An Analysis of Cross-Licensing With Asymmetric Patent Portfolios

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Introduction

With the advent of patent law, there was a shift in priority for antitrust organizations to not only include promotion of competition, but also a secondary objective: promotion of innovation (Beard and Kaserman, 2002). This is of course a result of the consensus that technological advancement is a crucial factor in economic growth for a society, and therefore the welfare of its members (Bakhoum and McEwin, 2013). Conventional justification for patents revolve around the idea that patents spur innovation by providing incentives for inventors, as well as protecting their intellectual properties. At the same time, however, many questions about the impact patents really have remain yet to be answered. Perhaps patents have a more complicated impact than simply spurring innovation through monopoly power incentive.

The current patent system has been heavily criticized in contemporary academic literature with suggestions that perhaps the patent system might actually be inducing the opposite effect on innovation (Lemley, 2000; Guellec et al, 2007; Choi, 2003). There is suggestion that the current system might be dissuading innovation via a patent thicket or restricting the effective dissemination of technology post-innovation. Amendments to the patent system have arisen allowing for more advanced methods of more quickly diffusing technology in response to some of these technological diffusion concerns and the possibility of addressing these issues. We will investigate the workings of the patent system, and the validity of these various claims. In particular, we shall focus on one specific type of tool to improve the diffusion of technology: Cross-Licensing.

As mentioned, there are now various additional, new and advanced methods of technology diffusion that exist in the patent space that have in recent years been developed and legalized. Methods include both those of the ex-ante variety (before innovation occurs) such as R&D Cartelization and RJV Cartelization, as well as ex-post (after innovation occurs), such as licensing, patent pooling and cross-licensing (De Laat, 1997). Originally these methods were seen as anti-competitive and collusive; however, they now have come to be approved in most major countries as the literature on the matter has evolved. Different degrees of legislative allowances have been made for them to successfully operate. These steps come as a result of much academic literature pointing out additional benefits of many of these methods and quelling many concerns about potential collusion (Shapiro, 2001). There is, however, still much to be understood about the full girth of the impacts they have on both the market and future innovation patterns. These new options must be evaluated based on merit of both of the antitrust agency priorities in the now much more crowded patent-space. Much of which will be discussed in the following pages.

This paper is divided into two parts. The first half of my paper will encompass a literature review. My objective is to review the major issues and criticisms surrounding contemporary patent law, as well, as to familiarize the reader with important academic progress on the subject. This should in turn, identify concerns for social planners and antitrust agencies. The emphasis of this section will be on cross-licensing and related topics of ex-post knowledge diffusion in the marketplace. The second half will be a theoretical framework building primarily on the work of Jeon and Lefouili (2015). It will deal with a mathematical analysis of the impacts of asymmetric patent portfolios on the cross-licensing process.

Part One: The Literature Review

Patents: An Overview

Foremost, what is a patent? A patent can be defined as a set of exclusive rights to a technology. They “provide a time-limited, legally protected, exclusive right to make, use and sell an invention.” (Canadian Intellectual Property Office Website). Patents are used as a system for rewarding innovation and protecting ownership of innovation (Gallini and Scotchmer, 2002). Other mechanisms to incentivize innovation have also been attempted. For instance, in France a monetary prize was offered for developing a workable water turbine (Reynolds, 1983). The purpose of this paper however is not to debate the merits of alternative incentive systems, instead, the objective is to review the workings of the patent system and to further investigate how cross-licensing operates within the patent system. Why do we need an incentive scheme at all for innovation? Arrow (1962) explains how the standard competitive market is not conducive to innovation. Arrow then reasons that it is the responsibility of government to induce more innovation. Today, the patent system is the most prevalent way that governments have done this globally.

One approach of looking at this undersupply of innovation is known as “Tragedy of the Anticommons” (Heller, 1998). This aptly named concept is analogue to the phenomenon known as “Tragedy of the Commons”. In the “Tragedy of the Commons”, a public resource is over-consumed since each agent is acting in a self-interested fashion. In reverse fashion, the “Tragedy of the Anticommons” suggests that total human knowledge, another public good is then under-produced if each agent is acting in a self-interested fashion (Buchanan and Yoon, 2000). This is due to the fact that individuals do not take into account the positive externalities that their innovation imparts on the rest of society. At the time of the patent system’s creation, this was the general consensus.

There have been several advances in the field of economics since Arrow, and as such we have a fuller grasp on innovation behaviour. We have a more robust view than simply firms statically maximizing their profit, with the ability to adjust their innovation. Examples include the field of behavioural economics which might suggest that innovation might not come solely from the firm, and is motivated by a non-monetary agenda. Kline and Rosenberg (1986) describe innovation as “complex, uncertain, somewhat disorderly and subject to changes of many sorts”. Other ways of analyzing innovation can be seen in Schumpeter (1946) where the market is treated as a dynamic game where firms also compete in innovation. However, even in Schumpeter’s model, the tragedy of the anticommons still prevails.

Over the past few decades especially, patents have come under intense scrutiny, with a large amount of literature analyzing their efficacy in facilitating the two antitrust priorities: innovation and competition. Criticism of the intellectual property system has grown alongside the explosive growth in the number of patents being awarded and massive changes in the patent landscape. This growth has often been attributed to the advances in the field of automotive, computing and electronic sectors (Kim and Marschke, 2004). Between 1995 and 2008 the total number of patent applications in the United States grew at an average of 6.8% annually (United States Patent and Trademark Office, 2015). On top of this growth, large changes in policy have changed the way patents are handled legally. In the United States specifically, this has taken several forms (see Table 1).

Table 1: Major Changes to the U.S. Patent System

Year	Event or Case	Result
1980	Diamond v Chakrabarty	Patentability of artificially engineered generic organisms
1980	Bayh-Dole legislation	Increase in university patenting
1981	Diamond v Diehr	Patentability of software
1982	Legislation	Creation of CAFC; patent validity more likely to be upheld
1984	Hatch-Waxman Act	Increased importance of patents for drugs vis a vis generic producers
1985/6	TI sues Japanese semiconductor firms	Wins suit; turns to suing U.S. semiconductor firms, funding R&D from licensing royalties
1986	Kodak-Polaroid	Decision on instant camera patent; final injunction against Kodak leading to $1B Judgement
1994	TRIPS agreement	Harmonisation drive begins
1998	State Street and ATT vs. Excel	Patentability of business methods

Source: Hall 2005

Looking at this table, we see that the patentability of new fields is likely a contributor to the major growth of patents. Beyond simply the amount of patents being awarded, we see that the patent landscape is affected dramatically by many of the other changes. Take for instance the Kodak-Polaroid case (Hall, 2015). This case set a precedent for the ability of a patent-holder to exercise their patent as a litigation tool effectively. The conclusion of the case led to a one-billion-dollar judgement and shut down Kodak's instant camera business. Patent applications increased immediately after the conclusion of the case. This demonstrates that patents are not only important as a technological good, but also valuable as potential profit from successfully litigating other firms. As well it serves as protection against other firms from being litigated themselves. Further into the paper we will discuss the impacts of litigation potential.

With these significant changes occurring in the development of the intellectual property space, it is no wonder questions are being posed if the system is going in the right direction and perhaps more critically, inherently flawed. Regardless of the direction policy is taking it is important to understand the way in which the patent system impacts various agents and aspects of the economy. The system under which we must analyze and study is inexorably more complex than its original incarnation.

Types of Patents

Patents often serve a myriad of different functions in the production process, and interact differently with different firms. Therefore, depending on the context patents may be modelled using different classifications depending on which function of the patent we are considering. For example, patents might be crucial in order for the product to be made in the first place. A patent might even be necessary to enter the market at all or a patent could be completely redundant for a firm.

One type of patent classification is the 'Necessary Patent'. A necessary patent is a patent that a firm requires to create their product and in many cases, enter the market at all. Later in the paper, we will analyze concerns with firms using necessary patents as a barrier to entry. It is important to note that a necessary patent might not be technically necessary to create the product but it is at the very least effectively so. This would be the case where the cost to innovate around the necessary patent is prohibitively high. Patent holders often hold a substantial amount of power in these situations and can leverage much of the firm's expected profits (Gilbert, 1990).

Another type of patent classification is the cost-reducing patent. A cost-reducing patent is a technology that streamlines an existing production process and makes the marginal cost of production cheaper. Most academic literature treats patents as the cost-reducing variety (Kamien and Tauman, 1986).

A similar classification to the cost-reducing patent is the value-adding patent. This classification assumes that instead of reducing cost, as the name suggests the patent adds value. In other words it makes the product more desirable (often assumed at zero cost). It might make the product higher quality, add additional features or make it safer. In models which consider product heterogeneity, this distinction becomes much more relevant. In most cases of product homogeneity, it is generally shown that this classification of patents is effectively equivalent to the cost-reducing patent (Jeon and Lefouili, 2015). This conclusion is based on the assumption of their models and the exact impact of a value-adding patent can be far more complicated than a cost-reducing patent when product differentiation is factored in (Filippini, 2006).

The way patents interact with the firm and the market can be dramatically different as discussed in the previous three classifications (Layne-Farrar and Lerner, 2011; Granstrand, 2004). Patents can also interact with each other in different ways. That is to say, existing technology might not be compatible with newly patented technology for all firms. Each firm might value a patent differently and depending on the ability of a patent to be a complement or substitute to other technological frameworks. Just as goods can be complements or substitutes in the goods market, patents can also be both complements and substitutes. Consider the fact that technologies aren't always fully compatible. In some cases, technologies could be perfect substitutes: two different methods for doing the same thing. In these instances, having access to both technologies would be redundant. Conversely technologies might also be complements, more useful together than the sum of their parts (Fershtman and Kamien, 1992). In extreme cases, perfect complements are technologies that cannot be used without each other. Most patents will obviously fall somewhere between these two extremes (Layne-Farrar and Lerner, 2011; Granstrand, 2004).

Problems with Patents: The Patent Thicket

Often identified as the major culprit behind the flaws in the patent system, the patent thicket is the most popular term used when identifying problems with the patent system. This has led to a gradient of different meanings for the same term. We will first look at Carl Shapiro's definition of the patent thicket.

Shapiro (2001) explains the patent thicket by first making the analogy of R&D being akin to the construction of a pyramid. The foundation of previous knowledge and intellectual contributions of others allow us to reach heights higher than we would be able to otherwise reach independently. Conventionally patents are thought to spur innovation by rewarding the innovator and thus giving him higher incentive to innovate. Building upon someone else's work and securing a royalty is thought to be a fair and just reward for the innovator as well as creating an incentive scheme that is beneficial for society.

The problem arises when new innovators must secure multiple licenses and pay multiple royalties. This can become a problem if we return to the pyramid analogy; the would-be innovator is held up at every individual brick of the pyramid. Constant growth is not sustainable, and therefore, major inefficiencies arise leading to a technological slowdown. This is the basic concept behind the patent thicket, where more patents create an increasingly more difficult landscape to innovate in, impeding new R&D. As it turns out there are more concerns about innovation obstacles than simply the accumulation of royalty costs. In the next few sections we will more closely examine these different aspects of the patent thicket in isolation.

Complements Problem

The first aspect of the patent thicket that we consider is the complements problem. Cournot (1838) explains the complements problem by discussing a brass manufacturer. The brass manufacturer requires both copper and zinc to produce his product. If the copper market was controlled by a single monopolist, and the zinc market was controlled by a different monopolist then the final price would be much higher than if both copper and zinc were owned by a single monopolist. The result is lower social welfare. Deadweight loss arrives through both lower consumer surplus and lower industry profits.

The same idea here applies to firms that control blocking patents, and thus create inefficiencies. When an entrant firm has developed a new product but requires several necessary patents from different patent holders, the entrant has to secure licensing from each individual patent holder leading to the final price of the new product being much higher. This implies also a larger deadweight loss. It is important to note that the complements problem exists even in a world with zero royalties, exactly what would be expected from Cournot's example. The key difference being that the cost of the final item would be much higher depending on the amount of necessary patents, instead of the amount of required primary goods. Aside from the problem of deadweight loss, new products might be crowded out altogether by the complements problem, which would suggest a feature of the patent system giving disincentives to innovate where it would be socially optimal to.

Holdup Problem

The other side of the patent thicket is the holdup problem (Lemley and Shapiro 2006; Ganglmair et al, 2012). The holdup problem can be broken down further into several different aspects. We first consider the impact of accumulating royalty payments. When a firm designs a new product it often has to attain and pay for many licenses. We can again look to Shapiro's pyramid analogy. Royalties individually distort the market as they can be thought of operating the same way as standard taxation. Accumulating multiple royalties makes it more and more costly to innovate for the firm. The higher prices also lead to more deadweight loss.

Secondly, there is also the threat of litigation, especially in a field where overwhelmingly more patents are awarded annually every year (i.e. semiconductors, computer, and software). For instance, in the semiconductor market Hall and Ziedonis (2001) find that the propensity to patent increases after 1982 up to 1993 where it levels out. This implies over the next ten years, the amount of patents applied for by firms, is higher per R&D dollar. In addition to the field having historically weak patent protection, it is often difficult for a firm to know whether or not it is actually infringing on a patent due to the sheer volume of patents. This fear of possible unknown costs via litigation acts as another deterrent for innovation. Consider how now that when the number of patents goes up, not only are firms disincentivized of designing a new product due to the requirement of attaining more patents but they are also now afraid of litigation.

Another part of the holdup problem to consider is timing. Often the process of acquiring a patent alone can cause costly delays (Shapiro, 2010). The delays created in securing a license (or conversely innovating around one) often create unfeasible costs for small firms and can act as a barrier to entry. Even for larger firms, the additional delay from innovation to market becomes increasingly costly. There are often threats on the horizon of the innovation becoming irrelevant by the time it hits market. As mentioned there is a threat of litigation, however, litigation also imposes a secondary cost of additional delay or additional "holdup" (Lemley and Shapiro, 2006).

Patent Legitimacy

When a patent is awarded, it is not always rightfully so. When challenged some patents don't maintain their legitimacy in court. Some patents do not have all of the criteria required even though they might have initially been awarded to an innovator. Patents then are not guaranteed rights to exclude but instead only confer "a right to try to exclude by asserting the patent in court." (Lemley and Shapiro 2005, p.1).

Due to this high level of uncertainty firms are constantly under threat of their product being litigated, and even their own patents possibly being ruled invalid. This pressure prevails beyond production and the hold-up problem, with these post-production difficulties further increasing the deterrent for firms to innovate. To escape these possibly costly threats, including simply the cost to defend themselves against litigation, firms have an incentive to engage in ex-post activities such as cross-licensing and patent pooling.

Choi (2007) addresses the holdup problem. For example, two firms with large patent portfolios could possibly engage in legal warfare wherein they excise their ability to litigate the other firm aggressively (as in a punishment strategy). These legal battles are costly and create unnecessary losses for both firms. Cross-licensing is a way to remove that possibility from happening, with both firms agreeing to opt out of legal warfare against each other.

Ménière and Parlane (2008) formulate a model to locate the equilibrium levels of innovation taking into account the threat of litigation. They find that the threat of patent holdup (previously discussed) has ambiguous effects on innovation. There is a positive effect on innovation that dominates when infringement fees are low due to industry concentration. When infringement fees are high however the risk that an innovator will be deprived of their profits has an effect that dominates the former.

Non-Practicing Entities

So far, we've only been considering active firms innovating and holding patents. It is also possible, however, that innovation can come from third parties. There is even the possibility that a separate third party entity, one that did not originally patent or develop a technology, could then acquire the patent. These third party patent holders are known as Non-Producing Entities (NPE) or Inactive Firms which hold patents but do not produce anything. In much of the modelling all firms considered are active producers, however, when analyzing patents, some firms still have to obtain licenses from NPEs. As patent holders NPEs also have the ability to litigate (and often liberally exercise this ability). NPEs therefore also apply pressure to the market, and influence the innovation behaviour of other firms even though they hold a zero percent market share. Non-practicing entities have become significantly more relevant as the patents being held become more numerously held by inactive firms in recent years. How significant are NPEs truly in the patent space? Approximately 60% of new patent lawsuits are filed by NPEs (Shwartz and Kesan, 2015). Areas where development has been moving at a fast pace and technological complementarity are high such as IT and Electronic companies have been frequent targets of NPE litigation whereas areas with low technological complementarity (i.e. pharmaceuticals) seldom are targeted (Rozek and Rainey, 2001).

There has been significant criticism of NPEs who obtain patents, as they are thought to be major contributors of the patent thicket by aggressively litigating. These NPEs are referred to colloquially as patent trolls and are characterized firstly by being an inactive firm, secondly by acquiring a large number of patents and thirdly by aggressively litigating . Patent trolls are generally distinguished in this way from independent research firms who are not regarded as the problem. Although some regard patent troll as a blanket term to encompass all NPEs this is not the primary usage of the word. Instead of trying to utilize these patents for their own production, these outside firms try to acquire as many patents as possible and assert their litigating power opportunistically to make a profit, unethically so in the opinion of critics. Most of their litigating threats are settled outside of court. Due to the court system in the United States, it is often expensive for firms to go to court even if they would win. In fact, approximately 1.5% of all patents are ever litigated and 0.1% are ever litigated to trial (Lemley and Shapiro, 2005).

Shwartz and Kesan (2015) use an empirical approach to evaluate how beneficial or harmful NPEs actually are. While "Patent Trolls" are met with the ire of the general public, the opinion of Shwartz and Kesan is that NPEs receive a disproportionate share of blame and are often used as a scapegoat for the costs of the patent enforcement system. Instead they believe that the focus should be purely on whether the patent claims are valid or not and not the source of the threat.

Bessen et al. (2011) find that from 1990 to 2010 NPE lawsuits resulted in half a trillion dollars of lost wealth. They also find that most of this money doesn't find its way to inventors. Their conclusion is that patent trolls hamper innovation and create a net loss of social welfare. Ford and Meurer (2011) arrive at the opposite conclusion that Schwartz and Kesan reach. There is still a lot of disagreement in both academia and the public view on many different facets of patent law, this being a prime example.

Unilateral Refusal

To increase welfare, diffusion of technology is very important. One of the issues therefore is that although patents bestow some market power onto the patent holder, damaging welfare, this is deemed an acceptable loss. Part of this effect however can still be mitigated via the ability to license patents. Theoretically then this should resolve much of the anticompetitive tendencies including barriers to entry that some patents could present. What happens then if a firm refuses to sell a license altogether? Is flat-out refusal to sell a license anti-competitive and should antitrust agencies be concerned about it?

Anti-trust agencies have been inconsistent in their stance. It is generally agreed that unilateral refusal to sell a patent has a negative impact on diffusion of technology, however, the problems for antitrust agencies encompass both possible infringement of a firms' patents rights and an inability to effectively monitor which refusals to sell a patent are indeed anti-competitive. Many cases exist where rulings have gone both in favour of absolute right to refuse and the opposite, with many being overturned. In MCI Communications Corporation v. AT&T (Gilbert, 1996), a firm was being sued for unilateral refusal. The supreme court ruled that a firm has the right to refuse unilaterally so long as there are legitimate competitive reasons and that the right is not absolute. The United States Department of Justice acknowledges that this has the potential to be anticompetitive but the eventual decision was that patents confer the ability to exclude and that firms can choose to refuse if they wish. It is important to note that this topic is still a point of contention. Gilbert and Shapiro (1996) cover unilateral refusal of patents with a look at specific antitrust laws in far more detail than is covered here.

Cross-Licensing

Cross-licensing has become an increasingly important part of the intellectual property space. The amount of cross-licensing arranged is highly dependent on the type of market. Generally cross-licensing becomes more prevalent in markets where the product is more complex. For example: Taylor and Silbertson (1973) report that the share of licensing done that is cross-licensing is 50% in the telecommunications and broadcasting industry, 25% in the electronic components sector, and 23% in the pharmaceutical industry.

Licensing agreements arise when two or more firms enter an agreement allowing each firm access to their patent portfolios in exchange for access to the patent portfolio of other firms. Cross-licensing would be the equivalent of trading entire recipe books. Sometimes these agreements involve royalties, fixed fees, a combination or none of the above (Poddar and Sougata, 2002).

Ideally cross-licensing under zero-transaction costs will occur when the joint profits are increased as a result. Shapiro (1985), however, points out that this isn't always the case. Three main problems impede this process.

Firstly, there is the issue of asymmetric information. Firms do not always have the same information about how valuable a patent is. The licensee might not know as much as the patentee regarding the technology, however, this can also take the reverse form. The patentee might not know how much benefit the licensee can extract from the patent and could lose profits if the licensee can use the technology to leverage a greater market position than expected.

Secondly there is the problem of revealing more than intended through cross-licensing a technology. Beard and Kaserman (2002) discuss how firms might choose not to patent their technology at all and keep it a trade secret. In this way they lose the ability to cross-license but retain any processes that might be revealed in the process of patenting. Cross-licensing an already patented technology might reveal even more than intended and the technological incumbent might prefer just to keep the technology to themselves due to this risk.

Thirdly, when employing a royalty in cross-licensing, measuring the royalty that should be awarded can be difficult or costly to monitor. A major concern about allowing firms to enter into cross-licensing agreements is the threat of using royalties to increase the industry cost and thus reduce quantity produced (Priest, 1977). This would effectively create a collusive monopoly outcome. From a welfare perspective, a per-unit royalty can create distortions, such as the licensee selling higher value goods sub-optimally to get around the per unit-tax.

This last effect captured by Priest (1977), proves to be the most heavily considered when dealing with cross-licensing. Jeon and Lefouili (2015) model this impact and attempt to identify its exact impact on upstream profit, downstream profit and social welfare. They make a recognition beyond Priest's original paper that the detrimental cons of cross-licensing might be in some cases outweighed by the pros.

Licensing Caveats

One of the possible caveats that can be included in a patent license is a grant-back. A grant-back is more-so a caveat for unilateral licensing. A grant-back is a clause that states that when a firm licenses its patents, any new R&D generated on the back of the licensee, the patent holder has a right to also use. In this way the licensor can ensure that he has access to any future technology generated (Chevigny, 1965). There has been criticism of the grant-back clause as it has been suggested that it can be used to abuse a monopolistic position by also creating a technology monopoly (Dunne, 1975). As well it has been proposed that it discourages smaller firms who are licensing from further innovating.

In cross-licensing the equivalent of a grant-back is including future patents in the agreement. This is common practice in the cross-licensing world whereas in unilateral licensing grant-backs are far rarer. There are several reasons why this is common practice. Firstly, since these deals are fairly large between large firms over long periods of time, including future patents simplifies things. There is then no requirement to have additional and unnecessary negotiations over the length of the agreement (Lemley and Shapiro 2006). Secondly, including future patents removes the threat of firms involved in the agreement to strategically delay major innovation. The firm in question could try and leverage a stronger market position, disadvantaging other included firms.

Much like grant-backs, cross-licensing of future patents has been suggested to reduce the incentive to innovate. The difference is that the threat of abusing monopolistic power here is far more difficult. Due to its nature and assuming larger firms innovate more, cross-licensing with future patents would actually weaken monopoly power (Shapiro, 1985). This is one of the reasons we see cross-licensing (in addition to asymmetric patent portfolios) generally created between similarly sized firms.

Duplication

When firms in the same market are working independently, often their R&D efforts may be redundant. For instance, in some cases they might be pursuing the same technology, and a patent race ensues. In a patent race, firms are researching the same technology (due to lack of coordination) and must invest resources in order to finish first. Regardless of the amount of effort invested, as long as firms have symmetric payouts, the outcome will be inefficient (Kultti et al., 2006). In the case of asymmetric firms, it is unclear whether or not it is efficient for firms to compete, especially if the first best is unattainable. For additional insights into patent races and their impact on R&D one can refer to Shapiro (1985) or Baye and Hoppe (2003).

In other cases, R&D contributions may be substitute technologies. Despite both firms acquiring a patent they lose the ability to efficiently cross-license with each other without pure collusion; in the case of perfect substitutes cross-licensing arrives at the purely collusive outcome. For the same reason why cross-licensing with each other does not create additional welfare, both patents cannot be part of the same patent pool (Kato, 2004). Regardless, both of these cases are socially inefficient.

Patent Pools

A patent pool is a separate ex-ante technological diffusion option from cross-licensing but is similar in many ways. Essentially a patent pool is a bundle of patents from multiple sources. The goal of a patent pool is to reduce transaction costs and allow related patents to be sold together.

In the literature, patent pools are often indistinguishable from multi-firm cross-licensing deals where each firm involved has a deal with every other member (Layne-Farrar and Lerner, 2011) We see this often because the models normalize all firms to only a single patent. These models also do not identify necessary patents and only model cost-reducing patents. In reality this is not necessarily the case. Foremost whereas a cross-licensing deal generally licenses one firm's entire patent portfolio to the other (and vice versa), a patent pool contains only necessary patents for a specific process (Lerner and Tirole, 2002). This is especially important as it is usually legally required that patents in a pool are in fact necessary (Carleson, 1985). Much in the same way as mergers, the creation of patent pools often requires approval from an antitrust agency. Without this requirement, firms could extract extra profit by selling licenses that would not normally be sold (whether or not they are actually used).

Additionally, patent pools are often managed by a third party, who helps establish the necessity of the patents so that they can successfully acquire approval. A patent pool also means that selling the patent pool to outside firms is much easier. It allows all the patents to be purchased in one place, dubbed one stop-shopping, reducing transaction costs and hold-up. It also means that unilateral refusal of licensing a patent isn't possible either. Note that unilateral refusal of licensing is something the Federal Trade Commission has established it won't intervene in regardless of its impact on competition (Shapiro, 2001).

Related Literature on Unilateral Licensing

While the focus of this paper is how cross-licensing operates with asymmetric firms, the work in that specific area is limited. The academic literature on unilateral patent licensing with asymmetries is however quite abundant and will be a major asset in establishing a model in the remainder of the paper. A reasonable question to ask is why cross-licensing requires different modelling than unilateral licensing. If cross-licensing is simply both firms licensing their patents to one another, is this simply not the same as two separate incidents of unilateral licensing? There are several caveats that make this not so. Foremost if we consider transaction costs, if two firms have a large portfolio it can be simply cheaper to cross-license their entire portfolios to each other. In this way, firms don't pick and choose which patents they get individually: they purchase the entire package. Some agreements include future patents as well, something that unilateral licensing cannot achieve. Unlike patent pools, however, there is no distinction between necessary patents and patents that are unused or unrequired to enter the market. Other additional parts of cross-licensing deals are that negotiating leverage is different as both firms have technological assets which are valuable to the opposing firm (Shapiro, 1985; Kwon and Nagaoka, 2004). This leverage varies with the value of the patent portfolios, the complementarity of technology between firms and size of firms. The important point being that the negotiating situation of firms undertaking cross-licensing will be in these ways dramatically different than with unilateral licensing.

In particular, two papers I found to be extremely valuable in constructing my model in the second half of this paper. Sinha (2015) investigates how a NPE licenses to an asymmetric duopoly. Sinha developed a mechanism under which patents are therefore valued differently to different firms based heavily on their market share. Wang and Yang (2004) investigate how a more dominant firm licenses to two weaker firms with asymmetric costs. They show that royalty-based licensing dominates fixed fee licensing, despite the asymmetries in costs.

Part Two: The Model

Objective

The objective of the model is to examine how, and if, including asymmetries impacts the existing theoretical framework laid out by Jeon and Lefouili (2015). In addition, my goal is also to discuss any relevant implications that come out of this extension. The direction of investigating portfolio asymmetries is important because the existing literature on cross-licensing for the most part makes the assumption that costs and patent portfolios are symmetric. In the actual marketplace, however, firms seldom if ever, have identically-valued patent portfolios. While a unilateral licensing literature with asymmetric costs does exist, to my knowledge this extension has not sufficiently been carried over to the cross-licensing literature. Furthermore, patent portfolios are something that the unilateral literature doesn't cover because, whereas, in unilateral licensing, patents are licensed individually, in cross-licensing all patents are licensed simultaneously.

In this section we will consider five cases. Foremost, we will consider the monopoly case. Next we will consider the duopoly case; both in a world in which we don't allow cross-licensing, and then one in which we do. Lastly we will consider three separate variations of the three firm case: No cross-licensing, multilateral licensing and bilateral licensing.

The Model Explained

Extending the framework constructed by Jeon and Lefouili (2015), in this section I seek to improve the robustness of their original model by deriving the results under asymmetric patent portfolios. The model follows a two-stage game.

Stage 1: In the first stage, we either allow or disallow cross-licensing depending on the case we are considering. In Jeon and Lefouili (2015), Lemma 1 (pp. 12-13) shows that as long as the condition of non-drastic technology holds, a cross-licensing agreement with a firm strictly dominates any position where there is no cross-licensing agreement. This also proves that firms are better off the more of them we allow into a single agreement. Using this lemma, it then follows that when we allow cross-licensing at different degrees of laterality, every firm will engage in a cross-licensing agreement to the maximum laterality with each other firm.

Stage 2: While we allow cooperation between firms in the first stage, cooperation in the second stage is not allowed. Any sort of cooperation in this stage would be collusion, which is widely banned by antitrust agencies for its anticompetitive properties. In this stage then firms simultaneously and independently choose quantities to maximize their individual profits, competing a la Cournot. As we progress through the model, we will often solve the second stage first and work backwards in order to derive our solutions.

When generating our profit functions for the various cases we can use one of the two following generalized formulas. The first possibility is in the case of no cross-licensing. The general form of the profit function for the individual firm will then be as follows:

$$\pi_i = Pq_i - (C - h_i)q_i$$

Here price is represented by P and the firm's quantity is represented by q_i. The first term then, represents the firm's revenue. The second term is the firm's total cost. It is characterized by quantity produced multiplied by C, the firm's marginal cost before technology, subtracted by the firm's cost-reducing technology h_i.

The formula for cross-licensing resembles the one without, save for some very important differences. Foremost, the firm has full access to the market's technology. The downstream revenue term $(C - H)q_i$ now includes H instead of h_i as a result. The firm however must pay a royalty for each technology utilized that it does not own the patent for expressed by the term $\sum_{j \neq i}^{n} h_j q_i$. Additionally, the firm generates a profit via licensing its own patents out to other active firms, represented in the final term $\sum_{j \neq i}^{n} h_i q_j$.

$$\pi_i = Pq_i - (C - H)q_i - \sum_{j \neq i}^{n} h_j q_i + \sum_{j \neq i}^{n} h_i q_j$$

Variables

α: Y-intercept of the inverse linear demand function

β: Slope of the inverse linear demand function

π: Profit

P: Price

Q: Total Market Quantity

q_i: Quantity produced by individual firm *i*

C: Marginal cost of the individual firm without technology

H: Total market technology

h_i: Patent portfolio of firm *i*

r_{ij}: The royalty rate agreed upon in the cross-licensing agreement between firms *i* and *j*

G: The ratio between h_j and h_k

CS: Consumer Surplus

W: Social Welfare

Value-Added vs Cost-Reducing Patents

There are two approaches to the way access to a patent is handled in a model. In this model we will assume that patents reduce firms' costs. The alternative method bolsters the value of the product instead of reducing costs and is known as the Value-Added approach. In Jeon and Lefouili (2015), it is proven that the Value-Added and Cost-Reduction methods are effectively equivalent. Nothing new introduced in this model should change that assumption hence the choice to utilize cost-reducing patents is irrelevant to the results. An advantage of using the cost-reducing approach is for simplicity in calculation as we retain product homogeneity. Introducing product differentiation would be necessary in the value-added approach and would add another dimension to the model.

Non-Drastic Technology

A key assumption we make is that technologies are non-drastic. We say that an innovation is drastic when one firm has access to a technology and another firm does not, the result is the latter firm reducing their production to zero (Cellini, 2008, p. 46). Additionally, we assume that the sum of all available and patentable market technologies is non-drastic. This means that an individual firm does not have a patent portfolio large enough to crowd out other active incumbent firms. It is also therefore impossible to completely crowd out other firms via multilateral exclusion of licensing (multiple firms committing to not dealing with a select firm). Mathematically this can be expressed by assuming $\propto$ is always sufficiently large enough to pass any thresholds. This often ensures the sign of terms that contain $\propto$ in cases where it would otherwise be ambiguous.

Perfect Information

This model assumes perfect information on the part of all firms. In the first stage all firms are aware of the production function, and the patent portfolios of other firms and can therefore derive the royalty rates on cross-licensing deals between other firms. In the second stage since all the previous information is known, there is no uncertainty on how much quantity to produce.

Fixed Fees

When firms negotiate a cross-licensing agreement, they choose both a royalty rate and a fixed fee. As it turns out that since in our model firms jointly maximize profit, the actual fee chosen has no impact on the optimal royalty rate, quantity produced, consumer welfare or industry profits. Note that the cost of the direct transfer of the first firm is cancelled out in equal and opposite measure by the gains of the transfer recipients. In a different model with greater distortionary forces the ratio between fixed fee and royalty might prove more important.

Case One - Monopoly

The first case we consider is the case of a monopoly. A single active firm is the sole producer of output, in addition, he is also the sole holder of all relevant industry patents. The monopolist's patent portfolio therefore represents total industry technology. As a result, h_i (the monopolist's patent portfolio) can simply be notated as *H*. While the additional variable patent portfolio extension should have no surprising results in comparison to the original model, this simplistic case will serve to establish a benchmark to compare the results of other cases to.

Stage One: Cross-Licensing

Normally, in the first stage of the model firms negotiate cross-licensing agreements between each other. In this case however there is only a single firm therefore cross-licensing is impossible and nothing occurs in this stage. An important consequence of not having any cross-licensing agreements is that royalties will be absent from all of our results under monopoly.

Stage Two: Firms Select Quantity

In this stage the monopolist maximizes profit, with quantity as his choice variable, we can define monopoly profit as follows:

$$\pi_m = PQ_m - (C - H)Q_m$$

We assume for all cases that the the demand of the market takes the form of a general inverse linear demand function:

$$P = [\alpha - \beta(Q)]$$

Substituting in our linear demand curve for P we arrive at the following equation:

$$\pi_m = [\alpha - \beta(Q_m)]Q_m - CQ_m + HQ_m$$

We then take the first order condition, maximizing π_m with $Q_m$as our choice variable.

$$Max\ \pi_m(Q_m) = [\alpha - \beta(Q_m)]Q_m - CQ_m + HQ_m$$

This gives us optimal choice of quantity for the firm in terms of purely exogenous parameters. Note that due to monopoly Q_m* also is representative of optimal total quantity *Q**.

$$Q_m{}^* = \frac{\alpha + H - C}{2\beta}$$

Let us investigate what happens to optimal quantity when we change the size of the patent portfolio.

$$\frac{\partial Q_m{}^*}{\partial H} = \frac{1}{2\beta} > 0$$

We see that quantity rises when we increase the size of the patent portfolio as β is positive. With this information we can describe price in terms of purely exogenous parameters as well. We begin with our linear demand curve and substitute $Q_m{}^*$ into Q_m.

$$P = \alpha - \beta(Q_m) = \frac{\alpha - H + C}{2}$$

We next investigate how price changes as we change the size of the patent portfolio. We discover that as we increase the size of the patent portfolio, price drops regardless of other factors.

$$\frac{\partial P}{\partial H} = -\frac{1}{2} < 0$$

Now we examine consumer surplus, using the following general expression for consumer surplus with a linear demand curve. We can express consumer surplus in terms of solely exogenous parameters which we can achieve by substituting in Q_m* for Q.

$$CS = \frac{(\alpha - P)Q}{2} = \frac{(\alpha + H - C)^2}{8\beta}$$

We investigate how profits change as we change the size of the patent portfolio.

$$\frac{\partial CS}{\partial H} = \frac{\alpha + H - C}{4\beta} > 0$$

We see that as we increase the size of the patent portfolio consumer surplus grows at an increasing rate. This is because we know α is sufficiently large for the numerator to be positive and that β is also positive. Therefore, consumer surplus increases with *H*.

Now that we've seen the monopoly results of consumer surplus, we can investigate its counterpart industry profit. Starting with the profit function of the monopolist we make the relevant substitutions and simplify until we have π_m in terms of only exogenous parameters.

$$\pi_m = [\alpha - \beta(Q_m)]Q_m - CQ_m + HQ_m$$

$$\pi_m = \frac{(\alpha + H - C)^2}{4\beta}$$

We investigate how industry profits change as we change the size of the patent portfolio. We can see that profits rise, as the patent portfolio grows at an increasing rate, similar to the interactions with consumer surplus.

$$\frac{\partial \pi_m}{\partial H} = \frac{6(\alpha + H - C)}{4\beta} > 0$$

With consumer surplus and industry profits calculated, we can sum both results to calculate social welfare. Similar steps as the previous equations are undertaken to express it in terms of exogenous parameters.

$$W = \pi_m + CS$$

$$W = \frac{3(\alpha + H - C)^2}{8\beta}$$

We then investigate how welfare changes with patent portfolio sizes.

$$\frac{\partial W}{\partial H} = \frac{6(\alpha + H - C)}{8\beta} > 0$$

Unsurprisingly with both consumer surplus and industry profit rising in relationship to *H*, so too does social welfare.

Case Two – Duopoly Without Cross-Licensing

In this case, we introduce a second firm into the market. The representative firms will be denoted as firms *i* and *j*. Both firms have asymmetrically sized patent portfolios h_i and h_j respectively. That is to say, unlike the base model, both these firms have portfolio sizes that deviate not only from the base value 1 but can also deviate from each other. Collectively h_i and h_j comprise the entirety of total market technology.

Stage One: Cross-Licensing

Once again, we disallow cross-licensing, meaning that nothing occurs in this stage. Just like the monopoly example, the implication is that royalties will be absent from our results.

Stage 2: Firms Select Quantity

Again we draw from our general form for firm profit in the absence of cross-licensing. This gives us the following equation:

$$\pi_i = (\alpha - \beta Q)q_i - (C - h_i)q_i$$

We take the first order condition of the profit function of firm *i*, and take q_i as our choice variable.

$$Max\ \pi_i(q_i) = (\alpha - \beta Q)q_i - (C - h_i)q_i$$

$$\frac{\partial \pi_i}{\partial q_i} = -2\beta q_i + \alpha - \beta q_j - C + h_i = 0$$

Setting the equation equal to zero and solving for q_i gives us the optimal quantity of firm *i*.

$${q_i}^* = \frac{\alpha - \beta q_j - C + h_i}{2\beta}$$

We can utilize symmetry to derive the optimal quantity of firm *j*.

$${q_j}^* = \frac{\alpha - \beta q_i - C + h_j}{2\beta}$$

Solving these two equations for firm q_i and q_j yields:

$${q_i}^* = \frac{\alpha - C - h_j + 2h_i}{4\beta - 4\beta^2} \qquad i, j = 1{,}2 \text{ and } i \neq j$$

Assuming β<1 , paired with our assumption of non-drastic technology, we know that α is sufficiently large to make $q_i{}^*$ positive.

Through symmetry we are able to combine $q_i{}^*$ and $q_j{}^*$ and arrive at the equilibrium market quantity:

$$Q^* = \frac{2\alpha - 2C + H}{4\beta - 4\beta^2}$$

Price

We take our linear demand curve and substitute *Q** for *Q* to give us a general form for price in terms of solely exogenous parameters. We can see that as total market technology *H* rises, price has the inverse effect and instead falls.

$$P = \alpha - \beta(Q) = \alpha - \left(\frac{2\alpha - 2C + H}{4 - 4\beta}\right)$$

$$\frac{\partial P}{\partial H} = -\left(\frac{1}{4 - 4\beta}\right) < 0$$

Consumer Surplus

We begin with the general form for consumer surplus with a linear demand curve. We substitute in our solution for price and optimal quantity *Q**. As price falls, consumer surplus increases with *H*.

$$CS = \frac{(\alpha - P)Q}{2} = \frac{(\frac{2\alpha - 2C + H}{4 - 4\beta})\frac{2\alpha - 2C + H}{4\beta - 4\beta^2}}{2} = \frac{(2\alpha - 2C + H)^2}{32\beta(\beta - 1)^2}$$

$$\frac{\partial CS}{\partial H} = \frac{2\alpha - 2C + H}{16\beta(\beta - 1)^2} > 0$$

Due to the assumption of non-drastic technology α is sufficiently large for the numerator to be positive. This indicates that as *H* rises, consumer surplus rises (it is also of interest to note that consumer surplus is invariant to the distribution of market technology).

Firm Profit

Solving for profit of the representative firm *i* we substitute *q* for *q** and *Q* for *Q** to get the following equation. We can't simply examine how firm profit is impacted by changes in total market technology as for the individual firm it matters how the growth in technology is distributed.

$$\pi_i = \left(\frac{(2\alpha + 2C - H)\big(\alpha - C - h_j + 2h_i\big)}{16\beta - 32\beta^2 + 16\beta^3}\right) + \left(\frac{C(C - \alpha + h_j - 3h_i) + h_i(\alpha - h_j + 2h_i)}{4\beta - 4\beta^2}\right)$$

$$\frac{\partial \pi_i}{\partial h_i} = \left(\frac{(3\alpha + 5C - 4hi - hj) + 4(1-\beta)\big(\alpha - 3c - h_j + 4h_i\big)}{16\beta(1-\beta)(1-\beta)}\right) > 0$$

$$\frac{\partial \pi_i}{\partial {h_i}^2} = \left(\frac{3 - 4\beta}{4\beta(1-\beta)(1-\beta)}\right)$$

Due to the assumption of non-drastic technology the sign of $\frac{\partial \pi_i}{\partial h_i}$ can be effectively identified by looking solely at α. Since α is positive everywhere in the numerator, and the denominator is also positive we can reason that $\frac{\partial \pi_i}{\partial h_i} > 0$. Taking the second derivative, we can also see whether there are increasing or diminishing returns. We find that since the denominator is positive the sign of $\frac{\partial \pi_i}{\partial {h_i}^2}$ depends on β. When $\beta > \frac{3}{4}$ the returns are diminishing, when $\beta < \frac{3}{4}$, the returns are increasing and when $\beta = \frac{3}{4}$ the returns are constant.

Industry profit

Industry profit is calculated by combining the profit of all firms in the market. We sum the above equations for firm *i* and firm *j* to arrive at the following equation to express industry profit solely in terms of exogenous parameters.

$$\pi_i + \pi_j = \left(\frac{(2\alpha + 2C - H)(2\alpha - 2C + H)}{16\beta(\beta - 1)^2}\right) + \left(\frac{C(2C - 2\alpha - 2H) + h_i\big(\alpha - h_j + 2h_i\big) + h_j(\alpha - h_i + 2h_j)}{4\beta(1-\beta)}\right)$$

We can then see how industry profit changes with respect to total market technology. This is not a calculation we can always make (and it will in fact be absent from future cases), however due to the symmetry of this particular problem, and the fact that we are dealing with an aggregated variable we can find the solution purely in terms of *H*.

$$\frac{\partial(\pi_i + \pi_j)}{\partial H} = \frac{(2H - 4C) - 8(\beta - 1)(\alpha + H - C)}{16\beta(\beta - 1)^2}$$

We see that industry profits actually vary with H depending on H and β. For high values of β, an increase in H might lower industry profits, whereas for low values of β the opposite occurs. Next we investigate what happens when we adjust the ratio between h_i and h_j.

$$\pi_i + \pi_j = \left(\frac{(2\alpha + 2C - H)(2\alpha - 2C + H)}{16\beta(1 - 2\beta + \beta^2)}\right) + \left(\frac{C(2C - 2\alpha - 2H) + h_i(\alpha - H + 3h_i) + (H - hi)(\alpha - 3h_i + 2(H)}{4\beta(1 - \beta)}\right)$$

$$\frac{\partial(\pi_i + \pi_j)}{\partial h_i} = \left(\alpha - 2h_j + 4h_i\right)$$

$$\frac{\partial(\pi_i + \pi_j)}{\partial(h_i - hj)} = \frac{6\left(h_i - h_j\right)}{4\beta(1 - \beta)}$$

As the difference in patent portfolios grows, so too does joint profit. Importantly this tells us that monopoly power in the patent space also generates similar monopoly results in the standard market. If a firm is already in possession of the larger patent portfolio joint profits rise. The larger the existing difference, the larger the gain. Likewise, if we made a distributional change to move technology from a larger portfolio to a smaller one, joint profits would be reduced.

Social Welfare

Social welfare is simply the sum of industry profit and consumer surplus. We sum the two and analyze the comparative statics.

$$SW = \left(\frac{(2\alpha + 2C - H)(2\alpha - 2C + H)}{16\beta(\beta - 1)^2}\right) + \left(\frac{C(2C - 2\alpha - 2H) + h_i\left(\alpha - h_j + 2h_i\right) + h_j(\alpha - h_i + 2h_j)}{4\beta(1 - \beta)}\right) + \frac{(2\alpha - 2C + H)^2}{32\beta(\beta - 1)^2}$$

$$= \left(\frac{2(2\alpha + 2C - H)(2\alpha - 2C + H) + (2\alpha - 2C + H)^2}{32\beta(\beta - 1)^2}\right) + \left(\frac{C(2C - 2\alpha - 2H) + h_i\left(\alpha - h_j + 2h_i\right) + h_j(\alpha - h_i + 2h_j)}{4\beta(1 - \beta)}\right)$$

$$\frac{\partial SW}{\partial h_i} = \left(\frac{(2\alpha + 2C - H)}{16\beta(\beta - 1)^2}\right) + \left(\frac{-2C + (\alpha + 4h_i)}{4\beta(1 - \beta)}\right)$$

When we increase the patent portfolio of an individual firm, social welfare increases but at a smaller rate the larger the opposing firm's patent portfolio, h_j is. Therefore the greater the disparity is between h_i and h_j, the greater is social welfare. The reasons for this are twofold. Firstly, as the disparity between the firms' patent portfolios increases, the lower the marginal cost of the firm, allowing for more efficient production than otherwise possible for the beneficiary firm. Secondly, even though the technology used has not changed, the presence of royalties inflates costs and shifting to a more unequal distribution of patents reduces this distortion. Therefore, greater inequality in patent portfolios creates a smaller impact on upstream profit. This lower cost creates lower price and therefore greater social welfare.

Case Three – Duopoly with Cross-Licensing

In case three, we investigate the duopoly case again. This time however we allow cross-licensing. Each firm in the first stage now jointly maximizes its profit, choosing the optimal royalty rate r_{ij}. As mentioned earlier the fixed-fee component of this agreement turns out to be irrelevant so will be excluded from the model. We begin with the profit function of firm *i* and work backwards from stage two.

Stage Two

We start with the profit function of a firm, derived from our general form.

$$\pi_i = Pq_i - \left(C - H + r_{ij}h_j\right)q_i + r_{ij}h_i q_j$$

We then take the first order condition, where firm *i*'s choice variable is q_i. Optimal quantity produced is then as follows:

$$Max_{q_i} \quad \pi_i = Pq_i - \left(C - H + r_{ij}h_j\right)q_i + r_{ij}h_i q_j$$

$${q_i}^* = \frac{\alpha - \beta q_j - C + H - r_{ij}h_j}{2\beta}$$

By symmetry, we can also derive firm *j*'s optimal quantity produced.

$${q_j}^* = \frac{\alpha - \beta q_i - C + H - r_{ij}h_i}{2\beta}$$

We then substitute q_j into q_i, to express q_i in terms of only exogenous parameters and r_{ij}.

$${q_i}^* = \frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}$$

Now that we have solved for the optimal quantity of firm *i*, it is also possible for us to derive total market quantity in equilibrium. Once again, by symmetry, we derive optimal quantity of firm *j*.

$${q_j}^* = \frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}$$

Combining optimal quantities of the individual firms, we find equilibrium joint quantity.

$$Q^* = \frac{2\alpha - 2C + 2H - r_{ij}H}{3\beta}$$

Stage One

We then return to the cross-licensing stage to solve for r_{ij}.

$$\pi_i + \pi_j = Pq_i - (C - H + r_{ij}h_j)q_i + r_{ij}h_i q_j + Pq_j - (C - H + r_{ij}h_i)q_j + r_{ij}h_j q_i$$

We substitute in price, and the quantities we just solved for.

$$\pi_i + \pi_j = P\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right) - (C - H + r_{ij}h_j)\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right)$$
$$+ r_{ij}h_i\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right) + P\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right)$$
$$- (C - H + r_{ij}h_i)\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right)$$
$$+ r_{ij}h_j\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right)$$

Substitute generalized price for the price specific to this case:

$$\pi_i + \pi_j = (\alpha - \beta(qi + qj))\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right)$$
$$- (C - H + r_{ij}h_j)\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right)$$
$$+ r_{ij}h_i\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right)$$
$$+ (\alpha - \beta(qi + qj))\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right)$$
$$- (C - H + r_{ij}h_i)\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right)$$
$$+ r_{ij}h_j\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right)$$

Now that we have this expanded form, we can maximize joint profits with r_{ij} as the choice variable.

$$Max_{r_{ij}} \quad \pi_i + \pi_j$$

Due to the complexity of the equation, we can solve this stage in several different parts. We will break it down into upstream profit, downstream revenue and downstream profit.

Upstream Profit

Denoting upstream profit as π_U, we take the joint profit of both firms and extract only the terms that represent either firm's direct profit from royalties. This is the combined upstream profit of the cross-licensing deal.

$$\pi_U = r_{ij}h_i\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right) + r_{ij}h_j\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right)$$

$$\pi_U = r_{ij}h_i\left(\frac{\alpha - C + H + (h_j - 2h_i)r_{ij}}{3\beta}\right) + r_{ij}h_j\left(\frac{\alpha - C + H + (h_i - 2h_j)r_{ij}}{3\beta}\right)$$

We can then solve the first order condition for this isolated component of upstream profit. Taking the partial derivative with respect to the royalty we get:

$$\frac{\partial \pi_U}{\partial r_{ij}} = h_i\left(\frac{\alpha - C + H + (h_j - 2h_i)r_{ij}}{3\beta}\right) + \frac{r_{ij}h_i(h_j - 2h_i)}{3\beta} h_j\left(\frac{\alpha - C + H + (h_i - 2h_j)r_{ij}}{3\beta}\right) + \frac{r_{ij}h_j(h_i - 2h_j)}{3\beta} > 0$$

The assumption of non-drastic technology keeps α sufficiently high to make this effect positive although the effect is diminishing with r_{ij}. Theoretically if this were the only component, the firms would have an incentive to raise the royalty rate until the effect consumed the entire demand function (though this would violate the assumption of non-drastic technology).

Downstream Revenue

Now considering downstream revenue, denoted as π_D, we extract the positive terms that have not been used in upstream profit. We then take the partial derivative of π_D with respect to the royalty.

$$\pi_D = \left(\frac{\alpha + 2C - 2H + r_{ij}(h_i + h_j)}{3}\right)\left(\frac{\alpha - C + H + (h_i - 2h_j)r_{ij}}{3\beta}\right)$$
$$+ \left(\frac{\alpha + 2C - 2H + r_{ij}(h_i + h_j)}{3}\right)\left(\frac{\alpha - C + H + (h_j - 2h_i)r_{ij}}{3\beta}\right)$$

$$\frac{\partial \pi_D}{\partial r_{ij}} = \frac{(h_i + h_j)}{3}\left(\frac{\alpha - C + H + (h_i - 2h_j)r_{ij}}{3\beta}\right) + \frac{(h_i - 2h_j)}{3\beta}\left(\frac{\alpha + 2C - 2H + r_{ij}(h_i + h_j)}{3}\right)$$
$$+ \frac{(h_i + h_j)}{3}\left(\frac{\alpha - C + H + (h_j - 2h_i)r_{ij}}{3\beta}\right) + \frac{(h_j - 2h_i)}{3\beta}\left(\frac{\alpha + 2C - 2H + r_{ij}(h_i + h_j)}{3}\right)$$

$$\frac{\partial \pi_D}{\partial r_{ij}} = (h_i + h_j)\left(\frac{2\alpha - 2C + 2H - r_{ij}(h_i + h_j)}{9\beta}\right) + (h_i - 2h_j)\left(\frac{\alpha + 2C - 2H + r_{ij}(h_i + h_j)}{9\beta}\right)$$
$$+ (h_j - 2h_i)\left(\frac{\alpha + 2C - 2H + r_{ij}(h_i + h_j)}{9\beta}\right)$$

We note that the sign of this effect is ambiguous and dependent on the existing value of r_{ij} relative to α, *C*, and *H*, as well as the ratio between h_j and h_i.

Downstream Cost

Lastly, we take the remainder of the terms, unused in downstream revenue and upstream profit. While these terms are ordinarily negative, they are expressed positively here in the context of a cost function. We note that the effect of r_{ij} on downstream cost is positive due to our assumption of non-drastic technology.

$$C_d = (C - H + r_{ij}h_j)\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right) + (C - H + r_{ij}h_i)\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right)$$

$$\frac{\partial C_d}{\partial r_{ij}} = h_j\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right) + \frac{hi - 2hj}{3\beta}(C - H + r_{ij}h_j) + h_i\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right) + \frac{hj - 2hi}{3\beta}(C - H + r_{ij}h_i) > 0$$

Summing the previous effects, we can calculate the total impact of a change in royalty on joint profit.

$$\frac{\partial(\pi_i + \pi_j)}{\partial r_{ij}} = h_i\left(\frac{\alpha - C + H + (h_j - 2h_i)r_{ij}}{3\beta}\right) + \frac{r_{ij}h_i(h_j - 2h_i)}{3\beta} + h_j\left(\frac{\alpha - C + H + (h_i - 2h_j)r_{ij}}{3\beta}\right) + \frac{r_{ij}h_j(h_i - 2h_j)}{3\beta} + (h_i + h_j)\left(\frac{2\alpha - 2C + 2H - r_{ij}(h_i + h_j)}{9\beta}\right) + (h_i - 2h_j)\left(\frac{\alpha + 2C - 2H + r_{ij}(h_i + h_j)}{9\beta}\right) + (h_j - 2h_i)\left(\frac{\alpha + 2C - 2H + r_{ij}(h_i + h_j)}{9\beta}\right) - h_j\left(\frac{\alpha - C + H - 2r_{ij}h_j + r_{ij}h_i}{3\beta}\right) - \frac{hi - 2hj}{3\beta}(C - H + r_{ij}h_j) - h_i\left(\frac{\alpha - C + H - 2r_{ij}h_i + r_{ij}h_j}{3\beta}\right) - \frac{hj - 2hi}{3\beta}(C - H + r_{ij}h_i)$$

We then solve for optimal royalty choice. Factoring out all terms with r_{ij}, we arrive at the following set of terms that are multiplied by r_{ij}.

$$\frac{2h_i(h_j - 2h_i)}{3\beta} + \frac{2h_j(h_i - 2h_j)}{3\beta} + \frac{(h_i + h_j)(-2h_i - 2h_j)}{9B} + \frac{h_j(h_i - 2h_j) + h_i(h_j - 2h_i)}{3b} + \frac{h_j(h_i - 2h_j)}{3B} + \frac{h_i(h_j - 2h_i)}{3B}$$

$$\frac{2h_i(h_j - 2h_i)}{3\beta} + \frac{2h_j(h_i - 2h_j)}{3\beta} + \frac{(h_i + h_j)(-2h_i - 2h_j)}{9\beta} - \frac{hj(2h_i - 4h_j) + h_i(2h_j - 4h_i)}{3\beta}$$

From here, we can simplify the above term into the following:

$$\frac{2(h_i + h_j)^2}{9\beta}$$

Next we deal with all the terms that do not have r_{ij} in them and simplify that term to the following.

$$(h_i + h_j)\left(\frac{\alpha - C + H}{9\beta}\right)$$

Isolating and solving for r_{ij} gives us the following equation for optimal royalty:

$$r_{ij}{}^{*} = \frac{\alpha - C + H}{2(H)}$$

We note that the optimal royalty value is not impacted by the ratio of patent portfolios. In fact, if we keep the total market technology available H constant, any changes in h_i are compensated by an equal and opposite change in h_j. So if there is no change in the ratio of patent portfolios, how does a change in total market technology impact it? We take the partial derivative of optimal two-firm royalty with respect to H.

$$\frac{\partial r_{ij}{}^{*}}{\partial H} = \frac{2(C - \alpha)}{4H^2} < 0$$

Since H is strictly positive, the sign of the change depends on whether α is greater than C. Since we assume that all the technologies are non-drastic then this will always be true. Therefore, increasing total technology lowers royalty rates in the two-firm setting.

Unlike Jeon and Lefouili (2015), firms in this setting have multiple patents so it is unclear if the firms are actually opting to reduce their upstream profits. The result merely explains that the royalty paid per patent goes down when more patents exist in the market. Keep in mind that all patents in this setting have a fixed homogenous cost-reducing value.

With optimal royalty calculated, we can then investigate what happens to individual quantity.

$$q_i{}^* = \frac{(h_i)\frac{\alpha - C + H}{(H)}}{2\beta}$$

$$\frac{\partial q_i{}^*}{\partial h_i} = \frac{\frac{\alpha - C + H}{H} + \frac{(h_i)(C - \alpha)}{H^2}}{2\beta}$$

The first term of this effect is positive and the second is negative, both due to the constraint of non-drastic technology. This indicates that the effect is ambiguous and dependent on the size of total market technology.

$$Q^* = \frac{\alpha - C + H}{2\beta}$$

$$\frac{\partial Q^*}{\partial H} = \frac{1}{2\beta} > 0$$

Price

When analyzing price, we can once again substitute quantity, to interpret price in terms of purely exogenous parameters. We discover that in this case, price is invariant to the distribution of patents. When taking the partial derivative of the aggregate, total market technology, we find that price has an inverse relationship with the level of technology.

$$P = \frac{\alpha + C - H}{2}$$

$$\frac{\partial P}{\partial H} = -\frac{1}{2} < 0$$

Consumer Surplus

We can substitute the above variables that were just solved for P and Q in order to find consumer surplus in terms of purely exogenous parameters.

$$CS = \frac{(\alpha - P)Q}{2} = \frac{(\alpha - \frac{\alpha + C - H}{2})\frac{\alpha - C + H}{2\beta}}{2} = \frac{(\alpha - C + H)^2}{8\beta}$$

We can note that $\frac{\partial CS}{\partial H} > 0$. Additionally, just as price is invariant to the distribution of patents so too is consumer surplus in this case.

Industry Profit

Next we investigate industry profit. First, we can solve the first half of the equation which only deals with the profit of firm *i*.

$$\pi_i + \pi_j = Pq_i - (C - H)q_i + Pq_j - (C - H)q_j$$

$$\pi_i = \frac{(h_i)\frac{\alpha - C + H}{H}(\alpha + C - H)}{4\beta} - \frac{(C - H)(2h_i)\frac{\alpha - C + H}{H}}{4\beta}$$

$$= \frac{(h_i)(\alpha + C - H) - (C - H)(2h_i)}{4\beta}$$

Using symmetry and combining the above terms we can come up with an equation for industry profit. Additionally, we can factor out like-terms as we are solely interested in the sign.

$$\pi_i = \left(h_i + h_j\right)(\alpha + C - H) - 2(C - H)\left(h_i + h_j\right)$$

$$(\alpha - C + H) > 0$$

We can conclude that industry profits are positive. With this relevant term in mind we can take the partial with respect to *H* and see that industry profits also rise with an increase in total market technology.

Social Welfare

Combining industry profit and consumer surplus, we can also calculate social welfare.

$$\pi_i + \pi_j + CS = \frac{(h_i)\frac{\alpha - C + H}{H}(\alpha + C - H)}{4\beta} - \frac{(C - H)(2h_i)\frac{\alpha - C + H}{H}}{4\beta} + \frac{(\alpha - C + H)^2}{8\beta}$$

Unsurprisingly we also find that social welfare rises with respect to total market technology.

Case Four – Three Firms Without Cross-Licensing

In case four, we add an additional firm. There are large differences moving between the two and three firm cases. Since we are mainly interested in working with jointly maximizing profit with two firms, cross-licensing agreements of three firm case should give us insights into cases with more than three firms. Another important distinction is that when we have at least three firms there is the possibility of cross-licensing agreements that do not span the entire market, as we will see in Case Six.

Stage One

With no cross-licensing, nothing occurs in stage one. We then proceed directly to the second stage.

Stage Two

We begin with the profit function of firm *i* and take the first order condition, with q_i as our choice variable.

$$Max_{q_i} \quad \pi_i = (\alpha - \beta Q)q_i - (C - h_i)q_i$$

$$\frac{\partial \pi_i}{\partial q_i} = \beta q_i + (\alpha - \beta Q) - C + h_i$$

$$q_i{}^* = \frac{(\alpha - \beta q_j - \beta q_k) - C + h_i}{2\beta}$$

Now that we have q_i* we can find the optimal quantities of the other firms through symmetry.

$$q_j{}^* = \frac{(\alpha - \beta q_i - \beta q_k) - C + h_j}{2\beta}$$

$$q_k{}^* = \frac{(\alpha - \beta q_i - \beta q_j) - C + h_k}{2\beta}$$

Using substitution, we can solve to get $q_i{}^*$ in terms of purely exogenous parameters.

$$q_j = \frac{2\alpha - 2\beta q_i - 2C + 2h_j}{4\beta} - \frac{(\alpha - \beta q_i - \beta q_j) - C + h_k}{4\beta}$$

$$q_j = \frac{\alpha - \beta q_i - C + 2h_j - h_k}{3\beta}$$

$$q_k = \frac{\alpha - \beta q_i - C + 2h_k - h_j}{3\beta}$$

$$q_i{}^* = \frac{\alpha - C - h_k - h_j + 3h_i}{4\beta}$$

We note how the equilibrium quantity rises as the firm's technology rises, and falls as the technology of competitors' rise. We can use symmetry to sum each of the optimal quantities to calculate optimal total market quantity.

$$Q^* = \frac{3\alpha - 3C + H}{4\beta}$$

Equilibrium market quantity is completely invariant to the distribution of technology, just as in all our previous cases without cross-licensing. With optimal market quantity we can then also solve for price.

$$P = \alpha - \beta(Q^*) = \frac{\alpha + 3C - H}{4}$$

Consequently, price is also invariant to the distribution of technology in this case. We can also calculate joint profit in terms of purely exogenous parameters.

$$\pi_{ijk} = (\alpha - \beta Q - C)Q + h_i q_i + h_j q_j + h_k q_k$$

$$= \left(\frac{(3\alpha - 3C + H)(\alpha - C - H)}{16\beta}\right) + h_i \left(\frac{\alpha - C - h_k - h_j + 3h_i}{4\beta}\right)$$

$$+ h_j \left(\frac{\alpha - C - h_k - h_i + 3h_j}{4\beta}\right) + h_k \left(\frac{\alpha - C - h_i - h_j + 3h_k}{4\beta}\right)$$

We have already seen that total quantity produced is invariant to the distribution of patents in the market. Here we will examine what happens to profit when we change the ratio of the distribution of total market technology. We will adjust h_i holding *H* constant. Unlike the two-firm case we cannot say exactly how h_j and h_k change when we change h_i. So we will introduce a rule for how a change in h_i affects a change in the other two patent portfolios.

Let *G* be a variable where $0<G<1$, and defined by the following two equations.

$h_j = H - (G - 1)\, h_i$

$h_k = H - (1 - G)\, h_i$

This rule states that when h_i changes the existing ratio between h_j and h_k is preserved. The variable *G* is then representative of the distribution of technology between h_j and h_k.

$$\frac{\partial \pi_{ijk}}{\partial h_i} = 6h_i - 2GH + 2G^2h_i + (2G-2)H + (2g-2)(G-1)h_i$$
$$-2H + 2Gh_i + 2Gh_i - 2(H + (G-1)h_i) + 2(1-G)h_i$$
$$-2[-G(H + (G-1)h_i)] - 2[(G-1)(H - Gh_i)]$$

$$= 6h_i + 2G^2h_i + (2g-2)(G-1)h_i$$
$$-4H + 4Gh_i - 2H - 2Gh_i + 2h_i + 2h_i - 2Gh_i$$
$$+2GH + 2G^2h_i - 2Gh_i + 2H - 2Gh_i - 2GH + 2G^2h_i$$

$$= \frac{2(1-G+G^2)h_i - h_j - h_k}{\beta}$$

We find that the closer that G is to 0.5 the smaller that industry profits will be. Additionally, we find that the larger that h_i is in comparison to the other patent portfolios, the greater the gains from further increasing h_i's share. However the opposite is also true; if h_i is insufficiently large (for example smaller than the other two patent portfolios) then increasing h_i will in fact decrease profits. Essentially the greater the technological inequality in all respects, the larger industry profits will be.

Next we can investigate consumer surplus. We can substitute our optimal aggregate quantity, as well as price, into the equation for consumer surplus with linear demand.

$$CS = \frac{(\alpha - P)Q}{2} = \frac{(\alpha - P)(3\alpha - 3C + H)}{8\beta} = \frac{(3\alpha - 3C + H)(3\alpha - 3C + H)}{32\beta}$$

Consumer surplus rises with total market technology and remains invariant to the distribution of patents. We can then combine consumer surplus and industry profits to analyze the impact on social welfare.

$$\pi_{ijk} + CS = \left(\frac{(3\alpha - 3C + H)(\alpha - C - H)}{16\beta}\right) + h_i\left(\frac{\alpha - C - h_k - h_j + 3h_i}{4\beta}\right) + h_j\left(\frac{\alpha - C - h_k - h_i + 3h_j}{4\beta}\right)$$
$$+ h_k\left(\frac{\alpha - C - h_i - h_j + 3h_k}{4\beta}\right) + \frac{(3\alpha - 3C + H)(3\alpha - 3C + H)}{32\beta}$$

We see that as the consumer surplus component is unaffected by the distribution of patents, the change in social welfare from changes in the distribution of technology will be identical to that of industry profits. It then is ambiguous whether changing total market technology will increase social welfare. It depends on how it impacts the ratio of patent portfolio sizes. A ratio-preserving increase in technology would however increase social welfare.

Case Five – Three Firms with Multilateral Cross-Licensing

This case makes the assumption that there are three active firms in the market. Every active firm then engages in a single multi-lateral cross-licensing agreement. The firms will then jointly choose a royalty that maximizes the joint profits of all three firms. In this case all firms are restricted to the single royalty of the licensing agreement and do not have the same freedom as the bilateral agreements of the next case. It is important to note that it has been shown in Jeon and Lefouili that when patents are symmetric multilateral cross-licensing agreements spanning the entire market lead to monopoly results. Previously we saw that with two firms, even with asymmetric patent portfolios this has been shown to be true. As a result of this, we can make the assumption that allowing this type of agreement to exist will result in a multilateral agreement with all firms that fully takes advantage of this allowance.

Firstly, we can reference the general form and define the profit function of the individual firm as follows. Note that in this model there is a singular royalty value as a result of there being only a singular agreement. This will be notated simply as r.

$$\pi_i = Pq_i - \left(C - H + rh_j + rh_k\right)q_i + rh_iq_j + rh_iq_k$$

Stage Two

Solving the firm's stage two problem we maximize profit with the firm's individual quantity as the choice variable.

$$Max_{q_i} \quad \pi_i = Pq_i - \left(C - H + rh_j + rh_k\right)q_i + rh_iq_j + rh_iq_k$$

$$\frac{\partial \pi_i}{\partial q_i} = -\beta q_i + (\alpha - \beta Q) - C + H - rh_j - rh_k$$

$$q_i = \frac{\alpha - \beta q_j - \beta q_k - C + H - rh_j - rh_k}{2\beta}$$

Via symmetry we can also find total market quantity Q. Unlike other cases, we will require just total quantity to solve the next stage as opposed to solving for individual quantity.

$$Q = \frac{3\alpha - 3C + 3H - 2rH}{4\beta}$$

Stage One

We return to stage one, and solve the first order condition for optimal royalty.

$$\pi_{ijk} = (\alpha - \beta Q)Q - CQ + HQ$$

$$Max_r \quad \pi_{ijk} = \left(\frac{\alpha + 3C - 3H + 2rH}{4}\right)\left(\frac{3\alpha - 3C + 3H - 2rH}{4\beta}\right)$$

$$+(H - C)\left(\frac{3\alpha - 3C + 3H - 2rH}{4\beta}\right)$$

$$\frac{\partial \pi_{ijk}}{\partial r} = \frac{2H}{4}\left(\frac{3\alpha - 3C + 3H - 2rH}{4\beta}\right) - \frac{2H}{4}\left(\frac{\alpha + 3C - 3H + 2rH}{4}\right) + (C - H)\frac{2H}{4}$$

$$r^* = \left(\frac{(3 - \beta)(\alpha - C + H)}{2(\beta + 1)H}\right)$$

If β >3 then the optimal royalty is zero. One of our constraints is however that β lies between 0 and 1. We note that outside of this case, the optimal royalty value is invariant to the composition of patent distribution and only dependent on the size of total market technology. We can identify how optimal royalty then changes in regard to total market technology.

$$\frac{\partial r}{\partial H} = \frac{(3 - \beta)(\beta + 1)2\,(\alpha - C)}{(2(\beta + 1)H)^2} > 0$$

We find that optimal royalty rises with *H*, but at a decreasing rate as *H* grows larger. Next we can solve for optimal total quantity by plugging in the optimal total royalty value.

$$Q^* = \frac{3\alpha - 3C + 3H - 2rH}{4\beta} = \frac{3\alpha - 3C + 3H + \left(\frac{(\beta - 3)(\alpha - C + H)}{(\beta + 1)}\right)}{4\beta} = \frac{(\alpha - C + H)}{(\beta + 1)}$$

Solving for price by substituting in optimal total market quantity *Q**, we find that price falls as total market technology rises.

$$P = (\alpha - \beta Q) = \left(\alpha - \beta\frac{(\alpha - C + H)}{(\beta + 1)}\right)$$

Solving for industry profit by substituting in optimal total market quantity we can find industry profit in terms of solely exogenous parameters. We find that industry profits rise as aggregate technology levels rise.

$$\pi_{ijk} = (\alpha - \beta Q)Q - CQ + HQ$$

$$= \left(\alpha - \beta\frac{(\alpha - C + H)}{(\beta + 1)} - C + H\right)\frac{(\alpha - C + H)}{(\beta + 1)}$$

$$= \frac{(\alpha - C + H)^2}{(\beta + 1)} - \beta \frac{(\alpha - C + H)^2}{(\beta + 1)^2} = \frac{(\alpha - C + H)^2}{(\beta + 1)^2}$$

Likewise, we can also calculate consumer surplus by substituting in the relevant terms. We find that consumer surplus rises at an increasing rate as total market technology rises.

$$CS = \frac{(\alpha - P)Q}{2} = \beta \frac{(\alpha - C + H)^2}{2(\beta + 1)^2}$$

$$\frac{\partial CS}{\partial H} = \beta \frac{\alpha - C + H}{(\beta + 1)^2} > 0$$

Social welfare is the sum of industry profits and consumer surplus and therefore we can calculate it by adding the two together. We find that social welfare also rises at an increasing rate as we raise the level of total market technology.

$$SW = CS + \pi_{ijk} = \frac{(\beta + 2)(\alpha - C + H)^2}{2(\beta + 1)^2}$$

Case Six – Three Firms with Bilateral Cross-Licensing

The sixth and final case is the most complex of all the cases studied in this paper. We have three firms that engage in cross-licensing, however, this time firms are limited by an upper limit of two on the amount of participants in a single agreement. A strong assumption we will make in this case, for the sake of simplicity, is to neglect the impact of royalty values on other royalty values. Without this assumption the equations become too complex to practically solve for the purpose of this extension.

Since we know that firms will always engage in cross-licensing agreements due to the assumption of non-drastic innovation we can write the following profit functions.

Stage One

$$\pi_i = Pq_i - \left(C - H + r_{ij}h_j + r_{ik}h_k\right)q_i + r_{ij}h_iq_j + r_{ik}h_iq_k$$

$$\pi_j = Pq_j - \left(C - H + r_{ij}h_i + r_{jk}h_k\right)q_j + r_{ij}h_jq_i + r_{jk}h_jq_k$$

Firms will bilaterally maximize their joint profits in the first stage by jointly selecting the optimal royalty value. By combining both profit functions we arrive at the following equation summing the profit of the two firms.

$$\pi_i + \pi_j = Pq_i - \left(C - H + r_{ij}h_j + r_{ik}h_k\right)q_i + r_{ij}h_iq_j + r_{ik}h_iq_k + Pq_j - \left(C - H + r_{ij}h_i + r_{jk}h_k\right)q_j + r_{ij}h_jq_i + r_{jk}h_jq_k$$

Stage Two

To proceed we must first solve the second stage. In other words, we must solve the problem of the individual firm by taking the first order condition. In this stage our choice variable is of course the firm's quantity produced.

$$Max\ \pi_i(q_i) = Pq_i - \left(C - H + r_{ij}h_j + r_{ik}h_k\right)q_i + r_{ij}h_iq_j + r_{ik}h_iq_k$$

We then can substitute in our linear demand function for P and expand out the terms in the equation.

$$Max\ \pi_i(q_i) = (\alpha - \beta Q)q_i - \left(C - H + r_{ij}h_j + r_{ik}h_k\right)q_i + r_{ij}h_iq_j + r_{ik}h_iq_k$$

$$= \left(\alpha - \beta\left(q_i + q_j + q_k\right)\right)q_i - \left(C - H + r_{ij}h_j + r_{ik}h_k\right)q_i + r_{ij}h_iq_j + r_{ik}h_iq_k$$

With q_i as our choice variable we can solve for the first order condition and express an equation which defines optimal q_i.

$$\frac{\partial \pi_i}{\partial q_i} = \alpha - \beta q_i - \beta(q_i + q_j + q_k) - (C - H + r_{ij}h_j + r_{ik}h_k)$$

$$0 = \alpha - 2\beta q_i - \beta q_j - \beta q_k - (C - H + r_{ij}h_j + r_{ik}h_k)$$

$$q_i = \frac{\alpha - \beta q_j - \beta q_k - (C - H + r_{ij}h_j + r_{ik}h_k)}{2\beta}$$

We use symmetry to find the quantity produced of firm *j*. Then we can substitute q_j into q_i.

$$q_j = \frac{\alpha - \beta q_i - \beta q_k - (C - H + r_{jk}h_k + r_{ij}h_i)}{2\beta}$$

$$q_i = \frac{\alpha - \frac{\alpha - \beta q_i - \beta q_k - (C - H + r_{ij}h_i + r_{jk}h_k)}{2} - \beta q_k - (C - H + r_{ij}h_j + r_{ik}h_k)}{2\beta}$$

$$q_i = \frac{2}{3\beta}\left[\alpha - \frac{1}{2}(\alpha - \beta q_k - C + H - r_{ij}h_i - r_{jk}h_k) - \beta q_k - (C - H + r_{ij}h_j + r_{ik}h_k)\right]$$

$$q_i = \frac{1}{3\beta}[(\alpha - C + H + r_{ij}h_i + r_{jk}h_k - 2r_{ij}h_j - 2r_{ik}h_k)] - \frac{q_k}{3}$$

Once again we use symmetry to find the quantity of firms *j* and *k* and continue with substitution.

$$q_j = \frac{1}{3\beta}[(\alpha - C + H + r_{jk}h_j + r_{ik}h_i - 2r_{jk}h_k - 2r_{ij}h_i)] - \frac{q_i}{3}$$

$$q_k = \frac{1}{3\beta}[(\alpha - C + H + r_{ik}h_k + r_{ij}h_j - 2r_{ik}h_i - 2r_{jk}h_j)] - \frac{q_j}{3}$$

Next we substitute q_j into q_k, then proceed to substitute q_k into q_i.

$$q_k = \frac{1}{3\beta}[(\alpha - C + H + r_{ik}h_k + r_{ij}h_j - 2r_{ik}h_i - 2r_{jk}h_j)]$$
$$- \frac{1}{9\beta}[(\alpha - C + H + r_{jk}h_j + r_{ik}h_i - 2r_{jk}h_k - 2r_{ij}h_i)] + \frac{q_i}{9}$$

$$q_i = \frac{1}{3\beta}[(\alpha - C + H + r_{ij}h_i + r_{jk}h_k - 2r_{ij}h_j - 2r_{ik}h_k)]$$
$$- \frac{1}{9\beta}[(\alpha - C + H + r_{ik}h_k + r_{ij}h_j - 2r_{ik}h_i - 2r_{jk}h_j)]$$
$$+ \frac{1}{27\beta}[(\alpha - C + H + r_{jk}h_j + r_{ik}h_i - 2r_{jk}h_k - 2r_{ij}h_i)] - \frac{q_i}{27}$$

$$q_i = \frac{9}{28\beta}\left[\left(\alpha - C + H + r_{ij}h_i + r_{jk}h_k - 2r_{ij}h_j - 2r_{ik}h_k\right)\right]$$
$$- \frac{3}{28\beta}\left[\left(\alpha - C + H + r_{ik}h_k + r_{ij}h_j - 2r_{ik}h_i - 2r_{jk}h_j\right)\right]$$
$$+ \frac{1}{28\beta}\left[\left(\alpha - C + H + r_{jk}h_j + r_{ik}h_i - 2r_{jk}h_k - 2r_{ij}h_i\right)\right]$$

By simplifying we have reduced q_i to be in terms of solely exogenous parameters. It is important to remember that for this model the effects of royalties on each other will be ignored and seen as exogenous. We can then use symmetry to solve for the quantity of the other firms as well.

$$q_i = \frac{\alpha - C + H + r_{ij}h_i + r_{jk}h_k - 3r_{ij}h_j - 3r_{ik}h_k + r_{ik}h_i + r_{jk}h_j}{4\beta}$$
$$q_j = \frac{\alpha - C + H + r_{jk}h_j + r_{ik}h_i - 3r_{jk}h_k - 3r_{ij}h_i + r_{ij}h_j + r_{ik}h_k}{4\beta}$$
$$q_k = \frac{\alpha - C + H + r_{ik}h_k + r_{ij}h_j - 3r_{ik}h_i - 3r_{jk}h_j + r_{jk}h_k + r_{ij}h_i}{4\beta}$$

By combining the above results we can also find total quantity.

$$Q = \frac{3\alpha - 3C + 3H - r_{ij}h_i - r_{jk}h_k - r_{ij}h_j - r_{ik}h_k - r_{ik}h_i - r_{jk}h_j}{4\beta}$$

With total quantity in hand, we can also calculate price using the formula for linear demand

$$P = (\alpha - \beta Q)$$
$$= \left(\frac{\alpha + 3C - 3H + r_{ij}h_i + r_{jk}h_k + r_{ij}h_j + r_{ik}h_k + r_{ik}h_i + r_{jk}h_j}{4}\right)$$

Returning to Stage One we can now solve for the optimal bilateral cross-licensing royalty

$$\pi_i + \pi_j = Pq_i - \left(C - H + r_{ij}h_j + r_{ik}h_k\right)q_i + r_{ij}h_iq_j + r_{ik}h_iq_k + Pq_j$$
$$- \left(C - H + r_{ij}h_i + r_{jk}h_k\right)q_j + r_{ij}h_jq_i + r_{jk}h_jq_k$$

$$\frac{\partial(\pi_i + \pi_j)}{\partial r_{ij}} = \frac{\left(h_i + h_j\right)q_i}{4} + \frac{\left(h_i - 3h_j\right)P}{4B} - h_jq_i - \frac{\left(h_i - 3h_j\right)\left(C - H + r_{ij}h_j + r_{ik}h_k\right)}{4B} + h_iq_j$$
$$+ \frac{r_{ij}h_i\left(h_j - 3h_i\right)}{4B} + \frac{r_{ik}h_i\left(h_j + h_i\right)}{4B} + \frac{\left(h_i + h_j\right)q_j}{4} + \frac{\left(h_j - 3h_i\right)P}{4B} - q_jh_j$$
$$- \frac{\left(h_j - 3h_i\right)\left(C - H + r_{ij}h_i + r_{jk}h_k\right)}{4B} + h_jq_i + \frac{r_{ij}h_j(hi - 3hj)}{4B} + \frac{rjkhj(hj + hi)}{4B}$$

$$r_{ij}{}^{*} = \frac{\left(3C - 3H - 4r_{jk}h_k - 4r_{ik}h_k + 4Br_{jk}hj + 4Br_{ik}hi + 4(hir_{ik}h_k - 3hjr_{ik}h_k) + 4\left(hjr_{jk}h_k - 3hir_{jk}h_k\right)\right)}{10\left(h_i + h_j\right)\beta}$$

$$= \frac{3C - 3H + 4\left(\left(h_i - 3h_j - 1\right)r_{ik}h_k + (h_j - 3h_i - 1)r_{jk}h_k + \beta r_{jk}h_j + \beta r_{ik}h_i\right)}{10\left(h_i + h_j\right)\beta}$$

At this point in Case Six, while the whole model is solvable, the length and complexity of the equations would overshadow the important takeaways of the case. Therefore, as we are solving the rest of the problem of firms *i* and *j* in stage one, we will leave the solutions more general as opposed to having everything purely in terms of portfolio size. For $r_{ij}{}^{*}$ the above solution is as deep as this paper will take the optimal royalty. We take the royalties of other cross-licensing agreements as exogenous though they are in reality endogenous.

With this in mind, we then calculate the change in royalty when changing the portfolio size in the licensing agreement, and then when changing the portfolio size of a firm outside of it. Note that the denominator will always be positive so when trying to determine the sign it can be safely ignored.

$$\frac{\partial r_{ij}{}^{*}}{\partial h_i} = -3 + 4r_{ik}h_k - 12r_{jk}h_k + 4\beta r_{ik} > 0 \text{ if } r_{ik}h_k + \beta r_{ik} > \frac{3}{4} + 3r_{jk}h_k$$

$$\frac{\partial r_{ij}{}^{*}}{\partial h_k} = -3 + 4\left(h_i - 3h_j - 1\right)r_{ik} + 4(h_j - 3h_i - 1)r_{jk}$$

$$> 0 \; if \text{ if } 4\left(h_i - 3h_j - 1\right)r_{ik} + 4(h_j - 3h_i - 1)r_{jk} > 3$$

We note that if both portfolios in the bilateral licensing agreement are equal, then an increase in the portfolio of an outside firm causes the royalty value to shrink. On the other hand, if the portfolios of both participating firms are relatively unequal then an increase in the value of an outside firm's patent portfolio actually causes the royalty to increase as well.

We also note that the effects of a change in any portfolio size on optimal royalty are actually ambiguous. Therefore, the change of any individual portfolio size creates ambiguous changes in quantity, price, industry profit and consumer surplus.

Conclusion

The Effects of Cross-Licensing

Whenever cross-licensing is introduced, industry profits rise. When considering different levels of scope of the cross-licensing agreement, the greater the scope (more firms allowed into an agreement), the greater the profits. This is due to the firm's ability to leverage more monopoly power by using their influence on the royalty rate to artificially inflate their own costs. When analyzing the impact on consumer surplus the results are less clear. Take as an example the consumer surplus equations of cases Two and Three (Duopoly with Cross-Licensing and Duopoly without Cross-Licensing).

With Cross-Licensing

$$CS = \frac{(\alpha - C + H)^2}{8\beta}$$

Without Cross-Licensing

$$CS = \frac{(2\alpha - 2C + H)^2}{32\beta(\beta - 1)^2}$$

Cross-Licensing vs. No Cross-Licensing

$$\frac{\alpha - C + H}{\sqrt{8}} \; vs \; \frac{2\alpha - 2C + H}{\sqrt{32}(\beta - 1)}$$

We can see that which result is larger depends largely on the slope of the demand function β and the actual value of α. There are conflicting forces at work. While the consolidation of technology allows for cheaper prices, it also means that firms more closely resemble the case of a monopolist.

Agreements that Span the Entire Market

Whenever a cross-licensing agreement spans the whole market what we see is that the end result is identical to monopoly. In these cases (Case One, Case Three and Case Five), we find that the results for price, consumer surplus and quantity produced are identical. Firms' costs are artificially inflated so that firms in the multilateral cross-licensing agreement can reap greater upstream profits. Industry profits will then rise due to the upstream profit gains until prices are monopoly prices. This is essentially the same as collusion in the second stage and should be seen as an undesirable outcome for antirust agencies in isolation. The choice made by the policy maker is to evaluate if the benefit of diffusing technology and therefore reducing prices is a net benefit to society.

We also note that the reason why firm profits differ from monopoly profits is that in the second stage firms are still competing a la Cournot. Despite the presence of monopoly prices, industry profits take a different shape depending on the number of active firms, since the restriction of non-drastic technology prevents changes in the number of active firms. Additionally, we discuss below the impact that differences in the portfolio size of firms has.

Upstream vs Downstream Profit

For the cases where we deal with cross-licensing, firms make both upstream and downstream profit. Upstream profit is the profit that firms make via royalty payments paid to them subtracted by the royalty payments they pay to other firms. In all cross-licensing agreements we see that upstream and downstream profits can be negatively or positively affected by changes in royalty value. The important thing to note is that effects (upstream and downstream) always have the opposite sign as the other one.

Distribution of Technology

When we analyze the distribution of technology, we see two types of classifications that matter. Firstly, when no cross-licensing exists, as the asymmetry or inequality of patent portfolio size increases so too do industry profits. As we give one firm more monopoly power, the market more greatly resembles a monopoly. The other classification of cases we consider is that in which we allow cross-licensing. Regarding individual firm profits, we find that as a firm's patent portfolio rises so too does its individual profit, however the joint profits of an agreement may not. It is largely dependent on whether or not a change in a firm's individual patent portfolio would increase or decrease inequality in technology. For this reason, it is unclear if increasing total market technology would increase or decrease industry profits, as it matters in which way the technology is distributed.

Regarding consumer surplus, it appears to be invariant to the distribution of patents regardless of whether cross-licensing is permitted or not. We can see this as only the aggregate variable H and not its individual components are present in the consumer surplus equation.

We note that many of these results that rely on the aggregate quantity become ambiguous when the limit of cross-licensing scope is less than the total amount in the market. As seen in Case Six as the results become more complicated so too does the ambiguity of these changes.

Future Research

As mentioned in the literature review section, the classification of a patent can play a huge role in the results of the model. Extensions of this model could involve introducing patent substitutes and complements, or even firm compatibility. In addition, patentees could be extended to also involve product differentiation which would affect both firm choice and be related to the complementarity of patents. In a more complicated model the usage of value-adding patents could be used, or even a mixture of value-adding and cost-reducing patents could be modelled.

One of the other extensions that could be done, is considering the problem using Bertrand instead of Cournot competition. Using Bertrand competition instead would allow also to consider using the value-added method over the cost-reducing one used in this paper, creating a different interpretation.

Table 2: Summary of the Six Cases

Case 1	**Equation**	$\frac{\partial(.)}{\partial H}$	$\frac{\partial(.)}{\partial h_i}$
Q_m	$\frac{\alpha + H - C}{2\beta}$	+	+
CS	$\frac{(\alpha + H - C)^2}{8\beta}$	+	+
π_m	$\frac{(\alpha + H - C)^2}{4\beta}$	+	+
Case 2			
q_i	$\frac{\alpha - \beta q_j - C + h_i}{2\beta}$		+
Q	$\frac{2\alpha - 2C + H}{4\beta - 4\beta^2}$	+	+
CS	$\frac{(2\alpha - 2C + H)^2}{32\beta(\beta - 1)^2}$	+	+
π_i	$\left(\frac{(2\alpha + 2C - H)(\alpha - C - h_j + 2h_i)}{16\beta - 32\beta^2 + 16\beta^3}\right) + \left(\frac{C(C - \alpha + h_j - 3h_i) + h_i(\alpha - h_j + 2h_i)}{4\beta - 4\beta^2}\right)$	$\pm$	+
$\pi_i + \pi_j$	$\left(\frac{(2\alpha + 2C - H)(2\alpha - 2C + H)}{16\beta(\beta - 1)^2}\right) + \left(\frac{C(2C - 2\alpha - 2H) + h_i(\alpha - h_j + 2h_i) + h_j(\alpha - h_i + 2h_j)}{4\beta(1 - \beta)}\right)$	+	+
Case 3			
q_i	$\frac{(h_i)\frac{\alpha - C + H}{(H)}}{2\beta}$	+	+
Q	$\frac{\alpha - C + H}{2\beta}$	+	+
CS	$\frac{(\alpha - C + H)^2}{8\beta}$	+	+

π_i	$\frac{(h_i)(\alpha + C - H) - (C - H)(2h_i)}{4\beta}$	±	+
$\pi_i + \pi_j$	$\frac{(H)(\alpha + C - H) - 2(C - H)(H)}{4\beta}$	+	+
Case 4			
q_i	$\frac{\alpha - C - h_k - h_j + 3h_i}{4\beta}$	±	+
Q	$\frac{3\alpha - 3C + H}{4\beta}$	+	+
CS	$\frac{(3\alpha - 3C + H)(3\alpha - 3C + H)}{32\beta}$	+	+
π_i	$\frac{\alpha - C - h_k - h_j + 3h_i}{4\beta}\left(\frac{\alpha + 3C - H}{4} - C + h_i\right)$	±	+
$\pi_i + \pi_j + \pi_k$	$\left(\frac{(3\alpha - 3C + H)(\alpha - C - H)}{16\beta}\right) + h_i\left(\frac{\alpha - C - h_k - h_j + 3h_i}{4\beta}\right) + h_j\left(\frac{\alpha - C - h_k - h_i + 3h_j}{4\beta}\right) + h_k\left(\frac{\alpha - C - h_i - h_j + 3h_k}{4\beta}\right)$	±	±
Case 5			
q_i	$\frac{\alpha - \beta q_j - \beta q_k - C + H - rh_j - rh_k}{2\beta}$	±	+
Q	$\frac{(\alpha - C + H)}{(\beta + 1)}$	+	+
CS	$\beta\frac{(\alpha - C + H)^2}{2(\beta + 1)^2}$	+	+
π_i	$Pq_i - (C - H + rh_j + rh_k)q_i + rh_i q_j + rh_i q_k$	±	+
$\pi_i + \pi_j + \pi_k$	$\frac{(\alpha - C + H)^2}{(\beta + 1)^2}$	+	+
Case 6			
q_i	$\frac{\alpha - C + H + r_{ij}h_i + r_{jk}h_k - 3r_{ij}h_j - 3r_{ik}h_k + r_{ik}h_i + r_{jk}h_j}{4\beta}$	±	+

Q	$\frac{3\alpha - 3C + 3H - r_{ij}h_i - r_{jk}h_k - r_{ij}h_j - r_{ik}h_k - r_{ik}h_i - r_{jk}h_j}{4\beta}$	$\pm$	$\pm$
CS	----	+	+
π_i	$Pq_i - (C - H + r_{ij}h_j + r_{ik}h_k)q_i + r_{ij}h_iq_j + r_{ik}h_iq_k$	$\pm$	+
$\pi_i + \pi_j + \pi_k$	----	$\pm$	$\pm$

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