# Geometric Realizations of the Basic Representation of the Affine General Linear Lie Algebra

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# Abstract

The realizations of the basic representation of $\widehat{\mathfrak{gl}}_r$ are well-known to be parametrized by partitions of $r$ and have an explicit description in terms of vertex operators on the bosonic/fermionic Fock space. In this thesis, we give a geometric interpretation of these realizations in terms of geometric operators acting on the equivariant cohomology of certain Nakajima quiver varieties.

# Résumé

Il est bien connu que les réalizations de la représentation de base de $\widehat{\mathfrak{gl}}_r$ sont paramétrisés par les partitions de $r$ et que chacune de ces réalization possède une description explicite en termes d'opérateurs de sommet qui agissent sur l'espace de Fock bosonique et fermionique. Dans cette thèse, nous donnons une interprétation géometrique de ces réalizations en termes d'opérateurs géometriques qui agissent sur la cohomologie équivariante des variétées de carquois de Nakajima.

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# Contents

# Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra and denote the corresponding untwisted affine Lie algebra by $\widehat{\mathfrak{g}}$. The basic representation of $\widehat{\mathfrak{g}}$, which we denote by $V_{\text{basic}} = V_{\text{basic}}(\widehat{\mathfrak{g}})$, is the irreducible highest weight representation whose highest weight is the fundamental weight corresponding to the additional node of the affine Dynkin diagram (compared to the corresponding finite Dynkin diagram). The basic representation is so-named since it is, in a sense, the simplest nontrivial representation of $\widehat{\mathfrak{g}}$. In the late 70's and early 80's mathematicians began constructing the first explicit realizations of $V_{\text{basic}}$. The first such realization was given by Lepowsky and Wilson in [14] for $V_{\text{basic}}(\widehat{\mathfrak{sl}_2})$. Their construction was later generalized to arbitrary simply-laced affine Lie algebras and twisted affine Lie algebras in [10], and this construction became known as the *principal* realization of $V_{\text{basic}}$. However, Frenkel and Kac in [6], and Segal in [25], gave an entirely different realization of $V_{\text{basic}}$; this construction was referred to as the *homogeneous* realization. While the principal and homogeneous realizations seemed completely unrelated, it was discovered by Kac and Peterson in [12], and by Lepowsky in [13], that the two realizations depend implicitly on the choice of a so-called maximal *Heisenberg* subalgebra of $\widehat{\mathfrak{g}}$. Indeed, one can associate a realization of $V_{\text{basic}}$ to each maximal Heisenberg subalgebra of $\widehat{\mathfrak{g}}$.

In this thesis, we will focus on the case where $\mathfrak{g} = \mathfrak{gl}_r$. While $\mathfrak{gl}_r$ is not semisimple, it is a one-dimensional central extension of the semisimple Lie algebra $\mathfrak{sl}_r$, and thus has a similar representation theory. Up to conjugacy under the adjoint action of the Kac-

Moody group, the maximal Heisenberg subalgebras of an affine Lie algebra are known to be parametrized by the conjugacy classes of the Weyl group of the corresponding finite-dimensional Lie algebra (see [12, Proposition of Section 9]). The Weyl group of $\mathfrak{gl}_r$ is the symmetric group on $r$ elements, $S_r$, and the conjugacy classes of $S_r$ are in one-to-one correspondence with *partitions* of $r$, i.e. $s$-tuples $(r_1, \ldots, r_s) \in (\mathbb{N}^+)^s$ such that

$$r = r_1 + \cdots + r_s, \quad \text{and} \quad r_1 \leq \cdots \leq r_s.$$

Thus, there exists a realization of $V_{\text{basic}}(\widehat{\mathfrak{gl}}_r)$ for each partition of $r$, the principal and homogeneous realizations corresponding to the two extreme partitions $(r)$ and $(1, \ldots, 1)$, respectively. The realizations for every partition of $r$ were described by ten Kroode and van de Leur in [26] using vertex operators acting on bosonic Fock space (a representation of the Heisenberg algebra) and fermionic Fock space (a representation of the Clifford algebra). More precisely, for each partition of $r$, there exists a precise vector space isomorphism between bosonic Fock space and fermionic Fock space (known as the boson-fermion correspondence), and thus the Heisenberg and Clifford algebras may be thought of as operators acting on a common space. The construction in [26] defines a representation of $\widehat{\mathfrak{gl}}_r$ on bosonic/fermionic Fock space in terms of vertex operators (i.e. formal power series) of Heisenberg and Clifford algebra operators. The so-called "zero-charge" subspace of bosonic/fermionic Fock space is then shown to be isomorphic, as a $\widehat{\mathfrak{gl}}_r$-representation, to $V_{\text{basic}}$.

In this thesis, we give a geometric interpretation of these algebraic realizations of $V_{\text{basic}}(\widehat{\mathfrak{gl}}_r)$. Our general strategy is as follows. We fix a partition $(r_1, \ldots, r_s)$ of $r$ and consider the moduli space of framed torsion-free sheaves of rank $r$ and second Chern class $n$, $\mathcal{M}(s, n)$. In [15], Licata and Savage showed that, under a suitable torus action, the localized equivariant cohomology of (a disjoint union of infinitely-many copies) of $\mathcal{M}(s, n)$ provides a suitable geometric analogue of bosonic/fermionic Fock space. This is accomplished by defining an action of the Heisenberg and Clifford algebras on this cohomology in terms of the top Chern classes of certain equivariant vector bundles on $\mathcal{M}(s,n)$. The construction given in [15] naturally corresponds to the homogeneous realization in [26], and thus we generalize their construction to an arbitrary partition. With this framework in place, we define a new set of operators using vector bundles on certain subvarieties of $\mathcal{M}(s,n)$. We then show that these operators may be expressed as vertex operators of our "geometric" Heisenberg and Clifford algebra operators, and that the formulas we obtain exactly match those found in the algebraic realizations of the representation of $\widehat{\mathfrak{gl}}_r$ on $V_{\text{basic}}$, thus giving us geometric realizations of $V_{\text{basic}}$.

The paper is organized into 5 chapters. In Chapter 1, we review the Heisenberg and Clifford algebra representations on bosonic and fermionic Fock space. We also briefly summarize the various algebraic realizations of $V_{\text{basic}}$ found in [26]. In Chapter 2, we recall some of the basic concepts from algebraic geometry, especially geometric invariant theory, that we will use in subsequent chapters. In Chapter 3, we will discuss our main geometric object of interest: Nakajima quiver varieties (of which the aforementioned moduli space $\mathcal{M}(s,n)$ is a special case). Chapter 4 will describe our method of constructing geometric operators on the localized equivariant cohomology of quiver varieties from equivariant vector bundles. Finally, in Chapter 5, we define our geometric analogues of the action of the Heisenberg algebra, Clifford algebra, and $\widehat{\mathfrak{gl}}_r$ on bosonic and fermionic Fock space. We also present our main theorem (Theorem 5.21), which is a geometric analogue of Theorem 1.20 (the main theorem of [26]).

# Chapter 1

# The Basic Representation of the Affine General Linear Lie Algebra

In this first chapter, we will summarize the known algebraic realizations of the basic representation of $\widehat{\mathfrak{gl}}_r$. In particular, the inequivalent realizations are parametrized by the different partitions of $r$. For a more in-depth treatment of this topic, the reader is encouraged to see [26] or [11]. The goal in the subsequent chapters will be to give a geometric construction of the representations presented here.

We begin with a description of the $s$-coloured oscillator algebra and $s$-coloured Clifford algebra, along with the associated $s$-coloured bosonic and fermionic Fock spaces.

**Definition 1.1.** ($s$-coloured oscillator algebra) For $s \in \mathbb{N}^+$, the *$s$-coloured oscillator algebra*, $\mathfrak{s}$, is the complex Lie algebra

$$\mathfrak{s} := \bigoplus_{\ell=1}^{s} \left( \bigoplus_{n \in \mathbb{Z}} \mathbb{C} P_\ell(n) \right) \oplus \mathbb{C}c,$$

with the Lie bracket determined by

$$[\mathfrak{s}, P_\ell(0)] = 0, \quad [P_\ell(n), P_k(m)] = \frac{1}{n} \delta_{\ell,k} \delta_{n+m,0} c, \ \ n \neq 0,$$

for all $\ell, k = 1, \ldots, s$ and $m, n \in \mathbb{Z}$.

**Remark 1.2.** Our presentation of the $s$-coloured oscillator algebra differs slightly from the one sometimes found elsewhere in the literature. In particular, in [26, Section 1], $\mathfrak{s}$ is defined as the complex Lie algebra with basis $\{\alpha_\ell(n)\}_{n\in\mathbb{Z},\ell=1,\dots,s} \cup \{c\}$ and Lie bracket given by

$$[\alpha_\ell(n), \alpha_k(m)] = n\delta_{\ell,k}\delta_{n+m,0}c.$$

The connection between the two presentations is made by setting

$$P_\ell(n) = \frac{1}{|n|}\alpha_\ell(n), \quad n \neq 0, \qquad \text{and} \qquad P_\ell(0) = \alpha_\ell(0).$$

We favour the presentation given in Definition 1.1 since this one turns out to be more natural from a geometric point of view.

**Definition 1.3.** ($s$-coloured Heisenberg algebra) The subalgebra

$$\mathfrak{s}_0 = \bigoplus_{\ell=1}^{s} \left( \bigoplus_{n\in\mathbb{Z}-\{0\}} \mathbb{C}P_\ell(n) \right) \oplus \mathbb{C}c,$$

of $\mathfrak{s}$ is the so-called *$s$-coloured Heisenberg algebra* and, as we will see later, it serves as the main ingredient in the various realizations of the basic representation of $\widehat{\mathfrak{gl}}_r$.

Let $\Lambda \subseteq \mathbb{C}[\![t_1, t_2, \dots]\!]$ denote the ring of symmetric functions in infinitely many variables. It is well-known that

$$\Lambda = \mathbb{C}[p_1, p_2, \dots],$$

where $p_n$ is the $n$-th power sum

$$p_n = \sum_{i=1}^{\infty} t_i^n.$$

We define the *$s$-coloured bosonic Fock space* to be the space

$$\mathbb{B} := B^{\otimes s}, \quad \text{where } B := \Lambda \otimes_{\mathbb{C}} \mathbb{C}[q, q^{-1}].$$

We define a $\mathbb{Z}$-grading on $B$ given by

$$B = \bigoplus_{c \in \mathbb{Z}} B_c, \quad B_c := \Lambda \otimes \mathbb{C}q^c.$$

This induces a $\mathbb{Z}^s$-grading on $\mathbb{B}$ given by

$$\mathbb{B} = \bigoplus_{\mathbf{c} \in \mathbb{Z}^s} \mathbb{B}_{\mathbf{c}}, \quad \mathbb{B}_{\mathbf{c}} := B_{\mathbf{c}_1} \otimes \cdots \otimes B_{\mathbf{c}_s}.$$

For $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_s) \in \mathbb{Z}^s$, we use the notation $|\mathbf{c}| = \mathbf{c}_1 + \cdots + \mathbf{c}_s$. We then have a $\mathbb{Z}$-grading on $\mathbb{B}$ given by

$$\mathbb{B} = \bigoplus_{c \in \mathbb{Z}} \mathbb{B}(c), \quad \mathbb{B}(c) = \bigoplus_{|\mathbf{c}|=c} \mathbb{B}_{\mathbf{c}}.$$

One can easily verify that the mapping

$$P_\ell(n) \mapsto 1^{\otimes \ell-1} \otimes \frac{\partial}{\partial p_n} \otimes 1^{\otimes s-\ell}, \quad n > 0,$$

$$P_\ell(-n) \mapsto 1^{\otimes \ell-1} \otimes \frac{1}{n} p_n \otimes 1^{\otimes s-\ell}, \quad n > 0,$$

$$P_\ell(0) \mapsto 1^{\otimes \ell-1} \otimes q\frac{\partial}{\partial q} \otimes 1^{\otimes s-\ell}, \quad c \mapsto 1,$$

defines an irreducible representation of $\mathfrak{s}$ on $\mathbb{B}$.

**Remark 1.4.** Again, our definition of $s$-coloured bosonic Fock space differs slightly from the definition sometimes found elsewhere in the literature. In particular, $s$-coloured bosonic Fock space is often only considered as a representation of the $s$-coloured Heisenberg algebra (rather than of the full oscillator algebra). The $s$-coloured Heisenberg subalgebra acts trivially on the $\mathbb{C}[q, q^{-1}]$ factors of bosonic Fock space, and so, as an $\mathfrak{s}_0$-module, $\mathbb{B}$ is isomorphic to a direct sum of infintely many copies of $\Lambda^{\otimes s}$. In [26, Section 1], bosonic Fock space is defined as

$$\mathbb{C}[x_1, x_2, \ldots]^{\otimes s},$$

with the action of the Heisenberg algebra given by

$$\alpha_\ell(n) \mapsto 1^{\otimes \ell-1} \otimes \frac{\partial}{\partial x_n} \otimes 1^{\otimes s-\ell}, \quad n > 0,$$

$$\alpha_\ell(-n) \mapsto 1^{\otimes \ell-1} \otimes n x_n \otimes 1^{\otimes s-\ell}, \quad n > 0,$$

$$c \mapsto 1.$$

However, one has an isomorphism (a priori, only of rings) $\Lambda \to \mathbb{C}[x_1, x_2, \dots]$, determined by the mapping $p_n \mapsto n x_n$. This in turn induces an isomorphism $\Lambda^{\otimes s} \to \mathbb{C}[x_1, x_2, \dots]^{\otimes s}$. One easily checks that the diagram

$$\begin{array}{ccc} \Lambda^{\otimes s} & \longrightarrow & \mathbb{C}[x_1, x_2, \dots]^{\otimes s} \\ {\scriptstyle |n| P_\ell(n)} \downarrow & & \downarrow {\scriptstyle \alpha_\ell(n)} \\ \Lambda^{\otimes s} & \longrightarrow & \mathbb{C}[x_1, x_2, \dots]^{\otimes s} \end{array}$$

commutes for all $\ell = 1, \dots, s$ and $n \in \mathbb{Z} - \{0\}$ (this justifies the statement from Remark 1.2 that $P_\ell(n) = \frac{1}{|n|}\alpha_\ell(n)$). Thus, the map $\Lambda^{\otimes s} \to \mathbb{C}[x_1, x_2, \dots]^{\otimes s}$ is in fact an isomorphism of $s$-coloured Heisenberg algebra representations.

**Remark 1.5.** Notice that the sets $\{1, \dots, s\} \times \mathbb{N}^+$ and $\mathbb{N}^+$ are countable. Pick a bijection $\varphi : \{1, \dots, s\} \times \mathbb{N}^+ \to \mathbb{N}^+$. Then the mapping

$$P_\ell(k) \mapsto P(\varphi(\ell, k)), \quad P_\ell(-k) \mapsto \frac{\varphi(\ell, k)}{k} P(-\varphi(\ell, k)), \quad c \mapsto c,$$

for all $\ell \in \{1, \dots, s\}$ and $k \in \mathbb{N}^+$, is an isomorphism from the $s$-coloured Heisenberg algebra to the 1-coloured Heisenberg algebra. Thus, we will often use the term "Heisenberg algebra" without specifying the number of colours. Moreover, one can show (see, for instance, [11, Lemma 14.4(a)]) that $\Lambda^{\otimes s}$ and $\Lambda$ are isomorphic as Heisenberg algebra modules. The reason for the $s$-coloured presentation of the Heisenberg algebra will become clear when we describe the various realizations of the basic representation of $\widehat{\mathfrak{gl}}_r$.

**Definition 1.6.** ($s$-coloured Clifford algebra) The *$s$-coloured Clifford algebra*, Cl, is the complex associative algebra generated by elements $\psi_\ell(i), \psi_\ell^*(i)$, $\ell = 1, \dots, s$, $i \in \mathbb{Z}$, and relations

$$\{\psi_\ell(i), \psi_k^*(j)\} = \delta_{ij}\delta_{\ell k}, \quad \{\psi_\ell(i), \psi_k(j)\} = \{\psi_\ell^*(i), \psi_k^*(j)\} = 0.$$

where $\{x, y\} = xy + yx$.

A *semi-infinite monomial* is an expression of the form

$$I = i_1 \wedge i_2 \wedge i_3 \wedge \cdots ,$$

where the $i_n$ are integers such that

$$i_1 > i_2 > i_3 > \dots, \quad i_{n+1} = i_n - 1, \text{ for } n \gg 0.$$

The *charge* of a semi-infinite monomial $I$ is the integer $c = c(I)$ such that

$$i_n = c - n + 1, \text{ for all } n \gg 0.$$

The *energy* of a semi-infinite monomial is

$$\sum_{n \in \mathbb{N}^+} i_n - (c - n + 1).$$

Let $F$ be the complex vector space with basis the set of all semi-infinite monomials. The charge induces a grading on $F$:

$$F = \bigoplus_{c \in \mathbb{Z}} F_c,$$

where $F_c$ is the subspace with basis the set of semi-infinite monomials of charge $c$. Define *wedging* and *contracting* operators $\psi(i)$ and $\psi^*(i)$, $i \in \mathbb{Z}$, on $F$ by

$$\psi(i)(i_1 \wedge i_2 \wedge \cdots) = \begin{cases} (-1)^k (i_1 \wedge \cdots \wedge i_k \wedge i \wedge i_{k+1} \wedge \cdots), & \text{if } i_k > i > i_{k+1}, \\ 0, & \text{if } i = i_n, \text{ for some } n, \end{cases}$$

$$\psi^*(i)(i_1 \wedge i_2 \wedge \cdots) = \begin{cases} (-1)^{k-1}(i_1 \wedge \cdots \wedge i_{k-1} \wedge i_{k+1} \wedge \cdots), & \text{if } i = i_k, \\ 0, & \text{if } i \neq i_n, \text{ for all } n. \end{cases}$$

It is easy to see that, when they do not act as zero, the wedging and contracting operators raise and lower the charge of a semi-infinite monomial by 1, and so

$$\psi(i) : F_c \to F_{c+1}, \quad \psi^*(i) : F_c \to F_{c-1},$$

for all $i, c \in \mathbb{Z}$.

We define *s-coloured fermionic Fock space* to be

$$\mathbb{F} := F^{\otimes s}.$$

By our decomposition of $F$ in terms of the charge, we have a $\mathbb{Z}^s$-grading

$$\mathbb{F} = \bigoplus_{\mathbf{c} \in \mathbb{Z}^s} \mathbb{F}_{\mathbf{c}}, \quad \mathbb{F}_{\mathbf{c}} := F_{\mathbf{c}_1} \otimes \cdots \otimes F_{\mathbf{c}_s}.$$

This induces a $\mathbb{Z}$-grading on $\mathbb{F}$ given by

$$\mathbb{F} = \bigoplus_{c \in \mathbb{Z}} \mathbb{F}(c), \quad \mathbb{F}(c) := \bigoplus_{|\mathbf{c}|=c} \mathbb{F}_{\mathbf{c}}.$$

One can show that we have a representation of Cl on $\mathbb{F}$ given by:

$$\psi_\ell(i)|_{\mathbb{F}_{\mathbf{c}}} \mapsto (-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{\ell-1}} \left(1^{\otimes \ell-1} \otimes \psi(i) \otimes 1^{\otimes s-\ell}\right),$$

$$\psi_\ell^*(i)|_{\mathbb{F}_{\mathbf{c}}} \mapsto (-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{\ell-1}} \left(1^{\otimes \ell-1} \otimes \psi^*(i) \otimes 1^{\otimes s-\ell}\right).$$

One can also show that $\mathbb{F}$ is a faithful, irreducible representation of Cl. In fact, $\mathbb{F}$ is generated by the so-called *vacuum vector*, $|0\rangle^{\otimes s}$, where

$$|0\rangle = 0 \wedge -1 \wedge -2 \wedge \cdots.$$

Note that $\mathbb{F}$ is referred to as the *spin module* in [26].

**Remark 1.7.** We have used here the convention that Clifford algebra generators of different colours anti-commute. However, the Clifford algebra is sometimes defined by letting generators of different colours commute (see for example [15, Section 1.2]). In this case it is necessary to modify the action of Cl on $\mathbb{F}$ to

$$\psi_\ell(i)|_{\mathbb{F}_\mathbf{c}} \mapsto 1^{\otimes \ell-1} \otimes \psi(i) \otimes 1^{\otimes s-\ell},$$

$$\psi^*_\ell(i)|_{\mathbb{F}_\mathbf{c}} \mapsto 1^{\otimes \ell-1} \otimes \psi^*(i) \otimes 1^{\otimes s-\ell}.$$

**Remark 1.8.** By an argument completely analogous to Remark 1.5, the $s$-coloured Clifford algebra is isomorphic to the 1-coloured algebra, and, by identifying the two algebras, $s$-coloured fermionic Fock space is isomorphic to 1-coloured fermionic Fock space. Thus, we will often refer to the Clifford algebra and fermionic Fock space without specifying the number of colours.

In the sequel, it will be useful to think of fermionic Fock space not only in terms of semi-infinite monomials, but also in terms of *Young diagrams*. Recall that a Young diagram is a finite collection of boxes arranged in rows and columns such that the number of boxes in the $i$-th row is greater than or equal to the number of boxes in the $(i+1)$-st row. There are different conventions regarding the way to draw Young diagrams, but we will choose the English notation. That is, our rows will be left-justified and subsequent rows are placed underneath the previous one, as illustrated in Figure 1.1. We say that a box is in the $(i,j)$-th *position* if it is in the $i$-th row and $j$-th column of the diagram. For example, in Figure 1.1, the box labelled with a "$\star$" is in the $(3,2)$-th position. We may thus view a Young diagram $\lambda$ as being a subset of $(\mathbb{N}^+)^2$, where $(i,j) \in \lambda$ if and only if $\lambda$ has a box in the $(i,j)$-th position. For any Young diagram $\lambda$ and any $k \in \mathbb{N}^+$, let $\lambda_k$ denote the number of boxes in its $k$-th row ($\lambda_k = 0$ if there are no boxes in the $k$-th row). For any $(i,j) \in (\mathbb{N}^+)^2$, define the *arm* and *leg length* of $(i,j)$ to be

$$a_\lambda(i,j) := \lambda_i - j \quad \text{and} \quad \ell_\lambda(i,j) := \max\{i' \in \mathbb{N}^+ \mid (i',j) \in \lambda\} - i,$$

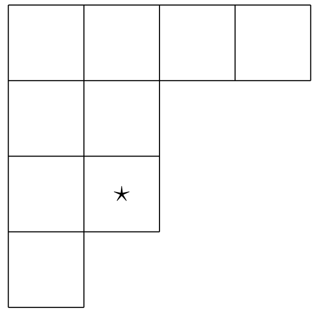


Figure 1.1: Young diagram

respectively. Intuitively, for each $(i,j) \in \lambda$, $a_\lambda(i,j)$ and $\ell_\lambda(i,j)$ count the number of boxes to the right of and below $(i,j)$, respectively. For two Young diagrams $\lambda$ and $\mu$, we define the *relative hook length* of $(i,j)$ to be

$$h_{\lambda,\mu}(i,j) := a_\lambda(i,j) + \ell_\mu(i,j) + 1. \tag{1.1}$$

If $\lambda = \mu$, we will simply write $h_\lambda(i,j) := h_{\lambda,\lambda}(i,j)$. Finally, we define the *residue* of box in the $(i,j)$-th position to be $j - i$.

Clearly, every weakly decreasing sequence of non-negative integers

$$(\lambda_1, \lambda_2, \lambda_3, \dots),$$

such that $\lambda_k = 0$ for $k \gg 0$ uniquely defines a Young diagram. Thus, one has a bijection between the set of semi-infinite monomials of a fixed charge and the set of Young diagrams. Indeed, suppose

$$I = i_1 \wedge i_2 \wedge i_3 \wedge \cdots,$$

is a semi-infinite monomial of charge $c$. Define $\lambda(I)$ to be the Young diagram determined by

$$\lambda(I)_k = i_k - c + k - 1. \tag{1.2}$$

**Lemma 1.9.** *The mapping $I \mapsto \lambda(I)$ is a bijection from the set of semi-infinite monomials of charge $c$ to the set of Young diagrams. Moreover, if $I$ is a semi-infinite monomial of energy $n$, then $\lambda(I)$ consists of $n$ boxes.*

Hence, by Lemma 1.9, we may think of $F_c$ as the complex vector space with basis the set of all Young diagrams, thus giving us a Young diagram interpretation of fermionic Fock space.

The description of the realizations of the basic representation of $\widehat{\mathfrak{gl}}_r$ will rely on the precise relationship between bosonic and fermionic Fock spaces known as the *boson-fermion correspondence*. It is well-known that the set of *Schur functions* $\{s_\lambda\}$, where $\lambda$ runs over all Young diagrams, is a basis of $\Lambda$ (see statement (3.3) of [17]). We then have a vector space isomorphism

$$\begin{aligned} B_c &\to F_c, \\ s_\lambda \otimes q^c &\mapsto \lambda, \end{aligned} \tag{1.3}$$

for each $c \in \mathbb{Z}$. This induces isomorphisms $\mathbb{B}(c) \xrightarrow{\cong} \mathbb{F}(c)$ and $\mathbb{B} \xrightarrow{\cong} \mathbb{F}$.

We turn our attention now to the representation theory of Lie algebras. Let $\mathfrak{g}$ be a matrix Lie algebra and let $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the loop algebra on $\mathfrak{g}$. The affine Lie algebra of $\mathfrak{g}$ is

$$\widehat{\mathfrak{g}} := \widetilde{\mathfrak{g}} \oplus \mathbb{C}c,$$

with Lie bracket given by

$$[\widehat{\mathfrak{g}}, c] = 0, \quad [x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n\delta_{n+m,0} \operatorname{tr}(xy)c. \tag{1.4}$$

Here we follow the conventions in [26, Equation (4.2.5)] (note that the two-cocycle $\mu$ in [26] simplifies to a multiple of the trace map when restricted to elements of the form $x \otimes t^n$).

**Remark 1.10.** In the literature, the Lie bracket on $\widehat{\mathfrak{g}}$ is usually defined as

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n\delta_{n+m,0}\langle x|y\rangle c,$$

where $\langle x|y\rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$ is the Killing form on $\mathfrak{g}$. For each of the classical Lie algebras, the Killing form is just a (nonzero) multiple of the trace map. In particular, for $\mathfrak{sl}_r$,

$$\langle x|y\rangle = 2r\,\mathrm{tr}(xy).$$

Thus, up to a scaling factor on $c$, there is no difference between using the Killing form and the trace map. Meanwhile, on $\mathfrak{gl}_r$, the Killing form is degenerate since

$$\langle I|x\rangle = 0,$$

for all $x \in \mathfrak{gl}_r$, whereas the trace map is nondegenerate, making it a more appropriate choice. This is our motivation for defining the Lie bracket on $\widehat{\mathfrak{g}}$ by (1.4).

**Remark 1.11.** The loop algebra $\widetilde{\mathfrak{g}}$ is sometimes defined as

$$\widetilde{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} e^{in\theta}\mathfrak{g},$$

where $\theta$ is a formal parameter. Of course, one has the obvious isomorphism of Lie algebras

$$\bigoplus_{n\in\mathbb{Z}} e^{in\theta}\mathfrak{g} \to \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}],$$
$$e^{in\theta}x \mapsto x\otimes t^n,$$

and so both definitions are equivalent. We will use this alternative description when it is more convenient.

Next, we describe the *basic representation*, $V_{\text{basic}} = V_{\text{basic}}(\widehat{\mathfrak{g}})$, of $\widehat{\mathfrak{g}}$ when $\mathfrak{g} = \mathfrak{sl}_r$ or $\mathfrak{gl}_r$, $r \in \mathbb{N}^+$ (although the description applies to other Lie algebras as well). Let $\{E_{i,j}\}_{1\le i,j\le r}$ be the standard basis of $\mathfrak{gl}_r$. That is $E_{i,j}$ is the $r\times r$ matrix with $(i,j)$-th entry 1 and zeros elsewhere. Recall that, for $r\ge 2$, $\widehat{\mathfrak{sl}}_r$ is generated by its *Chevalley generators* $E_i, F_i, H_i$, $i = 0,1,\ldots,r-1$, which are given by

$$E_i = E_{i,i+1}\otimes 1,\quad F_i = E_{i+1,i}\otimes 1,\quad H_i = (E_{i,i}-E_{i+1,i+1})\otimes 1,\quad i = 1,\ldots,r-1,$$

$$E_0 = E_{r,1}\otimes t,\quad F_0 = E_{1,r}\otimes t^{-1},\quad H_0 = ((E_{r,r}-E_{1,1})\otimes 1)+c.$$

The Chevalley generators of $\widehat{\mathfrak{sl}}_r$ satisfy the well-known *Kac-Moody relations* (for $\widehat{\mathfrak{sl}}_r$):

1. $[H_i, H_j] = 0$,
2. $[E_i, F_j] = \delta_{ij} H_i$,
3. $[H_i, E_j] = a_{ij} E_j$,
4. $[H_i, F_j] = -a_{ij} F_j$,
5. $(\mathrm{ad}\, E_i)^{(1-a_{ij})}(E_j) = 0, \quad \text{for } i \neq j$,
6. $(\mathrm{ad}\, F_i)^{(1-a_{ij})}(F_j) = 0, \quad \text{for } i \neq j$,

where

$$a_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -2, & \text{if } i \neq j, \end{cases} \quad \text{for } r = 2,$$

$$a_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } |i - j| = 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } r > 2.$$

Note that the $r \times r$ matrix $(a_{ij})$ is called the *generalized Cartan matrix* associated to $\widehat{\mathfrak{sl}}_r$. We define the *triangular decomposition* of $\widehat{\mathfrak{sl}}_r$:

$$\widehat{\mathfrak{sl}}_r = \widehat{\mathfrak{sl}}_r^{+} \oplus \mathfrak{h} \oplus \widehat{\mathfrak{sl}}_r^{-},$$

where $\widehat{\mathfrak{sl}}_r^{+}$ (resp. $\widehat{\mathfrak{sl}}_r^{-}$) is the subalgebra generated by the $E_i$ (resp. the $F_i$) and $\mathfrak{h}$ is the abelian subalgebra with basis $\{H_i\}$. The subalgebra $\mathfrak{h}$ is called a *Cartan subalgebra* of $\widehat{\mathfrak{sl}}_r$. Given $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{r-1}) \in \mathbb{Z}^r$, a *highest weight representation*, $V$, of $\mathfrak{sl}_r$ is a representation such that there exists a vector $v_0 \in V$ satisfying

$$E_i \cdot v_0 = 0, \quad \text{and} \quad H_i \cdot v_0 = \gamma_i v_0, \tag{1.5}$$

for all $i = 0, 1, \ldots, r-1$, and

$$U(\widehat{\mathfrak{sl}}_r) \cdot v_0 = V,$$

where $U(\widehat{\mathfrak{sl}}_r)$ is the universal enveloping algebra of $\widehat{\mathfrak{sl}}_r$. Note that, in terms of the triangular decomposition of $\widehat{\mathfrak{sl}}_r$, (1.5) is equivalent to

$$\widehat{\mathfrak{sl}}_r^+ \cdot v_0 = 0, \quad \text{and} \quad H_i \cdot v_0 = \gamma_i v_0.$$

The $r$-tuple $\gamma$ is called the *highest weight* of $V$ and the vector $v_0$ is called a *highest weight vector* of $V$. If $V$ is irreducible, one can show that any two highest weight vectors are proportional, thus, in this case, we will often abuse terminology and call $v_0$ *the* highest weight vector of $V$. Given a highest weight representation $V$ of highest weight $\gamma$, for each $\beta \in \mathbb{Z}^r$, define

$$V_\beta := \left\{ v \in V \;\middle|\; H_i \cdot v = \left( \gamma_i - \sum_{j=0}^{r-1} a_{ij}\beta_j \right) v \right\}.$$

Clearly, $v_0 \in V_{\mathbf{0}}$. Let $\mathbf{1}_i$ denote the $r$-tuple with 1 in its $i$-th entry and zeros elsewhere. It follows from the Kac-Moody relations that, restricted to $V_\beta$,

$$E_i, F_i : V_\beta \to V_{\beta \mp \mathbf{1}_i},$$

and so

$$V = \bigoplus_{\beta \in \mathbb{Z}^r} V_\beta. \tag{1.6}$$

We call the decomposition in (1.6) the *weight space decomposition* of $V$.

**Remark 1.12.** In our treatment of weights above, we are slightly abusing terminology. Given a semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, a *weight* is defined to be a linear map

$$\beta : \mathfrak{h} \to \mathbb{C},$$

and the associated weight space decomposition of a representation $V$ is

$$V = \bigoplus_\beta V_\beta, \quad V_\beta = \{ v \in V \mid h \cdot v = \beta(h)v \text{ for all } h \in \mathfrak{h} \}.$$

However, the Chevalley generators $H_0, H_1, \cdots H_{r-1}$ form a basis of our Cartan subalgebra $\mathfrak{h}$ and thus induce a basis of the vector space of weights. Our description merely identifies the weight with its coefficients with respect to this basis.

**Definition 1.13** (Basic representation of $\widehat{\mathfrak{sl}}_r$)**.** The *basic representation* of $\widehat{\mathfrak{sl}}_r$, which we denote $V_{\text{basic}}(\widehat{\mathfrak{sl}}_r)$, is the (unique) irreducible highest weight representation with highest weight $\gamma = (1, 0, \ldots, 0)$.

**Remark 1.14.** See [11, Proposition 9.3] for a proof that there is a unique irreducible highest weight representation of $\widehat{\mathfrak{sl}}_r$ with highest weight $\gamma = (1, 0, \ldots, 0)$.

There is another way to describe $V_{\text{basic}}(\widehat{\mathfrak{sl}}_r)$ that amounts to the same thing. Namely, one can define $V_{\text{basic}}(\widehat{\mathfrak{sl}}_r)$ to be the unique irreducible representation that admits a vector $v_0$ satisfying

$$(\mathfrak{sl}_r \otimes \mathbb{C}[t]) \cdot v_0 = 0, \quad \text{and} \quad c \cdot v_0 = v_0.$$

It is easy to check that this alternate description is equivalent to Definition 1.13 and that $v_0$ here plays the role of the highest weight vector. The advantage to this construction is that it naturally generalizes to $\widehat{\mathfrak{gl}}_r$.

**Definition 1.15** (Basic representation of $\widehat{\mathfrak{gl}}_r$)**.** The *basic representation* of $\widehat{\mathfrak{gl}}_r$, which we denote $V_{\text{basic}}(\widehat{\mathfrak{gl}}_r)$, is the (unique) irreducible representation that admits a vector $v_0$ (highest weight vector) such that

$$(\mathfrak{gl}_r \otimes \mathbb{C}[t]) \cdot v_0 = 0, \quad \text{and} \quad c \cdot v_0 = v_0.$$

From now on, unless otherwise specified, $V_{\text{basic}}$ will always mean $V_{\text{basic}}(\widehat{\mathfrak{gl}}_r)$.

**Remark 1.16.** The proof that $V_{\text{basic}}$ is unique is essentially the same as [11, Proposition 9.3].

**Remark 1.17.** There is a way to describe $V_{\text{basic}}$ that is more akin to Definition 1.13 by introducing a triangular decomposition of $\widehat{\mathfrak{gl}}_r$. Firstly, we observe that

$$\mathfrak{gl}_r = \mathfrak{sl}_r \oplus \mathbb{C}I,$$

and so

$$\widehat{\mathfrak{gl}}_r = \widehat{\mathfrak{sl}}_r \oplus \left(I \otimes \mathbb{C}[t, t^{-1}]\right).$$

Thus, we can define a triangular decomposition of $\widehat{\mathfrak{gl}}_r$ by extending the triangular decomposition of $\widehat{\mathfrak{sl}}_r$. Namely, we write

$$\widehat{\mathfrak{gl}}_r = \widehat{\mathfrak{gl}}_r^+ \oplus \mathfrak{h} \oplus \widehat{\mathfrak{gl}}_r^-,$$

where

$$\widehat{\mathfrak{gl}}_r^\pm := \widehat{\mathfrak{sl}}_r^\pm \oplus \left( \bigoplus_{k \in \mathbb{N}^+} \mathbb{C}(I \otimes t^{\pm k}) \right),$$

and $\mathfrak{h}$ is the Cartan subalgebra with basis $\{H_0, H_1, \ldots, H_{r-1}, (I \otimes 1)\}$. We can then define $V_{\text{basic}}$ to be the unique irreducible representation that admits a vector $v_0$ (highest weight vector) satisfying

$$\widehat{\mathfrak{gl}}_r^+ \cdot v_0 = 0, \quad H_i \cdot v_0 = \delta_{i,0} v_0, \quad (I \otimes 1) \cdot v_0 = 0.$$

Our next task is to describe the various vertex operator realizations of $V_{\text{basic}}$ as given in [26]. The motivation for these different realizations comes from the following proposition.

**Proposition 1.18.** *Let $V$ be a representation of the 1-coloured Heisenberg algebra such that $c$ acts as the identity and there exists an $N \in \mathbb{N}$ such that*

$$\alpha(n_1) \cdots \alpha(n_k) \cdot v = 0,$$

*for all $v \in V$ and $n_i \in \mathbb{N}^+$ such that $n_1 + \cdots + n_k > N$. Then $V$ is isomorphic to a direct sum of copies of $\Lambda$.*

**Proof:** This is [11, Lemma 14.4(b)] (where the fact that $c$ acts as the identity implies $a = 1$). ∎

Suppose that $\widehat{\mathfrak{g}}$ is an affine Lie algebra with underlying finite dimensional Lie algebra $\mathfrak{g}$ and $V$ is a representation of $\widehat{\mathfrak{g}}$. Furthermore, suppose that we have a (1-coloured) Heisenberg subalgebra $\mathfrak{s}_0 \subseteq \widehat{\mathfrak{g}}$ such that the restriction of $V$ to $\mathfrak{s}_0$ satisfies the hypotheses of Proposition 1.18. Then, by Proposition 1.18, one can, in principle, give a realization of $V$ completely in terms of (a direct sum of copies of) $\Lambda$. That is,

$$V \cong \bigoplus_{i \in \mathcal{I}} \Lambda,$$

as $\mathfrak{s}_0$-modules, for some indexing set $\mathcal{I}$. Define the *vacuum space*, $\Omega = \Omega(V)$ to be the subspace

$$\Omega := \left\{ v \in V \mid \alpha(k) \cdot v = 0 \text{ for all } k \in \mathbb{N}^+ \right\}.$$

It is an easy exercise to see that $\{1_i\}_{i \in \mathcal{I}}$ is a basis of $\Omega\left(\bigoplus_{i \in \mathcal{I}} \Lambda\right)$. Therefore,

$$V \cong \Omega(V) \otimes \Lambda,$$

as $\mathfrak{s}_0$-modules, where the action of $\mathfrak{s}_0$ on $\Omega(V) \otimes \Lambda$ is given by

$$\alpha(k) \cdot (v \otimes f) = v \otimes \alpha(k) \cdot f,$$

for all $k \in \mathbb{Z} - \{0\}$ and $v \otimes f \in \Omega(V) \otimes \Lambda$. Of course, in general, one has many Heisenberg subalgebras of $\widehat{\mathfrak{g}}$, and so we naturally have several realizations of $V$. When $\mathfrak{g}$ is semisimple, a complete classification of the Heisenberg subalgebras of $\widehat{\mathfrak{g}}$ is given in [12, Section 9]. We briefly summarize the classification below.

Let $W$ be the Weyl group of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and an element $w \in W$. Let $m$ be the order of $w$. Recall that there is an action of $W$ on $\mathfrak{h}$ induced by the action of $W$ on the root space of $\mathfrak{g}$. Write

$$\mathfrak{h} = \bigoplus_{k=0}^{m-1} \mathfrak{h}_k, \quad \mathfrak{h}_k := \{h \in \mathfrak{h} \mid w \cdot h = e^{-2\pi i k/m} h\}.$$

There exists an $x \in \mathfrak{g}$ such that $\mathrm{Ad}(\exp 2\pi i x)|_{\mathfrak{h}} = w$ and $[x, \mathfrak{h}_0] = 0$ (see, for instance, [8, Section 14.3] and [4, Theorem 2.5.5]). For each $h = (h_0, h_1, \ldots, h_{m-1}) \in \mathfrak{h}$ and $k \in \mathbb{Z}$, define $h(k)$ to be the loop

$$h(k) = e^{i\theta k/m} \,\mathrm{Ad}(\exp i\theta x) h_{\bar{k}} \in \widetilde{\mathfrak{g}}, \tag{1.7}$$

where $\bar{k}$ is the unique integer $0 \leq \bar{k} \leq m-1$ such that $\bar{k} \equiv k \mod m$. Then

$$\mathfrak{s}_0^w := \left( \bigoplus_{k \in \mathbb{Z}} \mathfrak{h}(k) \right) \oplus \mathbb{C}c, \tag{1.8}$$

is a maximal Heisenberg subalgebra of $\widehat{\mathfrak{g}}$. Let $W'$ be a set of representatives of the conjugacy classes of $W$. Then the subalgebras $\mathfrak{s}_0^w$, $w \in W'$, form a complete nonredundant list of maximal Heisenberg subalgebras of $\widehat{\mathfrak{g}}$, up to conjugacy under the adjoint action of the Lie group of $\widehat{\mathfrak{g}}$ (see [12, Proposition of Section 9]). Hence, the maximal Heisenberg subalgebras of $\widehat{\mathfrak{g}}$ are parametrized by the conjugacy classes of $W$.

Recall that the Weyl group of $\mathfrak{sl}_r$ is the symmetric group $S_r$. Define a *partition* of $r$ of length $s$ to be an $s$-tuple of positive integers $(r_1, \ldots, r_s)$ such that

$$r_1 \geq \cdots \geq r_s, \quad r_1 + \cdots + r_s = r.$$

Every $\sigma \in S_r$ can be written as a product of disjoint cycles, $\sigma = c_1 \cdots c_s$, and so $\sigma$ defines a partition of $r$ by setting $r_i$ equal to the length of the cycle $c_i$ (reordering the $c_i$ if necessary). Two elements in $S_r$ are conjugate if and only if they define the same partition of $r$. Thus, conjugacy classes in $S_r$, and hence Heisenberg subalgebras of $\widehat{\mathfrak{sl}}_r$, are parametrized by partitions of $r$.

We can extend these ideas to $\widehat{\mathfrak{gl}}_r$ (since the Weyl group of $\mathfrak{gl}_r$ is also $S_r$). That is, to each partition of $r$ we associate a Heisenberg subalgebra of $\widehat{\mathfrak{gl}}_r$ as in (1.7) and (1.8). The most well-known Heisenberg subalgebras of $\widehat{\mathfrak{gl}}_r$ are the so-called *principal Heisenberg subalgebra* and the *homogeneous Heisenberg subalgebra*, which correspond to the partitions $(r)$ and $(1, 1, \ldots, 1)$, respectively. The principal Heisenberg subalgebra is given by

$$\alpha(n) = \begin{cases} (E_0 + E_1 + \cdots + E_{r-1})^n, & \text{for } n > 0, \\ (F_0 + F_1 + \cdots + F_{r-1})^{-n} & \text{for } n < 0, \end{cases}$$

where the $E_k$ and $F_k$ are the Chevalley generators of $\widehat{\mathfrak{sl}}_r$ (which, of course, is a subalgebra of $\widehat{\mathfrak{gl}}_r$) and exponentiation is meant to be taken in the underlying associative algebra of $\widetilde{\mathfrak{gl}}_r$. Setting

$$\alpha(0) = I \otimes 1,$$

the $\alpha(n)$, $n \in \mathbb{Z}$, define a 1-coloured oscillator algebra. The homogeneous Heisenberg subalgebra, presented as an $r$-coloured algebra, is given by

$$\alpha_\ell(n) = E_{\ell,\ell} \otimes t^n,$$

for $n \in \mathbb{Z} - \{0\}$ and $\ell = 1, \ldots, r$. Setting

$$\alpha_\ell(0) = E_{\ell,\ell} \otimes 1,$$

defines an $r$-coloured oscillator algebra. The Heisenberg subalgebras associated to the other partitions of $r$ are "blends" of these two extremes. Namely, for an arbitrary partition $(r_1, \ldots, r_s)$, we divide matrices in $\mathfrak{gl}_r$ into $s^2$ blocks, with the $(i,j)$-th block of size $r_i \times r_j$ for all $i, j = 1, \ldots, s$. The $(\ell,\ell)$-th diagonal block corresponds to the subalgebra $\mathfrak{gl}_{r_\ell} \subseteq \mathfrak{gl}_r$, and on the level of affine algebras, $\widehat{\mathfrak{gl}}_{r_\ell} \subseteq \widehat{\mathfrak{gl}}_r$. Let $E_k^\ell, F_k^\ell, H_k^\ell$ be the Chevalley generators for $\widehat{\mathfrak{sl}}_{r_\ell} \subseteq \widehat{\mathfrak{gl}}_{r_\ell}$. The Heisenberg subalgebra corresponding to the partition $(r_1, \ldots, r_s)$, presented as an $s$-coloured algebra, is given by

$$\alpha_\ell(n) = \begin{cases} (E_0^\ell + E_1^\ell + \cdots + E_{r_\ell - 1}^\ell)^n, & \text{for } n > 0, \\ (F_0^\ell + F_1^\ell + \cdots + F_{r_\ell - 1}^\ell)^{-n}, & \text{for } n < 0, \end{cases}$$

for $\ell = 1, \ldots, s$. It is worth noting that, restricted to the $\ell$-th colour, the $\alpha_\ell(n)$ determine the principal Heisenberg subalgebra of $\widehat{\mathfrak{gl}}_{r_\ell}$. Set

$$\alpha_\ell(0) = I_\ell \otimes 1,$$

where $I_\ell$ is the identity matrix in $\mathfrak{gl}_{r_\ell}$. Then the $\alpha_\ell(n)$ define an $s$-coloured oscillator algebra.

To each partition of $r$, we have associated a Heisenberg (and oscillator) subalgebra of $\widehat{\mathfrak{gl}}_r$, and, as mentioned before, each Heisenberg subalgebra gives rise to a realization of $V_{\text{basic}}$. These various realizations were described explicitly in [26] in terms of vertex operators on bosonic and fermionic Fock space. We summarize the main theorem of that paper here.

Conceptually, it is easier to begin with the principal realization (corresponding to the partition $(r)$) before generalizing to the realizations corresponding to arbitrary partitions of $r$. One begins by defining *formal fermionic fields*

$$\psi(z) := \sum_{k\in\mathbb{Z}} \psi(k)z^k, \quad \psi^*(z) := \sum_{k\in\mathbb{Z}} \psi^*(k)z^{-k},$$

where $z$ is a formal variable. These are power series with coefficients in the (1-coloured) Clifford algebra. Define the *normal ordering* on the Clifford algebra generators

$$:\psi(i)\psi^*(j): = \begin{cases} \psi(i)\psi^*(j), & \text{if } j > 0, \\ -\psi^*(j)\psi(i), & \text{if } j \leq 0. \end{cases}$$

We also define *formal bosonic fields*

$$\alpha(z) :=: \psi(z)\psi(z): .$$

Expanded as a Laurent series $\alpha(z) = \sum_{k\in\mathbb{Z}} \alpha(k)z^{-k}$, one finds

$$\alpha(k) = \sum_{i\in\mathbb{Z}} :\psi(i)\psi^*(i+k): .$$

**Theorem 1.19.** *Let $\omega = e^{2\pi i/r}$. Then the homogeneous components of*

$$:\psi(\omega^p z)\psi^*(\omega^q z): -\frac{\omega^{p-q}}{1-\omega^{p-q}}, \quad 1 \leq p, q \leq r, \;\; p \neq q,$$

$$:\psi(z)\psi^*(z):,$$

*together with the identity operator, span a Lie algebra of operators on $\mathbb{F}$ (1-coloured fermionic Fock space) that is isomorphic to $\widehat{\mathfrak{gl}}_r$. Moreover, via this identification with $\widehat{\mathfrak{gl}}_r$,*

$$\mathbb{F}(0) \cong V_{\text{basic}},$$

*as* $\widehat{\mathfrak{gl}}_r$*-modules and the* $\alpha(k)$, $k \in \mathbb{Z} - \{0\}$ *(resp.* $k \in \mathbb{Z}$*), together with the identity operator, form a basis of the principal Heisenberg subalgebra (resp. oscillator subalgebra) of* $\widehat{\mathfrak{gl}}_r$*.*

**Proof:** This is a special case of Theorem 1.20 below. ∎

Note that the isomorphism $\mathbb{F}(0) \xrightarrow{\cong} V_{\text{basic}}$ in Theorem 1.19 is determined by $|0\rangle \mapsto v_0$. Also, recall that the idea behind the realizations of $V_{\text{basic}}$ was to describe $V_{\text{basic}}$ in terms of $\Lambda$ (or bosonic Fock space), whereas Theorem 1.19 describes $V_{\text{basic}}$ in terms of fermionic Fock space. However, via the boson-fermion correspondence (see (1.3)), $\mathbb{F}(0) \cong \mathbb{B}(0)$, and so Theorem 1.19 satisfies our goal of describing $V_{\text{basic}}$ in terms of bosonic Fock space.

We would now like to generalize Theorem 1.19 to an arbitrary partition of $r$. We fix, once and for all, a partition,

$$\underline{r} = (r_1, \dots, r_s),$$

of $r$. As with our construction of the Heisenberg subalgebra associated to $\underline{r}$, we divide matrices in $\mathfrak{gl}_r$ into $s^2$ blocks of size $r_i \times r_j$. The operators associated to the $(\ell, \ell)$-th diagonal blocks correspond to the subalgebra $\widehat{\mathfrak{gl}}_{r_\ell}$ and the $\ell$-th colour of the Heisenberg subalgebra is the principal Heisenberg subalgebra of $\widehat{\mathfrak{gl}}_{r_\ell}$. Thus, we can simply take $s$-copies of the principal realization above, which amounts to taking $s$-coloured versions of the Heisenberg and Clifford algebras as well as $s$-coloured versions of bosonic and fermionic Fock space. The operators associated to the off-diagonal blocks can be obtained by "mixing" Clifford algebra generators of different colours.

Let $R' = \operatorname{lcm}\{r_1, \dots, r_s\}$ and define

$$R = \begin{cases} R', & \text{if } R'\left(\frac{1}{r_i} + \frac{1}{r_j}\right) \in 2\mathbb{Z} \text{ for all } i, j, \\ 2R', & \text{if } R'\left(\frac{1}{r_i} + \frac{1}{r_j}\right) \notin 2\mathbb{Z} \text{ for some } i, j. \end{cases}$$

Define the *normal ordering* on the $s$-coloured Clifford algebra generators

$$:\psi_\ell(i)\psi_k^*(j): = \begin{cases} \psi_\ell(i)\psi_k^*(j), & \text{if } j > 0, \\ -\psi_k^*(j)\psi_\ell(i), & \text{if } j \leq 0. \end{cases}$$

We introduce the formal $s$-coloured fermionic and bosonic fields:

$$\psi_\ell(z) := \sum_{k\in\mathbb{Z}} \psi_\ell(i) z^{(R/r_\ell)k}, \quad \psi_\ell^*(z) := \sum_{k\in\mathbb{Z}} \psi_\ell^*(k) z^{-(R/r_\ell)k},$$

$$\alpha_\ell(z) = \sum_{k\in\mathbb{Z}} \alpha_\ell(k) z^{-(R/r_\ell)k} := :\psi_\ell(z)\psi_\ell^*(z):,$$

This brings us to the main theorem of [26] (as stated at the end of the introduction).

**Theorem 1.20.** *Let $\omega = e^{2\pi i/R}$. Then the homogeneous components of*

$$:\psi_\ell(\omega^p z)\psi_k^*(\omega^q z): -\delta_{k\ell}\frac{\omega^{(R/r_\ell)(p-q)}}{1-\omega^{(R/r_\ell)(p-q)}}, \qquad \begin{array}{c}(\ell \neq k,\ 1 \leq p \leq r_\ell,\ 1 \leq q \leq r_k) \\ \textit{or } (\ell = k,\ 1 \leq p \neq q \leq r_\ell),\end{array}$$

$$:\psi_\ell(z)\psi_\ell^*(z):, \quad 1 \leq \ell \leq s,$$

*together with the identity operator, span a Lie algebra of operators on $\mathbb{F}$ ($s$-coloured fermionic Fock space) that is isomorphic to $\widehat{\mathfrak{gl}}_r$. Moreover, via this identification with $\widehat{\mathfrak{gl}}_r$,*

$$\mathbb{F}(0) \cong V_{\text{basic}},$$

*as $\widehat{\mathfrak{gl}}_r$-modules and the $\alpha_\ell(k)$, $k \in \mathbb{Z} - \{0\}$ (resp. $k \in \mathbb{Z}$), together with the identity operator, give a basis of the Heisenberg subalgebra (resp. oscillator subalgebra) of $\widehat{\mathfrak{gl}}_r$ associated to the partition $\underline{r}$.*

As was the case with Theorem 1.19, the isomorphism $\mathbb{F}(0) \xrightarrow{\cong} V_{\text{basic}}$ in Theorem 1.20 is determined by $|0\rangle^{\otimes s} \mapsto v_0$.

As stated at the outset, our goal is to give a geometric version of the various realizations of $V_{\text{basic}}$. Our strategy will be as follows. We first construct geometric analogues the oscillator and Clifford algebra representations on bosonic and fermionic Fock space. Our method for doing this will be similar to [15, Section 3]. Next, we would like to construct an action of $\widehat{\mathfrak{gl}}_r$ on our geometric fermionic Fock space in the spirit of Theorem 1.20. To do this we recall a previous observation:

$$\widehat{\mathfrak{gl}}_r = \widehat{\mathfrak{sl}}_r \oplus \left(I \otimes \mathbb{C}[t, t^{-1}]\right),$$

and so $\widehat{\mathfrak{gl}}_r$ is generated by the Chevalley generators $E_k, F_k, H_k,\ k = 0, 1, \ldots, r-1$ of $\widehat{\mathfrak{sl}}_r$ and loops on the identity, $I \otimes t^n$, $n \in \mathbb{Z}$. Thus, to define an action of $\widehat{\mathfrak{gl}}_r$ on our geometric Fock space, it is enough to define the action of $E_k, F_k, H_k$ and $I \otimes t^n$. Algebraically, the action of $\widehat{\mathfrak{gl}}_r$ on fermionic Fock space is given by the homogeneous components of vertex operators as in Theorem 1.20. We explicitly describe the action of $E_k, F_k$ and $I \otimes t^n$ in the following Lemma 1.21 below (the action of $H_k$ is determined by $E_k$ and $F_k$ since $H_k = [E_k, F_k]$). First, we observe that for each $k = 0, 1, \ldots, r-1$, we can write

$$k = r_1 + \cdots + r_{\ell-1} + k',$$

for unique $1 \le \ell \le s$ and $0 \le k' < r_{\ell-1}$.

**Lemma 1.21.** *The action of $\widehat{\mathfrak{gl}}_r$ on $\mathbb{F}$ from Theorem 1.20 is given by the map $\rho : \widehat{\mathfrak{gl}}_r \to \mathfrak{gl}(\mathbb{F})$, described below. For each $k = 0, 1, \ldots, r-1$, write $k = r_1 + \cdots + r_{\ell-1} + k'$. If $k' \neq 0$, then*

$$\rho(E_k) = \sum_{i \in \mathbb{Z}} \psi_\ell(k' + ir_\ell)\psi_\ell^*(k' + ir_\ell + 1), \tag{1.9}$$

$$\rho(F_k) = \sum_{i \in \mathbb{Z}} \psi_\ell(k' + ir_\ell + 1)\psi_\ell^*(k' + ir_\ell). \tag{1.10}$$

*If $k' = 0$ and $\ell \neq 1$,*

$$\rho(E_k) = \sum_{i \in \mathbb{Z}} \psi_{\ell-1}((i+1)r_{\ell-1})\psi_\ell^*(ir_\ell + 1), \tag{1.11}$$

$$\rho(F_k) = \sum_{i \in \mathbb{Z}} \psi_\ell(ir_\ell + 1)\psi_{\ell-1}^*((i+1)r_{\ell-1}). \tag{1.12}$$

*If $k' = 0$ and $\ell = 1$, (i.e. $k = 0$)*

$$\rho(E_0) = \sum_{i\in\mathbb{Z}} \psi_s(ir_s)\psi_1^*(ir_1+1), \tag{1.13}$$

$$\rho(F_0) = \sum_{i\in\mathbb{Z}} \psi_1(ir_1+1)\psi_s^*(ir_s). \tag{1.14}$$

*Finally,*

$$\rho(I\otimes t^n) = \sum_{\ell=1}^{s} \alpha_\ell(nr_\ell) = \begin{cases} |n|\sum_{\ell=1}^{s} r_\ell P_\ell(nr_\ell), & \text{if } n\neq 0,\\ \sum_{\ell=1}^{s} P_\ell(0), & \text{if } n=0. \end{cases} \tag{1.15}$$

**Proof:** We will prove Equation (1.9) and leave the remaining equations up to the reader since they can be derived in a similar manner. The proof simply requires picking out the appropriate components from the vertex operators constructed in [26]. As before, we decompose the $r\times r$ matrices of $\mathfrak{gl}_r$ into $s^2$ blocks of size $r_i\times r_j$. For $1\le i,j\le s$ and $1\le p\le r_i$, $1\le q\le r_j$, define $E_{pq}^{ij}$ to be the matrix with 1 in the $(p,q)$-th entry of the $(i,j)$-th block and zeros elsewhere. Let $\omega_j = e^{2\pi i/r_j}$ be an $r_j$-th root of unity. As in [26, Equation (2.3.3)], define

$$A_{pq}^{ij} := \frac{1}{\sqrt{r_i r_j}}\sum_{a=1}^{r_i}\sum_{b=1}^{r_j} \omega_i^{pa}\omega_j^{-qb}E_{ab}^{ij}.$$

The $A_{pq}^{ij}$'s form a basis of $\mathfrak{gl}_n$. Thus, one can express the $E_{pq}^{ij}$'s in terms of the $A_{pq}^{ij}$'s by [26, Equation (2.3.7)],

$$E_{pq}^{ij} = \frac{1}{\sqrt{r_i r_j}}\sum_{a=1}^{r_i}\sum_{b=1}^{r_j} \omega_i^{-pa}\omega_j^{qb}A_{ab}^{ij}. \tag{1.16}$$

Now, as in [26, Equation (3.3.4)], define

$$h_{\underline{r}} = \sum_{\ell=1}^{s}\sum_{p=1}^{r_\ell} \frac{r_\ell - 2p + 1}{2r_\ell}E_{pp}^{\ell\ell}.$$

The element $h_{\underline{r}}$ induces a $\mathbb{Z}_R$-gradation of $\mathfrak{gl}_r$,

$$\mathfrak{gl}_r = \bigoplus_{\bar{n}\in\mathbb{Z}_R} \mathfrak{gl}_r^{(\bar{n})}, \quad \mathfrak{gl}_r^{(\bar{n})} := \{x\in\mathfrak{gl}_r \mid [h_{\underline{r}}, x] = \frac{n+mR}{R}x \text{ for some } m\in\mathbb{Z}\},$$

where we use the notation $\bar{n} = n \mod R$. Then the elements

$$x(n) := e^{-i\theta \operatorname{ad} h_{\underline{r}}} e^{i(n/R)\theta} x_{\bar{n}}, \quad x = (x_{\bar{n}}) \in \mathfrak{gl}_r, n \in \mathbb{Z}, \tag{1.17}$$

form a spanning set of $\widetilde{\mathfrak{gl}}_r$ (see [26, Equation (4.2.4)]). For each $x \in \mathfrak{gl}_r$ and $n \in \mathbb{Z}$, define

$$\widehat{x}(n) := x(n) - \delta_{n0} \operatorname{tr}(h_{\underline{r}} x) c, \tag{1.18}$$

as in [26, Equation (4.2.9)]. The map $\rho : \widehat{\mathfrak{gl}}_r \to \mathfrak{gl}(\mathbb{F})$ is given explicitly in terms of the elements $\widehat{A}^{ij}_{pq}(n)$. Namely, we define the formal power series

$$\widehat{A}^{ij}_{pq}(z) := \sum_{n \in \mathbb{Z}} \rho(\widehat{A}^{ij}_{pq}(n)) z^{-n},$$

and let $\rho$ be the map determined by setting

$$\widehat{A}^{ij}_{pq}(z) = \frac{z^{(R/2)((1/r_j)-(1/r_i))}}{\sqrt{r_i r_j}} : \psi_i(\omega^p z)\psi^*_j(\omega^q z) : -\frac{1}{r_i}\delta_{ij}\frac{\omega^{(R/r_i)(p-q)}}{1-\omega^{(R/r_i)(p-q)}}, \quad \text{if } p \neq q, \tag{1.19}$$

$$\widehat{A}^{ij}_{pp}(z) := \frac{z^{(R/2)((1/r_j)-(1/r_i))}}{\sqrt{r_i r_j}} : \psi_i(\omega^p z)\psi^*_j(\omega^p z) : . \tag{1.20}$$

Note that our equations vary slightly from [26, Equation 5.5.4] since we do not "absorb" the $z^{R/2r_i}$ terms into our definition of the fermionic fields.

Let $k = r_1 + \cdots + r_{\ell-1} + k' \in \{0, 1, \ldots, r-1\}$ such that $k' \neq 0$. Then

$$E_k = E_{k,k+1} = E^{\ell\ell}_{k',k'+1}.$$

By [26, Equation (3.4.5)],

$$[h_{\underline{r}}, E^{\ell\ell}_{k',k'+1}] = \left(\frac{k'+1}{r_\ell} - \frac{k'}{r_\ell} + \frac{1}{2r_\ell} - \frac{1}{2r_\ell}\right) E^{\ell\ell}_{k',k'+1} = \frac{1}{r_\ell} E^{\ell\ell}_{k',k'+1},$$

and so $E^{\ell\ell}_{k',k'+1} \in \mathfrak{gl}_r^{(\overline{R/r_\ell})}$ and, by Equations (1.17), (1.18) and (1.16),

$$E_k = E^{\ell\ell}_{k',k'+1}(R/r_\ell) = \widehat{E}^{\ell\ell}_{k',k'+1}(R/r_\ell) = \frac{1}{r_\ell}\sum_{p,q=1}^{r_\ell} \omega_\ell^{-(p-q)k'+q} \widehat{A}^{\ell\ell}_{pq}(R/r_\ell).$$

Now, $\rho(\widehat{A}^{\ell\ell}_{pq}(R/r_\ell))$ is equal to the coefficient of $z^{-R/r_\ell}$ in the vertex operators in Equations (1.19) and (1.20). A quick calculation then reveals that

$$\rho(\widehat{A}^{\ell\ell}_{pq}(R/r_\ell)) = \frac{1}{r_\ell}\sum_{i\in\mathbb{Z}} \omega_\ell^{(p-q)i-q}\psi_\ell(i)\psi^*_\ell(i+1).$$

Thus,

$$\rho(E_k) = \frac{1}{r_\ell^2}\sum_{p,q=1}^{r_\ell}\sum_{i\in\mathbb{Z}}\omega_\ell^{(i-k')(p-q)}\psi_\ell(i)\psi^*_\ell(i+1) = \frac{1}{r_\ell}\sum_{i\in\mathbb{Z}}\sum_{(p-q)=0}^{r_\ell-1}\omega_\ell^{(i-k')(p-q)}\psi_\ell(i)\psi^*_\ell(i+1).$$

Since,

$$\sum_{(p-q)=0}^{r_\ell-1}\omega_\ell^{(i-k')(p-q)} = \begin{cases} r_\ell, & \text{if } r_\ell \mid (i-k'), \\ 0, & \text{otherwise,}\end{cases}$$

one finds,

$$\rho(E_k) = \sum_{r_\ell\mid(i-k')}\psi_\ell(i)\psi^*_\ell(i+1) = \sum_{i\in\mathbb{Z}}\psi_\ell(k'+r_\ell i)\psi^*_\ell(k'+r_\ell i+1).$$

∎

# Chapter 2

# Geometric Invariant Theory

Our main geometric objects of interest will be the so-called *Nakajima quiver varieties*, which are examples of *geometric (invariant) quotients*. The purpose of this chapter will be to review some of the basic results from geometric invariant theory that we will need before studying quiver varieties in the following chapter. The material in this chapter is based largely on [19]. However, while [19] deals in full generality with schemes over an arbitrary field, we will deal exclusively with algebraic varieties over $\mathbb{C}$.

Throughout this chapter and all subsequent chapters, whenever $X$ is a set and $G$ is a group acting on $X$, we will denote the set of $G$-fixed points of $X$ by $X^G$.

Let $X$ be an algebraic variety and let $G$ an algebraic group acting on $X$, i.e. the group action

$$\begin{aligned} \sigma : G \times X &\to X, \\ (g, x) &\mapsto g \cdot x, \end{aligned}$$

is a morphism of varieties. Then for any $G$-invariant open set $U \subseteq X$, we have an induced action of $G$ on $\mathcal{O}_X(U)$ given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x),$$

for all $g \in G$, $f \in \mathcal{O}_X(U)$ and $x \in U$. Note that if $f \in \mathcal{O}_X(U)^G$, then $f$ is constant on the orbit $G \cdot x$ for each $x \in U$. Thus, $f$ induces a well-defined function on the orbit space $U/G$ given by

$$\begin{aligned} \bar{f} : U/G &\to \mathbb{C}, \\ G \cdot x &\mapsto f(x). \end{aligned} \tag{2.1}$$

This leads us to the definition of a geometric quotient. Note that our definition is a slightly simplified version of [19, Definition 0.6].

**Definition 2.1** (Geometric quotient)**.** Let $X$ be an algebraic variety and let $G$ an algebraic group acting on $X$ with group action $\sigma : G \times X \to X$. A pair $(Y, \phi)$, where $Y$ is an algebraic variety and $\phi : X \to Y$ is a morphism of varieties, is called a *geometric quotient* of $X$ by $G$ if the following conditions hold.

1. The diagram

   $$\begin{CD} G \times X @>{\sigma}>> X \\ @V{p}VV @VV{\phi}V \\ X @>>{\phi}> Y \end{CD},$$

   commutes, where the map $p$ is the projection onto the second factor.

2. The map $\phi$ is surjective and the fibres of $\phi$ are precisely the $G$-orbits of $X$ (i.e. for all $y \in Y$, there exists an $x \in X$ such that $\phi^{-1}(y) = G \cdot x$).

3. The variety $Y$ has the quotient topology induced by $\phi$.

4. The structure sheaf of $Y$ is given by

   $$\mathcal{O}_Y(U) = \left\{ \bar{f} : U \to \mathbb{C} \mid f \in \mathcal{O}_X(\phi^{-1}(U))^G \right\},$$

   where $\bar{f}$ is defined as in Equation (2.1).

**Remark 2.2.** Geometric quotients are not guaranteed to exist. However, if a geometric quotient $(Y, \phi)$ of $X$ by $G$ does exist, then by (2) and (3) of Definition 2.1, it is clear that, topologically, $Y \cong X/G$. Hence, whenever a geometric quotient $(Y, \phi)$ exists, we will identify it with $(X/G, \pi)$, where $\pi : X \twoheadrightarrow X/G$ is the canonical projection. Moreover, by a slight abuse of notation, we will simply refer to $X/G$ as the geometric quotient of $X$ by $G$, leaving the map $\pi$ implied.

We will, of course, also be interested in morphisms between geometric quotients. The next lemma will prove useful in constructing such morphisms.

**Lemma 2.3.** *Let $X$ and $Y$ be algebraic varieties and let $G_X$ and $G_Y$ be algebraic groups acting on $X$ and $Y$, respectively, such that $X/G_X$ and $Y/G_Y$ are geometric quotients. If $\varphi : X \to Y$ is a morphism of varieties and $\psi : G_X \to G_Y$ is a group homomorphism such that*

$$\begin{array}{ccc} G_X \times X & \longrightarrow & X \\ {\scriptstyle \psi \times \varphi}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ G_Y \times Y & \longrightarrow & Y \end{array} , \tag{2.2}$$

*commutes (where the horizontal arrows represent the group action), then the induced map*

$$\overline{\varphi} : X/G_X \to Y/G_Y,$$
$$G_X \cdot x \mapsto G_Y \cdot \varphi(x),$$

*is a well-defined morphism of varieties. Moreover, if $\varphi$ and $\psi$ are isomorphisms of varieties and groups, respectively, then $\overline{\varphi}$ is an isomorphism of varieties.*

**Proof:** The fact that $\overline{\varphi}$ is well-defined follows directly from the commutativity of Diagram (2.2). Clearly, $\overline{\varphi}$ is continuous. Thus, to show that $\overline{\varphi}$ is a morphism of varieties, it remains only to show that the mapping $\bar{f} \mapsto \bar{f} \circ \overline{\varphi}$ is a pullback

$$\mathcal{O}_{Y/G_Y}(U) \to \mathcal{O}_{X/G_X}(\overline{\varphi}^{-1}(U)),$$

for all open $U \subseteq Y$. Let $\pi_X : X \twoheadrightarrow X/G_X$ and $\pi_Y : Y \twoheadrightarrow Y/G_Y$ denote the canonical projections. By definition of geometric quotients, this is equivalent to showing that the mapping $f \mapsto f \circ \varphi$ is a map

$$\mathcal{O}_Y(\pi_Y^{-1}(U))^{G_Y} \to \mathcal{O}_X(\pi_X^{-1}(\overline{\varphi}^{-1}(U))^{G_X} = \mathcal{O}_X(\varphi^{-1}(\pi_Y^{-1}(U))^{G_X},$$

for all open $U \subseteq Y$. For any $f \in \mathcal{O}_Y(\pi_Y^{-1}(U))^{G_Y}$, $x \in \pi_X^{-1}(\overline{\varphi}^{-1}(U))$ and $g \in G_X$,

$$(g \cdot (f \circ \varphi))(x) = f(\varphi(g^{-1} \cdot x)) = f(\psi(g^{-1}) \cdot \varphi(x)) = f(\varphi(x)) = (f \circ \varphi)(x),$$

where the second equality follows from Diagram (2.2) and the third equality follows from the fact that $f \in \mathcal{O}_Y(\pi_Y^{-1}(U))^{G_Y}$. Therefore, we conclude that $\overline{\varphi}$ is a morphism of varieties.

In the case that $\varphi$ and $\psi$ are isomorphisms, we have the following commutative diagram:

$$\begin{array}{ccc} G_X \times X & \longrightarrow & X \\ {\scriptstyle \psi^{-1} \times \varphi^{-1}} \uparrow & & \uparrow {\scriptstyle \varphi^{-1}} \\ G_Y \times Y & \longrightarrow & Y \end{array}.$$

By repeating all the same arguments, we have a morphism of varieties

$$\begin{aligned} \overline{\varphi^{-1}} : Y/G_Y &\to X/G_X, \\ G_Y \cdot y &\mapsto G_X \cdot \varphi^{-1}(y), \end{aligned}$$

which is the inverse of $\overline{\varphi}$. Thus, $\overline{\varphi}$ is an isomorphism. ∎

We now turn our attention to tangent spaces, as they offer a very useful means for studying the local behaviour of a variety. We begin by recalling the definition. Let $X$ be an algebraic variety. Recall that the *stalk* of $x \in X$ is

$$\mathcal{O}_{X,x} := \varinjlim \mathcal{O}_X(U) \qquad \text{(for open sets } U \ni x)$$

$$= \{[f, U] \mid \text{open } U \ni x \text{ and } f \in \mathcal{O}_X(U)\},$$

where $[f, U]$ is the equivalence class of $(f, U)$ under the equivalence relation $(f, U) \sim (f', U')$ if and only if there exists an open $V \subseteq U \cap U'$ such that $x \in V$ and $f|_V = f'|_V$. By a slight abuse of notation, we will denote $[f, U]$ simply by $f$, leaving the open set $U$ implied. A *derivation* of $\mathcal{O}_{X,x}$ is a linear map $\delta : \mathcal{O}_{X,x} \to \mathbb{C}$ such that

$$\delta(fg) = \delta(f)g(x) + f(x)\delta(g),$$

for all $f, g \in \mathcal{O}_{X,x}$.

**Definition 2.4.** (Tangent space) Let $X$ be an algebraic variety. For $x \in X$, the *tangent space* of $X$ at $x$ is the set of derivations of $\mathcal{O}_{X,x}$, i.e.

$$\mathcal{T}_x(X) := \{\delta \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_{X,x}, \mathbb{C}) \mid \delta(fg) = \delta(f)g(x) + f(x)\delta(g), \text{ for all } f, g \in \mathcal{O}_{X,x}\}.$$

If $X$ is an affine variety, we may define the tangent space in more global terms by replacing the stalk $\mathcal{O}_{X,x}$ with the coordinate ring $\mathbb{C}[X]$ of $X$, i.e.

$$\mathcal{T}_x(X) = \{\delta \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}) \mid \delta(fg) = \delta(f)g(x) + f(x)\delta(g), \text{ for all } f, g \in \mathbb{C}[X]\}.$$

If $X$ and $Y$ are algebraic varieties and $\varphi : X \to Y$ is a morphism of varieties, then we have an induced map on the stalks

$$\begin{aligned} \varphi^* : \mathcal{O}_{Y,\varphi(x)} &\to \mathcal{O}_{X,x}, \\ f &\mapsto f \circ \varphi, \end{aligned}$$

for each $x \in X$ (in the affine case, we may replace the stalks $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,\varphi(x)}$ with the coordinate rings $\mathbb{C}[X]$ and $\mathbb{C}[Y]$). This induces a map on the tangent spaces

$$\begin{aligned} \mathrm{d}\varphi : \mathcal{T}_x(X) &\to \mathcal{T}_{\varphi(x)}(Y), \\ \delta &\mapsto \delta \circ \varphi^*, \end{aligned}$$

called the *differential* of $\varphi$ (at $x$). If $X$ is smooth, then we say that $\varphi$ is *étale* if $\mathrm{d}\varphi$ is an isomorphism. Note that this is not the original definition of an étale morphism (see [7, Definition 17.3.1]), but rather an equivalent characterization in the setting of smooth varieties (see [7, Corollary 17.11.2]).

Let $G$ be an algebraic group acting on $X$ and let $\sigma : G \times X \to X$ denote the group action. Define $\sigma_g := \sigma(g, -) : X \to X$ and $\sigma^x := \sigma(-, x) : G \to X$ for all $g \in G$ and $x \in X$. We then have induced maps on the tangent spaces

$$\mathrm{d}\sigma_g : \mathcal{T}_x(X) \to \mathcal{T}_{g\cdot x}(X) \quad \text{and} \quad \mathrm{d}\sigma^x : \mathcal{T}_1(G) \to \mathcal{T}_x(X).$$

For sufficiently nice varieties, the following lemma allows us to compute the tangent space of $X/G$.

**Lemma 2.5.** *Let $X$ be a smooth algebraic variety and let $G$ be a reductive algebraic group acting freely on $X$ such that the geometric quotient $X/G$ exists. Let $\pi : X \twoheadrightarrow X/G$ denote the canonical projection. Then, for all $x \in X$,*

$$0 \to \mathcal{T}_1(G) \xrightarrow{\mathrm{d}\sigma^x} \mathcal{T}_x(X) \xrightarrow{\mathrm{d}\pi} \mathcal{T}_{G\cdot x}(X/G) \to 0,$$

*is a short exact sequence.*

**Proof:** Luna's étale slice theorem (see [16, Section III, Théorème du slice étale]) implies that, for all $x \in X$, there exists a subvariety $V \subseteq X$, which we may assume to be smooth (see Remark 1 following the slice theorem in [16]), containing $x$ such that $\sigma|_{G\times V}$ is étale (note that since $G$ acts freely on $X$, the stabilizer $G_x$ is trivial and the orbit $G \cdot x$ is closed). Therefore,

$$\mathrm{d}\sigma : \mathcal{T}_{(1,x)}(G \times V) \to \mathcal{T}_x(X),$$

is an isomorphism. Luna's étale slice theorem also implies that the morphism $\bar{\sigma}$, induced from the following commutative diagram

$$\begin{array}{ccc} G \times V & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ (G \times V)/G \cong V & \overset{\bar{\sigma}}{\dashrightarrow} & X/G, \end{array}$$

where the vertical maps are the canonical projections, is étale. Thus,

$$\mathrm{d}\bar{\sigma} : T_{G\cdot(1,x)}((G \times V)/G \cong T_x(V) \to T_{G\cdot x}(X/G),$$

is an isomorphism. Notice that we have a commutative diagram:

$$\begin{array}{ccccc} G & \overset{i_1}{\hookrightarrow} & G \times V & \overset{\pi\circ\sigma}{\twoheadrightarrow} & X/G \\ \mathrm{id}\downarrow & & \sigma\downarrow & & \mathrm{id}\downarrow \\ G & \overset{\sigma^x}{\hookrightarrow} & X & \overset{\pi}{\twoheadrightarrow} & X/G, \end{array}$$

where $i_1(g) = (g, x)$ for all $g \in G$. This induces the following commutative diagram on the tangent spaces:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{T}_1(G) & \overset{\mathrm{d}i_1}{\hookrightarrow} & \begin{array}{c}\mathcal{T}_{(1,x)}(G \times V) \\ \cong \mathcal{T}_1(G) \oplus \mathcal{T}_x(V)\end{array} & \overset{\mathrm{d}\pi\circ\mathrm{d}\sigma}{\twoheadrightarrow} & \mathcal{T}_{G\cdot x}(X/G) & \to & 0 \\ & & \mathrm{id}\downarrow & & \mathrm{d}\sigma\downarrow & & \mathrm{id}\downarrow & & \\ 0 & \to & \mathcal{T}_1(G) & \overset{\mathrm{d}\sigma^x}{\hookrightarrow} & \mathcal{T}_x(X) & \overset{\mathrm{d}\pi}{\twoheadrightarrow} & \mathcal{T}_{G\cdot x}(X/G) & \to & 0. \end{array} \tag{2.3}$$

Clearly, the first row of Diagram (2.3) is a short exact sequence. Since the vertical arrows of Diagram (2.3) are all isomorphisms, it follows that the second row is also a short exact sequence. ∎

**Corollary 2.6.** *Let $X$ and $G$ be as in Lemma 2.5. Then, for all $x \in X$,*

$$\mathcal{T}_{G\cdot x}(X/G) \cong \mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x.$$

**Proof:** By Lemma 2.5 and the first isomorphism theorem, $\mathcal{T}_{G\cdot x}(X/G) = \operatorname{im}\mathrm{d}\pi \cong \mathcal{T}_x(X)/\ker\mathrm{d}\pi = \mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x$. ∎

Let us keep the assumptions of Lemma 2.5. Suppose $\varphi : X \to X$ is a $G$-equivariant morphism of varieties. Then we have a well-defined morphism

$$\begin{aligned} \overline{\varphi} : X/G &\to X/G, \\ G\cdot x &\mapsto G\cdot\varphi(x). \end{aligned}$$

Let $x \in X$ such that $G\cdot x \in X/G$ is a fixed point of $\overline{\varphi}$. Then, there exists a unique $g \in G$ such that $\varphi(x) = g^{-1}\cdot x$ and the map $\overline{\varphi}$ induces a map on $\mathcal{T}_{G\cdot x}(X/G)$:

$$\mathrm{d}\overline{\varphi} : \mathcal{T}_{G\cdot x}(X/G) \to \mathcal{T}_{G\cdot x}(X/G).$$

By Corollary 2.6, we may identify $\mathcal{T}_{G\cdot x}(X/G)$ with $\mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x$. Via this identification, $\mathrm{d}\overline{\varphi}$ induces a map on $T_x(X)/\operatorname{im}\mathrm{d}\sigma^x$, which we compute in the following lemma.

**Lemma 2.7.** *Let $X$, $G$, $\varphi$, $x$ and $g$ be as above. Let*

$$\psi : T_{G\cdot x}(X/G) \to T_x(X)/\operatorname{im}\mathrm{d}\sigma^x,$$

*be the isomorphism described in Corollary 2.6. Then we have well-defined maps*

$$\begin{aligned} \overline{\mathrm{d}\varphi} : \mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x &\to \mathcal{T}_{\varphi(x)}(X)/\operatorname{im}\mathrm{d}\sigma^{\varphi(x)}, \\ \delta + \operatorname{im}\mathrm{d}\sigma^x &\mapsto \mathrm{d}\varphi(\delta) + \operatorname{im}\mathrm{d}\sigma^{\varphi(x)}, \end{aligned}$$

*and*

$$\begin{aligned} \overline{\mathrm{d}\sigma_g} : \mathcal{T}_{\varphi(x)}(X)/\operatorname{im}\mathrm{d}\sigma^{\varphi(x)} &\to \mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x, \\ \delta + \operatorname{im}\mathrm{d}\sigma^{\varphi(x)} &\mapsto \mathrm{d}\sigma_g(\delta) + \operatorname{im}\mathrm{d}\sigma^x. \end{aligned}$$

*Moreover, the following diagram commutes.*

$$\begin{array}{ccc}
\mathcal{T}_{G\cdot x}(X/G) & \xrightarrow{\mathrm{d}\overline{\varphi}} & \mathcal{T}_{G\cdot x}(X/G) \\
\psi \downarrow & & \downarrow \psi \\
\mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x & \dashrightarrow & \mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x \\
\overline{\mathrm{d}\varphi} \searrow & & \nearrow \overline{\mathrm{d}\sigma_g} \\
& \mathcal{T}_{\varphi(x)}(X)/\operatorname{im}\mathrm{d}\sigma^{\varphi(x)} &
\end{array} \tag{2.4}$$

*In particular, the induced action of* $\mathrm{d}\overline{\varphi}$ *on* $\mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x$ *is given by the dashed line in Diagram* (2.4)

**Proof:** Let $\tau_g : G \to G$ be the map $\tau_g(h) = ghg^{-1}$. Note that $\tau_g$ is a morphism of varieties. We have the following commutative digram:

$$\begin{array}{ccccc}
G & \xhookrightarrow{\sigma^x} & X & \xrightarrow{\pi} & X/G \\
\mathrm{id} \downarrow & & \varphi \downarrow & & \overline{\varphi} \downarrow \\
G & \xhookrightarrow{\sigma^{\varphi(x)}} & X & \xrightarrow{\pi} & X/G \\
\tau_g \downarrow & & \sigma_g \downarrow & & \mathrm{id} \downarrow \\
G & \xhookrightarrow{\sigma^x} & X & \xrightarrow{\pi} & X/G,
\end{array}$$

which induces the following commutative diagram on the tangent spaces:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{T}_1(G) & \xhookrightarrow{\mathrm{d}\sigma^x} & \mathcal{T}_x(X) & \xrightarrow{\mathrm{d}\pi} & \mathcal{T}_{G\cdot x}(X/G) & \to & 0 \\
& & \mathrm{id} \downarrow & & \mathrm{d}\varphi \downarrow & & \mathrm{d}\overline{\varphi} \downarrow & & \\
0 & \to & \mathcal{T}_1(G) & \xhookrightarrow{\mathrm{d}\sigma^{\varphi(x)}} & \mathcal{T}_{\varphi(x)}(X) & \xrightarrow{\mathrm{d}\pi} & \mathcal{T}_{G\cdot x}(X/G) & \to & 0 \\
& & \mathrm{d}\tau_g \downarrow & & \mathrm{d}\sigma_g \downarrow & & \mathrm{id} \downarrow & & \\
0 & \to & \mathcal{T}_1(G) & \xhookrightarrow{\mathrm{d}\sigma^x} & \mathcal{T}_x(X) & \xrightarrow{\mathrm{d}\pi} & \mathcal{T}_{G\cdot x}(X/G) & \to & 0,
\end{array}$$

where every row is a short exact sequence by Lemma 2.5. Therefore, $\mathrm{d}\varphi(\operatorname{im}\mathrm{d}\sigma^x) \subseteq \operatorname{im}\mathrm{d}\sigma^{\varphi(x)}$ and $\mathrm{d}\sigma_g(\operatorname{im}\mathrm{d}\sigma^{\varphi(x)}) \subseteq \operatorname{im}\mathrm{d}\sigma^x$, thus proving that $\overline{\mathrm{d}\varphi}$ and $\overline{\mathrm{d}\sigma_g}$ are well-defined.

One then easily verifies the commutativity of Diagram (2.4). ∎

**Remark 2.8.** Let $X$ and $G$ be as in Lemma 2.7 and let $H$ be an algebraic group acting on $X$ such that the $G$ and $H$ actions commute. Then each $h \in H$ induces a $G$-equivariant morphism of varieties $\varphi_h : X \to X$, given by $x \mapsto h \cdot x$. Let $x \in X$ such that $G \cdot x \in (X/G)^H$. Then $H$ acts on $\mathcal{T}_{G\cdot x}(X/G)$ by

$$h \cdot \delta = \mathrm{d}\varphi_h(\delta),$$

for all $h \in H$ and $\delta \in \mathcal{T}_{G\cdot x}(X/G)$. By Corollary 2.6, we may identify $\mathcal{T}_{G\cdot x}(X/G)$ with $\mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x$. For each $h \in H$, let $g(h) \in G$ be the element such that $h\cdot x = g(h)^{-1}\cdot x$. By Lemma 2.7, the induced action of $H$ on $\mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x$ is given by

$$h \cdot (\delta + \operatorname{im}\mathrm{d}\sigma^x) = (\overline{\mathrm{d}\sigma_{g(h)}} \circ \overline{\mathrm{d}\varphi_h})(\delta + \operatorname{im}\mathrm{d}\sigma^x) = \left(\mathrm{d}\sigma_{g(h)} \circ \mathrm{d}\varphi_h\right)(\delta) + \operatorname{im}\mathrm{d}\sigma^x,$$

for all $h \in H$ and $\delta + \operatorname{im}\mathrm{d}\sigma^x \in \mathcal{T}_x(X)/\operatorname{im}\mathrm{d}\sigma^x$.

Let $X$ be an algebraic variety (not necessarily smooth) and $G$ an algebraic group (not necessarily reductive) acting on $X$. Let $g \in G$ and consider the morphism

$$\begin{aligned}\phi_g : X &\to X \times X,\\ x &\mapsto (x, g\cdot x).\end{aligned}$$

By definition of a variety, the diagonal

$$\Delta(X) = \{(x,x) \in X \times X\},$$

is a closed subset $X \times X$. Therefore,

$$\phi_g^{-1}(\Delta(X)) = X^g,$$

is closed in $X$. Hence,

$$X^G = \bigcap_{g\in G} X^g,$$

is a closed subvariety of $X$. For each $g \in G$, let

$$\varphi_g : X \to X,$$
$$x \mapsto g \cdot x.$$

Recall that, for each $x \in X^G$, the group $G$ acts on $\mathcal{T}_x(X)$ via

$$g \cdot \delta = \mathrm{d}\varphi_g(\delta),$$

for all $g \in G$ and $\delta \in \mathcal{T}_x(X)$. The following lemma shows that we may naturally identify $\mathcal{T}_x(X^G)$ with $\mathcal{T}_x(X)^G$.

**Lemma 2.9.** *Let $X$ be an algebraic variety and let $G$ be an algebraic group acting on $X$. Let*

$$i : X^G \hookrightarrow X,$$

*be the inclusion map. Then, for all $x \in X$, the differential* d$i$ *maps $\mathcal{T}_x(X^G)$ isomorphically onto $\mathcal{T}_x(X)^G$.*

**Proof:** Without loss of generality, we may assume that $X$ is affine. Suppose $X \subseteq \mathbb{A}^n$ and let $\mathbb{C}[X]$ and $\mathbb{C}[X^G]$ be the coordinate rings of $X$ and $X^G$, respectively. Both $\mathbb{C}[X]$ and $\mathbb{C}[X^G]$ are quotients of the polynomial ring $\mathbb{C}[t_1, \ldots, t_n]$. For each $g \in G$, we may consider the morphism $\varphi_g$ to be an $n$-tuple $(\varphi_g^1, \ldots, \varphi_g^n)$ of polynomials in $\mathbb{C}[X]$. Then

$$\mathbb{C}[X^G] \cong \mathbb{C}[X]/J,$$

where $J$ is the radical of the ideal generated by the set $\{\varphi_g^i - t_i \mid g \in G, i = 1, \ldots, n\}$. In particular, $\varphi_g^i + J = t_i + J$ in $\mathbb{C}[X^G]$ and so, $\varphi_g^*(f) + J = f + J$ for all $f + J \in \mathbb{C}[X^G]$. The pullback of $i$ is the map

$$i^* : \mathbb{C}[X] \twoheadrightarrow \mathbb{C}[X^G],$$
$$f \mapsto f + J.$$

Now, we consider the differential $\mathrm{d}i : \mathcal{T}_x(X^G) \to \mathcal{T}_x(X)$. Since $i^*$ is surjective, it is clear that $\mathrm{d}i$ is injective. Thus, it remains only to show that $\mathrm{d}i(\mathcal{T}_x(X^G)) = \mathcal{T}_x(X)^G$.

Let $\delta \in \mathcal{T}_x(X^G)$. Then for any $g \in G$ and $f \in \mathbb{C}[X]$,

$$(g \cdot (\mathrm{d}i(\delta)))(f) = \delta(\varphi_g^*(f) + J) = \delta(f + J) = (\mathrm{d}i(\delta))(f).$$

Thus, $\mathrm{d}i(\delta) \in \mathcal{T}_x(X)^G$ and so, $\mathrm{d}i(\mathcal{T}_x(X^G)) \subseteq \mathcal{T}_x(X)^G$. Conversely, let $\varepsilon \in \mathcal{T}_x(X)^G$. Notice that, for all $g \in G$ and $i \in \{1, \ldots, n\}$,

$$\varepsilon(\varphi_g^i - t_i) = \varepsilon(\varphi_g^i) - \varepsilon(t_i) = (g \cdot \varepsilon)(t_i) - \varepsilon(t_i) = \varepsilon(t_i) - \varepsilon(t_i) = 0.$$

Thus, $\varepsilon$ vanishes on the generators of $J$. Moreover, the generators of $J$ vanish on $x$, and so $\varepsilon(J) = 0$. Hence, we have a well-defined derivation

$$\begin{aligned} \delta : \mathbb{C}[X^G] &\to \mathbb{C}, \\ f + J &\mapsto \varepsilon(f), \end{aligned}$$

and $\mathrm{d}i(\delta) = \varepsilon$. Therefore, $\mathcal{T}_x(X)^G \subseteq \mathrm{d}i(\mathcal{T}_x(X^G))$, completing the proof. ∎

In the following chapters, we will commonly identify the tangent space of varieties with the middle cohomology of certain complexes. In light of the previous lemma, it will be useful for us to compute the fixed points of the middle cohomology of complexes.

**Lemma 2.10.** *Let $G$ be a finite-dimensional torus or a finite group and let*

$$X \overset{\alpha}{\hookrightarrow} Y \overset{\beta}{\twoheadrightarrow} Z$$

*be a complex of of $G$-modules such that $\alpha$ is injective and $\beta$ is surjective. Let $\alpha_G = \alpha|_{X^G}$ and $\beta_G = \beta|_{Y^G}$. Then* $\ker \beta_G / \operatorname{im} \alpha_G \cong (\ker \beta / \operatorname{im} \alpha)^G$ *as $G$-modules.*

**Proof:** We have the following commutative diagram

$$\begin{array}{ccccc} X^G & \xhookrightarrow{\alpha_G} & Y^G & \xrightarrow{\beta_G} & Z^G \\ \downarrow & & i \downarrow & & \downarrow \\ X & \xhookrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z, \end{array}$$

where the vertical arrows represent the inclusion maps. Note that the restrictions $\alpha_G$ and $\beta_G$ remain injective and surjective, respectively. By commutativity of the diagram, $i$ induces a $G$-module morphism

$$i_* : \ker \beta_G / \operatorname{im} \alpha_G \to \ker \beta / \operatorname{im} \alpha,$$
$$y + \operatorname{im} \alpha_G \mapsto i(y) + \operatorname{im} \alpha = y + \operatorname{im} \alpha.$$

Suppose $y + \operatorname{im} \alpha_G \in \ker i_*$. This implies that $y \in \operatorname{im} \alpha$, and so we have a (unique) $x \in X$ such that $\alpha(x) = y$. Because $y \in Y^G$, for every $g \in G$ we have

$$\alpha(g \cdot x) = g \cdot \alpha(x) = g \cdot y = y = \alpha(x).$$

Since $\alpha$ is injective, we have that $g \cdot x = x$. Therefore, $x \in X^G$ and so $y \in \operatorname{im} \alpha_G$. Thus $\ker i_* = 0$ and $i_*$ is injective. Hence, by the First Isomorphism Theorem, $\ker \beta_G / \operatorname{im} \alpha_G \cong \operatorname{im} i_*$.

We claim that $\operatorname{im} i_* = (\ker \beta / \operatorname{im} \alpha)^G$, which would complete the proof of the lemma. Clearly, $\operatorname{im} i_* \subseteq (\ker \beta / \operatorname{im} \alpha)^G$, thus it remains only to check the reverse inclusion. If $G = (\mathbb{C}^*)^d$ is a $d$-dimensional torus, then we have a weight space decomposition of $X$:

$$X = \bigoplus_{\mathbf{k} \in \mathbb{Z}^d} X_{\mathbf{k}}, \quad X_{\mathbf{k}} := \{x \in X \mid (g_1, \ldots, g_d) \cdot x = g_1^{\mathbf{k}_1} \cdots g_d^{\mathbf{k}_d} x, \text{ for all } (g_1, \ldots, g_d) \in G\}.$$

We also have a weight space decomposition $Y = \bigoplus_{\mathbf{k} \in \mathbb{Z}^d} Y_{\mathbf{k}}$, where the $Y_{\mathbf{k}}$ are defined in the same way as the $X_{\mathbf{k}}$. Suppose $y = (y_{\mathbf{k}}) \in \ker \beta$ such that $y + \operatorname{im} \alpha \in (\ker \beta / \operatorname{im} \alpha)^G$. Thus, for every $g \in G$, we have $(g \cdot y) - y \in \operatorname{im} \alpha$. Moreover, since $\alpha$ is a morphism of $G$-modules, $\alpha|_{X_{\mathbf{k}}} : X_{\mathbf{k}} \hookrightarrow Y_{\mathbf{k}}$, and so

$$((g \cdot y) - y)_{\mathbf{k}} = (g_1^{\mathbf{k}_1} \cdots g_d^{\mathbf{k}_d} - 1) y_{\mathbf{k}} \in \operatorname{im} \alpha,$$

for all $\mathbf{k} \in \mathbb{Z}^d$ and $g = (g_1, \ldots, g_d) \in G$. For each $\mathbf{k} \neq \mathbf{0}$, we may choose a $g = (g_1, \ldots, g_d) \in G$ such that $g^{\mathbf{k}_1} \cdots g^{\mathbf{k}_d} \neq 1$, and thus $y_{\mathbf{k}} \in \operatorname{im}\alpha$. Therefore,

$$y + \operatorname{im}\alpha = y_{\mathbf{0}} + \operatorname{im}\alpha \in \operatorname{im} i_*,$$

since $y_{\mathbf{0}} \in Y^G$.

If $G$ is a finite group, we again suppose $y \in \ker\beta$ such that $y + \operatorname{im}\alpha \in (\ker\beta/\operatorname{im}\alpha)^G$, i.e. $(g \cdot y) - y \in \operatorname{im}\alpha$ for all $g \in G$. Thus,

$$\frac{1}{|G|}\sum_{g\in G}(g \cdot y - y) = \left(\frac{1}{|G|}\sum_{g\in G} g \cdot y\right) - y \in \operatorname{im}\alpha,$$

and so

$$y + \operatorname{im}\alpha = \frac{1}{|G|}\sum_{g\in G} g \cdot y + \operatorname{im}\alpha \in \operatorname{im} i_*,$$

since $\frac{1}{|G|}\sum_{g\in G} g \cdot y \in Y^G$. ∎

We end this chapter with the following lemma, which gives a sufficient set of conditions to determine when a morphism of varieties is an isomorphism.

**Lemma 2.11.** *Let $X$ and $Y$ be smooth algebraic varieties and let $\varphi : X \to Y$ be a bijective étale morphism. Then $\varphi$ is an isomorphism (of varieties).*

**Proof:** The Inverse Function Theorem found in [18, Theorem 5.31] implies that $\varphi$ is birational. Then, since $\varphi$ is bijective and $Y$ is smooth (in particular, $Y$ is normal), by a version of Zariski's Main Theorem (see [20, Chapter III, Section 9, Theorem I]), it follows that $\varphi$ is an isomorphism. ∎

# Chapter 3

# Quiver Varieties

The key result of this chapter is Theorem 3.20, in which we show that the $\mathbb{Z}_m$-fixed point set of certain Nakajima quiver varieties is isomorphic to a disjoint union of Nakajima quiver varieties of type $\widehat{A}_{m-1}$. In Chapter 5, we will see that the equivariant cohomology of these varieties will provide us with a suitable geometric analogue of $V_{\text{basic}}$.

Let $Q = (Q_0, Q_1)$ be a quiver. For all $\rho \in Q_1$, write $t(\rho)$ and $h(\rho)$ for the tail and the head of $\rho$, respectively. Let $\widetilde{Q} = (\widetilde{Q}_0, \widetilde{Q}_1)$ be the double quiver of $Q$. That is,

$$\widetilde{Q}_0 = Q_0, \qquad \text{and} \qquad \widetilde{Q}_1 = Q_1 \cup \overline{Q}_1,$$

where $\overline{Q}_1$ is the set of arrows in $Q_1$ with orientation reversed. We then have a natural involution $^- : \widetilde{Q}_1 \to \widetilde{Q}_1$, which sends every arrow in $Q_1$ to its corresponding reverse arrow in $\overline{Q}_1$ and vice versa. We define the function $\varepsilon : \widetilde{Q}_1 \to \{\pm 1\}$ by

$$\varepsilon(\rho) = \begin{cases} 1, & \text{if } \rho \in Q_1, \\ -1, & \text{if } \rho \in \overline{Q}_1. \end{cases}$$

Let $V = \bigoplus_{k \in \widetilde{Q}_0} V_k$ and $W = \bigoplus_{k \in \widetilde{Q}_0} W_k$ be $\widetilde{Q}_0$-graded complex vector spaces and let

$n = \dim V$ and $s = \dim W$. Define

$$E_V = \bigoplus_{\rho \in \widetilde{Q}_1} \operatorname{Hom}(V_{t(\rho)}, V_{h(\rho)}), \qquad L_{V,W} = \bigoplus_{k \in \widetilde{Q}_0} \operatorname{Hom}(V_k, W_k).$$

We define a "multiplication" $E_V \times E_V \to L_{V,V}$ given by

$$(AB)_k = \sum_{\rho \in \widetilde{Q}_1, t(\rho)=k} A_{\bar{\rho}} B_\rho,$$

for all $A = (A_\rho), B = (B_\rho) \in E_V$. We then define

$$\mathbf{M} = \mathbf{M}(V, W) = E_V \oplus L_{W,V} \oplus L_{V,W}.$$

The function $\varepsilon : \widetilde{Q}_1 \to \{\pm 1\}$ induces a function $\varepsilon : E_V \to E_V$ given by

$$\varepsilon(C)_\rho = \varepsilon(\rho) C_\rho.$$

We have a symplectic form $\omega$ on $\mathbf{M}$ given by

$$\omega((C_1, i_1, j_1), (C_2, i_2, j_2)) = \operatorname{tr}(\varepsilon(C_1) C_2) + \operatorname{tr}(i_1 j_2 - i_2 j_1).$$

Let $G_V = \prod_{k \in \widetilde{Q}_0} \operatorname{GL}(V_k)$. Then $G_V$ acts on $\mathbf{M}$ via

$$g \cdot (C, i, j) = (gCg^{-1}, gi, jg^{-1}),$$

for all $g \in G_V$ and $(C, i, j) \in \mathbf{M}$. Then $G_V$ is an algebraic group whose action on $\mathbf{M}$ preserves the symplectic form $\omega$. The moment map vanishing at the origin is given by

$$\mu : \mathbf{M} \to L_{V,V},$$
$$(C, i, j) \mapsto \varepsilon(C) C + ij,$$

where the Lie algebra $L_{V,V}$ of $G_V$ is identified with its dual via the trace.

**Remark 3.1.** Note that the set $\mathbf{M}$ and the map $\mu$ do not depend on the orientation of the quiver $Q$. Thus, our construction above, as well as the definition of quiver varieties below, depend only on the underlying (undirected) graph of $Q$.

**Definition 3.2** (Invariant, Stable)**.** Let $S = \bigoplus_{k \in \widetilde{Q}_0} S_k$, where each $S_k$ is a subspace of $V_k$. For $C \in E_V$, we say that $S$ is $C$-*invariant* if $C_\rho(S_{t(\rho)}) \subseteq S_{h(\rho)}$ for all $\rho \in \widetilde{Q}_1$.

An element $(C, i, j) \in \mathbf{M}$ is called *stable* if the following condition holds: if $S$ is a $\widetilde{Q}_0$-graded subspace of $V$ such that $S$ is $C$-invariant and $\operatorname{im}(i) \subseteq S$, then $S = V$. We denote by $\mathbf{M}^{\mathrm{st}}$ the set of stable points of $\mathbf{M}$.

**Remark 3.3.** Recall that a path (of length $\ell$) in a quiver is a sequence of arrows $\rho_\ell \cdots \rho_2\rho_1$ such that $h(\rho_k) = t(\rho_{k+1})$. For any path $p = \rho_\ell \cdots \rho_2\rho_1$ and any $C \in E_V$, let $C^{(p)} := C_{\rho_\ell} \circ \cdots \circ C_{\rho_2} \circ C_{\rho_1}$, which is a linear map $V_{t(\rho_1)} \to V_{h(\rho_\ell)}$. For $(C, i, j) \in \mathbf{M}$, define

$$S(C, i, j) := \operatorname{span}\left\{ C^{(p)} i(x_k) \mid p \text{ is a path starting at } k,\ x_k \in W_k,\ k \in \widetilde{Q}_0 \right\}.$$

Notice that $(C, i, j)$ is stable if and only if $S(C, i, j) = V$. Indeed, suppose $(C, i, j)$ is stable. We have that $S(C, i, j)$ is a $\widetilde{Q}_0$-graded subspace with $k$-th component $S_k := S(C, i, j) \cap V_k$ for all $k \in \widetilde{Q}_0$. Moreover, $S(C, i, j)$ is $C$-invariant and $\operatorname{im}(i) \subseteq S(C, i, j)$, thus $S(C, i, j) = V$.

Conversely, suppose $S(C, i, j) = V$. Let $T$ be a $C$-invariant $\widetilde{Q}_0$-graded subspace of $V$ with $\operatorname{im}(i) \subseteq T$. Then we have $S(C, i, j) \subseteq T$, and so $T = V$. Thus, $(C, i, j)$ is stable.

**Lemma 3.4.** *The set* $\mathbf{M}_0^{\mathrm{st}} := \mu^{-1}(0) \cap \mathbf{M}^{\mathrm{st}}$ *is a quasi-affine algebraic variety.*

**Proof:** We will show that $\mathbf{M}^{\mathrm{st}}$ (resp. $\mu^{-1}(0)$) is a Zariski open (resp. closed) subset of $\mathbf{M}$. By choosing bases for $V_k$ and $W_k$ for each $k \in \widetilde{Q}_0$, we may identify $\mathbf{M}$ with a direct sum of matrix vector spaces, which can then be identified with affine space. Via this identification, it is clear that $\mu^{-1}(0)$ is the vanishing set of a set of polynomials whose variables are the entries of the matrices defined by $C$, $i$ and $j$, and thus $\mu^{-1}(0)$ is a Zariski closed set.

By Remark 3.3,

$$\mathbf{M}^{\mathrm{st}} = \{(C, i, j) \in \mathbf{M} \mid S(C, i, j) = V\}.$$

Suppose $x \in W_k$, for some $k \in \widetilde{Q}_0$, such that $i(x) \neq 0$, and $p = \rho_m \cdots \rho_2\rho_1$ is a path in $\widetilde{Q}$ starting at $k$ with $m \geq n = \dim V$. Let $p_\ell = \rho_\ell \cdots \rho_2\rho_1$, for all $1 \leq \ell \leq m$. Choose $\ell$ minimal such that $\{i(x), C^{(p_1)}i(x), \ldots, C^{(p_\ell)}i(x)\}$ is linearly dependent. Then

$$C^{(p_\ell)}i(x) = a_0 i(x) + a_1 C^{(p_1)}i(x) + \cdots + a_{\ell-1}C^{(p_{\ell-1})}i(x),$$

for some $a_q \in \mathbb{C}$, $q = 0, 1, \ldots, \ell - 1$, where we may choose $a_q = 0$ if $h(\rho_q) \neq h(\rho_\ell)$. Therefore,

$$C^{(p)}i(x) = (C_{\rho_m} \circ \cdots \circ C_{\rho_{\ell+1}})(a_0 i(x) + a_1 C^{(p_1)}i(x) + \cdots + a_{\ell-1}C^{(p_{\ell-1})}i(x)),$$

and so $C^{(p)}i(x) \in \operatorname{span}\{C^{(p')}i(x) \mid p'$ is a path of length smaller than $m\}$. Thus, by induction,

$$S(C,i,j) = \operatorname{span}\{C^{(p)}i(x_k) \mid p \text{ is a path starting at } k \text{ of length less than } n,\ x_k \in W_k,\ k \in \widetilde{Q}_0\}.$$

For each $k \in \widetilde{Q}_0$ let $i^{(k)}_\ell$, $1 \leq \ell \leq \dim W_k$, denote the images under $i$ of the basis elements of $W_k$. Let $X(C,i,j)$ be the matrix whose columns are $C^{(p_k)}i^{(k)}_{\ell_k}$, where we take all the paths $p_k$ starting at $k$ of length less that $n$, $1 \leq \ell_k \leq \dim W_k$, $k \in \widetilde{Q}_0$. By the above argument, $S(C,i,j)$ is the column space of $X(C,i,j)$, and so, by basic linear algebra, $S(C,i,j) = V$ if and only if $\operatorname{rk}(X(C,i,j)) = n$. The rank of a matrix may be defined as the size of the largest (square) submatrix with nonzero determinant. Hence, $\mathbf{M}^{\text{st}}$ may be described as

$$\mathbf{M}^{\text{st}} = \{(C,i,j) \in \mathbf{M} \mid \text{there is an } n \times n \text{ submatrix of } X(C,i,j) \text{ with } \det \neq 0\}.$$

The complement of $\mathbf{M}^{\text{st}}$, which we denote $\mathbf{M}^{\text{unst}}$ (the set of unstable points), is thus

$$\mathbf{M}^{\text{unst}} = \{(C,i,j) \in \mathbf{M} \mid \text{every } n \times n \text{ submtarix of } X(C,i,j) \text{ has } \det = 0\}.$$

Under our identification with affine space, the determinant of a submatrix yields a polynomial whose variables are the entries of the submatrix, hence $\mathbf{M}^{\text{unst}}$ is a vanishing set of polynomials, and thus a Zariski closed set. Therefore, $\mathbf{M}^{\mathrm{st}}$ is a Zariski open set.

Therefore, since $\mathbf{M}_0^{\mathrm{st}}$ is the intersection of an open subset and a closed subset of $\mathbf{M}$, it is a quasi-affine variety. ∎

**Lemma 3.5.** *The group $G_V$ acts freely on $\mathbf{M}^{\mathrm{st}}$.*

**Proof:** Let $(C,i,j) \in \mathbf{M}^{\mathrm{st}}$. Suppose $g = (g_k) \in G_V$ such that $g \cdot (C,i,j) = (C,i,j)$ (i.e. $gCg^{-1} = C$, $gi = i$, and $jg^{-1} = j$) and let $v \in V$. By Remark 3.3, $S(C,i,j) = V$, thus we may write $v$ as a finite sum:

$$v = \sum_q c_q C^{(p_q)} i(x_{k_q}),$$

where $c_q \in \mathbb{C}$, $p_q$ is a path in $\widetilde{Q}$ starting at $k_q$, $x_{k_q} \in W_{k_q}$, and $k_q \in \widetilde{Q}_0$, for each $q$. Applying $g$ to $v$ we get

$$g(v) = g\left(\sum_q c_q C^{(p_q)} i(x_{k_q})\right) = \sum_q c_q g C^{(p_q)} i(x_{k_q}) = \sum_q c_q C^{(p_q)} g i(x_{k_q})$$
$$= \sum_q c_q C^{(p_q)} i(x_{k_q}) = v.$$

Therefore, $g = 1$ and thus $G_V$ acts freely on $\mathbf{M}^{\mathrm{st}}$. ∎

**Definition 3.6** (Nakajima Quiver Variety)**.** The *Nakajima quiver variety* (associated to $Q$) is the geometric quotient

$$\mathfrak{M} = \mathfrak{M}(V,W) := \mathbf{M}_0^{\mathrm{st}}(V,W)/G_V.$$

We will write $[C,i,j]_{G_V}$ (or simply $[C,i,j]$ when there is no risk of confusion) to denote the $G_V$-orbit of a point $(C,i,j) \in \mathbf{M}_0^{\mathrm{st}}$.

As stated in Remark 2.2, geometric quotients are not guaranteed to exist, thus Defintion 3.6 requires some justification. The fact that there exists a geometric quotient of $\mathbf{M}_0^{\mathrm{st}}$ by $G_V$ is essentially proved by Nakajima in [21, Corollary 3.12], though our definition of quiver varieties in Definition 3.6 is seemingly different from Nakajima's definition in [21, Section 3.ii]. For convenience, we make precise the relation between the two definitions.

**Definition 3.7** (Costable)**.** An element $(C, i, j) \in \mathbf{M}$ is *costable* if whenever a $\widetilde{Q}_0$-graded $C$-invariant subspace $S \subseteq V$ satisfies $j(S) = 0$, then $S = 0$. Denote by $\mathbf{M}^{\mathrm{cost}}$ the set of costable points of $\mathbf{M}$ and by $\mathbf{M}_0^{\mathrm{cost}}$ the intersection $\mathbf{M}^{\mathrm{cost}} \cap \mu^{-1}(0)$.

**Remark 3.8.** Using arguments completely analogous to Lemmas 3.4 and Lemma 3.5, one can easily show that $\mathbf{M}_0^{\mathrm{cost}}$ is a quasi-affine variety and $G_V$ acts freely on $\mathbf{M}_0^{\mathrm{cost}}$.

**Definition 3.9** (Dual Nakajima quiver variety)**.** The *dual Nakajima quiver variety* (associated to $Q$) is the quotient

$$\mathfrak{M}^* = \mathfrak{M}^*(V, W) := \mathbf{M}_0^{\mathrm{cost}}(V, W)/G_V.$$

**Remark 3.10.** Note that in [21, Definition 3.9], Definition 3.7 is used as the definition of stability. By [21, Corollary 3.12], the Nakajima quiver variety defined in [21, Section 3.ii] corresponds to our definition of the *dual* Nakajima quiver variety in Definition 3.9. However the conditions of stability (in Definition 3.2) and costability (in Definition 3.7) are merely duals of each other, which yields an isomorphism of varieties between $\mathbf{M}_0^{\mathrm{st}}$ and $\mathbf{M}_0^{\mathrm{cost}}$. This in turn will allow us to naturally identify $\mathfrak{M}$ with $\mathfrak{M}^*$.

**Lemma 3.11.** *There exists an isomorphism of varieties* $\mathbf{M}_0^{\mathrm{st}}(V, W) \to \mathbf{M}_0^{\mathrm{cost}}(V^*, W^*)$ *and a group isomorphism* $G_V \to G_{V^*}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} G_V \times \mathbf{M}_0^{\mathrm{st}}(V,W) & \longrightarrow & \mathbf{M}_0^{\mathrm{st}}(V,W) \\ \downarrow & & \downarrow \\ G_{V^*} \times \mathbf{M}_0^{\mathrm{cost}}(V^*,W^*) & \longrightarrow & \mathbf{M}_0^{\mathrm{cost}}(V^*,W^*) \end{array},$$

*where the horizontal arrows represent the group action.*

**Proof:** Let $(C,i,j) \in \mathbf{M}$. For each $C_\rho \in \mathrm{Hom}(V_{t(\rho)}, V_{h(\rho)})$, we have the transpose map $(C_\rho)^* \in \mathrm{Hom}(V^*_{h(\rho)}, V^*_{t(\rho)})$, so we have a mapping $E_V \to E_{V^*}$ given by $C \mapsto C^*$, where

$$(C^*)_\rho = (C_{\bar\rho})^*,$$

for every $\rho \in \widetilde{Q}_1$. Moreover, since $i \in L_{W,V}$ and $j \in L_{V,W}$, we have $i^* \in L_{V^*,W^*}$ and $j^* \in L_{W^*,V^*}$. Hence, we have a map

$$\begin{aligned} \mathbf{M}(V,W) &\to \mathbf{M}(V^*,W^*), \\ (C,i,j) &\mapsto (C^*,j^*,i^*). \end{aligned} \tag{3.1}$$

We claim that above mapping induces a map $\mathfrak{M}(V,W) \to \mathfrak{M}^*(V^*,W^*)$. First, let $(C,i,j) \in \mathbf{M}_0^{\mathrm{st}}$. Note that $\varepsilon(C^*)C^* = C^*\varepsilon(C)^*$. Indeed, for every $\rho \in \widetilde{Q}_1$,

$$\varepsilon(C^*)_{\bar\rho}(C^*)_\rho = \varepsilon(\bar\rho)(C^*)_{\bar\rho}(C^*)_\rho = \varepsilon(\bar\rho)(C_\rho)^*(C_{\bar\rho})^* = (C_\rho)^*(\varepsilon(C)_{\bar\rho})^* = (C^*)_{\bar\rho}(\varepsilon(C)^*)_\rho.$$

Thus, for each $k \in \widetilde{Q}_0$, the $\mathrm{Hom}(V_k^*, V_k^*)$ component of $\varepsilon(C^*)C^*$ is

$$(\varepsilon(C^*)C^*)_k = \sum_{\rho\in\widetilde{Q}_1,\, t(\rho)=k} \varepsilon(C^*)_{\bar\rho}(C^*)_\rho = \sum_{\rho\in\widetilde{Q}_1,\, t(\rho)=k} (C^*)_{\bar\rho}(\varepsilon(C)^*)_\rho = (C^*\varepsilon(C)^*)_k.$$

Hence, applying the moment map to $(C^*,j^*,i^*)$ gives

$$\mu(C^*,j^*,i^*) = \varepsilon(C^*)C^* + j^*i^* = C^*\varepsilon(C)^* + j^*i^* = (\varepsilon(C)C + ij)^* = \mu(C,i,j)^* = 0,$$

where the final equality holds because $(C,i,j) \in \mu^{-1}(0)$. Next we show that $(C^*,j^*,i^*)$ is costable. Indeed, suppose $S = \bigoplus_{k\in\widetilde{Q}_0} S_k \subseteq V^*$ such that $S$ is $C^*$-invariant and

$i^*(S) = 0$. For each $k \in \widetilde{Q}_0$, let $U_k = \{v \in V_k \mid \varphi(v) = 0 \text{ for all } \varphi \in S_k\}$ and set $U = \bigoplus_{k \in \widetilde{Q}_0} U_k$. For any $\rho \in \widetilde{Q}_1$, $u \in U_{t(\rho)}$ and $\varphi \in S_{h(\rho)}$ we have

$$\varphi(C_\rho(u)) = (C_\rho)^*(\varphi)(u) = \underbrace{(C^*)_{\bar{\rho}}(\varphi)}_{\in S_{t(\rho)}}(u) = 0.$$

Therefore, $C_\rho(u) \in U_{h(\rho)}$, and so $U$ is $C$-invariant. Moreover, for any $i(x) \in \operatorname{im}(i)$ and $\varphi \in S$,

$$\varphi(i(x)) = \underbrace{i^*(\varphi)}_{=0}(x) = 0.$$

Thus, $i(x) \in U$, and so $\operatorname{im}(i) \subseteq U$. Because $(C, i, j)$ is stable, we have that $U = V$. By construction of $U$, $\varphi(V) = \varphi(U) = 0$ for every $\varphi \in S$, which means $S = 0$. Hence, $(C^*, j^*, i^*)$ is costable.

Therefore, the mapping $(C, i, j) \mapsto (C^*, j^*, i^*)$ is a mapping

$$\mathbf{M}_0^{\text{st}}(V, W) \to \mathbf{M}_0^{\text{cost}}(V^*, W^*).$$

Note that this mapping is a morphism of varieties (since by identifying $\mathbf{M}$ with affine space, this map corresponds to permuting coordinates and is thus a regular map). Furthermore, by a similar argument, one can show that the mapping

$$\begin{gathered} \mathbf{M}_0^{\text{cost}}(V^*, W^*) \to \mathbf{M}_0^{\text{st}}(V^{**}, W^{**}), \\ (C, i, j) \mapsto (C^*, j^*, i^*), \end{gathered} \tag{3.2}$$

is also a morphism of varieties. Via the canonical isomorphisms $V^{**} \cong V$ and $W^{**} \cong W$, one has a natural isomorphism of varieties

$$\mathbf{M}_0^{\text{st}}(V^{**}, W^{**}) \xrightarrow{\cong} \mathbf{M}_0^{\text{st}}(V, W). \tag{3.3}$$

The composition of the maps (3.2) and (3.3) is the inverse of map (3.1). Thus, the map (3.1) is an isomorphism of varieties $\mathbf{M}_0^{\text{st}}(V, W) \xrightarrow{\cong} \mathbf{M}_0^{\text{cost}}(V^*, W^*)$. Note also that under this mapping

$$g \cdot (C, i, j) = (gCg^{-1}, gi, jg^{-1}) \mapsto ((gCg^{-1})^*, (jg^{-1})^*, (gi)^*)$$

$$= ((g^*)^{-1} C^* g^*, (g^*)^{-1} j^*, i^* g^*)$$
$$= (g^*)^{-1} \cdot (C^*, j^*, i^*)$$

for all $g \in G_V$ and $(C, i, j) \in \mathbf{M}_0^{\mathrm{st}}(V, W)$. Hence, we have a commutative diagram

$$\begin{array}{ccc} G_V \times \mathbf{M}_0^{\mathrm{st}}(V,W) & \longrightarrow & \mathbf{M}_0^{\mathrm{st}}(V,W) \\ \downarrow & & \downarrow \\ G_{V^*} \times \mathbf{M}_0^{\mathrm{cost}}(V^*,W^*) & \longrightarrow & \mathbf{M}_0^{\mathrm{cost}}(V^*,W^*) \end{array},$$

where the horizontal arrows represent the group action and the mapping $G_V \times \mathbf{M}_0^{\mathrm{st}}(V, W) \to G_{V^*} \times \mathbf{M}_0^{\mathrm{cost}}(V^*, W^*)$ is given by $(g, (C, i, j)) \mapsto ((g^*)^{-1}, (C^*, j^*, i^*))$. ∎

**Corollary 3.12.** *The varieties $\mathfrak{M}(V, W)$ and $\mathfrak{M}^*(V^*, W^*)$ are isomorphic (as varieties).*

**Proof:** This follows from Lemma 3.11 and Lemma 2.3. ∎

**Corollary 3.13.** *The varietes $\mathbf{M}_0^{\mathrm{st}}$ and $\mathfrak{M}$ are smooth.*

**Proof:** By Corollary [21, Lemma 3.10, Corollary 3.12], we have that $\mathbf{M}_0^{\mathrm{cost}}$ and $\mathfrak{M}^*$ are smooth varieties. Thus, by Lemma 3.11 and Corollary 3.12, $\mathbf{M}_0^{\mathrm{st}}$ and $\mathfrak{M}$ are also smooth varieties. ∎

Let $(C, i, j) \in \mathbf{M}_0^{\mathrm{st}}$. The tangent space of $\mathbf{M}_0^{\mathrm{st}}$ at $(C, i, j)$ is

$$\mathcal{T}_{(C,i,j)}(\mathbf{M}_0^{\mathrm{st}}) = \mathcal{T}_{(C,i,j)}(\mu^{-1}(0)) = \ker \mathrm{d}\mu,$$

and one can easily check that the differential of the moment map $\mu$ at $(C, i, j)$ is

$$\begin{aligned} \mathrm{d}\mu : \mathbf{M} &\to L_{V,V}, \\ (D, a, b) &\mapsto \varepsilon(C)D + \varepsilon(D)C + ib + aj. \end{aligned}$$

By Corollary 2.6, the tangent space $\mathcal{T}_{[C,i,j]}(\mathfrak{M})$ can then be identified with the quotient $\ker \mathrm{d}\mu / \operatorname{im} \mathrm{d}\sigma^{(C,i,j)}$. Here, $\sigma^{(C,i,j)} : G_V \to \mathbf{M}_0^{\mathrm{st}}$ is the map given by $g \mapsto (gCg^{-1}, gi, jg^{-1})$, and so

$$\begin{aligned} \mathrm{d}\sigma^{(C,i,j)} : L_{V,V} &\to \mathcal{T}_{(C,i,j)}(\mathbf{M}_0^{\mathrm{st}}) = \ker \mathrm{d}\mu, \\ X &\mapsto (XC - CX, Xi, -jX). \end{aligned}$$

**Lemma 3.14.** *Let $(C, i, j) \in \mathbf{M}_0^{\mathrm{st}}$. Then*

1. *$\mathrm{d}\sigma^{(C,i,j)}$ is injective and $\mathrm{d}\mu$ (at $(C, i, j)$) is surjective, and*
2. *the tangent space of $\mathfrak{M}$ at $[C, i, j]$ may be identified with the middle cohomology of the following complex:*

$$L_{V,V} \xhookrightarrow{\mathrm{d}\sigma^{(C,i,j)}} E_V \oplus L_{W,V} \oplus L_{V,W} \xrightarrow{\mathrm{d}\mu} L_{V,V}. \tag{3.4}$$

**Proof:** The proof of (1) is completely analogous to [15, Lemma 3.2]. Part (2), as above, simply follows from Corollary 2.6. ∎

It is worth noting that, up to isomorphism, the varieties $\mathbf{M}_0^{\mathrm{st}}(V, W)$ and $\mathfrak{M}(V, W)$ are parametrized by the graded dimensions of $V$ and $W$. Indeed, let $\mathbf{v} = (\mathbf{v}_k), \mathbf{w} = (\mathbf{w}_k) \in \mathbb{N}^{\widetilde{Q}_0}$ and, for $i = 1, 2$, let $V^i$, $W^i$ be $\widetilde{Q}_0$-graded vector spaces such that $\dim V_k^i = \mathbf{v}_k$ and $\dim W_k^i = \mathbf{w}_k$ for all $k \in \widetilde{Q}_0$. Fix $\widetilde{Q}_0$-graded vector space isomorphisms $f : V^1 \to V^2$ and $h : W^1 \to W^2$. Then we define maps

$$\mathbf{M}_0^{\mathrm{st}}(V^1, W^1) \to \mathbf{M}_0^{\mathrm{st}}(V^2, W^2),$$

$$(C, i, j) \mapsto (fCf^{-1}, fih^{-1}, hif^{-1}),$$

and

$$G_{V^1} \to G_{V^2},$$
$$g \mapsto fgf^{-1}.$$

One can easily check that these maps are isomorphisms of varieties and groups, respectively. Thus, $\mathbf{M}_0^{\mathrm{st}}(V^1, W^1) \cong \mathbf{M}_0^{\mathrm{st}}(V^2, W^2)$ and, by Lemma 2.3, $\mathfrak{M}(V^1, W^1) \cong \mathfrak{M}(V^2, W^2)$. Hence, we define one standard representative for each isomorphism class:

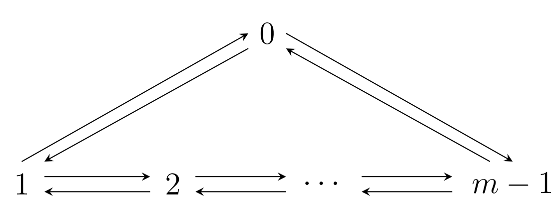
$$\mathbf{M}_0^{\mathrm{st}}(\mathbf{v}, \mathbf{w}) := \mathbf{M}_0^{\mathrm{st}}\left(\bigoplus_{k \in \widetilde{Q}_0} \mathbb{C}^{\mathbf{v}_k}, \bigoplus_{k \in \widetilde{Q}_0} \mathbb{C}^{\mathbf{w}_k}\right),$$

and

$$\mathfrak{M}(\mathbf{v}, \mathbf{w}) := \mathfrak{M}\left(\bigoplus_{k \in \widetilde{Q}_0} \mathbb{C}^{\mathbf{v}_k}, \bigoplus_{k \in \widetilde{Q}_0} \mathbb{C}^{\mathbf{w}_k}\right) = \mathbf{M}_0^{\mathrm{st}}(\mathbf{v}, \mathbf{w})/G_{\mathbf{v}},$$

where $G_{\mathbf{v}} = \prod_{k \in \widetilde{Q}_0} \mathrm{GL}_{\mathbf{v}_k}(\mathbb{C})$.

From now on, we restrict ourselves to the case where $Q$ is a quiver of type $\widehat{A}_{m-1}$, $m \in \mathbb{N}^+$ (with the case $m = 1$ corresponding to the quiver consisting of one vertex and one loop). We label the vertices of $Q$ by $\{0, 1, \ldots, m-1\}$, and hence we can identify $Q_0 = \widetilde{Q}_0$ with $\mathbb{Z}_m$. In this case, we have that $\widetilde{Q}$ is the quiver:




We denote by $\mathbf{M}(m; \mathbf{v}, \mathbf{w})$ (resp. $\mathfrak{M}(m; \mathbf{v}, \mathbf{w})$) the variety $\mathbf{M}(\mathbf{v}, \mathbf{w})$ (resp. $\mathfrak{M}(\mathbf{v}, \mathbf{w})$) corresponding to a quiver of type $\widehat{A}_{m-1}$ (recall that these varieties do not depend on the orientation of $Q$, thus $\mathbf{M}(m; \mathbf{v}, \mathbf{w})$ and $\mathfrak{M}(m; \mathbf{v}, \mathbf{w})$ are well-defined). Of particular interest to us will be the case $m = 1$, for which we introduce the special notation:

$$M(s, n) := \mathbf{M}(1; (n), (s)), \quad \mathcal{M}(s, n) := \mathfrak{M}(1; (n), (s)).$$

Note that in this case

$$M(s,n) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^s, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^s),$$

and so we will denote elements of $M(s,n)$ by $(A,B,i,j)$ and the elements of $\mathcal{M}(s,n)$ by $[A,B,i,j]$, where, by convention, $A$ represents the linear map associated to the loop in $Q_1$ and $B$ the linear map associated to the loop in $\overline{Q}_1$. The moment map $\mu$ then simplifies to

$$\mu(A,B,i,j) = [A,B] + ij.$$

Thus, by [22, Theorem 2.1], $\mathcal{M}(s,n)$ is isomorphic to the moduli space of framed torsion-free sheaves on $\mathbb{P}^2$ with rank $s$ and second Chern class $c_2 = n$.

Fix an $(s+1)$-dimensional torus

$$T = (\mathbb{C}^*)^s \times \mathbb{C}^*,$$

and denote elements of $T$ by $(e,t)$, where $e = (e_1, \dots, e_s) \in (\mathbb{C}^*)^s$ and $t \in \mathbb{C}^*$. We have a natural action of $(\mathbb{C}^*)^s$ on $\mathbb{C}^s$ given by

$$e \cdot (w_1, \dots, w_s) = (e_1 w_1, \dots, e_s w_s),$$

for all $e \in (\mathbb{C}^*)^s$ and $(w_1, \dots, w_s) \in \mathbb{C}^s$. For each $\mathbf{c} \in \mathbb{Z}^s$, we have an action of $T$ on $M(s,n)$ given by

$$(e,t) \cdot (A,B,i,j) = (tA, t^{-1}B, ie^{-1}t^{-\mathbf{c}}, et^{\mathbf{c}}j),$$

where

$$t^{\mathbf{c}} := (t^{\mathbf{c}_1}, \dots, t^{\mathbf{c}_s}).$$

The torus action preserves the space $M_0^{\mathrm{st}}(s,n)$ and commutes with the action of $\mathrm{GL}_n(\mathbb{C})$, thus we have a well-defined action on $\mathcal{M}(s,n)$:

$$(e,t) \cdot [A,B,i,j] = [tA, t^{-1}B, ie^{-1}t^{-\mathbf{c}}, et^{\mathbf{c}}j]. \tag{3.5}$$

Let $\mathcal{M}_{\mathbf{c}}(s,n)$ denote the moduli space $\mathcal{M}(s,n)$ with the torus action given by Equation (3.5).

We restrict our focus for now to the case $s = 1$. It is known that $(A,B,i,j) \in M_0^{\mathrm{st}}(1,n) \implies j = 0$ (see [22, Proposition 2.7]), thus we will simply write $(A,B,i)$ for an element $(A,B,i,0) \in M_0^{\mathrm{st}}(1,n)$. Let $\omega := e^{2\pi\sqrt{-1}/m}$. The finite cyclic group of order $m$ acts on $\mathcal{M}_c(1,n)$ via the embedding

$$\begin{aligned} \mathbb{Z}_m &\hookrightarrow T, \\ k &\mapsto (1,\omega^k). \end{aligned} \tag{3.6}$$

That is

$$k \cdot [A,B,i] = [\omega^k A, \omega^{-k} B, \omega^{-kc} i],$$

for all $k \in \mathbb{Z}_m$ and $[A,B,i] \in \mathcal{M}_c(1,n)$. Recall that the fixed point set $\mathcal{M}_c(1,n)^{\mathbb{Z}_m}$ is a closed subvariety of $\mathcal{M}_c(1,n)$ (see the discussion preceding Lemma 2.9). The goal for the remainder of this chapter will be to describe $\mathcal{M}_c(1,n)^{\mathbb{Z}_m}$ in terms of quiver varieties of type $\widehat{A}_{m-1}$.

**Remark 3.15.** Let

$$F_c(n) := \left\{(A,B,i) \in M_0^{\mathrm{st}}(1,n) \mid [A,B,i] \in \mathcal{M}_c(1,n)^{\mathbb{Z}_m}\right\},$$

i.e. we let $F_c(n)$ be the preimage of $\mathcal{M}_c(1,n)^{\mathbb{Z}_m}$ under the projection $M_0^{\mathrm{st}}(1,n) \twoheadrightarrow \mathcal{M}_c(1,n)$. Thus, $F_c(n)$ is a closed subvariety of $M_0^{\mathrm{st}}(1,n)$. We have that $(A,B,i) \in F_c(n)$ if and only if there exists a $g \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\begin{aligned} \omega A &= g^{-1} A g, \\ \omega^{-1} B &= g^{-1} B g, \\ \omega^{-c} i &= g^{-1} i. \end{aligned} \tag{3.7}$$

Note that since $\mathrm{GL}_n(\mathbb{C})$ acts freely on $M_0^{\mathrm{st}}$, such a $g$, if it exists, is unique. Assume that $(A,B,i) \in F_c(n)$. By Remark 3.3, every $v \in V$ may be written as a finite sum

$$v = \sum_{p,q} \lambda_{pq} A^p B^q i,$$

where $\lambda_{pq} \in \mathbb{C}$ (note that $A$ and $B$ commute since the moment map implies that $[A, B] = 0$) and we identify the map $i : \mathbb{C} \to \mathbb{C}^n$ with $i(1)$. For any $p$ and $q$,

$$g(\lambda_{pq} A^p B^q i) = \omega^{-p} \lambda_{pq} A^p g B^q i = \omega^{q-p} \lambda_{pq} A^p B^q g i = \omega^{q-p+c} \lambda_{pq} A^p B^q i.$$

Thus, we have a weight space decomposition

$$\mathbb{C}^n = \bigoplus_{k \in \mathbb{Z}_m} V_k(A, B, i), \quad V_k(A, B, i) := \{v \in \mathbb{C}^n \mid gv = \omega^k v\}.$$

Note that by the above calculation, it is easily seen that

$$V_k(A, B, i) = \operatorname{span}\{A^p B^q i \mid (q - p) \equiv (k - c) \mod m\}. \tag{3.8}$$

We observe that $i \in V_{\bar{c}}$ (where $\bar{c}$ is the equivalency class of $c$ in $\mathbb{Z}_m$) and the restrictions of $A$ and $B$ to $V_k$ yield maps

$$A|_{V_k(A,B,i)} : V_k(A, B, i) \to V_{k-1}(A, B, i) \quad \text{and} \quad B|_{V_k(A,B,i)} : V_k(A, B, i) \to V_{k+1}(A, B, i).$$

Conversely, suppose we have a weight space decomposition $\mathbb{C}^n = \bigoplus_{k \in \mathbb{Z}_m} V_k$ such that $A|_{V_k} : V_k \to V_{k-1}$ and $B|_{V_k} : V_k \to V_{k+1}$ and $i \in V_{\bar{c}}$. Set

$$g = \prod_{k \in \mathbb{Z}_m} \omega^k \operatorname{id}_{V_k}.$$

Then $g^{-1} A g = \omega A$, $g^{-1} B g = \omega^{-1} B$ and $g^{-1} i = \omega^{-c} i$, and so $(A, B, i) \in F_c(n)$. Moreover, $V_k(A, B, i) = V_k$ for each $k \in \mathbb{Z}_m$.

**Lemma 3.16.** *Let $(A, B, i) \in F_c(n)$ and $h \in \mathrm{GL}_n(\mathbb{C})$. Then*

$$V_k(h \cdot (A, B, i)) = h V_k(A, B, i),$$

*for all $k \in \mathbb{Z}_m$.*

**Proof:** Write

$$(A', B', i') = h \cdot (A, B, i) = (hAh^{-1}, hBh^{-1}, hi).$$

Let $g \in \mathrm{GL}_n(\mathbb{C})$ be the (unique) element satisfying (3.7) for $(A, B, i)$. Let $g' = hgh^{-1}$. Then

$$(g')^{-1}A'g' = (hg^{-1}h^{-1})(hAh^{-1})(hg^{-1}h^{-1}) = h(g^{-1}Ag)h^{-1} = \omega(hAh^{-1}) = \omega A'.$$

Similarly, $(g')^{-1}B'g' = \omega^{-1}B'$. Moreover,

$$(g')^{-1}i' = (hg^{-1}h^{-1})(hi) = h(g^{-1}i) = \omega^{-c}hi = \omega^{-c}i'.$$

Hence, $g'$ is the unique element in $\mathrm{GL}_n(\mathbb{C})$ satisfying (3.7) for $(A', B', i')$. Thus,

$$V_k(A, B, i) = \{v \in \mathbb{C}^n \mid gv = \omega^k v\}, \quad \text{and} \quad V_k(A', B', i') = \{v \in \mathbb{C}^n \mid g'v = \omega^k v\}.$$

For each $v \in V_k(A, B, i)$,

$$g'(hv) = (hgh^{-1})(hv) = h(gv) = \omega^k(hv),$$

and so $hv \in V_k(A', B', i')$. Thus, $h$ is a linear map $V_k(A, B, i) \to V_k(A', B', i')$. Since $h$ is invertible, $V_k(A', B', i') = hV_k(A, B, i)$. ∎

**Lemma 3.17.** *1. The variety $\mathcal{M}_c(1, n)^{\mathbb{Z}_m}$ is smooth.*

*2. The tangent space of $\mathcal{M}_c(1, n)^{\mathbb{Z}_m}$ at $[A, B, i]$ may be identified with the middle cohomology of the following complex:*

$$L \xhookrightarrow{\mathrm{d}\sigma^{(A,B,i)}} E^- \oplus E^+ \oplus \mathrm{Hom}(\mathbb{C}, V_{\bar{c}}) \oplus \mathrm{Hom}(V_{\bar{c}}, \mathbb{C}) \xrightarrow{\mathrm{d}\mu} L, \tag{3.9}$$

*where $L = \bigoplus_{k \in \mathbb{Z}_m} \mathrm{Hom}(V_k, V_k)$ and $E^{\pm} = \bigoplus_{k \in \mathbb{Z}_m} \mathrm{Hom}(V_k, V_{k\pm 1})$.*

**Proof:** For Part (1), note that $\mathbb{Z}_m$ is a compact Lie group, and thus, the category of finite-dimensional linear representations of $\mathbb{Z}_m$ is semisimple. Therefore, by [9, Proposition 1.3], $\mathcal{M}_c(1, n)^{\mathbb{Z}_m}$ is smooth. For Part (2), we first note that the tangent space of $\mathcal{M}_c(1,n)$ is obtained by setting $V = \mathbb{C}^n$ and $W = \mathbb{C}$ in Complex (3.4), i.e. the tangent space of $\mathcal{M}_c(1,n)$ is the middle cohomology of

$$\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n) \xhookrightarrow{\mathrm{d}\sigma^{(A,B,i)}} \begin{matrix} \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n) \\ \oplus \\ \mathrm{Hom}(\mathbb{C},\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}) \end{matrix} \xrightarrow{\mathrm{d}\mu} \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n). \tag{3.10}$$

Therefore, by Lemma 2.9, the tangent space of $\mathcal{M}_c(1,n)^{\mathbb{Z}_m} \cong (\ker \mathrm{d}\mu / \operatorname{im} \mathrm{d}\sigma^{(A,B,i)})^{\mathbb{Z}_m}$. Let $\mathbb{Z}_m$ act on $\mathbb{C}^n$ by

$$\omega \cdot v = gv,$$

for all $v \in V$, where $g$ is the element in $\mathrm{GL}_n(\mathbb{C})$ satisfying properties (3.7) for $(A,B,i)$. Likewise, let $\mathbb{Z}_m$ act on $\mathbb{C}$ by

$$\omega \cdot u = \omega^c u,$$

for all $u \in \mathbb{C}$. Then $\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)$, $\mathrm{Hom}(\mathbb{C},\mathbb{C}^n)$ and $\mathrm{Hom}(\mathbb{C}^n,\mathbb{C})$ naturally become $\mathbb{Z}_m$-modules. In order for $\mathrm{d}\sigma^{(A,B,i)}$ and $\mathrm{d}\mu$ to be $\mathbb{Z}_m$-module maps, we introduce the one-dimensional modules $\omega$ and $\omega^{-1}$, (on which $\omega$ acts by multiplication by $\omega$ and $\omega^{-1}$, respectively), and modify Complex (3.10) to

$$\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n) \xhookrightarrow{\mathrm{d}\sigma^{(A,B,i)}} \begin{matrix} \omega\,\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n) \oplus \omega^{-1}\,\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n) \\ \oplus \\ \mathrm{Hom}(\mathbb{C},\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}) \end{matrix} \xrightarrow{\mathrm{d}\mu} \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n),$$

where we use juxtaposition to indicate the tensor product. One checks that the action of $\mathbb{Z}_m$ on $\ker \mathrm{d}\mu / \operatorname{im} \mathrm{d}\sigma^{(A,B,i)}$ induced by the action on $\mathbb{C}^n$ and $\mathbb{C}$ matches the action described in Remark 2.8. Thus, by Lemma 2.10, $(\ker \mathrm{d}\mu / \operatorname{im} \mathrm{d}\sigma^{(A,B,i)})^{\mathbb{Z}_m} \cong \ker \mathrm{d}\mu_{\mathbb{Z}_m} / \operatorname{im} \mathrm{d}\sigma^{(A,B,i)}_{\mathbb{Z}_m}$, where $\mathrm{d}\mu_{\mathbb{Z}_m}$ and $\mathrm{d}\sigma^{(A,B,i)}_{\mathbb{Z}_m}$ are the restrictions of $\mathrm{d}\mu$ and $\mathrm{d}\sigma^{(A,B,i)}$ to the fixed points of their respective domains. Now, $f \in \mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)^{\mathbb{Z}_m}$ if and only if $f = gfg^{-1}$, which occurs if and only if $f|_{V_k} : V_k \to V_k$. Therefore,

$$\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)^{\mathbb{Z}_m} \cong \bigoplus_{k \in \mathbb{Z}_m} \mathrm{Hom}(V_k, V_k).$$

Similarly, one can show that

$$(\omega^{\pm 1}\operatorname{Hom}(\mathbb{C}^n,\mathbb{C}^n))^{\mathbb{Z}_m} \cong \bigoplus_{k\in\mathbb{Z}_m} \operatorname{Hom}(V_k, V_{k\mp 1}), \quad \operatorname{Hom}(\mathbb{C},\mathbb{C}^n)^{\mathbb{Z}_m} \cong \operatorname{Hom}(\mathbb{C}, V_{\bar{c}}),$$

$$\text{and} \quad \operatorname{Hom}(\mathbb{C}^n,\mathbb{C})^{\mathbb{Z}_m} \cong \operatorname{Hom}(V_{\bar{c}},\mathbb{C}),$$

which completes the proof. ∎

Let $(C,i,j) \in \mathbf{M}(m;\mathbf{v},\mathbf{1}_{\bar{c}})$. We identify $\bigoplus_{k\in\mathbb{Z}_m}\mathbb{C}^{\mathbf{v}_k}$ with $\mathbb{C}^{|\mathbf{v}|}$ by identifying $\mathbf{1}^k_\ell$ with $\mathbf{1}_{\mathbf{v}_0+\cdots+\mathbf{v}_{k-1}+\ell}$, where $\{\mathbf{1}^k_\ell\}_{\ell=1}^{\mathbf{v}_k}$ and $\{\mathbf{1}_\ell\}_{\ell=1}^{|\mathbf{v}|}$ denote the standard bases of $\mathbb{C}^{\mathbf{v}_k}$ and $\mathbb{C}^{|\mathbf{v}|}$, respectively. Let $A_C, B_C \in \operatorname{Hom}\left(\mathbb{C}^{|\mathbf{v}|},\mathbb{C}^{|\mathbf{v}|}\right)$ be the maps determined by

$$(A_C)|_{\mathbb{C}^{\mathbf{v}_k}} = \varepsilon(\rho)C_\rho \qquad \text{and} \qquad (B_C)|_{\mathbb{C}^{\mathbf{v}_k}} = C_\tau, \tag{3.11}$$

for each $k\in\mathbb{Z}_m$, $\rho : k \to k-1$ in $\widetilde{Q}_1$ and $\tau : k \to k+1$ in $\widetilde{Q}_1$ (note that in the $m = 2$ case, we simply make a choice as to which arrow of $Q_1$ corresponds to $k \to k+1$ and which one corresponds to $k \to k-1$). We thus have a mapping

$$\begin{aligned} \mathbf{M}(m;\mathbf{v},\mathbf{1}_{\bar{c}}) &\to M(1,|\mathbf{v}|), \\ (C,i,j) &\mapsto (A_C, B_C, i, j). \end{aligned} \tag{3.12}$$

Suppose that $\mu(C,i,j) = 0$. For every vertex $k \in \widetilde{Q}_0$, we have four arrows incident to $k$ as in the following diagram.

$$k-1 \underset{\alpha}{\overset{\bar{\alpha}}{\rightleftarrows}} k \underset{\beta}{\overset{\bar{\beta}}{\rightleftarrows}} k+1 \tag{3.13}$$

Thus, the $\operatorname{Hom}(\mathbb{C}^{\mathbf{v}_k},\mathbb{C}^{\mathbf{v}_k})$ component of $\varepsilon(C)C$ is

$$(\varepsilon(C)C)_k = \varepsilon(\bar{\alpha})C_{\bar{\alpha}}C_\alpha + \varepsilon(\beta)C_\beta C_{\bar{\beta}} = -C_{\bar{\alpha}}(\varepsilon(\alpha)C_\alpha) + (\varepsilon(\beta)C_\beta)C_{\bar{\beta}} = [A_C,B_C]|_{\mathbb{C}^{\mathbf{v}_k}}.$$

Hence, $\varepsilon(C)C = [A_C, B_C]$. Therefore,

$$[A_C,B_C] + ij = \varepsilon(C)C + ij = \mu(C,i,j) = 0.$$

Now, suppose $(C, i, j)$ is stable. Furthermore, suppose $S$ is a subspace of $\mathbb{C}^{|\mathbf{v}|}$ such that $A_C(S) \subseteq S$, $B_C(S) \subseteq S$ and $\operatorname{im}(i) \subseteq S$. For each $k \in \mathbb{Z}_m$, let $S_k$ be the vector space spanned by elements of the form:

$$D_{\ell_1} D_{\ell_2} \cdots D_{\ell_p} i(1),$$

where each $\ell_i \in \{1, 2\}$, $p \in \mathbb{N}$, $D_1 = A_C$, $D_2 = B_C$ and

$$\#\{\ell_i \mid \ell_i = 2\} - \#\{\ell_i \mid \ell_i = 1\} \equiv k - c \mod m.$$

Then $S = \bigoplus_{k \in \mathbb{Z}_m} S_k$ and, by construction, $S$ is thus a $C$-invariant subspace of $\mathbb{C}^{|\mathbf{v}|}$. Since $(C, i, j)$ is stable, $S = \mathbb{C}^{|\mathbf{v}|}$. Therefore, $(A_C, B_C, i, j)$ is stable. In particular, this means that if $(C, i, j) \in \mathbf{M}_0^{\mathrm{st}}(m; \mathbf{v}, \mathbf{1}_{\bar{c}})$, then $(A_C, B_C, i, j) \in M_0^{\mathrm{st}}(1, |\mathbf{v}|)$. In particular, by [22, Proposition 2.7], this implies $j = 0$. Since this fact turns out to be rather important, we highlight it with the following lemma.

**Lemma 3.18.** *If $(C, i, j) \in \mathbf{M}_0^{\mathrm{st}}(m; \mathbf{v}, \mathbf{w})$ with $|\mathbf{w}| = 1$, then $j = 0$.*

By the discussion above, the mapping (3.12) maps $\mathbf{M}_0^{\mathrm{st}} \to M_0^{\mathrm{st}}$. Thus, by Equation (3.11), Lemma 3.18 and Remark 3.15, we may define a morphism of varieties

$$\begin{aligned} \varphi_{\mathbf{v}} : \mathbf{M}_0^{\mathrm{st}}(m; \mathbf{v}, \mathbf{1}_{\bar{c}}) &\to F_c(|\mathbf{v}|), \\ (C, i, 0) &\mapsto (A_C, B_C, i). \end{aligned}$$

**Lemma 3.19.** *The map $\varphi_{\mathbf{v}}$ induces a morphism of varieties*

$$\begin{aligned} \overline{\varphi}_{\mathbf{v}} : \mathfrak{M}(m; \mathbf{v}, \mathbf{1}_{\bar{c}}) &\to \mathcal{M}_c(1, |\mathbf{v}|)^{\mathbb{Z}_m}, \\ [C, i, 0] &\mapsto [A_C, B_C, i]. \end{aligned}$$

*In particular, one has a morphism of varieties $\overline{\varphi} : \coprod_{|\mathbf{v}|=n} \mathfrak{M}(m; \mathbf{v}, \mathbf{1}_{\bar{c}}) \to \mathcal{M}_c(1, n)^{\mathbb{Z}_m}$, given by $\overline{\varphi} = \coprod_{|\mathbf{v}|=n} \overline{\varphi}_{\mathbf{v}}$.*

**Proof:** Let $\psi : G_{\mathbf{v}} \to \mathrm{GL}_{|\mathbf{v}|}(\mathbb{C})$ be the group homomorphism induced by our identification of $\bigoplus_{k\in\mathbb{Z}_m} \mathbb{C}^{\mathbf{v}_k}$ with $\mathbb{C}^{|\mathbf{v}|}$, i.e. partitioning $\mathrm{GL}_{|\mathbf{v}|}(\mathbb{C})$ into $m^2$ blocks of size $\mathbf{v}_i \times \mathbf{v}_j$, $i, j = 0, 1, \ldots, m-1$, we embed $\mathrm{GL}_{\mathbf{v}_k}(\mathbb{C})$ into the $(k, k)$-th diagonal block of $\mathrm{GL}_{|\mathbf{v}|}(\mathbb{C})$, for each $k = 0, 1, \ldots, m-1$. One then has the following commutative diagram:

$$\begin{array}{ccc} G_{\mathbf{v}} \times \mathbf{M}_0^{\mathrm{st}}(m; \mathbf{v}, \mathbf{1}_{\bar{c}}) & \longrightarrow & \mathbf{M}_0^{\mathrm{st}}(m; \mathbf{v}, \mathbf{1}_{\bar{c}}) \\ \psi \times \varphi \Big\downarrow & & \Big\downarrow \varphi \\ \mathrm{GL}_{|\mathbf{v}|}(\mathbb{C}) \times F_c(|\mathbf{v}|) & \longrightarrow & F_c(|\mathbf{v}|), \end{array}$$

where the horizontal arrows represent the group action. The result then follows by Lemma 2.3. ∎

**Theorem 3.20.** *The map $\overline{\varphi}$ from Lemma 3.19 is an isomorphism. Therefore,*

$$\mathcal{M}_c(1, n)^{\mathbb{Z}_m} \cong \coprod_{|\mathbf{v}|=n} \mathfrak{M}(m; \mathbf{v}, \mathbf{1}_{\bar{c}}),$$

*as varieties.*

**Proof:** We first begin by showing that $\overline{\varphi}$ is bijective. Suppose that $(C, i, 0) \in \mathbf{M}_0^{\mathrm{st}}(m; \mathbf{v}, \mathbf{1}_{\bar{c}})$ and $(D, a, 0) \in \mathbf{M}_0^{\mathrm{st}}(m; \mathbf{u}, \mathbf{1}_{\bar{c}})$, with $|\mathbf{v}| = |\mathbf{u}| = n$, are such that

$$\overline{\varphi}[C, i, 0] = \overline{\varphi}[D, a, 0],$$

i.e. $[A_C, B_C, i] = [A_D, B_D, a]$. Then there exists $g \in \mathrm{GL}_n(\mathbb{C})$ such that $(A_C, B_C, i) = g \cdot (A_D, B_D, a)$. For each $k \in \mathbb{Z}_m$,

$$\mathbb{C}^{\mathbf{v}_k} = V_k(A_C, B_C, i) = g(V_k(A_D, B_D, a)) = g(\mathbb{C}^{\mathbf{u}_k}),$$

where the second equality follows from Lemma 3.16. In particular, $\mathbf{v} = \mathbf{u}$. Moreover, since $g|_{\mathbb{C}^{\mathbf{v}_k}} : \mathbb{C}^{\mathbf{v}_k} \to \mathbb{C}^{\mathbf{v}_k}$, we may view $g$ as an element of $G_{\mathbf{v}}$. One then easily verifies that $(C, i, 0) = g \cdot (D, a, 0)$. Thus, $[C, i, 0] = [D, a, 0]$, and so $\overline{\varphi}$ is injective.

Now, let $(A, B, i) \in F_c(n)$. Set $\mathbf{v}_k = \dim V_k(A, B, i)$ and choose a graded linear isomorphism

$$f : \bigoplus_{k \in \mathbb{Z}_m} V_k(A, B, i) \to \bigoplus_{k \in \mathbb{Z}_m} \mathbb{C}^{\mathbf{v}_k}.$$

Then $[A, B, i] = [fAf^{-1}, fBf^{-1}, fi]$. Define $(C, a, 0) \in \mathbf{M}\,(m; \mathbf{v}, \mathbf{1}_{\bar{c}})$ by

$$C_\rho = \begin{cases} \varepsilon(\rho)(fAf^{-1})|_{\mathbb{C}^{\mathbf{v}_k}}, & \text{if } \rho : k \to k-1, \\ (fBf^{-1})|_{\mathbb{C}^{\mathbf{v}_k}}, & \text{if } \rho : k \to k+1, \end{cases} \quad \text{and} \quad a = fi,$$

for all $\rho \in \widetilde{Q}_1$. We claim that $(C, a, 0) \in \mathbf{M}_0^{\text{st}}(m; \mathbf{v}, \mathbf{1}_{\bar{c}})$. To show that $\mu(C, a, 0) = 0$, recall that for each $k \in \mathbb{Z}_m$, one has 4 arrows incident to $k$ as in Diagram (3.13). Thus, the $\text{Hom}(\mathbb{C}^{\mathbf{v}_k}, \mathbb{C}^{\mathbf{v}_k})$-component of $\varepsilon(C)C$ is

$$\begin{aligned}(\varepsilon(C)C)_k &= \varepsilon(\bar{\alpha})C_{\bar{\alpha}}C_\alpha + \varepsilon(\beta)C_\beta C_{\bar{\beta}} \\ &= \varepsilon(\bar{\alpha})\varepsilon(\alpha)(fBf^{-1})|_{\mathbb{C}^{\mathbf{v}_{k-1}}}(fAf^{-1})|_{\mathbb{C}^{\mathbf{v}_k}} + \varepsilon(\beta)^2(fAf^{-1})|_{\mathbb{C}^{\mathbf{v}_{k+1}}}(fBf^{-1})|_{\mathbb{C}^{\mathbf{v}_k}} \\ &= [fAf^{-1}, fBf^{-1}]|_{\mathbb{C}^{\mathbf{v}_k}}.\end{aligned}$$

Thus, applying the moment map to $(C, a, 0)$,

$$\mu(C, a, 0) = \varepsilon(C)C = [fAf^{-1}, fBf^{-1}] = f[A, B]f^{-1} = 0,$$

where the last equality follows from the fact that $\mu(A, B, i) = 0$. Therefore, $(C, a, 0) \in \mu^{-1}(0)$. To show that $(C, a, 0)$ is stable, suppose that $S = \bigoplus_{k \in \mathbb{Z}_m} S_k$ is a $C$-invariant subspace of $\bigoplus_{k \in \mathbb{Z}_m} \mathbb{C}^{\mathbf{v}_k}$ with $a \in S$. Then by construction, $A(f^{-1}(S)), B(f^{-1}(S)) \subseteq f^{-1}(S)$ and $i \in f^{-1}(S)$. Since $(A, B, i)$ is stable, $f^{-1}(S) = \bigoplus_{k \in \mathbb{Z}_m} V_k$, and thus $S = \bigoplus_{k \in \mathbb{Z}_m} \mathbb{C}^{\mathbf{v}_k}$. Hence, $(C, a, 0)$ is stable. Therefore, $(C, a, 0) \in \mathbf{M}_0^{\text{st}}(m; \mathbf{v}, \mathbf{1}_{\bar{c}})$ and

$$\overline{\varphi}[C, a, 0] = [fAf^{-1}, fBf^{-1}, fi] = [A, B, i].$$

Hence, $\overline{\varphi}$ is surjective.

Next, we show that $\overline{\varphi}$ is an étale morphism. Recall that we identify the tangent spaces of $\mathfrak{M}$ and $\mathcal{M}^{\mathbb{Z}_m}$ with the middle cohomologies of Complex (3.4) and Complex

(3.9), respectively. By construction of $\overline{\varphi}$, the induced map $\mathrm{d}\overline{\varphi}$ on the tangent spaces is

$$\mathrm{d}\overline{\varphi} : \mathcal{T}_{[C,i,0]}(\mathfrak{M}) \to \mathcal{T}_{[A_C,B_C,i]}(\mathcal{M}^{\mathbb{Z}_m}),$$
$$(D,a,b) + \operatorname{im} \mathrm{d}\sigma^{(C,i,j)} \mapsto (D^-, D^+, a, b) + \operatorname{im} \mathrm{d}\sigma^{(A_C,B_C,i)},$$

where $(D^-)_k = \varepsilon(\rho) D_{\rho:k\to k-1}$ and $(D^+)_k = D_{\rho:k\to k+1}$ for all $k \in \mathbb{Z}_m$. Clearly, $\mathrm{d}\overline{\varphi}$ is injective. Moreover, by Lemma 3.14 and Lemma 3.17, $\mathcal{T}_{[C,i,0]}(\mathfrak{M})$ and $\mathcal{T}_{[A_C,B_C,i]}(\mathcal{M}^{\mathbb{Z}_m})$ have the same dimension, and hence $\mathrm{d}\overline{\varphi}$ is an isomorphism. By Lemma 2.11, $\overline{\varphi}$ is an isomorphism. ∎

# Chapter 4

# Vector Bundles and Geometric Operators

In this chapter, we describe how to obtain so-called “geometric operators” on the (localized) equivariant cohomology of smooth algebraic varieties; this method was first introduced in [3] . The methods outlined in this chapter will serve as our main tool for constructing our geometric versions of the Clifford, Heisenberg and Chevalley operators in the next chapter. We do not review equivarient cohomology theory here, but instead refer the reader to such expository papers as [27] or [2]. Our constructions rely heavily on the Localization Theorem (see Theorem 4.1). The reader may wish to consult [1, Appendix to Chapter 6] for more information on localization.

Let $G = (\mathbb{C}^*)^d$ be a $d$-dimensional torus and, for each $j = 1, \ldots, d$, we denote the 1-dimensional $G$-module

$$(g_1, \ldots, g_d) \mapsto g_j,$$

by $g_j$. Let pt denote the space consisting of a single point equipped with the trivial action of the torus $G$. Let $t_j$ denote the first Chern class of

$$g_j \to \mathrm{pt},$$

for each $j = 1, \ldots, d$. Note that the $t_j$ are elements of degree 2. Recall that the *equivariant cohomology* of pt is

$$H_G^*(\mathrm{pt}) = \mathbb{C}[t_1, \ldots, t_d].$$

Let $X$ be a space with a $G$-action. Then $H_G^*(X)$ is an $H_G^*(\mathrm{pt})$-module. We consider the *localized equivariant cohomology* of $X$:

$$\mathcal{H}_G^*(X) := H_G^*(X) \otimes_{\mathbb{C}[t_1,\ldots,t_d]} \mathbb{C}(t_1, \ldots, t_d).$$

Unless otherwise noted, "cohomology" will always mean "localized equivariant cohomology". Let

$$i : X^G \hookrightarrow X,$$

be the inclusion of the $G$-fixed points and let

$$p : X^G \twoheadrightarrow \mathrm{pt}\,.$$

The advantage of localized equivariant cohomology over nonlocalized equivariant cohomology is that its study can be reduced to the cohomology of the $G$-fixed points. We will only be interested in the case where $X$ is a smooth variety with finitely many $G$-fixed points. In this situation, we have the following theorem.

**Theorem 4.1** (Localization Theorem)**.** *The following map is an isomorphism of algebras:*

$$\begin{aligned} \mathcal{H}_G^*(X) \to \mathcal{H}_G^*(X^G) &= \bigoplus_{x \in X^G} \mathcal{H}_G^*(\mathrm{pt}), \\ \alpha \mapsto & \left( \frac{i_x^*(\alpha)}{e_G(\mathcal{T}_x)} \right)_{x \in X^G}, \end{aligned} \tag{4.1}$$

*where $i_x : \{x\} \hookrightarrow X$, $\mathcal{T}_x$ is the tangent space of $x$ in $X$, and $e_G(\mathcal{T}_x)$ is the equivariant Euler class of $\mathcal{T}_x$. The inverse of (4.1) is given by the Gysin map $i_* : \mathcal{H}_G^*(X^G) \to \mathcal{H}_G^*(X)$.*

**Proof:** This is a restatement of [5, Proposition 9.1.2] in the case that $X$ has finitely many fixed points. ∎

Suppose now that $X$ has real dimension $4n$, for some $n \in \mathbb{N}$. We define a bilinear form $\langle -,- \rangle_X$ on the middle degree localized equivariant cohomology $\mathcal{H}_G^{2n}(X)$ by

$$\langle a, b \rangle_X := (-1)^n p_*(i_*)^{-1}(a \cup b), \tag{4.2}$$

where $i_*$ is invertible by the Localization Theorem. We can extend this idea to a product of varieties. Indeed, suppose $X_1, X_2$ are varieties of real dimension $4n_1$ and $4n_2$, respectively. We define a bilinear form $\langle -,- \rangle_{X_1 \times X_2}$ on $\mathcal{H}_G^{2(n_1+n_2)}(X_1 \times X_2)$ by

$$\langle a, b \rangle := (-1)^{n_2} p_*((i_1 \times i_2)_*)^{-1}(a \cup b), \tag{4.3}$$

where $i_1$ and $i_2$ are the inclusions of the $G$-fixed points into the first and second factors, respectively. An element $\alpha \in \mathcal{H}_G^{2(n_1+n_2)}(X_1 \times X_2)$ then defines an operator

$$\alpha : \mathcal{H}_G^{2n_1}(X_1) \to \mathcal{H}_G^{2n_2}(X_2), \tag{4.4}$$

by using the bilinear form to define structure constants:

$$\langle \alpha(x), y \rangle_{X_2} := \langle \alpha, x \otimes y \rangle_{X_1 \times X_2}.$$

Thus, an element $\alpha \in \mathcal{H}_G^{2(n_1+n_2)}(X_1 \times X_2)$ will be called a *geometric operator*.

Recall that the torus $T = (\mathbb{C}^*)^s \times \mathbb{C}^*$ acts on $\mathcal{M}_{\mathbf{c}}(s, n)$ via

$$(e, t) \cdot [A, B, i, j] = [tA, t^{-1}B, ie^{-1}t^{-\mathbf{c}}, et^{\mathbf{c}}j], \tag{4.5}$$

for all $(e, t) \in T$ and $[A, B, i, j] \in \mathcal{M}_{\mathbf{c}}(s, n)$. Let $T_\bullet = \mathbb{C}^*$ and define an action of $T_\bullet$ on $\mathcal{M}_{\mathbf{c}}(s, n)$ via the embedding

$$\begin{aligned} T_\bullet &\to T \\ z &\mapsto (1, z, z^2, \ldots, z^{s-1}, 1). \end{aligned}$$

Similar to Remark 3.15, one sees that $[A,B,i,j]\in\mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet}$ if and only if there exists a group homomorphism $g:T_\bullet\to\mathrm{GL}_n(\mathbb{C})$ such that

$$\begin{aligned} g(z)^{-1}Ag(z)&=A,\\ g(z)^{-1}Bg(z)&=B,\\ g(z)^{-1}i&=i(1,z^{-1},\dots,z^{1-s}),\\ jg(z)&=(1,z,\dots,z^{s-1})j. \end{aligned}\tag{4.6}$$

Define

$$V^k=V^k(A,B,i,j):=\{v\in\mathbb{C}^n\mid g(z)v=z^{k-1}v\}.\tag{4.7}$$

By the stability of $(A,B,i,j)$, we have that $\mathbb{C}^n=\bigoplus_{k=1}^s V^k$. Moreover,

$$A(V^k),B(V^k),i(\mathbb{C}\mathbf{1}_k)\subseteq V^k,\quad\text{and}\quad j(V_k)\subseteq\mathbb{C}\mathbf{1}_k,$$

for all $k=1,\dots,s$. Conversely, if there exists a decomposition $\mathbb{C}^n=\bigoplus_{k=1}^s U^k$ such that $A(U^k)$, $B(U^k)$, $i(\mathbf{1}_k)\subseteq U^k$ and $j(U^k)\subseteq\mathbb{C}\mathbf{1}_k$, then we may define a group homomorphism $g:T_\bullet\to\mathrm{GL}_n(\mathbb{C})$ by defining $g(z)|_{U^k}=z^{k-1}\,\mathrm{id}_{U^k}$. One easily checks that $g$ satisfies the conditions of (4.6) and $U^k=V^k(A,B,i,j)$. Thus, $[A,B,i,j]\in\mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet}$.

Now suppose $\mathbf{n}=(\mathbf{n}_1,\dots,\mathbf{n}_s)\in\mathbb{N}^s$ such that $|\mathbf{n}|=n$. Define

$$\mathcal{M}_{\mathbf{c}}(\mathbf{n}):=\mathcal{M}_{\mathbf{c}_1}(1,\mathbf{n}_1)\times\cdots\times\mathcal{M}_{\mathbf{c}_s}(1,\mathbf{n}_s).$$

Identify $\bigoplus_k\mathbb{C}^{\mathbf{n}_k}$ with $\mathbb{C}^n$ by identifying $\mathbf{1}_\ell^k$ with $\mathbf{1}_{\mathbf{n}_1+\cdots+\mathbf{n}_{k-1}+\ell}$, where $\{\mathbf{1}_\ell^k\}_{\ell=1}^{\mathbf{n}_k}$ is the standard basis of $\mathbb{C}^{\mathbf{n}_k}$. An element $([A_1,B_1,i_1],\dots,[A_s,B_s,i_s])\in\mathcal{M}_{\mathbf{c}}(\mathbf{n})$ then determines an element $[A,B,i,0]\in\mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet}$ by defining

$$A|_{\mathbb{C}^{\mathbf{n}_k}}:=A_k,\quad B|_{\mathbb{C}^{\mathbf{n}_k}}:=B_k,\quad i=i_1+\cdots+i_s.\tag{4.8}$$

One can then check that we have a well-defined map

$$\begin{aligned}\coprod_{|\mathbf{n}|=n}\mathcal{M}_{\mathbf{c}}(\mathbf{n})&\to\mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet},\\ ([A_1,B_1,i_1],\dots,[A_s,B_s,i_s])&\mapsto[A,B,i,0],\end{aligned}\tag{4.9}$$

where $[A,B,i,0]$ is defined as in (4.8).

**Lemma 4.2.** *The map (4.9) is an isomorphism of varieties. Thus,* $\mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet} \cong \coprod_{|\mathbf{n}|=n} \mathcal{M}_{\mathbf{c}}(\mathbf{n})$.

**Proof:** This is a straight-forward generalization of [23, Lemma 3.2]. ∎

We fix once and for all an $r \in \mathbb{N}^+$ and a partition of $r$ of length $s$,

$$\underline{r} := (r_1, \dots, r_s).$$

Let $R' = \operatorname{lcm}\{r_1, \dots, r_s\}$ and define

$$R := \begin{cases} R', & \text{if } R'\left(\frac{1}{r_k} + \frac{1}{r_\ell}\right) \in 2\mathbb{Z} \text{ for all } k, \ell, \\ 2R', & \text{otherwise.} \end{cases}$$

Consider the product variety $\mathcal{M}_{\mathbf{c}}(\mathbf{n}) = \mathcal{M}_{\mathbf{c}_1}(1,\mathbf{n}_1) \times \cdots \times \mathcal{M}_{\mathbf{c}_s}(1,\mathbf{n}_s)$. Each component $\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)$, for $\ell = 1, \dots, s$, carries with it the action of a 2-dimensional torus $T = \mathbb{C}^* \times \mathbb{C}^*$ (given by setting $s = 1$ in Equation (4.5)). Let $T_\star = (\mathbb{C}^*)^s \times \mathbb{C}^*$ and define a $T_\star$-action on $\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)$ via the map

$$T_\star \to T,$$
$$(e,t) \mapsto (e_\ell, t^{R/r_\ell}).$$

That is, $T_\star$ acts on $\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)$ by

$$(e,t) \star [A_\ell, B_\ell, i_\ell] = (e_\ell, t^{R/r_\ell}) \cdot [A_\ell, B_\ell, i_\ell] = [t^{R/r_\ell} A_\ell, t^{-R/r_\ell} B, i_\ell e_\ell^{-1} t^{-\mathbf{c}_\ell R/r_\ell}],$$

for all $(e,t) \in T_\star$ and $[A_\ell, B_\ell, i_\ell] \in \mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)$. Then $T_\star$ acts on the product $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$ by acting on each of its components, i.e.

$$(e,t) \star ([A_1,B_1,i_1], \dots, [A_s,B_s,i_s]) = ((e,t) \star [A_1,B_1,i_1], \dots, (e,t) \star [A_s,B_s,i_s]).$$

**Lemma 4.3.** *The set* $\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}$ *is in one-to-one correspondence with the set*

$$\{(I_1, \dots, I_s) \mid I_\ell \textit{ is a semi-infinite monomial of charge } \mathbf{c}_\ell \textit{ and energy } \mathbf{n}_\ell\}.$$

**Proof:** We first prove that

$$\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} = \mathcal{M}_{\mathbf{c}_1}(1, \mathbf{n}_1)^T \times \cdots \times \mathcal{M}_{\mathbf{c}_s}(1, \mathbf{n}_s)^T.$$

Let $([A_1, B_1, i_1], \dots, [A_s, B_s, i_s]) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}$. Fix $\ell \in \{1, \dots, s\}$. For all $(e, t) \in T = \mathbb{C}^* \times \mathbb{C}^*$, choose $\xi \in \mathbb{C}^*$ such that $\xi^{R/r_\ell} = t$. Then

$$(e, t) \cdot [A_\ell, B_\ell, i_\ell] = ((1, \dots, e, \dots, 1), \xi) \star [A_\ell, B_\ell, i_\ell] = [A_\ell, B_\ell, i_\ell].$$

Therefore, $\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \subseteq \mathcal{M}_{\mathbf{c}_1}(1, \mathbf{n}_1)^T \times \cdots \times \mathcal{M}_{\mathbf{c}_s}(1, \mathbf{n}_s)^T$. The reverse inclusion follows by construction of the action of $T_\star$ on $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$.

Now, by [21, Proposition 2.9], $\mathcal{M}_{\mathbf{c}_\ell}(1, \mathbf{n}_\ell)^T$ is in one-to-one correspondence with the set of Young diagrams of size $\mathbf{n}_\ell$, which is itself in one-to-one correspondence with the set of semi-infinite monomials of charge $\mathbf{c}_\ell$ and energy $\mathbf{n}_\ell$ by Lemma 1.9. ∎

In light of Lemma 4.3, we will henceforth identify points of $\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}$ with $s$-tuples of semi-infinite monomials.

Define an action of $\mathbb{Z}_R$ on $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$ via the embedding

$$\begin{aligned} \mathbb{Z}_R &\to T_\star, \\ k &\mapsto (1, \omega^k), \end{aligned}$$

where $\omega = e^{2\pi\sqrt{-1}/R}$. Then $\mathbb{Z}_R$ acts on the $\ell$-th component, $\mathcal{M}_{\mathbf{c}_\ell}(1, \mathbf{n}_\ell)$, of $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$ via the embedding

$$\begin{aligned} \mathbb{Z}_R &\to T, \\ k &\mapsto (1, \omega^{R/r_\ell}) = (1, e^{2\pi\sqrt{-1}/r_\ell}). \end{aligned}$$

Thus,

$$\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R} = \mathcal{M}_{\mathbf{c}_1}(1,\mathbf{n}_1)^{\mathbb{Z}_{r_1}} \times \cdots \times \mathcal{M}_{\mathbf{c}_s}(1,\mathbf{n}_s)^{\mathbb{Z}_{r_s}},$$

where the action of $\mathbb{Z}_{r_\ell}$ on $\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)$ is defined as in Equation (3.6). By Theorem 3.20, we know that $\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)^{\mathbb{Z}_{r_\ell}} \cong \coprod_{|\mathbf{v}^\ell|=\mathbf{n}_\ell} \mathfrak{M}(r_\ell;\mathbf{v}^\ell,\mathbf{1}_{\bar{\mathbf{c}}_\ell})$. Hence, we define

$$\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\ldots,\mathbf{v}^s) := \mathfrak{M}(r_1;\mathbf{v}^1,\mathbf{1}_{\bar{\mathbf{c}}_1}) \times \cdots \times \mathfrak{M}(r_s;\mathbf{v}^s,\mathbf{1}_{\bar{\mathbf{c}}_s}),$$

and obtain

$$\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R} \cong \coprod_{|\mathbf{v}^\ell|=\mathbf{n}_\ell} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\ldots,\mathbf{v}^s).$$

We summarize the various fixed point varieties with the following diagram of inclusions:

$$\begin{array}{cccc} \mathcal{M}_{\mathbf{c}}(s,n) \supseteq \mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet} \cong \coprod_{\mathbf{n}} \mathcal{M}_{\mathbf{c}}(\mathbf{n}) \supseteq & \coprod_{\mathbf{n}} \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R} & \supseteq \coprod_{\mathbf{n}} \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}. \\ & \wr\Vert & \\ & \coprod_{\mathbf{v}^\ell} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\ldots,\mathbf{v}^s) & \end{array} \tag{4.10}$$

We now consider the localized $T_\star$-equivariant cohomology of $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$. Denote the one-dimensional $T_\star$-modules

$$(e,t) \mapsto e_k, \quad \text{and} \quad (e,t) \mapsto t,$$

by $e_k$ and $t$, respectively, and denote the tensor product of such modules by juxtaposition. Moreover, we denote the first Chern classes of

$$e_k \mapsto \mathrm{pt}, \quad \text{and} \quad t \mapsto \mathrm{pt},$$

by $b_k$ and $\epsilon$, respectively. Thus,

$$\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})) = H_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})) \otimes_{\mathbb{C}[b_1,\ldots,b_s,\epsilon]} \mathbb{C}(b_1,\ldots,b_s,\epsilon).$$

Since $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$ has real dimension $4|\mathbf{n}|$, we define a bilinear form $\langle \text{-},\text{-}\rangle_{\mathbf{n},\mathbf{c}}$ on $\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))$ as in (4.2), induced by

$$i : \mathcal{M}_{\mathbf{c}}(s,n)^{T_\star} \hookrightarrow \mathcal{M}_{\mathbf{c}}(\mathbf{n}) \quad \text{and} \quad p : \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \twoheadrightarrow \mathrm{pt}\,.$$

We extend to a bilinear form $\langle \text{-},\text{-}\rangle$ on $\bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})) \cong \bigoplus_{n,\mathbf{c}} \mathcal{H}^{2n}_{T_\star}\left(\mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet}\right)$ by

$$\langle \text{-},\text{-}\rangle := \sum_{\mathbf{n},\mathbf{c}} \langle \text{-},\text{-}\rangle_{\mathbf{n},\mathbf{c}}.$$

One also defines a bilinear form $\langle \text{-},\text{-}\rangle_{\mathbf{n},\mathbf{m},\mathbf{c},\mathbf{d}}$ on $\mathcal{H}^{2(|\mathbf{n}|+|\mathbf{m}|)}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m}))$ as in Equation (4.3), which we extend to a bilinear form $\langle \text{-},\text{-}\rangle$ on $\bigoplus_{\mathbf{n},\mathbf{m},\mathbf{c},\mathbf{d}} \mathcal{H}^{2(|\mathbf{n}|+|\mathbf{m}|)}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m}))$ by

$$\langle \text{-},\text{-}\rangle := \sum_{\mathbf{n},\mathbf{m},\mathbf{c},\mathbf{d}} \langle \text{-},\text{-}\rangle_{\mathbf{n},\mathbf{m},\mathbf{c},\mathbf{d}}.$$

Our goal will be to construct geometric operators on $\bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))$ as in Equation (4.4). In order to simplify computations, it will be useful for us to introduce an orthonormal $\mathbb{C}(b_1,\dots,b_s,\epsilon)$-basis for $\mathcal{H}^{*}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))$. For each $\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}$, the $T_\star$-action on $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$ induces an action on the tangent space $\mathcal{T}_{\mathbf{I}} = \mathcal{T}_{\mathbf{I}}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))$. The decomposition of $\mathcal{T}_{\mathbf{I}}$ into one-dimensional $T_\star$ modules is given in the following lemma.

**Lemma 4.4.** *Let* $\mathbf{I} = (\mathbf{I}_1,\dots,\mathbf{I}_s) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}$. *Then, as a* $T_\star$*-module,*

$$\mathcal{T}_{\mathbf{I}} \cong \bigoplus_{\ell=1}^{s} \left( \bigoplus_{(i,j)\in\lambda(\mathbf{I}_\ell)} \left(t^{-h_{\lambda(\mathbf{I}_\ell)}(i,j)R/r_\ell} \oplus t^{h_{\lambda(\mathbf{I}_\ell)}(i,j)R/r_\ell}\right)\right),$$

*where* $\lambda(\mathbf{I}_\ell)$ *is the Young diagram associated to* $\mathbf{I}_\ell$ *and* $h_{\lambda(\mathbf{I})}$ *is the relative hook length (see equations* (1.2) *and* (1.1)*).*

**Proof:** We have that

$$\mathcal{T}_{\mathbf{I}}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})) \cong \bigoplus_{\ell=1}^{s} \mathcal{T}_{\mathbf{I}_\ell}(\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)).$$

The tangent space of $\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)$ at $\mathbf{I}_\ell$ may then be computed by replacing $t$ by $t^{R/r_\ell}$ in [15, Proposition 2.2]. ∎

It will be convenient to use the decomposition $\mathcal{T}_{\mathbf{I}} = \mathcal{T}_{\mathbf{I}}^+ \oplus \mathcal{T}_{\mathbf{I}}^-$, where

$$\mathcal{T}_{\mathbf{I}}^{\pm} := \bigoplus_{\ell=1}^{s}\left(\bigoplus_{(i,j)\in\lambda(\mathbf{I}_\ell)} t^{\pm h_{\lambda(\mathbf{I}_\ell)}(i,j)R/r_\ell}\right).$$

**Lemma 4.5.** *For $\mathbf{I}\in\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}$, the equivariant Euler classes of $\mathcal{T}_{\mathbf{I}}^+$ and $\mathcal{T}_{\mathbf{I}}^-$ are given by*

$$e_{T_\star}(\mathcal{T}_{\mathbf{I}}^+) = \prod_{\ell=1}^{s}\left(\prod_{(i,j)\in\lambda(\mathbf{I}_\ell)} h_{\lambda(\mathbf{I}_\ell)}(i,j)\frac{R}{r_\ell}\epsilon\right),$$

$$e_{T_\star}(\mathcal{T}_{\mathbf{I}}^-) = \prod_{\ell=1}^{s}\left(\prod_{(i,j)\in\lambda(\mathbf{I}_\ell)} -h_{\lambda(\mathbf{I}_\ell)}(i,j)\frac{R}{r_\ell}\epsilon\right) = (-1)^{|\mathbf{n}|}e_{T_\star}(\mathcal{T}_{\mathbf{I}}^+).$$

**Proof:** This follows directly from the definitions of $\mathcal{T}_{\mathbf{I}}^+$ and $\mathcal{T}_{\mathbf{I}}^-$. ∎

For each $\mathbf{I}\in\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}$, let

$$[\mathbf{I}] := \frac{i_*(1_{\mathbf{I}})}{e_{T_\star}(\mathcal{T}_{\mathbf{I}}^-)} \in \mathcal{H}_{T_\star}^{2|\mathbf{n}|}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})),$$

where $1_{\mathbf{I}}$ is the unit in $\mathcal{H}_{T_\star}^*(\mathrm{pt})$ and $e_{T_\star}(\mathcal{T}_{\mathbf{I}}^-)$ is to be interpreted as an invertible element in this ring. Since the elements $1_{\mathbf{I}}$ form a $\mathbb{C}(b_1,\ldots,b_s,\epsilon)$-basis of the cohomlogy $\bigoplus_{\mathbf{n},\mathbf{c}}\mathcal{H}_{T_\star}^*\left(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}\right)$, by the Localization Theorem (Theorem 4.1), the elements $[\mathbf{I}]$ form a $\mathbb{C}(b_1,\ldots,b_s,\epsilon)$-basis of $\bigoplus_{\mathbf{n},\mathbf{c}}\mathcal{H}_{T_\star}^*(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))$.

**Lemma 4.6.** *The $[\mathbf{I}]$ are orthonormal with respect to the bilinear form $\langle$-, -$\rangle$.*

**Proof:** The proof is completely analogous to [15, Proposition 2.4]. ∎

Define

$$\mathbf{A} = \mathrm{span}_{\mathbb{C}} \left\{ [\mathbf{I}] \mid \mathbf{I} \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star},\ \mathbf{n} \in \mathbb{N}^s,\ \mathbf{c} \in \mathbb{Z}^s \right\}. \tag{4.11}$$

Then $\mathbf{A}$ is a full $\mathbb{C}$-lattice in $\bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))$. It will be useful for us to consider the following gradings on $\mathbf{A}$:

$$\mathbf{A} = \bigoplus_{\mathbf{n},\mathbf{c}} \mathbf{A}_{\mathbf{c}}(\mathbf{n}), \quad \mathbf{A}_{\mathbf{c}}(\mathbf{n}) := \left\{ [\mathbf{I}] \mid \mathbf{I} \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \right\},$$

and

$$\mathbf{A} = \bigoplus_{c \in \mathbb{Z}} \mathbf{A}(c), \quad \mathbf{A}(c) := \left\{ [\mathbf{I}] \mid \mathbf{I} \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \text{ s.t. } |\mathbf{c}| = c \right\}.$$

**Corollary 4.7.** *The restriction of $\langle \text{-}, \text{-} \rangle$ to $\mathbf{A}$ is non-degenerate and $\mathbb{C}$-valued.*

**Proof:** This follows directly from Lemma 4.6. ∎

**Remark 4.8.** Notice that via the Künneth formula

$$\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})) \cong \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}_1}(1, \mathbf{n}_1)) \otimes \cdots \otimes \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}_s}(1, \mathbf{n}_s)),$$

the element $[\mathbf{I}] \in \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))$, where $\mathbf{I} = (\mathbf{I}_1, \ldots, \mathbf{I}_s)$, maps to

$$[\mathbf{I}_1] \otimes \cdots \otimes [\mathbf{I}_s] \in \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}_1}(1, \mathbf{n}_1)) \otimes \cdots \otimes \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}_s}(1, \mathbf{n}_s)).$$

Therefore,

$$\mathbf{A}_{\mathbf{c}}(\mathbf{n}) \cong \mathbf{A}_{\mathbf{c}_1}(\mathbf{n}_1) \otimes \cdots \otimes \mathbf{A}_{\mathbf{c}_s}(\mathbf{n}_s).$$

Let $X$ be an algebraic variety with a $T_\star$-action, and let $E \to X$ be an $T_\star$-equivariant vector bundle. We will denote the $k$-th equivariant Chern class of $E$ by $c_k(E)$. Note that $c_k(E) \in H^{2k}_{T_\star}(X)$. The following lemma will act as our main tool in constructing geometric operators in the following chapter.

**Lemma 4.9.** *Let* $\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}$ *and* $\mathbf{J} \in \mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star}$, *let* $E$ *be an equivariant vector bundle on* $\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})$ *and let* $\beta \in \mathcal{H}^{2k}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m}))$. *Then*

$$\langle \beta \cup c_{|\mathbf{n}|+|\mathbf{m}|-k}(E)[\mathbf{I}], [\mathbf{J}]\rangle = \frac{\beta_{\mathbf{I},\mathbf{J}} \cup c_{|\mathbf{n}|+|\mathbf{m}|-k}(E_{(\mathbf{I},\mathbf{J})})}{e_{T_\star}(\mathcal{T}^-_{\mathbf{I}})e_{T_\star}(\mathcal{T}^+_{\mathbf{J}})},$$

*where* $c_{|\mathbf{n}|+|\mathbf{m}|-k}(E_{(\mathbf{I},\mathbf{J})}) \in H^*_{T_\star}(\mathrm{pt}) = \mathbb{C}[b_1, \ldots, b_s, \epsilon]$ *is the polynomial given by the equivariant Chern class of the fiber* $E$ *over the point* $(\mathbf{I}, \mathbf{J})$ *and* $\beta_{\mathbf{I},\mathbf{J}} = i^*_{\mathbf{I},\mathbf{J}}(\beta)$, *where* $i_{\mathbf{I},\mathbf{J}} : \{(\mathbf{I}, \mathbf{J})\} \hookrightarrow \mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})$ *is the inclusion of the fixed point.*

**Proof:** See [15, Lemma 2.6]. ∎

**Remark 4.10.** The precise statement of [15, Lemma 2.6] differs slightly from ours (since we use the variety $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$ under the action of $T_\star$ rather than $\mathcal{M}_{\mathbf{c}}(s, n)$ under the action of $T$). However, every step in the proof of [15, Lemma 2.6] applies to Lemma 4.9; thus the two lemmas are essentially the same.

The next step will be to construct $T_\star$-equivariant vector bundles over $\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})$ whose Chern classes will define the appropriate geometric operators (as our ultimate goal is to construct geometric versions of the Heisenberg, Clifford and Chevalley operators from Chapter 1). We begin by defining vector bundles

$$\mathbb{C}^n \times_{\mathrm{GL}_n(\mathbb{C})} M_0^{\mathrm{st}}(s, n) \to \mathcal{M}_{\mathbf{c}}(s, n), \quad \text{and} \quad \mathbb{C}^s \times \mathcal{M}_{\mathbf{c}}(s, n) \to \mathcal{M}_{\mathbf{c}}(s, n),$$

which we denote by $\mathcal{V} = \mathcal{V}(\mathbf{c}, s, n)$ and $\mathcal{W} = \mathcal{W}(\mathbf{c}, s, n)$, respectively. Note that $\mathcal{V}$ and $\mathcal{W}$ are simply the associated bundles of the trivial $\mathrm{GL}_n(\mathbb{C})$-bundles

$$\mathbb{C}^n \times M_0^{\mathrm{st}}(s, n) \to M_0^{\mathrm{st}}(s, n), \quad \text{and} \quad \mathbb{C}^s \times M_0^{\mathrm{st}}(s, n) \to M_0^{\mathrm{st}}(s, n).$$

The bundles $\mathcal{V}$ and $\mathcal{W}$ are $T$-equivariant with respect to the trivial action of $T$ on $\mathbb{C}^n$ and the natural action of $T$ on $\mathbb{C}^s$, respectively. Consider the Hom-bundle $\mathrm{Hom}(\mathcal{V}, \mathcal{V})$

on $\mathcal{M}_{\mathbf{c}}(s,n)$. We can define a global section $s : \mathcal{M}_{\mathbf{c}}(s,n) \to \mathrm{Hom}(\mathcal{V},\mathcal{V})$ by defining $s[A,B,i,j]$ to be the (well-defined) linear map

$$\mathbb{C}^n \times_{\mathrm{GL}_n(\mathbb{C})} [A,B,i,j] \to \mathbb{C}^n \times_{\mathrm{GL}_n(\mathbb{C})} [A,B,i,j],$$
$$\mathrm{GL}_n(\mathbb{C}) \cdot (v,(A,B,i,j)) \mapsto \mathrm{GL}_n(\mathbb{C}) \cdot (Av,(A,B,i,j)).$$

By a slight abuse of notation, we will denote the section $s$ by $A$. We similarly define sections $B$, $i$ and $j$ of $\mathrm{Hom}(\mathcal{V},\mathcal{V})$, $\mathrm{Hom}(\mathcal{W},\mathcal{V})$ and $\mathrm{Hom}(\mathcal{V},\mathcal{W})$, respectively.

One can extend this construction to a product of moduli spaces. The bundle $\mathcal{V}(\mathbf{c},s,n) \to \mathcal{M}_{\mathbf{c}}(s,n)$ may be extended to a vector bundle over the product $\mathcal{M}_{\mathbf{c}}(s,n) \times \mathcal{M}_{\mathbf{d}}(s,m)$ by:

$$\mathcal{V}(\mathbf{c},s,n) \times \mathcal{M}_{\mathbf{d}}(s,m) \to \mathcal{M}_{\mathbf{c}}(s,n) \times \mathcal{M}_{\mathbf{d}}(s,m).$$

We denote this vector bundle by $\mathcal{V}_1 = \mathcal{V}_1(\mathbf{c},\mathbf{d},s,n,m)$. Likewise, we extend the bundles $\mathcal{W}(\mathbf{c},s,n) \to \mathcal{M}_{\mathbf{c}}(s,n)$, $\mathcal{V}(\mathbf{d},s,m) \to \mathcal{M}_{\mathbf{d}}(s,m)$ and $\mathcal{W}(\mathbf{d},s,m) \to \mathcal{M}_{\mathbf{d}}(s,m)$ to bundles $\mathcal{W}_1 = \mathcal{W}_1(\mathbf{c},\mathbf{d},s,n,m)$, $\mathcal{V}_2 = \mathcal{V}_2(\mathbf{c},\mathbf{d},s,n,m)$ and $\mathcal{W}_2 = \mathcal{W}_2(\mathbf{c},\mathbf{d},s,n,m)$ over $\mathcal{M}_{\mathbf{c}}(s,n) \times \mathcal{M}_{\mathbf{d}}(s,m)$. We then have bundles $\mathrm{Hom}(\mathcal{V}_k,\mathcal{V}_k)$, $\mathrm{Hom}(\mathcal{W}_k,\mathcal{V}_k)$ and $\mathrm{Hom}(\mathcal{V}_k,\mathcal{W}_k)$ as vector bundles over $\mathcal{M}_{\mathbf{c}}(s,n) \times \mathcal{M}_{\mathbf{d}}(s,m)$ with sections $A_k$, $B_k$, $i_k$ and $j_k$, where $k = 1,2$. We define a three-term, $T$-equivariant complex of vector bundles on $\mathcal{M}_{\mathbf{c}}(s,n) \times \mathcal{M}_{\mathbf{d}}(s,m)$ by

$$\mathrm{Hom}(\mathcal{V}_1,\mathcal{V}_2) \xrightarrow{\zeta} \begin{matrix} t\,\mathrm{Hom}(\mathcal{V}_1,\mathcal{V}_2) \oplus t^{-1}\,\mathrm{Hom}(\mathcal{V}_1,\mathcal{V}_2) \\ \oplus \\ \mathrm{Hom}(\mathcal{W}_1,\mathcal{V}_2) \oplus \mathrm{Hom}(\mathcal{V}_1,\mathcal{W}_2) \end{matrix} \xrightarrow{\tau} \mathrm{Hom}(\mathcal{V}_1,\mathcal{V}_2), \tag{4.12}$$

where

$$\zeta(X) = \begin{pmatrix} XA_1 - A_2X \\ XB_1 - B_2X \\ Xi_1 \\ -j_2X \end{pmatrix}, \quad \text{and} \quad \tau \begin{pmatrix} C \\ D \\ a \\ b \end{pmatrix} = A_2D - DA_1 + CB_1 - B_2C + i_2b + aj_1.$$

Note that here the modules $t^{\pm 1}$ are the one-dimensional $T$-modules $(e,t) \mapsto t^{\pm 1}$. One verifies that $\tau \circ \zeta = 0$.

**Remark 4.11.** We explain the presence of the "$t$" and "$t^{-1}$" terms in Complex (4.12); they are the result of the action of $T$ on the section $A_k$ and $B_k$. Let $\zeta_1 : \operatorname{Hom}(\mathcal{V}_1, \mathcal{V}_2) \to \operatorname{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ be the first component of $\zeta$. Denote the section "$A_1$" by $s$. To simplify notation, for every $(I_1, I_2) \in M_0^{\mathrm{st}}(s,n) \times M_0^{\mathrm{st}}(s,m)$, we will denote the corresponding pair of orbits in $\mathcal{M}_{\mathbf{c}}(s,n) \times \mathcal{M}_{\mathbf{d}}(s,m)$ by $(\overline{I_1}, \overline{I_2})$. Moreover, we will write $s_{(\overline{I_1},\overline{I_2})} := s(\overline{I_1}, \overline{I_2})$ and $\overline{v}_{(\overline{I_1},\overline{I_2})} := (\mathrm{GL}_n(\mathbb{C}) \cdot (v, I_1), \overline{I_2})$ for elements in the fiber of $\mathcal{V}_1$ over the point $(\overline{I_1}, \overline{I_2})$. We wish to compute the action of $T$ on $s$. For every $(e,t) \in T$ and $(I_1, I_2) \in M_0^{\mathrm{st}}(s,n) \times M_0^{\mathrm{st}}(s,m)$, with $I_1 = (A_1, B_1, i_1, j_1)$, the map

$$(e,t) \cdot s_{(\overline{I_1},\overline{I_2})} : (\mathbb{C}^n \times_{\mathrm{GL}_n(\mathbb{C})} \overline{I_1}) \times \overline{I_2} \to (\mathbb{C}^n \times_{\mathrm{GL}_n(\mathbb{C})} \overline{I_1}) \times \overline{I_2},$$

is given by

$$\begin{aligned} ((e,t) \cdot s_{(\overline{I_1},\overline{I_2})})(\overline{v}_{(\overline{I_1},\overline{I_2})}) &= (e,t) \cdot s_{(e,t)^{-1} \cdot (\overline{I_1},\overline{I_2})}((e,t)^{-1} \cdot \overline{v}_{(\overline{I_1},\overline{I_2})}) \\ &= (e,t) \cdot s_{(e,t)^{-1} \cdot (\overline{I_1},\overline{I_2})}(\overline{v}_{(e,t)^{-1} \cdot (\overline{I_1},\overline{I_2})}) \\ &= (e,t) \cdot \overline{t^{-1} A v}_{(e,t)^{-1} \cdot (\overline{I_1},\overline{I_2})} = \overline{t^{-1} A v}_{(\overline{I_1},\overline{I_2})}, \end{aligned}$$

for all $v \in \mathbb{C}^n$. Thus, the action of $T$ on $A_1$ (viewed as a section) is given by $(e,t) \cdot A_1 = t^{-1} A_1$. Similarly, $(e,t) \cdot A_2 = t^{-1} A_2$. Therefore,

$$\begin{aligned} \zeta_1((e,t) \cdot X) &= ((e,t) \cdot X) A_1 - A_2((e,t) \cdot X) \\ &= (e,t) \cdot (X((e,t)^{-1} \cdot A_1) - ((e,t)^{-1} \cdot A_2) X) \\ &= (e,t) \cdot (t(X A_1 - A_2 X)) = t(e,t) \cdot \zeta_1(X), \end{aligned}$$

for all $X \in \operatorname{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ and $(e,t) \in T$. Hence, $\zeta_1 : \operatorname{Hom}(\mathcal{V}_1, \mathcal{V}_2) \to t \operatorname{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ is $T$-equivariant, thus justifying the extra factor "$t$" in Complex (4.12). One can show, by a similar argument, that $\zeta$ and $\tau$ in Complex (4.12) are both $T$-equivariant.

**Lemma 4.12.** *The cohomology* $\ker\tau/\operatorname{im}\zeta$ *of Complex* (4.12) *is a vector bundle on* $\mathcal{M}_{\mathbf{c}}(s,n)\times\mathcal{M}_{\mathbf{d}}(s,m)$.

**Proof:** See [15, Lemma 3.2]. ∎

For $n,m\in\mathbb{N}$, we will denote the vector bundle

$$\ker\tau/\operatorname{im}\zeta\to\mathcal{M}_{\mathbf{c}}(s,n)\times\mathcal{M}_{\mathbf{d}}(s,m),$$

by $\mathcal{K}_{\mathbf{c},\mathbf{d}}(s,n,m)$. Notice that, by construction, one has the following vector bundle on $\mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet}\times\mathcal{M}_{\mathbf{d}}(s,n)^{T_\bullet}$:

$$(\ker\tau/\operatorname{im}\zeta)^{T_\bullet}\to\mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet}\times\mathcal{M}_{\mathbf{d}}(s,n)^{T_\bullet},$$

which we denote by $\mathcal{K}_{\mathbf{c},\mathbf{d}}(s,n,m)^{T_\bullet}$. By Lemma 4.2,

$$\mathcal{M}_{\mathbf{c}}(s,n)^{T_\bullet}\times\mathcal{M}_{\mathbf{d}}(s,m)^{T_\bullet}\cong\coprod_{|\mathbf{n}|=n,\ |\mathbf{m}|=m}\mathcal{M}_{\mathbf{c}}(\mathbf{n})\times\mathcal{M}_{\mathbf{d}}(\mathbf{m}),$$

and so we may identify these varieties and consider the restriction of $\mathcal{K}_{\mathbf{c},\mathbf{d}}(s,n,m)^{T_\bullet}$:

$$\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m}):=\mathcal{K}_{\mathbf{c},\mathbf{d}}(s,n,m)^{T_\bullet}|_{\mathcal{M}_{\mathbf{c}}(\mathbf{n})\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})},$$

which is a vector bundle on $\mathcal{M}_{\mathbf{c}}(\mathbf{n})\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})$. On $\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)\times\mathcal{M}_{\mathbf{d}_\ell}(1,\mathbf{m}_\ell)$, the product of the $\ell$-th components of $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$ and $\mathcal{M}_{\mathbf{d}}(\mathbf{m})$, one has the vector bundle $\mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1,\mathbf{n}_\ell,\mathbf{m}_\ell)$. Let

$$f_\ell:\mathcal{M}_{\mathbf{c}}(\mathbf{n})\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})\to\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)\times\mathcal{M}_{\mathbf{d}_\ell}(1,\mathbf{m}_\ell),$$

denote the canonical projection. Then the vector bundle pullback, $f_\ell^*\mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1,\mathbf{n}_\ell,\mathbf{m}_\ell)$, is a vector bundle on the full product $\mathcal{M}_{\mathbf{c}}(\mathbf{n})\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})$.

**Lemma 4.13.** *Let* $X,X_1,X_2$ *be smooth algebraic varieties,* $G,G_1,G_2$ *reductive algebraic groups acting freely on* $X,X_1,X_2$*, respectively, such that* $X/G,X_1/G_1,X_2/G_2$

*are smooth, and let $V, V_1, V_2$ be representations of $G, G_1, G_2$, respectively. Moreover, suppose that there exists an embedding $G_1 \times G_2 \hookrightarrow G$ such that $V \cong V_1 \oplus V_2$ as $G_1 \times G_2$-representations and that there exists a $G_1 \times G_2$-equivariant morphism $\varphi : X_1 \times X_2 \to X$ such that the induced morphism $\overline{\varphi} : X_1/G_1 \times X_2/G_2 \to X/G$ is an isomorphism. Let $f_i : X_1/G_1 \times X_2/G_2 \to X_i/G_i$ denote the projection. Then the vector bundles*

$$f_1^*(V_1 \times_{G_1} X_1) \oplus f_2^*(V_2 \times_{G_2} X_2) \to X_1/G_1 \times X_2/G_2, \quad \textit{and} \quad V \times_G X \to X/G,$$

*are isomorphic.*

**Proof:** For simplicity, we will view $G_1 \times G_2$ as a subgroup of $G$ and identify $V$ with $V_1 \oplus V_2$, as $G_1 \times G_2$-modules. Define

$$\begin{aligned} \psi : (V_1 \times X_1) \oplus (V_2 \times X_2) &\to V \times X, \\ (v_1, x_1) \oplus (v_2, x_2) &\mapsto (v_1 \oplus v_2, \varphi(x_1, x_2)). \end{aligned}$$

It is clear that $\psi$ is a morphism of varieties. Moreover, one checks that the following diagram commutes:

$$\begin{array}{ccc} (G_1 \times G_2) \times ((V_1 \times X_1) \oplus (V_2 \times X_2)) & \longrightarrow & (V_1 \times X_1) \oplus (V_2 \times X_2) \\ {\scriptstyle i \times \psi} \downarrow & & \downarrow {\scriptstyle \psi}, \\ G \times (V \times X) & \longrightarrow & V \times X \end{array}$$

where $i : G_1 \times G_2 \to G$ is the inclusion map and the horizontal arrows represent the group action. So, by Lemma 2.3, we have an induced morphism $\overline{\psi} : (V_1 \times_{G_1} X_1) \oplus (V_2 \times_{G_2} X_2) \to V \times_G X$, which yields the following commutative diagram:

$$\begin{array}{ccc} (V_1 \times_{G_1} X_1) \oplus (V_2 \times_{G_2} X_2) & \xrightarrow{\overline{\psi}} & V \times_G X \\ \downarrow & & \downarrow \\ X_1/G_1 \times X_2/G_2 & \xrightarrow[\overline{\varphi}]{} & X/G. \end{array}$$

Since, by assumption, $\overline{\varphi}$ is an isomorphism, it remains only to show that $\overline{\psi}$ is an isomorphism and that it induces linear maps on the corresponding fibers. The latter is obvious, and so we focus on the former. By virtue of $X/G$, $X_1/G_1$ and $X_2/G_2$ being smooth, we have that $(V_1 \times_{G_1} X_1) \oplus (V_2 \times_{G_2} X_2)$ and $V \times_G X$ are smooth. Thus, by Lemma 2.11, it suffices to show that $\overline{\psi}$ is bijective and étale. To show bijectivity, it suffices to look at the fibers. For every $(G_1 \cdot x_1, G_2 \cdot x_2) \in X_1/G_1 \times X_2/G_2$, the linear map induced by $\overline{\psi}$ on the fibers is given by

$$\begin{aligned}(V_1 \times_{G_1} G_1 \cdot x_1) \oplus (V_2 \times_{G_2} G_2 \cdot x_2) &\to V \times_G G \cdot \varphi(x_1, x_2),\\ G_1 \cdot (v_1, x_1) \oplus G_2 \cdot (v_2, x_2) &\mapsto G \cdot (v_1 \oplus v_2, \varphi(x_1, x_2)),\end{aligned}$$

which is clearly bijective. To show that $\overline{\psi}$ is étale, by Corollary 2.6,

$$\begin{aligned}\mathcal{T}_{G_1\cdot(v_1,x_1)\oplus G_2\cdot(v_2,x_2)}\left(\bigoplus_{k=1,2}(V_k \times_{G_k} X_k)\right) &\cong \bigoplus_{k=1,2} \mathcal{T}_{(v_k,x_k)}(V_k \times X_k)/\operatorname{im} \mathrm{d}\sigma^{(v_k,x_k)}\\ &\cong \bigoplus_{k=1,2} \left(\mathcal{T}_{v_k}(V_k) \oplus \mathcal{T}_{x_k}(X_k)\right)/\operatorname{im} \mathrm{d}\sigma^{(v_k,x_k)},\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_{G\cdot(v_1\oplus v_2,\varphi(x_1,x_2))}(V \times_G X) &\cong \mathcal{T}_{(v_1\oplus v_2,\varphi(x_1,x_2))}(V \times X)/\operatorname{im} \mathrm{d}\sigma^{(v_1\oplus v_2,\varphi(x_1,x_2))}\\ &\cong \left(\mathcal{T}_{(v_1\oplus v_2)}(V) \oplus T_{\varphi(x_1,x_2)}(X)\right)/\operatorname{im} \mathrm{d}\sigma^{(v_1\oplus v_2,\varphi(x_1,x_2))}.\end{aligned}$$

The differential $\mathrm{d}\overline{\psi}$ is given by

$$\begin{aligned}&((y_1 \oplus z_1) + \operatorname{im} \mathrm{d}\sigma^{(v_1,x_1)}) \oplus ((y_2 \oplus z_2) + \operatorname{im} \mathrm{d}\sigma^{(v_2,x_2)})\\ &\qquad \mapsto ((y_1 \oplus y_2) \oplus \mathrm{d}\varphi(z_1, z_2)) + \operatorname{im} \mathrm{d}\sigma^{(v_1\oplus v_2,\varphi(x_1,x_2))}.\end{aligned}$$

We will first show that $\mathrm{d}\overline{\psi}$ is surjective. Let $(y' \oplus z') + \operatorname{im} \mathrm{d}\sigma^{(v_1\oplus v_2,\varphi(x_1,x_2))} \in (\mathcal{T}_{v_1\oplus v_2}(V) \oplus \mathcal{T}_{\varphi(x_1,x_2)}(X))/\operatorname{im} \mathrm{d}\sigma^{(v_1\oplus v_2,\varphi(x_1,x_2))}$. Since $\mathrm{d}\overline{\varphi}$ is an isomorphism, there exists a $(z_1 + \operatorname{im} \mathrm{d}\sigma^{x_1}) \oplus (z_1 + \operatorname{im} \mathrm{d}\sigma^{x_2}) \in \mathcal{T}_{x_1}(X_1)/\operatorname{im} \mathrm{d}\sigma^{x_1} \oplus \mathcal{T}_{x_2}(X_2)/\operatorname{im} \mathrm{d}\sigma^{x_2}$ such that

$$\mathrm{d}\overline{\varphi}((z_1 + \operatorname{im} \mathrm{d}\sigma^{x_1}) \oplus (z_1 + \operatorname{im} \mathrm{d}\sigma^{x_2})) = \mathrm{d}\varphi(z_1, z_2) + \operatorname{im} \mathrm{d}\sigma^{\varphi(x_1,x_2)} = z' + \operatorname{im} \mathrm{d}\sigma^{\varphi(x_1,x_2)}.$$

Equivalently, $z' - \mathrm{d}\varphi(z_1, z_2) = \mathrm{d}\sigma^{\varphi(x_1,x_2)}(h)$, for some $h \in \mathcal{T}_1(G)$. We also note that $\sigma^{(v_1 \oplus v_2, \varphi(x_1,x_2))} : G \to V \times X$ is the restriction of

$$\sigma^{v_1 \oplus v_2} \times \sigma^{\varphi(x_1,x_2)} : G \times G \to V \times X,$$

to the diagonal of $G \times G$. Hence,

$$(y', z') + \mathrm{d}\sigma^{(v_1 \oplus v_2, \varphi(x_1,x_2))} = ((y' - \mathrm{d}\sigma^{v_1 \oplus v_2}(h)) \oplus \varphi(z_1, z_2)) + \mathrm{d}\sigma^{(v_1 \oplus v_2, \varphi(x_1,x_2))}.$$

Thus, $(y', z') + \mathrm{d}\sigma^{(v_1 \oplus v_2, \varphi(x_1,x_2))} \in \operatorname{im} \mathrm{d}\overline{\psi}$. By dimension counting, $\mathrm{d}\overline{\psi}$ is an isomorphism, and so $\overline{\psi}$ is étale. ∎

**Lemma 4.14.** *There is an isomorphism of vector bundles*

$$\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n}, \mathbf{m}) \cong \bigoplus_{\ell=1}^{s} f_\ell^*(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1, \mathbf{n}_\ell, \mathbf{m}_\ell)).$$

**Proof:** We identify each point

$$([A_1^1, B_1^1, i_1^1], \ldots, [A_1^s, B_1^s, i_1^s], [A_2^1, B_2^1, i_2^1], \ldots, [A_2^s, B_2^s, i_2^s]) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m}),$$

with its image, $([A_1, B_1, i_1, 0], [A_2, B_2, i_2, 0]) \in \mathcal{M}_{\mathbf{c}}(s, |\mathbf{n}|)^{T_\bullet} \times \mathcal{M}_{\mathbf{d}}(s, |\mathbf{m}|)^{T_\bullet}$, under the isomorphism given in Equation (4.9). By Lemma 4.13,

$$\mathcal{V}_k|_{\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})} \cong \bigoplus_{\ell=1}^{s} f_\ell^* \mathcal{V}_k^\ell,$$

where $\mathcal{V}_k^\ell = \mathcal{V}_k(\mathbf{c}_\ell, \mathbf{d}_\ell, 1, \mathbf{n}_\ell, \mathbf{m}_\ell)$. Likewise,

$$\mathcal{W}_k|_{\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})} \cong \bigoplus_{\ell=1}^{s} f_\ell^* \mathcal{W}_k^\ell,$$

where $\mathcal{W}_k^\ell = \mathcal{W}_k(\mathbf{c}_\ell, \mathbf{d}_\ell, 1, \mathbf{n}_\ell, \mathbf{m}_\ell)$. Thus, $\mathcal{K}_{\mathbf{c},\mathbf{d}}(s, |\mathbf{n}|, |\mathbf{m}|)$ restricted to $\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times$

$\mathcal{M}_{\mathbf{d}}(\mathbf{m})$ is the middle cohomology of

$$\bigoplus_{k,\ell=1}^{s} \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{V}_2^\ell) \xrightarrow{\zeta_{\mathrm{res}}} \bigoplus_{k,\ell=1}^{s} \begin{pmatrix} \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{V}_2^\ell) \oplus \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{V}_2^\ell) \\ \oplus \\ \mathrm{Hom}(f_k^*\mathcal{W}_1^k, f_\ell^*\mathcal{V}_2^\ell) \oplus \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{W}_2^\ell) \end{pmatrix} \xrightarrow{\tau_{\mathrm{res}}} \bigoplus_{k,\ell=1}^{s} \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{V}_2^\ell), \tag{4.13}$$

where $\zeta_{\mathrm{res}}$ and $\tau_{\mathrm{res}}$ are the restrictions of $\zeta$ and $\tau$ to the indicated domains (note that since we only consider the action of $T_\bullet$, the "$t$" and "$t^{-1}$" terms from Complex (4.12) may be safely omitted). It is easy to see that we have the following equality of sections:

$$A_k = \bigoplus_{\ell=1}^{s} f_\ell^* A_k^\ell, \quad B_k = \bigoplus_{\ell=1}^{s} f_\ell^* B_k^\ell, \quad i_k = \bigoplus_{\ell=1}^{s} f_\ell^* i_k^\ell,$$

for $k = 1, 2$. Therefore, Complex (4.13) decomposes as a direct sum of complexes:

$$\bigoplus_{k,\ell=1}^{s} \left( \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{V}_2^\ell) \xrightarrow{\zeta_{k\ell}} \begin{matrix} \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{V}_2^\ell) \oplus \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{V}_2^\ell) \\ \oplus \\ \mathrm{Hom}(f_k^*\mathcal{W}_1^k, f_\ell^*\mathcal{V}_2^\ell) \oplus \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{W}_2^\ell) \end{matrix} \xrightarrow{\tau_{k\ell}} \mathrm{Hom}(f_k^*\mathcal{V}_1^k, f_\ell^*\mathcal{V}_2^\ell) \right),$$

where

$$\zeta_{k\ell}(X) = \begin{pmatrix} X f_k^*(A_1^k) - f_\ell^*(A_2^\ell) X \\ X f_k^*(B_1^k) - f_\ell^*(B_2) X \\ X f_k^*(i_1^k) \\ 0 \end{pmatrix}, \text{ and}$$

$$\tau_{k\ell} \begin{pmatrix} C \\ D \\ a \\ b \end{pmatrix} = f_\ell^*(A_2^\ell) D - D f_k^*(A_1^k) + C f_k^*(B_1^k) - f_\ell^*(B_2^\ell) C + f_\ell^*(i_2^\ell) b.$$

We therefore have that

$$\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m}) = \bigoplus_{k,\ell=1}^{s} (\ker \tau_{k\ell}/\operatorname{im}\zeta_{k\ell})^{T_\bullet}.$$

The bundle $\ker \tau_{k\ell}/\operatorname{im}\zeta_{k\ell}$ is the vector bundle pullback of $\mathcal{K}_{\mathbf{c}_k,\mathbf{d}_\ell}(1,\mathbf{n}_k,\mathbf{m}_\ell)$ by the projection

$$\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m}) \to \mathcal{M}_{\mathbf{c}_k}(1,\mathbf{n}_k) \times \mathcal{M}_{\mathbf{d}_\ell}(1,\mathbf{m}_\ell).$$

Thus the task of computing $(\ker \tau_{k\ell}/\operatorname{im}\zeta_{k\ell})^{T_\bullet}$, reduces to computing $\mathcal{K}_{\mathbf{c}_k,\mathbf{d}_\ell}(1,\mathbf{n}_k,\mathbf{m}_\ell)^{T_\bullet}$. The $T_\bullet$-fixed points of $\mathcal{K}_{\mathbf{c}_k,\mathbf{d}_\ell}(1,\mathbf{n}_k,\mathbf{m}_\ell)$ may be computed (as a set) by computing the $T_\bullet$-fixed points of the fibers. Over the point $([A_1^k,B_1^k,i_1^k],[A_2^\ell,B_2^\ell,i_2^\ell]) \in \mathcal{M}_{\mathbf{c}_k}(1,\mathbf{n}_k) \times \mathcal{M}_{\mathbf{d}_\ell}(1,\mathbf{m}_\ell)$, we may identify the fiber of $\mathcal{V}_1^k$ with $\mathbb{C}^{\mathbf{n}_k}$ by

$$(\mathbb{C}^{\mathbf{n}_k} \times_{\mathrm{GL}_{\mathbf{n}_k}(\mathbb{C})} (A_1^k,B_1^k,i_1^k)) \times [A_2^\ell,B_2^\ell,i_2^\ell] \to \mathbb{C}^{\mathbf{n}_k},$$
$$\big(\mathrm{GL}_{\mathbf{n}_k}(\mathbb{C})\cdot(v,(A_1^k,B_1^k,i_1^k)),[A_2^\ell,B_2^\ell,i_2^\ell]\big) \mapsto v.$$

Similarly, we identify the fiber of $\mathcal{V}_2^\ell$ with $\mathbb{C}^{\mathbf{m}_\ell}$, the fiber of $\mathcal{W}_1^k$ with $\mathbb{C}\mathbf{1}_k$ and the fiber of $\mathcal{W}_2^\ell$ with $\mathbb{C}\mathbf{1}_\ell$. Via these identifications, the fiber of $\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m})$ is the middle cohomology of

$$\mathrm{Hom}(\mathbb{C}^{\mathbf{n}_k},\mathbb{C}^{\mathbf{m}_\ell}) \xrightarrow{\zeta_{k\ell}} \begin{matrix} \mathrm{Hom}(\mathbb{C}^{\mathbf{n}_k},\mathbb{C}^{\mathbf{m}_\ell}) \oplus \mathrm{Hom}(\mathbb{C}^{\mathbf{n}_k},\mathbb{C}^{\mathbf{m}_\ell}) \\ \oplus \\ \mathrm{Hom}(\mathbb{C}\mathbf{1}_k,\mathbb{C}^{\mathbf{m}_\ell}) \oplus \mathrm{Hom}(\mathbb{C}^{\mathbf{n}_k},\mathbb{C}\mathbf{1}_\ell) \end{matrix} \xrightarrow{\tau_{k\ell}} \mathrm{Hom}(\mathbb{C}^{\mathbf{n}_k},\mathbb{C}^{\mathbf{m}_\ell}),$$

where here we view $\zeta_{k\ell}$ and $\tau_{k\ell}$ as linear maps. By Lemma 2.10, $(\ker \tau_{k\ell}/\operatorname{im}\zeta_{k\ell})^{T_\bullet} \cong \ker \tau_{k\ell}|_{T_\bullet}/\operatorname{im}\zeta_{k\ell}|_{T_\bullet}$, where $\zeta_{k\ell}|_{T_\bullet}$ and $\tau_{k\ell}|_{T_\bullet}$ are the restrictions of $\zeta_{k\ell}$ and $\tau_{k\ell}$ to the $T_\bullet$-fixed points of their respective domains. Via our identification of $\mathcal{V}_1^k$ with $\mathbb{C}^{\mathbf{n}_k}$, we have that $T_\bullet$ acts on $\mathbb{C}^{\mathbf{n}_k}$ via multiplication by $z^{k-1}$ for all $z \in T_\bullet$. That is,

$$z \cdot v = z^{k-1} v,$$

for all $z \in T_\bullet$ and $v \in \mathbb{C}^{\mathbf{n}_k}$. Likewise, $T_\bullet$ acts on $\mathbb{C}^{\mathbf{m}_\ell}$ by multiplication by $z^{\ell-1}$, on $\mathbb{C}\mathbf{1}_k$ by multiplication by $z^{k-1}$, and on $\mathbb{C}\mathbf{1}_\ell$ by multiplication by $z^{\ell-1}$, for all $z \in T_\bullet$.

Therefore, for all $z \in T_\bullet$ and $f \in \mathrm{Hom}(\mathbb{C}^{\mathbf{n}_k}, \mathbb{C}^{\mathbf{m}_\ell})$,

$$z \cdot f = z^{k-\ell} f.$$

Thus,

$$\mathrm{Hom}(\mathbb{C}^{\mathbf{n}_k}, \mathbb{C}^{\mathbf{m}_\ell})^{T_\bullet} = \begin{cases} \mathrm{Hom}(\mathbb{C}^{\mathbf{n}_k}, \mathbb{C}^{\mathbf{m}_k}), & \text{if } k = \ell, \\ 0, & \text{if } k \neq \ell. \end{cases}$$

Similarly, $\mathrm{Hom}(\mathbb{C}\mathbf{1}_k, \mathbb{C}^{\mathbf{m}_\ell})$ and $\mathrm{Hom}(\mathbb{C}^{\mathbf{n}_k}, \mathbb{C}\mathbf{1}_\ell)$ are zero when $k \neq \ell$. Since the fibers of $\mathcal{K}_{\mathbf{c}_k,\mathbf{d}_\ell}(1, \mathbf{n}_k, \mathbf{m}_\ell)^{T_\bullet}$ are all zero when $k \neq \ell$, we conclude that $\mathcal{K}_{\mathbf{c}_k,\mathbf{d}_\ell}(1, \mathbf{n}_k, \mathbf{m}_\ell)^{T_\bullet} = 0$ for all $k \neq \ell$. Therefore,

$$\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n}, \mathbf{m}) = \bigoplus_{\ell=1}^{s} f_\ell^* \mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1, \mathbf{n}_\ell, \mathbf{m}_\ell).$$

∎

From our diagram of inclusions, (4.10), we have that

$$\begin{aligned} \mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m}) &\supseteq \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R} \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})^{\mathbb{Z}_R} \\ &\cong \coprod_{\substack{|\mathbf{v}^\ell| = \mathbf{n}_\ell \\ |\mathbf{u}^\ell| = \mathbf{m}_\ell}} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \dots, \mathbf{v}^\ell) \times \mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1, \dots, \mathbf{u}^s). \end{aligned}$$

We may thus view $\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \dots, \mathbf{v}^s) \times \mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1, \dots, \mathbf{u}^s)$ as a subvariety of $\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})$, and consider the restriction of $\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n}, \mathbf{m})$:

$$\mathfrak{K}_{\mathbf{c},\mathbf{d}}(\mathbf{v}^1, \mathbf{u}^1, \dots, \mathbf{v}^s, \mathbf{u}^s) := \mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n}, \mathbf{m})|_{\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s) \times \mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1,\dots,\mathbf{u}^s)},$$

which is a vector bundle on $\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \dots, \mathbf{v}^s) \times \mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1, \dots, \mathbf{u}^s)$. In the following section, the vector bundles $\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n}, \mathbf{m})$ and $\mathfrak{K}_{\mathbf{c},\mathbf{d}}(\mathbf{v}^1, \mathbf{u}^1, \dots, \mathbf{v}^s, \mathbf{u}^s)$ will allows us to construct geometric versions of the Heisenberg algebra, the Clifford algebra and $\widehat{\mathfrak{gl}}_r$.

# Chapter 5

# Geometric Realizations of the Basic Representation

In this chapter, we present the main theorem of this paper, Theorem 5.21, which describes our geometric realizations of the basic representation of $\widehat{\mathfrak{gl}}_r$. We will do so by constructing the oscillator algebra, the Clifford algebra and $\widehat{\mathfrak{gl}}_r$ as geometric operators (in the sense of (4.4)). Using the Localization Theorem, we will consider these geometric operators as operators on the same cohomology. We will show that these operators satisfy the same relations as those observed by their algebraic counterparts in Lemma 1.21, and that the cohomology on which they act (or more specifically the full $\mathbb{C}$-lattice $\mathbf{A}$) corresponds naturally to fermionic Fock space.

Throughout this chapter, for any $T_\star$-equivariant vector bundle $E$, we let $c_{\text{tnv}}(E)$ denote the *top nonvanishing* $T_\star$-equivariant Chern class of $E$. We will also frequently make use of the Künneth formula

$$\mathcal{H}^*_{T_\star}(X \times Y) \cong \mathcal{H}^*_{T_\star}(X) \otimes \mathcal{H}^*_{T_\star}(Y),$$

and simply identify the two rings.

We begin by constructing the geometric version of the oscillator/Heisenberg algebra. The dimension of $\mathcal{M}_{\mathbf{c}}(\mathbf{n})$ is $4|\mathbf{n}|$, thus elements of $\mathcal{H}^{2(|\mathbf{n}|+|\mathbf{m}|)}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m}))$

will define operators $\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})) \to \mathcal{H}^{2|\mathbf{m}|}_{T_\star}(\mathcal{M}_{\mathbf{d}}(\mathbf{m}))$. Consider the vector bundle $\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m})$ over $\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})$. The rank of $\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m})$ is $|\mathbf{n}|+|\mathbf{m}|$, which can be seen from the following lemma.

**Lemma 5.1.** *Let* $(\mathbf{I},\mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star}$. *The equivariant Euler class of the restriction of* $\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m})$ *to* $(\mathbf{I},\mathbf{J})$ *is*

$$e_{T_\star}(\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m})_{(\mathbf{I},\mathbf{J})}) = \prod_{\ell=1}^{s}\left(\prod_{x\in\lambda(\mathbf{I}_\ell)}(\mathbf{d}_\ell - \mathbf{c}_\ell - h_{\lambda(\mathbf{I}_\ell),\lambda(\mathbf{J}_\ell)}(x))\frac{R}{r_\ell}\epsilon\right) \times \left(\prod_{y\in\lambda(\mathbf{J}_\ell)}(\mathbf{d}_\ell - \mathbf{c}_\ell + h_{\lambda(\mathbf{J}_\ell),\lambda(\mathbf{I}_\ell)}(y))\frac{R}{r_\ell}\epsilon\right).$$

**Proof:** By Lemma 4.14,

$$\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m}) \cong \bigoplus_{\ell=1}^{s} f_\ell^*(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1,\mathbf{n}_\ell,\mathbf{m}_\ell)),$$

and so, by properties of the Euler class,

$$\begin{aligned} e_{T_\star}(\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m})) &= \prod_{\ell=1}^{s} f_\ell^*(e_{T_\star}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1,\mathbf{n}_\ell,\mathbf{m}_\ell))) \\ &= \prod_{\ell=1}^{s}\left(1^{\otimes \ell-1} \otimes e_{T_\star}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1,\mathbf{n}_\ell,\mathbf{m}_\ell)) \otimes 1^{\otimes s-\ell}\right). \end{aligned}$$

Now, by setting $r=1$ and replacing $\epsilon$ by $(R/r_\ell)\epsilon$ in [15, Lemma 3.3], we have that

$$e_{T_\star}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1,\mathbf{n}_\ell,\mathbf{m}_\ell)_{(\mathbf{I}_\ell,\mathbf{J}_\ell)}) = \left(\prod_{x\in\lambda(\mathbf{I}_\ell)}(\mathbf{d}_\ell - \mathbf{c}_\ell - h_{\lambda(\mathbf{I}_\ell),\lambda(\mathbf{J}_\ell)}(x))\frac{R}{r_\ell}\epsilon\right) \times \left(\prod_{y\in\lambda(\mathbf{J}_\ell)}(\mathbf{d}_\ell - \mathbf{c}_\ell + h_{\lambda(\mathbf{J}_\ell),\lambda(\mathbf{I}_\ell)}(y))\frac{R}{r_\ell}\epsilon\right).$$

The result follows. ∎

Recall from [15, Section 3.3], that the top nonvanishing Chern class of $\mathcal{K}_{c,c}(1,n,m)$ is

$$c_{\mathrm{tnv}}(\mathcal{K}_{c,c}(1,n,m)) = \begin{cases} c_{2n}(\mathcal{K}_{c,c}(1,n,n)), & \text{if } n = m, \\ c_{n+m-1}(\mathcal{K}_{c,c}(1,n,m)) & \text{if } n \neq m. \end{cases} \tag{5.1}$$

By Lemma 4.14, the top nonvanishing Chern class of $\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n},\mathbf{m})$ may be computed by

$$\begin{aligned} c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n},\mathbf{m})) = c_{\mathrm{tnv}}\left(\bigoplus_{\ell=1}^{s} f_\ell^*(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell}(1,\mathbf{n}_\ell,\mathbf{m}_\ell))\right) &= \prod_{\ell=1}^{s} f_\ell^*(c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell}(1,\mathbf{n}_\ell,\mathbf{m}_\ell))) \\ &= \prod_{\ell=1}^{s}\left(1^{\otimes \ell-1}\otimes c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell}(1,\mathbf{n}_\ell,\mathbf{m}_\ell))\otimes 1^{\otimes s-\ell}\right). \end{aligned}$$

Therefore, if $\mathbf{n}=\mathbf{m}$,

$$c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n},\mathbf{n})) = c_{2|\mathbf{n}|}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n},\mathbf{n})),$$

whereas if $\mathbf{n}$ differs from $\mathbf{m}$ in exactly one component,

$$c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n},\mathbf{m})) = c_{|\mathbf{n}|+|\mathbf{m}|-1}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n},\mathbf{m})).$$

For each $\ell = 1,\dots,s$, define $\alpha^\ell \in \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)^{T_\star}\times\mathcal{M}_{\mathbf{d}_\ell}(1,\mathbf{m}_\ell)^{T_\star}) = \bigoplus_{(I,J)}\mathcal{H}^*_{T_\star}(\mathrm{pt})$ to be the element with $(I,J)$-th component

$$\alpha^\ell_{(I,J)} = \begin{cases} \frac{\epsilon}{e_{T_\star}(\mathcal{T}_{(I,J)})}, & \text{if } \mathbf{n}_\ell = \mathbf{m}_\ell, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathcal{T}_{(I,J)}$ is the tangent space of $(I,J)$ in $\mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)\times\mathcal{M}_{\mathbf{d}_\ell}(1,\mathbf{m}_\ell)$. Let $\widetilde{\alpha}^\ell := i_*(\alpha^\ell)$, where $i : \mathcal{M}_{\mathbf{c}_\ell}(1,\mathbf{n}_\ell)^{T_\star}\times\mathcal{M}_{\mathbf{d}_\ell}(1,\mathbf{m}_\ell)^{T_\star} \hookrightarrow \mathcal{M}_{\mathbf{c}}(1,\mathbf{n}_\ell)\times\mathcal{M}_{\mathbf{d}}(1,\mathbf{m}_\ell)$ is the natural inclusion. Denote by $i_{I,J}$ the inclusion $\{(I,J)\}\hookrightarrow \mathcal{M}_{\mathbf{c}}(1,\mathbf{n}_\ell)\times\mathcal{M}_{\mathbf{d}}(1,\mathbf{m}_\ell)$. Then by [5, Equation (9.3)],

$$i^*_{I,J}(\widetilde{\alpha}^\ell) = (i^*_{I,J}\circ i_*)(\alpha^\ell) = e_{T_\star}(\mathcal{T}_{(I,J)})\cup\alpha^\ell_{(I,J)} = \begin{cases} \epsilon, & \text{if } \mathbf{n}_\ell = \mathbf{m}_\ell, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\gamma^\ell := f_\ell^*(\widetilde{\alpha}^\ell) = 1^{\otimes \ell-1}\otimes\widetilde{\alpha}^\ell\otimes 1^{\otimes s-\ell}\in\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})\times\mathcal{M}_{\mathbf{d}}(\mathbf{m}))$. Note that $\gamma^\ell$ is an element of degree 2.

**Definition 5.2** (Geometric oscillator/Heisenberg operators)**.** For $\ell = 1, \ldots, s$ and $k \in \mathbb{Z}$, define operators

$$\mathbf{P}_\ell(k) : \bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_\mathbf{c}(\mathbf{n})) \to \bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_\mathbf{c}(\mathbf{n})),$$

by

$$\mathbf{P}_\ell(k)|_{\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_\mathbf{c}(\mathbf{n}))} = \begin{cases} (R/r_\ell)\gamma^\ell \cup c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n}, \mathbf{n} - k\mathbf{1}_\ell)), & \text{if } k < 0, \\ -(R/r_\ell)\gamma^\ell \cup c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n}, \mathbf{n} - k\mathbf{1}_\ell)), & \text{if } k > 0, \end{cases}$$
$$\in \mathcal{H}^{2(2|\mathbf{n}|-k)}_{T}(\mathcal{M}_\mathbf{c}(\mathbf{n}) \times \mathcal{M}_\mathbf{c}(\mathbf{n} - k\mathbf{1}_\ell))$$

$$\mathbf{P}_\ell(0)|_{\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_\mathbf{c}(\mathbf{n}))} = \mathbf{c}_\ell \cdot c_{\text{tnv}}(\mathcal{K}_{c,c}(\mathbf{n}, \mathbf{n})) = \mathbf{c}_\ell \,\text{id}\,.$$

The $\mathbf{P}_\ell(k)$ will be called *geometric oscillators* (or, for $k \neq 0$, *geometric Heisenberg operators*).

**Theorem 5.3.** *The operators $\mathbf{P}_\ell(k)$ preserve the space $\mathbf{A}$ and satisfy the commutation relations*

$$[\mathbf{P}_\ell(k), \mathbf{P}_m(0)] = 0, \quad \textit{and} \quad [\mathbf{P}_\ell(k), \mathbf{P}_m(j)] = \frac{1}{k}\delta_{\ell,m}\delta_{k+j,0}\,\text{id},\ k \neq 0.$$

*In particular, the mapping*

$$P_\ell(k) \mapsto \mathbf{P}_\ell(k),$$

*defines a representation of the $s$-coloured oscillator algebra on $\mathbf{A}$ and the linear map determined by*

$$[\mathbf{I}] \mapsto \left(s_{\lambda(\mathbf{I}_1)} \otimes q^{c(\mathbf{I}_1)}\right) \otimes \cdots \otimes \left(s_{\lambda(\mathbf{I}_s)} \otimes q^{c(\mathbf{I}_s)}\right),$$

*is an isomorphism of $s$-coloured oscillator algebra representations $\mathbf{A} \to \mathbb{B}$. This isomorphism maps $\mathbf{A}(c) \to \mathbb{B}(c)$, for all $c \in \mathbb{Z}$.*

**Proof:** Take $\mathbf{m} = \mathbf{n} - k\mathbf{1}_\ell$. We know that

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n}, \mathbf{m})) = \textstyle\prod_{i=1}^s \left(1^{\otimes i-1} \otimes c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_i,\mathbf{c}_i}(1, \mathbf{n}_i, \mathbf{m}_i)) \otimes 1^{\otimes s-i}\right).$$

By [15, Lemma 3.10], for $i \neq \ell$, we have that $c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_i,\mathbf{c}_i}(1,\mathbf{n}_i,\mathbf{n}_i))$ is the identity as an operator $\mathcal{H}^{2\mathbf{n}_i}_{T_\star}(\mathcal{M}_{\mathbf{c}_i}(1,\mathbf{n}_i)) \to \mathcal{H}^{2\mathbf{n}_i}_{T_\star}(\mathcal{M}_{\mathbf{c}_i}(1,\mathbf{n}_i))$, i.e. $c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_i,\mathbf{c}_i}(1,\mathbf{n}_i,\mathbf{m}_i)) = 1$. Thus,

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n},\mathbf{m})) = 1^{\otimes \ell-1} \otimes c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell}(1,\mathbf{n}_\ell,\mathbf{n}_\ell - k)) \otimes 1^{\otimes s-\ell}.$$

Therefore, via the Künneth formula,

$$\bigoplus\nolimits_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})) \cong \bigoplus\nolimits_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2\mathbf{n}_1}_{T_\star}(\mathcal{M}_{\mathbf{c}_1}(1,\mathbf{n}_1)) \otimes \cdots \otimes \mathcal{H}^{2\mathbf{n}_s}_{T_\star}(\mathcal{M}_{\mathbf{c}_s}(1,\mathbf{n}_s)),$$

we have

$$\mathbf{P}_\ell(k)|_{\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))} = 1^{\otimes(\ell-1)} \otimes (R/r_\ell)\widetilde{\alpha}^\ell \cup c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell}(1,\mathbf{n}_\ell,\mathbf{n}_\ell - k)) \otimes 1^{\otimes(s-\ell)}.$$

By [15, Theorem 3.14], for a fixed $\ell \in \{1,\ldots,s\}$, the operators $\mathbf{P}_\ell(k)$ preserve the space $\bigoplus_{\mathbf{n}_\ell,\mathbf{c}_\ell} \mathbf{A}_{\mathbf{c}_\ell}(\mathbf{n}_\ell)$ (see Remark 4.8) and satisfy the 1-coloured oscillator algebra relations. The general result then follows by extension to the whole tensor product. ∎

For the geometric version of the Clifford algebra, we recall from [15, Section 3.2] that

$$c_{\text{tnv}}(\mathcal{K}_{c,c\pm 1}(1,n,m)) = c_{n+m}(\mathcal{K}_{c,c\pm 1}(1,n,m)). \tag{5.2}$$

Therefore, by equations (5.1) and (5.2), if $\mathbf{n}_i = \mathbf{m}_i$ for all $i \neq \ell$, then

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}\pm\mathbf{1}_\ell}(\mathbf{n},\mathbf{m})) = c_{|\mathbf{n}|+|\mathbf{m}|}(\mathcal{K}_{\mathbf{c},\mathbf{c}\pm\mathbf{1}_\ell}(\mathbf{n},\mathbf{m})).$$

**Definition 5.4** (Geometric Clifford operators)**.** For $\ell = 1,\ldots,s$ and $k \in \mathbb{Z}$, define operators

$$\mathbf{\Psi}_\ell(k), \mathbf{\Psi}^*_\ell(k) : \bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})) \to \bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})),$$

by

$$\mathbf{\Psi}_\ell(k)|_{\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))} := (-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{\ell-1}} c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}+\mathbf{1}_\ell}(\mathbf{n},\mathbf{n}+(k-\mathbf{c}_\ell-1)\mathbf{1}_\ell))$$

$$\in \mathcal{H}_{T_\star}^{2(2|\mathbf{n}|+k-\mathbf{c}_\ell-1)}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{c}+\mathbf{1}_\ell}(\mathbf{n}+(k-\mathbf{c}_\ell-1)\mathbf{1}_\ell)),$$

$$\begin{aligned}\mathbf{\Psi}_\ell^*(k)|_{\mathcal{H}_{T_\star}^{2|\mathbf{n}|}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))} &:= (-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{\ell-1}} c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}-\mathbf{1}_\ell}(\mathbf{n},\mathbf{n}-(k-\mathbf{c}_\ell)\mathbf{1}_\ell)) \\ &\in \mathcal{H}_{T_\star}^{2(2|\mathbf{n}|-k+\mathbf{c}_\ell)}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{c}-\mathbf{1}_\ell}(\mathbf{n}-(k-\mathbf{c}_\ell)\mathbf{1}_\ell)).\end{aligned}$$

The $\mathbf{\Psi}_\ell(k)$ and $\mathbf{\Psi}_\ell^*(k)$ will be called *geometric Clifford operators.*

**Theorem 5.5.** *The operators $\mathbf{\Psi}_\ell(k)$ and $\mathbf{\Psi}_\ell^*(k)$ preserve the space $\mathbf{A}$ and satisfy the relations*

$$\{\mathbf{\Psi}_\ell(k), \mathbf{\Psi}_j^*(i)\} = \delta_{ki}\delta_{\ell j}, \quad \{\mathbf{\Psi}_\ell(k), \mathbf{\Psi}_j(i)\} = \{\mathbf{\Psi}_\ell^*(k), \mathbf{\Psi}_j^*(i)\} = 0.$$

*In particular, the mapping*

$$\psi_\ell(k) \mapsto \mathbf{\Psi}_\ell(k), \quad \psi_\ell^*(k) \mapsto \mathbf{\Psi}_\ell^*(k),$$

*defines a representation of the $s$-coloured Clifford algebra on $\mathbf{A}$ and the linear map determined by*

$$[\mathbf{I}] \mapsto \mathbf{I},$$

*is an isomorphism of Clifford algebra representations $\mathbf{A} \to \mathbb{F}$. This isomorphism maps $\mathbf{A}(c) \to \mathbb{F}(c)$, for all $c \in \mathbb{Z}$.*

**Proof:** Using an argument analogous to the proof of Theorem 5.3, one can show that

$$\mathbf{\Psi}_\ell(k)|_{\mathcal{H}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))} = (-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{\ell-1}}(1^{\otimes \ell-1} \otimes c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell+1}(1,\mathbf{n}_\ell,\mathbf{n}_\ell+k-\mathbf{c}_\ell-1)) \otimes 1^{\otimes s-\ell}),$$

$$\mathbf{\Psi}_\ell^*(k)|_{\mathcal{H}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))} = (-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{\ell-1}}(1^{\otimes \ell-1} \otimes c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell-1}(1,\mathbf{n}_\ell,\mathbf{n}_\ell-k+\mathbf{c}_\ell)) \otimes 1^{\otimes s-\ell}).$$

By [15, Theorem 3.6], for fixed $\ell$, the operators $(-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{\ell-1}} c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell+1}(1,\mathbf{n}_\ell,\mathbf{n}_\ell + k-\mathbf{c}_\ell-1))$ and $(-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{\ell-1}} c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell-1}(1,\mathbf{n}_\ell,\mathbf{n}_\ell-k+\mathbf{c}_\ell))$ preserve the space $\bigoplus_{\mathbf{n}_\ell,\mathbf{c}_\ell} \mathbf{A}_{\mathbf{c}_\ell}(\mathbf{n}_\ell)$ (see Remark 4.8) and satisfy the 1-coloured Clifford algebra relations.

The general result then follows by extension to the whole tensor product. ∎

Recall from Diagram 4.10 that, since

$$\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R} \cong \coprod_{|\mathbf{v}^\ell|=n_\ell} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s),$$

we may view $\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s) \times \mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1,\dots,\mathbf{u}^s)$ as a subvariety of $\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})$, where $\mathbf{n}_\ell = |\mathbf{v}^\ell|$ and $\mathbf{m}_\ell = |\mathbf{u}^\ell|$. We therefore have an action of $T_\star$ on $\mathcal{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)$ and

$$\coprod_{\mathbf{v}^\ell} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)^{T_\star} = \coprod_{\mathbf{n}} \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}.$$

Let

$$\mathfrak{K}_{\mathbf{c}}(\mathbf{n};\ell,k)^{\pm} := \coprod_{|\mathbf{v}^i|=\mathbf{n}_i} \mathfrak{K}_{\mathbf{c},\mathbf{c}}(\mathbf{v}^1,\mathbf{v}^1,\dots,\mathbf{v}^\ell,\mathbf{v}^\ell \pm \mathbf{1}_k,\dots,\mathbf{v}^s,\mathbf{v}^s),$$

which is a vector bundle on $\coprod_{|\mathbf{v}^i|=\mathbf{n}_i} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s) \times \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^\ell \pm \mathbf{1}_k,\dots,\mathbf{v}^s)$. By construction, for any $(\mathbf{I},\mathbf{J}) \in \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)^{T_\star} \times \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^\ell \pm \mathbf{1}_k,\dots,\mathbf{v}^\ell)^{T_\star}$,

$$c_{\mathrm{tnv}}(\mathfrak{K}_{\mathbf{c}}(\mathbf{n};\ell,k)^{\pm}_{(\mathbf{I},\mathbf{J})}) = c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n},\mathbf{n}\pm\mathbf{1}_\ell)_{(\mathbf{I},\mathbf{J})}).$$

Thus,

$$c_{\mathrm{tnv}}(\mathfrak{K}_{\mathbf{c}}(\mathbf{n};\ell,k)^{\pm}) = c_{(2|\mathbf{n}|\pm 1)-1}(\mathfrak{K}_{\mathbf{c}}(\mathbf{n};\ell,k)^{\pm}).$$

By the same reasoning, for $(\mathbf{I},\mathbf{J}) \in \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)^{T_\star} \times \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)^{T_\star}$,

$$c_{\mathrm{tnv}}(\mathfrak{K}_{\mathbf{c},\mathbf{c}}(\mathbf{v}^1,\mathbf{v}^1,\dots,\mathbf{v}^s,\mathbf{v}^s)_{(\mathbf{I},\mathbf{J})}) = c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n},\mathbf{n})_{(\mathbf{I},\mathbf{J})}),$$

and so

$$c_{\mathrm{tnv}}(\mathfrak{K}_{\mathbf{c},\mathbf{c}}(\mathbf{v}^1,\mathbf{v}^1,\dots,\mathbf{v}^s,\mathbf{v}^s)) = c_{2|\mathbf{n}|}(\mathfrak{K}_{\mathbf{c},\mathbf{c}}(\mathbf{v}^1,\mathbf{v}^1,\dots,\mathbf{v}^s,\mathbf{v}^s)).$$

Let $h : \coprod_{\mathbf{v}^\ell,\mathbf{u}^\ell} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s) \times \mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1,\dots,\mathbf{u}^s) \cong \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R} \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})^{\mathbb{Z}_R} \hookrightarrow \mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})$ denote the natural inclusion and let $\beta^\ell := h^*(\gamma^\ell)$ for all $\ell = 1,\dots,s$. For $\ell = 1,\dots,s$ and $k = 0,\dots,r_\ell - 1$, we define

$$\mathfrak{E}^\ell_k(\mathbf{c},\mathbf{n}) := -(R/r_\ell)\beta^\ell \cup c_{\mathrm{tnv}}(\mathfrak{K}_{\mathbf{c}}(\mathbf{n};\ell,k)^{-})$$

$$\in \mathcal{H}_{T_\star}^{4|\mathbf{n}|-2}\left(\coprod_{|\mathbf{v}^i|=\mathbf{n}_i} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)\times\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^\ell-\mathbf{1}_k,\dots,\mathbf{v}^s)\right),$$

$$\begin{aligned}\mathfrak{F}_k^\ell(\mathbf{c},\mathbf{n}) &:= (R/r_\ell)\beta^\ell\cup c_{\text{tnv}}(\mathfrak{K}_{\mathbf{c}}(\mathbf{n};\ell,k)^+)\\ &\in \mathcal{H}_{T_\star}^{4|\mathbf{n}|+2}\left(\coprod_{|\mathbf{v}^i|=\mathbf{n}_i} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)\times\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^\ell+\mathbf{1}_k,\dots,\mathbf{v}^s)\right),\end{aligned}$$

$$\begin{aligned}\mathfrak{H}_k^\ell(\mathbf{c};\mathbf{v}^1,\dots,\mathbf{v}^s) &:= \left((\mathbf{1}_{\bar{c}})_k-\sum_{j=0}^{r_\ell-1}a_{kj}^\ell\mathbf{v}_j^\ell\right)c_{\text{tnv}}(\mathfrak{K}_{\mathbf{c},\mathbf{c}}(\mathbf{v}^1,\mathbf{v}^1,\dots,\mathbf{v}^s,\mathbf{v}^s))\\ &\in \mathcal{H}_{T_\star}^{4(|\mathbf{v}^1|+\cdots+|\mathbf{v}^s|)}(\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)\times\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)),\end{aligned}$$

where $a_{kj}^\ell$ is the $(k,j)$-th entry of the generalized Cartan matrix of type $\widehat{A}_{r_\ell-1}$, i.e.

$$a_{kj}^\ell = \begin{cases} 2, & \text{if } k=j,\\ -1, & \text{if } k=j\pm 1,\\ 0, & \text{otherwise,}\end{cases} \quad \text{if } r_\ell \geq 2,$$

$$a_{kj}^\ell = \begin{cases} 2, & \text{if } k=j,\\ -2 & \text{if } k=j\pm 1,\end{cases} \quad \text{if } r_\ell = 2,$$

$$a_{00}^\ell = 0, \quad \text{if } r_\ell = 1.$$

Note that the elements $\mathfrak{E}_k^\ell$, $\mathfrak{F}_k^\ell$ and $\mathfrak{H}_k^\ell$ do not define operators since they do not lie in the middle degree cohomology of their respective quiver varieties. However, we can push these elements forward to the middle degree cohomology of the appropriate moduli spaces; we describe this process below. For what follows, it will be convenient to view elements of $\mathcal{H}_{T_\star}^*(\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)\times\mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1,\dots,\mathbf{u}^s))$ as elements of $\mathcal{H}_{T_\star}^*(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})^{\mathbb{Z}_R})$. That is, by identifying

$$\mathcal{H}_{T_\star}^*(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})^{\mathbb{Z}_R}) \cong \mathcal{H}_{T_\star}^*\left(\coprod_{\substack{|\mathbf{v}^i|=\mathbf{n}_i\\|\mathbf{u}^i|=\mathbf{m}_i}} \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)\times\mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1,\dots,\mathbf{u}^s)\right)$$

$$\cong \bigoplus_{\substack{|\mathbf{v}^i|=\mathbf{n}_i \\ |\mathbf{u}^i|=\mathbf{m}_i}} \mathcal{H}^*_{T_\star}(\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\ldots,\mathbf{v}^s)\times\mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1,\ldots,\mathbf{u}^s)),$$

we will identify elements of $\mathcal{H}^*_{T_\star}(\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\ldots,\mathbf{v}^s)\times\mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1,\ldots,\mathbf{u}^s))$ with their images under the inclusion

$$\mathcal{H}^*_{T_\star}\left(\mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\ldots,\mathbf{v}^s)\times\mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1,\ldots,\mathbf{u}^s)\right)\hookrightarrow\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})^{\mathbb{Z}_R}).$$

In this way, we will consider

$$\mathfrak{E}^\ell_k(\mathbf{c},\mathbf{n})\in\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_{\mathbf{c}}(\mathbf{n}-\mathbf{1}_\ell)^{\mathbb{Z}_R}),$$

$$\mathfrak{F}^\ell_k(\mathbf{c},\mathbf{n})\in\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_{\mathbf{c}}(\mathbf{n}+\mathbf{1}_\ell)^{\mathbb{Z}_R}),$$

$$\mathfrak{H}^\ell_k(\mathbf{c};\mathbf{v}^1,\ldots,\mathbf{v}^s)\in\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}).$$

Recall from the Localization Theorem (4.1) that we have isomorphisms $f_1$ and $f_2$ as in the diagram below:

$$\begin{array}{ccc} \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})) & \dashleftarrow & \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})^{\mathbb{Z}_R}) \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & \mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star}). & \end{array}$$

Let

$$\mu:\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star})\to\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star}\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star}),$$

be the $\mathbb{C}(b_1,\ldots,b_s,\epsilon)$-linear map determined by

$$1_{(\mathbf{I},\mathbf{J})}\mapsto\frac{e_{T_\star}(\mathcal{T}'_{\mathbf{I},\mathbf{J}})}{e_{T_\star}(\mathcal{T}_{\mathbf{I},\mathbf{J}})}1_{(\mathbf{I},\mathbf{J})},$$

where $\mathcal{T}_{\mathbf{I},\mathbf{J}}$ and $\mathcal{T}'_{\mathbf{I},\mathbf{J}}$ are the tangent spaces of $(\mathbf{I},\mathbf{J})$ in $\mathcal{M}_{\mathbf{c}}(\mathbf{n})\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})$ and $\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_{\mathbf{d}}(\mathbf{m})^{\mathbb{Z}_R}$, respectively. Let $\eta:\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_{\mathbf{c}}(\mathbf{m})^{\mathbb{Z}_R})\to\mathcal{H}^*_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})\times\mathcal{M}_{\mathbf{d}}(\mathbf{m}))$ be the composition

$$\eta:=f_1^{-1}\circ\mu\circ f_2. \tag{5.3}$$

**Lemma 5.6.** *The map $\eta$ is degree-preserving. In particular, the images of $\mathfrak{E}_k^\ell(\mathbf{c},\mathbf{n})$, $\mathfrak{F}_k^\ell(\mathbf{c},\mathbf{n})$ and $\mathfrak{H}_k^\ell(\mathbf{c};\mathbf{v}^1,\ldots,\mathbf{v}^s)$ under $\eta$ lie in the middle degree cohomology of $\mathcal{M}_\mathbf{c}(\mathbf{n})\times\mathcal{M}_\mathbf{c}(\mathbf{n}\mp\mathbf{1}_\ell)$ and $\mathcal{M}_\mathbf{c}(\mathbf{n})\times\mathcal{M}_\mathbf{c}(\mathbf{n})$, respectively.*

**Proof:** The map $f_2$ decreases the degree by $\mathrm{rk}(e_{T_\star}(\mathcal{T}'_{\mathbf{I},\mathbf{J}}))$. The map $\mu$ increases the degree by $\mathrm{rk}(e_{T_\star}(\mathcal{T}'_{\mathbf{I},\mathbf{J}}))-\mathrm{rk}(e_{T_\star}(\mathcal{T}_{\mathbf{I},\mathbf{J}}))$. Finally, the map $f_1^{-1}$ increases the degree by $\mathrm{rk}(e_{T_\star}(\mathcal{T}_{\mathbf{I},\mathbf{J}}))$. Thus, the composition of these maps is degree preserving.

The second statement in the lemma follows from the fact that $\mathfrak{E}_k^\ell(\mathbf{c},\mathbf{n})$, $\mathfrak{F}_k^\ell(\mathbf{c},\mathbf{n})$ and $\mathfrak{H}_k^\ell$ are elements of degree $4|\mathbf{n}|-2$, $4|\mathbf{n}|+2$ and $4|\mathbf{n}|$, respectively. ∎

**Definition 5.7** (Geometric diagonal Chevalley operators)**.** For $\ell=1,\ldots,s$ and $k=0,\ldots,r_\ell-1$, define operators

$$\mathbf{E}_k^\ell,\mathbf{F}_k^\ell,\mathbf{H}_k^\ell:\bigoplus_{\mathbf{n},\mathbf{c}}\mathcal{H}_{T_\star}^{2|\mathbf{n}|}(\mathcal{M}_\mathbf{c}(\mathbf{n}))\to\bigoplus_{\mathbf{n},\mathbf{c}}\mathcal{H}_{T_\star}^{2|\mathbf{n}|}(\mathcal{M}_\mathbf{c}(\mathbf{n})),$$

by

$$\begin{aligned}\mathbf{E}_k^\ell|_{\mathcal{H}_{T_\star}^{2|\mathbf{n}|}(\mathcal{M}_\mathbf{c}(\mathbf{n}))}&:=\eta(\mathfrak{E}_k^\ell(\mathbf{c},\mathbf{n}))\in\mathcal{H}_{T_\star}^{4|\mathbf{n}|-2}(\mathcal{M}_\mathbf{c}(\mathbf{n})\times\mathcal{M}_\mathbf{c}(\mathbf{n}-\mathbf{1}_\ell)),\\ \mathbf{F}_k^\ell|_{\mathcal{H}_{T_\star}^{2|\mathbf{n}|}(\mathcal{M}_\mathbf{c}(\mathbf{n}))}&:=\eta(\mathfrak{F}_k^\ell(\mathbf{c},\mathbf{n}))\in\mathcal{H}_{T_\star}^{4|\mathbf{n}|+2}(\mathcal{M}_\mathbf{c}(\mathbf{n})\times\mathcal{M}_\mathbf{c}(\mathbf{n}+\mathbf{1}_\ell)),\\ \mathbf{H}_k^\ell|_{\mathcal{H}_{T_\star}^{2|\mathbf{n}|}(\mathcal{M}_\mathbf{c}(\mathbf{n}))}&:=\sum_{|\mathbf{v}^i|=\mathbf{n}_i}\eta(\mathfrak{H}_k^\ell(\mathbf{c};\mathbf{v}^1,\ldots,\mathbf{v}^s))\in\mathcal{H}_{T_\star}^{4|\mathbf{n}|}(\mathcal{M}_\mathbf{c}(\mathbf{n})\times\mathcal{M}_\mathbf{c}(\mathbf{n})).\end{aligned}$$

The $\mathbf{E}_k^\ell$, $\mathbf{F}_k^\ell$ and $\mathbf{H}_k^\ell$ will be called *geometric (diagonal) Chevalley operators.*

Our next goal will be to describe the geometric Chevalley operators in terms of geometric Clifford operators. To that end, we prove the following useful lemma.

**Lemma 5.8.** *Let $(\mathbf{I},\mathbf{J})\in\mathcal{M}_\mathbf{c}(\mathbf{n})^{T_\star}\times\mathcal{M}_\mathbf{d}(\mathbf{m})^{T_\star}$ and $\beta\in\mathcal{H}_{T_\star}^{2k}(\mathfrak{M}_\mathbf{c}(\mathbf{v}^1,\ldots,\mathbf{v}^s)\times\mathfrak{M}_\mathbf{d}(\mathbf{u}^1,\ldots,\mathbf{u}^s))$. Let*

$$\mathcal{M}_\mathbf{c}(\mathbf{n})\times\mathcal{M}_\mathbf{d}(\mathbf{m})\xleftarrow{i_1\times i_2}\mathcal{M}_\mathbf{c}(\mathbf{n})^{T_\star}\times\mathcal{M}_\mathbf{d}(\mathbf{m})^{T_\star}\xrightarrow{j_1\times j_2}\mathcal{M}_\mathbf{c}(\mathbf{n})^{\mathbb{Z}_R}\times\mathcal{M}_\mathbf{d}(\mathbf{m})^{\mathbb{Z}_R},$$

*be the natural inclusions and let* $(i_1 \times i_2)_{(\mathbf{I},\mathbf{J})}$ *and* $(j_1 \times j_2)_{(\mathbf{I},\mathbf{J})}$ *denote the restrictions of* $i_1 \times i_2$ *and* $j_1 \times j_2$*, respectively, to* $\{(\mathbf{I},\mathbf{J})\}$*. If*

$$(\mathbf{I},\mathbf{J}) \in \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\ldots,\mathbf{v}^s)^{T_\star} \times \mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1,\ldots,\mathbf{u}^s)^{T_\star},$$

*then*

$$\langle \eta(\beta \cup c_{|\mathbf{n}|+|\mathbf{m}|-k}(\mathfrak{K}_{\mathbf{c},\mathbf{d}}(\mathbf{v}^1,\mathbf{u}^1,\ldots,\mathbf{v}^s,\mathbf{u}^s))[\mathbf{I}],[\mathbf{J}]\rangle = \langle \beta' \cup c_{|\mathbf{n}|+|\mathbf{m}|-k}(\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m}))[\mathbf{I}],[\mathbf{J}]\rangle,$$

*where* $\beta'$ *is a preimage of* $(j_1 \times j_2)^*_{(\mathbf{I},\mathbf{J})}(\beta)$ *under the map* $(i_1 \times i_2)^*_{(\mathbf{I},\mathbf{J})}$*. Otherwise,*

$$\langle \eta(\beta \cup c_{|\mathbf{n}|+|\mathbf{m}|-k}(\mathfrak{K}_{\mathbf{c},\mathbf{d}}(\mathbf{v}^1,\mathbf{u}^1,\ldots,\mathbf{v}^s,\mathbf{u}^s)))[\mathbf{I}],[\mathbf{J}]\rangle = 0.$$

**Proof:** To simplify notation, we write $\mathfrak{K} = \mathfrak{K}_{\mathbf{c},\mathbf{d}}(\mathbf{v}^1,\mathbf{u}^1,\ldots,\mathbf{v}^s,\mathbf{u}^s)$, $\mathcal{K} = \mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m})$ and $c = c_{|\mathbf{n}|+|\mathbf{m}|-k}$. We begin by explicitly computing $\eta(\beta \cup c(\mathfrak{K}))$ (recall $\eta = f_1^{-1} \circ \mu \circ f_2$ as in (5.3)). By definition of $f_2$,

$$\begin{aligned} f_2(\beta \cup c(\mathfrak{K})) &= \left( \frac{(j_1 \times j_2)^*_{(\mathbf{K},\mathbf{L})}(\beta \cup c(\mathfrak{K}))}{e_{T_\star}(\mathcal{T}'_{(\mathbf{K},\mathbf{L})})} \right)_{(\mathbf{K},\mathbf{L}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star}} \\ &= \left( \frac{\beta_{\mathbf{K},\mathbf{L}} \cup (j_1 \times j_2)^*_{(\mathbf{K},\mathbf{L})}(c(\mathfrak{K}))}{e_{T_\star}(\mathcal{T}'_{(\mathbf{K},\mathbf{L})})} \right)_{(\mathbf{K},\mathbf{L}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star}}, \end{aligned}$$

where $\mathcal{T}'_{(\mathbf{K},\mathbf{L})}$ is the tangent space of $(\mathbf{K},\mathbf{L})$ in $\mathcal{M}_{\mathbf{c}}(\mathbf{n})^{\mathbb{Z}_R} \times \mathcal{M}_{\mathbf{c}}(\mathbf{m})^{\mathbb{Z}_R}$, and $\beta_{\mathbf{K},\mathbf{L}} = (j_1 \times j_2)^*_{(\mathbf{K},\mathbf{L})}(\beta)$. By applying $\mu$, we get

$$\mu \circ f_2(\beta \cup c(\mathfrak{K})) = \left( \frac{\beta_{\mathbf{K},\mathbf{L}} \cup (j_1 \times j_2)^*_{(\mathbf{K},\mathbf{L})}(c(\mathfrak{K}))}{e_{T_\star}(\mathcal{T}_{(\mathbf{K},\mathbf{L})})} \right)_{(\mathbf{K},\mathbf{L}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star}},$$

where $\mathcal{T}_{(\mathbf{K},\mathbf{L})}$ is the tangent space of $(\mathbf{K},\mathbf{L})$ in $\mathcal{M}_{\mathbf{c}}(\mathbf{n}) \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})$. By definition, the map $f_1^{-1} = (i_1 \times i_2)_*$, and thus,

$$\eta(\beta \cup c(\mathfrak{K})) = (i_1 \times i_2)_* \left( \frac{\beta_{\mathbf{K},\mathbf{L}} \cup (j_1 \times j_2)^*_{(\mathbf{K},\mathbf{L})}(c(\mathfrak{K}))}{e_{T_\star}(\mathcal{T}_{(\mathbf{K},\mathbf{L})})} \right)_{(\mathbf{K},\mathbf{L}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star}}.$$

By definition of the bilinear form,

$$\begin{aligned}\langle \eta(\beta \cup c(\mathfrak{K}))[\mathbf{I}], [\mathbf{J}]\rangle &= (-1)^{|\mathbf{m}|} p_*((i_1 \times i_2)_*)^{-1}(\eta(\beta \cup c(\mathfrak{K})) \cup [\mathbf{I}] \otimes [\mathbf{J}]) \\ &= (-1)^{|\mathbf{m}|} p_*((i_1 \times i_2)_*)^{-1} \left( \frac{\eta(\beta \cup c(\mathfrak{K}))}{e_{T_\star}(\mathcal{T}_{\mathbf{I}}^-) e_{T_\star}(\mathcal{T}_{\mathbf{J}}^-)} \cup (i_1 \times i_2)_*(1_{(\mathbf{I},\mathbf{J})}) \right),\end{aligned}$$

where the last equality comes from the definition of $[\mathbf{I}]$ and $[\mathbf{J}]$. By the projection formula,

$$\langle \eta(\beta \cup c(\mathfrak{K}))[\mathbf{I}], [\mathbf{J}]\rangle = (-1)^{|\mathbf{m}|} p_* \left( \frac{(i_1 \times i_2)^*(\eta(\beta \cup c(\mathfrak{K})))}{e_{T_\star}(\mathcal{T}_{\mathbf{I}}^-) e_{T_\star}(\mathcal{T}_{\mathbf{J}}^-)} \cup 1_{(\mathbf{I},\mathbf{J})} \right).$$

By [5, Equation 9.3], $(i_1 \times i_2)^*(i_1 \times i_2)_*$ is simply multiplication by the Euler class of the tangent space. Hence,

$$(i_1 \times i_2)^* \eta(\beta \cup c(\mathfrak{K})) = \left(\beta_{\mathbf{K},\mathbf{L}} \cup (j_1 \times j_2)^*_{(\mathbf{K},\mathbf{L})}(c(\mathfrak{K}))\right)_{(\mathbf{K},\mathbf{L}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{d}}(\mathbf{m})^{T_\star}}.$$

Thus,

$$\langle \eta(\beta \cup c(\mathfrak{K}))[\mathbf{I}], [\mathbf{J}]\rangle = (-1)^{|\mathbf{m}|} p_* \left( \frac{\beta_{\mathbf{I},\mathbf{J}} \cup (j_1 \times j_2)^*_{(\mathbf{I},\mathbf{J})}(c(\mathfrak{K}))}{e_{T_\star}(\mathcal{T}_{\mathbf{I}}^-) e_{T_\star}(\mathcal{T}_{\mathbf{J}}^-)} \right).$$

By construction, if

$$(\mathbf{I}, \mathbf{J}) \notin \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \ldots, \mathbf{v}^s)^{T_\star} \times \mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1, \ldots, \mathbf{u}^s)^{T_\star},$$

then

$$(j_1 \times j_2)^*_{(\mathbf{I},\mathbf{J})}(c(\mathfrak{K})) = 0.$$

On the other hand, if

$$(\mathbf{I}, \mathbf{J}) \in \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \ldots, \mathbf{v}^s)^{T_\star} \times \mathfrak{M}_{\mathbf{d}}(\mathbf{u}^1, \ldots, \mathbf{u}^s)^{T_\star},$$

then by functoriality of the Chern class and the construction of $\mathfrak{K}$,

$$(j_1 \times j_2)^*_{(\mathbf{I},\mathbf{J})}(c(\mathfrak{K})) = c(\mathfrak{K}_{(\mathbf{I},\mathbf{J})}) = c(\mathcal{K}_{(\mathbf{I},\mathbf{J})}).$$

Therefore,

$$\langle \eta(\beta \cup c(\mathfrak{K}))[\mathbf{I}], [\mathbf{J}]\rangle = (-1)^{|m|} \frac{\beta_{\mathbf{I},\mathbf{J}} \cup c(\mathcal{K}_{(\mathbf{I},\mathbf{J})})}{e_{T_\star}(\mathcal{T}_{\mathbf{I}}^-) e_{T_\star}(\mathcal{T}_{\mathbf{J}}^-)} = \frac{\beta_{\mathbf{I},\mathbf{J}} \cup c(\mathcal{K}_{(\mathbf{I},\mathbf{J})})}{e_{T_\star}(\mathcal{T}_{\mathbf{I}}^-) e_{T_\star}(\mathcal{T}_{\mathbf{J}}^+)} = \langle \beta' \cup c(\mathcal{K})[\mathbf{I}], [\mathbf{J}]\rangle,$$

where the last equality follows from Lemma 4.9. ∎

**Corollary 5.9.** *Let* $(\mathbf{I}, \mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{c}}(\mathbf{m})^{T_\star}$.

1. *For* $\mathbf{m} = \mathbf{n} - \mathbf{1}_\ell$, *if*

$$(\mathbf{I}, \mathbf{J}) \in \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \ldots, \mathbf{v}^s)^{T_\star} \times \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \ldots, \mathbf{v}^\ell - \mathbf{1}_k, \ldots, \mathbf{v}^s)^{T_\star},$$

*then*

$$\langle \mathbf{E}_k^\ell [\mathbf{I}], [\mathbf{J}] \rangle = \langle \mathbf{P}_\ell(1)[\mathbf{I}], [\mathbf{J}] \rangle.$$

*Otherwise*

$$\langle \mathbf{E}_k^\ell [\mathbf{I}], [\mathbf{J}] \rangle = 0.$$

2. *For* $\mathbf{m} = \mathbf{n} + \mathbf{1}_\ell$, *if*

$$(\mathbf{I}, \mathbf{J}) \in \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \ldots, \mathbf{v}^s)^{T_\star} \times \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \ldots, \mathbf{v}^\ell + \mathbf{1}_k, \ldots, \mathbf{v}^s)^{T_\star},$$

*then*

$$\langle \mathbf{F}_k^\ell [\mathbf{I}], [\mathbf{J}] \rangle = \langle \mathbf{P}_\ell(-1)[\mathbf{I}], [\mathbf{J}] \rangle.$$

*Otherwise*

$$\langle \mathbf{F}_k^\ell [\mathbf{I}], [\mathbf{J}] \rangle = 0.$$

3. *For* $\mathbf{m} = \mathbf{n}$,

$$\langle \mathbf{H}_k^\ell [\mathbf{I}], [\mathbf{J}] \rangle = \left( (\mathbf{1}_{\bar{\mathbf{c}}_\ell})_k - \sum_j a_{kj}^\ell \mathbf{v}_j^\ell \right) \delta_{\mathbf{I},\mathbf{J}}.$$

**Proof:** Statements (1) and (2) follow directly from Lemma 5.8 and the definitions of $\mathbf{P}_\ell(\pm 1)$ (note that, in the notation of Lemma 5.8, $(i_1 \times i_2)^*(\gamma^\ell) = (j_1 \times j_2)^*(\beta^\ell)$). The third statement follows from Lemma 5.8 and the fact that $c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(\mathbf{n}, \mathbf{n})) = \mathrm{id}$ as an operator on $\mathcal{H}_{T_\star}^{2|\mathbf{n}|}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))$ (see Definition 5.2). ∎

**Proposition 5.10.** *Let* $(\mathbf{I}, \mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{c}}(\mathbf{n} + \mathbf{1}_\ell)^{T_\star}$. *If* $\lambda(\mathbf{I}_\alpha) = \lambda(\mathbf{J}_\alpha)$ *for all* $\alpha \neq \ell$ *and* $\lambda(\mathbf{J}_\ell)$ *can be obtained by adding one box to* $\lambda(\mathbf{I}_\ell)$*, then*

$$\langle \mathbf{P}_\ell(-1)[\mathbf{I}], [\mathbf{J}] \rangle = 1.$$

*Otherwise,* $\langle \mathbf{P}_\ell(-1)[\mathbf{I}], [\mathbf{J}] \rangle = 0$.

**Proof:** This is a direct result of [15, Proposition 5.3 and the proof of Theorem 3.14]. Note that in our case, $\lambda(\mathbf{J}_\ell) - \lambda(\mathbf{I}_\ell)$ consists of a single box. ∎

**Lemma 5.11.** *If* $\mathbf{I} \in \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1, \ldots, \mathbf{v}^s)^{T_\star}$*, then* $\mathbf{v}^\ell_k$ *is equal to the number of boxes in* $\lambda(\mathbf{I}_\ell)$ *whose residue is congruent to* $k - \mathbf{c}_\ell$ *modulo* $r_\ell$.

**Proof:** Let $\mathbf{I}_\ell = [A, B, i] \in \mathcal{M}_{\mathbf{c}_\ell}(1, \mathbf{n}_\ell)^{T_\star}$. By [24, Proposition 2.9], $\lambda(\mathbf{I}_\ell)$ is obtained from $\mathbf{I}_\ell$ by drawing a box in the $(p, q)$-th position if $A^{p-1}B^{q-1}i \neq 0$ (note that our Young diagrams are rotated 90 clockwise from those in [24]). From Equation (3.8),

$$\mathbf{v}^\ell_k = \dim \operatorname{span}\{A^pB^qi \mid q - p \equiv k - \mathbf{c}_\ell \mod r_\ell\}.$$

Since the nonzero $A^pB^qi$ are linearly independent, the boxes $(p, q) \in \lambda(\mathbf{I}_\ell)$ whose residue is congruent to $k - \mathbf{c}_\ell \bmod r_\ell$ are in one-to-one correspondence with a basis of $\operatorname{span}\{A^pB^qi \mid q - p \equiv k - \mathbf{c}_\ell \bmod r_\ell\}$. Thus, $\mathbf{v}^\ell_k$ is equal to the number of such boxes. ∎

**Lemma 5.12.** *Let* $(\mathbf{I}, \mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{c}}(\mathbf{n} + \mathbf{1}_\ell)^{T_\star}$. *Then*

$$\langle \mathbf{F}^\ell_k[\mathbf{I}], [\mathbf{J}] \rangle = 1,$$

*if* $\lambda(\mathbf{I}_\alpha) = \lambda(\mathbf{J}_\alpha)$ *for all* $\alpha \neq \ell$ *and* $\lambda(\mathbf{J}_\ell)$ *can be obtained from* $\lambda(\mathbf{I}_\ell)$ *by adding one box whose residue is congruent to* $k - \mathbf{c}_\ell$ *modulo* $r_\ell$. *Otherwise,* $\langle \mathbf{F}^\ell_k[\mathbf{I}], [\mathbf{J}] \rangle = 0$.

**Proof:** This follows immediately from Corollary 5.9, Proposition 5.10 and Lemma 5.11. ∎

**Proposition 5.13.** *Let* $(\mathbf{I},\mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(\mathbf{n})^{T_\star} \times \mathcal{M}_{\mathbf{c}}(\mathbf{n}+1_\ell)^{T_\star}$. *Then for all* $i \in \mathbb{Z}$,

$$\langle \mathbf{\Psi}_\ell(i)\mathbf{\Psi}_\ell^*(i-1)[\mathbf{I}],[\mathbf{J}]\rangle = \begin{cases} 1, & \text{if } \mathbf{J} = \psi_\ell(i)\psi_\ell^*(i-1)\mathbf{I}, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof:** This follows immediately from Theorem 5.5. ∎

**Theorem 5.14.** *For* $\ell = 1,\dots,s$ *and* $k = 0,1,\dots,r_\ell - 1$,

$$\mathbf{E}_k^\ell = \sum_{i\in\mathbb{Z}} \mathbf{\Psi}_\ell(k+ir_\ell)\mathbf{\Psi}_\ell^*(k+ir_\ell+1),$$

$$\mathbf{F}_k^\ell = \sum_{i\in\mathbb{Z}} \mathbf{\Psi}_\ell(k+ir_\ell+1)\mathbf{\Psi}_\ell^*(k+ir_\ell),$$

*as operators on* $\bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}_{T_\star}^{2|\mathbf{n}|}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))$.

**Proof:** Let $\mathbf{I},\mathbf{J}$ be two $s$-tuples of semi-infinite monomials of charge $\mathbf{c}$. We first prove that

$$\langle \mathbf{F}_k^\ell[\mathbf{I}],[\mathbf{J}]\rangle = \left\langle \sum_{i\in\mathbb{Z}} \mathbf{\Psi}_\ell(k+ir_\ell+1)\mathbf{\Psi}_\ell^*(k+ir_\ell)[\mathbf{I}],[\mathbf{J}] \right\rangle. \tag{5.4}$$

If there exists an $\alpha \neq \ell$ such that $\mathbf{I}_\alpha \neq \mathbf{J}_\alpha$, then both sides of Equation (5.4) are zero and we are done. Thus, we assume that $\mathbf{I}_\alpha = \mathbf{J}_\alpha$ for all $\alpha \neq \ell$. Write

$$\mathbf{I}_\ell = i_1 \wedge i_2 \wedge i_3 \wedge \cdots, \quad \text{and} \quad \mathbf{J}_\ell = j_1 \wedge j_2 \wedge j_3 \wedge \cdots,$$

where $i_m, j_m \in \mathbb{Z}$. Recall that the number of boxes in the $m$-th row of $\lambda(\mathbf{I}_\ell)$ is $i_m - \mathbf{c}_\ell + m - 1$ (likewise for $\lambda(\mathbf{J}_\ell)$). Suppose that $\langle \mathbf{F}_k^\ell[\mathbf{I}],[\mathbf{J}]\rangle = 1$. Then by Lemma 5.12, $\lambda(\mathbf{J}_\ell)$ is obtained by adding one box of the appropriate residue to $\lambda(\mathbf{I}_\ell)$. This means that there exists an $m \in \mathbb{N}$ such that $j_n = i_n$ for all $n \neq m$ and $j_m = i_m + 1$. The position of the added box is thus $(m, j_m - \mathbf{c}_\ell + m - 1)$. The residue of the added box must be congruent to $k - \mathbf{c}_\ell$ modulo $r_\ell$, and so,

$$(j_m - \mathbf{c}_\ell + m - 1) - m = j_m - \mathbf{c}_\ell - 1 \equiv k - \mathbf{c}_\ell \mod r_\ell,$$

or equivalently

$$j_m - k - 1 \equiv 0 \mod r_\ell.$$

Thus, $j_m = k + ir_\ell + 1$, for some $i \in \mathbb{Z}$. Now, since $j_n = i_n$ for all $n \neq m$ and $j_m = i_m + 1$, we have that

$$\mathbf{J} = \psi_\ell(j_m)\psi_\ell^*(j_m - 1)\mathbf{I}, \quad \text{and} \quad \mathbf{J} \neq \psi_\ell(j)\psi_\ell^*(j-1)\mathbf{I}, \quad \text{for all } j \neq j_m.$$

Hence,

$$\left\langle \sum_{i \in \mathbb{Z}} \mathbf{\Psi}_\ell(k + ir_\ell + 1)\mathbf{\Psi}_\ell^*(k + ir_\ell)[\mathbf{I}], [\mathbf{J}] \right\rangle = \langle \mathbf{\Psi}_\ell(j_m)\mathbf{\Psi}_\ell^*(j_m - 1)[\mathbf{I}], [\mathbf{J}]\rangle = 1,$$

by Proposition 5.13. We have therefore proven that Equation (5.4) holds when $\langle \mathbf{F}_k^\ell[\mathbf{I}], [\mathbf{J}]\rangle = 1$.

Suppose now that $\langle \mathbf{F}_k^\ell[\mathbf{I}], [\mathbf{J}]\rangle = 0$. Then $\lambda(\mathbf{J}_\ell)$ cannot be obtained by adding one box to $\lambda(\mathbf{I}_\ell)$. In particular, $\mathbf{J} \neq \psi_\ell(i)\psi_\ell^*(i-1)\mathbf{I}$ for all $i \in \mathbb{Z}$. Thus,

$$\left\langle \sum_{i \in \mathbb{Z}} \mathbf{\Psi}_\ell(k + ir_\ell + 1)\mathbf{\Psi}_\ell^*(k + ir_\ell)[\mathbf{I}], [\mathbf{J}] \right\rangle = 0.$$

Therefore, Equation (5.4) holds in all cases, and so

$$\mathbf{F}_k^\ell = \sum_{i \in \mathbb{Z}} \mathbf{\Psi}_\ell(k + ir_\ell + 1)\mathbf{\Psi}_\ell^*(k + ir_\ell).$$

Now, since $\mathbf{P}_\ell(-1)$ and $\mathbf{P}_\ell(1)$ are adoint (see [15, Lemma 3.13]), it follows by Corollary 5.9 that $\mathbf{E}_k^\ell$ and $\mathbf{F}_k^\ell$ are adoint. Moreover, since $\mathbf{\Psi}_\ell(i)$ and $\mathbf{\Psi}_\ell^*(i)$ are adjoint (see [15, Lemma 3.5]), it follows that

$$\sum_{i \in \mathbb{Z}} \mathbf{\Psi}_\ell(k + ir_\ell + 1)\mathbf{\Psi}_\ell^*(k + ir_\ell), \quad \text{and} \quad \sum_{i \in \mathbb{Z}} \mathbf{\Psi}_\ell(k + ir_\ell)\mathbf{\Psi}_\ell^*(k + ir_\ell + 1),$$

are adjoint. Therefore, for all $s$-tuples of semi-infinite monomials $\mathbf{I}$, $\mathbf{J}$,

$$\langle \mathbf{E}_k^\ell[\mathbf{I}], [\mathbf{J}]\rangle = \langle [\mathbf{I}], \mathbf{F}_k^\ell[\mathbf{J}]\rangle = \left\langle [\mathbf{I}], \sum_{i\in\mathbb{Z}} \mathbf{\Psi}_\ell(k+ir_\ell+1)\mathbf{\Psi}_\ell^*(k+ir_\ell)[\mathbf{J}]\right\rangle$$
$$= \left\langle \sum_{i\in\mathbb{Z}} \mathbf{\Psi}_\ell(k+ir_\ell)\mathbf{\Psi}_\ell^*(k+ir_\ell+1)[\mathbf{I}], [\mathbf{J}]\right\rangle.$$

Thus,

$$\mathbf{E}_k^\ell = \sum_{i\in\mathbb{Z}} \mathbf{\Psi}_\ell(k+ir_\ell)\mathbf{\Psi}_\ell^*(k+ir_\ell+1).$$

∎

**Proposition 5.15.** *For all $\ell = 1,\dots,s$ and $k = 0,1,\dots,r_\ell-1$,*

$$[\mathbf{E}_k^\ell, \mathbf{F}_k^\ell] = \mathbf{H}_k^\ell.$$

**Proof:** If $r_\ell = 1$, then

$$\mathbf{E}_0^\ell = \mathbf{P}_\ell(1), \quad \mathbf{F}_0^\ell = \mathbf{P}_\ell(-1), \quad \mathbf{H}_0^\ell = \mathrm{id},$$

and the result follows from Theorem 5.3.

Now, fix an $\ell \in \{1,\dots,s\}$ such that $r_\ell \geq 2$. For any semi-infinite monomial $I$ of charge $c \in \mathbb{Z}$, we will say that a box $(p,q) \in \lambda(I)$ is a *$k$-box* if its residue is congruent to $k-c \bmod r_\ell$. We will say that a box $(p,q) \in \lambda(I)$ is *$k$-removable* if $(p,q)$ is a $k$-box and $(p,q)$ may be removed from $\lambda(I)$ to create a new Young diagram. Denote the set of $k$-removable boxes of $\lambda(I)$ by $R$. We will say that a box $(p,q) \notin \lambda(I)$ is *$k$-addable* if $(p,q)$ is a $k$-box and $(p,q)$ may be added to $\lambda(I)$ to create a new Young diagram. Denote the set of $k$-addable boxes of $\lambda(I)$ by $A$.

By Theorem 5.14, we have that, for all $\mathbf{I} \in \mathfrak{M}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)^{T_\star}$

$$\mathbf{E}_k^\ell[\mathbf{I}] = \sum_{\mathbf{J}} [\mathbf{J}], \quad \text{and} \quad \mathbf{F}_k^\ell[\mathbf{I}] = \sum_{\mathbf{K}} [\mathbf{K}],$$

where the $\mathbf{J}$ run over all semi-infinite monomials such that $\mathbf{J}_i = \mathbf{I}_i$ for all $i \neq \ell$ and $\mathbf{J}_\ell$ is obtained from $\mathbf{I}_\ell$ by removing a $k$-removable box, and the $\mathbf{K}$ run over all semi-infinite monomials such that $\mathbf{K}_i = \mathbf{I}_i$ for all $i \neq \ell$ and $\mathbf{K}_\ell$ is obtained from $\mathbf{I}_\ell$ by adding a $k$-addable box. Thus, it is easy to see that

$$[\mathbf{E}_k^\ell, \mathbf{F}_k^\ell][\mathbf{I}] = (|A| - |R|)[\mathbf{I}],$$

where here $A$ and $R$ refer to the sets of $k$-addable and $k$-removable boxes of $\mathbf{I}_\ell$, respectively. Therefore, it suffices to show that

$$|A|-|R| = (\mathbf{1}_{\bar{c}})_k - \sum_{j=0}^{r_\ell} a_{kj}^\ell \mathbf{v}_j^\ell = \delta_{\bar{c},\bar{k}} - 2\mathbf{v}_k^\ell + v_{k+1}^\ell + \mathbf{v}_{k-1}^\ell = \delta_{\bar{c},\bar{k}} + (\mathbf{v}_{k+1}^\ell - \mathbf{v}_k^\ell) + (\mathbf{v}_{k-1}^\ell - \mathbf{v}_k^\ell),$$

where, of course, the indices of $\mathbf{v}^\ell$ are taken modulo $r_\ell$.

For the remainder of the proof, we will identify $\mathbf{I}_\ell$ with its Young diagram $\lambda = \lambda(\mathbf{I}_\ell)$. The case where $\lambda$ is the empty Young diagram is trivial, so we assume $\lambda$ consists of at least one box. The $k$-*border* of $\lambda$ is the set

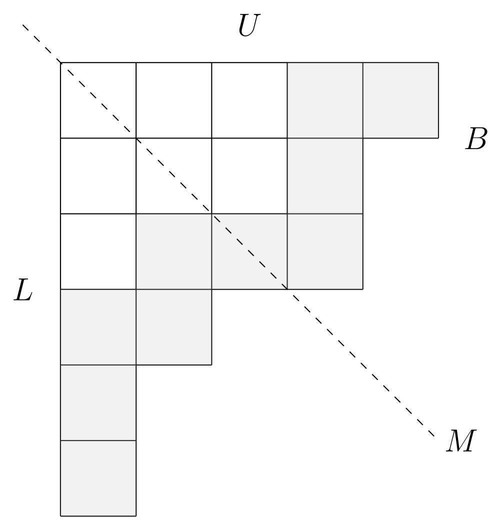
$$B := \{k\text{-boxes } (p,q) \in \lambda \mid (p+1,q), (p,q+1), \text{ or } (p+1,q+1) \notin \lambda\}.$$

We partition $B$ with respect to the main diagonal of $\lambda$ as follows:

$$B = U \,\dot{\cup}\, M \,\dot{\cup}\, L,$$

$$U = \{(p,q) \in B \mid q > p\}, \quad M = \{(p,p) \in B\}, \quad L = \{(p,q) \in B \mid p > q\}.$$

The sets $B, U, M$ and $L$ are illustrated in the following diagram:




Here $B$ is the set of $k$-boxes in the shaded region, $U$ (resp. $L$) is the set of boxes in $B$ that lie above (resp. below) the dashed line, and $M$ is the intersection of $B$ with the dashed line. Let $\Delta_\rightarrow$ (resp. $\Delta_\downarrow$) be the subset of $B$ consisting of boxes which have a right (resp. lower) neighbour. That is,

$$\Delta_\rightarrow = \{(p,q) \in B \mid (p,q+1) \in \lambda\}, \quad \text{and} \quad \Delta_\downarrow = \{(p,q) \in B \mid (p+1,q) \in \lambda\}.$$

Denote by $\Delta'_\rightarrow$ and $\Delta'_\downarrow$ the complements of $\Delta_\rightarrow$ and $\Delta_\downarrow$, respectively, in $B$.

We begin with the case where $k \not\equiv \mathbf{c}_\ell \bmod r_\ell$ (so that $M = \emptyset$). In the portion of $\lambda$ above the main diagonal, every $(k+1)$-box occurs as the right neighbour of a $k$-box. Conversely, every right neighbour of a $k$-box (if it has one) is a $(k+1)$-box. If a $k$-box does not have a right neighbour, it must therefore lie in $B$. Hence, the number of $k$-boxes less the number of $(k+1)$-boxes above the main diagonal is equal to $|U \cap \Delta'_\rightarrow|$. In the portion of $\lambda$ below and including the main diagonal, we can dualize this argument and conclude that the number of $(k+1)$-boxes less the number of $k$-boxes is equal to the number of $(k+1)$-boxes (on the border) with no lower neighbour. Any $(k+1)$-box not in the first column and without a lower neighbour has a left neighbour, which must be a $k$-box and lie in $L$ and $\Delta_\rightarrow$. Hence, the number of $(k+1)$-boxes less the number of $k$-boxes is equal to $|L \cap \Delta_\rightarrow| + \delta$, where $\delta = 1$ if the last box of the first column of $\lambda$ is a $(k+1)$-box and $\delta = 0$ otherwise. Since $\mathbf{v}_k^\ell$ (resp. $\mathbf{v}_{k+1}^\ell$) is the number of $k$-boxes (resp. $(k+1)$-boxes) in $\lambda$,

$$\mathbf{v}_{k+1}^\ell - \mathbf{v}_k^\ell = |L \cap \Delta_\rightarrow| + \delta - |U \cap \Delta'_\rightarrow|. \tag{5.5}$$

By a completely analogous argument,

$$\mathbf{v}_{k-1}^\ell - \mathbf{v}_k^\ell = |U \cap \Delta_\downarrow| + \delta' - |L \cap \Delta'_\downarrow|,$$

where $\delta' = 1$ if the last box of the first row of $\lambda$ is a $(k-1)$-box and $\delta' = 0$ otherwise. Now, we note that a $k$-box is $k$-removable if and only if it has no right or lower neighbours. Hence, $R = \Delta'_\rightarrow \cap \Delta'_\downarrow$ and so

$$|R| = |\Delta'_\rightarrow \cap \Delta'_\downarrow| = |B| - |\Delta_\rightarrow \cup \Delta_\downarrow|. \tag{5.6}$$

We can add a $k$-box at the end of the first row (resp. column) if and only if the last box of the first row (resp. column) is a $(k-1)$-box (resp. $(k+1)$-box). A $k$-box $(p,q)$, where $p, q \neq 1$, is $k$-addable if and only if both $(p-1,q)$ and $(p,q-1)$ are in $\lambda$, which occurs if and only if $(p-1,q-1) \in \Delta_\rightarrow \cap \Delta_\downarrow$. Hence,

$$|A| = |\Delta_\rightarrow \cap \Delta_\downarrow| + \delta + \delta' = |\Delta_\rightarrow| + |\Delta_\downarrow| - |\Delta_\rightarrow \cup \Delta_\downarrow| + \delta + \delta'. \tag{5.7}$$

Since $M = \emptyset$, we have that $B = U \mathbin{\dot\cup} L$, and so

$$|\Delta_{\rightarrow/\downarrow}| = |U \cap \Delta_{\rightarrow/\downarrow}| + |L \cap \Delta_{\rightarrow/\downarrow}| \tag{5.8}$$

Combining equations (5.6), (5.7) and (5.8), we get

$$\begin{aligned} |A| - |R| &= |U \cap \Delta_\rightarrow| + |L \cap \Delta_\rightarrow| + |U \cap \Delta_\downarrow| + |L \cap \Delta_\downarrow| - |B| + \delta + \delta' \\ &= |U| - |U \cap \Delta'_\rightarrow| + |L \cap \Delta_\rightarrow| + |U \cap \Delta_\downarrow| + |L| - |L \cap \Delta'_\downarrow| - |B| + \delta + \delta' \\ &= -|U \cap \Delta'_\rightarrow| + |L \cap \Delta_\rightarrow| + |U \cap \Delta_\downarrow| - |L \cap \Delta'_\downarrow| + \delta + \delta' \\ &= (\mathbf{v}_{k+1}^\ell - \mathbf{v}_k^\ell) + (\mathbf{v}_{k-1}^\ell - \mathbf{v}_k^\ell). \end{aligned}$$

In the case where $k \equiv \mathbf{c}_\ell \bmod r_\ell$, one only has to make a few modifications to the above arguments owing to the fact that $M$ now consists of a box. In the portion of $\lambda$ above and including the main diagonal, the number of $k$-boxes less the number of $(k+1)$-boxes is $|U \cap \Delta'_{\rightarrow}| + |M \cap \Delta'_{\rightarrow}|$. In the portion of $\lambda$ below the main diagonal, one again has that the number of $(k+1)$-boxes less the number of $k$-boxes is $|L \cap \Delta_{\rightarrow}| + \delta$. Hence,

$$\mathbf{v}^\ell_{k+1} - \mathbf{v}^\ell_k = |L \cap \Delta_{\rightarrow}| + \delta - |U \cap \Delta'_{\rightarrow}| - |M \cap \Delta'_{\rightarrow}|.$$

And again by a completely analogous argument,

$$\mathbf{v}^\ell_{k-1} - \mathbf{v}^\ell_k = |U \cap \Delta_{\downarrow}| + \delta' - |L \cap \Delta'_{\downarrow}| - |M \cap \Delta'_{\downarrow}|. \tag{5.9}$$

One can compute $|A|$ and $|R|$ exactly as in equations (5.7) and (5.6). The difference now is that, since $M \neq \emptyset$, we have $B = U \,\dot\cup\, M \,\dot\cup\, L$, and so

$$|\Delta_{\rightarrow/\downarrow}| = |U \cap \Delta_{\rightarrow/\downarrow}| + |M \cap \Delta_{\rightarrow/\downarrow}| + |L \cap \Delta_{\rightarrow/\downarrow}|.$$

Therefore, using similar calculations as before,

$$\begin{aligned}
|A| - |R| &= |U| - |U \cap \Delta'_{\rightarrow}| + |L \cap \Delta_{\rightarrow}| + |U \cap \Delta_{\downarrow}| + |L| - |L \cap \Delta'_{\downarrow}| - |B| + \delta + \delta' \\
&\qquad + |M \cap \Delta_{\rightarrow}| + |M \cap \Delta_{\downarrow}| \\
&= |U| - |U \cap \Delta'_{\rightarrow}| + |L \cap \Delta_{\rightarrow}| + |U \cap \Delta_{\downarrow}||L| - |L \cap \Delta'_{\downarrow}| - |B| + \delta + \delta' \\
&\qquad + |M| - |M \cap \Delta'_{\rightarrow}| + |M| - |M \cap \Delta'_{\downarrow}| \\
&= -|U \cap \Delta'_{\rightarrow}| + |L \cap \Delta_{\rightarrow}| + |U \cap \Delta_{\downarrow}| - |L \cap \Delta'_{\downarrow}| + \delta + \delta' - |M \cap \Delta'_{\rightarrow}| \\
&\qquad + |M| - |M \cap \Delta'_{\downarrow}| \\
&= 1 + (\mathbf{v}^\ell_{k+1} - \mathbf{v}^\ell_k) + (\mathbf{v}^\ell_{k-1} - \mathbf{v}^\ell_k),
\end{aligned}$$

which completes the proof. ∎

With the operators $\mathbf{E}^\ell_k$, $\mathbf{F}^\ell_k$ and $\mathbf{H}^\ell_k$, we are able to give the following partial result towards our goal of giving a geometric realization of $V_{\text{basic}}$.

**Proposition 5.16.** *The geometric diagonal Chevalley operators preserve the space* $\mathbf{A}$. *Moreover, when* $r_\ell \geq 2$, *the operators* $\mathbf{E}_k^\ell$, $\mathbf{F}_k^\ell$ *and* $\mathbf{H}_k^\ell$ *satisfy the Kac-Moody relations for* $\widehat{\mathfrak{sl}}_{r_\ell}$. *In particular, the mapping*

$$E_k \mapsto \mathbf{E}_k^\ell, \quad F_k \mapsto \mathbf{F}_k^\ell, \quad H_k \mapsto \mathbf{H}_k^\ell, \tag{5.10}$$

*for* $k = 0, 1, \ldots, r_\ell - 1$, *defines a representation of* $\widehat{\mathfrak{sl}}_{r_\ell}$ *on* $\mathbf{A}$ *and the linear isomorphism determined by* $[\mathbf{I}] \mapsto \mathbf{I}$, *is an isomorphism of* $\widehat{\mathfrak{sl}}_{r_\ell}$*-representations* $\mathbf{A} \to \mathbb{F}$. *The mapping* (5.10) *together with*

$$I \otimes t^n \mapsto |n| r_\ell \mathbf{P}_\ell(n r_\ell), \quad \text{and} \quad I \otimes 1 \mapsto \mathbf{P}_\ell(0),$$

*for* $n \in \mathbb{Z} - \{0\}$, *defines a representation of* $\widehat{\mathfrak{gl}}_{r_\ell}$ *on* $\mathbf{A}$ *and the mapping* $[\mathbf{I}] \mapsto \mathbf{I}$ *is then an isomorphism of* $\widehat{\mathfrak{gl}}_{r_\ell}$*-representations* $\mathbf{A} \to \mathbb{F}$.

*When* $r_\ell = 1$, *the mapping*

$$I \otimes t^n \mapsto |n| \mathbf{P}_\ell(n), \quad I \otimes 1 \mapsto \mathbf{P}_\ell(0), \quad c \mapsto \mathrm{id}_{\mathbf{A}},$$

*for all* $n \in \mathbb{Z} - \{0\}$, *defines a representation of* $\widehat{\mathfrak{gl}}_1$ *on* $\mathbf{A}$ *and the mapping* $[\mathbf{I}] \mapsto \mathbf{I}$ *is an isomorphism of* $\widehat{\mathfrak{gl}}_1$*-representations* $\mathbf{A} \to \mathbb{F}$.

**Proof:** The fact that the geometric diagonal Chevalley operators preserve $\mathbf{A}$ is clear from Corollary 5.9. By Theorem 5.5, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathbb{F} \\ {\scriptstyle \mathbf{\Psi}_\ell(i), \mathbf{\Psi}_\ell^*(i)} \downarrow & & \downarrow {\scriptstyle \psi_\ell(i), \psi_\ell^*(i),} \\ \mathbf{A} & \longrightarrow & \mathbb{F} \end{array} \tag{5.11}$$

for each $\ell = 1, \ldots, s$ and $i \in \mathbb{Z}$. Fix a value of $\ell \in \{1, \ldots, s\}$ such that $r_\ell \geq 2$. For each $k = 0, 1, \ldots, r_\ell - 1$, let

$$E_k^\ell = E_{r_1 + \cdots + r_{\ell-1} + k}, \quad F_k^\ell = F_{r_1 + \cdots + r_{\ell-1} + k}, \quad H_k^\ell = H_{r_1 + \cdots + r_{\ell-1} + k}.$$

Then by Lemma 1.21 and Theorem 5.14, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathbb{F} \\ {\scriptstyle \mathbf{E}_k^\ell, \mathbf{F}_k^\ell} \downarrow & & \downarrow {\scriptstyle E_k^\ell, F_k^\ell,} \\ \mathbf{A} & \longrightarrow & \mathbb{F} \end{array}$$

for all $k = 0, 1, \ldots, r_\ell - 1$. By Proposition 5.15, we therefore also have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathbb{F} \\ {\scriptstyle \mathbf{H}_k^\ell} \downarrow & & \downarrow {\scriptstyle H_k^\ell.} \\ \mathbf{A} & \longrightarrow & \mathbb{F} \end{array}$$

Thus, when $r_\ell \geq 2$, we have a representation of $\widehat{\mathfrak{sl}}_{r_\ell}$ on $\mathbf{A}$ and the map $\mathbf{A} \to \mathbb{F}$ is an isomorphism of $\widehat{\mathfrak{sl}}_{r_\ell}$-representations. The fact that the addition of the mapping

$$I \otimes t^n \mapsto |n| r_\ell \mathbf{P}_\ell(n r_\ell), \quad I \otimes 1 \mapsto \mathbf{P}_\ell(0),$$

gives us a representation of $\widehat{\mathfrak{gl}}_{r_\ell}$ on $\mathbf{A}$ follows from Lemma 1.21 and Theorem 5.3. The case where $r_\ell = 1$ is obvious. ∎

Proposition 5.16 shows that the operators $\mathbf{E}_k^\ell$, $\mathbf{F}_k^\ell$, $\mathbf{H}_k^\ell$ and $\mathbf{P}_\ell(n r_\ell)$ on $\mathbf{A}$ correspond to the operators associated to the $(\ell, \ell)$-th diagonal blocks of $\widehat{\mathfrak{gl}}_r$ on $\mathbb{F}$. Thus, as was the case algebraically, to complete our realization of $\widehat{\mathfrak{gl}}_r$, it remains only to construct the operators associated to the off-diagonal blocks.

Let $\mathbf{c} \in \mathbb{Z}^s$ and $i \neq j \in \{1, \ldots, s\}$. Suppose $\mathbf{n}, \mathbf{m} \in \mathbb{N}^s$ such that $\mathbf{n}_\ell = \mathbf{m}_\ell$ for all $\ell \neq i, j$. By equations (5.1) and (5.2), we have that

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}, \mathbf{c}+\mathbf{1}_i - \mathbf{1}_j}(\mathbf{n}, \mathbf{m})) = c_{|\mathbf{n}|+|\mathbf{m}|}(\mathcal{K}_{\mathbf{c}, \mathbf{c}+\mathbf{1}_i - \mathbf{1}_j}(\mathbf{n}, \mathbf{m})).$$

For $i = 1, \ldots, s-1$ and $k \in \mathbb{Z}$, define

$$\mathcal{K}_{\mathbf{c}}^+(\mathbf{n}; i, k) := \mathcal{K}_{\mathbf{c}, \mathbf{c}+\mathbf{1}_i - \mathbf{1}_{i+1}}(\mathbf{n}, \mathbf{n} + ((k+1) r_i - \mathbf{c}_i - 1)\mathbf{1}_i - (k r_{i+1} - \mathbf{c}_{i+1} + 1)\mathbf{1}_{i+1}),$$

$$\mathcal{K}^-_{\mathbf{c}}(\mathbf{n};i,k) := \mathcal{K}_{\mathbf{c},\mathbf{c}+\mathbf{1}_{i+1}-\mathbf{1}_i}(\mathbf{n},\mathbf{n}+((k-1)r_{i+1}-\mathbf{c}_{i+1})\mathbf{1}_{i+1}-(kr_i-\mathbf{c}_i)\mathbf{1}_i).$$

We also define

$$\mathcal{K}^+_{\mathbf{c}}(\mathbf{n};0,k) := \mathcal{K}_{\mathbf{c},\mathbf{c}+\mathbf{1}_s-\mathbf{1}_1}(\mathbf{n},\mathbf{n}+(kr_s-\mathbf{c}_s-1)\mathbf{1}_s-(kr_1-\mathbf{c}_1+1)\mathbf{1}_1),$$

$$\mathcal{K}^-_{\mathbf{c}}(\mathbf{n};0,k) := \mathcal{K}_{\mathbf{c},\mathbf{c}+\mathbf{1}_1-\mathbf{1}_s}(\mathbf{n},\mathbf{n}+(kr_1-\mathbf{c}_1)\mathbf{1}_1-(kr_s-\mathbf{c}_s)\mathbf{1}_s).$$

**Definition 5.17** (Geometric off-diagonal Chevalley operators)**.** For $i=0,1,\dots,s-1$, define operators

$$\mathbf{E}'_i,\mathbf{F}'_i,\mathbf{H}'_i : \bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})) \to \bigoplus_{\mathbf{n},\mathbf{c}} \mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n})),$$

by

$$\mathbf{E}'_i|_{\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))} = (-1)^{\mathbf{c}_i}\sum_{k\in\mathbb{Z}} c_{\mathrm{tnv}}(\mathcal{K}^+_{\mathbf{c}}(\mathbf{n};i,k)),$$
$$\mathbf{F}'_i|_{\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))} = (-1)^{\mathbf{c}_i}\sum_{k\in\mathbb{Z}} c_{\mathrm{tnv}}(\mathcal{K}^-_{\mathbf{c}}(\mathbf{n};i,k)),$$

for $i=1,\dots,s-1$, and

$$\mathbf{E}'_0|_{\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))} = (-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{s-1}}\sum_{k\in\mathbb{Z}} c_{\mathrm{tnv}}(\mathcal{K}^+_{\mathbf{c}}(\mathbf{n};0,k)),$$

$$\mathbf{F}'_0|_{\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_{\mathbf{c}}(\mathbf{n}))} = (-1)^{\mathbf{c}_1+\cdots+\mathbf{c}_{s-1}}\sum_{k\in\mathbb{Z}} c_{\mathrm{tnv}}(\mathcal{K}^-_{\mathbf{c}}(\mathbf{n};0,k)),$$

and finally, $\mathbf{H}'_i := [\mathbf{E}'_i,\mathbf{F}'_i]$, for all $i=0,\dots,s-1$.

**Remark 5.18.** For the diagonal Chevalley operators, the space $\mathbf{A}_{\mathbf{c}}(\mathbf{v}^1,\dots,\mathbf{v}^s)$ is a subspace of the the $\mathbf{v}^\ell$-weight space of $\widehat{\mathfrak{gl}}_{r_\ell}$, thus allowing us to define the operators $\mathbf{H}^\ell_k$ explicitly in terms of the $\mathbf{v}^\ell$. For the off-diagonal Chevalley operators, the $\mathbf{v}^\ell$ no longer encode information about the weights of $\widehat{\mathfrak{gl}}_r$. Hence, our defining $\mathbf{H}'_i$ simply as the commutator of $\mathbf{E}'_i$ and $\mathbf{F}'_i$ seems the best we can achieve.

**Lemma 5.19.** *As operators on* $\mathbf{A}$*,*

$$\mathbf{E}'_i = \sum_{k\in\mathbb{Z}} \mathbf{\Psi}_i((k+1)r_i)\mathbf{\Psi}^*_{i+1}(kr_{i+1}+1), \quad \mathbf{F}'_i = \sum_{k\in\mathbb{Z}} \mathbf{\Psi}_{i+1}((k-1)r_{i+1}+1)\mathbf{\Psi}^*_i(kr_i),$$

*for* $i = 1, \dots, s-1$, *and*

$$\mathbf{E}'_0 = \sum_{k\in\mathbb{Z}} \mathbf{\Psi}_s(kr_s)\mathbf{\Psi}^*_1(kr_1+1), \quad \mathbf{F}'_0 = \sum_{k\in\mathbb{Z}} \mathbf{\Psi}_1(kr_1+1)\mathbf{\Psi}^*_s(kr_s).$$

**Proof:** Fix $i \in \{1, \dots, s-1\}$. Consider the restriction of $\mathbf{E}'_i$ to $\mathcal{H}^{2|\mathbf{n}|}_{T_\star}(\mathcal{M}_\mathbf{c}(\mathbf{n}))$. To simplify notation, let $\mathbf{d} = \mathbf{c} + \mathbf{1}_i - \mathbf{1}_{i+1}$ and $\mathbf{m}^k = \mathbf{n} + ((k+1)r_i - \mathbf{c}_i - 1)\mathbf{1}_i - (kr_{i+1} - \mathbf{c}_{i+1} + 1)\mathbf{1}_{i+1}$. Then

$$\begin{aligned}\mathbf{E}'_i &= (-1)^{\mathbf{c}_i} \sum_{k\in\mathbb{Z}} c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{d}}(\mathbf{n},\mathbf{m}^k)) = (-1)^{\mathbf{c}_i} \sum_{k\in\mathbb{Z}} c_{\text{tnv}}\left(\bigoplus_{\ell=1}^{s} f^*_\ell \mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1,\mathbf{n}_\ell,(\mathbf{m}^k)_\ell)\right)\\ &= (-1)^{\mathbf{c}_i} \sum_{k\in\mathbb{Z}} \left(\prod_{\ell=1}^{s} (1^{\otimes \ell-1} \otimes c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1,\mathbf{n}_\ell,(\mathbf{m}^k)_\ell)) \otimes 1^{\otimes s-\ell})\right),\end{aligned}$$

where the second equality follows from Lemma 4.14 and the third equality is the application of the Künneth formula. For $\ell \neq i, i+1$,

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{d}_\ell}(1,\mathbf{n}_\ell,(\mathbf{m}^k)_\ell)) = c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_\ell,\mathbf{c}_\ell}(1,\mathbf{n}_\ell,\mathbf{n}_\ell)) = 1,$$

by [15, Lemma 3.10]. Thus,

$$\begin{aligned}\mathbf{E}'_i = (-1)^{\mathbf{c}_i} \sum_{k\in\mathbb{Z}} &\left(1^{\otimes i-1} \otimes c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_i,\mathbf{d}_i}(1,\mathbf{n}_i,(\mathbf{m}^k)_i)) \otimes 1^{\otimes s-i}\right) \times\\ &\left(1^{\otimes i} \otimes c_{\text{tnv}}(\mathcal{K}_{\mathbf{c}_{i+1},\mathbf{d}_{i+1}}(1,\mathbf{n}_{i+1},(\mathbf{m}^k)_{i+1})) \otimes 1^{\otimes s-i-1}\right)\\ = \sum_{k\in\mathbb{Z}} &\mathbf{\Psi}_i((k+1)r_i)\mathbf{\Psi}^*_{i+1}(kr_{i+1}+1).\end{aligned}$$

The remaining equalities can be proved in an analogous manner. ∎

For $k = 0, 1, \dots, r$, we can write

$$k = r_1 + \dots + r_{\ell-1} + k',$$

for unique $1 \le \ell \le s$ and $0 \le k' \le r_\ell - 1$. For all $k$ such that $k' \neq 0$, let

$$\mathbf{E}_k := \mathbf{E}^\ell_{k'}, \quad \mathbf{F}_k := \mathbf{F}^\ell_{k'}, \quad \mathbf{H}_k := \mathbf{H}^\ell_{k'}.$$

For all $k$ such that $\ell \neq 1$ and $k' = 0$, let

$$\mathbf{E}_k := \mathbf{E}'_{\ell-1}, \quad \mathbf{F}_k := \mathbf{F}'_{\ell-1}, \quad \mathbf{H}_k := \mathbf{H}'_{\ell-1}.$$

For $k = 0$,

$$\mathbf{E}_0 := \mathbf{E}'_0, \quad \mathbf{F}_0 := \mathbf{F}'_0, \quad \mathbf{H}_0 = \mathbf{H}'_0.$$

**Theorem 5.20.** *The operators $\mathbf{E}_k$, $\mathbf{F}_k$ and $\mathbf{H}_k$, $k = 0, 1, \ldots, s-1$, preserve the space $\mathbf{A}$ and satisfy the Kac-Moody relations for $\widehat{\mathfrak{sl}}_r$. In particular, the mapping*

$$E_k \mapsto \mathbf{E}_k, \quad F_k \mapsto \mathbf{F}_k, \quad H_k \mapsto \mathbf{H}_k,$$

*defines a representation of $\widehat{\mathfrak{sl}}_r$ on $\mathbf{A}$ and the linear map determined by $[\mathbf{I}] \to \mathbf{I}$ is an isomorphism of $\widehat{\mathfrak{sl}}_r$-representations $\mathbf{A} \mapsto \mathbb{F}$.*

**Proof:** This follows from the commutativity of Diagram (5.11) and comparison of Theorem 5.14 and Lemma 5.19 with Lemma 1.21. ∎

By Theorem 5.20, we have a geometric version of $\widehat{\mathfrak{sl}}_r$. As before, to get a similar geometric version of $\widehat{\mathfrak{gl}}_r$, we need to add operators corresponding to the loops on the identity. This leads us to our main theorem, from which we conclude that $\mathbf{A}(0)$ is a geometric version of $V_{\text{basic}}$.

**Theorem 5.21.** *For $k = 0, 1, \ldots, s-1$, and $n \in \mathbb{Z} - \{0\}$, the mapping*

$$E_k \mapsto \mathbf{E}_k, \quad F_k \mapsto \mathbf{F}_k, \quad H_k \mapsto \mathbf{H}_k, \quad I \otimes t^n \mapsto |n| \sum_{\ell=1}^{s} r_\ell \mathbf{P}_\ell(nr_\ell), \quad I \otimes 1 \mapsto \sum_{\ell=1}^{s} \mathbf{P}_\ell(0),$$

*defines a representation of $\widehat{\mathfrak{gl}}_r$ on the space $\mathbf{A}$ and the linear map determined by $[\mathbf{I}] \mapsto \mathbf{I}$ is an isomorphism of $\widehat{\mathfrak{gl}}_r$-representations $\mathbf{A} \to \mathbb{F}$. This isomorphism maps $\mathbf{A}(0) \to \mathbb{F}(0)$, and thus $\mathbf{A}(0) \cong V_{\text{basic}}$.*

**Proof:** From Theorem 5.20, the mapping

$$E_k \mapsto \mathbf{E}_k, \quad F_k \mapsto \mathbf{F}_k, \quad H_k \mapsto \mathbf{H}_k,$$

gives us a representation of $\widehat{\mathfrak{sl}}_r$ on $\mathbf{A}$. The additional mapping

$$I \otimes t^n \mapsto |n| \sum_{\ell=1}^{s} r_\ell \mathbf{P}_\ell(nr_\ell), \quad I \otimes 1 \mapsto \sum_{\ell=1}^{s} \mathbf{P}_\ell(0),$$

gives us a representation of $\widehat{\mathfrak{gl}}_r$ on $\mathbf{A}$ by Lemma 1.21. ∎

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