus b$.

(b) Meets are given by $a \wedge b = a \otimes (\neg a \oplus b)$.

Proof. (a) It is immediate that $b \leq (a \ominus b) \oplus b$. By (MV 4'), we also have $a \leq (b \ominus a) \oplus a = (a \ominus b) \oplus b$. Now suppose $c \in A$ such that $a \leq c$ and $b \leq c$. By Lemma 1.2.3(b) and (d), we have

$$\neg a \oplus c = 1, \tag{1.3.1}$$

$$c = b \oplus (c \ominus b). \tag{1.3.2}$$

Then,

$$\begin{aligned}
\neg((a \ominus b) \oplus b) \oplus c &= (\neg(a \ominus b) \otimes \neg b) \oplus (b \oplus (c \ominus b)) && (1.3.2)\\
&= ((\neg(a \ominus b) \ominus b) \oplus b) \oplus (c \ominus b) && \text{(by definition)}\\
&= ((b \ominus \neg(a \ominus b)) \oplus \neg(a \ominus b)) \oplus (c \ominus b) && \text{(MV 4')}\\
&= (b \ominus \neg(a \ominus b)) \oplus \neg a \oplus b \oplus (c \ominus b) && \text{(by definition)}\\
&= b \ominus \neg(a \ominus b) \oplus \neg a \oplus c && (1.3.2)\\
&= b \ominus \neg(a \ominus b) \oplus 1 && (1.3.1)\\
&= 1 && \text{(MV 3')}.
\end{aligned}$$

By Lemma 1.2.3 (b), we have $(a \ominus b) \oplus b \leq c$, so indeed $(a \ominus b) \oplus b$ is the least upper bound of a and b.

(b) We have

$$\begin{aligned} a \otimes (\neg a \oplus b) &= (b \oplus \neg a) \otimes a \\ &= \neg(\neg b \otimes a) \otimes a \\ &= \neg(\neg b \ominus \neg a) \otimes a \\ &= \neg((\neg b \ominus \neg a) \oplus \neg a) \\ &= \neg(\neg b \vee \neg a) \\ &= \neg(\neg a \vee \neg b). \end{aligned}$$

We already know joins exist, so $\neg a \leq \neg a \vee \neg b$ and $\neg b \leq \neg a \vee \neg b$. By Lemma 1.2.8 (a), we get $\neg(\neg a \vee \neg b) \leq a$ and $\neg(\neg a \vee \neg b) \leq b$. If $c \in A$ such that $c \leq a$ and $c \leq b$, then $\neg a \leq \neg c$ and $\neg b \leq \neg c$, whence $\neg a \vee \neg b \leq \neg c$. Then $c \leq \neg(\neg a \vee \neg b)$. Thus, $\neg(\neg a \vee \neg b) = a \otimes (\neg a \oplus b)$ is indeed the greatest lower bound of a and b.

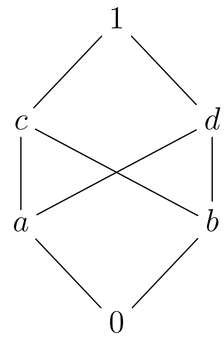
□

The following example shows that the preceding proposition does not go through, in general, for effect algebras.

Example 1.3.2. Let $H_6 = \{0, a, b, c, d, 1\}$. Define a partial binary operation $\widetilde{\oplus}$ on H_6 as follows.

$\widetilde{\oplus}$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	d	1	$\uparrow$	$\uparrow$
b	b	d	c	$\uparrow$	1	$\uparrow$
c	c	1	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
d	d	$\uparrow$	1	$\uparrow$	$\uparrow$	$\uparrow$
1	1	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$

A routine verification of the axioms shows that $(H_6, \widetilde{\oplus}, (-)^{\perp}, 0)$ is an effect algebra (with $0^{\perp} = 1$, $a^{\perp} = c$, and $b^{\perp} = d$). The Hasse diagram of H_6 with respect to $\leq$ is given as follows.



We see that H_6 is not lattice ordered; for instance, $c \wedge d$ does not exist, as a and b are both elements immediately below both c and d, but are incomparable.

Lemma 1.3.3. *Every lattice ordered effect algebra satisfies the De Morgan law, i.e.*

$$(a^{\perp} \wedge b^{\perp})^{\perp} = a \vee b$$

and

$$(a^{\perp} \vee b^{\perp})^{\perp} = a \wedge b.$$

Proof. Using Lemma 1.1.10 (e) together with a similar argument as Proposition 1.3.1(b) yields the result. □

Lemma 1.3.4. *Let A be an MV algebra. Then, $\otimes$ distributes over $\vee$ and $\oplus$ distributes over $\wedge$; that is, for all $a, b, c \in A$, the following hold.*

(a) $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$.

(b) $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$.

Proof. (a) By definition, $b \wedge c \leq b$ and $b \wedge c \leq c$, so Lemma 1.2.8(b) yields

$$a \oplus (b \wedge c) \leq a \oplus b, \;\; a \oplus (b \wedge c) \leq a \oplus c.$$

So $a \oplus (b \wedge c)$ is a lower bound for $a \oplus b$ and $a \oplus c$. Now suppose $x \in A$ such that $x \leq a \oplus b = b \oplus a$ and $x \leq a \oplus c = c \oplus a$. Applying Lemma 1.2.8(c), we get

$$x \otimes \neg a \leq b, \;\; x \otimes \neg a \leq c.$$

So then $x \otimes \neg a \leq b \wedge c$. Again using Lemma 1.2.8(c), we have

$$x \leq a \oplus (b \wedge c).$$

Therefore, $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$.

(b) Analogous to part (a). For an explicit proof, see [CDM00, p.11].

□

The following example shows that not every sub-effect algebra of a lattice ordered effect algebra need be lattice ordered.

Example 1.3.5. Define E to be the set of all functions $[0,1] \to [0,1]$. We define a partial operation $\widetilde{\oplus}$ on E as follows. For $f, g \in E$, we say $f \widetilde{\oplus} g \downarrow$ if and only if $(f+g)(x) \leq 1$ for all $x \in [0,1]$, in which case $f \widetilde{\oplus} g = f + g$. Define $f^{\perp}(x) = 1 - f(x)$, and write 0 for the zero function. Then, $(E, \widetilde{\oplus}, (-)^{\perp}, 0)$ is a lattice ordered effect algebra.

The subset F of all quadratic functions $[0,1] \to [0,1]$, i.e.

$$F = \{f \in E \mid f(x) = ax^2 + bx + c; a, b, c \in \mathbb{R}\}$$

is a sub-effect algebra of E but is not lattice ordered.

We refer the reader to [RMZ06] for more details on this example, and further discussion in general about lattice ordered (generalized) effect algebras.

Proposition 1.3.6. *Let A, B be MV algebras, $f\colon A \to B$ an MV homomorphism. Then, f preserves the lattice structure; that is, for all $a, b \in A$, we have $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$.*

Proof. We have

$$\begin{aligned}
f(a \vee b) &= f((a \ominus b) \oplus b) \\
&= f(\neg(\neg a \oplus b) \oplus b) \\
&= f(\neg(\neg a \oplus b)) \oplus f(b) \\
&= \neg(f(\neg a \oplus b)) \oplus f(b) \\
&= \neg(f(\neg a) \oplus f(b)) \oplus f(b) \\
&= \neg(\neg f(a) \oplus f(b)) \oplus f(b) \\
&= (f(a) \ominus f(b)) \oplus f(b) \\
&= f(a) \vee f(b)
\end{aligned}$$

and

$$\begin{aligned}
f(a \wedge b) &= f(a \otimes (\neg a \oplus b)) \\
&= f(\neg(\neg a \oplus \neg(\neg a \oplus b)) \\
&= \neg f(\neg a \oplus \neg(\neg a \oplus b)) \\
&= \neg(f(\neg a) \oplus f(\neg(\neg a \oplus b)) \\
&= \neg(\neg f(a) \oplus \neg f(\neg a \oplus b)) \\
&= \neg(\neg f(a) \oplus \neg(\neg f(a) \oplus f(b))) \\
&= f(a) \otimes (\neg f(a) \oplus f(b)) \\
&= f(a) \wedge f(b).
\end{aligned}$$

□

Recall that if A, B are MV algebras where every element is idempotent, they are boolean algebras (see Example 1.2.6). By the above lemma, every MV algebra homomorphism $f\colon A \to B$ is also a boolean algebra homomorphism. It is easy to see also that boolean algebra homomorphisms between boolean algebras also are MV algebra homomorphisms when considering boolean algebras as MV algebras. This gives us the following.

Corollary 1.3.7. *The full subcategory of* **MV** *of MV algebras where every element is idempotent is isomorphic to the category* **BA** *of boolean algebras and boolean algebra homomorphisms.*

1.4 MV-effect algebras

We now reach the main result of the first chapter — that, while a priori unrelated, effect algebras are in fact a generalization of MV algebras. More precisely, we will establish a categorical isomorphism between **MV** and a subcategory of **EA**.

Theorem 1.4.1. *Let E be a lattice ordered effect algebra. The following conditions are equivalent.*

(a) For all $a, b, c \in E$, $a \mathbin{\widetilde{\ominus}} (a \wedge b) = (a \vee b) \mathbin{\widetilde{\ominus}} b$.

(b) For all $a, b_1, b_2 \in E$, if $a \leq b_1 \mathbin{\widetilde{\oplus}} b_2$, then there exist $a_1, a_2 \in E$ such that $a = a_1 \mathbin{\widetilde{\oplus}} a_2$, $a_1 \leq b_1$, and $a_2 \leq b_2$.

(c) For all $a, b, c \in E$, if $b \mathbin{\widetilde{\oplus}} c \downarrow$, then $a \wedge (b \mathbin{\widetilde{\oplus}} c) \leq (a \wedge b) \mathbin{\widetilde{\oplus}} (a \wedge c)$.

(d) For all $a, b, c \in E$, if $b \mathbin{\widetilde{\oplus}} c \downarrow$, then $a \wedge (b \mathbin{\widetilde{\oplus}} c) \leq (a \wedge b) \mathbin{\widetilde{\oplus}} c$.

(e) For all $a, b \in E$, $a \leq (a \wedge b) \mathbin{\widetilde{\oplus}} (a \wedge b^{\perp})$.

(f) For all $a, b \in E$, if $a \wedge b = 0$, then $a \mathbin{\widetilde{\oplus}} b \downarrow$.

(g) For all $a, b, c \in E$, if $c \leq b$, then $(a \vee b) \mathbin{\widetilde{\ominus}} c \leq a \vee (b \mathbin{\widetilde{\ominus}} c)$.

(h) For all $a, b \in E$, there exist $x, y \in E$ such that $x \leq b$, $y \leq b^{\perp}$, and $a = x \mathbin{\widetilde{\oplus}} y$.

Proof. See [BF95, Theorem 3.11]. □

Condition (b) in the above theorem is known as the *Riesz decomposition property*, which will be discussed further in the next chapter. Conditon (f) is also known as the property of being *pseudoboolean*.

Definition 1.4.2 (MV-effect algebra)**.** An effect algebra is called an *MV-effect algebra* if it is lattice ordered and satisfies the equivalent conditions of Theorem 1.4.1.

Theorem 1.4.3. *There is a one-to-one correspondence between MV algebras and MV-effect algebras. In particular:*

(a) Given an MV-effect algebra $(E, \widetilde{\oplus}, (-)^{\perp}, 0)$*, we can form an MV algebra* $\mathcal{A}(E) = (E, \oplus, \neg, 0)$ *with* $\neg a = a^{\perp}$ *and*

$$a \oplus b = a \mathbin{\widetilde{\oplus}} (a^{\perp} \wedge b).$$

The natural orders of E *and* $\mathcal{A}(E)$ *coincide.*

(b) Given an MV algebra $(A, \oplus, \neg, 0)$*, we can form an MV-effect algebra* $\mathcal{E}(A) = (A, \widetilde{\oplus}, (-)^{\perp}, 0)$ *with* $a^{\perp} = \neg a$*, and*

$$a \mathbin{\widetilde{\oplus}} b = \begin{cases} a \oplus b & \text{if } a \leq \neg b, \\ \uparrow & \text{otherwise.} \end{cases}$$

The natural orders of A *and* $\mathcal{E}(A)$ *coincide.*

(c) $\mathcal{E} \circ \mathcal{A}(E) = E$ *and* $\mathcal{A} \circ \mathcal{E}(A) = A$*.*

Proof. (a) We first check that the operation $\oplus$ is indeed total. By definition, $a^{\perp} \wedge b \leq a^{\perp}$, so there exists $x \in E$ such that $(a^{\perp} \wedge b) \mathbin{\widetilde{\oplus}} x = a^{\perp}$. Then by (EA 1),

$$a \mathbin{\widetilde{\oplus}} ((a^{\perp} \wedge b) \mathbin{\widetilde{\oplus}} x) = a \mathbin{\widetilde{\oplus}} a^{\perp} = 1,$$

so by (PCM 2), $a \oplus b = a \mathbin{\widetilde{\oplus}} (a^{\perp} \wedge b) \downarrow$.

Now suppose that $a \leq_E b$. Then, there exists $x \in E$ such that $a \mathbin{\widetilde{\oplus}} x = b$. Then, $a \mathbin{\widetilde{\oplus}} (x \mathbin{\widetilde{\oplus}} b^{\perp}) = (a \mathbin{\widetilde{\oplus}} x) \mathbin{\widetilde{\oplus}} b^{\perp} = b \mathbin{\widetilde{\oplus}} b^{\perp} = 1$. By (EA 1), we get $x \mathbin{\widetilde{\oplus}} b^{\perp} = a^{\perp}$, so $x \leq a^{\perp}$. This implies $a^{\perp} \wedge x = x$. So then $a \oplus x = a \mathbin{\widetilde{\oplus}} (a^{\perp} \wedge x) = a \mathbin{\widetilde{\oplus}} x = b$, whence $a \leq_{\mathcal{A}(E)} b$.

Conversely, if $a \leq_{\mathcal{A}(E)} b$, then there is $x \in E$ such that $a \oplus x = a \mathbin{\widetilde{\oplus}} (a^{\perp} \wedge x) = b$, so $a \leq_E x$. Thus, the natural orders of E and $\mathcal{A}(E)$ coincide.

We now need to check the MV algebra axioms — (MV 3), (MV 4), and (MV 5) are immediately obvious.

- Commutativity (MV 2)

  Using Lemma 1.1.10 (d) and Lemma 1.3.3, we have

$$\begin{aligned} a \mathbin{\widetilde{\oplus}} (a^\perp \wedge b) &= b \mathbin{\widetilde{\oplus}} (b^\perp \wedge a) \\ &\Updownarrow \\ (a^\perp \mathbin{\widetilde{\ominus}} (a^\perp \wedge b))^\perp &= ((b^\perp \wedge a)^\perp \mathbin{\widetilde{\ominus}} b)^\perp \\ &\Updownarrow \\ a^\perp \mathbin{\widetilde{\ominus}} (a^\perp \wedge b) &= (b^\perp \wedge a)^\perp \mathbin{\widetilde{\ominus}} b \\ &= (b \vee a^\perp) \mathbin{\widetilde{\ominus}} b \\ &= (a^\perp \vee b) \mathbin{\widetilde{\ominus}} b. \end{aligned}$$

  The final equality holds by the equivalent condition (a) of Theorem 1.4.1. Thus, $a \oplus b = b \oplus a$.

- Associativity (MV 1)

  First observe that whenever $a, b \in E$ and $a \mathbin{\widetilde{\oplus}} b \downarrow$, then by Lemma 1.1.10(a) and (e), $a \leq b^\perp$ and $b \leq a^\perp$, and so $a^\perp \wedge b = b$, so that

$$a \oplus b = a \mathbin{\widetilde{\oplus}} (a^\perp \wedge b) = a \mathbin{\widetilde{\oplus}} b. \tag{1.4.1}$$

  We also have, that if $a \leq b$, then for any $x \in E$, $x^\perp \wedge a \leq x^\perp \wedge b$. So there exists a $y \in E$ such that $(x^\perp \wedge a) \mathbin{\widetilde{\oplus}} y = x^\perp \wedge b$, so then $x \mathbin{\widetilde{\oplus}} (x^\perp \wedge a) \mathbin{\widetilde{\oplus}} y = x \mathbin{\widetilde{\oplus}} (x^\perp \wedge b)$, and

$$a \oplus x = x \oplus a = x \mathbin{\widetilde{\oplus}} (x^\perp \wedge a) \leq x \mathbin{\widetilde{\oplus}} (x^\perp \wedge b) = x \oplus b = b \oplus x. \tag{1.4.2}$$

  (Note that the above observation is indeed necessary; we cannot automatically assume $\oplus$ is translation-invariant, because the proof for MV algebras relies on associativity already being established!)

As we already know $\oplus$ is commutative, we have

$$\begin{aligned}(a \oplus b) \oplus c &= (b \oplus a) \oplus c \\ &= (b \mathbin{\widetilde{\oplus}} (b^\perp \wedge a)) \mathbin{\widetilde{\oplus}} ((b \mathbin{\widetilde{\oplus}} (b^\perp \wedge a))^\perp \wedge c) \\ &= (b \mathbin{\widetilde{\oplus}} ((b \mathbin{\widetilde{\oplus}} (b^\perp \wedge a))^\perp \wedge c)) \mathbin{\widetilde{\oplus}} (b^\perp \wedge a) && \text{(PCM 1)} \\ &= (b \oplus ((b \oplus (b^\perp \wedge a))^\perp \wedge c)) \oplus (b^\perp \wedge a) && (1.4.1) \\ &\leq (b \oplus c) \oplus a && (1.4.2) \\ &= a \oplus (b \oplus c).\end{aligned}$$

By interchanging a and c in the above argument, we similarly deduce that

$$a \oplus (b \oplus c) = (b \oplus c) \oplus a \leq (b \oplus a) \oplus c = (a \oplus b) \oplus c.$$

By antisymmetry, we thus get $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

- Łukasiewicz axiom (MV 6)

  We have

$$\begin{aligned}\neg(\neg b \oplus a) \oplus a &= a \oplus \neg(a \oplus \neg b) \\ &= a \oplus (a \mathbin{\widetilde{\oplus}} (a^\perp \wedge b^\perp)^\perp) \\ &= a \mathbin{\widetilde{\oplus}} (a^\perp \wedge (a \mathbin{\widetilde{\oplus}} (a^\perp \wedge b^\perp))^\perp) \\ &= a \mathbin{\widetilde{\oplus}} (a^\perp \wedge (a^\perp \mathbin{\widetilde{\ominus}} (a^\perp \wedge b^\perp))) && \text{(Lemma 1.1.10(d))} \\ &= a \mathbin{\widetilde{\oplus}} (a^\perp \mathbin{\widetilde{\ominus}} (a^\perp \wedge b^\perp)) && (\text{as } a^\perp \mathbin{\widetilde{\ominus}} (a^\perp \wedge b^\perp) \leq a^\perp) \\ &= (a^\perp \mathbin{\widetilde{\ominus}} (a^\perp \mathbin{\widetilde{\ominus}} (a^\perp \wedge b^\perp))^\perp && \text{(Lemma 1.1.10(d))} \\ &= (a^\perp \wedge b^\perp)^\perp && \text{(Lemma 1.1.10(c))} \\ &= a \vee b\end{aligned}$$

  By interchanging a and b in the above calculation, we get

$$\neg(\neg a \oplus b) \oplus b = b \vee a = a \vee b,$$

  so indeed $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$.

(b) Since $\widetilde{\oplus}$ is a restriction of $\oplus$, then it is clear that the new operation is commutative, associative, and 0 is the identity, as long as we can verify all required sums are defined.

For (PCM 1), we have by Lemma 1.2.8(a),

$$a \mathbin{\widetilde{\oplus}} b \downarrow \Leftrightarrow a \leq \neg b \Leftrightarrow b \leq \neg a \Leftrightarrow b \mathbin{\widetilde{\oplus}} a \downarrow .$$

For (PCM 2), suppose $b \mathbin{\widetilde{\oplus}} c \downarrow$ and $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) \downarrow$. This means $b \leq \neg c$ and $a \leq \neg(b \oplus c)$ (equivalently, $c \leq \neg b$ and $b \oplus c \leq \neg a$). Now $b \leq b \oplus c$, so $\neg(b \oplus c) \leq \neg b$. By transitivity, $a \leq \neg b$, whence $a \mathbin{\widetilde{\oplus}} b \downarrow$. We have also that

$$\begin{aligned}
a \oplus b &\leq \neg(b \oplus c) \oplus b && \text{Lemma 1.2.8(b)} \\
&= \neg(c \oplus b) \oplus b && \text{(MV 2)} \\
&= (\neg c \ominus b) \oplus b && \text{(by definition)} \\
&= \neg c \vee b && \text{(by definition)} \\
&= \neg c && (\text{as } b \leq \neg c),
\end{aligned}$$

so $(a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c \downarrow$.

For (PCM 3), we have

$$a \mathbin{\widetilde{\oplus}} 0 \downarrow \Leftrightarrow a \leq \neg 0 \Leftrightarrow a \leq 1,$$

which is true for all $a \in A$.

For (EA 1), it is clear that $a \leq \neg\neg a$, and so $a \mathbin{\widetilde{\oplus}} a^{\perp} = a \oplus \neg a = 1$. Suppose now that $b \in A$ satisfies $a \mathbin{\widetilde{\oplus}} b = 1$. Then, we have that $a \leq \neg b$, so also $b \leq \neg a$. By Lemma 1.2.3(b), $a \oplus b = 1$ is equivalent to $\neg a \leq b$. Antisymmetry thus forces $a = \neg b$, so $b = \neg a = a^{\perp}$.

For (EA 2), we have

$$a \mathbin{\widetilde{\oplus}} 1 \downarrow \Leftrightarrow a \leq \neg 1 \Leftrightarrow a \leq 0 \Leftrightarrow a = 0.$$

Now if $a \leq_{\mathcal{E}(A)} b$, then there is $x \in A$ such that $a \mathbin{\widetilde{\oplus}} x = b$, so also $a \oplus x = b$ and $a \leq_A b$. Conversely, suppose $a \leq_A b$. Then,

$$(b \ominus a) \oplus a = a \vee b = b,$$

and $b \ominus a = \neg(\neg b \oplus a)$. Clearly $a \leq b \oplus \neg a$, so $a \mathbin{\widetilde{\oplus}} (b \ominus a) \downarrow$ with $a \mathbin{\widetilde{\oplus}} (b \ominus a) = b$, and so $a \leq_{\mathcal{E}(A)} b$. Thus, the natural orders of A and $\mathcal{E}(A)$ coincide, and $\mathcal{E}(A)$ inherits the lattice structure of A.

Finally, to show that $\mathcal{E}(A)$ is indeed an MV-effect algebra, we verify condition (f) of Theorem 1.4.1. Suppose $a \wedge b = 0$. We then have

$$
\begin{aligned}
a \otimes (\neg a \oplus b) &= 0, \\
\neg(a \otimes (\neg a \oplus b)) &= 1, \\
\neg a \oplus \neg(\neg a \oplus b) &= 1,
\end{aligned}
$$

and so $a \leq \neg(\neg a \oplus b) \leq \neg b$ (by Lemma 1.2.3(a) and (b)), whence $a \mathbin{\widetilde{\oplus}} b \downarrow$.

(c) By construction, $\mathcal{A}$ and $\mathcal{E}$ do not change the underlying sets, involution and orthocomplement operations, or zeroes. The only thing we must verify is that the composites $\mathcal{A} \circ \mathcal{E}$ and $\mathcal{E} \circ \mathcal{A}$ do not change the original binary operations.

Let E be an effect algebra, with its operation denoted by $\widetilde{\oplus}$ as usual and the operation of $\mathcal{A}(E)$ denoted by $\oplus$. Denote the operation of $\mathcal{E} \circ \mathcal{A}(E)$ by $\widehat{\oplus}$. Then,

$$
a \mathbin{\widehat{\oplus}} b = \begin{cases} a \oplus b & \text{if } a \leq_E b^{\perp}, \\ \uparrow & \text{otherwise.} \end{cases}
$$

Suppose that $a \leq_E b^{\perp}$. Then by (1.4.1), $a \mathbin{\widehat{\oplus}} b = a \oplus b = a \mathbin{\widetilde{\oplus}} b$.

On the other hand, if it is not the case that $a \leq_E b^{\perp}$, then by Lemma 1.1.10(a), $a \mathbin{\widetilde{\oplus}} b \uparrow$, coinciding with $a \mathbin{\widehat{\oplus}} b \uparrow$. In all cases we obtain $a \mathbin{\widehat{\oplus}} b = a \mathbin{\widetilde{\oplus}} b$.

Now let A be an MV algebra, with its operation denoted by $\oplus$ as usual and the operation of $\mathcal{E}(A)$ denoted by $\widetilde{\oplus}$. Denote the operation of $\mathcal{A} \circ \mathcal{E}(A)$ by $\bar{\oplus}$. Then,

$$
a \mathbin{\bar{\oplus}} b = a \mathbin{\widetilde{\oplus}} (a^{\perp} \wedge b) = \begin{cases} a \oplus (\neg a \wedge b) & \text{if } a \leq \neg(\neg a \wedge b), \\ \uparrow & \text{otherwise.} \end{cases}
$$

Now, $a \leq a \vee \neg b = \neg(\neg a \wedge b)$, so the operation is always defined, and using Lemma 1.3.4(a), we have

$$
a \mathbin{\bar{\oplus}} b = a \oplus (\neg a \wedge b) = (a \oplus \neg a) \wedge (a \oplus b) = 1 \wedge (a \oplus b) = a \oplus b. \qquad \square
$$

Remark 1.4.4. Theorem 1.4.3 is often stated in the present literature, with reference given to [CK97] for proof. However, this paper only mentions effect algebras in passing! They instead prove an equivalence between so-called *Boolean D-Posets* and MV algebras. It is known that D-Posets are equivalent to effect algebras (see, for instance, [PD00, p.21]), but it is not so obvious that the defining equation for a Boolean D-Poset is equivalent to those for an MV-effect algebra, or some parts of the proof would translate. This detail seems to have been frequently swept under the rug.

A proof similar in outline to the one we have given appears in [PD00, p.75], but there are enough nontrivial errors and/or typos in the explanations of associativity (in both directions) which render their proof impossible to parse.

It is natural to ask whether the above isomorphism on the objects of **EA** and **MV** extends to the level of morphisms. It is not difficult to see that the constructions in Theorem 1.4.3 are precisely abstractions of how one would turn $[0, 1]$ from its standard MV algebra structure to its standard effect algebra structure and back again, and that this also extends pointwise to $[0, 1]^2$.

We have already seen in Propositions 1.1.17(b) and 1.2.11(b) that

$$|\operatorname{Hom}_{\mathbf{MV}}([0,1]^2, [0,1])| = 2,$$

but

$$|\operatorname{Hom}_{\mathbf{EA}}([0,1]^2, [0,1])| = 2^{\aleph_0},$$

so there cannot be an isomorphism (or equivalence) between **MV** and the full subcategory of MV-effect algebras of **EA**. The issue is that effect algebra homomorphisms only preserve sums which are defined, so in some sense one can get away with more in defining these maps — once we turn them into MV-effect algebras and make the operation total, the conditions for MV algebra maps are more restrictive. This motivates the following definition.

Definition 1.4.5 (MV-effect homomorphism)**.** Let E, F be MV-effect algebras. Then, an effect algebra homomorphism $f \colon E \to F$ is called an *MV-effect homomorphism* if it additionally satisfies, for all $a, b \in E$, that

$$f(a \mathbin{\widetilde{\oplus}} (a^\perp \wedge b)) = f(a) \mathbin{\widetilde{\oplus}} (f(a)^\perp \wedge f(b)).$$

By direct computation, the above condition is equivalent to $f(a \oplus b) = f(a) \oplus f(b)$. Hence, any MV-algebra homomorphism $f\colon A \to B$ is also an MV-effect homomorphism $f\colon \mathcal{E}(A) \to \mathcal{E}(B)$ and conversely, any MV-effect homomorphism $f\colon E \to F$ is also an MV algebra homomorphism $f\colon \mathcal{A}(E) \to \mathcal{A}(F)$. Denote the *category of MV-effect algebras* with objects MV-effect algebras and arrows MV-effect homomorphisms by **MVEA**. From these results and observations, we thus immediately obtain the following.

Theorem 1.4.6. *Extending the maps $\mathcal{E}$ and $\mathcal{A}$ to morphisms by sending each MV algebra homomorphisms (respectively, each MV-effect homomorphism) to the same underlying function yields an isomorphism of categories* **MV** $\cong$ **MVEA**.

Proposition 1.4.7. *Let A, B be MV-effect algebras. The effect algebra homomorphisms $f\colon A \to B$ which are MV-effect homomorphisms are precisely those which preserve the lattice operations $\wedge$, $\vee$.*

Proof. Let $f\colon A \to B$ be an MV-effect homomorphism. Then, $\mathcal{A}(f)\colon \mathcal{A}(A) \to \mathcal{A}(B)$ is a MV homomorphism, so by Proposition 1.3.6, $\mathcal{A}(f)$ preserves the lattice operation on $\mathcal{A}(A)$. But as the natural orders of A and $\mathcal{A}(A)$ coincide, and f and $\mathcal{A}(f)$ are the same as functions on the underlying sets, we have that f preserves the lattice operations on A.

Conversely, let $f\colon A \to B$ be an effect algebra homomorphisms which preserves the lattice operation. Then, for all $a, b \in A$, we have

$$f(a \mathbin{\widetilde{\oplus}} (a^\perp \wedge b)) = f(a) \mathbin{\widetilde{\oplus}} f(a^\perp \wedge b) = f(a) \mathbin{\widetilde{\oplus}} (f(a^\perp) \wedge f(b)),$$

so that f is an MV-effect homomorphism. □

Example 1.4.8. Consider $f_{1/2}\colon [0,1]^2 \to [0,1]$ as defined in Proposition 1.1.17b. This is an effect algebra homomorphism which is not an MV-effect homomorphism and does not preserve the lattice structure. For instance, we have on the one hand

$$f_{1/2}((1/2, 0) \wedge (0, 1/2)) = f_{1/2}(0,0) = 0,$$

but on the other hand

$$f_{1/2}(1/2, 0) \wedge f_{1/2}(0, 1/2) = 1/4 \wedge 1/4 = 1/4.$$

Chapter 2

Connections to abelian groups and monoids

We turn our attention now to investigating some connections between effect algebras and more traditional algebraic structures in an attempt to better understand how they fit into the larger tapestry of mathematics.

2.1 Interval algebras and universal groups

In this section, we look at the connection between effect algebras and abelian groups via interval effect algebras. We write groups additively throughout.

Definition 2.1.1 (Partially ordered abelian group, positive element, positive cone)**.** An abelian group G is called *partially ordered* if it is equipped with a partial order $\leq$ that is *translation invariant*; that is, for all $x, y, z \in G$, if $x \leq y$, then $x + z \leq y + z$.

A *positive element* in a partially ordered abelian group G is an element $x \in G$ such that $x \geq 0$.

The *positive cone* of a partially ordered abelian group G, denoted G^+, is the subset of all positive elements (note this incluces 0 and is closed under $+$, so it is a cone — however, it is in general not a subgroup).

The following is stated without proof in [BF96].

Definition/Proposition 2.1.2 (Interval effect algebra)**.** Let G be a partially ordered abelian group and $u \in G^+$. Define the interval

$$G^+[0,u] = \{g \in G \mid 0 \leq g \leq u\}.$$

We can then form the *interval effect algebra* $(G^+[0,u], \widetilde{\oplus}, (-)^\perp, 0)$ where $x \widetilde{\oplus} y \downarrow$ if and only if $x + y \leq u$, in which case $x \widetilde{\oplus} y = x + y$, and the orthocomplement is given by $x^\perp = u - x$. The induced effect algebra partial order coincides with the original partial order of the group.

Proof. The conditions (PCM 1)-(PCM 2) follow directly from the fact that the group operation $+$ is commutative and associative, and similarly (PCM 3) follows from the fact that 0 is the group identity.

We now verify that if $a \in G^+[0,u]$, then also $a^\perp = u - a \in G^+[0,u]$. That $a \in G^+[0,u]$ means that $0 \leq a \leq u$. By translation invariance, we have $0 = a - a \leq u - a$. We also have $u - a = 0 + (u - a) \leq a + (u - a) = u$. So indeed, $u - a \in G^+[0,u]$.

For (EA 1), we have, by definition, that $a \widetilde{\oplus} a^\perp \downarrow$ if and only if $a + a^\perp \leq u$; that is, if $a + (u - a) = u \leq u$. This clearly holds, and $1 = 0^\perp = u$. Now if $x \in G^+[0,u]$ is any element such that $a \widetilde{\oplus} x = u$, then $a + x = u$ so $x = u - a$, proving uniqueness.

For (EA 2), if $a \widetilde{\oplus} 1 \downarrow$, that is, $a \widetilde{\oplus} u \downarrow$, then we have $a + u \leq u$. Since $a \in G^+[0,u]$, we know $0 \leq a \leq u$, so by translation invariance we have $a = (a + u) - u \leq u - u = 0$. So antisymmetry forces $a = 0$.

If by the induced effect algebra order we have $a \leq_E b$, then there exists $x \in G^+[0,u]$ such that $a \widetilde{\oplus} x = b$. So then $a + x = b$. But also $0 \leq x \leq u$, so translation invariance yields $a = 0 + a \leq x + a = b$, i.e. $a \leq b$.

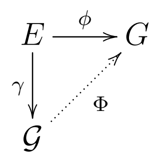
Conversely, if $a \leq b$, then we claim $x = b - a$ is the element such that $a \widetilde{\oplus} x = b$. That the equation holds is clear — we just need to verify that $b - a \in G^+[0,u]$. We have, in particular, $0 \leq a \leq b \leq u$. We have $b - a \leq u - a \leq u$ (recall $u - a = a^\perp$). Also, $0 = a - a \leq b - a$, so indeed $b - a \in G^+[0,u]$, and $a \leq_E b$. □

Definition 2.1.3 (G-valued measure)**.** Let E be an effect algebra and G an abelian group. A function $\phi \colon E \to G$ is called a *G-valued measure* if, for all $x, y \in E$, $x \widetilde{\oplus} y \downarrow$ implies that $\phi(x \widetilde{\oplus} y) = \phi(x) + \phi(y)$.

Definition 2.1.4 (Universal group of an effect algebra)**.** Let E be an effect algebra. A *universal group* for E is a pair $(\mathcal{G}, \gamma)$ where $\mathcal{G}$ is an abelian group and $\gamma\colon E \to \mathcal{G}$ is a $\mathcal{G}$-valued measure such that:

(a) $\gamma(E)$ generates $\mathcal{G}$.

(b) If G is any abelian group and $\phi\colon E \to G$ is a G-valued measure, then there exists a unique group homomorphism $\Phi\colon \mathcal{G} \to G$ such that $\phi = \Phi \circ \gamma$; i.e. the following diagram commutes.



If X is any set and G any abelian group, we define $G^{[X]}$ to be the set of all functions $\psi\colon X \to G$ with finite support; that is, $\psi(x) = 0$ for all but at most finitely many $x \in X$. Then, it is clear that $G^{[X]}$ is an abelian group under pointwise operations; i.e. $(\psi + \varphi)(x) = \psi(x) + \varphi(x)$.

Theorem 2.1.5. *Every effect algebra has a universal group which is unique up to isomorphism.*

Proof. Let E be an effect algebra. For each $p \in E$, define a map

$$\chi_p\colon E \to \mathbb{Z},$$
$$x \mapsto \begin{cases} 1, & \text{if } x = p, \\ 0, & \text{otherwise.} \end{cases}$$

Considering $\mathbb{Z}$ as an additive abelian group and E as simply a set, it is clear $\chi_p \in \mathbb{Z}^{[E]}$ for each $p \in E$.

Denote by H the subgroup of $\mathbb{Z}^{[E]}$ generated by the set

$$\{\chi_p + \chi_q - \chi_r \mid p \mathbin{\widetilde{\oplus}} q \downarrow \text{ and } p \mathbin{\widetilde{\oplus}} q = r\}.$$

Let $\mathcal{G} = \mathbb{Z}^{[E]}/H$, and let $\eta\colon \mathbb{Z}^{[E]} \to \mathcal{G}$ be the natural projection onto the quotient. Define a map

$$\begin{aligned} \gamma\colon E &\to \mathcal{G}, \\ x &\mapsto \eta(\chi_x). \end{aligned}$$

We claim $(\mathcal{G}, \gamma)$ is a universal group for E. First, if $p, q, r \in E$ such that $p \mathbin{\widetilde{\oplus}} q \downarrow$ and $p \mathbin{\widetilde{\oplus}} q = r$, then we have $\chi_p + \chi_q - \chi_r \in H$, so

$$\gamma(p) + \gamma(q) - \gamma(r) = \eta(\chi_p) + \eta(\chi_q) - \eta(\chi_r) = \eta(\chi_p + \chi_q - \chi_r) = 0,$$

from which it follows that $\gamma(p) + \gamma(q) = \gamma(r) = \gamma(p \mathbin{\widetilde{\oplus}} q)$. Hence, γ is a $\mathcal{G}$-valued measure.

Now, if $\psi \in \mathbb{Z}^{[E]}$, this means $\psi\colon E \to \mathbb{Z}$ has finite support, and we can write

$$\psi = \sum_{p \in \operatorname{supp}(\psi)} \psi(p) \cdot \chi_p,$$

and so we see that elements of the form χ_x for $x \in X$ generate $\mathbb{Z}^{[E]}$. Then,

$$\gamma(E) = \{\eta(\chi_x) \mid x \in E\} = \{\chi_x + H \mid x \in E\}, \tag{2.1.1}$$

which generates $\mathcal{G}$.

Let now G be any abelian group, and $\phi\colon E \to G$ a G-valued measure. Define a map

$$\begin{aligned} \Phi'\colon \mathbb{Z}^{[E]} &\to G, \\ \psi &\mapsto \sum_{p \in \operatorname{supp}(\psi)} \psi(p) \cdot \phi(p). \end{aligned}$$

That Φ' is a group homomorphism is clear. Observe also that for all $x \in E$, we have

$$\Phi'(\chi_x) = \sum_{p \in \operatorname{supp}(\chi_x)} \chi_x(p) \cdot \phi(p) = \chi_x(x) \cdot \phi(x) = \phi(x).$$

Furthermore, if $p, q, r \in E$ such that $p \mathbin{\widetilde{\oplus}} q \downarrow$ and $p \mathbin{\widetilde{\oplus}} q = r$, then using the fact that ϕ is a G-valued measure, we have

$$\Phi'(\chi_p + \chi_q - \chi_r) = \phi(p) + \phi(q) - \phi(r) = 0.$$

Thus, $H = \ker(\eta) \subseteq \ker(\Phi')$, and so this induces a well-defined group homomorphism

$$\Phi \colon \mathcal{G} \to G,$$
$$g + H \mapsto \Phi'(g).$$

Then, for $p \in E$, we have

$$\Phi \circ \gamma(p) = \Phi(\gamma(p)) = \Phi(\eta(\chi_p)) = \Phi(\chi_p + H) = \Phi'(\chi_p).$$

That is, the diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & G \\ {\scriptstyle \gamma}\big\downarrow & \nearrow_{\Phi} & \\ \mathcal{G} & & \end{array} \tag{2.1.2}$$

commutes.

Suppose now that $\tau \colon \mathcal{G} \to G$ is a group homomorphism making (2.1.2) commute in place of Φ. Since $\{\chi_x + H \mid x \in E\}$ (2.1.1) is a generating set for $\mathcal{G}$, to show $\tau = \Phi$, it suffices to show their action on this set is the same. Indeed, for $p \in E$, we have

$$\Phi(\chi_p + H) = \Phi'(\chi_p) = \phi(p) = \tau \circ \gamma(p) = \tau(\eta(\chi_p)) = \tau(\chi_p + H),$$

proving uniqueness of the induced map.

Now suppose $(\mathcal{G}, \gamma)$ and $(\mathcal{J}, \theta)$ are both universal groups for E. There are unique group homomorphisms $\overline{\theta}$ and $\overline{\gamma}$ making the following diagrams commute.

$$\begin{array}{ccc} E & \xrightarrow{\theta} & \mathcal{J} \\ {\scriptstyle \gamma}\big\downarrow & \nearrow_{\overline{\theta}} & \\ \mathcal{G} & & \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\gamma} & \mathcal{G} \\ {\scriptstyle \theta}\big\downarrow & \nearrow_{\overline{\gamma}} & \\ \mathcal{J} & & \end{array}$$

There must also be a unique group homomorphism $h \colon \mathcal{J} \to \mathcal{J}$ which makes

$$\begin{array}{ccc} E & \xrightarrow{\theta} & \mathcal{J} \\ {\scriptstyle \theta}\big\downarrow & \nearrow_{h} & \\ \mathcal{J} & & \end{array}$$

commute. One the one hand, it is clear that $h = \mathrm{id}_{\mathcal{J}}$ works. But also we have

$$\theta = \overline{\theta} \circ \gamma = \overline{\theta} \circ (\overline{\gamma} \circ \theta) = (\overline{\theta} \circ \overline{\gamma}) \circ \theta,$$

so $h = \overline{\theta} \circ \overline{\gamma}$ also works. Uniqueness thus forces $\overline{\theta} \circ \overline{\gamma} = \mathrm{id}_{\mathcal{J}}$. Arguing similarly, $\overline{\gamma} \circ \overline{\theta} = \mathrm{id}_{\mathcal{G}}$. Therefore, $\mathcal{G} \cong \mathcal{J}$. □

Remark 2.1.6. Definition 2.1.4 first appears in [BF94], but there appears to be a minor oversight — the uniqueness of the induced group homomorphism is not part of the definition! Theorem 2.1.5 appears in the same paper stated in the same way, but the proof of the uniqueness of the induced map is absent (as it is not part of their definition). However, the claim that the universal group is unique up to isomorphism is still present — but without uniqueness of the induced map, the present author does not see how the argument presented in the original paper suffices to support the claim.

Remark 2.1.7. One may naturally wonder if there are any further classifications of what kinds of effect algebras arise as interval algebras. It is well known that all MV algebras do, and so at least all MV-effect algebras do. We refer the reader to [CDM00, Chapters 2 & 7] for details.

2.2 The Riesz decomposition property

In this section we revisit the Riesz decomposition property, first mentioned in Theorem 1.4.1, for various structures.

Definition 2.2.1 (Riesz interpolation property)**.** Let X be a partially ordered set. We say that X has the *Riesz interpolation property* if whenever $a_1, a_2, b_1, b_2 \in X$ such that $a_i \leq b_j$ for $i, j = 1, 2$, then there is $c \in X$ such that $a_i \leq c \leq b_j$ for $i, j = 1, 2$.

Definition 2.2.2 (Riesz decomposition property)**.** Let M be a partial commutative monoid. We say that M has the *Riesz decomposition property* if whenever $a, b_1, b_2 \in M$ such that $a \leq b_1 \mathbin{\widetilde{\oplus}} b_2$, then there exist a_1, a_2 such that $a = a_1 \mathbin{\widetilde{\oplus}} a_2$, $a_1 \leq b_1$, and $a_2 \leq b_2$.

Definition 2.2.3 (Refinement property)**.** Let M be a partial commutative monoid. We say that M has the *refinement property* if whenever $a_1, a_2, b_1, b_2 \in M$ such that $a_1 \mathbin{\widetilde{\oplus}} a_2 = b_1 \mathbin{\widetilde{\oplus}} b_2$, then there exist c_{11}, c_{12}, c_{21}, $c_{22} \in M$ such that $a_i = c_{i1} \mathbin{\widetilde{\oplus}} c_{i2}$ and

$b_j = c_{1j} \widetilde{\oplus} c_{2j}$ for $i, j = 1, 2$. One can visualize this condition as a *refinement matrix*; that is, each row and column of the inner 2×2 matrix below respectively sums to the entry on the outside.

$$\begin{array}{c|cc} & a_1 & a_2 \\ \hline b_1 & c_{11} & c_{21} \\ b_2 & c_{12} & c_{22} \end{array}$$

It is well known that for abelian groups, the three conditions above are equivalent in the following sense.

Proposition 2.2.4. *Let G be a partially ordered abelian group. The following conditions are equivalent.*

(a) G has the Riesz interpolation property.

(b) The positive cone G^+ has the Riesz decomposition property.

(c) The positive cone G^+ has the refinement property.

Proof. See [PD00, p.58] □

A partially ordered abelian group satisfying the equivalent conditions of the preceding proposition is known as an *interpolation group*.

Proposition 2.2.5. *Let M be a cancellative partial commutative monoid (that is, a PCM satisfying (GEA 1)). Then, the following conditions are equivalent.*

(a) M has the Riesz decomposition property.

(b) M has the refinement property.

Proof. (a) $\implies$ (b): Let $a_1, a_2, b_1, b_2 \in M$ such that $a_1 \widetilde{\oplus} a_2 = b_1 \widetilde{\oplus} b_2$. Then, $a_1 \leq b_1 \widetilde{\oplus} b_2$. By hypothesis, there exist $c_{11}, c_{12} \in M$ such that $a_1 = c_{11} \widetilde{\oplus} c_{12}$, $c_{11} \leq b_1$, and $c_{12} \leq b_2$. Let $c_{21}, c_{22} \in M$ be the elements satisfying $c_{11} \widetilde{\oplus} c_{21} = b_1$ and $c_{12} \widetilde{\oplus} c_{22} = b_2$. We have

$$a_1 \widetilde{\oplus} a_2 = b_1 \widetilde{\oplus} b_2 = (c_{11} \widetilde{\oplus} c_{21}) \widetilde{\oplus} (c_{12} \widetilde{\oplus} c_{22}) = (c_{11} \widetilde{\oplus} c_{12}) \widetilde{\oplus} (c_{21} \widetilde{\oplus} c_{22}) = a_1 \widetilde{\oplus} (c_{21} \widetilde{\oplus} c_{22}).$$

It follows from (GEA 1) that $a_2 = c_{21} \mathbin{\widetilde{\oplus}} c_{22}$.

(b) $\implies$ (a): Suppose $a \leq b_1 \mathbin{\widetilde{\oplus}} b_2$. Let $a' \in M$ satisfying $a \mathbin{\widetilde{\oplus}} a' = b_1 \mathbin{\widetilde{\oplus}} b_2$. By hypothesis, there exist $c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that

$$\begin{array}{c|cc} & a & a' \\ \hline b_1 & c_{11} & c_{21} \\ b_2 & c_{12} & c_{22} \end{array}$$

is a refinement matrix. We have that $a = c_{11} \mathbin{\widetilde{\oplus}} c_{12}$, $c_{11} \leq b_1$, and $c_{12} \leq b_2$ as desired. □

Corollary 2.2.6. *For (generalized) effect algebras, the Riesz decomposition property and the refinement property are equivalent.*

Note that in the proof of Proposition 2.2.5, the cancellative property is only used in one direction. Thus we have also the following.

Corollary 2.2.7. *For partial commutative monoids (not necessarily cancellative), the refinement property implies the Riesz decomposition property.*

A PCM satisfying the refinement property (and thus the Riesz decomposition property) is known as a *partial refinement monoid*. However, the next example shows that the converse does not hold when we do not require our PCM to be cancellative.

Example 2.2.8. Let $M = \{0, 1, \infty\}$ with the following partial binary operation.

$\widetilde{\oplus}$	0	1	∞
0	0	1	∞
1	1	∞	∞
∞	∞	∞	$\uparrow$

We will show that M is a PCM which has the Riesz decomposition property but not the refinement property.

If $a = 0$, regardless of b_1, b_2, we may take $a_1 = a_2 = 0$ to satisfy the Riesz decomposition property. For $a = 1$ or $a = \infty$, the table below gives possible choices of a_1 and a_2 for all cases (since $\widetilde{\oplus}$ is commutative, we do not need both $b_1 \widetilde{\oplus} b_2$ and $b_2 \widetilde{\oplus} b_1$).

a	b_1	b_2	$b_1 \widetilde{\oplus} b_2$	a_1	a_2
1	1	0	1	1	0
1	1	1	∞	1	0
1	∞	1	∞	1	0
1	∞	0	∞	1	0
∞	1	1	∞	1	1
∞	∞	0	∞	∞	0
∞	∞	1	∞	∞	1

Now note that $1 \widetilde{\oplus} 1 = \infty = 1 \widetilde{\oplus} \infty$. Suppose there existed $c_{11}, c_{12}, c_{21}, c_{22}$ giving a refinement matrix as follows.

	1	∞
1	c_{11}	c_{21}
1	c_{12}	c_{22}

The only way to sum to 1 is $0 \widetilde{\oplus} 1 = 1 = 1 \widetilde{\oplus} 0$. So for $c_{11} \widetilde{\oplus} c_{12} = 1$ to hold, either $c_{11} = 1$ and $c_{12} = 0$, or $c_{11} = 0$ and $c_{12} = 1$. Without loss of generality, suppose it is the former.

Then, we have $1 = c_{11} \widetilde{\oplus} c_{21} = 1 \widetilde{\oplus} c_{21}$, forcing $c_{21} = 0$. But also $c_{21} \widetilde{\oplus} c_{22} = 0 \widetilde{\oplus} c_{22} = \infty$, forcing $c_{22} = \infty$. Finally, we have $\infty = 0 \widetilde{\oplus} \infty = c_{12} \widetilde{\oplus} c_{22} = 1$. This is a contradiction, and hence M does not have the refinement property.

Theorem 2.2.9. *(a) Every effect algebra with the Riesz decomposition property is (isomorphic to) an interval effect algebra.*

(b) An effect algebra can be embedded into an interpolation group if and only if it has the Riesz decomposition property.

Proof. See [Rav96, p.20]. □

Chapter 3

The category of effect algebras

In this chapter we develop the categorical structure of **EA**, inspired by work of Bart Jacobs, [Jac12] and [Jac14]. We will see, in particular, how the partiality of the operation causes some of the categorical constructs and proofs to be very complicated. In particular, the final theorem of the chapter is quite astounding in how simple the statement is in contrast with how convoluted the proof gets.

3.1 Some limits and colimits

We will show in this section that **EA** has finite products and coproducts, and equalizers, all of which are easy to describe. Coequalizers, on the other hand, will require some machinery to be developed in the following section.

Proposition 3.1.1. *The terminal object of* **EA** *is the trivial one-element effect algebra* $\{*\}$ *and the initial object of* **EA** *is the two-element effect algebra* $\{0, 1\}$ *with the only possible operation and orthocomplementation.*

Proof. Given any effect algebra E, it is clear the only effect algebra homomorphism into $\{*\}$ is the one mapping every element of E to $*$, and the only effect algebra homomorphism from $\{0, 1\}$ to E is the one sending 0 to 0_E and 1 to 1_E. □

Proposition 3.1.2. *Given two effect algebras* A *and* B*, the product* $A \times B$ *is the cartesian product of sets, with zero* $(0_A, 0_B)$ *and equipped with the pointwise partial* *operation*

$$(a_1, b_1) \widetilde{\oplus} (a_2, b_2) \downarrow \Leftrightarrow a_1 \widetilde{\oplus} a_2 \downarrow \text{ and } b_1 \widetilde{\oplus} b_2 \downarrow,$$

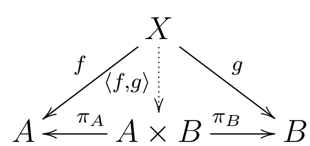
and when defined we have

$$(a_1, b_1) \widetilde{\oplus} (a_2, b_2) = (a_1 \widetilde{\oplus} a_2, b_1 \widetilde{\oplus} b_2).$$

The maps $\pi_A \colon A \times B \to A$ *and* $\pi_b \colon A \times B \to B$ *are the usual set-theoretic projections.*

Proof. Following an easy verification of the axioms, $A \times B$ with the described operation is an effect algebra, and π_A and π_B are effect algebra homomorphisms.

Suppose X is an effect algebra, $f \colon X \to A$ and $g \colon X \to B$ are effect algebra homomorphisms. Arguing as for products in **Set**, we know the unique map $\langle f, g \rangle \colon X \to A \times B$ making the diagram



commute on the level of sets and functions is $\langle f, g \rangle(x) = (f(x), g(x))$. It only remains to verify $\langle f, g \rangle$ is an effect algebra homomorphism. Indeed, $\langle f, g \rangle(1_X) = (f(1_X), g(1_X)) = (1_A, 1_B) = 1_{A \times B}$. Now suppose $x, y \in X$ with $x \widetilde{\oplus} y \downarrow$. Then,

$$\begin{aligned}
\langle f, g \rangle(x \widetilde{\oplus} y) &= (f(x \widetilde{\oplus} y), g(x \widetilde{\oplus} y)) \\
&= (f(x) \widetilde{\oplus} f(y), g(x) \widetilde{\oplus} g(y)) \\
&= (f(x), g(x)) \widetilde{\oplus} (f(y), g(y)) \\
&= \langle f, g \rangle(x) \widetilde{\oplus} \langle f, g \rangle(y).
\end{aligned}$$

□

Proposition 3.1.3. *Given two effect algebras A and B, the coproduct $A + B$ is the disjoint union with zeroes and units identified, that is, as a set, $A + B = \{(a, 0) \mid a \in A \backslash \{0_A, 1_A\}\} \cup \{(b, 1) \mid b \in B \backslash \{0_B, 1_B\}\} \cup \{0, 1\}$. The operation is given by*

$$(x, n) \widetilde{\oplus} (y, m) = \begin{cases} \uparrow & \text{if } n \neq m, \text{ or if } n = m \text{ and } x \widetilde{\oplus} y \uparrow, \\ (x \widetilde{\oplus} y, n) & \text{if } n = m \text{ and } x \widetilde{\oplus} y \downarrow. \end{cases}$$

and since 0 *is the zero and* $x \mathbin{\widetilde{\oplus}} 1 \downarrow$ *implies* $x = 0$*, the above is sufficient. The maps* $\iota_A \colon A \to A + B$ *and* $\iota_B \colon B \to A + B$ *are the usual set-theoretic inclusions (with* $\iota_A(0_A) = 0 = \iota_B(0_B)$ *and* $\iota_A(1_A) = 1 = \iota_B(1_B)$*).*

Proof. Following an easy verification of the axioms, $A + B$ with the described operation is an effect algebra, and ι_A and ι_B are effect algebra homomorphisms.

Suppose X is an effect algebra, $f \colon A \to X$ and $g \colon B \to X$ are effect algebra homomorphisms. Arguing similarly as for coproducts in **Set**, we see that the unique map $[f, g] \colon A + B \to X$ making the diagram

$$\begin{array}{ccccc} & & X & & \\ & \overset{f}{\nearrow} & \uparrow{\scriptstyle [f,g]} & \overset{g}{\nwarrow} & \\ A & \xrightarrow{\iota_A} & A + B & \xleftarrow{\iota_B} & B \end{array}$$

commute on the level of sets and functions is

$$\begin{aligned} [f, g] \colon 0 &\mapsto 0_X, \\ 1 &\mapsto 1_X, \\ (a, 0) &\mapsto f(a), \\ (b, 1) &\mapsto g(b). \end{aligned}$$

It only remains to verify $[f, g]$ is an effect algebra homomorphism. Suppose $a, b \in A + B$ with $a \mathbin{\widetilde{\oplus}} b \downarrow$. The cases where a or b are 0 or 1 are clear. So suppose $a = (x, n)$, $b = (y, m)$, and $(x, n) \mathbin{\widetilde{\oplus}} (y, m) \downarrow$. Then, $n = m$ and $x \mathbin{\widetilde{\oplus}} y \downarrow$. If $n = 0$, then

$$\begin{aligned} [f, g]((x, 0) \mathbin{\widetilde{\oplus}} (y, 0)) &= [f, g](x \mathbin{\widetilde{\oplus}} y, 0) \\ &= f(x \mathbin{\widetilde{\oplus}} y) \\ &= f(x) \mathbin{\widetilde{\oplus}} f(y) \\ &= [f, g](x, 0) \mathbin{\widetilde{\oplus}} [f, g](y, 0). \end{aligned}$$

The case $n = 1$ is similar. □

Proposition 3.1.4. *Given two effect algebra homomorphisms* $f, g \colon E \to F$*, their equalizer* $E_{f,g}$ *is the set* $\{x \in E \mid f(x) = g(x)\}$ *together with the inclusion* $i \colon E_{f,g} \to E$.

Proof. We verify that $E_{f,g}$ is a sub-effect algebra of E. For (SEA 1), we have $f(0_E) = 0_F = g(0_E)$, so $0_E \in E_{f,g}$.

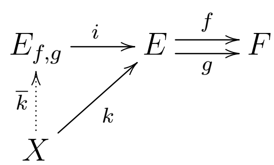
For (SEA 2), if $a \in E_{f,g}$, then $f(a) = g(a)$. Then by Lemma 1.1.16(c), $f(a^\perp) = f(a)^\perp = g(a)^\perp = g(a^\perp)$, so $a^\perp \in E_{f,g}$.

For (SEA 3), suppose $a, b \in E_{f,g}$ and $a \mathbin{\widetilde{\oplus}} b \downarrow$. Then,

$$f(a \mathbin{\widetilde{\oplus}} b) = f(a) \mathbin{\widetilde{\oplus}} f(b) = g(a) \mathbin{\widetilde{\oplus}} g(b) = g(a \mathbin{\widetilde{\oplus}} b),$$

so $a \mathbin{\widetilde{\oplus}} b \in E_{f,g}$.

It is clear that the inclusion $i\colon E_{f,g} \to E$ is an effect algebra homomorphism. Now suppose $k\colon X \to E$ is an effect algebra homomorphism such that $f \circ k = g \circ k$. Arguing as for equalizers in **Set**, we know the unique map that makes the diagram



commute on the level of sets and functions is $\overline{k}(x) = k(x)$. Since k is assumed to be an effect algebra homomorphism, $\overline{k}$ is obviously also an effect algebra homomorphism. □

Remark 3.1.5. For categorical logicians, it is natural to wonder whether any new category of interest is cartesian closed. For much the same reason as the category of Boolean algebras, **EA** is not — the product functor $A \times (-)$ clearly does not preserve the initial object $\{0, 1\}$ and so cannot be a left adjoint.

3.2 Congruences, BCMS, and coequalizers

We have seen that products, coproducts, and equalizers are in **EA** are fairly straightforward to describe. On the other hand, coequalizers are not so easy but they have been described in [Jac12]. The complication arises from the fact that it is very difficult to work with congruences when dealing with partial structures. We first describe what a congruence relation on an effect algebra needs to be in order for us to be able to quotient out by it, and see why this is not useful for our purposes.

Definition 3.2.1 ((Weak) effect algebra congruence)**.** An *effect algebra congruence*, or EA congruence, is an equivalence relation $\sim$ on an effect algebra E such that the following conditions hold.

(CE 1) For all $a_1, a_2, b_1, b_2 \in E$, if $a_1 \mathbin{\widetilde{\oplus}} b_1 \downarrow$, $a_2 \mathbin{\widetilde{\oplus}} b_2 \downarrow$, $a_1 \sim a_2$, and $b_1 \sim b_2$, then $a_1 \mathbin{\widetilde{\oplus}} b_1 \sim a_2 \mathbin{\widetilde{\oplus}} b_2$.

(CE 2) For all $a_1, a_2, b_1 \in E$, if $a_1 \mathbin{\widetilde{\oplus}} b_1 \downarrow$ and $a_1 \sim a_2$, then there exists $b_2 \in E$ such that $b_1 \sim b_2$ and $a_2 \mathbin{\widetilde{\oplus}} b_2 \downarrow$.

If only (CE 1) holds, then $\sim$ is a *weak effect algebra congruence*.

Proposition 3.2.2. *If E is an effect algebra and $\sim$ is an EA congruence on E, then $E/\sim$ is an effect algebra with $[a] \mathbin{\widetilde{\oplus}} [b] \downarrow \Leftrightarrow a \mathbin{\widetilde{\oplus}} b \downarrow$, in which case $[a] \mathbin{\widetilde{\oplus}} [b] = [a \mathbin{\widetilde{\oplus}} b]$.*

Proof. See [PD00, p.195] □

The issue is that we would like to, given a relation R on an effect algebra E, speak of the smallest congruence containing R or the congruence generated by R. However, neither of these notions is well-formed.

In the case of the former, the intersection of EA congruences need not be an EA congruence. Indeed, if $\sim_1$ and $\sim_2$ are congruences on E, we could have some $a_1, a_2, b_1 \in E$ such that $a_1 \mathbin{\widetilde{\oplus}} b_1 \downarrow$, and both $a_1 \sim_1 a_2$ and $a_1 \sim_2 a_2$. Then, (CE 2) guarantees existence of elements b_2, b_2' such that $b_1 \sim_1 b_2$ and $b_1 \sim_2 b_2'$, but there is no reason why there need be a single element for both relations!

In the case of the latter, (CE 2) again poses a problem - there is no way of choosing the elements that must exist to satisfy the condition. The best we could do is speak of the weak EA congruence generated by a relation, but this is not enough for the quotient to inherit the structure of an effect algebra.

What is required, then, is to *totalize* effect algebras into total structures, construct coequalizers in this new category, and via a certain type of adjunction, transfer them back to **EA**. We now summarize the definitions and results required to establish what coequalizers in **EA** are.

A *coreflection* is a pair of adjoint functors $F\colon \mathbf{C} \rightleftarrows \mathbf{D}\colon G$ (with $F \dashv G$) where the left adjoint F is full and faithful (equivalently, the unit of the adjuction η is a natural isomorphism). It is known that in such a situation, the existence of any type of limit or colimit in $\mathbf{D}$ implies the existence of the same type of limit or colimit in $\mathbf{C}$ in the following sense.

Theorem 3.2.3. *Let $F\colon \mathbf{C} \rightleftarrows \mathbf{D}\colon G$ be a coreflection as above and $\mathbf{J}$ be an index category. Let $D\colon \mathbf{J} \to \mathbf{C}$ be a diagram in $\mathbf{C}$. Then, if $l_j\colon L \to FD_j$ is a limit of FD in $\mathbf{D}$, then $\eta_{D_j}^{-1} \circ G(l_j)\colon G(L) \to D_j$ is a limit of D in $\mathbf{C}$. If $c_j\colon FD_j \to C$ is a colimit of $F \circ D$ in $\mathbf{D}$, then $G(c_j) \circ \eta_{D_j}\colon D_j \to G(C)$ is a colimit of D in $\mathbf{C}$.*

Proof. See [Bor94, Propositions 3.5.3 & 3.5.4]. □

Definition 3.2.4. A *barred commutative monoid*, or BCM, is a quadruple $(M, +, 0, u)$ wherein $(M, +, 0)$ is a commutative monoid and $u \in M$ is an element called the *bar* of M, such that the following hold for all $a, b, c \in M$.

(BCM 1) (Positivity): If $a + b = 0$, then $a = b = 0$.

(BCM 2) (Cancellation under the bar): If $a + b = a + c = u$, then $b = c$.

A BCM homomorphism $f\colon M \to N$ is a monoid homomorphism that additionally preserves the bar; i.e. $f(u_M) = u_N$. The *category of barred commutative monoids*, $\mathbf{BCM}$, has as objects BCMs and as arrows BCM homomorphisms.

We now define a *totalization* functor $\mathcal{T}o\colon \mathbf{EA} \to \mathbf{BCM}$ as follows. If $(E, \widetilde{\oplus}, (-)^{\perp}, 0)$ is an effect algebra, let $\mathcal{M}(E)$ be the free commutative monoid on the underlying set E; its elements are of the form $\sum_{i=1}^{k} n_i a_i$ with $n_i \in \mathbb{N}$, $a_i \in E$, which one can identify with functions $\phi\colon E \to \mathbb{N}$ with finite support and addition defined pointwise. Then, $\mathcal{T}o(E) = (\mathcal{M}(E)/\sim, 1(1_E))$, where $\sim$ is the smallest monoid congruence satisfying:

- For all $a, b \in E$ with $a \,\widetilde{\oplus}\, b \downarrow$, $1(a) + 1(b) \sim 1(a \,\widetilde{\oplus}\, b)$.
- $0_{\mathcal{M}(E)} \sim 1(0_E)$.

Elements of $\mathcal{T}o(E)$ are equivalence classes $[\phi]$, but when no confusion arises we will abuse notation and simply write ϕ. For an effect algebra homomorphism $f\colon E \to F$, we define

$$\mathcal{T}o(f)\colon \mathcal{T}o(E) \to \mathcal{T}o(F),$$
$$\sum_{i=1}^{k} n_i a_i \mapsto \sum_{i=1}^{k} n_i f(a_i).$$

In the other direction, we define a *partialization* functor $\mathcal{P}a\colon \mathbf{BCM} \to \mathbf{EA}$ by $\mathcal{P}a(M,u) = [0,u] = \{x \in M \mid x \leq u\}$, with $x \mathbin{\widetilde{\oplus}} y \downarrow \Leftrightarrow x + y \leq u$, in which case $x \mathbin{\widetilde{\oplus}} y = x + y$. Orthocomplementation is given by $x^{\perp} = u - x$. For a BCM homomorphism $f\colon M \to N$, we get an EA homomorphism $\mathcal{P}a(f)\colon \mathcal{P}a(M) \to \mathcal{P}a(N)$ via the restriction $f|_{[0,u]}$.

By following the more general construction for PCMs and DCMs (*downsets in commutative monoids*) in [Jac12], it is readily seen that $\mathcal{P}a\mathcal{T}o(E) = \{[1x] \mid x \in E\}$.

Theorem 3.2.5. *The functors $\mathcal{T}o\colon \mathbf{EA} \rightleftarrows \mathbf{BCM}\colon \mathcal{P}a$ form an adjunction with $\mathcal{T}o$ a full and faithful left adjoint. Hence, $\mathbf{EA} \hookrightarrow \mathbf{BCM}$ is a coreflection.*

Proof. See [Jac12, Theorem 6 & Proposition 3]. In particular, the unit of the adjunction $\eta\colon \mathrm{id}_{\mathbf{EA}} \to \mathcal{P}a\mathcal{T}o$ is an natural isomorphism, given for $E \in \mathrm{Ob}\,\mathbf{EA}$ by

$$\eta_E\colon E \to \mathcal{P}a\mathcal{T}o(E),$$
$$x \mapsto [1x].$$

□

Example 3.2.6. Consider the usual effect algebra $[0,1]$. Elements of $\mathcal{M}([0,1])$ are of the form $x = \sum_{i=1}^{k} n_i a_i$ with $n_i \in \mathbb{N}$, $a_i \in [0,1]$. If one takes this formal sum and simply treats it as a sum of products in $\mathbb{R}$, one gets a nonnegative real number, which we will call the *real value* of x. It is not difficult to see that if the real values of x and y are the same, then $x \sim y$ in $\mathcal{T}o([0,1])$. One can take, for any nonnegative real number r, $\lfloor r \rfloor(1) + 1(r - \lfloor r \rfloor)$ as a canonical representative of the class of all elements of $\mathcal{T}o([0,1])$ with real value r.

For example, consider the elements $1(0.3) + 1(0.8) + 1(0.4)$ and $3(0.5)$, both with real value 1.5. We have

$$1(0.3) + 1(0.8) + 1(0.4) \sim 1(0.3) + 1(0.7) + 1(0.1) + 1(0.4) \sim 1(1) + 1(0.5),$$

and

$$3(0.5) = 1(0.5) + 1(0.5) + 1(0.5) \sim 1(0.5 + 0.5) + 1(0.5) = 1(1) + 1(0.5).$$

That distinct real numbers are identified with distinct equivalence classes follows from the fact that $\sim$ is the smallest congruence satisfying the required conditions; i.e. it is generated by $1(a + b) \sim 1(a) + 1(b)$ for $a + b \leq 1$, together with the monoid congruence condition $a_1 \sim b_1$, $a_2 \sim b_2$ giving $a_1 + a_2 \sim b_1 + b_2$ and closed under equivalence relations, and two elements with distinct real values are never related by any of these.

Thus, we see that $\mathcal{T}o([0, 1]) \cong (\mathbb{R}_{\geq 0}, 1)$.

Examples 3.2.7. Reasoning similarly to the previous example, some more examples of totalizations of effect algebras are:

(a) $\mathcal{T}o(\{0, 1\}) \cong (\mathbb{N}, 1)$.

(b) $\mathcal{T}o([0, 1]^2) \cong (\mathbb{R}^2_{\geq 0}, (1, 1))$.

(c) Let X be a set and consider $\mathcal{P}(X)$ as an effect algebra. Elements of $\mathcal{T}o(\mathcal{P}(X))$ are formal sums of subsets of X which are in the same equivalence class if and only if the total number of times every element of X appears are the same.

For instance, let $X = \{x, y\}$. Then, consider the elements $a = 3\{x\} + 5\{y\}$ and $b = 2\{x, y\} + 1\{x\} + 3\{y\}$ of $\mathcal{T}o(\mathcal{P}(X))$. The element x appears a total of three times and the element y appears a total of 5 times in both a and b. We have

$$3\{x\} + 5\{y\} = 2\{x\} + 2\{y\} + 1\{x\} + 3\{y\} \sim 2\{x, y\} + 1\{x\} + 3\{y\}.$$

Thus an equivalence class in $\mathcal{T}o(\mathcal{P}(X))$ is determined by assigning a natural number to each element of X, and we see that $\mathcal{T}o(\mathcal{P}(X)) \cong (\mathbb{N}^X, f)$ where $f(x) = 1$ for all $x \in X$.

Definition 3.2.8 (BCM (pre)congruence)**.** Let M be a BCM. Then, a relation $\sim$ on M is a *BCM congruence* if it is an equivalence relation that satisfies the following conditions for all $a, b, c, d \in M$.

(CB 1) If $a \sim b$ and $c \sim d$, then $a + c \sim b + d$.

(CB 2) If $a + b \sim 0$, then $a \sim 0$ and $b \sim 0$.

(CB 3) If $a + b \sim u$ and $a + c \sim u$, then $b \sim c$.

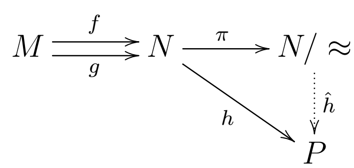
If $\sim$ satisfies only (CB 1) and (CB 2), then we call $\sim$ a *BCM precongruence*.

Proposition 3.2.9. *If M is a BCM and $\sim$ is a BCM congruence on M, then $M/\sim$ is a BCM with $[a] + [b] = [a + b]$ with zero $[0]$ and bar $[u]$.*

Proof. Well-definedness of the quotient operation follows from (CB 1). Similarly, (BCM 1) and (BCM 2) follow respectively from (CB 2) and (CB 3). $\square$

Proposition 3.2.10. *Let $f, g\colon M \to N$ be BCM homomorphisms. The coequalizer of f, g is $\pi\colon N \to N/\approx$, where $\approx$ is the smallest congruence on N containing $\{(f(x), g(x)) \mid x \in M\}$ and π is the natural projection onto the quotient.*

Proof. For all $a \in M$, we have $\pi \circ f(a) = [f(a)] = [g(a)] = \pi \circ g(a)$, so that $\pi \circ f = \pi \circ g$. Now let $h\colon N \to P$ be a BCM homomorphism such that $h \circ f = h \circ g$. Suppose there is a BCM homomorphism $\hat{h}\colon N/\approx \to P$ making the following diagram commute.



For all $x \in N$, we must have $h(x) = \hat{h} \circ \pi(x) = \hat{h}([x])$, so the only possible way to define $\hat{h}$ is by $\hat{h}([x]) = h(x)$.

We need to show $\hat{h}$ is well defined with respect to the quotient structure. We proceed by structural induction on all possible ways to obtain new relations in $\approx$. For the base case, we have that for all $x \in M$, $f(x) \approx g(x)$, and $\hat{h}([f(x)]) = h \circ f(x) = h \circ g(x) = \hat{h}([g(x)])$.

- Reflexivity and symmetry are trivial. Transitivity is also easy, and is similar to the argument for coequalizers in **Set**.

- For (CB 1), suppose $a, b, c, d \in N$ and $a \approx b$, $c \approx d$. We suppose already that $\hat{h}([a]) = \hat{h}([b])$ and $\hat{h}([c]) = \hat{h}([d])$. We obtain that $a + c \approx b + d$, and

$$\hat{h}([a+c]) = h(a+c) = h(a) + h(c) = h(b) + h(d) = h(b+d) = \hat{h}([b+d]).$$

- For (CB 2), suppose $a, b \in N$ and $a+b \approx 0$. We suppose already that $\hat{h}([a+b]) = \hat{h}([0])$. We obtain that $a \approx 0$, $b \approx 0$, and that $h(a)+h(b) = h(a+b) = h(0) = 0$. From (BCM 1), we get $h(a) = 0$ and $h(b) = 0$, so then $\hat{h}([a]) = h(a) = 0 = h(0) = \hat{h}([0])$ and similarly $\hat{h}([b]) = \hat{h}([0])$.

- For (CB 3), suppose $a, b, c \in N$ and $a + b \approx u$, $a + c \approx u$. We suppose already that $\hat{h}([a + b]) = \hat{h}([u_M]) = \hat{h}([a + c])$. We obtain that $b \approx c$, and

$$h(a) + h(b) = h(a + b) = h(u_M) = u_N = h(u_M) = h(a + c) = h(a) + h(c).$$

  It follows from (BCM 2) that $h(b) = h(c)$, so indeed $\hat{h}([b]) = \hat{h}([c])$.

Thus, $\hat{h}$ is well defined, and it is also clear that h being a BCM homomorphism entails that $\hat{h}$ is as well. □

Theorem 3.2.3 now allows us to describe coequalizers in **EA**.

Corollary 3.2.11. *Let $f, g\colon E \to F$ be effect algebra homomorphisms. Let $\pi\colon \mathcal{T}o(F) \to \mathcal{T}o(F)/\approx$ be the coequalizer in* **BCM** *of $\mathcal{T}o(f), \mathcal{T}o(g)\colon \mathcal{T}o(E) \to \mathcal{T}o(F)$.*

$$\mathcal{T}o(E) \underset{\mathcal{T}o(g)}{\overset{\mathcal{T}o(f)}{\rightrightarrows}} \mathcal{T}o(F) \xrightarrow{\pi} \mathcal{T}o(F)/\approx$$

Then, the coequalizer of f, g in **EA** *is $\mathcal{P}a(\pi) \circ \eta_F\colon F \to \mathcal{P}a(\mathcal{T}o(F)/\approx)$.*

$$E \underset{g}{\overset{f}{\rightrightarrows}} F \xrightarrow{\eta_F} \mathcal{P}a\mathcal{T}o(F) \xrightarrow{\mathcal{P}a(\pi)} \mathcal{P}a(\mathcal{T}o(F)/\approx)$$

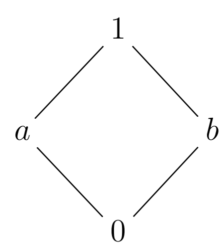
The constructions for products, coproducts, equalizers, and coequalizers we have given can be generalized from the binary/finite case to the arbitrary case in a straightforward manner, so that we have the following.

Corollary 3.2.12. **EA** *is complete and cocomplete.*

Example 3.2.13. Let $D_4 = \{0, a, b, 1\}$. Define a partial binary operation $\widetilde{\oplus}$ on D_4 as follows.

$\widetilde{\oplus}$	0	a	b	1
0	0	a	b	1
a	a	$\uparrow$	1	$\uparrow$
b	b	1	$\uparrow$	$\uparrow$
1	1	$\uparrow$	$\uparrow$	$\uparrow$

A routine verification of the axioms shows that $(D_4, \widetilde{\oplus}, (-)^{\perp}, 0)$ is an effect algebra, with $0^{\perp} = 1$, $a^{\perp} = b$ — in fact, D_4 is just the usual diamond Boolean algebra turned into an effect algebra via the construction described in Example 1.1.13:



A useful property of D_4 is that, given any effect algebra E and any element $x \in E$, there exists an effect algebra homomorphism $g_x \colon D_4 \to E$ defined by $g_x(0) = 0$, $g_x(a) = x$, $g_x(b) = x^{\perp}$, and $g_x(1) = 1$. This will be used in the next section.

Example 3.2.14. In $\mathcal{T}o(D_4)$, we have $1(a) + 1(a^{\perp}) \sim 1(a \,\widetilde{\oplus}\, a^{\perp}) = 1(1_{D_4})$, so $n(a) + n(a^{\perp}) \sim n(1)$ for all $n \in \mathbb{N}$. Given an arbitrary element $z = n_1(1) + n_2(a) + n_3(a^{\perp}) + n_4(0) \in \mathcal{M}(D_4)$, if $n_2 \leq n_3$, we have

$$z \sim n_1(1) + n_2(a) + n_2(a^{\perp}) + (n_3 - n_2)(a^{\perp}) \sim (n_1 + n_2)(1) + (n_3 - n_2)(a^{\perp}).$$

On the other hand if $n_2 > n_3$, then

$$z \sim n_1(1) + n_3(a) + (n_2 - n_3)(a) + n_3(a^{\perp}) \sim (n_1 + n_3)(1) + (n_2 - n_3)(a).$$

So the equivalence classes in $\mathcal{T}o(D_4)$ can be represented by

$$\{n_1(1) + n_2(a)\} \cup \{n_1(1) + n_2(a^{\perp})\} \cup \{n_1(1)\},$$

where $n_1 \in \mathbb{N}$, $n_2 \in \mathbb{N}^+$. It is clear each element of the above set is in a different equivalence class. If we identify the coefficients of (1) with natural numbers, coefficients of (a) with positive integers, and coefficients of $(a^\perp)$ with negative integers, we have $\mathcal{T}o(D_4) \cong (\mathbb{N} \times \mathbb{Z}, (1,0))$ with the operation given by

$$(n_1, z_1) + (n_2, z_2) = \begin{cases} (n_1 + n_2 + ||z_1| - |z_2||, z_1 + z_2), & \text{if } z_1, z_2 \text{ have opposite signs,} \\ (n_1 + n_2, z_1 + z_2), & \text{otherwise.} \end{cases}$$

For instance,

$$(2,2) + (2,-4) = (2 + 2 + ||2| - |-4||, 2 - 4) = (6, -2),$$

which corresponds to

$$\begin{aligned} (2(1) + 2(a)) + (2(1) + 4(a^\perp)) &= 4(1) + 2(a) + 2(a^\perp) + 2(a^\perp) \\ &= 4(1) + 2(1) + 2(a^\perp) \\ &= 6(1) + 2(a^\perp). \end{aligned}$$

Example 3.2.15. Let $X = \{x, y, z\}$ and consider $\mathcal{P}(X)$ as an effect algebra. We compute the coequalizer of $g_{\{x\}}, g_{\{y\}} \colon D_4 \to \mathcal{P}(X)$.

We first consider the coequalizer in **BCM**

$$\mathcal{T}o(D_4) \underset{\mathcal{T}o(g_{\{y\}})}{\overset{\mathcal{T}o(g_{\{x\}})}{\rightrightarrows}} \mathcal{T}o(\mathcal{P}(X)) \xrightarrow{\ \pi\ } \mathcal{T}o(\mathcal{P}(X))/\approx,$$

where $\approx$ is the BCM congruence generated by

$$\{\mathcal{T}o(g_{\{x\}})(Z), \mathcal{T}o(g_{\{y\}})(Z) \mid Z \in \mathcal{T}o(D_4)\}.$$

In the preceding example, we computed all the elements of $\mathcal{T}o(D_4)$. For elements of the form $n(1)$, we get

$$\mathcal{T}o(g_{\{x\}})(n(1)) = n(1) \approx n(1) = \mathcal{T}o(g_{\{y\}})(n(1)).$$

For elements of the form $n_1(1) + n_2(a)$, we have

$$\begin{aligned} \mathcal{T}o(g_{\{x\}})(n_1(1) + n_2(a)) &= n_1(1) + n_2(\{x\}) \\ &\sim n_1(1) + n_2(\{y\}) \\ &= \mathcal{T}o(g_{\{y\}})(n_1(1) + n_2(a)). \end{aligned}$$

For elements of the form $n_1(1)+n_2(a^\perp)$, we have

$$\begin{aligned}\mathcal{T}o(g_{\{x\}})(n_1(1)+n_2(a^\perp)) &= n_1(1)+n_2(\{y,z\})\\ &\sim n_1(1)+n_2(\{x,z\})\\ &= \mathcal{T}o(g_{\{y\}})(n_1(1)+n_2(a^\perp)),\end{aligned}$$

or $n_1(1)+n_2(\{y\})+n_2(\{z\}) \sim n_1(1)+n_2(\{x\})+n_2(\{z\})$.

Ultimately, we have just made the identification $1(\{x\}) \approx 1(\{y\})$. So then $\mathcal{P}a(\mathcal{T}o(\mathcal{P}(X)/\approx)) = \{Z \in \mathcal{T}o(\mathcal{P}(X)/\approx) \mid Z \leq 1[\{x,y,z\}]_\approx\} = \{Z \in \mathcal{T}o(\mathcal{P}(X)/\approx) \mid Z \leq 2[\{x\}]_\approx + 1[\{z\}]_\approx\}$.

So $\mathcal{P}a(\mathcal{T}o(\mathcal{P}(X)/\approx))$ consists of the six elements $n_1[\{x\}]+n_2[\{z\}]$, $n_1 = 0,1,2$, $n_2 = 0,1$, and the coequalizer of $g_{\{x\}}, g_{\{y\}}$ in **EA** is

$$\mathcal{P}a(\pi)\circ\eta_{\mathcal{P}(X)}\colon \mathcal{P}(X)\to \mathcal{P}a(\mathcal{T}o(\mathcal{P}(X)/\approx)).$$

Note that in the preceding example, if we simply took the quotient of $\mathcal{P}(X)$ by $\{(g_{\{x\}}(b), g_{\{y\}}(b)) \mid b \in D_4\}$ (i.e. the same construction as for coequalizers of sets), we would have ended up with the same (isomorphic) thing. Thus, it still seems somewhat mysterious what happens by going to BCMs, dividing out by a congruence, and then coming back to effect algebras. The following example shows what is gained from this construction.

Example 3.2.16. Let $E = \{0, 1/2, 1\}$ considered as a sub-effect algebra of $[0,1]$. Let $F = [0,1]+[0,1]$. Let $f, g\colon E \to F$ be defined by $f(1/2) = (1/2, 0)$ and $g(1/2) = (1/2, 1)$. If we simply try to take F and identify $(1/2,0) \sim (1/2,1)$ and call the resulting quotient structure F', what we end up with fails to be an effect algebra. For instance, we have

$$(1/4,0)\,\widetilde{\oplus}\,((1/4,1)\,\widetilde{\oplus}\,(1/4,1)) = (1/4,0)\,\widetilde{\oplus}\,(1/2,1) = (1/4,0)\,\widetilde{\oplus}\,(1/2,0) = (3/4,0),$$

but (PCM 2) fails as $(1/4,0)\,\widetilde{\oplus}\,(1/4,1)\uparrow$.

It turns out that we will "pick up" these elements which need to exist precisely by going to BCMs. Consider

$$\mathcal{T}o(E) \underset{\mathcal{T}o(g)}{\overset{\mathcal{T}o(f)}{\rightrightarrows}} \mathcal{T}o(F) \xrightarrow{\ \pi\ } \mathcal{T}o(F)/\approx,$$

where $\approx$ is the BCM congruence generated by $\{\mathcal{T}o(f)(z), \mathcal{T}o(g)(z) \mid z \in \mathcal{T}o(E)\}$.

Now, $\mathcal{P}a(\mathcal{T}o(F)/\approx)$, which is all the elements of $\mathcal{T}o(F)/\approx$ below $[1(1)]_\approx$, certainly contains all the elements $[1(x,i)]_\approx$ for $x \in [0,1]$, $i \in \{0,1\}$, but it also contains others — for instance, $[1(1/4,0)+1(1/4,1)]_\approx$ is in $\mathcal{P}a(\mathcal{T}o(F)/\approx)$, as

$$\begin{aligned}
[1(1/4,0)+1(1/4,1)]_\approx + [1(1/4,0)+1(1/4,1)]_\approx &= [2(1/4,0)+2(1/4,1)]_\approx \\
&= [1(1/2,0)+1(1/2,1)]_\approx \\
&= [1(1)]_\approx.
\end{aligned}$$

This fixes the problem we had before when trying the naïve approach of directly generalizing the coequalizers of sets — the "missing" element $(1/4,0) \mathbin{\widetilde{\oplus}} (1/4,1)$ now exists as a formal sum constructed in **BCM**, and is below the unit, and hence in $\mathcal{P}a(\mathcal{T}o(F)/\approx)$.

Observe, moreover, that the coequalizing map $\mathcal{P}a(\pi) \circ \eta_F \colon F \to \mathcal{P}a(\mathcal{T}o(F/\approx))$ is not surjective — all of these "new" elements are not in the image!

For another example, with finite effect algebras, where the coequalizing map fails to be surjective, see [Jac12, p.954] — in the example there, the codomain has more elements than the domain (this is harder to make precise in our preceding example, since we are dealing with uncountable underlying sets), so it is impossible that the construction of coequalizers is simply dividing out by some kind of an effect algebra congruence in disguise.

Remark 3.2.17. For categorical logicians, it is also of interest whether a given category is a regular category. By [Rey72, Proposition 1.7], in regular categories, an epimorphism is special if and only if it is a regular epimorphism (coequalizer). It is evident that for concrete categories, so long as images of maps are again objects of the category, the notion of a special epimorphism is just an arrow-theoretic formulation of a surjective function. Since we have seen that in **EA**, surjections and coequalizers do not coincide, **EA** cannot be a regular category.

3.3 Monomorphisms and subobjects

In this section, we characterize monomorphisms and dispel some misleading terminology that is currently in use.

Proposition 3.3.1. *Let $f\colon E \to F$ be an effect algebra homomorphism. Then, f is injective if and only if f is monic.*

Proof. Let $f\colon E \to F$ be monic. Suppose, on the contrary, that f is not injective. Then, there exist $x, y \in E$ with $x \neq y$ such that $f(x) = f(y)$. Now consider the maps $g_x, g_y\colon D_4 \to E$ as defined in Example 3.2.13. We have that

$$f \circ g_x(a) = f(x) = f(y) = f \circ g_y(a)$$

and

$$f \circ g_x(b) = f \circ g_x(a^{\perp}) = (f \circ g_x(a))^{\perp} = (f \circ g_y(a))^{\perp} = f \circ g_y(a^{\perp}) = f \circ g_y(b).$$

Both $f \circ g_x$ and $f \circ g_y$ must preserve 0 and 1, and so $f \circ g_x = f \circ g_y$. But, $g_x(a) = x \neq y = g_x(b)$, so $g_x \neq g_y$, contradicting the assumption that f is monic. Therefore, it must be the case that f is injective.

The converse, that f being injective implies that f is monic, follows the same standard proof as for **Set** or any other concrete category. □

Proposition 3.3.2. *Let $f\colon E \to F$ be an effect algebra homomorphism. The following conditions are equivalent.*

(a) For all $a, b \in E$, if $f(a) \leq f(b)$, then $a \leq b$.

(b) For all $a, b \in E$, if $f(a) \mathbin{\widetilde{\oplus}} f(b) \downarrow$, then $a \mathbin{\widetilde{\oplus}} b \downarrow$.

Proof. (a) $\implies$ (b): Suppose $f(a) \mathbin{\widetilde{\oplus}} f(b) \downarrow$. Then, by Lemma 1.1.10(a) $f(a) \leq f(b)^{\perp}$. By hypothesis, we have $a \leq b^{\perp}$, whence $a \mathbin{\widetilde{\oplus}} b \downarrow$.

(b) $\implies$ (a): Suppose $f(a) \leq f(b)$. Then, by Lemma 1.1.10(a), $f(a) \mathbin{\widetilde{\oplus}} f(b)^{\perp} \downarrow$. By hypothesis, $a \mathbin{\widetilde{\oplus}} b^{\perp} \downarrow$, whence $a \leq b$. □

Proposition 3.3.3. *An effect algebra homomorphism satisfying the equivalent conditions of Proposition 3.3.2 is injective, and thus a monomorphism.*

Proof. Let f be a homomorphism satisfying the conditions of Proposition 3.3.2. Now suppose $f(a) = f(b)$. Then, $f(a) \leq f(b)$, so $a \leq b$. Similarly, $f(b) \leq f(a)$, so $b \leq a$. Antisymmetry then yields $a = b$. □

In many sources in the present literature (for instance, in [PD00, p.20]), effect algebra homomorphisms satisfying the conditions of Proposition 3.3.2 are defined to be "monomorphisms". We have shown that these conditions *imply* that a map is a monomorphism, but the following example shows that the converse is *not* true.

Example 3.3.4. Consider the map $g_{1/4}\colon D_4 \to [0,1]$. We have $g_{1/4}(a) = 1/4$ and $g_{1/4}(b) = 3/4$, and clearly $1/4 \leq 3/4$ in [0,1]. However, a and b are incomparable elements of D_4, so $g_{1/4}$ fails condition (a) of Proposition 3.3.2 even though it is clearly injective.

We have given the definition of sub-effect algebras (Definition 1.1.9) as it currently appears in the literature, but this is also misleading — one would intuitively think that sub-effect algebras correspond to domains of the monics (or injective homomorphisms) and canonical representatives of subobjects, but this also fails to be true.

Consider again the previous example, but with $E = \{0, 1/4, 3/4, 1\}$ with operation inherited from $[0,1]$ in place of D_4 (these effect algebras are clearly isomorphic). Now consider the inclusion $i\colon E \hookrightarrow [0,1]$. Even though E is a subset of $[0,1]$ and also an effect algebra, it is not a sub-effect algebra; it fails (SEA 3) since $1/4 \mathbin{\widetilde{\oplus}} 1/4 \downarrow$ in $[0,1]$ but $1/4 \mathbin{\widetilde{\oplus}} 1/4 = 1/2 \notin E$.

However, the inclusion maps which additionally satisfy the conditions of Proposition 3.3.2 do clearly correspond to sub-effect algebras. We would now like to come up with a new definition that captures *all* subsets of effect algebras which are also effect algebras, but may not necessarily satisfy (SEA 3). Simply dropping (SEA 3), however, does not work either. Let $F = \{0, 1/8, 1/4, 1/2, 3/4, 7/8, 1\} \subseteq [0,1]$. Then F satisfies (SEA 1) and (SEA 2). But take $a = 1/4$, $b = c = 1/8$. Then, $b \mathbin{\widetilde{\oplus}} c = 1/4$ so $b \mathbin{\widetilde{\oplus}} c \downarrow$ in F, and $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) = 1/4 \mathbin{\widetilde{\oplus}} 1/4 = 1/2$, so also $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) \downarrow$ in F, but

$a \mathbin{\widetilde{\oplus}} b = 1/4 + 1/2 = 3/4 \notin F$, so $a \mathbin{\widetilde{\oplus}} b \uparrow$ in F. That is, F fails (EA 2) and is not an effect algebra. Thus, we need an axiom to force the existence of enough elements to satisfy (EA 2), motivating the following definition.

Definition 3.3.5 (Weak sub-effect algebra)**.** Let E be an effect algebra and $F \subseteq E$ a subset. Then, F is a *weak sub-effect algebra* of E if it satisfies (SEA 1), (SEA 2), and

(SEA 3') If $a, b, c \in E$ such that $b \mathbin{\widetilde{\oplus}} c \in F$ and $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) \in F$, then $a \mathbin{\widetilde{\oplus}} b \in F$ and $(a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c \in F$.

So injective homomorphisms correspond to weak sub-effect algebras, and those that additionally satisfy the conditions of Proposition 3.3.2 correspond to sub-effect algebras (and this is probably where the idea to call those maps "monomorphisms" came from).

Although it would make more sense from a categorical and mathematical point of view to call what we call a weak sub-effect algebra simply a sub-effect algebra and what we have called a sub-effect algebra a strong sub-effect algebra, we choose not to do this so as to not cause additional confusion with the existing literature. We will now see that there is an even nicer characterization of these maps.

Lemma 3.3.6. *Let R be an equivalence relation on a BCM M where $0 \neq u$, and let $\sim$ be the closure of R with respect to* (CB 1) *and* (CB 2)*. If $(0, x) \notin R$ for all $x \neq 0$, then $[0]_\sim = \{0\}$. That is, if the class of 0 in R is a singleton, then the same is true of the class of 0 in $\sim$.*

Proof. We proceed by structural induction. Suppose we have a relation $\sim'$ containing R where the class of 0 is a singleton. We then show that no applications of (CB 1) – (CB 2) change this.

- For (CB 1), suppose we have $a \sim' c$, $b \sim' d$ yielding $a+b \sim' c+d$ with $a+b = 0$. By (BCM 1), we have $a = b = 0$, and so by hypothesis $c = d = 0$, so $c + d = 0$ and we get nothing new in the class of 0.

- For (CB 2), if we have $a + b \sim' 0$ yielding $a \sim' 0$ and $b \sim' 0$, then by hypotheses $a + b = 0$, and by (BCM 1), $a = b = 0$ and we get nothing new in the class of 0.

□

Theorem 3.3.7. *The maps which satisfy the conditions of Proposition 3.3.2 are precisely the* regular monomorphisms *— i.e. equalizers.*

Proof. Let f be an equalizer of $g, h\colon E \to F$. As anything defined by universal property is unique up to a unique isomorphism, we may without loss of generality take the domain of f to be $E_{g,h} = \{x \in E \mid g(x) = h(x)\}$ and take $f\colon E_{g,h} \to E$ to be the inclusion. Suppose now that $f(a) \leq_E f(b)$. Let $x \in E$ be an element satisfying $a \mathbin{\widetilde{\oplus}} x = f(a) \mathbin{\widetilde{\oplus}} x = f(b) = b$. We also have $a, b \in E_{g,h}$ so that $g(a) = h(a)$ and $g(b) = h(b)$. Then,

$$g(a) \mathbin{\widetilde{\oplus}} g(x) = g(a \mathbin{\widetilde{\oplus}} x) = g(b) = h(b) = h(a \mathbin{\widetilde{\oplus}} x) = h(a) \mathbin{\widetilde{\oplus}} h(x) = g(a) \mathbin{\widetilde{\oplus}} h(x).$$

By (GEA 1), we have $g(x) = h(x)$, so also $x \in E_{g,h}$, and the equation $a \mathbin{\widetilde{\oplus}} x = b$ also holds in $E_{g,h}$, so $a \leq_{E_{g,h}} b$.

Conversely let $f\colon E \to F$ be an effect algebra homomorphism satisfying $f(a) \widetilde{\oplus} f(b)\downarrow \implies a \widetilde{\oplus} b\downarrow$ for all $a, b \in E$. Recall from Proposition 3.3.3 that this implies injectivity, so we may without loss of generality take E to be a sub-effect algebra of F and f to be the inclusion. Note that this means that

$$a, b \in E \text{ and } a \mathbin{\widetilde{\oplus}} b = x \in F \implies x \in E. \tag{3.3.1}$$

Equivalently, $f(a) \leq f(b) \implies a \leq b$, so also

$$a, b \in E \text{ and } a \mathbin{\widetilde{\oplus}} x = b \text{ (where } x \in F) \implies x \in E. \tag{3.3.2}$$

We know that if f is an equalizer, it should equalize its cokernel pair; that is, since we have coproducts and coequalizers, the square

$$\begin{array}{ccc} \mathcal{P}a(\mathcal{T}o(F+F)/\approx) & \xleftarrow{\;co\iota_1\;} & F \\ {\scriptstyle co\iota_2}\uparrow & & \uparrow{\scriptstyle f} \\ F & \xleftarrow{\quad f \quad} & E \end{array} \tag{3.3.3}$$

is a pushout, where c is the coequalizer in **EA** of $\iota_1 \circ f$ and $\iota_2 \circ f$

$$E \underset{\iota_2 \circ f}{\overset{\iota_1 \circ f}{\rightrightarrows}} F + F \xrightarrow{\eta_{F+F}} \mathcal{P}a\mathcal{T}o(F+F) \xrightarrow{\mathcal{P}a(\pi)} \mathcal{P}a(\mathcal{T}o(F+F)/\approx), \qquad c = \mathcal{P}a(\pi) \circ \eta_{F+F}$$

where π is the coequalizer in **BCM** of $\mathcal{T}o(\iota_1 \circ f)$ and $\mathcal{T}o(\iota_2 \circ f)$

$$\mathcal{T}o(E) \underset{\mathcal{T}o(\iota_2 \circ f)}{\overset{\mathcal{T}o(\iota_1 \circ f)}{\rightrightarrows}} \mathcal{T}o(F+F) \xrightarrow{\pi} \mathcal{T}o(F+F)/\approx,$$

where $\approx$ is the BCM congruence generated by

$$\{(\mathcal{T}o(\iota_1 \circ f)(z), \mathcal{T}o(\iota_2 \circ f)(z)) \mid z \in \mathcal{T}o(E)\}.$$

So, we must show that f is an equalizer of $c \circ \iota_1$ and $c \circ \iota_2$; or rather, of $\mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_1$ and $\mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_2$. It suffices to show that

$$E = \{x \in F \mid \mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_1(x) = \mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_2(x)\}.$$

First, if $x \in E$, then since the square (3.3.3) commutes, we have that

$$\mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_1 \circ f(x) = \mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_2 \circ f(x)$$

but as f is just the inclusion, then also

$$\mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_1(x) = \mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_2(x).$$

On the other hand, let $x \in F$ and $\mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_1(x) = \mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_2(x)$. If $x = 0$ or $x = 1$, then clearly $x \in E$ and $\mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_1(x) = \mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_2(x)$. So now let $x \neq 0$, $x \neq 1$.

Writing equivalence classes of $\mathcal{T}o(E) = (\mathcal{M}(E), \sim)$ as simply $[x]$, and equivalence classes of $\mathcal{T}o(F+F)/\approx$ as $[x]_\approx$ (this is a slight abuse of notation as these should be equivalence classes of equivalence classes $[[x]_\sim]_\approx$, but this is quite cumbersome to write), we get

$$\begin{aligned}
\mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_1(x) &= \mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_2(x), \\
\mathcal{P}a(\pi) \circ \eta_{F+F}(x, 0) &= \mathcal{P}a(\pi) \circ \eta_{F+F}(x, 1), \\
\mathcal{P}a(\pi)[1(x, 0)] &= \mathcal{P}a(\pi)[1(x, 1)], \\
[1(x, 0)]_\approx &= [1(x, 1)]_\approx.
\end{aligned}$$

That is, $([1(x,0)],[1(x,1)])$ is contained in the BCM congruence generated by

$$R = \{(\mathcal{T}o(\iota_1 \circ f)(z), \mathcal{T}o(\iota_2 \circ f)(z)) \mid z \in \mathcal{T}o(E)\}.$$

Suppose, towards a contradiction, that $x \notin E$.

Elements of $\mathcal{T}o(E)$ are equivalence classes $[n_0(1) + n_1(e_1) + \ldots + n_k(e_k)]$ with $n_i \in \mathbb{N}$ and $e_i \in E \backslash \{0,1\}$. So then $R = \{[n_0(1) + n_1(e_1,0) + \ldots + n_k(e_k,0)], [n_0(1) + n_1(e_1,1) + \ldots + n_k(e_k,1)] \mid n_0, \ldots, n_k \in \mathbb{N},\ e_1, \ldots, e_k \in E \backslash \{0,1\}\}$.

We now claim that R cannot contain $([1(x,0)],[1(x,1)])$. Since $x \notin E$ by hypothesis, the only way for this to happen is if, for some $n_0, n_1, \ldots, n_k \in \mathbb{N}$ and $e_1, \ldots, e_k \in E$, we have that in $\mathcal{T}o(F)$, $[1(x)] = [n_0(1) + n_1(e_1) + \ldots + n_k(e_k)]$. But this implies implies $n_0 = 0$ and $x = n_1 \cdot e_1 \mathbin{\widetilde{\oplus}} \ldots \mathbin{\widetilde{\oplus}} n_k \cdot e_k$ (where we write $n_i \cdot e_i$ to mean $\underbrace{e_i \mathbin{\widetilde{\oplus}} e_i \mathbin{\widetilde{\oplus}} \ldots \mathbin{\widetilde{\oplus}} e_i}_{n_i \text{ factors of } e_i}$). By iterating (3.3.1), we see the right hand side is in E, so this is not possible. So indeed, R does not contain $([1(x,0)],[1(x,1)])$. In fact, we see that for $y \in F$,

$$([1(y,0)],[1(y,1)]) \in R \Leftrightarrow y \in E.$$

We now need to show that $[1(x,0)]_{\approx} \neq [1(x,1)]_{\approx}$. Let $\approx_1$ be the equivalence relation generated by R. Since each element of R is related to at most one other element, then after taking the reflexive closure of R we have each element related to at most itself and one other element. Taking the symmetric closure does not change this, and so finally taking the transitive closure will not add anything new. So, we have that

$$\approx_1 = R \cup R^{\mathrm{op}} \cup \{(z,z) \mid z \in \mathcal{T}o(F+F)\}.$$

Now let $\approx_2$ be the BCM precongruence generated by R (which is the same thing as the BCM precongruence generated by $\approx_1$). It is clear that $\approx_1$ satisfies the hypothesis of Lemma 3.3.6, so $[0]_{\approx_2}$ is a singleton. Thus, no new elements are gained in passing from $\approx_1$ to $\approx_2$ by applying (CB 2) to the classes of $[1(x,0)]$ and $[1(x,1)]$.

We now see how (CB 1) can be used to obtain new elements in the class of $[1(x,0)]$ — we need to write $[1(x,0)]$ as a sum of two elements. This can happen if there exist $a, b \in F$ such that $a \mathbin{\widetilde{\oplus}} b = x$, and we have

$$[1(x,0)] = [1(a,0)] + [1(b,0)].$$

If $a \in E$ and $b \in E$, this would imply $a \widetilde{\oplus} b = x \in E$ contrary to hypothesis. If one of a, b is in E and the other is not in E — without loss of generality, suppose $a \notin E$, $b \in E$ — then we have $[1(b,0)] \approx_1 [1(b,1)]$, so that

$$[1(x,0)] \approx_2 [1(a,0)] + [1(b,1)]. \tag{3.3.4}$$

If $a \notin E$ and $b \notin E$, then thus far the classes of $[1(a,0)]$ and $[1(b,0)]$ are singletons and nothing new is gained. That is, we could write $[1(x,0)] \approx_2 [1(a,0)]+[1(b,0)]$, but $[1(a,0)]+[1(b,0)] = [1(a\widetilde{\oplus}b,0)] = [1(x,0)]$. Even if we could write $a = a_1 \widetilde{\oplus} a_2$ for $a_1 \notin E$, $a_2 \in E$, all we get is $[1(a_1,0)]+[1(a_2,1)]+[1(b,1)]$ and since $(a_2 \widetilde{\oplus} a_1) \widetilde{\oplus} b = x$, then $a_1 \widetilde{\oplus} b \downarrow$, we have just $[1(a_1 \widetilde{\oplus} b,0)]+[1(a_2,1)]$. Since $a_2 \in E$ and $a_2 \widetilde{\oplus} (a_1 \widetilde{\oplus} b) = x$, then it must be the case that $a_1 \widetilde{\oplus} b \notin E$, and we could have obtained this decomposition directly by the process in the preceding paragraph.

We will call an element of the form $[1(y,0)]$ with $y \in F$, $y \notin E$ a *0-rigid element*, and similarly an element of the form $[1(y,1)]$ with $y \in F$, $y \notin E$ a *1-rigid element*. If $y \in E$, then $[1(y,0)]$ and $[1(y,1)]$ will be called *nonrigid elements*.

For an element $y = \sum_{i=1}^{k} [1(a_i, n_i)]$, we call $a_1 \widetilde{\oplus} a_2 \widetilde{\oplus} \ldots \widetilde{\oplus} a_k$ the *raw sum* of y. Note that the raw sum is not necessarily defined, as this depends on the effect algebra operation of F. However, if it is defined, then we see that the raw sum is stable with respect to $\approx_1$ and $\approx_2$.

So we have seen in (3.3.4) that $[1(x,0)]_{\approx_2}$ contains elements of the form $[1(a,0)] + [1(b,1)]$ with the left summand 0-rigid and the right summand nonrigid, whose raw sum is x. A priori, it may be possible to further decompose these summands and obtain things of a different form, but we argue now that these are the only elements in $[1(x,0)]_{\approx_2}$.

Suppose there is a decomposition of the 0-rigid summand $[1(a,0)]$ in (3.3.4). By iterating the same argument as before, the only way to obtain new elements its equivalence class is if there exist $a_1, a_2 \in F$, $a = a_1 \widetilde{\oplus} a_2$, where $a_1 \notin E$ and $a_2 \in E$, so that $[1(a,0)] \approx_2 [1(a_1,0)] + [1(a_2,1)]$. Then,

$$[1(x,0)] \approx_2 [1(a,0)] + [1(b,1)] \approx_2 [1(a_1,0)] + [1(a_2,1)] + [1(b,1)].$$

But since $x = a \widetilde{\oplus} b = (a_1 \widetilde{\oplus} a_2) \widetilde{\oplus} b$, then also $a_2 \widetilde{\oplus} b \downarrow$, and moreover $a_2 \widetilde{\oplus} b \in E$. So then all we have is that $[1(x, 0)] \approx_2 [1(a_1, 0)] + [1(a_2 \mathbin{\widetilde{\oplus}} b, 1)]$, which is an element of the form already discussed.

Now suppose there is a decomposition of the nonrigid summand $[1(b, 1)]$ in (3.3.4), so that there are $b_1, b_2 \in F$ such that $b_1 \mathbin{\widetilde{\oplus}} b_2 = b$. Since $b \in E$, then by (3.3.2), it cannot be the case that one of b_1, b_2 is in E while the other is not. So the only possibilities are that both b_1, $b_2 \in E$, or both b_1, $b_2 \notin E$.

Suppose it is the case that both b_1, $b_2 \in E$. Then, we have the following decompositions.

- $[1(x, 0)] \approx_2 [1(a, 0)]+[1(b, 1)] \approx_2 [1(a, 0)]+[1(b_1, 0)]+[1(b_2, 0)] = [1(a \widetilde{\oplus} b_1 \widetilde{\oplus} b_2, 0)] = [1(x, 0)]$.
- $[1(x, 0)] \approx_2 [1(a, 0)] + [1(b, 1)] \approx_2 [1(a, 0)] + [1(b_1, 0)] + [1(b_2, 1)] = [1(a \mathbin{\widetilde{\oplus}} b_1, 0)] + [1(b_2, 1)]$. Here, $a \mathbin{\widetilde{\oplus}} b_1 \notin E$ and $b_2 \in E$.
- $[1(x, 0)] \approx_2 [1(a, 0)] + [1(b, 1)] \approx_2 [1(a, 0)] + [1(b_1, 1)] + [1(b_2, 0)] = [1(a \mathbin{\widetilde{\oplus}} b_2, 0)] + [1(b_1, 1)]$. Here, $a \mathbin{\widetilde{\oplus}} b_2 \notin E$ and $b_1 \in E$.
- $[1(x, 0)] \approx_2 [1(a, 0)] + [1(b, 1)] \approx_2 [1(a, 0)] + [1(b_1, 1)] + [1(b_2, 1)] = [1(a, 0)] + [1(b, 1)]$.

We never obtain anything by doing such a decomposition except something which is the sum of a 0-rigid element and a nonrigid element.

By a similar argument as before, nothing new is obtained in the case that b_1, $b_2 \notin E$. We thus see that $[1(x, 0)]_{\approx_2} = \{[1(a, 0)] + [1(b, 1)] \mid a \mathbin{\widetilde{\oplus}} b = x,\ a \notin E,\ b \in E\}$ (recall we have identified $0 = (0, 0) = (0, 1)$ so the element $[1(x, 0)]$ itself is obtained by taking $a = x$, $b = 0$).

By analogous reasoning with instances of 0 and 1 interchanged, we have $[1(x, 1)]_{\approx_2} = \{[1(a, 1)] + [1(b, 0)] \mid a \mathbin{\widetilde{\oplus}} b = x,\ a \notin E,\ b \in E\}$. It is clear that no element appears in both sets, so $[1(x, 0)]_{\approx_2} \neq [1(x, 1)]_{\approx_2}$.

We now claim that $[1(1)]_{\approx_2} = \{[1(a, 0)] + [1(a^{\perp}, 1)] \mid a \in E\}$ (again, we have identified $1 = (1, 0) = (1, 1)$, so the element $[1(1)]$ itself is obtained by taking $a = 1$). We know for every $a \in F$, we can write $[1(1)] = [1(a, 0)] + [1(a^{\perp}, 0)]$.

If $a \in E$, then we have $[1(1)] = [1(a,0)] + [1(a^\perp,0)] \approx_2 [1(a,0)] + [1(a^\perp,1)]$. We also have $[1(1)] \approx_2 [1(a,1)] + [1(a^\perp,0)]$, but this is just the same thing as $[1(a^\perp,0)] + 1[((a^\perp)^\perp,1)]$ (and $a \in E \Leftrightarrow a^\perp \in E$). Now we claim that further decomposition does not add anything new; indeed since $a \in E$, if $a = a_1 \mathbin{\widetilde{\oplus}} a_2$ for $a_1, a_2 \in F$, then by the same reasoning as before, either both $a_1, a_2 \notin E$ or both $a_1, a_2 \in E$. In the case of the former, no new elements are obtained. In the case of the latter, we could, for instance, write $[1(1)] \approx_2 [1(a_1,0)] + [1(a_2,1)] + [1(a^\perp,1)]$. But then $1 = (a_1 \mathbin{\widetilde{\oplus}} a_2) \mathbin{\widetilde{\oplus}} a^\perp = a_1 \mathbin{\widetilde{\oplus}} (a_2 \mathbin{\widetilde{\oplus}} a^\perp)$, so $a_2 \mathbin{\widetilde{\oplus}} a^\perp = a_1^\perp$, so all we have is $[1(a_1,0)] + [1(a_2 \mathbin{\widetilde{\oplus}} a^\perp,1)] = [1(a_1,0)] + [1(a_1^\perp,1)]$.

If $a \notin E$, we have only $[1(1)] = [1(a,0)] + [1(a^\perp,0)]$ (or $[1(1)] = [1(a,1)] + [1(a^\perp,1)]$, but we omit the reasoning for this decomposition as it is analogous). The only thing we could do is replace one of the summands by an element of its class in $\approx_2$, for instance, if $a_1 \mathbin{\widetilde{\oplus}} a_2 = a$ with $a_1 \notin E$, $a_2 \in E$, we have $[1(1)] = [1(a_1,0)] + [1(a_2,1)] + [1(a^\perp,0)] = [1(a_1 \mathbin{\widetilde{\oplus}} a^\perp,0)] + [1(a_2,1)]$. But since $1 = a \mathbin{\widetilde{\oplus}} a^\perp = (a_1 \mathbin{\widetilde{\oplus}} a_2) \mathbin{\widetilde{\oplus}} a^\perp = (a_1 \mathbin{\widetilde{\oplus}} a^\perp) \mathbin{\widetilde{\oplus}} a_2$, then $(a_1 \mathbin{\widetilde{\oplus}} a^\perp) = a_2^\perp$, which is already accounted for.

We now consider passing from $\approx_2$ to $\approx$. We claim that using (CB 3) cannot add new elements to $[0]_{\approx_2}$ or $[1(1)]_{\approx_2}$.

First, if (CB 3) were to add an element to $[1(1)]_{\approx_2}$, we must have some $b \in [1(1)]_{\approx_2}$, and some $a, c \in \mathcal{T}o(F+F)$ such that $a + b \approx_2 [1(1)]$ and $a + c \approx_2 [1(1)]$ to yield $[1(1)] \approx b \approx c$. It is evident that the raw sum of any element in $[1(1)]_{\approx_2}$ is 1, so this forces $a = 0$. But then, we already have by hypothesis that $c \approx_2 [1(1)]$, and nothing new is gained by applying (CB 3). Thus, $[1(1)]_{\approx_2} = [1(1)]_{\approx}$.

Next, we know $[0]_{\approx_2}$ is a singleton, so if (CB 3) were to add a new element to the class of 0, then we must have some elements $a, b \in \mathcal{T}o(F+F)$ such that $a + 0 \approx_2 [1(1)]$, $a + b \approx [1(1)]$, yielding $0 \approx b$. Again, since the raw sum of any element in $[1(1)]_{\approx}$ is 1, we have that the raw sum of a is 1. But $a+b$ must also have raw sum 1, forcing $b = 0$. So nothing new is gained, and $[0]_{\approx} = [0]_{\approx_2} = \{0\}$, so (CB 2) cannot add elements to $[1(x,0)]_{\approx}$ or $[1(x,1)]_{\approx}$.

Finally, we claim using (CB 3) does not add any new elements to $[1(x,0)]_{\approx_2}$ (or, by analogous reasoning, to $[1(x,1)]_{\approx_2}$). For it to do so, we would need $b \in [1(x,0)]_{\approx_2}$ and $a, c \in \mathcal{T}o(F+F)$ such that $a + b \approx_2 [1(1)]$ and $a + c \approx_2 [1(1)]$, yielding $b \approx_2 c$.

We know $b = [1(b_1,0)] + [1(b_2,1)]$ with $b_1 \mathbin{\widetilde{\oplus}} b_2 = x$, $b_1 \notin E$, $b_2 \in E$.

Then $a + [1(b_1,0)] + [1(b_2,1)] \approx_2 [1(1)]$. This means that for some $\ell \in E$, $a + [1(b_1,0)]+[1(b_2,1)] = [1(\ell,0)]+[1(\ell^{\perp},1)]$. Then, we must have $a = [1(a_1,0)]+[1(a_2,1)]$, with $a_1 \mathbin{\widetilde{\oplus}} b_1 = \ell$ and $a_2 \mathbin{\widetilde{\oplus}} b_2 = \ell^{\perp}$. Since $b_1 \notin E$, $\ell \in E$, we have $a_1 \notin E$. Since $b_2 \in E$ and $\ell \in E$, we have $a_2 \in E$. Since the raw sum of $a+b$ must be 1, this means that

$$1 = (a_1 \mathbin{\widetilde{\oplus}} a_2) \mathbin{\widetilde{\oplus}} (b_1 \mathbin{\widetilde{\oplus}} b_2) = (a_1 \mathbin{\widetilde{\oplus}} a_2) \mathbin{\widetilde{\oplus}} x,$$

forcing $a_1 \mathbin{\widetilde{\oplus}} a_2 = x^{\perp}$. Thus, $a \in [1(x^{\perp},0)]_{\approx_2}$. Now, applying the exact same reasoning to the fact that $c+a = a+c \approx_2 [1(1)]$ (with a playing the role of b and c playing the role of a in the above) , we see that we must already have had $c \in [1((x^{\perp})^{\perp},0)]_{\approx_2} = [1(x,0)]_{\approx_2}$.

Thus $[1(x,0)]_{\approx_2} = [1(x,0)]_{\approx}$ and $[1(x,1)]_{\approx_2} = [1(x,1)]_{\approx}$, so we deduce that $[1(x,0)]_{\approx} \neq [1(x,1)]_{\approx}$, which is a contradiction to hypothesis. So indeed, it must be the case that $x \in E$. Therefore, $E = \{x \in F \mid \mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_1(x) = \mathcal{P}a(\pi) \circ \eta_{F+F} \circ \iota_2(x)\}$ as claimed, and f is an equalizer. $\square$

Chapter 4

Coordinatization of MV algebras

In the final chapter, we introduce yet another generalization of boolean algebras — boolean inverse semigroups. Inverse semigroups are to partial symmetries what groups are to total symmetries, and those whose subset of idempotents form boolean algebras turn out to be related to MV algebras.

After giving an overview of inverse semigroups, the goal is to discuss the *coordinatization* of MV algebras — a theorem that links MV algebras to quotients of boolean inverse semigroups by their principal ideals. We give some new explicit examples of coordinatization and some new theorems that will help with future coordinatizations.

4.1 Background on inverse semigroups

Definition 4.1.1 (Inverse semigroup)**.** An *inverse semigroup* is a pair $(S, *)$ consisting of a set S and a binary operation $*\colon S \times S \to S$ satisfying (writing simply xy for $x * y$):

(IS 1) *Associativity*: for all $x, y, z \in S$, $(xy)z = x(yz)$.

(IS 2) *Pseudoinverse*: for all $x \in S$, there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

For an inverse semigroup S, we write $E(S) = \{x \in S \mid x \text{ is idempotent}\}$, and for $x, y \in S$, we define $x \leq y$ to mean there exists $e \in E(S)$ such that $x = ye$. When we henceforth refer to meets and joins in an inverse semigroup, we mean the greatest lower bound and least upper bound, respectively, with regard to this order.

Proposition 4.1.2. *Let S be an inverse semigroup. The following hold.*

(a) The relation $\leq$ defined above is a partial order.

(b) Idempotents commute (for $e, f \in E(S)$, we have $ef = fe$).

(c) Idempotents are pseudoinverse to themselves (for $e \in E(S)$, we have $e^{-1} = e$).

(d) If $e, f \in E(S)$ and $e \leq f$, then $e = fe$.

(e) For all $x, y \in S$, $(xy)^{-1} = y^{-1}x^{-1}$.

(f) For all $x \in S$, $(x^{-1})^{-1} = x$, and the elements xx^{-1} and $x^{-1}x$ are both idempotent.

(g) The idempotents $E(S)$, with respect to $\leq$, is a meet semilattice; i.e. for all $e, f \in E(S)$, $e \wedge f$ always exists and in particular, $e \wedge f = ef$.

Proof. See [Law98, Proposition 7, Section 1.4] for (a) and (d). See [Law98, Theorem 3, Section 1.1] for (b). See [Law98, Section 1.4, Proposition 1] for (c), (e), and (f). See [Law98, Proposition 8, Section 1.4] for (g). □

Lemma 4.1.3. *Let S be an inverse semigroup, and $x, y \in S$. The following are equivalent.*

(a) $x \leq y$.

(b) $x = ey$ for some idempotent e (that is, the side on which the idempotent appears does not matter).

(c) $x^{-1} \leq y^{-1}$.

(d) $x = yx^{-1}x$.

(e) $x = xx^{-1}y$.

Proof. See [Law98, Lemma 1.4.6] □

Lemma 4.1.4. *Let S be an inverse semigroup and $a, b \in S$. If there exists $c \in S$ such that $a, b \leq c$, then $a^{-1}b$ and ab^{-1} are both idempotent.*

Proof. Let $e, f \in E(S)$ satisfying $a = ce$, $b = cf$. So $a^{-1} = (ce)^{-1} = e^{-1}c^{-1} = ec^{-1}$. As idempotents commute, we then have

$$\begin{aligned}(a^{-1}b)(a^{-1}b) &= ec^{-1}cfec^{-1}cf \\ &= eec^{-1}cc^{-1}cff \\ &= ec^{-1}cf \\ &= a^{-1}b.\end{aligned}$$

Similarly, ab^{-1} is idempotent. □

Definition 4.1.5 (Compatible, orthogonal elements)**.** Let S be an inverse semigroup and $a, b \in S$. We say that a and b are *compatible*, and write $a \sim b$, to mean that ab^{-1} and $a^{-1}b$ are both idempotents.

If S has a zero (that is, an element 0 such that $0x = x0 = 0$ for all $x \in S$), we say that a and b are *orthogonal*, and write $a \perp b$, to mean that $ab^{-1} = 0 = ab^{-1}$.

Note that as a direct consequence of Lemma 4.1.4, if $a \vee b$ exists, then $a \sim b$. Note also that $a \perp b$ implies $a \sim b$.

Definition 4.1.6 ((Distributive, boolean) inverse monoid)**.** An *inverse monoid* is an inverse semigroup S together with an element $1 \in S$ such that $1x = x = x1$ for all $x \in S$. Note that it is *not* necessary for xx^{-1} to equal 1.

An inverse monoid with zero is *distributive* if the following hold.

(DIM 1) $E(S)$ is a distributive lattice.

(DIM 2) For all $a, b \in S$, if $a \sim b$, then $a \vee b$ exists.

(DIM 3) Multiplication distributes over binary joins; for all $a, b, c \in S$ such that $b \vee c$ exists, we have $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$.

A distributive inverse monoid S where $E(S)$ is a boolean algebra is a *boolean inverse monoid*. If binary meets also always exist, then S is a *boolean inverse* $\wedge$-*monoid*.

Example 4.1.7. Let X be a set. Then the *partial bijections* on X (that is, partially defined functions $X \to X$ which are injective), $\mathcal{I}(X)$, is an inverse semigroup called the *symmetric inverse monoid* on X. Given $f, g \colon \mathcal{I}(X)$, we have $gf = g \circ f$, where $\operatorname{dom}(g \circ f) = f^{-1}(\operatorname{dom} g \cap \operatorname{im} f)$, and when $x \in \operatorname{dom}(g \circ f)$, then $(g \circ f)(x) = g(f(x))$.

When X is a finite set of n elements, we write $\mathcal{I}(X)$ as $\mathcal{I}_n$. We call the elements of X *letters*.

The idempotents of $\mathcal{I}(X)$ are the partial identity maps; indeed, if $i \in \mathcal{I}(X)$ is idempotent, then $i^3 = i$, from which uniqueness of pseudoinverses forces $i^{-1} = i$, so $i = i^2 = i \circ i^{-1} = \mathrm{id}_{\operatorname{dom} i}$.

Now observe that $f \leq g$ if and only if there is a subset $A \subseteq X$ such that $f = g \circ \mathrm{id}_A$, if and only if $f(x) = g \circ \mathrm{id}_A(x) = g(x)$ for all $x \in \operatorname{dom} f$ if and only if the restriction of g to $\operatorname{dom} f$ coincides with f. That is, $f \leq g$ means g is an extension of f.

The meet, which is always defined, is given by

$$\operatorname{dom}(f \wedge g) = \{x \in X \mid f(x) = g(x)\}$$

and then $(f \wedge g)(x) = f(x) = g(x)$.

On the other hand, for the join, we have

$$f \vee g \downarrow \Leftrightarrow \forall x \in \operatorname{dom}(f) \cap \operatorname{dom}(g),\ \ f(x) = g(x),$$

in which case $\operatorname{dom}(f \vee g) = \operatorname{dom} f \cup \operatorname{dom} g$, and

$$f \vee g(x) = \begin{cases} f(x), & \text{if } x \in \operatorname{dom}(f), \\ g(x), & \text{if } x \in \operatorname{dom}(g). \end{cases}$$

The following result is the inverse semigroup theoretic analogue of Cayley's theorem for groups.

Theorem 4.1.8 (Wagner-Preston theorem)**.** *For every inverse semigroup S, there exists a set X and an injective homomorphism $i \colon S \hookrightarrow \mathcal{I}(X)$ such that, for $a, b \in S$, $a \leq b \Leftrightarrow i(a) \leq i(b)$.*

Proof. See [Law98, p.36]. $\square$

Definition 4.1.9 (Semisimple/fundamental/factorizable inverse monoid)**.** Let S be an inverse monoid. We say that S is

- *semisimple*, if it is isomorphic to a finite direct product of finite symmetric inverse monoids,
- *fundamental*, if the only elements that commute with every idempotent are the idempotents themselves,
- *factorizable*, if every element is beneath an element in the group of units (i.e. for all $x \in S$, there is $y \in S$ satisfying $yy^{-1} = 1 = y^{-1}y$, and $x \leq y$).

Example 4.1.10. Symmetric inverse monoids are boolean inverse $\wedge$-monoids. They are fundamental, and the finite ones are also factorizable. It follows that semisimple inverse monoids are fundamental and factorizable, and it is known that in fact the finite fundamental boolean inverse $\wedge$-monoids are *precisely* the semisimple inverse monoids. See [Law98] for further details.

Definition 4.1.11 (Green's relations)**.** We now define *Green's relations* $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{J}$ for an inverse semigroup S as follows. We say that, for $a, b \in S$, that $a\mathcal{J}b$ if the two-sided ideals SaS and SbS are equal.

We define $\mathbf{d}(a) = a^{-1}a$ and $\mathbf{r}(a) = aa^{-1}$ and call them the domain and range of a, respectively. We say that $a\mathcal{L}b$ if $\mathbf{d}(a) = \mathbf{d}(b)$ and $a\mathcal{R}b$ if $\mathbf{r}(a) = \mathbf{r}(b)$.

We say that $a\mathcal{D}b$ if there exists $c \in S$ such that $a = \mathbf{d}(c)$ and $b = \mathbf{r}(c)$, and write $a \xrightarrow{c} b$. Note that if $a\mathcal{D}b$, then $a = c^{-1}c$ and $b = cc^{-1}$ so that both a and b are idempotent.

We wish to extend the relation $\mathcal{D}$ to elements which are not necessarily idempotent. Define $a\mathcal{D}'b$ if $\mathbf{d}(a)\mathcal{D}\mathbf{d}(b)$.

Lemma 4.1.12. *The relation $\mathcal{D}'$ is an extension of $\mathcal{D}$; that is, if e, f are idempotent, then $e\mathcal{D}f$ if and only if $e\mathcal{D}'f$.*

Proof. Let e, f be idempotents in an inverse semigroup S. Suppose $e\mathcal{D}f$. Let $a \in S$ satisfy $e = a^{-1}a$ and $f = aa^{-1}$. From Proposition 4.1.2(c), $e^{-1} = e$ and $f^{-1} = f$. Thus,

$$\mathbf{d}(e) = e^{-1}e = ee = (a^{-1}a)(a^{-1}a) = a^{-1}a$$

and

$$\mathbf{d}(f) = f^{-1}f = ff = (aa^{-1})(aa^{-1}) = aa^{-1},$$

so also $e\mathcal{D}'f$.

Conversely, suppose $e\mathcal{D}'f$. Let $a \in S$ satisfy $\mathbf{d}(e) = e^2 = a^{-1}a$ and $\mathbf{d}(f) = f^2 = aa^{-1}$. Then,

$$e = e^3 = e^2e = a^{-1}ae = a^{-1}ae^2 = (a^{-1}a)(a^{-1}a) = a^{-1}a,$$

and similarly $f = aa^{-1}$, so that so $e\mathcal{D}f$. □

In view of the above lemma, we henceforth simply write $a\mathcal{D}b$ to mean $a\mathcal{D}'b$.

Definition 4.1.13 (Dedekind finite)**.** An inverse monoid S is said to be *Dedekind finite* if, for $a, b \in E(S)$, if $a\mathcal{D}b$ and $a \leq b$, then $a = b$. That is, the natural partial order restricted to $\mathcal{D}$ classes of idempotents is equality.

Remark 4.1.14. Previously in the literature (e.g. in [Law98]), Dedekind finiteness was instead called being *completely semisimple*. However, this has nothing to do with the already established notion of being semisimple and is rather confusing and counterintuitive, so we adopt the updated terminology.

The following shows why "Dedekind finite" is an appropriate choice of terminology.

Example 4.1.15. Let X be a set. We claim $\mathcal{I}(X)$ is Dedekind if and only if X is finite.

First, consider idempotents $\mathrm{id}_A, \mathrm{id}_B \in \mathcal{I}(X)$ (where $A, B \subseteq X$). Observe that

$$\mathrm{id}_A \,\mathcal{D}\, \mathrm{id}_B \Leftrightarrow \exists f \in \mathcal{I}(X).\ \mathrm{id}_A = f^{-1}f \text{ and } \mathrm{id}_B = ff^{-1} \Leftrightarrow |A| = |B|.$$

Now suppose $\mathrm{id}_A \leq \mathrm{id}_B$, $\mathrm{id}_A \,\mathcal{D}\, \mathrm{id}_B$. By the above, A and B have equal cardinality. But $\mathrm{id}_A \leq \mathrm{id}_B$ means id_B is an extension of id_A, so in particular $A \subseteq B$.

If X is finite, then A and B are also finite, and the only way for this to happen is $A = B$, whence $\mathrm{id}_A = \mathrm{id}_B$, so that $\mathcal{I}(X)$ is Dedekind finite. On the other hand, if X is infinite, then one could take $B = X$ and A to be a proper subset of X which is in bijection with X, so that $\mathcal{I}(X)$ is not Dedekind finite.

Lemma 4.1.16. *Let S be an inverse semigroup. The following hold.*

(a) $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, and $\mathcal{J}$ are equivalence relations.

(b) For all $a \in S$, $a\mathcal{L}a^{-1}a$ and $a^{-1}a\mathcal{L}a$.

(c) $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{D}$.

(d) $\mathcal{D} \subseteq \mathcal{J}$.

(e) The condition $\mathcal{D} = \mathcal{J}$ is equivalent to S being Dedekind finite.

Proof. (a) It is trivial that $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$ are equivalence relations. That $\mathcal{D}$ is an equivalence relation will follow from part (d), as equivalence relations are stable under composition.

(b) We have

$$\mathbf{d}(a^{-1}a) = a^{-1}aa^{-1}a = a^{-1}a = \mathbf{d}(a)$$

and so $a^{-1}a\mathcal{L}a$. Clearly also $a\mathcal{L}a^{-1}a$.

(c) We first prove $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. Suppose that $a, b \in S$ and $a(\mathcal{L} \circ \mathcal{R})b$. So there is some $c \in S$ such that $a\mathcal{R}c\mathcal{L}b$. That is, $aa^{-1} = cc^{-1}$ and $c^{-1}c = b^{-1}b$. Let $d = bc^{-1}a$. Then,

$$dd^{-1} = (bc^{-1}a)(bc^{-1}a)^{-1} = bc^{-1}aa^{-1}cb^{-1} = bc^{-1}cc^{-1}cb^{-1} = bc^{-1}cb^{-1} = bb^{-1},$$

and

$$d^{-1}d = (bc^{-1}a)^{-1}(bc^{-1}a) = a^{-1}cb^{-1}bc^{-1}a = a^{-1}cc^{-1}cc^{-1}a = a^{-1}cc^{-1}a = a^{-1}a.$$

Thus $a\mathcal{L}d\mathcal{R}b$, so $a(\mathcal{R} \circ \mathcal{L})b$. Therefore, $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$. The reverse inclusion is proved similarly.

Next we show $\mathcal{R} \circ \mathcal{L} = \mathcal{D}$ for idempotents. Let $e, f \in E(S)$. Suppose $e\mathcal{D}f$, with $e = a^{-1}a$ and $f = aa^{-1}$. We have

$$a^{-1}a = e = e^2 = e^{-1}e, \; aa^{-1} = f = f^2 = ff^{-1},$$

so that $a\mathcal{L}e\mathcal{R}f$, and $a(\mathcal{R} \circ \mathcal{L})f$. Conversely, if $e(\mathcal{R} \circ \mathcal{L})f$, then for some $a \in S$ we have $e\mathcal{L}a\mathcal{R}f$, and so

$$e = e^{-1}e = a^{-1}a, \; aa^{-1} = ff^{-1} = f$$

so that $e\mathcal{D}f$.

We now prove $\mathcal{R} \circ \mathcal{L} = \mathcal{D}$ for all elements. Let $a, b \in S$ and suppose that $a\mathcal{R} \circ \mathcal{L}b$. From part (b)4, we have $a^{-1}a\mathcal{L}a\mathcal{R}a$, so $a^{-1}a(\mathcal{R} \circ \mathcal{L})a$. Similarly, $b(\mathcal{R} \circ \mathcal{L})b^{-1}b$. By transitivity, we have $a^{-1}a(\mathcal{R} \circ \mathcal{L})b^{-1}b$. But these elements are idempotent, so from the previous argument we know $a^{-1}a\mathcal{D}b^{-1}b$. So there is some $c \in S$ such that $a^{-1}a = c^{-1}c$ and $cc^{-1} = b^{-1}b$ — this is precisely what $a\mathcal{D}b$ means.

Conversely, suppose that $a\mathcal{D}b$. Then, $a^{-1}a\mathcal{D}b^{-1}b$, so it follows that $a^{-1}a(\mathcal{R} \circ \mathcal{L})b^{-1}b$. Let $c \in S$ satisfying $a^{-1}a\mathcal{L}c\mathcal{R}b^{-1}b$, so (recall that idempotents are pseudoinverse to themselves) $a^{-1}a = c^{-1}c$ and $cc^{-1} = b^{-1}b$. Thus, $a\mathcal{L}c\mathcal{R}b$, and $a(\mathcal{R} \circ \mathcal{L})b$.

Hence, $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{D}$.

(d) Suppose $a, b \in S$ and $a\mathcal{D}b$. Let $c \in S$ satisfy $a^{-1}a = c^{-1}c$ and $b^{-1}b = cc^{-1}$. Now let $xay \in SaS$ be an arbitrary element ($x, y \in S$). Then,

$$xay = xaa^{-1}ay = xac^{-1}cy = xac^{-1}cc^{-1}cy = xac^{-1}b^{-1}bcy \in SbS,$$

so that $SaS \subseteq SbS$. Similarly, $SbS \subseteq SaS$, and so $a\mathcal{J}b$.

(e) See [Law98, pp.92–93].

□

Lemma 4.1.17. *Let S be an inverse semigroup. The following hold for all $a, b, c \in S$.*

(a) We have $a \sim b$ if and only if $a \wedge b$ exists, and if it does then $\mathbf{d}(a \wedge b) = \mathbf{d}(a) \wedge \mathbf{d}(b)$, and $\mathbf{r}(a \wedge b) = \mathbf{r}(a) \wedge \mathbf{r}(b)$.

(b) *If S is a distributive inverse monoid and if $a \vee b$ exists, then $\mathbf{d}(a \vee b) = \mathbf{d}(a) \vee \mathbf{d}(b)$, and $\mathbf{r}(a \vee b) = \mathbf{r}(a) \vee \mathbf{r}(b)$.*

(c) *If S is a distributive inverse monoid, $a \vee b$ and $c \wedge (a \vee b)$ both exist, then $c \wedge a$, $c \wedge b$ exist, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$.*

Proof. See [Law98, Proposition 1.4.17]. □

Lemma 4.1.18. *Let S be a boolean inverse monoid, and $a_1, a_2, b_1, b_2, u_1, u_2 \in S$ such that $a_1 \xrightarrow{u_1} b_1$ and $a_2 \xrightarrow{u_2} b_2$. Then if $a_1 \perp a_2$, it follows that $u_1 u_2^{-1} = 0$, and if $b_1 \perp b_2$, it follows that $u_2 u_1^{-1} = 0$. In particular, if both $a_1 \perp a_2$ and $b_1 \perp b_2$, then $u_1 \perp u_2$.*

Proof. Suppose $a_1 \perp a_2$. Then,

$$u_1 u_2^{-1} = u_1 u_1^{-1} u_1 u_2^{-1} u_2 u_2^{-1} = u_1 a_1^{-1} a_1 a_2^{-1} a_2 u_2^{-1} = u_1 a_1^{-1} 0 a_2 u_2^{-1} = 0.$$

Similarly, $b_1 \perp b_2$ implies $u_2^{-1} u_1 = 0$. □

If S is a boolean inverse monoid and $e \mathcal{D} f$ implies $\overline{e} \mathcal{D} \overline{f}$, we say that $\mathcal{D}$ *preserves complementation*.

Proposition 4.1.19. *Let S be a boolean inverse monoid where $\mathcal{D}$ preserves complementation. Then the following hold.*

(a) *S is factorizable.*

(b) *For all $e \in E(S)$, if $e \mathcal{D} 1$, then $e = 1$.*

(c) *S is Dedekind finite.*

Proof. (a) Let $a \in S$. Put $e = a^{-1}a$ and $f = aa^{-1}$, so that $e \mathcal{D} f$. By hypothesis, also $\overline{e} \mathcal{D} \overline{f}$, so let $b \in S$ satisfying $\overline{e} = b^{-1}b$ and $\overline{f} = bb^{-1}$. Note that $e\overline{e}^{-1} = e\overline{e} = e \wedge \overline{e} = 0$ and also $f\overline{f}^{-1} = 0$ so by Lemma 4.1.18, it follows that $a \perp b$. In particular, $a \sim b$, so by Lemma 4.1.17(a), the join $a \vee b = g$ exists. Now by Lemma 4.1.17(b), we have

$$g^{-1}g = \mathbf{d}(g) = \mathbf{d}(a \vee b) = \mathbf{d}(a) \vee \mathbf{d}(b) = e \vee \overline{e} = 1,$$

so g is a unit, and moreover $a \leq g$. Thus, S is factorizable.

(b) Suppose $e \in E(S)$ and $e\mathcal{D}1$. Then $\overline{e}\mathcal{D}\overline{1} = 0$, so there exists $b \in S$ such that $\overline{e} = b^{-1}b$ and $bb^{-1} = 0$. But then $\overline{e} = b^{-1}(bb^{-1})b = b^{-1}0b = 0$, so $e = 1$.

(c) Let $e, f \in E(S)$ such that $e\mathcal{D}f$ and $f \leq e$. By Lemma 4.1.2(d), $f = ef$. Let $a, b \in S$ satisfying $e = a^{-1}a$, $f = aa^{-1}$, $\overline{e} = b^{-1}b$, $\overline{f} = bb^{-1}$.

Observe that $\mathbf{d}(afa) = (afa)^{-1}(afa) = a^{-1}fa^{-1}afa = a^{-1}fefa = a^{-1}f^2a = a^{-1}fa = a^{-1}aa^{-1}a = a^{-1}a = e$. From Lemma 4.1.18, we deduce $(afa)b^{-1} = 0$.

On the other hand,

$$\begin{aligned}(afa)^{-1}b &= a^{-1}fa^{-1}b \\ &= a^{-1}aa^{-1}ab \\ &= a^{-1}a^{-1}b \\ &= a^{-1}a^{-1}aa^{-1}bb^{-1}b \\ &= a^{-1}a^{-1}(f\overline{f})b \\ &= a^{-1}a^{-1}(0)b \\ &= 0.\end{aligned}$$

Thus, $afa \perp b$, so the join $b \vee afa$ exists. Now, $\mathbf{d}(b \vee afa) = \mathbf{d}(b) \vee \mathbf{d}(afa) = \overline{e} \vee e = 1$. By part (b), we must also have

$$\begin{aligned}1 &= \mathbf{r}(b \vee afa) \\ &= \mathbf{r}(b) \vee \mathbf{r}(afa) \\ &= \overline{f} \vee (afa)(afa)^{-1} \\ &= \overline{f} \vee afaa^{-1}fa^{-1} \\ &= \overline{f} \vee af^3a^{-1} \\ &= \overline{f} \vee afa^{-1}.\end{aligned}$$

Next, observe that

$$(afa^{-1})(afa^{-1}) = afefa^{-1} = af^2a^{-1} = afa^{-1},$$

so that afa^{-1} is also idempotent. We have

$$\overline{f} \wedge afa^{-1} = \overline{f}afa^{-1} = \overline{f}aa^{-1}afa^{-1} = (\overline{f}f)afa^{-1} = 0.$$

By uniqueness of complements in a boolean algebra, it follows that $afa^{-1} = f$. Then,

$$af = aef = afe = afa^{-1}a = fa = aa^{-1}a = a.$$

Thus,

$$e = a^{-1}a = a^{-1}af = ef = f,$$

as desired. □

Proposition 4.1.20. *Let S be a factorizable boolean inverse monoid. Then, $\mathcal{D}$ preserves complementation.*

Proof. Suppose $e, f \in E(S)$ and $e\mathcal{D}f$. So for some $a \in S$, we have $e = a^{-1}a$ and $f = aa^{-1}$. By hypothesis, there is some $g \in S$ such that $a \leq g$ and $g^{-1}g = 1 = gg^{-1}$.

By Lemma 4.1.3 (d)-(e), we have $a = gaa^{-1} = ge$ and $a = aa^{-1}g = fg$. Put $b = g\overline{e}$. We have

$$\mathbf{r}(b) = bb^{-1} = (g\overline{e})(g\overline{e})^{-1} = g\overline{e}\overline{e}g^{-1} = g\overline{e}g^{-1},$$

and similarly $\mathbf{r}(a) = geg^{-1}$. Now observe that

$$f \wedge \mathbf{r}(b) = f\mathbf{r}(b) = fg\overline{e}g^{-1} = a\overline{e}g^{-1} = aa^{-1}a\overline{e}g^{-1} = a(e\overline{e})g^{-1} = 0.$$

On the other hand we have also

$$f \vee \mathbf{r}(b) = aa^{-1} \vee bb^{-1} = bb^{-1} \vee aa^{-1} = g\overline{e}g^{-1} \vee geg^{-1} = g(\overline{e} \vee e)g^{-1} = 1.$$

It follows that $b^{-1}b = \overline{f}$. Also, $bb^{-1} = (g\overline{e})^{-1}g\overline{e} = \overline{e}g^{-1}g\overline{e} = \overline{e}^2 = \overline{e}$. Thus, $\overline{e}\mathcal{D}\overline{f}$, whence $\mathcal{D}$ preserves complementation. □

As an immediate consequence of the preceding two propositions, we have the following.

Corollary 4.1.21. *A boolean inverse monoid is factorizable if and only if it is Dedekind finite and $\mathcal{D}$ preserves complementation.*

Definition 4.1.22 (Foulis monoid)**.** A *Foulis monoid* is a factorizable boolean inverse monoid.

4.2 The construction $S/\mathcal{D}$ and AF inverse monoids

In this section we define the constructions needed for coordinatization. We define Greene's $\mathcal{D}$-relation on inverse semigroups, towards showing that by taking a quotient by this relation, we obtain an MV algebra. We state the coordinatization theorem, which is that in fact every MV algebra arises as a quotient of an inverse semigroup this way. The types of inverse semigroups involved in this process are AF (approximately finite) inverse monoids, which are related to AF C*-algebras.

We begin with the more general notion of V-relation (named for Robert Vaught), which Greene's relations turn out to be an example of.

Definition 4.2.1 ((Conical, additive) V-relation, V-equivalence)**.** Let P, Q be PCMs and let $\Gamma \subseteq P \times Q$ be a binary relation. We say that Γ is (adopting the convention that partial sums, where written, are defined):

- a *left V-relation* if whenever $x_1, x_2 \in P$ and $y \in Q$ such that $x_1 \mathbin{\widetilde{\oplus}} x_2 \Gamma y$, then there exist $y_1, y_2 \in Q$ such that $y = y_1 \mathbin{\widetilde{\oplus}} y_2$ and $x_1 \Gamma y_1$, $x_2 \Gamma y_2$.
- a *right V-relation* if $\Gamma^{-1} \subseteq Q \times P$ is a left V-relation.
- a *V-relation* if it is both a left V-relation and a right V-relation.
- *left conical* if for $y \in Q$, $0_P \Gamma y$ implies $y = 0_Q$.
- *right conical* if Γ^{-1} is left conical.
- *conical* if it is both left conical and right conical.
- a *V-equivalence* if it is both a V-relation and an equivalence relation.
- *additive* if it as a partial submonoid of $P \times Q$ (that is, $0_P \Gamma 0_Q$, and for all $x_1, x_2 \in P$ and $y_1, y_2 \in Q$, if $x_1 \mathbin{\widetilde{\oplus}} x_2 \downarrow$, $y_1 \mathbin{\widetilde{\oplus}} y_2 \downarrow$, $x_1 \Gamma y_1$, and $x_2 \Gamma y_2$, then $x_1 \mathbin{\widetilde{\oplus}} x_2 \Gamma y_1 \mathbin{\widetilde{\oplus}} y_2$).

Example 4.2.2. If P is a PCM with the Riesz decomposition property, then $\leq$ is a left V-relation.

Lemma 4.2.3. *The composition of (left, right) V-relations is again a (left, right) V-relation. The composition of conical relations is again conical.*

Proof. We prove that the composition of left V-relations is a left V-relation. The proof for right V-relations is similar, and the statement then follows for V-relations.

Let P, Q, R be PCMs and let $\Gamma \subseteq P \times Q$ and $\Delta \subseteq Q \times R$ be left V-relations. Suppose that $x_1, x_2 \in P$, $y \in R$ such that $x_1 \mathbin{\widetilde{\oplus}} x_2 (\Delta \circ \Gamma) y$. This means there is $z \in Q$ such that $x_1 \mathbin{\widetilde{\oplus}} x_2 \Gamma z \Delta y$. Since Γ is a left V-relation, we must have $z_1, z_2 \in Q$ such that $z = z_1 \mathbin{\widetilde{\oplus}} z_2$, $x_1 \Gamma z_1$, and $x_2 \Gamma z_2$. But then, we have $z_1 \mathbin{\widetilde{\oplus}} z_2 \Delta y$. Since Δ is also a left V-relation, we must have $y_1, y_2 \in R$ with $y = y_1 \mathbin{\widetilde{\oplus}} y_2$, $z_1 \Delta y_1$, $z_2 \Delta y_2$. But then we have $x_1 (\Delta \circ \Gamma) y_2$ and $x_2 (\Delta \circ \Gamma) y_2$, and indeed $\Delta \circ \Gamma$ is a left V-relation.

Now suppose Γ, Δ are conical. Suppose $0_P (\Delta \circ \Gamma) x$ for $x \in R$. Then there is $y \in Q$ such that $0_P \Gamma y \Delta x$. That Γ is conical implies $y = 0_Q$, and that Δ is conical implies $x = 0_R$, as desired. $\square$

Lemma 4.2.4. *Let P be a PCM and $\Gamma \subseteq P \times P$ be an additive V-equivalence. Then, the quotient P/Γ is a PCM with the following operation.*

$$[x] \mathbin{\widetilde{\oplus}} [y] = \begin{cases} [x' \mathbin{\widetilde{\oplus}} y'], & \textit{if there exist } x' \in [x] \textit{ and } y' \in [y] \textit{ such that } x' \mathbin{\widetilde{\oplus}} y' \downarrow, \\ \uparrow, & \textit{otherwise.} \end{cases}$$

Moreover, we have the following.

(a) If both P and Γ are conical, then so is P/Γ.

(b) If P has the refinement property, then so does P/Γ.

Proof. See [Weh15, p.34]. $\square$

If S is a distributive inverse semigroup with zero, define the *orthogonal join* of two elements $a, b \in S$ to be

$$a \mathbin{\widetilde{\oplus}} b = \begin{cases} a \vee b, & \text{if } a \perp b, \\ \uparrow, & \text{otherwise.} \end{cases}$$

One can then regard S as a PCM with regard to $\widetilde{\oplus}$.

Proposition 4.2.5. *Let S be an inverse semigroup considered as a PCM as above. Then, $\mathcal{D}$ is a conical and additive V-equivalence, and so $S/\mathcal{D}$ is a PCM.*

Proof. We first prove that $\mathcal{L}$ is conical and a left V-equivalence. Suppose that $x \in S$ and $x\mathcal{L}0$. Then $xx^{-1} = 0$, so $x = xx^{-1}x = 0x = 0$. Thus, $\mathcal{L}$ is conical.

Now let $x, x_1, x_2, y \in S$ and suppose $x = x_1 \widetilde{\oplus} x_2$ with $x_1 \widetilde{\oplus} x_2 \mathcal{L} y$. So $\mathbf{d}(x) = \mathbf{d}(y)$. From Lemma 4.1.17, we have

$$\mathbf{d}(y) = \mathbf{d}(x) = \mathbf{d}(x_1 \widetilde{\oplus} x_2) = \mathbf{d}(x_1) \widetilde{\oplus} \mathbf{d}(x_2).$$

Put $y_1 = y\mathbf{d}(x_1) = yx_1^{-1}x_1$ and $y_2 = y\mathbf{d}(x_2) = yx_2^{-1}x_2$. We have

$$y = yy^{-1}y = y\mathbf{d}(y) = y(\mathbf{d}(x_1) \widetilde{\oplus} \mathbf{d}(x_2)) = y\mathbf{d}(x_1) \widetilde{\oplus} y\mathbf{d}(x_2) = y_1 \widetilde{\oplus} y_2.$$

Now, we have

$$\begin{aligned}
\mathbf{d}(y_1) &= (y\mathbf{d}(x_1))^{-1} y\mathbf{d}(x_1) \\
&= \mathbf{d}(x_1)\mathbf{d}(y)\mathbf{d}(x_1) \\
&= \mathbf{d}(x_1)\mathbf{d}(x_1)\mathbf{d}(y) \\
&= \mathbf{d}(x_1)\mathbf{d}(y) \\
&= \mathbf{d}(x_1)\mathbf{d}(x) \\
&= \mathbf{d}(x_1)\mathbf{d}(x_1 \widetilde{\oplus} x_2) \\
&= \mathbf{d}(x_1)(\mathbf{d}(x_1) \widetilde{\oplus} \mathbf{d}(x_2)) \\
&= \mathbf{d}(x_1) \widetilde{\oplus} \mathbf{d}(x_1)\mathbf{d}(x_2) \\
&= x_1^{-1}x_1 \vee x_1^{-1}x_1x_2^{-1}x_2 \\
&= x_1^{-1}x_1 \\
&= \mathbf{d}(x_1),
\end{aligned}$$

so that $x_1 \mathcal{L} y_1$. Similarly, $x_2 \mathcal{L} y_2$. Thus, $\mathcal{L}$ is a left V-equivalcnece.

By a similar argument, $\mathcal{L}$ is also a right V-equivalence and so is a V-equivalence. Analogously, $\mathcal{R}$ is conical and a V-equivalence. It follows from Lemma 4.2.3 that $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ is also conical and a V-equivalence.

Now we prove $\mathcal{D}$ is additive. It's clear that $0\mathcal{D}0$. Now suppose $a = a_1 \mathbin{\widetilde{\oplus}} a_2$, and $b = b_1 \mathbin{\widetilde{\oplus}} b_2$, with $a_1 \xrightarrow{u_1} b_1$ and $a_2 \xrightarrow{u_2} b_2$. Thus we have $a = a_1 \vee a_2$, $b = b_1 \vee b_2$, $a_1 \perp a_2$ and $b_1 \perp b_2$. By Lemma 4.2.3, we have $u_1 \perp u_2$, and since we are in a distributive boolean inverse monoid, the element $u = u_1 \mathbin{\widetilde{\oplus}} u_2$ exists. We have

$$\mathbf{d}(u) = \mathbf{d}(u_1 \mathbin{\widetilde{\oplus}} u_2) = \mathbf{d}(u_1) \mathbin{\widetilde{\oplus}} \mathbf{d}(u_2) = \mathbf{d}(a_1) \mathbin{\widetilde{\oplus}} \mathbf{d}(a_2) = \mathbf{d}(a_1 \vee a_2) = \mathbf{d}(a),$$

and similarly $\mathbf{r}(u) = \mathbf{r}(a)$. Thus, we have $a\mathcal{L}u\mathcal{R}b$, so $a(\mathcal{R} \circ \mathcal{L})b$, whence $a\mathcal{D}b$, as desired. ☐

Proposition 4.2.6. *If S is a Foulis monoid, then $S/\mathcal{D}$ is an effect algebra satisfying the refinement property. So, if additionally, $S/\mathcal{D}$ is a lattice, then $S/\mathcal{D}$ is an MV effect algebra.*

Proof. The proof can be found in [LS14, Sec. 2.3], together with Corollary 4.1.21. ☐

For an inverse semigroup S, it is clear that for all $a \in S$, we have $a\mathcal{D}\mathbf{d}(a)$, so it is enough to consider elements of the form $x/\mathcal{D}$ for idempotent x. We may thus simply think of $S/\mathcal{D}$ as

$$\{x/\mathcal{D} \mid x \in E(S)\},$$

with $z/\mathcal{D} = x/\mathcal{D} \mathbin{\widetilde{\oplus}} y/\mathcal{D}$ whenever $z = x \mathbin{\widetilde{\oplus}} y$.

We will now describe the construction of a directed colimit of a sequence

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots$$

of inverse monoids and injective homomorphism. For any explanation and justification we omit, see [LS14, Sec 3.1].

Note in what follows, [LS14] use the dual order on $\mathbb{N}$ (that is, $i < j$ means i is the larger number). The present author does not see the reason for this and finds it disorienting, so we will use the standard ordering.

For $i < j$, define $\tau^i_j \colon S_i \to S_j$ by $\tau^i_j = \tau_{j-1} \ldots \tau_{i+i}\tau_i$; i.e. $\tau^i_{i+1} = \tau_i$. Now let $S = \sqcup_{i=0}^{\infty} S_i$, and equip S with the following operation. If $a \in S_i$, $b \in S_j$, define

$$a \cdot b = \tau^i_{i\vee j}(a)\tau^j_{i\vee j}(b).$$

The multiplication on the right occurs in $S_{i\vee j}$. This endows S with the structure of an inverse monoid; for $a \in S$, we have $a \in S_i$ for some i, and its inverse in S is just its inverse in S_i. The identity of S is the identity of S_0.

Now denote the identity of S_i by e_i. Put $\mathcal{E} = \{e_i \mid i \in \mathbb{N}\}$. We define

$$a \equiv b \Leftrightarrow \exists e \in \mathcal{E} \text{ such that } ae = be \text{ for some } e \in \mathcal{E}.$$

Then, $\equiv$ is a monoid congruence on S, and $S/\equiv$, which we denote S_∞, is an inverse monoid with zero, where the zero is the $\equiv$-class of all the zeroes of the S_i.

Proposition 4.2.7. *Let*

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \dots$$

be a sequence of boolean inverse $\wedge$-monoids. Then,

(a) S_∞ as constructed above with the maps $\phi_i \colon S_i \to S$, $s \mapsto [s]$, is the directed colimit $\varinjlim S_i$ of the sequences, and is also a boolean $\wedge$-monoid.

(b) If all the S_i are fundamental, factorizable, Dedekind finite, or satisfy the property that $\mathcal{D}$ preserves complementation, then the same is also true of $\varinjlim S_i$.

Proof. See [LS14, Lemma 3.12]. □

Definition 4.2.8 (AF inverse monoid)**.** An inverse monoid constructed as a directed colimit as described above, where every S_i is a finite product of finite symmetric inverse monoids, is called an *AF (approximately finite) inverse monoid*.

We can now state what we mean by coordinatization and the coordinatization theorem precisely.

Definition 4.2.9 (Coordinatizable)**.** An MV algebra A is called *coordinatizable* if there is a Foulis monoid S such that $S/\mathcal{D} = S/\mathcal{J} \cong A$. We also say that S *coordinatizes A*.

Theorem 4.2.10 (Coordinatization of MV algebras)**.** *Every countable MV algebra A can be coordinatized. Moreover, S can be taken to be an AF inverse monoid.*

Proof. The proof of this theorem is the main subject of [LS14]. We have given some of the background and enough definitions to understand the statement but refer the reader to the original paper for details of the general proof. □

Remark 4.2.11. According to [Weh15, Theorem 5.2.10], the coordinatization theorem also extends to uncountable MV algebras, with what appears to be a direct generalization of AF inverse monoids applying at cardinality $\aleph_1$, but not beyond. However, this monograph is not about coordinatization or MV algebras per se, and there is a large amount of different definitions, machinery and terminology spanning over 100 pages from which the result falls out of. The present author believes the more general theorem to be true, but has found it difficult to compare the two approaches directly.

4.3 The coordinatization of $\mathbb{Q} \cap [0, 1]$

In [LS14], some concrete examples of coordinatization are given. The finite subalgebras of $[0, 1]$, the *Lukasiewicz chains*, $\text{Ł}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, are coordinatized by the symmetric inverse monoids $\mathcal{I}_n$.

Recall that $\mathbb{Q}_{\text{Dyad}}$, the *dyadic rationals*, are rational numbers of the form $\frac{a}{2^b}$. In [LS14, Sec. 5], $\mathbb{Q}_{\text{Dyad}} \cap [0, 1]$ is shown to be coordinatized by a construction called the *dyadic inverse monoid*, which turns out to be isomorphic to the directed colimit of

$$I_1 \xrightarrow{\tau_0} I_2 \xrightarrow{\tau_1} I_4 \xrightarrow{\tau_2} I_8 \xrightarrow{\tau_3} I_{16} \xrightarrow{\tau_4} \ldots,$$

where the τ_i are inclusion maps (we will make precise what this means shortly). The idea is that for $f \in \mathcal{I}_{2^l}$, the $\mathcal{D}$-class of idempotents of f is associated to the number $\frac{|\text{dom}(f)|}{2^l}$. We will now generalize this to the coordinatization of all rationals in $[0, 1]$.

Definition 4.3.1 (Omnidivisional sequence, D-canonical form)**.** A sequence $D = \{n_i\}_{i=1}^{\infty}$ of natural numbers is *omnidivisional* if it satisfies the following properties.

- For all i, $n_i \mid n_{i+1}$.
- For all $m \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $m \mid n_i$.

By the second condition, every rational number can be written with one of the n_i as the denominator. If $\frac{a}{b} \in \mathbb{Q}$ is in lowest terms, then there is c such that $bc = n_i$ and so $\frac{a}{b} = \frac{ac}{n_i}$.

For a fixed omnidivisional sequence D and $q \in \mathbb{Q}$, the smallest i such that $q = \frac{d}{n_i}$ for some $d \in \mathbb{N}$ is called the *D-canonical form* of q.

Example 4.3.2. The sequence $\{n!\}_{n=1}^{\infty}$ is easily seen to be omnidivisional.

Example 4.3.3. Let p_i be the i^{th} prime number. Then $\{\Pi_{i=1}^{n} p_i^{n-i+1}\}_{n=1}^{\infty}$ is an omnidivisional sequence. The first few members of the sequence are $2, 2^2 3, 2^3 3^2 5, 2^4 3^3 5^2 7, \ldots$.

Clearly each member of the sequence divides the next, and for $m \in \mathbb{N}$, if one looks at the prime factorization of m, then for sufficiently large k, the k^{th} member of the sequence will contain all prime factors of m, taking multiplicity into account.

We now describe what we mean by an inclusion map $\mathcal{I}_n \xrightarrow{\tau} \mathcal{I}_m$ for $n, m \in \mathbb{N}$ where $n \mid m$. Write $na = m$. We denote the underlying set of $\mathcal{I}_n$ by $X_n = \{x_1, \ldots, x_n\}$ and similarly for $\mathcal{I}_m$. We wish to identify each of the n elements of the X_n with a subset of a elements of X_m in a systematic way as follows.

$$\begin{aligned}
X_n \ni x_1 &\mapsto y_1 := \{x_a, x_{a-1}, \ldots, x_1\} \subseteq X_m \\
X_n \ni x_2 &\mapsto y_2 := \{x_{2a}, x_{2a-1}, \ldots, x_{a+1}\} \subseteq X_m \\
&\cdots \\
X_n \ni x_n &\mapsto y_n := \{x_{na}, x_{na-1}, \ldots, x_{(n-1)a+1}\} \subseteq X_m.
\end{aligned}$$

If f is a partial bijection on X_n and it maps x_i to x_j, then $\tau(f)$ should be a bijection from y_i to y_j in the obvious way. If $f(x_i) = x_j$, then we denote $f^*(i) = j$. For all $i \in \{1, \ldots, m\}$, we write $i = qa - r$ for some $1 \leq q \leq n$ and $0 \leq r < a$. Then, define

$$(\tau(f))(x_i) = (\tau(f))(x_{aq-r}) = x_{a(f^*(q))-r}.$$

This is clearly a partial bijection on X_m, and $|dom(\tau(f))| = a|dom(f)|$. If we identify f with its representing *rook matrix* (that is, the $n \times n$ matrix $(a_{i,j})$ with entries in $\{0, 1\}$ where $a_{i,j} = 1 \Leftrightarrow f(x_i) = x_j$), then the rook matrix for $\tau(f)$ is simply the

$na \times na$ expansion of the matrix for f obtained by replacing all 0s with the $a \times a$ zero matrix and replacing all 1s by the $a \times a$ identity matrix.

Theorem 4.3.4 (Coordinatization of the rationals)**.** *Let $D = \{n_i\}_{n=1}^{\infty}$ be an omnidivisional sequence. Then, the directed colimit of the sequence*

$$Q\colon \mathcal{I}_{n_1} \xrightarrow{\tau_1} \mathcal{I}_{n_2} \xrightarrow{\tau_2} \mathcal{I}_{n_3} \xrightarrow{\tau_3} \mathcal{I}_{n_4} \xrightarrow{\tau_4} \dots,$$

where the τ_i are inclusion maps in the sense described above, coordinatizes $\mathbb{Q} \cap [0, 1]$.

Proof. We denote the directed colimit of Q by Q_∞. We define a map $w\colon Q_\infty/\mathcal{D} \to \mathbb{Q} \cap [0, 1]$ as follows. For $s \in \mathcal{I}_{n_i}$, define

$$w([s]/\mathcal{D}) = \frac{|\operatorname{dom}(s)|}{n_i}.$$

We claim w is well defined on $\mathcal{D}$ classes, and is an isomorphism of MV effect algebras.

First, we will prove that, for $a \in \mathcal{I}_{n_i}$ and $b \in \mathcal{I}_{n_j}$, $[a] = [b]$ in Q_∞ implies $w([a]/\mathcal{D}) = w([b]/\mathcal{D})$. That $[a] = [b]$ implies the existence of $e_{n_k} \in \mathcal{I}_{n_k}$ such that $a \cdot e_{n_k} = b \cdot e_{n_k}$. This means

$$\tau^{n_i}_{n_i \vee n_k}(a)\tau^{n_k}_{n_i \vee n_k}(e_{n_k}) = \tau^{n_j}_{n_j \vee n_k}(b)\tau^{n_k}_{n_j \vee n_k}(e_{n_k}). \tag{4.3.1}$$

In the case that $n_i = n_j$, (4.3.1) yields $|\operatorname{dom}(\tau^{n_i}_{n_i \vee n_k}(a))| = |\operatorname{dom}(\tau^{n_i}_{n_i \vee n_k}(b))|$. Let $m \in \mathbb{N}$ such that $n_i m' = n_i \vee n_k$ (this exists because we chose these numbers to be from an omnidivisional sequence). Then, we have $m'|\operatorname{dom}(a)| = m'|\operatorname{dom}(b)|$, so $w([a]/\mathcal{D}) = \frac{|\operatorname{dom}(a)|}{n_i} = \frac{|\operatorname{dom}(b)|}{n_i} = w([b]/\mathcal{D})$.

Now suppose $n_i \neq n_j$. Since the left side of the equation (4.3.1) takes place in $S_{n_i \vee n_k}$ and the right side takes place in $S_{n_j \vee n_k}$ and these must be the same, we must have $n_i \vee n_k = n_j \vee n_k$, from which it follows that $n_k \geq n_i$ and $n_k \geq n_j$. Then, we have $|\operatorname{dom}(\tau^{n_i}_{n_k}(a))| = |\operatorname{dom}(\tau^{n_j}_{n_k}(b))|$. Pick $m_1, m_2 \in \mathbb{N}$ such that $m_1 n_i = n_k$ and $m_2 n_j = n_k$. Then, $m_1|\operatorname{dom}(a)| = m_2|\operatorname{dom}(b)|$. But $m_1 = \frac{n_k}{n_i}$ and $m_2 = \frac{n_k}{n_j}$, so it follows that $w([a]/\mathcal{D}) = \frac{|\operatorname{dom}(a)|}{n_i} = \frac{|\operatorname{dom}(b)|}{n_j} = w([b]/\mathcal{D})$.

Now we are ready to prove well definedness. If $[s]/\mathcal{D} = [t]/\mathcal{D}$, then there is some a such that $[s^{-1}s] = [a^{-1}a]$ and $[aa^{-1}] = [t^{-1}t]$. But then, by the above observation,

$$w(s) = w(s^{-1}s) = w(a^{-1}a) = w(aa^{-1}) = w(t^{-1}t) = w(t)$$

For injectivity, suppose $w([s]/\mathcal{D}) = w([t]/\mathcal{D})$. Then, if $s \in \mathcal{I}_{n_i}$ and $t \in \mathcal{I}_{n_j}$, and without loss of generality we let $n_i \leq n_j$, we have $\frac{|\operatorname{dom}(s)|}{n_i} = \frac{|\operatorname{dom}(t)|}{n_j}$. Write $n_i m = n_j$. We have

$$|\operatorname{dom}(\tau^{n_i}_{n_i \vee n_j}(s))| = |\operatorname{dom}(\tau^{n_i}_{n_j}(s))| = m|\operatorname{dom}(s)| = |\operatorname{dom}(t)|.$$

But then in $\mathcal{I}_{n_j}$, we know elements are $\mathcal{D}$-related if and only if they have the same cardinality, so $\tau^{n_i}_{n_j}(s)/\mathcal{D} = t/\mathcal{D}$. It follows that $[\tau^{n_i}_{n_j}(s)]/\mathcal{D} = [t]/\mathcal{D}$. Clearly,

$$e_{n_j} \cdot s = \tau^{n_i}_{n_j}(e_j)\tau^{n_i}_{n_j}(s) = e_{n_j}\tau^{n_i}_{n_j}(s),$$

so $[s] = [\tau^{n_i}_{n_j}(s)]$. So then $[s]/\mathcal{D} = [t]/\mathcal{D}$.

Surjectivity follows from omnidivisionality of the chosen sequence; if $q \in \mathbb{Q} \cap [0,1]$, we write $q = \frac{a}{n_q}$ in D-canonical form. Let $f \in \mathcal{I}_{n_q}$ be the partial identity on $\{x_1, \ldots, x_a\}$. Then, $w(f) = \frac{a}{n_q} = q$.

Finally, we argue that w is an MV-effect homomorphism and hence an MV homomorphism. Since MV algebras and homomorphisms are defined from an algebraic theory and we know w is bijective, it suffices to show that its inverse is an MV-effect homomorphism. Let $q \in \mathbb{Q} \cap [0,1]$ and write q in D-canonical form as $q = \frac{d}{n_i}$. Then, it is easy to see that

$$\begin{aligned} w^{-1} \colon \mathbb{Q} \cap [0,1] &\to Q_\infty/\mathcal{D}, \\ \frac{d}{n_i} &\mapsto [\mathrm{id}^{n_i}_{\{x_1,\ldots,x_d\}}]/\mathcal{D}, \end{aligned}$$

where by $\mathrm{id}^{n_i}_{\{x_1,\ldots,x_d\}}$ we mean the partial identity map on $\{x_1, \ldots, x_d\}$ considered as an element of $\mathcal{I}_{n_i}$, is really the two-sided inverse to w.

We have that $w(1) = w(\frac{n_1}{n_1}) = [\mathrm{id}_{I_{n_1}}]/\mathcal{D}$, which we argue is the top element of Q_∞. Clearly the τ maps take total identity functions to total identity functions, so for all i we have $[\mathrm{id}_{I_{n_1}}] = [\mathrm{id}_{I_{n_i}}]$; as it is enough to consider $\mathcal{D}$-classes of idempotents, if $s \in E(I_{n_i})$, then denoting the partial identity with domain $X \setminus \operatorname{dom}(s)$ by $s^\perp$, we have that $s \mathbin{\widetilde{\oplus}} s^\perp = s \vee s^\perp = \mathrm{id}_{X_{n_i}}$, so $s \leq \mathrm{id}_{X_{n_i}}$ whence $[s]/\mathcal{D} \leq [\mathrm{id}_{X_{n_1}}]/\mathcal{D}$.

Now let $a, b \in \mathbb{Q} \cap [0,1]$. Suppose $a \mathbin{\widetilde{\oplus}} b \downarrow$. Then $a + b \leq 1$. Write in D-canonical form $a = \frac{d_1}{n_i}$ and $b = \frac{d_2}{n_j}$. Without loss of generality, suppose $n_i \leq n_j$. Then $n_i \mid n_j$, so there is $m \in \mathbb{N}$ with $n_i m = n_j$, and we have $a = \frac{d_1 m}{n_j}$, and $d_1 m + d_2 \leq n_j$. We have

$$\begin{aligned} w^{-1}(a) \,\widetilde{\oplus}\, w^{-1}(b) &= [\mathrm{id}^{n_j}_{x_1,\ldots,x_{d_1 m}}]/\mathcal{D} \,\widetilde{\oplus}\, [\mathrm{id}^{n_j}_{x_1,\ldots,x_{d_2}}]/\mathcal{D} \\ &= [\mathrm{id}^{n_j}_{x_1,\ldots,x_{d_1 m}}]/\mathcal{D} \,\widetilde{\oplus}\, [\mathrm{id}^{n_j}_{x_{d_1 m+1},\ldots,x_{d_1 m+d_2}}]/\mathcal{D} \\ &= [\mathrm{id}^{n_j}_{x_1,\ldots,x_{d_1 m+d_2}}]/\mathcal{D} \\ &= w^{-1}\left(\frac{d_1 m + d_2}{n_j}\right) \\ &= w^{-1}(a \,\widetilde{\oplus}\, b). \end{aligned}$$

Now let a and b be written in D-canonical form as above. Then, we have that

$$w^{-1}(a) \vee w^{-1}(b) = [\mathrm{id}^{n_j}_{x_1,\ldots,x_{d_1 m}}]/\mathcal{D} \vee [\mathrm{id}^{n_j}_{x_1,\ldots,x_{d_2}}]/\mathcal{D}.$$

But the join of the above is the $\mathcal{D}$-class of the partial identity defined on the union of the domains, hence the above is equal to $w^{-1}(\max(a, b))$, which is $w^{-1}(a \vee b)$.

Repeating the above argument with meet in place of join, intersection in place of union, and min in place of max, yields that w^{-1} also preserves meets.

Thus, w^{-1} is an effect algebra isomorphism which preserves the lattice structure, hence an MV-effect isomorphism, and hence an MV isomorphism, and so Q_∞ coordinatizes $\mathbb{Q} \cap [0, 1]$. □

4.4 Coordinatization decomposition theorem

We now turn to generalizing the approach we took in coordinatizing $\mathbb{Q} \cap [0, 1]$. We knew the Łukasiewicz chains $Ł_n$ were coordinatized by $\mathcal{I}_n$, and if we fix an omnidivisional sequence $\{n_1, n_2, \ldots\}$, we know $\mathbb{Q} \cap [0, 1] = \bigcup_{i=1}^{\infty} Ł_{n_i}$ and its coordinatization turned out to be the direct limit of the sequence of I_{n_i}.

We have the following decomposition theorem, which gives us a general way to coordinatize MV algebras by writing them as unions of subalgebras which are coordinatized by a single inverse semigroup. Much of the proof is similar to that of Theorem 4.3.4.

Theorem 4.4.1 (Decomposition Theorem I)**.** *Let A be an MV algebra. Suppose that A has subalgebras forming a chain of inclusions*

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$$

such that $A = \bigcup_{i=1}^{\infty} A_i$ and that each A_i is coordinatized by an inverse semigroup S_i. Denote the inclusions by $\ell_i \colon A_i \to A_{i+1}$. Choose an explicit (MV algebra) isomorphism $f_i \colon A_i \to S_i/\mathcal{D}$ for each i. Suppose there are injective maps $\tau_i \colon S_i \to S_{i+1}$ such that the maps

$$\begin{aligned} t_i \colon S_i/\mathcal{D} &\to S_{i+1}/\mathcal{D}, \\ s/\mathcal{D} &\to \tau_i(s)/\mathcal{D} \end{aligned}$$

are well defined on $\mathcal{D}$-classes, and that we have $t_i = f_{i+1} \circ j_i \circ f_i^{-1}$; i.e. the following diagram commutes for all i.

$$\begin{array}{ccc} A_i & \xrightarrow{\ \ell_i\ } & A_{i+1} \\ {\scriptstyle f_i^{-1}}\uparrow & & \downarrow{\scriptstyle f_{i+1}} \\ S_i/\mathcal{D} & \xrightarrow[\ t_i\ ]{} & S_{i+1}/\mathcal{D} \end{array}$$

Then, A is coordinatized by the directed colimit of

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots .$$

Proof. We define $F \colon S_\infty/\mathcal{D} \to A$ as follows. Let $[s]/\mathcal{D} \in S_\infty/\mathcal{D}$. So $[s] \in S_\infty$, and hence $s \in S_i$ for some i. Define

$$F([s]/\mathcal{D}) = f_i^{-1}(s/\mathcal{D}).$$

We first claim that, for $s \in S_i$ and $u \in S_j$, that $[s] = [u]$ in S_∞ implies $F([s]/\mathcal{D}) = F([u]/\mathcal{D})$; that is, that $f_i^{-1}(s/\mathcal{D}) = f_j^{-1}(u/\mathcal{D})$. Without loss of generality, we let $i \leq j$. Now, $[s] = [u]$ means there exists $e_k \in S_k$ for some k such that $s \cdot e_k = u \cdot e_k$, where e_k is the identity of S_k. This means

$$\tau^i_{i\vee k}(s)\tau^k_{i\vee k}(e_k) = \tau^j_{j\vee k}(u)\tau^k_{j\vee k}(e_k). \tag{4.4.1}$$

If $i = j$, the above equation becomes $\tau^i_{i\vee k}(s) = \tau^i_{i\vee k}(u)$. But $\tau^i_{i\vee k}$ is a composite of injective maps and hence itself injective, so $s = u$, thus $f_i^{-1}(s/\mathcal{D}) = f_i^{-1}(u/\mathcal{D})$.

On the other hand, if $i \neq j$, the equation (4.4.1) occurs in $S_{i\vee k}$ on the left and $S_{j\vee k}$ on the right, so we must have $i \vee k = j \vee k = k$, and $k \geq i$, $k \geq j$. We have

$$\tau^j_k \tau^i_j(s) = \tau^i_k(s) = \tau^j_k(u).$$

By injectivity of the τ maps, we have $\tau^i_j(s) = u$. Then,

$$\begin{aligned}
f_j^{-1}(u/\mathcal{D}) &= f_j^{-1}(\tau^i_j(s)/\mathcal{D}) \\
&= f_j^{-1}(\tau_{j-1}\dots\tau_i(s)/\mathcal{D}) \\
&= f_j^{-1}(t_{j-1}\dots t_i(s/\mathcal{D})) \\
&= f_j^{-1}(f_j\ell_{j-1}f_{j-1}^{-1}f_{j-1}\dots f_{i+1}^{-1}f_{i+1}\ell_i f_i^{-1}(s/\mathcal{D})) \\
&= \ell_{j-1}\dots\ell_{i+1}\ell_i f_i^{-1}(s/\mathcal{D}) \\
&= f_i^{-1}(s/\mathcal{D}).
\end{aligned}$$

In all cases, the claim is proved.

We are now ready to prove F is well defined in $\mathcal{D}$ classes. Suppose now $s \in S_i$ and $u \in S_j$ such that $[s]/\mathcal{D} = [u]/\mathcal{D}$. That $[s]$ and $[u]$ are in the same $\mathcal{D}$ class means there exists $v \in S_k$ for some k such that

$$[s^{-1}s] = [v^{-1}v], \ [vv^{-1}] = [u^{-1}u].$$

We thus have

$$\begin{aligned}
F([s]/\mathcal{D}) &= f_i^{-1}(s/\mathcal{D}) && \text{(by definition)} \\
&= f_i^{-1}(s^{-1}s/\mathcal{D}) && \text{(same } \mathcal{D}\text{-class in } S_i) \\
&= f_k^{-1}(v^{-1}v/\mathcal{D}) && \text{(by above observation)} \\
&= f_k^{-1}(vv^{-1}/\mathcal{D}) && \text{(same } \mathcal{D}\text{-class in } S_k) \\
&= f_j^{-1}(u^{-1}u/\mathcal{D}) && \text{(by above observation)} \\
&= f_j^{-1}(u/\mathcal{D}) && \text{(same } \mathcal{D}\text{-class in } S_j) \\
&= F([u]/\mathcal{D}) && \text{(by definition).}
\end{aligned}$$

Next, we prove injectivity of F. Suppose $s \in S_i$, $u \in S_j$, and $F([s]/\mathcal{D}) = F([u]/\mathcal{D})$, so $f_i^{-1}(s/\mathcal{D}) = f_j^{-1}(u/\mathcal{D})$. We assume without loss of generality that $i \leq j$. We have that

$$e_j \cdot s = \tau_j^i(e_j)\tau_j^i(s) = e_j \cdot \tau_j^i(s),$$

so $[s] = [\tau_j^i(s)]$. We compute

$$\begin{aligned} f_j^{-1}(\tau_j^i(s)/\mathcal{D}) &= f_j^{-1}(\tau_{j-1} \dots \tau_{i+1}\tau_i(s)/\mathcal{D}) \\ &= f_j^{-1}(t_{j-1} \dots t_i(s/\mathcal{D})) \\ &= f_j^{-1} f_j \ell_{j-1} f_{j-1}^{-1} \dots \ell_i f_i^{-1}(s/\mathcal{D}) \\ &= f_i^{-1}(s/\mathcal{D}) \\ &= f_j^{-1}(u/\mathcal{D}). \end{aligned}$$

Since f_j^{-1} is by hypotheses an isomorphism, it is, in particular, injective, so $\tau_j^i(s)/\mathcal{D} = u/\mathcal{D}$. Thus,

$$[s]/\mathcal{D} = [\tau_j^i(s)]/\mathcal{D} = [u]/\mathcal{D}.$$

For surjectivity of F, let $a \in A$. Then let i be the smallest integer such that $a \in A_i$. Consider $f_i(a) \in S_i/\mathcal{D}$. Choose an element $s \in S_i$ which is in the $\mathcal{D}$-class $f_i(a)$. We have

$$F([s]/\mathcal{D}) = f_i^{-1}(s/\mathcal{D}) = f_i^{-1}(f_i(a)) = a.$$

That F is an MV algebra homomorphism follows directly from the fact that all the f_i are MV algebra homomorphisms. Thus, we have explicitly constructed a map giving us $S_\infty/\mathcal{D} \cong A$, and so A is coordinatized as stated. □

The converse of this theorem is also true, as given below.

Theorem 4.4.2 (Decomposition Theorem II)**.** *Suppose A is an MV algebra coordinatized by the directed colimit of*

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \dots.$$

Then, A has a sequence of subalgebras forming a chain of inclusions

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$$

such that $A = \bigcup_{i=1}^{\infty} A_i$, and each A_i is coordinatized by S_i.

Proof. Denote $S_\infty = \lim_\to(S_0 \to S_1 \to \dots)$ as usual. As A is coordinatized by S_∞, we choose an explicit MV isomorphism $F\colon S_\infty/\mathcal{D} \to A$. We put

$$A_i := \{F([a]/\mathcal{D}) \mid a \in S_i\}.$$

To show each A_i is a MV subalgebra of A, it suffices to show $0_A \in A_i$, and that A_i is closed under $\oplus$ and $\neg$.

Recall that the zero of S_∞, considered as a boolean inverse monoid, is the $\equiv$-class containing the zeroes of every S_i. In particular, as each S_i is boolean by hypothesis, we know, denoting z_i as the zero of S_i, that z_i is in 0_{S_∞}. From [LS14, Proposition 2.9 & Theorem 2.10], the $\mathcal{D}$-class of 0_{S_∞} is also the zero of $S_\infty/\mathcal{D}$ considered as an MV algebra. As MV maps preserve zeroes, we thus have for each i that $F^{-1}([z_i]/\mathcal{D}) = 0_A$, so $0_A \in A_i$.

Next we show closure under negation. Let $a \in S_i$ (without loss of generality we may take a to be an idempotent; replace it with $a^{-1}a$ if necessary, which is in the same $\mathcal{D}$-class). From [LS14, Theorem 2.10], in $S_i/\mathcal{D}$, $\neg(a/\mathcal{D}) = (\overline{a}/\mathcal{D})$, where $\overline{(-)}$ denotes boolean complementation. Since the restriction of the operation in $S_\infty/\mathcal{D}$ to $\mathcal{D}$-classes of S_∞-classes of elements of S_i coincides with the operation in S_i, we have in $S_\infty/\mathcal{D}$ that $\neg([a]/\mathcal{D}) = ([\overline{a}]/\mathcal{D})$. Thus, if $x \in A_i$, then $x = F([a]/\mathcal{D})$ for some $a \in S_i$. Since F preserves negation and $\overline{a} \in S_i$, we have

$$F([\overline{a}/\mathcal{D}]) = F(\neg([a]/\mathcal{D})) = \neg F([a]/\mathcal{D}) = \neg x \in A_i.$$

For closure under $\oplus$, let $x, y \in A_i$. So $x = F([a]/\mathcal{D})$ and $y = F([b]/\mathcal{D})$ for some $a, b \in S_i$. In $S_i/\mathcal{D}$, we know $a/\mathcal{D} \oplus b/\mathcal{D} = c/\mathcal{D}$ for some $c \in S_i$, so by the same remark about restriction as before, we have that

$$x \oplus y = F([a]/\mathcal{D}) \oplus F([b]/\mathcal{D}) = F([a]/\mathcal{D} \oplus [b]/\mathcal{D}) = F([c]/\mathcal{D}),$$

whence $x \oplus y \in A_i$. Thus, we see that each A_i is indeed an MV-subalgebra of A.

We now argue that $A_i \subseteq A_{i+1}$. Let $x \in A_i$. So $x = F([a]/\mathcal{D}$ for some $a \in S_i$. As argued in Theorem 4.4.1, $[a] = [\tau_i(a)]$, so

$$x = F([a]/\mathcal{D}) = F([\tau_i(a)]/\mathcal{D}) \in A_{i+1}.$$

It is clear that $\bigcup_{i=1}^{\infty} A_i \subseteq A$. For the reverse inclusion, suppose $x \in A$. Then, $F^{-1}(x) = [a]/\mathcal{D}$, where $a \in S_j$ for some j. Then,

$$F([a]/\mathcal{D}) = FF^{-1}(x) = x,$$

so $x \in A_j \subseteq \bigcup_{i=1}^{\infty} A_i$.

Finally, we must show $A_i \cong S_i/\mathcal{D}$ for each i. Define $f_i\colon A_i \to S_i/\mathcal{D}$ as follows. For $x \in A_i$, we know $x = F([a]/\mathcal{D})$ for some $a \in S_i$. Define $f_i(x) = a/\mathcal{D}$ – this is well defined because F is an isomorphism on $S_\infty/\mathcal{D}$. That the f_i are MV maps follows directly from the fact that F^{-1} is one.

For injectivity of f_i, if $f_i(x) = a/\mathcal{D} = b/\mathcal{D} = f_i(y)$, then $[a]/\mathcal{D} = [b]/\mathcal{D}$, and $x = F([a]/\mathcal{D}) = F([b]/\mathcal{D}) = y$. For surjectivity, let $a/\mathcal{D} \in S_i/\mathcal{D}$. Then, $F([a]/\mathcal{D}) \in A_i$, and $f_i(F[a]/\mathcal{D}) = a/\mathcal{D}$. □

4.5 The coordinatization of Chang's MV algebra

Recall Chang's MV algebra $\mathcal{C}$ from Example 1.2.7. We will give the coordinatization of this algebra, the details of which are quite straightforward once we come up with the right idea, but we highlight how the various theorems seen so far led to this idea and emphasize their usefulness.

A priori, we knew the coordinatization existed, but it could have been any AF inverse monoid, and the colimit of any number of inverse semigroups. However, it is readily seen that any nontrivial subalgebra of $\mathcal{C}$ (e.g. generated by $n \cdot c$ for some $n \in \mathbb{N}^+$) is isomorphic to $\mathcal{C}$ itself. As such, there is no meaningful way to write $\mathcal{C}$ as a union of successive subalgebras, whence the contrapositive of Theorem 4.4.2 tells us that we are in fact looking for a *single* inverse semigroup.

Next, Theorem 4.1.8 tells us every inverse semigroup is in fact a subsemigroup of some symmetric inverse monoid. As $\mathcal{C}$ is a countably infinite MV algebra, the natural (pun intended) place to start is $\mathcal{I}(\mathbb{N})$. We now need to figure out what $\mathcal{D}$-classes of partial bijections correspond to the elements of $\mathcal{C}$.

The obvious thing is that 0 should be the class of the empty function, 1 should be the class of total bijections, and $n \cdot c$ should be the class of partial bijections with domain size n. If we now look at the ordering of $\mathcal{C}$, we see that it looks like a copy of $\mathbb{N}$ at the bottom, with a mirror reflecting an upside-down $\mathbb{N}$ above it such that the two chains never meet. So, if the element $1 - n \cdot c$ is the complement to $n \cdot c$, then the complement to the $\mathcal{D}$-class of bijections on n elements should be the class of bijections on all but n elements. For brevity, we will henceforth in this section say "the co-size of X" to mean "the size of $\mathbb{N}\backslash X$".

There is, however, one additional condition — we only include the partial bijections with cofinite domain of the same co-size as the image. We will say a partial bijection that satisfies this property has *balanced cofinite* domain. Thus, we define $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ to be the subset of $\mathcal{I}(\mathbb{N})$ to be those partial bijections on $\mathbb{N}$ whose domain are either finite or balanced cofinite.

Lemma 4.5.1. *$\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ is a sub-inverse semigroup of $\mathcal{I}(\mathbb{N})$.*

Proof. We need to show closure under composition and pseudoinverses. It is clear the pseudoinverse of a partial bijection with finite domain also has finite domain, and the pseudoinverse of a function with cofinite domain also has cofinite domain.

Let $f, g \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$. Recall that $g \circ f$ has domain $f^{-1}(\operatorname{dom} g \cap \operatorname{im} f)$. If either or both of f, g have finite domain/image, then $\operatorname{dom} g \cap \operatorname{im} f$ is finite, and so is its inverse image under f, so $g \circ f \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$.

On the other hand, if both f, g have cofinite domain/image, then $\operatorname{dom} g \cap \operatorname{im} f$ is an intersection of two cofinite sets and is again cofinite. Furthermore, it is a subset of $\operatorname{im} f$ consisting of all but finitely many elements of $\operatorname{im} f$, so its inverse image under f is a cofinite set with finitely elements removed, and remains cofinite, so $g \circ f$ has cofinite domain.

Note that if we take a partial bijection on a balanced cofinite domain, and further restrict to a cofinite subset of its domain, the restricted map will also have balanced cofinite domain. Thus, as f and g are by hypothesis balanced, we have

$$\begin{aligned}|\mathbb{N}\backslash(\operatorname{dom} g \circ f)| &= |\mathbb{N}\backslash f^{-1}(\operatorname{dom} g \cap \operatorname{im} f)| \\ &= |\mathbb{N}\backslash(\operatorname{dom} g \cap \operatorname{im} f)| \\ &= |\mathbb{N}\backslash g(\operatorname{dom} g \cap \operatorname{im} f)| \\ &= |\mathbb{N}\backslash(\operatorname{im} g \circ f)|,\end{aligned}$$

whence $g \circ f \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$. □

Lemma 4.5.2. *The $\mathcal{D}$-classes of $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ are precisely as follows. For $f, g \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$, $f\mathcal{D}g$ if and only if either f, g have finite domains of the same size or f, g have balanced cofinite domains with the same co-size.*

Proof. Suppose $f\mathcal{D}g$, and suppose f has finite domain $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{N}$. So there is $h \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ such that

$$\mathrm{id}_X = f^{-1}f = h^{-1}h, \; hh^{-1} = g^{-1}g.$$

Then,

$$|\operatorname{dom} g| = |\operatorname{dom} g^{-1}g| = |\operatorname{dom} hh^{-1}| = |\operatorname{dom} h^{-1}h| = |\operatorname{dom} f^{-1}f| = n.$$

Now suppose f has balanced cofinite domain $\mathbb{N}\backslash X$ with $X = \{x_1, \ldots, x_n\}$. Then denote its image as $\mathbb{N}\backslash Y = \{y_1, \ldots, y_n\}$. If $f\mathcal{D}g$, then we have $h \in \mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ such that

$$\mathrm{id}_{\mathbb{N}\backslash X} = f^{-1}f = h^{-1}h, \; hh^{-1} = g^{-1}g.$$

This means h also has domain with co-size n. Since it is in $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$, it is balanced and has image with co-size n. Thus h^{-1} also has domain and image with co-size n, hence the same is true of $g^{-1}g$, and finally, of g.

Now suppose f has domain $X = \{x_1, \ldots, x_n\}$ and g has domain $Y = \{y_1, \ldots, y_n\}$. Define the partial bijection h by $x_i \mapsto y_i$ for $1 \leq i \leq n$. We see that $h^{-1}h$ is given by $x_i \mapsto y_i \mapsto x_i$, i.e. $h^{-1}h = \mathrm{id}_X = f^{-1}f$, and that hh^{-1} is given by $y_i \mapsto x_i \mapsto y_i$, i.e. $hh^{-1} = \mathrm{id}_Y = g^{-1}g$. So $f\mathcal{D}g$.

Now if X and Y as defined above are instead the complements of $\operatorname{dom} f$ and $\operatorname{dom} g$. We write $\operatorname{dom} f = \mathbb{N}\backslash X = \{x_{n+1}, x_{n+2}, \ldots\}$, where we enumerate all the elements of

$\mathbb{N}\backslash X$ in the reader's favorite order, and similarly $\operatorname{dom} g = \mathbb{N}\backslash Y = \{y_{n+1}, y_{n+2}, \ldots\}$. Define the partial bijection h by $x_{n+i} \mapsto y_{n+i}$ for $i \in \mathbb{N}^+$. We see that $f\mathcal{D}g$ via h. $\square$

Remark 4.5.3. The present author would like to thank Mark Lawson for the following observation. Initially, we defined $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ to contain all the partial bijections with cofinite domain, without the additional clause of being balanced. Prof. Lawson noted that the successor function $s\colon \mathbb{N} \to \mathbb{N}^+$, $n \mapsto n+1$, would be in our inverse semigroup and that it clearly cannot be extended to a total bijection. Hence with this definition of $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$, one obtains an inverse semigroup which is not factorizable and so not a Foulis monoid.

Moreover, denoting the pseudoinverse of s by $t\colon \mathbb{N}^+ \to \mathbb{N}$, $n \mapsto n-1$, observe that

$$\mathrm{id}_{\mathbb{N}^+} = st = ss^{-1},\ s^{-1}s = ts = \mathrm{id}_{\mathbb{N}},$$

so in fact we had a $\mathcal{D}$-relation between the total identity on $\mathbb{N}$ (with domain co-size 0), and the identity on $\mathbb{N}^+$ (with domain co-size 1), completely destroying our attempt to mirror the behaviour of complementary elements in $\mathcal{C}$!

Theorem 4.5.4. *$\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ coordinatizes $\mathcal{C}$.*

Proof. We first must prove that $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ is a Foulis monoid. All symmetric inverse monoids are boolean, and in particular $\mathcal{I}(\mathbb{N})$ is boolean. We begin by showing $E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$ is a boolean subalgebra of $E(\mathcal{I}(\mathbb{N}))$.

For a partial identity $\mathrm{id}_X \in E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$, either X is finite or cofinite (note that all partial identities on cofinite domains are balanced), and its boolean complement is $\mathrm{id}_{\mathbb{N}\backslash X}$. In either case, the complement is again defined on a finite or cofinite domain, so $E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$ is closed under complementation.

Next, we check closure under $\wedge$. Recall for partial bijections f, g, that $f \wedge g$ has domain $\{x \in X \mid f(x) = g(x)\}$ and $(f \wedge g)(x) = f(x) = g(x)$. So suppose $\mathrm{id}_X, \mathrm{id}_Y \in E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$. Since they are partial identities, the domain of $\mathrm{id}_X \wedge \mathrm{id}_Y$ is simply $X \cap Y$ and $\mathrm{id}_X \wedge \mathrm{id}_Y = \mathrm{id}_{X \cap Y}$. If at least one of X or Y is finite, then so is $X \cap Y$. On the other hand, if both are cofinite, then there are only finitely many elements of $\mathbb{N}$ missing from each of X and Y, hence only finitely many elements missing from either X or Y, and so $X \cap Y$ is cofinite.

Closure under $\vee$ now follow from DeMorgan's Law. Thus, $E(\mathcal{I}(\mathbb{N})_{\mathbf{fc}})$ is indeed a boolean algebra, from which conditions (DIM 1)–(DIM 3) are immediate. Finally, we need to show $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ is factorizable. The group of units are the total bijections. Any partial bijection with a finite domain can be extended to a total bijection by enumerating $\mathbb{N}\setminus \operatorname{dom} f = \{x_1, \ldots, x_n, \ldots\}$ and $\mathbb{N}\setminus \operatorname{im} f = \{y_1, \ldots, y_n, \ldots\}$ and sending $x_i \mapsto y_i$. Similarly, we can do this for partial bijections with balanced cofinite domain — the condition of being balanced being precisely what makes this possible. Thus, $\mathcal{I}(\mathbb{N})_{\mathbf{fc}}$ is a Foulis monoid.

Now define

$$\begin{aligned} H\colon \mathcal{C} &\to \mathcal{I}(\mathbb{N})_{\mathbf{fc}}/\mathcal{D}, \\ 0 &\mapsto\, !_{\emptyset}/\mathcal{D}, \\ n\cdot c &\mapsto \operatorname{id}_{\{0,1,\ldots,n-1\}}/\mathcal{D}, \\ 1 - n\cdot c &\mapsto \operatorname{id}_{\{n,n+1,n+2,\ldots\}}/\mathcal{D}, \\ 1 &\mapsto \operatorname{id}_{\mathbb{N}}/\mathcal{D}. \end{aligned}$$

Bijectivity of H is immediate from the preceding lemmas. It is also immediately obvious that H preserves zero and negation. We will check that it preserves $\oplus$.

Recall how the operation of $\mathcal{C}$ is defined in Example 1.2.7 — we will follow the same breakdown of cases. Recall also how the MV and MV-effect algebra operations are defined from each other (see Theorem 1.4.3) and how the operation on $\mathcal{D}$-classes is defined (following Lemma 4.2.4).

(Continued on next page...)

Let $x, y \in \mathcal{C}$.

- <u>Case 1: $x = n \cdot c$ and $y = m \cdot c$</u>

  We have that

$$\begin{aligned}
H(x) \oplus H(y) &= H(n \cdot c) \oplus H(m \cdot c) \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \oplus \mathrm{id}_{\{n,n+1,\ldots,n+m-1\}} /\mathcal{D} \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \,\widetilde{\oplus}\, (\mathrm{id}_{\{n,n+1,\ldots\}} /\mathcal{D} \wedge \mathrm{id}_{\{n,n+1,\ldots,n+m-1\}} /\mathcal{D}) \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \,\widetilde{\oplus}\, \mathrm{id}_{\{n,n+1,\ldots,n+m-1\}} /\mathcal{D} \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \vee \mathrm{id}_{\{n,n+1,\ldots,n+m-1\}} /\mathcal{D} \\
&= \mathrm{id}_{\{0,\ldots,n+m-1\}} /\mathcal{D} \\
&= H((n+m) \cdot c) \\
&= H((n \cdot c) \oplus (m \cdot c)) \\
&= H(x \oplus y).
\end{aligned}$$

- <u>Case 2: $x = 1 - n \cdot c$ and $y = 1 - m \cdot c$</u>

  We have that

$$\begin{aligned}
H(x) \oplus H(y) &= H(1 - n \cdot c) \oplus H(1 - m \cdot c) \\
&= \mathrm{id}_{\{n,n+1,\ldots\}} /\mathcal{D} \,\widetilde{\oplus}\, (\mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \wedge \mathrm{id}_{\{m,m+1,\ldots\}} /\mathcal{D}) \\
&= \mathrm{id}_{\{n,n+1,\ldots\}} /\mathcal{D} \,\widetilde{\oplus}\, \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \\
&= \mathrm{id}_{\{n,n+1,\ldots\}} /\mathcal{D} \vee \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \\
&= \mathrm{id}_{\mathbb{N}} /\mathcal{D} \\
&= H(1) \\
&= H((1 - n \cdot c) \oplus (1 - m \cdot c)) \\
&= H(x \oplus y).
\end{aligned}$$

- <u>Case 3: $x = n \cdot c$ and $y = 1 - m \cdot c$</u>

  First suppose $n < m$. Then,

$$\begin{aligned}
H(x) \oplus H(y) &= H(n \cdot c) \oplus H(1 - m \cdot c) \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \,\widetilde{\oplus}\, (\mathrm{id}_{\{n,n+1,\ldots\}} /\mathcal{D} \wedge \mathrm{id}_{\{m,m+1,\ldots\}} /\mathcal{D}) \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \,\widetilde{\oplus}\, \mathrm{id}_{\{m,m+1,\ldots\}} /\mathcal{D} \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \vee \mathrm{id}_{\{m,m+1,\ldots\}} /\mathcal{D} \\
&= \mathrm{id}_{\{0,\ldots,n-1,m,m+1,\ldots\}} /\mathcal{D} \\
&= \mathrm{id}_{\{m-n-1,m-n,\ldots,m-1,m,m+1,\ldots\}} /\mathcal{D} \\
&= H(1 - (m-n) \cdot c) \\
&= H(x \oplus y).
\end{aligned}$$

  On the other hand, if $n \geq m$, we have

$$\begin{aligned}
H(x) \oplus H(y) &= H(n \cdot c) \oplus H(1 - m \cdot c) \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \,\widetilde{\oplus}\, (\mathrm{id}_{\{n,n+1,\ldots\}} /\mathcal{D} \wedge \mathrm{id}_{\{m,m+1,\ldots\}} /\mathcal{D}) \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \,\widetilde{\oplus}\, \mathrm{id}_{\{n,n+1,\ldots\}} /\mathcal{D} \\
&= \mathrm{id}_{\{0,\ldots,n-1\}} /\mathcal{D} \vee \mathrm{id}_{\{n,n+1,\ldots\}} /\mathcal{D} \\
&= \mathrm{id}_{\mathbb{N}} /\mathcal{D} \\
&= H(1) \\
&= H(x \oplus y).
\end{aligned}$$

□

Remark 4.5.5. In the calculations of the preceding theorem, we have come full circle by using the operations of all three structures we have studied — the $\oplus$ of MV algebras, the $\widetilde{\oplus}$ of effect algebras, and the $\vee$ of boolean inverse semigroups.

Chapter 5

Future Directions

We end this thesis by giving some possible future directions of inquiry related to the work we have done.

Effect algebras — check stuff!

We have found many mathematical errors in definitions and proofs in the world of effect algebras, due largely in part to how much of the motivation comes from physics, and there is a whole wealth of literature that the present author has not looked at. It is statistically likely that there are many more errors that need to be corrected.

Coequalizers of effect algebras

In the examples at the end of Chapter 3.2, we were able to shed some light on what's going on behind the scenes of the very abstract and complicated construction of coequalizers, but it remains to come up with a nice general description.

It would be nice to obtain a general description of coequalizers that is intuitive and constructive and does not require involving an entire other category and taking multiple functors to describe. Beyond just being interesting to know in its own right, it would make proofs easier and expedite further research into categorical structures of effect algebras. Our proof of Theorem 3.3.7 was quite messy and complicated, in large part due to how hard it is to construct coequalizers.

Effect algebras and more general theories

Several examiner comments by Prof. Hofstra are in particular worth mentioning here. There is a more general theory of partial universal algebra (Peter Burmeister et al.) that effect algebras fit into. It would be a sensible next step to investigate which results in this thesis are special cases of more general facts.

Furthermore, the theory of term rewriting may be relevant to some of the calculations and categorical constructions we have done. In particular, they may be useful to the task of uncovering a cleaner description of coequalizers.

MV coordinatization — theorems

It is believed by researchers in the field that the coordinatization of MV algebras is functorial, but this needs to be explicitly written out. Furthermore, it would be desirable to reconciliate the coordinatization theorems of Wehrung with that of Lawson & Scott. In particular, since most of the work in coordinatization that has been done builds off of Lawson & Scott's version, it would be nice to have a direct extension of their theorem to uncountable MV algebras.

MV coordinatization — examples

By coordinatizating Chang's algebra, we gave our first example of a coordinatization of an MV algebra which does not embed into $[0, 1]$ — this is evident from its order type. However, all examples we have given are linear.

Some natural algebras to consider are free MV algebras. The free MV algebra on one generator, for instance, is already wildly nonlinear in its ordering. We know that it doesn't have any nontrivial subalgebras, so like the Chang algebra, we are looking for a single inverse semigroup to coordinatize it. The present author does not have the foggiest idea what that inverse semigroup would look like or where to even begin looking.

The table of examples on the following page, by Mundici, which also illustrates a connection between MV algebras and AF C-star algebras that we have not discussed, is a rich source of potential candidates for coordinatization.

Another interesting way of brute forcing examples would be to go the opposite direction — take specific Foulis monoids, and see what kind of MV algebra their coordinatization yields.

COUNTABLE MV ALGEBRA	ITS AF-ALGEBRAIC CORRESPONDENT
$\{0,1\}$	**C**, the complex numbers
finite	finite-dimensional
boolean	commutative
finite chain	$M_n(\mathbf{C})$, the C*-algebra of $n \times n$ matrices
dyadic rationals in the unit interval	CAR algebra of the Fermi gas
algebra generated by an irrational	Effros-Shen algebra
real algebraic numbers in $[0,1]$	Blackadar algebra
Chang algebra	Behncke-Leptin algebra $A_{0,1}$
atomless boolean	$C(X)$, with X the Cantor cube
rational simple algebra	uniformly hyperfinite, Glimm
the rationals in the unit interval	Glimm universal UHF algebra
totally ordered	with Murray-von Neumann comparability
free with countably many generators	the universal AF-algebra $\mathfrak{M}$ (more soon)
free with one generator	the Farey algebra $\mathfrak{M}_1$ (more soon)

Representation theorems

Although we have not touched on it explicitly in this thesis, now that we know MV algebras embed categorically into effect algebras, it is natural to wonder what sorts of known representation theorems for MV algebras (including coordinatization) can be generalized to wider classes of effect algebras.

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